SC223 - Linear Algebra

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Lecture 37

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Summary of Lecture 36

- Norm: For a vector space $(V, +, \cdot)$ over $\mathbb{R}$ or $\mathbb{C}$, **norm** is a function $||\cdot|| : V \to \mathbb{R}$ that satisfies:
  - Positive definiteness: $\forall x \in V, ||x|| \geq 0, ||x|| = 0 \Leftrightarrow x = \theta$
  - Absolute Homogeneity: $\forall x \in V, \forall a \in \mathbb{F}, ||a \cdot x|| = |a| \cdot ||x||$
  - Triangular Inequality: $\forall x, y \in V, ||x + y|| \leq ||x|| + ||y||$.
- A vector space $(V, +, \cdot)$ with a norm $||\cdot||$ is called a **Normed Vector space** (NVS).

Inner Product

- **Definition:** (Inner Product) Given a vector space $(V, +, \cdot)$ over $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$), an **inner product** is any mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that it satisfies the following properties:

    1. Positive definite: $\forall x \in V, \langle x, x \rangle \geq 0, = 0 \Leftrightarrow x = \theta$.

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   $\forall x, y, z \in V, \forall a, b \in \mathbb{F}, \langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

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  3. Conjugate symmetry: $\langle y, x \rangle = \overline{\langle x, y \rangle}$

$$\langle x, ay + bz \rangle = \overline{\langle ay + bz, x \rangle}$$

$$= \overline{a\langle y, x \rangle + b\langle z, x \rangle}$$

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- **Definition:** (Inner Product Space) A vector space V with an inner product is called an **Inner Product space**(IPS) and is denoted by $(V, \langle \cdot, \cdot \rangle)$.

Example 1: $V = \mathbb{R}^n$, $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$

$$\langle ax+by, z \rangle = \sum_{i=1}^{n} (ax_i + by_i) z_i$$
$$= a\sum_{i=1}^{n} x_i z_i + b\sum_{i=1}^{n} y_i z_i$$
$$= a\langle x, z \rangle + b\langle y, z \rangle$$

Examples of IPS

- $\mathbb{R}^n$: Euclidean inner product - $x, y \in \mathbb{R}^n, \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i = x^T y$, where x and y are written as column vectors.

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● $\mathcal{P}_n([0,1])$: L_2 inner product - $\forall p, q \in \mathcal{P}_n([0,1]), \langle p, q\rangle = \int_0^1 p(t)q(t)\, dt$.

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- $\mathcal{P}_n([0,1])$: L_2 inner product - $\forall p, q \in \mathcal{P}_n([0,1]), \langle p, q\rangle = \int_0^1 p(t)q(t)\,dt$.
- Let $G \in \mathbb{R}^{n\times n}$ be such that $\boxed{G = G^T}$ and $x^T G x > 0, \forall x \in \mathbb{R}^n, \neq \mathbf{0_n}$. Such a matrix G is ~~said to be~~ *Symmetric Positive-Definite* (SPD). Then, on $\mathbb{R}^n$, $\forall x, y \in \mathbb{R}^n$, $\boxed{\langle x, y\rangle = x^T G y}$ is a valid inner product.

$\forall x, y, \quad \langle x, y\rangle = x^T G y = \langle y, x\rangle = \underline{y^T G x}.$

$x^T G y = x^T G^T y. \qquad \Rightarrow \boxed{G = G^T}$

$\langle x, x\rangle = x^T G x \geq 0$ & if $x \neq \vec{0}$, $x^T G x > 0$

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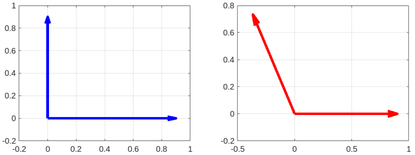

Figure: Orthogonal vectors with inner product: (left) $\langle x, y \rangle = x^T y$, (right) $\langle x, y \rangle = x^T G y, G = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Definitions

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$$

- Given an IPS $(V, \langle\cdot,\cdot\rangle), \forall x \in V, ||x|| = \sqrt{\langle x, x\rangle}$ is a valid norm, called the **induced norm**.

$$||x|| := \sqrt{\langle x,x\rangle}$$

$$||a\cdot x|| = \sqrt{\langle ax, ax\rangle} = \sqrt{a\langle x, ax\rangle}$$

$$= \sqrt{a\cdot\bar{a}\langle x,x\rangle} = \sqrt{|a|^2\langle x,x\rangle}$$

$$= |a|\cdot\sqrt{\langle x,x\rangle} = |a|\cdot||x||.$$

$$||x+y||^2 = \langle x, x+y\rangle + \langle y, x+y\rangle$$

$$= \langle x,x\rangle + \langle y,y\rangle + 2\mathrm{Re}\{\langle x,y\rangle\}.$$

$$\leq ||x||^2 + ||y||^2 + 2|\langle x,y\rangle|$$

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- A matrix $A \in \mathbb{R}^{n \times n}$ or $\mathbb{C}^{n \times n}$ is said to be an **orthogonal matrix** if all its n columns are orthonormal, i.e., $A^*A = I$, where A^* denotes the conjugate transpose of A. In this case, $A^{-1} = A^*$.

END OF CLASS

Properties

- **Proposition 23** (Pythagoras Theorem): In an IPS $(V, \langle \cdot, \cdot \rangle)$, if $\langle x, y \rangle = 0$, $||x + y||^2 = ||x||^2 + ||y||^2$.

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$$\langle w, y \rangle = 0$$

$$\langle x - a \cdot y, y \rangle = \langle x, y \rangle - a\langle y, y \rangle = 0$$

$$a = \frac{\langle x, y \rangle}{\langle y, y \rangle}$$

$$\text{Thus}, x = \frac{\langle x, y \rangle}{\langle y, y \rangle} \cdot y + \left(x - \frac{\langle x, y \rangle}{\langle y, y \rangle} \cdot y \right)$$

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- **Proposition 25** (Gram-Schmidt Procedure): Let $\{v_1, \ldots, v_m\}$ be a list of linearly independent vectors. Then there exists a list of orthonormal vectors $\{e_1, \ldots, e_m\}$ such that $span(\{v_1, \ldots, v_j\}) = span(\{e_1, \ldots, e_j\}), \forall j = 1, \ldots, m$.

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- Similarly, $e_3 = \frac{v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)}{||v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)||}$, and $e_k = \frac{v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i}{||v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i||}$.

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- Then $\forall l = 1, \ldots j$, with $\tilde{e_{j+1}} = v_{j+1} - \sum_{i=1}^{j} \langle v_{j+1}, e_i \rangle e_i$

$$\begin{aligned}
\langle e_{j+1}, e_l \rangle &= \frac{1}{||\tilde{e_{j+1}}||} \left(\langle v_{j+1}, e_l \rangle - \sum_{i=1}^{j} \langle v_{j+1}, e_i \rangle \langle e_i, e_l \rangle \right) \\
&= \frac{1}{||\tilde{e_{j+1}}||} \left(\langle v_{j+1}, e_l \rangle - \langle v_{j+1}, e_l \rangle \right) = 0
\end{aligned}$$

Orthogonal Complement

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$$U^{\perp} = \{v \in V \mid \langle v, u \rangle = 0, \forall u \in U\}$$

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- **Proposition 26:** Irrespective of whether U is a subspace of V or not, $U^{\perp}$ is a subspace.

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3. Idempotent: $(P_U)^2 =$

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2. Null$(P_U) = U^{\perp}$
3. Idempotent: $(P_U)^2 = P_U$
4. (Conjugate) Symmetric: If $U = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T =$

Orthogonal Projection

- Let U be a subspace of FD IPS V, and $V = U \oplus U^{\perp}$.
- Define $P_U \in \mathcal{L}(V)$ as $\forall v \in V$, if $v = u + w, u \in U, w \in U^{\perp}$, $P_U(v) = u$.
- P_U is said to be the *Orthogonal Projection Operator on* U.
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  4. (Conjugate) Symmetric: If $U = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T = P_U$ ($P_U^* = P_U$).
  5. $\forall v \in V, P_U(v) = \arg\min_{u \in U} ||u - v||^2$.