Computational Numerical Methods

CS 374

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LU Factorization / Decomposition

Diagonal matrix

Upper triangular.

Lower triangular.

$$\begin{bmatrix} a_{11} & \cdots & 0 \\ 0 & a_{22} \cdots & 0 \\ \vdots & & \\ 0 & 0 \cdots & a_{nn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$Ax = b.$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$Ax = b.$$

$$\begin{bmatrix} a_{11} & \cdots & & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$\cancel{A}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}}$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ m_{31} & m_{32} & 1 \cdots & 0 \\ m_{n1} & m_{n2} & \cdots & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \cdots & a_{3n}^{(3)} \\ \vdots & & & \\ 0 & 0 & \cdots 0 & a_{nn}^{(n)} \end{bmatrix}$$

$$\underline{A = LU}$$

Try to verify with any example.

$$Ax = b.$$

$$LUx = b.$$

$$Lz = b$$

$$Ux = z.$$

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$Ux = z.$$

Ex

$$\begin{cases} 6x + 18y + 3z = 3 \\ 2x + 12y + z = 19 \\ 4x + 15y + 3z = 0 \end{cases}$$

$$A = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$Ax = b$$

$$\begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix} x = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}$$

$$\overset{L}{\begin{bmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}.$$

let $\underline{\underline{Ux = z}}$.

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}$$

$$3z_1 = 3 \Rightarrow z_1 = 1$$

$$z_1 + 6z_2 = 19 \Rightarrow z_2 = \frac{18}{6} = 3$$

$$2z_1 + 3z_2 + z_3 = 0 \Rightarrow z_3 = -11$$

$$UX = Z$$

$$\Rightarrow \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -11 \end{bmatrix}$$

$$z = -11$$

$$y = 3$$

$$2x + 6y + z = 1$$

$$x = -3.$$

Next questions

1. Are these representations of L & U are unique.

2. How to find L & U.

$$A \cdot x = LU$$

$$= L \cdot D \cdot D^{-1} U.$$

$$= \tilde{L} \tilde{U}.$$

① Doolittle's method.

$$A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = L \cdot U.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}.$$

$$l_{21} u_{11} = a_{21} \Rightarrow l_{21} = \frac{a_{21}}{a_{11}}$$

$$- \; - \; -$$

$$l_{31} u_{13} + l_{32} \cdot u_{23} + u_{33} = a_{33}$$

$L \cdot U =$

Crout's method

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

~~Let~~ Decompose A ~~into~~ as $A = L\,U$.

where: $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = L \cdot U.$

Further take $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to solve $\underline{Ax = b}$.

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

L

$U = ($

$L\underline{Ux} = b.$ $\qquad$ $\underline{\underline{Ux = z}}.$ $\qquad$ $z = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$

$Lz = b.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$z_1 = 1$

$z_1 + z_2 = 1 \Rightarrow z_2 = 0.$

$\cancel{z_2 = 0}$

$-2z_1 + 3z_2 + z_3 = 1$

$z_3 = 3.$

So $$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

$\Rightarrow$ $2x_3 = 3 \Rightarrow x_3 = 3/2.$

$x_2 - x_3 = 0 \qquad x_2 = 3/2$

$x_1 + x_2 - x_3 = 1$

$\Rightarrow \quad x_3 = 1$

Cholesky's factorization