1

The relation $\sim$ is not an equivalence relation because it is not reflexive.

$$1 \in \mathbb{Z} \text{ however } \gcd(1,1) = 1$$

$$(1,1) \notin \sim$$

$$\implies \sim \text{ is not reflexive}$$

$$\therefore \sim \text{ is not an equivalence relation}$$

2

Yes the relation (R) is a POSET as it is reflexive, antisymmetric and transitive

- Reflexive:
  - $(a,a) \in R$
  - $(b,b) \in R$
  - $(c,c) \in R$
  - $(d,d) \in R$

  $\therefore \forall x \in A, (x,x) \in R$, so R is reflexive
- Antisymmetric: $\nexists (x,y) \in R, x \neq y$ such that $(y,x) \in R, \therefore R$ is antisymmetric
- Transitive:
  - $(a,a) \in R, (a,d) \in R, (a,d) \in R$
  - $(b,b) \in R, (b,d) \in R, (b,d) \in R$
  - $(c,c) \in R, (c,d) \in R, (c,d) \in R$
  - $(a,d) \in R, (d,d) \in R, (a,d) \in R$
  - $(b,d) \in R, (d,d) \in R, (b,d) \in R$
  - $(c,d) \in R, (d,d) \in R, (c,d) \in R$

  $\therefore R$ is transitive via proof by exhaustion

3

R is not an equivalence relation as $(6,6) \notin R \therefore R$ is not reflexive.

4

$R_1 \cup R_2$ is not an equivalence relation, consider

- $A = \{x, y, z\}$
- $R_1 = \{(x,x), (y,y), (z,z), (x,y), (y,x)\}$
- $R_2 = \{(x,x), (y,y), (z,z), (y,z), (z,y)\}$
- $R_1 \cup R_2 = \{(x,x), (y,y), (z,z), (x,y), (y,x), (y,z), (z,y)\}$

Now, $R_1 \cup R_2$ contains $(x,y), (y,z)$ however, $(x,z) \notin R_1 \cup R_2$

$$\therefore R_1 \cup R_2 \text{ is not transitive}$$

$$\therefore R_1 \cup R_2 \text{ is not an equivalence relation}$$

5

(a)
- Reflexive: $\forall (x,y) \in A, (x,y)R(x,y)$ as $xy = xy$
- Symmetric:

$$\begin{gathered} \forall (x_1,y_1),(x_2,y_2) \in A \\ (x_1,y_1)R(x_2,y_2) \implies (x_2,y_2)R(x_1,y_1) \\ (x_1y_1 = x_2y_2 \implies x_2y_2 = x_1y_1) \end{gathered}$$

- Transitive:

$$\begin{aligned} \forall (x_1,y_1),(x_2,y_2),(x_3,y_3) \in A, (x_1,y_1)R(x_2,y_2), &(x_2,y_2)R(x_3,y_3) \\ (x_1,y_1)R(x_2,y_2) &\implies x_1y_1 = x_2y_2 \\ (x_2,y_2)R(x_3,y_3) &\implies x_2y_2 = x_3y_3 \\ &\implies x_1y_1 = x_3y_3 \\ &\implies (x_1,y_1)R(x_3,y_3) \end{aligned}$$

$\therefore$ R is transitive

R is reflexive, symmetric and transitive $\therefore R$ is an equivalence relation.

(b) All equivalence classes are of the form

$$[(1,i)], \forall i \in \mathbb{N}$$

$$[(1,i)] = \{(x,y) | xy = i, \text{ and } x, y \in \mathbb{N}\}$$

(c) $[(1,2)]$ has two elements $(1,2),(2,1)$, infact all equivalence classes of the form $[(1,p)]$ have two elements where $p \in Primes$

(d) $[(1,4)]$ has three elements $(1,4),(2,2),(4,1)$, infact all equivalence classes of the form $[(1,p^2)]$ have two elements where $p \in Primes$

6

- Reflexive:
$$\forall x \in A, |x - x| = 0, 0 \text{ is even}$$
$$\therefore \forall x \in A, (x, x) \in R$$

- Symmetric:
$$|x - y| = |y - x|$$
$$\therefore (x, y) \in R \implies (y, x) \in R$$

- Transitive: Let $(x, y), (y, z) \in R$
$$\begin{aligned} |x - y| &= 2\lambda_1, \lambda_1 \in \mathbb{Z}^* \\ x - y &= 2\mu_1, \mu_1 \in \mathbb{Z} \\ |y - z| &= 2\lambda_2, \lambda_2 \in \mathbb{Z}^* \\ y - z &= 2\mu_2, \mu_2 \in \mathbb{Z} \\ \implies x - y + y - z &= 2(\mu_1 + \mu_2) \\ \implies x - z &= 2\mu_3, \mu_3 \in \mathbb{Z} \\ \implies |x - z| &= 2\lambda_3(\text{even}), \lambda_3 \in \mathbb{Z}^* \\ &\implies (x - z) \in R \end{aligned}$$

  $\therefore R$ is transitive

R is reflexive, symmetric and transitive so R is an equivalence relation.

7

Any equivalence relation ρ on set A induces a partition of A. So we can count partitions instead of equivalence relations

Type	Counts
4	$\binom{4}{0} = 1$
3,1	$\binom{4}{3} = 4$
2,2	$\frac{\binom{4}{2}}{2} = 3$
2,1,1	$\binom{4}{2} = 6$
1,1,1,1	$\binom{4}{4} = 1$

$\implies$ Number of Equivalence Relations on A = 1+4+3+6+1=15

S.No	Type	Equivalence Clases
1	4	$\{1,2,3,4\}$
2	$3,1$	$\{1,2,3\},\{4\}$
3	$3,1$	$\{1,2,4\},\{3\}$
4	$3,1$	$\{1,4,3\},\{2\}$
5	$3,1$	$\{4,2,3\},\{1\}$
6	$2,2$	$\{1,2\},\{3,4\}$
7	$2,2$	$\{1,3\},\{2,4\}$
8	$2,2$	$\{1,4\},\{3,2\}$
9	$2,1,1$	$\{1,2\},\{3\},\{4\}$
10	$2,1,1$	$\{1,3\},\{2\},\{4\}$
11	$2,1,1$	$\{1,4\},\{2\},\{3\}$
12	$2,1,1$	$\{3,2\},\{1\},\{4\}$
13	$2,1,1$	$\{4,2\},\{1\},\{3\}$
14	$2,1,1$	$\{3,4\},\{1\},\{2\}$
15	$1,1,1,1$	$\{1\},\{2\},\{3\},\{4\}$

8

The statement is true.
Proof:

- $\implies$ (If) We can prove this by its contrapositive
  If R is not antisymmetric,
  $$\exists (x,y),(y,x) \in R | x \neq y$$
  However, any closure of R would still contain $(x,y),(y,x)$ and would continue to remain antisymmetric $\implies$ no antisymmetric closure of R can exist

- $\impliedby$ (Only-If) R is antisymmetric $\implies$ the antisymmetric closure of R is itself, which exists

Total number of antisymmetric relations on a finite set of size n is given by $2^n \times 3^{\binom{n}{2}}$.
Proof:

- A relation R on a set A is antisymmetric if $\forall x,y \in A, (x,y),(y,x) \in R \implies x = y$.

- CASE 1: First we look at all pairs $(x,y)|x = y$. The number of such pairs is n, one for each element in A. We may have $(x,x) \in R$ or $(x,x) \notin R$. There are n such pairs, and 2 possibilities for each, so the total relations in this case are 2^n.

- CASE 2: Now we look at all pairs $(x,y)|x \neq y$. The number of such pairs is $\binom{n}{2}$, the number of ways of selecting 2 objects from a set of n objects. We may have
  - $(x,y) \in R, (y,x) \notin R$
  - $(y,x) \in R, (x,y) \notin R$
  - $(x,y),(y,x) \notin R$.

There are $\binom{n}{2}$ such pairs, and 3 possibilities for each, so the total relations in this case are $3^{\binom{n}{2}}$.

- CASE 1 and CASE 2 exhaust all possible pairs of elements. Using the multiplication rule of counting on the results of the two cases, the total number of antisymmetric relations on a finite set of size n are thus $2^n \times 3^{\binom{n}{2}}$.

9

Let $S = x, y$

$$R_1 = (x,x),(y,y),(x,y)$$

$$R_2 = (x,x),(y,y),(y,x)$$

$$R_1 \cup R_2 = (x,x),(y,y),(x,y),(y,x)$$

This is not antisymmetric $(x,y),(y,x) \in R_1 \cup R_2, x \neq y$
$\therefore R_1 \cup R_2$ is not a POSET on S

10

(a) Not a POSET,

$$(5,1) \preccurlyeq (5,2)$$

$$(5,2) \preccurlyeq (5,1)$$

$$(5,2) \neq (5,1)$$

$\therefore$ not antisymmetric, so not a POSET

(b) Is a POSET,

- Reflexive:

$$\forall (x,y) \in \mathbb{N} \times \mathbb{N}, (x,y) \preccurlyeq (x,y), x \leq x, y \geq y$$

$\therefore$ it is reflexive.

- Antisymmetric:

$$\text{Let } \exists (x_1,y_1),(x_2,y_2) \in \mathbb{N} \times \mathbb{N} : (x_1,y_1) \preccurlyeq (x_2,y_2),(x_2,y_2) \preccurlyeq (x_1,y_1)$$

$$\implies x_1 \leq x_2, x_2 \leq x_1, y_1 \geq y_2, y_2 \geq y_1$$

$$\implies x_1 = x_2, y_1 = y_2$$

$$\implies (x_1,y_1) = (x_2,y_2)$$

$\therefore$ it is antisymmetric.

- Transitive:

$$\text{Let } \exists (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{N} \times \mathbb{N} : (x_1, y_1) \preccurlyeq (x_2, y_2), (x_2, y_2) \preccurlyeq (x_3, y_3)$$

$$\implies x_1 \leq x_2, x_2 \leq x_3, y_3 \geq y_2, y_2 \geq y_1$$

$$\implies x_1 \leq x_3, y_3 \geq y_1$$

$$\implies (x_1, y_1) \preccurlyeq (x_3, y_3)$$

$\therefore$ it is transitive.

11

No this is not a POSET on $P(S)$

$$\text{Let } S = \{a_1, a_2, a_3\}$$

$$a_1 \preccurlyeq a_2 \ (|a_1| = 1 \leq |a_2| = 1)$$

$$a_2 \preccurlyeq a_1 \ (|a_2| = 1 \leq |a_1| = 1)$$

$$a_1 \neq a_2$$

$\therefore (P(S), \preccurlyeq)$ is not antisymmetric

12

a We know that $x \vee 1 = 1, x \vee 0 = x$, Let $M_R \vee I_n = S$

$$\therefore S[i][j] = \begin{cases} 1 & i = j \\ M_R[i][j] & i \neq j \end{cases}$$

$$\implies \forall i \, 0 \leq i < n, S[i][i] = 1$$

$\therefore$ the relation holds for all (x,x) in the set

b We know that $M_R[i][j] = M_R^t[j][i]$, Let $M_R \vee M_R^t = S$

$$S[i][j] = M_R^t[i][j] \vee M_R[i][j]$$

$$= M_R[j][i] \vee M_R[i][j]$$

$$= M_R[j][i] \vee M_R^t[j][i]$$

$$= S[j][i]$$

$$\therefore S[i][j] = S[j][i]$$

So if (x,y) $\in$ the relation $\iff$ (y,x) $\in$ the relation

13

(a) Let A be the set of all bit strings of length three or more.

- Reflexive: for some string y all bits of y agree, $\therefore (y, y) \in R$ thus R is reflexive
- Symmetric: Trivial to see that if all bits execpt the first three agree for some x and y, then both (x,y) and (y,x) would belong to R. Thus R is symmetric
- Transitive: Let $(x, y) \in R, (y, z) \in R$, so all bit after the third position agree for x and y, and for y and z, so they would agree for x and z, $\therefore (x, z) \in R$. Thus R is transitive

R is reflexive, symmetric,transitive so it is an equivalence relation

(b) Not true, $(0101, 0000) \in R, (0000, 0101) \in R$, but $0101 \neq 0000 \therefore$ R is not antisymmetric, so R is not a POSET on the given set.

14

For a finite totally ordered set, by definition all subsets(except empty) would have a least element, therefore a finite totally ordered set would be well ordered, so there exists no such set.

15

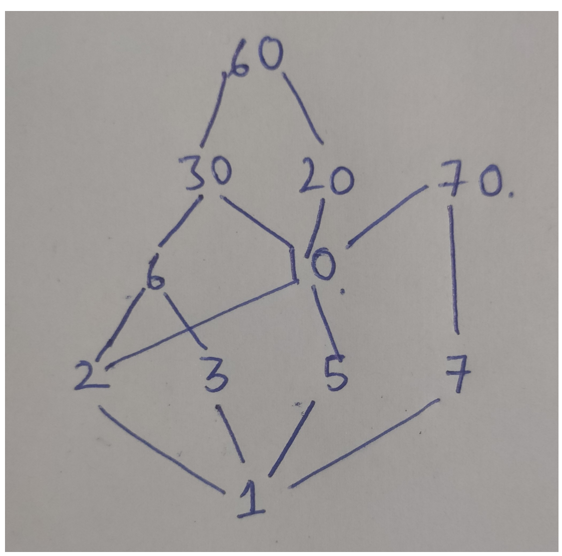

Figure 1: Hasse diagram

(a)

(b) Maximal elements : 60,70 [maximal elements have no sucessor]

(c) Minimal element : 1 [minimal elements have no predecessor]

(d) For greatest element(M) to exist $\forall x \in S, x \leq M$, where M is the greatest element.
There exist no element $M \in S$ that satisfies the above condition, hence the poset has no greatest element
Greatest element has all other elements as it's predecessors(direct/indirect)

(e) For least element(m) to exist $\forall x \in S, m \leq x$, where m is the least element.
The given poset has 1 as it's least element as it satisfies the above condition
Least element has all other elements as it's sucessors(direct/indirect)

(f) Upper bound of $\{2, 5\}$: $10, 20, 30, 60, 70$

(g) LUB of 2,5 : 10

(h) Lower bounds of 6,10 : 1,2

(i) GLB of 2,5 : 1

(j) This Poset is not a Lattice as many subsets have non-existent joins(LUB)
$\{20, 70\}, \{30, 70\}, \{60, 70\}$: LUB Does Not Exist (only some subsets listed)
$\therefore$ This Poset is not a lattice

16

~~$[A], [B], [C]$ are all lattices~~ For the solution, please refer to the end of the file

17

Property	(1)	(2)	(3)	(4)	(5)
Distributive	NO	NO	YES	YES	YES
Complemented	YES	YES	NO	YES	NO

DISTRIBUTIVE LATTICE CHECK :

- Every lattice element has atmost 1 complement

COMPLEMENTED LATTICE CHECK :

- Every lattice element has atleast 1 complement

(1)

x	$\bar{x}$
b	c,d
c	b
d	b

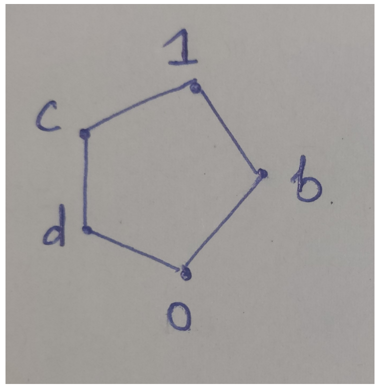

Figure 2: (q17 (1))

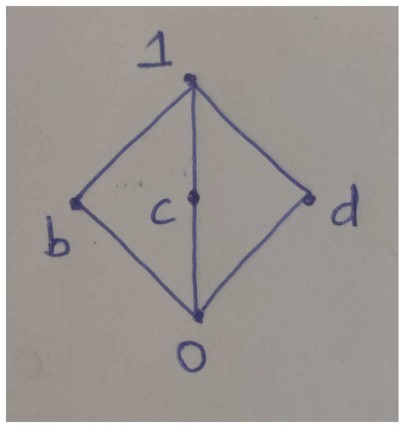

Figure 3: (q17 (2))

(2)

x	$\bar{x}$
b	c,d
c	b,d
d	b,c

(4)

x	$\bar{x}$
b	c
c	b

18

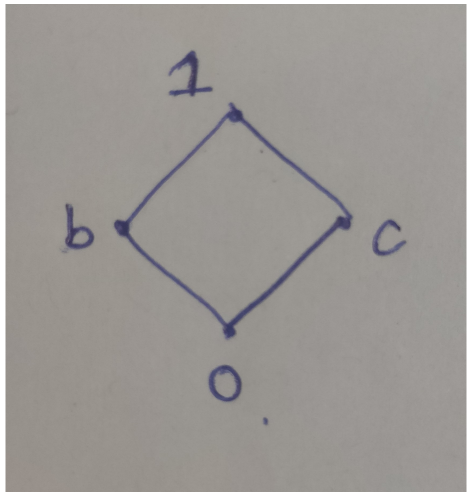

Figure 4: (q17 (4))

SUBSET	MAXIMAL(M)	MINIMAL(m)	GREATEST	LEAST	UB	LB	LUB	GLB
$\{d, k, f\}$	$\{k\}$	$\{d, f\}$	$\{k\}$	DNE	$\{k, l, m\}$	DNE	$\{k\}$	DNE
$\{b, h, f\}$	$\{h, f\}$	$\{b, f\}$	DNE	DNE	$\{l, m\}$	DNE	$\{k\}$	DNE
$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d, h, i, j, k, l, m\}$	$\{a, b, d\}$	$\{d\}$	$\{d\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	DNE	DNE	$\{k, l, m\}$	DNE	$\{k\}$	DNE
$\{l, m\}$	$\{l, m\}$	$\{l, m\}$	DNE	DNE	DNE	$\{a, b, c, d, e, f, g, h, k\}$	DNE	$\{k\}$

Table 1: Answers of 18

19

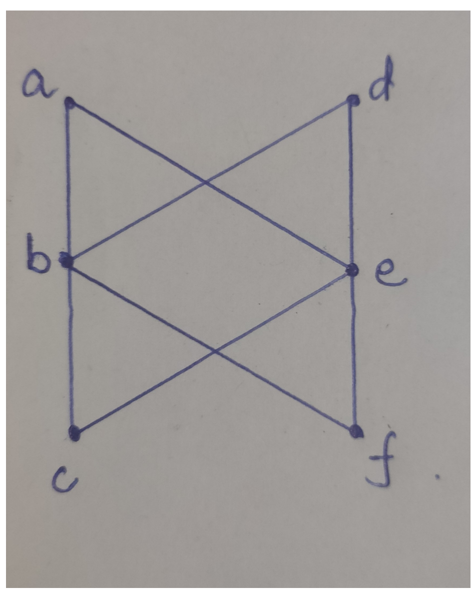

Figure 5: Poset P1 for (a),(d),(e)

- In the poset P1(figure), consider the subset $\{a, b, c, d, e, f\}$

It has :
2 maximal elements : a,d
2 minimal elements : c,f

- consider the poset $(\mathbb{Z}, \preccurlyeq)$, where $x \preccurlyeq y \iff x \leq y$
  and take the subset $(-\infty, 4]$
  Maximal element : 4
  Minimal element : Does Not exist

- Yes, as shown in the above example

- In the poset P1(figure), consider the subset $\{b, e\}$
  It has :
  Lower Bound : $\{c, f\}$
  GLB : Does Not exist

- In the poset P1(figure), consider the subset $\{b, e\}$
  It has :
  Upper Bound : $\{a, d\}$
  LUB : Does Not exist

20

(a) Let S be the set of divisors of 60. The given poset is a lattice as

$$\forall x, y \in S, x \vee y, x \wedge y \in S$$

(i.e) the meet and join exist and belong to the set, for all pairs of elements in S

Meet : $x \vee y \equiv \text{LCM}(x, y)$
Join : $x \wedge y \equiv \text{GCD}(x, y)$

(b) Let S be the power set of $\{0, 1, 2\}$. The given poset is a lattice as

$$\forall x, y \in S, x \vee y, x \wedge y \in S$$

(i.e) the meet and join exist and belong to the set, for all pairs of elements in S

Meet : $x \vee y \equiv x \cup y$
Join : $x \wedge y \equiv x \cap y$

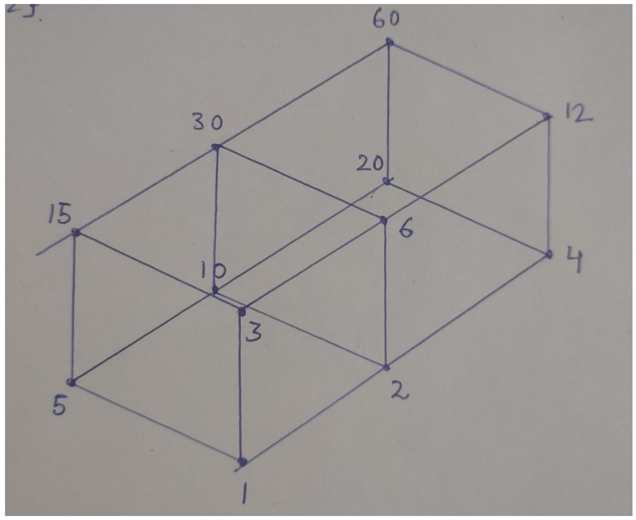

Figure 6: Hasse diagram for divisors of 60

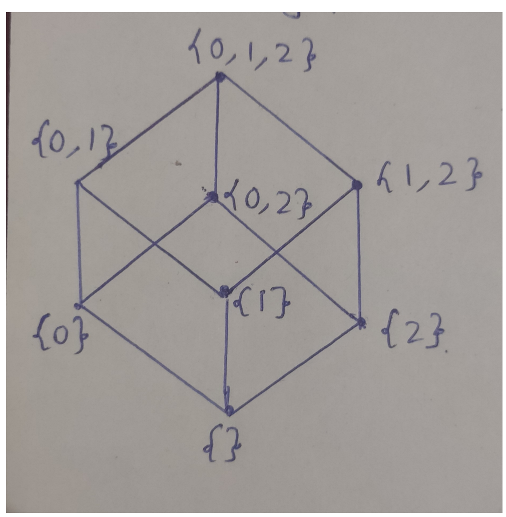

Figure 7: Hasse diagram for subsets of 0,1,2

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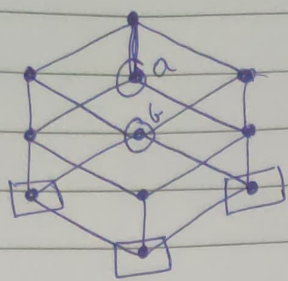
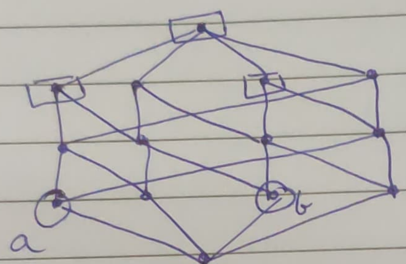


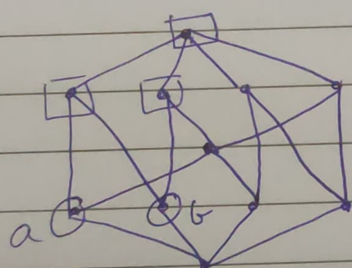
NOT A LATTICE

Consider elements a & b. Their LBs have been marked by squares. Clearly there is no GLB. Hence a, b dont have a meet.



NOT A LATTICE

Consider elements a & b Their UBs have been marked by squares. Clearly there is no LUB. Hence a, b don't have a join.



NOT A LATTICE

Consider elements a & b Their UBs have been marked by squares. Clearly there is no LUB. Hence a, b don't have a join