Problem on Differentiation Property

Lecture 43

Q. Find the current flowing through a circuit shown below when switch K is closed at $t=0$

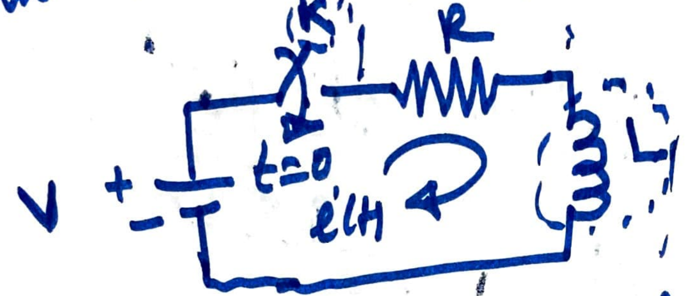


Fig. Series RL circuit

<u>Solution</u>: Apply KVL

$$V u(t) = V_R + V_L = R\,i(t) + L\frac{d\,i(t)}{dt}$$

→ Differential equation

Taking Laplace transform on both the sides,

$$\therefore \mathcal{L}\{V u(t)\} = \mathcal{L}\{R\,i(t)\} + L\,\mathcal{L}\left\{\frac{d\,i(t)}{dt}\right\}$$

$$V\left(\frac{1}{s}\right) = R\,I(s) + L\,s\,I(s)$$

$$I(s) = \frac{V\left(\frac{1}{s}\right)}{R + L s} = \frac{V}{L}\left\{\frac{1}{s\left(s+\frac{R}{L}\right)}\right\}$$

$$I(s) = \frac{A_1}{s} + \frac{A_2}{s+R/L} \quad (1)$$

[Partial fraction Expansion]

To find expression for current flowing in circuit, take $\mathcal{L}^{-1}\{\cdot\}$ on both the sides

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{A_1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{A_2}{s+R/2}\right\}$$

$$= A_1 [u(t)] + A_2 \left[e^{-(R/2)t} \cdot u(t)\right]$$

$$\boxed{i(t) = \frac{V}{R} \cdot \left[1 - e^{-(R/2)t}\right] \cdot u(t)}$$

Integration Property:→

Rt $x(t) \xleftrightarrow{L} X(s)$

then $\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{L} ?\ \left(\frac{1}{s}\right) \cdot X(s).$

Proof :: $\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$

$$L\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = L\{x(t) * u(t)\}$$

Convolution theorem

$$= L\{x(t)\} \cdot L\{u(t)\}$$

$$= X(s)\, \frac{1}{s}$$

(3.7) Analysis and Characterization of LTI Systems using the Laplace Transform

Application of convolution Theorem for LT.

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI} \\ \text{system} \\ h(t)\end{array}} \longrightarrow y(t)$$

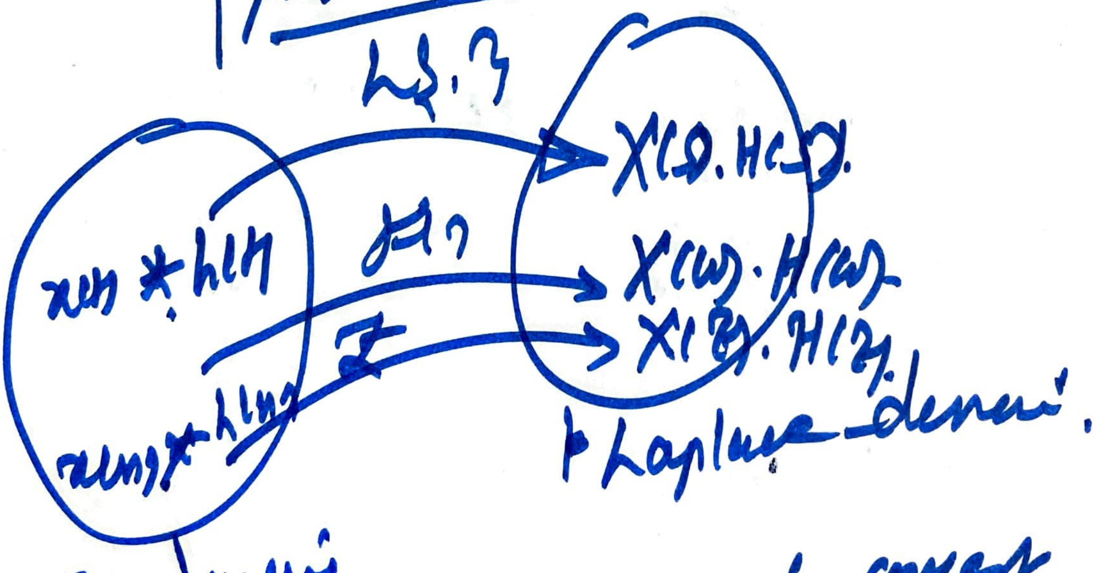
$$x(t) * h(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \cdot H(s)$$

$\therefore\ y(t) = x(t) * h(t)$

Proof. Homework.

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

$$\boxed{Y(s) = X(s) \cdot H(s)}$$



Convolution theorem helps us to convert computational complex task of convolution to a simple algebraic operation of multiplication in respective transform domains.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{L\{y(t)\}}{L\{x(t)\}}$$ ✓

System function or Transfer function [H(s)]:
Laplace transform of output divided by Laplace transform of input with zero initial conditions OR system being initially at rest.

Q. Why zero initial conditions or system initially at rest.

⇒ chapter 2 (DE)

System initially at rest ⇒ Underlying system is Linear, Time-Invariant & causal

$$H(s) = \frac{Y(s)}{X(s)}$$

$Y(s) = X(s) \cdot H(s)$ ← Convolution Theorem ← $y(t) = x(t) * h(t)$

1] Causality

$$h(t) = e^{-at} \cdot u(t) \xleftrightarrow{L} \frac{1}{s+a}$$

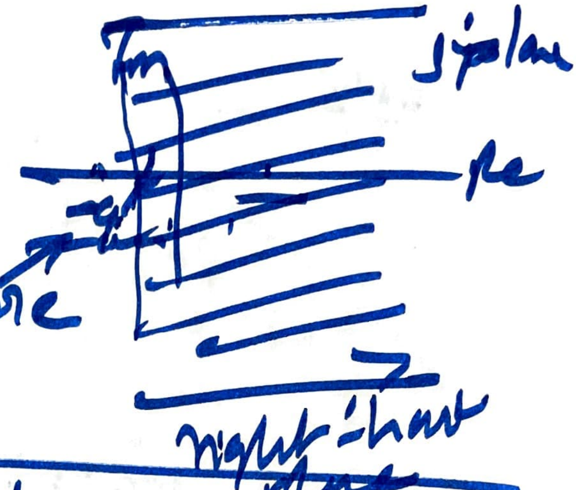
ROC : Re{s} > -a

causal system

causal signal

$$H(s) = \frac{1}{s+a}$$ rational

pole at $s = -a$



The ROC associated with the system function for a causal system is a right-half plane

system function = rational: causality of the system is equivalent to ROC being the right-half plane to the right of the right most pole

Problem $h(t) = e^{-|t|}$ → non-causal.

$h(t) = e^{-t}u(t) + e^{t}u(-t)$.

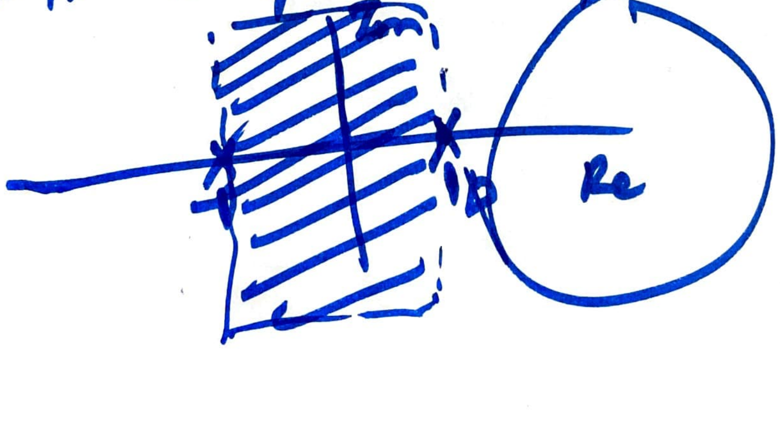
$H(s) = \frac{1}{s+1} + \frac{-1}{s-1} = \frac{-2}{s^2-1}$

⇒ Poles ⇒ $s = -1$, $s = 1$

ROC R_1 : $Re\{s\} > -1$ $R_2 \Rightarrow Re\{s\} < 1$

· ROC of $H(s) = R_1 \cap R_2 \Rightarrow -1 < Re\{s\} < +1$

$H(s)$ = is noncausal, it has an ROC that is not to the right of rightmost pole



(3) $H(s) = \frac{e^{s}}{s+1}$, $Re\{s\} > -1$

$e^{-t}\,u(t) \xleftrightarrow{L} \frac{1}{s+1}$

$e^{-(t+1)}\,u(t+1) \xleftrightarrow{L} \frac{e^{s}}{1+s}$

$h(t) = L^{-1}\left\{\frac{e^{s}}{s+1}\right\} = e^{-(t+1)}\,u(t+1)$

Non-causal

Inference:

Causality ⟹ ROC to the right of rightmost pole

Non-causal $\not\Leftarrow$ Not true in general

converse is NOT true.

Stability :→

If an LTI system stable → will satisfy absolute integrability

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Chapter 2 →

↓

Dirichlet condition satisfied

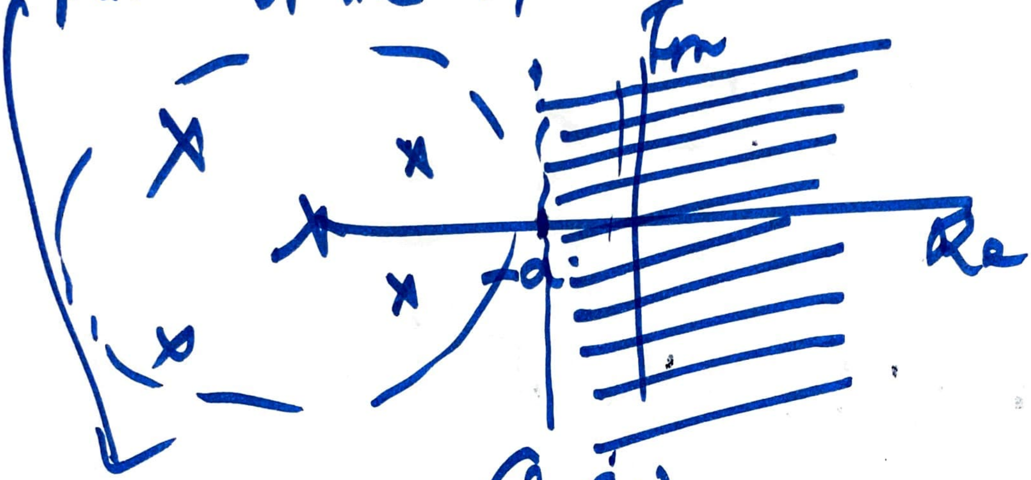
CTFT exists ← H(jw) is finite ← H(s)| s=jw = defined

↓

ROC contains s = jw - axis i.e. Re{s} = 0

An LTI system is stable if and only if the ROC of its system function H(s) includes the jw-axis [i.e. Re{S} = 0]

If an LTI system that stable is causal as well then ROC will include $s = j\omega$ axis and right half plane of the s-plane



$$H(s) = \frac{(s-a_1)}{(s-b_1)(s-b_2)\cdots(s-b_n)}$$

A causal system with rational system function $H(s)$ is stable if and only if all the poles of $H(s)$ lie in the left-half of the s-plane, i.e., all the poles have negative real parts.

Problem causal 2nd order system

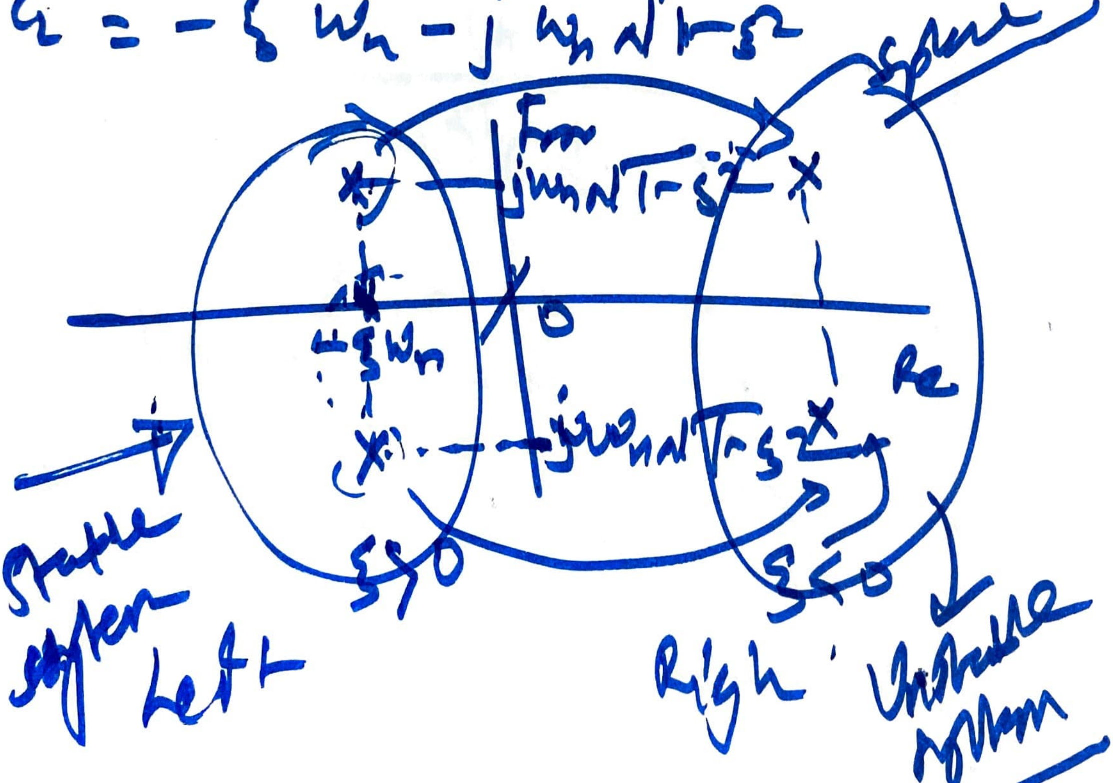
$$H(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$$

$s = j\omega \rightarrow$ chapter 6

$c_1, c_2 =$

$c_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$

$c_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$



LTI systems characterised by Linear Constant Coefficient Differential Equations

Chapter 2

$$\boxed{LCCDE}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

To find system function, $H(s)$.

Take Laplace transform on both sides

$$L\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = L\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

Linearity & differentiation property of LT

$$\sum_{k=0}^{N} a_k L\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k L\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Convolution theorem

$$H(s) = \frac{b_0 + b_1 s^1 + b_2 s^2 + \cdots + b_M s^M}{a_0 + a_1 s + a_2 s^2 + \cdots + a_N s^N}$$

$$H(s) = \frac{N(s)}{D(s)} = \frac{\rightarrow \text{roots of } N(s) \rightarrow \text{zeros of } H(s)}{\rightarrow \text{roots of poles of } H(s).}$$

<u>Problem</u> Consider an LTI system for which the input and output $y(t)$ satisfying the LCCDE,

$$\frac{dy(t)}{dt} + 3y(t) = x(t).$$

Find the impulse response, $h(t)$ of this system.

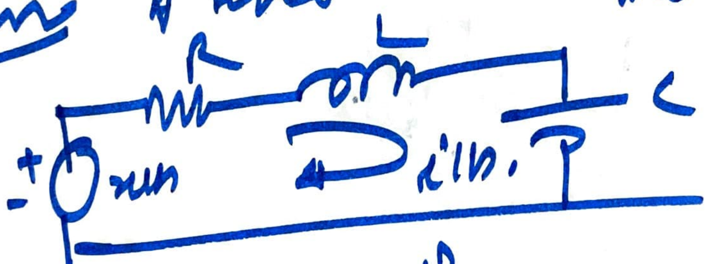
Soln: $\rightarrow$ $sY(s) + 3Y(s) = X(s)$

$$\therefore \boxed{H(s) = \frac{1}{s+3}}$$

$$\therefore h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{\begin{aligned} h(t) &= e^{-3t}u(t) \;;\; \text{ROC } Re\{s\} > -3. \\ \text{or} \\ &= -e^{-3t}u(-t) \;;\; \text{ROC } Re\{s\} < -3 \end{aligned}}$$

Problem A series RLC → find whether RLC ckt is stable?



$$(RC)\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = x(t)$$

$$H(s) = \frac{(1/LC)}{s^2 + (R/L)s + (1/LC)}$$

Roots of denominator polynomial of $H(s)$ will have negative real parts.

⇒ Poles of $H(s)$ will lie left-half of s-plane

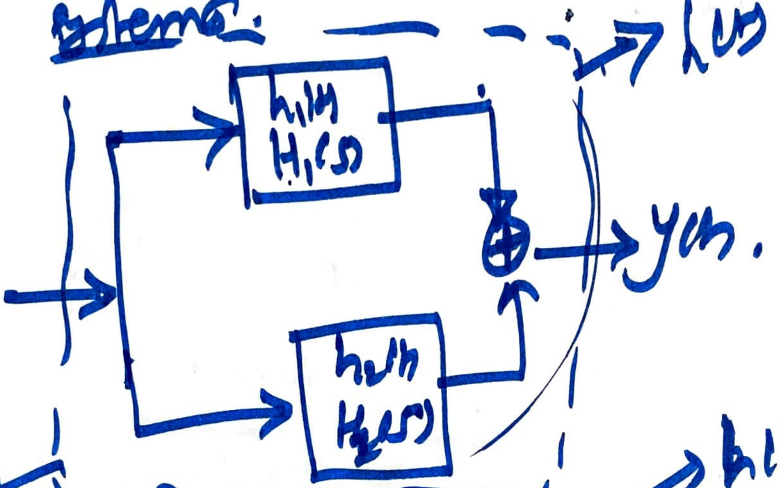
⇒ RLC ckt is stable.

(9.8) Control Systems

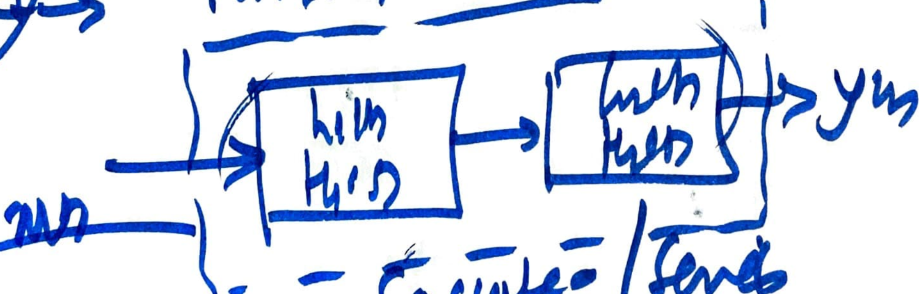
System Block Diagram and System Function Algebra

[Feedback Control Systems]

Background: System functions for Interconnections of LTI systems.



Chapter → Parallel



Cascade / Series

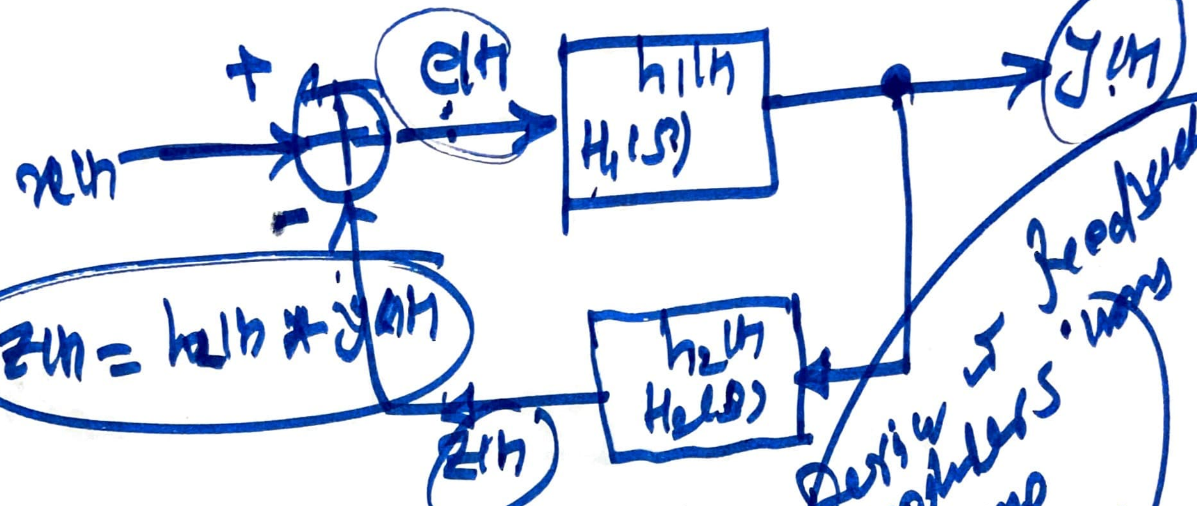
$h(t) = h_1(t) + h_2(t)$

$\therefore H(s) = H_1(s) + H_2(s)$

$h(t) = h_1(t) * h_2(t)$.

$H(s) = H_1(s) \cdot H_2(s)$ [Convolution Theorem]

Feedback Interconnection of two LTI system



$z(t) = h_2(t) * y(t)$

Design of Feedback amplifiers "uses" op-amp LM 741 ka

$Y(s) = H_1(s) \cdot E(s)$

$e(t) = x(t) - z(t)$

$\therefore E(s) = X(s) - Z(s)$

$Y(s) = H_1(s) [X(s) - Z(s)]$

$Z(s) = H_2(s) \cdot Y(s)$

overall of closed loop transfer function

$Y(s) = H_1(s) [X(s) - H_2(s) \cdot Y(s)]$

$Y(s) = H_1(s) \cdot X(s) - H_1(s) \cdot H_2(s) \cdot Y(s)$

$$\therefore \boxed{\frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s) \cdot H_2(s)} = H(s)}$$

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* Expression of $-3dB$ Bandwidth in S-plane:

In general, system function of 1st order filter in s-domain is given.

$$H(s) = \frac{-\sigma_p}{s - s_p}, \quad \sigma_p = Re\{s_p\}$$

where $s = \sigma + j\omega$

s_p = Pole of $H(s)$ in s-plane $= \sigma_p + j\omega_p$

$\therefore$ For frequency response, substitute, $s = j\omega$

$$\therefore H(s)\Big|_{s=j\omega} = H(j\omega) = \frac{-\sigma_p}{j\omega - (\sigma_p + j\omega_p)}$$

$$|H(j\omega)|^2 = \frac{(-\sigma_p)^2}{(-\sigma_p)^2 + (\omega - \omega_p)^2}$$

$\omega_p = 0$

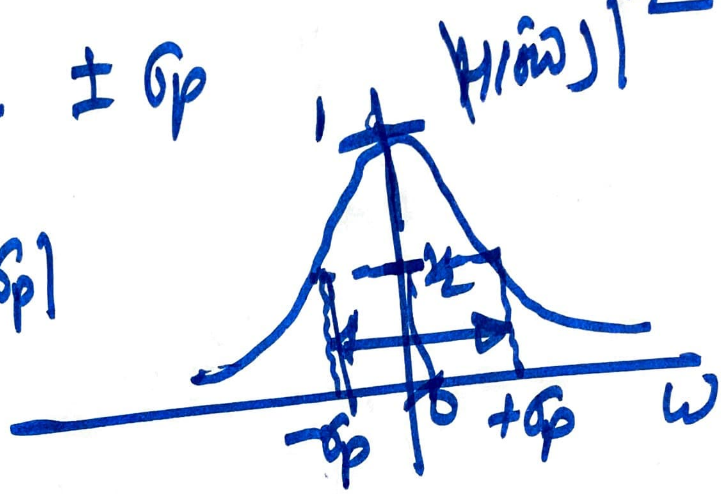
$$|H(j\omega)|^2 = \frac{\sigma_p^2}{\sigma_p^2 + \omega^2}$$

For half-power bandwidth,

$$|H(j\omega)|^2 = 1/2$$

$$\therefore \frac{\sigma_p^2}{\sigma_p^2 + \omega^2} = 1/2$$

$$\therefore \omega = \pm \sigma_p$$



$$\therefore \omega_{\text{Half power}} = 2|\sigma_p|$$

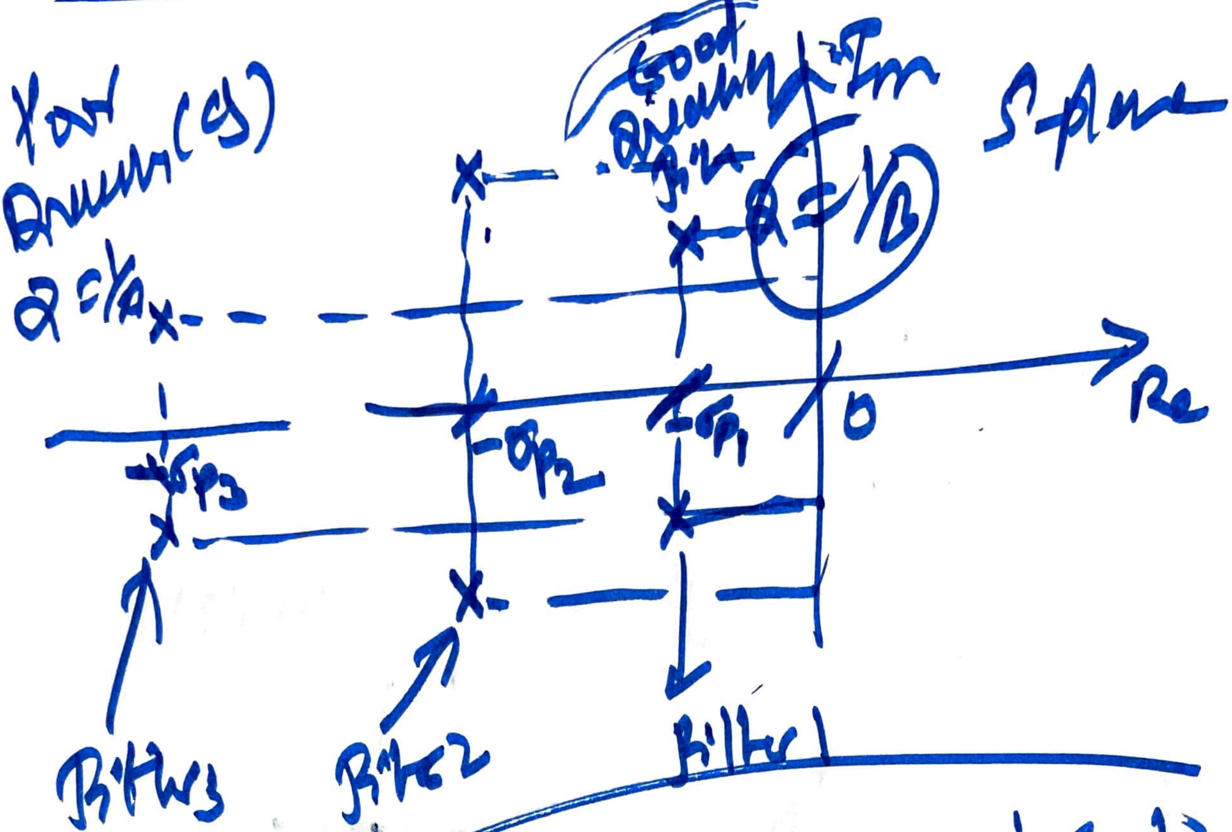
(rad/sec)

$$2\pi B \text{ (in Hz)} = 2|\sigma_p|$$

$$\therefore \boxed{B \text{ (in Hz) Bandwidth} = \frac{|\sigma_p|}{\pi}}$$

Half power or -3dB bandwidth

Inference: In Laplace domain, real part of the s-domain pole controls the -3dB Bandwidth of filter



$|\sigma_{p_1}|$, $|\sigma_{p_2}|$ $|\sigma_{p_3}|$

(minimum B (−3dB)). $B = \frac{|\sigma_p|}{k}$

$Q \propto \frac{1}{B}$ For $Q \uparrow \Rightarrow B \downarrow \Rightarrow |\sigma_p| \downarrow$