SC223 - Linear Algebra

Aditya Tatu

Lecture 40

November 22, 2023

Summary of Lecture 39

● **Proposition 26:** (Gram-Schmidt Procedure): Let $\{v_1, \ldots, v_m\}$ be a list of linearly independent vectors. Then there exists a list of orthonormal vectors $\{e_1, \ldots, e_m\}$ such that $span(\{v_1, \ldots, v_j\}) = span(\{e_1, \ldots, e_j\}), \forall j = 1, \ldots, m$.

● ~~Let $e_1 = \frac{v_1}{||v_1||}$. Define $e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{||v_2 - \langle v_2, e_1 \rangle e_1||}$.~~

● Let V be a FD IPS and let U be a subset of V. The **Orthogonal Complement** of U is defined as

$$U^{\perp} = \{v \in V \mid \langle v, u \rangle = 0, \forall u \in U\}$$

● Let U be a subspace of FD IPS V, and $V = U \oplus U^{\perp}$. Define $P_U \in \mathcal{L}(V)$ as $\forall v \in V$, if $v = u + w, u \in U, w \in U^{\perp}$, $P_U(v) = u$. P_U is said to be the *Orthogonal Projection Operator on U*.

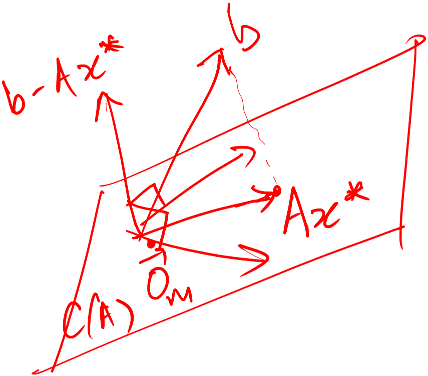
● Properties: (1) Range$(P_U) = U$, (2) Null$(P_U) = U^{\perp}$, (3) Idempotent: $(P_U)^2 = P_U$, (4) (Conjugate) Symmetric: If $V = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T = P_U$ ($P_U^* = P_U$), (5) $\forall v \in V, P_U(v) = \arg\min_{u \in U} ||u - v||^2$.

Least Squares

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $rank(A) = n$.
- Solve for $x \in \mathbb{R}^n$ in $Ax = b$ such that $b \notin C(A)$.
- Solution: $x^* = \arg\min_{x \in \mathbb{R}^n} ||Ax - b||^2$
- $Ax^* = P_{C(A)}(b)$

Least Squares

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $rank(A) = n$.
- Solve for $x \in \mathbb{R}^n$ in $Ax = b$ such that $b \notin C(A)$.
- Solution: $x^* = \arg\min_{x \in \mathbb{R}^n} ||Ax - b||^2$
- $Ax^* = P_{C(A)}(b)$, i.e., $\langle b - P_{C(A)}(b), Ax \rangle = 0, \forall x \in \mathbb{R}^n$.



Least Squares

$x^* = (A^T A)^{-1} A^T b.$

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $rank(A) = n$.
- Solve for $x \in \mathbb{R}^n$ in $Ax = b$ such that $b \notin C(A)$.
- Solution: $x^* = \arg\min_{x \in \mathbb{R}^n} ||Ax - b||^2$
- $Ax^* = P_{C(A)}(b)$, i.e., $\langle b - P_{C(A)}(b), Ax \rangle = 0, \forall x \in \mathbb{R}^n$.
- Let $a_{*i}, i = 1, \ldots, n$ denote the n column vectors. Then

$\langle b - P_{C(A)}(b), a_{*i} \rangle = 0, \forall i = 1, \ldots, n.$

$$a_{*i}^T (b - P_{C(A)}(b)) = 0, \quad i = 1, \ldots, n.$$

$$a_{*i}^T (b - Ax^*) = 0, \quad i = 1, \ldots, n.$$

$$\begin{bmatrix} - a_{*1}^T - \\ - a_{*2}^T - \\ \vdots \\ - a_{*n}^T - \end{bmatrix} \begin{bmatrix} b - Ax^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow A^T(b - Ax^*) = 0$$

Normal Equation

Least Squares

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $rank(A) = n$.
- Solve for $x \in \mathbb{R}^n$ in $Ax = b$ such that $b \notin C(A)$.
- Solution: $x^* = \arg\min_{x \in \mathbb{R}^n} ||Ax - b||^2$
- $Ax^* = P_{C(A)}(b)$, i.e., $\langle b - P_{C(A)}(b), Ax \rangle = 0, \forall x \in \mathbb{R}^n$.
- Let $a_{*i}, i = 1, \ldots, n$ denote the n column vectors. Then $\langle b - P_{C(A)}(b), a_{*i} \rangle = 0, \forall i = 1, \ldots, n$.
- $$\langle Ax^*, a_{*i} \rangle = \langle b, a_{*i} \rangle$$

Least Squares

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $rank(A) = n$.
- Solve for $x \in \mathbb{R}^n$ in $Ax = b$ such that $b \notin C(A)$.
- Solution: $x^* = \arg\min_{x \in \mathbb{R}^n} ||Ax - b||^2$
- $Ax^* = P_{C(A)}(b)$, i.e., $\langle b - P_{C(A)}(b), Ax \rangle = 0, \forall x \in \mathbb{R}^n$.
- Let $a_{*i}, i = 1, \ldots, n$ denote the n column vectors. Then $\langle b - P_{C(A)}(b), a_{*i} \rangle = 0, \forall i = 1, \ldots, n$.
-

$$\langle Ax^*, a_{*i} \rangle = \langle b, a_{*i} \rangle$$

Using Euclidean inner product

Least Squares

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $rank(A) = n$.
- Solve for $x \in \mathbb{R}^n$ in $Ax = b$ such that $b \notin C(A)$.
- Solution: $x^* = \arg\min_{x \in \mathbb{R}^n} ||Ax - b||^2$
- $Ax^* = P_{C(A)}(b)$, i.e., $\langle b - P_{C(A)}(b), Ax \rangle = 0, \forall x \in \mathbb{R}^n$.
- Let $a_{*i}, i = 1, \ldots, n$ denote the n column vectors. Then $\langle b - P_{C(A)}(b), a_{*i} \rangle = 0, \forall i = 1, \ldots, n$.
-

$$\langle Ax^*, a_{*i} \rangle = \langle b, a_{*i} \rangle$$

Using Euclidean inner product

$$(a_{*i})^T Ax^* = (a_{*i})^T b, \forall i = 1, \ldots, n$$

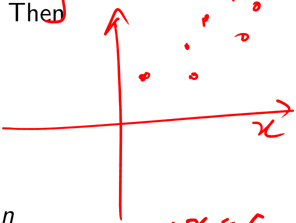
Least Squares

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix with $rank(A) = n$.
- Solve for $x \in \mathbb{R}^n$ in $Ax = b$ such that $b \notin C(A)$.
- Solution: $x^* = \arg\min_{x \in \mathbb{R}^n} ||Ax - b||^2$
- $Ax^* = P_{C(A)}(b)$, i.e., $\langle b - P_{C(A)}(b), Ax \rangle = 0, \forall x \in \mathbb{R}^n$.
- Let $a_{*i}, i = 1, \dots, n$ denote the n column vectors. Then $\langle b - P_{C(A)}(b), a_{*i} \rangle = 0, \forall i = 1, \dots, n$.
-

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c \\ m \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{b}$$

(x_i, y_i)

$i = 1, \dots, n$



$$\langle Ax^*, a_{*i} \rangle = \langle b, a_{*i} \rangle$$

Using Euclidean inner product

$$(a_{*i})^T Ax^* = (a_{*i})^T b, \forall i = 1, \dots, n$$

$$A^T Ax^* = A^T b \Rightarrow x^* = \underbrace{(A^T A)^{-1} A^T}_{A^\dagger} b,$$

$y = mx + c$

$y_i = mx_i + c$

$||Ax - b||^2 = \min \sum_{i=1}^{n} (y_i - mx_i - c)^2$

where $A^\dagger$ is known as the *Pseudo-inverse* of the matrix A.

$A^\#$

- **Theorem:** (Spectral Theorem for Hermitian Matrices) Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix, i.e., $A = A^*$.

● **Theorem:** (Spectral Theorem for Hermitian Matrices) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, i.e., $A = A^*$. Then (a) all it eigenvalues are real, (b) there exists an orthonormal basis of $\mathbb{C}^n$ containing eigenvectors of A.

$A \in \mathbb{C}^{n\times n}$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A.

$\langle x, y\rangle = y^* x$.

Let v be the associated evector

$$\underline{\langle Av, v\rangle} = \langle \lambda v, v\rangle = \underline{\lambda \|v\|^2}.$$

$$= v^* A v = v^* A^* v = \langle v, Av\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle = \underline{\bar{\lambda}\|v\|^2}$$

$\lambda_1, v_1,\ \lambda_2, v_2,\ \lambda_1 \neq \lambda_2$.

$$\underline{\langle Av_1, v_2\rangle} = \underline{\lambda_1\langle v_1, v_2\rangle} = v_2^* A v_1 = v_2^* A^* v_1 = \langle v_1, Av_2\rangle = \underline{\lambda_2\langle v_1, v_2\rangle}$$

- **Theorem:** (Spectral Theorem for Hermitian Matrices) Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix, i.e., $A = A^*$. Then (a) all it eigenvalues are real, (b) there exists an orthonormal basis of $\mathbb{C}^n$ containing eigenvectors of A.
- **Special Case:** (Spectral Theorem for Real Symmetric Matrices) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, i.e., $A = A^T$. Then (a) all it eigenvalues are real, (b) there exists an orthonormal basis of $\mathbb{R}^n$ containing eigenvectors of A.

- **Theorem:** (Spectral Theorem for Hermitian Matrices) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, i.e., $A = A^*$. Then (a) all it eigenvalues are real, (b) there exists an orthonormal basis of $\mathbb{C}^n$ containing eigenvectors of A.
- **Special Case:** (Spectral Theorem for Real Symmetric Matrices) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, i.e., $A = A^T$. Then (a) all it eigenvalues are real, (b) there exists an orthonormal basis of $\mathbb{R}^n$ containing eigenvectors of A.
- **General Case:** (Spectral Theorem for Self-Adjoint Operators) Let $T \in \mathcal{L}(V)$, where V is n-dimensional IP space. If $\forall x, y \in V, \langle Tx, y\rangle = \langle x, Ty\rangle$ then, (a) all eigenvalues of T are real, (b) there exists an orthonormal T-eigenbasis of V.

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m \times n}$ or $\mathbb{C}^{m \times n}$, can you diagonalize A?

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m \times n}$ or $\mathbb{C}^{m \times n}$, can you diagonalize A?
- What about $A^T A \in \mathbb{R}^{n \times n}$, $AA^T \in \mathbb{R}^{m \times m}$, and similarly $A^* A \in \mathbb{C}^{n \times n}$, $AA^* \in \mathbb{C}^{m \times m}$?

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, can you diagonalize A?
- What about $A^TA \in \mathbb{R}^{n\times n}$, $AA^T \in \mathbb{R}^{m\times m}$, and similarly $A^*A \in \mathbb{C}^{n\times n}$, $AA^* \in \mathbb{C}^{m\times m}$?
- Let $\{v_1, \ldots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of A^TA (or A^*A), with real eigenvalues $\lambda_1, \ldots, \lambda_n$.

$$A^TAv_i = \lambda_i v_i, \quad i = 1, \ldots, n.$$

$$A\,A^T(Av_i) = \lambda_i (Av_i), \quad i = 1, \ldots, n$$

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, can you diagonalize A?
- What about $A^T A \in \mathbb{R}^{n\times n}, AA^T \in \mathbb{R}^{m\times m}$, and similarly $A^* A \in \mathbb{C}^{n\times n}, AA^* \in \mathbb{C}^{m\times m}$?
- Let $\{v_1, \dots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of $A^T A$ (or $A^* A$), with real eigenvalues $\lambda_1, \dots, \lambda_n$.
- $$A^T A v_i = \lambda_i v_i, i = 1, \dots, n$$

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, can you diagonalize A?
- What about $A^TA \in \mathbb{R}^{n\times n}, AA^T \in \mathbb{R}^{m\times m}$, and similarly $A^*A \in \mathbb{C}^{n\times n}, AA^* \in \mathbb{C}^{m\times m}$?
- Let $\{v_1, \dots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of A^TA (or A^*A), with real eigenvalues $\lambda_1, \dots, \lambda_n$.
- $$\begin{aligned} A^TAv_i &= \lambda_i v_i, i = 1, \dots, n \\ (AA^T)Av_i &= \lambda_i(Av_i), i = 1, \dots, n \end{aligned}$$

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m \times n}$ or $\mathbb{C}^{m \times n}$, can you diagonalize A?
- What about $A^T A \in \mathbb{R}^{n \times n}, AA^T \in \mathbb{R}^{m \times m}$, and similarly $A^* A \in \mathbb{C}^{n \times n}, AA^* \in \mathbb{C}^{m \times m}$?
- Let $\{v_1, \ldots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of $A^T A$ (or $A^* A$), with real eigenvalues $\lambda_1, \ldots, \lambda_n$.
- $$A^T A v_i = \lambda_i v_i, i = 1, \ldots, n$$ $$(AA^T) A v_i = \lambda_i (A v_i), i = 1, \ldots, n$$
- Are $\{Av_1, \ldots, Av_n\}$ eigenvectors of AA^T?

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m \times n}$ or $\mathbb{C}^{m \times n}$, can you diagonalize A?
- What about $A^T A \in \mathbb{R}^{n \times n}, AA^T \in \mathbb{R}^{m \times m}$, and similarly $A^* A \in \mathbb{C}^{n \times n}, AA^* \in \mathbb{C}^{m \times m}$?
- Let $\{v_1, \dots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of $A^T A$ (or $A^* A$), with real eigenvalues $\lambda_1, \dots, \lambda_n$.
- $$A^T A v_i = \lambda_i v_i, i = 1, \dots, n$$
  $$(AA^T) A v_i = \lambda_i (A v_i), i = 1, \dots, n$$
- Are $\{Av_1, \dots, Av_n\}$ eigenvectors of AA^T?NO!

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, can you diagonalize A?
- What about $A^TA \in \mathbb{R}^{n\times n}, AA^T \in \mathbb{R}^{m\times m}$, and similarly $A^*A \in \mathbb{C}^{n\times n}, AA^* \in \mathbb{C}^{m\times m}$?
- Let $\{v_1, \ldots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of A^TA (or A^*A), with real eigenvalues $\lambda_1, \ldots, \lambda_n$.
-

$$A^TAv_i = \lambda_i v_i, i = 1, \ldots, n$$
$$(AA^T)Av_i = \lambda_i(Av_i), i = 1, \ldots, n$$

- Are $\{Av_1, \ldots, Av_n\}$ eigenvectors of AA^T?NO!
- Let $rank(A) = r$, and assume $\{Av_1, \ldots, Av_r\}$ be the eigenvectors of AA^T.

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, can you diagonalize A?
- What about $A^TA \in \mathbb{R}^{n\times n}, AA^T \in \mathbb{R}^{m\times m}$, and similarly $A^*A \in \mathbb{C}^{n\times n}, AA^* \in \mathbb{C}^{m\times m}$?
- Let $\{v_1, \ldots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of A^TA (or A^*A), with real eigenvalues $\lambda_1, \ldots, \lambda_n$.
- $$A^TAv_i = \lambda_i v_i, i = 1, \ldots, n$$
$$(AA^T)Av_i = \lambda_i(Av_i), i = 1, \ldots, n$$
- Are $\{Av_1, \ldots, Av_n\}$ eigenvectors of AA^T?NO!
- Let $rank(A) = r$, and assume $\{Av_1, \ldots, Av_r\}$ be the eigenvectors of AA^T.
- $||Av_i|| =?$.

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, can you diagonalize A?
- What about $A^T A \in \mathbb{R}^{n\times n}, AA^T \in \mathbb{R}^{m\times m}$, and similarly $A^*A \in \mathbb{C}^{n\times n}, AA^* \in \mathbb{C}^{m\times m}$?
- Let $\{v_1, \ldots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of $A^T A$ (or $A^* A$), with real eigenvalues $\lambda_1, \ldots, \lambda_n$.
-

$$A^T A v_i = \lambda_i v_i, i = 1, \ldots, n$$
$$(AA^T)Av_i = \lambda_i (Av_i), i = 1, \ldots, n$$

- Are $\{Av_1, \ldots, Av_n\}$ eigenvectors of AA^T?NO!
- Let $rank(A) = r$, and assume $\{Av_1, \ldots, Av_r\}$ be the eigenvectors of AA^T.
- $||Av_i|| =?$.
- Define $u_1 := \frac{Av_1}{\sqrt{\lambda_1}}, \ldots, u_r = \frac{Av_r}{\sqrt{\lambda_r}}$.

Singular Value Decomposition

- Given $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, can you diagonalize A?
- What about $A^TA \in \mathbb{R}^{n\times n}, AA^T \in \mathbb{R}^{m\times m}$, and similarly $A^*A \in \mathbb{C}^{n\times n}, AA^* \in \mathbb{C}^{m\times m}$?
- Let $\{v_1, \dots, v_n\}$ be the ONB of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing eigenvectors of A^TA (or A^*A), with real eigenvalues $\lambda_1, \dots, \lambda_n$.
- $$A^TAv_i = \lambda_i v_i, i = 1, \dots, n$$ $$(AA^T)Av_i = \lambda_i(Av_i), i = 1, \dots, n$$
- Are $\{Av_1, \dots, Av_n\}$ eigenvectors of AA^T?NO!
- Let $rank(A) = r$, and assume $\{Av_1, \dots, Av_r\}$ be the eigenvectors of AA^T.
- $||Av_i|| =?$.
- Define $u_1 := \frac{Av_1}{\sqrt{\lambda_1}}, \dots, u_r = \frac{Av_r}{\sqrt{\lambda_r}}$. One can extend this set to a ONB $\{u_1, \dots, u_r, \dots, u_m\}$ of $\mathbb{R}^m$ (or $\mathbb{C}^m$).

- Let $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}$. These are called the *singular values of A*.

- Let $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}$. These are called the *singular values of A*.
- Let

$$V_{n\times n} = \begin{bmatrix} | & | & | \\ v_1 & \ldots & v_n \\ | & | & | \end{bmatrix}, U_{m\times m} = \begin{bmatrix} | & | & | & | & | \\ u_1 = \frac{Av_1}{\sigma_1} & \ldots & u_r = \frac{Av_r}{\sigma_r} & \ldots & u_m \\ | & | & | & | & | \end{bmatrix}$$

- Let $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}$. These are called the *singular values of A*.
- Let

$$V_{n\times n} = \begin{bmatrix} | & & | \\ v_1 & \ldots & v_n \\ | & & | \end{bmatrix}, U_{m\times m} = \begin{bmatrix} | & & | & & | \\ u_1 = \frac{Av_1}{\sigma_1} & \ldots & u_r = \frac{Av_r}{\sigma_r} & \ldots & u_m \\ | & & | & & | \end{bmatrix}$$

- The vectors $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_m\}$ are called the *right* and *left singular vectors*.

$$\underbrace{\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_m & - \end{bmatrix}}_{A} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$x \in N(A)$

$AV =$

$$AV = A \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$$

$$=$$

$$
\begin{aligned}
AV = & \quad A \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} \\
= & \begin{bmatrix} | & & | \\ Av_1 & \dots & Av_n \\ | & & | \end{bmatrix} =
\end{aligned}
$$

$$
\begin{aligned}
AV &= A\begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ Av_1 & \dots & Av_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & | \\ \sigma_1 u_1 & \dots & \sigma_r u_r & 0u_{r+1} & 0u_m \\ | & & | & | & | \end{bmatrix} \\
&=
\end{aligned}
$$

$$AV = A\begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ Av_1 & \dots & Av_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & | \\ \sigma_1 u_1 & \dots & \sigma_r u_r & 0u_{r+1} & 0u_m \\ | & & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | & & | \\ u_1 & \dots & u_r & \dots & u_m \\ | & & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ | & | & | & | \\ \dots & \sigma_r & 0 & \dots \\ 0 & \dots & \dots & 0 \\ | & | & | & | \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$AV = U\Sigma$$

$$A_{m\times n} V_{n\times n} = U_{m\times m} \Sigma_{m\times n}.$$

$$
\begin{aligned}
AV &= A \begin{bmatrix} \vert & & \vert \\ v_1 & \dots & v_n \\ \vert & & \vert \end{bmatrix} \\
&= \begin{bmatrix} \vert & & \vert \\ Av_1 & \dots & Av_n \\ \vert & & \vert \end{bmatrix} = \begin{bmatrix} \vert & & \vert & \vert & \vert \\ \sigma_1 u_1 & \dots & \sigma_r u_r & 0 u_{r+1} & 0 u_m \\ \vert & & \vert & \vert & \vert \end{bmatrix} \\
&= \begin{bmatrix} \vert & & \vert & & \vert \\ u_1 & \dots & u_r & \dots & u_m \\ \vert & & \vert & & \vert \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ \vert & \vert & \vert & \vert \\ \dots & \sigma_r & 0 & \dots \\ 0 & \dots & \dots & 0 \\ \vert & \vert & \vert & \vert \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
AV &= U\Sigma \\
A &= U\Sigma V^T
\end{aligned}
$$

$y = V^T x.$

$Ax = b$

$U\Sigma y = b \quad \therefore \Sigma y = U^T b$

$U\Sigma V^T x = b.$

$$
\begin{aligned}
AV &= A\begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ Av_1 & \dots & Av_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & | \\ \sigma_1 u_1 & \dots & \sigma_r u_r & 0u_{r+1} & 0u_m \\ | & & | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & & | & & | \\ u_1 & \dots & u_r & \dots & u_m \\ | & & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ | & | & | & | \\ \dots & \sigma_r & 0 & \dots \\ 0 & \dots & \dots & 0 \\ | & | & | & | \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
AV &= U\Sigma \\
A &= U\Sigma V^T (\text{ or } A = U\Sigma V^*, \text{ if } A \in \mathbb{C}^{m\times n})
\end{aligned}
$$