SC223 - Linear Algebra

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Lecture 4

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A (better) way to understand Matrix multiplication

- Let $p, q \in \mathbb{R}^n$.
- We define
  - $\forall k \in \mathbb{R}, k \cdot p := \begin{bmatrix} kp_1 \\ kp_2 \\ \vdots \\ kp_n \end{bmatrix}$
  - $p + q := \begin{bmatrix} p_1 + q_1 \\ p_2 + q_2 \\ \vdots \\ p_n + q_n \end{bmatrix}$
  - Using the above operations:

$$\forall k_1, k_2 \in \mathbb{R}, p, q \in \mathbb{R}^n, k_1 \cdot p + k_2 \cdot q = \begin{bmatrix} k_1 p_1 + k_2 q_1 \\ k_1 p_2 + k_2 q_2 \\ \vdots \\ k_1 p_n + k_2 q_n \end{bmatrix}$$

- The above operations between vectors p and q are called **linear combination of p and q.**

- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.

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$$AB = \begin{bmatrix} \boxed{\begin{matrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{matrix}} & \boxed{\begin{matrix} a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{matrix}} & \begin{matrix} a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{matrix} \end{bmatrix}$$

$$b_{11} \cdot \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + b_{21} \cdot \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + b_{31} \cdot \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} = A b_{*1}$$

$$b_{12} \cdot \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + b_{22} \cdot \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + b_{32} \cdot \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} = A b_{*2}$$

$$AB = \begin{bmatrix} Ab_{*1} & Ab_{*2} & Ab_{*3} \end{bmatrix}$$

- $$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} - & a_{1*}^T B & - \\ - & a_{2*}^T B & - \end{bmatrix}$$

$$a_{11} \cdot [b_{11} \quad b_{12} \quad b_{13}] + a_{12} \cdot [b_{21} \quad b_{22} \quad b_{23}] + a_{13} [b_{31} \quad b_{32} \quad b_{33}]$$

$$= \underbrace{[a_{11} \quad a_{12} \quad a_{13}]}_{a_{1*}^T} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Elementary row transformations

- Can we encode ERO by matrices?

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$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 2 & 0 & 3 & 2 & 4 \\ -2 & 3 & -2 & 1 & 6 \\ 3 & -4 & 2 & 1 & 1 \end{array}\right]$$

A b

$R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 + 2R_1,\ R_4 \leftarrow R_4 - 3R_1$

$$\underline{E_1}\,[A \mid b] = [A' \mid b']$$

$$E_1 = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}}_{4\times 4} = \underbrace{\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 0 & 4 & 5 & 4 & 6 \\ 0 & -1 & -4 & -1 & 4 \\ 0 & 2 & 5 & 4 & 4 \end{array}\right]}_{4\times 5}$$

Elementary row transformations

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$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 2 & 0 & 3 & 2 & 4 \\ -2 & 3 & -2 & 1 & 6 \\ 3 & -4 & 2 & 1 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 + \frac{1}{4} R_2 \;,\; R_4 \leftarrow R_4 - \frac{1}{2} R_2$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{bmatrix}_{4 \times 4}$$

$$E_3 E_2 E_1 [A \mid b] = [U \mid c]$$

LU Decomposition

$$A = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}$$

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