SC223 - Linear Algebra

Aditya Tatu

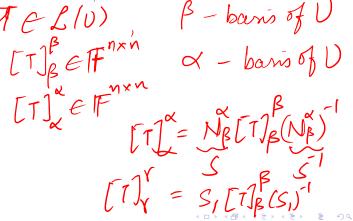
Lecture 30



October 18, 2023

Summary of Lecture 29

- **Similar matrices and Similarity transformation:** We say two matrices A and B are similar if there exists an invertible matrix, say S such that $B = SAS^{-1}$. The transformation $A \mapsto SAS^{-1}$ is said to be a similarity transformation of A by S.
- Let $T \in \mathcal{L}(U)$ be a linear operator. It is preferable to work with a basis of U, say β such that $[T]^{\beta}_{\beta}$ is diagonal/block-diagonal.



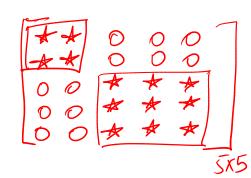
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.
 $U \in U$, $U = U_1 + U_2$.
 $Tu = Tu_1 + Tu_2$, $u_1 \in V_1$
 $u_2 \in V_2$.

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- If $T(V_1) \subseteq V_1$, $T(V_2) \subseteq V_2$, then in a basis $B = B_{V_1} \cup B_{V_2}$, $[T]_B^B = ?$

Ex: let dim (0)=5, dim
$$(V_1)=2$$
, dim $(V_2)=3$.



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 $[T]_B^B$ becomes block-diagonal.

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$$A \in \mathbb{R}^{n \times n}$$
, $A \in \mathcal{L}(\mathbb{R}^n)$
 $C(A)$, Let $y \in C(A^T)$, $Ay \in C(A)$.
 $\overline{C(A^T)}$: Let $y \in C(A^T) \Rightarrow y = A^T \times Ay = AZ$.

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- Other examples include N(T), R(T).

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$$\left(\begin{array}{c} T - \lambda I \end{array} \right) \left(\begin{array}{c} \hat{G} \end{array} \right) = \begin{array}{c} \theta \end{array}$$

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- ullet The polynomial c is called the **characteristic polynomial** of T.



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$$[T]_{\beta}^{\beta} E = E\Lambda \Rightarrow \Lambda = [T]_{F}^{E} = E^{-1} [T]_{\beta}^{\beta} E$$

• Eigenvalues of T are roots of $det([T]^{\beta}_{\beta} - \lambda I_n)$, and corresponding eigenvectors are linearly independent vectors in $\mathcal{N}(T - \lambda I)$.

Examples

 $R_{\theta}: \mathbb{R}^2 \to \mathbb{R}^2$

Results on Eigenvectors and Eigenvalues

• Proposition 19: Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T \in \mathcal{L}(U)$. Then the eigenvectors v_1, \ldots, v_m associated with these eigenvalues are linearly independent.