1. Which of the following functions are Linear transformations?

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (x, 0, 0), \forall(x, y, z) \in \mathbb{R}^3$
   Solution:

   - Additivity:
     $T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2, 0, 0) = (x_1, 0, 0) + (x_2, 0, 0) = T((x_1, y_1, z_1) + T(x_2, y_2, z_2)$.
   - Homogenity:
     $T(\lambda \cdot (x_1, y_1, z_1)) = T(\lambda \cdot x_1, \lambda \cdot y_1, \lambda \cdot z_1) = (\lambda \cdot x_1, 0, 0) = \lambda \cdot (x_1, 0, 0) = \lambda \cdot T(x_1, y_1, z_1)$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (5x, -x, 10y), \forall(x, y, z) \in \mathbb{R}^3$
   Solution:

   - Additivity:
     $T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (5x_1 + 5x_2, -x_1 - x_2, 10y_1 + 10y_2) = (5x_1, -x_1, 10y_1) + (5x_2, -x_2, 10y_2) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$.
   - Homogenity:
     $T(\lambda \cdot (x_1, y_1, z_1)) = T(\lambda \cdot x_1, \lambda \cdot y_1, \lambda \cdot z_1) = (5 \cdot \lambda x_1, -\lambda x_1, 10\lambda y_1) = \lambda(5x_1, -x_1, 10y_1) = \lambda T(x_1, y_1, z_1)$.

   (c) $T : \mathbb{R} \to \mathbb{R}, T(x) = ax + b, \forall x \in \mathbb{R}$, where a, b are some real-valued non-zero constants.
   Solution:
   $T(\theta) = a\theta + b \neq 0 = \theta$.
   Thus, since $T(\theta) \neq \theta$, it is not a linear transform.

   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (x, y, z) + (1, 2, -2), \forall(x, y, z) \in \mathbb{R}^3$
   Solution:
   $T(\theta) = T((0, 0, 0)) = (0, 0, 0) + (1, 2, -2) = (1, 2, -2) \neq (0, 0, 0) = \theta$. Since $T(\theta) \neq \theta$, it is not a linear transform.

   (e) $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R}), T(p) = q \cdot p \forall p \in \mathcal{P}_3(\mathbb{R})$, where $p \cdot q$ denotes multiplication between polynomials, and $q = q_0 + q_1 x + q_2 x^2 + q_3 x^3$, with $q_0, q_1, q_2, q_3 \in \mathbb{R}$ are fixed constants.
   Solution:

   - Additivity:
     Consider,
     $p_1 = \alpha_0 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3$
     and
     $p_1 = \beta_0 + \beta_1 \cdot x + \beta_2 \cdot x^2 + \beta_3 \cdot x^3$

     $T(p_1 + p_2) = T(\alpha_0 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3 + \beta_0 + \beta_1 \cdot x + \beta_2 \cdot x^2 + \beta_3 \cdot x^3)$
     $= T((\alpha_0 + \beta_0) + (\alpha_1 + \beta_1) \cdot x + (\alpha_2 + \beta_2) \cdot x^2 + (\alpha_3 + \beta_3) \cdot x^3)$
     $= q((\alpha_0 + \beta_0) + (\alpha_1 + \beta_1) \cdot x + (\alpha_2 + \beta_2) \cdot x^2 + (\alpha_3 + \beta_3) \cdot x^3)$
     $= q(\alpha_0 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3) + q(\beta_0 + \beta_1 \cdot x + \beta_2 \cdot x^2 + \beta_3 \cdot x^3)$
     $= q \cdot p_1 + q \cdot p_2$
     $= T(p_1) + T(p_2)$

- Homogenity:
  $p_1 = \alpha_0 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3$
  $T(\lambda p_1) = T(\lambda(\alpha_0 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3))$
  $= T(\lambda\alpha_0 + \lambda\alpha_1 \cdot x + \lambda\alpha_2 \cdot x^2 + \lambda\alpha_3 \cdot x^3)$
  $= q(\lambda\alpha_0 + \lambda\alpha_1 \cdot x + \lambda\alpha_2 \cdot x^2 + \lambda\alpha_3 \cdot x^3)$
  $= \lambda q(\alpha_0 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3)$
  $= \lambda T(p_1)$

(f) Let V be a vectors space, $T : V \to V, T(u) = w, \forall u \in V$, where $w \in V$ is a fixed non-zero vector.
Solution:
$T(\theta) = w \neq \theta$, hence it is not a linear transform.

(g) Consider the vector space $V = \{f : \mathbb{R} \to \mathbb{R}\}$ over $\mathbb{R}$. Let $T : V \to V, (T(f))(t) = \sin t \cdot f(t), \forall t \in \mathbb{R}, \forall f \in V$
Solution:

- Additivity:
  $T((f_1 + f_2)(t)) = \sin t \cdot (f_1 + f_2)(t) = \sin t \cdot f_1(t) + \sin t \cdot f_2(t) = T(f_1(t)) + T(f_2(t))$.
- Homogenity:
  $T(\lambda f(t)) = \sin t \lambda f(t) = \sin t(\lambda f)(t) = T((\lambda f)(t))$

(h) Let $T : \mathbb{C} \to \mathbb{C}$ be such that $T(x) = \bar{x}, \forall x \in \mathbb{C}$.
Solution:

- Additivity:
  $T(z_1 + z_2) = T((a_1 + b_1 i) + (a_2 + b_2 i))$
  $= T(a_1 + a_2 + (b_1 + b_2)i)$
  $= a_1 + a_2 - (b_1 + b_2)i$
  $= (a_1 - b_1 i) + (a_2 - b_2 i)$
  $= T(z_1) + T(z_2)$.
- Homogenity:
  $T(\lambda z) = T(\lambda(a + bi)) = T(\lambda a + \lambda bi) = \lambda a - \lambda bi = \lambda(a - bi) = \lambda T(z)$.

2. Let $T : U \to V$ be a linear transformation between vector spaces U and V. Show that, if W is a subspace of U, then the image $T(W)$ will be a subspace of V.
Solution:
The three conditions that confirm that W is a subspace of U are:

- $\theta \in W$
- $\forall x, y \in W, x + y \in W$
- $\forall \alpha \in F, \forall x \in W, \alpha \cdot x \in W$

Now we show that $T(W)$ is a subspace of V.

- $\forall x, y \in W$, $T(x + y) = T(x) + T(y)$, because T is a linear transform. Thus, $T(W)$ is closed under vector addition as $T(x)$ and $T(y)$ belong to $T(W)$, since $x, y \in W$ and we know that $T(x + y) = T(x) + T(y)$, and since W is a subspace, $x + y \in W$, and thus $T(x + y) \in T(W)$ but we know that $T(x + y) = T(x) + T(y)$, where $T(x), T(y) \in T(W)$.

- $\forall x \in W, T(\alpha \cdot x) = \alpha \cdot T(x)$, since T is a linear transform. However, since W is a subspace, $x, \alpha \cdot x \in W$ and thus $T(x), T(\alpha \cdot x) \in T(W)$, and $T(\alpha \cdot x) = \alpha \cdot T(x)$, thus $T(W)$ is closed under scalar multiplication.
- Since, W is a subspace it contains θ and for a linear transform $T(\theta) = \theta$, and thus $\theta \in T(W)$.

Since all three conditions hold, $T(W)$ is a subspace.

3. Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined as $T(A) = BA - AB, \forall A \in \mathbb{R}^{n\times n}$, where $B \in \mathbb{R}^{n\times n}$ is a fixed invertible matrix. Is T a linear transformation? Is it an isomorphism?
Solution:
Image of additive identity:
$T(\theta) = B\theta - \theta B = \theta - \theta = \theta$. Thus the image of the identity is the identity.

Additivity:
$T(A_1 + A_2) = B(A_1 + A_2) - (A_1 + A_2)B = BA_1 - A_1B + BA_2 - A_2B = T(A_1) + T(A_2)$.

Homogenity:
$T(\lambda A) = B(\lambda A) - (\lambda A)B = \lambda(BA) - \lambda(AB) = \lambda(BA - AB) = \lambda T(A)$.

Thus it is a linear transform.

It is an isomorphism if and only if θ has a unique preimage.

$BA - AB = \theta$.
$BA = AB$
Thus, for any A that commutes (w.r.t matrix multiplication) with B, $T(A) = \theta$. Simple examples of such matrices are $A = B^K, k = 1, 2, \ldots$. Thus, T is not an isomorphism.

4. Suppose V is a finite dimensional vector space over $\mathbb{R}$ and U is a non-trivial subspace of V. Corresponding to any vector v in the vector space V, we define the set $S_v(U) = \{v + u | u \in U\}$.

   (a) Show that any two such sets are either identical or disjoint.
   Solution:
   We define a relation R between two elements $x, y \in V$, and put $(x, y) \in R$ if and only if $x \in S_y$. We will prove that this is an equivalence relation and this from a basic theorem of discrete mathematics, implies a partition of V. Thus any two sets are either identical or disjoint.

   - Reflexive:
     We show that $x \in S_x$. This is clear because $x + \theta = x$ and $\theta \in U$, and thus $(x, x) \in R, \forall x \in V$.
   - Symmetric:
     We show that if $x \in S_y$, then $y \in S_x$. $x \in S_y$ means $x = y + u$, for some $u \in U$. It follows that $y = x + u^{-1}$ and we know that $u \in U \Rightarrow u^{-1} \in U$. Thus $y \in S_x$.
   - Transitive:
     Suppose $x \in S_y$ and $y \in S_z$. Thus, $x = u_1 + y$ and $y = u_2 + z$, for $u_1, u_2 \in U$. It means $x = u_1 + (u_2 + z) = (u_1 + u_2) + z$ and since $u_1 \in U$ and $u_2 \in U$, $u_1 + u_2 \in U$. Thus, $x \in S_z$.

   (b) Show that no such set is closed under vector addition, unless it is created by using an element of U.

solution:

Suppose $x, y \in S_v$, $x \neq y$ and $v \notin U$. We wish to argue that $x + y \notin S_v$. For some $u_1, u_2 \in U$, $x = v + u_1, y = v + u_2$. Thus $x + y = (v + u_1) + (v + u_2) = v(v + u_1 + u_2) = v + (v + u_3)$. Here $u_3 = u_1 + u_2$ is some element in U. $v + u_3 \in U$ if and only if $v \in U$. Thus, $x + y \in S_v$ when $x \in S_v$ and $y \in S_v$, if and only if $v \in U$.

(c) Let S_1, S_2 be two such sets (possibly identical). Define $S_1 + S_2 = \{s_1 + s_2 | s_1 \in S_1, s_2 \in S_2\}$. Show that any such sum of two sets is also a set generated in this way.

Solution:

$S_1 + S_2 = \{x + u + y + u | u \in U\} = \{(x + y) + 2u | u \in U\} = \{x + y + u | u\}$. Thus the sum of the sets associated with elements x and y is the set associated with their sum $x + y$.