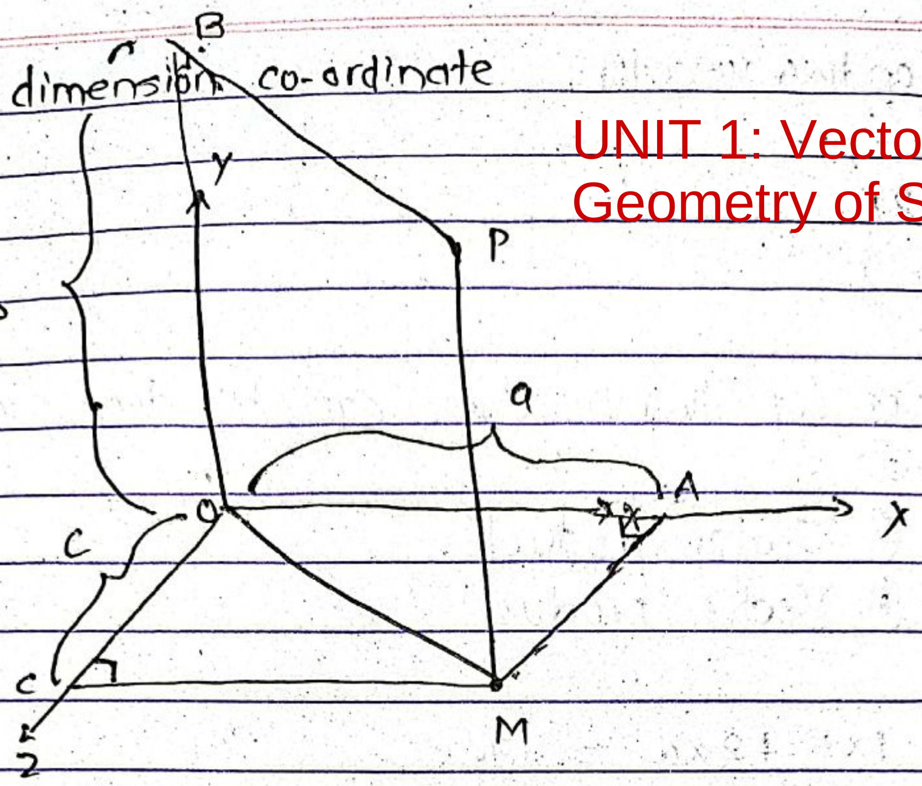
UNIT 1: Vector and Geometry of Space

Three dimension co-ordinate



Take O as origin P be a point on space.
Draw $\perp^r$ PM from P on XZ plane.
Join OM
Draw $\perp^r$ MA & MC from M on OX & OZ.
Draw $\perp^r$ PB from P on OY

Measure OA, OB, OC
if OA = a, OB = b & OC = c
Then the cordinate of P is (a, b, c)

Product of two vecotor

$\vec{a} = (1,2)$
$\vec{b} = (5,6)$

The product of two vectors can be done in following two ways:

(i) Scalar product
(ii) Vector product

$\vec{a} \cdot \vec{b} = 1 \times 5 + 2 \times 6 = 17$

(i) ~~Scalar~~ product (Dot)

$\vec{a} = (a_1, a_2)$ & $\vec{b} (b_1, b_2)$ be two vectors then their dot product is

$$\vec{a} \cdot \vec{b} = a_1 \times b_1 + a_2 \times b_2$$
$$= \boxed{a_1 b_1 + a_2 b_2}$$

which is scalar.
Hence the name scalar product is justified.

Similarly for space vector
$\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} (b_1, b_2, b_3)$

$$\vec{a} \cdot \vec{b} = (a_1 b + a_2 b_2 + a_3 b_3)$$

~~Geometrical meaning of dot~~

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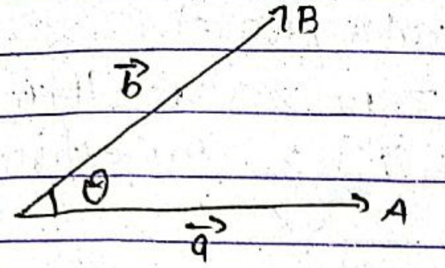
Angle between two vectors.



Let $OA = \vec{a}$ and $OB = \vec{b}$ then
if θ is angle betwⁿ OA and OB then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} \quad ; \quad \vec{a} \neq 0, \ \vec{b} \neq 0$$

Writing $\hat{a}$ = unit vector along $\vec{a}$

$$= \frac{\vec{a}}{|\vec{a}|}$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

Then,

$$\cos\theta = \hat{a}\cdot\hat{b}$$

Dot product of standard unit vectr

$\vec{i} = (1,0)$, $\vec{j}\,(0,1)$

Then. $\vec{i}\cdot\vec{i} = (1,0)\cdot(1,0) = 1+0 = 1$

$\vec{j}\cdot\vec{j} = 1$

$\vec{i}\cdot\vec{j} = (1,0)\cdot(0,1) = 0+0 = 0$

Note: if two vectors ar $\perp$ʳ then $\theta = 90°$ & $\cos 90 = 0$

$$0 = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

i.e. $\vec{a}\cdot\vec{b} = 0$

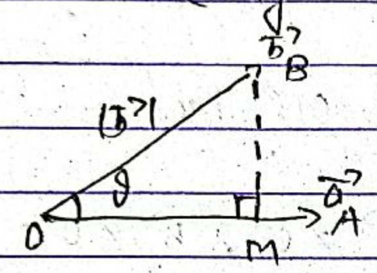
Geometrical meaning of scalar product

If $\vec{a}$ & $\vec{b}$ be non zero vectors then

$\vec{a} \cdot \vec{b}$ = (Magnitute of $\vec{a}$) x projection of $\vec{B}$ on $\vec{a}$

OR

= (Magnitude of $\vec{b}$) x projection of $\vec{a}$ on $\vec{b}$



$$\cos\theta = \frac{OM}{OB}$$

$\therefore$ $OM = OB \cos\theta$

Proof:

let $OA = \vec{a}$, $OB = \vec{b}$

Draw $\perp^r$ BM from B on OA

Then $\cos\theta = \frac{OM}{OB}$

$\therefore$ $OM = OB \cos\theta$

$OM = OB \cos\theta$

or, Projection of $\vec{OB}$ on $\vec{OA}$ = $|\vec{b}| \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \right)$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$\therefore$ $\vec{a} \cdot \vec{b} = |\vec{a}|$ x (Projection of B on A)

Similarly

$\vec{a}.\vec{b} = |\vec{b}| \times$ (Projection of $\vec{a}$ on $\vec{b}$)

Fact : scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

Vector projection $\vec{b}$ on $\vec{a}$ = scalar projection $\times$ unit vector along $\vec{a}$

$$= \frac{\vec{a}.\vec{b}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right)$$

* Class work *

Find scalar projection and vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a} = (1,2)$, $\vec{b} = (0,7)$

scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{(1,2).(0,7)}{\sqrt{0^2+7^2}}$$

$$= \frac{14}{7} = 2$$

Vector projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\left(\dfrac{\vec{b}}{|\vec{b}|}\right)$

$$= 2\left(\frac{(0,7)}{\sqrt{0^2+7^2}}\right)$$

$$= (0,2)$$

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}$$

Vector product (cross)

Let $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ be two space vectors. Then their vector product or cross product denoted by $\vec{a} \times \vec{b}$ & is defined by:

$\cancel{\vec{a} \times \vec{b} = (a_1 a_2 a_3}$

$$\vec{a} \times \vec{b} = (a_1 \quad a_2 \quad a_3) \quad (b_1 \quad b_2 \quad b_3)$$

$$= (a_2b_3 - b_2a_3,\ a_3b_1 - b_3a_1,\ a_1b_2 - a_2b_1)$$

~~Example~~ which is vector.

Since the product is vector. so the name 'vector product' is justified.

Vector Product in terms of determinants

If $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ then $\vec{a} \times \vec{b}$ can be expressed in terms of determinant as

$$\vec{a} \times \vec{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - b_2a_3)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - b_1a_2)\vec{k}$$

$$= (a_2b_3 - b_2a_3,\ a_3b_1 - a_1b_3,\ a_1b_2 - b_1a_2$$

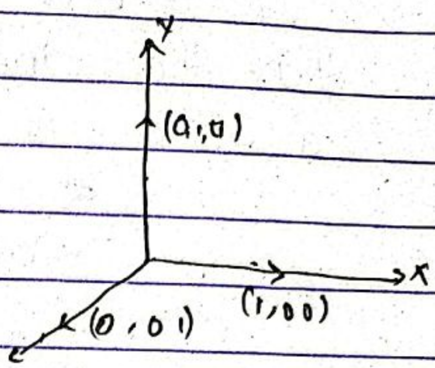
* Example:

Find cross product of $\vec{i} = (1,0,0)$ & $\vec{j} = (0,1,0)$

Now,

$$\vec{i}\times\vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

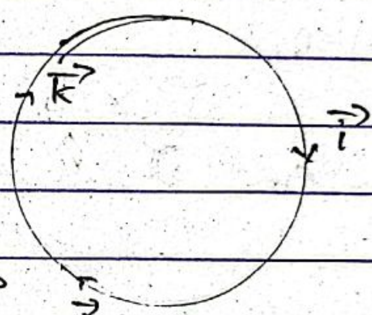
$$= \vec{k}$$



Note: $\vec{i}\times\vec{j} = \vec{k}$

$\vec{j}\times\vec{k} = \vec{i}$

& $\vec{k}\times\vec{j} = \vec{j}$



Thus, cross product of three standard unit vector $\vec{i}, \vec{j}, \vec{k}$ along x, y, z axis preserve cyclic order.

Similarly,

$\vec{j}\times\vec{i} = -(\vec{i}\times\vec{j}) = -\vec{k}$

$\vec{i}\times\vec{k} = -\vec{j}$

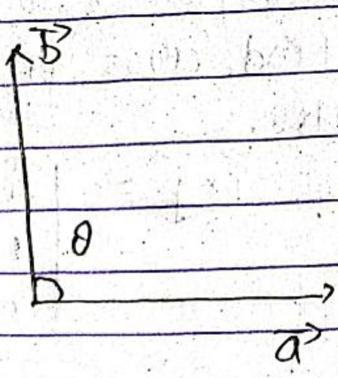
$\vec{k}\times\vec{j} = -\vec{i}$

Note: If $\vec{a}$ and $\vec{b}$ are two space vector then $\vec{a}\times\vec{b}$ alway represents the vector perpendicula to both $\vec{a}$ and $\vec{b}$.

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Sine angle between two vectors

If $\vec{a}$ and $\vec{b}$ be two vectors, then the sine angle between $\vec{a}$ & $\vec{b}$ is



$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

Note: $-1 \leq \sin\theta \leq +1$

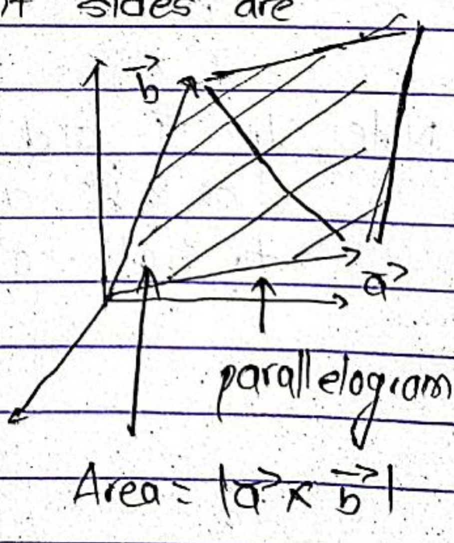
Also,

$$\vec{a} \times \vec{b} = \hat{n}\,|\vec{a}|\,|\vec{b}|\sin\theta$$

where $\hat{n}$ is the unit vector along $\vec{a} \times \vec{b}$

Geometrical meaning of cross product

Geometrically $|\vec{a} \times \vec{b}|$ always represent area of parallogram whose adjacent sides are $\vec{a}$ and $\vec{b}$



Also area of traingle determined by the vector $\vec{a}$ & $\vec{b}$ is

$$\frac{1}{2}|\vec{a} \times \vec{b}|$$

$\vec{a} \times \vec{b}$ = vector
$\vec{a} \cdot \vec{b}$ = scalor
$\vec{a}\vec{b}$ = undefined

$(\underbrace{\vec{a} \cdot \vec{b}}_{\text{scalar}}) \times \underbrace{\vec{c}}_{\text{vector}}$ = meaningless

Product of three vector

The product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ can be treated on the following concepts.

(i) $(\vec{a} \times \vec{b}) \times \vec{c} \rightarrow$ vector
(ii) $(\vec{a} \times \vec{b}) \cdot \vec{c} \rightarrow$ scalar
(iii) $(\vec{a} \cdot \vec{b}) \times \vec{c} \rightarrow$ meaningless
(iv) $(\vec{a}\,\vec{b}) \times \vec{c} \rightarrow$ undefined
(v) $(\vec{a} \times \vec{b})\,\vec{c} \rightarrow$ undefined
(vi) $\vec{a} \times (\vec{b} \times \vec{c}) \rightarrow$ vector
(vii) $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \rightarrow$ meaningless

Scalar product of three vectors

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors then the scalar triple product of $\vec{a}$, $\vec{b}$, $\vec{c}$ is denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$ or $(\vec{a}\,\vec{b}\,\vec{c})$ and is defined by

$$[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad \text{—— } (*)$$

Since in $(*)$
$\vec{b} \times \vec{c}$ is vector
and $\vec{a}$ is also vector
Hence the product
$\vec{a} \cdot (\vec{b} \times \vec{c})$ is also scalar.
Hence the name "scalar triple product" is justified.

Scalar triple product in terms of determinant.

Let $\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

& $\vec{c} = (c_1, c_2, c_3)$

Then their scalar triple product can be expressed in terms of determinant.

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hint: $\vec{b}\times\vec{c} = (b_2c_3 - c_2b_3,\ b_3c_1 - b_1c_3,\ \overset{b_1c_2 - b_2c_1}{\cancel{a_1b_2 - b_2a_1}})$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1, a_2, a_3)\cdot(\quad,\quad,\quad)$$

$$= a_1(b_2c_3 - c_2b_3) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of scalar triple product

(i) In the scalar triple product, the position of dot and cross can be interchanged.

Eg: $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

the position of

(ii) If any two vectors are interchanged then a minus sign is introduced.

Eg: $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$

Proof:

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\vec{b}\cdot(\vec{a}\times\vec{c})$$

$$= -[\vec{b}\ \vec{a}\ \vec{c}]$$

(iii) Scalar triple product of three standard unit vector $\vec{i}, \vec{j}\ \vec{k}$ is 1.

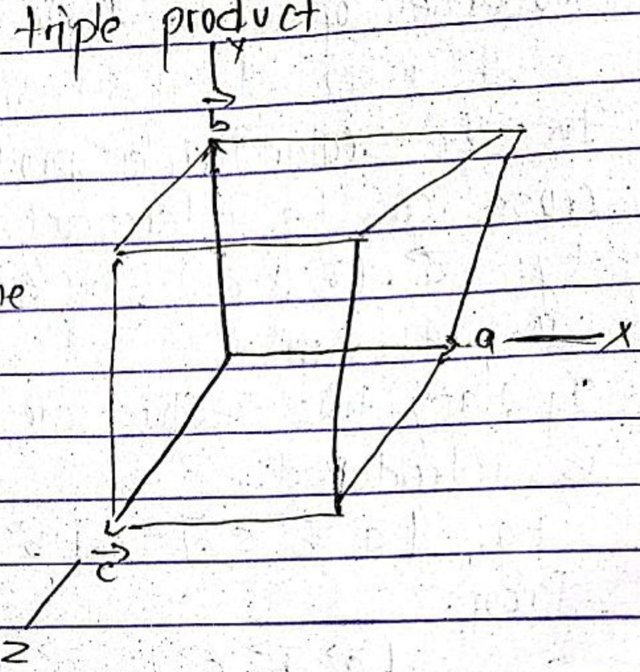
$$[\vec{i}\ \vec{j}\ \vec{k}] = \vec{i}\cdot(\vec{j}\times\vec{k})$$

$$= \vec{i}\cdot\vec{i}$$

$$= 1$$

Geometrical meaning of scalar triple product

Geometrically
$[\vec{a}\,\vec{b}\,\vec{c}]$ i.e. $\vec{a}\cdot(\vec{b}\times\vec{c})$
always represents the volume of parallelopied with sides $\vec{a}, \vec{b}$ & $\vec{c}$.



HW If $\vec{a} = (1,2,3)$, $\vec{b} = (2,0,1)$ & $\vec{c} = (4,5,1)$
Find
(i) Projection of $\vec{a}$ on $\vec{c}$ & $\vec{c}$ on $\vec{b}$
(ii) Area of Δ determined by $\vec{b}$ & $\vec{c}$
(iii) Area of parallogram with side $\vec{b}$ & $\vec{c}$
(iv) sine and cosine angles between $\vec{a}$ & $\vec{c}$
(v) Volume of parallelopied with side $\vec{a}, \vec{b}, \vec{c}$.

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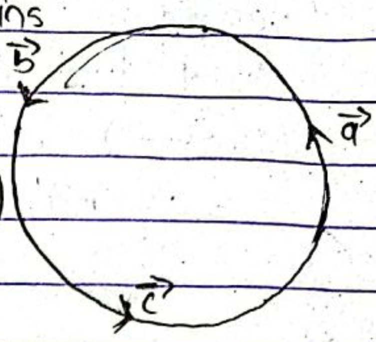
(iv) The value of scalar triple product remains unchanged by interchanging the vectors in cyclic order.

i.e. $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$



(v) The value of scalar triple product changes its sign if the position of any two vectors are interchanged.

$[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{b}\,\vec{a}\,\vec{c}]$

(vi) The value of scalar triple product is zero if two vectors are equal.

i.e $[\vec{a}\,\vec{a}\,\vec{c}] = 0$, $[\vec{b}\,\vec{c}\,\vec{c}] = 0$

(vii) The value of scalar triple product is zero if any two vectors are parallel.

More precisely

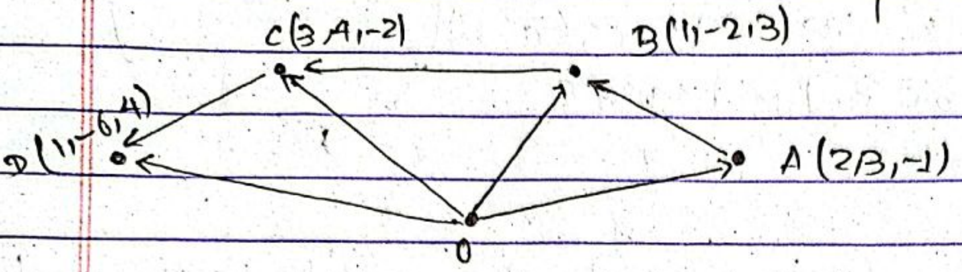
$[\vec{a}\,\vec{b}\,\vec{c}]$ if $\vec{b}$ and $\vec{c}$ are parallel

i.e. $\vec{b} = k\vec{c}$

(viii) The value of determinant is 0 if three vectors are coplanar and conversely

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ if and only if $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Example
Show that the four vectors $(2,3,-1)$, $(1,-2,3)$, $(3,4,-2)$ and $(1,-6,4)$ are coplanar.



Let O be the origin then
$\overrightarrow{OA} = (2,3,-1)$
$\overrightarrow{OB} = (1,-2,3)$
$\overrightarrow{OC} = (3,4,-2)$
$\overrightarrow{OD} = (1,-6,4)$

Now,
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= (1,-2,3) - (2,3,-1)$
$= (-1,-5,4)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$= (3,4,-2) - (1,-2,3)$
$= (2,6,-5)$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$
$= (1,-6,4) - (3,4,-2)$
$= (-2,-10,6)$

If four vector $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ are coplanar then three vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ are also coplanar.

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So, their scalar triple product is zero.

Now, $[\vec{AB}, \vec{BC}, \vec{CD}] = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 6 \end{vmatrix}$

$= -1 \begin{vmatrix} 6 & -5 \\ -10 & 6 \end{vmatrix} + 5 \begin{vmatrix} 2 & -5 \\ -2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 2 & 6 \\ -2 & -10 \end{vmatrix}$

$= 14 + 10 \cancel{+} 38$

$= -8$

So the points A, B, C and D are not coplanar.

HW
Example

1. Compute the scalar triple product

(a) $(2\vec{i} - 3\vec{j} + \vec{k}) \cdot (\vec{i} - \vec{j} + \vec{k}) \times (2\vec{i} + \vec{j} - \vec{k})$ $\boxed{\text{Ans: } -6}$

2. Show that the four points $(4,5,1)$, $(0,-1,1)$, $(3,4,9)$ & $(-4,4,4)$ lie on one plane.

3. Find the value of λ so that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} + \lambda\vec{j} + 5\vec{k}$ are coplanar.

Vector triple product

If $\vec{a}, \vec{b}, \vec{c}$ be three vectors then the product of the form

$\vec{a} \times (\vec{b} \times \vec{c})$ or $(\vec{a} \times \vec{b}) \times \vec{c}$ are called vector triple product.

Since in $\vec{a} \times (\vec{b} \times \vec{c})$

vector x vector

vector

Hence the name vector triple product is justified.

Formula: (Determination of $\vec{a} \times (\vec{b} \times \vec{c})$)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c}$$

* Example

Verify the formula

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c}$$

where

$\vec{a} = (1, 2, 0)$

$\vec{b} = (2, 3, -1)$

$\vec{c} = (0, 2, 3)$

$$RHS = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 0 & 2 & 3 \end{vmatrix}$$

$= 11\vec{i} - 6\vec{j} + 4\vec{k}$

$= (11, -6, 4)$

$L.H.S = \vec{a} \times \vec{b} \times \vec{c}$

$= (1, 2, 6) \times (11, -6, -4)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 11 & -6 & 4 \end{vmatrix}$$

$= (8, -4, -28)$

Equation of straight line in cartesian form

Slope intercept form

$$y = mx + c$$

$\Rightarrow$ $y - c = mx$

or, $\dfrac{y-c}{m} = \dfrac{x}{1}$

or, $\dfrac{x-0}{1} = \dfrac{y-c}{m}$

Two point form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$

Vector equation of straight line

Q. Find the vector equation of straight line passes through origin.

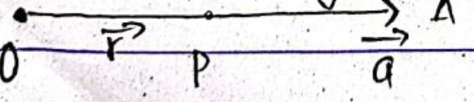
OR

Show that the vector equation of st. line with direction $\vec{a}$ and through origin is of the form

$$\vec{r} = t\vec{a}$$

$\rightarrow$ Soln:

Let O be the origin and $\vec{OA} = \vec{a}$ be the given vector



Let P be any point on $\vec{OA}$ such that

$$\vec{OP} = \vec{r}$$

Since $\vec{OP}$ and $\vec{OA}$ are collinear

$\therefore$ $\vec{OP} = t\,\vec{OA}$ for some scalar t.

or, $\vec{r} = t\vec{a}$, which is eqⁿ of st. line through origin.

Verification:

Let $\vec{r} = (x, y)$, $\vec{a} = (a_1, a_2)$.

Then $\vec{r} = t\vec{a}$ gives

$(x, y) = t(a_1, a_2)$

$(x, y) = (ta_1, ta_2)$

Equating

$x = ta_1$, $y = ta_2$

$\frac{x}{a_1} = t$, $\frac{y}{a_2} = t$

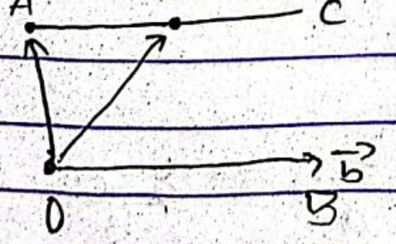
$$\boxed{\therefore \frac{x}{a_1} = \frac{y}{a_2}}$$

which is cartisian form

Form II

Vector equation of st. line passing through the point A (where $\vec{OA} = \vec{a}$) and parallel to vector $\vec{b}$ is

$\vec{r} = \vec{a} + \vec{b}$



Proof:

Let $\vec{OB} = \vec{b}$ be the given vector

Let $\vec{OA} = \vec{a}$

We find the equation of AC which is parallel to OB.

Now

From traingle law of vector addition

$\vec{OP} = \vec{OA} + \vec{AP}$

$\vec{r} = \vec{a} + \vec{AP}$ —— (1)

But $\vec{AP}$ is parallel to $\vec{OB}$

$\therefore \vec{AP} = t\,\vec{OB}$

$= t\,\vec{b}$

Hence equation (1) becomes

~~AP~~ $\boxed{\vec{r} = \vec{a} + t\vec{b}}$

Verification

Here $\vec{r} = \vec{a} + t\vec{b}$

or $(x, y) = (a_1, a_2) + t(b_1, b_2)$

or, $(x, y) - (a_1, a_2) = (tb_1, tb_2)$

or, $(x - a_1, y - a_2) = (tb_1, tb_2)$

$$\boxed{\begin{array}{l} y - y_1 = m(x - x) \\ \text{i.e. } \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} \end{array}}$$

Equating;

$x - a_1 = tb_1$ $\qquad$ $y - a_2 = tb_2$

or, $t = \dfrac{x - a_1}{b_1}$ $\qquad$ or, $t = \dfrac{y - a_2}{b_2}$

i.e. $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = t$

which is of the form $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b}$

In three dimension if
$\vec{r} = (x, y, z)$, $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then above equation becomes.

$$\boxed{\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}}$$

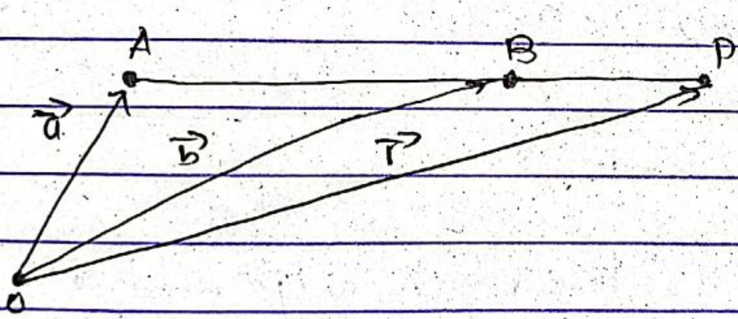
This looks like

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ as studies in our Bachelor level.

Form III

Vector equation of st. line passing through two points A and B (where $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

→ Soln;



let O be the origin. ~~AB~~ We find the vector equation of line AB
Here $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$
let P be any point on the line AB (Produced if necessary) such that $\vec{OP} = \vec{r}$
Then by traingle law of vector addition

$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\text{or, } \vec{r} = \vec{a} + \vec{AP} \quad \text{——— (i)}$$

But $\vec{AP} = t\,\vec{AB}$ [∵ $\vec{AP}$ & $\vec{AB}$ are collinear]

$$= t\,(\vec{OB} - \vec{OA})$$

$$= t\,(\vec{b} - \vec{a}) \quad \text{——— (ii)}$$

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Hence eq^n ① becomes

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$$

Verification

Here, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

let $\vec{r} = (x, y, z)$, $\vec{a}(a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

Then ① becomes

$(x, y, z) = (a_1, a_2, a_3) + t(b_1 - a_1, b_2 - a_2, b_3 - a_3)$

or, $(x - a_1, y - a_2, z - a_3) = (t(b_1 - a_1), t(b_2 - a_2), t(b_3 - a_3))$

Equating

$t(b_1 - a_1) = x - a_1$, $t(b_2 - a_2) = y - a_2$ $t(b_3 - a_3) = z - a_3$

$$\Rightarrow \frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3} = t$$

* Example

Find the equation of line which is parallel to $\vec{i} - 2\vec{j} + 3\vec{k}$ and passes through $(1, 2, 3)$

→ Soln:

Here the line passes through $A(1, 2, 3)$ i.e. $\vec{a} = (1, 2, 3)$

& parallel to $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$

The required eq^n is $\vec{r} = \vec{a} + t\vec{b}$

where, $(x, y, z) = (1, 2, 3) + t(1, -2, 3)$

or, ~~$(x, y, z) = (1 + t, 2 - 2t, 3 + 3t)$~~

or, $(x - 1, y - 2, z - 3) = (t, -2t, 3t)$

or, $x - 1 = t$, $y - 2 = -2t$, $z - 3 = 3t$

or, $x-1=t$, $\frac{y-2}{-2}=t$, $\frac{z-3}{3}=t$

$\therefore x-1=\frac{y-2}{-2}=\frac{z-3}{3}$

Example
Find the vector eqn where cartesian eqn is

$$\frac{x-2}{5}=\frac{y-6}{7}=\frac{z-1}{2}$$

→ Soln,

$$\frac{x-2}{5}=\frac{y-6}{7}=\frac{z-1}{2}=t \text{ (say)}$$

$\therefore x-2=5t$, $y-6=7t$, $z-1=2t$

or, $x=5t+2$, $y=7t+6$, $z=2t+1$

Let $\vec{r}=(x,y,z)$

Then

$$\vec{r}=(x,y,z)$$
$$=(2+5t, 6+7t, 1+2t)$$
$$=(2,6,1)+(5t, 7t, 2t)$$
$$=(2,6,1)+t(5,7,2)$$

Letting $(2,6,1)=\vec{a}$

& $(5,7,2)=\vec{b}$

Then above eqn looks like

$$\vec{r}=\vec{a}+t\vec{b}$$

~~Find the vector eqn of~~

Find by vector method, the eqn st. line in double intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

→ Proof

Let the st. line makes intercepts a and b on x-axis and y-axis at A and B.

$\therefore \vec{OA} = \vec{a} = (a, 0)$

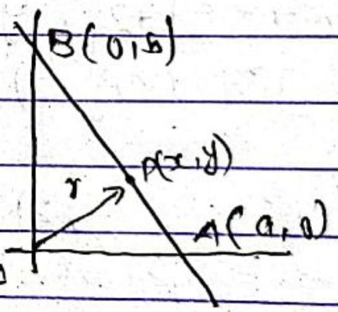
$\vec{OB} = \vec{b} = (0, b)$

Then the vectors eqn of st. line in two point form is

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

$= (a, 0) + t((0, b) - (a, 0))$

$= (a, 0) + (-at, bt)$



$\therefore (x, y) = (a - at, bt)$

$\therefore x = a - at \qquad y = tb$

$\frac{x}{a} = 1 - t$ —— (i) $\qquad t = y/b$ —— (ii)

Substituting $t = y/b$ in eqn (i) we get

$x/a = 1 - y/b$

$\frac{x}{a} + y/b = 1$

which is required eqn.

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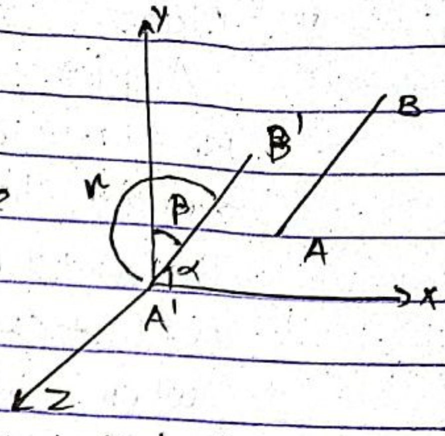
Note:

If a line AB makes angle α, β, γ with positive direction of x, y, and z axis then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosine of the line AB. It is denoted by l, m, n.

$$l^2+m^2+n^2=1$$



Direction ratios: Any three numbers that are proportional to direction cosines l, m, n are called direction ratio.

Thus if a, b, c are direction ratio, then

$$\frac{a}{l}=\frac{b}{m}=\frac{c}{n}=\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{l^2+m^2+n^2}}=\sqrt{a^2+b^2+c^2}$$

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}\ ;\quad m=\frac{b}{\sqrt{a^2+b^2+c^2}}\ ;\quad n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Note: We had

The eqn of st. line

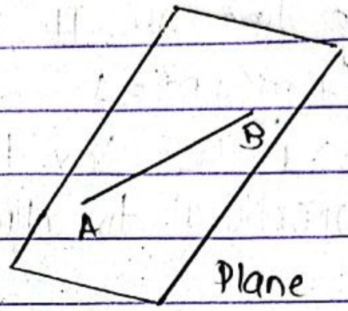
$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

where (x_1, y_1, z_1) is the point on the line & l, m, n are direction ratios of the line. (may be direction cosines)

Plane

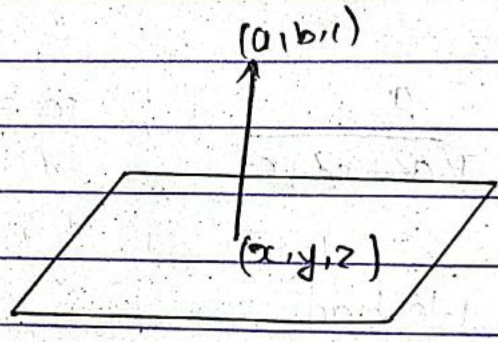
A plane is the locus of points in which if we take any two points A, B on the locus then straight line AB also lie on the locus.

The standard equation of plane is
$ax+by+cz+d=0$
where a, b, c are not all simultaneously zero.
Where a, b, c are called direction ratios of normal to the plane.



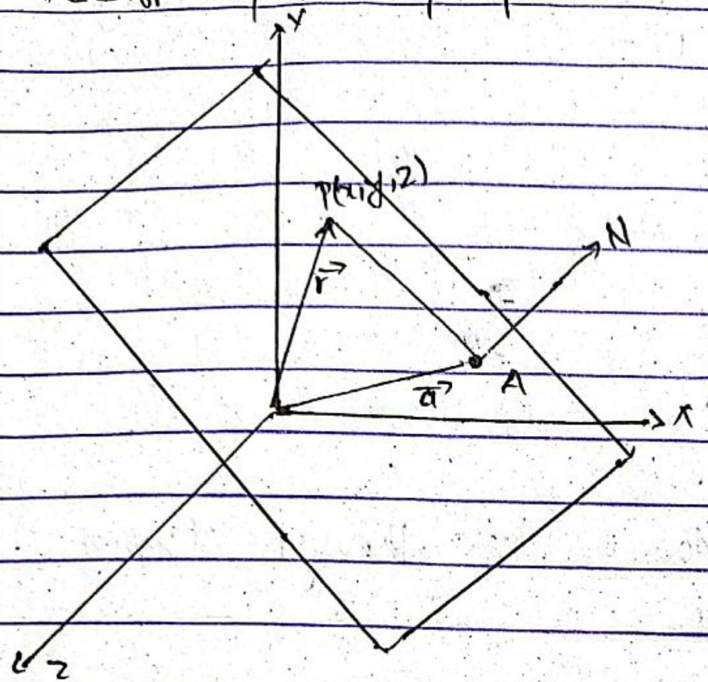
The equation of plane passing through origin is
$ax+by+cz=0$

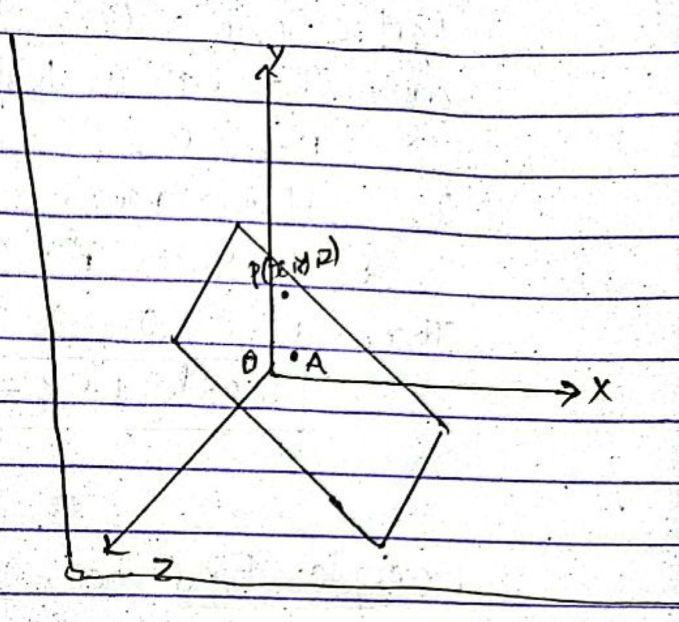
Fact: $(x,y,z).(a,b,c)=0$



Note: The direction ratio of the line joining two points $A(x_1,y_1,z_1)$ & $B(x_2,y_2,z_2)$ is $x_2-x_1, y_2-y_1, z_2-z_1$.

Vector equation of plane





To find the vector equation of plane passing through $A(x_0, y_0, z_0)$ and normal vector $\vec{x} = (a, b, c)$

→ Soln:

Let $\vec{a} = \vec{OA} = (x_0, y_0, z_0)$

let $P(x, y, z)$ be any point on plane so that

$$\vec{r} = \vec{OP} = (x, y, z)$$

so that $\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$

let $\vec{AN}$ be the normal vector to the plane such that

$$\vec{n} = \vec{AN} = (a, b, c)$$

Then $\vec{AP}$ and $\vec{AN}$ are $\perp^r$

$$\vec{AP} \cdot \vec{AN} = 0$$

or, $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

or, $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$ —— ①

which is eqn of plane.

Fact: If $\vec{a} = (0,0,0)$ then the plane passes through origin. Then eq^n (1) looks like

$$\vec{r}.\vec{n} = 0$$

Its cartisian form

$$\vec{r} = (x,y,z) \quad \vec{n} = (a,b,c)$$

Then $\vec{r}.\vec{n} = 0$ gives

$$\boxed{ax + by + cz = 0}$$

Example: Find the equation of plane through A(5,6,7) and normal vector (1,2,6)

→ Soln:

Here $\vec{OA} = \vec{a} = (5,6,7)$

Normal vector $= \vec{n} = (1,2,6)$

Let $\vec{r} = (x,y,z)$ be any point on the plane.

Then vector equation of plane is

$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

$$(x,y,z).(1,2,6) = (5,6,7).(1,2,6)$$

or, $x + 2y + 6z = 5 + 12 + 42$

or, $x + 2y + 6z = 59$

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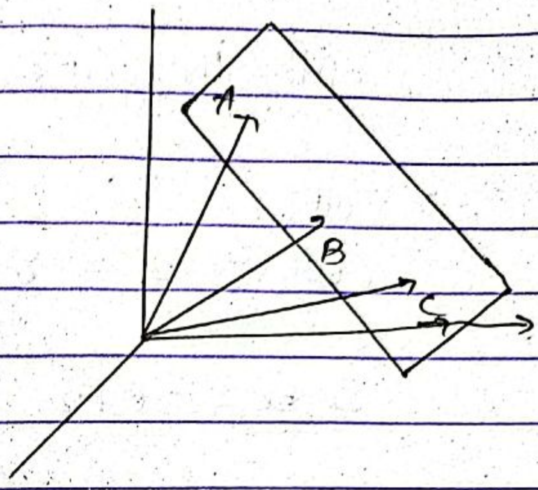
Plane through three points

Example: Find the equation of plane through three points $(3,-1,2)$, $(8,2,4)$ & $(-1,-2,-3)$



Soln:

$\cancel{\vec{AB} =}$ let O be the origin.

Now, $\vec{AB} = \vec{OB} - \vec{OA}$

$= (5, 3, 2)$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= (-9, -4, -7)$

Now

$\vec{AB} \times \vec{BC}$ is a vector perpendicular to both $\vec{AB}$ & $\vec{BC}$

$\therefore$ Normal vector to the plane

$$\vec{n} = \vec{AB} \times \vec{BC}$$

$$= \begin{vmatrix} i & j & k \\ 5 & 3 & 2 \\ -9 & -4 & -7 \end{vmatrix}$$

$$= -13i + 17j + 7j$$

$$= (-13, 17, 7)$$

Equation of plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$ where $\vec{a} = \vec{OA} = (3, -1, 2)$

$$-13x + 17y + 7z = -39 - 17 + 14$$

$$-13x + 17y + 7z = -42$$

$$13x - 17y - 7z = 42$$

is the required equation of plane passing through given points

Point of intersection of a line and plane

To find the point of intersection of line

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2} \quad \text{——— (i)}$$

with the plane $x+y-z=2$ ——— (ii)

→ Soln:

Let $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2}$ (t) say

$\therefore x = 3t+2, \; y = 4t+3, \; z = 2t+1$

For point of intersetion the point $(x,y,z) = (3t+2, 4t+3, 2t+1)$ should also lie on plane (ii)

$\therefore (3t+2) + (4t+3) - (2t+1) = 2$

or, $\cancel{3t+2}$ $5t = -2$

or, $t = -2/5$

Putting $t = -2/5$

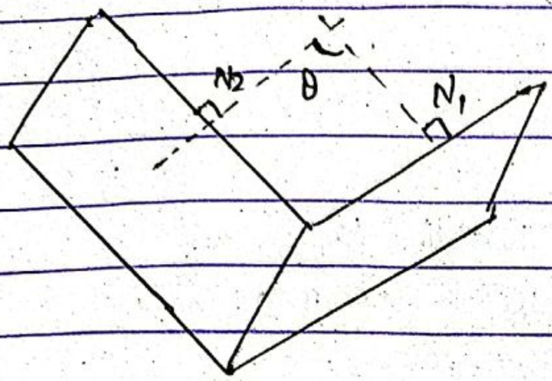
$(x,y,z) = \left(3\left(-2/5\right)+2, \; 4\left(-2/5\right)+3, \; 2\left(-2/5\right)+1\right)$

$= \left(4/5, 7/5, 1/5\right)$

$\therefore$ The required point on the plane is $\left(4/5, 7/5, 1/5\right)$

Angle between two planes

Angle between two planes is defined as the acute angle between their normals.



let $P_1 : 2x + 3y + 4z + 5 = 0$ ——— (i)

$P_2 : x + 5y + z + 1 = 0$ ——— (ii)

be two planes.

Then

Normal vectors to the plane are

$\vec{n_1} = (2, 3, 4)$

$\vec{n_2} = (1, 5, 1)$

If θ is the angle betw planes, then

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$= \frac{2 + 15 + 4}{\sqrt{29} \times \sqrt{27}}$$

$$= \frac{21}{\sqrt{29} \times \sqrt{27}}$$

$$\theta = \cos^{-1}\left(\frac{21}{\sqrt{29}\sqrt{27}}\right)$$

$$= 41.36^\circ$$

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Equation of line through the intersection of two planes:

Find the equation of the line of intersection of two planes
$$x+2y+3z=7$$
$$\& \; 3x+y-z=1$$
To find line, we need (i) Point on the line (x_1, y_1, z_1)
(ii) Direction ratio l, m, n, of the line

step I Here
$$P_1 : x+2y+3z=7 \quad \text{——— (i)}$$
$$P_2 : 3x+y-z=1 \quad \text{——— (ii)}$$

let $z=0$ be the point of intersection of (i) and (ii)
Then,
$$x+2y=7$$
$$3x+y=1$$
Solving, we get
$$y=4, \quad x=-1$$
$\therefore$ Point $(x_1, y_1, z_1) = (-1, 4, 0)$

step II To find direction ratio of required line.
Let l, m, n be the direction ratio of required line.
Since the normals to the planes are
$$\vec{n_1} = (1,2,3)$$
$$\vec{n_2} = (3,1,-1)$$
Since the line is perpendicular to the

normals to both the planes P_1 and P_2, so applying the condition of perpendicularity

$\therefore (1,2,3).(l,m,n) = 0$

& $(3,1,-1).(l,m,n) = 0$

i.e. $1l + 2m + 3n = 0$ —— (a)

$3l + m - n = 0$ —— (b)

Solving (a) and (b) for l, m, n by cross multiplication

$$\begin{array}{ccccc} & l & m & n & \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & -1 & 3 & 1 \end{array}$$

$$\frac{l}{-2-3} = \frac{m}{9+1} = \frac{n}{1-6}$$

$$\therefore \frac{l}{-5} = \frac{m}{10} = \frac{n}{-5}$$

or, $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = k$ (say)

$l = -k, \quad m = 2k, \quad n = -k$

$\therefore$ So, the required eq^n of line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

or, $\frac{x+1}{-k} = \frac{y-4}{2k} = \frac{z-0}{k}$

$\therefore \frac{x+1}{-1} = \frac{y-4}{2} = \frac{z}{-1}$ is the required eq^n of line.

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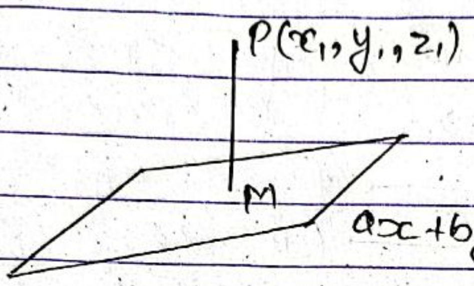
Length of $\perp^r$ from $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$



The $\perp^r$ length $PM = \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}$

Distance betw the two parallel planes.

$\dfrac{d_2 - d_1}{\sqrt{a^2+b^2+c^2}}$

Find the distance betwn parallel planes

$P_1: x + 2y + 3z = 7$ —— (i)

$P_2: x + 2y + 3z = 12$ —— (ii)

Any point on the plane (i) is

$x = 0, \; y = 0, \; z = 7/3$

$\therefore (0, 0, 7/3)$ is point on plane (i)

Then the length of $\perp^r$ from $(0, 0, 7/3)$ on plane P_2 is

$$= \left| \frac{1\times 0 + 2\times 0 + 3\times 7/3 - 12}{\sqrt{1^2+2^2+3^2}} \right|$$

$$= \left| \frac{-5}{\sqrt{14}} \right| = 5/\sqrt{14}$$

UNIT 2: Vector Function

Vector function

Vector function of scalar variable

Let t ~~$\in [a, b]$~~ be a scalar variable defined on some interval $[a, b]$ and let $\vec{r}$ is a vector depend on t where,

$$\vec{r} = (x(t), y(t))$$

then we say that $\vec{r}$ is a vector function of scalar variable t and we write $\vec{r} = \vec{r}(t)$.

Example:

Vector function of parabola

$$y^2 = 4ax \quad \text{——— (1)}$$

Here $x = at^2$ & $y = 2at$ satisfy ——— (1)

So we write

$$\vec{r} = (x(t), y(t))$$

$$\vec{r}(t) = (at^2, 2at)$$

* vector function of circle

$$x^2+y^2=a^2 \quad \text{--- } (1)$$

Here $x=a\cos t$ and $y=a\sin t$ satisfy (1)

squaring and adding

$$x^2+y^2=a^2$$

So we write

$$\vec{r}=(x(t),\, y(t))$$

$$\vec{r}(t)=(a\cos t,\, a\sin t)$$

By same argument

$$x=a\cos t,\; y=b\sin t$$

represents eqn of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

So, $\vec{r}=(a\cos t,\, b\sin t)$ is vector eqn of ellipse.

* Limit of vector function

A vector function $\vec{r}=\vec{r}(t)$ is said to have limit $\vec{L}$ as $t\to t_0$ if

$$\lim_{t\to t_0}\vec{r}(t)=\vec{L}$$

Example:

Let $\vec{r}(t)=(at^2,\, 2at)$

Then $\lim_{t\to 0}\vec{r}(t)$

$$=\lim_{t\to 0}(at^2,\, 2at)$$

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Page

$$= \left(\lim_{t\to 0} at^2,\ \lim_{t\to 0} at\right)$$
$$= (0, 0)$$
$$= \vec{0}$$

Continuity

A vector function $\vec{r} = \vec{r}(t)$ is said to be continuo at point $t = t_0$ if

$$\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$$

i.e. limiting value = functional value.

Derivative of a vector function

let $\vec{r} = \vec{r}(t)$ be a vector function of scalar variable t. Then the derivative of $\vec{r}$ at point $t = t_0$ is denoted by $\vec{r}'(t)$ or $\frac{d\vec{r}}{d(t)}$ and is defined by

$$\vec{r}'(t_0) = \lim_{t\to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} \quad \text{——} (1)$$

provided limit exists.

letting $t - t_0 = \delta t$, $t = t_0 + \delta t$

Also if $t \to t_0$, $\delta t \to 0$

Hence eqn (1) becomes

$$r'(t_0) = \lim_{\delta t\to 0} \frac{\vec{r}(t_0 - \delta t) - \vec{r}(t_0)}{\delta t}$$

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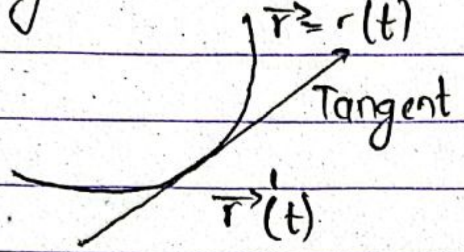
Geometrically, the derivative of the vector function $\vec{r} = \vec{r}(t)$ at $t = t_0$ represents the ~~slope~~ points of ~~tangent at~~ point ~~$t = t_0$~~

tangent. (when t increasing in direction of tangent)



Example

Let $\vec{r} = t^2\vec{i} - t\vec{j} + t\vec{k}$ —— ①

Find $\dfrac{d\vec{r}}{dt} \cdot \dfrac{d^2\vec{r}}{dt^2}$

→ Soln;

$$\frac{d\vec{r}}{dt} = 2t\vec{i} - \vec{j} + \vec{k}$$

& $$\frac{d^2\vec{r}}{dt^2}$$

$$= 2\vec{i} - 0\vec{j} + 0\vec{k}$$

$$\therefore \frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} = 2t \cdot 2 + 0 + 0 = 4t$$

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Some formulae on derivative

If $\vec{r}$, $\vec{r_1}$ & $\vec{r_2}$ be vector functions of scalar variable t.
let ϕ be the scalar function of t.
Then,

(i) $$\frac{d}{dt}(\vec{r_1} \pm \vec{r_2}) = \frac{d\vec{r_1}}{dt} \pm \frac{d\vec{r_2}}{dt}$$

(ii) $$\frac{d}{dt}\vec{a} = \vec{0}$$, where $\vec{a}$ is constant vector

(iii) $$\frac{d}{dt}(\phi\vec{r}) = \phi\frac{d\vec{r}}{dt} + r\frac{d\phi}{dt}\vec{r}$$

In particular if ϕ = constant = k (say)

$$\frac{d}{dt}(k\vec{r}) = k\left(\frac{d\vec{r}}{dt}\right)$$

(iv) $$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\cdot\frac{d\vec{r_2}}{dt} + \cancel{\vec{r_2}}\,\frac{d\vec{r_1}}{dt}\cdot\vec{r_2}$$

(derivative of dot product)

(v) $$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = r_1\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$$

(Derivative of cross product)

(vi) Derivative of scalar triple product

$$\frac{d}{dt}[\vec{r_1}\,\vec{r_2}\,\vec{r_3}] = \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \left[\vec{r_1}\,\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right] + \left[\vec{r_1}\,\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

(viii) Derivative of vector triple product

$$\frac{d}{dt}\left[\vec{r_1}\times(\vec{r_2}\times\vec{r_3})\right] = \left[\frac{d\vec{r_1}}{dt}\times(\vec{r_2}\times\vec{r_3})\right] + \left[\vec{r_1}\times\left(\frac{d\vec{r_2}}{dt}\times\vec{r_3}\right)\right]$$

$$+ \vec{r_1}\times\left(\vec{r_2}\times\frac{d\vec{r_3}}{dt}\right)$$

Q. Show that

$$\frac{d}{dt}[\vec{r_1}\,\vec{r_2}\,\vec{r_3}] = \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \left[\vec{r_1}\,\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right] + \left[\vec{r_1}\,\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

$$\text{L.H.S} = \frac{d}{dt}\left[\vec{r_1}\cdot(\vec{r_2}\times\vec{r_3})\right]$$

$$= \frac{d\vec{r_1}}{dt}\cdot(\vec{r_2}\times\vec{r_3}) + \vec{r_1}\cdot\frac{d}{dt}(\vec{r_2}\times\vec{r_3})$$

$$= \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \vec{r_1}\cdot\left[\frac{d\vec{r_2}}{dt}\,\vec{r_3} + \vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

$$= \frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3} + \vec{r_1}\cdot\left(\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right) + \vec{r_1}\cdot\left(\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right)$$

$$= \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \left[\vec{r_1}\,\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right] + \left[\vec{r_1}\,\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

Q.2. Show that (derivative of vector triple product)

$$\frac{d}{dt}[\vec{r_1} \times (\vec{r_2} \times \vec{r_3})] = \frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3}) + \vec{r_1} \times \frac{d}{dt}(\vec{r_2} \times \vec{r_3})$$

$$= \left(\frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3})\right) + \left(\vec{r_1} \times \left(\vec{r_2} \times \frac{d\vec{r_3}}{dt}\right)\right) + \left(\vec{r_1} \times \left(\frac{d\vec{r_2}}{dt} \times \vec{r_3}\right)\right)$$

$$= \left[\frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3})\right] + \left[\vec{r_1} \times \left(\frac{d\vec{r_2}}{dt} \times \vec{r_3}\right)\right] + \left[\vec{r_1} \times \left(\vec{r_2} \times \frac{d\vec{r_3}}{dt}\right)\right]$$

Vector with constant magnitude

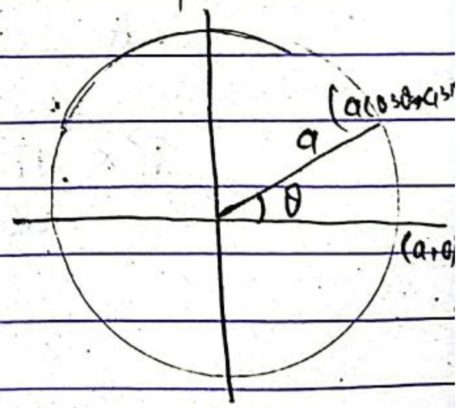
A vector ~~with~~ function $\vec{r}$ of scalar variable t is said to be constant magnitude if $|\vec{r}|$ is constant for all t.

In particular,

$$\vec{r} = (a\cos t, a\sin t)$$

Here, $|\vec{r}| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = a$

This represents circle with radius a.



Note:

A vector function $\vec{r}$ of scalar variable t has constant magnitude if $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$.

Vector with constant direction.

A vector function $\vec{r}$ of scalar variable t is said to be constant direction if $\vec{r}$ is collinear for all t.

In particular,

$$\vec{r} = (2t, 3t, 4t)$$

We get different point on line so $\vec{r}$ has constant direction.

Note: A vector function $\vec{r}$ of scalar variable t has constant direction if $\vec{r} \times \frac{d\vec{r}}{dt} = 0$

* Example:

$$\vec{r} = (2t, 3t, 4t)$$

$$\frac{d\vec{r}}{dt} = (2, 3, 4)$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 3t & 4t \\ 2 & 3 & 4 \end{vmatrix}$$

$$= t \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \cancel{t\vec{0}} \quad t\,\vec{0}$$

$$= \vec{0}$$

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Example

If $\vec{r}$ is a unit vector, prove that

$$\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$$

→ Soln:

Here $\vec{r}$ is a unit vector. So, $\vec{r}$ has constant magnitude.

$$\therefore \vec{r} \cdot \frac{d\vec{r}}{dt} = 0$$

This shows that $\vec{r}$ and $\frac{d\vec{r}}{dt}$ are perpendicular (i.e. have angle $90°$)

$$\frac{|a \times b|}{|\vec{a}||\vec{b}|} = \sin\theta$$

Now,

$$\text{L.H.S.} = \left|\vec{r} \times \frac{d\vec{r}}{dt}\right|$$

$$= |\vec{r}| \left|\frac{d\vec{r}}{dt}\right| \sin 90°$$

$$= 1 \left|\frac{d\vec{r}}{dt}\right| 1$$

$$= \left|\frac{d\vec{r}}{dt}\right|$$

Example

If $\vec{r} = \vec{a}e^{mt} + \vec{b}e^{nt}$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

Prove that

$$\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r} = \vec{0}$$

→ Soln;

$$\frac{d\vec{r}}{dt} = \vec{a}me^{mt} + \vec{b}ne^{nt}$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}m^2e^{mt} + \vec{b}n^2e^{nt}$$

$$\text{L.H.S.} = \frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r}$$

$$= \vec{a}m^2e^{mt} - (m+n)(\vec{a}me^{mt} + \vec{b}ne^{nt}) + mn(\vec{a}e^{mt} + \vec{b}e^{nt})$$

$$= 0$$

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Example:

If $\vec{r} = (a\cos t, a\sin t, 0)$

Find $\cancel{\frac{d}{dt}[\vec{r_1}\ \vec{r_2}\ \vec{r_3}]}$ $\left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right]$

→ Soln;

$\vec{r} = (a\cos t, a\sin t, 0)$

$\frac{d\vec{r}}{dt} = (-a\sin t, a\cos t, 0)$

$\frac{d^2\vec{r}}{dt^2} = (-a\cos t, -a\sin t, 0)$

Now,

$$\left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right] = \begin{vmatrix} a\cos t & a\sin t & 0 \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= 0 \quad [\because C_3 = 0]$$

Evaluate:

$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$, where $r = |\vec{r}|$

$\frac{d}{dt}(\phi\vec{r}) = \phi\frac{d\vec{r}}{dt} + \vec{r}\frac{d\phi}{dt}$

Soln

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{d}{dt}\left(\frac{1}{r}\vec{r}\right)$$

$$= \frac{1}{r}\frac{d\vec{r}}{dt} + \vec{r}\frac{d}{dt}\left(\frac{1}{r}\right)$$

$$= \frac{1}{r}\frac{d\vec{r}}{dt} + \vec{r}\left(-\frac{1}{r^2}\right)\frac{dr}{dt}$$

$$= \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}$$

Formula:

If $\vec{r}$ be the position vector of a particle at time t.

i.e. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Then, $\frac{d\vec{r}}{dt}$ & $\frac{d^2\vec{r}}{dt}$ always represents velocity & acceleration of moving particle at time t.

Example:

Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at time $t = \pi/2$. Find also their magnitude.

→ Soln:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= (\cancel{2\sin} \; 2\sin 3t\,\vec{i} + 2\cos 3t\,\vec{j} + 8t\,\vec{k})$$

$$\frac{d\vec{r}}{dt} = (6\cos 3t\,\vec{i} - 6\sin 3t\,\vec{j} + 8\vec{k})$$

At $t = \pi/2$

$$\frac{dr}{dt} = 6\cos 3\pi/2\,\vec{i} - 6\sin 3\pi/2\,\vec{j} + 8\vec{k}$$

$$= 0 - 6(-1)\vec{j} + 8\vec{k}$$
$$= 6\vec{j} + 8\vec{k}$$

$$\therefore \left|\frac{d\vec{r}}{dt}\right| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

Again diff. ① w.r.t. t (For accelration vector)

$$\frac{d^2\vec{r}}{dt^2} = -18\sin 3t\,\vec{i} - 18\cos 3t\,\vec{j} + 0\vec{k}$$

At $\pi/2$

$$\left(\frac{d^2\vec{r}}{dt^2}\right) = -18(-1)\vec{i} - 18\cdot 0\vec{j} + 0\vec{k}$$
$$= 18\vec{i}$$

$$\therefore \left|\frac{d^2\vec{r}}{dt^2}\right| = |18\vec{i}| = \sqrt{18^2 + 0^2 + 0^2}$$
$$= 18$$

CW

* If $\frac{d\vec{a}}{dt} = \vec{c}\times\vec{a}$, $\frac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$

Show that $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$

$$\text{L.H.S.} = \frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$
$$= \vec{a}\times(\vec{c}\times\vec{b}) + (\vec{c}\times\vec{a})\times\vec{b}$$
$$= (\vec{a}\cdot\vec{b})\vec{c} - \cancel{(\vec{a}\cdot\vec{c})\vec{b}} + \cancel{(\vec{c}\cdot\vec{b})}\ \cancel{(\vec{c}\cdot\vec{a})\vec{b}}$$
$$= (\vec{a}\cdot\vec{b})\vec{c} + (\vec{c}\cdot\vec{b})\vec{a}$$
$$= (\vec{a}\cdot\vec{b})\vec{c} -$$

$= (\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b} - \vec{b}\times(\vec{c}\times\vec{a})$

$= \cancel{(\vec{a}\cdot\vec{b})\vec{c}} - (\vec{a}\cdot\vec{c})\vec{b} - \cancel{(\vec{b}\cdot\vec{a})\vec{c}} + (\vec{b}\cdot\vec{c})\vec{a}$

$= (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

$= \vec{c}\times(\vec{a}\times\vec{b})$

$=$ R.H.S

Proved

Integration of vector function

We know, if $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$,

i.e. derivative of $\frac{d\vec{r}}{dt}$ w.r.t. t is $\frac{d^2\vec{r}}{dt^2}$

① Then we say that antiderivative of $\frac{d^2\vec{r}}{dt^2}$ is $\frac{d\vec{r}}{dt}$ and we write

$$\int \frac{d^2\vec{r}}{dt^2}\, dt = \frac{d\vec{r}}{dt} + c$$

② Similarly,

$$\cancel{\frac{d}{dr}(\vec{r_1}\cdot\vec{r_2})}$$

$$\frac{d}{dr}(\vec{r_1}\cdot\vec{r_2}) = \frac{d\vec{r_1}}{dt}\cdot\vec{r_2} + \vec{r_1}\frac{d\vec{r_2}}{dt}$$

Then,

$$\int\left(\frac{d\vec{r_1}}{dt}\cdot\vec{r_2} + \vec{r_1}\cdot\frac{d\vec{r_2}}{dt}\right)dt = \vec{r_1}\cdot\vec{r_2} + c$$

③ $$\frac{d}{dr}(\vec{r_1}\times\vec{r_2}) = \frac{d\vec{r_1}}{dt}\times\vec{r_2} + r_2\times\frac{d\vec{r_2}}{dt}$$

Then

$$\int\left(\frac{d\vec{r_1}}{dt}\times\vec{r_2} + \vec{r_1}\frac{d\vec{r_2}}{dt}\right)dt = \vec{r_1}\times\vec{r_2} + \vec{c}$$

If $\vec{r_1} = \vec{a}$, a constant vector

& $\vec{r_2} = \vec{r}$ so that $\frac{d\vec{r_1}}{dt} = \vec{0}$

Then the formula becomes

$$\int \vec{a}\times\frac{d\vec{r}}{dt} = (\vec{a}\times\vec{r}) + c$$

Q. If $\vec{r_1} = 2\vec{i} + t\vec{j} - \vec{k}$, $\vec{r_3} = 2\vec{i} - 3\vec{j} + 4\vec{k}$
$\vec{r_2} = t\vec{i} + 2\vec{j} + 3\vec{k}$

Find (a) $\int_0^2 (\vec{r_1} \times \vec{r_2})\, dt$

(b) $\int_0^2 [\vec{r_1}\ \vec{r_2}\ \vec{r_3}]\, dt$

→ Soln:

(a) $\vec{r_1} \times \vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & t & -1 \\ t & 2 & 3 \end{vmatrix}$

$= \vec{i}(3t+2) - \vec{j}(6+t) + \vec{k}(4-t^2)$

$\therefore \int_0^2 [\vec{r_1} \times \vec{r_2}]\, dt = \int_0^2 [i(3t+2) - \vec{j}(6+t) + k(4-t^2)]\, dt$

$= \left[\vec{i}\left(\frac{3t^2}{2}+2t\right) - \vec{j}\left(6t + \frac{t^2}{2}\right) + \vec{k}\left(4t - \frac{t^3}{3}\right)\right]_0^2$

$= \left[\vec{i}\left(\frac{3\times4}{2} + 2\times2\right) - \vec{j}\left(12 - \frac{2^2}{4}\right) + \vec{k}\left(8 - 8/3\right)\right] -$
$[\vec{i}\,0 - \vec{j}\,0 + \vec{k}\,0]$

$= 10\vec{i} - 14\vec{j} + 16/3\,\vec{k}$

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(5) $[\vec{r_1}\ \vec{r_2}\ \vec{r_3}] = \begin{vmatrix} 2 & t & -1 \\ t & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix}$

$= 34 + \cancel{} t(4t-6) \cancel{} -1(-3t-4)$

$= -4t^2 + 9t + 38$

$\therefore \int_0^2 (-4t^2 + 9t + 38)\,dt = \left[-\frac{4t^3}{3} + \frac{9t^2}{2} + 38t \right]_0^2$

$= 94 - 32/3$

$= 250/3$

Some Def^n

We know, if $\vec{r} = \vec{r}(t) = r_1\vec{i} + r_2\vec{j} + r_3\vec{k}$ where r_1, r_2, r_3 are scalar function of t.

Then $\frac{d\vec{r}}{dt} = \vec{r}'(t)$ alway represents the vector along the tangent.

Then the unit vector along the tangent is denoted by $\vec{T}(t)$ and is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

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Curvature

If $\vec{r} = \vec{r}(t)$ be a continuous curve,
Then the curvature measures how quickly the curve changes its direction of tangent.
It measures the bendness of the curve.

Formula:
The curvature of a curve is defined as the rate of change of unit tangent vector w.r.t arc length.
It is denoted by $\kappa(t)$.

Thus, $\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right|$

The radius of curvature is the reciprocal of curvature.

Some formula:
If $\vec{r} = \vec{r}(t)$ be ~~a~~ the vector function;
Then, the curvature $\kappa(t)$ is defined by

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Example

Find the curvature of the curve

$\vec{r}(t) = (t^2, t, 2t^2)$ at point ~~(1,2,0)~~ $(1,1,2)$

→ Soln:

$\vec{r}(t) = (t^2, t, 2t^2)$

$\vec{r}'(t) = (2t, 1, 4t)$

$\vec{r}''(t) = (2, 0, 4)$

Now,

$$|\vec{r}'(t) \times r''(t)| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 1 & 4t \\ 2 & 0 & 4 \end{vmatrix}$$

$$= \vec{i}(4-0) - \vec{j}(8t - 8t) + \vec{k}(0-2)$$

$$= 4\vec{i} - 0\vec{j} - 2\vec{k}$$

$$= (4, 0, -2)$$

$$\therefore \kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$= \frac{|(4,0,-2)|}{|2t, 1, 4t|^3}$$

$$= \frac{\sqrt{4^2 + 0^2 + (-2)^2}}{\left(\sqrt{4t^2 + 1 + (4t)^2}\right)^3}$$

$$= \frac{\sqrt{20}}{(21)^{3/2}}$$

At $t = 1$

$$\kappa(1) = \frac{\sqrt{20}}{(21)^{3/2}}$$

Radius of curvature = Reciprocal of curvature

$$= \frac{(21)^{3/2}}{7\sqrt{20}}$$

Example: Find curvature and radius of curvature

$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + 0\,\vec{k}$ at $t=0$

Ans $K(0) = 1/a$, and radius of curvature $= a$

Radius of Curvature formula

1. In cartesian form (x, y) form

If $y = f(x)$ be a function then the radius of curvature is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where } y_1 = \frac{dy}{dx},\ y_2 = \frac{d^2y}{dx^2},\ y_2 \neq 0$$

Similarly if $x = f(y)$ be the curve then

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \quad \text{where } x_1 = \frac{dx}{dy},\ x_2 = \frac{d^2x}{dy^2},\ x_2 \neq 0$$

$\cosh x = \frac{e^x + e^{-x}}{2}$ $\cosh^2 x - \sinh^2 x = 1$

$\sinh x = \frac{e^x - e^{-x}}{2}$

Exercise

③ Find the radius of curvature and curvature k at point (x,y) where $y = c\cosh\left(\frac{x}{c}\right)$

→ Soln:

$y = c\cosh\left(x/c\right)$

Diff. w.r.t. x

$y_1 = c\sinh\frac{x}{c} \cdot \frac{1}{c} = \sinh\frac{x}{c}$

$y_2 = \cosh\frac{x}{c} \cdot \frac{1}{c} = \frac{1}{c}\cosh\frac{x}{c}$

Now,

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{(1+(\sinh x/c)^2)^{3/2}}{1/c \cosh x/c}$

$= \frac{(\cosh^2 x/c)^{3/2}}{1/c \cosh x/c}$

$= \frac{(\cosh x/c)^3}{1/c \cosh x/c}$

$= \cosh^2 x/c \cdot c$

$= (y/c)^2 \cdot c$

$= y^2/c$

$\nabla \cdot \vec{J} \rightarrow$ divergence
$\nabla \times \vec{V} \rightarrow$ curl
$\nabla \phi \rightarrow$ gradient

Vector calculus

~~Diver~~ Vector differential operator:

The vector differential operator denoted by ∇ (nebla) and is given by

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

i.e. $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

1. ~~Dive~~ Gradient of scalar point function.

Let $\phi(x,y,z)$ be a scalar point function. Then its gradient is denoted by $\nabla\phi$ and is defined by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial x} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= \sum \vec{i}\frac{\partial \phi}{\partial x}$$

Example: Let $\phi = x^2+y^2+z^2$. Find $\nabla\phi$.

→ Soln;

Here, $\nabla\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$
$$+ \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

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$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$
$= 2(x\vec{i} + y\vec{j} + z\vec{k})$

Note: $\nabla\phi$ is a vector.

Divergent of vector point function
Let $\vec{V} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a vector point function. of $x, y, \& z$. Then its divergent is given by

$$\nabla\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{V}$$

$$= \vec{i}\cdot\frac{\partial\vec{V}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{V}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{V}}{\partial z}$$

$$= \sum\vec{i}\cdot\frac{\partial\vec{V}}{\partial x}$$

OR

$$\nabla\cdot\vec{V} = \vec{i}\frac{\partial v_1}{\partial x} + \vec{j}\frac{\partial v_2}{\partial y} + \vec{k}\frac{\partial v_3}{\partial z}$$

Eg: Let $\vec{V} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\nabla\cdot\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(x^2\vec{i} + y^2\vec{j} + z^2\vec{k})$$

$$= \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}$$

$$= 2x + 2y + 2z$$

Example:

let $\vec{v} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$

Find $\nabla\vec{v}$

$$\nabla\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(z^2x)$$

$$= 2xy + 2yz + 2zx$$

3. Curl of a vector point function

Let $v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a vector point function of x, y & z. Then its curl is denoted by $\nabla\times\vec{v}$ is given by

$$\nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{v}$$

$$= \vec{i}\times\frac{\partial\vec{v}}{\partial x} + \vec{j}\times\frac{\partial\vec{v}}{\partial y} + \vec{k}\,\frac{\partial\vec{v}}{\partial z}$$

$$= \sum\vec{i}\times\frac{\partial\vec{v}}{\partial x}$$

Calculation of $\nabla\times\vec{v}$

$$\nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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$$= \vec{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \vec{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \vec{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Example:

Let $\vec{V} = x^3 y \vec{i} + y^3 z \vec{j} + z^3 x \vec{k}$. find $\nabla \times \vec{V}$

→ Soln:

$$\nabla \times \vec{V} = \vec{i}\left(\frac{\partial z^3 x}{\partial y} - \frac{\partial y^3 z}{\partial z}\right) + \vec{j}\left(\frac{\partial x^3 y}{\partial z} - \frac{\partial z^3 x}{\partial x}\right)$$

$$+ \vec{k}\left(\frac{\partial y^3 z}{\partial x} - \frac{\partial x^3 y}{\partial y}\right)$$

$$= \vec{i}(0 - y^3) + \vec{j}(-z^3) + \vec{k}(0 - x^3)$$

$$= -\vec{i} y^3 - \vec{j} z^3 - \vec{k} x^3$$

$$= -y^3 \vec{i} - z^3 \vec{j} - x^3 \vec{k}$$

$$= -(y^3 \vec{i} + z^3 \vec{j} + x^3 \vec{k})$$

Example:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Then

(i) $\nabla \cdot \vec{r} = 3$ (iii) $\frac{\partial r}{\partial x} = \frac{x}{r}$

(ii) $\nabla \times \vec{r} = 0$

⇒ Soln;

$$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{\partial (x)}{\partial x} + \frac{\partial (y)}{\partial y} + \frac{\partial (z)}{\partial z}$$

$$= 1 + 1 + 1$$

$$= 3$$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial z \\ x & y & z \end{vmatrix}$

$= \vec{i}\,(0-0) + \vec{j}\,(0-0) + \vec{k}\,(0-0)$

$= \vec{0}$

(iii) $r = |\vec{r}| = |x\vec{i} + y\vec{j} + z\vec{k}|$

$= \sqrt{x^2+y^2+z^2}$

$\therefore \dfrac{\partial r}{\partial x} = \dfrac{\partial (x^2+y^2+z^2)^{1/2}}{\partial x}$

$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \times 2x$

$= \dfrac{x}{\sqrt{x^2+y^2+z^2}}$

$= \dfrac{x}{r}$

Simillory, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Example: If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Find

(i) $\nabla.(r^3\vec{r})$

(ii) $Div\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$

→ Soln;

We know, $\frac{\partial r}{\partial x} = \frac{x}{r}$

i) $\nabla.(r^3\vec{r}) = \sum \vec{i}.\frac{\partial(r^3\vec{r})}{\partial x}$ [$\because \nabla.\vec{v} = \sum \vec{i}.\frac{\partial v}{\partial x}$ by defination of divergence.

$= \sum \vec{i}.\left[r^3\frac{\partial \vec{r}}{\partial x} + \vec{r}\frac{\partial r^3}{\partial x}\right]$

$= \sum \vec{i}.\left[r^3\vec{i} + \vec{r}\left(3r^2\frac{\partial r}{\partial x}\right)\right]$

$= \sum \vec{i}.\left(r^3\vec{i} + \vec{r}\,3r^2\frac{x}{r}\right)$

$= \sum \vec{i}.\left(r^3\vec{i} + \vec{r}\,3xr\right)$

$= \sum r^3(\vec{i}.\vec{i}) + 3\sum \vec{i}.(\vec{r}\,3xr)$

$= \sum r^3(1) + 3\sum xr(\vec{i}.\vec{r})$

$= 3r^3 + 3\sum xrx$ $[\because \vec{i}.\vec{r} = x]$

$= 3r^3 + 3\sum x^2r$

$= 3r^3 + \cancel{3\sum x^2}\ 3r\sum x^2$ $\left[\sum x^2 = x^2+y^2+z^2 = r^2\right.$

$= 3r^3 + 3r.r^2$

$= 3r^3 + 3r^3$

$= 6r^3$

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$= r^{3/2} + \frac{3x}{2} r^{1/2}\, 2x + \frac{r^3 + 3}{2}\, y r^{1/2}\, 2y + \frac{r^3+3}{2}\, z r^{1/2}\, 2z$

$= r^{3/2} + 3x^2 r^{1/2} + 3y^2 r^{1/2} + 3z^2 r^{1/2}$

$= r^{3/2} + 3r^{1/2}(x^2 + y^2 + z^2)$

$= r^{3/2} + 3r^{1/2} . r^2$

$= r^{3/2} + 3r^{5/2}$

$= r^{3/2}(1 + 3r)$

$\Sigma x^2 = x^2 + y^2 + z^2$
$= r^2$

$= 3r^3 + 3r . r^2$

$= 3r^3 + 3r^3$

$= 6r^3$

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$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$= \Sigma \vec{i}\frac{\partial}{\partial x}$

4) ∇ is vector differential operator

$\nabla\phi = \text{grad}\ \phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$= \Sigma \vec{i}\frac{\partial\phi}{\partial x}$ vector

$\nabla . \vec{v} = \text{Div}\ \vec{v} = \Sigma \frac{\partial v_1}{\partial x}$

scalar

$\nabla \times \vec{v} = \text{Curl}\ \vec{v} = \Sigma \vec{i} \times \frac{\partial \vec{v}}{\partial x}$

Solenoidal and irrotational vector

A vector function $\vec{v}$ is said to be

i) solenoidal vector if $\nabla \cdot \vec{v} = 0$

i.e. $\text{div } \vec{v} = 0$

(Having no source and sink i.e. divergence free vector)

ii) irrotational vector if $\nabla \times \vec{v}$

i.e. $\text{curl } \vec{v} = \vec{0}$

(Fluid mechanics, no rotational fluid)

Free from rotational fluid

Fact: If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

then, $\nabla \times \vec{r} = \vec{0}$

$\therefore$ $\vec{r}$ is irrotational

EXAMPLE

Q. Find the values of constants p, q and r so that the vector $\vec{F} = (x + 2y + pz)\vec{i} + (qx - 3y - z)\vec{j} + (4x + ry + 2z)$ is irrotational.

Solution

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+pz & qx-3y-z & 4x+ry+2z \end{vmatrix}$$

$\Rightarrow \nabla \times \vec{F} = \Big| \vec{i} \qquad \Big|$

$\Rightarrow 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{i}(r+1) - \vec{j}(4-p) + \vec{k}(q-2)$

Thus,

$r+1=0$; $4-p=0$; $0=q-2$

$\therefore r=-1$; $\therefore p=4$; $\therefore q=+2$

$\therefore$ Required values of p, q, r are $4, +2, -1$ respectively.

EXAMPLE: Find the value of μ so that the vector $\vec{V}=(x+3y)\vec{i}+(y-2z)\vec{j}+(x+\mu z)\vec{k}$ is solenoidal.

Solution

$$\nabla\cdot\vec{V}=\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+\mu z)$$

$$\Rightarrow 0=1+1+\mu$$

$$\mu=-2$$

Laplacian Operator

The operator $\nabla=\nabla\cdot\nabla=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)$

$$=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

is called Laplacian operator

$\vec{k}$

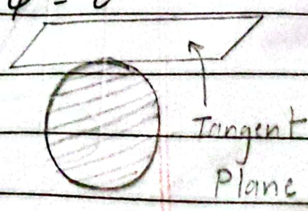
$$\nabla\cdot\nabla=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

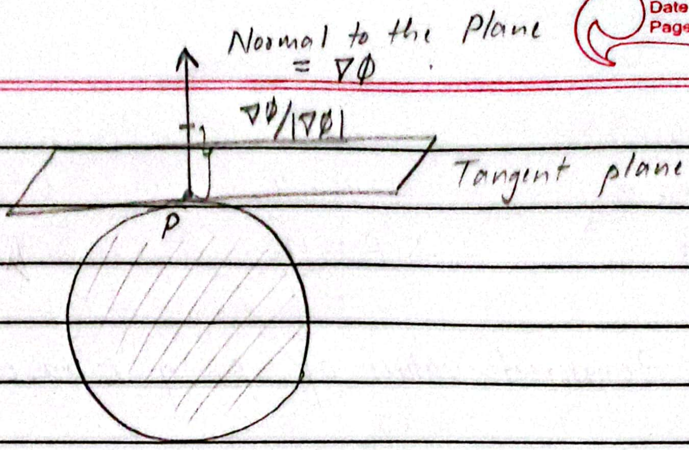
If ϕ is twice differentiable function, then $\nabla^2\phi=0$ is called Laplacian equation.

Note: If ϕ is a surface, i.e. $\phi(x,y,z)=0$, then $\nabla\phi$ i.e. gradient ϕ always represents the vector normal to the surface.

$\phi=x^2+y^2+z^2-a^2$

$\phi=0$





EXAMPLE

Q. Find the unit vector normal to the surface $z = x^2 + y^2$ at point $(-1, -2, 5)$.

Solution

Here,

$$\phi_{(x,y,z)} = x^2 + y^2 - z$$

Now,

$$\nabla \phi = \vec{i}\,\frac{\partial}{\partial x}(x^2+y^2-z) + \vec{j}\,\frac{\partial}{\partial y}(x^2+y^2-z) + \vec{k}\,\frac{\partial}{\partial z}(x^2+$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$\nabla\phi$ represents the vector normal to surface ϕ.

At point $(-1, -2, 5)$,

$$\nabla\phi = 2*(-1)\,\vec{i} + 2*-2\,\vec{j} - \vec{k}$$

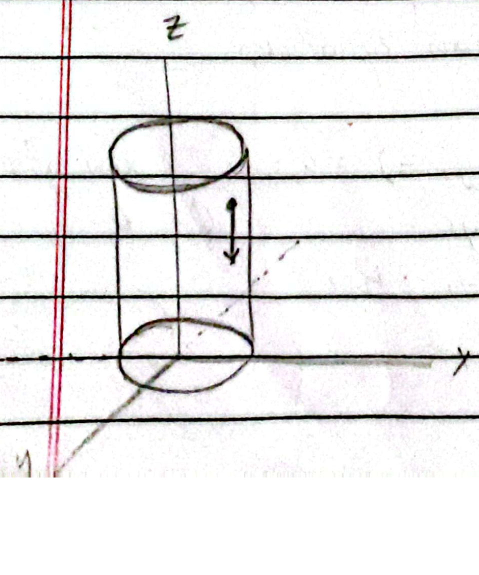
$$= -2\vec{i} - 4\vec{j} - \vec{k}$$

Now, unit vector normal to surface, $= \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{(-2, -4, -1)}{\sqrt{(-2)^2 + (-4)^2 + (-1)^2}}$$

$$= \frac{(-2, -4, -1)}{\sqrt{21}}$$

$$= \left(\frac{-2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}\right.$$

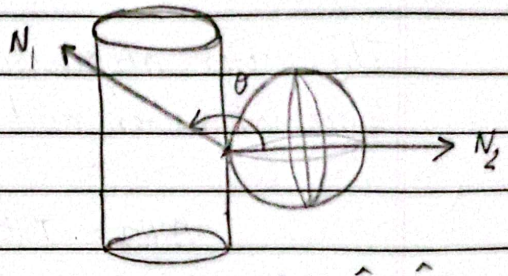


C/W Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at point $(2,-1,2)$.

Solution

Hint: $\phi_1 = x^2+y^2+z^2-9$

$\phi_2 = z - x^2 - y^2 + 3$



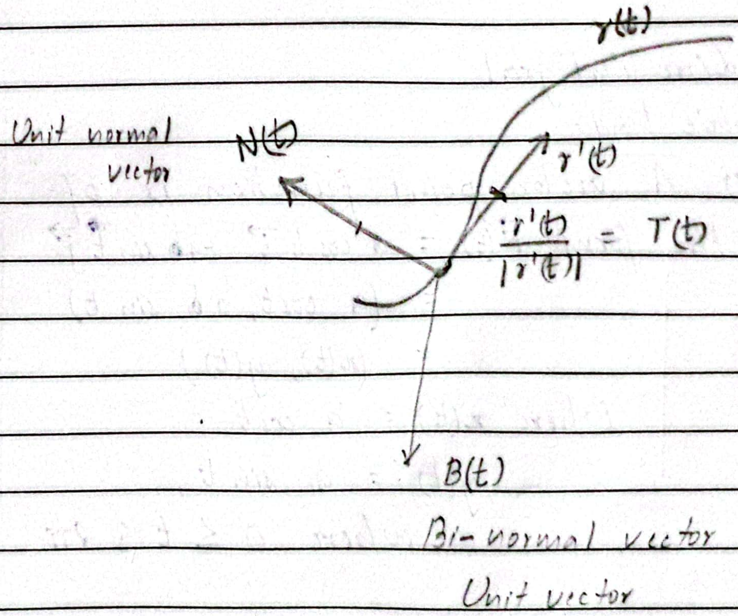
Find $\nabla\phi_1$ and $\nabla\phi_2$

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$

If θ is the angle between the normal vectors to the surface, then $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$

$\left[\text{Ans: } \cos\theta = \dfrac{-8}{3\sqrt{21}}\right]$

$y^2-z)$



Normal and bi-normal vector

For a smooth curve $\vec{r} = \vec{r}(t)$, there are many vectors orthogonal to unit tangent vector $\vec{T}(t) = \dfrac{r'(t)}{|r'(t)|}$

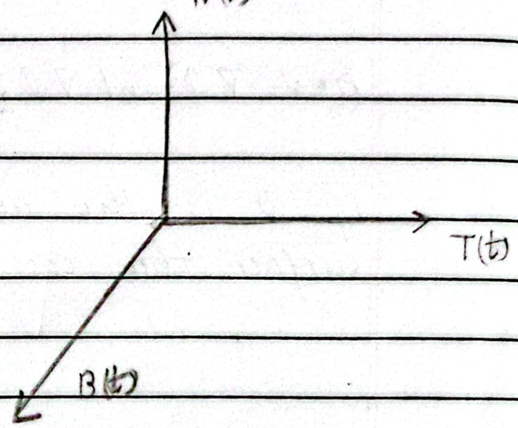
where, $|T(t)| = 1$.

The principal unit normal vector $N(t)$ is

$$N(t) = \frac{T'(t)}{|T'(t)|} \qquad \text{why } T(t) \cdot T'(t) = 0$$

The vector perpendicular to both $T(t)$ and $N(t)$ is called binormal vector and we write $B(t)$.

$$B(t) = T(t) \times N(t)$$



Unit 5 VECTOR CALCULUS

* Line integral

We had:

i) A vector point function is of the form $\vec{F}(t) = a \cos t \, \vec{i} + a\cancel{b} \sin t \, \vec{j}$

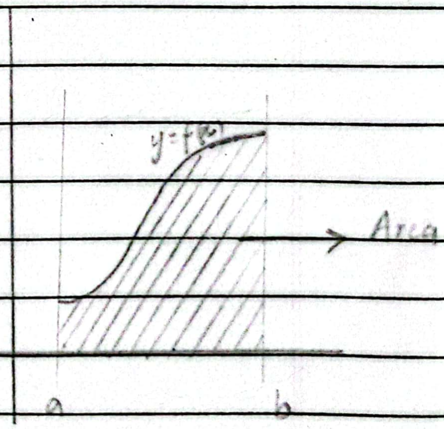
$= (a \cos t, a\cancel{b} \sin t)$

$= (x(t), y(t))$

where $x(t) = a \cos t$

$y(t) = a \sin t$,

where $0 \le t \le 2\pi$

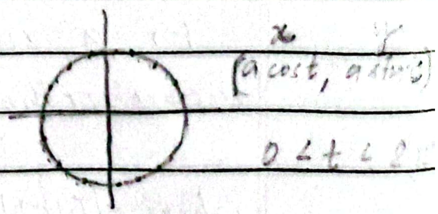


$$\int_a^b f(x) \, dx$$

More general, we write

$\vec{F}(t) = F_1(t) \, \vec{i} + F_2(t) \, \vec{j} + F_3(t) \, \vec{k}$

where $a \le t \le b$



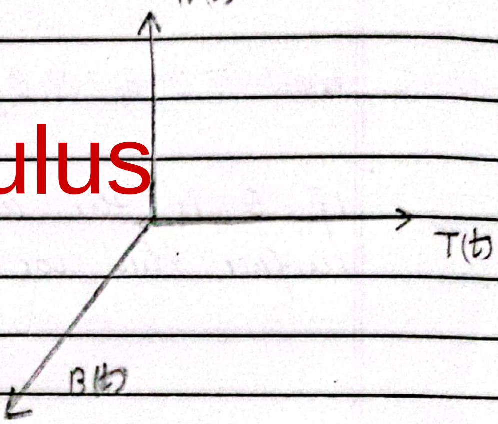
If $\vec{r} = (x, y, z)$ be a point on (1), then the

The principal unit normal vector $N(t)$ is

$N(t) = \dfrac{T'(t)}{|T'(t)|}$ why $T(t) \cdot T'(t) = 0$

The vector perpendicular to both $T(t)$ and $N(t)$ is called binormal vector and we write $B(t)$.

$$B(t) = T(t) \times N(t)$$



UNIT 5: Vector Calculus

Unit 5 VECTOR CALCULUS

* Line integral

We had:

i) A vector point function is of the form $\vec{F}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

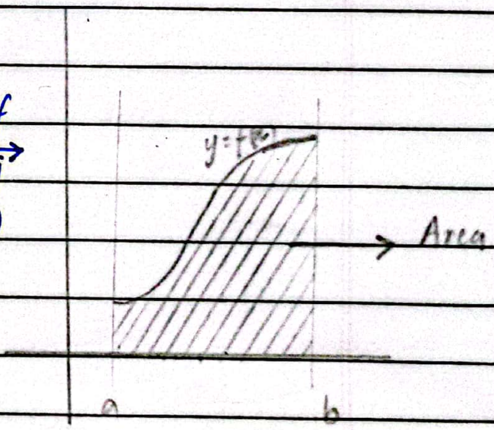
$= (a\cos t, a\sin t)$

$= (x(t), y(t))$

where $x(t) = a\cos t$.

$y(t) = a\sin t$,

where $0 \le t \le 2\pi$

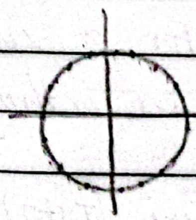


$\int_a^b f(x)\,dx$

More general, we write

$\vec{F}(t) = F_1(t)\,\vec{i} + F_2(t)\,\vec{j} + F_3(t)\,\vec{k}$

where $a < t < b$



$x \quad y$

$(a\cos t, a\sin t)$

$0 < t < 2\pi$

If $\vec{r} = (x, y, z)$ be a point on ①, then the

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$\vec{F} \cdot \vec{r}$ gives scalar.

Definition (Line integral)

Any integral which is evaluated along curve (or line) is called line integral

If $\vec{F}(t)$ be a vector point function, then the line integral of the function $\vec{F}(t)$ along the curve C is given by

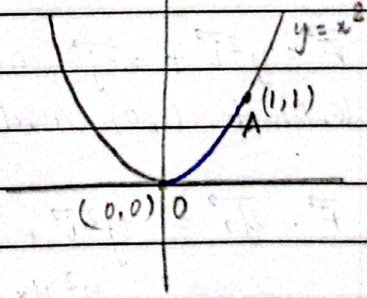
$$\int_C \vec{F} \cdot \vec{dr} \quad \text{where } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$$

$\boxed{\text{EXAMPLE}}$ Evaluate the line integral

$\int_C \vec{F} \cdot \vec{dr}$ where $\vec{F} = x^2\vec{i} + y^2\vec{j}$

& c is the arc of the parabola $y = x^2$ from (0, 0) to (1, 1)

Here,



$\vec{F} = x^2\vec{i} + y^2\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$\therefore d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$

Now,

$$\vec{F} \cdot \vec{dr} = (x^2\vec{i} + y^2\vec{j}) \cdot (\vec{i}\,dx + \vec{j}\,dy)$$
$$= x^2\,dx + y^2\,dy$$

$$\int_C \vec{F} \cdot \vec{dr} = \int_C (x^2\,dx + y^2\,dy)$$
$$= \int_{OA} (x^2\,dx + y^2\,dy) \quad \text{———} \; (i)$$

← change it to single variable

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Here, curve $y = x^2$

$\therefore dy = 2x\,dx$

At O, $x = 0$

and at A, $x = 1$

$$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_0^1 x^2 dx + (x^2)^2 \cdot 2x\,dx$$

$$= \int_0^1 x^2 dx + 2x^5\,dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^6}{6}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3}$$

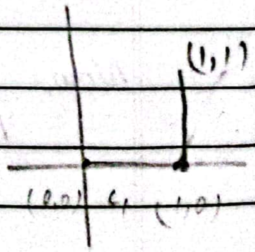
$$= \frac{2}{3}$$

C/W Evaluate the line integral $\int_C \vec{F}\cdot\vec{dr}$ where,

$\vec{F} = y^2\vec{i} + x^2\vec{j}$ and c is a straight line from (0, 0) to (1, 0) and from (1, 0) to (1, 1)

$$\vec{F}\cdot\vec{dr} = (y^2\vec{i} + x^2\vec{j})\cdot(\vec{i}dx + \vec{j}dy)$$

$$= y^2 dx + x^2 dy$$



Now, $$\int_C \vec{F}\cdot\vec{dr} = \int_{c_1} \vec{F}\cdot\vec{dr} + \int_{c_2} \vec{F}\cdot\vec{dr}$$

$y = 0$

$dy = 0$

$x = 1$

$dx = 0$

$$= \int_{c_1} y^2 dx + x^2 dy + \int_{c_2} y^2 dx + x^2 dy$$

$$= 0 + \int_0^1 1^2\, dy$$

$$= [y]_0^1$$

$$= (1-0)$$

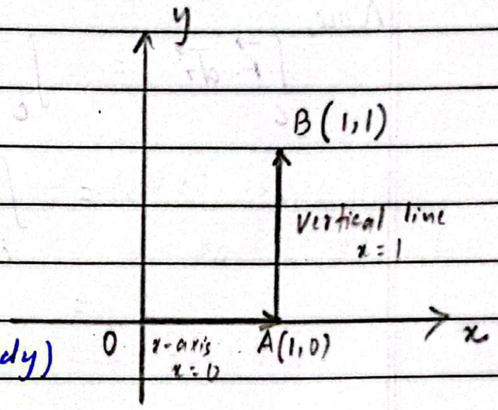
$$= 1$$

Line integral

$$= \int_C \vec{F} \cdot \vec{dr}$$

$$= \int_{OA} \vec{F} \cdot \vec{dr} + \int_{AB} \vec{F} \cdot \vec{dr}$$

$$= \int_{OA} (y^2 dx + x^2 dy) + \int_{AB} (y^2 dx + x^2 dy)$$



Here OA is $y = 0$

$dy = 0$

At O, $x = 0$

A, $x = 1$

$$= \int_0^1 (0\, dx + x^2 . 0)$$

Here AB is $x = 1$

$dx = 0$

At A, $y = 0$

B, $y = 1$

$$= \int_0^1 0^2 dx + x^2 dy$$

$$= \int_0^1 (1)^2 dy$$

$$= [y]_0^1$$

$$= (1-0)$$

$$= 1$$

$\therefore I = 0 + 1 = 1$

H/W In the above problem, find $\int_C \vec{F} \cdot \vec{dr}$ where C is

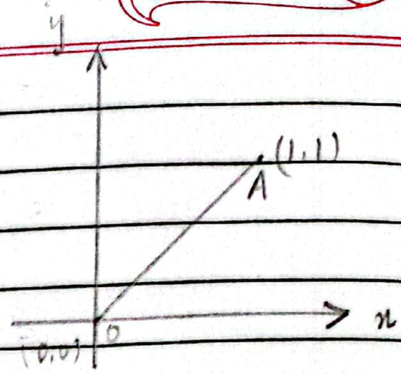
i) the straight line joining (0,0) and (1,1).

~~ii)~~

Hint:

Equation of OA is

$$y-0 = \frac{1-0}{1-0}(x-0)$$



$$\therefore y = x$$

$$\Rightarrow dy = dx$$

Now,

$$\int_C \vec{F}\cdot\vec{dr} = \int_C y^2\,dx + x^2\,dy$$

$$= \int_C x^2\,dx + x^2\,dx$$

$$= 2\int_0^1 x^2\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

$$= 2\times\frac{1^3}{3}$$

$$= 2/3$$

Evaluate the line integral $\int_C \vec{F}\cdot\vec{dr}$ from $(0,0,0)$ to $(1,1,1)$ where $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ and the path c is space curve $x=t$, $y=t^2$, $z=t^3$.

$$\vec{F}\cdot\vec{dr} = \left((3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}\right)\cdot(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \cancel{3x^3+6xy}$$

$$= (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

Now, $\int_C \vec{F}\cdot\vec{dr} =$

Now, $x = t$

$\therefore \cancel{y = x^2}$

$\therefore \cancel{z =} x^3$

$\therefore dx = dt$

$y = t^2$

$\therefore dy = 2t\,dt$

$z = t^3$

$\therefore dz = 3t^2\,dt$

$t = 0$ to 1

Now

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=1} (3t^2 + 6t^2)\,dt - 14t^2 \cdot t^3\, 2t\,dt + 20\,t\,t^6\, 3t^2\,dt$$

$$= \int_{t=0}^{t=1} 9t^2\,dt - 28t^6\,dt + 60\,t^9\,dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$$

$$= 3 \times (1)^3 - 4 * (1)^7 + 6 * (1)^{10}$$

$$= 3 - 4 + 6$$

$$= \boxed{5}$$

Q Ex ~~Find~~ In the above problem find $\int_C \vec{F} \cdot \vec{dr}$ where C is the straight line joining $(0,0,0)$ to $(2,4,8)$

~~Here, $t = 0$ to $t = 2$~~

~~$\therefore \int_C \vec{F} \cdot \vec{dr}$~~

Here line joining $(0,0,0)$ and $(2,4,8)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

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$$\Rightarrow \frac{x-0}{2} = \frac{y-0}{4} = \frac{z-0}{8}$$

$$\therefore \frac{x}{1} = \frac{y}{2} = \frac{z}{4} = t$$

$$\therefore x = t, \quad y = 2t, \quad z = 4t$$
$$dx = dt, \quad dy = 2dt, \quad dz = 4dt$$
$$\therefore t = 0 \text{ to } t = 2$$

Now,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^2 (3t^2 + 12t)\cdot dt - 14\cdot 2t\cdot 4t \cdot 2dt + 20t(4t)^2\, 4dt$$

$$= \int_0^2 (3t^2 + 12t)\,dt - 224\,t^2\,dt + 1280t^3\,dt$$

$$= \left[\frac{3t^3}{3} + \frac{12t^2}{2} - 224\frac{t^3}{3} + \frac{1280t^4}{4}\right]_0^2$$

$$= \left[8 + 6\times 4 - 224\times\frac{8}{3} + 320\times 16\right]$$

$$= \cancel{\boxed{4256}}\ 5752 - 448/3$$

$$= \frac{15008}{3}$$

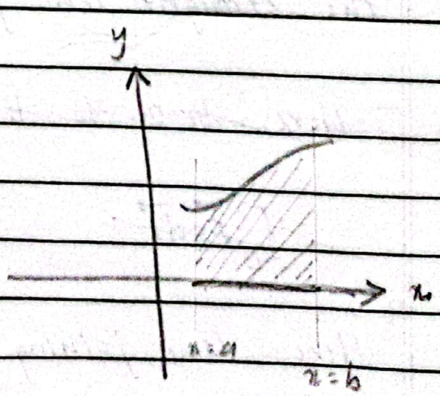
$$= \underline{5002.667}$$

$$= 32 + 5720 - 597.33$$

$$= \underline{4522.67}$$

$\boxed{\text{Sep 30, 2024}}$

$$\int_{x=a}^{x=b} f(x)\,dx$$



Line integral: $\int_C \vec{F}\cdot\vec{dr}$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

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Work done by a force

If a force $\vec{F}$ acts along on a particle which moves along the curve C, then,

The total work done by $\vec{F}$ = $\int_C \vec{F} \cdot \vec{dr}$

Problem

Find the work done by a moving particle around the circle $x^2+y^2=4$, $z=0$ if the force is $\vec{F} = (2x-y+2z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-5z)\vec{k}$

Solution

Here, the curve is $x^2+y^2=4$, $z=0$

Here, $x = 2\cos t$ and $y = 2\sin t$ satisfies the above equation.

$x^2+y^2=2^2$
$x = 2\cos t$
$y = 2\sin t$

$\therefore dx = -2\sin t\, dt$, $dy = 2\cos t\, dt$

Now,

$\vec{F} \cdot \vec{dr} = ((2x-y+2z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-5z)\vec{k}) \cdot (\vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz)$

$= (2x-y+2z)dx + (x+y-z)dy + (3x-2y-5z)dz$

$\therefore$ Total Work done $= \int_C \vec{F} \cdot \vec{dr}$

$= \int_C (2x-y+2z)dx + (x+y-z)dy + (3x-2y-5z)dz$

$= \int_C (4\cos t - 2\sin t)\, dx + (2\cos t + 2\sin t)\, dy$

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$x = 2\cos t$

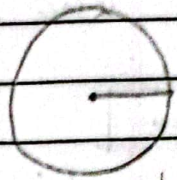
$\therefore dx = -2\sin t \, dt$

$y = 2\sin t$

$\therefore dy = 2\cos t \, dt$

$z = 0$

$\therefore dz = 0$



$t = 0$ to 2π

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C \{(4\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)\,2\cos t\}\,dt$$

$$= \int_0^{2\pi} (-8\cos t\sin t + 4\sin^2 t + 4\cos^2 t + 4\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (4 - 4\sin t\cos t)\,dt$$

$$= \oint^{2\pi} [4t]_0^{2\pi} + 4[\cos t]_0^{2\pi}$$

$$= 8\pi + 4 \times (-2)$$

$$= 8\pi - 8$$

$$= 8(\pi - 1)$$

$$= \int_0^{2\pi} 4 - 2\sin 2t \; dt$$

$$= 4[t]_0^{2\pi} + \left[\frac{2\cos 2t}{2}\right]_0^{2\pi}$$

$= 8\pi + (\cos 4\pi - \cos 0)$
$= 8\pi + (1-1)$
$= 8\pi$

We had: A vector function $\vec{F}$ was irrotational if $\text{curl } \vec{F} = 0$.

Example: Show that the vector $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$.

Also find the scalar function ϕ such that $\vec{F} = \nabla \phi$.

Solution

$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$= \cancel{\vec{i}(-x) \ \vec{j}(}$
$= \vec{i}(-x + x) - \vec{j}(-y + y) + \vec{k}(-z + z)$

$= \vec{0}$

$\nabla \phi = \vec{i}\dfrac{\partial \phi}{\partial x} + \vec{j}\dfrac{\partial \phi}{\partial y} + \vec{k}\dfrac{\partial \phi}{\partial z}$

$(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$

$\Rightarrow \dfrac{\partial \phi}{\partial x} = x^2 - yz \Rightarrow \partial \phi = (x^2 - yz)\partial x$

$\therefore \phi_1 = x^3/3 - xyz + c_1$

$y^2 - zx = \dfrac{\partial \phi}{\partial y}$

$\therefore \phi_2 = \dfrac{y^3}{3} - xyz + c_2$

$\therefore \phi_3 = \dfrac{z^3}{3} - xyz + c_3$

$\therefore \phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz + c$

LI - Exam (Sure)

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For 2^{nd} part,

$$\vec{F} = \nabla \phi \quad \text{———} \quad (1)$$

But, $\nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$

Hence eq^n (i) becomes,

$$(x^2 - yz)\vec{i} + (y^2 - zx)\vec{k} + (z^2 - xy)\vec{k}$$

$$= \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

Equating the co-efficient of $\vec{i}, \vec{j}, \vec{k}$.

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= x^2 - yz \\ \frac{\partial \phi}{\partial y} &= y^2 - zx \\ \frac{\partial \phi}{\partial z} &= z^2 - xy \end{aligned}\right\} \begin{aligned} &\Rightarrow \phi = \frac{x^3}{3} - xyz + c_1 \\ &\Rightarrow \phi = \frac{y^3}{3} - xyz + c_2 \\ &\Rightarrow \phi = \frac{z^3}{3} - xyz + c_3 \end{aligned}$$

Here, xyz is common to all. So, the solution is

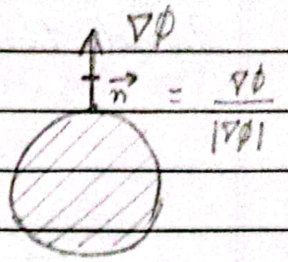
$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + C$$

Surface Integral

Any integral which is taken along the surface is called surface integral.

NOTE: If ϕ is a surface then the vector $\nabla \phi$ is a

vector normal to the surface.
So the unit vector normal to the surface $\vec{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$



Formula: (Surface Integral)

If S be the two sided surface. Taking ds as the area of small element in the surface and $\vec{n}$ is the unit vector normal to the surface, element ds in the surface. Then the surface integral of a vector function $\vec{F}$ ~~ove~~ over the surface S is given by.

$$= \iint_S \vec{F} \cdot d\vec{S}$$

where $d\vec{S} = \vec{n}\, ds$

$\therefore$ Surface integral $= \displaystyle\iint_S \vec{F} \cdot \vec{n}\, dS$

where $\vec{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$; ϕ is given in your problem

& $dS = \dfrac{dx\,dy}{|\vec{n} \cdot \vec{k}|}$ or $\dfrac{dy\,dz}{|\vec{n} \cdot \vec{i}|}$ or $\dfrac{dx\,dz}{|\vec{n} \cdot \vec{j}|}$

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Q. EXAMPLE

Find the surface integral

$$\iint_S \vec{F} \cdot \vec{n}\, ds$$

where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2 k$ and surface S is in the finite plane $x+y+z=1$ between co-ordinate axes.

Solution.

Now the surface integral $= \iint_S \vec{F} \cdot \vec{n}\, ds$ —— (i)

Also, $\phi = (x+y+z-1)$

~~Now~~

$\therefore$ ~~$\nabla\phi$~~ $=$

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}\phi + \vec{j}\frac{\partial}{\partial y}\phi + \vec{k}\frac{\partial}{\partial z}\phi$$

$$= \vec{i}\frac{\partial}{\partial x}(x+y+z-1) + \vec{j}\frac{\partial}{\partial y}(x+y+z-1) + \vec{k}\frac{\partial}{\partial z}(x+y+z-1)$$

$$= \vec{i} + \vec{j} + \vec{k}$$

$$\therefore \vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{(1)^2+(1)^2+(1)^2}}$$

$$= \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$$

$1/4 - 1/3 + 1/12$

$= \frac{3-4+1}{12}$

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Also,

$$ds = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \frac{dx\,dy}{|1/\sqrt{3}(\vec{i}+\vec{j}+\vec{k})\cdot\vec{k}|}$$

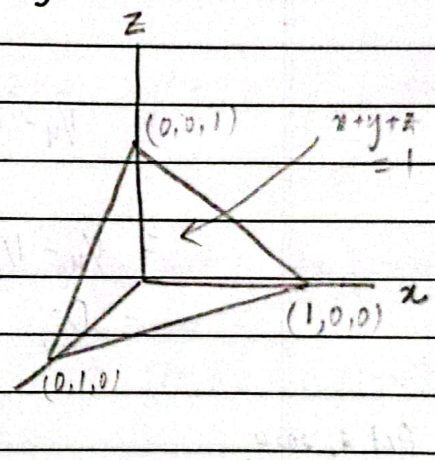
$$= \frac{dx\,dy}{1/\sqrt{3}}$$

$$= \sqrt{3}\,dx\,dy$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,ds = \iint_S (x^2\vec{i} + y^2\vec{j} + z^2k)\cdot\frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k})\sqrt{3}\,dx\,dy$$

Here $x+y+z=1$

$$= \iint (x^2 + y^2 + z^2)\,dx\,dy$$

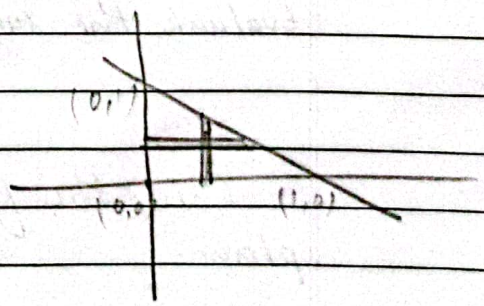


$z = 1-x-y$

$$= \int_0^1 \int_{x=0}^{x=(1-y)} (x^2+y^2+z^2)\,dx\,dy$$

$\left[\frac{x^3}{3} + y^2x\right]_0^{1-y}$

$$= \int_0^1\int_0^{1-y} (x^2+y^2)\,dx\,dy$$



$\frac{(1-y)^3}{3} + y^2(1-y)$

$$= \int_0^1 \left[x^3/3 + y^2x\right]_0^{1-y} dy$$

$\frac{(1-y)^3}{3} + y^3 - y^2$

$$= \int_0^1 \frac{(1-y)^3}{3} + y^2(1-y)\,dy$$

$\frac{(1-y)^4}{-12} + \frac{y^4}{4} - \frac{y^3}{3}$

$$= \int_0^1 \frac{(1-y)^3}{3} + y^3 - y^2\,dy$$

$\frac{1}{12} + 1/4 - 1/3$

$$= \left[\frac{(1-y)^4}{} + y^4/4 - y^3/3\right]_0^1 = 1/4 - 1/3 + 1/12$$

$$= \frac{3-4+1}{12}$$

$$= 0$$

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1/4

$$\int_0^1 \int_0^{1-y} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_0^{1-y} dy$$

$$= \int_0^1 \frac{(1-y)^3}{3} + y^2(1-y)\ dy$$

$$= \int_0^1 \frac{(1-y)^3}{3} + y^3 - y^2\ dy$$

$$= \left(\frac{(1-y)^4}{(-12)} + \frac{y^4}{4} - \frac{y^3}{3} \right]_0^1$$

$$= 1/4 - 1/3 - \left(-1/12 + 0 - 0 \right)$$

$$= 1/4 - 1/3 + 1/12$$

$$= 0.$$

Oct 2, 2024

Evaluate the surface integral $\iint_S \vec{F} \cdot \vec{n}\, ds$

$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$

S is the finite plane $x + y + z = 1$ between co-ordinate plane.

But, $z = 1 - x - y$

$$I = \iint_S \left\{ x^2 + y^2 + (1-x-y)^2 \right\} dx\, dy$$

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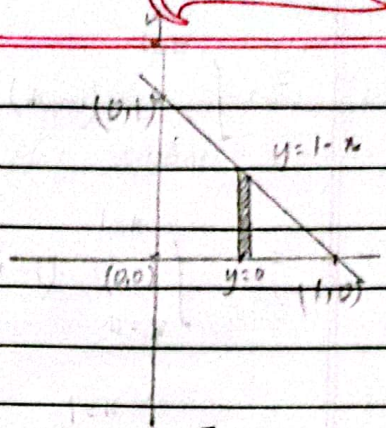
Now, the limit is $0 < y < 1-x$
$0 < x < 1$



$$I = \int_0^1 \left[\int_{y=0}^{y=1-x} x^2 + y^2 + (1-x-y)^2 \, dy\right] dx$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=1-x} x^2 + y^2 + 1 + x^2 + y^2 - 2x - 2y - 2xy \, dy\right] dx$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=1-x} 2x^2 + 2y^2 - 2xy - 2x - 2y + 1\right] dy \, dx$$

$$= \int_{x=0}^{x=1} \left[2x^2y + 2y^3/_3 - xy^2 - 2xy - y^2 + y\right]_0^{1-x} dx$$

$$= \int_{x=0}^{x=1} 2x^2(1-x) + \frac{2(1-x)^3}{3} - x(1-x)^2 - 2x(1-x) - (1-x)^2 + (1-x) \, dx$$

$$\cancel{= \int_{x=0}^{x=1} (1-x)\left\{2x^2 + \frac{2}{3}\,1-x\right.}$$

$$\cancel{= \int_{x=0}^{x=1} 2x^2 - 2x^3 + \frac{2(1-x)(1-x+x^2)}{3}}$$

$$= \int_{x=0}^{x=1} (1-x)\left\{2x^2 + \frac{2}{3}(1-x)^2 - x(1-x) - 2x - (1-x) + 1\right\} dx$$

$$= \int_{x=0}^{x=1} (1-x)\left\{2x^2 + \frac{2(1-2x+x^2)}{3} - x + x^2 - 2x - \cancel{1} + x + \cancel{1}\right) dx$$

$$= \int_{x=0}^{x=1} (1-x) \left\{ \frac{6x^2 + 2 - 4x + 2x^2}{3} - 2x + x^2 \right\} dx$$

$$= \int_{x=0}^{x=1} (1-x) \left\{ 3x^2 - 2x + \frac{2}{3}(1-x)^2 \right\}$$

$$= \int_{x=0}^{x=1} (1-x) \left\{ \frac{9x^2 - 6x + 2 - 4x + 2x^2}{3} \right\}$$

$$= \int_{x=0}^{x=1} (1-x) \left\{ \frac{11x^2 - 10x + 2}{3} \right\}$$

$$= \frac{1}{3} \int_{x=0}^{x=1} 11x^2 - 10x + 2 - 11x^3 + 10x^2 - 2x \; dx$$

$$= \frac{1}{3} \int_{x=0}^{x=1} 21x^2 - 12x - 11x^3 + 2 \quad dx$$

$$= \frac{1}{3} \left[7x^3 - 6x^2 - \frac{11x^4}{4} + 2x \right]_0^1$$

$$= \frac{1}{3} \left(7 - 6 - \frac{11}{4} + 2 \right)$$

$$= \frac{1}{3} \left(3 - \frac{11}{4} \right)$$

$$= \frac{1}{3} \left(\frac{44 - 11}{4} \right) \qquad = \frac{1}{3} \times \frac{12-11}{4}$$

$$= \cancel{\frac{1}{3} \times \frac{33}{4}} \qquad = \frac{1}{3} \times \frac{1}{4}$$

$$= \cancel{\frac{11}{4}}$$

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Q. Find the surface integral (flux) of the vector function $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ through the sphere $x^2+y^2+z^2=1$ in the first octant.

Sol^n.

$$\phi = x^2+y^2+z^2-1$$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-1)$$

$$= 2(ix + jy + kz)$$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2(ix+jy+kz)}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}$$

$$= \frac{ix+jy+kz}{\sqrt{x^2+y^2+z^2}}$$

$$= ix + jy + zk$$

Now, $\vec{n}\cdot\vec{k} = (ix+jy+zk)\cdot k$

$$= z$$

Now, Surface integral $= \iint_S \vec{F}\cdot\vec{n}\,dS$

$$= \iint_S (yzi + zxj + xyk)\cdot(xi+yj+zk)\frac{dx\,dy}{|z|}$$

$$= \iint_S (xyz + xyz + xyz)\frac{dx\,dy}{z}$$

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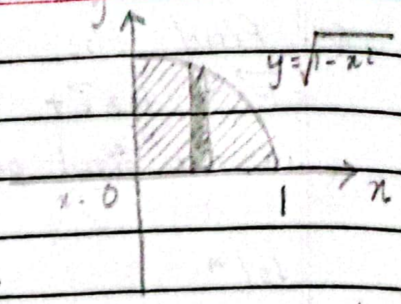
$$= \iint_R 3xyz \cdot \frac{1}{2} \, dx \, dy$$

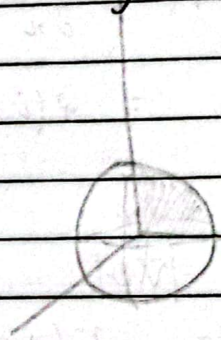


$$= \iint 3xy \, dx \, dy$$

Now the range is $0<x<1$ $\quad x^2+y^2=1 \quad (\because z=0)$

$0<y<\sqrt{1-x^2}$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} 3xy \, dy \, dx$$



$$= 3 \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx$$

$$= 3 \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2} \int_0^1 x(1-x^2) \, dx$$

$$= \frac{3}{2} \int_0^1 (x - x^3) \, dx$$

$$= \frac{3}{2} \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{3}{2} \times \frac{1}{4}$$

$$= \boxed{3/8}$$

Integral Transformation (Relation between line Integral and Surface Integral)

Green's Theorem in XY plane.

Statement: If S be a closed region in XY plane by simple closed curve C and let $\vec{F}(x, y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$ then, Green theorem state that

$$\underbrace{\int_C (F_1\,dx + F_2\,dy)}_{\text{Line integral}} = \underbrace{\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy}_{\text{Surface integral}}$$

$\int_C \vec{F}\cdot d\vec{r}$

$\vec{F} = F_1\vec{i} + F_2\vec{j}$

NOTE: If $\vec{F} = F_1\vec{i} + F_2\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F}\cdot d\vec{r} = F_1\,dx + F_2\,dy$

Q. Verify the Green theorem in the plane

$$\int_C (xy + y^2)\,dx + x^2\,dy$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$

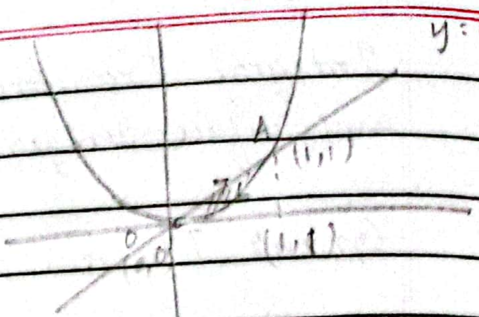
Here,

$F_1 = xy + y^2$

$\therefore \frac{\partial F_1}{\partial y} = x + 2y$

$F_2 = x^2$

$\therefore \frac{\partial F_2}{\partial x} = 2x$



$$\int_C F_1\,dx + F_2\,dy$$

$$= \int_C (xy+y^2)\,dx + x^2\,dy$$

$$= \int_{\text{line } OA} \underbrace{(xy+y^2)\,dx + x^2\,dy}_{y=x} + \int_{\substack{\text{curve } AO \\ y=x^2}} \underbrace{(xy+y^2)\,dx + x^2\,dy}_{y=x^2}$$

$$= \int (x^2+x^2)\,dx + x^2\,dx + \int (x^3+x^4)\,dx + x^2\cdot 2x\,dx$$

=

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{y=x^2}^{y=x} (2x - x - 2y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 [xy - y^2]_{x^2}^{x}\,dx$$

$$= \int_0^1 (x^2 - x^2 - x^3 + x^4)\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}$$

$$= \int_0^1 \int_{\sqrt{x}}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 [xy - y^2]_{\sqrt{x}}^{x}\,dx$$

$$= \int_0^1 (x^2 - x^2 - x\sqrt{x} + x)\,dx$$

$$= \int_0^1 (x - x^{3/2})\,dx$$

$$= \left[\frac{x^2}{2} - \frac{2}{5}x^{5/2}\right]_0^1$$

$$= \frac{1}{2} - \frac{2}{5}\times 1$$

$$= \frac{1}{2} - \frac{2}{5} = \frac{5-4}{10}$$

$$= \int_0 (x^2 - x^2 - x\sqrt{x} + x)$$

$$= \int_0^1 (x - x^{3/2})\,dx$$

$$= \frac{x^2}{2} - \frac{2}{5}x^{5/2}$$

$$= \frac{1}{2} - \frac{2}{5}$$

$$= \frac{5-4}{10} = \frac{1}{10}$$

$= 3/10$

$$= \int_{\text{line } OA} 2x^2\,dx + x^2\,dx + \int_{\substack{\text{Parabola} \\ y = x^2}} (\cancel{4}\,{}^{3}x^3 + x^4)\,dx$$

$$= \int_0^1 3x^2\,dx + \int_1^0 (4x^3 + x^4)\,dx$$

$$= \left[x^3\right]_0^1 + \left[x^4 + x^5/5\right]_1^0$$

$$= 1 + \left(-1 - 1/5\right)$$

$=$

$$= \left[x^3\right]_0^1 + \left[\frac{3}{4}x^4 + \frac{x^5}{5}\right]_1^0$$

$$= 1 - \frac{3}{4} - \frac{1}{5}$$

$$= \frac{20 - 15 - 4}{20}$$

$= 1/20$

$$\cancel{\int_0^1 \int_{x^2}^{x}}$$

$$\int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\cancel{x^2} - \cancel{x^2} - x^3 + x^4\right) dx$$

$= 1/20$ $\qquad = \left[x^5/5 - x^4/4\right]_0^1$

Numerically LHS = RHS

Hence, Green's theorem is verified.

Application of Green Theorem

The area enclosed by the closed curve C is given by $\frac{1}{2}\int [x\,dy - y\,dx]$

Proof: By Green Theorem, we have

$$\int_C [F_1\,dx + F_2\,dy] = \iint_S \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] dx\,dy \quad \longrightarrow (*)$$

In particular, put

$F_2 = x$ and $F_1 = -y$, then,

$$\frac{\partial F_2}{\partial x} = 1$$

$$\frac{\partial F_1}{\partial y} = -1$$

Then, eqn (*) becomes

$$\therefore \int_C [F_1\,dx + F_2\,dy] = \iint_S 2\,dx\,dy$$

$$\Rightarrow \int_C x\,dy - y\,dx = 2\iint_S dx\,dy$$

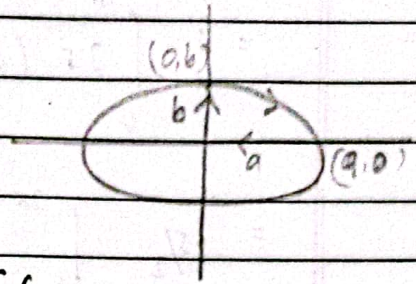
$$\Rightarrow \int_C x\,dy - y\,dx = 2\,(\text{Whole area of surface})$$

$$\therefore \text{Area} = \frac{1}{2}\int_C [x\,dy - y\,dx]$$

Example. Find the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\sqrt{1 - y^2/b^2}$

$$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx$$



$$\cancel{= \frac{1}{2}\left\{\int_{x=0} x\,dy - y\,dx + \int \int\right.}$$

Put $x = a\cos\theta \Rightarrow dx = -a\sin\theta\,d\theta$

$y = b\sin\theta \qquad dy = b\cos\theta\,d\theta$

Also θ varies from $\theta = 0$ to $\theta = 2\pi$

$$\text{Area of ellipse} = \frac{1}{2}\int_C [x\,dy - y\,dx]$$

$$= \frac{1}{2}\int_{\theta=2\pi} a\cos\theta\, b\cos\theta\, d\theta - b\sin\theta\,(-a)\sin\theta\, d\theta$$

$$= \frac{1}{2}\int_{\theta=0} ab(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi} d\theta$$

$$= \frac{ab}{2} \times 2\pi$$

$$= \boxed{\pi ab}$$

Find the area of circle $x^2 + y^2 = 25$

$x = 5\cos t \quad \therefore dx = -5\sin t$

$y = 5\sin t \quad \therefore dy = 5\cos t$

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$$A = \frac{1}{2}\int_C (5\cos t).\ 5\cos t\ dt - 5\sin t\ (-5\sin t)\ dt$$

$$= \frac{1}{2}\int_0^{2\pi} 25\ (\sin^2 t + \cos^2 t)\ dt$$

$$= 25/_2 \int_0^{2\pi} dt$$

$$= \frac{25}{2} \times 2\pi$$

$$= \underline{25\pi}$$

Stoke's Theorem

If $\vec{F}$ is continuous, differentiable vector point function and S be the surface bounded the closed curve c, then

$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{\text{line integral}} = \underbrace{\iint_S (\text{Curl } \vec{F})\cdot \hat{n}\ ds}_{\text{surface integral of curl } \vec{F} = \nabla \times \vec{F}}$$

$\int_C F\cdot dr = \iint_S (\text{Curl } \vec{F})\cdot \hat{n}\, ds$

Theorem

Prove that Green's theorem is a particular case of Stoke's theoremin XY plane.

If the surface S lies on XY plane, then the z-axis is the direction of normal. So, the unit vector normal to the surface $\vec{n} = \vec{K}$

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Now, by Stoke's theorem,

let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$

$\vec{F}\cdot d\vec{r} = F_1\,dx + F_2\,dy + F_3\,dz$

Then,

$$LHS = \int_c \vec{F}\cdot d\vec{r} = \int_c (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot(\vec{i}dx + \vec{j}dy + \vec{k}dz)$$

$$LHS = \int_c [F_1\,dx + F_2\,dy] \text{ ——— } (*)$$

RHS :-

$$(\text{Curl } \vec{F})\cdot\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} \cdot \vec{k}$$

$$= \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}\right]\cdot\vec{k}$$

$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\therefore RHS = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) ds = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\frac{dx\,dy}{|\vec{k}\cdot\vec{k}|}$$

$$= \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

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Hence, eqⁿ (i) becomes:

$$\int_c (F_1\,dx + F_2\,dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$\therefore \int_C \vec{r} \times d\vec{r} = \iint_S \vec{0} \cdot \vec{n}\, ds$$

$$= \boxed{0}$$

Gauss Divergence Theorem

Statement: If $\vec{F}$ is a vector differentiable function, then the normal surface integral of $\vec{F}$ over the closed surface S enclosed the volume V is equal to the volume integral of divergence of $\vec{F}$ over V.

i.e. $$\iint_S \vec{F} \cdot \vec{n}\, ds = \iiint_V (\nabla \cdot \vec{F})\, dv$$

Evaluate:

$\iint_S \vec{r} \cdot \vec{n}\, ds$ where S is the closed surface enclosed the volume V.

Solution

Replacing $\vec{F}$ by $\vec{r}$ in Gauss-Divergence theorem, we get:

$$\iint_S \vec{r} \cdot \vec{n}\, ds = \iiint_V (\nabla \cdot \vec{r})\, dv$$

$$= \iiint_V 3\, dV$$

$= \boxed{3V}$

Prove that

$$\iint_S \nabla r^2 \cdot \vec{n}\, ds = 6V$$

Proof

Replacing $\vec{F}$ by ∇r^2 in Gauss-Divergence theorem, we get:

$$\iint_S \nabla r^2 \cdot \vec{n}\, ds = \iiint_V (\nabla \cdot \nabla r^2)\, dV$$

Now,

~~$\nabla r^2 = \vec{i}\,\partial/\partial x\, x^2 \ \vec{j}\,\partial/\partial y\, y \ \vec{k}\,\partial/\partial z$~~

We know, $r^2 = x^2+y^2+z^2$

$$\nabla r^2 = \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla \cdot \nabla r^2 = \cancel{2x+2}\ 2+2+2 = 6$$

Now, $\iiint_V (\nabla \cdot \nabla r^2)\, dV = \iiint_V 6\, dV = 6\iiint_V dV$

$= \boxed{6V}$

EXAMPLE Use Gauss-Divergence theorem to find surface integral

$$\iint_S \vec{F}\cdot\vec{n}\,ds \quad \text{where}$$

$$\vec{F} = (2x+3z)\vec{i} - (xz+y)\vec{j} + (y^2+2z)\vec{k}$$

where S is the surface of the sphere $(x-3)^2+(y+1)^2+(z-2)^2=25$.

Solution

By Gauss-Divergence Formula:

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dV$$

~~Replacing~~

Now,

$$\nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left[(2x+3z)i - (xz+y)\vec{j} + (y^2+2z)\vec{k}\right]$$

$$= 2 + (-1) + 2$$

$$= 3$$

Now,

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V 3\,dV = 3V$$

$$= 3 * \frac{4}{3}\pi * 125$$

$$= \boxed{500\pi}$$

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EXAMPE With the help of Gauss-Divergence theorem

$$\iint_S (ax\vec{i} + by\vec{j} + cz\vec{k}) \cdot \vec{n}\, ds = \frac{4}{3}\pi(a+b+c)$$

where S is surface of the sphere $x^2+y^2+z^2=1$

Solution

$$\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$$

Thus,

$$\iint_S \vec{F}\cdot\vec{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dV$$

Now,

$$\nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(ax\vec{i} + by\vec{j} + cz\vec{k})$$

$$= (a+b+c)$$

$$\iiint_V (\nabla\cdot\vec{F})\, dV = \iiint_V (a+b+c)\, dV$$

$$= (a+b+c)V$$

$$= (a+b+c)\frac{4}{3}\pi \times (1)^3$$

$$= \frac{4}{3}\pi(a+b+c)$$

$$= RHS$$

proved

* Some extra problem:

For any closed surface S, prove that $\iint_S \text{Curl}\,\vec{F}\cdot\vec{n}\, ds = 0$

Proof

$$\iint_S \text{Curl}\,\vec{F}\cdot\vec{n}\,dS = \iiint_V (\nabla\cdot \text{Curl}\,\vec{F})\,dV \quad \text{---} \; (*)$$

$$\nabla\cdot\nabla\times\vec{F} = \cancel{\nabla\cdot\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & & \end{vmatrix}}$$

$$= \cancel{(\nabla\times\nabla)\cdot\vec{F}}$$

$$= \nabla\cdot\ \nabla\times\vec{F}$$

$$= 0$$

$$\iiint_V [\nabla\cdot(\nabla\times\vec{F})]\,dV = \iiint_V 0\,dv$$

$$= 0\iiint_V dv = 0V = \boxed{0}$$

$\boxed{\text{Proved}}$

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Parametric surfaces

In many cases, we can parametrize an equation of a curve or surface.

Eg: The parametrization of

i) circle $x^2+y^2=a^2$ is $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$[\because x = a\cos t,\ y = a\sin t,\ x^2+y^2=a^2]$

ii) ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j}$

iii) parabola $y^2=4ax$ is $\vec{r} = at^2\,\vec{i} + 2at\,\vec{j}$

$\because x = at^2,\ y = 2at$

Eliminiting $y^2 = 4ax$

In particular, if $\vec{r}$ is expressed in terms of two parameter u and v given by

$$\vec{r} = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$$

Eg:

$$\vec{r} = (u^2+2u)\,\vec{i} + uv\,\vec{j} + (6u+v)\,\vec{k}$$

Q Example Find the cartesian equation of surface whose parametric representation is:

$$\vec{F}(u,v) = u\,\vec{i} + u\cos v\,\vec{j} + u\sin v\,\vec{k}$$

Here,

$$\vec{F}(u,v) = x\vec{i} + y\vec{j} + z\vec{k}$$

where,

$$x = u,\ y = u\cos v,\ z = u\sin v$$

Here, squaring and adding y^2 and z^2,

$$y^2+z^2 = u^2(\cos^2 v + \sin^2 v)$$
$$= u^2 = x^2$$

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$\therefore x^2 = y^2 + z^2$, which is required cartesian form

Q. Find the parametric form of $x = 5y^2 + 2z^2 - 1$

Soln

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= (5y^2 + 2z^2 - 1)\vec{i} + y\vec{j} + z\vec{k}$

Some applications :

1) To find equation of tangent plane to surface $\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$ at point (u_0, v_0)

Steps :

I. Find partial derivatives $\vec{r_u}$ and $\vec{r_v}$ which gives the tangent vector to the surface .

II. Find $\vec{r_u} \times \vec{r_v}$ denote $\vec{n} = \vec{r_u} \times \vec{r_v}$ which gives direction of normal vector

III. The equation of Tangent plane at $\vec{a} = (u_0, v_0)$ with direction of normal $\vec{n}$ is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

where $\vec{r} = (x, y)$

Imp Example Find the equation of tangent plane to the surface given by

$$F(u,v) = u\vec{i} + 2v^2\vec{j} + (u^2+v)\vec{k}$$

at point $(2,2,3)$

Soln ⇒

$$F = u\vec{i} + 2v^2\vec{j} + (u^2+v)\vec{k}$$

Step I Differentiating w.r.t u and v partially

$$\vec{F_u} = 1\vec{i} + 2u\vec{k}$$

$$\vec{F_v} = 4v\vec{j} + \vec{k}$$

These represent direction of tangent in u & v complem

Step 2: Now, the normal vector

$$\vec{n} = \vec{F_u} \times \vec{F_v}$$

$$= \begin{array}{ccc} i & j & k \end{array}$$
$$\begin{vmatrix} 1 & 0 & 2u \\ 0 & 4v & 1 \end{vmatrix}$$

$$= -8uv\vec{i} - 1\vec{j} + 4v\vec{k}$$

$$\vec{n} = -8uv\vec{i} - \vec{j} + 4v\vec{k} \quad \text{——} \quad (*)$$

To find u & v in (*)

From question at $(2,2,3)$

$x = u, \quad y = 2v^2, \quad z = u^2+v$

$\Rightarrow 2 = u, \quad 2 = 2v^2, \quad 3 = u^2+v$

$\Rightarrow u = 2, \quad v = \pm 1$

we chose $v = -1$

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As $u = 2$ and $v = +1$ does not satisfy $u^2 + v = 3$
But $u = 2$ and $v = -1$ satisfy $u^2 + v = 3$
$\therefore u = 2$ and $v = -1$

$\therefore$ From (*) $\vec{n} = -8 \times 2(-1)\vec{i} - \vec{j} + 4(-1)\vec{k}$
$= 16\vec{i} - \vec{j} - 4\vec{k}$
$= (16, -1, -4)$

Therefore the eq^n of tangent plane at ~~pt~~ $\vec{a}\ (2,2,3)$ with normal vector $\vec{n} = (16, -1, -4)$

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$\Rightarrow [(x, y, z) - (2, 2, 3)] \cdot (16, -1, -4) = 0$

$\Rightarrow (x - 2) \cdot 16 + (y - 2) \cdot -1 + (z - 3) \cdot -4 = 0$

$\Rightarrow 16x - 32 - y + 2 - 4z + 12 = 0$
$\Rightarrow 16x - y - 4z = 18$

Q. Find the equation of tangent plane to the surface with parametric equation $x = s^2$, $y = t^2$, $z = s + 2t$ at point $\vec{i} + \vec{j} + 3\vec{k}$

Sol^n $\rightarrow$
Here,
$F(s,t) = x\vec{i} + y\vec{j} + z\vec{k}$
$= s^2\vec{i} + t^2\vec{j} + (s + 2t)\vec{k}$

Step I differentiating w.r.t s and t
$F_s(s,t) = 2s\vec{i} + 0\vec{j} + \vec{k}$

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$F_t = 0\vec{i} + 2t\vec{j} + 2\vec{k}$

Step2 → Now, normal vector $\vec{n} = F_s \times F_t$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2s & 0 & 1 \\ 0 & 2t & 2 \end{vmatrix}$$

$$= \vec{i}(-2t) - \vec{j}\,4s + \vec{k}\,4st \quad \text{——} \quad (*)$$

To find s and t, at (1, 1, 3)

$x = s^2 \qquad y = t^2 \qquad z = s + 2t$

$\Rightarrow s = \pm 1 \qquad t = \pm 1$

To satisfy $z = s + 2t$, ~~we~~ $s = 1$, $t = 1$

Equation of tangent plane at $\vec{a}$ (1,1,3) and ~~$\vec{n}$ (-2t, 4s, 4st)~~

$\vec{n} = -2\times1\,\vec{i} + (-4)\times1\,\vec{j} + 4\times1\times1\,\vec{k}$

$= 2\vec{i} + (-4)\vec{j} + 4\vec{k}$

$= (-2, -4, 4)$ is,

$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$\Rightarrow [(x, y, z) - (1, 1, 3)] \cdot (2, 4, 4) = 0$

$\Rightarrow (x-1)\cdot 2 + (y-1)\cdot 4 + (z-3)\cdot 4 = 0$

$\Rightarrow -2x + 2 + (-4y) + 4 + 4z - 12 = 0$

$\Rightarrow$ ~~$2x + 4y + 4z - 18 = 0$~~

$\Rightarrow -2x - 4y + 4z - 6 = 0$

$\Rightarrow -2x - 4y + 4z = 6$, is required eqn