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7 Given $A \in \mathbb{R}^{n \times n}$.

Now : To construct $L_{ij}$.

⇒ Let $y_j$ be the $j^{th}$ column of $L_{j-1} \cdots L_1 . A$ at the beginning of step $j$

⇒ choose $L_j$ such that $y_j = \begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{jj} \\ y_{j+1\,j} \\ \vdots \\ y_{nj} \end{bmatrix} \xrightarrow{L_j} L_j y_j = \begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

⇒ $l_{ij} = \dfrac{y_{ij}}{y_{jj}}$ ; $j \le i \le n$

$\therefore L_{ij} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & -l_{ij} & & \ddots \\ & & & & 1 \end{bmatrix}$ ⇒ All diagonal elements are 1. The element at $i,j$ is $-l_{ij}$ and all others are 0.

Get all such $L_{ij}$ and multiply them together, to get $L_j$

$\therefore L_j = \prod_{i=j+1}^{n} L_{ij} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & -l_{j+1\,j} & \ddots & \\ & & -l_{nj} & & 1 \end{bmatrix}$

This way, ~~get~~ we get all $L_j$ for $j = \{1, 2 \cdots, n-1\}$

Multiply all these $L_j$ with A ~~ase~~ to get $U$. (upper triangular matrix)

$\Rightarrow U = L_{n-1} \cdots L_2 L_1 A$

Now invert all $L_j$ by just changing the sign of the elements below the diagonal.

$$\Rightarrow L_j^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & l_{j+1\,j} & \ddots & \\ & & \vdots & & \\ & & l_{nj} & & 1 \end{bmatrix}$$

NOTE: $L_j$ are lower triangular matrices with diagonal elements $= 1$.
Thus, they are invertible.

Now, $L = L_1^{-1} \cdot L_2^{-1} \cdots L_{n-1}^{-1}$
This $L$ is a lower triangular matrix.

Now, we have $A, L, U$ such that, $A = LU$.
This is the LU Decomposition of A into L (lower triangular matrix) and U (Upper triangular matrix).

It ~~is~~ can be used to help solve systems of Linear Equations given by $Ax = b$ (A is invertible)

$\Rightarrow$ we have $A = LU$.

$\Rightarrow$ Also, $Ax = b$

$\Rightarrow (LU)x = b$

$\Rightarrow L(Ux) = b$

Let $Ux = y$.

$\Rightarrow Ly = b$. $\longrightarrow$ Solve by forward substitution and get $y$

$\Rightarrow Ux = y$ $\longrightarrow$ Solve by Backward substitution and get $x$.