MTH442 Assignment 1

Name - Kaushik Raj V Nadar
Roll No. - 208160499

1) Given, $X_t = \beta_1 + \beta_2 t + W_t$

$$\cancel{V_t} \quad V_t = \frac{1}{(2q+1)} \sum_{j=-q}^{q} X_{t-j}$$

The mean of moving average,

$$E[V_t] = \frac{1}{2q+1} \sum_{j=-q}^{q} E[X_{t-j}]$$

$$= \cancel{\frac{1}{2q+1} \sum_{j=-q}^{q} E[\beta_1 + \beta_2(t-j) + W_t]}$$

$$= \frac{1}{2q+1} \sum_{j=-q}^{q} E[\beta_1 + \beta_2(t-j) + W_{t-j}]$$

$$= \frac{1}{2q+1} \sum_{j=-q}^{q} [\beta_1 + \beta_2(t-j)] \qquad [\because W_{t-j} \sim N(0,\sigma_w^2)]$$

$$= \frac{1}{2q+1} \left[\beta_1(2q+1) + \beta_2 t(2q+1) - \beta_2 \sum_{j=-q}^{q} j\right]$$

$$= \frac{1}{2q+1} [(\beta_1 + \beta_2 t)(2q+1) - 0]$$

$$= \beta_1 + \beta_2 t$$

Auto covariance Function,

$$\gamma_V(h) = Cov(V_t, V_{t+h})$$

We know,

$$V_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j} \quad ; \quad V_{t+h} = \frac{1}{2q+1} \sum_{k=-q}^{q} X_{t+h-k}$$

$$\therefore Cov(V_t, V_{t+h}) = \frac{1}{(2q+1)^2} Cov\left(\sum_{j=-q}^{q} X_{t-j}, \sum_{k=-q}^{q} X_{t+h-k}\right)$$

$$= \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{k=-q}^{q} Cov(X_{t+j}, X_{t+h-k})$$

$$= \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{k=-q}^{q} Cov(W_{t+j}, W_{t+h-k})$$

$$= \frac{\sigma_W^2}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{k=-q}^{q} I(j = k-h)$$

Now, consider separate cases for h:

- When $h=0$, $j=k$, so $I(j-k=0)$ is 1 for every pair where $j=k$. ∴ There are $2q+1$ such pairs.

- When $h>0$, $j=k-h$, which implies k can range from $-q+h$ to q (for $-q \le j \le q$). The no. of such pairs is $2q+1-h$.

- When $h<0$, let $h=-m$, where $m>0$. Then $k=j+h=j-m \Rightarrow k$ ~~from~~ ranges from $-q$ to $q-m \Rightarrow$ There are $2q+1-|h|$ pairs

$$\therefore \sum_{j=-q}^{q} \sum_{k=-q}^{q} I(j-k=h) = 2q+1-|h| \quad \text{for } |h| \le 2q+1$$

$$\Rightarrow \gamma_Y(h) = \frac{\sigma_W^2}{(2q+1)^2}(2q+1-|h|)$$

$$= \frac{\sigma_W^2}{2q+1}\left[1 - \frac{|h|}{(2q+1)}\right]$$

for $|h| \le 2q+1$

M	T	W	T	F	S	S

Page No.:
Date:
YOUVA

Given,

2) $X_t = U_1 \sin(2\pi\omega_0 t) + U_2 \cos(2\pi\omega_0 t)$

We say a time series is ~~stationa~~ weakly stationary if all of the following satisfy:

- Variance of the process is finite at each time point
- The mean value function μ_t is constant and does not depend on t.
- The auto covariance function, $\gamma(s,t)$ depends on s and t only through their difference $|s-t|$.

Mean,

$E[X_t] = \sin(2\pi\omega_0 t) E[U_1]$
$\quad + \cos(2\pi\omega_0 t) E[U_2]$
$= 0 \qquad [\because E[U_1] = E[U_2] = 0]$

Further, it is given that $E[U_1^2] = E[U_2^2] = \sigma^2$

$\therefore Var(X_t) = Cov(X_t, X_t)$

Auto Covariance Function,

$\gamma_V(h) = Cov(X_t, X_{t+h})$
$= E[(X_t - E[X_t])(X_{t+h} - E[X_{t+h}])]$
$= E[X_t X_{t+h}] \qquad [\because X_t, X_{t+h}$ have zero means$]$

$= E[(U_1 \sin(2\pi\omega_0 t) + U_2 \cos(2\pi\omega_0 t))$
$\quad (U_1 \sin(2\pi\omega_0 (t+h)) + U_2 \cos(2\pi\omega_0 (t+h)))]$

$= E[U_1^2 \sin(2\pi\omega_0 t) \sin(2\pi\omega_0 (t+h))$
$\quad + U_2 U_1 \cos(2\pi\omega_0 t) \sin(2\pi\omega_0 (t+h))$
$\quad + U_1 U_2 \sin(2\pi\omega_0 t) \cos(2\pi\omega_0 (t+h))$
$\quad + U_2^2 \cos(2\pi\omega_0 t) \cos(2\pi\omega_0 (t+h))]$

$= E[U_1^2] \sin(2\pi\omega_0 t) \sin(2\pi\omega_0 (t+h))$
$\quad + E[U_2^2] \cos(2\pi\omega_0 t) \cos(2\pi\omega_0 (t+h))$

$[\because E[U_1 U_2] = E[U_1]E[U_2] = 0 \quad (\because U_1, U_2$ are independent$)]$

$$\Rightarrow \gamma_V(h) = \sigma^2 [\sin(2\pi w_0 t) \sin(2\pi w_0 (t+h)) + \cos(2\pi w_0 t) \cos(2\pi w_0 (t+h))]$$

$[\because E[U_1^2] = E[U_2^2] = \sigma^2]$

$$\Rightarrow \gamma_V(h) = \sigma^2 \cos(2\pi w_0 h)$$

$[\because \sin A \sin B + \cos A \cos B = \cos(A-B)]$

Also, $Var(X_t) = \gamma_V(0) = \sigma^2$

$\therefore$ $Var(X_t)$ is finite for every t.

$\therefore$ the time series X_t is weakly stationary.

3) Given, $X_t = W_t$

$Y_t = W_t - \theta W_{t-1} + U_t$

(a) Auto covariance Function,

$\gamma_Y(h) = Cov(Y_t, Y_{t+h})$

$= E[Y_t Y_{t+h}]$

$= E[(W_t - \theta W_{t-1} + U_t)(W_{t+h} - \theta W_{t+h-1} + U_{t+h})]$

$= E[W_t W_{t+h} - \theta W_t W_{t+h-1} + U_{t+h} W_t$
$- \theta W_{t-1} W_{t+h} + \theta^2 W_{t-1} W_{t+h-1} - \theta U_{t+h} W_{t-1}$
$+ U_t W_{t+h} - \theta U_t W_{t+h-1} + U_t U_{t+h}]$

$= E[W_t W_{t+h} - \theta W_t W_{t+h-1} - \theta W_{t-1} W_{t+h}$
$+ \theta^2 W_{t-1} W_{t+h-1} + U_t U_{t+h}]$

[$\because E[U_i W_j] = E[U_i] E[W_j] = 0$
as U_i and W_j are independent $\forall i, j$]

Now, $E[W_t W_{t+h}] = \sigma_W^2$ for $h=0$, 0 o.w.

$E[W_t W_{t+h-1}] = \sigma_W^2$ for $h=1$, 0 o.w.

$E[W_{t-1} W_{t+h}] = \sigma_W^2$ for $h=-1$, 0 o.w.

$E[W_{t-1} W_{t+h-1}] = \sigma_W^2$ for $h=0$, 0 o.w.

$E[U_t U_{t+h}] = \sigma_U^2$ for $h=0$, 0 o.w.

$\therefore$ ACF, For $h=0$:

$\gamma_Y(0) = \sigma_W^2 + \theta^2 \sigma_W^2 + \sigma_U^2$

For $h=1$,

$\gamma_Y(1) = -\theta \sigma_W^2$

For $h=-1$:

$\gamma_Y(-1) = -\theta \sigma_W^2$

$|h| \geq 2$:

$\gamma_Y(h) = 0$

Date:

$$\therefore \gamma_Y(h) = \begin{cases} \sigma_w^2 + \theta^2\sigma_w^2 + \sigma_u^2 & , \text{ for } h=0 \\ -\theta\sigma_w^2 & , \text{ for } |h|=1 \\ 0 & , \text{ for } |h|\geq 2 \end{cases}$$

$\therefore$ Auto correlation function,

$$\rho_Y(h) = \frac{\gamma_Y(h)}{\gamma_Y(0)}$$

$$= \begin{cases} 1 & , h=0 \\ \dfrac{-\theta\sigma_w^2}{\sigma_w^2(1+\theta^2)+\sigma_u^2} & , |h|=1 \\ 0 & , |h|\geq 2 \end{cases}$$

where $h \in \mathbb{Z}$

(b) For Cross - Correlation Function, $\rho_{X,Y}(h)$,
We find the Cross-covariance function,

$$\gamma_{X,Y}(h) = Cov(X_{t+h}, Y_t)$$
$$= E[(X_{t+h} - E[X_{t+h}])(Y_t - E[Y_t])]$$
$$= E[X_{t+h}\,Y_t]$$
$$= E[W_{t+h}W_t - \theta W_{t+h}W_{t-1} + W_{t+h}U]$$

$E[W_{t+h}W_t] = \sigma_w^2$ for $h=0$; 0 o.w.
$E[W_{t+h}W_{t-1}] = \sigma_w^2$ for $h=-1$, 0 o.w.
$E[W_{t+h}U] = 0$ for all h

$$\therefore \gamma_{X,Y}(h) = E[W_{t+h}W_t] - \theta E[W_{t+h}W_{t-1}]$$

$$= \begin{cases} \sigma_w^2 & \text{for } h=0 \\ -\theta\sigma_w^2 & \text{for } h=-1 \\ 0 & \text{for } h\in\mathbb{Z}\setminus\{0,-1\} \end{cases}$$

$$\gamma_X(0) = Cov(X_t, X_t) = Cov(W_t, W_t)$$
$$= \sigma_w^2$$

$\gamma_Y(0) = \sigma_w^2 + \theta^2\sigma_w^2 + \sigma_u^2$ [From (a)]

M T W T F S S
Page No.:
Date:
YOUVA

$\therefore$ CCF, $\rho_{X,Y}(h) = \dfrac{\gamma_{X,Y}(h)}{\sqrt{\gamma_X(0)\,\gamma_Y(0)}}$

$$= \begin{cases} \dfrac{\sigma_W}{\sqrt{\sigma_W^2(1+\theta^2)+\sigma_U^2}} & \text{for } h=0 \\ \dfrac{-\theta\,\sigma_W}{\sqrt{\sigma_W^2(1+\theta^2)+\sigma_U^2}} & \text{for } h=-1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\sqrt{1+\theta^2+\left(\dfrac{\sigma_U}{\sigma_W}\right)^2}} & \text{For } h=0 \\ \dfrac{-\theta}{\sqrt{1+\theta^2+\left(\dfrac{\sigma_U}{\sigma_W}\right)^2}} & \text{for } h=-1 \\ 0 & \text{for } h\in\mathbb{Z}\setminus\{0,-1\} \end{cases}$$

4) (a) Prove X_t is weakly stationary

Given, $X_t = \sin(2\pi U t)$ for $t = 1, 2, \dots$

- Mean,

$$E[X_t] = E[\sin(2\pi U t)]$$

$$= \int_0^1 \sin(2\pi u t)\, du \qquad [\because U \sim U(0,1)]$$

$$= \frac{1}{2\pi t}\left[-\cos(2\pi u t)\right]_0^1$$

$$= \frac{1}{2\pi t}\left[-\cos(2\pi t) + \cos(0)\right]$$

$$= \frac{1 - \cos(2\pi t)}{2\pi t}$$

But, $\cos(2\pi t) = 1 \;\forall\, t \in \mathbb{Z}$

$\therefore E[X_t] = 0 \quad \forall\, t \in \mathbb{Z}$

which satisfies the first condition for weak stationarity.

- Compute Variance

$$Var(X_t) = E[X_t^2] - (E[X_t])^2$$

Since $E[X_t] = 0$, we have,

$$Var(X_t) = E[X_t^2] = E[\sin^2(2\pi U t)]$$

$$= \int_0^1 \sin^2(2\pi u t)\, du$$

$$= \int_0^1 \frac{1 - \cos(4\pi u t)}{2}\, du$$

$$= \frac{1}{2} - \frac{1}{2}\int_0^1 \cos(4\pi u t)\, du$$

M	T	W	T	F	S	S

Page No.:
Date:
YOUVA

$$= \frac{1}{2} - \frac{1}{8\pi t}\left[\sin(4\pi u t)\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{8\pi t}\left[\sin 4\pi t - 0\right]$$

$$= \frac{1}{2} \qquad \left[\because \sin 4\pi t = 0 \quad \forall t \in \mathbb{Z}\right]$$

$\Rightarrow \mathrm{Var}(X_t) = \frac{1}{2}$ is constant and does not depend on t. This satisfies second condition for weak stationarity.

- Compute Auto covariance Function $\gamma_X(h)$

$$\gamma_X(h) = E\left[(X_t - E[X_t])(X_{t+h} - E[X_{t+h}])\right]$$
$$= E\left[X_t X_{t+h}\right]$$
$$= E\left[\sin(2\pi U t)\sin(2\pi U(t+h))\right]$$
$$= \frac{E}{2}\left[\cos(2\pi U h) - \cos(2\pi U(2t+h))\right]$$

$$= \frac{1}{2}\left[\frac{1}{2\pi h}\left[\sin(2\pi u h)\right]_0^1 - \frac{\left[\sin(2\pi u(2t+h))\right]_0^1}{2\pi(2t+h)}\right]$$

$$\gamma_X(h) = \frac{\sin 2\pi h}{4\pi h} - \frac{\sin(2\pi(2t+h))}{4\pi(2t+h)}$$

$$\gamma_X(0) = \lim_{h\to 0}\frac{\sin 2\pi h}{4\pi h} - \frac{\sin 4\pi t}{8\pi t}$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{\sin(2\pi h)}{(2\pi h)} - 0 \qquad \left[\because \sin 4\pi t = 0 \quad \forall t \in \mathbb{Z}\right]$$

$$= \frac{1}{2}$$

$\gamma_X(h) = 0$ for $h \neq 0$

$$\left[\because \sin 2\pi h = \sin(2\pi(2t+h)) = 0 \quad \forall t, h \in \mathbb{Z}\right]$$

So, the ACF does not depend on t. This satisfies 3rd condition for weakly stationary series.

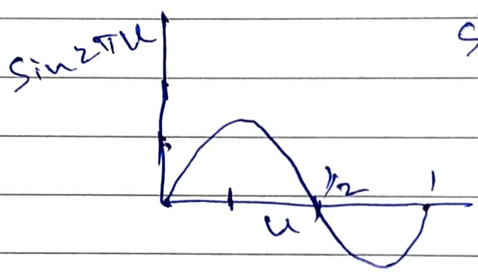
(b) A time series is strictly stationary if the joint distⁿ of any set of observations $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ depends only on the time differences, and not on the specific times $t_1, t_2, \ldots, t_n$.

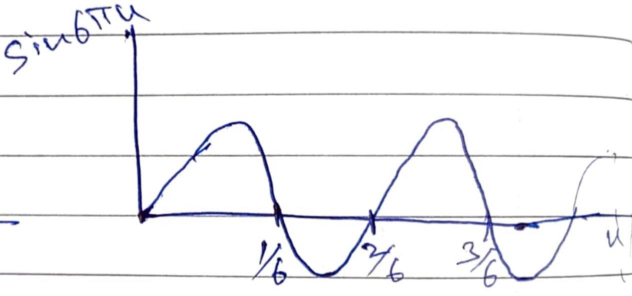
Consider the Example:

$P(X_1 \le 0, X_3 \le 0) = P(\sin(2\pi U) \le 0, \sin(6\pi U) \le 0)$

$= P(U \in [1/2, 1] \cap ([1/6, 1/3] \cup [1/2, 2/3] \cup [5/6, 1]))$

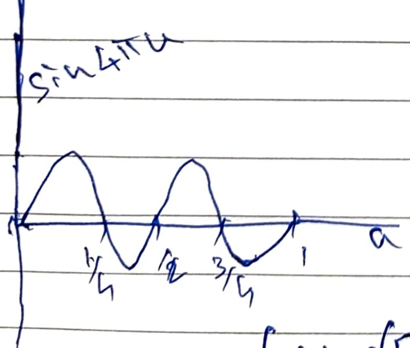
$= 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

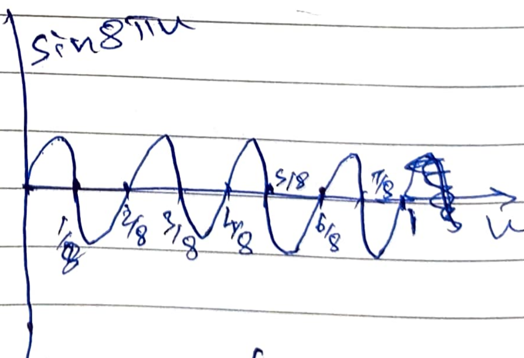




But, $P(X_2 \le 0, X_4 \le 0)$

$= P(\sin 4\pi U \le 0, \sin 8\pi U \le 0)$





$= P(U \in ([1/4, 1/2] \cup [3/4, 1]) \cap ([1/8, 2/8], [3/8, 4/8], [5/8, 6/8], [7/8, 1]))$

$= \frac{1}{4}$

$\therefore$ We can see that,
$P(X_1 \leq 0, X_3 \leq 0) \neq P(X_2 \leq 0, X_4 \leq 0)$

$\therefore$ X_t is not strictly stationary.

Date:

5) (a) Consider the Auto covariance matrix,

$$\Gamma = \{ \gamma(s-t);\ s, t = 1, \dots, n \}$$

Now, to prove ACF is non-negative Definite we need to prove.

$$\underset{\sim}{a}^T \Gamma \underset{\sim}{a} \geq 0 \quad \forall \underset{\sim}{a} \in \mathbb{R}^n.$$

For any $\underset{\sim}{a} \in \mathbb{R}^n$,

$$\underset{\sim}{a}^T \Gamma \underset{\sim}{a} = \sum_{s=1}^{n} \sum_{t=1}^{n} a_s \gamma(s-t) a_t$$

$$= \sum_{s=1}^{n} \sum_{t=1}^{n} a_s \, Cov(X_s, X_t) \, a_t$$

$$= Cov\left\{ \sum_s a_s X_s , \sum_t a_t X_t \right\}$$

$$= Var\left\{ \sum_{s=1}^{n} a_s X_s \right\}$$

$$\geq 0$$

[$\because$ Variance is always non-negative]

$$\therefore \underset{\sim}{a}^T \Gamma \underset{\sim}{a} \geq 0 \quad \forall \underset{\sim}{a} \in \mathbb{R}^n$$

(b) Consider the sample Auto covariance,

$$\hat{\gamma}(h) = \frac{1}{N} \sum_{t=1}^{N-h} (X_t - \bar{X})(X_{t+h} - \bar{X})$$

where $\bar{X} = \frac{1}{N} \sum_{t=1}^{N} X_t$

To prove sample ACF is non-negative definite, we need to prove,

$$\underset{\sim}{a}^T \hat{\Gamma}_n \underset{\sim}{a} \geq 0 \quad \forall \underset{\sim}{a} \in \mathbb{R}^n$$

where $\hat{\Gamma}_n = \{ \hat{\gamma}(s-t) \}_{s,t=1}^{n}$

Let $Y_t = X_t - \bar{X}$

$\therefore \hat{\gamma}(h) = \frac{1}{N} \sum_{t=1}^{N-h} Y_t Y_{t+h}$ for $h = 0, 1, \dots N-1$

Construct the $n \times 2n$ matrix

$$D = \begin{bmatrix} 0 & 0 & \cdots & 0 & Y_1 & Y_2 & \cdots & Y_N \\ 0 & \cdots & 0 & Y_1 & Y_2 & \cdots & Y_N & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & Y_1 & Y_2 & \cdots & Y_N & 0 & \cdots & 0 \end{bmatrix}$$

The matrix D is structured such that DD' captures the autocovariances ~~lags~~ at various lags for the mean-centered time series $Y_t = X_t - \bar{X}$ normalizing by n gives us the sample ACF matrix $\hat{\Gamma}_n$, which consists of the sample autocovariance $\hat{\gamma}(s-t)$ for each pair s, t.

$$\therefore \hat{\Gamma}_N = \frac{1}{N} DD'$$

Then, for any $\underset{\sim}{a} \in \mathbb{R}^n$,

$$\underset{\sim}{a}^T \Gamma_n \underset{\sim}{a} = \frac{1}{N} \underset{\sim}{a}' D D' \underset{\sim}{a} = \frac{1}{N} \underset{\sim}{c}' \underset{\sim}{c}$$

$$= \sum_{i=1}^{N} c_i^2 \geq 0$$

6) Given,

$$I(\underset{\sim}{\theta}_1;\underset{\sim}{\theta}_2) = T^{-1} E_1 \log \frac{f(\underset{\sim}{y};\underset{\sim}{\theta}_1)}{f(\underset{\sim}{y};\underset{\sim}{\theta}_2)}$$

$$\underset{\sim}{Y} \sim N(\underset{\sim}{Z}\underset{\sim}{\beta}, \sigma^2)$$

$\beta \in \mathbb{R}^{p\times 1}$

or $\underset{\sim}{Y} = Z\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ $\qquad Z \in \mathbb{R}^{T\times p}$

where $\underset{\sim}{\varepsilon} \sim N(0, \sigma^2)$

$$\therefore f_Y(\underset{\sim}{y};\underset{\sim}{\theta}_1) = \frac{1}{(2\pi\sigma_1^2)^{T/2}} \exp\left[-\frac{(Y-Z\beta_1)^T(Y-Z\beta_1)}{2\sigma_1^2}\right]$$

$$\log f_Y(y;\underset{\sim}{\theta}_1) = -\frac{T}{2}\log(2\pi\sigma_1^2) - \frac{(Y-Z\beta_1)^T(Y-Z\underset{\sim}{\beta}_1)}{2\sigma_1^2}$$

$$\log \frac{f_Y(y;\underset{\sim}{\theta}_1)}{f_Y(y;\underset{\sim}{\theta}_2)} = -\frac{T}{2}\log\left(\frac{\sigma_1^2}{\sigma_2^2}\right) - \frac{(Y-Z\beta_1)^T(Y-Z\beta_1)}{2\sigma_1^2} + \frac{(Y-Z\beta_2)^T(Y-Z\beta_2)}{2\sigma_2^2}$$

$$E_1\left[\log \frac{f(y;\theta_1)}{f(y;\theta_2)}\right] = -\frac{T}{2}\log\frac{\sigma_1^2}{\sigma_2^2} - \frac{1}{2\sigma_1^2} E_1\left[(Y-Z\beta_1)'(Y-Z\underset{\sim}{\beta}_1)\right] + \frac{1}{2\sigma_2^2} E_1\left[(Y-Z\beta_2)'(Y-Z\beta_2)\right]$$

$$E_1\left[(Y-Z\beta_1)'(Y-Z\beta_1)\right] = E_1\left[\varepsilon_1'\underset{\sim}{\varepsilon}_1\right] = T\sigma_1^2$$

$$E_1\left[(Y-Z\beta_2)'(Y-Z\underset{\sim}{\beta}_2)\right] = E_1\left[(Z\beta_1+\varepsilon_1-Z\beta_2)'(Z\beta_1+\underset{\sim}{\varepsilon}_1-Z\beta_2)\right]$$

$$= E\left[\left(Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})+\underset{\sim}{\varepsilon_1}\right)'\left(Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})+\underset{\sim}{\varepsilon_1}\right)\right]$$

$$= E\left[\underset{\sim}{\varepsilon_1}'\underset{\sim}{\varepsilon_1}\right] + \left\{Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})\right\}'\left\{Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})\right\}$$

$$= T\sigma_1^2 + (\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})'Z'Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})$$

$$\therefore I(\underset{\sim}{\theta_1};\underset{\sim}{\theta_2}) = \frac{1}{T}\left[-\frac{T}{2}\log\frac{\sigma_1^2}{\sigma_2^2} - \frac{1}{2\sigma_1^2}(T\sigma_1^2) + \frac{1}{2\sigma_2^2}\left(T\sigma_1^2 + (\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})'Z'Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})\right)\right]$$

$$= -\frac{1}{2}\log\frac{\sigma_1^2}{\sigma_2^2} - \frac{1}{2} + \frac{\sigma_1^2}{2\sigma_2^2} + \frac{(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})'Z'Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})}{2T\sigma_2^2}$$

$$= \frac{1}{2}\left(\frac{\sigma_1^2}{\sigma_2^2} - \log\left(\frac{\sigma_1^2}{\sigma_2^2}\right) - 1\right) + \frac{1}{2}\frac{(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})'Z'Z(\underset{\sim}{\beta_1}-\underset{\sim}{\beta_2})}{T\sigma_2^2}$$