The Wiener-Khintchine Theorem

Haykin's proof(!) of the Wiener-Khintchine theorem on pages 50-52 leaves a lot to be desired. Here is a reasonably complete proof, courtesy of B.P. Lathi, *Modern Digital and Analog Communication Systems*, and Prof. McEliece's EE160 course notes from a few years back.

Let $x(t)$ be a real wide-sense stationary process with autocorrelation function

$$R_x(\tau) = Ex(t)x(t+\tau).$$

Assume further that $R_x(\tau)$ satisfies the Dirichlet conditions (see e.g., pages 197-200 of *Signals and Systems*). This implies that $R_x(\tau)$ is absolutely integrable, i.e., that

$$\int_{-\infty}^{\infty} |R_x(\tau)| d\tau,$$

converges and that the Fourier transform

$$\int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau,$$

exists.

Now for each sample function $x(t)$, we can define the truncated Fourier transform

$$X_T(f) \triangleq \int_{-T/2}^{T/2} x(t) e^{-j2\pi ft} dt.$$

The corresponding truncated power spectral density is $\frac{1}{T}\left|X_T(f)\right|^2$. Since $x(t)$ is a random process, for each f, $\frac{1}{T}\left|X_T(f)\right|^2$ is a random variable. Let us denote its expectation by

$$S_T(f) \triangleq E\frac{1}{T}\left|X_T(f)\right|^2.$$

A natural definition for the power spectral density of the process $x(t)$ is therefore

$$S_x(f) = \lim_{T\to\infty} S_T(f). \tag{1}$$

Of course, we may worry about whether the above limit exists. The Wiener-Khintchine theorem, however, puts this issue to rest by asserting that the limit exists for all f, and by identifying its value.

Theorem 1 *For all f, the limit in (1) exists and*

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau. \tag{2}$$

Proof: Note that

$$
\begin{aligned}
E\left|X_T(f)\right|^2 &= E\left|\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt\right|^2 \\
&= E\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} x(t)x(\tau)e^{-j2\pi f(t-\tau)}dtd\tau \\
&= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} Ex(t)x(\tau)e^{-j2\pi f(t-\tau)}dtd\tau \\
&= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_x(t-\tau)e^{-j2\pi f(t-\tau)}dtd\tau.
\end{aligned}
$$

Now a simple exercise in calculus shows that

$$\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} f(t-\tau)dtd\tau = \int_{-T}^{T}(T-|\tau|)f(\tau)d\tau.$$

(Show this!) Applying the above formula to our problem yields

$$E\left|X_T(f)\right|^2 = \int_{-T}^{T}(T-|\tau|)R_x(\tau)e^{-j2\pi f\tau}d\tau,$$

and so

$$E\frac{1}{T}\left|X_T(f)\right|^2 = \int_{-T}^{T}(1-\frac{|\tau|}{T})R_x(\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} R_{x,T}(\tau)e^{-j2\pi f\tau}d\tau, \tag{3}$$

where we have defined

$$R_{x,T}(\tau) = \begin{cases} (1-\frac{|\tau|}{T})R_x(\tau) & |\tau| < T \\ 0 & |\tau| \geq T \end{cases}$$

Integrals of the form (3) appear to be tricky to analyze. However, their asymptotic behavior can be inferred from the following Lebesgue-dominated-convergence theorem (see, e.g., Bartle, *The Elements of Integration*).

Theorem 2 *Let (f_n) be a sequence of complex-valued measurable functions which converges almost everywhere to a complex-valued measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and, in fact,*

$$\int f d\mu = \lim_{n\to\infty}\int f_n d\mu. \tag{4}$$

If we take the f_n's as the complex-valued functions $R_{x,T}(\tau)e^{-j2\pi f\tau}$, then the corresponding limit is $f = R_x(\tau)e^{-j2\pi f\tau}$. Moreover, we can take the integrable function to be $g = |R_x(\tau)|$. Then from the definition of $R_{x,T}(\tau)$ it is straightforward to see that

$$\left|R_{x,T}(\tau)e^{-j2\pi f\tau}\right| \leq |R_x(\tau)|,$$

and the Lebesgue-dominated-convergence theorem applies. Therefore

$$
\begin{aligned}
S_x(f) = \lim_{T\to\infty} E\frac{1}{T}\left|X_T(f)\right|^2 &= \lim_{T\to\infty}\int_{-\infty}^{\infty} R_{x,T}(\tau)e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty}\lim_{T\to\infty} R_{x,T}(\tau)e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty}\mathcal{R}_x(\tau)e^{-j2\pi f\tau}d\tau.
\end{aligned}
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