Computational Numerical Methods

CS 374

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$$fl\left(\sqrt{100000}\right) = (-1)^0 \times 0.316228 \times 10^3.$$

$\therefore$ ~~$fl(fl(x+1)) - fl(x)$~~

$$fl\left(fl\sqrt{100001} - fl(100000)\right) = 0.1 \times 10^{-2}.$$

Finally

$$fl(f(100000)) = fl(100000) \times (0.1 \times 10^{-2})$$

$$= 100$$

Six digit chopping will give.

$$fl(f(100000)) = 200$$

$$\boxed{f(100000) = 158.113}$$

Machine Epsilon

The machine epsilon of a computer is the smallest +ve real number δ such that $fl(1+\delta) > 1$. Thus for any real number $0 < \bar{\delta} < \delta$ we have $fl(1+\bar{\delta}) = 1$ & $1+\bar{\delta}$ & 1 are identical within the computer arithmetic.

Errors

The error in a computed quantity is defined as

$E(x_A) =$ Error $=$ True value $-$ Approximate value.

Absolute error: The absolute value of the error.

$E_a(x_A)$

Relative error:

$$\text{Relative error} = \frac{\text{Error}}{\text{True value}}.$$

$E_r(x_A)$

Percentage error

$$= |\text{Relative error}| \times 100.$$

Propagation of error in arithmetic operations

Let $x_T = x_A + \varepsilon$.

$y_T = y_A + \delta$.

Addition | Subtraction.

$$E_r(x_A \pm y_A) = \frac{(x_T \pm y_T) - (x_A \pm y_A)}{(x_T \pm y_T)}$$

$$= \frac{(x_T \pm y_T) - \{(x_T - \varepsilon) \pm (y_T - \delta)\}}{x_T \pm y_T}$$

$$= \frac{\varepsilon \pm \delta}{x_T \pm y_T}.$$

Relative error propagates slowly for addition but might be very high for substraction

Propagation of error for multiplication

$$E_r(x_A * y_A) = \frac{x_T \cdot y_T - x_A \cdot y_A}{x_T \cdot y_T}.$$

$$= \frac{x_T y_T - (x_T - \varepsilon)(y_T - \delta)}{x_T y_T}$$

$$= \frac{\delta x_T + \varepsilon y_T - \varepsilon \delta}{x_T y_T}$$

$$= \frac{\varepsilon}{x_T} + \frac{\delta}{y_T} - \frac{\varepsilon \delta}{x_T y_T}$$

$$= E_r(x_A) + E_r(y_A) - E_r(x_A) E_r(y_A).$$

Relative error propagates slowly in case of multiplication

Division

Relative error propagates slowly unless error in y is ~~very~~ very high.

$E_r(x_A/y_A)$

$=$

E_r

$=$

$=$

$$= \frac{\varepsilon y_T - \delta x_T}{x_T(y_T - \delta)} = \frac{\varepsilon y_T - \delta x_T}{x_T y_T \left(1 - \frac{\delta}{y_T}\right)}$$

$$= \left(\frac{\varepsilon}{x_T} - \frac{\delta}{y_T}\right) \frac{1}{1 - \frac{\delta}{y_T}}$$

$\neq$

$$= \left(E_r(x_A) - E_r(y_A)\right) / \left(1 - E_r(y_A)\right)$$

How to avoid ~~s~~ huge error

$$f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right)$$

$$f(100000)$$

$$= \frac{x}{\sqrt{x+1} + \sqrt{x}}$$

Significant digit

$x = \frac{1}{3} = 0.333\ldots$

$x_A = 0.333$ has 3 significant digits.

Defn Let β be a radix.

If x_A is an approximation to x, then we say that x_A approximates x to r significant β-digits if r is the largest non-negative integer s.t

$$\frac{|x - x_A|}{|x|} \leq \frac{1}{2}\beta^{-r+1} \quad \text{with } x \neq 0$$

Consider two real numbers.

$x = 7.6545428 = 0.76545428 \times 10^1$

$y = 7.6544201 = 0.76544201 \times 10^1$

Let $x_A = 7.6545421 = 0.76545421 \times 10^1$

$y_A = 7.6544200 = 0.76544200 \times 10^1$

are approximate value of x & y correct upto 6th & 7th ~~of~~ significant digit.

$$z_A = x_A - y_A = 0.1221000 \times 10^{-3}$$

$$z = x - y = 0.1227000 \times 10^{-3}.$$

$$\Rightarrow \frac{|z - z_A|}{|z|} = 0.0049 < 0.5 \times 10^{-2}.$$

$$-r + 1 = -2.$$

$$r = 3$$

$$E_r(z_A) \approx 53581 \times E_r(x_A)$$