SC223 - Linear Algebra
Autumn 2023

Tutorial 2 — *LU* Decomposition

1. Find the LU factorization of the following matrices A (or PA as and when required):

   (a)
   $$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

   (b)
   $$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}, a, b, c, d \in \mathbb{R}$$

2. Determinant using LU decomposition:

   (a) Find the determinant of an upper or lower triangular $n \times n$ matrix.

   (b) Given that determinant of a matrix satisfies $det(AB) = det(A)\ det(B), \forall A, B \in \mathbb{R}^{n\times n}$, determine the determinant of a matrix A using its LU decomposition (assuming $A = LU$). How many computations would it require?

3. Let $E_1, \ldots, E_{n-1}$ denote the ERTs required to turn a matrix $A \in \mathbb{R}^{m\times n}$ into an upper triangular matrix $U \in \mathbb{R}^{m\times n}$, i.e., $E_{n-1} \cdot \ldots \cdot E_1 A = U$. We know that $L = E_1^{-1} \cdot E_2^{-1} \cdot \ldots \cdot E_{n-1}^{-1}$. How many computations will you require to obtain the matrix L? Why?

4. Let us encode a message in English by replacing each letter by its position in the English alphabet, i.e., $A \to 1, B \to 2, \ldots, Z \to 26, (Space) \to 27$. Denote this sequence of numbers that encodes a given message by $x_i, i = 1, \ldots, n$. Assuming that 4 divides n (if not, add spaces), the sequence can be seen as a sequence of quadruples: $x_{4i+1}, x_{4i+2}, x_{4i+3}, x_{4(i+1)}, i = 0, \ldots, \frac{n}{4} - 1$. Before transmitting, a further encoding is done by multiplying the following matrix to each quadruple (put in a column vector):

   $$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 3 & 3 & 2 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

   Each quadruple $y_{4i+1}, y_{4i+2}, y_{4i+3}, y_{4(i+1)}, i = 0, \ldots, \frac{n}{4} - 1$ is transmitted to the receiver. Assuming no distortion over the communication channel, and that the receiver knows the matrix A, decode the following message it has received:

   $66, 147, -45, 102, 98, 147, -22, 98, 46, 80, -11, 48, 82, 149, -22, 86, 28, 58, -18, 39, 57, 165$
   $-80, 136$