LECTURE 23

Lab work

$\ddot{x} = -\frac{g}{l}\sin x$, given $x(0) = x_0$, $\dot{x}(0) = 0$

$x(0) = 0,$ $\dot{x}(0) = v_0$.

$\ddot{x} = -\frac{g}{l}x$: $\longrightarrow$ Any set of initial conditions leads to oscillatory behaviour.

$\rightarrow$ Qualitative behaviour of the system is sensitive to the choice of initial conditions.

$$- \quad \begin{aligned} \dot{x} &= \\ \dot{y} &= \\ \dot{z} &= \end{aligned} \quad \underbrace{\begin{pmatrix} & - & \\ - & & \\ & & - \end{pmatrix}}$$

$\rightarrow$ note the failure/difficulty in using Standard methods.

This class of problems $\rightarrow$ stiff problem/ODEs.

'Radau'

Coupled oscillators

$$2\ddot{x} + \omega^2(5x-3y) = 0$$
$$2\ddot{y} + \omega^2(5y-3x) = 0 .$$

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} ;\quad K = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$$

Method 1: By inspection:

$$\boxed{\begin{aligned} (\ddot{x}+\ddot{y}) &= -\omega^2(x+y) \\ (\ddot{x}-\ddot{y}) &= -4\omega^2(x-y) \end{aligned}} .$$

Method 2: Write schematically in the for,

$$M\ddot{x} = -Kx$$

↳ contains off-diagonal terms responsible for coupling the eqns.

— Trial solⁿ: $\bar{z}(t) = \bar{a}\, e^{i\omega t}$.

Substituted, $(K - \omega^2 M)\, a = 0$.

— $|K - \omega^2 M| = 0 \rightarrow$ condition for non-trivial solⁿs to exist.

↳ solved for ω.

— Reformulation:- $\bar{M}\ddot{x} = -\bar{K}x$

Aim:- write K in a diagonal basis.

$\xi = O x$.

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$\Rightarrow \boxed{x = O^T \xi}$$

$$\bar{M} O^T \ddot{\xi} = -K O^T \xi$$

$$K_D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ's $\equiv$ eigenvalues of K.

$$\Rightarrow \quad O \bar{M} O^T \ddot{\xi} = -O K O^T \xi.$$

$$\Rightarrow \quad M O O^T \ddot{\xi} = -O K O^T \xi$$

$$\Rightarrow \quad M \ddot{\xi} = -K_D \xi.$$

$$\Rightarrow \quad M \ddot{\xi} = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \xi$$

$$\Rightarrow \quad \begin{pmatrix} \# & 0 \\ 0 & \# \end{pmatrix} \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \end{pmatrix} = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

$$\# \ddot{\xi}_1 = -\lambda_1 \xi_1 \qquad \# \ddot{\xi}_2 = -\lambda_2 \xi_2.$$

$$\left.\begin{aligned} K\begin{pmatrix} a \\ b \end{pmatrix} &= \lambda_1 \begin{pmatrix} a \\ b \end{pmatrix} \\ K\begin{pmatrix} c \\ d \end{pmatrix} &= \lambda_2 \begin{pmatrix} c \\ d \end{pmatrix} \end{aligned}\right\} \text{ Solve for } \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \begin{pmatrix} c \\ d \end{pmatrix}.$$

$$O = \underline{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}.$$

$$\underline{\xi = O^T x}.$$

Ex: $K = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$ $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. $\Rightarrow M^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Diagonalize K. → Secular eqn. to determine eigenvalues

$$\begin{vmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (5-\lambda)^2 = 9.$$

$$\Rightarrow 5-\lambda = \pm 3.$$

$$\left.\begin{aligned} 5-\lambda_1 &= 3 \\ 5-\lambda_2 &= -3. \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 8. \end{aligned}$$

$$M\ddot{x} = -Kx.$$

$$\Rightarrow \ddot{x} = -M^{-1}Kx.$$

$$M^{-1}K_D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$O = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

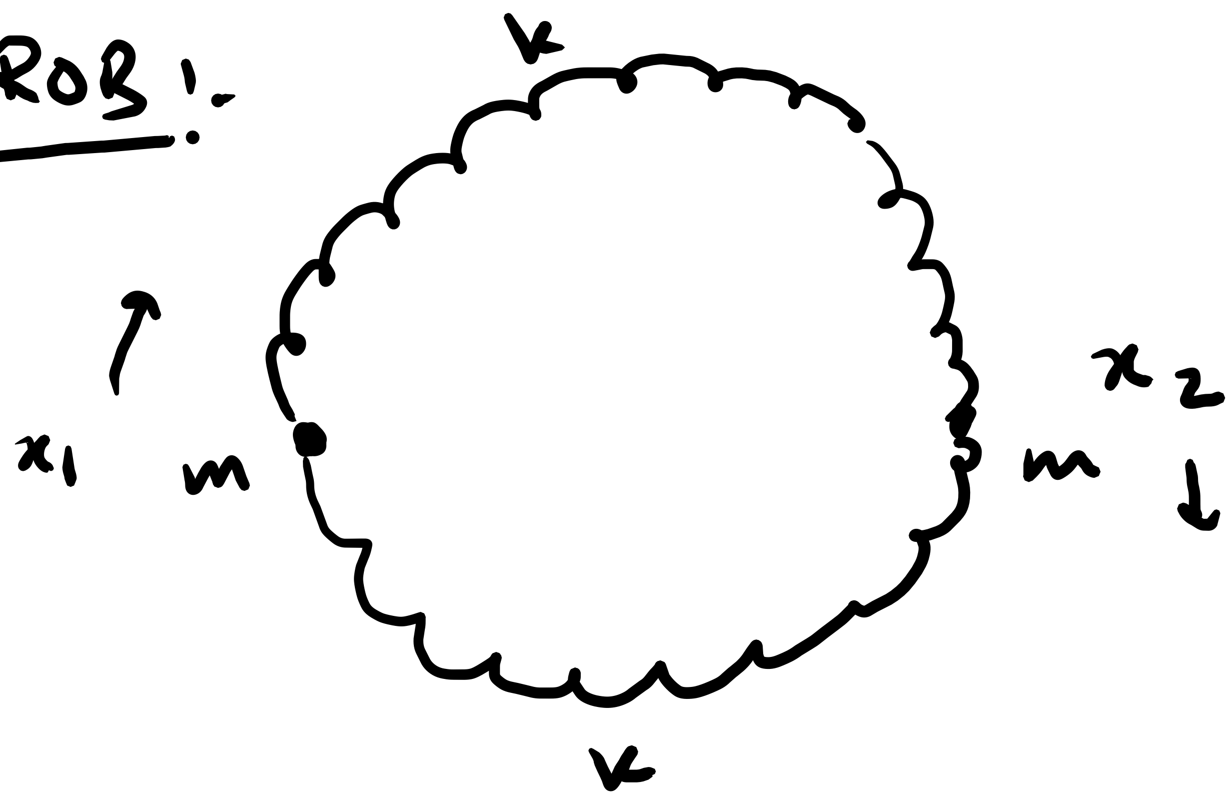
$$\left| \begin{aligned} \xi_1 &= (x_1 + x_2) \\ \xi_2 &= (x_1 - x_2) \end{aligned} \right|.$$

$O \rightarrow$ diagonalizes K $\qquad O^T K O$.

$cO \rightarrow$ will also diagonalize K.

IV OTHER EXAMPLES OF COUPLED OSCILLATORS

PROB :-



Identical masses connected by identical springs on a circle. One mass is subjected to a driving force $F_d \cos\omega_d t$.

EOM :-

$$m\ddot{x}_1 = -2k(x_1 - x_2) + F_d \cos\omega_d t .$$

$$m\ddot{x}_2 = -2k(x_2 - x_1) .$$

Rewrite, $m\ddot{x}_1 + 2k(x_1 - x_2) = F_d \cos\omega_d t$ & $m\ddot{x}_2 + 2k(x_2 - x_1) = 0$.

$$m\ddot{z}_1 + 2k(z_1 - z_2) = F_d e^{i\omega_d t} \qquad x = Re(z).$$

$$m\ddot{z}_2 + 2k(z_2 - z_1) = 0.$$

Trial soln:- $z_1 = A_1 e^{i\omega_d t}$

$z_2 = A_2 e^{i\omega_d t}$.

Sub:- $-\omega_d^2 A_1 + 2\omega^2 (A_1 - A_2) = F_d.$ ——— (1)

$-\omega_d^2 A_2 + 2\omega^2 (A_2 - A_1) = 0.$ ——— (2).

Above equs. can be solved for A_1 and A_2.

Solving (1) and (2),

$$x_1(t) = -\frac{F_d\,(2\omega^2-\omega_d^2)}{\omega_d^2(4\omega^2-\omega_d^2)}\cos\omega_d t.$$

$$x_2(t) = -\frac{2F_d\,\omega^2}{\omega_d^2(4\omega^2-\omega_d^2)}\cos\omega_d t.$$

Features:- (1) For $4\omega^2-\omega_d^2 = 0$, the solutions maximize / blow up.

$\Rightarrow \omega_d = 2\omega$.

$\boxed{\text{Resonance}}$

(2). For $\omega_d = \sqrt{2}\,\omega$, $x_1(t) = 0$. $\forall t$; $x_2(t)$ oscillates.