LECTURE 14

COUPLED OSCILLATORS.

- Coupled equations. Such as,

$$2\ddot{x} + \omega^2(5x - 3y) = 0 \quad \text{—— (1)}$$

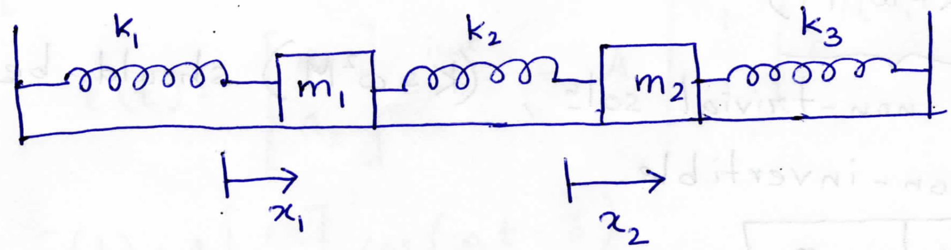
$$2\ddot{y} + \omega^2(5y - 3x) = 0 \quad \text{—— (2)}$$

$$(1) + (2) \Rightarrow \quad (\ddot{x} + \ddot{y}) = -\omega^2(x+y)$$

$$(1) - (2) \Rightarrow \quad (\ddot{x} - \ddot{y}) = -4\omega^2(x-y)$$

Linear combinations $(x+y)$ and $(x-y)$ oscillate with "pure" frequencies ω and 2ω respectively.

Systematic treatment.



EOM :- $\quad m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2$

$$m_2\ddot{x}_2 = k_2 x_1 - (k_2 + k_3)x_2$$

Above eqns. can be written in the matrix notation,

$$M\ddot{x} = -Kx, \quad \text{where } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Guessing that solns. should be oscillatory,

$x_1(t) = \alpha_1 \cos(\omega t - \delta_1)$ OR $y_1(t) = \alpha_1 \sin(\omega t - \delta_1)$

$x_2(t) = \alpha_2 \cos(\omega t - \delta)$ $y_2(t) = \alpha_2 \sin(\omega t - \delta_1)$

Also,

$$z_1(t) = x_1(t) + i y_1(t) = \alpha_1 e^{i(\omega t - \delta_1)} = a_1 e^{i\omega t}.$$

$$z_2(t) = a_2 e^{i\omega t}$$

Actual soln given by $Re(\bar{z}(t)) = \bar{x}(t)$.

$$\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \bar{a} e^{i\omega t}$$

where $\bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{bmatrix}$

Substitute, $-\omega^2 M a e^{i\omega t} = -K a e^{i\omega t}$.

$$\Rightarrow \underbrace{(\bar{K} - \omega^2 \bar{M})}\,\bar{a} = 0.$$

For non-trivial soln, $(K - \omega^2 M)$ should be non-invertible.

$$\therefore \boxed{|K - \omega^2 M| = 0}$$

Simplification :- $m_1 = m_2 = m$, $k_1 = k_2 = k$.

$$\bar{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \bar{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$(\bar{K} - \omega^2 \bar{M}) = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix}$$

$$|K - \omega^2 M| = 0$$

$$\Rightarrow (2k - m\omega^2) = \pm k.$$

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}.$$

Recall, solution to the eom. are $\bar{z}(t) = \bar{a} e^{i\omega t}$.

$$\bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \text{ satisfying } (\bar{K} - \omega^2 \bar{M}) \bar{a} = 0.$$

- FIRST NORMAL MODE.

$$(\bar{K} - \omega_1^2 \bar{M}) = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$ $\Rightarrow$ Note that $|\bar{K} - \omega_1^2 \bar{M}| = 0$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow \left.\begin{matrix} a_1 - a_2 = 0 \\ -a_1 + a_2 = 0 \end{matrix}\right\} \Rightarrow a_1 = a_2 = Ae^{-i\delta}$$

$$\therefore \bar{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_1 t - \delta)}$$

$$\bar{x}(t) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta).$$

$$\therefore \left.\begin{matrix} x_1(t) = A\cos(\omega_1 t - \delta) \\ x_2(t) = A\cos(\omega_1 t - \delta) \end{matrix}\right\} \rightarrow \text{Normal mode.}$$

- Second NORMAL MODE

$$(\bar{K} - \omega_2^2 \bar{M}) = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}$$

$$(\bar{K} - \omega_2^2 \bar{M}) a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_1 = -a_2 = Ae^{-i\delta}.$$

$$\therefore \quad z(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_2 t} = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_2 t - \delta)}$$

$$\left.\begin{aligned} x_1(t) &= A\cos(\omega_2 t - \delta) \\ x_2(t) &= -A\cos(\omega_2 t - \delta) \end{aligned}\right\} \rightarrow \text{Normal mode.}$$

– GENERAL SOLUTION.

$$\bar{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) \quad \text{and} \quad \bar{x}(t) = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

$$\bar{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

III Normal coordinates.

$$\xi_1 = \frac{1}{2}(x_1 + x_2)$$

$$\xi_2 = \frac{1}{2}(x_1 - x_2)$$

$$\left.\begin{aligned} \xi_1(t) &= A\cos(\omega_1 t - \delta) \\ \xi_2(t) &= 0 \end{aligned}\right\} \text{1st normal mode}$$

$$\left.\begin{aligned} \xi_1(t) &= 0 \\ \xi_2(t) &= A\cos(\omega_2 t - \delta) \end{aligned}\right\} \text{2nd normal mode.}$$

ξ_1 and ξ_2 are independent, each can oscillate independently.

III Alternative treatment.

Note:- Matrix $\bar{M}$ can be absorbed into K in R.H.S by considering $(M^{-1}K)$.

$$\bar{M}\ddot{x} = -\bar{K}x.$$

$$= \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} x.$$

- Aim, → write $\bar{K}$ in a basis where it is diagonal.

$$x' = Ox.$$

$$\Rightarrow x = O^T x'$$

$$O^T \bar{M}\ddot{x}' = -\bar{K} O^T x'$$

$$\Rightarrow \bar{M}\ddot{x}' = -O\bar{K}O^T x'$$

$$= -K_D x', \quad \text{where } K = O^T K_D O$$

Consider $\bar{M}^{-1}\bar{K} = \frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$\bar{S} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ → similarity transformation which diagonalizes $\bar{M}^{-1}\bar{K}$ with eigenvalues, $\left(\frac{3k}{m}, \frac{k}{m}\right)$

$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \bar{S}.\bar{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} -x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$

III WEAKLY COUPLED OSCILLATORS.

Take the limit $k_2 << k$ in the previous example:-

$$\bar{K} = \begin{bmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{bmatrix}$$

$$(\bar{K} - \omega^2 \bar{M}) = \begin{bmatrix} k+k_2-m\omega^2 & -k_2 \\ -k_2 & k+k_2-m\omega^2 \end{bmatrix}$$

$$|\bar{K} - \omega^2 \bar{M}| = (k+k_2-m\omega^2)^2 - k_2^2 = 0$$

$$\Rightarrow k+k_2-m\omega^2 = \pm k_2$$

$k+k_2-m\omega_1^2 = +k_2$ $\qquad$ $k+k_2-m\omega_2^2 = -k_2$

$\Rightarrow \omega_1 = \sqrt{\dfrac{k}{m_2}}$ $\qquad$ $\Rightarrow \omega_2 = \sqrt{\dfrac{k+2k_2}{m}}$

In phase mode.

Now, $k_2 << k$. So define $\omega_0 = \dfrac{\omega_1+\omega_2}{2}$

$\omega_1 = \omega_0 - \epsilon$

$\omega_2 = \omega_0 + \epsilon$

$$\therefore \bar{z}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0-\epsilon)t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0+\epsilon)t}$$

$$\bar{z}(t) = \underbrace{\left\{ C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\epsilon t} \right\}}_{\text{Since } \epsilon \text{ is small, varies slowly}} e^{i\omega_0 t}$$

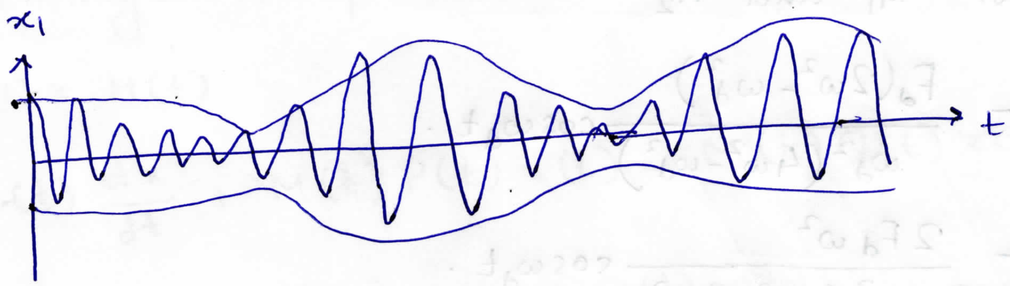
Since ϵ is small, varies slowly.

Say, $C_1 = C_2 = \frac{A}{2}$.

$$\bar{z}(t) = \frac{A}{2}\begin{bmatrix} e^{-i\epsilon t} + e^{i\epsilon t} \\ e^{-i\epsilon t} - e^{i\epsilon t} \end{bmatrix} e^{i\omega_0 t}$$

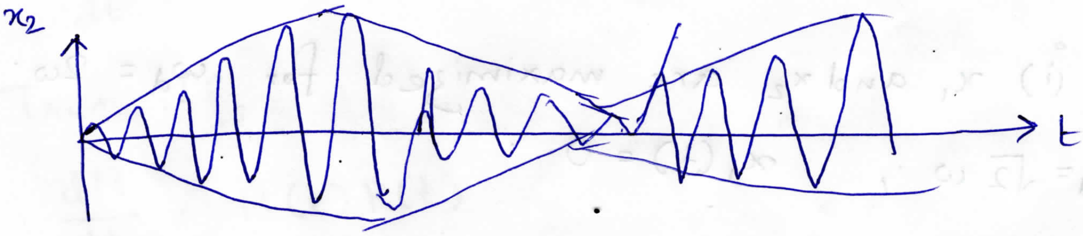
$$= A \begin{bmatrix} \cos\epsilon t \\ -i\sin\epsilon t \end{bmatrix} e^{i\omega_0 t}$$

$$\bar{x}(t) = \mathrm{Re}\,\bar{z}(t) \Rightarrow x_1(t) = A\cos\epsilon t\,\cos\omega_0 t$$
$$x_2(t) = A\sin\epsilon t\,\sin\omega_0 t.$$



$\xi_1 = \frac{1}{2}(x_1 + x_2)$

$\xi_2 = \frac{1}{2}(x_1 - x_2)$.



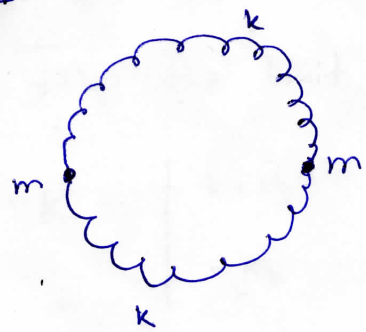
$x_1 = \xi_1(t) + \xi_2(t$

$x_2 = \xi_1 - \xi_2(t$

$$\xi_1(t) = \frac{1}{2}A\cos(\omega_0 - \epsilon)t = \frac{1}{2}A\cos\omega_1 t$$

$$\xi_2(t) = \frac{1}{2}A\cos(\omega_0 + \epsilon)t = \frac{1}{2}A\cos\omega_2 t.$$

DRIVEN MASS ON A CIRCLE



$$m\ddot{x}_1 + 2k(x_1 - x_2) = F_d \cos\omega_d t$$

$$m\ddot{x}_2 + 2k(x_2 - x_1) = 0.$$

Try, :- $z_1 = A_1 e^{i\omega_d t}$

$$z_2 = A_2 e^{i\omega_d t}.$$

$$-\omega_d^2 A_1 + 2\omega^2(A_1 - A_2) = F_d.$$

$$-\omega_d^2 A_2 + 2\omega^2(A_2 - A_1) = 0$$

Solving for A_1 and A_2 :-

$$x_1(t) = -\frac{F_d(2\omega^2 - \omega_d^2)}{\omega_d^2(4\omega^2 - \omega_d^2)}\cos\omega_d t.$$

$$x_2(t) = -\frac{2F_d\,\omega^2}{\omega_d^2(4\omega^2 - \omega_d^2)}\cos\omega_d t.$$

Remarks :- (i) x_1 and x_2 are maximized for $\omega_d = 2\omega$.

(ii) If $\omega_d = \sqrt{2}\,\omega$, $x_1(t) = 0$.

No driving force :- Normal modes.

$$m\ddot{x}_1 + 2k(x_1 - x_2) = 0 \quad \text{——— (1)}$$

$$m\ddot{x}_2 + 2k(x_2 - x_1) = 0 \quad \text{——— (2)}$$

(1) + (2),

$$m(\ddot{x}_1 + \ddot{x}_2) = 0 \Rightarrow \ddot{x}_1 + \ddot{x}_2 = 0$$

$$(1) - (2) \Rightarrow (\ddot{x}_1 - \ddot{x}_2) + 4\omega^2(x_1 - x_2) = 0$$

$$\Rightarrow \quad x_1 + x_2 = At + B$$

$$x_1 - x_2 = C\cos(2\omega t + \phi)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(At+B) + C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega t + \phi)$$

(redefined const).

☑ METHOD OF INTEGRATING FACTORS

$$\frac{dx}{dt} + P(t)\,x(t) = Q(t) \quad \text{—— (1)}$$

- Inhomogeneous eqn.
- (1)·× $M(t)$

$$M(t)\frac{dx}{dt} + M(t)P(t)\,x(t) = M(t)Q(t)$$

$$\Rightarrow \frac{d}{dt}\left(M(t)\,x(t)\right) = M(t)Q(t). \quad \text{—— (1)}$$

True iff,

$$\frac{dM}{dt} = M(t)P(t).$$

$$\Rightarrow \frac{dM}{M} = P(t)\,dt$$

$$\Rightarrow \ln M = \int dt\,P(t).$$

$$\Rightarrow M = \exp\left[\int dt\,P(t)\right].$$

From eqn. (1), $M(t)\,x(t) = \int dt\,M(t)Q(t)$.

$$\Rightarrow x(t) = M^{-1}\int dt\,M(t)Q(t)$$

— Eqn. with exponential R.H.S.

$(m-a)(m-b)y = F(x) = ke^{cx}$, $c \neq a, b$.

Example:- $(m-1)(m+5)y = 7e^{2x}$.

$(m^2+4m-5)y = 7e^{2x}$.

$y_p = Ce^{2x}$.

$y_p'' + 4y_p' - 5y_p = C\left(4e^{2x} + 8e^{2x} - 5e^{2x}\right) = 7e^{2x}$.

$\Rightarrow 7Ce^{2x} = 7e^{2x}$.

$\Rightarrow C = 1$.

~~— Solution of above eqn. by IF.~~

$\cancel{y'' + 4y' - 5y = 7e^{2x}}$.

~~Let~~ $\cancel{u = \left(\frac{d}{dx} + 2\right)y}$.

— Successive integration.

$(D^2 + D - 2)y = e^x$

$y'' + y' - 2y = e^x$.

$\Rightarrow (D-1)(D+2)y = e^x$.

Let $u = (D+2)y$.

$\therefore (D-1)u = e^x$.

$\Rightarrow u' - u = e^x$.

$P(x) = -1$ $\quad Q(x) = e^x$.

$IF = e^{-x} + c_1$ $\quad u_p = xe^x + c_1e^x$.

$\therefore (D+2)y = xe^{x} + c_1 e^{x}.$

$\Rightarrow y' + 2y = xe^{x} + c_1 e^{x}.$

$\cancel{I}$ $P(x) = 2$

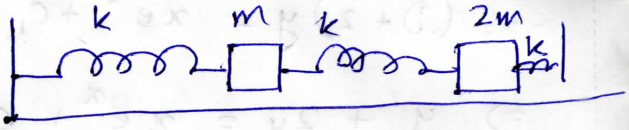
$IF = e^{2x}.$

$y = e^{-2x} \int dx\, e^{2x} (xe^{x} + c_1 e^{x}) + c_2.$

$= e^{-2x} \int dx\, (xe^{3x} + c_1 e^{3x}) + c_2.$

$= e^{-2x} \int dx\, e^{3x} (x + c_1) + c_2.$

<u>PROB</u>:- Unequal masses



$$\ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 = 0$$

$$2\ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 = 0 .$$

Let $x_1 = A_1 e^{i\alpha t}$
$x_2 = A_2 e^{i\alpha t}$.

$$\begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -2\alpha^2 + 2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow 2\alpha^4 - 6\alpha^2\omega^2 + 3\omega^4 = 0$$

$$\alpha_1 = \pm \omega\sqrt{\frac{3+\sqrt{3}}{2}} \qquad \alpha_2 = \pm \omega\sqrt{\frac{3-\sqrt{3}}{2}}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha_1 t + \phi_1)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha_2 t + \phi_2) .$$

Alternately, $\bar{M} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. $\bar{M}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$

$$\bar{K} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \bar{M}^{-1}K = \begin{pmatrix} 2 & -1 \\ -1/2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) - 1/2 = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) = 1/2$$

$$\Rightarrow 2 - 2\lambda - \lambda + \lambda^2 = 1/2$$

$$\Rightarrow \lambda^2 - 3\lambda + \frac{3}{2} = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-6}}{2}$$

$$= \frac{1}{2}(3 \pm \sqrt{3}).$$

$$0 = \left(\begin{pmatrix} -(1+\sqrt{3}) \\ 1 \end{pmatrix} \quad \begin{pmatrix} -(1-\sqrt{3}) \\ 1 \end{pmatrix} \right).$$

Going back,

$$\cancel{x_1 = (\sqrt{3}+1)\cos(\alpha_1 t + \phi_1)}$$
$$\cancel{x_2 = -\cos(\alpha_1 t + \phi_1).}$$

$$\cancel{x_1 + x_2 = (\sqrt{3}+1-1)\cos(\alpha_1 t + \phi_1) = \sqrt{3}\cos(\omega_1 t + \phi_1)}$$
$$\cancel{x_1 - x_2 = (\sqrt{3}+2)\cos(\alpha_1 t + \phi_1) = (\sqrt{3}+2)\cos(\omega_2 t + \phi_1)},$$

General sol^n is :-

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B_1 \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha_1 t + \phi_1) + B_2 \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha_2 t + \phi_2)$$

How to guess the normal modes from this ?

- How to guess the normal modes are "pure" frequency.
- Recall that the normal modes should be to find linear combinatio[ns]
- So, the procedure should be to find linear combinations which show either no $\dot{\alpha}_1$ dependence or no α_2 depende[nce]

These linear combinations are :- $x_1 + (\sqrt{3}+1)x_2$
$x_1 - (\sqrt{3}-1)x_2$

Obtained by solving, $B_1(\sqrt{3}+1) + B_2(\sqrt{3}-1) = x$
$-B_1 + B_2 = y$.

Example:- $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cos(\omega_1 t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} \cos(\omega_2 t + \phi_2)$

- What are the normal modes?
- The combinations, $(5x_1 + x_2)$
  $(2x_1 - 3x_2)$.
- Obtained by solving, $3B_1 + B_2 = x_1$
  $2B_1 - 5B_2 = x_2$

PROB:- Coupled + Damped :-

$$\left.\begin{aligned} 2\ddot{x} + \omega^2(5x - 3y) &= -b\dot{x} \\ 2\ddot{y} + \omega^2(5y - 3x) &= -b\dot{y} \end{aligned}\right\} \quad \begin{aligned} 2\ddot{x} + b\dot{x} + \omega^2(5x - 3y) &= 0 \\ 2\ddot{y} + b\dot{y} + \omega^2(5y - 3x) &= 0 \end{aligned}$$