ASSIGNMENT - III

Problems on partial derivatives

I. Compute the first-order partial derivatives.

1. $z = x^2 + y^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

2. $z = x^4 y^3$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^4 y^3) = y^3 \frac{\partial}{\partial x}(x^4) = y^3 . 4x^3 = 4x^3 y^3$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^4 y^3) = x^4 \frac{\partial}{\partial y}(y^3) = x^4 . 3y^2 = 3x^4 y^2$$

3. $z = x^4 y + x y^{-2}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^4 y + x y^{-2}) = \frac{\partial}{\partial x}(x^4 y) + \frac{\partial}{\partial x}(x y^{-2})$$

$$= y \cdot \frac{\partial}{\partial x}(x^4) + y^{-2} \frac{\partial (x)}{\partial x}$$

$$= 4x^3 y + y^{-2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^4 y + x y^{-2}) = \frac{\partial}{\partial y}(x^4 y) + \frac{\partial}{\partial y}(x y^{-2})$$

$$= x^4 \frac{\partial y}{\partial y} + x \cdot \frac{\partial}{\partial y}(y^{-2})$$

$$= x^4 + x . -2y^{-2-1}$$

$$= x^4 + (-2xy^{-3})$$

$$= x^4 - \frac{2x}{y^3}.$$

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4. $V = \pi r^2 h$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}(\pi r^2 h)$$

$$= \pi h \frac{\partial (r^2)}{\partial r}$$

$$= \pi h \cdot 2r$$

$$= 2\pi r h$$

$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}(\pi r^2 h)$$

$$= \pi r^2 \frac{\partial (h)}{\partial h}$$

$$= \pi r^2$$

5. $z = \frac{x}{x-y}$

Quotient rule: $\frac{d(u/v)}{dx} = \frac{v \cdot du/dx - u \cdot dv/dx}{v^2}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x-y}\right)$$

$$= \frac{(x-y) \cdot \frac{\partial x}{\partial x} - x \cdot \frac{\partial}{\partial x}(x-y)}{(x-y)^2}$$

$$= \frac{(x-y) \cdot 1 - x \cdot (1-0)}{(x-y)^2}$$

$$= \frac{x-y-x}{(x-y)^2}$$

$$= \frac{-y}{(x-y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x-y}\right)$$

$$= \frac{(x-y) \cdot \frac{\partial x}{\partial y} - x \cdot \frac{\partial}{\partial y}(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)\times 0 - x\cdot(0-1)}{(x-y)^2}$$

$$= \frac{x}{(x-y)^2}$$

6. $z = \dfrac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\sqrt{x^2+y^2}\cdot\frac{\partial x}{\partial x} - x\cdot\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right)}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{\sqrt{x^2+y^2}\cdot 1 - x * \frac{1}{2}(x^2+y^2)^{1/2-1}\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)}$$

$$= \frac{\sqrt{x^2+y^2} - \frac{x}{2\sqrt{x^2+y^2}}\cdot(2x+0)}{(x^2+y^2)}$$

$$= \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{\frac{(x^2+y^2) - x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{y^2}{\sqrt{x^2+y^2}} \times \frac{1}{(x^2+y^2)}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\sqrt{x^2+y^2} \cdot \frac{\partial x}{\partial y} - x \cdot \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right)}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{0 - x \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)}$$

$$= \frac{\frac{-x}{2\sqrt{x^2+y^2}} \cdot 2y}{(x^2+y^2)}$$

$$= \frac{-2xy}{2(x^2+y^2)^{1/2} * (x^2+y^2)}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{-xy}{(x^2+y^2)^{3/2}}$$

7. $S = \tan^{-1}(wz)$

$$\frac{\partial s}{\partial w} = \frac{\partial}{\partial w}\left(\tan^{-1}(wz)\right)$$

$$= \frac{1}{1+(wz)^2} \cdot \frac{\partial}{\partial w}(wz)$$

$$\therefore \frac{\partial s}{\partial w} = \frac{z}{1+w^2z^2}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(\tan^{-1}(wz)\right)$$

$$= \frac{1}{1+(wz)^2} \cdot \frac{\partial}{\partial z}(wz)$$

$$\therefore \frac{\partial s}{\partial z} = \frac{w}{1+w^2z^2}$$

8. $Q = re^{\theta}$

$$\frac{\partial Q}{\partial r} = \frac{\partial}{\partial r}(re^{\theta})$$

$$= e^{\theta} \cdot \frac{\partial r}{\partial r}$$

$$\therefore \frac{\partial Q}{\partial r} = e^{\theta}$$

$$\frac{\partial Q}{\partial \theta} = \frac{\partial}{\partial \theta}(re^{\theta})$$

$$= r\frac{\partial}{\partial \theta}(e^{\theta})$$

$$\therefore \frac{\partial Q}{\partial \theta} = r.e^{\theta}$$

9. $z = \ln(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\ln(x^2+y^2))$$

$$= \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} (2x+0)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\ln(x^2+y^2))$$

$$= \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot (0+2y)$$

$$\therefore \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$

10. $z = y^x$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(y^x)$

$\therefore \frac{\partial z}{\partial x} = y^x \ln(y)$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(y^x)$

$\therefore \frac{\partial z}{\partial y} = x \cdot y^{x-1}$

$\frac{d}{dx}(a^x) = a^x \ln(a)$

11. Show that the following functions satisfy the Laplace equation $u_{xx} + u_{yy} = 0$.

a. $u(x,y) = e^x \cos y$

$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^x \cos y)$

$= \cos y \frac{\partial}{\partial x}(e^x)$

$= e^x \cos y$

$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}(e^x \cos y)$

$= \cos y \frac{\partial}{\partial x}(e^x)$

$= e^x \cos y$

Now,

$u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y)$

$= e^x \frac{\partial}{\partial y}(\cos y)$

$= e^x(-\sin y)$

$= -e^x \sin y$

$$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}(-e^x \sin y)$$

$$= -e^x \frac{\partial}{\partial y}(\sin y)$$

$$= -e^x \cos y.$$

$$\therefore u_{xx} + u_{yy} = e^x \cos y - e^x \cos y$$

$$= 0 \quad \underline{\text{proved}}$$

b. $u(x,y) = \tan^{-1} yx$

Here,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\tan^{-1} yx)$$

$$= \frac{1}{1+(yx)^2} \cdot \frac{\partial}{\partial x}(yx)$$

$$= \frac{1}{1+x^2y^2} \cdot y \cdot 1$$

$$= \frac{y}{1+x^2y^2}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{1+x^2y^2}\right)$$

$$= \frac{(1+x^2y^2)\cdot \partial y/\partial x - y \cdot \frac{\partial}{\partial x}(1+x^2y^2)}{(1+x^2y^2)^2}$$

$$= \frac{(1+x^2y^2)\times 0 - y(0+y^2 \cdot 2x)}{(1+x^2y^2)^2}$$

$$= \frac{-2xy^3}{(1+x^2y^2)^2}$$

Now, $u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\tan^{-1} yx)$

$$= \frac{1}{1+(yx)^2} \cdot \frac{\partial}{\partial y}(yx) = \frac{x}{1+x^2y^2}$$

Now,

$$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{1+x^2y^2}\right)$$

$$= \frac{(1+x^2y^2)\cdot\frac{\partial x}{\partial y} - x\cdot\frac{\partial}{\partial y}(1+x^2y^2)}{(1+x^2y^2)^2}$$

$$= \frac{(1+x^2y^2)*0 - x\,(0+2x^2y^1)}{(1+x^2y^2)^2}$$

$$= \frac{-2x^3y}{(1+x^2y^2)^2}$$

c $u(x,y) = \ln(x^2+y^2)$

Here,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\ln(x^2+y^2)\right)$$

$$= \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{2x}{x^2+y^2}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\left(2x/x^2+y^2\right)$$

$$= \frac{(x^2+y^2)\cdot\frac{\partial(2x)}{\partial x} - 2x\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2) - 2x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

Now,

$$u_y = \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial y}\left(\ln (x^2+y^2)\right)$$

$$= \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{2y}{x^2+y^2}$$

$$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}\left(2y/x^2+y^2\right)$$

$$= \frac{(x^2+y^2)\cdot \frac{\partial (2y)}{\partial y} - (2y)\cdot \frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2) - 2y \cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\therefore u_{xx} + u_{yy} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= 0$$

$\therefore$ proved

III. Find all constants a, b such that $u(x,y) = \cos(ax)\, e^{by}$ satisfies the Laplace equation $u_{xx} + u_{yy} = 0$

Solution

Here, $u(x,y) = \cos(ax)\, e^{by}$

Now, $u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\cos(ax)\, e^{by})$

$= e^{by} \frac{d}{dx}(\cos(ax))$

$= e^{by}(-\sin ax) \cdot \frac{\partial}{\partial x}(ax)$

$= -a e^{by} \sin(ax)$

$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}(-a e^{by} \sin(ax))$

$= -a e^{by} \frac{\partial}{\partial x}(\sin(ax))$

$= -a e^{by} \cos(ax) \cdot \frac{\partial}{\partial x}(ax)$

$= -a^2 e^{by} \cos(ax)$

$u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\cos(ax)\, e^{by})$

$= \cos(ax) \frac{\partial}{\partial y}(e^{by})$

$= \cos(ax) \cdot e^{by} \frac{\partial (by)}{\partial y}$

$= b e^{by} \cos(ax)$

$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}(b e^{by} \cos(ax))$

$= b\cos(ax) \frac{\partial}{\partial y}(e^{by}) = b^2 e^{by} \cos(ax)$

Since, the equation satisfies Laplace equation, we have

$$u_{xx} + u_{yy} = 0$$

$$\Rightarrow -a^2 e^{by} \cos(ax) + b^2 e^{by} \cos ax = 0$$

$$\Rightarrow e^{by} \cos(ax) \{-a^2 + b^2\} = 0$$

$$\Rightarrow e^{by} \cos(ax) (b^2 - a^2) = 0$$

Either,

$b^2 - a^2 = 0$

$\therefore b = \pm a$

where $a \in R$

OR,

$e^{by} \cos(ax) = 0$

$\Rightarrow \cos(ax) = 0 \quad (\because e^{by} \neq 0)$

$\Rightarrow \cos ax = \cos\left(\frac{\pi}{2} + n\pi\right)$ where, $n \in W$

$\Rightarrow ax = \frac{\pi}{2} + n\pi$

$\Rightarrow ax = (2n+1)\frac{\pi}{2}$

$\therefore a = \frac{(2n+1)\pi}{2x}$

$\therefore b = \pm a$, where $a \in R$

or

$a = \frac{(2n+1)\pi}{2x}$ where $n \in W$

is the required value for the given equation to satisfy the Laplace equation.

IV. Show that the following functions satisfy the heat equation $u_t = u_{xx}$ for $t > 0$ and all x:

a. $z = e^{-t} \sin(x/c)$

Proof

$$\frac{dz}{dt} = \frac{d}{dt}\left[e^{-t}(\sin(x/c))\right]$$

$$= \sin(x/c) \cdot e^{-t} \cdot \frac{d(-t)}{dt}$$

$$\therefore u_t = -\sin(x/c)\, e^{-t}.$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(e^{-t}\sin(x/c)\right)$$

$$= e^{-t}\frac{\partial}{\partial x}\left(\sin(x/c)\right)$$

$$= e^{-t}\cos(x/c)\cdot\frac{\partial}{\partial x}(x/c)$$

$$= \frac{1}{c}e^{-t}\cos(x/c).$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(1/c\, e^{-t}\cos(x/c)\right).$$

$$= \frac{1}{c}e^{-t}\frac{\partial}{\partial x}\left(\cos(x/c)\right)$$

$$= \frac{1}{c}e^{-t}(-\sin x/c) * \frac{1}{c}$$

$$\therefore u_{xx} = \frac{-1}{c^2}e^{-t}\sin x/c$$

Now, we know the heat equation as

$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ where c is a constant

Thus,

$$-\sin\left(\frac{x}{c}\right)\cdot e^{-t} = c^2 * \left(-\frac{1}{c^2}\right)e^{-t}\sin(x/c)$$

$$\therefore -e^{-t}\sin\left(\frac{x}{c}\right) = -e^{-t}\sin(x/c)$$

proved.

Thus, heat equation is satisfied for all $t>0$.

b. $z = e^{-t}\cos(x/c)$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}\left(e^{-t}\cos(x/c)\right)$$

$$= \cos(x/c)\cdot e^{-t}\cdot(-1)$$

$$= -e^{-t}\cos(x/c)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(e^{-t}\cos x/c\right)$$

$$= e^{-t}\left[-\sin(x/c)\right] * \frac{1}{c}$$

$$= -\frac{1}{c}e^{-t}\sin\left(\frac{x}{c}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial z}\left[-\frac{1}{c}e^{-t}\sin(x/c)\right]$$

$$= -\frac{1}{c}e^{-t}\cos\left(\frac{x}{c}\right) * \frac{1}{c}$$

$$= -\frac{1}{c^2}e^{-t}\cos\left(\frac{x}{c}\right)$$

Now, we know heat equation to be

$$\frac{\partial z}{\partial t} = c^2\frac{\partial^2 z}{\partial x^2} \quad \text{where } c \text{ is a constant}$$

$$\Rightarrow -e^{-t}\cos\left(\frac{x}{c}\right) = c^2 * \left(-\frac{1}{c^2}\right)e^{-t}\cos(x/c)$$

$$\therefore -e^{-t}\cos(x/c) = -e^{-t}\cos(x/c)$$

Thus, heat equation is satisfied for all $t > 0$.

c. $$z = \frac{1}{2\sqrt{\pi t}}e^{-x^2/(4t)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2\sqrt{\pi t}}e^{-x^2/(4t)}\right)$$

$$= \frac{1}{2\sqrt{\pi t}}\frac{\partial}{\partial t}\left(e^{-x^2/4t}\right) + e^{-x^2/4t}\frac{\partial}{\partial t}\left(\frac{1}{2\sqrt{\pi t}}\right)$$

$$= \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}\cdot\frac{\partial}{\partial t}\left(-x^2/4t\right) + e^{-x^2/4t}\cdot\frac{1}{2\sqrt{\pi}}\cdot\left(\frac{-1}{2}\right)t^{-1/2-1}$$

$$= \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}\cdot\left(-\frac{x^2}{4}\right)\cdot(-1)*t^{-2} + e^{-x^2/4t}\cdot\left(\frac{-1}{4\sqrt{\pi}}\right)t^{-3/2}$$

$$= \frac{x^2 e^{-x^2/4t}}{8\sqrt{\pi t}\cdot t^2} - \frac{e^{-x^2/4t}}{4\sqrt{\pi}\, t\sqrt{t}} = \frac{e^{-x^2/4t}}{4t\sqrt{\pi t}}\left(\frac{x^2}{2t} - 1\right)$$

$$= \frac{e^{-x^2/4t}}{4\sqrt{\pi t}\cdot t}\left(\frac{x^2 - 2t}{2t}\right)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{\pi t}}\, e^{-x^2/(4t)}\right)$$

$$= \frac{1}{2\sqrt{\pi t}} \cdot e^{-x^2/4t} \cdot \frac{\partial}{\partial x}\left(-x^2/4t\right)$$

$$= \frac{-1}{4t\, 2\sqrt{\pi t}} \cdot e^{-x^2/4t} \cdot (-2x)$$

$$= \frac{x\, e^{-x^2/4t}}{4t\sqrt{\pi t}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x\, e^{-x^2/4t}}{4t\sqrt{\pi t}}\right)$$

$$= \frac{1}{4t\sqrt{\pi t}} \frac{\partial}{\partial x}\left(x\, e^{-x^2/4t}\right)$$

$$= \frac{1}{4t\sqrt{\pi t}} \left\{ x \frac{\partial}{\partial x} \cancel{x}\left(e^{-x^2/4t}\right) + e^{-x^2/4t} \cdot \frac{\partial x}{\partial x} \right\}$$

$$= \frac{1}{4t\sqrt{\pi t}} \left\{ x \cdot e^{-x^2/4t} \cdot \frac{(-2x)}{4t} + e^{-x^2/4t} \cdot 1 \right\}$$

$$= \frac{e^{-x^2/4t}}{4t\sqrt{\pi t}} \left(-\frac{2x^2}{4t} + 1\right)$$

$$= \frac{e^{-x^2/4t}}{4t\sqrt{\pi t}} \left(\frac{-2x^2 + 4t}{4t}\right)$$

$$= \frac{e^{-x^2/4t}}{4t\sqrt{\pi t}} \left(\frac{-x^2 + 2t}{2t}\right)$$

$$= -\frac{e^{-x^2/4t}}{4t\sqrt{\pi t}} \left(\frac{x^2 - 2t}{2t}\right)$$

Now,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \cancel{\frac{e^{-x^2/4t}}{4t\sqrt{\pi t}}} = \cancel{c^2} \cancel{\frac{e^{-x^2/4t}}{4t\sqrt{\pi t}}}$$

$$\Rightarrow \frac{e^{-x^2/4t}}{4t\sqrt{\pi t}}\left(\frac{x^2-2t}{2t}\right) = c^2\left[-\frac{e^{-x^2/4t}}{4t\sqrt{\pi t}}\left(\frac{x^2-2t}{2t}\right)\right]$$

$\therefore$ The heat equation is satisfied for $c^2 = -1$

i.e. $c = i$.

V. a) Show that a function of the form $u(x, t) = f(x+ct)$ satisfies the wave equation.

We know the wave equation to be

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here given,

$$u(x, t) = f(x+ct)$$

Now,
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(f(x+ct))$$
$$= f'(x+ct) \cdot \frac{\partial}{\partial t}(x+ct)$$
$$= c f'(x+ct)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(c f'(x+ct))$$
$$= c \frac{\partial}{\partial t}(f'(x+ct))$$
$$= c \cdot f''(x+ct) \frac{\partial}{\partial t}(x+ct)$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 f''(x+ct) \quad \cdots (i)$$

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$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(f(x+ct))$$

$$= f'(x+ct)\cdot\frac{\partial}{\partial x}(x+ct)$$

$$= f'(x+ct)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(f'(x+ct))$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = f''(x+ct) \quad \cdots (ii)$$

Now from (i), we have,

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x+ct)$$

$$= c^2 \frac{\partial^2 u}{\partial x^2} \quad [\text{from } (ii)]$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{proved.}$$

b) Show that a function of the form $u(x,t) = g(x-ct)$ satisfies the wave equation.

We know the wave equation to be.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here, $u(x,t) = g(x-ct)$

Thus, $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(g(x-ct))$

$$= g'(x-ct)\,\frac{\partial}{\partial t}(x-ct)$$

$$= g'(x-ct)\cdot(0-c)$$

$$= -c\,g'(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(-c\, g'(x-ct))$$

$$= -c\, g''(x-ct)\, \frac{\partial}{\partial t}(x-ct)$$

$$= -c \times -c\; g''(x-ct)$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 g''(x-ct) \quad \ldots (i)$$

Again,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(g(x-ct))$$

$$= g'(x-ct)\, \frac{\partial}{\partial x}(x-ct)$$

$$= g'(x-ct)\,(1-0)$$

$$= g'(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(g'(x-ct))$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = g''(x-ct) \quad \ldots (ii)$$

From (i), we have:

$$\frac{\partial^2 u}{\partial t^2} = c^2 g''(x-ct)$$

$$= c^2 \frac{\partial^2 u}{\partial x^2} \quad (\text{from (ii)})$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{proved.}$$

c. Show that the function of the form $u(x,t) = f(x+ct) + g(x-ct)$ satisfies the wave equation.

<u>Proof</u>: We know the wave equation to be

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here given,

$$u(x,t) = f(x+ct) + g(x-ct)$$

Now, $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}[f(x+ct) + g(x-ct)]$$

$$= \frac{\partial}{\partial t}(f(x+ct)) + \frac{\partial}{\partial t}(g(x-ct))$$

$$= f'(x+ct)\cdot\frac{\partial}{\partial t}(x+ct) + g'(x-ct)\frac{\partial}{\partial t}(x-ct)$$

$$= c f'(x+ct) + (-c)\, g'(x-ct)$$

$$= c\{f'(x+ct) - g'(x-ct)\}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}[c\{f'(x+ct) - g'(x-ct)\}]$$

$$= c\left[\frac{\partial}{\partial t}\{f'(x+ct)\} - \frac{\partial}{\partial t}\{g'(x-ct)\}\right]$$

$$= c\left[f''(x+ct)\frac{\partial}{\partial t}(x+ct) - g''(x-ct)\frac{\partial}{\partial t}(x-ct)\right]$$

$$= c[c\, f''(x+ct) - (-c)\, g''(x-ct)]$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2[f''(x+ct) + g''(x-ct)] \quad \cdots (i)$$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[f(x+ct) + g(x-ct)]$$

$$= \frac{\partial}{\partial x}(f(x+ct)) + \frac{\partial}{\partial x}(g(x-ct))$$

$$= f'(x+ct)\cdot 1 + g'(x-ct)\cdot 1$$

$$= f'(x+ct) + g'(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}[f'(x+ct) + g'(x-ct)]$$

$$= \frac{\partial}{\partial x}(f'(x+ct)) + \frac{\partial}{\partial x}(g'(x-ct))$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = f''(x+ct) + g''(x-ct) \quad \cdots (ii)$$

From (i), we have:

$$\frac{\partial^2 y}{\partial t^2} = c^2[f''(x+ct) + g''(x-ct)]$$

$$= c^2 . \frac{\partial^2 y}{\partial x^2} \quad (\text{from (ii)})$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{proved}$$

VI. Find the linear $L(x,y)$. Compare the value of the approximation $L(0.9, 0.2)$ with the exact value of the function $f(0.9, 0.2)$.

a. $f(x,y) = \sqrt{x+2y}$

Here, $f(x,y) = \sqrt{x+2y}$. The linear approximation of f at (a,b) is given by

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad \text{— (i)}$$

Now, consider (a,b) to be point close to given point $(0.9, 0.2)$.
Suppose, $(a,b) = (1,0)$.

Now,

$$f_x = \frac{\partial(\sqrt{x+2y})}{\partial x}$$

$$= \frac{\partial}{\partial x}(x+2y)^{1/2}$$

$$= \frac{1}{2}(x+2y)^{1/2-1} . \frac{\partial}{\partial x}(x+2y)$$

$$= \frac{1}{2\sqrt{x+2y}} . 1 = \frac{1}{2\sqrt{x+2y}}$$

$\therefore f_x(a,b) = f_x(1,0) = \dfrac{1}{2\sqrt{1+0}} = \dfrac{1}{2}$.

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{x+2y}\right)$$

$$= \frac{\partial}{\partial y}\left((x+2y)^{1/2}\right)$$

$$= \frac{1}{2\sqrt{x+2y}} \frac{\partial}{\partial y}(x+2y)$$

$$= \frac{2}{2\sqrt{x+2y}}$$

$$= \frac{1}{\sqrt{x+2y}}$$

$$\therefore f_y(a,b) = f_y(1,0) = \frac{1}{\sqrt{1+0}} = 1$$

$$f(a,b) = f(1,0) = \sqrt{1+2\times 0} = \sqrt{1} = 1$$

Thus, eqn (i) becomes

$L(x,y) = 1 + \frac{1}{2}(x-1) + 1(y-0)$ at $(a,b) = (1,0)$

$$\Rightarrow L(x,y) = \frac{2+x-1+2y}{2}$$

$$\Rightarrow L(x,y) = \frac{x+2y+1}{2}$$

Now, at $(x,y) = (0.9, 0.2)$,

$$L(x,y) = L(0.9, 0.2) = \frac{0.9 + 2\times 0.2 + 1}{2}$$

$$= \boxed{1.15}$$

$$f(x,y) = f(0.9, 0.2) = \sqrt{0.9 + 2\times 0.2}$$

$$= \sqrt{0.9+0.4}$$

$$= \sqrt{1.3}$$

$$= \boxed{1.14}$$

$\therefore f(0.9, 0.2) \approx L(0.9, 0.2)$

b. $f(x, y) = x^2 y$

Here, $f(x, y) = x^2 y$. The linear approximation of f at (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad \text{---(i)}$$

Consider a point $(1, 0)$ to be close to the given point $(0.9, 0.2)$ i.e. $(a, b) = (1, 0)$.

Now,

$$f_x = \frac{\partial}{\partial x}(x^2 y)$$

$$= y \cdot 2x$$

$$= 2xy$$

$f_x(a, b) = f_x(1, 0) = 2 * 1 * 0 = 0$

$$f_y = \frac{\partial}{\partial y}(x^2 y)$$

$$= x^2 . (1)$$

$$= x^2$$

$\therefore f_y(a, b) = f_y(1, 0) = (1)^2 = 1$

$f(a, b) = f(1, 0) = (1)^2 * 0 = 0$

Thus, equation (i) becomes

$L(x, y) = 0 + 0(x - 1) + 1(y - 0)$ at $(a, b) = (1, 0)$

$$= y$$

Now at $(x, y) = (0.9, 0.2)$,

$L(x, y) = L(0.9, 0.2) = 0.2$

$f(x, y) = f(0.9, 0.2) = (0.9)^2 * 0.2$

$= 0.162$

$\therefore f(0.9, 0.2) \approx L(0.9, 0.2)$

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c. $f(x, y) = xe^{-y}$

Here, $f(x, y) = xe^{-y}$. The linear approximation of f at (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \quad \cdots (i)$$

Consider $(a, b) = (1, 0)$ to be the point close to the given point $(0.9, 0.2)$.

Now,

$$f_x = \frac{\partial}{\partial x}(xe^{-y})$$

$$= e^{-y} \cdot \frac{\partial x}{\partial x}$$

$$= e^{-y}$$

$$\therefore f_x(a, b) = f_x(1, 0) = e^{-0} = 1$$

$$f_y = \frac{\partial}{\partial y}(xe^{-y})$$

$$= x \frac{\partial}{\partial y}(e^{-y})$$

$$= -xe^{-y}$$

$$f_y(a, b) = f_y(1, 0) = -1 * e^{-0} = (-1)$$

$$f(a, b) = f(1, 0) = 1e^{-0} = 1$$

Thus, equation (i) becomes

$$L(x, y) = 1 + 1(x-1) + (-1)(y-0)$$

$$= 1 + x - 1 - y$$

$$\therefore L(x, y) = x - y$$

Now at $(x, y) = (0.9, 0.2)$

$$L(x, y) = L(0.9, 0.2) = 0.9 - 0.2$$

$$= 0.7$$

$f(x,y) = f(0.9, 0.2) = 0.9 \times e^{-0.2}$

$= \frac{0.9}{e^{0.2}}$

$= 0.736$

$\therefore L(x,y) \approx f(x,y)$ at $(x,y) = (0.9, 0.2)$

d. $f(x,y) = e^{x} \sin y + e^{y} \sin x$

Here, $f(x,y) = e^{x} \sin y + e^{y} \sin x$. The linear approximation L of f at (a,b) is given by

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \cdots (i)$

Consider point $(a,b) = (1,0)$ to be close to the given point $(0.9, 0.2)$ where we make linear approximation.

Now,

$f_x = \frac{\partial}{\partial x}(e^{x} \sin y + e^{y} \sin x)$

$= \sin y \cdot e^{x} + e^{y} \cos x$

$\therefore f_x(a,b) = f_x(1,0) = \sin 0 \cdot e^{1} + e^{0} \cdot \cos 1$

$= 0 + 1 \times \cos 1$

$= 0.540$

$f_y = \frac{\partial}{\partial y}(e^{x} \sin y + e^{y} \sin x)$

$= e^{x} \cos y + \sin x \; e^{y}$

$= e^{x} \cos y + e^{y} \sin x$

$\therefore f_y(a,b) = f_y(1,0) = e^{1} \cos 0 + e^{0} \sin 1$

$= 3.559$

$f(a,b) = f(1,0) = e^{1} \sin 0 + e^{0} \sin 1$

$= 0 + 1 \times 0.841$

$= 0.841$

Now, equation (i) becomes:

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 0.841 + 0.540(x-1) + 3.559(y-0)$ where $(a,b) = (1,0)$

$\therefore L(x,y) = 0.540x + 3.559y + 0.301$

Now at $(x,y) = (0.9, 0.2)$,

$L(x,y) = L(0.9, 0.2)$

$= 0.540 * 0.9 + 3.559 * 0.2 + 0.301$

$= 1.4988$

$f(x,y) = f(0.9, 0.2) = e^{0.9} \sin 0.2 + e^{0.2} \sin 0.9$

$= 1.445$

$\therefore L(x,y) \approx f(x,y)$ at $(0.9, 0.2)$

e. $f(x,y) = \sin(x-1) \cos y$

Here, $f(x,y) = \sin(x-1)\cos y$. The Linear approximation of f i.e. L at (a,b) is given by:

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ —— (i)

Let us consider point (a,b) to be $(1,0)$ which is close to the given point $(0.9, 0.2)$.

Now,

$f_x = \frac{\partial}{\partial x}(\sin(x-1)\cos y)$

$= \cos y \frac{\partial}{\partial x}(\sin(x-1))$

$= \cos y \cos(x-1)$

$f_x(a,b) = f_x(1,0) = \cos 0 \cos(1-1)$

$= \cos 0 * \cos 0$

$= 1$

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$$f_y = \frac{\partial}{\partial y}(\sin(x-1)\cos y)$$

$$= \sin(x-1) \cdot \frac{\partial}{\partial y}(\cos y)$$

$$= -\sin(x-1)\sin y$$

$$f_y(a,b) = f_y(1,0) = -\sin(1-1)\sin 0$$

$$= -\sin 0 \cdot \sin 0$$

$$= 0$$

$$f(a,b) = f(1,0) = \sin(1-1)\cos 0$$

$$= \sin 0 \cos 0$$

$$= 0$$

Now, equation (i) becomes,

$$L(x,y) = 0 + 1(x-1) + 0(y-0)$$

$$\therefore L(x,y) = x-1$$

At point $(x,y) = (0.9, 0.2)$,

$$L(x,y) = L(0.9, 0.2) = 0.9-1$$

$$= -0.1$$

$$f(x,y) = f(0.9, 0.2) = \sin(0.9-1)\cos 0.2$$

$$= \sin(-0.1)\cos 0.2$$

$$= -0.097$$

$$\therefore L(0.9, 0.2) \approx f(0.9, 0.2)$$

VII. Compute the differential dz or dw of the function

1. $z = 7x - 2y$

Here $z = f(x,y)$ so

$$dz = f_x\,dx + f_y\,dy \quad \text{—— (i)}$$

Now,

$$f_x = \frac{\partial(7x-2y)}{\partial x} = 7 - 0 = 7$$

$$f_y = \frac{\partial}{\partial y}(7x - 2y)$$

$$= 0 - 2$$

$$= -2$$

from (i)

$\therefore \; dz = 7\,dx - 2\,dy$

2. $z = e^{xy}$

Here, $z = f(x, y)$

We know, differential dz is given by

$$dz = f_x\,dx + f_y\,dy \quad \text{—— (i)}$$

Here,

$$f_x = \frac{\partial}{\partial x}(e^{xy})$$

$$= e^{xy}\,\frac{\partial (xy)}{\partial x}$$

$$= y e^{xy}$$

$$f_y = \frac{\partial}{\partial y}(e^{xy})$$

$$= e^{xy} \cdot x$$

$$= x e^{xy}$$

Thus, from (i),

$$dz = y e^{xy}\,dx + x e^{xy}\,dy$$

3. $z = x^3 y^2$

Here,

$z = f(x, y)$.

We know that differential dz is given by

$$dz = f_x\,dx + f_y\,dy$$

Now,

$$f_x = \frac{\partial (x^3 y^2)}{\partial x}$$

$$= y^2 \cdot 3x^2$$

$$= 3x^2 y^2$$

$$f_y = \frac{\partial}{\partial y}(x^3y^2)$$
$$= x^3 \cdot 2y$$
$$= 2x^3y$$

Thus, from (i), we get:

$$dz = 3x^2y^2\,dx + 2x^3y\,dy$$

4. $z = \tan^{-1} xy$

Here,

$z = f(x, y)$. Thus, differential dz is given by

$$dz = f_x\,dx + f_y\,dy \quad \text{——— (1)}$$

Now,

$$f_x = \frac{\partial}{\partial x}(\tan^{-1} xy)$$
$$= \frac{1}{1+(xy)^2} \cdot \frac{\partial(xy)}{\partial x}$$
$$= \frac{y}{1+x^2y^2}$$

$$f_y = \frac{\partial}{\partial y}(\tan^{-1} xy)$$
$$= \frac{1}{1+(xy)^2}\ \frac{\partial}{\partial y}(xy)$$
$$= \frac{x}{1+x^2y^2}$$

Thus, from (i), we get:

$$dz = \frac{y}{1+x^2y^2}\,dx + \frac{x}{1+x^2y^2}\,dy$$

5. $z = e^{-3x} \cos 6y$

Here, $z = f(x, y)$. Thus, differential dz is given by

$$dz = f_x\,dx + f_y\,dy \quad \text{——— (1)}$$

Now,

$$f_x = \frac{\partial}{\partial x}(e^{-3x}\cos 6y)$$

$$= \cos 6y \frac{d}{dx}(e^{-3x})$$

$$= \cos 6y \cdot e^{-3x} \cdot (-3)$$

$$= -3\cos 6y \cdot e^{-3x}$$

$$f_y = \frac{\partial}{\partial y}(e^{-3x}\cos 6y)$$

$$= e^{-3x}\frac{\partial}{\partial y}(\cos 6y)$$

$$= e^{-3x}(-\sin 6y) \cdot \frac{\partial(6y)}{\partial y}$$

$$= -6e^{-3x}\sin 6y$$

Thus, from ①, we get:

$$dz = -3\cos 6y \cdot e^{-3x}\,dx - 6e^{-3x}\sin 6y$$

6. $w = 8x - 3y + 4z$

Here, $w = f(x, y, z)$, so the differential is given by

$$dw = f_x\,dx + f_y\,dy + f_z\,dz \quad \text{——} \ ①$$

Now,

$$f_x = \frac{\partial}{\partial x}(8x - 3y + 4z)$$

$$= 8$$

$$f_y = \frac{\partial}{\partial y}(8x - 3y + 4z)$$

$$= -3$$

$$f_z = \frac{\partial}{\partial z}(8x - 3y + 4z)$$

$$= 4$$

Thus, eqⁿ (i) becomes

$$dw = 8\,dx + (-3\,dy) + 4\,dz$$

$$\therefore dw = 8\,dx - 3\,dy + 4\,dz$$

7. $w = e^{xyz}$

Here,

$w = f(x, y, z)$, so the differential is given by

$$dw = f_x\,dx + f_y\,dy + f_z\,dz \quad \text{—— (i)}$$

Now,

$$f_x = \frac{\partial}{\partial x}(e^{xyz}) = e^{xyz} \cdot \frac{\partial (xyz)}{\partial x} = yz\,e^{xyz}$$

$$f_y = \frac{\partial}{\partial y}(e^{xyz}) = e^{xyz} \cdot \frac{\partial}{\partial y}(xyz) = xz\,e^{xyz}$$

$$f_z = \frac{\partial}{\partial z}(e^{xyz}) = e^{xyz} \frac{\partial}{\partial z}(xyz) = xy\,e^{xyz}$$

Thus, equation (i) becomes

$$dw = yz\,e^{xyz}\,dx + xz\,e^{xyz}\,dy + xy\,e^{xyz}\,dz$$

$$\therefore dw = e^{xyz}(yz\,dx + xz\,dy + xy\,dz)$$

8. $w = x^3 y^2 z$

Here,

$w = f(x, y, z)$, so the differential is given by

$$dw = f_x\,dx + f_y\,dy + f_z\,dz \quad \text{—— (i)}$$

Now,

$$f_x = \frac{\partial}{\partial x}(x^3y^2z)$$

$$= y^2 z \frac{\partial}{\partial x}(x^3)$$

$$= 3x^2y^2z$$

$$f_y = \frac{\partial}{\partial y}(x^3y^2z)$$

$$= x^3 z \frac{\partial}{\partial y}(y^2) = 2x^3yz$$

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$$f_z = \frac{\partial}{\partial z}(x^3y^2z)$$

$$= x^3y^2 \frac{\partial(z)}{\partial z}$$

$$= x^3y^2$$

Now, equation ① becomes,

$$dw = 3x^2y^2z\,dx + 2x^3yz\,dy + x^3y^2\,dz$$

VIII. Use the Chain Rule to calculate $\frac{d}{dt} f(c(t))$.

1. $f(x, y) = 3x - 7y$, $c(t) = (\cos t, \sin t)$, $t = 0$

Here,

$$f(x, y) = 3x - 7y$$

$$\& \; c(t) = (\cos t, \sin t)$$

$$\therefore x = x(t) = \cos t$$

$$y = y(t) = \sin t$$

Using Chain rule, we have,

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= (3)(-\sin t) + (-7)(\cos t)$$

$$= -3\sin t - 7\cos t$$

$$\therefore \left.\frac{d f(c(t))}{dt}\right|_{t=0} = -3\sin 0 - 7\cos 0$$

$$= -7 \times 1$$

$$= -7$$

2. $f(x, y) = 3x - 7y$, $c(t) = (t^2, t^3)$, $t = 2$.

Here,

$$f(x, y) = 3x - 7y$$

$$\& \; c(t) = (t^2, t^3)$$

$\therefore\ x = x(t) = t^2$
$y = y(t) = t^3$

Using chain rule, we have

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$
$$= (3)(2t) + (-7) \cdot 3t^2$$
$$= 6t - 21t^2$$

$$\therefore \frac{d}{dt} f(c(t))\Big|_{t=2} = 6 \times 2 - 21 \times 2^2$$
$$= 12 - 84$$
$$= -72$$

3. $f(x, y) = x^2 - 3xy$, $c(t) = (\cos t, \sin t)$, $t = 0$

Here,
$f(x, y) = x^2 - 3xy$
& $c(t) = (\cos t, \sin t)$ $\therefore\ x = x(t) = \cos t$; $y = y(t) = \sin t$

Using chain rule, we have

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$
$$= (2x - 3y) \cdot (-\sin t) + (-3x)(\cos t)$$

At $t = 0$,
$x = \cos t = \cos 0 = 1$
$y = \sin t = \sin 0 = 0$

$$\therefore \frac{d}{dt} f(c(t))\Big|_{t=0} = (2 \times 1 - 3 \times 0) \cdot (-\sin 0) - 3 \times 1 \times \cos 0$$
$$= 0 - 3 \times 1 \times 1$$
$$= -3$$

4. $f(x,y) = x^2 - 3xy, \quad c(t) = (\cos t, \sin t), \quad t = \pi/2$

Here, $f(x,y) = x^2 - 3xy$

& $c(t) = (\cos t, \sin t)$

$\therefore x = x(t) = \cos t$

$y = y(t) = \sin t$

Now, using chain rule,

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x - 3y) \cdot (-\sin t) + (-3x) \cdot (\cos t)$$

At $t = \pi/2$

$x = \cos t = \cos \frac{\pi}{2} = 0$

$y = \sin t = \sin \frac{\pi}{2} = 1$

$$\therefore \left. \frac{d}{dt} f(c(t)) \right|_{t = \pi/2} = -(2 \times 0 - 3 \times 1) \sin \frac{\pi}{2} - 3 \times 0 \times \cos \pi/2$$

$$= 3 * 1$$

$$= 3$$

5. $f(x,y) = \sin(xy), \quad c(t) = (e^{2t}, e^{3t}), \quad t = 0$

Here, $f(x,y) = \sin(xy)$

& $c(t) = (e^{2t}, e^{3t})$

$\therefore x = x(t) = e^{2t}$

$y = y(t) = e^{3t}$

Now, using Chain rule, we get:

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \cos xy \cdot y \cdot e^{2t} \cdot 2 + \cos xy \cdot x \cdot e^{3t} \cdot 3$$

$$= 2y \cos xy \, e^{2t} + 3x \cos xy \, e^{3t}$$

At $t = 0$,

$x = x(t) = e^{2t} = e^{2\times 0} = e^{0} = 1$

$y = e^{3t} = e^{0} = 1$

$$\therefore \left.\frac{d f(c(t))}{dt}\right|_{t=0} = 2*1*\cos(1\times1)*e^{0} + 3*1*\cos(1\times1)*e^{0}$$

$$= 2\cos 1 + 3\cos 1$$

$$= 5\cos 1$$

$$= \underline{2.701}$$

6. $f(x,y) = \cos(y-x)$, $c(t) = (t^2, t^2-4t)$, $t=0$

Here, $f(x,y) = \cos(y-x)$

& $c(t) = (t^2, t^2-4t)$

$\therefore$ $x = x(t) = t^2$

$y = y(t) = t^2 - 4t$

Now, using chain rule, we get:

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= -\sin(y-x)\cdot(-1)*2t + \{-\sin(y-x)\}\cdot(-1)*(2t-4)$$

At $t=0$,

$c(0) = (0^2, 0^2 - 4\times 0)$

$= (0, 0)$

$$\therefore \left.\frac{d}{dt} f(c(t))\right|_{t=0} = -\sin(0-0)\cdot(-1)*2*0 + \sin(0-0)*(2*0-4)$$

$$= 0 + 0$$

$$= 0$$

Q6. $f(x,y) = \cos(y-x)$, $c(t) = (e^t, e^{2t})$, $t = \ln 3$

Here, $f(x,y) = \cos(y-x)$

& $c(t) = (e^t, e^{2t})$.

$\therefore x = x(t) = e^t$

$y = y(t) = e^{2t}$

Now, using chain rule,

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= -\sin(y-x) \cdot (-1) \cdot e^t + -\sin(y-x) \cdot (1) \cdot e^{2t} \cdot 2$$

$$= e^t \sin(y-x) - 2e^{2t} \sin(y-x)$$

$$= \sin(y-x)(e^t - 2e^{2t})$$

At $t = \ln 3$,

$$c(\ln 3) = (e^{\ln 3}, e^{2\ln 3})$$

$$= (3, e^{\ln 3^2})$$

$$= (3, 3^2)$$

$$= (3, 9)$$

$$\therefore \frac{d}{dt} f(c(t)) \Big|_{t = \ln 3} = \sin(9-3)\left(e^{\ln 3} - 2e^{2\ln 3}\right)$$

$$= \sin 6 \,(3 - 2 * 9)$$

$$= \sin 6 \,(3 - 18)$$

$$= -15 \sin 6$$

$$= \underline{4.191}$$

7. ~~$f(x,y) = \ln x + \ln y$, $c(t) =$~~
$f(x,y) = xe^y$, $c(t) = (t^2, t^2-4t)$, $t=0$

Here,
$f(x,y) = xe^y$
& $c(t) = (t^2, t^2-4t)$
$\therefore x = x(t) = t^2$
$y = y(t) = t^2-4t$

Now using chain rule,

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= e^y . (2t) + xe^y (2t-4)$$

$$= e^y (2t + x(2t-4))$$

At $t=0$,
$c(0) = (0^2, 0^2-4\times 0)$
$= (0, 0)$

$$\therefore \left.\frac{d f(c(t))}{dt}\right|_{t=0} = e^0 (2*0 + 0(2*0-4))$$

$$= 1(0+0)$$

$$= 0$$

8. $f(x,y) = \ln x + \ln y$, $c(t) = (\cos t, t^2)$, $t = \pi/4$
Here, $f(x,y) = \ln x + \ln y$
& $c(t) = (\cos t, t^2)$
$\therefore x = x(t) = \cos t$
$y = y(t) = t^2$

Now using chain rule,

$$\frac{d}{dt} f(c(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{x} \cdot (-\sin t) + \frac{1}{y} \cdot 2t$$

$$= \frac{-\sin t}{x} + \frac{2t}{y}$$

At $t = \pi/4$

$$c(\pi/4) = \left(\cos \pi/4, \left(\frac{\pi}{4}\right)^2\right)$$

$$= \left(0.707, \frac{\pi^2}{16}\right)$$

$$\therefore \left.\frac{d}{dt} f(c(t))\right|_{t=\pi/4} = \frac{-\sin \pi/4}{0.707} + \frac{2 \times \pi/4}{\pi^2/16}$$

$$= -1 + \frac{2\pi}{\cancel{4}} \times \frac{\cancel{16}\ 4}{\pi^2}$$

$$= -1 + \frac{8}{\pi}$$

$$= \boxed{1.546}$$

IX. Find an equation of tangent plane at the given point

1. $f(x, y) = x^2y + xy^3, \quad (2, 1)$

The equation of tangent plane at point (a, b) to a given surface f is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad \text{— (i)}$$

Here, given surface S is

$$f(x, y) = x^2y + xy^3$$

$$f_x = \frac{\partial}{\partial x}(x^2y + xy^3)$$

$$= y \cdot (2x) + y^3 \cdot (1)$$

$$= 2xy + y^3$$

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$$f_y = \frac{\partial(x^2y + xy^3)}{\partial y}$$

$$= x^2 \cdot (1) + x \cdot (3y^2)$$

$$= x^2 + 3xy^2$$

Now given point $(a, b) = (2, 1)$. Thus,

$$f(a,b) = f(2,1) = 2^2 * 1 + 2 * (1)^3$$

$$= 4 + 2$$

$$= 6$$

$$f_x(a,b) = f_x(2,1) = 2 * 2 * 1 + (1)^3$$

$$= 4 + 1$$

$$= 5$$

$$f_y(a,b) = f_y(2,1) = (2)^2 + 3 * 2 * (1)^2$$

$$= 4 + 6$$

$$= 10$$

Thus, equation of tangent plane P to surface S given by equation (i) becomes,

$$z = 6 + 5(x-2) + 10(y-1)$$

$$= 6 + 5x - 10 + 10y - 10$$

$$= 5x + 10y + 6 - 20$$

$\boxed{\therefore z = 5x + 10y - 14}$ is the required equation of the plane.

2. $f(x,y) = \frac{x}{\sqrt{y}}$, $(4, 4)$

The equation of tangent plane at point (a,b) to a given surface S is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad \text{—— (1)}$$

Given surface S is

$$f(x,y) = \frac{x}{\sqrt{y}}$$

Thus, $f_x = \frac{\partial}{\partial x}(x/\sqrt{y}) = \frac{1}{\sqrt{y}} \cdot \frac{\partial x}{\partial x} = \frac{1}{\sqrt{y}}$

$$f_y = \frac{\partial}{\partial y}(x/\sqrt{y})$$

$$= x \cdot \frac{\partial}{\partial y}(y^{-1/2})$$

$$= -\frac{x \cdot 1}{2} \cdot y^{-1/2-1}$$

$$= \frac{-x}{2 \cdot y^{3/2}}$$

Now, given point $(a, b) = (4, 4)$. Thus,

$$f(a, b) = f(4, 4) = \frac{4}{\sqrt{4}} = \frac{4}{2} = 2$$

$$f_x(a, b) = f_x(4, 4) = \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$f_y(a, b) = f_y(4, 4) = -\frac{x}{2y^{3/2}} = \frac{-4}{2 \times (4)^{3/2}} = \frac{-4}{2 \times 8}$$

$$= -1/4$$

Eqn (i) which represents tangent plane becomes,

$$z = 2 + \frac{1}{2}(x-4) + \left\{\frac{-1}{4}(y-4)\right\}$$

$$\Rightarrow z = \frac{8 + 2(x-4) - (y-4)}{4}$$

$$\Rightarrow 4z = \frac{8 + 2x - 8 - y + 4}{1}$$

$$\Rightarrow 4z = 2x - y + 4$$

$\therefore$ $2x - y - 4z + 4 = 0$ is required equation of tangent plane.

3. $f(x, y) = x^2 + y^{-2}$ at $(4, 1)$

The equation of tangent plane P to a surface f is given by

$$z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \quad \text{—(i)}$$

at a point (a, b).

Here, given surface S is

$$f(x, y) = x^2 + y^{-2}$$

Now, $f_x = \frac{\partial}{\partial x}(x^2 + y^{-2})$

$= 2x + 0$

$= 2x$

$f_y = \frac{\partial}{\partial y}(x^2 + y^{-2})$

$= 0 + -2 \cdot y^{-2-1}$

$= \frac{-2}{y^3}$

At given point $(a, b) = (4, 1)$,

$f_x(a, b) = f_x(4,1) = 2 \times 4 = 8$

$f_y(a,b) = f_y(4,1) = \frac{-2}{(1)^3} = -2$

$f(a.b) = f(4,1) = (4)^2 + (1)^{-2}$

$= 16 + \frac{1}{1^2}$

$= 16 + 1$

$= 17$

Therefore, equation (i) becomes,

$z = 17 + 8(x-4) + \{-2(y-1)\}$

$\Rightarrow z = 17 + 8x - 32 - 2y + 2$

$\Rightarrow 8x - 2y - 13 = z$

$\therefore z = 8x - 2y - 13$ is the required equation of tangent plane

4. $G(u, w) = \sin(uw)$, $(\pi/6, 1)$

The equation of tangent plane at (a, b) to a surface S is given by

$z = f(a, b) + f_x(x-a) + f_y(y-b)$ —— (i)

Given surface S is

$G(u, w) = \sin(uw)$.

Eqn (i) then becomes,

$z = f(u, w) + f_u(u-a) + f_w(w-b)$ —— (ii)

Now,

$$f_u = \frac{\partial}{\partial u}(\sin(uw))$$

$$= \cos(uw) \cdot \frac{\partial (uw)}{\partial u}$$

$$\therefore f_u = w\cos(uw)$$

$$f_w = \frac{\partial}{\partial w}(\sin(uw))$$

$$= \cos(uw) \cdot \frac{\partial}{\partial w}(uw)$$

$$= u\cos(uw)$$

At given point $(a, b) = (\pi/6, 1)$, we have:

$$f(a,b) = f(\pi/6, 1) = \sin(\pi/6 \cdot 1)$$

$$= 1/2$$

$$f_u(a,b) = f_u(\pi/6, 1) = 1 * \cos(\pi/6 \cdot 1)$$

$$= \frac{\sqrt{3}}{2}$$

$$f_w(a,b) = f_w(\pi/6, 1) = \pi/6 * \cos(\pi/6 \cdot 1)$$

$$= \frac{\sqrt{3}\pi}{12}$$

Consequently, eq^n (ii) becomes,

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) + \frac{\sqrt{3}\pi}{12}(y-1)$$

$$\Rightarrow z = \frac{6 + 6\sqrt{3}(x - \pi/6) + \sqrt{3}\pi(y-1)}{12}$$

$\therefore 12z = 6 + 6\sqrt{3}(x - \pi/6) + \sqrt{3}\pi(y-1)$ is the required eq^n of the tangent plane.

5. $g(x,y) = e^{x/y}$ $(2,1)$

Here, given surface S is

$$g(x,y) = e^{x/y}$$

Thus, $g_x = \frac{\partial g}{\partial x}$

$$= \frac{\partial (e^{x/y})}{\partial x}$$

$$= e^{x/y} \cdot \frac{\partial}{\partial x}(x/y)$$

$$= \frac{1}{y} e^{x/y}$$

$$g_y = \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(e^{x/y})$$

$$= e^{x/y} \cdot \frac{\partial (x/y)}{\partial y}$$

$$= e^{x/y} \cdot x \frac{\partial (y^{-1})}{\partial y}$$

$$= \frac{-x \cdot e^{x/y}}{y^2}$$

The equation of tangent plane to a surface S is given by

$$z = g(a,b) + g_x(a,b)(x-a) + g_y(a,b)(y-b) \quad \text{at } (a,b) \quad \text{— (i)}$$

Now, $g(a,b) = g(2,1) = e^{2/1} = e^2$

$$g_x(a,b) = g_x(2,1) = \frac{1}{1} \cdot e^{2/1} = e^2$$

$$g_y(a,b) = g_y(2,1) = \frac{-2 \cdot e^{2/1}}{1^2} = -2e^2$$

Thus, (i) becomes,

$$z = e^2 + e^2(x-2) + \{-2e^2(y-1)\}$$

$\therefore$ $z = e^2 + e^2(x-2) - 2e^2(y-1)$ is required eqn of tangent plane at $(2,1)$.

6. $f(x, y) = \ln(4x^2 - y^2)$, $(1, 1)$

Here, given surface S is

$$f(x, y) = \ln(4x^2 - y^2)$$

$$\therefore f_x = \frac{\partial}{\partial x}(\ln(4x^2 - y^2))$$

$$= \frac{1}{(4x^2 - y^2)} \cdot \frac{\partial}{\partial x}(4x^2 - y^2)$$

$$= \frac{8x}{(4x^2 - y^2)}$$

$$\because f_y = \frac{\partial}{\partial y}(\ln(4x^2 - y^2))$$

$$= \frac{1}{(4x^2 - y^2)} \cdot \frac{\partial}{\partial y}(4x^2 - y^2)$$

$$= \frac{-2y}{(4x^2 - y^2)}$$

Now at point $(a, b) = (1, 1)$,

$$f(a, b) = f(1, 1) = \ln(4 \times 1^2 - 1^2)$$

$$= \ln(4 - 1)$$

$$= \ln 3$$

$$f_x(a, b) = f_x(1, 1) = \frac{8 * 1}{4 * 1^2 - 1^2}$$

$$= \frac{8}{4 - 1}$$

$$= 8/3$$

$$f_y(a, b) = f_y(1, 1) = \frac{-2 * 1}{(4 * 1^2 - 1^2)} = \frac{-2}{4 - 1} = \frac{-2}{3}$$

The equation of tangent plane P to a surface S is given by: $z = f(a, b) + \cancel{f_x(x+a)}\, f_x(a, b)(x - a) + f_y(a, b)(y - b)$

$\Rightarrow z = \ln 3 + \frac{8}{3}(x-1) + \left(-\frac{2}{3}\right)(y-1)$

$\Rightarrow 3z = 3\ln 3 + 8(x-1) - 2(y-1)$ is the required equation of tangent plane.

7. Find the linearization $L(x,y)$ of $f(x,y) = x^2y^3$ at $(a,b) = (2,1)$. Use it to estimate $f(2.01, 1.02)$ and $f(1.97, 1.01)$ and compare with actual values.

Here, given surface S is

$$f(x,y) = x^2y^3.$$

The linear approximation or linearization $L(x,y)$ to given surface S at point (a,b) is given by:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad \text{— (i)}$$

Now,

$$f_x = \frac{\partial}{\partial x}(x^2y^3)$$

$$= y^3 \frac{\partial}{\partial x}(x^2)$$

$$= 2xy^3$$

$$f_y = \frac{\partial}{\partial y}(x^2y^3)$$

$$= x^2 \cdot \frac{\partial (y^3)}{\partial y}$$

$$= 3x^2y^2$$

At point $(a,b) = (2,1)$, we get

$f_x(a,b) = f_x(2,1) = 2 \times 2 \times 1^3 = 4$

$f_y(a,b) = f_y(2,1) = 3 \times 2^2 \times 1^2 = 12$

$f(a,b) = (2)^2 * (1)^3 = 4$

Thus, eqn (i) becomes

$$L(x,y) = 4 + 4(x-2) + 12(y-1)$$

$\therefore L(x,y) = 4x + 12y - 16$ is the required linearization.

Now, estimation of given points is as follows.

$$f(2.01, 1.02) \approx L(2.01, 1.02)$$

$$= 4 * 2.01 + 12 * 1.02 - 16$$

$$= \underline{4.28}$$

$$f(1.97, 1.01) \approx L(1.97, 1.01)$$

$$= 4 * 1.97 + 12 * 1.01 - 16$$

$$= \underline{4}$$

Now actual values are given by

$$f(2.01, 1.02) = (2.01)^2 * (1.02)^3$$

$$= \underline{4.287}$$

$$f(1.97, 1.01) = (1.97)^2 * (1.01)^3$$

$$= \underline{3.998}$$

$$\therefore f(x, y) \approx L(x, y)$$