SC223 - Linear Algebra

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Lecture 8

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Why bother with LU decomposition?

- Solve $Ax = b$.
- Given the LU decomposition of A, solve: $LUx = b$.
- Letting $y := Ux$ *reduces* the original problem to two *simpler* linear systems:
  - Solve for y in $Ly = b$.
  - Solve for x in $Ux = y$.

$$A \cdot A^{-1} = I_n$$

$$A \begin{bmatrix} v_{*1} & \cdots & v_{*n} \end{bmatrix} = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix}$$

$$E_{n-1} \cdot E_{n-2} \cdots E_1 A = U$$

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GAUSS-JORDAN

$$E_{n-1} \cdot E_{n-2} \cdot \ldots \cdot E_1 \cdot I_n = L^{-1}$$

$$E_{n-1} \cdot E_{n-2} \cdots E_1 [A \mid I] = [U \mid L^{-1}]$$

$$\underbrace{Q_{n-1} \cdot Q_{n-2} \cdots Q_1}_{U^{-1}} [U \mid L^{-1}] = [I \mid A^{-1}]$$

$$U^{-1} \cdot U = I. \quad U^{-1} L^{-1} = (LU)^{-1} = A^{-1}$$

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$b_1 \in C(A)$
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$\forall p, q \in \mathbb{R}$ $p \cdot b_1 + q\, b_2 = p(Ax_1) + q(Ax_2)$
$= A(p \cdot x_1 + q \cdot x_2)$

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What about multiple solutions?

- Let $Ax = b$ and $Ay = b$, with $x \neq y$.

$$A(x - y) = 0_m$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$$

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● **Summary:** If $\exists z \in N(A), z \neq \mathbf{0}_n$, then $Ax = b$ will have infinitely many solutions, if one exists!

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● **Beware of the notation:** a_{i*} denotes the i^{th} row of A *written as a column matrix*.

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● **The Column Space, Row Space, Null Space and Left Nullspace are called the Four Fundamental Subspaces associated with a matrix.**