Introduction to Statistics (MAT 283)

Dipti Dubey

Department of Mathematics
Shiv Nadar University

Table of Contents

SOME SPECIAL CONTINUOUS DISTRIBUTIONS CONTINUED

The normal distribution plays a central role in probability and statistics and was discovered by a French mathematician Abraham DeMoivre (1667-1754). This distribution is also called the Gaussian distribution after Carl Friedrich Gauss, who proposed it as a model for measurement errors.

The normal distribution plays a central role in probability and statistics and was discovered by a French mathematician Abraham DeMoivre (1667-1754). This distribution is also called the Gaussian distribution after Carl Friedrich Gauss, who proposed it as a model for measurement errors.

3. The Normal (or Gaussian) Distribution:

A random variable X is said to have a normal distribution if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty$$

with parameters μ, where $-\infty < \mu < \infty$ and $\sigma > 0$.

The normal distribution plays a central role in probability and statistics and was discovered by a French mathematician Abraham DeMoivre (1667-1754). This distribution is also called the Gaussian distribution after Carl Friedrich Gauss, who proposed it as a model for measurement errors.

3. The Normal (or Gaussian) Distribution:

A random variable X is said to have a normal distribution if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty$$

with parameters μ, where $-\infty < \mu < \infty$ and $\sigma > 0$.

If X has a normal distribution with parameters μ and σ^2, then we write $X \sim N(\mu, \sigma^2)$.

The graph of a normal probability density function, shaped like the cross section of a bell.

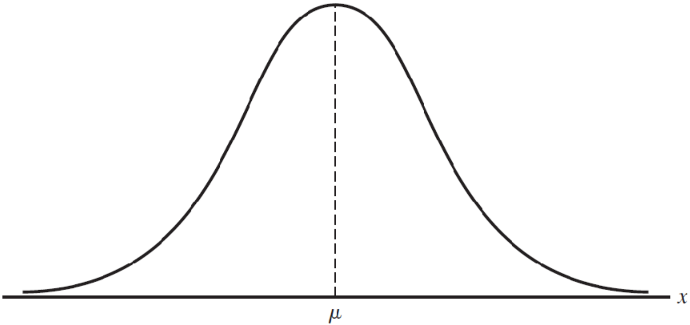


Figure Graph of normal distribution.

• From the form of the probability density function, we see that the density is symmetric about μ, $f(\mu - x) = f(\mu + x)$, where it has a maximum, and that the rate at which it falls off is determined by σ.

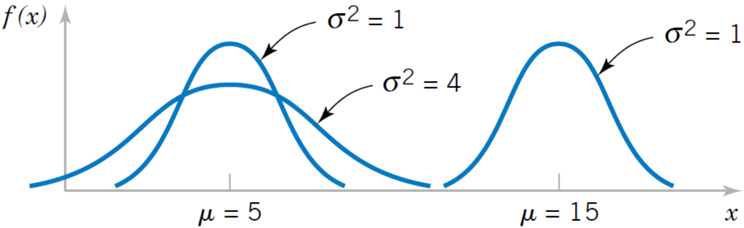


Figure Normal probability density functions for selected values of the parameters μ and σ^2.

Theorem: If $X \sim N(\mu, \sigma^2)$, then

$$E(X) = \mu_X = \mu$$
$$Var(X) = \sigma_X^2 = \sigma^2$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Proof.

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(-2xt\sigma^2+(x-\mu)^2\right)} dx
\end{aligned}
$$

Note that

$$
-2xt\sigma^2 + (x-\mu)^2 = [x - (\mu t + \sigma^2)]^2 - 2\mu t\sigma^2 - t^2\sigma^4,
$$

and hence

$$
M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \left\{ \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-(\mu t+\sigma^2)}{\sigma}\right)^2} dx \right\}
$$

Since the quantity inside the bracket is the integral from $-\infty$ to $-\infty$ of a normal probability density function with the parameters $\mu + t\sigma^2$ and σ, and hence is equal to 1, it follows that

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Further,

$$\begin{aligned} M_X'(t) &= (\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}\sigma^2 t^2}. \\ M_X''(t) &= \sigma^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2} + (\mu + \sigma^2 t)^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2}. \\ \implies E(X) = M_X'(0) &= \mu \ \text{ and } \ Var(X) = \sigma^2. \end{aligned}$$

Hence the proof.

Standard Normal Random Variable: A normal random variable is said to be standard normal, if its mean is zero and variance is one. We denote a standard normal random variable X by $X \sim N(0, 1)$ and it's probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty.$$

Example: If $X \sim N(0,1)$, what is the probability of the random variable X less than or equal to -1.72?

Example: If $X \sim N(0,1)$, what is the probability of the random variable X less than or equal to -1.72?

Solution: We have, using Standard Normal Table type I

$$P(X \leq -1.72) = 1 - P(X \leq 1.72)$$
$$= 1 - 0.9573$$
$$= 0.0427.$$

Example: If $X \sim N(0, 1)$, what is the probability of the random variable X less than or equal to -1.72?

Solution: We have, using Standard Normal Table type I

$$P(X \leq -1.72) = 1 - P(X \leq 1.72)$$
$$= 1 - 0.9573$$
$$= 0.0427.$$

Using Standard Normal Table type II,

$$P(X \leq -1.72) = 0.0427.$$

• Probabilities that are not of the form $P(Z \leq z)$ are found by using the basic rules of probability and the symmetry of the normal distribution.

• Probabilities that are not of the form $P(Z \leq z)$ are found by using the basic rules of probability and the symmetry of the normal distribution.

Examples:
1. $P(Z > 1.26)$
2. $P(Z > -1.37)$
3. $P(Z < -0.86)$
4. $P(-1.25 < z < 0.37)$

Theorem: If $X \sim N(\mu, \sigma^2)$ then the random variable $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$.

Theorem: If $X \sim N(\mu, \sigma^2)$ then the random variable $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.

Proof: We will show that Z is standard normal by finding the probability density function of Z. We compute the probability density of Z by first computing it's cumulative distribution function.

$$\begin{aligned}
F(z) &= P(Z \leq z) \\
&= P(\frac{X-\mu}{\sigma} \leq z) \\
&= P(X \leq \sigma z + \mu) \\
&= \int_{-\infty}^{\sigma z+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx
\end{aligned}$$

$$= \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}w^2} dw,$$

(where $w = \frac{x-\mu}{\sigma}$)

Hence

$$f(z) = F'(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

Hence the proof.

Example: If $X \sim N(3,16)$, then what is $P(4 \leq X \leq 8)$?

$$
\begin{aligned}
P(4 \leq X \leq 8) &= P(\frac{4-3}{4} \leq \frac{X-3}{4} \leq \frac{8-3}{4}) \\
&= P(\frac{1}{4} \leq Z \leq \frac{5}{4}) \\
&= P(Z \leq 1.25) - P(Z \leq 0.25) \\
&= 0.8944 - 0.5987 \\
&= 0.2957.
\end{aligned}
$$

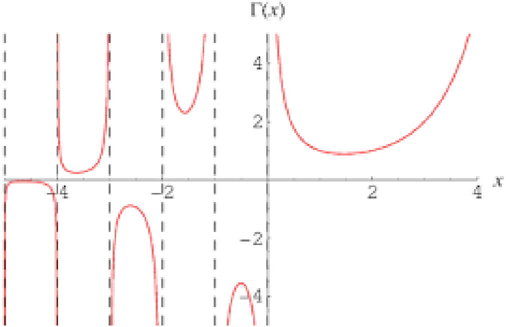
GAMMA DISTRIBUTION:

The gamma distribution involves the notion of gamma function. The gamma function, $\Gamma(z)$, is a generalization of the notion of factorial. The gamma function is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

where z is positive real number (that is, $z > 0$). The condition $z > 0$ is assumed for the convergence of the integral.
Although the integral does not converge for $z < 0$, it can be shown by using an alternative definition of gamma function that it is defined for all $z \in \mathbb{R}/\{0, -1, -2, -3, ...\}$.

The graph of the gamma function is shown below.



Properties of Gamma Function:

1. $\Gamma(1) = 1$.

Properties of Gamma Function:

1. $\Gamma(1) = 1$.

2.The gamma function $\Gamma(z)$ satisfies the functional equation $\Gamma(z) = (z-1)\Gamma(z-1)$ for all real number $z > 1$.

Properties of Gamma Function:

1. $\Gamma(1) = 1$.

2.The gamma function $\Gamma(z)$ satisfies the functional equation $\Gamma(z) = (z-1)\Gamma(z-1)$ for all real number $z > 1$.

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Properties of Gamma Function:

1. $\Gamma(1) = 1$.

2.The gamma function $\Gamma(z)$ satisfies the functional equation $\Gamma(z) = (z-1)\Gamma(z-1)$ for all real number $z > 1$.

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4. $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$.

Properties of Gamma Function:

1. $\Gamma(1) = 1$.

2.The gamma function $\Gamma(z)$ satisfies the functional equation $\Gamma(z) = (z-1)\Gamma(z-1)$ for all real number $z > 1$.

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4. $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$.

5. If n is a natural number then $\Gamma(n+1) = n!$.

GAMMA DISTRIBUTION:

A continuous random variable X is said to have a gamma distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ and $\alpha > 0$.

GAMMA DISTRIBUTION:

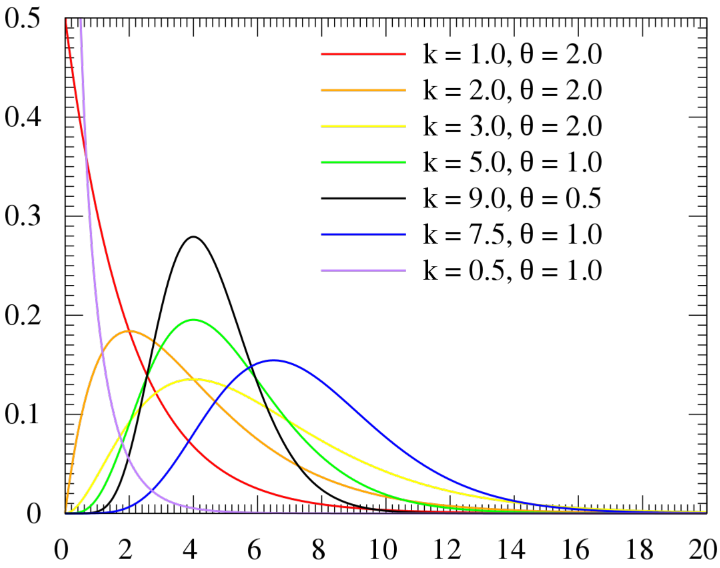
A continuous random variable X is said to have a gamma distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ and $\alpha > 0$.

We denote a random variable X with the Gamma distribution as $X \sim \text{GAM}(\theta, \alpha)$.

• Some special cases of the gamma distribution play important roles in statistics; for instance, for $\alpha = 1$ and $\beta = \theta$, we obtain the exponential distribution.



0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
18
20
k = 1.0, θ = 2.0
k = 2.0, θ = 2.0
k = 3.0, θ = 2.0
k = 5.0, θ = 1.0
k = 9.0, θ = 0.5
k = 7.5, θ = 1.0
k = 0.5, θ = 1.0

Here $k = \alpha$.

THEOREM: If $X \sim \mathrm{GAM}(\theta, \alpha)$, then the mean, variance and moment generating functions are respectively given by

$$E(X) = \mu_X = \theta\alpha$$

$$\mathrm{Var}(X) = \theta^2\alpha$$

$$M_X(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}, \text{ if } t < \frac{1}{\theta}.$$

BETA DISTRIBUTION:

The beta distribution involves the notion of beta function. First we explain the notion of the beta integral and some of its simple properties. Let α and β be any two positive real numbers. The beta function $B(\alpha, \beta)$ is defined as

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx.$$

Properties of beta function:

1. Let α and β be any two positive real numbers. Then

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

2. Let α and β be any two positive real numbers. Then

$$B(\alpha, \beta) = B(\beta, \alpha).$$

BETA DISTRIBUTION:

A continuous random variable X is said to have a beta distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

BETA DISTRIBUTION:
A continuous random variable X is said to have a beta distribution if its probability density function is given by

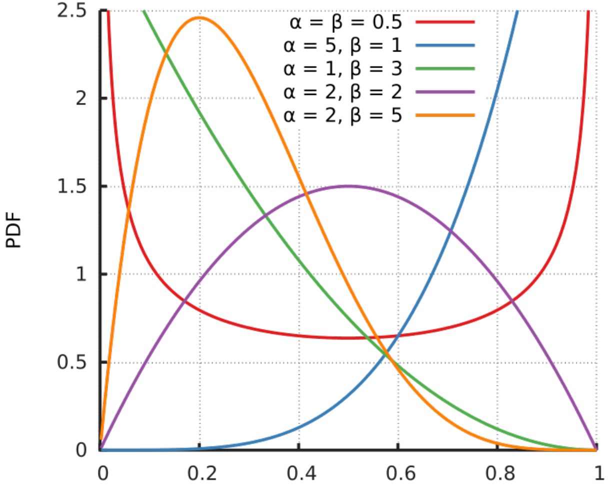
$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

We denote a random variable X with the Beta distribution as $X \sim \text{BETA}(\theta, \alpha)$.

• Some special cases of the beta distribution play important roles in statistics; for instance, for $\alpha = 1$ and $\beta = 1$, we obtain the uniform distribution over $(0, 1)$.

Graph of Beta Distribution:



THEOREM: If $X \sim \text{BETA}(\alpha, \beta)$, then the mean and variance are respectively given by

$$E(X) = \mu_X = \frac{\alpha}{\alpha + \beta}$$

$$\text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$