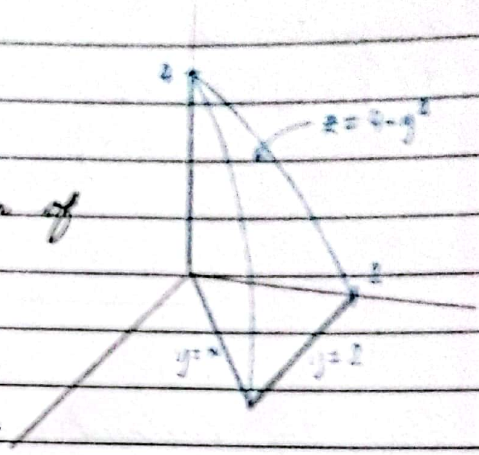
20.

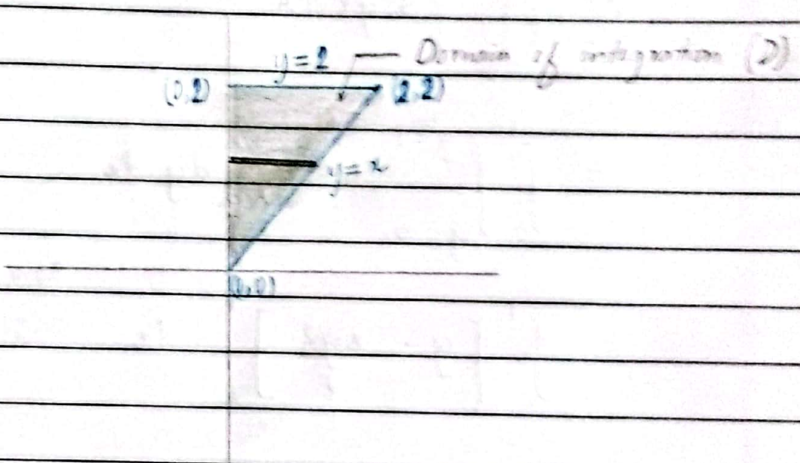
Here,

$$z = f(x,y) = 4 - y^2$$

Now, let's find the domain of integration D.



Sketching the domain in Euclidean plane as follows:



Thus, domain of integration is

$$D = \{(x,y) : 0 < x < y,\ 0 \le y < 2\}$$

Thus, enclosed volume V is given by:

$$V = \iint_D f(x,y)\, dA$$

$$= \int_0^2 \int_{x=0}^{x=y} (4-y^2)\, dx\, dy$$

$$= \int_0^2 \left[4x - xy^2\right]_0^y dy$$

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$$= \int_0^2 (4y - y^3)\, dy$$

$$= \left[2y^2 - \frac{y^4}{4}\right]_0^2$$

$$= 2 \times 2^2 - \frac{2^4}{4}$$

$$= 8 - 4$$

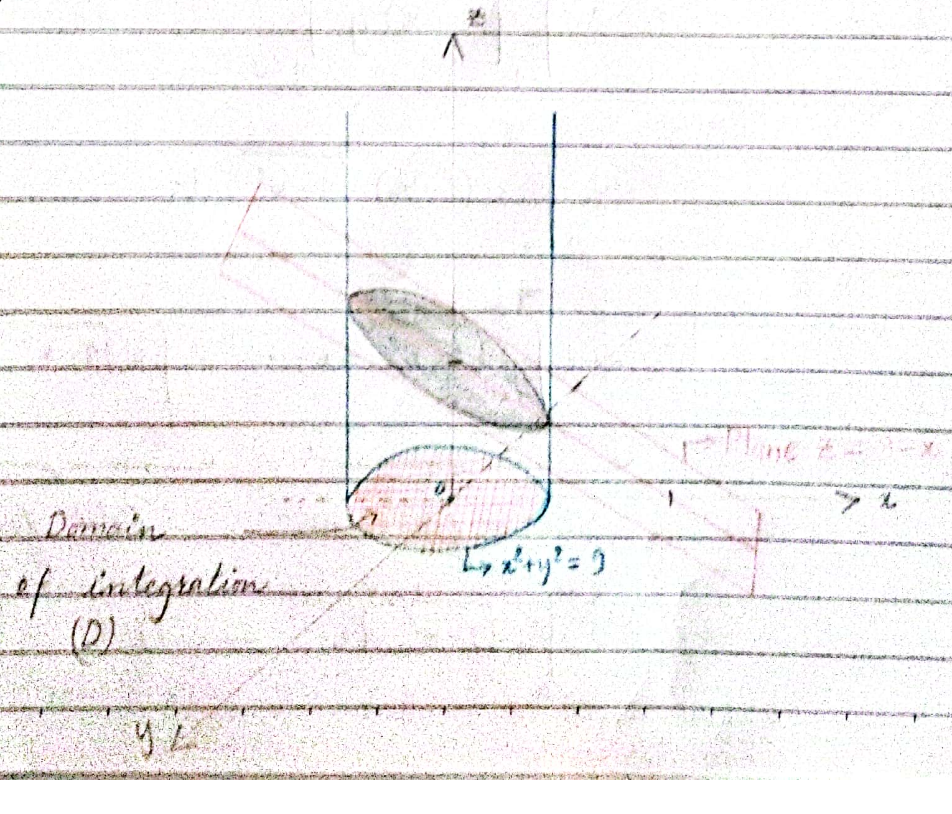
$$= \boxed{4}$$

Use double integration to find the volume of each solid.

21. The solid bounded by the cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=3-x$

Solution

Sketching above figures in Euclidean space as follows:



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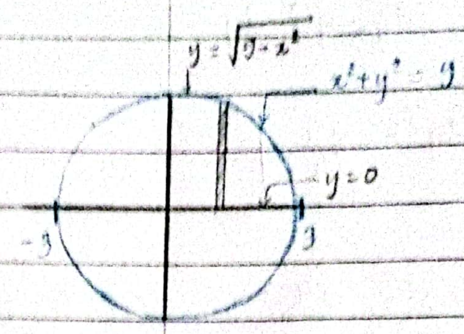


Domain of integration, $D = \{(x,y) : -3 \le x \le 3,$
$0 \le y \le \sqrt{9-x^2}\}$

Given function, $f(x,y) = 3 - x$

$\therefore$ Volume of enclosed surface,

$$V = \iint_D f(x,y)\, dA$$

$$= 2\int_{-3}^{3}\int_{y=0}^{y=\sqrt{9-x^2}} (3-x)\ dy\ dx$$

$$= 2\int_{-3}^{3}\Big[(3-x)\,y\Big]_{y=0}^{y=\sqrt{9-x^2}} dx$$

$$= 2\int_{-3}^{3} (3-x)\sqrt{9-x^2}\ dx$$

$$= \underbrace{2\cdot 3\int_{-3}^{3}\sqrt{9-x^2}\ dx}_{\text{Even function}} - \underbrace{2\int_{-3}^{3} x\sqrt{9-x^2}\ dx}_{\text{Odd function}}$$

$$= 6\cdot 2\int_{0}^{3}\sqrt{9-x^2}\ dx - 0$$

$$= 12\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= 12\left\{\frac{3}{2}\sqrt{9-9} + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right) - 0\right\}$$

$$= 12\left(0 + \frac{9\pi}{4} - 0\right)$$

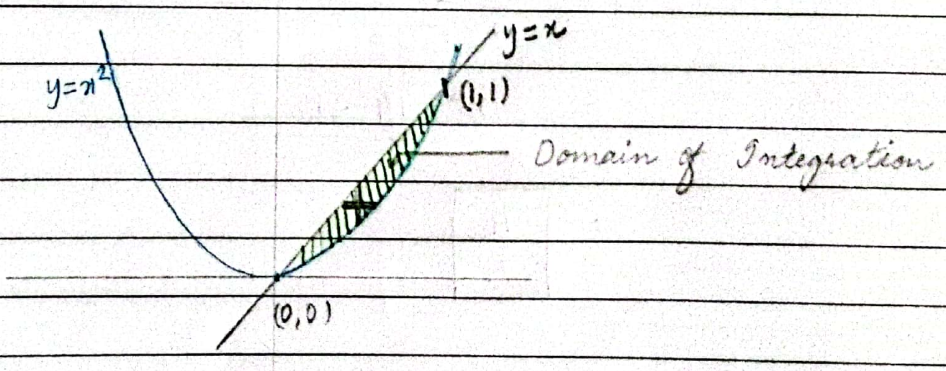
$$= \boxed{27\pi}$$

22. The solid in the first octant bounded above by the paraboloid $z = x^2 + 3y^2$, below by the plane $z = 0$, and laterally by $y = x^2$ and $y = x$.

Here

$z = f(x,y) = x^2 + 3y^2$

Sketching the domain of integration as follows:



Here, x ranges from 0 to 1

y ranges from $y = x^2$ to $y = x$

$\therefore D = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$

$\therefore$ Volume of enclosed solid is given by,

$$V = \iint_D f(x,y)\,dA$$

$$= \int_0^1 \int_{y=x^2}^{y=x} (x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1 \left[(x^2 y + y^3) \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 (x^3 + x^3 - x^4 - x^6)\,dx$$

$$= \int_0^1 (2x^3 - x^4 - x^6)\,dx$$

$$= \left[\frac{x^4}{2} - \frac{x^5}{5} - \frac{x^7}{7} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{5} - \frac{1}{7}$$

$$= \frac{35 - 14 - 10}{70}$$

$$= \boxed{\frac{11}{70}}$$

23. The solid bounded by the paraboloid $z = 9x^2 + y^2$ below by the plane $z = 0$, and laterally by the planes $x = 0$, $y = 0$, $x = 3$, and $y = 2$.

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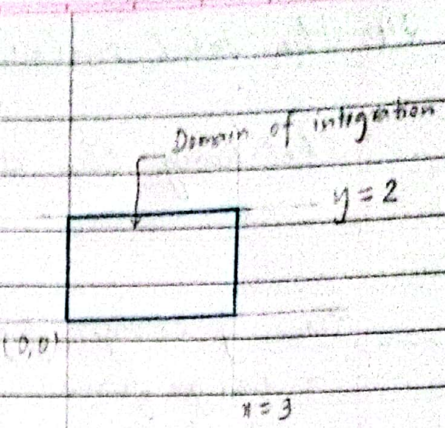
Here, x ranges from 0 to 3
y ranges from 0 to 2

$\therefore D = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le 2\}$



$\therefore$ Volume of solid enclosed,

$$V = \iint_D f(x,y)\, dA$$

$$= \int_0^3 \int_0^2 (9x^2 + y^2)\, dy\, dx$$

$$= \int_0^3 \left[9x^2 y + y^3/3\right]_0^2 dx$$

$$= \int_0^3 \left(18x^2 + 8/3\right) dx$$

$$= \left[\frac{18x^3}{3} + \frac{8x}{3}\right]_0^3$$

$$= \left[6x^3 + \frac{8}{3}x\right]_0^3$$

$$= \left(6 * 3^3 + \frac{8}{3} * 3\right)$$

$$= 27 * 6 + 8$$

$$= 162 + 8$$

$$= \boxed{170}$$

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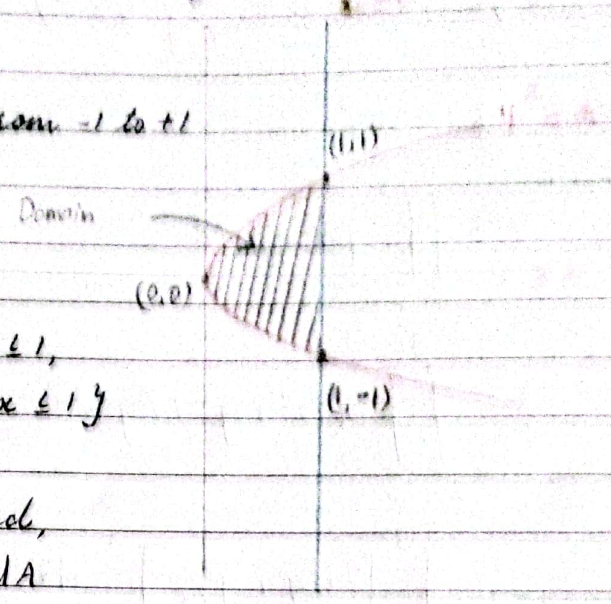
24. The solid enclosed by $y^2 = x$, $z = 0$, and $x + z = 1$

Sketching the domain of integration,



Here, y ranges from -1 to $+1$

x ranges from

$x = y^2$ to $x = 1$

$\therefore D = \{(x,y) : -1 \le y \le 1,$
$y^2 \le x \le 1\}$

$\therefore$ Volume enclosed,

$$V = \iint_D f(x,y)\, dA$$

$$f(x,y) = z = (1-x)$$

Thus,

$$V = \int_{-1}^{1}\int_{x=y^2}^{x=1} (1-x)\, dx\, dy$$

$$= \int_{-1}^{1} \left[x - \frac{x^2}{2} \right]_{y^2}^{1} dy$$

$$= \int_{-1}^{1} \left(1 - 1/_2 - y^2 + \frac{y^4}{2} \right) dy$$

$$= \underbrace{\int_{-1}^{1} \left(1/_2 - y^2 + y^4/_2 \right) dy}_{\text{Even function}}$$

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$$= 2\int_0^1 \left(\frac{1}{2} - y^2 - \frac{y^4}{2}\right) dy$$

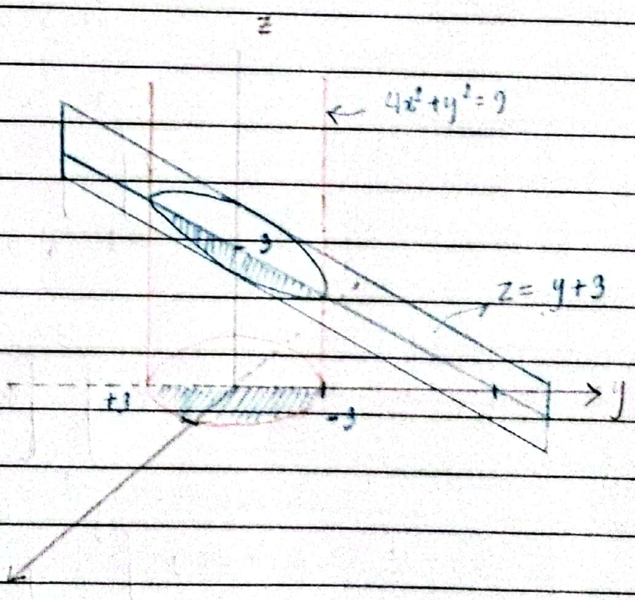
$$= 2\left[\frac{y}{2} - \frac{y^3}{3} - \frac{y^5}{10}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{10}\right)$$

$$= 2 * \frac{8}{30}$$

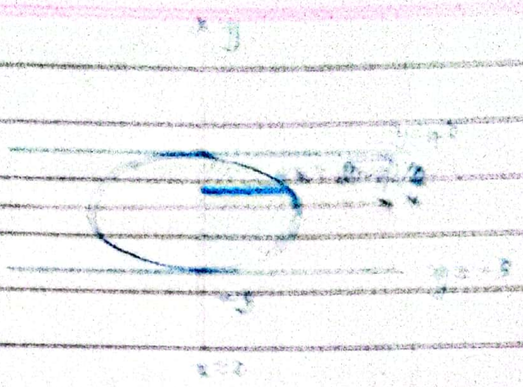
$$= \boxed{\frac{8}{15}}$$

25. The wedge cut from the cylinder $4x^2 + y^2 = 9$ by the planes $z = 0$ and $z = y + 3$.



Here, $z = f(x,y) = y+3$

Sketching the the domain of the above integration,



y ranges from -3 to $+3$.
x ranges from $x=0$ to $x=\dfrac{\sqrt{9-y^2}}{2}$

$$\therefore D=\left\{(x,y):\ -3\le y\le 3,\ \ 0\le x\le \frac{\sqrt{9-y^2}}{2}\right\}$$

Thus enclosed volume, $V=\ 2\displaystyle\int_{-3}^{3}\int_{x=0}^{x=\frac{\sqrt{9-y^2}}{2}} f(x,y)\,dx\,dy$

$$=2\int_{-3}^{3}\int_{x=0}^{x=\sqrt{9-y^2}/2}(y+3)\,dx\,dy$$

$$=2\int_{-3}^{3}\Big[x(y+3)\Big]_{0}^{\sqrt{9-y^2}/2}dy$$

$$=\underbrace{2\int_{-3}^{3}\frac{y\sqrt{9-y^2}}{2}}_{\text{odd function}}+\underbrace{2\cdot\frac{3}{2}\int_{-3}^{3}\sqrt{9-y^2}\,dy}_{\text{Even function}}$$

$$=0+3\times2\int_{0}^{3}\sqrt{9-y^2}\,dy$$

$$= 6\left[\frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}\left(\frac{y}{3}\right)\right]_0^3$$

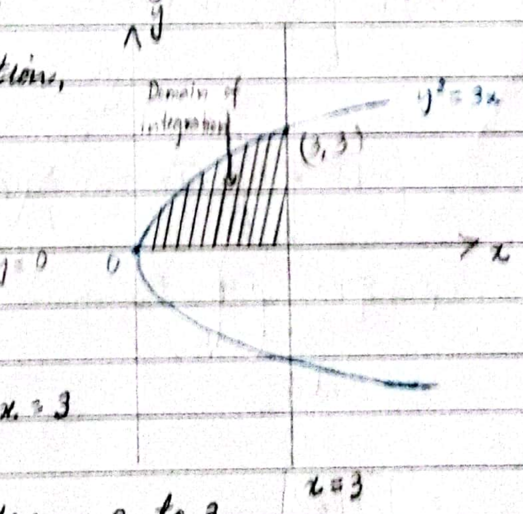
$$= 6\left(0 + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right) - 0\right)$$

$$= 6 \times \frac{9}{2} \times \frac{\pi}{2}$$

$$= \boxed{\frac{27\pi}{2}}$$

26 The solid in the first octant bounded above by $z = 9 - x^2$, below by $z = 0$, and laterally by $y^2 = 3x$.

Sketching the domain of above integration:



Here, given function,
$f(x,y) = 9 - x^2$

For $z = 0$,
$9 - x^2 = 0$
$\therefore x = \pm 3$
In first octant, $x = 3$

Thus, x ranges from 0 to 3,
y ranges from $y=0$ to $y = \sqrt{3x}$

$\therefore D = \{(x,y): 0 \le x \le 3, \ 0 \le y \le 3x\}$

Thus, the Volume is enclosed is given by:

$$V = \int_0^3 \int_{y=0}^{y=\sqrt{3x}} f(x,y)\, dA$$

$$= \int_0^3 \int_{y=0}^{y=\sqrt{3x}} (9-x^2)\, dy\, dx$$

$$= \int_0^3 \left[9y - x^2 y\right]_{y=0}^{y=\sqrt{3x}} dx$$

$$= \int_0^3 \left(9\sqrt{3x} - x^2\sqrt{3x}\right) dx$$

$$= \sqrt{3}\int_0^3 \left[9\sqrt{x} - x^{5/2}\right] dx$$

$$= \sqrt{3}\left[\frac{9\, x^{3/2}}{3/2} - \frac{x^{7/2}}{7/2}\right]_0^3$$

$$= \sqrt{3}\left[\frac{2}{3} * 9\, 3^{3/2} - \frac{2}{7} * 3^{7/2}\right]$$

$$= \left(6\sqrt{3} * 3 * \sqrt{3} - \frac{2}{7} * 3^3 * \sqrt{3} * \sqrt{3}\right)$$

$$= 54 - \frac{54 \cdot 3}{7}$$

$$= \frac{54 \cdot 4}{7}$$

$$= \boxed{\frac{216}{7}}$$

Sketch the region of integration and change the order of integration.

27. $\int_0^2 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx$

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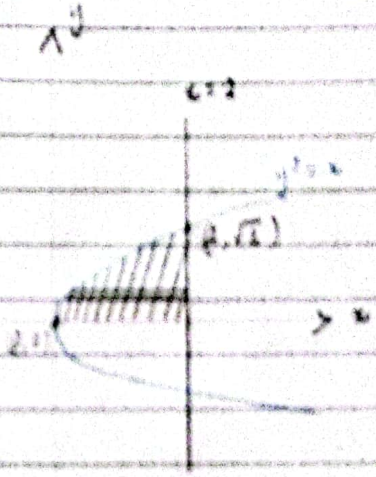
Here,

x varies from 0 to 2

y varies from $y=0$ to $y=\sqrt{x}$ (i.e. $y^2=x$)

Now, domain of integration is sketched as follows:



Changing the order of integration,

y ranges from 0 to $\sqrt{2}$.

x ranges from $x=y^2$ to $x=2$.

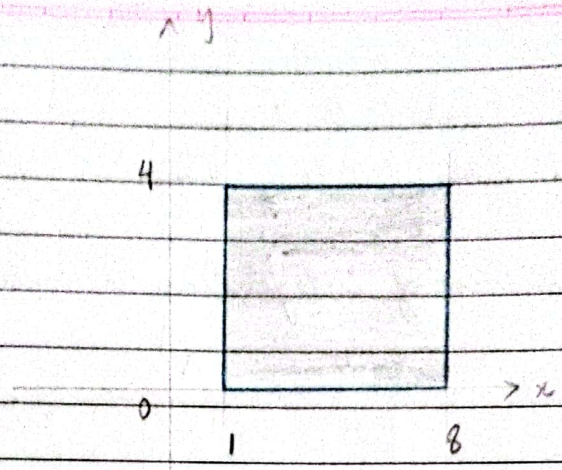
$\therefore$ The integral becomes $\displaystyle\int_0^{\sqrt{2}}\int_{x=y^2}^{x=2} f(x,y)\,dx\,dy$

28. $\displaystyle\int_0^4\int_1^8 f(x,y)\,dx\,dy$

Here y varies from 0 to 4

x varies from $x=1$ to $x=8$.

Sketching the region of integration:



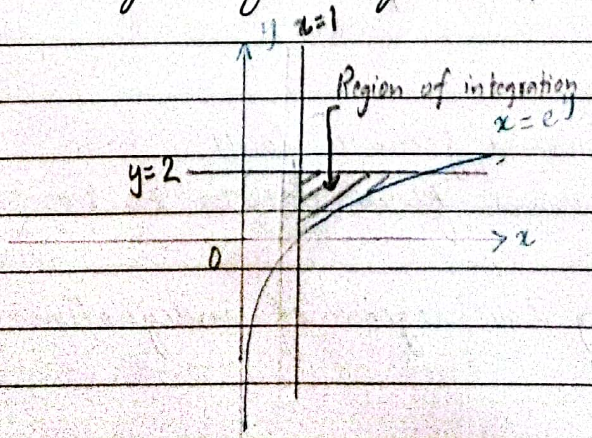
Changing the order of integration,
x ranges from 1 to 8
y ranges from 0 to 4

Thus, above integral becomes $\int_1^8 \int_0^4 f(x,y)\, dy\, dx$

29. $\int_0^2 \int_1^{e^y} f(x,y)\, dx\, dy$

Here, y varies from 0 to 2
x varies from $x=1$ to $x=e^y$

Sketching the region of integration,



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Changing the order of integration,

Solving $y = 2$ and $x = e^{y}$,
we get $x = e^2$
$\therefore x = e^2$

Now,
x varies from $x = 1$ to $x = e^2$
y varies from ~~$y = 0$ to~~ $y = \ln(x)$ to $y = 2$

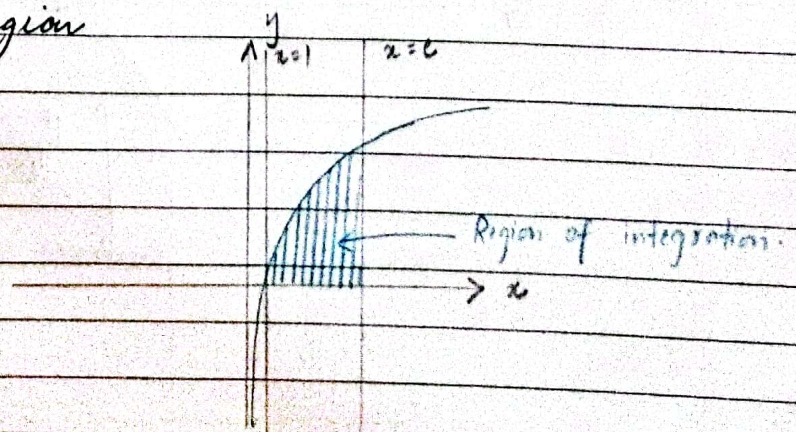
$\therefore$ Integral becomes, $\displaystyle\int_{1}^{e^2}\int_{\cancel{y=0}\; y=\ln(x)}^{\cancel{y=\ln(x)}\; y=2} f(x,y)\,dy\,dx$

$$= \int_{1}^{e^2}\int_{y=\ln(x)}^{y=2} f(x,y)\,dy\,dx$$

30. $\displaystyle\int_{1}^{e}\int_{0}^{\ln x} f(x,y)\,dy\,dx$

Here x varies from 1 to e
y varies from $y = 0$ to $y = \ln(x)$

Sketching the region



Changing order of integration,
Solving $y = \ln(x)$ and $x = e$,
we get, $y = \ln(e)$
$= 1$
$\therefore y = 1$

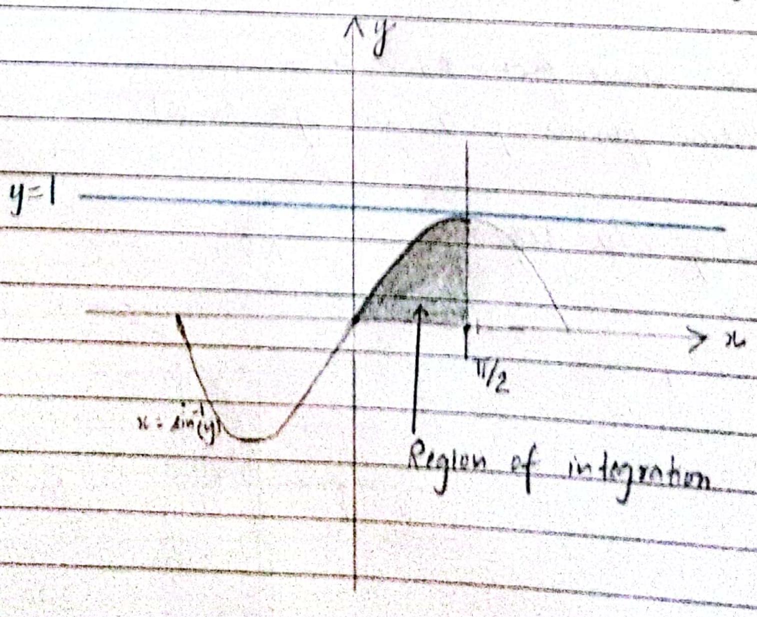
Thus, y ranges from 0 to 1.
x ranges from $x = e^y$ to $x = e$.

$\therefore$ The integral becomes, $\displaystyle\int_0^1 \int_{x=e^y}^{x=e} f(x,y)\, dx\, dy$

31. $\displaystyle\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} f(x,y)\, dx\, dy$

Here, y ranges from 0 to 1
x ranges from $x = \sin^{-1}(y)$ to $x = \pi/2$

Sketching the region of integration, we get:



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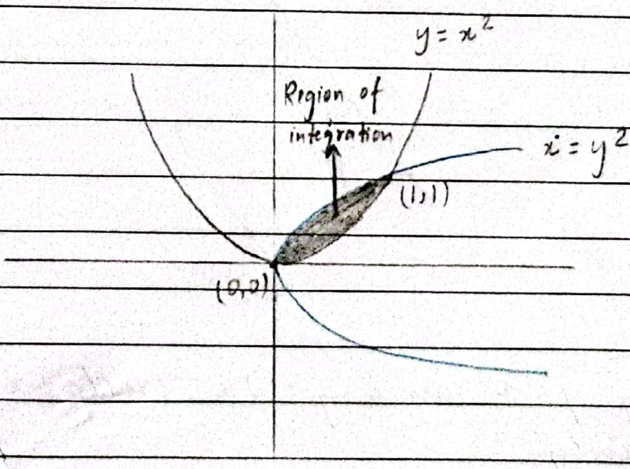
Changing the order of integration,
x ranges from 0 to $\pi/2$
y ranges from $y=0$ to $y = \sin x$.

Thus, above integral becomes, $\displaystyle\int_0^{\pi/2}\int_{y=0}^{y=\sin x} f(x,y)\,dy\,dx$.

32. $\displaystyle\int_0^1\int_{y^2}^{\sqrt{y}} f(x,y)\,dx\,dy$

Here, y ranges from 0 to 1,
x ranges from $x = y^2$ to $x = \sqrt{y}$ (i.e. $x^2 = y$).

Sketching the region,



Changing the order of integration,
x ranges from 0 to 1
y ranges from y~~x~~ $= \sqrt{x}$ to $y = x^2$

$\therefore$ Integral becomes, $\displaystyle\int_0^1\int_{\sqrt{x}}^{x^2} f(x,y)\,dy\,dx$

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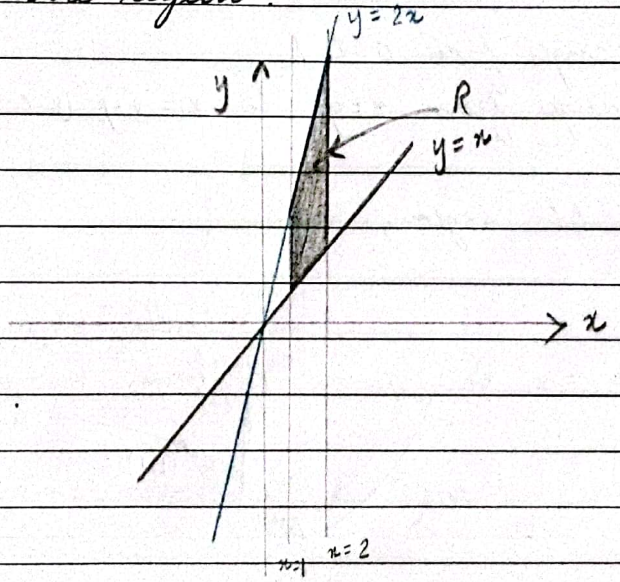
Set up integrals for both orders of integration. Use the more convenient order to evaluate the integral over the region R.

33. $\iint_R \frac{y}{x^2+y^2}\, dA$

R: trapezoid bounded by $y = x$, $y = 2x$, $x = 1$, $x = 2$

Solution.

Sketching the region:



Solving $y = x$ and $y = 2x$ with $x = 1$ and $x = 2$ respectively,

when $x = 1$,
$y = x$ becomes
$y = 1$
and $y = 2x$ becomes
$y = 2$

when $x = 2$,
$y = x$ becomes,
$y = 2$ and
$y = 2x$ becomes
$y = 2 * 2 = 4$

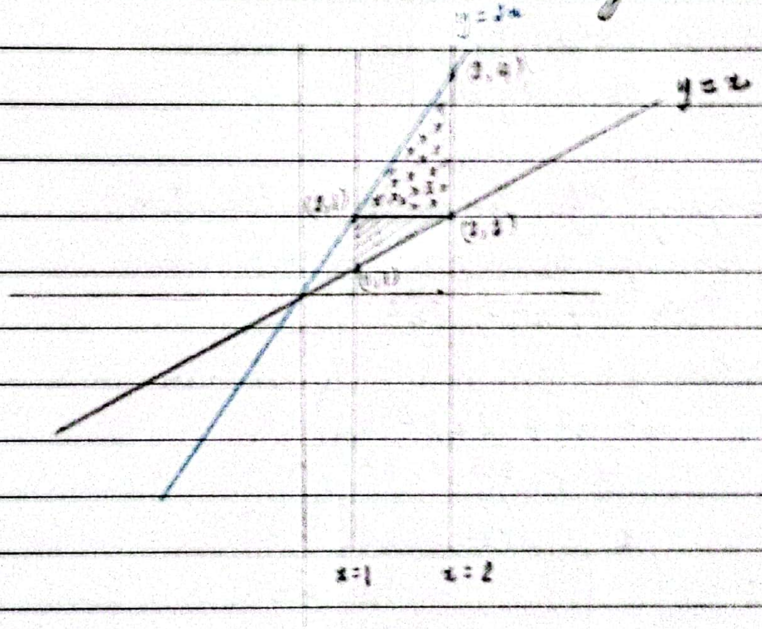
$\therefore$ Intersection points are $(1,1)$, $(2,2)$, $(1,2)$ and $(2,4)$

Type I integral
x ranges from 1 to 2,
y ranges from $y = x$ to $y = 2x$

$$\therefore I = \int_1^2 \int_{y=x}^{y=2x} \frac{y}{x^2+y^2} \, dy \, dx$$

Type II integral
y ranges from 1 to 4
x ranges from ?

Here, we have to subdivide the region



For region ▨, y ranges from 1 to 2
x ranges from $x=1$ to $x=y$

$$\therefore I_1 = \int_1^2 \int_{x=1}^{x=y} \frac{y}{x^2+y^2} \, dx \, dy$$

For region ⁘, y ranges from 2 to 4
x ranges from $y/2$ to 2

$$\therefore I_2 = \int_2^4 \int_{x=y/2}^{x=2} \frac{y}{x^2+y^2}\, dx\, dy$$

$$\therefore I = \int_1^2 \int_{x=1}^{x=y} \frac{y}{x^2+y^2}\, dx\, dy + \int_2^4 \int_{x=y/2}^{2} \frac{y}{x^2+y^2}\, dx\, dy$$

Type I integral is more convenient.

Thus,

$$I = \int_1^2 \int_{y=x}^{y=2x} \frac{y}{x^2+y^2}\, dy\, dx$$

Put $u = x^2 + y^2$

$\therefore du = 2y\, dy$

Changing the limit of integration,

when $y = x$,

$u = x^2 + x^2 = 2x^2$

when $y = 2x$,

$u = x^2 + (2x)^2$

$= 5x^2$

$$\therefore I = \int_1^2 \int_{u=2x^2}^{u=5x^2} \frac{du}{2u}\, dx$$

$$= \frac{1}{2} \int_1^2 \Big[\ln(u)\Big]_{2x^2}^{5x^2} dx$$

$$= \frac{1}{2} \int_1^2 \left(\ln 5x^2 - \ln 2x^2\right) dx$$

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$$= \frac{1}{2} \int_1^2 \ln\left(\frac{5x^2}{2x^2}\right) dx$$

$$= \frac{1}{2} \ln(5/2) \int_1^2 dx$$

$$= \frac{1}{2} \ln\left(\frac{5}{2}\right) \left[x\right]_1^2$$

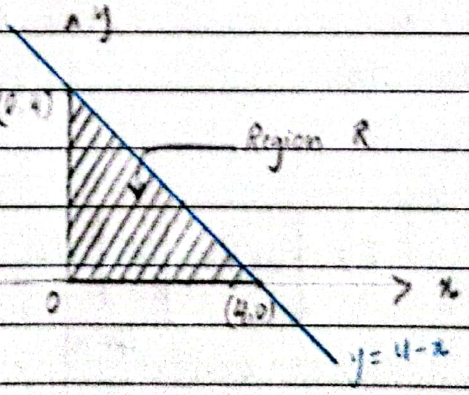
$$= \frac{1}{2} \ln\left(\frac{5}{2}\right) (2-1)$$

$$= \frac{1}{2} \ln\left(\frac{5}{2}\right)$$

34. $\iint_R x e^y \, dA$

R: triangle bounded by $y = 4-x$, $y = 0$, $x = 0$.

Sketching the region R



Type I Integral

x ranges from 0 to 4
y ranges from
$y=0$ to $y = 4-x$.

Thus,

$$I = \int_0^4 \int_0^{4-x} x e^y \, dy \, dx$$

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Type II integral,
y ranges from 0 to 4
x ranges from $x=0$ to $x=4-y$

$$\therefore I = \int_0^4 \int_0^{4-y} x e^y \, dx \, dy$$

Both integrals are convenient to solve. Let's go with the type I integral,

$$I = \int_0^4 \int_0^{4-x} x e^y \, dy \, dx$$

$$= \int_0^4 \left[x e^y \right]_0^{4-x} dx$$

$$= \int_0^4 (x e^{(4-x)} - x) dx = \underbrace{\int_0^4 x e^{4-x} dx}_{I'} - \underbrace{\int_0^4 x \, dx}_{I''}$$

For I'

Integration by parts,

$u = x$ $\qquad dv = e^{4-x} dx$

$\therefore du = dx$ $\qquad v = (-1) e^{4-x}$

Now,

$$I = \left[uv - \int v \right] du$$

$$= x(-1)e^{4-x} - \int (-1) e^{4-x} dx$$

$$= -x e^{4-x} + \int e^{4-x} dx$$

$$= -x e^{4-x} + \left(\frac{e^{4-x}}{-1} \right)$$

$$= -x e^{4-x} - e^{4-x} = e^{4-x}(-x-1)$$

$$\therefore I' = \left[e^{4-x}(-x-1)\right]_0^4$$

$$= e^{4-4}(-4-1) - e^{4-0}(-1-0)$$

$$= e^0 * (-5) - e^4 * (-1)$$
$$= e^4 - 5$$

Now,

$$I'' = \int_0^4 x \, dx$$

$$= \left[\frac{x^2}{2}\right]_0^4$$

$$= 8$$

$$\therefore I = I' - I''$$
$$= e^4 - 5 - 8$$
$$= (e^4 - 13)$$

35. $\iint_R (-2y) \, dA$

R: trapezoid bounded by $y = 4 - x^2$, $y = \sqrt{x}$, $x = 4$.

Sketching the region R