H/W-2

PROBLEMS ON LIMITS & CONTINUITY

Prove using definition that

1. $$\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$$

Let $\varepsilon > 0$. Then, $\forall (x,y) \in \mathbb{R}^2$ with

$0 < \|(x,y) - (0,0)\| < \delta$, we get:

$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$

Thus, $0 < \sqrt{x^2+y^2} < \delta$

We have,

$$|f(x,y) - L| = \left|\frac{5x^2y^2}{x^2+y^2} - 0\right|$$

$$= \left|\frac{5x^2y^2}{x^2+y^2}\right|$$

$$= 5y^2\left|\frac{x^2}{x^2+y^2}\right|$$

$$\leq 5y^2 \quad \left(\because \frac{x^2}{x^2+y^2} \leq 1\right)$$

$$\leq 5(x^2+y^2)$$

$$\leq 5\delta^2$$

Choosing $\delta = \sqrt{\frac{\varepsilon}{5}}$, we obtain

$$|f(x,y) - L| = \left|\frac{5x^2y^2}{x^2+y^2} - 0\right| < \varepsilon$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$$

proved.

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Q2. $\lim_{(x,y)\to(0,1)} \dfrac{x^2(y-1)^2}{x^2+(y-1)^2} = 0$

Let $\varepsilon > 0$, Then $\forall (x,y) \in \mathbb{R}^2$ with

$0 < \|(x,y) - (0,1)\| < \delta$

$\Rightarrow 0 < \sqrt{(x-0)^2 + (y-1)^2} < \delta$

$\Rightarrow 0 < \sqrt{x^2 + (y-1)^2} < \delta,$

we have,

$$|f(x,y) - L| = \left| \frac{x^2 (y-1)^2}{x^2+(y-1)^2} - 0 \right|$$

$$= (y-1)^2 \left| \frac{x^2}{x^2+(y-1)^2} \right|$$

$$\leq (y-1)^2 \qquad \left(\because \frac{x^2}{x^2+(y-1)^2} \leq 1 \right)$$

$$\leq x^2 + (y-1)^2$$

$$\leq \delta^2$$

choosing $\delta = \sqrt{\varepsilon}$, we obtain

$$|f(x,y) - L| = \left| \frac{x^2(y-1)^2}{x^2+(y-1)^2} - 0 \right| < \varepsilon$$

$$\therefore \lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2} = 0$$

proved.

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Q3. $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$

Let $\varepsilon > 0$. Then $\forall (x,y) \in \mathbb{R}^2$ with

$0 < \|(x,y) - (0,0)\| < \delta$

$\Rightarrow 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$

$\Rightarrow 0 < \sqrt{x^2+y^2} < \delta$

We have,

$$|f(x,y) - L| = \left|\frac{y^3}{x^2+y^2} - 0\right|$$

$$= \left|\frac{y \cdot y^2}{x^2+y^2}\right|$$

$$= |y| \left|\frac{y^2}{x^2+y^2}\right|$$

$$\leq |y| \qquad \left(\because \left|\frac{y^2}{x^2+y^2}\right| \leq 1\right)$$

$$= \sqrt{y^2}$$

$$\leq \sqrt{x^2+y^2}$$

$$< \delta^2$$

choosing $\delta = \sqrt{\varepsilon}$, we have:

$$|f(x,y) - L| = \left|\frac{y^3}{x^2+y^2} - 0\right| < \varepsilon$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$$

proved

Q4. $\lim_{(x,y)\to(4,-1)} x = 4$

choose $\varepsilon > 0$. Then, $\forall (x,y) \in \mathbb{R}^2$ with

$0 < \|(x,y) - (4,-1)\| < \delta$

$\Rightarrow 0 < \sqrt{(x-4)^2 + (y+1)^2} < \delta$

we have,

$$|f(x,y) - L| = |x-4|$$
$$= \sqrt{(x-4)^2}$$
$$\leq \sqrt{(x-4)^2 + (y+1)^2}$$
$$< \delta$$

choosing $\delta = \varepsilon$, we have

$|f(x,y) - L| = |x-4| < \varepsilon$

$\therefore \lim_{(x,y)\to(4,-1)} x = 4$ proved

Q5. $\lim_{(x,y)\to(1,-3)} y = -3$

Proof
choose $\varepsilon > 0$, then, $\forall (x,y) \in \mathbb{R}^2$ with

$0 < \|(x,y) - (1,-3)\| < \delta$

$\Rightarrow 0 < \sqrt{(x-1)^2 + (y+3)^2} < \delta$, we have

$$|f(x,y) - L| = |y-(-3)|$$
$$= |y+3|$$
$$= \sqrt{(y+3)^2}$$
$$\leq \sqrt{(x-1)^2 + (y+3)^2}$$
$$< \delta$$

choosing $\delta = \varepsilon$, we have

$|f(x,y) - L| < \varepsilon$.

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$$\therefore \lim_{(x,y)\to(1,-3)} y = -3$$

proved

Prove that each of the following limits does not exist.

Q6. $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Proof:

~~Taking~~ limit along the path $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2}$$

$$= 1$$

limit along the path $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2}$$

$$= -1$$

Since the limits are different along different paths, the limit of the given function does not exist.

Q7. $$\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$$

Proof

limit along the path $y = 2x$

$$\lim_{(x,y)\to(0,0)} \frac{x+2x}{x-2x}$$

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$$= \lim_{(x,y)\to(0,0)} \frac{3x}{-x}$$

$$= -3$$

Limit along the path $y = 3x$

$$\lim_{(x,y)\to(0,0)} \frac{x+3x}{x-3x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{4x}{-2x}$$

$$= -2$$

Since the limits of the function are different along different paths, the limit does not exist.

Q8. $\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{y^2+x}$

Limit along the path $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{mx^2 - m^2x^2}{m^2x^2 + x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^2(1-m)}{x(m^2x+1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx(1-m)}{1+m^2x}$$

$$= \frac{0(1-m)}{1+0}$$

$$= 0$$

Limit along the path $x = 0$

$$\lim_{(0,y)\to(0,0)} \frac{xy-y^2}{y^2+x}$$

$$= \lim_{(0,y)\to(0,0)} \frac{0-y^2}{y^2+0}$$

$$= \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$$

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Since the limits of the function are different along different paths, the limit does not exist.

Q 9. $\lim_{(x,y)\to(0,0)} \frac{\sin x^2}{y}$

Limit along the line $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{\sin x^2}{mx^2} \cdot x$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{\sin x^2}{x^2}\right) \cdot \frac{x}{m}$$

$$= 1 \times \frac{0}{m} \qquad \left(\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= 0$$

Limit along the path $y = x^2$

$$\lim_{(x,y)\to(0,0)} \frac{\sin x^2}{y}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin x^2}{x^2}$$

$$= 1 \qquad \left(\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

Since the limits along different paths are not equal, the limit does not exist.

Q10. $\lim_{(x,y)\to(1,2)} \frac{x+y-3}{x^2-1}$

Limit along the path $y = 2$

$$\lim_{(x,y)\to(1,2)} \frac{x+y-3}{x^2-1}$$

$$= \lim_{(x,y)\to(1,2)} \frac{x+2-3}{x^2-1}$$

$$= \lim_{(x,y)\to(1,2)} \frac{x-1}{(x-1)(x+1)}$$

$$= \lim_{(x,y)\to(1,2)} \frac{1}{x+1}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Limit along the path $y = x+1$

$$\lim_{(x,y)\to(1,2)} \frac{x+y-3}{x^2-1}$$

$$= \lim_{(x,y)\to(1,2)} \frac{x+x+1-3}{x^2-1}$$

$$= \lim_{(x,y)\to(1,2)} \frac{2x-2}{x^2-1}$$

$$= \lim_{(x,y)\to(1,2)} \frac{2(x-1)}{(x-1)(x+1)}$$

$$= \lim_{(x,y)\to(1,2)} \frac{2}{x+1}$$

$$= \frac{2}{1+1}$$

$$= \frac{2}{2}$$

$$= 1$$

Since, limits along different paths are not equal, the limit doesn't exist.

Q11. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y}$

Limit along the path $x=0$,

$$\lim_{(0,y)\to(0,0)} \frac{x^2+y^2}{y}$$

$$= \lim_{(0,y)\to(0,0)} \frac{y^2}{y}$$

$$= \lim_{(0,y)\to(0,0)} y$$

$$= 0$$

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Limit along the path $y = x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2+(x^2)^2}{x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(1+x^2)}{x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(1+x^2)}{1}$$

$$= 1+0^2$$

$$= 1$$

Thus, limits acorass different paths are not equal, thus limit does not exist.

Q12. $$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$$

Limit along the path $y=0$

$$= \lim_{(x,0)\to(0,0)} \frac{x}{x^2+y^2}$$

$$= \lim_{(x,0)\to(0,0)} \frac{x}{x^2}$$

$$= \lim_{(x,0)\to(0,0)} \frac{1}{x}$$

$$= \frac{1}{0} = \infty$$

Thus, limit does not exist.

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Find the limit and discuss the continuity of the function

13. $\lim_{(x,y)\to(2,1)} (2x^2+y)$

$= 2\times(2)^2+1$

$= 2\times 4+1$

$= 9$

Now $f(x,y)$ at $(x_0, y_0) = (2,1)$ is given by

$f(x_0,y_0) = 2x_0^2 + y_0$

$= 2\times 2^2+1$

$= 9$

Here, $\lim_{(x,y)\to(2,1)} (2x^2+y)$ exists and is equal to $f(x,y) = 2x^2+y$ at $(2,1)$. Thus, the functions is continuous at $(x_0, y_0) = (2,1)$.

14. $\lim_{(x,y)\to(0,0)} (x+4y+1)$

Here, $f(x,y) = x+4y+1$ and

$(x_0, y_0) = (0,0)$.

Now,

$\lim_{(x,y)\to(0,0)} (x+4y+1) = 0+4\times0+1$

$= 1$

Now $f(x_0,y_0) = x_0+4y_0+1$

$= 0+4\times0+1$

$= 1$

Here, limit of the function exists at $(0,0)$ and is equal to $f(x_0,y_0)$. Thus, the function is continuous at $(0,0)$.

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Q15. $\lim_{(x,y)\to(1,2)} e^{xy}$

Here, $f(x,y) = e^{xy}$

$(x_0, y_0) = (1, 2)$

Now,

$$\lim_{(x,y)\to(1,2)} e^{xy} = e^{1\times 2}$$

$$= e^2$$

$f(x,y)$ at $(x_0, y_0) = (1,2)$ is

$$f(x_0, y_0) = e^{x_0 y_0}$$

$$= e^{1\times 2}$$

$$= e^2$$

Thus, limit of the function exists at $(1,2)$; $f(x,y)$ also exists at $(1,2)$ and is equal to the limit. Thus, the given function is continuous at $(1,2)$.

Q16. $\lim_{(x,y)\to(2,4)} \frac{x+y}{x^2+1}$

Here, $f(x,y) = \frac{x+y}{x^2+1}$

& $(x_0, y_0) = (2, 4)$

Now,

$$\lim_{(x,y)\to(2,4)} \frac{x+y}{x^2+1} = \frac{2+4}{(2)^2+1}$$

$$= \frac{6}{4+1}$$

$$= \frac{6}{5}$$

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$$f(x,y) \text{ at } (x_0, y_0) = \frac{x_0 + y_0}{x_0^2 + 1}$$

$$= \frac{2+4}{(2)^2+1}$$

$$= \frac{6}{5}$$

The limit of the given function exists at (2, 4) and is equal to f at (2, 4). Thus, the function is continuous at (2, 4).

Q17. $\lim_{(x,y) \to (0,2)} \frac{x}{y}$

Here, $f(x,y) = \frac{x}{y}$

& $(x_0, y_0) = (0, 2)$

Now,

$$\lim_{(x,y) \to (0,2)} \frac{x}{y} = \frac{0}{2}$$

$$= 0$$

$$f(x,y) \text{ at } (x_0, y_0) = f(x_0, y_0)$$

$$= \frac{x_0}{y_0}$$

$$= \frac{0}{2}$$

$$= 0$$

The limit of given function exists at (0, 2); f(x, y) also exists at (0, 2) and is equal to the limit. Thus, f(x, y) is continuous at (0, 2).

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Q18. $\lim_{(x,y)\to(-1,2)} \frac{x+y}{x-y}$

Here, $f(x,y) = \frac{x+y}{x-y}$

& $(x_0, y_0) = (-1,2)$

Now, $\lim_{(x,y)\to(-1,2)} \frac{x+y}{x-y} = \frac{-1+2}{-1-2}$

$= \frac{1}{-3}$

$= -1/3$

$f(x,y)$ at $(x_0,y_0) = f(x_0,y_0)$

$= \frac{x_0+y_0}{x_0-y_0}$

$= \frac{-1+2}{-1-2}$

$= \frac{1}{-3}$

$= -\frac{1}{3}$

Since, $\lim_{(x,y)\to(-1,2)} \frac{x+y}{x-y} = f(x,y)$ at $(-1,2)$,

the given function is continuous.

Q19. $\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2}$

Here,

$f(x,y) = \frac{xy}{x^2+y^2}$

& $(x_0, y_0) = (1,1)$

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Now,
$$\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{1*1}{(1)^2+(1)^2}$$
$$= \frac{1}{1+1}$$
$$= \frac{1}{2}$$

$$f(x,y) \text{ at } (x_0, y_0) = f(x_0, y_0)$$
$$= \frac{x_0 y_0}{(x_0)^2+(y_0)^2}$$
$$= \frac{1*1}{(1)^2+(1)^2}$$
$$= \frac{1}{2}$$

The limit of f exists at (1, 1). f also exists at (1,1) and is equal to the limit of f. Thus, the given function f is continuous.

Q20. $$\lim_{(x,y)\to(1,1)} \frac{x}{\sqrt{x+y}}$$

Here, $f(x,y) = \dfrac{x}{\sqrt{x+y}}$

$f(x_0, y_0) = (1, 1)$

Now,
$$\lim_{(x,y)\to(1,1)} \frac{x}{\sqrt{x+y}} = \frac{1}{\sqrt{1+1}}$$
$$= \frac{1}{\sqrt{2}}$$

$$f(x,y) \text{ at } (x_0, y_0) = f(x_0, y_0)$$
$$= \frac{x_0}{\sqrt{x_0+y_0}}$$

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$$= \frac{1}{\sqrt{1+1}}$$

$$= \frac{1}{\sqrt{2}}$$

The limit of the given function exists at (1, 1). The function f (x, y) also exists at (1, 1) and is equal to the limit. Thus, the function f is continuous at (1, 1).

Q21. $\lim\limits_{(x,y) \to (\pi/4, 2)} y \cos xy$

Here,

$f(x, y) = y \cos y$

& $(x_0, y_0) = (\pi/4, 2)$

Now,

$$\lim_{(x,y) \to (\pi/4, 2)} y \cos xy = 2 \cos(2 \cdot \pi/4)$$

$$= 2 \cos(\pi/2)$$

$$= 2 \times 0$$

$$= 2$$

$$f(x, y) \text{ at } (x_0, y_0) = f(x_0, y_0)$$

$$= y_0 \cos x_0 y_0$$

$$= 2 \cos(\pi/4 \cdot 2)$$

$$= 2 \cos(\pi/2)$$

$$= 2 \times 0$$

$$= 0$$

∴ The limit exists at $(\pi/4, 2)$; f exists at $(\pi/4, 2)$ and both are equal. Thus, the function f is continuous at $(\pi/4, 2)$.

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Q22. $\lim_{(x,y)\to(2\pi,4)} \sin\frac{x}{y}$

Here, $f(x,y) = \sin\frac{x}{y}$

& $(x_0, y_0) = (2\pi, 4)$

Now,

$$\lim_{(x,y)\to(2\pi,4)} \sin\frac{x}{y} = \sin\frac{2\pi}{4}$$

$$= \sin\frac{\pi}{2}$$

$$= 1$$

$f(x,y)$ at $(x_0, y_0) = f(x_0, y_0)$

$$= \sin\frac{x_0}{y_0}$$

$$= \sin\frac{2\pi}{4}$$

$$= \sin\frac{\pi}{2}$$

$$= 1$$

The limit of given function f exists at $(2\pi, 4)$. f also exists at $(2\pi, 4)$ and both are equal. Thus, f is continuous at $(2\pi, 4)$.

Q23. $\lim_{(x,y)\to(0,1)} \frac{\arcsin xy}{1-xy}$

Here, $f(x,y) = \frac{\arcsin xy}{1-xy}$

& $(x_0, y_0) = (0,1)$

Now,

$$\lim_{(x,y)\to(0,1)} \frac{\arcsin xy}{1-xy} = \frac{\arcsin(0*1)}{1-0*1}$$

$$= \frac{\arcsin(0)}{1} = \frac{0}{1} = 0$$

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Now, $f(x,y)$ at $(x_0, y_0) = f(x_0, y_0)$

$$= \frac{\arcsin(x_0 y_0)}{1 - x_0 y_0}$$

$$= \frac{\arcsin(0*1)}{1-0*1}$$

$$= \frac{0}{1}$$

$$= 0$$

The limit exists at $(0,1)$, f also exists at $(0,1)$ and both are equal. Thus, the function is continuous at $(0,1)$.

Q24. $\lim_{(x,y)\to(0,1)} \frac{\arccos\left(\frac{x}{y}\right)}{1+xy}$

Here,

$$f(x,y) = \frac{\arccos\left(\frac{x}{y}\right)}{1+xy}$$

& $(x_0, y_0) = (0,1)$

Now,

$$\lim_{(x,y)\to(0,1)} \frac{\arccos\left(\frac{x}{y}\right)}{1+xy} = \frac{\arccos\left(\frac{0}{1}\right)}{1+0*1}$$

$$= \frac{\arccos(0)}{1}$$

$$= \frac{\pi/2}{1}$$

$$= \frac{\pi}{2}$$

Now, $f(x,y)$ at $(x_0, y_0) = f(x_0, y_0)$

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$$= \frac{\arccos\left(\frac{x_0}{y_0}\right)}{1 + x_0 y_0}$$

$$= \frac{\arccos\left(\frac{0}{1}\right)}{1 + 0 * 1}$$

$$= \frac{\arccos(0)}{1}$$

$$= \frac{\pi/2}{1}$$

$$= \pi/2$$

Thus, limit exists at $(0, 1)$ and f also exists at that point. Since both of them are equal, the function f is continuous at $(0, 1)$.

Q25. $$\lim_{(x,y,z) \to (1,3,4)} \sqrt{x+y+z}$$

Here, $f(x, y, z) = \sqrt{x+y+z}$

& $(x_0, y_0, z_0) = (1, 3, 4)$

Now,

$$\lim_{(x,y,z) \to (1,3,4)} \sqrt{x+y+z} = \sqrt{1+3+4}$$

$$= \sqrt{8}$$

Now, $f(x, y, z)$ at $(x_0, y_0, z_0) = f(x_0, y_0, z_0)$

$$= \sqrt{x_0 + y_0 + z_0}$$

$$= \sqrt{1+3+4}$$

$$= \sqrt{8}$$

Thus, limit exists at $(1, 3, 4)$ and is equal to f at the same point. Thus, the function f is continuous at $(1,3,4)$.

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Q26. $\lim_{(x,y,z)\to(-2,1,0)} x e^{yz}$

Here,

$$f(x,y,z) = x e^{yz}$$

$$\& \ (x_0, y_0, z_0) = (-2, 1, 0)$$

Now,

$$\lim_{(x,y,z)\to(-2,1,0)} x e^{yz} = (-2) * e^{1*0}$$

$$= (-2) * e^{0}$$

$$= -2 * 1$$

$$= -2$$

$$f(x,y,z) \text{ at } (x_0, y_0, z_0) = f(x_0, y_0, z_0)$$

$$= x_0 e^{y_0 z_0}$$

$$= (-2) * e^{1*0}$$

$$= (-2) * e^{0}$$

$$= (-2) * 1$$

$$= -2$$

Here, limit of the function f exists at $(-2, 1, 0)$ and is equal to f at $(-2, 1, 0)$. Thus, the function is continuous at $(-2, 1, 0)$.

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