MTH442 Assigment 4

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1) Given IMA (1, 1) model

$$X_t = X_{t-1} + W_t - \lambda W_{t-1}$$

Consider $Y_t = X_t - X_{t-1}$

$\therefore$ model becomes,

$$Y_t = W_t - \lambda W_{t-1}$$

The MA polynomial here is,

$$\theta(z) = 1 - \lambda z$$

$z_1 = \frac{1}{\lambda}$ is the root of the above polynomial

Given $|\lambda| < 1 \Rightarrow |z_1| = \frac{1}{|\lambda|} > 1$

$\therefore$ The above model is invertible.

Now,

$$Y_t = (1 - \lambda B) W_t$$

where B is the Backward shift operator.

$$W_t = (1-\lambda B)^{-1} Y_t$$

We expand $(1-\lambda B)^{-1}$ as a geometric series:

$$W_t = (1 + \lambda B + (\lambda B)^2 + \dots) Y_t$$

$$W_t = \sum_{j=0}^{\infty} \lambda^j B^j Y_t$$

$$W_t = \sum_{j=0}^{\infty} \lambda^j Y_{t-j}$$

$$\Rightarrow W_t = \sum_{j=0}^{\infty} \lambda^j (X_{t-j} - X_{t-1-j})$$

Rearranging,

$$W_t = X_t + (\lambda - 1) X_{t-1} + (\lambda^2 - \lambda) X_{t-2} + (\lambda^3 - \lambda^2) X_{t-3} + \dots$$

$$= X_t - \sum_{j=}^{\infty} (1-\lambda) \lambda^{j-1} X_{t-j}$$

$$\Rightarrow X_t = \sum_{j=1}^{\infty} (1-\lambda) \lambda^{j-1} X_{t-j} + W_t$$

$$Y_{T+2}^{T} = \delta + \phi\delta + \phi^2 Y_T$$

$$= \delta(1+\phi)$$

$$Y_{T+3}^{T} = \delta + \phi / \delta(1+\phi)$$

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2) (a) Given ARIMA (1,1,0) model:

$$(1-\phi B)(1-B) X_t = \delta + W_t$$

$$\Rightarrow \quad X_t - X_{t-1} - \phi(X_{t-1} - X_{t-2}) = \delta + W_t$$

Also, $Y_t = \nabla X_t = X_t - X_{t-1}$

$\therefore$ We have,

$$Y_t - \phi Y_{t-1} = \delta + W_t$$

$$\Rightarrow \quad Y_t = \phi Y_{t-1} + \delta + W_t$$

$\therefore$ This is an AR(1) model.

Formulate the predictor, (linear)

$$Y^T_{T+1} = a + b Y_T$$ [coeffs for lags ≥ 1 are zero assuming AR(1) model]

where a and b are constants.

Minimize the mean square error, and get the BLP,

$$E[(Y_{T+1} - Y^T_{T+1})^2]$$

Using the AR(1) model, we have,

$$Y_{T+1} = \delta + \phi Y_T + W_{T+1}$$

$$\therefore Y_{T+1} - Y^T_{T+1} = (\delta + \phi Y_T + W_{T+1}) - (a + b Y_T)$$

$$= (\delta - a) + (\phi - b) Y_T + W_{T+1}$$

To minimize the MSE, we want Y^T_{T+1} to be an unbiased predictor,

$$E[Y_{T+1} - Y^T_{T+1}] = 0$$

$$\Rightarrow \quad (\delta - a) + (\phi - b) E[Y_T] = 0$$

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$E[Y_T]$ will be a polynomial in δ since Y_T has a drift.

$\therefore$ on comparing the coefficients on the LHS with RHS we get.

$a = \delta$ and $b = \phi$

Thus, our linear predictor becomes,

$$Y^T_{T+1} = \delta + \phi Y_T$$

We can also say

$$Y^T_{T+2} = \delta + \phi Y^T_{T+1} = \delta + \phi(\delta + \phi Y_T)$$

$$Y^T_{T+3} = \delta + \phi Y^T_{T+2}, \ldots,$$

$$\Rightarrow Y^T_{T+j} = \delta + \phi Y^T_{T+j-1}$$

$$= \cancel{\delta + \phi(}$$

$$Y^T_{T+j} = \delta(1 + \phi + \ldots + \phi^{j-1}) + \phi^j Y_T$$

$$\Rightarrow Y^T_{T+j} = \delta \sum_{k=0}^{j-1} \phi^k + \phi^j Y_T$$

$$\Rightarrow Y^T_{T+j} = \delta \left(\frac{1-\phi^j}{1-\phi}\right) + \phi^j Y_T$$

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(b) We now have,

$$Y_{T+j}^{T} = \frac{\delta(1-\phi^{j})}{(1-\phi)} + \phi^{j} Y_T$$

(i) $$\sum_{j=1}^{M} Y_{T+j}^{T} = \sum_{j=1}^{M} (X_{T+j}^{T} - X_{T+j-1}^{T})$$

$$= (X_{T+1}^{T} - X_{T}^{T}) + (X_{T+2}^{T} - X_{T+1}^{T}) + \dots + (X_{T+M}^{T} - X_{T+M-1}^{T})$$

This forms a telescopic series and only 2 terms remain.

$$\therefore \sum_{T=1}^{M} Y_{T+j} = X_{T+M}^{T} - X_{T}^{T}$$

$$= X_{T+M}^{T} - X_T$$

(ii) $$\sum_{j=1}^{M} \frac{\delta(1-\phi^{j})}{(1-\phi)} = \frac{\delta}{(1-\phi)} \left[M - \sum_{j=1}^{M} \phi^{j} \right]$$

$$= \frac{\delta}{1-\phi} \left[M - \frac{\phi(1-\phi^{M})}{(1-\phi)} \right]$$

(iii) $$\sum_{j=1}^{M} \phi^{j} Y_T = Y_T \sum_{j=1}^{M} \phi^{j}$$

$$= Y_T \frac{\phi(1-\phi^{M})}{(1-\phi)}$$

Now,

$$Y_{T+j}^{T} = \frac{\delta(1-\phi^{j})}{(1-\phi)} + \phi^{j} Y_T$$

$$\Rightarrow \sum_{j=1}^{M} Y_{T+j}^{T} = \frac{\delta}{(1-\phi)} \sum_{j=1}^{M} + \sum_{j=1}^{M} \phi^{j} Y_T$$

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From (i), (ii) and (iii), we have,

$$X_{T+M}^{T} - X_T = \frac{\delta}{(1-\phi)}\left[M - \frac{\phi(1-\phi^M)}{(1-\phi)}\right] + Y_T\,\frac{\phi(1-\phi^M)}{(1-\phi)}$$

$$\Rightarrow X_{T+M}^{T} = X_T + \frac{\delta}{(1-\phi)}\left[M - \frac{\phi(1-\phi^M)}{(1-\phi)}\right] + (X_T - X_{T-1})\frac{\phi(1-\phi^M)}{(1-\phi)}$$

(c) To get the ψ^* coefficients, we need to solve,

~~$\psi^{2*}(z)(1-\phi z)$~~

$$\psi^*(z)\,\phi(z) = \theta(z)$$

where

$$\psi^*(z) = \psi_0^* + \psi_1^* z + \psi_2^* z^2 + \cdots$$

$$\phi(z) = (1-\phi z)(1-z)$$

$$\theta(z) = 1$$

$\therefore$ Equation becomes,

$$(\psi_0^* + \psi_1^* z + \psi_2^* z^2 + \cdots)(1-(1+\phi)z + \phi z^2) = 1$$

Comparing the coeffs of z^0, we get,

$$\psi_0^* = 1$$

Comparing the coefficients of z^1,

$$\psi_1^* - \psi_0^*(1+\phi) = 0$$

$$\Rightarrow \psi_1^* = (1+\phi)$$

Comparing the coeffs of z^2,

$$\psi_2^* - \psi_1^*(1+\phi) + \phi\,\psi_0^* = 0$$

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Comparing the coeffs of z^j, we have

$$\psi_j^* - \psi_{j-1}^*(1+\phi) + \phi\,\psi_{j-2}^* = 0 \qquad j \geq 2$$

Claim: $\psi_j^* = (1-\phi)^{-1}(1-\phi^{j+1}) \qquad j \geq 1$

Proof:

Use induction,

(i) For $j = 1$,

$$\psi_1^* = (1-\phi)^{-1}(1-\phi^2)$$

$$\psi_1^* = (1+\phi)$$

which is true

(ii) Assume $\psi_j^* = \dfrac{(1-\phi^{j+1})}{(1-\phi)}$

is true for $j = k$ and $k-1$, $k \geq 2$

(iii) Show $\psi_{k+1}^* = \dfrac{(1-\phi^{k+2})}{(1-\phi)}$ based on the above assumption.

We know, $\psi_{k+1}^* - \psi_k^*(1+\phi) + \phi\,\psi_{k-1}^* = 0$

$$\Rightarrow \psi_{k+1}^* = \frac{(1-\phi^{k+1})(1+\phi)}{(1-\phi)} - \frac{\phi(1-\phi^k)}{(1-\phi)}$$

$$= \frac{1}{1-\phi}\left[1-\phi^{k+1}+\phi-\phi^{k+2}-\phi+\phi^{k+1}\right]$$

$$\Rightarrow \psi_{k+1}^* = \frac{(1-\phi^{k+2})}{(1-\phi)}$$

Hence, Proved.

Thus, $\psi_j^* = \dfrac{(1-\phi^{j+1})}{(1-\phi)}$