Chapter 2

Lecture 15

Linear Time-Invariant (LTI) Systems.

Motivation: →

Chapter 1 :-

Linearity vs. Nonlinear.
TI vs TV
LTI Stable vs. Unstable
Causal vs. noncausal
Invertible vs. non-invertible.
Memory vs. memoryless

Chapter 2 → To study a specific class of systems that linear and time-invariant (LTI)

Why LTI ?? ⇒ Many practical systems of engineering importance can be reasonably modeled well as LTI.

Goal 1 ⇒ To develop a mathematical model or an analytical tool to describe property of LTI system.

Goal 2 : Use analytical tool of LTI system to model many physical systems.

Example of Systems representing LTI model.

1) Speech Production Mechanism

2) Seismic signal recording

3) Signal transmission over communication channel.

4) Reverberant acoustic room.

⋮ ⋮ ⋮ ⋮ ⋮

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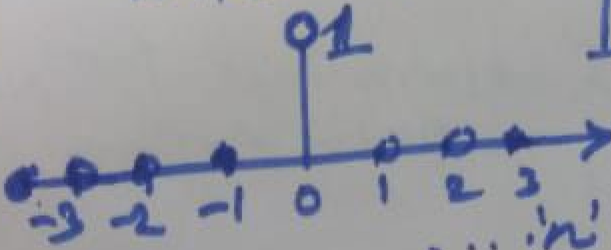
Concept of Impulse Response: → Output of a system to impulse-like excitation is called as impulse response

Impulse input

DTS

$$x[n] = \delta[n] \longrightarrow \boxed{\text{System } D\{\cdot\}} \longrightarrow y[n]$$



$y[n] = D\{x[n]\}$

If $x[n] = \delta[n]$

then $y[n] = h[n]$

$y[n] = D\{x[n]\}$

$$\boxed{h[n] = D\{\delta[n]\}}$$

Impulse response.

CTS

$$x(t) = \delta(t) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y(t) = H\{x(t)\}$$

$$\boxed{h(t) = H\{\delta(t)\}}$$

Impulse response.

Tutorial Problems on Impulse Response

1) Find the impulse response of following systems.

i) 3-point MA system.
(Noncausal)

$$x(n) \longrightarrow \boxed{\text{3-point MA system}} \longrightarrow y(n)$$

Step I) Find input-output relationships for a given system.

$$y(n) = \frac{1}{3}\sum_{k=-1}^{+1} x(n-k)$$

Step II) For impulse response, set $x(n) = \delta(n)$

$$h(n) = \frac{1}{3}\sum_{k=-1}^{+1} \delta(n-k)$$

$$h(n) = \frac{1}{3}\left[\delta(n+1) + \delta(n) + \delta(n-1)\right]$$

$$h(0) = h(n)\Big|_{n=0} = \frac{1}{3}\left[\delta(1) + \delta(0) + \delta(-1)\right]$$

$$= \frac{1}{3}\left[0 + 1 + 0\right]$$

$$= \frac{1}{3}$$

$h(0) = 1/3$, $h(-1) = 1/3$, $h(-2) = 0$, $h(-3) = 0$; $h(1) = 1/3$, $h(2) = 0$, $h(3) = 0$

(4)

③ Non-causal, 5-point MA system

$$h(n) = \begin{cases} 1/5 & n = -2, -1, 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

④ Causal - 5-point MA system.

$$h(n) = \begin{cases} 1/5 & n = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

5) Non causal 7-point MA system

6) Causal → 7 point MA system

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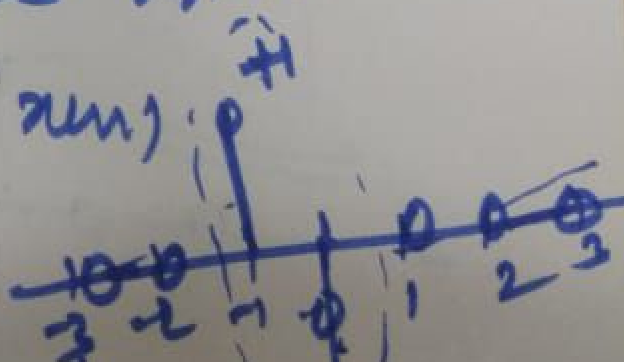
⑦ Forward difference system

↳ $y(n) = x(n+1) - x(n)$

$h(n) = \delta(n+1) - \delta(n)$

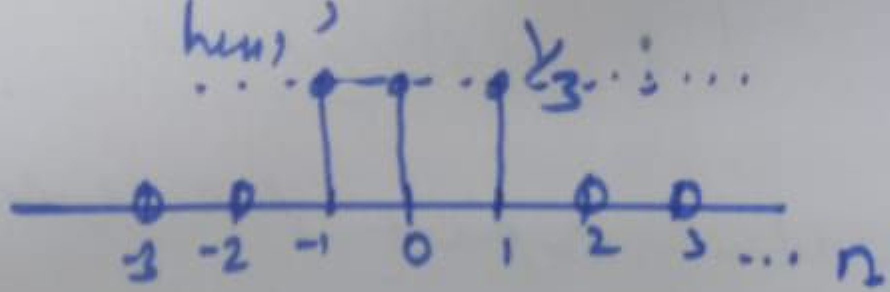


$h(0) = -1$; $h(-1) = 1$

$h(1) = 0$ | $h(-2) = 0$

$h(2) = 0$ | ⋮

$$h[n] = \begin{cases} 1/3, & n = -1, 0, 1 \\ 0 & \text{elsewhere} \end{cases}$$



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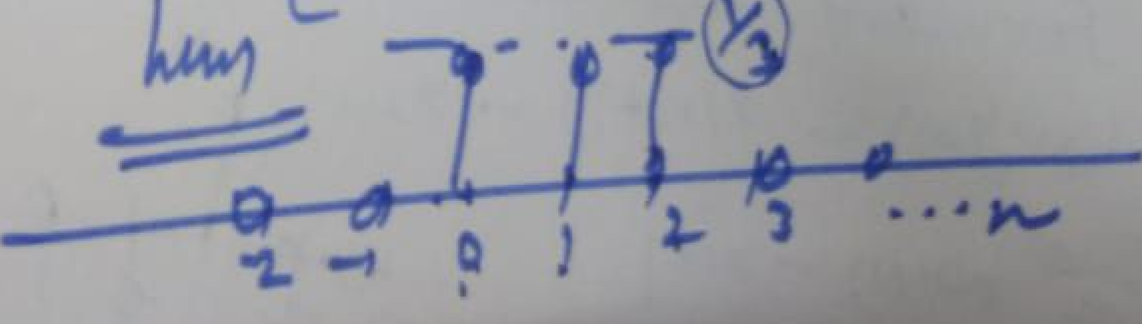
2) Causal 3 point MA system

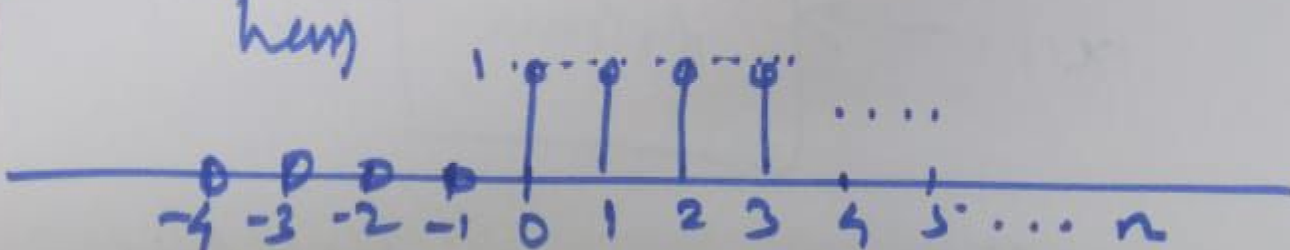
$$y[n] = \frac{1}{3} \sum_{k=0}^{2} x[n-k]$$

$\downarrow$

$$h[n] = \begin{cases} 1/3 & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$



$$h(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$\therefore \boxed{h(n) = u(n)}$$

Problem Find h(n) for finite-difference system.

Soln: $x(n) \rightarrow$ ~~Finite~~ Backward difference system $\rightarrow y(n)$

Step I) I/O relationship

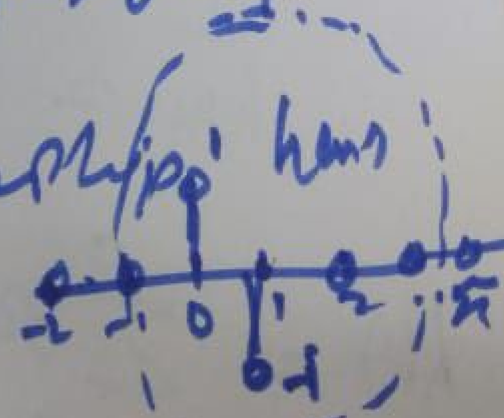
$$y(n) = x(n) - x(n-1).$$

Step II) for h(n), set $x(n) = \delta(n)$

$$\therefore h(n) = \delta(n) - \delta(n-1)$$

(2)

$h(0)=1$, $h(1)=-1$
$h(-1)=0$, $h(2)=0$
$h(-2)=0$, $h(3)=0$



7) Find the impulse response of an accumulator system.

$$x(n) \longrightarrow \boxed{\text{Accumulator system}} \longrightarrow y(n)$$

Step I) Find input-output relationship.

$$\therefore \; y(n) = \sum_{k=-\infty}^{n} x(k)$$

Step II) For impulse response set $x(n) = \delta(n)$

$$\therefore \; h(n) = \sum_{k=-\infty}^{n} \delta(k)$$

$$h(0) = \sum_{k=-\infty}^{0} \delta(k) = \underbrace{\cdots + \delta(-2) + \delta(-1)}_{0} + \underbrace{\delta(0)}_{1}$$

$h(0) = 1$ | $h(-1) = ?\ 0$; $h(1) = 1$

$h(-2) = ?\ 0$; $h(2) = 1$

$h(-3) = ?\ 0$; $\vdots$

$\vdots \quad 0$; $h(n) = 1 \quad n \geq 0$

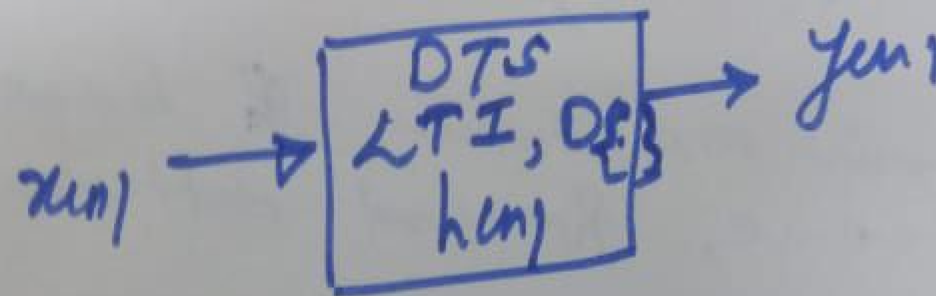
Systems having impulse response as non-zero values for finite number of independent variable 'n', these systems are called as finite-duration impulse response (FIR) systems.

e.g. 3-point MA system, backward difference, forward difference, etc.

On the other hand, there are systems (such as accumulator system) whose impulse response is non-zero for infinite duration (i.e. infinite values of independent variable 'n'). Such systems are called as Infinite-duration impulse response (IIR) systems.

* Development of Analytical Tool or Mathematical Model for LTI systems :

Case I) Discrete-time systems.



Step I) Input-output relationship.

$$y[n] = D\{x[n]\}, \quad \text{— (1)}$$

where $D\{\cdot\}$ is system property, in this case, $D\{\cdot\}$ is given to be LTI

Step II) By using sifting property of impulse,

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot \delta[n-k] \quad \text{— (2)}$$

Step III: Using eqn (2) in eqn (1), we get.

$$y(n) = D\{x(n)\} = D\left\{\sum_{k=-\infty}^{+\infty} x(k)\,\delta(n-k)\right\}$$

$$y(n) = D\{\cdots + \underbrace{x(-1)\,\delta(n+1)}_{x_1(n)} + \underbrace{x(0)\,\delta(n)}_{x_2(n)} + \cdots\}.$$

Given $D\{\cdot\}$ is LTI

i.e. $D\{\cdot\}$ is linear

$$D\{x_1(n) + x_2(n)\} = D\{x_1(n)\} + D\{x_2(n)\}$$

[Additivity] (3).

$$\therefore y(n) = \cdots + D\{x(-1)\,\delta(n+1)\} + D\{x(0)\,\delta(n)\} + \cdots$$

$$D\{\underbrace{x(-1)}_{\text{scalar}} \cdot \underbrace{\delta(n+1)}_{\text{signal}}\} = x(-1) \cdot D\{\delta(n+1)\}.$$

[Homogeneity].

(11)

$y(n) = D\{x(n)\}$

$\therefore D\{\delta(n)\} = h(n)$

$\because D\{\cdot\}$ is Time-invariant as well.

$\therefore D\{\delta(n+1)\} = h(n+1)$

$D\{\delta(n-1)\} = h(n-1)$

$\vdots \qquad \vdots \qquad \vdots$

$D\{\delta(n-k)\} = h(n-k)$

$\therefore$ Eqn (B) becomes,

$y(n) = \cdots + x(-1)\,h(n+1) + x(0)\,h(n) + x(1)\cdot h(n-1) + \cdots\cdots$

$$\therefore \boxed{y(n) = \sum_{k=-\infty}^{+\infty} x(k)\cdot h(n-k)}$$

$\rightarrow$ convolution operation.

* Convolution operation is not only due to linearity or time-invariance alone rather it is due to both linearity & Time-invariance

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Problem For an LTI system, impulse response $h[n]$ is given by,

$$h[n] = \begin{cases} 1/3 & -1 \leq n \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the input-output relationship.

Sol^n: $x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n]$

$\because$ The given system is LTI,

$$\therefore \; y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k].$$

$$x[n] * h[n] = h[n] * x[n]$$

[commutative property].

$$\delta(n) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(n)\end{matrix}} \longrightarrow y(n) = x(n) * h(n)$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

Convolution is a mathematical model to describe input-output relationship for an LTI system.

If the underlying system is time-variant (TV), then

$$D\{\delta(n-k)\} \neq h(n-k) \quad [\text{TV}]$$

Let $D\{\delta(n-k)\} = h(n,k)$

$$\therefore \boxed{y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n,k)}$$

Not a convolution!!

$$\therefore y(n) = \sum_{k=-\infty}^{+\infty} h(k) \cdot x(n-k)$$

$$= \sum_{k=-1}^{+1} \left(\frac{1}{3}\right) \cdot x(n-k)$$

$$\boxed{y(n) = \frac{1}{3} \sum_{k=-1}^{+1} x(n-k)} \rightarrow \text{Desired I/O relation}$$

$\Rightarrow$ 3-point non-causal MA system.

Inference: There is a unique relationship between impulse response and input-output relationship for an LTI system system.

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Problem

$$h[n] = \begin{cases} 1/5 & n = 0, 1, 2, 3, 4 \\ 0 & \text{elsewhere.} \end{cases}$$

System : LTI

Find Input_Output relationship

Soln: →

$$y[n] = \frac{1}{5} \sum_{k=0}^{4} x[n-k].$$

$\Downarrow$

5-point MA system . [causal].

Problem

$$h[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$y[n] = x[n] - x[n-1].$$