$$\Rightarrow (a_{11}-\lambda')(a_{22}-\lambda') = a_{12} * a_{21}$$

$$\Rightarrow (a_{11}-\lambda')(a_{22}-\lambda') = (a_{11}-\lambda)(a_{22}-\lambda) \quad \text{[From (i)]}$$

Since the constant terms of both LHS and RHS are equal, therefore,

$$\boxed{\lambda' = \lambda}$$

This proves that the eigenvalues of A and A^T are equal.

Q35.

Here,

$$A = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}$$

The eigenvalues of A are given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (5-\lambda)(1-\lambda) = 0$$

$$\therefore \lambda_1 = 5, \ \lambda_2 = 1$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be eigenvectors of A. Then,

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = 5$

$$\begin{pmatrix} 0 & 0 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$x_1 = 2x_2$

For example, the eigenvector $= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Unit eigenvector $v_{\lambda=5} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

<u>For $\lambda_2 = 1$</u>

$$\begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 = 0$

$\therefore x_2$ is a free variable.

For example, the eigenvector $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Unit eigenvector $v_{\lambda=1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eigenspace corresponding to each eigenvalue.

For $\lambda_1 = 5$,

$E_5 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 = 2x_2$, $x_1, x_2 \in \mathbb{R}$ & $x_1, x_2 \neq 0$

Then,

$$E_5 = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

<u>For $\lambda_2 = 1$</u>

$E_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where $x_1 = 0$ & $x_2 \in \mathbb{R} - \{0\}$

$$\therefore E_1 = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

34.

Here,

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}$$

Let us first find eigenvalue and eigenvector of A.

We know,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{vmatrix}$$

For eigenvalues,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(9-\lambda) - 16 = 0$$

$$\Rightarrow \lambda_1 = 11, \ \lambda_2 = 1$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be eigenvectors of A. Then,

$$(A - \lambda I) x = 0$$

$$\Rightarrow \begin{pmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = 11$

$$\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} -8x_1 + 4x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{cases}$$

This reduces to

For example, eigenvector $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Unit eigenvector, $v_{\lambda_1=11} = 1/\sqrt{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\lambda_2 = 1$

$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The above condition gives

$$\begin{cases} 2x_1 + 4x_2 = 0 \\ 4x_1 + 8x_2 = 0 \end{cases}$$

The above equation reduces to

$$x_1 = -2x_2$$

For example, eigenvector $= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Unit eigenvector, $v_{\lambda_2=1} = 1/\sqrt{5} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Now,

$$v_1 \cdot v_2 = v_1^T v_2 = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$= -2 + 2$$

$$= 0$$

$\therefore$ The two eigenvectors are orthogonal.

Now,

$$V = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$$

$$= 1/\sqrt{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$V^T = 1/\sqrt{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

A matrix 'A' is said to be orthogonally diagonalized when

$$\Lambda = V^T A V$$

where Λ is diagonal matrix formed by eigenvalues.

Now,

$$V^TAV = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 11 & -2 \\ 22 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 55 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \Lambda$$

Q 5.

$$A = \begin{pmatrix} -1 & 4 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

$A = PDP^{-1}$

$D = P^{-1}AP$

$\Rightarrow A^K = PD^KP^{-1}$

Diagnolizable criteria $\begin{cases} \bullet\ n \text{ LI eigenvectors} \\ \text{(or } n \text{ distinct eigenvalues)} \end{cases}$

i) Since the given matrix is a upper triangular matrix, the entries on the main diagonal are its eigenvalues.

$\therefore \lambda = 5, 4, -1$

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be eigenvector of A.

Then,

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{pmatrix} -1-\lambda & 4 & 0 \\ 0 & 4-\lambda & 3 \\ 0 & 0 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = 5$

$$\begin{pmatrix} -6 & 4 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} -6x_1 + 4x_2 = 0 & \dots (i) \\ -x_2 + 3x_3 = 0 & \dots (ii) \end{cases}$$

~~choose~~ By observation, eigenvector associated with $\lambda_1 = 5$ is

$$v_{\lambda_1 = 5} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Normalized eigenvector, $v_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

For $\lambda_2 = 4$

$$\begin{pmatrix} -5 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} -5x_1 + 4x_2 = 0 \\ 3x_3 = 0 \\ x_3 = 0 \end{cases}$$

By observation, eigenvector associated with $\lambda_2 = 4$ is

$$v_{\lambda_2 = 4} = \begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix}$$

Normalized eigenvector, $v_2 = \frac{\sqrt{41}}{5} \begin{pmatrix} 4/5 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = -1$

$$\begin{pmatrix} 0 & 4 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} 4x_2 = 0 \\ 5x_2 + 3x_3 = 0 \\ 5x_3 = 0 \end{cases}$$

By observation, eigenvector associated with $\lambda_3 = -1$ is

$$v_{\lambda_3=-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

ii)

$$P = (v_1 \quad v_2 \quad v_3)$$

$$= \cancel{\begin{pmatrix} 2/\sqrt{14} & \sqrt{41}/5 \cdot 4/5 & 1 \\ 3/\sqrt{14} & \sqrt{41}/5 & 0 \\ 1/\sqrt{14} & 0 & 0 \end{pmatrix}}$$

Not normalized vectors

$$= \begin{pmatrix} 2 & 4/5 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -4/5 & 2/5 \end{pmatrix}$$

Now,

$$D = P^{-1} A P$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -4/5 & 2/5 \end{pmatrix} \begin{pmatrix} -1 & 4 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{pmatrix} \cancel{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -4/5 & 2/5 \end{pmatrix}} \begin{pmatrix} 2 & 4/5 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -4/5 & 2/5 \end{pmatrix} \begin{pmatrix} 10 & -16/5 & -1 \\ 15 & 4 & 0 \\ 5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

iii) We know

$$A = P D P^{-1}$$

$$\Rightarrow A^{K} = P D^{K} P^{-1}$$

Putting $K = 4$,

$$A^{4} = P D^{4} P^{-1}$$

$$D^{4} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{4}$$

$$= \begin{pmatrix} 5^{4} & 0 & 0 \\ 0 & 4^{4} & 0 \\ 0 & 0 & (-1)^{4} \end{pmatrix}$$

$$= \begin{pmatrix} 625 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now,

$$A^{4} = \begin{pmatrix} 2 & 4/5 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 625 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4/5 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

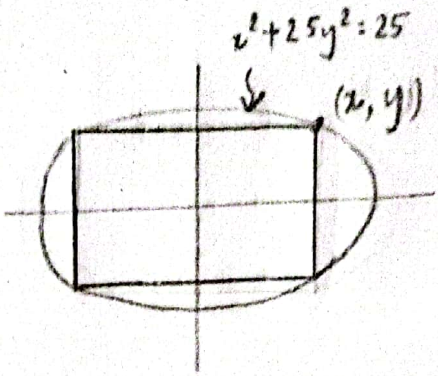
$$= \begin{pmatrix} 2 & 4/5 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1250 & 500 & 625 \\ 768 & 256 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3115.4 & 1204.8 & \text{[illegible]} \\ 4518 & 1756 & \text{[illegible]} \\ 1250 & 500 & \text{[illegible]} \end{pmatrix}$$

Q29.

Here, given ellipse is

$$x^2 + 25y^2 = 25 \quad \cdots (i)$$

The area, z of inscribed rectangle is given by



$$z = 4xy$$

We have to maximize $\boxed{z}$ with respect to the constraint (i).
The constraint is an ellipse.
Converting the ellipse to unit circle.

Dividing by 25 in eq^n(i):

$$\frac{x^2}{25} + \frac{y^2}{1} = 1 \quad \cdots (ii)$$

Put $x = 5x_1$ and $y = y_1$ in (ii), we get:

$$x_1^2 + y_1^2 = 1 \quad \cdots (iii)$$

Now,

$$z = 4 * 5 x_1 * y_1$$
$$= 20 x_1 y_1$$

Writing z in quadratic form:

$$z = (x_1, y_1) \begin{pmatrix} 0 & 10 \\ 10 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

where $x = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 10 \\ 10 & 0 \end{pmatrix}$

Calculating ~~eigenvectors~~ eigenvalues of A using characteristic equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{pmatrix} 0-\lambda & 10 \\ 10 & 0-\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \lambda^2 - 100 = 0$

$\therefore \lambda = \pm 10$

Let, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be eigenvector of A. Then,

For $\lambda_1 = 10$

$(A - \lambda I)\, x = 0$

$$\Rightarrow \begin{pmatrix} -\lambda & 10 \\ 10 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} -10x_1 + 10x_2 = 0 \\ 10x_1 - 10x_2 = 0 \end{cases}$$

This gives

$$x_1 = x_2$$

Eigenvector, for example $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Unit eigenvector $v_{\lambda = 10} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

The greater ~~eigenvect~~ of the two eigenvalues gives the maximum for given quadratic form. The eigenvector associated with the greater eigenvalue gives the point at which this maximum is achieved.

$\therefore$ Maximum area of rectangle inscribed, $z_{max} = \lambda_1$

$= \underline{10}$

This maximum is achieved at $\underline{(1/\sqrt{2},\ 1/\sqrt{2})}$.