(Power Series Solution for 2nd-order linear differential equation)

We focus on the 2nd-order ODE with the following structure.

$$y''(x) + \underbrace{P(x)}_{}\, y'(x) + \underbrace{Q(x)}_{}\, y = 0 \Leftarrow$$ Here, we treat x, y, $P(x)$, $Q(x)$ are complex.

coefficients are x-dependent.

Here, $P(x)$, $Q(x)$ are assumed to be non-divergent except for finite # of isolated singularities on $\mathbb{C}$.

In the following, the technical terms on differential eq. are adopted

- ✓ non-singular points : at $x = x_0$, $P(x_0)$ and $Q(x_0)$ are holomorphic (non divergent).
- ✓ singular points : at $x = x_0$, $P(x_0)$ and/or $Q(x_0)$ are divergent (not holomorphic).

☐ Power series solution around a non-singular point.

Here, we consider the expansion around $x=0$, where $P(x)$ and $Q(x)$ are Taylor expanded as

$$\begin{cases} P(x) = \sum_{n=0}^{\infty} p_n x^n, \\ Q(x) = \sum_{n=0}^{\infty} q_n x^n. \end{cases}$$

(Note: all the p_n & q_n are given).

$$\Rightarrow \quad y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Leftarrow$$ a_n are unknown!

$$\Downarrow$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$\Leftrightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{m=0}^{\infty} p_m x^m \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{m=0}^{\infty} q_m x^m \sum_{n=0}^{\infty} a_n x^n = 0$$

→ (the coefficient of x^0) : $2a_2 + p_0 a_1 + q_0 a_0 = 0$

$\Leftrightarrow a_2 = -\frac{1}{2}(p_0 a_1 + q_0 a_0)$, ← function of a_0, a_1

→ (" x^1) : $a_3 = -\frac{1}{6}(2p_0 a_2 + p_1 a_1 + q_0 a_1 + q_1 a_0)$

⋮

→ (" x^n) : $a_n =$ (a function of a_0, a_1)

So, $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution and well defined if the series is convergent.

(Ex.) We consider the Legendre's diff. eq.,

$$y'' - \frac{2x}{1-x^2} y' + \frac{\nu(\nu+1)}{1-x^2} y = 0 .$$

Here, $x=0$ is a non-singular point of this diff. eq. $\Rightarrow$ We try to solve it around $x=0$.

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n}$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \nu(\nu+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \left\{ (n+2)(n+1) a_{n+2} x^n - \underbrace{n(n-1) a_n x^n}_{\substack{= \\ 0 \\ \text{if } n=0 \text{ or } 1}} - \underbrace{2n a_n x^n}_{\substack{= \\ 0 \text{ if } n=0}} + \nu(\nu+1) a_n x^n \right\} = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} x^n \left\{ (n+2)(n+1) a_{n+2} - n(n-1) a_n - 2n a_n + \nu(\nu+1) a_n \right\} = 0$$

$(n+1)n + \cancel{n\nu} - \cancel{n\nu} - \nu(\nu+1)$

(the coefficient of x^n) = 0 $\rightsquigarrow$

$$a_{n+2} = \frac{(n+\nu+1)(n-\nu)}{(n+2)(n+1)} a_n \;\; //$$

When a_0 and a_1 are fixed, the other coefficients are determined as functions of a_0, a_1.

i) $a_0 = \cancel{0}\, 1,\ a_1 = 0$: the recursive relation tells us

$a_{odd} \propto a_1 = 0 \Rightarrow a_{odd} = 0$.

✓ $n = 2k-2$

$$\Rightarrow a_{2k} = \frac{(2k+\nu-1)(2k-\nu-2)}{2k(2k-1)} a_{2k-2}$$

$$= \left\{ \prod_{m=1}^{k} \left[\frac{(2m+\nu-1)(2m-\nu-2)}{2m(2m-1)} \right] \right\} \underset{\substack{= \\ 1}}{a_0}$$

$$= \frac{\prod_{m=1}^{k} (2m+\nu-1)(2m-\nu-2)}{(2k)!} \;\; //$$

If ν is an even positive integer, $a_\nu \neq 0$, $a_n (n>\nu) = 0$. $\Rightarrow$ reduced to a polynomial

ii) $a_0 = 0,\ a_1 = 1$: similar to the case i), we find $a_{even} = 0$.

$$✓\ a_{2k+1} = \frac{\prod_{m=1}^{k} (2m+\nu)(2m-\nu-1)}{(2k+1)!} \;\; //$$

If ν is an odd positive integer, $a_\nu \neq 0$, $a_n (n>\nu) = 0$. $\Rightarrow$ reduced to a polynomial.

☐ Power series solution around a regular singular point.

$\Rightarrow$ We can consider the series solutions around regular singular points (thanks to the Laurant expansion).

✓ [**Definion** of a regular singular point of a diff. eq.]

$$\left\{\begin{array}{l}\text{At } x = x_0,\ P(x) \text{ and/or } Q(x) \text{ are singular, where} \\ \text{the orders of the poles in } P(x) \text{ and } Q(x) \text{ are} \\ \text{"up to one", and "up to two", respectively.}\end{array}\right\}$$

associated

Fuch's theorem

If $x = x_0$ is a regular singular point of "$y'' + P(x)y' + Q(x)y = 0$", at least one solution can be represented in the form,

$$y = (x - x_0)^{\lambda} \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad (\lambda \in \mathbb{C}).$$

Here, when $x = 0$ is a regular singular point;

$$\left\{\begin{array}{l}\checkmark\ \text{We take } y(x) = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n \text{ with } a_0 \neq 0 \text{ (without loss of generality)}, \\ \checkmark\ P(x) = \sum_{n=-1}^{\infty} P_n x^n, \quad Q(x) = \sum_{n=-2}^{\infty} q_n x^n,\end{array}\right.$$

Laurant expansions

$y'' + P(x)y' + Q(x)y = 0$

$$\sum_{n=0}^{\infty} (\lambda+n)(\lambda+n-1) a_n x^{\lambda+n-2} + \sum_{m=-1}^{\infty} P_m x^m \sum_{n=0}^{\infty} (\lambda+n) a_n x^{\lambda+n-1} + \sum_{m=-2}^{\infty} q_m x^m \sum_{n=0}^{\infty} a_n x^{\lambda+n} = 0$$

$\odot$ (the "lowest-power" ($x^{\lambda-2}$)'s coefficient) $= [\lambda(\lambda-1) + P_{-1}\lambda + q_{-2}]\, a_0$

$$\Rightarrow \underbrace{\lambda(\lambda-1) + P_{-1}\lambda + q_{-2}}_{\equiv \phi(\lambda)} = 0$$ $\Leftarrow$ "the determining condition for λ"

called characteristic index

$\odot$ $\lambda = \frac{1}{2}\left(-(P_{-1}-1) \pm \sqrt{(1-P_{-1})^2 - 4Q_{-2}}\right)$ //

We parametrize two indices (λ_+, λ_-) under the rule: $\mathrm{Re}(\lambda_+) \geq \mathrm{Re}(\lambda_-)$

(If the root part is imaginary and Re[Root] < 0, flip the definition.)

$\odot$ (the coefficient of $x^{\lambda+n-2}$) $= (\lambda+n)(\lambda+n-1)a_n + P_{-1}(\lambda+n)a_n + Q_{-2}a_n + g(a_k\ (k<n), \lambda) = 0$

$$\Leftrightarrow \phi(\lambda+n)\, a_n + g(a_k\ (k<n), \lambda) = 0$$

$$\Leftrightarrow a_n = -\frac{1}{\phi(\lambda+n)}\, g(a_k\ (k<n), \lambda)$$

If $\phi(\lambda+n) \neq 0$, all of the coefficients are recursively determined. (consistently).

$\odot$ For $\lambda = \lambda_+$, $\phi(\lambda_+ + n) \neq 0$ //

If $\phi(\lambda_+ + n) = 0$,

$$\begin{cases} \lambda_+ + n = \lambda_+ & \leftarrow \text{inconsistent} \\ \text{or } \lambda_+ + n = \lambda_- & \leftarrow \text{contradict with } \mathrm{Re}(\lambda_+) \geq \mathrm{Re}(\lambda_-) \end{cases}$$

$y(x) = x^{\lambda_+} \sum_{n=0}^{\infty} a_n x^n$ is a solution //

The series is convergent in $|x| < R$ if both of $xP(x)$, $x^2Q(x)$ are holomorphic in $|x| < R$.

$\odot$ For $\lambda = \lambda_-$, $\phi(\lambda_- + n)$ can be zero if $\lambda_- + n = \lambda_+$ $(n = 0, 1, 2, 3 \dots)$

$$\Leftrightarrow \begin{cases} \lambda_- = \lambda_+, \\ \lambda_- = \lambda_+ - 1, \\ \lambda_- = \lambda_+ - 2, \\ \quad \vdots \end{cases}$$

In this case, the second series solution becomes ill-defined.

In such cases, we follow the Frobenius method.

$\hookrightarrow$ $y(x,\lambda) \equiv x^{\lambda} \sum_{n=0} a_n(\lambda)\, x^n$, where $a_0(\lambda)$ is determined as

$$a_0(\lambda) = \begin{cases} 1 & (\lambda_+ = \lambda_-), \\ \lambda - \lambda_- & (\lambda_+ - \lambda_- = 1, 2, 3, \cdots) \end{cases}.$$

(for others, as we previously did)

$\Rightarrow$ the two independent solutions are given as

$$\begin{cases} y_1 \equiv y(x, \lambda_+), \\ y_2 \equiv \dfrac{\partial}{\partial \lambda} y(x,\lambda)\Big|_{\lambda=\lambda_-}. \end{cases}$$

Here, we can determine a_n through the recursive relationship (with the inputs a_0, a_1), while the coefficient of $x^{\lambda-2}$ is nonvanishing

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + P(x)\frac{\partial}{\partial x} + Q(x)\right) y(x,\lambda) = \underbrace{\underset{(\lambda-\lambda_+)(\lambda-\lambda_-)}{\underset{\parallel}{\phi(\lambda)}}\, a_0(\lambda)}_{\propto\, (\lambda-\lambda_-)^2} x^{\lambda-2}$$

$$f(\lambda) = \begin{cases} 1 & (\lambda_+ = \lambda_-) \\ (\lambda - \lambda_+) & (\lambda_+ - \lambda_- = 1, 2, 3, \cdots) \end{cases}$$

$$= (\lambda-\lambda_-)^2 f(\lambda)\, x^{\lambda-2} \,//$$

$$\begin{cases} \checkmark \ \dfrac{\partial}{\partial \lambda}[\text{LHS}]\Big|_{\lambda=\lambda_-} = \left(\dfrac{\partial^2}{\partial x^2} + P(x)\dfrac{\partial}{\partial x} + Q(x)\right)\left[\dfrac{\partial y}{\partial \lambda}\right]\Big|_{\lambda=\lambda_-}, \\ \checkmark \ \dfrac{\partial}{\partial \lambda}[\text{RHS}]\Big|_{\lambda=\lambda_-} = \left\{ 2(\lambda-\lambda_-) f(\lambda)\, x^{\lambda-2} + (\lambda-\lambda_-)^2 \dfrac{\partial}{\partial \lambda}\left(f(\lambda)\, x^{\lambda-2}\right)\right\}_{\lambda\to\lambda_-} \\ \qquad = 0 \end{cases}$$

Therefore, y_2 provides another solution.

(Example)

$$y'' + \frac{1}{x}y' + y = 0.$$

$x=0$ is a regular singular point of this diff. eq.

We try to obtain the power series solution as $y = x^\lambda \sum_{n=0}^{\infty} a_n x^n$ //

$$\downarrow$$

$$\sum_{n=0}^{\infty}\left\{(\lambda+n)(\lambda+n-1)a_n x^{\lambda+n-2} + \frac{1}{x}(\lambda+n)a_n x^{\lambda+n-1} + a_n x^{\lambda+n}\right\} = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}\{(\lambda+n)(\lambda+n-1)a_n + (\lambda+n)a_n\}x^{\lambda+n-2} + \sum_{n=2}^{\infty} a_{n-2}x^{\lambda+n-2} = 0$$

∘ (the coefficient of $x^{\lambda-2}$) $= \overbrace{[\lambda(\lambda-1)+\lambda]}^{\equiv\phi(\lambda)} a_0 = 0$

$\Leftrightarrow \lambda^2 a_0 = 0 \Rightarrow \lambda^2 = 0$ and $\phi(n+\lambda) = (\lambda+n)^2$. //

∘ (" $x^{\lambda-1}$) $= \phi(\lambda+1)a_1 = 0. \Rightarrow \underline{a_1 = 0}$,

∘ (" $x^{\lambda+n-2}$ $(n=2,3,4,\dots)$) $= \phi(\lambda+n)a_n + a_{n-2} = 0$

$$\rightarrow a_n(\lambda) = -\frac{1}{(\lambda+n)^2}a_{n-2}.$$

$\underline{\text{under } a_0=1}$ $\Rightarrow a_{odd}(\lambda) = 0, \quad a_{2k}(\lambda) = \prod_{l=1}^{k}\left(\frac{-1}{(\lambda+2l)^2}\right)$ //

$$\left\{\begin{aligned} & x^\lambda \to \log(x^\lambda) = \lambda\log x \\ & \frac{\partial}{\partial\lambda} \downarrow \ \frac{1}{x^\lambda}\frac{\partial x^\lambda}{\partial\lambda} = \log x \\ & \Leftrightarrow \frac{\partial x^\lambda}{\partial\lambda} = x^\lambda \cdot \log x \end{aligned}\right.$$

Based on $y(x,\lambda) = \sum_{n=0}^{\infty} a_n(\lambda)x^{\lambda+n}$,

the first solution is written as

$$y_1(x) = y(x,\lambda=0) = \sum_{k=0}^{\infty}\left(\prod_{l=1}^{k}\frac{-1}{(2l)^2}\right)x^{2k}.$$

the second solution is

$$y_2(x) = \frac{\partial}{\partial\lambda}y(x,\lambda)\Big|_{\lambda=0} = \sum_{k=0}^{\infty}\frac{\partial}{\partial\lambda}a_{2k}(\lambda)\Big|_{\lambda=0}x^{2k} + \underbrace{\sum_{k=0}^{\infty}a_{2k}(0)\,x^{2k}\log x}_{=\log x \times y_1(x)}$$

$$\left\{\begin{aligned} & \checkmark\ \log a_{2k}(\lambda) = \sum_{m=1}^{k}\left(\log(-1) - 2\log(\lambda+2m)\right) \\ & \rightarrow \frac{1}{a_{2k}}\frac{\partial}{\partial\lambda}a_{2k} = \frac{\partial}{\partial\lambda}\log a_{2k} = -\sum_{m=1}^{k}\frac{2}{\lambda+2m} \\ & \rightarrow \frac{\partial}{\partial\lambda}a_{2k}\Big|_{\lambda=0} = -a_{2k}(0)\sum_{m=1}^{k}\frac{1}{m} = -\left(\prod_{l=1}^{k}\frac{-1}{(2l)^2}\right)\sum_{m=1}^{k}\frac{1}{m} \end{aligned}\right.$$

$$\Rightarrow y_2(x) = (\log x)\,y_1(x) - \sum_{k=0}^{\infty}\left(\prod_{l=1}^{k}\frac{-1}{(2l)^2}\right)\sum_{m=1}^{k}\frac{1}{m}x^{2k}$$

[Examples of diff. eq.]

□ (Gauss') hypergeometric differential eq. ← (regular singular points) $\Leftrightarrow x = 0, 1, \infty$.

$$y'' + \frac{c-(a+b+1)x}{x(1-x)}\, y' - \frac{ab}{x(1-x)}\, y = 0 \quad /\!/ \qquad y = x^\lambda \sum_{n=0}^{\infty} a_n x^n .$$

(the determining condition): $\phi(\lambda) = \lambda(\lambda-1) + c\lambda$
$= \lambda(\lambda-1+c)$.

$\phi(\lambda) = 0,\ 1-c.$

c is not a negative integer,
the solution ($\lambda=0$) always exists.

$$x(1-x) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \left(c - (a+b+1)x\right) \sum_{n=0}^{\infty} n a_n x^{n-1} - ab \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^n$$

$$\Leftrightarrow \sum_{n=0}^{\infty} x^n \left\{ (n+1) n a_{n+1} - n(n-1) a_n + c(n+1) a_{n+1} - (a+b+1) a_n \cdot n - ab\, a_n \right\} = 0$$

coefficients being zeros

$$\longrightarrow \quad (c+n)(1+n) a_{n+1} = (n+a)(n+b) a_n$$

$$\Leftrightarrow \quad a_{n+1} = \frac{(n+a)(n+b)}{(n+c)(n+1)} a_n$$

$$\xrightarrow{\text{under } a_0 = 1} \quad a_{n+1} = \frac{(a+n)(a+n-1)\cdots(a+0) \times (b+n)\cdots(b+0)}{(c+n)\cdots(c+0) \times (n+1)\cdots(1)} \times \underset{a_0}{\overset{1}{\Vert}}$$

$$\Rightarrow a_n = \frac{(a)_n (b)_n}{(c)_n\, n!}$$

with the Pochhammer symbol

$$(a)_n \equiv a(a+1)\cdots(a+(n-1)) = \Gamma(a+n)/\Gamma(a) \quad /\!/$$

Thus,

$$y(x) \left(\equiv {}_2F_1(a,b,c;x) \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n \quad /\!/$$

← In general, this series is convergent in $|x| < 1$.

This function is called the hypergeometric function.

(specific case)

$$\begin{cases} \bullet\ {}_2F_1(a,b,b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n = (1-x)^{-a}, \\ \bullet\ {}_2F_1(1,1,2;x) = \sum_{n=0}^{\infty} \frac{(1)_n \overbrace{(1)_n}^{n!}}{\underbrace{(2)_n}_{(n+1)!}\, n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n = -\frac{1}{x}\log(1-x), \end{cases}$$

□ Confluent hypergeometric diff. eq.

$$y'' + \left(\frac{c}{x} - 1\right) y' - \frac{a}{x} y = 0 \leftarrow \text{regular singular points at } x = 0, \infty.$$

[the hypergeometric case] $y'' + \dfrac{c-(a+b+1)x}{x(1-x)} y' - \dfrac{ab}{x(1-x)} y = 0$

$$\omega = bx \quad b^2 \frac{d^2y}{d\omega^2} + \frac{c-(a+b+1)\frac{\omega}{b}}{\frac{\omega}{b}\left(1-\frac{\omega}{b}\right)} \frac{\partial y}{\partial \omega} b - \frac{ab}{\frac{\omega}{b}\left(1-\frac{\omega}{b}\right)} y$$

$$\Longleftrightarrow \frac{d^2y}{d\omega^2} + \frac{c-(a+1)\frac{\omega}{b} - \omega}{\omega\left(1-\frac{\omega}{b}\right)} \frac{\partial y}{\partial \omega} - \frac{a}{\omega\left(1-\frac{\omega}{b}\right)} y$$

regular singular points at $x = 0, b, \infty$

$\Downarrow$ $b \to \infty$ — The two regular singular points merge at $x = \infty$.

$$\frac{d^2y}{d\omega^2} + \left(\frac{c-\omega}{\omega}\right)\frac{dy}{d\omega} - \frac{a}{\omega} y = 0$$

The ~~xxxx~~ corresponding solution to ${}_2F_1(a,b,c,x)$ is obtained as

$$\lim_{b\to\infty} {}_2F_1\left(a,b,c;\frac{x}{b}\right) = \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} \frac{(b)_n}{(b)^n} x^n = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n$$

$$\lim_{b\to\infty} \frac{b(b+1)\cdots(b+(n-1))}{b\cdots\cdots b} = 1$$

$$\equiv {}_1F_1(a,c;x)$$

This is called the confluent hypergeometric function.

$\checkmark$ ${}_1F_1(a,a;x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ //

(Note: the radius of the convergence is $|x| < \infty$ since the singularities are located only at $x = 0, \infty$)