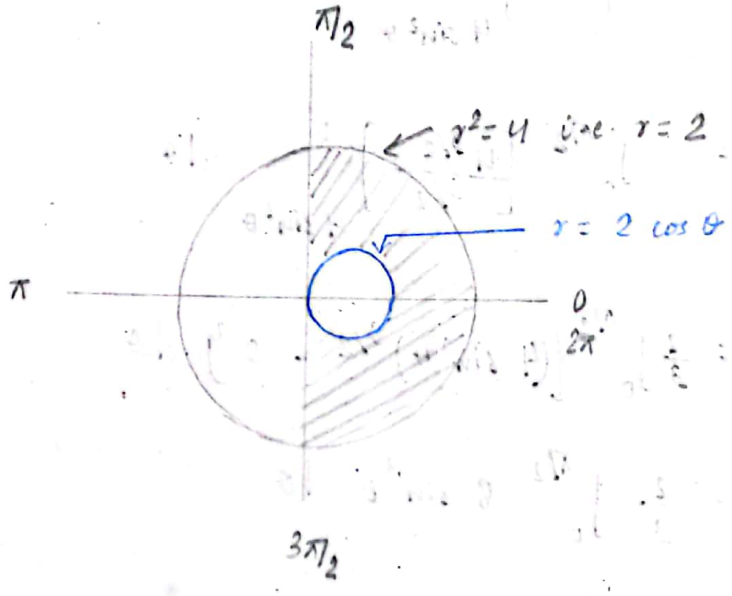
14. Inside the surface $r^2+z^2=4$ and outside the surface $r=2\cos\theta$.

Solution

Sketching the region in 2D plane, we get:



Here, $r^2+z^2=4$

$\Rightarrow z=\sqrt{4-r^2}$.

Volume occupied over shaded region,

$$V_1 = 2\int_0^{\pi/2}\int_{2\cos\theta}^{2} f(r,\theta)\, r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\int_{2\cos\theta}^{2} \sqrt{4-r^2}\, r\, dr\, d\theta$$

Put $4-r^2=u$

$\therefore du=-2r\,dr$

when $r=2\cos\theta,\ u=4\sin^2\theta,$

$r=2,\ u=0$

Thus, $V_1 = 2\int_0^{\pi/2}\int_{4\sin^2\theta}^{0} u^{1/2}\,\frac{du}{(-2)}\,d\theta$

$$= -\int_0^{\pi/2}\int_{4\sin^2\theta}^{0} u^{1/2}\,du\,d\theta$$

$$= -\int_0^{\pi/2}\left[\frac{u^{3/2}}{3/2}\right]_{4\sin^2\theta}^{0} d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2}\{(4\sin^2\theta)^{3/2} - 0\}\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} 8\sin^3\theta\,d\theta$$

$$= \frac{2}{3}*8\int_0^{\pi/2}\sin\theta\,(1-\cos^2\theta)\,d\theta$$

$$= \frac{16}{3}\int_0^{\pi/2}\sin\theta\,d\theta - \frac{16}{3}\underbrace{\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta}_{I'}$$

$$= \frac{16}{3}\left[-\cos\theta\right]_0^{\pi/2} - \frac{16}{3}I'$$

$$= \frac{16}{3}(1-0) - \frac{16}{3}I'$$

$$= \frac{16}{3} - \frac{16}{3}I'$$

For I', put $v = \cos\theta$

$\therefore dv = -\sin\theta\,d\theta$

when $\theta = 0$, $v = 1$

$\theta = \pi/2$, $v = 0$

$$\therefore I' = -\int_1^0 v^2 \, dv$$

$$= -\left[\frac{v^3}{3}\right]_1^0$$

$$= 1/3$$

$$\therefore V_1 = \frac{16}{3} - \frac{16}{9}$$

$$= \frac{48-16}{9}$$

$$= \frac{32}{9}$$

This volume represents volume above $z=0$. For the portion below $z=0$, the same volume should be added.

Thus total $V_1 = 2 * \frac{32}{9}$

$$= \frac{64}{9}$$

Now, volume for unshaded region is given by

$$V_2 = 4\int_0^{\pi/2} \int_0^2 \sqrt{4-r^2} \; r \, dr \, d\theta$$

Put $u = \sqrt{4-r^2}$

$u = 4 - r^2$

$\therefore du = -2r \, dr$

when $r=0,\ u=4$

$r=2,\ u=0$

Thus,

$$V_2 = \frac{-4}{2}\int_0^{\pi/2}\int_4^0 u^{1/2}\,du\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^4 u^{1/2}\,du\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{u^{3/2}}{3/2}\right]_0^4 d\theta$$

$$= 2\times\frac{2}{3}\int_0^{\pi/2}(4^{3/2}-0)\,d\theta$$

$$= \frac{4}{3} * 8\int_0^{\pi/2} d\theta$$

$$= \frac{32}{3} * [\theta]_0^{\pi/2}$$

$$= \frac{32}{3} * (\pi/2 - 0)$$

$$= \frac{16\pi}{3}$$

$\therefore$ Total Volume, $V = V_1 + V_2$

$$= \frac{64}{9} + \frac{16\pi}{3}.$$

Use polar co-ordinates to evaluate the double integral.

15. $\iint_R \sin(x^2+y^2)\, dA$, where R is the region enclosed by the circle $x^2+y^2=9$

Solution

Put $x^2+y^2=r^2$. Thus above integral becomes:

$$\int_0^{2\pi}\int_0^{r=3} \sin r^2 \; r \, dr \, d\theta$$

$$= 4\int_0^{\pi/2}\int_{r=0}^{r=3} \sin r^2 \; r \, dr \, d\theta$$

Put $u = r^2$

$\therefore du = 2r\, dr$

when $r=0$, $u=0$

$r=3$, $u=9$

Thus, above integral becomes:

$$\frac{4}{2}\int_0^{\pi/2}\int_0^{9} \sin u \, du \, d\theta$$

$$= 2\int_0^{\pi/2} \left[-\cos u\right]_0^9 \, d\theta$$

$$= 2\int_0^{\pi/2} (1-\cos 9)\, d\theta$$

$$= 2(1-\cos 9)\int_0^{\pi/2} d\theta$$

$$= 2(1-\cos 9)\left[\theta\right]_0^{\pi/2}$$

$$= 2(1-\cos 9) * \frac{\pi}{2}$$

$$= \pi(1-\cos 9)$$

16. $\iint_R \sqrt{9-x^2-y^2}\, dA$, where R is the region in the first quadrant within the circle $x^2+y^2=9$.

Solution

Put $x^2+y^2=r^2$. Then above integral becomes,

$$\int_0^{\pi/2}\int_0^{3} \sqrt{9-r^2}\; r\, dr\, d\theta$$

Let, $u = 9-r^2$

$\therefore du = -2r\,dr$

when $r=0$, $u=9$; $r=3$, $u=0$

Thus, above integral becomes:

$$-\frac{1}{2}\int_0^{\pi/2}\int_9^{0} u^{1/2}\, du\, d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{u^{3/2}}{3/2}\right]_9^0 d\theta$$

$$= -\frac{1}{2}\times\frac{2}{3}\int_0^{\pi/2}(0-9^{3/2})\,d\theta$$

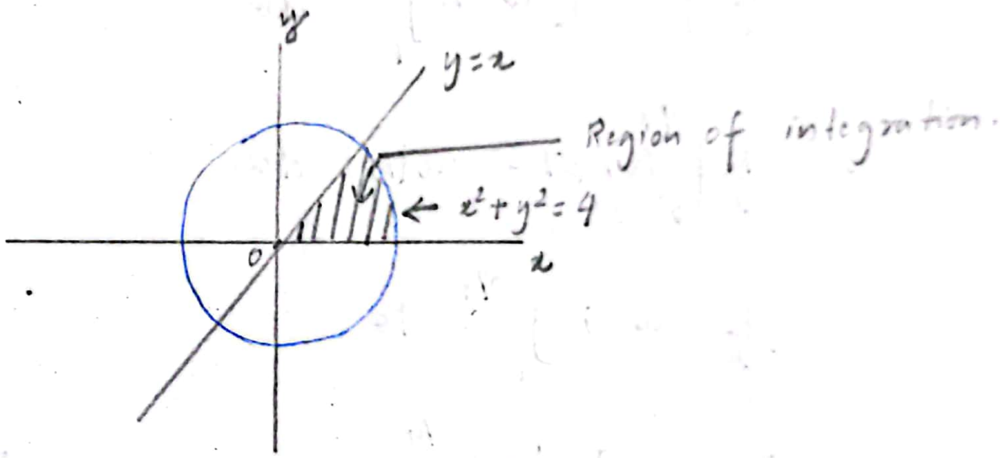
$$= -\frac{1}{3} * (-27)\int_0^{\pi/2} d\theta$$

$$= 9\,[\theta]_0^{\pi/2} = \boxed{9\pi/2}$$

17. $\iint_R \frac{1}{1+x^2+y^2}\, dA$, where R is the sector in the first quadrant bounded by $y=0$, $y=x$ and $x^2+y^2=4$.

<u>Solution</u>

Graphing the region of integration,



For $y=x$,

$\Rightarrow r\cos\theta = r\sin\theta$

$\Rightarrow \tan\theta = 1$

$\therefore \theta = \pi/4$

For $x^2+y^2=4$, we get

$r=2$

Thus, putting $x^2+y^2=r^2$, above integral becomes,

$$\int_0^{\pi/4}\int_0^2 \frac{1}{1+r^2}\, r\, dr\, d\theta$$

Put $u = 1+r^2$

$\therefore du = 2r\,dr$

when $r=0$, $u=1$; $r=2$, $u=5$

Thus, above integral becomes,

$$\frac{1}{2}\int_0^{\pi/4}\int_1^5 \frac{du}{u}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[\ln(u)\right]_1^5 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\ln(5) - \ln(1)\right) d\theta$$

$$= \frac{1}{2}\ln(5)\int_0^{\pi/4} d\theta$$

$$= \frac{1}{2}\ln(5)\left[\theta\right]_0^{\pi/4}$$

$$= \boxed{\pi/8 \ln(5)}$$

18.

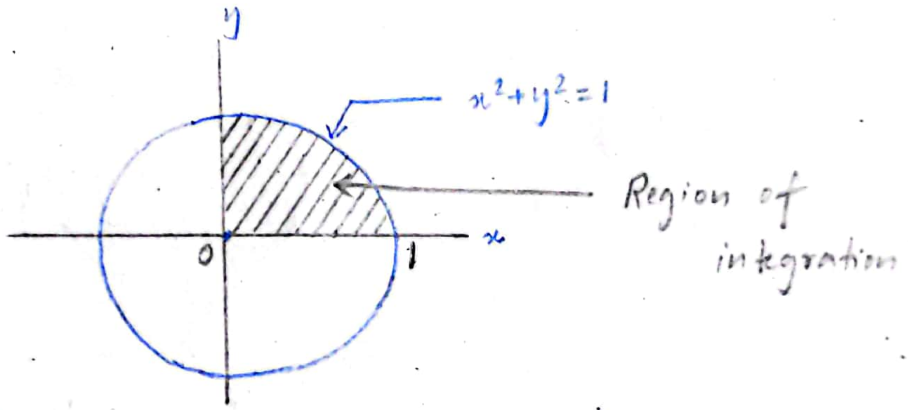
Evaluate the iterated integral by converting to polar coordinates.

19. $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

Sketching the region of integration:

Here y ranges from $y=0$ to $y=\sqrt{1-x^2}$

and x ranges from $x=0$ to $x=1$.



Put $x = r\cos\theta$ & $y = r\sin\theta$. Thus, above integral becomes

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 r^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^3\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi}{8}}$$

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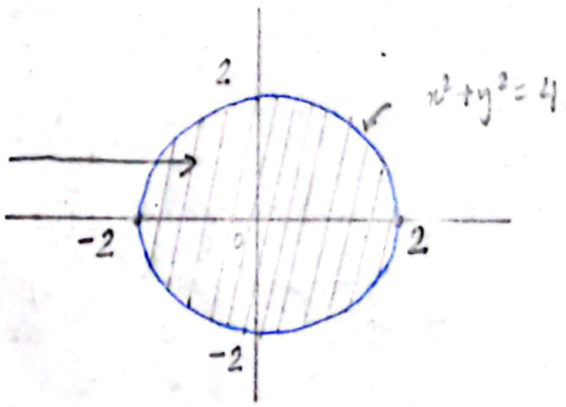
20. $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy$

Graphing the region of integration,

y ranges from -2 to $+2$

x ranges from $-\sqrt{4-y^2}$ to $+\sqrt{4-y^2}$.

Region of integration



Putting $x^2+y^2=r^2$, above integral becomes:

$$4\int_0^{\pi/2}\int_0^{2} e^{-r^2}\, r\, dr\, d\theta$$

Put ~~$u = e^{-r^2}$~~ $u = r^2$

$\therefore du = 2r\, dr$

when $r = 0$, $u = 0$

$r = 2$; $u = 2^2 = 4$

Thus, above integral becomes:

$\cancel{\frac{4}{2}\int_0^{\pi/2}\int_1^{e^{-4}}}$

$$\frac{4}{2}\int_0^{\pi/2}\int_0^{4} e^{-u}\, du\, d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{e^{-u}}{-1}\right]_0^4 d\theta$$

$$= -2\left(e^{-4} - e^{-0}\right)\int_0^{\pi/2} d\theta$$

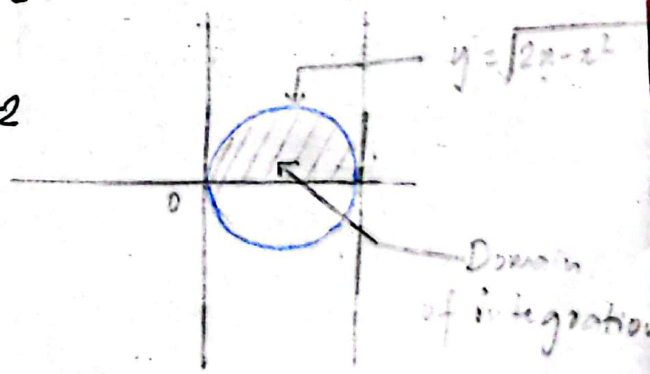
$$= \left(1 - e^{-4}\right) \cdot 2 * \left(\pi/2 - 0\right)$$

$$= \boxed{\left(1 - e^{-4}\right)\pi}$$

21. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Graphing the region of integration. Here,
y ranges from $y=0$ to $y=\sqrt{2x-x^2}$
& x ranges from 0 to 2.

~~Here, x ranges from 0 to~~ 2
~~and i.e. x~~



The shaded region in polar co-ordinates is given by:

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Here, r ranges from 0 to $2\cos\theta$
θ ranges from 0 to $\pi/2$.

Putting $x^2+y^2=r^2$, above integral becomes:

$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= 8/3 \int_0^{\pi/2} \cos^3\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} 4\cos^3\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} (\cos 3\theta + 3\cos\theta)\,d\theta$$

$$= \frac{2}{3}\left[\frac{\sin 3\theta}{3} + 3\sin\theta\right]_0^{\pi/2}$$

$$= \frac{2}{3}\left(-1/3 + 3\right)$$

$$= \frac{2}{3} * \frac{8}{3}$$

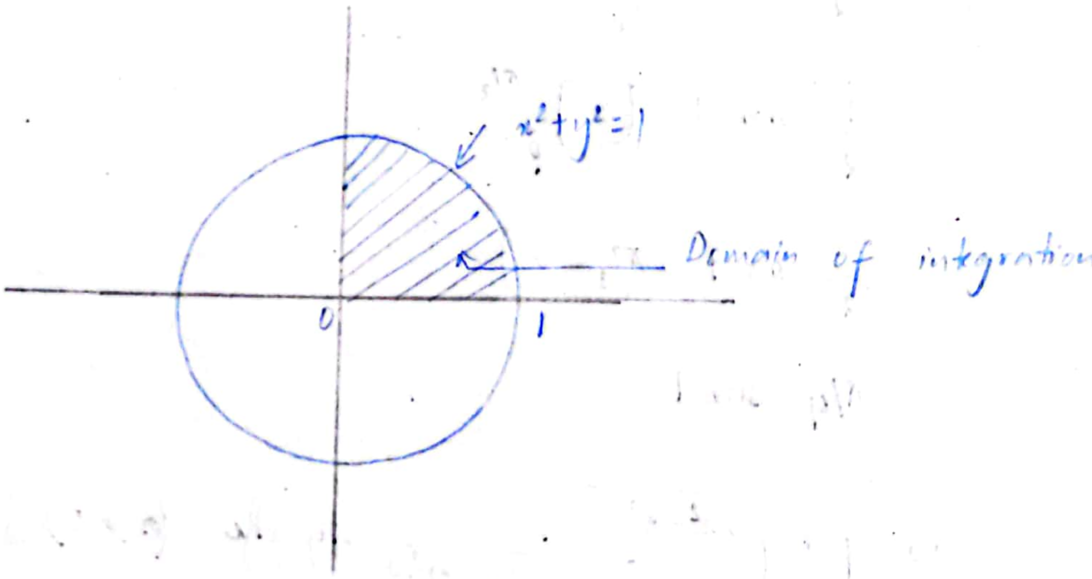
$$= \boxed{\frac{16}{9}}$$

22. $\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\, dx\, dy$

Here, x ranges from 0 to $x=\sqrt{1-y^2}$ &

y ranges from 0 to 1.

Graphing the region,



In polar co-ordinates,

r ranges from 0 to 1

θ ranges from 0 to $\pi/2$

Thus, above integral gets converted to:

$$\int_0^{\pi/2} \int_0^1 \cos r^2 \; r \, dr\, d\theta$$

Put $u = r^2$

$\therefore du = 2r$

when $r=0$, $u=0$

$r=1$, $u=1$

Thus, above integral becomes,

$$\frac{1}{2}\int_0^{\pi/2}\int_0^1 \cos u \, du \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\sin u\right]_0^1 d\theta$$

$$= \frac{1}{2}\sin 1 \int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}\sin 1 \left[\theta\right]_0^{\pi/2}$$

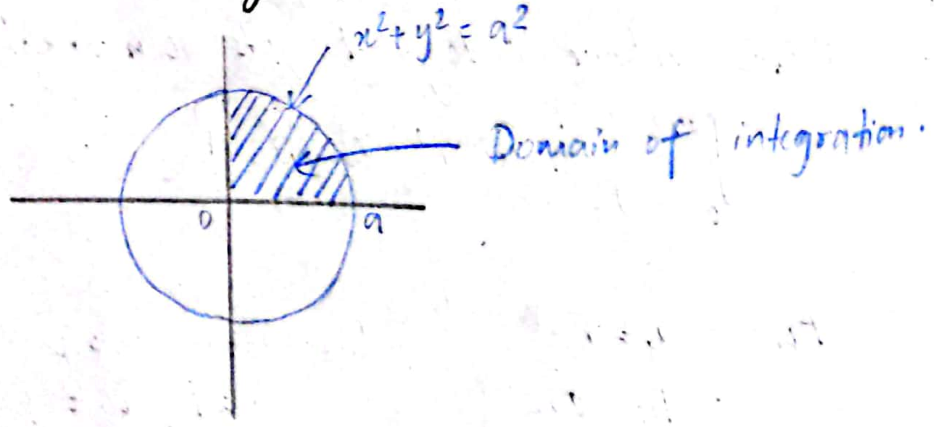
$$= \frac{1}{2}\sin 1 \left(\pi/2 - 0\right)$$

$$= \pi/4 \sin 1$$

23. $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{1}{(1+x^2+y^2)^{3/2}} \, dy \, dx \quad (a>0)$$

Here, y ranges from 0 to $y = \sqrt{a^2-x^2}$
& x ranges from 0 to a.

Graphing the above region,



Putting $x^2+y^2=r^2$ &

r ranges from 0 to a &

θ ranges from 0 to $\pi/2$, the above integral becomes:

$$\int_0^{\pi/2}\int_0^a \frac{1}{(1+r^2)^{3/2}}\, r\, dr\, d\theta$$

Put $u=1+r^2$

$\Rightarrow du = 2r\, dr$

when $r=0,\ u=1$

$r=a,\ u=1+a^2$

Thus, above integral becomes:

$$\frac{1}{2}\int_0^{\pi/2}\int_1^{1+a^2} u^{-3/2}\, du\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\frac{u^{-3/2+1}}{-3/2+1}\right]_1^{1+a^2} d\theta$$

$$= \frac{1}{2}\times(-2)\int_0^{\pi/2}\left[u^{-1/2}\right]_1^{1+a^2} d\theta$$

$$= -1\int_0^{\pi/2}\left\{(1+a^2)^{-1/2} - (1)^{-1/2}\right\} d\theta$$

$$= -1\int_0^{\pi/2}\left(\frac{1}{\sqrt{1+a^2}} - 1\right) d\theta$$

$$= \left(1-\frac{1}{\sqrt{1+a^2}}\right)\left[\theta\right]_0^{\pi/2}$$

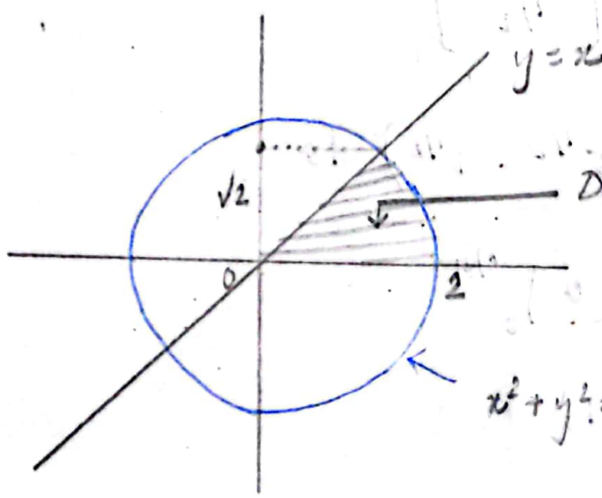
$$= \boxed{\pi/2\left(1-\frac{1}{\sqrt{1+a^2}}\right)}$$

25. $$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\, dx\, dy$$

Here, x ranges from $x = y$ to $x = \sqrt{4-y^2}$

& y ranges from 0 to $\sqrt{2}$.

Graphing the region:



For $y = x$, $\theta = \pi/4$

r ranges from 0 to 2. The above integral then becomes:

$$\int_0^{\pi/4} \int_0^{2} \frac{1}{\sqrt{1+r^2}}\, r\, dr\, d\theta \qquad (\text{Putting } r^2 = x^2+y^2)$$

$$= \int_0^{\pi/4} \int_0^{2} \frac{r}{\sqrt{1+r^2}}\, dr\, d\theta$$

Put $v = 1+r^2$

$\therefore dv = 2r\, dr$

when $r=0, \quad v=1$

$r=2, \quad v=5$

Thus, above integral becomes:

$$= \frac{1}{2}\int_0^{\pi/4}\int_1^5 v^{-1/2}\, dv\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[\frac{v^{1/2}}{1/2}\right]_1^5 d\theta$$

$$= \int_0^{\pi/4}\left(5^{1/2}-1^{1/2}\right) d\theta$$

$$= (\sqrt{5}-1)\,[\theta]_0^{\pi/4}$$

$$\boxed{= \pi/4\,(\sqrt{5}-1)}$$

26. $$\int_{-4}^{0}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 3x\, dy\, dx$$

Here y ranges from $-\sqrt{16-x^2}$ to $\sqrt{16-x^2}$

& x ranges from -4 to 0.

Graphing the region.

For polar co-ordinates:

r ranges from 0 to 4

θ ranges from $\pi/2$ to $3\pi/2$

Now, the above integral becomes,

$$\int_{\pi/2}^{3\pi/2} \int_0^4 3 * r\cos\theta \; r \, dr \, d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \int_0^4 3r^2 \cos\theta \, dr \, d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \left[\frac{3r^3}{3}\right]_0^4 \cos\theta \, d\theta$$

$$= 64 \int_{\pi/2}^{3\pi/2} \cos\theta \, d\theta$$

$$= 64 \left[\sin\theta\right]_{\pi/2}^{3\pi/2}$$

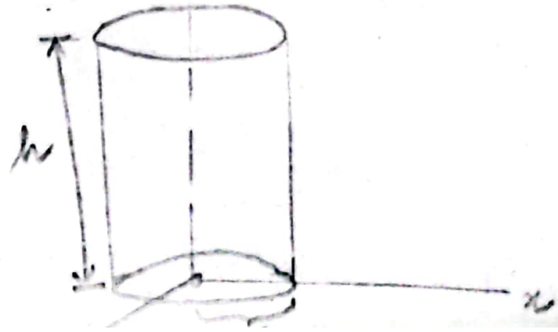
$$= 64(-1-1)$$

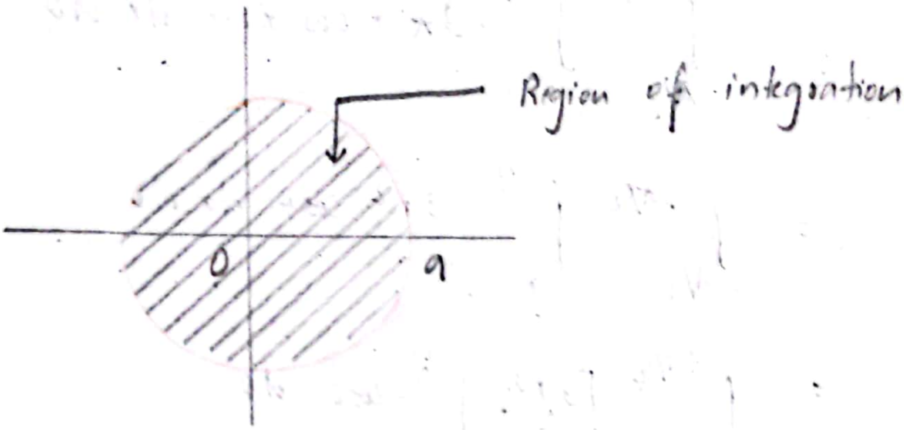
$$= \boxed{-128}$$

27. Use a double integral ~~to~~ in polar coordinates to find the volume of a cylinder of radius a and height h.

Solution

Let us sketch the domain of integration:





Here r ranges from 0 to a

θ ranges from 0 to 2π

Now,

$$\text{Volume}(V) = \int_0^{2\pi} \int_0^{a} f(r,\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{a} h\, r\, dr\, d\theta$$

$$= h\int_0^{2\pi} \int_0^{a} r\, dr\, d\theta$$

$$= h \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{a} d\theta$$

$$= h \int_0^{2\pi} \left(\frac{a^2}{2}\right) d\theta$$

$$= \frac{a^2 h}{2} \left[\theta\right]_0^{2\pi} = \frac{a^2 h}{2} \cdot 2\pi$$

$$= \boxed{\pi r^2 h}$$