LECTURE 27

AUTONOMOUS ODEs. (1st order)

$$\frac{dx}{dt} = f(t,x).$$

Autonomous :- $f(t,x) \equiv f(x)$.

$$\frac{dx}{dt} = f(x).$$ $\rightarrow$ In the R.H.S, there is no <u>explicit</u> dependence on t (independent variable).

Simple example of an autonomous ODE

$$\frac{dx}{dt} \propto x .$$

$$\Rightarrow \frac{dx}{dt} = \pm a x , \quad a > 0 .$$

Motivation: Population model.

$x(t) \equiv$ population of given species at time t.

$r(t,x) \equiv$ diff. in birth and death rates.

$$\frac{dx}{dt} = r(t,x)\, x(t)$$

Simplest model in which $r(t,x) = \text{constant}$.

$$- \quad \frac{dx}{dt} = \pm a x .$$

$$\Rightarrow \quad \frac{dx}{d(at)} = \pm x .$$

Define $T = at$.

$$\frac{dx}{dT} = \pm x$$

$$\Rightarrow \quad x(t) = A e^{\pm T}$$

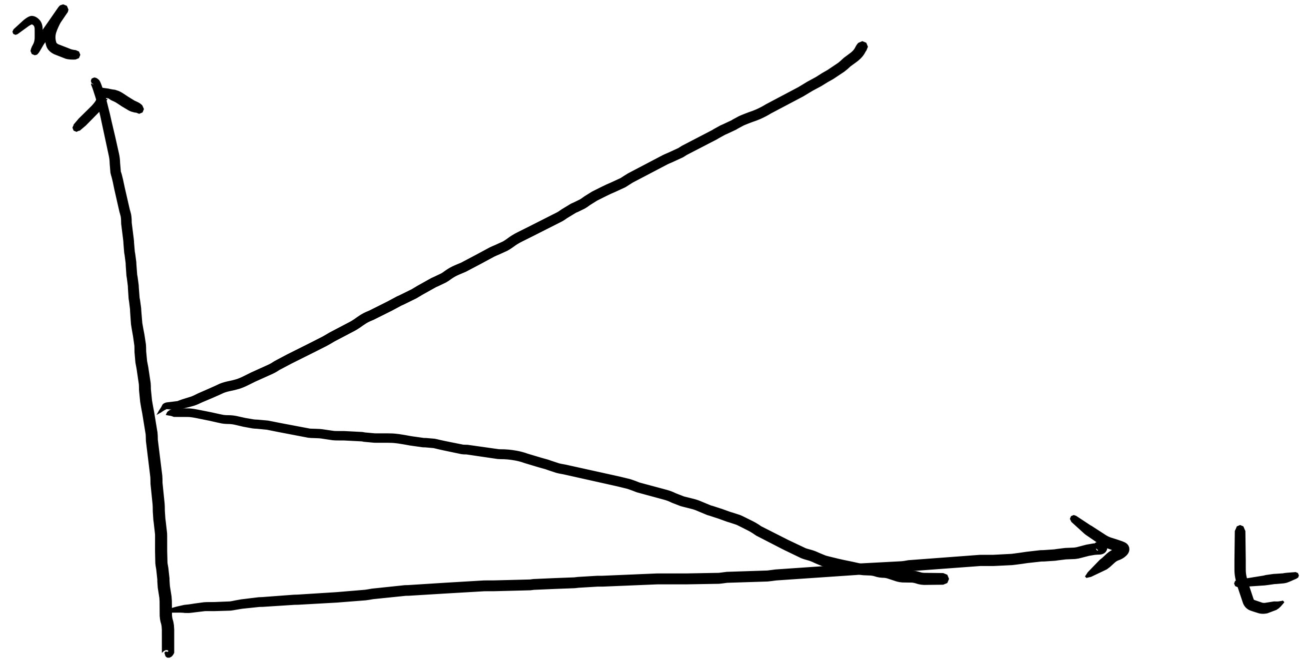
$$x(0) = x_0 \text{ (say)}$$

$$x(t) = x_0 e^{\pm T}$$

$$\Rightarrow \quad \frac{x}{x_0} = e^{\pm T}.$$

$$X = \frac{x}{x_0}$$

$$\Rightarrow \quad X = e^{\pm T}.$$



— Why "non-dimensionalization" is convenient.

$$\frac{dx}{dt} = a \pm bx .$$

Case I :- $\displaystyle \frac{dx}{dt} = a - bx = b\left(\frac{a}{b} - x\right).$

$$\Rightarrow \quad \frac{dx}{d(bt)} = \frac{a}{b} - x$$

Define $T = bt$, $\quad x_0 = \frac{a}{b}$.

$$\frac{dx}{dT} = x_0 - x = x_0\left(1 - \frac{x}{x_0}\right)$$

$$\frac{dx}{dT} = x_0\left(1 - \frac{x}{x_0}\right)$$

Define $X = \frac{x}{x_0}$.

$$\frac{d}{dT}\left(\frac{x}{x_0}\right) = 1 - \frac{x}{x_0}$$

$T = bt$

$$\Rightarrow \boxed{\frac{dX}{dT} = (1 - X).}$$

$$\Rightarrow \quad -\ln(1-X) = T + \text{const}$$

$$\Rightarrow \quad 1 - X = Ae^{-T}$$

Let $t = 0 \Rightarrow T = 0$, $x = 0 \Rightarrow X = 0 \Rightarrow 1 = A$.

$$\boxed{X = \left(1 - e^{-T}\right)}$$

$$\boxed{X = 1 - e^{-T}}$$ $\rightarrow$ dimensionless terms.

$$x = x_0 \left(1 - e^{-bt}\right)$$ $\rightarrow$ "dimensionfull" terms

$$= \frac{a}{b} \left(1 - e^{-bt}\right).$$

Small-time limit $T \ll 1$. $e^{-T} = 1 - T + \ldots$

$$X \approx T.$$

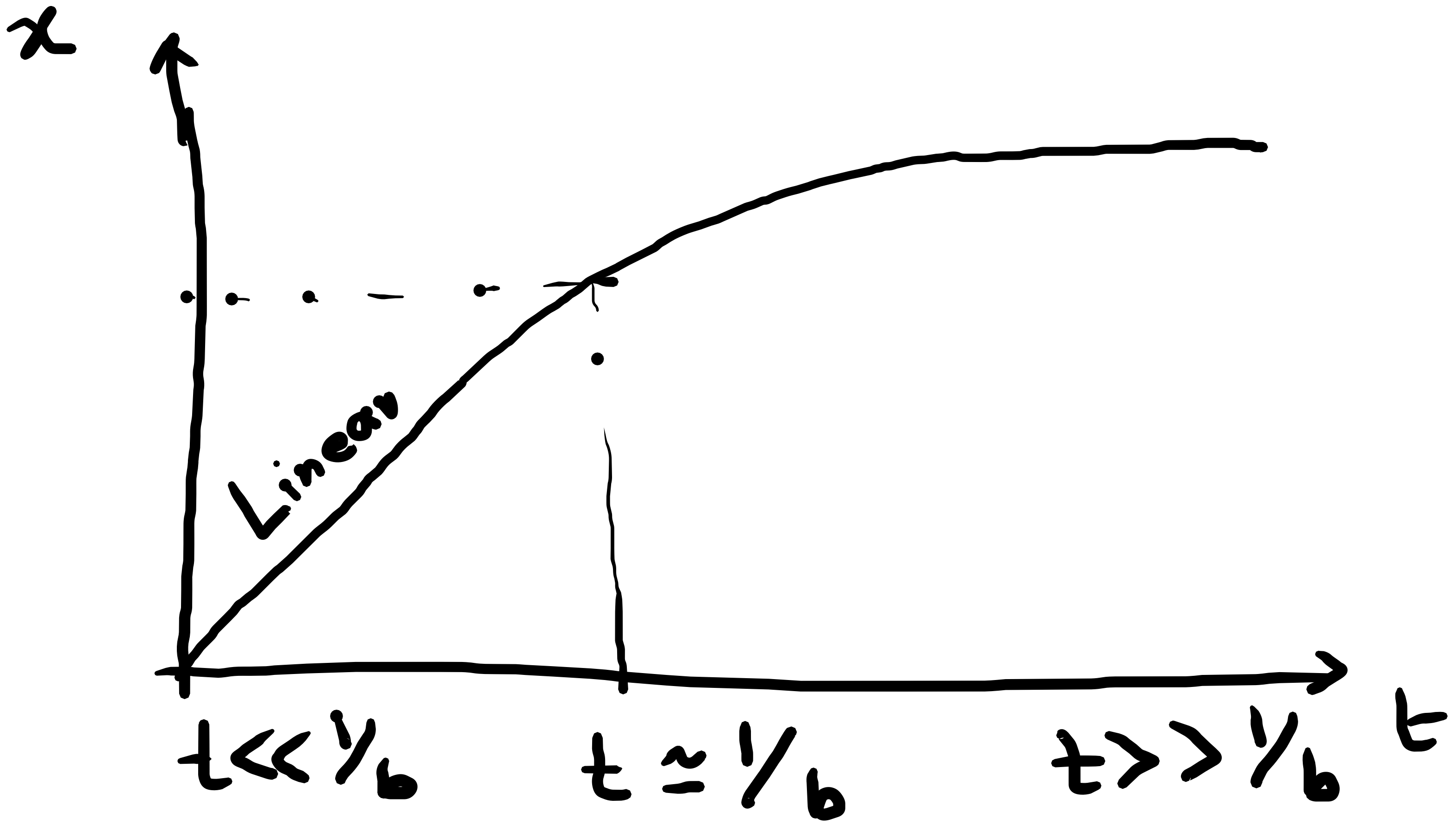
Large-time limit $T \gg 1$, $e^{-T} \rightarrow 0$.

$$X \simeq 1$$ (equivalent to $x = a/b$).

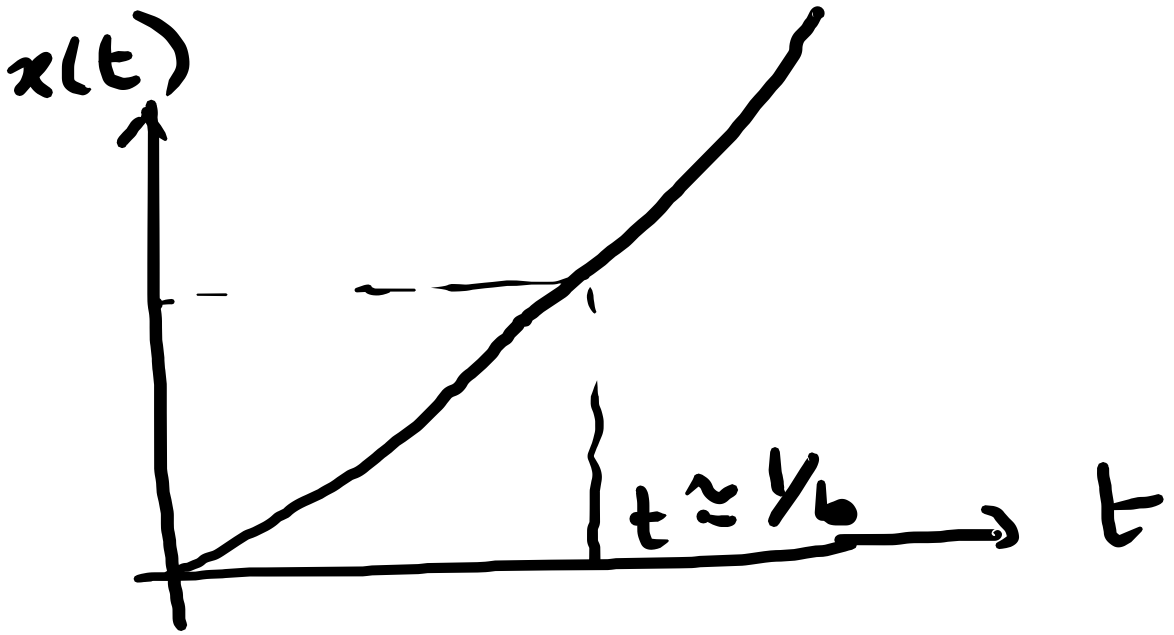
- Somewhere for some T, behaviour of X has changed.
- Change occurs at $T \approx 1$.

$\Rightarrow \quad \underline{t \simeq (1/b)}$. ↳ characteristic time scale at which behaviour of solution changes. depends on system parameter b.

- "Non-dimensionalization" is sometimes useful in identifying such characteristic values at which behaviour of the solution changes.



$T \simeq 1$



Case II: $\dfrac{dx}{dt} = a + bx$

$$x(t) = \frac{a}{-b}\left(1 - e^{bt}\right)$$

$$= \frac{a}{b}\left(e^{bt} - 1\right).$$

When $t << 1/b$,

$$x \simeq \frac{a}{b}(bt) \simeq at$$

When $t >> 1/b$,

$$x(t) \simeq \frac{a}{b} e^{bt}.$$

REFINEMENT OF THE MODEl.

$$\boxed{\frac{dx}{dt} = ax - bx^2}, \quad a > 0, \; b > 0.$$

$\downarrow$

LOGISTIC EQUATION

Motivation:- Suitable to model a large population, where members are competing for finite resources

$-bx^2 \rightarrow$ competition term.

$$\frac{dx}{dt} = (ax - bx^2) = ax\left(1 - \frac{x}{a/b}\right).$$

$$\Rightarrow \frac{dx}{d(at)} = x\left(1 - \frac{x}{a/b}\right).$$

Define $K = (a/b)$.

Define $T = at$

$X = \frac{x}{K}$.

$$\Rightarrow \frac{dx}{d(at)} = x\left(1 - \frac{x}{K}\right).$$

$$\Rightarrow \frac{dx}{dT} = x(1 - X)$$

$$\Rightarrow \frac{dX}{dT} = X(1 - X).$$

$$\Rightarrow \frac{dX}{X(1-X)} = dT.$$

Let $\dfrac{1}{X(1-X)} = \dfrac{A}{X} + \dfrac{B}{1-X} = \dfrac{(B-A)X + A}{X(1-X)}$

Comparing $A = 1$, $B = 1$.

$$\int dX \left(\frac{1}{X} + \frac{1}{1-X} \right) = \int dT.$$

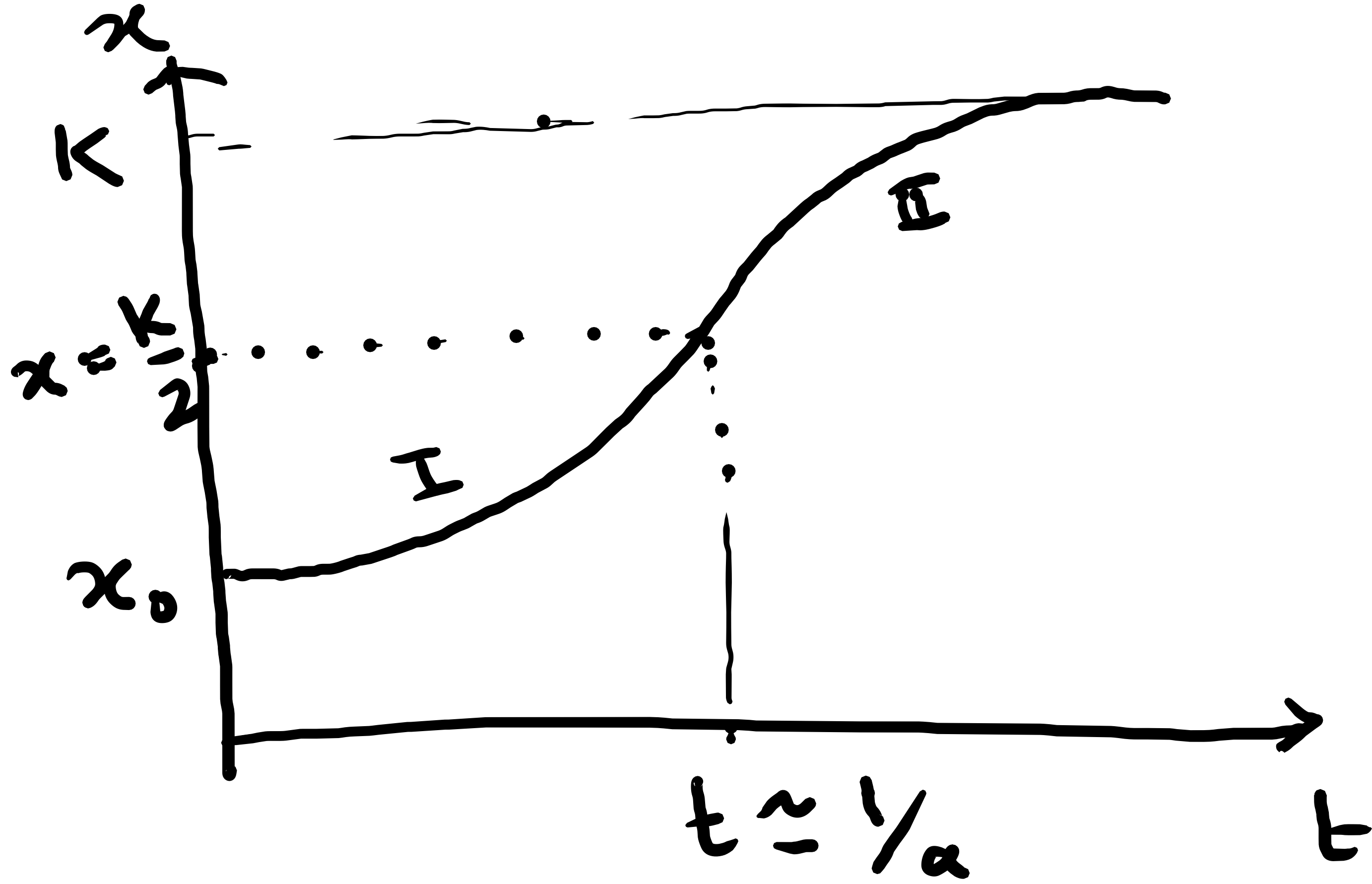
$$\Rightarrow \ln \frac{X}{1-X} = T + c$$

$$\Rightarrow \frac{X}{1-X} = A e^{T} \quad \Rightarrow \quad X = \frac{1}{1 + A^{-1} e^{-T}}.$$

At $t=0$, $X = X_0$ $(x = x_0)$.

$$\frac{X_0}{1-X_0} = A.$$

In "dimensionfull" terms, $X \simeq \frac{1}{2}$



$T \simeq 1$

$$x = \frac{K x_0}{x_0 + (K - x_0)\, e^{-at}}$$

At large-time limit,

$$x \simeq K$$

At small-time limit, $x \simeq \dfrac{x_0 e^{at}}{1 + \frac{x_0}{K} a t}$.

☑ More analysis.

$$\frac{dX}{dT} = X - X^2 = F(X).$$

— From graph, it is obvious that rate of growth of X (or equivalently x) has changed sign somewhere in between.

$$\frac{d^2X}{dT^2} = \frac{dF}{dX}\frac{dX}{dT} = F\frac{dF}{dX}.$$

— $\frac{d^2X}{dT^2} = 0$ when $\frac{dF}{dX} = 0 \Rightarrow 1 - 2X = 0$

$\Rightarrow X = \frac{1}{2}.$

- $\frac{dX}{dT}$ changes sign at $X = \frac{1}{2}$, $F(x)$ has a <u>turning point</u>.