Unit 3: Spectral theory I

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Summary

1. Eigenvectors and eigenvalues
2. Some simple examples
3. Using determinants to compute eigenvectors and eigenvalues
4. Basic properties of determinants

Eigenvectors and eigenvalues

Consider the scalar multiplication of a vector x by a scalar λ:

$$\lambda x.$$

This operation does not change the direction of the vector x and simply *striches* out the vector.

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There are many matrices that, when applied to the appropriate vector or collection of vectors, act exactly like scalar multiplication.

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$$Ae_1 = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for any $\lambda \in \mathbb{R}$.

Similarly, if one applies A to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then A changes the direction of it too:

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$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for any $\lambda \in \mathbb{R}$.

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i.e., the vector is stretched by a factor of 5 in the direction of the vector.

From the above discussion, in general, the present question of interest is:

Given an $n \times n$ matrix A, for what vectors $x \in \mathbb{R}^n$ and numbers $\lambda \in \mathbb{R}$ does $Ax = \lambda x$?

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Given an $n \times n$ matrix A, for what vectors $x \in \mathbb{R}^n$ and numbers $\lambda \in \mathbb{R}$ does $Ax = \lambda x$?

These vectors and numbers are *eigenvectors* and *eigenvalues* !

Eigenvectors and eigenvalues

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Note that

From the definition, it is clear that an eigenvector is a nonzero vector.

Example

If $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then evaluate Ax.

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$$Ax = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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$\square$

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i.e., the vector $3x$ is stretched by a factor of 5 in the direction of the vector.

In general, we have the following:

Theorem

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Due to this theorem, we usually look only for unit eigenvectors.

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$$\begin{aligned} A(kx) &= k(Ax) \\ &= k(\lambda x) \\ &= \lambda(kx). \end{aligned}$$

Hence kx is also an eigenvector corresponding to the eigenvalue λ. Since k is arbitrary, every non-zero scalar multiple of x is also an eigenvector of the matrix A corresponding to the eigenvalue λ. $\square$

Some simple examples

Before describing eigenvectors and eigenvalues in more detail, let's go through several examples of matrices and their eigenvectors and eigenvalues, which will illustrate more general properties that we will discuss in more detail below.

Diagonal matrix

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if x_1 and x_2 both are nonzero. So, in this case, the vector x cannot be eigenvector of the matrix A. The following figure illustrates this when $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$Ae_2 = e_2$
x
Ax
e_1
Ae_1

Note that in this case

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Thus,

Eigenvalue: $\lambda = 3$, Eigenvector: $v_{\lambda=3} = e_1$.

Similarly, if we apply A to the coordinate vector e_2, we have

$$Ae_2 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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Thus,

Eigenvalue: $\lambda = 1$, Eigenvector: $v_{\lambda=1} = e_2$.

Here, we have identified two eigenvalue-eigenvector pairs.

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1. The matrix A has two distinct eigenvalues and each is associated with an eigenvector.
2. The two eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. Hence they are linearly independent and span all of $\mathbb{R}^2$.
3. Each of the two directions is stretched out by a different amount under the action of A.

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Clearly, from this decomposition in view of (3), under the action of A the vector x will move to some other vector on $\mathbb{R}^2$ that is not just a stretch, but instead combines both directions.

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Clearly, from this decomposition in view of (3), under the action of A the vector x will move to some other vector on $\mathbb{R}^2$ that is not just a stretch, but instead combines both directions. So, x is not an eigenvector of A.

Identity matrix

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What this says is that any vector on $\mathbb{R}^2$ is an eigenvector of this A, with eigenvalue 1.

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If we are only considering normalized unit-length vectors, then what this says is that any vector on the L_2 unit ball is an eigenvector of this A, with eigenvalue 1. By the way, we can choose any pair of linearly independent vectors as eigen vectors.

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$$Ae_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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Thus,

Eigenvalue: $\lambda = 1$, Eigenvector: $v_{\lambda=1} = e_1$.

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Similarly, if we apply A to the coordinate vectors e_2, we have

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So, here are two vectors along the two coordinate axes, that are eigenvectors. Both have the same eigenvalue 1. When there are multiple eigenvectors associated with a given eigenvalue, it is common to call that a *degenerate eigenvalue*.

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- For this matrix, there are multiple eigenvectors for a given eigenvalue, which are not scalar multiples of one another. One can choose those 2 eigenvectors to be linearly independent, and even to be orthogonal to each other.
- The span of any two orthogonal eigenvectors of A is $\mathbb{R}^2$.

Reflection matrix

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$$Ax = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

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$$Ax = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

This matrix perfroms a reflection about the line $x_2 = x_1$.

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Let us apply A to the vectors $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$. We have

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Let us apply A to the vectors $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$. We have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 1$, Eigenvector: $v_{\lambda=1} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$.

Q2. *What are examples of eigenvalues/eigenvectors of this matrix?*

Ans.

Let us apply A to the vectors $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$. We have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 1$, Eigenvector: $v_{\lambda=1} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$.

Thus, if one considers a vector in the subspace defined by $x_2 = x_1$, then the vector is unchanged by the matrix.

Similarly, if we apply A to the vector $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$,

Similarly, if we apply A to the vector $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = (-1) \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}.$$

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Thus,
Eigenvalue: $\lambda = -1$,
Eigenvector: $v_{\lambda=1} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$.

Similarly, if we apply A to the vector $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = (-1) \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}.$$

Thus,
Eigenvalue: $\lambda = -1$,
Eigenvector: $v_{\lambda=1} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$.
Thus, if one considers a vector in the subspace perpendicular to $x_2 = x_1$, then the vector is "reflected through" the line $x_2 = x_1$.

So, here are two vectors along the two coordinate axes, that are eigenvectors.

So, here are two vectors along the two coordinate axes, that are eigenvectors. In this case, we can choose the orthonormal vectors $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ as eigen vectors relative to $\lambda = 1$ and $\lambda = -1$ respectively.

Q3. *What does the discussion for this example illustrate?*

Ans.

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- Eigenvalues can be positive or negative.

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Ans.

- Eigenvalues can be positive or negative.
- Associated with each distinct eigenvalue, there is an eigenvector.
- The two eigenvectors are perpendicular to each other.

Projection matrix

Consider the following matrix:

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Projection matrix

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$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Q1. What can we say about the action of A on an arbitrary vector ?

Q2. What are examples of eigenvalues and eigenvectors of this matrix ?

Q3. What does the discussion for this example illustrate?

Q1. *What can we say about the action of A on an arbitrary vector ?*

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Let us apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

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Ans.

Let us apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

We have

$$Ax = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2)/2 \\ (x_1 + x_2)/2 \end{pmatrix}.$$

Q1. *What can we say about the action of A on an arbitrary vector ?*

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We have

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That is, A "projects" the vector x onto the line $x_1 = x_2$.

Q1. *What can we say about the action of A on an arbitrary vector ?*

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Let us apply A to an arbitrary vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

We have

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That is, A "projects" the vector x onto the line $x_1 = x_2$. For example, we have

$$A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

It is illustrated in the following figure.

Q2. *What are examples of eigenvalues/eigenvectors of this matrix?*

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Ans.

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Let us apply A to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have

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Q2. *What are examples of eigenvalues/eigenvectors of this matrix?*

Ans.

Let us apply A to the vector $\begin{pmatrix}1\\1\end{pmatrix}$. We have

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 1$

Eigenvector: $v_{\lambda=1} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$, normalised vector relative to $\begin{pmatrix}1\\1\end{pmatrix}$.

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$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Similarly, if we apply A to the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, we have

$$A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus,

Eigenvalue: $\lambda = 0$

Eigenvector: $v_{\lambda=0} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$, normalised vector relative to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Observe that these are the same eigenvectors as for the reflection matrix considered earlier, but the eigenvalues are different.

Observe that these are the same eigenvectors as for the reflection matrix considered earlier, but the eigenvalues are different. These are also the same eigenvectors as one possible choice for the identity matrix considered earlier, but again the eigenvalues are different.

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Ans.

- Since the two columns are linearly dependent, A is a rank-deficient matrix.

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- Since the two columns are linearly dependent, A is a rank-deficient matrix. Thus, the output must lie in a lower-dimensional subspace, and that subspace is the span of the one linearly-independent column.

Q3. *What does the discussion for this example illustrate?*

Ans.

- Since the two columns are linearly dependent, A is a rank-deficient matrix. Thus, the output must lie in a lower-dimensional subspace, and that subspace is the span of the one linearly-independent column.
- Under the action of A on a vector, the vector is unchanged along one direction, while along the other direction, the vector is shrunk to the origin.

- This is an example of a "projection matrix," and so it really is doing a projection.

- This is an example of a "projection matrix," and so it really is doing a projection. (The reason for this is that the non-zero eigenvalue equals unity – more generally, an $n \times n$ matrix in which all the eigenvalues equal 0 or 1 is a projection matrix onto the subspace spanned by its columns.)

Using determinants to compute eigenvectors and eigenvalues

The examples of the previous section were sufficiently simple that we were able just to write down eigenvalue-eigenvector pairs by inspection, but in general they must be computed with some algorithm.

The examples of the previous section were sufficiently simple that we were able just to write down eigenvalue-eigenvector pairs by inspection, but in general they must be computed with some algorithm. There are several ways to compute eigenvalues and eigenvectors. In this course, we discuss two ways.

- **With determinants.** This approach is most appropriate for small 2×2 and 3×3 matrices, and it can be understood in terms of ideas like linear dependence and independence.

- **With determinants.** This approach is most appropriate for small 2×2 and 3×3 matrices, and it can be understood in terms of ideas like linear dependence and independence.
- **With quadratic forms.** This approach illustrates the basic algebraic and geometric ideas of how to compute eigenvectors and eigenvalues for larger more realistic problems, in a way that conveys the basic ideas more generally.

Basic properties of determinants

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

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The determinant of A is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

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Question: What does the determinant of a matrix geometrically mean?

Answer: The determinant $|A|$ represents the area of the parallelogram spanned by the columns of the matrix A.

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From the figure, we obtain

$$\begin{aligned} &\text{Area of parrallelogram} \\ &= (a_{11} + a_{12})(a_{21} + a_{22}) - a_{12}a_{22} \end{aligned}$$

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$$\begin{aligned} &\text{Area of parrallelogram} \\ &= (a_{11} + a_{12})(a_{21} + a_{22}) - a_{12}a_{22} \\ &\quad - a_{11}a_{21} - 2a_{21}a_{12} \end{aligned}$$

From the figure, we obtain

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From the figure, we obtain

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&\text{Area of parrallelogram}\\
&= (a_{11}+a_{12})(a_{21}+a_{22}) - a_{12}a_{22}\\
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&= a_{11}a_{21} + a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{22}\\
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&= a_{11}a_{21} + a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{22}\\
&\quad - a_{11}a_{22} - a_{11}a_{21} - 2a_{21}a_{12}\\
&= a_{11}a_{22} - a_{21}a_{12}.
\end{aligned}$$

Therfore, by definition,

$$\text{Area of parrallelogram} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

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$$\text{Area of parrallelogram} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Question: What does "the area = 0" mean?

Answer: When $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ is linearly dependent on $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$,

Answer: When $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ is linearly dependent on $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$, i.e., when

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \alpha \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{12} \\ \alpha a_{22} \end{pmatrix},$$

Answer: When $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ is linearly dependent on $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$, i.e., when

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we have

$$a_{11}a_{22} - a_{12}a_{21} = (\alpha a_{12})a_{22} - a_{12}(\alpha a_{22}) = 0.$$

Conversely, if Area $= 0$,

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$$\Rightarrow \quad \begin{cases} a_{11} = ka_{12} \\ a_{21} = ka_{22} \end{cases}$$

Conversely, if Area $= 0$, then

$$
\begin{aligned}
& a_{11}a_{22} - a_{12}a_{21} = 0 \\
\Rightarrow \quad & \frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} \\
\Rightarrow \quad & \begin{cases} a_{11} = ka_{12} \\ a_{21} = ka_{22} \end{cases} \\
\Rightarrow \quad & \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} ka_{12} \\ ka_{22} \end{pmatrix} = k \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}
\end{aligned}
$$

for some number k.

Conversely, if Area $= 0$, then

$$\begin{aligned} & a_{11}a_{22} - a_{12}a_{21} = 0 \\ \Rightarrow \quad & \frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} \\ \Rightarrow \quad & \begin{cases} a_{11} = ka_{12} \\ a_{21} = ka_{22} \end{cases} \\ \Rightarrow \quad & \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} ka_{12} \\ ka_{22} \end{pmatrix} = k \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \end{aligned}$$

for some number k. Hence the columns of A are linearly dependent.

Note that

These ideas can easily be generalized to 3×3 matrices and even to $n \times n$ matrices.

Some properties of determinants

- A square matrix A is invertible if and only if $|A| \neq 0$.

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Some properties of determinants

- A square matrix A is invertible if and only if $|A| \neq 0$.
- Equivalently, a square matrix A is not invertible if and only if $|A| = 0$.
- $|A| = 0$ if and only if the columns/rows of A are linearly dependent.
- If A is triangular, then $|A|$ is the product of the entries on the main diagonal of A.

1. Recall that A is invertible iff $|A| \neq 0$.

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1. Recall that A is invertible iff $|A| \neq 0$.
2. Equivalently, A is not invertible iff $|A| = 0$.
3. We know that A is invertible iff the equation $Ax = 0$ has only the solution $x = 0$ (trivial solution).
4. Equivalently, A is not invertible iff there is a nonzero vector x such that $Ax = 0$.
5. From (2) and (4), we see that there is a nonzero vector x such that $Ax = 0$ iff $|A| = 0$.

Thus,

There is a nonzero vector x such that $(A - \lambda I)x = 0$ iff $|A - \lambda I| = 0$.

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There is a nonzero vector x such that $(A - \lambda I)x = 0$ iff $|A - \lambda I| = 0$.

From this we get a method for finding eigenvalues of a matrix A.

Method for finding eigenvalues

Solve the equation $|A - \lambda I| = 0$ for eigenvalues λ.

Example

Find the eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

Solution:

We must find all scalars λ such that the matrix equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution.

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We must find all scalars λ such that the matrix equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. This problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is not invertible, i.e., $|A - \lambda I| = 0$.

We have

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix}$$

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$$= (2-\lambda)(-6-\lambda) - 3 \times 3$$

We have

$$
\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} \\
&= (2-\lambda)(-6-\lambda) - 3 \times 3 \\
&= \lambda^2 + 4\lambda - 21
\end{aligned}
$$

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&= (\lambda - 3)(\lambda + 7).
\end{aligned}
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\end{aligned}
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If $|A - \lambda I| = 0$, then

$$\lambda = 3 \text{ or } \lambda = -7.$$

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|A - \lambda I| &= \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} \\
&= (2-\lambda)(-6-\lambda) - 3 \times 3 \\
&= \lambda^2 + 4\lambda - 21 \\
&= (\lambda - 3)(\lambda + 7).
\end{aligned}
$$

If $|A - \lambda I| = 0$, then

$$\lambda = 3 \text{ or } \lambda = -7.$$

Therefore, the eigenvalues of A are 3 and -7.

Characteristic equation

The equation

$$|A - \lambda I| = 0$$

is called the **characteristic equation** for the equation $A = \lambda x$.

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Eigenvalue and characteristic equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$|A - \lambda I| = 0.$$

Eigenspace

λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. The set of all solutions of this equation together with the *zero vector* is a subspace of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to the eigenvalue λ.

Example.

Find the eigenvalues of $A = \begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

We have

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

We have

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$
$$= (5-\lambda)^2(3-\lambda)(1-\lambda).$$

If $|A - \lambda I| = 0$, then

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

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If $|A - \lambda I| = 0$, then

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 5, 3, 1.$$

Therefore, the eigenvalues of A are 5, 3, and 1.

In the last two examples , $|A - \lambda I|$ is a polynomial in λ.

In the last two examples , $|A - \lambda I|$ is a polynomial in λ. It can be shown that if A is an $n \times n$ matrix, then $|A - \lambda I|$ is a polynomial of degree n called the **characteristic polynomial** of A.

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The eigenvalue 5 in the last example is said to have **multiplicity 2**

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The eigenvalue 5 in the last example is said to have **multiplicity 2** because $5 - \lambda$ occurs two times as a factor of the characteristic polynomial.

In general, the (algebraic) multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation.

Each eigenvalue determines the corresponding eigenvector x by solving the system

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Solution:

We have found that the eigenvalues of A are 3 and -7.

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We have found that the eigenvalues of A are 3 and -7.

$\underline{\lambda = 3}$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = 3$. Then

$$(A - \lambda I)x = 0$$

Solution:

We have found that the eigenvalues of A are 3 and -7.

$\underline{\lambda = 3}$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = 3$. Then

$$(A - \lambda I)x = 0 \Rightarrow \left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Solution:

We have found that the eigenvalues of A are 3 and -7.

$\underline{\lambda = 3}$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = 3$. Then

$$(A - \lambda I)x = 0 \Rightarrow \left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Therefore, we obtain a system of equations

$$\begin{cases} -x_1 + 3x_2 = 0 \\ 3x_1 - 9x_2 = 0 \end{cases}.$$

Therefore, we obtain a system of equations

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However, both the equations represent the same equation:

$$-x_1 + 3x_2 = 0.$$

Therefore, we obtain a system of equations

$$\begin{cases} -x_1 + 3x_2 = 0 \\ 3x_1 - 9x_2 = 0 \end{cases}.$$

However, both the equations represent the same equation:

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The simplest solution is $x_1 = 3$ and $x_2 = 1$. Therefore, an eigenvector relative to the eigenvalue $\lambda = 3$ is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

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Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to $\lambda = -7$. Then

$$(A - \lambda I)x = 0 \Rightarrow \left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} + 7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

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$$\Rightarrow \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Therefore, we obtain a system of equations

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However, both the equations represent the same equation:

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The simplest solution is $x_1 = -1$ and $x_2 = 3$. Therefore, an eigenvector relative to the eigenvalue $\lambda = -7$ is $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Procedure for determining eigenvalues and eigenvectors

1. Solve the characteristic equation
$$|A - \lambda I| = 0$$
for the scalar λ.
2. For each eigenvalue λ determine the corresponding eigenvector x by solving the system
$$(A - \lambda I)x = 0.$$

Example

Determine the eigenvalues and corresponding eigenvectors of $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. Also sketch the effect of multiplying the eigenvectors by matrix A.

Solution:
We must find all scalars λ such that the matrix equation

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has a nontrivial solution. This problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is not invertible, i.e., $|A - \lambda I| = 0$.

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If $|A - \lambda I| = 0$, then

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Therefore, the eigenvalues of A are 2 and 3.

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$$(A - \lambda I)x = 0 \Rightarrow \left(\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

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$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Therefore, we obtain a system of equations

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Let $x_1 = s$ then $x_2 = -s$ where $s \neq 0$ and is a real number. Thus the eigenvectors belonging to $\lambda = 2$ are:

$$x = \begin{pmatrix} s \\ -s \end{pmatrix} \quad \text{where } s \neq 0.$$

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$$\Rightarrow \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

Therefore, we obtain a system of equations

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From these equations we must have $y_1 = 0$. We can choose y_2 to be any real number apart from zero because the eigenvector cannot be zero.

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$$y_2 = t, \quad \text{where } t \neq 0.$$

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From these equations we must have $y_1 = 0$. We can choose y_2 to be any real number apart from zero because the eigenvector cannot be zero. Thus, let

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Therefore, the general eigenvector belonging to $\lambda = 3$ is

$$\begin{pmatrix} 0 \\ t \end{pmatrix}.$$

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Plotting these eigenvectors, the effect of multiplying by the matrix A is shown in the figure given below.

Matrix A doubles ($\lambda = 2$) the eigenvector x and triples ($\lambda = 3$) the eigenvector y as you can see in the above figure. Matrix A does not change the direction of the eigenvectors.