SC223 - Linear Algebra

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Lecture 34

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Summary of Lecture 33

● Matrix Diagonalization: Existence of an eigenbasis, say E, is equivalent to $A = E\Lambda E^{-1}$, where Λ is the diagonal matrix.

● **Proposition 20**: Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T \in \mathcal{L}(U)$. Then the eigenvectors $v_1, \ldots, v_m$ associated with these eigenvalues are linearly independent.

Results on Eigenvectors and Eigenvalues

- **Proposition 21**: For $T \in \mathcal{L}(U)$, $dim(U) = n$, if $\sum_{i=1}^{m} AM(\lambda_i) = n$, and $GM(\lambda_i) = AM(\lambda_i), i = 1, \ldots, m$, then T is diagonalizable.

Let $\lambda \in \mathbb{F}$ s.t $C_T(\lambda) = 0$ & $C_T(x) = (x-\lambda)^k p(x)$ with $p(\lambda) \neq 0$.

$AM(\lambda) = k$; Assume $G.M(\lambda) = k+1 = dim(N(T-\lambda I))$

Let $\{v_1, \ldots, v_{k+1}\}$ be LI vectors from $N(T-\lambda I)$.

$Tv_i = \lambda v_i, i = 1, \ldots, k+1$.

Let $\beta = \{v_1, \ldots, v_{k+1}, u_1, \ldots, u_{n-(k+1)}\}$ be a basis of U.

$$[T]_\beta^\beta = \left[\begin{array}{cccc|ccc} \lambda & 0 & \cdots & 0 & * & \cdots & * \\ 0 & \lambda & \cdots & \vdots & * & \cdots & * \\ 0 & 0 & \cdots & 0 & \vdots & & \vdots \\ & (k+1)\times(k+1) & & \lambda & * & \cdots & * \\ \hline \vdots & \vdots & & 0 & * & & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{array}\right]_{n\times n} = \begin{bmatrix} \lambda I_{k+1} & A_{(k+1)\times(n-k-1)} \\ 0_{n-(k+1)\times(k+1)} & B_{(n-k-1)\times(n-k-1)} \end{bmatrix}$$

$$[T]_\beta^\beta = \begin{bmatrix} \lambda I_{k+1} & A \\ O & B \end{bmatrix}$$

$$C_{[T]_\beta^\beta}(x) = (x-\lambda)^{k+1} p(x)$$

$$[T]_\beta^\beta - x I_n = \begin{bmatrix} (x-\lambda) I_{k+1} & A \\ O & B - x I_{(n-k+1)} \end{bmatrix}$$

$$\det([T]_\beta^\beta - x I_n) = (x-\lambda)^{k+1} \det(B - x I_{n-k-1})$$

$$AM(\lambda) \geq k+1.$$

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● **Proposition 22**: For $T \in \mathcal{L}(U), dim(U) = n$, for any eigenvalue λ of T, $GM(\lambda) \leq AM(\lambda)$

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  Assume that the rate of change of predator population is $a \in \mathbb{R}$ times the current predator population and $b \in \mathbb{R}$ times that of the prey population. Similarly, the rate of change of prey population is $c \in \mathbb{R}$ times the current predator population and $d \in \mathbb{R}$ times the prey population, compute the predator-prey population at any time t, given some initial population.

$$\frac{d}{dt}x(t) = ax(t) + by(t)$$

$$\frac{d}{dt}y(t) = cx(t) + dy(t).$$

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$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

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$$\frac{d}{dt}x(t) = a\,x(t)$$

$$x(t) = e^{at}x(0)$$

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$$\frac{d}{dt} e^{At} = A e^{At}$$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\frac{d}{dt} X(t) = A\, X(t)$$

$$X(t) = \exp(At)\; X(0)$$

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \cdots$$

$$e^{At} = e^{\begin{bmatrix} at & bt \\ ct & dt \end{bmatrix}}$$

$$e^{At} := I_n + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

If $A = E\Lambda E^{-1}$

$$e^{At} = E\cancel{I}E^{-1} + E\Lambda E^{-1}t + E\Lambda^2E^{-1}\frac{t^2}{2!} + E\Lambda^3E^{-1}\frac{t^3}{3!} + \cdots = \sum_{n=0}^{\infty} E\Lambda^n E^{-1}\frac{t^n}{n!}$$

$$e^{At} = E\underbrace{\left(\sum_{n=0}^{\infty}\frac{\Lambda^n t^n}{n!}\right)}E^{-1} = E\,e^{\Lambda t}E^{-1}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & 0 \\ 0 & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

$$\frac{d}{dt}X(t) = AX(t) \Rightarrow \frac{d}{dt}X(t) = E\Lambda E^{-1}X(t)$$

$$\frac{d}{dt}\underbrace{\left(E^{-1}X(t)\right)}_{Y(t)} = \Lambda\left(E^{-1}X(t)\right)$$

$$\frac{d}{dt}Y(t) = \Lambda Y(t)$$

Applications

- Coupled Differential Equations:

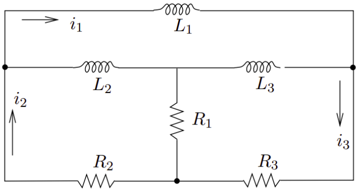

Figure: Source: Differential Equations and Linear Algebra, by GB Gustafson.

$$
\begin{aligned}
i_1' &= -\left(\frac{R_2}{L_1}\right) i_2 - \left(\frac{R_3}{L_1}\right) i_3, \\
i_2' &= -\left(\frac{R_2}{L_2} + \frac{R_2}{L_1}\right) i_2 + \left(\frac{R_1}{L_2} - \frac{R_3}{L_1}\right) i_3, \\
i_3' &= \left(\frac{R_1}{L_3} - \frac{R_2}{L_1}\right) i_2 - \left(\frac{R_1}{L_3} + \frac{R_3}{L_1} + \frac{R_3}{L_3}\right) i_3
\end{aligned}
$$

Applications

- Coupled Differential Equations:
- Epidemics Modeling:

▶ Population can be divided into: Susceptible/Healthy (S), Infected (I), and Dead (D) classes.

▶ Susceptible $\xrightarrow{-a}$ Infected $\xrightarrow{b}$ Susceptible: $\frac{d}{dt}S = -aS(t) + rI(t)$

▶ Infected $\xrightarrow{-d}$ Dead: $\frac{d}{dt}D = dI(t)$

▶ Infected: $\frac{d}{dt}I = aS(t) - dI(t) - rI(t)$

End of Class.

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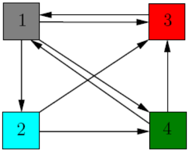

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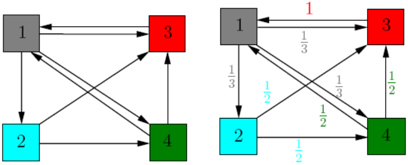

Figure: Model: If P_2 has links to P_3 and P_4, a surfer go to these pages with equal probability. Source: `pi.math.cornell.edu`

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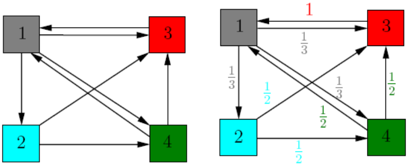

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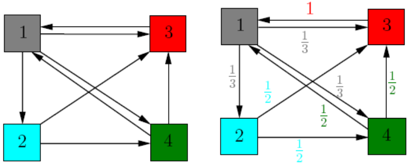

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- $\lim_{n\to\infty} A^n x_0 = y, Ay = y, y = [0.38, 0.12, 0.29, 0.19]^T$.

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- Properties:

Inner Products and Norms

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● Is this the only way to define length?

● What are the necessary conditions for a function on vector space for it be called *length*?

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- **Definition:** (Normed Vector Space) A normed vector space (NVS) is a vector space $(V, +, \cdot)$ over either $\mathbb{R}$ or $\mathbb{C}$ with a **norm**, a function $|| \cdot || : V \to \mathbb{R}$ which satisfies the following properties:

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  3. Triangular inequality: $\forall x, y \in V, ||x + y|| \leq ||x|| + ||y||$.
- A vector space V with a valid norm $||\cdot||$ is called a **Normed vector space** and is denoted by $(V, ||\cdot||)$.
- Also note that given a NVS $(V, ||\cdot||)$, we can define distance between two vectors x and y as $d(x, y) := ||x - y||$. Such a distance or metric is called the **induced metric**.

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- L_2 norm on $\mathcal{P}_n([-1,1])$: $||x||_{L_2} = \sqrt{\int_{-1}^1 (x(t))^2 \ dt}$.