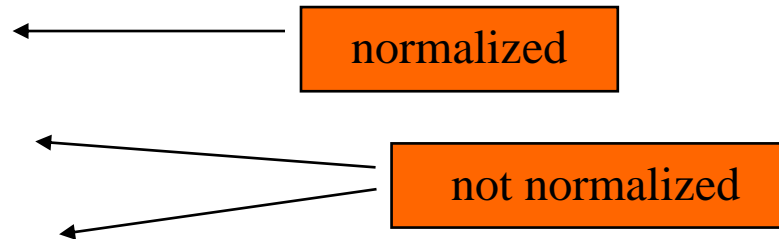

IEEE Floating Point Representation

Real Numbers

- Numbers with fractions
- Could be done in pure binary
 - $1001.1010 = 2^4 + 2^0 + 2^{-1} + 2^{-3} = 9.625$
- Where is the binary point?
- Fixed?
 - Very limited
- Moving?
 - How do you show where it is?

Floating Point

- Representation for non-integral numbers
 - Including very small and very large numbers
- Like scientific notation
 - -2.34×10^{56}
 - $+0.002 \times 10^{-4}$
 - $+987.02 \times 10^9$
- In binary
 - $\pm 1.xxxxxxx_2 \times 2^{yyy}$
- Types `float` and `double` in C



Floating Point Standard

- Defined by IEEE Std 754-1985
- Developed in response to divergence of representations
 - Portability issues for scientific code
- Now almost universally adopted
- Two representations
 - Single precision (32-bit)
 - Double precision (64-bit)

IEEE Floating-Point Format

single: 8 bits
double: 11 bits

single: 23 bits
double: 52 bits

S	Exponent	Fraction
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$$x = (-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$$

- S: sign bit (0 \Rightarrow non-negative, 1 \Rightarrow negative)
- Normalize significand: $1.0 \leq |\text{significand}| < 2.0$
 - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
 - Significand is Fraction with the "1." restored
- Exponent: excess representation: actual exponent + Bias
 - Ensures exponent is unsigned
 - Single: Bias = 127; Double: Bias = 1023

Single-Precision Range

- Exponents 00000000 and 11111111 reserved
- Smallest value
 - Exponent: 00000001
 \Rightarrow actual exponent = $1 - 127 = -126$
 - Fraction: 000...00 \Rightarrow significand = 1.0
 - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$
- Largest value
 - exponent: 11111110
 \Rightarrow actual exponent = $254 - 127 = +127$
 - Fraction: 111...11 \Rightarrow significand ≈ 2.0
 - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$

Double-Precision Range

- Exponents 0000...00 and 1111...11 reserved
- Smallest value
 - Exponent: 000000000001
 \Rightarrow actual exponent = $1 - 1023 = -1022$
 - Fraction: 000...00 \Rightarrow significand = 1.0
 - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$
- Largest value
 - Exponent: 111111111110
 \Rightarrow actual exponent = $2046 - 1023 = +1023$
 - Fraction: 111...11 \Rightarrow significand ≈ 2.0
 - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

Floating-Point Precision

- Relative precision
 - all fraction bits are significant
 - Single: approx 2^{-23}
 - Equivalent to $23 \times \log_{10} 2 \approx 23 \times 0.3 \approx 6$ decimal digits of precision
 - Double: approx 2^{-52}
 - Equivalent to $52 \times \log_{10} 2 \approx 52 \times 0.3 \approx 16$ decimal digits of precision

Floating-Point Example

- Represent -0.75
 - $-0.75 = (-1)^1 \times 1.1_2 \times 2^{-1}$
 - $S = 1$
 - Fraction = $1000\dots00_2$
 - Exponent = $-1 + \text{Bias}$
 - Single: $-1 + 127 = 126 = 01111110_2$
 - Double: $-1 + 1023 = 1022 = 01111111110_2$
- Single: $1011111101000\dots00$
- Double: $1011111111101000\dots00$

Floating-Point Example

- What number is represented by the single-precision float

11000000101000...00

—S = 1

—Fraction = 01000...00₂

—Exponent = 10000001₂ = 129

- $x = (-1)^1 \times (1 + 01_2) \times 2^{(129 - 127)}$
= $(-1) \times 1.25 \times 2^2$
= -5.0

Floating-Point Addition

- Consider a 4-digit decimal example
 - $9.999 \times 10^1 + 1.610 \times 10^{-1}$
- 1. Align decimal points
 - Shift number with smaller exponent
 - $9.999 \times 10^1 + 0.016 \times 10^1$
- 2. Add significands
 - $9.999 \times 10^1 + 0.016 \times 10^1 = 10.015 \times 10^1$
- 3. Normalize result & check for over/underflow
 - 1.0015×10^2
- 4. Round and renormalize if necessary
 - 1.002×10^2

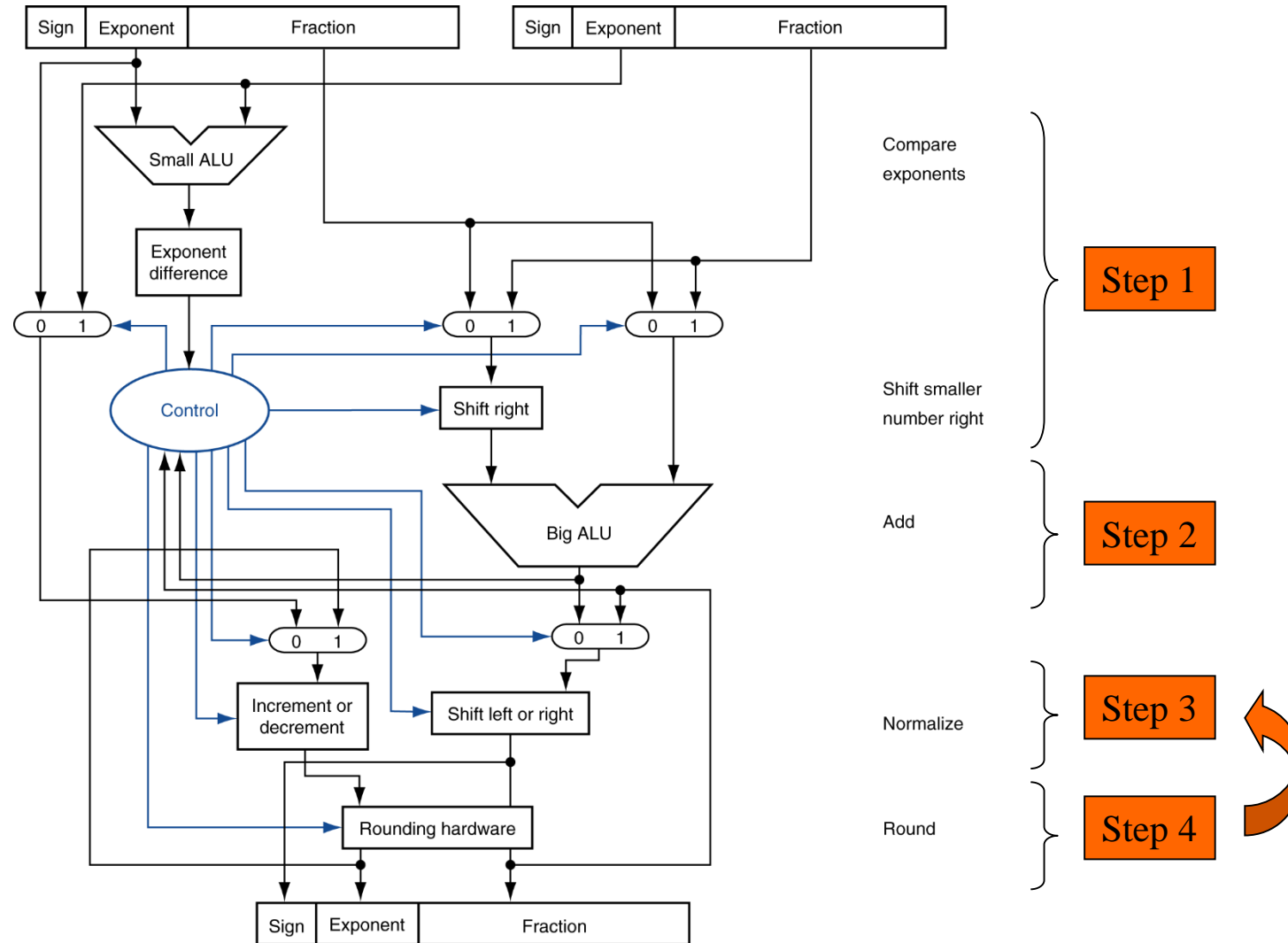
Floating-Point Addition

- Now consider a 4-digit binary example
 - $1.000_2 \times 2^{-1} + -1.110_2 \times 2^{-2}$ ($0.5 + -0.4375$)
- 1. Align binary points
 - Shift number with smaller exponent
 - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1}$
- 2. Add significands
 - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1} = 0.001_2 \times 2^{-1}$
- 3. Normalize result & check for over/underflow
 - $1.000_2 \times 2^{-4}$, with no over/underflow
- 4. Round and renormalize if necessary
 - $1.000_2 \times 2^{-4}$ (no change) = 0.0625

FP Adder Hardware

- Much more complex than integer adder
- Doing it in one clock cycle would take too long
 - Much longer than integer operations
 - Slower clock would penalize all instructions
- FP adder usually takes several cycles
 - Can be pipelined

FP Adder Hardware



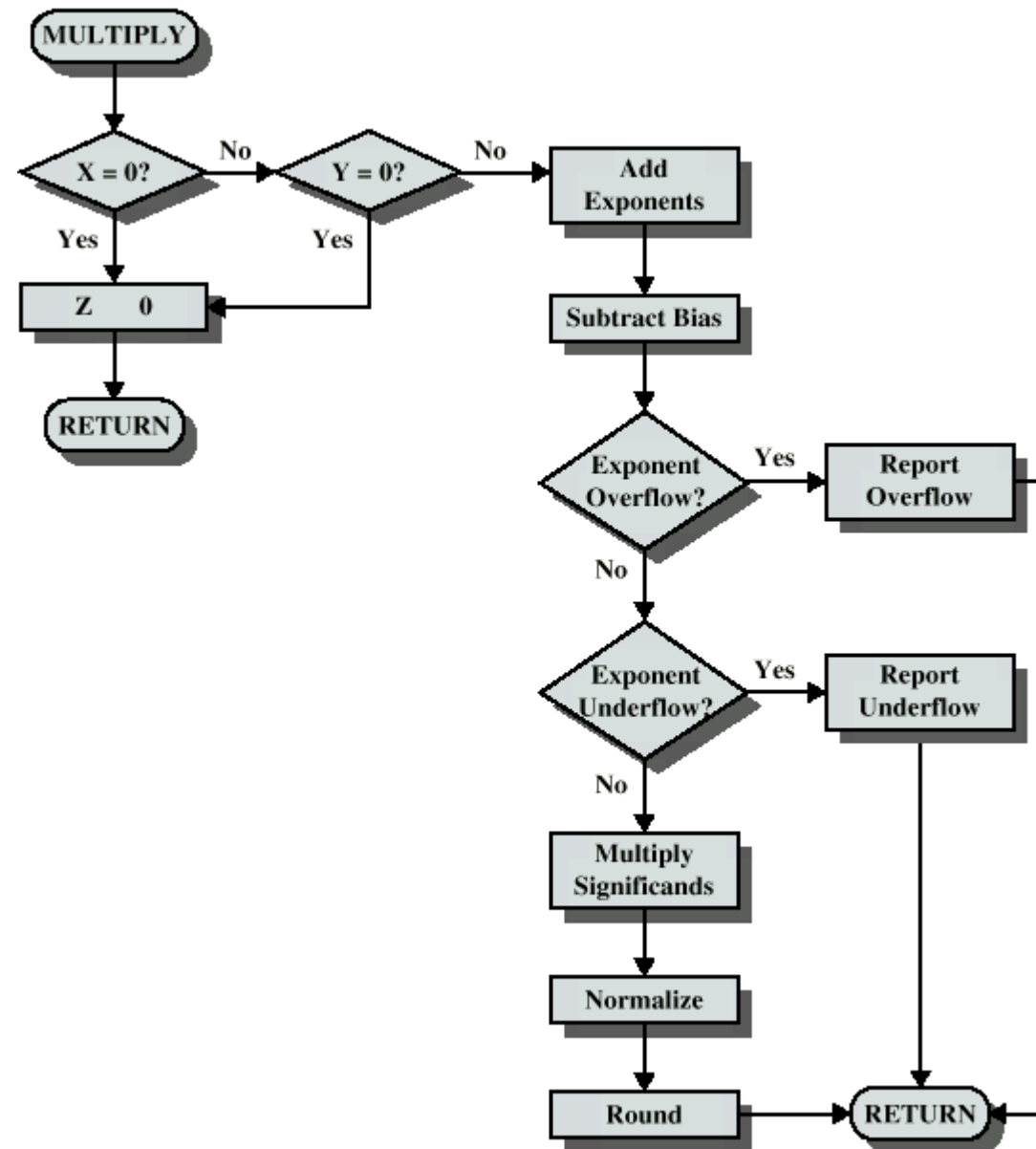
FP Arithmetic Hardware

- FP multiplier is of similar complexity to FP adder
 - But uses a multiplier for significands instead of an adder
- FP arithmetic hardware usually does
 - Addition, subtraction, multiplication, division, reciprocal, square-root
 - FP \leftrightarrow integer conversion
- Operations usually takes several cycles
 - Can be pipelined

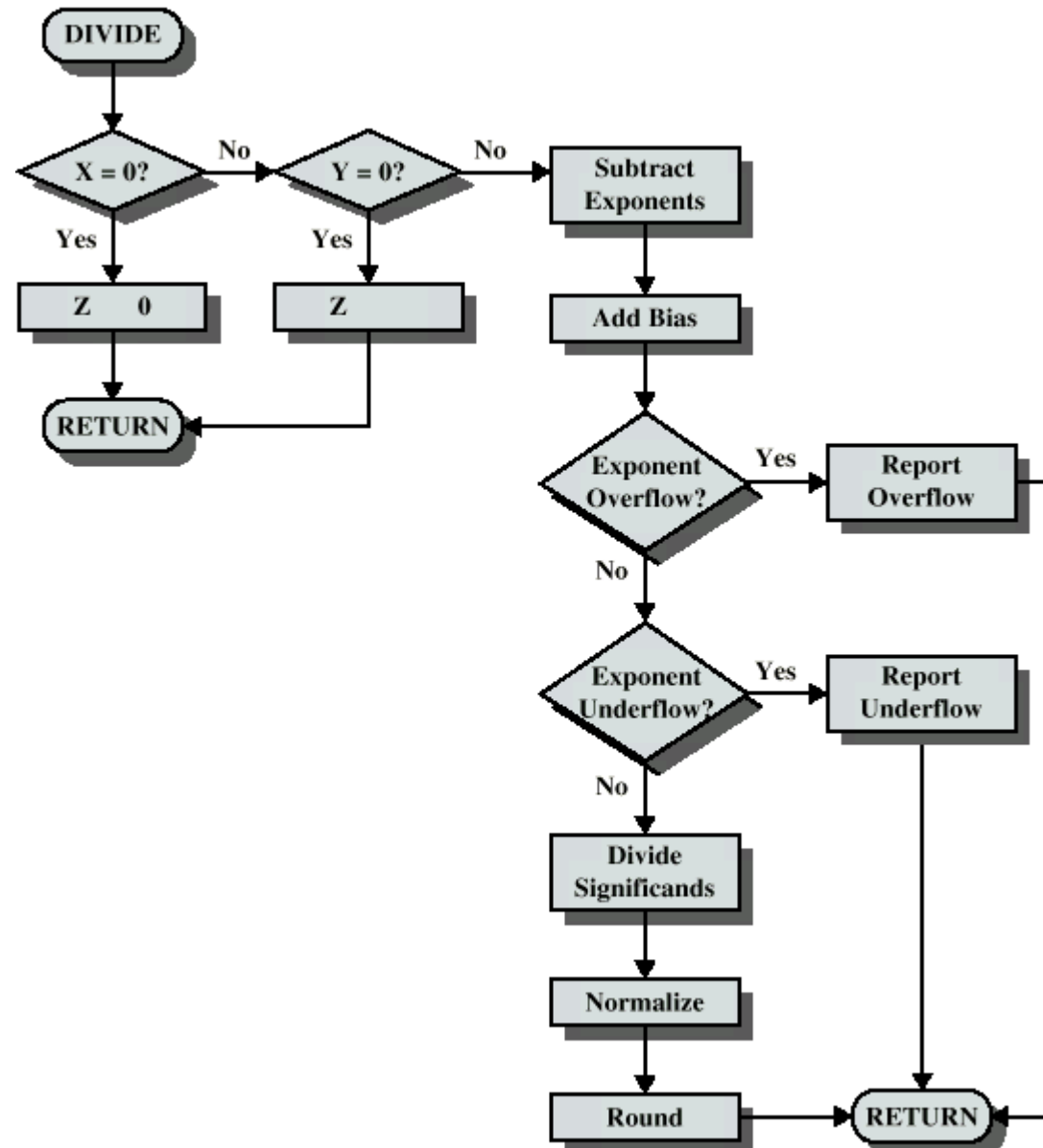
FP Arithmetic \times/\div

- Check for zero
- Add/subtract exponents
- Multiply/divide significands (watch sign)
- Normalize
- Round
- All intermediate results should be in double length storage

Floating Point Multiplication



Floating Point Division



Required Reading

- Stallings Chapter 9
- IEEE 754 on IEEE Web site