Calculus of Variations. Tut 3

→ The calculus of variations deals with problems where we search for a function or curve rather than a value of some variable, that makes a given quality stationary, usually an energy or Action integral. Because ~~these~~ a function is varied, these problems are called variational.

Variational Principles, such as those of D'Alemberts, Langrangs and Hamiltons have been developed in classical mechanics; Fermat's Principle in Electrodynamics etc.

Euler Equation:

The calculus of variations typically involves problems in which a quantity to be minimized (or maximized) appears as a functional, meaning that it is a quanility whose argument(s) are themselves functions, not just variables. The general case be, let J be a functional of y, defined as

$$J[y] = \int_{x_1}^{x_2} f\left(y(x), \frac{dy(x)}{dx}, x\right) dx \quad \to (1)$$

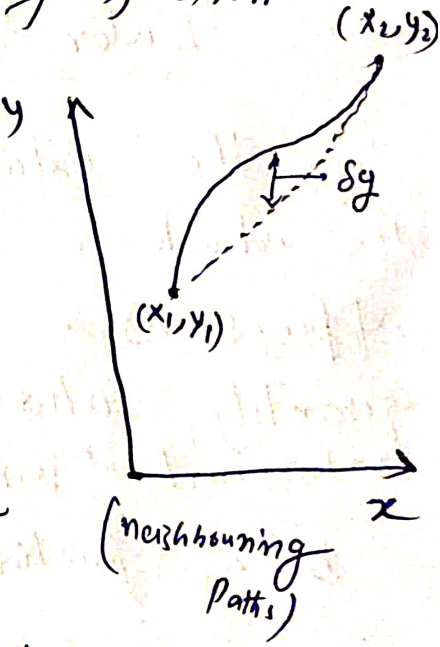
where f is fixed function of three variables $y, dy/dx$ & x
J is functional its value depends on behaviour of $y(x)$ throughout the entire range of x $(x_1 \leq x \leq x_2)$:
~~Typical~~ Typical problem in calculus of variations is to

a continous & differentiable ① function $y(x)$ that makes J stationary relative to small chang in y anywhere (or everywhere) in its range of definition. The stationary values of will in in many problems be minima or maxima, but they can also be saddle points.

We indicate the variation in J produced by a small variation ~~of~~ in y as

$$\delta J = \delta \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx \quad ; \dot{y} = dy/dx \qquad (2)$$

Note, δ is used instead of d or ∂, to remind us that the variation is that of function rather than that of variable.



Variation in the function y: for which the endpoints $y(x_1)$ & $y(x_2)$ are fixed is denoted by δy.

We describe δy as by introducing a new function, $\eta(x)$ & a scale factor α that controls magnitude of the variation. The function $\eta(x)$ is arbitray expect for being continous & differentiable, & to keep the endpoints fixed, with

$$\eta(x_1) = \eta(x_2) = 0 \longrightarrow (3)$$

So, our path, now a function of α, is

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x) \longrightarrow (4)$$

When $\alpha = 0$, neighbouring curves become $y = y(x)$ which is extremum path.

so, $\delta y = \alpha\eta(x)$. (3)

So, from (1)

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)\, dx.$$

Now, J is function of α rather than a functional of y.

To ~~procee~~ ~~make~~ obtain stationary value of J, i.e

$$\left[\frac{\partial J(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0$$

So,

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}\right) dx$$

$$\Rightarrow \frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right) dx = 0 \longrightarrow (5)$$

$\left\{\frac{\partial x}{\partial \alpha} = 0;\right.$ independent of $\left.\alpha\right\}$

from equation (4),

$$\frac{\partial y(x,\alpha)}{\partial \alpha} = \eta(x)$$

and $$\frac{\partial \dot{y}(x,\alpha)}{\partial \alpha} = \frac{d}{dx}\eta(x)$$

so, eqn (5) becomes

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial \dot{y}}\frac{d}{dx}\eta(x)\right) dx = 0$$

Integrating the second term by parts to get $\eta(x)$ as a common factor,

$$\int_{x_1}^{x_2} \frac{d}{dx}\eta(x)\,\frac{\partial f}{\partial \dot y}\,dx = \eta(x)\,\frac{\partial f}{\partial \dot y}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\,\frac{d}{dx}\frac{\partial f}{\partial \dot y}\,dx$$

The integrated part vanishes (by eqn 3)

So, we have

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\,\eta(x) - \eta(x)\frac{d}{dx}\frac{\partial f}{\partial \dot y}\right)dx = 0$$

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot y}\right)\right]\eta(x)\,dx = 0 \qquad (6)$$

or $\cancel{\frac{\partial J(\alpha)}{\partial}}$ $\int_{x_1}$ Multiplying by $\delta\alpha$, which gives upon using $\eta(x)\delta\alpha = \delta y$.

$$\delta J = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot y}\right)\delta y\,dx = 0$$

~~we~~ can ~~certainly set δJ as zero~~.

The condition δJ vanish for all δy implies that the quantity inside the brackets must vanish because δy is arbitrary. Recall that δy is a function of x and has arbitrary value at each point x, so δy cannot be pulled outside the integral; the bracketed quantity must indeed vanish for δJ to

Thus we are left with (5)

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

Known as Euler Equations.

This equation is 2nd order differential Equation.

If we have

$$\int_a^b f(x)g(x)\,dx = 0$$

for any $g(x)$

Then

$f(x) = 0 \quad \forall x \in [a,b]$.

Integrating Euler-Equation

Since f is $f(y, \dot{y}, x)$.

So, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial \dot{y}}\frac{d\dot{y}}{dx}$$

Using Euler equation $\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} = 0$

So, this gives

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \underbrace{\frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\frac{dy}{dx} + \frac{\partial f}{\partial \dot{y}}\frac{d\dot{y}}{dx}}_{\text{[Product rule]}}$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{d}{dx}\left(\dot{y}\frac{\partial f}{\partial \dot{y}}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \dot{y}\frac{\partial f}{\partial \dot{y}}\right) = \frac{\partial f}{\partial x}$$

If $f(y, \dot{y}, x) = f(y, \dot{y})$ i.e f is independent of x,

Then

$$\frac{d}{dx}\left(f - \dot{y}\frac{\partial f}{\partial \dot{y}}\right) = 0$$

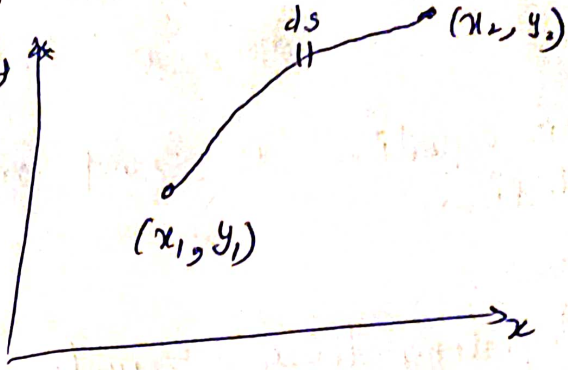
$$\Rightarrow \boxed{f - \dot{y}\frac{\partial f}{\partial \dot{y}} = C}$$

C = constant $\Rightarrow$ This is called Beltrami Identity

(b) Examples; Shortest distance between two points;

Given two points (x_1, y_1) & (x_2, y_2) in Euclidean spa[ce]

what is the path of the shortest distance.



The distance element is

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

Total length of path

$$S = \int_{x_1, y_1}^{x_2, y_2} \sqrt{(dx)^2 + (dy)^2}$$

$$S = \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2}\, dx$$

So, our f is $f = (1 + \dot{y}^2)^{1/2}$;

f is independent of y. ; $\frac{\partial f}{\partial y} = 0$.

from Euler equations, we get

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \quad \rightarrow ①$$

NOW $$\frac{\partial f}{\partial \dot{y}} = \frac{1}{2}(1 + \dot{y}^2)^{-1/2}\, 2\dot{y} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

from ①

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) = 0$$

$$\Rightarrow \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = c \qquad c = \text{constant}$$

$$\Rightarrow \dot{y} = c\sqrt{1 + \dot{y}^2}$$

$$\dot{y}^2 = c^2(1+\dot{y}^2)$$

$$\Rightarrow \quad (1-a^2)\dot{y}^2 = a^2 \qquad [\text{Put } c^2 = a$$

$$\Rightarrow \quad \dot{y} = \frac{a}{\sqrt{1-a^2}}$$

or

$$\dot{y} = \frac{a}{\sqrt{1-a^2}} = m$$

$$\Rightarrow \quad dy/dx = m$$

$$\boxed{y = mx + b} \qquad b = \text{constant.}$$

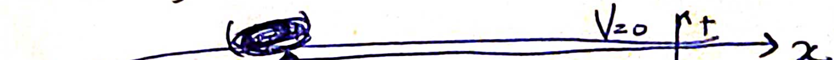
which is familiar equation of straight line.
The constants m & b are now choosen so that the line passes through two points (x_1, y_1) and (x_2, y_2).
Hence Euler equations predicts that the shortest distance between two fixed points in Euclidean space is a straight line.
The generalisation of this to curved four dimensional space-time leads to the important concept of the Geodesic in General Relativity.

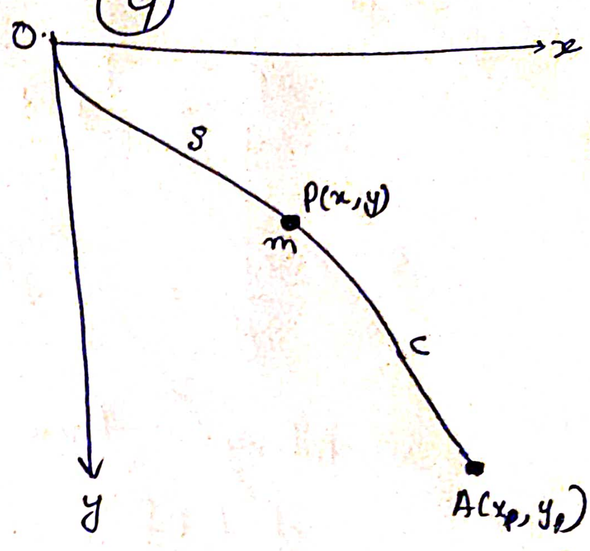
(4) Example 2: <u>Brachistochrone Problem.</u>

A Bead Slides without friction, along a wire bent, in a Shape $y(x)$ under gravity. Find the shape of the wire for which the time taken to reach endpoint is minimum.

Term Brachistochrone derived from Greek word, "brachistos" mens "shortest" & chronos means "time", delay".

The Brachistochrone problem was one of the earliest Problem posed in the Calculus of Variations, which originally Jacobi has posed as a open challenge to the Scientists at that time.





P(x,y) denote any position of the particle which we assume has mass m.

From Principle of conservation of energy, if we chose horizontal line through A as refrence level, we write

$$P.E. \text{ at } O + KE \text{ at } O = PE \text{ at } P + KE \text{ at } P$$

or $$mgy_0 + 0 = mg(y_0 - y) + \tfrac{1}{2} m (ds/dt)^2$$

ds/dt = instantaneous speed, at time t.

Then $$ds/dt = \sqrt{2gy}.$$

The total time taken to go from $y=0$ to $y=y_0$ is

$$T = \int_0^T dt = \int_{y=0}^{y_0} \frac{ds}{\sqrt{2gy}}$$

But $$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1+\dot{y}^2}\, dx$$

So, $$T = \frac{1}{\sqrt{2g}} \int_{y=0}^{y_0} \frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}}\, dx.$$

The integrand is clearly now a function that can be treated by variational technique we have, the function f is

$$f(y, \dot{y}) = \sqrt{\frac{1+\dot{y}^2}{y}}$$

Using Beltrami identity

$$f - \dot{y}\frac{df}{d\dot{y}} = c$$

$$\sqrt{\frac{1+\dot{y}^2}{2gy}} - \frac{\dot{y}^2(1+\dot{y}^2)^{-1/2}}{\sqrt{2gy}} = c$$

$\Rightarrow$

$$\Rightarrow \frac{1}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = c$$

$$\Rightarrow \frac{1}{\sqrt{y}\sqrt{1+\dot{y}^2}} = 2gc$$

$$\Rightarrow (1+\dot{y}^2)y = (2gc)^2$$

$$\Rightarrow \dot{y}^2 = \frac{(2gc)^2}{y} - 1$$

$$\Rightarrow \dot{y} = \sqrt{\frac{2a-y}{y}}$$ a = constant

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

or $$\int dx = \int \frac{\sqrt{y}}{\sqrt{2a-y}}\, dy$$

$$x - x_0 = \int \frac{\sqrt{y}}{\sqrt{2a-y}}\, dy$$

(11) let make trigonometric substitution (cycloid parametrization)

$$y = a(1-\cos\theta)$$
$$dy = a\sin\theta \, d\theta$$

$$x - x_0 = \int \frac{\sqrt{a(1-\cos\theta)}\; a\sin\theta \, d\theta}{\sqrt{2a - a + a\cos\theta}}.$$

$$= a\int \frac{\sqrt{1-\cos\theta}\,\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta$$

$$= a\int 2\sin^2\theta/2 = a\int (1-\cos\theta)\, d\theta$$

$$x - x_0 = a(\theta - \sin\theta) + c.$$
$$x = a(\theta - \sin\theta) + c$$

let us use initial conditions to get the constants; let we start from origin, $x=0, y=0$ gives $\theta = 0$

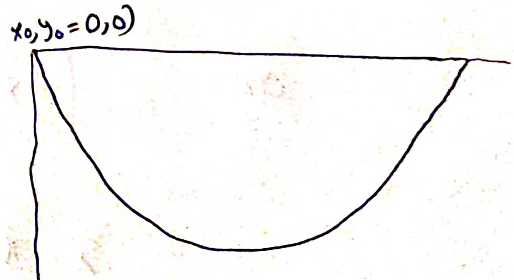
$$\Rightarrow c = 0$$

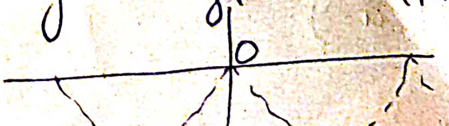
So, we have

$$x = a(\theta - \sin\theta)$$
$$\& \quad y = a(1-\cos\theta)$$

The result describes a cycloid.

The cycloid is the path taken by a fixed point on a circle as it rolls along a given line





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