Homework 2

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<u>Question 5:</u>

A.1. Exercise 1.12.2

(b) $p \to (q \land r)$

$\neg q$

$\therefore \neg p$

1.	$p \to (q \land r)$	Hypothesis
2.	$p \to q$	Simplification, 1
3.	$\neg q$	Hypothesis
4.	$\neg p$	Modus Tollens 2, 3

(e) $p \lor q$

$\neg p \lor r$

$\neg q$

$\therefore r$

1.	$p \lor q$	Hypothesis
2.	$\neg p \lor r$	Hypothesis
3.	$q \lor r$	Resolution 1, 2
4.	$\neg q$	Hypothesis
5.	r	Disjunctive Syllogism 3, 4

A.2. Exercise 1.12.3

(c) $p \lor q$

$\neg p$

$\therefore q$

1.	$p \lor q$	Hypothesis
2.	$\neg\neg p \lor q$	Double Negation Law, 1
3.	$\neg p \to q$	Conditional Identity, 2
4.	$\neg p$	Hypothesis
5.	q	Modus Ponens 3, 4

A.3. Exercise 1.12.5

(c) I will buy a new car and a new house only if I get a job.

I am not going to get a job.

∴ I will not buy a new car.

Define the propositional variables as,

c: I will buy a new car
h: I will buy a new house
j: I will get a job

Thus, the corresponding form on the argument is given by,

$(c \land h) \to j$

$\neg j$

$\therefore \neg c$

This argument is **invalid** because when c = T, and h = j = F, both the hypotheses are true and the conclusion is false.

(d) I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.

∴ I will not buy a new car.

Define the propositional variables as,

c: I will buy a new car
h: I will buy a new house
j: I will get a job

Thus, the corresponding form on the argument is given by,

$(c \wedge h) \rightarrow j$
$\neg j$
h

$\therefore \neg c$

1.	$(c \wedge h) \rightarrow j$	Hypothesis
2.	$\neg j$	Hypothesis
3.	$\neg(c \wedge h)$	Modus Tollens 1, 2
4.	$\neg c \vee \neg h$	De Morgan's Law, 3
5.	$\neg h \vee \neg c$	Commutative Law, 4
6.	h	Hypothesis
7.	$\neg\neg h$	Double Negation Law, 6
8.	$\neg c$	Disjunctive Syllogism 5, 7

Hence, the given argument is **valid**.

Question 5:

B.1. Exercise 1.13.3

(b) $\exists x(P(x) \lor Q(x))$

$\exists x \neg Q(x)$

$\therefore \exists x P(x)$

To show that the given argument is invalid, both the hypotheses must the true and the conclusion must be false.

	P(x)	**Q(x)**
a	F	T
b	F	F

$\exists x(P(x) \lor Q(x))$ is true because when x = a, $P(a) \lor Q(a)$ evaluates to true. Similarly, $\exists x \neg Q(x)$ is also true because when x = b, $\neg Q(b)$ evaluates to true. Finally, the conclusion $\exists x P(x)$ is false as there is no value of x that makes $P(x)$ true. Hence, the given argument is invalid.

B.2. Exercise 1.13.5

(d) Every student who missed class got a detention.

Penelope is a student in the class.

Penelope did not miss class.

$\therefore$ Penelope did not get a detention.

Define the predicates as follows,

D(x): x got a detention

M(x): x missed class

Thus, the corresponding form on the argument is given by,

$\forall x(M(x) \rightarrow D(x))$
Penelope is a student in the class
$\neg M(Penelope)$

$\therefore \neg D(Penelope)$

Since Penelope is a student in the class, when $M(Penelope) = \text{F}$ and $D(Penelope) = \text{T}$, all the hypotheses become true and the conclusion is false. Hence, the given argument is invalid.

(d) Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

∴ Penelope did not get a detention.

Define the predicates as follows,

D(x): x got a detention
M(x): x missed class
A(x): x got an A

Thus, the corresponding form on the argument is given by,

$\forall x((M(x) \lor D(x)) \rightarrow \neg A(x))$
Penelope is a student in the class
$A(Penelope)$

$\therefore \neg D(Penelope)$

Applying the rules of inference and the laws of propositional logic to the above argument,

1.	$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$	Hypothesis
2.	Penelope is a student in the class	Hypothesis
3.	$(M(Penelope) \vee D(Penelope)) \rightarrow \neg A(Penelope)$	Universal Instantiation 1, 2
4.	$A(Penelope)$	Hypothesis
5.	$\neg\neg A(Penelope)$	Double Negation Law, 4
6.	$\neg(M(Penelope) \vee D(Penelope))$	Modus Tollens 3, 5
7.	$\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan's Law, 6
8.	$\neg D(Penelope) \wedge \neg M(Penelope)$	Commutative Law, 7
9.	$\neg D(Penelope)$	Simplification, 8

Therefore, the given argument is valid.

Question 6:

Exercise 2.4.1

(d) The product of two odd integers is an odd integer.

Theorem: The product of two odd integers is an odd integer.

Proof: *Direct Proof.* Let x and y be two odd integers. We shall prove that xy is an odd integer.

Since x is odd, there is an integer k such that $x = 2k + 1$. Since y is also even, there is an integer j such that $y = 2j + 1$.

Substituting the values of x and y in xy results in,

$$xy = (2k + 1)(2j + 1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$$

Since k and j are integers, the expression $2kj + k + j$ is also an integer.

Since xy is of the form $2m + 1$ where $m = 2kj + k + j$ is an integer, xy is an odd integer. ■

Exercise 2.4.3

(b) If x is a real number and x ≤ 3, then $12 - 7x + x^2 \geq 0$.

Theorem: If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Proof: *Direct Proof.* Suppose that x is a real number such that $x \leq 3$. We shall prove that $12 - 7x + x^2 \geq 0$.

Subtract x from both sides of the inequality $x \leq 3$,

$$(x - x) \leq (3 - x)$$
$$0 \leq (3 - x)$$

We can rewrite the above equation as $(3 - x) \geq 0$. Since $4 - x$ is one larger than $3 - x$, then

$$(4 - x) > (3 - x) \geq 0$$

Since $(4 - x) \geq 0$ and $(3 - x) \geq 0$, their product is also greater than or equal to 0.

$$(4 - x)(3 - x) \geq 0$$

Multiplying out the two terms gives us,

$$(12 - 7x + x^2) \geq 0 \blacksquare$$

Question 7:

Exercise 2.5.1

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Theorem: For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof: *Proof by Contrapositive.* We assume that n is an even integer and prove that $n^2 - 2n + 7$ is odd.

Since n is an even integer, there is an integer k such that $n = 2k$.

Substituting the value of n into the equation $n^2 - 2n + 7$ gives us,

$$n^2 - 2n + 7 = (2k)^2 - 2(2k) + 7 = 4k^2 - 4k + 7 = 2(2k^2 - 2k) + 7$$

$$= 2(2k^2 - 2k + 3) + 1$$

Since k is an integer, $2k^2 - 2k$ is also an integer.

Since $n^2 - 2n + 7$ is of the form $2m + 1$ where $m = 2k^2 - 2k + 3$ is an integer, $n^2 - 2n + 7$ is odd. ■

Exercise 2.5.4

(a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Theorem: For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$

Proof: *Proof by Contrapositive.* We assume that for every pair of real numbers x and y, $x > y$ and show that $x^3 + xy^2 > x^2y + y^3$.

Since x and y are real numbers and $x > y$, either x or y is non-zero. Therefore, $x^2 + y^2$ is a real number such that $x^2 + y^2 \neq 0$ and $x^2 + y^2 > 0$.
Multiplying both sides of the inequality $x > y$ by $x^2 + y^2$ gives us,

$$x(x^2 + y^2) > y(x^2 + y^2)$$
$$x^3 + xy^2 > x^2y + y^3 \blacksquare$$

(b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Theorem: For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Proof: *Proof by Contrapositive.* We assume that for every pair of real numbers x and y, $x \leq 10$ and $y \leq 10$ and show that $x + y \leq 20$.

Since $x \leq 10$ and $y \leq 10$, we can add both the inequalities to get,

$$x + y \leq 20 \blacksquare$$

Exercise 2.5.5

(c) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Theorem: For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Proof: *Proof by Contrapositive.* Let x be a non-zero real number. We assume that $\frac{1}{x}$ is rational and show that x must be rational.

Every real number is either rational or irrational. Since $\frac{1}{x}$ is rational and $x \neq 0$, there exists two integers a and b such that,

$$\frac{1}{x} = \frac{a}{b} \; where \; b \neq 0$$

We know that $\frac{1}{x}$ cannot be 0 since $1 \neq 0. x$. Hence, a $\neq 0$.
Taking the reciprocal of the above equation on both sides,

$$x = \frac{b}{a}$$

Since $x = \frac{b}{a}$ and a $\neq 0$, it can be written as the ratio of two integers with a non-zero denominator. Thus, x is rational. ■

Question 8:

Exercise 2.6.6

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

Theorem: The average of three real numbers is greater than or equal to at least one of the numbers.

Proof: *Proof by Contradiction*. Let x, y and z be three real numbers. We assume that the average of three real numbers is less than all the three numbers.

$$\frac{(x+y+z)}{3} < x, \frac{(x+y+z)}{3} < y, \frac{(x+y+z)}{3} < z$$

Adding these three equations together yields,

$$\frac{3(x+y+z)}{3} < (x + y + z)$$

$$(x + y + z) < (x + y + z)$$

Since the sum of three real numbers cannot be less than their sum, this contradicts our assumption that the average of three real numbers is less than all the three numbers. Thus, the average of three real numbers is greater than or equal to at least one of the numbers. ■

(d) There is no smallest integer.

Theorem: There is no smallest integer.

Proof: *Proof by Contradiction*. We assume that there exists a smallest integer x.

Since x is an integer, there exists another integer $x - 1$ which is one lesser than the value of x.

$$(x - 1) < x$$

This contradicts our assumption that x is the smallest integer. Hence, we can conclude that there is no smallest integer. ■

Question 9:

Exercise 2.7.2

(b) If integers x and y have the same parity, then $x+y$ is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Theorem: If integers x and y have the same parity, then $x+y$ is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Proof: *Proof by Cases*.

Case 1: x and y are even integers.

Since x and y are even integers,

$$x = 2k \text{ for some integer } k$$
$$y = 2m \text{ for some integer } m$$

Adding x and y gives us,

$$x + y = 2k + 2m = 2(k + m)$$

Since k and m are integers, $k + m$ is also an integer. Therefore, $x + y$ is of the form 2n where $n = k + m$. Thus, $x + y$ is even.

Case 2: x and y are odd integers.

Since x and y are odd integers,

$$x = 2p + 1 \text{ for some integer } p$$
$$y = 2q + 1 \text{ for some integer } q$$

Adding x and y gives us,

$$x + y = 2p + 1 + 2q + 1 = 2p + 2q + 2 = 2(p + q + 1)$$

Since p and q are integers, $p + q + 1$ is also an integer. Therefore, $x + y$ is of the form $2l$ where $l = p + q + 1$. Thus, $x + y$ is even. ■