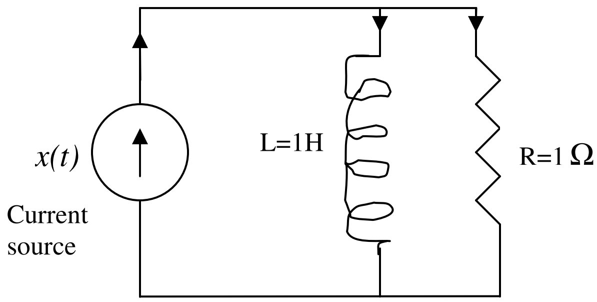
Signals and Systems (CT 203)

Tutorial Sheet-08 DA-IICT, Gandhinagar.

Problem 1:- Consider a causal LTI system implemented as the RL circuit shown in figure below. A current source produces an input current $x(t)$, and the system output is considered to be the current $y(t)$ flowing through the inductor ,



Solution:

(a) Find the differential equations relating $x(t)$ and $y(t)$.

From the above ckt voltage across the inductor is given by, $V_L = \frac{dy(t)}{dt}$.

Applying KCL to the circuit we get, $x(t) = \frac{L}{R}\frac{dy(t)}{dt} + y(t)$

R=1 and L=1,

Therefore we have, $\frac{dy(t)}{dt} + y(t) = x(t)$

(b) Determine the frequency response of this system by considering the output of the system to input of the form $x(t) = e^{j\omega t}$.

Input is an Eigen function so , $y(t) = H(j\omega)e^{j\omega t}$ hence we get

$$\frac{d}{dt}\left(H(j\omega)e^{j\omega t}\right) + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

So the system's frequency response is given by,

$$H(j\omega) = \frac{1}{1+j\omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

(c) Determine the output $y(t)$ if $x(t) = \cos(t)$.

x(t) is periodic with the period 2π

$$x(t) = \cos(t) = \frac{e^{jt} + e^{-jt}}{2} = \frac{e^{j\frac{2\pi}{2\pi}t} + e^{-j\frac{2\pi}{2\pi}t}}{2}$$

Fourier series representation of x(t) is the following

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$X_1=\frac{1}{2}, X_{-1}=\frac{1}{2}$$

Using the concept of the Fourier series and LTI system we can output as

$$y(t)=X_1 H(j\omega_0)e^{j\omega_0 t}+X_{-1}H(-j\omega_0)e^{-j\omega_0 t}$$

Since $\omega_0=2\pi/2\pi=1$

$$y(t)=\frac{1}{2}H(j)e^{jt}+\frac{1}{2}H(-j)e^{-jt} \text{ from equation (1)}$$

$$H(j)=\frac{1}{1+j}, H(-j)=\frac{1}{1-j}$$

We can get output $y(t)=\frac{1}{2}\left[\frac{e^{jt}}{1+j}+\frac{e^{-jt}}{1-j}\right]$

$$y(t)=\frac{1}{2}\left[\frac{(1-j)e^{jt}}{(1+j)}\frac{(1+j)e^{-jt}}{(1-j)}\right]$$

$$y(t)=\frac{1}{\sqrt{2}}\cos(t-\tfrac{\pi}{4})$$

Problem 2:- Consider a continuous time LTI system with impulse response $h(t)=e^{-4|t|}$.

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:-

Solution:

$$h(t)=e^{-4|t|}$$

$$h(t)=\begin{cases} e^{4t}, t<0 \\ e^{-4t}, t>0 \end{cases}$$

Taking FT, we get

$$H(j\omega)=\int_{-\infty}^{0} e^{4t}e^{-j\omega t}dt+\int_{0}^{\infty} e^{-4t}e^{-j\omega t}dt$$

$$H(j\omega)=\frac{1}{4-j\omega}+\frac{1}{4+j\omega}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

(a) $x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-n)$

$$x(t)=\sum_{n=-\infty}^{\infty}\delta(t-n)=\sum_{n=-\infty}^{\infty}\delta(t-nT), T=1$$

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}\ ,\ \ \omega_0=2\pi/T=2\pi$$

x(t) is the impulse train signal therefore X_n (i.e., it's Fourier series coefficients) will also be an impulse train,

$$X_n = \frac{1}{T} = 1, \forall n$$

From the Eigen value property of an LTI system, output y(t) can be written as

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(jn\omega_0) e^{-jn\omega_0 t}$$

where, $Y_n = X_n H(jn\omega_0) = Y_n = X_n H(jn2\pi)$ Fourier series coefficients of output signal $y(t)$,

$$\therefore Y_n = \frac{1}{4 + j2\pi n} + \frac{1}{4 - j2\pi n}$$

(b) $x(t) = \sum_{n=-\infty}^{+\infty} (-1)^n \delta(t - n)$

Here the period of the x(t), T=2; therefore $\omega_0 = 2\pi / 2 = \pi$.

$$X_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

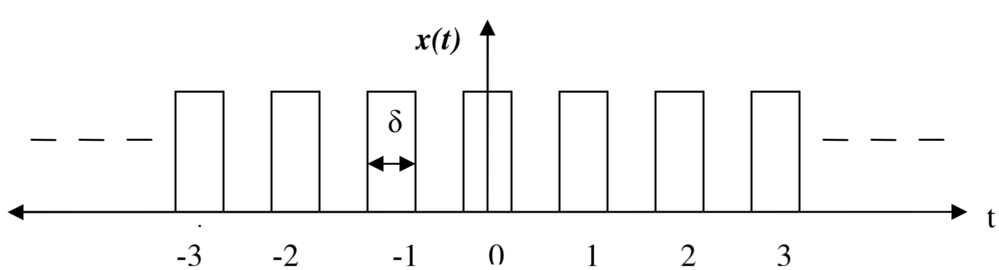
The Fourier series coefficients of the input sequence is given by

$$X_n = \begin{cases} 0, n = even \\ 1, n = odd \end{cases}$$

And the output Fourier series coefficients are,

$$Y_n = \begin{cases} 0, n = even \\ \dfrac{1}{4 + j2\pi n} + \dfrac{1}{4 - j2\pi n}, n = odd \end{cases}$$

(c) $x(t)$ is the periodic wave depicted in the figure bellow. Pulse width is $\delta = \frac{1}{2}$ and pulse height is 1.



Fourier Series coefficients for the Pulse train signal are given by the sinc function as following,

$$X_n = \left(\frac{A\delta}{T}\right)\frac{\sin\left(n\omega_0 \frac{\delta}{2}\right)}{n\omega_0}$$

$$T = 1 \Rightarrow \omega_0 = 2\pi / T = 2\pi$$

$$X_n = \begin{cases} \frac{A\delta}{T} = \frac{1}{2}, n = 0 \\ 0, n = even \\ \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}, n = odd \end{cases}$$

Output Fourier series coefficients are

$$Y_n = X_n H(jn\omega_0) = \begin{cases} \frac{1}{2}\left(\frac{1}{4 + j2\pi n} + \frac{1}{4 - j2\pi n}\right), n = 0 \\ 0, n = even \\ \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}\left[\frac{1}{4 + j2\pi n} + \frac{1}{4 - j2\pi n}\right], n = odd \end{cases}$$

3. Consider a continuous-time LTI system whose frequency response is

$$H\left(j\omega\right) = \int_{-\infty}^{+\infty} h\left(t\right) e^{-j\omega t} dt = \frac{\sin\left(4\omega\right)}{\omega}$$

If the input to this LTI system is a periodic signal

$$f\left(t\right) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$$

With period T=8, determine the corresponding system output.

Solution:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{8}\int_0^8 f(t) e^{-jn\frac{2\pi}{8}t} dt$$

$$= \frac{1}{8}\int_0^4 f(t) e^{-jn\frac{2\pi}{8}t} dt + \frac{1}{8}\int_4^8 f(t) e^{-jn\frac{2\pi}{8}t} dt$$

$$=\frac{1}{8}\int_{0}^{4}(1)e^{-jn\frac{2\pi}{8}t}dt+\frac{1}{8}\int_{4}^{8}(-1)e^{-jn\frac{2\pi}{8}t}dt$$

$$=\frac{1}{8}\left[\frac{e^{-jn\pi}-1-e^{-jn2\pi}+e^{-jn\pi}}{-jn\,{}^{2\pi}\!/_{8}}\right]$$

$$=\frac{1}{2n\pi j}\left[2-2e^{jn\pi}\right]$$

$$=\frac{1}{n\pi j}\left[1-e^{jn\pi}\right]$$

$$F_n=\begin{cases}\dfrac{2}{jn\pi}, n=0dd\\ 0, otherwise\end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty}X_nH(jn\omega_0)e^{-jn\omega_0 t}, \text{ where } \omega_0=\frac{2\pi}{T}=\frac{2\pi}{8}=\frac{\pi}{4}$$

Since F_n's are zero for the even values of n, we need to calculate $H\left(jn\omega_0\right)$ for only odd values of n those are non-zero.

$$H\left(jn\omega_0\right)=\frac{\sin\left(4n\omega_0\right)}{n\omega_0}$$

$$H\left(jn\omega_0\right)=\frac{\sin\left(4n\,{}^{\pi}\!/_{4}\right)}{n\,{}^{\pi}\!/_{4}}=0 \text{ ; n is to be odd integer}$$

$$\Rightarrow y(t)=\sum_{n=-\infty}^{\infty}X_nH(jn\omega_0)e^{-jn\omega_0 t}=\sum_{n=-\infty}^{\infty}X_n.0\,e^{-jn\omega_0 t}$$

$$\Rightarrow y(t)=0;\forall n$$