Computational Numerical Methods

CS 374

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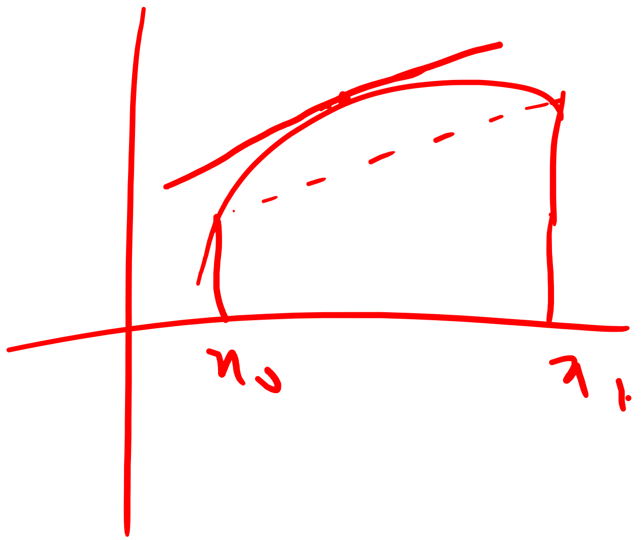
Divided difference

Let $y = f(x)$ for $x = x_0$, $x = x_1$ the

discrete derivative is.

$$\boxed{f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

it is first order divided difference of x.



Further: if x_0 & x_1 are close to each other.

$$f[x_0, x_1] \approx f'\left(\frac{x_0 + x_1}{2}\right)$$

Proof: Let $z = \frac{x_1 + x_0}{2}$ & $h = \frac{x_1 - x_0}{2}$.

$x_1 = z + h$, $x_0 = z - h$.

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(z+h) - f(z-h)}{2h}.$$

$$f(z+h) = f(z) + h f'(z) + \frac{h^2}{2!} f''(z) + \frac{h^3}{3!} f'''(z) \ldots$$

$$f(z-h) = f(z) - h f'(z) + \frac{h^2}{2!} f''(z) - \frac{h^3}{3!} f'''(z) \ldots$$

$$f[x_0, x_1] = f'(z) + \frac{1}{6}h^2 f'''(z)$$

Considering $h \to 0$ [as x_0 & x_1 are close to each other.
↳ assumption)

$$f[x_0, x_1] \approx f'(z) = f'\left(\frac{x_0+x_1}{2}\right)$$

Ex $f(x) = \cos x$. $x_0 = 0.2$, $x_1 = 0.3$

radian.

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\cos(0.3) - \cos(0.2)}{0.1} = -0.247300$$

$$f'\left(\frac{x_0+x_1}{2}\right) = -\sin\left(\frac{0.2+0.3}{2}\right) = -\sin(0.25) = -0.247404$$

Second order divided difference.

Let x_0, x_1, x_2 are distinct real numbers.

then

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

Third order divided difference.

For x_0, x_1, x_2, x_3 are distinct real numbers.

$$f[x_0, x_1, x_2, x_2] = \frac{f(x_1, x_2, x_3) - f[x_0, x_1, x_2]}{x_3 - x_0}.$$

n-th order divided difference.

For $x_0, x_1, \dots x_n$ ~~being~~ being $n+1$ distinct numbers

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1 \dots x_n] - f[x_0 \dots x_{n-1}]}{x_n - x_0}.$$

Th Let $n \geq 1$ and $f(x)$ is n-times differentiable on some interval $a \leq x \leq b$.

Let $x_0, x_1, \dots x_n$ be $n+1$ distinct numbers in $[a, b]$. Then.

$$f[x_0, x_1, \dots x_n] = \frac{1}{n!} f^{(n)}(c) \quad \text{where.}$$

c is some unknown point b/w the max & min of $x_0, x_1, \dots x_n$.

Ex For $f(x) = \cos x$. $x_0 = 0.2$, $x_1 = 0.3$, $x_2 = 0.4$.

$\therefore$ $f[x_0, x_1] = -0.2473009$

$f[x_1, x_2] = -0.3427550$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = -0.4772705$$

by previous Th.

$$f[x_0, x_1, x_2] \approx \frac{1}{2!} f^{(2)}(c) \qquad c = 0.3027.$$

$$-0.4772705 \approx -\frac{1}{2} \cos c$$

~~A P~~ A property

If the order of $x_0, x_1, \dots x_n$ is exchanged or permuted, then $f[x_0, x_1, \dots x_n]$ does not change its value.

① $$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f[x_1, x_0]$$

② $$f[x_0, x_1, x_2] = \frac{1}{x_2 - x_0}\Big[f[x_1, x_2] - f[x_0, x_1]\Big]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_0)}{(x_0 - x_2)(x_0 - x_1)}$$

$$- f(x_1)\left[\frac{1}{(x_2-x_0)(x_2-x_1)} + \frac{1}{(x_2-x_0)(x_1-x_0)}\right]$$

$$\Downarrow$$

$$\frac{f(x_1)}{x_2-x_0}\left[\frac{x_1-x_0+x_2-x_1}{(x_2-x_1)(x_1-x_0)}\right]$$

$$\frac{f(x_1)}{\cancel{x_2-x_0}}\left[\frac{\cancel{x_2-x_0}}{(x_2-x_1)(x_1-x_0)}\right]$$

$$\therefore f[x_0,x_1,x_2] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

$\Rightarrow$ Changing the order of x_0, x_1, x_2 will not change the value of $f[x_0, x_1, x_2]$

Newton's divided difference formula.

Let $P_n(x)$ denote the polynomial interpolation of $f(x_i)$ at x_i, $i = 0, 1, \dots n$.

$\therefore \quad P_n(x_i) = f(x_i)$ and ~~dge~~ degree $(P_n(x)) \le n$.

$$P_1(x) = f(x_0) + (x - x_0)\, f[x_0, x_1]$$

$$P_2(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$$

$\vdots$

$$P_n(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1})\, f[x_0, x_1, \dots x_n].$$

$$P_{k+1}(x) = P_k(x) + (x-x_0)(x-x_1)\cdots(x-x_k)\, f[x_0, x_1, \ldots x_{k+1}].$$

check when $x = x_0$ $\quad P_0(x_0) = f(x_0)$

$$x = x_1 \qquad P_1(x_1) = f(x_0) + (x_1-x_0)\frac{f(x_1)-f(x_0)}{x_1-x_0} = f(x_1)$$

$$x = x_2 \qquad P_2(x_2) = f(x_0) + (x_2-x_0)\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$+ (x_2-x_0)(x_2-x_1)\frac{1}{x_2-x_0}\left[\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}\right]$$

$$= f(x_2).$$