Schmidt Decomposition

A Programmed Journey

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Schmidt Decomposition

Schmidt decomposition

Suppose $|\psi_{AB}\rangle$ is a pure state of a composite system. Then there exist orthonormal states $|i_A\rangle$ for system A, and orthonormal states $|i_B\rangle$ of system B, such that

$$|\psi_{AB}\rangle = \sum \lambda_i |i_A\rangle \otimes |i_B\rangle \tag{1}$$

Singular Value Decomposition

Singular Value Decomposition

It can be applied to any matrix.

$$A = UDV^{\dagger} \tag{2}$$

where U and V are two unitary matrices and D is a Diagonal matrix.

Formulation

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Let us consider the state of a system consisting of two subsystems as

$$|\psi_{AB}\rangle = \sum_{i=1}^{N} \sum_{j=1}^{M} a_{ij} |i\rangle \otimes |j\rangle$$

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$$|\psi_{AB}\rangle = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}|i\rangle \otimes |j\rangle$$

$$|\psi_{AB}\rangle\langle\psi_{AB}| = (\sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}|i\rangle \otimes |j\rangle)(\sum_{k=1}^{N}\sum_{l=1}^{M} a_{kl}^{*}\langle k| \otimes \langle l|)$$

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Let us consider the state of a system consisting of two subsystems as

$$|\psi_{AB}\rangle = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}|i\rangle \otimes |j\rangle$$

$$\begin{aligned} |\psi_{AB}\rangle\langle\psi_{AB}| &= \Big(\sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}|i\rangle \otimes |j\rangle\Big)\Big(\sum_{k=1}^{N}\sum_{l=1}^{M} a^*_{kl}\langle k| \otimes \langle l|\Big) \\ |\psi_{AB}\rangle\langle\psi_{AB}| &= \sum_{i,j,k,l} a_{ij}a^*_{kl}|i\rangle\langle k| \otimes |j\rangle\langle l| \end{aligned} \tag{3}$$

Formulation

Reduced Density Matrix

Now let us take the partial trace with respect to B subsystem

Formulation

Reduced Density Matrix

Now let us take the partial trace with respect to B subsystem

$$tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_p \sum_{i,j,k,l} a_{ij}a^*_{kl}|i\rangle\langle k| \otimes \langle p|j\rangle\langle l|p\rangle$$

Formulation

Reduced Density Matrix

Now let us take the partial trace with respect to B subsystem

$$
\begin{aligned}
tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) &= \sum_p \sum_{i,j,k,l} a_{ij}a^*_{kl}|i\rangle\langle k| \otimes \langle p|j\rangle\langle l|p\rangle \\
&= \sum_p \sum_{i,k} a_{ip}a^*_{kp}|i\rangle\langle k|
\end{aligned}
$$

Formulation

Reduced Density Matrix

Now let us take the partial trace with respect to B subsystem

$$tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = \sum_p \sum_{i,j,k,l} a_{ij}a^*_{kl}|i\rangle\langle k| \otimes \langle p|j\rangle\langle l|p\rangle$$

$$= \sum_p \sum_{i,k} a_{ip}a^*_{kp}|i\rangle\langle k|$$

$$\rho_A = \sum_{i,k} \sum_p a_{ip}a^*_{kp}|i\rangle\langle k|$$

Formulation

Reduced Density Matrix

Now let us take the partial trace with respect to B subsystem

$$\begin{aligned} tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) &= \sum_p \sum_{i,j,k,l} a_{ij}a^*_{kl}|i\rangle\langle k| \otimes \langle p|j\rangle\langle l|p\rangle \\ &= \sum_p \sum_{i,k} a_{ip}a^*_{kp}|i\rangle\langle k| \\ \rho_A &= \sum_{i,k} \sum_p a_{ip}a^*_{kp}|i\rangle\langle k| \\ \rho_A = AA^\dagger \qquad & Similarly, \rho_B = A^\dagger A \end{aligned} \tag{4}$$

Formulation

Eigen Values

Now from Singular Value Decomposition, we can write

$$A = UDV^{\dagger}$$

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Eigen Values

Now from Singular Value Decomposition, we can write

$$A = UDV^{\dagger}$$

$$AA^{\dagger} = UDV^{\dagger}VDU^{\dagger}$$

$$AA^{\dagger} = UD^2U^{\dagger}$$

$$So, \sqrt{\lambda_i} \ are \ the \ eigenvalues \ of \ AA^{\dagger}$$

Formulation

Schmidt Decomposition

Singular Value Decomposition

Formulation of Schmidt Decomposition

Flowchart

References

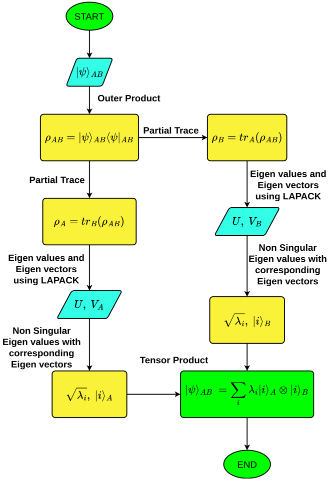
Eigen Values

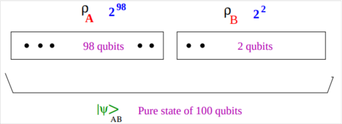
Now from Singular Value Decomposition, we can write

$$A = UDV^\dagger$$

$$\begin{aligned} AA^\dagger &= UDV^\dagger VDU^\dagger \\ AA^\dagger &= UD^2U^\dagger \\ So, \sqrt{\lambda_i}\ &are\ the\ eigenvalues\ of\ AA^\dagger \\ Similarly, A^\dagger A &= VD^2V^\dagger \\ So, \sqrt{\lambda_i}\ &are\ the\ eigenvalues\ of\ A^\dagger A \end{aligned} \tag{5}$$

Flowchart





$$|\psi\rangle_{AB} = \underbrace{\sum_{i}^{2^{100}} c_{AB}|\alpha_i\rangle_A|\beta_i\rangle_B}_{2^{100}\ terms} = \underbrace{\sum_{i}^{2^2} \sqrt{\lambda_i}|\psi_{98}\rangle_i|\psi_2\rangle_i}_{Only\ 2^2\ terms}$$

References

- Quantum Computation and Quantum Information: Michael A. Neilsen and L. Chuang, Cambridge University Press 2000
- Numerical Recipes in Quantum Information Theory and Quantum Computing: M. S. Ramkarthik, and Payal D. Solanki, CRC Press 2022

Thanks!