Unit 4: Multiple Integrals

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Contents

The integral of a function of two variables $f(x, y)$, called a **double integral**, is denoted

$$\iint_D f(x, y)\, dA$$

It represents the signed volume of the solid region between the graph of $f(x, y)$ and a domain D in the xy-plane (Figure 1), where the volume is positive for regions above the xy-plane and negative for regions below. There are many similarities between double integrals and the single

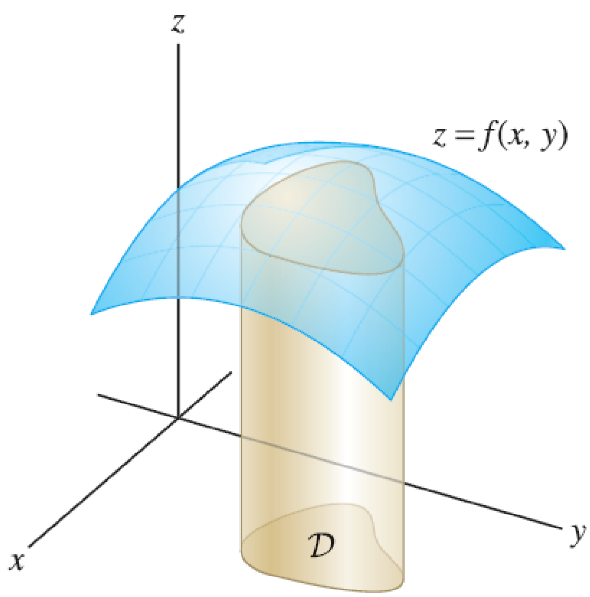


Figure 1: The double integral computes the volume of the solid region between the graph of $f(x, y)$ and the xy-plane over a domain $\mathcal{D}$.

integrals:

- Double integrals are defined as limits of sums.
- Double integrals are evaluated using the Fundamental Theorem of Calculus twice.

1 Double integrals

We consider a rectangle in $\mathbb{R}^2$ given by

$$\begin{aligned} R &= [a,b] \times [c,d] \\ &= \{(x,y) \in \mathbb{R}^2 : \ a \leq x \leq b, \ c \leq y \leq d\}. \end{aligned}$$

Let

$$A = \text{Area of } R.$$

Also, we will initially assume that $z = f(x,y) \geq 0$. The graph of f is a surface above the xy-plane with equation $z = f(x,y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x,y,z) \in \mathbb{R}^3 : \ 0 \leq z \leq f(x,y), \ (x,y) \in R\}.$$

Here is a sketch of this set up.

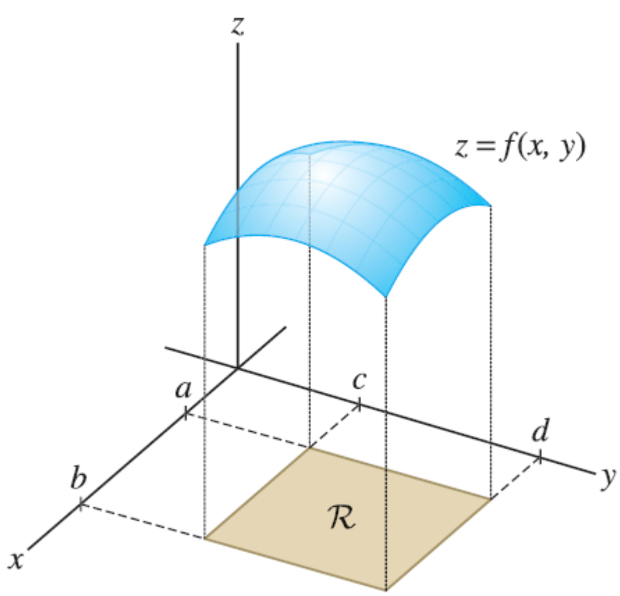


Figure 2: The solid that lies above $\mathcal{R}$ and under the surface of f.

Like integrals in one variable, double integrals are defined through a three-step process: **subdivision**, **summation**, and **passage to the limit**.

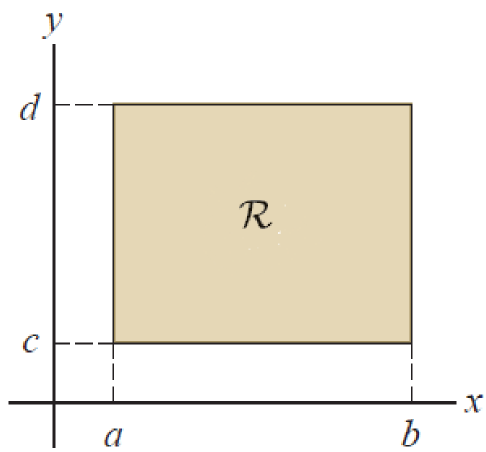


Figure 3: $\mathcal{R}$ = Domain of f

Subdivision

1. We first divide up $[a, b]$ into m subintervals of equal width $\Delta x = (b - a)/m$ and $[c, d]$ into n subintervals of equal width $\Delta y = (d - c)/n$ by choosing partitions:

$$a = x_0 < x_1 < ... < x_m = b, c = y_0 < y_1 < ... < y_n = d,$$

where m and n are positive integers.

2. Using this subdivision, create an $n \times m$ grid of subrectangles R_{ij}. The area of each subrectangle R_{ij} is given by

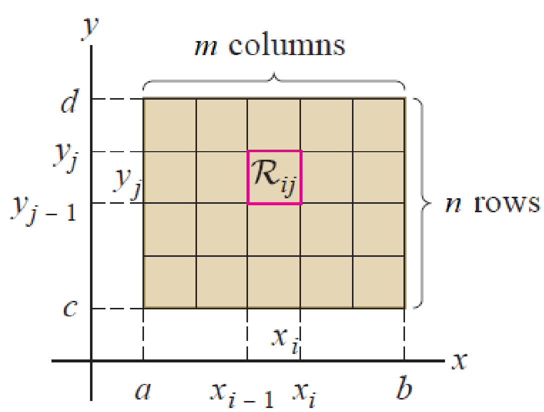


Figure 4: An $n \times m$ grid of $\mathcal{R}$

$$\Delta A = \Delta x \Delta y.$$

3. From each of these subrectangles we will choose a point (x_i^*, y_j^*), as shown in the figure given below.

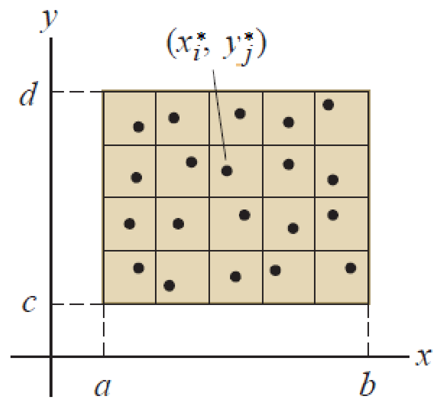


Figure 5: An $n \times m$ grid of R with sample points (x_i^*, y_j^*)

Now, over each of these subrectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$. See the figure given below. Each of the

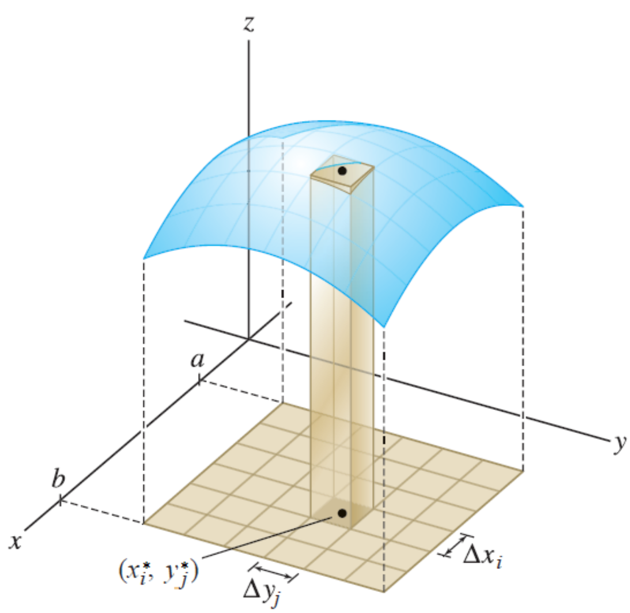


Figure 6: A box with volume $f(x_i^*, y_j^*)\Delta A$

boxes has a base area of ΔA and a height of $f(x_i^*, y_j^*)$ so the volume of each of these boxes is $f(x_i^*, y_j^*)\Delta A$.

Summation

The volume V of the solid S is now approximated by the sum of the volumes of all small boxes, that is :

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A.$$

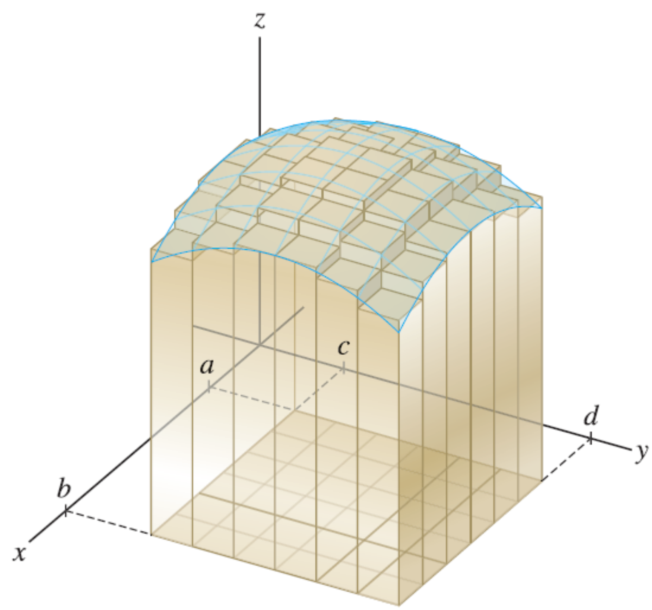


Figure 7: The solid S is approximated by the sum of the volumes of all boxes.

Double Riemann sum A **double Riemann sum** is defined as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A.$$

Passage to the limit

We have a double sum since we will need to add up volumes in both the x and y directions. To get a better estimation of the volume we will take n and m larger and larger and to get the exact volume we will need to take the limit as both n and m go to infinity. In other words,

$$V = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

This looks a lot like the definition of the integral of a function of single variable. In fact, this is also the definition of an integral of a function of two variables over a rectangle.

Here is the formal definition of a double integral of a function of two variables over a rectangle R as well as the notation that we'll use for it.

Double integral over a rectangle

$$\iint_R f(x,y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

The sample point (x_i^*, y_j^*) in the definition can be chosen to be any point (x_i, y_j) in the subrectangle.

Geometric interpretation

If f happens to be a positive function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 3, and is an approximation to the volume of the solid under the graph of f and above the rectangle R. Thus, we have the following difinition:

Geometric interpretation of a double integral

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$\text{Volume } = \iint_R f(x,y)\, dA.$$

Example 1. Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square . Sketch the solid and the approximating rectangular boxes.

Solution. The paraboloid is the graph of $f(x,y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. The squares are shown in Figure 8. Approximating the volume by the Riemann sum with $m = n = 2$, we

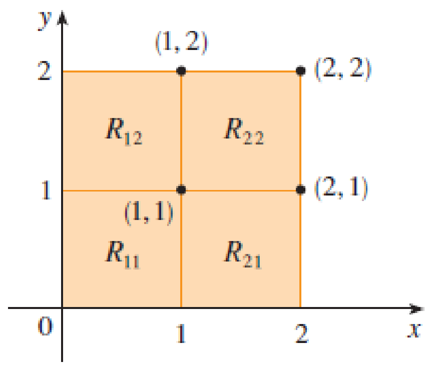


Figure 8

have

$$
\begin{aligned}
V &\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\Delta A \\
&= f(1,1)\Delta A + f(1,1)\Delta A + f(1,2)\Delta A + f(2,2)\Delta A \\
&= 13(1) + 7(1) + 10(1) + 4(1) = 34.
\end{aligned}
$$

This is the volume of the approximating rectangular boxes shown in Figure 9. ◀

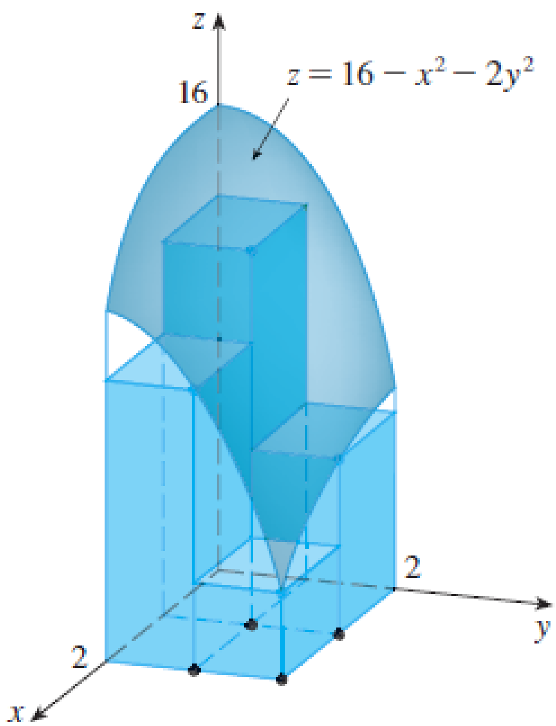


Figure 9

The Midpoint Rule

Let $\overline{x}_i$ be the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ be the midpoint of $[y_{j-1}, y_j]$. Then we can choose the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij} as the sample point (x_i^*, y_j^*).

Midpoint Rule for Double Integrals

$$\iint_A f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j) \Delta A,$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example 2. Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y)^2\, dA$, where $R = \{(x, y) |\ 0 \leq x \leq 2, 1 \leq y \leq 2\}$

Solution. In using the Midpoint Rule with $m = n = 2$, we evaluate

$f(x, y) = (x - 3y)^2$ at the centers of the four subrectangles shown in Figure 10.

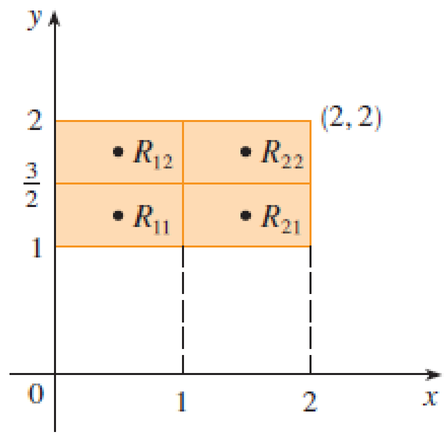


Figure 10

So $\overline{x}_1 = \frac{1}{2}, \overline{x}_2 = \frac{3}{2}, \overline{y}_1 = \frac{5}{4}$, and $\overline{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}
\iint_R (x - 3y)^2 \, dA &\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j) \Delta A \\
&= f(\overline{x}_1, \overline{y}_1)\Delta A + f(\overline{x}_1, \overline{y}_2)\Delta A + f(\overline{x}_2, \overline{y}_1)\Delta A \\
&\quad + f(\overline{x}_2, \overline{y}_2)\Delta A \\
&= f(\tfrac{1}{2}, \tfrac{5}{4})\Delta A + f(\tfrac{1}{2}, \tfrac{7}{4})\Delta A + f(\tfrac{3}{2}, \tfrac{5}{4})\Delta A \\
&\quad + f(\tfrac{3}{2}, \tfrac{7}{4})\Delta A \\
&= (-\tfrac{67}{16})\tfrac{1}{2} + (-\tfrac{130}{16})\tfrac{1}{2} + (-\tfrac{51}{16})\tfrac{1}{2} + (-\tfrac{123}{16})\tfrac{1}{2} \\
&= -\tfrac{95}{8} = -11.875
\end{aligned}$$

◀

Average Value

Let f be a function of two variables defined on a rectangle R. We define the average value of f to be

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA,$$

where $A(R)$ is the area of R.
If $f(x, y \geq 0$, the equation

$$A(R) \times f_{ave} = \iint_R f(x,y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

Properties of Double Integrals

Here are some properties of the double integral. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Some properties of the double integral

1.
$$\iint_R [f(x,y) + g(x,y)]\, dA$$
$$= \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

2. If c is a constant, then
$$\iint_R cf(x,y)\, dA = c \iint_R f(x,y)\, dA$$

3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in R$, then
$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$$

2 Iterated Integrals

Just like with the definition of a single integral, it is usually difficult to evaluate double integrals from first principles. So we need to start looking into how we actually compute double integrals.

Iterated integration

In the previous unit we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way.

For instance, suppose that $f_x(x, y) = 2xy$. We can treat y as staying constant and integrate to obtain $f(x, y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\ dx \\ &= \int 2xydx \\ &= \int x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y.

For instance, if

$$f(x, y) = x^2y + \sin y + y^3 + 17,$$

then $f_x(x, y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x,y) = \int f_x(x,y)\ dx = x^2y + C(y).$$

Using this process we can evaluate definite integrals.

Example 3. Evaluate the integral $\int_1^4 2xy \, dx$.

Solution. We consider y as a constant and integrate with respect to x:

$$\begin{aligned}\int_1^4 2xy \, dx &= x^2 y \Big|_1^4 \\ &= 4^2 y - 1^2 y \\ &= 15y.\end{aligned}$$

We have considered y to be a constant. So, the limits of the integral may be functions of y.

Example 4. Evaluate the integral $\displaystyle\int_1^{2y} 2xy \, dx$.

Solution. We consider y as a constant and integrate with respect to x:

$$\begin{aligned}\int_1^{2y} 2xy \, dx &= x^2 y \Big|_1^{2y} \\ &= (2y)^2 y - 1^2 y \\ &= 4y^3 - y.\end{aligned}$$

Remark

Note how the limits of the integral are from $x = 1$ to $x = 2y$ and that the final answer is a function of y.

Example 5. Evaluate the integral $\int_1^x (5x^3 y^{-3} + 6y^2) \, dy$.

Solution. Here, we consider x to be a constant and integrate with respect to y:

$$\begin{aligned}\int_1^x (5x^3y^{-3} + 6y^2)\, dy &= \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x \\ &= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3 + 2\right) \\ &= \frac{9}{2}x^3 - \frac{5}{2}x - 2.\end{aligned}$$

Remark

Note how the limits of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

We can integrate the result obtained in the previous example with respect to x as well. This process is known as **iterated integration**, or **multiple integration**.

Example 6. Evaluate the integral

$$\int_1^2 \left(\int_1^x (5x^3y^{-3} + 6y^2)\, dy\right) dx.$$

Solution. We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in the previous example.

$$\begin{aligned}
&\int_1^2 \left(\int_1^x (5x^3y^{-3} + 6y^2)\, dy\right) dx \\
= &\int_1^2 \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x dx \\
= &\int_1^2 \left(\frac{9}{2}x^3 - \frac{5}{2}x - 2\right)\, dx \\
= &\left(\frac{9}{8}x^4 - \frac{5}{4}x^2 - 2x\right)\Bigg|_1^2 \\
= &\frac{89}{8}.
\end{aligned}$$

Remark

Note how the limits of the integral are from $x = 1$ to $x = 2$ and that the final result was a number.

The previous example showed how we could perform something called an *iterated integral*; we do not yet know why we would be interested in doing so nor what the result, such as the number 89/8, means. Before we investigate these questions, we offer some definitions.

Partial integration and iterated integrals

We will continue to assume that we are integrating $f(x, y)$ over the rectangle

$$R = [a, b] \times [c, d].$$

If $x = x_0$ is kept fixed, we obtain a cross-section bounded by vertical lines $y = c$ and $y = d$, the horizontal line $z = 0$, and by the curve $z = f(x_0, y)$. The area of the cross-section is therefore given by

$$A(x_0) = \int_c^d f(x_0, y)\, dy.$$

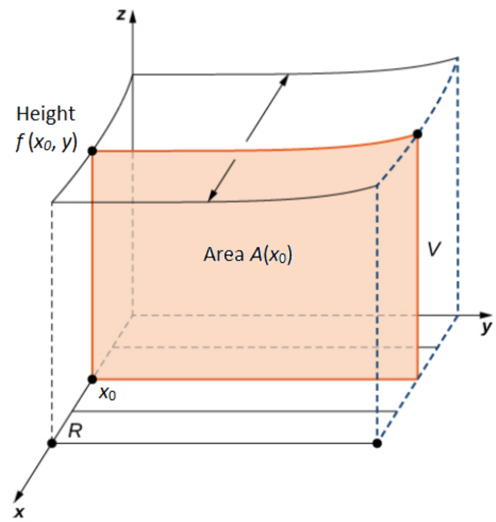


Figure 11: The cross-section $A(x_0)$.

Partial integration

We use the notation

$$\int_c^d f(x,y)\ dy$$

to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from c to d. This procedure is called **partial integration** with respect to y.

We see that the cross-section area $\int_c^d f(x,y)\ dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x,y)\ dy.$$

We now integrate the function A with respect to x from a to b. We then get

$$\int_a^b A(x)\ dx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx.$$

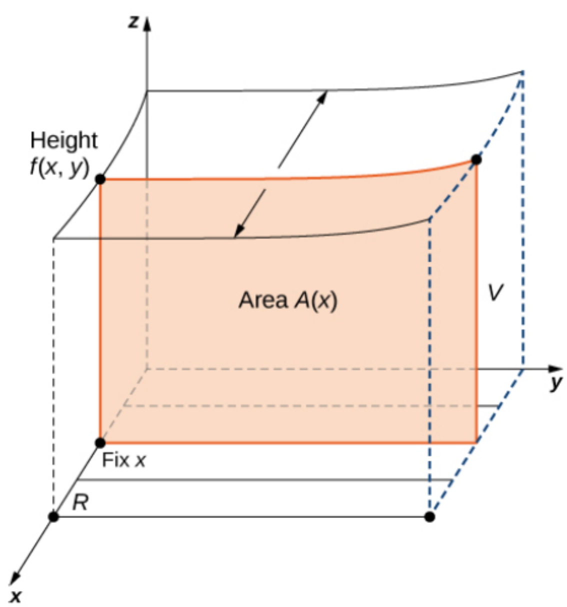
The integral on the right side is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\int_a^b \int_c^d f(x,y)\, dydx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx.$$

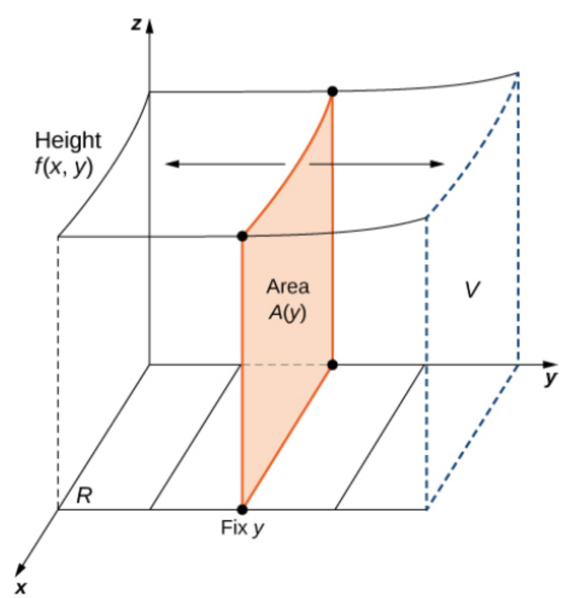
This means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, we define the iterated integral:

$$\int_c^d \int_a^b f(x,y)\, dxdy = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy.$$



Integrating first w.r.t. y and then w.r.t. x to find the area $A(x)$ and then the volume V.



Integrating first w.r.t. x and then w.r.t. y to find the area $A(y)$ and then the volume V.

Thus,

Iterated integrals

- $$\int_a^b \int_c^d f(x,y)\, dydx = \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx.$$
- $$\int_c^d \int_a^b f(x,y)\, dxdy = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy.$$

Example 7.

Evaluate the iterated integrals:

(a) $\int_0^1 \int_1^2 x^2 y\, dydx$ (b) $\int_1^2 \int_0^1 x^2 y\, dxdy$.

Solution. ... ◀

3 Double integral and iterated integral

The previous example illustrates a general fact: The value of an iterated integral does not depend on the order in which the integration is performed. This is a part of Fubini's Theorem. Even more important, Fubini's Theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral.

Fubini's theorem

Theorem 3.1. Suppose that $f(x,y)$ is is continuous over a rectangle

$$R = \{(x,y) \in \mathbb{R}^2 | a \le x \le b,\ c \le y \le d\}.$$

Then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy.$$

More generally, this is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves.

What Fubini's Theorem says is that

- The value of an iterated integral does not depend on the order in which the integration is performed.

$$\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy.$$

- A double integral can be calculated as an iterated integral.

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dydx$$

- The volume V can be calculated as the integral of the cross section perpendicular to the x or y-axis.

$$\iint_R f(x,y)\, dA = \int_a^b A(x)\, dx = \int_c^d A(y)\, dy.$$

Example 8. Compute the following double integral over the indicated rectangle.

$$\iint_R x\, dA,\ \ R = [0,2] \times [0,1].$$

Solution. Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration. ◀

Example 9. Compute the following double integral over the indicated rectangle.

$$\iint_R (2x - 4y^3)\, dA,\ \ R = [-5,4] \times [0,3].$$

Solution. Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration. ◀

Example 10. Find the volume under the surface $z = \sqrt{1-x^2}$ and above the triangle formed by $y = x, x = 1$, and the x-axis.

Solution. Let's consider the two possible ways to set this up:

$$\int_0^1 \int_0^x \sqrt{1-x^2}\, dydx \quad \text{or} \quad \int_0^1 \int_y^1 \sqrt{1-x^2}\, dxdy.$$

Which appears easier? In the first, the inner integral is easy, because we need an anti- derivative with respect to y, and the entire integrand $\sqrt{1-x^2}$ is constant with respect to y. Of course, the outer integral may be more difficult. In the second, the inner integral is mildly unpleasant – a trigonometric substitution. So let's try the first one, since the first step is easy, and see where that leaves us.

$$\begin{aligned} \int_0^1 \int_0^x \sqrt{1-x^2}\, dydx &= \int_0^1 y\sqrt{1-x^2}\Big|_0^x\, dx \\ &= \int_0^1 x\sqrt{1-x^2}\, dx. \end{aligned}$$

This is quite easy, since the substitution $u = 1 - x^2$ works:

$$\begin{aligned} \int x\sqrt{1-x^2}\, dx &= -\frac{1}{2}\int \sqrt{u}\, du \\ &= -\frac{1}{3}u^{2/3} \\ &= -\frac{1}{3}(1-x^2)^{2/3}. \end{aligned}$$

Therefore,
$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{2/3}\bigg|_0^1 = \frac{1}{3}.$$
◀

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it's usually worth considering both possibilities before going very far.

Example 11. Compute the following double integral over the indicated rectangle.
$$\iint_R y\sin(xy)\;dA,\;\; R=[1,2]\times[0,\pi].$$

Solution. It is easier to integrate first with respect to x and then with respect to y. ◀

Example 12. Find the volume of the solid S enclosed by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$ and $y=2$, and three coordinate planes.

Solution. ◀

Example 13. Find the volume of the solid enclosed by the planes $4x+2y+z=10, y=3x, z=0, x=0$.

Solution. Notice that the planes $4x+2y+z=10$ is the top of the volume and the planes $z=0$ and $x=0$ indicate that the plane

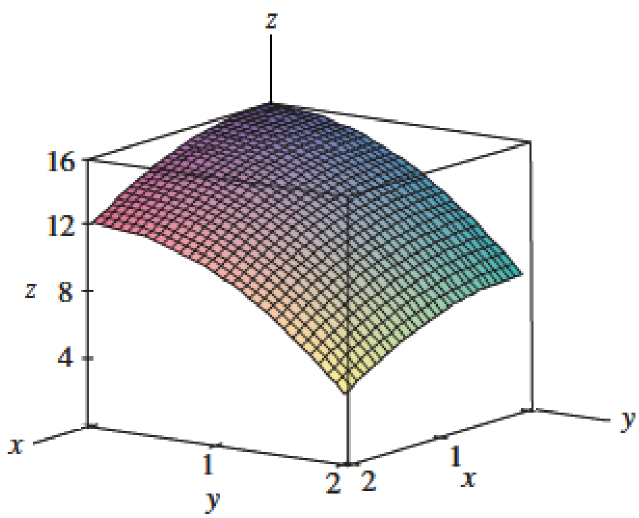


Figure 12

$4x + 2y + z = 10$ does not go past the xy-plane and the yz-plane. So we are really looking for the volume under the plane

$$z = 10 - 4x - 2y$$

and above the region R in the xy-plane. The second plane, $y = 3x$, gives one of the sides of the volume as shown below. The region R will be the

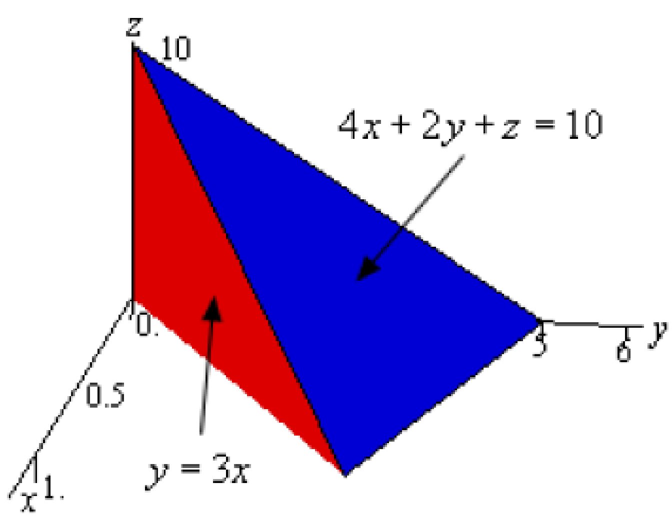


Figure 13

region in the xy-plane (i.e. $z = 0$) that is bounded by $y = 3x, x = 0$, and the line where $4x + 2y + z = 10$ intersects the xy-plane. We can determine where $4x + 2y + z = 10$ intersects the xy-plane by plugging $z = 0$ into it.

$$4x + 2y + 0 = 10 \Rightarrow 2x + y = 5 \Rightarrow y = -2x + 5$$

So, here is a sketch the region R. The region R is really where this

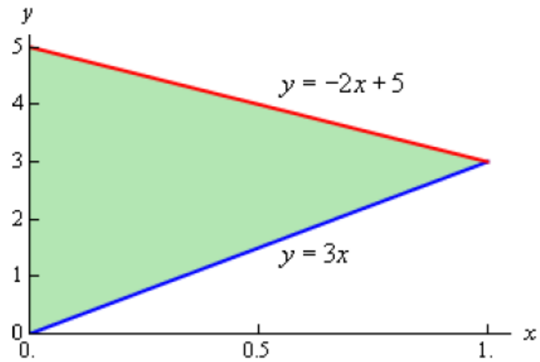


Figure 14

solid will sit on the xy-plane and here are the inequalities that define the region.

$$\begin{aligned} &0 \le x \le 1 \\ &3x \le y \le -2x + 5. \end{aligned}$$ ◀

A special case:

$$f(x, y) = f(x)h(y) \text{ on } R = [a, b] \times [c, d].$$

In this case,

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_a^b \int_c^d g(x)h(y)\, dydx \\ &= \int_a^b g(x)\, dx \int_c^d h(y)\, dy. \end{aligned}$$

Example 14. Compute the double integral of

$$f(x,y) = \frac{1+x^2}{1+y^2},$$

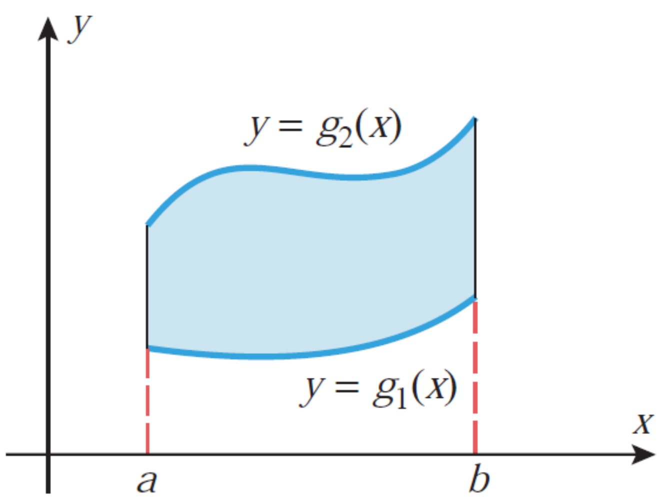
in the rectangular region $R = [0,2] \times [0,1]$.

4 Integration Regions between Two Curves

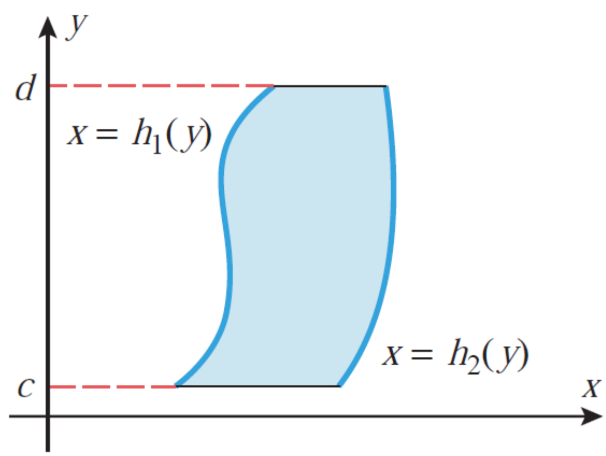
To this point, we restricted our attention to rectangular domains (in some cases, triangular domains). Now we shall treat the more general case of domains.

When D is a region between two curves in the xy-plane, we can evaluate double integrals over D as iterated integrals.

There are two types of regions that we need to look at. Here is a sketch of both of them.



(a) A type I region



(b) A type II region

We will often use set builder notation to describe these regions.

Types of regions

Type I: $D = \{(x, y)|\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

Type II: $D = \{(x, y)|\ h_1(x) \leq x \leq h_2(x), c \leq y \leq d\}$

This notation indicates that we are going to use all the points, (x, y), in which both of the coordinates satisfy the two given inequalities.

Area of a plane region

Although we usually think of double integrals as representing volumes, it is worth noting that we can express the area of a domain D in the plane as the double integral of the constant function $f(x, y) = 1$:

$$\text{Area}(D) = \iint_D 1\ dA. \tag{1}$$

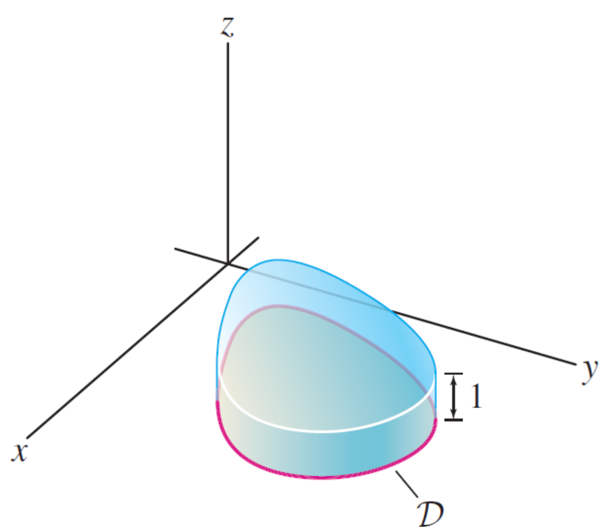


Figure 15

Indeed, as we see in Figure 15, the the area of D is equal to the volume of the "cylinder" of height 1 with D as base. More generally, for any constant C,

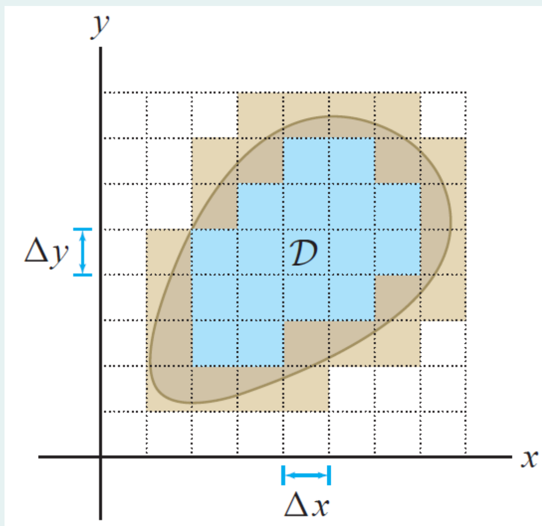
$$\iint_D C\ dA = C\ \text{Area}(D).$$

Conceptual insight

Equation (1) tells us that we can approximate the area of a domain D by a Riemann sum for

$$\iint_D 1 \, dA.$$

In this case, $f(x, y) = 1$, and we obtain a Riemann sum by adding up the areas $\Delta x_i \Delta y_j$ of those rectangles in a grid that are contained in D or that intersects the boundary of D (See the figure given below).



The finer the grid, the better the approximation. The exact area is the limit as the sides of the rectangles tend to zero.

A certain iterated integral can be viewed as giving the area of a plane region.

Theorem 4.1 (Area of a plane region)**.**

(a) Let a plane region be given by

$$D = \{(x, y)| \; a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

where g_1 and g_2 are continuous functions on $[a, b]$. Then the area

A of D is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dydx.$$

(b) Let a plane region be given by

$$D = \{(x, y)|\ h_1(y) \le x \le h_2(y), c \le y \le d\},$$

where h_1 and h_2 are continuous functions on $[c, d]$. Then the area A of D is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dxdy.$$

Proof.

Consider the type I region $D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$. We know that the area of D is given by

$$\int_a^b (g_2(x) - g_1(x))\ dx.$$

We can view the expression $g_2(x) - g_1(x)$ as

$$g_2(x) - g_1(x) = \int_{g_1(x)}^{g_2(x)} 1\ dy,$$

meaning we can express the area of D as an iterated integral:

$$\begin{aligned}\text{Area of } D &= \int_a^b (g_2(x) - g_1(x))\, dx \\ &= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} 1\, dy \right) dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dydx.\end{aligned}$$

Using a process similar to that above, the area of a type II region D could also be obtained. We have

$$\text{Area of } D = \int_c^d \int_{h_1(y)}^{h_2(y)} 1\, dxdy. \qquad \blacktriangleleft$$

Example 15. Find the area of the region enclosed by $y = 2x$ and $y = x^2$.

Solution. We'll find the area of the region using both orders of integration.

For the type I region, we have

$$2x = x^2 \Rightarrow x = 0, 2.$$

Thus,

$$0 \leq x \leq 2, \quad x^2 \leq y \leq 2x.$$

Therefore, the required area is

$$\int_0^2 \int_{x^2}^{2x} 1\, dy\, dx = \int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}.$$

For the type II region, we have

$$x = \frac{1}{2}y, \quad x = \sqrt{y}.$$

We then have

$$\frac{1}{2}y = \sqrt{y}.$$

This implies that

$$y^2 = 4y \Rightarrow y = 0,\ 4.$$

Therefore, the required area is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1\ dx\ dy = \int_0^4 (\sqrt{y} - y/2)\ dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}.$$

Theorem 4.2. Let $f(x, y)$ be continuous.

(a) If $D = \{(x, y)|\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_D f(x, y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dydx.$$

(b) If $D = \{(x, y)|\ h_1(x) \leq x \leq h_2(x), c \leq y \leq d\}$, then

$$\iint_D f(x, y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\ dxdy.$$

Setting up Limits of Integration

To apply the above theorem, it is helpful to start with a two-dimensional sketch of the region D. It is not necessary to graph $f(x, y)$. For a type I region, the limits of integration can be obtained as follows:

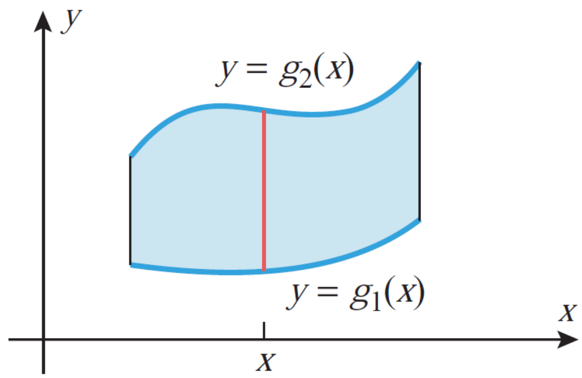


Fig. (a)

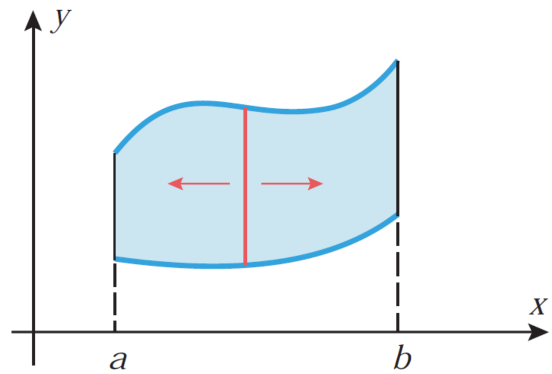


Fig. (b)

Determining Limits of Integration: Type I Region

1. x is held fixed for the first integration. We draw a vertical line through the region D at an arbitrary fixed value x (Figure (a)). This line crosses the boundary of D twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration over the type I region.

2. Imagine moving the line drawn in Step 1 first to the left and then to the right (Figure (b)). The leftmost position where the line intersects the region D is $x = a$, and the rightmost position where the line intersects the region D is $x = b$. This yields the limits for the x-integration over the type I region.

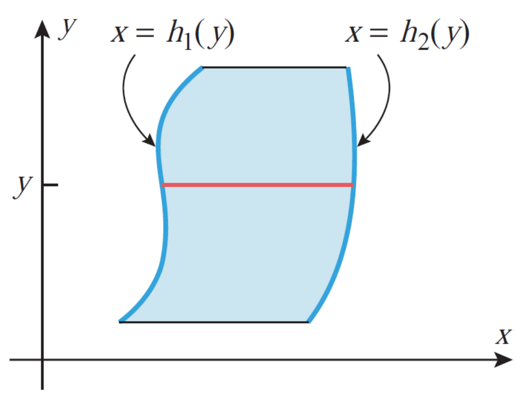


Fig. (c)

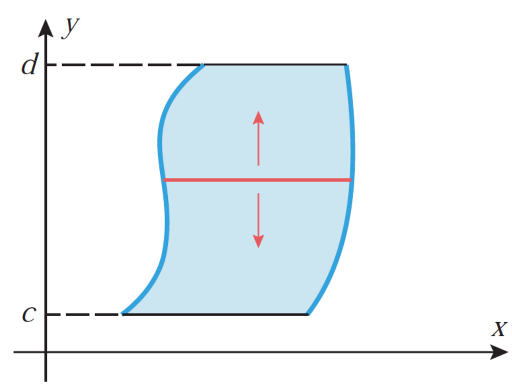


Fig. (d)

Determining Limits of Integration: Type II Region

1. y is held fixed for the first integration. We draw a horizontal line through the region D at a fixed value y (Figure (c)). This line crosses the boundary of D twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration over the type II region.

2. Imagine moving the line drawn in Step 1 first down and then up (Figure (d)). The lowest position where the line intersects the region D is $y = c$, and the highest position where the line intersects the region D is $y = d$. This yields the y-limits of integration over the type II region.

Calculating a double integral over a type I region

Example 16. Evaluate $\iint_D (x + 2y)\ dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution. We have

$$2x^2 = 1 + x^2 \Rightarrow x = \pm 1.$$

Then $y = 2$. Thus, the parabolas intersect at $(-1, 2)$ and $(1, 2)$. We see that the region D is given by

$$D = \{(x, y) : 1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

We also see that the lower boundary is $y = 2x^2$ and the upper boundary

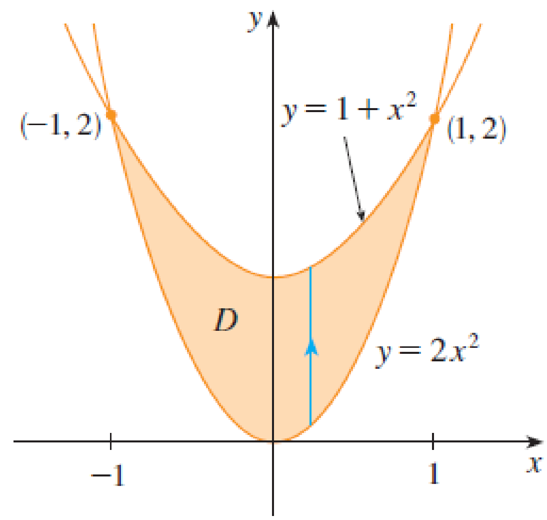


Figure 16: Type I region

is $y = 1 + x^2$. Therefore, we have

$$\begin{aligned}\iint_D (x + 2y)\ dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\ dydx \\ &= \frac{12}{15}.\end{aligned}$$

◀

Calculating a double integral over both type I and type II region

Example 17. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution. From the figure we see that D can be viewed as a type I region:

$$D = \{(x, y) : \ 0 \le x \le 2, x^2 \le y \le 2x\}$$

Therefore the volume under $z = x^2 + y^2$ and above D is

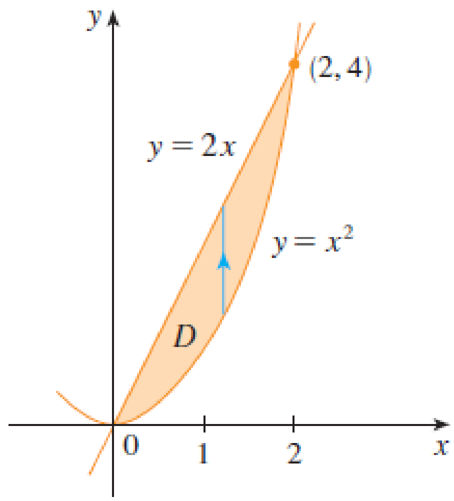


Figure 17: Type I region

$$
\begin{aligned}
V &= \iint_D (x^2 + y^2)\ dA \\
&= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\ dydx \\
&= \frac{216}{35}.
\end{aligned}
$$

◀

Alternatively,

Solution. From the figure we see that D can also be written as a type II region:

$$D = \{(x, y) :\ 0 \le y \le 4, \tfrac{1}{2}y \le x \le \sqrt{y}\}$$

Therefore another expression for V is

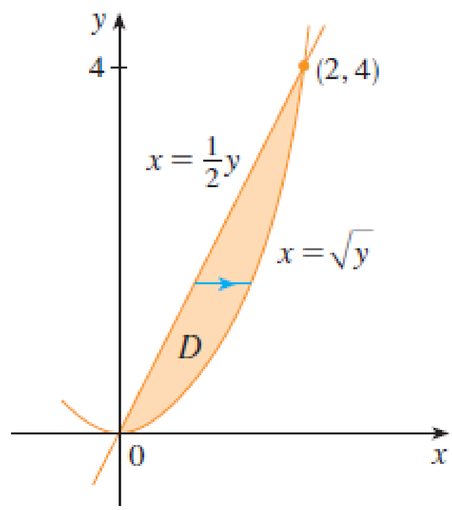


Figure 18: Type II region

$$
\begin{aligned}
V &= \iint_D (x^2 + y^2)\, dA \\
&= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dxdy \\
&= \frac{216}{35}.
\end{aligned}
$$

◀

Choosing the better description of a region

Example 18. Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution. The region D can be written as both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \{(x, y) : \ -2 \le y \le 4, \tfrac{1}{2}y^2 - 3 \le x \le y + 1\}$$

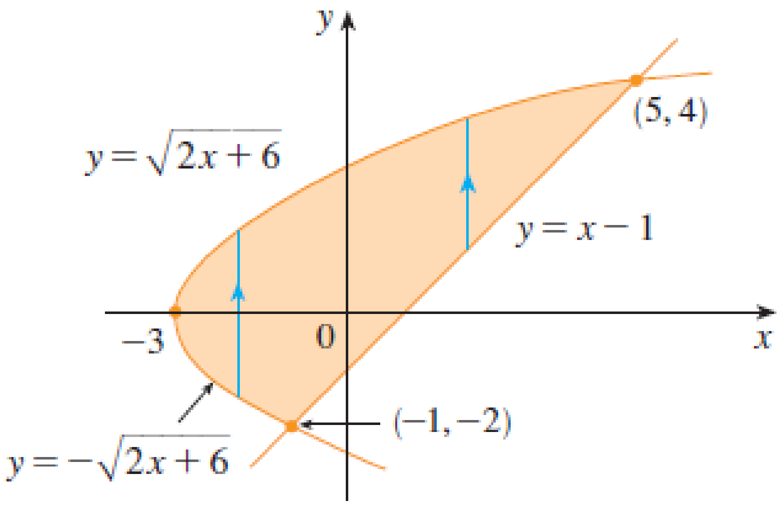


Figure 19: Type I region

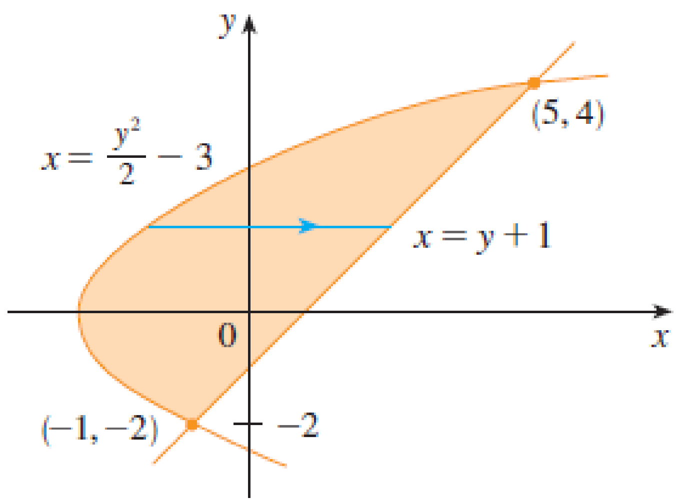


Figure 20: Type II region

Then we have

$$\iint_D xy\, dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\, dxdy$$
$$= 36.$$

If we had expressed D as a type I region using Figure 19, then we would have obtained

$$\iint\limits_D xy\ dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\ dydx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy\ dydx$$

$\blacktriangleleft$

Reversing the order of integration

Example 19. Find the iterated integral

$$\int_0^1 \int_1^x \sin(y^2)\ dy\, dx.$$

Solution. If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\ dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\ dy$ is not an elementary function. So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. We have

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\, dx = \iint\limits_D \sin(y^2)\ dA,$$

where $D = \{(x, y) :\ 0 \le x \le 1, x \le y \le 1\}$. The sketch of this region D is as follows:

An alternative description of D is as follows:

$$D = \{(x, y) :\ 0 \le y \le 1, 0 \le x \le y\}$$

This enables us to express the double integral as an iterated integral in

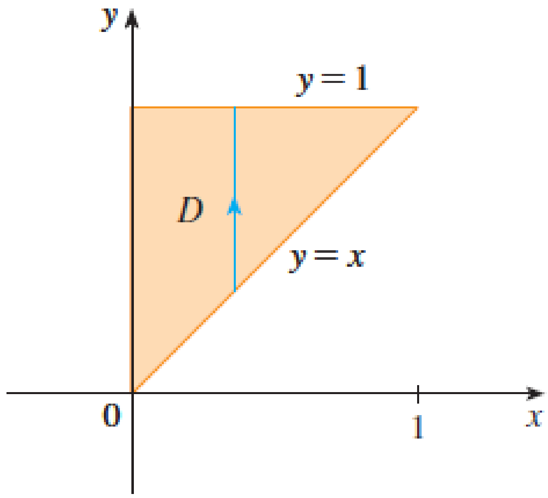


Figure 21: Type I region

the reverse order:

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx &= \iint_D \sin(y^2)\, dA \\ &= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy \\ &= \tfrac{1}{2}(1 - \cos 1).\end{aligned}$$

◀

Problem 1. Evaluate the integral:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\, dx\, dy.$$

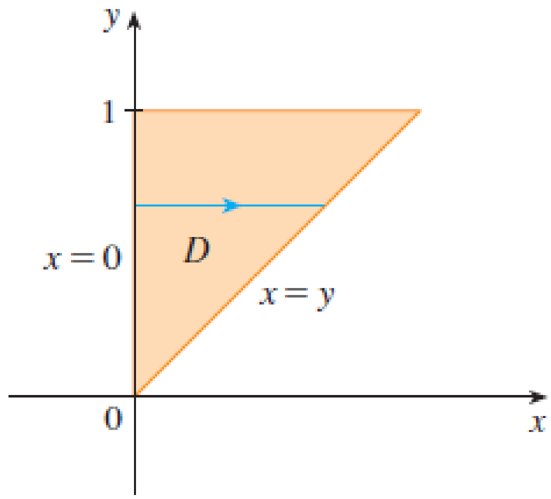


Figure 22: Type II region

Properties of Double Integrals

Here are some properties of the double integral. We assume that all of the following integrals exist. Note that all first three of these properties are really just generalizations of properties of double integrals over rectangles.

1. $\iint\limits_D [f(x,y)+g(x,y)]\,dA = \iint\limits_D f(x,y)\,dA + \iint\limits_D g(x,y)\,dA$

2. If c is a constant, then

$$\iint\limits_D cf(x,y)\,dA = c\iint\limits_D f(x,y)\,dA$$

3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in D$, then

$$\iint\limits_D f(x,y)\,dA \geq \iint\limits_D g(x,y)\,dA$$

Assume that $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries. See the figure.

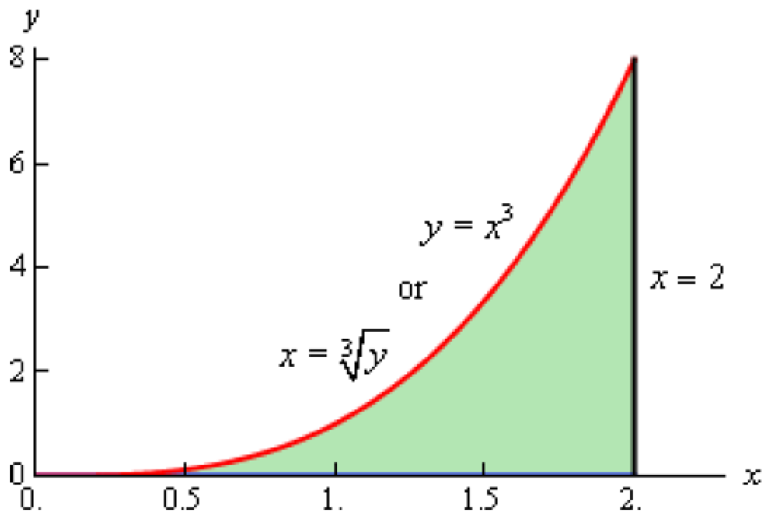
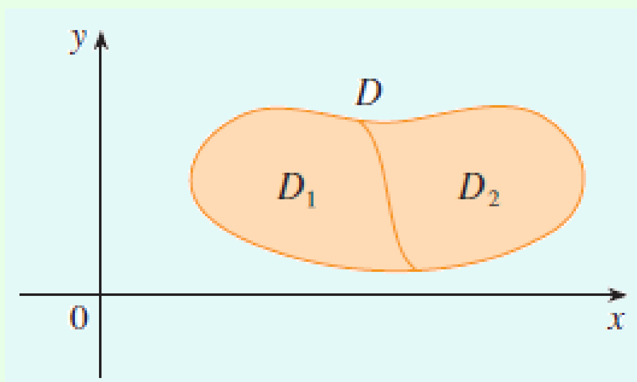


Figure 23: Domain of integration



Then

4. $\iint\limits_{D} [f(x, y) + g(x, y)]\ dA = \iint\limits_{D_1} f(x, y)\ dA + \iint\limits_{D_2} f(x, y)\ dA$

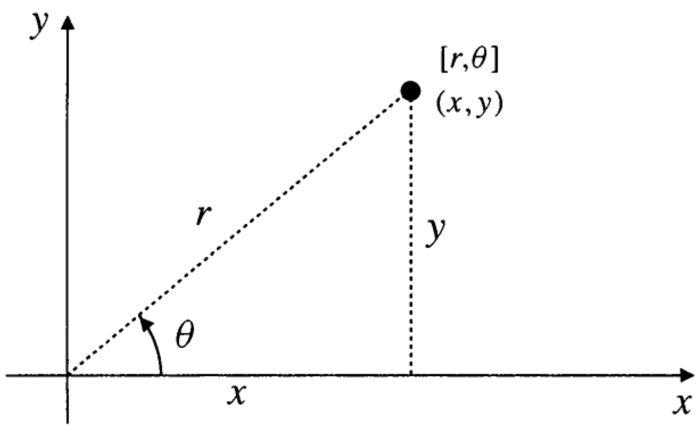
5.
$$\iint\limits_{D} 1\ dA = A(D),$$
where $A(D)$ is the area of D.

6. If $m \leq f(x, y) \leq M$ for all in $(x, y) \in D$, then
$$mA(D) \leq \iint\limits_{D} f(x, y)\ dA \leq MA(D)$$

5 Polar coordinates

Rectangular and Polar Coordinates

Recall that the polar representation of a point P is an ordered pair (r, θ), where r is the distance from the origin to P and θ is the angle that the ray through the origin and P makes with the positive x-axis.



Relation between Rectangular and Polar Coordinates

The polar coordinates r and θ of a point (x, y) in rectangular coordinates satisfy the following relations:

- $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.
- $x = r\cos\theta$ and $y = r\sin\theta$.

Polar coordinates are convenient when the domain of integration is an angular sector or a polar rectangle, as shown in the figure given below.

Moreover for many applications of double integrals, the integrand may be easier to integrate if it is in terms of polar coordinates than in terms of Cartesian coordinates.

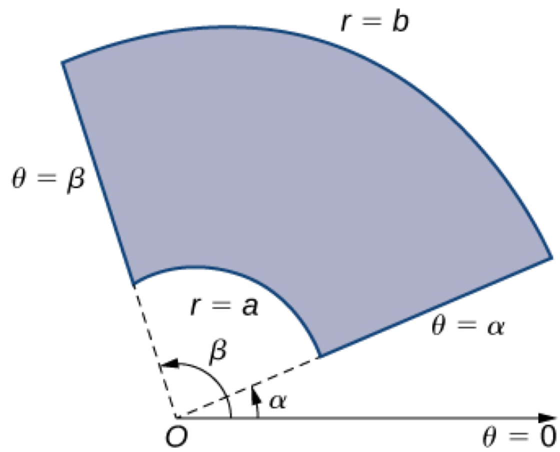


Figure 24: Polar rectangle

Example 20. Consider the double integral

$$\iint\limits_D e^{x^2+y^2}\, dA,$$

where D is the unit disk.

Note that we cannot directly evaluate this integral in rectangular coordinates. However, a change to polar coordinates will convert it to one we can easily evaluate.

First we establish the concept of a double integral in a polar rectangular region. Then we change rectangular coordinates to polar coordinates in double integrals.

Concept of a double integral in a polar rectangle

In polar coordinates, the shape we work with is a *polar rectangle*. See the figure given below.

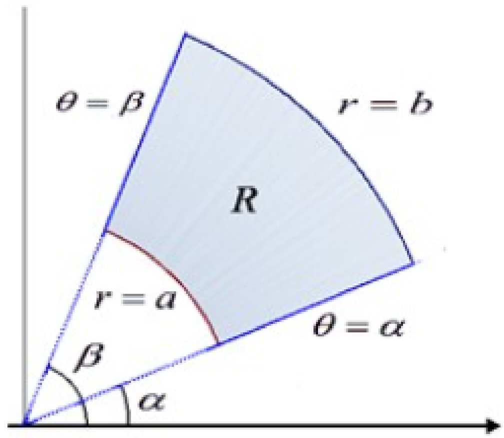


Figure 25: Polar rectangle

Polar rectangle

A polar rectangle is a region R given by

$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\},$$

where $0 \leq \beta - \alpha \leq 2\pi$.

Consider a function $f(r, \theta)$ over a polar rectangle R defined ablove.

Double integrals in polar coordinates are defined through a three-step process: **subdivision**, **summation**, and **passage to the limit**, as we did in the case of double integrals in rectangular coordinates.

Subdivision

We decompose R into an $n \times m$ grid of small polar subrectangles R_{ij} as follows:

We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length

$$\Delta r = (b - a)/m$$

and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width

$$\Delta\theta = (\beta - \alpha)/n$$

by choosing partitions:

$$a = r_0 < r_1 < ... < r_m = b, \ \alpha = \theta_0 < \theta_1 < ... < \theta_n = \beta,$$

where m and n are positive integers.

This means that the circles of radii $r = r_i$ and rays with angles $\theta = \theta_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} as in the figure given below.

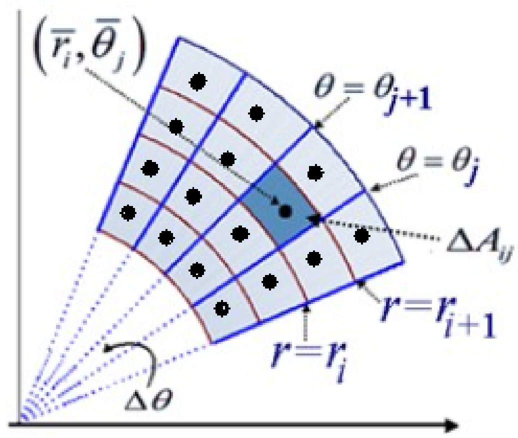


Figure 26: Polar grid

Choose the center $(\overline{r}_i, \overline{\theta}_l)$ of each polar subrectangle R_{ij} as a sample point. Then

$$\overline{r}_i = \frac{1}{2}(r_{i-1} + r_i), \quad \overline{\theta}_j = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

The area of the polar subrectangle R_{ij} is given by

$$\begin{aligned} \Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\ &= \overline{r}_i\Delta r\Delta\theta. \end{aligned}$$

So, the volume of each of the boxes with a base area of ΔA_i and a height of $f(\overline{r}_i, \overline{\theta}_j)$ is

$$f(\overline{r}_i, \overline{\theta}_j)\Delta A_i = f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i\Delta r\Delta\theta.$$

The volume of the solid with the vase R_{ij} is now approximated as follows:

$$\iint_{R_{ij}} f(r,\theta)\, dA \approx f(\overline{r}_i, \overline{\theta}_j)\Delta A_i = f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta.$$

Summation

The volume of the solid under the surface $z = f(r,\theta)$ with the vase R is now approximated as follows:

$$\begin{aligned}\iint_{R} f(r,\theta)\, dA &= \sum_{i=1}^{n}\sum_{j=1}^{m} \iint_{R_{ij}} f(r,\theta)\, dA \\ &\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\ \Delta A_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta.\end{aligned}$$

Riemann sum

The expression

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta$$

is called a **Riemann sum** for the double integral of $f(r,\theta)$ over the region

$$0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$

Passage to the limit

Double integral in polar coordinates

Let f be continuous on a polar rectangle R given by

$$0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$. The double integral $\iint_R f(r,\theta)\ dA$ is defined as follows:

$$\iint_R f(r,\theta)\ dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta.$$

Just as in Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence

$$\iint_R f(r,\theta)\ dA = \iint_R f(r,\theta) r\ dr\, d\theta = \int_\alpha^\beta \int_a^b f(r,\theta) r\ dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates. We have the following theorem.

Theorem

If f is continuous on a polar rectangle R given by

$$0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint\limits_D f(x,y)\ dA = \iint\limits_D f(r\cos\theta, r\sin\theta)\ r\, dr\, d\theta.$$

It is noteworthy that all the properties of the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Change to Polar Coordinates in a Double Integral

If we are given a double integral

$$\iint_D f(x,y)\,dA$$

in rectangular coordinates, we can write the corresponding iterated integral in polar coordinates by substitution.

Method for converting to integrals in polar coordinates

- Describe the domain of integration, R, and find bounds

  $$a \le r \le b \text{ and } \alpha \le \theta \le \beta,$$

  where $0 \le \beta - \alpha \le 2\pi$.

- Convert the function $z = f(x,y)$ to a function with polar coordinates with the substitutions

  $$x = r\cos\theta, y = r\sin\theta.$$

- Replace dA by

  $$r\,dr\,d\theta,$$

  to obtain

  $$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example 21. Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y):\ x^2+y^2 \le 1\}$. Evaluate $\iint_D f(x,y)\,dA$.

Solution. We have the unit disk

$$D = \{(x,y):\ x^2+y^2 \le 1\}.$$

We observe that

$$0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi.$$

Using

$$x = r\cos\theta, \ y = r\sin\theta, \ dA = r\,dr\,d\theta,$$

we then have

$$\begin{aligned}
\int_D e^{x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}e^{r^2}\Big|_0^1\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(e-1)\,d\theta \\
&= \pi(e-1).
\end{aligned}$$

◀

While there is no firm rule for when polar coordinates can or should be used, they are a natural alternative anytime the domain of integration may be expressed simply in polar form, and/or when the integrand involves expressions such as

$$\sqrt{x^2+y^2}.$$

Example 22. Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution. We know that the formula for finding the volume of a region is

$$V = \iint\limits_D f(x,y)\,dA.$$

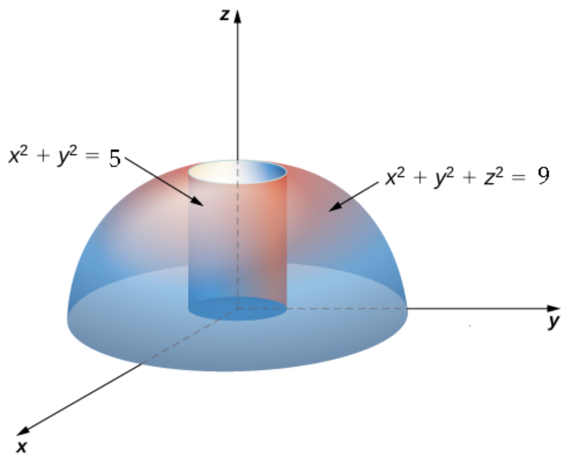
We have

$$f(x,y) = z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}.$$

The region D is the bottom of the cylinder given by $x^2+y^2=5$, that is, the disk

$$D=\{(x,y)|\ x^2+y^2\leq 5\}$$

in the xy-plane. So, the region that we want the volume for is really a cylinder with a cap that comes from the sphere. So, the region D in



polar coordinates is as follows:

$$D=\{(r,\theta):\ 0\leq r\leq\sqrt{5}, 0\leq\theta\leq 2\pi\}$$

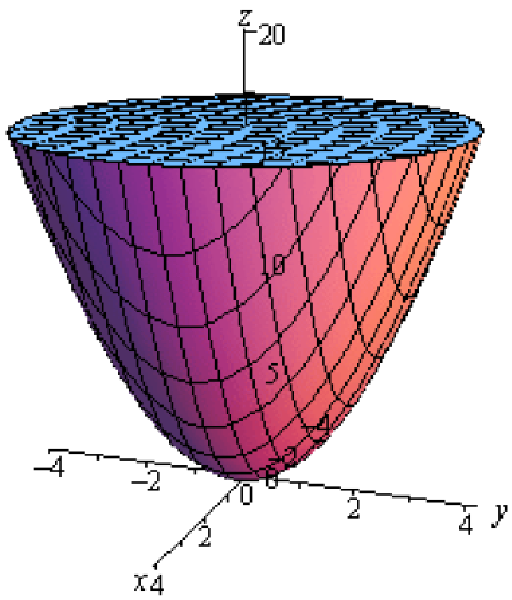
Now, the volume is

$$\begin{aligned}V&=\iint\limits_D\sqrt{9-x^2-y^2}\,dA\\&=\int_0^{2\pi}\int_0^{\sqrt{5}}r\sqrt{9-r^2}dr\,d\theta\\&=38\pi/3.\end{aligned}$$

◀

Example 23. Find the volume of the region that lies inside $z=x^2+y^2$ and below the plane $z=16$.

Solution. Let's start this example off with a sketch of the region.



Now, we see that the top of the region, where the elliptic paraboloid intersects the plane $z = 16$, is the widest part of the region. So, setting $z = 16$ in the equation of the paraboloid gives

$$16 = x^2 + y^2,$$

which is the equation of a circle of radius 4 centered at the origin. So, the domain of intgration, D, is given by

$$D = \{(r, \theta) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 4\}.$$

Notice that the formula

$$\iint_D 16\ dA.$$

will be the volume under plane $z = 16$ while the formula

$$\iint_D (x^2 + y^2)\ dA.$$

is the volume under the paraboloid $z = x^2 + y^2$, using the same D.

Hence the required volume is

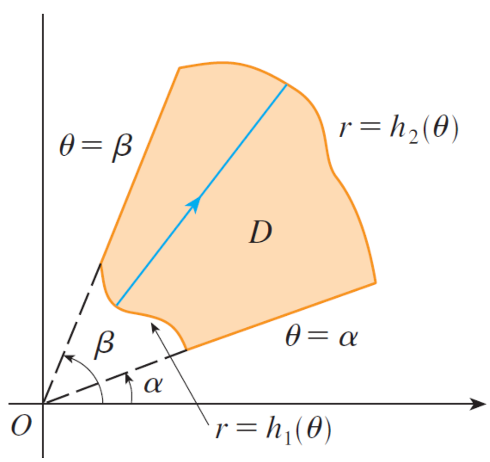
$$
\begin{aligned}
V &= \iint_D 16\, dA - \iint_D (x^2 + y^2)\, dA \\
&= \iint_D (16 - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^4 r(16 - r^2)\, dr\, d\theta \\
&= 128\pi.
\end{aligned}
$$

◀

6 General Polar Regions of Integration

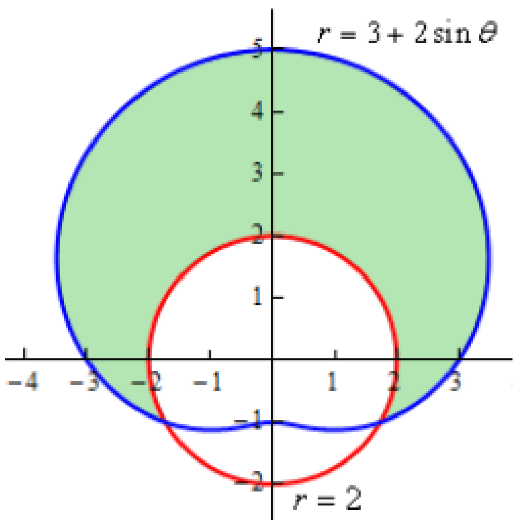
To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as

$$D = \{(r, \theta)|\ \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

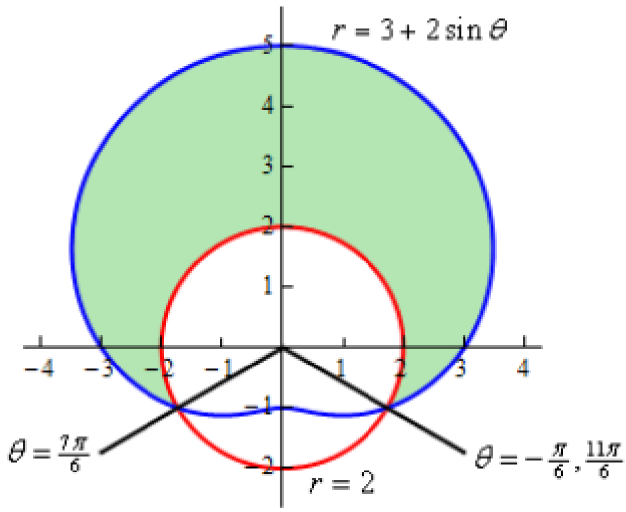


Example 24. Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution. Here is a sketch of the region, D, that we want to determine the area of (Figure (a)).



(a)



(b)

To determine this area we'll need to know that the value of θ for which the two curves intersect. We can determine these points by solving the two equations. We have

$$\begin{aligned} 3 + 2\sin\theta &= 2 \\ \Rightarrow \quad \sin\theta &= -\frac{1}{2} \\ \Rightarrow \quad \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}. \end{aligned}$$

Here is a sketch of the figure with these angles added (Figure (b)).

Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle $\frac{11\pi}{6} = 2\pi - \frac{\pi}{6}$. This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$, then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.

So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$$

$$2 \leq r \leq 3 + 2\sin\theta.$$

The area of the region D is then

$$\begin{aligned} A &= \iint\limits_D dA \\ &= \int_{-\pi/6}^{\pi/6} \int_2^{3+2\sin\theta} dr\, d\theta \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}. \end{aligned}$$

◀

Example 25. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

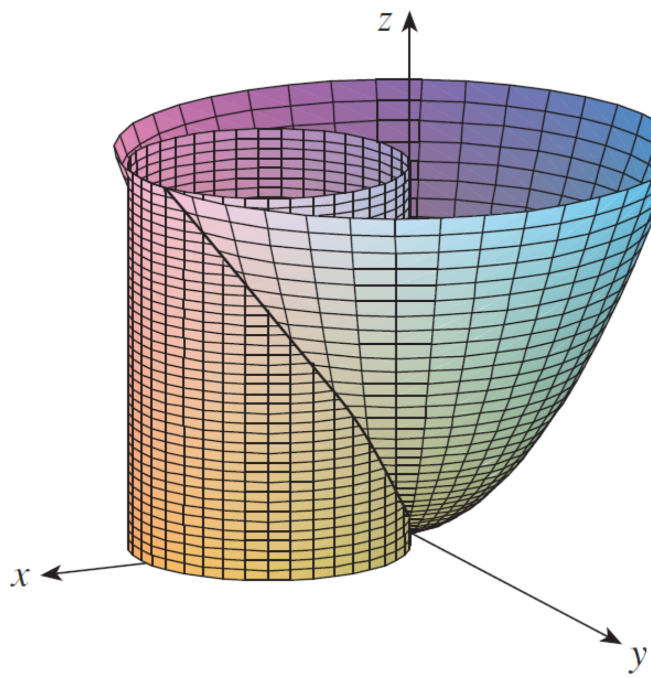
Solution. The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square, we get

$$(x-1)^2 + y^2 = 1$$

(See the figures given below.) To find the volume of the required solid, we have to evaluate the integral:

$$V = \iint_D (x^2 + y^2)\, dx\, dy.$$

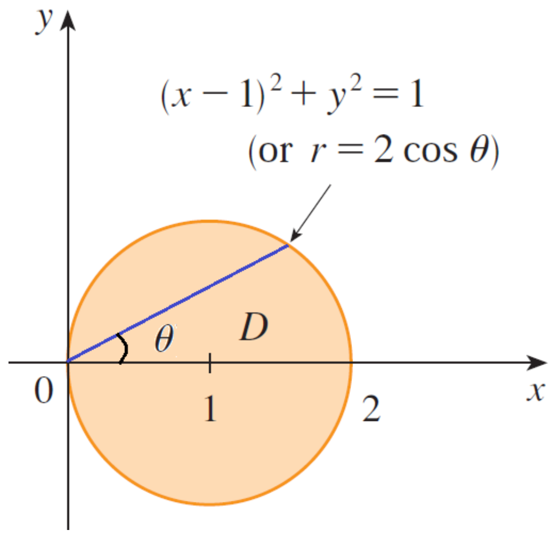
Converting the equation of the circle in polar form, we get



$$\begin{aligned} x^2 + y^2 = 2x &\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 2r\cos\theta \\ &\Rightarrow r^2 = 2r\cos\theta \\ &\Rightarrow r = 0 \text{ or } 2\cos\theta. \end{aligned}$$

Thus the disk D is given by

$$D = \{(r,\theta) : -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2\cos\theta\}.$$



Now, we have

$$
\begin{aligned}
V &= \iint_D (x^2+y^2)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 r\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta \\
&= 8\int_0^{\pi/2}\cos^4\theta\,d\theta = 8\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\
&= 2\int_0^{\pi/2}[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)]\,d\theta \\
&= 2\left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2(3/2)(\pi/2) \\
&= \frac{3\pi}{2}.
\end{aligned}
$$

◀

7 Applications of double integrals

Mass

We saw before that the double integral over a region of the constant function 1 measures the area of the region. If the region has uniform density 1, then the mass is the density times the area which equals the area. What if the density is not constant. Suppose that the density is given by the continuous function

$$\text{Density } = \rho(x, y)$$

In this case we can cut the region into tiny rectangles where the density is approximately constant. The area of mass rectangle is given by

$$\text{Mass } = (\text{ Density })(\text{ Area }) = \rho(x, y)\Delta x \, \Delta y$$

You probably know where this is going. If we add all to masses together and take the limit as the rectangle size goes to zero, we get a double integral.

Mass

Let $\rho(x, y)$ be the density of a lamina (flat sheet) D at the point (x, y). Then the total mass of the lamina is the double integral

$$\text{Mass } = \iint_D \rho(x, y) \, dy \, dx.$$

Example 26 (Finding the mass of a lamina with constant density).
Find the mass of a square lamina, with side length 1, with a density of $\rho = 3 \text{ gm/cm}^2$.

Solution. We represent the lamina with a square region in the plane as shown in the figure given below.

As the density is constant, it does not matter where we place the square. Following Definition 13.4.1, the mass M of the lamina is

$$M = \iint_R 3 \; dA = \int_0^1 \int_0^1 3 \; dx \, dy$$
$$= 3 \text{ gm.}$$

◀

This is all very straightforward.

Example 27 (Finding the mass of a lamina with variable density).
Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 13.4.2), with variable density $\rho(x, y) = (x + y + 2)$ gm/cm2.

Solution. The variable density ρ, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\rho(x, y)$ can be seen in Figure 13.4.3; notice how "same amount" of density is above $z = 3$ as below. We'll comment on the significance of this momentarily.

The mass M is found by integrating $\rho(x, y)$ over R. The order of integration is not important; we choose $dx\,dy$ arbitrarily. Thus

$$\begin{aligned}
\iint_R (x+y+2)\,dA &= \int_0^1 \int_0^1 (x+y+2)\,dx\,dy \\
&= \int_0^1 \left((1/2)x^2 + x(y+2)\right)\Big|_0^1 \,dy \\
&= \int_0^1 \left(\frac{5}{2}+y\right)\,dy \\
&= \left(\frac{5}{2}y + \frac{1}{2}y^2\right)\Big|_0^1 \\
&= 3 \text{ gm.}
\end{aligned}$$

◀

It turns out that since the density of the lamina is so uniformly distributed "above and below" $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in Examples 13.4.1 and 13.4.2 are graphed in Figure 13.4.3, which illustrates this concept.

Moments and Center of Mass

We know that the moments about an axis are defined by the product of the mass times the distance from the axis.

$$M_x = (\text{ Mass })(y), \qquad M_y = (\text{ Mass })(x).$$

If we have a region D with density function $\rho(x,y)$, then we do the usual thing. We cut the region into small rectangles for which the density is constant and add up the moments of each of these rectangles. Then take the limit as the rectangle size approaches zero. This will give us the total moment.

Moments of Mass and Center of Gravity

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the **moments of mass** are

$$M_x = \iint_D \rho(x, y)y \, dy \, dx, \qquad M_y = \iint_D \rho(x, y)x \, dy \, dx.$$

and if m is the mass of the lamina, then the **center of mass** or **center of gravity** is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

Example 28. Set up the integrals that give the center of mass of the rectangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ and density function proportional to the square of the distance from the origin.

Solution. Since the density function $\rho(x, y)$ is proportional to the square of the distance from the origin, $x^2 + y^2$, the mass is given by

$$m = \int_0^1 \int_0^1 k(x^2 + y^2) \, dy \, dx = \frac{2k}{3}.$$

The moments are given by

$$M_x = \int_0^1 \int_0^1 k(x^2 + y^2)y \, dy \, dx = 5k/12$$
$$M_y = \int_0^1 \int_0^1 k(x^2 + y^2)x \, dy \, dx = 5k/12$$

It should not be a surprise that the moments are equal since there is complete symmetry with respect to x and y. Finally, we divide to get

$$(\overline{x}, \overline{y}) = (5/8, 5/8)$$

This tells us that the metal plate will balance perfectly if we place a pin at (5/8 , 5/8). ◀

Example 29. Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$ if the density function is

$$\rho(x,y) = 1 + 3x + y.$$

Solution. content... ◀

Example 30. Find the center of mass of the lamina represented by the region R which is half an annulus with outer radius 6 and inner radius 5, with constant density 2 lb/ft^2.

Solution. Here, it is useful to represent R in polar coordinates. Using the description of R, we see that

$$R = \{(r,\theta) : \ 5 < r < 6,\ 0 < \theta < \pi\}.$$

As the lamina is symmetric about the y-axis, we should expect $M_y = 0$. We compute M, M_x and M_y: ◀

Moments of Inertia

We often call M_x and M_y the first moments. They have first powers of y and x in their definitions and help find the center of mass. We define the moments of inertia (or second moments) by introducing squares of y and x in their definitions. The moments of inertia help us find the kinetic energy in rotational motion. Below is the definition.

Moments of Inertia

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the moments of inertia about the x-axis and the y-axis are

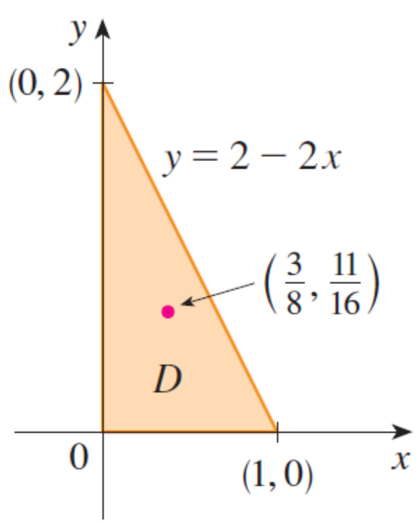
$$I_x = \iint_D \rho(x, y) y^2 \, dy \, dx, \qquad I_y = \iint_D \rho(x, y) x^2 \, dy \, dx.$$

It is also of interest to consider the moment of inertia about the origin, also called the **polar moment of inertia**:

$$I_x = \iint_D \rho(x, y)(x^2 + y^2) \, dy \, dx.$$

Example 31.

Find the moments of inertia for the square metal plate with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.



Solution. ◀

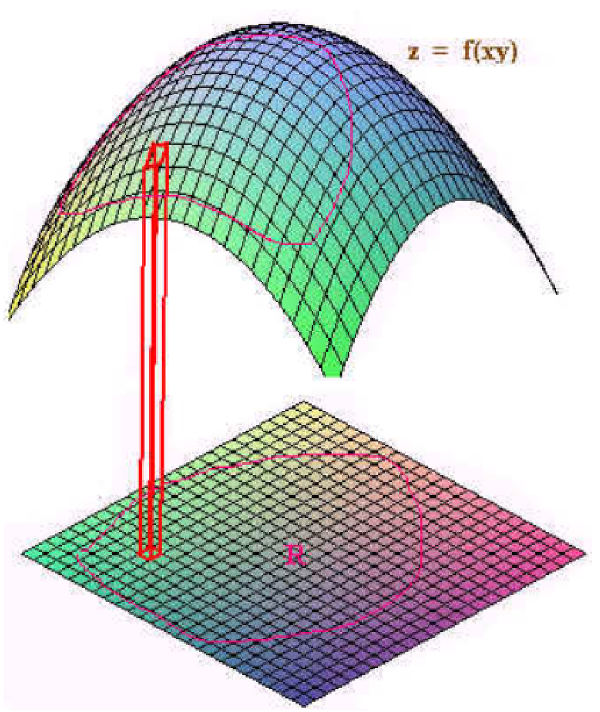
Example 32.

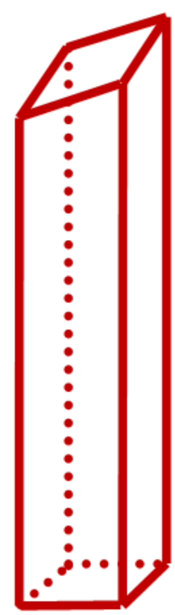
Find the moments of inertia I_x, I_y, and I_o of a homogeneous disk with density $\rho(x, y) = \rho$, center the origin, and radius a.

Solution. The boundary of D is the circle $x^2 + y^2 = a$ and in polar coordinates D is described by $0 \leq \theta \leq 2\pi, 0 \leq r \leq a$. First compute I_o. Then use $I_x + I_y = I_o$ and $I_x = I_y$ (due to the symmetry of the problem) to find I_x and I_y. ◀

8 Surface area

Let $z = f(x, y)$ be a surface in $\mathbb{R}^3$ defined over a region D in the xy-plane. cut the xy-plane into rectangles. Each rectangle will project vertically to a piece of the surface as shown in the figure below.





Although the area of the rectangle in D is

$$\text{Area } = \Delta y \Delta x.$$

The area of the corresponding piece of the surface will not be $\Delta y \Delta x$ since it is not a rectangle. Even if we cut finely, we will still not produce a rectangle, but rather will approximately produce a parallelogram. With a little geometry we can see that the two adjacent sides of the parallelogram are (in vector form)

$$u = \Delta x \vec{i} + f_x(x, y)\Delta x \vec{k}$$

and

$$v = f_y(x, y)\Delta y \vec{i} + \Delta y \vec{k}$$

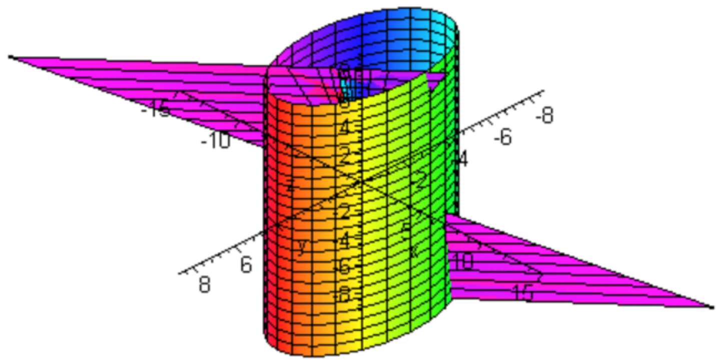
We can see this by realizing that the partial derivatives are the slopes in each direction. If we run Δx in the $\vec{i}$ direction, then we will rise $f_x(x, y)\Delta x$ in the $\vec{k}$ direction so that

$$\frac{\text{rise}}{\text{run}} = f_x(x, y),$$

which agrees with the slope idea of the partial derivative. A similar argument will confirm the equation for the vector v. Now that we know the adjacent vectors we recall that the area of a parallelogram is the magnitude of the cross product of the two adjacent vectors. We have

$$\begin{aligned}
|v \times w| &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x(x, y)\Delta x \\ 0 & \Delta y & f_y(x, y)\Delta y \end{vmatrix} \\
&= |-(f_y(x, y)\Delta y \Delta x)\vec{i} - (f_x(x, y)\Delta y \Delta x)\vec{j} + (\Delta y \Delta x)\vec{k}| \\
&= \sqrt{f_y^2(x, y)(\Delta y \Delta x)^2 + f_x^2(x, y)(\Delta y \Delta x)^2 + (\Delta y \Delta x)^2} \\
&= \sqrt{f_y^2(x, y) + f_x^2(x, y) + 1}\, \Delta y \Delta x.
\end{aligned}$$

This is the area of one of the patches of the quilt. To find the total area of the surface, we add up all the areas and take the limit as the rectangle



size approaches zero. This results in a double Riemann sum, that is a double integral. We state the definition below.

Definition of Surface Area
Let $z = f(x, y)$ be a differentiable surface defined over a region D. Then its surface area is given by

$$\text{Surface Area} = \iint_D \sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}\, dy\, dx.$$

Example 33.

Find the surface area of the part of the plane

$$z = 8x + 4y$$

that lies inside the cylinder

$$x^2 + y^2 = 16.$$

Solution. We calculate partial derivatives

$$f_x(x, y) = 8, \qquad f_y(x, y) = 4$$

so that

$$1 + f_x^2(x, y) + f_y^2(x, y) = 1 + 64 + 16 = 81$$

Taking a square root and integrating, we get

$$\iint_D 9 \, dy \, dx.$$

We could work this integral out, but there is a much easier way. The integral of a constant is just the constant times the area of the region. Since the region is a circle, we get

$$\text{Surface Area } = 9(16\pi) = 144\pi.$$

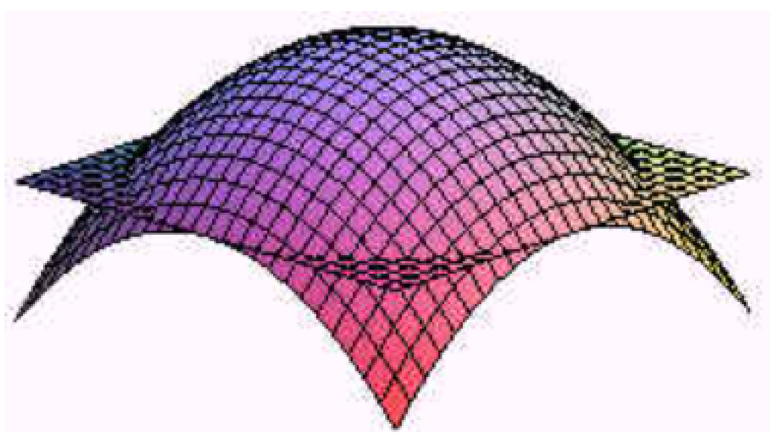
◀

Example 34.

Find the surface area of the part of the paraboloid

$$z = 25 - x^2 - y^2$$

that lies above the xy-plane.



Solution. We calculate partial derivatives

$$f_x(x, y) = -2x \qquad f_y(x, y) = -2y$$

so that

$$1 + f_x^2(x, y) + f_y^2(x, y) = 1 + 4x^2 + 4y^2.$$

Using "Polar Coordinates", we realize that the region is just the circle

$$r = 5$$

Now convert the integrand to polar coordinates to get

$$\int_0^{2\pi} \int_0^5 \sqrt{(}1 + 4r^2)\ rdr\ d\theta$$

Now let

$$u = 1 + 4r^2, \qquad du = 8rdr$$

and substitute

$$\frac{1}{8}\int_0^{2\pi} \int_1^{101} u^{1/2}\ du\ d\theta = \frac{1}{12}\int_0^{2\pi} \left[u^{3/2}\right]_1^{101}\ d\theta \approx 169.3\pi. \qquad \blacktriangleleft$$

9 Triple integrals

We have seen that the geometry of a double integral involves cutting the two dimensional region into tiny rectangles, multiplying the areas of the rectangles by the value of the function there, adding the areas up, and taking a limit as the size of the rectangles approaches zero. We have also seen that this is equivalent to finding the double iterated iterated integral.

We will now take this idea to the next dimension. Instead of a region in the xy-plane, we will consider a solid in xyz-space. Instead of cutting up the region into rectangles, we will cut up the solid into rectangular solids. And instead of multiplying the function value by the area of the rectangle, we will multiply the function value by the volume of the rectangular solid.

We can define the triple integral as the limit of the sum of the product of the function times the volume of the rectangular solids.

Instead of the double integral being equivalent to the double iterated integral, the triple integral is equivalent to the triple iterated integral.

Triple Integral

Let $f(x, y, z)$ be a continuous function of three variables defined over a solid B. Then the triple integral over B is defined as

$$\iiint_B f(x, y, z)\ dxdydz = \lim \sum f(x, y, z)\Delta x \Delta y \Delta z,$$

where the sum is taken over the rectangular solids included in the solid B and lim is taken to mean the limit as the side lengths of the rectangular solid.

This definition is only practical for estimating the triple integral when a data set is given. Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Fubini's Theorem for Triple Integrals

If f is continuous on a rectangular box

$$B = [a, b] \times [c, d] \times [r, s],$$

then

$$\iiint_B f(x, y, z)\ dx\, dy\, dz = \int_a^b \int_c^d \int_r^s f(x, y, z)\ dx\, dy\, dz.$$

Remark 1. As with double integrals the order of integration can be changed with care.

Triple integral over a general bounded region E

A solid region is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) : \ (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown below.

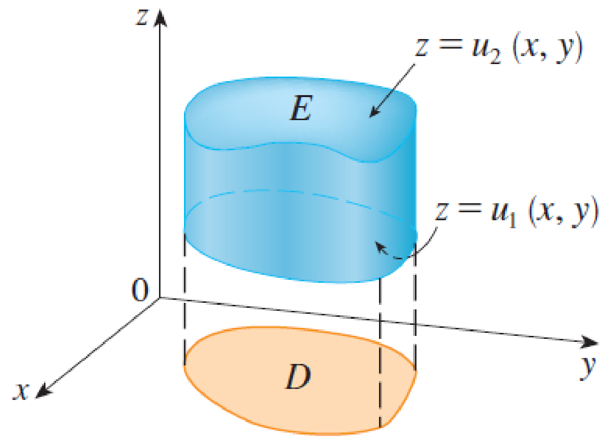


Figure 27: A type 1 solid region

Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$. Then the triple integral takes the form as given below.

$$\iiint\limits_{B} f(x, y, z)\ dxdydz$$

$$= \iint\limits_{D} \left[\int_{u_1(x)}^{u_2(x)} f(x, y, z)\ dz \right] dA.$$

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3),

then the triple integral becomes as in the following theorem:

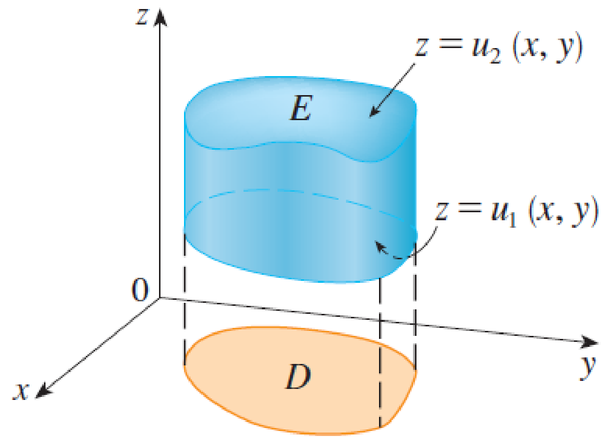


Figure 28: A type 1 solid region where the projection D is a type I plane region

Theorem for Evaluating Triple Integrals

Let $f(x, y, z)$ be a continuous function over a solid E defined by

$$E = \{(x, y, z) : \ a \leq x \leq b, g_1(x) \leq y \leq g_2(x), \\ u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Then the triple integral is equal to the triple iterated integral.

$$\iiint\limits_E f(x, y, z)\ dxdydz \\ = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dzdydx.$$

Example 35.

Evaluate

$$\iiint\limits_E f(x, y, z)\ dzdydx,$$

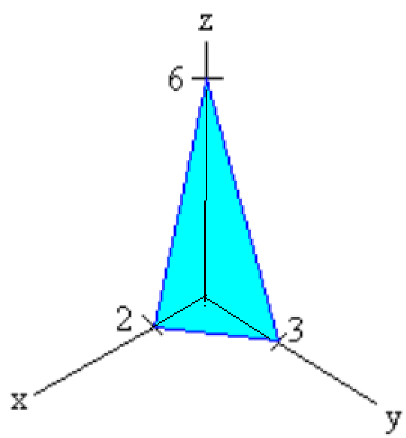
where

$$f(x, y, z) = 1 - x$$

and E is the solid that lies in the first octant and below the plain

$$3x + 2y + z = 6.$$

Solution. The picture of the region is The challenge here is to find



the limits. We work on the innermost limit first which corresponds with the variable "z". Think of standing vertically. Your feet will rest on the lower limit and your head will touch the higher limit. The lower limit is the xy-plane or

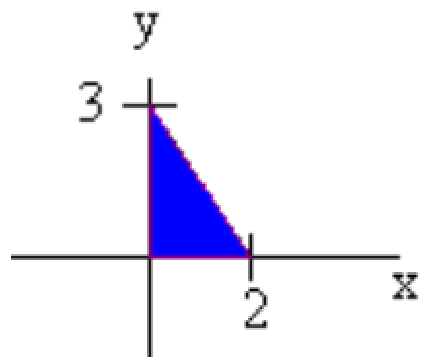
$$z = 0.$$

The upper limit is the given plane. Solving for z, we get

$$z = 6 - 3x - 2y.$$

Now we work on the middle limits that correspond to the variable "y". We look at the projection of the surface in the xy-plane. It is shown below. Now we find the limits just as we found the limits of double integrals. The lower limit is just

$$y = 0.$$



If we set $z = 0$ and solve for y, we get for the upper limit

$$y = 3 - (3/2)x.$$

Next we find the outer limits, corresponding to the variable "x". The lowest x gets is 0 and highest x gets is 2. Hence

$$0 < x < 2.$$

The integral is thus

$$\begin{aligned}
&\int_0^2 \int_0^{3-3x/2} \int_0^{6-3x-2y} (1-x)\ dzdydx \\
=&\int_0^2 \int_0^{3-3x/2} [z - xz]_{z=0}^{6-3x-2y}\ dydx \\
=&\int_0^2 \int_0^{3-3x/2} [(6-3x-2y) - (6x - 3x^2 - 2xy)]\ dydx \\
=&\int_0^2 \int_0^{3-3x/2} (6 - 9x - 2y + 3x^2 + 2xy)\ dydx \\
=&\int_0^2 \left[6y - 9xy - y^2 + 3x^2y + xy^2\right]_{y=0}^{3-3x/2}\ dx \\
=&\int_0^2 (9 - 18x + (45/4)x^2 - (9/4)x^3)\ dx \\
=&\left[9x - 9x^2 + (15/4)x^3 - (9/16)x^4\right]_0^2 \\
=&3.
\end{aligned}$$

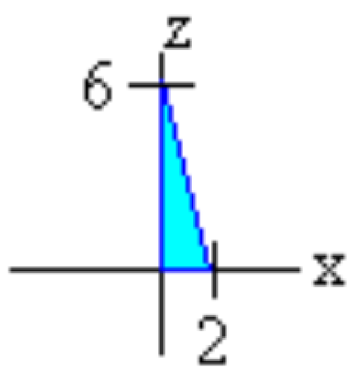
Example 36.

Switch the order of integration from the previous example so that $dydxdz$ appears.

Solution. This time we work on the "y" variable first. The lower limit for the y-variable is 0. For the upper limit, we solve for y in the plane to get

$$y = 3 - 3/2x - 1/2z$$

To find the "x" limits, we project onto the xz-plane as shown below The



lower limit for x is 0. To find the upper limit we set $y = 0$ and solve for x to get

$$x = 2 - (1/3)z$$

Finally, to get the limits for z, we see that the smallest z will get is 0 and the largest z will get is 6. We get

$$0 < z < 6$$

We can write

$$\int_0^6 \int_0^{2-z/3} \int_0^{6-3x/2-z=/3} (1-x)\, dydxdz.$$

10 Applications of Triple Integrals

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E :

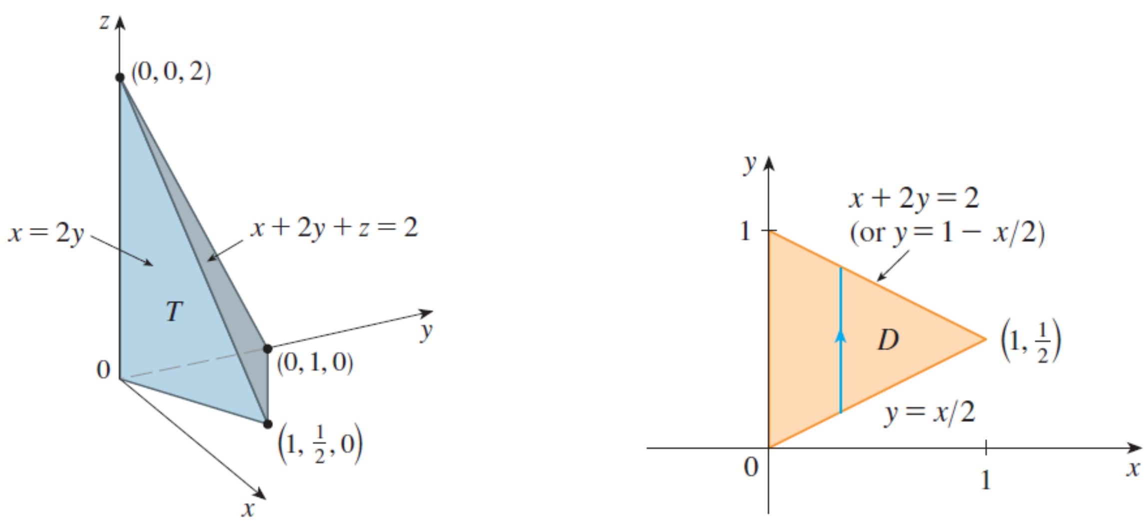
$$V(E) = \iiint_E dV.$$

Example 37.

Use a triple integral to find the volume of the tetrahedron T bounded by the planes

$$x + 2y + z = 2, \ x = 2y, \ z = 0.$$

Solution. The tetrahedron T and its projection D onto the XY-plane are shown in the figure. The lower boundary of T is the plane $z = 0$ and



the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

Therefore we have

$$V(E) = \iiint\limits_E dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dzdydx$$
$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dydx = \frac{1}{3}.$$

◀

Mass, Center of Mass, and Moments of Inertia

For a three dimensional solid E with constant density, the mass is the density times the volume. If the density is not constant but rather a continuous function of x, y, and z, then we can cut the solid into very small rectangular solids so that on each rectangular solid the density is approximately constant. The volume of the small rectangular solid is

$$\Delta\text{Mass} \;=\; (\text{Density})(\Delta\text{Volume}) \;= f(x, y, z)\Delta x\Delta y\Delta z$$

Now do the usual thing. We add up all the small masses and take the limit as the rectangular solids get small. This will give us the triple integral

$$\text{Mass} \;= \iiint\limits_E f(x, y, z)\; dzdydx.$$

We find the center of mass of a solid just as we found the center of mass of a lamina. Since we are in three dimensions, instead of the moments about the axes, we find the moments about the coordinate planes. We state the definitions from physics below.

Definition: **Moments and Center of Mass**
Let $\rho(x, y, z)$ be the density of a solid E. Then the first moments about the coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)\, dzdydx$$

$$M_{xz} = \iiint_E y\rho(x, y, z)\, dzdydx$$

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dzdydx$$

and the center of mass is given by

$$(x, y, z) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$$

Notice that letting the density function being identically equal to 1 gives the volume

$$\text{Volume } = \iiint_E \; dzdydx.$$

Just as with lamina, there are formulas for moments of inertial about the three axes. They involve multiplying the density function by the square of the distance from the axes. We have

Definition: Moments and Center of Mass
Let $\rho(x, y, z)$ be the density of a solid E. Then the first moments of inertia about the coordinate axes are

$$I_x = \iiint\limits_E (y^2 + z^2)\rho(x, y, z)\ dzdydx$$

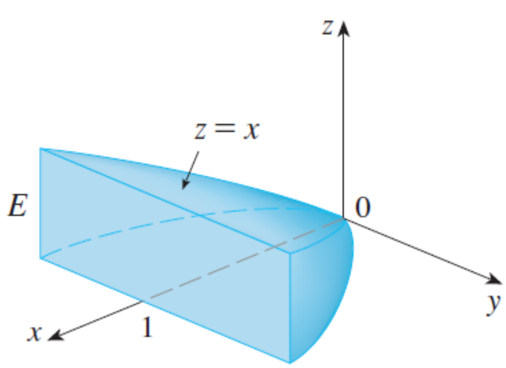
$$I_y = \iiint\limits_E (x^2 + z^2)\rho(x, y, z)\ dzdydx$$

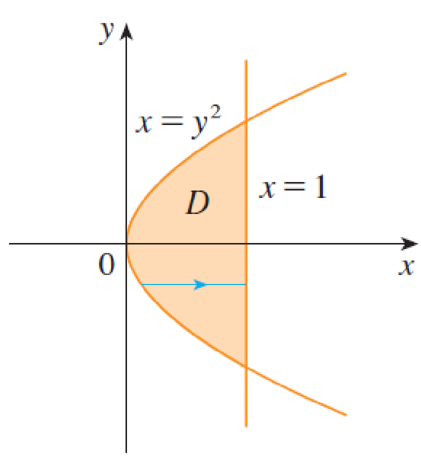
$$I_z = \iiint\limits_E (x^2 + y^2)\rho(x, y, z)\ dzdydx$$

Example 38.

Find the center of mass of a solid E of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0$, and $x = 1$.

Solution. The solid E and its projection D onto the xy-plane are shown in the figure given below.





The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe as a type 1 region:

$$E = \{(x, y, z) : \ -1 \leq y \leq 1, y^2 \leq x \leq 1, 0 \leq z \leq x\}.$$

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint\limits_E \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} \rho \, dz dy dx = \frac{4\rho}{5}.$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that M_{xz} and therefore $\overline{y} = 0$. The other moments are

$$M_{yz} = \iiint\limits_E x\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} x\rho \, dz dy dx = \frac{4\rho}{7}$$

$$M_{xy} = \iiint\limits_E z\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} z\rho \, dz dy dx = \frac{2\rho}{7}$$

◀

11 Change of variables in multiple integrals

Review of the Idea of Substitution

Consider the integral

$$\int_0^2 x \cos x^2 \, dx.$$

To evaluate this integral we use the substitution

$$u = x^2$$

This substitution sends the interval $[0, 2]$ onto the interval $[0, 4]$. We can see that there is stretching of the interval. The stretching is not uniform. In fact, the first part $[0, 0.5]$ is actually contracted. This is the reason why we need to find du.

$$\frac{du}{dx} = 2x \quad \text{or} \quad \frac{dx}{du} = \frac{1}{2x}$$

This is the factor that needs to be multiplied in when we perform the substitution. Notice for small positive values of x, this factor is greater than 1 and for large values of x, the factor is smaller than 1. This is how the stretching and contracting is accounted for.

Jacobians

We have seen that when we convert to polar coordinates, we use

$$dydx = rdrd\theta$$

With a geometrical argument, we showed why the "extra r" is included. Taking the analogy from the one variable case, the transformation to polar coordinates produces stretching and contracting. The "extra r" takes care of this stretching and contracting. The goal for this section is to be able to find the "extra factor" for a more general transformation.

We call this "extra factor" the *Jacobian of the transformation.* We can find it by taking the determinant of the 2×2 matrix of partial derivatives.

Jacobian

Let

$$x = g(u, v) \quad \text{and} \quad y = h(u, v)$$

be a transformation of the plane. Then the Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Remark 2. A useful fact is that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the original transformation.

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}}$$

This is a consequence of the fact that the determinant of the inverse of a matrix A is the reciprocal of the determinant of A.

Example 39.

Find the Jacobian of the polar coordinates transformation

$$x(r, \theta) = r\cos\theta, \qquad y(r, \theta) = r\sin\theta.$$

Solution. We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

◀

Remark 3. This example confirms that

$$dydx = rdrd\theta$$

Double Integration and the Jacobian

Theorem: **Integration and Coordinate Transformations**
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$x = g(u, v), \qquad y = h(u, v)$$

be a transformation on the plane that is one to one from a region S to a region R. If g and h have continuous partial derivatives such that the Jacobian is never zero, then

$$\iint_R f(x, y)\, dy\, dx = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dv\, du$$

Remark 4. Note that a small region $\Delta A = \Delta x \Delta y$ in the xy-plane is related to a small region in the uv-plane whose area is the product $\Delta u \Delta v$, that is,

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v.$$

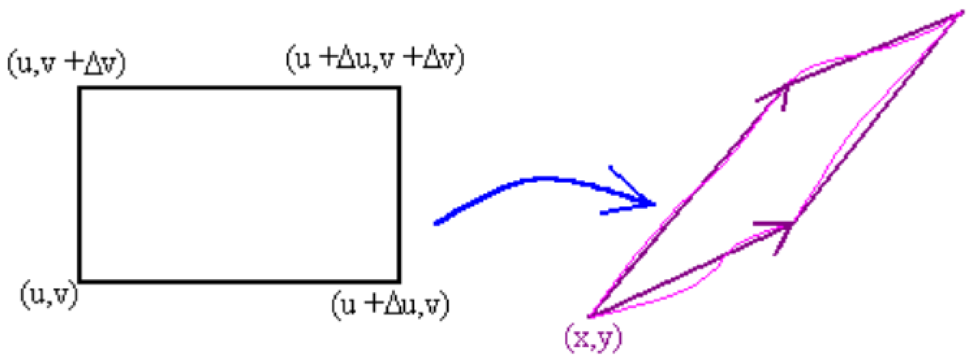
In the limiting case we have

$$\boxed{dx\, dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.}$$

The additional factor of $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ tells us how area changes under the map T.

Idea of the Proof

As usual, we cut S up into tiny rectangles so that the image under T of each rectangle is a parallelogram.



We need to find the area of the parallelogram. Considering differentials, we have

$$T(u+\Delta u, v) \approx T(u,v) + (x_u \Delta u, y_u \Delta u)$$
$$T(u, v+\Delta v) \approx T(u,v) + (x_v \Delta v, y_v \Delta v)$$

Thus the two vectors that make the parallelogram are

$$P = g_u \Delta u \vec{i} + h_u \Delta u \vec{j}$$
$$Q = g_v \Delta v \vec{i} + h_v \Delta v \vec{j}$$

To find the area of this parallelogram we just cross the two vectors.

$$\begin{aligned} |P \times Q| &= \text{ abs} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u \Delta u & y_u \Delta u & 0 \\ x_v \Delta v & y_v \Delta v & 0 \end{vmatrix} \\ &= |(x_u y_v - x_v y_u) \Delta u \Delta v| \\ &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta. \end{aligned}$$

and the extra factor is revealed.

Example 40.

Determining the image of a region under a transformation

A transformation is defined by the equations

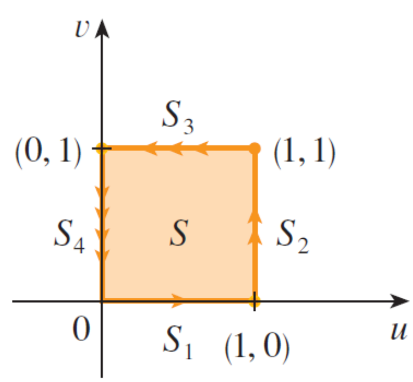
$$x = u^2 - v^2, \qquad y = 2uv.$$

Find the image of the square $S = \{(u, v) : \; 0 \leq u \leq 1, \; 0 \leq v \leq 1\}$.

Solution. The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side is given by

$$S_1 = \{(u, 0) : \; 0 \leq u \leq 1\}.$$

(See Figure 2.) From the given equations we have



$$x = u^2, \qquad y = 0$$

and so $0 \leq x \leq 1$. Thus, S_1 is mapped into the line segment from $(0, 0)$ to (1,0) in the xy-plane.

The second side is

$$S_2 = \{(1, v) : \; 0 \leq v \leq 1\}$$

and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2, \qquad y = 2v$$

Eliminating v, we obtain

$$x = 1 - \frac{y^2}{4} \qquad 0 \leq x \leq 1, \tag{2}$$

which is part of a parabola. Similarly, S_3 is given by

$$S_3 = \{(u, 1) : \ 0 \leq u \leq 1\},$$

whose image is the parabolic arc

$$x = \frac{y^2}{4} - 1 \qquad -1 \leq x \leq 0, \tag{3}$$

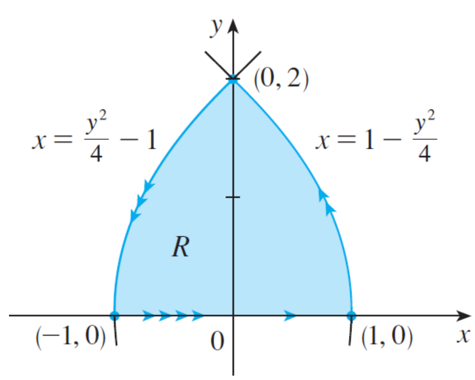
Finally, S_4 is given by

$$S_3 = \{(0, v) : \ 0 \leq v \leq 1\},$$

whose image is

$$x = -v^2, y = 0,$$

that is, $-1 \leq x \leq 0$. (Notice that as we move around the square in the



counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region (shown in Figure 2) bounded by the x-axis and the parabolas .

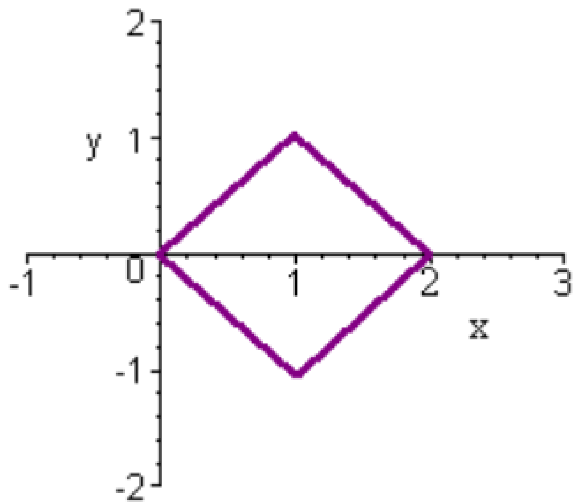
◀

Example 41.

Use an appropriate change of variables to find the volume of the region below

$$z = (x-y)^2$$

above the x-axis, over the parallelogram with vertices $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$.



Solution. We find the equations of the four lines that make the parallelogram to be

$$y = x, \quad y = x-2, \quad y = -x, \quad y = -x+2,$$

that is,

$$x-y=0, \quad x-y=2, \quad x+y=0, \quad x+y=2$$

The region is given by

$$0 < x-y < 2 \text{ and } 0 < x+y < 2$$

This leads us to the inverse transformation

$$u(x,y) = x-y, \quad v(x,y) = x+y$$

The Jacobian of the inverse transformation is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Since the Jacobian is the reciprocal of the inverse Jacobian we get

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

The region is given by

$$0 < u < 2 \text{ and } 0 < v < 2$$

and the function is given by

$$z = u^2$$

Putting this all together, we get the double integral

$$\begin{aligned} \int_0^2 \int_0^2 u^2 \frac{1}{2}\, du\, dv &= \int_0^2 \left[\frac{u^3}{6}\right]_0^2 dv \\ &= \int_0^2 \frac{4}{3}\, dv = \frac{8}{3}. \end{aligned}$$

◀

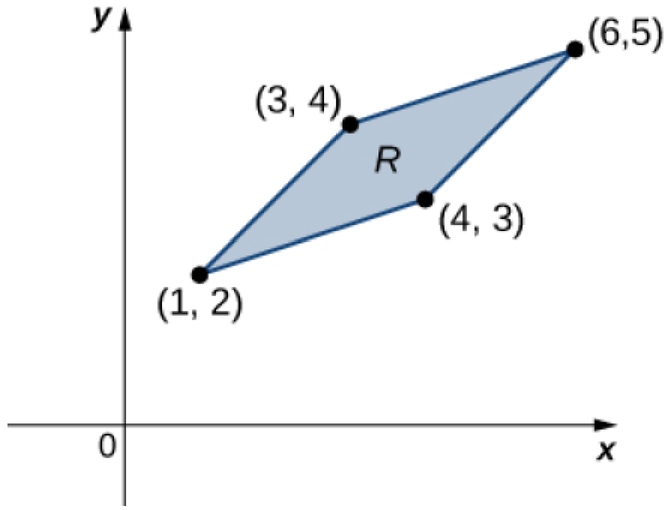
Example 42.

Changing Variables

Consider the integral

$$\iint_R (x-y)dy\, dx,$$

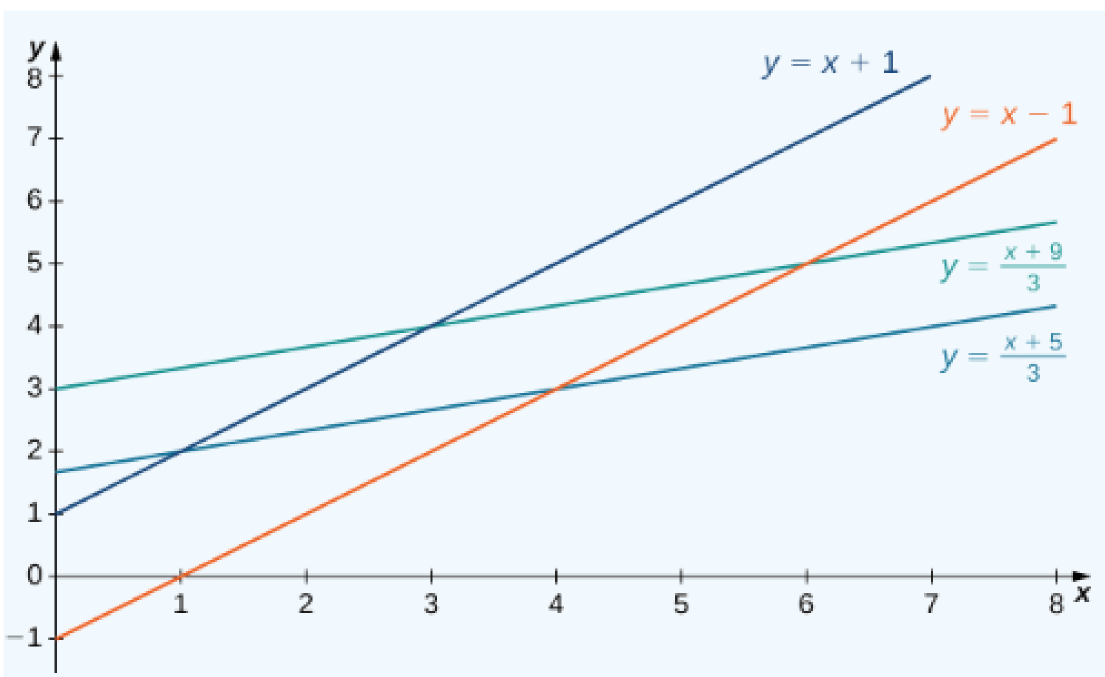
where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ (See the figure given below). Make appropriate changes of variables, and write the resulting integral.



Solution. First, we need to understand the region over which we are to integrate. The sides of the parallelogram are

$$x - y + 1,\ x - y - 1 = 0,\ x - 3y + 5 = 0 \text{ and } x - 3y + 9 = 0$$

See the figure.



Another way to look at them is

$$x - y = -1,\ x - y = 1,\ x - 3y = -5, \text{ and } x - 3y = -9.$$

Clearly the parallelogram is bounded by the lines

$$y = x + 1,\ y = x - 1,\ y = \frac{1}{3}(x + 5),\ y = \frac{1}{3}(x + 9).$$

Notice that if we were to make $u = x - y$ and $v = x - 3y$, then the limits on the integral would be

$$-1 \leq u \leq 1 \text{ and } -9 \leq v \leq -5.$$

To solve for x and y, we multiply the first equation by 3 and subtract the second equation,

$$3u - v = (3x - 3y) - (x - 3y) = 2x.$$

Then we have

$$x = \frac{3u - v}{2}.$$

Moreover, if we simply subtract the second equation from the first, we get

$$u - v = (x - y) - (x - 3y) = 2y \text{ and } y = \frac{u - v}{2}.$$

Thus, we can choose the transformation

$$T(u, v) = \left(\frac{3u - v}{2}, \frac{u - v}{2}\right)$$

and compute the Jacobian J(u,v). We have

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u,v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x-y)dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u,v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,$$

which is much simpler to compute. ◀

Jacobians and Triple Integrals

For transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, we define the Jacobian in a similar way

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

Example 43.

Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

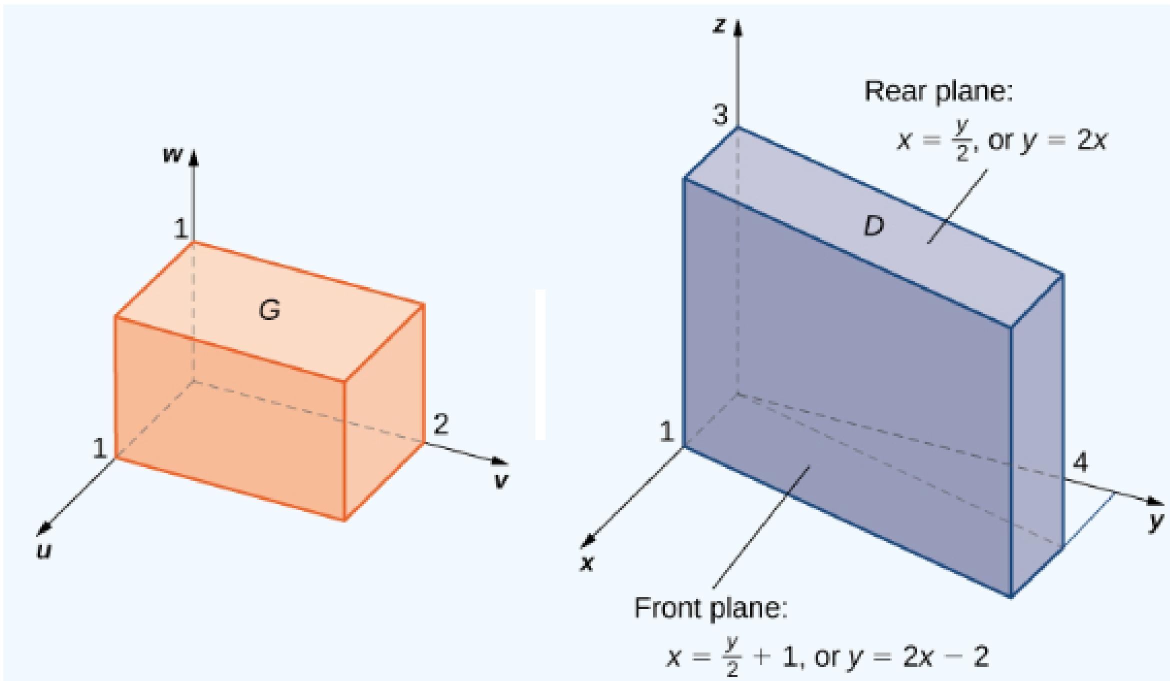
$$\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(x + \frac{z}{3}\right)dx\,dy\,dz$$

In xyz-space by using the transformation

$$u = (2x-y)/2,\; v = y/2, and w = z/3.$$

Then integrate over an appropriate region in uvw-space.

Solution. As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (see the figure PageIndex13). Clearly G in xyz-space is



bounded by the planes

$$x = y/2,\ x = (y/2) + 1,\ y = 0,\ y = 4,\ z = 0,\ \text{and } z = 4.$$

We also know that we have to use

$$u = (2x - y)/2,\ v = y/2,\ \text{and } w = z/3$$

for the transformations. We need to solve for x, y and z. Here we find that

$$x = u + v,\ y = 2v,\ \text{and } z = 3w.$$

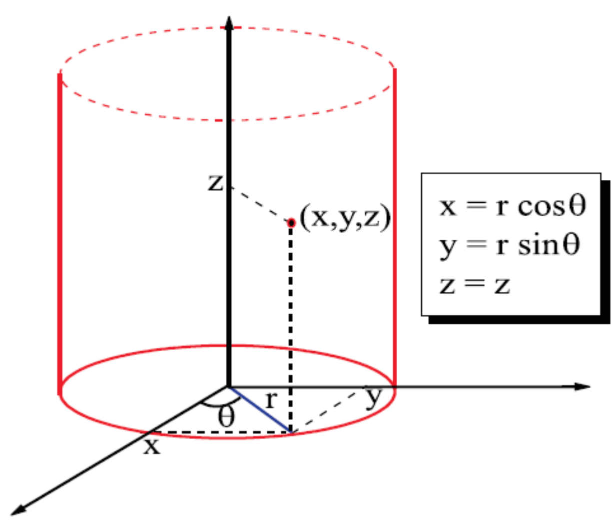
◀

Cylindrical Coordinates

When we were working with double integrals, we saw that it was often easier to convert to polar coordinates. For triple integrals we have been introduced to three coordinate systems. The rectangular coordinate system (x, y, z) is the system that we are used to. The other two systems are cylindrical coordinates (r, θ, z) and spherical coordinates (r, θ, ϕ).

Cylindrical coordinates are denoted by r, θ and z, and are defined by

$$\begin{aligned} r = & \text{ the distance from } (x, y, 0) \text{ to } (0, 0, 0) \\ = & \text{ the distance from } (x, y, z) \text{ to the } z\text{-axis} \\ \theta = & \text{ the angle between the positive } x\text{-axis and} \\ & \quad \text{the line joining } (x, y, 0) \text{ to } (0, 0, 0) \\ z = & \text{ the signed distance from } (x, y, z) \text{ to the } xy\text{-plane} \end{aligned}$$

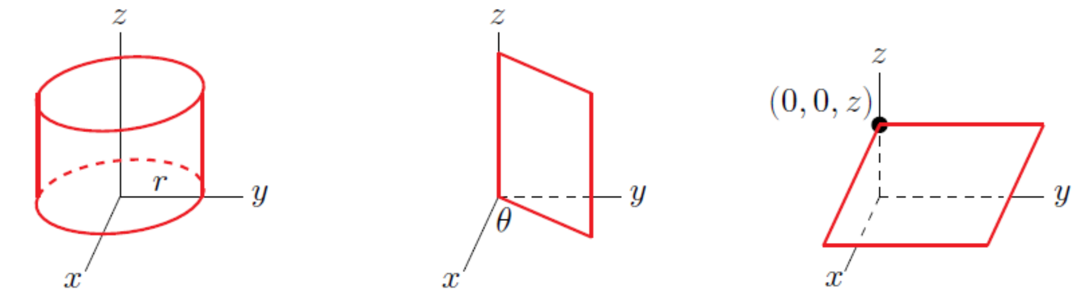


Here are sketches of surfaces of constant r, constant θ, and constant z.

The Cartesian and cylindrical coordinates are related by

Recall that cylindrical coordinates are most appropriate when the expression

$$x^2 + y^2$$



surface of constant r (a cylindrical shell) | surface of constant θ (a plane) | surface of constant z (a plane)

occurs. The construction is just an extension of polar coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

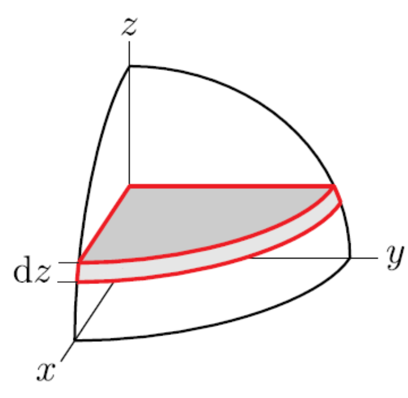
The Volume Element in Cylindrical Coordinates

We now establish that

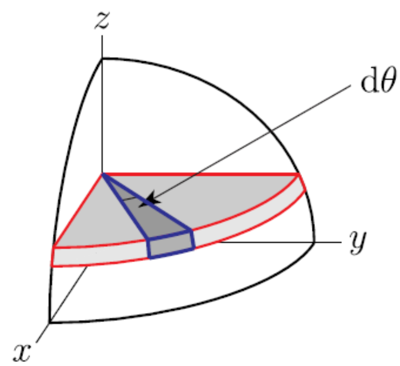
$$dV = r dr\, d\theta\, dz.$$

If we cut up a solid by

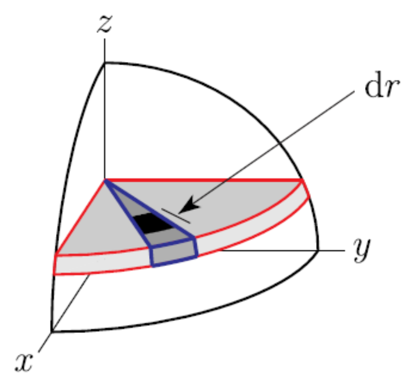
- first slicing it into horizontal plates of thickness dz by using planes of constant z,



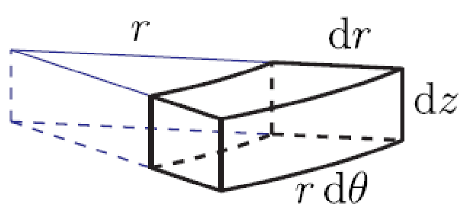
- and then subdividing the plates into wedges using surfaces of constant θ, say with the difference between successive θ's being $d\theta$,



- and then subdividing the wedges into approximate cubes using surfaces of constant r, say with the difference between successive r's being dr,



we end up with approximate cubes that look like



When we introduced slices using surfaces of

- constant r, the difference between the successive r's was dr, so the indicated edge of the cube has length dr.

- constant z, the difference between the successive z's was dz, so the vertical edges of the cube have length dz.
- constant θ the difference between the successive θ's was $d\theta$ so the remaining edges of the cube are circular arcs of radius essentially r that subtend an angle θ and so have length $rd\theta$.

Example 44.

Find the Jacobian for the cylindrical coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z$$

Solution. We compute the Jacobian

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{vmatrix} x_r & x_\theta & 0 \\ y_r & y_\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r. \end{aligned}$$

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This example indicates that in the case of the spherical coordinate transformation we have

$$dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

This leads us to the following theorem:

Theorem (Integration With Cylindrical Coordinates):
Let $f(x, y, z)$ be a continuous function on a solid B. Then

$$\iiint\limits_B f(x, y, z)\ dz\, dy\, dx$$

$$= \iiint\limits_B f(r\cos\theta, r\sin\theta, z) r\ dz\, dr\, d\theta.$$

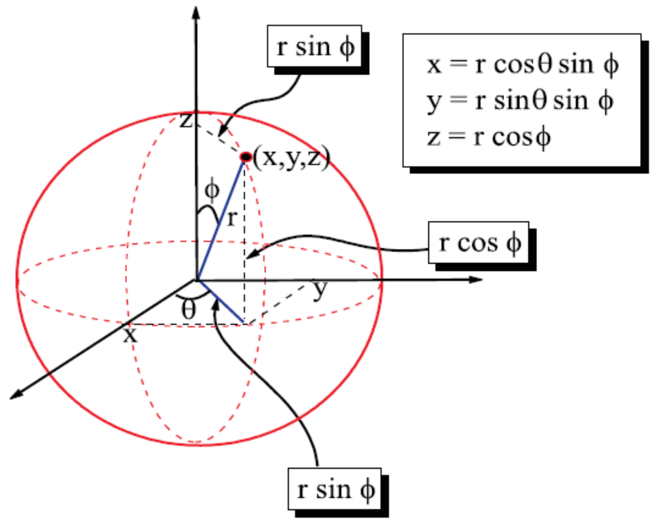
Triple Integrals in Spherical Coordinates

Another coordinate system that often comes into use is the spherical coordinate system.

Spherical coordinates are denoted by r, θ and ϕ, and are defined by

$$\begin{aligned} r =&\ \text{the distance from } (0,0,0) \text{ to } (x,y,z) \\ \theta =&\ \text{the angle between the positive } x\text{-axis and} \\ &\quad \text{the line joining } (x,y,0) \text{ to } (0,0,0) \\ \phi =&\ \text{the angle between the positive } z\text{-axis and} \\ &\quad \text{the line joining } (x,y,z) \text{ to } (0,0,0) \end{aligned}$$

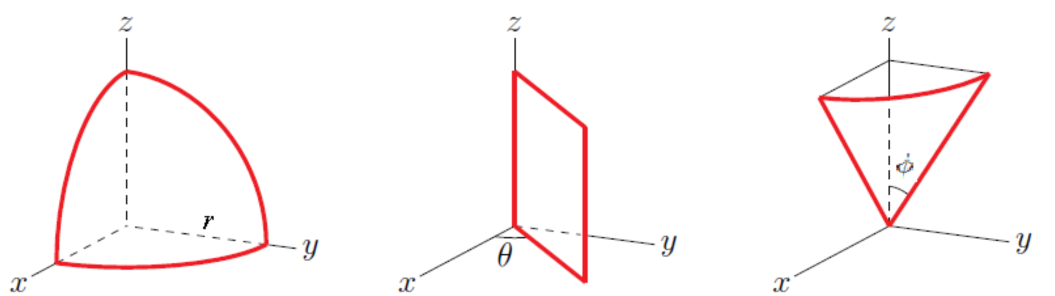
where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.



The spherical coordinate θ is the same as the cylindrical coordinate θ. The spherical coordinate ϕ is new. It runs from 0 (on the positive z-axis) to π (on the negative z-axis). The Cartesian and spherical coordinates are related by

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$
$$r^2 = x^2 + y^2 + z^2, \quad \theta = \arctan\tfrac{y}{x}, \qquad \phi = \arctan\frac{\sqrt{x^2+y^2}}{z}.$$

Here are sketches of surfaces of constant r, constant θ, and constant ϕ.

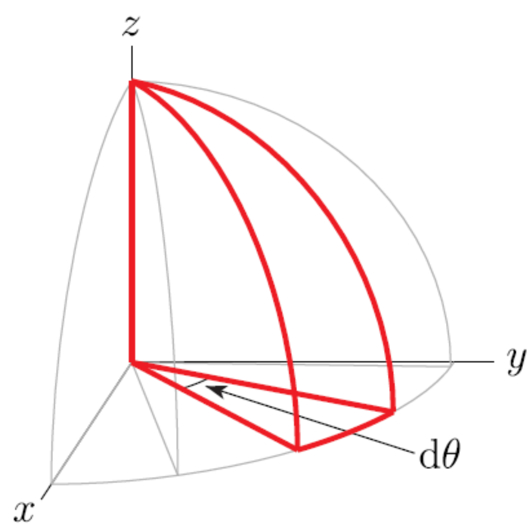


surface of constant r (a sphere) surface of constant θ (a plane) surface of constant ϕ (a cone)

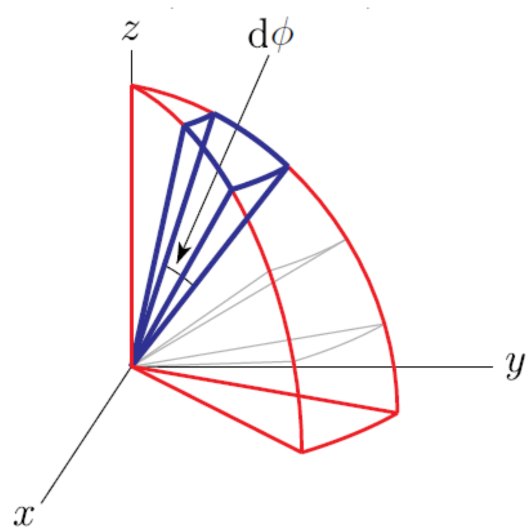
The Volume Element in Spherical Coordinates

If we cut up a solid by

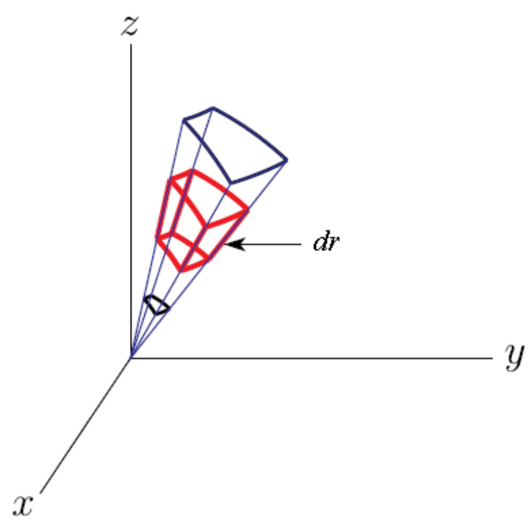
- first slicing it into segments (like segments of an orange) by using planes of constant θ, say with the difference between successive θ's being $d\theta$,



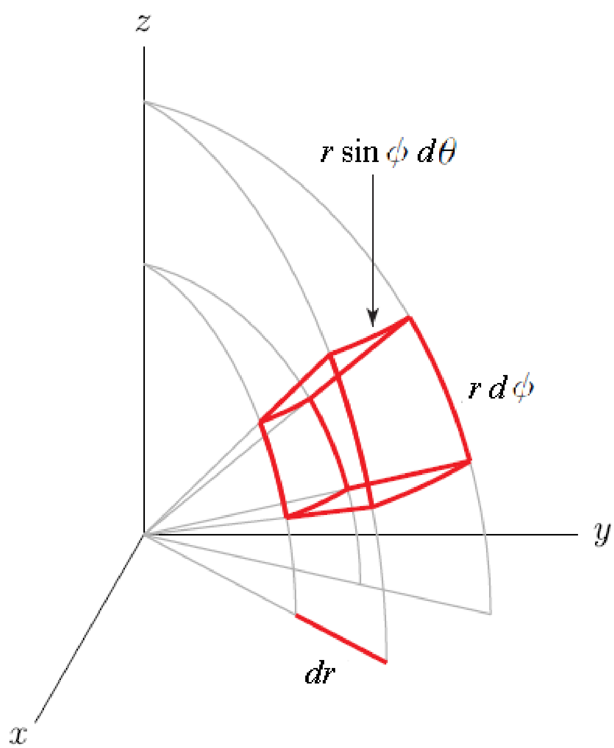
- and then subdividing the segments into "searchlights" (like the searchlight outlined in blue in the figure below) using surfaces of constant ϕ, say with the difference between successive ϕ's being$d\phi$,



- and then subdividing the searchlights into approximate cubes using surfaces of constant r, say with the difference between successive r's beingdr,



we end up with approximate cubes that look like the red one in the figure given above.



Example 45.

Find the Jacobian for the spherical coordinate transformation

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$

where $0 \le \theta \le 2$ and $0 \le \phi \le \pi$ and $r^2 = x^2 + y^2 + z^2$.

Solution. We compute the Jacobian

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} &= \begin{vmatrix}\cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\phi & 0 & -r\sin\phi\end{vmatrix}\\ &= r^2\sin\phi.\end{aligned}$$

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