SC223 - Linear Algebra

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Lecture 19

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Generating New subspaces from Old

- Let U, W be subspaces of V.
- $U \cap W$ is a subspace of V
- **Definition:** (Sum of subspaces): Let $U_1, \ldots, U_n$ be subspaces of V.

The **sum of subspaces** $U_1, \ldots, U_n$ is defined as:

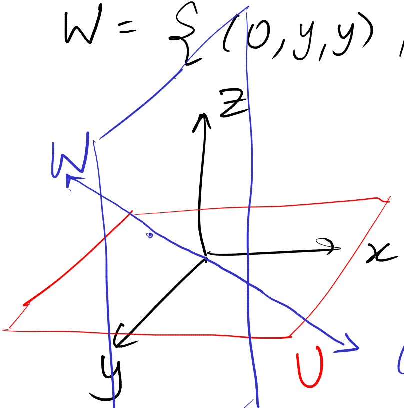
$$U_1 + \ldots + U_n =: \{u_1 + u_2 + \ldots + u_n \mid u_i \in U_i, i = 1, \ldots, n\}$$

- **Proposition 7:** The sum of subspaces $U_1, \ldots, U_n$ of V is a subspace.

- If $v = u_1 + \ldots + u_n$, $u_i \in U_i$, $i = 1, \ldots n$, we say that $(u_1, \ldots, u_n)$ is a decomposition of v.

$V = \mathbb{R}^3$, $U = \{(x,y,0) \mid \forall x,y \in \mathbb{R}\}$.

$W = \{(0,y,y) \mid \forall y \in \mathbb{R}\}$.



$(0,1,1) \in W$

$\boxed{(0,1,1) + u, \forall u \in U \in U + W}$

$(x,y,z) = (x, y-z, 0) + (0, z, z)$

$(x,y,z) = (x, y-2z, 0) + (0, \cancel{2z}, z)$

1. $V = \mathbb{R}^3$, $U_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ $U_1 + W_1 = \mathbb{R}^3$

$W_1 = \{(0, y, y) \mid y \in \mathbb{R}\}$.

2. $V \in \mathbb{R}^3$, $U_2 = \{(x, y, 0) \mid x, y \in \mathbb{R}\} = U_1$

$W_2 = \{(0, y, z) \mid y, z \in \mathbb{R}\}$.

$$U_2 + W_2 = \mathbb{R}^3$$

$$(x, y, z) = (x, y, 0) + (0, 0, z)$$

$$= (x, 0, 0) + (0, y, z)$$

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- **Definition:** (Direct Sum of Subspaces) In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.

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- **Definition:** (Direct Sum of Subspaces) In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.
- Direct sum notation: $W = U_1 \oplus U_2 \oplus \ldots \oplus U_n$.

- **Proposition 8:** Let $U_1, \dots, U_n$ be subspaces of V. Then $V = U_1 \oplus \dots \oplus U_n$ if and only if: (1) $V = U_1 + \dots + U_n$, and (2) The only decomposition of $\theta \in V$ is $(\theta, \dots, \theta)$.

$\Longrightarrow$

$\Longleftarrow$ If (1) & (2) are true $\Rightarrow V = \bigoplus_{i=1}^{n} U_i$

Let $w \in W$ be such that

$$w = u_1 + u_2 + \dots + u_n, \quad u_i \in U_i.$$
$$= v_1 + v_2 + \dots + v_n, \quad v_i \in U_i$$

At least one $u_i \neq v_i$.

$$u_1 + u_2 + \dots + u_n = v_1 + \dots + v_n$$

$$\underbrace{(u_1 - v_1)}_{\in U_1} + \underbrace{(u_2 - v_2)}_{\in U_2} + \dots + \underbrace{(u_n - v_n)}_{\in U_n} = \theta.$$

● **Proposition 9:** Let V be a VS with subspaces U_1, U_2. Then $V = U_1 \oplus U_2$ *iff* $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\theta\}$.

$\Rightarrow$

$\Leftarrow$ If (1) $V = U_1 + U_2$ & (2) $U_1 \cap U_2 = \{\theta\}$

$\Rightarrow V = U_1 \oplus U_2$

Let $\theta = \theta + \theta, \quad \theta \in U_1, \theta \in U_2$

$= v + w, \quad v \in U_1, w \in U_2.$

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- If $|U| = \infty$?
- Is $span(U)$ a subspace of V?
- **Proposition 10:** Let $U \subseteq V$. Then $span(U)$ is a subspace of V.

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- **Linearly independent set**: Let V be a vector space and let $W = \{v_1, \ldots, v_n\} \subset V$. We say that the set W is a set of linear independent vectors, if

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- What if $|W| = \infty$.