IT-216 - Design and analysis of algorithms

$x \leftarrow n$ digits no.
$y \leftarrow n$ " no.

multiply x and y.
$x * y$

$$\begin{array}{r} 5 \\ 6 \\ \hline 30 \end{array} \qquad \begin{array}{r} 25 \\ \times\ 11 \\ \hline 275 \end{array} \qquad \begin{array}{r} 567891102 \\ \times\ 123569 \\ \hline \end{array}$$

$$
\begin{array}{rrrrrrr}
 & x = & 5 & 6 & 7 & 8 & \\
 & y = \times & 1 & 2 & 3 & 4 & \\
\hline
 & & 2 & 2 & 7 & 1 & 2 \\
 & 1 & 7 & 0 & 3 & 4 & \times \\
1 & 1 & 3 & 5 & 6 & \times & \times \\
5 & 6 & 7 & 8 & \times & \times & \times \\
\hline
7 & 0 & 0 & 6 & 6 & 5 & 2
\end{array}
$$

3

3

2

$\underline{\underline{Q}}$: How many Premitive operations are there?

Sigle digit multiplication and addition are allowed.

$$\left.\begin{array}{l}\text{Each row takes} \rightarrow 2n \\ \#\text{ rows} - n\end{array}\right\}$$

total n rows takes $- 2n^2$

Adding two rows takes $\rightarrow$ $2n$ operations.

Adding n rows takes $\rightarrow$ $(n-1) \cdot 2n \cdot \leq 2n^2$

Total operations $\rightarrow$ $2n^2 + 2n^2 \leq 4n^2$

$\rightarrow$ constant $* \ n^2$

<u>Algorithm designer's mantra</u>

can we do better?

$x = {}_a\underline{5\,6}\;\underline{7\,8}_b$

$y = \underset{c}{\underline{1\,2}}\;\underset{d}{\underline{3\,4}}$

step 1: compute ac ⟵ $T(n/2)$

step 2: compute $b \cdot d$ ⟵ $T(n/2)$ $\quad 2 \cdot n/2$

step 3: compute $(a+b)\cdot(c+d)$ ⟵ $2 \cdot (\cancel{2 \cdot \tfrac{n}{2}}) + T(n/2)$

step 4: compute step 3 − step 1 − step 2 ⟵ constant · n

step 5: 10^4 step 1 $+ 10^2$ step 4 + step 2 ⟵ constant · n.

<u>Ans</u> 7006652 (H.W)

<u>overall time:</u>

$T(n) = 3T(n/2) + O(n)$

constant * n

$\boxed{\text{less than } 2n^2}$

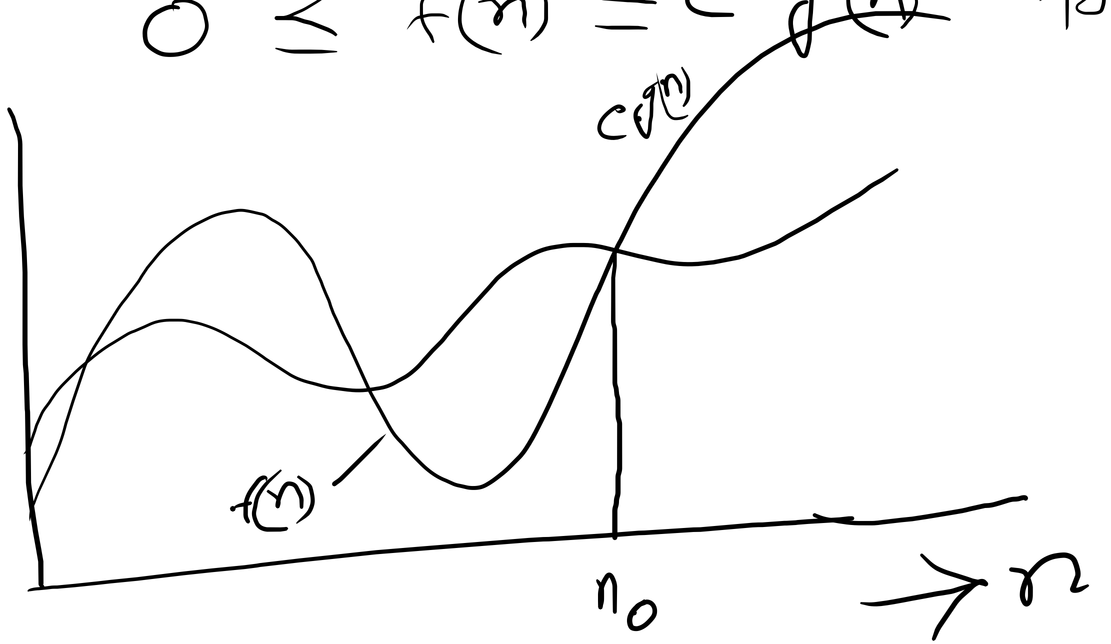
$T(n)$ ⟵ total number of operations to multiply two n digits number.

Asymptotic notations

Big 'O' notation (upper bound)

Question when is $f(n) = O(g(n))$?

$f(n) = O(g(n))$ if there exists constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le c\, g(n)$ for all $n \geq n_0$



$$x = 10^{n/2} a + b$$

$$y = 10^{n/2} c + d$$

$$x \cdot y = (10^{n/2} a + b)(10^{n/2} c + d)$$

$$= 10^{n} ac + 10^{n/2}(ad + bc) + bd$$

$$= (a+b)(c+d) - ac - bd$$

Exm

$2n^2 = O(n^3)$

$\left.\begin{array}{l} c = 1 \\ n_0 = 1 \end{array}\right\}$

$0 \le f(n) \le c\, g(n) \quad \forall\, n \ge n_0$

$2n_0^2 \le 1 \cdot n_0^3$

$2 \le 1$ ✗

$\left.\begin{array}{l} c = 1 \\ n_0 = 2 \end{array}\right\}$

$2n_0^2 \le 1\, n_0^3$

$2 \cdot 2^2 \le 1 \cdot 2^3$

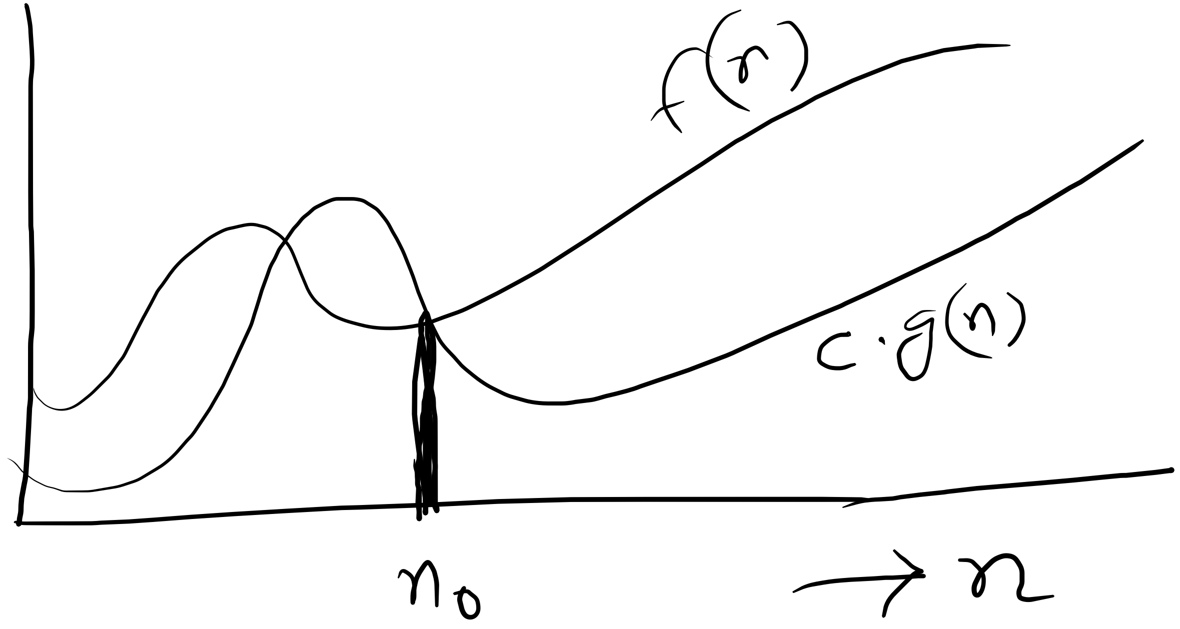
$8 \le 8$ ✓

$2n^2 = O(n^3)$

Big omega (Ω) notation

lower bound.

$f(n) = \Omega(g(n))$ if there exists constants $c > 0, n_0 > 0$ such that $0 \le c \cdot g(n) \le f(n) \ \forall n \ge n_0$



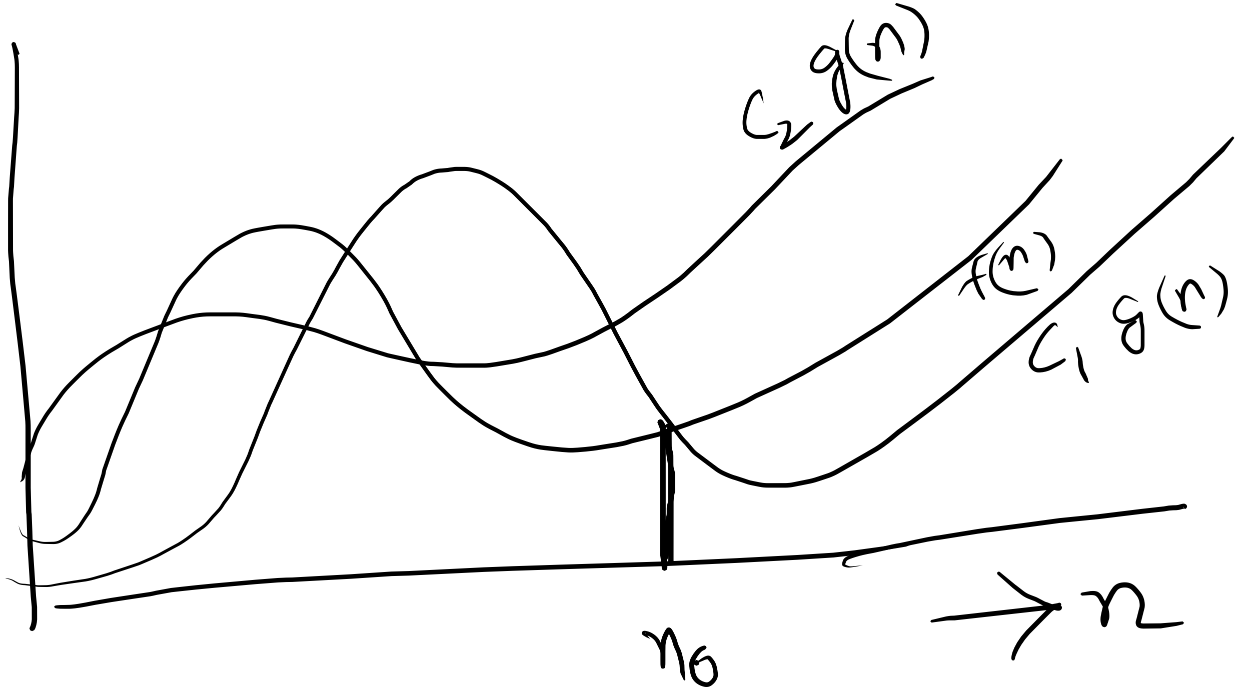
Ex^m $5n^2 + 3n = \Omega(n^2)$

Theta notation

Tight bound

$f(n) = \Theta(g(n))$ if $\exists$ constants $c_1 > 0, c_2 > 0, n_0 > 0$ such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n) \quad \forall n \ge n_0$$



Ex^m $3n+2 = \Theta(n)$

small 'o' and small ω.

$f(n) = o(g(n))$

$$0 \le f(n) < c\,g(n)$$

$f(n) = \omega(g(n))$

$$0 \le c\,g(n) < f(n)$$