SC223 - Linear Algebra

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Lecture 12

August 23, 2023

Structure

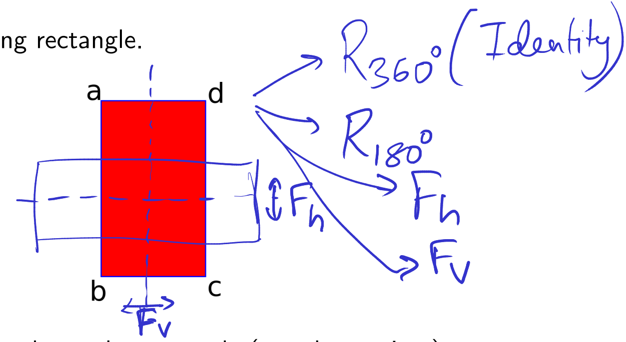
- What is *structure*?

Structure

- What is *structure*? Structure is the arrangement and relation between parts of an object, without getting into the particulars of an example or an instance.

Example of Structure in Math

- Consider the following rectangle.



- What transformations leave the rectangle (not the vertices) unchanged?

$S = \{I, R_{180°}, F_h, F_v\}$ with $\circ$

$(S, \circ)$, $\circ : S \times S \to ?$

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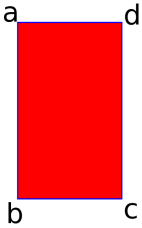

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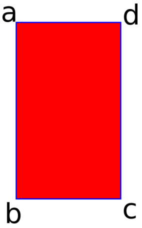

- What transformations leave the rectangle (not the vertices) unchanged?
- $S_r = \{I, F_h, F_v, R\}$. With Composition $\circ$, we have $(S_r, \circ)$:

Identity

Cayley Table.

$\circ$	I	F_h	F_v	R
I	I	F_h	F_v	R
F_h	F_h	I	R	F_v
F_v	F_v	R	I	F_h
R	R	F_v	F_h	I

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$+_2$	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

Identity

$a \in S$, $b \in S$, $a \circ b = \text{Identity}$.

1. $\circ$ is closed
2. Identity
3. Inverse.

- Compare the two:

$\circ$	I	F_h	F_v	R
I	I	F_h	F_v	R
F_h	F_h	I	R	F_v
F_v	F_v	R	I	F_h
R	R	F_v	F_h	I

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F_h	F_h	I	R	F_v
F_v	F_v	R	I	F_h
R	R	F_v	F_h	I

$+_2$	00	01	10	11
00	00	01	10	11
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- Consider the set of integers $\{1, 3, 5, 7\}$ with multiplication modulo-8.

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

- Compare the two:

$\circ$	I	F_h	F_v	R
I	I	F_h	F_v	R
F_h	F_h	I	R	F_v
F_v	F_v	R	I	F_h
R	R	F_v	F_h	I

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- Consider the set of integers $\{1, 3, 5, 7\}$ with multiplication modulo-8.
- Consider the set of matrices $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$, with matrix multiplication.

- Consider the set $\{e, a, b, a \cdot b\}$ with an operation $\cdot$

$\cdot$	e	a	b	$a \cdot b$
e	e	a	b	$a \cdot b$
a	a	e	$a \cdot b$	b
b	b	$a \cdot b$	e	a
$a \cdot b$	$a \cdot b$	b	a	e

	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b			
ab	ab			

$a \cdot a = ?$

$a^2 = b$

$\boxed{a^2 = a \times}$

$a^2 = e \to 1$

$a^2 = b \to 2$

$\boxed{a^2 = ab \times}$

$ab = b$

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- Consider the set $\{e, a, b, a \cdot b\}$ with an operation $\cdot$

$\cdot$	e	a	b	$a \cdot b$
e	e	a	b	$a \cdot b$
a	a	e	$a \cdot b$	b
b	b	$a \cdot b$	e	a
$a \cdot b$	$a \cdot b$	b	a	e

- Note that in these examples: $a^2 = b^2 = (ab)^2 = e$.

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$\times$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
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$\cdot$	e	a	b	$a \cdot b$
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- The first few examples can be abstracted as $S = \{e, a, b, a \cdot b\}$ with the operation $\cdot$ and

$\cdot$	e	a	b	$a \cdot b$
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- We denote the group by the tuple $(G, \cdot)$.

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$\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.

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- **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

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  - **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c, a \times (b +_F c) = a \times b +_F a \times c$

Examples of Fields

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- If the 3-tuple $(V, +, \cdot)$ with field $(\mathbb{F}, +_F, \times)$ satisfies all vector space axioms, we say that $(V, +, \cdot)$ forms a vector space over $\mathbb{F}$.

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● Any element of the vector space $(V, +, \cdot)$ will be referred to as a **vector**, and any element $a \in \mathbb{F}$ will be referred to as a **scalar**.

Examples of Vector spaces

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- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.
- $(\mathbb{L}_2(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathbb{L}_2(\mathbb{R})$ denotes the set of all square-integrable functions $f : \mathbb{R} \to \mathbb{R}$.