Exercise.4

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1 Dynamical Systems Theory in Machine Learning & Data Science

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3 1. Logistic Map

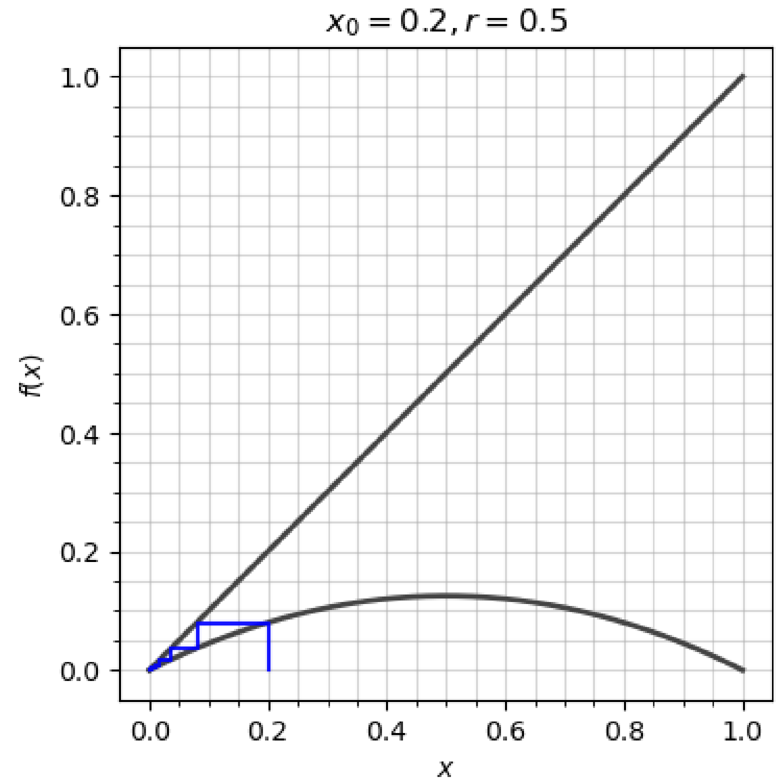
4 1.1 Plot the cobweb

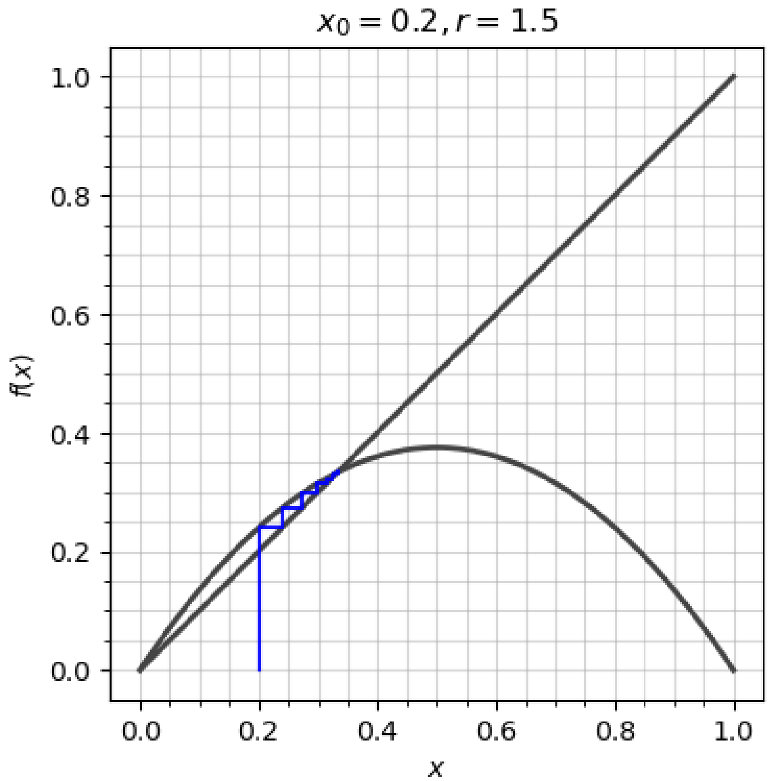
[1]:
```
import numpy as np
from matplotlib import rc
import matplotlib.pyplot as plt



def plot_cobweb(f, r, x0, nmax=30):
    """Make a cobweb plot.

    Plot y = f(x; r) and y = x for 0 <= x <= 1, and illustrate the behaviour of
    iterating x = f(x) starting at x = x0. r is a parameter to the function.

    """
    dpi = 100
    x = np.linspace(0, 1, 500)
    fig = plt.figure(figsize=(600/dpi, 450/dpi), dpi=dpi)
    ax = fig.add_subplot(111)

    # Plot y = f(x) and y = x
    ax.plot(x, f(x, r), c='#444444', lw=2)
    ax.plot(x, x, c='#444444', lw=2)

    # Iterate x = f(x) for nmax steps, starting at (x0, 0).
    px, py = np.empty((2,nmax+1,2))
    px[0], py[0] = x0, 0
    for n in range(1, nmax, 2):
        px[n] = px[n-1]
```

```python
        py[n] = f(px[n-1], r)
        px[n+1] = py[n]
        py[n+1] = py[n]

    # Plot the path traced out by the iteration.
    ax.plot(px, py, c='b', alpha=0.7)

    # Annotate and tidy the plot.
    ax.minorticks_on()
    ax.grid(which='minor', alpha=0.5)
    ax.grid(which='major', alpha=0.5)
    ax.set_aspect('equal')
    ax.set_xlabel('$x$')
    ax.set_ylabel('$f(x)$')
    ax.set_title('$x_0 = {:.1}, r = {:.3}$'.format(x0, r))
```

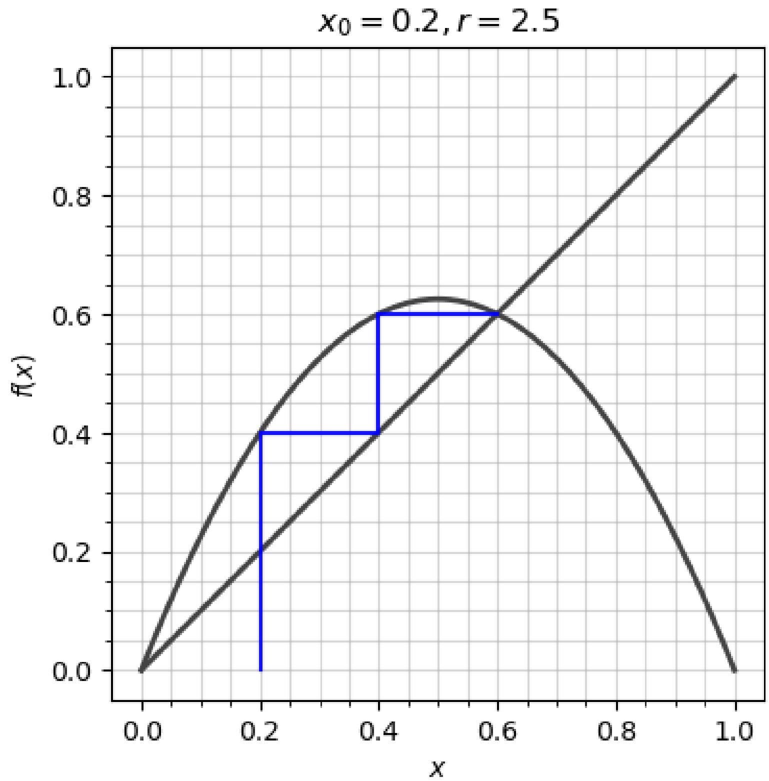
Define the logistic function

[2]:
```python
def f(x,r):
    return r*x*(1-x)

# Initial condition
x0=0.2
plot_cobweb(f,0.5,x0)
plot_cobweb(f,1.5,x0)
plot_cobweb(f,2.5,x0)
plot_cobweb(f,3.5,x0)
plot_cobweb(f,3.9,x0)
```

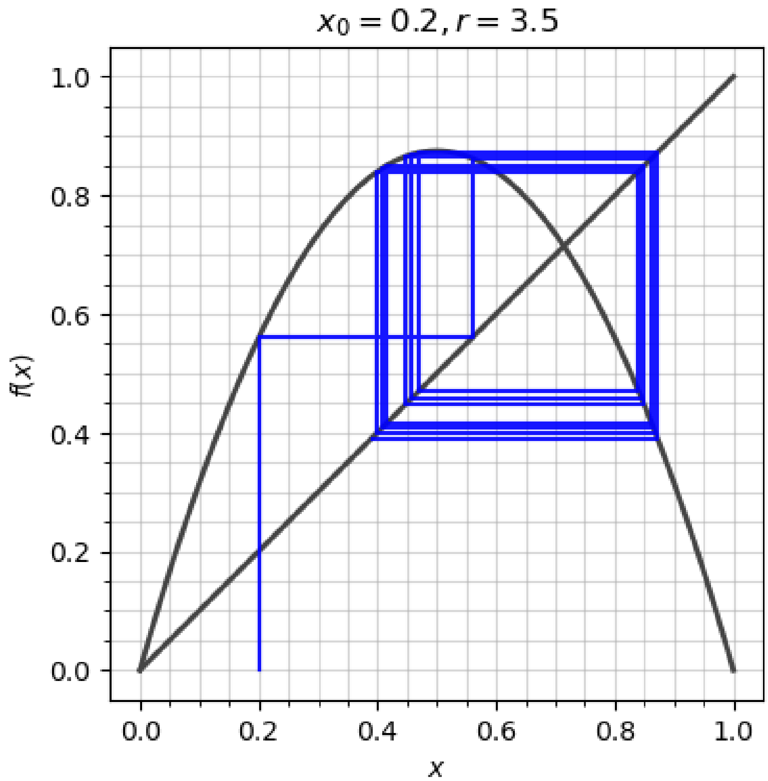

$x_0 = 0.2, r = 0.5$
1.0
0.8
0.6
0.4
0.2
0.0
$f(x)$
0.0
0.2
0.4
0.6
0.8
1.0
x



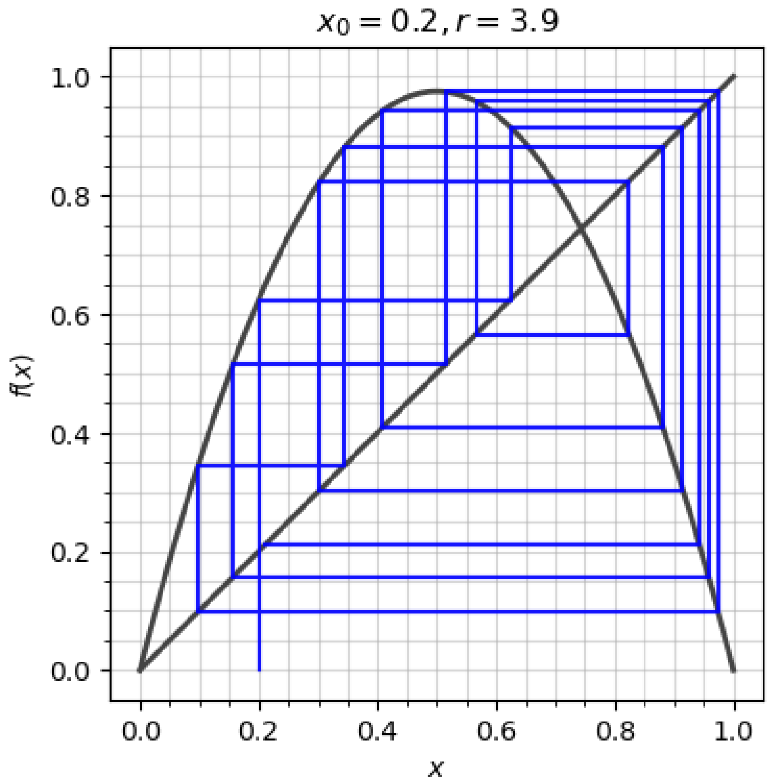
$x_0 = 0.2, r = 1.5$
1.0
0.8
0.6
0.4
0.2
0.0
$f(x)$
0.0
0.2
0.4
0.6
0.8
1.0
x



$x_0 = 0.2, r = 2.5$
1.0
0.8
0.6
0.4
0.2
0.0
f(x)
0.0
0.2
0.4
0.6
0.8
1.0
x



$x_0 = 0.2, r = 3.5$
1.0
0.8
0.6
0.4
0.2
0.0
$f(x)$
0.0
0.2
0.4
0.6
0.8
1.0
x

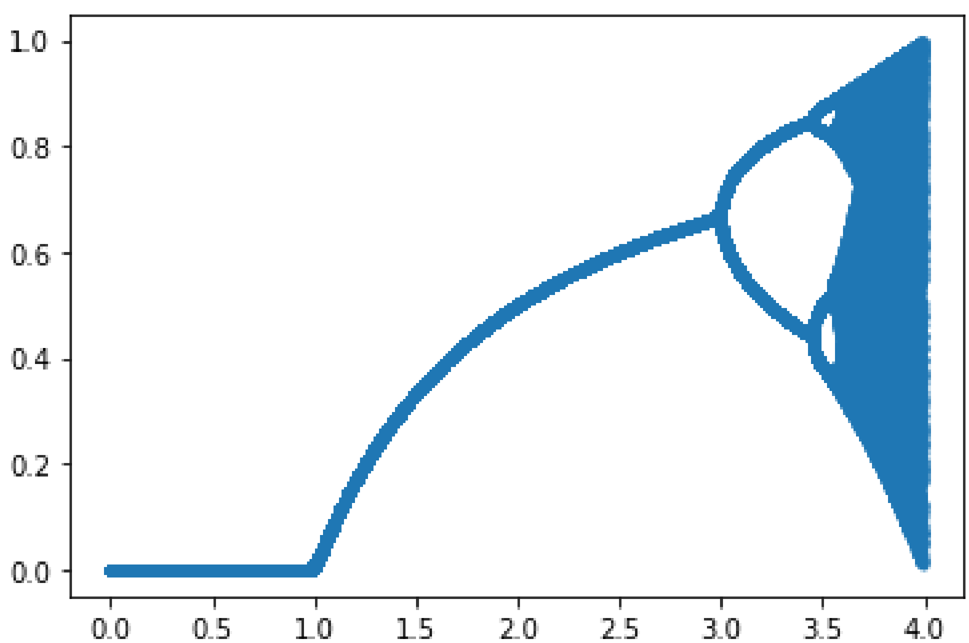


The plot can reveal stable, cyclic, or chaotic behaviour as convergence to a point, a repeating rectangle or by filling the plane with non-repeating line segments respectively.

5 5. Trajectories in the logistic map

[3]:
```
import numpy as np
import matplotlib.pyplot as plt
import pylab


def f(x, r):
    #logistic equation with parameter r
    return r*x*(1-x)

if __name__ == '__main__':
    # initial condition for x
    ys = []
    rs = np.linspace(0, 3.99, 2000)

    # Loop through `rs`. `r` is assigned the values in `rs` one at a time.
```

```
for r in rs:
    x = 0.2
    # Repeat this loop 100 times.
    for i in range(100):
        # Evaluate f at (x, r). The return value is assigned to x.
        # x is then fed back into f(x, r).
        # r remains fixed.
        x = f(x, r)

    # Do this 100 times
    for i in range(100):
        # Again make the x jump around according to the logistic equation
        x = f(x, r)
        # Save the point (r, x) in the list ys
        ys.append([r, x])

# ys is a list of lists.
# ys as a list of [r, x] point.
# This converts the list of lists into a 2D numpy array.
ys = np.array(ys)

# ys[:,0] is a 1D array of r values
# ys[:, 1] is a 1D array of x values
# This draws a scatter plot of (r, x) points.
pylab.plot(ys[:,0], ys[:,1], '.')
pylab.show()
```



```
[ ]:
```

Dynamical system

Exercise :- 4

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24/11/2021.

1. The logistic Map:-

$$x_{n+1} = r x_n (1 - x_n) \quad \text{for} \quad 0 \le x_n \le 1 \quad \text{and} \quad 0 \le r \le 4.$$

1. Plot a cobweb → soln is in jupyter notebook. attached along with this soln.

2. Find all the fixed points

Here $f(x) = r x (1-x)$

Conditions for fixed point : $f(x^*) = x^*$

Hence $r x^* (1 - x^*) = x^*$

$\Rightarrow x^* [1 - r(1 - x^*)] = 0$

$\Rightarrow x^* = 0$ or $1 - r(1 - x^*) = 0$ i.e, $r(1 - x^*) = 1$

$\therefore x^* = 1 - \frac{1}{r}$

Hence fixed points are

$x^* = 0$ and $x^* = 1 - \frac{1}{r}$ → is allowed value of x only if $r \ge 1$.

↓

for all r

Determine stability :-

Lemma:- Suppose that the map $f_\mu(x)$ has a fixed point at x^*. Then the fixed point is stable if

$$\left| \frac{d}{dx} f_\mu(x^*) \right| < 1$$

and is unstable if

$$\left| \frac{d}{dx} f_\mu(x^*) \right| > 1.$$

$\because f(x) = rx(1-x)$

$f'(x) = r(1-x) - rx = r - 2rx$

Using Lemma:

at $x^* = 0$

$f'(x^*) = r$

Hence $x^* = 0$ is stable if $0 < r < 1$

is unstable if $r > 1$.

at $x^* = 1 - \frac{1}{r}$

$f'(x^*) = r - 2r\left(1 - \frac{1}{r}\right) = r - 2r + 2 = 2 - r$

$\therefore f'(x^*) = 2 - r$ ~~$\therefore 2-r > 1 \Rightarrow r < 1$~~

~~and $2-r < 1 \Rightarrow$~~

$\because 0 \leq r \leq 4$

~~$\therefore x^* = 1 - \frac{1}{r}$ is stable if $1 < r < 3$~~

Now, $|f'(x^*)| < 1 \rightarrow$ stable

$\therefore |2-r| < 1 \Rightarrow -1 < 2-r < 1$

$\Rightarrow 1 > r-2 > -1$

$\Rightarrow 3 > r > 1$

and $|f'(x)| > 1 \Rightarrow |2-r| > 1 \rightarrow$ unstable

i.e. $2-r < -1$ or $2-r > 1$

i.e. $r > 3$ or $r < 1$

Hence $x^* = 1 - \frac{1}{r}$

is stable if $1 < r < 3$

and unstable if $r < 1$ or $r > 3$.

$\square$.

3. Show that the logistic map has a 2-cycle for $r > 3$.

Soln :- A 2-cycle exists if and only if there are two points p and q such that

$$f(p) = q \quad \text{and} \quad f(q) = p.$$

Equivalently, such a p must satisfy

$$f(f(p)) = p, \quad \text{where } f(x) = rx(1-x).$$

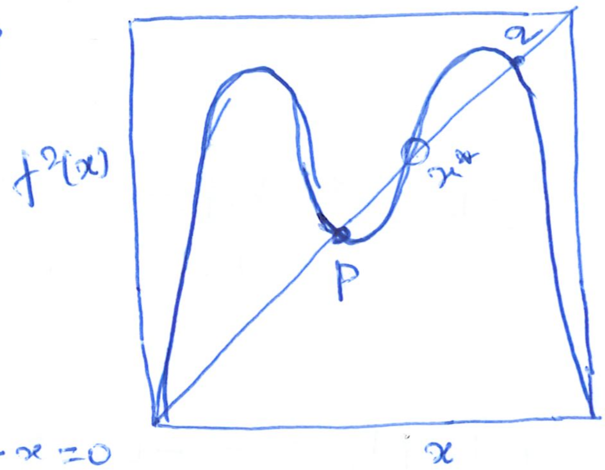
Hence p is a fixed point of the second-iterate map

$$f^2(x) = f(f(x))$$

$\because$ $f(x) \rightarrow$ quadratic polynomial,

$\therefore$ $f^2(x) \rightarrow$ 4th order polynomial (quartic)

Its graph for $r > 3$ $\longrightarrow$



To find p and q.

Solve the points where graph intersects the diagonal.

ie, $f^2(x) = x$

$$\Rightarrow f^2(x) - x = 0$$

$$\Rightarrow r^2 x(1-x)[1 - rx(1-x)] - x = 0$$

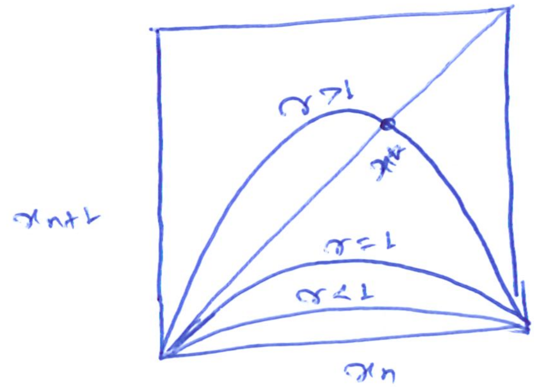
$$\Rightarrow x[r^2(1-x)[1 - rx(1-x)] - 1] = 0$$

$$\Rightarrow x^* = 0 \quad \text{or} \quad r^2(1-x)[1 - rx(1-x)] = 1$$

$$\Rightarrow x^* = 0 \quad \text{or} \quad \underbrace{r^2(1-x) - r^3 x(1-x)^2}_{(A)} = 1$$

If we try $x = 1-1/r$ in (A) $\Rightarrow$ $A = 1.$ $\Rightarrow$ $x = 1 - 1/r$ is trivial soln.

Differentiate for $r<1$, $r=1$ and $r>1$



For $r<1$ $\Rightarrow$ parabola lies below the diagonal and origin is the only fixed points. $x^* = 0$

As $r \rightarrow$ increases $\Rightarrow$ parabola gets biger and ~~tangents~~ going to touch tangent to the diagonal at $r=1$

For $r>1$ the parabola intersects the diagonal in a second fixed points $x^* = 1 - \frac{1}{r}$.

while origin loses stability.

At $r=1$, x^* bifercates from the origin in a transentical bifurcation.

and $f'(x^*) = -1$ at $r=3$ $\Rightarrow$ This resulting bifurcation is called flip bifuration.

Hence, x and $x-(1-1/r)$ are two factor of the $f^2(x)$.

So the resulting part is quadratic equation.

→ We use sympy to in Python to find out the factor

It is not easy to find the factor manually..

```
>>> from sympy.solvers import solve
>>> from sympy import symbols.
>>> x, r = symbols ('x r')
>> solve (f^2(x), x)
```

Thus we obtain the pair of roots for the remaining part.

$$p, q = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

Hence roots are real for $r > 3$.

Hence, a 2-cycle exists for all $r > 3$. □//

(4.) What is the stability of the 2-cycles from (3).

Soln
→ Analyze the stability of a ~~fixed~~ cycle, think about the fixed point.

∵ Both p and q are soln of $f^2(x) = x$. from (3).

Hence p and q are fixed point of the second iterated map.

Hence the 2-cycle is stable if p and q are stable fixed point.

We have to verify p and q are stable fixed point.

Find the derivative of f^2 for stability checking

$$\frac{d}{dx}\left(f^2(x)\right) = \frac{d}{dx}\left(f(f(x))\right)$$
$$= f'(f(x)) \cdot f'(x).$$

Hence at $x = p$ ~~and q~~

$$\frac{d}{dx}\left(f^2(x)\right)_{x=p} = f'(f(p))\, f'(p)$$

$\because f(p) = q$ from (3)
and $f(q) = p$

$$\Rightarrow \frac{d}{dx}\left(f^2(x)\right)_{x=p} = f'(q) \cdot f'(p).$$

similarly for $x = q$.

$$\frac{d}{dx}\left(f^2(x)\right)_{x=q} = f'(p)\, f'(q) = \lambda. \text{ (assume)}$$

for stable fixed point

$$\left|\frac{d}{dx}\left(f^2(x)\right)\right| < 1.$$

$\because f'(x) = r - 2rx.$

$f'(p) = r(1-2p)$
$f'(q) = r(1-2q)$

$$\therefore \left| r(1-2p)\, r(1-2q) \right| < 1$$

$\Rightarrow \left| r^2\left[1 - 2(p+q) + 4pq\right] \right| < 1$, we put the value for p and q from (2)

$$\Rightarrow \left| r^2\left[1 - 2\left(\frac{r+1}{r}\right) + 4\left(\frac{r+1}{r^2}\right)\right] \right| < 1$$

$$\Rightarrow |4 + 2r - r^2| < 1$$

$\therefore$ 2 cycles is linearly stable for $|4 + 2r - r^2| < 1$, ie, $3 < r < 1 + \sqrt{6}$ $\square$

yes, stability changes for $r > 3$.

Remarks:- yes stability changes for some $r > 3$ ie beyond $1+\sqrt{6}$

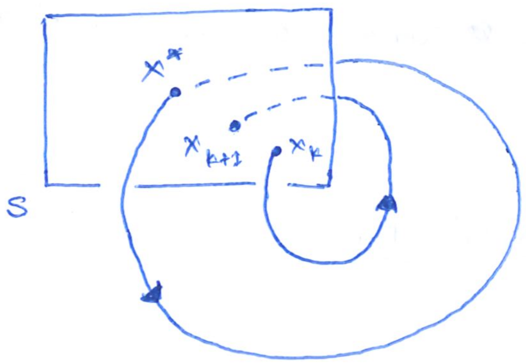
2. Poincaré Maps :-

soln! -

1. consider an n-dimensional system

$$\dot{x} = f(x)$$

Let S denotes an $(n-1)$ dimensional surface of section. S is required to be transverse to the flow.



The poincare map P is a mapping from S to itself, obtained by following trajectory from one intersection with S to the next. If $x_k \in S$ denotes the kth intersection, then poincare map is

$$x_{k+1} = P(x_k).$$

suppose x^* is a fixed point of P i.e, $P(x^*) = x^*$. Then a trajectory starting at x^* return to x^* after some time T. and is therefore a closed orbit for the original system $\dot{x} = f(x)$.

Yes, that is also a necessary condition. □//

(2) consider the system:-

$$\dot{r} = r(1-r)$$
$$\dot{\theta} = \frac{1}{2}$$

soln!- The origin is an unstable focus, and there is a limit cycle, of radius 1 centered at the origin.

System of ODE is solved, both differential equation are separable. The solution is given by.

$\therefore \dot{r} = r(1-r)$, $\dot{\theta} = 1/2$

$\Rightarrow \int \frac{dr}{r(1-r)} = \int dt$ $\quad \int d\theta = \int \frac{1}{2} dt$

$\Rightarrow r(t) = \frac{1}{1+ce^{-t}}$, $\theta(t) = \frac{1}{2}t + \theta_0$.

where c and θ_0 are constant.

The trajectories flow around the origin with period π

suppose that a trajectory start outside surface on S, say at $r_0 = 2$

The solution are given by

$$r(t) = \frac{1}{1-\frac{1}{2}e^{-t}}, \quad \theta(t) = \frac{1}{2}t$$

There fore a return map be expressed as

$$r_n = \frac{1}{1-\frac{1}{2}e^{-n\pi}}$$

where n is the natural number.

If the trajectory starts inside say $r_0 = 1/2$, then

$$\therefore r(t) = \frac{1}{1+e^{-t}}, \quad \theta(t) = \frac{t}{2}.$$

and return map is given by

$$r_n = \frac{1}{1+e^{-n\pi}}$$

3. Consider the system

$$\dot{y} = 2 - \sin\theta - y$$
$$\dot{\theta} = y$$

defined on the cylinder, $y \to$ height
$\theta \to$ angle.

(a) Show that there are no fixed points:-

Fixed points:-

$\dot{\theta} = 0$ and $\dot{y} = 0$

$\Rightarrow y = 0$ and $2 - \sin\theta - y = 0$

$\Rightarrow \sin\theta = 2 - y = 2$

$\Rightarrow \sin\theta = 2$ $\because y = 0$

which is impossible.

$\because \sin\theta \le 1$.

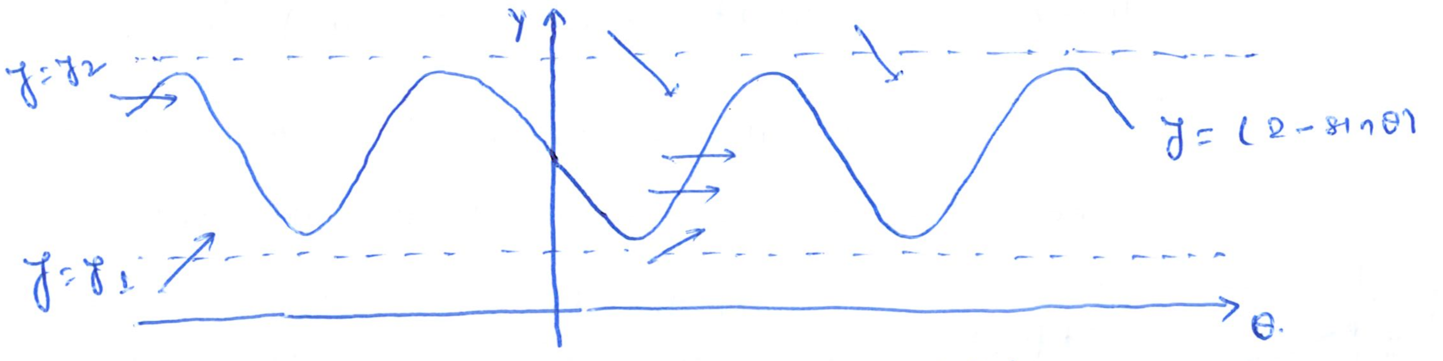
Hence, there doesn't exist any fixed points.

(b) Show there are $0 < y_1 < y_2$ such that for all trajectories

$$y_1 \le \lim_{t \to \infty} y < y_2.$$

Consider the nullclines: $y = 2 - \sin\theta$, where $\dot{y} = 0$.

Plot the curve with y versus θ.



The flow is downward above the nullclines and upward below it.

$\because \quad -1 \le \sin\theta \le 1$

$\Rightarrow \quad 1 \ge -\sin\theta \ge -1$

$\Rightarrow \quad 2+1 \ge 2-\sin\theta \ge 2-1$

$\Rightarrow \quad \underset{y_2}{3} \ge \underset{y}{2-\sin\theta} \ge \underset{y_1}{1}$

Here, all trajectories eventually enter the strip $y_1 \le y \le y_2$ and stay in there forever.

Here y_1 and y_2 are any fixed number such that

$0 < y_1 < 1$ and $y_2 > 1$,

Note Hence there exist $0 < y_1 < y_2$ such that all the trajectories: $y_1 < \lim_{t\to\infty} y < y_2$ $\square$

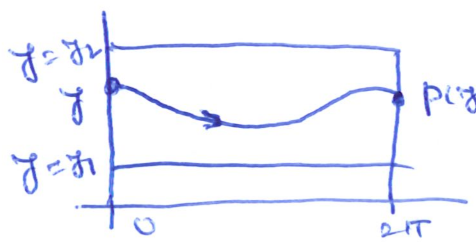
(c) choose $S = \{(y,\theta) : \theta = 0\}$ as a surface of section.

Show that poincare map must be continuous.

Soln:- Previous example (b), $\rightarrow$ Inside the strip, the flow is always to be right because $y > 0$ and $\dot\theta > 0$.

$\because$ $\theta = 0$ and $\theta = 2\pi$ are equivalent on the cylinder.

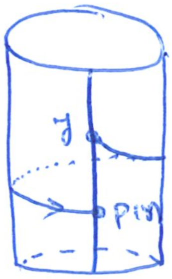
consider the rectangular box: $0 \le \theta \le 2\pi$, $y_1 \le y \le y_2$.



consider the trajectory start at height y and follow it until it intersects the right side of the box at some new height $P(y)$.

The mapping from y to $p(y)$ is called the poincare map.

The poincare map is also called first-return map.



$\theta = 0 \pmod{2\pi}$

$p(y)$ is a continuous function. This follows from the theorem that the soln of differential equation depend continuously on initial, ~~so~~ if the vector field is smooth enough. (from b).

(d). With (b) and (c) use the IVT → show that $\exists\, y \in (y_1, y_2)$ s.t. $p(y) = y$.

soln:- If we show that such a point y^* such that $p(y^*) = y^*$. then the corresponding trajectory is a closed orbit.

To show such a point y^* must exist, we need to know how the graph of $p(y)$ looks like.

consider the trajectory start at $y_0 = y_1$ and $\theta = 0$.

we claim that

$$p(y_1) > y_1$$

Since the flow is strictly upward at first time and trajectory won't return to the line $y = y_1$. Since the flow is upward every where. By the same argument.

$$p(y_2) < y_2.$$

Since $p(y)$ is continuous function.

Hence by Intermediate value theorem:- If $I = [a,b] \subseteq \mathbb{R}$ and a continuous function $f: I \to \mathbb{R}$ then $\exists$ a point $c \in [a,b]$ and.

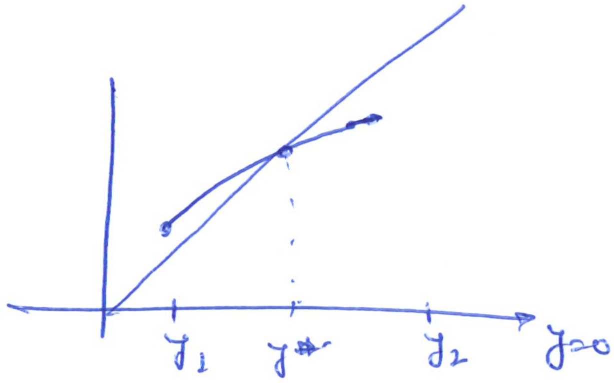
u is a number between $f(a)$ and $f(b)$. such that

$$\min(f(a), f(b)) < u < \max(f(a), f(b)).$$

such that $f(c) = u$.

Here $I = [y_1, y_2]$. so, $\exists\, y \in (y_1, y_2)$ such that $p(y) = y$. $\square$

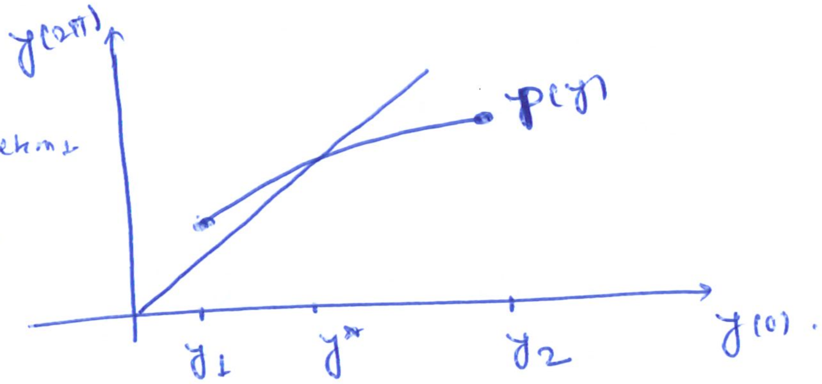
and $p(y_1) < p(y_2)$.



By intermediate value theorem the graph $p(y)$ must cross the $45°$ diagonal somewhere the intersection is the desired y^*.

(e) monotonic map.:

By drawing the trajectories for $p(y)$.

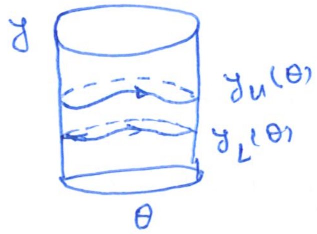


Proof by contradiction:
If $p(y)$ were not monotonic, two trajectories would cross., ~~This~~ is the

$\therefore$ This is the violation of Index theory.

(f). Show that there is exactly one y s.t $P(y) = y$.

Suppose that, there were two different rotations, the phase portrait on the cylinder looks like.



One of the rotation have to lie strictly above the other, because trajectory cannot cross.

Let $y_u(\theta)$ and $y_L(\theta)$ → denotes the upper and lower rotations.

where $y_u(\theta) > y_L(\theta)$ $\forall \theta$.

The existence of two such rotation leads to a contradiction by the following energy argument. Let

$$E = \frac{1}{2} y^2 - \cos\theta.$$

after one circuit around any rotation $y(\theta)$ changes in energy ΔE, must vanish

Hence

$$0 = \Delta E = \int_0^{2\pi} \frac{dE}{d\theta} \, d\theta$$

Here $\dfrac{dE}{d\theta} = y \dfrac{dy}{d\theta} + \sin\theta$

and $\dfrac{dy}{d\theta} = \dfrac{y'}{\theta'} = \dfrac{2 - \sin\theta - y}{y}$

$$\therefore \frac{dE}{d\theta} = 2 - \sin\theta - y + \sin\theta = 2 - y$$

Hence, $0 = \int_0^{2\pi} (2 - y)\, d\theta$ on any rotation.

$\Rightarrow \int_0^{2\pi} y(\theta)\, d\theta = 4\pi$ → any rotation must satisfy.

$\because y_u(\theta) > y_L(\theta) \Rightarrow \int_0^{2\pi} y_u(\theta)\, d\theta > \int_0^{2\pi} y_L(\theta)\, d\theta.$

Hence, the condition,

$$\int_0^{2\pi} \gamma(\theta)\,d\theta = 4\pi$$ - cann't hold for both rotation.

This contradicts ~~the~~ our claim.

Hence, there exist an unique limit cycles. $\square$//

(g):- Yes, we could also use the Poincaré - Bendixson theorem to prove the existence of closed cycles. $\square$//