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Q1.

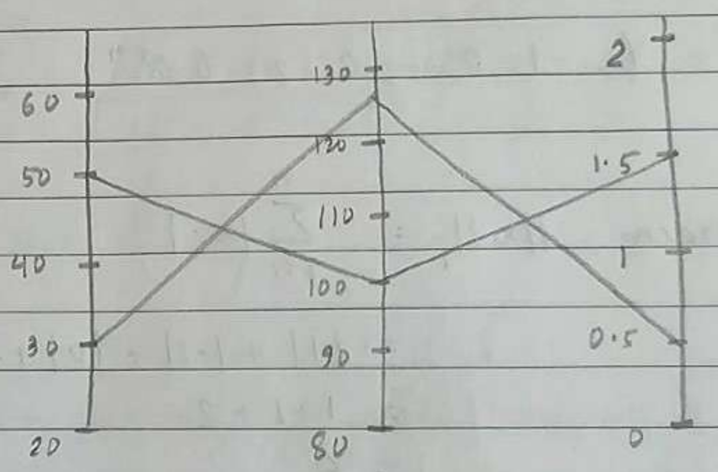
Parallel co-ordinate method is a method to visualize a set of points in n-dimension. In order to do so, equally spaced vertical lines are drawn in which points are represented as broken lines with vertices on the parallel axes.

For example:

if $x = (30, 115, 0.5)$,

$y = (50, 100, 1.5)$ be two points $x, y \in \mathbb{R}^3$.

Then, using parallel co-ordinate axis, we plot them as follows:



Usage in data science

- Used for multivariate numerical data
- Ideal for comparing many variables together and see relationship between them.

Q2.

L_1-norm on $\mathbb{R}^n$: If $\|\cdot\|$ represents a mapping from $\mathbb{R}^n \to \mathbb{R}$, then L_1-norm, represented by $\|x\|_1$ is defined as

$$\|X\|_1 = \sum_{i=1}^{n} |x_i|$$

where, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.

l_2-norm on $\mathbb{R}^n$: The l_2-norm, represented by $\|x\|_2$, is defined as

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

where, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

l_∞-norm on $\mathbb{R}^n$: The l_∞-norm, represented by $\|x\|_\infty$ is defined as

$$\|x\|_\infty = \max_{i \in \{1,2,\dots,n\}} |x_i|$$

where, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

Given, $x = (1, -1, 0, \dots, 0, 2) \in \mathbb{R}^n$.

Then,

$$l_1\text{-norm}, \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$= |1| + |-1| + |0| + \dots + |0| + |2|$$

$$= 1 + 1 + 2$$

$$= 4$$

$$l_2\text{-norm}, \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$$= \left((1)^2 + (-1)^2 + (0)^2 + \dots + (0)^2 + (2)^2\right)^{1/2}$$

$$= (6)^{1/2}$$

$$= 2.45$$

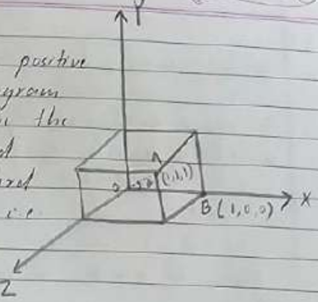
$$l_\infty\text{-norm}, \|x\|_\infty = \max_{i \in \{1,2,\dots,n\}} |x_i|$$

$$= \max\{|1|, |-1|, |0|, \dots, |0|, |2|\}$$

$$= 2$$

Q3. Consider a unit cube in positive orthant as shown in diagram alongside. Let $OA\ (1,1,1)$ be the diagonal of the cube and $OB\ (1,0,0)$ be the standard basis vector along x-axis i.e. e_1.

Angle between OA & OB = ?



Solution

$$\|OA\|_2 = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$\|OB\|_2 = \sqrt{(1)^2 + 0^2 + 0^2} = 1$$

$$OA \cdot OB = \sum_{i=1}^{3} x_i y_i$$

$$= 1\times1 + 1\times0 + 1\times0 = 1$$

From the property of dot product,

$$\cos\theta = \frac{OA \cdot OB}{\|OA\|_2 \|OB\|_2}$$

$$= \frac{1}{1\times\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}(1/\sqrt{3})$$

$$\therefore \theta = \cos^{-1}(1/\sqrt{3})$$

$\therefore$ The angle between diagonal and e_1 is $\cos^{-1}(1/\sqrt{3})$.

Q4

Given, $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

To prove orthogonality, let us find the dot product.

$u \cdot v = u_1 v_1 + u_2 v_2$
$= -1 \times -1 + 1 \times -1$
$= 1 - 1$
$= 0$

Since, $u \cdot v = 0$, u and v are orthogonal to each other.

To find orthonormal basis, we need to divide each vector by its corresponding L_2-norm.

Then,

$$\hat{u} = \frac{1}{\|u\|} u = \frac{1}{((-1)^2 + 1^2)^{1/2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\hat{v} = \frac{1}{\|v\|} v = \frac{1}{\sqrt{(-1)^2 + (-1)^2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Q5

Given eigen vectors v_1 and v_2 correspond to distinct eigenvalues λ_1 and λ_2 of matrix A.

Let us assume,

$$c_1 v_1 + c_2 v_2 = 0 \quad \dots (1)$$

Multiplying eqn (1) by A, we get:

$$c_1 A v_1 + c_2 A v_2 = 0$$

$$\Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \quad \dots (2) \quad \left[\because \begin{array}{l} A v_1 = \lambda v_1 \\ A v_2 = \lambda v_2 \end{array} \right]$$

Multiplying eqn (1) by λ_1, we get:

$$c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = 0 \quad \dots (3)$$

Subtracting eqn (3) from (2), we get:

$$c_2 (\lambda_1 - \lambda_2) v_2 = 0$$

Since, $\lambda_1 \neq \lambda_2$, $v_2 \neq 0$, we must have $c_2 = 0$

Similarly, we can show that $c_1 = 0$

Thus, eigenvectors v_1 and v_2 are linearly independent.

Q6.

a.

S

Domain $= \mathbb{R}^3$
Co-domain $= \mathbb{R}^2$

T

Domain $= \mathbb{R}^2$
Co-domain $= \mathbb{R}^3$

$$(S \circ T)(x) = S(T(x)) = S(Bx) = ABx$$

Since, co-domain of T and domain of S are equal, the composite transformation $S \circ T$ is defined.

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad S: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

Then,

$S \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Domain of $S \circ T = \mathbb{R}^2$

Co-domain of $S \circ T = \mathbb{R}^2$

b. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Determine $T(x)$

Solution

$$T(x) = Bx$$
$$= \begin{pmatrix} 3 & 0 \\ 5 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 3x_1 \\ 5x_1 - 2x_2 \\ x_2 \end{pmatrix}$$

c.

$$(S \circ T)(x) = S(T(x))$$
$$= S(Bx)$$
$$= ABx$$
$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 5 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3x_1 \\ 5x_1 - 2x_2 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x_1 + 10x_1 - 4x_2 \\ 5x_1 - 2x_2 + x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 13x_1 - 4x_2 \\ 5x_1 - x_2 \end{pmatrix}$$

d.

We know,

$(S \circ T)(x) = ABx$

$\Rightarrow Cx = ABx$

$\therefore C = AB$

Then,

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 5 & -2 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1\times3 + 2\times5 + 0\times0 & 1\times0 + 2\times-2 + 0\times1 \\ 0\times3 + 1\times5 + 1\times0 & 0\times0 + 1\times-2 + 1\times1 \end{pmatrix}$$
$$= \begin{pmatrix} 13 & -4 \\ 5 & -1 \end{pmatrix}$$

e.

A transformation T is said to be linear if

$T(x+y) = T(x) + T(y)$

$T(\alpha x) = \alpha\, T(x)$

where, 'α' is a scalar and x, y are matrices

Let a, b be two scalars and x, y be two matrices. Then,

$(S \circ T)(ax + by) = S(T(ax + by))$

$= S(B(ax+by))$ $[\because T(x) = Bx]$
$= S(aBx + bBy)$
$= A(aBx + bBy)$ $[\because S(y) = Ay]$
$= aABx + bABy$

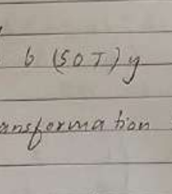
Put $AB = C$ where C represents the transformation $(S \circ T)$ i.e. $(S \circ T)(x) = Cx$, we get:

$$(S \circ T)(ax+by) = aCx + bCy$$
$$= a(S \circ T)x + b(S \circ T)y$$

This proves linearity of transformation $S \circ T$.

Q7.

Let $x, y \in \mathbb{R}^n$ and θ be the angle between them. Then, by the law of cosines,



$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta \quad \dots (1)$$

Now,
$$\|x-y\|^2 = (x-y)\cdot(x-y)$$
$$= x\cdot x - x\cdot y - y\cdot x - y\cdot y$$
$$= (x\cdot x) + (y\cdot y) - 2(x\cdot y)$$
$$= \|x\|^2 + \|y\|^2 - 2(x\cdot y) \quad \dots (2)$$

Comparing eq^n (1) & (2), we get:
$$x\cdot y = \|x\|\|y\|\cos\theta$$

Q8.

Here,
$$B \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Now,
$$1 + (-1) + 0 = 0$$
$$\& \quad 1 + 0 + (-1) = 0$$

Both points of B lie in V i.e. $B \in V$

Also, the two points of B are linearly independent.

Thus, B is a basis for V

To prove V is a subspace of $\mathbb{R}^3$. Let $V = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a point in $\mathbb{R}^3$.

Then,
$$x_1 + x_2 + x_3 = 0$$
$$\therefore x_1 = -x_2 - x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in \text{span}\{[-1, 1, 0], (-1, 0, 1)\}$$

$\subset \mathbb{R}^3$

$\therefore V$ is subspace of $\mathbb{R}^3$.

Q3.

Here,
$$A = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}$$

To find eigenvalues:

$$|A - \lambda I| = 0$$
$$\Rightarrow \left| \begin{pmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{pmatrix} \right| = 0$$
$$\Rightarrow (5-\lambda)(-4-\lambda) + 14 = 0$$
$$\Rightarrow -20 - 5\lambda + 4\lambda + \lambda^2 + 14 = 0$$
$$\Rightarrow \lambda^2 - \lambda - 6 = 0$$
$$\Rightarrow \lambda^2 - 3\lambda + 2\lambda - 6 = 0$$
$$\Rightarrow \lambda(\lambda - 3) + 2(\lambda - 3) = 0$$
$$\Rightarrow (\lambda+2)(\lambda-3) = 0$$
$$\therefore \lambda = -2, \lambda = 3$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be eigenvector. Then,

$$(A - \lambda I)x = 0$$
$$\Rightarrow \begin{pmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \ldots (1)$$

Put $\lambda = -2$ in (1), we get:

$$\begin{pmatrix} 7 & -2 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\Rightarrow \begin{pmatrix} 7x_1 - 2x_2 \\ 7x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above,

$$7x_1 - 2x_2 = 0$$
$$\Rightarrow x_1 = \frac{2}{7} x_2$$

Put $x_2 = t \in \mathbb{R}$, then,
$$x_1 = \frac{2}{7} t$$

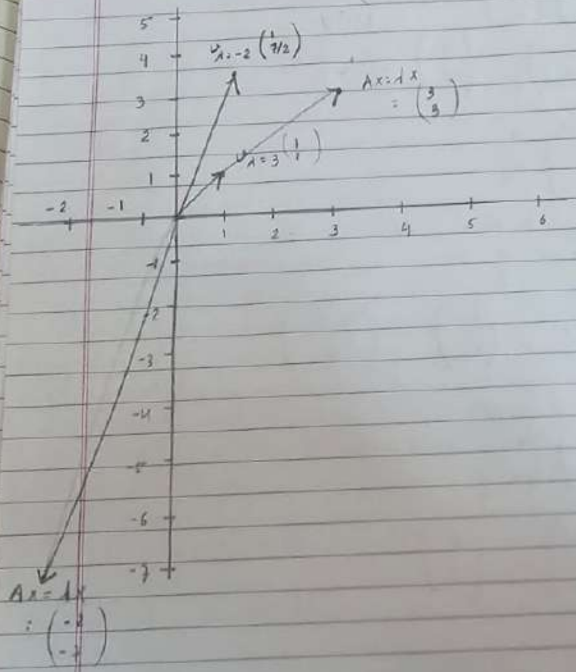
$$\therefore v_{\lambda = -2} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/7\, t \\ t \end{pmatrix}$$

Put $\lambda = 3$ in (1),
$$\begin{pmatrix} 2 & -2 \\ 7 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2x_1 - 2x_2 \\ 7x_1 - 7x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From above,
$$x_1 = x_2$$
Put, $x_2 = s \in \mathbb{R}$, then,
$$x_1 = s$$

$$\therefore v_{\lambda = 3} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$



Q10.

Given,

a. $Ax = \lambda x \quad \dots (i)$

Multiplying eq^n (i) by A, we get:

$A\,Ax = A(\lambda x)$

$\Rightarrow A^2 x = \lambda (Ax)$

$\Rightarrow A^2 x = \lambda (\lambda x) \quad$ (From (i), $Ax = \lambda x$)

$\Rightarrow A^2 x = \lambda^2 x \quad \dots (ii)$

Performing $(m-2)$ times multiplication of matrix A on both sides of eq^n (ii), we get

$A^m x = \lambda^m x \quad \dots (iii)$

Equation (iii) prove that λ^m is the eigenvalue of matrix A^m and corresponding eigenvector is x.

b. Here, matrix A is invertible i.e. A^{-1} exists

From (i), we have:

$Ax = \lambda x$

Multiplying both sides by A^{-1}, we get

$A^{-1} A x = A^{-1} \lambda x$

$\Rightarrow I x = A^{-1} \lambda x$

$\Rightarrow x = A^{-1} \lambda x$

$\Rightarrow A^{-1} x = \dfrac{1}{\lambda} x$

$\therefore A^{-1} x = \lambda^{-1} x \quad \dots (iv)$

Equation (iv) is in the form of (i), in which the eigen value of matrix A^{-1} is $1/\lambda$ (or λ^{-1}) and eigenvector is x.