Signals and Systems (CT 203)

Tutorial Sheet-11 DA-IICT, Gandhinagar.

1. Find the Discrete-time Fourier transform (DTFT) of

$$x(n) = (0.2)^n\, u(n)$$

Solution:-

$$x(n) = (0.2)^n\, u(n)$$

Taking DTFT, $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} (0.2)^n u(n) e^{-j\omega n}$

$$\therefore X\left(e^{j\omega}\right) = \sum_{n=0}^{+\infty} (0.2)^n e^{-j\omega n} \qquad \{\because u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}\}$$

$$\boxed{\therefore X\left(e^{j\omega}\right) = \sum_{n=0}^{+\infty} \left(0.2e^{-j\omega}\right)^n = \frac{1}{1-0.2e^{-j\omega}}} \; \underline{ANS} \quad \{\because \sum_{n=0}^{+\infty} a(r)^n = \frac{1}{1-r}, r<1\}$$

2. Prove that the DTFT of discrete-time signal $x(n)$, is periodic, i.e.,

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right) \qquad \text{where, } X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

Solution:-

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} e^{-j2\pi n} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X\left(e^{j\omega}\right).$$

$\{\because e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1 - 0 = 1, n \in \mathbb{Z}\}$

$$\therefore X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right).$$

Hence, proved. Inference: -> DTFT is periodic with period 2π in the frequency-domain.

3. Prove following *time* shifting and *frequency* shifting property of DTFT

$$x(n-n_o) \overset{F}{\leftrightarrow} e^{-j\omega n_o} X\left(e^{j\omega}\right) \quad \text{(Time shifting)}$$

$$e^{j\omega_o n} x(n) \overset{F}{\leftrightarrow} X\left(e^{j(\omega-\omega_o)}\right) \quad \text{(Frequency shifting)}$$

Solution:-

Time-shifting Property:-

$$F\left(x\left(n-n_0\right)\right)=\sum_{n=-\infty}^{+\infty} x\left(n-n_0\right)e^{-j\omega n}$$

Let $m=n-n_0$, $n=-\infty \Rightarrow m=-\infty, n=\infty \Rightarrow m=\infty$

$$\therefore F\left(x\left(n-n_0\right)\right)=\sum_{m=-\infty}^{+\infty} x\left(m\right)e^{-j\omega(m+n_0)}$$

$$=e^{-j\omega n_0}\sum_{m=-\infty}^{+\infty} x\left(m\right)e^{-j\omega m} \qquad \{\ \because\ e^{-j\omega n_0} \text{ is not dependent on summation variable } m\ \}$$

$$=e^{-j\omega n_0}X\left(e^{j\omega}\right) \qquad \{\text{ from DTFT definition }\}$$

$\therefore x\left(n-n_o\right)\overset{F}{\leftrightarrow}e^{-j\omega n_o}\ X\left(e^{j\omega}\right)$. Hence, proved.

Frequency-shifting Property:-

$$F\left(e^{j\omega_0 n}x\left(n\right)\right)=\sum_{n=-\infty}^{+\infty} e^{j\omega_0 n}x\left(n\right)e^{-j\omega n}=\sum_{n=-\infty}^{+\infty} x\left(n\right)e^{-j(\omega-\omega_0)n}$$

Let $\omega-\omega_0=\theta$ (temporary)

$$F\left(e^{j\omega_0 n}x\left(n\right)\right)=\sum_{n=-\infty}^{+\infty} x\left(n\right)e^{-j\theta n}=X\left(e^{j\theta}\right)=X\left(e^{j(\omega-\omega_0)}\right) \ \{\text{ again substitute } \theta=\omega-\omega_0\ \}$$

$\therefore e^{j\omega_o n}x\left(n\right)\overset{F}{\leftrightarrow}X\left(e^{j(\omega-\omega_o)}\right)$. Hence proved.

4. Prove *Parseval's* relation for DTFT $$\sum_{n=-\infty}^{+\infty}\left|x\left(n\right)\right|^2=\frac{1}{2\pi}\int_{2\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$ Explain the fundamental concept conveyed by this theorem.

Solution:-

LHS= $\displaystyle\sum_{n=-\infty}^{+\infty}\left|x\left(n\right)\right|^2=\sum_{n=-\infty}^{+\infty} x\left(n\right)x^*\left(n\right)$

$$= \sum_{n=-\infty}^{+\infty} x(n) \left\{ \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega \right\}^{*}$$ {Using, IDTFT definition, i.e.,

$x(n) = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$, in the second term}

$$= \sum_{n=-\infty}^{+\infty} x(n) \left\{ \frac{1}{2\pi} \int_{2\pi} X^{*}\left(e^{j\omega}\right) e^{-j\omega n} d\omega \right\}$$ {Interchange $\sum$ and $\int$ operators, carefully!!}

$$= \frac{1}{2\pi} \int_{2\pi} X^{*}\left(e^{j\omega}\right) \left\{ \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n} \right\} d\omega$$ {Using DTFT definition for the term which is inside the bracket}

$$= \frac{1}{2\pi} \int_{2\pi} X^{*}\left(e^{j\omega}\right) X\left(e^{j\omega}\right) d\omega = \frac{1}{2\pi} \int_{2\pi} \left| X\left(e^{j\omega}\right) \right|^{2} d\omega \ = \textbf{RHS}$$

Hence proved.

Inference -> This *Parseval's* relation says, the energy of the signal remains same in time-domain and frequency-domain, i.e., it is energy conservation principle in DTFT framework.

5. Prove the *convolution* property (theorem) for DTFT, if $y(n) = x(n) * h(n)$, then $Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$

Solution

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$Y\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} y(n) e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} x(k) h(n-k) \right) e^{-j\omega n}$$

$$\therefore Y\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} x(k) e^{-j\omega k} \right) h(n-k) e^{-j\omega n} e^{j\omega k}$$ {due to DTFT definition}

$$\therefore Y\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} \left(X\left(e^{j\omega}\right) \right) h(n-k) e^{-j\omega(n-k)} = X\left(e^{j\omega}\right) \sum_{n=-\infty}^{+\infty} h(n-k) e^{-j\omega(n-k)}$$

{$\because X\left(e^{j\omega}\right)$ is independent from the summation variable n}

Taking, $n-k=m$, $n=-\infty \Rightarrow m=-\infty, n=\infty \Rightarrow m=\infty$

$$\therefore Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)\sum_{m=-\infty}^{+\infty}h(m)e^{-j\omega m}=X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$\therefore Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$. Hence proved.

Inference-> Convolution in (discrete) time-domain corresponds to the multiplication in frequency-domain.

6. Prove the *multiplication* property for DTFT, if $y(n)=x(n)h(n)$, then

$$Y\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\theta}\right)H\left(e^{j(\omega-\theta)}\right)d\theta=\frac{1}{2\pi}\left[X\left(e^{j\omega}\right)*H\left(e^{j\omega}\right)\right]$$

Solution

$y(n)=x(n)h(n)$

Taking DTFT,

$$Y\left(e^{j\omega}\right)=\sum_{n=-\infty}^{+\infty}y(n)e^{-j\omega n}=\sum_{n=-\infty}^{+\infty}\left(x(n)h(n)\right)e^{-j\omega n}$$

Using, IDTFT definition for $x(n)$, i.e., $x(n)=\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\theta}\right)e^{j\theta n}d\theta$

$$\therefore Y\left(e^{j\omega}\right)=\sum_{n=-\infty}^{+\infty}\left(\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\theta}\right)e^{j\theta n}d\theta\right)h(n)e^{-j\omega n}$$

{Interchange $\sum$ and $\int$ operators, carefully!!}

$$\therefore Y\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\theta}\right)\left(\sum_{n=-\infty}^{+\infty}h(n)e^{-j(\omega-\theta)n}\right)d\theta$$

{Using DTFT definition for the term which is inside the braces, it is DTFT of $h(n)$ with shifted frequency by (θ) }

$\therefore Y\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{2\pi}X\left(e^{j\theta}\right)H\left(e^{j(\omega-\theta)}\right)d\theta=\frac{1}{2\pi}\left[X\left(e^{j\omega}\right)*H\left(e^{j\omega}\right)\right]$. Hence Proved.

7. A causal LTI system is characterized by the difference equation

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=2x(n)$$

Determine the *impulse response,* $h(n)$, of the LTI system.

Solution:-

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=2x(n)$$

Taking DTFT both sides and using time-shifting property…

$$\therefore Y\left(e^{j\omega}\right)-\frac{3}{4}e^{-j\omega}Y\left(e^{j\omega}\right)+\frac{1}{8}e^{-j2\omega}Y\left(e^{j\omega}\right)=2X\left(e^{j\omega}\right)$$

$$\therefore\left(1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}\right)Y\left(e^{j\omega}\right)=2X\left(e^{j\omega}\right)$$

Now,

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{2}{\left(1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}\right)}=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}=\frac{A}{\left(1-\frac{1}{2}e^{-j\omega}\right)}+\frac{B}{\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

By residue method for finding A and B

$$A=\left.\frac{\dfrac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}}{\left(1-\frac{1}{2}e^{-j\omega}\right)}\right|_{e^{-j\omega}=2}=\frac{2}{\left(1-\frac{1}{4}(2)\right)}=4$$

$$B=\left.\frac{\dfrac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}}{\left(1-\frac{1}{4}e^{-j\omega}\right)}\right|_{e^{-j\omega}=4}=\frac{2}{\left(1-\frac{1}{2}(4)\right)}=-2$$

$$H\left(e^{j\omega}\right)=\frac{4}{\left(1-\frac{1}{2}e^{-j\omega}\right)}+\frac{(-2)}{\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

Taking, IDTFT and using linearity property

$$h(n) = 4F^{-1}\left\{\frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)}\right\} - 2F^{-1}\left\{\frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)}\right\}$$

From, problem-1

$F\left\{(0.2)^n u(n)\right\} = \frac{1}{1-0.2e^{-j\omega}}$, in general, $F\left\{(r)^n u(n)\right\} = \frac{1}{1-re^{-j\omega}}$, $r<1$ for convergence

$\boxed{\therefore h(n) = 4\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{4}\right)^n u(n)}$ ***ANS***