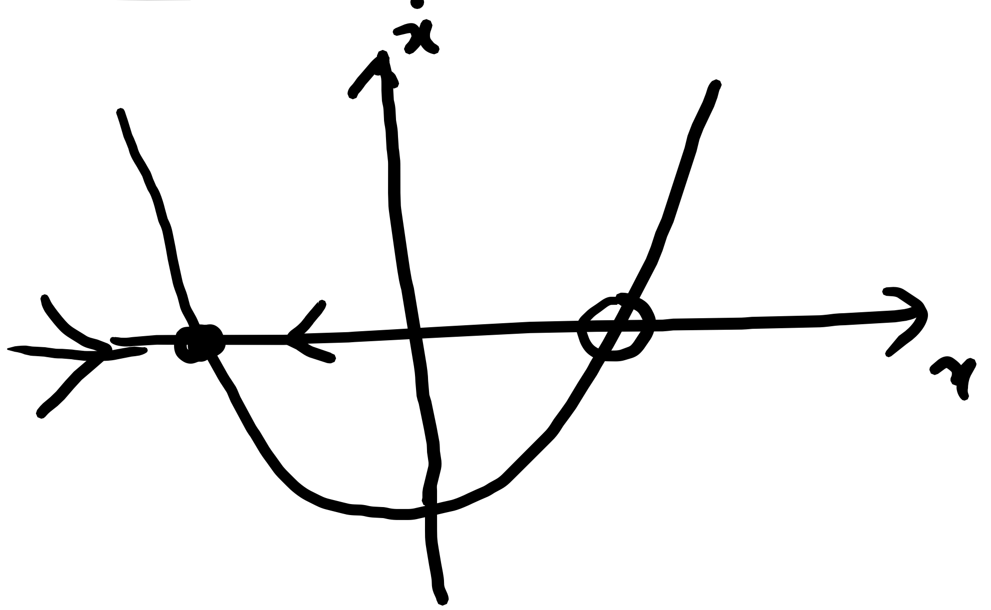
LECTURE 30

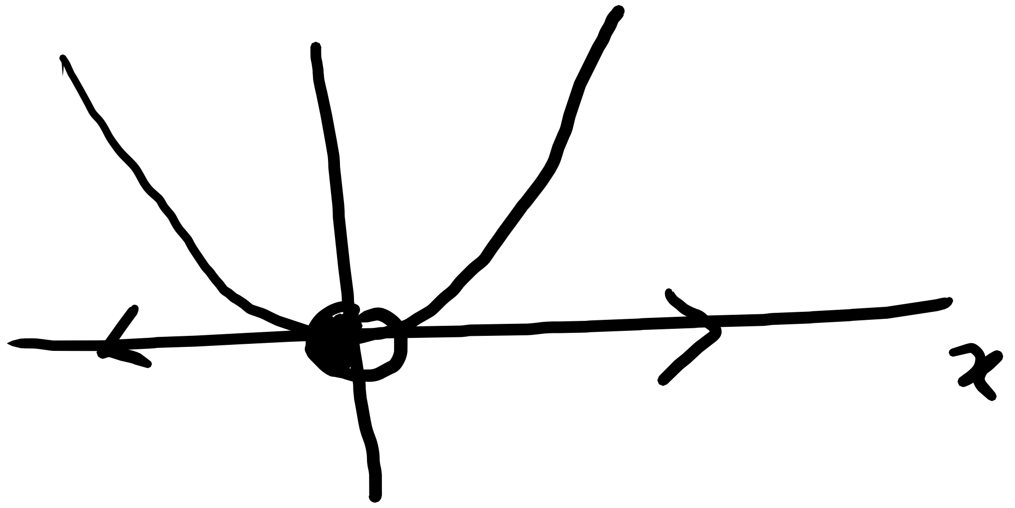
RECAP:- Bifurcations:- points where behaviour of a system undergoes fundamental qualitative change.

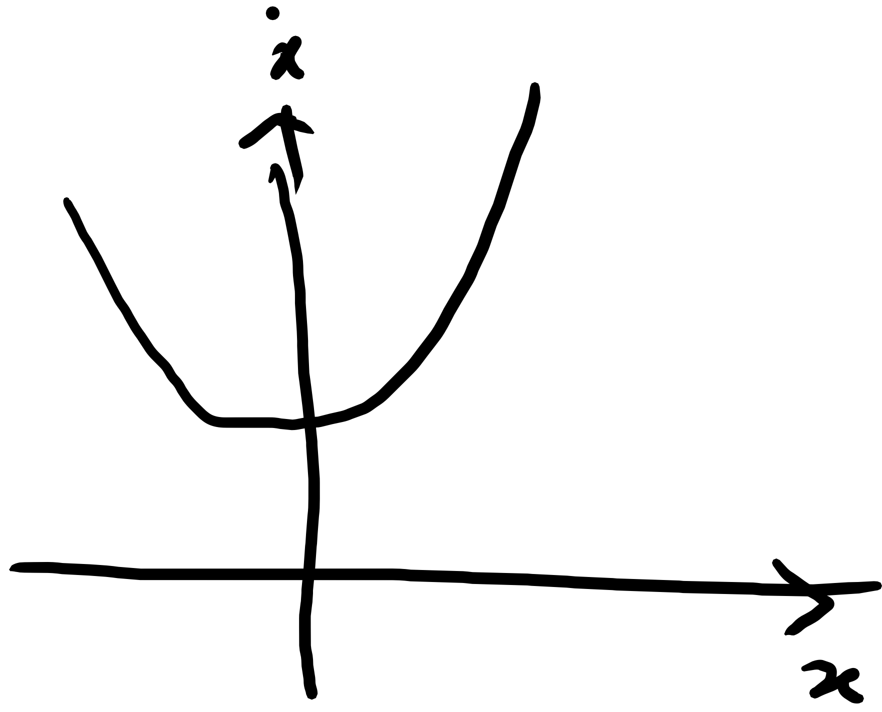
- Saddle node Bifurcation:

$$\boxed{\dot{x} = r + x^2,}$$

r, $\pm$ve or zero.







– $\dot{x} = r - x - e^{-x}$. $\longrightarrow$ exhibits a saddle node bifurcation at $x = 0$, $r = r_c = 1$.

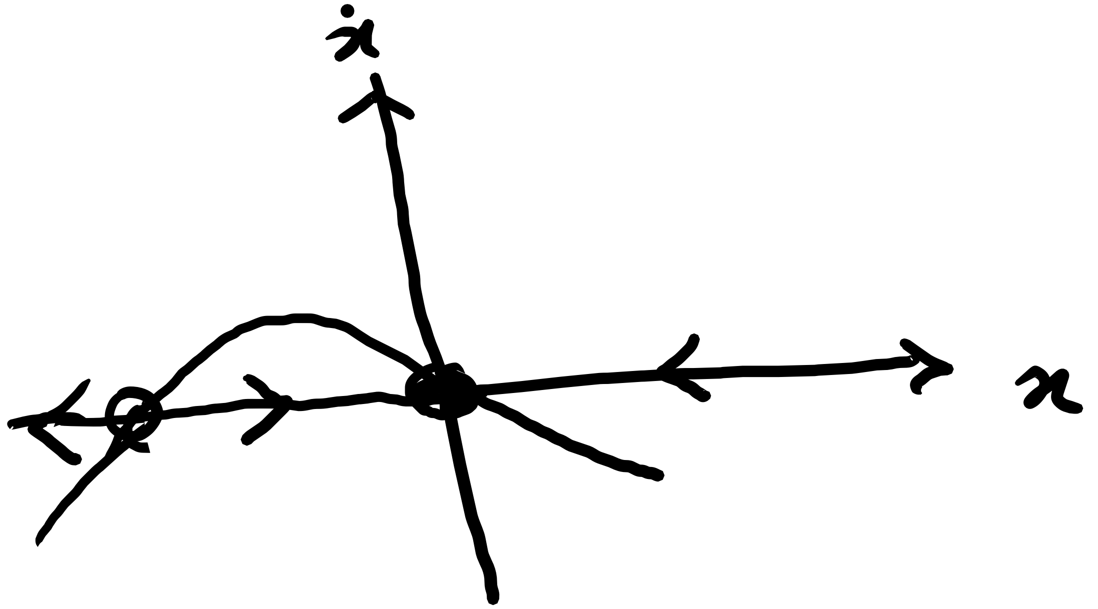
Expand around $x = 0$, $r = r_c = 1$.

$$\dot{x} \simeq (r - x) - \left[1 - x + \frac{x^2}{2!} + \cdots\right]$$

$$\simeq (r - 1) - \frac{x^2}{2!}$$

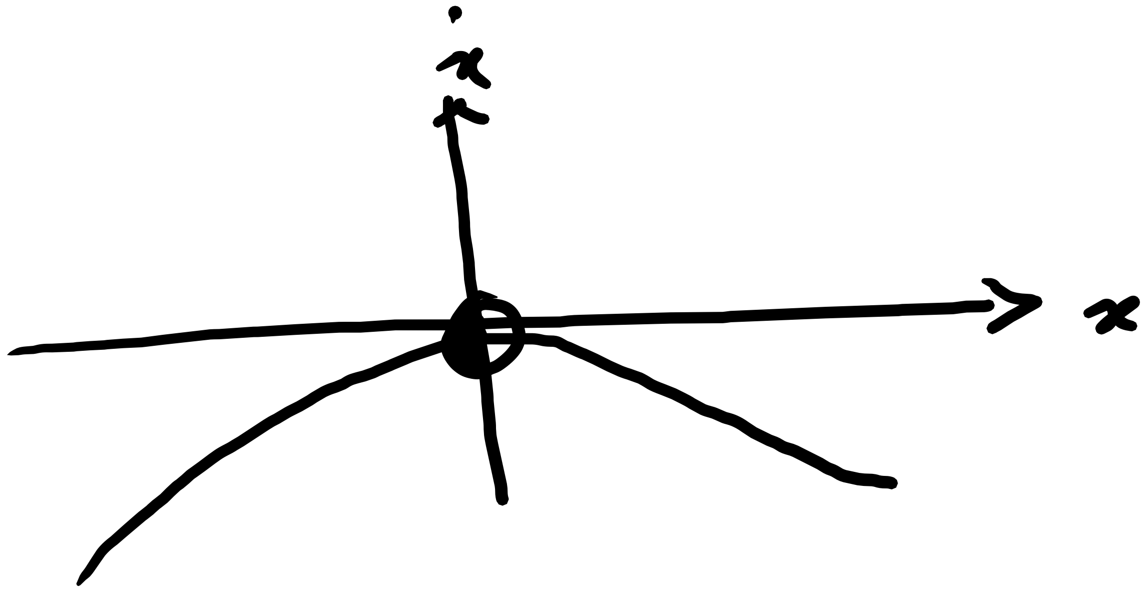
TRANSCRITICAL BIFURCATION

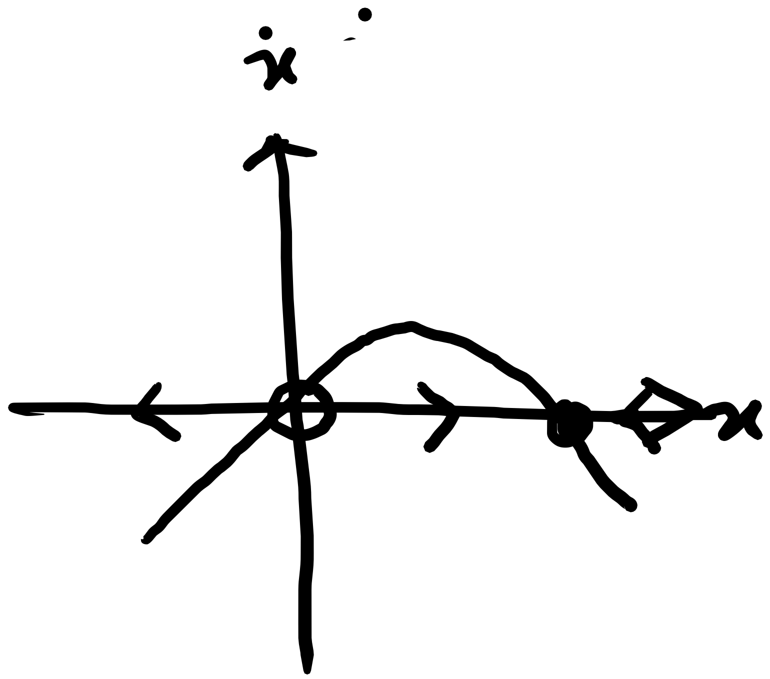
$\dot{x} = rx - x^2$. , both x and r allowed ±ve / zero.



$r < 0$

$r = 0$





$r > 0$

– Change of stability as r is tuned, but fixed points are **NOT** created or destroyed.

Ex:- $\dot{x} = x(1-x^2) - a(1-e^{-bx})$.

Fixed points $\Rightarrow$ $x^*(1-x^{*2}) - a(1-e^{-bx^*}) = 0$.

$x^* = 0$. $\forall\,(a,b)$.

$$1-e^{-bx} = 1 - \left[1 - bx + \frac{1}{2}b^2x^2 + \dots\right]$$

$$= bx - \frac{1}{2}b^2x^2 + O(x^3).$$

$$\dot{x} \simeq x - a\left(bx - \frac{1}{2}bx^2\right) + \dots = (1-ab)x + \frac{1}{2}ab^2x^2$$

$\dot{x} \simeq (1-ab)x + \frac{1}{2}ab^2x^2 \longrightarrow$ resembles normal form for transcritical bifurcation near $x^* = 0$.

Example:- $\dot{x} = \left(r\ln x + x - 1\right)$.

Schrödinger equ — Partial Differential Eqn.

PARTIAL DIFFERENTIAL EQNS

$\frac{dy}{dx} = f(x) \rightarrow$ ODE, only <u>ONE</u> independent variable.

$y \equiv y(x)$.

<u>PDE</u> :- More than ONE independent variable.

$u \equiv u(x, y)$

$u(x, t)$

- Upto linear 2nd order, the canonical form for PDE in.

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = 0 .$$

(i) $b^2 - 4ac > 0 \Rightarrow$ HYPERBOLIC

(ii) $b^2 - 4ac = 0 \Rightarrow$ PARABOLIC

(iii) $b^2 - 4ac < 0 \Rightarrow$ ELLIPTIC .

Examples:- $a u_{xx} + b u_{xy} + c u_{yy} + \underline{d u_x} + e u_y + f u = 0$

(i) WAVE EQN:- $u_{tt} - u_{xx} = 0$.

$a = 1, \quad b = 0, \quad c = -1$

$b^2 - 4ac = 0 - 4(1)(-1) = 4 > 0$. Hyperbolic.

(ii) LAPLACE EQN:- $u_{xx} + u_{yy} = 0$.

$b = 0, \quad a = 1, \quad c = 1$

$b^2 - 4ac = 0 - 4 < 0 \Rightarrow$ Elliptic.

(iii) HEAT EQN/DIFFUSION EQN.

$$u_t - u_{xx} = 0 \implies \text{PARABOLIC}.$$

$$b = c = 0, \quad a = -1$$

$$b^2 - 4ac = 0.$$

$$u_t - u_x = 0. \longrightarrow \text{ADVECTION EQN}$$

– Outside scope.

WAVE EQN.

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi . \qquad \frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi .$$

1d :- $$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} .$$

Claim :- $\boxed{\psi = f(x \pm ct)}$ is a class of solutions.

Let $z = x \pm ct$.

$$\frac{\partial z}{\partial x} = 1 \qquad \frac{\partial z}{\partial t} = \pm c .$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial f}{\partial z} .$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial z^2}$$

Also possible to write solns of $\psi = f\left(t \pm \frac{x}{c}\right)$.

$$\psi = A e^{i\omega(t - x/c)}.$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \pm c \frac{\partial f}{\partial z}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(\pm c \frac{\partial f}{\partial z}\right).$$

$$= \frac{\partial}{\partial z}\left(\pm c \frac{\partial f}{\partial z}\right)\frac{\partial z}{\partial t}$$

$$= \frac{\partial}{\partial z}\left(\pm c \frac{\partial f}{\partial z}\right)(\pm c).$$

$$= c^2 \frac{\partial^2 f}{\partial z^2}.$$

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2}$$

$$= c^2 \frac{\partial^2 f}{\partial z^2} - c^2 \frac{\partial^2 f}{\partial z^2}$$

$$= 0$$