Homework 4

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Question 9:

A. Exercise 4.1.3

(b) $f(x) = 1/(x^2 - 4)$

The function f is not well-defined for $x = 2$ and x = -2.

$\therefore f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 1/(x^2 - 4)$ is not a function.

(c) $f(x) = \sqrt{x^2}$

The function f has a well-defined value for every real number x.

$\therefore f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sqrt{x^2}$ is a function.

Range of f = $[0, \infty)$

B. Exercise 4.1.5

(b) Let A = {2, 3, 4, 5}. f: A → $\mathbb{Z}$ such that $f(x) = x^2$.

$\therefore$ Range of f = {4, 9, 16, 25}

(d) f: $\{0, 1\}^5 \rightarrow \mathbb{Z}$. For x $\in \{0, 1\}^5$, f(x) is the number of 1's that occur in x. For example f(01101) = 3, because there are three 1's in the string "01101".

$\therefore$ Range of f = {0, 1, 2, 3, 4, 5} since a 5-bit number can possibly have number of 1's ranging from 0 to 5.

(h) Let $A = \{1, 2, 3\}$. $f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x,y) = (y, x)$.

$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$\therefore \text{Range of } f = \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)\}$$

(i) Let $A = \{1, 2, 3\}$. $f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x,y) = (x,y+1)$.

$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$\therefore$ Range of $f = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

(l) Let $A = \{1, 2, 3\}$. $f: P(A) \rightarrow P(A)$. For $X \subseteq A$, $f(X) = X - \{1\}$.

$$P(A) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\} \}$$

$$\therefore \text{Range of } f = \{ \emptyset, \{2\}, \{3\}, \{2, 3\} \}$$

<u>Question 10:</u>

I. A. Exercise 4.2.2

(c) h: $\mathbb{Z} \to \mathbb{Z}$, $h(x) = x^3$

If $x \neq y$, then $x^3 \neq y^3$. Therefore, h is one-to-one. However, h is not onto because there is no integer x such that $x^3 = 5$.

∴ h is **one-to-one, but not onto**.

(g) f: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 1, 2y)$

f is one-to-one since every element in the domain maps to a unique element in the target. However, f is not onto because 2y is even for any integer y ∈ $\mathbb{Z}$. For example, there is no y ∈ $\mathbb{Z}$ such that $2y = 3$.

∴ f is **one-to-one, but not onto**.

(k) f: $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x, y) = 2^x + y$

f is not one-to-one because f(2, 1) = f(1, 3) = 5. Moreover, f is not onto since there is no $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $2^x + y = 1$.

∴ f is **neither one-to-one nor onto**.

I. B. Exercise 4.2.4

(b) $f:\{0,\ 1\}^3 \rightarrow \{0,\ 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.

> f(000) = f(100) = 100. Hence, f is not one-to-one. Let y be an element from the target such that y = 000. But, there is no x in the domain which makes f(x) = y = 000. Thus, f is not onto.

∴ f is **neither one-to-one nor onto**.

(c) $f:\{0,\ 1\}^3 \rightarrow \{0,\ 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example f(011) = 110.

> Every element in the domain $\{0,\ 1\}^3$ maps to a unique value of y in the target. Therefore, f is one-to-one. In addition, f is onto as the range of f is equal to the co-domain of f.

∴ f is **one-to-one and onto**.

(d) $f:\{0,\ 1\}^3 \rightarrow \{0,\ 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, f(100) = 1001.

> Every element in the domain $\{0,\ 1\}^3$ maps to a unique value of y in the target $\{0,\ 1\}^4$. Therefore, f is one-to-one. Let y be an element from the target such that y = 0001. However, there is no x in the domain which makes f(x) = y = 0001. Thus, f is not onto.

∴ f is **one-to-one, but not onto**.

(g) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8} and let B = {1}. f: P(A) → P(A). For X ⊆ A, f(X) = X - B. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

f({1, 3}) = f({3}) = {3}. Therefore, f is not one-to-one. Let Y be a set from the target such that it contains the element 1. But, there is no set X ⊆ A in the domain which makes f(X) = X - B = Y. For instance, when Y = {1, 3}, there exists no X ⊆ A that can make X - B = Y = {1, 3}. Thus, f is not onto.

∴ f is **neither one-to-one nor onto**.

II.

(a) f: $\mathbb{Z} \to \mathbb{Z}^+$, $f(x) = \{ 3x$ if $x > 0$; $3|x| + 1$ if $x \leq 0$

(b) f: $\mathbb{Z} \to \mathbb{Z}^+$, $f(x) = |x| + 1$

(c) f: $\mathbb{Z} \to \mathbb{Z}^+$, $f(x) = \{ 2x$ if $x > 0$; $2|x| + 1$ if $x \leq 0$

(d) f: $\mathbb{Z} \to \mathbb{Z}^+$, $f(x) = x^2 + 1$

Question 11:

A. Exercise 4.3.2

(c) f: $\mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$

Since f is both one-to-one and onto, it is a bijection. Therefore, the function has a well-defined inverse.

$$f^{-1}(x) = \frac{x-3}{2}$$

(d) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8}.

f: P(A) → {0, 1, 2, 3, 4, 5, 6, 7, 8}

For $X \subseteq A$, $f(X) = |X|$.

f({1, 2}) = f({2, 3}) = 2. So, the function f is not one-to-one and hence does not have a well-defined inverse.

(g) f:$\{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.

Since f is both one-to-one and onto, it is a bijection. Therefore, the function has a well-defined inverse.

$$f^{-1} = f = \{0, 1\}^3$$

(i) f: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 5, y - 2)$

Since f is both one-to-one and onto, it is a bijection. Therefore, the function has a well-defined inverse.

$$f^{-1}(x, y) = (x - 5, y + 2)$$

B. Exercise 4.4.8

(c) f o h

$$f \text{ o } h = f(h(x))$$

$$= f(x^2 + 1)$$

$$= 2(x^2 + 1) + 3$$

$$= 2x^2 + 5$$

(d) h o f

$$h \text{ o } f = h(f(x))$$

$$= h(2x + 3)$$

$$= (2x + 3)^2 + 1$$

$$= 4x^2 + 12x + 10$$

C. Exercise 4.4.2

(b) Evaluate (f o h)(52)

$$(f \text{ o } h)(52) = f(h(52))$$

$$= f(\lceil \tfrac{52}{5} \rceil)$$

$$= f(11)$$

$$= 11^2$$

$$= 121$$

(c) Evaluate (g o h o f)(4)

$(g \text{ o } h \text{ o} f)(4) \; = \; g(h(f(4))))$

$= \; g(h(4^2)))$

$= g(h(16))$

$= g(\lceil \frac{16}{5} \rceil)$

$= g(4)$

$= 2^4$

$= 16$

(d) Give a mathematical expression for h o f

$h \text{ o } f \; = \; h(f(x))$

$= \; h(x^2)$

$= \; \lceil \frac{x^2}{5} \rceil$

D. Exercise 4.4.6

(c) $(h \text{ o } f)(010)$

$(h \text{ o } f)(010) \; = \; h(f(010))$

$= h(110)$

$= 111$

(d) What is the range of h o f?

Range of h o f = { 101, 111 }

(e) What is the range of g o f?

Range of g o f = { 001, 101, 011, 111 }

E. Exercise 4.4.4

(c) **No.**

If f: X → Y is not one-to-one, then $\exists$ x_1, $x_2 \in X$: $x_1 \neq$ x2 and $f(x_1) = f(x_2)$. Moreover, $g(f(x_1)) = g(f(x_2))$ and $x_1 \neq x_2$. Therefore g o f is not one-to-one.

(d) **Yes.**

For example,