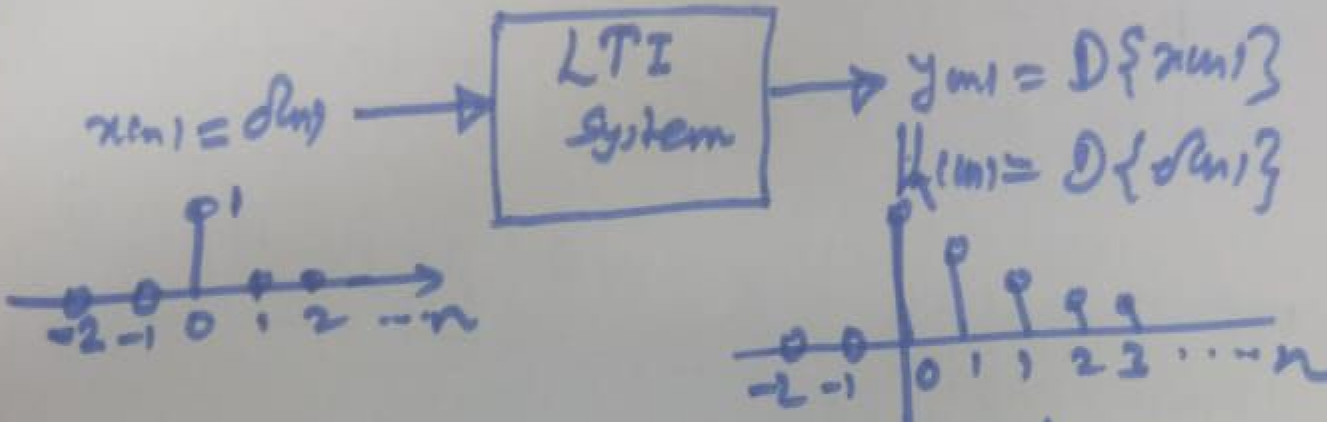
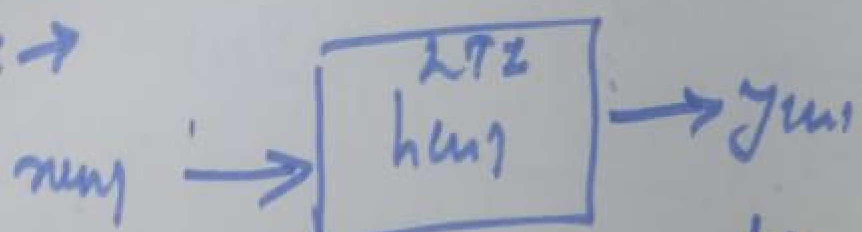
* Significance of Impulse Response of LTI System :→



Claim: The impulse response h(n) characterize an LTI system completely

<u>Proof</u>: →



$$y(n) = \underline{x(n) * h(n)} = \sum_{k=-\infty}^{+\infty} x(k)\cdot h(n-k)$$

Using sifting property of impulse signal,

$$x(n) = \sum_{k=-\infty}^{+\infty} x(k)\cdot \delta(n-k) = x(n) * \delta(n)$$

$\boxed{x[n] = x[n] * \delta[n]}$ → key result.

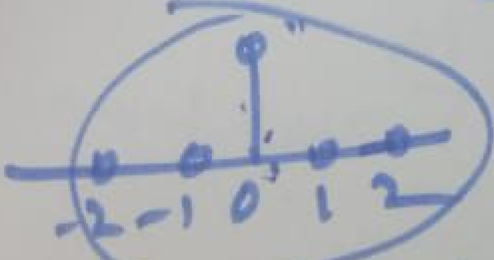
⇒ Unit impulse signal, $\delta[n]$, is an identity element under convolution.

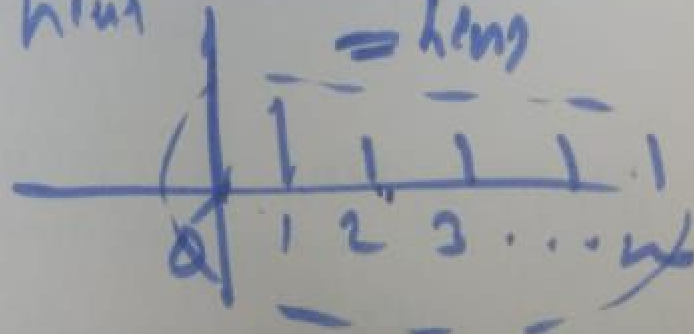
⇒ Impulse has NO property in its own right.

⇒ In signals and systems we are using impulse signal as excitation signal, that has NO property in its own right.

$$x[n] = \delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = D\{x[n]\} = D\{\delta[n]\} = h[n]$$

h[n]





No property

Lightening effect

∴ The output response (i.e., in this called impulse response) will have characteristic solely due to system and not due to excitation source.

Problem.

[2] Give a counterexample where problem ① is NOT valid for nonlinear systems

Sol^{n}:→ Consider two systems.

$$S1: \quad y(n) = [x(n) + x(n-1)]^2 \; ; \qquad$$

$$S_2: \quad y(n) = \max(x(n), x(n-1))$$

Nonlinear system.

→ $h_1(n)$ = Impulse response for system S_1

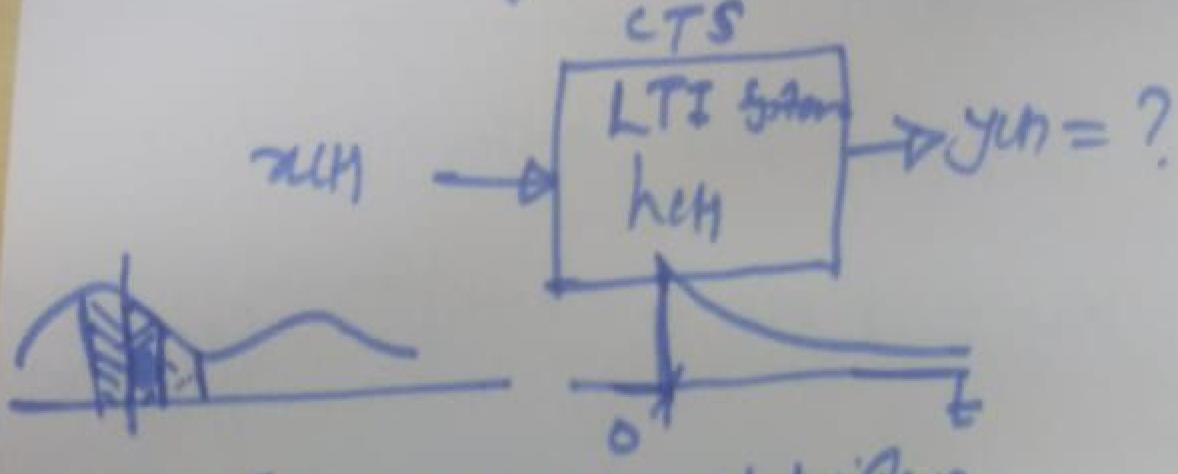
$$h_1(n) = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$h_2(n) = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Inference 1) There is NO unique relationship between system's impulse response and its input-output relationship, for a nonlinear system.

Inference 2) Impulse response of a nonlinear system cannot characterize the system completely ③

Case II Development of Mathematical Model or Analytical Tool for CTS:→



Step I Input-output relationship.

$$y(t) = H\{x(t)\} \quad —(1)$$

where $H\{\cdot\}$ = is system characterization, in this case, $H\{\cdot\}$ is given to be LTI

Step II) Using sifting property of continuous-time impulse function,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau \quad —(2)$$

Using fundamental theorem of integral calculus,

$$x(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{+\infty} x(k\Delta)\cdot\delta_\Delta(t-k\Delta)\cdot\Delta \quad —(3)$$

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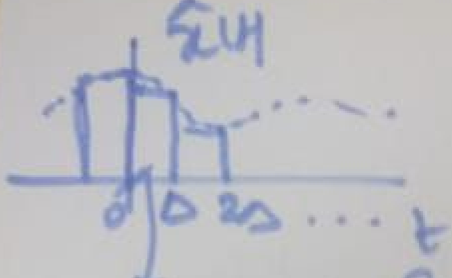
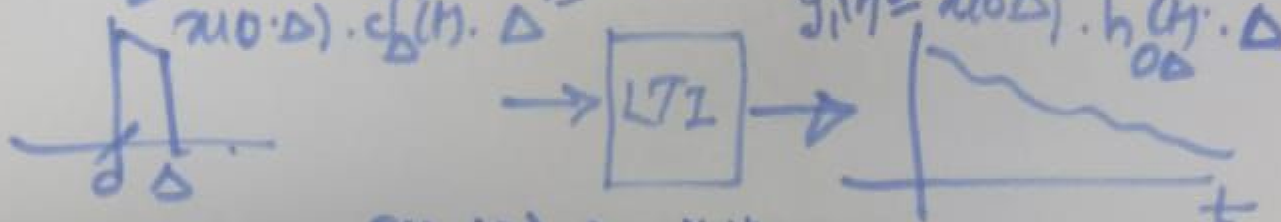
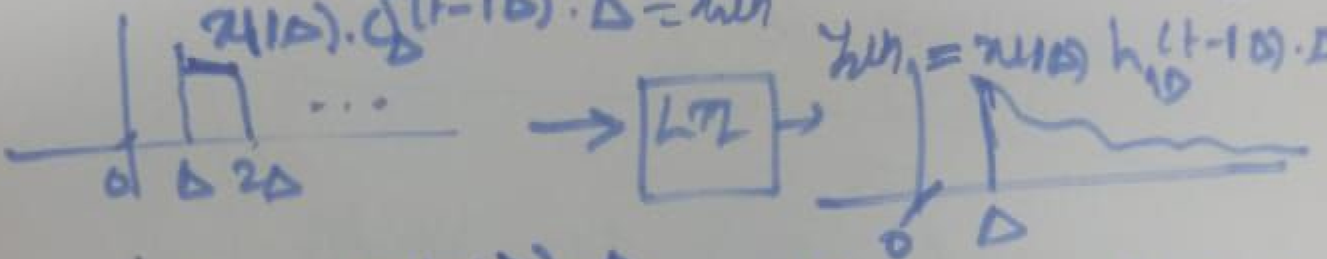


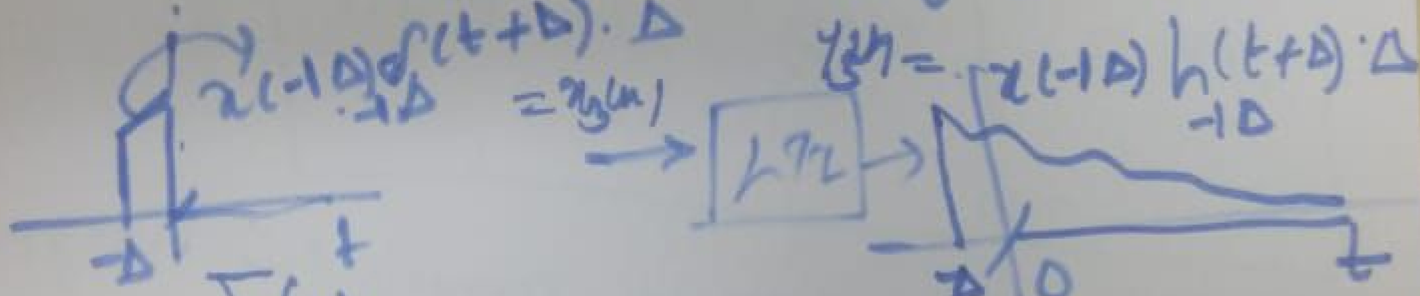
$x(0\cdot\Delta)\cdot\delta_\Delta(t)\cdot\Delta = x_1(t)$ → LTI → $y_1(t) = x(0\Delta)\cdot h_{0\Delta}(t)\cdot\Delta$



$x(1\Delta)\cdot\delta_\Delta(t-1\Delta)\cdot\Delta = x_2(t)$ → LTI → $y_2(t) = x(1\Delta)\, h_{1\Delta}(t-1\Delta)\cdot\Delta$

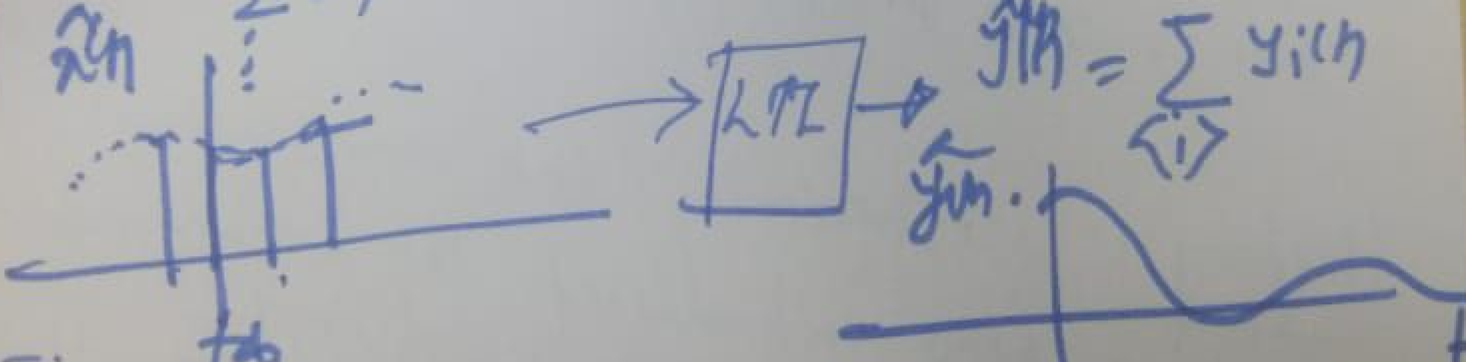


$x(-1\Delta)\delta_{\Delta}(t+\Delta)\cdot\Delta = x_3(t)$ → LTI → $y_3(t) = x(-1\Delta)\, h_{-1\Delta}(t+\Delta)\cdot\Delta$



$\sum(\cdot)$

$\hat{x}(t)$ → LTI → $\hat{y}(t) = \sum_{\langle i\rangle} y_i(t)$



$\hat{x}(t) \doteq \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$ → LTI → $\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\, h_{k\Delta}(t-k\Delta)\cdot\Delta$

Take $\lim_{\Delta\to 0}$

$x(t) = \lim_{\Delta\to 0}\hat{x}(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{+\infty} x(k\Delta)\cdot\delta_\Delta(t-k\Delta)\cdot\Delta$ → LTI → $y(t) = \lim_{\Delta\to 0}\hat{y}(t)$

$\therefore y(t) = \lim_{\Delta \to 0} \tilde{y}(t)$

$$= \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, h_{k\Delta}(t-k\Delta) \cdot \Delta$$

Applying limit,

$$\therefore \boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, d\tau}$$

$y(t) = x(t) * h(t)$ = convolution integral

<u>History</u>: Historically the convolution operation was introduced by a German mathematician, D. Gustav. The term convolution in German means ~~fal~~ faltung which means <u>folding</u>

$h(k) \rightarrow h(-k) \rightarrow x(k) \cdot h(-k)$

(6)

Problem ① For an LTI system given below find output $y(t)$.

$$x(t) = e^{-at}\cdot u(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI system} \\ h(t) = u(t)\end{array}} \longrightarrow y(t) = ??$$

Solution :- Since the given system is LTI, output response $y(t)$ can be written as

$$y(t) = x(t) * h(t)$$

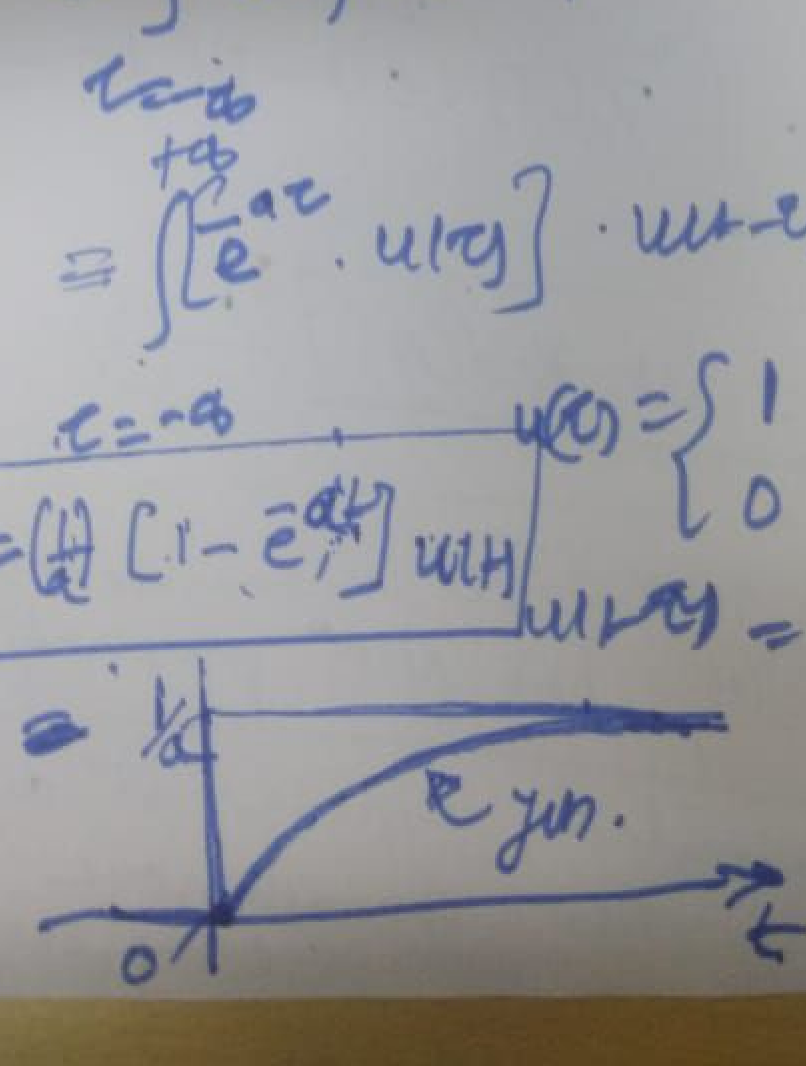
$$= \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{+\infty} \left[e^{-a\tau}\cdot u(\tau)\right]\cdot u(t-\tau)\, d\tau$$

$$u(\tau) = \begin{cases} 1, & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

$$\boxed{y(t) = \left(\frac{1}{a}\right)\left[1 - e^{-at}\right] u(t)}$$

$$u(t-\tau) = \{\ ?\ \ldots$$



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Problem ② Relate sifting property and convolution integral for CTS.

Soln:-

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad —①$$

Using sifting property of continuous time impulse function,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot \delta(t-\tau)\, d\tau = x(t) * \delta(t) \qquad —②$$

$$\therefore \boxed{x(t) = x(t) * \delta(t)} \qquad —③$$

Inference → The continuous time impulse signal, $\delta(t)$, has this property of its own.

⇒ The impulse response of a continuous time LTI system characterizes the system completely.

⑧

Properties of LTI systems:→

Since LTI systems are completely described by their corresponding impulse responses and LTI systems represents convolution sum/integral,

⇒ Properties of LTI system implies properties of convolution

⇒ It has implications on corresponding impulse responses.

Property 1: Commutative Property.

Example: Multiplication operation is commutative.

$$2 \cdot 3 = 6 = 3 \cdot 2$$

$$x(n) * h(n) = h(n) * x(n) = y(n)$$

Proof: → $L.H.S. = x(n) * h(n)$

$$= \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$$

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Using method of substitution

Let $n-k=m \Rightarrow k = n-m$

$$= \sum_{m=-\infty}^{+\infty} x(n-m) \cdot h(m)$$

$$= \sum_{m=-\infty}^{+\infty} h(m) \cdot x(n-m)$$

$m \rightarrow k$

$$= \sum_{k=-\infty}^{+\infty} h(k)\, x(n-k) = ??$$

$$= h(n) * x(n)$$

$$\boxed{LHS = RHS.}$$

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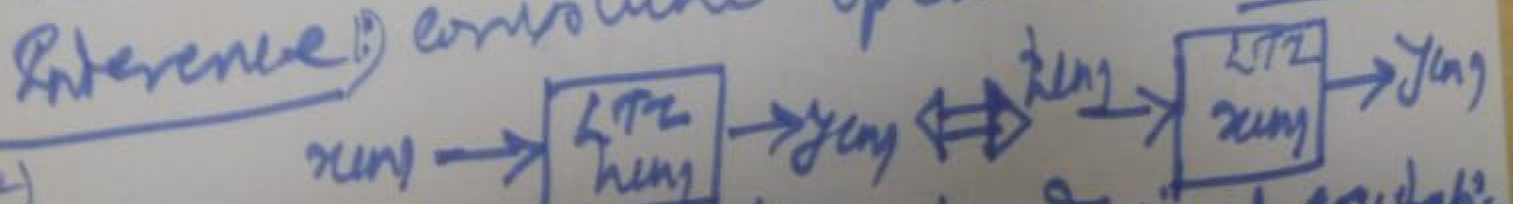
Inference: convolution operator is commutative



2)

For an LTI system, the role of input excitation source, x(n) and system's impulse response h(n) can be interchanged without affecting output y(n)

2) Distributive Property.

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

$$2 \times [3+5] = 2 \times 3 + 2 \times 5$$

→ Multiplication distributes over addition

Proof: LHS $= x(n) * [h_1(n) + h_2(n)]$

$$= \sum_{k=-\infty}^{+\infty} x(k)\,[h_1(n-k) + h_2(n-k)]$$

$$= \underbrace{\sum_{k=-\infty}^{+\infty} x(k)\,h_1(n-k)}_{} + \sum_{k=-\infty}^{+\infty} x(k)\,h_2(n-k)$$

$$= x(n) * h_1(n) + x(n) * h_2(n).$$

Only one convolution operation