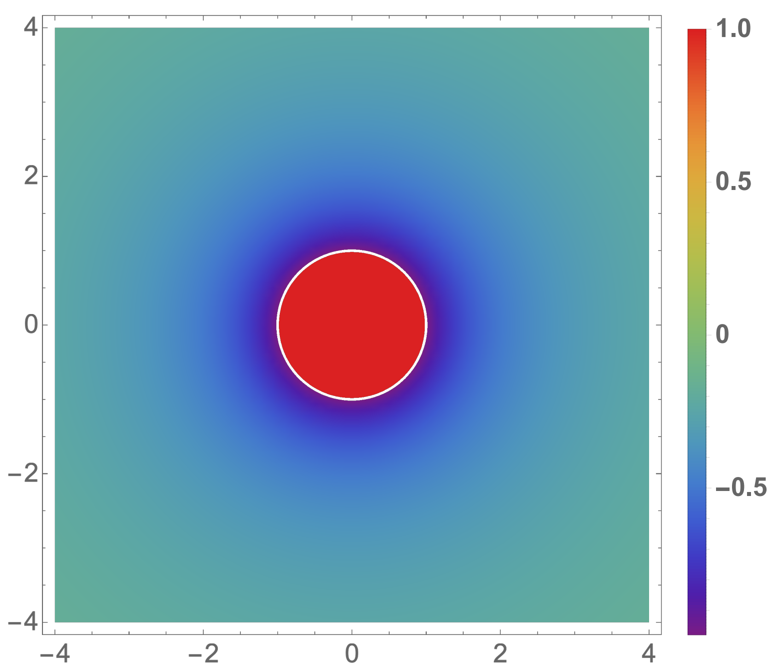
PHY 303: Classical Electrodynamics
MONSOON SEMESTER 2022
TUTORIAL 03

1. Consider an electrostatics problem in the half-space defined by $z \geq 0$, with Dirichlet boundary condition on the plane $z = 0$ (and at infinity). The potential on the plane $z = 0$ is specified to be the following in terms of the cylindrical coordinates (s, ϕ, z):

$$\Phi(s, \phi, 0) = \begin{cases} \Phi_0 & \text{for } 0 \leq s < R, \\ -\Phi_0 \, R/s & \text{for } s > R, \end{cases}$$

where Φ_0 is a constant (See the potential profile based on rainbow (VIBGYOR) colormap in the figure below). Moreover, the potential vanishes at infinity.



(a) Using the Green's function technique, find out the potential due to this system at an arbitrary point on the positive z-axis.

(b) How will the potential be modified if a point charge $+q$ is placed at $(0, 0, d)$?

Plot the electric potential along the positive z-axis in both cases by choosing some convenient values for the parameters.

2. Using Green's theorem, show that the solution of the Laplace equation outside ($r > a$) a sphere of radius a with the potential specified on its surface is given by

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\, \Phi(a,\theta',\phi') \frac{a(r^2-a^2)\sin\theta'}{(r^2+a^2-2ra\cos\gamma)^{3/2}},$$

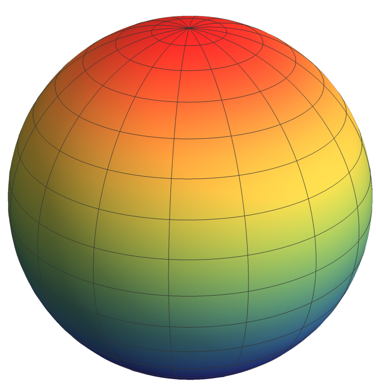
where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. Similarly, show that for inside ($r < a$) the sphere, the potential is

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\, \Phi(a,\theta',\phi') \frac{a(a^2-r^2)\sin\theta'}{(r^2+a^2-2ra\cos\gamma)^{3/2}}.$$

3. Use the result in the above problem to obtain the potentials in both the exterior and interior regions of the sphere, if $\Phi(a,\theta',\phi') = \Phi_0$ (i.e., if the sphere is kept at a constant potential). You may use some symmetry arguments in solving the integrals.

   Note that the potential in the interior region easily follows because of the averaging property of the Laplace equation.

4. Use the result in problem 2 again to write down the potential at an arbitrary point in the exterior region, if the potential on the surface is specified as $\Phi(a,\theta,\phi) = \Phi_0\cos\theta$, where Φ_0 is a constant (See the potential profile based on rainbow (VIBGYOR) colormap in the figure below). Consider the special case of determining the potential along the positive z-axis (for $z > a$) and perform the integrals to obtain a closed-form result.



Challenge Problem

Consider the electrostatic Green's function $G(\mathbf{r}, \mathbf{s})$, with $\nabla^2_{\mathbf{s}} G(\mathbf{r}, \mathbf{s}) = -4\pi\delta(\mathbf{r} - \mathbf{s})$, for Dirichlet and Neumann boundary conditions on the surface $\mathcal{S}$ bounding the volume $\mathcal{V}$. Apply Green's theorem with integration variable $\mathbf{s}$ and scalar fields $\phi(\mathbf{s}) = G(\mathbf{r}, \mathbf{s})$ and $\psi(\mathbf{s}) = G(\mathbf{r}', \mathbf{s})$. Find an expression for the difference $G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}', \mathbf{r})$ in terms of an integral over the boundary surface $\mathcal{S}$.

(a) For Dirichlet boundary condition, show that $G(\mathbf{r}, \mathbf{r}') = G_D(\mathbf{r}, \mathbf{r}')$ must be symmetric in $\mathbf{r}$ and $\mathbf{r}'$.

(b) For Neumann boundary condition, show that $G(\mathbf{r}, \mathbf{r}') = G_N(\mathbf{r}, \mathbf{r}')$ is not symmetric in general, but $G_N(\mathbf{r}, \mathbf{r}') + F(\mathbf{r})$ is symmetric in $\mathbf{r}$ and $\mathbf{r}'$, where

$$F(\mathbf{r}) = -\frac{1}{\mathcal{S}} \oint_{\mathcal{S}} G_N(\mathbf{r}, \mathbf{s}) da_{\mathbf{s}}$$

(c) Show that the addition of $F(\mathbf{r})$ to the Neumann Green's function does not affect the solution $\Phi(\mathbf{r})$.