LECTURE 11

(Ref :- Gregory :- Classical mechanics).

Recap.

- EOM for central force fields. $\vec{F} = f(r)\,\hat{r}$
  - (i) Moving to (r,θ) coordinates and deriving expressions for $\vec{v}$ and $\vec{a}$. In this approach, conservation of energy and angular momentum emerge automatically.
  - (ii) Using the fact that central force is conservative and thus preserves energy along with angular momentum EOM can be derived.

$$L = mr^2\dot{\theta} = mh = \text{const.}$$

$$E = \text{const.}$$

$r(t)$

$\theta(t)$

$$V(r) = -\int \vec{F}\cdot d\vec{r}$$

Using 2nd approach,

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\text{K.E} = T = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$T + V = E = \text{const.}$$

$$\Rightarrow \dot{r}^2 + r^2\dot{\theta}^2 = \frac{2(E-V)}{m}$$

$$\Rightarrow \dot{r}^2 + r^2\frac{L^2}{m^2r^4} = \frac{2(E-V)}{m}$$

$$\dot{r}^2 + \frac{L^2}{m^2r^2} = \frac{2(E-V)}{m}.$$

Non-trivial to solve for $r(t)$.

- Easier to determine $\underline{r \equiv r(\theta)}$

$r \equiv r(t)$

$\theta \equiv \theta(t)$.

- $$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = \frac{dr}{d\theta}\,\dot{\theta} = \frac{L}{mr^2}\left(\frac{dr}{d\theta}\right).$$

Substituting,

$$\frac{L^2}{m^2r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{L^2}{m^2r^2} = \frac{2(E-V)}{m}$$

$$\Rightarrow \frac{L^2}{m^2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] = \frac{2(E-V)}{m}$$

Qualitative features about orbit.

$$\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V = E$$

$$\Rightarrow \underbrace{\frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + V(r) = E}$$

Resembles the corr. eqn. for 1d motion with extra term $L^2/(2mr^2)$

$$V_{eff}(r) = \frac{L^2}{2mr^2} + V(r).$$

$\frac{L^2}{2mr^2}$ = angular momentum barrier.

$$\frac{1}{2}m\dot{r}^2 + V_{eff}(r) = E$$

— From EOM, (CROSS-CHECK)

$$m\ddot{r} - m r\dot{\theta}^2 = f(r)$$

$$V(r) = -\int dr\, f(r)$$

$$\Rightarrow \quad m\ddot{r} - mr \cdot \frac{L^2}{m^2 r^4} = f(r)$$

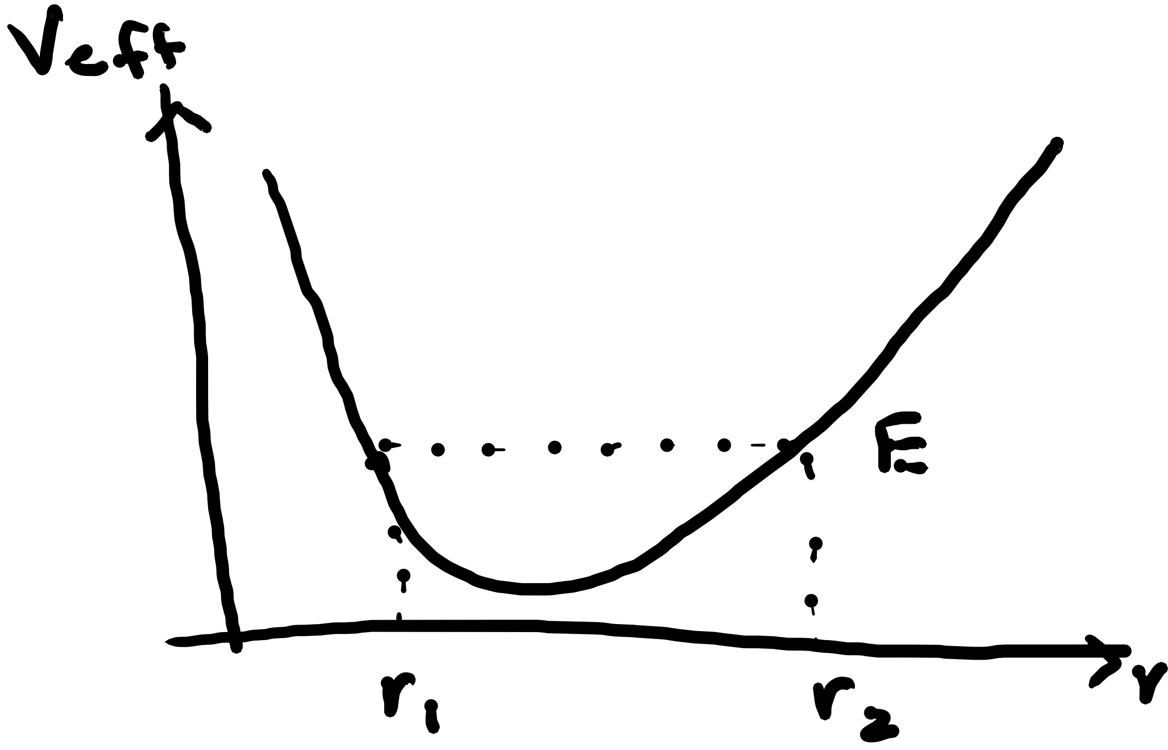
$$\Rightarrow \quad m\ddot{r} = \underbrace{\frac{L^2}{mr^3} + f(r)}.$$

$$V_{eff}(r) = \frac{L^2}{2mr^2} + V(r)$$

$$F_{eff} = -\frac{dV_{eff}(r)}{dr} = -\left(-\frac{L^2}{mr^3} - f(r)\right)$$

$$= \frac{L^2}{mr^3} + f(r)$$

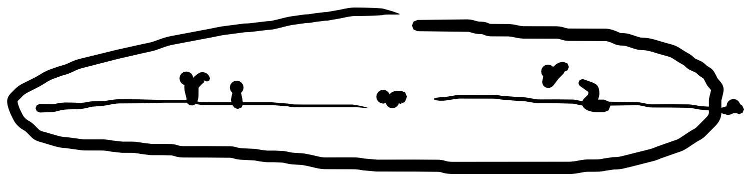
Example :- $V(r) = A r^2$



$$V_{eff}(r) = \frac{L^2}{2mr^2} + A r^2$$

$$\frac{dV_{eff}(r)}{dr} = -\frac{L^2}{mr^3} + 2Ar = 0$$

$$\Rightarrow \quad r^4 = \frac{L^2}{2Am}$$



Whenever, $E = V_{eff}$, K.E. $= 0$.

$\Downarrow$

vel. of particle $= 0$

$(r_1, r_2) \equiv$ turning points.

— If E is decreased, r_1 and r_2 approach each other.

— At minimum of V_{eff}, $r_1 = r_2 = r = \text{const}$.

Example 1: Potential satisfying a particular orbit $r = r_0 e^{a\theta}$ for a given energy $E = 0$

Since the EOM is written in terms of $\frac{dr}{dt}$,

$$\frac{dr}{dt} = r_0 a e^{a\theta} \dot{\theta} = a r \dot{\theta} = \frac{aL}{mr^2} \cdot r = \frac{aL}{mr}$$

EOM: $\frac{m}{2} \dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = E = 0$

$$\Rightarrow \frac{m}{2} \left(\frac{aL}{mr} \right)^2 + \frac{L^2}{2mr^2} + V(r) = 0$$

Solve for $V(r)$.

Example 3 : $V(r) = -V_0 e^{-\lambda^2 r^2}$

$$V_{eff}(r) = \frac{L^2}{2mr^2} - V_0 e^{-\lambda^2 r^2}$$

$$\frac{dV_{eff}(r)}{dr} = -\frac{L^2}{mr^3} + V_0 (2\lambda^2 r) e^{-\lambda^2 r^2}$$

$$\Rightarrow \quad 0 = -\frac{L^2}{mr^3} + 2V_0 \lambda^2 r e^{-\lambda^2 r^2}$$

$$\Rightarrow \quad L^2 = (2mV_0\lambda^2) r^4 e^{-\lambda^2 r^2}$$

$$\Rightarrow \quad \frac{L^2}{2mV_0\lambda^2} = r^4 e^{-\lambda^2 r^2}$$ → has maximum.