Problem : Dynamic System of an Actuated Pendulum Attached to an Actuated Collar

Consider the following dynamic system of an actuated pendulum attached to an actuated collar on a track, both with first order actuator dynamics

$$(m_x + m_\phi \sin^2(\phi))\ddot{x} = -c_x\dot{x} + m_\phi g \cos(\phi)\sin(\phi) + m_\phi l\dot{\phi}^2 \sin(\phi) + u_x \tag{1}$$

$$\dot{u_x} = -a_x u_x + \mu_x \tag{2}$$

$$m_\phi l^2 \ddot{\phi} = -c_\phi\dot{\phi} - m_\phi l g \sin(\phi) - m_\phi l \cos(\phi)\ddot{x} + u_\phi \tag{3}$$

$$\dot{u_\phi} = -a_\phi u_\phi + \mu_\phi \tag{4}$$

SOLUTION: This problem can be solved by gradient method and concurrent learning method. First, we will do the gradient method followed by concurrent learning method.

Method 1

Error equation for collar is,

$$e_x = x_d - x \tag{5}$$

$$\dot{e_x} = \dot{x_d} - \dot{x} \tag{6}$$

$$\ddot{e_x} = \ddot{x_d} - \ddot{x} \tag{7}$$

Error equation for pendulum is,

$$e_\phi = \phi_d - \phi \tag{8}$$

$$\dot{e_\phi} = \dot{\phi}_d - \dot{\phi} \tag{9}$$

$$\ddot{e_\phi} = \ddot{\phi}_d - \ddot{\phi} \tag{10}$$

Filtered tracking error for collar is,

$$r_x = \dot{e_x} + \alpha e_x \tag{11}$$

$$\dot{r_x} = \ddot{e_x} + \alpha \dot{e_x} \tag{12}$$

Filtered tracking error for pendulum is,

$$r_\phi = \dot{e_\phi} + \alpha e_\phi \tag{13}$$

$$\dot{r_\phi} = \ddot{e_\phi} + \alpha \dot{e_\phi} \tag{14}$$

Now, we know

$$x_d\ (t) = \bar{x_d}\ \sin(2\pi f_{xd}\, t) \tag{15}$$

$$\dot{x_d}\ (t) = (2\pi f_{xd})\ \bar{x_d}\ \cos(2\pi f_{xd}\, t) \tag{16}$$

$$\ddot{x_d}\ (t) = -(2\pi f_{xd})^2\ \bar{x_d}\ \sin(2\pi f_{xd}\, t) \tag{17}$$

$$\dddot{x_d}\ (t) = -(2\pi f_{xd})^3\ \bar{x_d}\ \cos(2\pi f_{xd}\, t) \tag{18}$$

$$\phi_d\ (t) = \bar{\phi_d}\ \sin(2\pi f_{\phi d}\, t) \tag{19}$$

$$\dot{\phi_d}\ (t) = (2\pi f_{\phi d})\ \bar{\phi_d}\ \cos(2\pi f_{\phi d}\ t) \tag{20}$$

$$\ddot{\phi_d}\ (t) = -(2\pi f_{\phi d})^2\ \bar{\phi_d}\ \sin(2\pi f_{\phi d}\ t) \tag{21}$$

$$\dddot{\phi_d}\ (t) = -(2\pi f_{\phi d})^3\ \bar{\phi_d}\ \cos(2\pi f_{\phi d}\ t) \tag{22}$$

Also, we know,

$$\tilde{u_x} = u_{dx} - u_x \tag{23}$$

$$\tilde{u_\phi} = u_{d\phi} - u_\phi \tag{24}$$

Adding and Subtracting u_{dx} in equation (1) and equation (3)

Therefore, we get

$$(m_x + m_\phi \sin^2(\phi))\ddot{x} = -c_x\dot{x} + m_\phi g\cos(\phi)\sin(\phi) + m_\phi l\dot{\phi}^2\sin(\phi) + u_{dx} - \tilde{u_x} \tag{25}$$

$$m_\phi l^2\ddot{\phi} = -c_\phi\dot{\phi} - m_\phi lg\sin(\phi) - m_\phi l\cos(\phi)\ddot{x} + u_{d\phi} - \tilde{u_\phi} \tag{26}$$

Also, from equation (7), we know $\ddot{e_x} = \ddot{x_d} - \ddot{x}$

Now, multiplying equation (7) by $(m_x + m_\phi\sin^2(\phi))$, we get,

For collar

$$(m_x + m_\phi \sin^2(\phi))\ddot{e_x} = (m_x + m_\phi \sin^2(\phi))(\ddot{x_d} - \ddot{x}) \tag{27}$$

$$(m_x + m_\phi \sin^2(\phi))\ddot{e_x} = (m_x + m_\phi \sin^2(\phi))\ddot{x_d} - (m_x + m_\phi \sin^2(\phi))\ddot{x} \tag{28}$$

From equation (25), we have

$$(m_x + m_\phi \sin^2(\phi))\ddot{x} = -c_x\dot{x} + m_\phi g \cos(\phi)\sin(\phi) + m_\phi l\dot{\phi}^2 \sin(\phi) + u_{dx} - \tilde{u_x}$$

Now, substituting equation (25) in equation (28), we get,

$$(m_x+m_\phi \sin^2(\phi))\ddot{e_x} = (m_x+m_\phi \sin^2(\phi))\ddot{x_d} - (-c_x\dot{x}+m_\phi g\cos(\phi)\sin(\phi)+m_\phi l\dot{\phi}^2 \sin(\phi)+u_{dx}-\tilde{u_x}) \tag{29}$$

$$(m_x+m_\phi \sin^2(\phi))\ddot{e_x} = (m_x+m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x}-m_\phi g\cos(\phi)\sin(\phi)-m_\phi l\dot{\phi}^2 \sin(\phi)-u_{dx}+\tilde{u_x}) \tag{30}$$

Now, from equation (12), we have, $\dot{r_x} = \ddot{e_x} + \alpha\dot{e_x}$

Now, multiplying equation (12) by $(m_x + m_\phi \sin^2(\phi))$, we get,

For collar,

$$(m_x + m_\phi \sin^2(\phi))\dot{r_x} \ = \ (m_x + m_\phi \sin^2(\phi))\ (\ddot{e_x} + \alpha\dot{e_x}) \tag{31}$$

$$(m_x + m_\phi \sin^2(\phi))\dot{r_x} \ = \ (m_x + m_\phi \sin^2(\phi))\ \ddot{e_x} + (m_x + m_\phi \sin^2(\phi))(\alpha\dot{e_x}) \tag{32}$$

Now, we have from equation (30), $(m_x + m_\phi \sin^2(\phi))\ddot{e_x}$

Substituting equation (30), in equation (32), we get,

$$(m_x+m_\phi \sin^2(\phi))\dot{r_x} \;=\; (m_x+m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x} - m_\phi g\cos(\phi)\sin(\phi) - m_\phi l\dot{\phi}^2\sin(\phi) - u_{dx} + \tilde{u_x}) + (m_x + m_\phi \sin^2(\phi))(\alpha\dot{e_x}) \quad (A)$$

Now, consider the lyapunov candidate,

$$V(\zeta,t) = \frac{1}{2}e_x^2 + \frac{1}{2}(m_x + m_\phi \sin^2(\phi))r_x^2 + \frac{1}{2}\tilde{u_x}^2 + \frac{1}{2}\tilde{\theta_x}^T\Gamma_x^{-1}\tilde{\theta_x} + \frac{1}{2\gamma_{ax}}\tilde{a_x}^2 + \frac{1}{2}e_\phi^2 + \frac{1}{2}(m_\phi l^2)r_\phi^2 + \frac{1}{2}\tilde{u_\phi}^2 + \frac{1}{2}\tilde{\theta_\phi}^T\Gamma_\phi^{-1}\tilde{\theta_\phi} + \frac{1}{2\gamma_{a\phi}}\tilde{a_\phi}^2 \quad (*)$$

Now for better analysis, let's divide the lyapunov candidate in two parts $V_x(\zeta)$ for collar and $v_\phi(\zeta)$ for pendulum.

Now, consider the lyapunov candidate for collar,

$$V_x(\zeta,t) = \frac{1}{2}e_x^2 + \frac{1}{2}(m_x + m_\phi \sin^2(\phi))r_x^2 + \frac{1}{2}\tilde{u_x}^2 + \frac{1}{2}\tilde{\theta_x}^T\Gamma_x^{-1}\tilde{\theta_x} + \frac{1}{2\gamma_{ax}}\tilde{a_x}^2 \qquad (33)$$

$$\dot{V_x}(\zeta) = e\dot{e} + m_\phi \sin(\phi)\cos(\phi)\,\dot{\phi}\,r_x^2 + (m_x+m_\phi \sin^2(\phi))\,\dot{r_x}(r_x) + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta_x}^T\Gamma_x^{-1}\dot{\tilde{\theta_x}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \quad (34)$$

Now, let's only consider, second term from eqaution (34),

$$m_\phi \sin(\phi)\cos(\phi)\,\dot{\phi}\,r_x^2 + (m_x + m_\phi \sin^2(\phi))\,\dot{r_x}(r_x)$$

Now, we have from equation (A), $(m_x + m_\phi \sin^2(\phi))\dot{r_x}$

Substituting equation (A), we get,

$$m_\phi \sin(\phi)\cos(\phi)\,\dot{\phi}\,r_x^2 + (m_x + m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x} - m_\phi g\cos(\phi)\sin(\phi) - m_\phi l\dot{\phi}^2\sin(\phi) - u_{dx} + \tilde{u_x}) + (m_x + m_\phi \sin^2(\phi))(\alpha\dot{e_x})$$

Now, taking r_x common, we get,

$$[m_\phi \sin(\phi)\cos(\phi)\,\dot{\phi}\,r_x + (m_x + m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x} - m_\phi g\cos(\phi)\sin(\phi) - m_\phi l\dot{\phi}^2\sin(\phi) - u_{dx} + \tilde{u_x}) + (m_x + m_\phi \sin^2(\phi))(\alpha\dot{e_x})]r_x$$

Taking the terms in bracket in the form of $Y_x\theta_x$, we get

$$Y_x = \begin{bmatrix} (\alpha\dot{e_x} + \ddot{x_d}) & (\sin(\phi)\cos(\phi)\dot{\phi}\,r_x + \sin^2(\phi)\alpha\dot{e_x} + \sin^2(\phi)\ddot{x_d} - g\cos(\phi)\sin(\phi)) & (\dot{x}) & (-\dot{\phi}^2\sin(\phi)) \end{bmatrix}$$

$$\theta_x = \begin{bmatrix} m_x \\ m_\phi \\ c_x \\ m_\phi l \end{bmatrix}$$

Therefore, equation (34), becomes

$$\dot{V}_x(\zeta) = e_x\dot{e_x} + r_x(Y_x\theta_x - u_{dx} + \tilde{u_x}) \ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta_x}^T\Gamma_x^{-1}\dot{\tilde{\theta_x}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{35}$$

Also, from equation (11), we have, $\dot{e_x} = r_x - \alpha_x e_x$

Substituting it in equation (35), we have,

$$\dot{V}_x(\zeta) = e_x(r_x - \alpha_x e_x) + r_x(Y_x\theta_x - u_{dx} + \tilde{u_x}) \ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta_x}^T\Gamma_x^{-1}\dot{\tilde{\theta_x}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{36}$$

Now, designing, $u_{dx} = Y_x\hat{\theta_x} + e_x + \beta_x r_x$

Now, substituting u_{dx} in equation (36)

$$\dot{V}_x(\zeta) = e_x(r_x - \alpha_x e_x) + r_x(Y_x\theta_x - (Y_x\hat{\theta_x} + e_x + \beta_x r_x) + \tilde{u_x}) \ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta_x}^T\Gamma_x^{-1}\dot{\tilde{\theta_x}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{37}$$

$$\dot{V}_x(\zeta) = e_x(r_x - \alpha_x e_x) + r_x(Y_x\theta_x - Y_x\hat{\theta_x} - e_x - \beta_x r_x + \tilde{u_x}) \ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta_x}^T\Gamma_x^{-1}\dot{\tilde{\theta_x}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{38}$$

$$\dot{V}_x(\zeta) = e_x r_x - \alpha_x e_x^2 + r_x Y_x\tilde{\theta_x} - e_x r_x - \beta_x r_x^2 + r_x\tilde{u_x} \ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta_x}^T\Gamma_x^{-1}\dot{\tilde{\theta_x}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{39}$$

Now, simplifying and cancelling some terms we get,

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta_x}^T\Gamma_x^{-1}\dot{\tilde{\theta_x}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{40}$$

Now, we have,

$$\tilde{u_x} = u_{dx} - u_x$$

Therefore,

$$\dot{\tilde{u_x}} = \dot{u_{dx}} - \dot{u_x}$$

$$\dot{\tilde{\theta}}_x = -\,\dot{\hat{\theta}}_x$$

$$\dot{\tilde{a_x}} = -\,\dot{\hat{a_x}}$$

Substituting all in equation (40), we get

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} - \dot{u_x}) - \tilde{\theta_x}^T \Gamma_x^{-1}\dot{\hat{\theta}}_x - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{41}$$

Now, from equation (2), we have, $\dot{u_x} = -a_x u_x + \mu_x$

Substituting it in equation (41), we get

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} - (-a_x u_x + \mu_x)) - \tilde{\theta_x}^T \Gamma_x^{-1}\dot{\hat{\theta}}_x - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{42}$$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} + a_x u_x - \mu_x) - \tilde{\theta_x}^T \Gamma_x^{-1}\dot{\hat{\theta}}_x - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{43}$$

we know, $\dot{u_{dx}} = \dot{Y_x}\hat{\theta_x} + Y_x\dot{\hat{\theta}}_x + \dot{e_x} + \beta_x\dot{r_x}$

Now, designing $\mu_x = \dot{u_{dx}} + \hat{a_x}u_x + s_x\tilde{u_x} + r_x$

where s_x is a positive constant.

Now, substituting μ_x in equation (43), we get,

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y_x \tilde{\theta_x} + r_x \tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} + a_x u_x - (\dot{u_{dx}} + \hat{a_x} u_x + s_x \tilde{u_x} + r_x)) - \tilde{\theta_x}^T \Gamma_x^{-1} \dot{\hat{\theta_x}} - \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{44}$$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y_x \tilde{\theta_x} + r_x \tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} + a_x u_x - \dot{u_{dx}} - \hat{a_x} u_x - s_x \tilde{u_x} - r_x) - \tilde{\theta_x}^T \Gamma_x^{-1} \dot{\hat{\theta_x}} - \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{45}$$

After, cancelling terms, and since

$$\tilde{a_x} = a_x - \hat{a_x} \tag{46}$$

we get,

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y_x \tilde{\theta_x} + \tilde{u_x} \tilde{a_x} u_x - s_x \tilde{u_x}^2 \ - \ \tilde{\theta_x}^T \Gamma_x^{-1} \dot{\hat{\theta_x}} \ - \ \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{47}$$

Now, we have, $r_x Y_x \tilde{\theta_x}$, which can also be written as (since r_x is a scalar):

$$r_x Y_x \tilde{\theta_x} = (r_x Y_x \tilde{\theta_x})^T = \tilde{\theta_x}^T Y_x^T r_x$$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 \ + \tilde{\theta_x}^T Y_x^T r_x + \tilde{u_x} \tilde{a_x} u_x - \ \tilde{\theta_x}^T \Gamma_x^{-1} \dot{\hat{\theta_x}} \ - \ \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{48}$$

Therefore,

Now, Designing $\dot{\hat{\theta_x}} = \Gamma_x Y_x^T r_x$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 + \tilde{\theta_x}^T Y_x^T r_x + \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta_x}^T \Gamma_x^{-1}(\Gamma_x Y_x^T r_x) - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \quad (49)$$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 + \tilde{\theta_x}^T Y_x^T r_x + \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta_x}^T Y_x^T r_x - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \quad (50)$$

Therefore, we have now after cancellation,

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \quad (51)$$

Now, Designing $\dot{\hat{a_x}} = \gamma_{ax}\tilde{u_x}u_x$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x - \frac{1}{\gamma_{ax}}\tilde{a_x}(\gamma_{ax}\tilde{u_x}u_x) \quad (52)$$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x - \tilde{u_x}\tilde{a_x}u_x \quad (53)$$

$$\dot{V_x}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 \quad (54)$$

Now, for pendulum, we have from equation (10), $\ddot{e_\phi} = \ddot{\phi_d} - \ddot{\phi}$

Now, multiplying equation (10) by $(m_\phi l^2)$, we get

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)(\ddot{\phi_d} - \ddot{\phi}) \quad (55)$$

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)\ddot{\phi}_d - (m_\phi l^2)\ddot{\phi} \tag{56}$$

Now, from equation (26), we have

$$m_\phi l^2\ddot{\phi} = -c_\phi\dot{\phi} - m_\phi lg\sin(\phi) - m_\phi l\cos(\phi)\ddot{x} + u_{d\phi} - \tilde{u_\phi}$$

Substituting equation (26) in equation (56), we get

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)\ddot{\phi}_d - (-c_\phi\dot{\phi} - m_\phi lg\sin(\phi) - m_\phi l\cos(\phi)\ddot{x} + u_{d\phi} - \tilde{u_\phi}) \tag{57}$$

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)\ddot{\phi}_d + c_\phi\dot{\phi} + m_\phi lg\sin(\phi) + m_\phi l\cos(\phi)\ddot{x} - u_{d\phi} + \tilde{u_\phi} \tag{58}$$

We have from equation (14), $\dot{r_\phi} = \ddot{e_\phi} + \alpha_\phi\dot{e_\phi}$

Now, multiplying equation (14) by $(m_\phi l^2)\ddot{\phi}$, we get,

For pendulum,

$$(m_\phi l^2)\dot{r_\phi} \;=\; (m_\phi l^2)\left(\ddot{e_\phi} + \alpha_\phi\dot{e_\phi}\right) \tag{59}$$

$$(m_\phi l^2)\dot{r_\phi} \;=\; (m_\phi l^2)\ddot{e_\phi} + (m_\phi l^2)(\alpha_\phi\dot{e_\phi}) \tag{60}$$

Now substituting equation (58) in equation (60)

$$(m_\phi l^2)\dot{r_\phi} \;=\; ((m_\phi l^2)\ddot{\phi}_d + c_\phi\dot{\phi} + m_\phi lg\sin(\phi) + m_\phi l\cos(\phi)\ddot{x} - u_{d\phi} + \tilde{u_\phi}) + (m_\phi l^2)(\alpha_\phi\dot{e_\phi}) \qquad (B)$$

Now equation (B) can be written as, $Y_\phi \theta_\phi$

$$Y_\phi \theta_\phi = \begin{bmatrix} (g\sin(\phi) + \cos(\phi)\ddot{x}) & (\ddot{\phi}_d + \alpha_\phi \dot{e_\phi}) & (\dot{\phi}) \end{bmatrix} \begin{bmatrix} m_\phi l \\ m_\phi l^2 \\ c_\phi \end{bmatrix}$$

Now, consider the lyapunov candidate for pendulum,

$$V_\phi(\zeta, t) = \frac{1}{2}e_\phi^2 + \frac{1}{2}(m_\phi l^2) r_\phi^2 + \frac{1}{2}\tilde{u_\phi}^2 + \frac{1}{2}\tilde{\theta_\phi}^T \Gamma_\phi^{-1} \tilde{\theta}_\phi + \frac{1}{2\gamma_{a\phi}}\tilde{a_\phi}^2 \tag{61}$$

$$\dot{V}_\phi(\zeta) = e_\phi \dot{e_\phi} + m_\phi l^2 \; \dot{r_\phi} \; r_\phi + \tilde{u_\phi}\dot{\tilde{u_\phi}} + \tilde{\theta_\phi}^T \Gamma_\phi^{-1} \dot{\tilde{\theta}}_\phi + \frac{1}{\gamma_{a\phi}} \tilde{a_\phi}\dot{\tilde{a_\phi}} \tag{62}$$

We know,

$$\dot{e_\phi} = r_\phi - \alpha_\phi e_\phi \tag{63}$$

$$\dot{\tilde{\theta}}_\phi \;=\; - \,\dot{\hat{\theta}}_\phi \tag{64}$$

$$\dot{\tilde{a_\phi}} = -\dot{\hat{a_\phi}} \tag{65}$$

$$\dot{\tilde{u_\phi}} = \dot{u_{d\phi}} - \dot{u_\phi} \tag{66}$$

Now, substituting equation $(63), (64), (65), (66)$ and (B) in equation (62), we get

$$\dot{V}(\zeta) = e_\phi(r_\phi - \alpha e_\phi) + (m_\phi l^2) r_\phi \; \dot{r_\phi} + \tilde{u_\phi}(\dot{u_{d\phi}} - \dot{u_\phi}) - \tilde{\theta_\phi}^T \Gamma_\phi^{-1} \dot{\hat{\theta}}_\phi - \frac{1}{\gamma_a} \tilde{a_\phi}\dot{\hat{a_\phi}} \tag{67}$$

Now, substituting equation (B), $(m_\phi l^2)\dot{r_\phi}$ in equation (67), we get

$$\dot{V}(\zeta) = e_\phi(r_\phi - \alpha_\phi e_\phi) + r_\phi\left((m_\phi l^2)\ddot{\phi_d} + c_\phi\dot{\phi} + m_\phi lg\sin(\phi) + m_\phi l\cos(\phi)\ddot{x} - u_{d\phi} + \tilde{u_\phi} + (m_\phi l^2)(\alpha_\phi\dot{e_\phi})\right) + \tilde{u_\phi}(\dot{u_{d\phi}} - \dot{u_\phi}) - \tilde{\theta_\phi}^T\Gamma_\phi^{-1}\dot{\hat{\theta}}_\phi - \frac{1}{\gamma_a}\tilde{a_\phi}\dot{\hat{a}}_\phi$$

Therefore,

$$\dot{V}(\zeta) = e_\phi(r_\phi - \alpha_\phi e_\phi) + r_\phi\left(Y_\phi\theta_\phi - u_{d\phi} + \tilde{u_\phi}\right) + \tilde{u_\phi}(\dot{u_{d\phi}} - \dot{u_\phi}) - \tilde{\theta_\phi}^T\Gamma_\phi^{-1}\dot{\hat{\theta}}_\phi - \frac{1}{\gamma_a}\tilde{a_\phi}\dot{\hat{a}}_\phi \quad (68)$$

Now, designing $u_{d\phi} = Y_\phi\hat{\theta_\phi} + e_\phi + \beta_\phi r_\phi$, we get

$$\dot{V}(\zeta) = e_\phi(r_\phi - \alpha_\phi e_\phi) + r_\phi\left(Y_\phi\theta_\phi - (Y_\phi\hat{\theta_\phi} + e_\phi + \beta_\phi r_\phi) + \tilde{u_\phi}\right) + \tilde{u_\phi}(\dot{u_{d\phi}} - \dot{u_\phi}) - \tilde{\theta_\phi}^T\Gamma_\phi^{-1}\dot{\hat{\theta}}_\phi - \frac{1}{\gamma_a}\tilde{a_\phi}\dot{\hat{a}}_\phi \quad (69)$$

$$\dot{V}(\zeta) = e_\phi r_\phi - \alpha_\phi e_\phi^2 + r_\phi\left(Y_\phi\theta_\phi - Y_\phi\hat{\theta_\phi} - e_\phi - \beta_\phi r_\phi + \tilde{u_\phi}\right) + \tilde{u_\phi}(\dot{u_{d\phi}} - \dot{u_\phi}) - \tilde{\theta_\phi}^T\Gamma_\phi^{-1}\dot{\hat{\theta}}_\phi - \frac{1}{\gamma_a}\tilde{a_\phi}\dot{\hat{a}}_\phi \quad (70)$$

Now, simplifying and cancelling some terms, we get

$$\dot{V}(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi\tilde{\theta_\phi} + r_\phi\tilde{u_\phi} + \tilde{u_\phi}(\dot{u_{d\phi}} - \dot{u_\phi}) - \tilde{\theta_\phi}^T\Gamma_\phi^{-1}\dot{\hat{\theta}}_\phi - \frac{1}{\gamma_a}\tilde{a_\phi}\dot{\hat{a}}_\phi \quad (71)$$

we know, $\dot{u_\phi} = -a_\phi u_\phi + \mu_\phi$

$$\dot{V}(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta_\phi} + r_\phi \tilde{u_\phi} + \tilde{u_\phi}(\dot{u_{d\phi}} - (-a_\phi u_\phi + \mu_\phi)) \ - \tilde{\theta_\phi}^T \Gamma_\phi^{-1} \dot{\hat{\theta_\phi}} - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{72}$$

$$\dot{V}(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta_\phi} + r_\phi \tilde{u_\phi} + \tilde{u_\phi}(\dot{u_{d\phi}} + a_\phi u_\phi - \mu_\phi) \ - \tilde{\theta_\phi}^T \Gamma_\phi^{-1} \dot{\hat{\theta_\phi}} - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{73}$$

Now, designing $\mu_\phi = \dot{u_{d\phi}} + \hat{a_\phi} u_\phi + r_\phi + s_\phi \tilde{u_\phi}$, we get

$$\dot{V}(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta_\phi} + r_\phi \tilde{u_\phi} + \tilde{u_\phi}(\dot{u_{d\phi}} + a_\phi u_\phi - (\dot{u_{d\phi}} + \hat{a_\phi} u_\phi + r_\phi + s_\phi \tilde{u_\phi})) - \tilde{\theta_\phi}^T \Gamma_\phi^{-1} \dot{\hat{\theta_\phi}} - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{74}$$

$$\dot{V}(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta_\phi} + r_\phi \tilde{u_\phi} + \tilde{u_\phi}(\dot{u_{d\phi}} + a_\phi u_\phi - \dot{u_{d\phi}} - \hat{a_\phi} u_\phi - r_\phi - s_\phi \tilde{u_\phi}) - \tilde{\theta_\phi}^T \Gamma_\phi^{-1} \dot{\hat{\theta_\phi}} - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{75}$$

Now, since $\tilde{a_\phi} = a_\phi - \hat{a_\phi}$, simplifying and cancelling some terms, we get,

$$\dot{V}(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 + r_\phi Y_\phi \tilde{\theta_\phi} + \tilde{u_\phi} \ \tilde{a_\phi} u_\phi \ - \ \tilde{\theta_\phi}^T \Gamma_\phi^{-1} \dot{\hat{\theta_\phi}} - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{76}$$

Now, designing $\dot{\hat{\theta_\phi}} = \Gamma_\phi Y_\phi^T r_\phi$

we have, $r_\phi Y_\phi \tilde{\theta_\phi}$, which can also be written as (since r_ϕ is a scalar):

$$r_\phi Y_\phi \tilde{\theta_\phi} = (r_\phi Y_\phi \tilde{\theta_\phi})^T = \tilde{\theta_\phi}^T Y_\phi^T r_\phi$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 + \tilde{\theta_\phi}^T Y_\phi^T r_\phi + \tilde{u_\phi}\ \tilde{a_\phi} u_\phi\ -\ \tilde{\theta_\phi}^T \Gamma_\phi^{-1} (\ \Gamma Y_\phi^T r_\phi)\ - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{77}$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 + \tilde{\theta_\phi}^T Y_\phi^T r_\phi + \tilde{u_\phi}\ \tilde{a_\phi} u_\phi\ -\ \tilde{\theta_\phi}^T Y_\phi^T r_\phi\ - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{78}$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 + \tilde{u_\phi}\ \tilde{a_\phi} u_\phi\ - \frac{1}{\gamma_a} \tilde{a_\phi} \dot{\hat{a_\phi}} \tag{79}$$

Now, designing $\dot{\hat{a_\phi}} = \gamma_{a\phi} \tilde{u_\phi} u_\phi$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 + \tilde{u_\phi}\ \tilde{a_\phi} u_\phi\ - \frac{1}{\gamma_a} \tilde{a_\phi} (\gamma_a \tilde{u_\phi} u_\phi) \tag{80}$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 + \tilde{u_\phi}\ \tilde{a_\phi} u_\phi\ - \tilde{a_\phi} \tilde{u_\phi} u_\phi \tag{81}$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 \tag{82}$$

Now from equation (54) and equation (82), collar and pendulum equations, we get,

$$\dot{V}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2\ - \alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2 \tag{83}$$

Therefore, $\dot{V}(\zeta)$ is Negative Semi Definite.

Now, let's try Lasalle,

$\dot{V}(\zeta) = 0$ implies $e_x, r_x, \tilde{u}_x, e_\phi, r_\phi, \tilde{u}_\phi = 0$

$\dot{\hat{\theta}} = \Gamma Y^T r$

since $r = 0$, $\dot{\hat{\theta}} = 0$

$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$, therefore $\dot{\tilde{\theta}} = 0$

$\tilde{\theta} = \theta - \hat{\theta}$

But with these, we cannot imply $\tilde{\theta} = 0$. So, we cannot use Lasalle neither for collar nor for pendulum.

Now, let's try Barbalet's Lemma, for general lyapunov candidate with collar and pendulum both.

(we will be using barbalet's lemma for general notion because it will be same for both collar and pendulum).

$$\dot{V}(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 \; - \alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2$$

$$\int_0^t \dot{V}(\zeta) \leq \int_0^t -\alpha_x e_x^2 - \beta_x r_x^2 - s_x \tilde{u_x}^2 \; - \alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2$$

$$V(\zeta(t)) - V(\zeta(0)) \leq -\int_0^t (\alpha_x e_x^2 + \beta_x r_x^2 + s_x \tilde{u_x}^2 \; + \alpha_\phi e_\phi^2 + \beta_\phi r_\phi^2 + s_\phi \tilde{u_\phi}^2)$$

Therefore, multiplying by (-1), we get

$$V(\zeta(0)) - V(\zeta(t)) \geq \int_0^t (\alpha_x e_x^2 + \beta_x r_x^2 + s_x \tilde{u_x}^2 \; + \alpha_\phi e_\phi^2 + \beta_\phi r_\phi^2 + s_\phi \tilde{u_\phi}^2)$$

Since, system is globally stable. Therefore,

$$V(\zeta(t)), V(\zeta(0)) \in L_\infty$$

so,

$$e_x, r_x, \tilde{u}_x, e_\phi, r_\phi, \tilde{u}_\phi \in L_2$$

Now,

$$r = \dot{e} + \alpha e$$

$$\dot{e} = r - \alpha e$$

since $V(\zeta(t))$ is bounded

$$e_x, r_x, e_\phi, r_\phi \in L_\infty$$

Therefore,

$$\dot{e} \in L_\infty$$

so, e is uniformly continuous (for both collar and pendulum).

$$\lim_{t \to \infty} e(t) = 0$$

Now,

θ is bounded ($\theta \in L_\infty$) so, $m_x, m_\phi \in L_\infty$

Since, $e_x, e_\phi \in L_\infty$ and x_d, ϕ_d is a constant.

$$x, \phi \in L_\infty$$

Since, $\dot{e_x}, \dot{e_\phi} \in L_\infty$

$$\dot{x}, \dot{\phi} \in L_\infty$$

Therefore,

$$Y_x, Y_\phi \in L_\infty$$

Now, general equation for u_{dx} and $u_{d\phi}$

$$u_d = e + \beta r + Y\hat{\theta}$$

$$\beta \in L_\infty$$

$$Y \in L_\infty$$

$$r, e = \in L_\infty$$

$$\dot{\hat{\theta}} = \Gamma Y^T r$$

Therefore,

$$\dot{\hat{\theta}}_x, \dot{\hat{\theta}}_\phi \in L_\infty$$

Since $\Gamma, Y, r \in L_\infty$

so, $u_d \in L_\infty$

$$\dot{r}_x, \dot{r}_\phi \in L_\infty$$

Therefore, r is uniformly continuous (for both collar and pendulum).

$$\lim_{t \to \infty} r(t) = 0$$

since, $u = u_d - \tilde{u}$

and

$u_d \in L_\infty, \tilde{u} \in L_\infty$

Therefore,

$$u_x, u_\phi \in L_\infty$$

and since $u \in L_\infty$,

$$\dot{u} \in L_\infty$$

and since $\dot{u}$ and $\dot{u_d} \in L_\infty$

$$\dot{\tilde{u}}_x, \dot{\tilde{u}}_\phi \in L_\infty$$

Therefore, $\tilde{u}$ is uniformly continuous (for both collar and pendulum).

$$\lim_{t \to \infty} \tilde{u}(t) = 0$$

Therefore, through barablet's lemma, We can say $e, r, \tilde{u}$ is converging to zero as time goes to ∞. Hence, We can conclude Global Asymptotic Tracking. (GAT)

We can also use, Lasalle - Yoshizawa's theorem to prove convergence.

$$\dot{V}(\zeta) \leq -(\alpha_x e_x^2 + \beta_x r_x^2 + s_x \tilde{u_x}^2 \; + \alpha_\phi e_\phi^2 + \beta_\phi r_\phi^2 + s_\phi \tilde{u_\phi}^2)$$

We can say,

$$\dot{V}(\zeta) \leq -(W)$$

where, $W = (\alpha_x e_x^2 + \beta_x r_x^2 + s_x \tilde{u_x}^2 \; + \alpha_\phi e_\phi^2 + \beta_\phi r_\phi^2 + s_\phi \tilde{u_\phi}^2)$ is a continuous function. Then all solutions of W are uniformly globally bounded and W is a semi definite function. As W goes to zero, $e, r, \tilde{u}$ also converge to zero as time goes to ∞. Therefore, global Asymptotic Tracking. (GAT)

Method 2 - Concurrent Learning

Consider the following dynamic system of an actuated pendulum attached to an actuated collar on a track, both with first order actuator dynamics

$$(m_x + m_\phi \sin^2(\phi))\ddot{x} = -c_x\dot{x} + m_\phi g\cos(\phi)\sin(\phi) + m_\phi l\dot{\phi}^2\sin(\phi) + u_x \tag{84}$$

$$\dot{u_x} = -a_x u_x + \mu_x \tag{85}$$

$$m_\phi l^2\ddot{\phi} = -c_\phi\dot{\phi} - m_\phi lg\sin(\phi) - m_\phi l\cos(\phi)\ddot{x} + u_\phi \tag{86}$$

$$\dot{u_\phi} = -a_\phi u_\phi + \mu_\phi \tag{87}$$

Solution:

Error equation for collar is,

$$e_x = x_d - x \tag{88}$$

$$\dot{e_x} = \dot{x_d} - \dot{x} \tag{89}$$

$$\ddot{e_x} = \ddot{x_d} - \ddot{x} \tag{90}$$

Error equation for pendulum is,

$$e_\phi = \phi_d - \phi \tag{91}$$

$$\dot{e_\phi} = \dot{\phi_d} - \dot{\phi} \tag{92}$$

$$\ddot{e_\phi} = \ddot{\phi_d} - \ddot{\phi} \tag{93}$$

Filtered tracking error for collar is,

$$r_x = \dot{e_x} + \alpha e_x \tag{94}$$

$$\dot{r_x} = \ddot{e_x} + \alpha \dot{e_x} \tag{95}$$

Filtered tracking error for pendulum is,

$$r_\phi = \dot{e_\phi} + \alpha e_\phi \tag{96}$$

$$\dot{r_\phi} = \ddot{e_\phi} + \alpha \dot{e_\phi} \tag{97}$$

Now, we know

$$x_d\ (t) = \bar{x_d}\ \sin(2\pi f_{xd}\, t) \tag{98}$$

$$\dot{x_d}\ (t) = (2\pi f_{xd})\ \bar{x_d}\ \cos(2\pi f_{xd}\, t) \tag{99}$$

$$\ddot{x_d}\ (t) = -(2\pi f_{xd})^2\ \bar{x_d}\ \sin(2\pi f_{xd}\, t) \tag{100}$$

$$\dddot{x_d}\ (t) = -(2\pi f_{xd})^3\ \bar{x_d}\ \cos(2\pi f_{xd}\, t) \tag{101}$$

$$\phi_d\ (t) = \bar{\phi_d}\ \sin(2\pi f_{\phi d}\, t) \tag{102}$$

$$\dot{\phi_d}\ (t) = (2\pi f_{\phi d})\ \bar{\phi_d}\ \cos(2\pi f_{\phi d}\, t) \tag{103}$$

$$\ddot{\phi_d}\ (t) = -(2\pi f_{\phi d})^2\ \bar{\phi_d}\ \sin(2\pi f_{\phi d}\, t) \tag{104}$$

$$\dddot{\phi}_d\ (t) = -(2\pi f_{\phi d})^3\ \bar{\phi}_d\ \cos(2\pi f_{\phi d}\ t) \tag{105}$$

Also, we know,

$$\tilde{u_x} = u_{dx} - u_x \tag{106}$$

$$\tilde{u_\phi} = u_{d\phi} - u_\phi \tag{107}$$

Adding and Subtracting u_{dx} in equation (84) and equation (86)

Therefore, we get

$$(m_x + m_\phi \sin^2(\phi))\ddot{x} = -c_x\dot{x} + m_\phi g \cos(\phi)\sin(\phi) + m_\phi l\dot{\phi}^2 \sin(\phi) + u_{dx} - \tilde{u_x} \tag{108}$$

$$m_\phi l^2\ddot{\phi} = -c_\phi\dot{\phi} - m_\phi lg\sin(\phi) - m_\phi l\cos(\phi)\ddot{x} + u_{d\phi} - \tilde{u_\phi} \tag{109}$$

Therefore,

$$Y_z\theta_z = \begin{bmatrix} (\ddot{x}) & (\sin^2(\phi)\ddot{x} - g\cos(\phi)\sin(\phi)) & (\dot{x}) & (-\dot{\phi}^2\sin(\phi)) \end{bmatrix} \begin{bmatrix} m_x \\ m_\phi \\ c_x \\ m_\phi l \end{bmatrix}$$

$$Y_z\theta_z = u_x \tag{110}$$

Multiplying by Y^T, throughout

$$Y_z^T Y_z\theta_z = Y_z^T u_x \tag{111}$$

Now, consider the lyapunov candidate for cart,

$$V_x(\zeta, t) = \frac{1}{2}e_x^2 + \frac{1}{2}(m_x + m_\phi \sin^2(\phi))r_x^2 + \frac{1}{2}\tilde{u_x}^2 + \frac{1}{2}\tilde{\theta}^T \Gamma_x^{-1}\tilde{\theta} + \frac{1}{2\gamma_{ax}}\tilde{a_x}^2 \tag{112}$$

$$\dot{V}_x(\zeta) = e_x\dot{e}_x + m_\phi \sin(\phi)\cos(\phi)\;\dot{\phi}\;r_x^2 + (m_x + m_\phi \sin^2(\phi))\;\dot{r_x}(r_x)\; + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta}^T \Gamma_x^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{113}$$

Now, multiplying equation (90) by $(m_x + m_\phi \sin^2(\phi))$, we get,

For collar

$$(m_x + m_\phi \sin^2(\phi))\ddot{e_x} = (m_x + m_\phi \sin^2(\phi))(\ddot{x_d} - \ddot{x}) \tag{114}$$

$$(m_x + m_\phi \sin^2(\phi))\ddot{e_x} = (m_x + m_\phi \sin^2(\phi))\ddot{x_d} - (m_x + m_\phi \sin^2(\phi))\ddot{x} \tag{115}$$

From equation (108), we have

$$(m_x + m_\phi \sin^2(\phi))\ddot{x} = -c_x\dot{x} + m_\phi g\cos(\phi)\sin(\phi) + m_\phi l\dot{\phi}^2\sin(\phi) + u_{dx} - \tilde{u_x}$$

Now, substituting equation (108) in equation (115), we get,

$$(m_x+m_\phi \sin^2(\phi))\ddot{e_x} = (m_x+m_\phi \sin^2(\phi))\ddot{x_d} - (-c_x\dot{x}+m_\phi g\cos(\phi)\sin(\phi)+m_\phi l\dot{\phi}^2\sin(\phi)+u_{dx}-\tilde{u_x}) \tag{116}$$

$$(m_x+m_\phi \sin^2(\phi))\ddot{e_x} = (m_x+m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x} - m_\phi g\cos(\phi)\sin(\phi) - m_\phi l\dot{\phi}^2\sin(\phi) - u_{dx} + \tilde{u_x}) \tag{117}$$

Now, from equation (95), we have, $\dot{r_x} = \ddot{e_x} + \alpha_x\dot{e_x}$

Now, multiplying equation (95) by $(m_x + m_\phi \sin^2(\phi))$, we get,

For collar,

$$(m_x + m_\phi \sin^2(\phi))\dot{r_x} \ = \ (m_x + m_\phi \sin^2(\phi))\ (\ddot{e_x} + \alpha_x\dot{e_x}) \tag{118}$$

$$(m_x + m_\phi \sin^2(\phi))\dot{r_x} \ = \ (m_x + m_\phi \sin^2(\phi))\ \ddot{e_x} + (m_x + m_\phi \sin^2(\phi))(\alpha_x\dot{e_x}) \tag{119}$$

Now, we have from equation (117), $(m_x + m_\phi \sin^2(\phi))\ddot{e_x}$

Substituting equation (117), in equation (119), we get,

$$\begin{aligned}(m_x+m_\phi \sin^2(\phi))\dot{r_x} \ = \ & (m_x+m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x} - m_\phi g\cos(\phi)\sin(\phi) - m_\phi l\dot{\phi}^2\sin(\phi) - u_{dx} + \tilde{u_x}) + \\ & (m_x + m_\phi \sin^2(\phi))(\alpha_x\dot{e_x}) \quad (AA)\end{aligned}$$

Now, let's only consider, second term from equation (113),

$$m_\phi \sin(\phi)\cos(\phi)\ \dot{\phi}\ r_x^2 + (m_x + m_\phi \sin^2(\phi))\ \dot{r_x}(r_x)$$

Now, we have from equation (AA), $(m_x + m_\phi \sin^2(\phi))\dot{r_x}$

Substituting equation (AA), we get,

$$m_\phi \sin(\phi)\cos(\phi)\,\dot{\phi}\,r_x^2 + [\,(m_x + m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x} - m_\phi g\cos(\phi)\sin(\phi) - m_\phi l\dot{\phi}^2\sin(\phi) - u_{dx} + \tilde{u_x}) + (m_x + m_\phi \sin^2(\phi))(\alpha_x \dot{e_x})\,]\,r_x$$

Now, taking r_x common, we get,

$$[m_\phi \sin(\phi)\cos(\phi)\,\dot{\phi}\,r_x + (m_x + m_\phi \sin^2(\phi))\ddot{x_d} + c_x\dot{x} - m_\phi g\cos(\phi)\sin(\phi) - m_\phi l\dot{\phi}^2\sin(\phi) - u_{dx} + \tilde{u_x}) + (m_x + m_\phi \sin^2(\phi))(\alpha_x \dot{e_x})]r_x$$

Taking the terms in bracket in the form of $Y_x\theta_x$, we get

$$Y_x = \begin{bmatrix}(\alpha\dot{e_x} + \ddot{x_d}) & (\sin(\phi)\cos(\phi)\dot{\phi}\,r_x + \sin^2(\phi)\alpha\dot{e_x} + \sin^2(\phi)\ddot{x_d} - g\cos(\phi)\sin(\phi)) & (\dot{x}) & (-\dot{\phi}^2\sin(\phi))\end{bmatrix}$$

$$\theta_x = \begin{bmatrix} m_x \\ m_\phi \\ c_x \\ m_\phi l \end{bmatrix}$$

Therefore, equation (113), becomes

$$\dot{V_x}(\zeta) = e_x\dot{e_x} + r_x(Y_x\theta_x - u_{dx} + \tilde{u_x}) \ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta}^T\Gamma_x^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{120}$$

Also, we have, $\dot{e_x} = r_x - \alpha_x e_x$

Substituting it in equation (120), we have,

$$\dot{V}_x(\zeta) = e_x(r_x - \alpha_x e_x) + r_x(Y_x\theta_x - u_{dx} + \tilde{u_x})\ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta}^T\Gamma_x^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{121}$$

Now, designing, $u_{dx} = Y_x\hat{\theta_x} + e_x + \beta_x r_x$

Now, substituting u_{dx} in equation (121)

$$\dot{V}_x(\zeta) = e_x(r_x - \alpha e_x) + r_x(Y_x\theta_x - (Y_x\hat{\theta_x} + e_x + \beta_x r_x) + \tilde{u_x})\ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta}^T\Gamma_x^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{122}$$

$$\dot{V}_x(\zeta) = e_x(r_x - \alpha e_x) + r_x(Y_x\theta_x - Y_x\hat{\theta_x} - e_x - \beta_x r_x + \tilde{u_x})\ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta}^T\Gamma_x^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{123}$$

$$\dot{V}_x(\zeta) = e_x r_x - \alpha e_x^2 + r_x Y_x\tilde{\theta_x} - e_x r_x - \beta_x r_x^2 + r_x\tilde{u_x}\ + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta}^T\Gamma_x^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{124}$$

Now, simplifying and cancelling some terms we get,

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}\dot{\tilde{u_x}} + \tilde{\theta}^T\Gamma_x^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\tilde{a_x}} \tag{125}$$

Now, we have,

$$\tilde{u_x} = u_{dx} - u_x$$

Therefore,

$$\dot{\tilde{u_x}} = \dot{u_{dx}} - \dot{u_x}$$

$$\dot{\tilde{\theta}}_x = -\,\dot{\hat{\theta}}_x$$

$$\dot{\tilde{a}}_x = -\,\dot{\hat{a}}_x$$

Substituting all in equation (125), we get

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} - \dot{u_x}) - \tilde{\theta}^T\Gamma_x^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{126}$$

Now, we have, $\dot{u_x} = -a_x u_x + \mu_x$

Substituting it in equation (126), we get

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} - (-a_x u_x + \mu_x)) - \tilde{\theta}^T\Gamma_x^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{127}$$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y\tilde{\theta} + r_x\tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} + a_x u_x - \mu_x) - \tilde{\theta}^T\Gamma_x^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{128}$$

we know, $\dot{u_{dx}} = \dot{Y_x}\hat{\theta_x} + Y_x\dot{\hat{\theta}}_x + \dot{e_x} + \beta_x\dot{r_x}$

Now, designing $\mu_x = \dot{u_{dx}} + \hat{a_x}u_x + s\tilde{u_x} + r_x$

Now, substituting μ_x in equation (128), we get,

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y_x\tilde{\theta}_x + r_x\tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} + a_x u_x - (\dot{u_{dx}} + \hat{a_x}u_x + s\tilde{u_x} + r_x)) - \tilde{\theta}^T\Gamma_x^{-1}\dot{\hat{\theta}} - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{129}$$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y_x \tilde{\theta}_x + r_x \tilde{u_x} + \tilde{u_x}(\dot{u_{dx}} + a_x u_x - \dot{u_{dx}} - \hat{a_x} u_x - s\tilde{u_x} - r_x) - \tilde{\theta}^T \Gamma_x^{-1} \dot{\hat{\theta}} - \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{130}$$

After, cancelling terms, and since

$$\tilde{a_x} = a_x - \hat{a_x} \tag{131}$$

we get,

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 + r_x Y_x \tilde{\theta}_x + \tilde{u_x} \tilde{a_x} u_x - s\tilde{u_x}^2 \ - \ \tilde{\theta}^T \Gamma_x^{-1} \dot{\hat{\theta}} \ - \ \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{132}$$

Where s is a positive constant.

Now, we have, $rY_x\tilde{\theta_x}$, which can also be written as (since r is a scalar):

$$r_x Y_x \tilde{\theta_x} = (r_x Y_x \tilde{\theta_x})^T = \tilde{\theta_x}^T Y_x^T r_x$$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 \ + \tilde{\theta_x}^T Y_x^T r_x + \tilde{u_x} \tilde{a_x} u_x - \ \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \ - \ \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{133}$$

Therefore,

Now, Designing $\dot{\hat{\theta}} = \Gamma_x Y_x^T r_x + \Gamma k_\theta \ (Y_z^T u_x - Y_z^T Y_z \hat{\theta})$ only for $(t < T)$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \tilde{\theta_x}^T Y_x^T r_x + \tilde{u_x} \tilde{a_x} u_x - \tilde{\theta_x}^T \Gamma_x^{-1} (\Gamma Y_x^T r_x + \Gamma k_\theta \ (Y_z^T u_x - Y_z^T Y_z \hat{\theta})) - \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a_x}} \tag{134}$$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \tilde{\theta_x}^T Y_x^T r_x + \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta_x}^T \Gamma_x^{-1}\Gamma_x Y_x^T r_x - \tilde{\theta_x}^T \Gamma_x^{-1}\Gamma_x k_\theta \left(Y_z^T u_x - Y_z^T Y_z \hat{\theta}\right) - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{135}$$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta}^T k_\theta \left(Y_z^T u_x - Y_z^T Y_z \hat{\theta}\right) - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{136}$$

Now, we know $Y_z^T u_x = Y_z^T Y_z \theta_z$

Therefore, substituting the value of $Y_z^T u_x$,

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta}^T k_\theta \left(Y_z^T Y_z \theta_z - Y_z^T Y_z \hat{\theta_z}\right) - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{137}$$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta}^T k_\theta \left(Y_z^T Y_z \tilde{\theta}\right) - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{138}$$

$$\dot{V}_x(\zeta) = -\alpha_x e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta}^T (k_\theta\, Y_z^T Y_z)\tilde{\theta} - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{139}$$

Now, we know after the design of $\dot{\hat{a}}$, we get a term, Where, $(k_\theta\, Y_z^T Y_z)$ is P.S.D

Therefore, for $t < T$, we can conclude that as $t \to \infty$, $e \to 0$, $r \to 0$ and $\tilde{u} \to 0$

But, let us desgin $\dot{\hat{\theta}}_x$ and $\dot{\hat{a_x}}$ after $t > T$,

Designing

$$\dot{\hat{\theta}}_x = \Gamma Y_x^T r_x + \Gamma k_\theta \left(Y^T u_x - Y^T Y \hat{\theta}\right) + \Gamma K_{cl} \left(\sum_{i=1}^{n} Y^T(ti)U(ti) - \sum_{i=1}^{n} Y^T(ti)Y(ti)\hat{\theta}(t)\right)$$

$$\dot{\hat{\theta}}_x = \Gamma Y_x^T r_x + \Gamma k_\theta \left(Y^T u_x - Y^T Y \hat{\theta}\right) + \Gamma K_{cl} \left(\sum_{i=1}^{n} Y^T(ti) Y(ti) \tilde{\theta}(t) \right)$$

$$\dot{\hat{\theta}}_x = \Gamma Y_x^T r_x + \Gamma k_\theta \left(Y^T Y \theta_x - Y^T Y \hat{\theta}\right)_x + \Gamma K_{cl} \left(\sum_{i=1}^{n} Y^T(ti) Y(ti) \tilde{\theta}(t) \right)$$

$$\dot{\hat{\theta}}_x = \Gamma Y_x^T r_x + \Gamma k_\theta \left(Y^T Y \tilde{\theta}\right)_x + \Gamma K_{cl} \left(\sum_{i=1}^{n} Y^T(ti) Y(ti) \tilde{\theta}(t) \right)$$

$$\dot{\hat{\theta}}_x = \Gamma Y_x^T r_x + \Gamma \left(k_\theta \, Y^T Y + K_{cl} \, \sum_{i=1}^{n} Y^T(ti) Y(ti) \right) \tilde{\theta}(t)$$

Substituting $\dot{\hat{\theta}}_x$ in equation (149),

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - s \tilde{u_x}^2 \; + \tilde{\theta_x}^T Y_x^T r_x + \tilde{u_x} \tilde{a_x} u_x - \\ \tilde{\theta_x}^T \Gamma^{-1} (\Gamma Y_x^T r_x + \Gamma \left(k_\theta \, Y^T Y + K_{cl} \, \sum_{i=1}^{n} Y^T(ti) Y(ti) \right) \tilde{\theta}(t) \; - \; \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a}}_x$$

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - s \tilde{u_x}^2 \; + \tilde{\theta_x}^T Y_x^T r_x - \; \tilde{\theta_x}^T Y_x^T r_x + \\ \tilde{u_x} \tilde{a_x} u_x + \tilde{\theta_x}^T \left(k_\theta \, Y^T Y + K_{cl} \, \sum_{i=1}^{n} Y^T(ti) Y(ti) \right) \tilde{\theta}(t) \; - \; \frac{1}{\gamma_{ax}} \tilde{a_x} \dot{\hat{a}}_x$$

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \tilde{u_x}\tilde{a_x}u_x + \tilde{\theta}_x^T\left(k_\theta\, Y^T Y + K_{cl}\, \sum_{i=1}^{n} Y^T(ti)Y(ti)\right)\tilde{\theta}(t) - \frac{1}{\gamma_{ax}}\tilde{a_x}\dot{\hat{a_x}} \tag{140}$$

From equation (84), we have,

$$\dot{u_x} = -a_x u_x + \mu_x \tag{141}$$

Therefore,

$$a_x u_x = \mu_x - \dot{u_x} \tag{142}$$

Now, considering it into $Y\theta$ form, we get,

$$Y_{ax}\theta_{ax} = \mu_x - \dot{u_x}$$

$$Y_{ax}\theta_{ax} = u_{ax}$$

Multiplying by Y_{ax}^T throughout, we get,

$$Y_{ax}^T\, Y_{ax}\, \theta_{ax} = Y_{ax}^T\, u_{ax}$$

Where $Y_{ax} = u_x$, $\theta_{ax} = a_x$ and $u_{ax} = \mu_x - \dot{u_x}$

Now, Designing $\dot{\hat{a_x}}$,

$$\dot{\hat{a_x}} = \gamma_{ax}\tilde{u_x}u_x + \gamma_{ax}\, K_{ax}\left(\sum_{i=1}^{n} Y_{ax}^T(u_{ax}) - \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\, \hat{\theta}_{ax}\right) \tag{143}$$

Now, substituting $Y_{ax}^T\, u_{ax} = Y_{ax}^T\, Y_{ax}\, \theta_{ax}$

$$\dot{\hat{a}_x} = \gamma_{ax}\tilde{u_x}u_x + \gamma_{ax}\ K_{ax}\Bigg(\sum_{i=1}^{n} Y_{ax}^T\ Y_{ax}\ \theta_{ax} - \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ \hat{\theta}_{ax}\Bigg) \tag{144}$$

Now, Equation (147) cannot be used for implementation as we do not the value of a_x and is unknown but we can used it for analysis whereas equation (146) can be used for implementation.

Now, substituting $\dot{\hat{a}_x}$ in equation (143), we get

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - \ s\tilde{u_x}^2 \ + \ \ \tilde{u_x}\tilde{a_x}u_x + \tilde{\theta_x}^T\Bigg(k_\theta\ Y^TY + K_{cl}\ \sum_{i=1}^{n} Y^T(ti)Y(ti)\Bigg)\tilde{\theta}_x(t) - \frac{1}{\gamma_{ax}}\tilde{a_x}\Bigg(\gamma_{ax}\tilde{u_x}u_x + \gamma_{ax}\ K_{ax}\Bigg(\sum_{i=1}^{n} Y_{ax}^T\ Y_{ax}\ \theta_{ax} - \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ \hat{\theta}_{ax}\Bigg)\Bigg)$$

Now, simplifying and later cancelling some terms, we get,

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 + \ \tilde{u_x}\tilde{a_x}u_x - \tilde{u_x}\tilde{a_x}u_x + \tilde{\theta_x}^T\Bigg(k_\theta\ Y^TY + K_{cl}\ \sum_{i=1}^{n} Y^T(ti)Y(ti)\Bigg)\tilde{\theta}_x(t) - \tilde{a_x}\ K_{ax}\Bigg(\sum_{i=1}^{n} Y_{ax}^T\ Y_{ax}\ \tilde{\theta}_{ax}\Bigg)$$

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - \ s\tilde{u_x}^2 \ + \tilde{\theta_x}^T\Bigg(k_\theta\ Y^TY + K_{cl}\ \sum_{i=1}^{n} Y^T(ti)Y(ti)\Bigg)\tilde{\theta}_x(t) - \tilde{a_x}\ K_{ax}\Bigg(\sum_{i=1}^{n} Y_{ax}^T\ Y_{ax}\ \tilde{\theta}_{ax}\Bigg)$$

where, $\Bigg(k_\theta\ Y^TY + K_{cl}\ \sum_{i=1}^{n} Y^T(ti)Y(ti)\Bigg)$ and $\Bigg(K_{ax}\ \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\Bigg)$ is P.S.D after $t > T$

Therefore, $\dot{V}_x(\zeta)$ is Negative Definite.

$$\dot{V}_x(\zeta) \leq -\alpha e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 \; - K_{cl} \; \underline{\lambda} \;_x \; \|\tilde{\theta}(t)\|^2 - K_{ax} \; \underline{\lambda} \;_{ax} \; \|\tilde{a}(t)\|^2 \tag{145}$$

$\lambda_1 = \text{min}\left(\alpha_x, \beta_x, \tilde{u}_x, K_{cl}\underline{\lambda}\;_x \;, K_{ax}\underline{\lambda}\;_{ax}\right)$

$$\dot{V}_x(\zeta) = -\lambda_1 \|\zeta\|^2 \tag{146}$$

Therefore, we can conclude that it is global exponential tracking for collar.

Another approach to this problem is,

From equation (84), we have,

$$\dot{u_x} = -a_x u_x + \mu_x \tag{147}$$

Therefore,

$$\dot{u_x} + a_x u_x = \mu_x \tag{148}$$

$$(Y_{ax})\;(\theta_{ax}) = \begin{bmatrix}(\dot{u_x}) & (-u_x)\end{bmatrix} \begin{bmatrix} 1 \\ a_x \end{bmatrix}$$

Therefore,

$$(Y_{ax})\;(\theta_{ax}) = \mu_x \tag{149}$$

Multiplying by Y^T, throughout, we get

$$Y_{ax}^T(Y_{ax})\;(\theta_{ax}) = Y_{ax}^T\;(\mu_x) \tag{150}$$

Now, Designing $\dot{\hat{a_x}} = \gamma_{ax}\tilde{u_x}u_x + \gamma_{ax}\ K_{ax}\left(\ \sum_{i=1}^{n} Y_{ax}^T(\mu_x) - \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ \hat{\theta}_{ax}\right)$

Also, we know, from equation (153)

$$Y_{ax}^T(Y_{ax})\ (a_{ax}) = Y_{ax}^T\ (\mu_x)$$

therefore we get,

$$\dot{\hat{a_x}} = \gamma_{ax}\tilde{u_x}u_x + \gamma_{ax}\ K_{ax}\left(\ \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ (\theta_{ax}) - \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ \hat{\theta}_{ax}\right)$$

$$\dot{\hat{a_x}} = \gamma_{ax}\tilde{u_x}u_x + \gamma_{ax}\ K_{ax}\left(\ \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ \tilde{\theta}_{ax}\right)$$

$$\dot{V_x}(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2\ + \tilde{u_x}\tilde{a_x}u_x -\ \tilde{\theta_x}^T\left(k_\theta\ Y^TY + K_{cl}\ \sum_{i=1}^{n} Y^T(ti)Y(ti)\right)\tilde{\theta}(t) - \frac{1}{\gamma_{ax}}\tilde{a_x}(\gamma_{ax}\tilde{u_x}u_x + \gamma_{ax}\ K_{ax}\left(\ \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ \tilde{\theta}_{ax}\right)$$

$$\dot{V_x}(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2\ + \tilde{u_x}\tilde{a_x}u_x -\ \tilde{u_x}\tilde{a_x}u_x - \tilde{\theta_x}^T\left(k_\theta\ Y^TY + K_{cl}\ \sum_{i=1}^{n} Y^T(ti)Y(ti)\right)\tilde{\theta}(t) - \tilde{a_x}K_{ax}\left(\ \sum_{i=1}^{n} Y_{ax}^T(Y_{ax})\ \tilde{\theta}_{ax}\right)$$

$$\dot{V}_x(\zeta) = -\alpha e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2 - \tilde{\theta_x}^T \left(k_\theta\, Y^T Y + K_{cl} \sum_{i=1}^{n} Y^T(ti) Y(ti) \right) \tilde{\theta}(t) - \tilde{a}_x \left(K_{ax}\, Y_{ax}^T(Y_{ax}) \right) \tilde{\theta}_{ax} \tag{151}$$

where, $\left(k_\theta\, Y^T Y + K_{cl}\ \sum_{i=1}^{n} Y^T(ti) Y(ti) \right)$ and $\left(K_{ax}\ \sum_{i=1}^{n} Y_{ax}^T(Y_{ax}) \right)$ is P.S.D after $t > T$

$$\dot{V}_x(\zeta) \leq -\alpha e_x^2 - \beta_x r_x^2 - s\tilde{u_x}^2\ - K_{cl}\, \underline{\lambda}_{\,x} \|\tilde{\theta}_x(t)\|^2 - K_{ax}\, \underline{\lambda}_{\,a} x\, \|\tilde{\theta}_{ax}(t)\|^2 \tag{152}$$

$\lambda_1 = \min \left(\alpha_x, \beta_x, \tilde{u}_x, K_{cl} \underline{\lambda}_{\,x}\ , K_{ax} \underline{\lambda}_{\,ax} \right)$

$$\dot{V}_x(\zeta) = -\lambda_1 \|\zeta\|^2 \tag{153}$$

Now for pendulum,

we have from equation (86),

$$m_\phi l^2 \ddot{\phi} = -c_\phi \dot{\phi} - m_\phi l g \sin(\phi) - m_\phi l \cos(\phi) \ddot{x} + u_\phi$$

$$m_\phi l^2 \ddot{\phi} + c_\phi \dot{\phi} + m_\phi l g \sin(\phi) + m_\phi l \cos(\phi) \ddot{x} = u_\phi$$

Therefore, we can write in $Y\theta = U$ form

$$Y_z \theta_z = \begin{bmatrix} (\cos(\phi)\ddot{x} + g\sin(\phi)) & (\dot{\phi}) & (\ddot{\phi})) \end{bmatrix} \begin{bmatrix} m_\phi l \\ c_\phi \\ m_\phi l^2 \end{bmatrix}$$

Therefore, we get, $Y_z \theta_z = u_\phi$

Multiplying by Y_z^T, throughout

$$Y_z^T Y_z \theta_z = Y_z^T u_\phi \tag{154}$$

Now, consider the lyapunov candidate for pendulum,

$$V_\phi(\zeta, t) = \frac{1}{2}e_\phi^2 + \frac{1}{2}(m_\phi l^2)r_\phi^2 + \frac{1}{2}\tilde{u_\phi}^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{1}{2\gamma_{a\phi}}\tilde{a_\phi}^2 \tag{155}$$

$$\dot{V}_\phi(\zeta) = e_\phi \dot{e}_\phi + (m_\phi l^2)\ \dot{r_\phi}(r_\phi)\ + \tilde{u_\phi}\dot{\tilde{u_\phi}} + \tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{ax}}\tilde{a_\phi}\dot{\tilde{a_\phi}} \tag{156}$$

Now we have, $\ddot{e_\phi} = \ddot{\phi}_d - \ddot{\phi}$

Multiplying it by $(m_\phi l^2)$, we get,

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)(\ddot{\phi}_d - \ddot{\phi})$$

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)\ddot{\phi}_d - (m_\phi l^2)\ddot{\phi}$$

Now we already have $(m_\phi l^2)\ddot{\phi}$, therefore substituting it, we get

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)\ddot{\phi}_d - (-c_\phi\dot{\phi} - m_\phi lg\sin(\phi) - m_\phi l\cos(\phi)\ddot{x} + u_{d\phi} - \tilde{u_\phi})$$

$$(m_\phi l^2)\ddot{e_\phi} = (m_\phi l^2)\ddot{\phi}_d + c_\phi\dot{\phi} + m_\phi lg\sin(\phi) + m_\phi l\cos(\phi)\ddot{x} - u_{d\phi} + \tilde{u}_\phi)$$

Now, we know $\dot{r_\phi} = \ddot{e_\phi} + \alpha\dot{e_\phi}$

Multiplying it by, $(m_\phi l^2)$, we get

$$(m_\phi l^2)\dot{r_\phi} = (m_\phi l^2)(\ddot{e_\phi} + \alpha\dot{e_\phi})$$

$$(m_\phi l^2)\dot{r_\phi} = (m_\phi l^2)\ddot{e_\phi} + (m_\phi l^2)\alpha\dot{e_\phi}$$

Now, we know $(m_\phi l^2)\ddot{e_\phi}$, therefore substituting it, we get

$$(m_\phi l^2)\dot{r_\phi} = (m_\phi l^2)\ddot{\phi}_d + c_\phi\dot{\phi} + m_\phi lg\sin(\phi) + m_\phi l\cos(\phi)\ddot{x} - u_{d\phi} + \tilde{u}_\phi) + (m_\phi l^2)\alpha\dot{e_\phi}$$

Therefore, writing it in $Y\theta$ form, we get

$$Y_z\theta_z = \begin{bmatrix}(\cos(\phi)\ddot{x} + g\sin(\phi)) & (\dot{\phi}) & (\ddot{\phi}_d + \alpha\dot{e_\phi})\end{bmatrix}\begin{bmatrix} m_\phi l \\ c_\phi \\ m_\phi l^2\end{bmatrix}$$

$$(m_\phi l^2)\dot{r_\phi} = (Y_\phi\theta_\phi - u_{d\phi} + \tilde{u}_\phi)$$

Now substituting this in $\dot{V}_\phi$, we get

$$\dot{V}_\phi(\zeta) = e_\phi\dot{e}_\phi + (Y_\phi\theta_\phi - u_{d\phi} + \tilde{u}_\phi)(r_\phi) \ + \tilde{u_\phi}\dot{\tilde{u}_\phi} + \tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{a\phi}}\tilde{a_\phi}\dot{\tilde{a}_\phi}$$

Now, we know, $\dot{e_\phi} = r_\phi - \alpha e_\phi$

Therefore, substituting it, we get

$$\dot{V}_\phi(\zeta) = e_\phi(r_\phi - \alpha e_\phi) + (Y_\phi\theta_\phi - u_{d\phi} + \tilde{u}_\phi)(r_\phi) \ + \tilde{u_\phi}\dot{\tilde{u}_\phi} + \tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}} + \frac{1}{\gamma_{a\phi}}\tilde{a_\phi}\dot{\tilde{a}_\phi}$$

Now, designing, $u_{d\phi} = Y_\phi \hat{\theta}_\phi + e_\phi + \beta_\phi r_\phi$

Substituting it now, we get

$$\dot{V}_\phi(\zeta) = e_\phi(r_\phi - \alpha e_\phi) + (Y_\phi \theta_\phi - (Y_\phi \hat{\theta}_\phi + e_\phi + \beta_\phi r_\phi) + \tilde{u}_\phi)(r_\phi) \ + \tilde{u_\phi}\dot{\tilde{u}}_\phi + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} + \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\tilde{a}}_\phi$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u}_\phi(r_\phi) \ + \tilde{u_\phi}\dot{\tilde{u}}_\phi + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} + \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\tilde{a}}_\phi$$

Now, we have,

$$\tilde{u_\phi} = u_{d\phi} - u_\phi$$

Therefore,

$$\dot{\tilde{u}}_\phi = \dot{u_{d\phi}} - \dot{u_\phi}$$

$$\dot{\tilde{\theta}}_\phi = - \dot{\hat{\theta}}_\phi$$

$$\dot{\tilde{a}}_\phi = - \dot{\hat{a}}_\phi$$

Keeping the notation as θ for better understanding of analysis. Substituting all, we get

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u}_\phi(r_\phi) \ + \tilde{u_\phi}(\dot{u_{d\phi}} - \dot{u_\phi}) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a}}_\phi$$

Now, we have, $\dot{u_\phi} = -a_\phi u_\phi + \mu_\phi$

Substituting it, we get

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u}_\phi(r_\phi) \ + \tilde{u_\phi}(\dot{u_{d\phi}} - (-a_\phi u_\phi + \mu_\phi)) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a}}_\phi$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u}_\phi(r_\phi) \ + \tilde{u_\phi}(\dot{u_{d\phi}} + a_\phi u_\phi - \mu_\phi) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a_\phi}}$$

we know, $\dot{u_{d\phi}} = \dot{Y_\phi} \hat{\theta_\phi} + Y_\phi \dot{\hat{\theta_\phi}} + \dot{e_\phi} + \beta_\phi \dot{r_\phi}$

Now, designing $\mu_\phi = \dot{u_{d\phi}} + \hat{a_\phi} u_\phi + s\tilde{u_\phi} + r_\phi$

Now, substituting μ_ϕ, we get,

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u}_\phi(r_\phi) + \tilde{u_\phi}(\dot{u_{d\phi}} + a_\phi u_\phi - (\dot{u_{d\phi}} + \hat{a_\phi} u_\phi + s\tilde{u_\phi} + r_\phi)) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a_\phi}}$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u}_\phi(r_\phi) + \tilde{u_\phi}(\dot{u_{d\phi}} + a_\phi u_\phi - \dot{u_{d\phi}} - \hat{a_\phi} u_\phi - s\tilde{u_\phi} - r_\phi) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a_\phi}}$$

Now simplifying and cancelling some terms we get,

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u}_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u_\phi} \tilde{a}_\phi u_\phi - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a_\phi}}$$

Where s_ϕ is a positive constant.

Now, we have, $r_\phi Y_\phi \tilde{\theta_\phi}$, which can also be written as (since r_ϕ is a scalar):

$$r_\phi Y_\phi \tilde{\theta_\phi} = (r_\phi Y_\phi \tilde{\theta_\phi})^T = \tilde{\theta_\phi}^T Y_\phi^T r_\phi$$

Designing $\dot{\hat{\theta}}$ after $t > T$,

$$\dot{\hat{\theta}} = \Gamma Y_\phi^T r_\phi + \Gamma k_\theta \left(Y_z^T u_\phi - Y_z^T Y_z \hat{\theta_z}\right) + \Gamma K_{cl} \left(\sum_{i=1}^{n} Y_z^T(ti) U_\phi(ti) - \sum_{i=1}^{n} Y_z^T(ti) Y(ti)_z \hat{\theta}(t)_z \right)$$

Since, $Y_z^T u_\phi = Y_z^T Y_z \theta_z$ Substituting that and simplifying we get,

$$\dot{\hat{\theta}} = \Gamma Y_\phi^T r_\phi + \Gamma k_\theta \left(Y_z^T Y_z \tilde{\theta_z}\right) + \Gamma K_{cl} \left(\sum_{i=1}^{n} Y_z^T(ti) Y(ti)_z \tilde{\theta}(t)_z \right)$$

Substituting $\dot{\hat{\theta}}$, we get

$$\begin{aligned} \dot{V}_\phi(\zeta) = {} & -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u}_\phi^2 + r_\phi Y_\phi \tilde{\theta}_\phi + \tilde{u_\phi} \tilde{a}_\phi u_\phi - \\ & \tilde{\theta}^T \Gamma^{-1} \left(\Gamma Y_\phi^T r_\phi + \Gamma k_\theta \left(Y_z^T Y_z \tilde{\theta_z}\right) + \Gamma K_{cl} \left(\sum_{i=1}^{n} Y_z^T(ti) Y(ti)_z \tilde{\theta}(t)_z \right) \right) - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a}}_\phi \end{aligned}$$

now, cancelling terms and simplifying it further we get,

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u}_\phi^2 + \tilde{u_\phi} \tilde{a}_\phi u_\phi - \tilde{\theta}^T \left(k_\theta \left(Y_z^T Y_z \tilde{\theta_z}\right) + K_{cl} \left(\sum_{i=1}^{n} Y_z^T(ti) Y(ti)_z \tilde{\theta}(t)_z \right) \right) - \frac{1}{\gamma_{a\phi}} \tilde{a_\phi} \dot{\hat{a}}_\phi$$

Now, we have $\dot{u_\phi} = -a_\phi u_\phi + \mu_\phi$

Therefore,

$$a_\phi u_\phi = \mu_\phi - \dot{u_\phi}$$

Now, considering it into $Y\theta$ form, we get,

$$Y_{a\phi}\theta_{a\phi} = \mu_\phi - \dot{u_\phi}$$

$$Y_{a\phi}\theta_{a\phi} = u_{a\phi}$$

Multiplying by $Y_{a\phi}^T$ throughout, we get,

$$Y_{a\phi}^T \, Y_{a\phi} \, \theta_{a\phi} = Y_{a\phi}^T \, u_{a\phi}$$

Where $Y_{a\phi} = u_\phi$, $\theta_{a\phi} = a_\phi$ and $u_{a\phi} = \mu_\phi - \dot{u_\phi}$

Now, Designing $\dot{\hat{a_\phi}}$,

$$\dot{\hat{a_\phi}} = \gamma_{a\phi}\tilde{u_\phi}u_\phi + \gamma_{a\phi} \, K_{a\phi}\left(\sum_{i=1}^{n} Y_{a\phi}^T(u_{a\phi}) - \sum_{i=1}^{n} Y_{a\phi}^T(Y_{a\phi}) \, \hat{\theta}_{a\phi}\right) \tag{157}$$

Now, substituting $Y_{a\phi}^T \, u_{a\phi} = Y_{a\phi}^T \, Y_{a\phi} \, \theta_{a\phi}$

$$\dot{\hat{a_\phi}} = \gamma_{a\phi}\tilde{u_\phi}u_\phi + \gamma_{a\phi} \, K_{a\phi}\left(\sum_{i=1}^{n} Y_{a\phi}^T \, Y_{a\phi} \, \theta_{a\phi} - \sum_{i=1}^{n} Y_{a\phi}^T(Y_{a\phi}) \, \hat{\theta}_{a\phi}\right) \tag{158}$$

Now, Equation (158) cannot be used for implementation as we do not the value of a_x and is unknown but we can used it for analysis whereas equation (157) can be used for implementation.

Now, substituting $\dot{\hat{a}_\phi}$, we get

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u}_\phi^2 + \tilde{u_\phi}\tilde{a}_\phi u_\phi - \tilde{\theta}^T\left(k_\theta\ (Y_z^T Y_z \tilde{\theta_z}) + K_{cl}\left(\sum_{i=1}^{n} Y_z^T(ti)Y(ti)_z\tilde{\theta}(t)_z\right)\right) - \frac{1}{\gamma_{a\phi}}\tilde{a_\phi}\left(\gamma_{a\phi}\tilde{u_\phi}u_\phi + \gamma_{a\phi}\ K_{a\phi}\left(\sum_{i=1}^{n} Y_{a\phi}^T\ Y_{a\phi}\ \theta_{a\phi} - \sum_{i=1}^{n} Y_{a\phi}^T(Y_{a\phi})\ \hat{\theta}_{a\phi}\right)\right)$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u}_\phi^2 + \tilde{u_\phi}\tilde{a}_\phi u_\phi - \tilde{\theta}^T\left(k_\theta\ (Y_z^T Y_z \tilde{\theta_z}) + K_{cl}\left(\sum_{i=1}^{n} Y_z^T(ti)Y(ti)_z\tilde{\theta}(t)_z\right)\right) - \frac{1}{\gamma_{a\phi}}\tilde{a_\phi}\left(\gamma_{a\phi}\tilde{u_\phi}u_\phi + \gamma_{a\phi}\ K_{a\phi}\left(\sum_{i=1}^{n} Y_{a\phi}^T(Y_{a\phi})\ \tilde{\theta}_{a\phi}\right)\right)$$

Now, simplifying and cancelling terms, we get,

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u}_\phi^2 - \tilde{\theta}^T\left(k_\theta\ (Y_z^T Y_z \tilde{\theta_z}) + K_{cl}\left(\sum_{i=1}^{n} Y_z^T(ti)Y(ti)_z\tilde{\theta}(t)_z\right)\right) - \tilde{a_\phi}\left(K_{a\phi}\left(\sum_{i=1}^{n} Y_{a\phi}^T(Y_{a\phi})\ \tilde{\theta}_{a\phi}\right)\right)$$

$$\dot{V}_\phi(\zeta) = -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u}_\phi^2 - \tilde{\theta}^T\left(k_\theta\ (Y_z^T Y_z) + K_{cl}\ \left(\sum_{i=1}^{n} Y_z^T(ti)Y(ti)_z\right)\right)\tilde{\theta} - \tilde{a_\phi}\left(\ K_{a\phi}\left(\sum_{i=1}^{n} Y_{a\phi}^T(Y_{a\phi})\ \tilde{\theta}_{a\phi}\right)\right)$$

where, $\left(k_\theta\ Y_z^T Y_z + K_{cl}\ \sum_{i=1}^{n} Y^T(ti)_z Y(ti)_z\right)$ and $\left(K_{a\phi}\ \sum_{i=1}^{n} Y_{a\phi}^T(Y_{a\phi})\right)$ is P.S.D after $t > T$

Therefore, $\dot{V}_\phi(\zeta)$ is Negative Definite.

$$\dot{V}_\phi(\zeta) \leq -\alpha_\phi e_\phi^2 - \beta_\phi r_\phi^2 - s_\phi \tilde{u_\phi}^2\ - K_{cl}\ \underline{\lambda}\ _\phi\ \|\tilde{\theta}(t)\|^2 - K_{a\phi}\ \underline{\lambda}\ _{a\phi}\ \|\tilde{a}(t)\|^2 \qquad (159)$$

$\lambda_1 = \min\left(\alpha_\phi, \beta_\phi, \tilde{u}_\phi, K_{cl}\underline{\lambda}\ _\phi\ , K_{a\phi}\underline{\lambda}\ _{a\phi}\right)$

$$\dot{V}_x(\zeta) = -\lambda_1\|\zeta\|^2 \qquad (160)$$

Therefore, we can conclude from the time derivative of both the lyapunov candidate for collar and pendulum that it is global exponential tracking.

b) Using the results of Part (1.a), we simulated the dynamics for 100 random values of $x_d, \phi_d, f_{xd}, f_{\phi d}$, $x(t=0), \phi(t=0)$ within our domain but keep

$$x(t=0) = 0$$

$$u_x(t=0) = 0$$

$$\phi(t=0) = 0$$

$$u_\phi(t=0) = 0$$

Attach below are the plots.

ii) We used two methods to design our problem. a) Integral Gradient method b) Concurrent Learning method The integral gradient method gives us the time derivative of Lyapunov function as negative semi definite and later using barbalet's lemma or Lasalle Yoshizawa theorem, we can only show global asymptotic tracking (GAT). The concurrent learning method gives us the time derivative of Lyapunov function as negative definite which we can take ahead to find bounds and show global exponential tracking.

iii) The plot of the norm of the tracking errors over time on a single plot is attached below. The plot of the norm of the estimation errors over time on a single plot is attached below.

iv) A. plot of the total input, the error feedback portions of the input, and the estimated feedforward portions of the input over time is attached below.

B. plot of the value of the Lyapunov function over time along with the best determined theoretical bound of the Lyapunov function over time is attached below.

v) plot of the norm of the difference between the total input and the error feedback portions of the input over time is attached below.

vi) plot of the norm of the difference between the total input and the estimated feedforward portions of the input over time is attached below.

Vii) We can conclude from the design and the plots that our simulation matches our stability results. As discussed above, we can achieve global asymptotic tracking from integral gradient method and global exponential tracking using concurrent learning method. Yes, the controller transitions predominantly from using error feedback terms to estimated feedforward portions as seen in the plot below. The bounds of my Lyapunov function are,

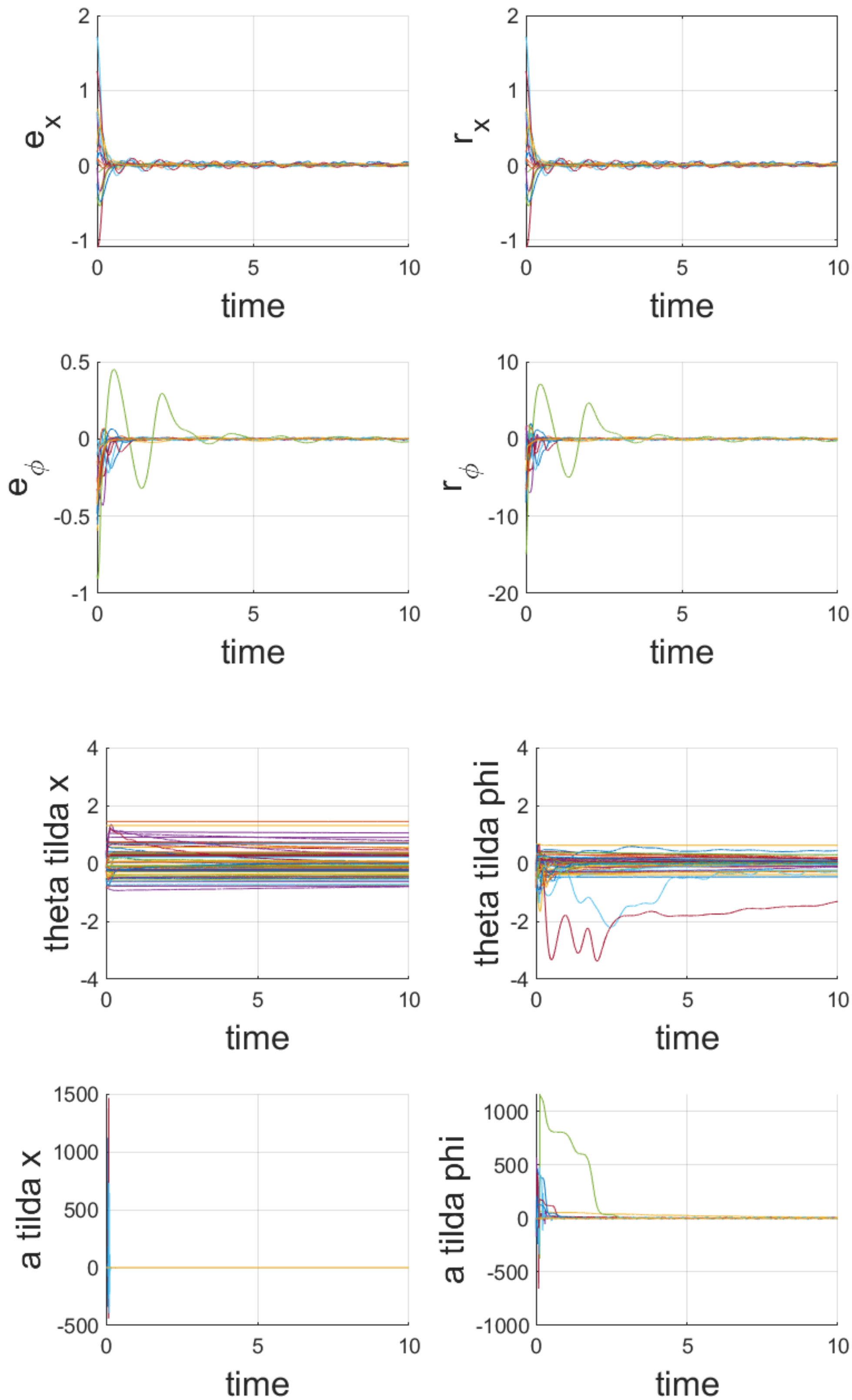
$$min\left\{\frac{1}{2},\frac{1}{2}m,\frac{1}{2},\frac{1}{2}min(\Gamma^{-1}),\frac{1}{2\gamma_a}\right\}\|\zeta\|^2 \leq V(\zeta) \leq max\left\{\frac{1}{2},\frac{1}{2}m,\frac{1}{2},\frac{1}{2}max(\Gamma^{-1}),\frac{1}{2\gamma_a}\right\}\|\zeta\|^2 \tag{161}$$

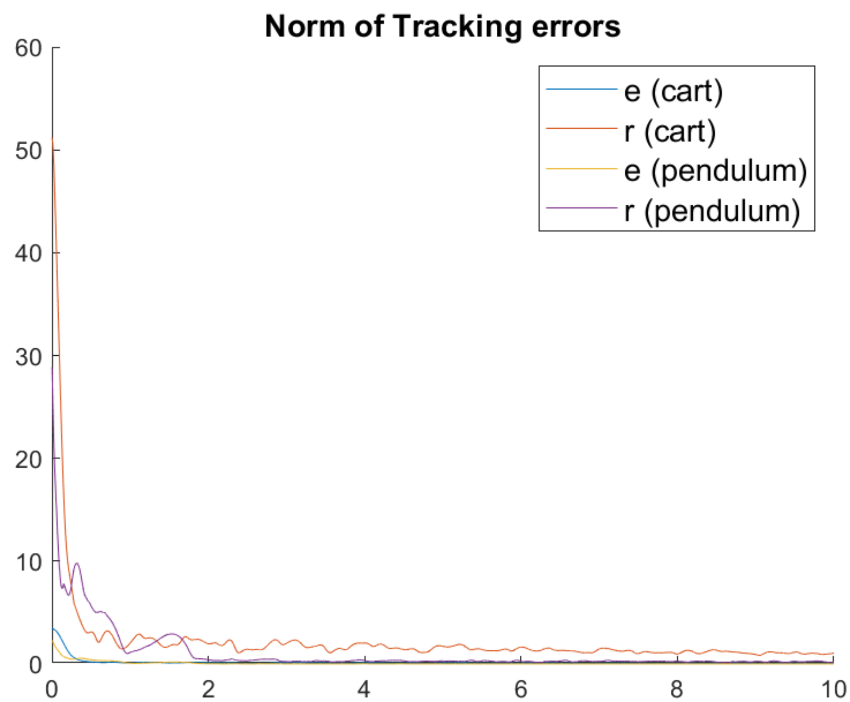
Now, multiplying by (-1) we get,

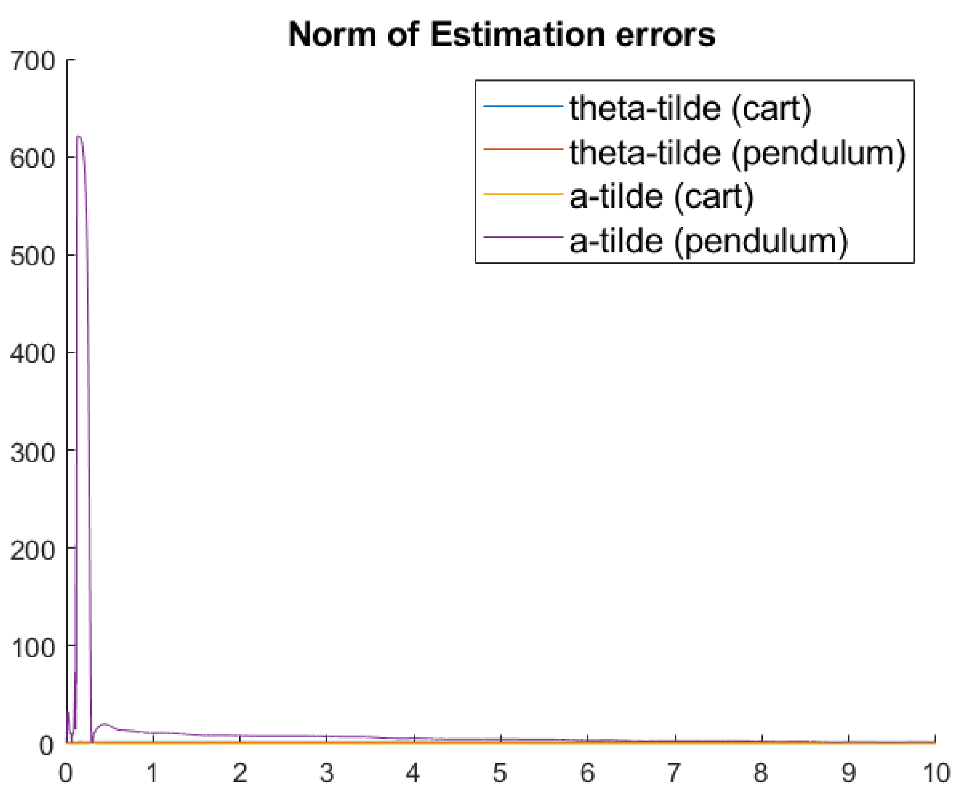
$$-min\left\{\frac{1}{2},\frac{1}{2}m,\frac{1}{2},\frac{1}{2}min(\Gamma^{-1}),\frac{1}{2\gamma_a}\right\}\|\zeta\|^2 \geq -V(\zeta) \geq -max\left\{\frac{1}{2},\frac{1}{2}m,\frac{1}{2},\frac{1}{2}max(\Gamma^{-1}),\frac{1}{2\gamma_a}\right\}\|\zeta\|^2$$

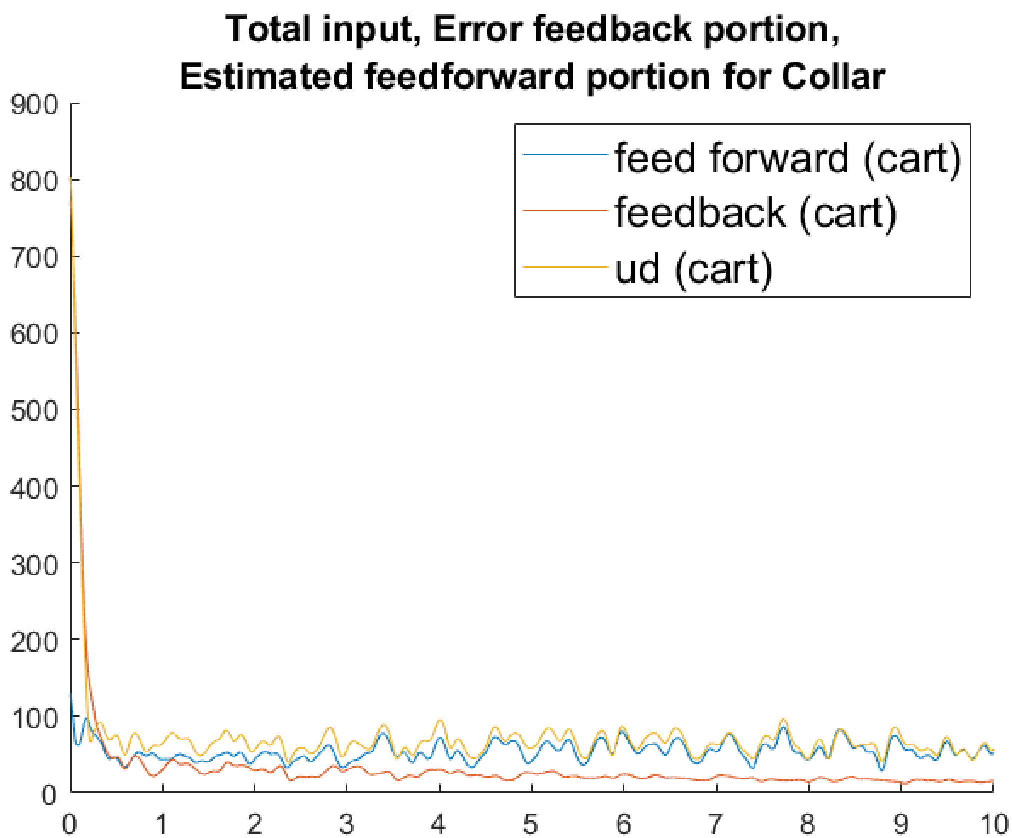
Now, Dividing by $max\left\{\frac{1}{2},\frac{1}{2}m,\frac{1}{2},\frac{1}{2}max(\Gamma^{-1}),\frac{1}{2\gamma_a}\right\}$ throughout, we get

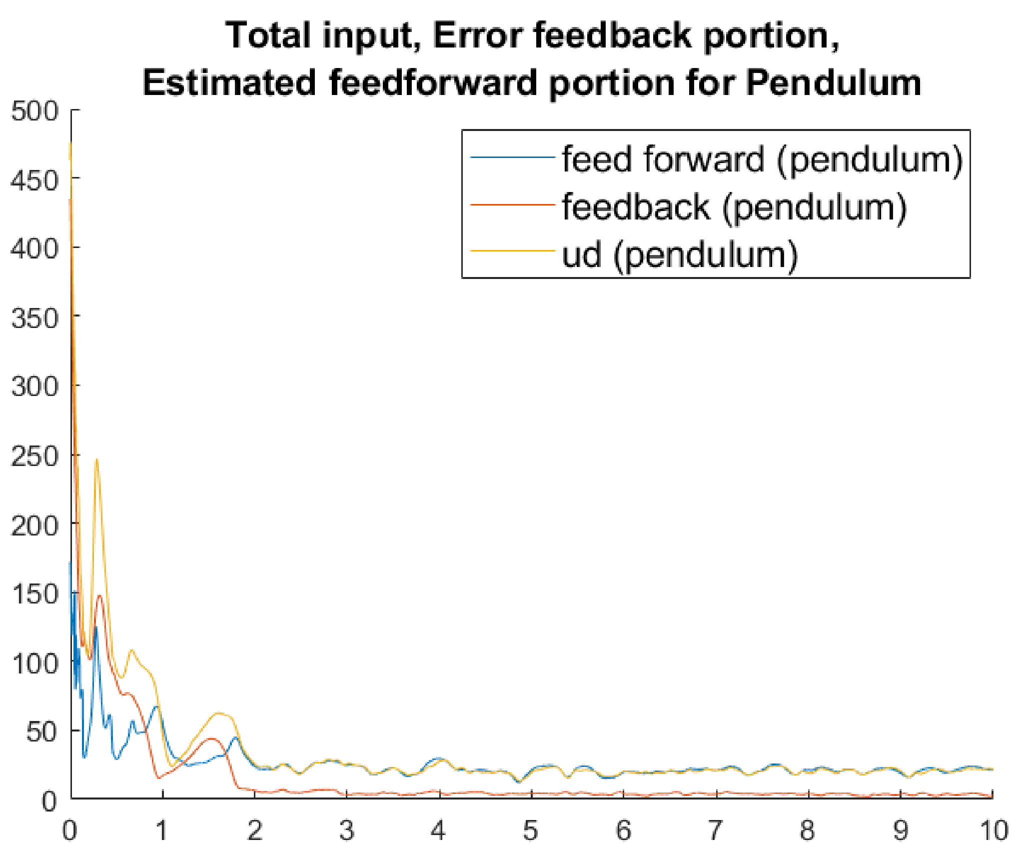
$$\frac{-1}{max\left\{\frac{1}{2},\frac{1}{2}m,\frac{1}{2},\frac{1}{2}max(\Gamma^{-1}),\frac{1}{2\gamma_a}\right\}}V(\zeta) \geq -\|\zeta\|^2$$

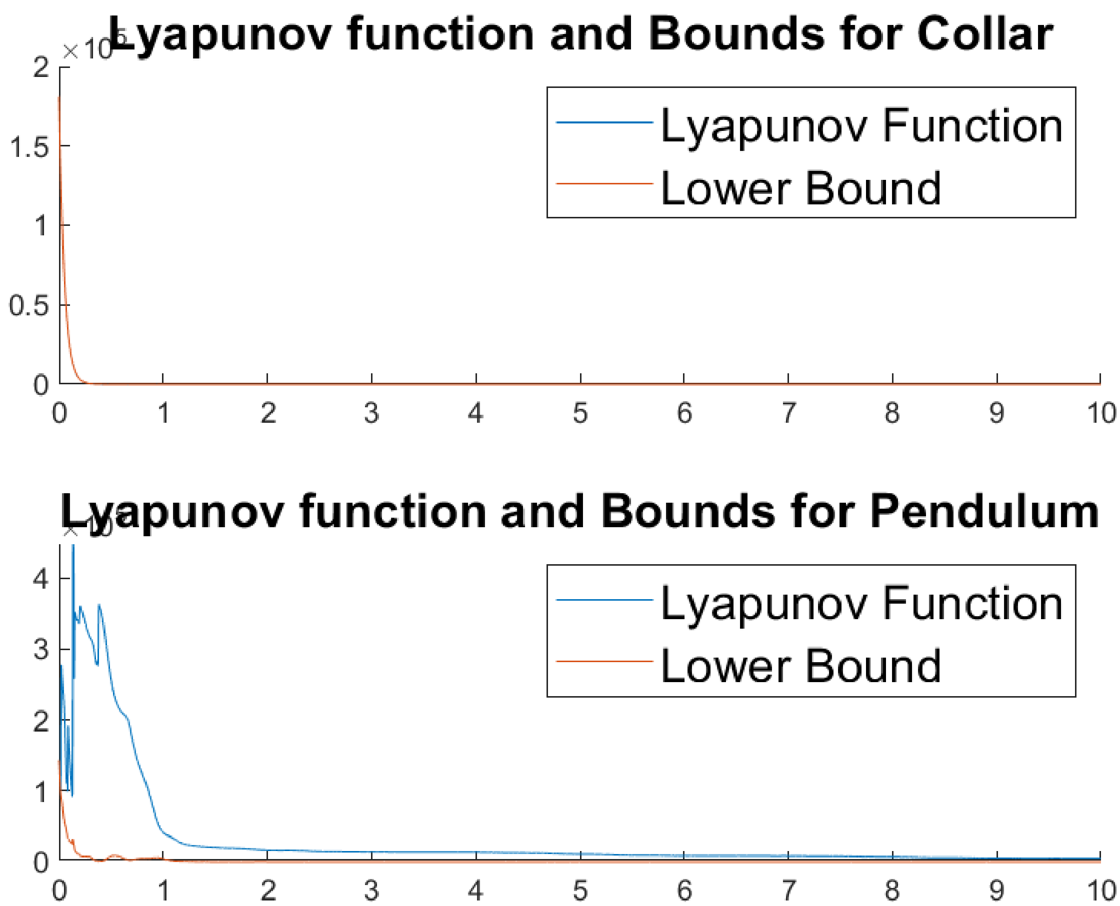

e_x
r_x
e_φ
r_φ
theta tilda x
theta tilda phi
a tilda x
a tilda phi
time

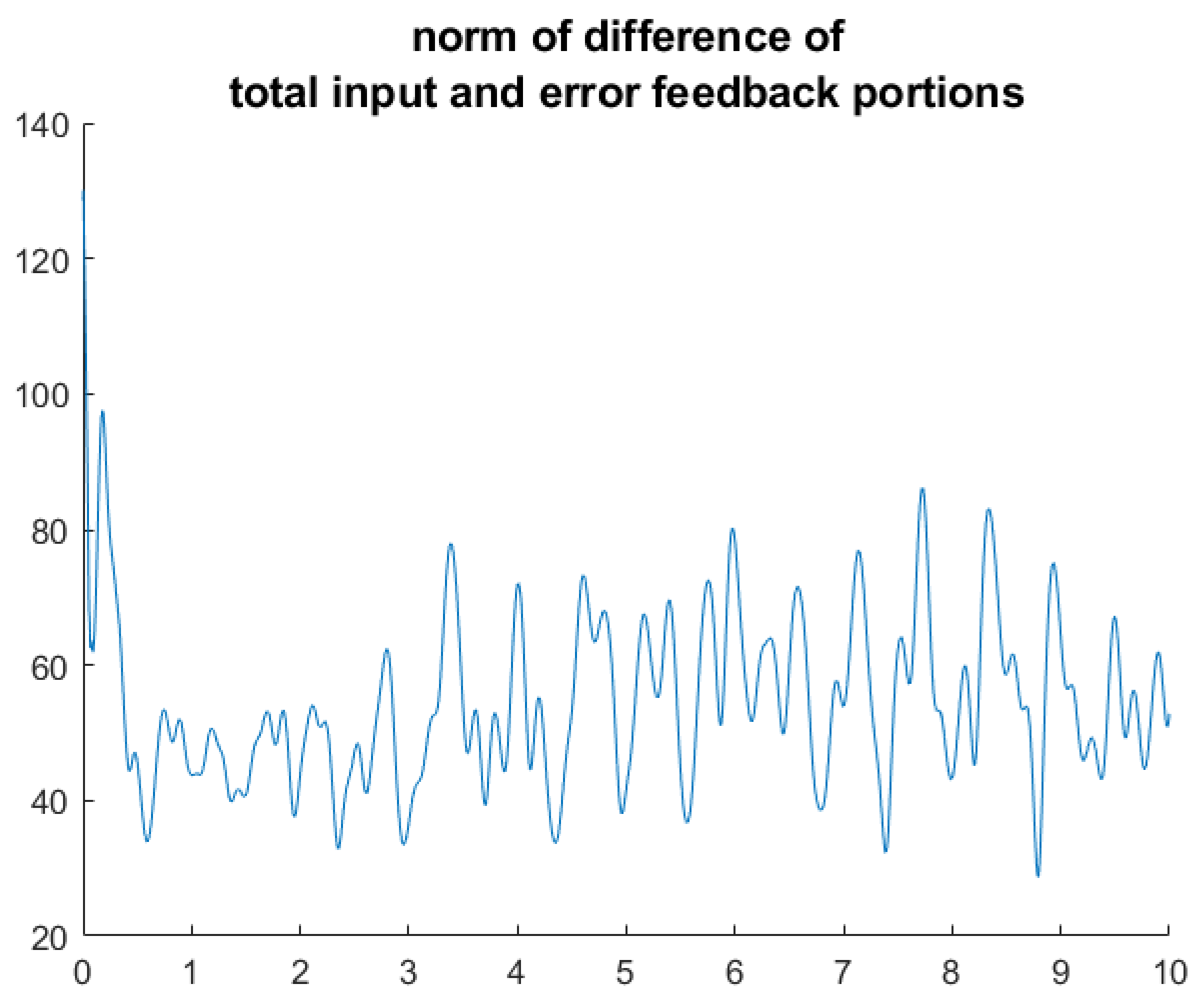

Norm of Tracking errors
e (cart)
r (cart)
e (pendulum)
r (pendulum)
60
50
40
30
20
10
0
0
2
4
6
8
10

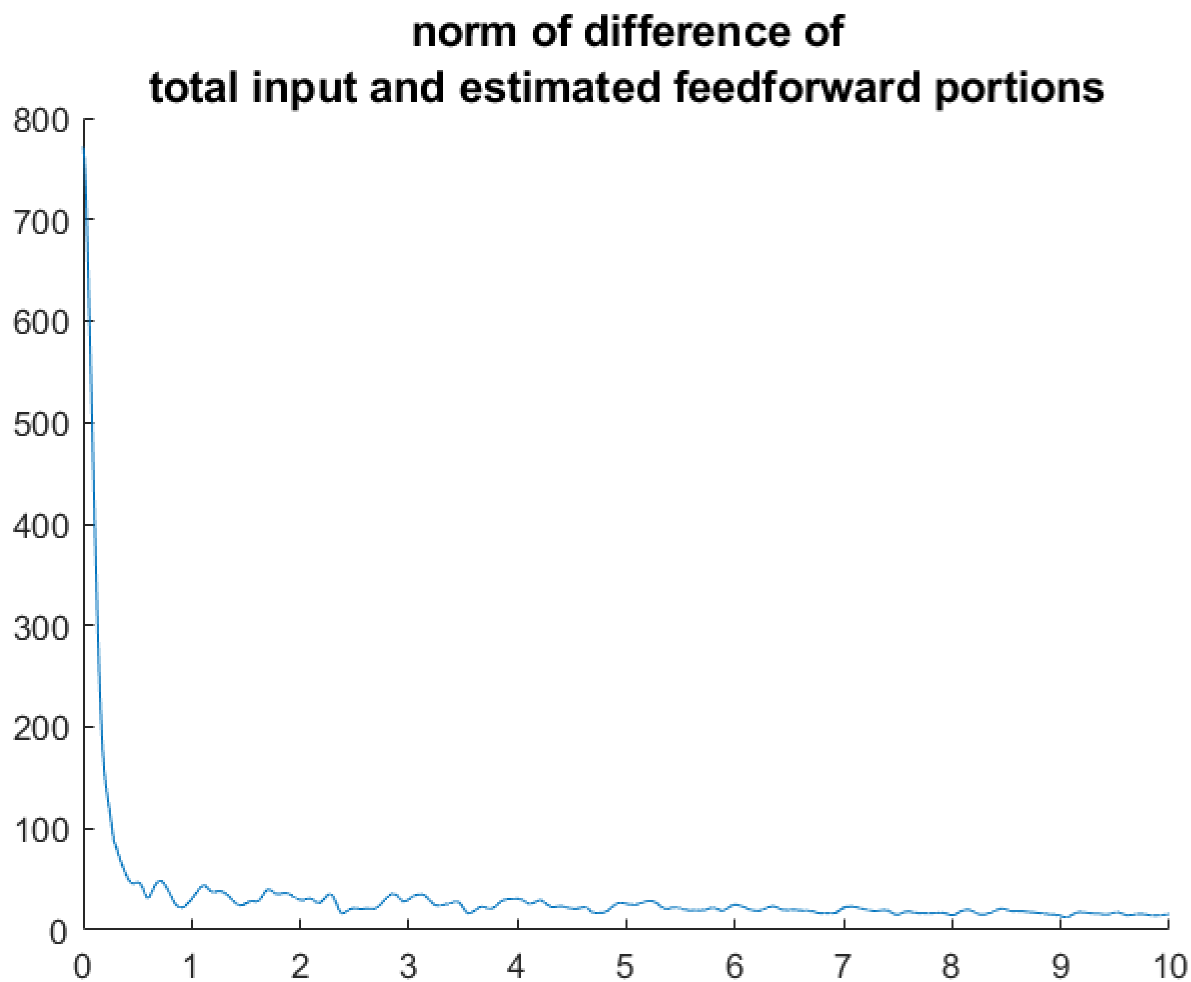

Norm of Estimation errors
theta-tilde (cart)
theta-tilde (pendulum)
a-tilde (cart)
a-tilde (pendulum)
700
600
500
400
300
200
100
0
0
1
2
3
4
5
6
7
8
9
10


Total input, Error feedback portion,
Estimated feedforward portion for Collar
feed forward (cart)
feedback (cart)
ud (cart)
900
800
700
600
500
400
300
200
100
0
0
1
2
3
4
5
6
7
8
9
10


Total input, Error feedback portion,
Estimated feedforward portion for Pendulum
feed forward (pendulum)
feedback (pendulum)
ud (pendulum)
500
450
400
350
300
250
200
150
100
50
0
0
1
2
3
4
5
6
7
8
9
10


Lyapunov function and Bounds for Collar
Lyapunov Function
Lower Bound
0
0.5
1
1.5
2
0 1 2 3 4 5 6 7 8 9 10
Lyapunov function and Bounds for Pendulum
Lyapunov Function
Lower Bound
0
1
2
3
4
0 1 2 3 4 5 6 7 8 9 10


norm of difference of
total input and error feedback portions
140
120
100
80
60
40
20
0
1
2
3
4
5
6
7
8
9
10


norm of difference of
total input and estimated feedforward portions
800
700
600
500
400
300
200
100
0
0
1
2
3
4
5
6
7
8
9
10