PHY 303: Classical Electrodynamics
MONSOON SEMESTER 2022
TUTORIAL 02

1. Consider a uniformly charged ring having a total charge Q and of radius a lying in the xy-plane, with its center coinciding with the origin. Find out the expression of the corresponding volume charge density $\rho(\mathbf{r})$ in cylindrical coordinates with the help of Dirac delta function(s).

2. The electric field due to a certain charge density $\rho(\mathbf{r})$ is given by

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \mathbf{0} & \text{for } r < R, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} & \text{for } r > R. \end{cases}$$

   With the aid of Gauss's law in differential form, find out the charge density $\rho(\mathbf{r})$ and interpret it physically. Moreover, based on your answer for the charge density, verify that the discontinuity in the electric field is as expected.

3. Derive Green's theorem by using the vector function $\mathbf{A}(\mathbf{r}) = \phi(\mathbf{r})\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla\phi(\mathbf{r})$ in the divergence theorem. Here, $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ are arbitrary scalar functions.

Challenge Problem:
Solve the differential equation $\nabla^2_{\mathbf{R}}\,\mathcal{G}(\mathbf{R}) = -4\pi\delta(\mathbf{R})$ using the Fourier transform method to show that $\mathcal{G}(\mathbf{R}) = 1/|\mathbf{R}|$. Here $\mathbf{R} = X\hat{\mathbf{x}} + Y\hat{\mathbf{y}} + Z\hat{\mathbf{z}}$ and $\nabla^2_{\mathbf{R}}$ represents the Laplacian involving the X, Y, Z variables.

PHY303 Tutorial 02 Solution (MONSOON 2022)

Q1 Sol. Working in the cylindrical coordinates system, clearly the charge distribution is restricted to $s = a$ and $z = 0$.

$$\therefore \quad \rho(\vec{r}) = A\,\delta(s-a)\,\delta(z),$$

where A is to be determined.

Since,

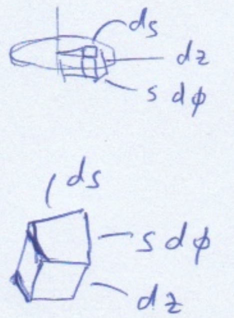
$$\int_{(\infty)} \rho(\vec{r})\, d^3r = Q$$

$$\Rightarrow \int_0^{\infty} ds \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \; s\, A\,\delta(s-a)\,\delta(z) = Q$$

$$\Rightarrow 2\pi a A = Q$$

$$\Rightarrow A = \frac{Q}{2\pi a}.$$

$\left\{ \because d^3r = s\, ds\, d\phi\, dz \right.$



Hence,

$$\rho(\vec{r}) = \frac{Q}{2\pi a}\,\delta(s-a)\,\delta(z)$$

$$\text{or,} \quad \frac{Q}{2\pi s}\,\delta(s-a)\,\delta(z)$$

Q2 Sol.

Notice that there is a discontinuity in the field at $r = R$. This must be taken into account while taking the divergence.

We can use theta function for this purpose, $\Theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$.

$$\therefore \vec{E}(\vec{r}) = \begin{cases} \vec{0} & \text{for } r < R \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{r} & \text{for } r > R \end{cases}$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \underbrace{\Theta(r-R)}_{\substack{0 \text{ for } r<R \\ 1 \text{ for } r>R}} \hat{r}$$

Now, $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \rho(\vec{r}) = \epsilon_0 \vec{\nabla}\cdot\vec{E}(\vec{r})$

$$\Rightarrow \rho(\vec{r}) = \cancel{\epsilon_0} \frac{1}{r^2}\frac{\partial}{\partial r}\left[\cancel{r^2} \frac{Q}{4\pi\cancel{\epsilon_0}\cancel{r^2}} \Theta(r-R)\right]$$

$$= \frac{Q}{4\pi r^2}\frac{\partial}{\partial r}\Theta(r-R)$$

$$= \frac{Q}{4\pi r^2}\delta(r-R)$$

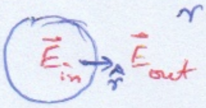
$$= \frac{Q}{4\pi R^2}\delta(r-R) \equiv \sigma\,\delta(r-R)$$

$\left\{ \vec{\nabla}\cdot(A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}) = \dfrac{1}{r^2}\dfrac{\partial(r^2 A_r)}{\partial r} \text{ for } A_\theta = 0 = A_\phi \right.$

This represents a charge Q uniformly distributed on a spherical shell of radius R. The surface charge density is clearly $\sigma = \dfrac{Q}{4\pi R^2}$.

- Discontinuity in $\vec{E}$ at the surface,

$$= \lim_{r\to R^+}\vec{E}(\vec{r}) - \lim_{r\to R^-}\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 R^2}\hat{r} - \vec{0} = \frac{\sigma}{\epsilon_0}\hat{r},$$



as expected.

Q3 Sol.

$$\vec{A} = \phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi$$

$$\therefore \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(\phi\vec{\nabla}\psi) - \vec{\nabla}\cdot(\psi\vec{\nabla}\phi)$$

$$= \cancel{\vec{\nabla}\phi\cdot\vec{\nabla}\psi} + \phi\nabla^2\psi - \cancel{\vec{\nabla}\psi\cdot\vec{\nabla}\phi} - \psi\nabla^2\phi$$

(Using $\vec{\nabla}(f\vec{F}) = \vec{\nabla}f\cdot\vec{F} + f\vec{\nabla}\cdot\vec{F}$)

$$= \phi\nabla^2\psi - \psi\nabla^2\phi \quad \text{—(1)}$$

& $$\vec{A}\cdot\hat{n} = \phi\vec{\nabla}\psi\cdot\hat{n} - \psi\vec{\nabla}\phi\cdot\hat{n}$$

$$= \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n} \quad \text{—(2)}$$

Here $\hat{n}$ is the outward normal unit vector on the surface S of a given/chosen volume V.

& $\frac{\partial}{\partial n}$ represents normal derivative

Divergence theorem asserts that

$$\int_V \vec{\nabla}\cdot\vec{A}\, d^3r = \oint_S \vec{A}\cdot\hat{n}\, da.$$

$$\Rightarrow \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3r = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da,$$

which is the GREEN'S THEOREM. {Using (1) & (2)}