Introduction to Statistics (MAT 283)

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PRODUCT MOMENTS OF BIVARIATE RANDOM VARIABLE

CONDITIONAL EXPECTATION AND VARIANCE

EXAMPLE: Let the random variables X and Y have the joint pmf

$$f(x,y) = \begin{cases} \frac{1}{4} & \text{if} \quad (x,y) = \{(0,1),\ (0,-1),\ (1,0),\ (-1,0)\} \\ \\ 0 & \quad \text{otherwise,} \end{cases}$$

What is the covariance of X and Y ? Are the random variables X and Y independent?

	Y=-1	Y=0	Y=1	P(X=x)
X= -1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
X= 0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{2}{4}$
X= 1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
P(Y=y)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	

Theorem: Let X and Y be any two random variables and let a and b be any two real numbers. Then

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y).$$

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Proof

$$\begin{aligned}
Var(aX + bY) &= E([aX + bY - E(aX + bY)]^2) \\
&= E([aX + bY - aE(X) - bE(Y)]^2) \\
&= E([a(X - E(X)) + b(Y - E(Y))]^2) \\
&= E(a^2(X - E(X))^2 + b^2(Y - E(Y))^2 \\
&\quad + 2ab(X - E(X))(Y - E(Y))) \\
&= a^2 E([X - E(X)]^2) + b^2 E([Y - E(Y)]^2) \\
&\quad + 2ab\, E[(X - E(X))(Y - E(Y))] \\
&= a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X, Y).
\end{aligned}$$

- In case of three random variables $X,\ Y,\ Z$, we have

$$\begin{aligned} Var(X+Y+Z) &= Var(X)+Var(Y)+Var(Z) \\ &\quad +2Cov(X,Y)+2Cov(Y,Z)+2Cov(Z,X) \end{aligned}$$

The functional dependency of the random variable Y on the random variable X can be obtained by examining the correlation coefficient.

CORRELATION COEFFICIENT:

Let X and Y be two random variables with variances σ_X^2 and σ_Y^2, respectively. Let the covariance of X and Y be $Cov(X, Y)$. Then the correlation coefficient ρ between X and Y is given by

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}.$$

Theorem: If X and Y are independent, the correlation coefficient between X and Y is zero.

• The converse of this theorem is not true. If the correlation coefficient of X and Y is zero, then X and Y are said to be uncorrelated.

LEMMA: If X^* and Y^* are the standardizations of the random variables X and Y, respectively, the correlation coefficient between X^* and Y^* is equal to the correlation coefficient between X and Y.

Theorem: For any random variables X and Y, the correlation coefficient ρ satisfies

$$-1 \leq \rho \leq 1,$$

and $\rho = 1$ or $\rho = -1$ implies that the random variable $Y = aX + b$ where a and b are arbitrary real constants with $a \neq 0$.

Proof: Let μ_X be the mean of X and μ_Y be the mean of Y, and σ_X^2 and σ_Y^2 be the variances of X and Y, respectively. Further, let

$$X^* = \frac{X - \mu_X}{\sigma_X} \text{ and } Y^* = \frac{Y - \mu_Y}{\sigma_Y}$$

be the standardization of X and Y, respectively. Then

$$\mu_{X^*} = 0 \text{ and } \sigma_X^2 = 1$$

$$\mu_{Y^*} = 0 \text{ and } \sigma_Y^2 = 1.$$

Thus

$$
\begin{aligned}
0 \le Var(X^* - Y^*) &= Var(X^*) + Var(Y^*) - 2Cov(X^*, Y^*) \\
&= \sigma_{X^*}^2 + \sigma_{Y^*}^2 - 2\rho^*\sigma_{X^*}\sigma_{Y^*} \\
&= 1 + 1 - 2\rho^* \\
&= 2 - 2\rho \qquad\qquad \text{(By above Lemma } \rho = \rho^*) \\
&= 2(1 - \rho)
\end{aligned}
$$

$$\Longrightarrow 1 - \rho \ge 0$$

$$\text{or} \quad \rho \le 1$$

Further,

$$\begin{aligned}
0 \le Var(X^* + Y^*) &= Var(X^*) + Var(Y^*) + 2Cov(X^*, Y^*) \\
&= \sigma^2_{X^*} + \sigma^2_{Y^*} + 2\rho^* \sigma_{X^*}\sigma_{Y^*} \\
&= 1 + 1 + 2\rho^* \\
&= 2 + 2\rho \qquad\qquad \text{(By above Lemma } \ \rho = \rho^*) \\
&= 2(1 + \rho)
\end{aligned}$$

$$\implies 1 + \rho \ge 0$$

or $\quad \rho \ge -1$

Now, we show that if $\rho = 1$ or $\rho = -1$, then Y and X are related through an affine transformation. Consider the case $\rho = 1$, then

$$Var(X^* - Y^*) = 0.$$

But if the variance of a random variable is 0, then all the probability mass is concentrated at a point (that is, the distribution of the corresponding random variable is degenerate). Thus $Var(X^* - Y^*) = 0$ implies $(X^* - Y^*)$ takes only one value. But $E(X^* - Y^*) = 0$. Thus we get

$$X^* - Y^* = 0 \text{ or } X^* = Y^*.$$

Hence

$$\frac{X - \mu_X}{\sigma_X} = \frac{Y - \mu_Y}{\sigma_Y}$$

Solving Y in terms of X we get,

$$Y = aX + b$$

where $a = \frac{\sigma_Y}{\sigma_X}$ and $b = \mu_Y - a\mu_X$.

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Thus if $\rho = 1$, then Y is a linear in X. Similarly, we can show for the case $\rho = -1$, the random variables X and Y are linearly related. This completes the proof of the theorem.

MOMENT GENERATING FUNCTION:

Let X and Y be two random variables with joint pdf or pmf f. A real valued function $M : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$M(s,t) = E(e^{sX+tY})$$

is called the joint moment generating function of X and Y if this expected value exists for all s is some interval $-h < s < h$ and for all t is some interval $-k < t < k$ for some positive h and k.

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Note that

$$M(s,0) = E(e^{sX})$$

$$M(0,t) = E(e^{tY}).$$

Further,

$$E(X^k) = \frac{\partial^k M(s,t)}{\partial s^k}\Big|_{(0,0)},$$

$$E(Y^k) = \frac{\partial^k M(s,t)}{\partial t^k}\Big|_{(0,0)},$$

for $k = 1, 2, 3, \ldots$; and

$$E(XY) = \frac{\partial^2 M(s,t)}{\partial s \partial t}\Big|_{(0,0)}.$$

EXAMPLE: Let the random variables X and Y have the joint pdf

$$f(x,y) = \begin{cases} e^{-y} & \text{for} \quad 0 < x < y < \infty \\ \\ 0 & \text{otherwise,} \end{cases}$$

then find the joint moment generating function of X and Y.

EXAMPLE: If the joint moment generating function of the random variables X and Y is

$$M(s,t) = e^{(s+3t+2s^2+18t^2+12st)}$$

what is the covariance of X and Y ?

Theorem: If X and Y are independent then

$$M_{aX+bY}(t) = M_X(at)M_Y(bt)$$

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Proof: Let $W = aX + bY$. Then

$$\begin{aligned} M_{aX+bY}(t) &= M_W(t) \\ &= E(e^{tW}) \\ &= E(e^{t(aX+bY)}) \\ &= E(e^{taX}e^{tbY}) \\ &= E(e^{taX})Ee^{tbY} \\ &= M_X(at)M_Y(bt). \end{aligned}$$

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CONDITIONAL EXPECTATION AND VARIANCE

Recall, let X and Y be any two random variables with joint pdf (or pmf) f and marginals f_X and f_Y. The conditional probability density function (or pmf) g of X, given (the event) $Y = y$, is defined as

$$g(x|y) = \frac{f(x,y)}{f_Y(y)},$$

provided $f_Y(y) > 0$.

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$$g(x|y) = \frac{f(x,y)}{f_Y(y)},$$

provided $f_Y(y) > 0$.

Similarly, the conditional probability density function (or pmf) h of Y, given (the event) $X = x$, is defined as

$$h(y|x) = \frac{f(x,y)}{f_X(x)},$$

provided $f_X(x) > 0$.

CONDITIONAL EXPECTED VALUE (or mean):

The conditional mean of X given $Y = y$ is defined as

$$\mu_{X|y} = E(X|y) = \begin{cases} \sum_{x \in R_X} x\, g(x|y) & \text{if X is discrete} \\ \int_{-\infty}^{\infty} x\, g(x|y) dx & \text{if X is continuous} \end{cases}$$

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and the conditional mean of Y given $X = x$ is defined as

$$\mu_{Y|x} = E(Y|x) = \begin{cases} \sum_{y \in R_Y} y\, h(y|x) & \text{if Y is discrete} \\ \int_{-\infty}^{\infty} y\, h(y|x) dy & \text{if Y is continuous.} \end{cases}$$

EXAMPLE: A fair coin is tossed two times; let X denote the number of heads on the first toss and Y the total number of heads. Sample space of this random experiment is

$$\Omega = \{HH, HT, TH, TT\}.$$

The joint PMF f of X and Y is as given in the following table:

	Y=0	Y=1	Y=2	P(X=x)
X= 0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{2}{4}$
X= 1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{4}$
P(Y=y)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	

Then find the conditional mean of X and Y.