Kaushal_Exercise3

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1 Exercise 1. Dynamical Systems Theory in Machine Learning & Data Science

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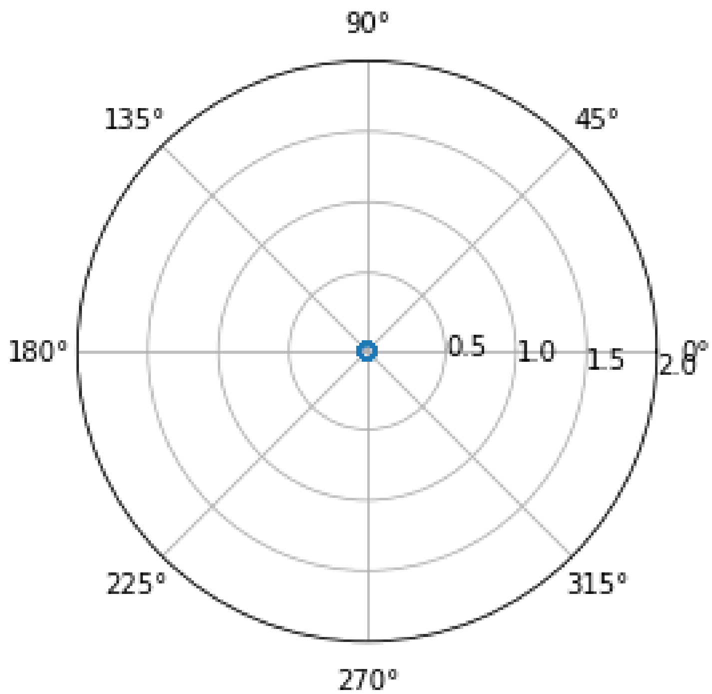
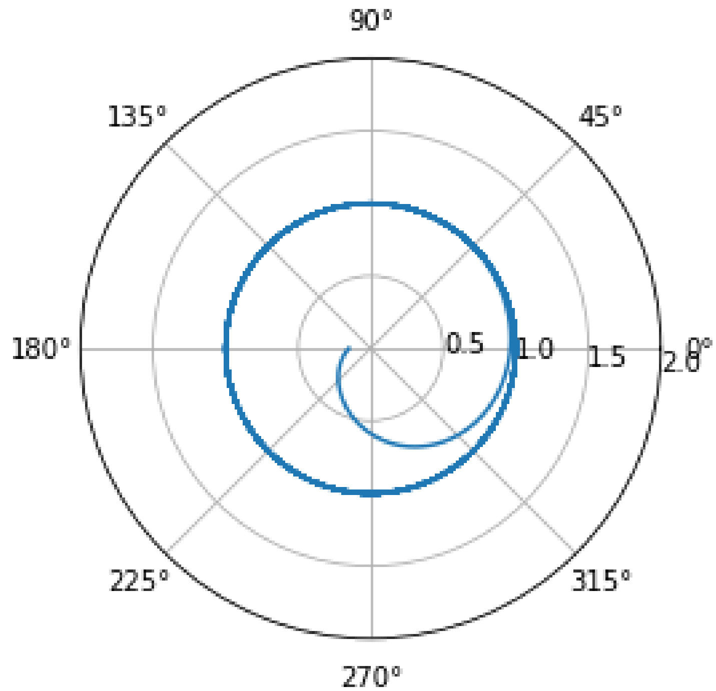
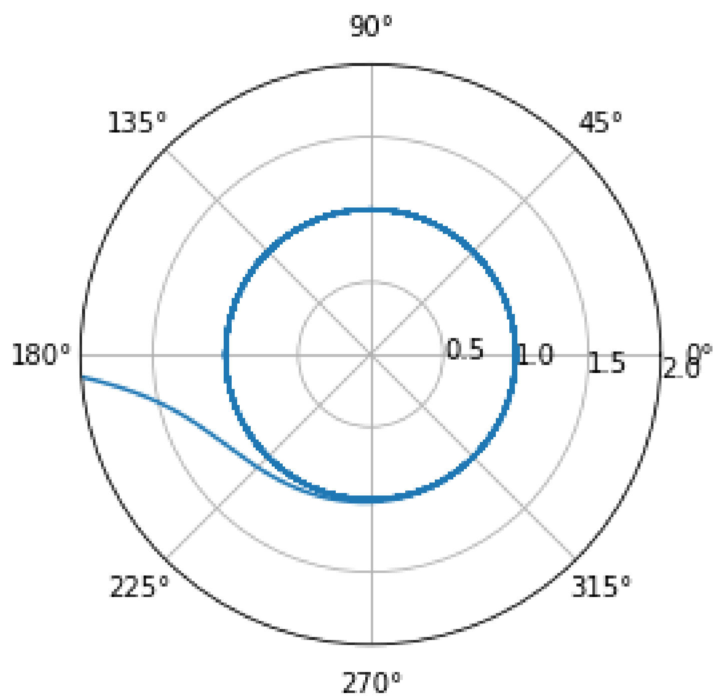
3 1. Poincare-Bendixson Theorem

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
dt=0.01
t=np.arange(0,100,dt)
n=len(t)

def f(r):
    return r*(1-r**2)

def g(theta):
    return 1

r=np.zeros(n)
theta=np.zeros(n)

ini=[0.00,0.15,3]
for k in range(len(ini)):
    r[0]=ini[k]

    theta[0]=np.pi

    for i in range(n-1):
        r[i+1]=r[i]+f(r[i])*dt
        theta[i+1]=theta[i]+g(theta[i])*dt


    fig, ax = plt.subplots(subplot_kw={'projection': 'polar'})
    ax.plot(theta, r)
```

```
    ax.set_rmax(2)
    ax.set_rticks([0.5, 1, 1.5, 2])  # Less radial ticks
    ax.set_rlabel_position(-5.5)  # Move radial labels away from plotted line
    ax.grid(True)


plt.show()
```







In the above plot, the top most plot is for initial condition at r=0. This is the unstalbe fixed point.

The Middle and the bottom plot, all the trajectories spiral asymptotically towards limit cycle at r=1. Here initial condition are 0.15 and 3 respectively.

[ ]:

```

```

4 3. FitzHugh-Nagumo Model

Details of analytic solution is attached with this jupyter as pdf.

for a=0, b=0.05.

The v-nullclines is$ w= v-v^{3}/3 +I$

The w- nullclines is $ w= 2v$

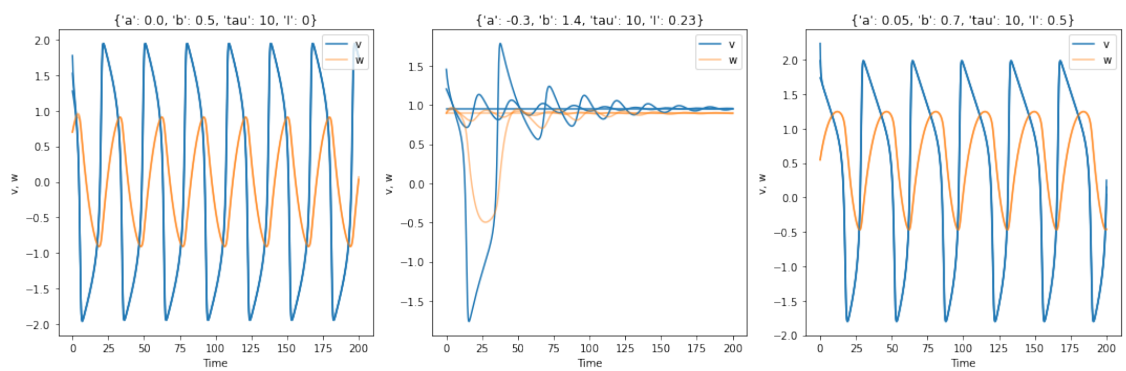
[2]:

```
from functools import partial
import numpy as np
import scipy.integrate
import scipy
import matplotlib.pyplot as plt
import matplotlib.patches as mpatches
%matplotlib inline

# simulate some trajectories.
# small perturbation of the resting potential as inital conditions.

scenarios = [
     {"a":0.0, "b":0.5, "tau":10, "I":0},
     {"a":-.3, "b":1.4, "tau":10, "I":0.23},
     {"a":0.05, "b":0.7, "tau":10, "I":0.5}
]
time_span = np.linspace(0, 200, num=1000)

def fitzhugh_nagumo(x, t, a, b, tau, I):
    return np.array([x[0] -x[0]**3/3 - x[1] + I,
                     (x[0] - a - b * x[1])/tau])

def get_displacement(param, dmax=0.5,time_span=np.linspace(0,200, 1000),␣
 ↪number=20):
    ic = scipy.integrate.odeint(partial(fitzhugh_nagumo, **param),
                                                      y0=[0.6,0.4],
                                                      t= np.linspace(0,999,␣
 ↪1000))[-1]
    # displacement of the potential.
    traj = []
    for displacement in np.linspace(0,dmax, number):
        traj.append(scipy.integrate.odeint(partial(fitzhugh_nagumo, **param),
```

```
                                                              y0=ic+np.
↪array([displacement,0]),
                                                              t=time_span))
    return traj
```

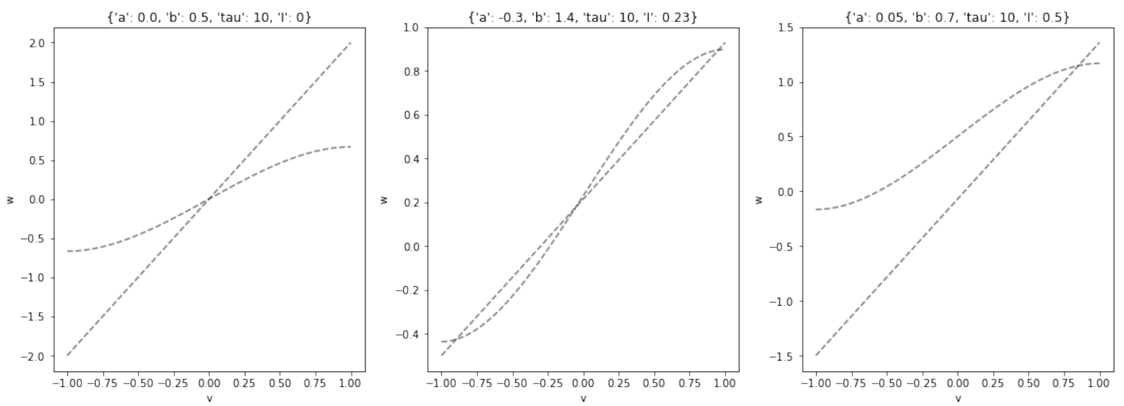
5 Trajectories at different values of I, a, b, tau

[3]:
```
# Do the numerical integration.
trajectories = {} # We store the trajectories in a dictionnary
for i,param in enumerate(scenarios):
    trajectories[i] = get_displacement(param, number=3, time_span=time_span,␣
↪dmax=0.5)

# Draw the trajectories.
fig, ax = plt.subplots(1, len(scenarios), figsize=(5*len(scenarios),5))
for i,param in enumerate(scenarios):
        ax[i].set(xlabel='Time', ylabel='v, w',
                     title='{}'.format(param))
        for j in range(len(trajectories[i])):
            v = ax[i].plot(time_span,trajectories[i][j][:,0], color='C0')
            w = ax[i].plot(time_span,trajectories[i][j][:,1], color='C1',␣
↪alpha=.5)
        ax[i].legend([v[0],w[0]],['v','w'])
plt.tight_layout()
```



6 Nullclines

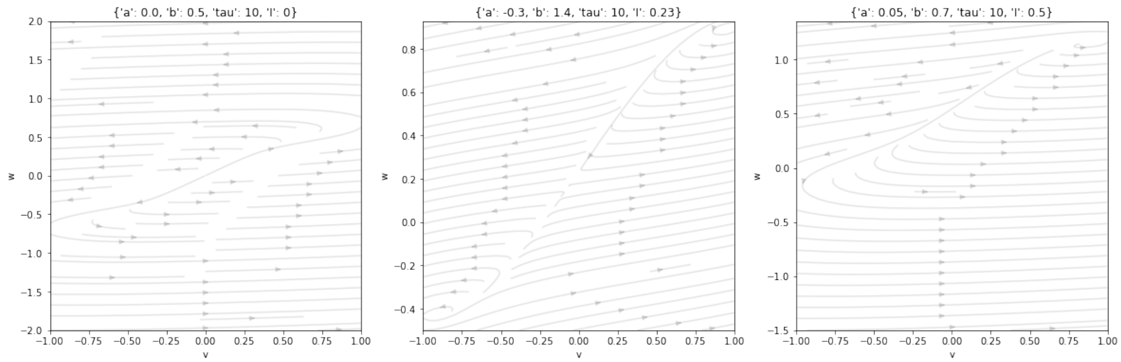
[4]:
```
def plot_isocline(ax, a, b, tau, I, color='k', style='--', opacity=.5, vmin=-1,vmax=1):
    v = np.linspace(vmin,vmax,100)
    ax.plot(v, v - v**3/3 + I, style, color=color, alpha=opacity)
    ax.plot(v, (v - a)/b, style, color=color, alpha=opacity)

fig, ax = plt.subplots(1, 3, figsize=(18, 6))
for i, sc in enumerate(scenarios):
    plot_isocline(ax[i], **sc)
    ax[i].set(xlabel='v', ylabel='w',
              title='{}'.format(sc))
```



7 Flow

[5]:
```
def plot_vector_field(ax, param, xrange, yrange, steps=50):
    # Compute the vector field
    x = np.linspace(xrange[0], xrange[1], steps)
    y = np.linspace(yrange[0], yrange[1], steps)
    X,Y = np.meshgrid(x,y)

    dx,dy = fitzhugh_nagumo([X,Y],0,**param)
    ax.streamplot(X,Y,dx, dy, color=(0,0,0,.1))

    ax.set(xlim=(xrange[0], xrange[1]), ylim=(yrange[0], yrange[1]))

fig, ax = plt.subplots(1, 3, figsize=(20, 6))
for i, sc in enumerate(scenarios):
    xrange = (-1, 1)
```

```
    yrange = [(1/sc['b'])*(x-sc['a']) for x in xrange]
    plot_vector_field(ax[i], sc, xrange, yrange)
    ax[i].set(xlabel='v', ylabel='w',
          title='{}'.format(sc))
```



8 Equilibrium points

[6]:
```
def find_roots(a,b,I, tau):
    # The coeficients of the polynomial equation are:
    # 1/3           * v**3
    # 0             * v**2
    # - (1/b - 1) * v**1
    # - (a/b + I) * v**0
    coef = [1/3, 0, 1/b - 1, - a/b - I]

    # We are only interested in real roots.
    # np.isreal(x) returns True only if x is real.
    # The following line filter the list returned by np.roots
    # and only keep the real values.
    roots = [np.real(r) for r in np.roots(coef) if np.isreal(r)]

    # We store the position of the equilibrium.
    return [[r, r - r**3/3 + I] for r in roots]

eqnproot = {}
for i, param in enumerate(scenarios):
    eqnproot[i] = find_roots(**param)


# You can use the following color code:
EQUILIBRIUM_COLOR = {'Stable node':'C0',
                    'Unstable node':'C1',
```

```
                    'Saddle':'C4',
                    'Stable focus':'C3',
                    'Unstable focus':'C2',
                    'Center':'C5'}

def jacobian_fitznagumo(v, w, a, b, tau, I):
    """ Jacobian matrix of the ODE system modeling Fitzhugh-Nagumo's excitable␣
↪system
    Args
    ====
        v (float): Membrane potential
        w (float): Recovery variable
        a,b (float): Parameters
        tau (float): Recovery timescale.
    Return: np.array 2x2"""
    return np.array([[- v**2 + 1 , -1],
                       [1/tau, -b/tau]])

# Symbolic computation of the Jacobian using sympy...
import sympy
sympy.init_printing()

# Define variable as symbols for sympy
v, w = sympy.symbols("v, w")
a, b, tau, I = sympy.symbols("a, b, tau, I")

# Symbolic expression of the system
dvdt = v - v**3/3 - w + I
dwdt = (v - a - b * w)/tau

# Symbolic expression of the matrix
sys = sympy.Matrix([dvdt, dwdt])
var = sympy.Matrix([v, w])
jac = sys.jacobian(var)

#  can convert jac to a function:
jacobian_fitznagumo_symbolic = sympy.lambdify((v, w, a, b, tau, I), jac,␣
↪dummify=False)

#jacobian_fitznagumo = jacobian_fitznagumo_symbolic
jac
```

[6]:

$$\begin{bmatrix} 1 - v^2 & -1 \\ \frac{1}{\tau} & -\frac{b}{\tau} \end{bmatrix}$$

[7]:

```
def stability(jacobian):
    """ Stability of the equilibrium given its associated 2x2 jacobian matrix.
```

```python
    Use the eigenvalues.
    Args:
        jacobian (np.array 2x2): the jacobian matrix at the equilibrium point.
    Return:
        (string) status of equilibrium point.
    """

    eigv = np.linalg.eigvals(jacobian)


    if all(np.real(eigv)==0) and all(np.imag(eigv)!=0):
        nature = "Center"
    elif np.real(eigv)[0]*np.real(eigv)[1]<0:
        nature = "Saddle"
    else:
        stability = 'Unstable' if all(np.real(eigv)>0) else 'Stable'
        nature = stability + (' focus' if all(np.imag(eigv)!=0) else ' node')
    return nature

def stability_alt(jacobian):
    """ Stability of the equilibrium given its associated 2x2 jacobian matrix.
    Use the trace and determinant.
    Args:
        jacobian (np.array 2x2): the jacobian matrix at the equilibrium point.
    Return:
        (string) status of equilibrium point.
    """

    determinant = np.linalg.det(jacobian)
    trace = np.matrix.trace(jacobian)
    if np.isclose(trace, 0):
        nature = "Center (Hopf)"
    elif np.isclose(determinant, 0):
        nature = "Transcritical (Saddle-Node)"
    elif determinant < 0:
        nature = "Saddle"
    else:
        nature = "Stable" if trace < 0 else "Unstable"
        nature += " focus" if (trace**2 - 4 * determinant) < 0 else " node"
    return nature

eqstability = {}
for i, param in enumerate(scenarios):
    eqstability[i] = []
    for e in eqnproot[i]:
        J = jacobian_fitznagumo(e[0],e[1], **param)
        eqstability[i].append(stability(J))
```

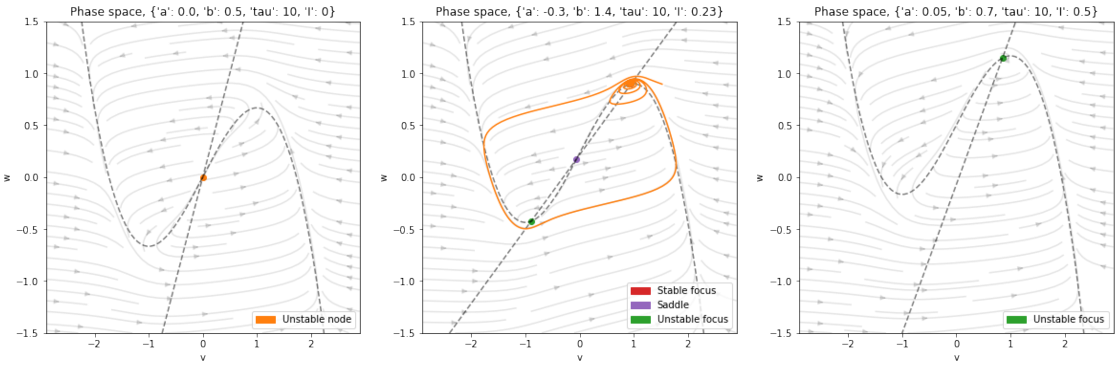
```
eqstability
```

[7]:
```
{0: ['Unstable node'],
 1: ['Stable focus', 'Unstable focus', 'Saddle'],
 2: ['Unstable focus']}
```

9 Phase diagram

[8]:
```
def plot_phase_diagram(param, ax=None, title=None):
    """Plot a complete Fitzhugh-Nagumo phase Diagram in ax.
    Including isoclines, flow vector field, equilibria and their stability"""
    if ax is None:
        ax = plt.gca()
    if title is None:
        title = "Phase space, {}".format(param)

    ax.set(xlabel='v', ylabel='w', title=title)

    # Isocline and flow...
    xlimit = (-2.9, 2.9)
    ylimit = (-1.5, 1.5)
    plot_vector_field(ax, param, xlimit, ylimit)
    plot_isocline(ax, **param, vmin=xlimit[0],vmax=xlimit[1])

    # Plot the equilibria
    eqnproot = find_roots(**param)
    eqstability = [stability(jacobian_fitznagumo(e[0],e[1], **param)) for e in␣
↪eqnproot]
    for e,n in zip(eqnproot,eqstability):
        ax.scatter(*e, color=EQUILIBRIUM_COLOR[n])

        # Show a small perturbation of the stable equilibria...
        time_span = np.linspace(0, 200, num=1500)
        if n[:6] == 'Stable':
            for perturb in (0.1, 0.6):
                ic = [e[0]+abs(perturb*e[0]),e[1]]
                traj = scipy.integrate.odeint(partial(fitzhugh_nagumo, **param),
                                                      y0=ic,
                                                      t=time_span)
                ax.plot(traj[:,0], traj[:,1])

    # Legend
    labels = frozenset(eqstability)
    ax.legend([mpatches.Patch(color=EQUILIBRIUM_COLOR[n]) for n in labels],␣
↪labels,
```

```
            loc='lower right')

fig, ax = plt.subplots(1,3, figsize=(20, 6))
for i, param in enumerate(scenarios):
    plot_phase_diagram(param, ax[i])
```



[ ]:

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Dynamical Systems Theory in Machine Learning & Data science

Exercise : 3. 17 Nov 2021.

1. Poincaré - Bendixson Theorem:-

System

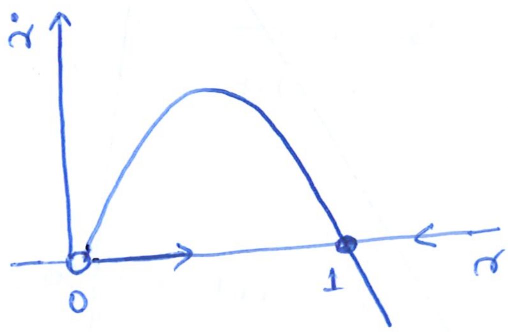
$$\dot{r} = r(1-r^2) + \varepsilon r \cos\theta \quad , \quad \dot{\theta} = 1$$

Soln:-

1. For $\varepsilon = 0$, system becomes

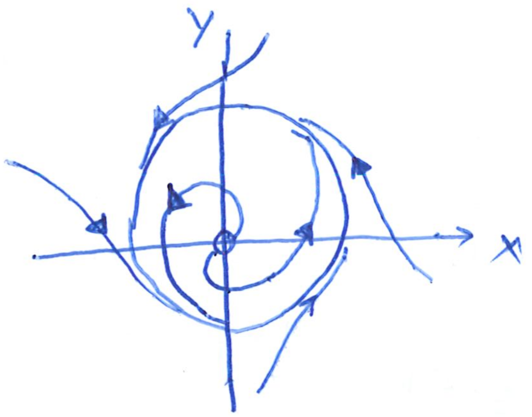
$$\dot{r} = r(1-r^2) \quad , \quad \dot{\theta} = 1$$

Treat $\dot{r} = r(1-r^2)$ as a vector line on the line.



One can observe that $r^* = 0$ is unstable fixed point and $r = 1$ is stable.

Here, in phase plane, all trajectories (except $r^* = 0$) approach the unit circle $r^* = 1$ monotonically. Since the motion in θ-direction is simply rotation at constant angular velocity and all the trajectories spiral asymptotically towards limit cycle at $r = 1$.



Note:- Along with this solution "I have attached Jupyter notebook for the phase analysis".

2. For $\varepsilon > 0$.

Soln:- Using hints given in exercise sheet: - construct a trapping region $\{(r, \theta): 0 \le r_1 \le r \le r_2\}$.

Construct a concentric circles with radius r_1 and r_2 such that $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle.

Hence the annulus $0 \le r_1 \le r \le r_2$ is our desired trapping region.

$\because \dot{\theta} = 1 > 0$, so there is no fixed point in the annulus.

Hence if we find r_1 and r_2, then the Poincare-Bendixson theorem will guarantee the existence of a closed orbit.

Find out r_1 and r_2.

For r_1, we require $\dot{r} = r(1-r^2) + \varepsilon r \cos\theta > 0 \quad \forall \theta$.

$$\because \cos\theta \ge -1$$

$$\therefore r(1-r^2) - \varepsilon r > 0$$

$$\Rightarrow r[1 - r^2 - \varepsilon] > 0$$

$$\Rightarrow r^2 < 1 - \varepsilon$$

$$\therefore r_1 < \sqrt{1-\varepsilon}$$

$\because$ as long as $\varepsilon < 1$ this $\sqrt{1-\varepsilon}$ makes sense.

So, choose r_1 as large as possible to put in the limit cycles.

Similarly, for the flow in the inward on the outer circles, if

$$\dot{r} = r(1-r^2) + \varepsilon r \cos\theta < 0 \quad \forall \theta, \quad \because \cos\theta \le 1$$

$$\therefore r(1-r^2) + \varepsilon r < 0$$

$$\Rightarrow 1 - r^2 + \varepsilon < 0 \Rightarrow r > \sqrt{1+\varepsilon} \quad \text{for } \varepsilon < 1$$

$\therefore$ min of r to be $r_2 > \sqrt{1+\varepsilon}$

Hence for a $\varepsilon < 1$, there exist r s.t

$$\sqrt{1-\varepsilon} < r < \sqrt{1+\varepsilon} \quad \square.$$

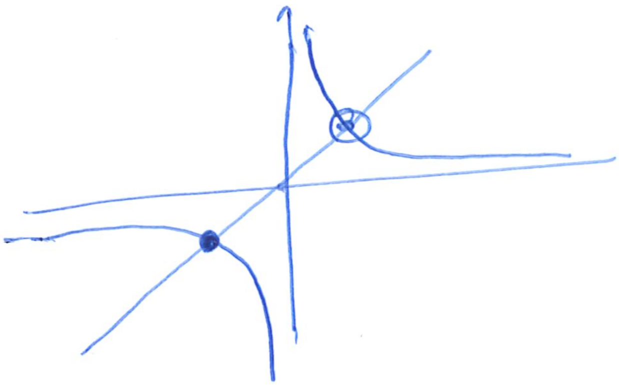
2. Example system.

(i) stable fixed point at $(-1,1)$ and unstable fixed point at $(1,1)$

Soln:-

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = xy - 1.$$

Nullclines are $y = x$ and $y = 1/x$



Find the Jacobian matrix

∴ calculate the eigenvalue

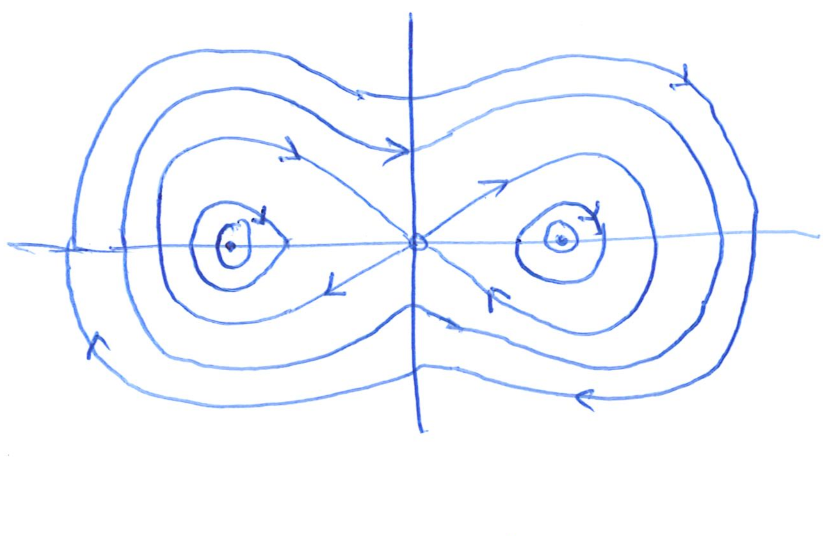
(ii) A linear system with fixed point other than origin

→ $\dot{x} = -x + x^3$
$\dot{y} = -2y$

Fixed points are $(0,0)$, $(1,0)$, $(-1,0)$.

(5) A conservative system with more than one cycle.

→ $\dot{x} = y$
$\dot{y} = x - x^3$



double well potential.

Equation of motion.

$\ddot{x} = x - x^3$

System is written as

$\dot{x} = y$
$\dot{y} = x - x^3$.

Saddle point at $(0,0)$ and center at $(1,0)$, $(-1,0)$.

3. FitzHugh - Nagumo model.

$$\dot{v} = v - \frac{1}{3}v^3 - w + I$$

$$\tau \dot{w} = v - a - bw.$$

with $a \in \mathbb{R}$, $b \in (0,1)$, $I \geq 0$, $\tau \gg 1$.

(1) Equation for the nullclines for $a = 0$, $b = 0.5$

System become

$$\dot{v} = v - \frac{1}{3}v^3 - w + I$$

$$\tau \dot{w} = v - 0.5w$$

→ For v - Nullclines, $\dot{v} = 0$

$$\Rightarrow v - \frac{1}{3}v^3 - w + I = 0$$

$$\Rightarrow \boxed{w = v - \frac{1}{3}v^3 + I}$$ → V- Nullcline.

→ For w - nullcline, $\dot{w} = 0$

$$\Rightarrow v - 0.5w = 0$$

$$\Rightarrow \boxed{w = 2v}$$ → w - Nullcline.

Calculation of fixed point

For fixed point $\dot{v} = 0$, $\dot{w} = 0$

⇒ (Intersection of Nullcline are the fixed points)

Hence $2v = v - \frac{1}{3}v^3 + I$

$$\Rightarrow \frac{1}{3}v^3 + v - I = 0$$

$$\therefore v^3 + 3v - 3I = 0$$

Solution of this cubic equation gives the fixed point $\boxed{v = f(I)}$