Tutorial, Class 1 $\boxed{\text{Lecture 04}}$

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

Prf.: Using Taylor's series. coeff. of polynom. → degree.

$$f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(v)}{k!} \underbrace{(x-u)^k}_{\text{polynomial}}$$

Taylor series: express a continuous function in terms of polynomial.

Let $v=0$

$$f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \text{[Maclaurin's series]}$$

$f(x) = e^x$

$f'(x) = \frac{d}{dx}(e^x) = e^x \Rightarrow f^{(1)}(x)\Big|_{x=0} = f^{(1)}(0) = 1$

$f''(x) = \frac{d^2}{dx^2}(e^x) = e^x \Rightarrow f^{(2)}(0) = 1.$

$\vdots$

$f^{(n)}(x)\Big|_{x=0} = \quad f^{(n)}(0) = 1$

①

$$f(x) = e^x = \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$e^x = \frac{1}{1}\cdot 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{x^n}{n!} + \cdots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots}$$

Let $x = j\theta$

$$\therefore e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots + \frac{(j\theta)^n}{n!} + \cdots$$

$$e^{j\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos(\theta)} + j\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta}$$

Algebraic Operation on Complex Numbers

1) $z_1 = x_1 + jy_1$, $z_2 = x_2 + jy_2$

$\downarrow$ $\downarrow$

$z_1 = r_1 e^{j\theta_1}$ $z_2 = r_2 e^{j\theta_2}$

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

2) $z_1 \cdot z_2 = (r_1 e^{j\theta_1}) \cdot (r_2 e^{j\theta_2})$

$$= (r_1 r_2)\left(e^{j\theta_1} \cdot e^{j\theta_2}\right)$$

$$= (r_1 r_2)\left(e^{j\theta_1 + j\theta_2}\right)$$

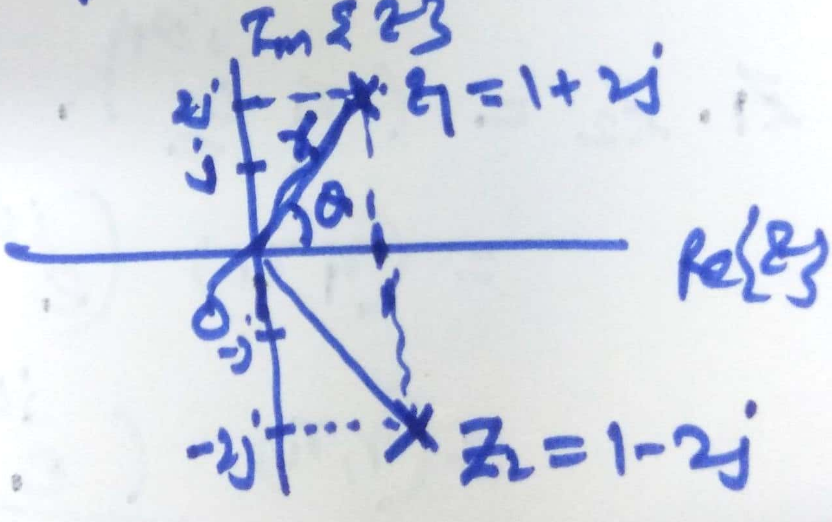
$$\boxed{z_1 \cdot z_2 = (r_1 r_2)\, e^{j(\theta_1 + \theta_2)}}$$

When two complex numbers are multiplied then their respective magnitudes are multiplied and phases are 'added'.

3) $$\boxed{\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) \cdot e^{j(\theta_1 - \theta_2)}}$$ key result.

⇒ When two complex nos. are divided, magnitude of Num. gets divided by magnitude of denominator and phase of Num. gets subtracted by phase of denominator

Problem ① $z_1 = 1 + 2j$, $z_2 = 1 - 2j$



① Polar $z_1 = r_1 e^{j\theta_1}$ Conversion

$r_1 = \sqrt{1^2 + 2^2} = \sqrt{5}$.

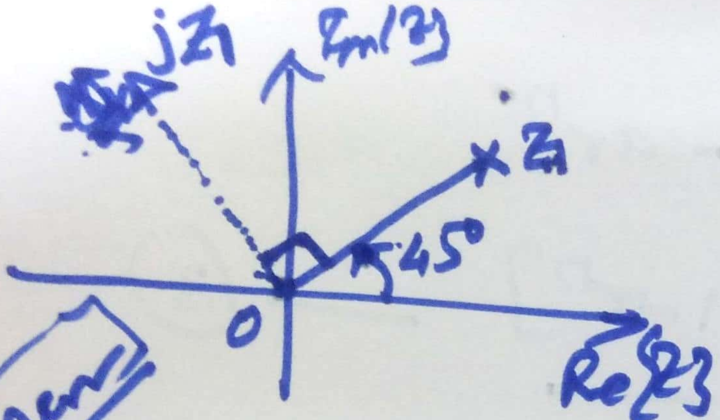
$\theta_1 = \tan^{-1}\left(\frac{2}{1}\right) =$

④

Problem Geometrical interpretation of $j = \sqrt{-1}$

j is a vector that has magnitude 1 and phase 90°

$z_1 = 1 + 2j$ $\qquad$ jz_1.



Problem

Sum of nth term of GP (Geometric Progression).

Term of GP: $a, ar, ar^2, \ldots, ar^{n-1}$

a = first term of GP.

r = common ratio $= \frac{ar}{a} = \frac{ar^2}{ar} = \cdots = \frac{ar^n}{ar^{n-1}}$

Let S_n = Sum of first nth term of GP

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ —(1)

Multiply eqn (1) by 'r'

We get

~~xS_n~~

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} \quad \text{---} (1)$$

$$- \quad rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n \quad \text{---} (2)$$

eqn (1) − eqn (2)

$$\therefore S_n - rS_n = a - ar^n.$$

$$S_n = \frac{a[1-r^n]}{1-r} \quad \text{---} (3)$$

Sum of n terms of GP

$$\boxed{\sum_{k=0}^{n-1} ar^k = a\left[\frac{1-r^n}{1-r}\right]}$$

Sum of infinite of terms of GP

Take $\lim_{n\to\infty}$ on both sides of eqn (3),

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{(1-r^n)}{1-r}$$

$$= \frac{a}{1-r}\left(1 - \lim_{n\to\infty} r^n\right)$$

If $|r| > 1$, $r^n \to \infty$

(6)

Problem 10: Prove that for a complex function $f(t)$ or $f(t)$

$$\int_{t=-\infty}^{t=+\infty} f^*(t)\,dt = \left[\int_{t=-\infty}^{+\infty} f(t)\,dt\right]^*$$

$\Rightarrow$ Integration commutes with complex conjugation.

Proof: $RHS = \left[\int_{t=-\infty}^{+\infty} f(t)\,dt\right]^*$ $\quad (a+jb)^* = a-jb$

Let $f(t) = a(t) + j\,b(t)$

$$= \left[\int_{-\infty}^{+\infty} [a(t) + j\,b(t)]\,dt\right]^*$$

$$= \left[\left(\int_{-\infty}^{+\infty} a(t)\,dt\right) + j\left(\int_{-\infty}^{+\infty} b(t)\,dt\right)\right]^*$$

$$= \int_{-\infty}^{+\infty} a(t)\,dt - j\int_{-\infty}^{+\infty} b(t)\,dt$$

$$= \int_{-\infty}^{+\infty} [a(t) - j\,b(t)]\,dt = \int_{-\infty}^{+\infty} f^*(t)\,dt$$

$= LHS.$

Continuity vs. Differentiability.

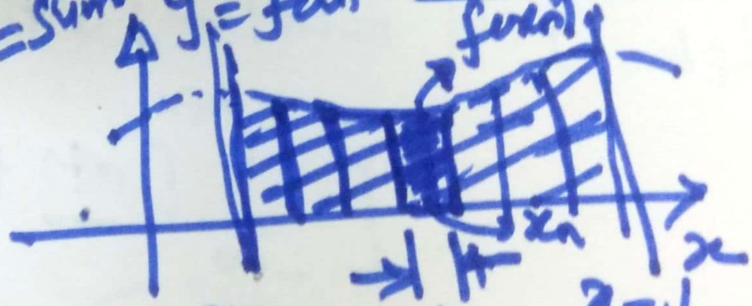
Q://

1) If a function is continuous then it is differentiable? → NO

2) If a function is differentiable at a point then function is continuous at that point.

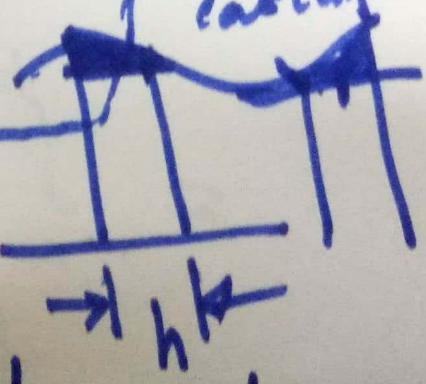
Geometrical Interpretation of Integration:

$$I = \int_{x=a}^{x=b} f(x)\,dx \rightarrow S = \text{Sum}$$



$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{h \to 0} \sum_{n=-\infty}^{\infty} \underline{f(x_n)} \cdot \underline{h}$$

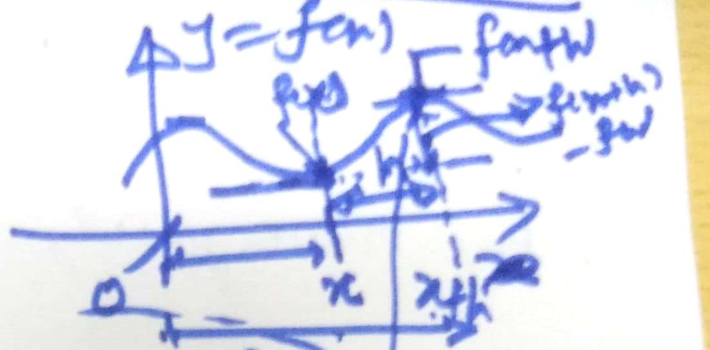
Fundamental theorem of integral calculus.



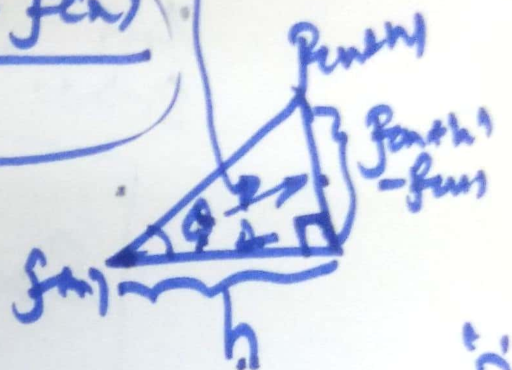
Integration means area under a curve $y = f(x)$ bounded by the lines $x=a$ to $x=b$.

Q: Geometrical Interpretation of Derivative

Laibnitz def^n of derivative by first principle,



$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$= \tan(\theta)$

= slope of function $f(x)$ or curve at a point

∴ Slope is a local or point point property (provided 'h' is small enough)

Geometrically derivative of a function at a point, is nothing slope of a curve at a point ⟹ Derivative is a localized phenomenon or point property.

Scanned by CamScanner

If $|r|>1 \Rightarrow r^n \uparrow$ as $n\to\infty$

$\therefore S_n \to$ diverge as $n\to\infty$

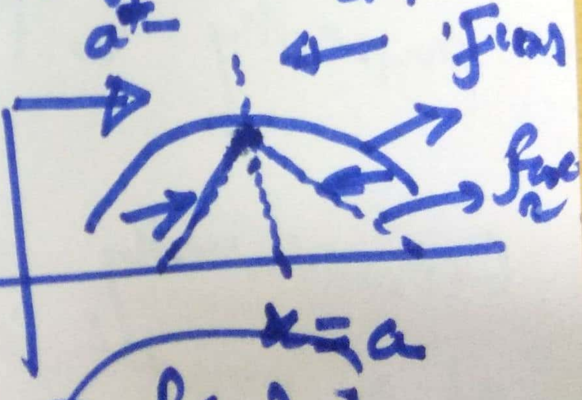
$\therefore$ For convergence, $|r|<1 \Rightarrow r^n \xrightarrow{n\to\infty} 0$

$\therefore \lim_{n\to\infty} S_n = S_\infty = \frac{a}{1-r}[1-0]$

$$\boxed{S_\infty = \sum_{k=0}^{+\infty} ar^k = \frac{a}{1-r}}$$

Continuity of a function

A function $f(x)$ is said to be continuous if the left hand limit and right hand limits are same and equal to function's value at the point of continuity



$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

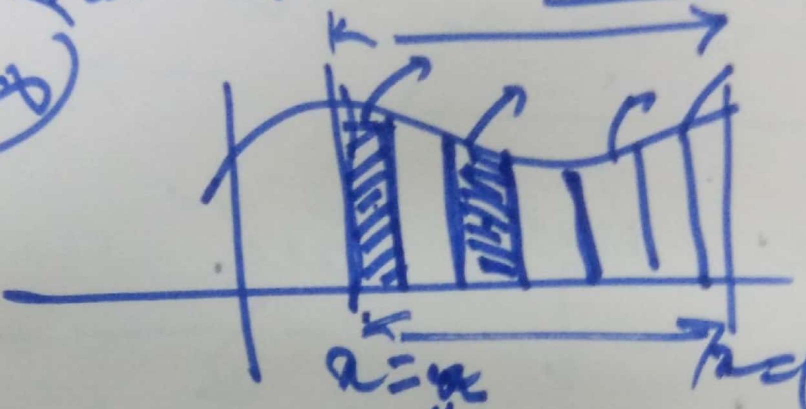
$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{h \to 0} \sum_{n=1}^{\infty} f(x_n)\,h$$

$x = b \to x = +\infty$

$f(x_n)\,h$ ↓ Area of strip

$x = a \to x = -\infty$ ↳ Area under a curve.

$(-\infty, +\infty)$



→ 'Riemann Integration'

* Limitation of Riemann Integration

↳ "Riemann integration requires bounded intervals"

$$I = \int_{t=-\infty}^{+\infty} f(t)\,dt$$

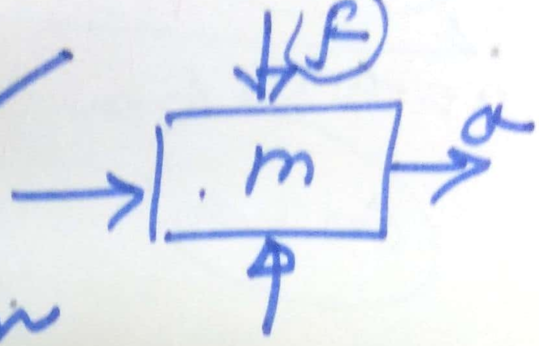
* Concept of Linear Combination

Physics,

Newton's second law of motion;

$$\boxed{f = m.a}$$

force mass acc.



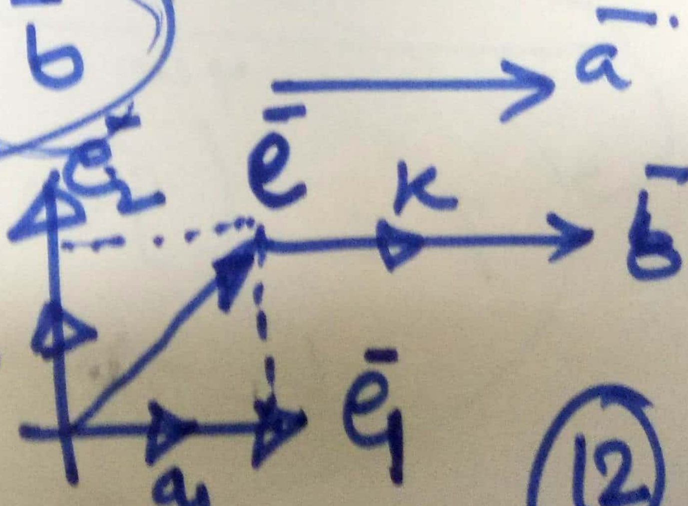
Vector 1 = Scalar × Vector 2

⇒ Newton's second law of motion says that we can express one vector (force) as a scalar multiple (mass) of another vector (acceleration).

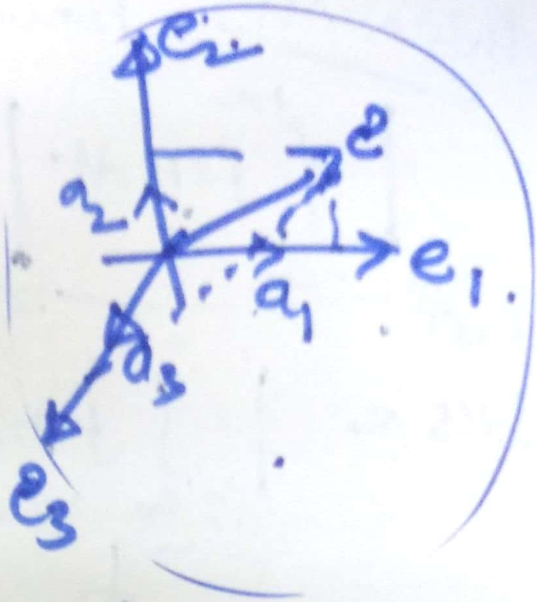
$$\bar{a} = k.\bar{b}$$

$$\boxed{\bar{e} = a_1\bar{e}_1 + a_2\bar{e}_2}$$

Linear combination



$e = a_1 e_1 + a_2 e_2$



$e = a_1 e_1 + a_2 e_2 + a_3 e_3$
$+ a_4 e_4$

$\vdots$

$e = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

$\vdots$

$e = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n + \cdots$
$+ \cdots$ infinite dimension. combina

$$e = \sum_{k=1}^{\infty} a_k e_k$$

→ Linear

e → Resultant vector.

a_k → coefficients of linear combination. → scalar

e_k → representative vectors. (or Basis vectors).

Problem 11. ~~Prove that for~~

$$\left| \int_{-\infty}^{+\infty} f(t)\,dt \right| \leq \int_{-\infty}^{+\infty} |f(t)|\,dt.$$

$$\left|\sum \cdot\right| \leq \sum |\cdot|$$

Proof:

$$LHS = \left| \int_{-\infty}^{+\infty} f(t)\,dt \right|$$

$$= \left| \lim_{h \to 0} \sum_{n=-\infty}^{+\infty} f(t_n) \cdot h \right|$$

$$= \lim_{h \to 0} \left| \sum_{\langle n \rangle} f(t_n) \cdot h \right|$$

Triangle inequality?

$$|a+b| \leq |a| + |b|$$

$$\left|\sum \cdot\right| \leq \sum |\cdot|$$

$$LHS \leq \lim_{h \to 0} \sum_{\langle n \rangle} |f(t_n)|\,|h|$$

$$\leq \int_{t=-\infty}^{+\infty} |f(t)|\,dt \rightarrow RHS$$

14