LECTURE 19

- **Recap** :- Oscillatory motion.
  - Simple harmonic motion with/without small θ-approximation.
  - Two-dimensional harmonic oscillator :- eqn. of path.

- Damped oscillations.

$$m\ddot{x} + b\dot{x} + kx = 0 \quad \rightarrow \text{resistive force} \propto \text{velocity}.$$

$$\Rightarrow \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \rightarrow \text{homogeneous linear ODE.}$$

Trial sol$\underline{\text{n}}$:- $\quad x(t) = e^{rt}$

$$\left(r^2 + 2\beta r + \omega_0^2\right)\underline{e^{rt}} = 0$$

$$r^2 + 2\beta r + \omega_0^2 = 0$$

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$$

$$r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = A e^{\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t} + B e^{\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t}.$$

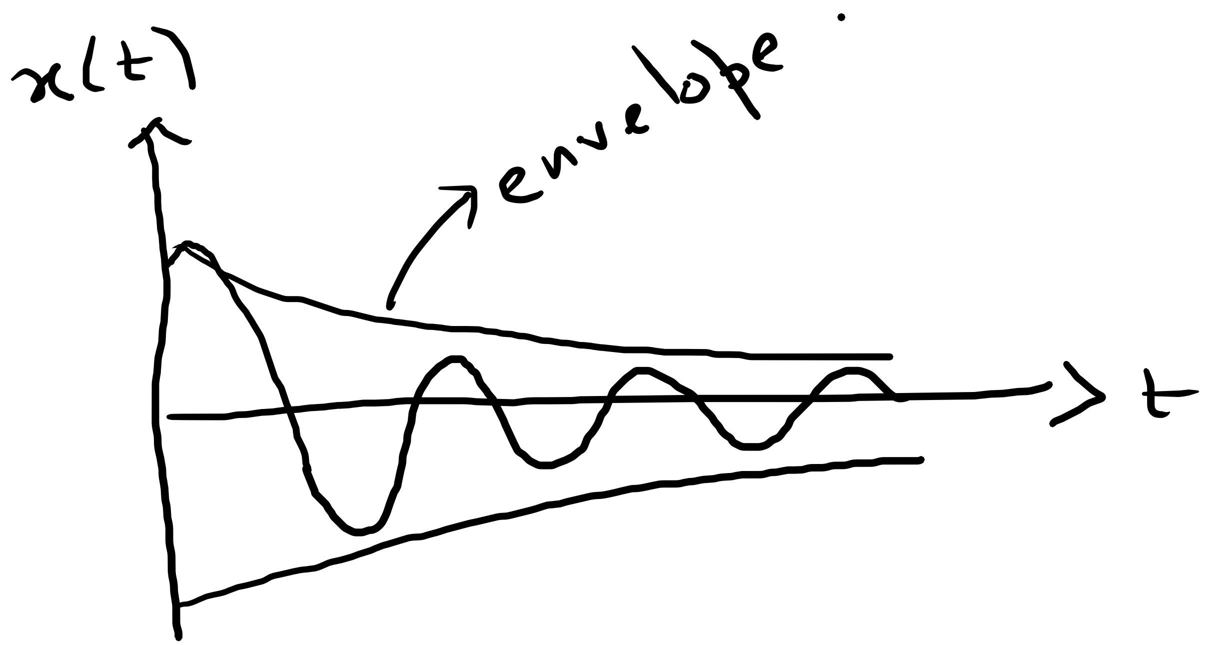
So far, nothing has been assumed regarding relative magnitudes of β and ω_0.

— Case 1 :- $\boxed{\beta < \omega_0}$ Underdamped oscillation.

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1$$

$$x(t) = e^{-\beta t}\left(A e^{i\omega_1 t} + B e^{-i\omega_1 t}\right)$$

$$= A\, e^{-\beta t} \cos(\omega_1 t - \delta)$$



— Oscillations $\left(\cos(\omega_1 t - \delta)\right)$ decay over time, but exhibit periodic behaviour.

$$x(t) = C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}.$$

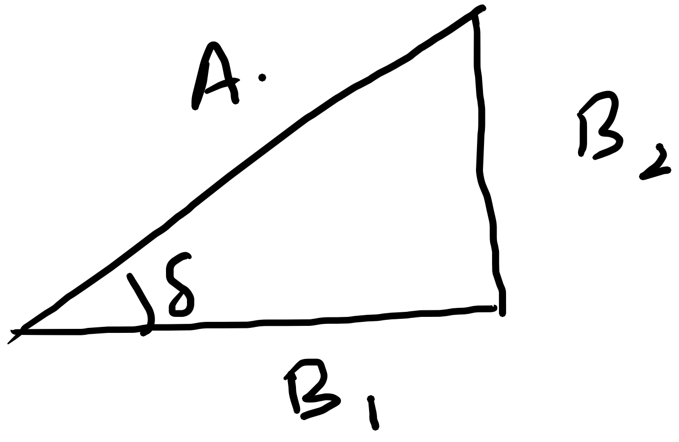
$$= C_1(\cos\omega_1 t + i\sin\omega_1 t) + C_2(\cos\omega_1 t - i\sin\omega_1 t)$$

$$= B_1 \cos\omega t + B_2 \sin\omega t.$$

$$= A\left[\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right]$$

$$= A\left[\cos\delta\cos\omega t + \sin\delta\sin\omega t\right]$$

$$= A\cos(\omega t - \delta)$$

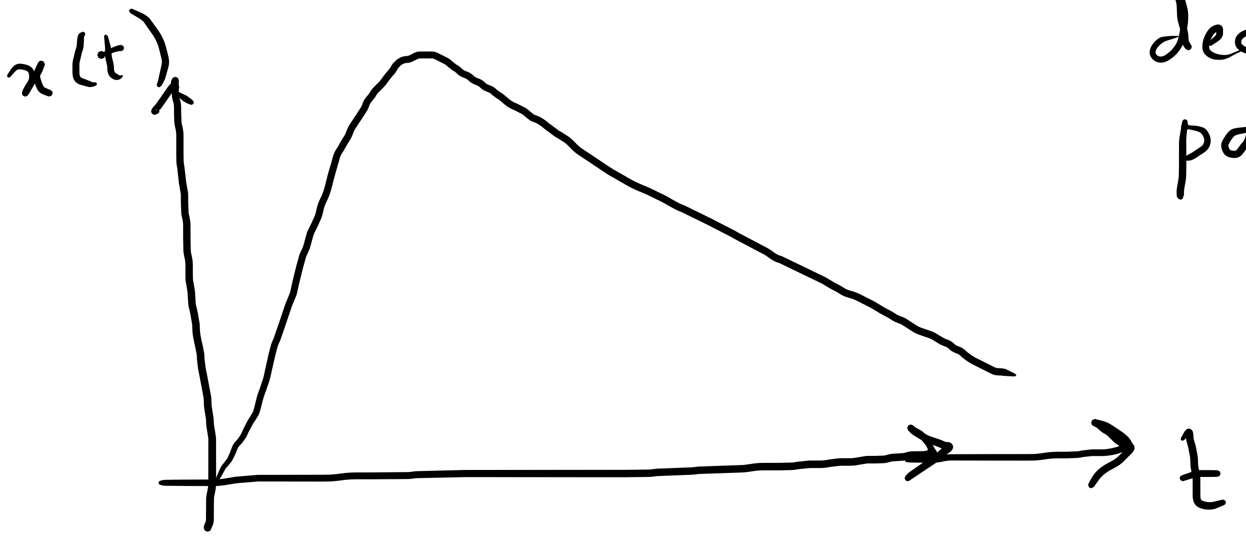


— $\boxed{\beta > \omega_0}$ Overdamped oscillations.

$$x(t) = e^{-\beta t}\left(A e^{\sqrt{\beta^2-\omega_0^2}\,t} + B e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)$$

$$= A e^{\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t} + B e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t}.$$

$$\underline{\left(-\beta+\sqrt{\beta^2-\omega_0^2}\right)t} \qquad \left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t.$$



decay parameter $= \left(\beta - \sqrt{\beta^2-\omega_0^2}\right)$

$$= \beta - \beta\sqrt{1-\frac{\omega_0^2}{\beta^2}}$$

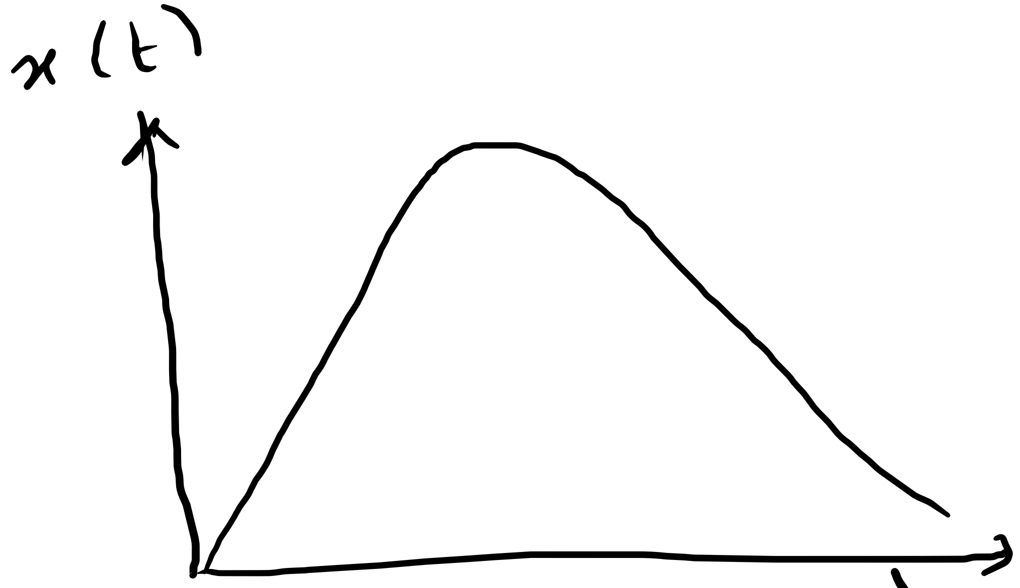
$$\approx \frac{\omega_0^2}{2\beta}.$$

— $\boxed{\beta = \omega_0}$ — Critical damping.

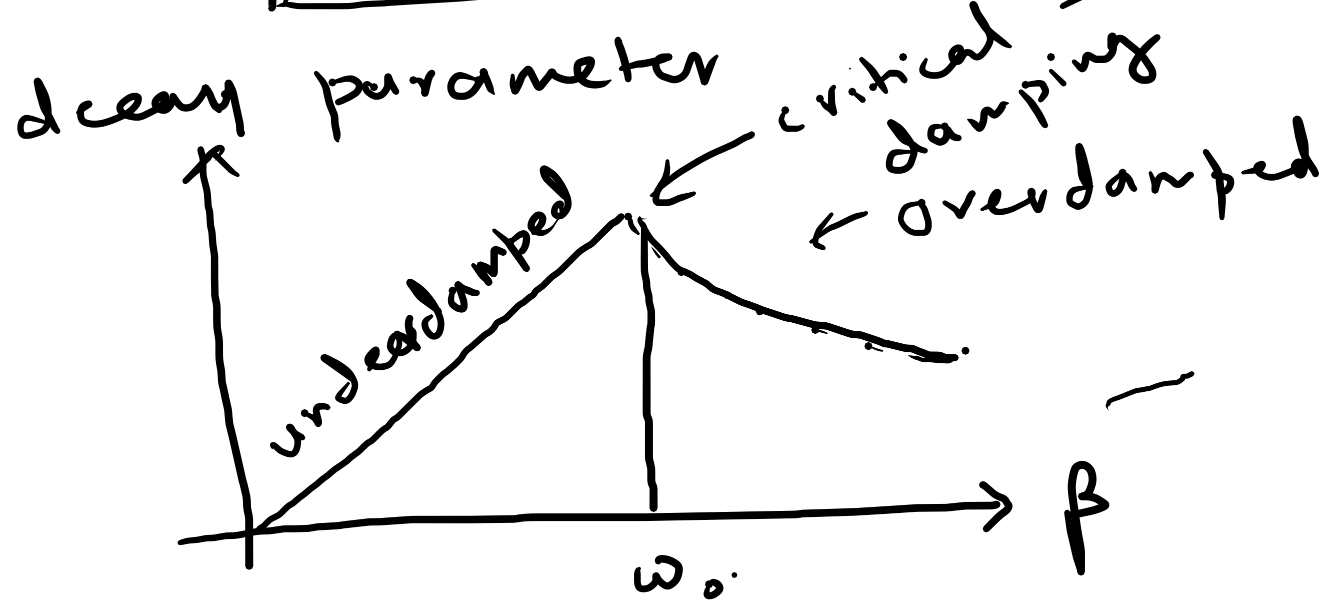
$x_1(t) = A e^{-\beta t}$. → only one solⁿ survives.

$x_2(t) = t e^{-\beta t}$.

$x(t) = (A + Bt) e^{-\beta t}$.



$A \, e^{-(\text{decay parameter}) t}$.



$A e^{-\beta t}$ → underdamped

$A e^{-(\beta - \sqrt{\beta^2 - \omega_0})t}$ → overdamped.

DRIVEN DAMPED OSCILLATOR

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t).$$

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

$D x_p = f$

$\hookrightarrow$ particular solution.

$D x_h = 0 \Rightarrow x_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

$$\underbrace{D(x_h + x_p)}_{\text{general sol}^{\underline{n}}} = Dx_p + Dx_h = f + 0 = f.$$

general sol$^{\underline{n}}$.

Algorithm

1. Find x_p
2. Find x_h for $Dx_h = 0$ and find $x = x_p + x_h$.

$-\quad f(t) = f_0 \cos\omega t \,.$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t \,.$$

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

$$\mathrm{Re}\left(\ddot{z} + 2\beta\dot{z} + \omega_0^2 z\right) = f_0 \cos\omega t \,.$$

$\rightarrow$ Trial sol^n :- $\quad z(t) = C e^{i\omega t} \,.$

$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right) C e^{i\omega t} = f_0 e^{i\omega t} \,.$$

$$\Rightarrow \quad C = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega} \,.$$

$$C = \frac{f_0}{(\omega_0^2-\omega^2)+2i\beta\omega} = \frac{f_0\left[(\omega_0^2-\omega^2)-2i\beta\omega\right]}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} = A\,e^{-i\delta}$$

$$= A\,(\cos\delta - i\sin\delta)$$

$$A^2 = C^* C = \frac{f_0^{\,2}}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

$$\tan\delta = \left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$$

$$C = \frac{f_0(\omega_0^2 - \omega^2)}{(\qquad)} - i\,\frac{2 f_0 \beta \omega}{(\qquad)} = A\cos\delta - i A \sin\delta$$

$$A\cos\delta = \frac{f_0(\omega_0^2 - \omega^2)}{(\quad)} \qquad\qquad \tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$A\sin\delta = \frac{2 f_0 \beta \omega}{(\qquad)}$$