SC223 - Linear Algebra

Aditya Tatu

Lecture 22

September 26, 2023

Summary: Lecture 21

- Let $U \subset V$. U is said to **span** V if $span(U) = V$
- $W = \{v_1, \ldots, v_n\} \subset V$ is a set of linear independent vectors, if

$$a_1 v_1 + \ldots + a_n v_n = \theta \Rightarrow a_i = 0, i = 1, \ldots, n$$

- **Proposition 11:** For any FDVS, the number of vectors in a linearly independent set of vectors cannot be more than the number of vectors in a spanning set.

Basis of a Vector space

- **Definition:** (Basis) Let V be a vector space. A subset $\beta \subset V$ is said to be a **Basis** of V if (1) $span(\beta) = V$, and (2) β is a set of linearly independent vectors.

Basis of a Vector space

- **Definition:** (Basis) Let V be a vector space. A subset $\beta \subset V$ is said to be a **Basis** of V if (1) $span(\beta) = V$, and (2) β is a set of linearly independent vectors.
- Examples:

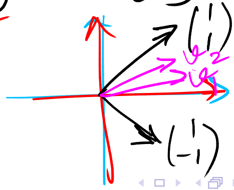
1. $V = \mathbb{R}^2$, $\beta_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. $\beta_2 = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$

$$\beta_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = k_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} k_2 = y \\ k_1 = x - y. \end{array}$$

$\mathbb{R}^2$

$$\beta_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

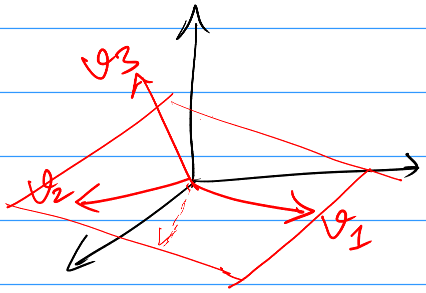


② $V = \mathbb{R}^3$

$$\beta_1 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}. \quad \beta_2 = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}.$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\mathbb{R}^3$



④ $V = \mathcal{P}(\mathbb{R})$

$$\beta_1 = \{1, x, x^2, x^3, \ldots\}.$$

$$\beta_2 = \{1, x+x^2, x^2, x^3, \ldots\}.$$

$$1\cdot(x+x^2) - 1\cdot x^2 = x.$$

⑤ $V = \mathbb{R}^{n\times n}$

CIRCULANT

$$U = \left\{ \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_{n-1} & h_0 & h_1 \cdots & h_{n-2} \\ & & \ddots & h_1 \\ h_1 & h_2 & \cdots & h_0 \end{bmatrix} \;\middle|\; \forall\, h_0, h_1, \ldots, h_{n-1} \in \mathbb{R} \right\}$$

$$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad D^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & & & & & 1 \\ 1 & 0 & \cdots & & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Any $H \in U$, $\quad H = h_0 I + h_1 D + h_2 D^2 + \cdots + h_{n-1} D^{n-1}$

$H \in \mathbb{R}^{n\times n}$, $\; x \in \mathbb{R}^n$

$$(Hx)(k) = \left((h_0 I + h_1 D + \cdots + h_{n-1} D^{n-1}) x \right)(k)$$

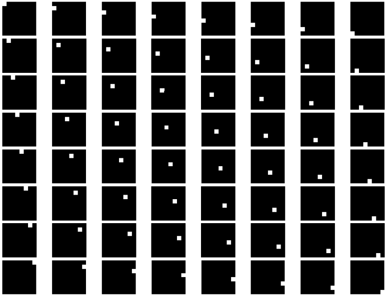
$$= \left(h_0 x + h_1 D x + \cdots + h_{n-1} D^{n-1} x\right)(k)$$

$$(Dx)(k) = x[(k+1) \bmod N].$$

$$= h_0 x(k) + h_1 x[(k+1) \bmod N] + \cdots + h_{n-1} x[(k+n-1) \bmod n]$$

Basis of a Vector space

- **Definition:** (Basis) Let V be a vector space. A subset $\beta \subset V$ is said to be a **Basis** of V if (1) $span(\beta) = V$, and (2) β is a set of linearly independent vectors.
- Examples: $V = \mathbb{R}^{8\times 8}$



Basis of a Vector space

- **Definition:** (Basis) Let V be a vector space. A subset $\beta \subset V$ is said to be a **Basis** of V if (1) $span(\beta) = V$, and (2) β is a set of linearly independent vectors.
- Examples:

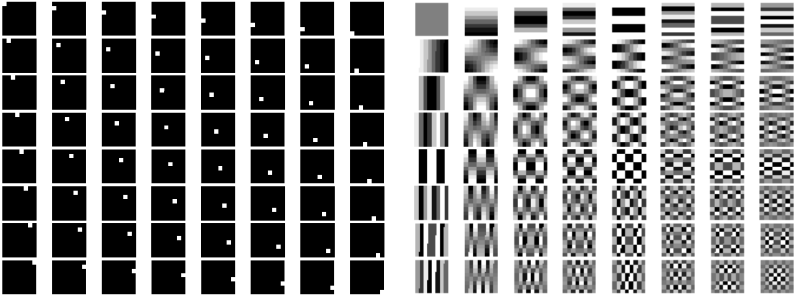


Figure: (left) Standard basis, (right) 2D-DCT basis for $\mathbb{R}^{8\times 8}$

- For speech/audio/1D signals:

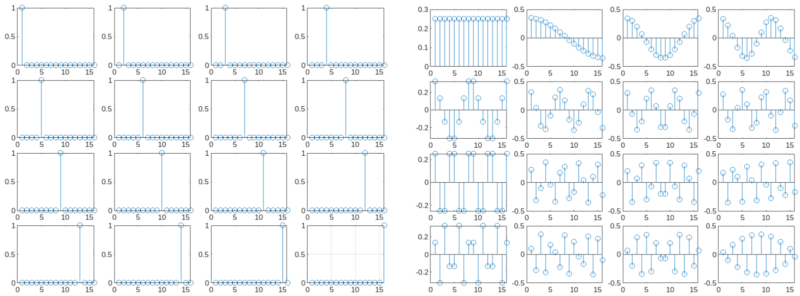


Figure: (left) Standard Basis, (right) DCT basis for $\mathbb{R}^{16}$

- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.

- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.
- **Proposition 14:** For a FDVS, every LI set can be extended to a basis.

- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.
- **Proposition 14:** For a FDVS, every LI set can be extended to a basis.
- **Proposition 15:** Every FDVS has a basis.

- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.
- **Proposition 14:** For a FDVS, every LI set can be extended to a basis.
- **Proposition 15:** Every FDVS has a basis.
- **Proposition 16:** A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n, a_i \in \mathbb{F}, i = 1, \ldots, n.$

- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.
- **Proposition 14:** For a FDVS, every LI set can be extended to a basis.
- **Proposition 15:** Every FDVS has a basis.
- **Proposition 16:** A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n, a_i \in \mathbb{F}, i = 1, \ldots, n.$. We call the vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$, the **representation of v in the basis U** and denote the same by $[v]_U$.

- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.
- **Proposition 14:** For a FDVS, every LI set can be extended to a basis.
- **Proposition 15:** Every FDVS has a basis.
- **Proposition 16:** A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n, a_i \in \mathbb{F}, i = 1, \ldots, n$.. We call the vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$, the **representation of v in the basis U** and denote the same by $[v]_U$.
- **Proposition 17:** Any set of basis vectors of a FDVS contains the same number of elements.

- **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.
- **Proposition 14:** For a FDVS, every LI set can be extended to a basis.
- **Proposition 15:** Every FDVS has a basis.
- **Proposition 16:** A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n, a_i \in \mathbb{F}, i = 1, \ldots, n.$. We call the vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$, the **representation of v in the basis** U and denote the same by $[v]_U$.
- **Proposition 17:** Any set of basis vectors of a FDVS contains the same number of elements.
- **Dimension of a Vector Space:** Let V be a FDVS. For any set of basis vectors β of V, we define the dimension of V as $dim(V) := |\beta|$.

● **Proposition 13:** For a FDVS, every spanning set can be reduced to a basis.

● **Proposition 14:** For a FDVS, every LI set can be extended to a basis.

● **Proposition 15:** Every FDVS has a basis.

● **Proposition 16:** A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n, a_i \in \mathbb{F}, i = 1, \ldots, n.$. We call the vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$, the **representation of v in the basis** U and denote the same by $[v]_U$.

● **Proposition 17:** Any set of basis vectors of a FDVS contains the same number of elements.

● **Dimension of a Vector Space:** Let V be a FDVS. For any set of basis vectors β of V, we define the dimension of V as $dim(V) := |\beta|$.

Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.

Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.
- **Proposition 18:** Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m\times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.
- **Proposition 18:** Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.
- **Theorem 4:** Let $A \in \mathbb{R}^{m\times n}$. $N(A) + C(A^T) =$

Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.
- **Proposition 18:** Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.
- **Theorem 4:** Let $A \in \mathbb{R}^{m \times n}$. $N(A) + C(A^T) = N(A) \oplus C(A^T) =$

Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.
- **Proposition 18:** Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.
- **Theorem 4:** Let $A \in \mathbb{R}^{m \times n}$. $N(A) + C(A^T) = N(A) \oplus C(A^T) = \mathbb{R}^n$.

Some beautiful results

● **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.

● **Proposition 18:** Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

● **Theorem 4:** Let $A \in \mathbb{R}^{m \times n}$. $N(A) + C(A^T) = N(A) \oplus C(A^T) = \mathbb{R}^n$. Similarly, $N(A^T) \oplus C(A) = \mathbb{R}^m$.

Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m\times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.
- **Proposition 18:** Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.
- **Theorem 4:** Let $A \in \mathbb{R}^{m\times n}$. $N(A) + C(A^T) = N(A) \oplus C(A^T) = \mathbb{R}^n$. Similarly, $N(A^T) \oplus C(A) = \mathbb{R}^m$.

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$