SC223 - Linear Algebra
Autumn 2023

In-Sem Examination 2 — October 14, 2023

Duration: 120 mins — Maximum Marks: 35

Note

1. You can use any result proved in lectures directly, but do state the result. Any other result or assumption made must be proved/justified.

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1. Let V be a finite dimensional vector space over the field $\mathbb{F}$, and let $Iso(V)$ denote the set of all isomorphisms from and to V. Considering $Iso(V)$ as a subset of $\mathcal{L}(V)$, the vector space of all linear operators on V, is $Iso(V)$ a subspace of $\mathcal{L}(V)$?

   Recall that the vector addition operation is defined as $(T_1 + T_2)(x) = T_1(x) + T_2(x), \forall x \in V$, and scalar multiplication is defined as $(\lambda T)(x) = \lambda \cdot T(x), \forall x \in V$ by the scalar. **[6]**

2. Let $A \in \mathbb{R}^{3\times 3}$ such that $A^2 \neq \mathbf{0}_{3\times 3}$, but $A^3 = \mathbf{0}_{3\times 3}$. Find the dimensions of $N(A)$,$N(A^2)$, $N(A^3)$. **[6]**

3. From the following given functions T between given vector spaces, find out which of them are linear. For the linear functions, find a matrix representation of the linear transformation. Clearly state your choice of basis, and derive the matrix appropriately. **[10]**

   (a) $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^n$ with $T(p) = r \in \mathbb{R}^n, \forall p \in \mathcal{P}_n(\mathbb{R})$, where $r = (r_1, \ldots, r_m, 0, \ldots, 0)$. The entries $r_i, 1 \leq i \leq m$ of r, with $m \leq n$ are the real roots of the polynomial p counting multiplicities. In case $m < n$, r is obtained by appending $n - m$ zeros.

   (b) Let $x_1, \ldots, x_n$ be arbitrary but fixed distinct real numbers. Let $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^n$ be defined as $T(p) = (p(x_1), \ldots, p(x_n)) \in \mathbb{R}^n$, where $p(x_i)$ is the value of the polynomial p at $x_i, i = 1, \ldots, n \in \mathbb{R}$.

   (c) Let U and V be two 2-dimensional vector spaces over the field $\mathbb{F}$. Let $T_w : \mathcal{L}(U, V) \to V$ be a function defined by $T_w(T_1) := T_1(w), \forall T_1 \in \mathcal{L}(U, V)$, where $w \in U$ is an arbitrary but fixed non-zero element.

4. Let $\mathcal{P}(\mathbb{R})$ denote the vector space of all polynomials with real number coefficients. Find the non-trivial subspace of $\mathcal{P}(\mathbb{R})$ for which applying the derivative operator followed by applying the transform of multiplying by the polynomial x, results in the same vector as the original one. **[6]**

5. Given any set S of vectors in a vector space, we define the function

$$Rep(S) = \Sigma_{v \in S} v$$

   Consider a finite dimensional vector space V of dimension n. Fix a basis $B = b_1, \ldots, b_n$. The power set of this basis set, denoted by 2^B, contains 2^n elements. We use S_n to denote arbitrary subsets of 2^B, of cardinality n. Associated with any such set, we define $B_n = \{Rep(x) | x \in S_n\}$. **[7]**

   Specify conditions on S_n such that B_n constitutes a basis of V.

Definitions and Propositions:

1. **Subspace** Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a subspace of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.

2. **Proposition:** A non-empty subset W of a vector space V is a subspace if and only if: (1) W is closed with respect to vector addition, and (2) W is closed with respect to scalar multiplication.

3. **Sum of subspaces**: Let $U_1, \ldots, U_n$ be subspaces of V. The sum of subspaces $U_1, \ldots, U_n$ is defined as: $U_1 + \ldots + U_n =: \{u_1 + u_2 + \ldots + u_n \mid \forall u_i \in U_i, i = 1, \ldots, n\}$

4. **Proposition:** The sum of subspaces $U_1, \ldots, U_n$ of V is a subspace of V.

5. **Direct Sum of Subspaces**: In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is uniquely expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$. Notation for Direct Sum: $U_1 \oplus U_2 \oplus \ldots \oplus U_n$.

6. **Proposition**: Let $U_1, \ldots, U_n$ be subspaces of V. Then $V = U_1 \oplus \ldots \oplus U_n$ if and only if: (1) $V = U_1 + \ldots + U_n$, and (2) The only decomposition of $\theta \in V$ is $(\theta, \ldots, \theta)$. The symbol θ denotes the additive identity of the vector space.

7. **Proposition**: Let V be a Vector Space with subspaces U_1, U_2. Then $V = U_1 \oplus U_2$ *iff* $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\theta\}$.

8. **Linearly independent set**: Let V be a vector space and let $W = \{v_1, \ldots, v_n\} \subset V$. We say that the set W is a set of linear independent vectors, if $a_1 v_1 + \ldots + a_n v_n = \theta \Rightarrow a_i = 0, i = 1, \ldots, n$.

9. **Proposition**: In a FDVS, the number of elements in a linearly independent set of vectors is always less than equal to the number of elements in a spanning set.

10. **Basis**: Let V be a vector space. A subset $\beta \subset V$ is said to be a Basis of V if (1) $span(\beta) = V$, and (2) β is a set of linearly independent vectors.

11. **Proposition**: A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n, a_i \in \mathbb{F}, i = 1, \ldots, n$.

12. **Dimension of a Vector Space**: Let V be a FDVS. For any set of basis vectors β of V, we define the dimension of V as $dim(V) := |\beta|$.

13. **Rank-Nullity Theorem**: Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.

14. **Proposition**: Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

15. **Theorem**: Let $A \in \mathbb{R}^{m \times n}$. $N(A) \oplus C(A^T) = \mathbb{R}^n$, and $N(A^T) \oplus C(A) = \mathbb{R}^m$.

16. **Linear Transformations**: Let U and V be vector spaces over the same field $\mathbb{F}$. A function $f : U \to V$ is said to be Linear transformation from U to V if it satisfies (1) Additivity: $\forall x, y \in U, f(x + y) = f(x) + f(y)$, and (2) Homogeneity $\forall a \in \mathbb{F}, \forall x \in U, f(a \cdot x) = a \cdot f(x)$.

17. **Isomorphic Vector Spaces/Isomorphism**: We say that two vector spaces over $\mathbb{F}$, U and V are isomorphic if there exists an invertible linear transformation between them. The invertible linear transformation between U and V is called an isomorphism.

18. **Proposition**: Two vector spaces U and V over $\mathbb{F}$ are isomorphic iff they have the same dimensions.

19. **Proposition**: The set of all linear transformations between two vector spaces U and V over the same field $\mathbb{F}$ is a vector space over the field $\mathbb{F}$.

Solutions

1. The set of all isomorphism from and to a vector space V does not form a subspace. Here's the proof:
Let $T \in Iso(V)$. Thus T is one-one and onto. Then it is also true that $-T$ (additive inverse of T in $\mathcal{L}(V)$ is also one-one and onto, implying that $-T \in Iso(V)$. Now, $T + (-T) = \Theta_{\mathcal{L}(V)}$, the zero-operator that maps every vector $v \in V$ to $\theta_V \in V$, since $\forall v \in V, (T + (-T))(v) = T(v) + (-T)(v) = T(v) + (-T(v)) = \theta_V$. This is not an isomorphism since it is neither one-one ($\Theta_{\mathcal{L}(V)}(v) = \theta_V, \forall v \in V$) nor onto ($\nexists v \in V$, such that $\Theta_{\mathcal{L}(V)}(v) = w, \forall q \neq \theta_V$).
One can arrive at the same conclusion via the scalar multiplication axiom: $\forall T \in Iso(V)$, $0 \in \mathbb{F}, 0 \cdot T = \Theta_{\mathcal{L}(V)}$, which as described above is not an isomorphism.

2. It is given that $A^2 \neq \mathbf{0}$, but $A^3 = \mathbf{0}$. This implies that $A \neq \mathbf{0}$. Since $A^3 = \mathbf{0}, \forall x \in \mathbb{R}^3, Ax = 0$. Thus $N(A^3) = \mathbb{R}^3$, and $dim(N(A^3)) = 3$.
Since $A, A^2 \neq \mathbf{0}, dim(N(A)), dim(N(A^2)) < 3$. If $dim(N(A)) = 0$, then A is invertible. But if A is invertible, so are A^2 and A^3, contradicting the given fact that $A^3 = \mathbf{0}$. Thus $dim(N(A)) > 0$.
Note that if $x \in N(A)$, then $Ax = 0$, and $A^2x = 0$. Thus $x \in N(A^2)$, thus $N(A) \subseteq N(A^2)$, and similarly $N(A^2) \subseteq N(A^3)$. We have: $0 < dim(N(A)) \leq dim(N(A^2)) < dim(N(A^3)) = 3$.
The only possibilities are (1) $dim(N(A)) = dim(N(A^2)) = 1$ or 2, or (2) $dim(N(A)) = 1, dim(N(A^2)) = 2$.
If $dim(N(A)) = dim(N(A^2))$, then since $N(A) \subset N(A^2)$, we have $N(A) = N(A^2)$. We will show that if this is true, than $N(A) = N(A^2) = N(A^3)$, which as per the question is not the case
Now, $\forall x \in N(A^3) = \mathbb{R}^3, A^3x = 0$. Re-writing gives us, $A^2(Ax) = 0$, thus $Ax \in N(A^2)$. With $N(A) = N(A^2)$, we get that $Ax \in N(A)$. Multiplying both sides by A, gives $A^2x = 0$, which implies that $x \in N(A^2)$. Thus $N(A^3) \subset N(A^2)$, which contradicts the fact that in this question, $N(A^2) \subsetneq N(A^3) = \mathbb{R}^3$.
Thus $dim(N(A)) = 1, dim(N(A^2)) = 2, dim(N(A^3)) = 3$.

3. Linear transformations and their matrix representations:

   (a) $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^n$ with $T(p) = r \in \mathbb{R}^n, \forall p \in \mathcal{P}_n(\mathbb{R})$, where $r = (r_1, \ldots, r_m, 0, \ldots, 0)$. The entries $r_i, 1 \leq i \leq m$ of r, with $m \leq n$ are the real roots of the polynomial p counting multiplicities. In case $m < n$, r is obtained by appending $n - m$ zeros.
   The question boils down to verifying whether adding two polynomials results in addition of their roots, and whether scaling a polynomial results in scaling of the roots. The second property(homogeneity) is easy to handle: Let $p(x) = (x-1)(x-2)$. we know that $T(p) = (1, 2, 0, \ldots, 0) \in \mathbb{R}^n$. But $\forall k \in \mathbb{R}, T(k \cdot p) = (1, 2, 0, \ldots, 0) \neq (k, 2k, 0, \ldots, 0) \in \mathbb{R}^n$. Thus homogeneity is not satisfied and hence T is not linear.
   One can show that T also does not satisfy additivity.
   As far as your argument is correct, whichever of two requirements: Additivity and Homogeneity, you have shown to be not satisfied, you will be awarded [2] marks.

   (b) Let $x_1, \ldots, x_n$ be arbitrary but fixed distinct real numbers. Let $T : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^n$ be defined as $T(p) = (p(x_1), \ldots, p(x_n)) \in \mathbb{R}^n$, where $p(x_i)$ is the value of the polynomial p at $x_i, i = 1, \ldots, n \in \mathbb{R}$.
   Let $x_1, \ldots, x_n$ be distinct real numbers.
   (1) **Additivity:** Now, $\forall p \in \mathcal{P}_n(\mathbb{R}), p(x_i) = p_0 + p_1x_i + p_2x_i^2 + \ldots + p_nx_i^n, \forall i = 1, \ldots, n$. Now, $\forall p, q \in \mathcal{P}_n(\mathbb{R}), (p+q)(x_i) = (p_0 + q_0) + (p_1 + q_1)x_i + (p_2 + q_2)(x_i)^2 + \ldots + (p_n + q_n)x_i^n = (p_0 + p_1x_i + p_2x_i^2 + \ldots + p_nx_i^n) + (q_0 + q_1x_i + q_2x_i^2 + \ldots + q_nx_i^n), \forall i = 1, \ldots, n$. Thus $T(p+q) = ((p+q)(x_1), (p+q)(x_2), \ldots, (p+q)(x_n)) = (p(x_1) + q(x_1), p(x_2) + q(x_2), \ldots, p(x_n) + q(x_n)) = (p(x_1), p(x_2), \ldots, p(x_n)) + (q(x_1), q(x_2), \ldots, q(x_n))$.
   (2) **Homogeneity:** $\forall k \in \mathbb{R}, \forall p \in \mathcal{P}_n(\mathbb{R}), T(k \cdot p) = ((k \cdot p)(x_1), (k \cdot p)(x_2), \ldots, (k \cdot p)(x_n)) = (k \cdot (p(x_1)), k \cdot (p(x_2)), \ldots, (k \cdot p)(x_n)) = k \cdot (p(x_1), p(x_2), \ldots, p(x_n)) = k \cdot T(p)$. Thus T is linear.

Let $\beta = \{1, x, x^2, \ldots, x^n\}$ and $\gamma = \{e_1, \ldots, e_n\}$ be the basis of $\mathcal{P}_n(\mathbb{R})$ and $\mathbb{R}^n$ respectively. Note that $e_i \in \mathbb{R}^n$ denotes the vector of length n with 1 in entry i, while the remaining entries are zeros. Now, since $T(1) = (1, \ldots, 1) \in \mathbb{R}^n, T(x) = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ldots, T(x^n) = (x_1^n, \ldots, x_n^n) \in \mathbb{R}^n$, the matrix representation of T in the chosen basis is:

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix}$$

Note that, in case you have chosen a different basis, you should end up with a different matrix.

The linearity part is worth [2] marks, deriving the matrix correctly is worth [2] marks.

(c) Let U and V be two 2-dimensional vector spaces over the field $\mathbb{F}$. Let $T_w : \mathcal{L}(U, V) \to V$ be a function defined by $T_w(T_1) := T_1(w), \forall T_1 \in \mathcal{L}(U, V)$, where $w \in U$ is an arbitrary but fixed non-zero element.

T **is linear**:

(1) **Additivity**: $\forall T_1, T_2 \in \mathcal{L}(U, V), T_w(T_1 + T_2) = (T_1 + T_2)(w) = T_1(w) + T_2(w) = T_w(T_1) + T_w(T_2)$.

(2) **Homogeneity**: $\forall k \in \mathbb{F}, \forall T_1 \in \mathcal{L}(U, V), T_w(k \cdot T_1) = (k \cdot T_1)(w) = k \cdot T_1(w) = k \cdot T_w(T_1)$.

Thus, T_w is linear. This is worth [2] marks.

In order to be able to write the matrix representation of T_w, we need to choose a basis for the domain: $\mathcal{L}(U, V)$ and co-domain V. Since $w \neq \theta_U$, let $\beta_U = \{w, x\}, \beta_V = \{y_1, y_2\}$ be the basis of U, V resp. Now let $T_1, T_2, T_3, T_4 \in \mathcal{L}(U, V)$ be defined as

$$\begin{aligned} T_1(w) = y_1, T_1(x) = \theta_V \\ T_2(w) = y_2, T_2(x) = \theta_V \\ T_3(w) = \theta_V, T_3(x) = y_1 \\ T_4(w) = \theta_V, T_4(x) = y_2 \end{aligned}$$

(1) Linear independence of $\{T_1, T_2, T_3, T_4\}$: Let $c_1, \ldots, c_4 \in \mathbb{F}$ such that $c_1T_1 + \ldots + c_4T_4 = \Theta_{\mathcal{L}(U,V)}$. Then $\forall u \in U, (c_1T_1 + \ldots + c_4T_4)(u) = \theta_V$. Since $u \in U, u = a_1w + a_2x$, thus giving $(c_1T_1 + \ldots + c_4T_4)(a_1w + a_2x) = a_1c_1T_1(w) + a_2c_1T_1(x) + a_1c_2T_2(w) + a_2c_2T_2(x) + a_1c_3T_3(w) + a_2c_3T_3(x) + a_1c_4T_4(w) + a_2c_4T_4(x) = a_1(c_1y_1 + c_2y_2) + a_2(c_3y_1 + c_4zy_2) = (a_1c_1 + a_2c_3)y_1 + (a_1c_2 + a_2c_4)y_2 = \theta_V$. Since $\{y_1, y_2\}$ are a basis, and the above equality should be true for any $a_1, a_2 \in \mathbb{F}$, this implies that $c_1 = c_2 = c_3 = c_4 = 0 \in \mathbb{F}$. Thus the set $\{T_1, \ldots, T_4\}$ is linearly independent.

(2) Span of $\{T_1, T_2, T_3, T_4\}$.

Let $T \in \mathcal{L}(U, V)$ be an arbitrary element. Let $T(w) = c_{11}y_1 + c_{21}y_2, T(x) = c_{12}y_1 + c_{22}y_2$. Then, $\forall u = a_1w + a_2x \in U$, $T(u) = a_1(c_{11}y_1 + c_{21}y_2) + a_2(c_{12}y_1 + c_{22}y_2)$. Substituting $T_1, \ldots, T_4$ appropriately gives:

$$\begin{aligned} T(u) =& a_1(c_{11}y_1 + c_{21}y_2) + a_2(c_{12}y_1 + c_{22}y_2) \\ =& a_1(c_{11}T_1(w) + c_{21}T_2(w)) + a_2(c_{12}T_3(x) + c_{22}T_4(x)) \\ =& c_{11}T_1(a_1w) + c_{21}T_2(a_1w) + c_{12}T_2(a_2x) + c_{22}T_4(a_2x) \\ =& c_{11}T_1(a_1w + a_2x) + c_{12}T_2(a_1w + a_2x) + c_{21}T_3(a_1w + a_2x) + c_{22}T_4(a_1w + a_2x) \\ =& c_{11}T_1(u) + c_{12}T_2(u) + c_{13}T_3(u) + c_{22}T_4(u), \forall u \in U \end{aligned}$$

Thus $T \in span(\{T_1, T_2, T_3, T_4\}), \forall T \in \mathcal{L}(U, V)$, giving us a basis of $\mathcal{L}(U, V)$.

Let $\beta_L = \{T_1, T_2, T_3, T_4\}$ be the basis of $\mathcal{L}(U, V)$, and $\beta_V = \{y_1, y_2\}$ be the basis of V. From the definition of $T_1, \ldots, T_4$ we know that $T_w(T_1) = y_1, T_w(T_2) = y_2, T_w(T_3) = \theta_V, T_w(T_4) = \theta_V$. Thus

$$[T_w]_{\beta_L}^{\beta_V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Common Errors in Q3:

i. Notation: $T(p(x))$. Should be $T(p)$

ii. Some students claim to have (dis)proved Additivity: $T(p+q) = T(p) + T(q)$ for the three parts in this question by simply writing the equation: $T(p+q) = T(p) + T(q)$, or $T(p+q) \neq T(p) + T(q)$, or stating the same in words. It is not (dis)proved unless you explicitly do so. Especially for part (a), additivity cannot be disproved simply by stating that roots of sum of two polynomials is not the same as sum of the roots of the two polynomials. This is exactly what you need to prove/disprove.

iii. Spelling mistake: It is *Homogeneity*, and not *Homoginity* or *Homogenity*.

4. We need to find the set $\{p \in \mathcal{P}(\mathbb{R}) \mid x \cdot \frac{d}{dx}\, p = p\}$, such that this set is a subspace of $\mathcal{P}(\mathbb{R})$. Let p be an arbitrary polynomial, $p(x) = p_0 + p_1 x + \ldots + p_n x^n$.
Let us try to solve the equation:

$$\begin{aligned} x \cdot \frac{d}{dx}\, p &= p \\ x \cdot (p_1 + 2p_2 x + 3p_3 x^2 + \ldots + np_n x^{n-1}) &= p_0 + p_1 x + p_2 x^2 + \ldots + p_n x^n \\ p_1 x + 2p_2 x^2 + 3p_3 x^3 + \ldots np_n x^n &= p_0 + p_1 x + p_2 x^2 + \ldots + p_n x^n \end{aligned}$$

Since $\{1, x, x^2, \ldots, x^n\}$ is a set of linearly independent vectors, the corresponding coefficients on both sides have to be equal, thus giving: $p_0 = 0, 2p_2 = p_2 \Rightarrow p_2 = 0, 3p_3 = p_3 \Rightarrow p_3 = 0, \ldots, p_n = 0$. Thus the set we are looking for is: $S = \{ax \mid \forall a \in \mathbb{R}\}$.
Note that $x \in S$, thus $S \neq \emptyset$. Also, for $a_1 x, a_2 x \in S, a_1 x + a_2 x = (a_1 + a_2)x \in S$, and $\forall k \in \mathbb{R}, \forall ax \in S, k \cdot ax = (ka) \cdot x \in S$. Thus S is a (non-trivial) subspace of $\mathcal{P}(\mathbb{R})$.
Mark distribution:
2 marks for coming up with the equation: $x \cdot \frac{d}{dx}\, p = p$.
3 marks for solving the equation correctly.
1 mark for showing that the set of polynomials that solve the above equation indeed forms a non-trivial subspace.

5. We need to determine if a candidate basis is indeed a basis. The candidate is an n element subset of the power set 2^B of the given basis $B = \{b_1, \ldots, b_n\}$.

   This set forms a basis if and only if the characteristic multi-vector of any subset of the original basis matches that of a disjoint such subset.

   If we order the new basis vectors as $b'_1, \ldots b'_n$, then the characteristic vector for a vector b_i of the original basis is a vector of dimension n where the j^{th} coordinate is 0, if $b_i \notin b'_j$ and 1 otherwise.

   For a subset of the original basis set, we define the characteristic multi-vector as the sum of the individual characteristic vectors.