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(1) (a) Let $P_{n_1}(x)$, $P_{n_2}(x)$, $P_{n_3}(x) \in P_n(\mathbb{R})$ and have degrees n_1, n_2 and n_3 respectively.

For $P_n(\mathbb{R})$ to be a Real vector space it must satisfy the following:

→ $P_{n_1}(x) + P_{n_2}(x) = (a_{n_1}x^{n_1} + \dots + a_1x + a_0) + (b_{n_2}x^{n_2} + \dots + b_1x + b_0)$

$= (b_{n_2}x^{n_2} + \dots + b_1x + b_0) + (a_{n_1}x^{n_1} + \dots + a_1x + a_0)$

$= P_{n_2}(x) + P_{n_1}(x)$

Thus, $P_n(\mathbb{R})$ is <u>commutative</u> in Addition

→ $\left(P_{n_1}(x) + P_{n_2}(x)\right) + P_{n_3}(x) = \left[(a_{n_1}x^{n_1} + \dots + a_1x + a_0) + (b_{n_2}x^{n_2} + \dots + b_1x + b_0)\right] + (c_{n_3}x^{n_3} + \dots + c_1x + c_0)$

$= a_{n_1}x^{n_1} + \dots + a_1x + a_0 + \left((b_{n_2}x^{n_2} + \dots + b_1x + b_0) + (c_{n_3}x^{n_3} + \dots + c_1x + c_0)\right)$

$= P_{n_1}(x) + (P_{n_2}(x) + P_{n_3}(x))$

Thus, $P_n(\mathbb{R})$ is <u>Associative</u> in Addition.

→ If $P_{n_2}(x)$ is the additive identity, then,

$P_{n_1}(x) + P_{n_2}(x) = P_{n_2}(x) + P_{n_1}(x) = P_{n_1}(x)$

↳ $(a_{n_1}x^{n_1} + \dots + a_1x + a_0) + (b_{n_2}x^{n_2} + \dots + b_1x + b_0) = (a_{n_1}x^{n_1} + \dots + a_1x + a_0)$

$\Rightarrow b_{n_2}x^{n_2} + \dots + b_1x + b_0 = 0$

Since x is indeterminate and can be anything, ~~the~~ for the polynomial to be equal to 0 always, all the coefficients should be equal to 0.

$\Rightarrow P_{n_2}(x) = b_{n_2}x^{n_2} + \dots + b_1x + b_0$; $[b_i = 0 \ \forall i \in \{0, 1, \dots, n_2\}]$

$\therefore$ <u>Additive identity</u> $= 0$ (Unique)

→ If $P_{n_2}(x)$ is additive inverse of $P_{n_1}(x)$, then

$P_{n_1}(x) + P_{n_2}(x) = 0$.

$\Rightarrow (a_{n_1}x^{n_1} + \dots + a_1x + a_0) + (b_{n_2}x^{n_2} + \dots + b_1x + b_0) = 0$

$\Rightarrow$ For the sum to be zero, all coefficients of $x^0, x^1, \dots$ should be 0. Also, $n_1 = n_2$.

$\therefore a_0 + b_0 = 0$, $a_1 + b_1 = 0$ ... $a_{n_1} + b_{n_1} = 0$

$\Rightarrow b_0 = -a_0$, $b_1 = -a_1$, ... $b_{n_1} = -a_{n_1}$

$\therefore b_i = -a_i \quad \forall i \in \{0, 1, \dots n_1\}$ & $n_1 = n_2$.

$\therefore$ Additive Inverse of $P_{n_1}(x) \Rightarrow -a_{n_1}x^{n_1} - \dots - a_1 x - a_0$. (Unique)

If $\alpha, \beta, \gamma \in \mathbb{R}$, then

$\rightarrow \alpha(\beta \cdot P_{n_1}(x)) = \alpha(\beta a_{n_1} x^{n_1} + \dots + \beta a_1 x + \beta a_0)$

$= \alpha(\beta(a_{n_1} x^{n_1} + \dots + a_1 x + a_0))$

$= \alpha\beta(a_{n_1} x^{n_1} + \dots + a_1 x + a_0)$

$= (\alpha\beta) P_{n_1}(x)$

$\rightarrow$ Let α be the multiplicative identity of $P_{n_1}(x)$.

$\Rightarrow \alpha \cdot P_{n_1}(x) = P_{n_1}(x)$

$\Rightarrow \alpha(a_{n_1}x^{n_1} + \dots + a_1 x + a_0) = (a_{n_1}x^{n_1} + \dots + a_1 x + a_0)$

$\Rightarrow \alpha = 1$

$\therefore$ Multiplicative Inverse of $P_{n_1}(x) \Rightarrow 1$

$\rightarrow (\alpha + \beta) P_{n_1}(x) = (\alpha+\beta)(a_{n_1}x^{n_1} + \dots + a_1 x + a_0)$

$= (\alpha a_{n_1} x^{n_1} + \dots + \alpha a_1 x + \alpha a_0) + (\beta a_{n_1} x^{n_1} + \dots + \beta a_1 x + \beta a_0)$

$= \alpha P_{n_1}(x) + \beta P_{n_1}(x)$.

$\rightarrow \alpha(P_{n_1}(x) + P_{n_2}(x)) = \alpha(a_{n_1}x^{n_1} + \dots + a_1 x + a_0 + b_{n_2}x^{n_2} + \dots + b_1 x + b_0)$

$= (\alpha a_{n_1} x^{n_1} + \dots + \alpha a_1 x + \alpha a_0) + (\alpha b_{n_2} x^{n_2} + \dots \alpha b_1 x + \alpha b_0)$

$= \alpha P_{n_1}(x) + \alpha P_{n_2}(x)$

$\therefore P_n(\mathbb{R})$ satisfies all the conditions of being a Real vector space

$\therefore \boxed{P_n(\mathbb{R}) \text{ is a Real Vector Space}}$

(b) we have $F : P_n(\mathbb{R}) \rightarrow \mathbb{R}$ defined as $F(p(x)) = \frac{d}{dx} p(x) \Big|_{x=0}$

To prove that this is a linear functional, we must show that it satisfies superposition. For $\alpha, \beta \in \mathbb{R}$ and $P_{n_1}(x), P_{n_2}(x) \in P(x)$

$\Rightarrow F(\alpha P_{n_1}(x) + \beta P_{n_2}(x)) = F((\alpha a_{n_1} x^{n_1} + \dots + \alpha a_1 x + \alpha a_0) + (\beta b_{n_2} x^{n_2} + \dots + \beta b_1 x + \beta b_0))$

$= \frac{d}{dx}((\alpha a_{n} x^{n_1} + \dots + \alpha a_1 x + \alpha a_0) + \beta b_{n_2} x^{n_2} + \dots + \beta b_1 x + \beta b_0) \Big|_{x=0}$

$$\Rightarrow \mathcal{F}(\alpha p_{n_1}(x) + \beta p_{n_2}(x)) = (\alpha a_{n_1} n_1 x^{n_1-1} + \dots + \alpha a_1) + (\beta b_{n_2} n_2 x^{n_2-1} \dots + \beta b_1)\Big|_{x=0}$$
$$= \alpha a_1 + \beta b_1$$

Now,
$$\alpha \mathcal{F}(p_{n_1}(x)) = \alpha \mathcal{F}(a_{n_1} x^{n_1} + \dots + a_1 x + a_0)$$
$$= \alpha \frac{d}{dx}(a_{n_1} x^{n_1} + \dots + a_1 x + a_0)\Big|_{x=0}$$
$$= \alpha \{a_{n_1} n_1 x^{n_1-1} + \dots + a_1\}\Big|_{x=0}$$
$$= \alpha a_1$$

$$\beta \mathcal{F}(p_{n_2}(x)) = \beta \mathcal{F}(b_{n_2} x^{n_2} + \dots + b_1 x + b_0)$$
$$= \beta \frac{d}{dx}(b_{n_2} x^{n_2} + \dots + b_1 x + b_0)\Big|_{x=0}$$
$$= \beta (b_{n_2} \cdot n_2 x^{n_2-1} + \dots + b_1)\Big|_{x=0}$$
$$= \beta b_1$$

$$\therefore \mathcal{F}(\alpha p_{n_1}(x) + \beta p_{n_2}(x)) = \alpha \mathcal{F}(p_{n_1}(x)) + \beta \mathcal{F}(p_{n_2}(x))$$

$\therefore$ $\mathcal{F}$ satisfies superposition. $\therefore$ $\boxed{\mathcal{F} \text{ is a linear functional}}$

(c) An inner product representation for the linear functional is

$\boxed{\mathcal{F}(p(x)) = e_2^T \cdot p(x)}$ where $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_1 \times 1}$ and $p_{n_1}(x) = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n_1} \end{pmatrix}_{n_1 \times 1}$

we can evaluate it, and get $\Rightarrow \mathcal{F}(p(x)) = (0\ 1\ 0 \cdots 0)\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n_1} \end{pmatrix}$

$$= a_1$$