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5. To show matrix multiplication is associative, ie,

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$ $\quad A \in \mathbb{R}^{p \times q}$, $B \in \mathbb{R}^{q \times r}$, $C \in \mathbb{R}^{r \times t}$

Now $A \cdot B = S$ $(S \in \mathbb{R}^{p \times r})$ s.t. $S_{ij} = \sum_{k=1}^{q} A_{ik} \cdot B_{kj}$

$\therefore (A \cdot B) \cdot C = S \cdot C = X$ $(X \in \mathbb{R}^{p \times t})$ s.t. $X_{ij} = \sum_{l=1}^{r} \left( \sum_{k=1}^{q} A_{ik} \cdot B_{kl} \right) \cdot C_{lj}$

$$= \sum_{l=1}^{r} \sum_{k=1}^{q} A_{ik} \cdot B_{kl} \cdot C_{lj}$$

Also $B \cdot C = T$ $(T \in \mathbb{R}^{q \times t})$ s.t $T_{ij} = \sum_{k=1}^{r} B_{ik} \cdot C_{kj}$

$\therefore A \cdot (B \cdot C) = A \cdot T = Y$ $(Y \in \mathbb{R}^{p \times t})$ s.t. $Y_{ij} = \sum_{l=1}^{q} A_{il} \sum_{k=1}^{r} B_{lk} C_{kj}$

$$= \sum_{l=1}^{q} \sum_{k=1}^{r} A_{il} \cdot B_{lk} \cdot C_{kj}$$

$\therefore X_{ij} = Y_{ij}$ $\quad \therefore (A \cdot B) \cdot C = A \cdot (B \cdot C)$ $\therefore$ Matrix multiplication is Associative

Now, to show matrix multiplication is not commutative.

$\Rightarrow A \cdot B \neq B \cdot A$

$A \cdot B = X \quad (X \in \mathbb{R}^{p \times r})$ and $B \cdot A = Y \quad (Y \in \mathbb{R}^{q \times q})$ ~~#~~ only when $(r = p)$

If $r \neq p$, then ~~A.B~~ $B \cdot A$ is not even possible.

So lets assume $r = p$.

Also, if $(p = r) \neq q$, then $A \cdot B$ and $B \cdot A$ don't have same dimensions. So they would not be equal.

So let's assume $p = q = r$.

$\therefore A \cdot B = X = \sum_{k=1}^{p} A_{ik} B_{kj}$

$B \cdot A = Y = \sum_{k=1}^{p} B_{ik} \cdot A_{kj}$

$j^{th}$ column of B $\qquad$ $j^{th}$ column of A

$$\Rightarrow \sum_{k=1}^{p} A_{ik} B_{kj} \neq \sum_{k=1}^{p} B_{ik} A_{kj} \quad (\text{usually})$$

$i^{th}$ row of A $\qquad$ $i^{th}$ row of B

$\therefore$ ~~digits~~ $X \neq Y$

$\Rightarrow A \cdot B \neq B \cdot C$ (usually)

$\therefore$ Matrix multiplication is not commutative.

$\Rightarrow$ Now, For $(AB) \cdot C$ to be more computationally efficient than $A \cdot (BC)$

$\Rightarrow$ while calculating $(AB) \cdot C$, Computation complexity $= (q \cdot p \cdot r) \cdot (r \cdot p \cdot t)$

no. of elements

for one element of $A \cdot B$ $\qquad$ for one element of $(A \cdot B) \cdot C$.

while calculating ~~AB C~~ $A(BC)$, computation complexity $= \underbrace{(r \cdot q \cdot t)}_{\text{for } B \cdot C} \cdot \underbrace{(q \cdot p \cdot t)}_{\text{for } A(B \cdot C)}$

we want complexity $[(AB)\, C] <$ Complexity $[A\,(BC)]$

$\Rightarrow (q p r)(r p t) < (r q t)(q p t)$

$\Rightarrow \boxed{rp < qt}$