Dynamical Systems Theory in Machine Learning & Data Science

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Exercise 1

To be uploaded before the exercise group on November 3rd, 2021

Task 1. A simple SIR model
In the first lecture you were introduced to a simple epidemiological model, the so-called SIR model, defined by

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta \frac{S\,I}{N} \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta \frac{S\,I}{N} - \gamma I \quad \frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I; \tag{I}$$

1. Assume a total population of $N = 1000$. Define a SRI model with $\beta = 2.4$, $\gamma = 1./20$, one initial sick patient, and 60 percent of the population susceptible to illness, and plot its time evolution until the dynamics run into a fixed point.
2. Adjust the model for the case that infections do not confer long-term immunity to the disease (this means that recovered patients instead fall back to the susceptible group). Depending on different values for β and γ, what are the two types of long-term behavior that this model predicts?
3. Assume a mortality rate of 1 percent, and include deceased patients into the model. Plot the time evolution of the new model.

Task 2. Higher order ODE systems

Show that any n-th order ODE of the form $f(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}^2t}, \cdots, \frac{\mathrm{d}^nx}{\mathrm{d}^nt}) = 0$ can always be transformed into a system of n first order ODEs.

Task 3. ODEs on the real line

Show why one-dimensional ODEs on the real line can never give rise to oscillatory behavior.

Task 4. Flow field of linear ODEs

Assume a linear ODE system, defined by $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A} \cdot \mathbf{x}$, where $A = \begin{pmatrix} 0.1 & -0.3 \\ 0.2 & -0.3 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

1. Determine the fixed points of this system by hand.
2. Plot the time evolution of the system for an initial vector of $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
3. Plot the flow field of the system in an appropriate range. What do you observe?
4. In the lecture you learned about 5 different types of dynamics that can be obtained from a two-dimensional linear dynamical system. Define system parameters of your choosing that satisfy the respective requirements, and plot the corresponding trajectories and flow fields.