Kaushik Jadav
47017653

Non Linear Problem

Problem : Switched Systems Approach to Tracking Control of Vehicles with Intermittent Feedback

Consider the following switched dynamic system of a vehicle moving through a field with intermittent feedback of the position

$$m\ddot{\eta} = -c\dot{\eta}(t) + \tau(t) + d(t) \tag{1}$$

Now, for better understanding of the problem, we will consider the problem from 3 different aspects.

1) No uncertainty, Disturbance exist and feedback is available all the time.

2) No uncertainty, Disturbance exist and feedback is NOT available all the time.

3) Uncertainty (Real parameters are not known) and disturbance exist and feedback is NOT available all the time. In this approach, we will consider stable region(σ) and unstable region(q) and assume we have intermittent feedback availability.

Approach 1

Let the tracking error be,

$$e = \eta_d - \eta \tag{2}$$

$$\dot{e} = \dot{\eta_d} - \dot{\eta} \tag{3}$$

$$\ddot{e} = \ddot{\eta_d} - \ddot{\eta} \tag{4}$$

Multiplying throughout by m, we get,

$$m\ddot{e} = m\ddot{\eta}_d - m\ddot{\eta} \tag{5}$$

Now, substituting the value of $m\ddot{\eta}$ from equation (1), we get,

$$m\ddot{e} = m\ddot{\eta}_d - (-c\dot{\eta}(t) + \tau(t) + d(t)) \tag{6}$$

$$m\ddot{e} = m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t)) \tag{7}$$

Let the filter tracking error be,

$$r = \dot{e} + \alpha e \tag{8}$$

$$\dot{r} = \ddot{e} + \alpha\dot{e} \tag{9}$$

Multiplying throughout by m, we get,

$$m\dot{r} = m\ddot{e} + m\alpha\dot{e} \tag{10}$$

Now, substituting the value of $m\ddot{e}$, we get,

$$m\dot{r} = m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) + m\alpha\dot{e} \tag{11}$$

Let the lyapunov candidate be,

$$V(\zeta(t)) = \frac{1}{2}e^T e + \frac{1}{2}r^T mr \tag{12}$$

Therefore, the time derivative of lyapunov candidate is,

$$\dot{V}(\zeta(t)) = e^T\dot{e} + r^T m\dot{r} \tag{13}$$

Now, substituting the values of $\dot{e}$ and $m\dot{r}$, we get

$$\dot{V}(\zeta(t)) = e^T(r - \alpha e) + r^T(m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) + m\alpha\dot{e}) \tag{14}$$

$$\dot{V}(\zeta(t)) = e^T r - \alpha\|e\|^2 + r^T(m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) + m\alpha\dot{e}) \tag{15}$$

Now, designing τ,

$$\tau = m\ddot{\eta}_d + c\dot{\eta}(t) + m\alpha\dot{e} + e + \beta r + \beta_d sgn(r) \tag{16}$$

Now, substituting the value of τ, we get

$$\dot{V}(\zeta(t)) = e^T r - \alpha\|e\|^2 + r^T(m\ddot{\eta}_d + c\dot{\eta}(t) - (m\ddot{\eta}_d + c\dot{\eta}(t) + m\alpha\dot{e} + e + \beta r + \beta_d sgn(r)) - d(t) + m\alpha\dot{e}) \tag{17}$$

$$\dot{V}(\zeta(t)) = e^T r - \alpha\|e\|^2 + r^T(m\ddot{\eta}_d + c\dot{\eta}(t) - m\ddot{\eta}_d - c\dot{\eta}(t) - m\alpha\dot{e} - e - \beta r - \beta_d sgn(r) - d(t) + m\alpha\dot{e}) \tag{18}$$

Now, simplifying and cancelling some terms, we get

$$\dot{V}(\zeta(t)) = -\alpha\|e\|^2 - \beta\|r\|^2 - r^T\beta_d sgn(r) - r^T d(t) \tag{19}$$

Therefore, we can take bounds here

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 - \beta_d\|r\| + \bar{d}\|r\| \tag{20}$$

So, if we take $\beta_d > \bar{d}$, then that term will always be less than 0.

Therefore,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 \tag{21}$$

Therefore, we can say that the time derivative of lyapunov candidate is Negative definite.

$$\dot{V}(\zeta(t)) \leq -2\,[\,\alpha,\ \beta\,]\,V(\zeta(t)) \tag{22}$$

$$\dot{V}(\zeta(t)) \leq -2\ K\ V(\zeta(t)) \tag{23}$$

Hence, since the time derivative of lyapunov candidate is Negative Definite and also the lyapunov function is radially unbounded. We can conclude that we have gloabally exponential tracking(GET) for approach 1.

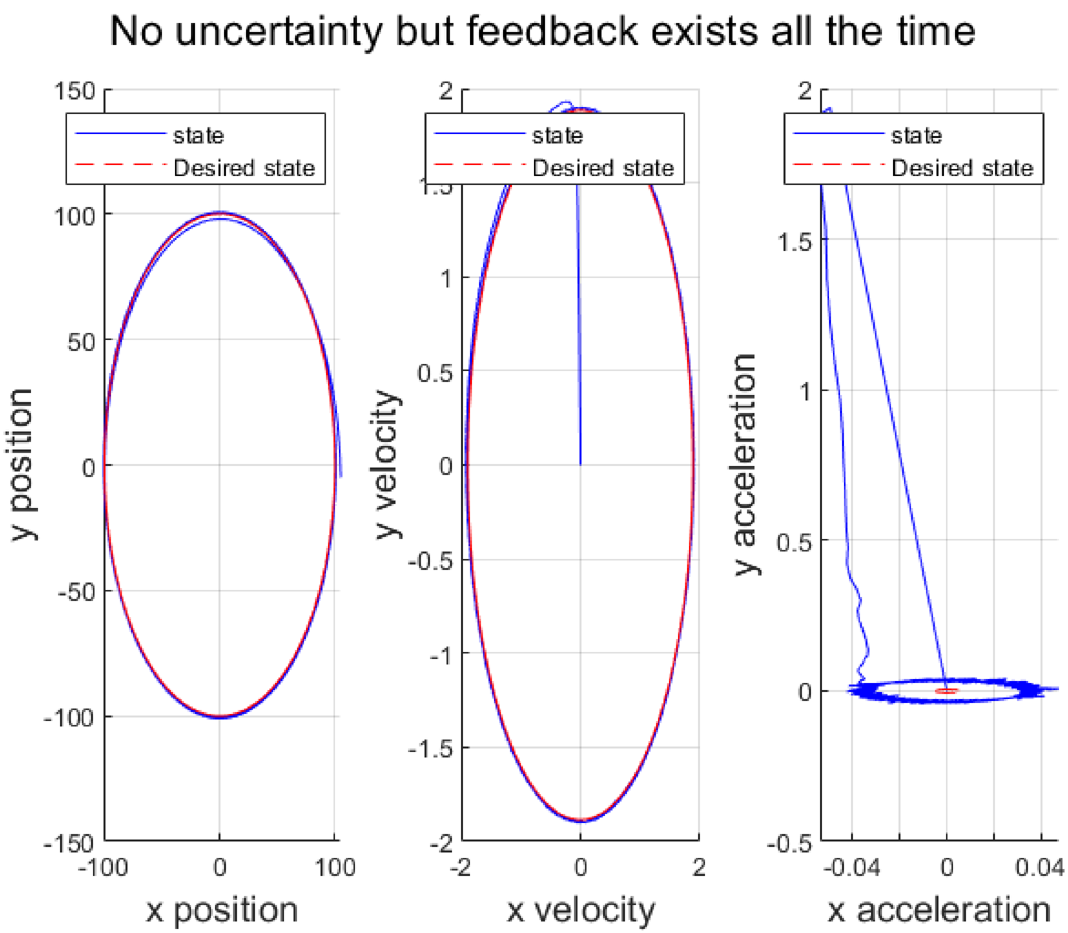

Figure 1: State and the Desired state trajectories when feedback is available all the time and the parameters are known.

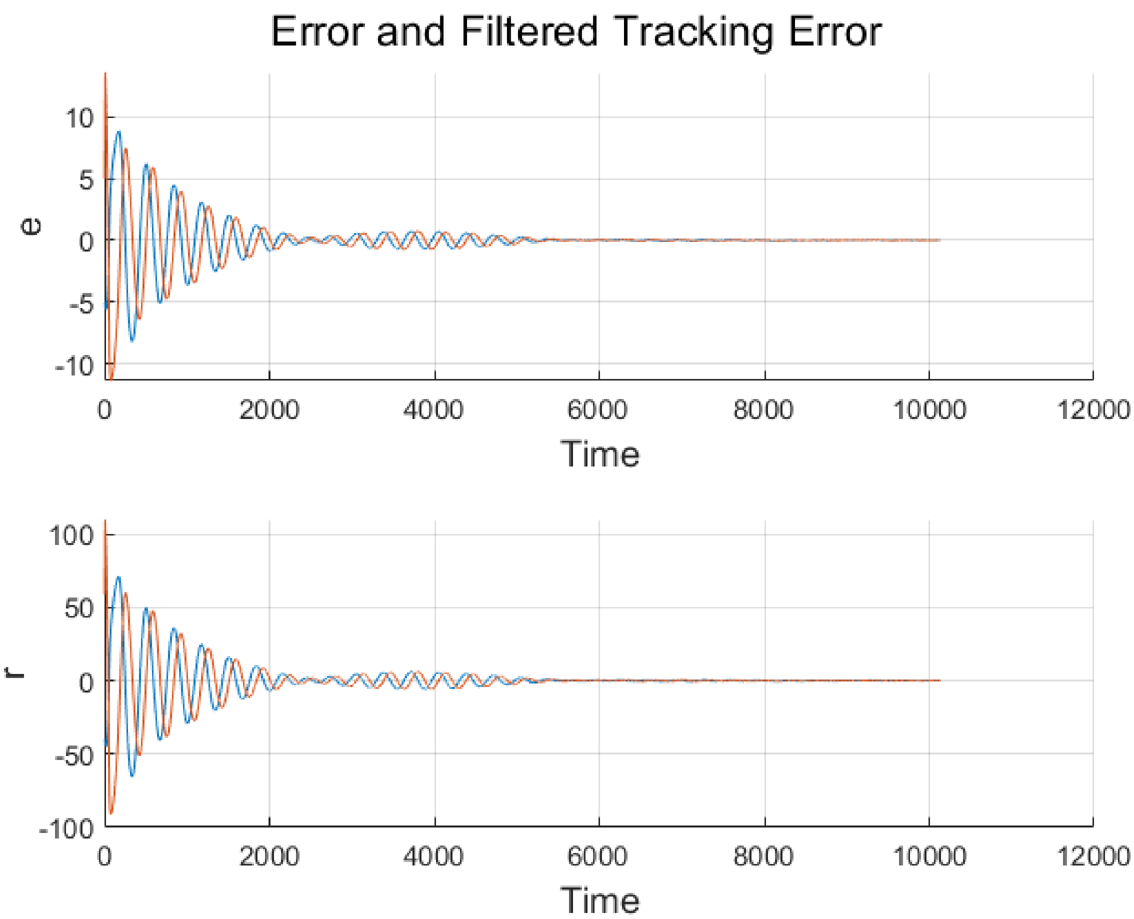

Figure 2: Error and Filtered Tracking Error.

Approach 2

No uncertainty, Disturbance exist and feedback is NOT available all the time. Therefore, state estimation is required.

Let the state estimation be,

$$\tilde{\eta} = \eta - \hat{\eta} \tag{24}$$

$$\dot{\tilde{\eta}} = \dot{\eta} - \dot{\hat{\eta}} \tag{25}$$

$$\ddot{\tilde{\eta}} = \ddot{\eta} - \ddot{\hat{\eta}} \tag{26}$$

Now, multiplying throughout by m, we get

$$m\ddot{\tilde{\eta}} = m\ddot{\eta} - m\ddot{\hat{\eta}} \tag{27}$$

Now, substituting the value of $m\ddot{\eta}$ from equation (1), we get

$$m\ddot{\tilde{\eta}} = (-c\dot{\eta}(t) + \tau(t) + d(t)) - m\ddot{\hat{\eta}} \tag{28}$$

$$m\ddot{\tilde{\eta}} = -c\dot{\eta}(t) + \tau(t) + d(t) - m\ddot{\hat{\eta}} \tag{29}$$

Let the lyapunov candidate be,

$$V(\zeta(t)) = \frac{1}{2}e^T e + \frac{1}{2}r^T m r + \frac{1}{2}\dot{\tilde{\eta}}^T m\dot{\tilde{\eta}} \tag{30}$$

Therefore, the time derivative of lyapunov candidate is,

$$\dot{V}(\zeta(t)) = e^T\dot{e} + r^T m\dot{r} + \dot{\tilde{\eta}}^T\ m\ddot{\tilde{\eta}} \tag{31}$$

Now, substituting the values of $\dot{e}$, $m\dot{r}$ and $m\ddot{\tilde{\eta}}$ we get

$$\dot{V}(\zeta(t)) = e^T(r-\alpha e)+r^T(m\ddot{\eta}_d+c\dot{\eta}(t)-\tau(t)-d(t)+m\alpha\dot{e})+\dot{\tilde{\eta}}^T(-c\dot{\eta}(t)+\tau(t)+d(t)-m\ddot{\tilde{\eta}}) \tag{32}$$

$$\dot{V}(\zeta(t)) = e^T r-\alpha\|e\|^2+r^T(m\ddot{\eta}_d+c\dot{\eta}(t)-\tau(t)-d(t)+m\alpha\dot{e})+\dot{\tilde{\eta}}^T(-c\dot{\eta}(t)+\tau(t)+d(t)-m\ddot{\tilde{\eta}}) \tag{33}$$

Now, designing τ,

$$\tau = m\ddot{\eta}_d + c\dot{\eta}(t) + m\alpha\dot{e} + e + \beta r + \beta_d sgn(r) \tag{34}$$

$$\begin{aligned}\dot{V}(\zeta(t)) = e^T r-\alpha\|e\|^2+r^T(m\ddot{\eta}_d+c\dot{\eta}(t)-(m\ddot{\eta}_d+c\dot{\eta}(t)+m\alpha\dot{e}+e+\beta r+\beta_d sgn(r))-d(t)+m\alpha\dot{e})+\\ \dot{\tilde{\eta}}^T(-c\dot{\eta}(t)+\tau(t)+d(t)-m\ddot{\tilde{\eta}})\end{aligned}$$

Now, simplifying and cancelling some terms, we get

$$\dot{V}(\zeta(t)) = -\alpha\|e\|^2 - \beta\|r\|^2 - r^T\beta_d sgn(r) - r^T d(t) + \dot{\tilde{\eta}}^T(-c\dot{\eta}(t) + \tau(t) + d(t) - m\ddot{\tilde{\eta}}) \tag{35}$$

Now designing $\ddot{\tilde{\eta}}$,

$$\ddot{\tilde{\eta}} = \frac{1}{m}(-c\dot{\eta}(t) + \tau(t) + \beta_s sgn(\dot{\tilde{\eta}}) + s\dot{\tilde{\eta}}) \tag{36}$$

where s is some positive constant.

$$\dot{V}(\zeta(t)) = -\alpha\|e\|^2 - \beta\|r\|^2 - r^T\beta_d sgn(r) - r^T d(t) + \dot{\tilde{\eta}}^T(-c\dot{\eta}(t) + \tau(t) + d(t) - m(\frac{1}{m})(-c\dot{\eta}(t) + \tau(t) + \beta_s sgn(\dot{\tilde{\eta}}) + s\dot{\tilde{\eta}} \tag{37}$$

$$\dot{V}(\zeta(t)) = -\alpha\|e\|^2 - \beta\|r\|^2 - r^T\beta_d sgn(r) - r^T d(t) + \dot{\tilde{\eta}}^T(-c\dot{\eta}(t) + \tau(t) + d(t) + c\dot{\eta}(t) - \tau(t) - \beta_s sgn(\dot{\tilde{\eta}}) - s\dot{\tilde{\eta}})) \tag{38}$$

$$\dot{V}(\zeta(t)) = -\alpha\|e\|^2 - \beta\|r\|^2 - r^T\beta_d sgn(r) - r^T d(t) + \dot{\tilde{\eta}}^T d(t) - \dot{\tilde{\eta}}^T\beta_s sgn(\dot{\tilde{\eta}}) - s\|\dot{\tilde{\eta}}\|^2 \tag{39}$$

Therefore, we can take bounds here

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 - s\|\dot{\tilde{\eta}}\|^2 - \beta_d\|r\| + \bar{d}\|r\| - \beta_s\|\dot{\tilde{\eta}}\| + \bar{d}\|\dot{\tilde{\eta}}\| \tag{40}$$

So, if we take $\beta_d, \beta_s > \bar{d}$, then both the terms will always be less than 0.

Therefore,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 - s\|\dot{\tilde{\eta}}\|^2 \tag{41}$$

Therefore, we can say that the time derivative of lyapunov candidate is Negative definite.

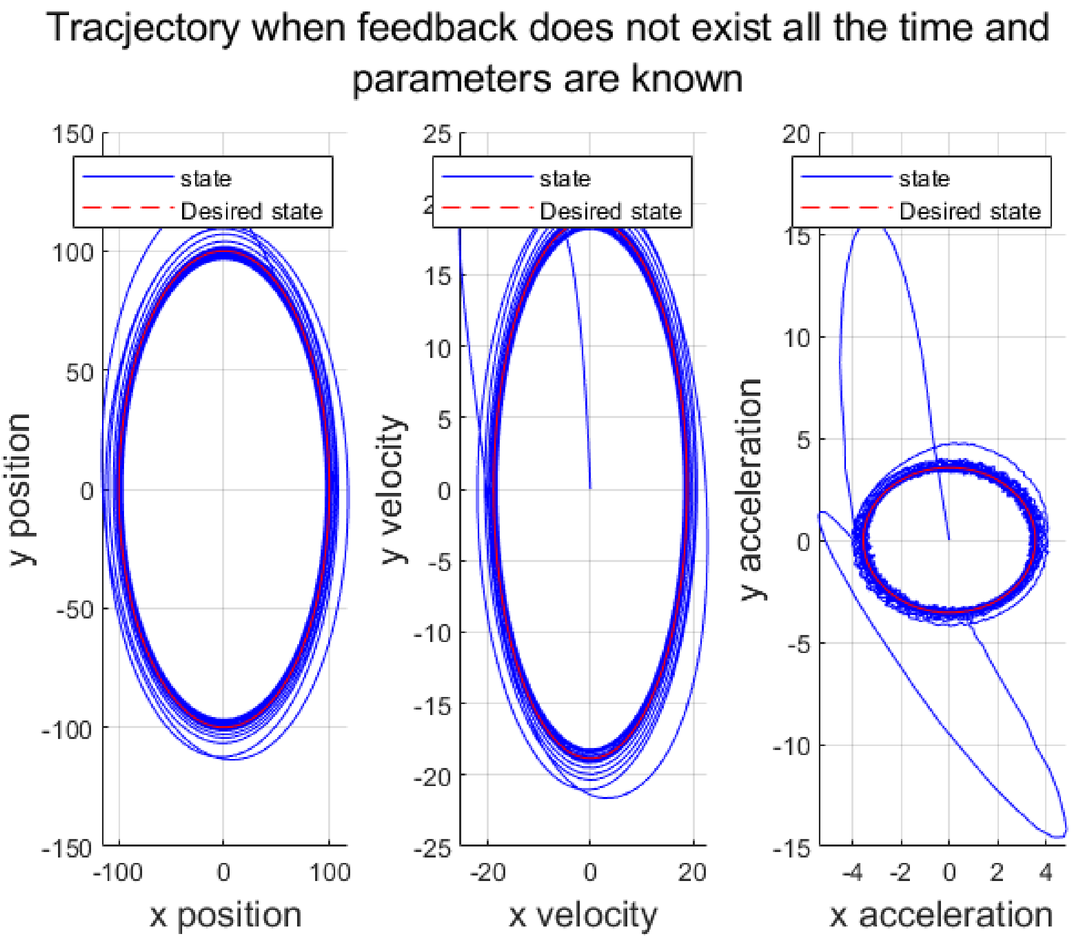

Figure 3: State and the Desired state trajectories when feedback is not available all the time and the parameters are known.

$$\dot{V}(\zeta(t)) \leq -2\,[\,\alpha,\ \beta\, s\,]\,V(\zeta(t)) \tag{42}$$

$$\dot{V}(\zeta(t)) \leq -2\,K\,V(\zeta(t)) \tag{43}$$

Hence, since the time derivative of lyapunov candidate is Negative Definite and also the lyapunov function is radially unbounded. We can conclude that we have global exponential tracking for approach 2.

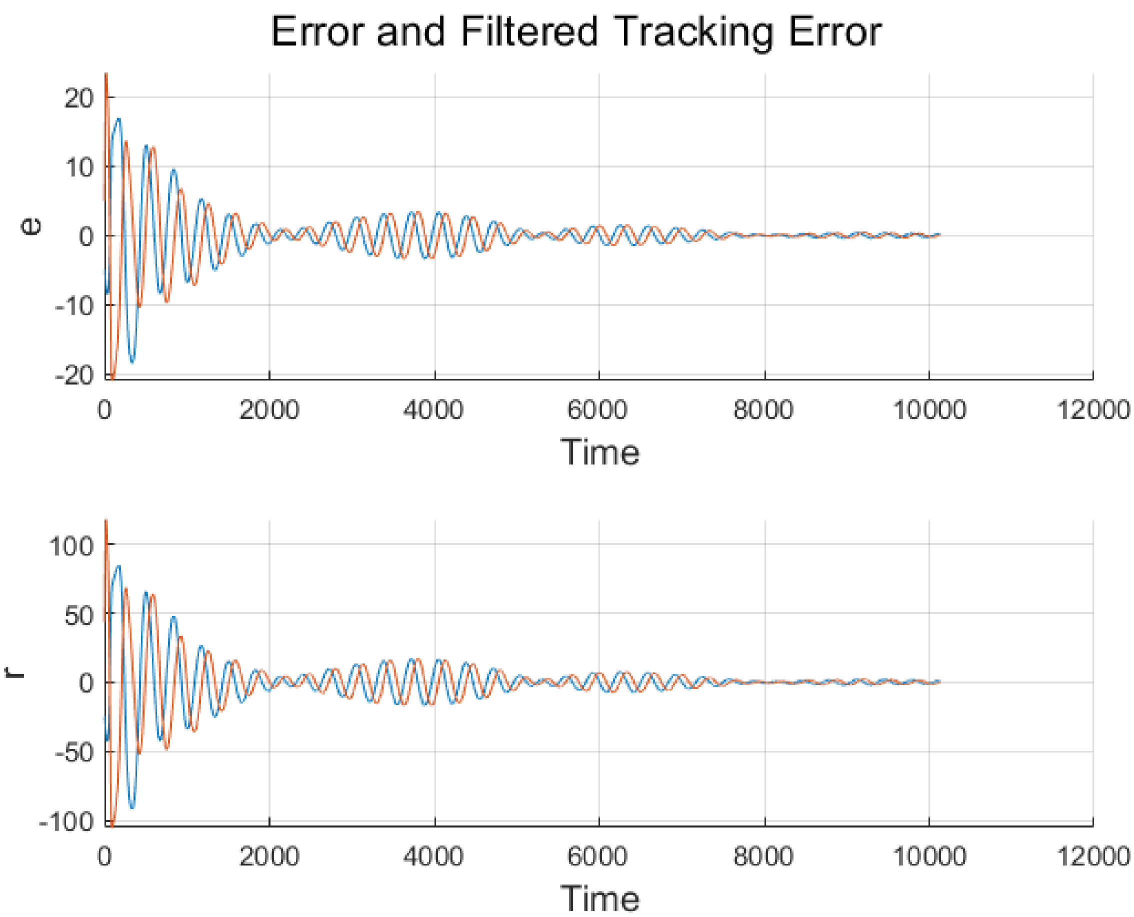

Figure 4: Error and Filtered Tracking Error.

Approach 3

Parameters are unknown(Uncertainty) and feedback is not available all the time.

Let the estimated tracking error be,

$$\hat{e} = \eta_d - \hat{\eta} \tag{44}$$

$$\dot{\hat{e}} = \dot{\eta_d} - \dot{\hat{\eta}} \tag{45}$$

which can also be written as,

$$\dot{\hat{e}} = \dot{\eta_d} - \dot{\eta_m} \tag{46}$$

$$\ddot{\hat{e}} = \ddot{\eta_d} - \ddot{\eta_m} \tag{47}$$

$$\ddot{\hat{e}} = \ddot{\eta_d} - \ddot{\eta}(t) - \omega_{\ddot{\eta}}(t) \tag{48}$$

Multiplying throughout by m, we get

$$m\ddot{\hat{e}} = m\ddot{\eta_d} - m\ddot{\eta}(t) - m\omega_{\ddot{\eta}}(t) \tag{49}$$

Now, substituting the value of $m\ddot{\eta}$ from equation (1), we get

$$m\ddot{\hat{e}} = m\ddot{\eta_d} - (-c\dot{\eta}(t) + \tau(t) + d(t)) - m\omega_{\ddot{\eta}}(t) \tag{50}$$

$$m\ddot{\hat{e}} = m\ddot{\eta_d} + c\dot{\eta}(t) - \tau(t) - d(t) - m\omega_{\ddot{\eta}}(t) \tag{51}$$

Let the state estimation be,

$$\tilde{\eta} = \eta - \hat{\eta} \tag{52}$$

$$\dot{\tilde{\eta}} = \dot{\eta} - \dot{\hat{\eta}} \tag{53}$$

which can also be written as,

$$\dot{\tilde{\eta}} = \dot{\eta} - \dot{\eta}_m \tag{54}$$

$$\ddot{\tilde{\eta}} = \ddot{\eta} - \ddot{\eta}_m \tag{55}$$

Now, multiplying throughout by m, we get

$$m\ddot{\tilde{\eta}} = m\ddot{\eta} - m\ddot{\eta}_m \tag{56}$$

Now, substituting the value of $m\ddot{\eta}$ from equation (1), we get

$$m\ddot{\tilde{\eta}} = (-c\dot{\eta}(t) + \tau(t) + d(t)) - m\ddot{\eta}_m \tag{57}$$

$$m\ddot{\tilde{\eta}} = -c\dot{\eta}(t) + \tau(t) + d(t) - m\ddot{\eta}_m \tag{58}$$

Now, let the estimated filtered tracking error be,

$$\hat{r} = \dot{\hat{e}} + \alpha\hat{e} \tag{59}$$

$$\dot{\hat{r}} = \ddot{\hat{e}} + \alpha\dot{\hat{e}} \tag{60}$$

Multiplying throughout by m, we get

$$m\dot{\hat{r}} = m\ddot{\hat{e}} + m\alpha\dot{\hat{e}} \tag{61}$$

Now, substituting the value of $m\ddot{\hat{e}}$, we get

$$m\dot{\hat{r}} = (m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) - m\omega_{\ddot{\eta}}(t)) + m\alpha\dot{\hat{e}} \tag{62}$$

Now, when $\sigma = s$ (stable region), $\tilde{\eta} \to \eta$ so estimation error is 0.

Now, let us consider the lyapunov candidate as,

$$V(\zeta(t)) = \frac{1}{2}e^T e + \frac{1}{2}r^T m r + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta} \tag{63}$$

Therefore,

$$\dot{V}(\zeta(t)) = e^T \dot{e} + r^T m\dot{r} + \tilde{\theta}^T \Gamma^{-1}\dot{\tilde{\theta}} \tag{64}$$

since,

$$\tilde{\theta} = \theta - \hat{\theta} \tag{65}$$

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}} \tag{66}$$

Now, substituting the values of $\dot{e}, m\dot{r}, \dot{\tilde{\theta}}$, we get,

$$\dot{V}(\zeta(t)) = e^T(r - \alpha e) + r^T(m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) + m\alpha\dot{e}) - \tilde{\theta}^T \Gamma^{-1}\dot{\hat{\theta}} \tag{67}$$

$$\dot{V}(\zeta(t)) = e^T r - \alpha\|e\|^2 + r^T(m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) + m\alpha\dot{e}) - \tilde{\theta}^T \Gamma^{-1}\dot{\hat{\theta}} \tag{68}$$

$$Y\theta = \begin{bmatrix} (\alpha\dot{e} + \ddot{\eta}_d) & \dot{\eta} \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix}$$

Therefore,

$$\dot{V}(\zeta(t)) = e^T r - \alpha\|e\|^2 + r^T(Y\theta - \tau(t) - d(t)) - \tilde{\theta}^T \Gamma^{-1}\dot{\hat{\theta}} \tag{69}$$

Now, designing, $\tau = Y\hat{\theta} + e + \beta r + \beta_d sgn(r)$

Therefore,

$$\dot{V}(\zeta(t)) = e^T r - \alpha\|e\|^2 + r^T(Y\theta - (Y\hat{\theta} + e + \beta r + \beta_d sgn(r)) - d(t)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{70}$$

$$\dot{V}(\zeta(t)) = e^T r - \alpha\|e\|^2 + r^T(Y\theta - Y\hat{\theta} - e - \beta r - \beta_d sgn(r) - d(t)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{71}$$

Now, simplifying and cancelling some terms we get,

$$\dot{V}(\zeta(t)) = -\alpha\|e\|^2 + r^T Y\tilde{\theta} - \beta\|r\|^2 - r^T\beta_d sgn(r) - r^T d(t) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{72}$$

Therefore,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 + r^T Y\tilde{\theta} - \beta\|r\|^2 - \beta_d\|r\| + \bar{d}\|r\| - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{73}$$

So, if we take $\beta_d > \bar{d}$, then that term will always be less than 0.

Therefore,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 + r^T Y\tilde{\theta} - \beta\|r\|^2 - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{74}$$

Now, we have from equation 1,

$$m\ddot{\eta} = -c\dot{\eta}(t) + \tau(t) + d(t) \tag{75}$$

which can be written as,

$$m\ddot{\eta} + c\dot{\eta}(t) = \tau(t) + d(t) \tag{76}$$

$$Y_z\theta_z = \begin{bmatrix} \ddot{\eta} & \dot{\eta} \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix}$$

$$Y_z\theta_z = \tau(t) + d(t) \tag{77}$$

Now, multiplying by Y_z^T, throughout, we get,

$$Y_z^T Y_z\theta_z = Y_z^T\tau(t) + Y_z^T d(t) \tag{78}$$

$$\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t) = \sum_{i=1}^{n} Y_z^T(ti)\tau(ti) + \sum_{i=1}^{n} Y_z^T(ti)d(ti) \tag{79}$$

$$\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) = \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t) - \sum_{i=1}^{n} Y_z^T(ti)d(ti) \tag{80}$$

Now, designing $\dot{\hat{\theta}} = \Gamma Y^T r$ for $(t < T)$

we have,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 + r^T Y\tilde{\theta} - \beta\|r\|^2 - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{81}$$

Now, $r^T Y\tilde{\theta}$ can be written as $\tilde{\theta}^T Y^T r$.

Therefore, simplifying and substituting values we get,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 + \tilde{\theta}^T Y^T r - \beta\|r\|^2 - \tilde{\theta}^T\Gamma^{-1}(\Gamma Y^T r) \tag{82}$$

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 \tag{83}$$

Therefore, we get $\dot{V}$ as Negative semi definite when we do not collect enough data.

Now, when we have enough data,

Designing $\dot{\hat{\theta}} = \Gamma Y^T r + \Gamma K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) - \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\Bigg)$

Now, substituting, we get,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2+\tilde{\theta}^T Y^T r-\beta\|r\|^2-\tilde{\theta}^T\Gamma^{-1}\Bigg(\Gamma Y^T r+\Gamma K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)\tau(ti)-\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\Bigg)\Bigg) \tag{84}$$

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 - \tilde{\theta}^T K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) - \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\Bigg) \tag{85}$$

Now we know,

$$\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) = \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t) - \sum_{i=1}^{n} Y_z^T(ti)d(ti) \tag{86}$$

Therefore,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2-\beta\|r\|^2-\tilde{\theta}^T K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t)-\sum_{i=1}^{n} Y_z^T(ti)d(ti)-\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\Bigg) \tag{87}$$

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 - \tilde{\theta}^T K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\tilde{\theta}_z(t) - \sum_{i=1}^{n} Y_z^T(ti)d(ti)\Bigg) \tag{88}$$

Now we can bound equations,

$$\dot{V}(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 - \underline{\mathbf{K}}_{CL}\,\underline{\lambda}\,\|\tilde{\theta}\|^2 \;+\; \bar{K}_{CL}\,\bar{Y}_z\,\bar{d}\,N\,\|\theta\| \tag{89}$$

Now using non linear damping property, we get,

Therefore,

$$-\frac{1}{2}\underline{\mathbf{K}}_{CL}\underline{\lambda}\,\|\tilde{\theta}\|^2 + \;\bar{K}_{CL}\,\bar{Y}_z\,\bar{d}\,N\,\|\theta\| \leq \frac{(\bar{K}_{CL}\,\bar{Y}_z\,\bar{d}\,N\,)^2}{2\,\underline{\mathbf{K}}_{CL}\underline{\lambda}} \tag{90}$$

Now, let this term be ϵ^s,

Therefore,

$$\dot{V}(\zeta(t))(\zeta(t)) \leq -\alpha\|e\|^2 - \beta\|r\|^2 - \frac{1}{2}\underline{\mathbf{K}}_{CL}\,\underline{\lambda}\,\|\tilde{\theta}\|^2 \;+\; \epsilon^s \tag{91}$$

Therefore, we get $\dot{V}$ as Negative Definite.

$$\dot{V}(\zeta(t)) \leq -KV + \epsilon^s \tag{92}$$

Therefore, as the lyapunov function is radially unbounded and the time derivative of lyapunov function is Negative Definite, we can say that we have globally exponentially tracking.

Now, for $\sigma = q$ (unstable region where feedback is not available),

Let the estimated tracking error be,

$$\hat{e} = \eta_d - \hat{\eta} \tag{93}$$

$$\dot{\hat{e}} = \dot{\eta_d} - \dot{\hat{\eta}} \tag{94}$$

which can also be written as,

$$\dot{\hat{e}} = \dot{\eta_d} - \dot{\eta_m} \tag{95}$$

$$\ddot{\hat{e}} = \ddot{\eta_d} - \ddot{\eta_m} \tag{96}$$

$$\ddot{\hat{e}} = \ddot{\eta_d} - \ddot{\eta}(t) - \omega_{\ddot{\eta}}(t) \tag{97}$$

Multiplying throughout by m, we get

$$m\ddot{\hat{e}} = m\ddot{\eta_d} - m\ddot{\eta}(t) - m\omega_{\ddot{\eta}}(t) \tag{98}$$

Now, substituting the value of $m\ddot{\eta}$ from equation (1), we get

$$m\ddot{\hat{e}} = m\ddot{\eta_d} - (-c\dot{\eta}(t) + \tau(t) + d(t)) - m\omega_{\ddot{\eta}}(t) \tag{99}$$

$$m\ddot{\hat{e}} = m\ddot{\eta_d} + c\dot{\eta}(t) - \tau(t) - d(t) - m\omega_{\ddot{\eta}}(t) \tag{100}$$

Now, let the estimated filtered tracking error be,

$$\hat{r} = \dot{\hat{e}} + \alpha\hat{e} \tag{101}$$

$$\dot{\hat{r}} = \ddot{\hat{e}} + \alpha\dot{\hat{e}} \tag{102}$$

Multiplying throughout by m, we get

$$m\dot{\hat{r}} = m\ddot{\hat{e}} + m\alpha\dot{\hat{e}} \tag{103}$$

Now, substituting the value of $m\ddot{\hat{e}}$, we get

$$m\dot{\hat{r}} = (m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) - m\omega_{\ddot{\eta}}) + m\alpha\dot{\hat{e}} \tag{104}$$

$$m\dot{\hat{r}} = m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) - m\omega_{\ddot{\eta}} + m\alpha\dot{\hat{e}} \tag{105}$$

Therefore,

$$V_{\eta}^{q} = \frac{1}{2}\tilde{\eta}^T\tilde{\eta} = \frac{1}{2}\|\tilde{\eta}\|^2 \tag{106}$$

$$\dot{V}_{\eta}^{q} = \tilde{\eta}^T\dot{\tilde{\eta}} \tag{107}$$

Now, substituting the value of $\dot{\tilde{\eta}}$,

$$\dot{V}_{\eta}^{q} = \tilde{\eta}^T(\dot{\eta} - \dot{\hat{\eta}}) \tag{108}$$

$$\dot{V}_{\eta}^{q} = \tilde{\eta}^T(\dot{\eta} - \dot{\eta}_m) \tag{109}$$

Now, substituting the value $\dot{\eta_m}$,

$$\dot{V}_{\eta}^{q} = \tilde{\eta}^T(\dot{\eta} - (\dot{\eta} + \omega_{\dot{\eta}})) \tag{110}$$

$$\dot{V}_{\eta}^{q} = \tilde{\eta}^T(\omega_{\dot{\eta}}) \tag{111}$$

$$\dot{V}_{\eta}^{q} = -\tilde{\eta}^T\omega_{\dot{\eta}} \tag{112}$$

$$-\tilde{\eta}^T\omega_{\dot{\eta}} \leq \|\tilde{\eta}\|\|\omega_{\dot{\eta}}\| \tag{113}$$

$$\dot{V}_\eta^q \leq \|\tilde{\eta}\|\|\omega_{\dot{\eta}}\| \tag{114}$$

Now, we know the young's inequality,

$$\|a\|\|b\| \leq \frac{1}{2}\|a\|^2 + \frac{1}{2}\|b\|^2 \tag{115}$$

$$\|\tilde{\eta}\|\|\omega_{\dot{\eta}}\| \leq \frac{1}{2}\|\tilde{\eta}\|^2 + \frac{1}{2}\|\omega_{\dot{\eta}}\|^2 \tag{116}$$

$$\dot{V}_\eta^q \leq \frac{1}{2}\|\tilde{\eta}\|^2 + \frac{1}{2}\|\omega_{\dot{\eta}}\|^2 \tag{117}$$

Now, we know,

$$V_\eta^q = \frac{1}{2}\tilde{\eta}^T\tilde{\eta} = \frac{1}{2}\|\tilde{\eta}\|^2 \tag{118}$$

and let,

$$\frac{1}{2}\|\omega_{\dot{\eta}}\|^2 = \epsilon_\eta^q \tag{119}$$

Therefore, we get

$$\dot{V}_\eta^q \leq V_\eta^q + \epsilon_\eta^q \tag{120}$$

$$V_\eta^q(\tilde{\eta}(t),t) \;\leq\; (V_\eta^q(\tilde{\eta}(t_j^q),t_j^q) + \epsilon_\eta^q)\, exp\,(t - t_j^q) - \epsilon_\eta^q \tag{121}$$

Now, for maximum dwell time,

$$V_\eta^q(\tilde{\eta}(t_{j+1}^s),t_{j+1}^s) \leq \bar{V} \tag{122}$$

$$V_\eta^q(\tilde{\eta}(t_j^q),t_j^q) \leq \underline{\mathrm{V}} \tag{123}$$

$$\bar{V} \leq (\underline{\mathbf{V}} + \epsilon_\eta^q)\ exp\ (t_{j+1}^s - t_j^q) - \epsilon_\eta^q \tag{124}$$

$$\bar{V} \leq (\underline{\mathbf{V}} + \epsilon_\eta^q)\ exp\ (\Delta t^q) - \epsilon_\eta^q \tag{125}$$

$$\bar{V} + \epsilon_\eta^q \leq (\underline{\mathbf{V}} + \epsilon_\eta^q)\ exp\ (\Delta t^q) \tag{126}$$

Therefore,

$$\Delta t^q \leq \log \frac{\bar{V} + \epsilon_\eta^q}{\underline{\mathbf{V}} + \epsilon_\eta^q} \tag{127}$$

Now, to calculate the numerical value for dwell time, substituting the values of $\bar{V}$, $\underline{\mathbf{V}}$ and ϵ_η^q, we get

$$\Delta t^q \leq \log \frac{50 + 1.25e - 5}{0.005 + 1.25e - 5} \tag{128}$$

$$\Delta t^q \leq 9.2078 \tag{129}$$

Now, let us consider the lyapunov candidate as,

$$V_e^q = \frac{1}{2}\hat{e}^T\hat{e} + \frac{1}{2}\hat{r}^T m\hat{r} + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{130}$$

Therefore,

$$\dot{V}_e^q = \hat{e}^T\dot{\hat{e}} + \hat{r}^T m\dot{\hat{r}} + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}} \tag{131}$$

Here

Now, substituting the value of $\dot{\hat{e}}$, $m\dot{\hat{r}}$ and $\dot{\tilde{\theta}}$

$$\dot{V}_e^q = \hat{e}^T(\hat{r} - \alpha\hat{e}) + \hat{r}^T(m\ddot{\eta}_d + c\dot{\eta}(t) - \tau(t) - d(t) - m\omega_{\ddot{\eta}} + m\alpha\dot{\hat{e}}) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{132}$$

$$Y\theta = \begin{bmatrix}(\alpha\dot{\hat{e}} + \ddot{\eta}_d) & \dot{\eta}\end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix}$$

Therefore,

$$\dot{V}_e^q = \hat{e}^T(\hat{r} - \alpha\hat{e}) + \hat{r}^T(Y\theta - \tau(t) - d(t)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{133}$$

Now, designing, $\tau = Y\hat{\theta} + \hat{e} + \beta\hat{r} + \beta_d sgn(\hat{r})$

Therefore,

$$\dot{V}_e^q = \hat{e}^T(\hat{r} - \alpha\hat{e}) + \hat{r}^T(Y\theta - (Y\hat{\theta} + \hat{e} + \beta\hat{r} + \beta_d sgn(\hat{r})) - d(t)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{134}$$

$$\dot{V}_e^q = \hat{e}^T(\hat{r} - \alpha\hat{e}) + \hat{r}^T(Y\theta - Y\hat{\theta} - \hat{e} - \beta\hat{r} - \beta_d sgn(\hat{r})) - d(t)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{135}$$

$$\dot{V}_e^q = \hat{e}^T\hat{r} - \alpha\|\hat{e}\|^2 + \hat{r}^T(Y\tilde{\theta} - \hat{e} - \beta\hat{r} - \beta_d sgn(\hat{r})) - d(t)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{136}$$

Now, simplifying and cancelling some terms we get,

$$\dot{V}_e^q = -\alpha\|\hat{e}\|^2 + \hat{r}^TY\tilde{\theta} - \beta\|\hat{r}\|^2 - \hat{r}^T\beta_d sgn(\hat{r}) - \hat{r}^Td(t) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{137}$$

Therefore,

$$\dot{V}_e^q \leq -\alpha\|\hat{e}\|^2 + \hat{r}^T Y\tilde{\theta} - \beta\|\hat{r}\|^2 - \beta_d\|\hat{r}\| + \bar{d}\|\hat{r}\| - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{138}$$

So, if we take $\beta_d > \bar{d}$, then that term will always be less than 0.

Therefore,

$$\dot{V}_e^q \leq -\alpha\|\hat{e}\|^2 + \hat{r}^T Y\tilde{\theta} - \beta\|\hat{r}\|^2 - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{139}$$

Now, we have from equation 1,

$$m\ddot{\eta} = -c\dot{\eta}(t) + \tau(t) + d(t) \tag{140}$$

which can be written as,

$$m\ddot{\eta} + c\dot{\eta}(t) = \tau(t) + d(t) \tag{141}$$

$$Y_z\theta_z = \begin{bmatrix} \ddot{\eta} & \dot{\eta} \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix}$$

$$Y_z\theta_z = \tau(t) + d(t) \tag{142}$$

Now, multiplying by Y_z^T, throughout, we get,

$$Y_z^T Y_z\theta_z = Y_z^T\tau(t) + Y_z^T d(t) \tag{143}$$

$$\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t) = \sum_{i=1}^{n} Y_z^T(ti)\tau(ti) + \sum_{i=1}^{n} Y_z^T(ti)d(ti) \tag{144}$$

$$\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) = \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t) - \sum_{i=1}^{n} Y_z^T(ti)d(ti) \tag{145}$$

Now, designing $\dot{\hat{\theta}} = \Gamma Y^T \hat{r}$ for $(t < T)$

we have,

$$\dot{V}_e^q \leq -\alpha\|\hat{e}\|^2 + \hat{r}^T Y\tilde{\theta} - \beta\|\hat{r}\|^2 - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \tag{146}$$

Now, $\hat{r}^T Y\tilde{\theta}$ can be written as $\tilde{\theta}^T Y^T \hat{r}$.

Therefore, simplifying and substituting values we get,

$$\dot{V}_e^q \leq -\alpha\|\hat{e}\|^2 + \tilde{\theta}^T Y^T \hat{r} - \beta\|\hat{r}\|^2 - \tilde{\theta}^T\Gamma^{-1}(\Gamma Y^T \hat{r}) \tag{147}$$

$$\dot{V}_e^q \leq -\alpha\|\hat{e}\|^2 - \beta\|\hat{r}\|^2 \tag{148}$$

Therefore, we get $\dot{V}$ as Negative semi definite when we do not collect enough data.

Now, when we have enough data,

Designing $\dot{\hat{\theta}} = \Gamma Y^T \hat{r} + \Gamma K_{CL}\bigg(\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) - \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\bigg)$

Now, substituting, we get,

$$\dot{V}_e^q \leq -\alpha\|\hat{e}\|^2+\tilde{\theta}^T Y^T \hat{r}-\beta\|\hat{r}\|^2-\tilde{\theta}^T\Gamma^{-1}\Bigg(\Gamma Y^T \hat{r}+\Gamma K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)\tau(ti)-\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\Bigg)\Bigg) \tag{149}$$

$$\dot{V}_e^q \le -\alpha\|\hat{e}\|^2 - \beta\|\hat{r}\|^2 - \tilde{\theta}^T K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) - \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\Bigg) \tag{150}$$

Now we know,

$$\sum_{i=1}^{n} Y_z^T(ti)\tau(ti) = \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t) - \sum_{i=1}^{n} Y_z^T(ti)d(ti) \tag{151}$$

Therefore,

$$\dot{V}_e^q \le -\alpha\|\hat{e}\|^2 - \beta\|\hat{r}\|^2 - \tilde{\theta}^T K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\theta_z(t) - \sum_{i=1}^{n} Y_z^T(ti)d(ti) - \sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\hat{\theta}(t)\Bigg) \tag{152}$$

$$\dot{V}_e^q \le -\alpha\|\hat{e}\|^2 - \beta\|\hat{r}\|^2 - \tilde{\theta}^T K_{CL}\Bigg(\sum_{i=1}^{n} Y_z^T(ti)Y_z(ti)\tilde{\theta}_z(t) - \sum_{i=1}^{n} Y_z^T(ti)d(ti)\Bigg) \tag{153}$$

Now we can bound equations,

$$\dot{V}_e^q \le -\alpha\|\hat{e}\|^2 - \beta\|\hat{r}\|^2 - \underline{\mathrm{K}}_{CL}\, \underline{\lambda}\, \|\tilde{\theta}\|^2 \;+\; \bar{K}_{CL}\, \bar{Y}_z\, \bar{d}\, N\, \|\theta\| \tag{154}$$

Now using non linear damping property, we get,

Therefore,

$$-\frac{1}{2}\underline{\mathrm{K}}_{CL}\underline{\lambda}\, \|\tilde{\theta}\|^2 + \;\bar{K}_{CL}\, \bar{Y}_z\, \bar{d}\, N\, \|\theta\| \le \frac{(\bar{K}_{CL}\, \bar{Y}_z\, \bar{d}\, N\,)^2}{2\, \underline{\mathrm{K}}_{CL}\underline{\lambda}} \tag{155}$$

Now, let this term be ϵ^q,

Therefore,

$$\dot{V}_e^q \leq -\alpha\|\hat{e}\|^2 - \beta\|\hat{r}\|^2 - \frac{1}{2}\underline{\mathbf{K}}_{CL}\,\underline{\lambda}\,\|\tilde{\theta}\|^2 \,+\, \epsilon^q \tag{156}$$

Therefore, we get $\dot{V}$ as Negative Definite.

$$\dot{V}_e^q \leq -KV + \epsilon^q \tag{157}$$

Therefore, as the lyapunov function is radially unbounded and the time derivative of lyapunov function is Negative Definite, we can say that we have globally exponentially tracking for approach 3, where we have intermittent feedback availability.

c) i) We took three different approaches to understand the problem as mentioned above.

$1)$ No uncertainty, Disturbance exist and feedback is available all the time.

$2)$ No uncertainty, Disturbance exist and feedback is NOT available all the time.

$3)$ Uncertainty (Real parameters are not known) and disturbance both exist and feedback is NOT available all the time. In this approach, we considered stable region(σ) and unstable region(q) and assume we have intermittent feedback availability.

In all the three approaches, we were able to track the reference properly. We used ADAPTIVE Controller for the system. All the plots of the trajectories and errors are attached below each and every approach.

c) ii) The norm of the estimation error is attached below. A) Yes, The position error was less than the upper limit. B) Yes, at times it gets close to upper limit. C) Yes, Based on A and B the dwell time was conservative.

c) iii) The norm of the estimated tracking errors over time and the norm of the true tracking error over time is attached below. Based on the plots, we can say that the controller tracks the desired trajectory perfectly.

c) iv) Yes, we did an adaptive approach (Concurrent Learning) and the norm of parametric error is attached below. The parameters did converge and the parametric error converges to zero.

c) v) The norm of the total input, the feedback portions of the input, and the feedforward portions of the input is attached below.

c) vi) Some extra plots are attached at the end. A) We can conclude that our simulation matches our stability results and with our adaptive controller, as the lyapunov function is radially unbounded and its time derivative is Negative definite, We have achieved global exponential tracking.

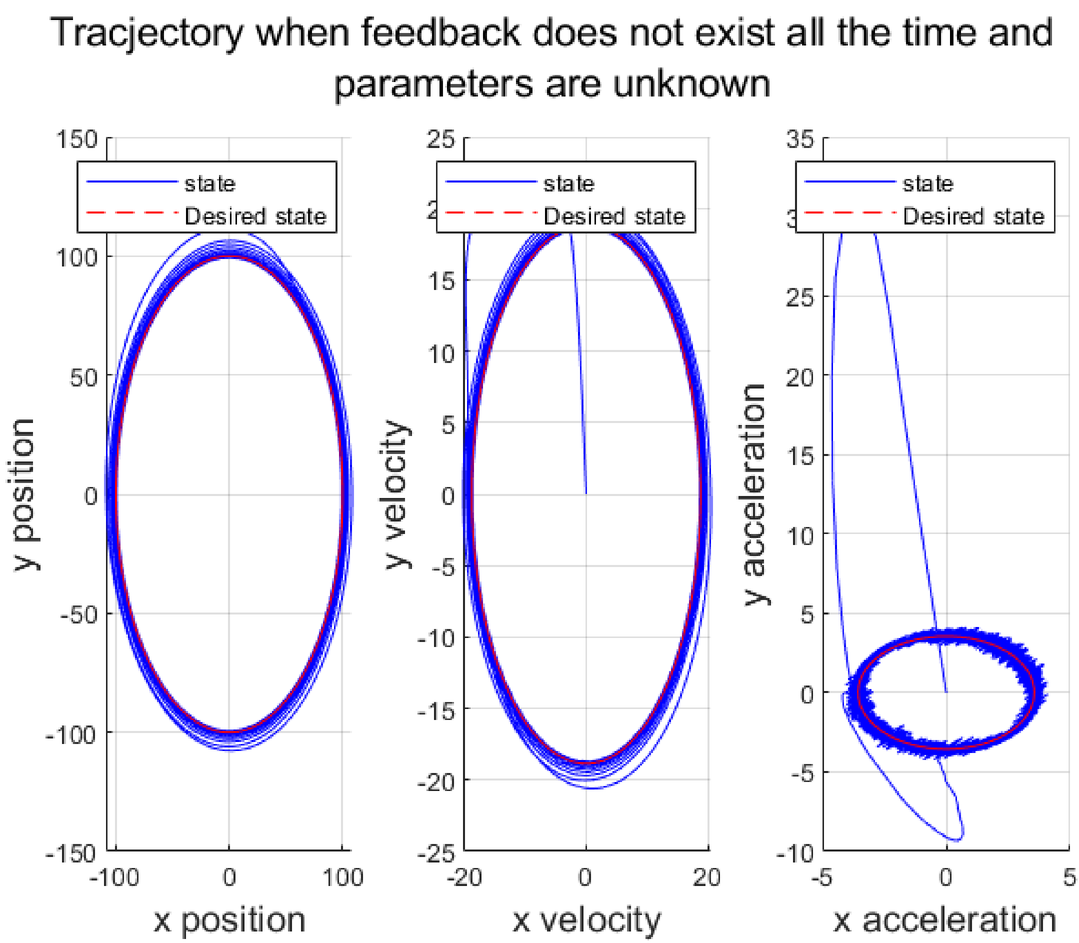

Figure 5: State and the Desired state trajectories when feedback is not available all the time and the parameters are unknown.

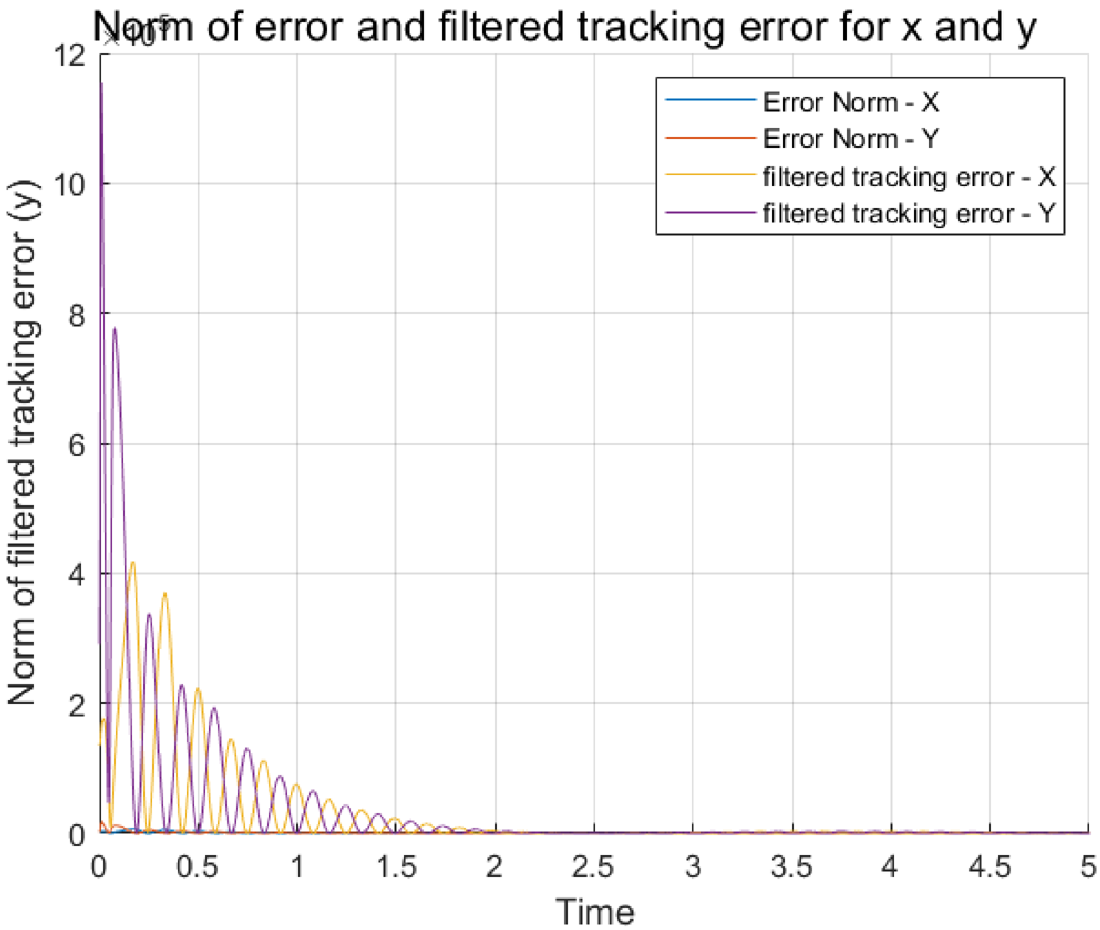

Figure 6: Norm of Error and Filtered Tracking Error.

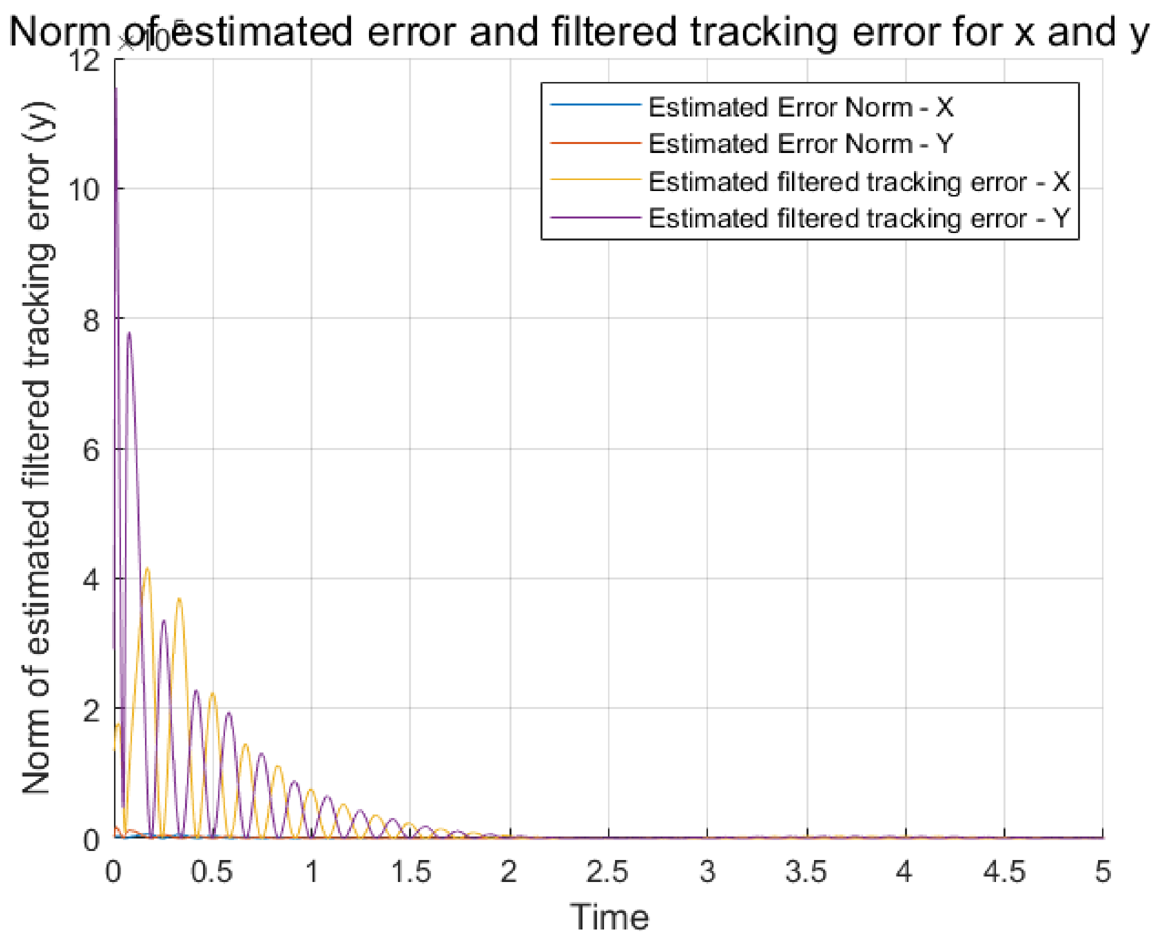

Figure 7: Norm of Estimation error and Filtered tracking error.

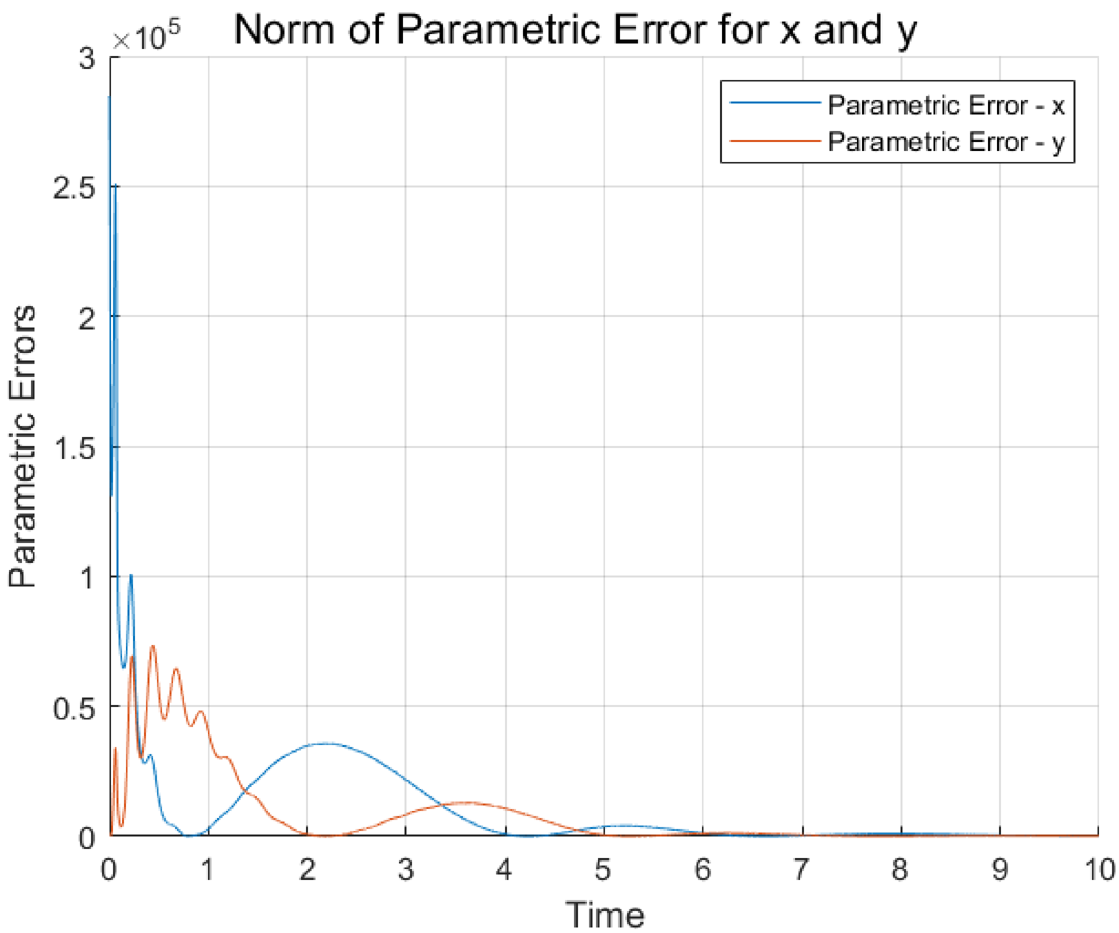

Figure 8: Norm of Parametric Errors.

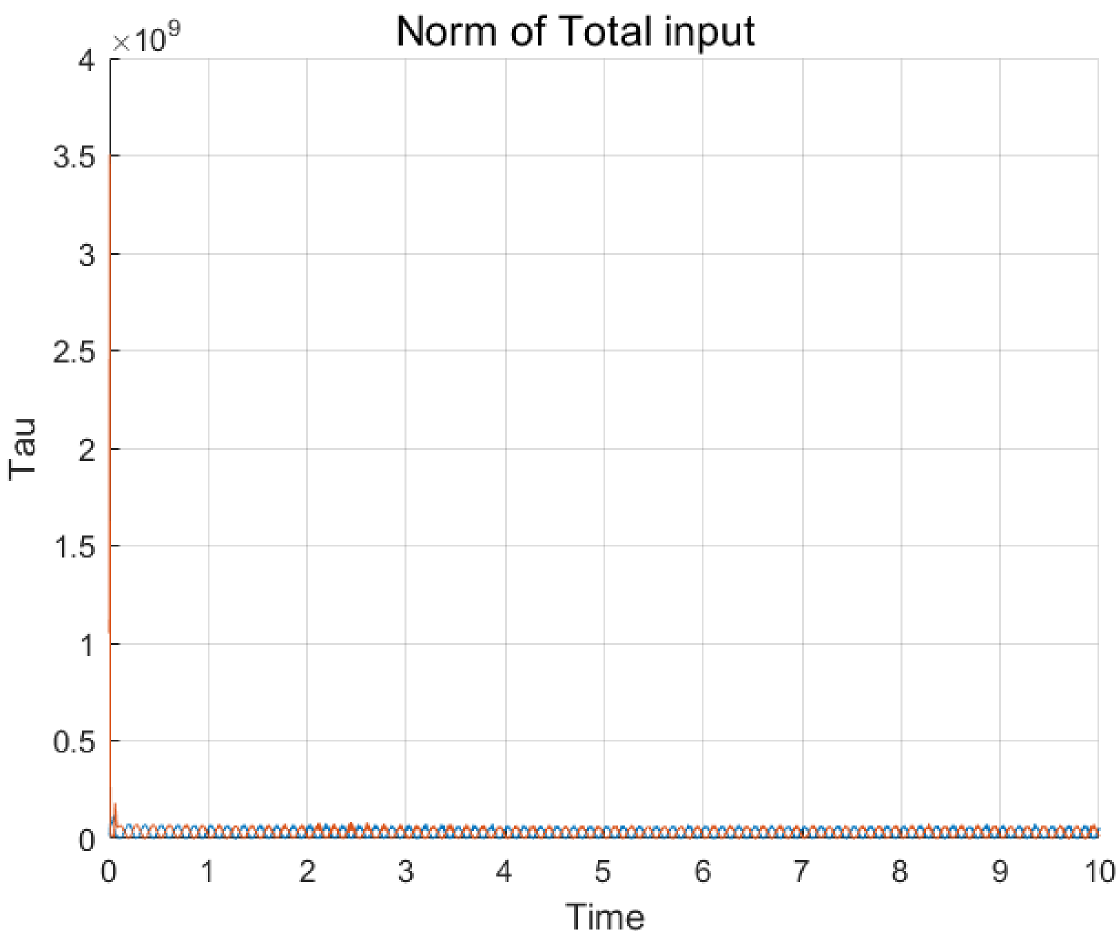

Figure 9: Norm of Total Input.

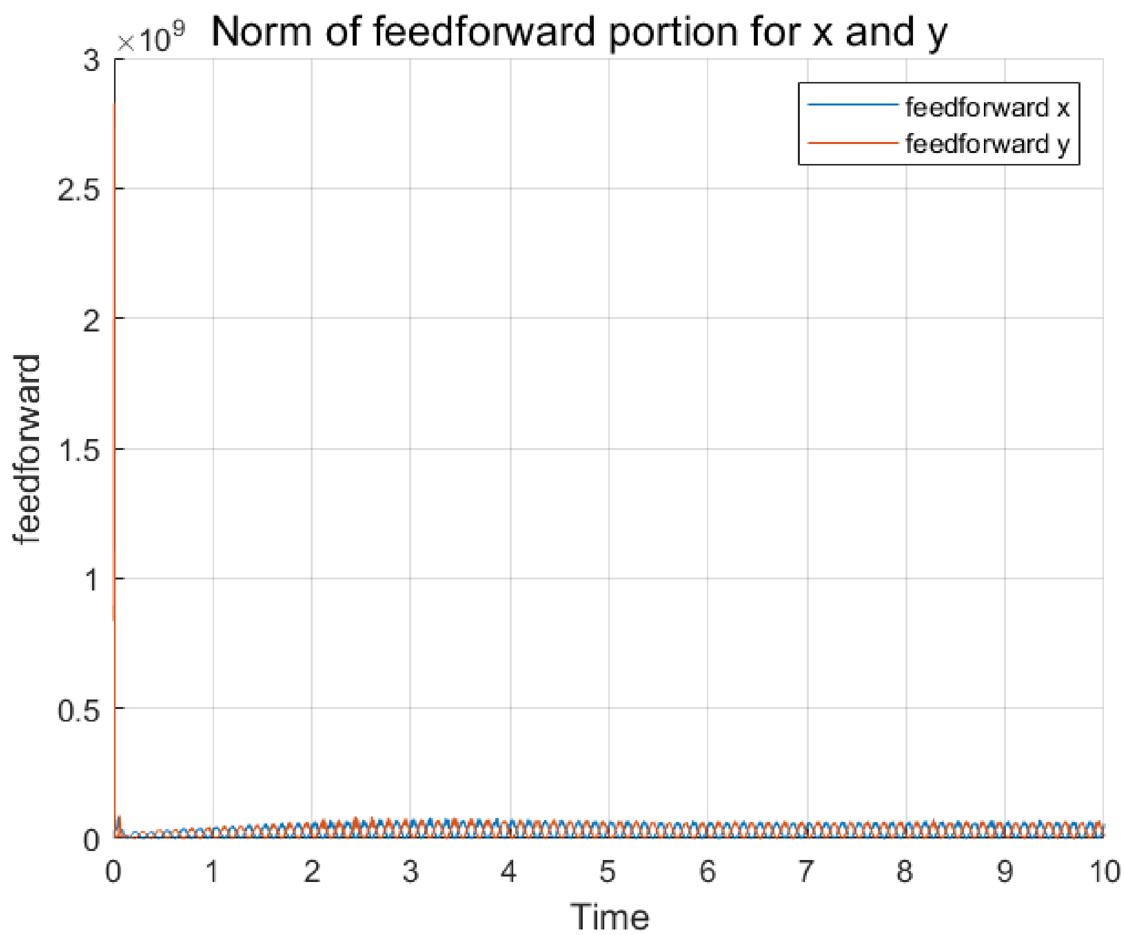

Figure 10: feedforward portion of X and Y.

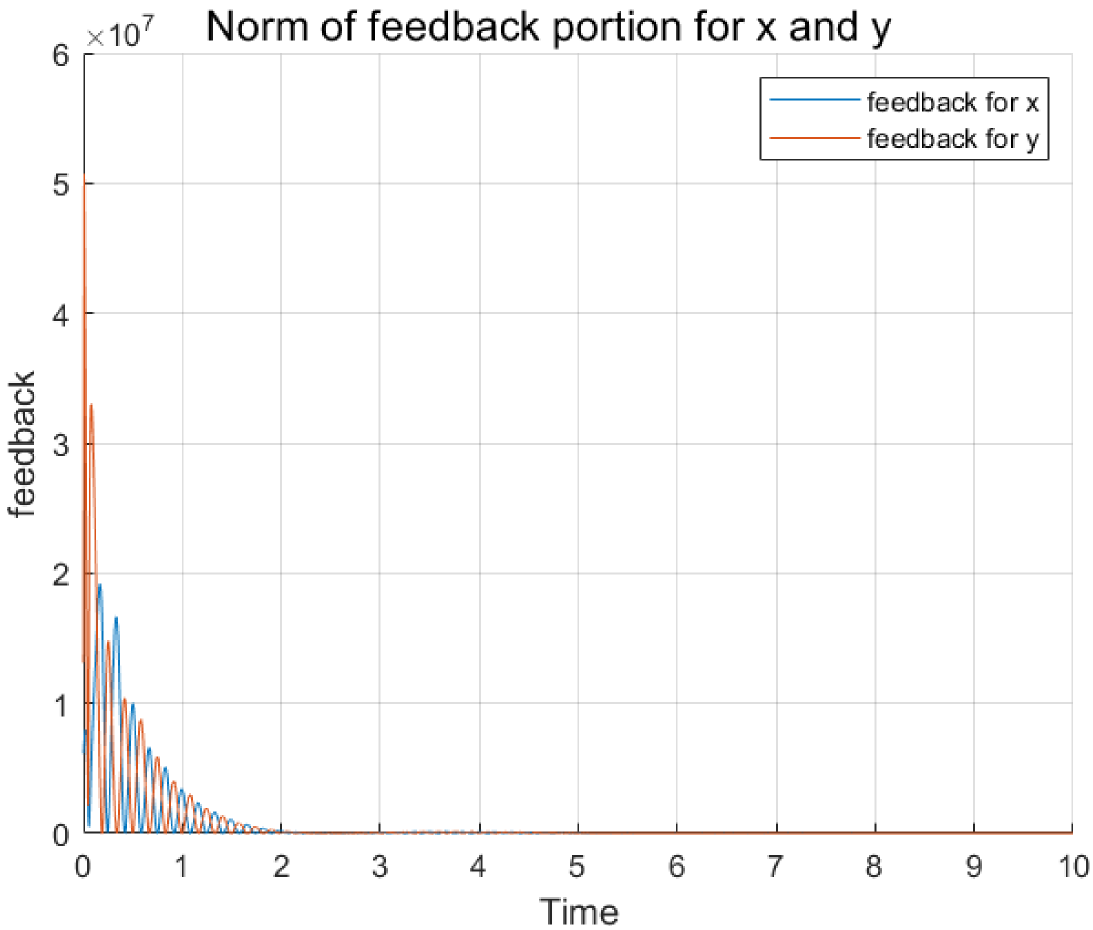

Figure 11: feedback portion of X and Y.

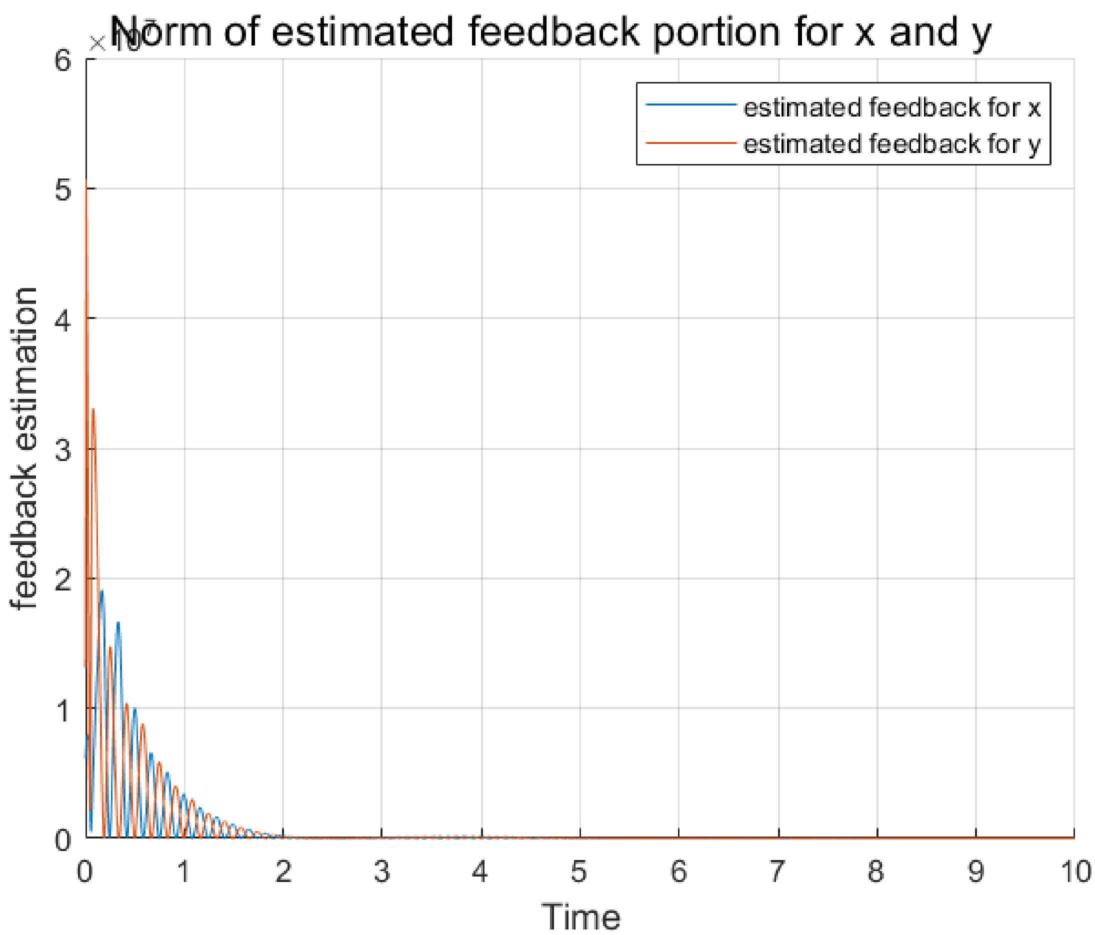

Figure 12: Estimated feedback portion of X and Y.

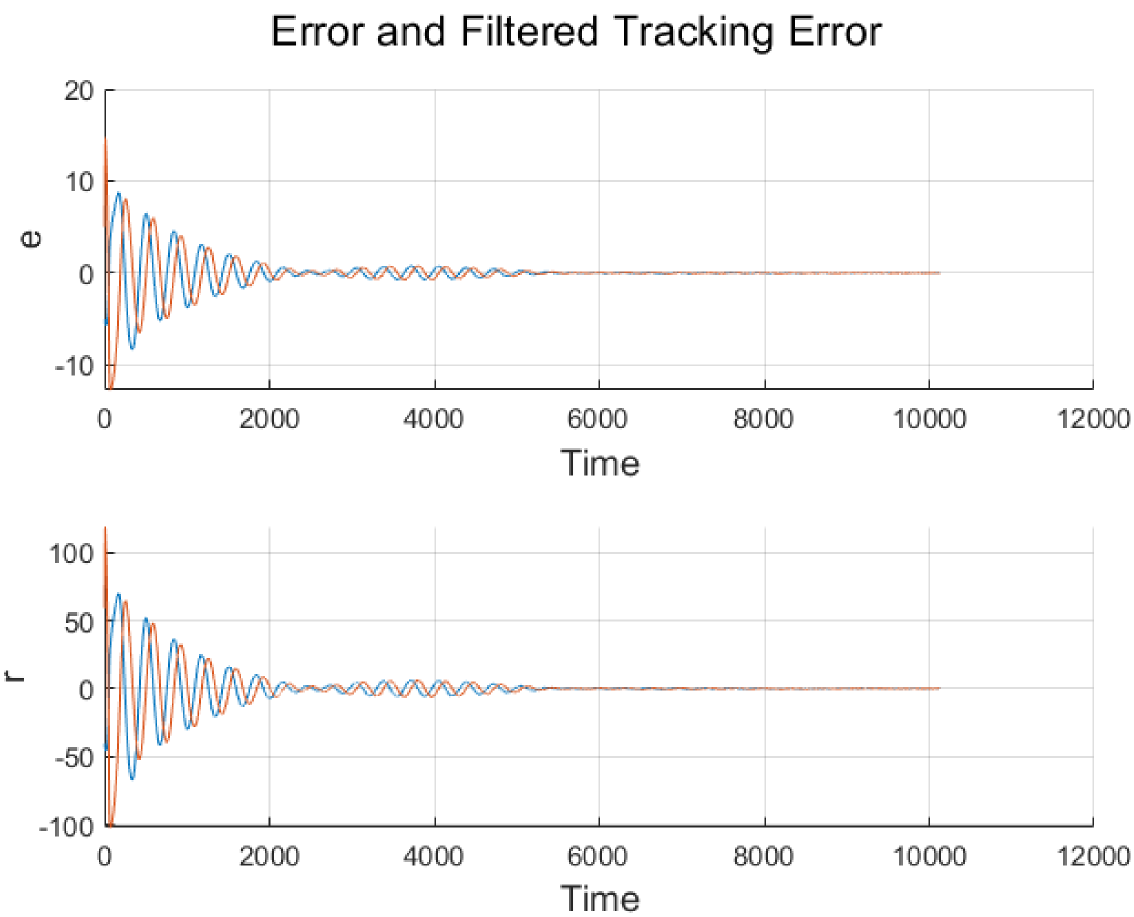

Figure 13: Error and Filtered Tracking Error.

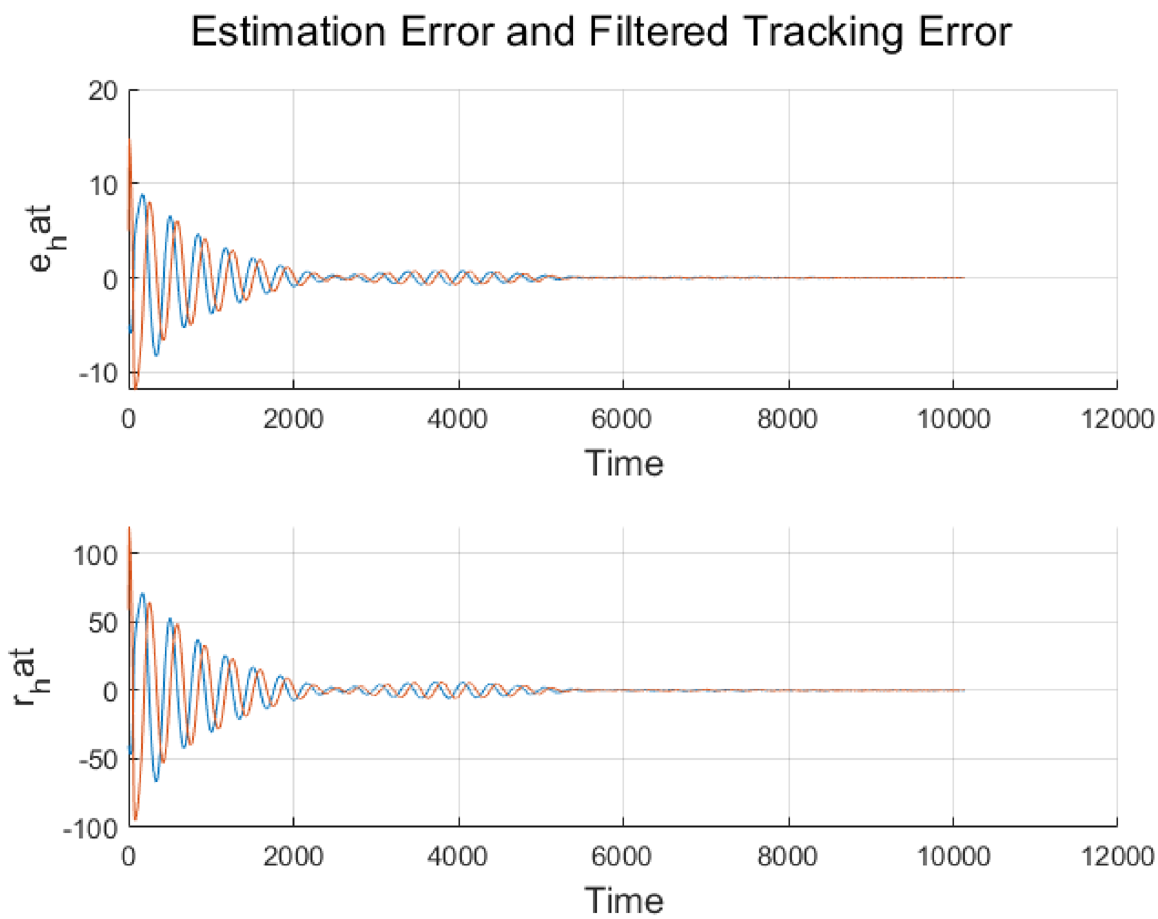

Figure 14: Estimation error and filter tracking error.

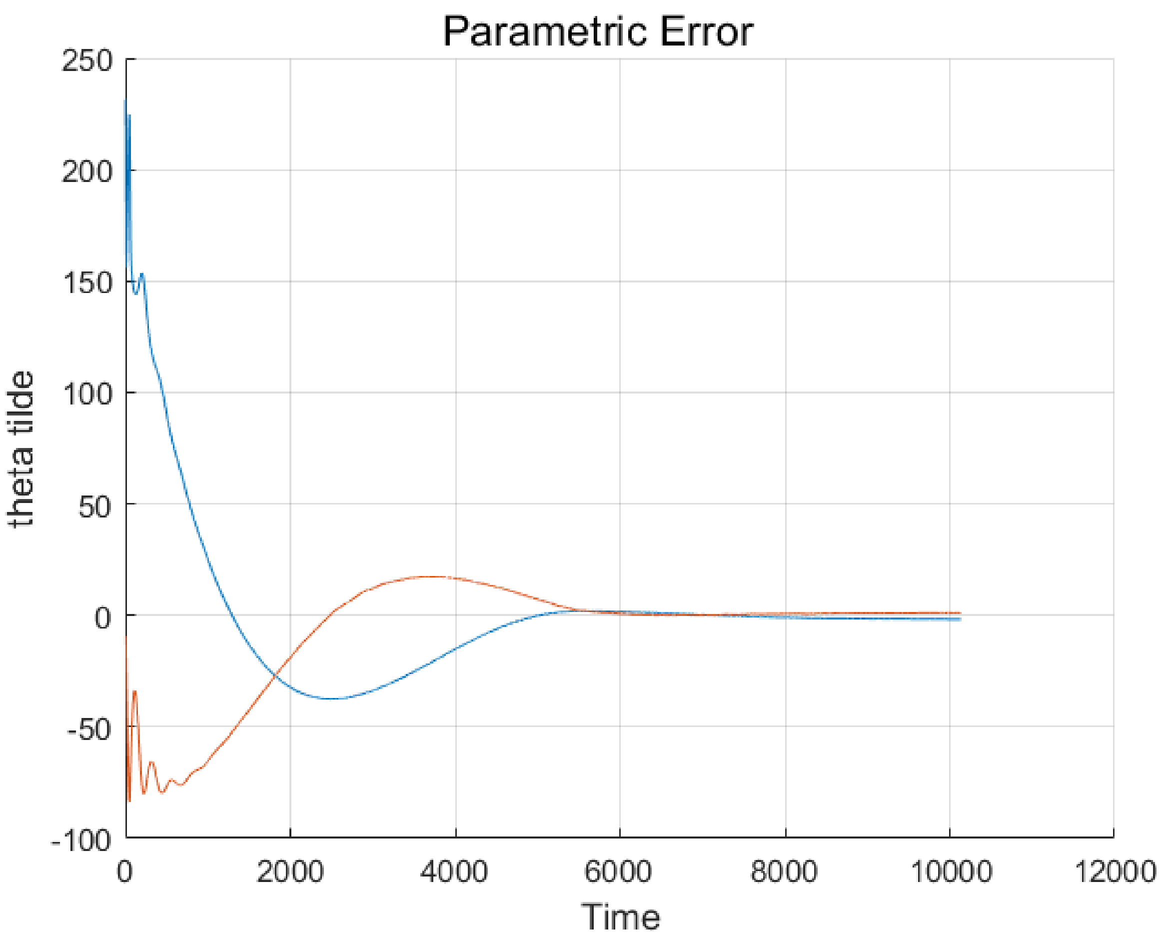

Figure 15: Parametric Error.

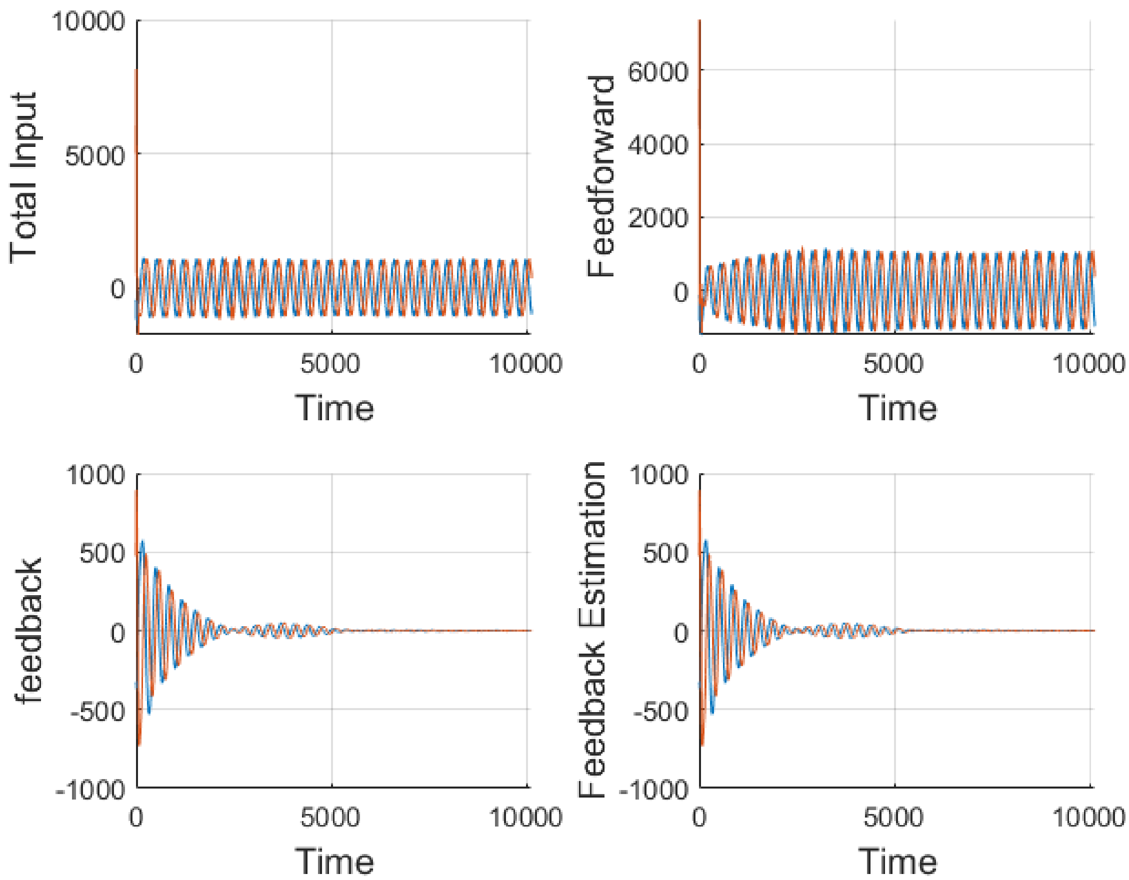

Figure 16: Total input, feedback portion, feedforward portion, Estimation feedback.