LECTURE 15

VII FIRST-ORDER AUTONOMOUS SYSTEMS.

$\frac{dx}{dt} \propto x$

$\Rightarrow \frac{dx}{dt} = \pm ax.$

Motivation :- Population model. Let $x(t)$ denote the population of given species at time t and let $r(t,x)$ denote the diff. between birth and death rate.

If this population is isolated; $\frac{dx}{dt} = r(t,x)\, x(t)$.

For a simple model, assume, $r(t,x) = a = \text{const}$.

$$\boxed{\frac{dx}{dt} = \pm ax.}$$

$[a] = [T]^{-1}$

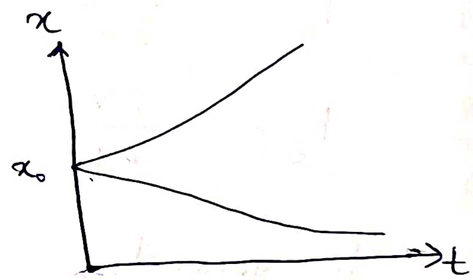
$\Rightarrow [at] \rightarrow$ dimensionless.

$\Rightarrow \frac{dx}{d(at)} = \pm x.$

$\Rightarrow \int \frac{dx}{x} = \pm \int dT$, where $T = at$.

$\Rightarrow \ln x = \pm T + c.$

$\Rightarrow x = Ae^{\pm T}$



At $t = 0$, $x = x_0$.

$\therefore x = x_0 e^{\pm T}.$

$\Rightarrow \left(\frac{x}{x_0}\right) = e^{\pm T}.$

$\Rightarrow X = e^{\pm T}.$

Dimensionless.

– Consider a more complicated system,

$\frac{dx}{dt} = a \pm bx.$

Case I :- $\frac{dx}{dt} = a - bx = b\left(\frac{a}{b} - x\right)$.

$$\Rightarrow \frac{dx}{d(bt)} = \left(\frac{a}{b} - x\right)$$

$[b] = [T]^{-1}$

$\Rightarrow [bT] \rightarrow$ dimensionless.

$T = bt$. $\quad x_0 = \frac{a}{b}$.

$$\frac{dx}{dT} = (x_0 - x) = x_0\left(1 - \frac{x}{x_0}\right).$$

$$\Rightarrow \frac{d(x/x_0)}{dT} = \left(1 - \frac{x}{x_0}\right)$$

$$\Rightarrow \frac{dX}{dT} = (1 - X).$$

$$\Rightarrow -\ln(1-X) = T + c$$

$$\Rightarrow (1-X) = A e^{-T}.$$

At $t = 0 \Rightarrow T = 0$, $x = 0 \Rightarrow X = 0$.

$(1 - 0) = A$.

$\Rightarrow A = 1$

$\therefore (1 - X) = e^{-T}$

$$\Rightarrow \underbrace{X = (1 - e^{-T})}_{\text{Dimensionless.}}$$

In dimensionfull terms, the soln is,

$$x = x_0(1 - e^{-bt})$$

$$= \frac{a}{b}(1 - e^{-bt}).$$

Small-time limit ($T \ll 1$)

$e^{-T} \simeq 1 - T + \ldots$

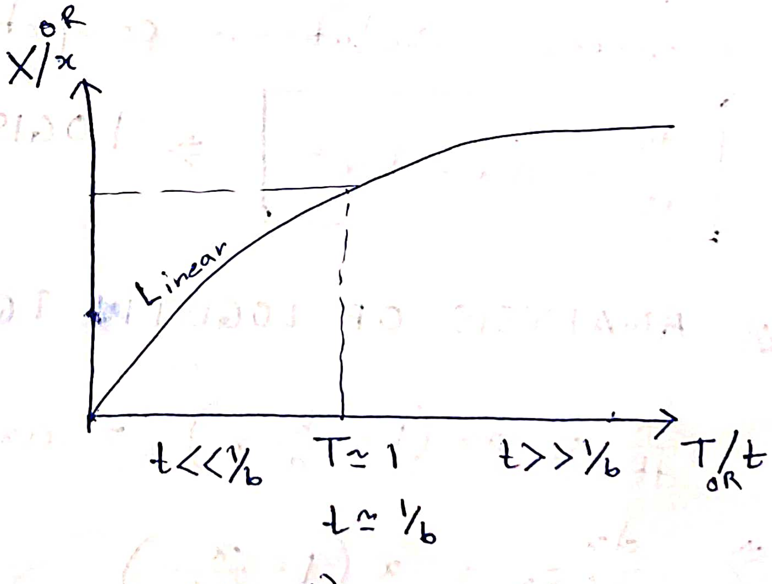
$X \simeq (1 - 1 - T) = T$.

$\therefore x = at$.

Large-time limit ($T \gg 1$),

$e^{-T} \rightarrow 0$.

$X \simeq 1 \Rightarrow x \simeq x_0 = a/b$.



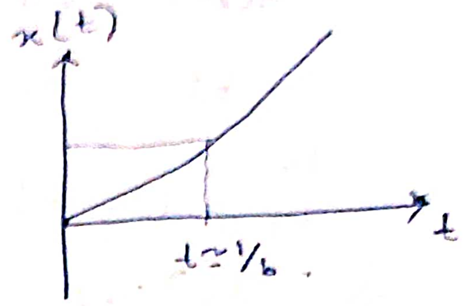
$\boxed{t \simeq 1/b}$ $\rightarrow$ point where the behaviour of the soln changes

$\hookrightarrow$ depends on parameter b of system.

At $t \simeq 1/b$, $x \simeq 0.63\,x_0$. $\Rightarrow x = x_0(1 - e^{-1})$.

"Non-dimensionalizing" the eqn. makes it easier to identify natural scale of the system.

<u>Case II</u> :- $\frac{dx}{dt} = a + bx.$



$x(t) = \frac{a}{-b}(1 - e^{bt}) = \frac{a}{b}(e^{bt} - 1).$

When $t << 1/b$, $x \simeq \frac{a}{b}(bt) \simeq at$

When $t >> 1/b$, $x(t) \simeq \frac{a}{b} e^{bt}$.

PROBLEMS WITH SIMPLE MODEL FOR POPULATION GROWTH.

$x(t) = x_0 e^{at}$ $\rightarrow$ exponential growth, solution blows up.

Linear models for population growth are suitable as long as population is not too large. But for very large population, members are competing with each other for resources. Solution :- competition term,

$$\boxed{\frac{dx}{dt} = ax - bx^2.}$$ $\Rightarrow$ LOGISTIC EQUATION

ANALYSIS OF LOGISTIC EQUATION.

$$\frac{dx}{dt} = ax\left(1 - \frac{b}{a}x\right) = ax\left(1 - \frac{x}{a/b}\right).$$

$$\Rightarrow \frac{dx}{d(at)} = x\left(1 - \frac{x}{a/b}\right).$$ Define $K = (a/b)$.

$$\Rightarrow \frac{dx}{d(at)} = x(1 - x/K)$$ Define $T = at$, $X = \frac{x}{K}$.

$$\frac{dx}{dT} = x(1-X) = K\frac{x}{K}(1-X)$$

$$\Rightarrow \frac{dX}{dT} = X(1-X).$$

$$\Rightarrow \frac{dX}{X(1-X)} = dT$$

Let $\dfrac{1}{X(1-X)} = \dfrac{A}{X} + \dfrac{B}{1-X}$ Solving, $A=1$, $B=1$.

$$= \frac{A+(B-A)X}{X(1-X)}$$

$$\int\left(\frac{dX}{X} + \frac{dX}{1-X}\right) = T + c$$

$$\Rightarrow \ln X - \ln(1-X) = T + c$$

$$\Rightarrow \ln\frac{X}{1-X} = T + c.$$

$$\Rightarrow \frac{X}{1-X} = A e^{T}.$$

$$\Rightarrow X = A e^{T}(1-X)$$

$$\Rightarrow X(1+Ae^{T}) = Ae^{T}.$$

$$\Rightarrow X(T) = \frac{A e^{T}}{1+Ae^{T}}$$

$$\Rightarrow X = \frac{1}{1+A^{-1}e^{-T}}$$

$$\Rightarrow X = \frac{1}{1+\frac{(1-X_0)}{X_0}e^{-T}}$$

$$\Rightarrow X = \frac{X_0}{X_0+(1-X_0)e^{-T}}$$

At $t=0$, $X = X_0$ $(x = x_0)$.

$$\frac{X_0}{1-X_0} = A.$$

$$\cancel{\Rightarrow X_0 = A - AX_0}.$$

$$\cancel{x = \frac{K^2 x_0}{K x_0 + (1-K/x_0)e^{-dt}}}$$

$$x = \frac{\cancel{K}\, x_0}{\frac{x_0}{K} + \left(1-\frac{x_0}{K}\right)e^{-T}}$$

$$\Rightarrow x = \frac{K x_0}{x_0 + (K-x_0)e^{-T}}$$

At large-time limit,

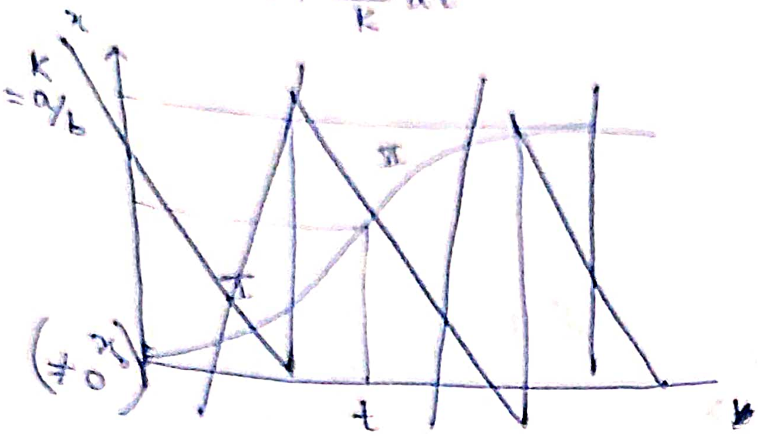
$x \to K = a/b$ (for any initial value x_0).

At small-time limit,

$$x \simeq \frac{K x_0}{x_0(1-e^{-T}) + Ke^{-T}}$$

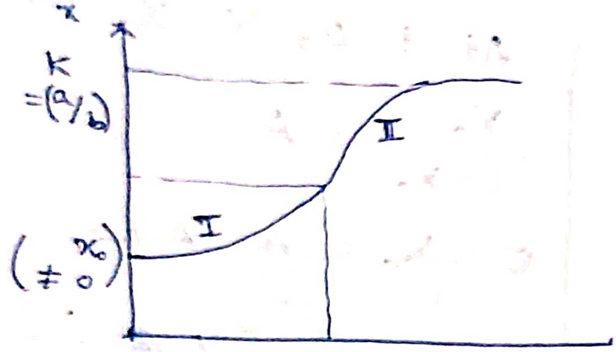
$\Rightarrow$

$$x \simeq \frac{x_0 e^{at}}{1 + \frac{x_0}{K} e^{at}} \longrightarrow \text{dominates time evolution}$$



I $\to$ early-time exponential growth

II $\to$ growth at decreasing rate at later time.



Growth saturates at $K = (a/b)$ irrespective of initial x_0. This is termed as the carrying capacity.

$$\frac{dX}{dT} = F(X) = X - X^2$$

$$\frac{d^2X}{dT^2} = \frac{dF}{dX}\frac{dX}{dT} = F\frac{dF}{dX}. \Rightarrow \frac{d^2X}{dT^2} = 0 \text{ when } \frac{dF}{dX} = 0$$

$$\frac{dF}{dX} = 1 - 2X = 0 \Rightarrow X = \frac{1}{2}$$

$\frac{d^2X}{dT^2}$ changes sign at $X = \frac{1}{2}$ $\Rightarrow$ $F(X)$ has turning point.

When $X > \frac{1}{2}$, $\frac{d^2X}{dT^2} < 0$, so growth occurs at decreasing rate

When $X < \frac{1}{2}$, $\frac{d^2X}{dT^2} > 0$, so growth occurs at increasing rate.

Physical interpretation - At $X = 1/2$, the non-linear term $(-bx^2)$ begins to affect.

$$X = \frac{1}{2} = \frac{1}{1 + A^{-1} e^{-T_{n1}}} \Rightarrow 2 = 1 + A^{-1} e^{-T_{n1}}.$$

$$\Rightarrow T_{n1} = \ln\left(\frac{1}{A}\right).$$

$$= \ln\left(\frac{1 - X_0}{X_0}\right).$$

$$\Rightarrow a t_{n1} = \ln\left(\frac{K - x_0}{x_0}\right)$$

$$\Rightarrow t_{n1} = \frac{1}{a} \ln\left(\frac{K}{x_0} - 1\right).$$

Constraint, $t_{n1} > 0$.

$$\frac{K}{x_0} - 1 > 1.$$

$$\Rightarrow x_0 < K/2$$

$$\Rightarrow X_0 < 1/2.$$

recall that this the turning point

Hence if $\frac{K}{2} < x_0 < K$, growth occurs at decreasing rate only.

VII WAYS TO GUESS QUALITATIVE FEATURES OF FIRST-ORDER AUTONOMOUS ODE ?

$\dot{x} = f(x) \rightarrow$ general form.

- Visualise t as time, x as the position of a particle moving along x-axis, and $\dot{x}$ is the velocity of same.
- ~~Poo spoody~~ When $\dot{x} = 0$, such points are termed "FIXED POINTS".
- There are two types of fixed points - stable and unstable. / (attractors and repellers).

Example:- Consider the logistic eqn:-

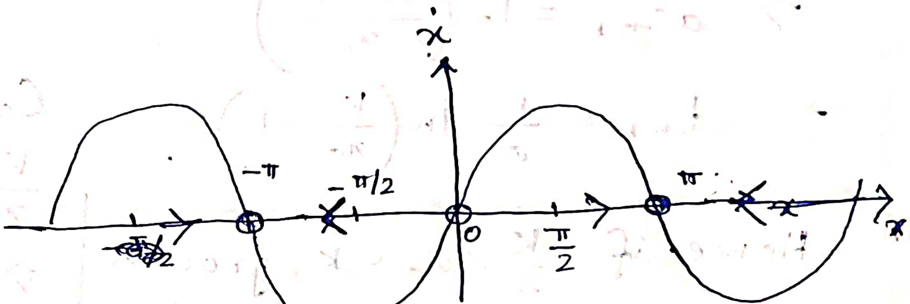
$$\dot{x} = ax - bx^2 = x(a - bx).$$

The fixed points are $x^* = 0, \left(\frac{a}{b}\right)$. Has been observed that at long-times, solutions tend towards (a/b) for given initial values.

Example:- $\dot{x} = \sin x$.

$$\Rightarrow \frac{dx}{dt} = \sin x$$

$$\Rightarrow \frac{dx}{\sin x} = dt$$



$$\Rightarrow t = -\ln|\operatorname{cosec} x + \cot x| + C.$$

Say, $x = x_0$ at $t = 0$,

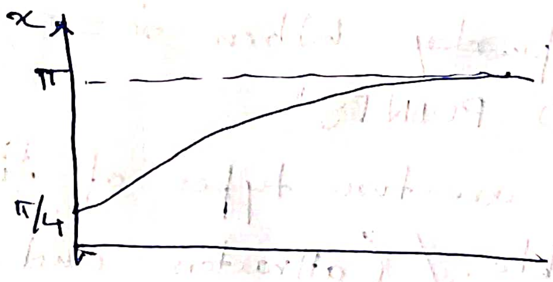
$$t = \ln\left|\frac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x}\right|.$$

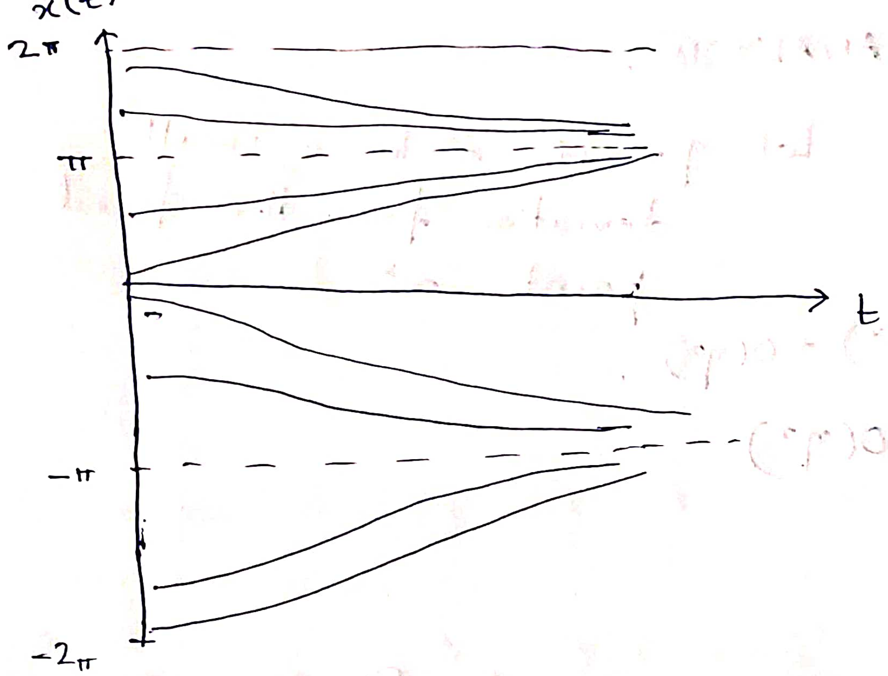
Fixed points are given by,

$$\sin x^* = 0$$

$$\Rightarrow x^* = 0, \pm\pi, \ldots$$

Reminder:- visualise particle moving along x-axis at velocity $\dot{x}$ at $t=0$. If $\dot{x} > 0$, it means velocity of particle is increasing with x and vice versa. If the particle tends to move back towards a fixed point, then the fixed points is stable, else it is unstable.





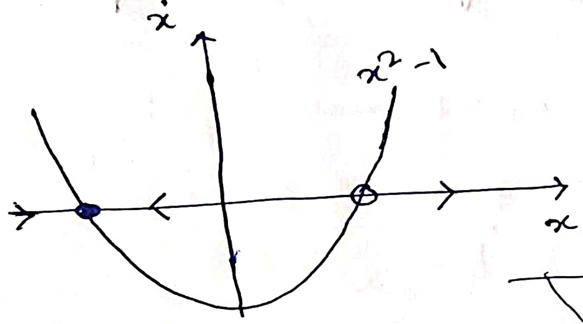
Fixed points: represent equilibrium solutions.

Example:- $f(x) = x^2 - 1$

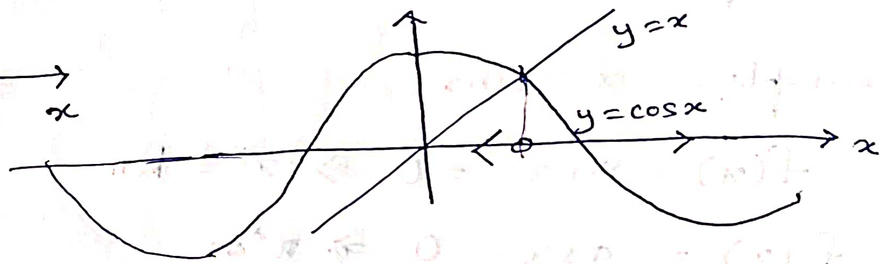
$f(x) = x(x-1)$

$f(x^*) = 0$

$\Rightarrow x^* = 0, 1$



Example:- $f(x) = x - \cos x$.

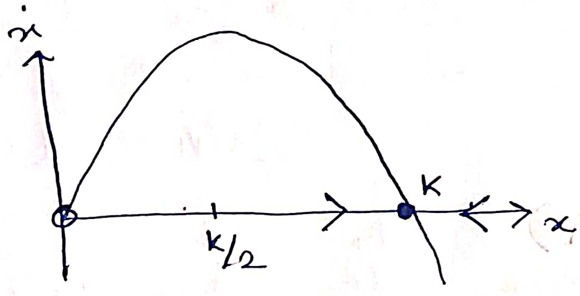


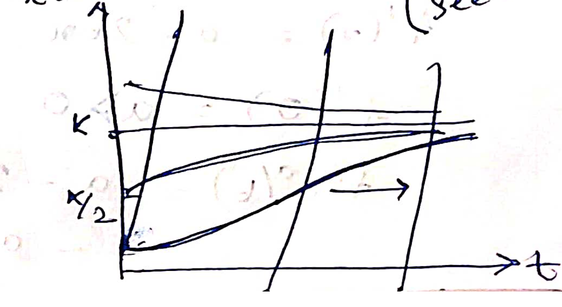
Example:- Logistic eqn. $\dot{x} = ax - bx^2$.

OR $\dot{X} = X - X^2$.

Fixed points are $x^*(a - bx^*) = 0 \Rightarrow x^* = 0, (a/b)$.

$= 0, K$ (see later)





LINEAR STABILITY ANALYSIS.

$\dot{x} = f(x)$

Let $\eta = x(t) - x^*$ be a small deviation from the fixed point x^*.

$\Rightarrow \dot{\eta} = f(x^* + \eta)$.

$\Rightarrow \dot{\eta} = f(x^*) + \eta f'(x^*) + O(\eta^2)$.

$= \eta f'(x^*) + O(\eta^2)$

Linearise,

$\dot{\eta} = \eta f'(x^*)$.

If $f'(x^*) > 0$, $\eta(t)$ grows and if $f'(x^*) < 0$, $\eta(t)$ decays.

Example:- $\dot{x} = \sin x$.

$f(x) = \sin x = 0 \Rightarrow x^* = k\pi$

$f'(x) = \cos x = 0$ $\cancel{\Rightarrow x^*}$

$\Rightarrow f'(x^*) = \cos k\pi = 0$ $\cancel{\Rightarrow}$ $\begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd.} \end{cases}$

$\Rightarrow$ x^* is unstable if k is even and stable if k is odd.

Example:- $f(x) = ax - bx^2$

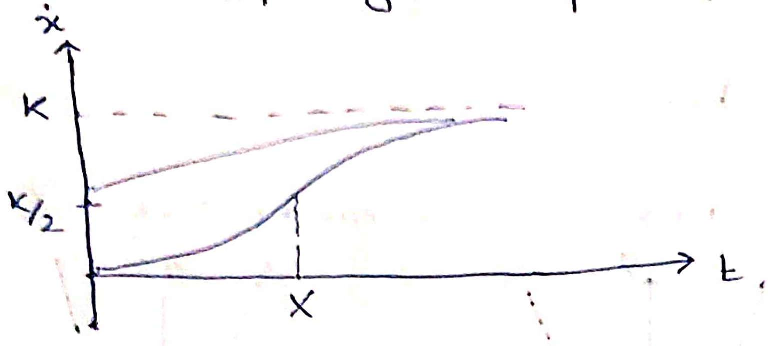
$f'(x) = a - 2bx$

$f'(0) = a > 0$

$f'(a/b) = a - 2(b)(a/b)$

$= -a < 0$.

Solution of logistic eqn. for different initial conditions.



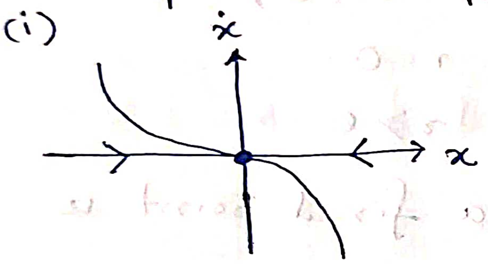
– Linear stability analysis cannot say anything conclusive when $f'(x^*) = 0$.

Consider (i) $\dot{x} = -x^3$; (ii) $\dot{x} = x^3$, (iii) $\dot{x} = x^2$,

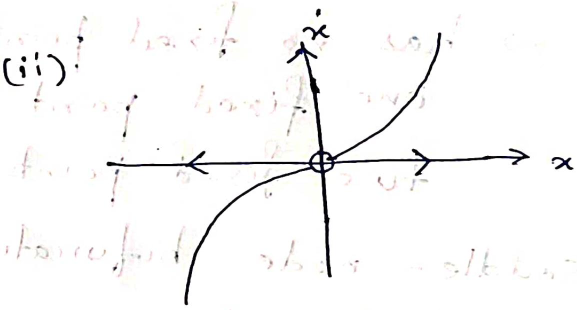
Above examples all have $f'(x^*) = 0$ for $x^* = 0$.

Graphical analysis:-

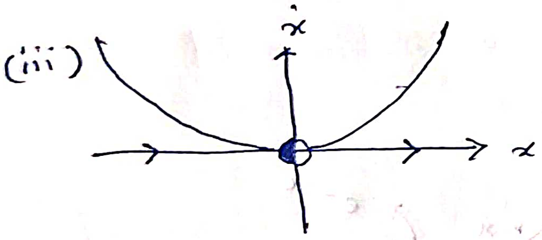
(i)



(ii)



(iii)



POTENTIAL PICTURE.

– Visualize a particle sliding down a potential well:-

$$f(x) = -\frac{dV}{dx}$$

$$\frac{dV}{dt} = \frac{dV}{dx}\dot{x}$$

$$= -\left(\frac{dV}{dx}\right)^2 \leq 0.$$

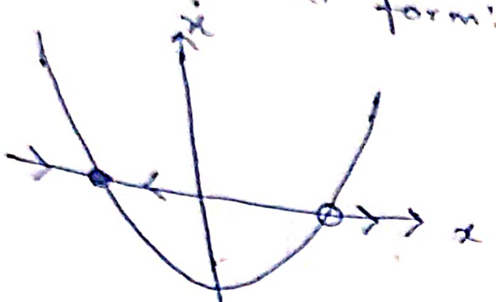
$\Rightarrow V(t)$ decreases along trajectories except if $\frac{dV}{dx} = 0$ (equilibrium)

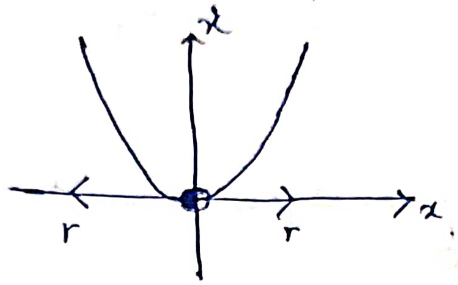
So, local minima of $V(x)$ corresponds to stable fixed points.

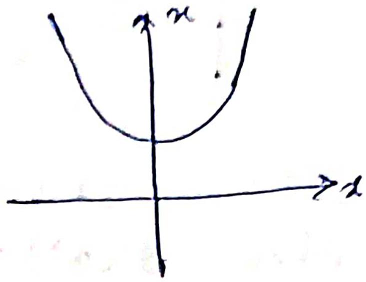
BIFURCATIONS

Saddle-Node Bifurcation.

- Normal form:- $\dot{x} = r + x^2$, r can be $\pm$ve or 0.







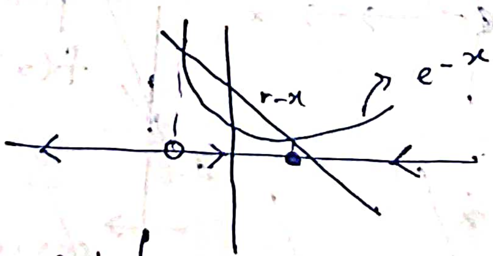
- Situation changes qualitatively as $r=0$ is crossed.
- BIFURCATION.

$\dot{x} = r - x^2 \rightarrow$ has no fixed points for $r<0$,
one fixed point for $r=0$.
two fixed points for $r>0$.

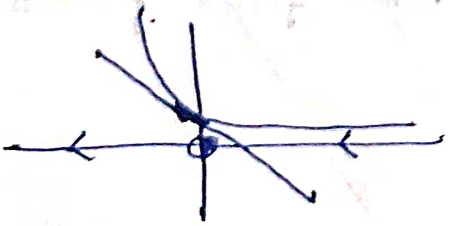
- In saddle-node bifurcation, a fixed point is created or destroyed.

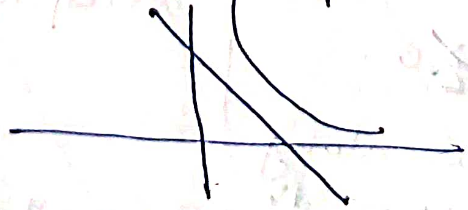
Example:- $\dot{x} = r - x - e^{-x}$.

$f(x^*) = r - x^* - e^{-x^*}$



Consider $(r-x)$ and e^{-x}.





To find bifurcation point r_c, impose the condition that $(r-x)$ and e^{-x} intersect tangentially.

$$e^{-x} = r - x$$

$$\frac{d}{dx}(e^{-x}) = \frac{d}{dx}(r-x)$$

$$\Rightarrow \quad e^{-x} = 1$$

$$\Rightarrow \quad x = 0$$

$\Rightarrow \quad r_c = 1.$

and it occurs at $x = 0$.

Also, $\dot{x} = r - x - e^{-x}$

Expand near $x = 0$,

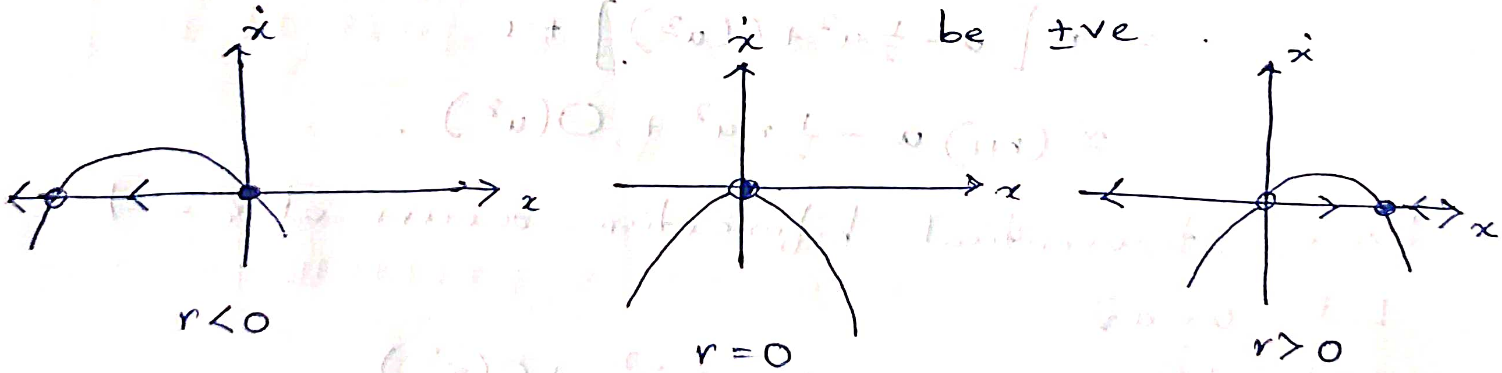
$$\dot{x} \simeq (r-x) - \left[1 - x + \frac{x^2}{2!} + \cdots\right]$$

$$\simeq (r-1) - \frac{x^2}{2!} + \cdots$$

↳ same algebraic form as $\dot{x} = r - x^2$.

– Transcritical Bifurcation:-

– Normal form:- $\dot{x} = rx - x^2$, allow x and r to be $\pm$ve.



– Fixed point is not created/destroyed, but stability changes.

Example :- $\dot{x} = x(1-x^2) - a(1-e^{-bx})$

$\dot{x} = 0 \Rightarrow x(1-x^2) - a(1-e^{-bx}) = 0 \rightarrow$ satisfied for $x = 0$ $\forall (a,b)$

$\Rightarrow$

$$1 - e^{-bx} = 1 - \left[1 - bx + \frac{1}{2}b^2x^2 + O(x^3)\right]$$

$$= bx - \frac{1}{2}b^2x^2 + O(x^3).$$

$$\therefore \dot{x} \simeq x - a\left(bx - \frac{1}{2}b^2x^2\right) + O(x^3)$$

$$= (1-ab)x + \frac{1}{2}ab^2x^2 + O(x^3)$$ $\rightarrow$ resembles normal form for transcritical bifurcation.

Example :- $\dot{x} = r\ln x + x - 1.$

$\dot{x} = 0 = r\ln x + x - 1. \rightarrow$ satisfied for $x = 1 \ \forall r.$

Define $u = x - 1$

$$\dot{u} = \dot{x} = r\ln(1+u) + u$$

$$= r\left[u - \frac{1}{2}u^2 + O(u^3)\right] + u$$

$$\simeq (r+1)u - \frac{1}{2}ru^2 + O(u^3).$$

Hence, transcritical bifurcation occurs at $r_c = -1$

Let $u = av$.

$$a\dot{v} = a(r+1)v - \frac{1}{2}ra^2v^2 + O(v^3).$$

$$\Rightarrow \dot{v} = (r+1)v - \frac{1}{2}rav^2 + O(v^3).$$

Choose $R = r+1$ and $X = v$.

$$\dot{X} \approx RX - X^3 + O(X^3).$$

$$X = v = \frac{u}{a} = \frac{1}{2} r(x-1).$$

~~Solved~~

1.4.1 $\dot{x} = a - bx^2$, ~~and~~ $a, b > 0$.

~~Rescale, $\frac{dx}{d(at)} = 1 - \frac{x^2}{(a/b)}$.~~

~~Let $X = \frac{x}{K}$, where $K = a/b$,~~ ~~T~~ $T' = \sqrt{ab}\,t$

~~$K \frac{dX}{d(at)} = 1 - X^2$.~~

~~$\Rightarrow \frac{dX}{}$~~

$$\dot{x} = a\left(1 - \frac{x^2}{a/b}\right)$$

$$\Rightarrow \frac{dx}{d(at)} = 1 - \frac{x^2}{a/b}$$

$$X = \frac{x}{\sqrt{a/b}}$$

$$\Rightarrow \frac{dx}{d(at)} = 1 - X^2.$$

$$\Rightarrow \sqrt{\frac{a}{b}} \frac{dX}{d(at)} = 1 - X^2$$

$$\Rightarrow \frac{dX}{d(\sqrt{ab}\,t)} = 1 - X^2$$

$$\Rightarrow \frac{dX}{dT} = 1 - X^2$$

PARTIAL DIFFERENTIAL EQUATIONS

$\frac{dy}{dx} = f(x)$. $\rightarrow$ ODE, only ONE independent variable

PDE :- More than one independent variable.

Ex:- $u \equiv u(x, y)$

OR,

$u \equiv u(x, t)$.

- Upto linear second order, the canonical form for a PDE is,

$$\boxed{a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = 0}$$

(i) $b^2 - 4ac > 0 \rightarrow$ HYPERBOLIC

(ii) $b^2 - 4ac = 0 \rightarrow$ PARABOLIC

(iii) $b^2 - 4ac < 0 \rightarrow$ ELLIPTIC.

WAVE EQN :- $u_{tt} - u_{xx} = 0$

In this case, $a = 1$, $b = 0$, $c = -1$.

$b^2 - 4ac = 0 - 4(1)(-1) = 4 > 0 \Rightarrow \boxed{\text{HYPERBOLIC}}$

LAPLACE EQN :- $u_{xx} + u_{yy} = 0$.

$a = 1$, $c = 1$, $b = 0$

$b^2 - 4ac = 0 - 4(1)(1) < 0 \Rightarrow \boxed{\text{ELLIPTIC}}$

HEAT EQN :- $u_t - u_{xx} = 0$.

$a = -1$, $c = 0$, $b = 0$

$b^2 - 4ac = 0 \Rightarrow \boxed{\text{PARABOLIC}}$.

WAVE EQN

$$\frac{\partial^2\psi}{\partial t^2} = c^2\nabla^2\psi$$

1d :- $$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}.$$

Claim :- $\psi = f(x \pm ct)$ is a sol^n

Let $u = x \pm ct$.

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial t} = \pm c.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x} = \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} = \pm c\cdot\frac{\partial f}{\partial u}$$

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial u^2}$$

Substitute. $$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2}$$

$$= c^2\frac{\partial^2 f}{\partial u^2} - c^2\frac{\partial^2 f}{\partial u^2} = 0 \quad \text{(Proved)}$$

$\psi = f(t \pm \frac{x}{c})$ also a solution :-

$f(t - \frac{x}{c})$ represents wave travelling in x-dir^n

$f(t + \frac{x}{c})$ " " " " $-$ve x-dir^n.

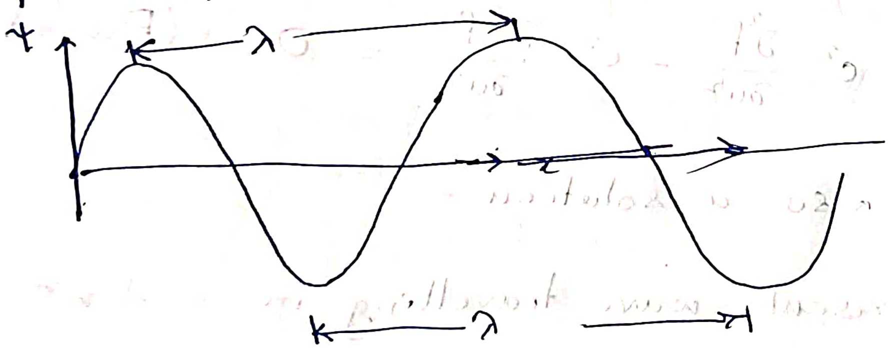
$y = Ae^{-i\omega(t - x/c)}$. $\to$ sol^n to wave eqn.

$$\begin{tikzcd} \text{WAVES} \arrow[rr] \arrow[dr] & & \text{PARTICLES} \\ & \text{QUANTUM MECHANICS} \arrow[ur] & \end{tikzcd}$$

De-Broglie hypothesis:- Any particle moving with momentum p, has a wave-length associated with it.
But a "free" wave extends from $-\infty$ to $+\infty$. Hence difficult to associate any notion of a particle with it.

~~Wave packets~~ :-

— Wavelength - Spatial distance between corresponding points of a periodic solution.



De Broglie

$$p = \frac{h}{\lambda}$$

$$E = \frac{hc}{\lambda}$$

$$\boxed{h = 6.62 \times 10^{-34}\ m^2 kg/s}$$

$$\lambda = \frac{c}{f}$$

$$\psi(x,t) = A\cos\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

$$\psi(x+2\pi, t) \equiv \psi(x,t).$$

$$\psi(x,t) \equiv A\cos(kx - \omega t),$$

$$kc = \omega$$

$$\Rightarrow \lambda = \frac{c}{f}$$

~~[illegible]~~ $\psi(x, t + 1/f) \equiv \psi(x, t)$

$\Rightarrow \boxed{c = \lambda f}$

WAVE PACKET (WITHOUT CONSIDERING t, FOR NOW)

- Key to localization.
- Wave packet - Add waves of different wave no. k's.

Ex:- $y(x) = A_1 \cos k_1 x + A_2 \cos k_2 x$

$= A_1 \cos\left(\frac{2\pi}{\lambda_1} x\right) + A_2 \cos\left(\frac{2\pi}{\lambda_2} x\right)$

Say, $A_1 = A_2 = A$.

$y(x) = A\left[\cos\left(\frac{2\pi}{\lambda_1} x\right) + \cos\left(\frac{2\pi}{\lambda_2} x\right)\right]$

$= 2A \left\{ \cos\left[\cancel{\tfrac{1}{2}}\left(\frac{2\pi}{\lambda_1} x + \frac{2\pi}{\lambda_2} x\right)\right] \cos\left[\left(\frac{\pi}{\lambda_1} x - \frac{\pi}{\lambda_2} x\right)\right]\right.$

$= 2A \cos\left(\frac{\pi x}{\lambda_1} - \frac{\pi x}{\lambda_2}\right) \cos\left(\frac{\pi x}{\lambda_1} + \frac{\pi x}{\lambda_2}\right)\Big]$

If λ_1 and λ_2 are close together, $\lambda_1 - \lambda_2 = \Delta\lambda \ll \lambda_1, \lambda_2$

$\lambda_{avg} = \frac{\lambda_1 + \lambda_2}{2} \approx \lambda_1$

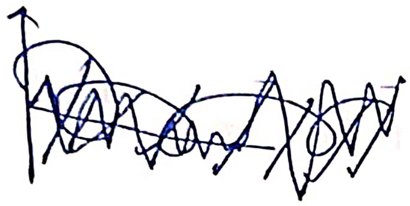
~~$y(x) = 2A\cos$~~ $\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \approx \frac{1}{\lambda_1} - \frac{1}{\lambda_1 - \Delta\lambda}$

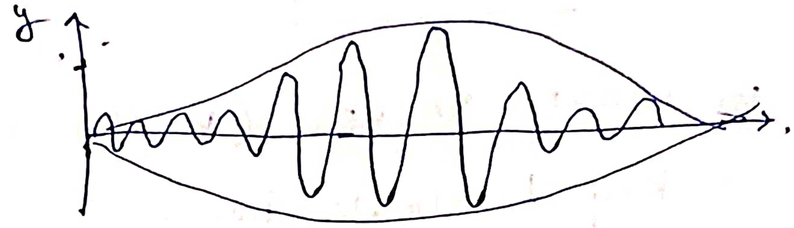
$\Rightarrow \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \approx \frac{1}{\lambda_1} - \frac{1}{\lambda_1}\left(1 - \frac{\Delta\lambda}{\lambda_1}\right)^{-1}$

$= \frac{\Delta\lambda}{\lambda_1^2} \approx \frac{\Delta\lambda}{\lambda_{avg}}$

$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \cancel{2/\lambda_{avg}} \approx \frac{2}{\lambda_{avg}}$

$\therefore$ $$\boxed{y(x) \approx 2A \cos\left(\frac{\Delta\lambda \pi x}{\lambda_{avg}^2}\right) \cos\left(\frac{2\pi x}{\lambda_{av}}\right)} \Rightarrow \text{Beats.}$$





— Some change (diminution of amplitude), but still extends from $-\infty$ to $+\infty$.

— Any finite combinations of waves with discrete wavelengths will ~~be~~ have same outcome.

— $$y(x) = \int dk\, A(k) \cos kx.$$

Suppose we have a range of k's from $\left(k_0 - \frac{\Delta k}{2}\right)$ to $\left(k_0 + \frac{\Delta k}{2}\right)$ in a continuous distribution,

with $A(k) = A_0 = \text{const}$ i.e., $A(k) = A_0, \quad k_0 - \frac{\Delta k}{2} \le k \le k_0 + \frac{\Delta k}{2}$

$= 0$, otherwise.

$$y(x) = A_0 \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \Delta k/2} dk \cos kx.$$

$$= \frac{A_0}{x} \sin kx \Big|_{k_0 - \frac{\Delta k}{2}}^{k_0 + \Delta k/2}$$

$$= \frac{A_0}{x}\left[\sin\left(k_0 x + \frac{\Delta k}{2} x\right) - \sin\left(k_0 x - \frac{\Delta k}{2} x\right)\right]$$

$$= \frac{2A_0}{x} \cos(k_0 x) \sin\left(\frac{\Delta k\, x}{2}\right)$$

$$\Rightarrow \boxed{y(x) = \frac{2A_0}{x}\sin\left(\frac{\Delta k}{2}x\right)\cos(k_0 x)}$$

– This has the desirable property that $y(x) \to 0$ as $x \to \pm\infty$, BUT, blows up at $x = 0$.

— Better approximation:- $A(k) = A_0 e^{-(k-k_0)^2/2(\Delta k)^2}$.

$$y(x) = A_0 \Delta k \sqrt{2\pi}\, e^{-(\Delta k x)^2/2} \cos k_0 x .$$

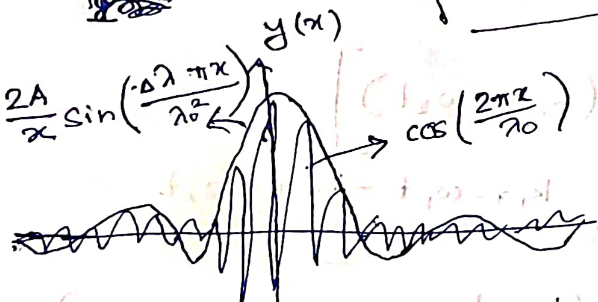
PROOF :- $y(x) = A_0 \int_{-\infty}^{+\infty} dk\, e^{-(k-k_0)^2/2(\Delta k)^2} \cos(kx)$.

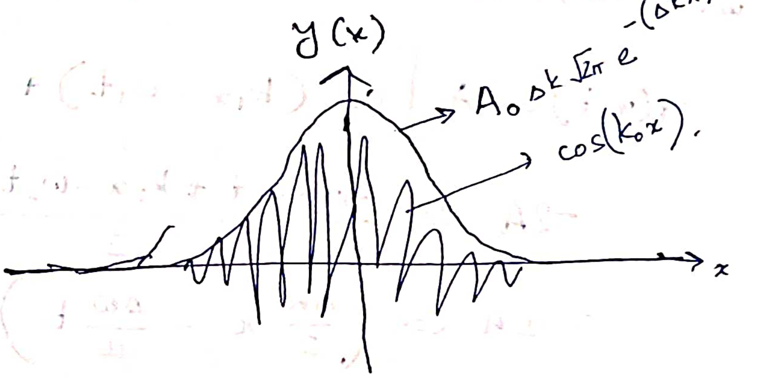
Consider,

$$I = A_0 \int_{-\infty}^{+\infty} dk\, e^{-(k-k_0)^2/2(\Delta k)^2} e^{ikx}.$$

Complete the square and take $\cos(kx) = \mathrm{Re}(e^{ikx})$.

$$\boxed{y(x) = \frac{2A}{x}\sin\left(\frac{\Delta\lambda\, \pi x}{\lambda_0^2}\right)\cos\left(\frac{2\pi x}{\lambda_0}\right)}$$





- Summary :- superposition of waves succeeds in confining a wave-packet to a specific region of space

- Again consider the function,

$$y(x) = A_0 \Delta k \, e^{-(\Delta k x)^2/2} \cos(k_0 x).$$

$$k = \frac{2\pi}{\lambda} \Rightarrow \Delta k = -\frac{2\pi}{\lambda^2} \Delta\lambda.$$

$$y(x) = A_0 \Delta k \, e^{-\left(\frac{2\pi}{\lambda^2} x \Delta\lambda\right)^2/2} \cos(k_0 x).$$

Concentrate on factor $e^{-\left(\frac{2\pi}{\lambda^2}\Delta\lambda x\right)^2} = e^{-\left(\frac{2\pi}{\lambda^2}\Delta\lambda\right)^2 x^2}$

$$= e^{-\alpha^2 x^2}.$$

$\alpha \rightarrow$ inversely proportional to "width" of gaussian. (Why?)

So, decreasing width means increasing $\left(\frac{\Delta\lambda}{\lambda}\right)$ or (Δk).

MOTION OF WAVE PACKET.

$$\psi(x,t) = A\left[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\right]$$

$$= 2A \cos\frac{k_1 x - \omega_1 t + k_2 x - \omega_2 t}{2} \cos\frac{k_1 x - \omega_1 t - k_2 x + \omega_2 t}{2}$$

$$= 2A \cos\left(\frac{\Delta k}{2} x - \frac{\Delta\omega}{2} t\right) \cos\left(\frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t\right).$$

- For any individual wave, $\underbrace{c = \lambda f = \left(\frac{2\pi}{k}\right)\left(\frac{\omega}{2\pi}\right) = \frac{\omega}{k}}_{\text{Phase speed } v_p}$.

The combination $\psi(x,t) = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)$

travels with velocity of "envelope". (group velocity).

$$v_g = \frac{\Delta\omega}{\Delta k} \iff v_g = \frac{d\omega}{dk}.$$

VII Group velocity of de Broglie wave.

$$p = \frac{h}{\lambda} \qquad E = hf = \hbar\omega$$

$$= \hbar k.$$

$$v_g = \frac{d\omega}{dk} = \frac{dE/\hbar}{dp/\hbar} = \frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{p}{m} = v.$$

$$\Rightarrow \boxed{v_g = v_{particle}}$$

<u>EX:-</u> Waves on an ocean travel with a phase velocity

$$v_{phase} = \sqrt{\frac{g\lambda}{2\pi}} \qquad v_g = ?$$

<u>Soln:-</u>

$$\left.\begin{array}{l} v_{phase} = \dfrac{\omega}{k} \\ \Rightarrow \sqrt{\dfrac{g}{k}} = \dfrac{\omega}{k} \\ \Rightarrow \omega = \sqrt{gk}. \end{array}\right\} \Rightarrow v_g = \frac{d\omega}{dk} = \sqrt{g}\,\frac{1}{2}\,k^{-1/2} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

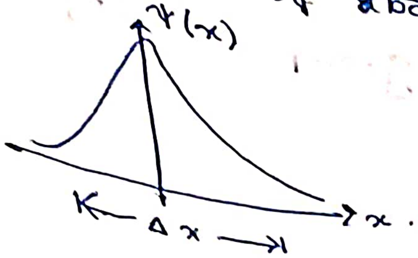
$$= \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}.$$

<u>Summary (so far)</u> :- – Possible to localize a wave to a region, but not to a specific point.

– So, there is always going to be some uncertainty associated with a <u>wave-packet</u>.

$$\psi(x) = A_0 \Delta k \cos(k_0 x) \times e^{-(\Delta k x)^2/2}.$$

~~Width of above G~~



— Intuitively, (Δk) may be regarded as the "uncertainty" in k and (Δx) may be regarded as the "uncertainty" in x.

— (Δx) can be ~~increase~~ decreased, only at the cost of increasing $\Delta p_x = \hbar \Delta k_x$.

— This is encapsulated in the form of the

III HEISENBERG UNCERTAINTY PRINCIPLE

$$\Delta k \Delta x > 1/2 \iff \boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$$

— Not the only uncertainty relation, $\Delta E \Delta t \geq \frac{\hbar}{2}$.

— Can be formulated between "complementary" variables.

IV PROBABILISTIC INTERPRETATION OF $\psi(x,t)$ / BORN INTERPETATION

Probability of finding a particle associated with $\psi(x,t)$ in the region between x and $x+dx$ is,

$$P(x,t)\, dx = |\psi(x,t)|^2 dx$$

So, $$\int_{-\infty}^{+\infty} dx\, |\psi(x,t)|^2 = 1.$$

$\Rightarrow$ $|\psi(x,t)|^2$ is a probability density / probability distribution function.

— ~~A free particle can be anywhere, hence is impossible to localize. Hence it is described by a monochromatic plane wave~~

— $\psi(x,t) = \int_{-\infty}^{+\infty} dk\, A(k)\, e^{i(kx-\omega t)}$ (note inclusion of t).

Now, by de-Broglie hypothesis, $E = \frac{hc}{\lambda} = \cancel{\frac{h c k}{2\pi}}$

$= hf$

$= \hbar\omega$

$p = \hbar k$

Again, $\lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$

$\Rightarrow \frac{2\pi}{k} = \frac{2\pi c}{\omega}$

$\Rightarrow \omega = ck$

$\therefore \psi(x,t) = \int_{-\infty}^{+\infty} dk\, A(k)\, e^{i(px-Et)/\hbar}$

$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp\, \phi(p)\, e^{i(px-Et)/\hbar}$

— Consider two cases:

(I) Free particle, $E = p^2/2m$, $\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp\, \phi(p)\, e^{i(px-E(p)t)/\hbar}$

(II) $E = \frac{p^2}{2m} + V(x)$. $\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp\, \phi(p)\, e^{i(px-(\;)/\hbar}\, e^{iV(x)/\hbar}$

does not affect $|\psi(x,t)|^2$ which is the physical quantity of interest.

— Naive attempt fails. Need to find an PDE eqn. for $\psi(x,t)$ and then modify it to accept $V(x)$.

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \, \phi(p) \, E(p) \, e^{i(px - Et)/\hbar}.$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \, \phi(p) \, \frac{p^2}{2m} \, e^{i(px - Et)/\hbar}$$

$$\left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial x^2} \psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \, \phi(p) \, p^2 \, e^{i(px - Et)/\hbar}$$

- So, for free particle,

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

- In presence of a potential,

$$\boxed{i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\psi(x)} \Rightarrow \text{Schrödinger eqn.}$$

FACTS ABOUT x, p etc.

- Have already seen that there is uncertainty associated with x, p etc.
- $|\psi|^2$ is probability density.
- In QM, p is interpreted as differential operator. $\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$
- $\langle p \rangle = \int dx \, \psi^* \hat{p} \, \psi$ $\qquad$ $\langle x \rangle = \int dx \, \psi^* x \, \psi$

$$= \int dx \, \psi^* \left(\frac{\hbar}{i}\right) \frac{\partial \psi}{\partial x}$$

$$= \left(\frac{\hbar}{i}\right) \int dx \, \psi^* \frac{\partial \psi}{\partial x}.$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx\, x |\psi|^2$$

- Hints about operators.

~~For a free particle~~

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

$$p = \frac{\hbar}{i} \frac{\partial}{\partial x} \Rightarrow p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}.$$

$$E\psi = \left(\frac{p^2}{2m} + V(x)\right)\psi$$

- $$\langle G(x) \rangle = \int_{-\infty}^{+\infty} dx\, G(x) |\psi|^2$$

- $$\langle E \rangle = \int_{-\infty}^{+\infty} dx\, \psi^* \hat{E} \psi = \int_{-\infty}^{+\infty} dx\, \psi^* \left(i\hbar \frac{\partial \psi}{\partial t}\right)$$

$$= i\hbar \int \psi^* \frac{\partial \psi}{\partial t}.$$

- $$\langle G(x,p) \rangle = \int_{-\infty}^{+\infty} dx\, \psi^* \hat{G} \psi.$$

VII SCHRODINGER EQN.

- Separation of variables.
- Let $V(x,t) \equiv V(x)$. $\Leftrightarrow$ $\psi(x,t) = X(x)T(t)$

$$i\hbar X \frac{dT}{dt} = -\frac{\hbar^2}{2m} T \frac{d^2X}{dx^2} + V(x) X(x) T(t)$$

$$\Rightarrow i\hbar \frac{1}{T}\frac{dT}{dt} = -\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + V(x) = E = \text{const.}$$

$$T(t) = e^{-iEt/\hbar}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2 X}{dx^2} + V(x)X(x) = EX(x)} \Rightarrow \text{TIME INDEPENDENT SCHRÖDINGER EQN.}$$

Eigenvalue problem.

Ex:- Take an operator $\hat{G} = \frac{d^2}{dx^2}$. Eigenfunction $\psi = e^{2x}$.

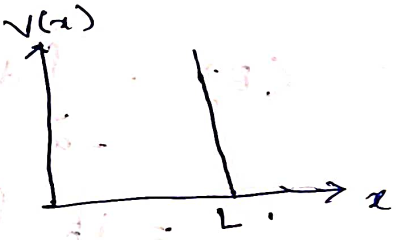
$$\hat{G}\psi = \frac{d^2}{dx^2} e^{2x} = 4e^{2x} = 4\psi$$

- Eigenvalue is 4.

$$\int P(x,t)\,dx = \int |\phi(x)|^2 e^{-iEt/\hbar} e^{iEt/\hbar}\,dx = \int |\phi(x)|^2 dx$$

PARTICLE IN A BOX.

$$V(x) = 0, \quad 0 < x < L$$
$$= \infty, \quad \text{otherwise}$$



$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V(x)\phi(x) = E\phi(x)$$

- Energy is infinite outside of the box, which is impossible. So, the particle cannot exist outside the box $\Rightarrow \psi = 0$ outside the box.

$$\therefore -\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \frac{2mE}{\hbar^2}\phi = 0$$

$$\Rightarrow \phi(x) = A\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + B\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)$$

– Boundary conditions, $\phi(0) = \phi(L) = 0$

$\Rightarrow B = 0$. from $\phi(0) = 0$

$$\phi(x) = A \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right)$$

Now $\phi(L) = A \sin\left(\frac{\sqrt{2mE}}{\hbar} L\right) = 0$

$$\Rightarrow \frac{\sqrt{2mE}}{\hbar} L = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \frac{2mE}{\hbar^2} L^2 = n^2\pi^2$$

$$\Rightarrow E_n = \left(\frac{n^2\pi^2\hbar^2}{2mL^2}\right),$$

$$\phi_n = A \sin\left(\frac{\sqrt{2mE_n}}{\hbar} x\right) = A \sin\left(\frac{n\pi x}{L}\right).$$

To determine const. A,

$$\int_{-\infty}^{+\infty} dx\, |\phi_n(x)|^2 = A^2 \int_0^L dx \sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{A^2}{2}\left[\int_0^L dx - \int_0^L dx \cos\left(\frac{2\pi n x}{L}\right)\right]$$

$$= \frac{A^2}{2}(L) = 1$$

$$\Rightarrow A = \sqrt{\frac{2}{L}}$$

$$\therefore \boxed{\phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right). \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}$$

~~applications~~

Simpler examples:-

Solutions for constant potential energy. V_0.

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V_0\phi = E\phi$$

$$\Rightarrow \frac{d^2\phi}{dx^2} = -\frac{2m}{\hbar^2}(E-V_0)\phi = -k^2\phi.$$

$$\frac{d^2\phi}{dx^2} + k^2\psi = 0.$$

$$\psi(x) = A\cos kx + B\sin kx.$$

Forbidden region:- $E < V_0$.

$$\frac{d^2\phi}{dx^2} = k'^2\phi, \quad \text{with} \quad k' = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\phi''(x) - k'^2\phi(x) = 0$$

$$\Rightarrow \phi(x) = Ae^{k'x} + Be^{-k'x}.$$

- Free particle:- $V_0 = 0$

$$-\frac{\hbar^2}{2m}\phi''(x) = E\phi(x)$$

$$\Rightarrow \phi''(x) + \frac{2mE}{\hbar^2}\phi(x) = 0.$$

$$\phi(x) = Ae^{ikx} + Be^{-ikx}.$$

$$\psi(x,t) = (Ae^{ikx} + Be^{-ikx})e^{-i\omega t}$$

$$= Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}.$$

Expectation values for particle in a box.

$\phi_n = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$

$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx\, x\, |\phi|^2 = \frac{2}{L}\int_0^L dx\; x \sin^2\frac{n\pi x}{L}.$$

$$= \frac{2}{L}\int_0^L dx.\ \frac{x}{2}\left[1 - \cos\left(\frac{2\pi n x}{L}\right)\right]$$

$$= \frac{2}{L}\int_0^L dx.\ \frac{x}{2}. - \frac{1}{L}\int_0^L dx\; x\cos\left(\frac{2\pi n x}{L}\right).$$

$$= \frac{\cancel{2}}{L}.\ \frac{x^2}{\cancel{4}2}\Big|_0^L - \frac{1}{L}\left[x\int_0^L dx\cos\left(\frac{2\pi n x}{L}\right) - \int_0^L dx.\int dx \cos\frac{2\pi n x}{L}\right]$$

$$= \frac{L}{2}$$

$\therefore \langle x \rangle = \frac{L}{2}.$

$$\langle p \rangle = \int_{-\infty}^{+\infty} dx\, \phi^* p\, \phi$$

$$= \left(\frac{\hbar}{i}\right)\int_0^L dx.\left(\frac{2}{L}\right)\sin\frac{n\pi x}{L}\frac{d}{dx}\left(\sin\frac{n\pi x}{L}\right)$$

$$= \left(\frac{\hbar}{i}\right)\left(\frac{2}{L}\right).\frac{n\pi}{2L}\int_0^L dx\, 2\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

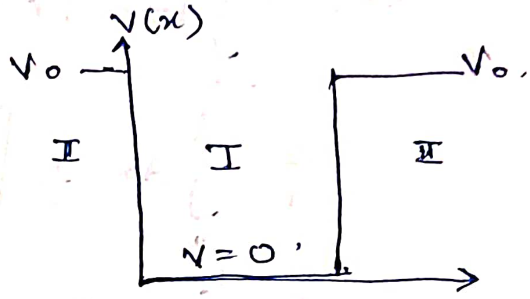
$$= 0$$

$$E_n = \frac{p_n^2}{2m}$$

$$\Rightarrow p_n = \pm \frac{n\pi\hbar}{L}$$

Avg. value $= \dfrac{+\frac{n\pi h}{L} - \frac{n\pi h}{L}}{2} = 0$.

(iii) FINITE POTENTIAL WELL.



$V(x) = 0, \quad 0 \le x \le L$

$= V_o, \quad x<0, \quad x>L.$

Region I :-

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi$$

$$\Rightarrow \phi(x) = A\cos kx + B\sin kx, \quad 0 \le x \le L.$$

Region II :-

$$-\frac{\hbar^2}{2m}\phi''(x) + V_o\phi(x) = E\phi(x)$$

$$\Rightarrow \phi(x) = Ce^{k'x} + De^{-k'x}, \quad \text{where } k' =$$

Now, at $x \to -\infty$, $e^{-k'x} \to \infty \Rightarrow D = 0$.

$\phi_{x<0}(x) = Ce^{k'x}$.

$\phi_{x>L}(x) = De^{-k'x}$.

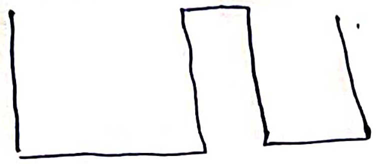
$$k'^2 = \frac{2m(V_o - E)}{\hbar^2}$$

As $V_o \to \infty$, $\phi_{x<0 | x>L} \to 0$.

To determine (A, B, C, D), ψ must be continuous, as well as $\left(\frac{d\psi}{dx}\right)$.

TUNNEL EFFECT

— PURELY QUANTUM PHENOMENON, HAS <u>NO</u> CLASSICAL ANALOGUE



— Barrier of finite thickness.

— Particle may 'tunnel' through for a sufficiently thin barrier.

<u>PROB</u>:- For a particle in a box, calculate $\Delta x \, \Delta p$.

<u>Soln</u>:- $\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$.

$\langle x \rangle = \frac{L}{2}$ $\qquad \langle p \rangle = 0$.

$$\langle x^2 \rangle = \frac{2}{L} \int_0^L dx \, x^2 \sin^2\left(\frac{n\pi x}{L}\right).$$

$$= \frac{1}{L} \int_0^L dx \, x^2 \left[1 - \cos\left(\frac{2\pi n x}{L}\right)\right]$$

$$= \frac{1}{L} \int_0^L dx \, x^2 - \frac{1}{L} \int_0^L dx \, x^2 \cos\left(\frac{2\pi n x}{L}\right).$$

$$= \frac{L^2}{3} - \frac{1}{\cancel{L}} \, x^2 \sin\left(\frac{2\pi n x}{L}\right) \frac{1}{\frac{2\pi n}{\cancel{L}}} \Bigg|_0^L + \frac{1}{L} \int_0^L 2x \, \frac{\sin\left(\frac{2\pi n x}{L}\right)}{\frac{2\pi n}{L}} \, dx$$

$$= \frac{L^2}{3} + \frac{1}{\pi n} \int_0^L dx \, x \sin\left(\frac{2n\pi x}{L}\right).$$

$$\cancel{\frac{L^2}{3} + \frac{L^2}{2\pi^2 n^2} + \frac{1}{\pi n} \left[\frac{x \cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} \Bigg|_0^L - \int_0^L dx \, \frac{\cos\left(\frac{2\pi n x}{L}\right)}{\frac{2\pi n}{L}} \right]}$$

$$\langle x^2\rangle = \frac{L^2}{3} + \frac{1}{\pi n}\left[-x\,\frac{\cos\left(\frac{2\pi n x}{L}\right)}{\frac{2\pi n}{L}}\Bigg|_0^L + \int_0^L dx\,\frac{\cos\left(\frac{2\pi n x}{L}\right)}{2\pi n/L}\right]$$

$$= \frac{L^2}{3} + \frac{1}{\pi n}\left[-\frac{L^2}{2\pi n}\cos(2\pi n) + \frac{L}{2\pi n}\,\frac{\sin\left(\frac{2\pi x}{L}\right)}{\frac{2\pi n}{L}}\Bigg|_0^L \right]$$

$$= \frac{L^2}{3} - \frac{L^2}{2\pi^2 n^2}.$$

$$\langle p^2\rangle = -\hbar^2 \int_0^L dx\,\phi_n^*(x)\,\frac{d^2\phi_n}{dx^2}$$

$$= +\hbar^2\left(\frac{2}{L}\right)\frac{n^2\pi^2}{L^2}\int_0^L dx\,\sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2n^2\pi^2\hbar^2}{L^3}\left(\frac{L}{2}\right) = \frac{n^2\pi^2\hbar^2}{L^2}.$$

$$\Delta p_1 = \frac{\pi^2\hbar^2}{L^2}$$

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2} = \frac{\Delta p_1^2}{2m}$$

$$\Delta x\,\Delta p = \sqrt{\langle x^2\rangle - \langle x\rangle^2}\,\sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

$$= \sqrt{\frac{L^2}{3} - \frac{L^2}{2\pi^2 n^2} - \frac{L^2}{4}}\,\sqrt{\frac{n^2\pi^2\hbar^2}{L^2} - 0}$$

$$= \sqrt{\frac{L^2}{12} - \frac{L^2}{2\pi^2 n^2}}\,\sqrt{\frac{n^2\pi^2\hbar^2}{L^2}}$$

$$= \left(\frac{n\pi\hbar}{L}\right)\frac{L}{2}\sqrt{\frac{1}{6} - \frac{1}{\pi^2 n^2}}$$

$$= \frac{\hbar}{2}\left(n\pi\right)\sqrt{\frac{1}{6} - \frac{1}{\pi^2 n^2}} = \left(\frac{\hbar}{2}\right) n\pi\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} \geq \frac{\hbar}{2}$$

(induction).

PROB:- Probability that a particle is trapped in a box of length L can be found between 0.5L and L ?

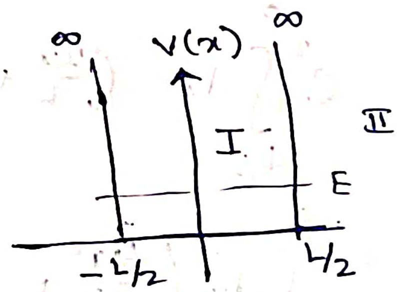
Soln:- $P_{x_1, x_2} = \int_{x_1}^{x_2} dx \, |\phi_n(x)|^2$

$$= \frac{2}{L} \int_{x_1}^{x_2} dx \, \sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{L} \int_{x_1}^{x_2} dx \left[1 - \cos\left(\frac{2\pi n x}{L}\right)\right]$$

$$= \left(\frac{x_2 - x_1}{L}\right)$$

PROB:- Symmetric square well:-



$$V(x) = \begin{cases} +\infty, & x < -L/2 \\ 0, & -L/2 \le x \le L/2 \\ +\infty, & x > L/2 \end{cases}$$

Region I:- $-\frac{\hbar^2}{2m}\phi''(x) = E\phi(x)$

$\times \frac{\hbar^2}{2m}$

$$\Rightarrow \phi''(x) + \left(\frac{2mE}{\hbar^2}\right)\phi(x) = 0$$

$$\phi(x) = A\cos(kx) + B\sin(kx)$$

Region II:- $\phi(x) = 0$

$$\phi(-L) = A\cos(kL) - B\sin(kL) = 0 \quad \Rightarrow A = 0$$

$$\phi(L) = A\cos(kL) + B\sin(kL) = 0$$

PROB :-

Consider,

$$\phi(x) = 0, \quad x < -L/2$$
$$= C\left(\frac{2x}{L}+1\right), \quad -\frac{L}{2} < x < 0$$
$$= C\left(-\frac{2x}{L}+1\right), \quad 0 < x < L/2$$
$$= 0, \quad x > +L/2$$

$C = ?$

Soln :-

$$\int_{-L/2}^{+L/2} dx\, |\phi(x)|^2 = 1$$

$$\Rightarrow C^2 \int_{-L/2}^{0} dx \left(\frac{2x}{L}+1\right)^2 dx + C^2 \int_{0}^{L/2} dx \left(-\frac{2x}{L}+1\right)^2 = 1$$

$$\Rightarrow C^2 \int_{-L/2}^{0} dx \left(\frac{4x^2}{L^2} + \frac{4x}{L} + 1\right) + C^2 \int_{0}^{L/2} dx \left(\frac{4x^2}{L^2} - \frac{4x}{L} + 1\right) = 1$$

$$\Rightarrow C^2 \int_{-L/2}^{+L/2} dx \left(\frac{4x^2}{L}\right) + C^2 \int_{-L/2}^{0} dx \frac{4x}{L} - C^2 \int_{0}^{L/2} dx \frac{4x}{L} + C^2 \int_{-L/2}^{+L/2} dx = 1$$

$$\Rightarrow \frac{4C^2}{L} \left.\frac{x^3}{3}\right|_{-L/2}^{+L/2} + \frac{4C^2}{L} \left.\frac{x^2}{2}\right|_{-L/2}^{0} - \frac{4C^2}{L} \left.\frac{x^2}{2}\right|_{0}^{L/2} + C^2 . L = 1$$

$$\Rightarrow \frac{4C^2}{3\cancel{L}} \frac{\cancel{L^3}\, L^2}{\cancel{4}} + \frac{4C^2}{L} \frac{L^2}{4} - \frac{4C^2}{L} \frac{L^2}{4} + C^2 L = 1$$

$$\Rightarrow 2C^2 L = 1 \Rightarrow C^2 = \frac{1}{2L} \Rightarrow C = \frac{1}{\sqrt{2L}}$$