Application of Convolution Theorem [Contd.]

Lecture 33

Analysis of an LTI system using CTFT

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ \text{system}\\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t)$$

The dynamics of this LTI system is described by N^{th} order linear constant coefficient differential equation (LCCDE)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} (b_k) \cdot \frac{d^k x(t)}{dt^k}$$

Take CTFT on the both the sides

$$F\left\{ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} \right\} = F\left\{ \sum_{k=0}^{M} (b_k) \cdot \frac{d^k x(t)}{dt^k} \right\}$$

Using linearity and differentiation property of CTFT,

$$\sum_{k=0}^{N} F\left\{ a_k \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^{M} F\left\{ b_k \frac{d^k x(t)}{dt^k} \right\}$$

constants

$$\sum_{k=0}^{N} a_k (j\omega)^k \cdot Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k \cdot X(\omega)$$

$$\therefore \boxed{\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k \cdot (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}}$$

For the given LTI system,

$y(t) = x(t) * h(t)$.

Using convolution theorem,

$\therefore \mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\}$

$Y(\omega) = X(\omega) \cdot H(\omega)$ [Convolution Theorem]

Frequency response of LTI system $\therefore$ $\boxed{\frac{Y(\omega)}{X(\omega)} = H(\omega)}$

$$\therefore H(\omega) = \frac{b_0 (j\omega)^0 + b_1 (j\omega)^1 + b_2 (j\omega)^2 + \dots + b_M (j\omega)^M}{a_0 (j\omega)^0 + a_1 (j\omega)^1 + a_2 (j\omega)^2 + \dots + a_N (j\omega)^N}$$

function of frequency, ω. (2)

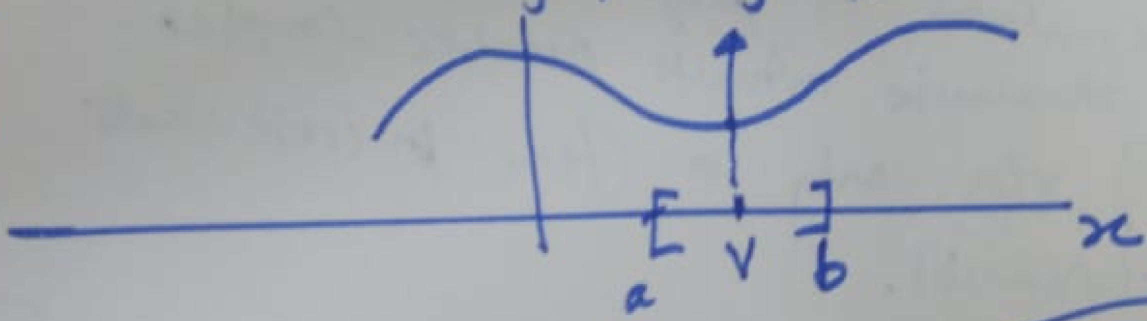
$\Rightarrow H(\omega) = \frac{N(\omega)}{D(\omega)}$. polynomials in ω.

$\therefore$ function $\cong$ polynomials (in ω).
(ω)

(Why ??)

<u>Theorem</u> [Real Analysis] <u>Weierstrass–Stone</u> Approximation Theorem: If a function $f(x)$ is continuous over the interval $[a, b]$ then it can be approximated well by the polynomial; $P_n(x)$

$$f(x)\big|_{x=v} = P_n(x).$$



$$P_1(x) \quad P_2(x) \quad P_3(x) \; \ldots\ldots \; P_n(x) \xrightarrow{n \to \infty} f(x)$$

$$\boxed{\lim_{n \to \infty} \underset{\sim}{P_n(x)} = f(x)}$$

B

$\Rightarrow$ Polynomials can be used to approximate continuous functions.

$f(x) = 2x^2 + 3x + 5$

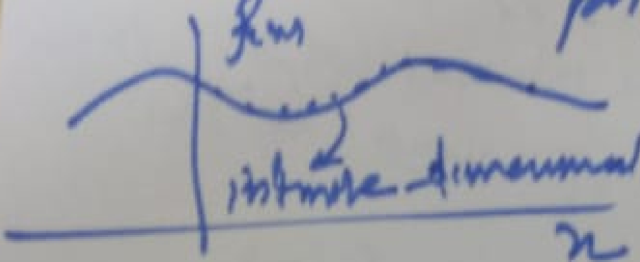
function $g(x) = 7x^2 + 4x + 3$

polynomial

$$\begin{array}{ccc} x^2 & x^1 & x^0 \\ \end{array}$$

$$\begin{bmatrix} 7 & 4 & 3 \end{bmatrix}$$

finite number.



⇒ Polynomial structure

⇒ We ~~[illegible]~~ are able to have a compact representation of ~~poly~~ a function (that needs otherwise infinite number samples points) via only a few co-efficients of polynomial.

→ [$\int$ polynomial, $\frac{d}{dx}$ polynomial, +, −, ×, ÷]

↳ polynomial

⇒ Polynomials are analytically tractable

⇒ They maintain their structure

[7] Modulation Theorem :→

[6]. Multiplication or Property of CTFT.

If $m(t) \xleftrightarrow{F} M(\omega)$, and $c(t) \xleftrightarrow{F} C(\omega)$.

then $y(t) = m(t) \cdot c(t) \xleftrightarrow{F(\cdot)} \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$

$\therefore \quad Y(\omega) = F\{m(t) \cdot c(t)\} = \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$

Proof :→ $R.H.S. = \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$

$$= \frac{1}{2\pi}\left[\int_{\xi=-\infty}^{\xi=+\infty} M(\xi)\, C(\omega-\xi)\, d\xi\right]$$

running variable for frequency 'ω'

where, $M(\xi) = F\{m(t)\} = \int_{-\infty}^{+\infty} m(t) \cdot e^{-j\xi t}\, dt$

Similarly $C(\omega-\xi) = \int_{-\infty}^{+\infty} c(t) \cdot e^{-j(\omega-\xi)t}\, dt.$

$$RHS = \frac{1}{2\pi}\left\{\int \underbrace{\left[\int_{-\infty}^{+\infty} m(t)\, e^{-j\xi t}\, dt\right]}_{M(\xi)} \underbrace{\left[\int_{-\infty}^{+\infty} c(t) \cdot e^{-j(\omega-\xi)t}\, dt\right]}_{\cancel{M(t)}\ C(\omega-\xi)} d\xi\right\}.$$

(5)

$$RHS = \frac{1}{2\pi}\left\{\left[\int_{-\infty}^{+\infty} m(t)\,c(t)\,e^{-j\xi t}\,dt\right]\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} e^{-j(\omega-\xi)t}\,dt\right]d\xi\right.$$

$$\int_{-\infty}^{+\infty} e^{-j\omega t}\,dt = 2\pi\,\delta(\omega).$$

$$\therefore \int_{-\infty}^{+\infty} e^{-j(\omega-\xi)t}\,dt = 2\pi\,\delta(\omega-\xi)$$

$$RHS = \left[\frac{1}{2\pi}\right]\left\{\int_{-\infty}^{+\infty} c(t)\,m(t)\,e^{-j\xi t}\,dt\right]\left[\int_{-\infty}^{+\infty} 2\pi\,\delta(\omega-\xi)\,d\xi\right]$$

$$\int_{-\infty}^{+\infty} \underbrace{\delta(\omega-\xi)}\,d\xi$$

impulse in frequency domain

Impulse function is symmetric

$$\delta(\omega) = \delta(-\omega)$$

$$\delta(\omega-\xi) = \delta(-(\omega-\xi))$$

$$= \delta(\xi-\omega)$$

$$\xi - \omega = \omega_1$$

$$d\xi = d\omega_1$$

$$\int_{\xi=-\infty}^{+\infty} \delta(\xi-\omega)\,d\xi = 1$$

$$RHS = \int_{-\infty}^{+\infty} m(t) \cdot c(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= Y(\omega) = F\{y(t)\} = RHS.$$

$$m(t) \cdot c(t) \xleftrightarrow{F(\cdot)} \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$$

Hence proved.

Let $f(x, y) \rightarrow$ 2-D function of x & y.

$$\iint f(x, y)\, dx\, dy = \int\left[\int f(x, y)\, dx\right] dy$$

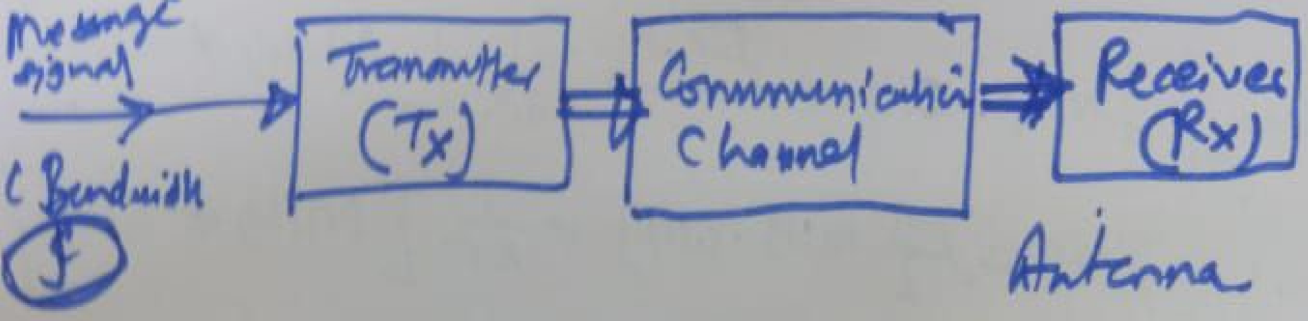
Fubini's theorem

$$= \int\left[\int f(x, y)\, dy\right] dx$$

Application of Modulation theorem...

Amplitude Modulation (AM): →

Analog Communication System.



$$(\text{Size of Antenna}) \propto \frac{1}{\text{Bandwidth of Transmitter..}}$$

$$l \downarrow \propto \frac{1}{f \uparrow}$$

Design..

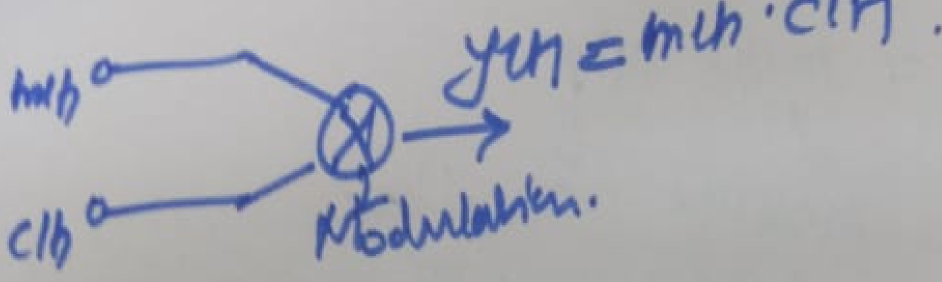
Engineering

2. How to increase BW of 'f' artificially of AM

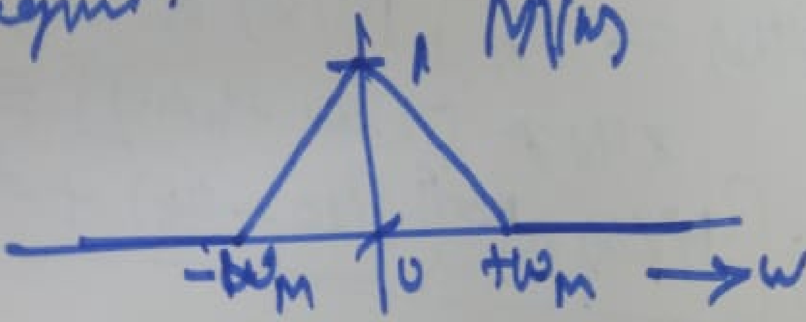
⇒ Design

We employ a carrier signal, $c(t)$

Let $c(t) = \cos(w_c t)$



Let the message signal occupy low frequency region.



$$C(w) = F\{c(t)\} = F\{\cos(w_c t)\}$$

$$= F\left\{\frac{1}{2} e^{jw_c t} + \frac{1}{2} e^{-jw_c t}\right\}$$

$$= \left(\frac{1}{2}\right) F\{e^{jw_c t}\} + \frac{1}{2} F\{e^{-jw_c t}\}.$$

$$= \left(\frac{1}{2}\right) 2\pi\, \delta(w - w_c) + \frac{1}{2} 2\pi\, \delta(w + w_c)$$

$$\therefore C(w) = \pi\, [\delta(w - w_c) + \delta(w + w_c)]$$

To increase BW of m(t) artificially one employ modulation of m(t) via a carrier signal, i.e.

$$y(t) = m(t) \cdot c(t)$$

modulating signal, message, carrier

Using Modulation Theorem,

$$\therefore Y(\omega) = \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$$

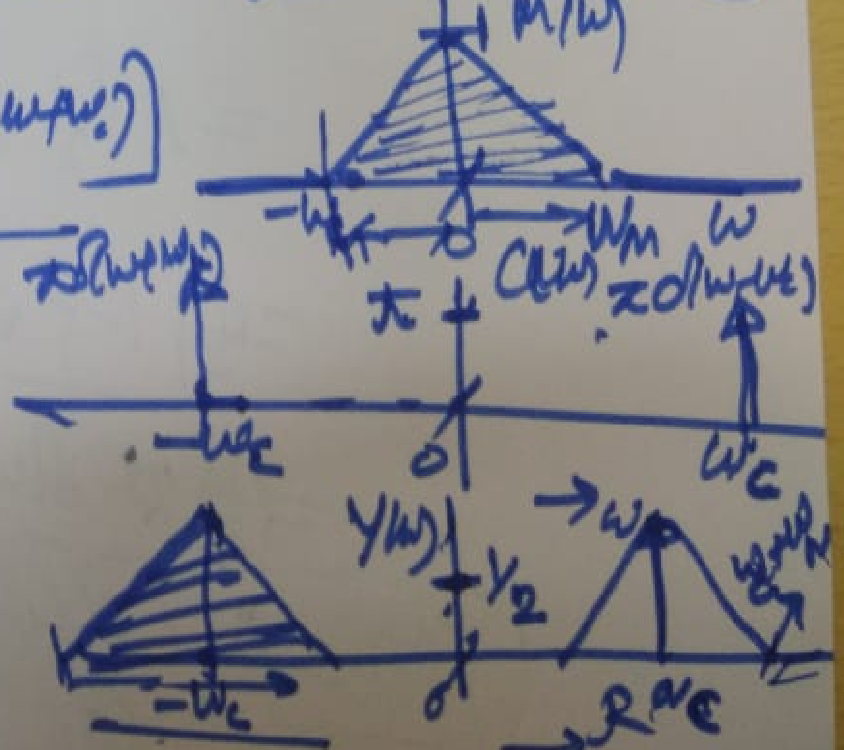
$x(t) h_2(t)$ + $x(t) * h_1(t)$

$x(t) * [h_1(t) + h_2(t)] =$

$$Y(\omega) = \frac{1}{2\pi}\left[M(\omega) * \pi\left(\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right)\right]$$

$$Y(\omega) = \frac{1}{2}\left[M(\omega) * \delta(\omega-\omega_c) + M(\omega) * \delta(\omega+\omega_c)\right]$$

$$Y(\omega) = \frac{1}{2}\left[M(\omega-\omega_c) + M(\omega+\omega_c)\right]$$



[11] Integration Property of CTFT

If $f(t) \xleftrightarrow{F} F(\omega)$

then $\displaystyle\int_{-\infty}^{t} f(\tau)\,d\tau \xleftrightarrow{F(\cdot)} \frac{F(\omega)}{j\omega} + ?\,\pi F(0)\delta(\omega)$.

Proof: →

$f(t) \longrightarrow \boxed{\text{Integrator LTI } h(t)} \longrightarrow y(t) = \displaystyle\int_{-\infty}^{t} f(\tau)\,d\tau.$

∴ For integrator system, $h(t) = u(t)$.

$\therefore\ y(t) = f(t) * h(t) = f(t) * u(t)$

$$y(t) = \int_{-\infty}^{+\infty} f(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{t} f(\tau)\,d\tau$$

$$F\{y(t)\} = F\left\{\int_{-\infty}^{t} f(\tau)\,d\tau\right\} = \underline{F\{f(t) * u(t)\}}$$

$$Y(\omega) = F(\omega) \cdot U(\omega) = F(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$Y(\omega) = \frac{F(\omega)}{j\omega} + \pi F(\omega)\,\delta(\omega)$$

Impulse in frequency dom

Using sampling property of impulse funct,

$x(t)\,\delta(t) = x(0)\cdot\delta(t)$. [Chapter 1]

$\therefore F(\omega)\cdot\delta(\omega) = F(0)\cdot\delta(\omega)$

$$\therefore \boxed{Y(\omega) = \frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)}$$

$$F(0) = ? \Rightarrow F(\omega)\Big|_{\omega=0} = \int_{-\infty}^{+\infty} f(t)\cdot e^{-j\omega t}\,dt\Big|_{\omega=0}$$

$$F(0) = \int_{-\infty}^{+\infty} f(t)\,dt$$

= Average value of a signal f(t)

$F(0)$ = d.c. value / d.c. component.

$\therefore$ For signal having 0 d.c. component,

$$Y(\omega) = \frac{F(\omega)}{j\omega} + \pi(0)\,\delta(\omega) = \frac{F(\omega)}{j\omega}.$$

Problem: Find the Fourier transform $x(t) = u(t)$ by using integration property.

Proof: → Let $g(t) = \delta(t) \xleftrightarrow{F} G(\omega) = 1$

We know that $\delta(t)$ and $u(t)$ are related by

$$x(t) = u(t) = \int_{\tau=-\infty}^{t} \delta(\tau)\, d\tau$$

$$\therefore F\{x(t)\} = F\{u(t)\} = F\left\{\int_{-\infty}^{t} \delta(\tau)\, d\tau\right\}$$

Using integration property

$$U(\omega) = \frac{F\{\delta(t)\}}{j\omega} + \pi\, \delta(\omega)\Big|_{\omega=0} \cdot \delta(\omega)$$

$$= \frac{1}{j\omega} + \pi \cdot 1 \cdot \delta(\omega)$$

$$\boxed{U(\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)}$$

$$\delta(t) = \frac{d}{dt}[u(t)]$$

(13) $$F\{\delta(t)\} = F\left\{\frac{d}{dt} u(t)\right\}$$

$$\delta(\omega) = (j\omega)\left[U(\omega)\right]$$

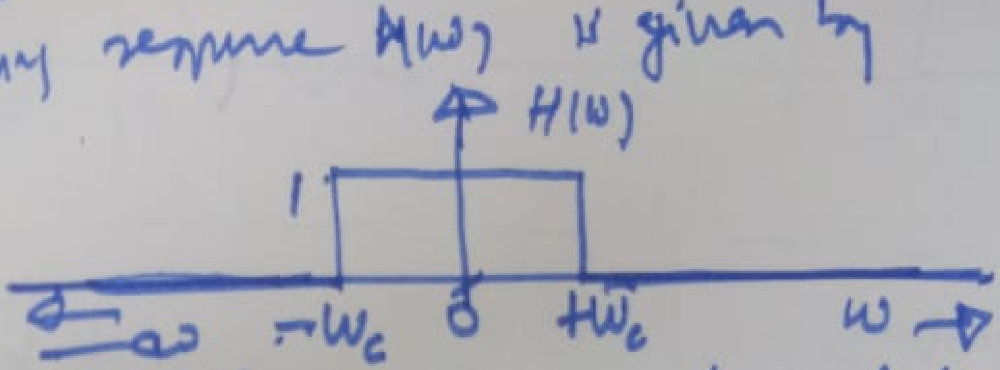
$$= (j\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= 1 + \underbrace{\pi(j\omega)\delta(\omega)}$$

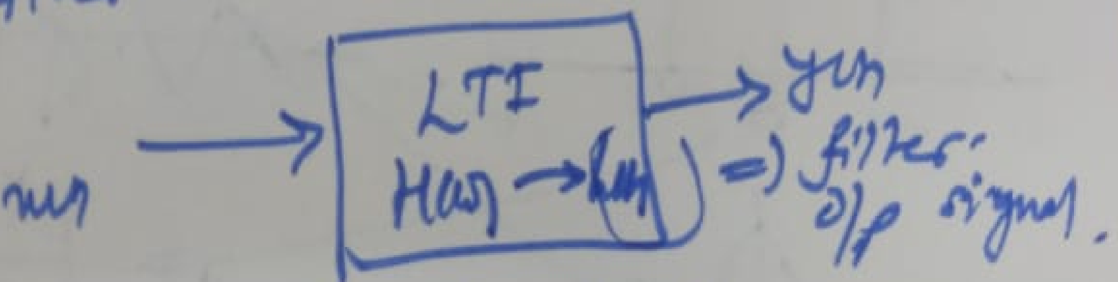
$$\omega\delta(\omega) = 0 \;\Big|_{\omega=0}$$

$$\therefore \boxed{\delta(\omega) = 1}$$

Problem:- Consider an ideal LPF whose frequency response H(ω) is given by



Find the impulse response, h(t) of this LPF filter.

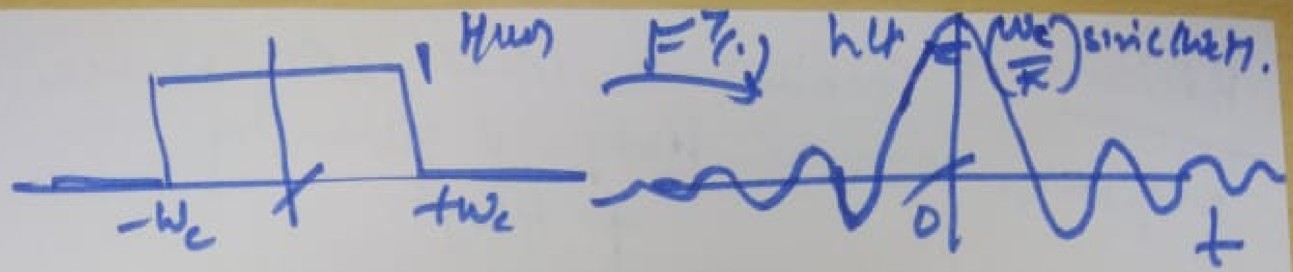


Using Inverse Fourier transform, i.e.,

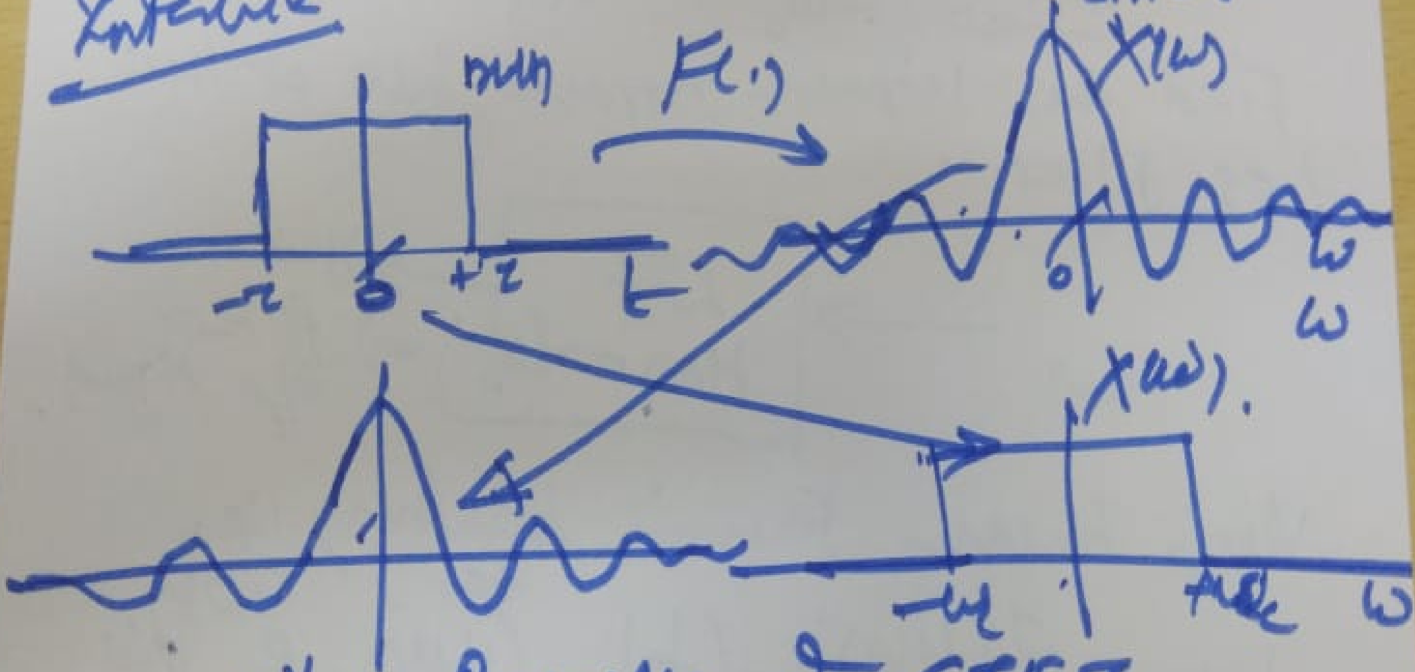
$$h(t) = F^{-1}\{H(\omega)\} = \left(\frac{1}{2\pi}\right)\cdot\int_{\omega=-\infty}^{+\infty} H(\omega)\cdot e^{j\omega t}\,d\omega$$

$$h(t) = \left(\frac{1}{2\pi}\right)\left\{\int_{-W_c}^{+W_c} 1\cdot e^{j\omega t}\,dt\right]$$

$$h(t) = \left(\frac{W_c}{\pi}\right)\cdot\left[\frac{\sin(W_c t)}{W_c t}\right]$$ → sinc function



Inference



Duality Property of CTFT.

$$x(t) * h(t) \xleftrightarrow{F(\cdot)} X(\omega)\cdot H(\omega)$$

$$x(t)\cdot h(t) \quad \frac{1}{2\pi}\left[X(\omega) * H(\omega)\right]$$

$$x(t-t_0) \xleftrightarrow{F(\cdot)} e^{-j\omega t_0}\cdot X(\omega)$$

$$e^{j\omega_0 t}\, x(t) \quad X(\omega-\omega_0)$$

$$\frac{d}{dt}[x(t)] \xleftrightarrow{F(\cdot)} (j\omega)\, X(\omega)$$

$$(-jt) \cdot x(t) \longleftrightarrow \frac{d}{d\omega}[X(\omega)]$$

$$\int_{\tau=-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{F(\cdot)} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

$$-\frac{x(t)}{jt} + \pi x(0) \cdot \delta(t) \xrightarrow{F(\cdot)} \int_{\xi=-\infty}^{\omega} X(\xi) \cdot d\xi$$

I wish you Happy Diwali !!