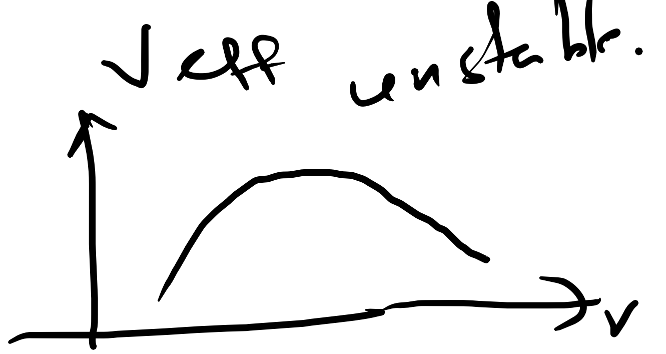
LECTURE 16

Recap :-

- Studied perturbations of a circular orbit. If the perturbation grew with time, orbit was unstable, otherwise stable.
- Circular orbit was a result of $\frac{dV_{eff}}{dr} = 0$ having a solution.

Natural to expect stability / unstable nature of orbit being related to $\operatorname{sgn}\left(\left.\frac{d^2 V_{eff}}{dr^2}\right|_{r=a} \right)$



— $\text{sgn}\left(3 + \frac{a f'(a)}{f(a)}\right)$

— Should be possible that above quantity related to

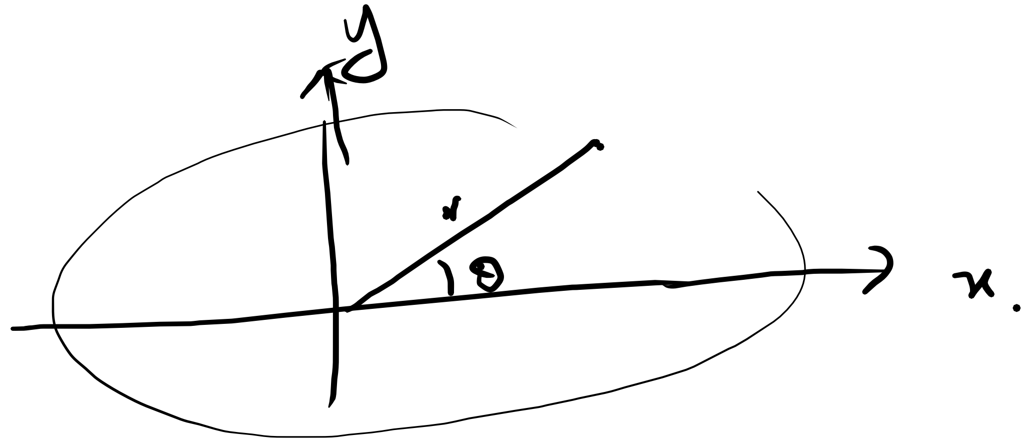
$$\left.\frac{d^2 V_{eff}}{dr}\right|_{r=a}$$

— Exercise :- $f(r) = k r^{\nu}$.

III Computational approach.

$$\frac{d^2u}{d\theta^2} + \cdots$$

$$\left(\frac{du}{d\theta}\right)^2 + \cdot \quad -$$



— Above equns. may be relatively more difficult to handle.

— $$\frac{d^2x}{dt^2} = F_x = F\cos\theta = F\,\frac{x}{r} \qquad r = \sqrt{x^2+y^2}$$

$$\frac{d^2y}{dt^2} = F_y = F\sin\theta = F\,\frac{y}{r}.$$

$F \rightarrow$ can be derived from potential.

x^2+y^2

— $x(0)$

$y(0)$

$v_x(0)$

$v_y(0)$

$E = K.E. + P.E$

$= K.E. + V(r).$

↳ can be evaluated from initial conditions and remains const. throughout.

$$\epsilon = \sqrt{1 + \frac{2EL^2}{(\ \ )}}$$

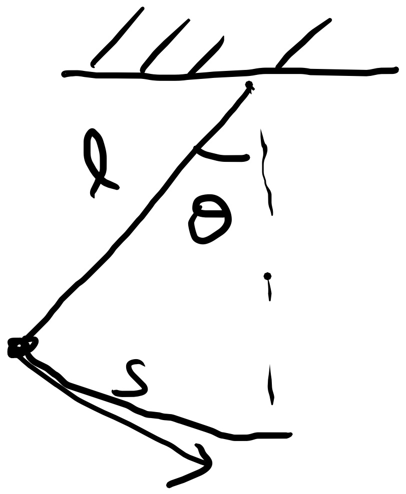
- $$\left.\begin{aligned} \frac{dx}{dt} &= v_x \\ \frac{dv_x}{dt} &= F_x \\ \frac{dy}{dt} &= v_y \\ \frac{dv_y}{dt} &= F_y \end{aligned}\right\} \Rightarrow$$

Discretise by Euler's method.

↳ unsuitable for oscillatory problem.

— Problems with Euler's method for oscillatory systems.

— Consider simple pendulum.



In small-angle approx. $\sin\theta \simeq \theta$.

$$\ddot{s} = -g\sin\theta = -g\theta$$

$$\Rightarrow l\ddot{\theta} = -g\theta$$

$$\Rightarrow \ddot{\theta} + \underbrace{\frac{g}{l}}_{\sim\ \Omega^2}\theta = 0$$

$$\theta = A\cos\Omega t + B\sin\Omega t.$$

$$- \quad E = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta) \approx \frac{1}{2} m l^2 \omega^2 + \frac{1}{2} mgl\,\theta^2$$

(expanding $\cos\theta$).

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\boxed{\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\frac{g}{l}\theta \end{aligned}} \Rightarrow$$

Discretise by Euler

$$\theta_{n+1} = \theta_n + \omega_n \Delta t.$$

$$\omega_{n+1} = \omega_n - \frac{g}{l}\theta_n \Delta t.$$

$$E_{n+1} = \frac{ml^2}{2}\left[\omega_{n+1}^2 + \frac{g}{l}\theta_{n+1}^2\right]$$

$$= \frac{ml^2}{2}\left[\left(\omega_n - \frac{g}{l}\theta_n \Delta t\right)^2 + \frac{g}{l}\left(\theta_n + \omega_n \Delta t\right)^2\right]$$

$$= \frac{ml^2}{2}\left[\omega_n^2 + \frac{g^2}{l^2}\theta_n^2 \Delta t^2 + \frac{g}{l}\theta_n^2 + \frac{g}{l}\omega_n^2 \Delta t^2\right]$$

$$= \frac{ml^2}{2}\left(\omega_n^2 + \frac{g}{l}\theta_n^2\right) + (\quad)(\quad)\Delta t^2$$

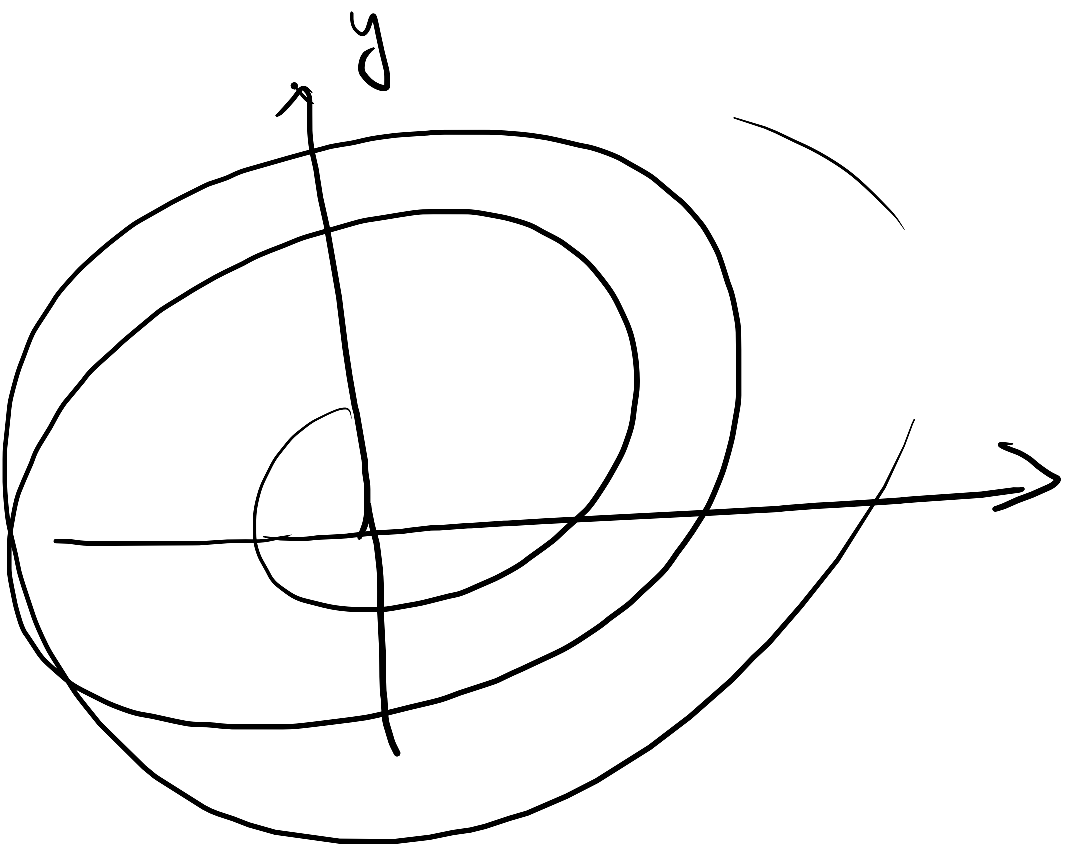
$$= E_n + (\quad)\Delta t^2$$

$$\Rightarrow E_{n+1} - E_n = (\quad)\Delta t^2 \neq 0.$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m g l \theta^2$$

$$\frac{dE}{dt} = m l^2 \dot{\theta}\ddot{\theta} + m g l \theta \dot{\theta} = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$



— Euler's method does not conserve energy.

— symplectic methods conserve energy.

— Modifications of Euler's algorithm.

$$\frac{dy}{dt} = f(t, y)$$

$$\frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1}, y_{n+1})$$

$$\omega_{n+1} = \omega_n - \frac{g}{l}\theta_n \Delta t$$

$$\theta_{n+1} = \theta_n + \omega_n \Delta t \ . \rightarrow \text{Euler}$$

Modified $\theta_{n+1} = \theta_n + \omega_{n+1} \Delta t \ . \rightarrow$ Euler-Cromer method

— Substitute Euler-Cromer discretisation,

$$E_{n+1} - E_n = (\quad)\Delta t^2 - (\,)\Delta t^3 - (\quad)\Delta t^4$$

— Still not very accurate.

— Need to look at other integration methods.

<u>Taylor series</u>:-

$$\frac{dy}{dt} = f(t, y)$$

$$y_{n+1} = y_n + \Delta t\, f(t_n, y_n).$$

Alternative, expand R.H.S. in a Taylor series

$$y_{n+1} = y_n + \Delta t \frac{dy}{dt} + \frac{\Delta t^2}{2}\frac{d^2y}{dt^2} + \dots$$

$\frac{d^n y}{dt^n}$ can be evaluated from the eqn.

— Tedious to evaluate derivatives.

▨ Runge-Kutta (RK) methods

$$\frac{dy}{dt} = f(t,y).$$

$$\Rightarrow \quad dy = dt\, f(t,y).$$

$$\int_{y_n}^{y_{n+1}} dy = \int_{t_n}^{t_{n+1}} dt\, f(t,y).$$

$$\Rightarrow \quad y_{n+1} = y_n + \underbrace{\int_{t_n}^{t_{n+1}} dt\, f(t,y)}_{\text{approximate integral numerically.}}$$

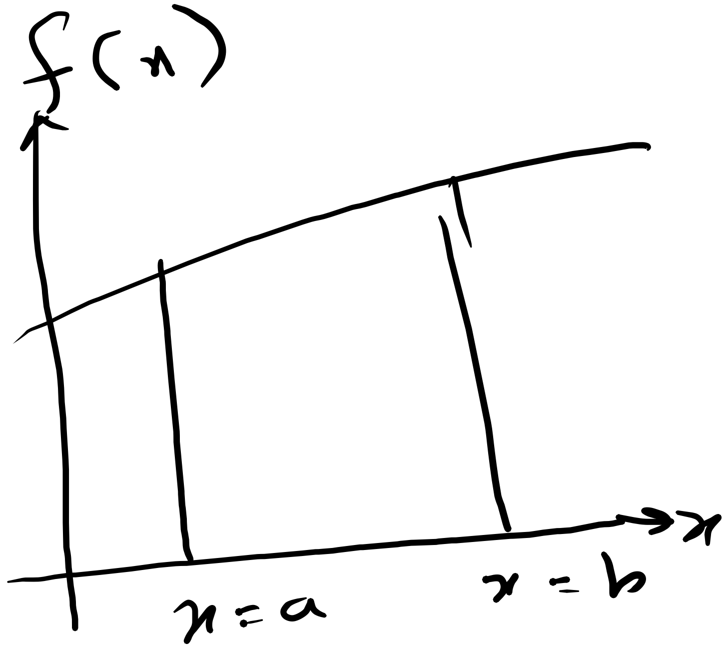
— $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} dt\, f(t, y)$.

$= y_n + f(t_n, y_n) \int_{t_n}^{t_{n+1}} dt$

$= y_n + f(t_n, y_n)\Delta t$. $\longrightarrow$ Euler's rule.

— Trapezoidal rule.

$$\int_a^b dx\, f(x) = \left(\frac{b-a}{2}\right)\left[f(a) + f(b)\right]$$



Using above result,

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

↳ Eqn. is implicit in y_{n+1} hence difficult to solve.

— Solution is, $y_{n+1} = y_n + \Delta t \underbrace{f(t_n, y_n)}_{k_1}$

$$f(t_{n+1}, y_{n+1}) = f(t_n + \Delta t, y_n + \Delta t k_1) = k_2$$

$$y_{n+1} = y_n + \frac{\Delta t}{2}(k_1 + k_2).$$

$\Downarrow$

2nd order RK method.

— Simpson's rule:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \tfrac{1}{2}\Delta t,\ y_n + \tfrac{1}{2}\Delta t\, k_1)$$

$$k_3 = f(t_n + \tfrac{1}{2}\Delta t,\ y_n + \tfrac{1}{2}\Delta t\, k_2)$$

$$k_4 = f(t_n + \Delta t,\ y_n + \Delta t\, k_3)$$

$$\boxed{y_{n+1} = y_n + \tfrac{1}{6}\Delta t\left(k_1 + 2k_2 + 2k_3 + k_4\right)}$$

$$\Downarrow$$

4th order RK4 method.