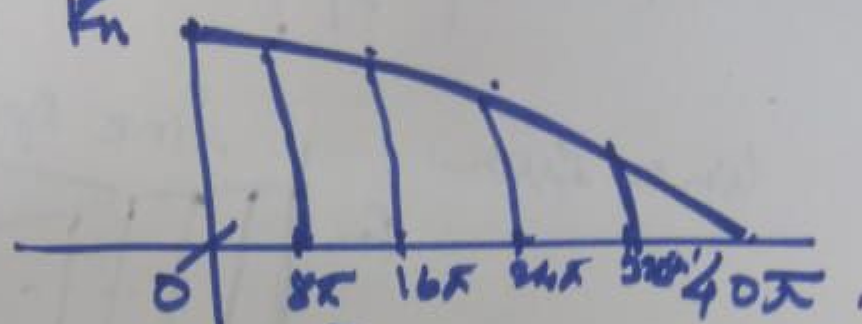
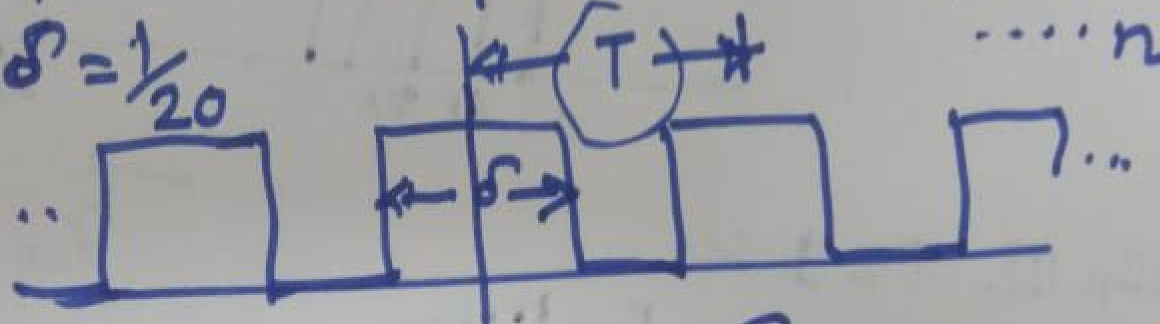
CTFS Representation for Pulse-Train Signal

Lecture 28



Let $\delta = \frac{1}{20}$



$\omega_n = k\cdot\pi$

$\omega_n \rightarrow n\omega_0$

$n\omega_0 \cdot \delta/2 = k\pi$

$\delta/2 \rightarrow 1/40$

$\omega_n = 40\pi k$

$\omega_n = n\cdot\omega_0$

$\omega_n = n \times \left(\frac{2\pi}{T}\right)$

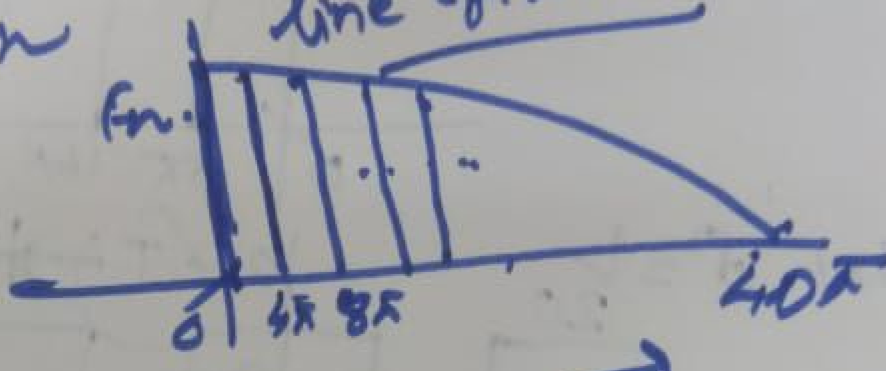
Case I) $T = 1/4$ sec.

$\omega_n = n \times \frac{2\pi}{1/4} = 8\pi n$

Case II) $T = \frac{1}{2}$ sec.

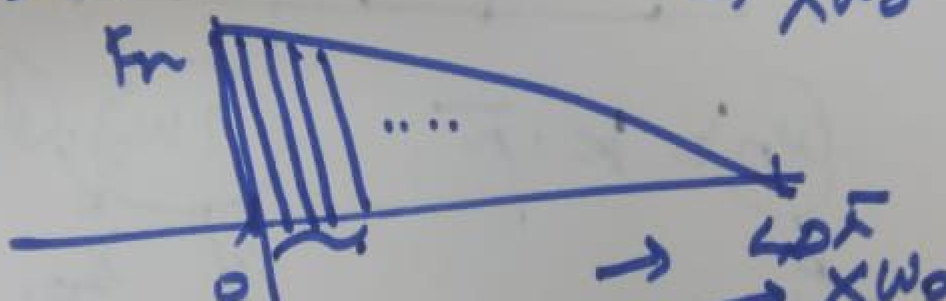
$$\omega_n = n\left(\frac{2\pi}{T}\right) = n\left(\frac{2\pi}{1/2}\right)$$

$$\omega_n = 4\pi n$$



Case III $T = 1$ sec.

$$\omega_n = 2\pi n$$



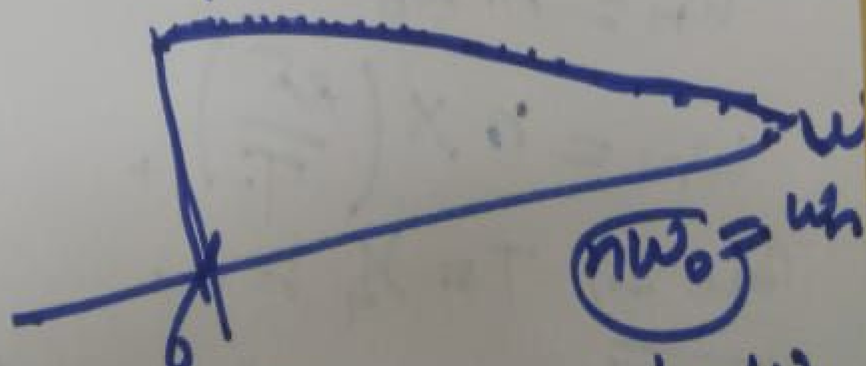
spectrum.

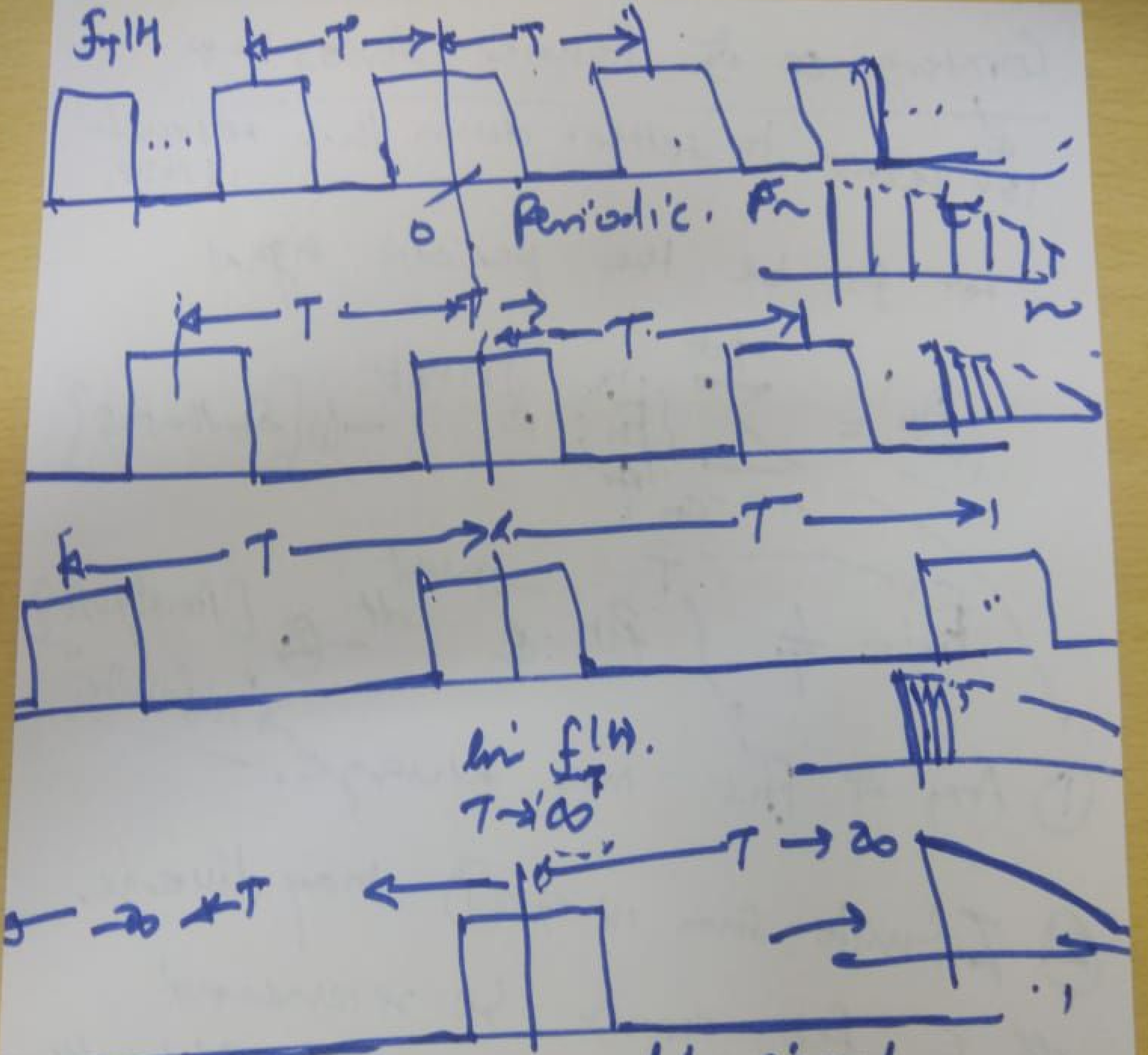
continuous

$\lim_{T\to\infty} f_T(t) =$

$T = 2$ sec

$= 4$ sec





aperiodic signal

$\lim_{T\to\infty} f_T(t) = f(t) \Rightarrow$ line spectrum $\Rightarrow$ Continuous Spectrum

$f_T(t)$ ↓ Periodic signal

$f(t)$ ↓ aperiodic

line spectrum ↓ Fourier series ↙ chapter 3

Continuous Spectrum ↓ Fourier Transform ↙ Chapter 4.

3

Convergence of Fourier Series: →

150 years to settle down this research issue.

Let $f(t)$ be the periodic signal

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{jn\omega_0 t} \quad -(1) \; [\text{Synthesis}]$$

$$F_n = \frac{1}{T}\int_0^T f(t) \cdot e^{-jn\omega_0 t} \, dt \quad -(2) \; [\text{Analysis}].$$

(1) Any of F_ns may diverge. → "infinite"

(2) Infinite sum in eqn (1) may diverge.

If so, $f(t)$ cannot be recovered from its Fourier series coefficients

$$\Rightarrow \quad \begin{tikzcd} f(t) \arrow[r] & F_n \arrow[l, bend left, "\times"] \end{tikzcd}$$

Cont

(3) $\{F_n\}_{n \in \mathbb{Z}}$ $\Longrightarrow$ There are infinite number of Fourier series coefficients

However, in computer, we cannot define infinite-dimensional array.

$\therefore$ We are forced to truncate Fourier series coefficients

$$\tilde{f}_N(t) = \sum_{n=-N}^{+N} F_n \cdot e^{jn\omega_0 t}$$

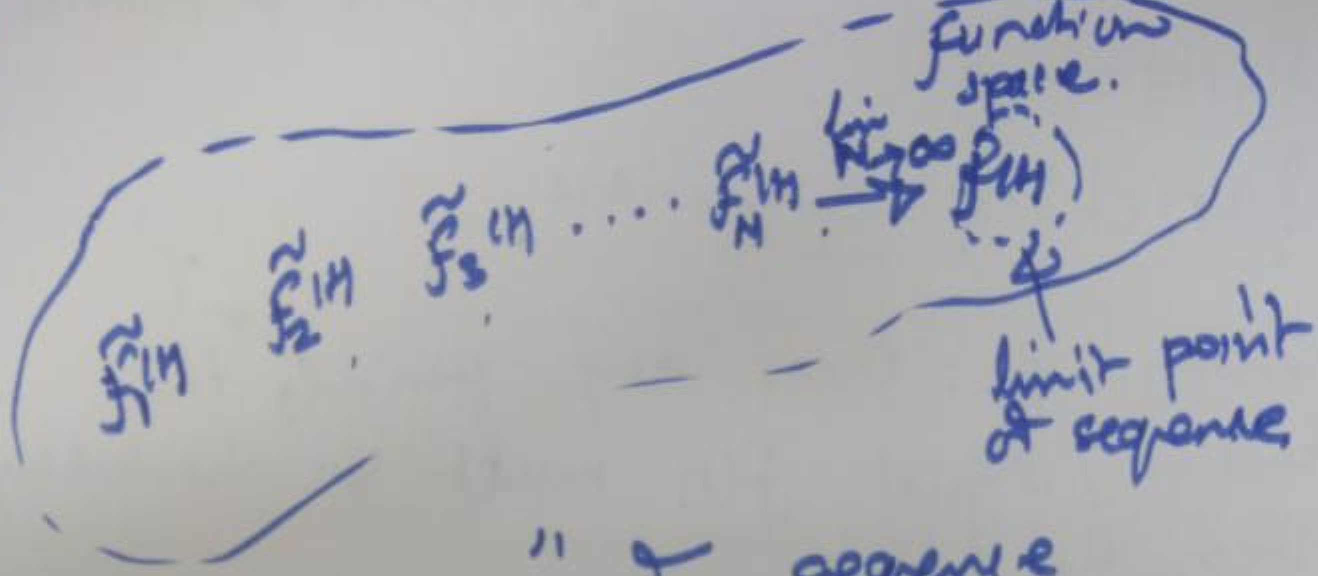
Approximated signal.

$$f(t) - \tilde{f}_N(t) = e_N(t)$$

Approximation error.

Objective: To minimize the error, $e_N(t)$.

$$\lim_{N \to \infty} \tilde{f}_N(t) \to f(t)$$



"Convergence" of sequence

Let $x_{1n} = 1 + \frac{1}{n}$

$x_{2n} = 1 + \frac{1}{n^2}$

$x_{3n} = 1 + \frac{1}{n!}$

$\Big\} \rightarrow \lim_{n\to\infty} x_n = 1$

Key difference: The sequence, x_{3n} reaches much faster than x_{1n} and x_{2n} to the limit point, i.e., 1.

Thus, convergence in sequence x_{3n} is relatively much faster.

6

Conditions for the Existence of Fourier Series :→

Dirichlet Conditions

Condition 1 : Over any period 'T' the function / signal f(t) must be absolutely integrable i.e.,

$$\int_T |f(t)|\, dt < +\infty .$$

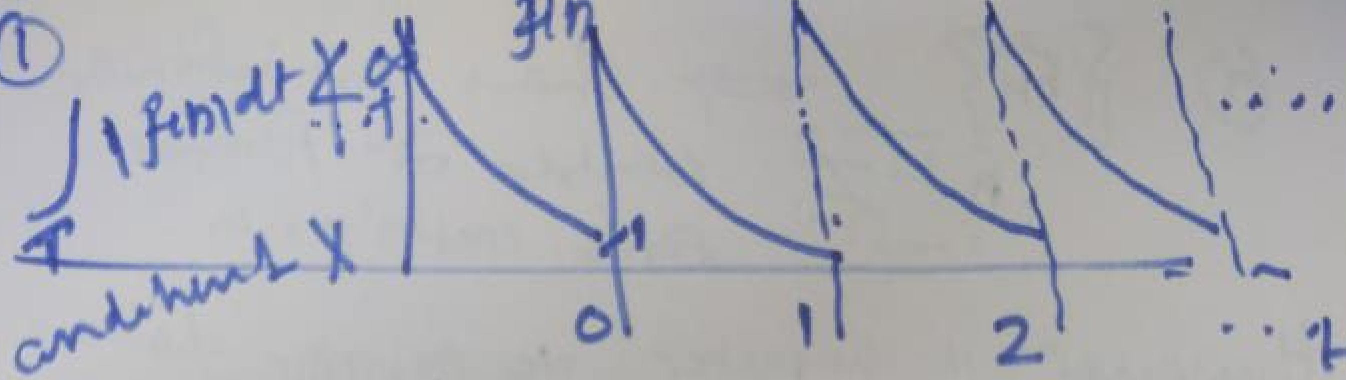
$$F_n = \frac{1}{T}\int_0^T f(t) \cdot e^{-jn\omega_0 t}\, dt .$$

$$|F_n| = \left|\frac{1}{T}\int_T f(t) \cdot e^{-jn\omega_0 t}\, dt\right|$$

$$\therefore |F_n| \le \frac{1}{T}\int_T |f(t) \cdot e^{-jn\omega_0 t}|\, dt .$$

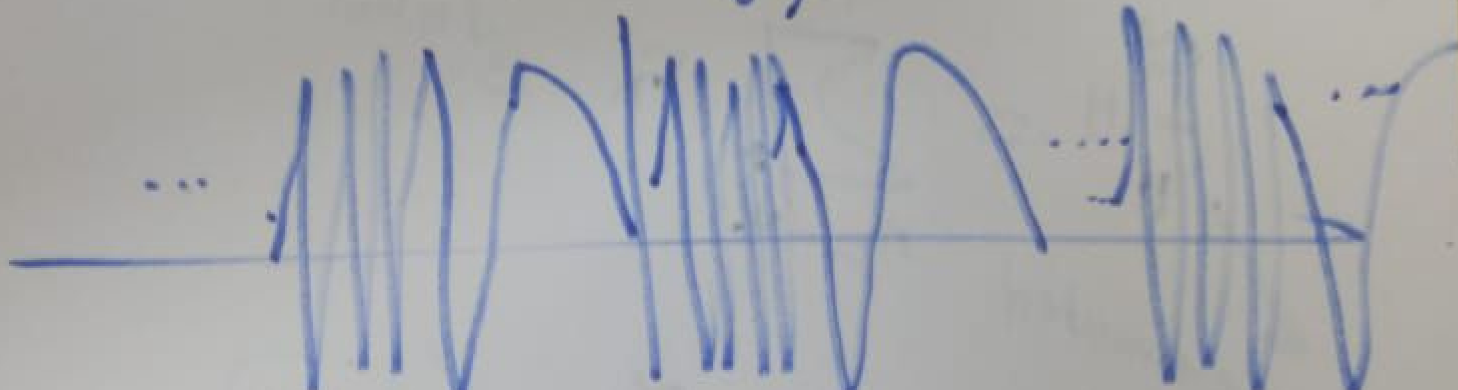
$$|F_n| \le \left(\frac{1}{T}\right) \cdot \int_T |f(t)|\, dt \quad (*)$$

①



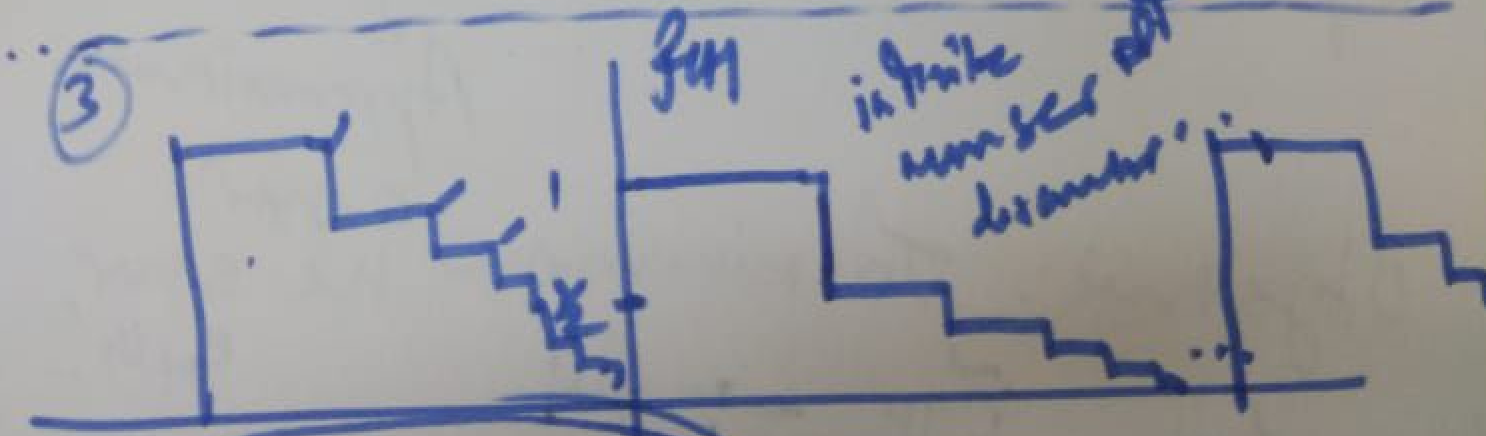
$f(t) = 1/t \;,\; 0 < t \leq 1.$

② $f(t) = \sin\left(\frac{2\pi}{t}\right) \;;\; 0 < t \leq 1$



Condition 2 ✗

③



Condition 3 ✗

$\therefore$ If $\int_T |f(t)|\, dt < +\infty$

$\Rightarrow |F_n| < +\infty$; $\forall\, n \in \mathbb{Z}$

$\therefore$ Condition 1 ensures that each of the fourier series coefficient is finite

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Condition 2: In any finite interval of time, the function / signal f(t) is of bounded variation (BV), i.e., there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3 — In finite interval of time there are only a finite number of discontinuities. Furthermore each of these discontinuities is finite.

Pulse train signal

Since there are finite number of discontinuities in given time period 'T', Fourier series exists.

Condition 3 is satisfied

Michelson's Effect :→

1892 → American Physicist.

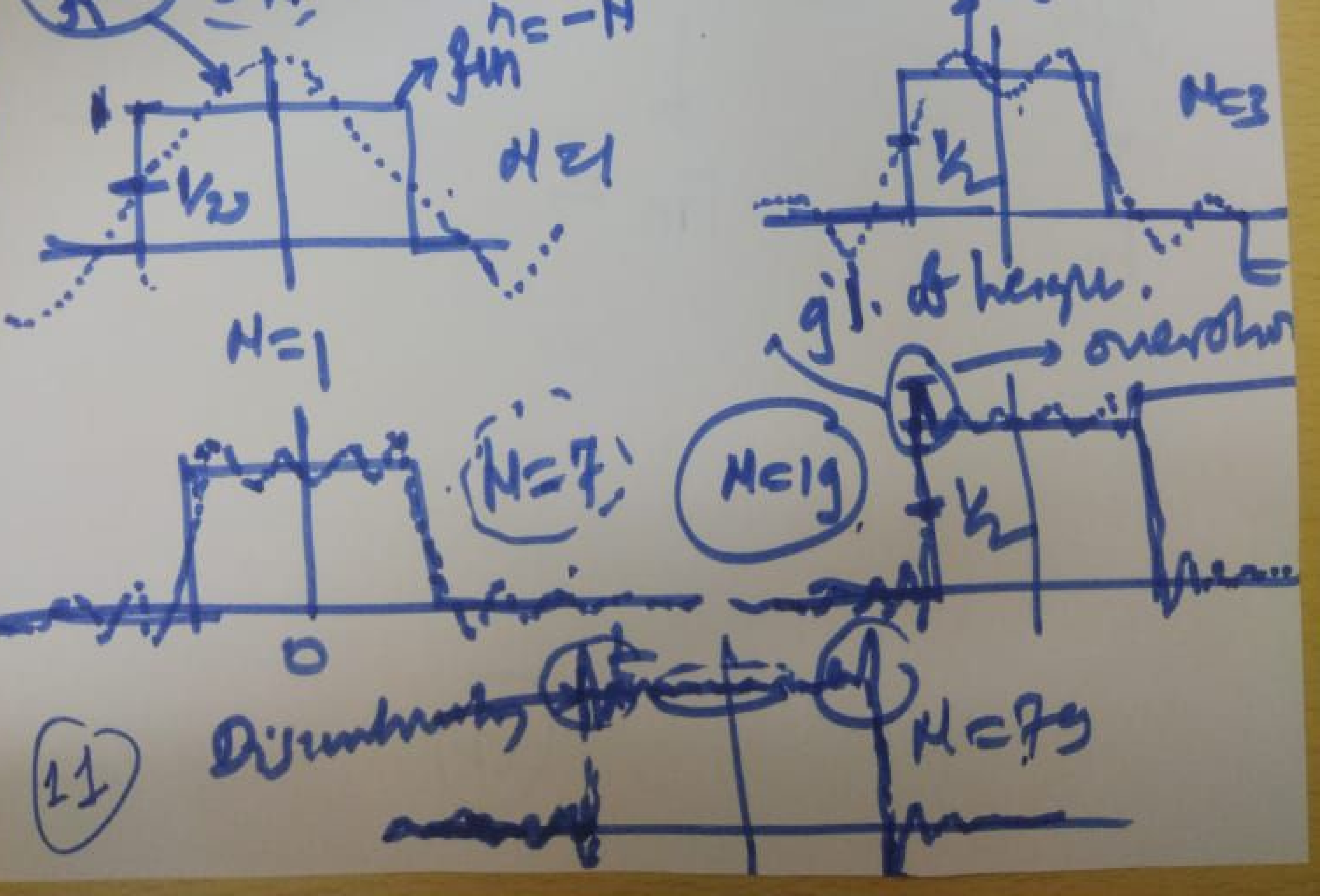
$f(t)$ → Periodic signal

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \cdot e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_T f(t)\, e^{-jn\omega_0 t}\, dt.$$

$$f_N(t) = \sum_{n=-N}^{+N} F_n\, e^{jn\omega_0 t}$$

$\tilde{f}_1(t)$



J. Gibbs

Gibbs phenomenon → Design of FIR filters

$$\lim_{N \to \infty} \tilde{f}_N(t) \to f(t)$$

$\boxed{\text{Property 4}}$ Time Scaling:

$$f(t) \longrightarrow f_1(t) = f(at)$$

$$\downarrow \qquad\qquad \downarrow$$

$F_n \qquad\qquad F_{1n} =$

Properties of CTFS

CTFS [Continuous-time Fourier Series]

f(t) and g(t) two periodic signal

$f(t) \xleftrightarrow{FS} F_n$ | $g(t) \xleftrightarrow{FS} G_n$

$f(t)$ → signal (periodic)

F_n → Fourier series coefficients.

① Linearity :→

$$z(t) = A \cdot f(t) + B \cdot g(t) \xleftrightarrow{FS} A \cdot F_n + B \cdot G_n$$

$$Z_n = \frac{1}{T} \int z(t) \cdot e^{-jn\omega_0 t} dt$$

$$A \underbrace{\left[\frac{1}{T} \int_T f(t) e^{-jn\omega_0 t} dt \right]}_{F_n} + B \underbrace{\left[\frac{1}{T} \int_T g(t) e^{-jn\omega_0 t} dt \right]}_{G_n}$$

[2] Time Shifting :-

$f_1(t) = f(t-t_0)$

$\therefore F_{1n} = \frac{1}{T}\int_T f_1(t)\cdot e^{-jn\omega_0 t}\,dt$

$= \frac{1}{T}\int_T f(t-t_0)\cdot e^{-jn\omega_0 t}\,dt$

$t - t_0 = \tau \Rightarrow t = \tau + t_0$

$= \underbrace{\left[\frac{1}{T}\int_T f(\tau)\, e^{-jn\omega_0 \tau}\,d\tau\right]}_{F_n} \times e^{-jn\omega_0 t_0}$

$\therefore \boxed{f(t-t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0}\cdot F_n}$

$\because |e^{-jn\omega_0 t_0}\cdot F_n| = |F_n|$

(14)

⇒ When a periodic signal is shifted in time, the magnitudes of its Fourier series coefficients remain unaltered. rather phase spectrum is changed.

Property 3 Time Reversal

$$f(t) \Longrightarrow f_1(t) = f(-t)$$

$$f(t) \downarrow F_n \qquad f_1(t) \downarrow F_{1n} = F_{-n}$$

$$f(-t) \xleftrightarrow{FS} F_{-n}$$

Inference: Time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients.

If $f(t)$ = even.

$$\Rightarrow f(t) = f(-t)$$

$$\downarrow \qquad \downarrow$$

$$F_n = F_{-n}$$

$\Rightarrow$

$\Rightarrow$ If a signal is even, then its Fourier series coefficient are also "even"

- - - - - - - - - - - -

If $f(t)$ = odd

$$\Rightarrow f(t) = -f(-t)$$

$$F_n = -F_{-n}$$

$\Rightarrow$ Fourier series coefficients are also odd.

(16)

$\boxed{\text{Property 4}}$ Time Scaling.

$$\begin{array}{ccc} f(t) & \longrightarrow & f_1(t) = f(\alpha t) \\ \downarrow & & \downarrow \\ F_n & & F_n = ? \end{array}$$

Property 5 Multiplication

$$f(t) \cdot g(t) \overset{FS}{\longleftrightarrow} ?$$

Property 6 Conjugation and Symmetry

$$f(t) \overset{FS}{\longleftrightarrow} F_n \Rightarrow f^*(t) \overset{FS}{\longleftrightarrow} ?$$

Property 7| Parseval's Relation

For a continuous-time periodic signal $f(t)$,

$$\underbrace{\frac{1}{T}\int_T |f(t)|^2 dt}_{\text{Average Power in time domain}} = \underbrace{\sum_{n=-\infty}^{+\infty} |F_n|^2}_{\text{Average power in frequency.}}$$

⇒ The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components;

Fourier Series Representation for Discrete-Time Periodic Signals.

$f(t) = f(t+T)$ $\qquad$ $f(n) = f(n+N)$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \longrightarrow f(n) = \sum_{\langle N \rangle} F_k e^{jk\omega_0 n}$$

$$F_n = \frac{1}{T} \int_T f(t) e^{-jn\omega_0 t} dt \longrightarrow F_k = \frac{1}{N} \sum_{n=\langle N \rangle} f(n) e^{-jk\omega_0 n} \ ??$$

$$e^{j\omega t} \longrightarrow e^{j\omega n}$$

$$F_k = \frac{\langle f(n), e^{jk\omega_0 n} \rangle}{\langle e^{jk\omega_0 n}, e^{jk\omega_0 n} \rangle}$$

N