Group 'A'

Q1.

Here,

$\lambda = 5, 1$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector of the given matrix $A = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}$

Then, we have:

$(A - \lambda I)x = 0$

$$\Rightarrow \begin{pmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \cdots (i)$$

For $\lambda = \lambda_1 = 5$,

$$\begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} 2x_1 + 4x_2 = 0 \\ -3x_1 - 6x_2 = 0 \end{cases}$$

These reduce to

$x_1 = -2x_2$

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Thus,

$$v_{\lambda=5} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$= \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

$$\therefore E_5 = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

Basis for eigenspace $= \left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

<u>for $\lambda = 1$</u>

From (i)

$$\begin{pmatrix} 6 & 4 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} 6x_1 + 4x_2 = 0 \\ -3x_1 - 2x_2 = 0 \end{cases}$$

This reduces to

$$-\frac{3}{2}x_1 = x_2$$

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Thus,

$$v_{\lambda=1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ -3/2\, x_1 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ -3/2 \end{pmatrix} \right\}$$

$$\therefore E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ -3/2 \end{pmatrix} \right\}$$

$$\therefore \text{Basis for } E_1 = \left\{ \begin{pmatrix} 1 \\ -3/2 \end{pmatrix} \right\}$$

2.5

Q2.

Writing quadratic form in matrix form:

$$Q = x^T A x$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Eigenvector for $\lambda = 9$.

We have

$$(A - \lambda I)\, x = 0$$

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$$\Rightarrow \begin{pmatrix} 9-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 9$, we get:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} -5x_2 = 0 \\ -5x_3 = 0 \end{cases}$$

x_1 is a free variable. So put $x_1 = 0$

$$\therefore v_{\lambda=9} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = u_1$$

Now,

Maximum value of $Q = ?$

Given constraints are:

$x^T x = 1$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 = 1$

Now for $u_1 = (1, 0, 0)$, we have $x_1 = 1$, $x_2 = 0$, $x_3 = 0$

$\Rightarrow x_1^2 + 0 + 0 = 1$

$\therefore x_1^2 = 1$

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$\therefore$ Maximum value of $Q = 9x_1^2 + 4x_2^2 + 4x_3^2$

$\leq 9x_1^2 + 9x_2^2 + 9x_3^2$

$= 9(x_1^2 + x_2^2 + x_3^2)$

$= 9(1^2 + 0 + 0)$

$= 9$

$\therefore$ Maximum value $= 9$

Q3.

let, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

$A^T A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$

Now, let's find eigenvalues of $A^T A$.

We have,

$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix}$

$= (2-\lambda)(3-\lambda) - 4$

For eigenvalues,

$$|A - \lambda I| = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) - 4 = 0$$

$$\Rightarrow 6 - 2\lambda - 3\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 2 = 0$$

$$\Rightarrow \lambda = 4.56, 0.438$$

$$\therefore \lambda = 4.56, 0.438$$

$$\therefore \lambda_1 = 4.56, \lambda_2 = 0.438$$

Now,

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{4.56} = 2.13$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{0.438} = 0.66$$

3

$\therefore$ Singular values of A are $\sigma_1 = 2.13$
$\sigma_2 = 0.66$

Q4.

Writing the given quadratic form in matrix form,

$$Q = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x^T A x$$

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Finding eigenvectors of A

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & -1-\lambda & 4 \\ -2 & 4 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left((-1-\lambda)(-1-\lambda)-16\right) - 2\left(2(-1-\lambda)+8\right) - 2\left(8+2(-1-\lambda)\right)$$

$$= (2-\lambda)\left(1+2\lambda+\lambda^2-16\right) - 2\left(-2-2\lambda+8\right) - 2\left(8-2-2\lambda\right)$$

$$= (2-\lambda)\left(\lambda^2+2\lambda-15\right) - 2\left(6-2\lambda\right) - 2\left(6-2\lambda\right)$$

$$= \cancel{2\lambda^2} + 4\lambda - 30 - \lambda^3 - \cancel{2\lambda^2} + 15\lambda - 24 + 8\lambda$$

$$= -\lambda^3 + 27\lambda - 54$$

Now,

$$|A-\lambda I| = 0$$

$$\Rightarrow \lambda^3 - 27\lambda + 54 = 0$$

$\Rightarrow \lambda = -6, 3$ ~~and other~~ ~~and other values~~.

Here eigenvalue is negative & positive so it is ~~negative~~ indefinite ~~(as $\lambda \neq 0$)~~.

2

Q5.

Writing given system of linear equation in augmented form:

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right)$$

Performing $R_2 \longrightarrow R_2 + R_1$

$$\sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right)$$

$R_3 \longrightarrow R_3 - 2R_1$

$$\sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right)$$

$R_3 \longrightarrow R_3 + 3R_2$

$$\sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

On converting the given matrix to upper-triangular matrix (using Gaussian Elimination technique), we

see that the co-efficient of x_3 is 0 implying that it is a free variable. The second row also shows that x_2 is always 0. Thus, the solution of this equation is not bound i.e. there are many solutions to the problem.

The above problem has arisen due to the fact that eq^n (ii) and (iii) are basically the same i.e. they are dependent. Thus, the system has many <u>solutions</u>.

1

Group 'B'

Q6.

a. We have

$$V = (v_1 \quad v_2)$$

$$= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

Now,

$$AV = A(v_1 \quad v_2)$$

$$= (Av_1 \quad Av_2)$$

$$= \begin{pmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} \\ \lambda_1 v_{21} & \lambda_2 v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= (v_1 \quad v_2)\Lambda$$

$$= V\Lambda$$

Multiplying both sides by V^{-1} on the right hand side:

$$AVV^{-1} = V\Lambda V^{-1}$$

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$\Rightarrow A = V \Lambda V^T$ (The matrix formed by eigenvectors of a symmetric matrix is orthogonal. Thus, $V^{-1} = V^T$)

$\therefore A = V \Lambda V^T$ proved

b.

Here,

$Av = \lambda v$

$A = ?$

We know,

$$A = V \Lambda V^T$$

$$= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \lambda_1/\sqrt{2} & \lambda_1/\sqrt{2} \\ \lambda_2/\sqrt{2} & -\lambda_2/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda_1+\lambda_2)/2 & (\lambda_1-\lambda_2)/2 \\ (\lambda_1-\lambda_2)/2 & (\lambda_1+\lambda_2)/2 \end{pmatrix}$$

6

is the required matrix.

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Q7b.

Here

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

A matrix 'A' is said to be diagonalizable if it is similar to a diagonal matrix.
A matrix is B is similar to A if there exists an invertible matrix P such that

$$B = P^{-1} A P.$$

Let us calculate eigenvectors of A.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)(4-\lambda) + 1$$

For eigenvalues,

$$|A - \lambda I| = 0$$

$$\Rightarrow (2-\lambda)(4-\lambda) + 1 = 0$$

$$\Rightarrow \lambda = 3$$

$$\therefore \lambda = 3$$

Here, the number of eigenvalue is 1. For a $n \times n$ matrix to be diagonalizable, it should have

n linearly independent eigenvectors implying that it should have n distinct eigenvalues. In our case, we should have had 2 eigenvalues. Thus, matrix A is not diagonalizable.

7a)

A matrix is said to be diagonalizable if it is similar to a diagonal matrix defined by

$D = P^{-1} A P$

where P is an invertible matrix. P is formed from eigenvectors of A.

Here Order of $A = n \times n$

If A has n linearly independent eigenvectors, then,

Order of $P = n \times n$

$\Rightarrow$ Order of $P^{-1} = n \times n$

3

$\therefore$ Multiplication of $P^{-1} A P$ is possible as orders of all matrices are same. Hence, we get:

$D = P^{-1} A P$.

If there are lesser number of vectors, the size of P decreases and order also changes. Thus, multiplication may not be possible. Thus A can't be confirmed to be diagonalizable.

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Q8.

a. Here given

$$B_A(v, w) = v^T A w \quad \ldots (i) \text{ and } A \text{ is}$$

a symmetric matrix.

To prove $B_A(cv, w) = c\, B_A(v, w)$ where c is an arbitrary constant.

Now,

$$B_A(cv, w) = (cv)^T A w \qquad [\text{From } (i)]$$

$$= c\, v^T A w$$

$$= c\,(v^T A w)$$

$$= c\, B_A(v, w)$$

$\therefore$ This proves that $B_A(v, w)$ is linear in the first variable. We can further do as follows:

$$B_A(au + bv, w) = (au + bv)^T A w \quad \text{where } u, v, w \in V$$

$$= ((au)^T + (bv)^T) A w$$

$$= a u^T A w + b v^T A w$$

$$= a\,(u^T A w) + b\,(v^T A w)$$

$$= a\, B_A(u, w) + b\, B_A(v, w)$$

$\therefore$ We prove that B_A is linear in first variable.

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b)

Writing the term in matrix form:

$$Q = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x^T A x$$

Now finding eigenvalues of A

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 10-\lambda & -4 \\ -4 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (10-\lambda)(4-\lambda) - 16 = 0$$

$$\Rightarrow \lambda = 12, 2$$

Calculation of eigenvectors

For $\lambda_1 = 12$

$$(A - \lambda I) x = 0$$

$$\Rightarrow \begin{pmatrix} 10-12 & -4 \\ -4 & 4-12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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From above, we get:

$x_1 = -2x_2$.

$\therefore$ Eigenvector, for example $= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Unit eigenvector(v_1) $v_{\lambda=12} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

<u>For $\lambda_2 = 2$</u>

$$\begin{pmatrix} 10-2 & -4 \\ -4 & 4-2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$2x_1 = x_2$

Eigenvector, for example $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore$ Unit eigenvector(v_2) $v_{\lambda=2} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

Now,

$$v_{\lambda=12} \cdot v_{\lambda=2} = -2/\sqrt{5} \times 1/\sqrt{5} + 1/\sqrt{5} \times 2/\sqrt{5}$$

$$= 0$$

Thus, the eigenvectors are orthogonal.

$V = (v_1 \quad v_2)$.

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Now,

$$\Lambda = V^T A V$$

$$= \cancel{\begin{pmatrix} -2/\sqrt{5} & \\ 1/\sqrt{5} & \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix}}$$

$$= \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} * \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -24 & 2 \\ 12 & 4 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 60 & 0 \\ 0 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix}$$

Suppose $x = Vy$ where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Then,

$$x^T A x = (Vy)^T A (Vy)$$

$$= y^T V^T A V y$$

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$$= y^T \Lambda y$$

$$= (y_1 \quad y_2) \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$= 12y_1^2 + 2y_2^2$ is the required term without cross product terms.

6

Q 9.

let, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

lets find eigenvectors of $A^T A$. We have,

$|A^T A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

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$$(2-\lambda)(5-\lambda) - 4 = 0$$

$$\Rightarrow \lambda = 6, 1$$

<u>For $\lambda_1 = 6$</u>

$$(A - \lambda I)\, x = 0$$

$$\Rightarrow \begin{pmatrix} 2-7 & 2 \\ 2 & 5-7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-6 & 2 \\ 2 & 5-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$2x_1 = x_2$$

$\therefore$ Unit eigenvector $v_{\lambda=6} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

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For $\lambda_2 = 1$

$$\begin{pmatrix} 2-1 & 2 \\ 2 & 5-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$x_1 = -2x_2$

$\therefore$ Unit eigenvector $v_{\lambda=1} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

Now,

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{6}$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$= \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

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u_3 is orthogonal to u_1 & u_2. let $u_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Thus,

$u_1 \cdot u_3 = 0$

$\Rightarrow x + 2y + 5z = 0 \quad \ldots (i)$

$u_2 \cdot u_3 = 0$

$\Rightarrow -2x + y = 0 \quad \ldots (ii)$

Let, $x = 1$. Then, $y = 2x$

$= 2 \times 1 = 2$

Then, $z = \dfrac{-x - 2y}{5}$

$= \dfrac{-1 - 2 \times 2}{5}$

$= -5/5 = -1$

$\therefore u_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Normalized $u_3 = \dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

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Now,

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

$$U \Sigma V^T = \begin{pmatrix} 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{6}/\sqrt{5} & 2\sqrt{6}/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 + 4/5 & 2/5 - 2/5 \\ 2/5 - 2/5 & 4/5 + 1/5 \\ 5/5 & 10/5 + 0 + 0 \end{pmatrix}$$

6

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} = A \quad \therefore A = U \Sigma V^T$$

proved

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Q10.

A matrix is said to be in reduced row echelon form if it satisfies following four properties:

i) A row containing all zero element must be present at the bottom of the matrix

ii) The first non-zero entry of a column must be 1. This 1 is called leading 1.

iii) Two consecutive non-zero rows should have 1 move from top left to bottom right.

iv) If a column contains leading 1 element, it should not contain any other non-zero element.

Eg: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Example of augmented matrix of order 4x5:

Eg: $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right)$

Above matrix is in reduced echelon form.

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$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4 & 3 & 2 & -3 \\ 6 & -5 & 3 & -14 \end{array}\right)$$

$R_2 \longrightarrow R_2 - 2R_1$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 6 & -5 & 3 & -14 \end{array}\right)$$

$R_3 \longrightarrow R_3 - 3R_1$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & -8 & 6 & -20 \end{array}\right)$$

$R_2 \longleftrightarrow R_3$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -8 & 0 & -20 \\ 0 & 1 & 4 & -7 \end{array}\right)$$

$R_3 \longrightarrow R_3 + R_2/8$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -8 & 0 & -20 \\ 0 & 0 & 4 & \cancel{-7 \; 20/8} \; -19/2 \end{array}\right)$$

$R_3 \longrightarrow R_3 / 4$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -8 & 0 & -20 \\ 0 & 0 & 1 & -19/8 \end{array}\right)$$

$R_3 \longrightarrow R_3 + 8R_1$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 38 & -76 \end{array}\right)$$

$R_3 \longrightarrow R_3 / 38$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$R_1 \longrightarrow R_1 + R_3, \quad R_2 \longrightarrow R_2 - 4R_3$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

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$R_1 \longrightarrow R_1 - R_2$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$R_1 \longrightarrow R_1/2$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$\therefore \quad x = -1/2$

$y = 1$

$z = -2$

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