19CS10039
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(2) Given $A \in \mathbb{R}^{n \times n}$ is an invertible matrix.

We define Maximum magnification of A as

$$\Rightarrow \quad \text{maxmag}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|A \cdot x\|_2$$

We define Minimum magnification of A as

$$\Rightarrow \quad \text{minmag}(A) = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \min_{\|x\|_2 = 1} \|A \cdot x\|_2$$

Since $\text{maxmag}(A) = \max_{\|x\|_2 = 1} \|A \cdot x\|_2$, it is half length of the major axis of the ellipse that got ~~converted~~ transformed from the unit circle $\|x\|_2 = 1$

Since ~~minmag~~ $\text{minmag}(A) = \min_{\|x\|_2 = 1} \|A \cdot x\|_2$, it is half length of the minor axis of the ellipse that got transformed from the unit circle $\|x\|_2 = 1$

Also, $\text{maxmag}(A) = \|A\|_2$ ~~and~~ (By definition, as $\text{maxmag}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2$

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We define cond(A) $= \|A\|_2 \cdot \|A^{-1}\|_2$

It is the condition number of A, for $Ax = b$.

It ~~scales~~ scales the perturbation in b.

$\Rightarrow$ A small value of the condition number means that x is less sensitive to perturbations in b.

$\Rightarrow$ A large value of the condition number means that x is more sensitive to perturbations in b.

(a) To prove : $\text{maxmag}(A) = \dfrac{1}{\text{minmag}(A^{-1})}$

Let $Ax = y$. $\Rightarrow x = A^{-1} \cdot y$ (A is invertible)

Now, $$\text{maxmag}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$= \max_{y \neq 0} \frac{\|y\|_2}{\|A^{-1}y\|_2}$$

$$= \frac{1}{\min\limits_{y \neq 0} \dfrac{\|A^{-1}y\|_2}{\|y\|_2}}$$

$$= \frac{1}{\text{minmag}(A^{-1})}$$

$\therefore \text{maxmag}(A) = \dfrac{1}{\text{minmag}(A^{-1})}$

(b) To prove : $\text{cond}(A) = \dfrac{\text{maxmag}(A)}{\text{minmag}(A)}$

Since $\text{cond}(A) = \|A\|_2 \cdot \|A^{-1}\|_2$

$= \text{maxmag}(A) \cdot \text{maxmag}(A^{-1})$ (by definition)

$= \dfrac{\text{maxmag}(A)}{\text{minmag}(A)}$ (proved in A)