Kaushal Banthia
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(3) For a function $\|\cdot\|_w$ to be called as norm, it must satisfy

→ <u>Non Negative Homogenity</u>: For $\gamma \in \mathbb{R}$, $x \in \mathbb{R}^n$

$$\|\gamma x\|_w = \sqrt{\sum_{i=1}^{n} w_i (\gamma x_i)^2} = \sqrt{\gamma^2 \sum_{i=1}^{n} w_i x_i^2} = |\gamma| \sqrt{\sum_{i=1}^{n} w_i x_i^2} = |\gamma| \, \|x\|_w$$

$$\therefore \underline{\|\gamma x\|_w = |\gamma| \cdot \|x\|_w}$$

→ <u>Triangle Inequality</u>: For $x, y \in \mathbb{R}^n$;

$$\|x+y\|_w = \sqrt{\sum_{i=1}^{n} w_i (x_i+y_i)^2}$$

$$\Rightarrow \|x+y\|_w^2 = \sum_{i=1}^{n} w_i (x_i^2 + 2x_iy_i + y_i^2)$$

$$\|x+y\|_w^2 = \sum_{i=1}^{n} w_i x_i^2 + \sum_{i=1}^{n} 2 x_i y_i w_i + \sum_{i=1}^{n} w_i y_i^2$$

$$\|x+y\|_w^2 = \|x\|_w^2 + \sum_{i=1}^{n} 2 x_i y_i w_i + \|y\|_w^2 \quad -(1)$$

Using cauchy-schwartz Inequality,

$$x^T y \le \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}$$

$$\Rightarrow (\sqrt{w}\, x)^T (\sqrt{w}\, y) \le \sqrt{\sum_{i=1}^{n} (\sqrt{w}\, x)^2} \sqrt{\sum_{i=1}^{n} (\sqrt{w}\, y)^2}$$

$$\Rightarrow \sum w_i x_i y_i \le \sqrt{\sum_{i=1}^{n} w x^2} \sqrt{\sum_{i=1}^{n} w y^2}$$

$\Rightarrow$ from (1),
$$\|x+y\|_w^2 \le \|x\|_w^2 + \|y\|_w^2 + 2\sqrt{\sum_{i=1}^{n} w x^2} \sqrt{\sum_{i=1}^{n} w y^2}$$
$$\le \|x\|_w^2 + \|y\|_w^2 + 2\|x\|_w \|y\|_w$$
$$\le (\|x\|_w + \|y\|_w)^2$$

$$\therefore \underline{\|x+y\|_w \le \|x\|_w + \|y\|_w}$$

→ <u>Non Negativity</u>: $\forall x \in \mathbb{R}^n$, $\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i x_i^2}$

here $w_i > 0$ and $x_i^2 \ge 0$ $\therefore$ $\underline{\|x\|_w \ge 0}$.

→ <u>Definiteness</u>: If $\|x\|_w = 0 \Rightarrow \sqrt{\sum_{i=1}^{n} w_i x_i^2} = 0 \Rightarrow \sum_{i=1}^{n} w_i x_i^2 = 0$

Since $w_i > 0$ for $i = \{1, \ldots, n\}$

$\therefore$ for $\|x\|_w$ to be 0, all $x_i = 0$ for $i = \{1, \ldots, n\}$

$\therefore$ $\underline{x = 0}$

$\therefore$ $\boxed{\|\cdot\|_w \text{ is a norm}}$