Some notes on the Power Spectral Density of Random Processes

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1 Energy/Power Spectral Density for Continuous Deterministic Signals

Given a generic signal $x(t)$ its normalized instantaneous power is defined as $|x(t)|^2$. If $x(t)$ is a voltage $|x(t)|^2$ represents the instantaneous power dissipated applying $x(t)$ to a resistor of 1Ω. The normalized energy over the period T is defined (considering T centered around 0) as $\int_{-T/2}^{T/2} |x(t)|^2 dt$, while the normalized mean power is given by $\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$.

Considering $T \to \infty$, for a signal with finite normalized energy, we have $E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt$. In this case the mean normalized power is, obviously, null.

A signal has finite mean normalized power if the following limit exists: $\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |x(t)|^2 dt$. In this case the normalized energy does not exists (or in informal words is infinite).

In the following the terms "normalized" and "mean" with reference to energy and power will be discharged for notation simplicity.

For a finite energy signal $x(t)$ the following expression of its energy (Parseval theorem) holds:

$$E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

where $X(f)$ is the Fourier transform of $x(t)$: $X(f) = \mathcal{F}(x(t))$. $|X(f)|^2$ is called the Energy Spectral Density of the signal $x(t)$ and it describes how the energy of $x(t)$ is distributed at the various frequencies. Consider a linear system characterized by an impulse response $h(t)$ to which corresponds a transfer function $H(f) = \mathcal{F}(h(t))$ that is 0 every where except in a very small frequency interval Δf centered around f_0 in which it is equal to 1 . The Fourier transform of the system output will be $Y(f) = X(f)H(f)$. For the Parseval theorem $E_{y(t)} = \int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 df = \int_{f_0-\Delta f}^{f_0+\Delta f} |X(f)|^2 df \overset{\Delta f \to 0}{=} |X(f_0)|^2 \Delta f$. Therefore $|X(f_0)|^2$ is the Energy Spectral Density of $x(t)$ at $f = f_0$ (the unique frequency component that is not zeroed by the considered system).

We can write $|X(f)|^2$ as $X(f) \cdot X(f)^* = \mathcal{F}(x(t) \star x^*(-t)$ therefore the Energy Spectral Density of the deterministic signal $x(t)$ can be seen as the Fourier Transform of the deterministic autocorrelation of the signal: $\mathcal{R}_{x(t)}(\tau) = \int_{-\infty}^{\infty} x^*(t)x(t+\tau)dt$.

Given a signal with finite power (characterized by "infinite" energy) we can consider a truncated version of it (between $-T/2$ e $T/2$): $x_T(t)$. This new signal has finite energy and it is possible to evaluate its Energy Spectral Density: $|X_T(f)|^2$ where $X_T(f) = \mathcal{F}(x_T(t))$. We can define the Power Spectral Density of $x_T(t)$ as

$$S_{x_T(t)} \equiv \frac{1}{T}|X_T(f)|^2 = \frac{1}{T}\left|\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt\right|^2$$

Increasing T till infinite we obtain the Power Spectral Density of $x(t)$:

$$S_{x(t)} = \lim_{T\to\infty} S_{x_T(t)} = \lim_{T\to\infty} \frac{1}{T}\left|\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt\right|^2$$

Also the Power Spectral Density can be seen as the Fourier Transform of the autocorrelation function of the considered deterministic signal, but it is necessary to define the autocorrelation in a slightly different way with respect to what defined before:

$\mathbb{R}_{x(t)}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} x^*(t)x(t+\tau)dt$. This new definition is in any case necessary because the signal has now finite power and not finite energy.

2 Energy/Power Spectral Density for Discrete Deterministic Signals

It is interesting to define the energy and power associate do a discrete deterministic signal given by the sequence $x(nT)$ with n that spans from $-\infty$ to ∞, where nT are the time instants at which the samples are available. The energy and power of this discrete signal can be defined as the energy of the continuous signal that can interpolated from $x(nT)$ by using an ideal reconstruction filter (that has impulse response $\psi(t) = \frac{sin(\frac{\pi}{T}t)}{\frac{\pi}{T}t}$:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\psi(t-nT)$$

the energy of this signal results:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} x(nT)\psi(t-nT) \right|^2 dt$$

to evaluate this integral it is necessary to solve integrals of the type:

$$\int_{-\infty}^{\infty} \psi(t-iT)\psi(t-mT)dt$$

The Fourier transform of $\psi(t)$ is a box function in the frequency domain centered in the origin with $1/T$ width and T height. Applying the Parseval theorem in its general formulation:

$$\int_{-\infty}^{\infty} x(t)y(t)dt = \int_{-\infty}^{\infty} X(-f)Y(f)df = \int_{-\infty}^{\infty} X(f)Y(-f)df$$

it is possible to write:

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(t-iT)\psi(t-mT)dt &= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} (Te^{-j2\pi fiT})^*(Te^{-j2\pi fmT})df = \\ T^2 \int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{-j2\pi f(m-i)T} df &= T^2 \int_{-\frac{1}{2T}}^{\frac{1}{2T}} cos(2\pi f(m-i)T)\end{aligned}$$

the considered integral has value $T^2/T = T$ per $i = m$ and 0 for $i \neq m$.

Therefore we may write:

$$\int_{-\infty}^{\infty} |x(T)|^2 dt = \int_{-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} x(nT)\psi(t-nT) \right|^2 dt = T \sum_{n=-\infty}^{\infty} |x(nT)|^2$$

The energy of $x(t)$ is equal to the energy of a step-wise signal where each step has value $x(nT)$. In the case that $T\sum_{n=-\infty}^{\infty}|x(nT)|^2$ assumes a finite value we consider this as the energy of the discrete signal $x(nT)$.

When $x(nT)$ has no finite energy, we can try to evaluate its power. At first we evaluate

$$\int_{-t_0}^{t_0}|x(t)|^2 dt = T\sum_{n=-t_0/T}^{t_0/T}|x(nT)|^2$$

with $t_0 = KT$ where K is very big. This condition on K is necessary to ensure that the area of the square values of the tails of the sinc functions associated to each sample that fall outside $-t_0/2)$ and $t_0/2$ is negligible with respect to $\int_{-t_0}^{t_0}|x(t)|^2 dt$. Obviously it is also true

$$\frac{1}{2t_0}\int_{-t_0}^{t_0}|x(t)|^2 dt = \frac{T}{2t_0}\sum_{n=-t_0/T}^{t_0/T}|x(nT)|^2$$

The previous relation may be written also as $\overline{|x(t)|^2} = \overline{|x(nT)|^2}$ that is valid also for $t_0 \to \infty$. Therefore the power of $x(t)$ is equal to the mean of the square values of the samples used to generate it. The mean of the sample square values is defined also as the power of the discrete signal $x(nT)$. *Note that the power of a sample sequence is independent from to the sampling rate.*

Considering a discrete deterministic signal $x(nT)$ with finite energy ($T\sum_{n=-\infty}^{\infty}|x(nT)|^2$), in coherence with what shown for continuous signal, we have an Energy Spectral Density (ESD) that is $|X(f)|^2$ where $X(f)$ is the Discrete Time Fourier Transform (DTFT) of $x(nT)$: $X(f) = T\sum_{n=-\infty}^{\infty}x(nT)e^{-j2\pi fnT}$. The ESD can be seen as the DTFT of the autocorrelation function of the signal: $\mathcal{R}_{x(nT)}(kT) = T\sum_{n=-\infty}^{\infty}x^*(nT)x((n+k)T) = x(kT)\star x^*(-kT)$. $X(f)$ is a periodic function, with period $1/T$. The Inverse DTFT (IDTFT) is therefore defined as: $x(nT) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}}X(f)e^{-j2\pi fnT}df$. The energy of $x(nT)$ can be evaluated integrating the ESD in the same frequency interval: $\int_{-\frac{1}{2T}}^{\frac{1}{2T}}|X(f)|^2 df$.

If we work with normalized time/frequency axes ($t_n = t/T$, $f_n = fT$) the scaling factor T disappears in the DTFT definition and the ESD must be integrated between $-1/2$ and $1/2$ to evaluate the total energy, but it is necessary to multiply the ESD by T to obtain the correct result is the real sampling rate is T.

3 Energy/Power Spectral Density for Deterministic Signals through LTI systems

It is easy to show that for both continuous and discrete deterministic signal the ESD and PSD resoect the following relationship when the signals are applied to an LTI system with transfer function $H(f)$ or $H(e^{j2\pi f})$:

$$\begin{aligned} ESD/PSD_{output}(f) &= ESD/PSD_{input}(f)\cdot|H(f)|^2 \\ ESD/PSD_{output}(e^{j2\pi f}) &= ESD/PSD_{input}(e^{j2\pi f})\cdot|H(e^{j2\pi f})|^2 \end{aligned}$$

4 Random Processes

As known a real random variable x is a mapping between the sample space $\mathbf{S}$ and the real line $\Re$. That is, $x : \mathbf{S} \to \Re$.

A real continuous random process (also known as stochastic process) is a mapping from the sample space into an ensemble of time functions (known as sample functions). To every $\rho \in \mathbf{S}$ there corresponds a function of time (a sample function) $x(t;\rho)$. Often, from the notation, we drop the ρ variable, and write just $x(t)$. However, the sample space ρ variable is always there, even if it is not shown explicitly.

For a random process $x(t)$ the first-order probability distribution function is defined as

$$F_{x(t)}(a;t) = P[x(t) \leq a]$$

and the first-order density function is defined as

$$f_{x(t)}(a;t) \equiv \frac{dF(a;t)}{da}$$

These definition can be generalized to n-th order case. In particular the n-th order distribution function is:

$$F_{x(t)}(a_1, a_2, \ldots a_n; t_1, t_2, \ldots, t_n) = P[x(t_1) \leq a_1, x(t_2) \leq a_2, \ldots, x(t_n) \leq a_n]$$

and the n-th order density function is

$$f_{x(t)}(a_1, a_2, \ldots, a_n; t_1, t_2, \ldots, t_n) = \frac{\partial F_{x(t)}(a_1, a_2, \ldots a_n; t_1, t_2, \ldots, t_n)}{\partial a_1 \partial a_2 \ldots \partial a_n}$$

In general a complete statistical description of a random process requires knowledge of all order distribution function.

A process is said to be stationary if its statistical properties do not change with time. More precisely $x(t)$ is stationary if:

$$F_{x(t)}(a_1, a_2, \ldots a_n; t_1, t_2, \ldots, t_n) = F_{x(t)}(a_1, a_2, \ldots a_n; t_1 + c, t_2 + c, \ldots, t_n + c)$$

For a random process it is possible to define also statistical averages.

The first order statistical average is

$$\mu(t) = E[x(t)] = \int_{-\infty}^{\infty} a f_{x(t)}(a;t) da$$

the autocorrelation function is defined for a real process as

$$R(t_1, t_2) = E[x(t_1)x(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1 a_2 f_{x(t)}(a_1, a_2; t_1, t_2) da_1 da_2$$

A process $x(t)$ is said to be wide-sense stationary if:

- the mean $\mu(t) = E[x(t)]$ is a constant, it does not depend on t;

- the autocorrelation function $R(t, t+\tau) = E[x(t)x(t+\tau)]$ depends only on the time difference (τ): $R(t, t+\tau) = R(\tau)$

The stationarity of a process implies the wide-sense stationarity, however the converse is not true.

A process is said to be ergodic if all orders os statistical and time averages are interchangeable. For these processes the mean, autocorrelation and other statistics can be computed by using any sample function of the process. That is, for the mean and autocorrelation:

$$\mu_{x(t)} = E[x(t)] = \int_{-\infty}^{\infty} a f_{x(t)}(a;t)da = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t;\rho)dt$$

$$R(\tau) = E[x(t)x(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1 a_2 f_{x(t)}(a_1, a_2; t, t+\tau)da_1 da_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t;\rho)x(t+\tau;\rho)dt$$

where $x(t;\rho)$ is a process sample function.

What has been said for real continuous random processes can be easily extended to complex and discrete processes.

In particular for a complex continuous process the autocorrelation function is defined as

$$R(t, t+\tau) = E[x(t)^* x(t+\tau)]$$

where $x(t)^*$ indicates the process that is the complex conjugate of $x(t)$ (each sample function of $x(t)^*$ is the complex conjugate of the correspondent sample function of $x(t)$).

And for a complex discrete process the autocorrelation function becomes:

$$R(n, n+k) = E[x(n)^* x(n+k)]$$

For a complex process that is stationary al least in the wide-sense it is possible to define the statistical power as:

$$P_{x(t)} = E[|x(t)|^2] = E[x(t)^* x(t)] = R_{x(t)}(0)$$

The extension of the definition for a complex discrete wide sense stationary process is straightforward.

The principal properties of the autocorrelation function can summarized as:

$$R_{x(t)}(0) = P_{x(t)} \geq 0$$

$$|R_{x(t)}(\tau)| \leq R_{x(t)}(0)$$

$$R_{x(t)}(\tau) = R^*_{x(t)}(-\tau)$$

for discrete processes the same properties can be written as

$$R_{x(n)}(0) = P_{x(n)} \geq 0$$

$$|R_{x(n)}(k)| \leq R_{x(n)}(0)$$

$$R_{x(n)}(k) = R^*_{x(n)}(-k)$$

Due to the wide-sense stationarity of the considered processes the statistical power can be seen also defined (in the continuous case) as:

$$P_{x(t)}(\rho) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t;\rho)^* x(t;\rho) dt$$

$$P_{x(t)} = E[P_{x(t)}(\rho)]$$

where ρ is used to select a specific sample function of the process.

Therefore the statistical power of a process can be interpreted as a statistical mean of the deterministic power associated to each sample function.

In several situations it could be useful to know (in a statistical way) what fraction of the total statistical power of a process is associated to a specific range of frequencies.

5 Random Processes and LTI systems

A Linear Time Invariant (LTI) system, with one input and one output, is characterized by its impulse response ($h(t)$ or $h(n)$) that is, in the more general case, a complex function. Each sample function of a random process $x(t)$ applied to a LTI system generates an output signal that can be seen as a sample function of another random process $y(t)$.

In the case that the input process $x(t)$ is WSS also the output process $y(t)$ will be WSS. The following expressions hold for the continuous case:

$$\mu_{y(t)} = \mu_{x(t)} \cdot \int_{-\infty}^{\infty} h(t) dt = \mu_{x(t)} \cdot H(0) \qquad H(f) = \mathcal{F}(h(t))$$

$$R_{y(t)}(\tau) = R_{x(t)}(\tau) * h(\tau) * h^*(-\tau) = R_{x(t)}(\tau) * \mathcal{R}_{h(t)}(\tau)$$

where $H(f) = \mathcal{F}(h(t))$ is the Fourier transform of the impulse response and $\mathcal{R}_{h(t)}(\tau) = \int_{-\infty}^{\infty} h^*(\alpha)h(\tau+\alpha)d\alpha$ is the deterministic autocorrelation of the impulse response.

Analogous considerations can be made for discrete WSS processes.

6 Power Spectral Density for Continuous Random Processes

Let $x(t)$ be a continuous wide-sense stationary random process with autocorrelation function:

$$R_{x(t)}(\tau) = E[x^*(t)x(t+\tau)]$$

for each sample function of the process $x(t;\rho)$ we can define the truncated Fourier transform:

$$X_T(f) \equiv \int_{-T/2}^{T/2} x(t) e^{-j2\pi f t} dt$$

The corresponding truncated power spectral density is $\frac{1}{T}|X_T(f)|^2$. Since $x(t)$ is a random process, for each f, $\frac{1}{T}|X_T(f)|^2$ is a random variable. Let us denote its expectation by

$$S_{x(t),T}(f) \equiv E[\frac{1}{T}|X_T(f)|^2]$$

and a natural definition of the power spectral density of the process is therefore

$$S_{x(t)}(f) = \lim_{T\to\infty} S_{x(t),T}(f)$$

The Wiener-Khintchine asserts that the limit indicated in the previous formula exists for all f and its value is

$$S_{x(t)}(f) = \int_{-\infty}^{\infty} R_{x(t)}(\tau)e^{-j2\pi f\tau}d\tau$$

The unique required condition is that the Fourier Transform of the autocorrelation function exists.

Proof:

$$\begin{aligned}
E\left[|X_T(f)|^2\right] &= E\left[\left|\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt\right|^2\right] \\
&= E\left[\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} x(t)x(\tau)^* e^{-j2\pi(t-\tau)}dtd\tau\right] \\
&= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} E\left[x(t)x(\tau)^*\right]e^{-j2\pi(t-\tau)}dtd\tau \\
&= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_{x(t)}(t-\tau)e^{-j2\pi(t-\tau)}dtd\tau
\end{aligned}$$

We have to solve and integral of the type:

$$\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} f(t-\tau)dtd\tau$$

with the change of variable $\alpha = t - \tau$ the integral becomes:

$$\int_{-T/2}^{T/2}\int_{-T/2-\tau}^{T/2-\tau} f(\alpha)d\alpha d\tau$$

The domain over which the integral is evaluated is reported in figure 1.

Following the previous expression the integral is made along vertical strips (along which $f(\alpha)$ is not a constant). It is more useful to revert the order of the two integral (in this case the integration is made along horizontal strips, for which $f(\alpha)$ is constant). Therefore it is

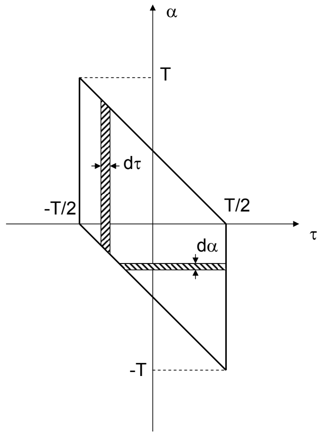

Figure 1: Integration domain of $\int_{-T/2}^{T/2}\int_{-T/2-\tau}^{T/2-\tau} f(\alpha)d\alpha d\tau$

possible to write:

$$\int_{-T/2}^{T/2}\int_{-T/2-\tau}^{T/2-\tau} f(\alpha)d\alpha d\tau =$$
$$= \int_{-T}^{0}\int_{-T/2-\alpha}^{T/2} f(\alpha)d\tau d\alpha + \int_{0}^{T}\int_{-T/2}^{T/2-\alpha} f(\alpha)d\tau d\alpha =$$
$$= \int_{-T}^{0} f(\alpha)\int_{-T/2-\alpha}^{T/2} d\tau d\alpha + \int_{0}^{T} f(\alpha)\int_{-T/2}^{T/2-\alpha} d\tau d\alpha =$$
$$= \int_{-T}^{0}(T+\alpha)f(\alpha)d\alpha + \int_{0}^{T}(T-\alpha)f(\alpha)d\alpha =$$
$$= \int_{-T}^{T}(T-|\alpha|)f(\alpha)d\alpha$$

This result is obvious looking at figure 1 and remembering that along horizontal strips $f(\alpha) = cost$.
Therefore

$$\begin{aligned} E\left[|X_T(f)|^2\right] &= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_{x(t)}(t-\tau)e^{-j2\pi(t-\tau)}dtd\tau \\ &= \int_{-T}^{T}(T-|\alpha|)R_{x(t)}(\alpha)e^{-j2\pi f\tau}d\alpha \end{aligned}$$

normally the independent variable of the autocorrelation function of a WSS process is τ, and so changing α in τ:

$$E\left[\frac{1}{T}|X_T(f)|^2\right] = \int_{-T}^{T}(1-\frac{|\tau|}{T})R_{x(t)}(\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} R_{x(t),T}(\tau)e^{-j2\pi f\tau}d\tau$$

where we have defined

$$R_{x(t),T}(\tau) = \begin{cases} (1 - \frac{|\tau|}{T})R_{x(t)} & |\tau| < T \\ 0 & |\tau| \geq T \end{cases}$$

From an intuitive point of view considering the previous formula it is therefore easy to accept that

$$S_{x(t)}(f) = \lim_{T\to\infty} S_{x(t),T}(f) = \lim_{T\to\infty} E[\frac{1}{T}|X_T(f)|^2] =$$
$$\lim_{T\to\infty} \int_{-T}^{T} (1 - \frac{|\tau|}{T})R_{x(t)}(\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} R_{x(\tau)}e^{-j2\pi f\tau}d\tau$$

in fact for $T \to \infty$ the term $(1 - \frac{|\tau|}{T}) \approx 1$ for each significant value of τ.

Formally we can use the Lebesgue-dominated-convergence theorem to prove the previous expression. The theorem says that: let (f_n) be a sequence of complex-valued measurable functions which converges almost everywhere to a complex-valued measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and, in fact,

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

If we take the f_n's as the complex-valued functions $R_{x(t),T}(\tau)e^{-j2\pi f\tau}$, then the corresponding limit is $f = R_{x(t)}(\tau)e^{-j2\pi f\tau}$. Moreover, we can take the integrable function to be $g = |R_{x(t)}(\tau)|$. Then from the definition of $R_{x(t),T}(\tau)$ it is straightforward to see that

$$|R_{x(t),T}(\tau)e^{-j2\pi f\tau}| \leq |R_{x(t)}(\tau)|$$

and the Lebesgue-dominated-convergence theorem applies. Therefore

$$\begin{aligned} S_{x(t)}(f) &= \lim_{T\to\infty} E[\frac{1}{T}|X_T(f)|^2] = \\ \lim_{T\to\infty} \int_{-T}^{T} (1 - \frac{|\tau|}{T})R_{x(t)}(\tau)e^{-j2\pi f\tau}d\tau &= \int_{-\infty}^{\infty} R_{x(\tau)}e^{-j2\pi f\tau}d\tau \end{aligned}$$

7 Power Spectral Density for Discrete Random Processes

Let $x(n)$ be a discrete wide-sense stationary random process with autocorrelation function:

$$R_{x(n)}(k) = E[x^*(n)x(n+k)]$$

for each sample function of the process $x(n;\rho)$ we can define the truncated Fourier transform:

$$X_T(e^{j2\pi f}) \equiv \sum_{n=-N}^{N} x(n)e^{-j2\pi fn}$$

The corresponding truncated power spectral density is $\frac{1}{2N+1}|X_N(f)|^2$. Since $x(n)$ is a random process, for each $e^{j2\pi f}$, $\frac{1}{2N+1}|X_T(e^{j2\pi f})|^2$ is a random variable. Let us denote its expectation by

$$S_{x(t),N}(e^{j2\pi f}) \equiv E\left[\frac{1}{2N+1}|X_N(e^{j2\pi f}|^2\right]$$

and also in this case a natural definition of the power spectral density of the process is therefore

$$S_{x(n)}(e^{j2\pi f}) = \lim_{N\to\infty} S_{x(n),N}(e^{j2\pi f}$$

The Wiener-Khintchine asserts that the limit indicated in the previous formula exists for all $e^{j2\pi f}$ and its value, for a discrete process, is

$$S_{x(n)}(e^{j2\pi f}) = \sum_{k=-\infty}^{\infty} R_{x(n)}(k)e^{-j2\pi fk}$$

The unique required condition is that the Fourier Transform of the autocorrelation function exists.

Proof

$$
\begin{aligned}
S_{x(n)}(e^{j2\pi f}) &= \lim_{N\to\infty} E\left[\frac{1}{2N+1}|X_N(e^{j2\pi f}|^2\right] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} E\left[\left(\sum_{n=-N}^{N} x(n)e^{-j2\pi n}\right)\left(\sum_{m=-N}^{N} x(m)e^{-j2\pi m}\right)^*\right] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} E\left[\sum_{n=-N}^{N}\sum_{m=-N}^{N} x(n)x(m)^* e^{-j2\pi(n-m)}\right] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\sum_{m=-N}^{N} E[x(n)x(m)^*]e^{-j2\pi(n-m)} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\sum_{m=-N}^{N} R_x(n-m)e^{-j2\pi(n-m)} \\
&= \lim_{N\to\infty}\lim_{M\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\sum_{m=-M}^{M} R_x(n-m)e^{-j2\pi(n-m)} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\lim_{M\to\infty}\sum_{m=-M}^{M} R_x(n-m)e^{-j2\pi(n-m)} \overset{k=n-m}{\rightarrow} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\left(\sum_{k=-\infty}^{\infty} R_x(k)e^{-j2\pi k}\right) \\
&= \left(\sum_{k=-\infty}^{\infty} R_x(k)e^{-j2\pi k}\right)\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 \\
&= \left(\sum_{k=-\infty}^{\infty} R_x(k)e^{-j2\pi k}\right)
\end{aligned}
$$

In coherence with the definition of the Discrete Time Fourier Transform:

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn}$$

$$x(n) = \int_{-1/2}^{1/2} X(f)e^{j2\pi fm}df$$

it is

$$P_{x(n)} = R_{x(n)}(0) = \int_{-1/2}^{1/2} S_{x(n)}(e^{j2\pi f})df$$

In all the previous discussions on the PSD associated to discrete processes it has been assumed that the sampling rate of $x(n)$ was unitary. In the case that the sampling time is T (and not 1) the previous relations hold considering to work with normalized time and frequency axes ($t_n = t/T$, $f_n = fT$).

On the basis of what shown in the section 5 it is immediate to understand how is modified the PSD of a WSS process (continuous or discrete) when it is applied as input to a LTI system:

$$\begin{aligned} S_{output}(f) &= S_{input}(f) \cdot |H(f)|^2 \\ S_{output}(e^{j2\pi f}) &= S_{input}(e^{j2\pi f}) \cdot |H(e^{j2\pi f})|^2 \end{aligned}$$

this result is exactly the same obtained for deterministic signals.