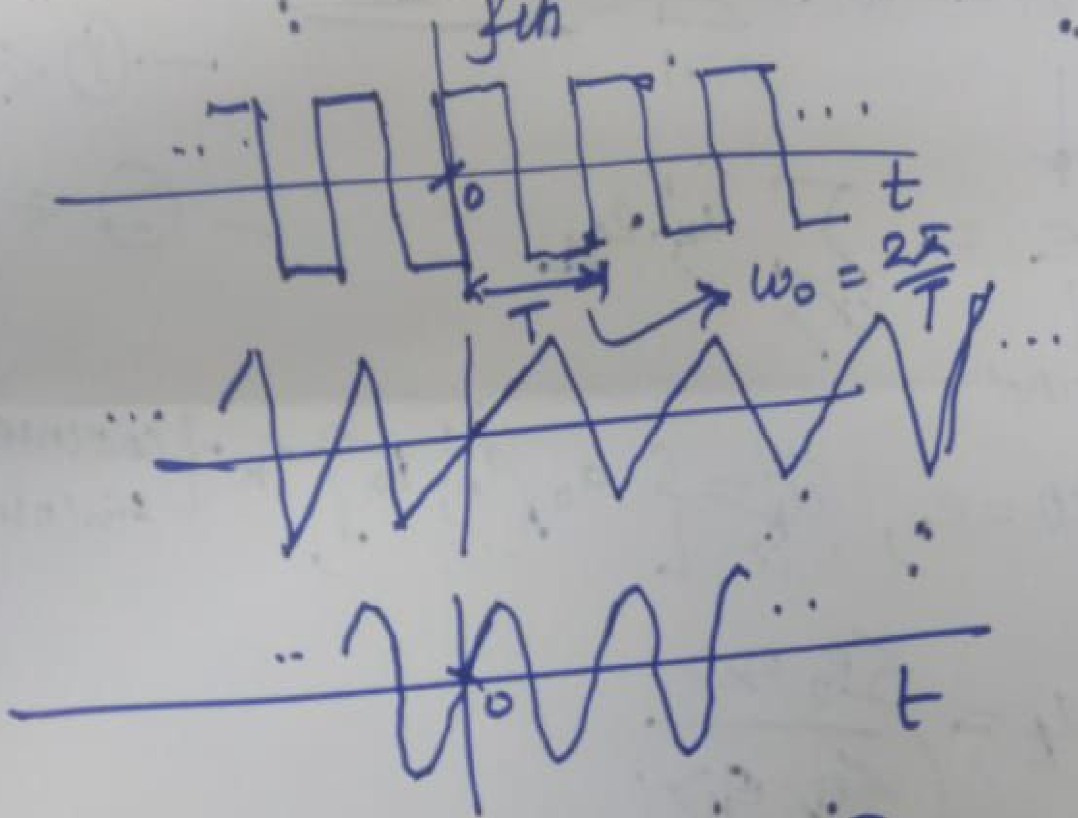
Lecture 26

Fourier Series: → Used to represent a periodic signal as a linear combination of sines and cosines.



$\{w_0, 2w_0, 3w_0, \cdots, nw_0\}\ n \in \mathbb{Z}$

Fourier Series

↓ Trigonometric Fourier Series

↓ Exponential Fourier Series.

1) Trignometric Fourier Series Representation

Let $f(t)$ be the periodic signal

$$\underset{\text{signal}}{e}\ (f(t)) = a_0 + \sum_{n=1}^{+\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \quad \text{— (1)}$$

$$e = \sum_{\langle K \rangle} a_K \cdot e_K \quad \text{— (2)}$$

vector

$f(t) = e$, $a_K = \{a_0, a_n, b_n\}$, $e_K = \{\cos(n\omega_0 t), \sin(n\omega_0 t)\}$.

$$a_K = \frac{\langle e, e_K \rangle}{\langle e_K, e_K \rangle}$$

$$a_n = \frac{\langle f(t), \cos(n\omega_0 t) \rangle}{\langle \cos(n\omega_0 t), \cos(n\omega_0 t) \rangle}$$

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{+\infty} f(t) \cdot g^*(t)\, dt$$

$$\therefore \langle f(t), \cos(n\omega_0 t) \rangle = \int_{-\infty}^{+\infty} f(t) \cdot \cos(n\omega_0 t)\, dt$$

$$\langle \cos(n\omega_0 t), \cos(n\omega_0 t)\rangle = \int_{-\infty}^{+\infty} \cos^2(n\omega_0 t)\,dt$$

$$= \int_0^{T} \cos^2(n\omega_0 t)\,dt$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow T = 2\pi/\omega_0$$

$$= \int_0^{2\pi/\omega_0} \left[\cos^2(n\omega_0 t)\right] dt$$

??

$$= \pi/\omega_0 \text{ or } \frac{\pi}{2\pi/T} = T/2$$

$$\boxed{a_n = \left(\frac{2}{T}\right) \int_0^T f(t)\cos(n\omega_0 t)\,dt}$$

$$a_0 = a_n\Big|_{n=0} = \left(\frac{2}{T}\right) \int_0^T f(t)\,\underline{\cos(0.\omega_0 t)}\,dt$$

$$\boxed{a_0 = \left(\frac{2}{T}\right) \cdot \int_0^T f(t)\,dt}$$

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$$b_n = \frac{\langle f(t), \sin(n\omega_0 t) \rangle}{\langle \sin(n\omega_0 t), \sin(n\omega_0 t) \rangle}$$

$$\boxed{b_n = \left(\frac{2}{T}\right) \cdot \int_0^T f(t) \cdot \sin(n\omega_0 t)\, dt}$$

$\{a_0, a_n, b_n\} \Rightarrow$ Fourier series coefficient.

2) Exponential Fourier Series Representation

Let $f(t)$ = periodic signal.

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \cdot (e^{jn\omega_0 t}) \quad \text{--- (3)}$$

$$e^{jn\omega_0 t} = \cos(n\omega_0 t) + j\sin(n\omega_0 t)$$

[Euler's relation]

$$\therefore F_n = \frac{\langle f(t), e^{jn\omega_0 t} \rangle}{\langle e^{jn\omega_0 t}, e^{jn\omega_0 t} \rangle}$$

(4)

$$\boxed{F_n = \left(\frac{1}{T}\right) \cdot \int_0^T f(t) \cdot e^{-jn\omega_0 t} \, dt}$$

$$\langle e^{jn\omega_0 t}, e^{+jn\omega_0 t} \rangle = \int_0^T e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t} \, dt$$

$$= \int_0^T dt = T$$

Exponential Fourier Series Representation

$f(t)$ = periodic signal

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \cdot e^{jn\omega_0 t} \rightarrow \text{Synthesis Eq}^n.$$

$$F_n = \left(\frac{1}{T}\right) \int_0^T f(t) \cdot e^{-jn\omega_0 t} \, dt \rightarrow \text{Analysis Eq}^n.$$

Fourier series coefficients

ω_0 = fundamental frequency

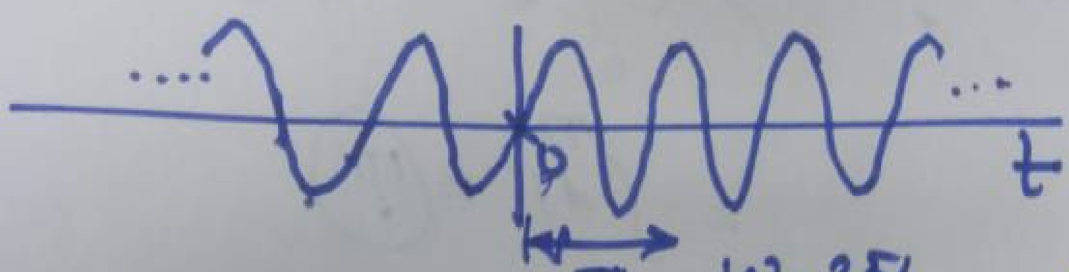
$\{n\omega_0\}_{n \in \mathbb{Z}}$ = Harmonics $\rightarrow$ Music literature

$\hookrightarrow$ integer multiple of fundamental frequency

Tutorial Problems

For the following signals, find the **exponential** Fourier series representation.

Problem 1) $f(t) = \sin(\omega_0 t)$



$T \longrightarrow \omega_0 = 2\pi/T.$

$$f(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_0 t}$$

$$\boxed{F_1 = \frac{1}{2j}, \quad F_{-1} = \frac{-1}{2j}}$$

$$F_2 = F_{-2} = F_3 = F_{-3} = F_0 = \dots = 0$$

$$F_n = 0 \quad \forall n \notin \cancel{[-1,1]} \{-1, 1\}$$

Observation: Fourier series coefficients F_n, in general, <u>complex numbers</u>

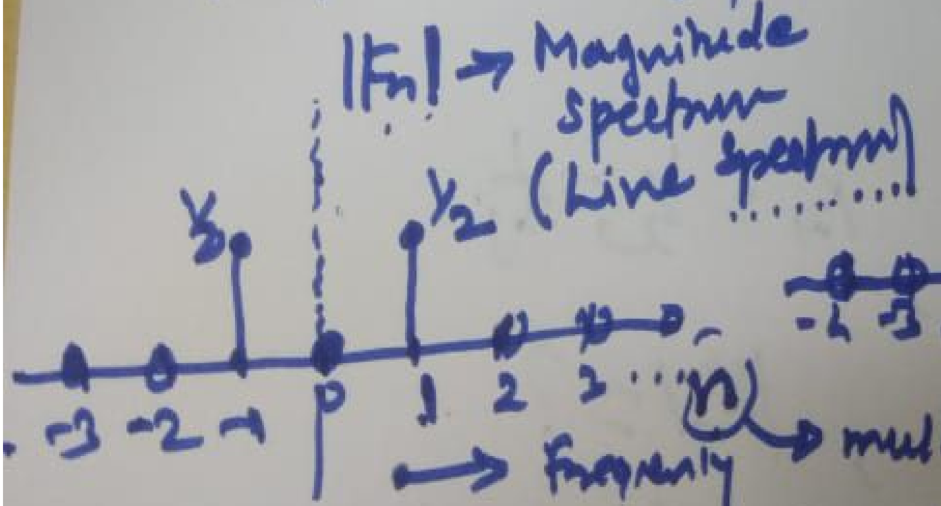
∴ F_n's are represented via magnitude and phase information.

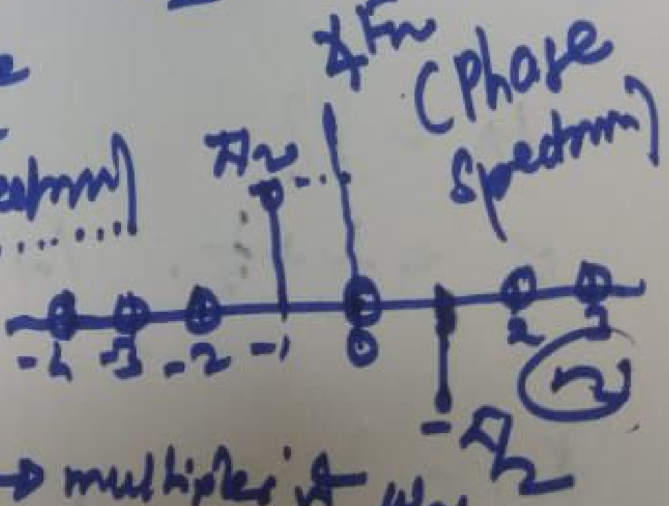
$$F_1 = \frac{1}{2j} \Rightarrow |F_1| = \left|\frac{1}{2j}\right| = \frac{1}{2}$$

$$F_{-1} = \frac{-1}{2j} \Rightarrow |F_{-1}| = \left|\frac{-1}{2j}\right| = 1/2$$

$$\angle F_1 = \tan^{-1}\left(\frac{-1/2}{0}\right) = -\pi/2$$

$$\angle F_{-1} = \tan^{-1}\left(\frac{1/2}{0}\right) = \pi/2$$





→ Frequency

(n) → multiples of w_0.

$$F_n = \frac{1}{T}\int_0^T f(t)\cdot e^{-jnw_0 t}\,dt$$

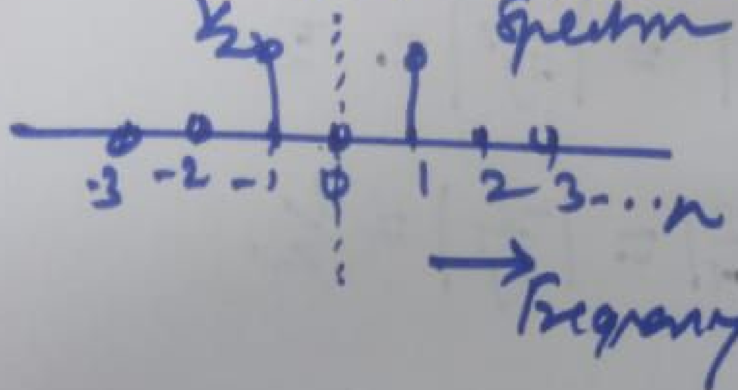
(?)

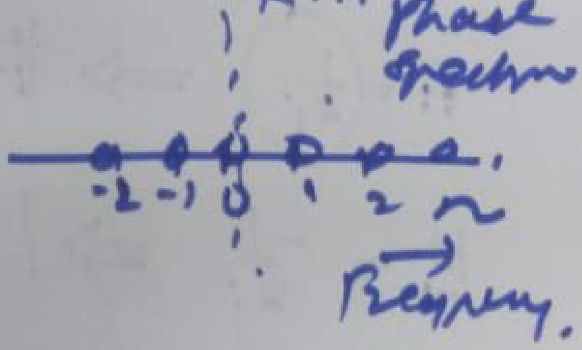
Problem (2)

$f(t) = \cos(\omega_0 t)$

$F_n = ??$ $F_1 = F_{-1} = \frac{1}{2}$, $F_n = 0 \ \forall n \notin \{-1,1\}$.

$|F_n|$ Magnitude Spectrum , $\measuredangle F_n$ phase spectrum





Problem (3)

$f(t) = \sin(\underline{\omega_0 t}) + \cos(\underline{\omega_0 t})$.

$F_n = ??$

$F_1 = \frac{1}{2} + \frac{1}{2j}$, $F_{-1} = \frac{1}{2} - \frac{1}{2j}$

$|F_1| = ?$, $|F_{-1}| = ?$

$\measuredangle F_1 = ?$ $\measuredangle F_{-1} = ??$

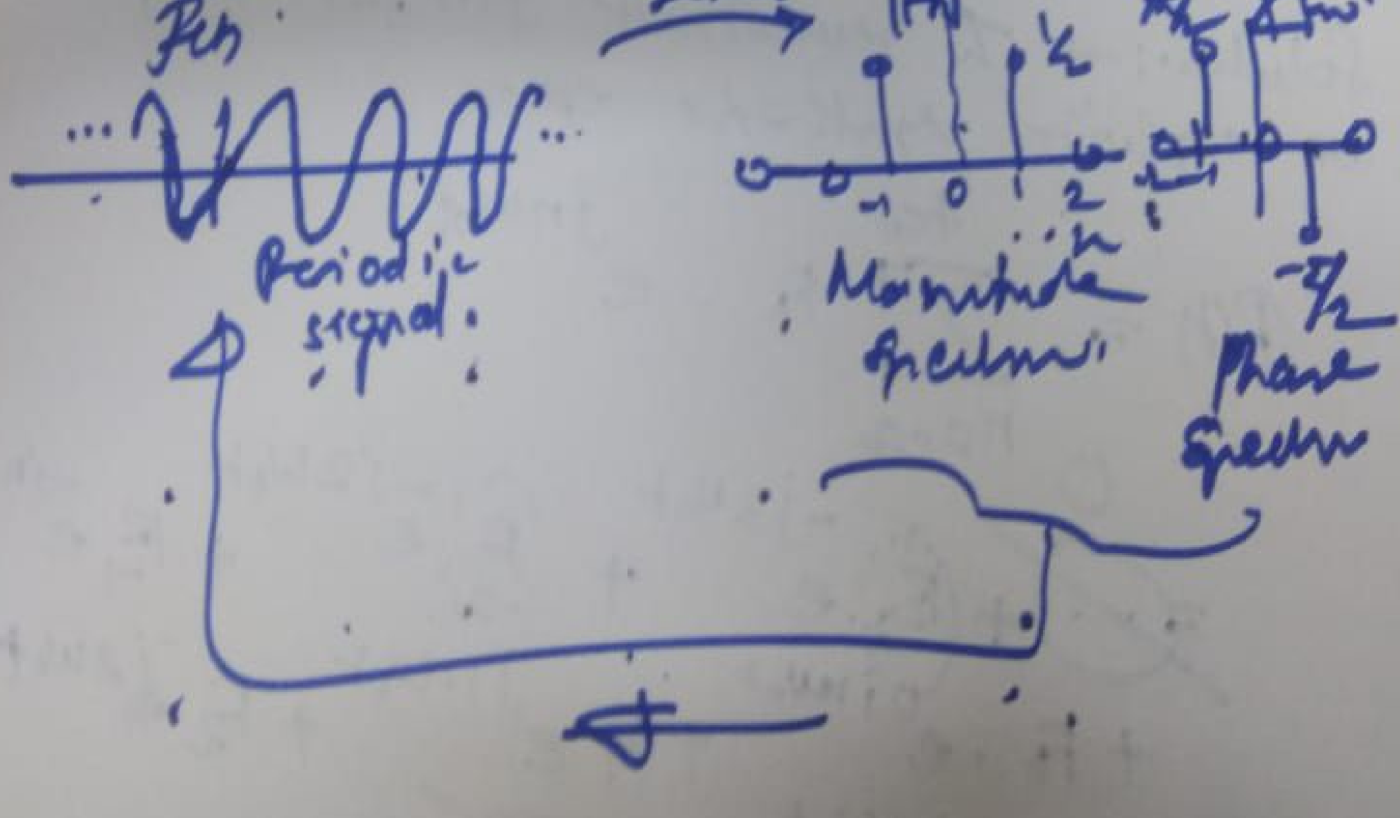
Problem (4)

$f(t) = \sin(2\omega_0 t) + \cos(1\omega_0 t) + \cos(2\omega_0 t) + \sin(1\omega_0 t)$

<u>Homework</u>.

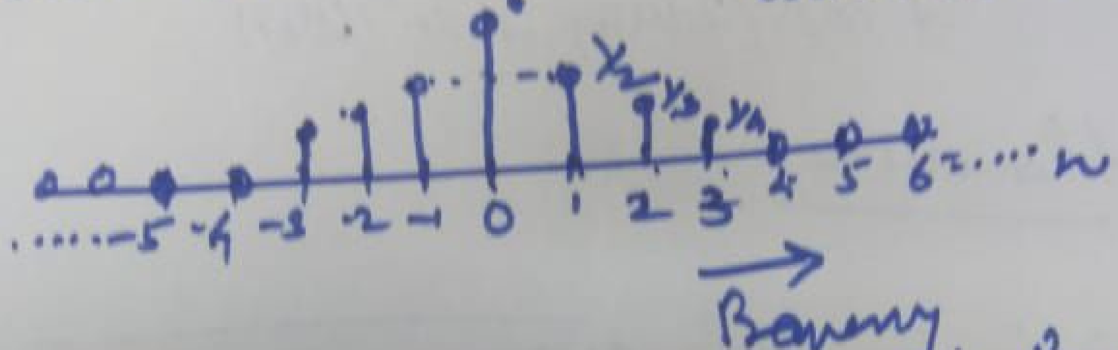
Fourier series (??)

Time-domain . $\xrightarrow{\text{Fourier series}}$ Frequency domain



Problem (5) For a periodic signal $f(t)$ exponential Fourier representation is given by

$F_n \rightarrow$ Fourier series coefficient.



Reconstruct periodic signal $f(t)$ from F_n.

Solution: $\rightarrow$ To reconstruct $f(t)$ from F_n, we exploit synthesis eqn.

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \cdot e^{jn\omega_0 t}$$

$$= \underbrace{\cancel{\dots}}_{0} + F_{-3}\, e^{-j3\omega_0 t} + F_{-2}\, e^{-j2\omega_0 t} + F_{-1}\, e^{-j1\omega_0 t}$$

$$+ F_0 \cdot e^{j0\omega_0 t} + F_1\, e^{j1\cdot\omega_0 t} + F_2\, e^{j2\omega_0 t}$$

$$+ F_3\, e^{j3\omega_0 t} + \underbrace{\cancel{\dots}}_{0}$$

$$f(t) = 0 + \frac{1}{4} e^{-j3\omega_0 t} + \left(\frac{1}{3}\right) e^{-j2\omega_0 t}$$

$$+ \left(\frac{1}{2}\right) e^{-j\omega_0 t} + 1 + \left(\frac{1}{2}\right) e^{j\omega_0 t} + \left(\frac{1}{3}\right) e^{j2\omega_0 t}$$

$$+ \left(\frac{1}{4}\right) e^{+j3\omega_0 t} + 0$$

$$f(t) = \left(\frac{1}{4}\right) \cdot 2 \left[\frac{e^{j3\omega_0 t} + e^{-j3\omega_0 t}}{2}\right]$$

$$+ \left(\frac{1}{3}\right) \cdot 2 \left[\frac{e^{j2\omega_0 t} + e^{-2j\omega_0 t}}{2}\right]$$

$$+ \left(\frac{1}{2}\right) \cdot 2 \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$+ 1$$

$$\boxed{f(t) = 1 + (1) \cos(\omega_0 t) + \left(\frac{2}{3}\right) \cos(2\omega_0 t) + \left(\frac{1}{2}\right) \cos(3\omega_0 t)}$$

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