SC223 - Linear Algebra

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Lecture 7

August 8, 2023

LU Decomposition

$$A = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}$$

- $E_3E_2E_1A = EA = U \quad \Rightarrow A = E^{-1}U$.
- By Theorem 2, E^{-1} is a lower triangular matrix. Define $L := E^{-1}$.

Thus $A = LU$, known as the **LU decomposition**.

- For this example:

$$\underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1/4 & 1 & 0 \\ 3 & 1/2 & -10/11 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 4 & 5 & 4 \\ 0 & 0 & -11/4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U}$$

Is $A = LU$ always possible?

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$$E_3' E_2' E_1' \underbrace{(P_{14} \cdot P_{23})} A = U$$

$$\begin{pmatrix} P_{1 \leftrightarrow 4} \\ 2 \leftrightarrow 3 \end{pmatrix} A = \underbrace{(E_1')^{-1} (E_2')^{-1} (E_3')^{-1}}_{L} U$$

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  - Solve for y in $Ly = b$. $\sim O(n^2)$
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● **Beware of the notation:** a_{i*} denotes the i^{th} row of A *written as a column matrix*.

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- **The Column Space, Row Space, Null Space and Left Nullspace are called the Four Fundamental Subspaces associated with a matrix.**