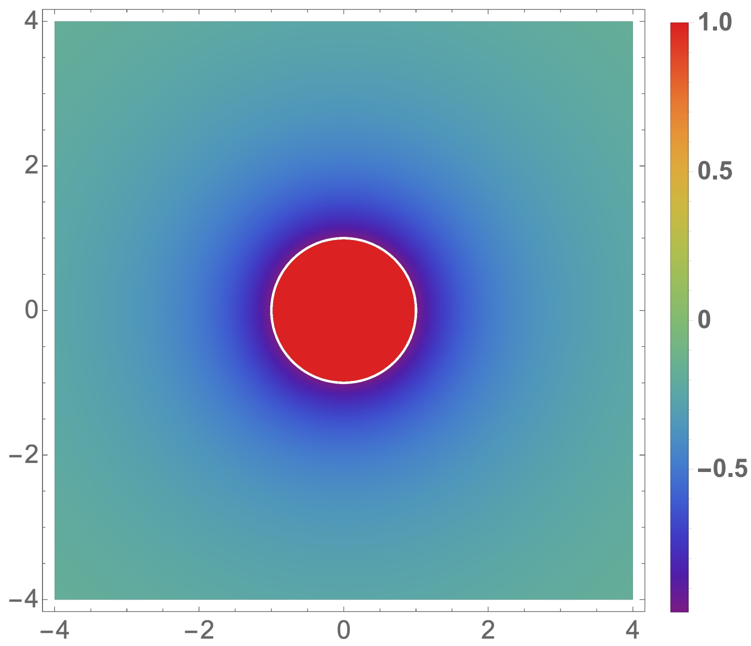
PHY 303: Classical Electrodynamics
MONSOON SEMESTER 2022
TUTORIAL 03

1. Consider an electrostatics problem in the half-space defined by $z \geq 0$, with Dirichlet boundary condition on the plane $z = 0$ (and at infinity). The potential on the plane $z = 0$ is specified to be the following in terms of the cylindrical coordinates (s, ϕ, z):

$$\Phi(s, \phi, 0) = \begin{cases} \Phi_0 & \text{for } 0 \leq s < R, \\ -\Phi_0 \, R/s & \text{for } s > R, \end{cases}$$

where Φ_0 is a constant (See the potential profile based on rainbow (VIBGYOR) colormap in the figure below). Moreover, the potential vanishes at infinity.



(a) Using the Green's function technique, find out the potential due to this system at an arbitrary point on the positive z-axis.

(b) How will the potential be modified if a point charge $+q$ is placed at $(0, 0, d)$?

Plot the electric potential along the positive z-axis in both cases by choosing some convenient values for the parameters.

2. Using Green's theorem, show that the solution of the Laplace equation outside ($r > a$) a sphere of radius a with the potential specified on its surface is given by

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\, \Phi(a,\theta',\phi') \frac{a(r^2-a^2)\sin\theta'}{(r^2+a^2-2ra\cos\gamma)^{3/2}},$$

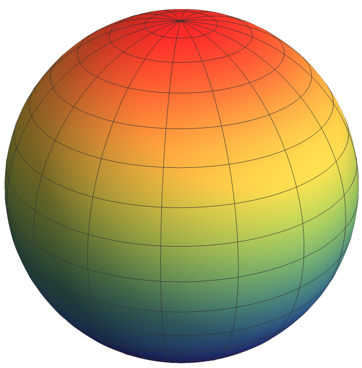
where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. Similarly, show that for inside ($r < a$) the sphere, the potential is

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\, \Phi(a,\theta',\phi') \frac{a(a^2-r^2)\sin\theta'}{(r^2+a^2-2ra\cos\gamma)^{3/2}}.$$

3. Use the result in the above problem to obtain the potentials in both the exterior and interior regions of the sphere, if $\Phi(a,\theta',\phi') = \Phi_0$ (i.e., if the sphere is kept at a constant potential). You may use some symmetry arguments in solving the integrals.

   Note that the potential in the interior region easily follows because of the averaging property of the Laplace equation.

4. Use the result in problem 2 again to write down the potential at an arbitrary point in the exterior region, if the potential on the surface is specified as $\Phi(a,\theta,\phi) = \Phi_0\cos\theta$, where Φ_0 is a constant (See the potential profile based on rainbow (VIBGYOR) colormap in the figure below). Consider the special case of determining the potential along the positive z-axis (for $z > a$) and perform the integrals to obtain a closed-form result.



Challenge Problem

Consider the electrostatic Green's function $G(\mathbf{r}, \mathbf{s})$, with $\nabla_{\mathbf{s}}^2 G(\mathbf{r}, \mathbf{s}) = -4\pi\delta(\mathbf{r} - \mathbf{s})$, for Dirichlet and Neumann boundary conditions on the surface $\mathcal{S}$ bounding the volume $\mathcal{V}$. Apply Green's theorem with integration variable $\mathbf{s}$ and scalar fields $\phi(\mathbf{s}) = G(\mathbf{r}, \mathbf{s})$ and $\psi(\mathbf{s}) = G(\mathbf{r}', \mathbf{s})$. Find an expression for the difference $G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}', \mathbf{r})$ in terms of an integral over the boundary surface $\mathcal{S}$.

(a) For Dirichlet boundary condition, show that $G(\mathbf{r}, \mathbf{r}') = G_D(\mathbf{r}, \mathbf{r}')$ must be symmetric in $\mathbf{r}$ and $\mathbf{r}'$.

(b) For Neumann boundary condition, show that $G(\mathbf{r}, \mathbf{r}') = G_N(\mathbf{r}, \mathbf{r}')$ is not symmetric in general, but $G_N(\mathbf{r}, \mathbf{r}') + F(\mathbf{r})$ is symmetric in $\mathbf{r}$ and $\mathbf{r}'$, where

$$F(\mathbf{r}) = -\frac{1}{\mathcal{S}} \oint_{\mathcal{S}} G_N(\mathbf{r}, \mathbf{s}) da_{\mathbf{s}}$$

(c) Show that the addition of $F(\mathbf{r})$ to the Neumann Green's function does not affect the solution $\Phi(\mathbf{r})$.

PHY303 Tutorial 03 Solution (MONSOON 2022)

Q1 Sol. This is a Dirichlet problem. The solution is given in terms of the Green's function $G_D(\vec{r},\vec{r}\,')$ as

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}\,')\, G_D(\vec{r},\vec{r}\,')\, d^3r' - \frac{1}{4\pi}\oint_S \Phi(\vec{r}\,')\frac{\partial G_D}{\partial n'}\, da' \quad -(1)$$

As already discussed in the lectures, the $G_D(\vec{r},\vec{r}\,')$ appropriate to this problem is

$$G_D(\vec{r},\vec{r}\,') = G(x,y,z;x',y',z')$$

$$= \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \quad -(2)$$

It vanishes on the XY plane $(z'=0)$ and also as $|\vec{r}\,'|\to\infty$ in V $(z>0)$

Note that $\Phi(\vec{r}\,')$ on S is nonzero only on the XY plane $(z'=0)$. For this we also calculated,

$$\left.\frac{\partial G_D}{\partial n'}\right|_{z'=0} = -\left.\frac{\partial G_D}{\partial z'}\right|_{z'=0} = \frac{-2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} \quad -(3)$$

(a) For this part $\rho(\vec{r}\,') = 0 \rightarrow$ There is no charge source in the volume of interest $(z > 0)$
We just have the surface contribution.

$$\therefore \Phi(\vec{r}) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \, \Phi(\vec{r}\,') \frac{2z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}$$

On the z-axis, $x=0$, $y=0$
[Desired observation region]

$$\Rightarrow \Phi(\vec{r}) \rightarrow \Phi(z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{\Phi(x', y', 0)}{[x'^2 + y'^2 + z^2]^{3/2}} \quad z' = 0.$$

Switch to cylindrical coordinates.

$$\Rightarrow \Phi(z) = \frac{z}{2\pi} \int_0^{\infty} ds' \int_0^{2\pi} d\phi' \frac{s' \, \Phi(s', \phi', 0)}{(s'^2 + z^2)^{3/2}}$$

$$= z \left[\int_0^R ds' \frac{\Phi_0 \cdot s'}{(s'^2 + z^2)^{3/2}} + \int_R^{\infty} ds' \frac{\left(-\frac{\Phi_0 R}{s'}\right) s'}{(s'^2 + z^2)^{3/2}} \right]$$

As $\Phi(s', \phi', 0) = \begin{cases} \Phi_0 & \text{for } 0 \le s' < R \\ -\frac{\Phi_0 R}{s'} & \text{for } s' > R. \end{cases}$

$$= z\Phi_0 \left[\int_0^R ds' \frac{s'}{(s'^2 + z^2)^{3/2}} - R \int_R^{\infty} ds' \frac{1}{(s'^2 + z^2)^{3/2}} \right]$$

$$= z\Phi_0 \left[\left\{ \frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}} \right\} - R \left\{ \frac{1}{z^2} - \frac{R}{z^2 \sqrt{z^2 + R^2}} \right\} \right]$$

$$= \Phi_0 \left[1 - \frac{z}{\sqrt{z^2 + R^2}} - \frac{R}{z} + \frac{R^2}{z\sqrt{z^2 + R^2}} \right]$$

(4)

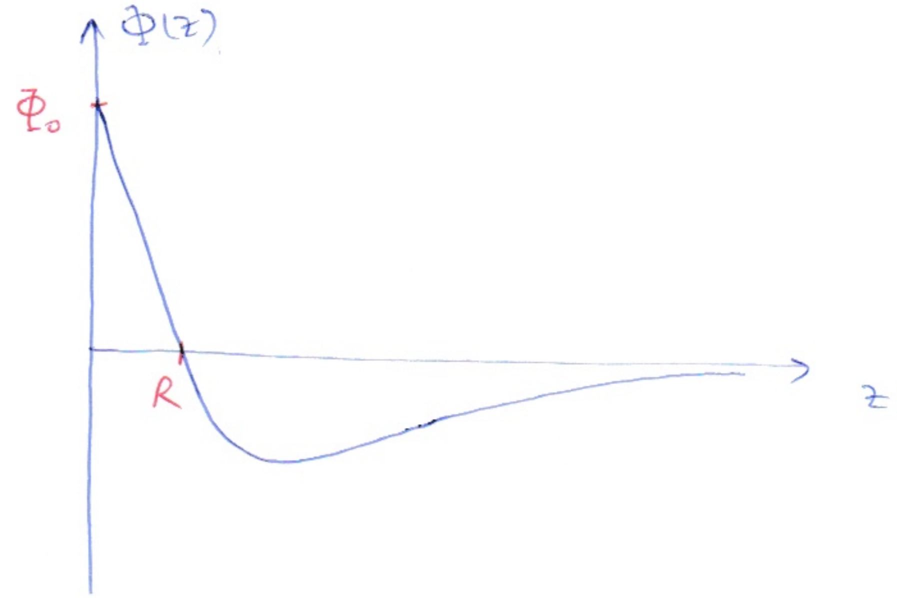
Note that for $z \to 0$, $\Phi(z) = \Phi_0$

for $z = R$, $\Phi(z) = \Phi_0 \left[1 - \frac{R}{\sqrt{2}R} - \frac{R}{R} + \frac{R}{\sqrt{2}R} \right]$
$= 0$

[$\Phi(z)$ changes sign at $z = R$]

for $z \to \infty$, $\Phi(z) = \Phi_0 [1 - 1 - 0 + 0] = 0$.

Plot.



(b) In this case we have volume contribution along with the surface contribution as found in eq (4).

From eq (1), the volume contribution is

$$\frac{1}{4\pi\epsilon_0} \int_V \rho(\bar{r}') \, G_D(\bar{r}, \bar{r}') \, d^3r'$$

Here, $\rho(\vec{r}') = q\,\delta(x')\,\delta(y')\,\delta(z'-d)$

↑ A point charge at $(0,0,d)$

$$\therefore \frac{1}{4\pi\epsilon_0}\int_V q\,\delta(x')\,\delta(y')\,\delta(z'-d)\,G_D(x,y,z;x',y',z')\,d^3r'$$

$$= \frac{q}{4\pi\epsilon_0}\,G_D(x,y,z;0,0,d)$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

(Using (2))

[Recall the image problem].

$\therefore$ ~~Electrostatic potential~~ Along the z-axis $(z>0)$, this gives

$$\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(z-d)^2}} - \frac{1}{\sqrt{(z+d)^2}}\right]$$

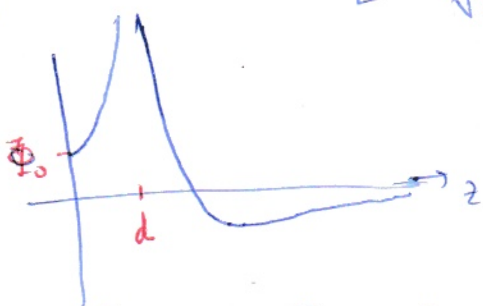
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|z-d|} - \frac{1}{z+d}\right]$$

$\therefore$ Total potential along $z\,(>0)$,

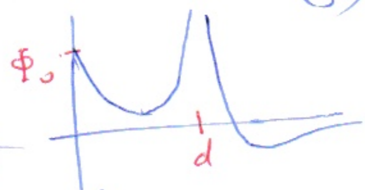
$$\Phi(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|z-d|} - \frac{1}{z+d}\right] + \Phi_0\left[1 - \frac{z}{\sqrt{z^2+R^2}} - \frac{R}{z} + \frac{R^2}{z\sqrt{z^2+R^2}}\right] \quad (5)$$

Complicated Plot

$d \leq R$



$d > R$



or

Competition between R & d.

Q2 Sol. The Dirichlet Green's function for the spherical case is (refer to the Lecture notes),

$$G_D(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{a}{r'\left|\vec{r} - \frac{a^2}{r'^2}\vec{r}\,'\right|}$$

$$= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma}},$$

where γ is the angle between $\vec{r}$ and $\vec{r}\,'$:

$\{\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')\}$

The above vanishes for $r = a$ (or $r' = a$).

[Recall that $G_D(\vec{r},\vec{r}\,')$ is symmetric in $\vec{r}$ & $\vec{r}\,'$]

Also, the above can be used for solving problems both in the exterior ($r > a$) and interior ($r < a$) regions.

This can be seen as follows. If the source is placed at $\vec{r}\,'$ with $|\vec{r}\,'| = r' > a$, then its image is at $\frac{a^2}{r'^2}\vec{r}\,'$.

Clearly, $\left|\frac{a^2}{r'^2}\vec{r}\,'\right| = \frac{a^4}{r'^3} = \left(\frac{a}{r'}\right)^3 a < a$

So the image lies inside the sphere.

$$\therefore \nabla^2 G_D(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,')$$

from the $\frac{1}{|\vec{r}-\vec{r}\,'|}$ term

$$\nabla^2 \frac{a}{r'\left|\vec{r} - \frac{a^2}{r'^2}\vec{r}\,'\right|} = 0$$

inside V (exterior)

Similarly, if the source is placed at $\vec{r}'$ with $|\vec{r}'| = r' < a$ i.e. inside, then its image lies at a distance $\left(\frac{a}{r'}\right)^3 a > a$, i.e. outside the sphere. So we still have

$$\nabla^2 G_D(\vec{r}, \vec{r}') = -4\pi\,\delta(\vec{r} - \vec{r}') \quad \text{for } V \text{ (interior)}.$$

$\Big\{$ Due to $\vec{r} \leftrightarrow \vec{r}'$ symmetry the argument can be repeated for $\vec{r}$ as the location of source & then $\nabla'^2 G_D(\vec{r}, \vec{r}') = -4\pi\,\delta(\vec{r}' - \vec{r})$ etc. $\Big\}$

The thing that distinguishes the two is the normal direction $\rightarrow -\hat{r}$ for exterior V
$+\hat{r}$ for interior V.

- Now, we are solving Laplace's eq. in V, $\rho(\vec{r}') = 0$. So the general solution gives.

$$\Phi(\vec{r}) = -\frac{1}{4\pi} \oint_S \Phi(\vec{r}')\, \frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'}\, dS'$$

$\Big\{$ Using dS' for area element.

For exterior region, $\dfrac{\partial}{\partial n'} \rightarrow -\dfrac{\partial}{\partial r'}$ 

$$\therefore \quad \frac{\partial G_D}{\partial n'} = -\frac{\partial}{\partial r'}\left[\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma}}\right]$$

$$= \frac{-\left(-\frac{1}{2}\right)(2r' - 2r\cos\gamma)}{\left[r^2 + r'^2 - 2rr'\cos\gamma\right]^{3/2}} + \frac{\left(-\frac{1}{2}\right)\left(\frac{2r^2}{a^2}r' - 2r\cos\gamma\right)}{\left[\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right]^{3/2}}$$

$$= \frac{(r' - r\cos\gamma)}{[\qquad]^{3/2}} - \frac{\left(\frac{r^2}{a^2}r' - r\cos\gamma\right)}{[\quad]^{3/2}}$$

At the surface ($r' = a$),

$$\left.\frac{\partial G_D}{\partial n'}\right|_{r'=a} = \frac{(a - r\cos\gamma)}{(r^2+a^2-2ra\cos\gamma)^{3/2}} - \frac{\left(\frac{r^2}{a} - r\cos\gamma\right)}{(r^2+a^2-2ra\cos\gamma)^{3/2}}$$

$$= \frac{(a^2-r^2)}{a(r^2+a^2-2ra\cos\gamma)^{3/2}}$$

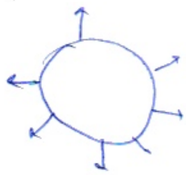
Also, $dS' = a^2\sin\theta'\,d\theta'\,d\varphi'$ on $r' = a$
& $\Phi(\bar{r}') = \Phi(a, \theta', \varphi')$ on $r' = a$.

Hence,

$$\Phi(\vec{r}) = -\frac{1}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi'\,(a^2\sin\theta')\,.\,\frac{(a^2-r^2)\,\Phi(a,\theta',\varphi')}{a(r^2+a^2-2ra\cos\gamma)^{3/2}}$$

$$= \frac{1}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi'\,\frac{a(r^2-a^2)\sin\theta'.\,\Phi(a,\theta',\varphi')}{(r^2+a^2-2ra\cos\gamma)^{3/2}},$$

For interior region, $\frac{\partial}{\partial n'} \rightarrow \frac{\partial}{\partial r'}$



$$\therefore \left.\frac{\partial G_D}{\partial n'}\right|_{r'=a} = \frac{(r^2-a^2)}{a(r^2+a^2-2ra\cos\gamma)^{3/2}}$$

$$\therefore \Phi(\vec{r}) = \frac{1}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi'\,\frac{a(a^2-r^2)\sin\theta'.\,\Phi(a,\theta',\varphi')}{(r^2+a^2-2ra\cos\gamma)^{3/2}}$$

Q3 Sol. In this problem, $\Phi(a, \theta', \varphi') = \Phi_0$.

$\therefore$ For $r > a$,

$$\Phi(\vec{r}) = \frac{1}{4\pi} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \frac{\Phi_0 \, a (r^2 - a^2) \sin\theta'}{[r^2 + a^2 - 2ra\cos\gamma]^{3/2}},$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$.

Since this problem has complete spherical symmetry, the answer will depend only on r & not on θ and φ.

$\therefore$ We may as well set $\theta = 0$
(Potential along the z-axis)

In this case, $\cos\gamma = \cos\theta'$

$$\therefore \Phi(\vec{r}) = \frac{\Phi_0 \, a (r^2 - a^2)}{4\pi} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \frac{\sin\theta'}{(r^2 + a^2 - 2ra\cos\theta')^{3/2}}$$

$$= \frac{\Phi_0 \, a (r^2 - a^2)}{2} \int_0^{\pi} d\theta' \frac{\sin\theta'}{(r^2 + a^2 - 2ra\cos\theta')^{3/2}}$$

$$= \frac{\Phi_0 \, a (r^2 - a^2)}{2} \int_{-1}^{1} \frac{du}{(r^2 + a^2 - 2rau)^{3/2}}$$

where $u = \cos\theta$.

$$\Rightarrow \quad \Phi(\vec{r}) = \frac{\Phi_0 a (r^2 - a^2)}{2} \cdot \frac{1}{ra} \int_{-1}^{1} d \frac{1}{(r^2+a^2-2rau)^{\frac{1}{2}}}$$

$$= \frac{\Phi_0}{2r}(r^2-a^2) \cdot \left[\frac{1}{(r^2+a^2-2ra)^{\frac{1}{2}}} - \frac{1}{(r^2+a^2+2ra)^{\frac{1}{2}}} \right]$$

$$= \frac{\Phi_0}{2r}(r^2-a^2)\left[\frac{1}{|r-a|} - \frac{1}{r+a} \right]$$

$$= \frac{\Phi_0}{2r}(r^2-a^2)\left[\frac{1}{r-a} - \frac{1}{r+a} \right]$$

$$(\because r > a)$$

$$= \frac{\Phi_0}{2r}(r^2-a^2) \frac{2a}{(r^2-a^2)}$$

$$= \frac{\Phi_0 a}{r} \quad //$$

Now For $r < a$, again setting $\theta = 0$.

$$\Rightarrow \quad \Phi(\vec{r}) = \frac{\Phi_0 a (a^2 - r^2)}{4\pi} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \frac{\sin\theta'}{(r^2+a^2-2ra\cos\theta')^{3/2}}$$

$$= \frac{\Phi_0}{2r}(a^2-r^2)\left[\frac{1}{|r-a|} - \frac{1}{r+a} \right]$$

$$= \frac{\Phi_0}{2r}(a^2-r^2)\left(\frac{1}{a-r} - \frac{1}{a+r} \right)$$

$$(\because r < a)$$

$$\therefore \quad \Phi(\vec{r}) = \frac{\Phi_0 (a^2 - r^2)}{2r} \cdot \frac{2r}{(a^2 - r^2)} = \Phi_0$$

i.e. inside the sphere, the potential is Φ_0 everywhere, as expected from the averaging property of Laplace' eq. solution.

Contd...

Q4 Sol.: *In this problem, we have to solve for the exterior region with*

$$\Phi(a,\theta,\varphi) = \Phi_0 \cos\theta$$

Using the result of problem 2 we have

$$\Phi(r,\theta,\phi) = \frac{\Phi_0}{4\pi} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \, \frac{a^2 \sin\theta' \quad (r^2 - a^2)\cos\theta'}{a\,[r^2 + a^2 - 2ra\cos\gamma]^{3/2}}$$

$$\left\{ \begin{array}{l} ds' = a^2 \sin\theta' \, d\theta' \, d\phi' \\ \text{on } S' \end{array} \right.$$

$$= \frac{\Phi_0\, a\,(r^2 - a^2)}{4\pi} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \, \frac{\sin\theta' \cos\theta'}{[r^2 + a^2 - 2ra\cos\gamma]^{3/2}},$$

Task for students: Try to solve this for general case.

where $\cos\gamma = \sin\theta \sin\theta' \cos(\varphi - \varphi') + \cos\theta\cos\theta'$.

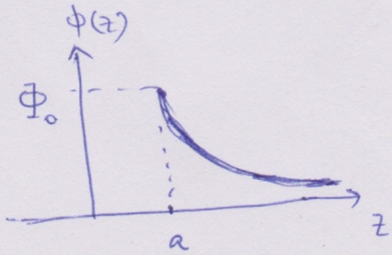
Now, if we consider the potential along the z-axis, we have $\theta = 0 \Rightarrow \cos\gamma = \cos\theta'$ (3)

$$\therefore \quad \Phi(r, \theta = 0, \phi) \equiv \Phi(z)$$

$$= \frac{\Phi_0\, a\,(r^2 - a^2)}{4\pi} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \, \frac{\sin\theta' \cos\theta'}{[r^2 + a^2 - 2ra\cos\theta']^{3/2}}$$

$$= \frac{\Phi_0\, a\,(r^2 - a^2)}{4\pi} \cdot 2\pi \cdot \int_{-1}^{1} \frac{u\, du}{[r^2 + a^2 - 2rau]^{3/2}}$$

$$= \frac{\Phi_0\, a\,(r^2 - a^2)}{2} \cdot \frac{2a}{r^2 (r^2 - a^2)} \qquad \textit{(For } r > a\textit{)}$$



$$= \frac{\Phi_0\, a^2}{r^2}$$

Challenge Problem Sol. Green's theorem (with $\vec{s}$ as the integration variable) is,

$$\int_V \left[\phi(\vec{s})\, \nabla^2_{\vec{s}}\, \psi(\vec{s}) - \psi(\vec{s})\, \nabla^2_{\vec{s}}\, \phi(\vec{s})\right] d^3s$$

$$= \oint_S \left[\phi(\vec{s})\, \frac{\partial \psi(\vec{s})}{\partial n_{\vec{s}}} - \psi(\vec{s})\, \frac{\partial \phi(\vec{s})}{\partial n_{\vec{s}}}\right] da_{\vec{s}}$$

Not to be confused with integration variable

Set $\phi(\vec{s}) = G(\vec{r}, \vec{s})$

& $\psi(\vec{s}) = G(\vec{r}', \vec{s})$

$$\Rightarrow \int_V \left[G(\vec{r}, \vec{s})\, \nabla^2_{\vec{s}}\, G(\vec{r}', \vec{s}) - G(\vec{r}', \vec{s})\, \nabla^2_{\vec{s}}\, G(\vec{r}, \vec{s})\right] d^3s$$

$$= \oint_S \left[G(\vec{r}, \vec{s})\, \frac{\partial G(\vec{r}', \vec{s})}{\partial n_{\vec{s}}} - G(\vec{r}', \vec{s})\, \frac{\partial G(\vec{r}, \vec{s})}{\partial n_{\vec{s}}}\right] da_{\vec{s}}$$

$$\Rightarrow \int_V \left[G(\vec{r}, \vec{s})\{-4\pi\, \delta(\vec{r}' - \vec{s})\} - G(\vec{r}', \vec{s})\{-4\pi\, \delta(\vec{r} - \vec{s})\}\right] d^3s$$

$$= \oint_S [\cdots]\, da_{\vec{s}}$$

$$\Rightarrow -4\pi\left[G(\vec{r}, \vec{r}') - G(\vec{r}', \vec{r})\right] = \oint_S [\cdots]\, da_{\vec{s}}$$

$$\Rightarrow \boxed{\begin{aligned} &G(\vec{r}, \vec{r}') - G(\vec{r}', \vec{r}) \\ &= -\frac{1}{4\pi} \oint_S \left[G(\vec{r}, \vec{s})\, \frac{\partial G(\vec{r}', \vec{s})}{\partial n_{\vec{s}}} - G(\vec{r}', \vec{s})\, \frac{\partial G(\vec{r}, \vec{s})}{\partial n_{\vec{s}}}\right] da_{\vec{s}} \end{aligned}}$$

(1)

(a) For DIRICHLET boundary condition, the Green's function $G(\vec{r},\vec{s}) = G_D(\vec{r},\vec{s})$ is constructed in such a way that it vanishes on the surface,

i.e. $G(\vec{r},\vec{s}) = G_D(\vec{r},\vec{s}) = 0$ for $\vec{s}$ lying on S. —(2)

The potential in this case is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{s})\, G_D(\vec{r},\vec{s})\, d^3 s - \frac{1}{4\pi}\oint_S \Phi(\vec{s})\frac{\partial G_D(\vec{r},\vec{s})}{\partial n_{\vec{s}}}\, da_{\vec{s}} \quad \text{—(3)}$$

Using information (2) in eq (1), we get

$$G_D(\vec{r},\vec{r}') - G_D(\vec{r}',\vec{r})$$

$$= -\frac{1}{4\pi}\oint_S \left[(0)\cdot\frac{\partial G_D(\vec{r}',\vec{s})}{\partial \vec{n}_s} - (0)\cdot\frac{\partial G_D(\vec{r},\vec{s})}{\partial \vec{n}_s}\right] da'$$

$$\Rightarrow \quad G_D(\vec{r},\vec{r}') - G_D(\vec{r}',\vec{r}) = 0$$

$$\therefore \boxed{G_D(\vec{r},\vec{r}') = G_D(\vec{r}',\vec{r})} \quad \text{—(4)}$$

i.e. G is symmetric in $\vec{r}$ and $\vec{r}'$.

(b) For NEUMANN boundary condition, we set

$$\frac{\partial G_N(\vec{r},\vec{s})}{\partial n_{\vec{s}}} = -\frac{4\pi}{S} \quad \text{on the surface} \qquad (5)$$

(S is total surface area of boundary)

for $G(\vec{r},\vec{s}) = G_N(\vec{r},\vec{s})$, and the potential solution is

$$\Phi(\vec{r}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{s})\, G_N(\vec{r},\vec{s})\, d^3s + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{s})}{\partial n_{\vec{s}}}\, G_N(\vec{r},\vec{s})\, da_{\vec{s}} \qquad (6)$$

Using (5) in (1), we obtain

$$G_N(\vec{r},\vec{r}\,') - G_N(\vec{r}\,',\vec{r})$$

$$= -\frac{1}{4\pi}\oint_S \left[G_N(\vec{r},\vec{s})\left(-\frac{4\pi}{S}\right) - G_N(\vec{r}\,',\vec{s})\left(-\frac{4\pi}{S}\right)\right] da_{\vec{s}}$$

$$\Rightarrow G_N(\vec{r},\vec{r}\,') - G_N(\vec{r}\,',\vec{r})$$

$$= \frac{1}{S}\oint_S \left[G_N(\vec{r},\vec{s}) - G_N(\vec{r}\,',\vec{s})\right] da_{\vec{s}} \qquad (7)$$

i.e. the Neumann Green's function is not automatically symmetric in $\vec{r}$ and $\vec{r}\,'$.

However, (7) gives

$$G_N(\vec{r},\vec{r}\,') - \frac{1}{S}\oint_S G_N(\vec{r},\vec{s})\,da_{\vec{s}}$$
$$= G_N(\vec{r}\,',\vec{r}) - \frac{1}{S}\oint_S G_N(\vec{r}\,',\vec{s})\,da_{\vec{s}}$$

ie, $$G_N(\vec{r},\vec{r}\,') + F(\vec{r}) = G_N(\vec{r}\,',\vec{r}) + F(\vec{r}\,') \quad (8)$$

where $F(\vec{r}) = -\frac{1}{S}\oint_S G_N(\vec{r},\vec{s})\,da_{\vec{s}}$

Clearly $\tilde{G}_N(\vec{r},\vec{r}\,') = G_N(\vec{r},\vec{r}\,') + F(\vec{r})$

is symmetric in $\vec{r}$ and $\vec{r}\,'$, since

$$\tilde{G}_N(\vec{r}\,',\vec{r}) = G_N(\vec{r}\,',\vec{r}) + F(\vec{r}\,') = G(\vec{r},\vec{r}\,') + F(\vec{r})$$
$$= \tilde{G}_N(\vec{r},\vec{r}\,') \quad \text{(using (8))}$$

(c) We have to show that addition of $F(\vec{r})$ to $G_N(\vec{r},\vec{s})$ does not change the solution (6), i.e. $\tilde{G}_N(\vec{r},\vec{s})$ gives same result as $G_N(\vec{r},\vec{s})$.

Let $$\tilde{\Phi}(\vec{r}) = \langle \Phi_S \rangle + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{s})\,\tilde{G}_N(\vec{r},\vec{s})\,d^3s$$
$$+ \frac{1}{4\pi}\oint_S \frac{\partial \Phi(\vec{s})}{\partial n_{\vec{s}}}\,\tilde{G}_N(\vec{r},\vec{s})\,da_{\vec{s}}$$

$$\Rightarrow \tilde{\Phi}(\vec{r}) = \langle \Phi_s \rangle + \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{s}) \left[G_N(\vec{r},\vec{s}) + F(\vec{r}) \right] d^3s$$

$$+ \frac{1}{4\pi} \oint_S \frac{\partial \Phi(\vec{s})}{\partial n_{\vec{s}}} \left[G_N(\vec{r},\vec{s}) + F(\vec{r}) \right] da_{\vec{s}}$$

$$= \left. \begin{aligned} &\langle \Phi_s \rangle + \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{s})\, G_N(\vec{r},\vec{s})\, d^3s \\ &+ \frac{1}{4\pi} \oint_S \frac{\partial \Phi(\vec{s})}{\partial n_{\vec{s}}} G_N(\vec{r},\vec{s})\, da_{\vec{s}} \end{aligned} \right\} \begin{aligned} &= \Phi(\vec{r}) \\ &\text{using (6)} \end{aligned}$$

$$+ \frac{F(\vec{r})}{4\pi} \left[\frac{1}{\epsilon_0} \int_V \rho(\vec{s})\, d^3s + \oint_S \frac{\partial \Phi(\vec{s})}{\partial n_{\vec{s}}} da_{\vec{s}} \right]$$

$$\Rightarrow \tilde{\Phi}(\vec{r}) = \Phi(\vec{r}) + \frac{F(\vec{r})}{4\pi} \left[\frac{1}{\epsilon_0} \int_V \rho(\vec{s})\, d^3s + \oint_S \vec{\nabla}_{\vec{s}} \Phi(\vec{s}) \cdot \hat{n}_{\vec{s}}\, da_{\vec{s}} \right]$$

$$= \Phi(\vec{r}) + \frac{F(\vec{r})}{4\pi} \left[\frac{1}{\epsilon_0} \int_V \rho(\vec{s}) d^3s + \underbrace{\int_V \vec{\nabla}_s \cdot (\vec{\nabla}_s \Phi(\vec{s}))\, d^3s}_{\text{Using divergence theorem}} \right]$$

$$= \Phi(\vec{r}) + \frac{F(\vec{r})}{4\pi} \left\{ \int_V \underbrace{\left[\frac{\rho(\vec{s})}{\epsilon_0} + \nabla_{\vec{s}}^2 \Phi(\vec{s}) \right]}_{=0 \text{ using Poisson eq.}} d^3s \right\}$$

$$= \Phi(\vec{r})$$

$\therefore$ Addition of $F(\vec{r})$ to $G_N(\vec{r},\vec{s})$ does not change the solution. Therefore although $G_N(\vec{r},\vec{s})$ is not symmetric in $\vec{r}$ & $\vec{s}$, one can use $\tilde{G}_N(\vec{r},\vec{s}) = G_N(\vec{r},\vec{s}) + F(\vec{r})$ which is symmetric in $\vec{r}$ & $\vec{s}$.