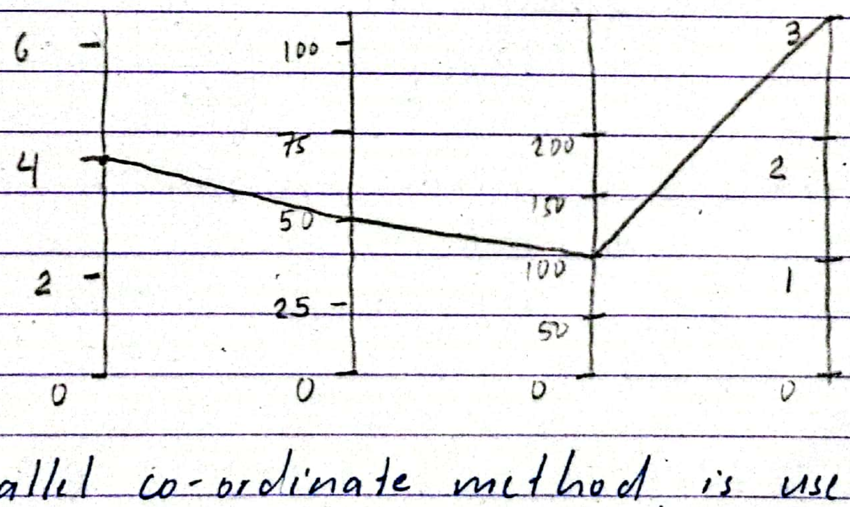
Group 'A'

Q1.

Parallel coordinates method is a system of representing n-dimensional tuples/ co-ordinates using n-parallel lines. Such lines are spaced in a consistent pattern i.e. the distance between two lines is fixed. The scale associated with an individual line can vary from one line to another.

For instance, we can represent the point (4, 50, 100, 3) using parallel coordinate method as follows:



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Parallel co-ordinate method is useful in data science to find the relation among various variables present in a tuple/ data-points. Visual description of relationship such as co-relation can be seen among variables using this method.

Q2.

A mapping $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a norm if it satisfies following properties:

i) Non-negativity
ii) Positive scalability
iii) Triangle inequality

L_1 norm: L_1 norm for any $x \in \mathbb{R}^n$ is defined as:

$$\|X\|_1 = \sum_{i=1}^{n} |X_i|$$

L_2 norm: L_2 norm for any $x \in \mathbb{R}^n$ is defined as:

$$\|X\|_2 = \left(\sum_{i=1}^{n} X_i^2 \right)^{1/2}$$

L_∞ norm: L_∞ norm for any $x \in \mathbb{R}^n$ is defined as:

$$\|X\|_\infty = \max_{i \in \{1, 2, \dots, n\}} |X_i|$$

Here, $x = (1, -1, 0, \dots, 0, 2) \in \mathbb{R}^n$

L_1-norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

$$= |x_1| + |x_2| + \dots + |x_n|$$
$$= |1| + |-1| + |0| + \dots + |0| + |2|$$
$$= 1 + 1 + 2 = \boxed{4}$$

L_2 - norm : $\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

$$= \left(x_1^2 + x_2^2 + \dots + x_n^2\right)^{1/2}$$
$$= \left(1^2 + (-1)^2 + 0^2 + \dots + 0^2 + 2^2\right)^{1/2}$$
$$= (6)^{1/2} = \boxed{2.449}$$

L_∞ - norm : $\|x\|_\infty = \max_{i \in \{1,2,\dots,n\}} \{|x_i|\} = \max\{|1|, |-1|, |0|, \dots, |0|, |2|\}$

$$= \max\{1, 1, 0, \dots, 0, 2\}$$
$$= \boxed{2}$$

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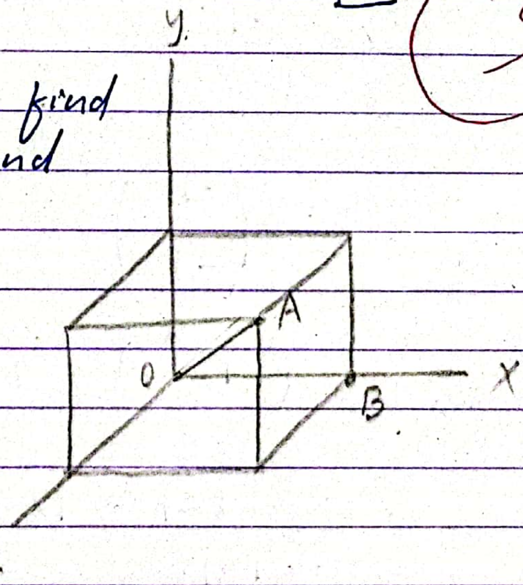
Q3.

Here, we have to find angle between OA and OB.



Co-ordinates of $A = (1, 1, 1)$

Co-ordinates of $B = (1, 0, 0)$

Let, $OA = u = (1, 1, 1)$

$OB = v = (1, 0, 0)$

Angle between two vectors θ is given by:

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

$$= \frac{(1,1,1) \cdot (1,0,0)}{\sqrt{1^2+1^2+1^2} \cdot \sqrt{1^2+0^2+0^2}}$$

$$= \frac{1+0+0}{\sqrt{3} \cdot \sqrt{1}}$$

$$= \frac{1}{\sqrt{3}}$$

$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \underline{54.75^\circ}$ is the angle between diagonal and e_1.

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Q4.

Here,

$$u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$v = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$u \cdot v = (-1, 1) \cdot (-1, -1)$$

$$= -1 \times -1 + 1 \times -1$$

$$= 1 - 1$$

$$= 0$$

This implies that u, v are orthogonal.

The corresponding orthonormal basis in $\mathbb{R}^2$ is given by:

$$\frac{u}{||u||}, \quad \frac{v}{||v||}$$

Let, w, z be corresponding orthonormal bases to u & v.

Then, $w = \frac{u}{\|u\|}$

$= \frac{(-1, 1)}{\sqrt{(-1)^2 + (1)^2}}$

$= \frac{1}{\sqrt{2}} (-1, 1)$

$= (-1/\sqrt{2}, 1/\sqrt{2})$

$= \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Also, $z = \frac{v}{\|v\|}$

$= \frac{(-1, -1)}{\sqrt{(-1)^2 + (-1)^2}}$

$= \frac{(-1, -1)}{\sqrt{2}}$

$= (-1/\sqrt{2}, -1/\sqrt{2})$

$= \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

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Q 5.

For ~~Suppose~~ v_1 and v_2 to be ~~are~~ linearly independent,

~~Then,~~

$c_1 v_1 + c_2 v_2 = 0 \quad \dots\dots (i)$

Multiplying eq^n (i) by A,

$c_1 A v_1 + c_2 A v_2 = 0$

$\Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \quad$ (i.e. $Av = \lambda v$)

$\therefore c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \dots (ii)$

Multiplying (i) by λ_2, we get:

$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0 \quad \dots (iii)$

Subtracting (iii) from (ii), we get:

$c_1 v_1 (\lambda_1 - \lambda_2) = 0$

Since, $\lambda_1 \neq \lambda_2$ and $v_1 \neq 0$,

$c_1 = 0$

Similarly, we can get $c_2 = 0$ by multiplying eq^n (i) by λ_1:

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$\therefore c_1 = c_2 = 0$

This proves that vectors v_1, v_2 are linearly independent.

Group 'B'

Q6.

a.

	Domain	Co-domain
S	$\mathbb{R}^3$	$\mathbb{R}^2$
T	$\mathbb{R}^2$	$\mathbb{R}^3$

The composite transformation SoT is defined because the co-domain of T and domain of S are same.

i.e. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$S: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$\therefore (S \circ T)(x) = S(T(x))$ is defined.

Domain of $(S \circ T)(x) = \mathbb{R}^2$
Co-domain of $(S \circ T)(x) = \mathbb{R}^2$

b. Given,
$T(x) = Bx$
Thus,
$$T(x) = \begin{pmatrix} 3 & 0 \\ 5 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\therefore\ T(x) = \begin{pmatrix} 3x_1 \\ 5x_1 - 2x_2 \\ x_2 \end{pmatrix}$$

c.

$$(S \circ T)(x) = S\left(T(x)\right)$$

$$= \cancel{S\left(B(x)\right)}$$

$$= \cancel{S\ 3x}$$

$$=$$

$$= S(Bx)$$

$$= ABx$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3x_1 \\ 5x_1 - 2x_2 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x_1 + 10x_1 - 4x_2 \\ 5x_1 - 2x_2 + x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 13x_1 - 4x_2 \\ 5x_1 - x_2 \end{pmatrix}$$

d.

Given,

$$(S \circ T)(x) = Cx$$

$$\Rightarrow Cx = \begin{pmatrix} 13x_1 - 4x_2 \\ 5x_1 - x_2 \end{pmatrix}$$

$$\Rightarrow Cx = \begin{pmatrix} 13 & -4 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 13 & -4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow C = \begin{pmatrix} 13 & -4 \\ 5 & -1 \end{pmatrix}$$

$$\therefore C = \begin{pmatrix} 13 & -4 \\ 5 & -1 \end{pmatrix}$$

e.

$\cancel{(S \circ T)(x) =}$

$$(S \circ T)(ax + by) = S(T(ax + by))$$

$$= S(T(ax) + T(by))$$

$(\because T \text{ is linear})$

$$= S(a\,T(x) + b\,T(y))$$

$= S(aBx + bBy)$

$= S(aBx) + S(bBy)$ $(\because S \text{ is linear})$

$= a\,S(Bx) + b\,S(By)$

$= a\,ABx + b\,ABy$

Put $C = AB$, $= a\,Cx + b\,Cy$

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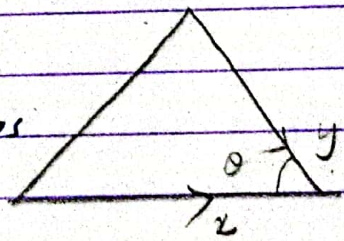
$\therefore (S \circ T)(ax + by) = a\,Cx + b\,Cy$

This proves that $(S \circ T)$ is linear.

Q7.

Here, θ is the angle between two non-zero vectors x, y in $\mathbb{R}^n$.



Then, according to Cosine law,

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta \quad \cdots (i)$$

Also, from the definition of norm and dot product,

$$\|x - y\|^2 = (x - y)\cdot(x - y)$$

$$= x \cdot x - x \cdot y - y \cdot x + y \cdot y$$

$$= ||x||^2 - 2x \cdot y + ||y||^2 \quad (\because ||x||^2 = x \cdot x)$$

$$\therefore ||x-y||^2 = ||x||^2 + ||y||^2 - 2x \cdot y \quad \cdots (ii)$$

Equating (i) & (ii), we get:

$$2x \cdot y = 2||x||||y|| \cos\theta$$

$$\therefore x \cdot y = ||x||||y|| \cos\theta$$

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The above result holds true even if $y = cx$.

?

Q 8.

V is a subspace of $\mathbb{R}^3$ because:

i) V is closed under vector addition

Let $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ & $v_2 = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ be two vectors $\in V$. Then,

$$v_1 + v_2 = \begin{pmatrix} -1+0 \\ 1-2 \\ 0+2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \in V$$

ii) V is closed under scalar multiplication

Let $\alpha \in \mathbb{R}$ be a scalar. Then,

$$\alpha v_1 = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha \\ \alpha \\ 0 \end{pmatrix} \in V \quad [\text{as } -\alpha + \alpha + 0 = 0]$$

To prove B is a basis for V.

i) Proof that $B \in V$

$$b_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in V \quad \text{as} \quad 1 - 1 + 0 = 0$$

$$b_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in V \quad \text{as} \quad 1 + 0 - 1 = 0$$

ii) Proof that basis vectors are linearly independent

Let, c_1 & c_2 be two scalars. Then,

$$c_1 b_1 + c_2 b_2 = 0 \quad \cdots (i)$$

$$\Rightarrow c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} c_1 \\ -c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \\ -c_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 \\ -c_1 \\ -c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$-c_1 = 0$; $-c_2 = 0$

$\therefore c_1 = 0$ $\therefore c_2 = 0$

$$c_1 + c_2 = 0$$

$$\therefore c_1 = c_2 = 0$$

This proves that eq^n (i) is true when $c_1 = c_2 = 0$.

iii) Proof that span$\{(1,-1,0), (1,0,-1)\} = V$

Let $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V$. Then, according to the question:

$$v_1 + v_2 + v_3 = 0$$

$$\therefore \quad v_1 = -v_2 - v_3 \quad \cdots (ii)$$

Thus,

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} -v_2 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -v_3 \\ 0 \\ v_3 \end{pmatrix}$$

$$= -v_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - v_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\in \operatorname{span}\{(1,-1,0),\ (1,0,-1)\}$$

Thus, B is a basis for V.

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Q9.

Here,

$$A = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}$$

To find eigenvalues, we use characteristic equation:

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 5 & -2 \\ 7 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(-4-\lambda) + 14 = 0$$

$$\Rightarrow -(5-\lambda)(4+\lambda) + 14 = 0$$

$$\Rightarrow -(20 + 5\lambda - 4\lambda - \lambda^2) + 14 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 20 + 14 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2\lambda - 6 = 0$$

$$\Rightarrow \lambda(\lambda - 3) + 2(\lambda - 3) = 0$$

$$\Rightarrow (\lambda + 2)(\lambda - 3) = 0$$

$\Rightarrow \lambda = -2$ or $\lambda = 3$

For eigen vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have,

$(A - \lambda I)x = 0$

$$\Rightarrow \begin{pmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} (5-\lambda)x_1 - 2x_2 \\ 7x_1 + x_2(-4-\lambda) \end{bmatrix} = 0 \quad \ldots (i)$$

Put $\lambda = -2$ in (i)

$\Rightarrow 7x_1 - 2x_2$

$7x_1$

We know,

$Ax = \lambda x$

where x is an eigenvector.

$\Rightarrow Ax =$

To find eigenvectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have

$(A - \lambda I)x = 0$

$$\Rightarrow \left(\begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 (5-\lambda) - 2x_2 \\ 7x_1 + x_2(-4-\lambda) \end{pmatrix} = 0 \quad \ldots (i)$$

Put $\lambda = -2$ in (i), we get:

$$\begin{pmatrix} 7x_1 - 2x_2 \\ 7x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 7x_1 - 2x_2 = 0$$

$$\Rightarrow x_1 = \frac{2}{7} x_2 \quad \ldots (ii)$$

Put $x_2 = t$, $\in \mathbb{R}$ then,

$$x_1 = \frac{2}{7} t$$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/7 \, t \\ t \end{pmatrix} \text{ when, } t \in \mathbb{R}$$

Put $\lambda = 3$ in (i), we get:

$$\begin{pmatrix} 2x_1 - 2x_2 \\ 7x_1 - 7x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x_1 = x_2$

Let $x_2 = s \in \mathbb{R}$

$\therefore \; x_1 = s$

$\therefore \; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$, where $s \in \mathbb{R}$

$\therefore$ Eigenvalues, $\lambda = \underline{-2, \; 3}$

Eigenvectors; $x_{\lambda=-2} = \begin{pmatrix} 2/7 \, t \\ t \end{pmatrix}$, $x_{\lambda=3} = \begin{pmatrix} s \\ s \end{pmatrix}$

where, $t, s \in \mathbb{R}$

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Q10.

a. Given,

$$Au = \lambda u \quad \cdots (i)$$

where A is square matrix, u is eigenvector and λ is the eigenvalue.

Multiplying both sides by matrix A, we get:

$$A\,Au = A\lambda u$$
$$\Rightarrow A^2 u = \lambda (Au)$$
$$\Rightarrow A^2 u = \lambda \lambda u \quad (\text{From (i)}, Au = \lambda u)$$
$$\therefore A^2 u = \lambda^2 u \quad \cdots (ii)$$

Similarly multiplying eq^n(i) $(m-1)$ number of times, we get the relation as follows:

$$A^m u = \lambda^m u \quad \cdots (iii)$$

The result in (iii) follows the result in (ii) as seen above. Since eq^n (iii) itself follows the equation (i) pattern, we can say that matrix A^m where m is a natural number has eigenvalue of λ^m and the eigenvector u remains unchanged.

b.

Again, $Au = \lambda u \quad \cdots (i)$

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Here, matrix A is invertible i.e. A^{-1} exists.

Multiplying both sides of (i) by A^{-1}, we get:

$$A^{-1}Au = A^{-1}\lambda u$$

$$\Rightarrow Iu = \lambda(A^{-1}u) \qquad (\because AA^{-1} = I)$$

$$\Rightarrow \lambda(A^{-1}u) = u \qquad (\because Iu = u)$$

$$\Rightarrow A^{-1}u = \frac{1}{\lambda}u$$

$$\Rightarrow A^{-1}u = \lambda^{-1}u \quad \ldots \text{(ii)}$$

Equation (ii) is in the form of equation (i). Thus, eigenvalue of $A^{-1} = \lambda^{-1}$ and eigenvector $= u$.

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