Second Assessment 2081

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Group A

1. If $\lambda = 1, 5$ eigenvalues of the matrix $\begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}$, find a basis for the eigenspace corresponding to each eigenvalue.

Solution

Given,

$$\lambda = 1, 5$$

Let

$$A = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}$$

$$\lambda_1 = 5 \ \& \ \lambda_2 = 1$$

For $\lambda_1 = 5$

to find the eigen vector; we have,

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{pmatrix} 7-5 & 4 \\ -3 & -1-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\Rightarrow \cancel{(7-5)(-1-\lambda)+12=0}$$

$$\Rightarrow \begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 + 4x_2 = 0 \\ -3x_1 - 6x_2 = 0 \end{cases}$$

from above,

$$x_1 = -2x_2$$

For example

$v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is the eigen vector

$v_1 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ is the unit eigen vector corresponding to the eigen value $\lambda_1 = 5$

for $\lambda = 1$

to find the eigen vector; we have

$$\begin{pmatrix} 7-1 & 4 \\ -3 & -1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 6x_1 + 4x_2 = 0 \\ -3x_1 - 2x_2 = 0 \end{cases}$$

from above;

$$x_1 = -\frac{2}{3}x_2$$

for example;

$$v_2 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$$ is the eigen vector

$$v_2 = \begin{pmatrix} -\frac{2}{\sqrt{13}} \\ \frac{1}{\sqrt{13}} \end{pmatrix}$$ is the eigen vector corresponding to the eigen value $\lambda_1 = 1$

Basis for the eigenspaces

$$E_5 = \left\{ \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \right\}$$

is the basis for eigenspace correspond to eigenvalue $\lambda_1 = 5$

$$E_1 = \left\{ \begin{pmatrix} \frac{2}{\sqrt{13}} \\ \frac{1}{\sqrt{23}} \end{pmatrix} \right\}$$

is the basis for eigenspace correspond to eigenvalue $\lambda_2 = 1$

2. Find the maximum value of $9x_1^2+4x_2^2+4x_3^2$ subject to the constraints $x^Tx=1$ and $x^Tu_1=0$, where $u_1=(1,0,0)$.
Find x where it is attained. Here u_1 is ~~the~~ a unit eigen vector corresponding to greatest eigenvalue $\lambda=9$ of the matrix of the quadratic form.

Solution

If the coordinates of x are x_1, x_2, x_3 then the constraints $x^Tu_1=0$ means simply that $x_1=0$. means simply that $x_1=0$. For such a unit vector, $x_2^2+x_3^2=1$ and

$$9x_1^2+4x_2^2+4x_3^2 = 4x_2^2+4x_3^2$$
$$=4(x_2^2+x_3^2)$$
$$=4$$

Thus the constrained maximum of the quadratic form does not exceed 4. And this value is attained for $x_1=(0,1,0)$, which is an eigenvector for the second greatest eigenvalue of the matrix of the quadratic form

3) Find the singular values of the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

so

$$A^TA = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+1 & 1+0-1 \\ 1+0-1 & 1+1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

To find the eigen value; we have

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) = 0$$

$$\therefore \lambda_1 = 3 \;\&\; \lambda_2 = 2$$

for $\lambda_1 = 3$

to find the eigen vector; we have

$$\begin{pmatrix} 2-3 & 0 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \cancel{\lambda_1 = 0}$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$

$\sqrt{3}$ & $\sqrt{2}$ are the singular valu

of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$.

4. Consider the quadratic form

$$Q(x) = 2x_1^2 + 4x_1x_2 - 4x_3x_1 - x_2^2 + 8x_3x_2 - x_3^2$$

Decide whether this quadratic form is Positive, negative or indefinite.

Soln:

Let A be an $n\times n$ symmetric matrix and $Q(x) = x^TAx$ is the corresponding quadratic form. Then Q is

a) Positive definite if $x^TAx > 0, \forall x \neq 0$

b) negative definite if $x^TAx < 0, \forall x \neq 0$

c) Indefinite if $x^TAx > 0$ for some x and $x^TAx < 0$ for others.

d) Positive semidefinite if $x^TAx \geq 0, \forall x = 0$

e) negative semidefinite if $x^TAx \leq 0, \forall x = 0$

The quadratic form $Q(x) = 2x_1^2 + 4x_1x_2 - 4x_3x_1 - x_2^2 + 8x_3x_2 - x_3^2$ can be expressed as matrix form

$$(x_1\ x_2\ x_3)\begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

whee $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$

to find eigen value of A, we know

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & -1-\lambda & 4 \\ -2 & 4 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2-\lambda \begin{vmatrix} -1-\lambda & 4 \\ 4 & -1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 \\ -2 & -1-\lambda \end{vmatrix}$$

$$-2 \begin{vmatrix} 2 & -1-\lambda \\ -2 & 4 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(-1-\lambda)^2 - 16\right] - 2(2(-1-\lambda)+8)$$

$$-2(8+2(-1-\lambda)) = 0$$

$-2(-1)(-1)$

$\Rightarrow \cancel{(2-\lambda)(1-}$

$\Rightarrow (2-\lambda)(1+2\lambda+\lambda^2-16)-2(-2-2\lambda+8)$

$-2(8-2-2\lambda)=0$

$\Rightarrow (2-\lambda)(\lambda^2+2\lambda-15)-2(-2\lambda+6)$

$-2(6-2\lambda)=0$

$\Rightarrow \cancel{2\lambda^2}+4\lambda-30-\lambda^3-\cancel{2\lambda^2}+15\lambda$

$+4\lambda-12-12+4\lambda=0$

$\Rightarrow -\lambda^3+27\lambda-54$

$\Rightarrow \lambda^3-27\lambda+54=0$

$$A = \begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & 1-\lambda & 4 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$$C_2 \rightarrow C_2 + C_3$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & -2 \\ 2 & 3-\lambda & 4 \\ -2 & 3-\lambda & -1-\lambda \end{vmatrix} = 0$$

$$R_2 \rightarrow R_2 - R_3$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & -2 \\ 4 & 0 & 5+\lambda \\ -2 & 3-\lambda & -1-\lambda \end{vmatrix} = 0$$

expanding along C_2

$$(3-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ 4 & 5+\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left((2-\lambda)(5+\lambda)+8\right) = 0$$

$$\Rightarrow (3-\lambda)\left(10+2\lambda-5\lambda-\lambda^2+8\right) = 0$$

$$\Rightarrow (3-\lambda)\left(\lambda^2+3\lambda-18\right) = 0$$

$$\Rightarrow (3-\lambda)\left(\lambda^2+6\lambda-3\lambda-18\right) = 0$$

$$\Rightarrow (3-\lambda)\left(\lambda(\lambda+6)-3(\lambda+6)\right) = 0$$

$$\Rightarrow (3-\lambda)(\lambda+6)(\lambda-3) = 0$$

$\therefore \lambda = -6, 3$ $\therefore$ Indefinite

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5. Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$3x_1 + 5x_2 - 4x_3 = 0$, $-3x_1 - 2x_2 + 4x_3 = 0$,

$6x_1 + x_2 - 8x_3 = 0$.

Sol^n:

The augmented matrix is

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right)$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right)$$

$\cancel{R_3 \Rightarrow \frac{R_3}{-9}}$

$R_3 \to R_3 + 3R_2$

$$\sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

from above;

$x_2 = 0;$

$3x_1 + 0 - 4x_3 = 0$

$x_1 = \frac{4}{3} x_3$

nontrivial solution of given homogeneous system has $\left(\frac{4}{3}a, 0, a\right)$.

where $a \in R$ and $a \neq 0$

It ~~has~~ has infinitely many solution

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Group B

6) Consider the matrix; $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

a) What can we say about the action of A on an arbitrary vector?

b) What are examples of eigenvalues and eigenvectors of this matrix?

c) What does the discussion for this example illustrate?

a) Let's apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have

$$Ax = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix}$$

we observe that there is no $\lambda \in \mathbb{R}$ such that

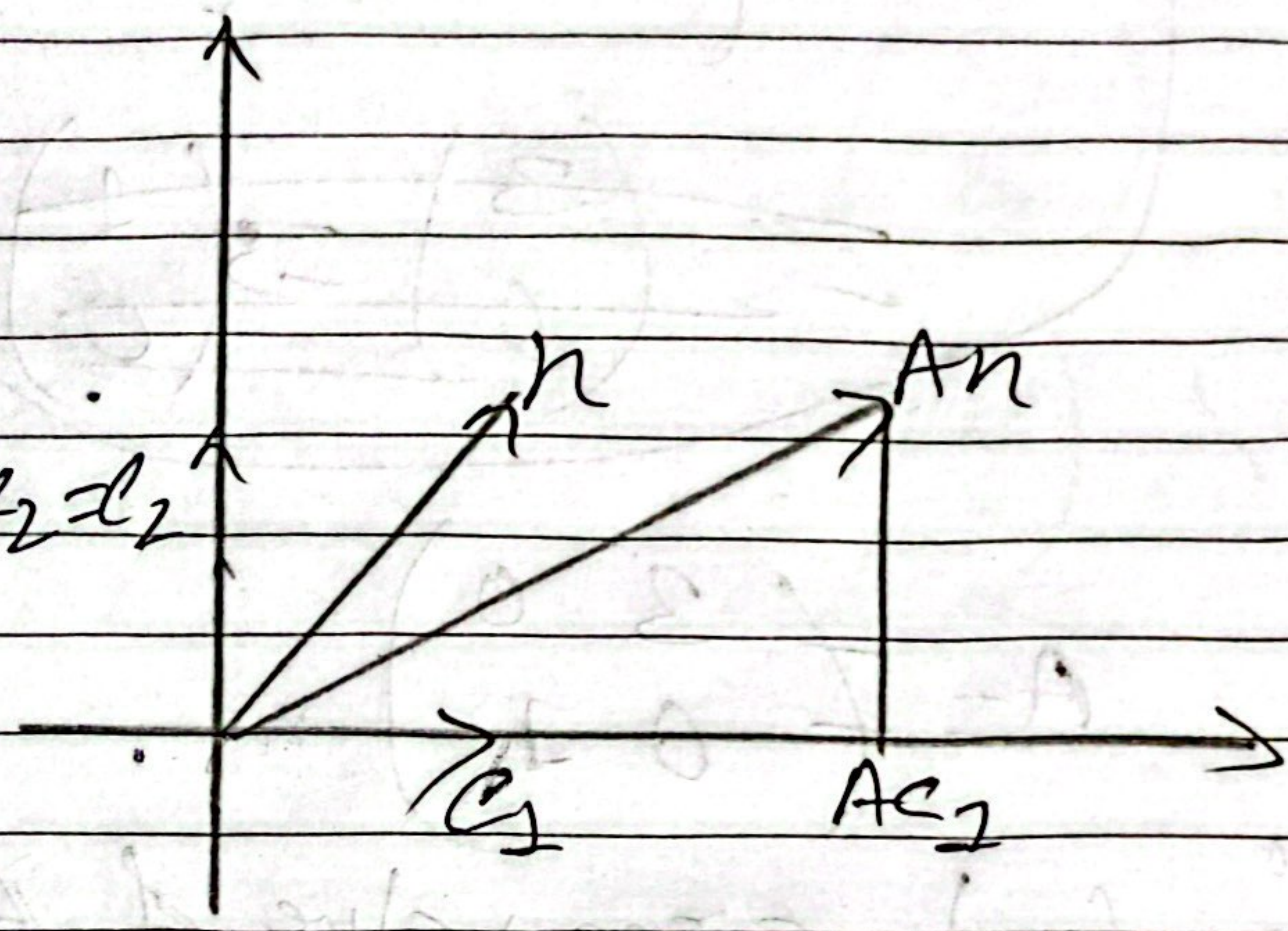
$$\begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix} \neq \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

if x_1 and x_2 both are nonzero

$\therefore$ So, in this case the vector x cannot be eigenvector of the matrix A.

The following figure illustrates this when

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



Note that in this case

$$Ax = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} + A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= Ae_1 + Ae_2$$

b) What are examples of eigenvalue and eigenvectors of this matrix?

⇒ Let us apply A to the coordinate vectors e_1 and e_2 we have

$$Ae_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$|A-\lambda I|$

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$3-\lambda$, $1-\lambda = 0$

to find eigen value of A

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(1-\lambda) = 0$$

$\therefore \lambda_1 = 3$ & $\lambda_2 = 1$

for $\lambda_1 = 3$

to find eigen vector

$$\begin{pmatrix} 3-3 & 0 \\ 0 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -2x_2 = 0$$

$$\therefore \quad x_2 = 0$$

for $\lambda_2 = 1$
to find eigen vector

$$\begin{pmatrix} 3-1 & 0 \\ 0 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$2x_1 = 0$$

6

Q) a) Let v_1, v_2 be the eigenvectors associated with the eigenvalues λ_1, λ_2 of a 2X2 symmetric matrix A respectively. Prove that if $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $V = (v_1, v_2)$ then $A = V \Lambda V^T$.

Sol^n

we have

$$AV = (Av_1 \; Av_2)$$

$$= \begin{pmatrix} \cancel{Av_{11}} & \cancel{Av_{21}} \\ \cancel{Av_{21}} & \cancel{Av_{22}} \end{pmatrix}$$

$$= (\lambda v_1 \; \lambda v_2)$$

$$= \begin{pmatrix} \lambda v_{11} & \lambda v_{12} \\ \lambda v_{21} & \lambda v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= V \Lambda$$

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~~therefore~~
~~thus is~~

therefore, from this in view of $VV^T = I$, we obtain

$$A = U\Lambda U^T \quad \text{--- (a)}$$

this theorem provides a decomposition of the matrix A into the product of three matrices;

→ an orthogonal matrix V (consisting of the eigenvectors of A),

→ a diagonal matrix Λ (consisting of the eigenvalues of A) and

→ the transpose of that orthogonal matrix, V^T

We can write above equation

$$A = V\Lambda V^T = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} U_{11} & U_{21} \\ U_{12} & U_{22} \end{pmatrix}$$

6 (b) Find all 2X2 matrices A which admit the normalized eigenvectors $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with the corresponding eigenvalues λ_1 and λ_2.

Solution

Given

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A = V \Lambda V^T$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

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$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \lambda_1+\lambda_2 & \lambda_1 \cancel{\lambda} - \lambda_2 \\ \lambda_1-\lambda_2 & \lambda_1-\lambda_2 \end{pmatrix}$$

7 a) Let A be an $n \times n$ matrix. Prove that if A has n linearly independent eigenvectors, then A is diagonalizable.

Solution

Let $\{u_1, \ldots, u_n\}$ be a linearly independent eigenvectors of A corresponding to eigenvalues $\lambda_1 \ldots \lambda_n$.

then $V = \{u_1, \ldots u_n)$ is non-singular

and

$$AV = [Au_1, \ldots, Au_n]$$

$$= [\lambda_1 u_1 \ldots \lambda_2 u_n]$$

$$= V\Lambda$$

where $\Lambda = \text{dia}(\lambda_1 \ldots \lambda_n)$.

Therefore

$$V^{-1}AV = \Lambda$$

This implies that A is diagonalizable

7 (b) Show that the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$ is not diagonalizable.

Solution

Given that $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$

to find the eigenvalue A, we have

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(4-\lambda) + 1 = 0$$

$$\cancel{\lambda_1 = 3 \; \& \; \lambda_2 =}$$

$$\Rightarrow 8 - 2\lambda - 4\lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 3\lambda + 9 = 0$$

$$\Rightarrow \lambda(\lambda-3) - 3(\lambda-3) = 0$$

$$\Rightarrow (\lambda-3)(\lambda-3) = 0$$

$$\therefore \lambda_1 = 3 \ \& \ \lambda_2 = 3$$

To find eigen vector; we have

$$\begin{pmatrix} 2-3 & -1 \\ 1 & 4-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -x_1 - x_2 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

from this

$$x_1 = -x_2$$

for example

$$v_{\lambda=3} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ this is the unit eigen vector correspon to the eigen value $\lambda =$

∵ Eigen value λ_1 & λ_2 are equal, So. eigen vector also equal. therefore they are linearly dependent and they are not diagonalizable.

8. a) Prove that if A is a Symmetric $n \times n$ matrix and $B_A(v,w) = v^T A w$, then $B_A(v,w)$ is linear in the first variable v.

Solution

For any Real number a

$$B_A(av, w) = (av)^T A w$$
$$= a v^T A w$$
$$= a B_A(v,w)$$

Also

$$B_A(u+v, w) = (u+v)^T A w$$
$$= (u^T + v^T) A w$$
$$= u^T A w + v^T A w$$
$$= B_A(u,w) + B_A(v,w)$$

Therefore, $B_A(v,w) = v^T A w$ is linear in the first variable v.

b) Write the quadratic form $10x_1^2 - 8x_1x_2 + 4x_2^2$ as $x^T A x$. Transform it into a quadratic form without the cross product term using eigenvalues and eigenvectors.

Solution

The quadratic form $Q = 10x_1^2 - 8x_1x_2 + 4x_2^2$ can be expressed in matrix form as

$$(x_1, x_2)\begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $A = \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix}$

to find the eigen values of A, we have

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 10-\lambda & -4 \\ -4 & 4-\lambda \end{vmatrix} = 0$$

$x^T A x = (Py)^T A (Py)$
$= Py^T P' A P y$ $= y' x y$

$\Rightarrow (10-\lambda)(4-\lambda)-16=0$

$\lambda_1 = 12 \quad \lambda_2 = 2$

for $\lambda_1 = 12$
to find the eigen vector; we have

$(A-\lambda I)x = 0$

$$\Rightarrow \begin{pmatrix} 10-12 & -4 \\ -4 & 4-12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -2x_1 - 4x_2 = 0 \\ -4x_1 - 8x_2 = 0 \end{cases}$$

From above;

$$\cancel{2}\ x_1 = -2x_2$$

for example

$$v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ is eigen vector}$$

$$v_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$ is the unit eigen vector corresponding to eigen value $\lambda = 12$.

For $\lambda = 2$
to find the eigen vector; we have

$$\begin{pmatrix} 10-2 & -4 \\ -4 & 4-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 8x_1 - 4x_2 = 0 \\ -4x_1 + 2x_2 = 0 \end{cases}$$

from above

$$2x_1 = x_2$$

for example

$U_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the eigen vector

$U_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ is the unit eigen vector crossponding to eigenvalue $\lambda = 2$

The matrix of eigenvector $P = (U_1 \; U_2)$

$$= \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$P^T = \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

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from the definition of Λ, we have

$\cancel{A}$

$$\Lambda = P^{\dagger} A P$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -20-4 & 8+4 \\ 10-8 & -4+8 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -24 & 12 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

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$$= \cancel{\frac{1}{5}\begin{pmatrix} 32-4 & 16-8 \\ 4+4 & 2+8 \end{pmatrix}}$$

$$= \cancel{\frac{1}{5}\begin{pmatrix} 28 & 8 \\ 8 & 10 \end{pmatrix}}$$

$$= \frac{1}{5}\begin{pmatrix} 48+12 & -24+24 \\ -4+4 & 2+8 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 60 & 0 \\ 0 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix}$$

we know

$$x = Py$$

or $x^T A x = (Py)^T A (Py)$

$$= P^T y^T A P y$$

$$= \cancel{P^T y^T y}\; y^T P^T A P y$$

$$= y^T \Lambda y \quad (\because P^T A P = \Lambda)$$

the quadratic form ~~[illegible]~~ without the Cross Product term.

$$\therefore Q' = y^T \Lambda y$$

$$= (y_1 \; y_2) \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$= 12y_1^2 + 2y_2^2$ is the required quadratic form without the crossproduct term using eigenvalues and eigenvectors.

9) Find an SVD of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$

Solution

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$

so, $A^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+0+1 & 0+0+2 \\ 0+0+2 & 0+1+4 \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

to find the eigen value; we have

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(5-\lambda)-4=0$$

$$\therefore \lambda_1 = 6 \text{ and } \lambda_2 = 1$$

for $\lambda_1 = 6$

to find the eigen vector; we have

$$\begin{pmatrix} 2-6 & 2 \\ 2 & 5-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -4x_1+2x_2=0 \\ 2x_1-x_2=0 \end{cases}$$

from above; we get

$$2x_1 = x_2$$

for example,

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$ is the eigen vecto

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$ is the unit eigen vector crossponding the eigen value $\lambda_1 = 6$

for $\lambda = 1$

to find the eigen vector; we have

$$\begin{pmatrix} 2-1 & 2 \\ 2 & 5-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases}$$

from above; we get

$$x_1 = -2x_2$$

for example.

$$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$ is eigen vector

$$v_2 = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$ is the unit eigen vector crossponding to eigen value $\lambda = 1$

Now

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{6}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

we know;

$$u_1 = \frac{1}{\sigma_1} A v_1$$

$$= \frac{1}{\sqrt{6}} \underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}} \frac{1}{\sqrt{5}} \underset{2\times 2}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$= \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2$$

$$= \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$-\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\therefore u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

~~to find u_3~~

Let's expand $\{u_1, u_2\}$ to $\{u_1, u_2, u_3\}$

u_3 is orthogonal to both u_1 & u_2

$$\text{Let } u_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$u_1 . u_2 = 0$$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$5 5z = ...$
$z = -1 \quad -2 \times 1 ... \quad y = 2$

$\Rightarrow x + 2y + 5z = 0 \quad \text{--- (i)}$

$u_2 \cdot u_3 = 0$

$$\Rightarrow \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-2x + y = 0 \quad \text{--- (ii)}$

from eqn (i) & (ii) we get

$x = 1$
$y = 2$
$z = -1$

$$\therefore u_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$1 + 1$

normalized $u_3 =$

10. What is reduced row echelon form? Illustrate with an example of an augmented matrix of order 4×5. Solve the following linear system by placing the augmented matrix in reduced row echelon form.

$2x + y - z = 2$, $4x + 3y + 2z = -3$,

$6x - 5y + 3z = -14$

Solution

A matrix is in reduced row echelon form, normally abbreviated to rref, if it satisfies all the following conditions

1. If there are any rows containing only zero entries then they are located in the bottom part of the matrix

2. If a row contains non-zero entries then the first non-zero entry is a 1. This 1 is called a leading 1.

3. The leading 1's of two consecutive non-zero rows go strictly from top left to bottom right of the matrix

4. The only non-zero entry in a column containing a leading 1 is the leading 1.

The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4 & 3 & 2 & -3 \\ 6 & -5 & 3 & -14 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & -8 & 6 & -20 \end{array}\right)$$

$R_3 \to R_3 + 8R_2$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 38 & -76 \end{array}\right)$$

$$R_3 \to \frac{R_3}{38}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$R_1 \to R_1 + R_3$$

$$R_2 \to R_2 - 4R_3$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$R_1 \to \frac{R_1}{2}$$

$\sim$

$R_1 \to R_1 - R_2$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right)$$

$R_1 \to \frac{R_1}{2}$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right)$$

$$\therefore \left. \begin{array}{l} x = -\frac{1}{2} \\ y = 1 \\ z = -2 \end{array} \right\} \text{Ans}$$