DIRECTIONAL DERIVATIVES AND GRADIENT

1. Find ∇f if

a) $f(x,y) = -x^2y + xy^2 + xy$

We know,

$$\nabla f = (f_x(x,y),\ f_y(x,y))$$

$$f_x(x,y) = \frac{\partial}{\partial x}(-x^2y + xy^2 + xy)$$

$$= -2xy + y^2 + y$$

$$f_y(x,y) = \frac{\partial}{\partial y}(-x^2y + xy^2 + xy)$$

$$= -x^2 + 2xy + x$$

$$\therefore \nabla f = (-2xy + y^2 + y,\ -x^2 + 2xy + x)$$

b) $f(x,y) = \sin x \cos y$

We know,

$$\nabla f = (f_x(x,y),\ f_y(x,y))$$

$$f_x(x,y) = \frac{\partial}{\partial x}(\sin x \cos y)$$

$$= \cos y \cdot \cos x$$

$$= \cos x \cos y$$

$$\cancel{f_y(x,y) = \frac{\partial}{\partial y}(\sin x \sin y)}$$

$$\cancel{= \sin y \cdot \cos y}$$

$$f_y(x,y) = \frac{\partial}{\partial y}(\sin x \cos y)$$

$$= \sin x \frac{\partial}{\partial y}(\cos y)$$

$$= \sin x(-\sin y)$$

$$= -\sin x \sin y$$

$\therefore \nabla f = (\cos x \cos y, -\sin x \sin y)$

2. A function $z = f(x,y)$ and a point P are given. Find the directional derivative of f in the indicated directions.

a. $f(x,y) = \frac{1}{x^2+y^2+1}$, $P = (1,1)$.

i) In the direction of $v = (1,-1)$.

Here,

$$f(x,y) = \frac{1}{x^2+y^2+1}$$

we know, $D_u f(x,y) = \nabla f(x,y) \cdot u$

where u is a unit vector.

Calculation of $\nabla f(x,y)$.

$\nabla f(x,y) = (f_x(x,y), f_y(x,y))$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2+1}\right)$$

$$= \frac{\partial (x^2+y^2+1)^{-1}}{\partial x}$$

$$= \frac{-1}{(x^2+y^2+1)^2} \cdot \frac{\partial (x^2+y^2+1)}{\partial x}$$

$$= \frac{-2x}{(x^2+y^2+1)^2}$$

$$\therefore f_x(1,1) = \frac{-2\times 1}{((1)^2+(1)^2+1)^2}$$

$$= \frac{-2}{3^2}$$

$$= \frac{-2}{9}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2+1}\right)$$

$$= \frac{\partial}{\partial y}(x^2+y^2+1)^{-1}$$

$$= \frac{-1}{(x^2+y^2+1)^2} \cdot \frac{\partial}{\partial y}(x^2+y^2+1)$$

$$= \frac{-2y}{(x^2+y^2+1)^2}$$

$$\therefore f_y(1,1) = \frac{-2\times 1}{(1^2+1^2+1)^2}$$

$$= \frac{-2}{9}$$

$$\therefore \nabla f(1,1) = (f_x(1,1), f_y(1,1))$$

$$= (-2/9, -2/9)$$

Given, $v = (1,-1)$

Unit vector along v is given by $\hat{v} = \frac{v}{|v|}$

$$= \frac{(1,-1)}{\sqrt{1^2+(-1)^2}}$$

$$= (1/\sqrt{2}, -1/\sqrt{2})$$

$$\therefore D_{\hat{v}} f(1,1) = \nabla f(1,1) \cdot \hat{v}$$

$$= (-2/9, -2/9) \cdot (1/\sqrt{2}, -1/\sqrt{2})$$

$$= -\frac{2}{9\sqrt{2}} + \frac{2}{9\sqrt{2}}$$

$$= 0$$

ii) In the direction toward the point $Q = (-2,-2)$

Now,

$$\vec{PQ} = (-2,-2) - (1,1)$$

$$= (-3,-3)$$

$\therefore$ Unit vector u along $\vec{PQ} = \frac{(-3,-3)}{\sqrt{(-3)^2+(-3)^2}}$

$$= \frac{(-3, -3)}{\sqrt{18}}$$

$$= \left(\frac{-3}{\sqrt{18}}, -\frac{3}{\sqrt{18}}\right)$$

$$\therefore D_u f(1,1) = \nabla f(1,1) \cdot u$$

$$= (-2/9, -2/9) \cdot \left(-3/\sqrt{18}, -3/\sqrt{18}\right)$$

$$= \frac{6}{9\sqrt{18}} + \frac{6}{9\sqrt{18}}$$

$$= \frac{12}{9\sqrt{18}}$$

$$= \frac{12}{27\sqrt{2}}$$

$$= \frac{4}{9\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{9}$$

b) $f(x,y) = -4x + 3y, \quad P = (5, 2)$

i) In the direction of $v = (3, 1)$

Here,

$$f(x,y) = -4x + 3y$$

$$\therefore f_x(x,y) = \frac{\partial}{\partial x}(-4x + 3y)$$

$$= -4$$

$$\therefore f_y(x,y) = \frac{\partial}{\partial y}(-4x + 3y)$$

$$= 3$$

$$\therefore \nabla f(x,y) = (f_x(x,y), f_y(x,y))$$

$$= (-4, 3)$$

$$\therefore \nabla f(5,2) = (-4, 3)$$

Now, given vector $v = (3,1)$

Unit vector u along $v = \dfrac{(3,1)}{\sqrt{(3)^2+(1)^2}}$

$$= \frac{(3,1)}{\sqrt{10}}$$

$$= \left(3/\sqrt{10},\ 1/\sqrt{10}\right)$$

$\therefore D_u f(5,2) = \nabla f(5,2) \cdot u$

$$= (-4,3) \cdot \left(3/\sqrt{10},\ \frac{1}{\sqrt{10}}\right)$$

$$= \frac{-12}{\sqrt{10}} + \frac{3}{\sqrt{10}}$$

$$= \frac{-9}{\sqrt{10}}$$

ii) In the direction toward the point $Q = (2,7)$.

Here, $\overrightarrow{PQ} = (2,7) - (5,2)$

$$= (-3,5)$$

Unit vector along $\overrightarrow{PQ}$, $u = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

$$= \frac{(-3,5)}{\sqrt{(-3)^2+(5)^2}}$$

$$= \frac{(-3,5)}{\sqrt{34}}$$

$$= \left(-3/\sqrt{34},\ 5/\sqrt{34}\right)$$

$\therefore D_u f(5,2) = \nabla f(5,2) \cdot u$

$$= (-4,3) \cdot \left(\frac{-3}{\sqrt{34}},\ \frac{5}{\sqrt{34}}\right)$$

$$= \frac{12}{\sqrt{34}} + \frac{15}{\sqrt{34}}$$

$$= \frac{27}{\sqrt{34}}.$$

3. If

a) $f(x,y) = x^2 + 2y^2 - xy - 7x$, $P = (4,1)$

then

i) Find the direction of maximal increase of f at P.

Here, $f(x,y) = x^2 + 2y^2 - xy - 7x$ and $P(4,1)$.

The direction of maximal increase of f at P is given by $\frac{\nabla f(P)}{\|\nabla f(P)\|}$ or $\nabla f(P)$.

Here,

$$f_x(x,y) = 2x - y - 7$$

$$\therefore f_x(4,1) = 2\times4 - 1 - 7$$
$$= 0$$

$$f_y(x,y) = \frac{\partial}{\partial y}(x^2 + 2y^2 - xy - 7x)$$

$$= 2\times2y - x$$
$$= 4y - x$$

$$\therefore f_y(4,1) = 4\times1 - 4$$
$$= 0$$

$$\therefore \nabla f(P) = (f_x(P), f_y(P)]$$

$$= (0,0)$$

Thus, the required direction is $(0,0)$.

ii) What is the maximal value of $D_u f$ at P?

Maximal value of $D_u f$ at P is given by $\|\nabla f(P)\|$.

$$\|\nabla f(P)\| = \sqrt{(0)^2 + (0)^2}$$
$$= \sqrt{0}$$
$$= 0.$$

iii)

iv) Give a direction u such that $D_u f = 0$ at P.

Here,

$$D_u f = 0 \text{ at } P$$

$$\Rightarrow \nabla f(P) \cdot u = 0 \quad \text{———— (i)}$$

We know that $\nabla f(P) = 0$. Thus for any value of u, the above eqn(i) is satisfied. Thus, for all directions u, $D_u f(P) = 0$.

b) $f(x, y) = x^2 y^3 - 2x$, $P = (1, 1)$,

i) Find the direction of maximal increase of f at P.

Here, $f(x, y) = x^2 y^3 - 2x$, $P = (1, 1)$

Thus,

$$f_x(x,y) = \frac{\partial}{\partial x}(x^2y^3 - 2x)$$

$$= 2xy^3 - 2$$

$$\therefore f_x(1,1) = 2\times1\times1^3 - 2$$

$$= 0$$

$$f_y(x,y) = \frac{\partial}{\partial y}(x^2y^3 - 2x)$$

$$= 3x^2y^2$$

$$\therefore f_y(1,1) = 3\times1^2\times1^2$$

$$= 3$$

$$\therefore \nabla f(P) = (f_x(1,1), f_y(1,1))$$

$$= (0,3)$$

The direction of maximal increase is given by

$$\frac{\nabla f(P)}{\|\nabla f(P)\|} = \frac{(0,3)}{\sqrt{0^2+3^2}}$$

$$= \frac{(0,3)}{3}$$

$$= (0,1)$$

ii) What is the maximal value of $D_u f$ at P?

The maximal value of $D_u f$ at P is given by $\|\nabla f(P)\|$.

$$\|\nabla f(P)\| = \sqrt{(0)^2 + (3)^2}$$

$$= \sqrt{0+9}$$

$$= \sqrt{9}$$

$$= 3.$$

iii) Find the direction of minimal increase of f at P.

iv) Give a direction u such that $D_u f = 0$ at P.

Here,

$$D_u f(P) = 0$$

$$\Rightarrow \nabla f(P) \cdot u = 0$$

$$\Rightarrow (0, 3) \cdot u = 0$$

$$\Rightarrow (0, 3) \cdot (u_1, u_2) = 0$$

$$\Rightarrow 0 + 3u_2 = 0$$

$$\therefore u_2 = 0$$

$\therefore$ Required direction $u = (u_1, u_2)$

$= (u_1, 0)$

Since u is a unit vector,

$u = (1, 0)$.

4. A function $w = f(x, y, z)$, a vector v and a point P are given.

a) $f(x, y, z) = 3x^2z^3 + 4xy - 3z^2$, $v = (1, 1, 1)$, $P = (3, 2, 1)$.

i) Find $\nabla f(x, y, z)$

Here,

$$f(x, y, z) = 3x^2z^3 + 4xy - 3z^2$$

Now,

$$f_x(x, y, z) = \frac{\partial}{\partial x}(3x^2z^3 + 4xy - 3z^2)$$

$$= 6xz^3 + 4y$$

$$f_y(x, y, z) = \frac{\partial}{\partial y}(3x^2z^3 + 4xy - 3z^2)$$

$$= 4x - 0$$

$$= 4x$$

$$f_z(x, y, z) = \frac{\partial}{\partial z}(3x^2z^3 + 4xy - 3z^2)$$

$$= 9x^2z^2 - 6z$$

$$\therefore \nabla f(x, y, z) = (6xz^3 + 4y,\ 4x,\ 9x^2z^2 - 6z)$$

$$\therefore \nabla f(P) = \nabla f(3, 2, 1) = (6 \times 3 \times 1^3 + 4 \times 2,\ 4 \times 3,\ 9 \times 3^2 \times 1^2 - 6 \times 1)$$

$$= (26, 12, 75)$$

ii) Find the directional derivative of f at P in the direction of v.

$$D_v f(P) = ?$$

Here, $v = (1, 1, 1)$

Unit vector along $v = \frac{v}{||v||}$

$$= \frac{(1, 1, 1)}{\sqrt{(1)^2 + (1)^2 + (1)^2}}$$

$$= \frac{(1, 1, 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\therefore D_{\hat{v}} f(P) = \nabla f(P) \cdot \hat{v}$$

$$= (26, 12, 75) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{26 + 12 + 75}{\sqrt{3}}$$

$$= \frac{113}{\sqrt{3}}$$

b) $f(x, y, z) = \sin(x) \cos(y) e^{z}$, $v = (2, 2, 1)$, $P = (0, 0, 0)$.

i) Find $\nabla f(x, y, z)$

Here,

$$f(x, y, z) = \sin(x) \cos(y) e^{z}$$

$$f_x(x, y, z) = \frac{\partial}{\partial x} (\sin(x) \cos(y) e^{z})$$

$$= \cos(y) e^{z} \cos(x)$$

$$= \cos(x) \cos(y) e^{z}$$

$$f_y(x, y, z) = \frac{\partial}{\partial y} (\sin(x) \cos(y) e^{z})$$

$$= \sin(x) e^{z} (-\sin(y))$$

$$= -\sin(x) \sin(y) e^{z}$$

$$f_z(x, y, z) = \frac{\partial}{\partial z} (\sin(x) \cos(y) e^{z})$$

$$= \sin(x) \cos(y) e^{z}$$

$$\therefore \nabla f(x, y, z) = (f_x, f_y, f_z)$$

$$= (\cos(x) \cos(y) e^{z}, -\sin(x) \sin(y) e^{z}, \sin(x) \cos(y) e^{z})$$

ii) Find the directional derivative of f at P in the direction of v.

Here,

$D_v f(P) = ?$

Given $v = (2, 2, 1)$ and $P = (0, 0, 0)$.

Unit vector along $v = \frac{v}{||v||}$

$= \frac{(2, 2, 1)}{\sqrt{(2)^2 + (2)^2 + (1)^2}}$

$= \frac{(2, 2, 1)}{3}$

$= (2/3, 2/3, 1/3)$

$\nabla f(P) = \left(\cos(0)\cos(0)e^0, -\sin(0)\sin(0)e^0, \sin(0)\cos(0)e^0\right)$

$= (1, 0, 0)$

$\therefore D_v f(P) = \nabla f(P) \cdot \hat{v}$

$= (1, 0, 0) \cdot \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$= \frac{2}{3}$

5. Find the directional derivative of $f(x, y, z) = xy + z^3$ at $P = (3, -2, -1)$ in the direction pointing to the origin.

Here,

$f(x, y, z) = xy + z^3$

$\vec{PO} = (0, 0, 0) - (3, -2, -1)$

$= (-3, 2, 1)$

Unit vector u along $\vec{PO} = \frac{(-3, 2, 1)}{\sqrt{(-3)^2 + (2)^2 + (1)^2}}$

$$= \frac{(-3, 2, 1)}{\sqrt{14}}$$

$$= \left(\frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right)$$

Now,

$$f_x(x, y, z) = \frac{\partial}{\partial x}(xy + z^3)$$

$$= y$$

$$f_y(x, y, z) = \frac{\partial}{\partial y}(xy + z^3)$$

$$= x$$

$$f_z(x, y, z) = \frac{\partial}{\partial z}(xy + z^3)$$

$$= 3z^2$$

$$\therefore \nabla f(x, y, z) = (y, x, 3z^2)$$

$$\therefore \nabla f(P) = \nabla f(3, -2, -1) = (-2, 3, 3 \times (-1)^2)$$

$$= (-2, 3, 3)$$

$\therefore$ Directional derivative $D_u f(P) = \nabla f(P) \cdot u$

$$= (-2, 3, 3) \cdot \left(\frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right)$$

$$= \frac{6 + 6 + 3}{\sqrt{14}}$$

$$= \frac{15}{\sqrt{14}}.$$

6. A bug located at $(3, 9, 4)$ begins walking in a straight line toward $(5, 7, 3)$. At what rate is the bug's temperature changing if the temperature is $T(x, y, z) = xe^{y-z}$? Units are in meters and degrees Celsius.

Here,

Temperature function, $T(x, y, z) = xe^{y-z}$

starting point, $P = (3, 9, 4)$; Directed point $Q = (5, 7, 3)$

Direction vector, $u = \overrightarrow{PQ}$

$= (5, 7, 3) - (3, 9, 4)$

$= (2, -2, -1)$

$\therefore$ Unit vector $\hat{u} = \frac{u}{\|u\|}$

$= \frac{(2, -2, -1)}{\sqrt{(2)^2 + (-2)^2 + (-1)^2}}$

$= \frac{(2, -2, -1)}{\sqrt{9}}$

$= \left(2/3, -2/3, -1/3\right)$

Now,

$T_x(x, y, z) = \frac{\partial}{\partial x}(xe^{y-z})$

$= e^{y-z}$

$T_y(x, y, z) = \frac{\partial}{\partial y}(xe^{y-z})$

$= xe^{y-z}$

$T_z(x, y, z) = \frac{\partial}{\partial z}(xe^{y-z})$

$= -xe^{y-z}$

$\therefore \nabla T(x, y, z) = \left(e^{y-z}, xe^{y-z}, -xe^{y-z}\right)$

$\therefore \nabla T(P) = \nabla T(3, 9, 4) = \left(e^{9-4}, 3e^{9-4}, -3e^{9-4}\right)$

$= (e^5, 3e^5, -3e^5)$

Now,

$$D_{\hat{u}}\, T(P) = \nabla T(P) \cdot \hat{u}$$

$$= (e^5, 3e^5, -3e^5) \cdot (2/3, -2/3, -1/3)$$

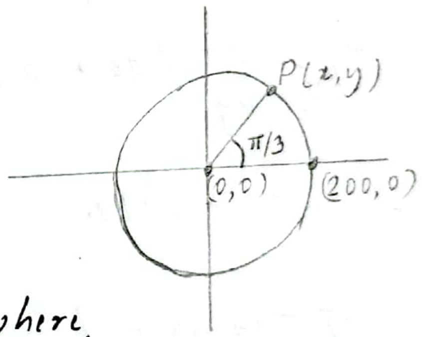
$$= \frac{2e^5 + (-6e^5) + 3e^5}{3}$$

$$= \frac{-e^5}{3}$$

$$\approx -49.47$$

$\therefore$ Rate of change of temperature is $-49.47°C$.

Q7. The temperature at location (x, y) is $T(x, y) = 20 + 0.1(x^2 - xy)$ (degree celcius). Beginning at $(200, 0)$ at time $t = 0$ (second), a bug travels along a circle of radius 200 cm centered at the origin, at a speed of 3 cm/s. How fast is the temperature changing at time $t = \pi/3$?



The position $P(x, y)$ of a circle with center $(0, 0)$ is given in parametric form by

$$r(t) = (200 \cos t,\ 200 \sin t) \quad \text{—(i)} \quad \text{where, } 0 \leq t \leq 2\pi$$

At $t = \pi/3$,

$$r(\pi/3) = (200 \times \cos \pi/3,\ 200 \sin \pi/3)$$

$$= (100,\ 100\sqrt{3})$$

Differentiating (i), with respect to t, we get:

$$r'(t) = (-200 \sin t,\ 200 \cos t)$$

At $t = \pi/3$,

$$v = r'(t) = r'(\pi/3) = (-200 \times \sin \pi/3,\ 200 \times \cos \pi/3)$$

$$= (-100\sqrt{3},\ 100)$$

The unit vector u associated with v is given by

$$u = \frac{v}{\|v\|}$$

$$= \frac{(-100\sqrt{3},\ 100)}{\sqrt{(-100\sqrt{3})^2 + (100)^2}}$$

$$= \frac{(-100\sqrt{3},\ 100)}{200}$$

$$= \left(-\sqrt{3}/2,\ 1/2\right)$$

Now, for gradient of Temperature function,

Here, $T(x, y) = 20 + 0.1\,(x^2 - xy)$

$$T_x = \frac{\partial}{\partial x}\left(20 + 0.1\,(x^2 - xy)\right)$$

$$= 0.2x - \cancel{y}\, 0.1y$$

$$T_y = \frac{\partial}{\partial y}\left(20 + 0.1\,(x^2 - xy)\right)$$

$$= -0.1x$$

$$\therefore \nabla T(x, y) = (T_x,\ T_y)$$

$$= (0.2x - 0.1y,\ -0.1x)$$

$$\therefore \nabla T(100, 100\sqrt{3}) = (0.2 \times 100 - 0.1 \times 100\sqrt{3},\ -0.1 \times 100)$$

$$= (20 - 10\sqrt{3},\ -10)$$

The rate of change of temperature at the point $t = \pi/3$ is given by directional directive:

$$D_u T(\pi/3) = \nabla T(100, 100\sqrt{3}) \cdot u$$

$$= (20 - 10\sqrt{3},\ -10) \cdot \left(\frac{-\sqrt{3}}{2},\ \frac{1}{2}\right)$$

$$= -10\sqrt{3} + 15 - 5$$

$= 10 - 10\sqrt{3}$

≈ -7.32

$\therefore$ The change in temperature is $\underline{-7.32^\circ C}$.

8. Suppose that $\nabla f_P = (2, -4, 4)$. Is f increasing or decreasing at P in the direction $v = (2, 1, 3)$?

Here,

$v = (2, 1, 3)$

Unit vector u in direction of $v = \dfrac{v}{\|v\|}$

$$= \frac{(2,1,3)}{\sqrt{(2)^2+(1)^2+(3)^2}}$$

$$= \frac{(2,1,3)}{\sqrt{14}}$$

$$= \left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

$$\therefore D_u f(P) = \nabla f_P \cdot u$$

$$= (2,-4,4) \cdot \left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

$$= \frac{4-4+12}{\sqrt{14}}$$

$$= \frac{12}{\sqrt{14}} > 0$$

$\therefore$ f is increasing at P in the direction of v.

9. Let $f(x, y, z) = \sin(xy + z)$ and $P = (0, -1, \pi)$. Calculate $D_u f(P)$, where u is a unit vector making an angle $\theta = 30^\circ$ with ∇f_P.

Here,

$$f(x, y, z) = \sin(xy + z)$$

$$P = (0, -1, \pi)$$

Given u is a unit vector making an angle $\theta = 30^\circ$ with ∇f_P.

Now,

$$f_x(x, y, z) = \frac{\partial}{\partial x}(\sin(xy + z))$$

$$= y\cos(xy + z)$$

$$f_y(x, y, z) = \frac{\partial}{\partial y}(\sin(xy + z))$$

$$= x\cos(xy + z)$$

$$f_z(x, y, z) = \frac{\partial}{\partial z}(\sin(xy + z))$$

$$= \cos(xy + z)$$

Thus, $\nabla f(x, y, z) = (y\cos(xy + z),\ x\cos(xy + z),\ \cos(xy + z))$

$$\therefore \nabla f_P = \nabla f(0, -1, \pi) = (-1\cos\pi,\ 0\cos\pi,\ \cos\pi)$$

$$= (1, 0, -1).$$

Now,

$$D_u f(P) = \nabla f_P \cdot u$$

$$= \|\nabla f_P\| \|u\| \cos\theta$$

$$= \sqrt{(1)^2 + (0)^2 + (-1)^2}\ (1) \cos 30^\circ$$

$$= \sqrt{2} * \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{6}}{2}$$

10. Find an equation of tangent plane to the surface at the given point.

a) $x^2 + 3y^2 + 4z^2 = 20$, $P = (2, 2, 1)$

Here,

$$F(x, y, z) = x^2 + 3y^2 + 4z^2$$

Now,

$$F_x(x, y, z) = \frac{\partial}{\partial x}(x^2 + 3y^2 + 4z^2)$$

$$= 2x$$

$$F_y(x, y, z) = \frac{\partial}{\partial y}(x^2 + 3y^2 + 4z^2)$$

$$= 6y$$

$$F_z(x, y, z) = \frac{\partial}{\partial z}(x^2 + 3y^2 + 4z^2)$$

$$= 8z$$

$\therefore F_x(2, 2, 1) = 2 \times 2 = 4$

$\therefore F_y(2, 2, 1) = 6 \times 2 = 12$

$\therefore F_z(2, 2, 1) = 8 \times 1 = 8$

Now equation of tangent plane to a given surface S at point P is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad \text{— (i)}$$

where (x_0, y_0, z_0) is the given point P.

For $P = (2, 2, 1)$, eq^n (i) becomes:

$$F_x(2, 2, 1)(x - 2) + F_y(2, 2, 1)(y - 2) + F_z(2, 2, 1)(z - 1) = 0$$

$\Rightarrow 4(x - 2) + 12(y - 2) + 8(z - 1) = 0$

$\Rightarrow 4x + 12y + 8z - 40 = 0$ is the required equation of tangent plane.

6. $xz + 2x^2y + y^2z^3 = 11$, $P(2,1,1)$

Here,

$$F(x,y,z) = xz + 2x^2y + y^2z^3, \quad P(2,1,1)$$

Now,

$$F_x(x,y,z) = \frac{\partial}{\partial x}(xz + 2x^2y + y^2z^3)$$

$$= z + 4xy$$

$$F_y(x,y,z) = \frac{\partial}{\partial y}(xz + 2x^2y + y^2z^3)$$

$$= 2x^2 + 2yz^3$$

$$F_z(x,y,z) = \frac{\partial}{\partial z}(xz + 2x^2y + y^2z^3)$$

$$= x + 3y^2z^2$$

At Point $P = (2,1,1)$

$\therefore F_x(2,1,1) = 1 + 4 \times 2 \times 1 = 9$

$\therefore F_y(2,1,1) = 2 \times 2^2 + 2 \times 1 \times 1^3 = 10$

$\therefore F_z(2,1,1) = 2 + 3 \times 1^2 \times 1^2 = 5$

Now, equation of tangent plane to a given surface S at point $P(x_0, y_0, z_0) = P(2,1,1)$ is given by:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$\Rightarrow F_x(2,1,1)(x-2) + F_y(2,1,1)(y-1) + F_z(2,1,1)(z-1) = 0$$

$$\Rightarrow 9(x-2) + 10(y-1) + 5(z-1) = 0$$

$\therefore 9x + 10y + 5z - 33 = 0$ is required equation of tangent plane P' for Surface S at point $P(2,1,1)$.

11. Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f at P in that direction.

a) $f(x, y) = 4x^3y^2$; $P(-1, 1)$

Here,

$$f(x, y) = 4x^3y^2$$

Then, $f_x(x, y) = \frac{\partial}{\partial x}(4x^3y^2)$

$$= 12x^2y^2$$

$$f_y(x, y) = \frac{\partial}{\partial y}(4x^3y^2)$$

$$= 8x^3y$$

At $P = (-1, 1)$

$\therefore f_x(-1, 1) = 12 * (-1)^2 * (1)^2 = 12$

$\therefore f_y(-1, 1) = 8 * (-1)^3 * 1 = -8$

$\therefore \nabla f_P = \nabla f(-1, 1) = (f_x(-1, 1), f_y(-1, 1))$

$$= (12, -8)$$

We know that the directional derivative is maximum along the direction of ∇f_P for a given point P. Thus, f increases most rapidly towards the direction of ∇f_P for a given point P.

$\therefore$ The required unit vector is given by, $u = \frac{\nabla f_P}{\|\nabla f_P\|}$

$$= \frac{(12, -8)}{\sqrt{(12)^2 + (-8)^2}}$$

$$= \frac{(12, -8)}{\sqrt{208}}$$

$$= \left(\frac{12}{\sqrt{208}}, \frac{-8}{\sqrt{208}}\right)$$

Required rate of change at P in that direction is given by $\|\nabla f_p\|$.

$$\|\nabla f_p\| = \sqrt{(12)^2 + (-8)^2}$$
$$= \sqrt{144 + 64}$$
$$= \sqrt{208}$$
$$= \underline{14.422}$$

b) $f(x, y) = 3x - \ln y$; $P(2, 4)$

Here,

$$f(x, y) = 3x - \ln y$$

Then, $$f_x(x, y) = \frac{\partial}{\partial x}(3x - \ln y)$$
$$= 3$$

$$f_y(x, y) = \frac{\partial}{\partial y}(3x - \ln y)$$
$$= -\frac{1}{y}$$

At Point $P = (2, 4)$,

$$f_x(x, y) = f_x(2, 4) = 3$$
$$f_y(x, y) = f_y(2, 4) = -1/4$$

$$\therefore \nabla f_p = \nabla f(2, 4) = (f_x(2, 4), f_y(2, 4))$$
$$= (3, -1/4)$$

We know that for a given point P, f increases most rapidly towards the direction of ∇f_p. Now, unit vector associated with ∇f_p is $u = \frac{\nabla f_p}{\|\nabla f_p\|}$

$$= \frac{(3, -1/4)}{\sqrt{(3)^2 + (-1/4)^2}}$$

$$= \frac{(3, -1/4)}{\sqrt{9 + \frac{1}{16}}}$$

$$= \frac{(3, -1/4)}{\sqrt{\frac{145}{16}}}$$

$$= \left(\frac{12}{\sqrt{145}}, \frac{-1}{\sqrt{145}}\right)$$

The rate of change of f at point P in that direction is given by $\|\nabla f_P\|$.

$$\therefore \|\nabla f_P\| = \sqrt{(3)^2 + (-1/4)^2}$$

$$= \sqrt{9 + \frac{1}{16}}$$

$$= \frac{\sqrt{145}}{4}$$

12. Find a unit vector in which f decreases most rapidly at P, and find the rate of change of f at P in that direction.

a) $f(x, y) = 20 - x^2 - y^2$; $P(-1, -3)$

Here,

$$f(x, y) = 20 - x^2 - y^2$$

Then, $f_x(x, y) = \frac{\partial}{\partial x}(20 - x^2 - y^2)$

$$= -2x$$

$$f_y(x, y) = \frac{\partial}{\partial y}(20 - x^2 - y^2)$$

$$= -2y$$

At point $P(-1, -3)$,

$f_x(x, y) = f_x(-1, -3) = -2x-1 = 2$

$f_y(x, y) = f_y(-1, -3) = -2x-3 = 6$

$\therefore \nabla f_P = \nabla f(-1, -3) = (f_x(-1, -3), f_y(-1, -3))$

$= (2, 6)$

We know that for a given point P, f decreases the most in the direction of $-\nabla f_P$. Thus, unit vector u associated with $-\nabla f_P$ is $u = \dfrac{-\nabla f_P}{\|-\nabla f_P\|}$

$$= \frac{(-2, -6)}{\sqrt{(-2)^2 + (-6)^2}}$$

$$= \frac{(-2, -6)}{\sqrt{4+36}}$$

$$= \left(\frac{-2}{\sqrt{40}}, \frac{-6}{\sqrt{40}}\right)$$

The rate of change in that direction is given by $-\|\nabla f_P\|$.

$$\therefore -\|\nabla f_P\| = -\sqrt{(2)^2 + (6)^2}$$

$$= -\sqrt{40}$$

$$= -2\sqrt{10}.$$

b) $f(x, y) = e^{xy}$; $P(2, 3)$.

Here, $f(x, y) = e^{xy}$

Then, $f_x(x, y) = \dfrac{\partial}{\partial x}(e^{xy})$

$= e^{xy} \cdot y$

$$= y e^{xy}$$

$$f_y(x, y) = \frac{\partial}{\partial y}(e^{xy})$$

$$= e^{xy} \cdot x$$

$$= x e^{xy}$$

At Point $P = (2, 3)$,

$$f_x(x, y) = f_x(2, 3) = 3e^{2*3} = 3e^6$$

$$f_y(x, y) = f_y(2, 3) = x e^{xy} = 2e^{2*3} = 2e^6$$

$$\because \nabla f_P = \nabla f(2, 3) = (f_x(2, 3), f_y(2, 3))$$

$$= (3e^6, 2e^6)$$

We know that for a given point P, f decreases the most in the direction of $-\nabla f_P$. Thus, unit vector u associated with $-\nabla f_P$ is $u = \frac{-\nabla f_P}{\|-\nabla f_P\|}$

$$= \frac{(-3e^6, -2e^6)}{\sqrt{(-3e^6)^2 + (-2e^6)^2}}$$

$$= \frac{(-3e^6, -2e^6)}{\sqrt{13}\, e^6}$$

$$= \left(-3/\sqrt{13}, -\frac{2}{\sqrt{13}}\right)$$

Now, maximum rate of decrease $= -\|\nabla f_P\|$

$$= -\sqrt{(3e^6)^2 + (2e^6)^2}$$

$$= -\sqrt{13 e^{12}}$$

$$= \underline{-\sqrt{13}\, e^6}$$

13. Find the directional derivative of $f(x,y) = x^2 + 4y^2$ at $P = (3,2)$ in the direction pointing to the origin.

Here,

$$f(x,y) = x^2 + 4y^2$$

Now,

$$f_x(x,y) = \frac{\partial}{\partial x}(x^2 + 4y^2)$$

$$= 2x$$

$$f_y(x,y) = \frac{\partial}{\partial x}(x^2 + 4y^2)$$

$$= 8y$$

Now, at point $P = (3,2)$,

$$f_x(x,y) = f_x(3,2) = 2 \times 3 = 6$$

$$f_y(x,y) = f_y(3,2) = 8 \times 2 = 16$$

$$\therefore \nabla f_P = \nabla f(3,2) = (f_x(3,2), f_y(3,2))$$

$$= (6, 16)$$

Now vector $\overrightarrow{PO} = (0,0) - (3,2)$

$$= (-3, -2)$$

Unit vector u along the direction of $\overrightarrow{PO} = \dfrac{(-3,-2)}{\sqrt{(-3)^2 + (-2)^2}}$

$$= \frac{(-3,-2)}{\sqrt{13}}$$

$$= \left(\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right)$$

$$\therefore D_u f(P) = D_u f(3,2) = \nabla f(3,2) \cdot u$$

$$= (6,16) \cdot \left(\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right)$$

$$= \frac{-18-32}{\sqrt{13}}$$

$$= \frac{-50}{\sqrt{13}}$$

14. Find an equation of the tangent plane and find a set of symmetric equations for the normal line to the surface at the given point.

a) $z = x^2 - y^2$, $(3, 2, 5)$

Here,

$$z = x^2 - y^2$$

$$\Rightarrow x^2 - y^2 - z = 0$$

$$\therefore F(x, y, z) = x^2 - y^2 - z$$

Now,

$$F_x(x, y, z) = \frac{\partial}{\partial x}(x^2 - y^2 - z)$$

$$= 2x$$

$$F_y(x, y, z) = \frac{\partial}{\partial y}(x^2 - y^2 - z)$$

$$= -2y$$

$$F_z(x, y, z) = \frac{\partial}{\partial z}(x^2 - y^2 - z)$$

$$= -1$$

At $(3, 2, 5)$, i.e. $(x_0, y_0, z_0) = (3, 2, 5)$

$\nabla f(x_0, y_0, z_0)$

$$F_x(3, 2, 5) = 2 \times 3 = 6$$

$$F_y(3, 2, 5) = -2 \times 2 = -4$$

$$F_z(3, 2, 5) = -1$$

Now equation of tangent plane to a ~~plane~~ surface at point P is given by:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$\Rightarrow F_x(3, 2, 5)(x - 3) + F_y(3, 2, 5)(y - 2) + F_z(3, 2, 5)(z - 5) = 0$$

$$\Rightarrow 6(x - 3) + (-4)(y - 2) + (-1)(z - 5) = 0$$

$$\Rightarrow 6x - 4y - z - 18 + 8 + 5 = 0$$

$\therefore$ $6x - 4y - z = 5$ is required equation of tangent plane.

Symmetric equation of normal line is given by:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$$\Rightarrow \frac{x - 3}{6} = \frac{y - 2}{-4} = \frac{z - 5}{-1}$$

$\therefore$ $\frac{x-3}{6} = \frac{y-2}{-4} = \frac{z-5}{-1}$ is required equation of normal line.

b) $xy - z = 0$, $(-2, -3, 6)$

Here,

$$F(x, y, z) = xy - z$$

$$\therefore F_x(x, y, z) = y$$

$$\therefore F_y(x, y, z) = x$$

$$\therefore F_z(x, y, z) = -1$$

At point $P(x_0, y_0, z_0) = (-2, -3, 6)$

$$F_x(-2, -3, 6) = -3$$

$$F_y(-2, -3, 6) = -2$$

$$F_z(-2, -3, 6) = -1$$

Now eq^n of tangent plane at point $P(x_0, y_0, z_0)$ is given by:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$\Rightarrow -3(x+2) - 2(y+3) + (-1)(z-6) = 0$$

$\therefore$ $3x + 2y + z + 6 = 0$ is required equation of tangent plane.

Symmetric eq^n of normal line is given by:

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$\therefore$ $\frac{x+2}{3} = \frac{y+3}{2} = \frac{z-6}{1}$ is required eq^n of normal line.

c. $xyz = 10$, $(1, 2, 5)$

Here,

$$F(x,y,z) = xyz$$

$$\therefore F_x(x,y,z) = \frac{\partial}{\partial x}(xyz) = yz$$

$$\therefore F_y(x,y,z) = \frac{\partial}{\partial y}(xyz) = xz$$

$$\therefore F_z(x,y,z) = \frac{\partial}{\partial z}(xyz) = xy$$

At point $(x_0, y_0, z_0) = (1, 2, 5)$,

$$F_x(1,2,5) = 2\times 5 = 10$$

$$F_y(1,2,5) = 1\times 5 = 5$$

$$F_z(1,2,5) = 1\times 2 = 2$$

Now, eq^n of tangent plane is given by:

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

$\Rightarrow 10(x-1) + 5(y-2) + 2(z-5) = 0$

$\therefore$ $10x + 5y + 2z - 30 = 0$ is required equation of tangent plane.

Equation of normal line at (x_0, y_0, z_0) is given by

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$\Rightarrow \frac{x-1}{10} = \frac{y-2}{5} = \frac{z-5}{2}$ is required equation of normal line.

d) $z = ye^{2xy}$, $(0, 2, 2)$

Here,

$z = ye^{2xy}$

$\therefore$ $ye^{2xy} - z = 0$

Thus,

$F(x, y, z) = ye^{2xy} - z$

Now,

$F_x(x, y, z) = \frac{\partial}{\partial x}(ye^{2xy} - z)$

$= ye^{2xy} \cdot 2y$

$= 2y^2 e^{2xy}$

$F_y(x, y, z) = \frac{\partial}{\partial y}(ye^{2xy} - z)$

$= y \frac{\partial}{\partial y}(e^{2xy}) + e^{2xy} \cdot \frac{\partial y}{\partial y} - 0$

$$= ye^{2xy} \cdot 2x + e^{2xy}$$

$$= 2xye^{2xy} + e^{2xy}$$

$$F_z(x,y,z) = \frac{\partial}{\partial z}(ye^{2xy} - z)$$

$$= -1$$

Now at point $(x_0, y_0, z_0) = (0, 2, 2)$,

$$F_x(x_0, y_0, z_0) = F_x(0,2,2) = 2 \times 2^2 \times e^0 = 8$$

$$F_y(x_0, y_0, z_0) = F_y(0,2,2) = 2 \times 0 \times 2 \times e^0 + e^0 = 0 + 1 = 1$$

$$F_z(x_0, y_0, z_0) = -1$$

Equation of tangent plane at $(x_0, y_0, z_0) = (0, 2, 2)$ is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$\Rightarrow 8(x - 0) + 1(y - 2) + (-1)(z - 2) = 0$$

$\therefore$ $8x + y - z = 0$ is required equation of tangent plane.

Symmetric eqn of normal line at (x_0, y_0, z_0) is given by:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Put $(x_0, y_0, z_0) = (0, 2, 2)$, we get:

$$\frac{x - 0}{8} = \frac{y - 2}{1} = \frac{z - 2}{-1}$$

$\therefore$ $\frac{x}{8} = \frac{y-2}{1} = \frac{z-2}{-1}$ is required symmetric equation of normal line.

e) $y \ln(xz^2) = 2$, $(e, 2, 1)$

Here,

$$F(x, y, z) = y \ln(xz^2)$$

Now,

$$F_x(x, y, z) = \frac{\partial}{\partial x}\left(y \ln(xz^2)\right)$$

$$= y \frac{1}{xz^2} \cdot \frac{\partial (xz^2)}{\partial x}$$

$$= \frac{yz^2}{xz^2}$$

$$= y/x$$

$$F_y(x, y, z) = \frac{\partial}{\partial y}\left(y \ln(xz^2)\right)$$

$$= \ln(xz^2)$$

$$F_z(x, y, z) = \frac{\partial}{\partial z}\left(y \ln(xz^2)\right)$$

$$= y \frac{1}{xz^2} \cdot \frac{\partial}{\partial z}(xz^2)$$

$$= \frac{2xyz}{xz^2}$$

$$= \frac{2y}{z}$$

At point $(x_0, y_0, z_0) = (e, 2, 1)$,

$$F_x(x_0, y_0, z_0) = F_x(e, 2, 1) = 2/e$$

$$F_y(x_0, y_0, z_0) = F_y(e, 2, 1) = \ln(e(1)^2)$$

$$= \ln e = 1$$

$$F_z(x_0, y_0, z_0) = F_z(e, 2, 1) = \frac{2 \times 2}{1} = 4$$

The equation of tangent plane is given by:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$\Rightarrow \frac{2}{e}(x - e) + 1(y - 2) + 4(z - 1) = 0$$

$$\Rightarrow \frac{2}{e}x + y + 4z = 8$$

$\therefore \frac{2}{e}x + y + 4z = 8$ is required equation of tangent plane.

Now, symmetric equation of normal line at (x_0, y_0, z_0) is given by:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$$\Rightarrow \frac{x - e}{2/e} = \frac{y - 2}{1} = \frac{z - 1}{4}$$

$\therefore \frac{x - e}{2/e} = \frac{y - 2}{1} = \frac{z - 1}{4}$ is required symmetric equation of normal line.