# LECTURE 31

## RECAP.

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} \qquad \text{OR} \qquad \frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

where $c/v \equiv$ vel. of wave.

— Had proved $\psi = f(x \pm ct)$

$= f\left(t \pm \frac{x}{c}\right)$ are always solutions.

— $\psi = e^{-i\omega(t - x/v)}$.

$= \begin{matrix}\cos\\ \sin\end{matrix}(kx - \omega t)$

- $\psi(x,t) = A\cos\left[\frac{2\pi}{\lambda}(x-ct)\right]$. $\lambda = [L]$.

$\lambda \equiv$ Wavelength.

$$\psi(x+\lambda, t) = A\cos\left[\frac{2\pi}{\lambda}(x+\lambda-ct)\right]$$

$$= A\cos\left[\frac{2\pi}{\lambda}(x-ct) + 2\pi\right]$$

$$= \psi(x,t).$$

$f = \frac{c}{\lambda} \longrightarrow$ frequency. $\omega = 2\pi f \to$ circular frequency.

$[f] = [T]^{-1}$

$$\psi\left(x, t + \frac{1}{f}\right) = A\cos\left[\frac{2\pi}{\lambda}\left(x - c\left(t + \frac{1}{f}\right)\right)\right]$$

$$= A\cos\left[\frac{2\pi}{\lambda}(x - ct) + \frac{2\pi c}{\lambda f}\right]$$

$\boxed{c = \lambda f}$ $\Rightarrow$ Relⁿ between vel, wavelength & frequency.

$k = \frac{2\pi}{\lambda} \longrightarrow$ wave no.

$\psi(x,t) = \cos(kx - \omega t)$.

$\omega = 2\pi f$.

## ☒ De Broglie hypothesis.

For any particle moving with momentum $p$, there is a wave associated with it, where

$$p = \frac{h}{\lambda} \qquad E = hf.$$

$h$ = Planck's const $= 6.62 \times 10^{-34}\ m^2\,kg/s$.

$$\hbar = \frac{h}{2\pi}$$

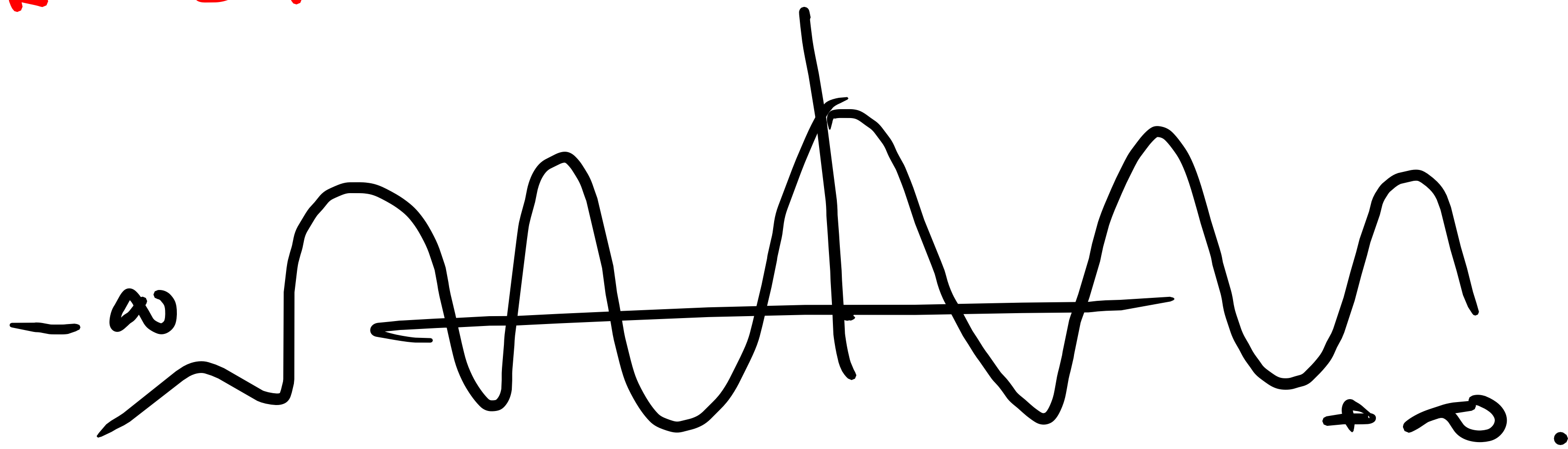
Hypothesis confirmed experimentally.

WAVES $\longrightarrow$ PARTICLES

QUANTUM MECHANICS

SCHRÖDINGER EQN

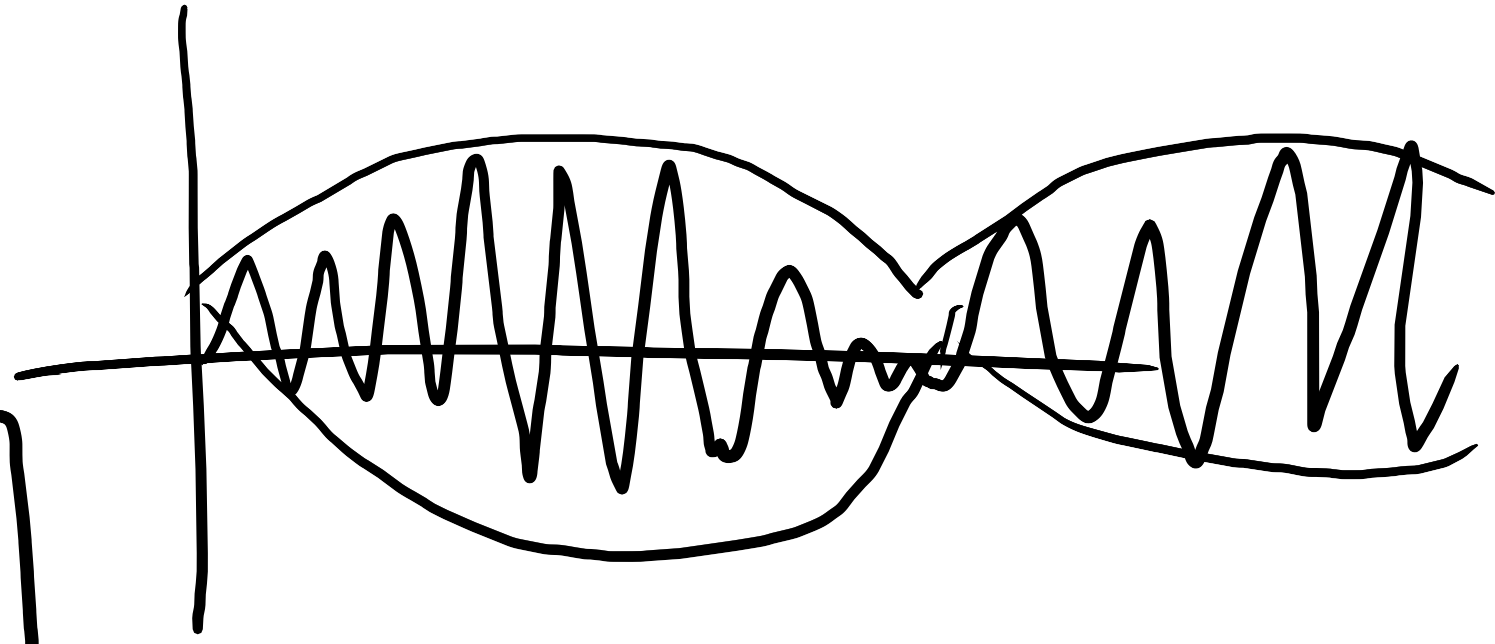
$$\psi(x,t) = A\cos(kx - \omega t).$$




Without $t$, $\psi(x) = A\cos kx$.

— $\psi(x,t)$ extends from $\pm\infty$., but the notion of a particle is that it is localised to one region.

— Localization of waves.



$$\psi(x) = A\left[\cos(k_1 x) + \cos(k_2 x)\right]$$

$$= A\left[\cos\left(\frac{2\pi}{\lambda_1}x\right) + \cos\left(\frac{2\pi}{\lambda_2}x\right)\right]$$

$$= 2A\cos\left(\frac{\pi x}{\lambda_1} - \frac{\pi x}{\lambda_2}\right)\cos\left(\frac{\pi x}{\lambda_1} + \frac{\pi x}{\lambda_2}\right).$$

$$\lambda_1 - \lambda_2 = \Delta\lambda << \lambda_1, \lambda_2 .$$

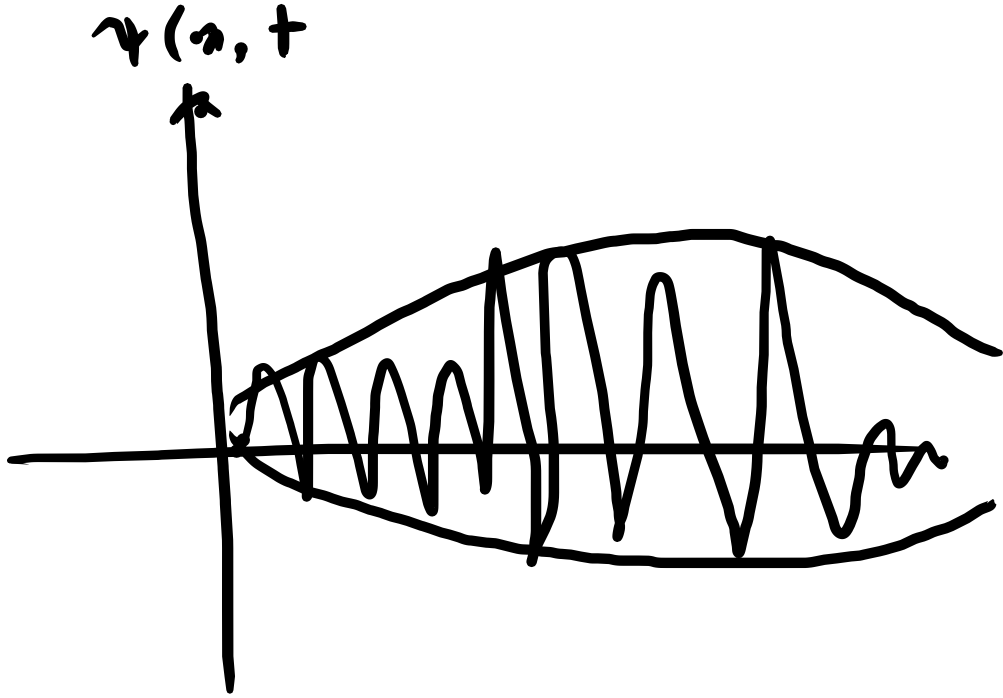
$$\frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{\lambda_1} - \frac{1}{\lambda_1 - \Delta\lambda}.$$

$$\lambda_{avg} = \frac{\lambda_1 + \lambda_2}{2}.$$

$$\simeq \frac{1}{\lambda_1} - \frac{1}{\lambda_1}\left(1 - \frac{\Delta\lambda}{\lambda_1}\right)^{-1}.$$

$$\lambda_{avg} \simeq \lambda_1 \text{ or } \lambda_2.$$

$$\simeq \frac{\Delta\lambda}{\lambda_1^2}$$




Similarly for $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{2}{\lambda_{avg}}$.

$$\Psi(x) = 2A\cos\left(\frac{\Delta\lambda\pi x}{\lambda_{avg}^2}\right)\cos\left(\frac{2\pi x}{\lambda_{avg}}\right)$$

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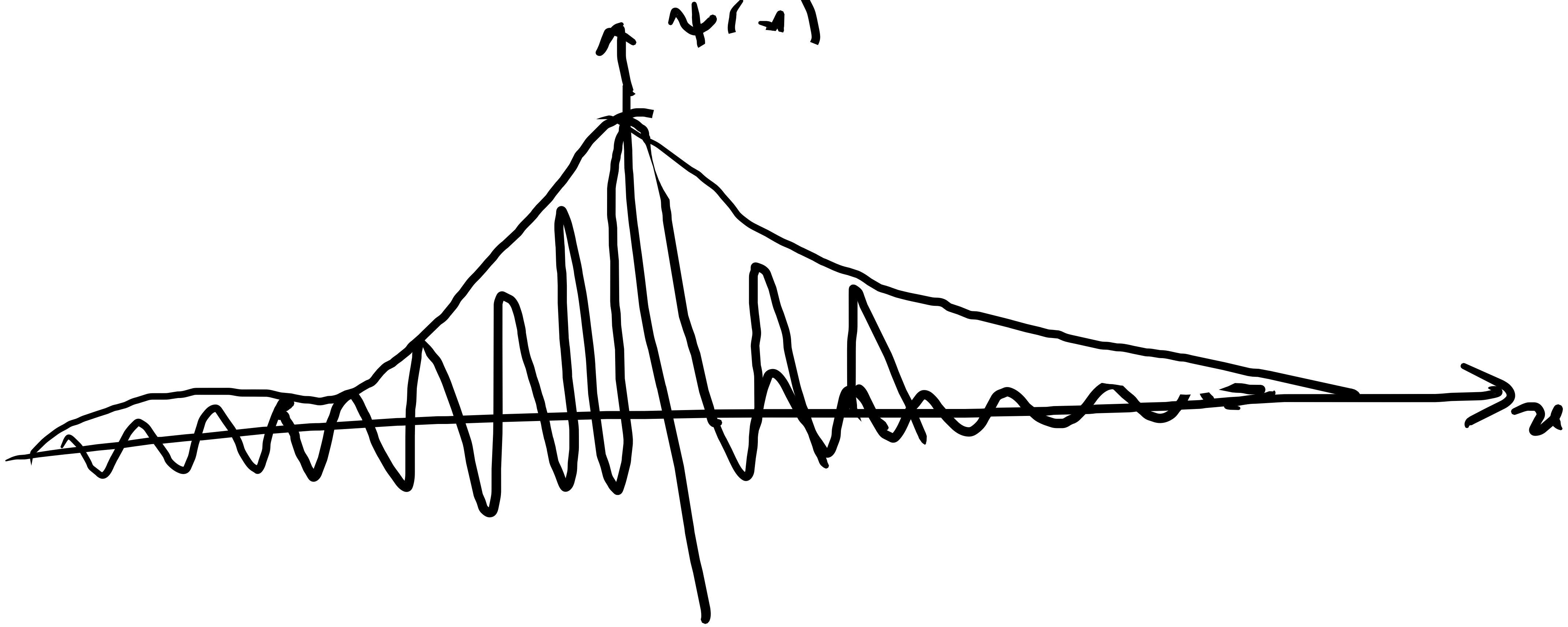
- 'Problem persists $\longrightarrow$ this still extends from $-\infty$ to $+\infty$.

- Superposition/addition of two cosine sol$^{\underline{n}}$s has produced an amplitude which is a f$^{\underline{n}}$ of $x$.
- Possibility of constructing an amplitude f$^{\underline{n}}$ which is non-zero in one region and zero elsewhere? (Localization).
- Consider the f$^{\underline{n}}$ $\psi(x) = \frac{2A}{x} \cos\left(\frac{\Delta\lambda \pi x}{\lambda^2}\right) \cos\left(\frac{2\pi}{\lambda} x\right)$

$\psi(\pm\infty) \to 0$., but $\psi(0) \to \infty$.
- Refinement :- $y(x) = \frac{2A}{x} \sin\left(\frac{\Delta\lambda \pi x}{\lambda^2}\right) \cos\left(\frac{2\pi x}{\lambda}\right)$




$$\psi(x) = \sum_i A(k_i) \cos(k_i x)$$

$$\psi(x) = \int dk\, A(k) \cos kx \,.$$

$\longrightarrow$ necessary for localization.

– Suppose, range of k's from $\left(k_0 - \frac{\Delta k}{2}\right)$ to $\left(k_0 + \frac{\Delta k}{2}\right)$.

with $A(k) = A_0 = \text{const}$.

$$A(k) = A, \qquad k_0 - \frac{\Delta k}{2} \leq k \leq k_0 + \frac{\Delta k}{2}$$

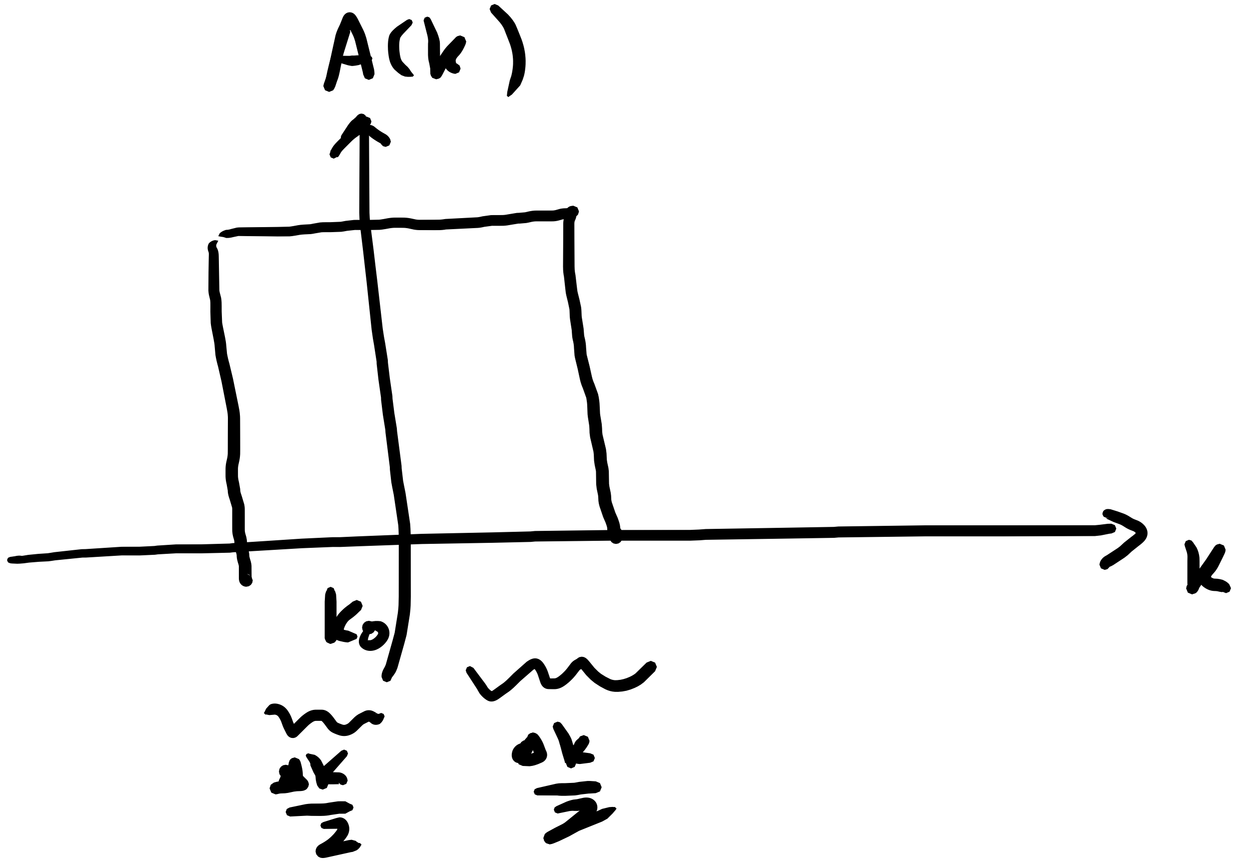
$$= 0, \qquad \text{otherwise}.$$

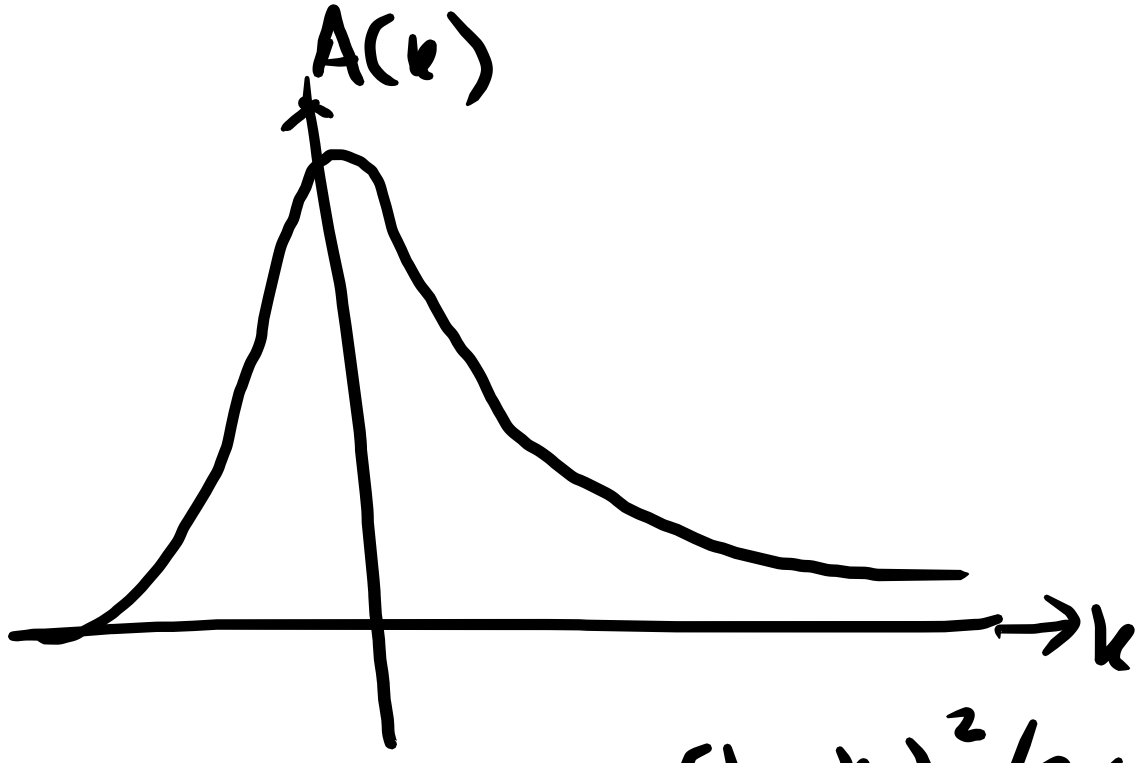
$$\psi(x) = \int dk\, A(k) \cos kx \quad -$$

$$= A_0 \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \Delta k/2} dk \cos(kx). \quad = \frac{A_0}{x} \sin kx \Big|_{k_0 - \Delta k/2}^{k_0 + \Delta k/2}.$$

$$= \frac{A_0}{x}\left[\sin\left(k_0 x + \frac{\Delta k}{2}x\right) - \sin\left(k_0 x - \frac{\Delta k}{2}x\right)\right]$$

$$= \frac{2A_0}{x}\sin\left(\frac{\Delta k}{2}x\right)\cos(k_0 x).$$




$$A(k) = A_0 e^{-(k-k_0)^2/2(\Delta k)^2}$$

Task :-

evaluate $\psi(x) = \int dk \, e^{-(k-k_0)^2/2(\Delta x)^2} \cos kx$ .