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PROJECT 1

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Dynamics

Consider the two-link dynamic system

$$M(\phi)\ddot{\phi} + C(\phi, \dot{\phi}) + G(\phi) = \tau$$

where, $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$, $\dot{\phi} = \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$, $\ddot{\phi} = \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix}$, $\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \in \mathbb{R}^2$

$$M(\phi) = \begin{bmatrix} m_1 l_1^2 + m_2(l_1^2 + 2l_1 l_2 c_2 + l_2^2) & m_2(l_1 l_2 c_2 + l_2^2) \\ m_2(l_1 l_2 c_2 + l_2^2) & m_2 l_2^2 \end{bmatrix}$$

$$C(\phi, \dot{\phi}) = \begin{bmatrix} -2m_2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\phi_2}^2 \\ m_2 l_1 l_2 s_2 \dot{\phi_1}^2 \end{bmatrix}$$

$$G(\phi) = \begin{bmatrix} (m_1 + m_2) g l_1 c_1 + m_2 g l_2 c_{12} \\ m_2 g l_2 c_{12} \end{bmatrix}$$

$$c_i = \cos(\phi_i),$$

$$s_i = \sin(\phi_i),$$

$$c_{12} = \cos(\phi_1 + \phi_2),$$

$$s_{12} = \sin(\phi_1 + \phi_2).$$

Design of Gradient Adaptive Controller

Let the tracking error be,

$$e = \phi_d - \phi$$

$$\dot{e} = \dot{\phi}_d - \dot{\phi}$$

$$\ddot{e} = \ddot{\phi}_d - \ddot{\phi}$$

Let the filtered tracking error be,

$$r = \dot{e} + \alpha e$$

$$\dot{r} = \ddot{e} + \alpha \dot{e}$$

Now, multiplying throughout by $M(\phi)$, we get

$$M(\phi)\dot{r} = M(\phi)\ddot{e} + M(\phi)\alpha\dot{e}$$

Now substituting the value of $\ddot{e}$, we get

$$M(\phi)\dot{r} = M(\phi)(\ddot{\phi}_d - \ddot{\phi}) + M(\phi)\alpha\dot{e}$$

$$M(\phi)\dot{r} = M(\phi)\ddot{\phi}_d - M(\phi)\ddot{\phi} + M(\phi)\alpha\dot{e}$$

Now, from the dynamics we know,

$$M(\phi)\ddot{\phi} + C(\phi, \dot{\phi}) + G(\phi) = \tau$$

Therefore,

$$M(\phi)\ddot{\phi} = -C(\phi, \dot{\phi}) - G(\phi) + \tau$$

Now, substituting the value of $M(\phi)\ddot{\phi}$ in $M(\phi)\dot{r}$, we get

$$M(\phi)\dot{r} = M(\phi)\ddot{\phi}_d - (-C(\phi, \dot{\phi}) - G(\phi) + \tau) + M(\phi)\alpha\dot{e}$$

$$M(\phi)\dot{r} = M(\phi)(\ddot{\phi}_d + \alpha\dot{e}) + C(\phi, \dot{\phi}) + G(\phi) - \tau$$

Now, the parametric error $\theta \in \mathbb{R}^p$

$$\tilde{\theta} = \theta - \hat{\theta}$$

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$$

Now let the lyapunov candidate be,

$$V(\zeta) = \frac{1}{2}e^T e + \frac{1}{2}r^T M(\phi) r + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta}$$

Now, the time derivative of lyapunov candidate is,

$$V\dot{(\zeta)} = e^T\dot{e} + \frac{1}{2}r^T\dot{M}(\phi)r + r^T M(\phi)\dot{r} + \tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}}$$

$$\dot{M}(\phi) = \frac{d}{dt}M(\phi)$$

$$\dot{M}(\phi) = \begin{bmatrix} m_2(-2l_1l_2s_2\dot{\phi}_2) & m_2(-l_1l_2s_2\dot{\phi}_2) \\ m_2(-l_1l_2s_2\dot{\phi}_2) & 0 \end{bmatrix}$$

$$V(\dot{\zeta}) = e^T(r-\alpha e) + \frac{1}{2}r^T\dot{M}(\phi)r + r^T(M(\phi)(\ddot{\phi}_d + \alpha\dot{e}) + C(\phi,\dot{\phi}) + G(\phi) - \tau) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

$$V(\dot{\zeta}) = e^Tr - e^T\alpha e + r^T(M(\phi)(\ddot{\phi}_d + \alpha\dot{e}) + C(\phi,\dot{\phi}) + G(\phi) + \frac{1}{2}\dot{M}(\phi)r - \tau) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

We want to show,

$$Y\theta = M(\phi)(\ddot{\phi}_d + \alpha\dot{e}) + C(\phi,\dot{\phi}) + G(\phi) + \frac{1}{2}\dot{M}(\phi)r$$

We need to find the minimum realization of θ

Let $\ddot{\phi}_d + \alpha\dot{e} = \eta$

Therefore,

$$Y\theta = M(\phi)\eta + C(\phi,\dot{\phi}) + G(\phi) + \frac{1}{2}\dot{M}(\phi)r$$

Now, let's consider only the $M(\phi)\eta$ part,

$$M(\phi)\eta = \begin{bmatrix} m_1l_1^2 + m_2(l_1^2 + 2l_1l_2c_2 + l_2^2) & m_2(l_1l_2c_2 + l_2^2) \\ m_2(l_1l_2c_2 + l_2^2) & m_2l_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$M(\phi)\eta = \begin{bmatrix} m_1l_1^2 + m_2l_1^2 + m_2l_2^2 + m_22l_1l_2c_2 & m_2l_1l_2c_2 + m_2l_2^2 \\ m_2l_1l_2c_2 + m_2l_2^2 & m_2l_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

Let,

$$\theta_1 = m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2$$

$$\theta_2 = m_2 l_1 l_2$$

$$\theta_3 = m_2 l_2^2$$

Therefore,

$$M(\phi)\eta = \begin{bmatrix} \theta_1 + 2\theta_2 c_2 & \theta_2 c_2 + \theta_3 \\ \theta_2 c_2 + \theta_3 & \theta_3 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$M(\phi)\eta = \begin{bmatrix} (\theta_1 + 2\theta_2 c_2)\eta_1 + (\theta_2 c_2 + \theta_3)\eta_2 \\ (\theta_2 c_2 + \theta_3)\eta_1 + (\theta_3)\eta_2 \end{bmatrix}$$

$$M(\phi)\eta = \begin{bmatrix} \eta_1 & 2c_2\eta_1 + c_2\eta_2 & \eta_2 \\ 0 & c_2\eta_1 & (\eta_1 + \eta_2) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Now, let's consider $C(\phi, \dot{\phi})$,

$$C(\phi, \dot{\phi}) = \begin{bmatrix} -2m_2 l_1 l_2 s_2 \dot{\phi_1}\dot{\phi_2} - m_2 l_1 l_2 s_2 \dot{\phi_2}^2 \\ m_2 l_1 l_2 s_2 \dot{\phi_1}^2 \end{bmatrix}$$

Therefore,

$$C(\phi, \dot{\phi}) = \begin{bmatrix} -2\theta_2 s_2 \dot{\phi_1}\dot{\phi_2} - \theta_2 s_2 \dot{\phi_2}^2 \\ \theta_2 s_2 \dot{\phi_1}^2 \end{bmatrix}$$

$$C(\phi,\dot{\phi}) = \begin{bmatrix} 0 & (-2s_2\dot{\phi}_1\dot{\phi}_2 - s_2\dot{\phi}_2^{\ 2}) & 0 \\ 0 & s_2\dot{\phi}_1^{\ 2} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Now, look at $G(\phi)$,

$$G(\phi) = \begin{bmatrix} (m_1+m_2)gl_1c_1 + m_2gl_2c_{12} \\ m_2gl_2c_{12} \end{bmatrix}$$

$$G(\phi) = \begin{bmatrix} (m_1l_1+m_2l_1)gc_1 + m_2gl_2c_{12} \\ m_2gl_2c_{12} \end{bmatrix}$$

Now, let

$$m_1l_1 + m_2l_1 = \theta_4$$

$$m_2l_2 = \theta_5$$

$$G(\phi) = \begin{bmatrix} gc_1 & gc_{12} \\ 0 & gc_{12} \end{bmatrix} \begin{bmatrix} \theta_4 \\ \theta_5 \end{bmatrix}$$

Now, consider $\dot{M}(\phi,\dot{\phi})r$, we get

$$\dot{M}(\phi) = \begin{bmatrix} -2m_2l_1l_2s_2\dot{\phi}_2 & -m_2l_1l_2s_2\dot{\phi}_2 \\ -m_2l_1l_2s_2\dot{\phi}_2 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\dot{M}(\phi) = \begin{bmatrix} -2\theta_2s_2\dot{\phi}_2r_1 - \theta_2s_2\dot{\phi}_2r_2 \\ -\theta_2s_2\dot{\phi}_2r_1 \end{bmatrix}$$

$$\dot{M}(\phi) = \begin{bmatrix} 0 & (-2s_2\dot{\phi}_2 r_1 - s_2\dot{\phi}_2 r_2) & 0 \\ 0 & -s_2\dot{\phi}_2 r_1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Therefore,

$$M(\phi)\eta = \begin{bmatrix} \eta_1 & 2c_2\eta_1 + c_2\eta_2 & \eta_2 & 0 & 0 \\ 0 & c_2\eta_1 & (\eta_1 + \eta_2) & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = Y_M\theta$$

$$C(\phi, \dot{\phi}) = \begin{bmatrix} 0 & (-2s_2\dot{\phi}_1\dot{\phi}_2 - s_2\dot{\phi}_2^{\,2}) & 0 & 0 & 0 \\ 0 & s_2\dot{\phi}_1^{\,2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = Y_C\theta$$

$$G(\phi) = \begin{bmatrix} 0 & 0 & 0 & gc_1 & gc_{12} \\ 0 & 0 & 0 & 0 & gc_{12} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = Y_G\theta$$

$$\dot{M}(\phi) = \begin{bmatrix} 0 & (-2s_2\dot{\phi}_2 r_1 - s_2\dot{\phi}_2 r_2) & 0 & 0 & 0 \\ 0 & -s_2\dot{\phi}_2 r_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} = Y_{\dot{M}}\theta$$

$$Y\theta = (Y_M + Y_C + Y_G + Y_{\dot{M}})\,\theta$$

We need to determine regressor Y and show that it is linear in unknown parameters.

$$\dot{V}(\zeta) = e^T r - e^T\alpha e + r^T(Y\theta - \tau) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

Now, designing,

$\tau = Y\hat{\theta} + e + \beta r$, we get

$$\dot{V}(\zeta) = e^T r - e^T\alpha e + r^T(Y\theta - (Y\hat{\theta} + e + \beta r)) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

$$\dot{V}(\zeta) = e^T r - e^T\alpha e + r^T(Y\theta - Y\hat{\theta} - e - \beta r) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

we can write $e^T r$ as $r^T e$,

Therefore,

$$\dot{V}(\zeta) = -e^T\alpha e + r^T Y\tilde{\theta} - r^T\beta r - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

we can write $r^T Y\tilde{\theta}$ as $\tilde{\theta}^T Y^T r$, Therefore,

$$\dot{V}(\zeta) = -e^T\alpha e + r^T Y\tilde{\theta} - r^T\beta r - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}$$

Now, designing $\dot{\hat{\theta}} = \Gamma Y^T r$, we get

$$\dot{V}(\zeta) = -e^T\alpha e + \tilde{\theta}^T Y^T r - r^T\beta r - \tilde{\theta}^T\Gamma^{-1}(\Gamma Y^T r)$$

$$\dot{V}(\zeta) = -e^T\alpha e - r^T\beta r$$

$$\dot{V}(\zeta) \leq -\underline{\alpha}\|e\|^2 - \underline{\beta}\|r\|^2$$

where, $\underline{\alpha}, \underline{\beta}$ are the minimum eigen values of α and β

Now, using Barbalet's lemma, we can say that,

$e \to 0$ and $r \to 0$ as $t \to \infty$

Therefore, as the lyapunov function is radially unbounded and the time derivative of lyapunov function is Negative Semi Definite, then using barbalet's lemma we can conclude that we have Globally Asymptotic Tracking(GAT).

2) SIMULATIONS:

(a) The norm of the tracking error and filtered tracking error over time is attached below.

i. Based on the plots, the designed controller tracks the desired trajectory aysmptotically. We need to have to adaptive learning controller to estimate the parameters properly when there is uncertainty. With the gradient controller, we can only acheive global asymptotic tracking.

(b) The norm of the parametric error over time is attached below. i. Based on the plots, the estimator is not able to approximate the unknown parameters. As mentioned above, we need to have adaptive controller to learn and estimate the parameters.

ii. The norm of the total input, the feedback portions of the input, and the feedforward portions of the input are attached below.

(c) Based on the plots, we can conclude that our stability result matches our simulation result. We achieved global asymptotic stability through barbalet's lemma because the time derivative of our lyapunov candidate was Negative Semi Definite and the lyapunov function was radially unbounded. To improve the result, we can use concurrent learning which will estimate the unknown parameters and the paramteric error will go to 0 as time goes to ∞

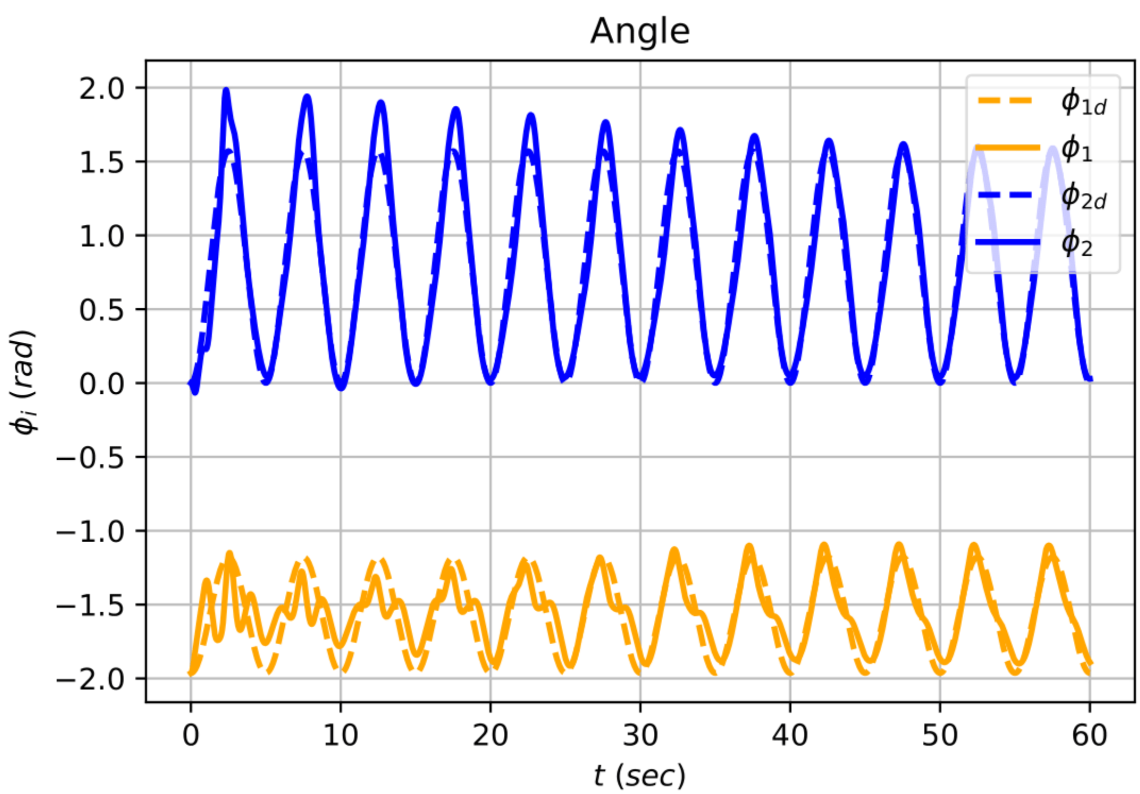

Figure 1: Angle Plot.

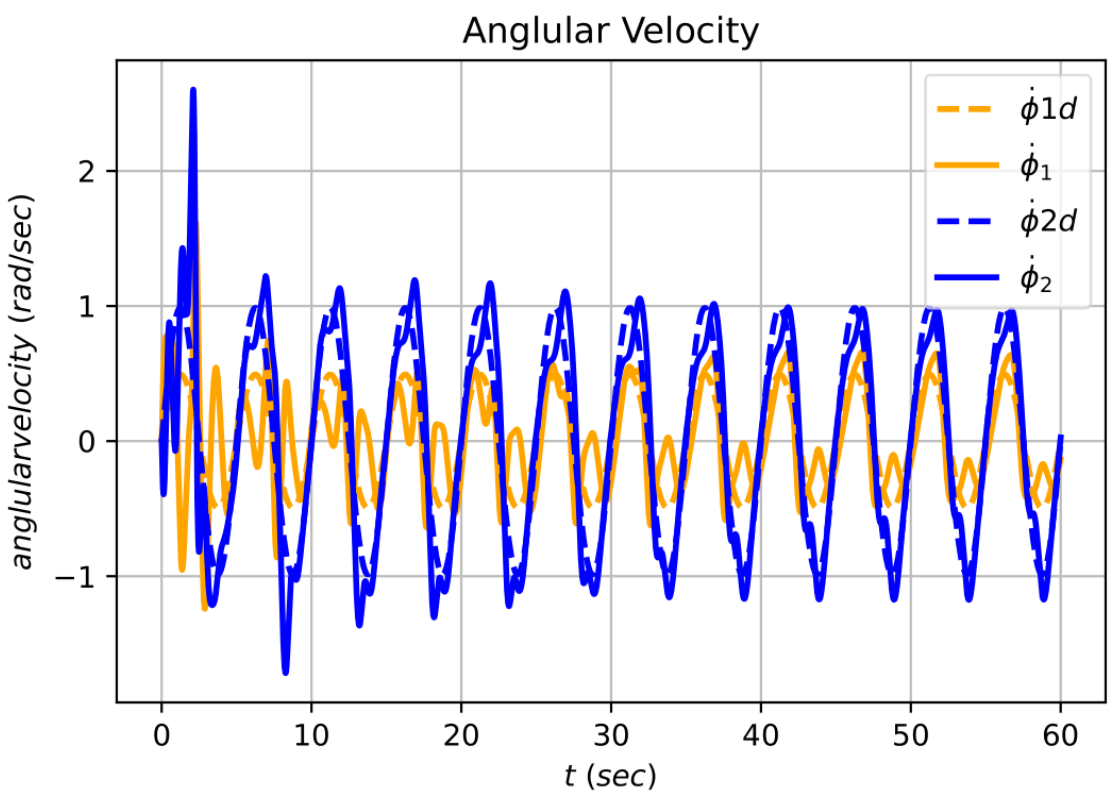

Figure 2: Angular Velocity Plot.

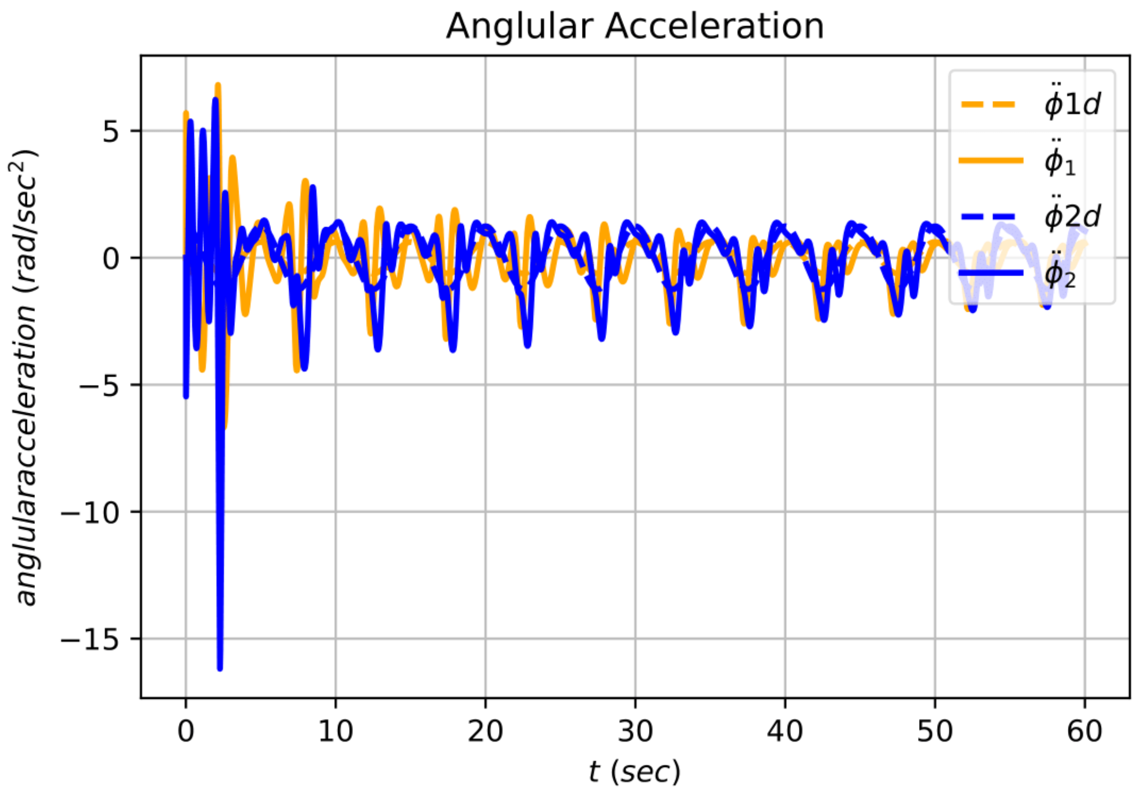

Figure 3: Angular Acceleration Plot.

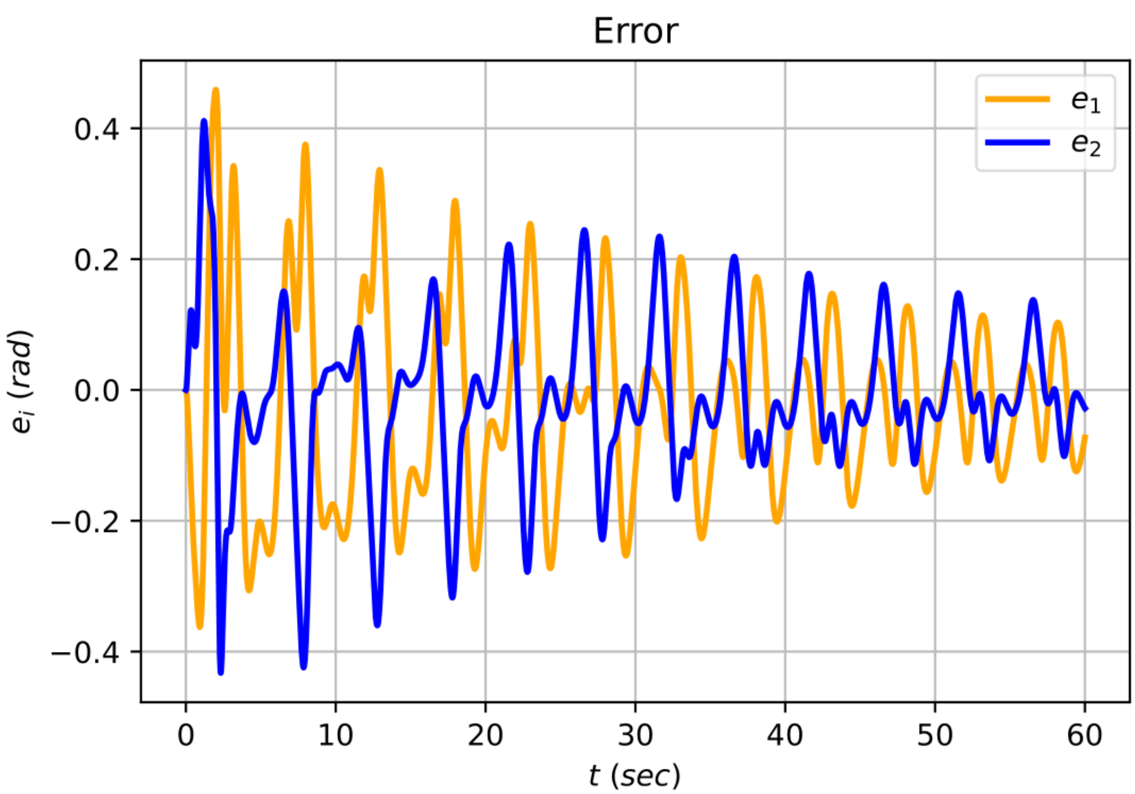

Figure 4: Error Plot.

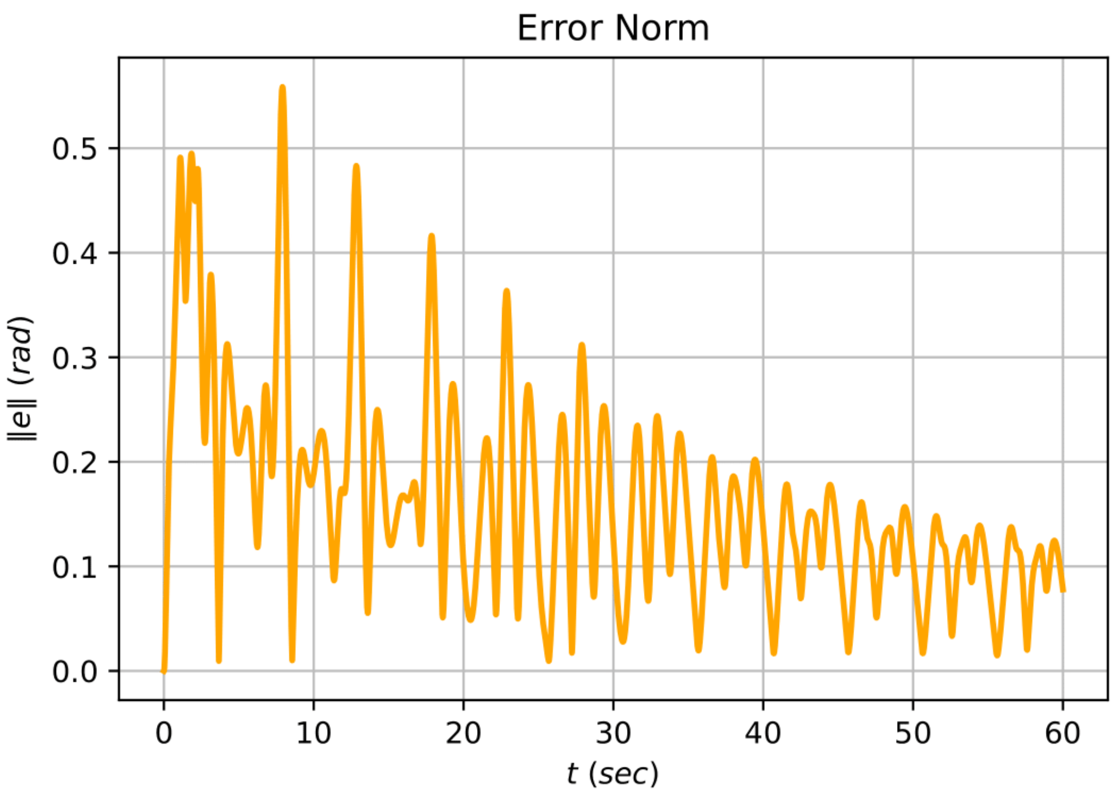

Figure 5: Error Norm Plot.

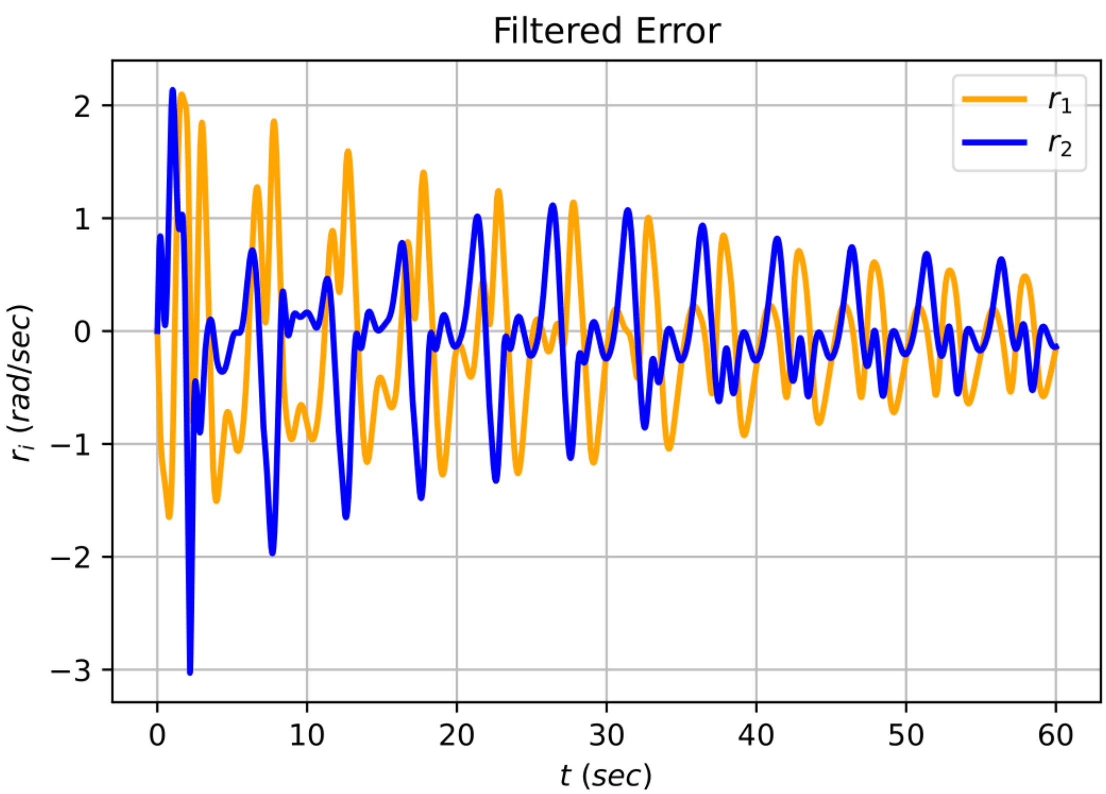

Figure 6: Filtered Tracking Error Plot.

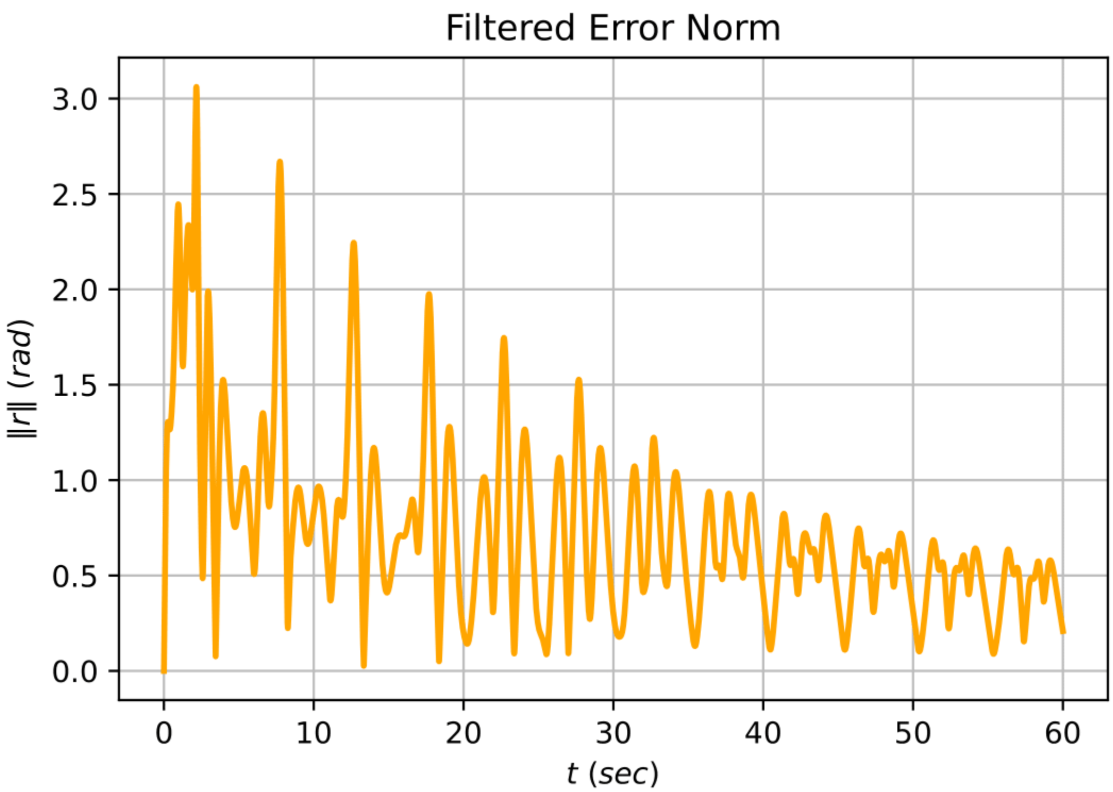

Figure 7: Filtered Tracking Error Norm Plot.

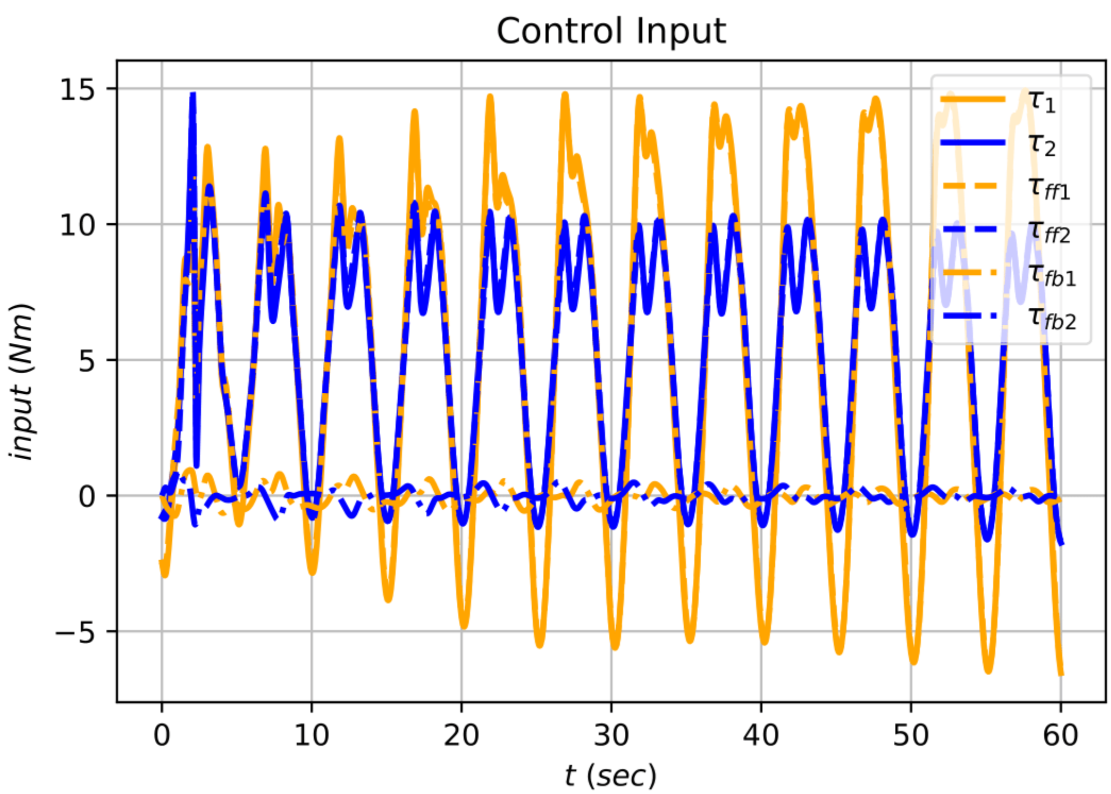

Figure 8: Input Plot.

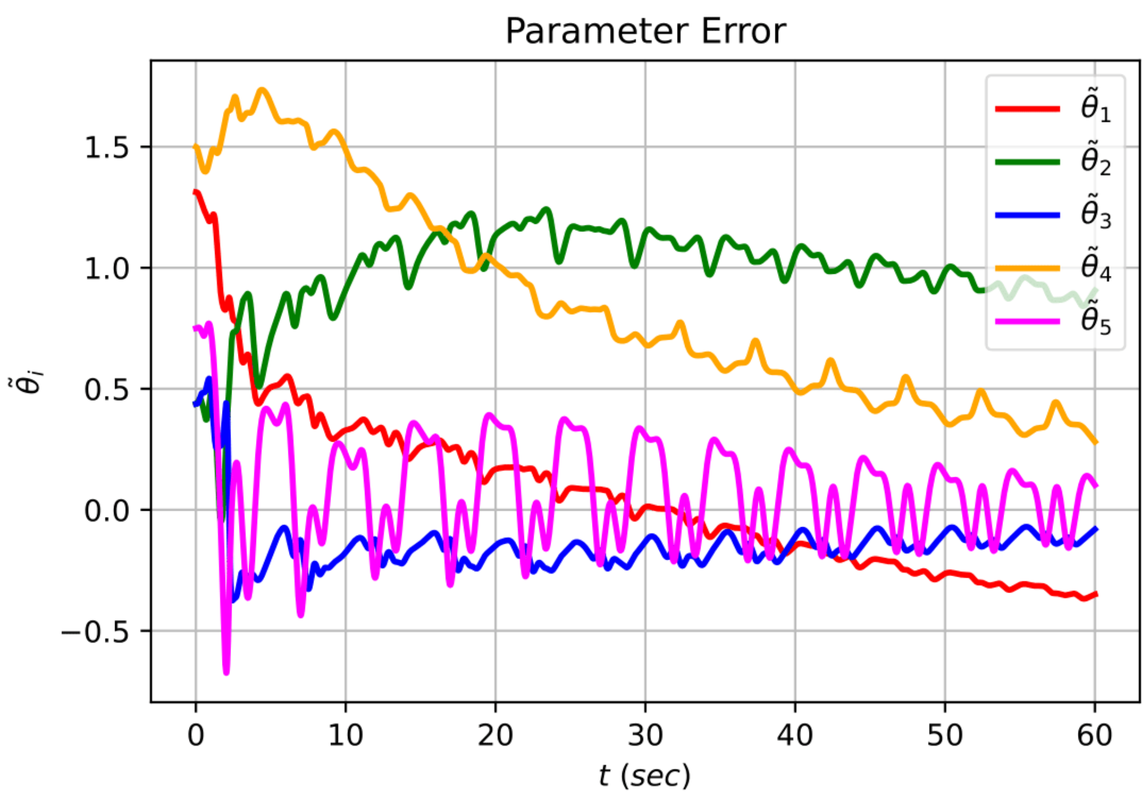

Figure 9: Theta Tilde Plot.

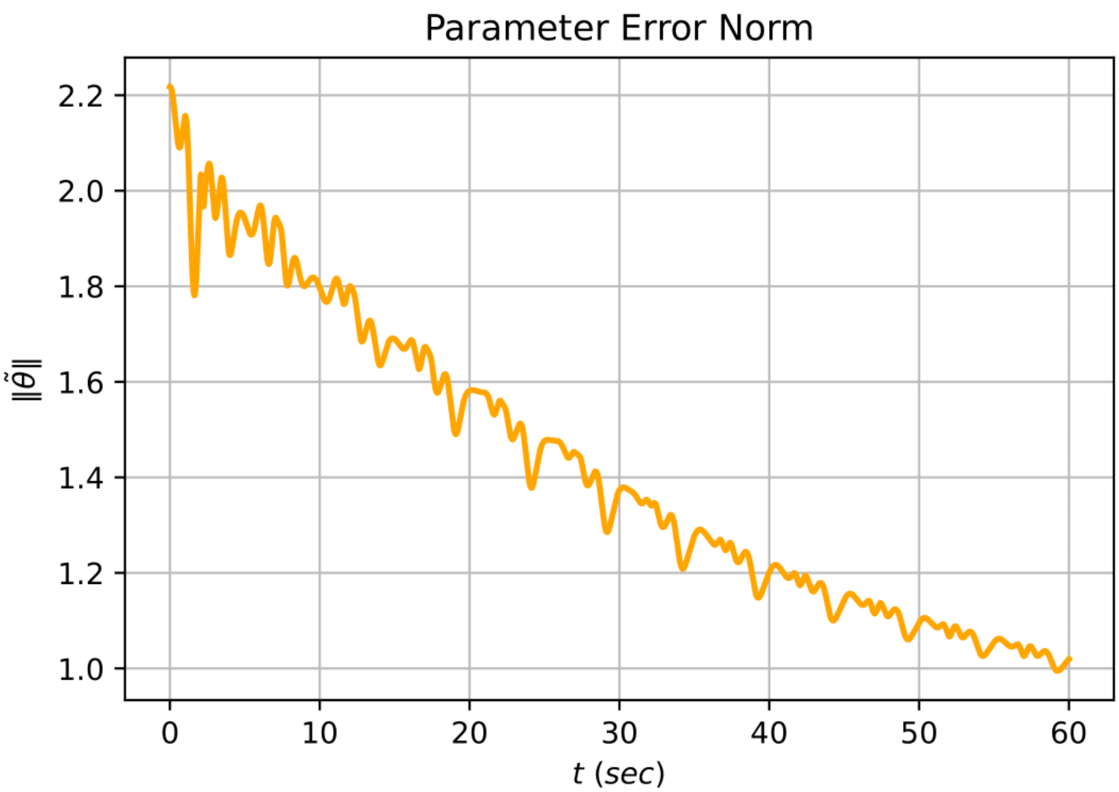

Figure 10: Theta Tilde Norm Plot.