SC223 - Linear Algebra

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Lecture 6

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- **Theorem 2:**
  1. Using the convention for ERO, any ERT will be a lower triangular matrix. (except row exchange)
  2. The product of any two lower triangular matrices is a lower triangular matrix.
  3. Any ERT is an invertible matrix.
  4. The inverse of any invertible lower triangular matrix is also a lower triangular matrix.

$$E = E_3 E_2 E_1$$

$A \in \mathbb{R}^{n \times n}$, $\quad B \in \mathbb{R}^{n \times n}$ s.t $AB = BA = I_n$.

$$E \cdot \underbrace{(E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1})}_{E^{-1}} = I = \underbrace{(E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1})}_{E^{-1}} \cdot E$$

$$\underbrace{\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}} = I_n$$

$$I_n = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix}$$

$$Lv_1 = e_1$$

$$\boxed{Lv_2 = e_2}$$

$$\vdots$$

$$Lv_n = e_n$$

Lower Δular!!

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i^{th} pos^n

$$L v_2 = e_2$$

$$\begin{bmatrix} l_{*1} & l_{*2} & \cdots & l_{*n} \\ & & & \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow v_{21} l_{*1} + v_{22} l_{*2} + \cdots + v_{2n} l_{*n} = \underline{e_2}.$$

$$v_{21} l_{11} + v_{22} \cdot 0 + \cdots + v_{2n} \cdot 0 = 0$$

$$\Rightarrow v_{21} = 0$$

$L v_k = e_k$:

$$v_{k1} l_{*1} + v_{k2} l_{*2} + \cdots + v_{kn} l_{*n} = e_k.$$

$$i<k, \quad v_{k1} l_{11} + \cdots + v_{k,k-1} l_{i\,k-1} = 0 \Rightarrow v_{k1} = \cdots = v_{k,k-1} = 0$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 & 0 \\ l_{21} & l_{22} & & \vdots & \\ \vdots & & l_{33} & 0 & \\ & & & l_{kk} & \\ & & & l_{k+1,k} & \\ & & & \vdots & \\ l_{n1} & l_{n2} & & l_{nk} & l_{nn} \end{bmatrix} \qquad \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}$$

k^{th} column

$$v_{k1} \begin{bmatrix} l_{11} \\ \vdots \\ l_{41} \end{bmatrix} + v_{k2} \begin{bmatrix} 0 \\ l_{22} \\ \vdots \\ l_{42} \end{bmatrix} + v_{k3} \begin{bmatrix} 0 \\ 0 \\ l_{33} \\ l_{43} \end{bmatrix} + v_{k4} \begin{bmatrix} 0 \\ 0 \\ 0 \\ l_{44} \end{bmatrix}$$

LU Decomposition

$$A = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}$$

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- $E_3 E_2 E_1 A = EA = U$

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- $E_3E_2E_1A = EA = U \;\Rightarrow A = E^{-1}U$.
- By Theorem 2, E^{-1} is a lower triangular matrix. Define $L := E^{-1}$.

Thus $A = LU$, known as the **LU decomposition**.

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3/2 & 1/4 & 1 & 0 \\ -7/11 & -3/11 & 10/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & -10/11 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 \quad x_2$

$= 1$

LU Decomposition

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- For this example:

$$\underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1/4 & 1 & 0 \\ 3 & 1/2 & -10/11 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 4 & 5 & 4 \\ 0 & 0 & -11/4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U}$$

Is $A = LU$ always possible?

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 3 \\ 2 & 4 & 0 & 1 & 4 \\ -2 & -3 & 0 & 5 & -4 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right]$$

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- $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

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- $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_1 A = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & -2 & 5 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

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- $P_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

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- $E_2 P_{23} E_1 A = U$.

$$A = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 4 & 0 & 1 \\ -2 & -3 & 0 & 5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

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