Motion of the Foucault Pendulum: Glimpse of Coriolis effect

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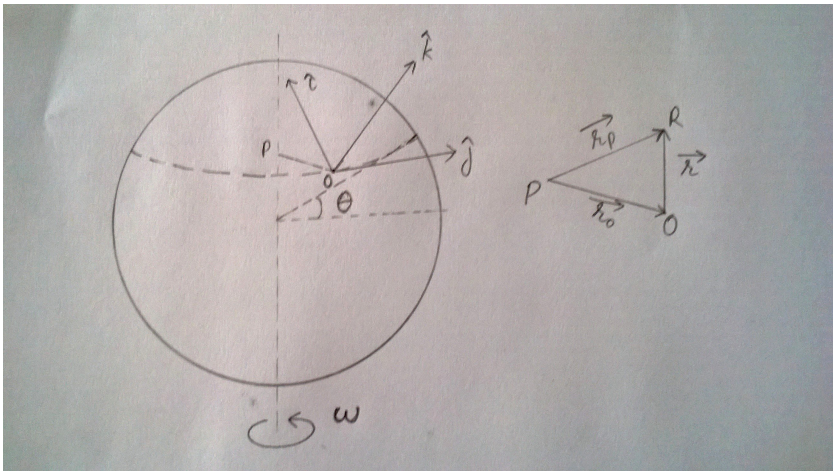
Abstract

The Foucault's Pendulum demonstrates the rotation of the Earth. It's easy enough to understand the motion of a pendulum suspended at one of the Earth's poles, because in this case the point of suspension is not accelerating. But at anyother latitude, the point of suspension accelerates(Earth is a non intetial frame) and this introduces some more time-dependent forces acting on the pendulum. At the poles, the (apparent) precession period is 24 hours, whereas elsewhere it is longer by a factor of the reciprocal sine of the latitude.

The following report shows what are the other forces acting on the pendulum and their effects. To understand it in an even better manner, the trajectory of the pendulum at various latitudes has been plotted.

Let us consider the Earth, which is rotating with an angular velocity ω about its own axis with respect to an inertial frame of reference. Assume a point O on the surface, such that the latitude at that point is θ. We can construct a cartesian coordinate system with x *axis* pointing North, y *axis* horizontal and pointing East and z *axis* pointing upward.



In the diagram shown above, P is the point closest to O on the Earth's axis and $\vec{r_o}$ is the vector joining the points P and O. Also, the frame of reference at O is non inertial, it is rotating with the Earth. This means that $\vec{r_o}$ and $\hat{i}, \hat{j}, \hat{k}$ are all varying with time.

Let $\vec{r}$ be the position vector of any moving particle R relative to O. Therefore the position vector of R relative to P can be given as

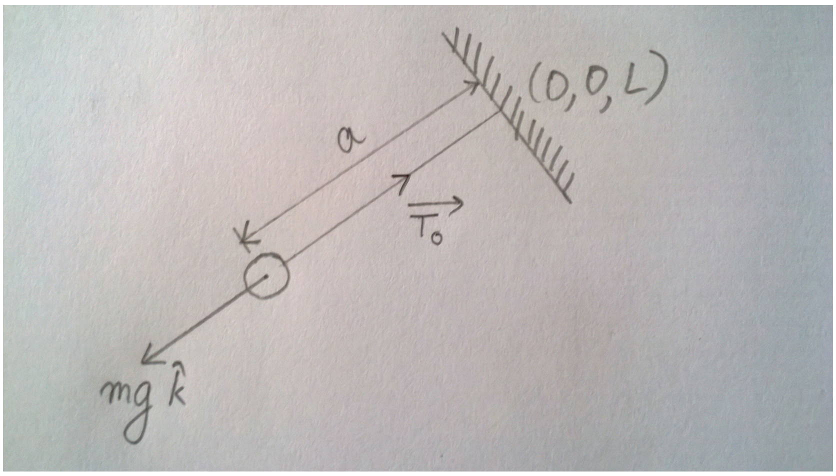
$$\vec{r_p} = \vec{r_o} + \vec{r}$$

Further, let us assume $\vec{r}$ to be significantly small so that the gravitational field doesn't vary significantly between O and R.

If we see the pendulum from the frame of reference P then the forces can be balanced as follows.

$$m\vec{r_p''} = m\vec{G} + \vec{F}$$

Here $\vec{G}$ is the Gravitational field of the Earth and $\vec{F}$ are the other forces acting on the pendulum and dashes signify derivative w.r.t time.



Let us consider the pendulum to be at rest w.r.t. to frame O. The pendulum is suspended at the coordinates $(0,0,L)$ and the length of the string is a The forces acting on the pendulum are the tension due to the string ($\vec{T}$) and the weight of the bob (m$\vec{g}$) in the opposite direction. Thus the forces are balanced as follows.

$$\vec{T_o} = mg\hat{k}$$

For the verticle position of pendulum

$$m\vec{r''} = m\vec{G} + \vec{T_o} - m\vec{\omega} \times (\vec{\omega} \times \vec{r_o})$$

Since the pendulum is stationary w.r.t. frame O

$$\vec{r'} = \vec{r''} = 0$$

Therefore

$$\begin{aligned} m\vec{G} + \vec{T_o} - m\vec{\omega} \times (\vec{\omega} \times \vec{r_o}) = 0 \\ \Rightarrow \vec{G} + g\hat{k} = \vec{\omega} \times (\vec{\omega} \times \vec{r_o}) \quad (1) \end{aligned}$$

If we see the pendulum in motion from frame P

$$m\vec{r''_p} = \vec{T} + m\vec{G} \quad (2)$$

We know that $\vec{r_p} = \vec{r_o} + \vec{r}$ and also $\left(\frac{d\vec{B}}{dt}\right)_{inertial} = \left(\frac{d\vec{B}}{dt}\right)_{non-inertial} + (\vec{\omega} \times \vec{B})$ where $\vec{\omega}$ is the angular velocity of the non inertial frame.

$$\begin{aligned} \vec{r'_p} &= \vec{r'_o} + (\vec{r})' \\ \vec{r'_p} &= \vec{r'_o} + \vec{r'} + (\vec{\omega} \times \vec{r}) \\ \vec{r''_p} &= \vec{r''_o} + (\vec{r'})' + (\vec{\omega} \times \vec{r})' \\ \vec{r''_p} &= \vec{r''_o} + \vec{r''} + (\vec{\omega} \times \vec{r'}) + \underbrace{(\vec{\omega'} \times \vec{r})}_{0} + [\omega \times (\vec{r})'] \\ &= \vec{r''_o} + \vec{r''} + (\vec{\omega} \times \vec{r'}) + [\vec{\omega} \times (\vec{r'} + (\vec{\omega} \times \vec{r}))] \\ &= \vec{r''_o} + \vec{r''} + (\vec{\omega} \times \vec{r'}) + [(\vec{\omega} \times \vec{r'}) + \vec{\omega} \times (\vec{\omega} \times \vec{r})] \end{aligned}$$

Since $\vec{r} \ll \vec{r_0}$

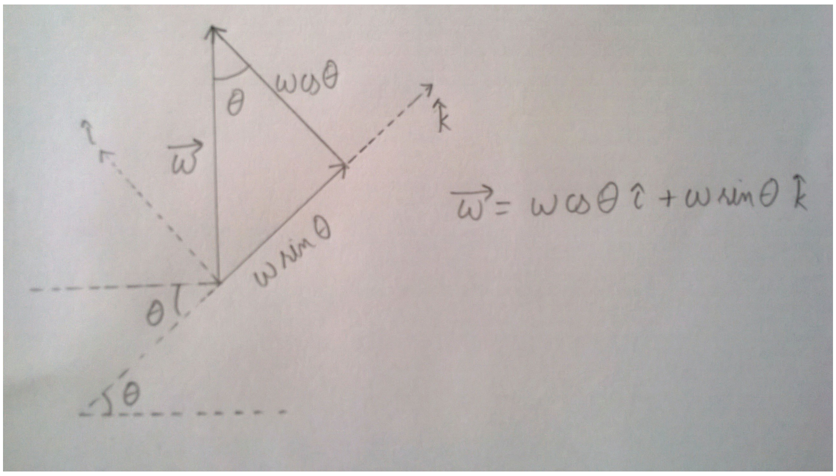
$$\begin{aligned} \vec{\omega} \times (\vec{\omega} \times \vec{r}) \approx 0 \\ \Rightarrow \vec{r''_p} = \vec{r''_o} + \vec{r} + 2(\vec{\omega} \times \vec{r'}) \end{aligned}$$

But $\vec{r''_o} = \vec{\omega} \times (\vec{\omega} \times \vec{r_o})$

$$\begin{aligned} \Rightarrow &\quad \vec{r''_p} = (\vec{G} + g\hat{k}) + \vec{r''} + 2(\vec{\omega} \times \vec{r'}) \\ \Rightarrow &\quad m\vec{r''_p} = m(\vec{G} + g\hat{k}) + m\vec{r''} + 2m(\vec{\omega} \times \vec{r'}) \\ \Rightarrow &\quad \vec{T} + m\vec{G} = m\vec{G} + mg\hat{k} + m\vec{r''} + 2m(\vec{\omega} \times \vec{r'}) \\ \\ \Rightarrow &\quad \vec{T} = mg\hat{k} + m\vec{r''} + 2m(\vec{\omega} \times \vec{r'}) \end{aligned}$$

Let

$$\begin{aligned}\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k}\\ \vec{r'} &= x'\hat{i} + y'\hat{j} + z'\hat{k}\\ \vec{r''} &= x''\hat{i} + y''\hat{j} + z''\hat{k}\\ \vec{T} &= T_x\hat{i} + T_y\hat{j} + T_z\hat{k}\end{aligned}$$



The angular velocity vector can be represented in the frame O as

$$\vec{\omega} = \omega\cos\theta\hat{i} + \omega\sin\theta\hat{k}$$

and hence

$$\begin{aligned}\vec{\omega}\times\vec{r'} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos\theta & 0 & \sin\theta\\ x' & y' & z'\end{vmatrix}\\ &= \omega[\hat{i}(-y'\sin\theta) - \hat{j}(z'\cos\theta - x'\sin\theta) + \hat{k}(y'\cos\theta)]\end{aligned}$$

Therefore $T_x\hat{i} + T_y\hat{j} + T_z\hat{k} =$
$mg\hat{k} + m(x''\hat{i} + y''\hat{j} + z''\hat{k}) + 2m\omega[\hat{i}(-y'\sin\theta) - \hat{j}(z'\cos\theta - x'\sin\theta) + \hat{k}(y'\cos\theta)]$

$$\begin{aligned}\Rightarrow mx'' &= T_x + 2m\omega\sin\theta y' \quad (3)\\ my'' &= T_y + 2m(\cos\theta z' - \sin\theta x') \quad (4)\\ mz'' &= T_z - (mg + 2m\omega\cos\theta y') \quad (5)\end{aligned}$$

But for a swing of small angle, z $\ll$ a, motion in the verticle(z) direction is very small and thus we can safely assume z' and z'' to be approximately zero.

Therefore $T_z = mg + 2m\omega\cos\theta y' \quad (6)$

For small oscillations (x,y,z all $\ll$ a) the components of tension T in string are $\quad T_x = -T\left(\frac{x}{a}\right)$;

$T_y = -T\left(\frac{y}{a}\right)$; and $T_z = T\left(\frac{L-z}{a}\right)$

Taking advantage of approximation $z' = 0$ we can say

$$\begin{aligned} mx'' &= T_x + 2m\omega \sin\theta y' \qquad (7) \\ my'' &= T_y - 2m\omega \sin\theta x' \qquad (8) \end{aligned}$$

Substituting T_z in equation (6).

$$\begin{aligned} T\left(\frac{L-z}{a}\right) &= mg + 2m\omega \cos\theta y' \\ \left(\frac{T}{a}\right) &= \frac{mg + 2m\omega \cos\theta y'}{L-z} \end{aligned}$$

We know that

$$T_x = -T\left(\frac{x}{a}\right)$$
$$\Rightarrow T_x = \left(\frac{T}{a}\right)(-x)$$

Substituting the value of $\left(\frac{T}{a}\right)$ in T_x.

$$\begin{aligned} T_x &= \frac{m(g + 2\omega \cos\theta y')}{L-z}(-x) \qquad (9) \\ similarly & \\ T_y &= \frac{m(g + 2\omega \cos\theta y')}{L-z}(-y) \qquad (10) \end{aligned}$$

Let us assume that $\omega^2 = (\frac{g}{a})$
Substituting the value of g in (9)

$$\begin{aligned} T_x &= \frac{m(\omega^2 a + 2\omega \cos\theta y')}{L-z}(-x) \\ &= \frac{m\omega^2(a + 2(\frac{\cos\theta y'}{\omega}))}{L-z}(-x) \end{aligned}$$

The length of the pendulum(a) is approximately equal to the height(L) of the point of suspension above the ground i.e $a \approx L$ and also $z \ll L$, therefore it can be safely approximated that

$$\frac{a + 2(\frac{\cos\theta y'}{\omega})}{L-z} \approx 1$$

$$\Rightarrow T_x = -m\omega^2 x \quad and \quad T_y = -m\omega^2 y$$

hence (7) can be rewritten as

$$mx'' = -m\omega^2 x + 2m\omega \sin\theta y'$$
$$\Rightarrow x'' = -\omega^2 x + 2\omega \sin\theta y'$$

$\Rightarrow x'' - 2\omega \sin\theta y' + \omega^2 x = 0 \quad (11)$

similarly

$y'' + 2\omega \sin\theta x' + \omega^2 y = 0 \quad (12)$

Let us replace the Earth's *real* frame of reference with a *complex* frame, therefore we set $z = x + \iota y$.
$(11) + \iota(12)$

$$\begin{aligned} x'' + \iota y'' - 2\omega \sin\theta y' + 2\iota \sin\theta x' + \omega^2 x + \iota\omega^2 y &= 0 \\ (x'' + \iota y'') - 2\omega \sin\theta(y' - \iota x') + \omega^2(x + \iota y) &= 0 \\ z'' + 2\iota\omega \sin\theta z' + \omega^2 z &= 0 \quad (13) \end{aligned}$$

Since $\omega \neq 0$ and also $\theta \neq 0$, the complex vector rotates (in the frame O) with an angular frequency $-\omega \sin\theta$.
Therefore the period can be determined as follows.

$$T_p = \frac{2\pi}{\omega \sin\theta}$$

$$T_p = \frac{T_{earth}}{\sin\theta}$$

To solve the the differential equation obtained, let us make an aribitary yet logical assumption.

$$\begin{aligned} z(t) &= Z_0(t)exp(-\iota\omega \sin\theta t) \\ and\ \omega \sin\theta &= \alpha \\ \Rightarrow z(t) &= Z_0(t)exp(-\iota\alpha t) \end{aligned}$$

Subsitituting $z(t) = Z_0(t)exp(-\iota\alpha t)$ in (13) and simplifying further we get

$$exp(-\iota\alpha t)[Z_0''(t) + \alpha^2 Z_0(t) + \omega^2 Z_0(t)] = 0$$

Thus we obtain the differential equation

$$Z''(t) + (\alpha^2 + \omega^2)Z(t) = 0$$

But $\alpha^2 \ll \omega^2$ thus it can be safely neglected.

$$\Rightarrow Z''(t) + \omega^2 Z(t) = 0$$

Let us use the general expression $Z(t) = Aexp(\iota\omega t) + Bexp(-\iota\omega t)$
therefore the solution to our problem is

$$z = exp(-\iota\alpha t)[Aexp(\iota\omega t) + Bexp(-\iota\omega t)]$$

The two special solutions which correspond to the harmonic oscillations of the pendulum are $A = B$ and $A = -B$ where A and B are real.

CASE 1: If $A = B$,

$$\begin{aligned} z &= exp(-\iota\alpha t)[Aexp(\iota\omega t) + Aexp(-\iota\omega t)] \\ &= Aexp(-\iota\alpha t)[exp(\iota\omega t) + exp(-\iota\omega t)] \\ &= Aexp(-\iota\alpha t)[\cos\omega t + \iota\sin\omega t + \cos\omega t - \iota\sin\omega t] \\ &= Aexp(-\iota\alpha t)(2\cos\omega t) \\ z &= 2A\cos\omega t exp(-\iota\alpha t) \qquad (13) \end{aligned}$$

CASE 2: If $A = -B$

$$\begin{aligned} z &= exp(-\iota\alpha t)[Aexp(\iota\omega t) - Aexp(-\iota\omega t)] \\ &= Aexp(-\iota\alpha t)[exp(\iota\omega t) - exp(-\iota\omega t)] \\ &= Aexp(-\iota\alpha t)[\cos\omega t + \iota\sin\omega t - \cos\omega t + \iota\sin\omega t] \\ &= Aexp(-\iota\alpha t)(2\sin\omega t) \\ z &= 2A\sin\omega t exp(-\iota\alpha t) \qquad (14) \end{aligned}$$

Applying Euler's formula to (13)

$$\begin{aligned} z &= 2A\cos\omega t exp(\iota\alpha t) \\ &= 2A\cos\omega t[\cos\alpha t - \iota\sin\omega t] \\ &= 2A\cos\omega t\cos\alpha t - 2A\iota\cos\omega t\sin\alpha t \\ x + \iota y &= 2A\cos\omega t\cos\alpha t - 2A\iota\cos\omega t\sin\alpha t \end{aligned}$$

$$\begin{aligned} x &= 2A\cos\omega t\cos\alpha t \\ y &= -2A\cos\omega t\sin\alpha t \end{aligned}$$

Similarly, applying Euler's formula to (14) gives the following result

$$\begin{aligned} x &= 2A\sin\omega t\sin\alpha t \\ y &= 2A\sin\omega t\cos\alpha t \end{aligned}$$

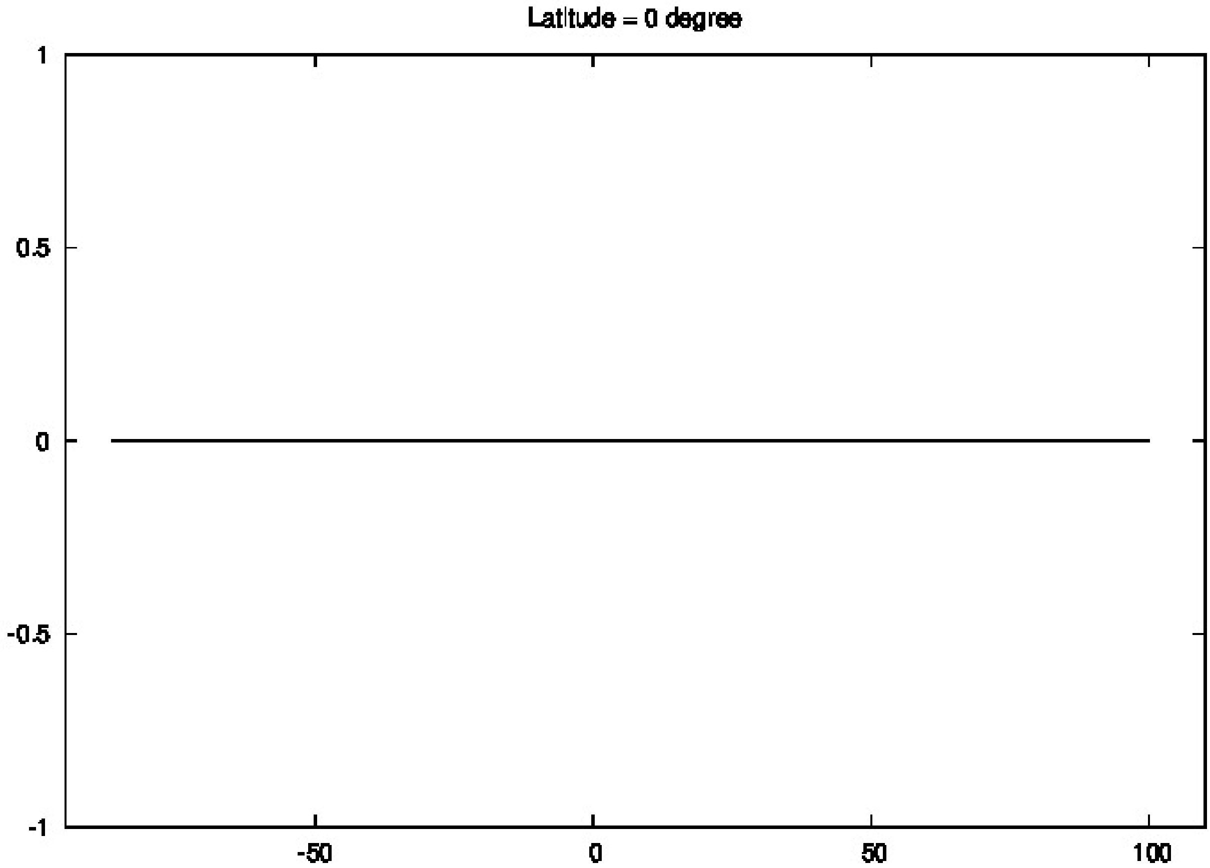
The result from (13) correspond to the initial condition $z(t = 0) = 2A$ and the result from (14) corresponds to the initial condition $z(t = 0) = 0$

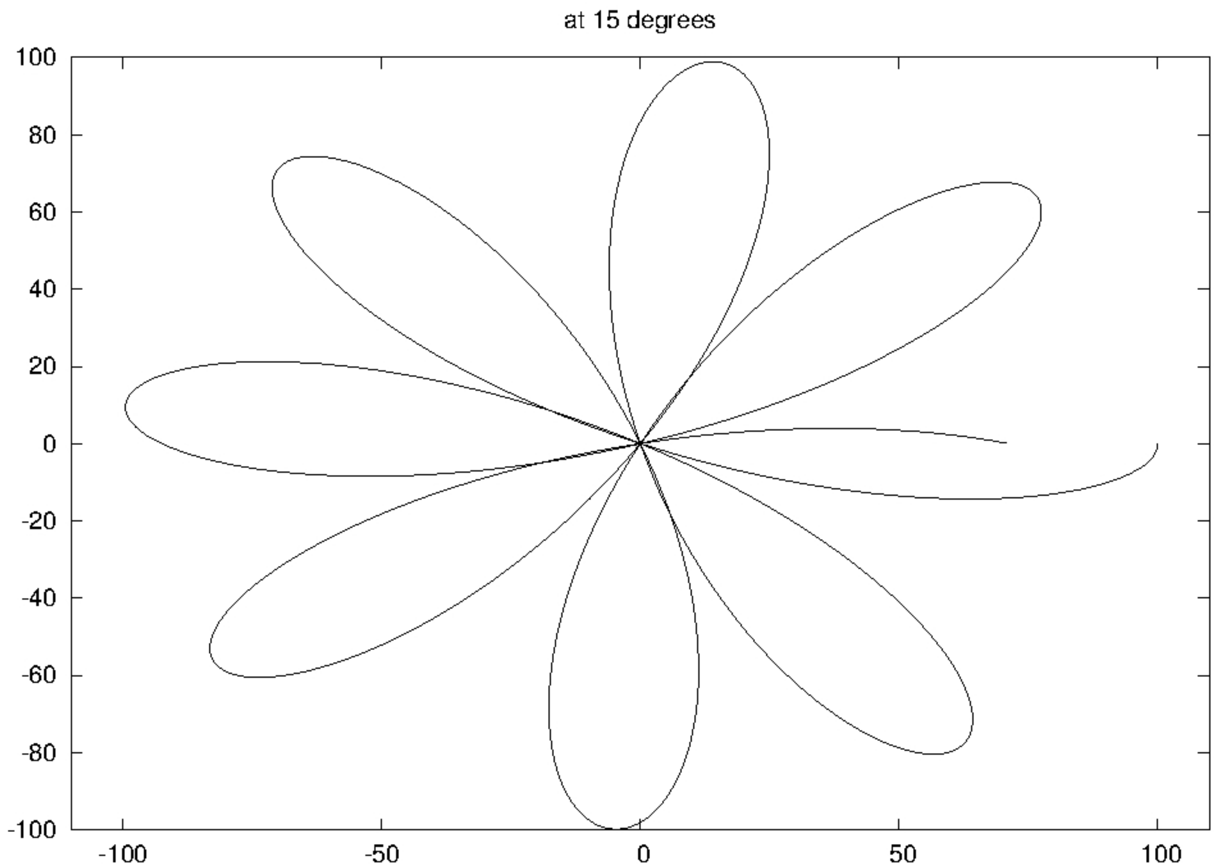
The following part of the report consists of the trajectory (as seen on Earth) of the pendulum at various lattitudes on Earth. The lattitutes correspond to various famous cities across the globe. For convenience the following assumptions have been made.

$$\begin{aligned} A &= 50 \\ x(t = 0) &= 100 \\ y(t = 0) &= 0 \end{aligned}$$

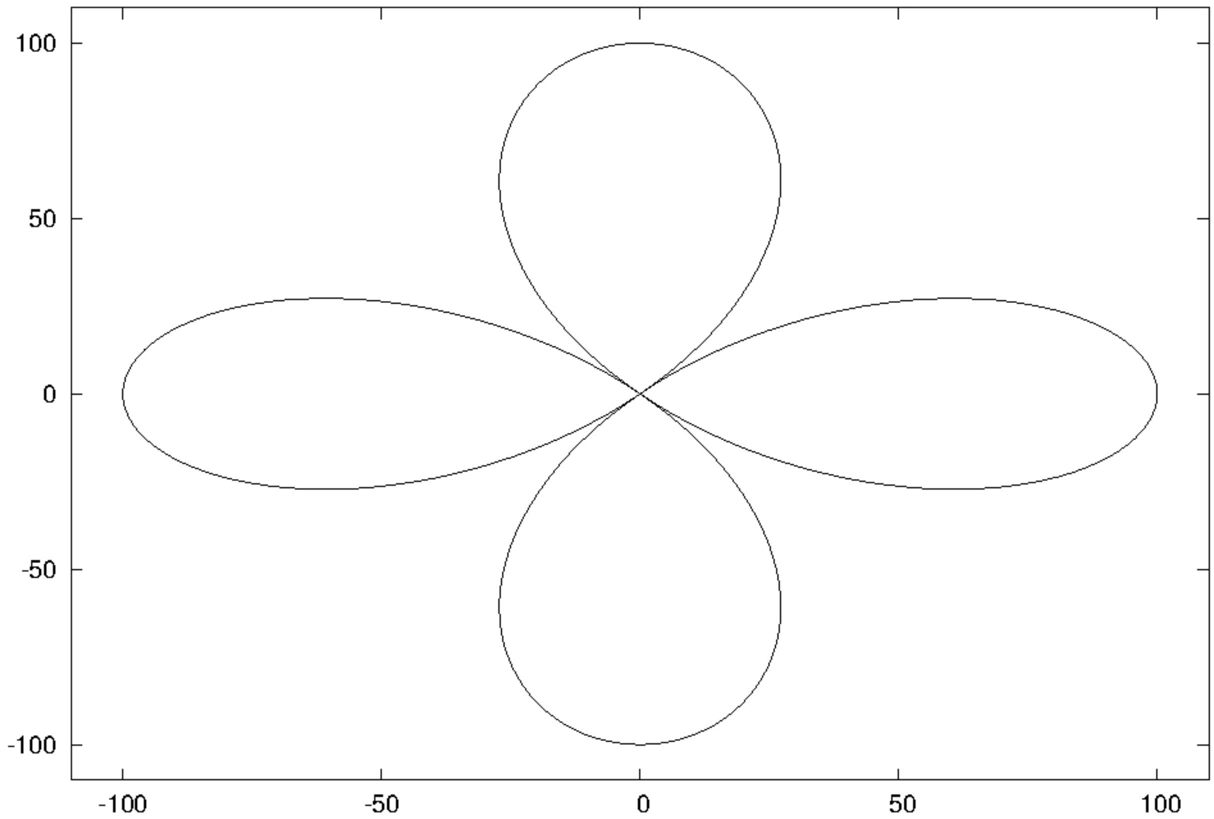
$$\begin{aligned} \Rightarrow x &= 100\cos\omega t\cos\alpha t \\ y &= -100\cos\omega t\sin\alpha t \end{aligned}$$

The following is the trajectory of Foucault's pendulum at different lattitudes.

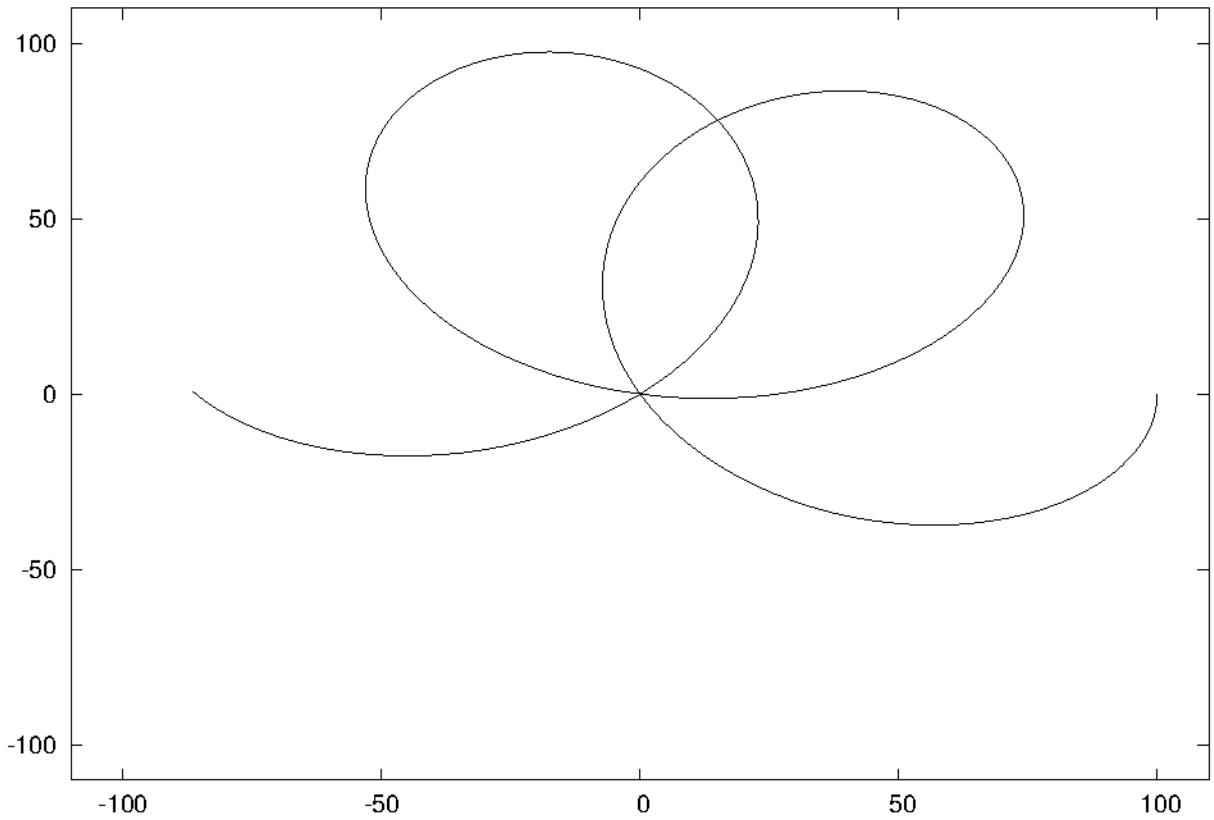




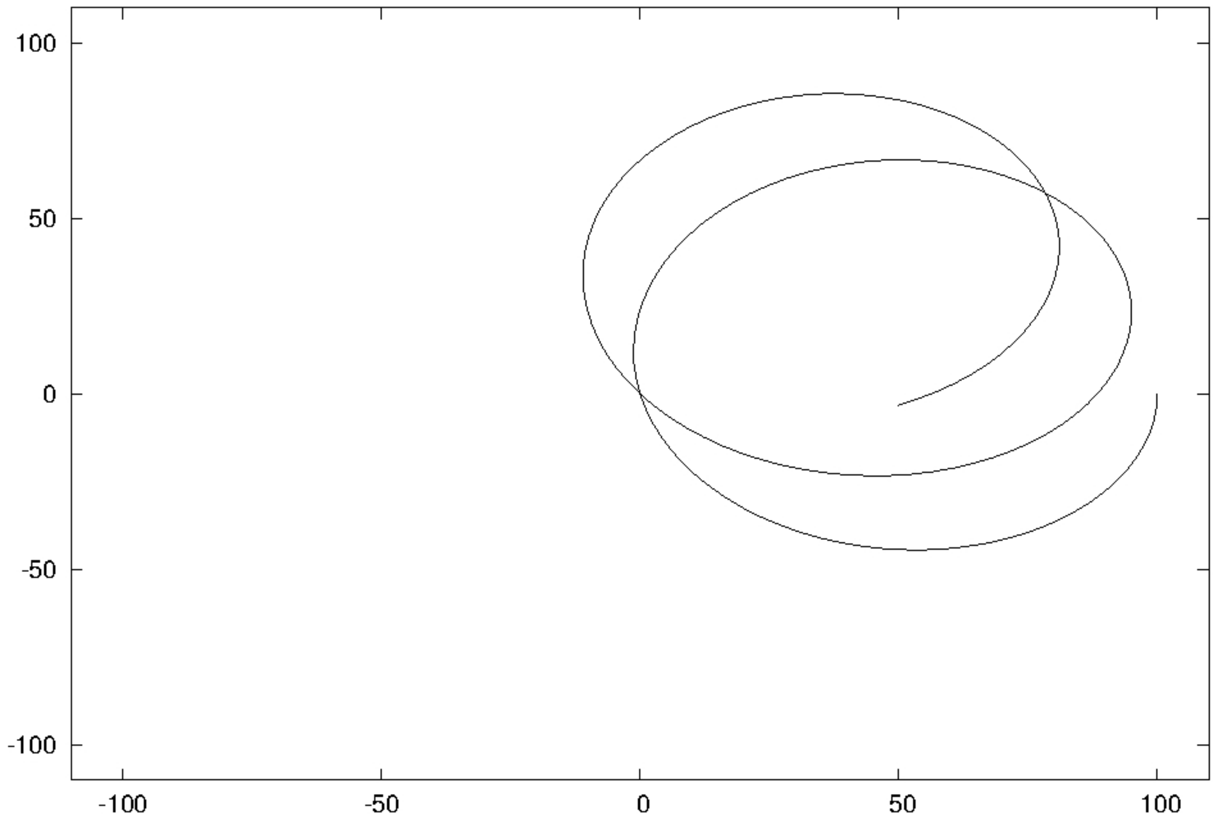
at 30 degrees



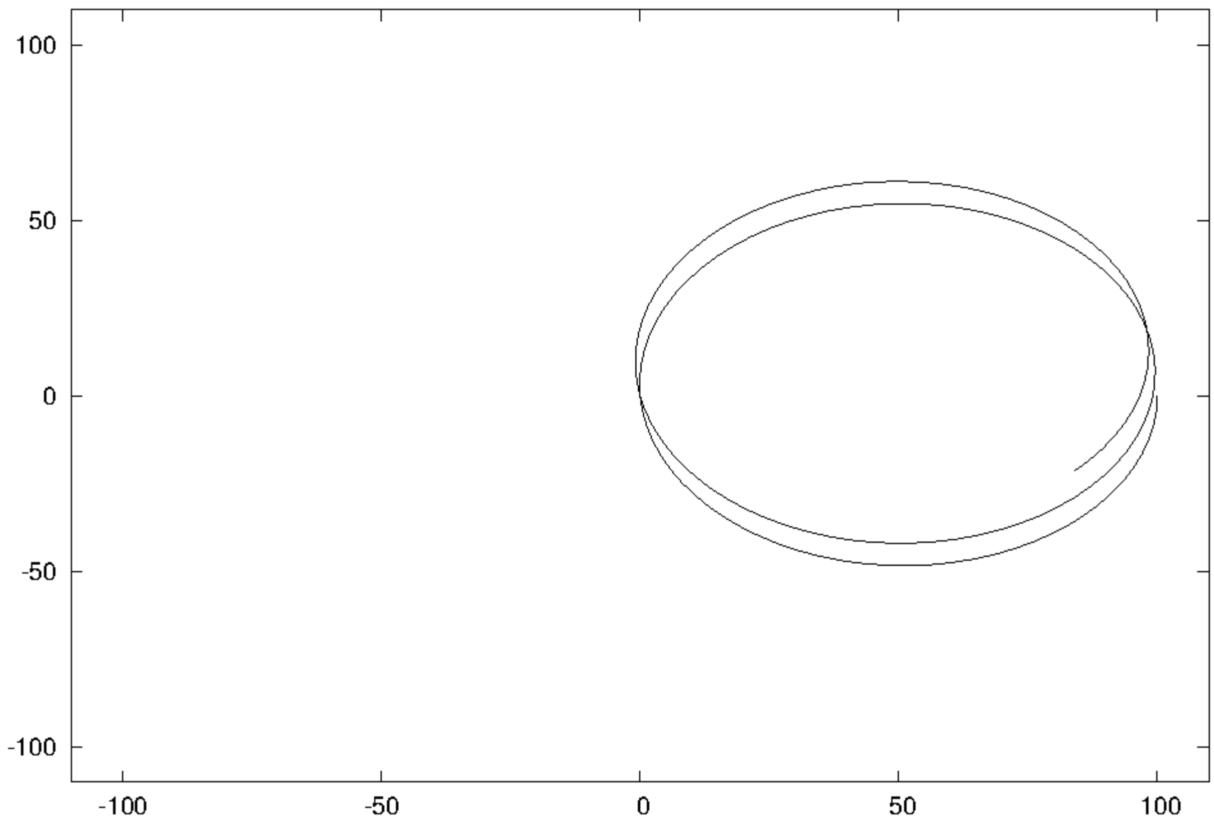
at 45 degrees

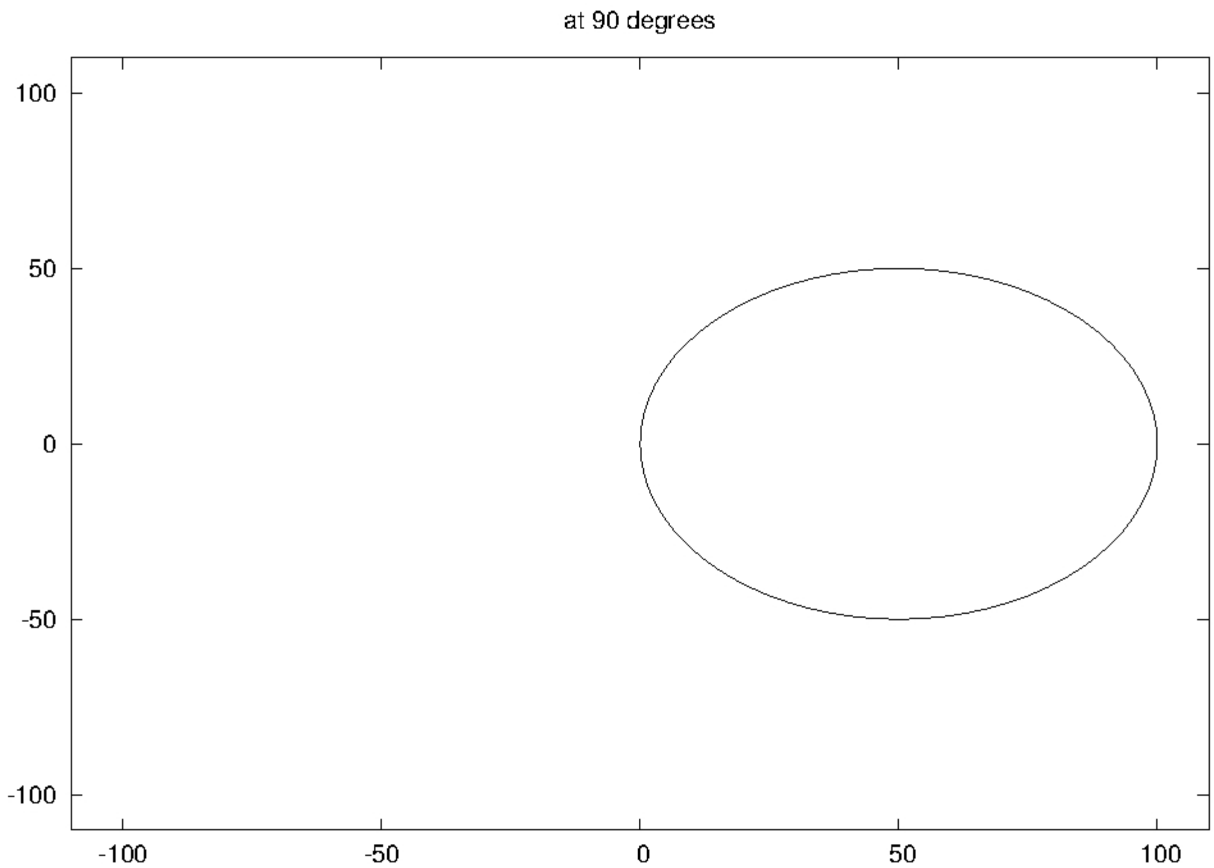


at 60 degrees



at 75 degrees





at 90 degrees
100
50
0
-50
-100
-100
-50
0
50
100