SC223 - Linear Algebra

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Lecture 23

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Summary: Lecture 22

- **Definition:** (Basis) Let V be a vector space. A subset $\beta \subset V$ is said to be a **Basis** of V if (1) $span(\beta) = V$, and (2) β is a set of linearly independent vectors.
- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.
- **Proposition 14:** For a FDVS, every LI set can be extended to a basis.
- **Proposition 15:** Every FDVS has a basis.

● **Proposition 16:** A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as
$v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n, a_i \in \mathbb{F}, i = 1, \ldots, n.$

" $\Rightarrow$ "

" $\Leftarrow$ " ① $\forall v \in V$, $v = a_1 u_1 + \cdots + a_n u_n$.

$\Rightarrow \text{span}(U) = V$.

② $\forall v \in V$, $\exists$ a unique LC :

$v = a_1 u_1 + \cdots + a_n u_n$.

Contradiction ① Assume U is LD.

② $\exists a_1, \ldots, a_n \in \mathbb{F}$, not all zeros s.t.

$a_1 u_1 + \cdots + a_n u_n = \theta$.

Since $0 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n = \theta$ & $\exists i$, $a_i \neq 0$. $\Rightarrow\Leftarrow$

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● **Proposition 17:** Any set of basis vectors of a FDVS contains the same number of elements.

Let $\beta_1, \beta_2 \subseteq V$ be two sets of basis vectors

β_1 is LI, β_2 spans V, $\therefore |\beta_1| \leq |\beta_2|$

β_1 spans V, β_2 is LI, $\therefore |\beta_2| \leq |\beta_1|$

$\Rightarrow |\beta_1| = |\beta_2|$

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● **Dimension of a Vector Space:** Let V be a FDVS. For any set of basis vectors β of V, we define the dimension of V as $dim(V) := |\beta|$.

Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.

$$A : \mathbb{R}^n \to \mathbb{R}^m.$$

$$n = dim(\mathbb{R}^n) = dim\ (\text{domain VS})$$

$$A \in \mathbb{R}^{4 \times 6}$$

$$\text{Any } x \in N(A) = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} - \\ - \\ - \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} - \\ - \\ - \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} - \\ - \\ - \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

Proof (Rank-Nullity Theorem). $A: \mathbb{R}^n \to \mathbb{R}^m$.

$$r + \dim(N(A)) = n.$$

$$N(A) \subseteq \mathbb{R}^n.$$

Let $\beta_{N(A)} = \{w_1, \dots, w_k\}$ be a basis of $N(A)$.

Extend $\beta_{N(A)}$ to $\beta \subset \mathbb{R}^n$ so that β is a basis of $\mathbb{R}^n$

$$\beta = \{\underline{w_1, \dots, w_k}, p_1, \dots, p_{n-k}\} \text{ is a basis of } \mathbb{R}^n$$

$$Aw_i = \vec{0} \quad i = 1, \dots, k.$$

$$Ap_i \neq \vec{0}.$$

$$\{Ap_1, \dots, Ap_{n-k}\} \Rightarrow LI.$$

$$\sum_{i=1}^{n-k} c_i A p_i = \vec{0} = A\left(\sum_{i=1}^{n-k} c_i p_i\right) = \vec{0}$$

$$\Rightarrow c_i = 0, \; i = 1, \dots, n-k.$$

$$\forall x \in \mathbb{R}^n, \quad x = \sum_{i=1}^{k} c_i w_i + \sum_{i=1}^{n-k} b_i p_i$$

$$\text{Any } y \in C(A), \quad y = Ax = \sum_{i=1}^{n-k} b_i A p_i$$

$$\therefore n - k = r = (\text{rank}(A))$$

$$\Rightarrow \boxed{n - \dim(N(A)) = r}$$

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$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$