Q32.

The given quadratic form can be written as

$$Q = x^T A x$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$ is a symmetric matrix.

The above quadratic form can be expressed with no cross-terms in the form of

$$Q' = y^T \Lambda y \quad \text{where, } x = Py,$$

~~where~~ P is a ~~diagonal~~ matrix defined as

$$\Lambda = P^T A P.$$

P is an orthogonal matrix formed by eigenvectors of A.

To find eigenvectors of A,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - 25 = 0$$

This gives,

$$1 - \lambda = -5$$

$$\therefore \lambda_1 = 6$$

Or,

$$1 - \lambda = +5$$

$$\therefore \lambda_2 = -4$$

For $\lambda_1 = 6$

$$\begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$x_1 = x_2$

Let $x_2 = 1$, then, $x_1 = 1$

For example, $v_{\lambda_1=6} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Unit eigenvector $u_{\lambda_1=6} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

For $\lambda_2 = -4$

$$\begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$x_1 = -x_2$

For example, the eigenvector $v_{\lambda=-4} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Unit eigenvector $u_{\lambda=-4} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Two vectors are orthogonal if $u_1 \cdot u_2 = 0$

$$\Rightarrow u_1^T u_2 = 0$$

Now, $u_{\lambda_1=6} \cdot u_{\lambda_2=-4} = u_{\lambda_1=6}^T \, u_{\lambda_2=-4}$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$= 0$$

$\therefore$ The two eigenvectors are orthogonal.

Now,

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 6 & -4 \\ 6 & 4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 12 & 0 \\ 0 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$$

Thus,

$$x^T A x = (Py)^T A (Py)$$

$$= y^T (P^T A P) y$$

$$= y^T \Lambda y$$

$$= (y_1 \;\; y_2) \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \boxed{6y_1^2 - 4y_2^2}$$

Q29.

SVD

$A = U \Sigma V^T$

$u_n = \frac{1}{\sigma_n} A v_n$

$U = (u_1 \; u_2 \ldots u_n)$

$\sigma = \sqrt{\lambda}$

$$A = \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix}$$

$$\rightarrow A^T A = \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix}^T \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 16 & 24 \\ 24 & 52 \end{pmatrix}$$

To find eigenvalues of $A^T A$,

$$\begin{vmatrix} 16-\lambda & 24 \\ 24 & 52-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (16-\lambda)(52-\lambda) - (24)^2 = 0$$

$$\Rightarrow \lambda_1 = 64, \quad \lambda_2 = 4$$

For $\lambda_1 = 64$

$$\begin{pmatrix} -48 & 24 \\ 24 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -48x_1 + 24x_2 = 0 \\ 24x_1 - 12x_2 = 0 \end{cases}$$

It follows that

$$2x_1 = x_2$$

For example, the eigenvector $\cancel{x}_{\lambda=64} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Unit ~~Normal~~ eigenvector $v_{\lambda=64} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

For $\lambda_2 = 4$

$$\begin{pmatrix} 12 & 24 \\ 24 & 48 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} 12x_1 + 24x_2 = 0 \\ 24x_1 + 48x_2 = 0 \end{cases}$$

The above follows that

$$x_1 = -2x_2$$

For example, the eigenvector $x_{\lambda=4} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Unit eigenvector $v_{\lambda=4} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

Now,

$$\sigma_1 = \sqrt{\lambda_1} = 8$$

$$\sigma_2 = \sqrt{\lambda_2} = 2$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{8} \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$= \frac{1}{8\sqrt{5}} \begin{pmatrix} 16 \\ 8 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$U = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

$$U \Sigma V^T = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 8 & 16 \\ -4 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 20 & 30 \\ 0 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 \\ 0 & 4 \end{pmatrix}$$

$$= A$$

CamScanner

Q28

let, $A = \begin{pmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{pmatrix}$

$A^T A = \begin{pmatrix} 7 & 5 & 0 \\ 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 74 & 32 \\ 32 & 26 \end{pmatrix}$

Let's compute its eigenvalues and find corresponding eigenvectors.

$A - \lambda I = \begin{pmatrix} 74-\lambda & 32 \\ 32 & 26-\lambda \end{pmatrix}$

For eigenvalues,

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 74-\lambda & 32 \\ 32 & 26-\lambda \end{vmatrix} = 0$

$\Rightarrow (74-\lambda)(26-\lambda) - (32)^2 = 0$

$\Rightarrow \lambda_1 = 90 \quad , \quad \lambda_2 = 10$

For $\lambda_1 = 90$

$\begin{pmatrix} 74-90 & 32 \\ 32 & 26-90 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -16 & 32 \\ 32 & -64 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ From above, we get:

$$\begin{cases} -16x_1 + 32x_2 = 0 \\ 32x_1 - 64x_2 = 0 \end{cases}$$

This means,

$x_1 = 2x_2$

For example, eigenvector for $\lambda_1 (= 90) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Unit eigenvector $v_{\lambda_1 = 90} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 10$

$$\begin{pmatrix} 74-10 & 32 \\ 32 & 26-10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 64 & 32 \\ 32 & 16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} 64x_1 + 32x_2 = 0 \\ 32x_1 + 16x_2 = 0 \end{cases}$$

This means,

$2x_1 = -x_2$

For example, eigenvector for $\lambda_2 (= 10) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Unit eigenvector $v_{\lambda_2 = 10} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Now,

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{90} = 3\sqrt{10}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{10}$$

Now,

$$u_1 = \frac{1}{\sigma_1} A v_1$$

$$= \frac{1}{3\sqrt{10}} \begin{pmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3\sqrt{50}} \begin{pmatrix} 15 \\ 15 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2$$

$$= \frac{1}{\sqrt{10}} \begin{pmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 5/\sqrt{50} & 5/\sqrt{50} \\ 5/\sqrt{50} & -5/\sqrt{50} \\ 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \end{pmatrix}$$

$$V = (v_1 \quad v_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$V^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

Now,

$$U \Sigma V^T = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 & 5 \\ 5 & -5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \frac{1}{5\sqrt{10}} \begin{pmatrix} 5 & 5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} 6\sqrt{10} & 3\sqrt{10} \\ \sqrt{10} & -2\sqrt{10} \end{pmatrix}$$

$$= \frac{1}{5\sqrt{10}} \begin{pmatrix} 35\sqrt{10} & 5\sqrt{10} \\ 25\sqrt{10} & 25\sqrt{10} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{pmatrix}$$

$$= A$$

$$= \begin{pmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{pmatrix}$$

$$= A$$

Steps: i) Compute A^TA

ii) Compute λ for A^TA

iii) Compute $V = (v_1 \; v_2 \dots)$

iv) Compute $u_i = \frac{1}{\sigma_i} A v_i$

v) SVD: $A = U \Sigma V^T$

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

$$U = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 & 5 \\ 5 & -5 \\ 0 & 0 \end{pmatrix}$$

$$U^T = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 & 5 & 0 \\ 5 & -5 & 0 \end{pmatrix}$$

$$U \Sigma U^T = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 & 5 \\ 5 & -5 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \end{pmatrix} \frac{1}{\sqrt{50}} \begin{pmatrix} 5 & 5 & 0 \\ 5 & -5 & 0 \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 5 & 5 \\ 5 & -5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 15\sqrt{10} & 15\sqrt{10} & 0 \\ 5\sqrt{10} & -5\sqrt{10} & 0 \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 100\sqrt{10} & 50\sqrt{10} & 0 \\ 50\sqrt{10} & 100\sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{10} & \sqrt{10} & 0 \\ \sqrt{10} & 2\sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq A$$

Q31.

Here,

$Q(x) = x_1^2 + x_2^2 - 10x_1 x_2$ can be written in matrix form as

$$Q = x^T A x$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where, $A = \begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix}$

Let's find eigenvalues and eigenvectors of A.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - 25$$

To compute eigenvalues, $|A - \lambda I| = 0$. Thus,

$(1-\lambda)^2 - 25 = 0$

$\Rightarrow (1-\lambda)^2 = (\pm 5)^2$

$\therefore \lambda_1 = 6, \quad \lambda_2 = -4$

Let, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be eigenvector. Then,

For $\lambda_1 = 6$

$(A - \lambda I)x = 0$

$$\Rightarrow \begin{pmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$\{ -5x_1 - 5x_2 = 0$

$\therefore x_1 = -x_2$

for example, eigenvector $x_{\lambda=6} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Unit eigenvector $v_{\lambda=6} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

The maximum value of $Q(x)$ subject to the constraint $x^T x$ is attained at the corresponding unit eigenvector associated with the greater eigenvalue (i.e. $\lambda = 6$). The maximum value of $Q(x)$ is equal to that of the greater eigenvalue.

$\therefore$ Maximum of $Q(x) = \lambda_1$
$= 6$

Unit vector u where this maximum is attained is $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\boxed{\text{check}}$

Put $x_1 = -1/\sqrt{2}$ & $x_2 = 1/\sqrt{2}$ in eq^n (i),

$Q(x) = x_1^2 + x_2^2 - 10 x_1 x_2$

$= \left(-1/\sqrt{2}\right)^2 + \left(1/\sqrt{2}\right)^2 - 10 \times -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{2} + \frac{1}{2} + \frac{10}{2}$

$= 5 + 1$

$= 6$

$= \lambda_1$

This verifies our theorem.

Q26.

A definite quadratic form is a quadratic form over some real vector space V that has the same sign (always positive or always negative) for every non-zero vector of V.

Let A be an $n\times n$ symmetric matrix and $Q(x) = x^T A x$ is the corresponding quadratic form. Then Q is

a) positive definite if $x^T A x > 0, \ \forall x \neq 0$

b) negative definite if $x^T A x < 0, \ \forall x \neq 0$

c) indefinite if $x^T A x > 0$ for some x and $x^T A x < 0$ for others.

d) positive semidefinite if $x^T A x \geq 0, \ \forall x \neq 0$

e) negative semidefinite if $x^T A x \leq 0, \ \forall x \neq 0$

Here,

$$Q(x) = 2x_1^2 - 4x_1 x_2 - x_2^2 \quad \text{—— (i)}$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x^T A x$$

Thus,

$$A = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}$$

lets find eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -2 \\ -2 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda) - 4$$

- For eigenvalues' computation,

$$|A - \lambda I| = 0$$

$$\Rightarrow -2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2\lambda - 6 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 2) = 0$$

$$\therefore \lambda_1 = 3, \ \lambda_2 = -2$$

Since both the eigenvectors are positive, the given quadratic form $Q(x)$ is positive definite.

Let's calculate the eigenvectors

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be eigenvector of A. Then,

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{pmatrix} 2-\lambda & -2 \\ -2 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = 3$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} -x_1 - 2x_2 = 0 \\ -2x_1 - 4x_2 = 0 \end{cases}$$

The above equations reduce to

$$x_1 = -2x_2$$

For example, the eigenvector $x_{\lambda_1 = 3} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\therefore$ Unit eigenvector $v_{\lambda_1=3} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

For $\lambda_2 = -2$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} 4x_1 - 2x_2 = 0 \\ -2x_1 + x_2 = 0 \end{cases}$$

which gives,

$$2x_1 = x_2$$

For example, the eigenvector $x_{\lambda=-2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Unit eigenvector $v_{\lambda=-2} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

$$v_{\lambda=3} \cdot v_{\lambda=-2} = v^T_{\lambda=3}\, v_{\lambda=-2}$$

$$= \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$= -2/\sqrt{5} + 2/\sqrt{5}$$

$$= 0$$

Thus the unit eigenvectors are orthogonal.

Then,

Matrix of eigenvectors, $P = (v_1 \quad v_2)$

$$= \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

The given quadratic form (i) can be expressed in quadratic form with no cross terms by substituting

$$x = Py$$

where P is matrix of unit eigenvectors and the corresponding quadractic form is:

$$Q = y^T \Lambda y$$

From definition of Λ, we have

$$\Lambda = P^T A P$$

$$= \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -6 & -2 \\ 3 & -4 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 15 & 0 \\ 0 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\therefore Q = y^T \Lambda y$$

$$= (y_1 \;\; y_2) \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$= 3y_1^2 - 2y_2^2$ is the required quadratic form.

Q1.

$$A = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}$$

Let, λ be eigenvalue and x be the eigenvector of matrix A such that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then,

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{vmatrix}$$

$$= (5-\lambda)(-4-\lambda) + 14$$

For eigenvalue,
The above expression is 0 i.e.

$|A-\lambda I| = 0$

$\Rightarrow (5-\lambda)(-4-\lambda) + 14 = 0$

$\Rightarrow -20 - 5\lambda + 4\lambda + \lambda^2 + 14 = 0$

$\Rightarrow \lambda^2 - \lambda - 6 = 0$

$\therefore \lambda_1 = 3, \ \lambda_2 = -2$

To calculate eigenvectors, $(A-\lambda I)x = 0$

For $\lambda_1 = 3$

$$\begin{pmatrix} 2 & -2 \\ 7 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} 2x_1 - 2x_2 = 0 \\ 7x_1 - 7x_2 = 0 \end{cases}$$

The above equation reduces to $x_1 = x_2$. For example,

$v_{\lambda=3} = \cancel{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = -2$

$$\begin{pmatrix} 7 & -2 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$7x_1 - 2x_2 = 0$$

$$\Rightarrow x_1 = \frac{2}{7} x_2$$

For example, the eigenvector

$$v_{\lambda=-2} = \begin{pmatrix} 2/7 \\ 1 \end{pmatrix}$$

Effect of multiplying the eigenvector by matrix A

$$Av_{\lambda_1=3} = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5-2 \\ 7-4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$= 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= 3 v_{\lambda_1=3}$$

$$Av_{\lambda_2=-2} = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix} \begin{pmatrix} 2/7 \\ 1 \end{pmatrix} = \begin{pmatrix} 10/7 - 2 \\ 2 - 4 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 3/7 \\ -2 \end{pmatrix}} = \begin{pmatrix} -4/7 \\ -2 \end{pmatrix}$$

$$= -2 \begin{pmatrix} 2/7 \\ 1 \end{pmatrix}$$

$$= -2 v_{\lambda_2=-2}$$

Therefore, it shows that the effect of multiplying the eigenvector by A is scaling up the eigenvector in the same or opposite direction. This confirms that

$$\boxed{Ax = \lambda x}$$

where A is an arbitrary matrix and other symbols have their usual meanings.

Plotting of Eigenspaces E_λ

<u>For $\lambda = 3$</u>

$E_3 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 = x_2$. Therefore, it can be written as:

$$E_3 = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \text{span}\{(1, 1)\}$$

$\therefore$ Basis for $E_3 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

<u>For $\lambda = -2$</u>

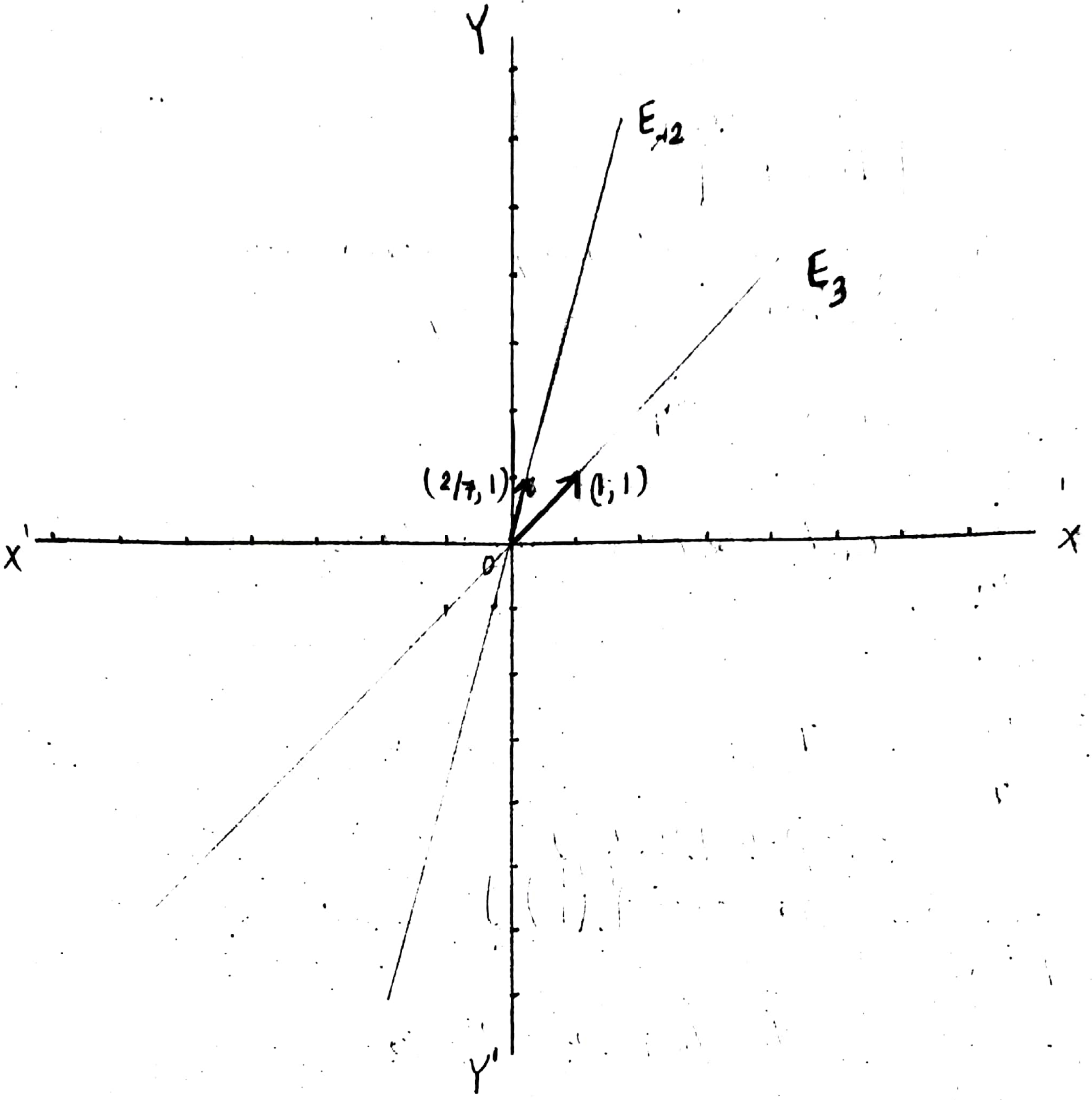
$E_{-2} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 = 2/7 \, x_2$

$$\therefore E_{-2} = \begin{pmatrix} 2/7 \, x_2 \\ x_2 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} 2/7 \\ 1 \end{pmatrix}$$

$$= \text{span}\,(2/7, 1)$$

$\therefore$ Basis for $E_{-2} = \left\{ \begin{matrix} 2/7 \\ 1 \end{matrix} \right\}$



Basis for $E_3 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

Basis for $E_{-2} = \left\{ \begin{pmatrix} 2/7 \\ 1 \end{pmatrix} \right\}$

Q3.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & 9 & 1 \end{pmatrix}$$

For eigenvalues and eigenvectors, we use characteristic equation:

$|A - \lambda I| = 0$

Now,

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -3 & 1-\lambda & 0 \\ 7 & 9 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(1-\lambda)$$

Now, we know,

$|A - \lambda I| = 0$

$\Rightarrow (1-\lambda)(1-\lambda)(1-\lambda) = 0$

$\therefore \lambda = 1$

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigenvector. Then,

$(A - \lambda I)x = 0$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 7 & 9 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From above, we get:

$$\begin{cases} -3x_1 = 0 \\ 7x_1 + 9x_2 = 0 \end{cases}$$

$\therefore x_1 = 0 \quad x_2 = 0$

x_3 is a free variable. So, we can choose any value of x_3.

For example, eigenvector $v_{\lambda=1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Q4. Let us consider a $2*2$ matrix A which is given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Let, λ be the eigenvalue of the above matrix A. Then,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12} * a_{21} = 0$$

$$\therefore (a_{11}-\lambda)(a_{22}-\lambda) = a_{12} * a_{21} \quad \cdots \text{(i)}$$

Now, lets compute eigenvalues of transpose matrix A^T. i.e.

$$|A^T - \lambda' I| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda' & a_{21} \\ a_{12} & a_{22}-\lambda' \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda')(a_{22}-\lambda') - a_{12} * a_{21} = 0.$$

$$\Rightarrow (a_{11}-\lambda')(a_{22}-\lambda') = a_{12} * a_{21}$$

$$\Rightarrow (a_{11}-\lambda')(a_{22}-\lambda') = (a_{11}-\lambda)(a_{22}-\lambda) \quad [\text{From (i)}]$$

Since the constant terms of both LHS and RHS are equal, therefore,

$$\boxed{\lambda' = \lambda}$$

This proves that the eigenvalues of A and A^T are equal.