SC223 - Linear Algebra

Aditya Tatu

Lecture 17

September 12, 2023

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$.
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$.
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.
- **Closure with respect to Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$.
- **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.
- **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.
- **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

Properties of Vector Spaces

- **Proposition 1:** Every vector space has a unique additive identity.
- **Proposition 2:** Every vector in a vector space has a unique additive inverse.
- **Proposition 3:** $\forall v \in V, 0 \cdot v = \theta$
- **Proposition 4:** $\forall a \in \mathbb{F}, a \cdot \theta = \theta$.
- **Proposition 5:** $\forall v \in V, (-1) \cdot v = -v$.

Since $W \neq \emptyset$, $\exists\, v \in W$.

Since W is closed w.r.t $\cdot$,

$0 \cdot v \in W$

$\Rightarrow \theta \in W$

Subspace

- **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.

Subspace

- **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.
- Note on notation: Capitals(U) for subspaces, small letters(u) for vectors(elements).

Subspace

- **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.
- Note on notation: Capitals(U) for subspaces, small letters(u) for vectors(elements).
- Examples:

Subspace

● **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.

● Note on notation: Capitals(U) for subspaces, small letters(u) for vectors(elements).

● Examples:

► For any vector space V, V and $\{\theta\}$ are always subspaces. These are called **trivial subspaces**.

Subspace

● **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.

● Note on notation: Capitals(U) for subspaces, small letters(u) for vectors(elements).

● Examples:

► For any vector space V, V and $\{\theta\}$ are always subspaces. These are called **trivial subspaces**.

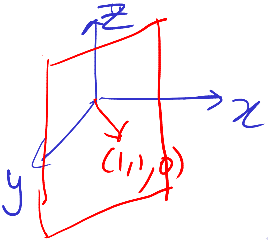
► $V = \mathbb{R}^2, W = \{(x, y) \in V \mid ax + by = c\}$.

Subspace

- **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.
- Note on notation: Capitals(U) for subspaces, small letters(u) for vectors(elements).
- Examples:
  - ► For any vector space V, V and $\{\theta\}$ are always subspaces. These are called **trivial subspaces**.
  - ► $V = \mathbb{R}^2, W = \{(x, y) \in V \mid ax + by = c\}.W$ is a subspace iff $c = 0$.

$V = \mathbb{R}^3,\ W = \{(x, y, z) \in V \mid ax + by + cz = d\}.$

$\underline{d = 0}$



$W = N(A)$

$A = [a \quad b \quad c]$

Subspace

● **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.

● Note on notation: Capitals(U) for subspaces, small letters(u) for vectors(elements).

● Examples:

► For any vector space V, V and $\{\theta\}$ are always subspaces. These are called **trivial subspaces**.

► $V = \mathbb{R}^2, W = \{(x, y) \in V \mid ax + by = c\}.W$ is a subspace iff $c = 0$.

► $V = \mathcal{P}(\mathbb{R}), W = \mathcal{P}_n(\mathbb{R})$, where $\mathcal{P}_n(\mathbb{R})$ denotes the set of all polynomials of one variable with real coefficients with degree at most n.

Subspace

● **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.

● Note on notation: Capitals(U) for subspaces, small letters(u) for vectors(elements).

● Examples:

► For any vector space V, V and $\{\theta\}$ are always subspaces. These are called **trivial subspaces**.

► $V = \mathbb{R}^2, W = \{(x, y) \in V \mid ax + by = c\}.W$ is a subspace iff $c = 0$.

► $V = \mathcal{P}(\mathbb{R}), W = \mathcal{P}_n(\mathbb{R})$, where $\mathcal{P}_n(\mathbb{R})$ denotes the set of all polynomials of one variable with real coefficients with degree at most n.

► $V = \mathcal{L}^2(\mathbb{R}), W = \{f \in V \mid \int_{-\infty}^{\infty} f(t)\, dt = 0\}$.

- **Proposition 6:** A non-empty subset W of a vector space V is a subspace if and only if

● **Proposition 6:** A non-empty subset W of a vector space V is a subspace if and only if

- W is closed with respect to vector addition, and

● **Proposition 6:** A non-empty subset W of a vector space V is a subspace if and only if

- W is closed with respect to vector addition, and
- W is closed with respect to scalar multiplication.

- **Proposition 6:** A non-empty subset W of a vector space V is a subspace if and only if
  - W is closed with respect to vector addition, and
  - W is closed with respect to scalar multiplication.
- Familiar examples of Subspaces: Let $A \in \mathbb{R}^{m \times n}$.

● **Proposition 6:** A non-empty subset W of a vector space V is a subspace if and only if

- W is closed with respect to vector addition, and
- W is closed with respect to scalar multiplication.

● Familiar examples of Subspaces: Let $A \in \mathbb{R}^{m \times n}$. Then, $C(A), N(A^T)$ and $N(A), C(A^T)$ are subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Four fundamental Subspaces of A.

Generating New subspaces from Old

- Let U, W be subspaces of V.
- Is $U \cup W$ a subspace of V?

Generating New subspaces from Old

- Let U, W be subspaces of V.
- Is $U \cup W$ a subspace of V? No.
- Is $U \cap W$ a subspace of V?

Generating New subspaces from Old

- Let U, W be subspaces of V.
- Is $U \cup W$ a subspace of V? No.
- Is $U \cap W$ a subspace of V? Yes.

Generating New subspaces from Old

- Let U, W be subspaces of V.
- Is $U \cup W$ a subspace of V? No.
- Is $U \cap W$ a subspace of V? Yes.
- **Definition:** (Sum of subspaces): Let $U_1, \ldots, U_n$ be subspaces of V.

The **sum of subspaces** $U_1, \ldots, U_n$ is defined as:

$$U_1 + \ldots + U_n =: \{u_1 + u_2 + \ldots + u_n \mid u_i \in U_i, i = 1, \ldots, n\}$$

Generating New subspaces from Old

- Let U, W be subspaces of V.
- Is $U \cup W$ a subspace of V? No.
- Is $U \cap W$ a subspace of V? Yes.
- **Definition:** (Sum of subspaces): Let $U_1, \ldots, U_n$ be subspaces of V.

The **sum of subspaces** $U_1, \ldots, U_n$ is defined as:

$$U_1 + \ldots + U_n =: \{u_1 + u_2 + \ldots + u_n \mid u_i \in U_i, i = 1, \ldots, n\}$$

- **Proposition 7:** The sum of subspaces $U_1, \ldots, U_n$ of V is a subspace.

- If $v = u_1 + \ldots + u_n, u_i \in U_i, i = 1, \ldots n$, we say that $(u_1, \ldots, u_n)$ is a decomposition of v.

- If $v = u_1 + \ldots + u_n, u_i \in U_i, i = 1, \ldots n$, we say that $(u_1, \ldots, u_n)$ is a decomposition of v.
- Is this decomposition unique?

- If $v = u_1 + \ldots + u_n, u_i \in U_i, i = 1, \ldots n$, we say that $(u_1, \ldots, u_n)$ is a decomposition of v.
- Is this decomposition unique?
- **Definition:** (Direct Sum of Subspaces) In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.

- If $v = u_1 + \ldots + u_n, u_i \in U_i, i = 1, \ldots n$, we say that $(u_1, \ldots, u_n)$ is a decomposition of v.
- Is this decomposition unique?
- **Definition:** (Direct Sum of Subspaces) In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.
- Direct sum notation: $W = U_1 \oplus U_2 \oplus \ldots \oplus U_n$.

- **Proposition 8:** Let $U_1, \ldots, U_n$ be subspaces of V. Then $V = U_1 \oplus \ldots \oplus U_n$ if and only if: (1) $V = U_1 + \ldots + U_n$, and (2) The only representation of $\theta \in V$ is $(\theta, \ldots, \theta)$.

- **Proposition 9:** Let V be a VS with subspaces U_1, U_2. Then $V = U_1 \oplus U_2$ *iff* $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\theta\}$.