PROBLEMS ON DOUBLE INTEGRALS IN POLAR COORDINATES

Evaluate the iterated integral

1. $\int_0^{\pi/2} \int_0^{\sin\theta} r \cos\theta \, dr \, d\theta$

$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$

$= \frac{1}{2} \int_0^{\pi/2} \cos\theta \sin^2\theta \, d\theta$

Put $u = \sin\theta$

$\therefore du = \cos\theta \, d\theta$

changing the limits;

when $\theta = 0$, $u = 0$

$\theta = \pi/2$, $u = 1$

Thus, the above integral becomes,

$= \frac{1}{2} \int_0^{\pi/2} \cos\theta \sin^2\theta \, d\theta$

$= \frac{1}{2} \int_0^1 u^2 \, du$

$= \frac{1}{2} \left[\frac{u^3}{3}\right]_0^1$

$= \frac{1}{2} \left(\frac{1}{3} - 0\right)$

$= \frac{1}{6}$

2. $\int_0^{\pi} \int_0^{1+\cos\theta} r \, dr \, d\theta$

$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta$

$= \frac{1}{2} \int_0^{\pi} (1+\cos\theta)^2 \, d\theta$

$= \frac{1}{2} \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta$

$= \frac{1}{2} \left[\theta + \frac{2(-\sin\theta)}{1}\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} \frac{1+\cos 2\theta}{2} \, d\theta$

$= \frac{1}{2}(\pi - 0) + \frac{1}{2} \times \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$

$= \frac{1}{2} * \pi + \frac{1}{4}(\pi - 0)$

$= \pi/2 + \pi/4$

$= \frac{3\pi}{4}$

3. $\int_0^{\pi/2} \int_0^{a\sin\theta} r^2 \, dr \, d\theta$

$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{a\sin\theta} d\theta$

$= \frac{a^3}{3} \int_0^{\pi/2} \sin^3\theta \, d\theta$

$$= \frac{a^3}{3} \int_0^{\pi/2} \frac{(3\sin\theta - \sin 3\theta)}{4}\, d\theta$$

$$= \frac{a^3}{12} \left[-3\cos\theta + \frac{\cos 3\theta}{3} \right]_0^{\pi/2}$$

$$= \frac{a^3}{12} \left(-1/3 + 3 \right)$$

$$= \frac{a^3}{12} * \frac{8}{3}$$

$$= \frac{2a^3}{9}$$

4. $$\int_0^{\pi/6} \int_0^{\cos 3\theta} r\, dr\, d\theta$$

$$= \int_0^{\pi/6} \left[\frac{r^2}{2} \right]_0^{\cos 3\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/6} \cos^2 3\theta\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/6} \left(\frac{1 + \cos 6\theta}{2} \right) d\theta$$

$$= \frac{1}{4} \int_0^{\pi/6} (1 + \cos 6\theta)\, d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/6}$$

$$= \frac{1}{4} \left(\pi/6 - 0 \right)$$

$$= \pi/24.$$

5. $\int_0^{\pi} \int_0^{1-\sin\theta} r^2 \cos\theta \, dr \, d\theta$

$= \int_0^{\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{1-\sin\theta} d\theta$

$= \frac{1}{3} \int_0^{\pi} (1-\sin\theta)^3 \cos\theta \, d\theta$

Put, $u = 1 - \sin\theta$

$\therefore du = -\cos\theta \, d\theta$

when $\theta = 0, \; u = 1$

$\theta = \pi, \; u = 1$

Thus, above integral becomes

$-\frac{1}{3} \int_1^1 u^3 \, du$

$= 0 \qquad \left(\because \int_a^a f(x)\, dx = 0\right)$

6. $\cancel{\int_0^{\pi/2} \int_0^{a \sin\theta} r^2 \, dr \, d\theta}$

$\cancel{= \int\int_0^{\pi/2}}$

6. $\int_0^{\pi/2} \int_0^{\cos\theta} r^3 \, dr \, d\theta$

$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{\cos\theta} d\theta$

$= \frac{1}{4} \int_0^{\pi/2} \cos^4\theta \, d\theta$

$$= \frac{1}{4}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{16}\int_0^{\pi/2} (1+ 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \frac{1}{16}\left[\theta + -\frac{2\sin 2\theta}{2}\right]_0^{\pi/2} + \frac{1}{16}\int_0^{\pi/2} \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{16}(\pi/2 - 0) + \frac{1}{32}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi}{32} + \frac{1}{32}(\pi/2 - 0)$$

$$= \frac{\pi}{32} + \frac{\pi}{64}$$

$$= \frac{3\pi}{64}$$

Use a double integral in polar co-ordinates to find the area of the region described.

7. The region enclosed by the cardioid $r = 1 - \cos\theta$.

<u>Solution</u>

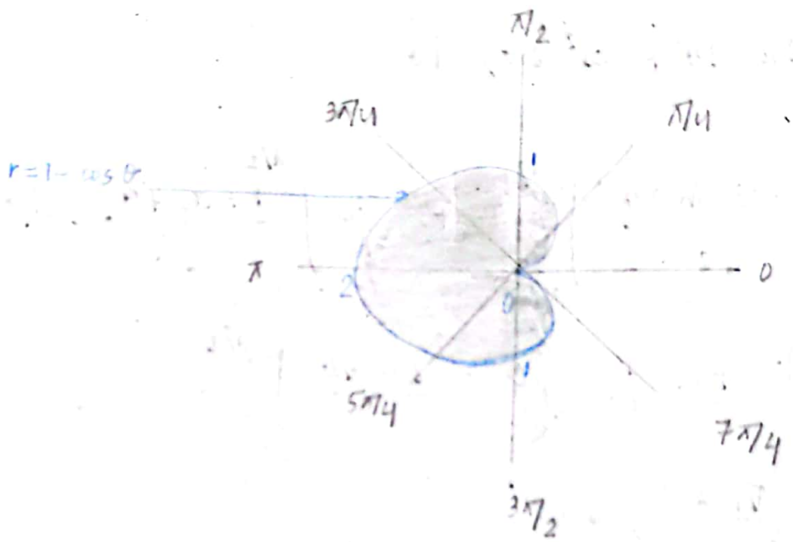
Let's sketch the cardioid in polar plane.

when $\theta = 0$, $r = 1 - \cos 0 = 0$

$\theta = \pi/2$, $r = 1 - \cos \pi/2 = 1 - 0 = 1$

$\theta = \pi$, $r = 1 - \cos\pi = 1+1 = 2$

Here, r ranges from 0 to 2 and θ ranges from 0 to 2π.



Area of the cardioid is given by

$$A = \int_0^{2\pi} \int_0^{1-\cos\theta} r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \frac{1}{2}\left[\theta + 2\sin\theta\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}(2\pi + 0 - 0) + \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \pi + \frac{1}{4}\left(2\pi - 0\right)$$

$$= \pi + \pi/2$$

$$= \frac{3\pi}{2}$$

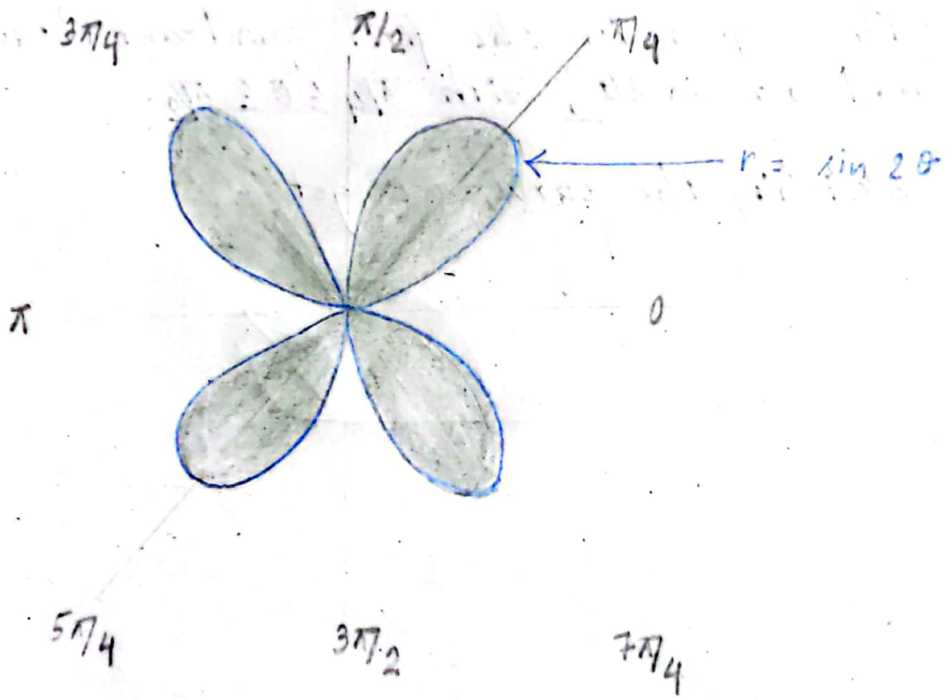
8. The region enclosed by rose $r = \sin 2\theta$.

Here

θ ranges from 0 to 2π.

and r ranges from 0 to r.

Sketching the curve in polar co-ordinate plane, we get:



Now, area of shaded rose is given by:

$$A = \int_0^{2\pi} \int_0^{\sin 2\theta} r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{\sin 2\theta} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{2} \left[\frac{1 - \cos 4\theta}{2}\right]_0^{2\pi}$$

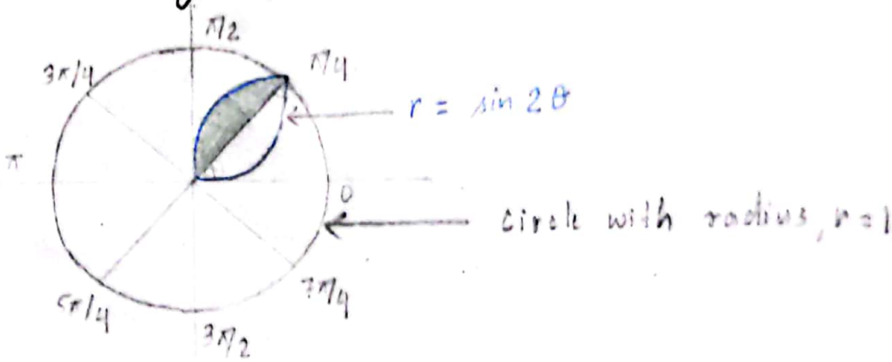
$$= \frac{1}{4} (1 - \cos 4\theta)$$

$$= \frac{1}{4} (2\pi - 0)$$

$$= \pi/2$$

9. The region in the first quadrant bounded by $r = 1$ and $r = \sin 2\theta$, with $\pi/4 \leq \theta \leq \pi/2$.

Sketching the curve, we get:



Area of shaded petal, A is given by:

$$A = \int_{\pi/4}^{\pi/2} \int_{r = \sin 2\theta}^{1} r \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2}\right]_{\sin 2\theta}^{1} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} (1 - \sin^2 2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} d\theta - \frac{1}{2}\int_{\pi/4}^{\pi/2} \sin^2 2\theta\, d\theta$$

$$= \frac{1}{2}\left(\pi/2 - \pi/4\right) - \cancel{\frac{1}{2}\left[\frac{1-\cos 4\theta}{2}\right]_{\pi/4}^{\pi/2}} - \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{2} * \frac{\pi}{4} - \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2}$$

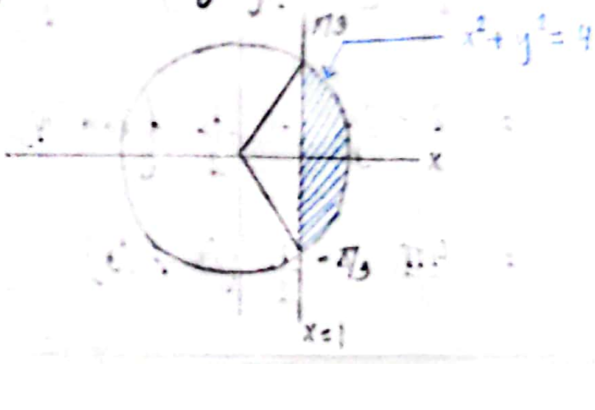
$$= \frac{\pi}{8} - \frac{1}{4}\left(\pi/2 + 0 - \pi/4 - 0\right)$$

$$= \pi/8 - \pi/16$$

$$= \pi/16$$

10. The region inside the circle $x^2+y^2=4$ and to the right of the line $x=1$.

Sketching the region, we get:



$x^2+y^2=4$ gets converted to $r=2$ over $\theta : 0$ to 2π in polar co-ordinates. For $x=1$,

Putting $x = r\cos\theta = 1$, we get:

$r\cos\theta = 1$

For $r=2$,

$\cos\theta = \frac{1}{2}$

$\therefore \theta = -\pi/3 , \pi/3$

$\therefore$ line $x=1$ becomes $r\cos\theta = 1$ over $\theta = -\pi/3$ to $+\pi/3$.

Thus, area of shaded region, A is given by:

$$A = \int_{-\pi/3}^{\pi/3} \int_{r=1/\cos\theta}^{r=2} r \, dr \, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left[\frac{r^2}{2}\right]_{\sec\theta}^{2} d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4 - \sec^2\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4 \, d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} \sec^2\theta \, d\theta$$

$$= \frac{1}{2} * 4\left[\pi/3 - (-\pi/3)\right] - \frac{1}{2}\left[\tan\theta\right]_{-\pi/3}^{\pi/3}$$

$$= 2 * \frac{2\pi}{3} - \frac{1}{2}\left\{\tan \pi/3 - \tan(-\pi/3)\right\}$$

$$= \frac{4\pi}{3} - \frac{1}{2}\left(\sqrt{3} + \sqrt{3}\right)$$

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$$= \frac{4\pi}{3} - \sqrt{3}$$

$$= \left(\frac{4\pi}{3} - \sqrt{3}\right)$$

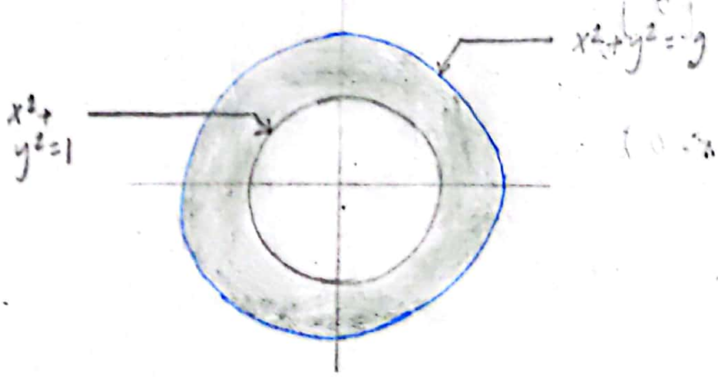
Find the volume of the solid discribed:

11. Inside of $x^2+y^2+z^2=9$, outside of $x^2+y^2=1$.

We know, in polar co-ordinates,

$x = r\cos\theta$, $y = r\sin\theta$.

Thus, the surface $x^2+y^2+z^2=9$ becomes $z = \sqrt{9-r^2}$. The domain of integration in polar co-ordinates is given by:



Now, required volume is given by:

$$V = \int_0^{2\pi}\int_{r=1}^{r=3} f(r,\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_{r=1}^{r=3} \sqrt{9-r^2}\; r\, dr\, d\theta$$

Put $u = 9-r^2$

$\therefore du = -2r\,dr$

when $r=1$, $u = 9-1^2 = 8$

$r=3$, $u = 9-9 = 0$

Thus the integral becomes,

$$= -\frac{1}{2}\int_0^{2\pi}\int_8^0 u^{1/2}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^8 u^{1/2}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{u^{3/2}}{3/2}\right]_0^8 d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}(16\sqrt{2}-0)\,d\theta$$

$$= \frac{1}{3} * 16\sqrt{2}\,\left[\theta\right]_0^{2\pi}$$

$$= \frac{16\sqrt{2}}{3}(2\pi - 0)$$

$$= \frac{32\sqrt{2}\,\pi}{3}$$

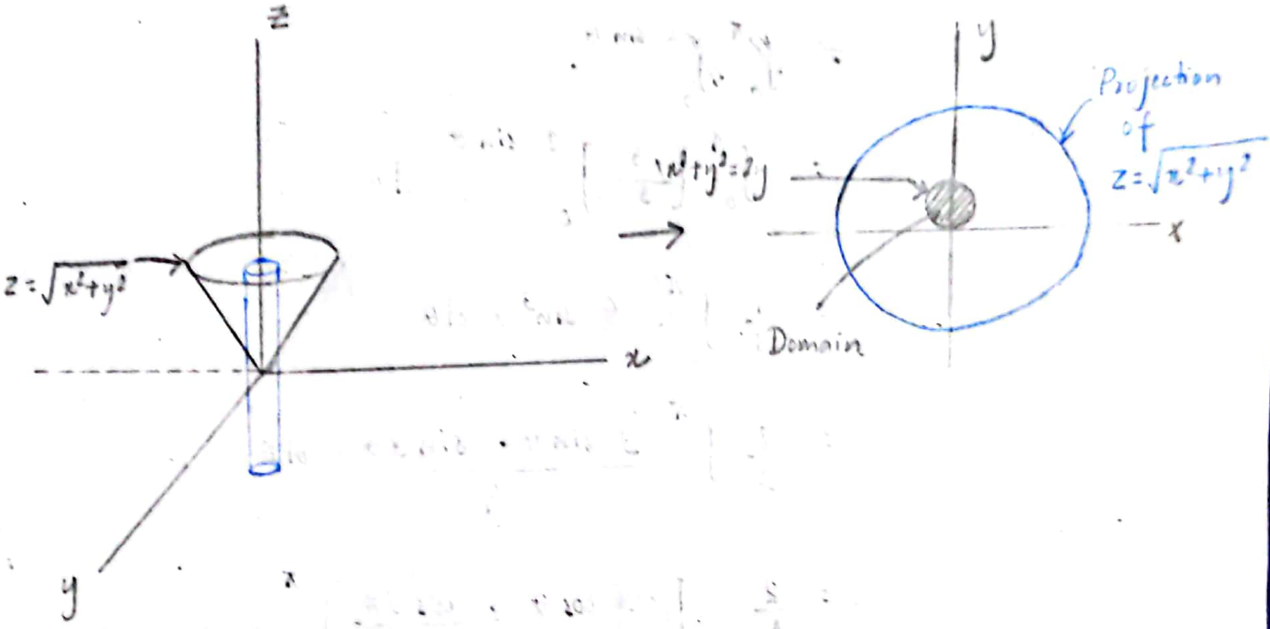
This gives the volume within the hemisphere. Total volume $V' = 2V$

$$= \frac{2 * 32\sqrt{2}\,\pi}{3}$$

$$= \frac{64\pi\sqrt{2}}{3}$$

12. Below $z = \sqrt{y^2 + x^2}$, inside of $x^2 + y^2 = 2y$, above $z = 0$.

Sketching the region in 3D,



Now, we have $x = r\cos\theta$ and $y = r\sin\theta$. Thus, $z = \sqrt{x^2+y^2}$ becomes,

$$z = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \sqrt{r^2}$$
$$= r$$

$x^2 + y^2 = 2y$ becomes $r^2(\cos^2\theta + \sin^2\theta) = 2 * r\sin\theta$

$$\therefore \; r = 2\sin\theta$$

Now, Required volume is given by

$$V = \int_0^{\pi}\int_0^{2\sin\theta} f(r,\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} r\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} r^2\, dr\, d\theta$$

$$\cancel{= \int_0^{\pi} \int_0^{2\sin\theta}}$$

$$= \int_0^{\pi} \left[\frac{r^3}{3}\right]_0^{2\sin\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi} 8 \sin^3\theta\, d\theta$$

$$= \frac{8}{3} \int_0^{\pi} \frac{3\sin\theta - \sin 3\theta}{4}\, d\theta$$

$$= \frac{2}{3} \left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\pi}$$

$$= \frac{2}{3} \left\{3 - \frac{1}{3} - \left(-3 + 1/3\right)\right\}$$

$$= \frac{2}{3} \left(3 - \frac{1}{3} + 3 - 1/3\right)$$

$$= \frac{2}{3} \left(6 - \frac{2}{3}\right)$$

$$= \frac{2}{3} * \frac{16}{3}$$

$$= \frac{32}{9}$$

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