SC223 - Linear Algebra

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Lecture 36

November 7, 2023

Summary of Lecture 35

- Projection Operator: $T \in \mathcal{L}(V)$ where $V = U \oplus W$, is a projection on subspace U if $\forall v \in V, v = u + w, u \in U, w \in W, T(v) = u$.
- A projection operator:
  - is Idempotent,
  - is Diagonalizable,
  - has eigenvalues 0 and 1.
- Any operator $T \in \mathcal{L}(V)$ such that $T^2 = T$ is a projection operator.

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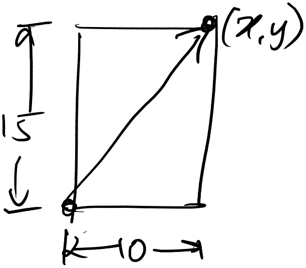

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● Is this the only way to define length?

● What are the necessary conditions for a function on vector space for it be called *length*?

Normed Vector space

- **Definition:** (Normed Vector Space) A normed vector space (NVS) is a vector space $(V, +, \cdot)$ over either $\mathbb{R}$ or $\mathbb{C}$ with a **norm**, a function $||\cdot|| : V \to \mathbb{R}$ which satisfies the following properties:

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  1. Positive definiteness: $||x|| \geq 0, \forall x \in V$ and $||x|| = 0 \Leftrightarrow x = \theta$.

length = norm

$||\cdot||$

$||v||$

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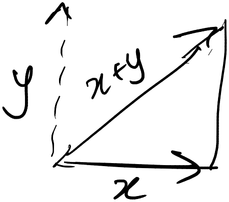
Triangular Inequality

$$||x+y|| \leq ||x|| + ||y||$$



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- A vector space V with a valid norm $|| \cdot ||$ is called a **Normed vector space** and is denoted by $(V, || \cdot ||)$.
- Also note that given a NVS $(V, || \cdot ||)$, we can define distance between two vectors x and y as $d(x, y) := ||x - y||$. Such a distance or metric is called the **induced metric**.

Examples of NVS

- 1-norm on $\mathbb{R}^n$: $||x|| = \sum_{i=1}^{n} |x_i|$. This is denoted by $||x||_1$.

(a) Positive definiteness

- $\forall x \in \mathbb{R}^n$, $||x|| = \sum_{i=1}^{n} |x_i|$, Since $|a| \geq 0, \forall a \in \mathbb{R}$

  $|x_i| \geq 0$. $\therefore \sum_{i=1}^{n} |x_i| \geq 0$.

- Let $x = \theta = (0, \ldots, 0)$. $||\theta|| = \sum_{i=1}^{n} |0| = \sum_{i=1}^{n} 0 = 0$.

  Let $x \in \mathbb{R}^n$ s.t. $||x|| = \sum_{i=1}^{n} |x_i| = 0$.

  Since $|x_i| \geq 0, \forall x_i \in \mathbb{R}$, $||x|| = 0 \Rightarrow x_i = 0$

  $\therefore x = (0, \ldots, 0)$.

ⓒ Triangular Inequality

Let $x, y \in \mathbb{R}^n$

$$\|x+y\| = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|)$$

$$\leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\| + \|y\|.$$

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- max or the sup norm on $\mathbb{R}^n$: $||x|| = \max_{i=1}^{n}\{|x_i|\}$. This is denoted by $||x||_{\sup}$ or $||x||_{\max}$.

Triangular Ineq. :

$\max_i \{|x_i+y_i|\}$ $\max_i \{|x_i|\} + \max_i \{|y_i|\}$.

$$|x_i+y_i| \leq |x_i| + |y_i|$$

$$\max |x_i+y_i| \leq \max\{|x_i|+|y_i|\}$$

$$\leq \max\{|x_i|\}+\max\{|y_i|\}$$

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- L_2 norm on $\mathcal{P}_n([-1,1])$: $||x||_{L_2} = \sqrt{\int_{-1}^1 (x(t))^2\ dt}$.

$$||f||_1 = \int_{-1}^{1} |f(x)|\,dx$$

$$||f||_\infty = \sup_{x\in[-1,1]}\{|f(x)|\}$$

— END OF CLASS —

Inner Product

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2. Linear in first argument:
$\forall x, y, z \in V, \forall a, b \in \mathbb{F}, \langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

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- **Definition:** (Inner Product Space) A vector space V with an inner product is called an **Inner Product space**(IPS) and is denoted by $(V, \langle \cdot, \cdot \rangle)$.

Examples of IPS

- $\mathbb{R}^n$: Euclidean inner product - $x, y \in \mathbb{R}^n, \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y$, where x and y are written as column vectors.

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● $\mathbb{C}^n$: Euclidean inner product - $x, y \in \mathbb{C}^n, \langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} = y^* x$, where x and y are written as column vectors, and y^* denotes the *conjugate transpose* of y.

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- $\mathcal{P}_n([0,1])$: L_2 inner product - $\forall p, q \in \mathcal{P}_n([0,1]), \langle p, q\rangle = \int_0^1 p(t)q(t)\ dt$.

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- $\mathcal{P}_n([0,1])$: L_2 inner product - $\forall p, q \in \mathcal{P}_n([0,1]), \langle p, q\rangle = \int_0^1 p(t)q(t)\, dt$.
- Let $G \in \mathbb{R}^{n\times n}$ be such that $G = G^T$, and $x^T G x > 0, \forall x \in \mathbb{R}^n, \neq \mathbf{0_n}$. Such a matrix G is said to be *Symmetric Positive-Definite* (SPD). Then, on $\mathbb{R}^n$, $\forall x, y \in \mathbb{R}^n, \langle x, y\rangle = x^T G y$ is a valid inner product.

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- A matrix $A \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ is said to be an **orthogonal matrix** if all its n columns are orthonormal, i.e., $A^*A = I$, where A^* denotes the conjugate transpose of A. In this case, $A^{-1} = A^*$.

Properties

- **Proposition 22** (Pythagoras Theorem): In an IPS $(V, \langle \cdot, \cdot \rangle)$, if $\langle x, y \rangle = 0$, $||x+y||^2 = ||x||^2 + ||y||^2$.

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$$\langle w, y \rangle = 0$$
$$\langle x - a \cdot y, y \rangle = \langle x, y \rangle - a\langle y, y \rangle = 0$$
$$a = \frac{\langle x, y \rangle}{\langle y, y \rangle}$$
$$\text{Thus, } x = \frac{\langle x, y \rangle}{\langle y, y \rangle} \cdot y + \left(x - \frac{\langle x, y \rangle}{\langle y, y \rangle} \cdot y \right)$$

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- From previous proposition, $x = \frac{\langle x,y\rangle}{\langle y,y\rangle} \cdot y + \left(x - \frac{\langle x,y\rangle}{\langle y,y\rangle} \cdot y\right) = \frac{\langle x,y\rangle}{\langle y,y\rangle} \cdot y + w$, with $w \perp y$.

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Gram-Schmidt Procedure

- **Proposition 24** (Gram-Schmidt Procedure): Let $\{v_1, \ldots, v_m\}$ be a list of linearly independent vectors. Then there exists a list of orthonormal vectors $\{e_1, \ldots, e_m\}$ such that $span(\{v_1, \ldots, v_j\}) = span(\{e_1, \ldots, e_j\}), \forall j = 1, \ldots, m$.

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- Let $e_1 = \frac{v_1}{||v_1||}$. Define $e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{||v_2 - \langle v_2, e_1 \rangle e_1||}$.

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- Similarly, $e_3 = \frac{v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)}{||v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)||}$, and $e_k = \frac{v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i}{||v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i||}$.

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- Observe that $span(\{v_1, \ldots, v_j\}) = span(\{e_1, \ldots, e_j\})$.
- It is easy to see that $e_1 \perp e_2$. Assume that $\{e_1, \ldots, e_j\}$ are orthonormal.

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- **Proposition 24** (Gram-Schmidt Procedure): Let $\{v_1, \dots, v_m\}$ be a list of linearly independent vectors. Then there exists a list of orthonormal vectors $\{e_1, \dots, e_m\}$ such that $span(\{v_1, \dots, v_j\}) = span(\{e_1, \dots, e_j\}), \forall j = 1, \dots, m$.
- Let $e_1 = \frac{v_1}{||v_1||}$. Define $e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{||v_2 - \langle v_2, e_1 \rangle e_1||}$.
- Similarly, $e_3 = \frac{v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)}{||v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)||}$, and $e_k = \frac{v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i}{||v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i||}$.
- Observe that $span(\{v_1, \dots, v_j\}) = span(\{e_1, \dots, e_j\})$.
- It is easy to see that $e_1 \perp e_2$. Assume that $\{e_1, \dots, e_j\}$ are orthonormal.
- Then $\forall l = 1, \dots j$, with $\tilde{e_{j+1}} = v_{j+1} - \sum_{i=1}^{j} \langle v_{j+1}, e_i \rangle e_i$

$$\begin{aligned}
\langle e_{j+1}, e_l \rangle &= \frac{1}{||\tilde{e_{j+1}}||} \left(\langle v_{j+1}, e_l \rangle - \sum_{i=1}^{j} \langle v_{j+1}, e_i \rangle \langle e_i, e_l \rangle \right) \\
&= \frac{1}{||\tilde{e_{j+1}}||} \left(\langle v_{j+1}, e_l \rangle - \langle v_{j+1}, e_l \rangle \right) = 0
\end{aligned}$$

Orthogonal Complement

- Let V be a FD IPS and let U be a subset of V. The **Orthogonal Complement** of U is defined as

$$U^{\perp} = \{v \in V \mid \langle v, u \rangle = 0, \forall u \in U\}$$

- **Proposition 25:** Irrespective of whether U is a subspace of V or not, $U^{\perp}$ is a subspace.

Orthogonal Projection

- Let U be a subspace of FD IPS V, and $V = U \oplus U^{\perp}$.

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1. Range$(P_U) = U$
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3. Idempotent: $(P_U)^2 = P_U$
4. (Conjugate) Symmetric: If $U = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T = P_U$ ($P_U^* = P_U$).

Orthogonal Projection

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- Define $P_U \in \mathcal{L}(V)$ as $\forall v \in V$, if $v = u + w, u \in U, w \in U^{\perp}$, $P_U(v) = u$.
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- It has the following properties:

1. $\text{Range}(P_U) = U$
2. $\text{Null}(P_U) = U^{\perp}$
3. Idempotent: $(P_U)^2 = P_U$
4. (Conjugate) Symmetric: If $U = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T = P_U$ ($P_U^* = P_U$).
5. $\forall v \in V, P_U(v) = \arg\min_{u \in U} ||u - v||^2$.