September 10, 2024

* Hypothesis Testing

α

β

$(1-\beta)$

Simple: Hypo complete info

Composite: Incomplete

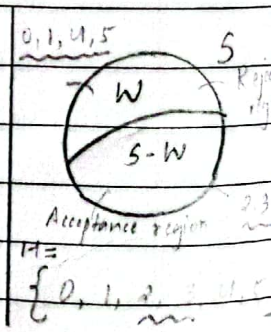
Simple Hypothesis

A hypothesis which completely specifies the distribution of random variables is known as simple hypothesis. For example, $X \sim N(\mu, \sigma^2)$ [i.e. X follows normal distribution with parameters μ and σ^2] and the hypothesis to be tested as $H_0: \mu = \mu_0, \sigma^2 = \sigma_0^2$, then under this hypothesis the form of the distribution is completely specified.

Composite Hypothesis

A hypothesis which ~~dis~~ does not completely ~~specified~~ specify the form of the distribution is said to be composite hypothesis. For example, $X \sim N(\mu, \sigma^2)$ and the hypothesis to be tested as $H_0: \mu = \mu_0$ then under this hypothesis σ^2 is not specified.

Critical Region



Let us define a Sample Space 's' and we divide the whole sample space into two mutually exclusive (disjoint) subsets say, W and S-W such that,

$$W \cup (S - W) = S$$

$$\text{and } W \cap (S-W) = \phi \quad (\text{Null set})$$

If the sample points fall in the region W, then we can reject the null hypothesis (H_0) and the regio[n] is called region of rejection or it is also called critica[l] region and if the sample point fall in the region (S-W then we accept the null hypothesis and the region is called region of acceptance.

Errors in Hypothesis Testing

Type I error:

Consumer's Risk
The error committed in rejecting H_0 when it is true is called Type I error. The probability of commiting Type I error is called size of the test or size of the critical region and denoted by α and is given by

$\alpha = P(x \in W / H_0)$

$$\alpha = Prob(\text{type I error})$$
$$= Prob(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$
$$= Prob(x \in W / H_0)$$
$$= \int_{x \in W} f(x/H_0)\, dx \quad [\text{Continuous}]$$
$$= \sum_{x \in W} P(x/H_0) \quad [\text{Discrete}]$$

Producer's Risk
Type II error: The error committed in accepting H_0 when H_0 is false called type second error. The probability of commiting type II^{nd} error is called size of the type II error and it is denoted by β and is given by:

$$\beta = Prob(\text{Type II error})$$
$$= Prob(\text{accept } H_0 \text{ when } H_0 \text{ is false})$$
$$= Prob(\text{accept } H_0 \text{ when } H_1 \text{ is True})$$
$$= P(x \in S-W / H_1)$$
$$= \int_{x \in S-W} f(x/H_1)\, dx$$

$$= \sum_{x \in S-W} P(x/H_1)$$

Power of the Test $(1-\beta)$

The complementary part probability of committing type II error is known as power of the test or power of the function and it is denoted by $(1-\beta)$ i.e.

$P(A/B) + P(A'/B) = 1$

$$1-\beta = 1 - \text{Prob}(\text{accept } H_0 \text{ when } H_0 \text{ is false})$$
$$= 1 - \text{Prob}(\text{accept } H_0 \text{ when } H_1 \text{ is true})$$
$$= \text{Prob}(\text{reject } H_0 \text{ when } H_1 \text{ is true})$$
$$= \text{Prob}(\text{accept } H_1 \text{ when } H_1 \text{ is true})$$
$$= \text{Prob}(x \in W/H_1)$$
$$= \int_{x \in W} f(x/H_1)\, dx \quad \text{(Continuous)}$$

$$\Rightarrow (1-\beta) = \sum_{x \in W} P(x/H_1) \quad \text{(Discrete)}$$

$$\Rightarrow (1-\beta) = \text{Prob}(\text{Correct decision})$$

Power of the test is also defined as the probability of correct decision.

In any test, we would like to make the probability of correct decision as large as possible. In fact, we would like to make $1-\beta = 1$ i.e. the probability of committing type II error as minimum as possible.

Q. Toss a coin 5 times, if 2 or 3 times head turn up then the coin is accepted as an unbiased coin. Find α, β and $(1-\beta)$ when $H_0: p = 1/2$ against $H_1: p = 3/4$. Find α, β.

$\alpha = \text{Prob}(x \in W/H_0)$

$\beta = \text{Prob}(x \in S-W/H_1)$

Solution

Let X = number of heads.

Since, a coin is tossed five times

$S = \{0, 1, 2, 3, 4, 5\}$

Now, $S - W = \{2, 3\}$

and $W = \{0, 1, 4, 5\}$

Here, $X \sim B(n, p)$ so $P(X = x) = {}^nC_x \, p^x \, q^{n-x}$; $x = 0, 1, \dots, n$

We have, $n = 5$ and $H_0 : P = 1/2$ ~~and~~ vs $H_1 : P = 3/4$

$\therefore \alpha = Prob\left(x \in W / H_0\right)$

$= P\left(x \in 0, 1, 4, 5 / p = 1/2\right)$

$= P\left(X = 0 / p = 1/2\right) + P\left(X = 1 / p = 1/2\right) + P\left(X = 4 / p = 1/2\right)$

$+ P\left(X = 5 / p = 1/2\right)$

$= {}^5C_0 (1/2)^0 (1/2)^{5-0} + {}^5C_1 (1/2)^1 (1/2)^{5-1} + {}^5C_4 (1/2)^4$
$(1/2)^{5-1} + {}^5C_5 (1/2)^5 (1/2)^{5-5}$

$= 0.375$

$\beta = P\left(x \in S - W / H_1\right)$

$= P\left(x \in 2, 3 / H_1\right)$

$= P\left(X = 2 / p = 3/4\right) + P\left(X = 3 / p = 3/4\right)$

$= {}^5C_2 (3/4)^2 (1/4)^{5-2} + {}^5C_3 (3/4)^3 (1/4)^{5-3}$

$= 0.087 + 0.263 = 0.35$

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Sep 12, 2024

$\alpha =$ Prob (reject H_0 / H_0)
$= P(x \in W / H_0)$

$\beta =$ Prob (accept H_0 / H_1)
$= P(x \in S - W / H_1)$

Q. If $X \sim P(\lambda)$ and critical region is defined as $X > 2$. To test the null hypothesis $H_0 : \lambda = 1$ against $H_1 : \lambda = 3$; find α, β and power of the test.

Solution:

Since, $X \sim P(\lambda)$ so $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$; $x = 0, 1, 2, \ldots$; $\lambda > 0$

$X \sim P(\lambda)$
$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$; $x = 0, 1, 2, \ldots$; $\lambda > 0$

Given, $W = \{x : x > 2\}$
$S - W = \{x : x \le 2\}$

Now, $P(X / H_0) = \frac{e^{-1} 1^x}{x!}$ and $P(X / H_1) = \frac{e^{-3} 3^x}{x!}$

Probability of type I error,

$$\begin{aligned}
\alpha &= P(x \in W / H_0) \\
&= P(x > 2 / H_0) \\
&= 1 - P(x \le 2 / H_0) \\
&= 1 - \left[P(X = 0 / H_0) + P(X = 1 / H_0) + P(X = 2 / H_0) \right] \\
&= 1 - \left[\frac{e^{-1} 1^0}{0!} + \frac{e^{-1} 1^1}{1!} + \frac{e^{-1} 1^2}{2!} \right] \\
&= 1 - 0.919 \\
&= \cancel{0.0808} \; 0.0803
\end{aligned}$$

Probability of type II error,

$$\beta = P\left(x \in S-W / H_1\right)$$

$$= P\left(x \le 2 / H_1\right)$$

$$= P\left(X=0 / H_1\right) + P\left(X=1 / H_1\right) + P\left(X=2 / H_1\right)$$

$$= P\left(X=0 / \lambda=3\right) + P\left(X=1 / \lambda=3\right) + P\left(X=2 / \lambda=3\right)$$

$$= \frac{e^{-3} 3^0}{0!} + \frac{e^{-3} 3^1}{1!} + \frac{e^{-3} 3^2}{2!}$$

$$= \underline{0.423}$$

Power of the test,

$$1-\beta = 1-0.423$$

$$= \underline{0.576}$$

Q. If $x \ge 1$ is the critical region for testing $H_0: \theta = 2$ against $H_1: \theta = 1$ on the basis of single observation from the population,

$$f(x, \theta) = \theta e^{-\theta x} \quad ; \quad 0 \le x \le \infty$$

Obtain the value of type I, type II errors. Also, find the power of the test.

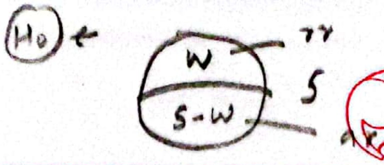
Solution

Given, $f(x, \theta) = \theta e^{-\theta x} \quad ; \quad x > 0$

and $H_0: \theta = 2$ against $H_1: \theta = 1$

$$W = \{x : x \ge 1\} \text{ and } S-W = \{x : x < 1\}$$

Type I error



I II

A

$$\alpha = Prob(x \in W / H_0)$$

$$= \int_1^\infty f(x/H_0)\, dx$$

$x \in S\text{-}W / H_1$

$$\Rightarrow \alpha = \int_1^\infty 2e^{-2x}\, dx$$

$$\Rightarrow \alpha = 2\left[\frac{e^{-2x}}{-2}\right]_1^\infty$$

$$\Rightarrow \alpha = -\left[e^{-2x}\right]_1^\infty$$

$$\Rightarrow \alpha = -\left(0 - \frac{1}{e^2}\right)$$

$$\Rightarrow \alpha = \frac{1}{e^2}$$

$$\therefore \alpha = \underline{0.135}$$

Type II error

$$\beta = Prob\left(x \in S\text{-}W / H_1\right) = P\left(x < 1 / H_1\right)$$

Continuous; so allowed ⟵

$$\Rightarrow \beta = \int_0^1 f(x/H_1)\, dx$$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$\Rightarrow \beta = \int_0^1 1e^{-x}\, dx$$

$$\Rightarrow \beta = \left[\frac{e^{-x}}{-1}\right]_0^1$$

$$\Rightarrow \beta = \left[e^{-x}\right]_1^0$$

$$\Rightarrow \beta = e^0 - \frac{1}{e} \qquad \therefore \beta = \underline{0.632}$$

$\therefore$ Power of the test $= 1 - \beta = 1 - 0.632 = \underline{0.368}$

Q. Given the freq^n function,

$$f(x,\theta) = \begin{cases} 1/\theta \; ; & 0 \le x \le \theta \\ 0 \; ; & \text{otherwise} \end{cases}$$

and that you are testing null hypothesis $H_0 : \theta = 1$ against $H_1 : \theta = 2$.

What would be the size of the type I and type II errors, if you choose the interval

(i) $0.5 \le x$ (ii) $1 \le x \le 1.5$ as the critical regions?

Also obtain the power function of the test.

$W = \{x : x \ge 0.5\}$

$S - W = \{x : x < 0.5\} \; x \ge 0$

Solution

$$\alpha = \int_{x \in W} f(x/H_0)\, dx$$

$$\beta = \int_{x \in S-W} f(x/H_1)\, dx$$

$$\alpha = \int_{x \in W} f(x/H_0)\, dx$$

$$= \int_{0.5}^{\infty} f(x/H_0)\, dx$$

$$= \int_{0.5}^{1} 1/\theta \, dx + \cancel{\int_{1}^{\infty} 0 \, dx}^{\,0}$$

$$= \frac{1}{\theta}\left[x\right]_{0.5}^{1}$$

$$= \frac{1}{\theta}(1 - 0.5)$$

$$= \frac{1}{1}(0.5)$$

$$= 0.5$$

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Solution

Given $f(x, \theta) = \frac{1}{\theta}$; $0 \le x \le \theta$

$H_0 : \theta = 1$ vs $H_1 : \theta = 2$

(i) Given, $w = \{x : x \ge 0.5\}$

$S - W = \{x : x < 0.5\}$

Probability of type I error,

$$\alpha = \int_{x \in W} f(x/H_0)\, dx$$

$$= \int_{0.5}^{\theta} f(x/\theta = 1)\, dx$$

$$= \int_{0.5}^{1} 1/1\, dx$$

$$= [x]_{0.5}^{1}$$

$= 1 - 0.5$
$= 0.5$

Python
?

$$\beta = \int_{x \in S-W} f(x/H_1)\, dx$$

$$= \int_{0}^{0.5} f\, x/\theta_1$$

$$= \int_0^{0.5} f(x/\theta=2)\, dx$$

$$= \int_0^{0.5} \frac{1}{2}\, dx$$

$$= \frac{1}{2}\left[x\right]_0^{0.5}$$

$$= \frac{1}{2} * 0.5$$

$$= \underline{0.25}$$

Power of the test $= 1-\beta$

$= 1-0.25$

$= 0.75$

(ii) Given, $w = \{x : x \quad 1 \le x \le 1.5\}$

$S-w = \{x : x$

$\alpha = P(x \in w / H_0)$

x Doesn't lie in this interval

$$= \int_{\boxed{1}}^{\boxed{1.5}} f(x/\theta=1)\, dx$$

x is only defined for 0 to θ i.e. 0 to 1

$\Rightarrow \alpha = 0$ [Since probability does not exists) $H_0 : \theta = 1$ and ~~thisa~~ $f(x) = \frac{1}{\theta};\ 0 \le x \le \theta$]

$\beta = P(x \in S-w / H_1)$

$$= \int_{x \in S-w} f(x/H_1)\, dx \qquad = 1 - \int_{x \in w} f(x/H_1)\, dx$$

NOTE: $\beta = P(x \in S-w \mid H_1)$

$\beta = 1 - P(x \in w \mid H_1)$

classmate

Date

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$$= 1 - \int_{1}^{1.5} f(x/_{\theta=2})\, dx$$

$$= 1 - \int_{1}^{1.5} \frac{1}{2}\, dx$$

$$= 1 - \frac{1}{2}\left[x\right]_{1}^{1.5}$$

$$= 1 - \frac{(1.5-1)}{2}$$

$$= 1 - 0.25$$

$$= \underline{0.75}$$

$\therefore$ The power of the test,

$$1-\beta = 1 - 0.75 = \underline{\underline{0.25}}$$

$\boxed{\text{Sep 14, 2024}}$

Normal Distribution

If a continuous random variable X follows normal distribution with parameters μ and σ^2, then the probability density function of the distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-1/2\left(\frac{X-\mu}{\sigma}\right)^2} \;; \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

Standard Normal Distribution

If $X \sim N(\mu, \sigma^2)$ then $Z \sim N(0,1)$ and then pdf of z is given by

$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-1/2\, z^2} \;; \quad -\infty < z < \infty$$

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where $z = \frac{x - \mu}{\sigma}$ = standard normal variate

$E(X) = \mu$

$V(X) = \sigma^2$

$E(Z) = 0$

$V(Z) = 1$

$$E(Z) = E\left[\frac{X-\mu}{\sigma}\right]$$

$$= \frac{1}{\sigma}\left[E(X) - \mu\right]$$

$$= \frac{1}{\sigma}(\mu - \mu)$$

$$= 0$$

$V(aX) = a^2 V(X)$

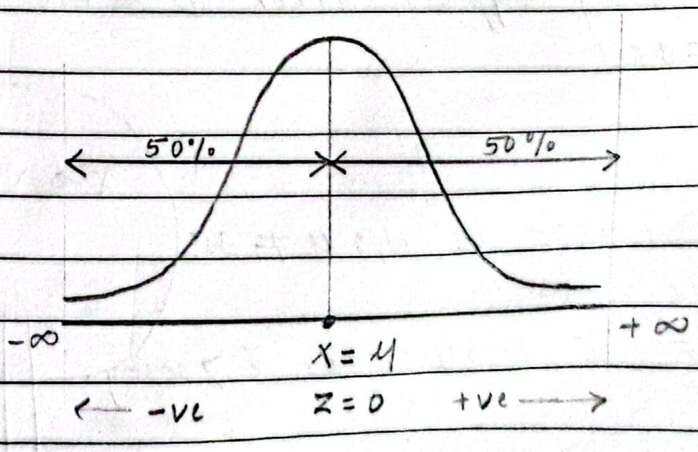
$$V(Z) = V\left[\frac{X-\mu}{\sigma}\right]$$

$$= \frac{1}{\sigma^2}\left[V(X) - V(\mu)\right]$$

$$= \frac{1}{\sigma^2}(\sigma^2 - 0)$$

$$= \sigma^2/\sigma^2 = 1$$

Properties:



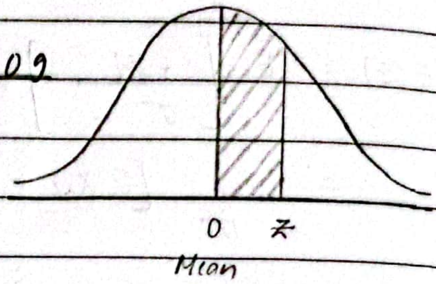
68.2%

95.4%

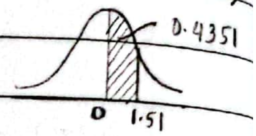
99.7%

Area under normal curve

Z	0.00	0.01	...	0.09
0.0				
0.1				
.				
.		0.4345		
.				
3.6				



$P(0 < Z < 1.51)$



Q. Suppose it is desired to test the hypothesis $H_0: \mu = 35$ against the alternative hypothesis $H_1: \mu \neq 35$ on the basis of random sample of size 16 from a normal pop^n $N(\mu, 1)$. The decision rule is; reject H_0 if the sample mean $\bar{x} < 34.65$ or $\bar{x} > 35.35$

(a) Find the prob of type I error

(b) Find the prob of type II error (i) when $\mu = 36$ and (ii) when $\mu = 36.1$

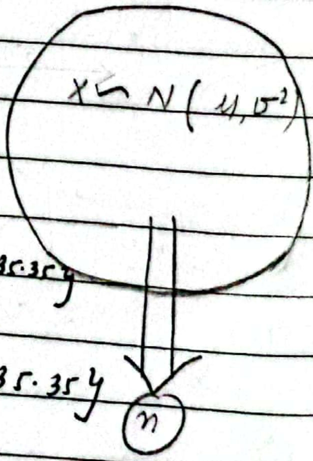
Solution

Given, $H_0: \mu = 35$ vs $H_1: \mu \neq 35$

$n = 16$

and $w = \{\bar{x} : \bar{x} < 34.65 \text{ or } \bar{x} > 35.35\}$

$\therefore S - w = \{\bar{x} : 34.65 \leq \bar{x} \leq 35.35\}$



$Z = \frac{X - \mu}{\sigma}$

If $X \sim N(\mu, \sigma^2)$ then $\bar{x} \sim N(\mu, \sigma^2/n)$ $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$\therefore \bar{x} \sim N\left(\mu, \frac{1}{n}\right) \qquad [\because x \sim N(\mu, 1)]$

ⓐ Probability of type I error

$$\alpha = Prob(x \in w / H_0)$$

$$= Prob(\bar{x} < 34.65 \text{ or } \bar{x} > 35.35 / \mu = 35)$$

$$\Rightarrow \alpha = P(\bar{x} < 34.65 / \mu = 35) + P(\bar{x} > 35.35 / \mu = 35)$$

$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$$\Rightarrow \alpha = P\left(z < \frac{(34.65 - 35)}{1/\sqrt{16}}\right) + P\left(z > \frac{35.35 - 35}{1/\sqrt{16}}\right)$$

$$\Rightarrow \alpha = P(z < -1.4) + P(z > 1.4)$$

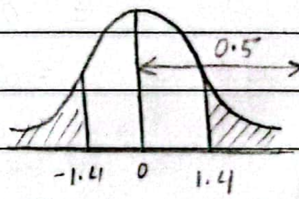
$$\Rightarrow \alpha = 2P(z > 1.4) \quad \text{(By symmetry)}$$

$$\Rightarrow \alpha = 2(0.5 - P(0 < z < 1.4))$$

$$\Rightarrow \alpha = 2(0.5 - 0.4192)$$

$$\Rightarrow \alpha = 0.1616$$

$$\therefore \alpha = 0.1616$$

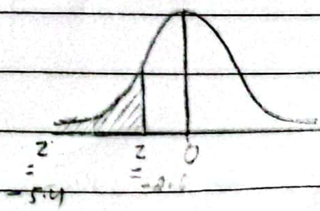


ⓑ Prob. of type II error when (i) $\mu = 36$ and (ii) $\mu = 36.1$

(i) $\mu = 36$

$$\beta = Prob(x \in S - w / H_1)$$

$$= P(\bar{x} : 34.65 \leq \bar{x} \leq 35.35 / \mu = 36)$$



$$= P(34.65 \leq \bar{x} \leq 35.35 / \mu = 36)$$

$$= P\left(\frac{34.65 - 36}{1/\sqrt{16}} \leq z \leq \frac{35.35 - 36}{1/\sqrt{16}}\right)$$

-5.4 -2.6

$$= P(-5.4 \leq z \leq -2.6)$$

$$= P(z \leq -2.6)$$

$$= 0.5 - P(-2.6 < Z < 0)$$
$$= 0.5 - 0.4953$$
$$= 0.0047$$

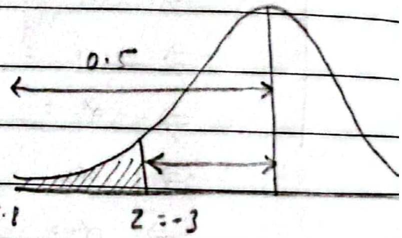
(ii) $\mu = 36.1$

$$\beta = Prob(x \in S-W / H_1)$$

$$= P(\bar{x} : 34.65 < \bar{x} < 35.35 \; / \; \mu = 36.1)$$

$$= P\left(\frac{34.65 - 36.1}{1/\sqrt{16}} < Z < \frac{35.35 - 36.1}{1/\sqrt{16}} \right)$$

$$= P(-5.8 < Z < -3)$$
$$= P(Z < -3)$$
$$= 0.5 - P(-3 < Z < 0)$$
$$= 0.5 - 0.4987$$
$$= 0.0013$$



[STAT] $\boxed{\text{iid}}$ Sep 21, 2024

Q. Let, x_1, x_2, x_3, x_4, x_5 be a random sample of size 5 drawn from a $N(\mu, 4)$, μ unknown. If you choose $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \geq 0$ as a critical region for testing $H_0: \mu = -1$ against $H_1: \mu = 1$, find the size of the critical region and power of the test.

$E(X_1 + X_2) = E(X_1) + E(X_2)$

$V(X_1 \pm X_2) = V(X_1) + V(X_2) \pm 2\,Cov(X_1, X_2)$

Solution

Given $x_i \sim N(\mu, 4)$ $\forall\, i = 1, 2, 3, 4, 5$

Let $y = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5$

$$E(y) = E(x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5)$$
$$= E(x_1) + 2E(x_2) + 3E(x_3) + 4E(x_4) + 5E(x_5)$$
$$= \mu + 2\mu + 3\mu + 4\mu + 5\mu$$
$$= 15\mu$$

and
$$V(y) = V(x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5)$$
$$= V(x_1) + 4V(x_2) + 9V(x_3) + 16V(x_4) + 25V(x_5)$$
$$= 4 + 4(4) + 9(4) + 16(4) + 25(4)$$
$$= 220$$

$\therefore y \sim N(15\mu, 220)$

Hence, $w = \{y : y \geq 0\}$ $s - w = \{y : y < 0\}$

$$\boxed{\text{Now, } Z = \frac{y - \mu}{\sigma}}$$

Prob. of type I error,

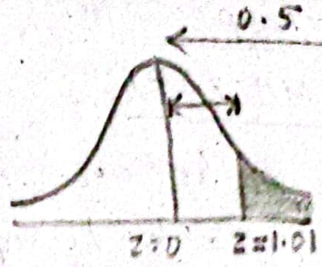
$$\alpha = Prob(x \in w / H_0)$$
$$= P(y \geq 0 / \mu = -1)$$
$$= P\left(Z \geq \frac{0 - 15(-1)}{\sqrt{220}}\right)$$

$$\Rightarrow \alpha = P(Z \geq 1.011)$$

$$= 0.5 - P(0 \leq Z \leq 1.011)$$

$$= 0.5 - 0.3438$$

$$= \underline{0.1562}$$



Probability of type II error,

$$\beta = Prob(x \in S-W / H_1)$$

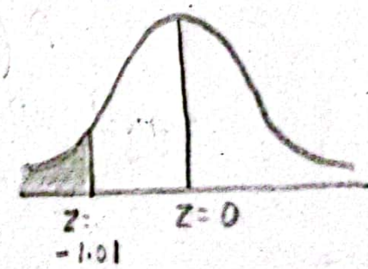
$$= Prob(y < 0 / \mu = 1)$$

$$= P\left(Z < \frac{0-15}{\sqrt{220}}\right)$$

$$= P(Z < -1.011) \quad \text{(By symmetry)}$$

$$= P(Z > 1.011) = 0.5 - 0.3438$$

$$= \underline{0.1562}$$



β = minimum

$(1-\beta)$ = maximum

Best Critical Region

A critical region of low size α is said to be ~~the~~ a best critical region if it has minimum β or maximum $(1-\beta)$ among all critical regions.

A test which minimizes β or maximizes the power $(1-\beta)$ for a desired low level of significance α, is known as best test.

Most powerful critical region (MPCR) and most powerful test (MP test)

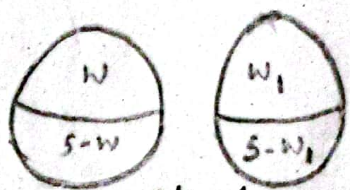
Let x be an arbitrary sample point in a sample space 'S'. Then, a critical region W is said to be a most powerful critical region (MPCR or BCR) of size α for testing a simple

↓ Best Critical Region

null hypothesis $H_0 : \theta = \theta_0$ against a simple alternative hypothesis $H_1 : \theta = \theta_1$ if

$$P(x \in W / H_0) = \alpha \quad \text{———} \quad (i)$$

and $P(x \in W / H_1) \geq P(x \in W_1 / H_1)$



for every other critical region W_1 satisfying (i) and a test corresponding to this MPCR or BCR of size α is called most powerful (MP) test of size α.

(IMP)

Theorem ↓

Neyman Pearson Lemma (N-P Lemma)

N-P lemma provides the general method of finding a most powerful test of simple hypothesis against the simple alternative hypothesis.

Statement:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size 'n' from popⁿ with prob. function $f(x; \theta)$. Let K be any non-negative constant and W be a critical region for testing a simple hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, then the most powerful critical region (MPCR) W of size α (fixed) is given by

$$\frac{L(x/H_1)}{L(x/H_0)} \geq k \quad \forall\ x \in W$$

or, $$\frac{L(x/H_1)}{L(x/H_0)} < k \quad \forall\ x \in S-W$$

where, $L(x/H_0)$ and $L(x/H_1)$ are the likelihood functions of the sample observations under H_0 and H_1 respectively.

Proof : Since, α is the probability of ~~fire~~ type I error

$$\therefore \alpha = \text{Prob}(\text{Type I error}) = \int_{x \in W} L(x/H_0)\,dx \quad \text{(i)}$$

$$= P(x \in W / H_0) \quad \text{(iv)}$$

and β is the probability of type II error,

$$\therefore \beta = \text{Prob}(\text{Type II error})$$

$$\Rightarrow \beta = P(x \in S-W / H_1)$$

$$= \int_{x \in S-W} L(x/H_1)\,dx$$

$$\Rightarrow \beta = 1 - \int_{x \in W} L(x/H_1)\,dx \quad \text{(ii)}$$

Let W^* be any other critical region with prob. of type I error α_1 (which is less than or equal to α) and prob. of type II error β_1

$$\therefore \alpha_1 = \text{Prob}(\text{Type I error})$$

$$= P(x \in W^* / H_0)$$

$$\Rightarrow \alpha = \int_{x \in W^*} L(x/H_0)\,dx \quad \text{(iii)}$$

and β_1 = Prob (Type II error)

$= \text{Prob}\left(x \in S - w^* / H_1 \right)$

$\Rightarrow \beta_1 = 1 - P\left(x \in w^* / H_1 \right)$ ——— (iv)

Since $\alpha \geq \alpha_1$

$$\Rightarrow \int_{x \in w} L(x/H_0)\, dx \geq \int_{x \in w^*} L(x/H_0)\, dx \quad \text{——— (v)}$$

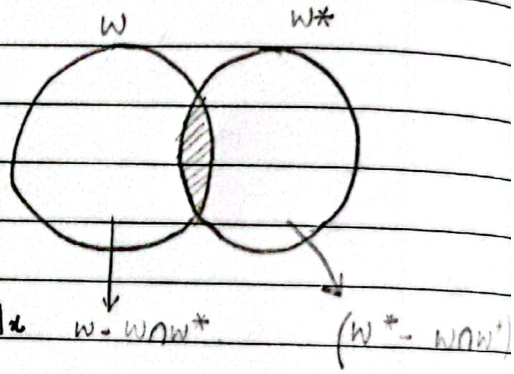
Oct 1, 2024

N-P Lemma (Continue)

From eqns (ii) and (v)

$$\beta_1 - \beta = 1 - \int_{x \in W^*} L(x/H_1)\, dx - 1 + \int_{x \in W} L(x/H_1)\, dx$$

$$\Rightarrow \beta_1 - \beta = \int_{x \in W} L(x/H_1)\, dx - \int_{x \in W^*} L(x/H_1)\, dx$$



$$\Rightarrow \beta_1 - \beta = \int_{x \in W - W \cap W^*} L(x/H_1)\, dx + \cancel{\int_{x \in W \cap W^*} L(x/H_1)\, dx} - \int_{x \in W^* - W \cap W^*} L(x/H_1)\, dx - \cancel{\int_{x \in W \cap W^*} L(x/H_1)\, dx}$$

$$\Rightarrow \beta_1 - \beta = \int_{x \in W - W \cap W^*} L(x/H_1)\, dx - \int_{x \in W^* - W \cap W^*} L(x/H_1)\, dx \quad \text{——— (V)}$$

From N-P Lemma,

$$\frac{L(x/H_1)}{L(x/H_0)} > k \quad \forall\, x \in W$$

$$1 - \beta_1 \leq 1 - \beta$$

$$\Rightarrow L(x/H_1) \geq k\, L(x/H_0)$$

Now,

$$\int_{x \in W - W \cap W^*} L(x/H_1)\, dx \geq k \int_{x \in W - W \cap W^*} L(x/H_0)\, dx \quad \text{——— (vi)}$$

and

$$\int_{x \in W^* - W \cap W^*} L(x/H_1)\, dx \geq k \int_{x \in W^* - W \cap W^*} L(x/H_0)\, dx \quad \text{——— (vii)}$$

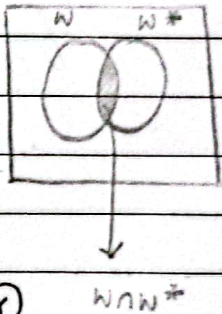
$$\cancel{\Rightarrow \beta_1 - \beta = \int_{x \in W - W\cap W^*} L(x/H_1)\,dx - \int_{x \in W^* - W\cap W^*} L(x/H_1)\,dx} \quad \text{— (vii)}$$

Using equations (vi) and (vii) in eqn (v), we get:

$$\beta_1 - \beta \geq K \int_{x \in W - W\cap W^*} L(x/H_0)\,dx - K \int_{x \in W^* - W\cap W^*} L(x/H_0)\,dx \quad \text{— (viii)}$$

From eqn (iii),

$$\int_{x \in W^*} L(x/H_0)\,dx \leq \int_{x \in W} L(x/H_0)\,dx$$



$$\Rightarrow \int_{x \in W - W\cap W^*} L(x/H_0)\,dx - \int_{x \in W^* - W\cap W^*} L(x/H_0)\,dx \geq 0 \quad \text{— (ix)}$$

$$\cancel{\Rightarrow \int_{x \in W - W\cap W^*} L(x/H_1)\,dx - \int_{x \in W^* - W\cap W^*} L(x/H_1)\,dx \geq 0}$$

From equation (ix) and (v), we get:

$$\beta_1 - \beta \geq 0$$

$$\Rightarrow \beta_1 \geq \beta$$

$$\Rightarrow 1 - \beta_1 \leq 1 - \beta$$

$$\boxed{\therefore\ 1 - \beta_1 \leq 1 - \beta}$$

$1 - \beta \geq 1 - \beta_1$. Hence, the critical region W of the fixed α is the most powerful critical region (MPCR or BCR) of the size α and the test is the most powerful test.

* **Theorem**

Statement: In testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_1$, the power of the MP critical region (MPCR or BCR) i.e. the power of the MP test is never less than the size of the BCR.

$$\therefore 1 - \beta \geq \alpha$$

Proof

Let w be a MP critical region or a BCR of size α (fixed) in testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, then by definition, we have,

$$\alpha = \int_{x \in w} L(x/H_0)\, dx \quad \text{and}$$

$$\beta = \int_{x \in S-w} L(x/H_1)\, dx$$

For N-P lemma, the BCR w is given by

$$\frac{L(x/H_1)}{L(x/H_0)} \geq k \quad \forall x \in w \Rightarrow L(x/H_1) \geq k\, L(x/H_0)$$

$$\Rightarrow \int_{x \in w} L(x/H_1)\, dx \geq k \int_{x \in w} L(x/H_0)\, dx$$

$$\Rightarrow (1-\beta) \geq k\alpha \quad \text{--- (i)}$$

Further,

$$\frac{L(x/H_1)}{L(x/H_0)} < k \quad \forall x \in S-w$$

$$\Rightarrow L(x/H_1) < k\, L(x/H_0)$$

$$\Rightarrow \int_{x \in S-w} L(x/H_1)\, dx < k \int_{x \in S-w} L(x/H_0)\, dx$$

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$\Rightarrow \beta \leq (1-\alpha) K$ ——— (ii)
$\Rightarrow (1-\alpha) K \geq \beta$

From (i) & (ii),
$(1-\beta)(1-\alpha) K \geq K \alpha \beta$
$\Rightarrow 1-\alpha-\beta+\alpha\beta \geq \alpha\beta$
$\Rightarrow 1-\alpha-\beta \geq 0$
$\Rightarrow \cancel{-\alpha - \beta \geq 0}$
$\boxed{1-\beta \geq \alpha}$

<u>Proved</u>

Hence, the power of the MP test is never less than the size of the MP critical region (BCR).

November 13, 2024

(IMP) Q. Let $x_1, x_2, \ldots, x_n$ be a random sample of size 'n' drawn from a normal population $N(\mu, \sigma^2)$ where σ^2 is known. Then for testing $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$, find best critical region (BCR) (i) If $\mu_1 > \mu_0$ and (ii) If $\mu_1 < \mu_0$. Find power of the tests for the above two cases.

Solution: If $X \sim N(\mu, \sigma^2)$ then the pdf the dist^n is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}; -\infty < x < \infty$$

Now, the likelihood function of $X = x_1, x_2, \ldots, x_n$

$\alpha =$ Prob (Reject H_0 / H_0 is true)
$= P(x \in w / H_0)$

$\beta =$ Prob (accept H_0 / H_1 is true)
$= P(x \in S-w / H_1)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$-\infty < x < \infty$
$-\infty < \mu < \infty$
$\sigma > 0$

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Date
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$$L = \prod_{i=1}^{n} f(x_i)$$

$$= \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2 \left(\frac{x_1 - \mu}{\sigma}\right)^2} \right\} \cdots \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2 \left(\frac{x_n - \mu}{\sigma}\right)^2} \right\}$$

$$\Rightarrow L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-1/2\sigma^2 \sum_{i=1}^{n} (x_i - \mu)^2}$$

Now, $L\left(x/H_0\right) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-1/2\sigma^2 \sum_{i=1}^{n}(x_i - \mu_0)^2}$ [$\because$ Under H_0, $\mu = \mu_0$]

and $L\left(x/H_1\right) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-1/2\sigma^2 \sum_{i=1}^{n}(x_i - \mu_1)^2}$ [$\because$ Under H_1, $\mu = \mu_1$]

Now, from N-P lemma, the best critical region 'w' is given by:

$$\frac{L(x/H_1)}{L(x/H_0)} \geq k \quad \forall x \in w$$

$$\Rightarrow \frac{e^{-1/2\sigma^2 \sum_{i=1}^{n}(x_i - \mu_1)^2}}{e^{-1/2\sigma^2 \sum_{i=1}^{n}(x_i - \mu_0)^2}} \geq k$$

$$\Rightarrow e^{-\frac{1}{2\sigma^2}\left[\Sigma(x_i - \mu_1)^2 - \Sigma(x_i - \mu_0)^2\right]} \geq k$$

Taking log on both sides, we get:

$$\frac{-1}{2\sigma^2}\left[\Sigma(x_i - \mu_1)^2 - \Sigma(x_i - \mu_0)^2\right] > \log k$$

$$\Rightarrow \cancel{\frac{-1}{2\sigma^2} \Sigma (x_i - \mu_1 + x_i - \mu_0)(x_i - \mu_1 - x_i + \mu_0)}$$

$$\Rightarrow \frac{-1}{2\sigma^2}\left[\Sigma\left(x_i^2 - 2x_i\mu_1 + \mu_1^2\right) - \Sigma\left(x_i^2 - 2x_i\mu_0 + \mu_0^2\right)\right] \geq \log k$$

$$\Rightarrow \frac{-1}{2\sigma^2}\left[\cancel{\Sigma x_i^2} - 2\mu_1(\Sigma x_i) + n\mu_1^2 - \cancel{\Sigma x_i^2} + 2\mu_0(\Sigma x_i) - n\mu_0^2\right] \geq \log k$$

$$\Rightarrow \frac{-1}{2\sigma^2}\left[-2n\mu_1\bar{x} + 2n\mu_0\bar{x} + n(\mu_1^2 - \mu_0^2)\right] \geq \log k$$

$$\Rightarrow \frac{-1}{2\sigma^2}\left[-2n\bar{x}(\mu_1 - \mu_0) + n(\mu_1^2 - \mu_0)^2\right] \geq \log k$$

$$\Rightarrow n\bar{x}(\mu_1 - \mu_0) - \frac{n}{2}(\mu_1^2 - \mu_0)^2 \geq \sigma^2 \log k$$

$$\Rightarrow n\bar{x}(\mu_1 - \mu_0) \geq \sigma^2 \log k + \frac{n}{2}(\mu_1^2 - \mu_0^2)$$

$$\Rightarrow \bar{x}(\mu_1 - \mu_0) \geq \frac{\sigma^2 \log k}{n} + \frac{(\mu_1^2 - \mu_0^2)}{2} \quad \text{--- (i)}$$

Case I

If $\mu_1 > \mu_0$, the BCR w is given by

$$\bar{x} \geq \frac{\sigma^2}{n}\frac{\log k}{(\mu_1 - \mu_0)} + \frac{(\mu_1 + \mu_0)}{2}$$

$$\Rightarrow \bar{x} \geq k_1$$

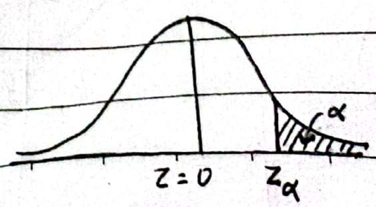
where, $k_1 = \frac{\sigma^2}{n}\frac{\log k}{(\mu_1 - \mu_0)} + \frac{(\mu_1 + \mu_0)}{2}$

$\therefore$ BCR is $w = \{x : \bar{x} \geq k_1\}$

Where the value of k_1 is obtained from fixed α

Now, $\alpha = Prob(x \in w / H_0)$

$\Rightarrow \alpha = Prob(\bar{x} \geq k_1 / H_0)$



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$\left[\bar{x} \pm Z_{\frac{\alpha}{2}} \, S.E.(\bar{x})\right]$

$$\Rightarrow \alpha = P\left(Z \geq \frac{k_1 - \mu_0}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow \frac{k_1 - \mu_0}{\sigma/\sqrt{n}} = z_\alpha$$

$$\Rightarrow k_1 = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Hence, the BCR, $w = \left\{ x : \bar{x} \geq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$

Now, the power of the test

$$1-\beta = P(x \in w / H_1)$$

$$\Rightarrow 1-\beta = P\left(\bar{x} \geq k_1 / \mu = \mu_1\right)$$

$$\Rightarrow 1-\beta = P\left(z > \frac{k_1 - \mu_1}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow 1-\beta = P\left(z \geq \frac{\mu_0 + z_\alpha \sigma/\sqrt{n} - \mu_1}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow 1-\beta = P\left(z \geq z_\alpha - \frac{(\mu_1 - \mu_0)}{\sigma/\sqrt{n}}\right)$$

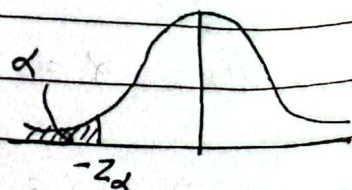
Case II :

$\mu_1 < \mu_0$ (left tailed), the BCR is given by

$$\bar{x} \leq \frac{\sigma^2}{n} \frac{\log k}{(\mu_1 - \mu_0)} + \cancel{\frac{(\mu_1 + \mu_0)}{2}} \; \frac{(\mu_1 + \mu_0)}{2}$$

$$\Rightarrow \bar{x} \leq k_2$$

where $k_2 = \frac{\sigma^2 \log k}{n(\mu_1 - \mu_0)} + \frac{(\mu_1 + \mu_0)}{2}$



$\therefore$ The BCR $w = \{ x : \bar{x} \leq k_2 \}$ and the value of k_2 is obtained from fixed α.

$$\alpha = \text{Prob}\left(x \in w / H_0\right)$$

$$= \text{Prob}\left(\bar{x} \in k_2 / \mu = \mu_0\right)$$

$$= \text{Prob}\left(Z \leq \frac{k_2 - \mu_0}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow \frac{k_2 - \mu_0}{\sigma/\sqrt{n}} = -Z_\alpha$$

$$\Rightarrow k_2 = \mu_0 - Z_\alpha \frac{\sigma}{\sqrt{n}}$$

$\therefore$ The BCR $(w) = \left\{ x : \bar{x} \leq \mu_0 - Z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$.

Now the power of the test:

$$1-\beta = P\left(x \in w / H_1\right)$$

$$\Rightarrow 1-\beta = P\left(\bar{x} \leq k_2 / \mu = \mu_1\right)$$

$$\Rightarrow 1-\beta = P\left(Z \leq \frac{k_2 - \mu_1}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow 1-\beta = P\left(Z \leq -Z_\alpha + \frac{(\mu_0 - \mu_1)}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow 1-\beta = P\left(Z < \frac{(\mu_0 - \mu_1)}{\sigma/\sqrt{n}} - Z_\alpha\right)$$

1%	5%	10%

$z_{0.01} = 2.33$ $z_{0.05} = 1.645$ $z_{0.1} = 1.28$ (1T) 1.96, 2.576, (2T)

3 marks

NOTE: When $\mu_1 < \mu_0$

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_1 - \mu_0)^2}$$

and $\mu_1 < \mu_0$

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_0 - \mu_1)^2}$$

Q. Let a random sample of size 40 is selected from a $N(\mu, 16)$. In testing $H_0: \mu = 50$ against $H_1: \mu = 52$ at 5% level of significance, find

(i) Best critical region of size α

(ii) Prob. of type II error

(iii) Power of the test

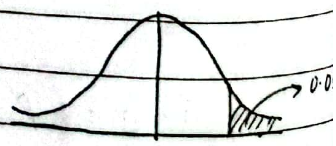
(iv) Also is this test unbiased?

Solution

Given, $n = 40$, $\sigma^2 = 16$, $\alpha = 0.05$

For testing $H_0: \mu = 50$ against $H_1: \mu = 52$ (>50), the right tailed test at α level of significance, the BCR of size α is given by:

$$w = \{ x : \bar{x} \geq k_1 \}$$



$Z_{0.05} = 1.645$

where, $k_1 = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$

$= 50 + 1.645 * \frac{4}{\sqrt{40}}$

$= 50 + 1.04$

$= \boxed{51.0411}$

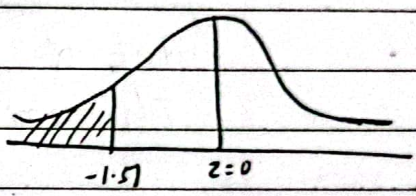
(ii) The prob. of type II error is given by,

$$\beta = Prob\left(x \in S-W / H_1\right)$$

$$= Prob\left(x < 57.0411 / \mu_1 = 52\right)$$

$$= Prob\left(z < \frac{57.0411 - 52}{4/\sqrt{40}}\right)$$

$$= Prob\left(z < -1.57\right)$$



$$= 0.5 - P(-1.5 < z < 0)$$

$$= 0.5 - 0.4357$$

$$= \boxed{0.0643}$$

(iii)

Power of the test $(1-\beta) = 1 - 0.0643$

$$= \boxed{0.9357}$$

(iv) Since, power of the test $(1-\beta) = 0.9357$ is very large as compared to $\alpha = 0.05$, so the test is unbiased.

November 14, 2024

Q. Suppose we want to test $H_0: \mu = 10$ against $H_1: \mu = 8$ using a random sample of size 'n' from a normal population $N(\mu, 4)$ and we reject the null hypothesis H_0 if $\bar{x} < c$. If $\alpha = 0.05$ and $\beta = 0.10$, find sample size 'n' and constant 'c'.

<u>Solution</u>

given, $X_i \sim N(\mu, 4), \quad \forall\, i = 1, 2, \ldots, n$

$\alpha = 0.05$

$\boxed{Z_\alpha = 1.645}$

$\beta = 0.10$

$\boxed{Z_\beta = 1.28}$

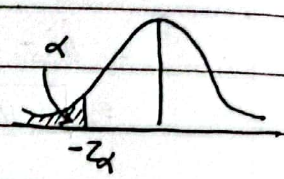
$$\Rightarrow \bar{x} \sim N\left(\mu, 4/n\right)$$

Here, we want to test $H_0: \mu = 10$ against $H_1: \mu = 8\ (< 10)$, left-tailed test at α level of significance.

The critical region, $w = \{x : \bar{x} < c\}$

$$\alpha = \text{Prob}\left(x \in w / H_0\right)$$

$$= P\left(\bar{x} < c / \mu = 10\right)$$



$$= \frac{c-10}{2/\sqrt{n}}$$

$$\Rightarrow \alpha = P\left(Z < \frac{c-10}{2/\sqrt{n}}\right)$$

$$\Rightarrow \frac{c-10}{2/\sqrt{n}} = -Z_\alpha$$

$$\Rightarrow c = 10 - \frac{2}{\sqrt{n}} Z_\alpha$$

$$\Rightarrow c = 10 - \frac{2}{\sqrt{n}}(1.645)$$

$$\therefore c = 10 - \frac{2}{\sqrt{n}}(1.645)$$

Now, the sample size,

$$n = \frac{(Z_\alpha + Z_\beta)^2 \sigma^2}{(\mu_0 - \mu_1)^2}$$

$$= \frac{(Z_{0.05} + Z_{0.1})\, 4}{(10-8)^2}$$

$= 8.55$

$\therefore \boxed{n \approx 9}$ (Take ceiling value)

Now,

$$c = 10 - \frac{2}{\sqrt{9}} (1.645)$$

$$= 10 - \frac{2}{3} * (1.645)$$

$$= \boxed{8.90}$$

H/W Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a $N(\mu, 100)$ and $w = \{x : \bar{x} \geq c\}$ is a BCR of size α for testing $H_0: \mu = 75$ against $H_1: \mu = 78$ at α level. If the probability of type I and type II errors are fixed at $\alpha = 0.05$ and $\beta = 0.10$, find the sample size 'n' and the constant 'c'.

Ans
$n = 95$
$c = 76.69$

Q. Obtain BCR and power function of test for testing $H_0: \theta = 1$ against $H_1: \theta = \theta_1 (>1)$ on the basis of a sample size 1 from the population with pdf:

$$f(x, \theta) = \theta x^{\theta - 1} \; ; \; 0 < x < 1$$
$$= 0 \; ; \; \text{otherwise}$$

Solution

Given, $f(x, \theta) = \theta x^{\theta - 1} \; ; \; 0 < x < 1$

We have to test $H_0: \theta = 1$ vs $H_1: \theta = \theta_1 \; (>1)$.

Now, $f(x/H_0) = 1 x^{1-1} = 1$

$\theta_1 - 1 \; ; \; 0 < x < 1$

Now, from NP-lemma, the best critical region is obtained as

$$\frac{f(x/H_1)}{f(x/H_0)} \geq k, \quad \forall x \in W$$

$$\Rightarrow \frac{\theta_1 x^{\theta_1 - 1}}{1} \geq k$$

$$\Rightarrow x^{\theta_1 - 1} \geq \frac{k}{\theta_1}$$

$$\Rightarrow \log(x^{\theta_1 - 1}) \geq \log\left(\frac{k}{\theta_1}\right)$$

$$\Rightarrow (\theta_1 - 1) \log x \geq \log\left(\frac{k}{\theta_1}\right)$$

$$\Rightarrow \log x \geq \frac{\log(k/\theta_1)}{(\theta_1 - 1)}$$

$$\Rightarrow x \geq \text{antilog}\left[\frac{\log(k/\theta_1)}{(\theta_1 - 1)}\right] = c \text{ (say)}$$

$$\therefore W = \{x : x \geq c\}$$

The value of 'c' can be determined from fixed α.

$$\therefore \alpha = \text{Prob}(x \in W / H_0)$$

$$\Rightarrow \alpha = P(x \geq c / H_0)$$

$$\Rightarrow \alpha = \int_c^1 f(x/H_0)\, dx$$

$$\Rightarrow \alpha = \int_c^1 1\, dx$$

$$\Rightarrow \alpha = [x]_c^1$$

$$\Rightarrow \alpha = 1 - c$$

$$\boxed{\therefore c = 1 - \alpha}$$

$\therefore W = \{ x : x \geq (1-\alpha) \}$

Power of the test

$$\beta = P(x \in W / H_1)$$

$$= P(x \geq c / H_1)$$

$$= \int_c^1 f(x/H_1)\, dx$$

$$= \int_c^1 \theta_1 x^{\theta_1 - 1}\, dx$$

$$= \theta_1 \left[\frac{x^{\theta_1}}{\theta_1} \right]_c^1$$

$$= \left[x^{\theta_1} \right]_c^1$$

$$= \cancel{x - x^c}$$

$$= 1^{\theta_1} - c^{\theta_1}$$

$$\therefore \boxed{\beta = (1 - c^{\theta_1})}$$

$$\therefore 1 - \beta = 1 - (1 - c^{\theta_1})$$

$\therefore$ Power of the test, $\boxed{\therefore 1-\beta = c^{\theta_1}}$

LIKELIHOOD RATIO TEST (LRT)

NP Lemma based on the ratio of the two ~~probabilities~~ probability density functions provides the best test for simple null hypothesis against simple alternative hypothesis. In this case, the best depends on the nature of the population distribution and the form of the alternative hypothesis considered. But, likelihood ratio test suggest a sample method of test construction which is closely related to MLE (Maximum Likelihood Estimator), hence the likelihood ratio (LR) gives the method of testing composite hypothesis.

① Likelihood ratio test for mean of a normal population when variance is known.

z-test	t-test
$n > 30$ or popⁿ s.d. σ is known	$n \leq 30$ and popⁿ s.d. σ is unknown

variance is known $\Rightarrow$ z-test
variance is unknown $\Rightarrow$ t-test
$n > 30$ (z-test)
$n \leq 30$ (t-test)

Procedure:

$H_0: \mu = \mu_0$ i.e. there is no significant difference between sample mean and population mean.

$H_1: \mu \neq \mu_0$ i.e. there is a significant difference between sample mean and population mean. (Two-Tailed Test)

or, $\left.\begin{matrix} H_1: \mu > \mu_0 \\ H_1: \mu < \mu_0 \end{matrix}\right\}$ (One Tailed Test)

Test-statistic: Under H_0,

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Decision: If $|Z_{cal}| > |Z_{tab}|$ then we reject H_0 otherwise do not reject H_0.

[**NOTE**: If α is not given, consider it as 5%.]

(ii) LRT for mean of a normal population when variance is unknown.

Test-statistic: Under H_0,

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

where, s = sample s.d. $= \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} = \sqrt{\frac{\sum x^2 - n\bar{x}^2}{n-1}}$ (For making estimate unbiased)

Critical Value: $t_{tab} = t_{n-1, \alpha}$

where, $n-1$ = degree of freedom.

Decision: If $|t_{cal}| \geq |t_{tab}|$, then we reject H_0, otherwise do not reject H_0.

[3 marks]

Q1. Given $\mu = 400$,
$\sigma = 20$,
$n = 100$,
$\bar{x} = 390$
$\alpha = 5\% = 0.05$

Notation

n = Sample size
N = Population size
σ = Pop^n s.d.
s = sample s.d.
μ = Pop^n mean
$\bar{x}$ = Sample mean

H_0: $\mu = 400$ i.e. the claim is valid / correct.

H_1: $\mu \neq 400$ i.e. the claim is incorrect.

Test-Statistic: Under H_0,

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{390-400}{20/\sqrt{100}} = \frac{-10(10)}{20}$$

$$\Rightarrow z_{cal} = -5$$

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Critical value:

We have, $\alpha = 0.05$

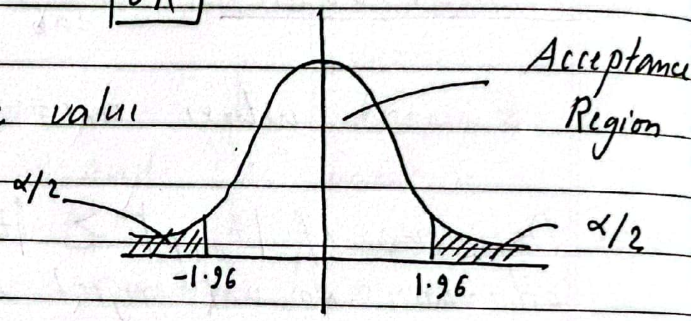
$\therefore Z_{tab} = Z_{0.05}$ (Two tailed test) $= 1.96$

$-3.99 < Z < 3.99$

Decision: Since $|Z_{cal}| = 5 > Z_{tab} = 1.96$, so we reject H_0.
Hence, the claim is incorrect.

$\boxed{OR}$

Since, the test-statistic value lies in the rejection region, so we reject H_0.



Q3. Given,

$\mu = 75$

$n = 25$

$\bar{x} = 83.40$

$s = 11.83$

$\alpha = 0.05$

$H_0: \mu = 75$ i.e. the average balance is not more than \$75.

$H_1: \mu > 75$ i.e. the average balance is more than \$75. (Right or one-tailed test)

Test-statistic: Under H_0,

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$= \frac{83.40 - 75}{11.83/\sqrt{n}}$$

$= 3.55$

Critical value:

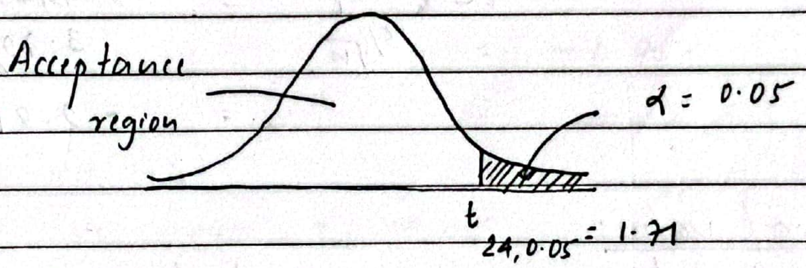
We have, $\alpha = 0.05$ and dof $= n-1 = 25-1 = 24$

$\therefore t_{tab} = t_{24,\,0.05}$ (One Tailed Test)

$= 1.71$

Decision: Since, $t_{cal} > t_{tab}$, so we reject H_0.
Hence, the ~~avg.~~ average monthly balance is more than \$75.

OR



Since, the test-statistic value lies in the rejection region so we reject H_0.

Q.4

Let x = length of stay (in days).

$\Sigma x = 10+15+11+5+7+4+8+14+11 = 85$ and $\Sigma x^2 = 100+225+121+25+49+16+64+196+121 = 917$

$\therefore n = 9$

$\bar{x} = \frac{\Sigma x}{n} = \frac{85}{9} = 9.44$

$\mu = 13$

$$s.d.\ (s) = \sqrt{\frac{\Sigma x^2 - n\bar{x}^2}{n-1}}$$

$$= \sqrt{\frac{917 - 9(9.44)^2}{9-1}}$$

$$= \underline{3.79}$$

$H_0: \mu = 13$ i.e. the average length of stay of tourist is 13 days

$H_1: \mu \neq 13$ i.e. the average length of stay of tourist is not 13 days (Two-Tailed Test)

__Test-statistic__ : Under H_0,

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{9.44 - 13}{3.79/\sqrt{9}}$$

$$= -2.81.$$

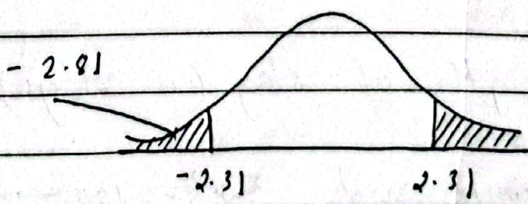
__Critical value:__

We have, $\alpha = 0.05$ and dof $= n-1 = 9-1 = 8$

$\therefore t_{tab} = t_{8,\,0.05}$ (Two-Tailed Test)

$= 2.31$

__Decision__



Since, test-statistic value lies in the rejection region so we reject H_0.

Hence, the claim is incorrect.

Q15

Given, $\mu = 2.8$, $\sigma = 0.20$

$n = 25$, $\bar{x} = 2.7$

$\alpha = 0.05$

H_0: $\mu \geq 2.8$ i.e. the production equipment does not need to be adjusted.

H_1: $\mu < 2.8$ i.e. the production equipment needs to be adjusted. (Left-Tailed Test)

Test-statistic: Under H_0,

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{2.73 - 2.8}{0.20/\sqrt{25}}$$

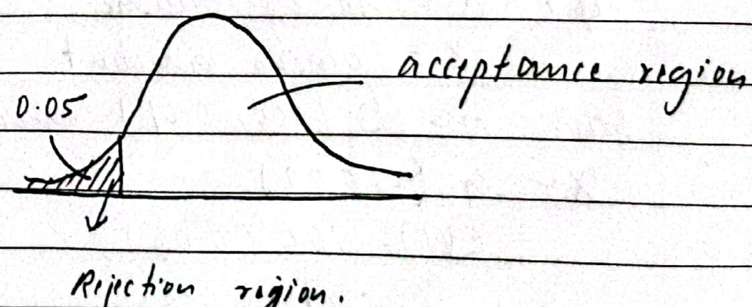
$$= -1.75$$

Critical value:

We have, $\alpha = 0.05$

$\therefore Z_{tab} = Z_{0.05}$ (Left-tailed Test) $= -1.645$

Decision:



Since, test-statistic value lies in the rejection region, so we reject Ho. Hence the production equipment needs to be adjusted.

Assignment: Q17, Q16, Q9, Q6