Introduction to Statistics (MAT 283)

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Transformation Method for Bivariate Case:

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Theorem 2. Let X and Y be two continuous random variables with joint pdf f. Let $U = P(X, Y)$ and $V = Q(X, Y)$ be functions of X and Y . If the functions $P(x, y)$ and $Q(x, y)$ have single valued inverses, say $X = R(U, V)$ and $Y = S(U, V)$, then the joint pdf g of U and V is given by

$$g(u, v) = |J| f(R(u, v), S(u, v)),$$

where J denotes the Jacobian and given by

$u = p(x, y)$

$v = q(x, y)$

$$J = det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

X, Y

$U = P(X, Y)$

$V = Q(X, Y)$

EXAMPLE: Let X and Y have the joint probability density function

$$f(x,y) = \begin{cases} 8xy & \text{for} \quad 0 < x < y < 1; \\ \\ 0 & \quad \text{otherwise,} \end{cases}$$

What is the joint density of $U = \frac{X}{Y}$ and $V = Y$?

$$U = \frac{X}{Y} \quad , \quad V = Y$$

$$u = \frac{x}{y} \quad , \quad v = y$$

$$\Rightarrow \quad x = uy \quad , \quad v = y$$

$$x = uv \qquad y = S(u,v)$$

$$= R(u,v) \qquad = v$$

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} = v$$

$$g(u,v) = |J|\, f\big(R(u,v), S(u,v)\big)$$

$$= v\, 8(uv)\, v$$

$$= \begin{cases} 8uv^3, & 0 < v < 1, \ 0 < u < 1 \\ 0, & \text{otherwise} \end{cases}$$

$0 < x < y < 1$

$0 < uv < v < 1$

$0 < v < 1$

$0 < u < 1$

Moment Generating Function Method:

Moment Generating Function Method:

We know that if X and Y are **independent** random variables, then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

This result can be used to find the distribution of the sum $X + Y$.

Example. Let $X \sim POI(\lambda_1)$ and $Y \sim POI(\lambda_2)$. What is the probability density function of $X + Y$ if X and Y are independent?

Solution: Since, $X \sim POI(\lambda_1)$ and $Y \sim POI(\lambda_2)$, we get

$$M_X(t) = e^{\lambda_1(e^t - 1)}$$

and

$$M_X(t) = e^{\lambda_2(e^t - 1)}.$$

Further, since X and Y are independent, we have

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

that is, $X + Y \sim POI(\lambda_1 + \lambda_2)$.

Hence the pdf h of $Z = X + Y$ is given by

$$h(z) = \begin{cases} \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^z}{z!} & \text{for} \quad \text{z=0,1,2,3,\ldots;} \\ \\ 0 & \text{otherwise,} \end{cases}.$$

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Distribution of sample mean and variance :

Tossing a fair coin

1) X_1 , $R_{X_1} = \{0, 1\}$ Random experiment

2) X_2 , $R_{X_2} = \{0, 1\}$ X_1

3) X_3 , $R_{X_3} = \{0, 1\}$ X_2

$\vdots$

X_n

RANDOM SAMPLE :

Consider a random experiment. Let X be the random variable associated with this experiment. Let f be the probability density function of X.

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Then the collection of the random variables $\{X_1, X_2, ..., X_n\}$ is a **random sample** of size n. The random variables $X_1, X_2, ..., X_n$ are independent and identically distributed with the common probability density function f.

For a random sample, functions such as the sample mean

$$(\bar{X}_n \text{ or})\ \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

the sample variance

$$(S_n^2 \text{ or})\ S^2 = \frac{1}{\underset{n-1}{\cancel{1-n}}}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

are called **statistics**.

Theorem. If $X_1, X_2, ..., X_n$ are mutually independent random variables with respective means $\mu_1, \mu_2, ..., \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, then the mean and variance of $Y = \sum_{i=1}^n a_i X_i$, where $a_1, a_2, ..., a_n$ are real constants, are given by

$$\mu_Y = \sum_{i=1}^n a_i \mu_i$$

and

$$\sigma_Y^2 = \sum_{i=1}^n a_i^2 \sigma_i^2 .$$

$$Y = \sum_{i=1}^n a_i X_i$$

$$E(Y) = \mu_Y = E\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i E(X_i)$$

$$= \sum a_i \mu_i$$

Example. Let the independent random variables X_1 and X_2 have means $\mu_1 = -4$ and $\mu_2 = 3$, respectively and variances $\sigma_1^2 = 4$ and $\sigma_2^2 = 9$. What are the mean and variance of $Y = 3X_1 - 2X_2$?

$$\mu_Y = 3\mu_{X_1} - 2\mu_{X_2}$$

$$= 3 \times (-4) - 2 \times 3 = -18$$

$$\sigma_Y^2 = (3)^2 \sigma_1^2 + (-2)^2 \sigma_2^2$$

$$= 9 \times 4 + 4 \times 9 = 72.$$

Example. Let $X_1, X_2, \ldots, X_{50}$ be a random sample of size 50 from a distribution with pdf

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{for} \quad x > 0; \\ \\ 0 & \text{otherwise,} \end{cases}.$$

What are the mean and variance of the sample mean $\bar{X}$?

X is exponential random variable

$\mu_X = \theta, \quad \sigma_X^2 = \theta^2$

$E(\bar{X}) = E\left(\frac{1}{50} \sum_{i=1}^{50} X_i\right)$

$$= \frac{1}{50}\sum_{i=1}^{50} E(X_i) = \frac{1}{50}\sum_{i=1}^{50} \theta = \theta$$

$$Var(\bar{X}) = Var\left(\sum_{i=1}^{50} \frac{1}{50} X_i\right)$$

$$= \left(\frac{1}{50}\right)^2 \sum_{i=1}^{50} \sigma_{X_i}^2$$

$$= \left(\frac{1}{50}\right)^2 \left(\sum_{i=1}^{50} \sigma^2\right) = \frac{\sigma^2}{50}$$