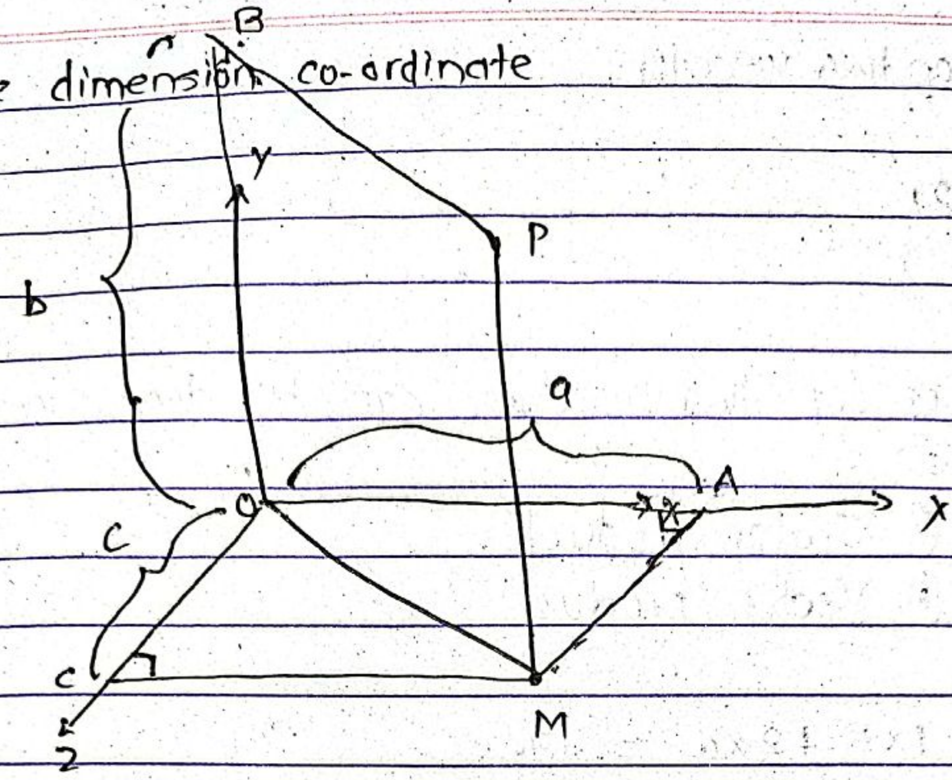
Three dimension co-ordinate




Take O as origin P be a point on space.
Draw $\perp^r$ PM from P on ~~xy~~ XZ plane.
Join OM
Draw $\perp^r$ MA & MC from M on OX & OZ.
Draw $\perp^r$ PB from P on OY

Measure OA, OB, OC
if OA = a, OB = b & OC = c
Then the cordinate of P is (a, b, c)

# Product of two vecotor

$\vec{a} = (1,2)$
$\vec{b} = (5,6)$

The product of two vectors can be done in following two ways:
(i) Scalar product
(ii) Vector product

$\vec{a}.\vec{b} = 1\times5 + 2\times6 = 17$

(i) ~~Scalar~~ product (Dot)
$\vec{a} = (a_1, a_2)$ & $\vec{b}$ $(b_1, b_2)$ be two vectors then their dot product is

$$\vec{a}\cdot\vec{b} = a_1 \times b_1 + a_2 \times b_2$$
$$= \boxed{a_1 b_1 + a_2 b_2}$$

which is scalar.
Hence the name scalar product is justified.

Similarly for space vector
$\vec{a} = (a_1, a_2, a_3)$ & $\vec{b}$ $(b_1, b_2, b_3)$

$$\vec{a}.\vec{b} = (a_1 b + a_2 b_2 + a_3 b_3)$$

~~Geometrical meaning of dot~~

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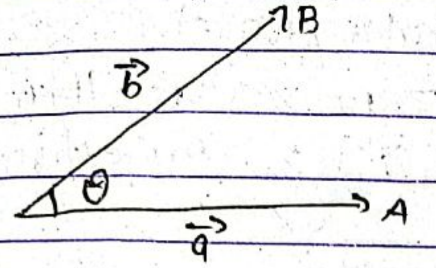
Angle between two vectors.



Let $OA = \vec{a}$ and $OB = \vec{b}$ then
if $\theta$ is angle betwⁿ OA and OB then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|} \quad , \quad \vec{a} \neq 0 \, , \, \vec{b} \neq 0$$

Writing $\hat{a}$ = unit vector along $\vec{a}$

$$= \frac{\vec{a}}{|\vec{a}|}$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

Then,

$$\cos\theta = \hat{a} \cdot \hat{b}$$

Dot product of standard unit vector

$\vec{i} = (1,0)$ , $\vec{j}$ $(0,1)$

Then. $\vec{i} \cdot \vec{i} = (1,0).(1,0) = 1 + 0 = 1$

$\vec{j} \cdot \vec{j} = 1$

$\vec{i} \cdot \vec{j} = (1,0) \cdot (0,1) = 0 + 0 = 0$

Note: if two vectors ar $\perp^r$ then $\theta = 90°$ & $\cos 90 = 0$

$$0 = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|.|\vec{b}|}$$

i.e. $\vec{a} \cdot \vec{b} = 0$

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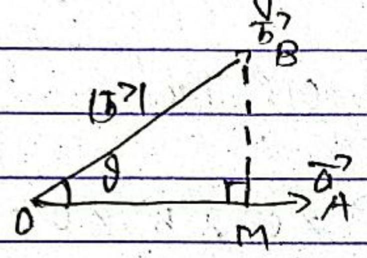
# Geometrical meaning of scalar product

If $\vec{a}$ & $\vec{b}$ be non zero vectors then

$\vec{a} \cdot \vec{b}$ = (Magnitute of $\vec{a}$) x projection of $\vec{B}$ on $\vec{a}$

OR

= (Magnitude of $\vec{b}$) x projection of $\vec{a}$ on $\vec{b}$




$$\cos\theta = \frac{OM}{OB}$$

$\therefore$ OM = OB cos $\theta$

Proof:

let OA = $\vec{a}$ , OB = $\vec{b}$

Draw $\perp^r$ BM from B on OA

Then $\cos\theta = \frac{OM}{OB}$

$\therefore$ OM = OB cos $\theta$

OM = OB cos $\theta$

or, Projection of $\vec{OB}$ on $\vec{OA}$ = $|\vec{b}| \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \right)$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$\therefore$ $\vec{a} \cdot \vec{b} = |\vec{a}|$ x (Projection of B on A)

Similarly

$\vec{a}.\vec{b} = |\vec{b}| \times (\text{Projection of } \vec{a} \text{ on } \vec{b})$

Fact : scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

Vector projection $\vec{b}$ on $\vec{a}$ = scalar projection × Unit vector along $\vec{a}$

$$= \frac{\vec{a}.\vec{b}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right)$$

* Class work *

Find scalar projection and vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a} = (1,2)$ , $\vec{b} = (0,7)$

scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{(1,2).(0,7)}{\sqrt{0^2+7^2}}$$

$$= \frac{14}{7} = 2$$

Vector projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\left(\dfrac{\vec{b}}{|\vec{b}|}\right)$

$$= 2\left(\frac{(0,7)}{\sqrt{0^2+7^2}}\right)$$

$$= (0,2)$$

$$\begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$$

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# Vector product (cross)

Let $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ be two space vectors. Then their vector product or cross product denoted by $\vec{a} \times \vec{b}$ & is defined by:

~~$\vec{a} \times \vec{b} = (a_1 a_2 a_3$~~

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$$

$$= (a_2 b_3 - b_2 a_3, \; a_3 b_1 - b_3 a_1, \; a_1 b_2 - a_2 b_1)$$

~~Example~~ which is vector.

Since the product is vector. so the name 'vector product' is justified.

# Vector Product in terms of determinants

If $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ then $\vec{a} \times \vec{b}$ can be expressed in terms of determinant as

$$\vec{a} \times \vec{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - b_2 a_3)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - b_1 a_2)\vec{k}$$

$$= (a_2 b_3 - b_2 a_3, \; a_3 b_1 - a_1 b_3, \; a_1 b_2 - b_1 a_2$$

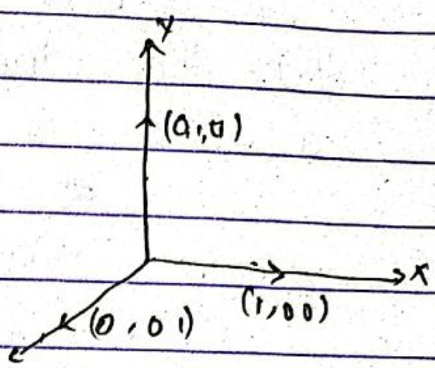
* Example:

Find cross product of $\vec{i} = (1, 0 0)$ & $\vec{j} = (0, 1, 0)$

Now,

$$\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

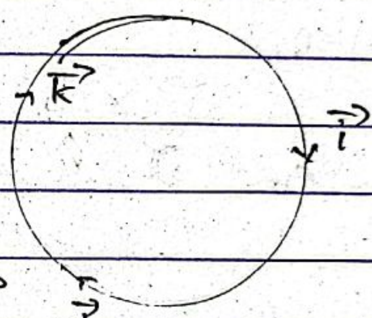
$$= \vec{k}$$




Note: $\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

& $\vec{k} \times \vec{j} = \vec{j}$




~~[illegible]~~ Thus, cross product of three standard unit vector $\vec{i}, \vec{j}, \vec{k}$ along $x, y, z$ axis preserve cyclic order.

Similarly,

$\vec{j} \times \vec{i} = -(\vec{i} \times \vec{j}) = -\vec{k}$

$\vec{i} \times \vec{k} = -\vec{j}$

$\vec{k} \times \vec{j} = -\vec{i}$

Note: If $\vec{a}$ and $\vec{b}$ are two space vector then $\vec{a} \times \vec{b}$ alway represents the vector perpendicula to both $\vec{a}$ and $\vec{b}$.

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

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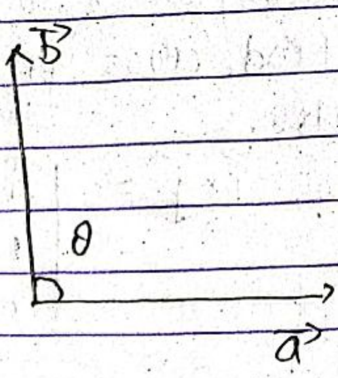
# Sine angle between two vectors



If $\vec{a}$ and $\vec{b}$ be two vectors, then the sine angle between $\vec{a}$ & $\vec{b}$ is

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

Note: $-1 \leq \sin\theta \leq +1$

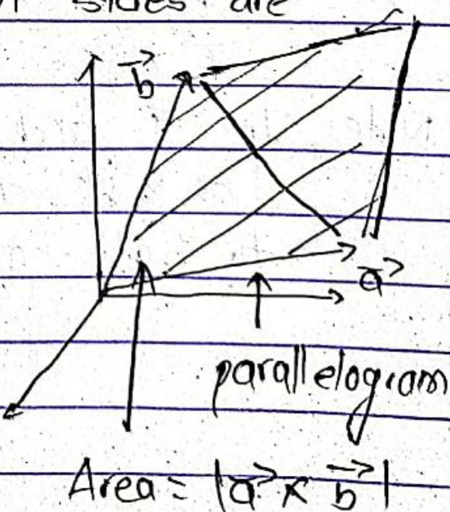
Also,

$$\vec{a} \times \vec{b} = \hat{\eta}\,|\vec{a}|\,|\vec{b}|\sin\theta$$

where $\hat{\eta}$ is the unit vector along $\vec{a} \times \vec{b}$

# Geometrical meaning of cross product

Geometrically $|\vec{a} \times \vec{b}|$ always represent area of parallogram whose adjacent sides are $\vec{a}$ and $\vec{b}$




Also area of traingle determined by the vector $\vec{a}$ & $\vec{b}$ is

$$\frac{1}{2}|\vec{a} \times \vec{b}|$$

$\vec{a} \times \vec{b} =$ vector
$\vec{a} \cdot \vec{b} =$ scalar
$\vec{a}\,\vec{b} =$ undefined

$(\underbrace{\vec{a} \cdot \vec{b}}_{\text{scalar}}) \times \underbrace{\vec{c}}_{\text{vector}} =$ meaningless

# Product of three vector

The product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ can be treated on the following concepts.

(i) $(\vec{a} \times \vec{b}) \times \vec{c} \rightarrow$ vector
(ii) $(\vec{a} \times \vec{b}) \cdot \vec{c} \rightarrow$ scalar
(iii) $(\vec{a} \cdot \vec{b}) \times \vec{c} \rightarrow$ meaningless
(iv) $(\vec{a}\,\vec{b}) \times \vec{c} \rightarrow$ undefined
(v) $(\vec{a} \times \vec{b})\,\vec{c} \rightarrow$ undefined
(vi) $\vec{a} \times (\vec{b} \times \vec{c}) \rightarrow$ vector
(vii) $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \rightarrow$ meaningless

# Scalar product of three vectors

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors then the scalar triple product of $\vec{a}$, $\vec{b}$, $\vec{c}$ is denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$ or $(\vec{a}\,\vec{b}\,\vec{c})$ and is defined by

$$[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad \text{—— } (*)$$

Since in $(*)$
$\vec{b} \times \vec{c}$ is vector
and $\vec{a}$ is also vector
Hence the product
$\vec{a} \cdot (\vec{b} \times \vec{c})$ is also scalar.
Hence the name "scalar" triple product" is justified.

# Scalar triple product in terms of determinant.

Let $\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

& $\vec{c} = (c_1, c_2, c_3)$

Then their scalar triple product can be expressed in terms of determinant.

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hint: $\vec{b}\times\vec{c} = (b_2c_3 - c_2b_3,\ b_3c_1 - b_1c_2,\ \overset{b_1c_2 - b_2c_1}{\cancel{a_1b_2 - b_2a_1}})$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1, a_2, a_3)\cdot(\ \ ,\ \ ,\ \ )$$

$$= a_1(b_2c_3 - c_2b_3) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

# Properties of scalar triple product

(i) In the scalar triple product, the position of dot and cross can be interchanged.

Eg: $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

(ii) If the position of any two vectors are interchanged then a minus sign is introduced.

Eg: $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$

Proof:

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\vec{b}\cdot(\vec{a}\times\vec{c})$$

$$= -[\vec{b}\ \vec{a}\ \vec{c}]$$

(iii) Scalar triple product of three standard unit vector $\vec{i}, \vec{j}, \vec{k}$ is 1.

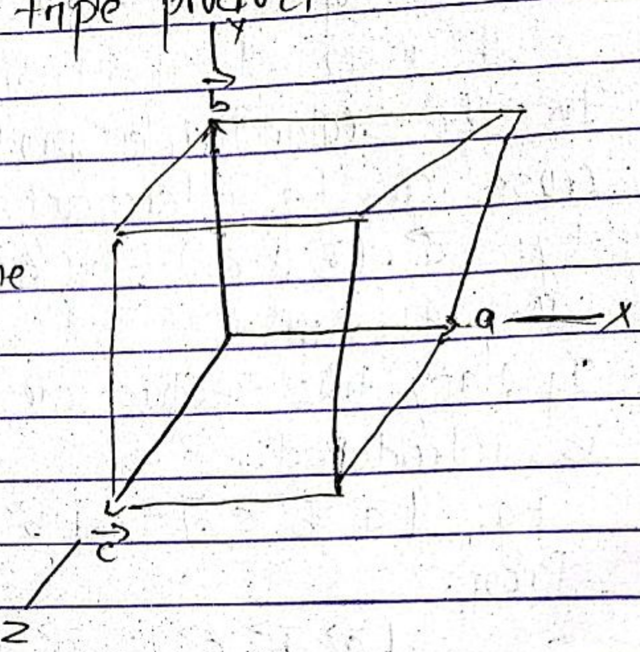
$$[\vec{i}\ \vec{j}\ \vec{k}] = \vec{i}\cdot(\vec{j}\times\vec{k})$$

$$= \vec{i}\cdot\vec{i}$$

$$= 1$$

# Geometrical meaning of scalar triple product

Geometrically
$[\vec{a}\,\vec{b}\,\vec{c}]$ i.e. $\vec{a}\cdot(\vec{b}\times\vec{c})$
always represents the volume of parallelopied with sides $\vec{a}$, $\vec{b}$ & $\vec{c}$.




HW If $\vec{a} = (1,2,3)$, $\vec{b} = (2,0,1)$ & $\vec{c} = (4,5,1)$
Find
(i) Projection of $\vec{a}$ on $\vec{c}$ & $\vec{c}$ on $\vec{b}$
(ii) Area of $\Delta$ determined by $\vec{b}$ & $\vec{c}$
(iii) Area of parallogram with side $\vec{b}$ & $\vec{c}$
(iv) sine and cosine angles between $\vec{a}$ & $\vec{c}$
(v) Volume of parallelopied with side $\vec{a}, \vec{b}, \vec{c}$.

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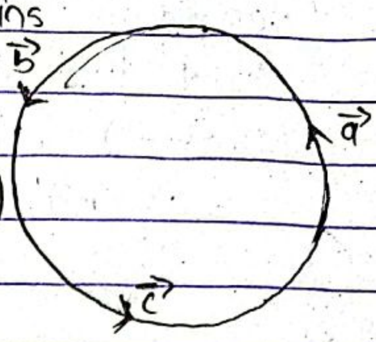
(iv) The value of scalar triple product remains unchanged by interchanging the vectors in cyclic order.

i.e. $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$




(v) The value of scalar triple product changes its sign if the position of any two vectors are interchanged.

$[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{b}\,\vec{a}\,\vec{c}]$

(vi) The value of scalar triple product is zero if two vectors are equal.

i.e $[\vec{a}\,\vec{a}\,\vec{c}] = 0$ , $[\vec{b}\,\vec{c}\,\vec{c}] = 0$

(vii) The value of scalar triple product is zero if any two vectors are parallel.

More precisely

$[\vec{a}\,\vec{b}\,\vec{c}]$ if $\vec{b}$ and $\vec{c}$ are parallel

i.e. $\vec{b} = k\vec{c}$

(viii) The value of determinant is 0 if three vectors are coplanar and conversely

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ if and only if $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

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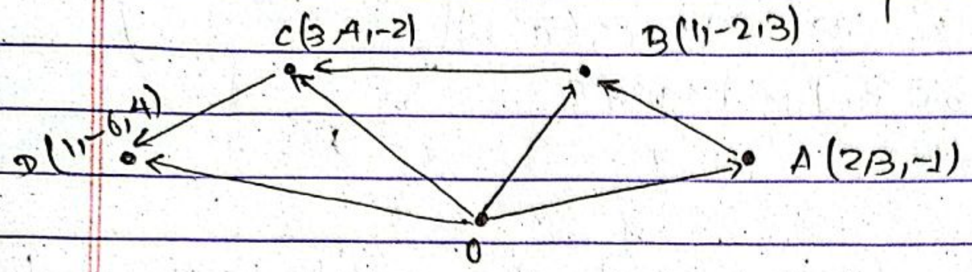
Example

Show that the four vectors $(2,3,-1)$, $(1,-2,3)$, $(3,4,-2)$ and $(1,-6,4)$ are coplanar.




Let O be the origin then

$\overrightarrow{OA} = (2,3,-1)$

$\overrightarrow{OB} = (1,-2,3)$

$\overrightarrow{OC} = (3,4,-2)$

$\overrightarrow{OD} = (1,-6,4)$

Now,

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (1,-2,3) - (2,3,-1)$

$= (-1,-5,4)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (3,4,-2) - (1,-2,3)$

$= (2,6,-5)$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= (1,-6,4) - (3,4,-2)$

$= (-2,-10,6)$

If four vector $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ are coplanar then three vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ are also coplanar.

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So, their scalar triple product is zero.

Now, $[\vec{AB}, \vec{BC}, \vec{CD}] = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 6 \end{vmatrix}$

$= -1\begin{vmatrix} 6 & -5 \\ -10 & 6 \end{vmatrix} + 5\begin{vmatrix} 2 & -5 \\ -2 & 6 \end{vmatrix} + 4\begin{vmatrix} 2 & 6 \\ -2 & -10 \end{vmatrix}$

$= 14 + 10 \cancel{+} 38$

$= -8$

So the points A, B, C and D are not coplanar.

HW

Example

1. Compute the scalar triple product

(a) $(2\vec{i} - 3\vec{j} + \vec{k}) \cdot (\vec{i} - \vec{j} + \vec{k}) \times (2\vec{i} + \vec{j} - \vec{k})$ $\boxed{\text{Ans: } -6}$

2. Show that the four points $(4,5,1)$, $(0,-1,1)$, $(3,4,9)$ & $(-4,4,4)$ lie on one plane.

3. Find the value of $\lambda$ so that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} + \lambda\vec{j} + 5\vec{k}$ are coplanar.

# Vector triple product

If $\vec{a}, \vec{b}, \vec{c}$ be three vectors then the product of the form

$$\vec{a} \times (\vec{b} \times \vec{c}) \text{ or } (\vec{a} \times \vec{b}) \times \vec{c}$$

are called vector triple product.

Since in $\vec{a} \times (\vec{b} \times \vec{c})$

$$\vec{a} \downarrow \qquad \downarrow (\vec{b} \times \vec{c})$$

vector x vector

vector

Hence the name vector triple product is justified.

<u>Formula:</u> (Determination of $\vec{a} \times (\vec{b} \times \vec{c})$)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$$

* Example

Verify the formula

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$$

where

$\vec{a} = (1, 2, 0)$

$\vec{b} = (2, 3, -1)$

$\vec{c} = (0, 2, 3)$

$$RHS = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 0 & 2 & 3 \end{vmatrix}$$

$= 11\vec{i} - 6\vec{j} + 4\vec{k}$

$= (11, -6, 4)$

L.H.S $= \vec{a} \times \vec{b} \times \vec{c}$

$= (1, 2, 6) \times (11, -6, -4)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 11 & -6 & 4 \end{vmatrix}$$

$= (8, -4, -28)$

# Equation of straight line in cartesian form

Slope intercept form

$y = mx + c$

$\Rightarrow \; y - c = mx$

or, $\dfrac{y-c}{m} = \dfrac{x}{1}$

or, $\dfrac{x-0}{1} = \dfrac{y-c}{m}$

Two point form

$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

or, $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$

# Vector equation of straight line

Q. Find the vector equation of straight line passes through origin.

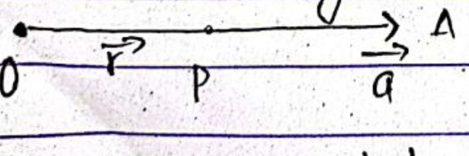
OR

Show that the vector equation of st. line with direction $\vec{a}$ and through origin is of the form

$$\vec{r} = t\vec{a}$$

→ Soln:

Let O be the origin and $\vec{OA} = \vec{a}$ be the given vector




Let P be any point on $\vec{OA}$ such that $\vec{OP} = \vec{r}$

Since $\vec{OP}$ and $\vec{OA}$ are collinear

$\therefore \vec{OP} = t\,\vec{OA}$ for some scalar t.

or, $\vec{r} = t\vec{a}$, which is eqn of st. line through origin.

# Verification:

Let $\vec{r} = (x, y)$, $\vec{a} = (a_1, a_2)$.

Then $\vec{r} = t\vec{a}$ gives

$(x, y) = t(a_1, a_2)$

$(x, y) = (ta_1, ta_2)$

Equating

$x = ta_1$, $y = ta_2$

$\frac{x}{a_1} = t$, $\frac{y}{a_2} = t$

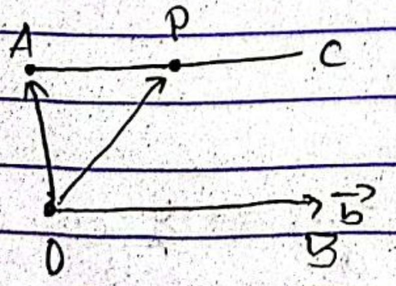
$$\boxed{\therefore \frac{x}{a_1} = \frac{y}{a_2}}$$

which is cartisian form

# Form II

Vector equation of st. line passing through the point A (where $\vec{OA} = \vec{a}$) and parallel to vector $\vec{b}$ is

$\vec{r} = \vec{a} + \vec{b}$



Proof:

Let $\vec{OB} = \vec{b}$ be the given vector

Let $\vec{OA} = \vec{a}$

We find the equation of AC which is parallel to OB.

Now

From traingle law of vector addition

$\vec{OP} = \vec{OA} + \vec{AP}$

$\vec{r} = \vec{a} + \vec{AP}$ ——— (1)

But $\vec{AP}$ is parallel to $\vec{OB}$

$\therefore \vec{AP} = t\,\vec{OB}$

$= t\,\vec{b}$

Hence equation (1) becomes

~~AP~~ $\boxed{\vec{r} = \vec{a} + t\vec{b}}$

Verification

Here $\vec{r} = \vec{a} + t\vec{b}$

or $(x,y) = (a_1, a_2) + t(b_1, b_2)$

or, $(x,y) - (a_1, a_2) = (tb_1, tb_2)$

or, $(x - a_1, y - a_2) = (tb_1, tb_2)$

$$\boxed{\begin{array}{l} y - y_1 = m(x - x) \\ \text{i.e. } \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} \end{array}}$$

Equating;

$x - a_1 = tb_1$ $\qquad$ $y - a_2 = tb_2$

or, $t = \dfrac{x - a_1}{b_1}$ $\qquad$ or, $t = \dfrac{y - a_2}{b_2}$

i.e. $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = t$

which is of the form $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b}$

In three dimension if

$\vec{r} = (x, y, z)$, $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then above equation becomes.

$$\boxed{\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}}$$

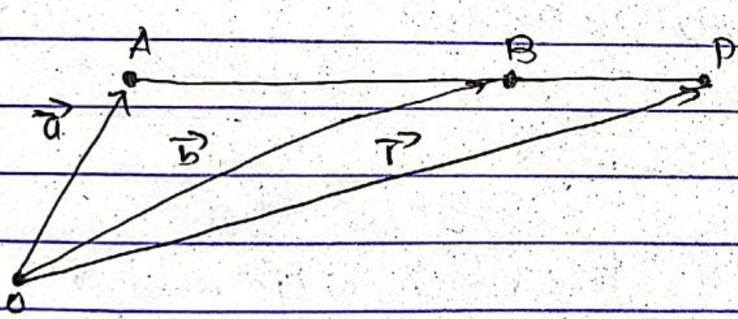
This looks like

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ as studies in our Bachelor level.

Form III

Vector equation of st. line passing through two points A and B (where $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

→ Soln;




let O be the origin. ~~AB~~ We find the vector equation of line AB

Here $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

let P be any point on the line AB (Produced if necessary) such that $\vec{OP} = \vec{r}$

Then by traingle law of vector addition

$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\text{or, } \vec{r} = \vec{a} + \vec{AP} \quad \text{——— (i)}$$

But $\vec{AP} = t\,\vec{AB}$ [$\because$ $\vec{AP}$ & $\vec{AB}$ are collinear]

$$= t(\vec{OB} - \vec{OA})$$

$$= t(\vec{b} - \vec{a}) \quad \text{——— (ii)}$$

Hence eq$^n$ ① becomes
$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$$

# Verification

Here, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

let $\vec{r} = (x, y, z)$, $\vec{a}(a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

Then ① becomes

$(x, y, z) = (a_1, a_2, a_3) + t(b_1 - a_1, b_2 - a_2, b_3 - a_3)$

or, $(x - a_1, y - a_2, z - a_3) = (t(b_1 - a_1), t(b_2 - a_2), t(b_3 - a_3))$

Equating

$t(b_1 - a_1) = x - a_1$, $t(b_2 - a_2) = y - a_2$ $t(b_3 - a_3) = z - a_3$

$$\Rightarrow \frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3} = t$$

* Example

Find the equation of line which is parallel to $\vec{i} - 2\vec{j} + 3\vec{k}$ and passes through $(1, 2, 3)$

→ Soln:

Here the line passes through $A(1, 2, 3)$ i.e. $\vec{a} = (1, 2, 3)$

& parallel to $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$

The required eq$^n$ is $\vec{r} = \vec{a} + t\vec{b}$

where, $(x, y, z) = (1, 2, 3) + t(1, -2, 3)$

or, ~~$(x, y, z) = (1+t, 2-2t, 3+3t)$~~

or, $(x - 1, y - 2, z - 3) = (t, -2t, 3t)$

or, $x - 1 = t$, $y - 2 = -2t$, $z - 3 = 3t$

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or, $x-1=t$ , $\frac{y-2}{-2}=t$, $\frac{z-3}{3}=t$

$\therefore x-1=\frac{y-2}{-2}=\frac{z-3}{3}$

# Example

Find the vector eqⁿ where cartesian eqⁿ is

$$\frac{x-2}{5}=\frac{y-6}{7}=\frac{z-1}{2}$$

→ Solⁿ,

$$\frac{x-2}{5}=\frac{y-6}{7}=\frac{z-1}{2}=t \text{ (say)}$$

$\therefore x-2=5t, \; y-6=7t, \; z-1=2t$

or, $x=5t+2$ , $y=7t+6$ , $z=2t+1$

Let $\vec{r}=(x,y,z)$

Then

$$\vec{r}=(x,y,z)$$
$$=(2+5t, 6+7t, 1+2t)$$
$$=(2,6,1)+(5t, 7t, 2t)$$
$$=(2,6,1)+t(5,7,2)$$

Letting $(2,6,1)=\vec{a}$

& $(5,7,2)=\vec{b}$

Then above eqⁿ looks like

$$\vec{r}=\vec{a}+t\vec{b}$$

~~Find the vector eqn of~~

Find by vector method, the eqn st. line in double intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

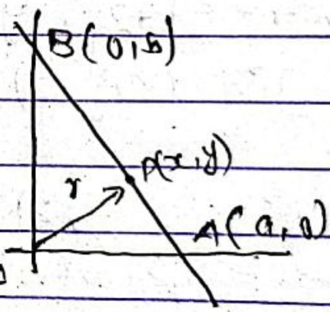
→ Proof

Let the st. line makes intercepts a and b on x-axis and y-axis at A and B.

$$\therefore \vec{OA} = \vec{a} = (a,0)$$

$$\vec{OB} = \vec{b} = (0,b)$$

Then the vectors eqn of st. line in two point form is




$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$$

$$= (a,0) + t((0,b) - (a,0))$$

$$= (a,0) + (-at, bt)$$

$$\therefore (x,y) = (a-at, bt)$$

$$\therefore x = a - at \qquad y = tb$$

$$\frac{x}{a} = 1-t \quad \text{—(i)} \qquad t = y/b \quad \text{—(ii)}$$

Substituting $t = y/b$ in eqn (i) we get

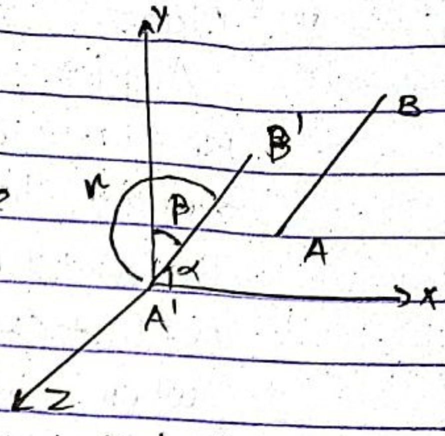
$$x/a = 1 - y/b$$

$$\frac{x}{a} + y/b = 1$$

which is required eqn.

**Note:**

If a line AB makes angle $\alpha$, $\beta$, $\gamma$ with positive direction of x, y, and z axis then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosine of the line AB. It is denoted by $l, m, n$.




$$l^2 + m^2 + n^2 = 1$$

**Direction ratios**: Any three numbers that are proportional to direction cosines $l, m, n$ are called direction ratio.

Thus if $a, b, c$ are direction ratio, then

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{l^2+m^2+n^2}} = \sqrt{a^2+b^2+c^2}$$

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} \; ; \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}} \; ; \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

**Note**: We had

The eq$^n$ of st. line

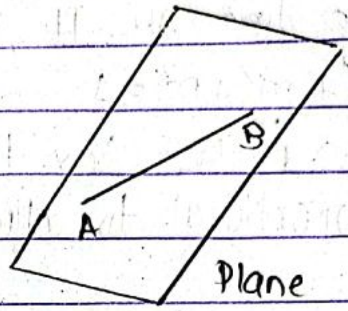
$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

where $(x_1, y_1, z_1)$ is the point on the line & $l, m, n$ are direction ratios of the line. (may be direction cosines)

# Plane

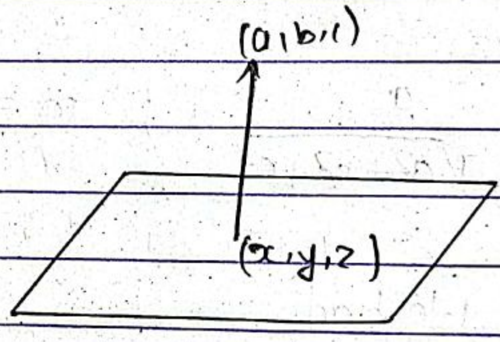
A plane is the locus of points in which if we take any two points A, B on the locus then straight line AB also lie on the locus simultaneously.

The standard equation of plane is
$ax+by+cz+d=0$
where $a, b, c$ are not all zero.
Where $a, b, c$ are called direction ratios of normal to the plane.



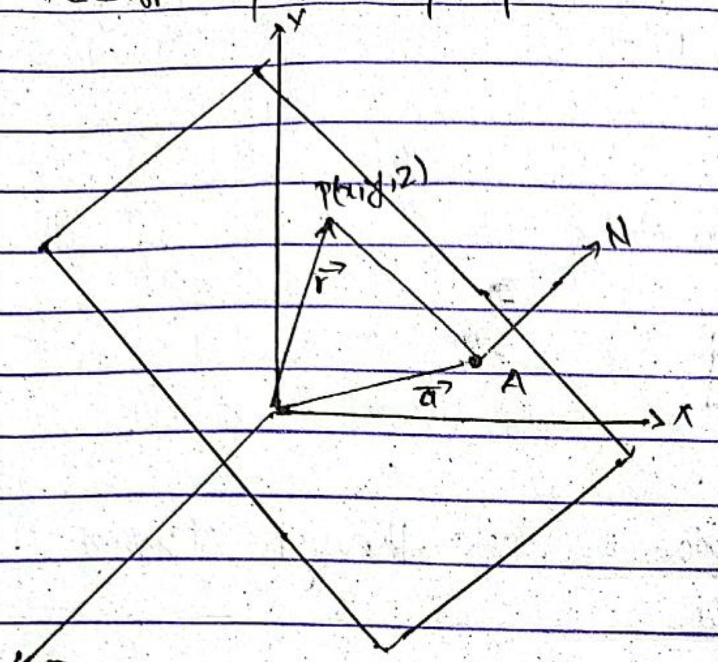

The equation of plane passing through origin is
$$ax+by+cz=0$$

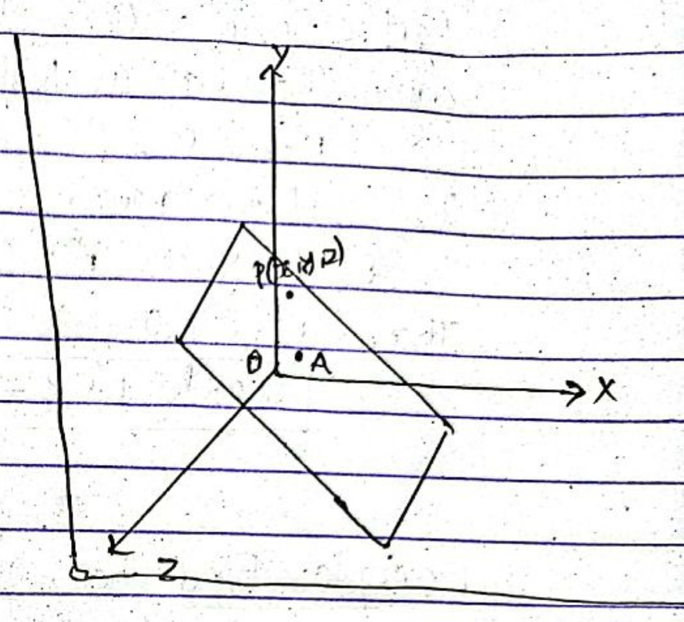
<u>Fact:</u> $(x, y, z) \cdot (a, b, c) = 0$




<u>Note:</u> The direction ratio of the line joining two points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ is $x_2-x_1, y_2-y_1, z_2-z_1$

# Vector equation of plane




To find the vector equation of plane passing through $A(x_0, y_0, z_0)$ and normal vector $\vec{x} = (a, b, c)$

→ Soln:

Let $\vec{a} = \vec{OA} = (x_0, y_0, z_0)$

Let $P(x, y, z)$ be any point on plane so that

$$\vec{r} = \vec{OP} = (x, y, z)$$

so that $\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$

let $\vec{AN}$ be the normal vector to the plane such that

$$\vec{n} = \vec{AN} = (a, b, c)$$

Then $\vec{AP}$ and $\vec{AN}$ are $\perp^r$

$$\vec{AP} \cdot \vec{AN} = 0$$

or, $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

or, $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$ —— ①

which is eq$^n$ of plane.

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Fact: If $\vec{a} = (0,0,0)$ then the plane passes through origin. Then eqn (1) looks like

$$\vec{r}.\vec{n} = 0$$

Its cartesian form

$$\vec{r} = (x,y,z) \quad \vec{n} = (a,b,c)$$

Then $\vec{r}.\vec{n} = 0$ gives

$$\boxed{ax+by+cz=0}$$

Example: Find the equation of plane through $A(5,6,7)$ and normal vector $(1,2,6)$

→ Soln:

Here $\vec{OA} = \vec{a} = (5,6,7)$

Normal vector $= \vec{n} = (1,2,6)$

Let $\vec{r} = (x,y,z)$ be any point on the plane.

Then vector equation of plane is

$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

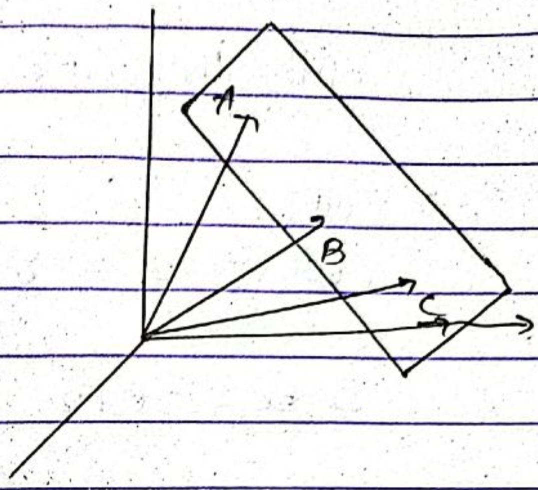
$$(x,y,z).(1,2,6) = (5,6,7).(1,2,6)$$

or, $x+2y+6z = 5+12+42$

or, $x+2y+6z = 59$

Plane through three points

Example: Find the equation of plane through three points $(3,-1,2)$, $(8,2,4)$ & $(-1,-2,-3)$




Soln:

$\cancel{\overrightarrow{AB} =}$ let O be the origin.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (5, 3, 2)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (-9, -4, -7)$

Now

$\overrightarrow{AB} \times \overrightarrow{BC}$ is a vector perpendicular to both $\overrightarrow{AB}$ & $\overrightarrow{BC}$

$\therefore$ Normal vector to the plane

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC}$$

$$= \begin{vmatrix} i & j & k \\ 5 & 3 & 2 \\ -9 & -4 & -7 \end{vmatrix}$$

$$= -13i + 17j + 7j$$

$$= (-13, 17, 7)$$

Equation of plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$ where $\vec{a} = \overrightarrow{OA} = (3, -1, 2)$

$$-13x + 17y + 7z = -39 - 17 + 14$$

$$-13x + 17y + 7z = -42$$

$$13x - 17y - 7z = 42$$

is the required equation of plane passing through given points

# Point of intersection of a line and plane

To find the point of intersection of line

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2} \quad \text{—— (i)}$$

with the plane $x+y-z=2$ —— (ii)

→ Soln:

Let $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2}$ (t) say

$\therefore x = 3t+2, \; y = 4t+3, \; z = 2t+1$

For point of intersetion the point $(x,y,z) = (3t+2, 4t+3, 2t+1)$ should also lie on plane (ii)

$\therefore (3t+2) + (4t+3) - (2t+1) = 2$

or, $\cancel{3t+2}\; 5t = -2$

or, $t = -2/5$

Putting $t = -2/5$

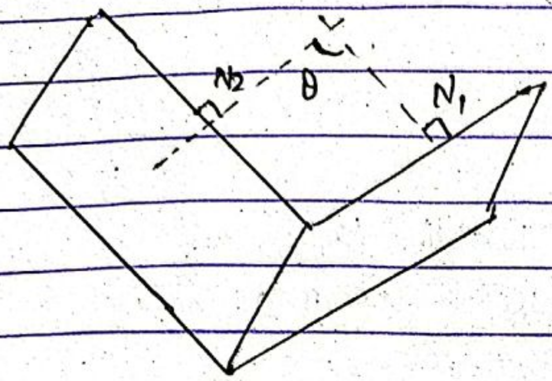
$(x,y,z) = \left(3\left(-2/5\right)+2, \; 4\left(-2/5\right)+3, \; 2\left(-2/5\right)+1\right)$

$= \left(4/5, \; 7/5, \; 1/5\right)$

$\therefore$ The required point on the plane is $\left(4/5, 7/5, 1/5\right)$

# Angle between two planes

Angle between two planes is defined as the acute angle between their normals.




let $P_1: 2x+3y+4z+5=0$ —— (i)

$P_2: x+5y+z+1=0$ —— (ii)

be two planes.

Then

Normal vectors to the plane are

$\vec{n_1} = (2,3,4)$

$\vec{n_2} = (1,5,1)$

If $\theta$ is the angle betw planes, then

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$= \frac{2+15+4}{\sqrt{29}\times\sqrt{27}}$$

$$= \frac{21}{\sqrt{29}\times\sqrt{27}}$$

$$\theta = \cos^{-1}\left(\frac{21}{\sqrt{29}\sqrt{27}}\right)$$

$$= 41.36^{\circ}$$

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# Equation of line through the intersection of two planes:

Find the equation of the line of intersetion of two planes

$$x+2y+3z=7$$

$$\& \ 3x+y-z=1$$

To find line, we need (i) Point on the line $(x_1, y_1, z_1)$

(ii) Direction ratio $l, m, n$, of the line

Step I Here

$$P_1 : x+2y+3z=7 \quad \text{——— (i)}$$

$$P_2 : 3x+y-z=1 \quad \text{——— (ii)}$$

let $z=0$ be the point of intersection of (i) and (ii)

Then,

$$x+2y=7$$

$$3x+y=1$$

Solving, we get

$$y=4, \quad x=-1$$

$\therefore$ Point $(x_1, y_1, z_1) = (-1, 4, 0)$

Step II To find direction ratio of required line.

Let $l, m, n$ be the direction ratio of required line.

Since the normals to the planes are

$$\vec{n_1} = (1, 2, 3)$$

$$\vec{n_2} = (3, 1, -1)$$

Since the line is perpendicular to the

normals to both the planes $P_1$ and $P_2$, so applying the condition of perpendicularity

$\therefore (1,2,3).(l,m,n) = 0$

& $(3,1,-1).(l,m,n) = 0$

i.e. $1l + 2m + 3n = 0$ —— ⓐ

$3l + m - n = 0$ —— ⓑ

Solving ⓐ and ⓑ for $l, m, n$ by cross multiplication

$$\begin{array}{ccccc} & l & m & n & \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & -1 & 3 & 1 \end{array}$$

$$\frac{l}{-2-3} = \frac{m}{9+1} = \frac{n}{1-6}$$

$$\therefore \frac{l}{-5} = \frac{m}{10} = \frac{n}{-5}$$

or, $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = k$ (say)

$l = -k, \quad m = 2k, \quad n = -k$

$\therefore$ So, the required eqn of line is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

or, $\frac{x+1}{-k} = \frac{y-4}{2k} = \frac{z-0}{k}$

$\therefore \frac{x+1}{-1} = \frac{y-4}{2} = \frac{z}{-1}$ is the required eqn of line.

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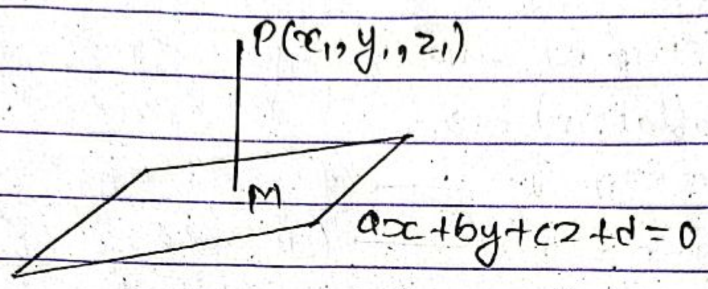
# Length of $\perp^r$ from $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$




The $\perp^r$ length $PM = \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$

Distance betw the two parallel planes.

$$\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}$$

Find the distance betw$^n$ parallel planes

$P_1 : x + 2y + 3z = 7$ —— (i)

$P_2 : x + 2y + 3z = 12$ —— (ii)

Any point on the plane (i) is

$x = 0,\ y = 0,\ z = 7/3$

$\therefore (0, 0, 7/3)$ is point on plane (i)

Then the length of $\perp^r$ from $(0, 0, 7/3)$ on plane $P_2$ is

$$= \left| \frac{1\times 0 + 2\times 0 + 3\times 7/3 - 12}{\sqrt{1^2+2^2+3^2}} \right|$$

$$= \left| \frac{-5}{\sqrt{14}} \right| = 5/\sqrt{14}$$

# Vector function

Vector function of scalar variable

Let $t$ ~~$\in [a, b]$~~ be a scalar variable defined on some interval $[a, b]$ and let $\vec{r}$ is a vector depend on $t$ where,

$$\vec{r} = (x(t), y(t))$$

then we say that $\vec{r}$ is a vector function of scalar variable $t$ and we write $\vec{r} = \vec{r}(t)$.

Example:

Vector function of parabola

$$y^2 = 4ax \quad \text{---} \quad (1)$$

Here $x = at^2$ & $y = 2at$ satisfy —— (1)

So we write

$$\vec{r} = (x(t), y(t))$$

$$\vec{r}(t) = (at^2, 2at)$$

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* vector function of circle

$$x^2 + y^2 = a^2 \quad \text{—— (1)}$$

Here $x = a\cos t$ and $y = a\sin t$ satisfy (1)

Squaring and adding

$$x^2 + y^2 = a^2$$

So we write

$$\vec{r} = (x(t), y(t))$$

$$\vec{r}(t) = (a\cos t, a\sin t)$$

By same argument

$$x = a\cos t, \; y = b\sin t$$

represents eqn of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So, $\vec{r} = (a\cos t, b\sin t)$ is vector eqn of ellipse.

* Limit of vector function

A vector function $\vec{r} = \vec{r}(t)$ is said to have limit $\vec{L}$ as $t \to t_0$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

Example:

Let $\vec{r}(t) = (at^2, 2at)$

Then $\lim_{t \to 0} \vec{r}(t)$

$$= \lim_{t \to 0} (at^2, 2at)$$

$$= \left(\lim_{t\to 0} at^2, \lim_{t\to 0} at\right)$$
$$= (0, 0)$$
$$= \vec{0}$$

# Continuity

A vector function $\vec{r} = \vec{r}(t)$ is said to be continuous at point $t = t_0$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

i.e. limiting value = functional value.

# Derivative of a vector function

Let $\vec{r} = \vec{r}(t)$ be a vector function of scalar variable $t$. Then the derivative of $\vec{r}$ at point $t = t_0$ is denoted by $\vec{r}\,'(t)$ or $\frac{d\vec{r}}{d(t)}$ and is defined by

$$\vec{r}\,'(t_0) = \lim_{t \to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} \quad \text{——} (1)$$

provided limit exists.

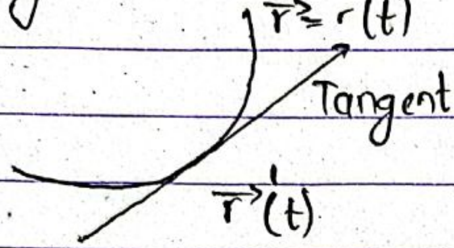
Letting $t - t_0 = \delta t$, $t = t_0 + \delta t$

Also if $t \to t_0$, $\delta t \to 0$

Hence eqn (1) becomes

$$r'(t_0) = \lim_{\delta t \to 0} \frac{\vec{r}(t_0 - \delta t) - \vec{r}(t_0)}{\delta t}$$

Geometrically, the derivative of the vector function $\vec{r} = \vec{r}(t)$ at $t = t_0$ represents the ~~slope~~ points of ~~tangent at~~ point ~~$t = t_0$~~
tangent. (when t increasing in direction of tangent)




# Example

let $\vec{r} = t^2\vec{i} - t\vec{j} + t\vec{k}$ —— (1)

Find $\dfrac{d\vec{r}}{dt} \cdot \dfrac{d^2\vec{r}}{dt^2}$

→ Soln;

$$\frac{d\vec{r}}{dt} = 2t\vec{i} - \vec{j} + \vec{k}$$

& $$\frac{d^2\vec{r}}{dt^2}$$

$$= 2\vec{i} - 0\vec{j} + 0\vec{k}$$

$$\therefore \frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} = 2t \cdot 2 + 0 + 0 = 4t$$

# Some formulae on derivative

If $\vec{r}$, $\vec{r_1}$ & $\vec{r_2}$ be vector functions of scalar variable $t$.
Let $\phi$ be the scalar function of $t$.
Then,

(i) $$\frac{d}{dt}(\vec{r_1} \pm \vec{r_2}) = \frac{d\vec{r_1}}{dt} \pm \frac{d\vec{r_2}}{dt}$$

(ii) $\frac{d}{dt}\vec{a} = \vec{0}$, where $\vec{a}$ is constant vector

(iii) $$\frac{d}{dt}(\phi\vec{r}) = \phi\frac{d\vec{r}}{dt} + r\frac{d\phi}{dt}\vec{r}$$

In particular if $\phi$ = constant = $k$ (say)

$$\frac{d}{dt}(k\vec{r}) = k\left(\frac{d\vec{r}}{dt}\right)$$

(iv) $$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\cdot\frac{d\vec{r_2}}{dt} + \cancel{\vec{r_2}}\,\frac{d\vec{r_1}}{dt}\cdot\vec{r_2}$$

(derivative of dot product)

(v) $$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = r_1\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$$

(Derivative of cross product)

(vi) Derivative of scalar triple product

$$\frac{d}{dt}[\vec{r_1}\,\vec{r_2}\,\vec{r_3}] = \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \left[\vec{r_1}\,\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right] + \left[\vec{r_1}\,\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

(viii) Derivative of vector triple product

$$\frac{d}{dt}\left[\vec{r_1} \times (\vec{r_2} \times \vec{r_3})\right] = \left[\frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3})\right] + \left[\vec{r_1} \times \left(\frac{d\vec{r_2}}{dt} \times \vec{r_3}\right)\right]$$

$$+ \vec{r_1} \times \left(\vec{r_2} \times \frac{d\vec{r_3}}{dt}\right)$$

Q. Show that

$$\frac{d}{dt}[\vec{r_1}\,\vec{r_2}\,\vec{r_3}] = \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \left[\vec{r_1}\,\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right] + \left[\vec{r_1}\,\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

$$L.H.S = \frac{d}{dt}\left[\vec{r_1} \cdot (\vec{r_2} \times \vec{r_3})\right]$$

$$= \frac{d\vec{r_1}}{dt} \cdot (\vec{r_2} \times \vec{r_3}) + \vec{r_1} \cdot \frac{d}{dt}(\vec{r_2} \times \vec{r_3})$$

$$= \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \vec{r_1} \cdot \left[\frac{d\vec{r_2}}{dt}\,\vec{r_3} + \vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

$$= \frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3} + \vec{r_1} \cdot \left(\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right) + \vec{r_1} \cdot \left(\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right)$$

$$= \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \left[\vec{r_1}\,\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right] + \left[\vec{r_1}\,\vec{r_2}\,\frac{d\vec{r_3}}{dt}\right]$$

Q.2. Show that (derivative of vector triple product)

$$\frac{d}{dt}\left[\vec{r_1} \times (\vec{r_2} \times \vec{r_3})\right] = \frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3}) + \vec{r_1} \times \frac{d}{dt}(\vec{r_2} \times \vec{r_3})$$

$$= \left(\frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3})\right) + \left(\vec{r_1} \times \left(\vec{r_2} \times \frac{d\vec{r_3}}{dt}\right)\right) + \left(\vec{r_1} \times \left(\frac{d\vec{r_2}}{dt} \times \vec{r_3}\right)\right)$$

$$= \left[\frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3})\right] + \left[\vec{r_1} \times \left(\frac{d\vec{r_2}}{dt} \times \vec{r_3}\right)\right] + \left[\vec{r_1} \times \left(\vec{r_2} \times \frac{d\vec{r_3}}{dt}\right)\right]$$

# Vector with constant magnitude

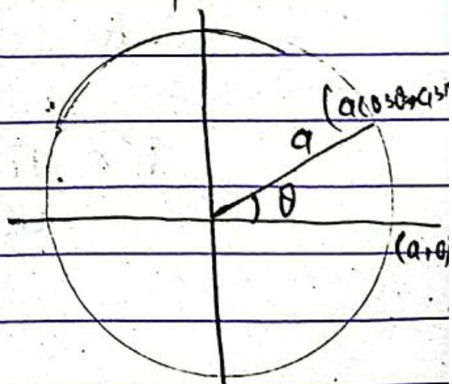
A vector ~~with~~ function $\vec{r}$ of scalar variable $t$ is said to be constant magnitude if $|\vec{r}|$ is constant for all $t$.

In particular,

$$\vec{r} = (a\cos t, a\sin t)$$

Here, $|\vec{r}| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = a$

This represents circle with radius $a$.




Note:

A vector function $\vec{r}$ of scalar variable $t$ has constant magnitude if $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$.

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# Vector with constant direction

A vector function $\vec{r}$ of scalar variable $t$ is said to be constant direction if $\vec{r}$ is collinear for all $t$.

In particular,

$$\vec{r} = (2t, 3t, 4t)$$

We get different point on line so $\vec{r}$ has constant direction.

Note: A vector function $\vec{r}$ of scalar variable $t$ has constant direction if $\vec{r} \times \frac{d\vec{r}}{dt} = 0$

* Example:

$$\vec{r} = (2t, 3t, 4t)$$

$$\frac{d\vec{r}}{dt} = (2, 3, 4)$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 3t & 4t \\ 2 & 3 & 4 \end{vmatrix}$$

$$= t \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \cancel{t\,\vec{0}} \quad t\,\vec{0}$$

$$= \vec{0}$$

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Example

If $\vec{r}$ is a unit vector, prove that

$$\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$$

→ Soln:

Here $\vec{r}$ is a unit vector. So, $\vec{r}$ has constant magnitude.

$$\therefore \vec{r} \cdot \frac{d\vec{r}}{dt} = 0$$

This shows that $\vec{r}$ and $\frac{d\vec{r}}{dt}$ are perpendicular (i.e. have angle $90^\circ$)

$$\frac{|a \times b|}{|\vec{a}||\vec{b}|} = \sin\theta$$

Now,

$$\text{L.H.S.} = \left|\vec{r} \times \frac{d\vec{r}}{dt}\right|$$

$$= |\vec{r}| \left|\frac{d\vec{r}}{dt}\right| \sin 90^\circ$$

$$= 1 \left|\frac{d\vec{r}}{dt}\right| 1$$

$$= \left|\frac{d\vec{r}}{dt}\right|$$

Example

If $\vec{r} = \vec{a}e^{mt} + \vec{b}e^{nt}$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

Prove that

$$\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r} = \vec{0}$$

→ Soln;

$$\frac{d\vec{r}}{dt} = \vec{a}me^{mt} + \vec{b}ne^{nt}$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}m^2e^{mt} + \vec{b}n^2e^{nt}$$

$$\text{L.H.S.} = \frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r}$$

$$= \vec{a}m^2e^{mt} - (m+n)(\vec{a}me^{mt} + \vec{b}ne^{nt}) + mn(\vec{a}e^{mt} + \vec{b}e^{nt})$$

$$= 0$$

# Example:

If $\vec{r} = (a\cos t, a\sin t, 0)$

Find $\cancel{\frac{d}{dt}[\vec{r_1}\ \vec{r_2}\ \vec{r_3}]}$ $\left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right]$

→ Soln;

$\vec{r} = (a\cos t, a\sin t, 0)$

$\frac{d\vec{r}}{dt} = (-a\sin t, a\cos t, 0)$

$\frac{d^2\vec{r}}{dt^2} = (-a\cos t, -a\sin t, 0)$

Now,

$$\left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right] = \begin{vmatrix} a\cos t & a\sin t & 0 \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= 0 \quad [\because c_3 = 0]$$

Evaluate:

$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$, where $r = |\vec{r}|$

$\frac{d}{dt}(\phi\vec{r}) = \phi\frac{d\vec{r}}{dt} + \vec{r}\frac{d\phi}{dt}$

Soln

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{d}{dt}\left(\frac{1}{r}\vec{r}\right)$$

$$= \frac{1}{r}\frac{d\vec{r}}{dt} + \vec{r}\frac{d}{dt}\left(\frac{1}{r}\right)$$

$$= \frac{1}{r}\frac{d\vec{r}}{dt} + \vec{r}\left(-\frac{1}{r^2}\right)\frac{dr}{dt}$$

$$= \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}$$

Formula:

If $\vec{r}$ be the position vector of a particle at time t.

i.e. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Then, $\frac{d\vec{r}}{dt}$ & $\frac{d^2\vec{r}}{dt}$ always represents velocity & acceleration of moving particle at time t.

# Example:

Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at time $t = \pi/2$. Find also their magnitude.

→ Soln:

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= (~~2\sin~~\ 2\sin 3t\ \vec{i} + 2\cos 3t\ \vec{j} + 8t\ \vec{k})$

$\frac{d\vec{r}}{dt} = (6\cos 3t\ \vec{i} - 6\sin 3t\ \vec{j} + 8\vec{k})$

At $t = \pi/2$

$\frac{dr}{dt} = 6\cos 3\pi/2\ \vec{i} - 6\sin 3\pi/2\ \vec{j} + 8\vec{k}$

$$= 0 - 6(-1)\vec{j} + 8\vec{k}$$
$$= 6\vec{j} + 8\vec{k}$$

$$\therefore \left|\frac{d\vec{r}}{dt}\right| = \sqrt{6^2+8^2} = \sqrt{100} = 10$$

Again diff. ① w.r.t. t (For accelration vector)

$$\frac{d^2\vec{r}}{dt^2} = -18\sin 3t\,\vec{i} - 18\cos 3t\,\vec{j} + 0\vec{k}$$

At $\pi/2$

$$\left(\frac{d^2\vec{r}}{dt^2}\right) = -18(-1)\vec{i} - 18\cdot 0\vec{j} + 0\vec{k}$$
$$= 18\vec{i}$$

$$\therefore \left|\frac{d^2\vec{r}}{dt^2}\right| = |18\vec{i}| = \sqrt{18^2+0^2+0^2}$$
$$= 18$$

CW *

If $\dfrac{d\vec{a}}{dt} = \vec{c}\times\vec{a}$, $\dfrac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$

Show that $\dfrac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$

$$\text{L.H.S.} = \frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$
$$= \vec{a}\times(\vec{c}\times\vec{b}) + (\vec{c}\times\vec{a})\times\vec{b}$$
$$= \cancel{(\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b} + (\vec{c}\cdot\vec{b})}$$
$$\cancel{(\vec{c}\cdot\vec{a})\vec{b}}$$
$$= (\vec{a}\cdot\vec{b})\vec{c} + (\vec{c}\cdot\vec{b})\vec{a}$$
$$= (\vec{a}\cdot\vec{b})\vec{c} -$$

$= (\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b} - \vec{b}\times(\vec{c}\times\vec{a})$

$= \cancel{(\vec{a}\cdot\vec{b})\vec{c}} - (\vec{a}\cdot\vec{c})\vec{b} - \cancel{(\vec{b}\cdot\vec{a})\vec{c}} + (\vec{b}\cdot\vec{c})\vec{a}$

$= (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

$= \vec{c}\times(\vec{a}\times\vec{b})$

$=$ R.H.S

Proved

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# Integration of vector function

We know, if $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$,

i.e. derivative of $\frac{d\vec{r}}{dt}$ w.r.t. $t$ is $\frac{d^2\vec{r}}{dt^2}$

(1) Then we say that antiderivative of $\frac{d^2\vec{r}}{dt^2}$ is $\frac{d\vec{r}}{dt}$ and we write

$$\int \frac{d^2\vec{r}}{dt^2}\,dt = \frac{d\vec{r}}{dt} + c$$

(2) Similarly,

$$\frac{d}{dr}(\vec{r_1}\cdot\vec{r_2}) = \frac{d\vec{r_1}}{dt}\cdot\vec{r_2} + \vec{r_1}\frac{d\vec{r_2}}{dt}$$

Then,

$$\int\left(\frac{d\vec{r_1}}{dt}\cdot\vec{r_2} + \vec{r_1}\cdot\frac{d\vec{r_2}}{dt}\right)dt = \vec{r_1}\cdot\vec{r_2} + c$$

(3) $\frac{d}{dr}(\vec{r_1}\times\vec{r_2}) = \frac{d\vec{r_1}}{dt}\times\vec{r_2} + r_2 \times \frac{d\vec{r_2}}{dt}$

Then

$$\int\left(\frac{d\vec{r_1}}{dt}\times\vec{r_2} + \vec{r_1}\frac{d\vec{r_2}}{dt}\right)dt = \vec{r_1}\times\vec{r_2} + \vec{c}$$

If $\vec{r_1} = \vec{a}$, a constant vector

& $\vec{r_2} = \vec{r}$ so that $\frac{d\vec{r_1}}{dt} = \vec{0}$

Then the formula becomes

$$\int \vec{a}\times\frac{d\vec{r}}{dt} = (\vec{a}\times\vec{r}) + c$$

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Q. If $\vec{r_1} = 2\vec{i} + t\vec{j} - \vec{k}$, $\vec{r_3} = 2\vec{i} - 3\vec{j} + 4\vec{k}$
$\vec{r_2} = t\vec{i} + 2\vec{j} + 3\vec{k}$

Find (a) $\int_0^2 (\vec{r_1} \times \vec{r_2})\,dt$

(b) $\int_0^2 [\vec{r_1}\ \vec{r_2}\ \vec{r_3}]\,dt$

→ Soln:

(a) $\vec{r_1} \times \vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & t & -1 \\ t & 2 & 3 \end{vmatrix}$

$= \vec{i}(3t+2) - \vec{j}(6+t) + \vec{k}(4-t^2)$

$\therefore \int_0^2 [\vec{r_1} \times \vec{r_2}]\,dt = \int_0^2 [i(3t+2) - \vec{j}(6+t) + k(4-t^2)]\,dt$

$= \left[\vec{i}\left(\frac{3t^2}{2}+2t\right) - \vec{j}\left(6t + \frac{t^2}{2}\right) + \vec{k}\left(4t - \frac{t^3}{3}\right)\right]_0^2$

$= \left[\vec{i}\left(\frac{3\times4}{2} + 2\times2\right) - \vec{j}\left(12 - \frac{2^2}{4}\right) + \vec{k}\left(8 - 8/3\right)\right] -$
$[\vec{i}\,0 - \vec{j}\,0 + \vec{k}\,0]$

$= 10\vec{i} - 14\vec{j} + 16/3\,\vec{k}$

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(5) $[\vec{r_1}\ \vec{r_2}\ \vec{r_3}] = \begin{vmatrix} 2 & t & -1 \\ t & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix}$

$= 34 - t(4t-6) - 1(-3t-4)$

$= -4t^2 + 9t + 38$

$\therefore \int_0^2 (-4t^2 + 9t + 38)\,dt = \left[-\frac{4t^3}{3} + \frac{9t^2}{2} + 38t\right]_0^2$

$= 94 - 32/3$

$= 250/3$

# Some Def$^n$

We know, if $\vec{r} = \vec{r}(t) = r_1\vec{i} + r_2\vec{j} + r_3\vec{k}$ where $r_1, r_2, r_3$ are scalar function of t.

Then $\frac{d\vec{r}}{dt} = \vec{r}'(t)$ alway represents the vector along the tangent.

Then the unit vector along the tangent is denoted by $\vec{T}(t)$ and is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

# Curvature

If $\vec{r} = \vec{r}(t)$ be a continuous curve,
Then the curvature measures how quickly the curve changes its direction of tangent.
It measures the bendness of the curve.

Formula:
The curvature of a curve is defined as the rate of change of unit tangent vector w.r.t arc length.
It is denoted by $\kappa(t)$.

Thus, $\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right|$

The radius of curvature is the reciprocal of curvature.

Some formula:
If $\vec{r} = \vec{r}(t)$ be the vector function.
Then, the curvature $\kappa(t)$ is defined by

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

# Example

Find the curvature of the curve

$\vec{r}(t) = (t^2, t, 2t^2)$ at point ~~(1,2,0)~~ (1,1,2)

→ Soln:

$\vec{r}(t) = (t^2, t, 2t^2)$

$\vec{r}'(t) = (2t, 1, 4t)$

$\vec{r}''(t) = (2, 0, 4)$

Now,

$$|\vec{r}'(t) \times r''(t)| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 1 & 4t \\ 2 & 0 & 4 \end{vmatrix}$$

$$= \vec{i}(4-0) - \vec{j}(8t - 8t) + \vec{k}(0-2)$$

$$= 4\vec{i} - 0\vec{j} - 2\vec{k}$$

$$= (4, 0, -2)$$

$$\therefore \kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$= \frac{|(4,0,-2)|}{|2t, 1, 4t|^3}$$

$$= \frac{\sqrt{4^2 + 0^2 + (-2)^2}}{\left(\sqrt{4t^2 + 1 + (4t)^2}\right)^3}$$

$$= \frac{\sqrt{20}}{(21)^{3/2}}$$

At $t = 1$

$$\kappa(1) = \frac{\sqrt{20}}{(21)^{3/2}}$$

Radius of curvature = Reciprocal of curvature

$$= \frac{(21)^{3/2}}{7\sqrt{20}}$$

# Example: Find curvature and radius of curvature

$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + 0\,\vec{k}$ at $t = 0$

Ans $K(0) = 1/a$, and radius of curvature $= a$

# Radius of Curvature formula

1. In cartesian form $(x, y)$ form

If $y = f(x)$ be a function then the radius of curvature is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where } y_1 = \frac{dy}{dx},\ y_2 = \frac{d^2y}{dx^2},\ y_2 \neq 0$$

Similarly if $x = f(y)$ be the curve then

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \quad \text{where } x_1 = \frac{dx}{dy},\ x_2 = \frac{d^2x}{dy^2},\ x_2 \neq 0$$

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$\cosh x = \frac{e^{x}+e^{-x}}{2}$ $\cosh^2 x - \sinh^2 x = 1$

$\sinh x = \frac{e^{x}-e^{-x}}{2}$

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Exercise

③ Find the radius of curvature and curvature k at point $(x,y)$ where $y = c\cosh\left(\frac{x}{c}\right)$

→ Soln:

$$y = c\cosh(x/c)$$

Diff. w.r.t. $x$

$$y_1 = c\sinh\frac{x}{c} \cdot \frac{1}{c} = \sinh\frac{x}{c}$$

$$y_2 = \cosh\frac{x}{c} \cdot \frac{1}{c} = \frac{1}{c}\cosh\frac{x}{c}$$

Now,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+(\sinh x/c)^2)^{3/2}}{1/c \cosh x/c}$$

$$= \frac{(\cosh^2 x/c)^{3/2}}{1/c \cosh x/c}$$

$$= \frac{(\cosh x/c)^3}{1/c \cosh x/c}$$

$$= \cosh^2 x/c \cdot c$$

$$= (y/c)^2 \cdot c$$

$$= y^2/c$$

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$\nabla \cdot \vec{v} \to$ divergence
$\nabla \times \vec{v} \to$ curl
$\nabla \phi \to$ gradient

# Vector calculus

~~Diver~~ Vector differential operator:

The vector differential operator denoted by $\nabla$ (nebla) and is given by

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

i.e. $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

1. ~~Dive~~ Gradient of scalar point function.

let $\phi(x, y, z)$ be a scalar point function. Then its gradient is denoted by $\nabla\phi$ and is defined by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial x} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \sum \vec{i}\frac{\partial\phi}{\partial x}$$

Example: Let $\phi = x^2+y^2+z^2$. Find $\nabla\phi$.

$\to$ Soln;

Here, $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$

$= 2(x\vec{i} + y\vec{j} + z\vec{k})$

Note: $\nabla\phi$ is a vector.

Divergent of vector point function

Let $\vec{V} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be a vector point function. of $x, y, \& z$. Then its divergent is given by

$$\nabla\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{V}$$

$$= \vec{i}\cdot\frac{\partial\vec{V}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{V}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{V}}{\partial z}$$

$$= \sum \vec{i}\cdot\frac{\partial\vec{V}}{\partial x}$$

OR

$$\nabla\cdot\vec{V} = \vec{i}\frac{\partial v_1}{\partial x} + \vec{j}\frac{\partial v_2}{\partial y} + \vec{k}\frac{\partial v_3}{\partial z}$$

Eg: Let $\vec{V} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\nabla\cdot\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(x^2\vec{i} + y^2\vec{j} + z^2\vec{k})$$

$$= \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}$$

$$= 2x + 2y + 2z$$

Example:

Let $\vec{v} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$

Find $\nabla\vec{v}$

$$\nabla\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(z^2x)$$

$$= 2xy + 2yz + 2zx$$

3. Curl of a vector point function

Let $\vec{v_1}\vec{i} + \vec{v_2}\vec{j} + v_3\vec{k}$ be a vector point function of $x, y$ & $z$. Then its curl is denoted by $\nabla\times\vec{v}$ is given by

$$\nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{v}$$

$$= \vec{i}\times\frac{\partial\vec{v}}{\partial x} + \vec{j}\times\frac{\partial\vec{v}}{\partial y} + \vec{k}\,\frac{\partial\vec{v}}{\partial z}$$

$$= \sum\vec{i}\times\frac{\partial\vec{v}}{\partial x}$$

Calculation of $\nabla\times\vec{v}$

$$\nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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$$= \vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \vec{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Example:

Let $\vec{v} = x^3 y\vec{i} + y^3 z\vec{j} + z^3 x\vec{k}$. find $\nabla \times \vec{v}$

→ Soln;

$$\nabla \times \vec{v} = \vec{i}\left(\frac{\partial z^3x}{\partial y} - \frac{\partial y^3z}{\partial z}\right) + \vec{j}\left(\frac{\partial x^3y}{\partial z} - \frac{\partial z^3x}{\partial x}\right)$$
$$+ \vec{k}\left(\frac{\partial y^3z}{\partial x} - \frac{\partial x^3y}{\partial y}\right)$$

$$= \vec{i}(0 - y^3) + \vec{j}(-z^3) + \vec{k}(0 - x^3)$$

$$= -\vec{i}\,y^3 - \vec{j}\,z^3 - \vec{k}\,x^3$$
$$= -y^3\vec{i} - z^3\vec{j} - x^3\vec{k}$$
$$= -(y^3\vec{i} + z^3\vec{j} + x^3\vec{k})$$

Example:

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Then

(i) $\nabla \cdot \vec{r} = 3$ (iii) $\frac{\partial r}{\partial x} = \frac{x}{r}$

(ii) $\nabla \times \vec{r} = 0$

⇒ Soln;

$$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z}$$

$$= 1 + 1 + 1$$
$$= 3$$

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(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial z \\ x & y & z \end{vmatrix}$

$= \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0)$

$= \vec{0}$

(iii) $r = |\vec{r}| = |x\vec{i} + y\vec{j} + z\vec{k}|$

$= \sqrt{x^2+y^2+z^2}$

$\therefore \frac{\partial r}{\partial x} = \frac{\partial (x^2+y^2+z^2)^{1/2}}{\partial x}$

$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \times 2x$

$= \frac{x}{\sqrt{x^2+y^2+z^2}}$

$= \frac{x}{r}$

Simillorly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Example : If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Find

(i) $\nabla.(r^3\vec{r})$

(ii) $Div\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$

→ Soln;

We know, $\frac{\partial r}{\partial x} = \frac{x}{r}$

i) $\nabla.(r^3\vec{r}) = \sum \vec{i}.\frac{\partial(r^3\vec{r})}{\partial x}$ $\left[ \because \nabla.\vec{v} = \sum \vec{i}.\frac{\partial v}{\partial x} \right.$ by defination of divergence.

$= \sum \vec{i}.\left[ r^3\frac{\partial \vec{r}}{\partial x} + \vec{r}\frac{\partial r^3}{\partial x} \right]$

$= \sum \vec{i}.\left[ r^3\vec{i} + \vec{r}\left(3r^2\frac{\partial r}{\partial x}\right)\right]$

$= \sum \vec{i}.\left( r^3\vec{i} + \vec{r}\, 3r^2\frac{x}{r}\right)$

$= \sum \vec{i}.\left( r^3\vec{i} + \vec{r}\, 3xr\right)$

$= \sum r^3(\vec{i}.\vec{i}) + 3\sum \vec{i}.(\vec{r}\, 3xr)$

$= \sum r^3(1) + 3\sum xr(\vec{i}.\vec{r})$

$= 3r^3 + 3\sum xrx$ $\quad [\because \vec{i}.\vec{r} = x]$

$= 3r^3 + 3\sum x^2 r$

$= 3r^3 + 3r\sum x^2$ $\quad \left[ \sum x^2 = x^2+y^2+z^2 = r^2 \right.$

$= 3r^3 + 3r.r^2$

$= 3r^3 + 3r^3$

$= 6r^3$