Lecture-19

* Linear Constant Coefficient Differential Equation (L.C.C.D.E): →

* Differential Equation (D.E) to model Physical systems.

Q.1) What a DE represent?

- Radioactive decay process → 1^{st} order DE
- Simple Harmonic Motion
- Series RLC circuit. ✓

→ 2^{nd} order DE

→ DE represents dynamics of underlying systems

For example, in chapter 1, we saw first order DE

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

- RC circuit
- automobile
- level control

$$\boxed{\frac{dy(t)}{dt} + a \cdot y(t) = b \cdot x(t).}$$

DE ⇒ mathematical model of first order systems.

Q.2) Why to solve DE ?

Sol ⇒
$$\frac{dy(t)}{dt} + a \cdot y(t) = b \cdot x(t)$$

⇒ DE represent implicit relationship between the input and the output of system.

∴ To get output $y(t)$ explicitly in terms of input, $x(t)$, we need to solve DE

Q.3) How to solve DE ?. OR what kind of solutions a DE represent??

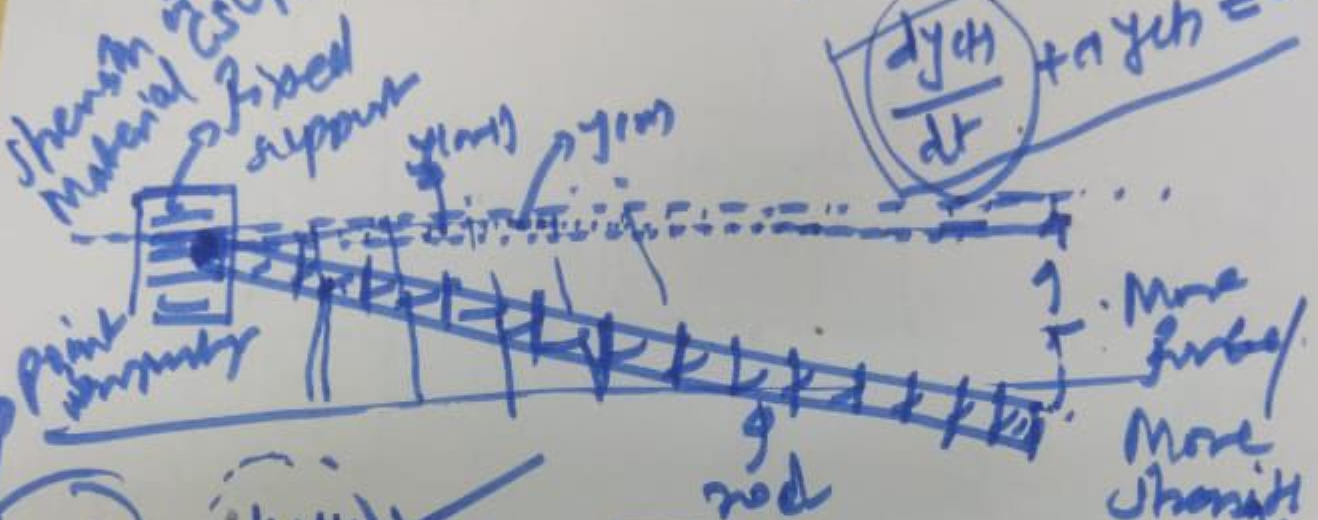
Homogenous solution
↓
General solution

Particular solution
↓
Soln of DE depends upon boundary or conditions or initial conditions

Q.4) What physical interpretation of a DE?

⇒ We claim that a DE represent "point property" of a system.

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t)$$



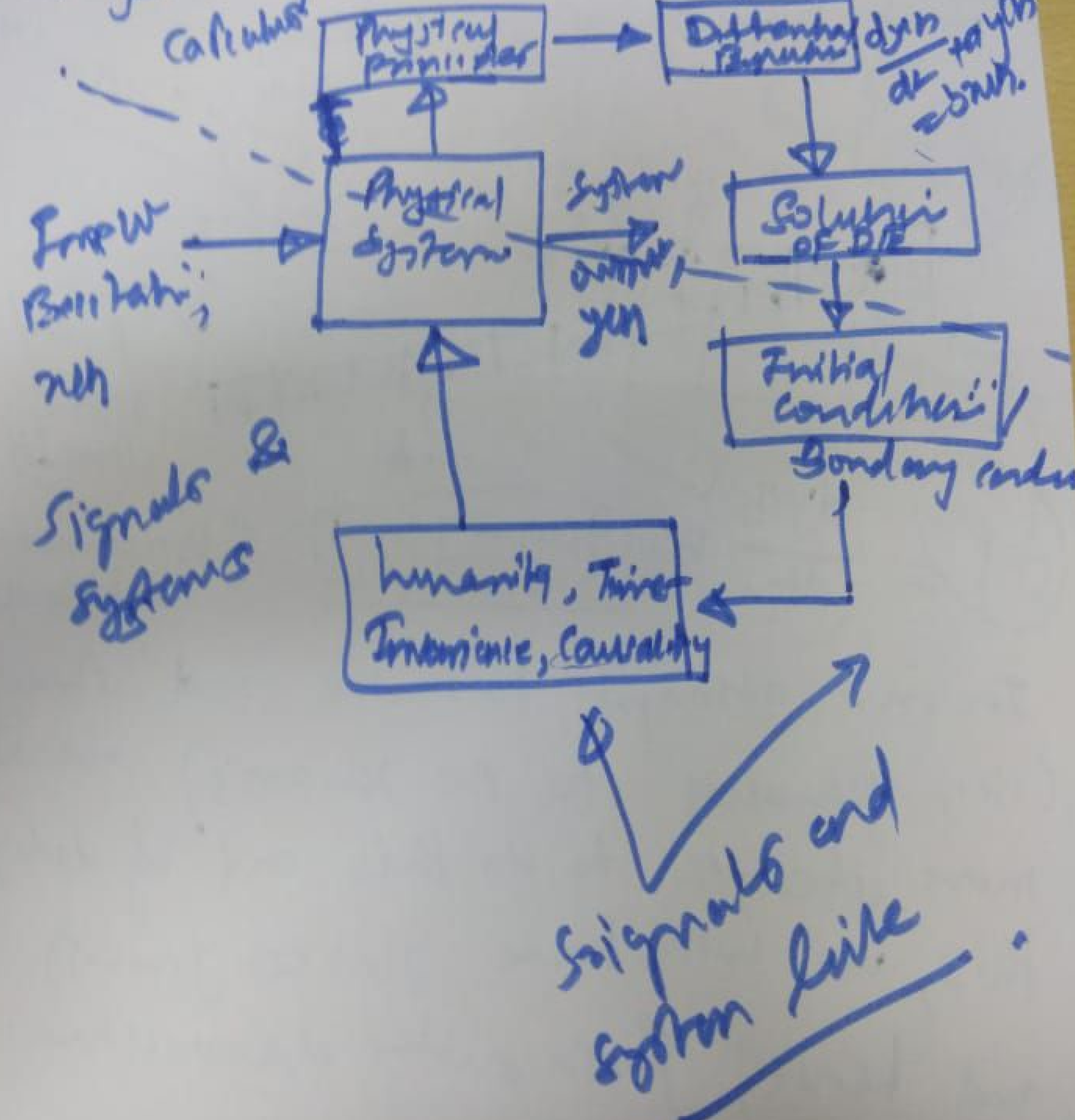
$$\text{slope} = \frac{dy(t)}{dt} \approx \overline{y(n) - y(n-1)}$$ [Backward difference]

In our attempt to make 'rod' straight (i.e., parallel to the x-axis) we need more strength to do this and by doing this, we will have $y(n) \approx y(n-1)$ and hence, first order derivative,

i.e. $$\frac{dy(n)}{dt} \approx y(n) - y(n-1) = 0$$

$$\frac{d^2y(t)}{dt^2} = \cdots = \frac{d^n y(t)}{dt^n} = 0 \quad (3)$$

* Connection between DE, solution of DE, and system properties (Linearity, TI, Causality)



* Computer simulation of Differential Equations:-

$$\boxed{\frac{dy(t)}{dt} + a \cdot y(t) = b\,x(t)}$$

Q. How to simulate this DE into computer?

$$\underline{\underline{y(n) - y(n-1)}} + \underline{\underline{a}}\, y(n) = b \cdot x(n)$$

$$(1+a) \cdot y(n) + (-1) \cdot y(n-1) = b \cdot x(n)$$

$$\boxed{a_0\, y(n) + a_1\, y(n-1) = b_0\, x(n)}$$

Difference equation for discrete-time implementation of Differential Equation

Differential Equation $\Longrightarrow$ Continuous-time systems.

Difference Equation $\Longrightarrow$ Discrete-time systems.

2nd order Differential Equations: →

$$a_0 \frac{d^2y(n)}{dt^2} + a_1 \frac{dy(n)}{dt} + a_2 y(n) = b_0 x(n) + b_1 \frac{dx(n)}{dt}$$

$$\frac{d^2y(n)}{dt^2} = \frac{d}{dt}\left(\frac{dy(n)}{dt}\right) \cong \frac{d}{dt}(y(n) - y(n-1))$$

$$\cong \frac{d}{dt}[y(n)] - \frac{d}{dt}[y(n-1)]$$

$$= [y(n) - y(n-1)] - [y(n-1) - y(n-2)]$$

$$\frac{d^2y(n)}{dt^2} \cong y(n) - 2y(n-1) + y(n-2)$$

$$a_0[y(n) - 2y(n-1) + y(n-2)] + a_1 [y(n) - y(n-1)]$$

$$+ a_2 y(n) = b_0 x(n) + b_1 (x(n) - x(n-1)).$$

$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2)$

$= b_0 x(n) + b_1 x(n-1)$

⋮

3rd order DE

4th order DE

⋮

Nth order DE

Nth order linear constant coefficient difference equation:

$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + \dots + a_k y(n-k) + \dots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)$ → DTS

$$\boxed{\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)} \quad \text{— (A)}$$

$\neq$

(2) $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \text{— (B)}$$

Nth order linear constant coefficient differential equation → CTS

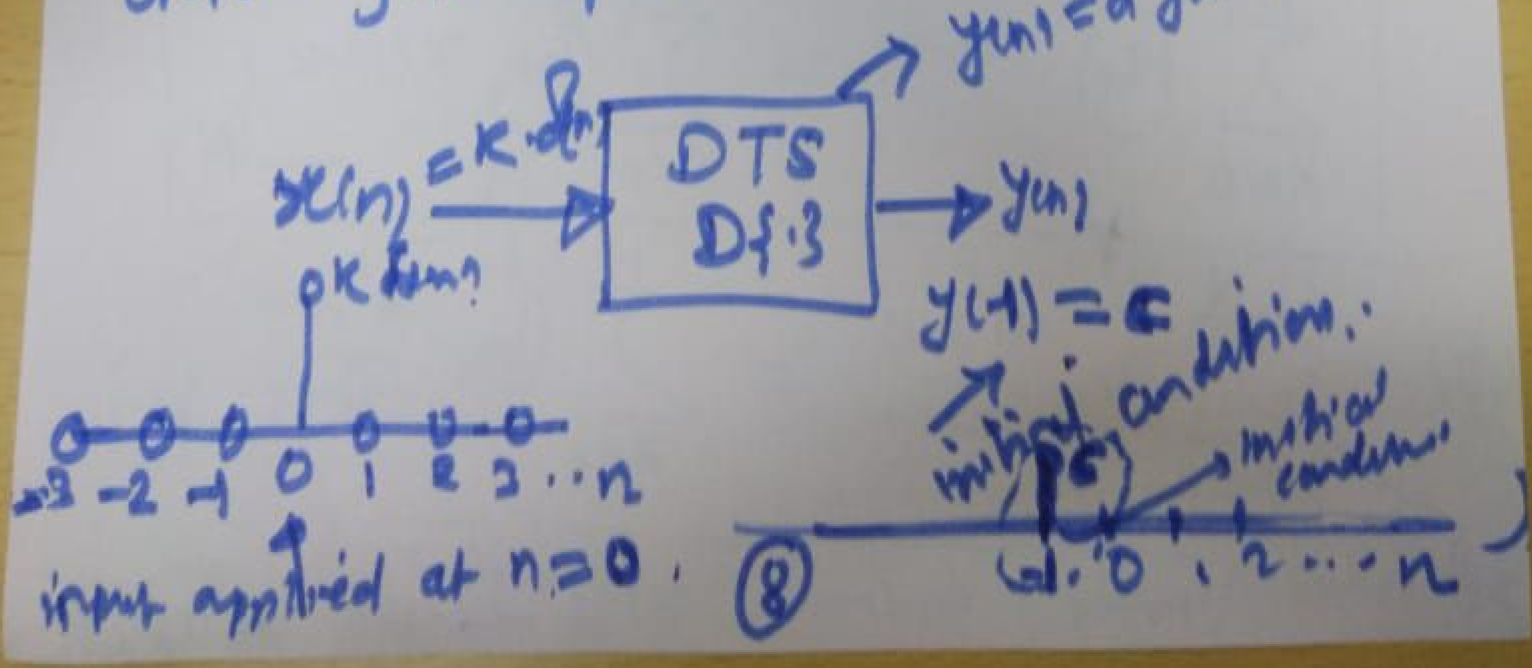
The set $\{a_k, b_k\}$ are different in eqn (A) and in eqn (B).

Problem: For a discrete-time system, the input-output relationship is described by difference equation

$$y(n) = a \cdot y(n-1) + x(n)$$

with initial condition $y(-1) = C$ and input signal, $x(n) = k \cdot \delta(n)$.

Find solution of DE and investigate significance of initial condition, $y(-1) = C$ onto system properties??



input applied at n=0.

Solution : → Solution of difference equation

Given DE : $y(n) = a\,y(n-1) + x(n)$

$n \geq 0$	$n < 0$
$y(n) = a\,y(n-1) + x(n)$	$a \cdot y(n-1) = y(n) - x(n)$
$y(n) = a \cdot y(n-1) + k \cdot \delta(n)$	$\therefore y(n-1) = \frac{1}{a}[y(n) - x(n)]$
	$n \rightarrow n+1$ Non-causal process
$y(0) = a \cdot y(0-1) + k \cdot \delta(0)$	$\therefore \boxed{y(n) = \left(\frac{1}{a}\right)[y(n+1) - x(n+1)]}$
$= a\,c + k \times 1$	put $n = -1$
$y(0) = ac + k$	$y(-1) = \left(\frac{1}{a}\right)[y(0) - x(0)]$
$y(1) = a^2 c + ak$	$y(-1) = a^{-1}[(ac+k) - k] = c$
$y(2) = a^3 c + a^2 k$	$y(-2) = a^{-1} \cdot c$
$\vdots$	$y(-3) = a^{-2} c$
$\vdots$	$\vdots$
$y(n) = a^{n+1} \cdot c + a^n \cdot k$ for $n \geq 0$	$y(n) = a^{n+1} \cdot c \;;\; n < 0$

$$\boxed{y(n) = a^{n+1} \cdot c + a^n \cdot k\,u(n) \,;\; \forall n}$$

→ Solution of Difference equation

Investigation on role of initial conditions

$y(-1) = c$?

$$y[n] = a^{n+1} \cdot c + a^n \cdot k \cdot u[n], \quad \forall n$$

1) System is causal or noncausal ??

2) $x[n] = 0 \;\forall n$

$y[n] = a^n \cdot c \neq 0 \;; \forall n$

$c \neq 0$, $y(-1) = c \neq 0$

Using proposition 1.1, the system $D\{\cdot\}$ is nonlinear.

3) Time-invariance

$y[n] = D\{x[n]\} = a^{n+1} \cdot c + a^n \cdot k\, u[n] \quad \forall n$

return

$D\{x[n-n_0]\} = a^{n+1} \cdot c + a^{n-n_0} \cdot k\, u[n-n_0]$ — (A)

$y(n-n_0) = a^{n-n_0+1} \cdot c + a^{n-n_0} \cdot k\, u[n-n_0]$ — (B)

(10)

(A1) & (B1) imply

$y(n-n_0) \neq D\{x(n-n_0)\}$

$\therefore$ Underlying system is time-variant.

$y(-1) = c = 0$ Zero initial condition

Then

$y(n) = D\{x(n)\} = \underline{a^n \cdot k \cdot u(n)}; \forall n$

1) If $x(n) = 0 \ \forall n$,

$y(n) = 0 \ \forall n.$

$\therefore$ The system $D\{\cdot\}$ is linear.

2) The Causal system

3) TI vs. TV ?? $\rightarrow D\{x(n-n_0)\} = a^{n-n_0} k \, u(n-n_0)$

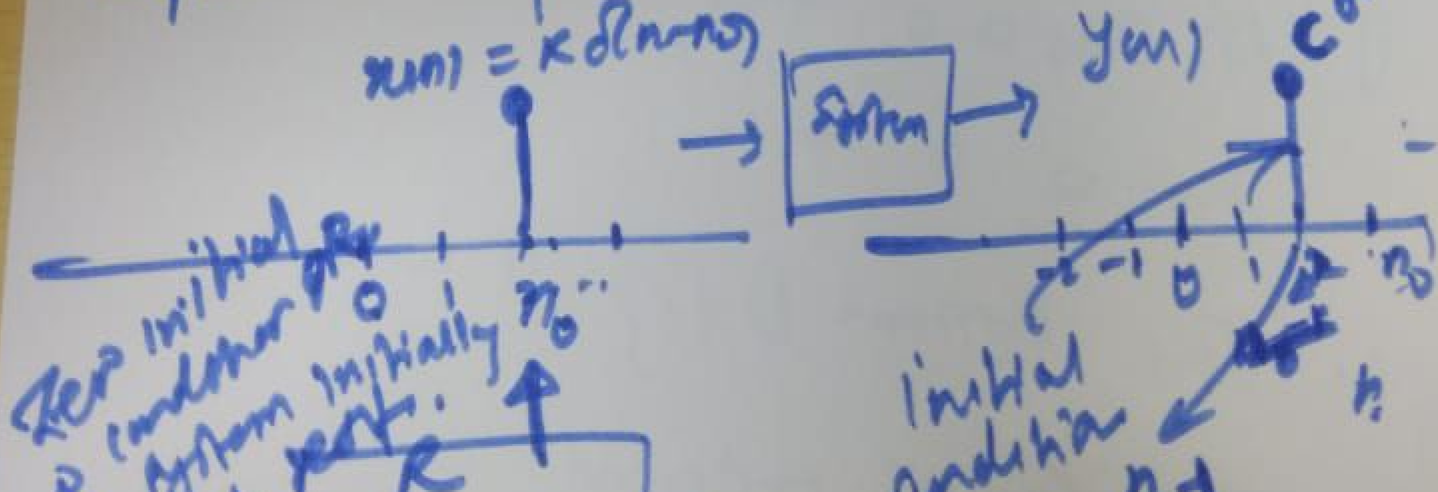
$\rightarrow y(n-n_0) = a^{n-n_0} \cdot k \, u(n-n_0)$

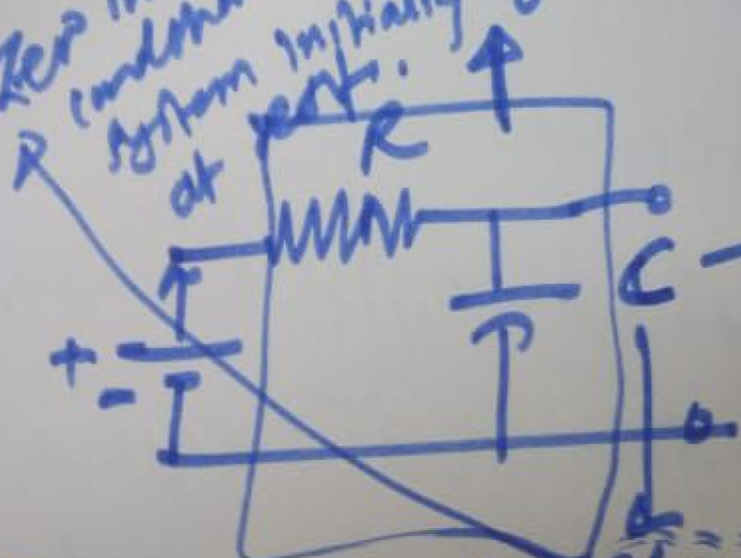
$\therefore$ System $D\{\cdot\}$ is time-invariant.

(ii)

Key result: → The initial condition (i.e., before input is applied, what is the output) dictates system properties such as linearity, time-invariance and causality.

→ Initial condition need not be limited to only y(-1) rather it depends only when input applied.





→ when capacitor is NOT discharged before application of i/p
↳ system (RC) is nonlinear ⇒ Non zero initial condition

If C discharged ⇒ ckt ⇒ linear

chapter 1

* Block Representation of DE

$$y(n) + a\,y(n-1) = b\,x(n)$$

$$\boxed{y(n) = -a\,y(n-1) + b \cdot x(n)}$$

Q. How to implement this difference equation via hardware??

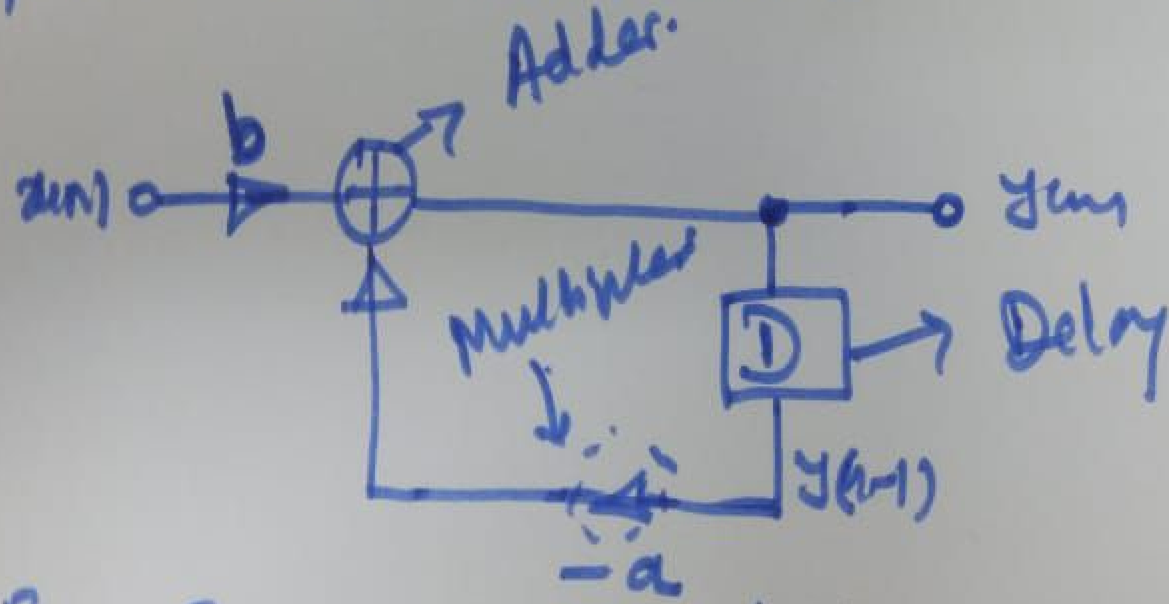


Fig. Block diag. of difference eqtn

Hardware elements: →

1) Adder
2) Delay
3) Multiplier

→ used to implement a difference equation