LECTURE 15

Recap:

- Solved the eqn. for the path/orbit under a central force with few convenient forms of $V(r)$ / $F(r)$. Forms of $V(r)$/ $F(r)$ dictated by whether the eqn. for orbit is integrable or not.

- Consider general $V(r)$ with extremum at $r = r_0$.

$$V(r) \simeq \underbrace{V(r_0)}_{\text{const.}} + (r-r_0) \underbrace{\cancel{\frac{dV}{dr}}}_{0}\bigg|_{r=r_0} + \underbrace{\frac{1}{2}(r-r_0)^2 \frac{d^2V}{dr^2}\bigg|_{r=r_0}} + \cdots$$

Near $r = r_0$,

$$V(r) \simeq \frac{1}{2} k \, (r - r_0)^2$$

$$k = \left. \frac{d^2 V}{dr^2} \right|_{r = r_0}.$$

☑ Try to work out some facts about orbit under a general force field having an extremum.

$$\vec{F} = f(r) \, \hat{r}$$

In the context of a central force $V \equiv V_{eff}(r)$.

$$\omega_r = \sqrt{\frac{k}{m}} = \sqrt{V''_{eff}(r_0)}$$

EOM :-

$$\ddot{r} - r\dot{\theta}^2 = f(r)$$

$$\Rightarrow \ddot{r} - r\frac{L^2}{r^4} = f(r)$$

$$\Rightarrow \ddot{r} - \frac{L^2}{r^3} = f(r).$$

$$L = r^2\dot{\theta} \quad \Rightarrow Lu^2 = \dot{\theta}$$

$$r = \frac{1}{u}$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta}$$

$$= -L\frac{du}{d\theta}$$

$$\ddot{r} = -L^2u^2\frac{d^2u}{d\theta^2}$$

Sub :-

$$-L^2u^2\frac{d^2u}{d\theta^2} - L^2u^3 = f(1/u)$$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{L^2u^2}}$$

$\Rightarrow$ 2nd order ODE for orbit.

— Physically, presence of extremum in the context of central force implies presence of circular orbit: Since, small deviations from the extremum $O[(r-r_0)^2]$ are considered, geometrically, this corresponds to small deviations/perturbations from circular orbit.

— Particle moving in a circular orbit of radius a under some attractive central force. $\Rightarrow -f(r)$.

$$\frac{v^2}{a} = f(a)$$

$$L = va \Rightarrow L^2 = a^3 f(a).$$

Radius of the circular orbit = a.

$r = a + \epsilon(t)$ where ϵ is small.

$u = \frac{1}{a}\left(1 + \xi(\theta)\right)$, ξ is small.

Eqn. for orbit,

$$\frac{d^2u}{d\theta^2} + u = + \frac{f(1/u)}{L^2u^2}$$

Sub: $u = \frac{1}{a}\left(1+\xi(\theta)\right)$

$$\frac{1}{a}\frac{d^2\xi}{d\theta^2} + \frac{1}{a}\left(1+\xi(\theta)\right) = \frac{f\left(\frac{a}{1+\xi(\theta)}\right)}{L^2u^2}$$

$$\Rightarrow \frac{d^2\xi}{d\theta^2} + 1 + \xi = \frac{a}{L^2\tilde{u}^2} f\left(\frac{a}{1+\xi}\right) = \frac{a^3}{L^2}(1+\xi)^{-2} f\left(\frac{a}{1+\xi}\right)$$

$$= \frac{(1+\xi)^{-2} f\left(\frac{a}{1+\xi}\right)}{f(a)}.$$

$$f\left(\frac{a}{1+\xi_\phi}\right) = f\left(a - \frac{a\xi_\phi}{1+\xi_\phi}\right).$$

$$= f(a) - \frac{a\xi_\phi}{1+\xi_\phi} f'(a) + O\left(\frac{\xi_\phi}{1+\xi_\phi}\right)^2$$

$$\simeq f(a) - a f'(a)\xi_\phi + O(\xi_\phi^2)$$

$$(1+\xi_\phi)^{-2} \simeq 1 - 2\xi_\phi + O(\xi_\phi^2).$$

$$(1+\xi_\phi)^{-2} f\left(\frac{a}{1+\xi_\phi}\right) \simeq (1-2\xi_\phi)\left(f(a) - a f'(a)\xi_\phi\right)$$

$$\simeq f(a) - a f'(a)\xi_\phi - 2\xi_\phi f(a).$$

$$\frac{(1+\xi_\phi)^{-2} f\left(\frac{a}{1+\xi_\phi}\right)}{f(a)} = 1 - a\frac{f'(a)}{f(a)}\xi_\phi - 2\xi_\phi.$$

Eqn. for path :

$$\frac{d^2\xi}{d\theta^2} + \cancel{1} + \xi = \cancel{1} - a\,\frac{f'(a)}{f(a)}\,\xi - 2\xi$$

$$\Rightarrow \quad \frac{d^2\xi}{d\theta^2} + \left(3 + \frac{a\,f'(a)}{f(a)}\right)\xi = 0 .$$

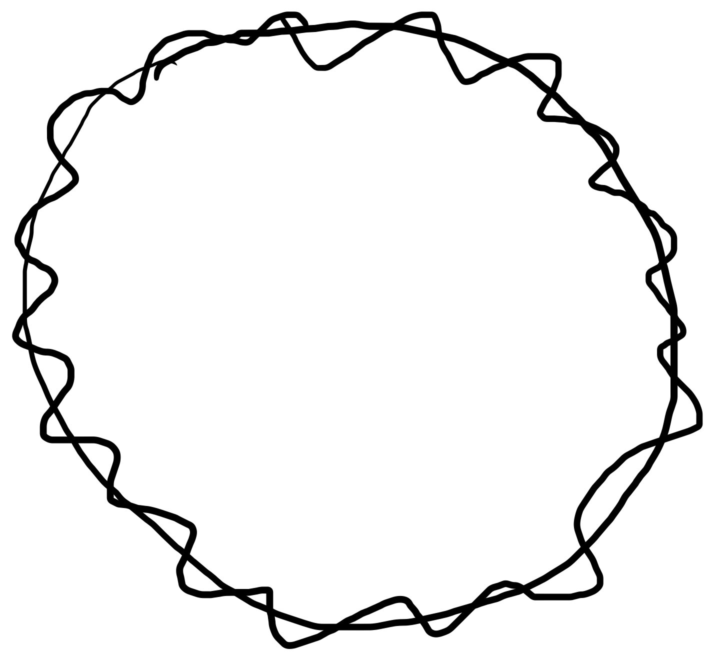
<u>Case 1</u> : $\quad 3 + \dfrac{a f'(a)}{f(a)} < 0 .$

$$\xi = (\quad)\, e^{m_1\theta} + (\quad)\, e^{-m_1\theta}$$

ξ grows with time $\Rightarrow$ orbit does not remain circular .

Case 2 :- $\Omega^2 = 3 + \dfrac{a f'(a)}{f(a)} > 0$.

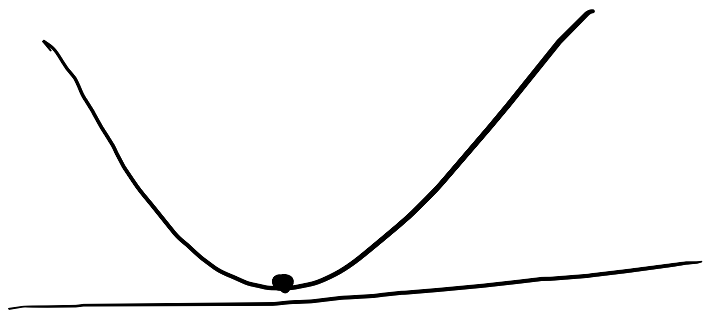
$$\frac{d^2\xi}{d\theta^2} + \Omega^2 \xi = 0.$$

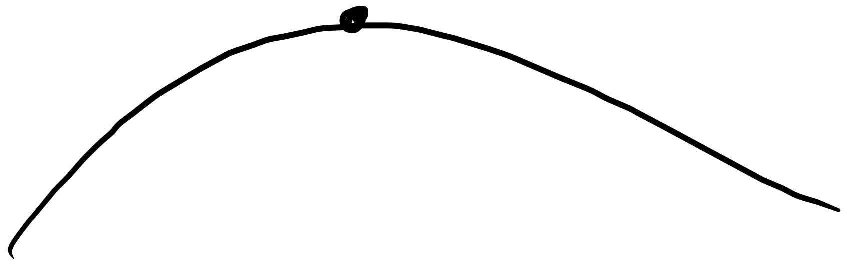


$$\xi = A \cos \Omega\theta + B \sin \Omega\theta$$

Circular orbit retains its shape.

For the perturbed orbit to close :- $\Omega \to$ +ve integer.





– Consider $f(r) = k r^{\nu}$.

$$\Omega^2 = 3 + \frac{a f'(a)}{f(a)} = 3 + \frac{a \cancel{k} \nu a^{\nu-1}}{\cancel{k} a^{\nu}} = \nu + 3 > 0$$

$$\Rightarrow \nu > -3.$$

For closure of orbit.

$f(r) = k r^{\nu}$

$\nu + 3 = m^2 \rightarrow$ +ve integer.

$V(r) = - \int dr\, f(r)$.

$\Rightarrow \nu = m^2 - 3$.

$$\boxed{V_{eff}(r) = \frac{L^2}{2r^2} + k r^{\nu}}$$

$$\omega_r = \sqrt{V''_{eff}(a)}$$

$$\frac{d^2 \xi_\phi}{d\theta^2} + \omega_r^2 \xi_\phi^2 = 0 .$$

Exercise:- Stability analysis for $V'_{eff}(r)$ leads to same results as linearisation of eqn. for orbit.

$$\int \frac{dx}{\sqrt{a^2 - x^2}}$$

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