LECTURE 21

$$\ddot{X} + \omega^2 X = 0 .$$

Coupled eqns.

$$2\ddot{x} + \omega^2(5x - 3y) = 0 \qquad \text{——— (1)}$$

$$2\ddot{y} + \omega^2(5y - 3x) = 0 . \qquad \text{——— (2)}.$$

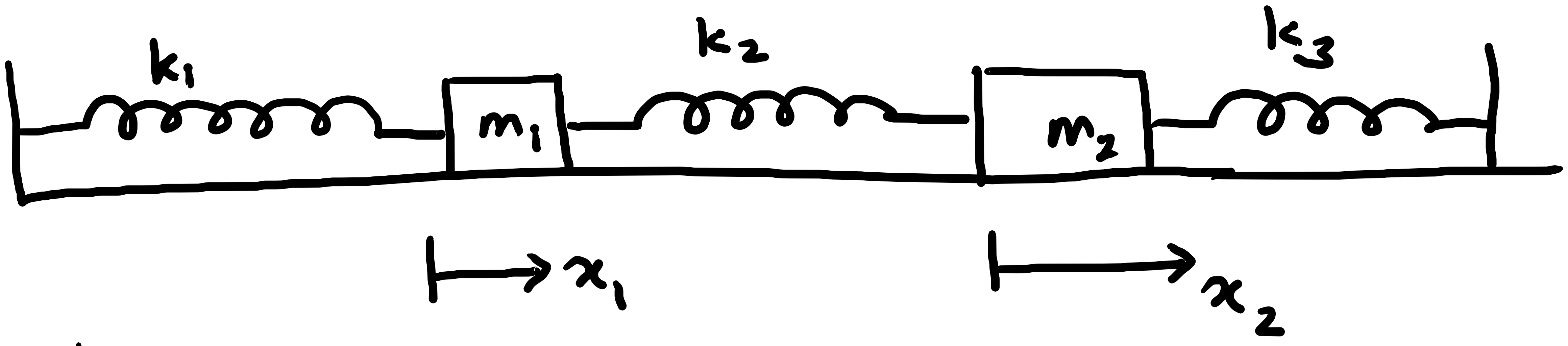
(1) + (2), $\quad (\ddot{x} + \ddot{y}) = -\omega^2 (x+y)$

(1) − (2), $\quad (\ddot{x} - \ddot{y}) = -4\omega^2 (x-y)$.

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} (x+y) \\ (x-y) \end{matrix}$

Linear combinations $(x+y)$ and $(x-y)$ oscillate with frequencies ω and 2ω respectively.

Systematic treatment.



EOM:-

$$\boxed{\begin{aligned} m_1\ddot{x}_1 &= -(k_1+k_2)x_1 + k_2 x_2 \\ m_2\ddot{x}_2 &= k_2 x_1 - (k_2+k_3)x_2 \end{aligned}} \quad \begin{aligned} &= -k_1 x_1 - k_2(x_1 - x_2) \\ &= -k_2(x_2 - x_1) - k_3 x_2 \end{aligned}$$

Coupled oscillators give rice to coupled eqns.

$$m_1 \ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2$$

$$m_2 \ddot{x}_2 = k_2 x_1 - (k_2+k_3)x_2$$

Matrix notation:

$$\bar{M}\ddot{\bar{x}} = -\bar{K}\bar{x}, \quad \text{where} \quad \bar{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix},$$

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$M\ddot{x} = -Kx \longrightarrow$ Differential eqn. in matrix notation.

Trial solution:- $x_1(t) = \alpha_1 \cos(\omega t - \delta_1)$ OR $y_1(t) = \alpha_1 \sin(\omega t - \delta_1)$

$x_2(t) = \alpha_2 \cos(\omega t - \delta)$. $y_2(t) = \ldots\ldots$

$$z_1 = x_1 + i y_1$$

$$M\ddot{z} = -Kz$$

Actual solⁿ: given by $\bar{x}(t) = \mathrm{Re}(\bar{z}(t))$.

$$\bar{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \bar{a}\, e^{i\omega t}$$

$$\bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{bmatrix}$$

$\alpha \equiv$ amplitude

$\delta \equiv$ phase.

$\bar{z}(t) = \bar{a}\, e^{i\omega t}$.

Substitute, $\bar{M}\ddot{z} = -\bar{K}z$

$$\Rightarrow \underbrace{(\bar{K} - \omega^2 \bar{M})}_{B.}\, a = 0 \longrightarrow \text{matrix eqn.}$$

For a non-trivial sol^n to exist, matrix $(\bar{K} - \omega^2 \bar{M})$ should be non-invertible.

$$\Downarrow$$

$$|\bar{K} - \omega^2 M| = 0.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$$= \left(\frac{1}{ad - bc}\right) \cdot \begin{pmatrix} & \end{pmatrix}$$

— Effectively, problem is reduced to

$$\boxed{\left|\bar{K} - \omega^2 \bar{M}\right| = 0}$$

— For ease of calculation, put $k_1 = k_2 = k_3 = k$

$m_1 = m_2 = m$

$$\bar{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \bar{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$\bar{K} - \omega^2 \bar{M} = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix}$$

NORMAL MODES.

$$|\bar{K} - \omega^2 \bar{M}| = 0$$

$$\Rightarrow (2k - m\omega^2)^2 - k^2 = 0$$

$$\Rightarrow (2k - m\omega^2) = \pm k.$$

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

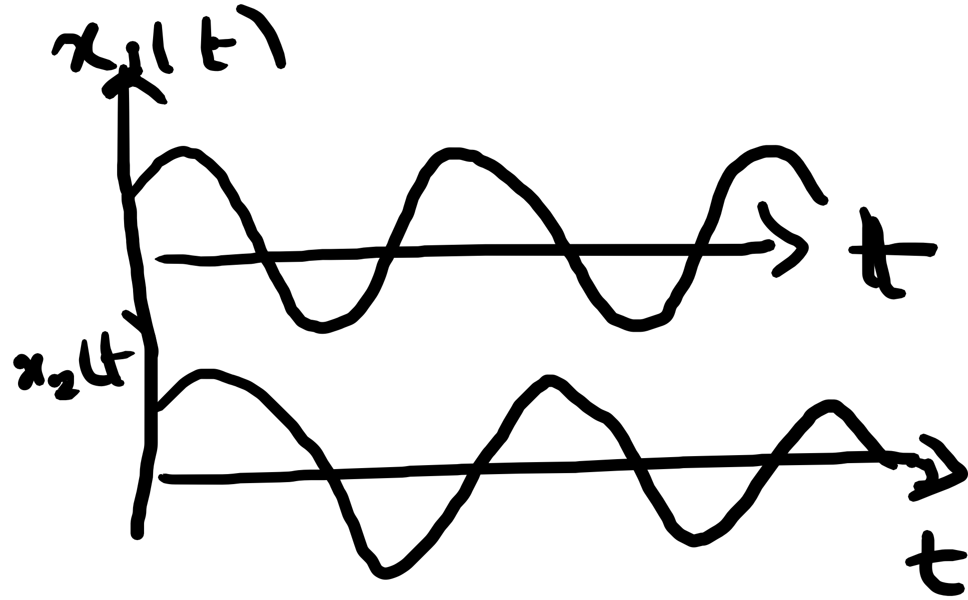
$$\bar{z}(t) = \bar{a}\, e^{i\omega t}.$$

— First normal mode

$$(\bar{K} - \omega_1^2 \bar{M}) = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$(\bar{K} - \omega_1^2 \bar{M})\,\bar{a} = 0$$

$$\Rightarrow \quad k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \quad \Rightarrow \quad a_1 = a_2 = A e^{-i\delta}$$



$$\bar{z}(t) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_1 t - \delta_1)}$$

$$\bar{x}(t) = \mathrm{Re}(\bar{z}(t)) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1).$$

$$x_1(t) = A\cos(\omega_1 t - \delta_1)$$

$$x_2(t) = A\cos(\omega_1 t - \delta_1).$$

$\rightarrow$ 1st normal mode

— Second normal mode

$$\left(\bar{K} - \omega_2^2 \bar{M}\right) = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} = -k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left(\bar{K} - \omega_2^2 \bar{M}\right) a = 0 .$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

$$a_1 = -a_2 = A e^{-i\delta_2}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\vec{z}(t) = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_2 t - \delta_2)}$$

$$x_1(t) = A\cos(\omega_2 t - \delta_2)$$

$$x_2(t) = -A\cos(\omega_2 t - \delta_2)$$