Introduction to Statistics (MAT 283)

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Bivariate Random Variable

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JOINT PPROBABILITY MASS FUNCTION :

Given a pair of discrete random variables X and Y , their joint probability mass function $f : R_X \times R_Y \to \mathbb{R}$ is defined by

$$f(x, y) = P(X = x, Y = y).$$

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EXAMPLES:

1. A fair coin is tossed two times; let X denote the number of heads on the first toss and Y the total number of heads.
Sample space of this random experiment is

$$\omega = \{HH, HT, TH, TT\}.$$

We have

$$R_X = \{0, 1\}$$

$$R_Y = \{0, 1, 2\}$$

and $R_X \times R_Y = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$. Then

$$f(0,0) = P(X = 0, Y = 0) = \frac{1}{4}$$

and

$$f(0,1) = P(X = 1, Y = 0) = 0.$$

The joint PMF of X and Y is as given in the following table:

	Y=0	Y=1	Y=2
X= 0	$\frac{1}{4}$	$\frac{1}{4}$	0
X= 1	0	$\frac{1}{4}$	$\frac{1}{4}$

2. A fair die is rolled, and a fair coin is tossed independently. Let X be the face value on the die, and let $Y = 0$ if a tail turns up and $Y = 1$ if a head turns up. Find the joint PMF of X and Y?

MARGINAL PPROBABILITY MASS FUNCTION :

Given a pair of discrete random variables X and Y and their joint probability mass function f. The marginal PMF of X, $f_X : R_X \to \mathbb{R}$ is defined by

$$f_X(x) = \sum_{y \in R_Y} f(x, y).$$

Similarly, the function $f_Y : R_Y \to \mathbb{R}$ defined by

$$f_Y(y) = \sum_{x \in R_X} f(x, y).$$

is called the marginal PMF of Y .

EXAMPLE: A fair coin is tossed two times; let X denote the number of heads on the first toss and Y the total number of heads. Sample space of this random experiment is

$$\omega = \{HH, HT, TH, TT\}.$$

The joint PMF f of X and Y is as given in the following table:

	Y=0	Y=1	Y=2	P(X=x)
X= 0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{2}{4}$
X= 1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{4}$
P(Y=y)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	

Then the marginal PMF of X is given by

$$f_X(0) = f(0,0) + f(0,1) + f(0,2) = 1/4 + 1/4 + 0 = 2/4$$

and

$$f_X(1) = f(1,0) + f(1,1) + f(1,2) = 0 + 1/4 + 1/4 = 2/4.$$

Similarly, the marginal PMF of Y is given by

$$f_Y(0) = f(0,0) + f(1,0) = 1/4 + 0 = 1/4$$

$$f_Y(1) = f(0,1) + f(1,1) + = 1/4 + 1/4 = 2/4$$

and

$$f_Y(2) = f(0,2) + f(1,2) = 0 + 1/4 = 1/4.$$

JOINT CUMULATIVE DISTRIBUTION FUNCTION :

Given a pair of discrete random variables X and Y and their joint probability mass function f. The joint CDF of X and Y , is a function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(x, y) = P(X \leq x, Y \leq y),$$

or

$$F(x, y) = \sum_{s \leq x} \sum_{t \leq y} f(s, t).$$

JOINT PPROBABILITY DENSITY FUNCTION :

A bivariate random variable (X, Y) is said to be of continuous type , if there exists a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

- $f(x, y) \geq 0$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$
- For any subset $A \subseteq \mathbb{R} \times \mathbb{R}$,

$$P((X, Y) \in A) = \int \int_A f(x, y)dxdy.$$

The function f is called the joint PDF of (X, Y).

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The function f is called the joint PDF of (X, Y).

EXAMPLE: Let X and Y have the joint density function

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2 + 2xy) & \text{if} \quad 0 < x < 1, 0 < y < 1 \\ \\ 0 & \text{otherwise,} \end{cases}$$

What is $P(X \leq Y)$?

MARGINAL PPROBABILITY DENSITY FUNCTION :

Let (X, Y) be a continuous bivariate random variable. Let f be the joint probability density function of X and Y. The function $f_X : \mathbb{R} \to \mathbb{R}$ defined by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy.$$

is called the marginal probability density function of X. Similarly, the function $f_Y : \mathbb{R} \to \mathbb{R}$ defined by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx,$$

is called the marginal probability density function of Y .

EXAMPLE: Let (X, Y) be jointly distributed with PDF

$$f(x,y) = \begin{cases} 2 & \text{if} \quad 0 < x < y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

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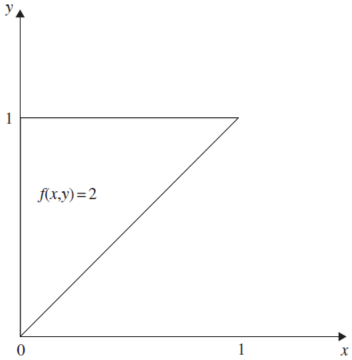

Fig. $f(x,y) = 2, 0 < x < y < 1.$

Then

$$f_X(x) = \int_x^1 2dy = \begin{cases} 2 - x & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \int_0^y 2dx = \begin{cases} 2y & \text{if} \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

are the two marginal pdfs.

JOINT CUMULATIVE DISTRIBUTION FUNCTION :

Given a pair of continuous random variables X and Y and their joint probability density function f. The joint CDF of X and Y , is a function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t) ds\, dt.$$

Note that

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y},$$

wherever these partial derivative exists.

EXAMPLE: Let (X, Y) be jointly distributed with CDF

$$F(x,y) = \begin{cases} \frac{1}{5}(2x^3y + 3x^2y^2) & \text{if} \quad 0 < x < y < 1 \\ \\ 0 & \text{elsewhere,} \end{cases}$$

Then what is the joint pdf of X and Y?