SC223 - Linear Algebra

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Lecture 32

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Summary of Lecture 31

- If V is a 1-dimensional $T-$ invariant subspace, $\forall u \in V, u \neq \theta$, $Tu = \lambda u$. λ is the eigenvalue associated with eigenvector u.
- To compute eigenvalues and eigenvectors of T, find roots of the characteristic polynomial: $c(x) = det([T]_\beta^\beta - xI_n)$. For any root λ, $u \neq \theta, u \in N(T - \lambda I)$ is an eigenvector.
- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{u_1, \ldots, u_n\}$.
- Since $Tu_i = \lambda_i u_i$, $[T]_e^e = \Lambda = diag(\lambda_1, \ldots, \lambda_n)$, and in any other basis β, $[T]_\beta^\beta [u_i]_\beta = \lambda_i [u_i]_\beta$.
- Thus,

$$[T]_\beta^\beta E = E\Lambda \Rightarrow \Lambda = [T]_e^e = E^{-1}[T]_\beta^\beta E$$

- The process of similarity transformation on a matrix A, using eigenvectors as columns of a matrix, say E, to get a diagonal matrix Λ: $\Lambda = E^{-1}AE$ is called **matrix diagonalization**.

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Denoted as $AM(\lambda)$
- **Geometric Multiplicity of λ**: $GM(\lambda) = dim(N(T - \lambda I))$.

Examples

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$[R_\theta]_\beta^\beta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$c(x) = \det\left(\begin{bmatrix} \cos\theta - x & -\sin\theta \\ \sin\theta & \cos\theta - x \end{bmatrix} \right) = 1 - 2\cos\theta\, x + x^2$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$= \cos\theta \pm i\sin\theta$$

$$\lambda = e^{\pm i\theta}$$

$\underline{R_\theta : \mathbb{C}^2 \to \mathbb{C}^2}$

$\lambda = e^{i\theta}$

$$[R_\theta - \lambda I_2]_\beta^\beta = \begin{bmatrix} \cos\theta - e^{i\theta} & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta - e^{i\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

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$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$[R_\theta]_e^e = \Lambda = E^{-1} [R_\theta]_\beta^\beta E = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

Example

- Let $S_N = \{x : \mathbb{Z} \to \mathbb{C} \mid x[n+N] = x[n], \forall n \in \mathbb{Z}, x[n] \in \mathbb{C}, \forall n \in \mathbb{Z}\}$.

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- Let $\beta = \{\delta_0, \ldots, \delta_{N-1}\}$ denote the basis of S^N, where $\forall k = 0, \ldots, N-1$,

$$\begin{aligned} \delta_k[n] &= 1, n = k + m \cdot N, m \in \mathbb{Z} \\ &= 0, \text{ else } . \end{aligned}$$

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$N = 3$ $\quad \ldots, x[2], x[0], x[1], x[2], x[0], x[1], \ldots$

$n=0$

$Dx \quad \ldots x[0], x[1], x[2], x[0], x[1], x[2], x[0], \ldots$

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$$[Dx]_\beta[n] = x[(n-1) \bmod N]$$

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- The last equation is called *Circular Convolution*, and is denoted by $x \circledast h$.
- By change of variables, $m = (n-k) mod N$, i.e. $n - k = m + pN$, we get $y[n] = \sum_{m=0}^{N-1} x[(n-m) mod N]h[m], n = 0, \ldots, N-1$.
- Circular convolution is thus commutative: $x \circledast h = h \circledast x$.

- $$[T]_{\beta}^{\beta} = \begin{bmatrix} h[0] & h[N-1] & \dots & h[1] \\ h[1] & h[0] & \dots & h[2] \\ \vdots & \vdots & \vdots & \vdots \\ h[N-1] & h[N-2] & \dots & h[0] \end{bmatrix}$$
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End of Class.

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- Let $H(p) := \left(\sum_{k=0}^{N-1} h[k] w^{-kp}\right), p = 0, \ldots, N-1$.
- We have shown that $Tf_p = H(p) f_p, p = 0, \ldots, N-1$, and so $[T]_\beta^\beta [f_p]_\beta = H(p)[f_p]_\beta, p = 0, \ldots, N-1$.

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- Assuming (for now) $F = \{f_0, \ldots, f_{N-1}\}$ are linearly independent, F forms an eigenbasis for any shift-invariant linear operator on S_N, and thus

$$[T]_\beta^\beta \begin{bmatrix} | & | & \cdots & | \\ [f_0]_\beta & [f_1]_\beta & \cdots & [f_{N-1}]_\beta \\ | & | & \cdots & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & \cdots & | \\ [f_0]_\beta & [f_1]_\beta & \cdots & [f_{N-1}]_\beta \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} H(0) & 0 & \cdots & 0 \\ 0 & H(1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & H(N-1) \end{bmatrix}$$

- Assuming (for now) $F = \{f_0, \ldots, f_{N-1}\}$ are linearly independent, F forms an eigenbasis for any shift-invariant linear operator on S_N, and thus

$$[T]_\beta^\beta \begin{bmatrix} | & | & \ldots & | \\ [f_0]_\beta & [f_1]_\beta & \ldots & [f_{N-1}]_\beta \\ | & | & \ldots & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & \ldots & | \\ [f_0]_\beta & [f_1]_\beta & \ldots & [f_{N-1}]_\beta \\ | & | & \ldots & | \end{bmatrix} \begin{bmatrix} H(0) & 0 & \ldots & 0 \\ 0 & H(1) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & H(N-1) \end{bmatrix}$$

- $H(p) = \sum_{n=0}^{N-1} h[n] w^{-np}, p = 0, \ldots, N-1$ is called the *Discrete Fourier Transform(DFT)* of the sequence h.

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$$= \begin{bmatrix} | & | & \cdots & | \\ [f_0]_\beta & [f_1]_\beta & \cdots & [f_{N-1}]_\beta \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} H(0) & 0 & \cdots & 0 \\ 0 & H(1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & H(N-1) \end{bmatrix}$$

- $H(p) = \sum_{n=0}^{N-1} h[n] w^{-np}, p = 0, \ldots, N-1$ is called the *Discrete Fourier Transform(DFT)* of the sequence h.
- Also, in the basis F,

$$[T]_F^F = \begin{bmatrix} H(0) & 0 & \cdots & 0 \\ 0 & H(1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & H(N-1) \end{bmatrix}$$

Results on Eigenvectors and Eigenvalues

- **Proposition 20**: Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T \in \mathcal{L}(U)$. Then the eigenvectors $v_1, \ldots, v_m$ associated with these eigenvalues are linearly independent.