PROBLEMS ON DOUBLE INTEGRALS OVER RECTANGLES

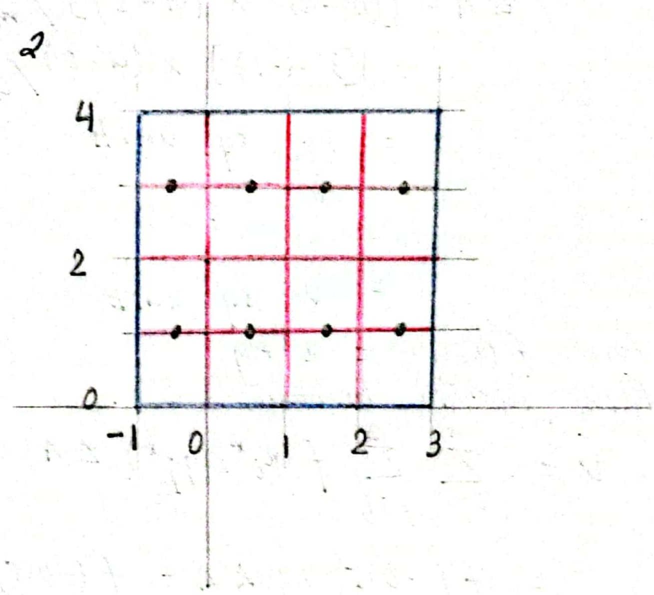
1. Use the Midpoint Rule to estimate the volume under $f(x,y) = x^2+y$ and above the rectangle given by $-1 \le x \le 3,\ 0 \le y \le 4$ in the xy-plane. Use 4 subdivisions in the x direction and 2 subdivisions in the y-direction.

Solution

Here, given rectangle is $R = [-1, 3] \times [0, 4] = [a, b] \times [c, d]$

Divisions in x direction, $m = 4$

Divisions in y direction, $n = 2$



Now, $\Delta x = \dfrac{b-a}{m}$

$= \dfrac{3-(-1)}{4}$

$= 1$

$\Delta y = \dfrac{d-c}{n}$

$= \dfrac{4-0}{2}$

$= 2$

Calculation of sample points (x_i^*, y_j^*) is as follows:

$x_1 = \dfrac{-1+0}{2} = -0.5$ (By using mid-point rule)

$y_1 = \dfrac{0+2}{2} = 1$

We repeat the process for other sample points and tabulate the data from computation as follows:

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x_i^*	$x_1=-0.5$	$x_2=0.5$	$x_3=1.5$	$x_4=2.5$	$x_1=-0.5$	$x_2=0.5$	$x_3=1.5$	$x_4=2.5$
y_j^*	$y_1=1$	$y_1=1$	$y_1=1$	$y_1=1$	$y_2=3$	$y_2=3$	$y_2=3$	$y_2=3$

Using Riemann sum, we know volume is given by,

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta A \quad \text{———} \quad (i)$$

where ΔA is the area of sub-divided rectangle.

Now,

$$\Delta A = [(b-a) \times (d-c)] / m*n$$
$$= [3-(-1)] \times (4-0) / 4*2$$
$$= \frac{16}{8} \text{ sq units}$$

$$= 2 \text{ sq units.}$$

And $f(x,y) = x^2 + y$

Now, eq^n (i) becomes,

$$V = \sum_{i=1}^{4} \sum_{j=1}^{2} f(x_i^*, y_j^*)\, \Delta A$$

$$= f(-0.5, 1)\Delta A + f(-0.5, 3)\Delta A + f(0.5, 1)\Delta A + f(0.5, 3)\Delta A + f(1.5, 1)\Delta A + f(1.5, 3)\Delta A + f(2.5, 1)\Delta A + f(2.5, 3)\Delta A$$

$$= ((-0.5)^2+1)\Delta A + ((-0.5)^2+3)\Delta A + ((0.5)^2+1)\Delta A + ((0.5)^2+3)\Delta A + ((1.5)^2+1)\Delta A + ((1.5)^2+3)\Delta A + ((2.5)^2+1)\Delta A + ((2.5)^2+3)\Delta A$$

$$= 34\,\Delta A$$

$$= 34 \times 2$$

$= 68$ ~~sq. un~~ cubic units.

$\therefore$ Approx volume is 68 cubic units.

Q2.a) Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle $R = [0,6] \times [0,4]$. Use Riemann sum with $m = 3$, $n = 2$ and take the sample points to be the upper right corner of each square.

Solution

Here, given function is $f(x,y) = xy$ and rectangle $R = [0,6] \times [0,4]$.

Also, $m = 3$, $n = 2$

Now, $\Delta x = \frac{6-0}{3} = 2$

$\Delta y = \frac{4-0}{2} = 2$

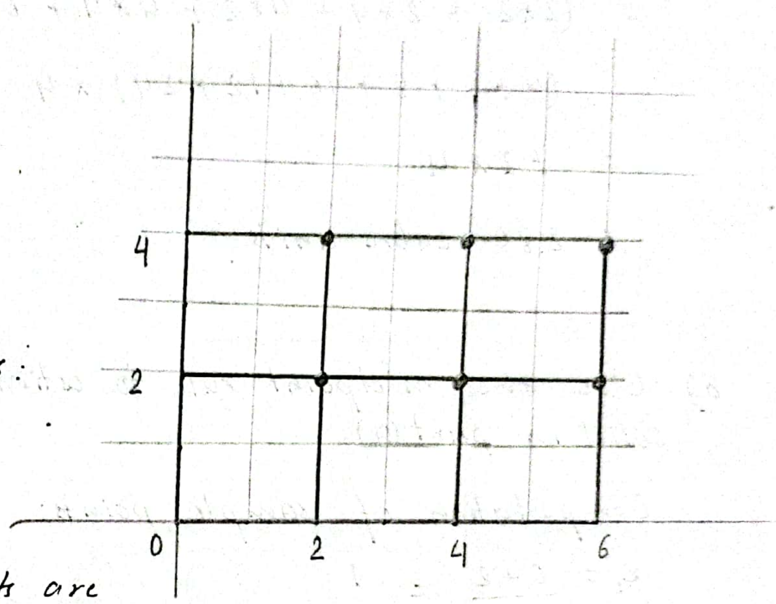

Fig: Rectangle with sample points

Calculation of sample points:

$x_1 = 0 + \Delta x = 0 + 2 = 2$

$y_1 = 0 + \Delta y = 0 + 2 = 2$

Similarly, other sample points are computed and data are tabulated as follows:

i	j	x_i^*	y_j^*
$i=1$	$j=1$	2	2
	$j=2$	2	4
$i=2$	$j=1$	4	2
	$j=2$	4	4
$i=3$	$j=1$	6	2
	$j=2$	6	4

Now, Area of sub-divided rectangle, $\Delta A = \frac{[(b-a) * (c-d)]}{m * n}$

$= \frac{(6-0) * (4-0)}{3 * 2} = 4$

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Volume of a surface V is given by Riemann sum as

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i^*, y_j^*) \Delta A$$

$$= f(2,2)\Delta A + f(2,4)\Delta A + f(4,2)\Delta A + f(4,4)\Delta A + f(6,2)\Delta A + f(6,4)\Delta A$$

$$= (2*2 + 2*4 + 4*2 + 4*4 + 6*2 + 6*4)\Delta A$$

$$= (4 + 8 + 8 + 16 + 12 + 24) * 4$$

$$= 72 * 4$$

$$= 288 \text{ cubic units}$$

b) Use the midpoint rule to estimate the volume of the solid in part (a).

Computation of sample points:

$$x_1 = \frac{0+2}{2} = 1$$

$$y_1 = \frac{0+2}{2} = 1$$

Tabulating other sample points as follows:

i	j	x_i^*	y_j^*
$i=1$	$j=1$	1	1
	$j=2$	1	3
$i=2$	$j=1$	3	1
	$j=2$	3	3
$i=3$	$j=1$	5	1
	$j=2$	5	3

Area of sub-divided rectangle, $\Delta A = \frac{6*4}{3*2} = 4$ sq units.

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Now, Volume using Riemann sum is given by

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i^*, y_j^*) \Delta A$$

$$= f(1,1) \Delta A + f(1,3) \Delta A + f(3,1) \Delta A + f(3,3) \Delta A + f(5,1) \Delta A + f(5,3) \Delta A$$

$$= (1*1 + 1*3 + 3*1 + 3*3 + 5*1 + 5*3) \Delta A$$

$$= 36 * 4$$

$$= 144 \text{ cubic units.}$$

3. If $R = [0,4] \times [-1,2]$, use a Riemann sum with $m=2$, $n=3$ to estimate the value of $\iint_R (1 - xy^2)\, dA$. Take the sample points to be a) the lower right corners and b) the upper left corners of the rectangles.

Solution

Here, given function, $f(x,y) = (1 - xy^2)$ and rectangle $R = [0,4] \times [-1,2]$.

Also, $m = 2$, $n = 3$.

Now, $\Delta x = \frac{4-0}{2} = 2$

$\Delta y = \frac{2-(-1)}{3} = 1$

The estimated volume V of under the given surface is given by Riemann sum as

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A \quad \text{——— (i)}$$

where, (x_i^*, y_j^*) are sample points and ΔA is the area of sub-divided rectangle.

$\Delta A = (4*3)/(2*3) = 2$ sq units.

Computation of sample points.

a) Sample points taken as lower right corners

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$x_1 = 0 + \Delta x = 0 + 2 = 2$

$y_1 = c = -1$ and increments for $y_2 = y_1 + \Delta y$ follow accordingly.

Other sample points are computed and data tabulated as follows:

i	j	x_i^*	y_j^*
$i=1$	$j=1$	2	-1
	$j=2$	2	0
	$j=3$	2	1
$i=2$	$j=1$	4	-1
	$j=2$	4	0
	$j=3$	4	1

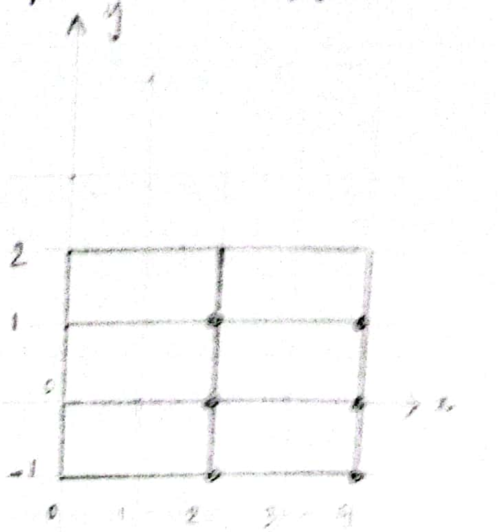

fig: Rectangles with sample points

Now, from (i)

$$V = \sum_{i=1}^{2} \sum_{j=1}^{3} f(x_i^*, y_j^*)\,\Delta A$$

$$= f(2,-1)\Delta A + f(2,0)\Delta A + f(2,1)\Delta A + f(4,-1)\Delta A + f(4,0)\Delta A + f(4,1)\Delta A$$

$$= [2 - 2(-1)^2]\Delta A + [2 - 2(0)^2]\Delta A + [2 - 2(1)^2]\Delta A + [4 - 4(-1)^2]\Delta A + [4 - 4(0)^2]\Delta A + 4 - 4$$

$$= [1 - 2(-1)^2]\Delta A + [1 - 2(0)^2]\Delta A + [1 - 2(1)^2]\Delta A + [1 - 4(-1)^2]\Delta A + [1 - 4(0)^2]\Delta A + [1 - 4(1)^2]\Delta A$$

$$= [(1-2) + (1-0) + (1-2) + (1-4) + (1-0) + (1-4)]\Delta A$$

$$= -6 \times 2$$

$$= -12 \text{ cubic units.}$$

$$\therefore \iint_R (1 - xy^2)\,dA \approx -12$$

b) Sample point taken as upper left corner of rectangles

Here,

$x_1 = $ ~~a+..~~ $= a = 0$; increaments for x is Δx.

$y_1 = c + \Delta y = -1 + 1 = 0$

Other sample point are computed and data tabulated as follows:

i	j	x_i^*	y_j^*
$i=1$	$j=1$	0	0
	$j=2$	0	1
	$j=3$	0	2
$i=2$	$j=1$	2	0
	$j=2$	2	1
	$j=3$	2	2

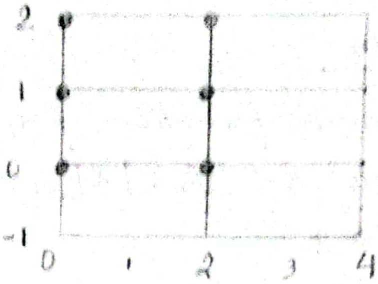

fig: Rectangles with sample points

Now, from (i),

$$V = \sum_{i=1}^{2} \sum_{j=1}^{3} f(x_i^*, y_j^*) \Delta A$$

$$= f(0,0)\Delta A + f(0,1)\Delta A + f(0,2)\Delta A + f(2,0)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A$$

$$= [1 - 0(0)^2]\Delta A + [1 - 0(1)^2]\Delta A + [1 - 0(2)^2]\Delta A + [1 - 2(0)^2]\Delta A + [1 - 2(1)^2]\Delta A + [1 - 2(2)^2]\Delta A$$

$$= [(1-0)+(1-0)+(1-0)+(1-0)+(-1)+(1-8)]\Delta A$$

$$= -4 * 2$$

$$= -8 \text{ cubic units}$$

$$\therefore \iint_R (1 - xy^2)\, dA = -8$$

4. a) Use Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0, 2] \times [0, 1]$. Take the sample points to be upper right corners.

Solution

Here, given function $f(x, y) = xe^{-xy}$ and rectangle $R = [0, 2] \times [0, 1]$ and

$m = n = 2$

Now, $\Delta x = \frac{2-0}{2} = 1$

$\Delta y = \frac{1-0}{2} = 0.5$

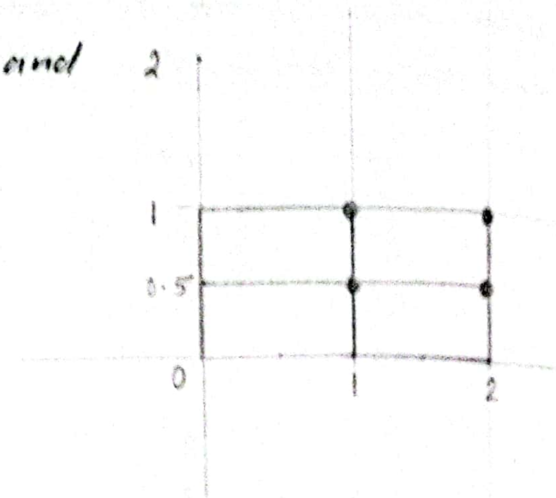

Fig: Rectangle with sample points

Computation of sample points.
for example,

$x_1 = a + \Delta x = 0 + 1 = 1$

$y_1 = c + \Delta y = 0 + 0.5 = 0.5$

Similarly, other sample points are calculated and tabulated below:

i	j	x_i^*	y_j^*
$i=1$	$j=1$	1	0.5
	$j=2$	1	1
$i=2$	$j=1$	2	0.5
	$j=2$	2	1

Now, the volume of given surface is estimated by Riemann sum,

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta A$$

where, $\Delta A = \frac{2 * 1}{2 * 2} = \frac{2}{4} = 0.5$ sq units.

Now, $V = \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i^*, y_j^*)\, \Delta A$

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$$= f(1, 0.5)\,\Delta A + f(1,1)\,\Delta A + f(2, 0.5)\,\Delta A + f(2,1)\,\Delta A$$

$$= \left(1e^{-1*0.5} + 1e^{-1*1} + 2e^{-2*0.5} + 2e^{-2*1}\right)\Delta A$$

$$= \left(e^{-0.5} + e^{-1} + 2e^{-1} + 2e^{-2}\right)\Delta A$$

$$= 1.980 * 0.5$$

$$= 0.990$$

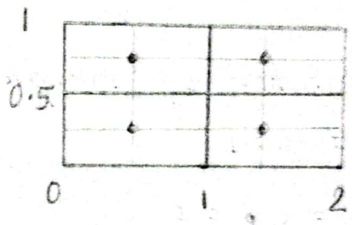
$$\therefore \iint_R x e^{-xy}\, dA \approx V = 0.990$$

b) Use the Midpoint rule to estimate the integral in part (a).

Computation of sample points

$$x_1 = \frac{0+1}{2} = 0.5$$

$$y_1 = \frac{0+0.5}{2} = 0.25$$



Similarly, other sample points are computed and values are tabulated as follows:

i	j	x_i^*	y_j^*
$i=1$	$j=1$	0.5	0.25
	$j=2$	0.5	0.75
$i=2$	$j=1$	1.5	0.25
	$j=2$	1.5	0.75

Now,

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i^*, y_j^*)\,\Delta A$$

$$= f(0.5, 0.25)\,\Delta A + f(0.5, 0.75)\,\Delta A + f(1.5, 0.25)\,\Delta A + f(1.5, 0.75)\,\Delta A$$

$$= \left(0.5e^{-0.5*0.25} + 0.5e^{-0.5*0.75} + 1.5e^{-1.5*0.25} + 1.5e^{-1.5*0.75}\right)\Delta A$$

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$= 2.302 * 0.5$

$= 1.151$

5a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1, 2] \times [0, 3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower ~~ri~~ left corners.

Solution

Here, given function $f(x, y) = 1 + x^2 + 3y$ and rectangle $R = [1, 2] \times [0, 3]$ with $m = n = 2$.

Now, $\Delta x = \frac{2-1}{2} = 0.5$

$\Delta y = \frac{3-0}{2} = 1.5$

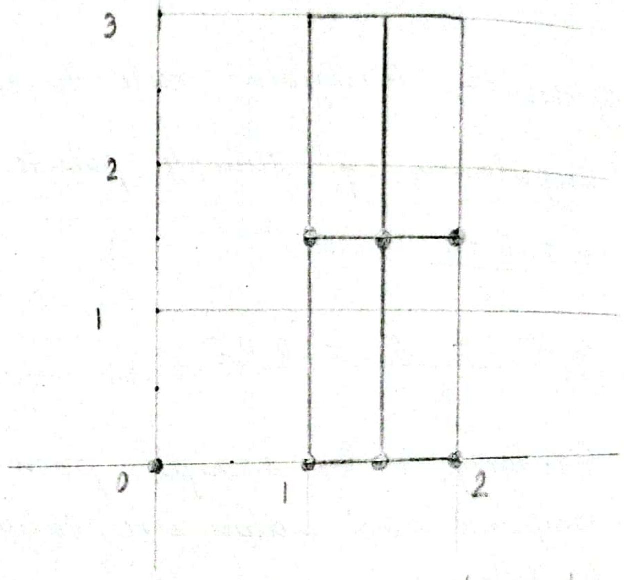

Fig: Rectangle with sample points

Calculation of sample points

$x_1 = a = 1$

$y_1 = c = 0$

$x_2 = a + \Delta x = 1 + 0.5 = 1.5$

$y_2 = c + \Delta y = 0 + 1.5 = 1.5$

~~$y_2 = c + \Delta y = 0 + 1.5 = 1.5$~~

~~$y_2 = c = 0$ [for x_2~~

Other sample points are calculated and tabulated as follows:

i	j	x_i^*	y_j^*
1	1	1	0
	1	1	1.5
2	1	1.5	0
	1	1.5	1.5

Now, Area of subdivided rectangle $= \frac{1 * 3}{2 * 2} = 0.75$

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Now,

Volume of surface using Riemann sum,

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i^*, y_j^*) \Delta A$$

$$= f(1,0)\Delta A + f(1,1.5)\Delta A + f(1.5,0)\Delta A + f(1.5,1.5)\Delta A$$

$$= [(1+1^2+3*0) + (1+1^2+3*1.5) + 1+(1.5)^2+3*0 + (1+1.5^2+3*1.5)]\Delta A$$

$$= (2+6.5+3.25+7.75)*0.75$$

$$= 14.625$$

b) Use mid-point rule to estimate the volume in part (a)

Calculation of sample points

$$x_1 = \frac{1+\cancel{0}1.5}{2} = \cancel{0.175}\ 1.25$$

$$y_1 = \frac{0+1.5}{2} = 0.75$$

other sample points are calculated and tabulated as follows:

i	j	x_i^*	y_j^*
1	1	1.25	0.75
	2	1.25	2.25
2	1	1.75	0.75
	2	1.75	2.25

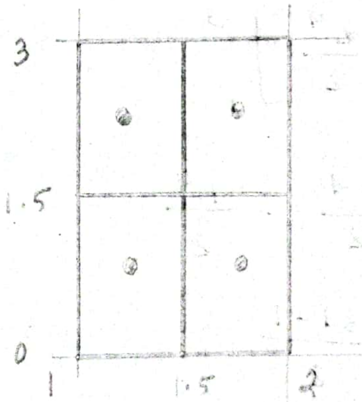

fig: Rectangle with sample points

Now, Volume $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$

$$= f(1.25, 0.75)\Delta A + f(1.25, 2.25)\Delta A + f(1.75, 0.75)\Delta A + f(1.75, 2.25)\Delta A$$

$$= [1+(1.25)^2 + 3*0.75 + 1+(1.25)^2 + 3*2.25 + 1+(1.75)^2 + 3*0.75 + 1+(1.75)^2 + 3*2.25]\Delta A$$

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$= 31.25 * 0.75$

$= 23.4375$

Evaluate the iterated integrals

6. $\int_1^3 \int_0^2 x^3 y \, dy \, dx$

$= \int_1^3 \left[\frac{x^3 y^2}{2}\right]_0^2 dx$

$= \int_1^3 \left[\frac{4x^3}{2}\right]_0^2 dx$

$= \int_1^3 2x^3 \, dx$

$= \left[\frac{2x^4}{4}\right]_1^3$

$= \left[\frac{x^4}{2}\right]_1^3$

$= \frac{3^4}{2} - \frac{1^4}{2}$

$= \frac{81-1}{2}$

$= 40$

7. $\int_0^2 \int_1^3 x^3 y \, dx \, dy$

$= \int_0^2 \left[\frac{x^4 y}{4}\right]_1^3 dy$

$= \int_0^2 \left[\frac{3^4 y}{4} - \frac{1^4 y}{4}\right] dy$

$= \int_0^2 \left(\frac{81y}{4} - \frac{y}{4}\right) dy$

$= \int_0^2 20y \, dy$

$= 20\left[\frac{y^2}{2}\right]_0^2$

$= 10\,[y^2]_0^2$

$= 10\,(4-0)$

$= 40$

8. $\int_{-1}^1 \int_0^\pi x^2 \sin y \, dy \, dx$

$= \int_{-1}^1 \left[-x^2 \cos y\right]_0^\pi dx$

$= \int_{-1}^1 \left[+x^2 \cos y\right]_\pi^0 dx$

$$= \int_{-1}^{1} (x^2 \cos 0 - x^2 \cos \pi)\, dx$$

$$= \int_{-1}^{1} (x^2 + x^2)\, dx$$

$$= \int_{-1}^{1} 2x^2\, dx$$

$$= \frac{2}{3} \left[x^3\right]_{-1}^{1}$$

$$= \frac{2}{3} (1+1)$$

$$= \frac{4}{3}$$

9. $\int_2^6 \int_1^4 x^2\, dx\, dy$

$$= \int_2^6 \left[\frac{x^3}{3}\right]_1^4 dy$$

$$= \int_2^6 \left(\frac{4^3}{3} - \frac{1^3}{3}\right) dy$$

$$= \int_2^6 \left(\frac{63}{3}\right) dy$$

$$= 21 \left[y\right]_2^6$$

$$= 21(6-2)$$

$$= 84$$

10. $\int_0^1 \int_0^2 (x + 4y^3)\, dx\, dy$

$$= \int_0^1 \left[\frac{x^2}{2} + 4y^3 x\right]_0^2 dy$$

$$= \int_0^1 (2 + 8y^3)\, dy$$

$$= \left[2y + \frac{8y^4}{4}\right]_0^1$$

$$= \left[2y + 2y^4\right]_0^1$$

$$= (2\times 1 + 2\times 1^4 - 0)$$

$$= 4$$

11. $\int_0^4 \int_0^9 \sqrt{x+4y}\, dx\, dy$

$$= \int_0^4 \int_0^9 (x+4y)^{1/2}\, dx\, dy$$

$$= \int_0^4 \left[\frac{(x+4y)^{1/2+1}}{1/2+1}\right]_0^9 dy$$

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$$= \int_0^4 \left[\frac{2}{3}(x+4y)^{3/2}\right]_0^9 dy$$

$$= \int_0^4 \left[\frac{2}{3}(9+4y)^{3/2} - \frac{2}{3}(0+4y)^{3/2}\right] dy$$

$$= \frac{2}{3}\int_0^4 \left[(9+4y)^{3/2} - (4y)^{3/2}\right] dy$$

$$= \frac{2}{3}\left[\frac{(9+4y)^{5/2}}{5/2 \cdot 4}\right]_0^4 - \frac{2}{3}\left[\frac{(4y)^{5/2}}{5/2 \cdot 4}\right]_0^4$$

$$= \frac{2}{3}\left\{\frac{(9+4\times 4)^{5/2}}{10} - \frac{9^{5/2}}{10}\right\} - \frac{2}{3}\left\{\frac{16^{5/2}}{10} - 0\right\}$$

$$= \frac{2}{3}\left(\frac{3125 - 243}{10}\right) - \frac{2}{3}\left(\frac{1024}{10}\right)$$

$$= \frac{2}{3} * \frac{2882}{10} - \frac{2}{3} * \frac{1024}{10}$$

$$= \frac{2}{3}\left(\frac{2882 - 1024}{10}\right)$$

$$= \frac{2}{3} * \frac{1858}{10}$$

$$= \frac{1858}{15}$$

12. $\int_0^{\pi/4}\int_{\pi/4}^{\pi/2} \cos(2x+y)\, dy\, dx$

$$= \int_0^{\pi/4}\left[\sin(2x+y)\right]_{\pi/4}^{\pi/2} dx$$

$$= \int_0^{\pi/4} \left[\sin(2x+y) \right]_{\pi/2}^{\pi/4} dx$$

$$= \int_0^{\pi/4} \left(\sin(2x + \pi/2) - \sin(2x + \pi/4) \right) dx$$

$$= \left[- \frac{\cos(2x + \pi/2)}{2} \right]_0^{\pi/4} - \left[- \frac{\cos(2x + \pi/4)}{2} \right]_0^{\pi/4}$$

$$= \left[\frac{\cos(2x + \pi/2)}{2} \right]_{\pi/4}^{0} + \left[\frac{\cos(2x + \pi/4)}{2} \right]_0^{\pi/4}$$

$$= \left(\frac{\cos \pi/2}{2} - \frac{\cos \pi}{2} \right) + \left(\frac{\cos 3\pi/4}{2} - \frac{\cos \pi/4}{2} \right)$$

$$= \left\{ 0 - \left(\frac{-1}{2} \right) \right\} + \left(\frac{-1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \right)$$

$$= \frac{1}{2} - \frac{2}{2\sqrt{2}}$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2}}$$

$$= -0.207$$

13. $\int_1^2 \int_0^4 \frac{dy\, dx}{x+y}$

$$= \int_1^2 \int_0^4 \frac{1}{x+y}\, dy\, dx$$

$$= \int_1^2 \left[\ln(x+y) \right]_0^4 dx$$

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$$= \int_1^2 [\ln(x+4) - \ln(x+0)]\, dx$$

$$= \int_1^2 \cancel{\ln}\left(\cancel{\frac{x+4}{x}}\right) \cancel{dx}$$

$$= \int_1^2 \ln(x+4)\, dx - \int_1^2 \ln x\, dx$$

$$= \Big[(x+4)\{\ln(x+4) - 1\}\Big]_1^2 - \Big[x\{\ln(x) - 1\}\Big]_1^2$$

$$= 6(\ln 6 - 1) - \cancel{2(\ln 2 - 1)}$$
$$\cancel{-5(\ln 5 - 1)}$$

$$= 6(\ln(6) - 1) - 5(\ln(5) - 1) - 2(\ln(2) - 1)$$
$$+ 1(\ln(1) - 1)$$

$$= 6\ln(6) - 6 - 5\ln(5) + 5 - 2\ln(2) + 2 + 0 - 1$$

$$= 6\ln(6) - 2\ln(2) - 5\ln(5)$$

$$= 1.317$$

14. $\int_1^2 \int_2^4 e^{3x-y}\, dy\, dx$

$$= \int_1^2 \left[\frac{e^{3x-y}}{-1}\right]_2^4 dx$$

$$= \int_1^2 \left[e^{3x-y}\right]_4^2 dx$$

$$= \int_1^2 \left(e^{3x-2} - e^{3x-4}\right) dx$$

$$= \left[\frac{e^{3x-\cancel{2}}}{\cancel{2}} - \frac{e^{3x-\cancel{4}}}{\cancel{4}}\right]_1^2$$

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$$= \frac{e^{6-2}}{3} - \frac{e^{6-4}}{4} - \frac{e^{1}}{3} + \frac{e^{-1}}{4}$$

$$= \frac{e^{4}}{3} - \frac{e^{2}}{4} - \frac{e^{1}}{3} + \frac{e^{-1}}{4}$$

$$= \left[\frac{e^{3x-2}}{3} - \frac{e^{3x-4}}{3}\right]_1^2$$

$$= \frac{1}{3}\left[e^{3x-2} - e^{3x-4}\right]_1^2$$

$$= \frac{1}{3}\left(e^{6-2} - e^{6-4} - e^{1} + e^{-1}\right)$$

$$= \frac{1}{3}\left(e^{4} - e^{2} - e^{1} + e^{-1}\right)$$

15. $$\int_0^4 \int_0^5 \frac{dy\,dx}{\sqrt{x+y}}$$

$$= \int_0^4 \int_0^5 (x+y)^{-1/2}\,dy\,dx$$

$$= \int_0^4 \left[\frac{(x+y)^{-1/2+1}}{-1/2+1}\right]_0^5 dx$$

$$= \int_0^4 2\left[(x+y)^{1/2}\right]_0^5 dx$$

$$= 2\int_0^4 \left[(x+5)^{1/2} - x^{1/2}\right] dx$$

$$= 2\left[\frac{(x+5)^{1/2+1}}{1/2+1} - \frac{x^{1/2+1}}{1/2+1}\right]_0^4$$

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$$= 2\left[\frac{2}{3}(x+5)^{3/2} - \frac{2}{3}x^{3/2}\right]_0^4$$

$$= \frac{4}{3}\left[(x+5)^{3/2} - x^{3/2}\right]_0^4$$

$$= \frac{4}{3}\left[9^{3/2} - 4^{3/2}\right] - 5^{3/2} + 0^{3/2}\Big]$$

$$= \frac{4}{3} * \left\{(27-8) - 5\sqrt{5}\right\}$$

$$= \frac{4}{3}(19 - 5\sqrt{5})$$

$$= 10.426$$

16. $\displaystyle\int_0^8 \int_2^1 \frac{x\,dx\,dy}{\sqrt{x^2+y}}$

$$= \int_0^8 \int_2^1 (x^2+y)^{-1/2}\,x\,dx\,dy$$

Put $u = x^2+y$

$du = 2x\,dx$

$\therefore x\,dx = \frac{du}{2}$

when $x = 2,\quad u = 4+y$

$x = 1,\quad u = 1+y$

$\therefore$ Above integral becomes,

$$\int_0^8 \int_{4+y}^{1+y} \frac{u^{-1/2}}{2}\,du\,dy$$

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Q16. $\int_0^8 \int_1^2 \frac{x}{\sqrt{x^2+y}} \, dx \, dy$

Put $x^2+y = u$ when $x=1, \; u = 1+y$

$\therefore \; du = 2x \, dx$ $\quad x = 2, \; u = 4+y$

Thus, above integral becomes,

$$\int_0^8 \int_{1+y}^{4+y} u^{-1/2} \frac{du}{2} \, dy$$

$$= \frac{1}{2} \int_0^8 \int_{1+y}^{4+y} u^{-1/2} \, du \, dy$$

$$= \frac{1}{2} \int_0^8 \left[\frac{u^{-1/2+1}}{-1/2+1} \right]_{1+y}^{4+y} dy$$

$$= \frac{1}{2} \int_0^8 2 \left[u^{1/2} \right]_{1+y}^{4+y} dy$$

$$= \int_0^8 \left\{ (4+y)^{1/2} - (1+y)^{1/2} \right\} dy$$

$$= \left[\frac{(4+y)^{3/2}}{3/2} - \frac{(1+y)^{3/2}}{3/2} \right]_0^8$$

$$= \frac{2}{3} \left[(4+y)^{3/2} - (1+y)^{3/2} \right]_0^8$$

$$= \frac{2}{3} \cdot \left(12^{3/2} - 9^{3/2} - 4^{3/2} + 1^{3/2} \right)$$

$$= \frac{2}{3} \left\{ (2^2 \cdot 3)^{3/2} - 3^3 - 2^3 + 1 \right\}$$

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$$= \frac{2}{3}\left(8 \cdot 3\sqrt{3} - 27 - 8 + 1\right)$$

$$= \frac{2}{3}\left(24\sqrt{3} - 34\right)$$

$$= \left(\frac{48}{\sqrt{3}} - \frac{68}{3}\right)$$

$$= \frac{48}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} - \frac{68}{3}$$

$$= \frac{48\sqrt{3}}{3} - \frac{68}{3}$$

$$= \left(16\sqrt{3} - \frac{68}{3}\right)$$

17. $\int_1^2 \int_1^3 \frac{\ln xy}{y}\, dy\, dx$

Put $v = \ln xy$

Now, $\frac{dv}{dy} = \frac{d}{dy}(\ln xy)$

$$\Rightarrow \frac{dv}{dy} = \frac{1}{xy} \cdot \frac{d}{dy}(xy)$$

$$\Rightarrow \frac{dv}{dy} = \frac{1}{xy} \cdot x \quad (\text{Taking } x \text{ as constant})$$

$$\therefore \frac{dy}{y} = dv$$

when $y = 1$, $v = \ln x$

when $y = 3$, $v = \ln 3x$

Thus, above integral becomes,

$$\int_1^2 \int_{\ln x}^{\ln 3x} v \, dv \, dx$$

$$= \int_1^2 \left[\frac{v^2}{2}\right]_{\ln x}^{\ln 3x} dx$$

$$= \frac{1}{2}\int_1^2 (\ln 3x)^2 - (\ln x)^2 \quad dx$$

$$= \frac{1}{2} \int_1^2 (\ln 3x + \ln x)(\ln 3x - \ln x) \, dx$$

$$= \frac{1}{2} \int_1^2 \ln 3x^2 \cdot \ln 3 \, dx$$

$$= \frac{\ln 3}{2} \int_1^2 \ln 3x^2 \, dx$$

$$= \frac{1}{2} (\ln 3) \underbrace{\int_1^2 1 \cdot \ln 3x^2 \, dx}_{I'}$$

for integral I', put $u = \ln 3x^2$ and $dv = 1 \cdot dx$

Thus,

$$\frac{du}{dx} = \frac{1}{3x^2} \cdot 6x$$

$$\therefore \quad du = \frac{2}{x} dx$$

And $dv = 1 dx$

$$\therefore \quad v = x$$

Now, $I' = uv - \int v \, du$

$$= x \ln(3x^2) - \int x \cdot \frac{2}{x} dx$$

$$= x \ln(3x^2) - 2x$$

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$$\therefore \int_1^2 \int_1^3 \frac{\ln xy}{y}\, dy\, dx = \frac{1}{2}\ln(3)\left[x \ln(3x^2) - 2x\right]_1^2$$

$$= \frac{1}{2}\ln(3)\left(2\ln(12) - 2\times 2 - \ln(3) + 2\times 1\right)$$

$$= \frac{1}{2}\ln(3)\left(\ln 12^2 - \ln 3 - 2\right)$$

$$= \frac{1}{2}\ln(3)\left[\ln(144/3) - 2\right]$$

$$= \frac{1}{2}\ln(3)\left(\ln(48) - 2\right)$$

$$= 1.0278$$

18. $\int_0^1 \int_2^3 \frac{1}{(x+4y)^3}\, dx\, dy$

$$= \int_0^1 \int_2^3 (x+4y)^{-3}\, dx\, dy$$

$$= \int_0^1 \left[\frac{(x+4y)^{-3+1}}{-3+1}\right]_2^3 dy$$

$$= \int_0^1 \left[\frac{(x+4y)^{-2}}{-2}\right]_2^3 dy$$

$$= -\frac{1}{2}\int_0^1 \left\{(3+4y)^{-2} - (2+4y)^{-2}\right\} dy$$

$$= -\frac{1}{2}\left[\frac{(3+4y)^{-2+1}}{(-2+1)\cdot 4} - \frac{(2+4y)^{-2+1}}{(-2+1)\cdot 4}\right]_0^1$$

$$= -\frac{1}{2}\left[-\frac{1}{4}(3+4y)^{-1} + \frac{1}{4}(2+4y)^{-1}\right]_0^1$$

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$$= \frac{1}{8}\left[\frac{1}{(3+4y)} - \frac{1}{(2+4y)}\right]_0^1$$

$$= \frac{1}{8}\left(\frac{1}{3+4} - \frac{1}{6} - \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{8}\left(\frac{1}{7} + \frac{1}{2} - \frac{1}{3} - \frac{1}{6}\right)$$

$$= \frac{1}{8}\left(\frac{1}{7} + \frac{3-2-1}{6}\right)$$

$$= \frac{1}{8} \times \frac{1}{7}$$

$$= \frac{1}{56}$$

Evaluate the double integral over the rectangular region R.

19. $\iint_R 4xy^3\, dA$; $R = \{(x,y) : -1 \leq x \leq 1,\ -2 \leq y \leq 2\}$

Thus,

$$\iint_R 4xy^3\, dA = \int_{-1}^{1}\int_{-2}^{2} 4xy^3\, dy\, dx$$

$$= \int_{-1}^{1} \left[\frac{4xy^4}{4}\right]_{-2}^{2} dx$$

$$= \int_{-1}^{1} x\{2^4 - (-2)^4\}\, dx$$

$$= \int_{-1}^{1} 0\, dx$$

$$= 0$$

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20. $\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\, dA$; $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$

Solution

$$\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\, dA = \int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\, dx\, dy$$

Put $u = x^2+y^2+1$

Then, $\frac{du}{dx} = 2x$

$\therefore \quad x\,dx = \frac{du}{2}$

when $x = 0$, $u = 0^2 + y^2+1$

$\therefore u = y^2+1$

when $x = 1$, $u = 1^2+y^2+1$

$\therefore u = y^2+2$

Thus, above integral becomes

$$\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\, dx\, dy = \int_0^1\int_{y^2+1}^{y^2+2} y\,\frac{du}{2\sqrt{u}}\, dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{y\, u^{-1/2+1}}{-1/2+1}\right]_{y^2+1}^{y^2+2} dy$$

$$= \int_0^1 \left[y\sqrt{u}\right]_{y^2+1}^{y^2+2} dy$$

$$= \underbrace{\int_0^1 y\sqrt{y^2+2}\, dy}_{I_1} - \underbrace{\int_0^1 y\sqrt{y^2+1}\, dy}_{I_2}$$

For I_1

Put $v = y^2+2$

$\therefore y\,dy = dv/2$

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when $y=0$, when $y=1$,
$v = 0^2+2 = 2$ $v = 1^2+2 = 3$

$$\therefore I_1 = \frac{1}{2}\int_2^3 \sqrt{v}\, dv$$

$$= \frac{1}{2}\left[\frac{v^{3/2}}{3/2}\right]_2^3$$

$$= \frac{1}{3}\left[v^{3/2}\right]_2^3$$

$$= \frac{1}{3}\left(3^{3/2} - 2^{3/2}\right)$$

for I_2

Put $v = y^2+1$

$$\therefore y\,dy = \frac{dv}{2}$$

when $y=0$, $v=1$
$y=1$, $v = 1^2+1 = 2$

$$\therefore I_2 = \frac{1}{2}\int_1^2 \sqrt{v}\, dv$$

$$= \frac{1}{2}\left[\frac{v^{3/2}}{3/2}\right]_1^2$$

$$= \frac{1}{3}\left(2^{3/2} - 1^{3/2}\right)$$

$$\therefore \iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\, dA = I_1 - I_2$$

$$= \frac{1}{3}\left(3^{3/2} - 2^{3/2}\right) - \frac{1}{3}\left(2^{3/2} - 1^{3/2}\right)$$

$$= 0.179$$

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21. $\iint_R x\sqrt{1-x^2}\, dA$; $R = \{(x,y) : 0 \le x \le 1,\ 2 \le y \le 3\}$

$$\iint_R x\sqrt{1-x^2}\, dA = \int_2^3 \int_0^1 x\sqrt{1-x^2}\, dx\, dy$$

Put $u = 1-x^2$

$\Rightarrow \frac{du}{dx} = -2x$

$\therefore x\, dx = -\frac{du}{2}$

when $x=0$, $u = 1-0^2 = 1$

$x = 1$, $u = 1-1^2 = 0$

$$\therefore \int_2^3 \int_0^1 x\sqrt{1-x^2}\, dx\, dy = \int_2^3 \int_1^0 \sqrt{u}\left(-\frac{du}{2}\right) dy$$

$$= \frac{-1}{2}\int_2^3 \int_1^0 u^{1/2}\, du\, dy$$

$$= -\frac{1}{2}\int_2^3 \left[\frac{u^{3/2}}{3/2}\right]_1^0 dy$$

$$= -\frac{1}{2} \times \frac{2}{3} \int_2^3 (0-1)\, dy$$

$$= \frac{-1}{3}\int_2^3 -dy$$

$$= \frac{1}{3}\int_2^3 dy$$

$$= \frac{1}{3}[y]_2^3$$

$$= \frac{1}{3} * (3-2) = \frac{1}{3}$$

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22. $\iint_R (x \sin y - y \sin x)\, dA$; $R = \{(x,y) : 0 \le x \le \pi/2,\ 0 \le y \le \pi/3\}$

$$\iint_R (x \sin y - y \sin x)\, dA = \int_0^{\pi/3}\int_0^{\pi/2} (x \sin y - y \sin x)\, dx\, dy$$

$$= \int_0^{\pi/3} \left[\sin y \frac{x^2}{2} - y(-\cos x)\right]_0^{\pi/2} dy$$

$$= \int_0^{\pi/3} \left[\frac{x^2}{2} \sin y + y \cos x\right]_0^{\pi/2} dy$$

$$= \int_0^{\pi/3} \left(\frac{\pi^2}{8} \sin y + 0 - 0 - y\right) dy$$

$$= \int_0^{\pi/3} \left(\frac{\pi^2}{8} \sin y - y\right) dy$$

$$= \left[\frac{\pi^2}{8}(-\cos y) - \frac{y^2}{2}\right]_0^{\pi/3}$$

$$= \left[\frac{\pi^2}{8} \cos y + \frac{y^2}{2}\right]_{\pi/3}^{0}$$

$$= \left(\frac{\pi^2}{8} \cos 0 + 0 - \frac{\pi^2}{8} \cos \frac{\pi}{3} - \frac{\pi^2}{18}\right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{16} - \frac{\pi^2}{18}$$

$$= \frac{18\pi^2 - 9\pi^2 - 8\pi^2}{144}$$

$$= \frac{\pi^2}{144}$$

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Use a double integral to find volume

23. The volume under the plane $z = 2x + y$ and over the rectangle $R = \{(x, y) : 3 \le x \le 5,\ 1 \le y \le 2\}$

Volume of certain solid $V = \iint_R f(x,y)\, dA$

$$= \int_1^2 \int_3^5 (2x+y)\, dx\, dy$$

$$= \int_1^2 \left[\frac{2x^2}{2} + xy\right]_3^5 dy$$

$$= \int_1^2 \left[x^2 + xy\right]_3^5 dy$$

$$= \int_1^2 (25 + 5y - 9 - 3y)\, dy$$

$$= \int_1^2 (16 + 2y)\, dy$$

$$= \left[16y + \frac{2y^2}{2}\right]_1^2$$

$$= \left[16y + y^2\right]_1^2$$

$$= (32 + 4) - (16 + 1)$$

$$= 36 - 17$$

$= 19$ cubic units.

24. The volume under the surface $z = 3x^3 + 3x^2y$ and over the rectangle $R = \{(x, y) : 1 \le x \le 3,\ 0 \le y \le 2\}$

Volume under the surface is given by:

$$V = \iint_R f(x,y)\, dA$$

$$= \int_0^2 \int_1^3 (3x^3 + 3x^2y)\, dx\, dy$$

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$$= \int_0^2 \int_1^3 (3x^3 + 3x^2 y)\, dx\, dy$$

$$= \int_0^2 \left[\frac{3x^4}{4} + 3y\frac{x^3}{3}\right]_1^3 dy$$

$$= \int_0^2 \left[\frac{3x^4}{4} + x^3 y\right]_1^3 dy$$

$$= \int_0^2 \left[\frac{3(3)^4}{4} + 3^3 y - \frac{3(1)^4}{4} - 1^3 y\right] dy$$

$$= \int_0^2 \left(\frac{243}{4} + 27y - \frac{3}{4} - y\right) dy$$

$$= \left[\frac{243}{4} y + \frac{27y^2}{2} - \frac{3y}{4} - \frac{y^2}{2}\right]_0^2$$

$$= \left[\frac{243}{4} * 2 + \frac{27 * 2^2}{2} - \frac{6}{4} - 2\right]$$

$$= \left(\frac{486 - 6}{4} + 54 - 2\right)$$

$$= \left(\frac{480}{4} + 52\right)$$

$$= 120 + 52$$

$$= 172 \text{ cubic units.}$$

25. The volume of the solid enclosed by the surface $z = x^2$ and the planes $x = 0$, $x = 2$, $y = 3$, $y = 0$ and $z = 0$.

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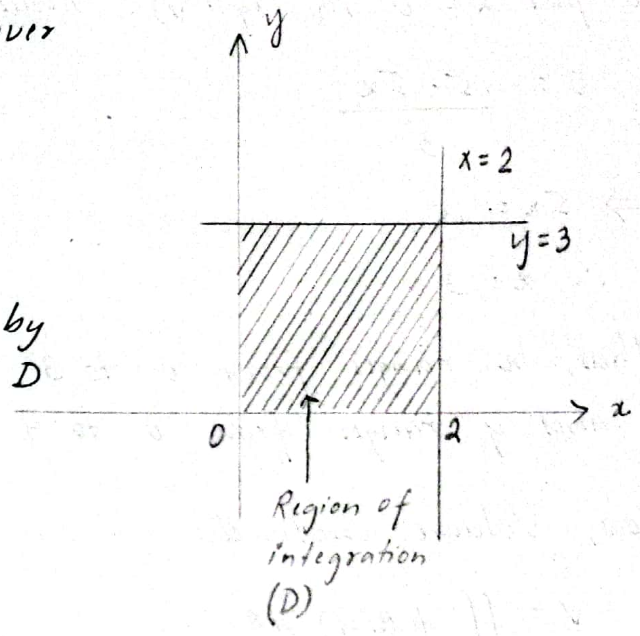
Solution

Let us draw the region (D) over which we have to perform integration.

Given,

$f(x,y) = x^2$



Now volume of solid enclosed by surface f and over the region D is given by

$$V = \iint_D f(x,y)\, dA$$

$$= \int_0^3 \int_0^2 x^2 \, dx \, dy$$

$$= \int_0^3 \left[\frac{x^3}{3}\right]_0^2 dy$$

$$= \int_0^3 \frac{8}{3} \, dy$$

$$= \frac{8}{3} [y]_0^3$$

$$= \frac{8}{3} * 3$$

= 8 cubic units.

26. The volume in the first octant bounded by the co-ordinate planes, the plane $y = 4$ and the plane $(x/3) + (z/5) = 1$

Solution

Given plane under which volume has ~~have~~ to be found is

$$\frac{x}{3} + \frac{z}{5} = 1$$

$$\Rightarrow 5x + 3z = 15$$

$$\therefore z = \frac{15 - 5x}{3} \quad \ldots\ldots (i)$$

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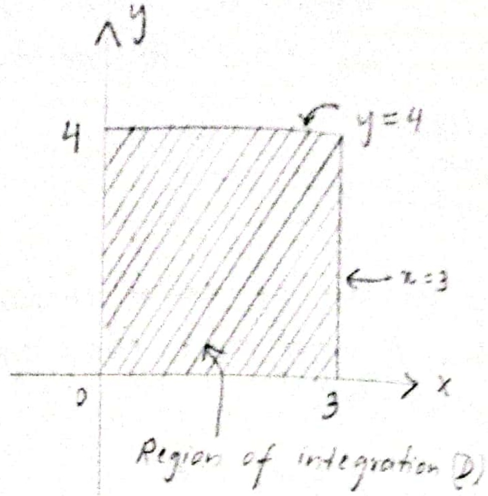
To find the region over which we have to integrate, we put $z = 0$ in eqn (i). Thus,

$$0 = \frac{15-5x}{3}$$

$$\Rightarrow 5x = 15$$

$$\therefore x = 3$$

Thus, x ranges from 0 to 3 and y ranges from 0 to 4.



Now, Volume enclosed,

$$V = \iint_D f(x,y)\, dA$$

$$= \int_0^4 \int_0^3 \left(\frac{15-5x}{3}\right) dx\, dy$$

$$= \frac{1}{3}\int_0^4 \left[\frac{(15-5x)^2}{2\cdot(-5)}\right]_0^3 dy$$

$$= -\frac{1}{30}\int_0^4 \left[(15-25x)^2\right]_0^3 dy$$

$$= -\frac{1}{30}\int_0^4 (0 - 225)\, dy$$

$$= \frac{225}{30}\int_0^4 dy$$

$$= \frac{225}{30} * [y]_0^4$$

$$= \frac{225}{30} * (4-0)$$

$$= 30 \text{ cubic units.}$$

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27. Find the average value of $f(x,y) = xy^2$ over the rectangle $[0,8] \times [0,6]$

Solution

Here, Rectangle $(R) = [0,8] \times [0,6]$

$\therefore$ Area of rectangle, $A(R) = 8 \times 6 = 48$

& $f(x,y) = xy^2$

We know,

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

$$= \frac{1}{48} \int_0^6 \int_0^8 xy^2 \, dx \, dy$$

$$= \frac{1}{48} \int_0^6 \left[\frac{x^2y^2}{2}\right]_0^8 dy$$

$$= \frac{1}{48} \int_0^6 \left[\frac{64y^2}{2}\right] dy$$

$$= \frac{1}{48} \int_0^6 32y^2 \, dy$$

$$= \frac{32}{48} \int_0^6 y^2 \, dy$$

$$= \frac{32}{48} \left[\frac{y^3}{3}\right]_0^6$$

$$= \frac{32}{48} * \frac{6^3}{3}$$

$$= \boxed{48}$$

28. Find the average value of $f(x,y) = x^2 + 7y$ over the rectangle $[0,3] \times [0,6]$

Solution

Rectangle $(R) = [0,3] \times [0,6]$

$\therefore$ Area of rectangle, $A(R) = 3 \times 6 = 18$

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$f(x,y) = x^2 + 7y$

Now, average value of f is given by:

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

$$= \frac{1}{18} \int_0^6 \int_0^3 (x^2 + 7y)\, dx\, dy$$

$$= \frac{1}{18} \int_0^6 \left[\frac{x^3}{3} + 7xy\right]_0^3 dy$$

$$= \frac{1}{18} \int_0^6 (9 + 21y)\, dy$$

$$= \frac{1}{18} \left[9y + \frac{21}{2} y^2\right]_0^6$$

$$= \frac{1}{18} \left(54 + \frac{21}{2} * 6^2\right)$$

$$= \frac{1}{18} * 432$$

$$= \boxed{24}$$

29. Find the average value of $f(x,y) = y \sin xy$ over the rectangle $[0,1] \times [0, \pi/2]$

Solution

Here, given rectangle, $R = [0,1] \times [0, \pi/2]$

Area of rectangle, $A(R) = 1 \times \pi/2$

$= \pi/2$

Given function, $f(x,y) = y \sin xy$

Now, $f_{ave} = ?$

We have,

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$$f_{ave} = \frac{\iint_R f(x,y)\,dA}{A(R)}$$

$$= \frac{1}{(\pi/2)} \int_0^{\pi/2}\int_0^1 y \sin xy \, dx\, dy$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \left[\left(-\frac{\cos xy}{y}\right) y\right]_0^1 dy$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} [\cos xy]_0^1 \, dy$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} (\cos y - \cos 0)\, dy$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} (\cos y - 1)\, dy$$

$$= -\frac{2}{\pi} [\sin y - y]_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left(\sin \pi/2 - \pi/2 - \sin 0 + 0\right)$$

$$= \frac{-2}{\pi}\left(1 - \frac{\pi}{2}\right)$$

$$= \left(1 - \frac{2}{\pi}\right)$$

30. Find the average value of $f(x,y) = x(x^2+y)^{1/2}$ over the rectangle $[0,1] \times [0,3]$.

Solution

Here, given rectangle $R = [0,1] \times [0,3]$

Area of rectangle, $A(R) = 1*3$

$= 3$

Given function $f(x,y) = x(x^2+y)^{1/2}$

Now, $f_{ave} = ?$

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$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

$$= \frac{1}{3} \int_0^3 \int_0^1 x(x^2+y)^{1/2}\, dx\, dy$$

$$= \frac{1}{3} \int_0^3 \int_0^1 x(x^2+y)^{1/2}\, dx\, dy$$

Put $u = x^2+y$

$$\frac{du}{dx} = 2x$$

$\therefore x\,dx = \frac{du}{2}$

when $x=0$, $u=y$

when $x=1$, $u=1+y$

Now,

$$\int_0^3 \int_0^1 x(x^2+y)^{1/2}\, dx\, dy = \int_0^3 \int_y^{1+y} u^{1/2} \frac{du}{2}\, dy$$

$$= \frac{1}{2} \int_0^3 \left[\frac{u^{3/2}}{3/2}\right]_y^{1+y} dy$$

$$= \frac{1}{2} * \frac{2}{3} \int_0^3 \left((1+y)^{3/2} - y^{3/2}\right) dy$$

$$= \frac{1}{3} \left[\frac{(1+y)^{5/2}}{5/2} - \frac{y^{5/2}}{5/2}\right]_0^3$$

$$= \frac{2}{3*5} \left[(1+y)^{5/2} - y^{5/2}\right]_0^3$$

$$= \frac{2}{15} \left\{(1+3)^{5/2} - 3^{5/2} - (1+0)^{5/2} + 0\right\}$$

$$= \frac{2}{15} \left(32 - 3^{5/2} - 1\right)$$

$$= \frac{2}{15} \left(31 - 9\sqrt{3}\right)$$

$$\therefore f_{ave} = \frac{1}{3} * \frac{2}{15} (31 - 9\sqrt{3})$$

$$= \frac{2}{45} (31 - 9\sqrt{3})$$

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31. Suppose that the temperature in degree celcius at a point (x, y) on a metal plate is $T(x,y) = 10 - 8x^2 - 2y^2$, where x and y are in meters. Find the average temperature of the rectangular portion of the plate for which $0 \le x \le 1$ and $0 \le y \le 2$.

Solution

Given rectangle $R = [0,1] \times [0,2]$

Area of rectangle, $A(R) = 1 * 2$

$= 2$

Given function, $T(x,y) = 10 - 8x^2 - 2y^2$

Now, $T_{ave} = ?$

~~Solution~~ We have,

$$T_{ave} = \frac{1}{A(R)} \iint_R T(x,y)\, dA$$

$$= \frac{1}{2} \int_0^1 \int_0^2 (10 - 8x^2 - 2y^2)\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \left[10y - 8x^2y - \frac{2y^3}{3} \right]_0^2 dx$$

$$= \frac{1}{2} \int_0^1 \left(20 - 16x^2 - \frac{16}{3} \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{44}{3} - 16x^2 \right) dx$$

$$= \frac{1}{2} \left[\frac{44}{3} x - \frac{16x^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left(\frac{44}{3} - \frac{16}{3} \right)$$

$$= \frac{1}{2} * \frac{28}{3}$$

$$= \left(\frac{14}{3} \right) {}^\circ C$$

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32. Show that if $f(x, y)$ is constant on the rectangle $R = [a, b] \times [c, d]$, say $f(x, y) = k$, then

$$f_{ave} = k \text{ over } R$$

Here, Given rectangle, $R = [a, b] \times [c, d]$

Area of R, $A(R) = (b-a) * (d-c)$

$f(x, y) = k$

Now,

$$f_{ave} = \iint_R f(x, y)\, dA$$

$$= \frac{1}{A(R)} \int_a^b \int_c^d k \; dy \; dx$$

$$= \frac{k}{(b-a) * (d-c)} \int_a^b [y]_c^d \, dx$$

$$= \frac{k}{(b-a) * (d-c)} \int_a^b (d-c) \, dx$$

$$= \frac{k(d-c)}{(b-a) * (d-c)} \int_a^b dx$$

$$= \frac{k}{(b-a)} [x]_a^b$$

$$= \frac{k(b-a)}{(b-a)}$$

$$= k$$

$$= RHS$$

proved.

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