Computational Numerical Methods

CS 374

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Cubic Spline interpolation

1. $S(x)$ is a polynomial of degree ≤ 3 on an subinterval $[x_{j-1}, x_j]$ $j = 2, 3, \ldots n$

2. $S(x)$, $S'(x)$, $S''(x)$ are continuous in $[a, b]$ $a = x_1$, $b = x_n$.

3. $S''(x_1) = S''(x_n) = 0$.

* $S(x)$ is cubic $\Rightarrow$ $S''(x)$ = linear function

Introduce M_i $(i = 1, \ldots n)$ such that

$$\cancel{M_i(} \quad M_i = S''(x_i)$$

$\therefore$ On an interval $[x_{j-1}, x_j]$.

$$S''(x_{j-1}) = M_{j-1} \quad , \quad S''(x_j) = M_j$$

With these two values we interpolate a linear function $S''(x)$ b/w x_{j-1} & x_j.

$\therefore$ The line joining $(x_{j-1}, S''(x_{j-1}))$, $(x_j, S''(x_j))$

is given as.

$$S''(x) = \frac{(x_j - x)\, S''(x_{j-1}) + (x - x_{j-1})\, S''(x_j)}{x_j - x_{j-1}}$$

$$= \frac{(x_j - x)\, M_{j-1} + (x - x_{j-1})\, M_j}{x_j - x_{j-1}}$$

On integrating we get

$$S'(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \times \left(- \frac{(x_j - x)^2}{2}\right) + \frac{M_j}{x_j - x_{j-1}} \times \frac{1}{2}(x - x_{j-1})^2 + A.$$

Here A is an arbitrary integrating constant.

$S'(x)$ gives a quadratic function.

Integrating again

$$S(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \times \frac{(x_j - x)^3}{6} + \frac{M_j}{x_j - x_{j-1}} \frac{(x - x_{j-1})^3}{6}.$$

$$+ Ax + B.$$

A & B are integrating constants.

We assume

$$Ax + B = \frac{C(x_j - x) + D(x - x_{j-1})}{}$$

$A = D - C$, $B = Cx_j - Dx_{j-1}$

$$\therefore S(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \frac{(x_j - x)^3}{6} + \frac{M_j}{x_j - x_{j-1}} \frac{(x - x_{j-1})^3}{6}$$

$$+ C(x_j - x) + D(x - x_{j-1})$$

As $S(x)$ is an interpolating polynomial

$$S(x_{j-1}) = y_{j-1} \quad \& \quad S(x_j) = y_j$$

when $x = x_j$

$$S(x_j) = y_j = \frac{M_j}{x_j - x_{j-1}} \frac{(x_j - x_{j-1})^3}{6} + D(x_j - x_{j-1})$$

$$D = \frac{y_j}{x_j - x_{j-1}} - \frac{M_j}{6}(x_j - x_{j-1})$$

for $x = x_{j-1}$

$$S(x_{j-1}) = y_{j-1} = \frac{M_{j-1}}{6}(x_j - x_{j-1})^3 + c(x_j - x_{j-1})$$

$$\therefore \quad c = \frac{y_{j-1}}{x_j - x_{j-1}} - \frac{M_{j-1}}{6}(x_j - x_{j-1})$$

$$A = \frac{y_j - y_{j-1}}{x_j - x_{j-1}} - \frac{(x_j - x_{j-1})}{6}(M_j - M_{j-1})$$

$$S'(x) = -\frac{M_{j-1}}{x_j - x_{j-1}}\frac{(x_j - x)^2}{2} + \frac{M_j}{x_j - x_{j-1}}\frac{(x - x_{j-1})^2}{2}$$

$$+ \frac{y_j - y_{j-1}}{x_j - x_{j-1}} - \frac{x_j - x_{j-1}}{6}(M_j - M_{j-1})$$

For the interval $[x_{j-1}, x_j]$ $S(x_j)$ must be equal to the value of $S(x_j)$ for the interval $[x_j, x_{j+1}]$

The function $S'(x)$ has been obtained for the interval $[x_{j-1}, x_j]$. For the interval $[x_j, x_{j+1}]$ we simply substitute $j-1 \to j$ & $j \to j+1$

$$\therefore S'(x) = -\frac{(x_{j+1}-x)^2 M_j + (x-x_j)^2 M_{j+1}}{2(x_{j+1}-x_j)} + \frac{y_{j+1}-y_j}{x_{j+1}-x_j} - \frac{x_{j+1}-x_j}{6}(M_{j+1}-M_j)$$

The former interval gives. $[x_{j-1}, x_j]$

$$S'(x_j) = \frac{1}{2} M_j (x_j - x_{j-1}) + \frac{y_j - y_{j-1}}{x_j - x_{j-1}} - \frac{x_j - x_{j-1}}{6}(M_j - M_{j-1})$$

For later interval $[x_j, x_{j+1}]$

$$S'(x_j) = -\frac{1}{2} M_j (x_{j+1} - x_j) + \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{x_{j+1} - x_j}{6}(M_{j+1} - M_j)$$

$\therefore$ $S'(x_j) = S'(x_j)$ for two diff intervals.

$$\therefore \quad \frac{1}{2} M_j (x_j - x_{j-1}) + \frac{y_j - y_{j-1}}{x_j - x_{j-1}} - \frac{x_j - x_{j-1}}{6} (M_j - M_{j-1})$$

$$= -\frac{1}{2} M_j (x_{j+1} - x_j) + \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{x_{j+1} - x_j}{6} (M_{j+1} - M_j)$$

$$\Rightarrow \quad \frac{1}{2} M_j (\cancel{x_j} - x_{j-1} + x_{j+1} - \cancel{x_j}) - \frac{1}{6} M_j (\cancel{x_j} - x_{j-1} + x_{j+1} - \cancel{x_j})$$

$$+ \frac{M_{j-1}}{6} (x_j - x_{j-1}) + \frac{M_{j+1}}{6} (x_{j+1} - x_j)$$

$$= \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$$\Rightarrow \frac{M_{j-1}}{6}(x_j - x_{j-1}) + \frac{1}{3} M_j (x_{j+1} - x_{j-1}) + \frac{M_{j+1}}{6}(x_{j+1} - x_j) = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

For n data points $x_1, x_2 \ldots x_n$ the above eqn matches derivatives at $x_2, x_3 \ldots x_{n-1}$ i.e. $n-2$ ~~data~~ ~~pts~~ eqns along with $M_1 = M_n = 0$.

One can solve the eqns to find out M_j's & establish the a cubic spline interpolation.

Example Find the natural cubic spline to interpolate.

$(1,1)$, $(2, 1/2)$, $(3, 1/3)$, $(4, 1/4)$.

All $x_j - x_{j-1} = 1$ $\quad x_{j+1} - x_{j-1} = 2$.

$x_1 = 1$, $x_2 = 2$, $x_3 = 3$

$x_4 = 4$.

j=2

$$\frac{M_1}{6} + \frac{2M_2}{3} + \frac{M_3}{6} = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3}.$$

for j=3.

$$\frac{M_2}{6} + \frac{2M_3}{3} + \frac{1}{6}M_4 = \frac{1}{4} - \frac{1}{3} - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{12}.$$

& $M_1 = M_4 = 0$.

$M_1 = 0$

$M_2 = 1/2$

$M_3 = 0$

$M_4 = 0$

$$S(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \frac{(x_j - x)^3}{6} + \frac{M_j}{x_j - x_{j-1}} \frac{(x - x_{j-1})^3}{6}$$

$$+ C(x_j - x) + D(x - x_{j-1})$$