MTH 442 ASSIGNMENT 2

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Saathi

1) Given MA(1) model: $X_t = W_t + \theta W_{t-1}$

ACF for lag 1:

$\gamma(1) = Cov(X_{t+1}, X_t)$

$= Cov[(W_{t+1} + \theta W_t)(W_t + \theta W_{t-1})]$

$= E[(W_{t+1} + \theta W_t)(W_t + \theta W_{t-1})]$ $[\because E[X_t] = 0]$

$= \theta E[W_t^2] = \sigma_w^2 \theta$

$\gamma(0) = Cov(X_t, X_t)$

$= \cancel{Cov} E[X_t X_t]$

$= E[(W_t + \theta W_{t-1})(W_t + \theta W_{t-1})]$

$= E[W_t^2] + \theta^2 E[W_{t-1}^2]$

$= \sigma^2(1+\theta^2)$

$\therefore \rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{1+\theta^2}$

We know that AM $\geq$ GM,

$\therefore \frac{\left[\theta + \frac{1}{\theta}\right]}{2} \geq \sqrt{\theta \cdot \frac{1}{\theta}}$

$\Rightarrow \left[\frac{\theta^2+1}{\theta}\right] \geq 2$

$\Rightarrow \left[\frac{\theta}{\theta^2+1}\right] \leq \frac{1}{2}$

$\Rightarrow \rho(1) \leq \frac{1}{2}$

$\therefore$ lag 1 ACF is bounded above by 0.5.

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2) Given, $\{W_t ; t = 0, 1, \ldots\}$: white noise process

$|\phi| < 1$

$X_0 = W_0$ and $X_t = \phi X_{t-1} + W_t$; $t = 1, 2, \ldots$

(a) We can write iteratively as

$$
\begin{aligned}
X_t &= \phi X_{t-1} + W_t \\
&= \phi(\phi X_{t-2} + W_{t-1}) + W_t \\
&= \phi^2 X_{t-2} + (\phi W_{t-1} + W_t) \\
&\vdots \\
&= \phi^k X_{t-k} + \sum_{j=0}^{k-1} \phi^j W_{t-j}
\end{aligned}
$$

for $k = t$.

$$X_t = \phi^t X_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}$$

$[\because X_0 = W_0]$

$$\Rightarrow X_t = \sum_{j=0}^{t} \phi^j W_{t-j}$$

(b)
$$
\begin{aligned}
E(X_t) &= \sum_{j=0}^{t} E[\phi^j W_{t-j}] \\
&= \sum_{j=0}^{t} \phi^j E[W_{t-j}] \\
&= 0
\end{aligned}
$$

(c)
$$
\begin{aligned}
Var(X_t) &= Var\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right) \\
&= \cancel{\sum_{j=0}^{t} \phi^j} E\left[\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right] \\
&= \sum_{j=0}^{t} \phi^{2j} E[W_{t-j}^2] \\
&= \sigma_w^2 \sum_{j=0}^{t} \phi^{2j} = \sigma_w^2 \left[\frac{1 - \phi^{2(t+1)}}{1 - \phi^2}\right]
\end{aligned}
$$

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(d) $Cov(X_{t+h}, X_t) = E(X_{t+h} X_t)$

$[\because E[X_{t+h}] = E[X_t] = 0]$

$= E\left[\left(\sum_{j=0}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t+h-j}\right)\right]$

$= E\left[\left(\sum_{j=0}^{h-1} \phi^j W_{t+h-j} + \sum_{j=h}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right]$

$= E\left[\left(\sum_{j=0}^{h-1} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right] + E\left[\left(\sum_{j=h}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right]$

$= 0 + E\left[\left(\sum_{j=h}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right]$

$= E\left[\left(\phi^h W_t + \phi^{h+1} W_{t-1} + \phi^{h+2} W_{t-2} + \dots + \phi^{t+h} W_0\right)\left(\phi^0 W_t + \phi W_{t-1} + \dots + \phi^t W_0\right)\right]$

$= \phi^h E[W_t^2] + \phi^{h+2} E[W_{t-1}^2] + \dots + \phi^{h+2t} E[W_0^2]$

$= \sigma_w^2 \left[\phi^h + \phi^{h+2} + \dots + \phi^{h+2t}\right]$

$= \phi^h \sigma_w^2 (1 + \phi^2 + \dots + \phi^{2t})$

$= \phi^h \sigma_w^2 \cdot \frac{(1-(\phi^2)^{t+1})}{1-\phi^2}$

$= \phi^h Var(X_t)$

(e) X_t is not stationary since $Cov(X_{t+h}, X_t)$ depends on t.

(f) ~~As $t \to \infty$~~ $Cov(X_{t+h}, X_t) = \phi^h Var(X_t)$

$\lim_{t \to \infty} Cov(X_{t+h}, X_t) = \lim_{t\to\infty} \phi^h \sigma_w^2 \frac{(1-\phi^{2(t+1)})}{1-\phi^2}$

$= \frac{\phi^h \sigma_w^2}{1-\phi^2}$ $[\because |\phi| < 1]$

Now, the $Cov(X_{t+h}, X_t) = \gamma(h)$ is independent of t and depends only on h.

Further, $E[X_t] = 0$ and $Var(X_t)$ is finite

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for every $t \geq 0$.

$\therefore$ X_t is asymptotically stationary.

(g) We can generate ~~n obser~~ more than n observations like $n+n_0$, where n_0 is fairly large and discard the first n_0.

The remaining n observations can then be used to simulate asymptotically stationary AR(1) model

(h) Consider $X_0 = \dfrac{W_0}{\sqrt{1-\phi^2}}$

Now, $X_t = \phi X_{t-1} + W_t$

$$= \phi^t X_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}$$

$$\Rightarrow X_t = \phi^t \frac{W_0}{\sqrt{1-\phi^2}} + \sum_{j=0}^{t-1} \phi^j W_{t-j}$$

$$\Rightarrow E[X_t] = \frac{\phi^t}{\sqrt{1-\phi^2}} E[W_0] + \sum_{j=0}^{t-1} \phi^j E[W_{t-j}]$$

$$\Rightarrow E[X_t] = 0$$

$\therefore$ $E[X_t]$ is constant.

$$\gamma(h) = Cov(X_{t+h}, X_t) = E[X_{t+h} X_t]$$

$$= E\left[\left(\frac{\phi^{t+h}}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t+h-1} \phi^j W_{t+h-j}\right)\left(\frac{\phi^t}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}\right)\right]$$

$$= E\left[\left(\frac{\phi^{t+h}}{\sqrt{1-\phi^2}} W_0 + \sum_{j=h}^{t+h-1} \phi^j W_{t+h-j} + \sum_{j=0}^{h-1} \phi^j W_{t+h-j}\right)\left(\frac{\phi^t}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}\right)\right]$$

$$= E\left[\left(\frac{\phi^{t+h}}{\sqrt{1-\phi^2}} W_0 + \sum_{j=h}^{t+h-1} \phi^j W_{t+h-j}\right)\left(\frac{\phi^t}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}\right)\right]$$

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$$= \phi^h E[W_t^2] + \phi^{h+2} E[W_{t-1}^2] + \cdots + \phi^{h+2(t-1)} E[W_1^2]$$
$$+ \frac{\phi^{h+2t}}{1-\phi^2} E[W_0^2]$$

$$= \sigma_W^2 \phi^h \left(1 + \phi^2 + \cdots + (\phi^2)^{t-1} + \frac{(\phi^2)^t}{1-\phi^2}\right)$$

$$= \sigma_W^2 \phi^h \cdot \left[\frac{(1-\phi^{2t})}{1-\phi^2} + \frac{\phi^{2t}}{1-\phi^2}\right]$$

$$= \sigma_W^2 \phi^h \cdot \frac{1}{1-\phi^2} = \frac{\phi^h \sigma_W^2}{1-\phi^2}$$

$$Var(X_t) = \gamma(0) = \frac{\phi^h \sigma_W^2}{1-\phi^2} < \infty$$

We can see that $\gamma(h)$ is independent of t and depends only on h. Further, $Var(X_t)$ is finite ~~bat~~ for all t.

$\therefore$ X_t is stationary when $X_0 = \frac{W_0}{\sqrt{1-\phi^2}}$

3) Given AR(2) model:

$$(1 - \phi_1 B - \phi_2 B^2) X_t = W_t$$

The AR polynomial will be:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

roots of $\phi(z)$ are:

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_1}$$

The given model is Causal if and only if:

$$\left| \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_1} \right| > 1$$

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The roots of $\phi(z)$ may be real and distinct, real and equal, or a complex conjugate pair.
Roots are real if $\phi_1^2 + 4\phi_2 \geq 0$ else roots are complex conjugates.

Consider z_1, z_2 as the roots of $\phi(z)$ such that $z_1, z_2 \in \mathbb{C}$. ~~and $z_1 = z_2$~~

Model is causal if and only if $|z_1| > 1$ and $|z_2| > 1$

We can write $\phi(z)$ as

$$\phi(z) = (1 - z_1^{-1} z)(1 - z_2^{-1} z)$$

$$\Rightarrow \phi(z) = 1 - (z_1^{-1} + z_2^{-1}) z + z_1^{-1} z_2^{-1} z$$

Comparing the above expression with

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

we get,

$$\phi_1 = z_1^{-1} + z_2^{-1}, \qquad \phi_2 = -(z_1 z_2)^{-1}$$

Let $u_1 = z_1^{-1}$ and $u_2 = z_2^{-1}$

$$\therefore \cancel{\phi_1 = z_1} \Rightarrow |u_1| = |z_1^{-1}| \text{ and } |u_2| = |z_2^{-1}|$$

$$\Rightarrow |u_1| = \frac{1}{|z_1|} < 1 \quad \text{and} \quad |u_2| = \frac{1}{|z_2|} < 1$$

Also,

$$\phi_2 = \frac{-1}{z_1 z_2} \Rightarrow |\phi_2| = \frac{1}{|z_1||z_2|} < 1$$

$$\phi_1 + \phi_2 - 1 = z_1^{-1} + z_2^{-1} - (z_1 z_2)^{-1} - 1$$

$$= u_1 + u_2 - u_1 u_2 - 1$$

$$= u_1(1 - u_2) + u_2 - 1$$

$$= -(1 - u_1)(1 - u_2)$$

$$\cancel{\Rightarrow \phi_1 + \phi_2 < 1}$$

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$$\phi_2 - \phi_1 - 1 = -z_1^{-1} - z_2^{-1} - (z_1 z_2)^{-1} - 1$$
$$= -u_1 - u_2 - u_1 u_2 - 1$$
$$= -(1+u_1)(1+u_2)$$

Consider the case when $z_1, z_2 \in \mathbb{R}$.

$\therefore \phi_2 + \phi_1 - 1 = -(1-u_1)(1-u_2) < 0$

[$\because |u_1|$ and $|u_2| < 1$]

$\phi_2 - \phi_1 - 1 = -(1+u_1)(1+u_2) < 0$

$\therefore$ We have following conditions when roots are real,

$|\phi_2| < 1$, $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$

Now, consider the case when $z_1, z_2 \in \mathbb{C}$ and $z_1 = \overline{z_2}$.

$$\phi_2 + \phi_1 - 1 = -(1-u_1)(1-u_2)$$
$$= -(1-u_1)(1-\overline{u_1}) \quad [\because z_2 = \overline{z_1} \Rightarrow u_2 = \overline{u_1}]$$
$$= -(1-u_1)(\overline{1-u_1})$$
$$= -|(1-u_1)|^2 < 0$$

$$\phi_2 - \phi_1 - 1 = -(1+u_1)(1+u_2)$$
$$= -(1+u_1)(1+\overline{u_1})$$
$$= -(1+u_1)(\overline{1+u_1})$$
$$= -|(1+u_1)|^2 < 0$$

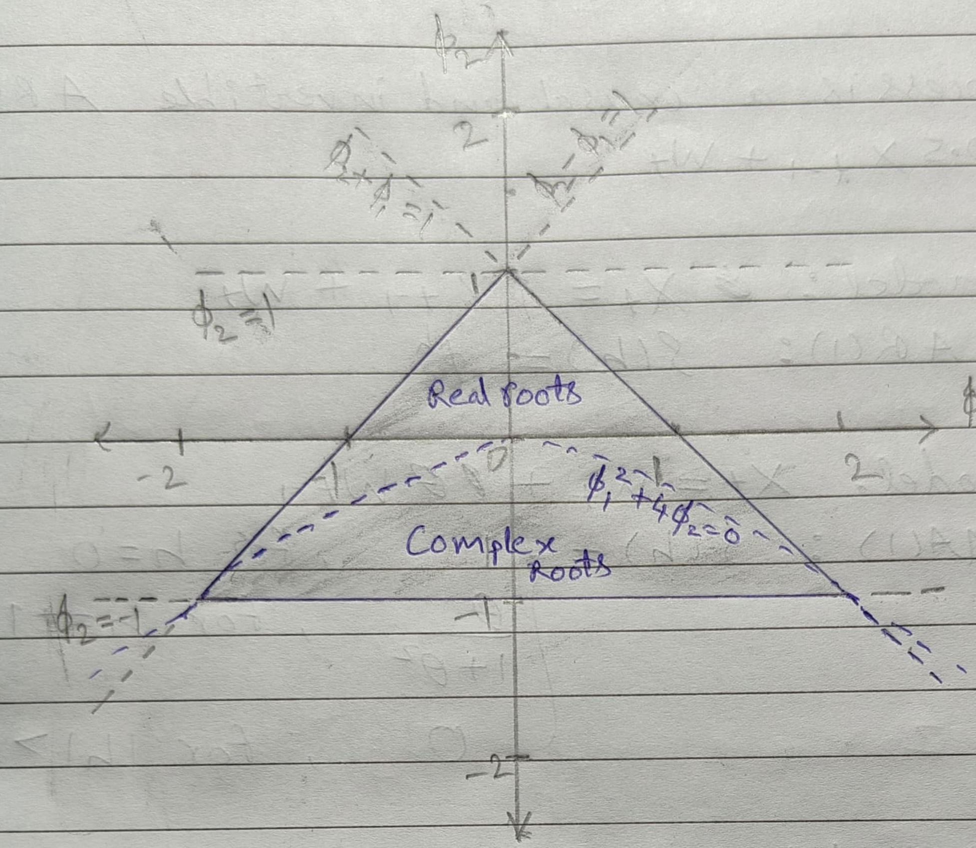
$\therefore$ We again have following conditions when roots are complex conjugates,

$|\phi_2| < 1$, $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$

Thus, we have the following conditions in the general case to establish causality,

$|\phi_2| < 1$, $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$

This causality condition specifies a triangular region in the parameter space.



ϕ_2
2
$\phi_2 + \phi_1 = 1$
$\phi_2 - \phi_1 = 1$
1
$\phi_2 = 1$
Real roots
ϕ_1
-2
-1
1
2
$\phi_1^2 + 4\phi_2 = 0$
Complex Roots
$\phi_2 = -1$
-1
-2

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4) (i) Given, AR(2) model :

$X_t = 0.25\, X_{t-2} + W_t$

AR polynomial is given by,

$\phi(z) = 1 - 0.25\, z^2$

Roots of $\phi(z)$:

$\phi(z) = 0$

$\Rightarrow 1 - 0.25 z^2 = 0$

$\Rightarrow (z - 2)(z + 2) = 0$

$\Rightarrow z = \pm 2$

$X_t = 0.25\, X_{t-2} + W_t$

Multiply each side by X_{t-h} for $h > 0$ and take expectation

$E[X_t X_{t-h}] = 0.25\, E[X_{t-2} X_{t-h}] + E[W_t X_{t-h}]$

$\Rightarrow \gamma(h) = 0.25\, \gamma(h-2) \quad , h = 1, 2, \ldots$

Dividing by $\gamma(0)$,

$\rho(h) = 0.25\, \rho(h-2)$

$\Rightarrow \rho(h) - 0.25\, \rho(h-2) = 0$

This is a difference equation with order 2.

$\therefore$ Solution of $\rho(h)$ is

$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$

Also, Initial conditions :

(i) $\rho(0) = \gamma(0)/\gamma(0) = 1$

(ii) $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = 0.25\, \rho(1-2)$

$\Rightarrow \rho(1) - 0.25\, \rho(-1) = 0$

$\Rightarrow \rho(1)[1 - 0.25] = 0 \qquad [\because \rho(-1) = \rho(1)]$

$\Rightarrow \rho(1) = 0$

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Using $\rho(0)=1$,

$$\rho(0) = c_1 z_1^{-0} + c_2 z_2^{-0}$$

$$\Rightarrow \quad c_1 + c_2 = 1$$

Using $\rho(1) = 0$,

$$\rho(1) = c_1 z_1^{-1} + c_2 z_2^{-1} = 0$$

$$\Rightarrow \quad \frac{c_1}{2} - \frac{c_2}{2} = 0$$

$$\Rightarrow \quad c_1 = c_2$$

$\therefore$ We get, $c_1 = 1/2$ and $c_2 = 1/2$

$\therefore$ ~~General~~ $\rho(h)$ becomes,

$$\rho(h) = \frac{(2)^{-h}}{2} + \frac{(-2)^{-h}}{2}$$

$$\Rightarrow \rho(h) = [1+(-1)^h]\, 2^{-(h+1)}$$

(ii) Given AR(2) model:

$$X_t = -0.9\, X_{t-2} + W_t$$

AR polynomial is given by:

$$\phi(z) = 1 + 0.9 z^2$$

Roots of $\phi(z)$:

$$\phi(z) = 0$$

$$\Rightarrow \quad 1 + 0.9 z^2 = 0$$

$$\Rightarrow \quad z^2 = -(0.9)^{-1}$$

$$\Rightarrow \quad z = \pm \frac{i}{\sqrt{0.9}}$$

$$\arg(z) = \pi/2$$

$$\therefore \quad z = \pm \frac{1}{\sqrt{0.9}}\, e^{i\pi/2}$$

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$$X_t = -0.9 X_{t-2} + W_t$$

Multiply each side by X_{t-h} for $h>0$, and take expectation,

$$E[X_t X_{t-h}] = -0.9\, E[X_{t-2} X_{t-h}] + E[W_t X_{t-h}]$$

$$\Rightarrow \gamma(h) = -0.9\, \gamma(h-2) \quad ; \quad h = 1, 2, \ldots$$

Dividing by $\gamma(0)$ gives,

$$\rho(h) = -0.9\, \rho(h-2)$$

$$\Rightarrow \rho(h) + 0.9\, \rho(h-2) = 0$$

This is a difference equation with order 2 and $\alpha_1 = 0, \alpha_2 = 0.9$.

$\therefore$ Solution of $\rho(h)$ is

$$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}.$$

$\because$ z_1 and z_2 are a complex conjugate pair, then

$c_2 = \bar{c}_1$ [$\because \rho(h)$ is real]

$$\Rightarrow \rho(h) = c_1 z_1^{-h} + \bar{c}_1 \bar{z}_1^{-h}$$

Writing $z_1 = |z_1| \exp(i\theta)$ in the polar representation,

$$\rho(h) = a|z_1|^{-h} \cos(h\theta + b) = a(\sqrt{0.9})^h \cos\left(\frac{h\pi}{2} + b\right)$$

Initial conditions,

(i) $\rho(0) = \gamma(0)/\gamma(0) = 1$

(ii) $\rho(1) + 0.9\, \rho(1-2) = 0$

$\Rightarrow \rho(1) + 0.9\, \rho(1) = 0$ $\quad$ [$\because \rho(-1) = \rho(1)$]

$\Rightarrow \rho(1) = 0$

Using $\rho(0) = 1$,

$$\rho(0) = a(0.9)^0 \cos\left(\frac{0 \times \pi}{2} + b\right) = a \cos b$$

$$\Rightarrow \quad a \cos b = 1$$

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Using $\rho(1) = 0$

$\rho(1) = a\sqrt{0.9}\cos\left(\frac{\pi}{2}+b\right)$

$\Rightarrow \cos\left(\frac{\pi}{2}+b\right) = 0 \Rightarrow \boxed{b=0}$

$\because a\cos b = 1 \Rightarrow \boxed{a=1}$

$\therefore \rho(h)$ becomes,

$\rho(h) = (0.9)^{h/2}\cos\left(\frac{h\pi}{2}\right)$

5) Given ARMA model:

~~$X_t = 0.25X_{t-2} + w$~~

$X_t = 0.80X_{t-1} - 0.15X_{t-2} + W_t - 0.30W_{t-1}$

$\Rightarrow X_t - 0.8X_{t-1} + 0.15X_{t-2} = W_t - 0.30W_{t-1}$

AR polynomial:

$\phi(z) = 1 - 0.8z + 0.15z^2 = (1-0.3z)(1-0.5z)$

MA polynomial:

$\theta(z) = 1 - 0.3z$

We can see that the factor $(1-0.3z)$ is common to both the AR and MA polynomials. Therefore, yes, there is parameter redundancy.

The model reduces to an AR(1) process:

$(1-0.5B)X_t = W_t$

The roots of AR polynomial is $z_\phi = 2 > 1$

$\therefore$ The process is causal.

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And there is no root such that $|z_\theta| \leq 1$ in the MA polynomial, hence, the process is invertible.

$\therefore$ The process is a causal and invertible AR(1):

$X_t = 0.5 X_{t-1} + W_t$

b) AR(1) model: $X_t = \phi X_{t-1} + W_t$

ACF for AR(1): $\rho(h) = \phi^h$

MA(1) model: $X_t = W_t + \theta W_{t-1}$

ACF for MA(1): $\rho(h) = \begin{cases} 1 & , \text{for } h=0 \\ \dfrac{\theta}{1+\theta^2} & , \text{for } h=\pm 1 \\ 0 & , \text{for } |h|>1 \end{cases}$

ARMA(1,1) model: $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$

ACF for ARMA(1,1): $\rho(h) = \dfrac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2} \phi^{h-1}, \quad h \geq 1$

We need to plot these functions for all three cases with $\phi = 0.6$ and $\theta = 0.9$.