Quantum Entanglement and Thermodynamics in Spin Systems

Project Report: Phase 1

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Outline

Entropy

Boltzmann Entropy:

In Statistical Mechanics, Boltzmann's Equation is given by:

$$S = k_B \ln(\omega)$$

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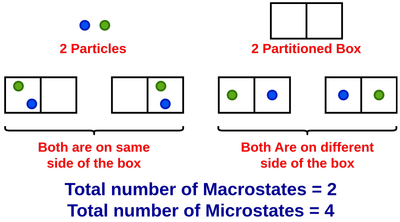

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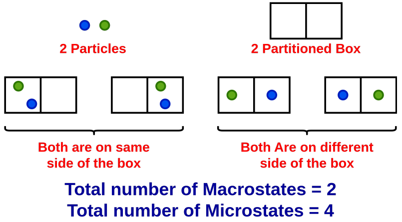

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von Neumann Entropy:

In a quantum Mechanical system represented by a Density matrix ρ,
von Neumann entropy is:

$$S = -tr(\rho \ln \rho)$$

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Decoherence

Closed System:

Let us consider a pure state $|\psi\rangle$, Now if we let it evolve with time,

$$U(t)|\psi\rangle \rightarrow |\phi\rangle$$

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Open System:

Now let us consider the same state $|\psi\rangle$ interacting with the environment. So the state of the system will now become $|\psi'\rangle|e\rangle$ Now if we let it evolve with time,

$$U(t)|\psi'\rangle|e\rangle \rightarrow |F\rangle$$

Then, $|F\rangle$ may not be a pure state.

Outline

Dephasing Channel

Representation:

- To remove the coherence caused by the interaction of the system with the environment, we require a Dephasing channel, and the process is called Decoherence.
- For Computational Basis (i.e. $|0\rangle$ & $|1\rangle$), the Dephasing operation is given by:
$$D \to (1-p)\rho_{AB} + p\sigma_z\rho_{AB}\sigma_z,$$
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Operation:

$$\begin{bmatrix} a & x_1 & x_2 & x_4 \\ x_7 & b & x_3 & x_5 \\ x_8 & x_9 & c & x_6 \\ x_{10} & x_{11} & x_{12} & d \end{bmatrix} \xrightarrow{D} \begin{bmatrix} a & y_1 & y_2 & y_4 \\ y_7 & b & y_3 & y_5 \\ y_8 & y_9 & c & y_6 \\ y_{10} & y_{11} & y_{12} & d \end{bmatrix} \cdots\cdots \xrightarrow{D^n} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

Outline

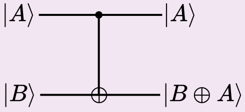
Controlled NOT Gate

Representation:

It is a 2-qubit gate represented as

$$|A, B\rangle \rightarrow |A, A \oplus B\rangle$$

It flips the 2^{nd} qubit, if the 1^{st} qubit is $|1\rangle$.



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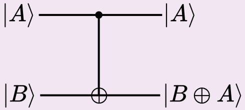


C-NOT Gate

Operation:

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Truth Table:

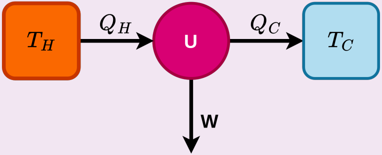
Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Outline

First Law:

It is just another form of conservation of energy.

$$\Delta U = Q - W$$



First Law of Thermodynamics

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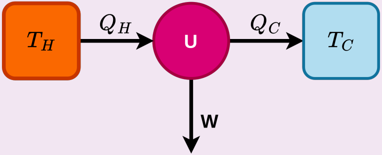

Isothermal Process:

For an Isothermal Process, $\Delta U = 0$.

$$So,\ Q = W\ and\ Q_{rev} = T\Delta S$$

$$W = T\Delta S = K_B T \ln(\omega)$$

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Thermodynamical Approach to Quantifying Quantum Correlations

TAQQC Paper:

VOLUME 89, NUMBER 18 PHYSICAL REVIEW LETTERS 28 OCTOBER 2002

Thermodynamical Approach to Quantifying Quantum Correlations

Jonathan Oppenheim,[1,2] Michał Horodecki,[2] Paweł Horodecki,[3] and Ryszard Horodecki[2]
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[3]*Faculty of Applied Physics and Mathematics, Technical University of Gdańsk, 80–952 Gdańsk, Poland*
(Received 11 February 2002; published 11 October 2002)

We consider the amount of work which can be extracted from a heat bath using a bipartite state ρ shared by two parties. In general it is less then the amount of work extractable when one party is in possession of the entire state. We derive bounds for this "work deficit" and calculate it explicitly for a number of different cases. In particuar, for pure states the work deficit is exactly equal to the distillable entanglement of the state. A form of complementarity exists between physical work which can be extracted and distillable entanglement. The work deficit is a good measure of the quantum correlations in a state and provides a new paradigm for understanding quantum nonlocality.

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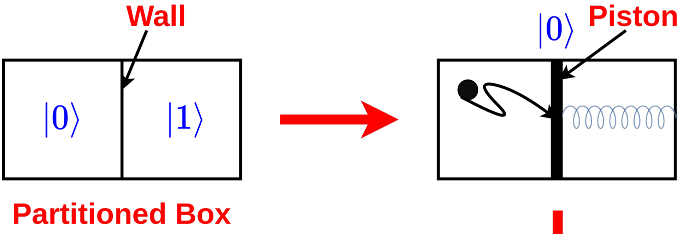
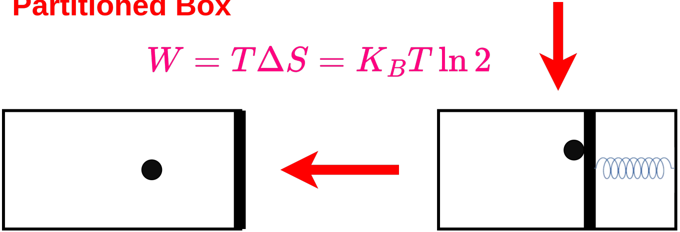
Work deficit is defined as the difference between the work extracted from taking the entire state with one party and the work extracted from taking the state shared with two parties.

Outline

Classical Model



$$W = T\Delta S = K_B T \ln 2$$



$a|0\rangle + b|1\rangle$ $a|0\rangle + b|1\rangle$

Work done by particle on the piston

Classical Model

Work Deficit:

1	2	....	n
$\boxed{\|0\rangle}\boxed{\|1\rangle}$	$\boxed{\|0\rangle}\boxed{\|1\rangle}$	$\bullet\bullet\bullet\bullet$	$\boxed{\|0\rangle}\boxed{\|1\rangle}$

$$H(X) = -\sum_{x=1}^{n} p_x log_2(p_x)$$

$$H(X) = -\sum_{x=1}^{n} \frac{1}{2^n} log_2(\frac{1}{2^n}) = n$$

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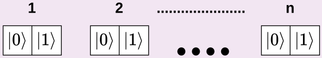

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Classical Model

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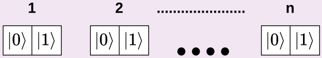

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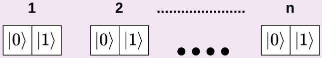

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- Now if we replace Shannon Entropy with von Neumann Entropy, $W_t = n - S(\rho)$.

Classical Model

Work Deficit:

1   2   ....   n

$\boxed{|0\rangle}\boxed{|1\rangle}$ $\boxed{|0\rangle}\boxed{|1\rangle}$ •••• $\boxed{|0\rangle}\boxed{|1\rangle}$

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- Now if we replace Shannon Entropy with von Neumann Entropy, $W_t = n - S(\rho)$.
- Now, If W_l is the work extracted by Alice and Bob after performing LOCC operations.
  Then work deficit, $\Delta = W_t - W_l$.

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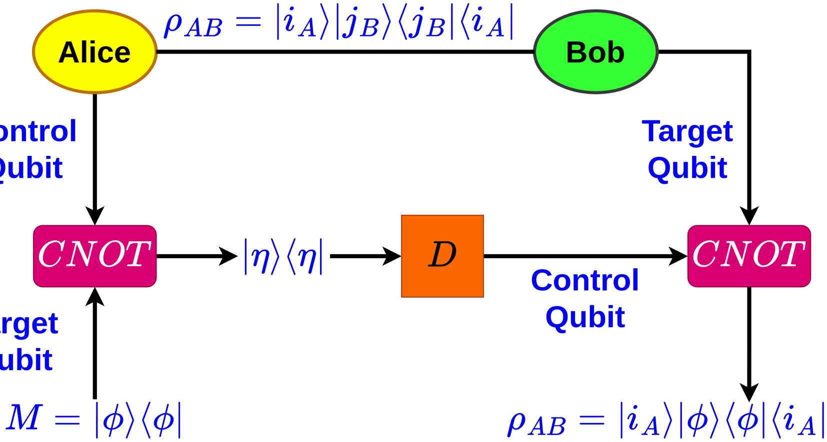
Quantum Model (LOCC)



Now Since Bob's qubit is known, so 1 bit of work can be extracted from Bob's qubit.

Fortran Program

```
!Generate Random Density Matrix
call RANDDMR(d,state)
!Compute Eigen Values
call DSYEV(JOBZ, UPLO, N, state, LDA, W,
   WORK, LWORK, INFO)
! Calculate the von Neumann entropy
sum = 0.0D0
DO i = 0, d - 1
    IF (W(i) > 0.0D0) THEN
        sum = sum - W(i) *
           LOG(W(i))/LOG(2.0d0)
    END IF
END DO
!Applying CNOT on rho
csys=0.0d0
call cnot(sys, csys)
```

Fortran Program

```
!Initialising parameters for PTRDMR
site(0) = 1
call PTRDMR(s, s1, site, csys, rdm)
! Call DSYEV to compute eigenvalues and
   eigenvectors
call DSYEV(JOBZ, UPLO, M, rdmb, LDA1, RW,
   RWORK, LWORK1, INFO)
! Calculate the von Neumann entropy
sum = 0.0D0
DO i = 0, m - 1
    IF (RW(i) > 0.0D0) THEN
        sum = sum - RW(i) *
            LOG(RW(i))/LOG(2.0d0)
    END IF
END DO
c_ent = ent_rho - ent_red
write(100,*) c_ent
```

Output

```
RHO :
  4.8770983126898163E-002   3.2789837842840362E-002  -8.2253410255965928E-002  -3.7497881267235578E-002
  3.2789837842840362E-002  0.20979556004068756      -0.14060519364323271      -0.10482907089261517
 -8.2253410255965928E-002 -0.14060519364323271       0.19387716076973360       0.17690209306155602
 -3.7497881267235578E-002 -0.10482907089261517       0.17690209306155602       0.54755629606268064
EV of RHO :
  1.3720352237903246E-003   8.0458945030873377E-002  0.23726620104507512       0.68090281870026104
Entropy of RHO :
  1.1755379513478428
CNOT_RHO :
  4.8770983126898163E-002   3.2789837842840362E-002  -3.7497881267235578E-002  -8.2253410255965928E-002
  3.2789837842840362E-002  0.20979556004068756      -0.10482907089261517      -0.14060519364323271
 -3.7497881267235578E-002 -0.10482907089261517       0.54755629606268064       0.17690209306155602
 -8.2253410255965928E-002 -0.14060519364323271       0.17690209306155602       0.19387716076973360
EV of CNOT_RHO :
  1.3720352237903068E-003   8.0458945030873349E-002  0.23726620104507512       0.68090281870026115
Entropy of CNOT_RHO :
  1.1755379513478423
RED_CNOT_RHO :
 0.59632727918957884       0.20969193090439639
 0.20969193090439639       0.40367272081042116
EV of RHO_B :
 0.26924114187645748       0.73075885812354247
Entropy of RHO_B :
 0.84037361376372455
Change in Entropy :
 0.33516433758411823
```

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- TAQQC Paper review.
- Made code for calculating the change in entropy and then Work deficit for 2 qubit cases.

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Future Work

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- Because this leads to two spin half's for every site, the result must be the wavefunction of a spin 1 system.
- Understanding the implications of work deficit in multiqubit scenarios.

References

References:

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Thanks For Your Attention!