SC223 - Linear Algebra

Aditya Tatu

Lecture 18

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Subspace

- **Definition:** (Subspace) Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$. A subset $W \subseteq V$ is said to be a **subspace** of V if $(W, +, \cdot)$ is a Vector space over $\mathbb{F}$.
  - For any vector space V, V and $\{\theta\}$ are always subspaces. These are called **trivial subspaces**.
- **Proposition 6:** A non-empty subset W of a vector space V is a subspace if and only if
  - W is closed with respect to vector addition, and
  - W is closed with respect to scalar multiplication.

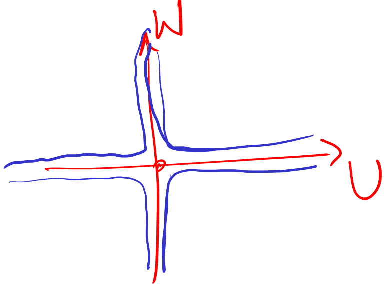
Generating New subspaces from Old

- Let U, W be subspaces of V.
- Is $U \cup W$ a subspace of V?

$V = \mathbb{R}^2$.

$U = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix}, \forall x \in \mathbb{R} \right\}$.

$W = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix}, \forall y \in \mathbb{R} \right\}$.



Generating New subspaces from Old

- Let U, W be subspaces of V.
- Is $U \cup W$ a subspace of V? No.
- Is $U \cap W$ a subspace of V?

Let $x, y \in U \cap W$.

$\Rightarrow x \in U$ and $x \in W$

$\Rightarrow y \in U$ and $y \in W$

$x + y \in U$ (because U is a subspace)

$x + y \in W$ (" W ").

$\Rightarrow x + y \in U \cap W$.

Let U be a subspace of V.

$a \cdot U = \{a \cdot u \mid \forall u \in U\} = U$

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- **Definition:** (Sum of subspaces): Let $U_1, \ldots, U_n$ be subspaces of V.

The **sum of subspaces** $U_1, \ldots, U_n$ is defined as:

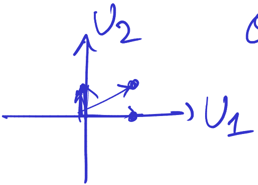
$$U_1 + \ldots + U_n =: \{u_1 + u_2 + \ldots + u_n \mid u_i \in U_i, i = 1, \ldots, n\}$$

$\underline{n=2}$:

$$U_1 + U_2 := \{ u_1 + u_2 \mid \forall u_1 \in U_1, \forall u_2 \in U_2 \}$$

① Since $0 \in U_i$, $i = 1, \ldots, n$,

$$0 + 0 + \cdots + 0 = 0 \in U_1 + \cdots + U_n.$$



② Let $p, q \in U_1 + \cdots + U_n$.

$\Rightarrow p = p_1 + p_2 + \cdots + p_n$, $p_i \in U_i$.

$q = q_1 + q_2 + \cdots + q_n$, $q_i \in U_i$

Generating New subspaces from Old

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- Is $U \cup W$ a subspace of V? No.
- Is $U \cap W$ a subspace of V? Yes.
- **Definition:** (Sum of subspaces): Let $U_1, \ldots, U_n$ be subspaces of V.

The **sum of subspaces** $U_1, \ldots, U_n$ is defined as:

$$U_1 + \ldots + U_n =: \{u_1 + u_2 + \ldots + u_n \mid u_i \in U_i, i = 1, \ldots, n\}$$

- **Proposition 7:** The sum of subspaces $U_1, \ldots, U_n$ of V is a subspace.

Examples: $V = \mathbb{R}^{n \times n}$

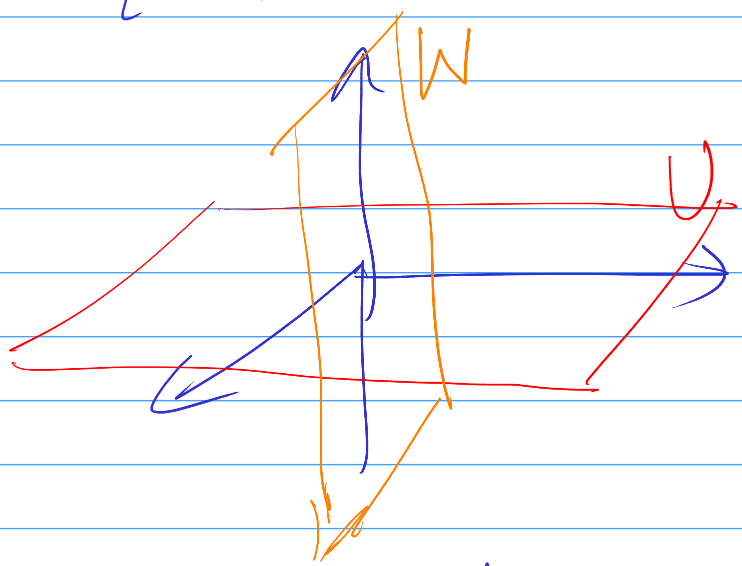
$$U = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & & 0 \end{bmatrix}, \forall a_{1i}, a_{2i} \in \mathbb{R} \right\}$$

$$W = \left\{ \begin{bmatrix} 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}, \forall a_{2i}, a_{3i} \in \mathbb{R} \right\}.$$

(2) $V = \mathbb{R}^3$.

$U = \{ (x, y, 0) \in \mathbb{R}^3 \mid \forall x, y \in \mathbb{R} \}$.

$W = \{ (0, y, z) \in \mathbb{R}^3 \mid \forall y, z \in \mathbb{R} \}$.



$U + W = ?$

Let $(x, y, z) \in V$.

$$(x, y, z) = \underbrace{(x, y, 0)}_{\in U} + \underbrace{(0, 0, z)}_{\in W}$$

$$= (x, 2y, 0) + (0, -y, z)$$

(3) $V = \{ f : \mathbb{R} \to \mathbb{R} \}$.

$U = \{ f \in V \mid f(t) = f(-t), \forall t \in \mathbb{R} \}$

$W = \{ f \in V \mid f(t) = -f(-t), \forall t \in \mathbb{R} \}$

Show. $U + W = V$

- If $v = u_1 + \ldots + u_n, u_i \in U_i, i = 1, \ldots n$, we say that $(u_1, \ldots, u_n)$ is a decomposition of v.

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- Is this decomposition unique?
- **Definition:** (Direct Sum of Subspaces) In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.

- If $v = u_1 + \ldots + u_n, u_i \in U_i, i = 1, \ldots n$, we say that $(u_1, \ldots, u_n)$ is a decomposition of v.
- Is this decomposition unique?
- **Definition:** (Direct Sum of Subspaces) In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.
- Direct sum notation: $W = U_1 \oplus U_2 \oplus \ldots \oplus U_n$.

- **Proposition 8:** Let $U_1, \ldots, U_n$ be subspaces of V. Then $V = U_1 \oplus \ldots \oplus U_n$ if and only if: (1) $V = U_1 + \ldots + U_n$, and (2) The only decomposition of $\theta \in V$ is $(\theta, \ldots, \theta)$.

- **Proposition 9:** Let V be a VS with subspaces U_1, U_2. Then $V = U_1 \oplus U_2$ *iff* $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\theta\}$.