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6 Left Inverse of a matrix A is another matrix X, such that it satisfies $XA = I$.
$A \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{n\times m}$

Left Inverse exists in those cases, when the columns of matrix A are linearly independent.

(a) $A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ The columns are linearly independent, thus, A has a left inverse X.

$\Rightarrow X = [x_1 \ x_2 \ x_3 \ x_4 \ x_5]$

$\Rightarrow XA = I \Rightarrow [x_1 \ x_2 \ x_3 \ x_4 \ x_5]\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = [1]$

$\Rightarrow x_1 + x_4 = 1$

$\Rightarrow x_4 = 1 - x_1$

$\therefore X = [x_1 \ x_2 \ x_3 \ 1-x_1 \ x_5]$

$$\boxed{X = x_1 e_1^T + x_2 e_2^T + x_3 e_3^T + (1-x_1) e_4^T + x_5 e_5^T \quad \forall\, x_1, x_2, x_3, x_5 \in \mathbb{R}}$$

X is all left inverses of A given 1 left inverse exists

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 3 & 3 \end{bmatrix}$ The columns are linearly indepedent, thus A has a left inverse X.

$x_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = 0$

$\Rightarrow 2x_1 + 0 = 0$

$0 - 2x_2 = 0 \Rightarrow x_1 = x_2 = 0$

$3x_1 + 3x_2 = 0$

~~$\Rightarrow X = x_1 \ x_2$~~

$\Rightarrow X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \Rightarrow XA = I \Rightarrow \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore 2x_{11} + 3x_{13} = 1 \quad -(1)$ $\qquad 2x_{21} + 3x_{23} = 0 \quad -(3)$

$-2x_{12} + 3x_{13} = 0 \quad -(2)$ $\qquad -2x_{22} + 3x_{23} = 1 \quad -(4)$

$\Rightarrow$ From ①, $\cancel{x_{13} = \frac{1-2x_{11}}{3}}$ $\quad x_{11} = \dfrac{1-3x_{13}}{2}$

From ②, $\quad x_{12} = \dfrac{3x_{13}}{2}$

From ③, $\quad x_{21} = -\dfrac{3x_{23}}{2}$

From ④, $\quad x_{22} = \dfrac{3x_{23}-1}{2}$.

$$\therefore X = \begin{bmatrix} \frac{1-3x_{13}}{2} & \frac{3x_{13}}{2} & x_{13} \\ -\frac{3x_{23}}{2} & \frac{3x_{23}-1}{2} & x_{23} \end{bmatrix} \qquad \forall x_{13}, x_{23} \in \mathbb{R}.$$

$$X = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix} + x_{13}\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} + x_{23}\begin{bmatrix} 0 & 0 & 0 \\ -\frac{3}{2} & \frac{3}{2} & 1 \end{bmatrix} \qquad \forall x_{13}, x_{23} \in \mathbb{R}.$$

X is all left inverses of A, given 1 left inverse exists.