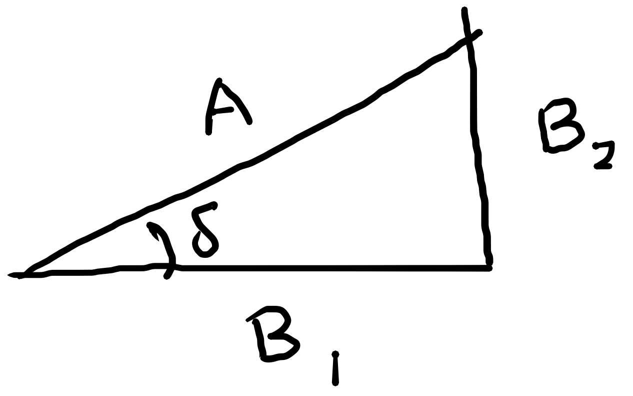
LECTURE 18



Recap:- Oscillatory motion.

$$\ddot{x} + \omega^2 x = 0.$$

$$\Rightarrow x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

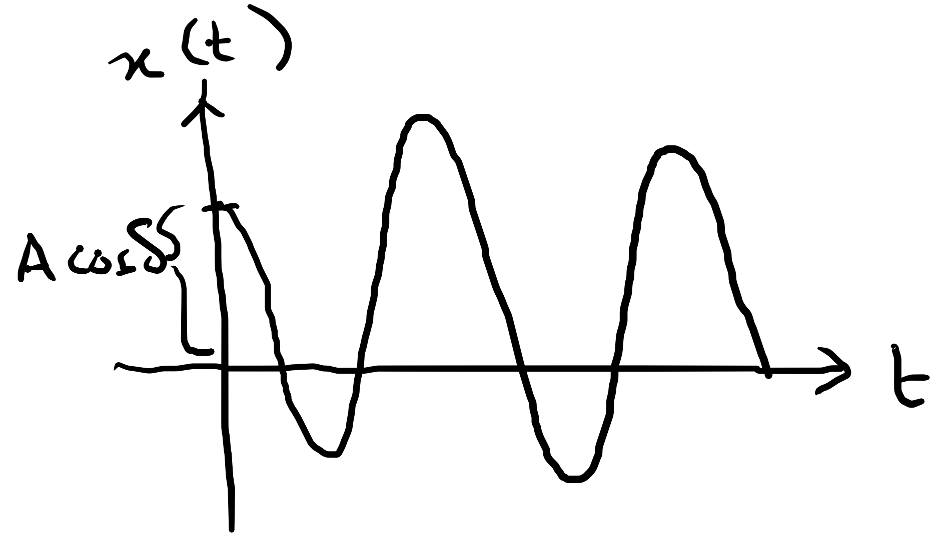
$$= B_1 \cos\omega t + B_2 \sin\omega t.$$

$$= A\left[\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right]$$

$$= A\left[\cos\delta\cos\omega t + \sin\delta\sin\omega t\right]$$

$$= A\cos(\omega t - \delta)$$

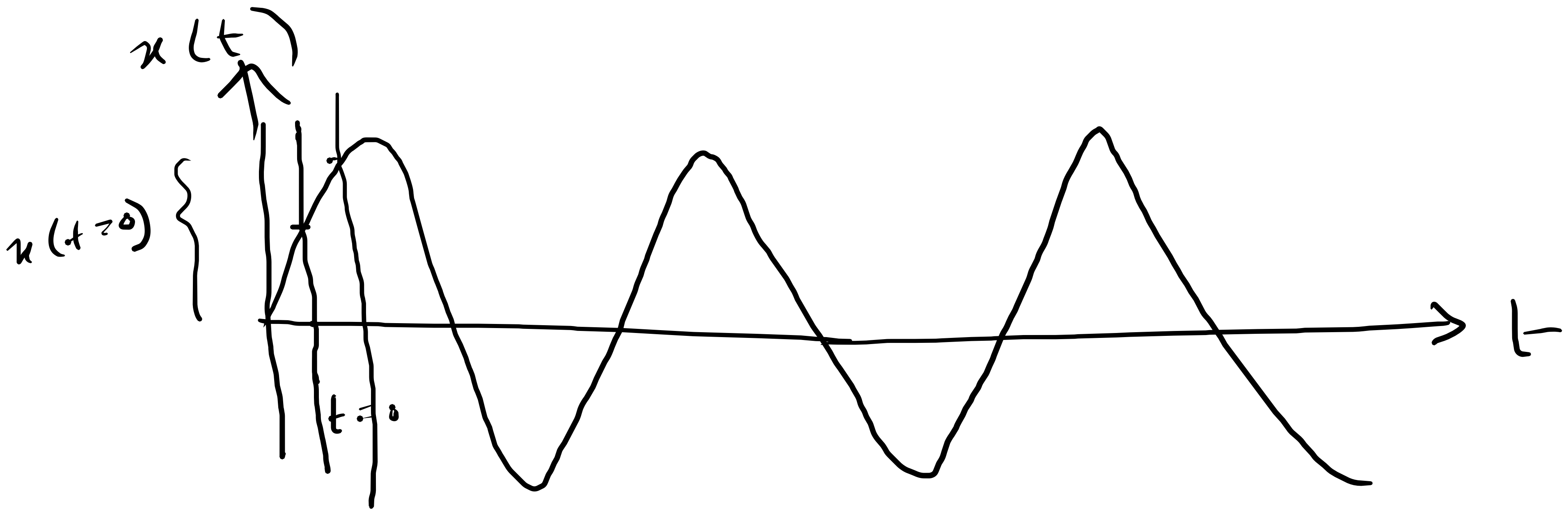
$x_{max} = A$.

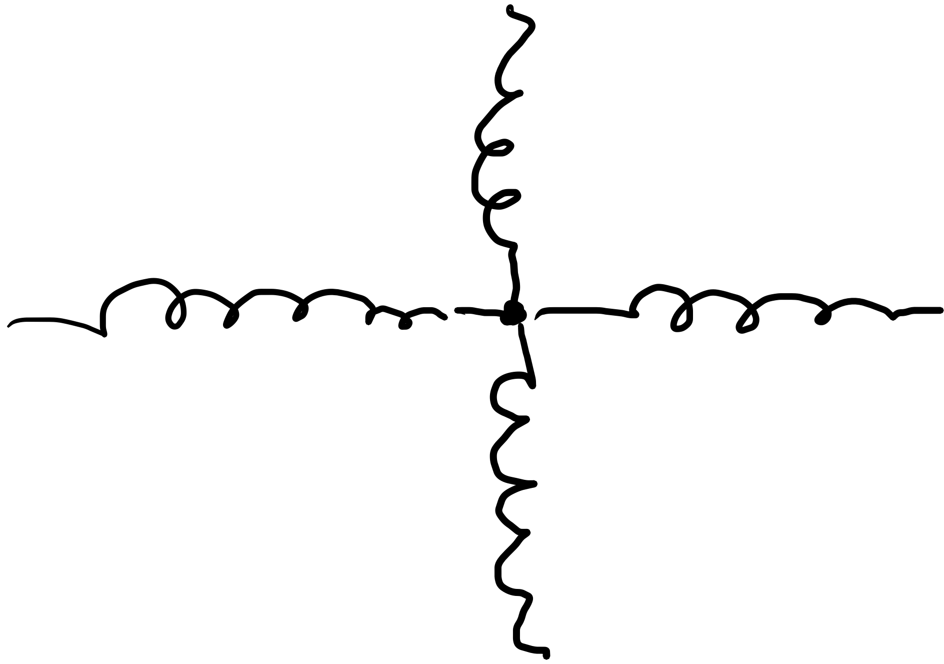


$\delta \equiv$ phase

$x(t=0) = A\cos\delta$.

$\boxed{\delta \rightarrow \text{depends on where } t=0 \text{ is placed.}}$

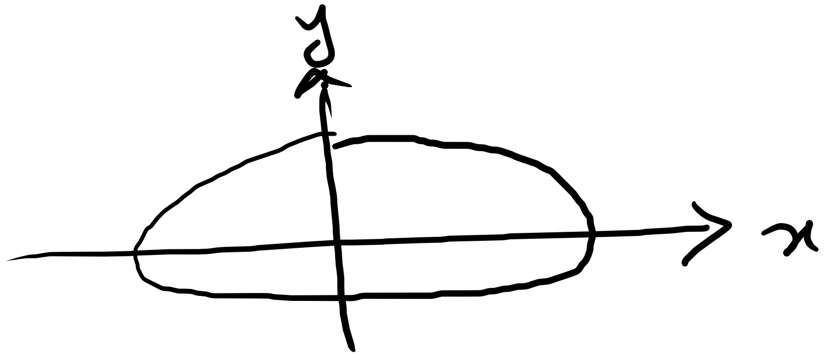




$$x(t) = A\cos(\omega t - \delta)$$

$$F = -kx$$

$$\vec{F} = -kx\,\hat{x} - ky\,\hat{y}$$



$\delta = \pi/2$

$$x(t) = A_x \cos(\omega t - \delta_x)$$

$$y(t) = A_y \cos(\omega t - \delta_y).$$

$$\delta = \delta_x - \delta_y$$

$\delta = \pi/4$

TWO-DIMENSIONAL OSCILLATOR.

$$\left\{\begin{array}{l} x = a\cos\omega t + c\sin\omega t \\ y = b\cos\omega t + d\sin\omega t\,. \end{array}\right\} \quad \begin{array}{l} \cos\omega t = \dfrac{dx - cy}{ad - bc} \\ \sin\omega t = \dfrac{ay - bx}{ad - bc}\,. \end{array}$$

$$\cos^2\omega t + \sin^2\omega t = 1$$

$$\Rightarrow (dx - cy)^2 + (ay - bx)^2 = (ad - bc)^2$$

$$\Rightarrow \underline{(b^2 + d^2)}x^2 + \underline{(c^2 + a^2)}y^2 - \underline{2(ab + cd)}xy = (ad - bc)^2$$

$$\Rightarrow \underbrace{Ax^2 + Bxy + Cy^2 = D}\,.$$

Can be cast in the form of an ellipse if,

$$B^2 - 4AC < 0\,.$$

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$$\Rightarrow 4(ab+cd)^2 - 4(b^2+d^2)(c^2+a^2) < 0.$$

$$\Rightarrow a^2\cancel{b^2} + c^2\cancel{d^2} + 2abcd - (b^2c^2 + a^2\cancel{b^2} + c^2\cancel{d^2} + a^2d^2) < 0$$

$\Rightarrow -(ad-bc)^2 < 0$ ∴, always true, provided $ad \neq bc$.

$\Rightarrow$ If $ad \neq bc$, the orbit in the xy plane is always going to be elliptical.

Simple pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta .$$

$$\Rightarrow \ddot{\theta} + \omega^2\theta = 0 .$$ → relies on small-θ approximation.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Master eqn:- $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta .$$

SIMPLE PENDULUM WITHOUT SMALL ANGLE APPROXIMATION

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta .$$

Let $u = \frac{d\theta}{dt}$.

$$\frac{du}{dt} = -\frac{g}{l}\sin\theta$$

$$\Rightarrow \frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -\frac{g}{l}\sin\theta .$$

$$\Rightarrow u\frac{du}{d\theta} = -\frac{g}{l}\sin\theta$$

$$\Rightarrow \frac{u^2}{2} = -\frac{g}{l}\int_{\theta_0}^{\theta} d\theta \sin\theta$$

$$\Rightarrow u^2 = \frac{2g}{l}\cos\theta \Big|_{\theta_0}^{\theta}$$

$$\Rightarrow u = \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$$

$$\Rightarrow \frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$$

$$\Rightarrow dt = \pm\frac{d\theta}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}$$

Say T = Time period.

$$T = -4\sqrt{\frac{l}{2g}}\int_{\theta_0}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} .$$

$$T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\cos\theta = 1 - 2\sin^2\theta$$

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

Let, $\sin\frac{\theta}{2} = \underbrace{\left(\sin\frac{\theta_0}{2}\right)}_{k} \sin\phi$

$= k\sin\phi$

$$\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}} = k\cos\phi$$

$$\frac{1}{2}\cos\frac{\theta}{2}\, d\theta = k\cos\phi\, d\phi$$

$$\Rightarrow \sqrt{1-k^2\sin^2\phi}\, d\theta = 2k\cos\phi\, d\phi$$

$$\Rightarrow d\theta = \frac{2k\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\, d\phi .$$

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \longrightarrow \text{elliptic integral.}$$

Taylor expansion in terms of ϕ.

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \cdots\cdots$$

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} d\phi \left(1 + \frac{1}{2}k^2\sin^2\phi + \cdots\right)$$

$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \frac{1}{2}k^2\int_0^{\pi/2} d\phi \sin^2\phi + \cdots\right]$$

$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \frac{1}{2}k^2\frac{\pi}{4} + \cdots\right]$$

$$= 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \cdots\right]$$

Systematic method of including corrections to the small-θ approximation.

DAMPED OSCILLATIONS.

— Consider a pendulum oscillating in a viscous medium.

$$m\frac{d^2\theta}{dt^2} = -k\theta - b\dot{\theta}$$

$$\Rightarrow \quad m\ddot{\theta} + b\dot{\theta} + k\theta = 0.$$

$$\Rightarrow \quad \ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{k}{m}\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\theta = 0.$$

Trial sol$\underline{n}$:- $\theta(t) = e^{rt}$.

$$(r^2 + 2\beta r + \omega_0^2)\, e^{rt} = 0$$

$$\Rightarrow \quad r^2 + 2\beta r + \omega_0^2 = 0.$$

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$$

$$r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}.$$

$$\theta(t) = e^{-\beta t}\left(A e^{\sqrt{\beta^2-\omega_0^2}\,t} + B e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right).$$

$\beta < \omega_0$ $\beta > \omega_0$ $\beta = \omega_0$.