Master's theorem

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$$T(n) = n^3 + 2T\left(\frac{n}{4}\right)$$

$$T(n) = n^d + a \cdot T\left(\frac{n}{b}\right) \qquad a, b, d$$

$$= n^d + a \cdot \left[\frac{n^d}{b^d} + a \cdot T\left(\frac{n}{b^2}\right)\right]$$

$$T(n) = n^d + \frac{a}{b^d} \cdot n^d + a^2 T\left(\frac{n}{b^2}\right)$$

$$T(n) = n^d + \frac{a}{b^d} \cdot n^d + a^2\left(\frac{n^d}{b^{d2}} + a\,T\left(\frac{n}{b^3}\right)\right)$$

$$T(n) = n^d + \frac{a}{b^d} \cdot n^d + \left(\frac{a}{b^d}\right)^2 \cdot n^d + a^3 T\left(\frac{n}{b^3}\right)$$

$$T(n) = n^d\left(1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \left(\frac{a}{b^d}\right)^3 + \dots + \left(\frac{a}{b^d}\right)^{k-1}\right) + a^k T\left(\frac{n}{b^k}\right)$$

STOP WHEN $n = b^k$ or $k = \log_b n$

$$T(n) = n^d\left(1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \dots + \left(\frac{a}{b^d}\right)^{\log_b n - 1}\right) + \cancel{a^{\log_b n}}\; n^{\log_b a}$$

$\boxed{\text{Compare } d \text{ to } \log_b a}$

1) $d = \log_b a$, $\quad b^d = a \qquad \frac{a}{b^d} = 1$

$$T(n) = n^d \cdot \log_b n + n^d \in \Theta(n^d \log n)$$

2) $d < \log_b a$, $\quad b^d < a \quad$ or $\quad 1 < \frac{a}{b^d}$

$$n^d \cdot \left(\frac{\left(\frac{a}{b^d}\right)^{\log_b n} - 1}{\frac{a}{b^d} - 1}\right) = n^d \cdot \frac{\left(\frac{a^{\log_b n}}{n^d} - 1\right)}{c}$$

$$b^{d \cdot \log_b n} = \left(b^{\log_b n}\right)^d = n^d$$

$$T(n) = \frac{a^{\log_b n} - n^d}{c} + n^d$$

$$T(n) = \frac{1}{c} n^{\log_b a} + n^d - \frac{n^d}{c} \in \Theta(n^{\log_b a})$$

3) $d > \log_b a$, $\quad b^d > a$, $\quad \boxed{\frac{a}{b^d} < 1}$

$$T(n) = c \cdot n^d + n^{\log_b a} \in \Theta(n^d)$$

$$\boxed{1 + r + r^2 + \dots = \frac{1}{1-r}}$$

$$T(n) = n^d + a \cdot T\left(\frac{n}{b}\right)$$ Master's ~~theorem~~

compare d versus $\log_b a$

1. $d > \log_b a$, $T(n) \in \Theta(n^d)$
2. $d < \log_b a$, $T(n) \in \Theta(n^{\log_b a})$
3. $d = \log_b a$, $T(n) \in \Theta(n^d \cdot \log n)$

$$T(n) = n + 2T\left(\frac{n}{2}\right) \in \Theta(n \cdot \log n)$$

$1 \quad \log_2 2$

$$T(n) = n + 4T\left(\frac{n}{4}\right) \in \Theta(n \log n)$$

$1 \quad \log_4 4$

$$T(n) = n^2 + 4T\left(\frac{n}{2}\right) \in \Theta(n^2 \log n)$$

$2 = \log_2 4$

$$T(n) = n^2 + 3T\left(\frac{n}{2}\right) \in \Theta(n^2)$$

$d = 2 > \log_2 3$

$$T(n) = n + 3T\left(\frac{n}{2}\right) \in \Theta(n^{\log_2 3})$$

$1 < \log_2 3$

$$T(n) = 1 + 2T\left(\frac{n}{2}\right) \in \Theta(n)$$

$$0 < \log_2 2$$

$$T(n) = 2T(n-1) + n$$

$$T(n) = 2\left[2T(n-2) + n-1\right] + n$$

$$= 2^2 T(n-2) + 2(n-1) + n$$

$$T(n) = 2^2\left(2T(n-3) + (n-2)\right) + 2(n-1) + n$$

$$= 2^3 T(n-3) + 2^2(n-2) + 2(n-1) + n$$

$$T(n) = 2^k T(n-k) + 2^{k-1}(n-k-1) + \dots + n$$

$$T(n) = 2^n + 2^{n-1} \cdot (n-n+1) + 2^{n-2} \cdot (n-n+2) + \dots$$

$$= 2^n + 2^{n-1} \cdot 1 + 2^{n-2} \cdot 2 + \dots + 2^1 \cdot (n-1) + n$$

$$T(n) \in O(n \cdot 2^n)$$

$$T(n) \in \Omega(2^n)$$