[Orthogonal polynomial sequence]

◎ We have learnt that the (complex) functions are considered as infinite-dimensional vectors in the region $[a, b]$ ($a, b \in \mathbb{R}$), where the inner product of the two functions $f(x)$ and $g(x)$ are defined as

$$(f, g) = \int_a^b dx\, \rho(x) f^*(x) g(x),$$

where $\rho(x) > 0$ (in $[a,b]$) is called the weight function.

▷ when $a = 0$, $b = \pi$, $\rho(x) = 1$, the Fourier-sine series

$f_n(x) = \sin(nx)$ $(n = 1, 2, 3, \cdots)$ works as orthogonal bases

(for $k_n > 0$, $(f_n, f_m) = k_n \times \delta_{nm}$) to represent functions in $[0, \pi]$

with the boundary conditions ($= 0$ @ $x = 0$ and $x = \pi$).

Here, we will learn a class of such orthogonal polynomials.

▨ For $[a, b]$ and $\rho(x)$, the following conditions are considered

✓ $\dfrac{\rho'(x)}{\rho(x)} = \dfrac{A(x)}{B(x)}$ with $\underbrace{A(x) = a_0 + a_1 x}_{A(x)\text{ is up to a linear polynomial.}}$, $\underbrace{B(x) = b_0 + b_1 x + b_2 x^2}_{B(x)\text{ is up to a quadratic polynomial.}}$, ($a_0, a_1, b_0, b_1, b_2$: constants)

✓ $\lim_{x \to a} B(x)\rho(x) = \lim_{x \to b} B(x)\rho(x) = 0.$

→ If all of the above conditions are fulfilled, the forms

$$F_n(x) \equiv \rho(x)^{-1} \left(\frac{d}{dx}\right)^n \left(\rho(x) B(x)^n\right) \quad (n = 0, 1, 2, \cdots)$$

are order-n polynomials and they are orthogonal each other $\underline{(F_n, F_m) = 0 \text{ if } n \neq m}$.

Note: normalization factors are determined by (F_n, F_n) //

[Proof]

- For $Q \equiv \rho^{-1}_{(x)} \frac{d}{dx} \rho(x)$, if $g(x)$ is an arbitrary order-l polynomial,

$$Q\big(g(x)\, \underset{k \ge 1}{B^k(x)}\big) = \big(\text{a polynomial up to the } (l+1) \text{ order}\big) \times B^{k-1}_{(x)},$$

since $Q(gB^k) = \rho^{-1}\frac{d}{dx}(\rho g B^k) = \rho^{-1}\left(\rho' g B^k + \rho g' B^k + \rho g\, k \cdot B^{k-1} B'\right)$

$$= \underbrace{(Ag + g'B + kgB')}_{\substack{A,\ B \text{ are up to order-1 and order-2 and } g(x) \text{ is order-}l. \\ \Rightarrow \text{ This part is up to order-}(l+1).}} B^{k-1} \;//$$

- $F_n(x)$ can be represented as $F_n(x) = Q^n B^n(x)$ //

The above statement tells us

$$F_n(x) = Q^n B^n = Q^{n-1}\big[(\text{polynomial up to order-1}) \times B^{n-1}\big]$$

$$= \cdots = (\text{polynomial up to order-}n) \times B^0 \;//$$

$\Rightarrow$ Thus, $F_n(x)$ is a polynomial whose order is up to order-n.

- (F_m, F_n) [with $m > n$] $= \int_a^b dx\, \rho(x) F_m(x) F_n(x)$

$$= \int_a^b dx\, \cancel{\rho} \left(\cancel{\rho^{-1}} \frac{d^m}{dx^m}(\rho B^m)\right) F_n$$

$$= \underbrace{\left[\frac{d^{m-1}}{dx^{m-1}}(\rho B^m) F_n\right]_{x=a}^{b}}_{\lim_{x\to b}\left(\frac{d^{m-1}}{dx^{m-1}}(\rho B^m)\right) F_n \,-\, \lim_{x\to a}\left(\frac{d^{m-1}}{dx^{m-1}}(\rho B)\right) F_n} - \int_a^b dx\, \frac{d^{m-1}}{dx^{m-1}}(\rho B^m) \frac{d}{dx} F_n$$

$$\frac{d^{m-1}}{dx^{m-1}}(\rho B^m) = \underbrace{\rho^{(m-1)} B^m + \rho^{(m-2)} m B^{m-1} B' + \cdots + \rho B B' (m!)}_{= 0 \text{ since } (\lim_{x\to b}(\rho B) = 0)}, \qquad \lim_{x\to a}\left(\frac{d^{m-1}}{dx^{m-1}}(\rho B)\right) F_n = 0$$

The surface terms vanish at the two end points due to $\lim_{x\to a}(\rho B) = 0 = \lim_{x\to b}(\rho B)$.

$$= (-1)^m \int_a^b dx\, \rho(x) B^m(x) \underbrace{\frac{d^m}{dx^m} F_n(x)}_{= 0 \text{ since } m > n}$$

$$= 0 \;//$$

(Famous examples of orthogonal polynomial sequences belonging to this class)

✓ $\boxed{\text{the Legendre polynomials}}$:

in the domain $[a=-1, b=+1] = [-1,+1]$, with $\rho(x)=1$,

where $A(x)$ and $B(x)$ are taken as $A(x)=0$, $B(x)=(x^2-1)$ //

$$\Rightarrow P_n(x) = \underbrace{\frac{1}{2^n n!}}_{\substack{\text{this is the well-adopted} \\ \text{normalization factor.}}} \left(\frac{d}{dx}\right)^n (x^2-1)^n \quad (n=0,1,2,\cdots) \,//$$

✓ $\boxed{\text{the Laguerre polynomials}}$:

For $a=0$, $b=+\infty$, $\rho(x)=e^{-x}$, where $\frac{\rho'}{\rho}=-1$ and

we take $A(x)=-x$, $B(x)=x$.

$$\Rightarrow L_n(x) = e^x \left(\frac{d}{dx}\right)^n (e^{-x} x^n) \quad (n=0,1,2,\cdots) \,//$$

✓ $\boxed{\text{the Hermite polynomials}}$:

For $a=-\infty$, $b=+\infty$, $\rho(x)=e^{-x^2}$, where $\frac{\rho'}{\rho}=-2x$ and

we take $A(x)=-2x$, $B(x)=1$.

$$\Rightarrow H_n(x) = \underbrace{(-1)^n}_{\substack{\text{This is the well-adopted} \\ \text{normalization factor.}}} e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} \quad (n=0,1,2,\cdots) \,//$$

- An example to dertermine $|F_n|^2 = (F_n, F_n)$

Even though $(F_n, F_m) = 0$ $(n \neq m)$ is shown, the normalization (F_n, F_n) is requested to be clarified when we utilize orthogonal polynomials. We will see the concrete example of the Legendre polynomials.

$$\rightarrow (P_n, P_n) = \int_{-1}^{1} dx \cdot \left[\left(\frac{d}{dx}\right)^n \cdot (x^2-1)^n\right]\left[\left(\frac{d}{dx}\right)^n (x^2-1)^n\right] \times \frac{1}{(2^n n!)^2}$$

$$= (-1)^n \int_{-1}^{1} dx \, (x^2-1)^n \underbrace{\left(\frac{d}{dx}\right)^{2n} (x^2-1)^n}_{=(2n)!} \times \frac{1}{(2^n n!)^2}$$

$x^2-1 = (x+1)(x-1)$

$z = \frac{1-x}{2}$

$$= (-1)^n (2n)! \frac{1}{(2^n n!)^2} \underbrace{\int_{1}^{0} (-2\,dz)(-2z)^n (2(1-z))^n}$$

$$= \int_0^1 dz \, (-1)^n \, 2^{2n+1} z^n (1-z)^n$$

the beta function

$$= (-1)^n 2^{2n+1} B(n+1, n+1)$$

$$= (-1)^n 2^{2n+1} \times \boxed{\frac{(\Gamma(n+1))^2}{\Gamma(2n+2)}} = \frac{(n!)^2}{(2n+1)!}$$

$$= \frac{(2n)!}{(2n+1)!} \frac{1}{\cancel{2^n 2^n (n!)^2}} \cancel{2^{2n+1}} \cancel{(n!)^2} = \frac{2}{2n+1} \quad //$$

(The differential eq. which the above-class polynomials obey)

The polynomials (in $[a,b]$ with the weight function $\rho(x)$, where

$$\begin{cases} \checkmark \ \dfrac{\rho'(x)}{\rho(x)} = \dfrac{A(x)}{B(x)} \quad \text{with } A(x) = a_0 + a_1 x, \ B(x) = b_0 + b_1 x + b_2 x^2, \\ \checkmark \ \lim_{x\to a} B(x)\rho(x) = 0 = \lim_{x\to b} B(x)\rho(x) . \end{cases}$$

) obey

the differential equation.

$$B(x) F_n''(x) + (A(x) + B'(x)) F_n'(x) - \alpha_n F_n(x) = 0$$

with $\alpha_n \equiv n(n+1) b_2 + n a_1$. //

[Proof]

The part " $B(x) F_n''(x) + (A(x) + B'(x)) F_n'(x)$ " is a polynomial up to the order-n, and so, it can be represented as a linear combination as

$$B(x) F_n''(x) + (A(x) + B'(x)) F_n'(x) = \sum_{k=0}^{n} C_k F_k(x) .$$

In the following, we show the two statements.

(i) $C_n = \alpha_n$

parametrization

$$F_n = h_n x^n + (\text{less than the order } x^n), \quad (h_n \neq 0)$$

$$\Rightarrow \begin{cases} B(x) F_n'' + (A + B') F_n' = b_2 n(n-1) h_n x^n + (a_1 + 2b_2) n h_n x^n + (\text{others}) \\ \qquad = h_n \cdot x^n (b_2 \cdot n(n+1) + n a_1) + (\text{other orders}) \\ \alpha_n F_n(x) = h_n \alpha_n x^n + (\text{other orders}) \end{cases}$$

$$\therefore \ \alpha_n = C_n = n(n+1) b_2 + n a_1 .$$

(ii) $C_k = 0 \ (k = 0, 1, \cdots, n-1)$

$$\frac{\rho'}{\rho} = \frac{A}{B}$$

$$g(x) \equiv B(x) F_n''(x) + (A(x) + B'(x)) F_n'(x) = \frac{1}{\rho} \frac{d}{dx} (\rho B F_n')$$

$$\rightarrow \underbrace{(g, F_k)}_{C_k} = \int_a^b dx\, \rho(x) g(x) F_k(x) = \int_a^b dx \frac{d}{dx} (\rho B F_n') F_k$$

$$= -\int_a^b dx\, \rho B F_n' F_k' = \int_a^b dx\, F_n \frac{d}{dx} (\rho B F_k') = (F_n, G)$$

$$G = \frac{1}{\rho} \frac{d}{dx} (\rho B F_k') = Q(B F_k')$$

$\Rightarrow$ a polynomial up to the order k.

Since $n > k$, $\underbrace{(F_n, G) = 0}$ $\underbrace{= (g, F_k) = C_k}$ //

The concrete examples.

- for the Legendre polynomials ($(a,b)=(-1,1)$, $\rho(x)=1$, $A(x)=0$, $B(x)=x^2-1$)

$$\Rightarrow \underbrace{(x^2-1)P_n''(x) + (0+2x)P_n'(x) - [n(n+1)\cdot 1]\, P_n(x) = 0}$$ //

- for the Laguerre polynomials ($(a,b)=(0,\infty)$, $\rho(x)=e^{-x}$, $A(x)=-x$, $B(x)=x$)

$$\Rightarrow x L_n''(x) + (-x+1)L_n'(x) \underbrace{- [n(-1)]\, L_n(x)}_{=+n L_n(x)} = 0$$

- for the Hermite polynomials ($(a,b)=(-\infty,\infty)$, $\rho(x)=e^{-x^2}$, $A(x)=-2x$, $B(x)=1$)

$$\Rightarrow H_n''(x) + (-2x+0)H_n'(x) - [-2n]\, H_n(x) = 0$$

$$\Leftrightarrow H_n''(x) - 2x H_n'(x) + 2n\, H_n(x) = 0$$ //

(Generating functions of orthogonal polynomials)

Associated with a series of orthogonal polynomials f_n ($n=0,1,2,\cdots$),

the following function is called the generating function,

$$G(x,t) \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} f_n(x)$$ ← $f_n(x)$ including normalization factors

(with/without such combinatoric factors depending on the normalization of $f_n(x)$) //

For the three kinds of polynomials, we can get

$$G(x,t) = \begin{cases} = \displaystyle\sum_{n=0}^{\infty} P_n(x)\, t^n = \frac{1}{\sqrt{1-2xt+t^2}} & \text{(for the Legendre polynomials),} \\ = \displaystyle\sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!} = \frac{e^{\frac{tx}{t-1}}}{1-t} & \text{(for the Laguerre "),} \\ = \displaystyle\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}. \end{cases}$$

How to derive this ?

$$F_n(x) = \frac{1}{\rho(x)}\left(\frac{d}{dx}\right)^n\left(\rho(x)B^n(x)\right),$$

$f(\xi)$ is holomorphic inside C_x.

$$\downarrow \quad f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C_x}\frac{f(\xi)}{(\xi-x)^{n+1}}d\xi$$

$\downarrow$ (the Goursat formula)

$$F_n(x) = \frac{1}{\rho}\frac{n!}{2\pi i}\oint_{C_x}d\xi\,\frac{\rho(\xi)B^n(\xi)}{(\xi-x)^{n+1}}$$

$$G(x,t) \equiv \sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!} = \frac{1}{2\pi i\rho}\oint_{C_x}d\xi\,\sum_{n=0}^{\infty}\frac{\rho(\xi)B^n(\xi)}{(\xi-x)^{n+1}}t^n$$

$$\| \quad \frac{\rho(\xi)}{(\xi-x)}\sum_{n=0}^{\infty}\frac{B^n(\xi)}{(\xi-x)^n}t^n$$

$$\boxed{\text{if } \left|\frac{B(\xi)t}{\xi-x}\right|<1} \Rightarrow \frac{\rho(\xi)}{\xi-x}\times\frac{1}{1-\frac{B(\xi)t}{\xi-x}}$$

$$\Rightarrow \frac{1}{2\pi i\rho}\oint_{C_x}d\xi\,\frac{\rho(\xi)}{\xi-x}\times\frac{1}{1-\frac{B(\xi)t}{\xi-x}}$$

$$= \frac{1}{2\pi i\rho}\oint_{C_x}d\xi\,\frac{\rho(\xi)}{\xi-x-B(\xi)t}$$

For example, the Legendre polynomials' case is calculated as follows.

$\cdot\ \rho(x)=1,\ B(x)=x^2-1.$ $\Rightarrow$ (the poles of the integrand)

$$\Rightarrow \xi-x-(\xi^2-1)t=0$$

$$\Leftrightarrow t\xi^2-t-\xi+x = t\xi^2-\xi+x-t=0$$

$$\xi=\xi_\pm = \frac{1\pm\sqrt{1-4t(x-t)}}{2t}$$

$$\checkmark\ \frac{B(\xi)t}{\xi-x} = \frac{(\xi^2-1)t}{\xi-x}$$

$$\xrightarrow{t\ll1} \frac{1\pm(1-2t(x-t))}{2t} = \begin{cases}\frac{2-2t(x-t)}{2t} = \frac{1}{t}-x+t\\ \frac{2t(x-t)}{2t} = x-t\end{cases}$$

$t\ll1$

$$\to \xi_\pm - x \simeq \begin{cases}\frac{1}{t}-2x+t \xrightarrow{t\to0} \infty\\ -t \xrightarrow{t\to0} 0\end{cases}$$

ξ_+

So we will take the path, where only the ξ_- is located inside.

$$\Rightarrow G(x,t) = \mathrm{Res}\left[\frac{1}{-t(\xi-\xi_+)(\xi-\xi_-)},\ \xi=\xi_-\right]$$

$$= \frac{1}{-t(\xi_- - \xi_+)} = \frac{1}{-t\times(-)\frac{\sqrt{1-4t(x-t)}}{t}}$$

$$= \frac{1}{\sqrt{1-4tx+4t^2}}$$

When we focus on the Legendre polynomials normalized as

$$P_n(x) = \frac{1}{2^n n!} F_n(x)$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} P_n(x) (2t)^n = \frac{1}{\sqrt{1-4tx+4t^2}}$$

$$\xrightarrow{2t \to t} \Rightarrow \sum_{n=0}^{\infty} P_n(x) (t)^n = \frac{1}{\sqrt{1-2tx+t^2}} \quad //$$