Homework 5

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<u>Question 8:</u>

A. Exercise 5.1.2

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

Let D be the set of digits, L be the set of letters and S be the set of special characters such that $|D| = 10$, $|L| = 26$ and $|S| = 4$. The set of all allowed characters is $C = D \cup L \cup S$. Since $D \cap L \cap S = \emptyset$, the sum rule can be applied to find the cardinality of C i.e. $|C| = 10 + 26 + 4 = 40$.

Let A_j denote the strings of length j over the alphabet C. By the product rule, $|A_j| = 36^j$. For $j \neq k$, A_j and A_k are disjoint because a string can not have length j and length k at the same time. If the user must select a password of length 7 or 8 or 9, then the sum rule applies:

$$\therefore |A_7 \cup A_8 \cup A_9| = |A_7| + |A_8| + |A_9| = 40^7 + 40^8 + 40^9$$

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

Let D be the set of digits, L be the set of letters and S be the set of special characters such that $|D| = 10$, $|L| = 26$ and $|S| = 4$. The set of all allowed characters is $C = D \cup L \cup S$. Since $D \cap L \cap S = \emptyset$, the sum rule can be applied to find the cardinality of C i.e. $|C| = 10 + 26 + 4 = 40$.

Since the first character cannot be a letter, there are only 14 possible choices for the first character and each of the remaining characters in the string can be any of the 40 characters.

$$\therefore |A_7 \cup A_8 \cup A_9| = |A_7| + |A_8| + |A_9| = (14 \times 40^6) + (14 \times 40^7) + (14 \times 40^8)$$

B. Exercise 5.3.2

(a) The set contains three letters a, b and c to choose from. Since no two consecutive characters can be the same letter, the first character can be filled in 3 ways while the remaining nine characters can only be filled in 2 ways.

$$\therefore \text{Total possible strings} = 3 \times 2^{9}$$

C. Exercise 5.3.3

(b) How many license plate numbers are possible if no digit appears more than once?

There are 10 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit. Each of the remaining letters can be filled in 26 ways.

$$\therefore \text{Total number of license plates} = 3 \times 2^{9}$$

(c) How many license plate numbers are possible if no digit or letter appears more than once?

There are 10 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit. Similarly, there are 26 choices for the first letter, 25 choices for the second letter, 24 choices for the third letter and 23 choices for the fourth letter.

$$\therefore \text{Total number of license plates} = 10 \times 9 \times 8 \times 26 \times 25 \times 24 \times 23$$

D. Exercise 5.2.3

(a) To prove that there exists a bijection between the set of binary strings with 9 bits, B^9, and the set of binary strings with 10 bits that have an even number of 1's, E_{10}, we need to find a function that maps each element of B^9 to a unique element of E_{10}, and vice versa.

Now, let's define the function f: $B^9 \to E_{10}$.

$$\therefore f(b) = \{ b0 \text{ if } b \in E_9;\ b1 \text{ if } b \notin E_9$$

Therefore, for each element of E_{10}, there is exactly one element of B^9 to which it is uniquely mapped. Similarly, for any element in B^9, an element of E_{10} can be uniquely formed by adding either a 0 or a 1 to the end of the element in B^9. This operation - appending either 0 or 1 to create parity - maps each element of B^9 to a unique element of E_{10}. Since there is a unique mapping from all E_{10} to B^9, and from all B^9 to E_{10}, there exists a bijection between B^9 and E_{10}.

(b) Since there is a bijection between B^9 and E_{10}, we know that $|E_{10}| = |B^9|$. Therefore $|B^9| = 2^9$, since each of the 9 bits in B^9 can be filled with one of 2 values (0 or 1).

$$\therefore \text{Therefore, } |E_{10}| = 2^9.$$

Question 9:

A. Exercise 5.4.2

(a) Since the first 3 digits start with either 824 or 825, we have 2 possible choices for the first 3 digits. The last 4 digits can be chosen from any of the 10 digits.

$$\therefore \text{Total possible phone numbers} = 2 \times 10 \times 10 \times 10 \times 10 = 2 \times 10^4.$$

(b) Since the first 3 digits start with either 824 or 825, we have 2 possible choices for the first 3 digits. For the last 4 digits to be chosen without repetition, we have $10 \times 9 \times 8 \times 7$ choices.

$$\therefore \text{Total possible phone numbers} = 2 \times 10 \times 9 \times 8 \times 7.$$

B. Exercise 5.5.3

(a) 2^{10}

(b) 2^7

(c) $2^7 + 2^8$

(d) 2^8

(e) 10 choose 6

(f) 9 choose 6

(g) (5 choose 1) $\times$ (5 choose 3)

C. Exercise 5.5.5

(a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

$$(30 \text{ choose } 10) \times (35 \text{ choose } 10)$$

D. Exercise 5.5.8

(c) There are 13 hearts and 13 diamonds in the deck. Hence, the total number of ways to select five-card hands that are made entirely of hearts and diamonds is,

$$= 26 \text{ choose } 5$$

(d) To select 4 cards of the same rank, we need to select a rank from 13 ranks and pick all 4 cards of that rank. Then, we need to select the rank for the 5th card from the remaining 12 ranks and then select the suit for that 5th card. Therefore, five-card hands with 4 cards of the same rank will be,

$$= 13 \times 48 = 624$$

(e) The number of ways to select 2 ranks from a set of 13 ranks is 13 choose 2. Since, there are 2 ranks in a full house from which we select one, the number of ways to do this is 2 choose 1. As there are four suits for each rank, we have 4 choose 3 ways for the three-card combination and 4 choose 2 ways for the two-card combination. Therefore, the number of five-card hands that contain a full house is,

$$= (13 \text{ choose } 2) \times (2 \text{ choose } 1) \times (4 \text{ choose } 3) \times (4 \text{ choose } 2) = 3744$$

(f) How many five-card hands do not have any two cards of the same rank?

The number of ways to choose 5 different ranks is 13 choose 5. For each rank, there are 4 different cards available to choose from the deck. Therefore the total number of five-card hands that do not have any two cards of the same rank is,

$$= (13 \text{ choose } 5) \times 4^5$$

E. Exercise 5.6.6

(a) How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

In order to select a committee of 10 senate members with the same number of Demonstrators and Repudiators, we choose 5 members from each of the parties. Therefore, the total number of ways is,

$$(44 \text{ choose } 5) \times (56 \text{ choose } 5)$$

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

There are a total of 56 choices to select a speaker and 55 choices to select a vice speaker from the Repudiators party. Similarly, there are a total of 44 choices to select a speaker and 43 choices to select a vice speaker from the Demonstrators party. Therefore, the total number of ways that two speakers and two vice speakers can be selected is,

$$= (56 \text{ choose } 1) \times (55 \text{ choose } 1) \times (44 \text{ choose } 1) \times (43 \text{ choose } 1)$$

Question 10:

A. Exercise 5.7.2

(a) Number of five-card hands having at least one club = Total number of five-card hands combination formed using 52 cards - Number of five-card hands combination formed without a club.

∴ Total number of five-card hands with at least one club is,

$$= (52 \text{ choose } 5) - (39 \text{ choose } 5)$$

(b) Number of five-card hands have at least two cards with the same rank = Total number of five-card hands combination formed using 52 cards - Total number of five-card hands with no same rank.

∴ Total number of five-card hands that have at least two cards with the same rank is,

$$= (52 \text{ choose } 5) - ((13 \text{ choose } 5) \times 4^5)$$

B. Exercise 5.8.4

(a) Here, 20 books can be distributed to 5 kids with each book having 5 ways.

∴ Total number of ways to distribute the comic books = 5^{20}

(b) Since the 20 books are divided evenly among 5 kids such that each kid gets 4 books, we have,

∴ Total number of ways to distribute the comic books = $\frac{20!}{4!4!4!4!4!}$

Question 11:

(a) Since the domain has 5 elements which is one greater than the target with 4 elements, it is not possible to have a one-to-one function since one of the five elements in the domain will either be unmapped or mapped to an already paired element in the target.

(b) Since the domain and the target have equal number of elements, each element in the domain gets mapped to a unique value in the target.

$$\therefore \text{Total possible one-to-one functions} = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

(c) Since the domain has 5 elements and the target has 6 elements, each element in the domain gets mapped to a unique value in the target for a function to be one-to-one.

$$\therefore \text{Total possible one-to-one functions} = 6 \times 5 \times 4 \times 3 \times 2 = 720$$

(d) Since the domain has 5 elements and the target has 7 elements, each element in the domain gets mapped to a unique value in the target for a function to be one-to-one.

$$\therefore \text{Total possible one-to-one functions} = 7 \times 6 \times 5 \times 4 \times 3 = 2520$$

Question 7:

A. Exercise 8.2.2

(b) Let $f(n) = n^3 + 3n^2 + 4$ and $g(n) = n^3$.

Choose $n_0 = 1$ and assume $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{n^3+3n^2+4}{n^3} < \frac{n^3+3n^3+4n^3}{n^3} = 8$$

Choose $c = 8$. Also, $3n^2 < 3n^3$ and $4 < 4n^3$.

Therefore, when $c = 8$, $n_0 = 1$ and $n > 1$, $f(n) = n^3 + 3n^2 + 4 \le 8n^3$

Hence, $f = \theta(n^3)$

B.

We shall prove that $c1g(n) \le \sqrt{7n^2 + 2n - 8} \le c1g(n)$. Let us take $c1 = 1$.

If $n = 1$, $1 \le \sqrt{7 + 2 - 8}$

If $n = 2$, $2 \le \sqrt{28 + 4 - 8}$

Therefore, $c1n \le \sqrt{7n^2 + 2n - 8}$ for $c1 = 1$. Now, let us take $c2 = 6$.

If $n = 1$, $\sqrt{7 + 2 - 8} \le 6$

If $n = 2$, $\sqrt{28 + 4 - 8} \le 12$

Hence, $c1n \le \sqrt{7n^2 + 2n - 8} \le c2n$ for $c1 = 1$, $c2 = 6$, and $n = 1$.

So, $\sqrt{7n^2 + 2n - 8} = \Theta(n)$

C. Exercise 8.3.5

(a) The algorithm sorts all the numbers in the sequence less than p on one side and sorts the numbers greater than p on the other side.

(b)The amount of time i=i+1 and j=j-1 runs depends on the actual numbers in the sequence. i runs when the current value is lesser than P and j runs when the current value is greater than P.

<u>Maximum & Minimum counts for i:</u>

If P is the largest number in the sequence ($p = A_n$), then i will run n times.

If P is the smallest number in the sequence ($p = A_0$), then i will run 0 times.

<u>Maximum & Minimum counts for j:</u>

If P is the smallest number in the sequence ($p = A_0$), then j will run n times.

If P is the largest number in the sequence ($p = A_n$), then j will run 0 times.

(c) The amount of swap operations executed will depend on the actual numbers in the sequence. If the sequence is full of negative numbers and p is 0, then no swap will happen. Similarly, if the sequence is full of positive numbers and p is 0, then no swap will happen. Maximum swap operations that can happen is $\frac{n}{2}$ because one swap operation can put two numbers in the right place.

(d) The time complexity for the algorithm is n because the loops go through the sequence only once. Each number in the sequence is checked only once by the algorithm.

$$\text{Lower Bound} = O(n)$$

(e) The upper bound for the mystery algorithm is O(n).