SC223 - Linear Algebra

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Lecture 10

August 18, 2023

Sets associated with a matrix $A \in \mathbb{R}^{m \times n}$, $A : \mathbb{R}^n \to \mathbb{R}^m$

- **Column Space:**

$$C(A) = \{Ax \mid \forall x \in \mathbb{R}^n\}, C(A) \subseteq \mathbb{R}^m$$

- **Nullspace:**

$$N(A) := \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}_m\}, N(A) \subseteq \mathbb{R}^n$$

- **Rowspace:**

$$C(A^T) = \{A^T y \mid \forall y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

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● All the above sets $S \subseteq \mathbb{R}^k$, $k = m, n$ as appropriate, satisfy: (1) $\mathbf{0}_k \in S$, (2) $\forall p, q \in S, \forall k_1, k_2 \in \mathbb{R}, k_1 p + k_2 q \in S$.

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● **Row/Column Rank of a Matrix:** The number of linearly independent rows/columns of a matrix is called the Row/Column Rank.

● LD columns of $A \Leftrightarrow \exists z \neq 0$ such that $Az = \mathbf{0}_m$.

Row Space and Nullspace/Column Space and Left Nullspace

- Who lives in $C(A^T) \cap N(A)$? $\Rightarrow$ $C(A^T) \cap N(A) = \{\vec{0}_n\}$.

$C(A) \cap N(A^T) = \{\vec{0}_m\}$

Let $x \in C(A^T) \cap N(A)$.

$\Rightarrow x \in C(A^T)$

$\exists\, y_1 \in \mathbb{R}^m$ s.t $A^T y_1 = x \rightarrow (1)$

$\Rightarrow x \in N(A)$

$\Rightarrow Ax = \vec{0} \rightarrow (2)$

Substitute (1) in (2)

$$\Rightarrow \underbrace{(A(A^T y_1))}_{m\times 1} = \vec{0}_{m\times 1}$$

$$\Rightarrow \underbrace{y_1^T A A^T y_1}_{(A^T y_1)^T (A^T y_1) = 0} = y_1^T \vec{0} = 0$$

$\Rightarrow x^T x = 0 \Rightarrow x = \vec{0}_n$

Row rank $<, >, =$ Column Rank?

- Let $A \in \mathbb{R}^{m \times n} = \begin{bmatrix} | & \cdots & | \\ a_{*1} & \cdots & a_{*n} \\ | & \cdots & | \end{bmatrix}$

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Row rank $<, >, =$ Column Rank?

- Let $A \in \mathbb{R}^{m \times n} = \left[\begin{array}{ccc} | & \dots & | \\ a_{*1} & \dots & a_{*n} \\ | & \dots & | \end{array} \right]$
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- $$A = \left[\begin{array}{ccccc} | & & | & & | \\ a_{*1} & \dots & a_{*k} & \dots & a_{*n} \\ | & & | & & | \end{array} \right]$$

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$$= A_k \underbrace{\begin{bmatrix} I_k & b_{*,k+1} & \dots & b_{*,n} \end{bmatrix}_{k \times n}}_{B}$$

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$$\begin{bmatrix} - & a_{1*}^T & - \\ \vdots & \vdots & \vdots \\ - & a_{m*}^T & - \end{bmatrix} =$$

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  $$A = A_k B$$
  $$\begin{bmatrix} - & a_{1*}^T & - \\ \vdots & \vdots & \vdots \\ - & a_{m*}^T & - \end{bmatrix} = A_k \begin{bmatrix} - & b_{1*}^T & - \\ \vdots & \vdots & \vdots \\ - & b_{k*}^T & - \end{bmatrix}$$
- Thus row rank of A

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- Thus row rank of $A \leq k$.

- Let row rank of $A = r < k$.

$$A_{m\times n} = \underbrace{\begin{bmatrix} \begin{matrix} 1 & 0 & \cdots & 0 \\ & I_r & & \end{matrix} \\ \hline - \; c_{r+1,*}^T \; - \\ \vdots \\ - \; c_{m,*}^T \; - \end{bmatrix}}_{m\times r} \underbrace{\begin{bmatrix} - \; a_{1*}^T \; - \\ \vdots \\ - \; a_{r*}^T \; - \end{bmatrix}}_{r\times n}$$

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- Thus col rank of $A \leq r$, which is a contradiction!.

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- **Rank of a matrix** = Col rank = Row Rank.

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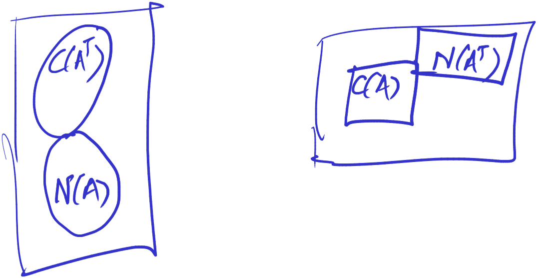
$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$



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- For $y_1, y_2 \in C(A^T)$, what if $Ay_1 = Ay_2$?

$$Ay_1 = Ay_2$$

$$Ay_1 - Ay_2 = \vec{0}_m$$

$$A(y_1 - y_2) = \vec{0}_m$$

$$y_1 - y_2 \in N(A)$$

$$y_1, y_2 \in C(A^T), \quad y_1 - y_2 \in C(A^T)$$

$$y_1 - y_2 = \vec{0}_n \Rightarrow y_1 = y_2$$

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$$\begin{bmatrix} A_{LI} \\ A_{LD} \end{bmatrix} A_{LI}^T y = \begin{bmatrix} A_{LI} \\ A_{LD} \end{bmatrix} x$$

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- Consider the restriction $A|_{C(A^T)} : C(A^T) \to C(A) \subseteq \mathbb{R}^m$.
- For $y_1, y_2 \in C(A^T)$, what if $Ay_1 = Ay_2$?
- Thus $A|_{C(A^T)}$ is **injective**.
- Is $A|_{C(A^T)}$ **surjective**?i.e., $\forall Ax, \exists \bar{y} \in \mathbb{R}^m$ such that $AA^T\bar{y} = Ax$.
- Let $rank(A) = r$. $A = \begin{bmatrix} A_{LI\,r\times n} \\ A_{LD\,(m-r)\times n} \end{bmatrix}$.
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- Find a solution (i.e., $y \in \mathbb{R}^r$ for any given $x \in \mathbb{R}^n$) to $A_{LI}A_{LI}^T y = A_{LI}x$.

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- Thus, $A|_{C(A^T)} : C(A^T) \to C(A)$ is **invertible**!

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$