# SC223 - Linear Algebra

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# Vector Spaces

- **Definition:** A Vector space is a set $V$ with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
  - $(V, +)$ is an **Abelian group**:
    - $\forall x, y \in V, x + y \in V$.
    - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$.
    - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote $y$ by $-x$.
    - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
    - $\forall x, y \in V, x + y = y + x$.
  - **Closure with respect to Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$.
  - **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.
  - **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.
  - **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

# Field

- **Definition:**(Field). A field is a set $\mathbb{F}$ with two binary operations, addition $+_F$ and multiplication $\times$ that satisfy the following axioms:
  - $(\mathbb{F}, +_F)$ is an **Abelian group**. The additive identity will be denoted by 0.
  - $(\mathbb{F} - \{0\}, \times)$ is an **Abelian group**. The mutiplicative identity will be denoted by 1.
  - **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c$.

# Properties of Vector Spaces

- **Proposition 1:** Every vector space has a unique additive identity.

$(V, +) \rightarrow \underline{\underline{\theta}}$ $\quad (G, *) \longrightarrow$ unique identity element.

* Let $e_1, e_2$ be two identity elements in $(G, *)$

$$e_1 = e_1 * e_2 = e_2$$

# Properties of Vector Spaces

- **Proposition 1:** Every vector space has a unique additive identity.
- **Proposition 2:** Every vector in a vector space has a unique additive inverse.

Let $a \in (G, *)$ and let $b_1, b_2 \in (G, *)$

such that $a * b_1 = b_1 * a = e \rightarrow (1)$

$a * b_2 = b_2 * a = e \rightarrow (2)$

$$b_1 = b_1 * e = b_1 * (a * b_2) = (b_1 * a) * b_2$$

$$= e * b_2 = b_2$$

## Properties of Vector Spaces

- **Proposition 1:** Every vector space has a unique additive identity.
- **Proposition 2:** Every vector in a vector space has a unique additive inverse.
- **Proposition 3:** $\forall v \in V, 0 \cdot v = \theta$

$$0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v$$

$$\theta = \theta + 0 \cdot v = 0 \cdot v$$

# Properties of Vector Spaces

- **Proposition 1:** Every vector space has a unique additive identity.
- **Proposition 2:** Every vector in a vector space has a unique additive inverse.
- **Proposition 3:** $\forall v \in V, 0 \cdot v = \theta$
- **Proposition 4:** $\forall a \in \mathbb{F}, a \cdot \theta = \theta$.

$$a \cdot \theta = a \cdot (\theta + \theta) = a \cdot \theta + a \cdot \theta$$

$$\theta = \theta + a \cdot \theta \Rightarrow \boxed{a \cdot \theta = \theta}$$

# Properties of Vector Spaces

- **Proposition 1:** Every vector space has a unique additive identity.
- **Proposition 2:** Every vector in a vector space has a unique additive inverse.
- **Proposition 3:** $\forall v \in V, 0 \cdot v = \theta$
- **Proposition 4:** $\forall a \in \mathbb{F}, a \cdot \theta = \theta$.
- **Proposition 5:** $\forall v \in V, (-1) \cdot v = -v$.

$$
\begin{aligned}
v + (-1)\cdot v &= 1\cdot v + (-1)\cdot v \\
&= (1+(-1))\cdot v \\
&= 0\cdot v \\
&= \theta
\end{aligned}
$$

# Tutorial 3

$A \in \mathbb{R}^{4 \times 6}$

Any $x \in \mathbb{R}^6$, s.t $Ax = \vec{0}$ is of the form

$$x = \begin{bmatrix} 2p+q \\ p \\ 3q-r \\ q \\ r \\ 4p+2r \end{bmatrix}, \quad \forall\, p, q, r \in \mathbb{R}$$

$p = 1, q = r = 0$

$p = 0, q = 0, r = 1$

$p = 0, q = 1, r = 0$

$\{a_{*1}, a_{*2}, a_{*6}\} \rightarrow LD$

Since $\exists\, x \neq \vec{0}$, s.t $Ax = \vec{0}$

$\Rightarrow$ col rank $A < 6$.

$\Rightarrow$ Consider all subsets of cols of $A$ with 5 elements

$\{a_{*1}, a_{*2}, \dots, a_{*5}\}$ $\{a_{*1}, \dots, a_{*4}, a_{*6}\}$

---- $\{a_{*2}, a_{*3}, \dots, a_{*6}\}$

$p = \cancel{2}, q = r = 0$

$$x = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \rightarrow Ax = 2a_{*1} + 1a_{*2} + 4a_{*6} = \vec{0}$$

$\underline{\{a_{*1}, a_{*2}, a_{*6}\}}$ LD

$p = 0, q = 1, r = 0,$ $\{a_{*1}, a_{*3}, a_{*4}\}$ LD

$p = q = 0, r = 1,$ $\{a_{*3}, a_{*4}, a_{*6}\}$ LD

$${}^6C_4 = 15$$

$$N(A) \subseteq \mathbb{R}^6$$

$$\{x_1, x_2, x_3\} \in N(A)$$

$$z_1, z_2, z_3 \in \mathbb{R}^6$$

$\{x_1, x_2, \cdots x_3, z_1, z_2, z_3\}$ are LI

$$Az_1, Az_2, Az_3$$