SC223 - Linear Algebra

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Lecture 9

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Column Space and Nullspace

- **Column Space:** The set of all possible linear combinations of columns of A is called the Column space of matrix A, and is denoted by $C(A)$.

$$C(A) = \{Ax \mid \forall x \in \mathbb{R}^n\}, C(A) \subseteq \mathbb{R}^m$$

- **Nullspace:** For a matrix $A \in \mathbb{R}^{m \times n}$, the *Nullspace* is the set of vectors that get mapped to $\mathbf{0}_m$, and is denoted by $N(A)$.

$$N(A) := \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}_m\}, N(A) \subseteq \mathbb{R}^n$$

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$A : x \in \mathbb{R}^n \longmapsto Ax \in \mathbb{R}^m$$

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- **Properties:**
- $\mathbf{0}_m \in C(A), \mathbf{0}_n \in N(A)$
- $\forall b_1, b_2 \in C(A), \forall n_1, n_2 \in N(A), \forall p, q \in \mathbb{R}, pb_1 + qb_2 \in C(A), pn_1 + qn_2 \in N(A)$.

$$\begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} \xrightarrow{LC} \begin{pmatrix} x_3 \\ y_3 \\ 0 \end{pmatrix}$$

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● $C(A)$ helps to show existence of solutions, $N(A)$ helps to show existence of infinitely many solutions.

- If there are multiple solutions to $Ax = b \Rightarrow \exists z \neq \mathbf{0}_n$ such that $Az = \mathbf{0}_m$.
- Re-writing, we get $\sum_{i=1}^{n} z_i a_{*i} = \mathbf{0}_m$, with at least one non-zero entry in z, say z_k.
- $a_{*k} = \sum_{i=1, i\neq k}^{n} \frac{-z_i}{z_k} a_{*i}$. Thus we can write column k as a *linear combination* of other columns.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad z = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow Az = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow Az = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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- **Column Rank of a Matrix:** The number of linearly independent columns of a matrix is called the Column Rank.

- **Matrix Transpose:** For $A \in \mathbb{R}^{m\times n}$ given by

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

the matrix

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{a_{1n}}_{a_{1*}} & \underbrace{a_{2n}}_{a_{2*}} & \dots & \underbrace{a_{mn}}_{a_{m*}} \end{bmatrix} \in \mathbb{R}^{n\times m},$$

is called the *transpose of* A, and is denoted by A^T.

- **Beware of the notation:** a_{i*} denotes the i^{th} row of A *written as a column matrix*.

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- Since rows of A are columns of A^T, notation for rowspace is $C(A^T)$, and can be written as

$$C(A^T) = \{A^T y \mid \forall y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

Proofs & Refutations, Imre Lakatos

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- **Properties:**
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  - If $x_1, x_2 \in C(A^T)$, then $\forall p, q \in \mathbb{R}, p \cdot x_1 + q \cdot x_2 \in C(A^T)$.

Is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ LD or LI?

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- **Row Rank of a Matrix:** The number of linearly independent rows of a matrix is called the Column Rank.

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- **Summary:** Column Space, Nullspace, Row Space, Left Nullspace are four important subsets associated with any matrix A, that have *similar* properties.

Row Space and Nullspace/Column Space and Left Nullspace

- Who lives in $C(A^T) \cap N(A)$?

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$