PROBLEMS ON UNCONSTRAINED OPTIMIZATION

In the problems 1-13, locate all relative maxima, relative minima, and saddle points, if any.

1. $f(x,y) = y^2 + xy + 3y + 2x + 3$

Here,

$$f(x,y) = y^2 + xy + 3y + 2x + 3$$

Then,

$$f_x(x,y) = \frac{\partial}{\partial x}(y^2 + xy + 3y + 2x + 3)$$

$$= y + 2$$

$$f_y(x,y) = \frac{\partial}{\partial y}(y^2 + xy + 3y + 2x + 3)$$

$$= 2y + x + 3$$

Now, for computing critical point,

$f_x = 0$

$\Rightarrow y + 2 = 0$

$\therefore y = -2$

Also, $f_y = 0$

$\Rightarrow 2y + x + 3 = 0$

$\Rightarrow 2(-2) + x + 3 = 0$

$\therefore x = 1$

$\therefore$ Critical point $(a,b) = (x,y) = (1,-2)$

Now,

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(y+2) = 0$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}(2y + x + 3) = 2$$

$$f_{xy}(x,y) = \frac{\partial}{\partial x}(2y + x + 3) = 1$$

Now, let's compute the discriminant D of the function f.

We know,

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= f_{xx} f_{yy} - (f_{xy})^2$$

$$= 0 * 2 - (1)^2$$

$$= -1$$

Since $D < 0$, the function f has a saddle point at $(1, -2)$.

2. $f(x,y) = x^2 + xy - 2y - 2x + 1$

Here,

$$f(x,y) = x^2 + xy - 2y - 2x + 1$$

Then, $f_x = \frac{\partial}{\partial x}(x^2 + xy - 2y - 2x + 1)$

$$= 2x + y - 2$$

$$f_y = \frac{\partial}{\partial y}(x^2 + xy - 2y - 2x + 1)$$

$$= x - 2$$

At critical point, $f_x = f_y = 0$. Thus,

$2x + y - 2 = 0$ —— (i)

& $x - 2 = 0$ —— (ii)

From (ii), we get

$$x = 2$$

Putting value of x in (i), we get:

$$2 \times 2 + y - 2 = 0$$

$$\therefore y = -2$$

$\therefore$ Critical point $(a, b) = (2, -2)$

Now, computing second order derivatives,

$$f_{xx} = \frac{\partial}{\partial x}(2x + y - 2)$$

$$= 2$$

$$f_{yy} = \frac{\partial}{\partial y}(x - 2)$$

$$= 0$$

$$f_{xy} = \frac{\partial}{\partial x}(x - 2)$$

$$= 1$$

Now, discriminant D of given function f is computed as

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 0 - (1)$$

$$= -1$$

Since $D < 0$, the function f has a saddle point $(2, -2)$.

3. $f(x, y) = x^2 + xy + y^2 - 3x$

Here,

$$f(x,y) = x^2 + xy + y^2 - 3x$$

Then, $f_x = \frac{\partial}{\partial x}(x^2 + xy + y^2 - 3x)$

$= 2x + y - 3$

$f_y = \frac{\partial}{\partial y}(x^2 + xy + y^2 - 3x)$

$= 2y + x$

At critical point, $f_x = f_y = 0$.

Then,

$2x + y - 3 = 0$ —— (i)

$x + 2y = 0$ —— (ii)

Solving (i) & (ii) using elimination method:

$2x + \cancel{y} = 3 \quad \times 2$

$x + \cancel{2y} = 0$

$(-) \quad (-) \quad (-)$

$3x = 6$

$\therefore x = 2$

Putting value of x in eq^n (i), we get:

$2(2) + y - 3 = 0$

$\therefore y = -1$

$\therefore$ Critical point $(a, b) = (x, y) = (2, -1)$.

Calculation of second order derivatives,

$f_{xx} = \frac{\partial}{\partial x}(2x + y - 3)$

$= 2$

$f_{yy} = \frac{\partial}{\partial y}(2y + x)$

$= 2$

$f_{xy} = \frac{\partial}{\partial x}(2y + x)$

$= 1$

Now, discriminant of function f is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 4 - 1$$

$$= 3$$

Here, $D > 0$ and $f_{xx}(a,b) = f_{xx}(2,-1) = 2 > 0$, thus the function f has relative minimum at $(2,-1)$.

4. $f(x,y) = xy - x^3 - y^2$

Here,

$$f(x,y) = xy - x^3 - y^2$$

Then,

$$f_x = \frac{\partial}{\partial x}(xy - x^3 - y^2)$$

$$= y - 3x^2$$

$$f_y = \frac{\partial}{\partial y}(xy - x^3 - y^2)$$

$$= x - 2y$$

At critical point, $f_x = f_y = 0$. Then,

$$y - 3x^2 = 0$$

$$\Rightarrow y = 3x^2 \quad \text{--- (i)}$$

$$x - 2y = 0$$

$$\Rightarrow x - 6x^2 = 0 \quad \text{(Putting value of } y \text{ from (i))}$$

$$\Rightarrow x(1 - 6x) = 0$$

Either,

$x = 0$

OR,

$1 - 6x = 0$

$\therefore x = 1/6$

$y = 3(0)^2$
$= 0$

when $x = 1/6$,
$y = 3 \times \left(\frac{1}{6}\right)^2$
$= \frac{1}{12}$

$\therefore$ Critical points are $(0,0)$ and $(1/6, 1/12)$.

calculation of second order derivatives of f.

$f_{xx} = \frac{\partial}{\partial x}(y - 3x^2)$

$= -6x$

$f_{yy} = \frac{\partial}{\partial y}(x - 2y)$

$= -2$

$f_{xy} = \frac{\partial}{\partial x}(x - 2y)$

$= 1$

Now, discriminant D of function f is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} -6x & 1 \\ 1 & -2 \end{vmatrix}$$

$= 12x - 1$

At point $(0,0)$

$f_{xx}(0,0) = -6 \times 0 = -6 < 0$

$$D = 12 \times 0 - 1$$
$$= -1 < 0$$

Here, $D < 0$ so the function f has a saddle point at $(0,0)$.

At point $(1/6, 1/12)$

$$f_{xx}(1/6, 1/12) = -6 \times 1/6 = -1 < 0$$

$$D = 12 \times 1/6 - 1 = 2 - 1 = 1 > 0$$

Since $D > 0$ and $f_{xx}(a,b) < 0$ at $(1/6, 1/12)$, the function f has a relative maximum at $(1/6, 1/12)$.

5. $f(x,y) = x^2 + y^2 + 2xy$

Here,

$$f(x,y) = x^2 + y^2 + 2xy$$

Then,

$$f_x = \frac{\partial}{\partial x}(x^2 + y^2 + 2xy)$$

$$= 2x + 2y$$

$$f_y = \frac{\partial}{\partial y}(x^2 + y^2 + 2xy)$$

$$= 2y + 2x$$

At critical point, $f_x = f_y = 0$. Then,

$$2x + 2y = 0$$

$\Rightarrow x = -y$ where $y \in \mathbb{R}$

Suppose $y = 1$. Then,

$$x = -1$$

$\therefore$ Critical point $(a,b) = (x,y) = (-1, 1)$

Now, we have to compute second order derivatives,

$$f_{xx} = \frac{\partial}{\partial x}(2x+2y)$$
$$= 2$$
$$f_{yy} = \frac{\partial}{\partial y}(2y+2x)$$
$$= 2$$
$$f_{xy} = \frac{\partial}{\partial x}(2y+2x)$$
$$= 2$$

Now, discriminant of given function f is given by

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}$$

$$= 4-4 = 0$$

Since $D=0$, this test yields no information about what happens at $(a,b) = (-1,1)$.

However, since $f(x,y) = x^2+2xy+y^2 = (x+y)^2 \geq 0$ for all $x, y \in \mathbb{R}$,

so the minimum value of f is 0.

At critical point $(-1,1)$,

$$f(-1,1) = (-1+1)^2$$
$$= 0$$

$\therefore$ The function f has a <u>relative minimum</u> at all the <u>points</u> along the critical line $x+y=0$.

6. $f(x,y) = 2x^2 - 4xy + y^4 + 2$

Here,

$$f(x,y) = 2x^2 - 4xy + y^4 + 2$$

Then,

$$f_x = \frac{\partial}{\partial x}(2x^2 - 4xy + y^4 + 2)$$

$$= 4x - 4y$$

$$f_y = \frac{\partial}{\partial y}(2x^2 - 4xy + y^4 + 2)$$

$$= -4x + 4y^3$$

At critical point, $f_x = f_y = 0$. Then,

$4x - 4y = 0$

$\therefore x = y$ —— (i)

$-4x + 4y^3 = 0$

$\Rightarrow -4y + 4y^3 = 0$ (Putting value of x from (i))

$\Rightarrow 4y(y^2 - 1) = 0$

$\Rightarrow y(y^2-1) = 0$

Either,

$y = 0$

OR,

$y^2 - 1 = 0$

$\Rightarrow y^2 = 1$

$\therefore y = \pm 1$

when $y = 0$, $x = 0$

when $y = 1$, $x = 1$

& when $y = -1$, $x = -1$

$\therefore$ Critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$.

Computation of second order derivatives:

$$f_{xx} = \frac{\partial}{\partial x}(4x - 4y)$$

$$= 4$$

$f_{yy} = \dfrac{\partial}{\partial y}(-4x + 4y^3)$

$= 12y^2$

$f_{xy} = \dfrac{\partial}{\partial x}(-4x + 4y^3)$

$= -4$

Calculation of Hessian matrix's determinant is as follows:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 4 & -4 \\ -4 & 12y^2 \end{vmatrix}$$

$= 48y^2 - 16$

At point $(0, 0)$

$D = 48(0)^2 - 16 = -16 < 0$

$f_{xx}(0,0) = 4 > 0$

$\therefore f$ has a saddle point at $(0, 0)$.

At point $(1, 1)$

$D = 48(1)^2 - 16 = 32 > 0$

$f_{xx}(1,1) = 4 > 0$

$\therefore f$ has a relative minima at $(1,1)$

At point $(-1, -1)$

$D = 48(-1)^2 - 16 = 32 > 0$

$f_{xx}(-1,-1) = 4 > 0$

$\therefore f$ has a relative minima at $(-1, -1)$.

7. $f(x, y) = x e^y$

Here,

$$f(x, y) = x e^y$$

Then,

$$f_x = \frac{\partial}{\partial x}(x e^y)$$

$$= e^y$$

$$f_y = \frac{\partial}{\partial y}(x e^y)$$

$$= x e^y$$

At critical point, $f_x = f_y = 0$. Then,

$$e^y = 0 \quad \text{—— (i)}$$

$$x e^y = 0 \quad \text{—— (ii)}$$

Eq^n(i) can't be solved as there are no any values of $y \in \mathbb{R}$ such that $e^y = 0$. Thus, there is no critical point.

8. $f(x, y) = x^2 + y - e^y$

Here,

$$f(x, y) = x^2 + y - e^y$$

Then,

$$f_x = \frac{\partial}{\partial x}(x^2 + y - e^y)$$

$$= 2x$$

$$f_y = \frac{\partial}{\partial y}(x^2 + y - e^y)$$

$$= (1 - e^y)$$

At critical point, $f_x = f_y = 0$. Then,

$$2x = 0$$

$$\therefore x = 0$$

And, $1 - e^y = 0$

$\Rightarrow 1 - e^y = 0$

$\Rightarrow e^y = 1 \Rightarrow e^y = e^0$

$\therefore y = 0$

$\therefore$ Critical point $(a, b) = (x, y) = (0, 0)$.

Now, calculation of second order derivative:

$$f_{xx} = \frac{\partial}{\partial x}(2x)$$

$$= 2$$

$$f_{yy} = \frac{\partial}{\partial y}(1 - e^y)$$

$$= -e^y$$

$$f_{xy} = \frac{\partial}{\partial x}(1 - e^y)$$

$$= 0$$

Now, the discriminant D of given function f is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 0 & -e^y \end{vmatrix}$$

$$= -2e^y - 0$$

$$= -2e^y$$

At point $(a, b) = (0, 0)$,

$$D = -2e^{(0)} = -2 \times 1 = -2 < 0$$

Since, $D < 0$, the function f has saddle point at $(0, 0)$.

9. $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

Here,

$$f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$$

Then,

$$f_x = \frac{\partial}{\partial x}\left(xy + 2/x + 4/y\right)$$

$$= y - \frac{2}{x^2}$$

$$f_y = \frac{\partial}{\partial y}\left(xy + 2/x + 4/y\right)$$

$$= x - \frac{4}{y^2}$$

At critical point, $f_x = f_y = 0$. Then,

$$y - 2/x^2 = 0$$

$$\Rightarrow y = 2/x^2$$

$$\therefore y = \frac{2}{x^2} \quad \text{——— (i)}$$

Also, $x - \frac{4}{y^2} = 0$

$$\Rightarrow x - \frac{4}{(2/x^2)^2} = 0$$

$$\Rightarrow x - \frac{4x^4}{4} = 0$$

$$\Rightarrow x - 2x^4 = 0$$

$$\Rightarrow x(1 - 2x) = 0$$

Either,

$x = 0$

OR,

$1 - 2x = 0$

$\therefore x = 1/2$

when $x = 0$,

$y = \infty$ (Rejected)

when $x = 1/2$,

$$y = \frac{2}{(1/2)^2}$$

$$= \frac{2}{(1/4)}$$

$$= 8$$

$\therefore$ Critical point $(a, b) = (x, y) = (1/2, 8)$

$\Rightarrow x - \frac{4x^4}{4} = 0$

$\Rightarrow x - x^4 = 0$

$\Rightarrow x(1 - x^3) = 0$

Either,

$x = 0$

OR,

$1 - x^3 = 0$

$\therefore x = 1$

when $x = 0$,

$y = 2/0^2$

$\therefore y = \infty$ (Rejected)

when $x = 1$,

$$y = \frac{2}{(+1)^2}$$

$$= 2$$

when $x = -1$,

$$y = \frac{2}{(-1)^2}$$

$$= 2$$

$\therefore$ Critical point is ~~are~~ $(1, 2)$. ~~and $(-1, 2)$~~.

Calculation of second order derivatives:

$$f_{xx} = \frac{\partial}{\partial x}\left(y - 2/x^2\right)$$

$$= \frac{4}{x^3}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(x - 4/y^2\right)$$

$$= \frac{8}{y^3}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(x - 4/y^2\right) = 1$$

The discriminant D of given function f is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 4/x^3 & 1 \\ 1 & 8/y^3 \end{vmatrix}$$

$$= \frac{32}{x^3y^3} - 1$$

At critical point $(1, 2)$,

$$D = \frac{32}{(1)^3(2)^3} - 1$$

$$= \frac{32}{8} - 1$$

$$= 4 - 1$$

$$= 3 > 0$$

$$f_{xx}(1, 2) = \frac{4}{(1)^3} = 4 > 0$$

$\therefore f$ has a relative minimum at $(1, 2)$.

At critical point $(-1, 2)$,

$$D = \frac{32}{(-1)^3(2)^3} - 1$$

$$= -4 - 1$$

$$= -5 < 0$$

$\therefore f$ has a saddle point at $(-1, 2)$.

10. $f(x,y) = e^x \sin y$

Here,
$$f(x,y) = e^x \sin y$$

Then,
$$f_x = \frac{\partial}{\partial x}(e^x \sin y)$$
$$= e^x \sin y$$
$$f_y = \frac{\partial}{\partial y}(e^x \sin y)$$
$$= e^x \cos y$$

At critical point(s), $f_x = f_y = 0$. Then,
$$e^x \sin y = 0 \quad \text{—— (i)}$$
$$e^x \cos y = 0 \quad \text{—— (ii)}$$

Squaring and adding (i) & (ii), we get:
$$e^{2x}(\sin^2 y + \cos^2 y) = 0$$
$$\Rightarrow e^{2x}(1) = 0$$
$$\Rightarrow e^{2x} = 0$$

There are no solutions for above equation i.e. there is no $x \in \mathbb{R}$ that satisfies $e^{2x} = 0$.

Thus, there are no critical points.

11. $f(x,y) = y \sin x$

Here,
$$f(x,y) = y \sin x$$

Then,
$$f_x = \frac{\partial}{\partial x}(y \sin x)$$
$$= y \cos x$$
$$f_y = \frac{\partial}{\partial y}(y \sin x)$$

$= \sin x$

For critical point(s), $f_x = f_y = 0$. Thus,

$y \cos x = 0$ ——— (i)

$\sin x = 0$ ——— (ii)

From (i)

$y \cos x = 0$

Either,

$y = 0$

OR,

$\cos x = 0$

$\Rightarrow \cos x = \cos (2n+1)\pi/2$ where, $n \in W$

$\therefore x = (2n+1)\pi/2$ where $n \in W$

Putting value of x in (ii),

$\sin (2n+1)\pi/2 = \sin \pi/2$ (For $n = 0 \in W$)

$= 1 \neq 0$

Thus, eq^n (ii) is not satisfied. Now, let's start with eq^n(ii),

$\sin x = 0$

$\Rightarrow \sin x = \sin n\pi$ where $n \in W$

$\therefore x = n\pi$

Putting this value of x in (i), we get

$y \cos n\pi = 0$ ——— (iii)

The value of $\cos n\pi$ is either $+1$ or -1 depending upon value of n. Thus for eq^n (iii) to be true, $y = 0$.

$\therefore$ Critical point $(a, b) = (x, y) = (n\pi, 0)$ where, $n \in W$.

Calculation of second order derivatives:

$f_{xx} = \frac{\partial}{\partial x}(y \cos x)$

$= -y \sin x$

$f_{yy} = \frac{\partial}{\partial y}(\sin x)$

$= \cancel{\cos x} \quad 0$

$f_{xy} = \frac{\partial}{\partial x}(\sin x)$

$= \cos x$

Now, the discriminant D of given function f is given by:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} -y\sin x & \cos x \\ \cos x & 0 \end{vmatrix}$$

$$= -\cos^2 x$$

At critical point $(n\pi, 0)$

$D = -\cos^2(n\pi)$

$= -[\cos(n\pi)]^2$

$= -1 \quad (\because \cos^2 n\pi = 1, \text{ for all } n \in W)$

$\therefore D < 0$

Hence, f has a saddle point at $(n\pi, 0)$.

12. $f(x,y) = e^{-(x^2+y^2+2x)}$

Here, $f(x,y) = e^{-(x^2+y^2+2x)}$

$= e^{-(x^2+y^2+2x)}$

Now,

$f_x = \frac{\partial}{\partial x}\left(e^{-(x^2+y^2+2x)}\right)$

$= e^{-(x^2+y^2+2x)} \frac{\partial}{\partial x}(-(x^2+y^2+2x))$

$$= e^{-(x^2+y^2+2x)} \cdot (2x+2)$$

$$= -(2x+2)\, e^{-(x^2+y^2+2x)}$$

$$f_y = \frac{\partial}{\partial y}\left(e^{-(x^2+y^2+2x)}\right)$$

$$= e^{-(x^2+y^2+2x)} \cdot \frac{\partial}{\partial y}\left(-(x^2+y^2+2x)\right)$$

$$= -2y\, e^{-(x^2+y^2+2x)}$$

At critical point, $f_x = f_y = 0$. Then,

$$-(2x+2)\, e^{-(x^2+y^2+2x)} = 0$$

$$\Rightarrow -(2x+2) = 0 \quad \left(\because e^{-(x^2+y^2+2x)} \neq 0\right)$$

$$\therefore x = -1$$

$$-2y\, e^{-(x^2+y^2+2x)} = 0$$

$$\Rightarrow -2y = 0$$

$$\therefore y = 0$$

$\therefore$ Critical point $(a, b) = (x, y) = (-1, 0)$

calculation of second order derivatives,

$$f_{xx} = \frac{\partial}{\partial x}\left\{-(2x+2)\, e^{-(x^2+y^2+2x)}\right\}$$

$$= -(2x+2) \cdot \frac{\partial}{\partial x} e^{-(x^2+y^2+2x)} + e^{-(x^2+y^2+2x)} \cdot \frac{\partial}{\partial x}[-(2x+2$$

$$= -(2x+2) \cdot e^{-(x^2+y^2+2x)} \cdot (-2x+2) + e^{-(x^2+y^2+2x)} \cdot (-2)$$

$$= \cancel{(2x+2)^2\{e^{-(x^2+y^2+2x)}\} - 2\}} = e^{-(x^2+y^2+2x)}\left\{(2x+2)^2 - 2\right\}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(-2y\, e^{-(x^2+y^2+2x)}\right)$$

$$= -2y \cdot e^{-(x^2+y^2+2x)} \cdot (-2y) + e^{-(x^2+y^2+2x)} \cdot (-2)$$

$$= 4y^2\, e^{-(x^2+y^2+2x)} + e^{-(x^2+y^2+2x)} \cdot (-2)$$

$$= e^{-(x^2+y^2+2x)} \{ 4y^2 - 2 \}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(-2y\, e^{-(x^2+y^2+2x)}\right)$$

$$= -2y \times -(2x+2) \cdot e^{-(x^2+y^2+2x)}$$

$$= e^{-(x^2+y^2+2x)} \cdot 4y(x+1)$$

At $(a, b) = (-1, 0)$

$$f_{xx} = e^{-((-1)^2+0^2+2\times -1)} \{ (2\times -1+2)^2 - 2 \}$$

$$= e^1 \times (-2)$$

$$= -2e^1$$

$$f_{yy} = e^{-((-1)^2+0^2+2\times -1)} \{ 4\times 0^2 - 2 \}$$

$$= -2e^1$$

$$f_{xy} = e^{-((-1)^2+0^2+2\times -1)} \cdot 4\times 0(-1+1)$$

$$= 0$$

Thus, discriminant D is given by

$$D = \begin{vmatrix} -2e^1 & 0 \\ 0 & -2e^1 \end{vmatrix} = 4e^2 > 0$$

& $f_{xx}(-1, 0) = -2e^1 < 0$

$\therefore$ f has relative maximum at $(-1, 0)$.

13. $f(x,y) = xy + \frac{a^3}{x} + \frac{b^3}{y} \quad (a \neq 0,\ b \neq 0)$

Here,

$$f(x,y) = xy + \frac{a^3}{x} + \frac{b^3}{y}$$

Then,

$$f_x = \frac{\partial}{\partial x}\left(xy + a^3/x + b^3/y\right)$$

$$= y - \frac{a^3}{x^2}$$

$$f_y = \frac{\partial}{\partial y}\left(xy + a^3/x + b^3/y\right)$$

$$= x - \frac{b^3}{y^2}$$

At critical point, $f_x = f_y = 0$. There fore,

$$y - \frac{a^3}{x^2} = 0$$

$$\Rightarrow y = \frac{a^3}{x^2} \quad \text{——— (i)}$$

Also, $x - \frac{b^3}{y^2} = 0$

$$\Rightarrow x - \frac{b^3}{(a^3/x^2)^2} = 0 \quad (\text{Putting value of } x \text{ from (i)})$$

$$\Rightarrow x - \frac{b^3 x^4}{a^6} = 0$$

$$\Rightarrow x\left(1 - \frac{b^3}{a^6}x^3\right) = 0$$

Either, $x = 0$

OR,

$$1 - \frac{b^3}{a^6}x^3 = 0$$

$$\Rightarrow 1 = \left(\frac{b}{a^2}x\right)^3$$

$$\Rightarrow \frac{b}{a^2}x = 1$$

$$\therefore x = a^2/b$$

Putting value of x in (i), we get:

$$y = \frac{a^3}{(a^2/b)^2}$$

$$= \frac{a^3 \cdot b^2}{a^4}$$

$$= b^2/a$$

$\therefore$ Critical point $(c, d) = (a^2/b,\ b^2/a)$

Calculation of second order derivatives:

$$f_{xx} = \frac{\partial}{\partial x}\left(y - a^3/x^2\right)$$

$$= \frac{2a^3}{x^3}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(x - \frac{b^3}{y^2}\right)$$

$$= \frac{2b^3}{y^3}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(x - b^3/y^2\right)$$

$$= 1$$

Now, discriminant of function f is given by:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 2a^3/x^3 & 1 \\ 1 & 2b^3/y^3 \end{vmatrix}$$

$$= \frac{4a^3b^3}{x^3y^3} - 1$$

At $(a^2/b, b^2/a)$,

$$D = \frac{4a^3b^3}{\left(\frac{a^2}{b}\right)^3\left(\frac{b^2}{a}\right)^3} - 1$$

$$= \frac{4a^3b^3 \cdot a^3b^3}{a^6b^6} - 1$$

$$= 4 - 1$$

$$= 3 > 0$$

$$f_{xx}(a^2/b, b^2/a) = \frac{2a^3}{(a^2/b)^3}$$

$$= \frac{2a^3b^3}{a^6}$$

$$= \frac{2b^3}{a^3} > 0$$

$\therefore f$ has relative ~~maximum~~ minimum at $(a^2/b, b^2/a)$.

14. A company manufactures running shoes and basketball shoes. The total revenue from x_1 units of running shoes and x_2 units of basketball shoes is

$$R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximize the revenue.

Solution.

Here, revenue function $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

Now, first order partial derivatives of given revenue function are:

$$R_{x_1} = \frac{\partial}{\partial x_1}\left(-5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2\right)$$

$$= -10x_1 - 2x_2 + 42$$

$$R_{x_2} = \frac{\partial}{\partial x_2}\left(-5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2\right)$$

$$= -16x_2 - 2x_1 + 102$$

For critical points, $f_x = f_y = 0$. Thus,

$$R_{x_1} = 0$$

$$\Rightarrow -10x_1 - 2x_2 + 42 = 0$$

$$\therefore\ 10x_1 + 2x_2 = 42 \quad \text{—— (i)}$$

Also, $R_{x_2} = -16x_2 - 2x_1 + 102 = 0$

$$\therefore\ 2x_1 + 16x_2 = 102 \quad \text{—— (ii)}$$

Multiplying eqn (ii) by 5 and subtracting from (i), we get:

$$\begin{array}{rcrcr} \cancel{10x_1} & + & 2x_2 & = & 42 \\ \cancel{10x_1} & + & 80x_2 & = & 510 \\ (-) & & (-) & & (-) \\ \hline & & -78x_2 & = & -468 \end{array}$$

$$\therefore\ x_2 = 6$$

Putting value of x_2 in (i), we get:

$$10x_1 + 2(6) = 42$$

$$\Rightarrow 10x_1 = 30$$

$$\therefore\ x_1 = 3$$

$\therefore$ Critical point $(a, b) = (x_1, x_2) = (3, 6)$

Calculation of second order dirivatives are as follows:

$$R_{x_1 x_1} = \frac{\partial}{\partial x_1}(-10x_1 - 2x_2 + 42)$$

$$= -10$$

$$R_{x_2 x_2} = \frac{\partial}{\partial x_2}(-16x_2 - 2x_1 + 102)$$

$$= -16$$

$$R_{x_1 x_2} = \frac{\partial}{\partial x_1}(-16x_2 - 2x_1 + 102)$$

$$= -2$$

Now, discriminant of given function R is given by:

$$D = \begin{vmatrix} R_{x_1 x_1} & R_{x_1 x_2} \\ R_{x_1 x_2} & R_{x_2 x_2} \end{vmatrix}$$

$$= \begin{vmatrix} -10 & -2 \\ -2 & -16 \end{vmatrix}$$

$$= 160 - 4$$

$$= 156 > 0$$

Now

$$R_{x_1 x_1}(x_1, x_2) = R_{x_1 x_1}(3, 6) = -10 < 0$$

$\therefore$ The revenue function has maxima at $(x_1, x_2) = (3, 6)$.

$\therefore$ $x_1 = 3000$ units

$x_2 = 6000$ units.

15. A corporation manufactures candles at two locations. The cost of producing x_1 units at location 1 is

$$C_1 = 0.02x_1^2 + 4x_1 + 500$$

and the cost of producing x_2 units at location 2 is

$$C_2 = 0.05x_2^2 + 4x_2 + 275$$

The candle sells for \$15 per unit. Find the quantity that should be produced at each location to maximize the profit.

$$P = 15(x_1 + x_2) - C_1 - C_2 .$$

Solution

Here, $C_1 = 0.02x_1^2 + 4x_1 + 500$; $C_2 = 0.05x_2^2 + 4x_2 + 275$ where x_1 and x_2 are units of candles produced at location 1 and location 2 respectively.

The profit function $P(x_1, x_2)$ is given by:

$$P = 15(x_1 + x_2) - C_1 - C_2$$

$$= 15x_1 + 15x_2 - 0.02x_1^2 - 4x_1 - 500 - 0.05x_2^2 - 4x_2 - 275$$

$$\therefore P(x_1, x_2) = -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 + 775$$

Computation of first order derivative:

$$P_{x_1} = \frac{\partial}{\partial x_1}(-0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 + 775)$$

$$= (-0.04x_1 + 11)$$

$$P_{x_2} = \frac{\partial}{\partial x_2}(-0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 + 775)$$

$$= (-0.1x_2 + 11)$$

At critical point, $Px_1 = Px_2 = 0$. Thus,

$$P_{x_1} = 0$$

$$\Rightarrow -0.04x_1 + 11 = 0$$

$$\Rightarrow 0.04x_1 = 11$$

$$\therefore x_1 = 275$$

$P_{x_2} = 0$

$\Rightarrow -0.1x_2 + 11 = 0$

$\Rightarrow 0.1x_2 = 11$

$\therefore x_2 = 110$

$\therefore$ Critical point $(x_1, x_2) = (275, 110)$.

Calculation of second order derivatives:

$$P_{x_1x_1} = \frac{\partial}{\partial x_1}(-0.04x_1 + 11)$$

$$= -0.04$$

$$P_{x_2x_2} = \frac{\partial}{\partial x_2}(-0.1x_2 + 11)$$

$$= -0.1$$

$$P_{x_1x_2} = \frac{\partial}{\partial x_1}(-0.1x_2 + 11)$$

$$= 0$$

The discriminant of given function $P(x_1, x_2)$ is given by:

$$D = \begin{vmatrix} P_{x_1x_1} & P_{x_1x_2} \\ P_{x_1x_2} & P_{x_2x_2} \end{vmatrix}$$

$$= \begin{vmatrix} -0.04 & 0 \\ 0 & -0.1 \end{vmatrix}$$

$$= (-0.04)(-0.1) - 0$$

$$= 0.004 > 0$$

& $P_{x_1x_1}(x_1, x_2) = P_{x_1x_1}(275, 110) = -0.04 < 0$

$\therefore P(x_1, x_2)$ has a maxima at $(x_1, x_2) = (275, 110)$.

Thus, Profit is maximized for $\left.\begin{aligned} x_1 &: 275 \text{ units} \\ x_2 &: 110 \text{ units} \end{aligned}\right\}$

16. A company manufactures two items which are sold in two separate markets where it has a monopoly. The quantities, q_1 and q_2, demanded by consumers, and the prices p_1 and p_2 (in dollars), of each item are related by

$$p_1 = 600 - 0.3q_1 \quad \text{and} \quad p_2 = 500 - 0.2q_2$$

Thus, if the price for either item increases, the demand for it decreases. The company's total production cost is given by

$$C = 16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2$$

To maximize its total profit, how much of each product should be produced? What is the maximum profit?

Solution

Here, price for q_1 and q_2 are given as:

$$p_1 = 600 - 0.3q_1$$

$\therefore$ Revenue from product 1, $R_1 = p_1 q_1$

$$= 600q_1 - 0.3q_1^2$$

$$p_2 = 500 - 0.2q_2$$

$\therefore$ Revenue from product 2, $R_2 = p_2 q_2$

$$= 500q_2 - 0.2q_2^2$$

$\therefore$ Total revenue, $R = R_1 + R_2$

$$= 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2$$

$\therefore$ Profit function(P) is given by,

$$P = R - C$$

$$= 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2 - (16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2)$$

$$\therefore P(q_1, q_2) = 598.8q_1 + 498.5q_2 - 0.3q_1^2 - 0.2q_2^2 - 16 - 0.2q_1q_2 \quad \text{—— (i)}$$

Now, to maximize the Profit function, let's compute the critical point:

First of all, we compute first order partial derivatives:

$$P_{q_1} = \frac{\partial}{\partial q_1}(598.8q_1 + 498.5q_2 - 0.3q_1^2 - 0.2q_2^2 - 16 - 0.2q_1q_2)$$

$$= 598.8 - 0.6q_1 - 0.2q_2$$

$$P_{q_2} = \frac{\partial}{\partial q_2}(P(q_1, q_2))$$

$$= 498.5 - 0.4q_2 - 0.2q_1$$

For critical point(s), $P_{q_1} = P_{q_2} = 0$. Thus,

$$P_{q_1} = 0$$

$$\Rightarrow 598.8 - 0.6q_1 - 0.2q_2 = 0$$

$$\therefore 0.6q_1 + 0.2q_2 = 598.8 \quad \text{—— (i)}$$

$$P_{q_2} = 0$$

$$\Rightarrow 498.5 - 0.4q_2 - 0.2q_1 = 0$$

$$\therefore 0.2q_1 + 0.4q_2 = 498.5 \quad \text{—— (ii)}$$

On solving (i) & (ii), we get:

$q_1 = 699.1$

$q_2 = 896.7$

Now, critical point $(a, b) = (699.1, 896.7)$

calculation of second order partial derivatives:

$$P_{q_1q_1} = \frac{\partial}{\partial q_1}(598.8 - 0.6q_1 - 0.2q_2)$$

$$= -0.6$$

$$P_{q_2q_2} = \frac{\partial}{\partial q_2}(498.5 - 0.4q_2 - 0.2q_1)$$

$$= -0.4$$

$$P_{q_1q_2} = \frac{\partial}{\partial q_1}(498.5 - 0.4q_2 - 0.2q_1)$$

$$= -0.2$$

Now the discriminant D of function P is given by

$$D = \begin{vmatrix} P_{q_1q_1} & P_{q_1q_2} \\ P_{q_1q_2} & P_{q_2q_2} \end{vmatrix}$$

$$= \begin{vmatrix} -0.6 & -0.2 \\ -0.2 & -0.4 \end{vmatrix}$$

$$= (-0.6) * (-0.4) - (0.2)^2$$

$$= 0.2 > 0$$

Also, $P_{q_1q_1}(a, b) = \cancel{+N16}\ P_{q_1q_1}(699.1, 896.7) = -0.6 < 0$

Therefore, $P(q_1, q_2)$ is maximized at $(699.1, 896.7)$.

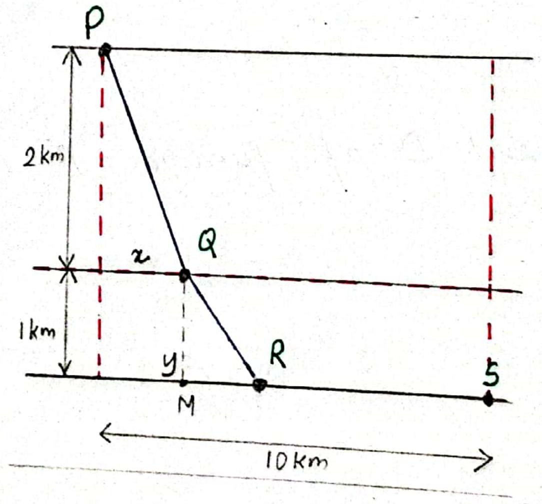
Now substituting the value of $q_1 = 699.1$ and $q_2 = 896.7$ in (i), we get the maximum profit.

$$\therefore P_{max} = 598.8 * 699.1 + 498.5 * 896.7 - 0.3 * (699.1)^2 - 0.2 * (896.7)^2 - 16 - 0.2 * 699.1 * 896.7$$

$$= \$ 432,979.015$$

$$\approx \$ 433,000.$$

17. A water line is to be built from point P to S point and must pass through regions where construction costs differ (see figure). The cost per kilometers (in dollars) is 3k from P to Q, 2k from Q to R and k from R to S. Find x and y such that the total cost C will be minimized.



Solution

Here, $PQ = \sqrt{2^2 + x^2}$

$QR = \sqrt{1^2 + (y-x)^2}$

$RS = (10-y)$

Now, the cost function of building water line from point P to point S is given by multiplying distance with the respective costs. Thus, total cost C is given by:

$$C = PQ * 3k + QR * 2k + RS * k$$

$$\Rightarrow C = \left(\sqrt{2^2 + x^2}\right) 3k + \left(\sqrt{1 + (y-x)^2}\right) 2k + (10-y)k \qquad \text{—— (i)}$$

We have to ~~maximize~~ minimize eqn (i). Let's compute partial derivatives first.

$$C_x = \frac{\partial}{\partial x}(C) = \frac{\partial}{\partial x}\left[\left(\sqrt{2^2+x^2}\right) 3k + \left(\sqrt{1+(y-x)^2}\right) 2k + (10-y)k\right]$$

$$\Rightarrow C_x = \frac{6xk}{2\sqrt{2^2+x^2}} + \left(-\frac{2k(y-x)}{\sqrt{1+(y-x)^2}}\right)$$

$$\therefore C_x = \frac{3xk}{\sqrt{2^2+x^2}} - \frac{2k(y-x)}{\sqrt{1+(y-x)^2}}$$

$$C_y = \frac{\partial}{\partial y}(C)$$

$$= \frac{\partial}{\partial y}\left[\left(\sqrt{2^2+x^2}\right) 3k + \left(\sqrt{1+(y-x)^2}\right) 2k + (10-y)k\right]$$

$$= \frac{2k}{2\sqrt{1+(y-x)^2}} * 2(y-x) - k$$

$$= \frac{2k(y-x)}{\sqrt{1+(y-x)^2}} - k$$

For critical point, $C_x = C_y = 0$. Thus,

$$C_y = 0$$

$$\Rightarrow \frac{2k(y-x)}{\sqrt{1+(y-x)^2}} - k = 0$$

$$\therefore \frac{(y-x)}{\sqrt{1+(y-x)^2}} = \frac{1}{2} \qquad \text{—— (i)}$$

$C_x = 0$

$$\Rightarrow \frac{3xk}{\sqrt{4+x^2}} - \frac{2k(y-x)}{\sqrt{1+(y-x)^2}} = 0$$

$$\Rightarrow \frac{3xk}{\sqrt{4+x^2}} - 2k * \frac{1}{2} = 0 \quad \text{(Putting value from eq}^n\text{(i))}$$

$$\Rightarrow \frac{3xk}{\sqrt{4+x^2}} = K$$

$$\Rightarrow \frac{3x}{\sqrt{4+x^2}} = 1$$

Squaring on both sides,

$$9x^2 = 4 + x^2$$

$$\Rightarrow 8x^2 = 4$$

$$\Rightarrow x^2 = 1/2$$

$\therefore x = \frac{1}{\sqrt{2}}$ (Distance can't be negative so $-1/\sqrt{2}$ not considered).

Putting value of x in eq^n(i), we get:

$$\frac{y - 1/\sqrt{2}}{\sqrt{1+\left(y-\frac{1}{\sqrt{2}}\right)^2}} = 1/2$$

$$\Rightarrow \left(y - 1/\sqrt{2}\right)^2 = \frac{1}{4}\left\{1 + \left(y - \frac{1}{\sqrt{2}}\right)^2\right\}$$

$$\Rightarrow \frac{3}{4}\left(y - \frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4}$$

$$\Rightarrow \left(y - \frac{1}{\sqrt{2}}\right)^2 = \frac{1}{3}$$

$$\Rightarrow \left(y - \frac{1}{\sqrt{2}}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2$$

$$\therefore y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}}$$

$\therefore$ Critical point $(a,b) = (x,y) = \left(1/\sqrt{2}, \frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}}\right)$.

Now, calculating second order derivatives of cost function.

$$c_{xx} = \frac{\partial}{\partial x}(c_x)$$

$$= \frac{\partial}{\partial x}\left(\frac{3x.k}{\sqrt{4+x^2}} - \frac{2k(y-x)}{\sqrt{1+(y-x)^2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3k}{\frac{\sqrt{4+x^2}}{x}} - \frac{2k}{\frac{\sqrt{1+(y-x)^2}}{(y-x)}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3k}{\sqrt{4/x^2+1}} - \frac{2k}{\sqrt{\frac{1}{(y-x)^2}+1}}\right)$$

$$= 3k \cdot \frac{(-1)\times 4\times(-2).x^{-2-1}}{2\sqrt{4/x^2+1}} - 2k\cdot\left(\frac{-1}{2}\right)\times\frac{(-2)(y-x)^{-3}\cdot(-1)}{\sqrt{1/(y-x)^2+1}}$$

$$= \frac{24k\,x^{-3}}{2\sqrt{1+4/x^2}} + \frac{4k\,(y-x)^{-3}}{2\sqrt{\frac{1}{(y-x)^2}+1}}$$

$$= \frac{12k}{x^3\sqrt{1+4/x^2}} + \frac{2k}{(y-x)^3\sqrt{1+\frac{1}{(y-x)^2}}}$$

$$c_{yy} = \frac{\partial}{\partial y}(c_y)$$

$$= \frac{\partial}{\partial y}\left(\frac{2k(y-x)}{\sqrt{1+(y-x)^2}} - k\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2k}{\sqrt{\frac{1}{(y-x)^2}+1}} - k\right)$$

$$= \frac{2k}{(-2)\sqrt{1/(y-x)^2+1}}\cdot -2(y-x)^{-3}\cdot 1 - 0$$

$$= \frac{2k}{(y-x)^3\sqrt{1+\frac{1}{(y-x)^2}}}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{2k(y-x)}{\sqrt{1+(y-x)^2}} - k\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2k}{\sqrt{\frac{1}{(y-x)^2}+1}} - k\right)$$

$$= \frac{2k}{(-2)\cdot\sqrt{1/(y-x)^2+1}}\cdot(-2)(y-x)^{-3}\cdot(-1) \quad - \quad 0$$

$$= \frac{-2k}{(y-x)^3\sqrt{1+1/(y-x)^2}}$$

At critical point $\left(1/\sqrt{2}, \frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}}\right)$

$$f_{xx} = f_{xx} = \frac{12k}{(1/\sqrt{2})^3\sqrt{1+\frac{4}{(1/\sqrt{2})^2}}} + \frac{2k}{\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)^3\sqrt{1+\frac{1}{\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}}-\frac{1}{\sqrt{2}}\right)^2}}}$$

$$= \frac{12k}{\frac{1}{2\sqrt{2}}\sqrt{9}} + \frac{2k}{\left(\frac{\sqrt{3}+\sqrt{2}-\sqrt{3}}{\sqrt{6}}\right)^3\sqrt{1+\frac{1}{\left(\frac{\sqrt{3}+\sqrt{2}-\sqrt{3}}{\sqrt{6}}\right)^2}}}$$

$$= \frac{12k}{\frac{3}{2\sqrt{2}}} + \frac{2k}{\left(\frac{\sqrt{2}}{\sqrt{6}}\right)^3 \times \sqrt{1+\frac{1}{2/6}}}$$

$$= \frac{12k \times 2\sqrt{2}}{3} + \frac{2k}{\left(\frac{\sqrt{2}}{\sqrt{6}}\right)^3 * 2}$$

$$= 8\sqrt{2}k + \frac{6\sqrt{6}\,k}{2\sqrt{2}}$$

$$= 8\sqrt{2}k + 3\sqrt{3}k$$

$$= (8\sqrt{2} + 3\sqrt{3})k$$

$$C_{yy} = \frac{2K}{\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)^2 \sqrt{1 + \frac{1}{\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)^2}}}$$

$$= 3\sqrt{3}\,K$$

$$C_{xy} = \frac{-2K}{\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)^2 \sqrt{1 + \frac{1}{\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)^2}}}$$

$$= -3\sqrt{3}\,K$$

Now discriminant D of function C is given by

$$D = \begin{vmatrix} C_{xx} & C_{xy} \\ C_{xy} & C_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} (8\sqrt{2}+3\sqrt{3})K & -3\sqrt{3}K \\ -3\sqrt{3}K & 3\sqrt{3}K \end{vmatrix}$$

$$= (24\sqrt{6} + 27)K^2 - 27K^2$$

$$= 24\sqrt{6}\,K^2 > 0$$

Also, $C_{xx}(a, b) = (8\sqrt{2} + 3\sqrt{3})K > 0$

$\therefore$ The function $C(x, y)$ has a minimum value at (x, y)

$$= \left(1/\sqrt{2},\ \frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}}\right)$$

$\therefore x = 1/\sqrt{2}$ km $= 0.707$ km.

$\therefore y = (\sqrt{3}+\sqrt{2})/\sqrt{6} = 1.284$ km.

18. Let $f(x,y) = 3x^2 + ky^2 + 9xy$. Determine the values of k (if any) for which the critical point at $(0,0)$ is:

a) A saddle point

b) A local maximum

c) A local minimum.

Here given $f(x,y) = 3x^2 + ky^2 + 9xy$

Now,

$$f_x = \frac{\partial}{\partial x}(3x^2 + ky^2 + 9xy)$$

$$\therefore f_x = 6x + 9y \quad \text{—} (i)$$

$$f_y = \frac{\partial}{\partial y}(3x^2 + ky^2 + 9xy)$$

$$\therefore f_y = 2ky + 9x \quad \text{—} (ii)$$

The point $(0,0)$ satisfies both $f_x = 0, f_y = 0$. Thus, it is a critical point.
Now, computation of second order derivatives are as follows:

$$f_{xx} = \frac{\partial}{\partial x}(f_x)$$

$$= \frac{\partial}{\partial x}(6x + 9y)$$

$$= 6$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y)$$

$$= 2k$$

$$f_{xy} = \frac{\partial}{\partial x}(f_y)$$

$$= \frac{\partial}{\partial x}(2ky + 9x)$$

$$= 9$$

The discriminant D of given function f is given by:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 9 \\ 9 & 2k \end{vmatrix}$$

$$= 12k - 81$$

a) For point (0, 0) to be a saddle point,

$D < 0$

$\Rightarrow 12k - 81 < 0$

$\Rightarrow 12k < 81$

$\therefore k < \frac{81}{12}$

b) For point (0,0) to be a local minimum,

$f_{xx}(0,0) > 0$

We have $f_{xx}(0,0) = 6 > 0$. Thus, for point (0,0) to be local minimum, $D > 0$

$\Rightarrow 12k - 81 > 0$

$\therefore k > 81/12$

c) A local ~~minimum~~ maximum

For point (0,0) to be local maximum,

$f_{xx}(0,0) < 0$ & $D > 0$.

Since $f_{xx}(0,0) = 6 > 0$, thus point (0,0) can never be a local maximum for any value of k.

19. Let $f(x,y) = x^3 + ky^2 - 5xy$. Determine the value of k (if any) for which the critical point at $(0,0)$ is:

a) A saddle point

b) A local maximum

c) A local minimum.

Here, $f(x,y) = x^3 + ky^2 - 5xy$

Now, $f_x = \frac{\partial}{\partial x}(x^3 + ky^2 - 5xy)$

$= 3x^2 - 5y$

$f_y = \frac{\partial}{\partial y}(x^3 + ky^2 - 5xy)$

$= 2ky - 5x$

$f_{xx} = \frac{\partial}{\partial x}(3x^2 - 5y)$

$= \cancel{2}\ 6x$

$f_{yy} = \frac{\partial}{\partial y}(2ky - 5x)$

$= 2k$

$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2ky - 5x)$

$= -5$

Put $x = 0$ and $y = 0$ in f_x and f_y.

We get:

$f_x(0,0) = 3(0)^2 - 5(0) = 0$

$f_y(0,0) = 2k(0) - 5(0) = 0$

Thus $(0,0)$ is a critical point of f.

Now, the discriminant D of function f is calculated as:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 3x^2 & -5 \\ -5 & 2k \end{vmatrix}$$

$$= \cancel{4kx^2 + 2/5} \; 6kx^2 - 25$$

$\therefore$ for $(0,0)$, $D = 0 - 25 = -25$

a) A saddle point

For $(0,0)$ to be a saddle point,

$D < 0$

$\Rightarrow \cancel{4k - 25 < 0}$ Here, $D < 0$. Thus, the given point $(0,0)$

$\therefore \cancel{k = 25/4}$ is a saddle point for any value of $k \in \mathbb{R}$.

b) A local maximum

For a point (a, b) to be local maximum, we must have

$f_{xx}(a,b) < 0$ &

$D > 0$.

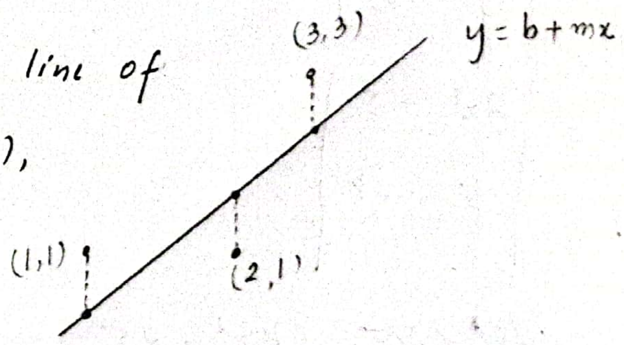
Here $f_{xx}(0,0) = 0$ & $D < 0$. Thus, for no any value of k, the point $(0,0)$ can be a local maximum.

c) A local minimum

Here, $D < 0$, so for no value of k, can the critical point $(0,0)$ be a local minimum.

20. Find a least squares line for the following data points $(1,1)$, $(2,1)$ and $(3,3)$.

<u>Solution</u>



Let, $y = b+mx$ be the best line of fit for the given data points $(1,1)$, $(2,1)$ and $(3,3)$ where,

b: intercept of line

& m: slope of line.

Now for given points $(1,1)$, $(2,1)$ and $(3,3)$

$$y = b+mx$$

for $x_1 = 1$,

$$y_1 = b + m(1) = b+m$$

for $x_2 = 2$,

$$y_2 = b + m(2) = b+2m$$

for $x_3 = 3$,

$$y_3 = b + m(3) = b+3m$$

The corresponding errors are:

$$e_1 = 1-(b+m)$$

$$e_2 = 1-(b+2m)$$

$$e_3 = 3-(b+3m)$$

The square of errors are

$$E = e_1^2 + e_2^2 + e_3^2$$

$$= [1-(b+m)]^2 + [1-(b+2m)]^2 + [3-(b+3m)]^2$$

$$\therefore E = [1-(b+m)]^2 + [1-(b+2m)]^2 + [3-(b+3m)]^2$$

Since square of error E is parameterized by, b, m. We can write E as $f(b,m)$. Thus,

$$f(b,m) = [1-(b+m)]^2 + [1-(b+2m)]^2 + [3-(b+3m)]^2 \quad \text{--- (i)}$$

Now, we calculate partial derivatives:

$$f_b = \frac{\partial}{\partial b}[f(b,m)]$$

$$= 2[1-(b+m)]\cdot(-1) + 2[1-(b+2m)]\cdot(-1) + 2[3-(b+3m)]\cdot(-1)$$

$$= (2-2b-2m)\times(-1) + (2-2b-4m)(-1) + (6-2b-6m)(-1)$$

$$= -2+2b+2m-2+2b+4m-6+2b+6m$$

$$= -10+6b+12m$$

$$= (12m+6b-10)$$

$$f_m = \frac{\partial}{\partial m}[f(b,m)]$$

$$= (-2)[1-(b+m)] - 4[1-(b+2m)] - 6[3-(b+3m)]$$

$$= -2+2b+2m-4+4b+8m-18+6b+18m$$

$$= (28m+12b-24)$$

For critical point,

$f_b = 0$; $f_m = 0$. Thus,

$$12m + \cancel{6b} - 10 = 0 \quad \times 2$$

$$28m \cancel{+12b} - 24 = 0$$

$$(-) \quad (-) \quad (+)$$

$$-4m + 4 = 0$$

$$\therefore m = 1$$

Putting value of $m=1$ in

$$12m + 6b - 10 = 0$$

$$\Rightarrow 12(1) + 6b - 10 = 0$$

$$\therefore b = -1/3$$

classmate

Date

Page

$\therefore$ Critical point $(b, m) = (-1/3, 1)$.

Now, we calculate second order derivatives,

$$f_{bb} = \frac{\partial}{\partial b}(f_b)$$

$$= \frac{\partial}{\partial b}(12m + 6b - 10)$$

$$= 6$$

$$f_{mm} = \frac{\partial}{\partial m}(28m + 12b - 24)$$

$$= 28$$

$$f_{bm} = \cancel{}\frac{\partial}{\partial b}(f_m)$$

$$= \frac{\partial}{\partial b}(28m + 12b - 24)$$

$$= 12$$

Now, we compute discriminant of given function f as follows:

$$D = \begin{vmatrix} f_{bb} & f_{bm} \\ f_{bm} & f_{mm} \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 12 \\ 12 & 28 \end{vmatrix}$$

$$= 6 \times 28 - 144$$

$$= 24 > 0$$

$$f_{bb}(-1/3, 1) = 6 > 0$$

Thus the function f has a minimum value at $(-1/3, 1)$.

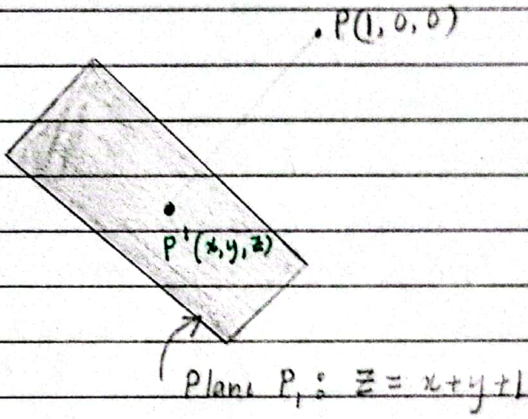
$\therefore$ least square line is

$y = b + mx$

$$\boxed{\therefore\ y = \frac{-1}{3} + x}$$

21. Find the point on the plane $z = x+y+1$ closest to the point $P = (1, 0, 0)$.

Let $P'(x, y, z)$ be a point on plane $P_1 : z = x+y+1$ such that its distance from point $P(1, 0, 0)$ is minimum.



f

Using distance formula:

$P'P = \sqrt{(x-1)^2 + y^2 + z^2}$

$\Rightarrow d^2 = (x-1)^2 + y^2 + (x+y+1)^2$ —— (i)

Writing eq^n (i) as

$f(x, y) = (x-1)^2 + y^2 + (x+y+1)^2$

Now,

$f_x = 2(x-1) + 2(x+y+1)$

$= 4x + 2y$

$f_y = 2y + 2(x+y+1)$

$= 2x + 4y + 2$

For critical point, $f_x = 0$; $f_y = 0$.

Thus,

$4x + 2y = 0$

$\therefore\ y = -2x$ —— (ii)

Also, $2x + 4y + 2 = 0$

$\Rightarrow\ 2x + 4*(-2x) + 2 = 0$

$\Rightarrow\ -6x + 2 = 0$

$\therefore\ x = 1/3$

Putting value of x in eqn (ii),

$y = -2 * \left(\frac{1}{3}\right)$

$= -2/3$

$\therefore$ Critical point $(a, b) = \left(1/3, -2/3\right)$

Now, calculating second order derivatives:

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(4x + 2y) = 4$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(2x + 4y + 2) = 4$$

$$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2x + 4y + 2) = 2$$

Now, calculating the discriminant of f,

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

classmate
Date
Page

$$= \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}$$

$= 16 - 4$

$= 12 > 0$

And $f_{xx}(a, b) = f_{xx}\left(1/3, -2/3\right) = 4 > 0$

Thus, function f has minimum at critical point $(a, b) = \left(1/3, -2/3\right)$.

Now, we have

$z = x + y + 1$

$= 1/3 + \left(-2/3\right) + 1$

$= 1 - 1/3$

$= 2/3$

$\therefore$ Required point on plane $(x, y, z) = \left(1/3, -2/3, 2/3\right)$.