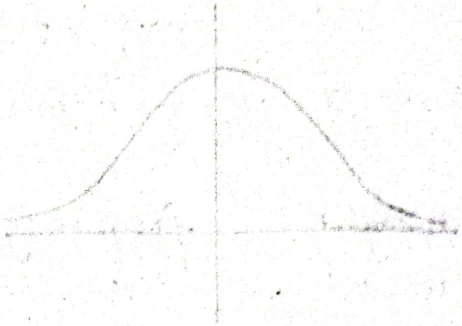
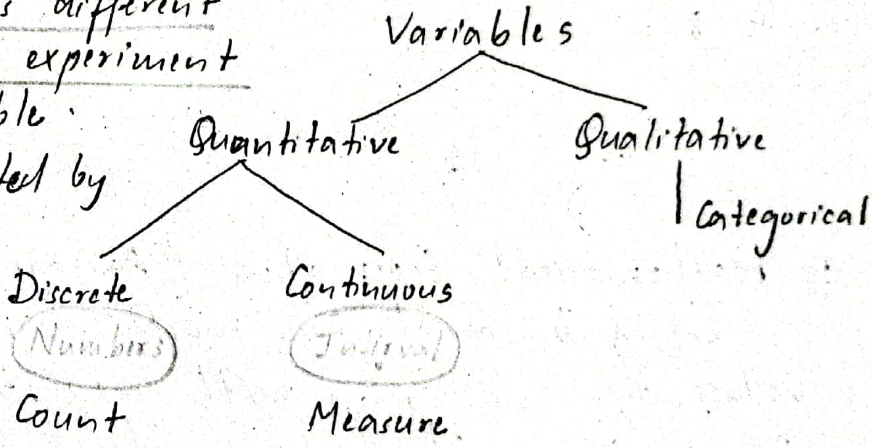
i) Multinomial Distribution
ii) Extreme value distribution
iii) Generalized Power Series Distribution
iv) Prior and posterior distribution
v) Compound Mixed type distribution
vi) Non-parametric test
⇓
No specific distribution

vii) Testing of Hypothesis



Random Variable:

The variable which takes different values generated by random experiment is known as random variable. Random variables are denoted by X, Y, and Z.



For example: if we ~~toss a coin~~ toss a coin three times, then total number of possible cases
= 8

no. of totals (events)
$= 2^3 = 8$

Sample space

	HH	HT	TH	TT
H	HHH	HHT	HTH	HTT
T	THH	THT	TTH	TTT

X = number of heads

X	P(X)
3	1/8
2	3/8
1	3/8
0	1/8
	$\sum P(X) = 1$

where, X is a random variable.

Weight	frequency	Probability → Continuous
45-55	5	5/35
55-65		
65-75		
75-85		
85+		
	35	$\sum = 1$

Mathematical Expectation (Average value)

Let, X be a discrete random variable which takes the values $x_1, x_2, \ldots, x_n$ with respective probabilities $P(x_1)$, $P(x_2), \ldots, P(x_n)$ then the mathematical expectation of random variable X is given by

$$E(X) = x_1 P(x_1) + x_2 P(x_2) + \ldots + x_n P(x_n)$$

$$= \sum x P(x) \rightarrow \text{Probability mass function} \quad ; 1 \le x \le n$$

If X is continuous random variable, then

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx \quad ; \quad -\infty < x < \infty$$

$f(x)$ = Probability density function

Suppose a dice is rolled once, then total number of possible cases = 6

$\cdot$ X = values on the face of dice

X	$P(X)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
	$\Sigma P(X) = 1$

$$E(X) = 1 \cdot 1/6 + \cdots + 6 \cdot 1/6$$
$$= 1/6 [1+2+\cdots+6]$$
$$= \frac{\Sigma X}{n}$$
$$= \bar{X}$$

Variance in terms of mathematical expectation

$$Var(X) = E[X - E(X)]^2$$

$$\frac{\Sigma(X-\bar{X})^2}{n}$$
= Average of square of deviations.

where, $E(X) = \Sigma X P(X)$

$E(X^2) = \Sigma X^2 P(X)$

If X is continuous,

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

Second order moment about origin.

$$\frac{\Sigma x^2}{n} - \bar{x}^2$$
$$= \overline{x^2} - \bar{x}^2$$
$$= E(X^2) - [E(X)]^2$$
$$= E(X^2) - [E(X)]^2$$

x $\bar{x}$ $y = (x-\bar{x})^2$

$\bar{Y} = \Sigma y/n = \frac{\Sigma(x-\bar{x})^2}{n}$

$V(X)$

$$E(Y) = E[X-\bar{X}]^2$$
$$= E[X - E(X)]^2$$

Covariance in terms of mathematical expectation

$$Cov(X, Y) = E\left[\{X - E(X)\}\{Y - E(Y)\}\right]$$

$$= E(XY) - E(X)E(Y)$$

$$r = \frac{Cov(X, Y)}{sd(X) * sd(Y)}$$

$$-1 \le r \le +1$$

$$Cov(X, Y) = \frac{\Sigma (X - \bar{X})(Y - \bar{Y})}{n}$$

$$= \frac{\Sigma Z}{n}$$

$$= \bar{Z}$$

$$E(Z) = E[(X - \bar{X})(Y - \bar{Y})]$$

$$= E[X - E(X)][Y - E(Y)]$$

$$= \frac{\Sigma XY}{n} - \bar{X}\bar{Y}$$

Moments

The average of various powers of deviations taken from chosen value is known as moments.

Mean | Any arbitrary value

eg: $\frac{\Sigma (x - a)^r}{n}$ ← Raw moment

If the deviations are taken from mean, then it is called central moments and if from any arbitrary value (assumed mean) then it is known as raw moments.

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Central moments:

The r^{th} order central moment is denoted by μ_r and is given by

$$\mu_r = \frac{1}{n} \Sigma (x - \bar{x})^r$$

First four central moments

$$\mu_1 = \frac{1}{n} \Sigma (x - \bar{x})^1 = 0$$

$$\mu_2 = \frac{1}{n} \Sigma (x - \bar{x})^2 = \text{Variance}$$

$$\mu_3 = \frac{1}{n} \Sigma (x - \bar{x})^3$$

$$\mu_4 = \frac{1}{n} \Sigma (x - \bar{x})^4$$

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In terms of expectation,

$$\mu_r = E[X - E(X)]^r$$

Raw moments

The r^{th} order raw moment about any arbitrary value 'a' is denoted by μ_r' and is given by:

$$\mu_r' = \frac{1}{n} \Sigma (x-a)^r$$

The r^{th} raw moment about origin

$$\mu_r' = \frac{1}{n} \Sigma x^r \quad [\because a = 0]$$

In terms of expectation,

$$\mu_r' = E[X - a]^r$$

$$\mu_r' = E(X)^r \quad (\text{if } a = 0)$$

NOTE: $\mu_2 = \mu_2' - (\mu_1')^2$

Moment Generating Function

The moment generating function of random variable X is given by

$$M_X(t) = E[e^{tX}]$$

$$= \sum_x e^{tx} P(x) \quad (\text{for discrete})$$

pmf

$$= \int_x e^{tx} f(x)\, dx \quad (\text{for continuous})$$

Pdf

NOTE: $M_X(t) = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \dots + \frac{t^rX^r}{r!} + \dots\right]$

$$= \left[1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \dots + \frac{t^r}{r!}E(X^r) + \dots\right]$$

Expansion of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$E(aX) = a\,E(X)$

$\mu_1' = E(X) = \text{mean}$

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$$\Rightarrow M_X(t) = \left[1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots\right]$$

$\Rightarrow \mu_r' =$ Coefficient of $\frac{t^r}{r!}$ in the expansion of $M_X(t)$.

$$\left.\frac{d\,M_X(t)}{dt}\right|_{t=0} = \left. 0 + \mu_1' + \frac{2t}{2!}\mu_2' + \dots + \frac{r\,t^{r-1}}{r!}\mu_r' + \dots \right|_{t=0}$$

$$= \mu_1'$$

$$= E(X)$$

Further,

$$\mu_r' = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}$$

$$\therefore \mu_1' = \left.\frac{d\,M_X(t)}{dt}\right|_{t=0} = E(X)$$

$$\therefore \mu_2' = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = E(X^2)$$

NOTE:

$$\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu_r'$$

$$Var(X) = E[X^2] - [E(X)]^2$$

$$E[X^2] = E[X-0]^2$$

$\mu_r = E[X-\bar{X}]^r$

$\mu_r' = E[X-a]^r$

Put $a = 0$

~~as~~ r^{th} order raw moment about origin,

$\mu_r' = E[X^r]$

$\mu_1' = E[X] =$ mean

$\mu_2' = E[X^2] =$

$\mu_2 = E[X^2] - [E(X)]^2$

$= \mu_2' - (\mu_1')^2$

Characteristics Function

The characteristic function of random variable X is given by

$$\phi_X(t) = E\left[e^{itX}\right]$$

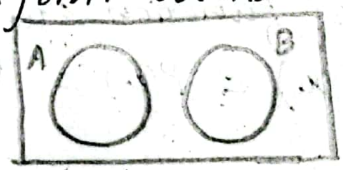
$$= \sum_x e^{itx} p(x) \quad [\text{For discrete}]$$

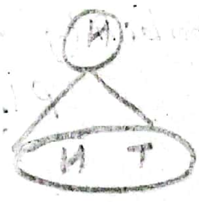
$$= \int_x e^{itx} f(x)\, dx \quad [\text{For continuous}]$$

Generalized Power Series Distribution

i) Binomial Distribution
ii) Poisson distribution
iii) Negative binomial distribution

* Independent events : Occurence of first one does not influence second event.
* Mutually exclusive events : Disjoint events





A or A'

$n = 10 \rightarrow$ fixed (no. of trials)

* Events are independent
* Events are mutually exclusive
* Probability of success (p) is constant

$X =$ values

i) Binomial Distribution

Under the following conditions, the binomial distribution is applied :

a. Number of trials 'n' is fixed.
b. Events are statistically independent.
c. Events are mutually exclusive i.e. only two possible outcomes like success or failure, head or tail, either male or female etc.

d. The probability of success (p) is constant in each trial.

e. $p+q=1$

Coin toss

(H) Success or Failure (T)

$1/2$ $1/2$

Product $\longrightarrow$ Defective or non-defective

0.01 0.99

n, p are parameters of the distribution.

Definition

Let, X be a discrete random variable which follows binomial distribution with parameters 'n' and 'p', then the probability mass function of the distribution is given by,

$$P(X=x) = P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

where $x = 0, 1, 2, \ldots, n$

$p+q=1$

It gives the probability of getting 'x' successes out of 'n' independent trials

$X =$ no. of success

$p =$ probability of success in each trial

$q = 1-p =$ probability of failure in each trial

Example:

$n = 5$

Coin tossed 5 times, $p = 1/2$, $q = 1/2$

probability of getting 2 heads is: $P(X=2) = {}^{5}C_{2}\,(0.5)^{2}\,(0.5)^{3}$

Generalized Power Series Distribution

A discrete random variable X is said to follow a general power series distribution (GPSD) if its probability mass function is given by

$$P(X=x) = P(x) = \begin{cases} \dfrac{a_x \theta^x}{f(\theta)} & ; x = 0, 1, 2, \ldots \\ & \;\; a_x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Power series

where, $\sum_{x \in S^*} a_x \theta^x = f(\theta)$; $\theta \geq 0$

so that $f(\theta)$ is positive, finite and differentiable and 'S' is non-empty countable subset of non-negative integers.

Some special cases of GPSD

i) Binomial distribution:

In GPSD, take $\theta = \dfrac{p}{1-p}$ and $f(\theta) = (1+\theta)^n$ and $S = \{0, 1, 2, \ldots, n\}$

The probability mass function of GPSD is $P(X=x) = P(x) = \dfrac{a_x \theta^x}{f(\theta)}$.

$$\boxed{(a+b)^n = \sum_{x=0}^{n} {}^nC_x b^x a^{n-x}}$$

where, $f(\theta) = \sum_{x \in S} a_x \theta^x$ ... (i)

Now, $f(\theta) = (1+\theta)^n = \sum_{x=0}^{n} {}^nC_x \theta^x 1^{n-x}$

$= \sum_{x=0}^{n} {}^nC_x \theta^x$ ... (ii)

From (i) and (ii), $a_x = {}^nC_x$

$$\therefore P(X=x) = \frac{{}^nC_x \left(\frac{p}{1-p}\right)^x}{\left(1 + (p/1-p)\right)^n} = \frac{{}^nC_x p^x (1-p)^{n-x}}{(1-p+p)^n}$$

* sample space

$$= {}^nC_x\, p^x q^{n-x} \quad ; \quad x = 0, 1, 2, \ldots, n$$

This is the probability mass function of the binomial distribution.

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$$(a+b)^n = \sum_{x=0}^{n} {}^nC_x\, b^x a^{n-x}$$

* Mean and Variance of binomial distribution

If $X \sim B(n,p)$ [i.e. X follows binomial distribution with parameters 'n' and 'p'], then

$$P(X=x) = {}^nC_x\, p^x q^{n-x} \quad ; \quad x = 0, 1, 2, \ldots, n$$
$$p+q=1$$

Mean: $$E(X) = \sum_{x=0}^{n} x\, P(X=x)$$

$$= \sum_{x=0}^{n} x\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

$$\Rightarrow E(X) = \sum_{x=1}^{n} x \frac{n(n-1)!}{(n-x)!\,x(x-1)!} p^{x-1} p\, q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!\,(x-1)!} p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x-1=0}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(q+p)^{n-1} \qquad (\because p+q=1)$$

$$\boxed{\therefore E(X) = np}$$

$n! = n \times (n-1)!$

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\Rightarrow Var(X) = E(X^2) - (np)^2 \text{ ——— (ii)}$$

Now, $$E(X^2) = \sum_{x=0}^{n} x^2 P(X=x)$$

$$= \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1)+x] \cdot \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\,x!} p^x q^{n-x} +$$

$$\underbrace{\sum_{x=0}^{n} x \frac{n!}{(n-x)!\,x!} p^x q^{n-x}}_{np} \longleftarrow E(X) = np$$

$$= \sum_{x=2}^{n} \frac{x(x-1)\, n(n-1)(n-2)!}{(n-x)!\, x(x-1)(x-2)!} \, p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!\,(x-2)!} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2} \, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$\therefore E(X^2) = n(n-1)p^2 + np \qquad [\because p+q=1]$$

From (ii), $$Var(X) = n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np(1-p)$$

$$\boxed{\therefore Var(X) = npq}$$

Hence, mean and variance of binomial distribution with ~~are~~ parameters n and p are respectively np and npq.

* Moment Generating function of binomial distribution

If $X \sim B(n,p)$ then $P(X=x) = {}^nC_x p^x q^{n-x}$; $x = 0,1,2,\ldots,n$
$p+q=1$

Now, moment generating function of X,

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \sum_{x=0}^{n} e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$\boxed{\therefore M_X(t) = (q + pe^t)^n}$$

Mean and Variance using MGF (mgf)

$$\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu_r' \leftarrow E[X-0]^r$$

μ_r' → r^{th} order raw moment about origin

We know that, $\mu_r' = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}$

If $X \sim B(n,p)$ then $M_X(t) = (q+pe^t)^n$

Now,

$$\mu_1' = E(X) = \left.\frac{d M_X(t)}{dt}\right|_{t=0} = n(q+pe^t)^{n-1} \cdot pe^t$$

Putting $t=0$,

$$= n(q+pe^0)^{n-1} \cdot pe^0$$

$= n(q+p)^{n-1} \cdot p$

$\boxed{\therefore E(X) = np}$ $\quad (\because p+q=1)$

Further, $\mu_2' = E(X^2)$: Second order raw moment about origin.

$$= \frac{d^2 M_x(t)}{dt^2}\Big|_{t=0}$$

$$= \frac{d}{dt}\left[n(q+pe^t)^{n-1} pe^t\right]\Big|_{t=0}$$

$$= np\left[e^t(n-1)(q+pe^t)^{n-2} pe^t + (q+pe^t)^{n-1} \cdot e^t\right]_{t=0}$$

$$= np\left[e^0(n-1)(q+pe^0)^{n-2} pe^0 + (q+pe^0)^{n-1} e^0\right]$$

$$= np\left[(n-1)p + 1\right]$$

$$= np(np - p + 1)$$

$\therefore E(X^2) = (n^2p^2 - np^2 + np)$

Now, $\quad V(X) = E[X^2] - [E(X)]^2$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$\boxed{\therefore \mu_2 = npq}$

Q2. *PROBLEMS*

Given,

$n = 7$ and $p = 0.2$, $q = 0.8$

(a) $P(X = 5)$

If $X \sim B(n, p)$ then $P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$; $x = 0, 1, \dots, n$

$p + q = 1$

(a) $P(X=5) = {}^{7}C_{5}\, 0.2^{5}\, 0.8^{2} = \underline{0.0043}$

(b) $P(X > 2) = 1 - P(X \leq 2)$

$\sum P(x) = 1$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left({}^{7}C_{0}\, 0.2^{0}\, 0.8^{7} + {}^{7}C_{1}\, 0.2^{1}\, 0.8^{6} + {}^{7}C_{2}\, 0.2^{2}\, 0.8^{5}\right)$

$= 1 - (0.2097 + 0.3670 + 0.2752)$

$= \underline{0.148}$

(c) $P(X \geq 3)$

(d) $P(X \geq 4)$

Q4.

$n = 6$

$P(X=3) = 0.2457$

$P(X=4) = 0.089$

$p = ?$

If $X \sim B(n, p)$ then, $P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$; $x = 0, 1, 2, \dots, n$

$p + q = 1$

Then,

$$\frac{P(X=3)}{P(X=4)} = \frac{{}^{6}C_{3}\, p^{3}\, q^{6-3}}{{}^{6}C_{4}\, p^{4}\, q^{6-4}}$$

$$\Rightarrow \frac{0.2457}{0.089} = \frac{{}^6C_3\, p^3 q^3}{{}^6C_4\, p^4 q^2}$$

$$\Rightarrow 2.76 = \frac{20q}{15p} \quad \Rightarrow \frac{1-p}{p} = \frac{2.76 * 15}{20}$$

$$\Rightarrow \frac{1-p}{p} = 2.07$$

$$\Rightarrow 1/p = 3.07$$

$$\therefore p = \underline{0.325}$$

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Q5.
$n \rightarrow$ fixed
$p \rightarrow$ constant
Events are independent and mutually exclusive
$p+q=1$

p = probability of withdraw = 0.20
$q = 1-p = 1-0.20 = 0.80$
X = number of withdraw
$n = 18$

Here, $X \sim B(n,p)$, then $P(X=x) = {}^nC_x\, p^x q^{n-x}$; $x = 0,1,\dots,n$
$p+q=1$

(a) Probability that none will withdraw,
$$P(X=0) = {}^{18}C_0\, (0.20)^0 (0.80)^{18-0}$$
$$= \underline{0.018}$$

(b) $P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$
$= 1 - 0.018$
$= \underline{0.982}$

(c) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= {}^{18}C_0 (0.2)^0 (0.8)^{18-0} + {}^{18}C_1 (0.2)^1 (0.8)^{18-1} + {}^{18}C_2 (0.2)^2 (0.8)^{18-2}$

$= 0.018 + 0.0810 + 0.1722$

$= \underline{0.2712}$

Assignment
Q6, Q7, Q13, Q16, Q20, Q24

<u>QNo10</u>

$p = 0.05$

$q = 1 - 0.05 = 0.95$

$n = \cancel{90}\ 3$

X: No. of car that needs warranty repair in first 90 days.

Here, $X \sim B(n, p)$, so $P(X=x) = {}^{n}C_x\, p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$; $p + q = 1$

(a) $P(X=0) = {}^{3}C_0 (0.05)^0 (0.95)^{3-0}$

$= \underline{0.8573}$

(b) $P(X \ge 1) = 1 - P(X < 1)$

$= 1 - P(X=0)$

$= 1 - 0.8573$

$= \underline{0.1427}$

(c) $P(X > 1) = 1 - [P(X=0) + P(X=1)]$

$= 1 - \left(0.8573 + {}^{3}C_1 (0.05)^1 (0.95)^{3-1}\right)$

$= 1 - (0.8573 + 0.1353)$

$= \underline{0.0074}$

QN14

(a) $n = 10$

$p = 0.3$

$q = 1-p = 1-0.3 = 0.7$

X = number of liberals

Here, $X \sim B(n, p)$ then, $P(X = x) = {}^nC_x \, p^x q^{n-x}$; $x = 0, 1, 2, \ldots n$
$p+q = 1$

$$P(X=4) = {}^{10}C_4 (0.3)^4 (0.7)^{10-4}$$
$$= 210 \times 0.0081 \times 0.1176$$
$$= \underline{0.200}$$

(b) X = number of conservative

$p = 0.55$, $q = 1-p = 1-0.55 = 0.45$
$n = 10$

Here, $X \sim B(n,p)$, then $P(X = x) = {}^nC_x \, p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$
$p+q = 1$

$$\therefore P(X=0) = {}^{10}C_0 (0.55)^0 (0.45)^{10-0}$$
$$= 1 \times 1 \times 0.00034$$
$$= \underline{0.00034}$$

(c) X = number of middle of the road

$p = 0.15$

$q = 1-p = 1-0.15 = 0.85$
$n = 10$

Here, $X \sim B(n,p)$, then $P(X = x) = {}^nC_x \, p^x q^{n-x}$; $x = 0, 1, 2, \ldots n$,

$\therefore P(X=2) = {}^{10}C_2 \, (0.15)^2 (0.85)^{10-2}$

$= \underline{0.2758}$

(d) X = number of liberals

$P(X \geq 8) = ?$

$P(X \geq 8) = P(X=8) + P(X=9) + P(X=10)$

$= {}^{10}C_8 (0.3)^8 (0.7)^{10-8} + {}^{10}C_9 (0.3)^9 (0.7)^{10-9}$

$+ {}^{10}C_{10} (0.3)^{10} (0.7)^{10-10}$

$=$

QN 20

Here, $np = 0.4$ —— (i)

$\sqrt{npq} = 0.6$

$\Rightarrow npq = 0.36$ —— (ii)

Dividing (i) by (ii)

$\frac{1}{q} = \frac{0.4}{0.36}$

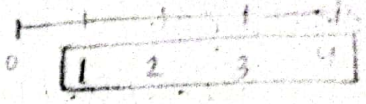
$\therefore q = 0.9$

$\therefore p = 0.1$ $\therefore n = 0.4/0.1 = 4$

If $X \sim B(n,p)$, then $P(X=x) = {}^nC_x \, p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$

$p+q=1$

$$\therefore P(X \geq 1) = 1 - P(X=0)$$
$$= 1 - {}^4C_0 (0.1)^0 (0.9)^{4-0}$$



$$= 1 - 1 \times 1 \times 0.6561$$
$$= \underline{0.3439}$$

Q 23

$p = 1/4 = 0.25$

$q = 1 - 0.25 = 0.75$

$n = 5$

X = number of correct questions

Here, $X \sim B(n,p)$, then $P(X=x) = {}^nC_x \, p^x q^{n-x}$; $x = 0, 1, \ldots, n$

$p+q=1$

ⓐ $P(X=5) = {}^5C_5 (0.25)^5 (0.75)^{5-5}$

$$= 1 \times 0.00097 \times 1$$
$$= \cancel{0.00485} \quad \underline{0.00097}$$

ⓑ $P(X \geq 4) = P(X=4) + P(X=5)$

$$= {}^5C_4 (0.25)^4 (0.75)^{5-4} + {}^5C_5 (0.25)^5 (0.75)^{5-5}$$
$$= 0.01464 + \cancel{0.00485} \; 0.00097$$
$$= \underline{0.01561}$$

POISSON'S DISTRIBUTION (Use for rare events)

λ = Average number of calls per hour $\quad X = 0, 1, 2, \ldots$

= 5 (say)

X = Number of calls per hour

$P(X \geq 1) = ?$

$= 1 - P(X < 1)$

$= 1 - P(X = 0)$

$$\frac{e^{-\lambda} \lambda^{x}}{x!}$$

$n \longrightarrow \infty$

$p \longrightarrow 0$

$\lambda = np$

$X = 0, 1, \ldots$

Let X be a discrete random variable which follows Poisson's distribution with parameter 'λ' then, the probability mass function (pmf) of the distribution is given by

$$P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \quad ; \quad x = 0, 1, 2, \ldots$$

$$\lambda > 0$$

where, λ = average value

Poisson distribution using GPSD

$$P(X = x) = \frac{a_x \theta^{x}}{f(\theta)}$$

where, $f(\theta) = \sum_{x \in S} a_x \theta^{x}$

$\theta \geq 0$

$x \in S$

$\downarrow$ $0, 1, 2, \ldots, \infty$

$\theta = ?$

$f(\theta) = ?$

$S = ?$

In GPSD, put $\theta = \lambda$, $f(\theta) = e^{\theta}$
and $S = \{0, 1, 2, \ldots, \infty\}$

Now, $f(\theta) = e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \cdots$

$$f(\theta) = \sum_{x=0}^{\infty} \frac{\theta^x}{x!} \quad \text{———— (i)}$$

Since, $f(\theta) = \sum_{x=0}^{\infty} a_x \theta^x$ ———— (ii)

From (i) and (ii), $a_x = \frac{1}{x!}$

The probability mass function of Generalized Power Series Distribution is

$$P(X=x) = \frac{a_x \theta^x}{f(\theta)}$$

$$= \frac{\frac{1}{x!} \lambda^x}{e^{\lambda}}$$

$$\boxed{\therefore P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}} \; ; \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

This is the probability mass function of Poisson's distribution with parameter λ.

Mean and Variance of the distribution

If $X \sim P(\lambda)$, then $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, \ldots$; $\lambda > 0$

Mean:

$$E(X) = \sum_{x=0}^{\infty} x\, P(x) = \sum_{x=0}^{\infty} x \left\{ \frac{e^{-\lambda}\lambda^x}{x!} \right\} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \cdot \lambda^x}{x(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda \lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{\therefore E(X) = \lambda}$$

July 18, '24

Q24) No. of trials, $(n) = 5$

[7, 11 or 12] (If a pair of dice is rolled)

p = probability that a player gets audited

No. of possible cases $= 6^2$

$= 36$

$$\therefore p = \frac{9}{36} = \frac{1}{4} = 0.25$$

$q = 1 - p = 1 - 0.25 = 0.75$

X = no. of times she gets audited

Here, $X \sim B(n, p)$, then $P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$; $x = 0, 1, 2, \ldots$
$p + q = 1$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - {}^{5}C_{0} (0.25)^{0} (0.75)^{5-0}$$

$$= 0.7627$$

CamScanner

Variance of Poisson's distribution

$$V(X) = E(X^2) - [E(X)]^2$$

$$= E(X^2) - \lambda^2 \quad \text{———} \quad (i)$$

Now,

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \, P(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] \, P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \, P(x) + \sum_{x=0}^{\infty} x \, P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} x \, P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \cdot \lambda^x}{x(x-1)(x-2)!} + \lambda$$

$$= e^{-\lambda} \cdot \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots}$$

$$= e^{-\lambda} \cdot \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots \right] + \lambda$$

$$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

From (i),

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\therefore V(X) = \lambda}$$

Moment Generating Function

If $X \sim P(\lambda)$, then $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$; $x = 0, 1, 2, \dots$
$\lambda > 0$

Now, mgf of X,

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \left\{\frac{e^{-\lambda}\lambda^{x}}{x!}\right\}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^{t}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \cdots\right]$$

$$= e^{-\lambda}\left(e^{\lambda e^{t}}\right)$$

$$= \cancel{e^{\lambda}(e^{t}}$$

$$\therefore M_X(t) = e^{\lambda(e^{t}-1)} \quad \text{------} \quad (i)$$

Mean and variance using mgf:

If $X \sim P(\lambda)$, then $M_X(t) = e^{\lambda(e^{t}-1)}$

Since $\mu_r' = \dfrac{d^r M_X(t)}{dt^r}\Big|_{t=0}$

Mean: $\mu_1' = E(X) = \dfrac{d\, M_X(t)}{dt}\Big|_{t=0}$

$$= \frac{d}{dt}\left(e^{\lambda(e^{t}-1)}\right)\Big|_{t=0}$$

$$= \cancel{e^{\lambda}\cdot e^{t-1}} = e^{\lambda(e^{t}-1)} \cdot \lambda \cdot e^{t}\Big|_{t=0}$$

$$= e^{\lambda(e^0-1)} \cdot \lambda e^0$$

$$= e^{\lambda \times 0} \cdot \lambda e^0$$

$$= 1 \cdot \lambda \cdot 1$$

$$\boxed{\therefore \underset{(\mu_1')}{\text{Mean}} = \lambda}$$

<u>Variance</u> : $V(X) = E(X^2) - [E(X)]^2$

$$\mu_2' \overset{? E(X^2)}{=} \left. \frac{d^2 M_X(t)}{dt^2} \right|_{t=0}$$

$$\frac{d^2 M_X(t)}{dt^2} = \frac{d}{dt}\left(e^{\lambda(e^t-1)} \cdot \lambda e^t\right)$$

$$= \lambda \frac{d}{dt}\left(e^{\lambda(e^t-1)} \cdot e^t\right)$$

$$= \lambda \left[e^{\lambda(e^t-1)} \cdot \frac{d}{dt}(e^t) + e^t \cdot \frac{d}{dt}\left(e^{\lambda(e^t-1)}\right) \right]$$

$$= \lambda \left[e^{\lambda(e^t-1)} \cdot e^t + e^t \left\{ e^{\lambda(e^t-1)} \cdot \lambda \cdot e^t \right\} \right]$$

$$\therefore E(X^2) = \left. \frac{d^2 M_X(t)}{dt^2} \right|_{t=0} = \lambda \left[e^{\lambda(e^0-1)} \cdot e^0 + e^0 \left\{ e^{\lambda(e^0-1)} \cdot \lambda e^0 \right\} \right]$$

$$= \lambda \left[e^{\lambda \times 0} \times 1 + 1 \times e^{\lambda \times 0} \cdot \lambda \times 1 \right]$$

$$= \lambda (1 + \lambda)$$

$$= \lambda^2 + \lambda$$

$$\therefore V(X) = E(X^2) - [E(X)]^2 = \mu_2' - (\mu_1')^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\therefore V(X) = \lambda}$$

PROBLEMS

Q1. $\lambda = 4.2$

(a) $P(X \geq 5)$

$= 1 - P(X < 5) \qquad \left[\because P(X = x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!} ; \quad x = 0, 1, 2, \ldots \quad \lambda > 0 \right]$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)]$

$$= 1 - \left[\frac{e^{-4.2} \, 4.2^{0}}{0!} + \frac{e^{-4.2} \, 4.2^{1}}{1!} + \frac{e^{-4.2} \, 4.2^{2}}{2!} + \frac{e^{-4.2} \, 4.2^{3}}{3!} + \frac{e^{-4.2} \, 4.2^{4}}{4!} \right]$$

$= 1 - (e^{-4.2} / e^{4.2})$

$= 1 - 1/e$

$= 1 - e^{-4.2} [1 + 4.2 + 8.82 + 12.34 + 12.96]$

$= 1 - 0.589$

$= \underline{0.410}$

(b) $P(X < 2) = P(X=0) + P(X=1)$

$= e^{-4.2} [1 + 4.2]$

$= \underline{0.0779}$

QN3) X = No. of price hikes every three years
λ = average number of price hikes every 3 years
= 4 (every three years)

Here, $X \sim P(\lambda)$ so, $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$; $x = 0, 1, 2, \dots$; $\lambda > 0$

X = no. of successes
p = prob. of success

(a) $P(X=0) = \dfrac{e^{-4}4^{0}}{0!} = e^{-4} = \underline{0.0183}$

(b) $P(X=2)$

(c) $P(X=4)$

(d) $P(X \geq 5) = 1 - P(X < 5)$
$=$

Q5) X = Number of claims per hour

λ = Average number of claims per hour

$= 3.1$

Here, $X \sim P(\lambda)$ so, $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$; $x = 0, 1, 2, \ldots$; $\lambda > 0$

(a) $P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$$= \frac{e^{-3.1}\, 3.1^{0}}{0!} + \frac{e^{-3.1}\, 3.1^{1}}{1!} + \frac{e^{-3.1}\, (3.1)^{2}}{2!}$$

$$= e^{-3.1}\left(1 + 3.1 + \cancel{4.80}\right)$$

$$= e^{-3.1} * \cancel{8.9}$$

$$= 0.045 * \cancel{8.9}\ 9.22$$

$$= \cancel{0.400}\ \ 0.4149$$

$$= e^{-3.1}\left(1 + 3.1 + 4.8\right)$$

$$= e^{-3.1} * 8.9$$

$$= \underline{0.400}$$

(b) $P(X=3) = \dfrac{e \quad 3.1'}{3!} = \underline{0.224}$

(c) $P(X \geq 3) = 1 - P(X < 3)$

$= 1 - 0.400$

$= \underline{0.600}$

(d) $P(X > 3) = 1 - P(X \leq 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - [P(X<3) + P(X=3)]$

$= 1 - [0.401 + 0.224]$

$= \underline{0.375}$

Q7.

λ = Average number of arrivals in one minute

$= 6$

for (e), (f)

$\lambda = 12$

= Average number of arrivals in two minutes

X = Number of arrivals in one minute.

Poisson approximation to Binomial

The binomial distribution reduces to Poisson distribution under the following conditions:

(i) If no. of trials 'n' is very large i.e. $n \to \infty$ $(n > 20)$

(ii) If probability of success 'p' is very small i.e. $p \to 0$ $(p \leq 0.05)$

$$\therefore \lambda = np$$

$$\Rightarrow P(X = x) = \frac{e^{-np} (np)^x}{x!}$$