PHY 303: Classical Electrodynamics
MONSOON SEMESTER 2022
TUTORIAL 04

1. Solve the problems 3 and 4 of Tutorial 3 again with the help of the general solution of Laplace equation in spherical coordinates. Note that with this approach, you may be able to obtain a closed-form solution for the potential at an arbitrary point even for problem 4, without much effort.

2. Consider the Taylor series expansion of $1/|\mathbf{r} - \mathbf{r}'|$ about $\mathbf{r}' = 0$ (or, equivalently, about $1/|\mathbf{r}|$) in the potential expression,
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|},$$
to obtain the corresponding multipole expansion:
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q_m}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \cdots \right).$$
The summation indices i, j in the sum above run over $1, 2, 3$, and give the cartesian coordinates as $x_1 = x, x_2 = y, x_3 = z$. Moreover, in the above expression, $q_m = \int_{\mathcal{V}} d^3r' \rho(\mathbf{r}')$ is the total charge, $\mathbf{p} = \int_{\mathcal{V}} d^3r' \, \mathbf{r}' \rho(\mathbf{r}')$ is the electric dipole moment vector, and $Q_{ij} = \int_{\mathcal{V}} d^3r' \, (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\mathbf{r}')$ is the quadrupole moment tensor.

3. Verify, for the quadrupole moment tensor, the following properties:
   (a) Symmetry: $Q_{ij} = Q_{ji}$,
   (b) Tracelessness: $\sum_i Q_{ii} = 0$.

4. Consider a given volume charge density $\rho(\mathbf{r}) = q\,\delta(\mathbf{r}) - \frac{q}{4\pi r^2} \delta(r - R)$, where q, R are constants of appropriate units and $\delta(u)$ represents the Dirac-delta function. Physically what kind of charge distribution does this density represent? Calculate the monopole and dipole moments associated with this charge distribution, i.e.,
$$q_m = \int d^3r' \rho(\mathbf{r}') \quad \text{and} \quad \mathbf{p} = \int d^3r' \, \mathbf{r}' \rho(\mathbf{r}').$$

5. A *localized** distribution of charge has a density
$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2 \theta.$$
Make a multipole expansion of the potential due to this charge density and determine all the non-vanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

(* *Note that the charge density decays in r faster than a power law.*)

PHY303 Tutorial 04 Solution (MONSOON 2022)

Q1 Sol. The general solution of Laplace eq. for azimuthal symmetry is

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty} \left(C_l r^l + D_l r^{-l-1}\right) \underbrace{P_l(\cos\theta)}_{\text{Legendre polynomial}} \quad -(1)$$

(Refer to Lecture notes)

* For PROBLEM 3, $\Phi(a, \theta, \varphi) = \Phi_0$

Exterior Region ($r > a$)

As $\Phi(\vec{r}) \to 0$ when $r \to \infty$, we must have $C_l = 0$ for all l ($l = 0, 1, 2, \ldots$).

[If one or more of $C_1, C_2, C_3, \ldots$ is/are nonzero, Φ will diverge as $r \to \infty$. On the other hand, C_0 will give $\Phi = C_0$ as $r \to \infty$. Since we want $\Phi \to 0$, we also set $C_0 = 0$.]

$\therefore$ Eq (1) reduces to

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos\theta) \quad -(2)$$

At $r = a$, this gives

$$\Phi_0 = \frac{D_0}{a} \underbrace{P_0(\cos\theta)}_{=1} + \sum_{l=1}^{\infty} \frac{D_l}{a^{l+1}} P_l(\cos\theta)$$

Due to orthogonality of $P_j(\cos\theta)$ with respect to measure $\sin\theta\, d\theta$ on $\theta \in [0, \pi]$, we may compare the coefficients of $P_j(\cos\theta)$ on both sides.

$\Rightarrow \quad D_0 = \Phi_0 a \quad \& \quad D_l = 0 \quad$ for $l = 1, 2, \ldots$

$\therefore \quad \Phi(\vec{r}) = \dfrac{\Phi_0 a}{r}$. using (2).

Note that $\Phi(a, \theta, \varphi) = \sum_{l=0}^{\infty} \dfrac{D_l}{a^{l+1}} P_l(\cos\theta)$

$$\Rightarrow \int_0^{\pi} \Phi(a,\theta,\varphi) P_m(\cos\theta) \sin\theta \, d\theta = \sum_{l=0}^{\infty} \frac{D_l}{a^{l+1}} \underbrace{\int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta}$$

$$= \sum_{l=0}^{\infty} \frac{D_l}{a^{l+1}} \underbrace{\delta_{lm} \left(\frac{2}{2l+1}\right)}$$

$$\Rightarrow D_m = a^{m+1} \int_0^{\pi} \Phi(a,\theta,\varphi) P_m(\cos\theta) \sin\theta \, d\theta \cdot \left(\frac{2m+1}{2}\right)$$

As $\Phi(a, \theta, \varphi) = \Phi_0 = \Phi_0 P_0(\cos\theta)$

$\therefore \quad D_m = a^{m+1} \Phi_0 \delta_{m,0} \cdot \left(\frac{2}{2*0+1}\right) \cdot \left(\frac{2m+1}{2}\right)$

$\therefore \quad D_0 = a\Phi_0 \quad \& \quad D_m = 0$ for $m = 1, 2, \ldots$

Interior Region $(r < a)$

In this case, $\Phi(\vec{r})$ must be well behaved at $r = 0$. Therefore $D_l = 0$ for $l = 0, 1, \ldots$ in eq (1).

$$\Rightarrow \Phi(\vec{r}) = \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta) = C_0 + \sum_{l=1}^{\infty} C_l r^l P_l(\cos\theta)$$

At $r = a$, we get $\qquad \qquad \llcorner (3) \quad (\because P_0(u) = 1)$

$$\Phi_0 = C_0 + \sum_{l=1}^{\infty} C_l a^l P_l(\cos\theta)$$

Comparing coefficients of $P_j(\cos\theta)$ gives,

$$C_0 = \Phi_0 \quad \& \quad C_\ell = 0 \text{ for } \ell = 1, 2 \ldots$$

$\therefore$ Using (3), $\quad \Phi = \Phi_0$ everywhere inside.

✡ For PROBLEM 4

$$\Phi(a, \theta, \varphi) = \Phi_0 \cos\theta$$

& we are solving for $r > a$.

Using argument above, we must have

$C_\ell = 0$ for $\ell = 0, 1, 2 \ldots$ in eq (1).

$$\therefore \Phi(\vec{r}) = \sum_{\ell=0}^{\infty} D_\ell \, r^{-\ell-1} P_\ell(\cos\theta) \quad \{\leftarrow \text{Eq } (2)\}$$

At $r = a$, this gives

$$\Phi_0 \cos\theta = \frac{D_0}{a} P_0(\cos\theta) + \frac{D_1}{a^2} P_1(\cos\theta) + \sum_{\ell=2}^{\infty} \frac{D_\ell}{a^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow \Phi_0 P_1(\cos\theta) = \frac{D_0}{a} P_0(\cos\theta) + \frac{D_1}{a^2} P_1(\cos\theta) + \sum_{\ell=2}^{\infty} \frac{D_\ell}{a^{\ell+1}} P_\ell(\cos\theta)$$

$\{\because P_1(u) = u\}$

Compare coefficients of $P_j(\cos\theta)$

$$\Rightarrow D_0 = 0, \; D_1 = \Phi_0 a^2, \; D_2 = D_3 = \cdots = 0$$

$\therefore$ From eq (2),

$$\Phi(\vec{r}) = \Phi_0 \frac{a^2}{r^2} \cos\theta$$

$\{$ Along z-axis $\Phi_0 \frac{a^2}{r^2}$, as found in Q5.

Q2 Sol. We have to expand $f(\vec{r},\vec{r}\,') = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$ about $\vec{r}\,'=0$.

In cartesian coordinates,

$$f(\vec{r},\vec{r}\,') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$\therefore$ About $\vec{r}\,'=0$,

$$f(\vec{r},\vec{r}\,') = f(\vec{r},\vec{0}) + \sum_{j=1}^{3} x_j' \left.\frac{\partial f}{\partial x_j'}\right|_{\vec{r}\,'=0} + \frac{1}{2!}\sum_{j=1}^{3}\sum_{k=1}^{3} x_j' x_k' \left.\frac{\partial^2 f}{\partial x_j' \partial x_k'}\right|_{\vec{r}\,'=0} + \cdots$$

Here $x_1' = x'$, $x_2' = y'$, $x_3' = z'$.

Now
$$\frac{\partial f}{\partial x_j'} = \frac{\partial}{\partial x_j'} \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{(x_j - x_j')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

$$\therefore \left.\frac{\partial f}{\partial x_j'}\right|_{\vec{r}\,'=0} = \left.\frac{\partial f}{\partial x_j'}\right|_{\substack{x'=0\\ y'=0\\ z'=0}} = \frac{x_j}{[x^2+y^2+z^2]^{3/2}} = \frac{x_j}{r^3}.$$

&
$$\frac{\partial^2 f}{\partial x_j' \partial x_k'} = \frac{3(x_j - x_j')(x_k - x_k')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{5/2}} - \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\delta_{jk}$$

$$\therefore \quad \left.\frac{\partial^2 f}{\partial x_j' \partial x_k'}\right|_{\vec{r}'=0} = \frac{3x_j x_k}{r^5} - \frac{\delta_{jk}}{r^3}$$

$$\therefore \quad \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + \sum_{j=1}^{3} x_j' \frac{x_j}{r^3} + \frac{1}{2!}\sum_{j=1}^{3}\sum_{k=1}^{3} x_j' x_k' \left[\frac{3x_j x_k}{r^5} - \frac{\delta_{jk}}{r^3}\right] + \cdots \quad (1)$$

$$= \frac{1}{r} + \sum_{j=1}^{3} x_j' \frac{x_j}{r^3} + \frac{1}{2}\sum_{j,k=1}^{3} \frac{3x_j' x_k' x_j x_k}{r^5} - \frac{1}{2}\sum_{j,k=1}^{3} \frac{x_j' x_k' \delta_{jk}}{r^3} + \cdots$$

focus on the last sum:-

$$\sum_{j,k=1}^{3} \frac{x_j' x_k' \delta_{jk}}{r^3} = \sum_{j=1}^{3} \frac{x_j'^2}{r^3} = \frac{x_1'^2 + x_2'^2 + x_3'^2}{r^3}$$

$$= \frac{r'^2}{r^3} = \frac{r'^2}{r^5} r^2 = \frac{r'^2}{r^5} \sum_{j=1}^{3} x_j^2$$

$$= \frac{r'^2}{r^5} \sum_{j,k=1}^{3} x_j x_k \delta_{jk}$$

Hence, $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + \sum_{j=1}^{3} \frac{x_j x_j'}{r^3} + \frac{1}{2}\sum_{j,k=1}^{3} \frac{x_j x_k}{r^5}\left(3x_j' x_k' - r'^2 \delta_{jk}\right)$$

$$= \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \frac{1}{2}\sum_{j,k} \frac{x_j x_k}{r^5}\left(3x_j' x_k' - r'^2 \delta_{jk}\right)$$

Plug this in the potential expression.

$$\Phi(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r' \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_V d^3r' \left[\frac{1}{r} + \frac{\vec{r}'\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{j,k}\frac{x_j x_k}{r^5}(3x_j' x_k' - r'^2\delta_{jk}) + \cdots\right] \times \rho(\bar{r}')$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int_V d^3r' \rho(\bar{r}') + \frac{1}{r^3}\left(\int_V d^3r' \bar{r}'\rho(\bar{r}')\right)\cdot\vec{r} + \frac{1}{2}\sum_{j,k}\frac{x_j x_k}{r^5}\int_V d^3r' (3x_j' x_k' - r'^2\delta_{jk})\rho(\bar{r}') + \cdots\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{j,k} Q_{jk}\frac{x_j x_k}{r^5} + \cdots\right).$$

(See an alternative method ahead.)

Q3 Sol. (a)

$$Q_{ij} = \int_V d^3r' (3x_i' x_j' - r'^2\delta_{ij})\rho(\bar{r}')$$

$$= \int_V d^3r' (3x_j' x_i' - r'^2\delta_{ji})\rho(\bar{r}')$$

$$= Q_{ji} \Rightarrow \text{Symmetry.}$$

(b)

$$\sum_i Q_{ii} = \sum_i \int_V d^3r' (3x_i' x_i' - r'^2\underbrace{\delta_{ii}}_{=1})\rho(\bar{r}')$$

$$= \int_V d^3r' \left[3\sum_{i=1}^{3} x_i'^2 - r'^2\sum_{i=1}^{3}(1)\right]\rho(\bar{r}')$$

$$= \int_V d^3r' [3r'^2 - 3r'^2]\rho(\bar{r}') = 0$$

$\Rightarrow$ Traceless.

$\Rightarrow$ Only 5 independent elements in $\overleftrightarrow{Q}$

Q2 Sol. Aliter: Consider expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ about $\frac{1}{|\vec{r}|}$

Here we have $f(\vec{r}) = \frac{1}{|\vec{r}|}$, & we need

$$f(\vec{r}-\vec{r}') \equiv f(x-x', y-y', z-z')$$

In cartesian coordinates,

$$f(x-x', y-y', z-z') = f(x,y,z) - \sum_{j=1}^{3} x_j' \frac{\partial f}{\partial x_j} + \frac{1}{2!}\sum_{j,k=1}^{3} x_j' x_k' \frac{\partial^2 f}{\partial x_j \partial x_k} + \cdots$$

We have

$$\frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{-x_j}{(x^2+y^2+z^2)^{3/2}} = -\frac{x_j}{r^3}$$

&

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{3x_j x_k}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\delta_{jk}$$

$$= \frac{3x_j x_k}{r^5} - \frac{\delta_{jk}}{r^3}$$

$$\therefore \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{|\vec{r}|} - \sum_{j=1}^{3} x_j'\left(-\frac{x_j}{r^3}\right) + \frac{1}{2!}\sum_{j,k=1}^{3} x_j' x_k'\left(\frac{3x_j x_k}{r^5} - \frac{\delta_{jk}}{r^3}\right)$$

$$= \frac{1}{r} + \sum_{j=1}^{3} \frac{x_j' x_j}{r^3} + \frac{1}{2!}\sum_{j,k=1}^{3} x_j' x_k'\left(\frac{3x_j x_k}{r^5} - \frac{\delta_{jk}}{r^3}\right) + \cdots$$

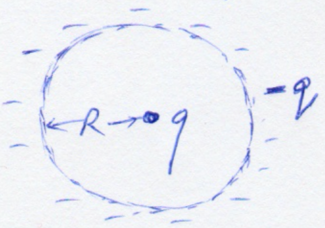
which is same as (1) in the "other approach."

And then one can obtain the expansion, as done earlier.

Q4 Sol.

Given, $\rho(\vec{r}) = q\,\delta(\vec{r}) - \dfrac{q}{4\pi r^2}\,\delta(r-R)$.

The first term of $\rho(\vec{r})$ signifies a point charge q at the origin. The second term represents a total charge $-q$ uniformly distributed on a spherical surface of radius R with center at origin.



[Note that
$\dfrac{q}{4\pi r^2}\delta(r-R) \equiv \dfrac{q}{4\pi R^2}\delta(r-R)$]

Now, we calculate the multipole moments.

Monopole moment,

$$q_m = \int_{(\infty)} d^3r\, \rho(\vec{r}) = \int d^3r \left[q\,\delta(\vec{r}) - \frac{q}{4\pi r^2}\delta(r-R)\right]$$

$$= q\int d^3r\,\delta(\vec{r}) - q\int_0^\infty \underbrace{dr\,.\cancel{4\pi r^2}}_{\leftarrow d^3r \text{ for spherical symmetry}} \cdot \frac{1}{\cancel{4\pi r^2}}\,\delta(r-R)$$

$$= q - q = 0$$

$\leftarrow$ Total charge

Dipole moment (about the origin),

$$\vec{p} = \int d^3r\,\vec{r}\,\rho(\vec{r}) = \int d^3r\,\vec{r}\left[q\,\delta(\vec{r}) - \frac{q}{4\pi r^2}\delta(r-R)\right]$$

$$= \underbrace{\int d^3r\,\vec{r}\,q\,\delta(\vec{r})} - \int d^3r\,\vec{r}\,\frac{q}{4\pi r^2}\,\delta(r-R)$$

$$= 0 - \frac{q}{4\pi}\int d^3r\,\frac{\vec{r}}{r^2}\,\delta(r-R)$$

or $\vec{p} = -\frac{q}{4\pi} \int d^3r \, \frac{(x\hat{x} + y\hat{y} + \cancel{z}\hat{z})}{r^2} \, \delta(r-R)$

$\equiv p_x \hat{x} + p_y \hat{y} + p_z \hat{z}$

Note that $\vec{r} = r\hat{r}$. However, it changes within the volume, so we can't have $\vec{p} = \hat{r} \int (\;) \, d^3r$

where,

$$p_x = -\frac{q}{4\pi} \int d^3r \, \frac{x}{r^2} \, \delta(r-R)$$

$$= -\frac{q}{4\pi} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \; r^2 \sin\theta \cdot \frac{r \sin\theta \cos\phi}{r^2} \, \delta(r-R)$$

$$= -\frac{qR}{4\pi} \int_0^\pi d\theta \, \sin^2\theta \underbrace{\int_0^{2\pi} d\phi \, \cos\phi}_{=0} = 0$$

$$p_y = -\frac{q}{4\pi} \int d^3r \, \frac{y}{r^2} \, \delta(r-R)$$

$$= -\frac{q}{4\pi} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \; r^2 \sin\theta \cdot \frac{r \sin\theta \sin\phi}{r^2} \, \delta(r-R)$$

$$= -\frac{qR}{4\pi} \int_0^\pi d\theta \, \sin^2\theta \underbrace{\int_0^{2\pi} d\phi \, \sin\phi}_{=0} = 0$$

$$p_z = -\frac{q}{4\pi} \int d^3r \, \frac{z}{r^2} \, \delta(r-R)$$

$$= -\frac{q}{4\pi} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \; r^2 \sin\theta \cdot \frac{r \cos\theta}{r^2} \, \delta(r-R)$$

$$= -\frac{qR}{4\pi} \underbrace{\int_0^\pi \underbrace{d\theta \, \sin\theta \cos\theta}_{= d(\sin\theta)}}_{=0} \int_0^{2\pi} d\phi = 0$$

$\therefore \vec{p} = \vec{0}$ ✓

← Expected as the charge is symmetrically distributed about the origin. (No separation of effective '+' & '−' charges about $\vec{0}$.

Shortcut for dipole moment:-

$$\vec{p} = \int_{(\infty)} d^3r \; \vec{r} \; \rho(\vec{r})$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \vec{r} \; \rho(\vec{r}) \qquad \text{—(1)}$$

Consider, $\vec{r} \to -\vec{r}$ i.e. $x \to -x$, $y \to -y$, $z \to -z$ { Magnitude of $\vec{r}$ remains same i.e. $r \to r$

$$\therefore \rho(\vec{r}) = q\,\delta(\vec{r}) - \frac{q}{4\pi r^2}\,\delta(r-R)$$

$$\to \underbrace{q\,\delta(-\vec{r})}_{=\delta(\vec{r})} - \frac{q}{4\pi r^2}\,\delta(r-R) = \rho(\vec{r})$$

$$\therefore \vec{p} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; (-\vec{r})\,\rho(\vec{r}) = -\vec{p} \quad \text{[Using (1)]}$$

$$\therefore \vec{p} = \vec{0}.$$

//

{ $dx \to -dx$

$\pm\infty \to \mp\infty$

$\therefore \int_{-\infty}^{\infty} dx \to \int_{\infty}^{-\infty} (-dx) = \int_{-\infty}^{\infty} dx$

||ly for y & z.

Q5 Sol.

Multipole expansion in terms of spherical harmonics is,

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{q_{lm}}{(2l+1)} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

for $r > r'$.

Here,
$$q_{lm} = \int Y^*_{lm}(\theta',\phi')\, r'^l \rho(\vec{r}')\, d^3r'$$

We have,
$$\rho(\vec{r}') = \frac{1}{64\pi} r'^2 e^{-r'} \sin^2\theta'$$

Note that ρ actually extends up to ∞. However, due to $e^{-r'}$ term it's negligible for large r! Therefore, we may use the above multipole expansion which is based on the assumption $r > r'$.

(It decays faster than any power law.)

The calculation will be easier if we express angular part of $\rho(\vec{r}')$ in terms of $Y_{lm}(\theta',\phi')$. Note that there is no ϕ dependence, so we must have $m=0$.

We find that
$$Y_{00}(\theta',\phi') = \frac{1}{\sqrt{4\pi}}$$

$$\& \quad Y_{2,0}(\theta',\phi') = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$$

$$\Rightarrow \cos^2\theta = \frac{1}{3}\sqrt{\frac{16\pi}{5}}\, Y_{20} + \frac{1}{3}$$

$$\therefore \sin^2\theta' = 1-\cos^2\theta' = 1-\frac{4}{3}\sqrt{\frac{\pi}{5}}Y_{20}-\frac{1}{3}$$

$$= \frac{2}{3}-\frac{4}{3}\sqrt{\frac{\pi}{5}}Y_{20} = \frac{2}{3}\sqrt{4\pi}\,Y_{00}-\frac{4}{3}\sqrt{\frac{\pi}{5}}Y_{20}$$

$$\therefore\quad q_{lm} = \int_{(\infty)} Y^*_{lm}(\theta',\phi')\,r'^l\,\frac{1}{64\pi}r'^2 e^{-r'}$$

$$\left[\frac{2}{3}\sqrt{4\pi}\,Y_{00}(\theta',\phi')-\frac{4}{3}\sqrt{\frac{\pi}{5}}\,Y_{20}(\theta',\phi')\right]d^3r'$$

$$= \frac{1}{64\pi}\int_0^\infty dr'\,r'^{l+4}e^{-r'}\left[\frac{2\sqrt{4\pi}}{3}\int d\Omega'\,Y^*_{lm}(\theta',\phi')\,Y_{00}(\theta',\phi')\right.$$

$$\left.-\frac{4}{3}\sqrt{\frac{\pi}{5}}\int d\Omega'\,Y^*_{lm}(\theta',\phi')\,Y_{20}(\theta',\phi')\right]$$

$$= \frac{1}{64\pi}\int_0^\infty dr'\,r'^{l+4}e^{-r'}\left[\frac{2\sqrt{4\pi}}{3}\delta_{l0}\delta_{m0}\right.$$

$$\left.-\frac{4}{3}\sqrt{\frac{\pi}{5}}\,\delta_{l2}\delta_{m0}\right]$$

$\therefore$ Only q_{00} & q_{20} are nonzero.

$$q_{00} = \frac{1}{64\pi}\int_0^\infty dr'\,r'^4 e^{-r'}\cdot\frac{2\sqrt{4\pi}}{3}$$

$$= \frac{1}{64\pi}\cdot\frac{2\sqrt{4\pi}}{3}\cdot 4! = \frac{1}{2\sqrt{\pi}}.$$

&

$$q_{20} = \frac{1}{64\pi}\int_0^\infty dr'\,r'^6 e^{-r'}\left[-\frac{4}{3}\sqrt{\frac{\pi}{5}}\right]$$

$$= \frac{1}{64\pi}\left(-\frac{4}{3}\sqrt{\frac{\pi}{5}}\right)(6!)$$

$$= -3\sqrt{\frac{5}{\pi}}.$$

& so, $\Phi(\vec{r}) = \frac{1}{\epsilon_0} \frac{q_{00}}{(1)} \frac{Y_{00}(\theta,\phi)}{r} + \frac{1}{\epsilon_0} \frac{q_{20}}{5} \frac{Y_{20}(\theta,\phi)}{r^3}$

Express Y_{l0} directly in terms of $P_l(\cos\theta)$ or

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0} \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{r} \cdot \frac{1}{\sqrt{4\pi}}$$

$$+ \frac{1}{\epsilon_0} \cdot \frac{1}{5} \left(-3\sqrt{\frac{5}{\pi}}\right) \frac{1}{r^3} \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right)$$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0 r} - \frac{3}{4\pi\epsilon_0 r^3}\left(3\cos^2\theta - 1\right)$$

Now $P_0(\cos\theta) = 1$ & $P_2(\cos\theta) = \frac{3\cos^2\theta - 1}{2}$.

$$\therefore \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0 r} P_0(\cos\theta) - \frac{3}{2\pi\epsilon_0 r^3} P_2(\cos\theta)$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{P_0(\cos\theta)}{r} - \frac{6}{r^3} P_2(\cos\theta)\right].$$

Valid for large r, i.e. in a region where $\rho(\vec{r}\,')$ is negligible (due to $e^{-r'}$ factor).

(Note that there are no more additional terms in this potential expansion.)

— x —