MTH 442 ASSIGNMENT 2

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1) Given MA(1) model: $X_t = W_t + \theta W_{t-1}$

ACF for lag 1:

$\gamma(1) = Cov(X_{t+1}, X_t)$

$= Cov[(W_{t+1} + \theta W_t)(W_t + \theta W_{t-1})]$

$= E[(W_{t+1} + \theta W_t)(W_t + \theta W_{t-1})]$ $\quad [\because E[X_t] = 0]$

$= \theta E[W_t^2] = \sigma_w^2 \theta$

$\gamma(0) = Cov(X_t, X_t)$

$= \cancel{Cov} E[X_t X_t]$

$= E[(W_t + \theta W_{t-1})(W_t + \theta W_{t-1})]$

$= E[W_t^2] + \theta^2 E[W_{t-1}^2]$

$= \sigma^2(1+\theta^2)$

$\therefore \rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{1+\theta^2}$

We know that AM $\geq$ GM,

$\therefore \frac{\left[\theta + \frac{1}{\theta}\right]}{2} \geq \sqrt{\theta \cdot \frac{1}{\theta}}$

$\Rightarrow \left[\frac{\theta^2+1}{\theta}\right] \geq 2$

$\Rightarrow \left[\frac{\theta}{\theta^2+1}\right] \leq \frac{1}{2}$

$\Rightarrow \rho(1) \leq \frac{1}{2}$

$\therefore$ lag 1 ACF is bounded above by 0.5.

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2) Given, $\{W_t ; t = 0, 1, \ldots\}$: white noise process

$|\phi| < 1$

$X_0 = W_0$ and $X_t = \phi X_{t-1} + W_t$; $t = 1, 2, \ldots$

(a) We can write iteratively as

$$X_t = \phi X_{t-1} + W_t$$
$$= \phi(\phi X_{t-2} + W_{t-1}) + W_t$$
$$= \phi^2 X_{t-2} + (\phi W_{t-1} + W_t)$$
$$\vdots$$
$$= \phi^k X_{t-k} + \sum_{j=0}^{k-1} \phi^j W_{t-j}$$

for $k = t$.

$$X_t = \phi^t X_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}$$

$[\because X_0 = W_0]$

$$\Rightarrow X_t = \sum_{j=0}^{t} \phi^j W_{t-j}$$

(b)
$$E(X_t) = \sum_{j=0}^{t} E[\phi^j W_{t-j}]$$
$$= \sum_{j=0}^{t} \phi^j E[W_{t-j}]$$
$$= 0$$

(c)
$$Var(X_t) = Var\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)$$
$$= E\left[\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right]$$
$$= \sum_{j=0}^{t} \phi^{2j} E[W_{t-j}^2]$$
$$= \sigma_w^2 \sum_{j=0}^{t} \phi^{2j} = \sigma_w^2 \left[\frac{1 - \phi^{2(t+1)}}{1 - \phi^2}\right]$$

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(d) $Cov(X_{t+h}, X_t) = E(X_{t+h} X_t)$

$[\because E[X_{t+h}] = E[X_t] = 0]$

$$= E\left[\left(\sum_{j=0}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t+h-j}\right)\right]$$

$$= E\left[\left(\sum_{j=0}^{h-1} \phi^j W_{t+h-j} + \sum_{j=h}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right]$$

$$= E\left[\left(\sum_{j=0}^{h-1} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right] + E\left[\left(\sum_{j=h}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right]$$

$$= 0 + E\left[\left(\sum_{j=h}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right)\right]$$

$$= E\left[(\phi^h W_t + \phi^{h+1} W_{t-1} + \phi^{h+2} W_{t-2} + \dots + \phi^{t+h} W_0)(\phi^0 W_t + \phi W_{t-1} + \dots + \phi^t W_0)\right]$$

$$= \phi^h E[W_t^2] + \phi^{h+2} E[W_{t-1}^2] + \dots + \phi^{h+2t} E[W_0^2]$$

$$= \sigma_w^2 [\phi^h + \phi^{h+2} + \dots + \phi^{h+2t}]$$

$$= \phi^h \sigma_w^2 (1 + \phi^2 + \dots + \phi^{2t})$$

$$= \phi^h \sigma_w^2 \cdot \frac{(1 - (\phi^2)^{t+1})}{1 - \phi^2}$$

$$= \phi^h Var(X_t)$$

(e) X_t is not stationary since $Cov(X_{t+h}, X_t)$ depends on t.

(f) ~~As $t \to \infty$~~ $Cov(X_{t+h}, X_t) = \phi^h Var(X_t)$

$$\lim_{t \to \infty} Cov(X_{t+h}, X_t) = \lim_{t \to \infty} \phi^h \sigma_w^2 \frac{(1 - \phi^{2(t+1)})}{1 - \phi^2}$$

$$= \frac{\phi^h \sigma_w^2}{1 - \phi^2} \quad [\because |\phi| < 1]$$

Now, the $Cov(X_{t+h}, X_t) = \gamma(h)$ is independent of t and depends only on h.

Further, $E[X_t] = 0$ and $Var(X_t)$ is finite

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for every $t \geq 0$.

$\therefore X_t$ is asymptotically stationary.

(g) We can generate ~~n obser~~ more than n observations like $n+n_0$, where n_0 is fairly large and discard the first n_0.

The remaining n observations can then be used to simulate asymptotically stationary AR(1) model

(h) Consider $X_0 = \dfrac{W_0}{\sqrt{1-\phi^2}}$

Now, $X_t = \phi X_{t-1} + W_t$

$$= \phi^t X_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}$$

$$\Rightarrow X_t = \phi^t \frac{W_0}{\sqrt{1-\phi^2}} + \sum_{j=0}^{t-1} \phi^j W_{t-j}$$

$$\Rightarrow E[X_t] = \frac{\phi^t}{\sqrt{1-\phi^2}} E[W_0] + \sum_{j=0}^{t-1} \phi^j E[W_{t-j}]$$

$$\Rightarrow E[X_t] = 0$$

$\therefore E[X_t]$ is constant.

$\gamma(h) =$ $Cov(X_{t+h}, X_t) = E[X_{t+h} X_t]$

$$= E\left[\left(\frac{\phi^{t+h}}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t+h-1} \phi^j W_{t+h-j}\right)\left(\frac{\phi^t}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}\right)\right]$$

$$= E\left[\left(\frac{\phi^{t+h}}{\sqrt{1-\phi^2}} W_0 + \sum_{j=h}^{t+h-1} \phi^j W_{t+h-j} + \sum_{j=0}^{h-1} \phi^j W_{t+h-j}\right)\left(\frac{\phi^t}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}\right)\right]$$

$$= E\left[\left(\frac{\phi^{t+h}}{\sqrt{1-\phi^2}} W_0 + \sum_{j=h}^{t+h-1} \phi^j W_{t+h-j}\right)\left(\frac{\phi^t}{\sqrt{1-\phi^2}} W_0 + \sum_{j=0}^{t-1} \phi^j W_{t-j}\right)\right]$$

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$$= \phi^h E[W_t^2] + \phi^{h+2} E[W_{t-1}^2] + \dots + \phi^{h+2(t-1)} E[W_1^2] + \frac{\phi^{h+2t}}{1-\phi^2} E[W_0^2]$$

$$= \sigma_w^2 \phi^h \left(1 + \phi^2 + \dots + (\phi^2)^{t-1} + \frac{(\phi)^{2t}}{1-\phi^2}\right)$$

$$= \sigma_w^2 \phi^h \cdot \left[\frac{(1-\phi^{2t})}{1-\phi^2} + \frac{\phi^{2t}}{1-\phi^2}\right]$$

$$= \sigma_w^2 \phi^h \cdot \frac{1}{1-\phi^2} = \frac{\phi^h \sigma_w^2}{1-\phi^2}$$

$$Var(X_t) = \gamma(0) = \frac{\phi^h \sigma_w^2}{1-\phi^2} < \infty$$

We can see that $\gamma(h)$ is independent of t and depends only on h. Further, $Var(X_t)$ is finite ~~for~~ for all t.

$\therefore$ X_t is stationary when $X_0 = \frac{W_0}{\sqrt{1-\phi^2}}$

3) Given AR(2) model:

$$(1 - \phi_1 B - \phi_2 B^2) X_t = W_t$$

The AR polynomial will be:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

roots of $\phi(z)$ are:

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_1}$$

The given model is Causal if and only if:

$$\left|\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_1}\right| > 1$$

The roots of $\phi(z)$ may be real and distinct, real and equal, or a complex conjugate pair.
Roots are real if $\phi_1^2 + 4\phi_2 \geq 0$ else roots are complex conjugates.

Consider z_1, z_2 as the roots of $\phi(z)$ such that $z_1, z_2 \in \mathbb{C}$ ~~and $z_1 = z_2$~~.

Model is causal if and only if $|z_1| > 1$ and $|z_2| > 1$

We can write $\phi(z)$ as

$$\phi(z) = (1 - z_1^{-1}z)(1 - z_2^{-1}z)$$

$$\Rightarrow \phi(z) = 1 - (z_1^{-1} + z_2^{-1})z + z_1^{-1}z_2^{-1}z$$

Comparing the above expression with

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

we get,

$$\phi_1 = z_1^{-1} + z_2^{-1}, \qquad \phi_2 = -(z_1 z_2)^{-1}$$

Let $u_1 = z_1^{-1}$ and $u_2 = z_2^{-1}$

$\therefore$ ~~$\phi_1 = z_1$~~ $\Rightarrow |u_1| = |z_1^{-1}|$ and $|u_2| = |z_2^{-1}|$

$\Rightarrow |u_1| = \frac{1}{|z_1|} < 1$ and $|u_2| = \frac{1}{|z_2|} < 1$

Also,

$$\phi_2 = \frac{-1}{z_1 z_2} \Rightarrow |\phi_2| = \frac{1}{|z_1||z_2|} < 1$$

$$\begin{aligned}
\phi_1 + \phi_2 - 1 &= z_1^{-1} + z_2^{-1} - (z_1 z_2)^{-1} - 1 \\
&= u_1 + u_2 - u_1 u_2 - 1 \\
&= u_1(1 - u_2) + u_2 - 1 \\
&= -(1 - u_1)(1 - u_2)
\end{aligned}$$

~~$\Rightarrow \phi_1 + \phi_2 < 1$~~

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$$\phi_2 - \phi_1 - 1 = -z_1^{-1} - z_2^{-1} - (z_1 z_2)^{-1} - 1$$
$$= -u_1 - u_2 - u_1 u_2 - 1$$
$$= -(1+u_1)(1+u_2)$$

Consider the case when $z_1, z_2 \in \mathbb{R}$.

$\therefore \phi_2 + \phi_1 - 1 = -(1-u_1)(1-u_2) < 0$

$[\because |u_1|$ and $|u_2| < 1]$

$\phi_2 - \phi_1 - 1 = -(1+u_1)(1+u_2) < 0$

$\therefore$ We have following conditions when roots are real,

$|\phi_2| < 1$, $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$

Now, consider the case when $z_1, z_2 \in \mathbb{C}$ and $z_1 = \overline{z_2}$.

$$\phi_2 + \phi_1 - 1 = -(1-u_1)(1-u_2)$$
$$= -(1-u_1)(1-\overline{u_1}) \quad [\because z_2 = \overline{z_1} \Rightarrow u_2 = \overline{u_1}]$$
$$= -(1-u_1)(\overline{1-u_1})$$
$$= -|(1-u_1)|^2 < 0$$

$$\phi_2 - \phi_1 - 1 = -(1+u_1)(1+u_2)$$
$$= -(1+u_1)(1+\overline{u_1})$$
$$= -(1+u_1)(\overline{1+u_1})$$
$$= -|(1+u_1)|^2 < 0$$

$\therefore$ We again have following conditions when roots are complex conjugates,

$|\phi_2| < 1$, $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$

Thus, we have the following conditions in the general case to establish causality,

$|\phi_2| < 1$, $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$

This causality condition specifies a triangular region in the parameter space.

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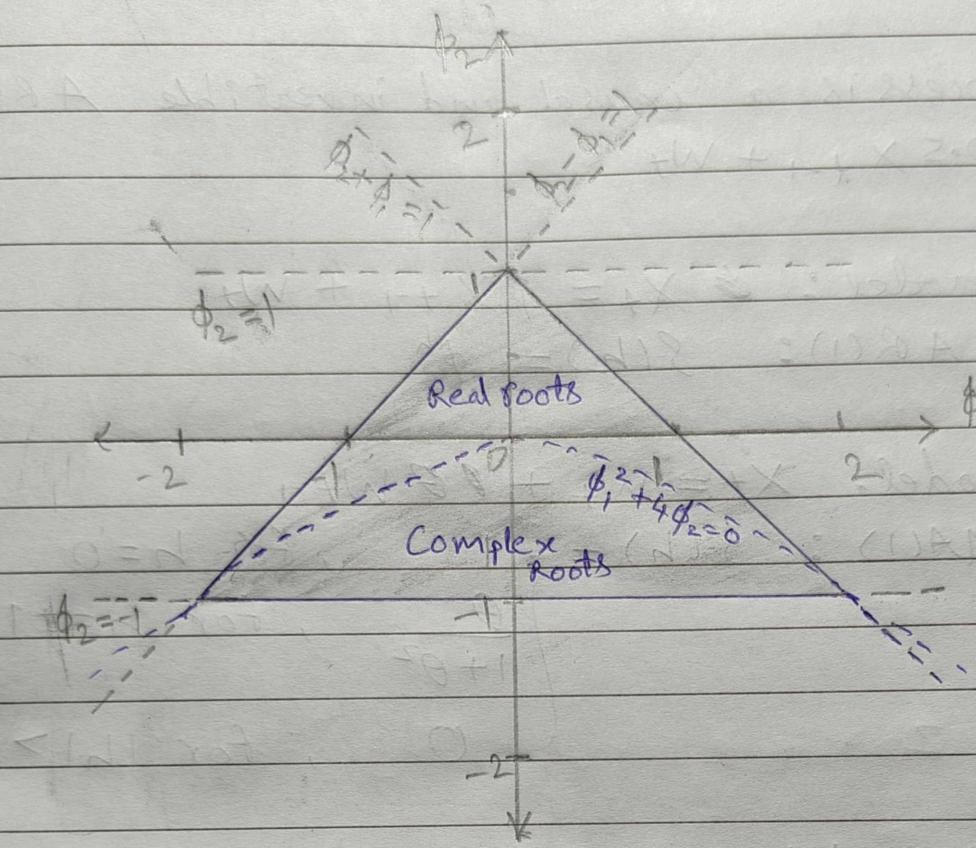

ϕ_2
2
$\phi_2+\phi_1=1$
$\phi_2-\phi_1=1$
1
$\phi_2=1$
Real roots
ϕ_1
-2
2
$\phi_1^2+4\phi_2=0$
Complex roots
$\phi_2=-1$
-1
-2

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4) (i) Given, AR(2) model :

$$X_t = 0.25 X_{t-2} + W_t$$

AR polynomial is given by,

$$\phi(z) = 1 - 0.25 z^2$$

Roots of $\phi(z)$:

$$\phi(z) = 0$$

$$\Rightarrow 1 - 0.25 z^2 = 0$$

$$\Rightarrow (z - 2)(z + 2) = 0$$

$$\Rightarrow z = \pm 2$$

$$X_t = 0.25 X_{t-2} + W_t$$

Multiply each side by X_{t-h} for $h > 0$ and take expectation

$$E[X_t X_{t-h}] = 0.25 E[X_{t-2} X_{t-h}] + E[W_t X_{t-h}]$$

$$\Rightarrow \gamma(h) = 0.25 \gamma(h-2) \quad , h = 1, 2, \ldots$$

Dividing by $\gamma(0)$,

$$\rho(h) = 0.25 \rho(h-2)$$

$$\Rightarrow \rho(h) - 0.25 \rho(h-2) = 0$$

This is a difference equation with order 2.

$\therefore$ Solution of $\rho(h)$ is

$$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$$

Also, Initial conditions:

(i) $\rho(0) = \gamma(0)/\gamma(0) = 1$

(ii) $\rho(1) = \dfrac{\gamma(1)}{\gamma(0)} = 0.25 \rho(1-2)$

$$\Rightarrow \rho(1) - 0.25 \rho(-1) = 0$$

$$\Rightarrow \rho(1)[1 - 0.25] = 0 \qquad [\because \rho(-1) = \rho(1)]$$

$$\Rightarrow \rho(1) = 0$$

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Using $\rho(0)=1$,

$$\rho(0) = c_1 z_1^{-0} + c_2 z_2^{-0}$$

$$\Rightarrow \quad c_1 + c_2 = 1$$

Using $\rho(1)=0$,

$$\rho(1) = c_1 z_1^{-1} + c_2 z_2^{-1} = 0$$

$$\Rightarrow \quad \frac{c_1}{2} - \frac{c_2}{2} = 0$$

$$\Rightarrow \quad c_1 = c_2$$

$\therefore$ We get, $c_1 = 1/2$ and $c_2 = 1/2$

$\therefore$ ~~General~~ $\rho(h)$ becomes,

$$\rho(h) = \frac{(2)^{-h}}{2} + \frac{(-2)^{-h}}{2}$$

$$\Rightarrow \rho(h) = [1+(-1)^h]\, 2^{-(h+1)}$$

(ii) Given AR(2) model:

$$X_t = -0.9\, X_{t-2} + W_t$$

AR polynomial is given by:

$$\phi(z) = 1 + 0.9 z^2$$

Roots of $\phi(z)$:

$$\phi(z) = 0$$

$$\Rightarrow 1 + 0.9 z^2 = 0$$

$$\Rightarrow z^2 = -(0.9)^{-1}$$

$$\Rightarrow z = \pm \frac{i}{\sqrt{0.9}}$$

$$\arg(z) = \pi/2$$

$$\therefore z = \pm \frac{1}{\sqrt{0.9}} e^{i\pi/2}$$

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$X_t = -0.9 X_{t-2} + W_t$

Multiply each side by X_{t-h} for $h>0$, and take expectation,

$E[X_t X_{t-h}] = -0.9\, E[X_{t-2} X_{t-h}] + E[W_t X_{t-h}]$

$\Rightarrow \gamma(h) = -0.9\, \gamma(h-2)$; $h = 1, 2, \ldots$

Dividing by $\gamma(0)$ gives,

$\rho(h) = -0.9\, \rho(h-2)$

$\Rightarrow \rho(h) + 0.9\, \rho(h-2) = 0$

This is a difference equation with order 2 and $\alpha_1 = 0, \alpha_2 = 0.9$.

$\therefore$ Solution of $\rho(h)$ is

$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$.

$\because$ z_1 and z_2 are a complex conjugate pair, then

$c_2 = \bar{c}_1$ [$\because \rho(h)$ is real]

$\Rightarrow \rho(h) = c_1 z_1^{-h} + \bar{c}_1 \bar{z}_1^{-h}$

Writing $z_1 = |z_1| \exp(i\theta)$ in the polar representation,

$\rho(h) = a |z_1|^{-h} \cos(h\theta + b) = a(\sqrt{0.9})^h \cos\left(h\frac{\pi}{2} + b\right)$

Initial conditions,

(i) $\rho(0) = \gamma(0)/\gamma(0) = 1$

(ii) $\rho(1) + 0.9\, \rho(1-2) = 0$

$\Rightarrow \rho(1) + 0.9\, \rho(1) = 0$ [$\because \rho(-1) = \rho(1)$]

$\Rightarrow \rho(1) = 0$

Using $\rho(0) = 1$,

$\rho(0) = a (0.9)^0 \cos\left(\frac{0 \times \pi}{2} + b\right) = a \cos b$

$\Rightarrow a \cos b = 1$

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Using $\rho(1) = 0$

$\rho(1) = a\sqrt{0.9}\cos\left(\frac{\pi}{2} + b\right)$

$\Rightarrow \cos\left(\frac{\pi}{2} + b\right) = 0 \Rightarrow \boxed{b = 0}$

$\because a\cos b = 1 \Rightarrow \boxed{a = 1}$

$\therefore \rho(h)$ becomes,

$\rho(h) = (0.9)^{h/2}\cos\left(\frac{h\pi}{2}\right)$

5) Given ARMA model:

~~$X_t = 0.25 X_{t-2} + W$~~

$X_t = 0.80 X_{t-1} - 0.15 X_{t-2} + W_t - 0.30 W_{t-1}$

$\Rightarrow X_t - 0.8 X_{t-1} + 0.15 X_{t-2} = W_t - 0.30 W_{t-1}$

AR polynomial:

$\phi(z) = 1 - 0.8z + 0.15z^2 = (1 - 0.3z)(1 - 0.5z)$

MA polynomial:

$\theta(z) = 1 - 0.3z$

We can see that the factor $(1 - 0.3z)$ is common to both the AR and MA polynomials. Therefore, yes, there is parameter redundancy.

The model reduces to an AR(1) process:

$(1 - 0.5B) X_t = W_t$

The roots of AR polynomial is $z_\phi = 2 > 1$

$\therefore$ The process is causal.

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And there is no root such that $|z_{\theta}| \leq 1$ in the MA polynomial, hence, the process is invertible.

$\therefore$ The process is a causal and invertible AR(1):

$$X_t = 0.5 X_{t-1} + W_t$$

b) AR(1) model: $X_t = \phi X_{t-1} + W_t$

ACF for AR(1): $\rho(h) = \phi^h$

MA(1) model: $X_t = W_t + \theta W_{t-1}$

ACF for MA(1): $\rho(h) = \begin{cases} 1 & , \text{for } h=0 \\ \dfrac{\theta}{1+\theta^2} & , \text{for } h=\pm 1 \\ 0 & , \text{for } |h|>1 \end{cases}$

ARMA(1,1) model: $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$

ACF for ARMA(1,1): $\rho(h) = \dfrac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2} \phi^{h-1}, \quad h \geq 1$

We need to plot these functions for all three cases with $\phi = 0.6$ and $\theta = 0.9$.

MTH442 Assignment 2

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27 September, 2024

Setup

```r
knitr::opts_chunk$set(echo = TRUE)

# Load the required packages
library(ggplot2)
library(astsa)
library(dplyr)
library(tidyr)
```

Problem 4

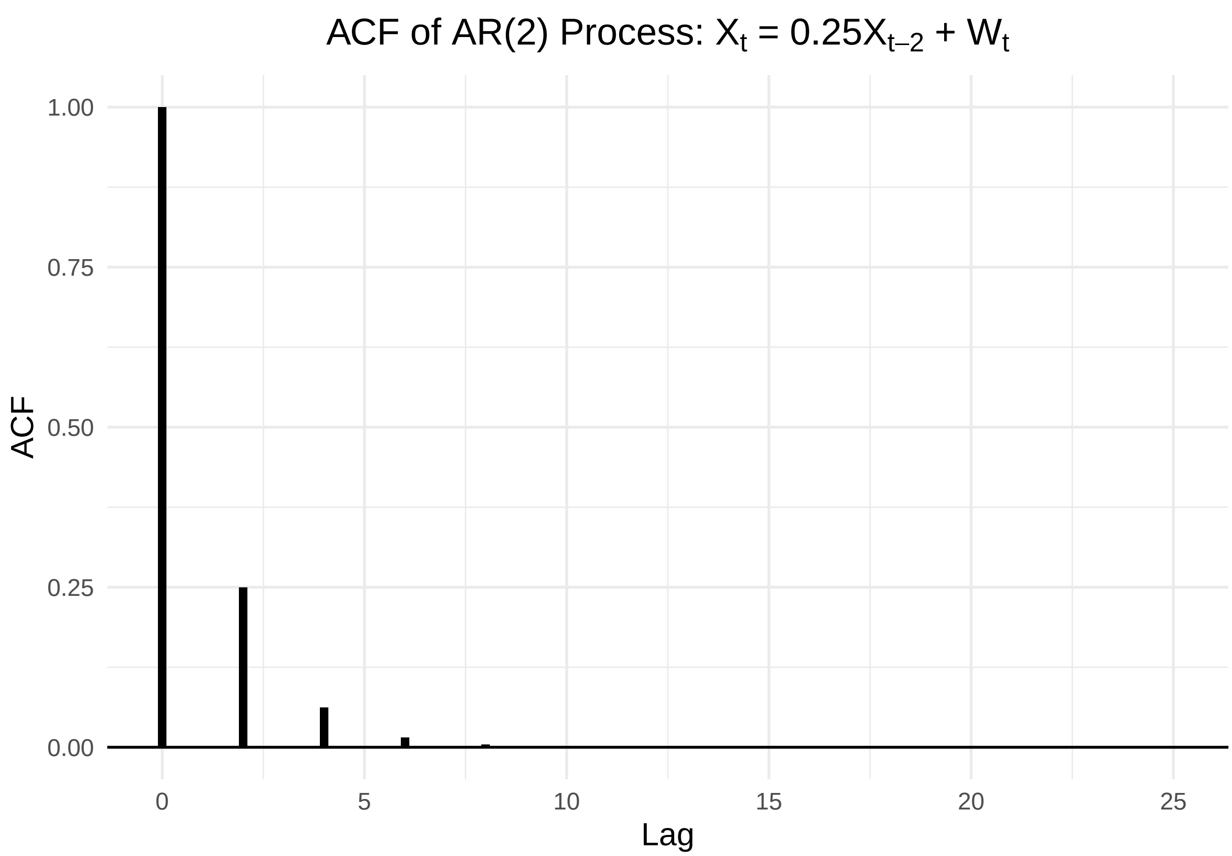
4. For the AR(2) models given by $X_t = 0.25X_{t-2} + W_t$ and $X_t = -0.9X_{t-2} + W_t$, find the roots of the autoregressive polynomials, and then plot their ACFs, $\rho(h)$. $(0.5 \times 2 = 1$ point)

AR(2) Model: $X_t = .25X_{t-2} + W_t$

$$\rho(h) = \frac{[1 + (-1)^h]}{2^{h+1}}; h \geq 1$$

```r
# Define the function to compute and plot ACF for AR(2)
compute_ACF_AR2_1 <- function(lag.max) {
  lags <- seq(from = 0, to = lag.max)

  # Function to compute ACF for each lag
  compute_acf <- function(h) {
    if (h == 0) return(1)
    acf <- (1+(-1)^h)/(2^(h+1))
    return(acf)
  }

  # Compute ACF for all lags
  acfs <- sapply(lags, compute_acf)

  # Create a data frame for plotting
  acf_data <- data.frame(Lag = lags, ACF = acfs)
```

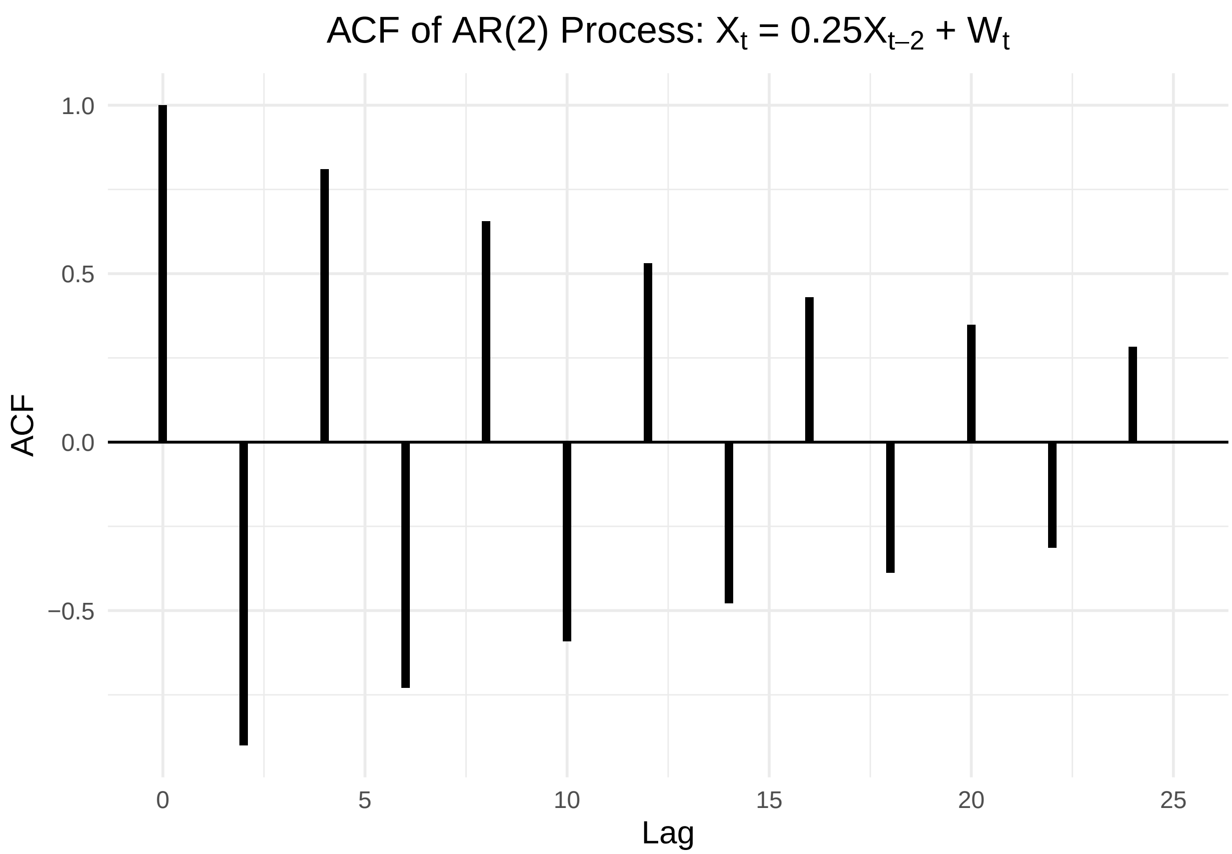
```r
  # Create the ggplot
  ggplot(acf_data, aes(x = Lag, y = ACF)) +
    geom_hline(yintercept = 0, color = "black") + # Line at y = 0
    geom_bar(stat = "identity", fill = "black", width = 0.2) + # Bar plot for ACF values
    labs(
      title = expression(paste("ACF of AR(2) Process: ", X[t], " = 0.25", X[t-2], " + ", W[t])),
      x = "Lag",
      y = "ACF"
    ) +
    theme_minimal() + # Minimal theme for a clean look
    theme(
      plot.title = element_text(hjust = 0.5, face = "bold", size = 14), # Centered and bold title
      axis.title.x = element_text(size = 12), # X-axis label size
      axis.title.y = element_text(size = 12)  # Y-axis label size
    )
}

# Call the function with lag.max = 25
compute_ACF_AR2_1(25)
```



AR(2) Model: $X_t = -0.9X_{t-2} + W_t$

$$\rho(h) = 0.9^{\frac{h}{2}} cos(h\frac{\pi}{2}); h \geq 1$$

```r
# Define the function to compute and plot ACF for AR(2)
compute_ACF_AR2_2 <- function(lag.max) {
  lags <- seq(from = 0, to = lag.max)

  # Function to compute ACF for each lag
  compute_acf <- function(h) {
    if (h == 0) return(1)
    acf <- (0.9^(h/2))*cos(h*pi/2)
    return(acf)
  }

  # Compute ACF for all lags
  acfs <- sapply(lags, compute_acf)

  # Create a data frame for plotting
  acf_data <- data.frame(Lag = lags, ACF = acfs)

  # Create the ggplot
  ggplot(acf_data, aes(x = Lag, y = ACF)) +
    geom_hline(yintercept = 0, color = "black") + # Line at y = 0
    geom_bar(stat = "identity", fill = "black", width = 0.2) + # Bar plot for ACF values
    labs(
      title = expression(paste("ACF of AR(2) Process: ", X[t], " = 0.25", X[t-2], " + ", W[t])),
      x = "Lag",
      y = "ACF"
    ) +
    theme_minimal() + # Minimal theme for a clean look
    theme(
      plot.title = element_text(hjust = 0.5, face = "bold", size = 14), # Centered and bold title
      axis.title.x = element_text(size = 12), # X-axis label size
      axis.title.y = element_text(size = 12)  # Y-axis label size
    )
}

# Call the function with lag.max = 25
compute_ACF_AR2_2(25)
```


ACF of AR(2) Process: $X_t = 0.25X_{t-2} + W_t$
1.0
0.5
0.0
−0.5
ACF
0
5
10
15
20
25
Lag

Problem 6

6. Plot the theoretical ACFs (all derived in the class) of the three series AR(1)=ARMA(1,0) $X_t = \phi X_{t-1} + W_t$, MA(1)=ARMA(0,1) $X_t = W_t + \theta W_{t-1}$, and ARMA(1,1) $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$ on the same graph for $\phi = 0.6$, $\theta = 0.9$, and comment on the diagnostic capabilities of the ACF in this case. (1 point)

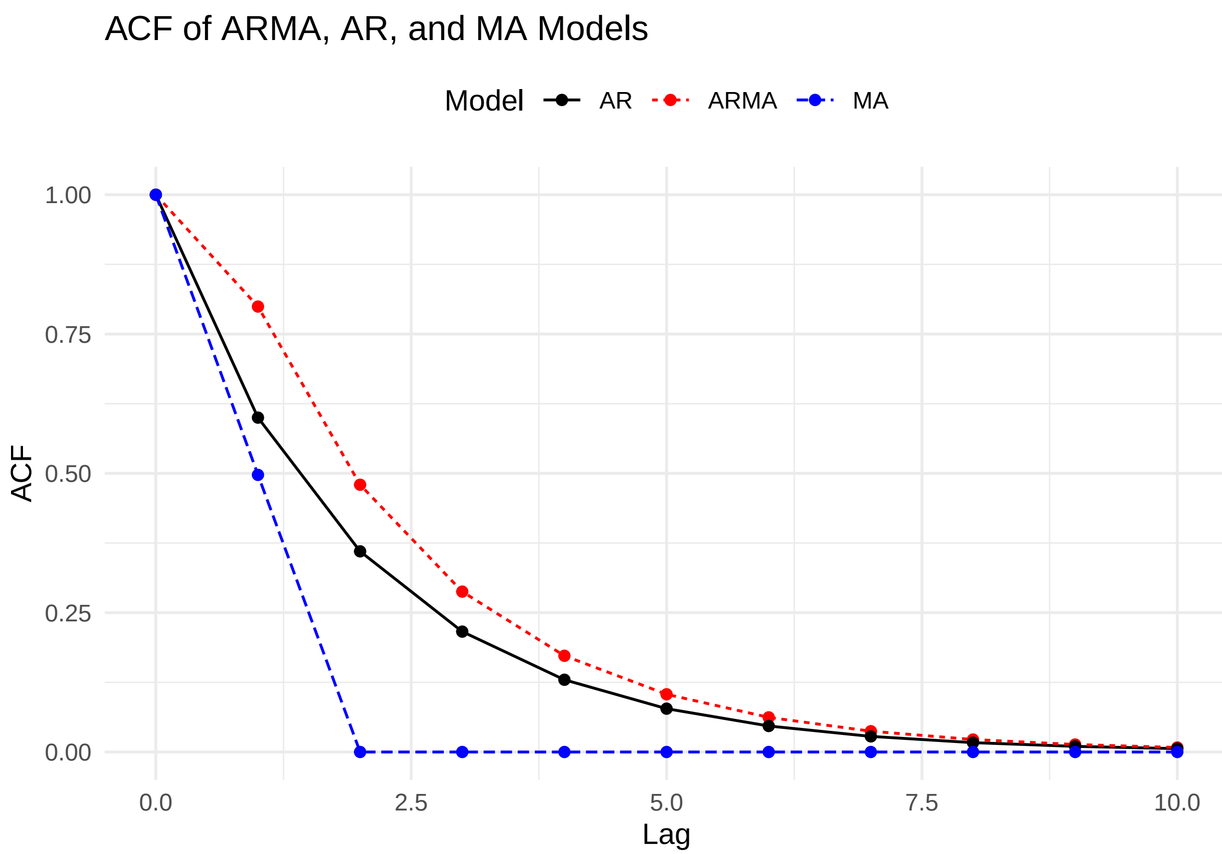
ACF of ARMA(1,1):

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\phi^{h-1}; h \geq 1$$

ACF of AR(1) can be obtained by putting $\theta = 0$ in the above formula.

ACF of MA(1) can be obtained by putting $\phi = 0$ in the above formula.

```
compute_ACF_ARMA <- function(phi, theta, lag.max) {
  lags <- seq(from=0, to=lag.max)
  compute_acf <- function(h) {
    if(h==0) return(1)
    acf <- ((phi^(h-1))*(1+theta*phi)*(phi+theta))/(1+2*theta*phi+theta*theta)
    return(acf)
  }
  acfs <- sapply(lags, compute_acf)
  return(acfs)
}

# Generate ACF values for the models
u1 <- compute_ACF_ARMA(0.6, 0.9, lag.max=10)
u2 <- compute_ACF_ARMA(0.6, 0, lag.max=10)
u3 <- compute_ACF_ARMA(0, 0.9, lag.max=10)

# Create a data frame for ggplot
df <- data.frame(
  Lag = 0:10,
  ARMA = u1,
  AR = u2,
  MA = u3
) %>%
  pivot_longer(cols = c("ARMA", "AR", "MA"), names_to = "Model", values_to = "ACF")

# Plot using ggplot
ggplot(df, aes(x = Lag, y = ACF, color = Model)) +
  geom_line(aes(linetype = Model)) +
  geom_point() +
  labs(title = "ACF of ARMA, AR, and MA Models", x = "Lag", y = "ACF") +
  scale_color_manual(values = c("black", "red", "blue")) +
  theme_minimal() +
  theme(legend.position = "top")
```

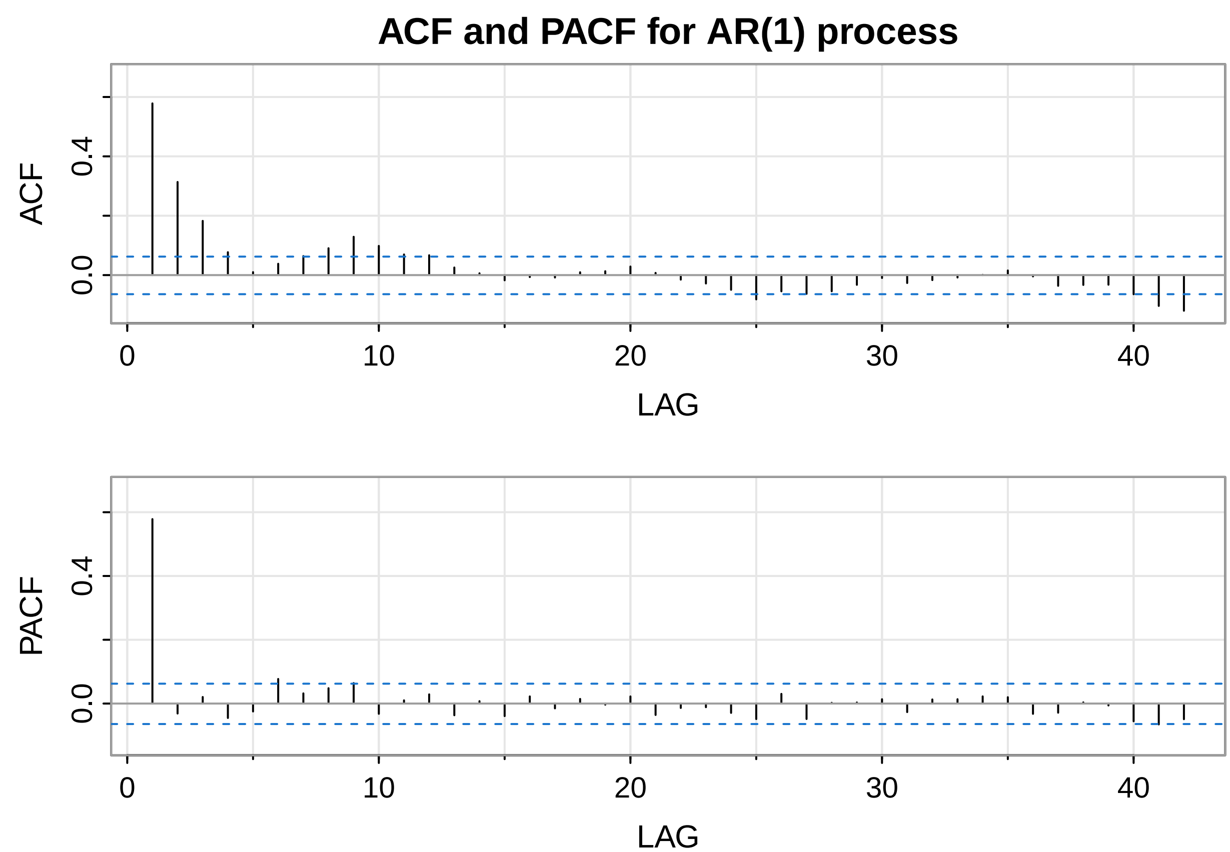

ACF of ARMA, AR, and MA Models
Model
AR
ARMA
MA
ACF
1.00
0.75
0.50
0.25
0.00
0.0
2.5
5.0
7.5
10.0
Lag

Problem 7

7. Generate $n = 1000$ observations from each of the models in the previous question and comment on the important patterns in the ACF and PACF plots. (1 point)

AR(1) Process

```
ar = arima.sim(list(order=c(1,0,0), ar=.6), n=1000)
acf2(ar, main="ACF and PACF for AR(1) process")
```



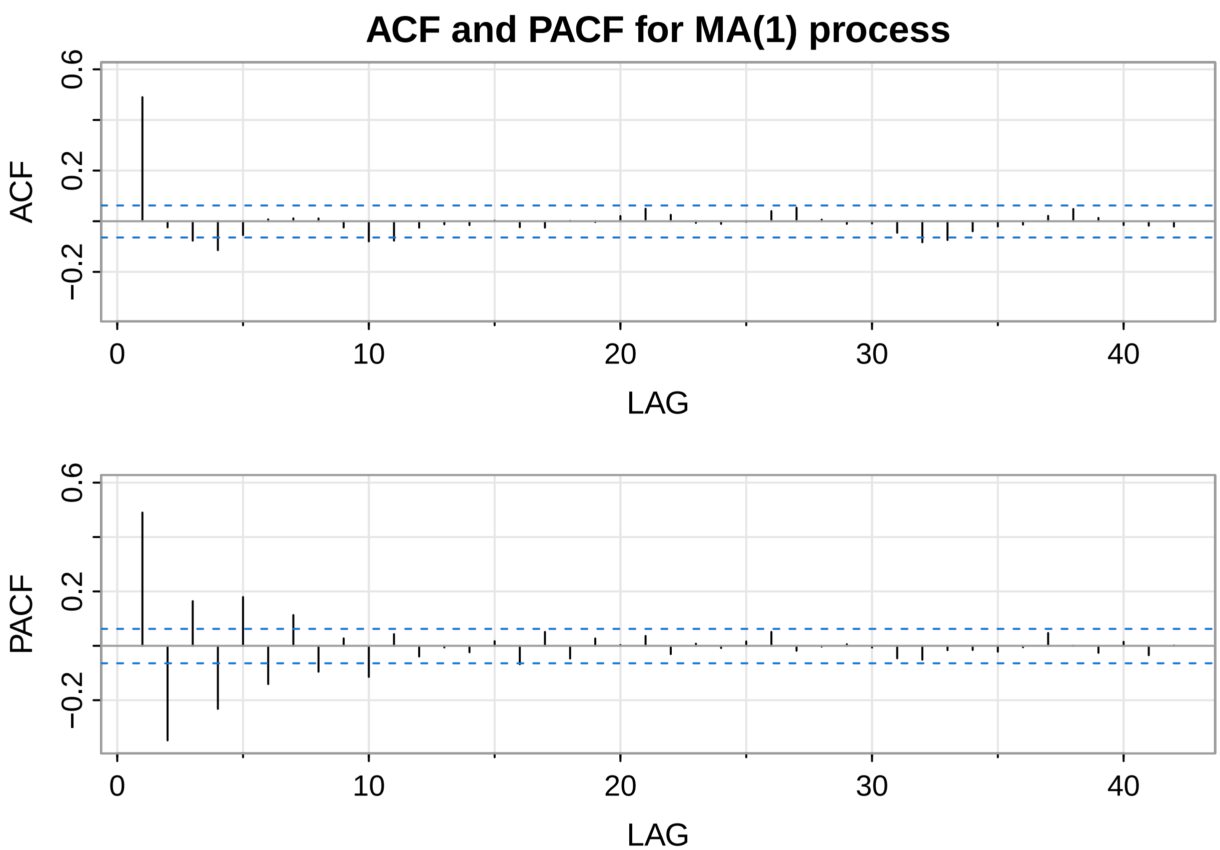

```
##       [,1]  [,2] [,3]  [,4]  [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13]
## ACF  0.58  0.31 0.18  0.08  0.01 0.04 0.06 0.09 0.13  0.10  0.07  0.07  0.03
## PACF 0.58 -0.03 0.02 -0.05 -0.02 0.08 0.03 0.05 0.06 -0.03  0.01  0.03 -0.04
##      [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
## ACF   0.01 -0.02 -0.01 -0.01  0.01  0.01  0.03  0.01 -0.02 -0.03 -0.05 -0.08
## PACF  0.01 -0.04  0.02 -0.02  0.01  0.00  0.02 -0.04 -0.01 -0.01 -0.03 -0.05
##      [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
## ACF  -0.05 -0.06 -0.05 -0.03 -0.01 -0.03 -0.02 -0.01  0.00  0.02  0.00 -0.04
## PACF  0.03 -0.05  0.00  0.00  0.01 -0.03  0.01  0.01  0.02  0.02 -0.03 -0.03
##      [,38] [,39] [,40] [,41] [,42]
## ACF  -0.03 -0.03 -0.06 -0.10 -0.12
## PACF  0.00 -0.01 -0.06 -0.07 -0.05
```

Comments on AR(1):

- **ACF**: The ACF plot shows a gradual decrease or tapering off of the autocorrelation coefficients as the lag increases. This pattern is characteristic of an autoregressive (AR) process, where the current value of the series is dependent on its past values.
- **PACF**: The PACF plot exhibits a sharp cutoff after the first lag, indicating that the partial autocorrelations beyond the first lag are effectively zero. This is a typical signature of an AR(1) or first-order autoregressive process, where the current value of the series is linearly dependent only on the immediately preceding value.

MA(1) Process

```
ma = arima.sim(list(order=c(0,0,1), ma=.9), n=1000)
acf2(ma, main="ACF and PACF for MA(1) process")
```




```
##       [,1]  [,2]  [,3]  [,4]  [,5]  [,6] [,7]  [,8]  [,9] [,10] [,11] [,12]
## ACF  0.49 -0.02 -0.08 -0.11 -0.05  0.01 0.01  0.01 -0.02 -0.08 -0.08 -0.03
## PACF 0.49 -0.35  0.16 -0.23  0.18 -0.14 0.11 -0.10  0.03 -0.11  0.04 -0.04
##      [,13] [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24]
## ACF  -0.01 -0.02  0.00 -0.02 -0.03  0.00  0.00  0.02  0.05  0.03 -0.01 -0.01
## PACF -0.01 -0.02  0.02 -0.07  0.05 -0.05  0.03  0.00  0.04 -0.03  0.01 -0.01
##      [,25] [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36]
## ACF   0.00  0.04  0.05  0.01 -0.01 -0.01 -0.05 -0.08 -0.07 -0.04 -0.02 -0.01
```

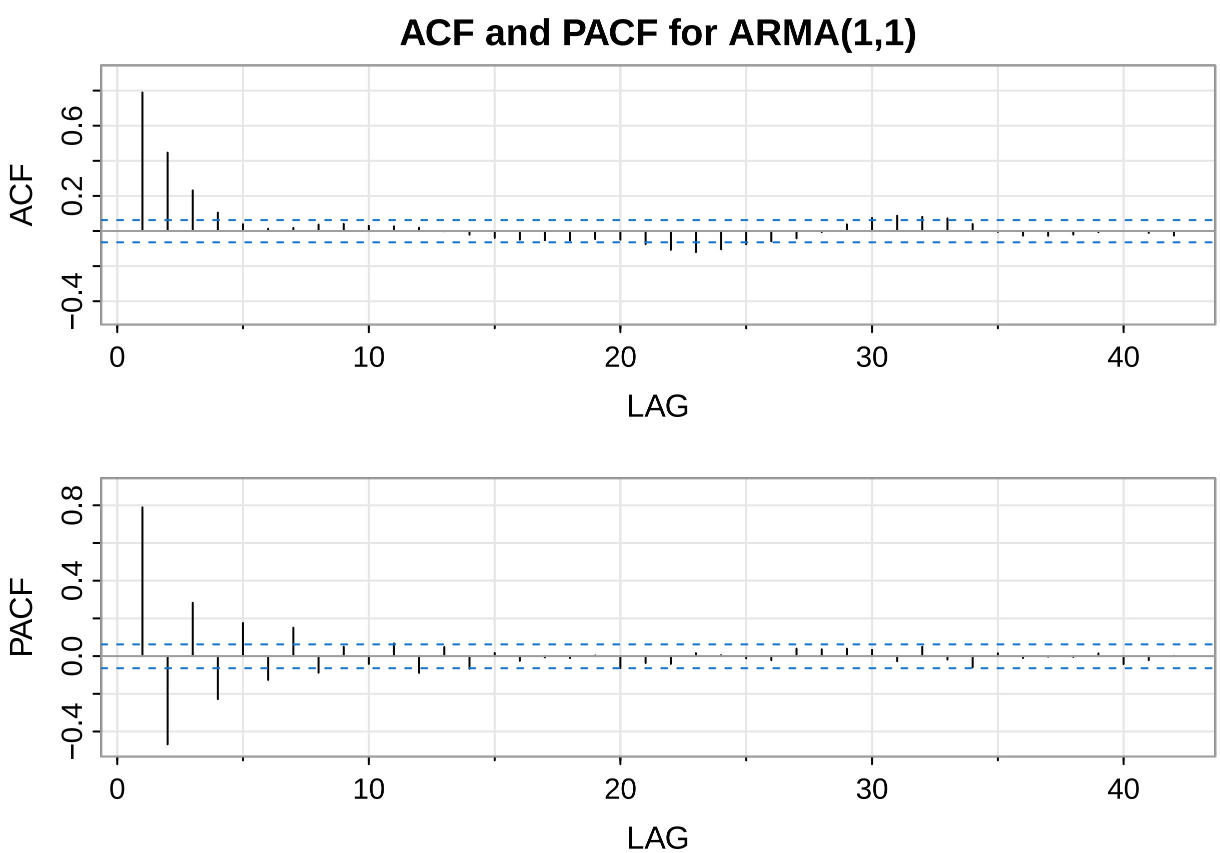
```
## PACF  0.02  0.05 -0.02  0.00  0.01 -0.01 -0.05 -0.05 -0.02 -0.02 -0.02 -0.01
##      [,37] [,38] [,39] [,40] [,41] [,42]
## ACF   0.02  0.05  0.01 -0.01 -0.02 -0.02
## PACF  0.05  0.00 -0.03  0.01 -0.03  0.00
```

Comments on MA(1):

- **ACF**: The ACF plot exhibits a sharp cutoff after the first lag, with the autocorrelation coefficients becoming effectively zero beyond this point. This pattern is characteristic of a moving average (MA) process, where the current value of the series is dependent on the immediate past random shocks or errors.
- **PACF**: The PACF plot shows a gradual tapering off of the partial autocorrelation coefficients as the lag increases. This behavior is consistent with an MA process, where the current value depends linearly on the immediately preceding random shock or error term.

ARMA(1,1) Process

```
arma = arima.sim(list(order=c(1,0,1), ar=.6, ma=.9), n=1000)
acf2(arma, main="ACF and PACF for ARMA(1,1)")
```




```
##      [,1]  [,2] [,3]  [,4] [,5]  [,6] [,7]  [,8] [,9] [,10] [,11] [,12] [,13]
## ACF  0.79  0.45 0.23  0.10 0.04  0.01 0.02  0.04 0.04  0.03  0.03  0.02  0.00
```

```
## PACF 0.79 -0.47 0.28 -0.23 0.18 -0.13 0.15 -0.09 0.05 -0.04  0.07 -0.09  0.05
##      [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
## ACF  -0.02 -0.04 -0.05 -0.05 -0.05 -0.05 -0.05 -0.07 -0.11 -0.12  -0.1 -0.08
## PACF -0.07  0.02 -0.03 -0.01 -0.01  0.00 -0.06 -0.04 -0.04  0.02   0.0 -0.01
##      [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
## ACF  -0.06 -0.04 -0.01  0.04  0.07  0.09  0.08  0.07  0.04 -0.01 -0.03 -0.03
## PACF -0.02  0.04  0.04  0.04  0.03 -0.03  0.05 -0.02 -0.06  0.02 -0.01  0.00
##      [,38] [,39] [,40] [,41] [,42]
## ACF  -0.02 -0.01  0.00 -0.01 -0.03
## PACF  0.00  0.01 -0.04 -0.02  0.00
```

Comments on ARMA(1,1):

- **ACF**: The ACF plot shows a gradual tapering off of the autocorrelation coefficients as the lag increases. This pattern is characteristic of a combined autoregressive and moving average (ARMA) process, where the current value of the series is dependent on both its past values and past random shocks or errors.
- **PACF**: The PACF plot also exhibits a gradual decline in the partial autocorrelation coefficients as the lag increases. This behavior is consistent with an ARMA process, where the current value depends linearly on both its own past values and the immediately preceding random shocks or error terms.