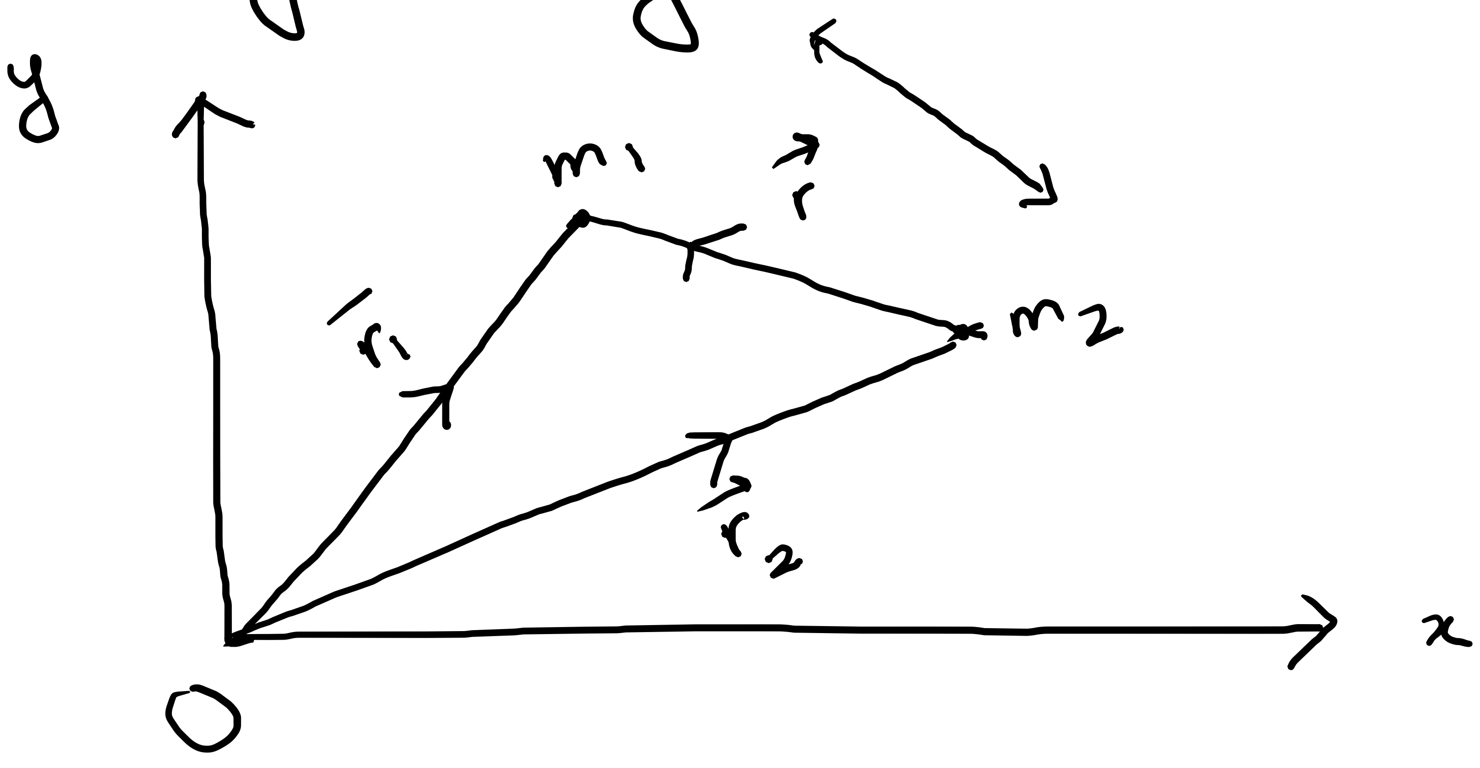
LECTURE 17

TWO-BODY PROBLEM

Two self-gravitating bodies



$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

$$= -\frac{Gm_1m_2}{r^3}\vec{r}$$

So far, have considered only central force directed towards the origin.

EOM :-

$$m_1\ddot{\vec{r}}_1 = \vec{F}_{12}$$

$$m_2\ddot{\vec{r}}_2 = \vec{F}_{21}$$

$\vec{F}_{ij}$ ≡ force on body i due to body j

$$\vec{F}_{12} = -\vec{F}_{21}$$

Aim:- to reduce this problem to a "one-body" problem, so far considered.

Define, $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$ $\longrightarrow$ Centre of mass

$\vec{r} = \vec{r}_1 - \vec{r}_2$ $\longrightarrow$ relative separation.

Helpful to know the relⁿs between (R, r) and $(\bar{r}_1, \bar{r}_2)$

$$\boxed{\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$= \frac{m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r})}{m_1 + m_2}$$

$$= \vec{r}_1 - \frac{m_2 \vec{r}}{m_1 + m_2}$$

$$\Rightarrow \quad \vec{r}_1 = \vec{R} + \frac{m_2 \vec{r}}{m_1 + m_2} \quad //$$

$$\vec{r}_2 = \vec{r}_1 - \vec{r} = \vec{R} + \frac{m_2 \vec{r}}{m_1 + m_2} - r$$

$$= R - \frac{m_1 \vec{r}}{m_1 + m_2} \quad //$$

- EOM in the new 'variables',

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_{12} \quad \text{———} (1)$$

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$m_2 \ddot{\vec{r}}_2 = \vec{F}_{21} \quad \text{———} (2)$$

$m_2 \times (1) - m_1 \times (2)$,

$$m_1 m_2 (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) = m_2 \vec{F}_{12} - m_1 \vec{F}_{21} = (m_1 + m_2) \vec{F}_{12}$$

$$\Rightarrow \frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{r}} = \vec{F}_{12}$$

Effectively represents a "one-body problem" if

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

and $\vec{r}$ is the position vector.

$$\boxed{\mu \ddot{\vec{r}} = \vec{F}_{12}}$$

$$\left.\begin{aligned} m_1 \ddot{\vec{r}}_1 &= \bar{F}_{12} \\ m_2 \ddot{\vec{r}}_2 &= \bar{F}_{21} \end{aligned}\right\} \quad \frac{m_1 \ddot{\vec{r}}_1}{m_1+m_2} + \frac{m_2 \ddot{\vec{r}}_2}{m_1+m_2} = \frac{\bar{F}_{12}}{m_1+m_2} + \frac{\bar{F}_{21}}{m_1+m_2}$$

$$\Rightarrow \frac{m_1 \ddot{\bar{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1+m_2} = 0$$

$$\Rightarrow \ddot{\vec{R}} = 0 . \Rightarrow \dot{\vec{R}} = \text{const} .$$

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\bar{r}}_2^{\,2}$$

$$= \frac{1}{2} m_1 \left[\bar{R} + \frac{m_2 \dot{\vec{r}}}{m_1+m_2} \right]^2 + \frac{1}{2} m_2 \left[\bar{R} - \frac{m_1 \dot{\bar{r}}}{m_1+m_2} \right]^2$$

$$= \frac{1}{2}(m_1+m_2) \dot{\bar{R}}^2 + \frac{1}{2} m_1 \frac{m_2^2 \dot{\bar{r}}^2}{(m_1+m_2)^2} + \frac{1}{2} m_2 \frac{m_1^2 \dot{\bar{r}}^2}{(m_1+m_2)^2}$$

$$= \frac{1}{2}(m_1+m_2) \dot{\bar{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{(m_1+m_2)^{\cancel{2}}} \cancel{(m_1+m_2)} \dot{\bar{r}}^2$$

$$= \underbrace{\frac{1}{2}(m_1 + m_2)}_{\text{const.}} \dot{\bar{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} \dot{\bar{r}}^2$$

$$= \text{const} + \underbrace{\frac{1}{2} \mu \dot{\bar{r}}^2}$$

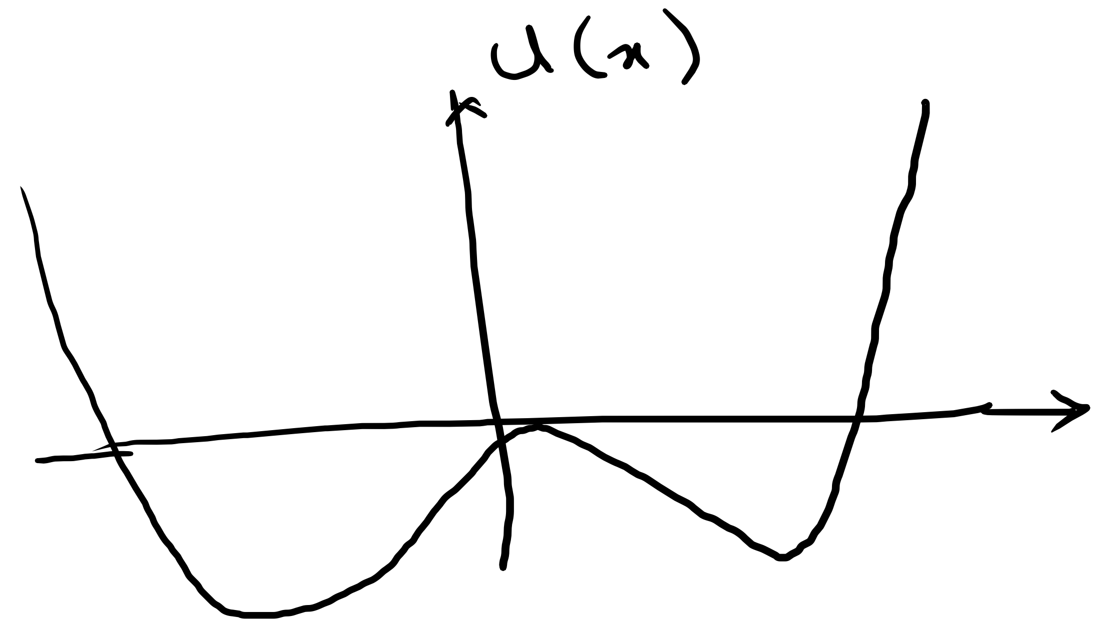
$$U(|\bar{r}|)$$

$$E = \frac{1}{2} \mu \dot{\bar{r}}^2 + U(|\bar{r}|) \, .$$

→ expression similar to conv. equ. for one-body problem.

Small oscillations about extrema of a potential.

$$U(x) = U_0(-ax^2 + bx^4), \qquad U_0, a, b > 0.$$



$$\frac{dU(x)}{dx} = U_0(-2ax + 4bx^3) = 0$$

$$\Rightarrow \quad 2x(a - 2bx^2) = 0$$

$$x = 0, \pm\sqrt{\frac{a}{2b}}$$

$$U(0) = 0$$

$$U\left(x = \pm\sqrt{\frac{a}{2b}}\right) = U_0\left[-a\frac{a}{2b} + \cancel{b}\cdot\frac{a^2}{4b^{\cancel{2}}}\right] = U_0\left(-\frac{a^2}{4b}\right)$$

$$\frac{d^2U}{dx^2} = U_0\left(-2a + 12bx^2\right)$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=0} = -2aU_0 < 0 .$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=\pm\sqrt{\frac{a}{2b}}} = U_0\left(-2a + \cancel{12}^{\,6}\,\cancel{b}\cdot\frac{a}{\cancel{2b}}\right) = 4aU_0 > 0$$

$$U(x) = \underline{U(x_0)} + \underbrace{\left.\frac{dU}{dx}\right|_{x_0}}_{0}(x - x_0) + \underbrace{\frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0}(x-x_0)^2}$$

$$F(x) = -\frac{dU(x)}{dx} = -\underbrace{\left.\frac{d^2U}{dx^2}\right|_{x_0}}_{k}(x - x_0) .$$

$$m\ddot{x} = -k(x - x_0)$$

$$\Rightarrow \ddot{x} + \frac{k}{m}(x - x_0) = 0$$

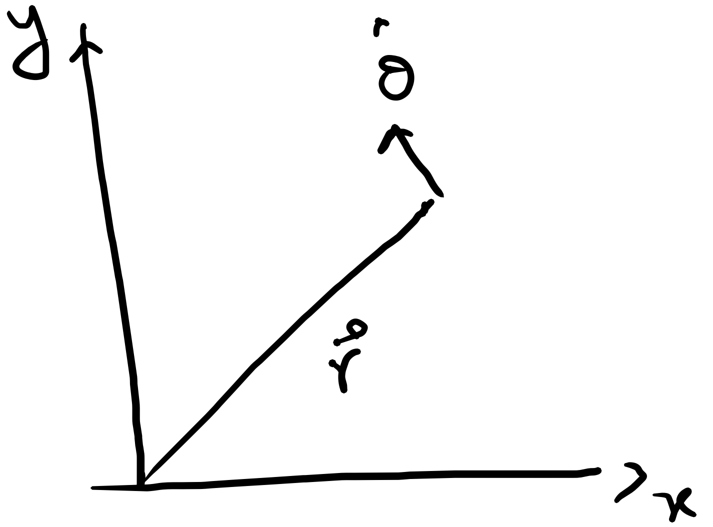
$$\Rightarrow \ddot{x} + \omega^2(x - x_0) = 0$$

$$\omega = \sqrt{4aU_0} .$$

Different type of force.

$$\vec{F} = \frac{A}{r}\hat{\theta}$$

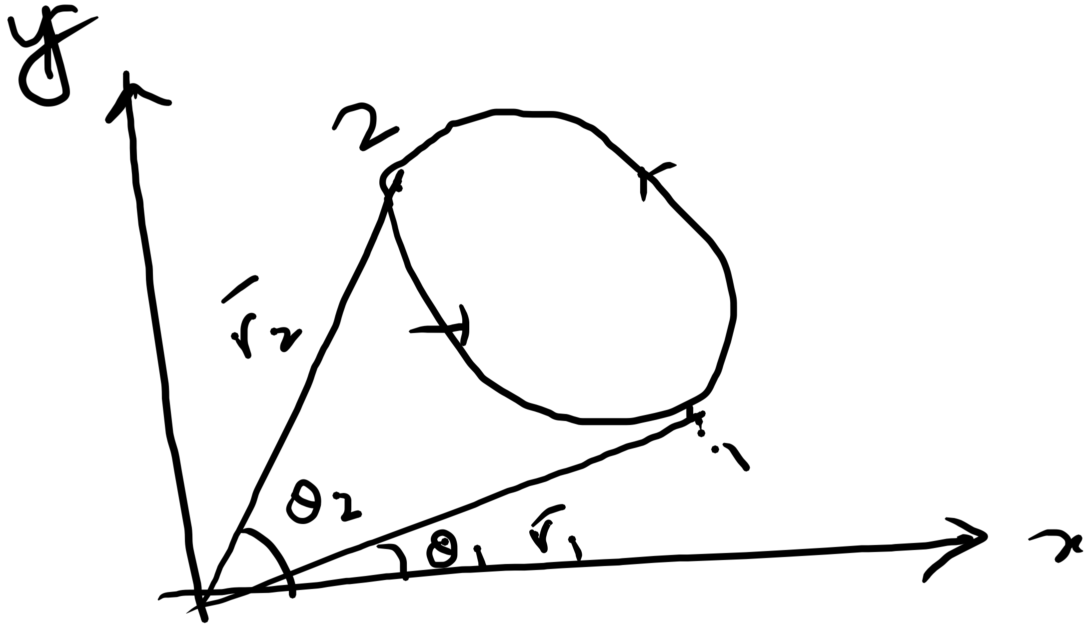
$$d\vec{r} = dr\hat{r} + r\,d\theta\,\hat{\theta}$$

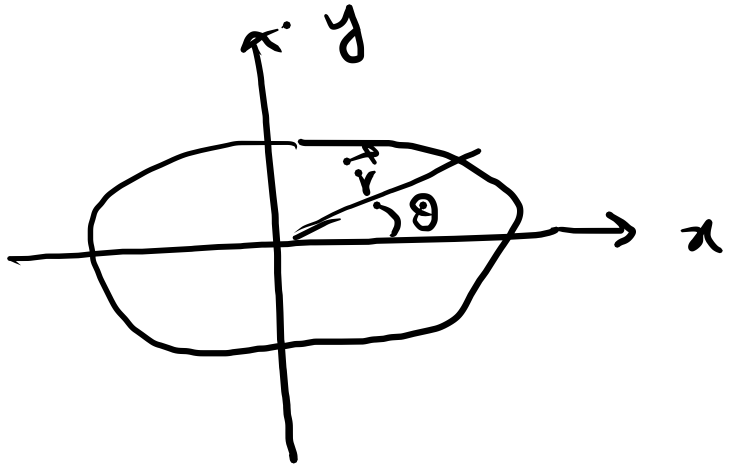


$$dW = \vec{F}\cdot d\vec{r}$$

$$= \frac{A}{r} r\,d\theta$$

$$= A\,d\theta.$$

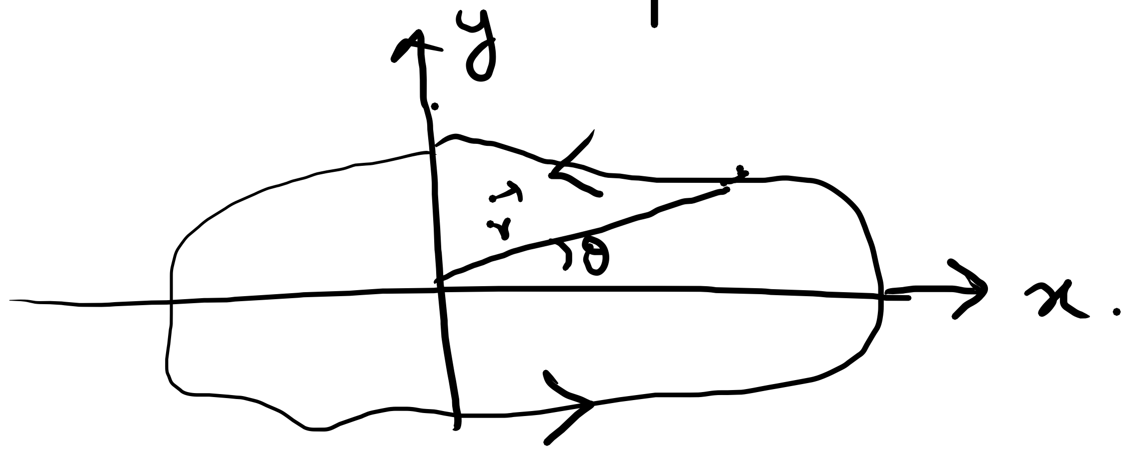




$$W = \int_1^2 dW + \int_2^1 dW.$$

$$= A(\theta_2 - \theta_1) + A(\theta_1 - \theta_2) = 0$$

Another closed path

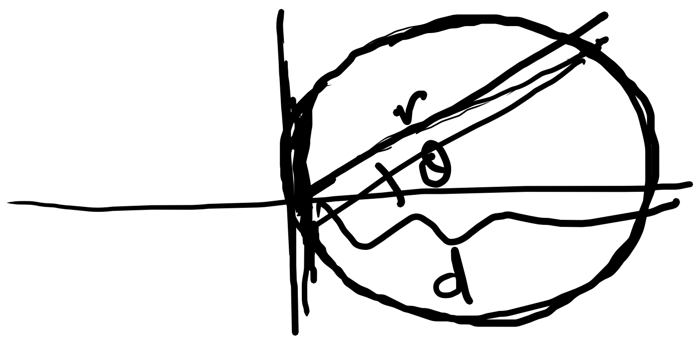


$$W = A\int_0^{2\pi} d\theta = 2\pi A \neq 0 .$$

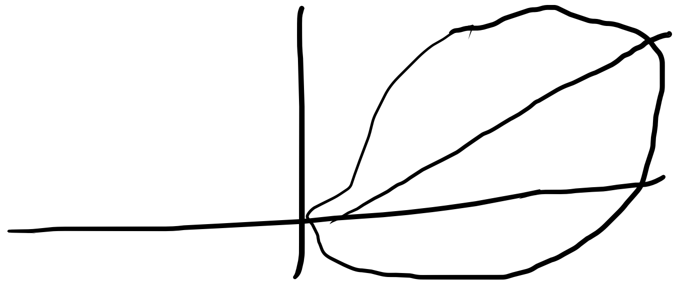
$$\vec{F} = \frac{A}{r}\hat{\theta}$$

$\hookrightarrow$ not defined at the origin.

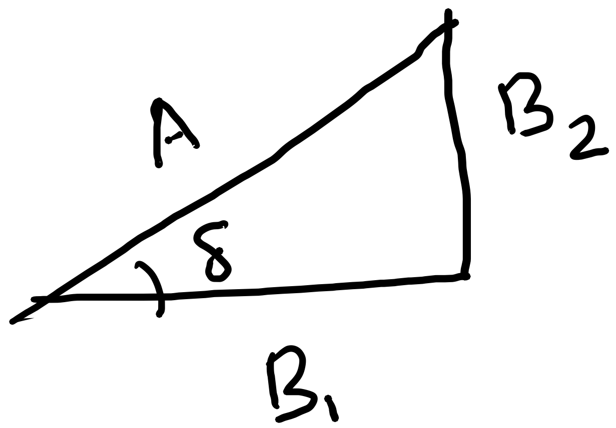
So, any closed path containing the origin is treated differently from closed path not enclosing the origin.



$r = d\cos\theta$



OSCILLATORY MOTION



$$\ddot{x} + \omega^2 x = 0 .$$

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} .$$

$$= B_1 \cos\omega t + B_2 \sin\omega t .$$

$$= A\left(\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right)$$

$$= A\left(\cos\omega t \cos\delta + \sin\omega t \sin\delta\right)$$

$$= A\cos(\omega t - \delta) .$$

phase.