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Non Linear Control
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Problem : Consider an autonomous tilted mass-spring-damper system with first order actuator dynamics

$$m\dot{x} = -c\dot{x} - kx + mg\sin(\phi) - \tilde{u} + u_d$$

$$\dot{u} = -au + \mu$$

Also,

$$e = x_d - x \tag{1}$$

$$r = \dot{e} + \alpha e \tag{2}$$

$$\tilde{u} = u_d - u \tag{3}$$

and the parametric errors are,

$$\tilde{\theta} = \theta - \hat{\theta} \tag{4}$$

$$\tilde{a} = a - \hat{a} \tag{5}$$

where,

$$\theta = \begin{bmatrix} m \\ c \\ k \end{bmatrix}, \hat{\theta} = \begin{bmatrix} \hat{m} \\ \hat{c} \\ \hat{k} \end{bmatrix}$$

(a) let

$$\zeta = \begin{bmatrix} e \\ r \\ \tilde{u} \\ \tilde{\theta} \\ \tilde{a} \end{bmatrix}$$

and consider the Lyapunov candidate

$$V(\zeta) = \frac{1}{2}e^2 + \frac{1}{2}mr^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{1}{2\gamma_a}\tilde{a}^2$$

i. What are the bounds of $V(\zeta)$ using the Rayleigh Ritz theorem.

$$min\left\{\frac{1}{2}, \frac{1}{2}m, \frac{1}{2}, \frac{1}{2}min(\Gamma^{-1}), \frac{1}{2\gamma_a}\right\}\zeta^2 \leq V(\zeta) \leq max\left\{\frac{1}{2}, \frac{1}{2}m, \frac{1}{2}, \frac{1}{2}max(\Gamma^{-1}), \frac{1}{2\gamma_a}\right\}\zeta^2 \tag{6}$$

Now, multiplying by (-1) we get,

$$-min\left\{\frac{1}{2}, \frac{1}{2}m, \frac{1}{2}, \frac{1}{2}min(\Gamma^{-1}), \frac{1}{2\gamma_a}\right\}\zeta^2 \geq -V(\zeta) \geq -max\left\{\frac{1}{2}, \frac{1}{2}m, \frac{1}{2}, \frac{1}{2}max(\Gamma^{-1}), \frac{1}{2\gamma_a}\right\}\zeta^2$$

Now, Dividing by $max\left\{\frac{1}{2}, \frac{1}{2}m, \frac{1}{2}, \frac{1}{2}max(\Gamma^{-1}), \frac{1}{2\gamma_a}\right\}$ throughout, we get

$$\frac{-1}{max\left\{\frac{1}{2}, \frac{1}{2}m, \frac{1}{2}, \frac{1}{2}max(\Gamma^{-1}), \frac{1}{2\gamma_a}\right\}}V(\zeta) \geq -\zeta^2$$

ii)

$$\dot{e} = -\dot{x}$$

$$\ddot{e} = -\ddot{x}$$

$$m\ddot{e} = -m\ddot{x}$$

Substituting the value of $m\ddot{x}$

$$m\ddot{e} = -(-c\dot{x} - kx + mg\sin(\phi) - \tilde{u} + u_d)$$

$$m\ddot{e} = c\dot{x} + kx - mg\sin(\phi) + \tilde{u} - u_d$$

$$\dot{r} = \ddot{e} + \alpha\dot{e}$$

$$m\dot{r} = m\ddot{e} + m\alpha\dot{e}$$

$$m\dot{r} = c\dot{x} + kx - mg\sin(\phi) + \tilde{u} - u_d + m\alpha\dot{e}$$

$$Y\theta = \begin{bmatrix} (\alpha\dot{e} - g\sin(\phi)) & \dot{x} & x \end{bmatrix} \begin{bmatrix} m \\ c \\ k \end{bmatrix}$$

Therefore,

$$m\dot{r} = Y\theta + \tilde{u} - u_d \tag{7}$$

and we know the Lyapunov candidate is,

$$V(\zeta) = \frac{1}{2}e^2 + \frac{1}{2}mr^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{1}{2\gamma_a}\tilde{a}^2$$

$$\dot{V}(\zeta) = e\dot{e} + mr\dot{r} + \tilde{u}\dot{\tilde{u}} + \tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_a}\tilde{a}\dot{\tilde{a}}$$

$$\dot{V}(\zeta) = e(r - \alpha e) + r(Y\theta + \tilde{u} - u_d) + \tilde{u}(\dot{u}_d - \dot{u}) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

since,

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$$

and

$$\dot{\tilde{a}} = -\dot{\hat{a}}$$

Now, Designing $u_d = e + \beta r + Y\hat{\theta}$, we get

$$\dot{V}(\zeta) = er - \alpha e^2 + r(Y\theta + \tilde{u} - (e + \beta r + Y\hat{\theta})) + \tilde{u}(\dot{u_d} - \dot{u}) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

Simplifying further and cancelling some terms we get,

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 + rY\tilde{\theta} + r\tilde{u} + \tilde{u}(\dot{u_d} - \dot{u}) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

Now, we know

$$u_d = e + \beta r + Y\hat{\theta}$$

Therefore,

$$\dot{u_d} = \dot{e} + \beta\dot{r} + Y\dot{\hat{\theta}} + \dot{Y}\hat{\theta}$$

and we know,

$$\dot{u} = -au + \mu$$

Therefore, substituting $\dot{u_d}$ and $\dot{u}$ in $\dot{V}(\zeta)$, we get

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 + rY\tilde{\theta} + r\tilde{u} + \tilde{u}((\dot{e} + \beta\dot{r} + Y\dot{\hat{\theta}} + \dot{Y}\hat{\theta}) - (-au + \mu)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

Now, Designing $\mu = \dot{u}_d + \hat{a}u + r + s\tilde{u}$, we get

$$\mu = \dot{e} + \beta\dot{r} + Y\dot{\hat{\theta}} + \dot{Y}\hat{\theta} + \hat{a}u + r + s\tilde{u}$$

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 + rY\tilde{\theta} + r\tilde{u} + \tilde{u}((\dot{e} + \beta\dot{r} + Y\dot{\hat{\theta}} + \dot{Y}\hat{\theta}) + au - (\dot{e} + \beta\dot{r} + Y\dot{\hat{\theta}} + \dot{Y}\hat{\theta} + \hat{a}u + r + s\tilde{u}) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

since,

$$\tilde{a} = a - \hat{a}$$

Simplifying further and cancelling some terms we get,

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 + rY\tilde{\theta} - s\tilde{u}^2 + \tilde{u}\tilde{a}u - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}} \tag{8}$$

Now, Designing $\dot{\hat{\theta}} = \Gamma Y^T r$

Also, from equation (30), we have, $rY\tilde{\theta}$, which can also be written as (since r is a scalar):

$$rY\tilde{\theta} = (rY\tilde{\theta})^T = \tilde{\theta}^T Y^T r$$

Therefore,

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 + \tilde{\theta}^T Y^T r - s\tilde{u}^2 + \tilde{u}\tilde{a}u - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

After substituting $\dot{\hat{\theta}}$, we get,

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 + \tilde{\theta}^T Y^T r - s\tilde{u}^2 + \tilde{u}\tilde{a}u - \tilde{\theta}^T\Gamma^{-1}(\Gamma Y^T r) - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 - s\tilde{u}^2 + \tilde{u}\tilde{a}u - \frac{1}{\gamma_a}\tilde{a}\dot{\hat{a}}$$

Now, Designing $\dot{\hat{a}} = \gamma_a \tilde{u} u$

Substituting, $\dot{\hat{a}}$ in $\dot{V}$, we get

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 - s\tilde{u}^2 + \tilde{u}\tilde{a}u - \frac{1}{\gamma_a}\gamma_a \tilde{u}\tilde{a}u$$

$$\dot{V}(\zeta) = -\alpha e^2 - \beta r^2 - s\tilde{u}^2$$

Therefore, we can say $\dot{V}(\zeta)$ is Negative Semi Definite. Since, $V(\zeta)$ is radially unbounded, we can say that the equilibrium points are globally stable.

Now, let's try Lasalle,

$\dot{V} = 0$ implies $e, r, \tilde{u} = 0$

$\dot{\hat{\theta}} = \Gamma Y^T r$

since $r = 0$, $\dot{\hat{\theta}} = 0$

$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$, therefore $\dot{\tilde{\theta}} = 0$

$\tilde{\theta} = \theta - \hat{\theta}$

But with these, we cannot imply $\tilde{\theta} = 0$. So, we cannot use Lasalle.

Now, let's try Barbalet's Lemma,

$$\dot{V}(\zeta) \leq -\alpha e^2 - \beta r^2 - s\tilde{u}^2$$

$$\int_0^t \dot{V}(\zeta) \leq \int_0^t -\alpha e^2 - \beta r^2 - s\tilde{u}^2$$

$$V(\zeta(t)) - V(\zeta(0)) \leq -\int_0^t (\alpha e^2 + \beta r^2 + s\tilde{u}^2)$$

Therefore, multiplying by (-1), we get

$$V(\zeta(0)) - V(\zeta(t)) \geq \int_0^t (\alpha e^2 + \beta r^2 + s\tilde{u}^2)$$

Since, system is globally stable. Therefore,

$$V(\zeta(t)), V(\zeta(0)) \in L_\infty$$

so,

$$e, r, \tilde{u} \in L_2$$

Now,

$$r = \dot{e} + \alpha e$$

$$\dot{e} = r - \alpha e$$

since $V(\zeta(t))$ is bounded

$$e, r \in L_\infty$$

Therefore,

$$\dot{e} \in L_\infty$$

so, e is uniformly continuous.

$$\lim_{t \to \infty} e(t) = 0$$

Now,

$$m\dot{r} = Y\theta + \tilde{u} - u_d$$

θ is bounded ($\theta \in L_\infty$) so, $m \in L_\infty$

$$Y = [(\alpha\dot{e} - g\sin(\phi)) \;\; \dot{x} \;\; x]$$

Since, $e \in L_\infty$ and x_d is a constant.

$$x \in L_\infty$$

Since, $\dot{e} \in L_\infty$

$$\dot{x} \in L_\infty$$

Therefore,

$$Y \in L_\infty$$

Now,

$$u_d = e + \beta r + Y\hat{\theta}$$

$$\beta \in L_\infty$$

$$Y \in L_\infty$$

$$r, e = \in L_\infty$$

$$\dot{\hat{\theta}} = \Gamma Y^T r$$

Therefore,

$$\dot{\hat{\theta}} \in L_\infty$$

Since $\Gamma, Y, r \in L_\infty$

so, $u_d \in L_\infty$

$$\dot{r} \in L_\infty$$

Therefore, r is uniformly continuous.

$$\lim_{t \to \infty} r(t) = 0$$

since, $u = u_d - \tilde{u}$

and

$u_d \in L_\infty, \tilde{u} \in L_\infty$

Therefore,

$$u \in L_\infty$$

and since $u \in L_\infty$,

$$\dot{u} \in L_\infty$$

and since $\dot{u}$ and $\dot{u_d} \in L_\infty$

$$\dot{\tilde{u}} \in L_\infty$$

Therefore, $\tilde{u}$ is uniformly continuous.

$$\lim_{t \to \infty} \tilde{u}(t) = 0$$

Therefore, through barablet's lemma, We can say $e, r, \tilde{u}$ is converging to zero as time goes to ∞.

We can also use, Lasalle - Yoshizawa's theorem to prove convergence.

$$\dot{V}(\zeta) \leq -(\alpha e^2 + \beta r^2 + s\tilde{u}^2)$$

We can say,

$$\dot{V}(\zeta) \leq -(W)$$

where, $W = (\alpha e^2 + \beta r^2 + s\tilde{u}^2)$ is a continuous function. Then all solutions of W are uniformly globally bounded and W is a semi definite function. As W goes to zero, $e, r, \tilde{u}$ also converge to zero as time goes to ∞.

P4) b) i) Using the above results, Simulated the dynamics from 100 random values for x_d, $x(t = 0)$, $\hat{\theta}(t = 0)$, and $\hat{a}(t = 0)$ within the domain using the randn function in the matlab keeping

$$\dot{x}(t = 0) = 0$$

$$u(t = 0) = 0$$

Used random values for m, c, k, a, ϕ using the rand function in the matlab for positive random values.

ii) Below are the resulting trajectories of Monte Carlo simulation for $e, r, \tilde{u}, \mu, \tilde{\theta}, \tilde{a}$

The trajectories are as follows $(\alpha, \beta, s = 0.1)$:

e for $t = 50, 100, 500$

r for $t = 50, 100, 500$

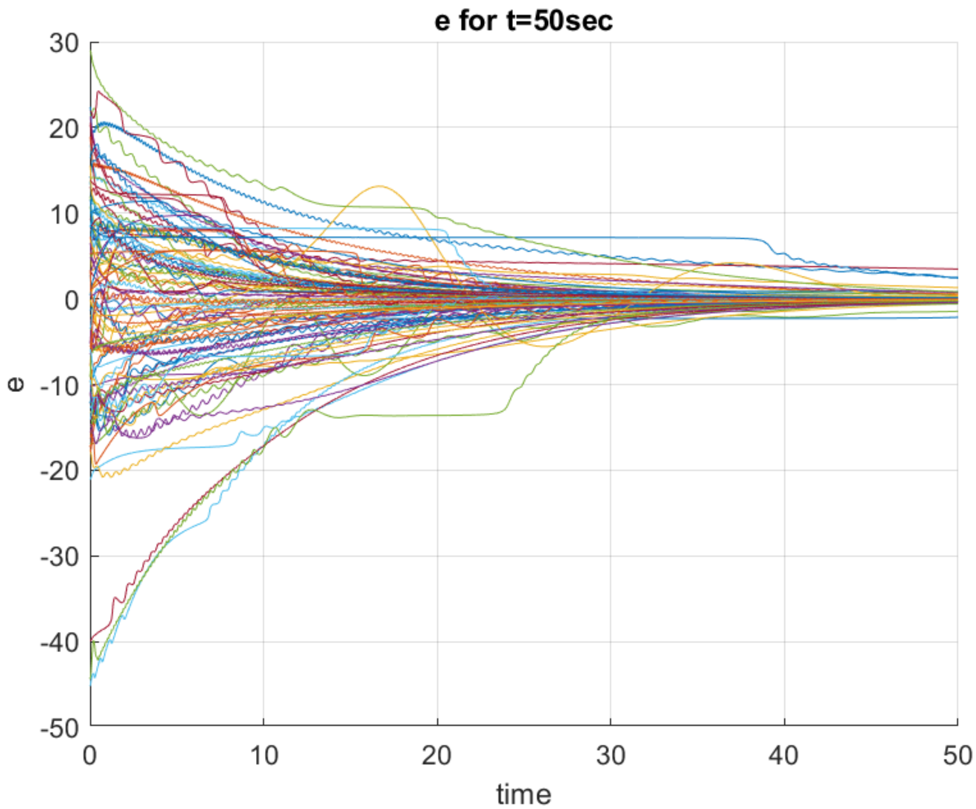
$\tilde{u}$ for $t = 50, 100, 500$

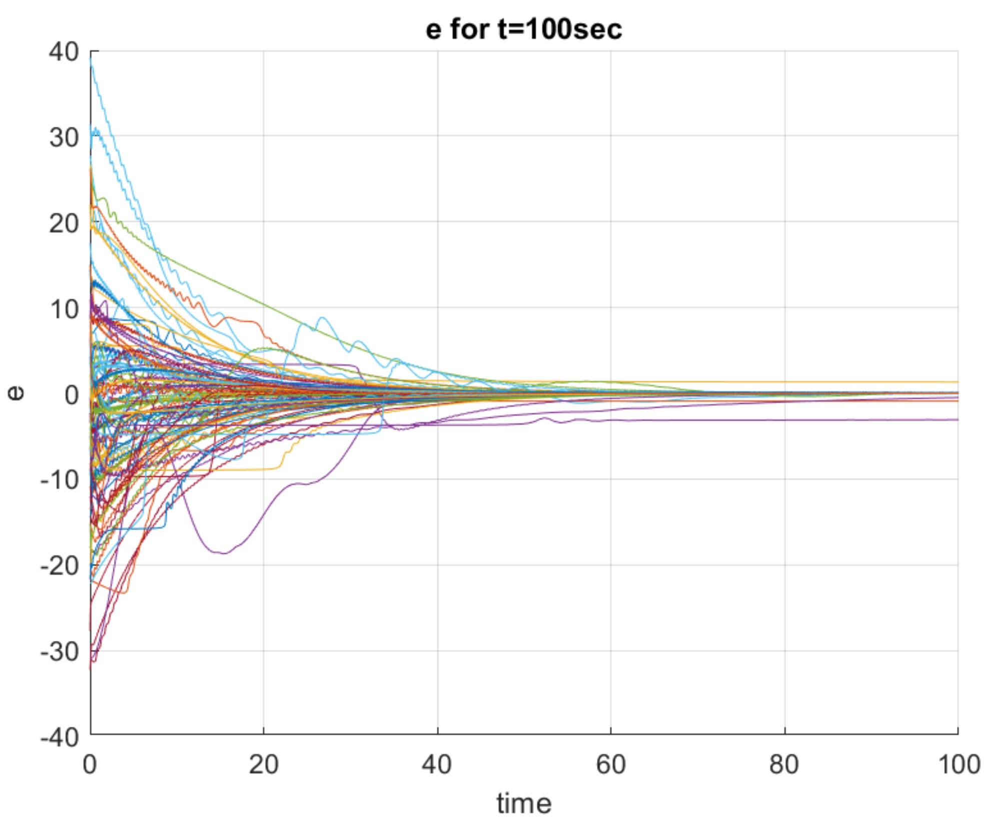
μ for $t = 50, 100, 500$

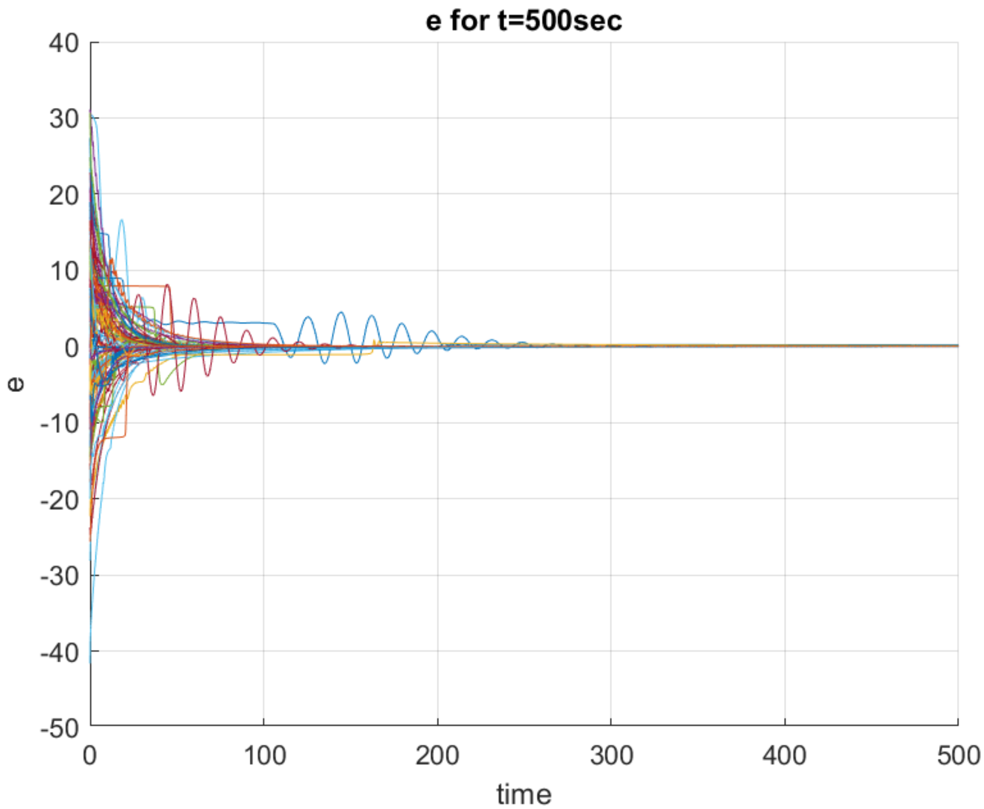
$\tilde{\theta}$ for $t = 50, 100, 500$

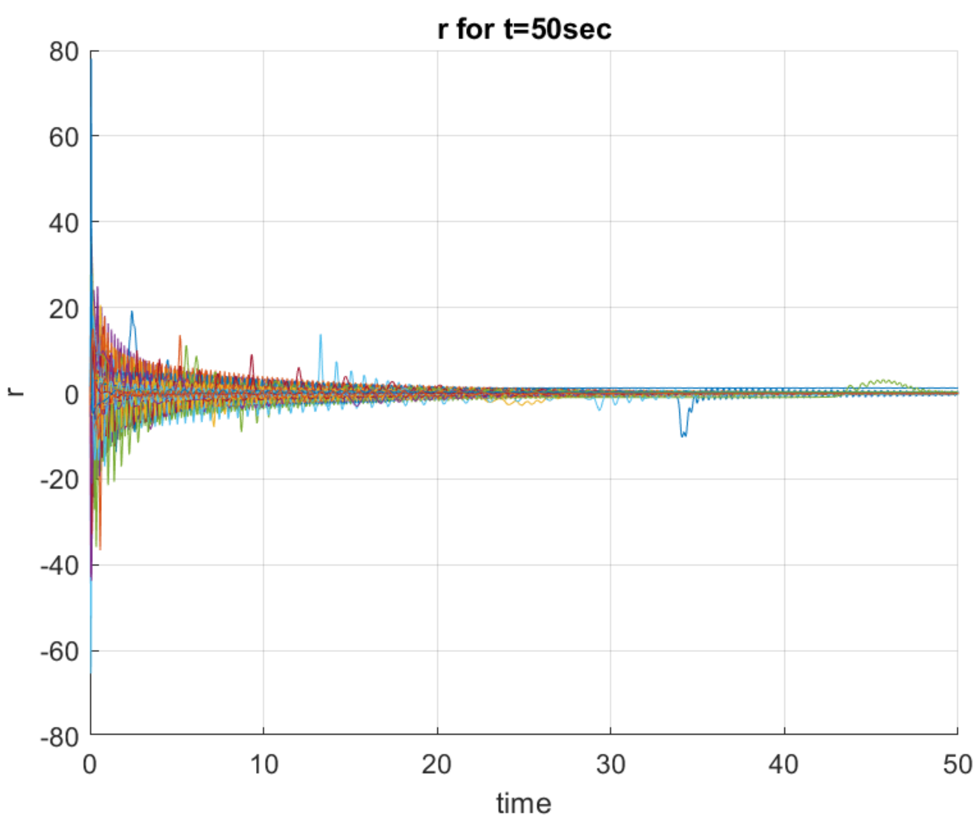
$\tilde{a}$ for $t = 50, 100, 500$

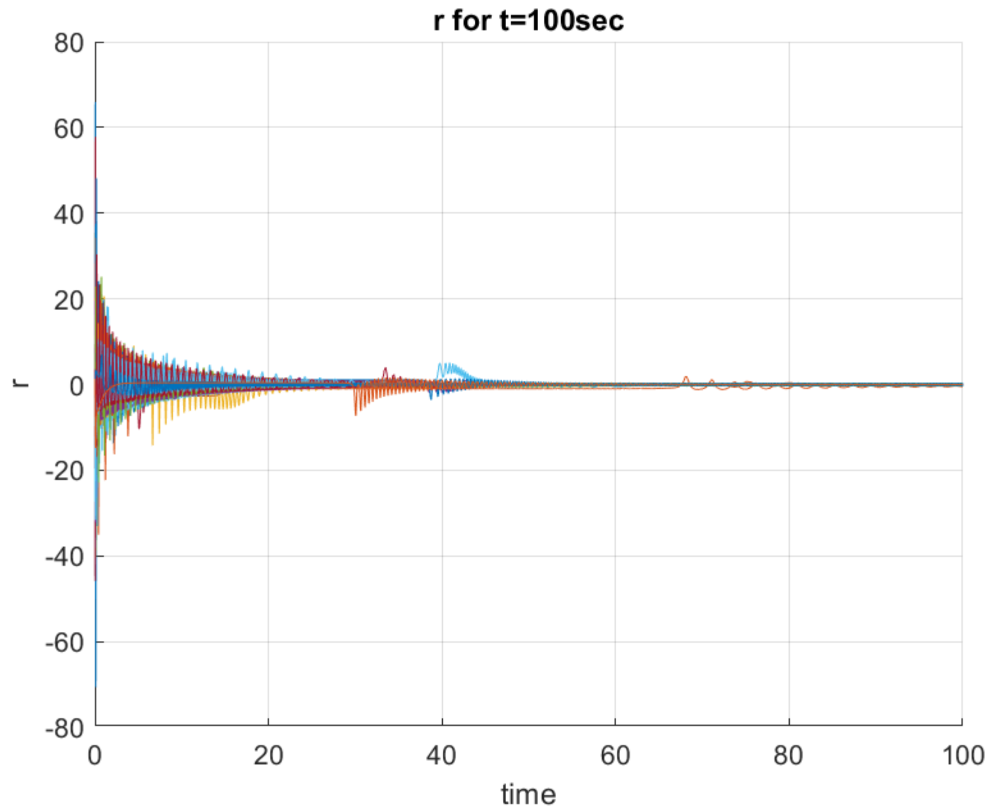
iii) The simulation matches our stability results. We showed convergence for $e, r, \tilde{u}$ using barbalet's lemma and we can see the convergence in simulations as well for $e, r, \tilde{u}$. As it is a regulation problem, $\tilde{\theta}$ is not converging. But, it finds a value which helps us reach our goal. Therefore, we can see that $\mu, \tilde{a}$ are converging and we can conclude the equilibrium points are globally stable with $e, r, \tilde{u}, \mu, \tilde{a}$ converging to zero.


e for t=50sec
30
20
10
0
-10
-20
-30
-40
-50
e
0
10
20
30
40
50
time


e for t=100sec
40
30
20
10
0
-10
-20
-30
-40
e
0
20
40
60
80
100
time


e for t=500sec
40
30
20
10
0
-10
-20
-30
-40
-50
e
0
100
200
300
400
500
time


r for t=50sec
80
60
40
20
0
-20
-40
-60
-80
r
0
10
20
30
40
50
time


r for t=100sec
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-20
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r
0
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40
60
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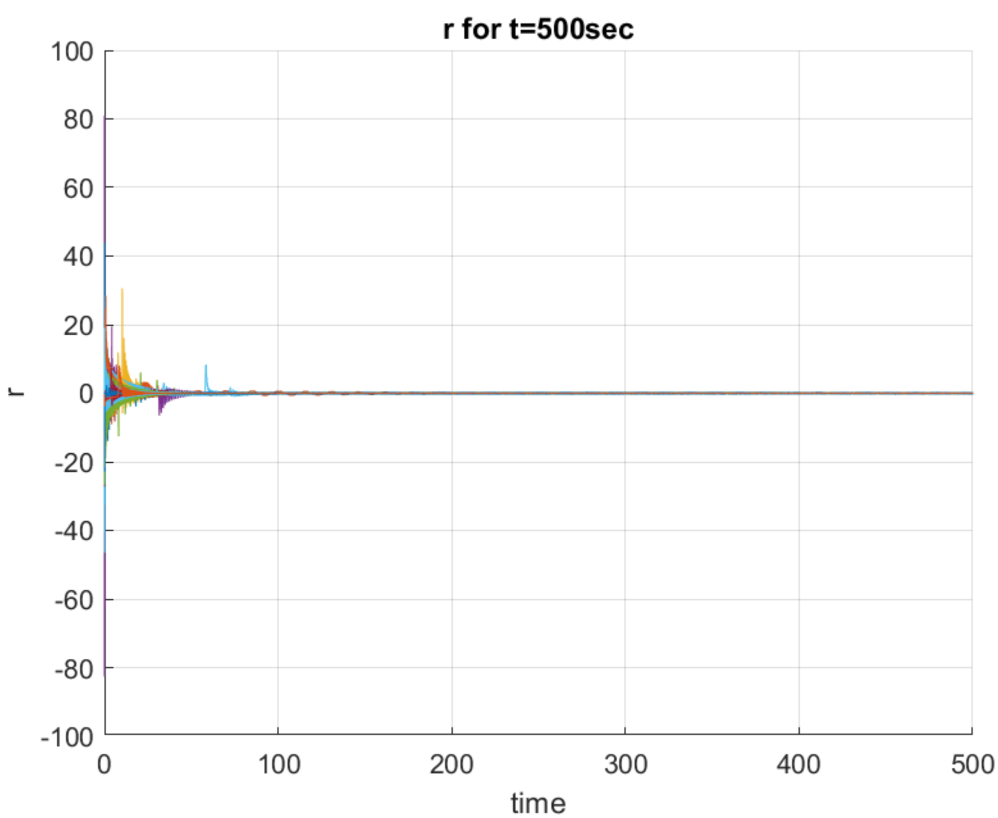
time


r for t=500sec
100
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40
20
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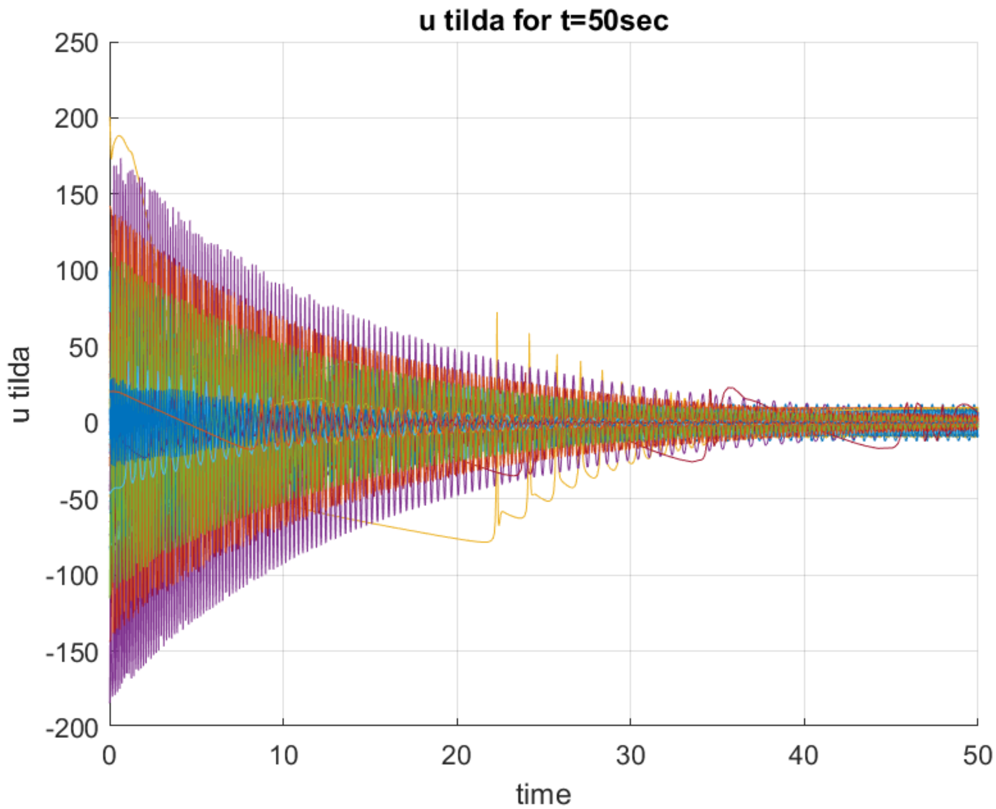
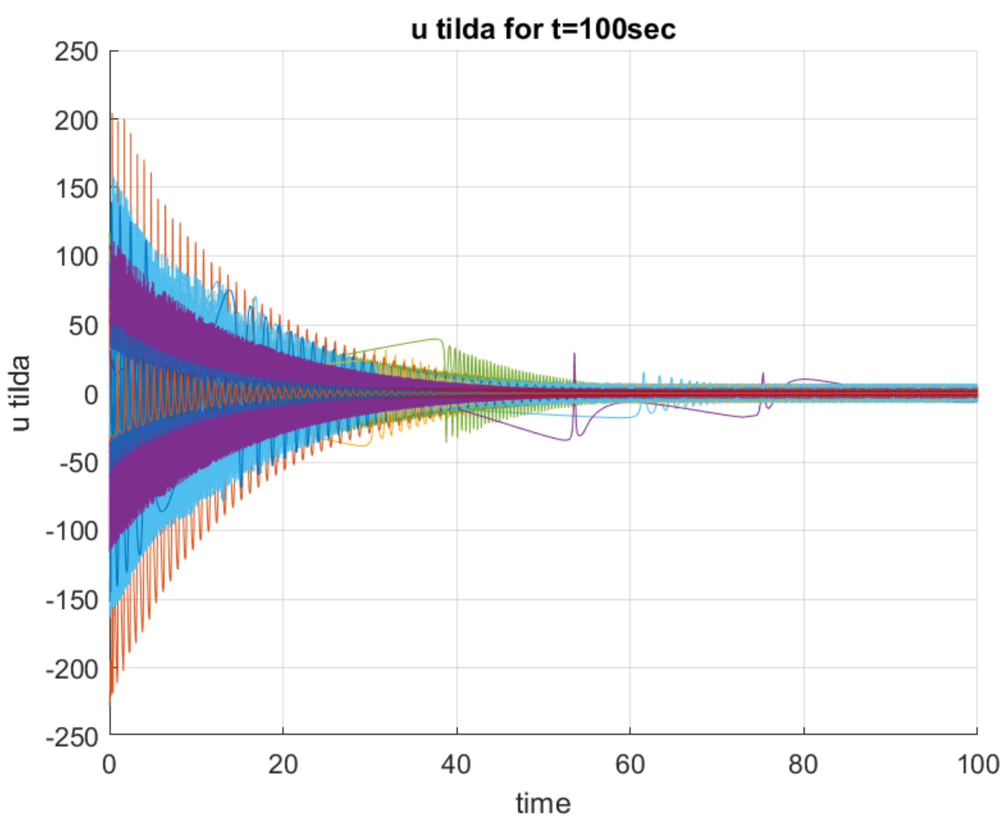
-40
-60
-80
-100
r
0
100
200
300
400
500
time


u tilda for t=50sec
250
200
150
100
50
0
-50
-100
-150
-200
u tilda
0
10
20
30
40
50
time


u tilda for t=100sec
250
200
150
100
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-150
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-250
u tilda
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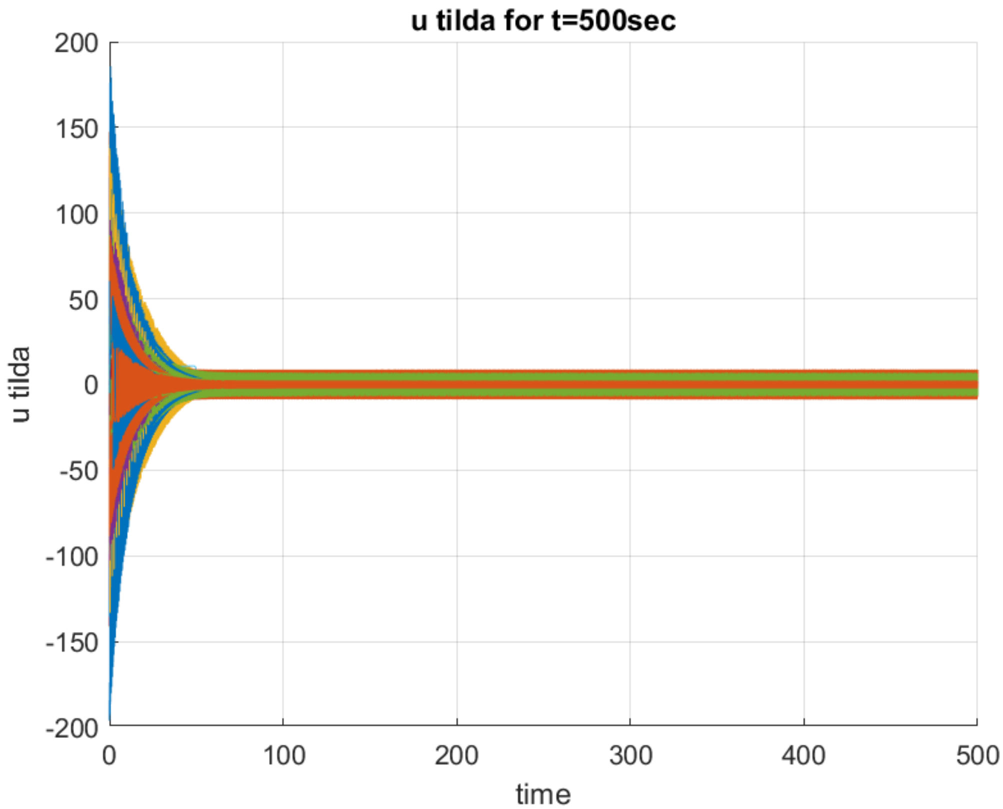
60
80
100
time


u tilda for t=500sec
200
150
100
50
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-50
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-150
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100
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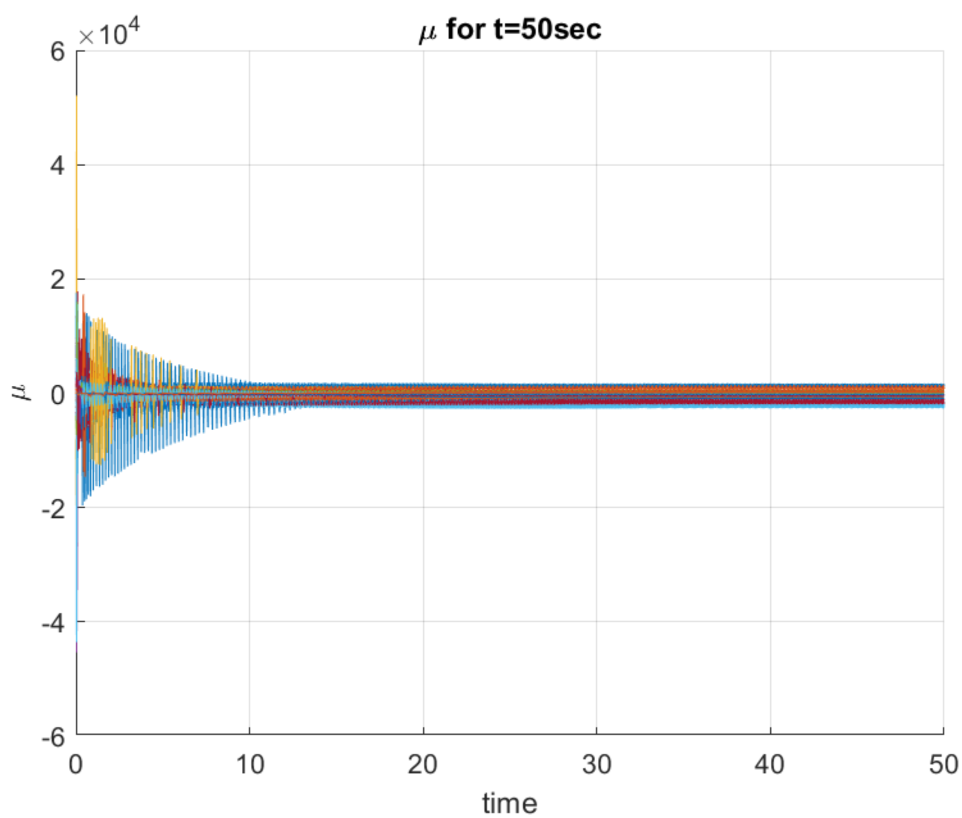
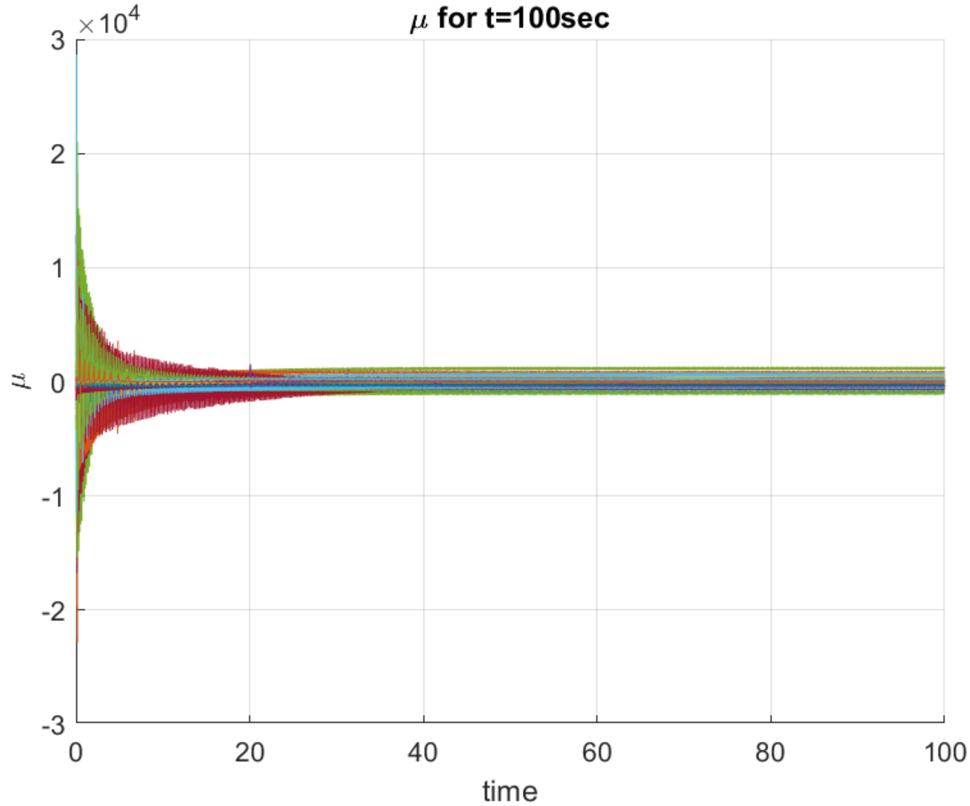
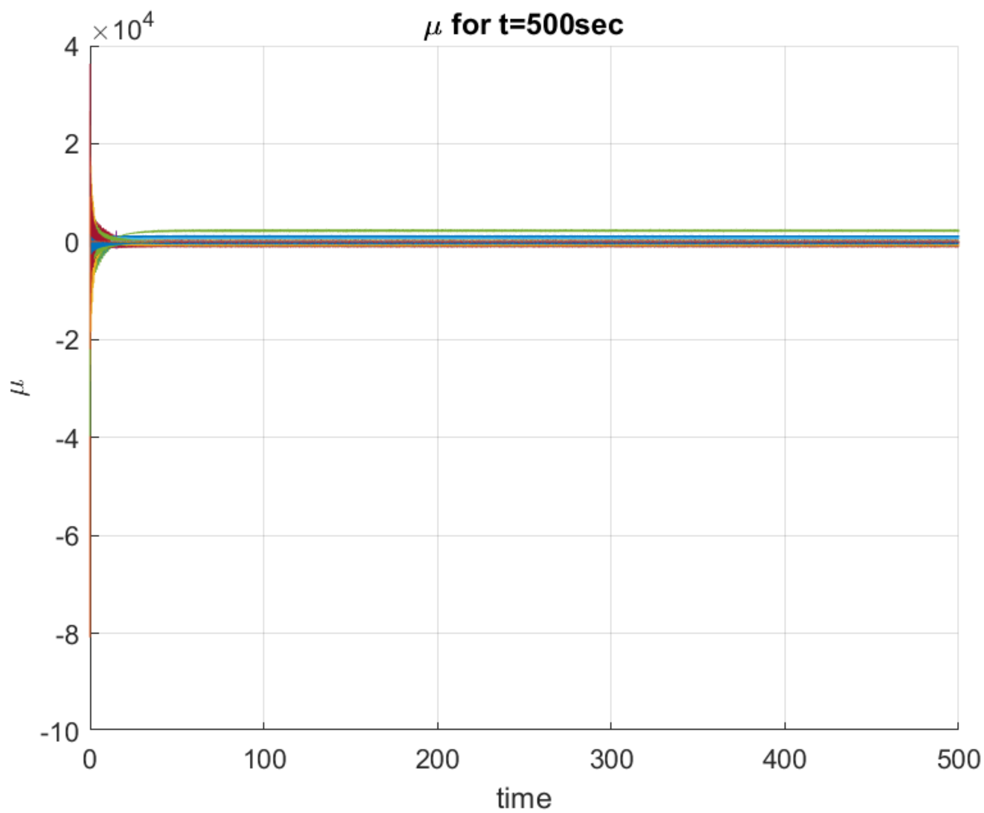
300
400
500
time
u tilda


μ for t=50sec
×10^4
6
4
2
0
-2
-4
-6
0
10
20
30
40
50
time
μ


μ for t=100sec
$\times 10^4$
3
2
1
0
-1
-2
-3
μ
0
20
40
60
80
100
time


μ for t=500sec
$\times 10^4$
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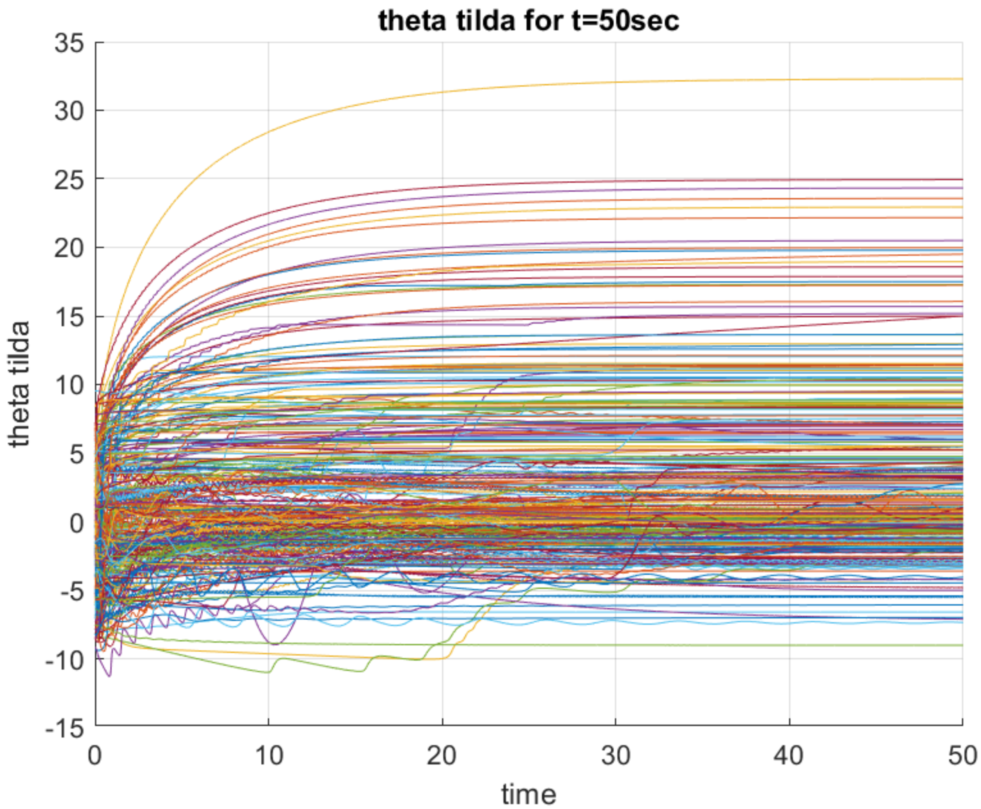
2
0
-2
-4
-6
-8
-10
μ
0
100
200
300
400
500
time


theta tilda for t=50sec
theta tilda
time
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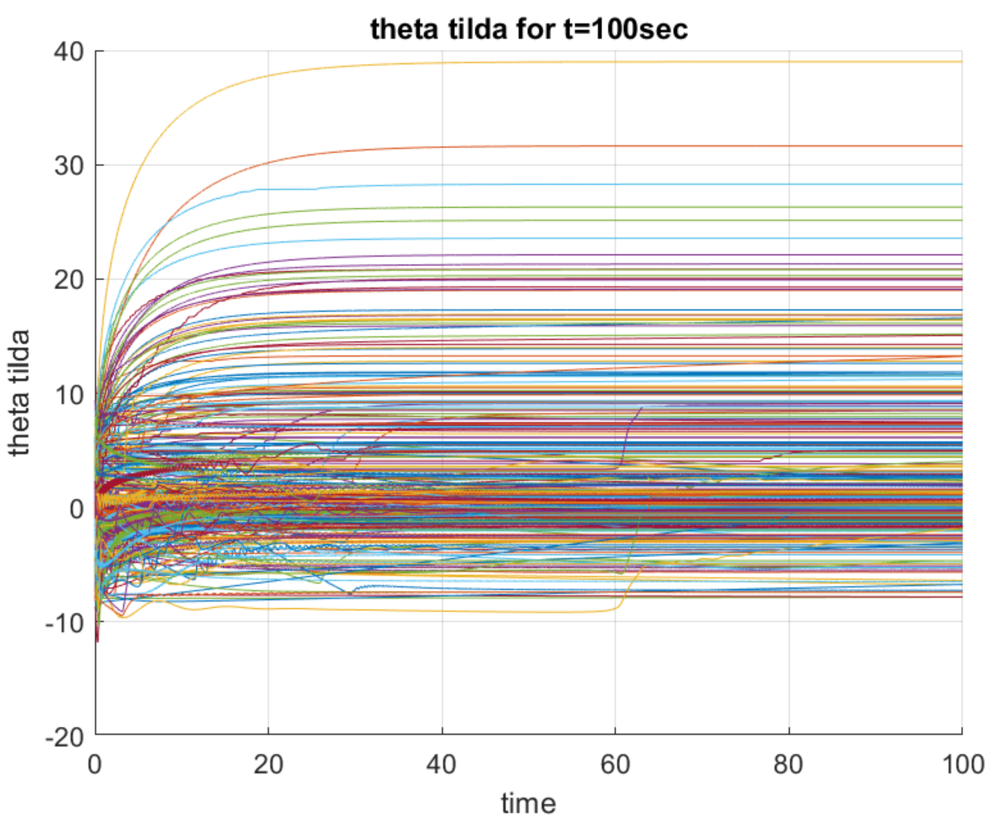
theta tilda for t=100sec
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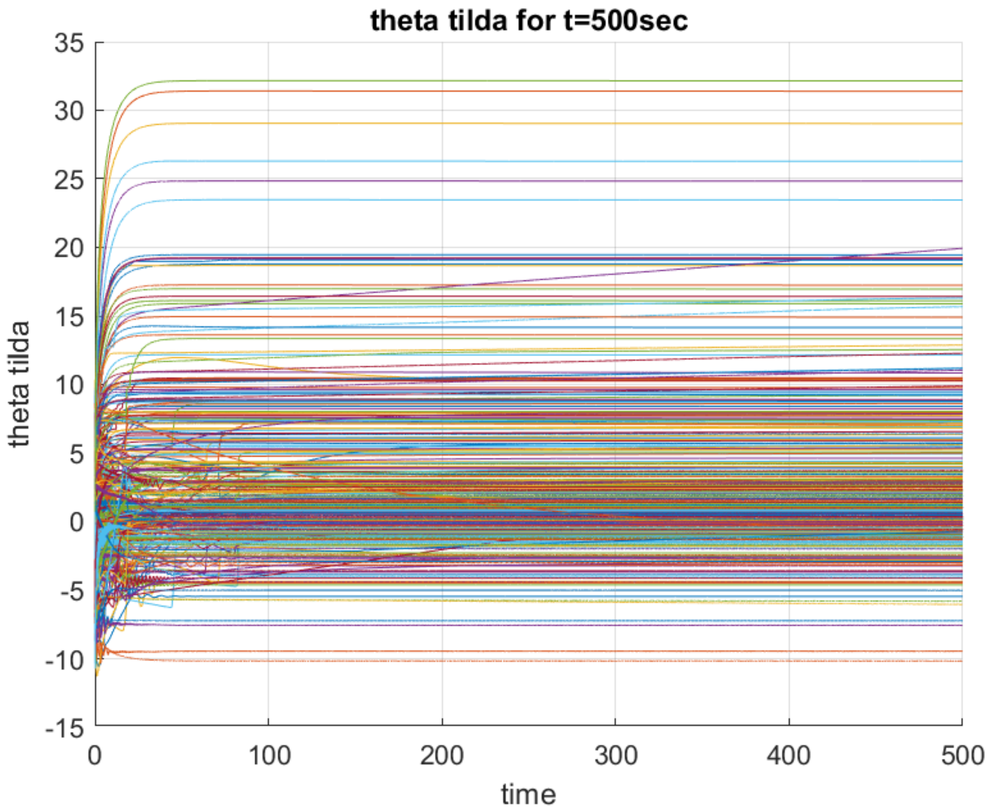
theta tilda for t=500sec
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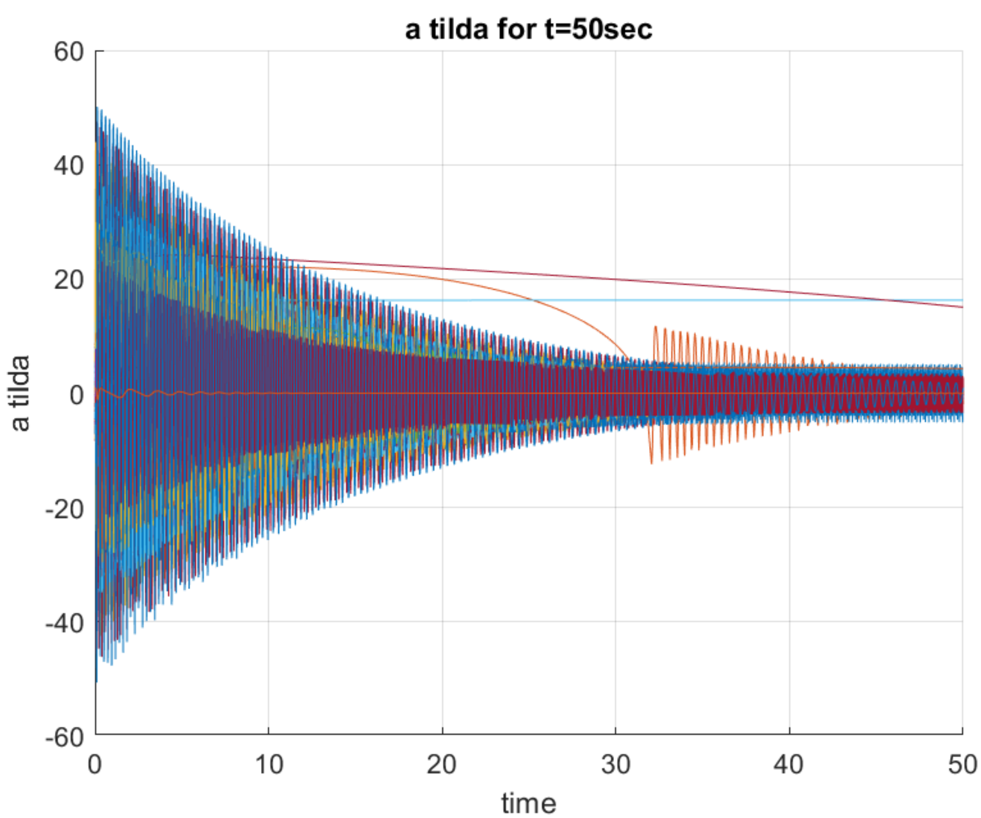
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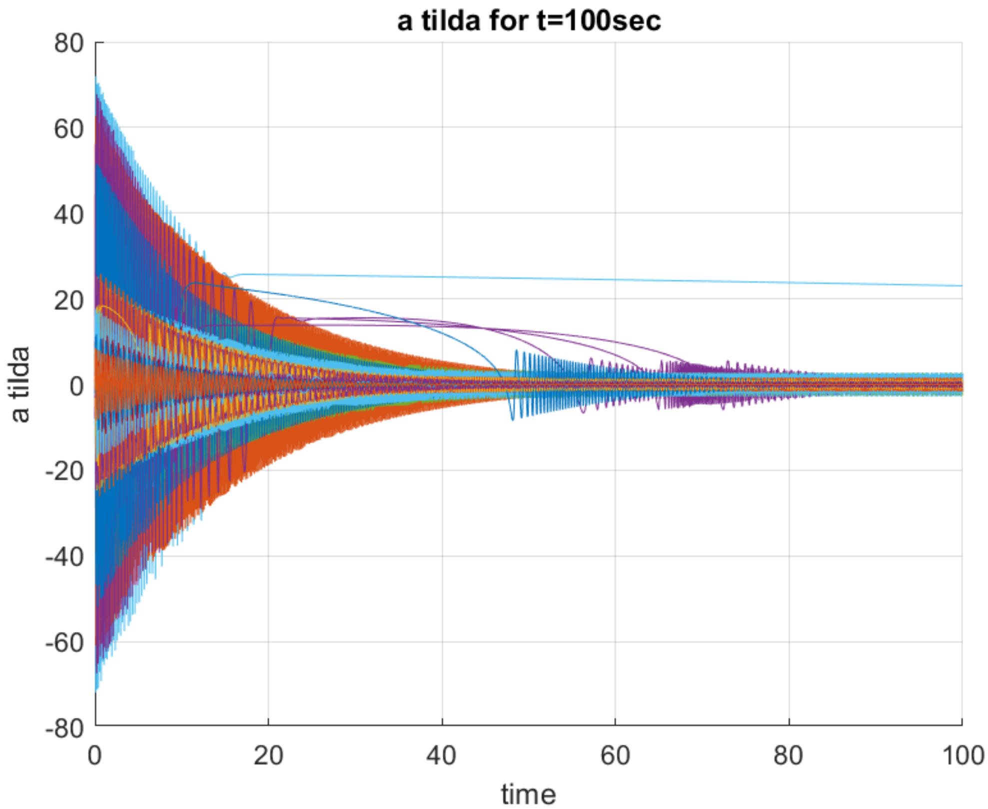
a tilda for t=100sec
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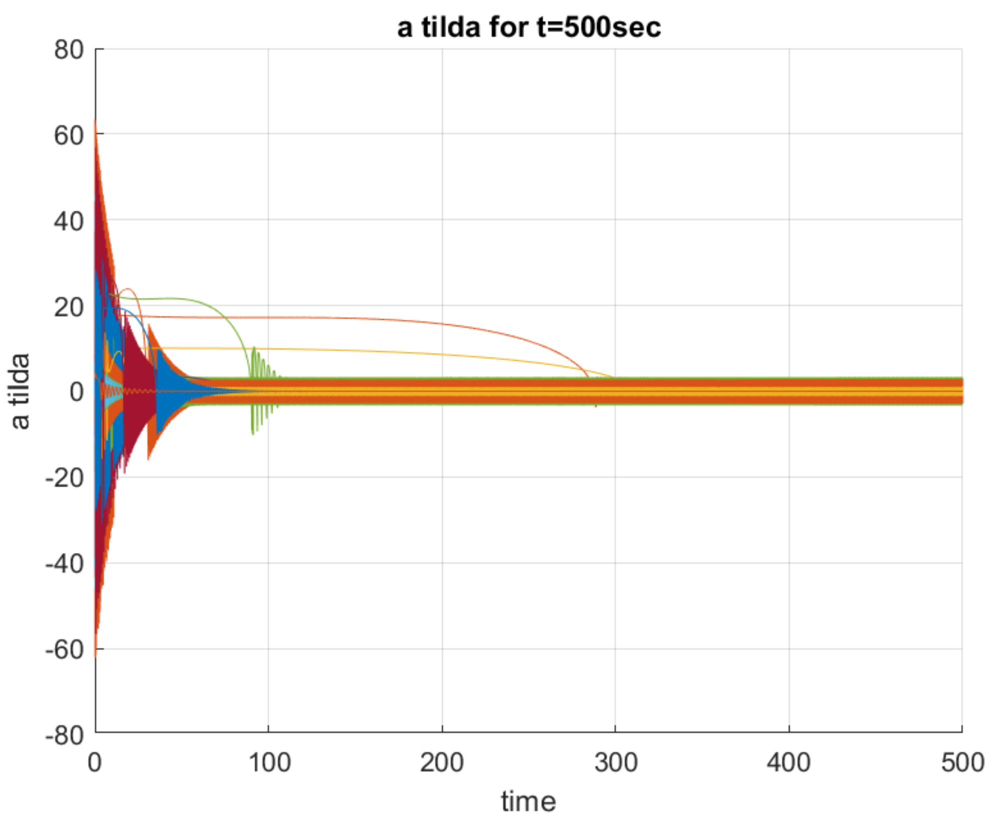
a tilda
time


a tilda for t=500sec
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a tilda
time