LECTURE 3

RECAP

- Dimensional analysis ✓
- Function approximation.
  - Useful in analysis of dynamical systems described by ODEs.

$$\boxed{\frac{dy}{dx} = f(x)}. \Rightarrow y(x) = \int f(x)\,dx.$$

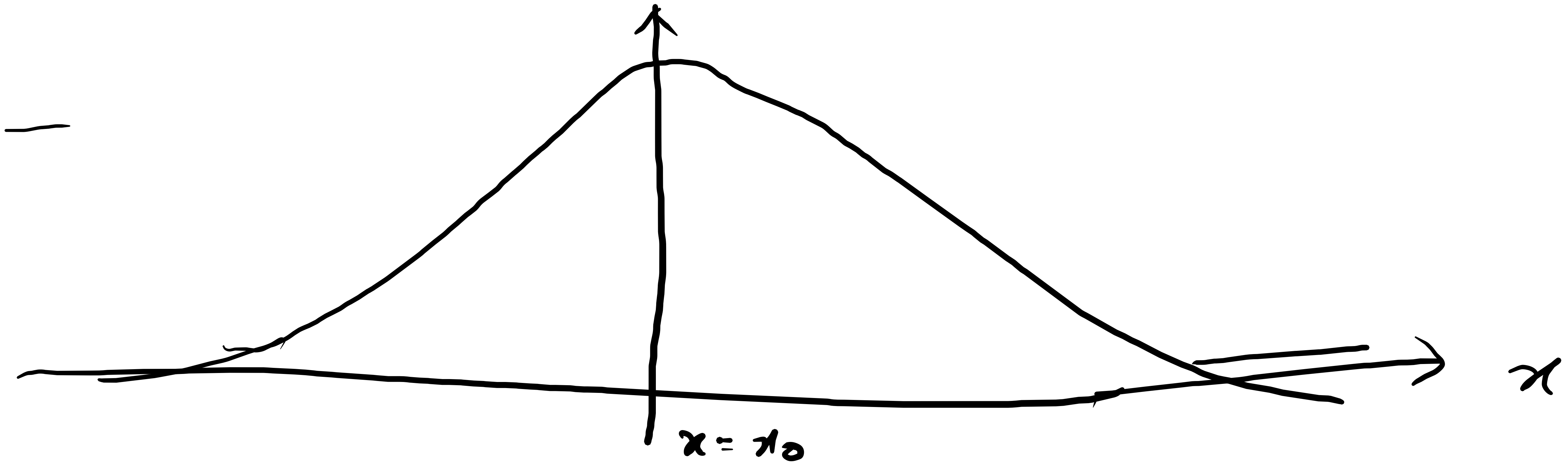
DYNAMICAL SYSTEM.

- Not all dynamical systems are first order, but can often be reduced to a system/collection of first order ODEs.

- $$\frac{d^2y}{dx^2} = f(x). \longrightarrow \text{2nd order ODE.}$$

$$\boxed{\begin{aligned} y_1 &= \frac{dy}{dx} \\ f(x) &= \frac{dy_1}{dx} \end{aligned}}$$

— $f(x) = e^{-\alpha(x-x_0)^2}$, $-\infty < x < \infty$.

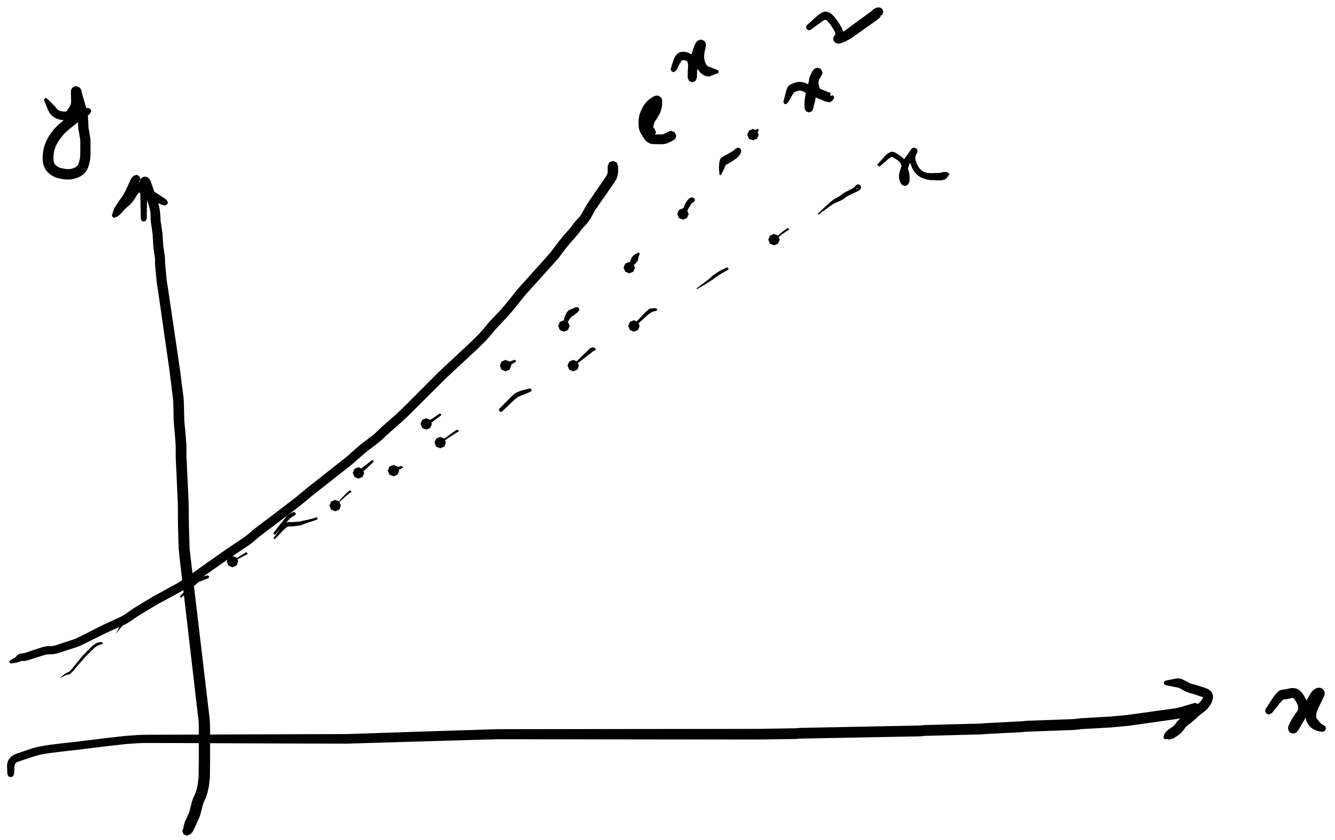


$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha(x-x_0)^2}$$

— Taylor polynomial.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$\cdots$



$$e^x = 1 + x + \cdots$$

– REMAINDER THEOREM

If $f(x)$ has $(n+1)$ continuous derivatives in $\alpha \leq x \leq \beta$ and $x = a$ belongs to this interval, then $R_n(x) \equiv f(x) - p_n(x)$ is given by,

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \qquad a < \xi < x.$$

– $$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!} + \cdots$$

$$= p_n(x) + \underbrace{\frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!} + \cdots}_{R_n(x) = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}}$$

$$= p_n(x) + \underbrace{\frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!} + \cdots}_{R_n(x) = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}}$$

<u>Example</u>: $f(x) = \cos x$. Approximate $f(x)$ in $-\pi/4 \le x \le \pi/4$ with an error $\le 10^{-5}$.

<u>Soln</u>: $R_n(x) = \dfrac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$

$$f(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \frac{x^{2(n+1)}}{(2n+2)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{(even terms survive).}$$

$$\left| R_{2n+1}(x) \right| = \left| \frac{x^{2n+2}}{(2n+2)!} \underbrace{f^{(\,)}(\xi)} \right| \le \left| \frac{x^{2n+2}}{(2n+2)!} \right|$$

Example :- $f(x) = \cos x$. Evaluated at $x = 0.6$, for $n = 2$, estimate the error. Assume $a = 0$

Soln :- $n = 2$ is given.

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

$$\left| R_2(x) \right| \Big|_{x=0.6} = \left| \frac{f^{(3)}(\xi)}{3!} x^3 \right| \Big|_{x=0.6} \leq \frac{(0.6)^3}{3!}$$

- <u>Example</u>:- Consider,

$$\operatorname{Sint} x = \frac{1}{x}\int_0^x dt\, \frac{\sin t}{t}.$$

Between $-1 < x < 1$, derive a Taylor polynomial approximatn s.t. error $\leq 10^{-9}$.

<u>Soln</u>:-

$$\operatorname{Sint} x = \frac{1}{x}\int_0^x dt\, \frac{1}{t}\left[t - \frac{t^3}{3!} + \cdots\cdots (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!} + (-1)^n \frac{t^{2n+1}}{(2n+1)!}\cos\xi\right]$$

$$= \frac{1}{x}\int_0^x dt\left[1 - \frac{t^2}{3!} + \cdots\cdots + (-1)^{n-1}\frac{t^{2n-2}}{(2n-1)!}\right]dt + R_{2n-2}(x)$$

Example:- Consider, $I = \int_0^{1/3} dx\, e^{-x^2}$. Approximate. s.t.

error $\leq 10^{-6}$

Integrating term by term,

$$\mathrm{Sint}\ x = \frac{1}{x}\left[x - \frac{x^3}{3!\,3} + \dots (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!\,(2n-1)} \right] + R_{2n-2}(x)$$

Explicitly,

$$R_{2n-2}(x) = \frac{1}{x}\int_0^x dt\ (-1)^n \frac{t^{2n}}{(2n+1)!} \cos(\xi) .$$

$$\left| R_{2n-2}(x) \right| \leq \left| \frac{1}{x} \int_0^x dt\ \frac{t^{2n}}{(2n+1)!} \right|$$

$$= \left| \frac{x^{2n}}{(2n+1)!\,(2n+1)} \right|$$

Worst case seenario is $x = \pm 1$

$$\frac{1}{(2n+1)!\,(2n+1)} \le 10^{-9}.$$

Solve for n.

Worst case scenario (max. possible error) is given by

$$\left| \frac{(\pi/4)^{2(n+1)}}{(2n+2)!} \right| \leq 10^{-5} \quad \Rightarrow \quad \boxed{n \geq 3}$$

Example:- Consider, $I = \int_0^{1/3} dx\, e^{-x^2}$. Approximate. s.t. error $\leq 10^{-6}$

Soln:- $e^x = 1 + x + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{\xi}$

$$e^{-x^2} = 1 - x^2 + \dots + (-1)^n \frac{x^{2n}}{n!} + (-1)^{n+1} \frac{x^{2(n+1)}}{(n+1)!} e^{\xi}.$$

As in the previous problem,

$I = \int_0^{1/3} dx\, e^{-x^2} \longrightarrow$ integrate term by term.

$$\int_0^{1/3} dx\, e^{-x^2} = \int_0^{1/3} dx \left[\left(1 - x^2 + \frac{x^4}{2!} + \cdots + (-1)^n \frac{x^{2n}}{n!}\right) + (-1)^{n+1} \frac{x^{2n+2}}{(n+1)!} e^{\xi} \right]$$

$$\int_0^{1/3} dx \frac{e^{\xi} x^{2(n+1)}}{(n+1)!}, \quad \text{wher } \xi < 1/3$$

$$e^{1/3} < 1$$

$$< \frac{1}{(n+1)!\,(2n+3)} < 10^{-6} \longrightarrow \text{solve for } n.$$

$\boxed{\text{Ref: Atkinson. Numerical analysis.}}$