SC223 - Linear Algebra
Autumn 2023

Tutorial 4 — Rank of a matrix

1. For the following matrices A, B, find the rank of matrices A, B and AB:

   (a) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, B = A^T$.

   (b) $A = B^T$, where $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -1 & 0 & -1 \\ -1 & 3 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} -2 & -2 & 0 & 2 \\ 6 & -4 & 10 & -16 \\ 10 & 0 & 10 & -20 \\ 0 & 10 & -10 & 10 \end{bmatrix}$

2. Let $A \in \mathbb{R}^{10\times 50}$ be a matrix. Is it possible for a matrix $D \in \mathbb{R}^{50\times 10}$ to exist such that $DA = I_{50}$? Why or Why not? Similarly, is it always possible for a matrix $C \in \mathbb{R}^{50\times 10}$ to exist such that $AC = I_{10}$? Justify.

3. Let $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 1 & 2 & 7 \\ 2 & 4 & 6 & 8 \\ 2 & 1 & 4 & 7 \end{bmatrix}$

   (a) What is the rank of A?

   (b) Can you alter the rank of A by changing the entry at position $A[3,4]$ to any number of your choice, and leaving everything else unchanged? If so what number will you put at $A[3,4]$? If not possible, explain why.

   (c) Can you alter the rank of A by changing the entry at $A[4,4]$ to any number of your choice, and leaving everything else unchanged? If so what number will you put at $A[4,4]$? If not possible, explain why.

4. (a) By how much can the rank of a matrix change when one alters one of its entries?

   (b) Show that there **always** exists at least one position whose alteration can alter the rank of a **square** matrix.

   (c) If we wish to alter the rank of a matrix by k, what is the fewest number of entries that must be changed? What is the condition on the location of the entries within the matrix for this to potentially work?