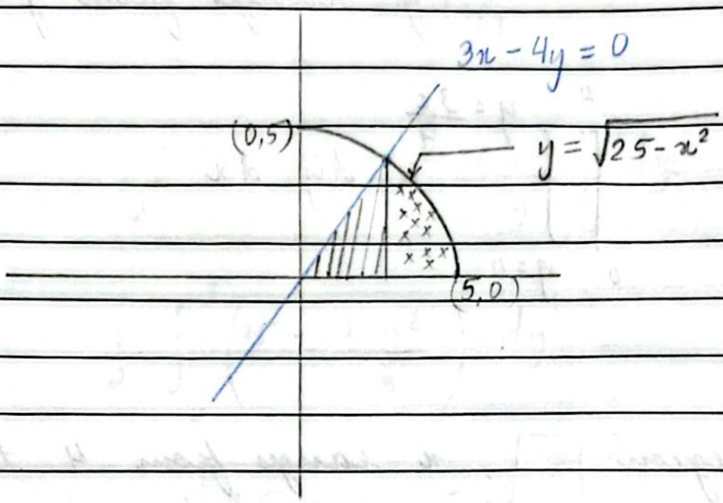
37. $\iint x \, dA$

R: sector of a circle in the first quadrant bounded by $y = \sqrt{25 - x^2}$, $y = 0$, $3x - 4y = 0$.

Solution

Graphing the region:



Solving $3x - 4y = 0$ and $y = \sqrt{25 - x^2}$, we get:

$3x - 4y = 0$

$\therefore y = \frac{3}{4}x$ ——— (i)

And $y = \sqrt{25 - x^2}$ ——— (ii)

Equating (i) and (ii), we get:

$\frac{3x}{4} = \sqrt{25 - x^2}$

$\Rightarrow \frac{9x^2}{16} = 25 - x^2$

$\Rightarrow \frac{25x^2}{16} = 25$

$\therefore x = \pm 4$

When $x = 4$ (positive value),

$$y = \frac{3 \times 4}{4}$$
$$= 3$$

Type I integral

For region , x ranges from 0 to 4

y ranges from $y = 0$ to $y = \frac{3x}{4}$

$$\therefore I_1 = \int_0^4 \int_{y=0}^{y=\frac{3x}{4}} x \, dy \, dx$$

For region , x ranges from 4 to 5

y ranges from $y = 0$ to $y = \sqrt{25 - x^2}$.

$$\therefore I_2 = \int_4^5 \int_{y=0}^{y=\sqrt{25-x^2}} x \, dy \, dx$$

Thus, $$\iint_R x \, dA = \int_0^4 \int_{y=0}^{y=\frac{3x}{4}} x \, dy \, dx + \int_4^5 \int_{y=0}^{y=\sqrt{25-x^2}} x \, dy \, dx$$

Type II integral,

y ranges from 0 to 3,

x ranges from $x = \frac{4y}{3}$ to $x = \sqrt{25 - y^2}$

$$\therefore \iint_R x \, dA = \int_0^3 \int_{x=4y/3}^{x=\sqrt{25-y^2}} x \, dx \, dy$$

Solving using Type II integral,

$$\iint_R x \, dA = \int_0^3 \int_{x=4y/3}^{x=\sqrt{25-y^2}} x \, dx \, dy$$

$$= \int_0^3 \left[\frac{x^2}{2}\right]_{x=4y/3}^{x=\sqrt{25-y^2}} dy$$

$$= \frac{1}{2}\int_0^3 \left(25 - y^2 - \frac{16y^2}{9}\right) dy$$

$$= \frac{1}{2}\int_0^3 \left(25 - \frac{25y^2}{9}\right) dy$$

$$= \frac{1}{2}\left[25y - \frac{25y^3}{27}\right]_0^3$$

$$= \frac{1}{2}(75 - 25)$$

$$= \boxed{25}$$

38. $\iint_R (x^2 + y^2) \, dA$

R: semicircle bounded by $y = \sqrt{4 - x^2}$, $y = 0$.

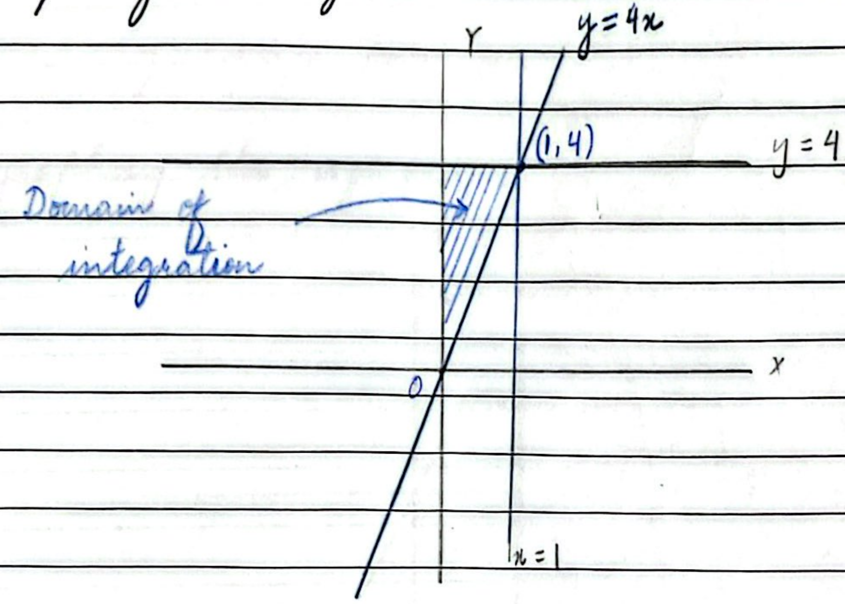
Evaluate the integral by first reversing the order of integration:

39. $\int_0^1 \int_{4x}^{4} e^{-y^2}\, dy\, dx$

Solution

Here, y ranges from $y = 4x$ to $y = 4$ and
x ranges from $x = 0$ to $x = 1$.

Graphing the region:



Changing the order of integration,
y ranges from 0 to 4,
x ranges from $x = 0$ to $x = y/4$

Thus, above integral becomes,

$$I = \int_0^4 \int_{x=0}^{x=y/4} e^{-y^2}\, dx\, dy$$

$$= \int_0^4 \left[e^{-y^2} \cdot x \right]_{x=0}^{x=y/4} dy$$

$$= \int_0^4 \frac{e^{-y^2} y}{4} \, dy$$

$$= \frac{1}{4} \int_0^4 y e^{-y^2} \, dy$$

Put $u = e^{-y^2}$

$$\therefore du = -2y e^{-y^2} dy$$

when $y = 0, \; u = 1$

$y = 4, \; u = e^{-16}$

Thus, above integral becomes.

$$I = \frac{1}{4} \int_1^{e^{-16}} \left(\frac{-1}{2}\right) u \, du$$

$$= -\frac{1}{8} \int_1^{e^{-16}} u \, du$$

$$= -\frac{1}{16} \left[u^2\right]_1^{e^{-16}}$$

$$= \frac{-1}{16} \left(e^{-32} - 1\right)$$

$$= \frac{1 - e^{-32}}{16}$$

$$I = \frac{1}{4} \int_1^{e^{-16}} \left(\frac{-1}{2}\right) du$$

$$= \frac{-1}{8} \int_1^{e^{-16}} du = \frac{-1}{8} \left[u\right]_1^{e^{-16}}$$

$$= -\frac{1}{8}\left(e^{-16}-1\right)$$

$$= \boxed{\frac{1-e^{-16}}{8}}$$

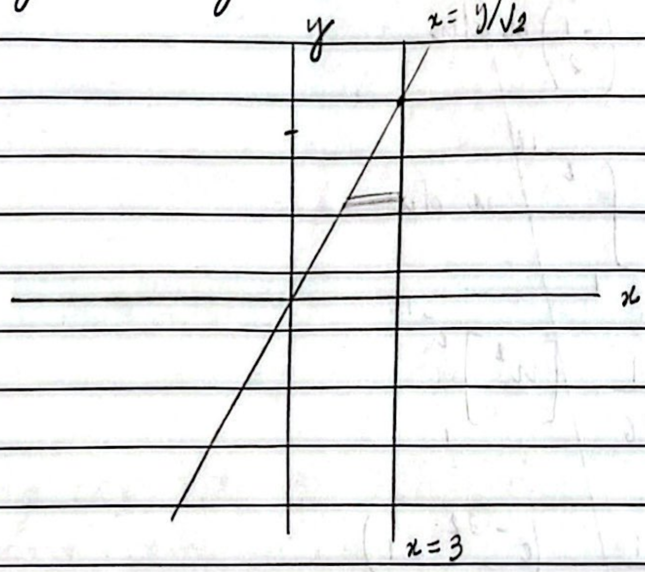
40. $\displaystyle\int_0^2\int_{y/\sqrt{2}}^3 \cos x^2 \, dx \, dy$

Solution

Here, x ranges from $x = \frac{y}{\sqrt{2}}$ to $x = 3$ and

y ranges from $y = 0$ to $y = 2$.

Graphing the region,

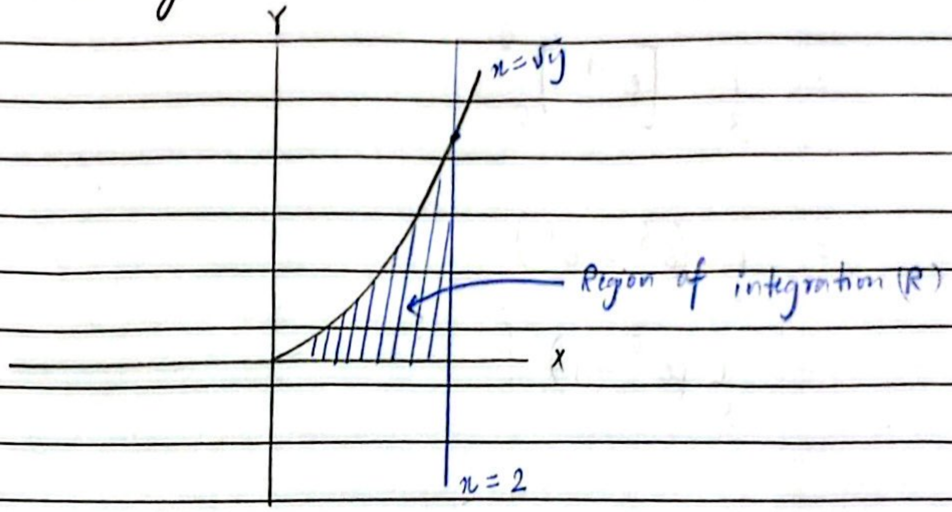


41. $\displaystyle\int_0^4\int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$

Solution

Here, x ranges from $x=\sqrt{y}$ to $x=2$.
y ranges from $y=0$ to $y=4$.

Graphing the region:



when $x=2$, $y=x^2=2^2=4$

Thus, changing the order of integration,
x ranges from 0 to 2,
y ranges from $y=0$ to $y=x^2$.

Thus, above integral becomes,

$$\int_0^2\int_{y=0}^{y=x^2} e^{x^3}\,dy\,dx\;\cancel{dy} = \int_0^2 \left[e^{x^3}\cdot y\right]_0^{x^2} dx$$

$$= \int_0^2 x^2\,e^{x^3}\,dx$$

Put $u = x^3$

$\therefore du = 3x^2\, dx$

when $x = 0$, $u = 0$

$x = 2$, $u = 8$. Thus, above integral becomes,

$$\int_0^8 e^u \frac{du}{3}$$

$$= \frac{1}{3} \left[e^u\right]_0^8$$

$$= \frac{1}{3}\left(e^8 - e^0\right)$$

$$= \boxed{\frac{1}{3}\left(e^8 - 1\right)}$$

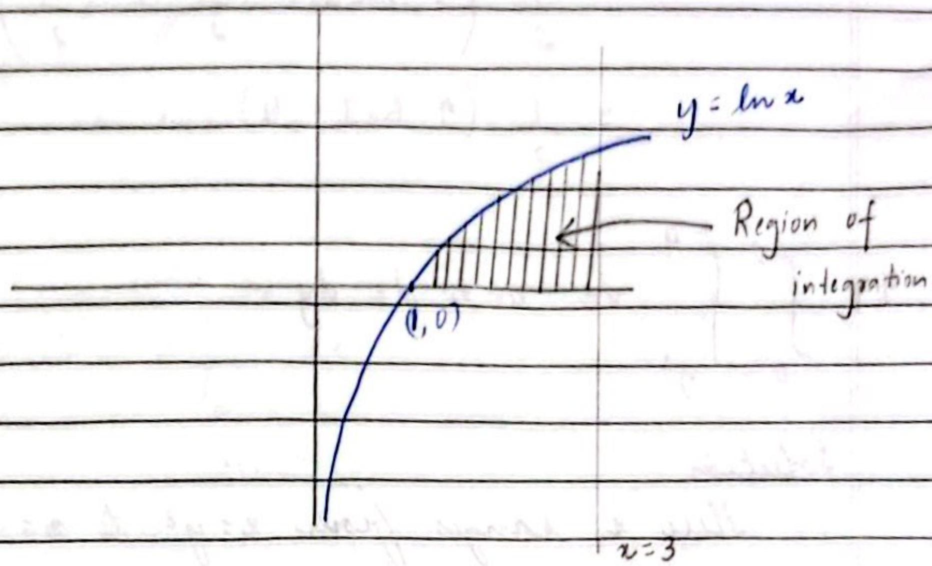
42. $$\int_1^3 \int_0^{\ln x} x \, dy \, dx$$

Solution

Here y ranges from $y = 0$ to $y = \ln x$

x ranges from $x = 1$ to $x = 3$.

Graphing the region as follows:



Changing the order of integration,

x ranges from $x = e^y$ to $x = 3$

y ranges from 0 to ln 3.

Thus, above integral becomes,

$$\int_0^{\ln 3}\int_{x=e^y}^{x=3} x \, dx \, dy$$

$$= \int_0^{\ln 3} \left[\frac{x^2}{2}\right]_{e^y}^{3} dy$$

$$= \frac{1}{2}\int_0^{\ln 3} (9 - e^{2y}) \, dy$$

$$= \frac{1}{2}\left[9y - \frac{e^{2y}}{2}\right]_0^{\ln 3}$$

$$= \frac{1}{2}\left(9 \ln 3 - \frac{e^{2\ln 3}}{2} - 0 + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(9 \ln 3 - \frac{9}{2} + \frac{1}{2}\right)$$

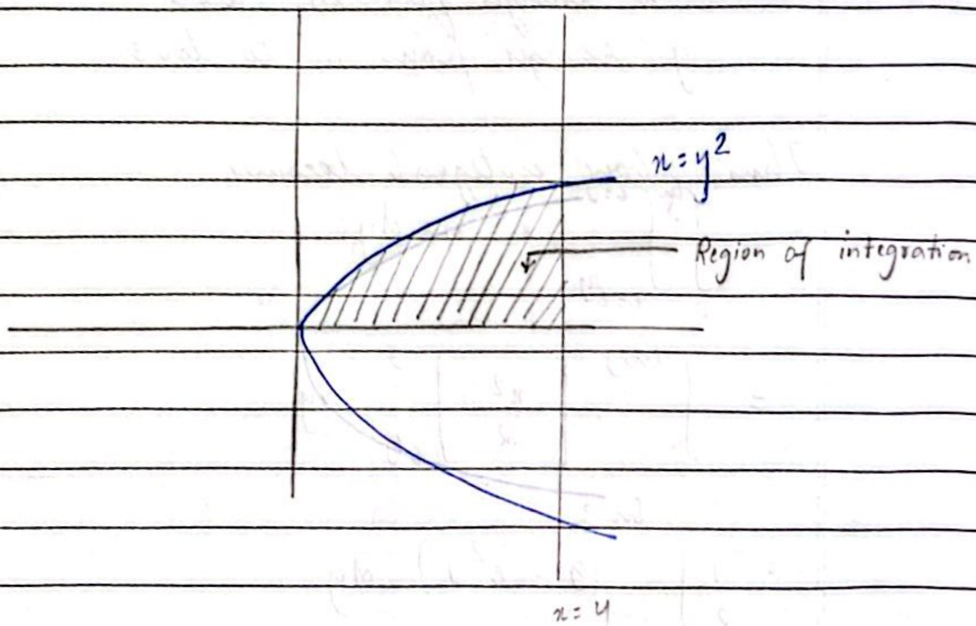
$$= \frac{1}{2}\left(9 \ln 3 - 4\right)$$

44. $\displaystyle\int_0^2 \int_{y^2}^4 \sqrt{x} \sin x \, dx \, dy$

Solution

Here x ranges from $x = y^2$ to $x = 4$ &
y ranges from $y = 0$ to $y = 2$.

Graphing the region of integration,



When $x = 4$, $y^2 = 4$, $\therefore y = 2$

Changing the order of integration,

x ranges from 0 to 4
y ranges from $y = 0$ to $y = \sqrt{x}$

Now, above integral becomes,

$$\int_0^4 \int_{y=0}^{y=\sqrt{x}} \sqrt{x} \sin x \, dy \, dx$$

$$= \int_0^4 \left[\sqrt{x} \sin x \cdot y \right]_0^{\sqrt{x}} dx$$

$$= \int_0^4 x \sin x \, dx$$

Put, $u = x$ and $dv = \sin x \, dx$

Integrating by parts, above integral becomes,

$$uv - \int v \, du$$

$$= \left[x(-\cos x) \right]_0^4 - \int_0^4 (-\cos x) \, dx$$

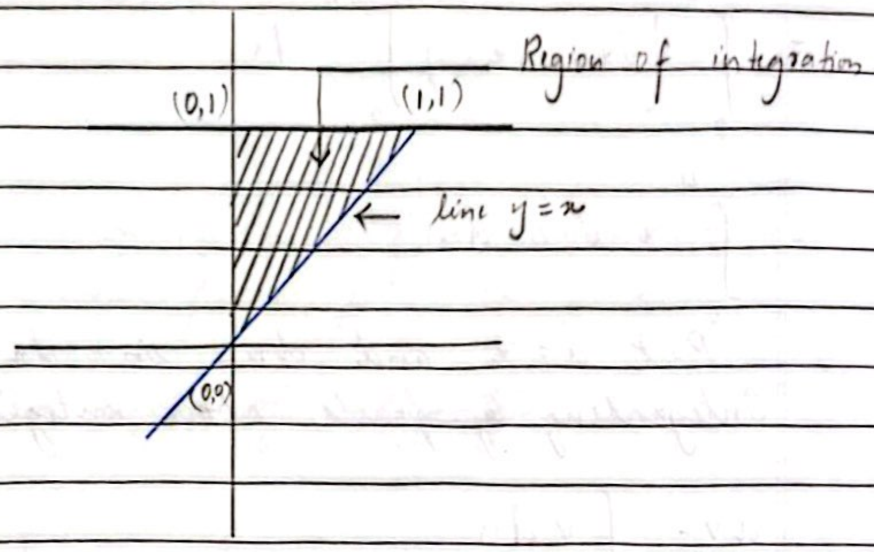
$$= \left[-x \cos x \right]_0^4 + \left[\sin x \right]_0^4$$

$$= \boxed{\sin 4 - 4 \cos 4}$$

45. Find the average value of $1/(1+x^2)$ over the triangular region with vertices $(0,0)$, $(1,1)$ and $(0,1)$.

Solution

Drawing the region of integration:



Setting up integral

x ranges from 0 to 1

y ranges from $y=x$ to $y=1$

Thus,

$$\iint_R \frac{1}{1+x^2}\, dA = \int_0^1 \int_{y=x}^{y=1} \frac{1}{1+x^2}\, dy\, dx$$

$$= \int_0^1 \frac{1}{1+x^2} \Big[y \Big]_x^1 dx$$

$$= \int_0^1 \frac{1-x}{1+x^2}\, dx$$

$$= \int_0^1 \frac{1}{1+x^2} - \int_0^1 \frac{x}{1+x^2}\, dx$$

$$= \left[\tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= \pi/4 - \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$$

$$= \pi/4 - \frac{\ln(2)}{2}$$

Now,

$$f_{av} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

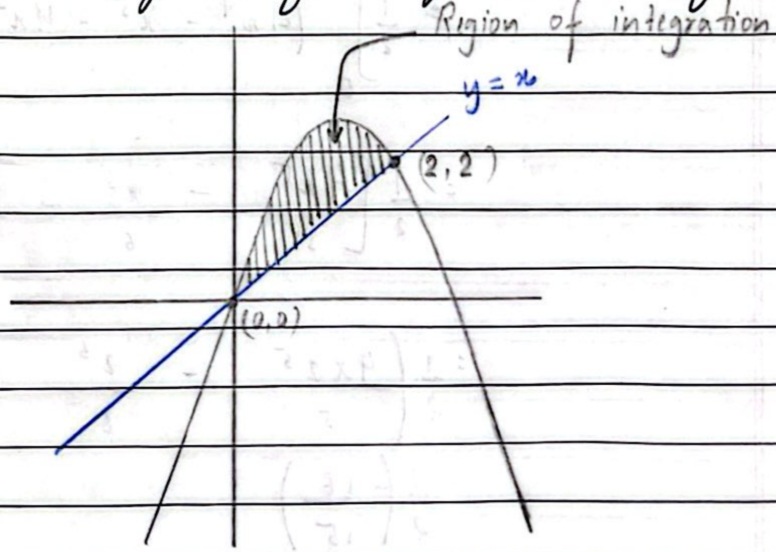
$$= \frac{1}{1/2 * 1 * 1}\left(\pi/4 - \frac{\ln(2)}{2}\right)$$

$$= \left(\pi/2 - \ln(2)\right)$$

46. Find the average value of $f(x,y) = x^2 - xy$ over the region enclosed by $y = x$, and $y = 3x - x^2$.

Solution

Graphing the region of integration as follows:



The point of intersection of two curves is (0,0) & (2,2). Now, setting up integral.

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{y=x}^{y=3x-x^2} f(x,y)\,dy\,dx$$

$$= \int_0^2 \int_{y=x}^{y=3x-x^2} (x^2-xy)\,dy\,dx$$

$$= \int_0^2 \left[x^2y - \frac{xy^2}{2}\right]_{y=x}^{y=3x-x^2} dx$$

$$= \int_0^2 \left(x^2(3x-x^2) - \frac{x}{2}(3x-x^2)^2 - x^3 + \frac{x^3}{2}\right) dx$$

$$= \int_0^2 \left(3x^3 - x^4 - \frac{x(9x^2-6x^3+x^4)}{2} - \frac{x^3}{2}\right) dx$$

$$= \frac{1}{2}\int_0^2 (6x^3 - 2x^4 - 9x^3 + 6x^4 - x^5 - x^3)\,dx$$

$$= \frac{1}{2}\int_0^2 (4x^4 - x^5 - 4x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{4x^5}{5} - \frac{x^6}{6} - \frac{4x^4}{4}\right]_0^2$$

$$= \frac{1}{2}\left(\frac{4\times 2^5}{5} - \frac{2^6}{6} - 2^4\right)$$

$$= \frac{1}{2}\left(-\frac{16}{15}\right)$$

$$A(R) = \int_0^2 \int_{y=x}^{y=3x-x^2} dy\, dx$$

$$= \int_0^2 \left[y \right]_x^{3x-x^2} dx$$

$$= \int_0^2 (3x - x^2 - x)\, dx$$

$$= \int_0^2 (2x - x^2)\, dx$$

$$= \left[x^2 - \frac{x^3}{3} \right]_0^2$$

$$= 4 - 8/3$$

$$= \frac{4}{3}$$

$$\therefore f_{avg} = \frac{-16/15 * 1/2}{4/3}$$

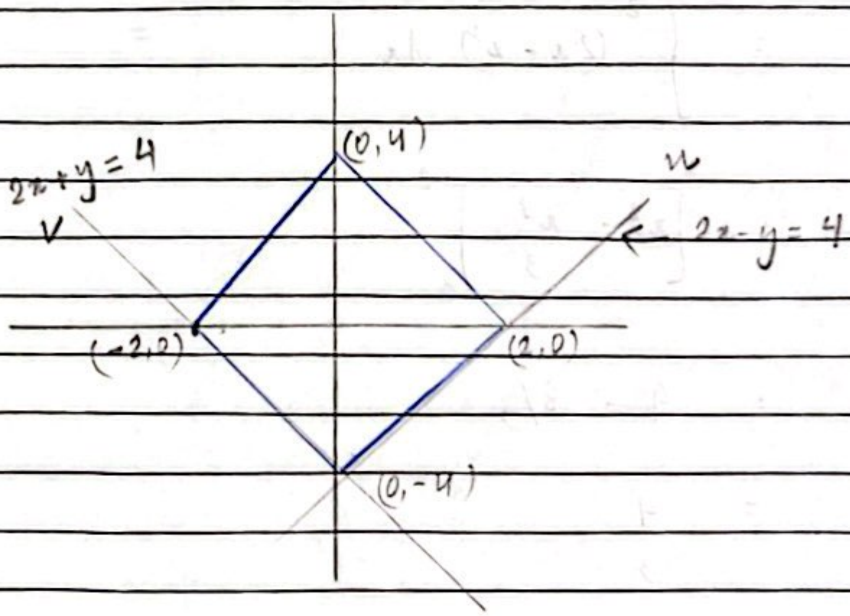
$$= \frac{-16}{15} \times \frac{3}{4} \times \frac{1}{2}$$

$$= \boxed{\frac{-2}{5}}$$

47. Suppose that the temperature in degree Celcius is at a point (x, y) on a flat metal plate is $T(x,y) = 5xy + x^2\cancel{y^2}$, where x and y are in meters. Find the average temperature of the diamond-shaped portion of the plate for which $|2x+y| \le 4$ and $|2x-y| \le 4$

Solution

Drawing the region:



Setting up Jacobian(J):

Put $u = 2x - y$

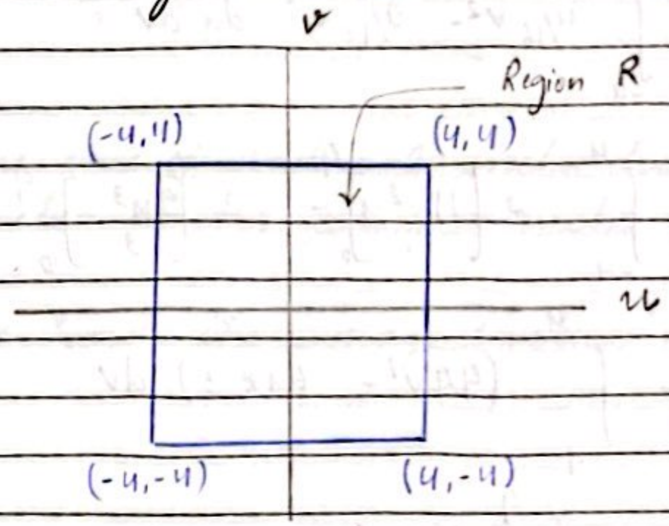
$v = 2x + y$

$\therefore x = \dfrac{u+v}{4} \; ; \; y = \dfrac{v-u}{2}$

Mapping of points in new co-ordinate system:

x	y	u	v
0	-4	4	-4
2	0	4	4
0	4	-4	4
-2	0	-4	-4

Sketching the region:



$$\text{Jacobian, } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \end{vmatrix}$$

$$= 1/8 + 1/8$$

$$= 1/4$$

$$\therefore \text{New integral} = \iint_R f(u,v)\, J \, du\, dv$$

$$= J \int_{-4}^{4} \int_{-4}^{4} 5\left(\frac{u+v}{4}\right)\left(\frac{u-v}{2}\right) + \left(\frac{u+v}{4}\right)^2 du\, d$$

$$= \frac{1}{4}\int_{-4}^{4}\int_{-4}^{4}\left\{ 5/8\,(v^2-u^2) + 1/16\,(u^2+2uv+v^2) \right\} du\,dv$$

$$= \frac{1}{4}\int_{-4}^{4}\int_{-4}^{4} 5/8\,v^2 + 1/16\,v^2 - 5/8\,u^2 + 1/16\,u^2 + \underbrace{\frac{uv}{8}}_{\text{odd function}} \; du\,dv$$

$$= \frac{1}{4}\int_{-4}^{4}\int_{-4}^{4} 11/16\,v^2 - 9/16\,u^2 \; du\,dv$$

$$= \frac{1}{16*4}\int_{-4}^{4} 2\left[11v^2u\right]_0^4 - 2\left[\frac{9u^3}{3}\right]_0^4 dv$$

$$= \frac{2}{64}\int_{-4}^{4}(44v^2 - 64*3)\,dv$$

$$= \frac{1}{32}*2\int_0^4 (44v^2 - 64*3)\,dv$$

$$= \frac{1}{16}\left[\frac{44v^3}{3} - 64*3*v\right]_0^4$$

$$= \frac{1}{16}\left(\frac{44*4^3}{3} - 64*3*4\right)$$

$$= \frac{1}{16}*\frac{512}{3}$$

$$= \frac{32}{3}$$

Area of region $A(R) = J\int_{-4}^{4}\int_{-4}^{4} du\,dv$

$$= \frac{1}{4} * 64$$

$$= 16$$

$$\therefore f_{ave} = \frac{32/3}{16}$$

$$= \frac{2}{3} \, ^\circ C$$

48. A circular lens of radius 2 inches has thickness $1 - \left(\frac{r^2}{4}\right)$ inches at all points r inches from the center of the lens. Find the average thickness of the lens.

Solution

Here r ranges from 0 to 2

θ ranges from 0 to 2π.

And thickness $t = f(r) = 1 - \frac{r^2}{4}$

Now,

$$\iint_R f(r,\theta)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \left(1 - \frac{r^2}{4}\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(r - \frac{r^3}{4}\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{16}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} (2-1)\, d\theta$$

$$= \int_0^{2\pi} d\theta$$

$$= 2\pi$$

Area of region, $A(R) = \pi r^2$

$$= \pi(2)^2$$

$$= 4\pi$$

$\therefore$ Average thickness, $t = \dfrac{2\pi}{4\pi}$

$$= \frac{1}{2}''$$