SC223 Linear Algebra
Autumn 2023-24
Practice Questions

Note: The following questions have been taken from various online sources, and are meant for your practice. Based on these questions, do not make any assumptions about the kind of questions that you will face in the examination. Do not expect me to share solutions of these questions.

1. Show that similar matrices have the same rank and trace.

2. Let $A \in \mathbb{C}^{n\times n}$. Show that if n is odd, and $SAS^{-1} = -A$ for an invertible matrix $S \in \mathbb{C}^{n\times n}$, then 0 is an eigenvalue of A. [Hint: $det(AB) = det(A)\,det(B)$].

3. Are the matrices $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ similar?

4. For a non-zero vector $v \in \mathbb{R}^2$, derive/find the linear operator on $\mathbb{R}^2$ that reflects any vector of $\mathbb{R}^2$ about the line passing through the origin and v. Assume the usual inner product on $\mathbb{R}^2$.

5. Show that any matrix $A \in \mathbb{R}^{2\times 2}$ such that $A = A^T$ is diagonalizable.

6. Let P be an invertible $n \times n$ matrix. For any $n \times n$ matrix A, let $T_P(A) = P^{-1}AP$. Verify the following:

   (a) $T_P(I) = I$
   (b) $T_P(AB) = T_p(A)T_P(B)$
   (c) $T_P(A + B) = T_p(A) + T_P(B)$
   (d) $T_P(r \cdot A) = rT_p(A), \forall r \in \mathbb{F}$
   (e) $T_P(A^k) = (T_P(A))^k, k \geq 1$
   (f) If A is invertible, $T_P(A^{-1}) = (T_P(A))^{-1}$
   (g) If Q is invertible, $T_Q(T_P(A)) = T_{PQ}(A)$.

7. Let a sequence be defined by the following linear recurrence relation: $x_{n+m} = a_0x_n + a_1x_{n+1} + \ldots + a_{n+m-1}x_{n+m-1}, \forall n \geq 0$ with given initial values $x_0, \ldots, x_{m-1}$. Re-write this as a matrix-vector relation on $\mathbb{R}^m$.

8. Let $T \in \mathcal{L}(V)$ be a projection operator on subspace U along subspace W. Show that $I - T$ is also a projection operator. Find the range and nullspace of $I = T$.

9. Derive the matrix representation of the projection operator $T \in \mathcal{L}(\mathbb{R}^2)$ on subspace $U = \{k \cdot (1, 0) \mid \forall k \in \mathbb{R}\}$ along subspace $W = \{k \cdot (1, 1) \mid \forall k \in \mathbb{R}\}$ in the standard basis.

10. Derive the matrix representation of the projection operator $T \in \mathcal{L}(\mathbb{R}^2)$ on subspace $U = \{k \cdot (\cos\theta, \sin\theta) \mid \forall k \in \mathbb{R}\}$ along subspace $W = \{k \cdot (-\sin\theta, \cos\theta) \mid \forall k \in \mathbb{R}\}$ in the standard basis.

11. Let $T \in \mathcal{L}(U, V)$ where $(U, ||\cdot||_U)$ and $(V, ||\cdot||_V)$ are NVS. Show that

$$\forall T \in \mathcal{L}(U, V), ||T|| = \sup_{x \in U, x \neq \theta_U} \frac{||Tx||_V}{||x||_U}$$

defines a norm on $\mathcal{L}(U, V)$.

12. For any vector space $(U, ||\cdot||_U)$, the set $S = \{x \in U \mid ||x||_U \leq 1\}$ is called the closed unit ball in U. For the vector space $\mathbb{R}^2$, draw the closed unit ball with respect to the following norms:

    (a) $||(x, y)|| = \sqrt{x^2 + y^2}$
    (b) $||(x, y)|| = \max\{|x|, |y|\}$
    (c) $||(x, y)|| = |x| + |y|$

13. Let $(U, ||\cdot||_U)$ be a NVS over $\mathbb{R}$. A subset $S \subset U$ is said to be convex if $\forall x, y \in S, t \cdot x + (1-t) \cdot y \in S, \forall t \in [0, 1] \in \mathbb{R}$. Show that the closed unit ball in any finite dimensional NVS $(U, ||\cdot||_U)$ is convex.