Exercise 2.

November 9, 2021

1 Dynamical Systems Theory in Machine Learning & Data Science

2 Name:- Kaushal Kumar

Remark: I am submitting this exercise group individually.

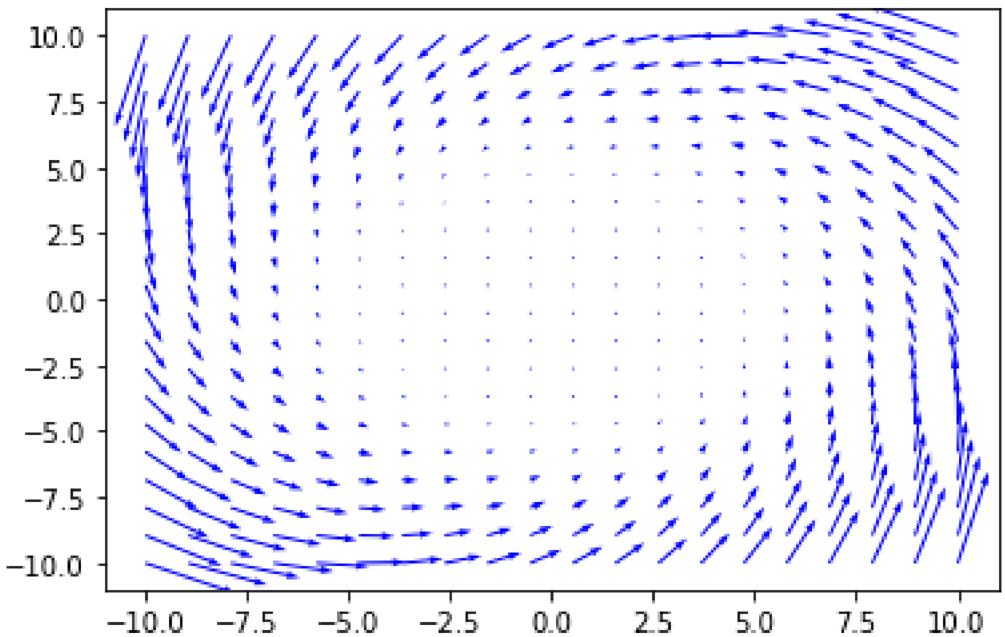
3 EX.4:- Visualization of 2D-systems

1. Plot the vector field of the system.

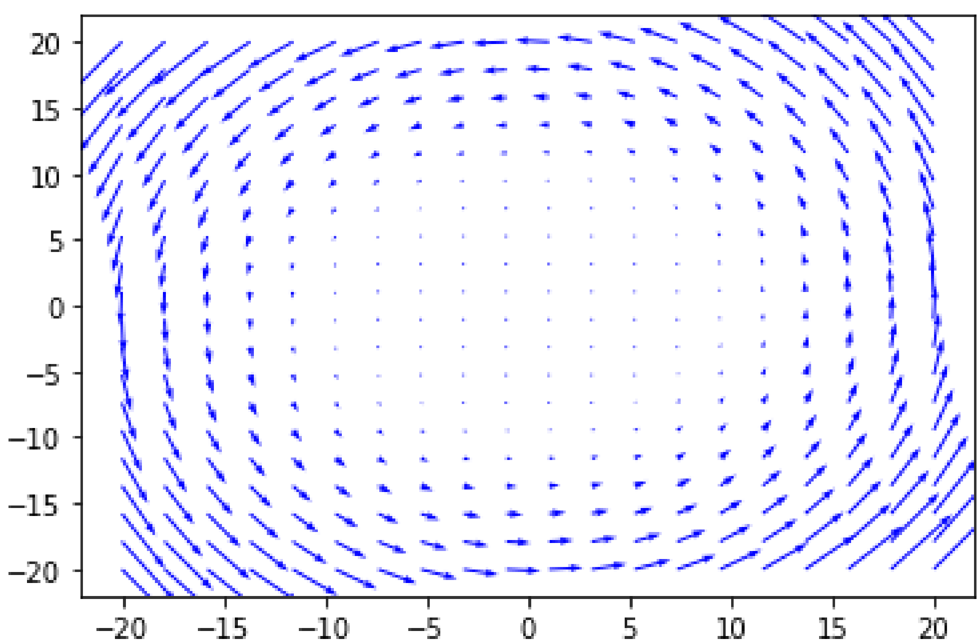
[1]:
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
import pylab as pl
%matplotlib inline
```

[2]:
```python
# case 1. when "a" is negative , choose a= -0.5
X,Y = np.mgrid[-10:10:20j, -10:10:20j]
u1=-Y+(-0.5*X-Y)*(X**2+Y**2)
v1= X+(X-0.5*Y)*(X**2+Y**2)
pl.quiver(X,Y,u1,v1,color='b')
plt.xlabel('Case.1: a < 0')
plt.show()

# case 2. when "a" is zero , choose a= 0.0
X,Y = np.mgrid[-20:20:20j, -20:20:20j]
u2=-Y+(-Y)*(X**2+Y**2)
v2= X+(X)*(X**2+Y**2)
pl.quiver(X,Y,u2,v2,color='b')
plt.xlabel('Case.2: a = 0')
plt.show()

# case 3. when a is positive, take a=0.5
X,Y = np.mgrid[-20:20:20j, -20:20:20j]
u3= -Y+(0.5*X-Y)*(X**2+Y**2)
```

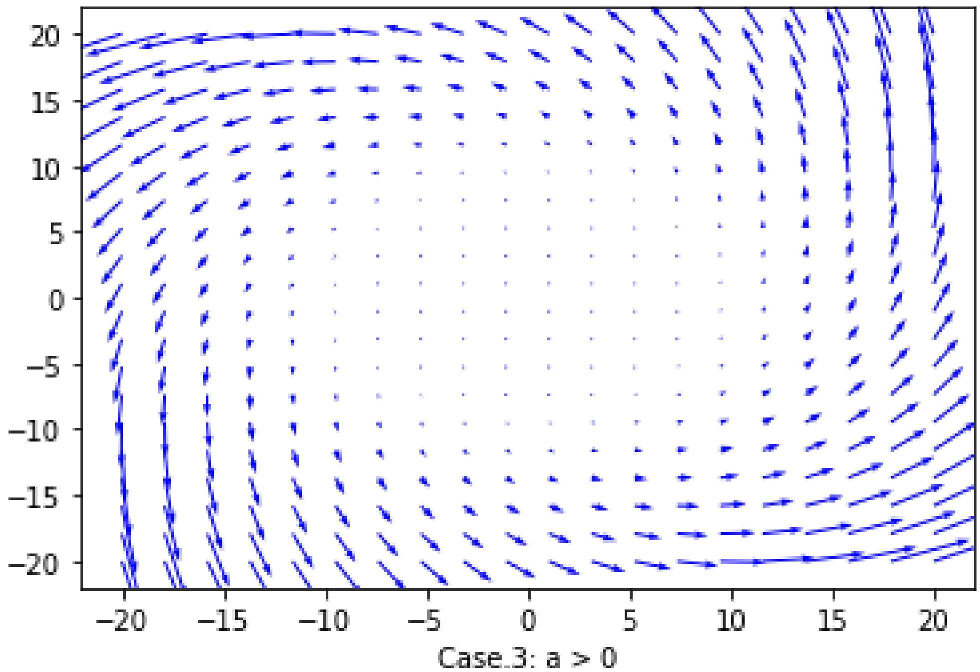
```
v3= X+(X+0.5*Y)*(X**2+Y**2)
pl.quiver(X,Y,u3,v3,color='b')
plt.xlabel('Case.3: a > 0')
plt.show()
```



Case.1: a < 0



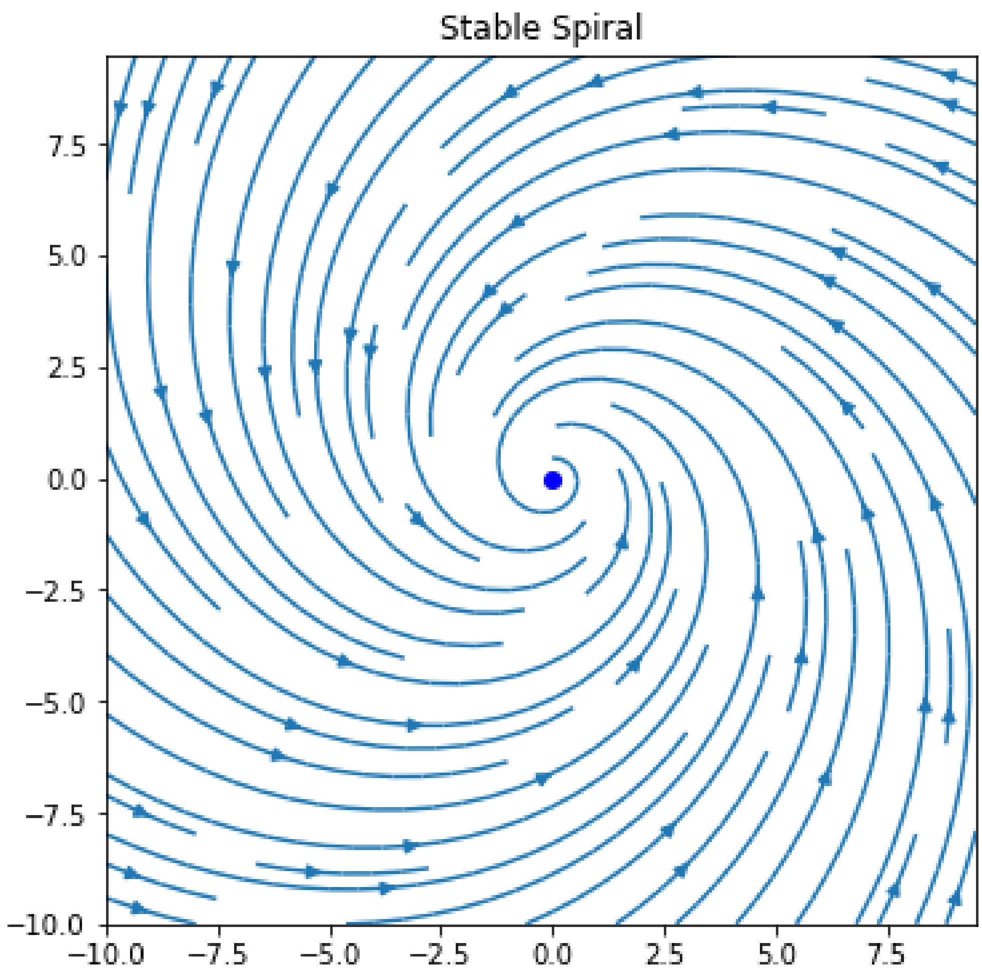
Case.2: a = 0

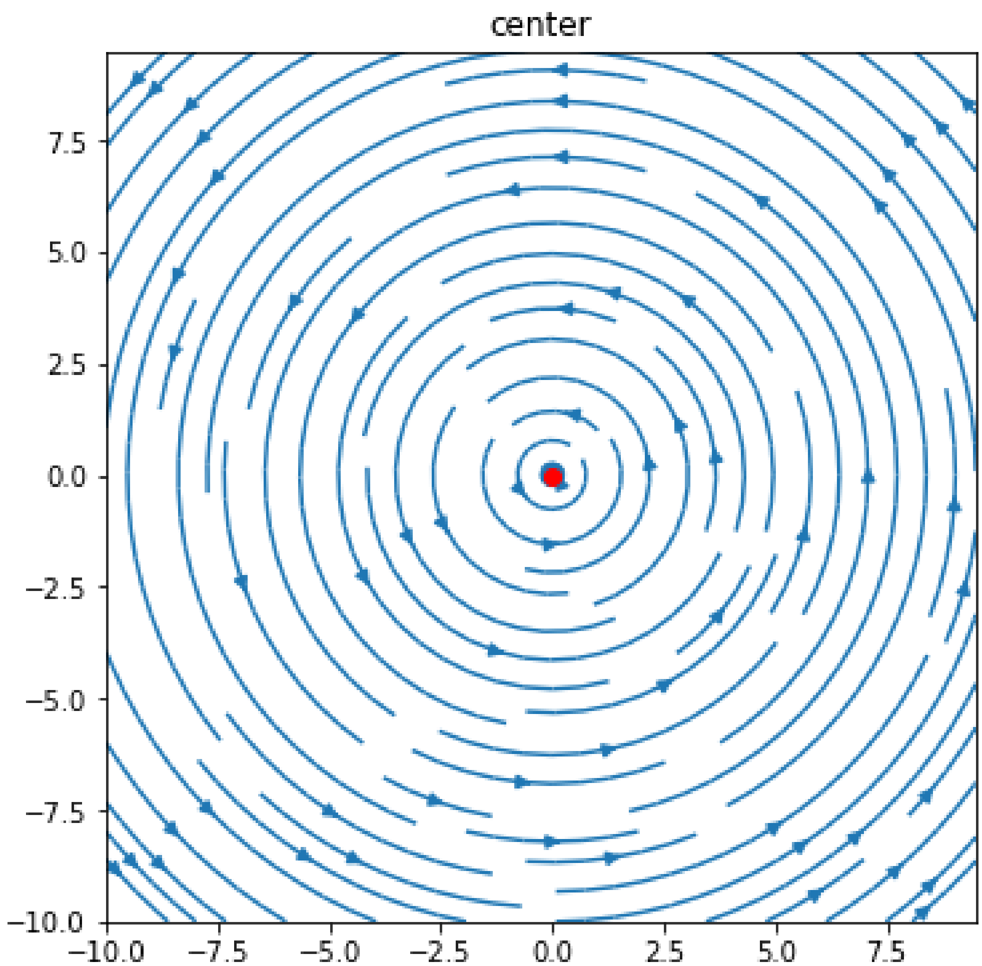


2. Plot the fixed point and their stability

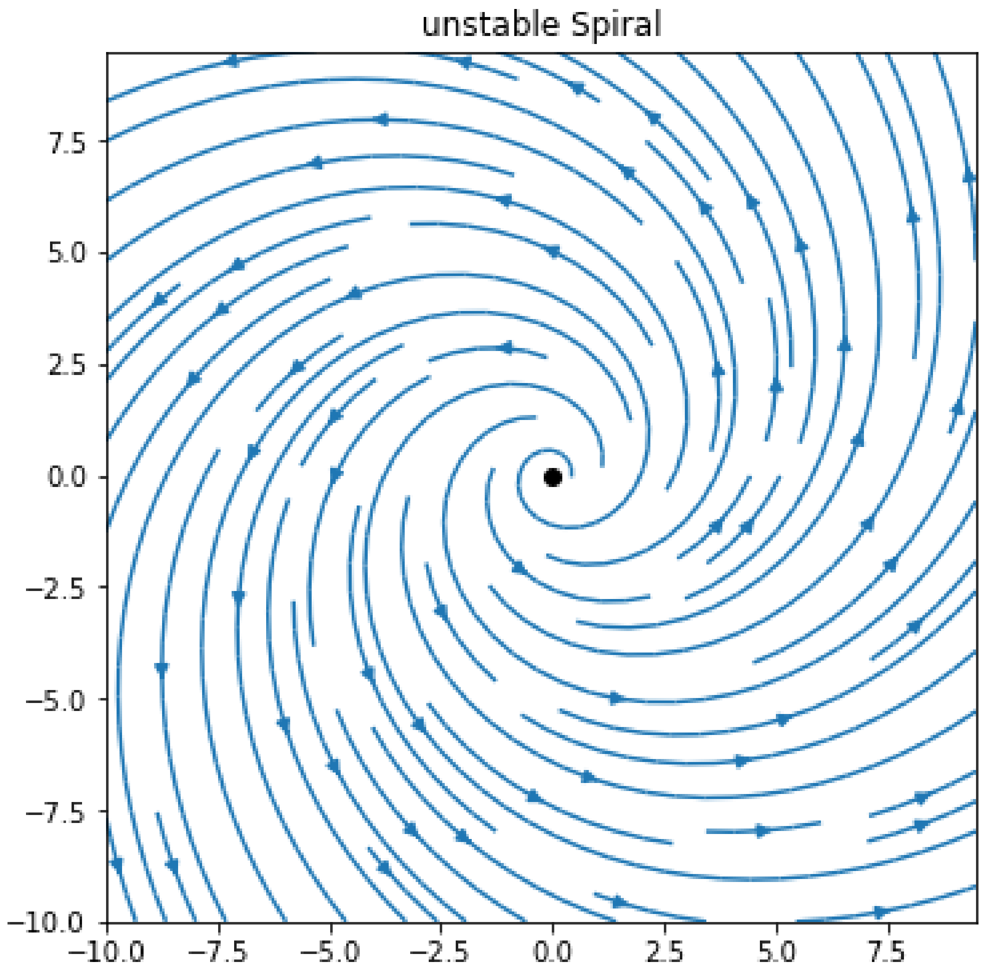
[3]:
```
# case 1. when "a" is negative , choose a= -0.5
x = np.arange(-10,10,0.5)
y = np.arange(-10,10,0.5)
X,Y = np.meshgrid(x,y)
Ex = -Y+(-0.5*X-Y)*(X**2+Y**2)
Ey = X+(X-0.5*Y)*(X**2+Y**2)
fig, ax = plt.subplots(figsize=(6,6))
ax.streamplot(X,Y,Ex,Ey)
ax.set_aspect('equal')
ax.plot(0,0,'-ob')
ax.set_title('Stable Spiral')
plt.show()


# case 2. when "a" is zero , choose a= 0.0
Ex2 = -Y+(-Y)*(X**2+Y**2)
Ey2 = X+(X)*(X**2+Y**2)
fig, ax = plt.subplots(figsize=(6,6))
ax.streamplot(X,Y,Ex2,Ey2)
ax.set_aspect('equal')
ax.plot(0,0,'-or')
```

```
ax.set_title('center')
plt.show()



# case 3. when a is positive, take a=0.5
Ex3 = -Y+(0.5*X-Y)*(X**2+Y**2)
Ey3 = X+(X+0.5*Y)*(X**2+Y**2)
fig, ax = plt.subplots(figsize=(6,6))
ax.streamplot(X,Y,Ex3,Ey3)
ax.set_aspect('equal')
ax.plot(0,0,'-ok')
ax.set_title('unstable Spiral')
plt.show()
```





center
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5



3. Fixed point

[4]:
```
# when 'a' ia negative , i.e a=-0.5

# define system in terms of separated differential equations
def f(x,y):
    return -y+(-0.5*x-y)*(x**2+y**2)
def g(x,y):
    return x+(x-0.5*y)*(x**2+y**2)

# brute force: iterate through possibility space (r)
fp = []

def find_fixed_points(r):
    for x in range(r):
        for y in range(r):
            if ((f(x,y) == 0) and (g(x,y) == 0)):
```

```
                fp.append((x,y))
                print('The system has a fixed point in %s,%s' % (x,y))
    return fp

find_fixed_points(10)
```

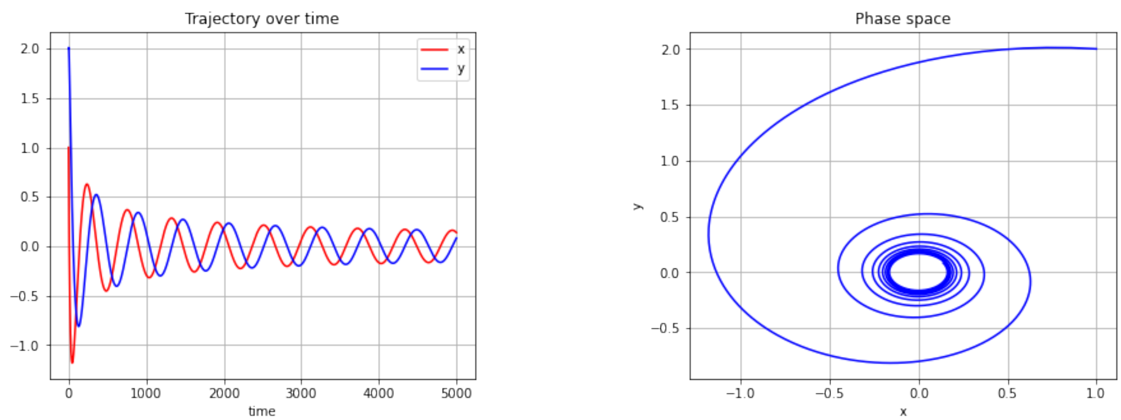
The system has a fixed point in 0,0

[4]: [(0, 0)]

4. Visualize the trajectory over time from a random initial condition.

[5]:
```
# when 'a' ia negative , i.e a=-0.5

# define system in terms of separated differential equations
def f(x,y):
    return -y+(-0.5*x-y)*(x**2+y**2)
def g(x,y):
    return x+(x-0.5*y)*(x**2+y**2)

# initialize lists containing values
x = []
y = []

#iv1, iv2 = initial values, dt = timestep, time = range
def sys(iv1, iv2, dt, time):
    # initial values:
    x.append(iv1)
    y.append(iv2)
    #z.append(iv3)
    # compute and fill lists
    for i in range(time):
        x.append(x[i] + (f(x[i],y[i])) * dt)
        y.append(y[i] + (g(x[i],y[i])) * dt)
        #z.append(z[i] + (h(x[i],y[i],z[i])) * dt)
    return x, y

sys(1, 2, 0.01, 5000)

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(x, 'r-', label='x')
ax1.plot(y, 'b-', label='y')
```

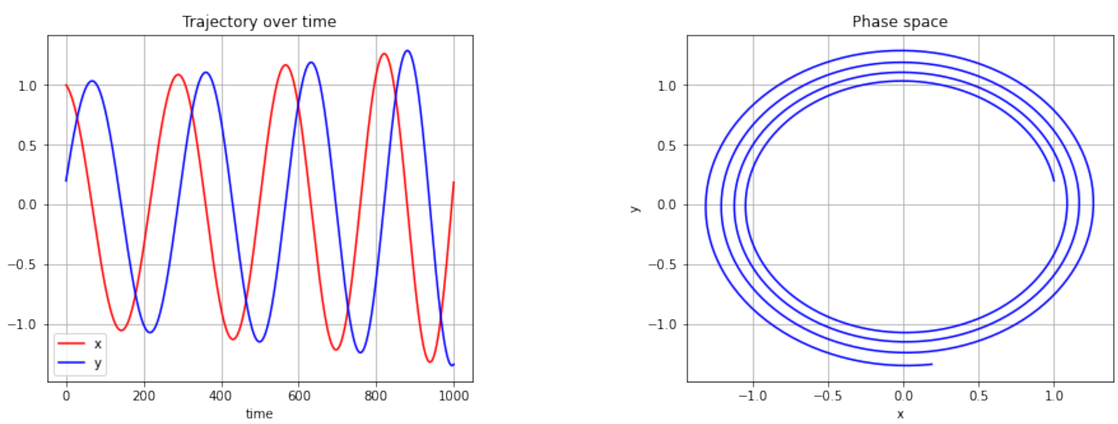
```
ax1.set_title("Trajectory over time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x, y, color="blue")
ax2.set_xlabel("x")
ax2.set_ylabel("y")
ax2.set_title("Phase space")
ax2.grid()
```



[6]:

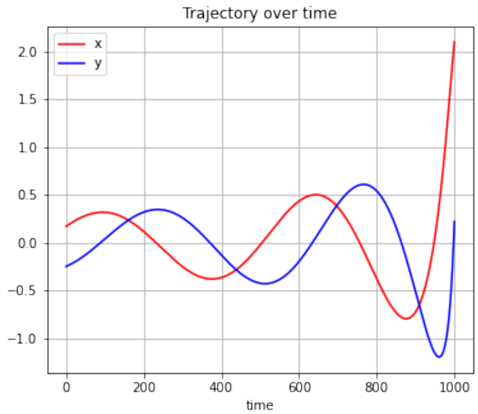
```
# when 'a' is zero, i.e a=0.0

# define system in terms of separated differential equations
def f(x,y):
    return -y+(-y)*(x**2+y**2)
def g(x,y):
    return x+(x)*(x**2+y**2)

# initialize lists containing values
x = []
y = []

#iv1, iv2 = initial values, dt = timestep, time = range
def sys(iv1, iv2, dt, time):
    # initial values:
    x.append(iv1)
    y.append(iv2)
    #z.append(iv3)
    # compute and fill lists
    for i in range(time):
        x.append(x[i] + (f(x[i],y[i])) * dt)
```

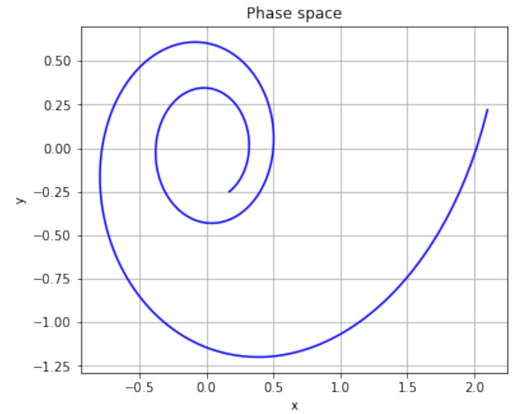
```
        y.append(y[i] + (g(x[i],y[i])) * dt)
        #z.append(z[i] + (h(x[i],y[i],z[i])) * dt)
    return x, y

sys(1, 0.2, 0.01, 1000)

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(x, 'r-', label='x')
ax1.plot(y, 'b-', label='y')
ax1.set_title("Trajectory over time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x, y, color="blue")
ax2.set_xlabel("x")
ax2.set_ylabel("y")
ax2.set_title("Phase space")
ax2.grid()
```



[7]:
```
# when 'a' is positive , i.e a=0.5

# define system in terms of separated differential equations
def f(x,y):
    return -y+(0.5*x-y)*(x**2+y**2)
def g(x,y):
    return x+(x+0.5*y)*(x**2+y**2)
```

```python
# initialize lists containing values
x = []
y = []

#iv1, iv2 = initial values, dt = timestep, time = range
def sys(iv1, iv2, dt, time):
    # initial values:
    x.append(iv1)
    y.append(iv2)
    #z.append(iv3)
    # compute and fill lists
    for i in range(time):
        x.append(x[i] + (f(x[i],y[i])) * dt)
        y.append(y[i] + (g(x[i],y[i])) * dt)
        #z.append(z[i] + (h(x[i],y[i],z[i])) * dt)
    return x, y

sys(0.169,-0.25, 0.01, 1000)

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(x, 'r-', label='x')
ax1.plot(y, 'b-', label='y')
ax1.set_title("Trajectory over time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x, y, color="blue")
ax2.set_xlabel("x")
ax2.set_ylabel("y")
ax2.set_title("Phase space")
ax2.grid()
```


Trajectory over time
x
y
time


Phase space
x
y

Dynamical system, Kaushal Kumar.

Exercise 2. Date:- 10 November 2021.

1. Linearization Fails.

$$\dot{x} = -y + (ax - y)(x^2+y^2)$$
$$\dot{y} = x + (x + ay)(x^2+y^2)$$

1. Show that for all $a \in \mathbb{R}$, $(0,0)$ is a linear ~~system~~ center.

Soln:- Calculate the Jacobian matrix at a general point (x, y).

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix}$$

Now,

$$\frac{\partial \dot{x}}{\partial x} = a(x^2+y^2) + (ax-y) \times 2x = a(x^2+y^2) + 2x(ax-y)$$

$$\frac{\partial \dot{x}}{\partial y} = -1 - (x^2+y^2) + (ax-y) \times 2y = -1 - (x^2+y^2) + 2y(ax-y)$$

$$\frac{\partial \dot{y}}{\partial x} = 1 + (x^2+y^2) + (x+ay)2x = 1 + (x^2+y^2) + 2x(x+ay)$$

$$\frac{\partial \dot{y}}{\partial y} = a(x^2+y^2) + (x+ay)2y = a(x^2+y^2) + 2y(x+ay).$$

Hence the Jacobian at $(x^*, y^*) = (0,0)$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The characteristic equation of the given system is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$\therefore \lambda = \pm i$ (→ imaginary roots)

$\therefore \quad \tau = \lambda_1 + \lambda_2 \rightarrow$ trace

$= 0$

Δ = determinant $= \lambda_1 \lambda_2 = 1 > 0$

So, origin is always a center according to Linearization. $\Box$

Alternative,

Linearization about $(x^*, y^*) = (0,0)$

we can take the following short cut, For any system with a fixed point at origin, x and y represents derivatives from the fixed points, since

$$u = x - x^* = x \text{ and } v = y - y^* = y;$$

Hence we can ~~simply~~ Linearize by simply omitting nonlinear term in x and y. Thus the linearized system is

$$\dot{x} = -y$$
$$\dot{y} = x$$

Jacobian is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\tau = 0, \quad \Delta = 1 > 0$

$\therefore (0,0)$ is always a center.

$\Box$

2. To prove that it is not alway a true center,

Find the equivalent differential equation for
$r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

Soln:-

Let

$x = r\cos\theta$

$y = r\sin\theta$

$\therefore\ r^2 = x^2+y^2$

differentiate both side, we have

$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$

$\Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

substitute the values for $\dot{x}$ and $\dot{y}$

$$\Rightarrow r\dot{r} = x\left[-y + (ax-y)(x^2+y^2)\right] + y\left[x + (x+ay)(x^2+y^2)\right]$$

$$= -\cancel{xy} + (ax^2-xy)(x^2+y^2) + \cancel{xy} + (xy+ay^2)(x^2+y^2)$$

$$= (ax^2 - \cancel{xy} + \cancel{xy} + ay^2)(x^2+y^2)$$

$$= a(x^2+y^2)(x^2+y^2) = a(x^2+y^2)^2$$

$\therefore\ r\dot{r} = ar^4$

$\Rightarrow \boxed{\dot{r} = ar^3}$ —— (*)

similarly differentiate for θ

$\because\ \theta = \tan^{-1}(y/x)$

Let $y/x = u$

$\therefore\ \theta = \tan^{-1}(u)$

differentiating both side

$$\Rightarrow \dot{\theta} = \frac{1}{1+u^2}\, du$$

$$\because u = y/x$$

$$\Rightarrow du = \frac{x\,dy - y\,dx}{x^2}$$

Hence, $\dot{\theta} = \frac{1}{1+y^2/x^2} \times \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{\cancel{x^2}}{(x^2+y^2)} \times \frac{(x\dot{y} - y\dot{x})}{\cancel{x^2}}$

$$\Rightarrow \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

substitute for $\dot{x}, \dot{y}$

$$\therefore \dot{\theta} = \frac{x[x + (x+ay)(x^2+y^2)] - y[-y + (ax-y)(x^2+y^2)]}{r^2}$$

$$= \frac{x^2 + (x^2+axy)(x^2+y^2) + y^2 + (-axy + y^2)(x^2+y^2)}{r^2}$$

$$= \frac{(x^2+y^2) + (x^2+y^2)(x^2 + \cancel{axy} - \cancel{axy} + y^2)}{r^2}$$

$$= \frac{(x^2+y^2) + (x^2+y^2)^2}{r^2} = \frac{\cancel{(x^2+y^2)}[1 + (x^2+y^2)]}{\cancel{r^2}}$$

$$\boxed{\therefore\ \dot{\theta} = 1 + r^2} \text{ —— } \cancel{Ans}$$

Hence, $\left.\begin{array}{l} \dot{r} = ar^3 \\ \dot{\theta} = 1 + r^2 \end{array}\right\} \rightarrow$ equivalent differential equation.

(3) Find $a \in \mathbb{R}$, such that $(0,0)$ is a stable equilibrium / spiral, unstable spiral or center.

$\because \dot{r} = ar^3$

and $\dot{\theta} = 1 + r^2$

Assume that $r \ll 1$ and write the system in polar co-ordinates. when $O(r)$ terms in $\dot{\theta}$ are dropped

Hence the system in polar co-ordinates is

$$\dot{r} = ar^3, \quad \dot{\theta} \approx 1. \quad \overset{\text{soln}}{\Rightarrow} \quad r^2(t) = \frac{1}{\frac{1}{r_0^2} - 2at}, \quad \theta(t) = \theta_0 + t$$

where $r(0) = r_0$, $\theta(0) = \theta_0$

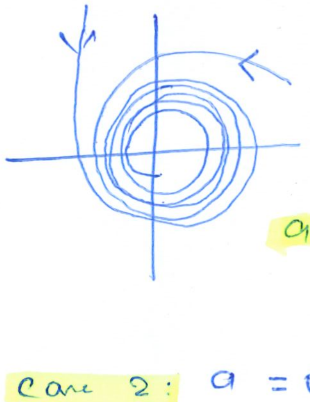
Hence the radial and angular momentum are independent, so all trajectories rotate about the origin with constant angular velocity $\dot{\theta} = 1$

The radial motion depends on a,

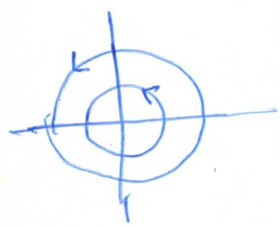
Case 1. If $a < 0$ then $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \frac{1}{\sqrt{\frac{1}{r_0^2} - 2at}} \to 0 \Rightarrow$ stable

($-2at \searrow \infty$)

and $\lim_{t\to\infty} |\theta(t)| = \lim_{t\to\infty} |t + \theta_0| \to \infty \Rightarrow$ spiral.



$a < 0 \Rightarrow$ Stable spiral at $(0,0)$

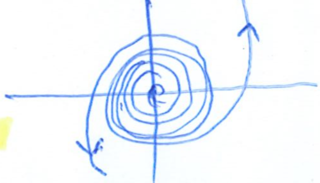


$a = 0$ center

Case 2: $a = 0$ then $\lim_{t\to\infty} r(t) = r_0$ and origin is center

Case 3:- If $a > 0$ then $r(t) \to \infty$ monotonically and the origin is an unstable spiral

$a > 0$ unstable spiral



(4) Find the stable set and unstable sets.

case 1. If $a<0$, $(0,0) \rightarrow$ stable spiral

Hence stable set $\Omega^s = (-\infty, 0)$

case 3. If $a>0$, $(0,0)$ unstable spiral

Hence unstable set $\Omega^u = (0, \infty)$.

2. Cycles:-

Decide if the system either has no cycles or at least one. Explain

2. 1. $\dot{x} = f(x) \in \mathbb{R}^2$. with fixed point x^*. and

$V : \mathbb{R}^2 \to \mathbb{R}$ — cts differtiable function such that

(a) $V(x) > 0 \;\forall x \neq x^*$ (b) $V(x^*) = 0$ and $\frac{dV}{dt}(x) < 0 \;\forall x \neq x^*$.

Soln:- If the above condition holds, then x^* is globally asymptotically stable: for all initial conditions, $x(t) \to x^*$ as $t \to \infty$.

The system has no closed orbits.

The intuition is that all trajectories move monotonically down the graph of $V(x)$ toward x^*. in figure.

The solution can't get stuck anywhere else because if the did, V would stop changing, but by assumption $\dot{V} < 0$ everywhere except at x^*.

$\square$ //

2. 2. SIR model

$$\dot{S} = -\frac{\beta}{N} SI, \quad \dot{I} = \frac{\beta}{N} SI - \gamma I, \quad \dot{R} = \gamma I$$

If this is conservative system then It must have at least one cycles.

2. (3) system :-

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y)$$

where $f(x,-y) = -f(x,y)$ and $g(x,-y) = g(x,y)$ $\forall x,y \in \mathbb{R}$

Any system of the above form is called reversible system.

Hence from Lemma : Reversible system is sufficiently close to the origin and all trajectories are closed curves.

Hence this system has at least one cycles.

2.

4. $\dot{x} = f(x) \in \mathbb{R}^2$, $V \to \mathbb{R}^2 \to \mathbb{R}$, continuously differentiable function.

s.t (a) $\frac{dV}{dt} = 0$ and (b) $\dot{x} = -\nabla V$.

such system is a gradient system

soln:-

closed orbits are impossible in gradient system:-

We obtain a contradiction

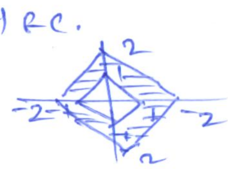
proof:- suppose there is a closed orbit. by considering the change in V after one circuit. on the other hand $\Delta V \equiv 0$ since V is single-valued but on the other hand

$$\Delta V = \int_0^T \frac{dV}{dt} dt = \int_0^T (\nabla V \cdot \dot{x}) dt = -\int_0^T \|\dot{x}\|^2 dt < 0$$

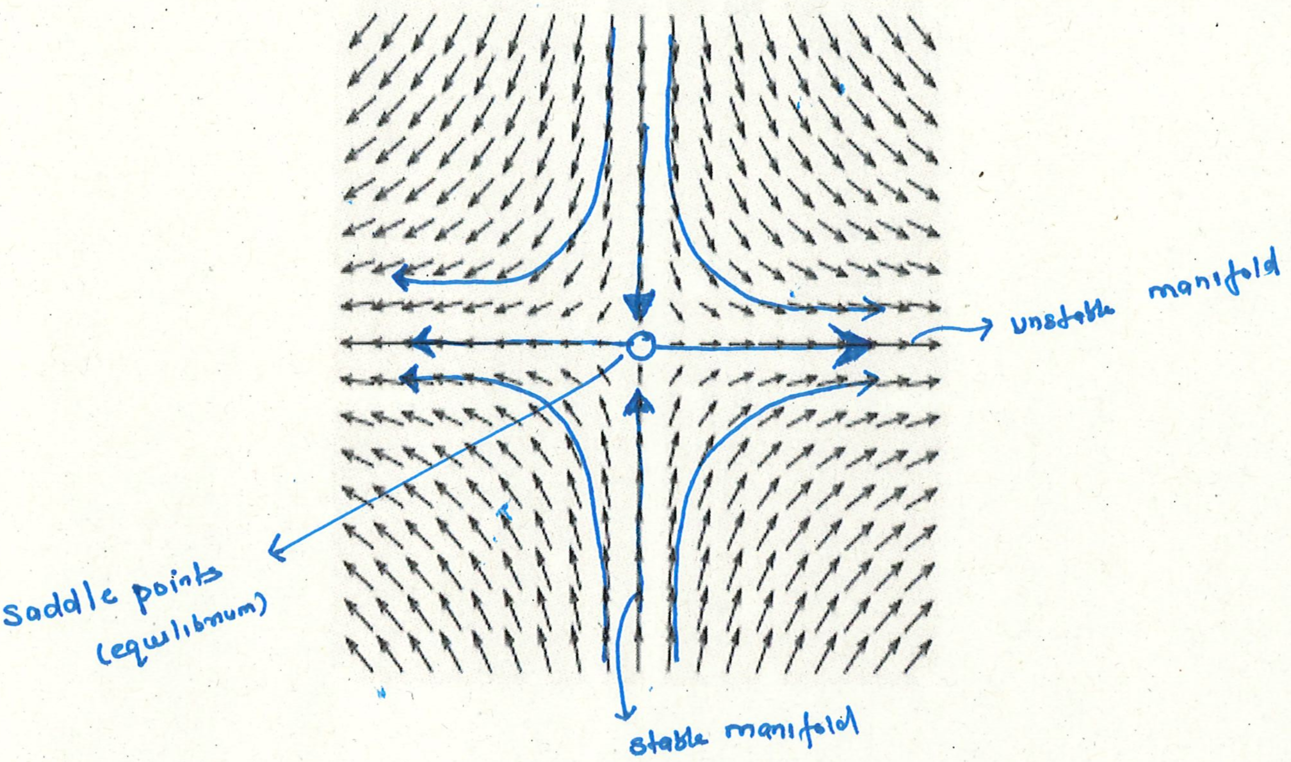
(unless $\dot{x} \equiv 0$, in which case the trajectory is a fixed point not a closed orbit). This is a contradiction, that closed orbit can't exist in gradient system.

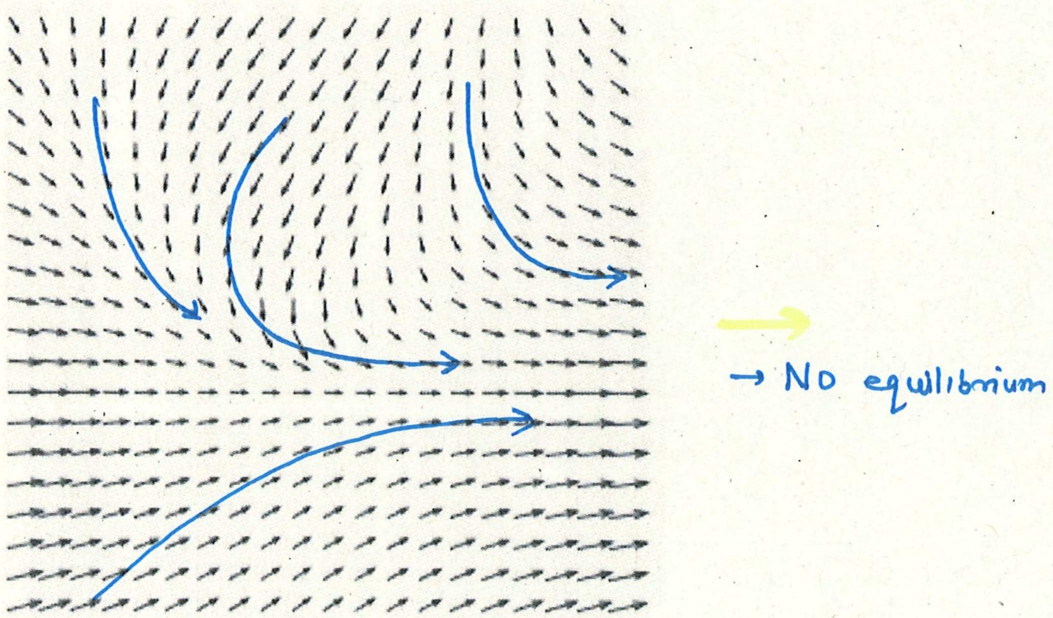
5. Assume $\dot{x} = f(x) \in \mathbb{R}^2$ has no fixed point in the set $C = \{x \in \mathbb{R}^2 \mid 1 \le \|x\| \le 2\}$ and there exist a trajector x such that $\forall t \in \mathbb{R}$, $x(t) \in C$.

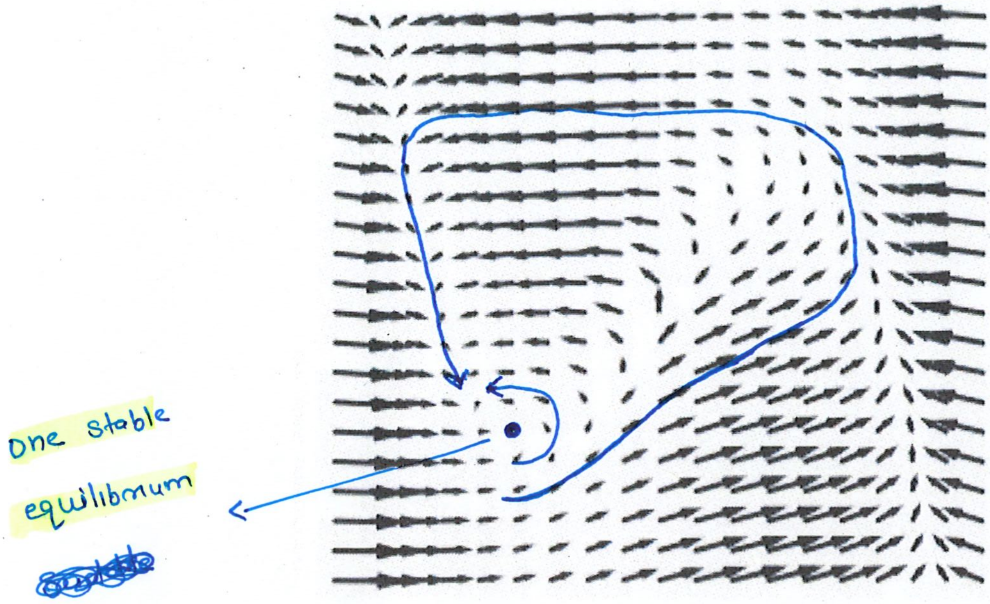
soln:- It must have closed orbits. → from above explain (4).

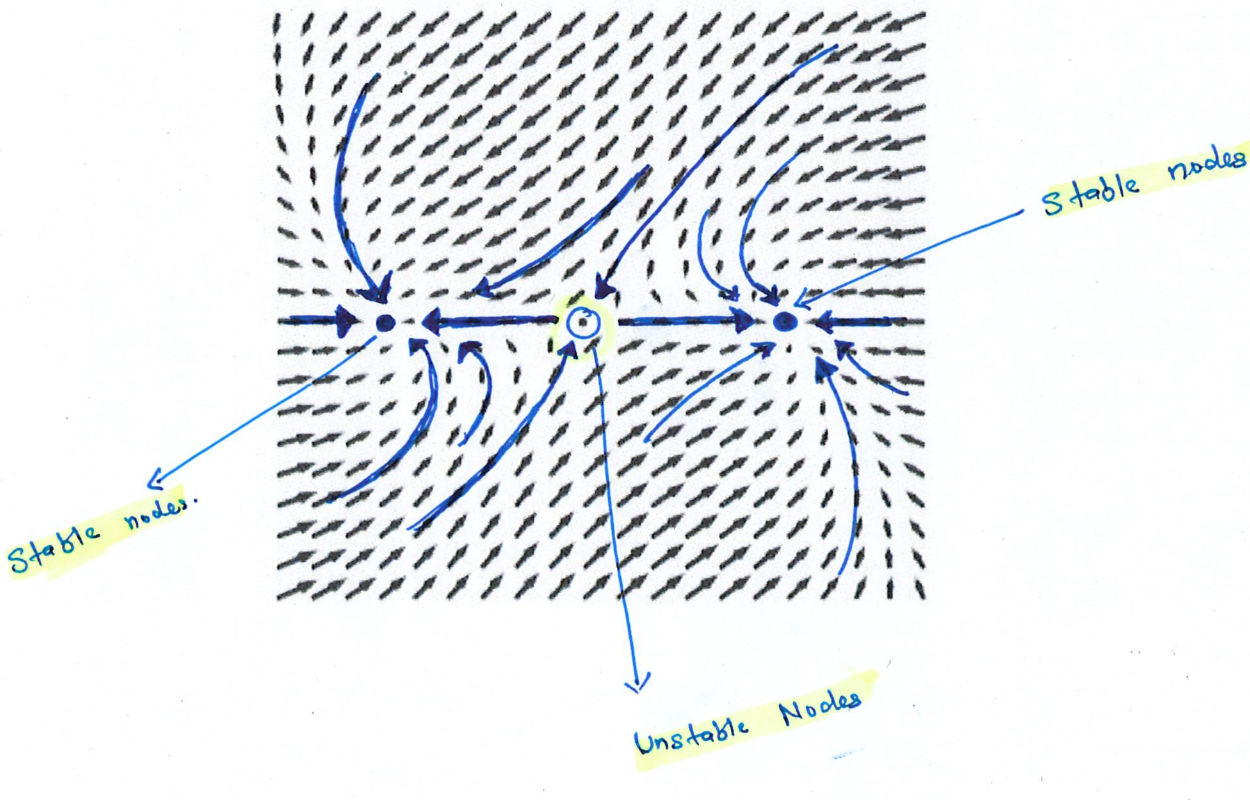


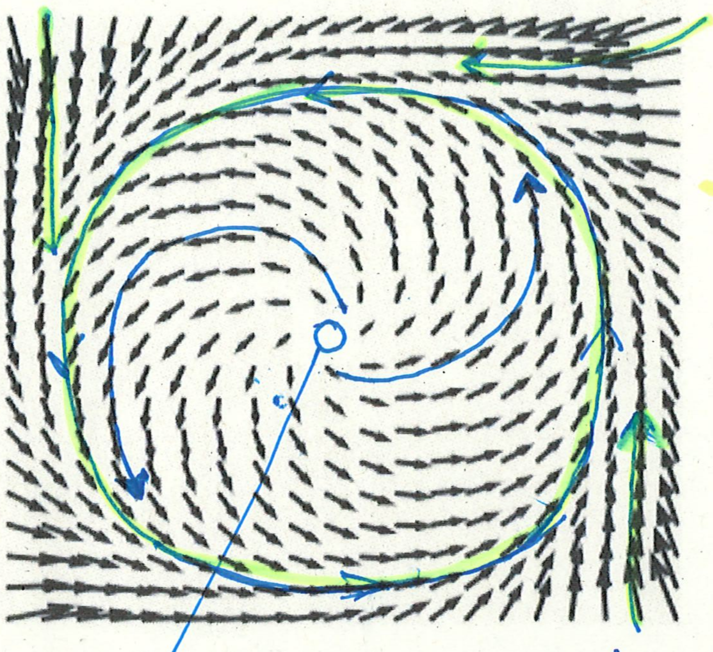
3. Graphical Analysis.





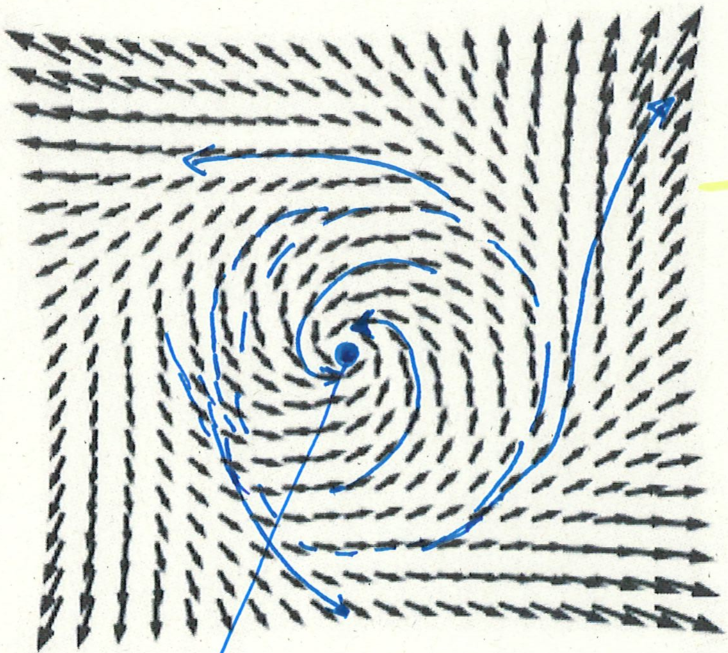

One stable
equilibrium


Stable nodes
Stable nodes.
Unstable Nodes



Unstable focous inside a a stable limit cycle

unstable fixed point



Ans

stable focous inside an unstable limit cycle.

stable fixed point.