Dynamical Systems Theory in Machine Learning & Data Science

lecturers: Daniel Durstewitz, Zahra Monfared
tutors: Manuel Brenner, Daniel Kramer, **Janik Fechtelpeter**, **Max Thurm**, Jonas Mikhaeil, Unai Fischer
WS2021/22

Exercise 2

To be uploaded before the exercise group on November 10, 2021

1 Linearization Fail

Note that for any $a \in \mathbb{R}$ the following system has a fixed point at $(0, 0)$:

$$\dot{x} = -y + (ax - y)(x^2 + y^2)$$
$$\dot{y} = x + (x + ay)(x^2 + y^2)$$

1. Show that for all $a \in \mathbb{R}$, $(0, 0)$ is a linear center.
2. To prove that it is not always a true center, transfer the system into polar coordinates. This means that you have to find equivalent differential equations for $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$
3. Find $a \in \mathbb{R}$ such that $(0, 0)$ is a stable spiral, unstable spiral, or center.
4. For each of these cases, find the stable set Ω^s and the unstable set Ω^u.

2 Cycles

A cycle or closed orbit is a trajectory $x(t)$ in a dynamical system such that there are $t_2 > t_1$ with $x(t_2) = x(t_1)$, and for all $t \in (t_1, t_2)$, $x(t) \neq x(t_1)$. A center is a fixed point such that all trajectories sufficiently close to it are cycles.
For the following cases, decide if the system either has no cycle or at least one. Explain why.

1. Consider the system $\dot{x} = f(x)$ in the $\mathbb{R}^2$ space. $x^\star$ is a fixed point and $V : \mathbb{R}^2 \to \mathbb{R}$ a continuously differentiable function such that (a) $V(x) > 0$ for all $x \neq x^\star$, (b) $V(x^\star) = 0$, and (c) $\frac{dV}{dt}(x) < 0$ for all $x \neq x^\star$. (Such a function is called a Liapunov function.)
2. Consider the SIR model from last exercise sheet:

   $$\dot{S} = -\frac{\beta}{N} S I, \quad \dot{I} = \frac{\beta}{N} S I - \gamma I, \quad \dot{R} = \gamma I.$$

   Show that it is a conservative system.
3. Consider the system

   $$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

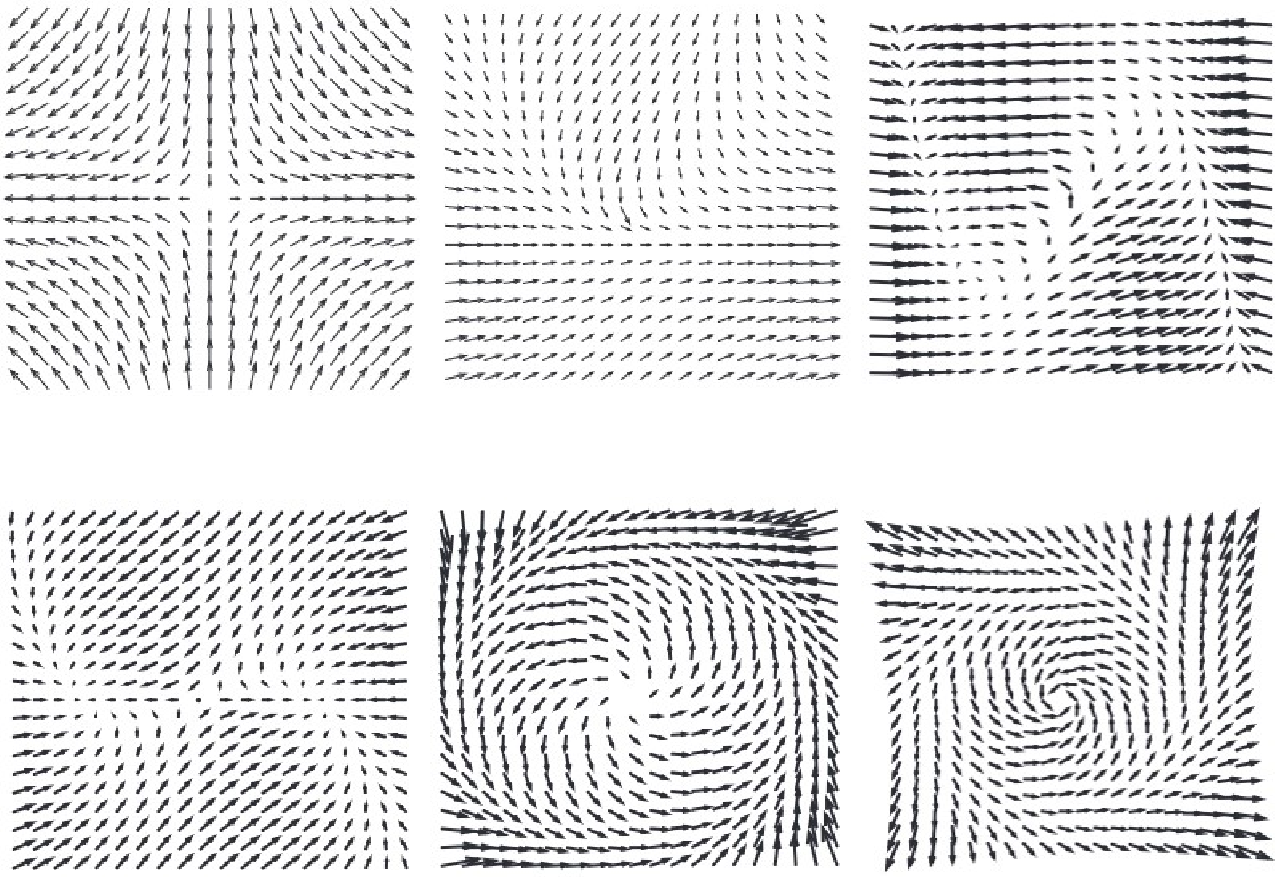
   where $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$ for all $x, y \in \mathbb{R}$.

4. Consider $\dot{x} = f(x)$ in the $\mathbb{R}^2$ space. Assume there exists a continuously differentiable function $V : \mathbb{R}^2 \to \mathbb{R}$ such that (a) $\frac{dV}{dt} \equiv 0$, and (b) the system can be written as $\dot{x} = -\nabla V$. (Such a system is called a gradient system.)

5. Assume $\dot{x} = f(x)$ in the $\mathbb{R}^2$ space has no fixed point in the set $C = \{x \in \mathbb{R}^2 | 1 \leq \|x\| \leq 2\}$, and that there exists a trajectory x such that for all $t \in \mathbb{R}$, $x(t) \in C$ (For this task, an intuitive explanation suffices. A rigorous one follows in the next lecture).

3 Graphical Analysis

Use a pencil or a graphical tool to fill the following vector fields with phase portraits. Specifically draw fixed points (indicating their stability), nullclines and, if possible, cycles.



4 Visualization of 2D-systems

Write a simple graphical tool (in Python or Matlab) to create vector fields from system equations. Use the system from Ex. 1 as an example. Hint (if you are working in Python): use `meshgrid` and `quiver`. Please provide code preferably as a Jupyter notebook.

1. Plot the vector field of the system from Ex.1 for different a.

2. Plot the fixed points and their stability to the plots from part 1.

3. Add nullclines and cycles.

4. Add a subplot that visualizes the trajectory over time from a random initial condition.