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MTH 442 Assignment 3

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~~Given MA(1) Model~~

1) Assume there is a Γ_n that is not positive definite.
Since $\gamma(0) > 0$, $\Gamma_1 = \{\gamma(0)\}$ is non-singular.
$\therefore$ We can consider a sequence $\Gamma_1, \Gamma_2, \ldots$ and suppose Γ_{r+1} is the first ~~sign~~ singular Γ_n in the sequence.

Claim: If cov matrix of $\underset{\sim}{X}$ is not p.d., then wp 1, components of $\underset{\sim}{X}$ are linearly related

Proof: If $\Sigma \not> 0$, then $\exists$ an $\underset{\sim}{\alpha} \in \mathbb{R}^p$ ($\underset{\sim}{\alpha} \neq 0$) $\ni$

$$0 = \underset{\sim}{\alpha}' \Sigma \underset{\sim}{\alpha} = Var(\underset{\sim}{\alpha}' \underset{\sim}{X})$$

$$\Rightarrow P(\alpha' \underset{\sim}{X} = \alpha' \underset{\sim}{\mu}) = 1$$

$$\Rightarrow P(\alpha'(\underset{\sim}{X} - \underset{\sim}{\mu}) = 0) = 1$$

i.e. $\sum_i \alpha_i (X_i - \mu_i) = 0$ w.p. 1 for not all $\alpha_i = 0$

ie. wp. 1 X_is are linearly related.

$\therefore$ By the above claim, we can say that Γ_{r+1} not being positive definite $\Rightarrow$ X_{r+1} is a linear combination of $\underset{\sim}{X} = (X_1, \ldots, X_r)'$

$\Rightarrow X_{r+1} = \underset{\sim}{b}' \underset{\sim}{X}$ where $\underset{\sim}{b} = (b_1, \ldots, b_r)'$

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4) Consider the prediction equations,
$$\Gamma_h \phi_h = \gamma_h$$
Dividing both sides by $\gamma(0)$, we obtain,
$$R_h \phi_h = \rho_h$$
Partition ϕ_h s.t. $\phi_h = [\phi'_{h-1}, \phi_{h,h})'$
$$\begin{bmatrix} R_{h-1} & \tilde{\rho}_{h-1} \\ \tilde{\rho}'_{h-1} & 1 \end{bmatrix} \begin{bmatrix} \phi_{h-1} \\ \phi_{hh} \end{bmatrix} = \begin{bmatrix} \rho_{h-1} \\ \rho(h) \end{bmatrix}$$
$\therefore$ We have the following equations,
$$R_{h-1}\phi_{h-1} + \tilde{\rho}_{h-1}\phi_{hh} = \rho_{h-1} \quad \text{—— (1)}$$
$$\tilde{\rho}'_{h-1}\phi_{h-1} + \phi_{hh} = \rho(h) \quad \text{—— (2)}$$
Finding ϕ_{h-1} using equation (1),
$$\phi_{h-1} = R_{h-1}^{-1}(\rho_{h-1} - \tilde{\rho}_{h-1}\phi_{hh})$$
Substitute this ϕ_{h-1} in eq^n (2) to find ϕ_{hh},
$$\phi_{hh} = \frac{\rho(h) - \tilde{\rho}'_{h-1} R_{h-1}^{-1}\rho_{h-1}}{1 - \tilde{\rho}'_{h-1} R_{h-1}^{-1}\tilde{\rho}_{h-1}}$$

Now, we need to show that the PACF,
$$\frac{E(\epsilon_t \delta_{t-h})}{\sqrt{E(\epsilon_t^2) E(\delta_{t-h}^2)}}$$
Can be written in the form of eq^n (3).

Consider $E(\epsilon^2) = E\left[\left(X_t - \sum_{i=1}^{h-1} a_i X_{t-i}\right)^2\right]$

minimize $E[\epsilon^2]$ wrt $a_1, \dots, a_{h-1}$

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$$\frac{\partial E[\epsilon_i]}{\partial a_k} = E\left[2\left(X_t - \sum_{i=1}^{h-1} a_i X_{t-i}\right)(-X_{t-k})\right] = 0$$

for $k = 1, \ldots, h-1$

$$\Rightarrow \gamma(k) - \sum_{i=1}^{h-1} a_i \gamma(k-i) = 0$$

$$\Rightarrow \sum_{i=1}^{h-1} a_i \gamma(k-i) = \gamma(k) \quad \text{for } k = 1, \ldots, h-1$$

Now, writing these ~~~~ $h-1$ equations in the form of matrix:

$$\begin{bmatrix} \gamma(0) & \cdots & \gamma(h-1) \\ \vdots & & \\ \gamma(h-1) & \cdots & \gamma(0) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{h-1} \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(h-1) \end{bmatrix}$$

$$\Rightarrow \Gamma_{h-1} \underset{\sim}{a} = \gamma_{h-1}$$

$$\Rightarrow \underset{\sim}{a} = \Gamma_{h-1}^{-1} \gamma_{h-1}$$

where $\underset{\sim}{a} = (a_1, \ldots, a_{h-1})'$

Consider $E[\delta^2_{t-h}] = E\left[\left(X_{t-h} - \sum_{j=1}^{h-1} b_j X_{t-j}\right)^2\right]$

To minize $E[\delta^2_{t-h}]$ w.r.t. $b_1, \ldots, b_{h-1}$

$$\frac{\partial E[\delta^2_{t-h}]}{\partial b_k} = E\left[2\left(X_{t-h} - \sum_{j=1}^{h-1} b_j X_{t-j}\right)(-X_{t-k})\right] = 0$$

for $k = 1, \ldots, h-1$

$$\Rightarrow \gamma(h-k) - \sum_{j=1}^{h-1} b_j \gamma(j-k) = 0$$

$$\Rightarrow \sum_{j=1}^{h-1} b_j \gamma(j-k) = \gamma(h-k)$$

Now, write all the $h-1$ equations in matrix form,

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(h-1) \\ \vdots & & & \\ \gamma(h-1) & \cdots & \cdots & \gamma(0) \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_{h-1} \end{bmatrix} = \begin{bmatrix} \gamma(h-1) \\ \vdots \\ \gamma(1) \end{bmatrix}$$

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$$\Rightarrow \Gamma_{h-1}\, \underset{\sim}{b} = \tilde{\gamma}_{h-1}$$

where $\underset{\sim}{b} = (b_1, \ldots, b_{h-1})'$

$\therefore$ The residuals ~~of~~ will become,

$$\epsilon_t = X_t - \gamma_{h-1}' \Gamma_{h-1}^{-1} \underset{\sim}{X}$$

$$\delta_{t-h} = X_{t-h} - \tilde{\gamma}_{h-1}' \Gamma_{h-1}^{-1} \underset{\sim}{X}$$

where $\underset{\sim}{X} = (X_{t-1}, \ldots, X_{t-h+1})'$

[$\because$ The regression of X_t on $\underset{\sim}{X}$ is $(\Gamma_{h-1}^{-1}\gamma_{h-1})'\underset{\sim}{X}$
& regression of X_{t-h} on $\underset{\sim}{X}$ is $(\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1})'\underset{\sim}{X}$.]

~~$E[\epsilon_t \delta_{t-h}] = Cov$~~

$$E[\epsilon_t] = E[X_t - \sum_{i=1}^{h-1} a_i X_{t-i}] = 0$$

Similarly, $E[\delta_{t-h}] = 0$.

$$\therefore E[\epsilon_t \delta_{t-h}] = Cov(\epsilon_t, \delta_{t-h})$$

$$= Cov(X_t - \gamma_{h-1}'\Gamma_{h-1}^{-1}\underset{\sim}{X},\ X_{t-h} - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\underset{\sim}{X})$$

$$= \cancel{Cov(X_t, X_{t-h})} -$$

$$= \gamma(h) - \tilde{\gamma}_{h-1}' \Gamma_{h-1}^{-1} \gamma_{h-1}$$

$$E[\delta_{t-h}^2] = Var(\delta_{t-h}) = \gamma(0) - \tilde{\gamma}_{h-1}' \Gamma_{h-1}^{-1} \tilde{\gamma}_{h-1}$$

$$E[\epsilon_t^2] = Var(\epsilon_t) = \gamma(0) - \gamma_{h-1}' \Gamma_{h-1}^{-1} \gamma_{h-1}$$

Further, regressing X_t on $\underset{\sim}{\tilde{X}}$, where $\underset{\sim}{\tilde{X}} = (X_{t-h+1}, \ldots, X_{t-1})'$, gives residuals as

$$X_t - \sum_{i=1}^{h-1} c_i X_{t-h+i}$$

$$= X_t - \underset{\sim}{c}' \underset{\sim}{\tilde{X}}$$

which is equal to

$$\epsilon_t = X_t - \sum_{i=1}^{h-1} a_i X_{t-i}$$

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$\therefore$ After fitting the ~~test~~ model in new form, the residuals becomes,

$$\epsilon_t = X_t - (\Gamma_{h-1}^{-1} \tilde{\gamma}_{h-1})' \tilde{X}$$

And, $E[\epsilon_t^2] = Var[\epsilon_t]$

$$= \gamma(0) - \gamma_{h-1}' \Gamma_{h-1}^{-1} \gamma_{h-1}$$

$$= \gamma(0) - \tilde{\gamma}_{h-1}' \Gamma_{h-1}^{-1} \tilde{\gamma}_{h-1}$$

$$= E[\delta_{t-h}^2]$$

$\therefore$ The PACF,

$$\phi_{h,h} = \frac{E[\epsilon_t \delta_{t-h}]}{\sqrt{E(\epsilon_t^2) E(\delta_{t-h}^2)}} = \frac{\gamma(h) - \tilde{\gamma}_{h-1}' \Gamma_{h-1}^{-1} \gamma_{h-1}}{\sqrt{(\gamma(0) - \tilde{\gamma}_{h-1}' \Gamma_{h-1}^{-1} \tilde{\gamma}_{h-1})^2}}$$

Dividing by $\gamma(0)$ in numerator and Denominator,

$$= \frac{\rho(h) - \tilde{\rho}_{h-1}' R_{h-1}^{-1} \rho_{h-1}}{1 - \tilde{\rho}_{h-1}' R_{h-1}^{-1} \tilde{\rho}_{h-1}}$$

$$= \alpha_{h,h}$$

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5) (a) We need to find $g(x)$ such that $E[(y-g(x))^2]$ is minimized.

We can write,

$$E[(y-g(x))^2] = E[E[(y-g(x))^2 | x]]$$

~~Now, to minimize wrt $g(x)$,~~

$$\cancel{\frac{\partial E[(y-g(x))^2]}{\partial g(x)} = E\left[\frac{\partial}{\partial g(x)} E[(y-g(x))^2 | x]\right]}$$

$\Rightarrow$ ~~0 =~~

Now, to minimize wrt $g(x)$, we can minimize the inner expectation.

$$\therefore \frac{\partial}{\partial g(x)} E[(y-g(x))^2 | x] = 0$$

Consider $g(x) = a$

and $f(a) = E[(y-a)^2 | X=x]$

$$= E[y^2 | X=x] - 2aE(y|x) + a^2$$

$$\Rightarrow f'(a) = -2E(y|x) + 2a = 0$$

$$\Rightarrow E[y|x] = a$$

and $f''(a) = 2$

$\therefore$ $g(x) = E[y|x]$ gives the minimum value of $E[(y-g(x))^2 | x]$

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Given, $Y = X^2 + Z$, where X and Z are independent zero-mean normal variables with variance one.

Let $g(x) = a + bx$

Using prediction equations,

(i) $E[y - g(x)] = 0$

$\Rightarrow E[y] = E[a + bx] \Rightarrow E[y] = a + b\,E[x]$

(ii) $E[(y - g(x))x] = 0$

$\Rightarrow E[xy] = E[(a+bx)x]$

We know, $E[x] = 0$ and

$E[y] = E[x^2] + E[Z] = 1$

So, from (i), $a = 1$

From (ii), $E[xy] = E[ax + bx^2]$

$\Rightarrow E[xy] = a\,E[x] + b\,E[x^2]$

$\Rightarrow E[x(x^2+z)] = b$

$\Rightarrow b = E[x^3] + E[x]E[z]$

$b = 0 + 0 = 0$

$\left[\because X \sim N(0,1) \text{ is symmetric around } 0 \Rightarrow x^3 \text{ is also symmetric around } 0\right]$

$M_x(t) = E[e^{tx}] = e^{t^2/2}$ [$\because x$ is normal]

$M_x'(0) = E[x] = e^{0^2/2} \cdot (0) = 0$

$M_x'''(0) = E[x^3] = 0$

$M_x''''(0) = E[x^4] = 3$

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Finally, $g(x) = a + bx = 1$
and

$$MSE = E(y-1)^2$$
$$= E(y^2) - 1$$
$$= E[x^4] + E(z^2) - 1$$
$$= 3 + 1 - 1 = 3$$
$$[\because E[x^4] = M_x^4(0) = 3]$$

$\therefore$ The best linear predictor has three times the error of optimal predictor (conditional expectation).