Chapter 4 | Lecture 34

Application of Duality Property of CTFT

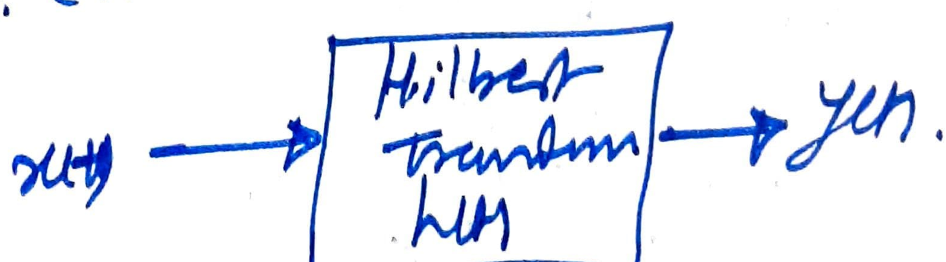
↳ Development of Hilbert Transform

Case I: Continuous-time case

Motivation.

1. $\frac{\pi}{2}$-phase shifter
2. Single sideband Modulation (SSB)
3. Frequency Modulation
4. To generate analytic signal and analytic (instantaneous phase)
5. To develop algorithm for instantaneous frequency (IF).

Defn: (Hilbert Transform)



$$y(t) = x(t) * h(t),$$

where $h(t) = \frac{1}{\pi t}$.

$y(t) = $ Hilbert Transform $\{x(t)\}$

$$y(t) = x(t) * \frac{1}{\pi t} \rightarrow \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot \frac{1}{\pi(t-\tau)}\, d\tau}$$

When $\tau \rightarrow t$, the integrand $x(\tau) \times \frac{1}{\pi(t-\tau)} \rightarrow \infty$

∴ Such integrals are called as singular integrals

∴ Evaluation in time domain is difficult

∴ We work in frequency domain

$$y(t) = x(t) * \frac{1}{\pi t}$$

$$F\{y(t)\} = F\{x(t) * \frac{1}{\pi t}\}$$

$$Y(\omega) = X(\omega) \cdot \underbrace{F\{\frac{1}{\pi t}\}}_{??}$$

Goal is to find $F\{h(t)\} = F\left\{\frac{1}{\pi t}\right\} = ??$

We have, $F\{u(t)\} = U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

$\therefore F\{2u(t)-1\} = 2F\{u(t)\} - F\{1\}$

$= 2\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - 2\pi\delta(\omega)$

$= \frac{2}{j\omega} + \underline{2\pi\delta(\omega) - 2\pi\delta(\omega)}$

$$\boxed{F\{2u(t)-1\} = \frac{2}{j\omega}}$$

$2u(t) - 1 \xleftrightarrow{F(\cdot)} \left(\frac{2}{j\omega}\right)$

Duality

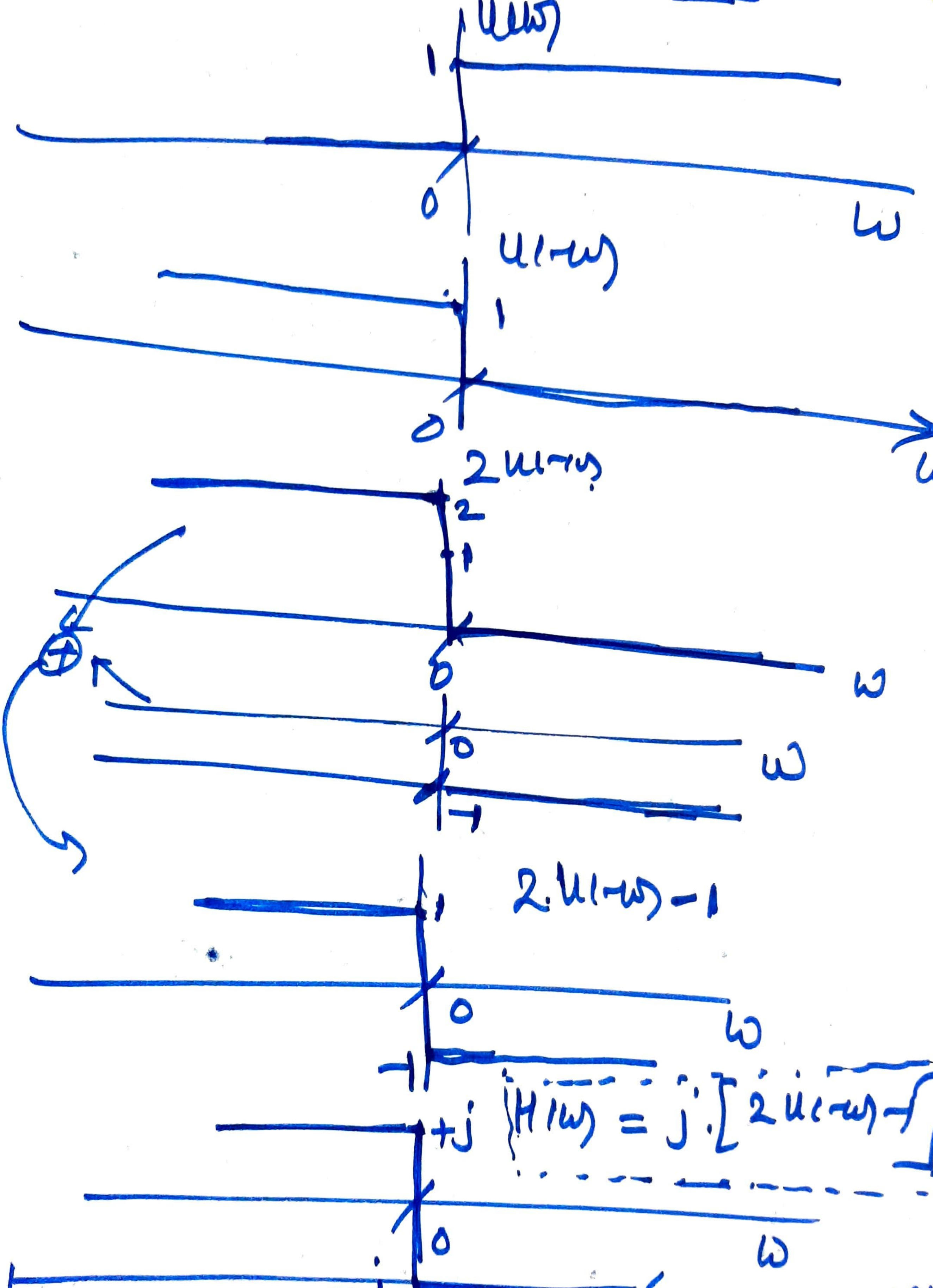
$\frac{2}{jt} \xleftrightarrow{F(\cdot)} 2\pi[2u(-\omega)-1]$

$F\left\{\frac{2}{jt}\right\} = 2\pi\left[2u(-\omega)-1\right]$

$\frac{1}{\pi t} \xleftrightarrow{F(\cdot)} \underline{j\left[2u(-\omega)-1\right]}$

3

$$\mathcal{F}\left\{h(t) = \frac{1}{\pi t}\right\} = j\,[\,2\,u(-\omega) - 1\,]$$



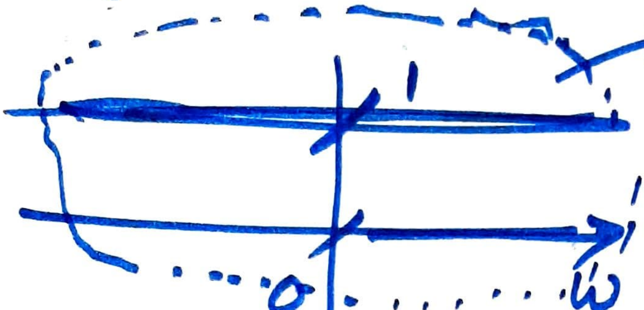
$$H(\omega) = j\cdot[\,2\,u(-\omega) - 1\,]$$

$$H(\omega) = \begin{cases} -j, & \omega > 0 \\ +j, & \omega < 0 \end{cases}$$ → frequency response of Hilbert transform

$$|H(\omega)| = \begin{cases} |-j| = 1 & \omega > 0 \\ |+j| = 1, & \omega < 0 \end{cases}$$

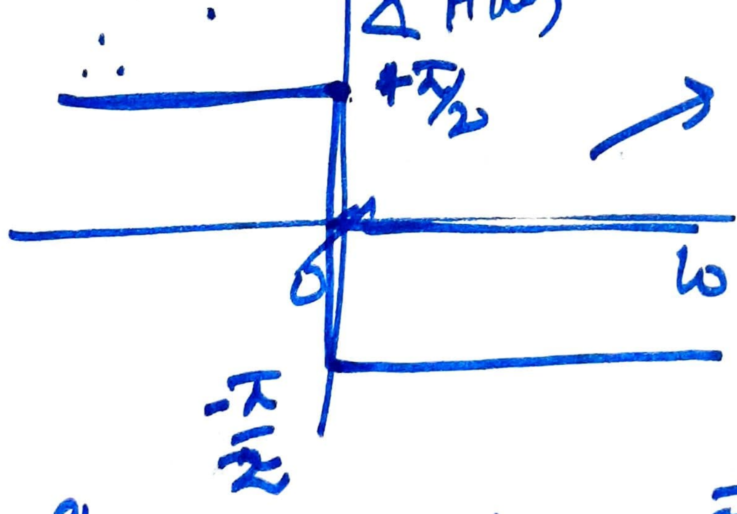
$$|H(\omega)| = 1 \quad \forall \omega$$

→ allpass "filter"



Magnitude response

$$\measuredangle H(\omega) = \begin{cases} \tan^{-1}\left(\frac{-1}{0}\right) = -\pi/2, & \omega > 0 \\ \tan^{-1}\left(1/0\right) = +\frac{\pi}{2}, & \omega < 0 \end{cases}$$



Phase response

→ Hilbert transform shift the phase of the input signal by $\pi/2$ in frequency domain.

⟹ Hence, Hilbert transform is called as $\pi/2$-phase shifter and filter.

Problems:

(1) Find the Hilbert transform of $f(t) = \cos(\omega_c t)$.

$$x(t) = \cos(\omega_c t) \longrightarrow \boxed{\begin{matrix}\text{Hilbert} \\ \text{Transform} \\ h(t)\end{matrix}} \longrightarrow y(t) = \text{Hilbert}\{x(t)\},$$

$$y(t) = ? = \sin(\omega_c t)$$

Solution:-

$$y(t) = x(t) * h(t)$$

$$\therefore \quad Y(\omega) = X(\omega) \cdot H(\omega)$$

$$x(t) = \cos(\omega_c t) \xleftrightarrow{\mathcal{F}(\cdot)} \pi[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)]$$

$$Y(\omega) = \pi[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)] \times H(\omega) = \begin{cases} -j, & \omega > 0 \\ +j, & \omega < 0 \end{cases}$$

$$Y(\omega) = \pi[-j\,\delta(\omega-\omega_c) + j\,\delta(\omega+\omega_c)]$$

$$\frac{j \times j}{j}[\pi\,\delta(\omega-\omega_c) + \pi\,\delta(\omega+\omega_c)]$$

$$Y(\omega) = \frac{1}{j}[2\pi\,\delta(\omega+\omega_c) - 2\pi\,\delta(\omega-\omega_c)]$$

$$\mathcal{F}^{-1}\{Y(\omega)\} = \frac{1}{j}\left\{\mathcal{F}^{-1}\{2\pi\delta(\omega+\omega_c)\} - \mathcal{F}^{-1}\{2\pi\delta(\omega-\omega_c)\}\right\}$$

$$= \frac{e^{+j\omega_c t} - e^{-j\omega_c t}}{2j}$$

$\therefore\ y(t) = \sin(\omega_c t) \rightarrow \frac{\pi}{2}$ - phase shifts

② Find Hilbert $\{x(t) = \sin(\omega_c t)\}$.

ANS : $\cos(\omega_c t)$.

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③ Application of Hilbert Transform

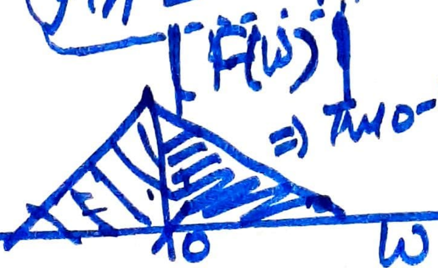
Generation of analytic signal

Let $f(t)$ = real signal and $f_a(t)$ be the corresponding analytic signal of $f(t)$.

$\because\ f(t)$ = real $\rightarrow$ using property of FT (conjugate symmetric)

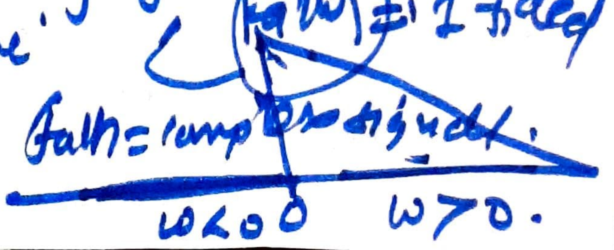
$F(\omega)$ $\Rightarrow$ two-sided amplitude

$F_a(\omega)$ = 1 sided



?

$f_a(t)$ = complex signal.



Analytic signal has '0' values for $\omega < 0$

$$\boxed{F_a(\omega) = 0 \quad \text{for} \quad \omega < 0}$$

causal signal in frequency domain

Q. How to generate $f_a(t)$ from $f(t)$?

$\because$ $f_a(t)$ is a complex signal, so let

$$f_a(t) = f(t) + j\hat{f}(t)$$

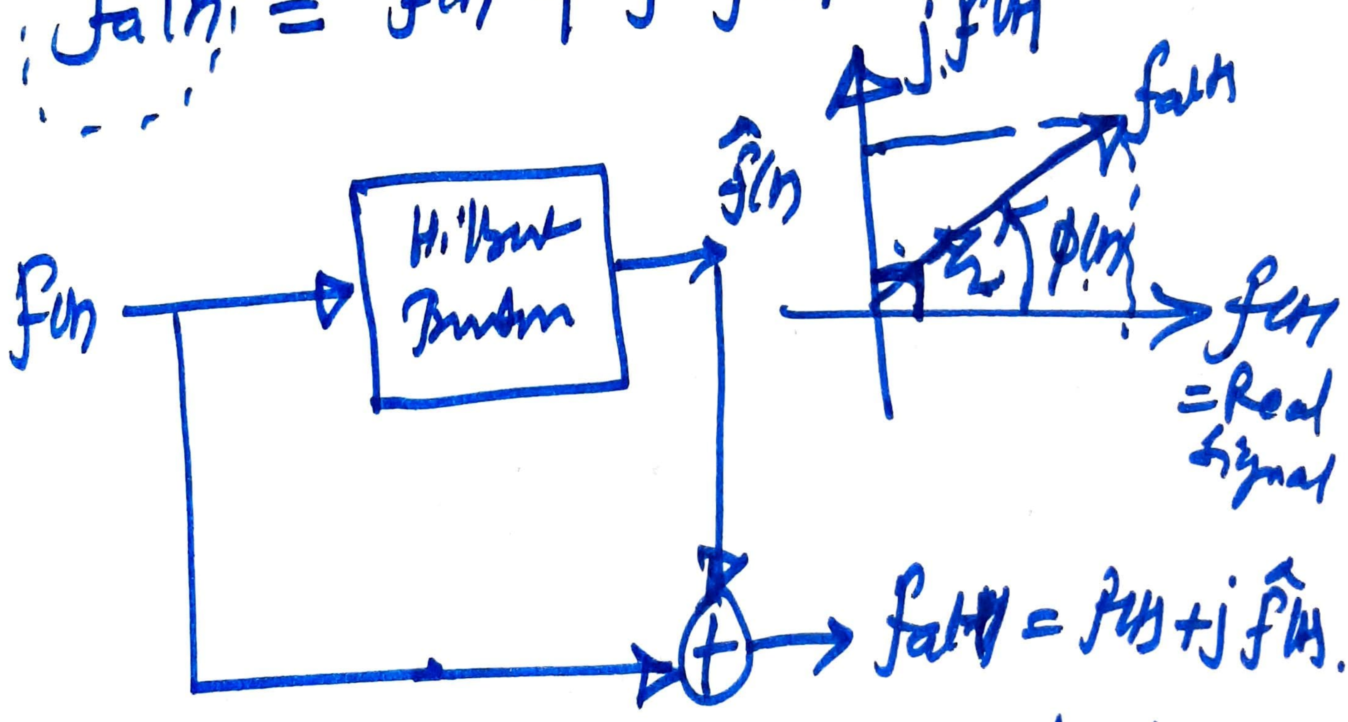


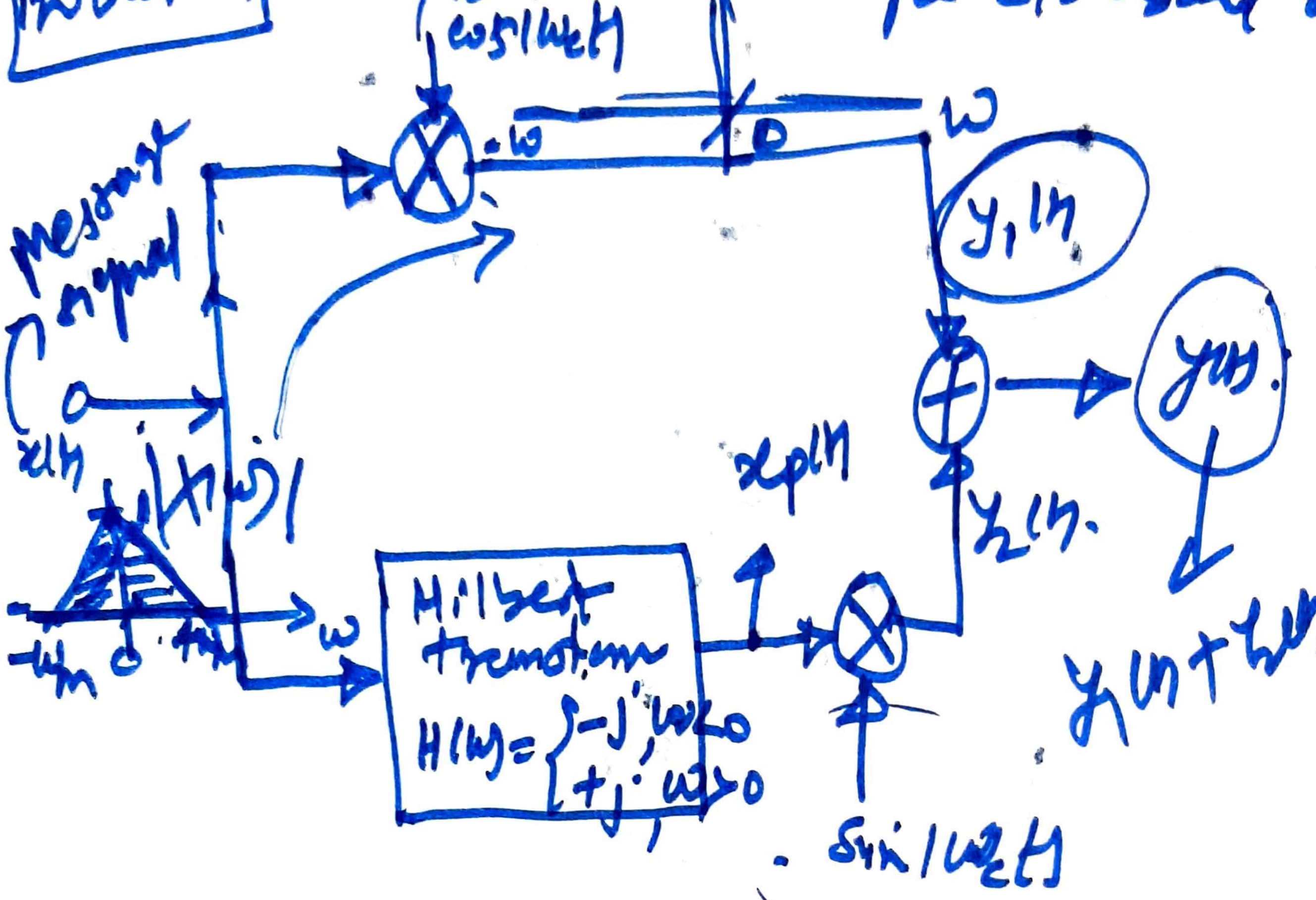
Fig.. Generation of analytic signal.

$$|f_a(t)| = \sqrt{f^2(t) + \hat{f}^2(t)} = \text{Hilbert envelope}$$

$$\angle f_a(t) = \tan^{-1}\left(\frac{\hat{f}(t)}{f(t)}\right) = \text{analytic phase, } \phi(t) \text{ or instantaneous phase}$$

:: Instantaneous frequency (IF) of $f(t) = \frac{d[\phi(t)]}{dt}$

Problem 4 Generation of Single Sideband Sign

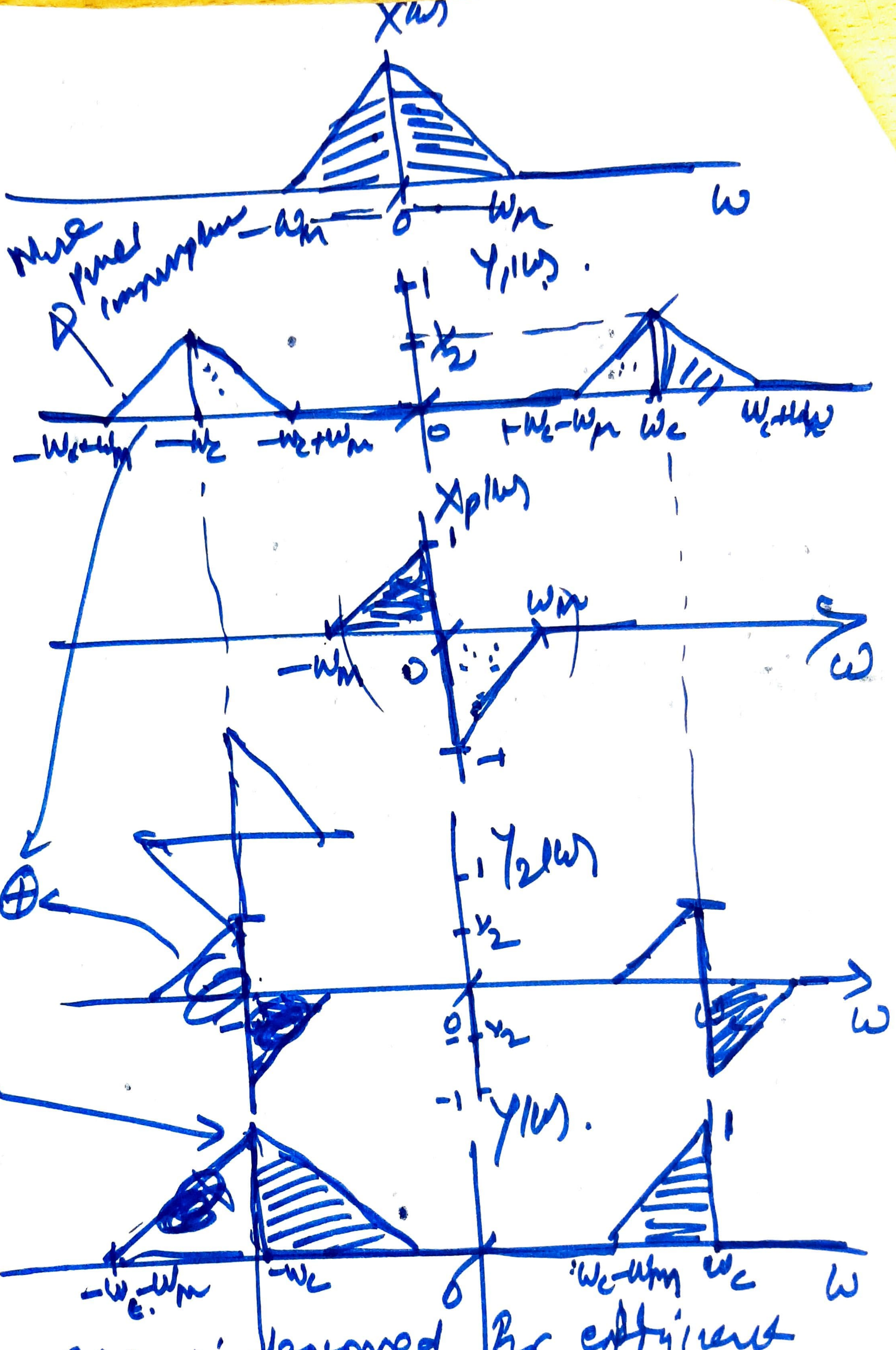


SSB Network.

AM →

$$F\{\cos(w_c t)\} = 2\pi\,[\delta(w-w_c) + \delta(w+w_c)]$$

$$F\{\sin(w_c t)\} = \left(\frac{2\pi}{j}\right)[\delta(w-w_c) - \delta(w+w_c)]$$



SSB is designed for efficient power consumption.

Time Scaling Property :→

$$f(t) \xleftrightarrow{F} F(\omega).$$

$$f(at) \xleftrightarrow{F(\cdot)} ?$$

Ans. $F\{f(at)\} = \int_{-\infty}^{+\infty} f(at) \cdot e^{-j\omega t} \, dt.$

Let $at = t_1$ [Method of substitution]

$a \cdot dt = dt_1$

$$= \int_{-\infty}^{+\infty} f(t_1) \cdot e^{-j\omega(t_1/a)} \cdot \left(\frac{dt_1}{a}\right)$$

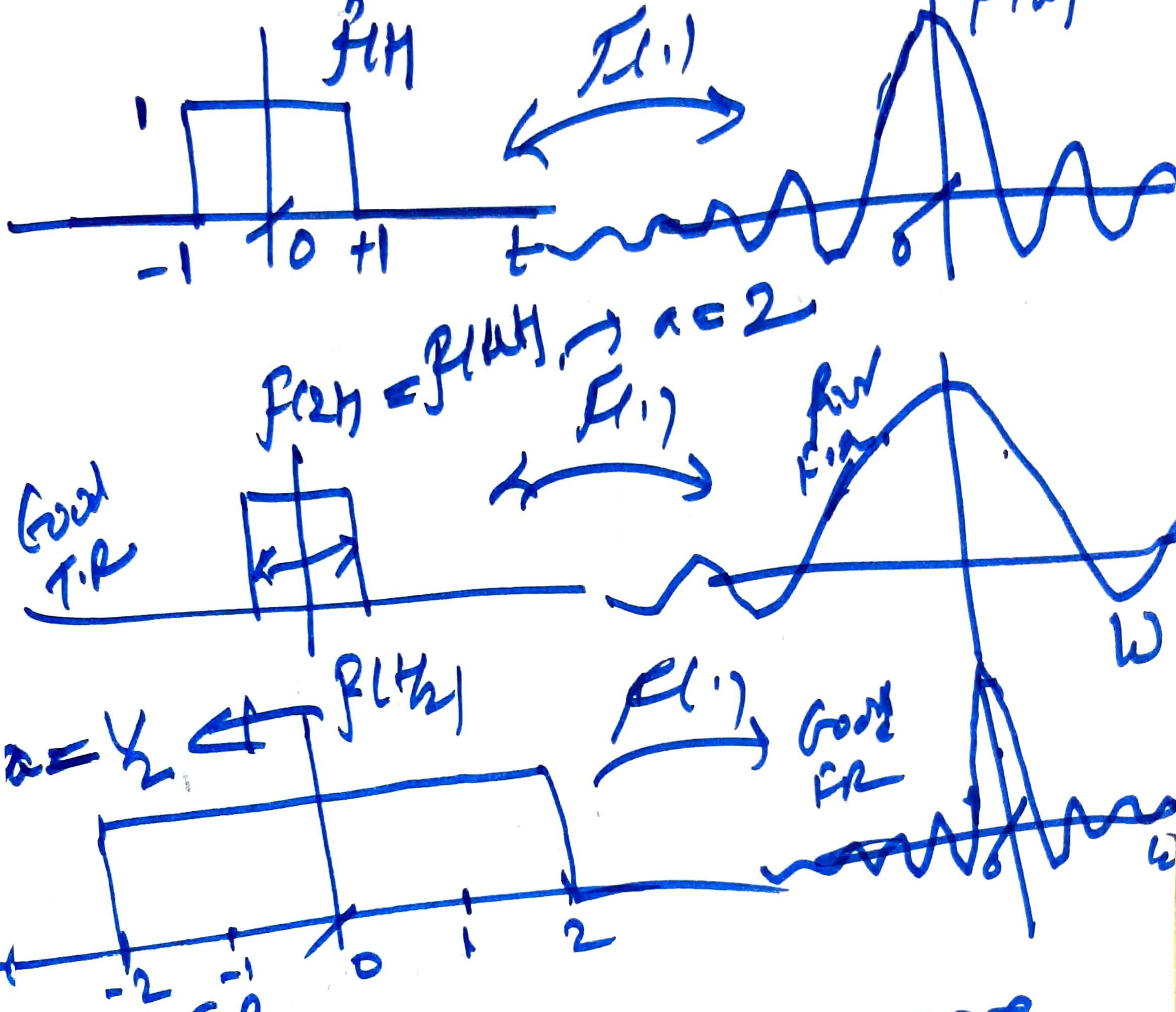
$$= \left(\frac{1}{a}\right) \left\{ \int_{-\infty}^{+\infty} f(t_1) \, e^{-j\omega(t_1/a)} \, dt_1 \right\}$$

$$F\{f(at)\} = \left(\frac{1}{a}\right) F(\omega).$$

$$F\{f(at)\} = \begin{cases} \frac{1}{a} F(\omega/a), & a > 0 \\ -\frac{1}{a} F(\omega/a), & a < 0 \end{cases}$$

11

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right); \forall a.$$



Time-scaling property of CTFT manifests the Heisenberg's uncertainty principle in signal processing framework

Property 10) Parseval's Theorem for CTFT

$$\boxed{\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega.}$$

Proof:→ L.H.S. $= \int_{-\infty}^{+\infty} |f(t)|^2 dt.$

$$= \int_{-\infty}^{+\infty} f(t) \cdot f^*(t)\, dt$$

$$= \int_{-\infty}^{+\infty} f(t) \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega) \cdot e^{j\omega t} d\omega\right]^* dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t) \left[\int_{-\infty}^{+\infty} F^*(\omega) \cdot e^{-j\omega t} d\omega\right] dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^*(\omega) \left(\int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt\right) d\omega$$

$\rightarrow F(\omega).$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^*(\omega) \cdot F(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega$$

= RHS

Inference: Energy of a signal, $f(t)$ in time-domain, (i.e. $\int |f(t)|^2 dt$) is conserved in frequency domain (i.e. $\frac{1}{2\pi}\int |F(\omega)|^2 d\omega$), $t \in \mathbb{R}$, $\omega \in \mathbb{R}$

$\Rightarrow$ Parseval's theorem for CTFT is manifestation of energy conservation principle in physics.

Application of Parseval's theorem -

Signal, $f(t)$

$\downarrow$ $f(t)$ = aperiodic: Finite energy

$\int |f(t)|^2 dt < +\infty$, $t \in \mathbb{R}$

$\downarrow$

Energy spectral Density (ESD)

$\downarrow$ $f(t)$ = Periodic

$\downarrow$

energy = infinite

$\therefore$ Finite average power

$$P_{av} = \lim_{T \to +\infty} \left(\frac{1}{2T}\right) \cdot \int_{-T}^{+T} |f(t)|^2 dt$$

$\downarrow$

Power spectral Density (PSD).