LECTURE 12

Recap :-

- Perturbation ~~[illegible]~~ applied to circular orbit. If it grows with time, orbit was unstable otherwise stable.

- Circular orbit is a result of $V_{eff}(r)$ having extremum. Intuitively natural to expect stability/instability of orbit to be related to ~~[illegible]~~

$$\text{sgn}\left(\left.\frac{d^2 V_{eff}}{dr^2}\right|_{r=a} \right)$$

- Considered. $f(r) = k r^{\nu}$.

$$V(r) = -\int dr\, f(r) = -\frac{k r^{1+\nu}}{1+\nu}$$

$\boxed{\text{VII}}$ Computational approach to central force problem.

$$\frac{d^2x}{dt^2} = F_x = -\frac{\alpha}{r^2}\cos\theta.$$

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}.$$

$$\frac{d^2y}{dt^2} = F_y = -\frac{\alpha}{r^2}\sin\theta$$

Rewrite as coupled 1st order ODE.

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_x}{dt} = F_x = F\cos\theta = \frac{x}{r}$$

$$\frac{dy}{dt} = v_y$$

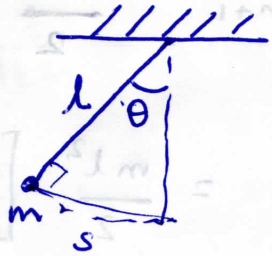
$$\frac{dv_y}{dt} = F_y = F\sin\theta = \frac{y}{r}$$

$\Rightarrow$ Discretise via Euler's method.

$\Rightarrow$ recall that the eqns. for a central force depend on conservation of energy and angular momentum.

— Euler's method is problematic for oscillatory systems.

— Simple example:- pendulum:-



$$m\ddot{s} = -F_\theta = -mg\sin\theta = mg\theta \quad \text{(small angle approx).}$$

$$\Rightarrow m l\ddot{\theta} = -mg\theta$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\theta(t) = A\cos\Omega t + B\sin\Omega t$$

$$- \; E = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta) \approx \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} mgl\theta^2 .$$

– Euler algorithm :–

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta .$$

$$\boxed{\begin{aligned} \frac{d\theta}{dt} &= \omega . \\ \frac{d\omega}{dt} &= -\frac{g}{l}\theta . \end{aligned}}$$

$$\Rightarrow \cancel{\theta^{n+1} = \theta^n + \theta_1 \Delta t}$$

$$\Rightarrow \cancel{\theta_1^{n+1} = \theta_1^n - \frac{g}{l}\theta \Delta t}$$

$$\cancel{t^{n+1} = t^n + \Delta t}$$

$$\boxed{\begin{aligned} E &= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} mgl\theta^2 \\ \dot{E} &= m l^2 \dot{\theta}\ddot{\theta} + mgl\theta\dot{\theta} = 0 \\ &\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0 \end{aligned}}$$

$$\hookrightarrow \theta_{n+1} = \theta_n + \omega^n \Delta t .$$

$$\omega_{n+1} = \omega_n - \frac{g}{l}\theta^n \Delta t .$$

$$t_{n+1} = t^n + \Delta t .$$

$$E_{n+1} = \frac{ml^2}{2}\left[\omega_{n+1}^2 + \frac{g}{l}\theta_{n+1}^2 \right]$$

$$= \frac{ml^2}{2}\left[\left(\omega_n - \frac{g}{l}\theta_n \Delta t\right)^2 + \frac{g}{l}\left(\theta_n + \omega^n \Delta t\right)^2 \right]$$

$$= \frac{ml^2}{2}\left[\omega_n^2 + \frac{g^2}{l^2}\theta_n^2 \Delta t^2 + \frac{g}{l}\theta_n^2 + \frac{g}{l}(\omega^n)^2 \Delta t^2 \right]$$

$$= \frac{ml^2}{2}\left(\omega_n^2 + \frac{g}{l}\theta_n^2\right) + \frac{ml^2}{2}\left[\frac{g^2}{l^2}\theta_n^2 \Delta t^2 + \frac{g}{l}\omega_n^2 \right]\Delta t^2$$

$$= E_n + \frac{mgl}{2}\left(\frac{g}{l}\theta_n^2 + \omega_n^2\right)\Delta t^2 > E_n .$$

- Euler's method does not conserve energies.

$$\omega_{n+1} = \omega_n - \frac{g}{l}\theta_n \Delta t$$

$$\theta_{n+1} = \theta_n + \underbrace{\omega^{n+1}}_{\text{instead of } \omega_n} \Delta t$$

- Intuitively, ~~both~~ should be slightly better.

$$E_{n+1} - E_n = \left(\omega_n^2 - \frac{g}{l}\theta_n^2\right)\Delta t^2 - (\ )\Delta t^3 + (\ )\Delta t^4.$$

- Still not very accurate.

- Other, more accurate schemes of integration?

Taylor Method

$$\frac{dy}{dt} = f(t, y).$$

$$y_{n+1} = y_n + \Delta t \frac{dy}{dt} + \frac{\Delta t^2}{2}\frac{d^2y}{dt^2} + \dots$$

$$\Rightarrow y_{n+1} = y_n + \Delta t f(t, y) + \frac{\Delta t^2}{2}\frac{df}{dt} + \dots$$

- Can be extended to any order of accuracy.
- But tedious to compute derivatives.

☒ Runge-Kutta (RK) methods

$$y(t_{n+1}) - y(t_n) = \underbrace{\int_{t_n}^{t_{n+1}} dt\, f(t, y(t))}$$

Approximate this integral.

1. $$\int_{t_n}^{t_{n+1}} dt\, f(t_n, y(t)) = f(t_n, y_n)\,\Delta t.$$

$$\Rightarrow \quad y_{n+1} = y_n + f(t_n, y_n)\Delta t \longrightarrow \text{Euler's method.}$$

2. Trapezoidal rule.

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\left[f(a) + f(b)\right]$$

$$y_{n+1} - y_n = \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

Implicit equ., hard to solve.

Solution: Evaluate $f(t_{n+1}, y_{n+1})$ by solving y_{n+1} using Euler method.

$$f(t_{n+1}, y_{n+1}) \cancel{= f(t_{n+1}, y_n}$$

$$= f(t_n + \Delta t, y_n + \Delta t\, k_1), \quad \text{where } k_1 = f(t_n, y_n)$$

$$= k_2. \quad (\text{say}).$$

$\therefore\ y_{n+1} - y_n = \frac{\Delta t}{2}\left[\cancel{f(t_n, y_n)} + k_1 + k_2\right]$

$$\Rightarrow \boxed{y_{n+1} = y_n + \frac{\Delta t}{2}(k_1 + k_2)}$$

$\Downarrow$

2nd order Runge-Kutta method.

— Simpson's rule:-

$k_1 = f(t_n, y_n)$

$k_2 = f(t_n + \frac{1}{2}\Delta t,\ y_n + \frac{1}{2}\Delta t\, k_1)$

$k_3 = f(t_n + \frac{1}{2}\Delta t,\ y_n + \frac{1}{2}\Delta t\, k_2)$

$k_4 = f(t_n + \Delta t,\ y_n + \Delta t\, k_3)$.

$$\boxed{y_{n+1} = y_n + \frac{1}{6}\Delta t\,(k_1 + 2k_2 + 2k_3 + k_4)}$$

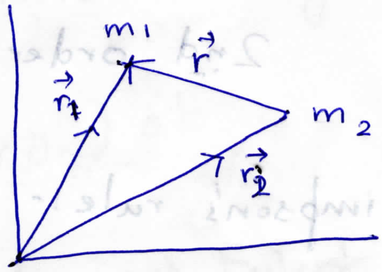
$\Downarrow$

4th order RK method.

LECTURE 13

TWO-BODY PROBLEM

Consider 2 self-gravitating bodies moving under self-interacting central force.



$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

$$= -\frac{Gm_1m_2}{r^3}\vec{r}.$$

Eqns. of motion are :-

$$m_1\ddot{\vec{r_1}} = \vec{F}_{12}$$

$$m_2\ddot{\vec{r_2}} = \vec{F}_{21}.$$

Obviously, $\vec{F}_{12} = -\vec{F}_{21}$

Define $\vec{R} = \dfrac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1+m_2}$. $\rightarrow$ coordinate of CM.

$\vec{r} = \vec{r_1} - \vec{r_2} \longrightarrow$ relative separation.

Now, $\vec{R} = \dfrac{m_1\vec{r_1} + m_2(\vec{r_1} - \vec{r})}{m_1+m_2}$

$$\Rightarrow \vec{R} = \vec{r_1} - \frac{m_2\vec{r}}{m_1+m_2}$$

$$\Rightarrow \vec{r_1} = \vec{R} + \frac{m_2\vec{r}}{m_1+m_2}$$

$$\vec{r_2} = \vec{r_1} - \vec{r} = \vec{R} + \frac{m_2\vec{r}}{m_1+m_2} - \vec{r}$$

$$\Rightarrow \vec{r_2} = \vec{R} + \frac{m_2\vec{r}}{m_1+m_2} - \vec{r} = \vec{R} - \frac{m_1\vec{r}}{m_1+m_2}$$

EOM are:-

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_{12} \quad \text{——— (1)}$$

$$m_2 \ddot{\vec{r}}_2 = \vec{F}_{21} \quad \text{——— (2)}$$

$m_2 \times (1) - m_1 \times (2)$,

$$m_1 m_2 \left(\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2\right) = m_2 \vec{F}_{12} - m_1 \vec{F}_{21} = (m_1 + m_2) \vec{F}_{12}^{\,int}$$

$$\Rightarrow \frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{r}} = \vec{F}_{12}$$

$$\Rightarrow \mu \ddot{\vec{r}} = \vec{F}_{12}^{\,int.}$$

$$\frac{m_1 \ddot{\vec{r}}_1}{m_1 + m_2} + \frac{m_2 \ddot{\vec{r}}_2}{m_1 + m_2} = \frac{\vec{F}_{12}}{m_1 + m_2} + \frac{\vec{F}_{21}}{m_1 + m_2}$$

$$\Rightarrow \ddot{\vec{R}} = 0 \quad \Rightarrow \quad \dot{\vec{R}} = \text{const.}$$

$$T = \frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2}$$

$$\Rightarrow T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$

$$\Rightarrow T = \frac{1}{2} m_1 \left[\dot{\vec{R}} + \frac{m_2 \dot{\vec{r}}}{m_1 + m_2}\right]^2 + \frac{1}{2} m_2 \left[\dot{\vec{R}} - \frac{m_1 \dot{\vec{r}}}{m_1 + m_2}\right]^2$$

$$= \frac{1}{2}(m_1 + m_2) \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)^2} \left(m_2 \dot{\vec{r}}^2 + m_1 \dot{\vec{r}}^2\right)$$

$$= \frac{1}{2}(m_1 + m_2) \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2$$

$$= \text{const} + \frac{1}{2} \mu \dot{\vec{r}}^2$$

$$U = U(|\vec{r}|)$$

$$E = \frac{1}{2} \mu \dot{\vec{r}}^2 + U(|\vec{r}|).$$

III Small oscillations about extrema of a potential.

$$U(x) = U_0(-ax^2 + bx^4), \qquad U_0, a, b > 0.$$

$$\frac{dU}{dx} = 0$$

$$\Rightarrow U_0(-2ax + 4bx^3) = 0$$

$$\Rightarrow x(-a + 2bx^2) = 0$$

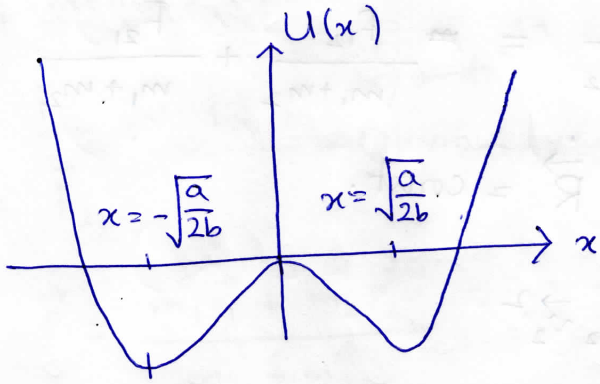
$$\Rightarrow x = 0, \pm\sqrt{\frac{a}{2b}}.$$

$$U(x=0) = 0$$

$$U\left(x = \pm\sqrt{\frac{a}{2b}}\right) = U_0\left(-a\frac{a}{2b} + b\cdot\frac{a^2}{4b^2}\right)$$

$$= U_0\left(-\frac{a^2}{2b} + \frac{a^2}{4b}\right)$$

$$= U_0\left(-\frac{a^2}{4b}\right)$$



$$\frac{d^2U}{dx^2} = U_0(-2a + 12bx^2)$$

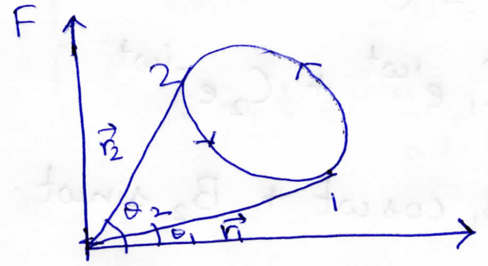
$$\left.\frac{d^2U}{dx^2}\right|_{x=0} = -2U_0 a < 0.$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=\pm\sqrt{\frac{a}{2b}}} = U_0\left(-2a + 12b\frac{a}{2b}\right) = 4aU_0 > 0.$$

$\cancel{E = \frac{1}{2}m\dot{x}^2 + U(x)}$ $\quad \omega = \sqrt{2aU_0}$

Example of non-conservative force.

$\vec{F} = \frac{A}{r}\hat{\theta}$. Recall, $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$



$dW = \vec{F}.d\vec{r}$

$= \frac{A}{r}.r\,d\theta$

$= A\,d\theta$

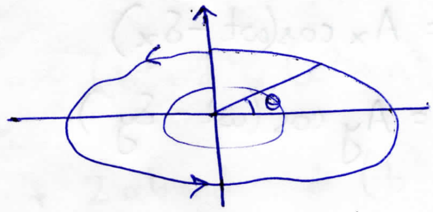
Work done in moving from θ_1 to θ_2 is given by,

$W = A(\theta_2 - \theta_1)$

Work done in moving around a complete path is,

$$W = \int_1^2 A\,d\theta + \int_2^1 A\,d\theta = A(\theta_2 - \theta_1) + A(\theta_1 - \theta_2) = 0$$

However,



$W = 2\pi A \neq 0$.

$\vec{\nabla} \times \vec{F} \neq 0$ at origin.

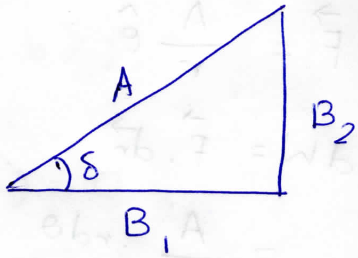
$$\cancel{\vec{\nabla} \times \vec{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r F_\theta) - \frac{\partial F_r}{\partial \theta}\right]}$$

$$\cancel{= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{A}{r}\right)}$$

$$\cancel{= \frac{1}{r}\frac{\partial A}{\partial r}}$$

OSCILLATORY MOTION

$$\ddot{x} + \omega^2 x = 0$$



$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$$= B_1 \cos\omega t + B_2 \sin\omega t.$$

$$x(t) = A\left[\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right]$$

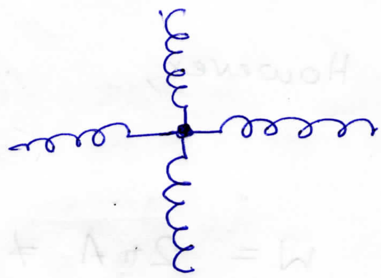
$$= A\left[\cos\delta \cos\omega t + \sin\delta \sin\omega t\right]$$

$$= A\cos(\omega t - \delta)$$

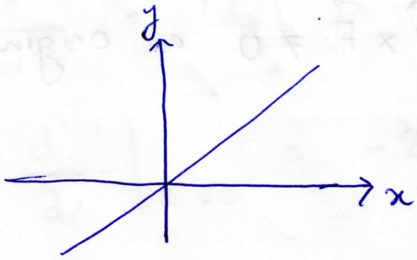
Consider 2-d oscillatory motion in xy plane.

$\ddot{x} = -\omega^2 x \qquad x(t) = A_x \cos(\omega t - \delta_x)$

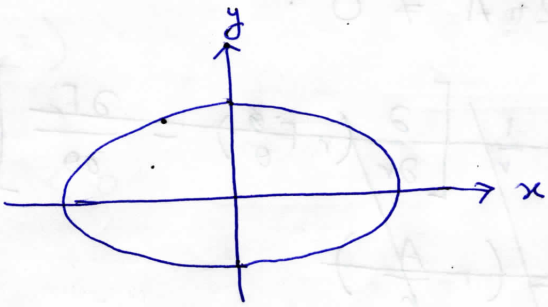
$\ddot{y} = -\omega^2 y \qquad y(t) = A_y \cos(\omega t - \delta_y)$



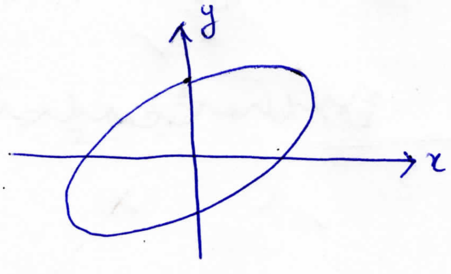
$$\delta = \delta_x - \delta_y$$



$\delta = 0$



$\delta = \pi/2$



$\delta = \pi/4.$

TWO DIMENSIONAL OSCILLATOR.

$$\left.\begin{aligned} x &= a\cos\omega t + c\sin\omega t \\ y &= b\cos\omega t + d\sin\omega t \end{aligned}\right\} \Rightarrow \begin{aligned} \cos\omega t &= \frac{dx - cy}{ad - bc} \\ \sin\omega t &= \frac{ay - bx}{ad - bc}. \end{aligned}$$

$$\cos^2\omega t + \sin^2\omega t = 1$$

$$\Rightarrow (dx - cy)^2 + (ay - bx)^2 = (ad - bc)^2.$$

$$\Rightarrow (b^2 + d^2)x^2 + (c^2 + a^2)y^2 - 2(ab + cd)xy = (ad - bc)^2.$$

$$(ad \neq bc)$$

$$\Rightarrow Ax^2 + Bxy + Cy^2 = D.$$

Above eqn. represents an ellipse if

$$B^2 - 4AC < 0$$

$$\Rightarrow 4(ab + cd)^2 - 4(b^2 + d^2)(c^2 + a^2) < 0.$$

$$\Rightarrow a^2b^2 + c^2d^2 + 2abcd - (b^2c^2 + a^2b^2 + c^2d^2 + d^2a^2) < 0$$

$$\Rightarrow \cancel{a^2b^2} + \cancel{c^2d^2} + 2abcd - b^2c^2 - \cancel{a^2b^2} - \cancel{c^2d^2} - a^2d^2 < 0$$

$$\Rightarrow -(ad - bc)^2 < 0.$$

SIMPLE PENDULUM WITHOUT SMALL ANGLE APPROXIMATION.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

Now, $u = \frac{d\theta}{dt}$

$$\therefore \frac{du}{dt} = -\frac{g}{l}\sin\theta.$$

$$\Rightarrow \frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -\frac{g}{l}\sin\theta.$$

$$\Rightarrow u\,du = -\frac{g}{l}\sin\theta\,d\theta.$$

Integrate,

$$\frac{u^2}{2} = -\frac{g}{l}\int_{\theta_0}^{\theta} d\theta\,\sin\theta$$

$$\Rightarrow u^2 = \frac{2g}{l}\cos\theta\Big|_{\theta_0}^{\theta}.$$

$$\Rightarrow u = \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$$

$$\Rightarrow \frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$$

$$\Rightarrow dt = \pm\frac{d\theta}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}$$

$$\frac{T}{4} = -\sqrt{\frac{l}{2g}}\int_{\theta_0}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\Rightarrow T = -4\sqrt{\frac{l}{2g}}\int_{\theta_0}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\Rightarrow T = 4\sqrt{\frac{l}{2g}}\int_{0}^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

$$\cos\theta = 1 - 2\sin^2\theta.$$

$$T = 2\sqrt{\frac{l}{g}}\int_{0}^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

Let $\sin\frac{\theta}{2} = \left(\sin\frac{\theta_0}{2}\right)(\sin\phi)$

$$\Rightarrow \sin\frac{\theta}{2} = k\sin\phi$$

$$\cancel{T = 2\sqrt{\frac{l}{g}}\int_{0}^{\pi/2}} \quad \frac{1}{2}\cos\frac{\theta}{2}\,d\theta = k\cos\phi\,d\phi$$

$$\Rightarrow d\theta = \frac{2k\cos\phi}{\cos\frac{\theta}{2}}\,d\phi.$$

$$\Rightarrow d\theta = \frac{2k\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi.$$

$$\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}} = k\cos\phi.$$

$$T = 2\sqrt{\frac{l}{g}}\,2\int_{0}^{\pi/2}\frac{\cancel{k\cos\phi}\,d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$\Rightarrow T = 4\sqrt{\frac{l}{g}}\int_{0}^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \cdots$$

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2\phi + \cdots\right) d\phi$$

$$\Rightarrow T = 4\sqrt{\frac{l}{g}} \left[\frac{\pi}{2} + \frac{1}{2}k^2 \int_0^{\pi/2} d\phi \sin^2\phi + \cdots\right].$$

$$\Rightarrow T = 4\sqrt{\frac{l}{g}} \left[\frac{\pi}{2} + \frac{1}{2}k^2 \frac{\pi}{4} + \cdots\right]$$

$$\Rightarrow T = 2\pi\sqrt{\frac{l}{g}} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \cdots\right].$$

$\boxed{\text{///}}$ DAMPED OSCILLATION.

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Trial solⁿ:- $x(t) = e^{rt}$.

$$r^2 + 2\beta r + \omega_0^2 = 0$$

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \qquad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}.$$

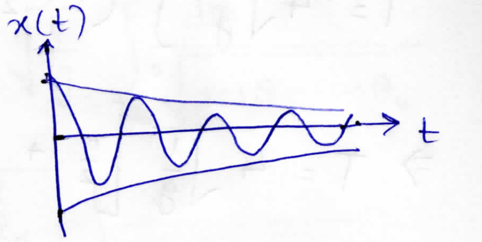
$$x(t) = e^{-\beta t}\left(A e^{\sqrt{\beta^2 - \omega_0^2}\, t} + B e^{-\sqrt{\beta^2 - \omega_0^2}\, t}\right)$$

- Underdamped oscillation :-

$$\boxed{\beta < \omega_0} \qquad \sqrt{\beta^2-\omega_0^2} = i\sqrt{\omega_0^2-\beta^2} = i\,\omega_1$$

$$x(t) = e^{-\beta t}\left(A e^{i\omega_1 t} + B e^{-i\omega_1 t}\right).$$

$$= A e^{-\beta t}\cos(\omega_1 t - \delta).$$



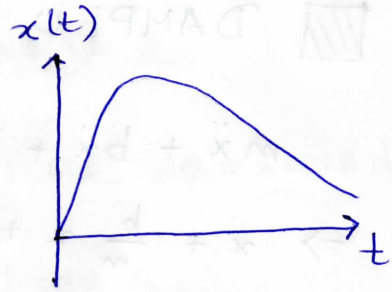
- Overdamped oscillation :-

$$\boxed{\beta > \omega_0} \qquad x(t) = e^{-\beta t}\left(A e^{\sqrt{\beta^2-\omega_0^2}\,t} + B e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right).$$

$$= A e^{(-\beta + \sqrt{\beta^2-\omega_0^2})t} + B e^{-(\beta+\sqrt{\beta^2-\omega_0^2})t}.$$

$$= \underbrace{A e^{-(\beta-\sqrt{\beta^2-\omega_0^2})t}} + B e^{-(\beta+\sqrt{\beta^2-\omega_0^2})t}.$$

decreases more slowly, hence long term evolution is dominated by this term.



decay parameter $= \beta - \sqrt{\beta^2-\omega_0^2}$

$$\beta - \sqrt{\beta^2-\omega_0^2} = \beta - \beta\sqrt{1-\frac{\omega_0^2}{\beta^2}}.$$

$$= \beta - \beta\left(1 - \frac{\omega_0^2}{2\beta^2} + \cdots\right)$$

$$= \frac{\omega_0^2}{2\beta}.$$

→ note that large value of β leads to smaller decay parameter.

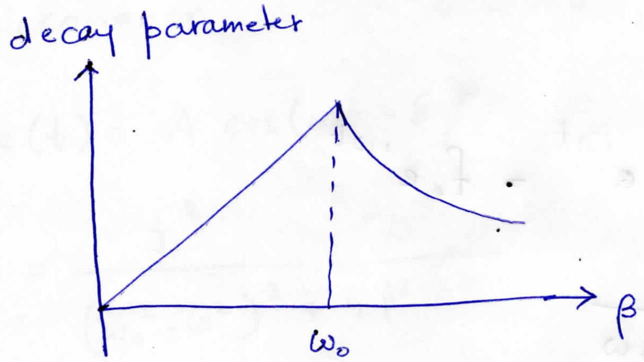
- Critically damped oscillation:-

$\boxed{\beta = \omega_0}$ $\quad x(t) = A e^{-\beta t} \longrightarrow$ note that this is only a single solution.

It can be explicitly verified that $x(t) = t e^{-\beta t}$ is another solution.

$\therefore \; x(t) = \cancel{A(1+t)e^{-\beta t}} \; (A + Bt) e^{-\beta t}$.

damping	β	decay parameter
none	0	0
under	$\beta < \omega_0$	β
critical	$\beta = \omega_0$	β
over	$\beta > \omega_0$	$\beta - \sqrt{\beta^2 - \omega_0^2}$



VIII DRIVEN DAMPED OSCILLATOR.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t).$$

↳ Inhomogeneous linear ODE.

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

$$Dx = f$$

$$Dx_p = f$$

↳ particular solⁿ.

$$Dx_h = 0 \Rightarrow x_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

$$\underbrace{D(x_p + x_h)}_{\text{general solution.}} = Dx_p + Dx_h = f + 0 = f.$$

$$f(t) = f_0 \cos\omega t.$$

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}.$$

Try, $z(t) = Ce^{i\omega t}$.

$$(-\omega^2 + 2\beta i\omega + \omega_0^2)Ce^{i\omega t} = f_0 e^{i\omega t}$$

$$\Rightarrow C = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega}$$

$$= \frac{f_0\left[(\omega_0^2 - \omega^2) - 2i\beta\omega\right]}{\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]}$$

$$C = Ae^{-i\delta} = (A\cos\delta - iA\sin\delta)$$

$$A^2 = C^* C$$

$$= \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$\Rightarrow \delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

$$z(t) = C e^{i\omega t} = A e^{i(\omega t - \delta)}.$$

$$x_p(t) = A\cos(\omega t - \delta)$$

$$x(t) = A\cos(\omega t - \delta) + \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}_{\text{transient}}.$$

Resonance.

$$x(t) = A\cos(\omega t - \delta)$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

A^2 is maximum when $\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]$ is minimum.

<u>Case 1</u> :- Vary ω_0 with ω fixed $\rightarrow$ $\omega = \omega_0$ minimises denominator

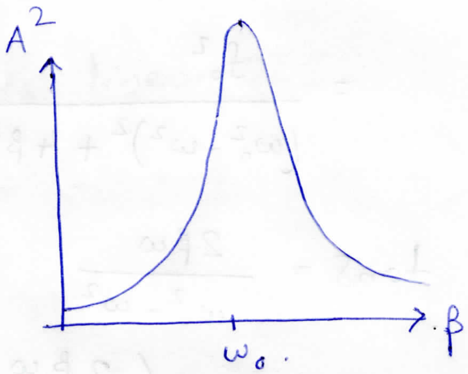
<u>Case 2</u> :- Vary ω with ω_0 fixed.

$$\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right] = 2(\omega_0^2-\omega^2)(-2\omega)+8\beta^2\omega = 0$$

$$\Rightarrow \quad \omega_0^2-\omega^2 = 2\beta^2$$

$$\Rightarrow \quad \omega = \sqrt{\omega_0^2-2\beta^2} = \omega_2 \text{ (say)}$$

When $\beta \ll \omega_0$, $\omega_2 \approx \omega_0$



– Width of resonance.

FULL WIDTH AT HALF MAXIMUM (FWHM)

$$A_{max}^2 \approx \frac{f_0^2}{4\beta^2\omega_0^2}$$

$$\Rightarrow \quad A_{max} \approx \frac{f_0}{2\beta\omega_0}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} = \frac{f_0^2}{8\beta^2\omega_0^2}.$$

$$\Rightarrow \quad 8\beta^2\omega_0^2 = (\omega_0^2-\omega^2)^2+4\beta^2\omega^2.$$

$$\cancel{\Rightarrow \quad 8\beta^2\omega_0^2 \quad 4\beta^2\omega^2 = (\omega_0^2-\omega^2)}$$

$$\cancel{\Rightarrow \quad 4\beta^2} \quad \Rightarrow \quad [(\omega_0+\omega)(\omega_0-\omega)]^2 = 8\beta^2\omega_0^2-4\beta^2\omega^2$$

$$\Rightarrow \quad [2\omega_0(\omega_0-\omega)]^2 = 8\beta^2\omega_0^2-4\beta^2\omega^2$$

$$\Rightarrow \quad 4\omega_0^2(\omega_0-\omega)^2 = 8\beta^2\omega_0^2-4\beta^2\omega^2.$$

$$\Rightarrow \quad 4\omega_0^2(\omega-\omega_0)^2+4\beta^2\omega^2 = 8\beta^2\omega_0^2$$

$$\Rightarrow \quad \omega_0^2(\omega-\omega_0)^2+\beta^2\omega_0^2 \approx 2\beta^2\omega_0^2$$

$$\Rightarrow \quad (\omega-\omega_0)^2 \approx \beta^2 \quad \Rightarrow \quad \omega-\omega_0 = \pm\beta \Rightarrow \boxed{\omega = \omega_0 \pm \beta.}$$

FWHM $\approx 2\beta$ $\Rightarrow$ sharpness of resonance peak. related to FWHM.

$$Q = \frac{\omega_0}{2\beta} \rightarrow \text{Q-factor}$$

↳ large Q indicates narrow resonance. and vice versa.

$Ae^{-\beta t} \rightarrow$ falls to e^{-1} of its initial value in $t \approx 1/\beta$.

$$(\text{decay time}) = \frac{1}{\beta}$$

$$\text{period} = \frac{2\pi}{\omega_0} \quad (\beta << \omega_0)$$

$$Q = \frac{\omega_0}{2\beta} = \pi \frac{1/\beta}{\frac{2\pi}{\omega_0}} = \pi \frac{\text{decay time}}{\text{period}}$$

At resonance, $\omega_0 \approx \omega$. At this point, the ODE to be solved is

$$\ddot{x} + \omega^2 x = f_0 \cos\omega t$$

The corresponding homogeneous ODE is,

$$\ddot{x}_h + \omega^2 x_h = 0$$

$$\Rightarrow x_h = A\cos\omega t + B\sin\omega t$$

Assume, $x_p = t(c_1\cos\omega t + c_2\sin\omega t)$.

$$\dot{x}_p = (c_1\cos\omega t + c_2\sin\omega t) + t(-\omega c_1\sin\omega t + \omega c_2\cos\omega t)$$

$$\cancel{\ddot{x}_p = (-\omega c_1\sin\omega t + \omega c_2\cos\omega t) + t(-\omega c_1\sin\omega t + \omega c_2\cos\omega t)}$$

$$\ddot{x}_p = \cancel{(-\omega c_1\sin\omega t + \omega c_2\cos\omega t)} + t(-\omega^2 c_1\cos\omega t \cancel{+ \omega c_2 \cos})$$

$$\ddot{x}_p = (-\omega c_1 \sin\omega t + \omega c_2 \cos\omega t) + (-\omega c_1 \sin\omega t + \omega c_2 \cos\omega t) +$$

$$t(-\omega^2 c_1 \cos\omega t - \omega^2 c_2 \sin\omega t)$$

$$= +2\omega(-c_1 \sin\omega t + c_2 \cos\omega t) - t\omega^2(c_1\cos\omega t + c_2 \sin\omega t)$$

$$= 2\omega(-c_1 \sin\omega t + c_2\cos\omega t) - \cancel{t}\,\omega^2 x_p(t)$$

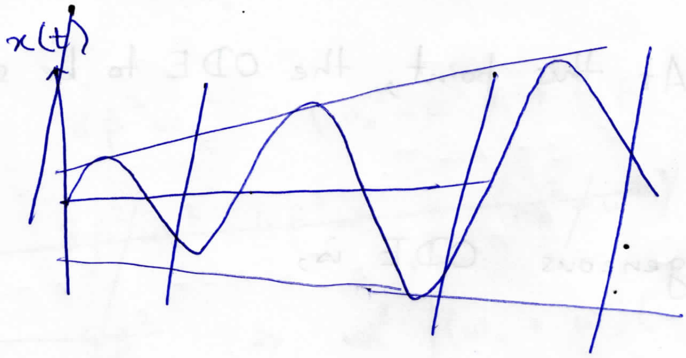
$\Rightarrow$ ~~x_p~~ Substituting,

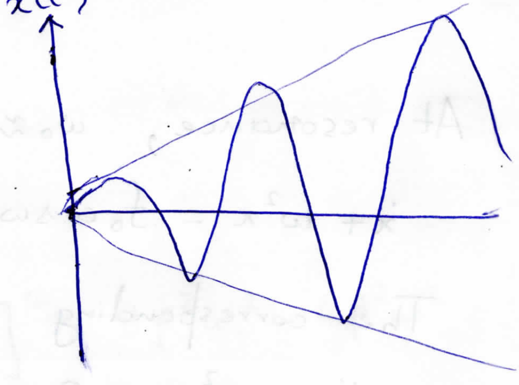
$$\ddot{x}_p + \omega^2 x_p = f_0 \cos\omega t$$

$$\Rightarrow \quad 2\omega(-c_1 \sin\omega t + c_2 \cos\omega t) = f_0 \cos\omega t$$

$$\Rightarrow \quad \therefore c_1 = 0 \qquad 2\omega c_2 = f_0 \Rightarrow c_2 = \frac{f_0}{2\omega}$$

$$\therefore \boxed{x(t) = A\cos\omega t + B\sin\omega t + \left(\frac{f_0}{2\omega}\right) t \sin\omega t}$$





- Phase shift at resonance

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

If $\omega << \omega_0$, $\quad \delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2}\right) \rightarrow$ small quantity.

At $\omega \approx \omega_0$, $\quad \delta = \pi/2$.

If $\omega > \omega_0$, $\quad \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) < 0$ and appoaches zero gradually.

III COUPLED OSCILLATORS.

Consider the coupled ODEs,

$$2\ddot{x} + \omega^2(5x - 3y) = 0 \quad \text{——— (1)}$$

$$2\ddot{y} + \omega^2(5y - 3x) = 0 \quad \text{——— (2)}$$

(1) + (2),

$$2(\ddot{x} + \ddot{y}) + 2\omega^2(x + y) = 0$$

$$\Rightarrow (\ddot{x} + \ddot{y}) + \omega^2(x + y) = 0$$

$$x + y = A_1 \cos(\omega t + \phi_1)$$

(1) − (2),

$$(\ddot{x} - \ddot{y}) + 4\omega^2(x - y) = 0$$

$$x - y = A_2 \cos(2\omega t + \phi_2).$$

$$\therefore \; x(t) = B_1 \cos(\omega t + \phi_1) + B_2 \cos(2\omega t + \phi_2)$$

$$y(t) = B_1 \cos(\omega t + \phi_1) - B_2 \cos(2\omega t + \phi_2).$$