$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

S&S Lecture 07

$$\omega_0 = \frac{2\pi}{T} \text{ or } -\frac{2\pi}{T}.$$

50 Hz vs. $-$50 Hz

w.r.t. a reference.

$x(t) = \sin(\omega_0 t)$

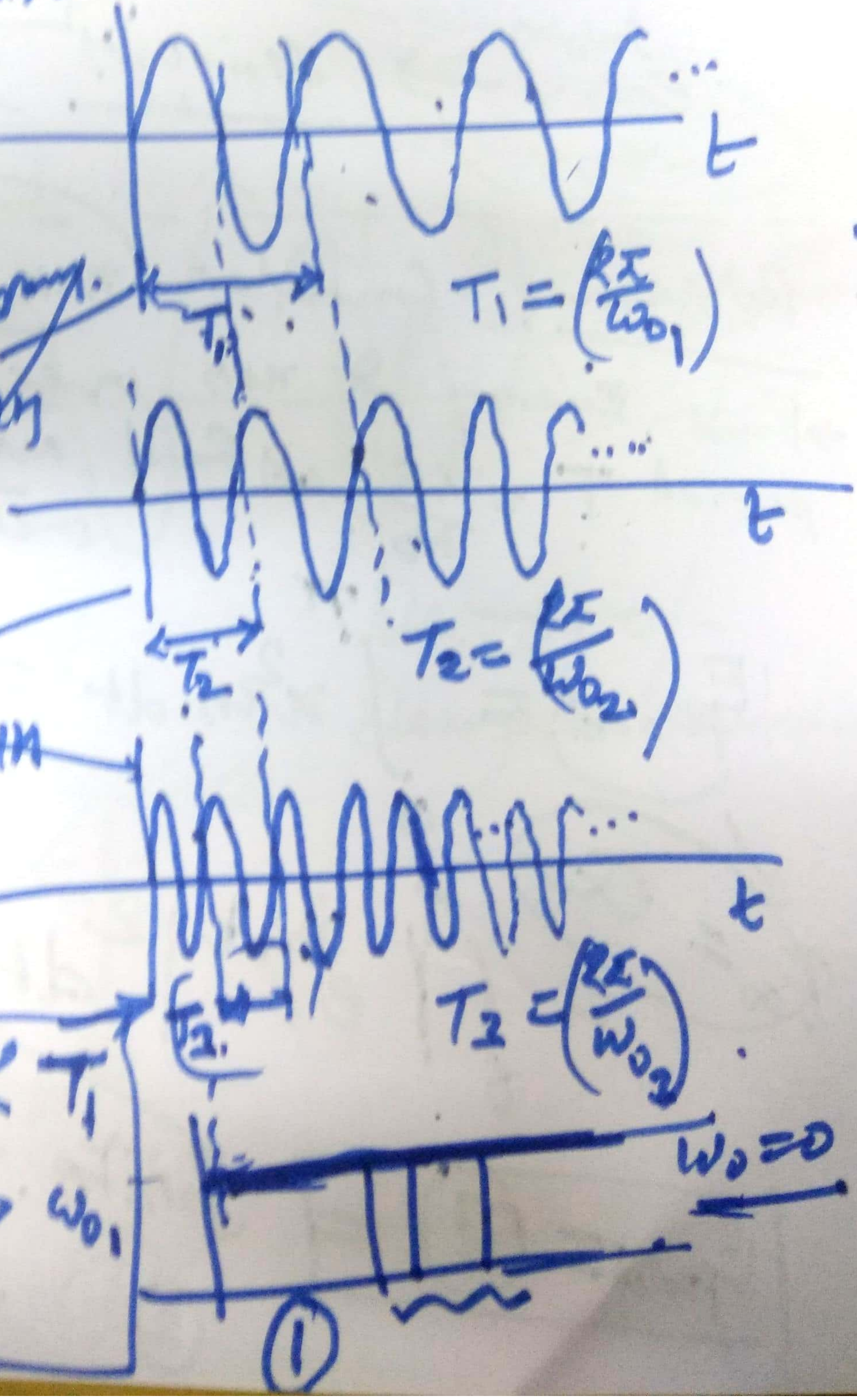
$x_1(t)$

$T = \frac{2\pi}{\omega_0}$

$\omega_0 = 0 \Rightarrow T$ = arbitrary.

$T_2 < T_1$

Distinct periodic signals.



$T_3 < T_2 < T_1$

$\omega_{0_3} > \omega_{0_2} > \omega_{0_1}$

For "continuous time" periodic complex exponential signal, we get distinct periodic signal for arbitrary choice of fundamental period T (and also fundamental frequency $\omega_0 = \frac{2\pi}{T}$).

$$e^{j\omega_0 t} \rightarrow \text{"Uniqueness"} ; \omega_0 = \frac{2\pi}{T}$$

Problem. Consider $x(t) = e^{j\omega_0 t}$?

→ Find energy of $x(t)$ over its fundamental period $T = \frac{2\pi}{\omega_0}$ and find average power over T.

$$E_{period} = \int_0^T x^2(t) \cdot dt = \int_0^T |x(t)|^2 dt.$$

$$E_\infty = \infty$$

$$= \int_0^T |e^{j\omega_0 t}|^2 dt \quad E_\infty = T + T + \ldots \text{ infinite times}$$

$$\boxed{E_{period} = T = \text{finite}} \quad \boxed{\therefore E_\infty = \infty}$$

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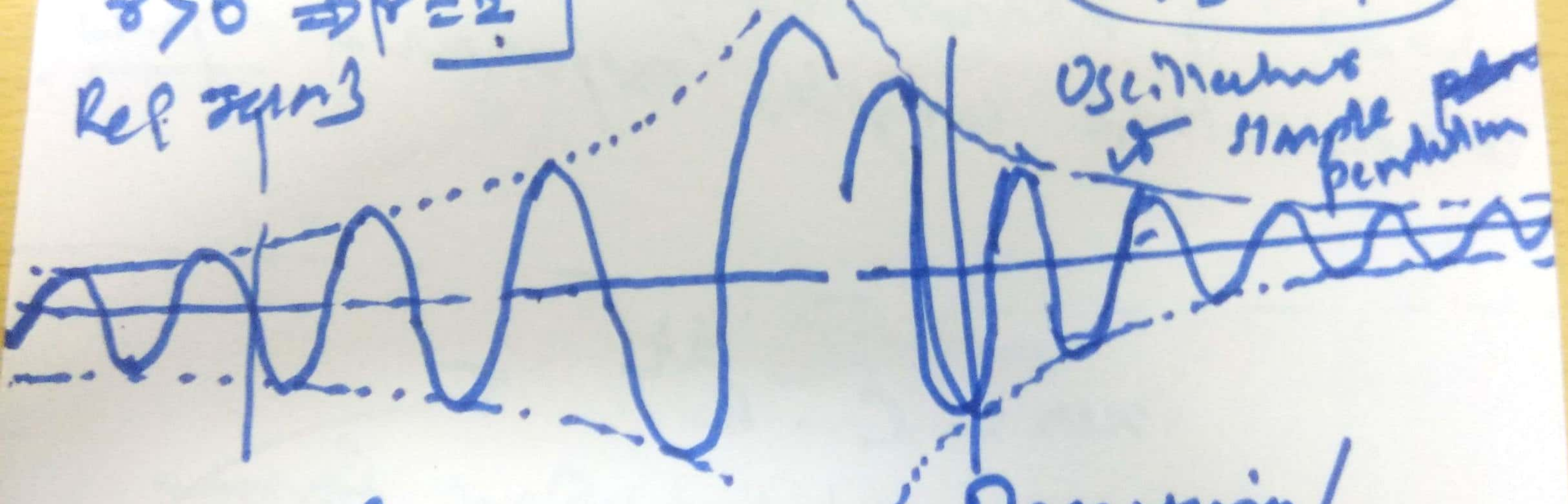
$$x(n) = |C| e^{rt} \cos(\omega_0 t + \theta) + j|C| e^{rt} \sin(\omega_0 t + \theta)$$

$$Re\{x(n)\} = |C| e^{rt} \cos(\omega_0 t + \theta).$$

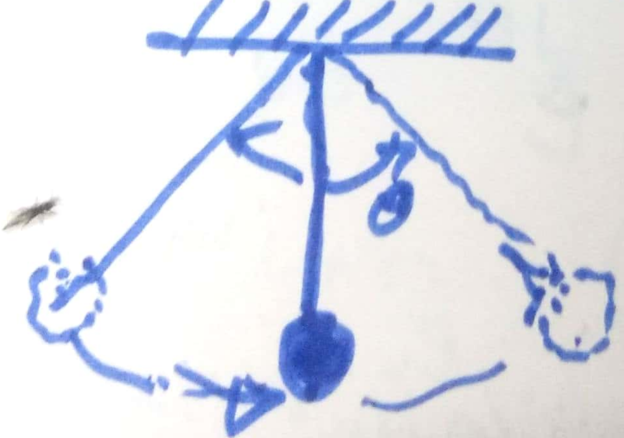
$r > 0 \Rightarrow \boxed{r = 2}$

$r < 0$
$r = -1$



Growing Sinusoid

Decaying/
Damped Sinusoid



In music literature, the integer multiple of fundamental frequency (e.g. harmonium) harmonics

$\therefore \{\phi_n(t)\}_{n \in Z} = \{e^{jn\omega_0 t}\}_{n \in Z}$

$\Rightarrow$ Set of harmonically related periodic continuous-time complex exponential signals.

$x(t) = C \cdot e^{at}$

Let us express 'C' in polar form and 'a' in rectangular

Let $C = |C| \cdot e^{j\theta}$ and $a = r + j\omega_0$

$\therefore x(t) = (|C| e^{j\theta}) \cdot e^{(r + j\omega_0)t}$

$= |C| e^{rt} e^{j(\theta + \omega_0 t)}$ → Euler's relation

$= (|C| e^{rt}) [\cos(\theta + \omega_0 t) + j \sin(\theta + \omega_0 t)]$

$x(t) = e^{j\omega_0 t}$

$$|x(t)| = |e^{j\omega_0 t}| = |(\cos(\omega_0 t) + j\sin(\omega_0 t))|$$

$$= \sqrt{\cos^2\theta + \sin^2\theta}$$

$$|e^{j\omega_0 t}| = 1$$

$$x(t) = e^{j\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$\omega_n = n.\omega_0 = n\left(\frac{2\pi}{T}\right); \; n \in \mathbb{Z}$$

$$\phi_n(t) = e^{jn\omega_0 t}$$

$$\phi_n(t) = \left\{e^{j\omega_n t}\right\}_{n \in \mathbb{Z}} = \left\{\ldots, e^{j\omega_1 t}, e^{j\omega_2 t}, e^{j\omega_3 t}, \ldots\right\}$$

Each of the signals in set $\{e^{j\omega_n t}\}$ have same fundamental period and their fundamental frequency is integer multiple of the others.

$$P_{average} = \frac{1}{T}\left[\int_0^T |x(t)|^2 dt\right]$$

$$= \frac{1}{T} \times T$$

$$\boxed{P_{period\,(T)} = 1}$$

Average ~~over~~ power over ~~infinite~~ time interval, $e^{j\omega t}$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\left\{\int_{-T}^{+T} |x(t)|^2 dt\right\}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left\{\int_{-T}^{+T} 1\, dt\right\}$$

$$\boxed{P_\infty = 1}$$

$x(t) = e^{j\omega t}$ power signal

$\Rightarrow$ Complex exponential signal $x(t)$ has finite average power and infinite energy over entire time interval

6

Discrete-Time Complex Exponential and Sinusoidal Signals

$$x(t) = C e^{at} \Longrightarrow x[n] = C \cdot \alpha^{n}$$

[Continuous-time] [Discrete-Time].

Discrete-time complex exponential signal

where 'C' & α are complex numbers.

Let $\alpha = e^{\beta}$

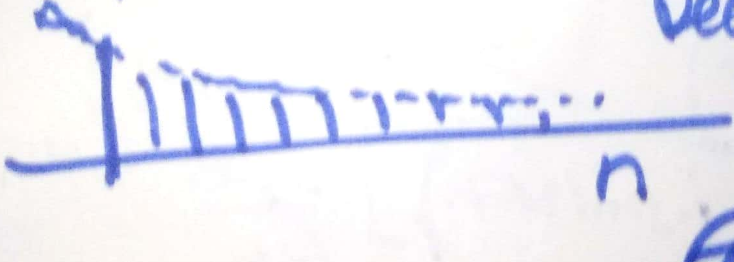
$$\therefore \boxed{x[n] = C e^{\beta \cdot n}}$$

β = real

Real exponential signal

Growing exponential

Decaying exponential



β = complex

β = purely imaginary

$\beta = j\omega_0$, $C = 1$

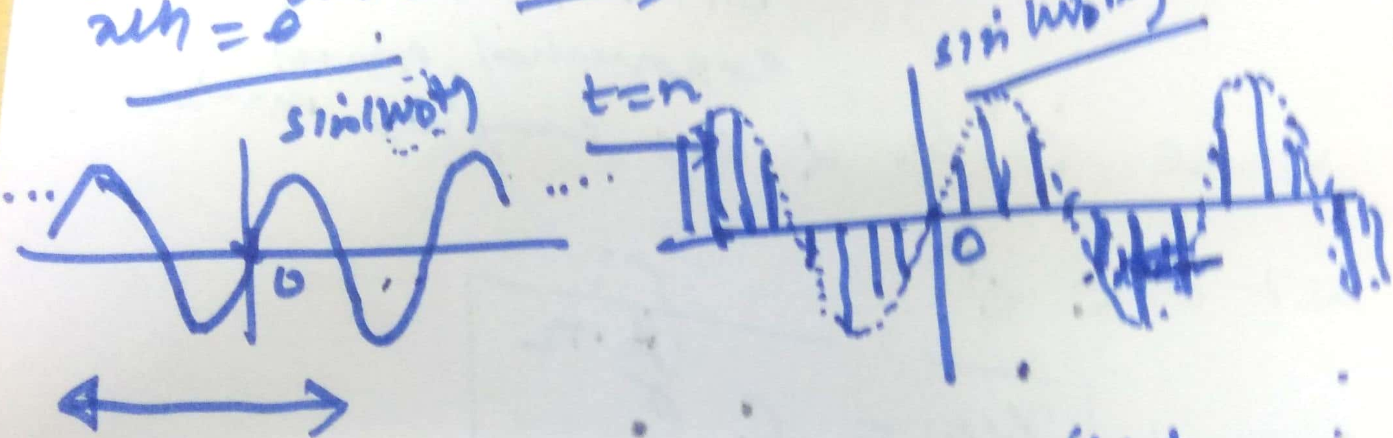
$$x[n] = e^{j\omega_0 n}$$

$$\boxed{x(t) = e^{j\omega_0 t} \quad \text{vs.} \quad x(n) = e^{j\omega_0 n}}$$

Problem: Investigate difference betw periodicity property of $e^{j\omega_0 t}$ vs. $e^{j\omega_0 n}$

Solution

$x(t) = e^{j\omega_0 t}$ $\longrightarrow$ $x(n) = e^{j\omega_0 n}$



$e^{j\omega_0 n} = x(n)$

$x(t) = e^{j\omega_0 t}$

$x(t) = \cos(\omega_0 t) + j\sin(\omega_0 t)$, $t \in \mathbb{R}$

$\downarrow$

$\omega_0 \to \omega_0 + 2\pi$

$x(n) = \cos(\omega_0 n) + j\sin(\omega_0 n)$, $n \in \mathbb{Z}$

$\downarrow$

$\omega_0 \to \omega_0 + 2\pi$

$x_1(n) = e^{j(\omega_0 + 2\pi)n}$

$= e^{j\omega_0 n} \cdot e^{j2\pi n} \to 1$

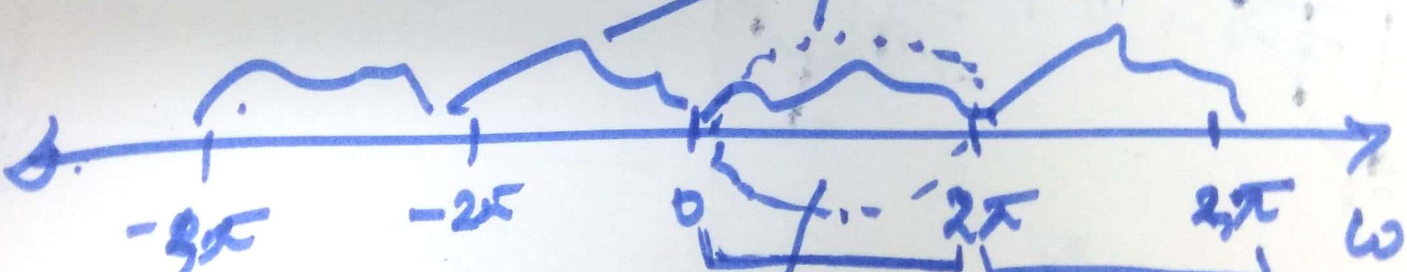
$= e^{j\omega_0 n}[\cos(2\pi n) + j\sin(2\pi n)] = e^{j\omega_0 n} = x(n)$

$$\boxed{e^{j(\omega_0+2\pi)n} = \underbrace{e^{j\omega_0 n}}_{x(n)}} \text{ , } n \in Z;$$

$$\boxed{e^{j(\omega_0+2\pi)t} \neq \underbrace{e^{j\omega_0 t}}_{x(t)} \text{ ; } t \in R}$$

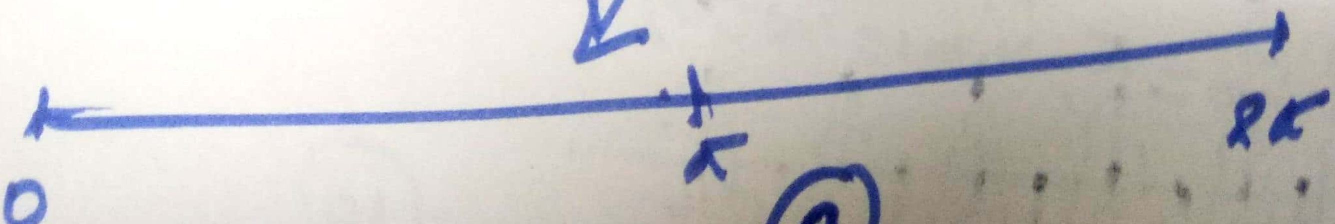
⇒ Discrete-time periodic complex exponential signal is periodic in frequency-domain, with a period of 2π.

Periodicity in frequency domain



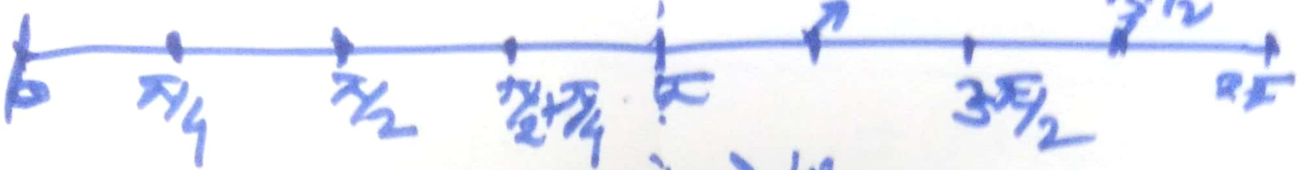
Req. Periodic for discrete-time signal.

$x(n) = e^{j\omega_0 n}$; → period = 2π



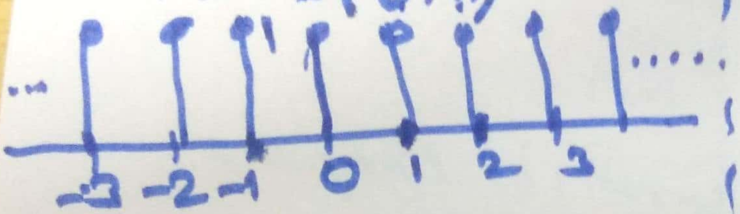
$x(n) = e^{j\omega_0 n}$

$\boxed{x_1(n) = \cos(\omega_0 n)}$



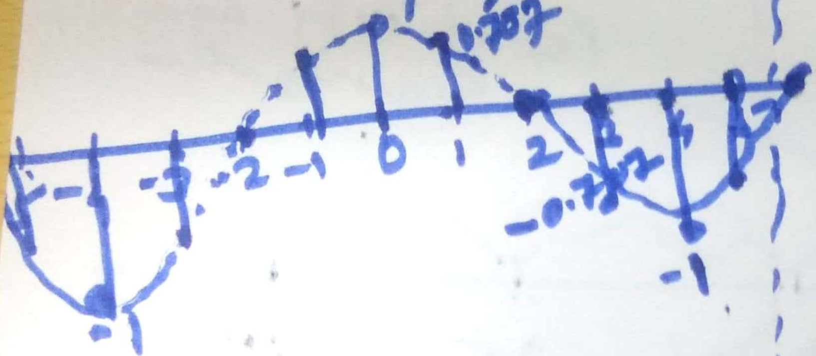
Case I) $\omega_0 = 0$

$x_1(n) = \cos(0 \cdot n)$



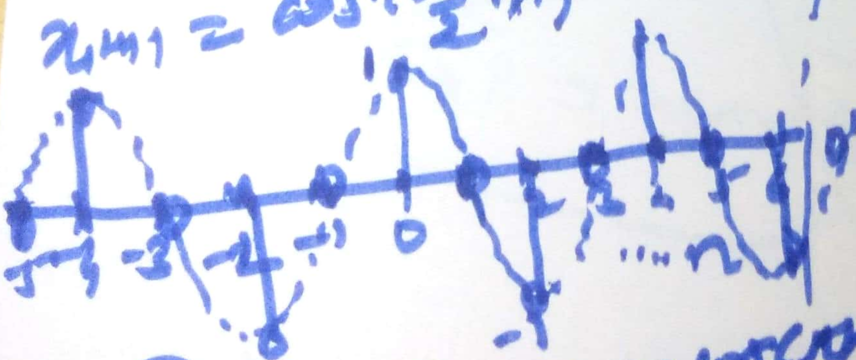
Case II $\omega_0 = \pi/4$

$x_1(n) = \cos(\frac{\pi}{4} \cdot n)$

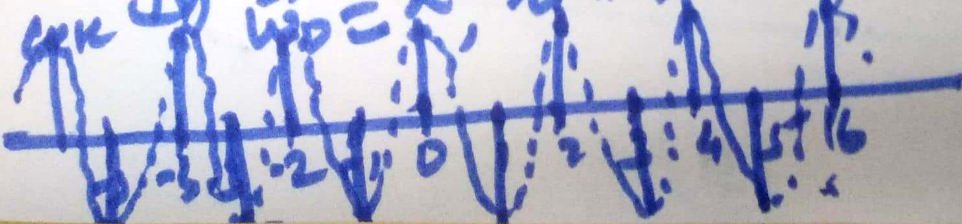


Case III) $\omega_0 = \pi/2$

$x_1(n) = \cos(\frac{\pi}{2} \cdot n)$



Case IV) $\omega_0 = \pi$, $x_1(n) = \cos(\pi n)$



$\omega_0 = \frac{\pi}{2} + \pi/4$

$\omega_0 = 3\pi/2$

$\omega_0 = \frac{3\pi}{2} + \pi/4$

$\omega_0 = 2\pi$

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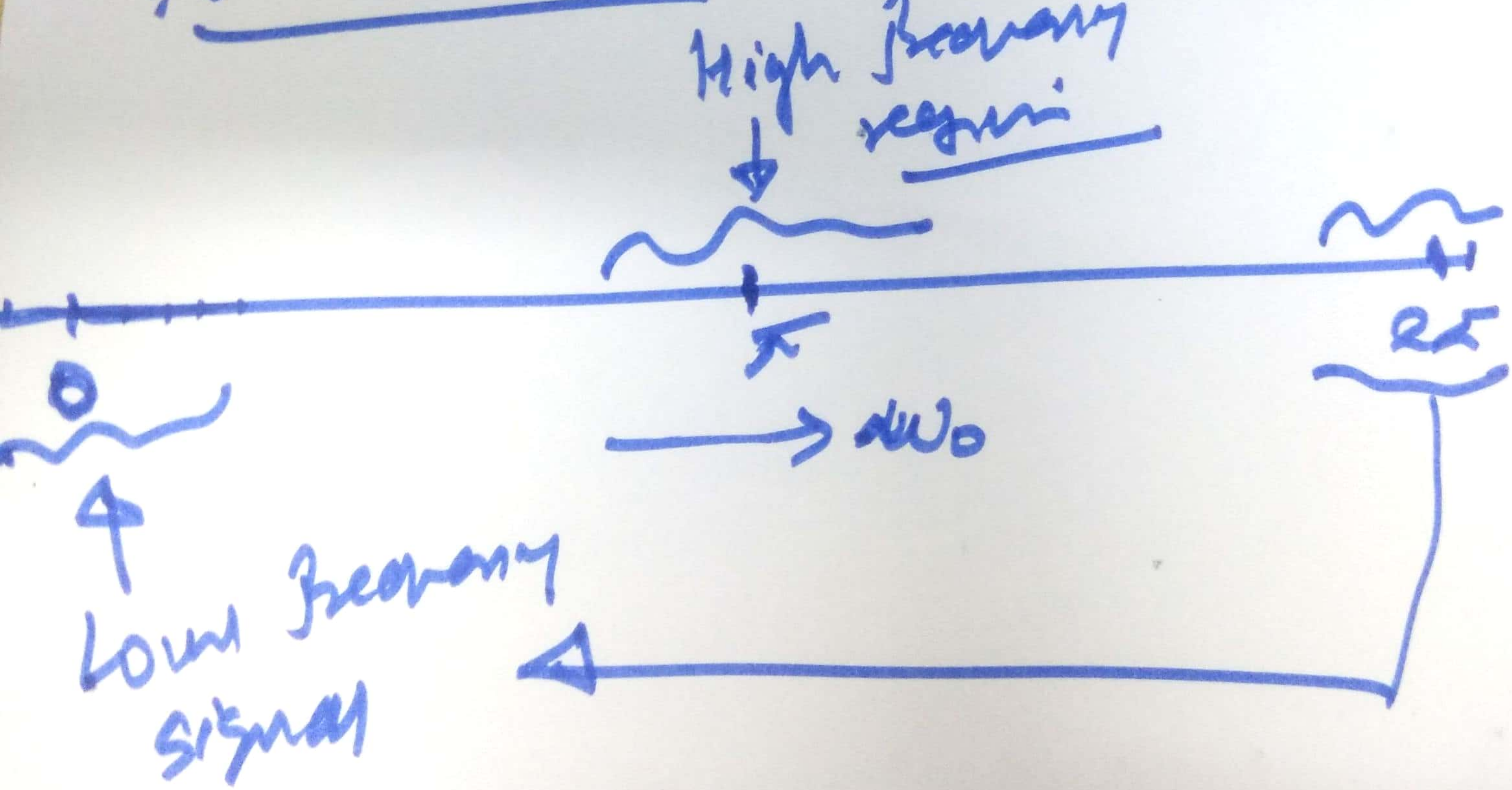
Observations

1) As we move from $\omega_0 = 0$ to $\omega_0 = \pi$ number of oscillations in $x_1(n) = \cos(\omega_0 n)$ is increasing and hence, frequency ω_0 is increasing

2) As we move from $\omega_0 = \pi$ to $\omega_0 = 2\pi$ the number of oscillations in $x_1(n) = \cos(\omega_0 n)$ is decreasing and hence, frequency ω_0 decreases



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