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MTH442 Assigment 4

Name - Kaushik Raj V Nadar

Roll no - 208160499

Email - nkaushik20@iitk.ac.in

Date - 30/10/2024

1) Given IMA (1, 1) model

$$X_t = X_{t-1} + W_t - \lambda W_{t-1}$$

Consider $Y_t = X_t - X_{t-1}$

$\therefore$ model becomes,

$$Y_t = W_t - \lambda W_{t-1}$$

The MA polynomial here is,

$$\theta(z) = 1 - \lambda z$$

$z_1 = \frac{1}{\lambda}$ is the root of the above polynomial

Given $|\lambda| < 1 \Rightarrow |z_1| = \frac{1}{|\lambda|} > 1$

$\therefore$ The above model is invertible.

Now,

$$Y_t = (1 - \lambda B) W_t$$

where B is the Backward shift operator.

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$$W_t = (1-\lambda B)^{-1} Y_t$$

We expand $(1-\lambda B)^{-1}$ as a geometric series:

$$W_t = (1 + \lambda B + (\lambda B)^2 + \dots) Y_t$$

$$W_t = \sum_{j=0}^{\infty} \lambda^j B^j Y_t$$

$$W_t = \sum_{j=0}^{\infty} \lambda^j Y_{t-j}$$

$$\Rightarrow W_t = \sum_{j=0}^{\infty} \lambda^j (X_{t-j} - X_{t-1-j})$$

Rearranging,

$$W_t = X_t + (\lambda - 1) X_{t-1} + (\lambda^2 - \lambda) X_{t-2} + (\lambda^3 - \lambda^2) X_{t-3} + \dots$$

$$= X_t - \sum_{j=1}^{\infty} (1-\lambda) \lambda^{j-1} X_{t-j}$$

$$\Rightarrow X_t = \sum_{j=1}^{\infty} (1-\lambda) \lambda^{j-1} X_{t-j} + W_t$$

$$Y_{T+2}^{T} = \delta + \phi \delta + \phi^2 Y_T$$

$$= \delta(1+\phi)$$

$$Y_{T+3}^{T} = \delta + \phi / \delta(1+\phi)$$

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2) (a) Given ARIMA (1,1,0) model:

$$(1-\phi B)(1-B) X_t = \delta + W_t$$

$$\Rightarrow \quad X_t - X_{t-1} - \phi(X_{t-1} - X_{t-2}) = \delta + W_t$$

Also, $Y_t = \nabla X_t = X_t - X_{t-1}$

$\therefore$ We have,

$$Y_t - \phi Y_{t-1} = \delta + W_t$$

$$\Rightarrow \quad Y_t = \phi Y_{t-1} + \delta + W_t$$

$\therefore$ This is an AR(1) model.

Formulate the predictor, (linear)

$$Y^T_{T+1} = a + b Y_T$$ [coeffs for lags ≥ 1 are zero assuming AR(1) model]

where a and b are constants.

Minimize the mean square error and get the BLP,

$$E[(Y_{T+1} - Y^T_{T+1})^2]$$

Using the AR(1) model, we have,

$$Y_{T+1} = \delta + \phi Y_T + W_{T+1}$$

$$\therefore Y_{T+1} - Y^T_{T+1} = (\delta + \phi Y_T + W_{T+1}) - (a + b Y_T)$$

$$= (\delta - a) + (\phi - b) Y_T + W_{T+1}$$

To minimize the MSE, we want Y^T_{T+1} to be an unbiased predictor,

$$E[Y_{T+1} - Y^T_{T+1}] = 0$$

$$\Rightarrow \quad (\delta - a) + (\phi - b) E[Y_T] = 0$$

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$E[Y_T]$ will be a polynomial in δ since Y_T has a drift.

$\therefore$ on comparing the coefficients on the LHS with RHS we get.

$$a = \delta \quad \text{and} \quad b = \phi$$

Thus, our linear predictor becomes,

$$Y_{T+1}^{T} = \delta + \phi Y_T$$

We can also say

$$Y_{T+2}^{T} = \delta + \phi\, Y_{T+1}^{T} = \delta + \phi\,(\delta + \phi Y_T)$$

$$Y_{T+3}^{T} = \delta + \phi\, Y_{T+2}^{T} \quad , \ldots ,$$

$$\Rightarrow Y_{T+j}^{T} = \delta + \phi\, Y_{T+j-1}^{T}$$

$$= \cancel{\delta + \phi (}$$

$$Y_{T+j}^{T} = \delta(1 + \phi + \cdots + \phi^{j-1}) + \phi^{j} Y_T$$

$$\Rightarrow Y_{T+j}^{T} = \delta \sum_{k=0}^{j-1} \phi^{k} + \phi^{j} Y_T$$

$$\Rightarrow Y_{T+j}^{T} = \delta \left(\frac{1 - \phi^{j}}{1 - \phi} \right) + \phi^{j} Y_T$$

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(b) We now have,

$$Y_{T+j}^{T} = \frac{\delta(1-\phi^{j})}{(1-\phi)} + \phi^{j} Y_T$$

(i) $$\sum_{j=1}^{M} Y_{T+j}^{T} = \sum_{j=1}^{M} (X_{T+j}^{T} - X_{T+j-1}^{T})$$

$$= (X_{T+1}^{T} - X_{T}^{T}) + (X_{T+2}^{T} - X_{T+1}^{T}) + \dots + (X_{T+M}^{T} - X_{T+M-1}^{T})$$

This forms a telescopic series and only 2 terms remain.

$$\therefore \sum_{T=1}^{M} Y_{T+j} = X_{T+M}^{T} - X_{T}^{T}$$

$$= X_{T+M}^{T} - X_T$$

(ii) $$\sum_{j=1}^{M} \frac{\delta(1-\phi^{j})}{(1-\phi)} = \frac{\delta}{(1-\phi)}\left[M - \sum_{j=1}^{M} \phi^{j}\right]$$

$$= \frac{\delta}{1-\phi}\left[M - \frac{\phi(1-\phi^{M})}{(1-\phi)}\right]$$

(iii) $$\sum_{j=1}^{M} \phi^{j} Y_T = Y_T \sum_{j=1}^{M} \phi^{j}$$

$$= Y_T \frac{\phi(1-\phi^{M})}{(1-\phi)}$$

Now,

$$Y_{T+j}^{T} = \frac{\delta(1-\phi^{j})}{(1-\phi)} + \phi^{j} Y_T$$

$$\Rightarrow \sum_{j=1}^{M} Y_{T+j}^{T} = \frac{\delta}{(1-\phi)} \sum_{j=1}^{M} + \sum_{j=1}^{M} \phi^{j} Y_T$$

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From (i), (ii) and (iii), we have,

$$X_{T+M}^{T} - X_T = \frac{\delta}{(1-\phi)}\left[M - \frac{\phi(1-\phi^M)}{(1-\phi)}\right] + Y_T \frac{\phi(1-\phi^M)}{(1-\phi)}$$

$$\Rightarrow X_{T+M}^{T} = X_T + \frac{\delta}{(1-\phi)}\left[M - \frac{\phi(1-\phi^M)}{(1-\phi)}\right] + (X_T - X_{T-1})\frac{\phi(1-\phi^M)}{(1-\phi)}$$

(c) To get the ψ^* coefficients, we need to solve,

$\cancel{\psi^{2*}(z)(1-\phi z)}$

$$\psi^*(z)\,\phi(z) = \theta(z)$$

where

$$\psi^*(z) = \psi_0^* + \psi_1^* z + \psi_2^* z^2 + \cdots$$

$$\phi(z) = (1-\phi z)(1-z)$$

$$\theta(z) = 1$$

$\therefore$ Equation becomes,

$$(\psi_0^* + \psi_1^* z + \psi_2^* z^2 + \cdots)(1-(1+\phi)z + \phi z^2) = 1$$

Comparing the coeffs of z^0, we get,

$$\psi_0^* = 1$$

Comparing the coefficients of z^1,

$$\psi_1^* - \psi_0^*(1+\phi) = 0$$

$$\Rightarrow \psi_1^* = (1+\phi)$$

Comparing the coeffs of z^2,

$$\psi_2^* - \psi_1^*(1+\phi) + \phi\,\psi_0^* = 0$$

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Comparing the coeffs of z^j, we have

$$\psi_j^* - \psi_{j-1}^*(1+\phi) + \phi\,\psi_{j-2}^* = 0 \qquad j \geq 2$$

Claim: $\psi_j^* = (1-\phi)^{-1}(1-\phi^{j+1}) \qquad j \geq 1$

Proof:

Use induction,

(i) For $j = 1$,

$$\psi_1^* = (1-\phi)^{-1}(1-\phi^2)$$

$$\psi_1^* = (1+\phi)$$

which is true

(ii) Assume $\psi_j^* = \dfrac{(1-\phi^{j+1})}{(1-\phi)}$

is true for $j = k$ and $k-1$, $k \geq 2$

(iii) Show $\psi_{k+1}^* = \dfrac{(1-\phi^{k+2})}{(1-\phi)}$ based on the above assumption.

We know, $\psi_{k+1}^* - \psi_k^*(1+\phi) + \phi\,\psi_{k-1}^* = 0$

$$\Rightarrow \psi_{k+1}^* = \frac{(1-\phi^{k+1})(1+\phi)}{(1-\phi)} - \frac{\phi(1-\phi^k)}{(1-\phi)}$$

$$= \frac{1}{1-\phi}\left[1-\phi^{k+1}+\phi-\phi^{k+2}-\phi+\phi^{k+1}\right]$$

$$\Rightarrow \psi_{k+1}^* = \frac{(1-\phi^{k+2})}{(1-\phi)}$$

Hence, Proved.

Thus, $\psi_j^* = \dfrac{(1-\phi^{j+1})}{(1-\phi)}$

MTH442 Assignment 4

Kaushik Raj V Nadar (208160499)

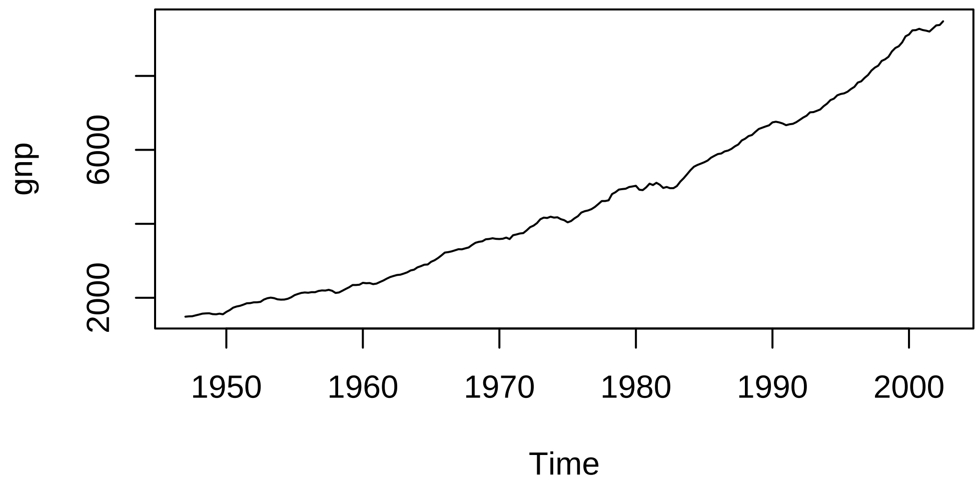
2024-10-31

Problem 3

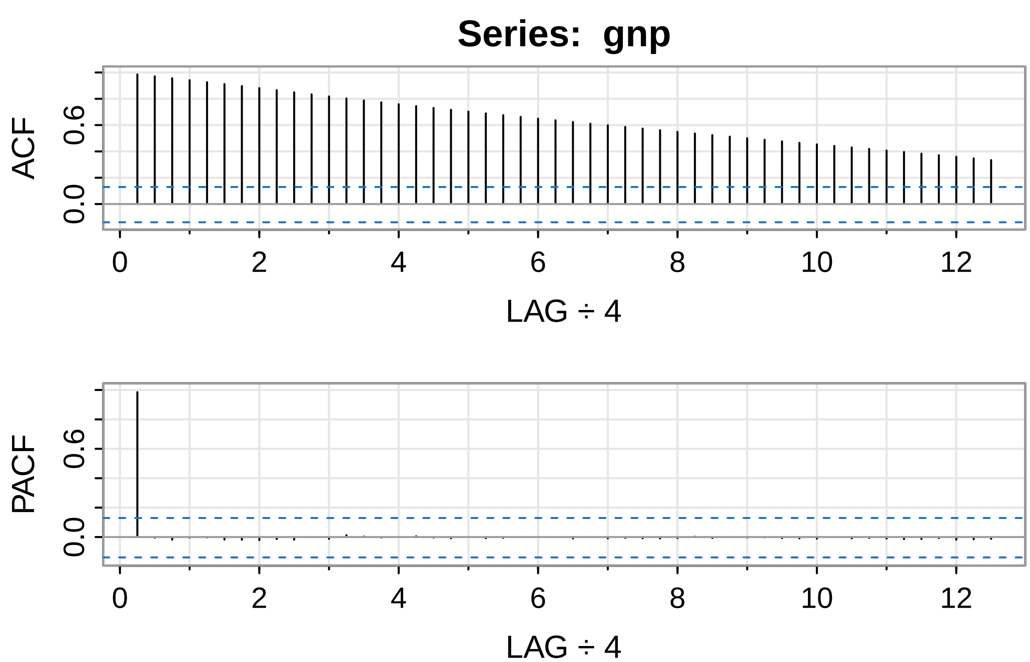
We discussed quarterly United States Gross National Product (GNP) data analysis in the lab. We said that AR(1) and MA(2) are two possible models for the differenced log GNP data. We discussed the model diagnostics for MA(2) model but also pointed out that AR(1) is preferable. Show the model diagnostics for AR(1) model. Repeat the diagnostics for ARMA(1, 2) model and compare the results.

Plot the data

```r
library(astsa)
plot(gnp)
```




```
acf2(gnp, 50)
```

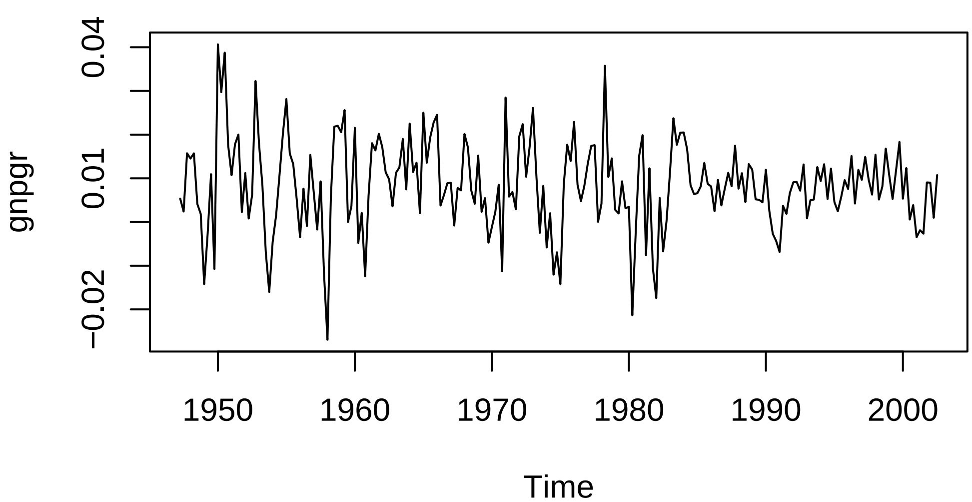



```
[,1] [,2]  [,3] [,4] [,5]  [,6]  [,7]  [,8]  [,9] [,10] [,11] [,12] [,13]
```

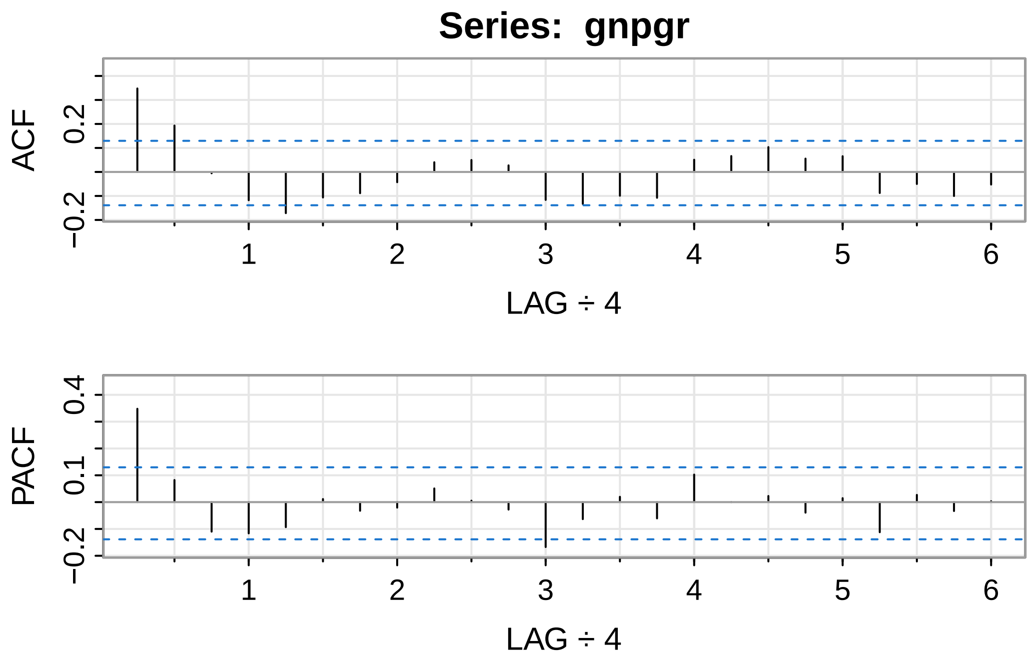
```
ACF  0.99 0.97  0.96 0.94 0.93  0.91  0.90  0.88  0.87  0.85  0.83  0.82  0.80
PACF 0.99 0.00 -0.02 0.00 0.00 -0.02 -0.02 -0.02 -0.01 -0.02  0.00 -0.01  0.01
     [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
ACF   0.79  0.77  0.76  0.74  0.73  0.72   0.7  0.69  0.68  0.66  0.65  0.64
PACF  0.00  0.00  0.00  0.01  0.00 -0.01   0.0 -0.01 -0.01  0.00  0.00  0.00
     [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
ACF   0.62  0.61  0.60  0.59  0.57  0.56  0.55  0.54  0.52  0.51   0.5  0.49
PACF -0.01  0.00 -0.01 -0.01 -0.01 -0.01 -0.01  0.00 -0.01  0.00   0.0  0.00
     [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48] [,49]
ACF   0.48  0.47  0.45  0.44  0.43  0.42  0.41  0.40  0.38  0.37  0.36  0.35
PACF -0.01 -0.01 -0.01  0.00 -0.01 -0.01 -0.01 -0.01 -0.01 -0.01 -0.02 -0.02
     [,50]
ACF   0.33
PACF -0.01
```

Plot Differenced Log GNP

```
gnpgr = diff(log(gnp)) # growth rate
plot(gnpgr)
```



```r
acf2(gnpgr, 24)
```



```
     [,1] [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8] [,9] [,10] [,11] [,12] [,13]
ACF  0.35 0.19 -0.01 -0.12 -0.17 -0.11 -0.09 -0.04 0.04  0.05  0.03 -0.12 -0.13
PACF 0.35 0.08 -0.11 -0.12 -0.09  0.01 -0.03 -0.02 0.05  0.01 -0.03 -0.17 -0.06
     [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24]
ACF  -0.10 -0.11  0.05  0.07  0.10  0.06  0.07 -0.09 -0.05 -0.10 -0.05
PACF  0.02 -0.06  0.10  0.00  0.02 -0.04  0.01 -0.11  0.03 -0.03  0.00
```

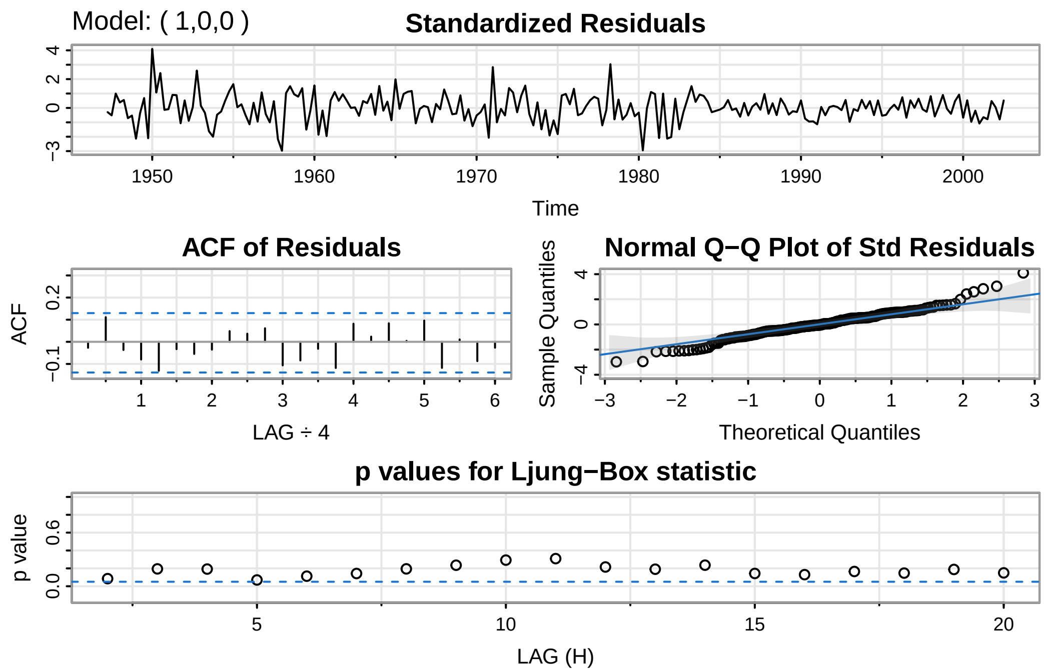
Model Diagnostics for AR(1)

```r
# Fit AR(1) model
cat("\nAR(1) Model Diagnostics:\n")
```

```
AR(1) Model Diagnostics:
```

```r
ar1_fit = sarima(gnpgr, 1, 0, 0)
```

```
initial  value -4.589567
iter   2 value -4.654150
iter   3 value -4.654150
iter   4 value -4.654151
iter   4 value -4.654151
iter   4 value -4.654151
final  value -4.654151
converged
initial  value -4.655919
iter   2 value -4.655921
iter   3 value -4.655922
iter   4 value -4.655922
iter   5 value -4.655922
iter   5 value -4.655922
iter   5 value -4.655922
final  value -4.655922
converged
<><><><><><><><><><><><><><>

Coefficients:
      Estimate     SE t.value p.value
ar1     0.3467 0.0627  5.5255       0
xmean   0.0083 0.0010  8.5398       0

sigma^2 estimated as 9.029569e-05 on 220 degrees of freedom

AIC = -6.44694  AICc = -6.446693  BIC = -6.400958
```




```r
# Display AR(1) coefficients and standard errors
cat("\nAR(1) Model Summary:\n")
```

```
AR(1) Model Summary:
```

```r
ar1_fit$ttable
```

```
      Estimate     SE t.value p.value
ar1     0.3467 0.0627  5.5255       0
xmean   0.0083 0.0010  8.5398       0
```

The above figure displays a plot of the standardized residuals, the ACF of the residuals, a boxplot of the standardized residuals, and the p-values associated with the Q-statistic at lags H= 2 through H= 20.

Like the MA(2) model, inspecting the time plot of standardized residuals in Figure 3.16 reveals no clear patterns. However, some outliers are apparent, with a few values exceeding 3 standard deviations in magnitude. The autocorrelation function (ACF) of these residuals does not indicate any deviation from the model assumptions, and the Q-statistics at the shown lags are not statistically significant. Additionally, the normal Q-Q plot of residuals suggests the assumption of normality is largely valid, aside from the presence of potential outliers.

Model Diagnostics for ARMA(1,2)

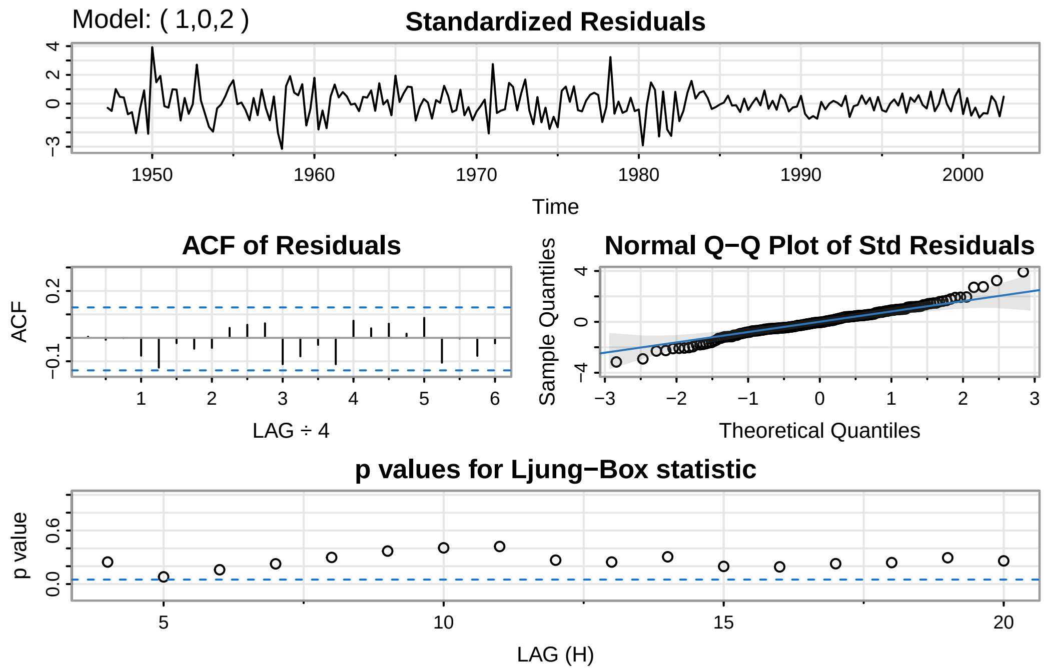
```
# Fit ARMA(1,2) model
cat("\nARMA(1,2) Model Diagnostics:\n")
```

```
ARMA(1,2) Model Diagnostics:
```

```
arma12_fit = sarima(gnpgr, 1, 0, 2)
```

```
initial  value -4.589567
iter   2 value -4.593469
iter   3 value -4.661378
iter   4 value -4.662245
iter   5 value -4.662354
iter   6 value -4.662395
iter   7 value -4.662567
iter   8 value -4.662643
iter   9 value -4.662676
iter  10 value -4.662678
iter  10 value -4.662678
final  value -4.662678
converged
initial  value -4.664308
iter   2 value -4.664311
iter   3 value -4.664312
iter   4 value -4.664314
iter   5 value -4.664315
iter   6 value -4.664316
iter   7 value -4.664316
iter   8 value -4.664317
iter   9 value -4.664317
iter   9 value -4.664317
iter   9 value -4.664317
final  value -4.664317
converged
<><><><><><><><><><><><><><>

Coefficients:
      Estimate     SE t.value p.value
ar1     0.2407 0.2066  1.1651  0.2453
```

```
ma1      0.0761 0.2026  0.3754  0.7077
ma2      0.1623 0.0851  1.9084  0.0577
xmean    0.0083 0.0010  8.0774  0.0000

sigma^2 estimated as 8.877466e-05 on 218 degrees of freedom

AIC = -6.445712  AICc = -6.444882  BIC = -6.369075
```




```r
# Display ARMA(1,2) coefficients and standard errors
cat("\nARMA(1,2) Model Summary:\n")
```

```
ARMA(1,2) Model Summary:
```

```r
arma12_fit$ttable
```

```
      Estimate     SE t.value p.value
ar1     0.2407 0.2066  1.1651  0.2453
ma1     0.0761 0.2026  0.3754  0.7077
ma2     0.1623 0.0851  1.9084  0.0577
xmean   0.0083 0.0010  8.0774  0.0000
```

The results are very similar to the AR(1) and MA(2). Q-Q plot here is a bit better than the previous cases with most of the residuals following the normality assumption.

Comparing AR(1) and ARMA(1,2) using Information Criteria

```
# Calculate AIC and BIC for AR(1)
ar1_aic = ar1_fit$ICs[1]
ar1_aicc = ar1_fit$ICs[2]
ar1_bic = ar1_fit$ICs[3]

# Calculate AIC and BIC for ARMA(1,2)
arma12_aic = arma12_fit$ICs[1]
arma12_aicc = arma12_fit$ICs[2]
arma12_bic = arma12_fit$ICs[3]

# Compare AIC and BIC
cat("AR(1) AIC:", ar1_aic,  "AICc:", ar1_aicc, "BIC:", ar1_bic, "\n")
```

```
AR(1) AIC: -6.44694 AICc: -6.446693 BIC: -6.400958
```

```
cat("ARMA(1,2) AIC:", arma12_aic, "AICc:", arma12_aicc, "BIC:", arma12_bic, "\n")
```

```
ARMA(1,2) AIC: -6.445712 AICc: -6.444882 BIC: -6.369075
```

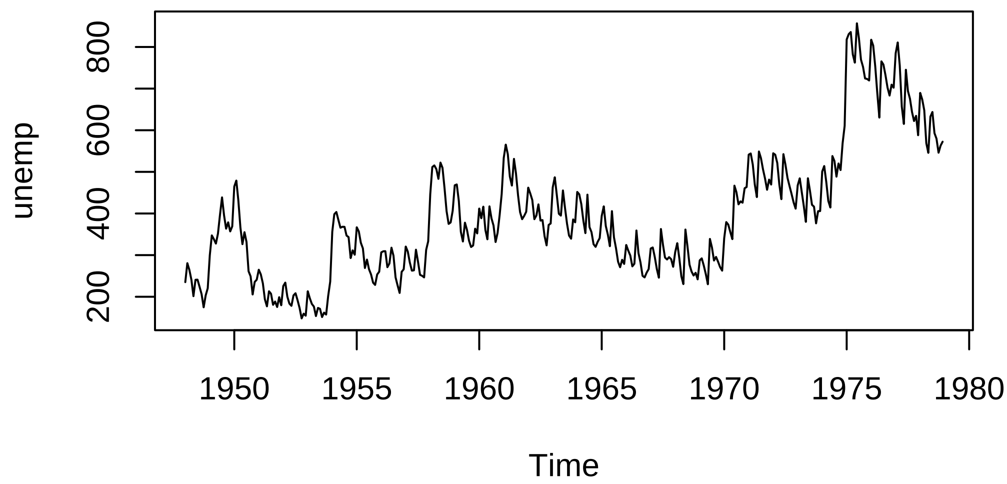
We can see that the AIC, AICc, as well as BIC all prefer the AR(1) model over ARMA(1,2). The AR(1) can considered as the preferred model in this case. Moreover, pure auto-regressive models are easier to work with.

Problem 4

Fit a seasonal ARIMA model of your choice to the unemployment data in **unemp** from the R package *astsa*. Use the estimated model to forecast the next 12 months.

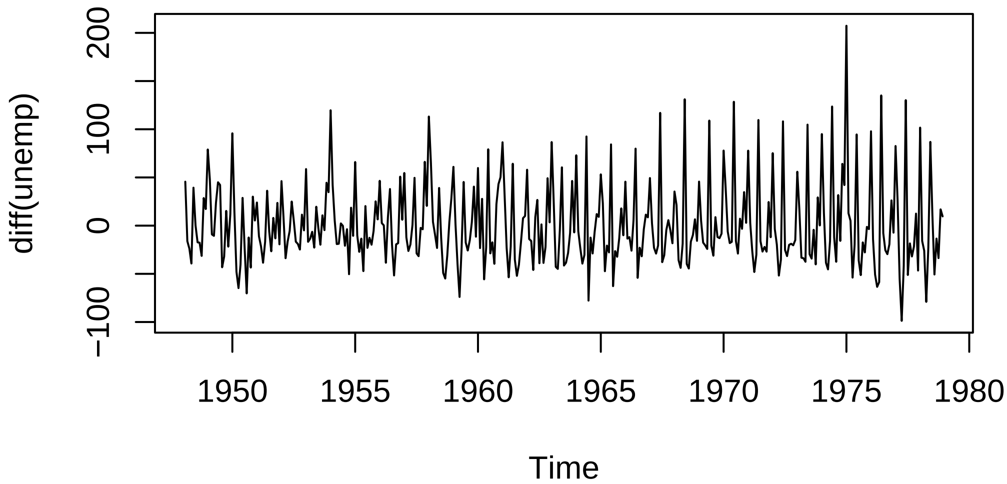
Plot the unemp data

```
plot(unemp)
```



Plot the Differenced unemp

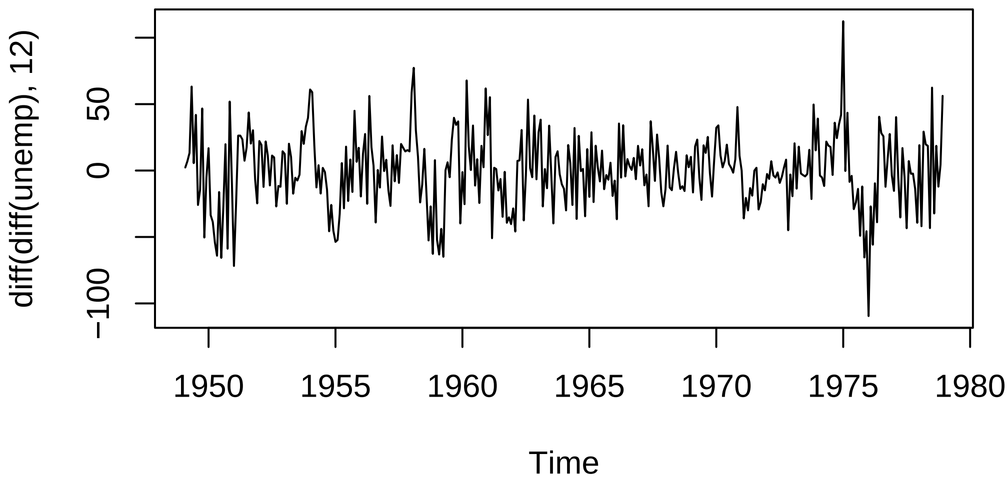
```
plot(diff(unemp))
```



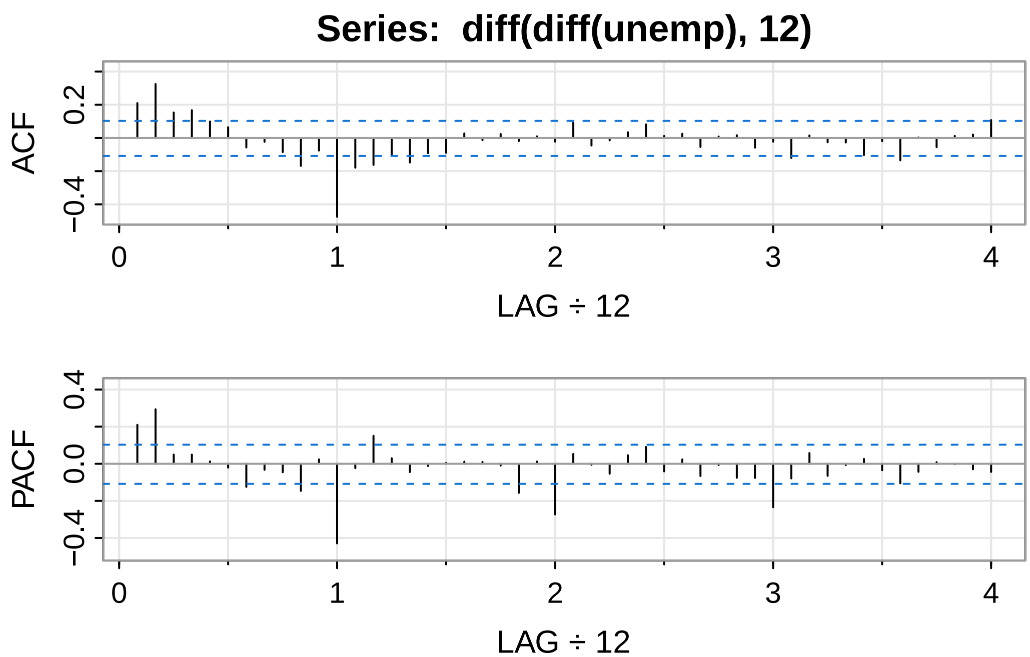
Plot Difference of Differenced unemp at lag 12

Since the given data varies over months, thus there may be a seasonal pattern over months of a year. Therefore, it is reasonable to assume that Differencing at lag 12 may give us a stationary time with which can continue our analysis.

```
plot(diff(diff(unemp),12))
```




```
acf2(diff(diff(unemp), 12))
```




```
[,1] [,2] [,3] [,4] [,5]  [,6]  [,7]  [,8]  [,9] [,10] [,11] [,12] [,13]
```

```
ACF  0.21 0.33 0.15 0.17 0.10  0.06 -0.06 -0.02 -0.09 -0.17 -0.08 -0.48 -0.18
PACF 0.21 0.29 0.05 0.05 0.01 -0.02 -0.12 -0.03 -0.05 -0.15  0.02 -0.43 -0.02
     [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
ACF  -0.16 -0.11 -0.15 -0.09 -0.09  0.03 -0.01  0.02 -0.02  0.01 -0.02  0.09
PACF  0.15  0.03 -0.04 -0.01  0.00  0.01  0.01 -0.01 -0.16  0.01 -0.27  0.05
     [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
ACF  -0.05 -0.01  0.03  0.08  0.01  0.03 -0.05  0.01  0.02 -0.06 -0.02 -0.12
PACF -0.01 -0.05  0.05  0.09 -0.04  0.02 -0.07 -0.01 -0.08 -0.08 -0.23 -0.08
     [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48]
ACF   0.01 -0.03 -0.03 -0.10 -0.02 -0.13  0.00 -0.06  0.01  0.02  0.11
PACF  0.06 -0.07 -0.01  0.03 -0.03 -0.11 -0.04  0.01  0.00 -0.03 -0.04
```

The ACF and PACF confirm our assumption of seasonal pattern. Here, we can observe the seasonal lags in the ACF cut off after lag 12, whereas the seasonal lags in the PACF tail off at lags 12, 24, 36, and so on. This indicates a clear SMA(1) pattern.

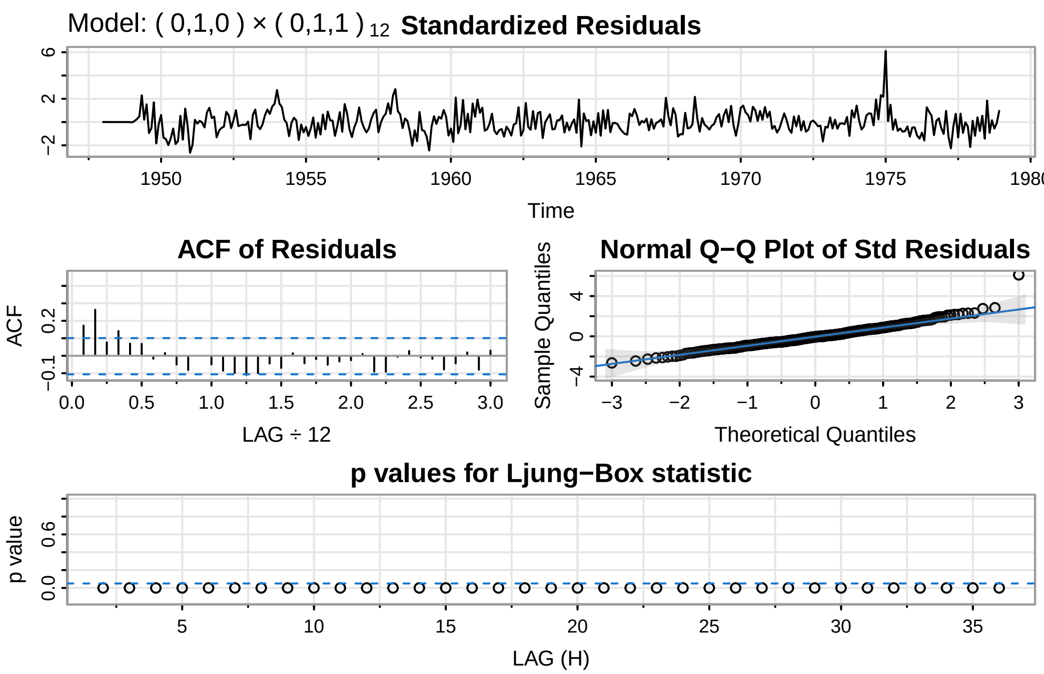
Fit a SMA(1) model

Now, wefit an SARIMA$(0,1,0) \times (0,1,1)_{12}$ to xt and look at the ACF and PACF of the residuals

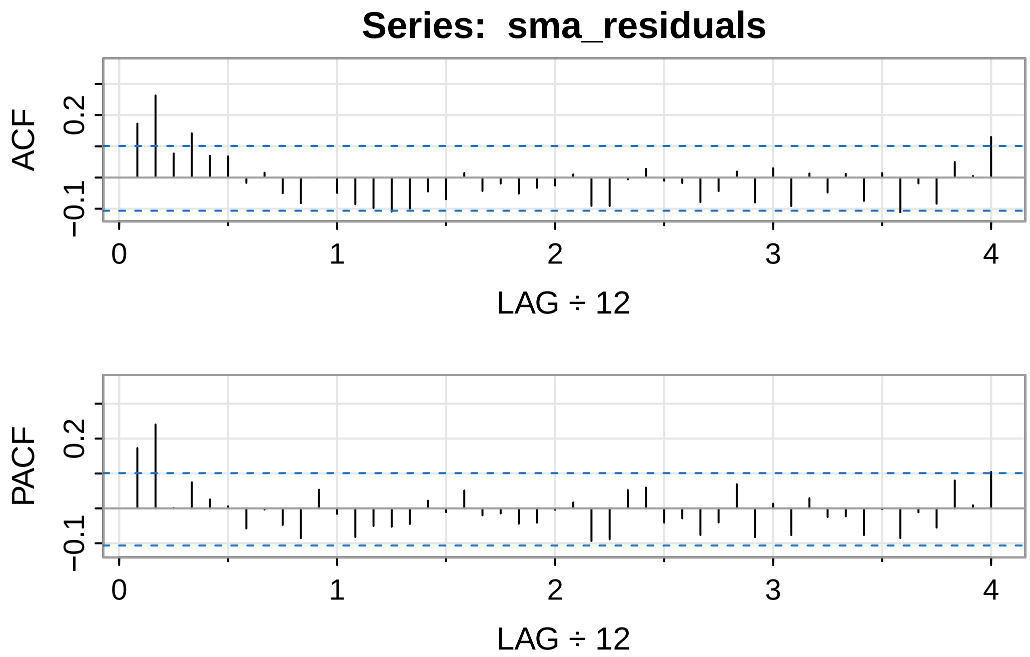
```r
sma_fit <- sarima(unemp, 0,1,0, 0,1,1,12)
```

```
initial  value 3.338111
iter   2 value 3.154910
iter   3 value 3.119632
iter   4 value 3.113045
iter   5 value 3.112067
iter   6 value 3.108906
iter   7 value 3.108728
iter   8 value 3.108723
iter   9 value 3.108720
iter   9 value 3.108720
iter   9 value 3.108720
final  value 3.108720
converged
initial  value 3.111599
iter   2 value 3.111596
iter   3 value 3.111594
iter   3 value 3.111594
iter   3 value 3.111594
final  value 3.111594
```

```
converged
<><><><><><><><><><><><><><>

Coefficients:
     Estimate     SE t.value p.value
sma1  -0.7334 0.0366 -20.025       0

sigma^2 estimated as 491.4653 on 358 degrees of freedom

AIC = 9.072207  AICc = 9.072239  BIC = 9.093842
```




```r
# Extract residuals
sma_residuals <- sma_fit$fit$residuals

acf2(sma_residuals)
```




```
     [,1] [,2] [,3] [,4] [,5] [,6]  [,7] [,8]  [,9] [,10] [,11] [,12] [,13]
ACF  0.17 0.26 0.08 0.14 0.07 0.07 -0.02 0.02 -0.05 -0.08  0.00 -0.05 -0.09
PACF 0.17 0.24 0.00 0.07 0.03 0.01 -0.06 0.00 -0.05 -0.09  0.05 -0.02 -0.08
     [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
ACF  -0.10 -0.11 -0.10 -0.04 -0.07  0.01 -0.04 -0.02 -0.05 -0.03 -0.03  0.01
PACF -0.05 -0.05 -0.04  0.02 -0.01  0.05 -0.02 -0.01 -0.04 -0.04  0.00  0.02
     [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
ACF  -0.09 -0.09 -0.01  0.03 -0.01 -0.02 -0.08 -0.04  0.02 -0.08  0.03 -0.09
PACF -0.09 -0.09  0.05  0.06 -0.04 -0.03 -0.08 -0.04  0.07 -0.08  0.01 -0.08
     [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48]
ACF   0.01 -0.05  0.01 -0.07  0.01 -0.11 -0.02 -0.08  0.05  0.00  0.13
PACF  0.03 -0.02 -0.02 -0.08  0.00 -0.09 -0.01 -0.05  0.08  0.01  0.10
```

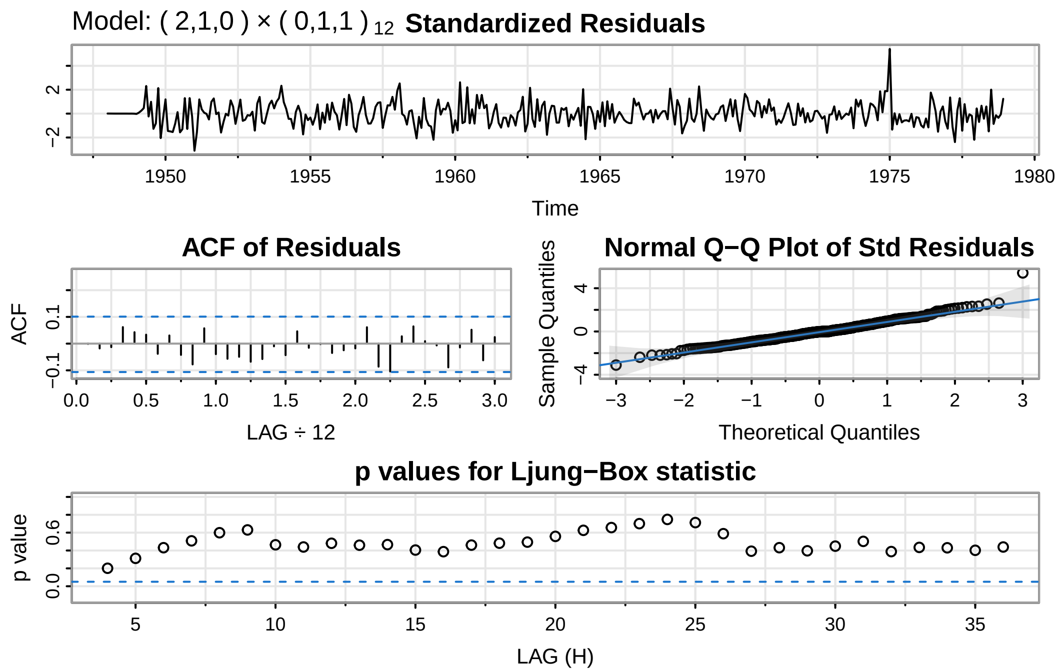
The within-season portion of the ACF gradually tapers off, while the PACF either cuts off at lag 2 or also tapers. These observations indicate that an AR(2) or ARMA(1,1) model could be suitable for the within-season component of the model.

Fit SARIMA with AR(2) component

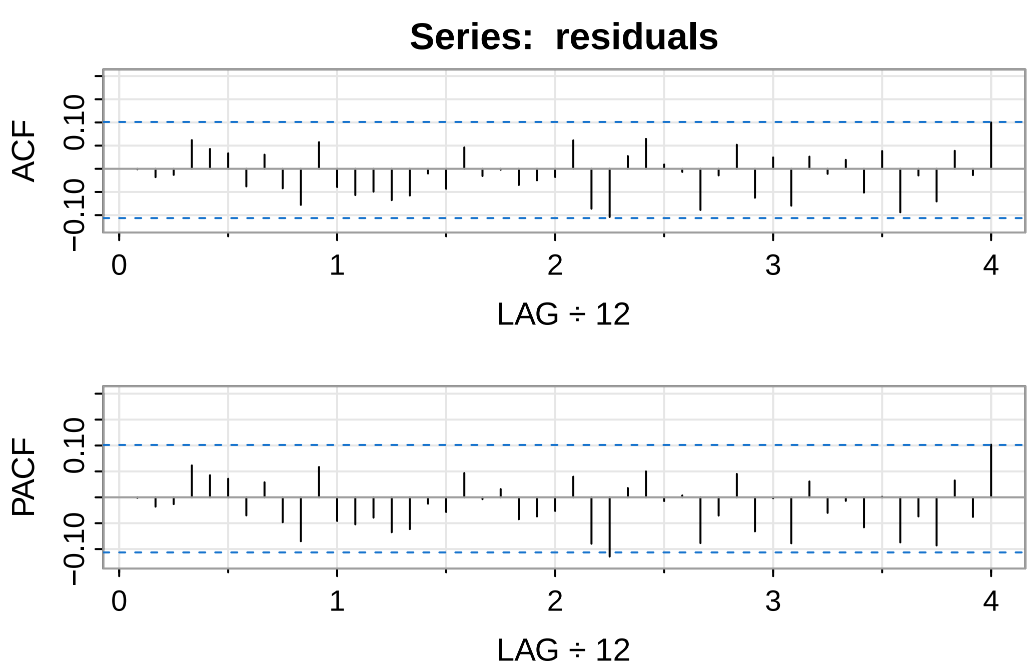
$\text{SARIMA}(2,1,0) \times (0,1,1)_{12}$

```r
sarima_model <- sarima(unemp, 2, 1, 0, 0, 1, 1, 12)
```

```
initial  value 3.340809
iter   2 value 3.105512
iter   3 value 3.086631
iter   4 value 3.079778
iter   5 value 3.069447
iter   6 value 3.067659
iter   7 value 3.067426
iter   8 value 3.067418
iter   8 value 3.067418
final  value 3.067418
converged
initial  value 3.065481
iter   2 value 3.065478
iter   3 value 3.065477
iter   3 value 3.065477
iter   3 value 3.065477
final  value 3.065477
converged
<><><><><><><><><><><><><><>

Coefficients:
     Estimate     SE  t.value p.value
ar1    0.1351 0.0513   2.6326  0.0088
ar2    0.2464 0.0515   4.7795  0.0000
sma1  -0.6953 0.0381 -18.2362  0.0000

sigma^2 estimated as 449.637 on 356 degrees of freedom

AIC = 8.991114  AICc = 8.991303  BIC = 9.034383
```




```
# Extract residuals
residuals <- sarima_model$fit$residuals
acf2(residuals)
```



```
     [,1]  [,2]  [,3] [,4] [,5] [,6]  [,7] [,8]  [,9] [,10] [,11] [,12] [,13]
ACF     0 -0.02 -0.01 0.06 0.04 0.03 -0.04 0.03 -0.04 -0.08  0.06 -0.04 -0.06
PACF    0 -0.02 -0.01 0.06 0.04 0.04 -0.03 0.03 -0.05 -0.08  0.06 -0.05 -0.05
     [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
ACF  -0.05 -0.07 -0.06 -0.01 -0.04  0.05 -0.02  0.00 -0.03 -0.02 -0.02  0.06
PACF -0.04 -0.07 -0.06 -0.01 -0.03  0.05  0.00  0.02 -0.04 -0.04 -0.03  0.04
     [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
ACF  -0.09 -0.10  0.03  0.06  0.01 -0.01 -0.09 -0.01  0.05 -0.06  0.02 -0.08
PACF -0.09 -0.11  0.02  0.05 -0.01  0.00 -0.09 -0.04  0.05 -0.07  0.00 -0.09
     [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48]
ACF   0.03 -0.01  0.02 -0.05  0.04 -0.09 -0.01 -0.07  0.04 -0.01   0.1
PACF  0.03 -0.03 -0.01 -0.06  0.00 -0.09 -0.04 -0.09  0.03 -0.04   0.1
```

```r
# Print MSE
mse <- mean(residuals^2)
mse
```

```
[1] 433.9248
```

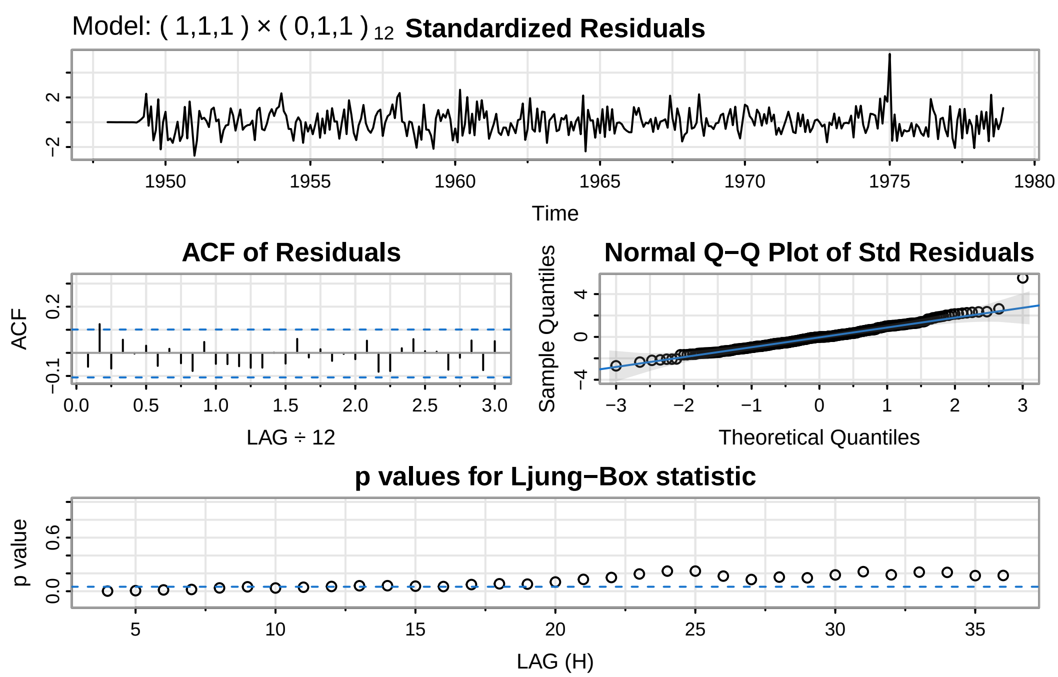
Fit SARIMA with ARMA(1,1) component

$\text{SARIMA}(1, 1, 1) \times (0, 1, 1)_{12}$

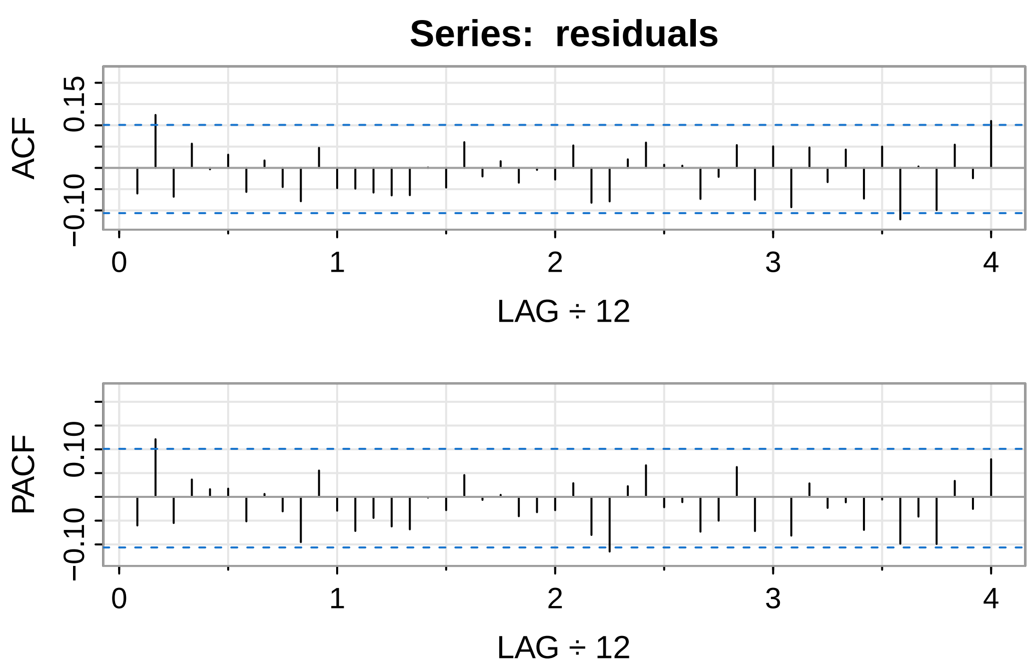
```r
sarima_model <- sarima(unemp, 1, 1, 1, 0, 1, 1, 12)
```

```
initial  value 3.339497
iter   2 value 3.185867
iter   3 value 3.137572
iter   4 value 3.103005
iter   5 value 3.102720
iter   6 value 3.096553
iter   7 value 3.095284
iter   8 value 3.093866
iter   9 value 3.093270
iter  10 value 3.091924
iter  11 value 3.084816
iter  12 value 3.078887
iter  13 value 3.076774
iter  14 value 3.075302
iter  15 value 3.075003
iter  16 value 3.074831
```

```
iter  17 value 3.074802
iter  18 value 3.074783
iter  19 value 3.074770
iter  20 value 3.074768
iter  21 value 3.074767
iter  21 value 3.074767
iter  21 value 3.074767
final  value 3.074767
converged
initial  value 3.074581
iter   2 value 3.074580
iter   3 value 3.074578
iter   4 value 3.074578
iter   5 value 3.074577
iter   6 value 3.074577
iter   6 value 3.074577
iter   6 value 3.074577
final  value 3.074577
converged
<><><><><><><><><><><><><><>

Coefficients:
     Estimate     SE  t.value p.value
ar1    0.7756 0.0763  10.1655       0
ma1   -0.5978 0.0922  -6.4810       0
sma1  -0.7005 0.0376 -18.6099       0

sigma^2 estimated as 457.8124 on 356 degrees of freedom

AIC = 9.009315  AICc = 9.009503  BIC = 9.052583
```




```r
# Extract residuals
residuals <- sarima_model$fit$residuals
acf2(residuals)
```



```
      [,1] [,2]  [,3] [,4] [,5] [,6]  [,7] [,8]  [,9] [,10] [,11] [,12] [,13]
ACF  -0.06 0.12 -0.07 0.06 0.00 0.03 -0.06 0.02 -0.05 -0.08  0.05 -0.05 -0.05
PACF -0.06 0.12 -0.05 0.04 0.02 0.02 -0.05 0.01 -0.03 -0.10  0.06 -0.03 -0.07
     [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
ACF  -0.06 -0.06 -0.06     0 -0.05  0.06 -0.02  0.02 -0.03  0.00 -0.03  0.05
PACF -0.04 -0.06 -0.07     0 -0.03  0.05 -0.01  0.00 -0.04 -0.03 -0.03  0.03
     [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
ACF  -0.08 -0.08  0.02  0.06  0.01  0.01 -0.07 -0.02  0.05 -0.07  0.05 -0.09
PACF -0.08 -0.12  0.02  0.07 -0.02 -0.01 -0.07 -0.05  0.06 -0.07  0.00 -0.08
     [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48]
ACF   0.05 -0.03  0.04 -0.07  0.05 -0.12  0.00  -0.1  0.05 -0.02  0.11
PACF  0.03 -0.02 -0.01 -0.07 -0.01 -0.10 -0.04  -0.1  0.03 -0.03  0.08
```

```r
# Print MSE
mse <- mean(residuals^2)
mse
```

```
[1] 441.8145
```

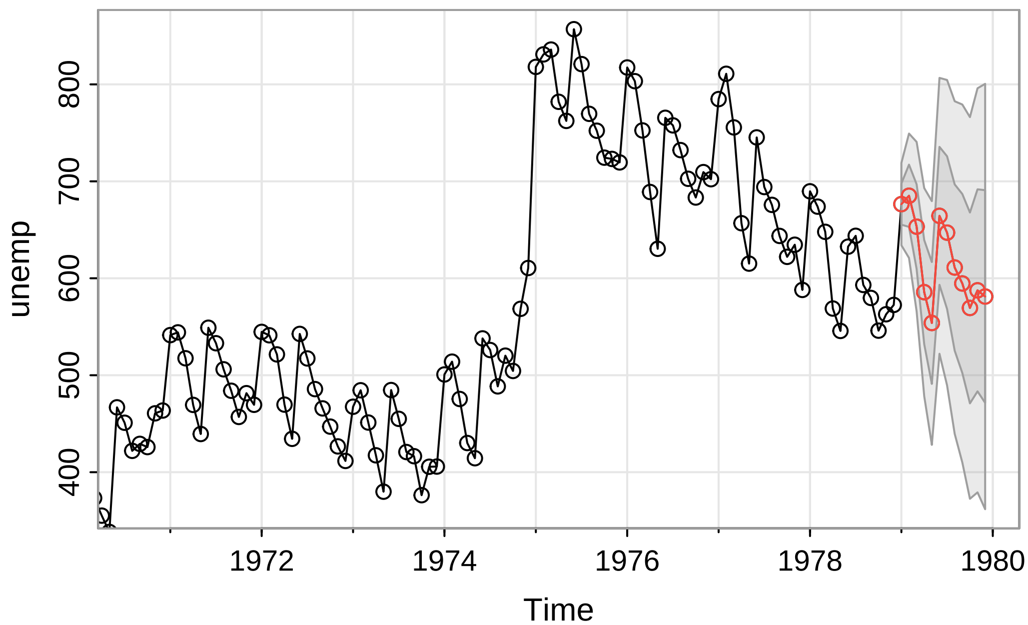
From the above two plots we can see that the first model is way better than the second one as the first one has MSE lower than the second model. Also the Q-statistic in the first model is not significant compared to the second which implies that the residuals are more likely to be white noise in the first case.

Therefore, we prefer the first SARIMA model.

Forecasting

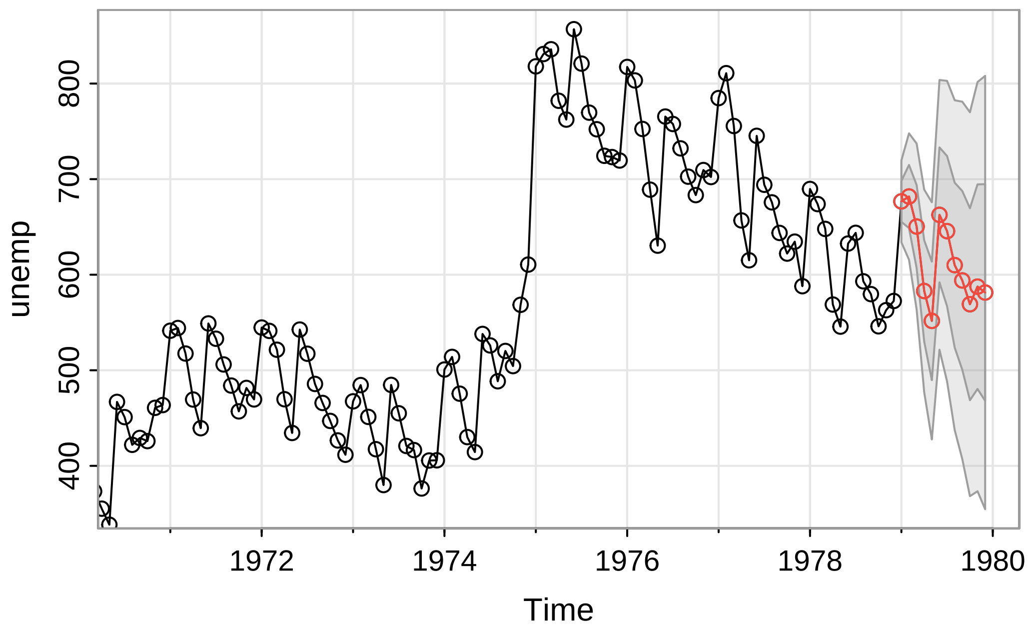
Forecasting using both models for the next 12 months.

```r
# SARIMA(2,1,0) ×(0,1,1)12
sarima.for(unemp, 12, 2, 1, 0, 0, 1, 1, 12)
```




```
$pred
          Jan      Feb      Mar      Apr      May      Jun      Jul      Aug
1979 676.4664 685.1172 653.2388 585.6939 553.8813 664.4072 647.0657 611.0828
          Sep      Oct      Nov      Dec
1979 594.6414 569.3997 587.5801 581.1833

$se
           Jan       Feb       Mar       Apr       May       Jun       Jul
1979  21.20465  32.07710  43.70167  53.66329  62.85364  71.12881  78.73590
           Aug       Sep       Oct       Nov       Dec
1979  85.75096  92.28663  98.41329 104.19488 109.67935
```

```
# SARIMA(1,1,1) ×(0,1,1)12
sarima.for(unemp, 12, 1, 1, 1, 0, 1, 1, 12)
```

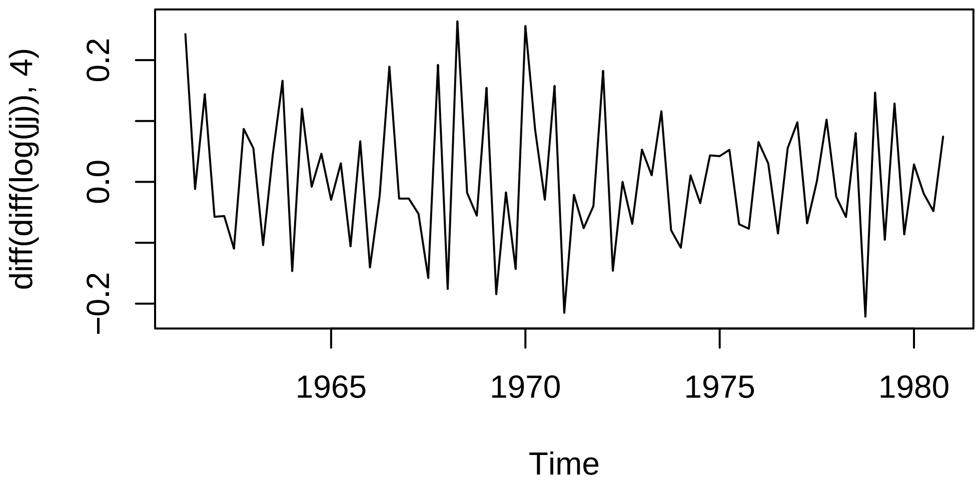



```
$pred
          Jan      Feb      Mar      Apr      May      Jun      Jul      Aug
1979 676.7140 681.8208 650.4030 582.9454 551.6973 662.6317 645.5813 610.0253
          Sep      Oct      Nov      Dec
1979 593.9988 569.1150 587.5246 581.3489

$se
           Jan       Feb       Mar       Apr       May       Jun       Jul
1979  21.39655  33.05952  43.42253  53.03046  62.04995  70.56057  78.61490
           Aug       Sep       Oct       Nov       Dec
1979  86.25504  93.51784 100.43660 107.04161 113.36045
```

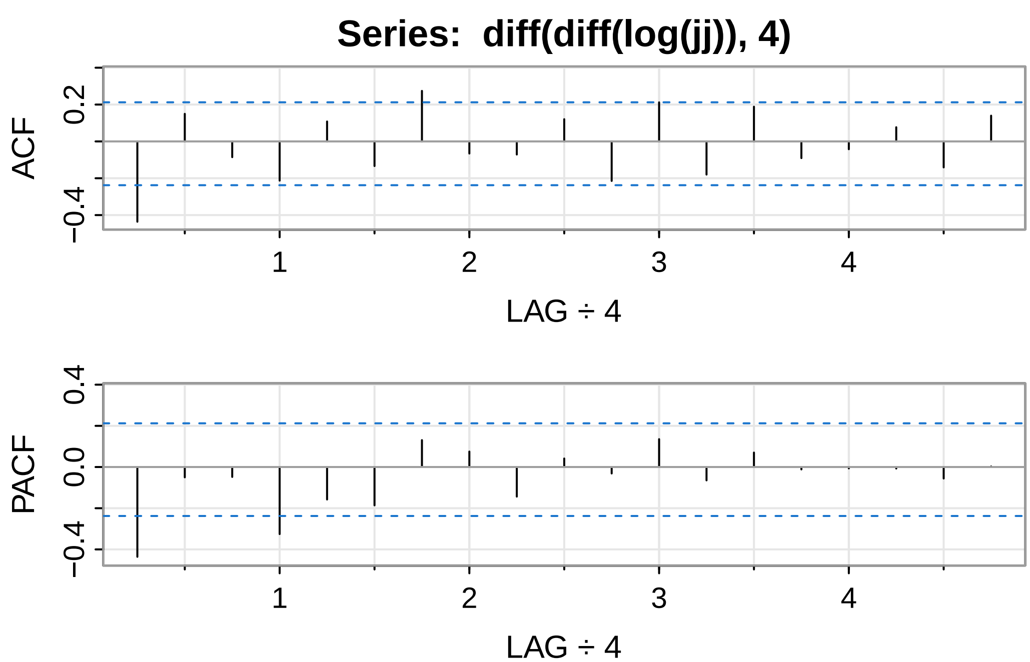
Problem 5

Fit an appropriate seasonal ARIMA model to the log-transformed Johnson and Johnson earnings series (**jj** from the R package *astsa*) discussed in Lecture 2. Use the estimated model to forecast the next 4 quarters.

```
plot(diff(diff(log(jj)),4))
```




```
acf2(diff(diff(log(jj)),4))
```

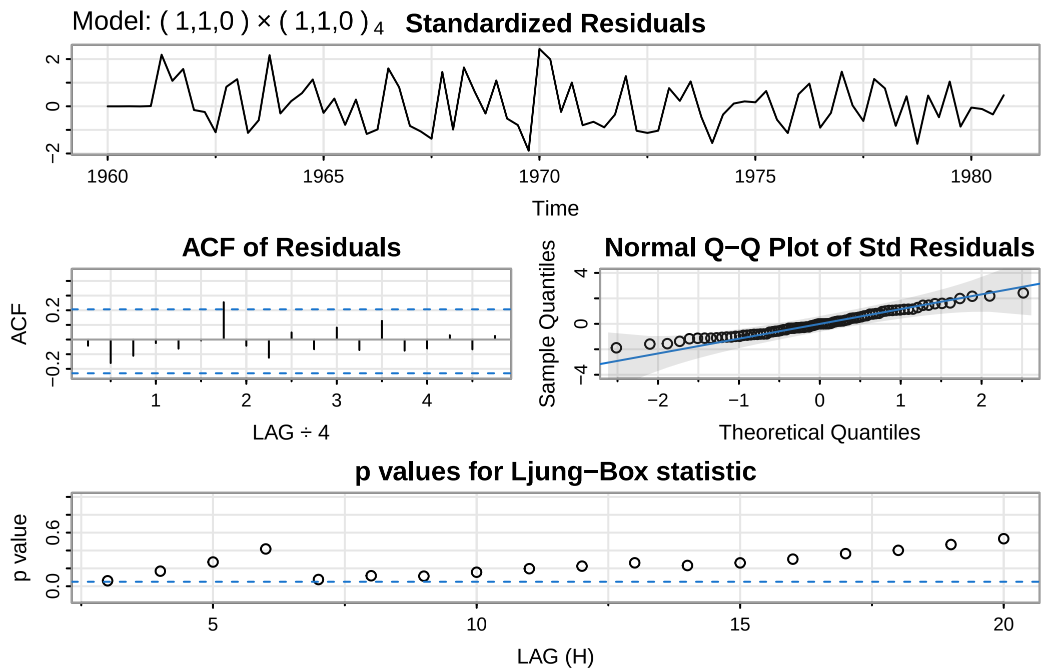



```
[,1]   [,2]   [,3]   [,4]   [,5]   [,6]  [,7]   [,8]   [,9]  [,10]  [,11]  [,12]
```

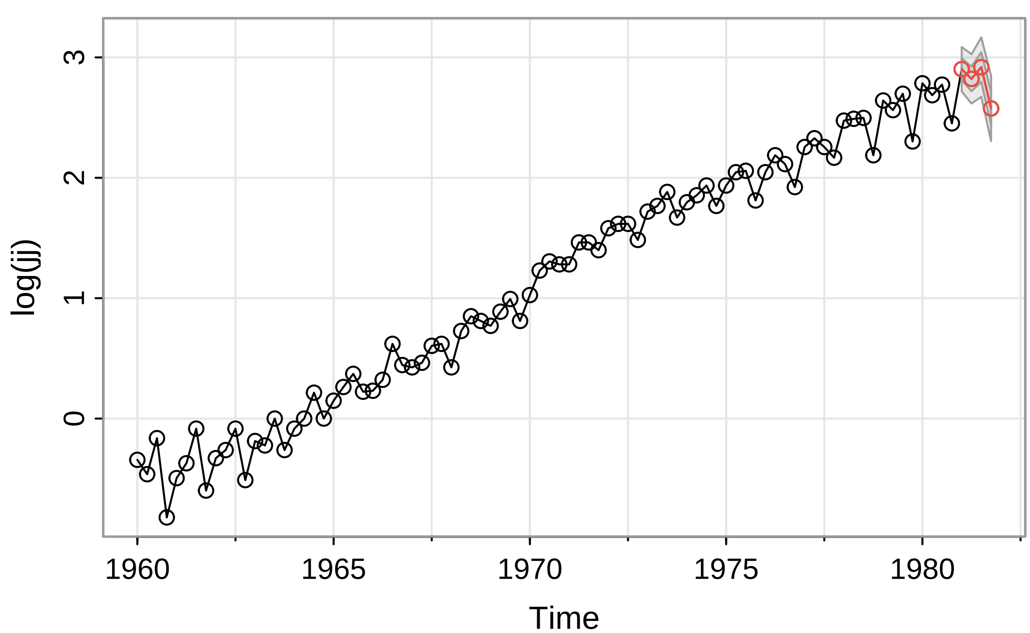
```
ACF  -0.44  0.15 -0.09 -0.21  0.11 -0.13 0.27 -0.07 -0.07  0.12 -0.21  0.21
PACF -0.44 -0.05 -0.05 -0.33 -0.16 -0.19 0.13  0.08 -0.14  0.04 -0.03  0.14
     [,13] [,14] [,15] [,16] [,17] [,18] [,19]
ACF  -0.18  0.19 -0.09 -0.04  0.08 -0.14  0.14
PACF -0.06  0.07 -0.01 -0.01 -0.01 -0.06  0.00
```

```
sarima(log(jj),1,1,0,1,1,0,4)
```

```
initial  value -2.232392
iter   2 value -2.403794
iter   3 value -2.409520
iter   4 value -2.410263
iter   5 value -2.410266
iter   6 value -2.410266
iter   6 value -2.410266
final  value -2.410266
converged
initial  value -2.381009
iter   2 value -2.381164
iter   3 value -2.381165
iter   3 value -2.381165
iter   3 value -2.381165
final  value -2.381165
converged
<><><><><><><><><><><><><><>

Coefficients:
     Estimate     SE t.value p.value
ar1   -0.5152 0.1009 -5.1064   0.000
sar1  -0.3294 0.1109 -2.9697   0.004

sigma^2 estimated as 0.008467914 on 77 degrees of freedom

AIC = -1.848505  AICc = -1.846506  BIC = -1.758525
```




```
sarima.for(log(jj),4,1,1,0,1,1,0,4)
```

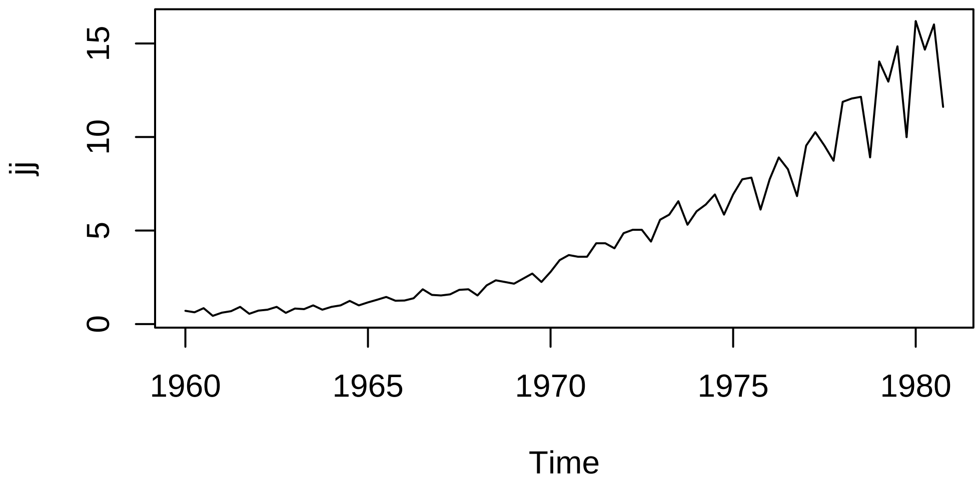



```
$pred
```

```
          Qtr1     Qtr2     Qtr3     Qtr4
1981 2.902126 2.821452 2.919034 2.575784

$se
            Qtr1       Qtr2       Qtr3       Qtr4
1981 0.09202127 0.10226343 0.12338542 0.13568573
```

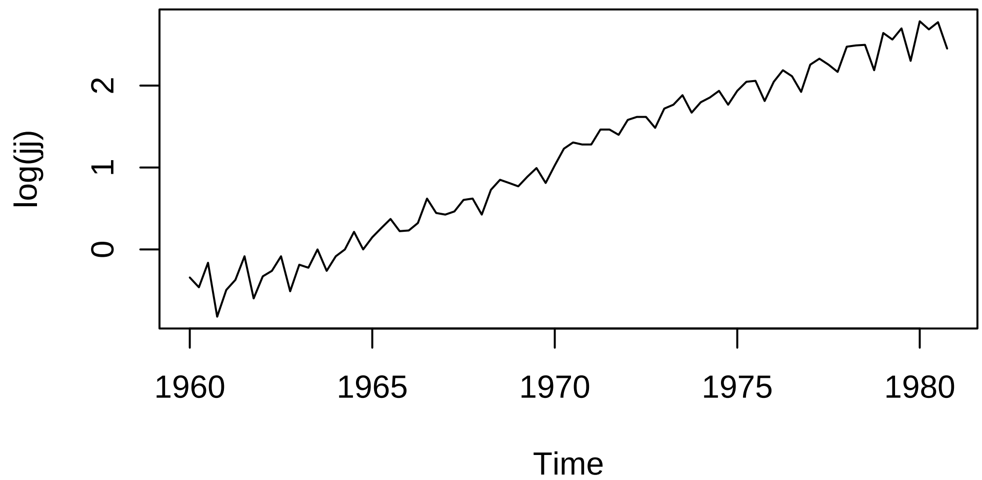
Plot the jj data

```
plot(jj)
```



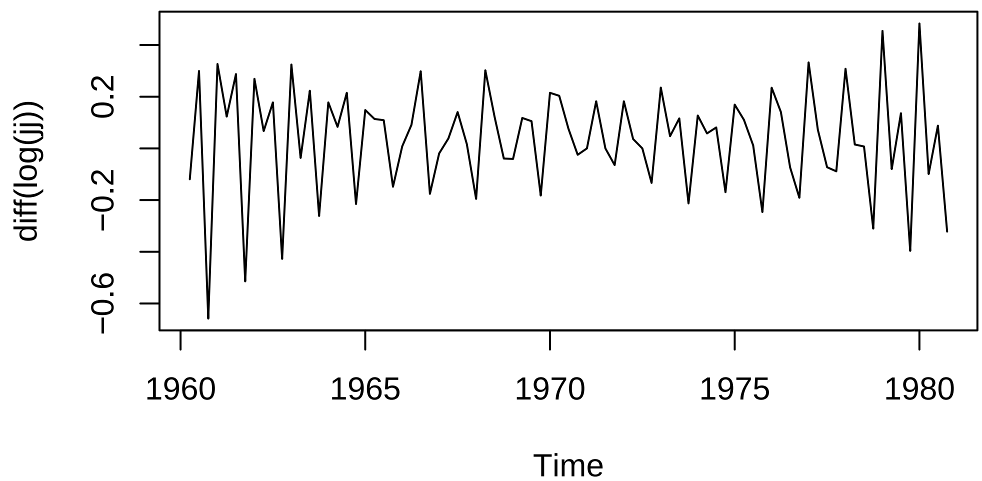
Apply Log transformation

```
plot(log(jj))
```



Plot the Differenced log jj data to remove trend

```
plot(diff(log(jj)))
```

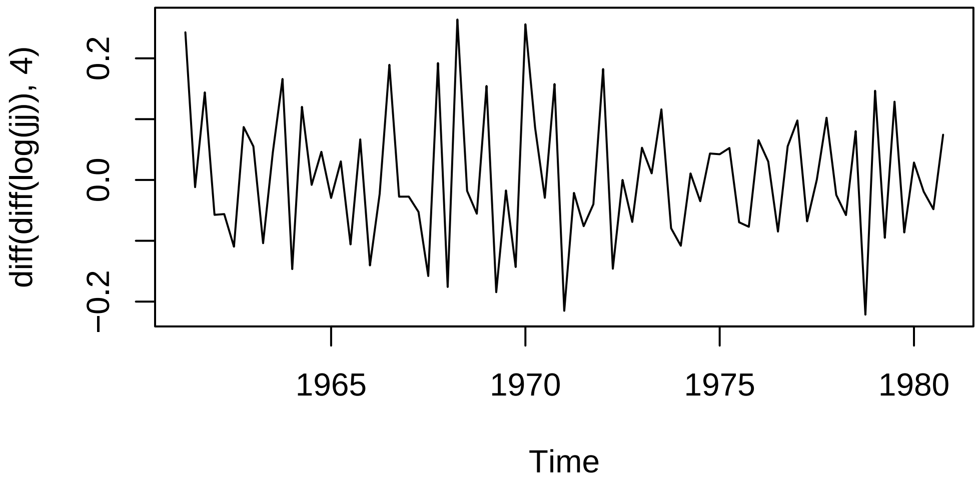


Difference at lag 4 over the Differenced log data

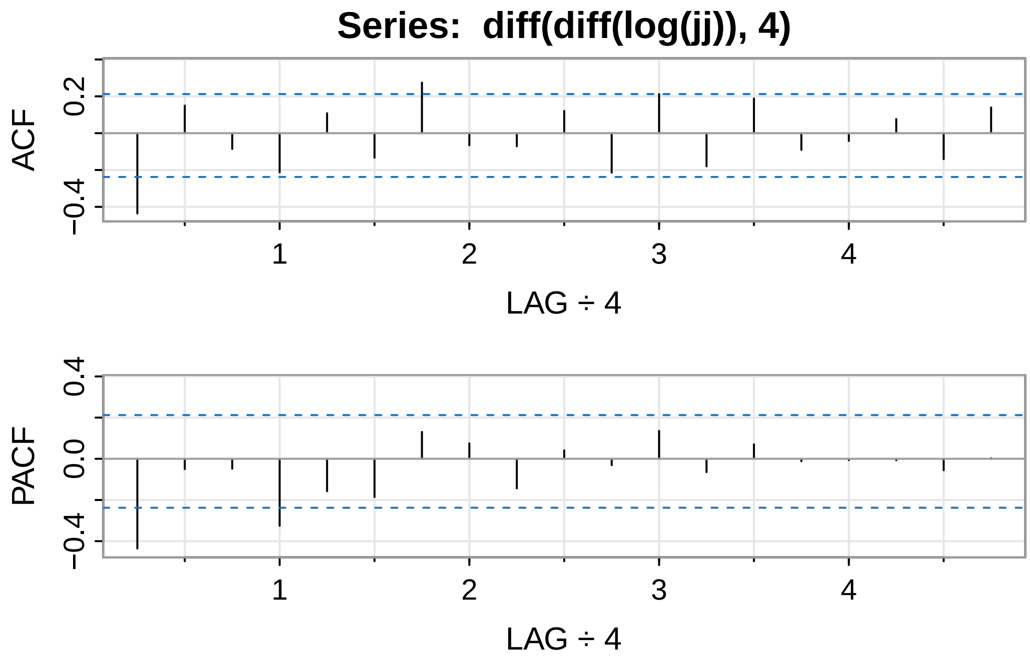
Since the data varies over quarters, there may be seasonal patterns over the quarters of each year.

Thus we take difference over lag 4.

```
plot(diff(diff(log(jj)), 4))
```




```
acf2(diff(diff(log(jj)), 4))
```



```
     [,1]  [,2]  [,3]  [,4]  [,5]  [,6] [,7]  [,8]  [,9] [,10] [,11] [,12]
ACF  -0.44  0.15 -0.09 -0.21  0.11 -0.13 0.27 -0.07 -0.07  0.12 -0.21  0.21
PACF -0.44 -0.05 -0.05 -0.33 -0.16 -0.19 0.13  0.08 -0.14  0.04 -0.03  0.14
     [,13] [,14] [,15] [,16] [,17] [,18] [,19]
ACF  -0.18  0.19 -0.09 -0.04  0.08 -0.14  0.14
PACF -0.06  0.07 -0.01 -0.01 -0.01 -0.06  0.00
```

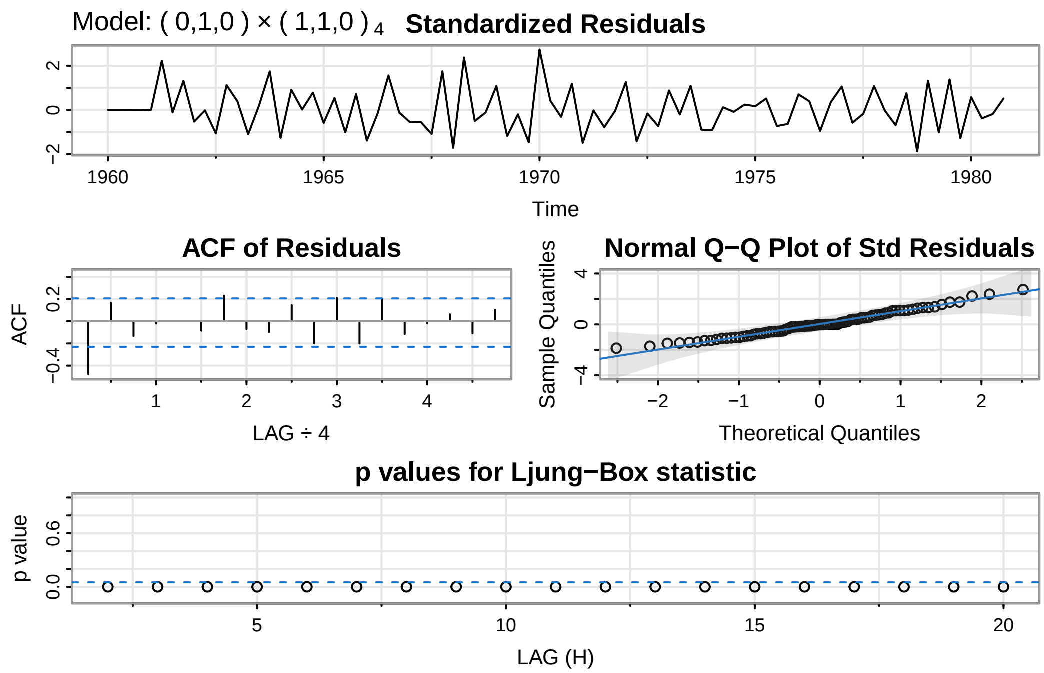
The PACF reveals a large correlation at the seasonal lag 4, so an SAR(1) seems appropriate.

Fit SAR(1)

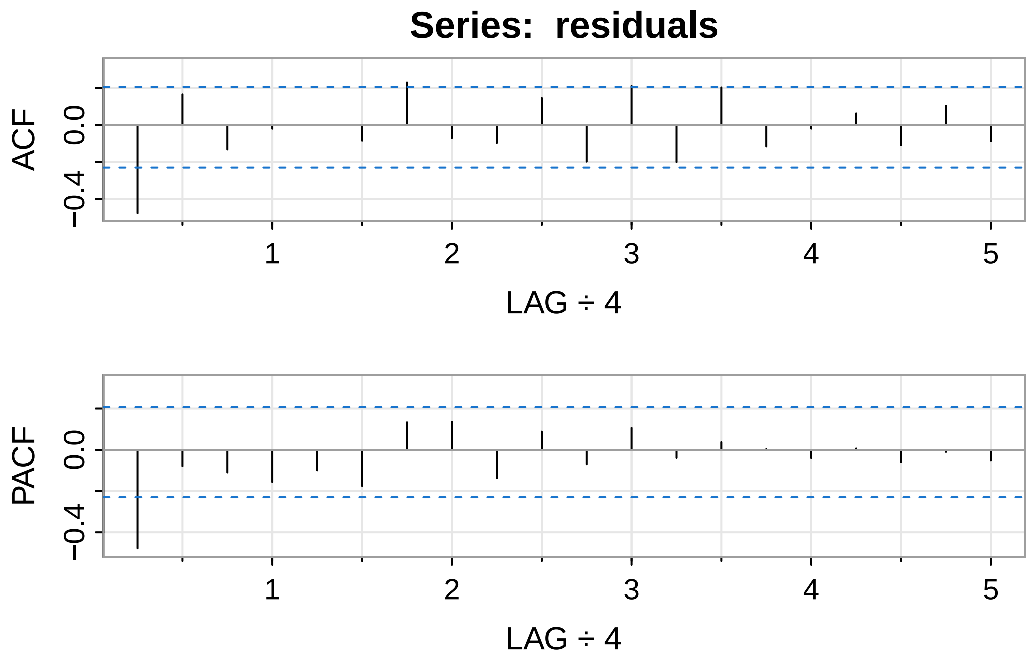
```r
sar_model <- sarima(log(jj), 0,1,0, 1,1,0, 4)
```

```
initial  value -2.237259
iter   2 value -2.262782
iter   3 value -2.262979
iter   4 value -2.262980
iter   4 value -2.262980
final  value -2.262980
converged
initial  value -2.241179
```

```
iter   2 value -2.241215
iter   3 value -2.241216
iter   3 value -2.241216
iter   3 value -2.241216
final  value -2.241216
converged
<><><><><><><><><><><><><><>

Coefficients:
     Estimate     SE t.value p.value
sar1  -0.2234 0.1121 -1.9928  0.0498

sigma^2 estimated as 0.01127663 on 78 degrees of freedom

AIC = -1.593921  AICc = -1.593264  BIC = -1.533935
```




```r
#Residuals
residuals <- sar_model$fit$residuals
acf2(residuals)
```




```
      [,1]  [,2]  [,3]  [,4] [,5]  [,6] [,7]  [,8]  [,9] [,10] [,11] [,12]
ACF  -0.48  0.17 -0.13 -0.02  0.0 -0.08 0.23 -0.07 -0.10  0.15 -0.20  0.21
PACF -0.48 -0.08 -0.11 -0.16 -0.1 -0.18 0.13  0.14 -0.14  0.09 -0.07  0.11
     [,13] [,14] [,15] [,16] [,17] [,18] [,19] [,20]
ACF  -0.20  0.20 -0.12 -0.02  0.06 -0.11  0.10 -0.09
PACF -0.04  0.04  0.00 -0.04  0.01 -0.06 -0.01 -0.05
```

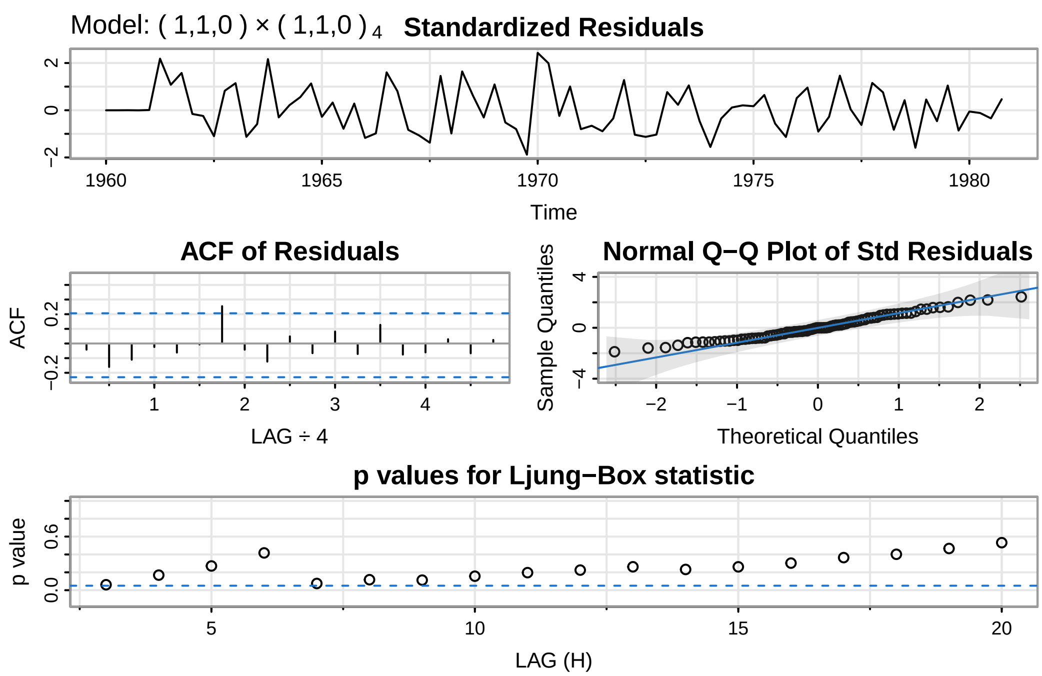
As the ACF and PACF of the residuals tails off at seasonal lag 4, this reveals an ARMA(1,1) correlation structure for the within the seasons.

Fit SARIMA(1,1,0)x(1,1,0) and SARIMA(1,1,1)x(1,1,0) and compare them

```
# SARIMA(1,1,0)x(1,1,0)4
sarima_model1 <- sarima(log(jj), 1,1,0, 1,1,0, 4)
```

```
initial  value -2.232392
iter   2 value -2.403794
iter   3 value -2.409520
iter   4 value -2.410263
iter   5 value -2.410266
iter   6 value -2.410266
```

```
iter   6 value -2.410266
final  value -2.410266
converged
initial  value -2.381009
iter   2 value -2.381164
iter   3 value -2.381165
iter   3 value -2.381165
iter   3 value -2.381165
final  value -2.381165
converged
<><><><><><><><><><><><><><>

Coefficients:
     Estimate     SE t.value p.value
ar1   -0.5152 0.1009 -5.1064   0.000
sar1  -0.3294 0.1109 -2.9697   0.004

sigma^2 estimated as 0.008467914 on 77 degrees of freedom

AIC = -1.848505  AICc = -1.846506  BIC = -1.758525
```



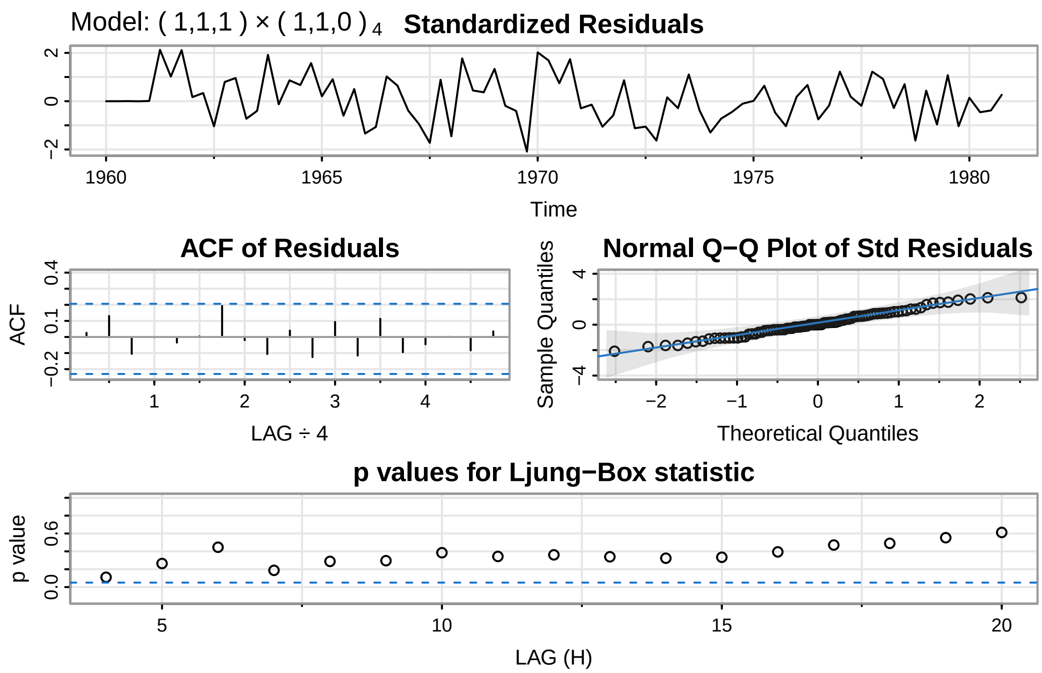
```r
sarima_model1$ICs[3]
```

```
      BIC
-1.758525
```

```r
# SARIMA(1,1,1)x(1,1,0)4
sarima_model2 <- sarima(log(jj), 1,1,1, 1,1,0, 4)
```

```
initial  value -2.232392
iter   2 value -2.397976
iter   3 value -2.442269
iter   4 value -2.447299
iter   5 value -2.452117
iter   6 value -2.454541
iter   7 value -2.455805
iter   8 value -2.455938
iter   9 value -2.455938
iter  10 value -2.455941
iter  11 value -2.455942
iter  12 value -2.455943
iter  13 value -2.455943
iter  13 value -2.455943
iter  13 value -2.455943
final  value -2.455943
converged
initial  value -2.411490
iter   2 value -2.411708
iter   3 value -2.412085
iter   4 value -2.412085
iter   5 value -2.412087
iter   5 value -2.412087
iter   5 value -2.412087
final  value -2.412087
converged
<><><><><><><><><><><><><><>

Coefficients:
     Estimate     SE t.value p.value
ar1   -0.0141 0.2221 -0.0635  0.9495
ma1   -0.6700 0.1814 -3.6940  0.0004
sar1  -0.3265 0.1320 -2.4728  0.0156
```

```
sigma^2 estimated as 0.007912779 on 76 degrees of freedom

AIC = -1.885031  AICc = -1.880981  BIC = -1.765059
```



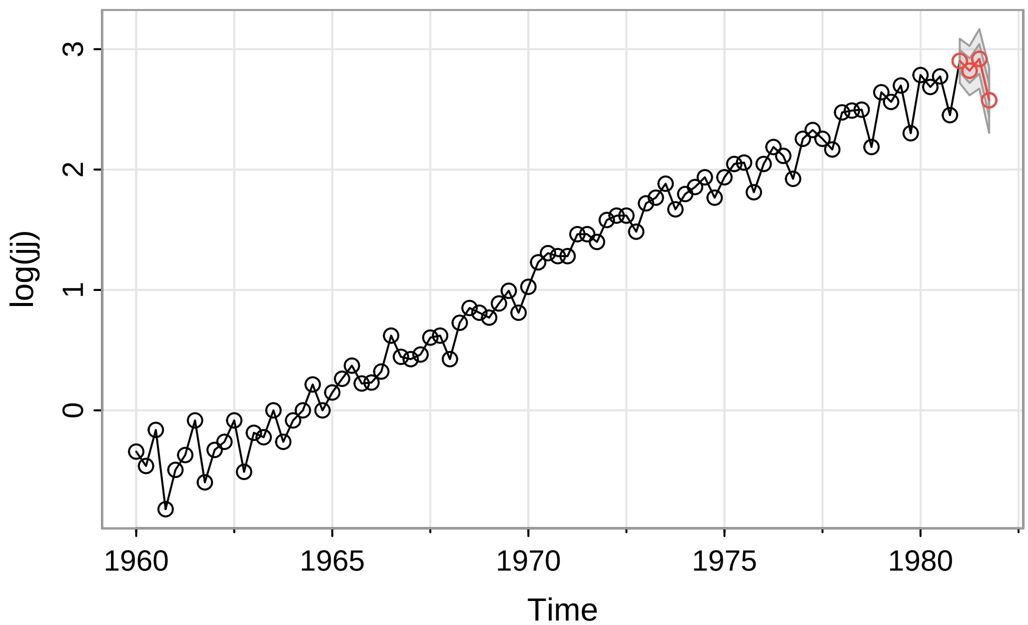

```
sarima_model2$ICs[3]
```

```
      BIC
-1.765059
```

Both the models perform well. Moreover, the second model has lower BIC, thus we prefer the SARIMA$(1, 1, 1) \times (1, 1, 0)_4$ model

Forecasting

```
sarima.for(log(jj), 4,1,1,0, 1,1,0, 4)
```




```
$pred
         Qtr1     Qtr2     Qtr3     Qtr4
1981 2.902126 2.821452 2.919034 2.575784

$se
           Qtr1       Qtr2       Qtr3       Qtr4
1981 0.09202127 0.10226343 0.12338542 0.13568573
```