LECTURE 33

- $$\psi(x,t) = \int_{-\infty}^{+\infty} dk \; A(k)\, e^{i(kx - \omega t)}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \; \phi(p)\, e^{i(px - Et)/\hbar}.$$

- Free particle : $$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp\, \phi(p)\, e^{i\left(px - \frac{p^2}{2m}t\right)/\hbar}.$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp\; \phi(p)\, e^{i\left(px - \frac{p^2}{2m}t\right)/\hbar} \; e^{iV(x)/\hbar}.$$

$P(x,t)\,dx = |\psi(x,t)|^2\,dx$

- Need a way to systematically evaluate $\psi(x,t)$ for any situation.

$$\boxed{i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\,\psi(x,t)}$$

↳ Time-dependent Schrödinger eqn.

— Uncertainty associated with x, p etc.

— $|\psi(x,t)|^2$ is a probability density.

$$E[x] = \langle x \rangle = \int_{-\infty}^{+\infty} dx\, x\, |\psi(x,t)|^2$$

— In QM, p, E are intepreted as differential operators.

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad E = i\hbar \frac{\partial}{\partial t}.$$

$$\langle p \rangle = \int_{-\infty}^{+\infty} dx\, \psi^* \, p\, \psi = \frac{\hbar}{i}\int_{-\infty}^{+\infty} dx\, \psi^* \frac{\partial \psi}{\partial x} \ldots$$

$$\langle p \rangle = \frac{\hbar}{i} \cancel{\int dx\, |\psi|^2 \frac{\partial}{\partial x}} \qquad \langle p \rangle = \frac{\hbar}{i} \int_{-\infty}^{+\infty} dx\, \psi^* \frac{\partial \psi}{\partial x}.$$

$$\langle E \rangle = i\hbar \int_{-\infty}^{+\infty} dt\, \psi^* \frac{\partial \psi}{\partial t}.$$

Beiser.

$$E = i\hbar \frac{\partial}{\partial t}$$

$$p = \frac{\hbar}{i} \frac{\partial}{\partial x} \Rightarrow p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \Rightarrow \frac{p^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x,t)\psi \Rightarrow E\psi = \left(\frac{p^2}{2m} + V\right)\psi$$

Soln of Schrödinger's eqn

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x,t)\psi$$

— In many situations, $V(x,t) \equiv V(x)$.

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t).$$

— Separation of variables, $\psi(x,t) = \phi(x)T(t)$.

$$i\hbar \phi \frac{dT}{dt} = -\frac{\hbar^2}{2m} T \frac{d^2\phi}{dx^2} + V(x)\phi(x)T(t).$$

$$\Rightarrow \boxed{i\hbar \frac{1}{T}\frac{dT}{dt}} = -\frac{\hbar^2}{2m}\frac{1}{\phi}\frac{d^2\phi}{dx^2} + V(x)$$

$$\boxed{i\hbar \frac{1}{T}\frac{dT}{dt}} = \text{L.H.S.}$$

$$i\hbar \frac{dT}{dt} = E = \text{const.}$$

$$\Rightarrow \quad T = e^{-iEt/\hbar}.$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V(x)\phi(x) = E\phi(x).} \Rightarrow \text{TIME INDEPENDENT SCHRÖDINGER EQN.}$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\phi(x) = E\phi(x).$$

Solve for $\phi(x)$, $\quad \psi(x,t) = e^{-i(\text{const})t/\hbar}\,\phi(x).$

$$|\psi(x,t)|^2 = |\phi(x)|^2 .$$

$$P = \int dx \, |\phi(x)|^2 .$$

$$\hat{G} = \frac{d^2}{dx^2} .$$

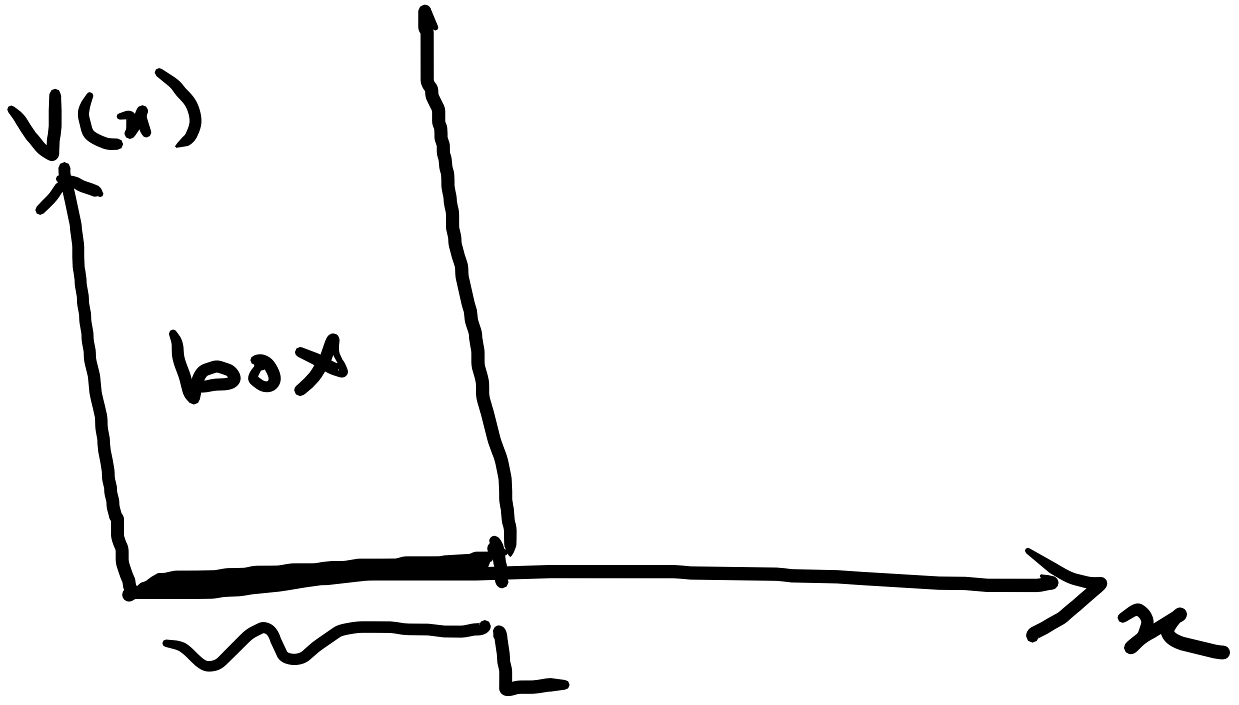
$\psi = e^{2x} . \longrightarrow$ eigenvector of $\hat{G}$

$$\hat{G}\psi = \frac{d^2}{dx^2} e^{2x} = 4 e^{2x} = 4\psi .$$

$$\left(-\frac{\hbar^2}{2m} \frac{d^2\phi}{dx^2} + V(x) \right) \phi(x) = E \phi(x)$$

▨ PARTICLE IN A BOX.

$$V(x) = 0, \quad 0 < x < L$$
$$= \infty, \quad \text{otherwise.}$$



ϕ has to be zero outside the "box", since it cannot have infinite potential energy.

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + \underbrace{V\,\phi(x)}_{=0.} = E\,\phi(x).$$

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi .$$

$$\Rightarrow \quad \frac{d^2\phi}{dx^2} + \left(\frac{2mE}{\hbar^2}\right)\phi = 0$$

$$\phi(x) = A\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + B\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) .$$

$$\phi(0) = \phi(L) = 0 .$$

$$\left.\begin{aligned} &\phi(0) = 0 \\ \Rightarrow \quad &A = 0 , \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &\phi(x) = B\sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) . \\ &\phi(L) = B\sin\left(\frac{\sqrt{2mE}}{\hbar}L\right) = 0 . \end{aligned}$$

$$\frac{\sqrt{2mE}}{\hbar} L = n\pi, \qquad n = 1, 2, \ldots \ldots$$

$$\Rightarrow \frac{2mE}{\hbar^2} L^2 = n^2 \pi^2$$

$$\Rightarrow E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}, \quad n = 1, 2, \ldots \quad .$$

↳ Energy is quantized.

$$\phi_n = B \sin\left(\frac{\sqrt{2mE}}{\hbar} x\right) = B \sin\left(\frac{n\pi}{L} x\right)., \quad n = 1, 2, 3, \ldots$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\phi_n = E_n \phi_n$$

$$\boxed{\begin{aligned}\phi_n &= \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\\ E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2}\end{aligned}}$$

$$\phi_n = B\sin\left(\frac{n\pi x}{L}\right).$$

$$\int_0^L dx\,|\phi_n|^2 = 1.$$

$$\Rightarrow B^2\int_0^L dx\,\sin^2\left(\frac{n\pi x}{L}\right) = 1.$$

$$\Rightarrow B^2\int_0^L dx\,\frac{1}{2}\left[1 - \cos\left(\frac{2\pi n x}{L}\right)\right] = 1.$$

$$\Rightarrow \frac{B^2}{2}L = 1 \Rightarrow B = \sqrt{\frac{2}{L}}$$