1. For the following vector spaces V and their corresponding subsets U, find whether U is a subspace of V or not.

   (a) $V = \mathbb{R}, U = \mathbb{Q}$.
   Solution:

   - Presence of θ:
     $\theta = 0$ is indeed a rational number, thus the identity for vector addition is in the candidate subspace.
   - Closure under vector addition:
     The sum of two rational numbers is a rational number, thus this condition is satisfied.
   - Closure under scalar multiplication:
     Since the underlying field is $\mathbb{R}$, this axiom fails, as multiplying a vector corresponding to a non-zero rational number, by a irrational number from the field gives us an irrational number which is not a vector in the candidate vector space.

   This is not a subspace.

   (b) $V = \mathbb{C}$ over $\mathbb{R}$, $U = \mathbb{R}$.
   Solution:

   - Presence of θ:
     $\theta = 0$ is a real number and is thus in the subset.
   - Closure under vector addition:
     The sum of two real numbers is again a real number.
   - Closure under scalar multiplication:
     The product of a real number vector with a real number scalar from the field is again a real number vector.

   This is a subspace.

   (c) $V = \mathbb{R}^{n\times n}, U = \{A \in V, A^T = A\}$.
   Solution:

   - Presence of θ:
     θ here is the matrix of size $n \times n$ with all entries zero. This is indeed a symmetric matrix, and thus present within the subset.
   - Closure under vector addition:
     The sum of two symmetric $n \times n$ matrices is again a symmetric $n \times n$ matrix. To see this, call the two individual matrices A, B and the sum matrix C.
     $c_{j,i} = a_{j,i} + b_{j,i}$
     $= a_{i,j} + b_{i,j}$
     $= c_{i,j}$.
   - Closure under scalar multiplication:
     A scalar multiple of a symmetric matrix is also symmetric. To see this,
     $(\lambda \dot{A})_{J,i} = \lambda \cdot (A_{j,i})$
     $= \lambda \cdot (A_{i,j})$
     $= (\lambda \dot{A})_{i,j}$

This is a subspace.

(d) $V = \mathbb{R}^n$. For a fixed $A \in \mathbb{R}^{n\times n}$, $U_\lambda = \{x \in V \mid Ax = \lambda x\}$, where $\lambda \in \mathbb{R}$.
Solution:

- Presence of θ:
  $A\theta = \theta = \lambda \cdot \theta$ and thus $\theta \in U_\lambda$.
- Closure under vector addition:
  $A(x_1 + x_2) = Ax_1 + Ax_2$
  $= \lambda x_1 + \lambda x_2$
  $= \lambda(x_1 + x_2)$.
  Thus, additive closure holds.
- Closure under scalar multiplication:
  $\alpha(\lambda x) = \alpha(Ax)$
  $= (\alpha A)x$. Thus this closure also holds.

It is a vector subspace.

(e) $V = \{f : \mathbb{R} \to \mathbb{R}\}, U_{t_0} = \{f \in V \mid f(t_0) = 0\}$ for some $t_0 \in \mathbb{R}$.
Solution:

- Presence of θ:
  θ is a function that takes value 0, everywhere, thus it also takes value 0 at t_0, in particular. Thus, it belongs to the candidate subspace.
- Closure under vector addition:
  The sum of two functions that take each value 0 at t_0, also takes value 0 at t_0, since we define vector addition as pointwise addition of the function values.
- Closure under scalar multiplication:
  If a function f that takes value 0 at t_0 then the function λf takes value $\lambda \cdot 0 = 0$ at t_0. Thus a scalar multiple of a function that belongs to the candidate subspace, also belongs to the candidate subspace.

This is a subspace.

(f) Let $V = \{(x_i)_{i=0}^\infty \mid x_i \in \mathbb{R}\}$, i.e., the set of all real-valued sequence beginning at index 0. $U = \{(x_i)_{i=0}^\infty \in V \mid x_0 = a, x_1 = b, x_n = x_{n-1} + x_{n-2}, n \geq 2\}$.
Solution:

- Presence of θ:
  Clearly, for the all zero sequence, $S_i = 0 = 0 + 0 = S_{i-1} + S_{i-2}, \forall i \geq 2$. Thus θ belongs to the candidate subspace.
- Closure under vector addition:
  $z_i = x_i + y_i$
  $= x_{i-1} + x_{i-2} + y_{i-1} + y_{i-2}$
  $= x_{i-1} + y_{i-1} + x_{i-2} + y_{i-2}$
  $= z_{i-1} + z_{i-2}$ which shows that z also belongs to the candidate subspace.
- Closure under scalar multiplication:
  $\alpha x_i = \alpha(x_{i-1} + x_{i-2}) = \alpha x_{i-1} + \alpha x_{i-2}$. This also belongs to the candidate subspace.

This is a subspace.

2. For the following vector spaces V, and subspaces U, W, find $U + W$. Also find if the sum of U and W is a Direct sum or not.

(a) $V = \mathbb{R}^{n\times n}, U = \{A \in V \mid A^T = A\}, W = \{A \in V \mid A^T = -A\}$
Solution:
This is a direct sum. This can be seen from the fact that the only vector that is both symmetric and skew symmetric (the intersection of the two subspaces) is the matrix with all entries 0. A matrix that is both symmetric and skew-symmetric, $a_{i,j} = a_{j,i}$ and $a_{i,j} = -a_{j,i}$.
For any matrix A, $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$. It can be easily verified that the former is a symmetric matrix, while the latter is a skew-symmetric matrix.

(b) $V = \mathcal{P}(\mathbb{R}), U_{x_0} = \{p \in V, p(x_0) = 0\}, W_{x_1} = \{p \in V, p(x_1) = 0\}, x_0, x_1 \in \mathbb{R}, x_0 \neq x_1$.
Solution:
Take any non-zero polynomial that takes value 0 at x_0 and x_1 and its negative. Their sum yields the θ polynomial and this is clearly non-uniquely representable, since the number of polynomials taking value 0 at x_0 and x_1 is infinite. Any polynomial with factor $(x - x_0)(x - x_1)$ is such a polynomial.

(c) Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 2 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}, V = \mathbb{R}^4, U = C(A), W = C(A^T)$.
Solution:
Here the fourth row is equal to the first row and the third row is the second row minus the first row. The second row is not a multiple of the first row. It follows that the rank of this matrix is 2. The row space has as a basis $(1,2,1,0)(3,1,1,2)$
Also the first column is the sum of the third and fourth columns. The second column is two times the third column minus half the fourth column. The column space has a basis as $(1,1,0,1)(0,2,2,0)$.
The first vector in the row space basis we have given has first coordinate 1 and the only way that can be generated from the column space basis is using 1 times the first basis vector plus any multiple of the second basis vector. However, that leads to a value of 1 for the fourth coordinate which is a mismatch. Thus the first basis vector of the column space is not a part of the row space. The only possible way to get the fourth coordinate of the second basis vector of the row space from a linear combination of the other row space basis vector and both column space basis vectors is two times the first vector in the basis of the column space and any multiples of the other two. Subject to this we can see that to get a match in the first coordinate, we must add the other vector in the row space basis with multiplier 1. Now, subject to these two choices, in order to get a match on the third coordinate, we must add 0 times the second basis vector in the column space. However, with all these forced choices, we get a mismatch in the second coordinate. This shows that the two basis vectors of the row space and the two basis vectors of the column space are all linearly independent. They span all of $\mathbb{R}^4$. Since the basis vectors of the two subspaces are independent of each other, the only vector in the intersection is those that can be generated as linear combinations of each of the bases. All multipliers must be 0, from the fact that they are linearly independent, and thus, the only vector in the intersection is θ. This establishes that it is a direct sum.

(d) For the above matrix A, $V = \mathbb{R}^4, U = C(A), W = N(A)$.
Solution:
The null space equation is:

$Ax = 0$. The reduced form of the matrix after guassian elimination is:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -5 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is:

$$\begin{bmatrix} 9x_3 - 4x_4 \\ 2x_4 - 2x_3 \\ 5x_3 \\ 5x_4 \end{bmatrix}$$

Thus a basis for the null space could be $(9, -2, 1, 0), (-4, 2, 0, 1)$. Thus this is also a direct sum, since the basis vectors of the null space and the column space are independent.

3. Let U_1, U_2, U_3 be three subspaces of the vector space V. If $U_i \cap U_j = \{\theta\}, 1 \leq i, j \leq 3, i \neq j$, is the sum of these three subspaces a Direct sum?
Solution:
This need not be a direct sum. Take for example The subspaces of R^2, given by $x = 0$, $y = 0$ and $y = x$. These three subspaces each satisfy the conditions given in the question, but R^2 is not a direct sume of these three subspaces. This can be confirmed easily by working out more than on representation for any vector, including the zero vector.

4. Let V be a vector space over $\mathbb{F}$. Let $S = \{W \mid W \text{ is a subspace of } V\}$ be equipped with the following binary operations: $+$ denoting addition of subspaces in V and $\cdot$ denoting scalar multiplication defined by: $\forall a \in \mathbb{F}, \forall W \in S, a \cdot W = \{a \cdot w \mid \forall w \in W\}$, where $a \cdot w$ denotes the scalar multiplication defined on V. Is $(S, +, \cdot)$ a vector space?
Solution:
This is not a vector space, which can be established by showing that some of the axioms fail. Only one is necessary, but we enumerate all, for the sake of greater exposure to techniques:

   - Closure under vector addition operation:
     Consider any two vectors in the set, say x and y. In effect, we are talking about two subspaces (not necessarily distinct), of the vector space V. As proved in the lectures of both sections, the sum of two vector subspaces is indeed a vector space. You are also aware that it is the smallest subspace containing both these subspaces as subsets. Apart from the fact that this was proved in the lectures, you may verify it for yourself, using another result proved in the lectures on the subspace test, which consists of the three conditions:

     (a) Presence of the identity for vector addition in the set
     (b) Closure of the set under vector addition
     (c) Closure of the set under scalar multiplication

     Since the set we defined is the set of all subspaces, we see that the closure axiom for vector addition holds.

   - Existence of identity for vector addition:
     The identity element for vector addition is the subspace consisting of only the vector which is the identity for vector addition. This follows, since it is contained in every subspace and the sum of two subspaces, one of which is contained in the other is the larger subspace.

- Existence of inverses:
  This axiom fails, as the sum of two subspaces always contains both individual subspaces as subsets. The inverse when operated on an element must give us the identity element which in this case is the subspace consisting of only the identity for vector addition, in V. This will not happen in general as this subspace does not contain any other subspace within it.
- Associativity:
  Here, we are adding three subspaces. Given any triple of vectors from the three subspaces, the order of simplification doesn't impact the final answer, since the three vectors re from a vector space which satisfies the associativity axiom. Since the sum of our three subspaces is the collection of all vectors obtained by adding the vectors of triples, one from each of the three subspaces, the vector addition operation in our newly defined (candidate) vector space, is associative.
- Commutativity:
  Since the sum of two subspaces is the collection of vectors obtained by adding the vectors of a pair of vectors one from each of the two subspaces, taken over all choices of pairs; and the addition of individual vector pairs is commutative (since they are part of a vector space),the sum of subspaces operation is also commutative.
- Scalar multiplicative closure: Not only is the scalar multiple of a subsoace a subspace itself, but in fact it is either the subspace consisting of only the vector that is the additive identity (if the scalar is 0) or the original subspace, if the scalar is any other number.

  These follow because the scalar 0 times any vector gives us the identity for vector addition. Also, if we take a scalar multiple of any vector in a subspace, the resulting vector must also belong to the same subspace, as dictated by the scalar multiplicative closure property in V.
- Identity for scalar multiplication
  Here, any non zero scalar serves as identity, because multiplying the set of all vectors in a subspace by a fixed scalar α generates all vectors of the same subspace and nothing more. Suppose we wish to generate the vector v in the subspace, we just multiply the vector $\frac{1}{\alpha}v$ by the scalar α, and both these entities lie in the respective sets. In order to get the zero vector we just multiply the constant α with the zero vector.
- Distributivity axiom (two scalars and a vector):
  As we have already seen, if the the two scalars are no-zero and totalling to non-zero distributivity must hold. This is because multiplying a subspace by a non zero scalar is an idempotent operation resulting in the same subspace. If we have two non-zero constants totalling zero, this axiom fails. As the individual scalar multiples give the same vector space and adding the vector space toitself results in the same vector space. However multiplying zero will make it collapse to the vector space containing only the identity for vector addition in V. Thus, this axiom fails.
- Distributivity axiom (one scalar and two vectors)
  This axiom holds. If the scalar is 0 then the individual vector spaces or their sum each collapse to the vector subspace containing only the identity vector for addition in V. Thus the order of computation doesn't alter the result. If the scalar is non-zero, then it is an idempotent element and doesnt alter either the individual spaces nor the sum. Thus this distributivity axiom holds.

- Compatibility:
  Consider the product $\alpha_1\alpha_2x$, where x is a subspace of V. if both are non-zero. then the product is non-zero and multiplying the two individual constants in order with the subspace is two successive idempotent operations and multiplying with the product is also an idempotent operation thus the answer is the same subspace x. If either or both constants are 0, then the product is zero and multiplying x by it results in the subspace of V containing only the additive identity. This will also be the result when we eventually multiply by 0, whether it occurs first or second in the product.

  We have mentioned and established the axioms that hold and those that fail. Since there is at least one failed axiom, this is not a vector space.