SC223 - Linear Algebra

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Lecture 25

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Summary of Lecture 24

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.
- **Proposition 18:** Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.
- **Theorem 4:** Let $A \in \mathbb{R}^{m \times n}$. $N(A) + C(A^T) = N(A) \oplus C(A^T) = \mathbb{R}^n$. Similarly, $N(A^T) \oplus C(A) = \mathbb{R}^m$.
- **Module-3: Linear Transformations**
- Let U and V be vector spaces over the same field $\mathbb{F}$. A function $f : U \to V$ is said to be **Linear transformation** from U to V if

$$Additive : \forall x, y \in U, f(x + y) = f(x) + f(y)$$
$$Homogeneous : \forall a \in \mathbb{F}, \forall x \in U, f(a \cdot x) = a \cdot f(x).$$

- **Linear Operators**: A LT with same domain and co-domain vector spaces.

Vector space Homomorphism & Isomorphism

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$T : U \to V$. defined as $T(u) = \theta_V$. $\forall u \in U$

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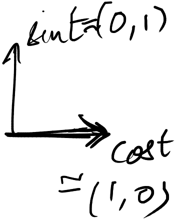
$T : U \to V$, T is invertible



$U = \text{span}(\{\cos t, \sin t\})$

$V = \mathbb{R}^2$.

Define $T(\cos t) = \begin{pmatrix}1\\0\end{pmatrix}$, $T(\sin t) = \begin{pmatrix}0\\1\end{pmatrix}$

& T is linear.

$$T(a\cos t + b\sin t) = aT(\cos t) + bT(\sin t) = \begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{R}^2.$$

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- If T is an invertible linear transformation between U and V, then we say that T is an **isomorphism** between U and V.
- **Proposition 19:** Show that two vector spaces U and V over $\mathbb{F}$ are isomorphic iff they have the same dimensions.

Proof: "$\Rightarrow$" $U \cong V$.

Let $T: U \to V$ be an ILT. betn U & V.

Let $\beta_U = \{u_1, \ldots, u_n\}$ be a basis of U.

$\{Tu_1, \ldots, Tu_n\}$

① $\{Tu_1, \dots, Tu_n\}$ spans V

Let $v \in V$. $\exists\, u \in U$ s.t $Tu = v$.

$u = \sum_{i=1}^{n} a_i u_i \Rightarrow Tu = v = \sum a_i Tu_i$

$\Rightarrow v \in \text{span}(\{Tu_1, \dots, Tu_n\})$.

② $\{Tu_1, \dots, Tu_n\}$ is LI

$\boxed{\text{"}\Leftarrow\text{"} \quad \dim(U) = \dim(V) = n \Rightarrow U \cong V.}$

$\beta_U = \{u_1, \dots, u_n\}$ ~~is~~ a basis of U

$\beta_V = \{v_1, \dots, v_n\}$ is a basis of V

~~Let~~ $Tu_1, \dots, Tu_n$

Let $x \in U$. $x = \sum_{i=1}^{n} x_i u_i$, $Tx = T\left(\sum_{i=1}^{n} x_i u_i\right)$

$Tx = \sum_{i=1}^{n} x_i Tu_i$

Define $T: U \to V$, linear as: $Tu_i = v_i$, $i = 1, \dots, n$.

$\forall y \in V$, $y = \sum y_i v_i = \sum y_i Tu_i = T\left(\sum y_i u_i\right)$

$T\left(\sum z_i u_i\right) = T\left(\sum y_i u_i\right)$

$\Rightarrow T\left(\sum (z_i - y_i) u_i\right) = \theta_V$.

Representation of Linear Transformations between FDVS

- Let $T : U \to V$ be a LT, and let $\beta_U := \{u_1, \ldots, u_n\}$ and $\beta_V = \{v_1, \ldots, v_m\}$ denote the basis of U and V respectively.

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- For $k \in \{1, \dots, m\}$, $b_k = \sum_{i=1}^{n} c_{ki} a_i$, or,

$$\underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{[y]_{\beta_V}} = \underbrace{\begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \dots & c_{mn} \end{bmatrix}}_{[T]_{\beta_U}^{\beta_V}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{[x]_{\beta_U}}$$

- The matrix $[T]_{\beta_U}^{\beta_V}$ is called the matrix representation of the linear tranformation T with respect to the basis β_U and β_V.

Examples

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- $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.
- Let $p \in \mathcal{P}_3(\mathbb{R})$ be such that $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$. Define $T_p : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ by $T_p(q) = p \cdot q, \forall q \in \mathcal{P}_3(\mathbb{R})$, where $\cdot$ represents multiplication between polynomials.