PROBLEMS ON DOUBLE INTEGRALS OVER GENERAL REGION:

Evaluate the iterated integral.

1. $\int_0^1 \int_{x^2}^{x} xy^2 \, dy \, dx$

$= \int_0^1 \left[\frac{xy^3}{3} \right]_{x^2}^{x} dx$

$= 1/3 \int_0^1 \left[xy^3 \right]_{x^2}^{x} dx$

$= 1/3 \int_0^1 (x^4 - x^7) \, dx$

$= 1/3 \left[x^5/5 - x^8/8 \right]_0^1$

$= 1/3 \, (1/5 - 1/8)$

$= 1/3 * \left(\frac{8-5}{40} \right)$

$= 1/40$

2. $\int_1^{3/2} \int_y^{3-y} y \, dx \, dy$

$= \int_1^{3/2} [xy]_y^{3-y} \, dy$

$= \int_1^{3/2} [(3-y)y - y^2] \, dy$

$= \int_1^{3/2} (3y - y^2 - y^2) \, dy$

$$= \int_1^{3/2} (3y - 2y^2)\, dy$$

$$= \left[\frac{3y^2}{2} - \frac{2y^3}{3} \right]_1^{3/2}$$

$$= \frac{3(3/2)^2}{2} - \frac{2(3/2)^3}{3} - \frac{3}{2} + \frac{2}{3}$$

$$= \frac{81 - 54 - 36 + 16}{24}$$

$$= \frac{7}{24}$$

3. $$\int_0^3 \int_0^{\sqrt{9-y^2}} y\, dx\, dy$$

$$= \int_0^3 \left[yx \right]_0^{\sqrt{9-y^2}} dy$$

$$= \int_0^3 \left(y\sqrt{9-y^2} \right) dy$$

Put $u = 9 - y^2$

$\Rightarrow du = -2y\, dy$

Changing the limit of integration,

when $y = 0$, $u = 9 - 0^2 = 9$

when $y = 3$, $u = 9 - (3)^2 = 0$

Thus, above integral becomes,

$$-\frac{1}{2}\int_9^0 u^{1/2}\, du$$

$$= -\frac{1}{2}\int_{9}^{0} u^{1/2}\, du$$

$$= \frac{-1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{9}^{0}$$

$$= \frac{-1}{2} * \frac{2}{3}\left(0^{3/2} - 9^{3/2}\right)$$

$$= -\frac{1}{3} * (-27)$$

$$= 9$$

4. $$\int_{1/4}^{1}\int_{x^2}^{x} \sqrt{\frac{x}{y}}\, dy\, dx$$

$$= \int_{1/4}^{1}\int_{x^2}^{x} \sqrt{x}\cdot y^{-1/2}\, dy\, dx$$

$$= \int_{1/4}^{1} \sqrt{x}\cdot\left[\frac{y^{-1/2+1}}{-1/2+1}\right]_{x^2}^{x} \cancel{dy}\, dx$$

$$= 2\int_{1/4}^{1}\left[\sqrt{x}\sqrt{y}\right]_{x^2}^{x} dx$$

$$= 2\int_{1/4}^{1}\left(\sqrt{x}\cdot\sqrt{x} - \sqrt{x}\cdot\sqrt{x^2}\right) dx$$

$$= 2\int_{1/4}^{1}\left(x - x\sqrt{x}\right) dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^{5/2}}{5/2}\right]_{1/4}^{1}$$

$$= 2\left[\frac{x^2}{2} - \frac{2}{5}x^{5/2}\right]_{1/4}^{1}$$

$$= 2\left(1/2 - 2/5 - 1/32 + \frac{2}{5} * \frac{1}{32}\right)$$

$$= 2 * \left(\frac{80-64-5+2}{32*5}\right)$$

$$= 2 * \frac{13}{32*5}$$

$$= \frac{13}{80}$$

6. $\int_{-1}^{1}\int_{-x^2}^{x^2}(x^2-y)\,dy\,dx$

$$= \int_{-1}^{1}\left[x^2y - \frac{y^2}{2}\right]_{-x^2}^{x^2} dx$$

$$= \int_{-1}^{1}\left(x^2x^2 - \cancel{\frac{x^4}{2}} + x^4 + \cancel{\frac{x^4}{2}}\right) dx$$

$$= \int_{-1}^{1} 2x^4\,dx$$

$$= 2\int_{-1}^{1} x^4\,dx$$

$$= 2\left[\frac{x^5}{5}\right]_{-1}^{1}$$

$$= 2\left(\frac{1}{5} + \frac{1}{5}\right) = \frac{2*2}{5} = \frac{4}{5}$$

7. $\int_0^1 \int_0^x y\sqrt{x^2-y^2}\, dy\, dx$

Let, $I = \int_0^1 \int_0^x y\sqrt{x^2-y^2}\, dy\, dx$

Put $u = x^2-y^2$

$\Rightarrow du = -2y\, dy$

Changing the limit,

when $y=0$, $u = x^2-0 = x^2$

when $y=x$, $u = x^2-x^2 = 0$.

Thus, above integral becomes,

$$I = -\frac{1}{2}\int_0^1 \int_{x^2}^0 u^{1/2}\, du\, dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{u^{3/2}}{3/2}\right]_{x^2}^0 dx$$

$$= -\frac{1}{3}\int_0^1 (0 - x^3)\, dx$$

$$= \frac{1}{3}\int_0^1 x^3\, dx$$

$$= \frac{1}{3}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} * \left(\frac{1}{4} - 0\right)$$

$$= \frac{1}{12}$$

8. $\int_1^2 \int_0^{y^2} e^{x/y^2} \, dx \, dy$

$= \int_1^2 \left[y^2 e^{x/y^2} \right]_0^{y^2} dy$

$= \int_1^2 (y^2 e - y^2) \, dy$

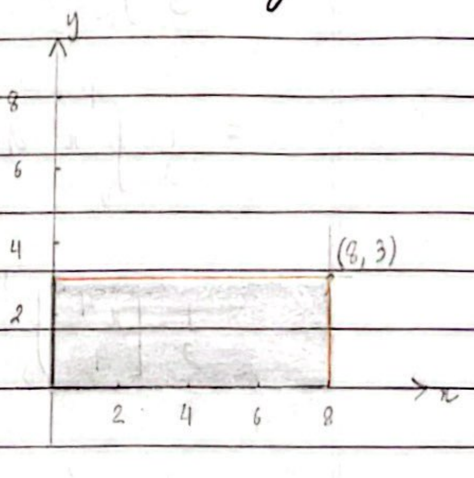
$= (e-1) \int_1^2 y^2 \, dy$

$= (e-1) \left[\frac{y^3}{3} \right]_1^2$

$= (e-1) \left(\frac{2^3}{3} - \frac{1^3}{3} \right)$

$= \frac{7}{3} (e-1)$

9. Use an iterated integral to find the area of the shaded region.



Here x ranges from 0 to 8 and y ranges from 0 to 3

Thus, domain of integration,
$D = \{(x, y) : 0 \leq x \leq 8, \ 0 \leq y \leq 3\}$

Now, area of shaded region is given by,

$A = \iint_D dx \, dy$

$= \int_0^3 \int_0^8 dx \, dy$

$= \int_0^3 [x]_0^8 \, dy$

$$= \int_0^3 (8-0)\,dy$$

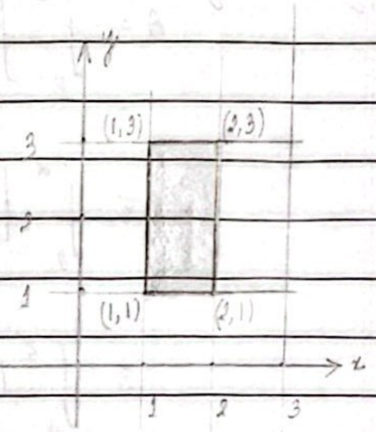
$$= 8\,[y]_0^3$$

$$= 8\,(3-0)$$

$$= 24$$

10.

Here, x ranges from 1 to 2.
y ranges from 1 to 3.



Thus, domain of integration
$D = \{(x,y) : \cancel{0}\ 1 \le x \le 2,\ 1 \le y \le 3\}$

The area of shaded region is given by,

$$A = \iint_D dy\,dx$$

$$= \int_1^2\int_1^3 dy\,dx$$

$$= \int_1^2 [y]_1^3\,dx$$

$$= \int_1^2 (3-1)\,dx$$

$$= 2\int_1^2 dx$$

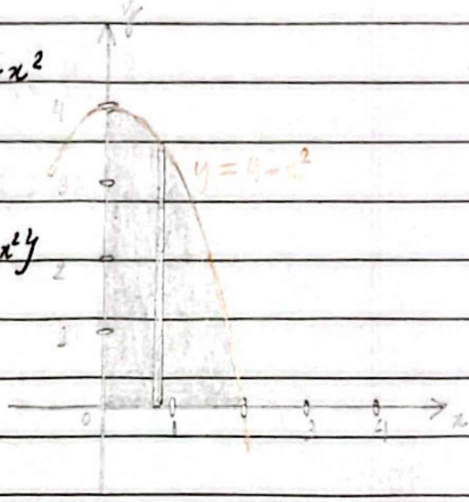
$$= 2\,[x]_1^2$$

$$= 2(2-1) = \boxed{2}$$

11.

Here, x ranges from 0 to 2
y ranges from $y=0$ to $y=4-x^2$

$\therefore$ Domain of integration,
$D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 4-x^2\}$



Thus, area of shaded region is given by

$$A = \int_{0}^{2}\int_{y=0}^{y=4-x^2} dy\,dx$$

$$= \int_{0}^{2} \left[y \right]_{0}^{4-x^2} dx$$

$$= \int_{0}^{2} (4-x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3} \right]_{0}^{2}$$

$$= 4*2 - \frac{(2)^3}{3}$$

$$= 8 - \frac{8}{3}$$

$$= \frac{16}{3}$$

Use double integration to find the area of the plane region enclosed by the given curves.

12. $y = \sin x$ and $y = \cos x$, for $0 \le x \le \pi/4$

Area, A is given by,

$$A = \int_0^{\pi/4} \int_{y=\sin x}^{y=\cos x} dy\, dx$$

$$= \int_0^{\pi/4} \left[y \right]_{\sin x}^{\cos x} dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx$$

$$= \left[\sin x + \cos x \right]_0^{\pi/4}$$

$$= \sin \pi/4 + \cos \pi/4 - \sin 0 - \cos 0$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$= \frac{2}{\sqrt{2}} - 1$$

$$= (\sqrt{2} - 1)$$

13. $y^2 = -x$ and $3y - x = 4$

Finding intersection point of given curves,

$y^2 = -x$ .... (i)

$3y - x = 4$ ... (ii)

Putting value of $-x$ from (i) in (ii), we get:

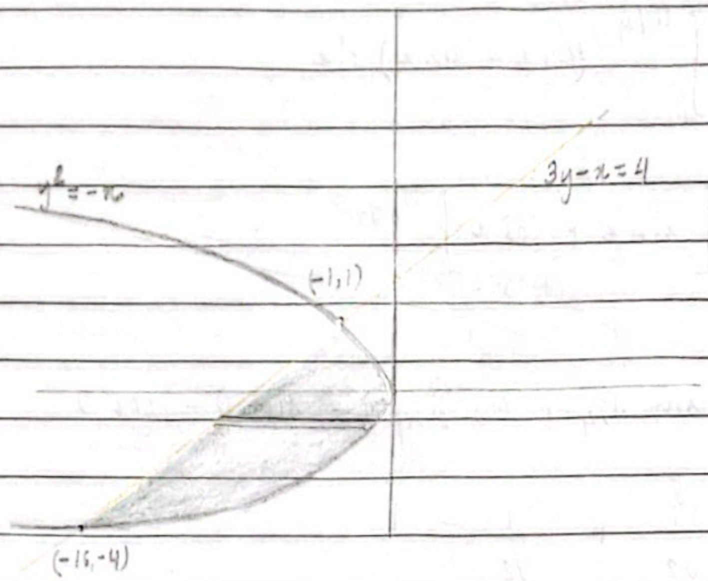
$3y + y^2 = 4$

$\Rightarrow y^2 + 3y - 4 = 0$

$\Rightarrow (y+4)(y-1) = 0$

$\therefore y = -4$ or 1

Sketching the given curves in Euclidean plane as follows:



Domain of integration:

y ranges from -4 to 1,

x ranges from $x = 3y-4$ to $x = -y^2$

$\therefore D = \{(x,y) : 3y-4 \le x \le -y^2, \ -4 \le y \le 1\}$

Thus, area of shaded region is given by,

$$A = \iint_D dx\, dy$$

$$= \int_{-4}^{1} \int_{x=3y-4}^{x=-y^2} dx\, dy$$

$$= \int_{-4}^{1} \left[x\right]_{3y-4}^{-y^2} dy$$

$$= \int_{-4}^{1} (-y^2 - 3y + 4)\, dy$$

$$= \int_{-4}^{1} (4 - 3y - y^2)\, dy$$

$$= \left[4y - \frac{3y^2}{2} - \frac{y^3}{3}\right]_{-4}^{1}$$

$$= \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(-16 - 24 + \frac{64}{3}\right)$$

$$= 4 + 16 + 24 - \frac{3}{2} - \frac{1}{3} - \frac{64}{3}$$

$$= 44 - \frac{9 + 2 + 64 * 2}{6}$$

$$= 44 - \frac{139}{6}$$

$$= \boxed{\frac{125}{6}}$$

14. $y^2 = 9 - x$ and $y^2 = 9 - 9x$

Here, $y^2 = 9 - x$ —— (i)

$y^2 = 9 - 9x$ —— (ii)

On solving (i) and (ii), we get:

$9 - x = 9 - 9x$

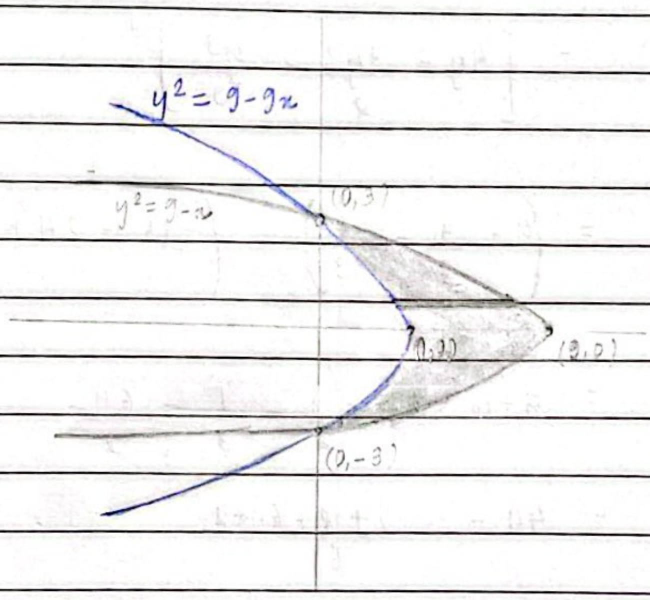
$\Rightarrow x = 0$

Putting $x = 0$, in (1), we get:

$y^2 = 9 - 0$

$\therefore y = \pm 3$

Sketching the given curves in Euclidean plane:



Domain of integration,

y ranges from -3 to $+3$

x ranges from $x = \frac{9-y^2}{9}$ to $9 - y^2$

$\therefore$ Domain $= \left\{ (x, y) : \frac{9-y^2}{9} \le x \le 9 - y^2, \ -3 \le y \le 3 \right\}$

Thus, area of shaded region,

$$A = \int_{-3}^{3} \int_{x = \frac{9-y^2}{9}}^{x = 9-y^2} dx\, dy$$

$$= \int_{-3}^{3} \left[x \right]_{\frac{9-y^2}{9}}^{9-y^2} dy$$

$$= \int_{-3}^{3} \left\{ (9-y^2) - \frac{(9-y^2)}{9} \right\} dy$$

$$= \frac{8}{9} \int_{-3}^{3} (9-y^2)\, dy$$

$$= \frac{8}{9} \left[9y - \frac{y^3}{3} \right]_{-3}^{3}$$

$$= \frac{8}{9} \left\{ \left(27 - \frac{27}{3} \right) - \left(-27 + \frac{27}{3} \right) \right\}$$

$$= \frac{8}{9} * (27 - 9 + 27 - 9)$$

$$= \frac{8}{9} * 36$$

$$= \boxed{32}$$

Use a double integral to find the volume of the indicated solid.

15.

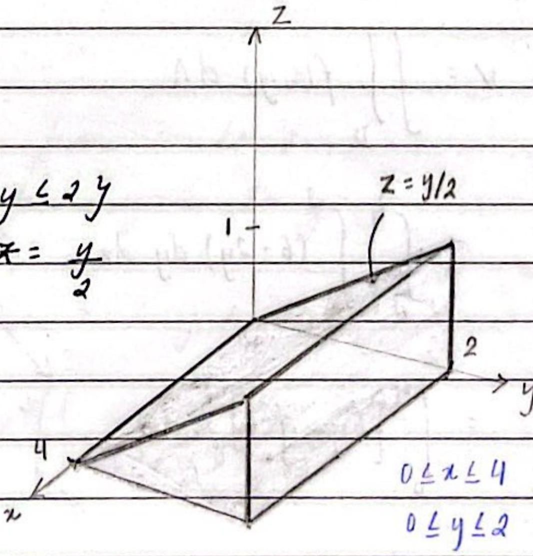
Domain of integration,

$D = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le 2\}$

Given function $f(x,y) = z = \frac{y}{2}$

Now, volume of the shaded solid,

$$V = \iint_D f(x,y)\, dA$$



$$= \int_0^4 \int_0^2 y/_2 \, dy \, dx$$

$$= \int_0^4 \left[\frac{y^2}{4}\right]_0^2 dx$$

$$= \frac{1}{4} \int_0^4 (2^2 - 0) \, dx$$

$$= \frac{4}{4} \int_0^4 dx$$

$$= [x]_0^4$$

$$= (4 - 0)$$

$$= \boxed{4}$$

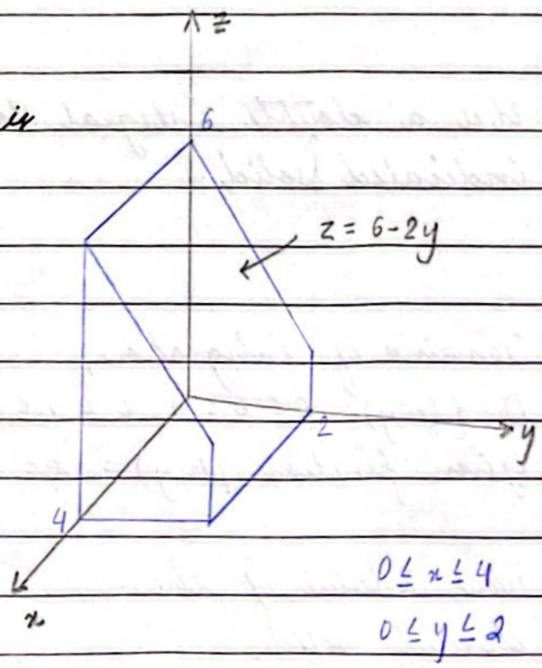
16.

Volume of given solid is given by,

$$V = \iint_D f(x, y) \, dA$$

$$= \int_0^4 \int_0^2 (6 - 2y) \, dy \, dx$$

$$= \int_0^4 \left[6y - y^2\right]_0^2 dx$$



$0 \le x \le 4$

$0 \le y \le 2$

$$= \int_0^4 (12-4)\, dx$$

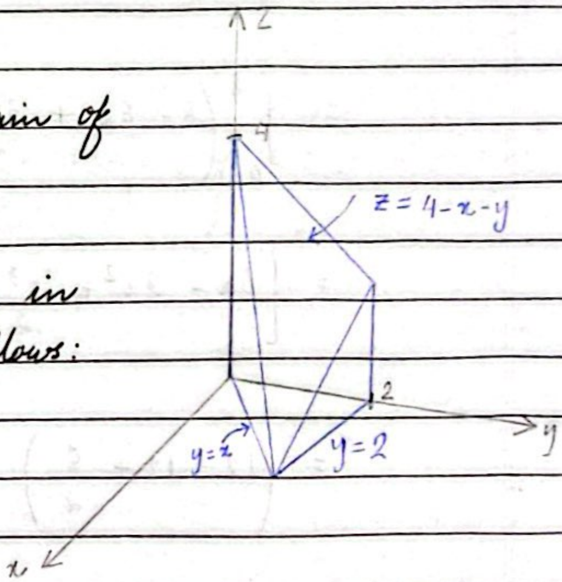
$$= 8\,[x]_0^4$$

$$= 8 * (4-0)$$

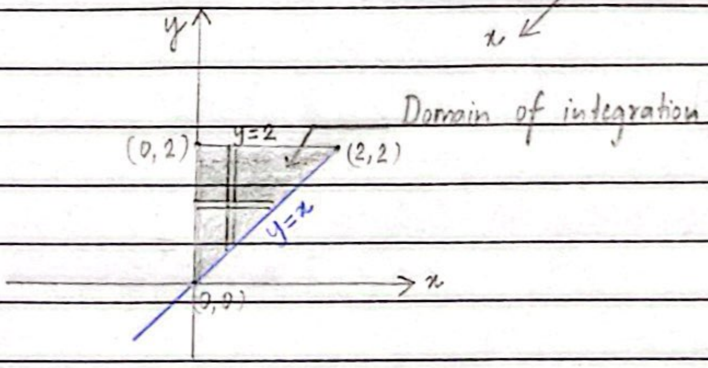
$$= \boxed{32}$$

17. Here, $z = f(x,y) = 4-x-y$

Let's find out the domain of integration D.



Sketching the constraints in Euclidean plane as follows:



Here, value of x ranges from 0 to 2

y ranges from $y=x$ to $y=2$

$\therefore D = \{(x,y) : 0 \le x \le 2,\ x \le y \le 2\}$

$\therefore$ Volume enclosed, $V = \iint_D f(x,y)\, dA$

$$= \int_0^2 \int_{y=x}^{y=2} (4-x-y)\, dy\, dx$$

$$= \int_0^2 \left[4y - xy - \frac{y^2}{2}\right]_x^2 dx$$

$$= \int_0^2 \left\{(8 - 2x - 2) - \left(4x - x^2 - \frac{x^2}{2}\right)\right\} dx$$

$$= \int_0^2 \left(6 - 6x + \frac{3x^2}{2}\right) dx$$

$$= \left[6x - 3x^2 + \frac{x^3}{2}\right]_0^2$$

$$= \left(12 - 12 + \frac{8}{2}\right) - 0$$

$$= \boxed{4}$$

18.

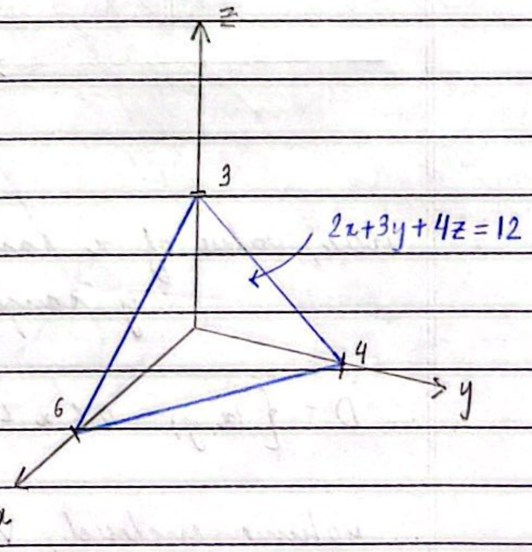
Here,

$2x + 3y + 4z = 12$ ——— (i)

$\Rightarrow 4z = 12 - 2x - 3y$

$\Rightarrow z = \dfrac{12 - 2x - 3y}{4}$

$\therefore f(x,y) = \dfrac{12 - 2x - 3y}{4}$



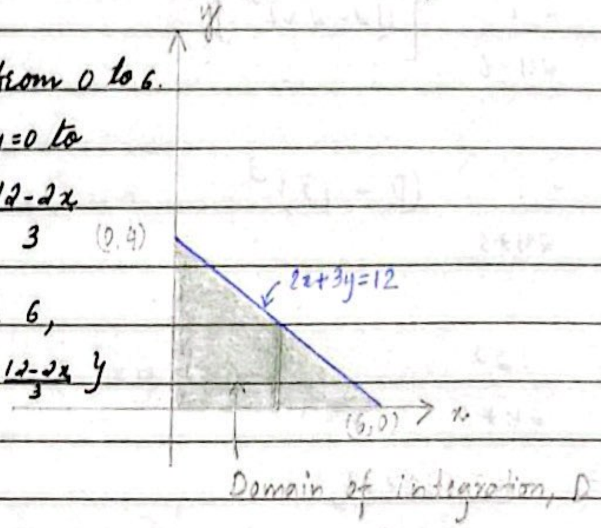
Now let's find out the domain of integration, D.

Putting $z=0$ in equation (i), we get:
$2x+3y=12$ —— (ii)

Plotting equation (ii) in Euclidean plane, we get:

Here, x ranges from 0 to 6.
y ranges from $y=0$ to
$y=\frac{12-2x}{3}$

$\therefore D=\{(x,y): 0\le x\le 6,$
$0\le y\le \frac{12-2x}{3}\}$



Now,
Enclosed volume,

$$V=\iint_D f(x,y)\,dA$$

$$=\int_0^6\int_0^{y=(12-2x)/3}\frac{12-2x-3y}{4}\,dy\,dx$$

$$=\frac{1}{4}\int_0^6\left[12x-2xy-\frac{3y^2}{2}\right]_0^{(12-2x)/3}dx$$

$$=\frac{1}{4}\int_0^6\left[4(12-2x)-2x\frac{(12-2x)}{3}-\frac{(12-2x)^2}{6}\right]dx$$

$$=\frac{1}{4}\int_0^6(12-2x)\left\{4-\frac{2x}{3}-\frac{(12-2x)}{6}\right\}dx$$

$$=\frac{1}{4}\int_0^6(12-2x)\left(\frac{24-2\times 2x-12+2x}{6}\right)dx$$

$$= \frac{1}{24} \int_0^6 (12-2x)^2 \, dx$$

$$= \frac{1}{24} \left[\frac{(12-2x)^3}{3*(-2)} \right]_0^6$$

$$= \frac{-1}{24*6} \left[(12-2x)^3 \right]_0^6$$

$$= \frac{-1}{24*6} (0-12)^3$$

$$= \frac{12^3}{24*6}$$

$$= \frac{12*12*12}{12*12}$$

$$= \boxed{12}$$

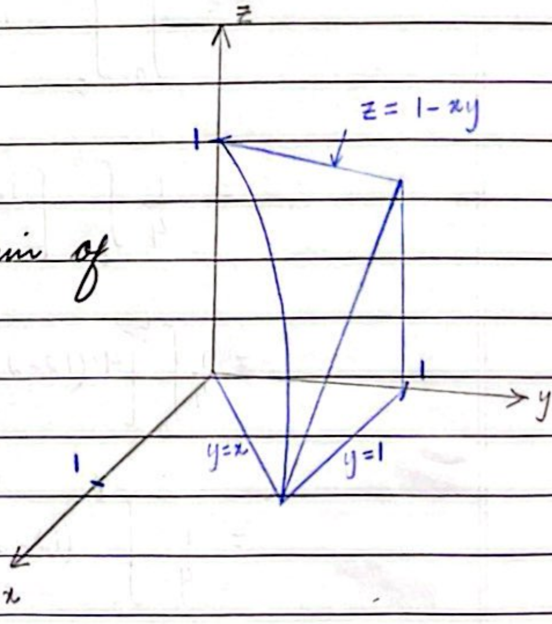
19.

Here, $z = 1 - xy$

$\therefore f(x,y) = 1 - xy$

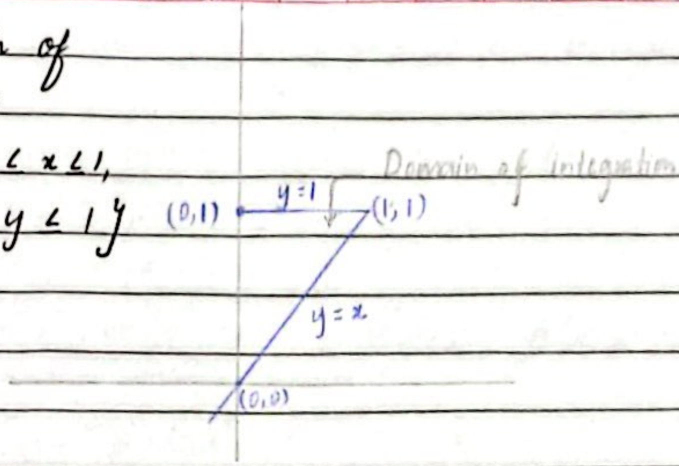
Now, let's find the domain of integration D,

Sketching constraints in Euclidean plane as follows:



Thus, domain of integration is

$D = \{(x,y) : 0 \le x \le 1, \; x \le y \le 1\}$



Thus, the enclosed volume is given by:

$$V = \iint_D f(x,y)\, dA$$

$$= \int_0^1 \int_{y=x}^{y=1} (1 - xy)\, dy\, dx$$

$$= \int_0^1 \left[y - \frac{xy^2}{2} \right]_x^1 dx$$

$$= \int_0^1 \left\{ \left(1 - \frac{x}{2}\right) - \left(x - \frac{x^3}{2}\right) \right\} dx$$

$$= \int_0^1 \left(1 - \frac{3x}{2} + \frac{x^3}{2}\right) dx$$

$$= \left[x - \frac{3x^2}{4} + \frac{x^4}{8} \right]_0^1$$

$$= 1 - \frac{3}{4} + \frac{1}{8}$$

$$= \frac{8 - 6 + 1}{8}$$

$$= \boxed{3/8}$$