Lecture 31

Understanding (Reading) of Fourier Transform :→

1) Graphical Interpretation :→

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \, dt$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \left[\cos(\omega t) - j \sin(\omega t)\right] dt$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot \cos(\omega t) \, dt - j \int_{-\infty}^{+\infty} f(t) \sin(\omega t) \, dt.$$

$$\boxed{F(\omega) = F_R(\omega) + j F_I(\omega)}$$

① ∴ Fourier transform of a signal $f(t)$, is in general a complex number, i.e., it has real part (i.e., $F_R(\omega)$) and an imaginary part (i.e., $F_I(\omega)$)

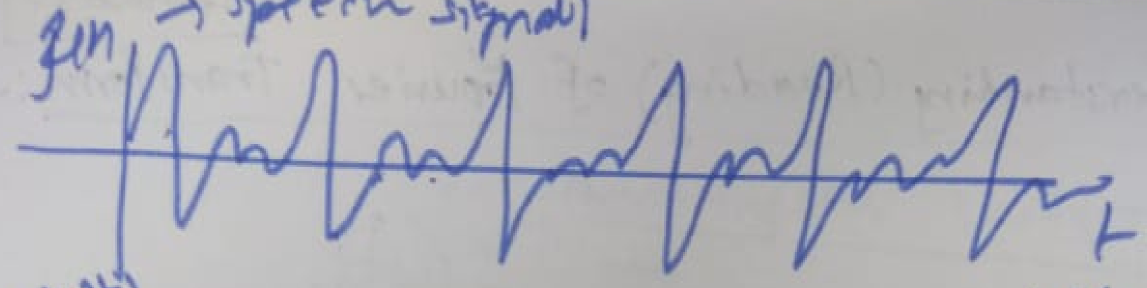
② $F(\omega)$ has two attribute, namely

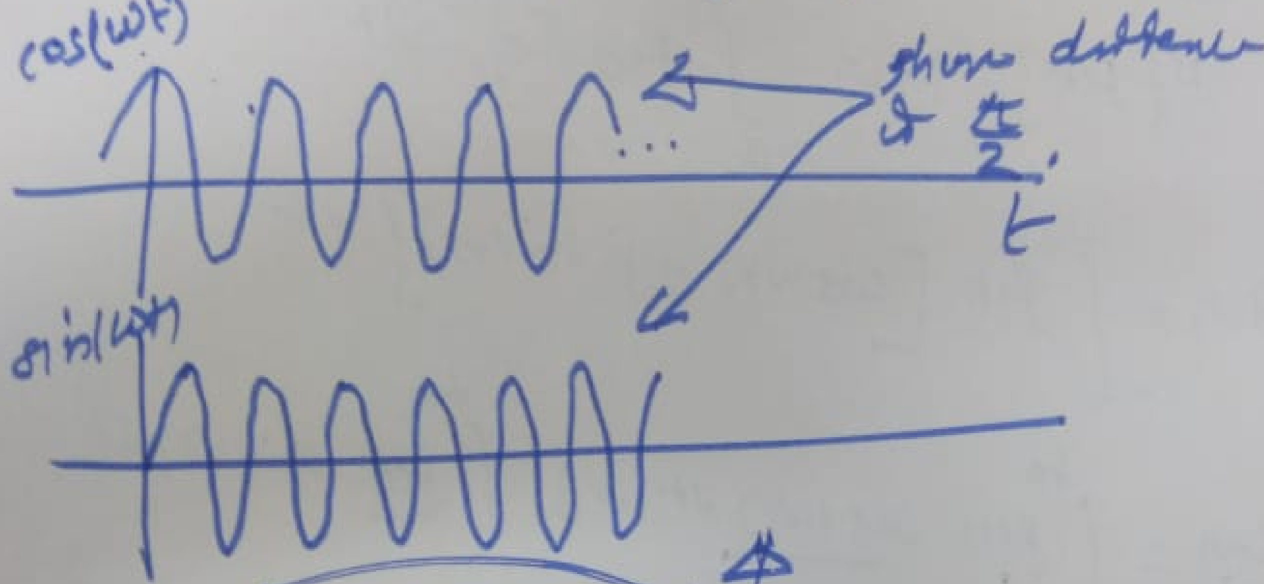
a) Magnitude spectrum $|F(\omega)| = \sqrt{F_R^2(\omega) + F_I^2(\omega)}$

b) Phase Spectrum, $\angle F(\omega) = \tan^{-1}\left(\frac{F_I(\omega)}{F_R(\omega)}\right)$

①

$f(t) \rightarrow$ speech signal



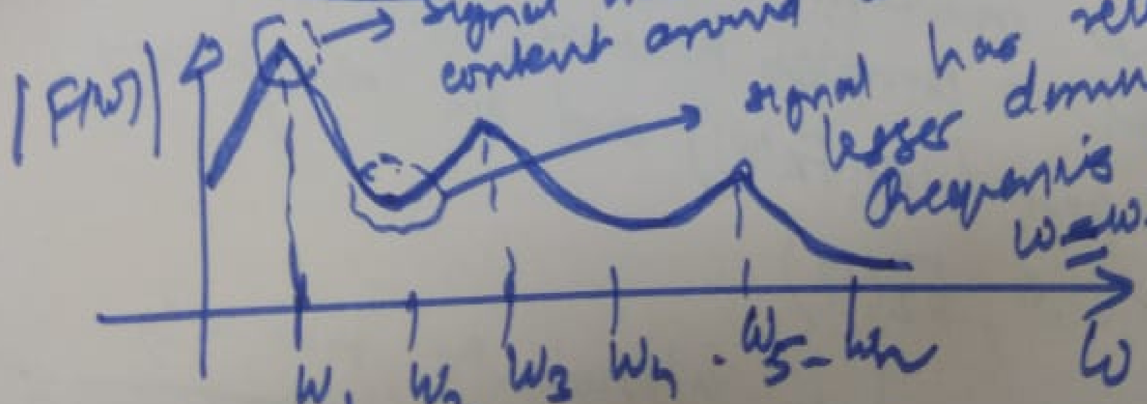


$$\int_{-\infty}^{\infty} f(t) \cdot \cos(\omega t)$$

$\omega = 2\pi f$ frequency

Signal has dominant content around $\omega = \omega_1$

signal has relatively lesser dominant frequencies around $\omega = \omega_2$



② Inner Product Interpretation :→

Let f(t), g(t) are two signals, then

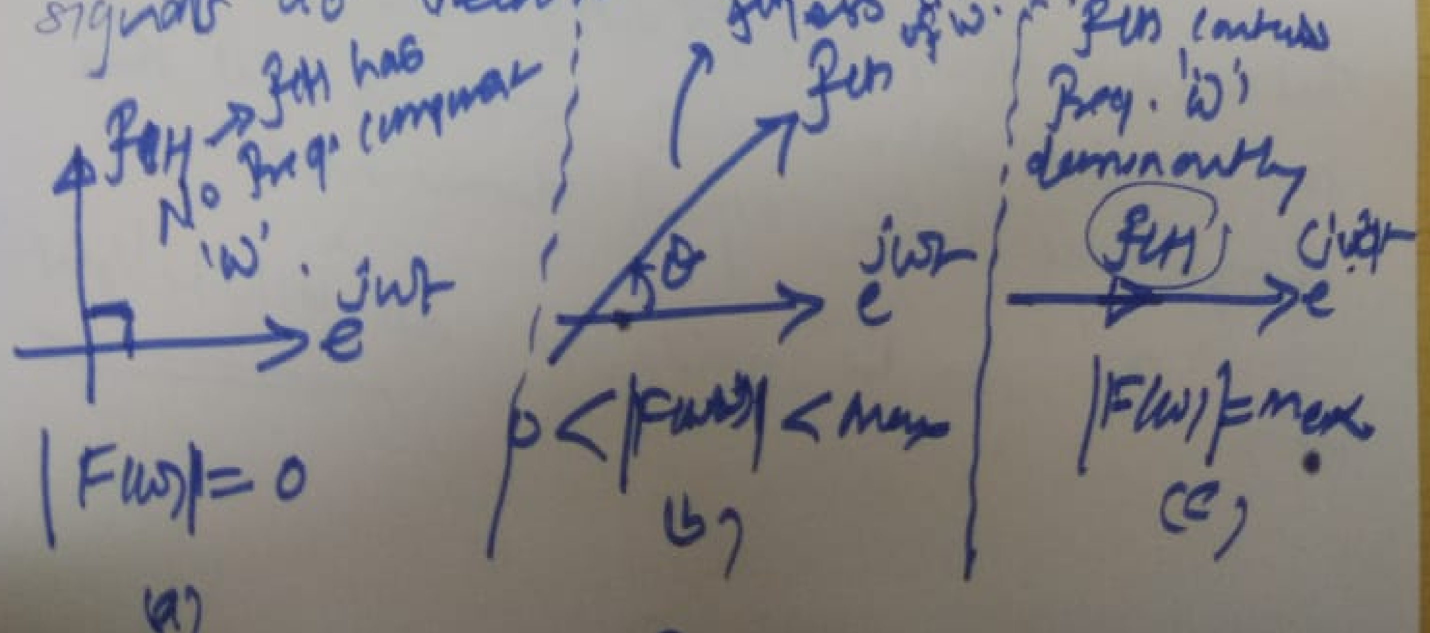
$$\langle f(t), g(t)\rangle = \int_{-\infty}^{+\infty} f(t)\cdot g^*(t)\, dt \quad \text{—} ①$$

$$F\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\cdot e^{-j\omega t}\, dt \quad \text{—} ②$$

$$\boxed{F(\omega) = \langle f(t), e^{j\omega t}\rangle}$$

geometric structure.

In signals and systems, we model signals as vectors



Problems on Fourier Transform

① Find Fourier transform of $f(t) = \delta(t)$.

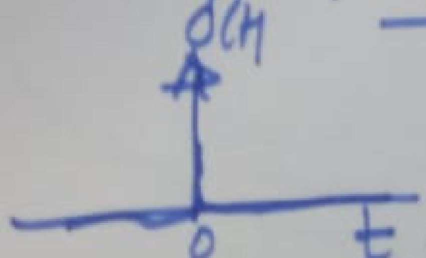
Soln: $f(t) = \delta(t)$

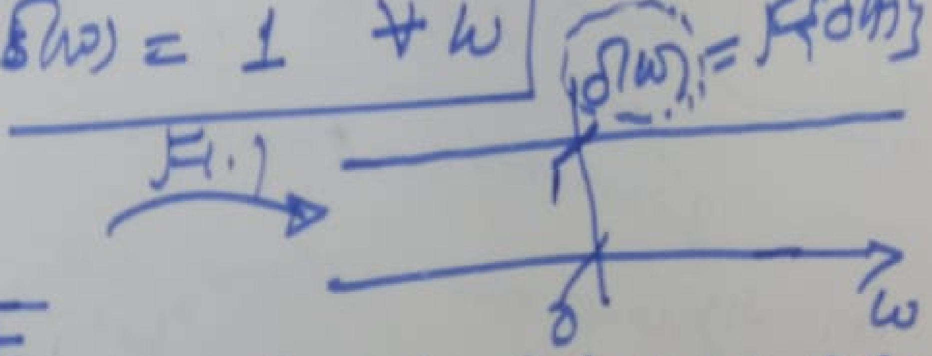
$$\therefore F(\omega) = F\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\therefore F\{\delta(t)\} = \delta(\omega) = \int_{-\infty}^{+\infty} \delta(t) \cdot e^{-j\omega t}\, dt$$

– impulse function

$$\boxed{\delta(\omega) = 1 \quad \forall\, \omega}$$

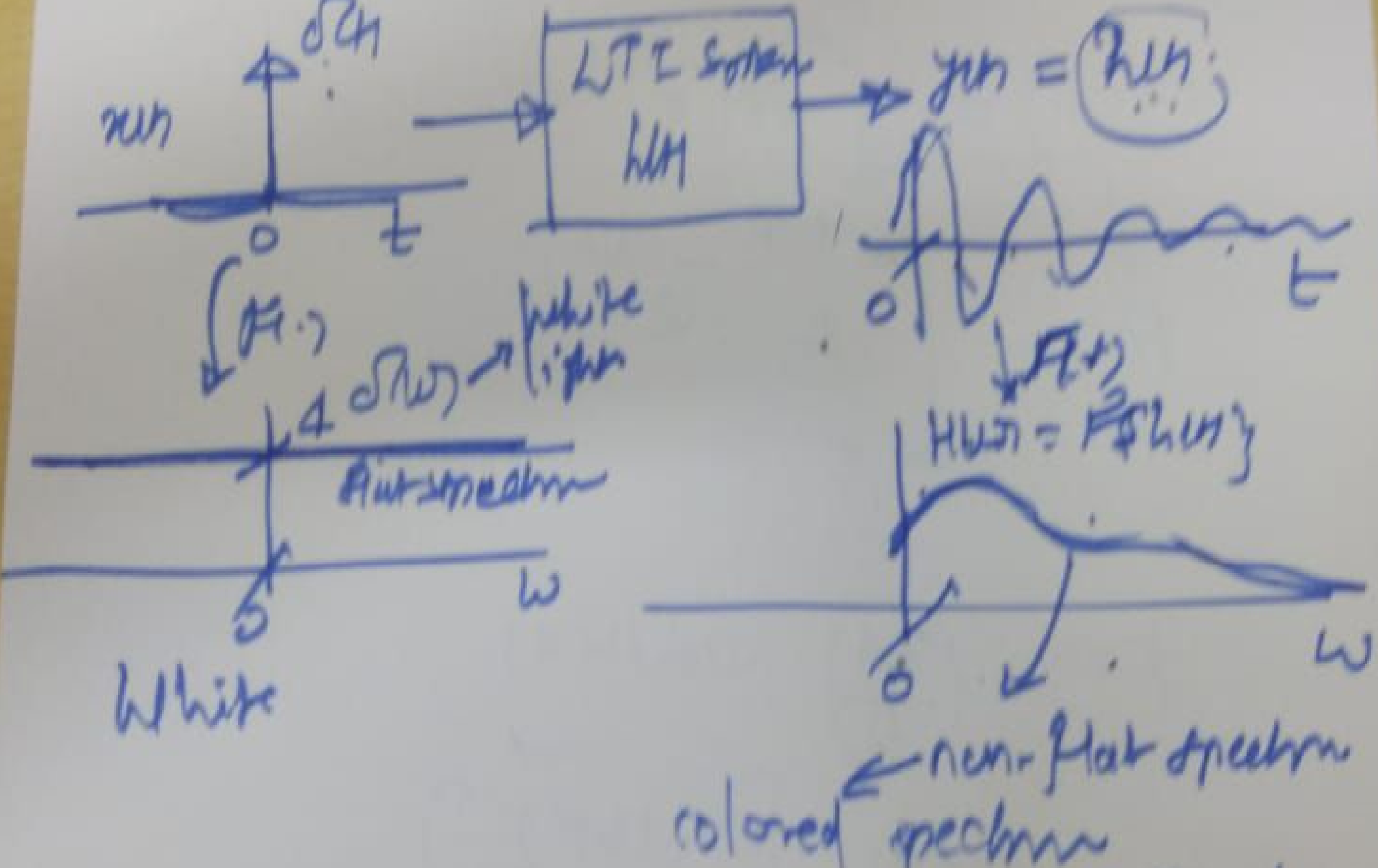




Inference: $\delta(\omega)$ i.e. $F\{\delta(t)\}$ is $1\ \forall\, \omega$ means the impulse signal $\delta(t)$ contains all the frequencies with 'equal dominance'

$\Rightarrow$ '$\delta(t)$' can be considered as "white light"

(2) Using Fourier transform of $\delta(t)$, prove that impulse response of an LTI system characterises an LTI system completely.

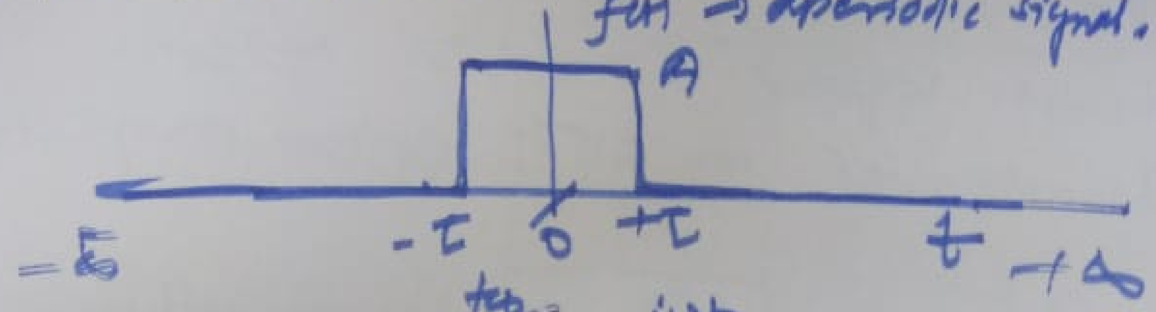


⇒ The LTI system under consideration is responsible to introduce spectral colours in its impulse response, $h(t)$.

⇒ Hence, $h(t)$ completely characterise LTI system.

(5) Find $F(\omega)$ for pulse signal

$f(t) \rightarrow$ aperiodic signal.



$$F\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

$\downarrow$

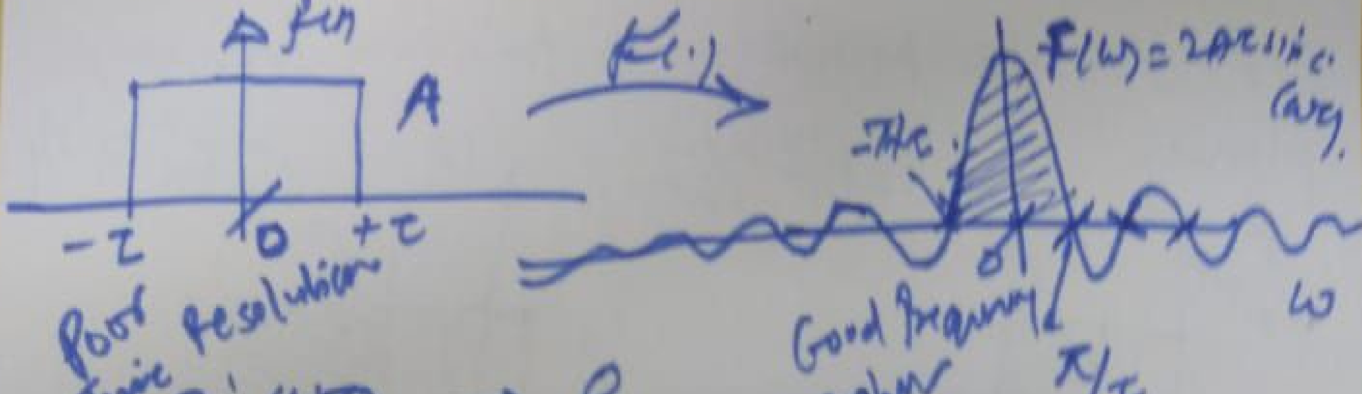
$$F(\omega) = \int_{-\tau}^{+\tau} A \cdot e^{-j\omega t}\, dt$$

$\vdots$

$$F(\omega) = \left(\frac{2A}{\omega}\right) \sin(\omega\tau)$$

$$= (2A\tau) \cdot \left[\frac{\sin(\omega\tau)}{(\omega\tau)}\right]$$

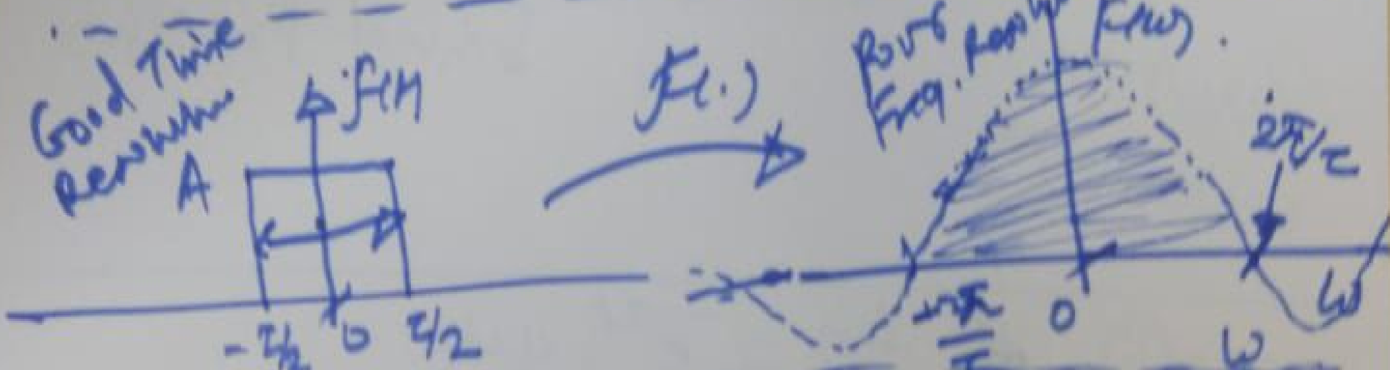
$$\boxed{F(\omega) = (2A\tau)\ \mathrm{sinc}(\omega\tau)}$$



$\sin(\omega\tau) \to 0$

$\omega\tau = \pi \cdot n$

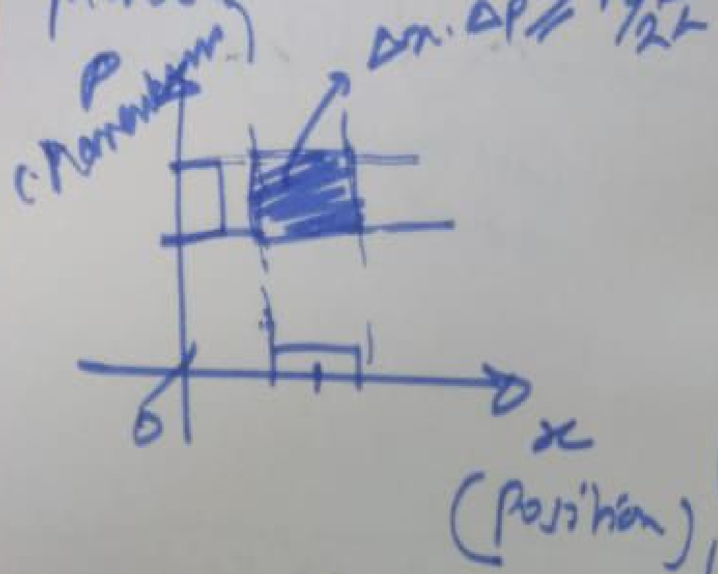
$= \left(\frac{\pi}{\tau}\right) \cdot n$



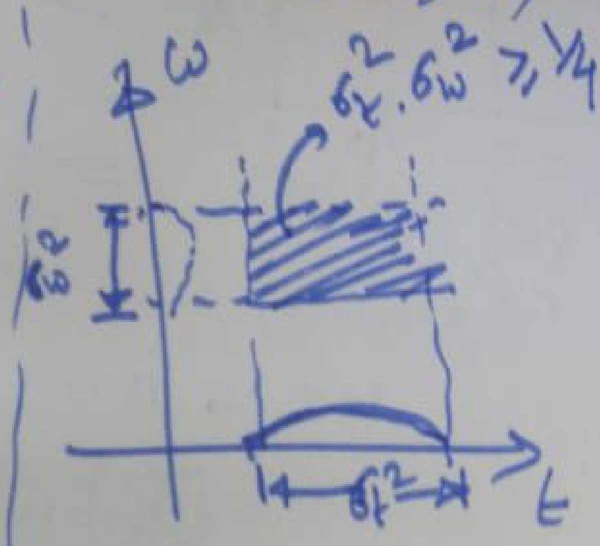
Inference: We cannot achieve good time resolution and good frequency resolution at the same time.

Reminds Heisenberg's Uncertainty Principle in Signal Processing Framework

Let $f(t)$ be particle in time and frequency plane.



Quantum Mechanics



Signal Processing

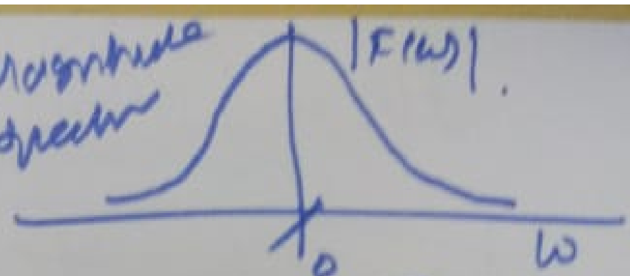
Problem (4) $f(t) = e^{-at} . u(t)$

Sol^n: $\rightarrow F\{f(t)\} = F(\omega) = \int_{-\infty}^{+\infty} f(t) . e^{-j\omega t} dt$

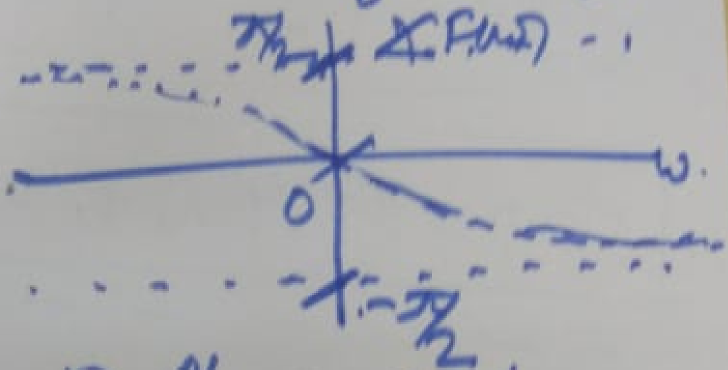
$$\therefore F\{f(t)\} = F\{e^{-at} . u(t)\} = \int_{-\infty}^{+\infty} [e^{-at} . u(t)] . e^{-j\omega t} dt$$

$$\boxed{F(\omega) = \frac{1}{a+j\omega}} \rightarrow \text{RC circuit}$$

$$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$ (a) Magnitude spectrum



$$\angle F(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



(b) Phase spectrum.

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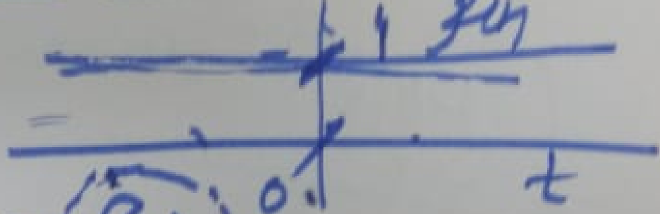
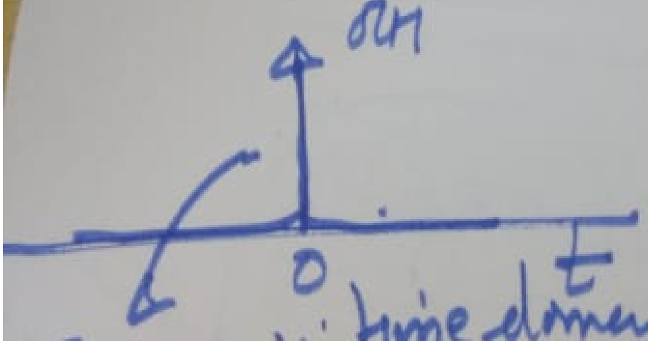
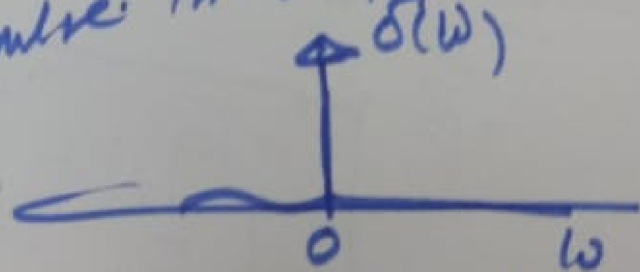
(5) $f(t) = 1, \ \forall t$



Let $F(\omega) = \mathcal{F}\{f(t)\} = 2\pi\,\delta(\omega)$

Impulse in frequency domain





Impulse in time domain

We have $\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \left(\frac{1}{2\pi}\right)\int_{\omega=-\infty}^{\omega=+\infty} F(\omega)\, e^{j\omega t}\, d\omega$

(9) $$f(t) = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} (2\pi)\,\delta(\omega)\, e^{j\omega t}\, d\omega$$

$$f(t) = \int_{-\infty}^{+\infty} \delta(\omega) \cdot e^{j\omega t} d\omega$$

$$f(t) = 1 \quad \forall t$$ impulse

$$\therefore \boxed{F(\omega) = \mathcal{F}\{f(t)\} = 2\pi\delta(\omega)}$$

$$\mathcal{F}\{1\} = 2\pi\delta(\omega)$$

$$\int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt = 2\pi\delta(\omega)$$

$$\therefore \int_{-\infty}^{+\infty} 1 \cdot e^{-j\omega t} dt = 2\pi\delta(\omega)$$

$$\boxed{\int_{-\infty}^{+\infty} e^{-j\omega t} dt = 2\pi\delta(\omega)}$$

Impulse in frequency domain

LIATE rule

6

$$\omega \rightarrow \omega - \omega_0$$

$$\therefore \int_{-\infty}^{+\infty} e^{-j(\omega-\omega_0)t} \, dt = 2\pi \, \delta(\omega-\omega_0)$$

$$\omega - n\omega_0 .$$

$$\int_{-\infty}^{+\infty} e^{-j(\omega - n\omega_0)t} \, dt = 2\pi \, \delta(\omega - n\omega_0)$$

$$F\{e^{jn\omega_0 t}\} = \underline{2\pi \, \delta(\omega - n\omega_0)}$$

$$F\{F_n \, e^{jn\omega_0 t}\} = 2\pi F_n \, \delta(\omega - n\omega_0)$$

$$n=1 \rightarrow F\{F_1 \, e^{j1.\omega_0 t}\} = 2\pi F_1 \, \delta(\omega - 1.\omega_0)$$

$$n=2 \rightarrow F\{F_2 \, e^{j2\omega_0 t}\} = 2\pi F_2 \, \delta(\omega - 2\omega_0)$$

$$F\{F_1 \, e^{j1.\omega_0 t} + F_2 \, e^{j2\omega_0 t}\} = 2\pi F_1 \, \delta(\omega - \omega_0) + 2\pi F_2 \, \delta(\omega - 2\omega_0)$$

$$\boxed{F\left\{\sum_{n=-\infty}^{+\infty} F_n \cdot e^{jn\omega_0 t}\right\} = \sum_{n=-\infty}^{+\infty} 2\pi F_n \, \delta(\omega - n\omega_0)}$$

$f(t)$

If $f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_0 t}$, where F_n's are called as Fourier Series coefficients.

then we can even find Fourier transform of periodic signal, $f(t)$

$$F(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi \cdot F_n \, \delta(\omega - n\omega_0)$$

where, $\omega_0 = \frac{2\pi}{T}$, where T = fundamental period of periodic signal $f(t)$

with a condition that impulses are allowed to be represented in frequency domain $\{\delta(\omega - n\omega_0)\}_{n \in \mathbb{Z}}$ are permitted.

Problem 6 Find the Fourier transform of impulse-train signal., $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT)$

$p(t)$

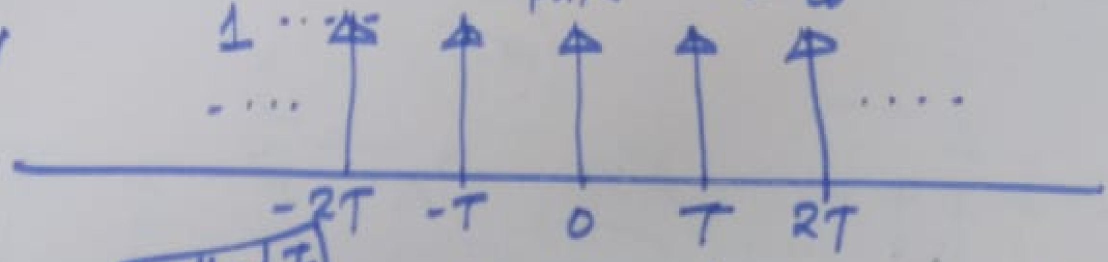


Fig. Impulse-train signal

Solution Method I

$$\mathcal{F}\{p(t)\} = P(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi P_n \cdot \delta(\omega - n\omega_0)$$

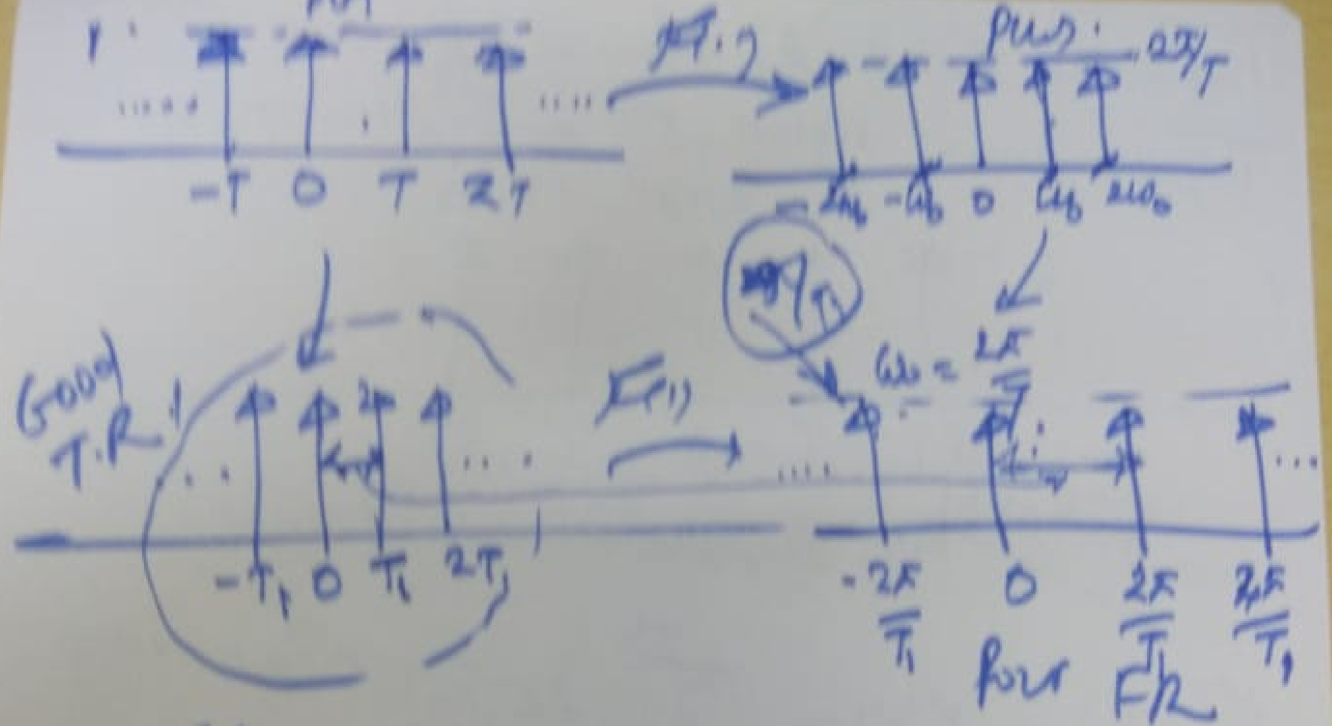
where P_n = Fourier series coefficients of $p(t)$.

$$P_n = \frac{1}{T}, \ \forall n$$

$$P(\omega) = \sum_{n=-\infty}^{+\infty} 2\pi\left(\frac{1}{T}\right) \cdot \delta(\omega - n\omega_0)$$

$$\boxed{P(\omega) = \left(\frac{2\pi}{T}\right) \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)}$$

Impulses in freq. domain



⇒ Inference: →

Manifestation of Heisenberg's uncertainty phenomena in signal processing domain

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Method (II) Via Fourier transform expression

$$P(\omega) = \mathcal{F}\{p(t)\} = \int_{-\infty}^{+\infty} p(t)\, e^{-j\omega t}\, dt$$

$$\therefore P(\omega) = \int_{-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} \delta(t - nT)\right] e^{-j\omega t}\, dt .$$

Assuming $\int \sum = \sum \int$

$$P(\omega) = \sum_{n=-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \delta(t-nT) \cdot e^{-j\omega t}\, dt \right\}$$

$$\int_{-\infty}^{+\infty} \delta(t) \cdot e^{-j\omega t}\, dt = F\{\delta(t)\} = 1$$

$$F\{\delta(t-nT)\} = \cancel{e}\; e^{-jn\omega T}$$

$$\boxed{P(\omega) = \sum_{n=-\infty}^{+\infty} e^{-j\omega nT}}$$

$$P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)$$

$$\therefore \boxed{\sum_{n=-\infty}^{+\infty} e^{-j\omega nT} = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_0)}$$

Poisson's formula

Properties of Fourier Transform:→

Notation. Fourier transform pairs.

$$f(t) = e^{-at}.u(t) \longrightarrow F(\omega) = \frac{1}{a+j\omega}$$

$$f(t) \xleftrightarrow{F} F(\omega) \qquad e^{-at}.u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}$$

$$\boxed{\begin{array}{l} f(t) = e^{-at}u(t) + e^{-bt}u(t) \\ F(\omega) = \frac{1}{a+j\omega} + \frac{1}{b+j\omega} \end{array}}$$

1) Linearity.

If $f(t) \xleftrightarrow{F} F(\omega)$ and $g(t) \xleftrightarrow{F} G(\omega)$

then $z(t) = a_1 f(t) + a_2 g(t)$

$$F[z(t)] = \int_{-\infty}^{+\infty} z(t).\, e^{-j\omega t} dt = \int_{-\infty}^{+\infty} [a_1 f(t) + a_2 g(t)] \times e^{-j\omega t} dt$$

$$Z(\omega) = a_1 \left[\int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt\right] + a_2 \left[\int_{-\infty}^{+\infty} g(t) e^{-j\omega t} dt\right]$$

$$Z(\omega) = a_1 F(\omega) + a_2 G(\omega)$$

$$\sum_{\langle k \rangle} a_k f_k(t) \xleftrightarrow{F(\cdot)} \sum_{\langle k \rangle} a_k F_k(\omega)$$