SC223 - Linear Algebra
Autumn 2023

End Semester Examination — December 5, 2023

Duration: 120 mins — Maximum Marks: 50

Note
1. You can use any result proved in lectures directly, but do state the result. Any other result or assumption made must be proved/justified.
2. You may get penalized for every nonsensical/tangential argument you write.

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1. Prove or disprove the following statements: (1) Two matrices $A, B \in \mathbb{R}^{n\times n}$ are similar if they have the same eigenvalues, each with same multiplicity. (2) If two matrices $A, B \in \mathbb{R}^{n\times n}$ are similar then they have the same eigenvalues, each with same multiplicity. **[6]**

2. Let us consider the IPS: $\mathcal{L}_2([-1,1])$, the vector space of square integrable real-valued functions defined on the interval $[-1,1] \subset \mathbb{R}$, with the $\mathcal{L}_2$ inner product: $\forall f, g \in \mathcal{L}_2([-1,1]), \langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$. Let $\mathcal{P}_2([-1,1])$ denote the subspace of polynomials with real coefficient of degree at most 2, considered as functions on the interval: $[-1,1]$. Find the orthogonal projection of $1 + 2x + 3x^2 + 4x^3$ on this subspace. **[6]**

3. Let S be the subspace of $\mathbb{R}^4$ with basis $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. **[8]**

   (a) Given that we are interested in finding a matrix A such that $N(A) = S$, are there any conditions on the size of such a matrix A? If yes, then state those conditions, precisely.

   (b) Systematically (without guess work, or trial and error) find a matrix A (of any valid size) such that $N(A) = S$.

   (c) For the chosen size, is this matrix unique, given that we consider two matrices whose rows are scalar multiples of each other the same (different rows can be multiplied with different scalars)?

4. From the information given about each of the following matrix (denoted by A), find its rank, **with proper justification**. **[15]**

   (a) Let $A \in \mathbb{R}^{n\times n}$. The characteristic polynomial for A is $c_A(x) = x^2(x-3)(x+1)(x+5)(x-5)$, and A is not diagonalizable.

   (b) Let $A \in \mathbb{R}^{7\times 5}$. $dim(N(A^T)) = 3$

   (c) Let $A : \mathbb{R}^{5\times 7}$. $Rank(A^T A) = 4$.

   (d) Let $\{x_i \in \mathbb{R}^n, i = 1, \ldots, m\}$ and $\{y_i \in \mathbb{R}^n, i = 1, \ldots, m\}$ be two linearly independent set of vectors. $A = \sum_{i=1}^{m} x_i y_i^T$.

   (e) Let $\{x_1, \ldots, x_k\}, x_i \in \mathbb{R}^n$ be a set of linearly independent vectors. Let $A \in \mathbb{R}^{k\times k}$, defined as $A_{i,j} = \langle x_j, x_i\rangle, 1 \le i, j \le k$, where $\langle \cdot, \cdot \rangle$ represents the usual inner product.

5. Answer the following questions pertaining to Projection operators/matrices on $\mathbb{R}^n$, with appropriate proofs. **[15]**

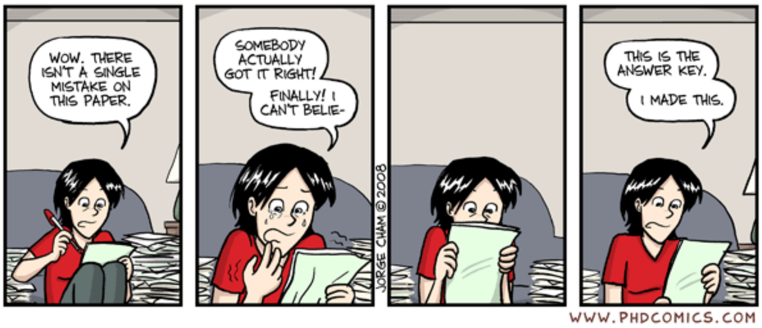
   (a) Let $S \subset \mathcal{L}(\mathbb{R}^n)$ denote the set of all projection operators. Is S a subspace of $\mathcal{L}(\mathbb{R}^n)$.

(b) Is it true that $\forall x \in \mathbb{R}^n, ||Px|| \leq ||x||$, for any projection operator P. What if we assume that P is an orthogonal projection operator (to any subspace of $\mathbb{R}^n$)?

(c) Is it true that a non-orthogonal Projection operator cannot have an orthonormal set of eigenvectors?

(d) A projection operator can have eigenvalues 1 and 0 only. Is the converse true, i.e., is any operator on $\mathbb{R}^n$ with eigenvalues 0 and 1 a projection operator?

(e) Let $A = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$ Is A a matrix that represents a projection operator on a subspace of $\mathbb{R}^3$? If yes, (1) is it an orthogonal projection, and (2) find the basis of the subspace to which it projects.



Source: phdcomics.com

Definitions and Propositions:

1. **Column Space and Nullspace of a matrix:** For $A \in \mathbb{R}^{m \times n}$, Column space is denoted by $C(A)$, and defined as $C(A) = \{Ax \mid \forall x \in \mathbb{R}^n\}$. Similarly, the Nullspace is denoted by $N(A)$, and is defined as $N(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$.

2. **Rank of a matrix:** The number of linearly independent rows or columns of a matrix is called Rank of the matrix.

3. **Group:** A set G with a binary operation $*$ is called a group if the following axioms are satisfied: (1)**Closure:** $\forall x, y \in G, x * y \in V$, (2) **Existence of Identity:** $\exists e \in G, \forall x \in G, x * e = e * x = x$, (3) **Existence of Inverse:** $\forall x \in G, \exists y \in G, x * y = y * x = e$, (4) **Associativity:** $\forall x, y, z \in G, (x * y) * z = x * (y * z)$.

4. **Abelian Group:** A group $(G, *)$ is called an Abelian group if $\forall x, y \in G, x * y = y * x$.

5. **Field:** A set $\mathbb{F}$ with two binary operations: $+_F, \times$ is said to be Field if the following axioms are satisfied: (1) $(\mathbb{F}, +_F)$ is an **Abelian group**. The additive identity is denoted by 0, (2) $(\mathbb{F} - \{0\}, \times)$ is an **Abelian group**. The multiplicative identity is denoted by 1, (3) **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c$.

6. **Vector Spaces:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms: (1) $(V, +)$ is an **Abelian group**, (2) **Closure with respect to Scalar multiplication:** $\forall a \in \mathbb{F}, \forall u \in V, a \cdot u \in V$, (3) **Scalar Multiplication identity:** $\exists e \in \mathbb{F}$ such that $e \cdot v = v, \forall v \in V$, (4) **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$, (5) **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

7. **Subspace** Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a subspace of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.

8. **Proposition:** A non-empty subset W of a vector space V is a subspace if and only if: (1) W is closed with respect to vector addition, and (2) W is closed with respect to scalar multiplication.

9. **Sum of subspaces**: Let $U_1, \ldots, U_n$ be subspaces of V. The sum of subspaces $U_1, \ldots, U_n$ is defined as: $U_1 + \ldots + U_n =: \{u_1 + u_2 + \ldots + u_n \mid \forall u_i \in U_i, i = 1, \ldots, n\}$

10. **Proposition:** The sum of subspaces $U_1, \ldots, U_n$ of V is a subspace of V.

11. **Direct Sum of Subspaces**: In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is uniquely expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$. Notation for Direct Sum: $U_1 \oplus U_2 \oplus \ldots \oplus U_n$.

12. **Proposition**: Let $U_1, \ldots, U_n$ be subspaces of V. Then $V = U_1 \oplus \ldots \oplus U_n$ if and only if: (1) $V = U_1 + \ldots + U_n$, and (2) The only decomposition of $\theta \in V$ is $(\theta, \ldots, \theta)$. The symbol θ denotes the additive identity of the vector space.

13. **Proposition**: Let V be a Vector Space with subspaces U_1, U_2. Then $V = U_1 \oplus U_2$ *iff* $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\theta\}$.

14. **Span of a set**: Given a vector space V, let $U = \{u_1, \ldots, u_n\} \subset V$. We define the **Span of** U as $span(U) := \{a_1 v_1 + \ldots + a_n v_n \mid \forall a_1, \ldots, a_n \in \mathbb{F}\}$.

15. **Spanning set**: Let V be a VS, and let $W \subset V$. If $span(W) = V$, we say that W is a **spanning set** of V, or W **spans** V.

16. **Linearly independent set**: Let V be a vector space and let $W = \{v_1, \ldots, v_n\} \subset V$. We say that the set W is a set of linear independent vectors, if $a_1 v_1 + \ldots + a_n v_n = \theta \Rightarrow a_i = 0, i = 1, \ldots, n$.

17. **Basis**: Let V be a vector space. A subset $\beta \subset V$ is said to be a Basis of V if (1) $span(\beta) = V$, and (2) β is a set of linearly independent vectors.

18. **Proposition**: A subset $U = \{u_1, \ldots, u_n\}$ of VS V is a basis of V if and only if every $v \in V$ can be uniquely written as $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n, a_i \in \mathbb{F}, i = 1, \ldots, n$.

19. **Proposition**: Any set of basis vectors of a FDVS contains the same number of elements.

20. **Dimension of a Vector Space**: Let V be a FDVS. For any set of basis vectors β of V, we define the dimension of V as $dim(V) := |\beta|$.

21. **Rank-Nullity Theorem**: Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.

22. **Proposition**: Let U, W be subspaces of FDVS V. Then, $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

23. **Theorem**: Let $A \in \mathbb{R}^{m \times n}$. $N(A) \oplus C(A^T) = \mathbb{R}^n$, and $N(A^T) \oplus C(A) = \mathbb{R}^m$.

24. **Linear Transformations**: Let U and V be vector spaces over the same field $\mathbb{F}$. A function $f : U \to V$ is said to be Linear transformation from U to V if it satisfies (1) Additivity: $\forall x, y \in U, f(x + y) = f(x) + f(y)$, and (2) Homogeneity $\forall a \in \mathbb{F}, \forall x \in U, f(a \cdot x) = a \cdot f(x)$.

25. **Proposition**: The set of all linear transformations between two vector spaces U and V over the same field $\mathbb{F}$ is a vector space over the field $\mathbb{F}$.

26. **Similar matrices and Similarity transformation:** We say two matrices A and B are similar if there exists an invertible matrix, say S such that $B = SAS^{-1}$. The transformation $A \mapsto SAS^{-1}$ is said to be a similarity transformation of A by S.

27. **Matrix Diagonalization**: The process of similarity transformation on a matrix $A \in \mathbb{C}^{n\times n}$, using all n linearly independent eigenvectors as columns of a matrix, say E, to get a diagonal matrix Λ, where $\Lambda = E^{-1}AE$ is called matrix diagonalization.
28. **Algebraic mutiplicity of** λ: Multiplicity of λ as a root of the characteristic polynomial of an operator T. Denoted as $AM(\lambda)$.
29. **Geometric Multiplicity of** λ: $GM(\lambda) = dim(N(T - \lambda I))$.
30. **Proposition**: Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T \in \mathcal{L}(U)$. Then the eigenvectors $v_1, \ldots, v_m$ associated with these eigenvalues are linearly independent.
31. **Proposition**: For $T \in \mathcal{L}(U), dim(U) = n$, if $\sum_{i=1}^{m} AM(\lambda_i) = n$, and $GM(\lambda_i) = AM(\lambda_i), i = 1, \ldots, m$, then T is diagonalizable.
32. **Proposition**: Let $T \in \mathcal{L}(U), dim(U) = n$. For any eigenvalue λ of T, $GM(\lambda) \leq AM(\lambda)$.
33. **Projection**: Let V be a vector space with subspace $V = U \oplus W$. The projection operator on U along W, denoted as $P_U \in \mathcal{L}(U)$ is defined as $\forall v = u + w, u \in U, w \in W, P_U(v) = u$.
34. **Proposition**: Let $P_U \in \mathcal{L}(V)$ be a projection operator on subspace U along subspace W. Then, (1) $P_U^2 = P_U$, (2) $U = Range(P_U), W = N(P_U)$, (3) P_U can have eigenvalues only 0 and/or 1, (4) P_U is diagonalizable.
35. **Definition:** (Normed Vector Space) A normed vector space (NVS) is a vector space $(V, +, \cdot)$ over either $\mathbb{R}$ or $\mathbb{C}$ with a **norm**, a function $||\cdot|| : V \to \mathbb{R}$ which satisfies the following properties: (1) Positive definiteness: $||x|| \geq 0, \forall x \in V$ and $||x|| = 0 \Leftrightarrow x = \theta$, (2) Absolute homogeneity: $\forall x \in V, \forall a \in \mathbb{F}, ||a \cdot x|| = |a| \, ||x||$, and (3) Triangular inequality: $\forall x, y \in V, ||x + y|| \leq ||x|| + ||y||$.
36. **Definition:** (Inner Product Space): An Inner product space(IPS) is a vector space $(V, +, \cdot)$ over $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$), with an **inner product**, which is any mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ that it satisfies the following properties: (1) Positive definite: $\forall x \in V, \langle x, x \rangle \geq 0, = 0 \Leftrightarrow x = \theta$, (2) Linear in first argument: $\forall x, y, z \in V, \forall a, b \in \mathbb{F}, \langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$, (3) Conjugate symmetry: $\langle y, x \rangle = \overline{\langle x, y \rangle}$.
37. **Definition**: Given an IPS $(V, \langle \cdot, \cdot \rangle)$, a set of vectors $\{v_1, \ldots, v_n\} \in V$ is said to be **orthogonal** if $\langle v_i, v_j \rangle = 0, \forall i \neq j$, and is said to be **orthonormal** if $\langle v_i, v_j \rangle = 0, \forall i \neq j$, $||v_i|| = 1, \forall i$.
38. **Definition**: A matrix $A \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ is said to be an **orthogonal matrix** if all its n columns are orthonormal, i.e., $A^*A = I$, where A^* denotes the conjugate transpose of A.
39. **Proposition**: (Cauchy-Schwartz inequality): In an IPS $(V, \langle \cdot, \cdot \rangle)$, $|\langle x, y \rangle| \leq ||x|| \, ||y||$.
40. **Definition:** (Orthogonal Complement) Let V be a FD IPS and let U be a subset of V. The **Orthogonal Complement** of U is defined as $U^\perp = \{v \in V \mid \langle v, u \rangle = 0, \forall u \in U\}$
41. **Definition**: Let U be a subspace of FD IPS V, and $V = U \oplus U^\perp$. Orthogonal Projection on subspace U, defined as $P_U \in \mathcal{L}(V)$, such that $\forall v \in V$, if $v = u + w, u \in U, w \in U^\perp$, $P_U(v) = u$.
42. **Proposition**: For a FD IPS V with $V = U \oplus U^\perp$, the orthogonal projection operator on U, P_U satisfies all properties of a projection operator, and the following aditional properties: (1) (Conjugate) Symmetric: If $V = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T = P_U$ ($P_U^* = P_U$), (2) $\forall v \in V, P_U(v) = \arg\min_{u \in U} ||u - v||^2$.
43. **Theorem:** (Spectral Theorem for Hermitian and Real-Symmetric Matrices) Let $A \in \mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$) be a Hermitian matrix (or real-symmetric), i.e., $A = A^*$ (or $A^T = A$). Then (1) all its eigenvalues are real, (2) there exists an orthonormal basis of $\mathbb{C}^n$ (or $\mathbb{R}^n$) containing eigenvectors of A.

Solutions

1. (1) Let $A, B \in \mathbb{R}^{n\times n}$ be two similar matrices. Thus, there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $A = SBS^{-1}$. Then,

$$\begin{aligned}
c_A(x) =& det(A - xI)\\
=& det(SBS^{-1} - xI)\\
=& det(S(B - xI)S^{-1})\\
=& det(S^{-1}S(B - xI))\\
=& det(B - xI)\\
=& c_B(x)
\end{aligned}$$

Since the characteristic polynomials of A and B are the same, their eigenvalues and the corresponding multiplicities will also be same.
Note that one cannot assume that the matrices A, B are diagonalizable.
(2) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $B = I_2$, the 2×2 identity matrix. It is easy to see that $c_A(x) = (x - 1)^2 = c_B(x)$. Thus both the matrices have same eigenvalue (1) with multiplicity 2. But since $A - 1 \cdot I_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has rank 1, thus $GM(1) = dim(N(A - 1I_2) = 1 < AM(1) = 2$. This implies that A is not diagonalizable, while $B = I_2$ is diagonal, and thus A and B cannot be similar.
One should not assume that A and B are diagonalizable.

2. Let $V = \mathcal{L}_2([-1,1])$ and $U = \mathcal{P}_2([-1,1])$. Let $p \in V, p = 1 + 2x + 3x^2 + 4x^3$. We have been asked to compute $P_U(p) \in U$. Note that since P_u is a linear operator, we have the following simiplification:

$$\begin{aligned}
P_U(p) =& P_U(1 + 2x + 3x^2 + 4x^3)\\
=& P_U(1 + 2x + 3x^2) + P_U(4x^3)\\
=& 1 + 2x + 3x^2 + 4P_U(x^3)
\end{aligned}$$

Note that since $P_U(x^3) \in U$, thus $P_U(x^3) = a + bx = cx^2, a, b, c \in \mathbb{R}$. Also by definition of orthogonal projection, $\langle x^3 - P_U(x^3), 1\rangle = \langle x^3 - P_U(x^3), x\rangle = \langle x^3 - P_U(x^3), x^2\rangle = 0$. Thus we have three (linear) equations in three unknowns (a, b, c):

$\langle x^3 - P_U(x^3), 1\rangle = 0$
$\langle x^3, 1\rangle = \langle P_U(x^3), 1\rangle = \langle a + bx + cx^2, 1\rangle = a\langle 1, 1\rangle + b\langle x, 1\rangle + c\langle x^2, 1\rangle$
Similarly,
$\langle x^3, x\rangle = a\langle 1, x\rangle + b\langle x, x\rangle + c\langle x^2, x\rangle$
and,
$\langle x^3, x^2\rangle = a\langle 1, x^2\rangle + b\langle x, x^2\rangle + c\langle x^2, x^2\rangle$

Collecting the equations into a matrix form gives:

$$\begin{bmatrix} \langle 1,1\rangle & \langle x,1\rangle & \langle x^2,1\rangle \\ \langle 1,x\rangle & \langle x,x\rangle & \langle x^2,x\rangle \\ \langle 1,x^2\rangle & \langle x,x^2\rangle & \langle x^2,x^2\rangle \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \langle x^3,1\rangle \\ \langle x^3,x\rangle \\ \langle x^3,x^2\rangle \end{bmatrix}$$

Evaluting the entries of the matrix:

$$\langle 1,1\rangle = \int_{-1}^{1} 1\,dx = 2.$$
$$\langle x,1\rangle = \int_{-1}^{1} x\,dx = 0, \text{ because } x \text{ is an odd function.}$$
$$\langle x^2,1\rangle = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$
$$\langle x,x\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \text{ (as shown above).}$$
$$\langle x,x^2\rangle = \int_{-1}^{1} x^3\,dx = 0, \text{ because } x^3 \text{ is an odd function.}$$
$$\langle x^2,x^2\rangle = \int_{-1}^{1} x^4\,dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5}.$$

Similarly, evaluating the entries of the right hand side:

$$\langle x^3,1\rangle = \int_{-1}^{1} x^3\,dx = 0, \text{ because } x^3 \text{ is an odd function.}$$
$$\langle x^3,x\rangle = \int_{-1}^{1} x^4\,dx = \frac{2}{5}, \text{ (as shown above).}$$
$$\langle x^3,x^2\rangle = \int_{-1}^{1} x^5\,dx = 0, \text{ because } x^5 \text{ is an odd function.}$$

We now have the following matrix equation:

$$\begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{5} \\ 0 \end{bmatrix}$$

It is easy to see that row 2 is linearly independent of rows 1 and 3, and then row 1 is linearly independent of row 3. We can conclude that the matrix in invertible, and so we will have a unique solution. Moreover, the right-hand side vector is just a scalar multiple of column 2, thus giving us the solution: $a = 0, b = \frac{3}{5}, c = 0$.
We finally have:

$$\begin{aligned} P_U(p)) =& 1 + 2x + 3x^2 + 4P_U(x^3) \\ =& 1 + 2x + 3x^2 + 4(0\cdot 1 + \frac{3}{5}x + 0\cdot x^2) \\ =& 1 + \frac{22}{5}x + 3x^2 \end{aligned}$$

3. Let S be the subspace of $\mathbb{R}^4$ with basis $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (a) For any matrix $A \in \mathbb{R}^{m\times n}$, $N(A) \subset \mathbb{R}^n$. In this question, $N(A) = S \subset \mathbb{R}^4$, thus $n = 4$. Moreover, $dim(N(A)) = 2$, so $dim(R(A)) = 2$. Since $dim(C(A)) = rank(A) = dim(R(A)) = 2$, thus $m \geq 2$, since $C(A) \subset \mathbb{R}^m$.

(b) Letting $m = 2, A = \begin{bmatrix} a_{1*}^T \\ a_{2*}^T \end{bmatrix}$, and $\beta = \{b_1, b_2\}$, we get the following conditions: $a_{1*}^T b_1 = a_{1*}^T b_2 = a_{2*}^T b_1 = a_{2*}^T b_2 = 0$. Note that the left hand sides can be interpreted as the Euclidean inner product, implying: $\langle a_{1*}, b_1\rangle = \langle a_{1*}, b_2\rangle = \langle a_{2*}, b_1\rangle = \langle a_{2*}, b_2\rangle = 0$. Thus, the two rows of the desired matrix A are (any) two linearly independent vectors in $\beta^\perp$. We need to choose the two rows of A to be linearly independent because otherwise the nullspace of A will contain additional vectors not listed in β. Since the inner product is symmetric, letting

$$B = \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix}$$

implies that the two rows of A are (any) basis of $N(B)$. Using ERT:

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_2 - R_1]{} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -2 \end{bmatrix}$$

Thus any $x = (x_1, x_2, x_3, x_4) \in N(B)$ has to satisfy: $-2x_2 + x_3 - 2x_4 = 0$, and $x_1 + x_2 + 2x_4 = 0$. Letting $x_3 = a, x_4 = b$ to be the free variables gives $x_2 = \frac{a-2b}{2}, x_1 = -\frac{a+2b}{2}$. Thus we get one choice of rows of A by choosing $a = 2, b = 0$, $a_{1*}^T = [-1, 1, 2, 0]$, and $a = 0, b = 1$, $a_{2*}^T = [-1, -1, 0, 1]$. Thus

$$A = \begin{bmatrix} -1 & 1 & 2 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

Both rows are linearly independent, hence it is trivial to verify that $N(A) = S$.

(c) Once it is clear that the row vectors are any two linearly independent vectors orthogonal to the given set β, i.e., any two linearly independent vectors in $\beta^\perp$, there are infinitely many choices for row vectors of A. One choice for example is with $a = 2, b = 1$ and $a = -2, b = 1$ that gives

$$A = \begin{bmatrix} -2 & 0 & 2 & 1 \\ 0 & -2 & -2 & 1 \end{bmatrix}$$

4. Find the rank of the given matrix A:

(a) Since $deg(c_A) = n = 6, A \in \mathbb{R}^{6\times 6}$. All eigenvalues except 0 have $AM = 1$, Thus $GM = AM$ for these eigenvalues. Now for the eigenvalue 0, $AM(\lambda = 0) = 2$. The only way A is not going to be diagonalizable is if $GM(0) < AM(0)$. Thus, $GM(0) = 1 = dim(N(A - 0 \cdot I_6) = dim(N(A)) = 1$. From the rank-nullity theorem, we get that $rank(A) = 5$.
Commonly found incorrect answers:

i. Since there are 5 distinct eigenvalues, the rank is 5. **[0]**

ii. Since there are 6 eigenvalues, the rank is 6. **[0]**

(b) $N(A^T) \subset \mathbb{R}^m$, and $C(A) \oplus N(A^T) = \mathbb{R}^m$. Since $dim(N(A^T)) = 3, dim(C(A)) = m - 3 = 4$ (from rank-nullity theorem). Thus $rank(A) = dim(C(A)) = 4$.
Commonly found incorrect answers:

i. $dim(N(A)) = dim(N(A^T)) = 3$. Therefore from rank-nullity theorem, rank = 2. **[0]**

(c) In one of the tutorials, we have proved that $N(A) = N(A^TA)$. Thus $dim(N(A)) = dim(N(A^TA)) = 7 - 4 = 3$ (from rank-nullity theorem). Thus, $rank(A) = 7 - 3 = 4$ (again, using rank-nullity theorem).
Commonly found incorrect answers:

i. Since $Rank(A) = Rank(A^T) = Rank(A^TA)$, so $Rank(A) = 4$. Unless you prove this statement, you will get a 0.

ii. Since $rank(AB) = min(rank(A), rank(B))$, using $B = A^T$ gives $rank(A) = 4$. Please note that the correct statement is $rank(AB) \leq min(rank(A), rank(B))$. You will be awarded 0 marks if you have written such an argument.

iii. Since $rank(A^TA) \leq min(rank(A), rank(A^T)$, and $rank(A^TA) = 4$, thus $rank(A) \geq 4$. This is not enough, will get a 0.

(d) $A = \sum_{i=1}^{m} x_i y_i^T \in \mathbb{R}^{n \times n}$. Let $X = \begin{bmatrix} | & | & | \\ x_1 & \dots & x_m \\ | & | & | \end{bmatrix}$, and $Y = \begin{bmatrix} | & | & | \\ y_1 & \dots & y_m \\ | & | & | \end{bmatrix}$. Then, it can be verified that $A = X_{n\times m}(Y_{n\times m})^T$. It is given that $rank(X) = rank(Y) = m$, thus $N(X) = \{\mathbf{0}\}$, and $dim(N(Y^T)) = n - m$. Thus $dim(N(A)) = n - m$, and using the rank-nullity theorem, we get $rank(A) = m$.

Commonly found incorrect answers:

i. Since the vectors $\{x_i, i = 1, \dots, m\}$ and vectors in $\{y_i, i = 1, \dots, m\}$ are linearly independent, multiplication (or some such similar operation) between them will result in m linearly independent vectors. Thus $rank(A) = m$. You need to give a proper proof, else 0.

ii. A lot of students have just written that since the vectors $\{x_i\}$ and $\{y_i\}$ are linearly independent, hence their multiplication/combination will give all linearly independent vectors. These kind of solutions will get a 0.

(e) Let $A_{i,j} = \langle x_j, x_i \rangle$. Let us try to examine the nullspace of A. For $c \in N(A), \sum_{j=1}^{k} c_j \langle x_j, x_i \rangle = 0, i = 1, \dots, k$.

$$\sum_{j=1}^{k} c_j \langle x_j, x_i \rangle = 0, i = 1, \dots, k$$
$$\langle \sum_{j=1}^{k} c_j x_j, x_i \rangle = 0, i = 1, \dots, k$$

Taking a linear combination of the k equations above, each with a scalar $c_i, i = 1, \dots, k$, we get

$$\begin{aligned} \sum_{i=1}^{k} c_i \langle \sum_{j=1}^{k} c_j x_j, x_i \rangle &= 0 \\ \Rightarrow \langle \sum_{j=1}^{k} c_j x_j, \sum_{i=1}^{k} c_i x_i \rangle &= 0 \\ \Rightarrow || \sum_{j=1}^{k} c_j x_j ||^2 &= 0 \\ \sum_{j=1}^{k} c_j x_j &= 0 \\ c_1 = c_2 = \dots = c_k &= 0. \end{aligned}$$

The last result comes from the linear indepedence of $\{x_1, \dots, x_k\}$. Thus, $N(A) = \{\mathbf{0}\}$, implying that $rank(A) = k$.

Commonly found incorrect answers:

i. Since the vectors $\{x_i, i = 1, \dots, k\}$ and vectors in $\{y_i, i = 1, \dots, k\}$ are linearly independent, multiplication (or inner product, or some such similar operation) between them will result in k linearly independent vectors. Thus $rank(A) = k$. You need to give a proper proof, else 0.

ii. Since the vectors $\{x_i, i = 1, \dots, k\}$ and vectors in $\{y_i, i = 1, \dots, k\}$ are linearly independent, they are orthogonal. Thus A is a diagonal matrix with non-zero entries on the diagonal, proving that $rank(A) = k$. This is incorrect, and will be awarded 0 marks.

5. Projection operators:

(a) Let $S \subset \mathcal{L}(\mathbb{R}^n)$. Let $A \in S$, i.e., $A^2 = A$. Now $\forall k \neq 0, k \neq 1, k \in \mathbb{R}, (kA)^2 = k^2A^2 = k^2A \neq kA$. Thus S is not closed under scalar multiplication. While this is enough, one can show that S is also not closed under addition. Let $u_1, u_2 \in \mathbb{R}^n$ be two distinct unit vectors ($||u_1|| = ||u_2|| = 1$). Let $A, B \in \mathcal{L}(\mathbb{R}^n)$ be the orthogonal projection matrices to $span\{u_1\}, span\{u_2\}$ resp. We have $A = u_1u_1^T, B = u_2u_2^T$. We need to verify $(A+B)^2 = A + B$:

$$\begin{aligned}(A+B)^2 =& (u_1u_1^T + u_2u_2^T)^2 \\ =& u_1u_1^Tu_1u_1^T + u_2u_2^Tu_1u_1^T + u_1u_1^Tu_2u_2^T + u_2u_2^Tu_1u_1^T \\ =& u_1u_1^T + u_2u_2^T + u_1^Tu_2(u_1u_2^T + u_2u_1^T) \\ =& A + B + u_1^Tu_2(u_1u_2^T + u_2u_1^T)\end{aligned}$$

Thus, unless $u_1^Tu_2 = \langle u_1, u_2 \rangle = 0$, $(A+B)^2 \neq (A+B)$.

(b) Is $||Px|| \leq ||x||$? Let us take the example of $V = \mathbb{R}^2$. Let $U = span\{(1,1)\}, W = span\{(1,0)\}$. We know that $V = U \oplus W$. Let P_U denote the projection operator on U along W. Now for $x = (0,1), x = u + w, u \in U, w \in W$ with $u = (1,1), w = (-1,0)$, we have $P_U((0,1)) = u = (1,1)$. So $||P_Ux|| = ||u|| = \sqrt{2}$, while $||x|| = ||(0,1)|| = 1$, thus $||P_Ux|| \nleq ||x||$.
For orthogonal projection operators, with $V = U \oplus U^\perp$, for all $x \in V$, we have $x = u + w, u \in U, w \in U^\perp$ such that $\langle u, w \rangle = 0$. Thus for the orthogonal projection operator P_U on subspace U, we have $P_Ux = u$, and using Pythagoras theorem, $||x||^2 = ||u||^2 + ||w||^2 = ||P_Ux||^2 + ||w||^2 \geq ||P_Ux||^2$. Thus $||P_Ux|| \leq ||x||$ for all $x \in V$ and for all orthogonal projection operators.

(c) Let $P \in \mathbb{R}^{n \times n}$ be the matrix of a non-orthogonal projection. Let us assume that $\{u_1, \dots, u_n\}$ form an orthonormal basis of $\mathbb{R}^n$ entirely consisting of eigenvectors of P. Letting U denote the $n \times n$ matrix whose column i contains $u_i, i = 1, \dots, n$, we get $P = U\Lambda U^{-1} = U\Lambda U^T$, where $\Lambda \in \mathbb{R}^{n \times n}$ diagonal matrix containing the eigenvalues 1 and 0. We see that $P^T = (U\Lambda U^T)^T = U\Lambda U^T = P$. Thus, $P^2 = P, P^T = P$ which implies that P is a matrix representing an orthogonal projection.

(d) Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The eigenvalues of A is 0 with multiplicity 2. But note that $A^2 = \mathbf{0}_{2\times 2}$. Thus A is not a projection operator.
If one additionally assumes that the operator is diagonalizable (an incorrect assumption as far as this question is concerned), then $A^2 = (S\Lambda S^{-1})^2 = S\Lambda^2 S^{-1} = S\Lambda S^{-1} = A$, since Λ is a diagonal matrix containing 0 and 1.

(e) One can compute A^2 explcitly, and see that $A^2 = A$, thus implying that A is indeed a projection matrix. (1) Since $A^T = A$, it is also an orthogonal projection matrix. (2) Using ERT we get the row echelon form of A as $\frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 0 & \frac{6}{5} & \frac{12}{5} \\ 0 & 0 & 0 \end{bmatrix}$, implying that $rank(A) = 2$.
We know that a projection matrix projects to its column space, hence we can pick any two linearly independent columns to get the basis of the subspace. Choosing columns 1 and 2, gives the subspace to which A projects as $span\left\{\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\right\}$. One can also

argue as follows: Since $det(A) = 0, rank(A) < 3$. It is easy to see that columns 1 and 2 are linearly independent, thus giving the same answer.
