Introduction to Statistics (MAT 283)

Dipti Dubey

Department of Mathematics
Shiv Nadar University

Feburary 16, 2022

Table of Contents

Example 1: Let $X \sim N(0,1)$. Find the distribution of $Y = X^2$.
Solution: We have

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt \\
&= 2\int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt
\end{aligned}
$$

Changing the variable $t^2 = u$, we get

$$
F_Y(y) = 2\int_0^y \frac{1}{2\sqrt{2\pi u}} e^{\frac{-u}{2}} dt
$$

and hence

$$g(y) = \frac{dF_Y}{dy} = \frac{1}{2\sqrt{\pi u}} e^{\frac{-u}{2}} dt,$$

for $y > 0$.

and hence

$$g(y) = \frac{dF_Y}{dy} = \frac{1}{2\sqrt{\pi u}} e^{\frac{-u}{2}} dt,$$

for $y > 0$.

A random variable with the pdf g defined above is said to have a χ^2-distribution. It is very important in statistics, and we will this distribution in detail later.

Transformation Method for Univariate Case:

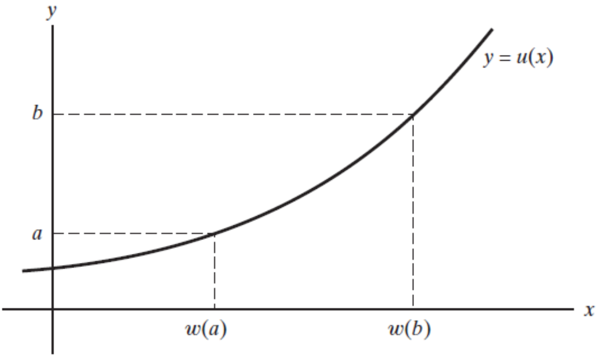
Transformation Method for Univariate Case:

Theorem 1: Let $f(x)$ be the value of the probability density function of the continuous random variable X at x. If the function given by $y = u(x)$ is differentiable and either increasing or decreasing for all values within the range of X for which $f(x) \neq 0$, then, for these values of x, the equation $y = u(x)$ can be uniquely solved for x to give $x = w(y)$, and for the corresponding values of y the probability density of $Y = u(X)$ is given by

$$g(y) = f(w(y)).|w'(y)|$$

provided $u'(x) \neq 0$. Elsewhere, $g(y) = 0$.

Proof: First, let us prove the case where the function given by $y = u(x)$ is increasing. As can be seen from Figure given below, X must take on a value between $w(a)$ and $w(b)$ when Y takes on a value between a and b. Hence,



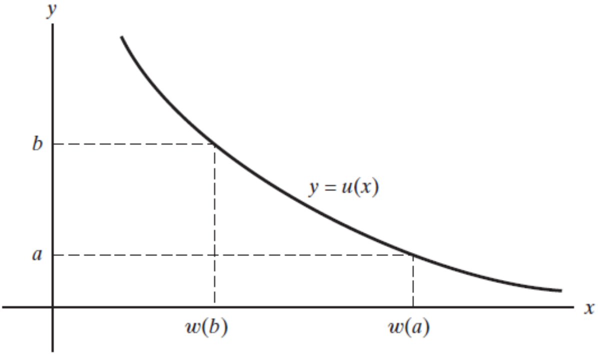
Increasing function

$$
\begin{aligned}
P(a < Y < b) &= P(w(a) < X < w(b)) \\
&= \int_{w(a)}^{w(b)} f(x)dx \\
&= \int_a^b f(w(y))w'(y)dy
\end{aligned}
$$

where we performed the change of variable $y = u(x)$, or equivalently $x = w(y)$, in the integral. The integrand gives the probability density function of Y as long as $w'(y)$ exists, and we can write

$$g(y) = f(w(y))w'(y).$$

When the function given by $y = u(x)$ is decreasing, it can be seen from Figure given below that X must take on a value between $w(b)$ and $w(a)$ when Y takes on a value between a and b. Hence,



Decreasing function

$$
\begin{aligned}
P(a < Y < b) &= P(w(b) < X < w(a)) \\
&= \int_{w(b)}^{w(a)} f(x)dx \\
&= \int_{b}^{a} f(w(y))w'(y)dy \\
&= -\int_{a}^{b} f(w(y))w'(y)dy
\end{aligned}
$$

where we performed the same change of variable as before, and it follows that

$$g(y) = -f(w(y))w'(y).$$

Since $w'(y) = \frac{dx}{dy} = \frac{1}{dy/dx}$ is positive when the function given by $y = u(x)$ is increasing, and $-w'(y)$ is positive when the function given by $y = u(x)$ is decreasing, we can combine the two cases by writing

$$g(y) = f(w(y)).|w'(y)|.$$

EXAMPLE: If X has the exponential distribution given by

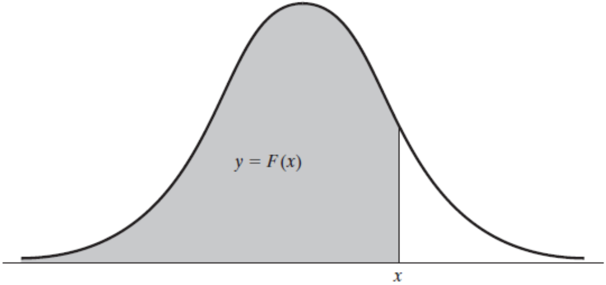
$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

find the probability density function of the random variable $Y = \sqrt{x}$.

EXAMPLE: If $F(x)$ is the value of the distribution function of the continuous random variable X at x, find the probability density of $Y = F(X)$.

EXAMPLE: If $F(x)$ is the value of the distribution function of the continuous random variable X at x, find the probability density of $Y = F(X)$.

Solution. As can be seen from Figure given below, the value of Y corresponding to any particular value of X is given by the area under the curve, that is, the area under the graph of the pdf of X to the left of x.



Differentiating $y = F(x)$ with respect to x, we get $\frac{dy}{dx} = F'(x) = f(x)$ and hence

$$\frac{dx}{dy} = 1/\frac{dy}{dx} = \frac{1}{f(x)}$$

provided $f(x) \neq 0$.

It follows from Theorem 1 that $g(y) = f(x)\frac{1}{f(x)} = 1$ for $0 < y < 1$, and we can say that y has the uniform pdf with $a = 0$ and $b = 1$.

EXAMPLE: Let $Y = -lnX$. If $X \sim UNIF(0,1)$, then what is the density function of Y where nonzero?