PHY 303: Classical Electrodynamics
MONSOON SEMESTER 2022
TUTORIAL 04

1. Solve the problems 3 and 4 of Tutorial 3 again with the help of the general solution of Laplace equation in spherical coordinates. Note that with this approach, you may be able to obtain a closed-form solution for the potential at an arbitrary point even for problem 4, without much effort.

2. Consider the Taylor series expansion of $1/|\mathbf{r} - \mathbf{r}'|$ about $\mathbf{r}' = 0$ (or, equivalently, about $1/|\mathbf{r}|$) in the potential expression,
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|},$$
to obtain the corresponding multipole expansion:
$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q_m}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \cdots \right).$$
The summation indices i, j in the sum above run over $1, 2, 3$, and give the cartesian coordinates as $x_1 = x, x_2 = y, x_3 = z$. Moreover, in the above expression, $q_m = \int_{\mathcal{V}} d^3r' \rho(\mathbf{r}')$ is the total charge, $\mathbf{p} = \int_{\mathcal{V}} d^3r' \, \mathbf{r}' \rho(\mathbf{r}')$ is the electric dipole moment vector, and $Q_{ij} = \int_{\mathcal{V}} d^3r' \, (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\mathbf{r}')$ is the quadrupole moment tensor.

3. Verify, for the quadrupole moment tensor, the following properties:
   (a) Symmetry: $Q_{ij} = Q_{ji}$,
   (b) Tracelessness: $\sum_i Q_{ii} = 0$.

4. Consider a given volume charge density $\rho(\mathbf{r}) = q\,\delta(\mathbf{r}) - \frac{q}{4\pi r^2} \delta(r - R)$, where q, R are constants of appropriate units and $\delta(u)$ represents the Dirac-delta function. Physically what kind of charge distribution does this density represent? Calculate the monopole and dipole moments associated with this charge distribution, i.e.,
$$q_m = \int d^3r' \rho(\mathbf{r}') \quad \text{and} \quad \mathbf{p} = \int d^3r' \, \mathbf{r}' \rho(\mathbf{r}').$$

5. A *localized** distribution of charge has a density
$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2 \theta.$$
Make a multipole expansion of the potential due to this charge density and determine all the non-vanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

(* *Note that the charge density decays in r faster than a power law.*)