LECTURE 28

RECAP.

- Studied 1st order autonomous ODEs.

$$\frac{dx}{dt} = f(x)$$

R.H.S. is purely a function of x.

- LOGISTIC EQN.

$$\frac{dx}{dt} = (ax - bx^2).$$

$$- \quad \frac{dx}{dt} = ax - bx^2 = x(a-bx).$$

$$\frac{dX}{dT} = X - X^2, \qquad \text{where } T = at$$

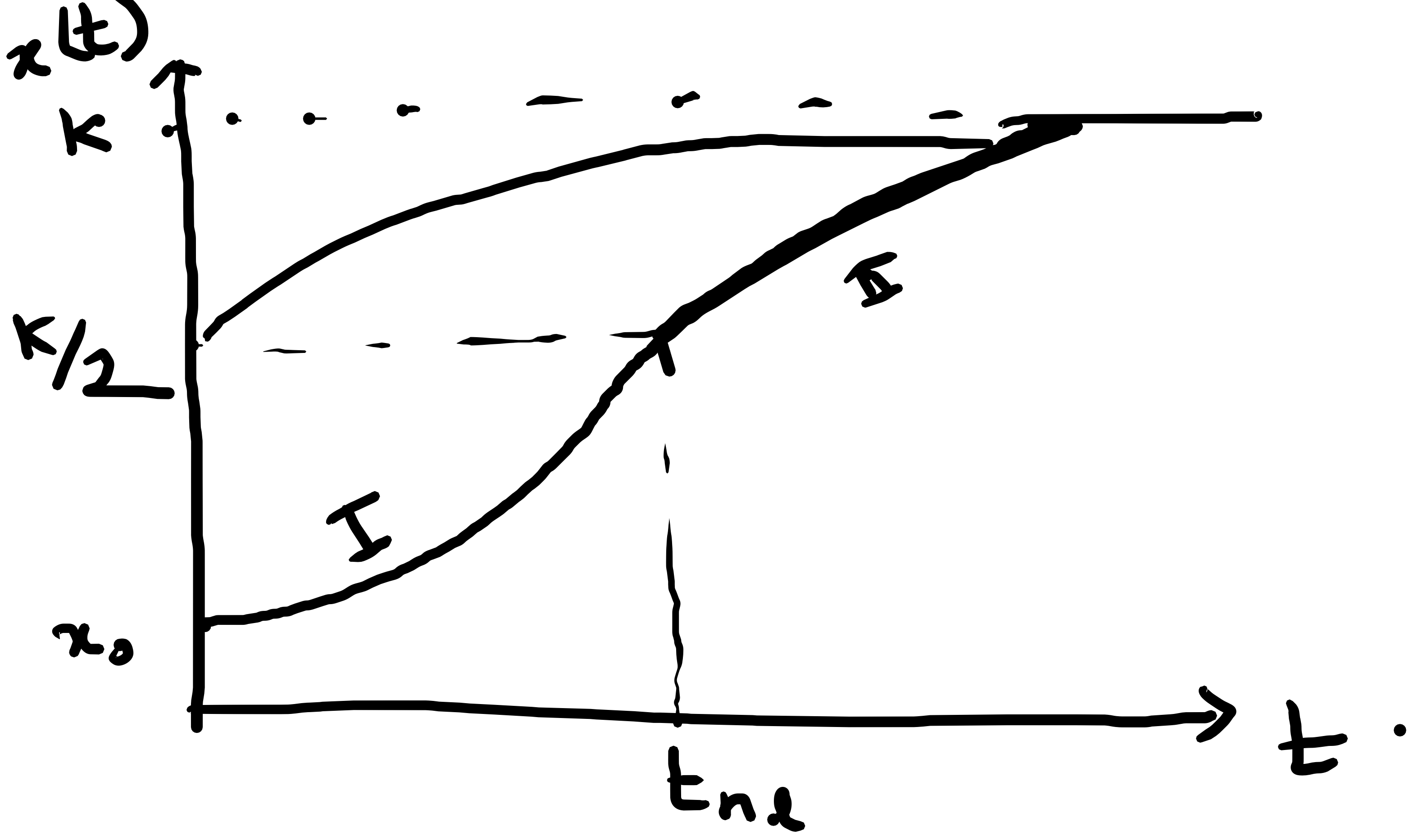
$$X = \left(x/k\right) \quad \text{where } k = \left(a/b\right).$$

In the small-time limit,

$$x(t) \approxeq \frac{x_0 e^{at}}{1 + \frac{x_0}{k} at} \longrightarrow \text{dominating behaviour.}$$

In the large-time limit

$$x(t) \simeq K.$$



$$\frac{dX}{dT} = X - X^2 = F(X)$$

$$\frac{d^2X}{dT^2} = \frac{dF}{dX}\,\frac{dX}{dT} = F\,\frac{dF}{dX}$$

$$\frac{dF}{dx} = 0$$

$$\Rightarrow 1 - 2X = 0$$

$$\Rightarrow X = 1/2$$

$$\Rightarrow x = \frac{K}{2}$$

When $X < \frac{1}{2}$, $\frac{d^2X}{dT^2} > 0$ — (I)

When $X > \frac{1}{2}$, $\frac{d^2X}{dT^2} < 0$ — (II)

(I) growth occurring at increasing rate

(II) " " " decreasing ".

— Complete solⁿ:

$$X = \frac{1}{1 + A^{-1} e^{-T}}$$

$$X = \frac{1}{2} = \frac{1}{1 + A^{-1} e^{-T_{n2}}}$$

$$\Rightarrow \quad 1 + A^{-1} e^{-T_{n2}} = 2$$

$$\Rightarrow \quad T_{n2} = \ln\left(\frac{1}{A}\right) = \ln\left(\frac{1 - X_0}{X_0}\right)$$

$$\Rightarrow \quad a t_{n2} = \ln\left(\frac{K - x_0}{x_0}\right) = \frac{1}{a} \ln\left(\frac{K}{x_0} - 1\right).$$

INFER QUALITATIVE FEATURES OF 1st ORDER AUTONOMOUS ODEs.

$\dot{x} = f(x)$. → general form.

— Visualise t as time, x as position of a particle at instant t, $\dot{x}$ = velocity of the particle.

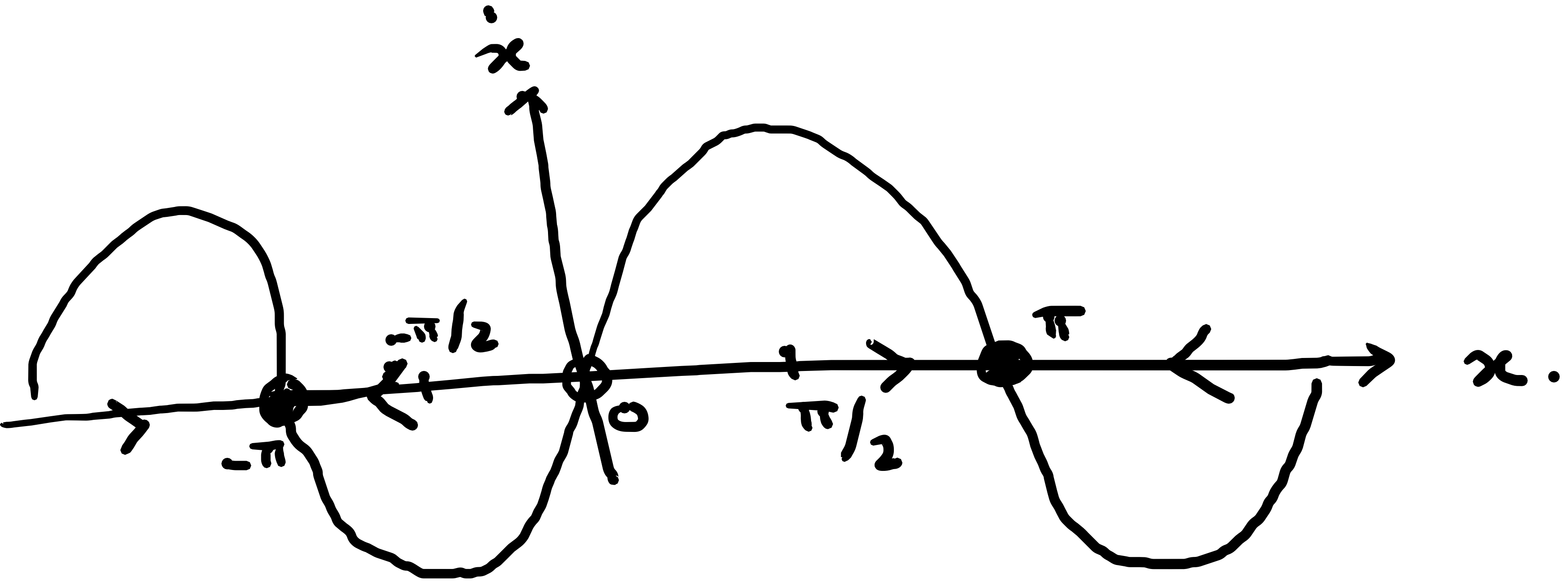
— When $\dot{x} = 0$, such points are "FIXED POINTS" (can be stable or unstable)

attractors repellers.

Example :- $\dot{x} = \sin x$.

$$\Rightarrow \quad \operatorname{cosec} x \, dx = dt$$

$$t = \ln \left| \frac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x} \right| .$$



Fixed points :-

$$\sin x^* = 0 .$$

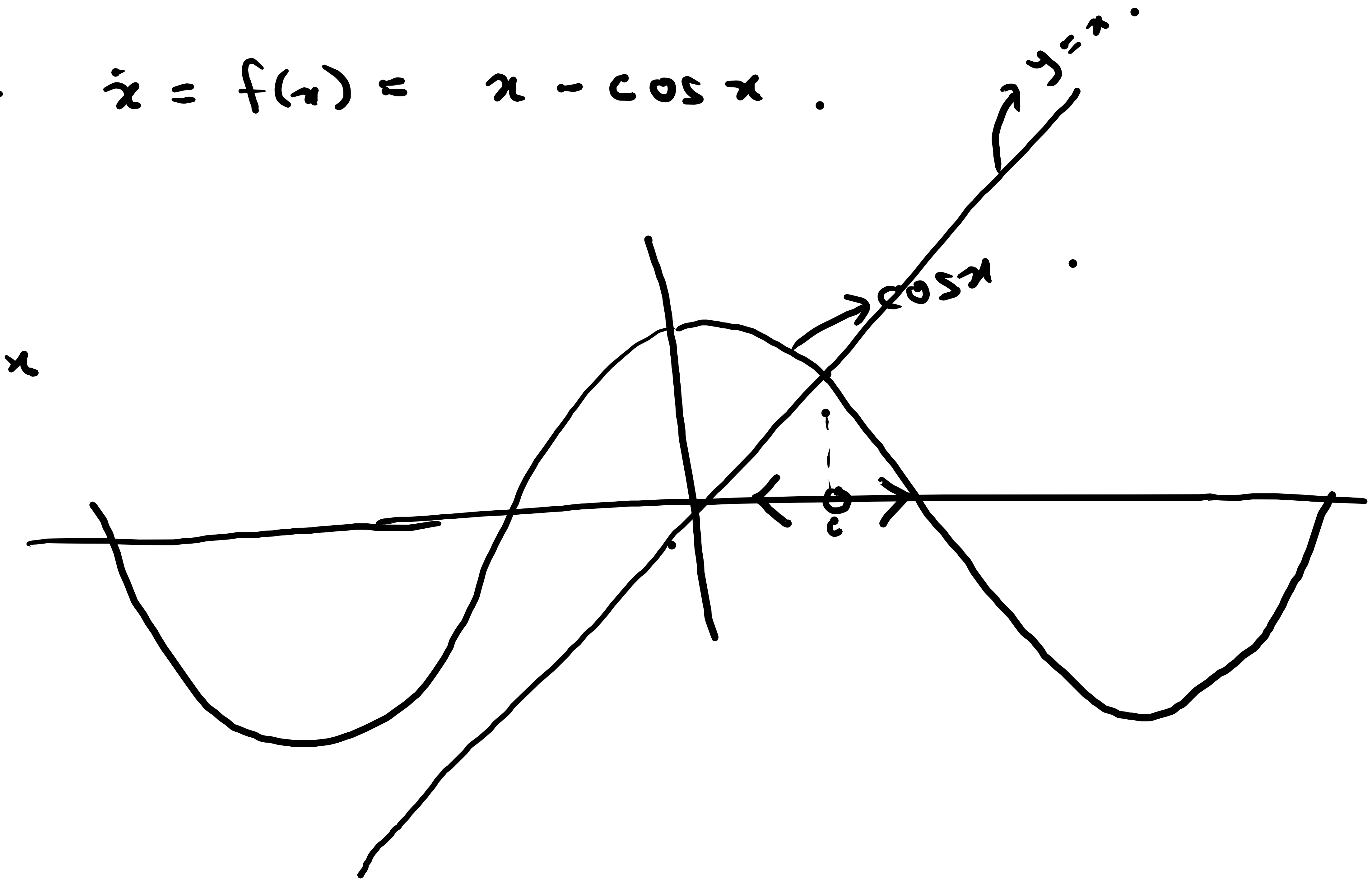
$$\Rightarrow x^* = 0, \pm \pi, \ldots$$

If particle tends to move back towards a fixed point then fixed point is stable, else it is unstable.

Example :- $\dot{x} = f(x) = x - \cos x$.

$y = x$

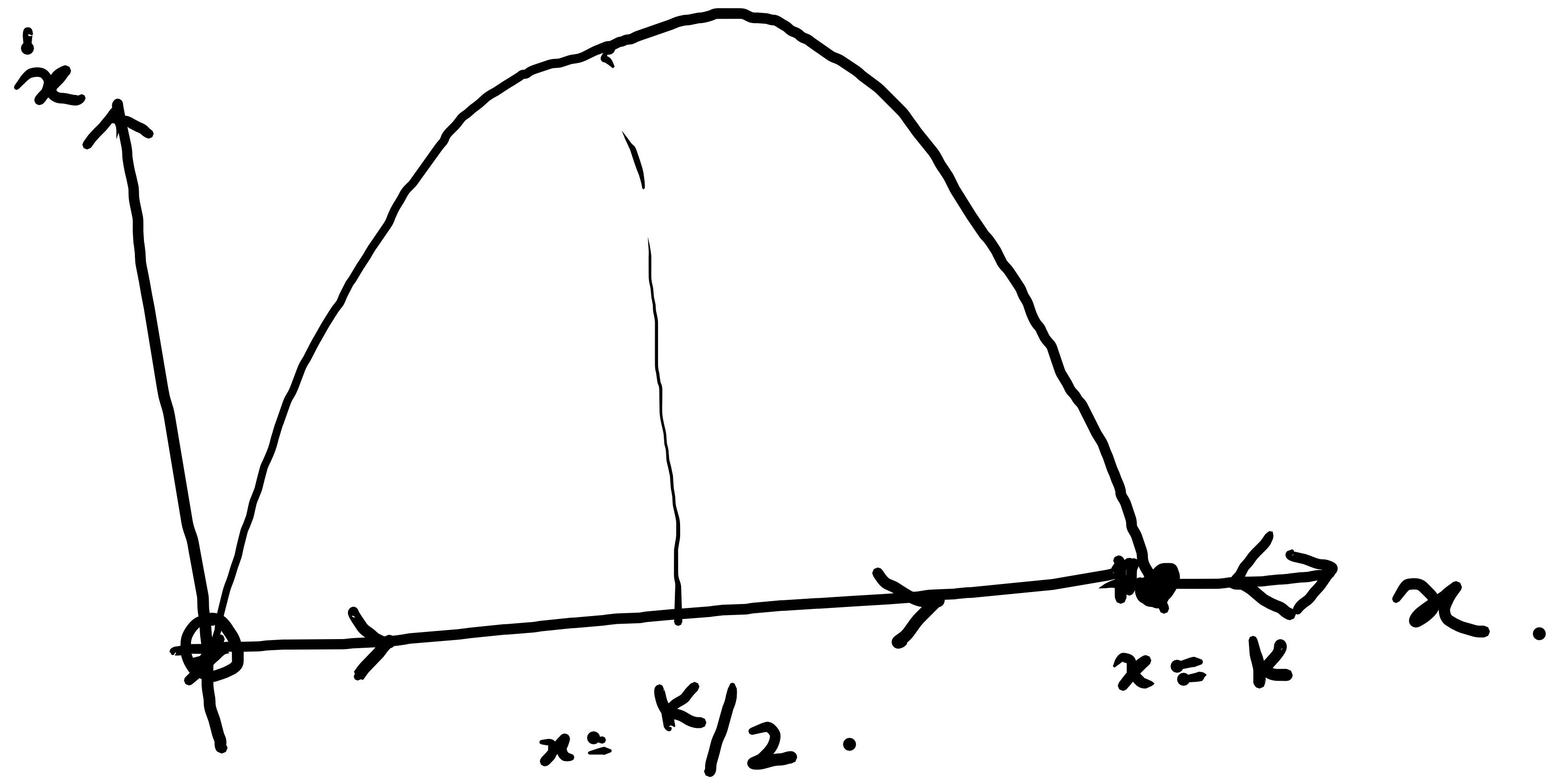
$y = \cos x$

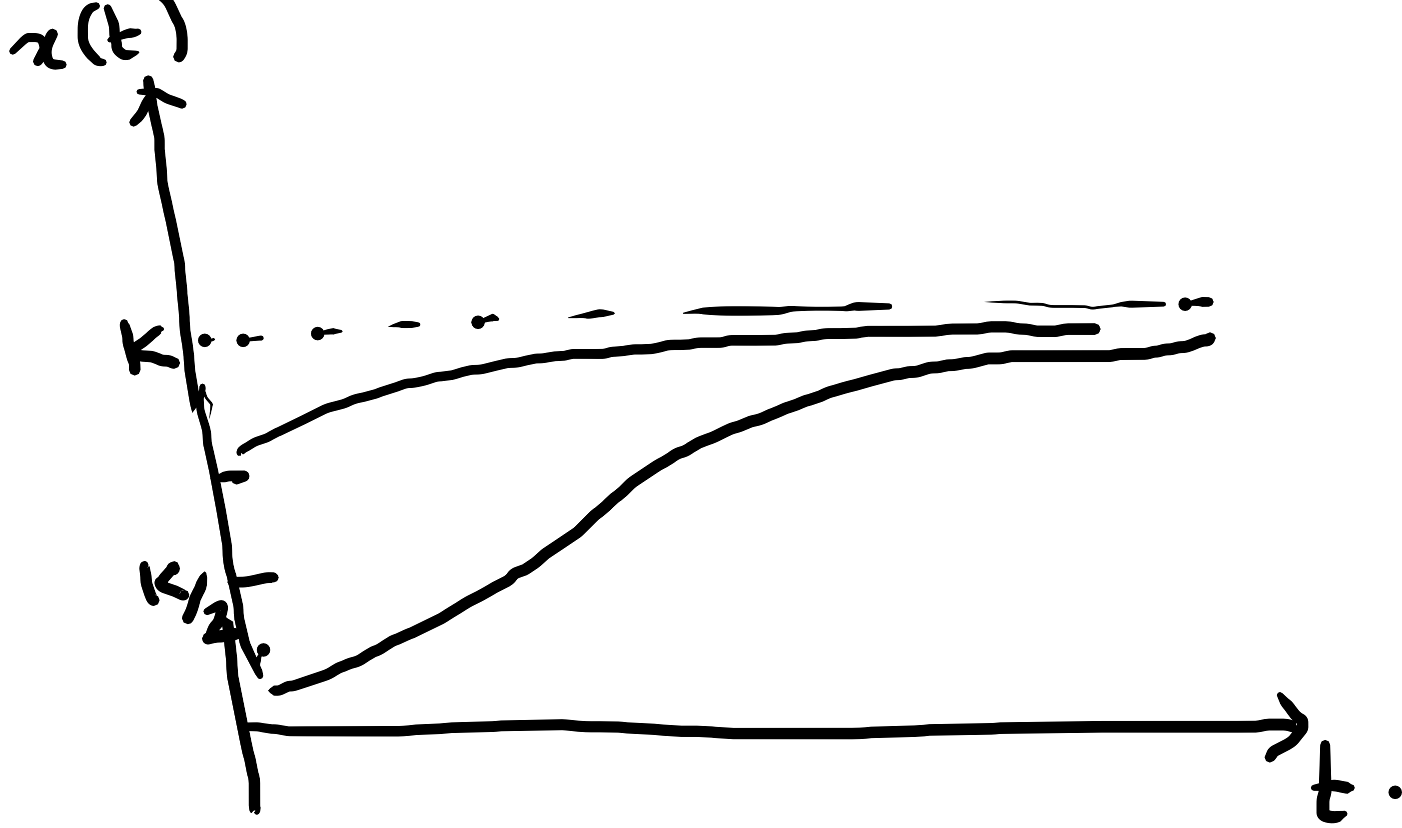


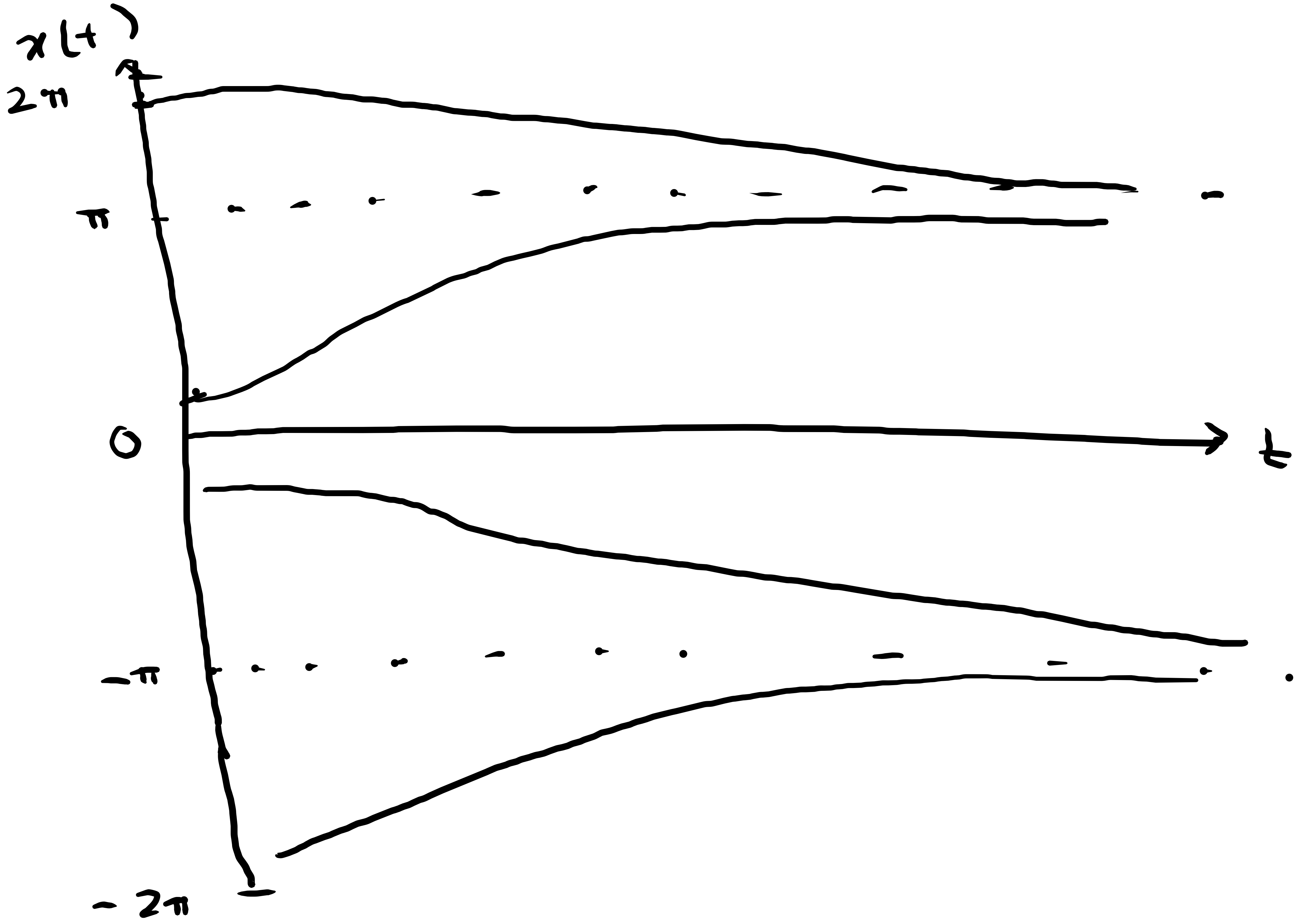
Example:- LOGISTIC EQN.

$$\dot{x} = ax - bx^2 \qquad x^* = 0,\ (a/b) = K.$$

$$\dot{X} = X - X^2. \qquad \dot{X} = 0,\ 1.$$







x(t)
2π
π
0
t
-π
-2π