SC223 - Linear Algebra

Aditya Tatu

Lecture 3

July 28, 2023

Gaussian Elimination

- $[A \mid b] \xrightarrow{\text{Elem. Row Oper.}} [U \mid c]$.
- Elementary row operations(ERO) consists of: (1) Replacing a row with sum of the row and scalar times another row., (2) Multiplying a row with any $k \in \mathbb{R}$, (3) Row exchange.
- Solve $[U \mid c]$ via back-substitution.
- A matrix A is said to be in **row-echelon form(REF)** if
  - All non-zero rows are above any rows of all zeros
  - Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  - All entries in a column below a leading entry are zeros.
- For the previous example, U is upper-triangular, and $[U \mid c]$ is in REF.

Why does elimination work?

Why does elimination work?

- **Theorem** Let $z \in \mathbb{R}^n$ be a solution to $Ax = b$, with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Then z also is a solution to $Ux = c$, where the Augmented matrix $[U \mid c]$ is obtained via EROs.

1). Let $1 \leq p \leq m$

$$\sum_{k=1}^{n} a_{pk} x_k = b_p.$$

Since $Az = b$, $\therefore$ $\sum_{k=1}^{n} a_{pk} z_k = b_p$.

For any $q \in \mathbb{R}$, $q\left(\sum_{k=1}^{n} a_{pk} z_k\right) = q b_p$.

$$\Rightarrow \sum_{k=1}^{n} (q a_{pk}) z_k = q b_p.$$

More examples

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 2 & 4 & 0 & 1 & 4 \\ -2 & -3 & 0 & 5 & -4 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 + 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 0 & 0 & -2 & 5 & -2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 0 & \boxed{1} & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & 5 & -2 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 5 & -2 \end{array}\right]$$

$R_3 \longleftrightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Back-Substitution

Eqn 3: $-2x_3 + 5x_4 = -2 \Rightarrow x_3 = 1 + \frac{5}{2}x_4$

Eqn 2: $x_2 + 2x_3 + x_4 = 2 \Rightarrow x_2 = -6x_4$

Eqn 1: $x_1 + 2x_2 + x_3 - 2x_4 = 3$

$$x_1 = \frac{23}{2}x_4 + 2$$

$$x = \begin{pmatrix} 2 + \frac{23}{2} x_4 \\ -6 x_4 \\ 1 + 5/2\, x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k \cdot \begin{pmatrix} 23/2 \\ -6 \\ 5/2 \\ 1 \end{pmatrix}$$

$\forall k \in \mathbb{R}.$

More examples

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 2 & 4 & 0 & 1 & 4 \\ -2 & -3 & 0 & 5 & -2 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right]$$

REF

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$0 \neq -2 \Rightarrow$ No Solutions!

"INCONSISTENT"

A (better) way to understand Matrix multiplication

- Let $p, q \in \mathbb{R}^n$.

A (better) way to understand Matrix multiplication

- Let $p, q \in \mathbb{R}^n$.
- We define
  - $\forall k \in \mathbb{R}, k \cdot p := \begin{bmatrix} kp_1 \\ kp_2 \\ \vdots \\ kp_n \end{bmatrix}$

A (better) way to understand Matrix multiplication

- Let $p, q \in \mathbb{R}^n$.
- We define
  - $\forall k \in \mathbb{R}, k \cdot p := \begin{bmatrix} kp_1 \\ kp_2 \\ \vdots \\ kp_n \end{bmatrix}$
  - $p + q := \begin{bmatrix} p_1 + q_1 \\ p_2 + q_2 \\ \vdots \\ p_n + q_n \end{bmatrix}$

A (better) way to understand Matrix multiplication

- Let $p, q \in \mathbb{R}^n$.
- We define
  - $\forall k \in \mathbb{R}, k \cdot p := \begin{bmatrix} kp_1 \\ kp_2 \\ \vdots \\ kp_n \end{bmatrix}$
  - $p + q := \begin{bmatrix} p_1 + q_1 \\ p_2 + q_2 \\ \vdots \\ p_n + q_n \end{bmatrix}$
  - Using the above operations:

$$\forall k_1, k_2 \in \mathbb{R}, p, q \in \mathbb{R}^n, k_1 \cdot p + k_2 \cdot q := \begin{bmatrix} k_1 p_1 + k_2 q_1 \\ k_1 p_2 + k_2 q_2 \\ \vdots \\ k_1 p_n + k_2 q_n \end{bmatrix}$$

- The above operations between vectors p and q are called **linear combination of** p **and** q.

- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.

- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.
-

$$AB =$$

- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.
-

$$AB =$$
$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

Elementary row transformations

- Can we encode ERO by matrices?