ASSIGNMENT - I

Q1. Find the L_1, L_2 and L_∞ norms of the following vectors.

i) $(5, 2)$

Let, $x = (x_1, x_2) = (5, 2) \in \mathbb{R}^2$. Then, from the definition of norms, we have,

$$L_1 \text{ norm} = \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$L_2 \text{ norm} = \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$$L_\infty \text{ norm} = \|x\|_\infty = \max_{i \in [n]} |x_i|$$

For $\mathbb{R}^2$, $n = 2$. Thus, we can calculate the norms accordingly.

$$\|x\|_1 = \sum_{i=1}^{2} |x_i|$$

$$= |x_1| + |x_2|$$

$$= |5| + |2|$$

$$= 5 + 2$$

$$= \underline{7}$$

$$\|x\|_2 = \left(\sum_{i=1}^{2} x_i^2\right)^{1/2}$$

$$= (x_1^2 + x_2^2)^{1/2}$$

$$= ((5)^2 + (2)^2)^{1/2}$$

$$= (29)^{1/2}$$

$$= \underline{5.38}$$

$$\|X\|_\infty = \max_{i \in \{1,2\}} |x_i|$$

$$= \max\{x_1, x_2\}$$

$$= \max(5, 2)$$

$$= \underline{5}$$

ii) $(-4, 2, 3)$

Let, $x = (x_1, x_2, x_3) = (-4, 2, 3) \in \mathbb{R}^3$. Then,

$$\|X\|_1 = \sum_{i=1}^{3} |x_i|$$

$$= |x_1| + |x_2| + |x_3|$$

$$= |-4| + |2| + |3|$$

$$= \underline{9}$$

$$\|X\|_2 = \left(\sum_{i=1}^{3} x_i^2\right)^{1/2}$$

$$= (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$= ((-4)^2 + (2)^2 + (3)^2)^{1/2}$$

$$= (16 + 4 + 9)^{1/2}$$

$$= (29)^{1/2}$$

$$= \underline{5.38}$$

$$\|X\|_\infty = \max_{i \in \{1,2,3\}} |x_i|$$

$$= \max\{|x_1|, |x_2|, |x_3|\}$$

$$= \max\{|-4|, |2|, |3|\}$$

$$= \max\{4, 2, 3\}$$

$$= \underline{4}$$

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iii) $(1, 2, 3, 4)$

Let, $x = (x_1, x_2, x_3, x_4) = (1, 2, 3, 4) \in \mathbb{R}^4$. Then,

$$\|x\|_1 = \sum_{i=1}^{4} |x_i|$$

$$= |x_1| + |x_2| + |x_3| + |x_4|$$

$$= |1| + |2| + |3| + |4|$$

$$= 1 + 2 + 3 + 4$$

$$= \underline{10}$$

$$\|x\|_2 = \left(\sum_{i=1}^{4} x_i^2\right)^{1/2}$$

$$= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2}$$

$$= \left((1)^2 + (2)^2 + (3)^2 + (4)^2\right)^{1/2}$$

$$= (1 + 4 + 9 + 16)^{1/2}$$

$$= (30)^{1/2}$$

$$= \underline{5.47}$$

$$\|x\|_\infty = \max_{i \in \{1, 2, 3, 4\}} |x_i|$$

$$= \max\{|x_1|, |x_2|, |x_3|, |x_4|\}$$

$$= \max\{|1|, |2|, |3|, |4|\}$$

$$= \max\{1, 2, 3, 4\}$$

$$= \underline{4}$$

iv) $(4, -2, 1, 3)$

Let, $x = (x_1, x_2, x_3, x_4) = (4, -2, 1, 3) \in \mathbb{R}^4$. Then,

$$\|x\|_1 = \sum_{i=1}^{4} |x_i|$$

$$= |x_1| + |x_2| + |x_3| + |x_4|$$

$$= |4| + |-2| + |1| + |3|$$

$$= 4 + 2 + 1 + 3$$

$$= \underline{10}$$

$$\|x\|_2 = \left(\sum_{i=1}^{4} x_i^2 \right)^{1/2}$$

$$= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2}$$

$$= \left((4)^2 + (-2)^2 + (1)^2 + (3)^2\right)^{1/2}$$

$$= (16 + 4 + 1 + 9)^{1/2}$$

$$= (30)^{1/2}$$

$$= \underline{5.47}$$

$$\|x\|_\infty = \max_{i \in \{1, 2, 3, 4\}} |x_i|$$

$$= \max\{|x_1|, |x_2|, |x_3|, |x_4|\}$$

$$= \max\{|4|, |-2|, |1|, |3|\}$$

$$= \max\{4, 2, 1, 3\}$$

$= \underline{4}$

v) $(0, 0, 0, 7, 0, 0)$

Let, $x = (x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 7, 0, 0) \in \mathbb{R}^6$.

Then,

$$\|x\|_1 = \sum_{i=1}^{6} |x_i|$$

$$= |x_1| + |x_2| + |x_3| + |x_4| + |x_5| + |x_6|$$

$$= 0 + 0 + 0 + 7 + 0 + 0$$

$$= \underline{7}$$

$$\|x\|_2 = \left(\sum_{i=1}^{6} x_i^2 \right)^{1/2}$$

$$= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \right)^{1/2}$$

$$= \left(0^2 + 0^2 + 0^2 + 7^2 + 0^2 + 0^2 \right)^{1/2}$$

$$= (7^2)^{1/2}$$

$$= \underline{7}$$

$$\|x\|_\infty = \max_{i \in \{1, 2, 3, 4, 5, 6\}} |x_i|$$

$$= \max\{ |x_1|, |x_2|, |x_3|, |x_4|, |x_5|, |x_6| \}$$

$$= \max\{0, 0, 0, 7, 0, 0\}$$

$$= \underline{7}$$

Q2. Show that the L_1-norm satisfies each of the conditions in the definition of a norm. First, do this for $\mathbb{R}^2$, and then do this for $\mathbb{R}^n$.

Solution

A mapping $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a norm, if it satisfies following properties:

i. $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$.
ii. $\|x\| = 0$ iff $x = 0$
iii. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.
iv. $\|x+y\| \leq \|x\| + \|y\|$

For $\mathbb{R}^2$

Let us consider a vector $x = (x_1, x_2) \in \mathbb{R}^2$. Then, L_1-norm is defined as:

$$L_1\text{-norm} = \|x\|_1 = |x_1| + |x_2| \quad \cdots\cdots\cdots (*)$$

Proof of (i)

$\|x\|_1 = |x_1| + |x_2|$ shows that $\|x\|_1$ can never be negative as it is the result of sum of two absolute values. However, in a scenario where $x = 0$, all the components are zero i.e. $x_1 = x_2 = 0$. In such case, the L_1-norm can acquire a minimum value of zero. Therefore, $\|x\|_1 \geq 0$.

Proof of (ii)

If $x = 0$, then, $x_1 = x_2 = 0$, so $\|x\|_1 = 0$

Conversely, if $\|x\|_1 = 0$, $|x_1| = |x_2| = 0$ because of the non-negativity property of absolute value. This implies $x = 0$

$$\therefore \|x\|_1 = 0 \Leftrightarrow x = 0.$$

Proof of (iii)

Let, $\alpha \in \mathbb{R}$ be a scalar. Then, from the definition of L_1 norm, we get,

$$\|\alpha x\|_1 = |\alpha x_1| + |\alpha x_2|$$

$$= |\alpha|\,|x_1| + |\alpha|\,|x_2| \quad (\because |ab| = |a|\,|b|)$$

$$= |\alpha|\left(|x_1| + |x_2|\right)$$

$$= |\alpha|\,\|x\|_1 \quad (\text{From } \circledast)$$

$$\therefore \|\alpha x\|_1 = |\alpha|\,\|x\|_1$$

Proof of (iv)

Let, $y = (y_1, y_2) \in \mathbb{R}^2$. Then, from the definition of L_1-norm, we have,

$$\|x+y\|_1 = |x_1+y_1| + |x_2+y_2|$$

$$\leq |x_1| + |y_1| + |x_2| + |y_2| \quad (\because |a+b| \leq |a| + |b|)$$

$$= |x_1| + |x_2| + |y_1| + |y_2|$$

$$= \|x\|_1 + \|y\|_1$$

$$\therefore \|x+y\|_1 \leq \|x\|_1 + \|y\|_1$$

For $\mathbb{R}^n$

Let us consider $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then, L_1-norm for $\mathbb{R}^n$ is defined as:

$$L_1\text{-norm} = \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Proof of (i)

$||X||_1 = \sum_{i=1}^{n} |x_i|$ shows that the L_1-norm is the sum of n absolute values. Thus, it can never be negative. However, when all the components are zero i.e. $x_1 = x_2 = \cdots x_n = 0$, then, in such scenario, L_1-norm becomes zero.

$$\therefore ||X||_1 \geq 0.$$

Proof of (ii)

If $X = \vec{0}$, then $x_i = 0$ for all values of i, so, $||X||_1 = 0$. Conversely, if $||X||_1 = 0$, $|x_i| = 0$ for all values of i because of the non-negative property of absolute value. Thus, $X = 0$.

$$\therefore ||X||_1 = 0 \Leftrightarrow X = 0.$$

Proof of (iii)

Let $\alpha \in \mathbb{R}$ be a scalar. Then, from definition of L_1-norm, we have,

$$||\alpha X||_1 = \sum_{i=1}^{n} |\alpha x_i|.$$

$$= |\alpha| \sum_{i=1}^{n} |x_i|$$

$$= |\alpha| \, ||X||_1$$

$$\therefore ||\alpha X||_1 = |\alpha| \, ||X||_1.$$

Proof of (iv)

Let $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. Then,

$$||X + y||_1 = \sum_{i=1}^{n} |x_i + y_i|$$

$$\leq \sum_{i=1}^{n} (|x_i| + |y_i|) \qquad (\because |a+b| \leq |a| + |b|)$$

$$= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

$$= \|x\|_1 + \|y\|_1$$

$$\therefore \|x+y\|_1 \leq \|x\|_1 + \|y\|_1$$

This completes our proof.

Q3. Show that L_∞-norm satisfies each of the conditions in the definition of a norm. First do this for $\mathbb{R}^2$, and then do this for $\mathbb{R}^n$.

Solution

A mapping $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a norm, if it satisfies following properties.

i. $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$.
ii $\|x\| = 0$ iff $x = 0$
iii. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.
iv. $\|x+y\| \leq \|x\| + \|y\|$

For $\mathbb{R}^2$

Let us consider a vector $x = (x_1, x_2) \in \mathbb{R}^2$. Then, L_∞-norm is defined as:

$$L_\infty\text{-norm} = \|x\|_\infty = \max_{i \in \{1,2\}} |x_i|$$

$$= \max\{|x_1|, |x_2|\}$$

Proof of (i)

$\|x\|_\infty = \max\{|x_1|, |x_2|\}$ shows that L_∞ norm

can never be a negative value as it is the result of selection of maximum of absolute value of the vector components. However, when both the components are zero (i.e. $x_1 = x_2 = 0$), then, L_∞-norm can achieve the minimum possible value of zero.

$$\therefore \|X\|_\infty \geq 0.$$

Proof of (ii)
If $X = 0$, then, $x_1 = x_2 = 0$, so $\|X\|_\infty = 0$.
Conversely, if $\|X\|_\infty = 0$, $|x_1| = |x_2| = 0$ because the maximum value can only be zero if both the components are zero. For any other values, $\|X\|_\infty \neq 0$.
This implies that $X = 0$ for $\|X\|_\infty = 0$

$$\therefore \|X\|_\infty = 0 \Leftrightarrow X = 0$$

Proof of (iii)
Let $\alpha \in \mathbb{R}$ be a scalar. Then, from definition of L_∞-norm,

$$\|\alpha X\|_\infty = \max\{|\alpha x_1|, |\alpha x_2|\}$$

$$= \max\{|\alpha||x_1|, |\alpha||x_2|\}$$

$$= |\alpha| \max\{|x_1|, |x_2|\}$$

$$= |\alpha| \|X\|_\infty$$

$$\therefore \|\alpha X\|_\infty = |\alpha| \|X\|_\infty$$

Proof of (iv)
Let, $y = (y_1, y_2) \in \mathbb{R}^2$ be a vector. Then,

$$\|X + y\|_\infty = \max\{|x_1 + y_1|, |x_2 + y_2|\}$$

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$$\leq \max\{|x_1|+|y_1|, |x_2|+|y_2|\}$$

$$= \max\{|x_1|, |x_2|\} + \max\{|y_1|, |y_2|\}$$

$$= \|x\|_\infty + \|y\|_\infty$$

$$\therefore \|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$$

For $\mathbb{R}^n$

Consider $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then, L_∞-norm on $\mathbb{R}^n$ is defined as:

$$L_\infty\text{-norm} = \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Proof of (i)

Since $\|x\|_\infty = \max_{i \in [n]} |x_i|$, $\|x\|_\infty$ can never acquire a negative value owing to the non-negative property of absolute value. However, when all the components of x are zero, $\|x\|_\infty$ can acquire a minimum value of zero.

$$\therefore \|x\|_\infty \geq 0.$$

Proof of (ii)

If $x = 0$, then, $x_1 = x_2 = \ldots x_n = 0$, so $\|x\|_\infty = 0$.
Conversely, if $\|x\|_\infty = 0$, $|x_1| = |x_2| = \ldots |x_n| = 0$ because the maximum value can only be zero if all the components of x are zero. For any other values, $\|x\|_\infty \neq 0$. This implies that $x = 0$ for $\|x\|_\infty = 0$.

$$\therefore \|x\|_\infty = 0 \Leftrightarrow x = 0.$$

Proof of (iii)

Let, $\alpha \in \mathbb{R}$ be a scalar. Then, from definition of L_∞ - norm,

$$\|\alpha x\|_\infty = \max_{1 \le i \le n} |\alpha x_i|$$

$$= |\alpha| \max_{1 \le i \le n} |x_i|$$

$$= |\alpha| \, \|x\|_\infty$$

$$\therefore \|\alpha x\|_\infty = |\alpha| \, \|x\|_\infty .$$

Proof of (iv)

Let, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. Then, we have:

$$\|x+y\|_\infty = \max_{1 \le i \le n} |x_i + y_i| \le \max_{1 \le i \le n} (|x_i| + |y_i|)$$

$$= \max_{1 \le i \le n} |x_i| + \max_{1 \le i \le n} |y_i|$$

$$= \|x\|_\infty + \|y\|_\infty$$

$$\therefore \|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$$

This completes our proof.

Q4. Let $x = (1, 1/2, 1/3)$ in $\mathbb{R}^3$. Compute L_1, L_2, and L_∞ - norms.

Here, $x = (1, 1/2, 1/3) \in \mathbb{R}^3$. Then, from definition of norms, we have:

$$L_1\text{-norm} = \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$L_2\text{-norm} : \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

L_∞-norm $= \max_{i \in [n]} |x_i|$

For R^3, $n:3$, we get results accordingly

$$\therefore \|x\|_1 = \sum_{i=1}^{3} |x_i|$$

$$= |x_1| + |x_2| + |x_3|$$

$$= |1| + |1/2| + |1/3|$$

$$= 1 + 1/2 + 1/3$$

$$= \underline{1.833}$$

$$\|x\|_2 = \left(\sum_{i=1}^{3} x_i^2\right)^{1/2}$$

$$= \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$$

$$= \left((1)^2 + (1/2)^2 + (1/3)^2\right)^{1/2}$$

$$= \left(1 + 1/4 + 1/9\right)^{1/2}$$

$$= \underline{1.166}$$

$$\|x\|_\infty = \max_{i \in \{1,2,3\}} |x_i|$$

$$= \max\{|x_1|, |x_2|, |x_3|\}$$

$$= \max\{|1|, |1/2|, |1/3|\}$$

$$= \max\{1, 1/2, 1/3\}$$

$$= 1$$

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