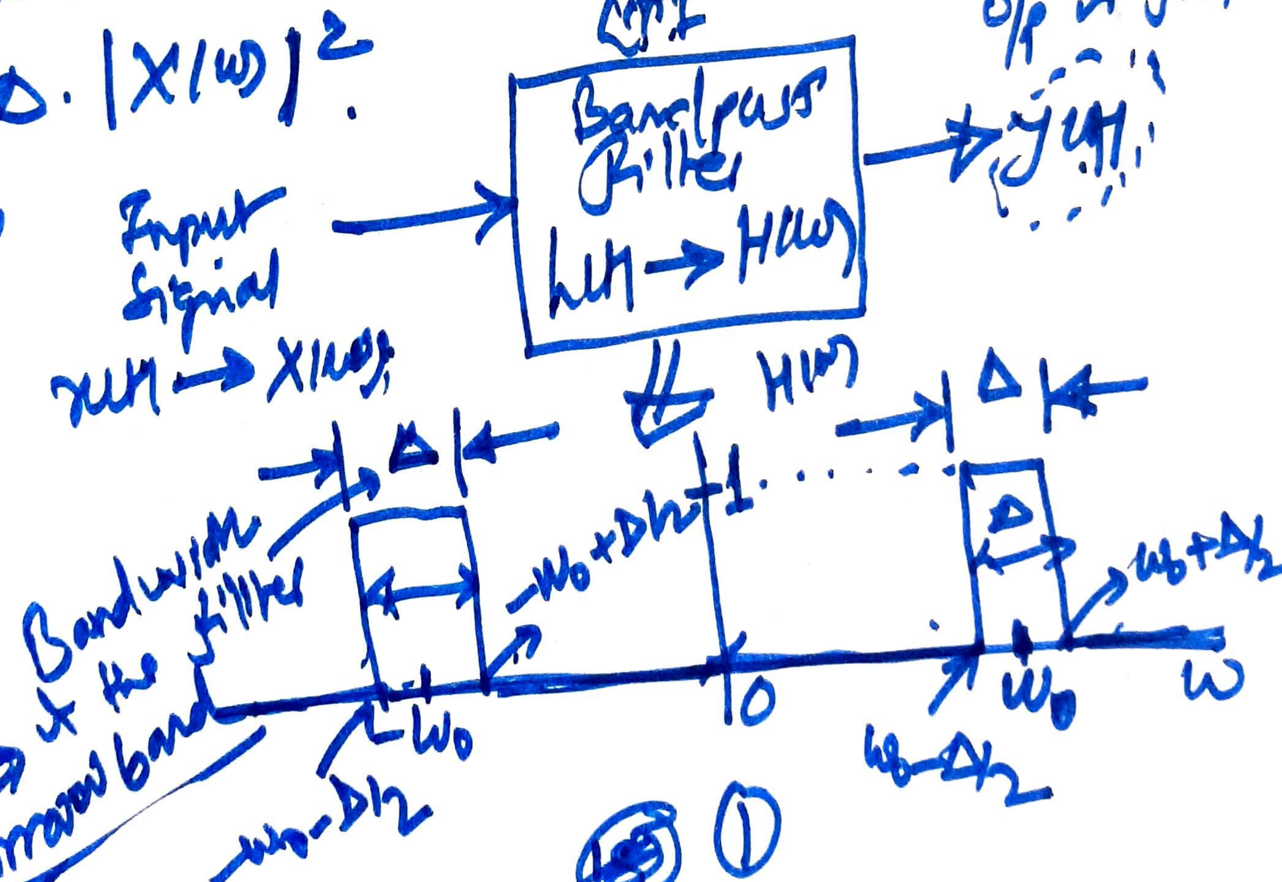
Problem to Understand concept of ESD.

Lecture 35

Problem: Consider a real-valued signal, $x(t)$ processed by ideal bandpass, $H(\omega)$ shown below. Express the energy in the output signal $y(t)$ as an integration over frequency of $|X(\omega)|^2$.

For Δ sufficiently small so that $|X(\omega)|$ is approximately constant over a frequency interval of width Δ, show that the energy in the output, $y(t)$ of the bandpass filter is approximately proportional to $\Delta \cdot |X(\omega)|^2$.

Sol^n:



∵ $x(t)$ is passed through an LTI bandpass filter, $h(t)$, output is given by.

$$y(t) = x(t) * h(t).$$

Convolution theorem

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{\underbrace{x(t) * h(t)}\}$$

$$Y(\omega) = X(\omega) \cdot H(\omega) \quad \text{—} \ (1)$$

$$Y^*(\omega) = X^*(\omega) \cdot H^*(\omega) \quad \text{—} \ (2)$$

$$\underline{Y(\omega) \cdot Y^*(\omega)} = [\underbrace{X(\omega) \cdot X^*(\omega)}] \cdot [\underbrace{H(\omega) \cdot H^*(\omega)}]$$

$$|Y(\omega)|^2 = |X(\omega)|^2 \cdot |H(\omega)|^2 \quad \text{—} \ (3)$$

To find the energy in the output, $y(t)$ we employ Parseval's theorem.

$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y(\omega)|^2 d\omega$$

(2)

am c3 in nd...

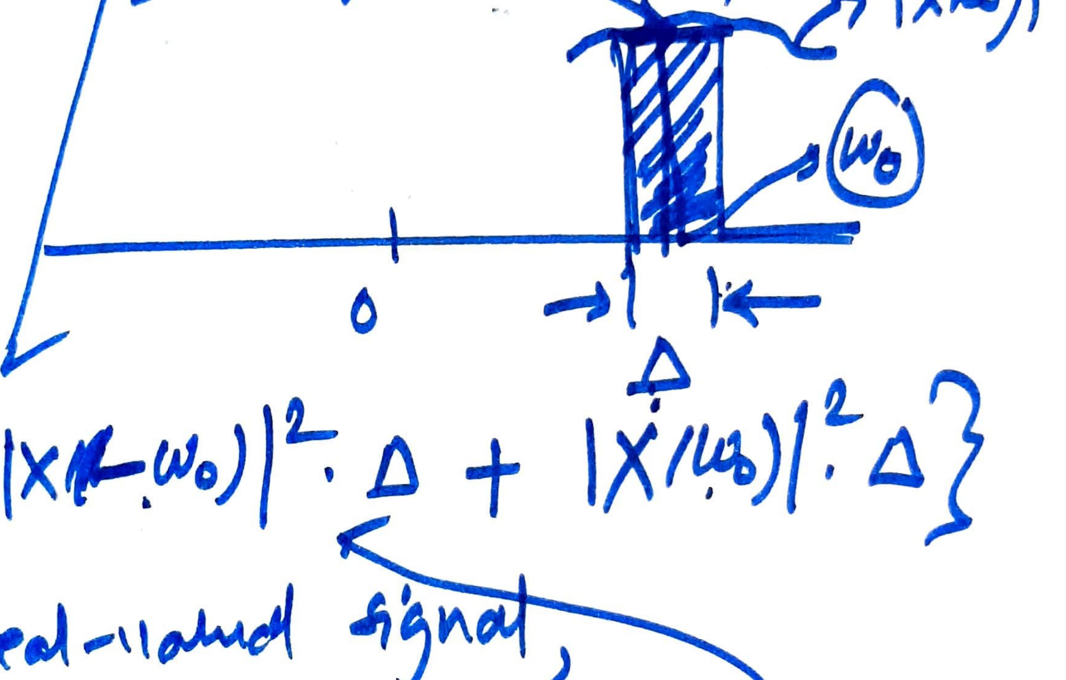
Using eqn (3),

E_y

$$E_y = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=+\infty} \underbrace{|X(\omega)|^2}_{\text{i/p signal}} \cdot \underbrace{|H(\omega)|^2}_{\text{Filter}} \, d\omega$$

$$= \frac{1}{2\pi}\left\{\int_{-\omega_0-\Delta/2}^{-\omega_0+\Delta/2} |X(\omega)|^2\cdot 1\, d\omega + \int_{\omega_0-\Delta/2}^{\omega_0+\Delta/2}|X(\omega)|^2 \cdot 1\, d\omega\right\}$$

integrand $|X(\omega_0)|^2$



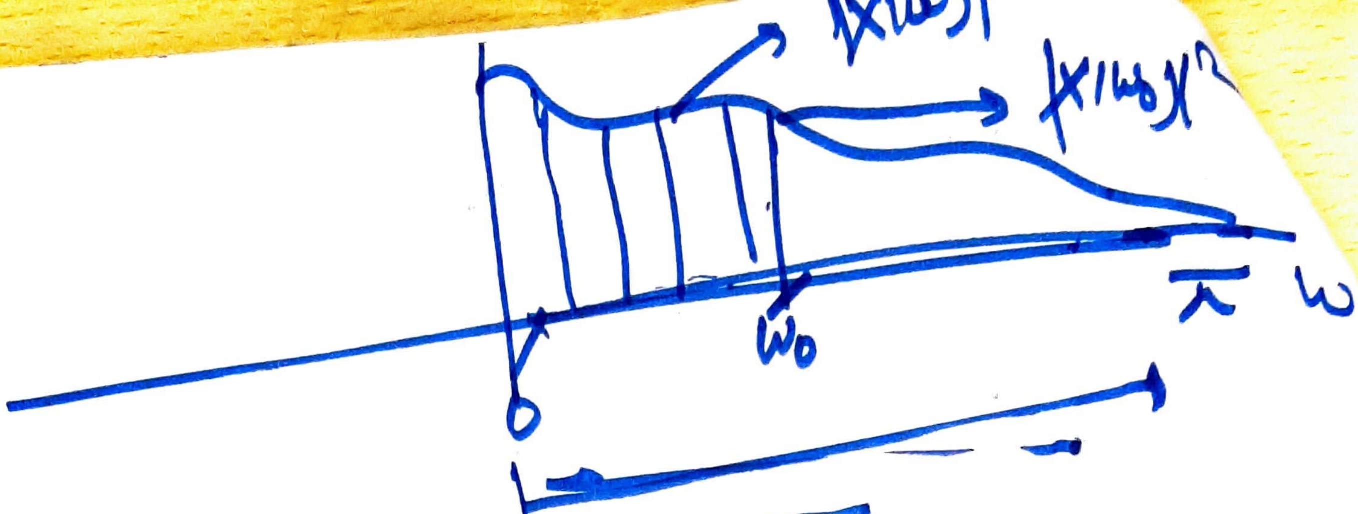
$$= \frac{1}{2\pi}\left\{|X(-\omega_0)|^2\cdot\Delta + |X(\omega_0)|^2\cdot\Delta\right\}$$

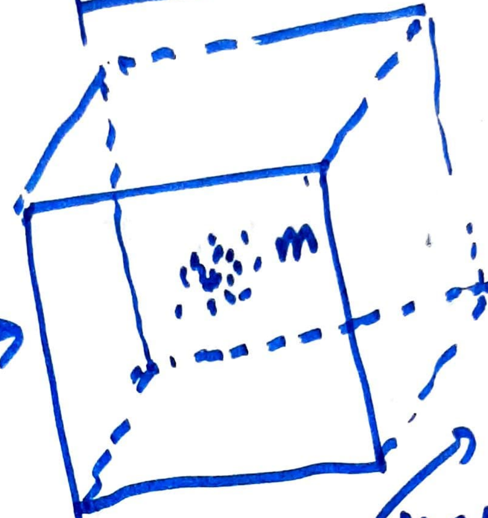
$\because x(t)$ = real-valued signal,

$\therefore |X(-\omega_0)|^2 = |X(\omega_0)|^2$

$$= \frac{1}{2\pi}\times 2\,|X(\omega_0)|^2\cdot\Delta$$

$$\boxed{\therefore \frac{E_y}{\Delta} = \frac{|X(\omega_0)|^2}{\pi}} \quad (3)$$





cube of volume V

instantaneous mass

$$\text{Density of fluid} = \frac{\text{Mass of fluid}}{\text{volume of cube}}$$

Ref.

$$= \frac{|X(\omega)|^2}{\pi}$$

Energy spectral Density (ESD)

$f(t) \to F(\omega) \to |F(\omega)|^2 \to \frac{|F(\omega)|^2}{\pi}$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega$$

even function

$$= \left(\frac{1}{2\pi}\right) \left[2 \int_{0}^{+\infty} |X(\omega)|^2 \, d\omega \right]$$

$$= \int_{0}^{+\infty} \underbrace{\left(\frac{|X(\omega)|^2}{\pi} \right)}_{ESD_x(\omega)} d\omega$$

$$E_x = \int_{0}^{+\infty} ESD_x(\omega) \, d\omega .$$

$$E_{x[\omega_1, \omega_2]} = \int_{\omega_1}^{+\omega_2} E_x(\omega) \, d\omega .$$

→ Periodic signal ⟶ concept of Power Spectral Density (PSD)

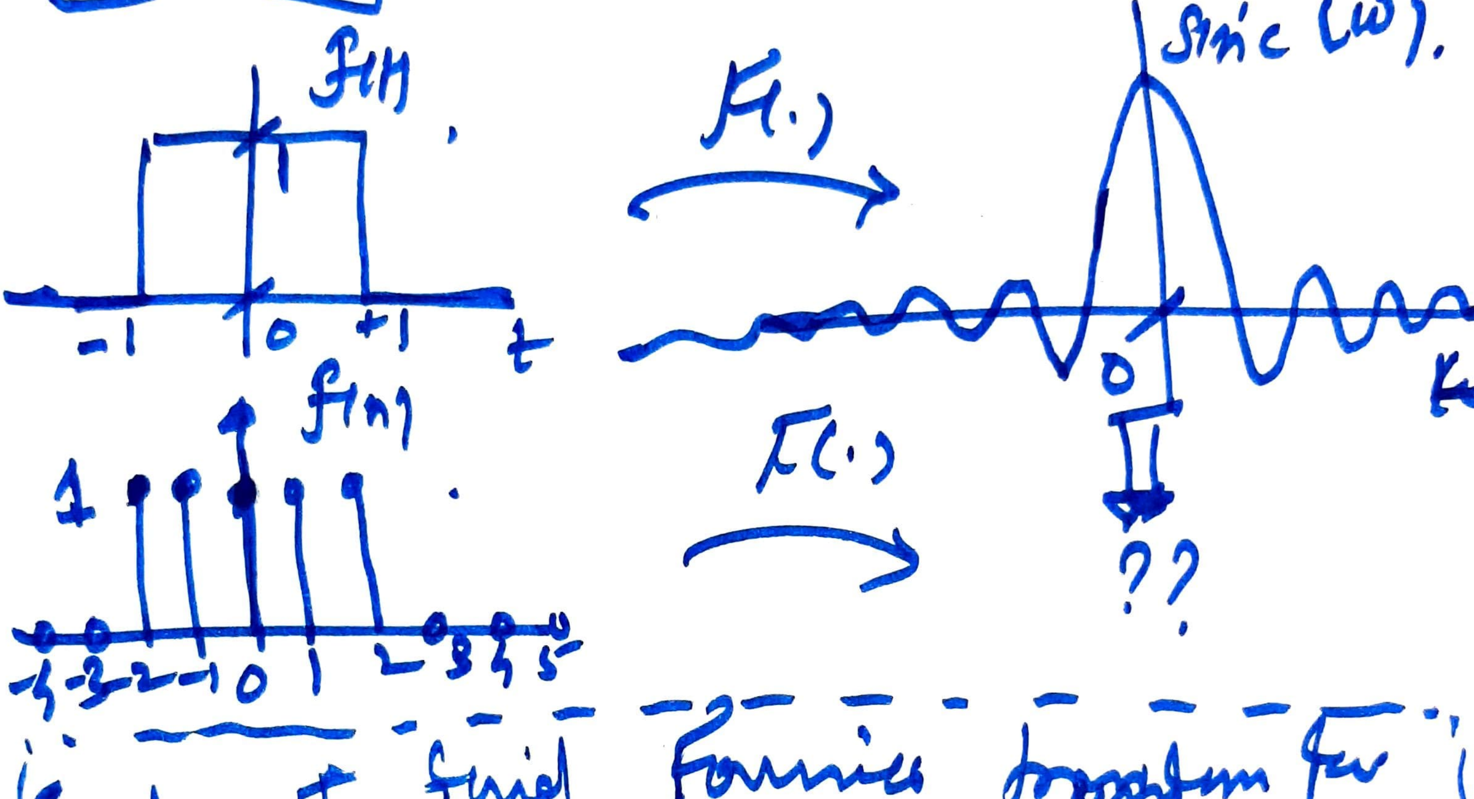
(5)

$\boxed{\text{Chapter 5}}$

The Discrete-Time Fourier Transform (DTFT)

<u>Motivation</u>: → Why to study DTFT?

$\boxed{\text{Chapter 4}}$ → continuous-time signal, $f(t)$



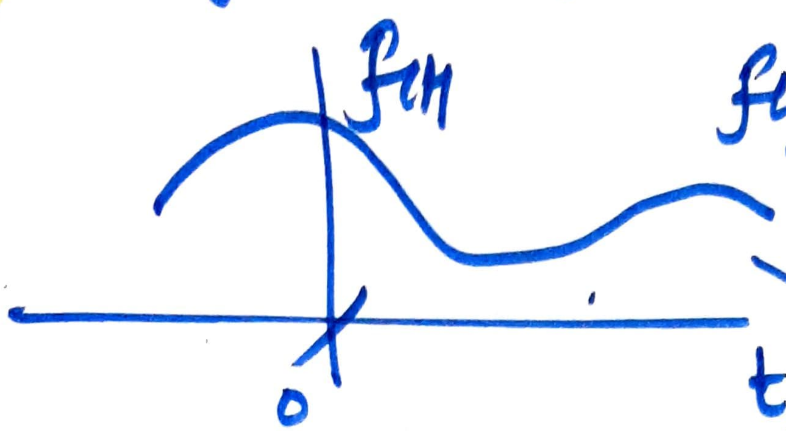
Goal: To find Fourier transform for discrete-time signal, $f(n)$

$f(t) \longrightarrow F(w)$

$F(w) \longrightarrow$??

Sampling of continuous-time signal:

$f(t) \rightarrow \hat{f}(t)$??



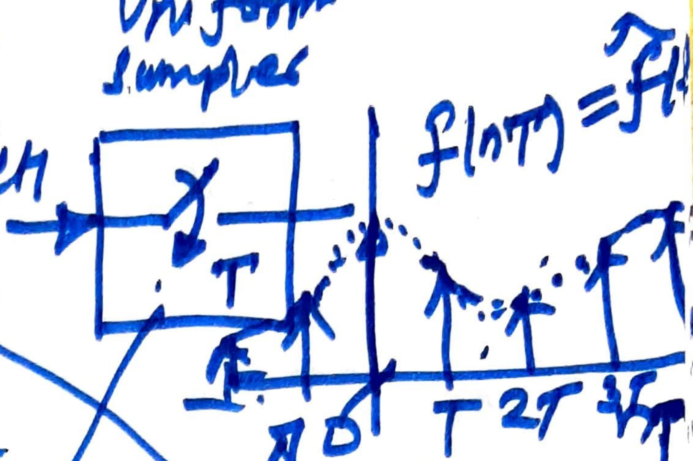


Fig. (a) Continuous-time signal

Fig. Discrete-time signal.

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT) \qquad T = \text{sampling interval}$$

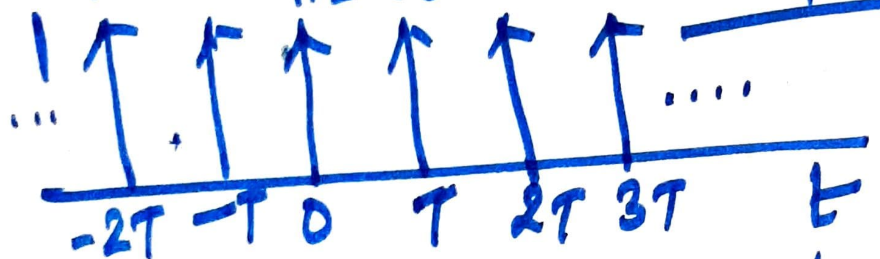


Fig. Impulse-train signal

$$\hat{f}(t) = f(t) \cdot p(t)$$

$$= f(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

$$\hat{f}(t) = \sum_{n=-\infty}^{+\infty} \underline{f(t) \cdot \delta(t-nT)}$$

Using sampling property of impulse function (Chapter 1) ⓟ

$$\hat{f}(t) = \sum_{n=-\infty}^{+\infty} f(nT) \cdot \delta(t-nT)$$

$$\left[\because x(t) \cdot \delta(t) = x(0) \cdot \delta(t) \right]$$

Take CTFT on both sides

$$F\{\hat{f}(t)\} = \int_{-\infty}^{+\infty} \hat{f}(t) \cdot e^{-j\omega t} \, dt$$

$$F\{f(nT)\} = \int_{-\infty}^{+\infty} \left\{ \sum_{n=-\infty}^{+\infty} f(nT) \cdot \delta(t-nT) \right\} e^{-j\omega t} \, dt$$

Assuming $\int \sum = \sum \int$

$$F\{f(nT)\} = \sum_{n=-\infty}^{+\infty} f(nT) \underbrace{\left\{ \int_{-\infty}^{+\infty} \delta(t-nT) \cdot e^{-j\omega t} \, dt \right\}}$$

$$\therefore F\{f(nT)\} = \sum_{n=-\infty}^{+\infty} f(nT) \times F\{\delta(t-nT)\}$$

$$⑧ \quad = \sum_{n=-\infty}^{n=+\infty} f(nT) \, e^{-j\omega nT} \times F\{\delta(t)\}$$

1

$$\therefore \mathcal{F}\{f(nT)\} = \sum_{n=-\infty}^{+\infty} f(nT).\, e^{-j\omega nT}$$

Let $T = 1$ sec [Time normalisation]

$$\therefore \mathcal{F}\{f(n)\} = \sum_{n=-\infty}^{+\infty} f(n).\, e^{-j\omega n}$$

$$F(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} f(n).\, e^{-j\omega n} \quad \text{--- [A]}$$

[Analysis Equation]

Now, we will develop synthesis equation.

Multiply eqn (A) by $e^{j\omega m}$ on both the sides,

$$F(e^{j\omega}).\, e^{j\omega m} = \sum_{n=-\infty}^{+\infty} f(n)\, e^{-j\omega n} \times e^{j\omega m}$$

Integrate on both the sides w.r.t 'ω' with limits [ω_0 to $\omega_0 + 2\pi$]

$$\int_{\omega_0}^{\omega_0+2\pi} F(e^{j\omega}).\, e^{j\omega m}.\, d\omega = \int_{\omega_0}^{\omega_0+2\pi} \left\{ \sum_{n=-\infty}^{+\infty} f(n)\, e^{-j\omega n}.\, e^{j\omega m} \right\} d\omega$$

Assuming $\int \sum = \sum \int$

$$\int_{\omega_0}^{\omega_0+2\pi} F(e^{j\omega}) \cdot e^{j\omega m} \, d\omega = \sum_{n=-\infty}^{+\infty} f(n) \underbrace{\left\{ \int_{\omega_0}^{\omega_0+2\pi} e^{-j\omega(n-m)} \, d\omega \right\}}_{I}$$

where, $I = \int_{\omega_0}^{\omega_0+2\pi} e^{-j\omega(n-m)} \, d\omega = \begin{cases} 2\pi, & n=m \\ 0 & n \neq m \end{cases}$

$$= \sum_{n=-\infty}^{+\infty} f(n) \cdot I \to \begin{cases} 2\pi & n=m \\ 0 & \forall\, n \neq m \end{cases}$$

$$= 0 + 0 + f(n)\Big|_{n=m} \times 2\pi + 0 + 0 + 0$$

$$\therefore f(m) = \frac{1}{2\pi} \int_{\omega_0}^{\omega_0+2\pi} F(e^{j\omega}) \cdot e^{j\omega m} \, d\omega$$

$m \rightarrow n$

$$f(n) = \left(\frac{1}{2\pi}\right) \int_{\omega_0}^{\omega_0+2\pi} F(e^{j\omega}) \cdot e^{j\omega n} \, d\omega$$

Case I) if $\omega_0 = 0$

$$f[n] = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} F(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega \quad \text{— (B1)}$$

Case II if $\omega_0 = -\pi$

$$f[n] = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{+\pi} F(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \text{— (B2)}$$

Let $x[n]$ be the discrete-time signal then

$$X(e^{j\omega}) = DTFT\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j\omega n} \quad \text{— (A)}$$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\omega=+\pi} X(e^{j\omega}) \cdot e^{j\omega n}\, d\omega \quad \text{— (B)}$$

DTFT pair

Tutorial Problems on DTFT.

Problem 1 Find the DTFT of $x(n) = a^n \cdot u(n)$ → |a|

Solution: $\text{DTFT}\{x(n)\} = X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}$

$$\therefore X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left[a^n \, u(n)\right] e^{-j\omega n}$$

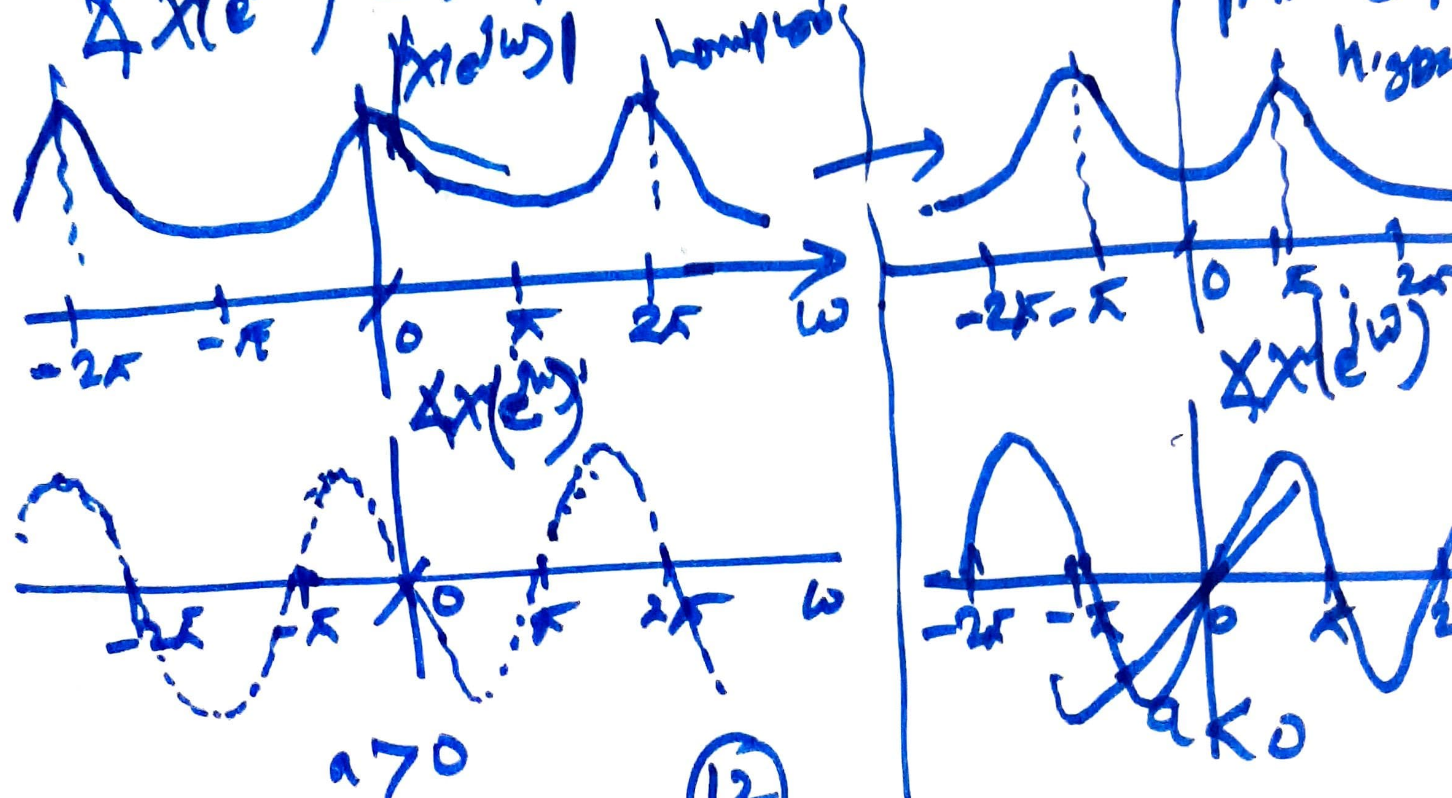
$=$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$|X(e^{j\omega})|$ → Magnitude spectrum

$\measuredangle X(e^{j\omega})$ → Phase spectrum

$|X(e^{j\omega})|$



orem (2)

$x(n) = a^{|n|}$; $|a| < 1$

Find $X(e^{j\omega})$??

Solution: → $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} \cdot e^{-j\omega n}$$

$$|n| = \begin{cases} +n, & n > 0 \\ -n, & n < 0 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} a^{-n} \cdot e^{-j\omega n} + \sum_{n=0}^{+\infty} a^{+n} \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} (a e^{j\omega})^{-n} + \sum_{n=0}^{+\infty} (a e^{-j\omega})^{n}$$

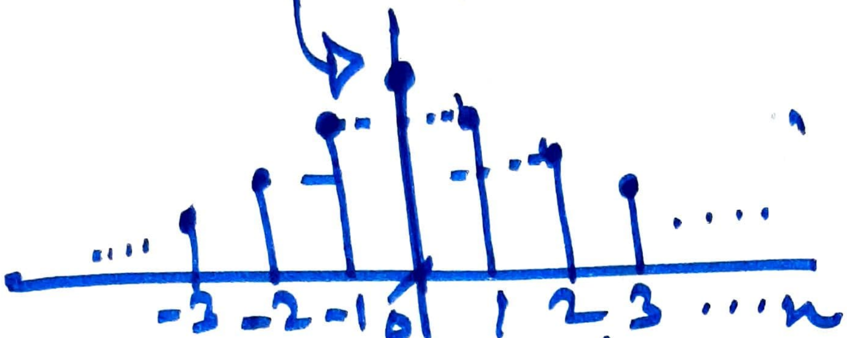
$n = -m$

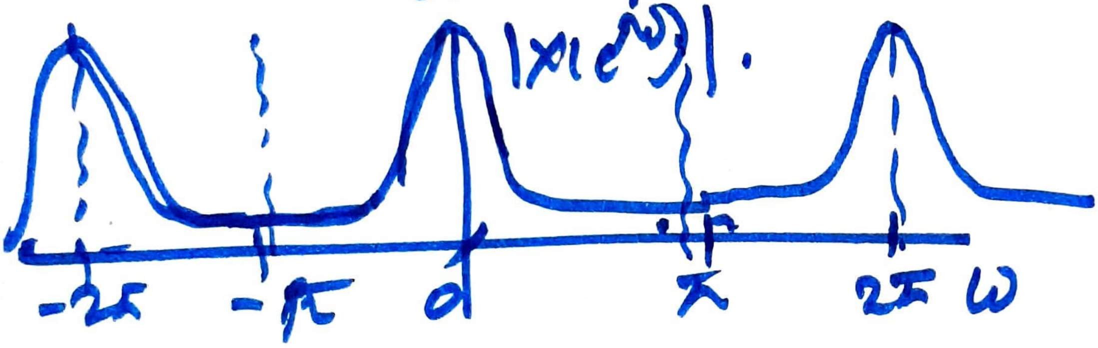
$$= \sum_{m=1}^{+\infty} (a e^{j\omega})^{m} + \sum_{n=0}^{+\infty} (a e^{-j\omega})^{n}$$

$$X(e^{j\omega}) = \frac{a e^{j\omega}}{1-a e^{j\omega}} + \frac{1}{1-a e^{-j\omega}}$$

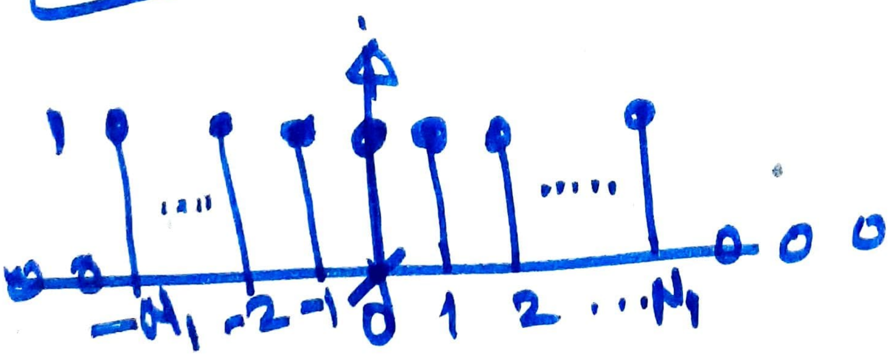
$$X(e^{j\omega}) = \frac{1-a^2}{1-2a\cos(\omega)+a^2}$$

$x[n] = a^{|n|}$; $|a|<1$





$\boxed{\text{Problem 3}}$ $x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{+N_1} 1\, e^{-j\omega n}$$

Let $m = n + N_1$, $n = m - N_1$

$n = -N_1$, $m = 0$

$n = N_1 \rightarrow m = 2N_1$

$$X(e^{j\omega}) = \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)}$$

$$= e^{j\omega N_1} \left\{ \sum_{n=0}^{2N_1} e^{-j\omega n} \right\}$$

$$= e^{j\omega N_1} \times 1 \times \left\{ \frac{1 - \left(e^{-j\omega}\right)^{2N_1+1}}{1 - e^{-j\omega}} \right\}$$

$$= \frac{e^{j\omega N_1} \times e^{-j\omega(2N_1+1)/2}}{e^{-j\omega/2}} \times \frac{\left\{ e^{j\omega(2N_1+1)/2} - e^{-j\omega\frac{(2N_1+1)}{2}} \right\}}{\left\{ e^{j\omega/2} - e^{-j\omega/2} \right\}}$$

15

$$\boxed{X(e^{j\omega}) = \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}}$$

---

* Convergence Issues Associated with DTFT.

The analysis eqn (A) $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ will converge either if sequence $x[n]$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < +\infty$$

Or if the sequence $x[n]$ has finite energy

i.e.,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < +\infty$$

Problem: $x[n] = \delta[n] \longrightarrow$ DTFT.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n] \cdot e^{-j\omega n} = 0 + 0 + 1 \underbrace{(e^{-j\omega n})}_{n=0} + 0 \cdots$$

$$= 1$$

$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$ (16) $\quad DTFT\{\delta[n]\} = 1$