PHY 303: Classical Electrodynamics
MONSOON SEMESTER 2022
TUTORIAL 05

1. A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = kr\,\mathbf{r},$$

where k is a constant, $\mathbf{r}$ is the vector from the center, and $r = |\mathbf{r}|$. Calculate the bound charges σ_b and ρ_b.

2. Calculate the electric potential produced by a uniformly polarized sphere of radius a and center at the origin with electric polarization (dipole moment per unit volume) given by $P_0\,\hat{\mathbf{z}}$, using the result

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\left(\rho(\mathbf{r}') - \boldsymbol{\nabla}'\cdot\mathbf{P}(\mathbf{r}')\right)}{|\mathbf{r}-\mathbf{r}'|} d^3r' + \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\mathbf{P}(\mathbf{r}')\cdot d\mathbf{a}'}{|\mathbf{r}-\mathbf{r}'|}.$$

Note that for this problem there is no (free) charge involved and outside the given sphere the polarization is zero. Obtain results for both interior ($|\mathbf{r}| < a$) and exterior ($|\mathbf{r}| > a$) regions using the following three approaches:

(a) Consider the volume ($\mathcal{V}$) and the enclosing surface ($\mathcal{S}$) used in the integrals above as the ones approaching the surface and volume of the polarized sphere from inside, i.e., surface and volume same as the given sphere in a limiting sense, approaching from inside.

(b) Consider $\mathcal{V}$ and $\mathcal{S}$ as the ones approaching the surface and volume of the polarized sphere from outside.

(c) Consider $\mathcal{V}$ to be infinite.

Q1 Sol. The surface bound charge density is given by,

$$\sigma_b = \vec{P} \cdot \hat{n} \quad \text{at the surface.}$$

Here $\hat{n} = \hat{r}$, the unit vector in the radial direction. Also at the surface of the sphere $r = R$.

$$\therefore \sigma_b = k r \vec{r} \cdot \hat{r} \Big|_{r=R}$$

$$= k r^2 \hat{r} \cdot \hat{r} \Big|_{r=R}$$

$$= kR^2$$

Volume bound charge density is given by,

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

We know that $\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r)$ when $\vec{v}$ has only r dependence i.e. $\vec{v} = v_r \hat{r}$.

This is the case with $\vec{P}$.

$$\vec{P} = k r \vec{r} = k r^2 \hat{r}$$

$$\therefore P_r = k r^2$$

$$\therefore \rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 k r^2)$$

$$\text{a.} \quad \rho_b = -\frac{k}{r^2}(4r^3) = -4kr$$

Varies in the volume.

Alternative approach We can get total bound charge density using just $-\vec{\nabla}\cdot\vec{P}$, provided we define $\vec{P}$ such that it holds everywhere

(Recall the discussion in the class)

Overall, $\vec{P}(\vec{r}) = \begin{cases} kr\vec{r} & \text{inside the sphere of radius } R \\ \vec{0} & \text{outside.} \end{cases}$

We can write this using Heaviside theta function,

$$\vec{P}(\vec{r}) = kr\vec{r}\,\Theta(R-r)$$

$$= \underbrace{kr^2\,\Theta(R-r)}\,\hat{r}$$

$\Theta(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } x>0 \end{cases}$

$\therefore$ Overall,

$$\tilde{\rho}_b = -\vec{\nabla}\cdot\vec{P}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\cdot kr^2\,\Theta(R-r)\right]$$

$$= -\frac{k}{r^2}\frac{d}{dr}\left[r^4\,\Theta(R-r)\right]$$

$$= -\frac{k}{r^2}\left[4r^3\,\Theta(R-r) - r^4\,\delta(R-r)\right]$$

$$= -4kr\,\Theta(R-r) + kr^2\,\delta(R-r)$$

$$= \underbrace{-4kr\,\Theta(R-r)}_{\substack{\text{Volume contribution} \\ \text{(Nonzero for } r<R)}} + \underbrace{kR^2\,\delta(r-R)}_{\substack{\text{Surface contribution} \\ \text{(Nonzero for } r=R)}}$$

$-4kr$ as found for ρ_b in earlier method

kR^2 as found in earlier method for σ_b

Q2 Sol. As there is no free charge involved, $\rho(\vec{r}') = 0$.

The polarization is $P_0 \hat{z}$ within the sphere $(r < a)$, and outside $(r > a)$ it is zero.

So overall, $\vec{P}(\vec{r}') = P_0 \hat{z}\, \Theta(a - r')$

$$\therefore \vec{\nabla}' \cdot \vec{P}(\vec{r}') = \left(\hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}\right) \cdot P_0 \Theta(a - r')\hat{z}$$

$$= P_0 \frac{\partial}{\partial z'} \Theta(a - r')$$

$$= P_0 \frac{\partial r'}{\partial z'} \frac{\partial}{\partial r'} \Theta(a - r')$$

$$= P_0 \frac{\partial}{\partial z'}(x'^2 + y'^2 + z'^2)^{\frac{1}{2}} \left(-\delta(a - r')\right)$$

$$= -\frac{P_0}{2} \frac{2z'}{(x'^2 + y'^2 + z'^2)^{\frac{1}{2}}} \delta(r' - a)$$

$$= -P_0 \frac{z'}{r'} \delta(r' - a)$$

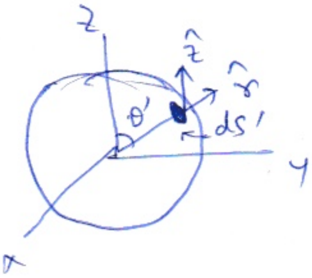
$$= -P_0 \frac{r'\cos\theta'}{r'} \delta(r' - a)$$

$$= -P_0 \cos\theta' \, \delta(r' - a)$$

(1)

Moreover, $\vec{P}(\vec{r}')\cdot d\vec{S}' = P_0\,\Theta(a-r')\,\hat{z}\cdot dS'\,\hat{r}$

$$= P_0\,\Theta(a-r')\,dS'\cos\theta' \qquad -(2)$$



Now, (a) [APPROACH 1]

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V^-}\frac{[\rho(\vec{r}') - \vec{\nabla}'\cdot\vec{P}(\vec{r}')]}{|\vec{r}-\vec{r}'|}\,d^3r'$$

$$+ \frac{1}{4\pi\epsilon_0}\oint_{S^-}\frac{\vec{P}(\vec{r}')\cdot d\vec{S}'}{|\vec{r}-\vec{r}'|}$$

(I’ve used here dS’ instead of da’ for the area element to avoid confusion with the radius of the sphere.)

For V^-, r' will vary from 0 to $a-\delta$, and for S^-, $r' = a-\delta$, with $\delta\to 0$.

(δ not to be confused with the δ(.) of the Dirac delta function.)

$$\therefore\ \Phi(\vec{r}) = \lim_{\delta\to 0}\frac{1}{4\pi\epsilon_0}\int_0^{a-\delta}dr'\int_0^{\pi}d\theta'\int_0^{2\pi}d\varphi'\,\frac{[0 + P_0\cos\theta'\,\delta(r'-a)]\,r'^2\sin\theta'}{|\vec{r}-\vec{r}'|}$$

$$+ \lim_{\delta\to 0}\frac{1}{4\pi\epsilon_0}\int_0^{\pi}d\theta'\int_0^{2\pi}d\varphi'\,\frac{[P_0\,\Theta(\cancel{a}-\cancel{a}+\delta)\cos\theta']\,(a-\delta)^2\sin\theta'}{|\vec{r}-\vec{r}'|_{r'=a-\delta}}$$

Clearly, in the first term, $r' \neq a$ is not achieved (We are considering $r'\to a$), so the delta-function vanishes, thereby making the integral zero.

In the second term $\Theta(\delta) = 1$, and then $r' = a$ (ie $\delta = 0$) can be substituted for the limit $\delta\to 0$.

$$\therefore \quad \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \, \frac{P_0 \, a^2 \sin\theta' \cos\theta'}{|\vec{r} - \vec{r}'|_{r'=a}} \qquad -(3)$$

$\Bigg\{$ Note that in this case, the above can be obtained by simply arguing that for V^- & S^-, $\vec{\nabla}' \cdot \vec{P}(\vec{r}') = 0 \Rightarrow$ No volume contribution

while, the surface term contributes.

Now using the addition theorem for spherical harmonics,

$$\frac{1}{|\vec{r} - \vec{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

$$\text{where } r_< = \min(r, r') \qquad -(4)$$

$$\& \quad r_> = \max(r, r')$$

$\therefore$ From (3) & (4),

$$\Phi(\vec{r}) = \frac{P_0 a^2}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r_<^{\,l}}{r_>^{\,l+1}} \right)_{r'=a} Y_{lm}(\theta, \varphi)$$

$$\times \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \sin\theta' \cos\theta' \, Y_{lm}^*(\theta', \varphi')$$

Noting that $\cos\theta' = \sqrt{\frac{4\pi}{3}} \, Y_{10}(\theta', \varphi')$, we have

$$\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \sin\theta' \cos\theta' \, Y_{lm}^*(\theta', \varphi') = \sqrt{\frac{4\pi}{3}} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \sin\theta' \, Y_{10}(\theta', \varphi') Y_{lm}^*(\theta', \varphi')$$

$$= \sqrt{\frac{4\pi}{3}} \, \delta_{l,1} \, \delta_{m,0} \qquad \text{[Using orthonormality]}$$

$$\therefore\ \Phi(\vec{r}) = \frac{P_0 a^2}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r_<^{\,l}}{r_>^{\,l+1}}\right)_{r'=a} Y_{lm}(\theta,\varphi)$$

$$\times \sqrt{\frac{4\pi}{3}}\ \delta_{l,1}\ \delta_{m,0}$$

$$= \frac{P_0 a^2}{\epsilon_0} \frac{1}{(2\times 1+1)} \left(\frac{r_<^{\,1}}{r_>^{\,1+1}}\right)_{r'=a} Y_{10}(\theta,\varphi) \cdot \sqrt{\frac{4\pi}{3}}$$

(Out of the summations only $l=1$ & $m=0$ survive)

$$= \frac{P_0 a^2}{3\epsilon_0} \left(\frac{r_<}{r_>^2}\right)_{r'=a} \cdot \sqrt{\frac{3}{4\pi}} \cos\theta \cdot \sqrt{\frac{4\pi}{3}}$$

$$= \frac{P_0 a^2}{3\epsilon_0} \left(\frac{r_<}{r_>^2}\right)_{r'=a} \cos\theta \qquad -(5)$$

Now for interior region, $r_< = r$ & $r_> = r'$

$$\therefore\ \Phi_{r<a}(\vec{r}) = \frac{P_0 a^2}{3\epsilon_0} \left(\frac{r}{r'^2}\right)_{r'=a} \cos\theta$$

$$= \frac{P_0 a^2}{3\epsilon_0 a^2} r\cos\theta = \frac{P_0}{3\epsilon_0} r\cos\theta. \qquad -(6)$$

For exterior region, $r_< = r'$ & $r_> = r$

$$\therefore\ \Phi_{r>a}(\vec{r}) = \frac{P_0 a^2}{3\epsilon_0} \left(\frac{r'}{r^2}\right)_{r'=a} \cos\theta = \frac{P_0 a^3}{3\epsilon_0 r^2} \cos\theta \qquad -(7)$$

Note that, as Φ is continuous, at $r=a$, both (6) & (7) give identical answers, viz $\frac{P_0 a}{3\epsilon_0}\cos\theta$.

Now, (b) $\Phi(\vec{r}) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int_{V^+} \dfrac{[\rho(\vec{r}\,') - \vec{\nabla}' \cdot \vec{P}(\vec{r}\,')]}{|\vec{r} - \vec{r}\,'|} d^3 r'$

[APPROACH 2]

$$+ \frac{1}{4\pi\epsilon_0} \oint_{S^+} \frac{\vec{P}(\vec{r}\,') \cdot d\vec{S}'}{|\vec{r} - \vec{r}\,'|}$$

For V^+, r' will vary from 0 to $a+\delta$,
and for S^+, $r' = a + \delta$, with $\delta \to 0$

$$\therefore \Phi(\vec{r}) = \lim_{\delta \to 0} \frac{1}{4\pi\epsilon_0} \int_0^{a+\delta} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \frac{[0 + P_0 \cos\theta' \delta(r' - a)] r'^2 \sin\theta'}{|\vec{r} - \vec{r}\,'|}$$

$$+ \lim_{\delta \to 0} \frac{1}{4\pi\epsilon_0} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \frac{[P_0\, \Theta(a - a - \delta) \cos\theta'] (a+\delta)^2 \sin\theta'}{|\vec{r} - \vec{r}\,'|_{r' = a+\delta}}$$

In the first term, since r' goes up to $a+\delta$, the delta-function contributes $\rightsquigarrow$ Use $\displaystyle\int_0^{a+\delta} f(r')\, \delta(r' - a)\, dr' = f(a)$,

In the second term, $\Theta(-\delta) = 0$,
so it vanishes.

$$\therefore \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \frac{P_0\, a^2 \cos\theta' \sin\theta'}{|\vec{r} - \vec{r}\,'|_{r' = a}}$$

This is same as eq (3), and so the final answer will be as obtained using the APPROACH 1.

Hence, in this case (V^+ & S^+) the surface integral vanishes as $\vec{P}(\vec{r}\,') = 0$ just outside the sphere. However, $\vec{\nabla}' \cdot \vec{P}(\vec{r}\,')$ picks up the discontinuity of $\vec{P}$ on the surface & leads to identical result.

(c) In this case, $\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{(\infty)} \frac{[\rho(\vec{r}') - \vec{\nabla}' \cdot \vec{P}(\vec{r}')]}{|\vec{r} - \vec{r}'|} d^3 r'$
[APPROACH 3]

(As derived in class notes*.)

This will give same result as (b) & hence finally same answer will be obtained.

This is because here r' varies from 0 to ∞, so the delta function in $\vec{\nabla}' \cdot \vec{P}(\vec{r}')$ again picks up the discontinuity at surface S.

Also, there is no explicit surface term here to worry about.

(*And if we do consider the surface integral, that will be zero using the same reasoning as in the APPROACH 2.)

[That information is already there in the volume term itself.]