Midterm Exam 1

Date	:	Wednesday 15^{th} February, 2023, from 4:30 pm to 6:30 pm
Total Marks	:	50
Notes:	:	All questions are scored (no optionals).

Name: ________________________ **Student ID**: ________________________

Reference:

- The binary Entropy function $H_b(p)$ is plotted in Fig. 1 as a function of p.
- The model of the binary assymetric channel BASC(p_0, p_1) is provided in Fig. 2.
- The notation λ_n stands for the posterior likelihood ratio in favor of $X = 1$, and it is the ratio of the conditional probability that $X = 1$ given the observation n to the conditional probability that $X = 0$ given this observation. The prior likelihood ratio is the probability that $X = 1$ to the probability that $X = 0$.
- $\log_{10}(2) \approx 0.3$

1 Fill In the Blanks:

Instructions

▷ There are total 12 questions in this section that carry total of 18 marks.

▷ There is no negative marking in this section.

Questions

1. (1 point) Write an expression of E_b/N_0 as a function of the bandwidth efficiency η_B:

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   ....

2. (2 points) When the interest rate is $10 \times \log_{10}(p)$ (dB) and the compounding of interest is continuous and you invest Rs. 1 today, what is the amount you will have in your account one year from today.[1]?

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   ....

[1]Hint: the correct answer is some function of p and e_d, where e_d is defined as exp(1) in dB, i.e., $e_d = 10 \times \log_{10}(\exp(1))$.

3. (2 points) Write the full forms of (1) LRT, (2) MAP, (3) ML, and (4) AWGN.

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4. (1 point) The MAP and the ML are two forms of

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5. (1 point) The MAP and the ML differ in the following way:

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6. (1 point) Suppose the decoder of a channel code of length N bits can correct up to t_c bits in error. This code is transmitted over BSC(p). Write an expression for the probability of correct detection p_c and probability of detection error p_e at the output of the decoder as a function of p, N and t_c?

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7. (2 points) Suppose a binary word of length N bits is transmitted without any channel coding over BSC(p). What is the probability of word detection error at the receiver as a function of p and N?[2]

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8. (2 points) Determine $H(\tilde{X}_1)$ (i.e., $H(X \mid Y = 1)$), and $I(X; Y = 1) = H(X) - H(\tilde{X}_1)$ for the joint probability distribution $P(X, Y)$ in Table 1?

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Table 1: Joint probability $P(X, Y)$ for Problem 8

$P(X,Y)$	$X = 0$	$X = 1$
$Y = 0$	2/5	1/2
$Y = 1$	1/10	0

[2]Note: since there is no channel coding, the binary word will be detected wrong if any one or more number of N bits is received in error over the BSC.

9. (2 points) The input bits to a BSC(p) follow the Bernoulli($q = 0.5$) distribution. Suppose the output of the BSC is $Y = 1$. In this case, (a) what is the posterior likelihood ratio in favor of $X = 1$? (b) what is the probability of bit detection error at the receiver when the receiver uses the ML detection?

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10. (2 points) Define a new logarithmic function (not the standard logarithmic function which is in decible scale) of SNR λ such that, when $\lambda \gg 1$, the SNR at the output of your logarithmic function becomes *equal* to the spectral efficiency η_B in bps/Hz.

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11. (1 points) The SNR per symbol and the SNR per bit are related as $E_S/N_0 = \alpha \times E_b/N_0$. Here $\alpha =$:

....

....

12. (1 points) A power value in the decibel scale of 30 dBm equals how many watts?

....

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2 Multiple Choice Questions:

Instructions

- ▷ This is MCQ Version I of IV.
- ▷ There are total 20 questions in this section that carry a maximum of 32 marks.
- ▷ Tick mark the correct option.
  - → Total marks for a correctly answered question are indicated for each question. If the question is answered incorrectly, negative marking will apply, with negative one-fourth of the total marks awarded to a wrong answer.
  - → If more than one option is tick-marked, zero marks will be given for the question even if one of the tick-marked options is correct.

Questions

1. (1 points) The RV X at the input to a BSC(p) is the Bernoulli($q = 0.5$) RV. Each bit X is repeated three times. Suppose the output of the BSC is $\{0, 1, 0\}$. In this case, what is the a-posterior likelihood ratio in favor of $X = 1$?

   (a) $\left(\dfrac{p}{1-p}\right)^3$

   (b) $\dfrac{1-p}{p}$

(c) $\left(\dfrac{1-p}{p}\right)^3$
(d) $\dfrac{p}{1-p}$

2. (3 points) The RV X at the input to a BSC(p) is the Bernoulli($q = 0.5$) RV. Each bit X is repeated three times. Suppose the output of the BSC is $\{0, 1, 0\}$. In this case, what is the probability of bit detection error at the receiver when the receiver uses the ML (i.e., the majority voting) detection?

   (a) p
   (b) $3p^2(1-p) + p^3$
   (c) $3(1-p)^2 p + p^3$
   (d) p^3

3. (2 points) When the input X to the BSC(p) channel is the Bernoulli(q) RV, the notation that we have used in the class for the conditional probability $P(X = 0 \,|\, Y = 1)$ is

   (a) $\tilde{q}_0$
   (b) $\tilde{q}_1$
   (c) $1 - \tilde{q}_1$
   (d) $1 - \tilde{q}_0$

4. (2 points) When the input X to the BSC(p) channel is the Bernoulli($q = 0.5$) RV, the conditional probability $P(X = i \,|\, Y = j)$ (i and j take values from the binary set $\{0, 1\}$) equals

   (a) p if $i = j$
   (b) $1 - p$ if $i \neq j$
   (c) $1 - p$ if $i = j$
   (d) $\tilde{q}_i$ regardless of j

5. (2 points) [Maximum Likelihood Detection] When the input X to the BSC(p) channel is the Bernoulli($q = 0.5$) RV, when does the Maximum Likelihood Bayesian Detector of $X = i$ *flip* the bit $Y = j$ (i and j take values from the binary set $\{0, 1\}$) received at the output of the BSC to make the decision regarding X?

   (a) if $(1-p)/p < 1$
   (b) if $i = j$
   (c) if $i \neq j$
   (d) if $(1-p)/p > 1$

6. (2 points) When the input to the BSC($p = 0.5$) is the Bernoulli(q) RV, the posterior likelihood ratio λ_0 equals

   (a) 1
   (b) $\dfrac{1-p}{p}$

(c) $\dfrac{1-q}{q}$
(d) $\dfrac{p}{1-p}$
(e) $\dfrac{q}{1-q}$

7. (1 points) Suppose the information is sent at a data rate of 1000 bits per second over a BSC(p). If $p = 0.1$, the data rate at which the information can be successfully received at the receiver equals:

   (a) ≈ 100 bps
   (b) ≈ 900 bps
   (c) ≈ 500 bps
   (d) Will depend on the exact pattern of the errors introduced by the BSC

8. (1 points) Let the rate of a channel code that repeats each bit n times be denoted as r_n. $\lim_{n\to\infty} r_n =$

   (a) 0
   (b) 0.5
   (c) 1
   (d) k/n

9. (2 points) Suppose the SNR $\lambda \gg 1$. In this case, $\lambda_d = 10 \times \log_{10}(\lambda)$ dB is proportional to the spectral efficiency η_B in bits per second per Hertz. The constant of proportionality equals

   (a) $\log_e(2)$
   (b) 1/3
   (c) 1/2
   (d) $\log_2(e)$

10. (2 points) Suppose the spectral efficiency η_B obtained is x bps/Hz when the SNR λ equals y dB. Suppose $y \gg 1$. In this case, if the SNR increases to $y+6$ dB, the spectral efficiency η_B becomes

    (a) $4x$ bps
    (b) $x + 6$ bps
    (c) $x + 2$ bps
    (d) $2x$ bps

11. (1 points) Suppose the information is sent at a data rate of 1000 bits per second over a BSC(p). For some value of p, the data rate at which the information can be successfully (i.e., reliably) received at the receiver becomes zero. This p equals

    (a) 1
    (b) 0
    (c) 0.5
    (d) Depends on the conditional probability $\tilde{q}_n$.

12. (1 points) When the input to the BSC(p) is the Bernoulli($q = 0.5$) RV, the posterior likelihood ratio λ_0 equals

   (a) $\dfrac{p}{1-p}$
   (b) $\dfrac{1-p}{p}$
   (c) 0
   (d) 1

13. (1 points) Suppose the information source is the Bernoulli RV(q) whose entropy is shown in Fig. 1 for different q. What is the maximum rate at which the information can be transmitted reliably over the BSC($p = 0.3$).

   (a) ≈ 0.9
   (b) ≈ 0.1
   (c) ≈ 0.3
   (d) ≈ 0.7

14. (1 points) Suppose an informative source generates one of five symbols whose probabilities are $\{1 - q, \alpha \times q, \alpha \times q, \alpha \times q, \alpha \times q\}$. The value of α equals:

   (a) 0.25
   (b) 0.2
   (c) 0.4
   (d) 0.1

15. (1 points) Suppose an informative source generates one of five symbols whose probabilities are $\{1 - q, \alpha \times q, \alpha \times q, \alpha \times q, \alpha \times q\}$. The value of q at which the source becomes maximally informative equals:

   (a) 0.75
   (b) 0.2
   (c) 0.25
   (d) 0.8

16. (2 points) A binary assymetric channel BASC(p_0, p_1) is shown in Fig. 2. When the input X to this channel is the Bernoulli(q) RV, the conditional probability $P(X = 1 \,|\, Y = 1)$ equals

   (a) $\dfrac{(1-p_0)\,(1-q)}{(1-p_0)\,(1-q) + p_1\,q}$
   (b) $\dfrac{p_0\,q}{(1-p_1)\,(1-q) + p_0\,q}$
   (c) $\dfrac{p_0\,(1-q)}{(1-p_0)\,(1-q) + (1-p_1)\,q}$
   (d) $\dfrac{(1-p_1)\,q}{(1-p_1)\,q + p_0\,(1-q)}$

17. (1 points) When the input X to the binary assymetric channel BASC(p_0, p_1) in Fig. 2 is the Bernoulli$(q = 0.5)$ RV, what is the posterior likelihood ratio λ_0?

   (a) $\dfrac{p_1}{1-p_0}$
   (b) $\dfrac{(1-p_0)}{p_0}$
   (c) $\dfrac{1-p_1}{p_1}$
   (d) $\dfrac{p_0}{1-p_1}$

18. (2 points) Suppose[8] an observation of $Y = j$ at the output of the BSC$(p < 0.5)$ results in the conditional entropy $H(\tilde{X}_j) < 1$ bit. When the input to this BSC is the Bernoulli$(q = 0.5)$ RV X, the posterior (after the observation) λ_j as compared to the prior (before the observation) likelihood ratio λ

   (a) can either increase or decrease
   (b) cannot increase
   (c) cannot decrease
   (d) becomes 1 regardless of λ

19. (2 points) Suppose an observation of $Y = 1$ at the output of the BSC$(p < 0.5)$ results in the conditional entropy $H(\tilde{X}_{j=1}) < 1$ bit. When the input to this BSC is the Bernoulli$(q = 0.5)$ RV X, the posterior λ_j as compared to the prior λ

   (a) moves closer to 1
   (b) cannot decrease
   (c) moves toward ∞
   (d) becomes 1 regardless of λ

20. (2 points) The input to the BSC$(p < 0.5)$ is the Bernoulli$(q > 0.5)$ RV X, and an observation of $Y = 1$ at the output of the BSC results in the conditional entropy $H(\tilde{X}_j) > H(X)$ bit. The posterior λ_j as compared to the prior λ

   (a) moves away from 1
   (b) moves closer to 1
   (c) moves closer to 0
   (d) moves closer to ∞
   (e) becomes 1 regardless of λ

[8] This question is about the similarity between the posterior likelihood ratio, conditional entropy and mutual information. On observing an output from a communication channel (evidence), what happens to the likelihood if the conditional entropy reduces. Do you become more certain about the state of the transmitter or less?

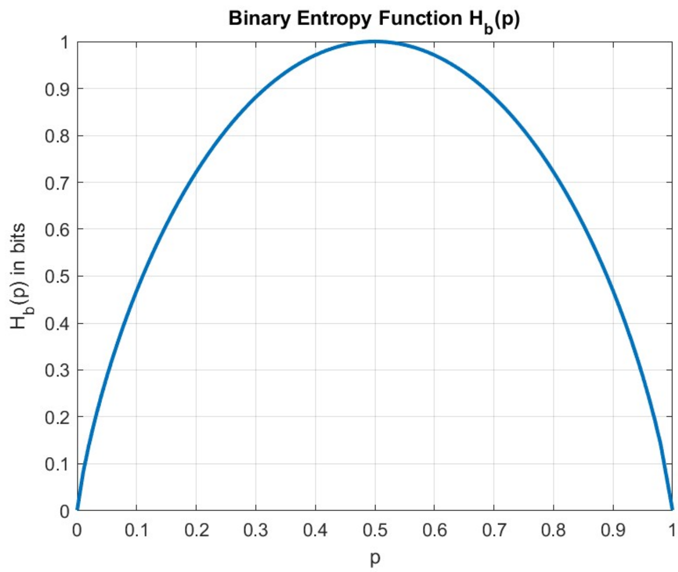

Figure 1: Binary entropy function.

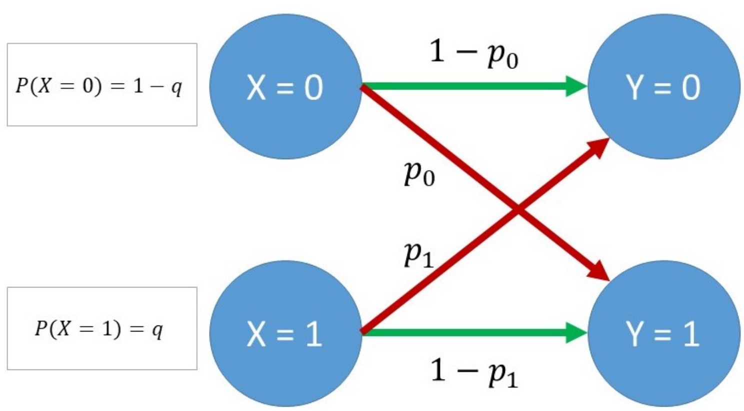

Figure 2: Binary Assymetric Channel BASC(p_0, p_1).