Dynamical Systems Theory in Machine Learning & Data Science

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Exercise 3

To be uploaded before the exercise group on November 17, 2021

1 Poincaré-Bendixson Theorem

Consider the system

$$\dot{r} = r(1 - r^2) + \varepsilon r \cos\theta, \quad \dot{\theta} = 1$$

which is defined in polar coordinates.

1. Show that, for $\varepsilon = 0$, the system has a stable fixed point at $r = 1$. Argue why this proves the existence of a stable limit cycle.
2. Now let $\varepsilon > 0$ - the fixed point is gone. Use the Poincaré-Bendixson theorem to show that for small enough ε, the limit cycle still exists. Hint: construct a trapping region that has the shape $\{(r, \theta) : 0 < r_1 \leq r \leq r_2\}$, i.e. it looks like a doughnut.

2 Example Systems

For each of the following conditions, provide an example of a dynamical system defined in $\mathbb{R}^2$ which fulfills them (with proof that it does, the proof may be graphical). The examples have to be nontrivial, i.e. no derivative can be 0 everywhere. Use either cartesian, polar, or cylinder coordinates. If you think that no such system exists, explain why.

1. A stable fixed point at $(-1, 1)$ and an unstable fixed point at $(1, 1)$. No other fixed points.
2. A linear system with a fixed point other than the origin, and no line attractor (a.k.a. non-isolated fixed point).
3. Two saddle nodes at $(1, 0)$ and $(0, 1)$ and an unstable node.
4. A half-stable ”saddle” cycle around the origin.
5. A conservative system with more than one cycle.
6. Four saddle nodes at $(\pm\sqrt{2}, \pm\sqrt{2})$ and a half-stable cycle with radius 1 around $(1, 1)$. No other fixed points.
7. Infinitely many fixed points, but no line attractor.

3 FitzHugh-Nagumo Model

This model as defined in the lecture imitates the generation of neuronal action potentials:

$$\dot{v} = v - \frac{1}{3}v^3 - w + I, \quad \tau\dot{w} = v - a - bw$$

with $a \in \mathbb{R}, b \in (0,1), I \geq 0, \tau \gg 1$. v mimics the membrane potential and w a recovery outward current.

1. Write down equations for the nullclines for $a = 0, b = 0.5$. Calculate the fixed point and classify its stability as a function of the input current I.
2. Make a plot with the nullclines for $I = 0, \tau = 10$, and a trajectory starting from any initial conditions except the origin. To plot the trajectory, use a numerical integrator (e.g. in Python `scipy.integrate.solve_ivp`). Please attach the code.
3. On your plot, you should see a limit cycle, which partly follows the v nullcline. Make an educated guess about the time the trajectory flows along the nullcline compared to the time it spends apart from it, e.g. $t_{nullcline} \approx O(xyz)$.
4. Numerically ascertain that the cycle vanishes for some $a < 0$, but returns if you increase I. Find a condition for a and I that guarantees the existence or non-existence of the cycle.