# SC223 - Linear Algebra

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Lecture 5



August 2, 2023

• Can we encode ERO by matrices?

$$\left[\begin{array}{cccc|cccc}
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2 & 0 & 3 & 2 & 4 \\
-2 & 3 & -2 & 1 & 6 \\
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● Elementary Row Transformations(ERT) - Matrix representation of ERO

$$E_1 = \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{array} \right], E_2 = \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{array} \right],$$

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 $\bullet$   $E_k$ 's are lower triangular!

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- How about  $E = E_3 E_2 E_1$ ?

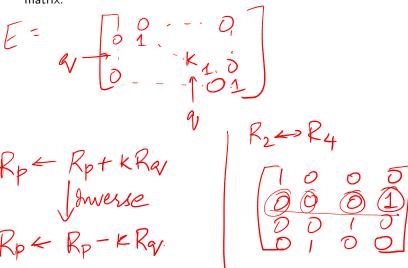
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$$E = \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3/2 & 1/4 & 1 & 0 \\ -7/11 & -3/11 & 10/11 & 1 \end{array} \right]$$

#### Theorem 2:

1. Using the convention for ERO, any ERT will be a lower triangular matrix.



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- 1. Using the convention for ERO, any ERT will be a lower triangular matrix. (except row exchange)
- 2. The product of any two lower triangular matrices is a lower triangular matrix.

A, B 
$$\rightarrow$$
 lower A .  $n \times n$   
 $C = AB$   
 $Cij = \sum_{K=1}^{n} Aik Bkj$ 

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#### • Theorem 2:

- 1. Using the convention for ERO, any ERT will be a lower triangular matrix. (except row exchange)
- 2. The product of any two lower triangular matrices is a lower triangular matrix.
- 3. Any ERT is an invertible matrix.
- 4. The inverse of any invertible lower triangular matrix is also a lower triangular matrix.

$$A = \left[ \begin{array}{rrrr} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{array} \right]$$

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- By Theorem 2,  $E^{-1}$  is a lower triangular matrix. Define  $L := E^{-1}$ . Thus A = LU, known as the **LU decomposition**.

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For this example:

$$\underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1/4 & 1 & 0 \\ 3 & 1/2 & -10/11 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 4 & 5 & 4 \\ 0 & 0 & -11/4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U}$$