SC223 - Linear Algebra

Aditya Tatu

Lecture 30

October 18, 2023

Summary of Lecture 29

● **Similar matrices and Similarity transformation:** We say two matrices A and B are similar if there exists an invertible matrix, say S such that $B = SAS^{-1}$. The transformation $A \mapsto SAS^{-1}$ is said to be a similarity transformation of A by S.

● Let $T \in \mathcal{L}(U)$ be a linear operator. It is preferable to work with a basis of U, say β such that $[T]_\beta^\beta$ is diagonal/block-diagonal.

$T \in \mathcal{L}(U)$ $\quad$ β - basis of U

$[T]_\beta^\beta \in \mathbb{F}^{n\times n}$ $\quad$ α - basis of U

$[T]_\alpha^\alpha \in \mathbb{F}^{n\times n}$

$$[T]_\alpha^\alpha = \underbrace{N_\beta^\alpha}_{S} [T]_\beta^\beta \underbrace{(N_\beta^\alpha)^{-1}}_{S^{-1}}$$

$$[T]_\gamma^\gamma = S_1 [T]_\beta^\beta (S_1)^{-1}$$

Invariant Subspaces

- Let $T \in \mathcal{L}(U)$, and $U = V_1 \oplus V_2$.

Invariant Subspaces

- Let $T \in \mathcal{L}(U)$, and $U = V_1 \oplus V_2$.
- Consider $T|_{V_1}$, $T|_{V_2}$.

$u \in U$, $\quad u = v_1 + v_2$.

$Tu = Tv_1 + Tv_2$, $\quad v_1 \in V_1$

$v_2 \in V_2$.

Invariant Subspaces

- Let $T \in \mathcal{L}(U)$, and $U = V_1 \oplus V_2$.
- Consider $T|_{V_1}$, $T|_{V_2}$.
- If $T(V_1) \subseteq V_1$, $T(V_2) \subseteq V_2$, then in a basis $B = B_{V_1} \cup B_{V_2}$, $[T]_B^B = ?$

Ex: Let $\dim(U) = 5$, $\dim(V_1) = 2$, $\dim(V_2) = 3$.

$$[T]_B^B \in \mathbb{F}^{5\times 5} = \begin{bmatrix} \star & \star & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & \star & \star & \star \end{bmatrix}_{5\times 5}$$

$B = B_{V_1} \cup B_{V_2}$

$B_{V_1} = \{P_1, P_2\}$

$B_{V_2} = \{q_1, q_2, q_3\}$.

Invariant Subspaces

- Let $T \in \mathcal{L}(U)$, and $U = V_1 \oplus V_2$.
- Consider $T|_{V_1}$, $T|_{V_2}$.
- If $T(V_1) \subseteq V_1$, $T(V_2) \subseteq V_2$, then in a basis $B = B_{V_1} \cup B_{V_2}$, $[T]_B^B = ?$

$$[T]_B^B = \begin{bmatrix} [T_{V_1}]_{B_{V_1}}^{B_{V_1}} & 0 \\ 0 & [T_{V_2}]_{B_{V_2}}^{B_{V_2}} \end{bmatrix},$$

$[T]_B^B$ becomes block-diagonal.

Invariant Subspaces

- Let $T \in \mathcal{L}(U)$, and $U = V_1 \oplus V_2$.
- Consider $T|_{V_1}, T|_{V_2}$.
- If $T(V_1) \subseteq V_1, T(V_2) \subseteq V_2$, then in a basis $B = B_{V_1} \cup B_{V_2}$, $[T]_B^B =$?

$$[T]_B^B = \begin{bmatrix} [T_{V_1}]_{B_{V_1}}^{B_{V_1}} & 0 \\ 0 & [T_{V_2}]_{B_{V_2}}^{B_{V_2}} \end{bmatrix},$$

$[T]_B^B$ becomes block-diagonal.

- **Definition:** Let $T \in \mathcal{L}(U)$. If V is a subspace of U such that $\forall u \in V, Tu \in V$, then V is said to be an *invariant subspace* of T, or T*-invariant* subspace.

Invariant Subspaces

- Let $T \in \mathcal{L}(U)$, and $U = V_1 \oplus V_2$.
- Consider $T|_{V_1}$, $T|_{V_2}$.
- If $T(V_1) \subseteq V_1$, $T(V_2) \subseteq V_2$, then in a basis $B = B_{V_1} \cup B_{V_2}$, $[T]_B^B =$?

$$[T]_B^B = \begin{bmatrix} [T_{V_1}]_{B_{V_1}}^{B_{V_1}} & 0 \\ 0 & [T_{V_2}]_{B_{V_2}}^{B_{V_2}} \end{bmatrix},$$

$[T]_B^B$ becomes block-diagonal.

- **Definition:** Let $T \in \mathcal{L}(U)$. If V is a subspace of U such that $\forall u \in V$, $Tu \in V$, then V is said to be an *invariant subspace* of T, or T*-invariant* subspace.
- U and $\{\theta\}$ are trivial examples of invariant subspaces.

$A \in \mathbb{R}^{n\times n}$, $A \in \mathcal{L}(\mathbb{R}^n)$

$\underline{C(A)}$, Let $y \in C(A)$, $Ay \in C(A)$.

$C(A^T)$: Let $y \in C(A^T) \Rightarrow y = A^T x$. $Ay = A^T z$.

Invariant Subspaces

- Let $T \in \mathcal{L}(U)$, and $U = V_1 \oplus V_2$.
- Consider $T|_{V_1}, T|_{V_2}$.
- If $T(V_1) \subseteq V_1$, $T(V_2) \subseteq V_2$, then in a basis $B = B_{V_1} \cup B_{V_2}$, $[T]_B^B =$?

$$[T]_B^B = \begin{bmatrix} [T_{V_1}]_{B_{V_1}}^{B_{V_1}} & 0 \\ 0 & [T_{V_2}]_{B_{V_2}}^{B_{V_2}} \end{bmatrix},$$

$[T]_B^B$ becomes block-diagonal.

- **Definition:** Let $T \in \mathcal{L}(U)$. If V is a subspace of U such that $\forall u \in V, Tu \in V$, then V is said to be an *invariant subspace* of T, or T*-invariant* subspace.
- U and $\{\theta\}$ are trivial examples of invariant subspaces.
- Other examples include $N(T), R(T)$.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x =$

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx =$

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u$

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$,

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u =$

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.

$$(T - \lambda I)(\theta) = \theta$$

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is not invertible, and *det*

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is not invertible, and $det([T]_\beta^\beta - \lambda I_n) =$

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is not invertible, and $det([T]_\beta^\beta - \lambda I_n) = 0$.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is not invertible, and $det([T]_\beta^\beta - \lambda I_n) = 0$.
- Define $c(\lambda) = det([T]_\beta^\beta - \lambda I)$. Eigenvalues of T are roots of the polynomial $c(\lambda)$, and the eigenvector is a non-zero vector belonging to $\mathcal{N}(T - \lambda I)$.

1 dimensional Invariant Subspaces

- Let V be a 1-dimensional invariant subspace for $T \in \mathcal{L}(U)$.
- $V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta$. Let $x \neq \theta, x \in V$, i.e., $x = a_1 u$ for some non-zero $a_1 \in \mathbb{F}$.
- Since V is $T-$invariant, $Tx \in V$, therefore $Tx = a_2 u \Rightarrow Tu = \frac{a_2}{a_1} u$.
- We usually write this as $Tu = \lambda u$, and we call λ **as the eigenvalue associated with eigenvector** u.
- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is not invertible, and $det([T]_\beta^\beta - \lambda I_n) = 0$.
- Define $c(\lambda) = det([T]_\beta^\beta - \lambda I)$. Eigenvalues of T are roots of the polynomial $c(\lambda)$, and the eigenvector is a non-zero vector belonging to $\mathcal{N}(T - \lambda I)$.
- The polynomial c is called the **characteristic polynomial** of T.

End of class.

● For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \ldots, v_n\}$.

- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \ldots, v_n\}$.
- Since $Tv_i = \lambda_i v_i$, $[T]_e^e =$

- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \ldots, v_n\}$.
- Since $Tv_i = \lambda_i v_i$, $[T]_e^e = diag(\lambda_1, \ldots, \lambda_n)$.

- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \ldots, v_n\}$.
- Since $Tv_i = \lambda_i v_i$, $[T]_e^e = diag(\lambda_1, \ldots, \lambda_n)$.
- Note that for any other basis β, $[T]_\beta^\beta [v_i]_\beta =$

- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \dots, v_n\}$.
- Since $Tv_i = \lambda_i v_i$, $[T]_e^e = diag(\lambda_1, \dots, \lambda_n)$.
- Note that for any other basis β, $[T]_\beta^\beta [v_i]_\beta = \lambda_i [v_1]_\beta$.

- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \ldots, v_n\}$.
- Since $Tv_i = \lambda_i v_i$, $[T]_e^e = diag(\lambda_1, \ldots, \lambda_n)$.
- Note that for any other basis β, $[T]_\beta^\beta [v_i]_\beta = \lambda_i [v_1]_\beta$.
- Thus,

$$[T]_\beta^\beta \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ [v_1]_\beta & [v_2]_\beta & \ldots & [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \lambda_1 [v_1]_\beta & \lambda_2 [v_2]_\beta & \ldots & \lambda_n [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$=$$

● For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \dots, v_n\}$.
● Since $Tv_i = \lambda_i v_i$, $[T]_e^e = diag(\lambda_1, \dots, \lambda_n)$.
● Note that for any other basis β, $[T]_\beta^\beta [v_i]_\beta = \lambda_i [v_1]_\beta$.
● Thus,

$$[T]_\beta^\beta \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ [v_1]_\beta & [v_2]_\beta & \dots & [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \lambda_1 [v_1]_\beta & \lambda_2 [v_2]_\beta & \dots & \lambda_n [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ [v_1]_\beta & [v_2]_\beta & \dots & [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & \vdots \\ 0 & \lambda_2 & \dots & 0 \\ 0 & \dots & \vdots & \lambda_n \end{bmatrix}}_{\Lambda}$$

- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{v_1, \ldots, v_n\}$.
- Since $Tv_i = \lambda_i v_i$, $[T]_e^e = diag(\lambda_1, \ldots, \lambda_n)$.
- Note that for any other basis β, $[T]_\beta^\beta [v_i]_\beta = \lambda_i [v_1]_\beta$.
- Thus,

$$[T]_\beta^\beta \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ [v_1]_\beta & [v_2]_\beta & \ldots & [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \lambda_1 [v_1]_\beta & \lambda_2 [v_2]_\beta & \ldots & \lambda_n [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ [v_1]_\beta & [v_2]_\beta & \ldots & [v_n]_\beta \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \ldots & \vdots \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & \ldots & \vdots & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$[T]_\beta^\beta E = E\Lambda \Rightarrow \Lambda = [T]_E^E = E^{-1}[T]_\beta^\beta E$$

- Eigenvalues of T are roots of $det([T]_\beta^\beta - \lambda\ I_n)$, and corresponding eigenvectors are linearly independent vectors in $\mathcal{N}(T - \lambda\ I)$.

Examples

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

Results on Eigenvectors and Eigenvalues

- Proposition 19: Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T \in \mathcal{L}(U)$. Then the eigenvectors $v_1, \ldots, v_m$ associated with these eigenvalues are linearly independent.