LECTURE 11

Recap :- $V = -\alpha/r$

$$\frac{1}{r} = \frac{m\alpha}{L^2}(1 + \epsilon\cos\theta). \quad \text{where } \epsilon = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}.$$

$\epsilon = 0. \Rightarrow \quad \frac{1}{r} = \text{const}$

$$V_{eff}(r) = \frac{L^2}{2mr^2} - \frac{\alpha}{r}.$$

$$V_{eff}^{min}(r) = -\frac{m\alpha^2}{2L^2}.$$

$\epsilon > 1 \Rightarrow \quad r_{min} = \frac{L^2}{m\alpha(1+\epsilon)}$

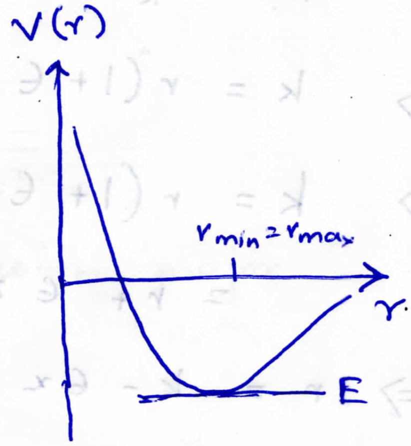
$$r_{max} = \frac{L^2}{m\alpha(1-1)} = \infty.$$

$\epsilon < 1 \Rightarrow \quad r_{min} = \frac{L^2}{m\alpha(1+\epsilon)}$

$$r_{max} = \frac{L^2}{m\alpha(1-\epsilon)}.$$

Now, $\quad \epsilon = 0, \quad 0 = 1 + \frac{2EL^2}{m\alpha^2}$

$$\Rightarrow E = -\frac{m\alpha^2}{2L^2}.$$

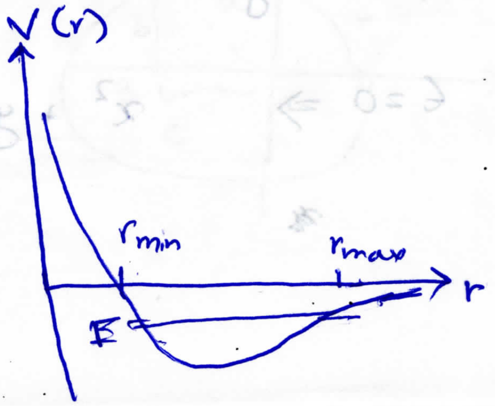


$0 < \epsilon < 1$

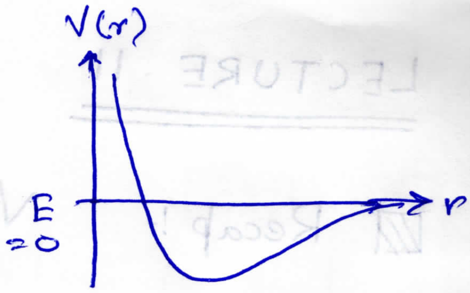
$$\Rightarrow 0 < 1 + \frac{2EL^2}{m\alpha^2} < 1$$

$$\Rightarrow -1 < \frac{2EL^2}{m\alpha^2} < 0$$

$$\Rightarrow -\frac{m\alpha^3}{2L^2} < E < 0.$$



$\epsilon = 1$:- $1 + \frac{2EL^2}{m\alpha^2} = 1$



$\Rightarrow E = 0$.

$v(r\to\infty) = 0$

$\epsilon > 1$:- $\Rightarrow E > 0$.

$\frac{1}{2} m v(r\to\infty)^2 = E$

$\Rightarrow v(r\to\infty) = \sqrt{\frac{2E}{m}}$.

CONIC SECTIONS

$\frac{1}{r} = \frac{m\alpha}{L^2}(1+\epsilon\cos\theta)$. $k = \frac{L^2}{m\alpha}$

$\Rightarrow \frac{1}{r} = \frac{1}{k}(1+\epsilon\cos\theta)$

$\Rightarrow k = r(1+\epsilon\cos\theta)$

$\Rightarrow k = r(1+\epsilon\, x/r)$

$= r + \epsilon x$.

$\Rightarrow r = k - \epsilon x$

$\Rightarrow x^2 + y^2 = k^2 + \epsilon^2 x^2 - 2\epsilon k x$.

$\epsilon = 0 \Rightarrow x^2 + y^2 = k^2 = \text{const}$.

$0 < \epsilon < 1$:

$$x^2 + y^2 = k^2 + \epsilon^2 x^2 - 2\epsilon k x.$$

$$\Rightarrow (1-\epsilon^2)x^2 + 2\epsilon k x + y^2 = k^2.$$

$$\Rightarrow x^2 + 2x\frac{k\epsilon}{1-\epsilon^2} + \frac{y^2}{1-\epsilon^2} = \frac{k^2}{1-\epsilon^2}$$

$$\Rightarrow x^2 + \frac{2\epsilon k x}{1-\epsilon^2} + \left(\frac{k\epsilon}{1-\epsilon^2}\right)^2 - \left(\frac{k\epsilon}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{k^2}{1-\epsilon^2}$$

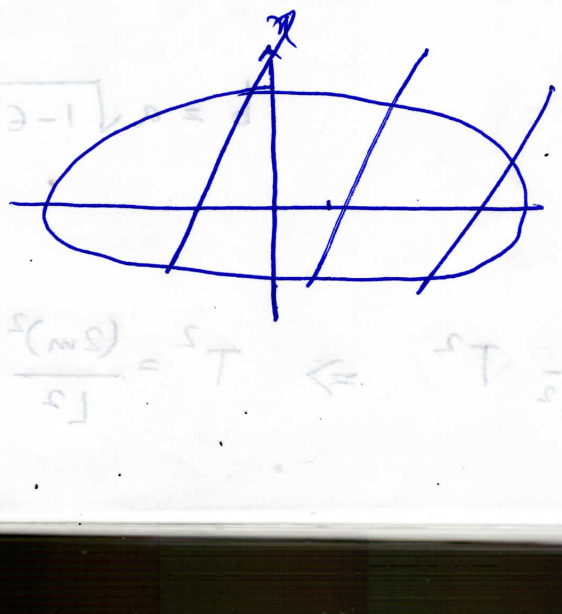
$$\Rightarrow \left(x + \frac{k\epsilon}{1-\epsilon}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{k^2}{1-\epsilon^2}\left[1 + \frac{\epsilon^2}{1-\epsilon^2}\right] = \frac{k^2}{(1-\epsilon^2)^2}$$

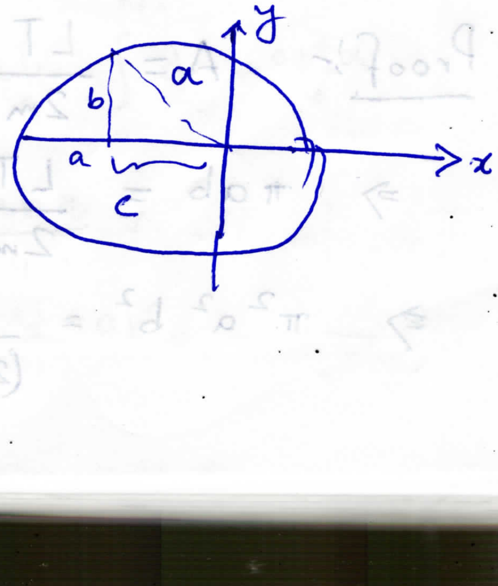
$$\Rightarrow \frac{\left(x + k\epsilon/(1-\epsilon)\right)^2}{k^2/(1-\epsilon^2)^2} + \frac{y^2}{\frac{k^2}{1-\epsilon^2}} = 1.$$

$$\Rightarrow \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = \dfrac{k}{1-\epsilon^2}$, $b = \dfrac{k}{\sqrt{1-\epsilon^2}}$.

This is the equ. for an ellipse. with major axis a and minor axis b.



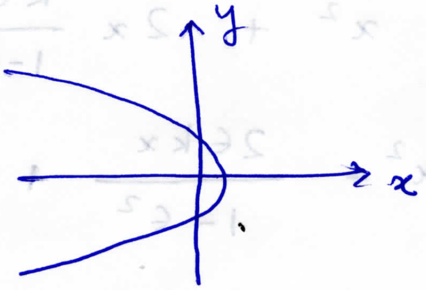


$\underline{\epsilon = 1}$:-

$$x^2 + y^2 = k^2 + x^2 - 2\epsilon k x$$

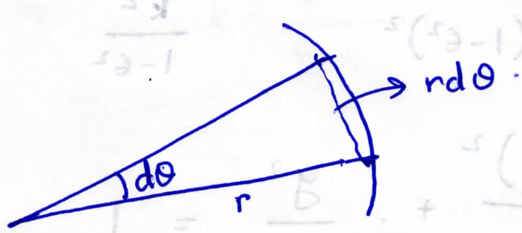
$$\Rightarrow \quad y^2 = k^2 - 2kx.$$

$\underline{\epsilon > 1}$:-



$$\frac{\left(x - \frac{k\epsilon}{\epsilon^2 - 1}\right)^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Areal velocity :-



$$dA = \frac{1}{2} r (r\, d\theta)$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$= \frac{L}{2m} = \text{const.} = \text{areal velocity}.$$

$$\boxed{T^2 \propto a^3}$$

$\underline{\text{Proof}}$:- $A = \frac{LT}{2m}$ $\qquad b = a\sqrt{1-\epsilon^2}$.

$$\Rightarrow \pi a b = \frac{LT}{2m}$$

$$\Rightarrow \pi^2 a^2 b^2 = \frac{L^2}{(2m)^2} T^2 \quad \Rightarrow \quad T^2 = \frac{(2m)^2}{L^2} \pi^2 a^4 (1 - \epsilon^2)$$

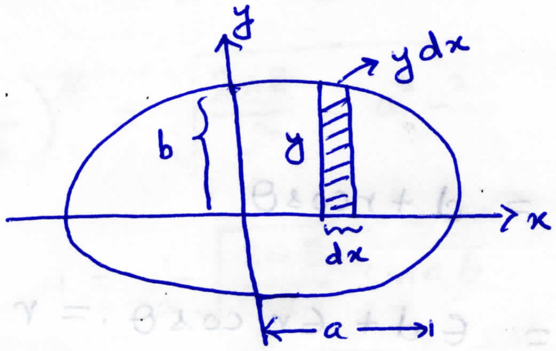
$$\Rightarrow \pi^2 a^4 = \left(\frac{L^2}{m(1-\epsilon^2)}\right)\frac{T^2}{4m} \qquad \text{Now, } L^2 = m\alpha k.$$

$$\Rightarrow \pi^2 a^4 = \frac{m\alpha k}{m(1-\epsilon^2)}\,\frac{T^2}{4m}$$

$$\Rightarrow \pi^2 a^4 = \alpha a \frac{T^2}{4m}$$

$$\Rightarrow T^2 \propto a^3 \longrightarrow \text{Kepler's 3rd law.}$$

— Area of ellipse :-



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$\frac{A}{4} = \int_0^a dx\, y$$

$$\Rightarrow \frac{A}{4} = b\int_0^a dx\,\sqrt{1 - \frac{x^2}{a^2}}$$

$$\Rightarrow \frac{A}{4} = \frac{b}{a}\int_0^a dx\,\sqrt{a^2 - x^2}$$

Sub:- $x = a\sin\alpha$

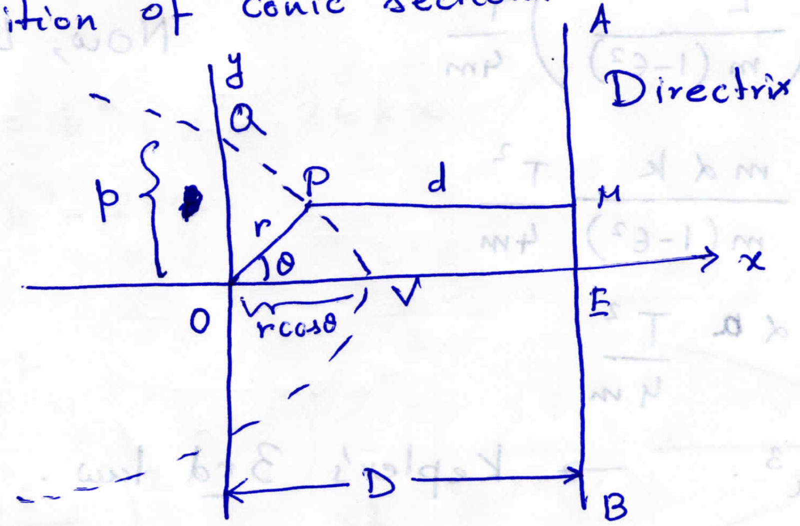
$$\frac{A}{4} = \frac{b}{a}\int_0^{\pi/2} (a^2\cos^2\alpha)\, d\alpha$$

$$= ab\int_0^{\pi/2} d\alpha \cos^2\alpha.$$

$$= \frac{ab\pi}{4}$$

$$\Rightarrow A = \pi ab$$

Definition of conic section.



Focus O ≡ fixed

$$\frac{r}{d} = \epsilon = \text{const.} = \frac{p}{D}$$

$$D = d + r\cos\theta .$$

$$\Rightarrow \quad p = \epsilon(d + r\cos\theta) = \epsilon d + \epsilon r\cos\theta = r(1+\epsilon\cos\theta)$$

$$\Rightarrow \quad r = \left(\frac{p}{1+\epsilon\cos\theta}\right) \rightarrow \text{polar equ. for a conic.}$$

$V(r) = \beta/r^2$

$$V_{eff}(r) = \frac{L^2}{2mr^2} + \frac{\beta}{r^2} = \frac{L^2}{2mr^2}\left(1 + \frac{2m\beta}{L^2}\right) = \frac{a^2L^2}{2mr^2}$$

where $a^2 = \left(1 + \frac{2m\beta}{L^2}\right)$.

Equ. for the path is:-

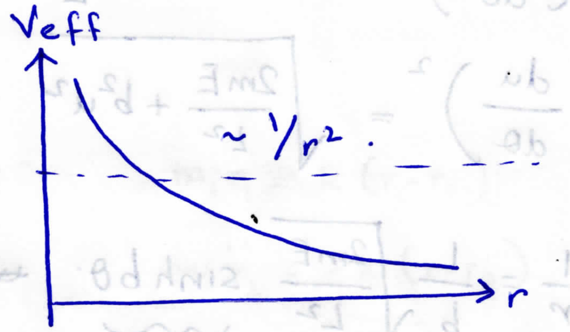
$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{L^2}(E - V)$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2mE}{L^2} - \frac{2m}{L^2}\beta u^2.$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + \left(1 + \frac{2m\beta}{L^2}\right)u^2 = \frac{2mE}{L^2}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + a^2u^2 = \frac{2mE}{L^2}.$$

Case 1:- $a^2 > 0$



$$\left(1 + \frac{2m\beta}{L^2}\right) > 0$$

$$\Rightarrow \beta > -\frac{L^2}{2m}.$$

$$\left(\frac{du}{d\theta}\right) = \sqrt{\frac{2mE}{L^2} - a^2u^2}$$

$$\Rightarrow u = \frac{1}{a}\sqrt{\frac{2mE}{L^2}}\sin a\theta$$

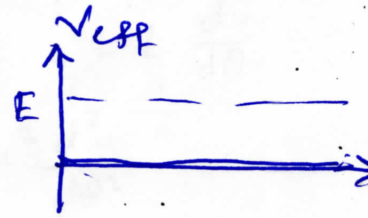
$$\Rightarrow \frac{1}{r} = \frac{1}{a}\sqrt{\frac{2mE}{L^2}}\sin a\theta.$$

$\theta = 0 \Rightarrow$ $r_{max} = \infty$

$$r_{min} = a\sqrt{\frac{L^2}{2mE}}.$$

Case 2:- $a = 0$. $\quad 1 + \frac{2m\beta}{L^2} = 0 \Rightarrow \beta = -\frac{L^2}{2m}.$

$V_{eff} = 0.$



$$\left(\frac{du}{d\theta}\right)^2 = \frac{2mE}{L^2}$$

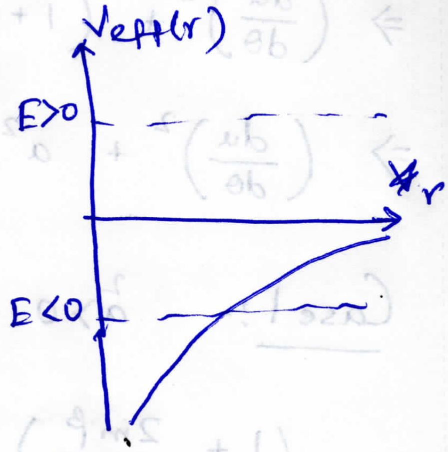
$$\Rightarrow \left(\frac{du}{d\theta}\right) = \sqrt{\frac{2mE}{L^2}}$$

$$\Rightarrow r = \frac{1}{\theta}\sqrt{\frac{L^2}{2m}}$$

Case 3 :- $a^2 < 0$.

Let $-a^2 = b^2$

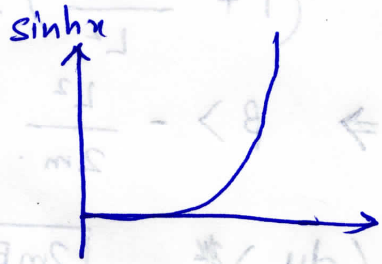
$$\Rightarrow \beta < -\frac{L^2}{2m}.$$



E>0 :-

$$\left(\frac{du}{d\theta}\right)^2 - b^2u^2 = \frac{2mE}{L^2}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \sqrt{\frac{2mE}{L^2} + b^2u^2}$$



$$\frac{1}{r} = \frac{1}{b}\sqrt{\frac{2mE}{L^2}}\,\sinh b\theta.$$

has no maximum.

$\therefore r \to 0$ at some finite θ.

E<0 :-

$$\left(\frac{du}{d\theta}\right)^2 - b^2u^2 = -\frac{2m|E|}{L^2}$$

$$\Rightarrow b^2u^2 - \left(\frac{du}{d\theta}\right)^2 = \frac{2m|E|}{L^2}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{b}\sqrt{\frac{2m|E|}{L^2}}\,\cosh(b\theta).$$

has no maximum

but has finite minimum unity at $\theta = 0$

$$\therefore r_{max} = b\sqrt{\frac{L^2}{2m|E|}}$$

Small oscillations

$$U(r) = U(r_0) + \underbrace{(r-r_0)\cancel{\frac{dU}{dr}}\Big|_{r=r_0}}_{0} + \frac{1}{2}(r-r_0)^2\frac{d^2U}{dr^2}\Big|_{r=r_0} + \dots$$

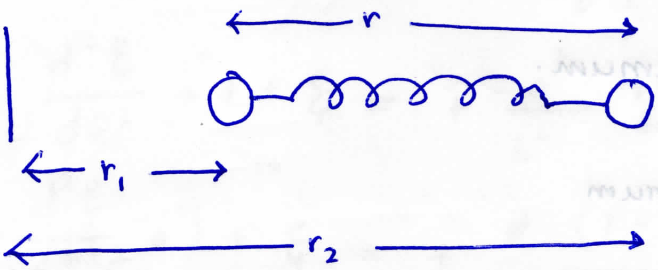
$$\approx \frac{1}{2} k (r-r_0)^2$$

- Diatomic molecule

$$m_1\ddot{r}_1 = k(r-r_0)$$

$$m_2\ddot{r}_2 = -k(r-r_0)$$

where $r = r_2 - r_1$.



$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r-r_0)$$

$$\Rightarrow \ddot{r} = -\frac{k}{\mu}(r-r_0)$$

where $\mu = \frac{1}{m_1} + \frac{1}{m_2}$ is the reduced mass.

Remark:- Retaining up to quadratic term in potential is equivalent to linearising force.

- Derivation of second order equ. for path:-

The EOM for central force is given by, $L = r^2\dot{\theta} \Rightarrow \dot{\theta} = Lu^2$

$$\ddot{r} - r\dot{\theta}^2 = f(r)$$

$$\Rightarrow \ddot{r} - r\frac{L^2}{r^4} = f(r)$$

$$\Rightarrow \ddot{r} - \frac{L^2}{r^3} = f(r)$$

$$r = \frac{1}{u}$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta} = -L\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{d\theta}(\dot{r})\,\dot{\theta} = -L^2u^2\frac{d^2u}{d\theta^2}$$

Substituting,

$$-L^2u^2\frac{d^2u}{d\theta^2} - L^2u^3 = f(1/u)$$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{L^2u^2}}$$

VI PERTURBATIONS OF CIRCULAR ORBIT:

– $\frac{dV}{dr} = 0 \rightarrow$ extremum.

$\frac{d^2V}{dr^2} > 0 \rightarrow$ minimum

$\frac{d^2V}{dr^2} < 0 \rightarrow$ maximum.

Particle moves in a circular orbit of radius a under the ~~radial~~ attractive force $f(r)$, which must equal the centripetal force.

$$\Rightarrow \frac{v^2}{a} = f(a)$$

$$L = va$$

$$\Rightarrow L^2 = v^2a^2 = a^3 f(a)$$

Suppose the particle is subjected to a small radial impulse, which leaves angular momentum unchanged.

$u = \frac{1}{a}(1+\xi(\theta))$, where ξ is assumed to be small.

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{L^2u^2}$$

$$\Rightarrow \cancel{\frac{d^2u}{d\theta}}\ \frac{1}{a}\frac{d^2\xi}{d\theta^2} + \frac{1}{a}(1+\xi(\theta)) = +\frac{f(1/u)}{L^2u^2}$$

$$= +\frac{f\left(\frac{a}{1+\xi(\theta)}\right)}{L^2u^2}$$

$$\Rightarrow \cancel{a}\ \frac{d^2\xi}{d\theta^2} + 1 + \xi = +\frac{a}{L^2u^2} f\left(\frac{a}{1+\xi}\right)$$

$$\Rightarrow \frac{d^2\xi}{d\theta^2} + 1 + \xi = +\frac{a^3}{L^2}(1+\xi)^{-2} f\left(\frac{a}{1+\xi}\right)$$

$$\Rightarrow \frac{d^2\xi}{d\theta^2} + 1 + \xi = +\frac{(1+\xi)^{-2}}{f(a)} f\left(\frac{a}{1+\xi}\right)$$

$$\cancel{\Rightarrow \frac{d^2\xi}{d\theta^2} + 1 + \xi = }$$

$$f\left(\frac{a}{1+\xi}\right) = f\left(a - \frac{a\xi}{1+\xi}\right)$$

$$= f(a) - \frac{a\xi}{1+\xi} f'(a) + O\left(\frac{\xi}{1+\xi}\right)^2$$

$$\approx f(a) - a f'(a)\xi + O(\xi^2).$$

$$(1+\xi)^{-2} = 1 - 2\xi + O(\xi^2).$$

$$(1+\xi)^{-2} f\left(\frac{a}{1+\xi}\right) = (1-2\xi)(f(a) - af'(a)\xi)$$

$$\approx f(a) - af'(a)\xi - 2\xi f(a).$$

$$\Rightarrow \frac{(1+\xi)^{-2} f\left(\frac{a}{1+\xi}\right)}{f(a)} \approx 1 - a\frac{f'(a)}{f(a)}\xi - 2\xi$$

Substituting,

$$\frac{d^2\xi}{d\theta^2} + \cancel{1} + \xi = \cancel{1} - a\,\frac{f'(a)}{f(a)}\,\xi - 2\xi$$

$$\Rightarrow \boxed{\frac{d^2\xi}{d\theta^2} + \left(3 + \frac{a f'(a)}{f(a)}\right)\xi = 0} \Rightarrow$$ linear in perturbation.

Case 1:- $3 + \frac{a f'(a)}{f(a)} < 0$.

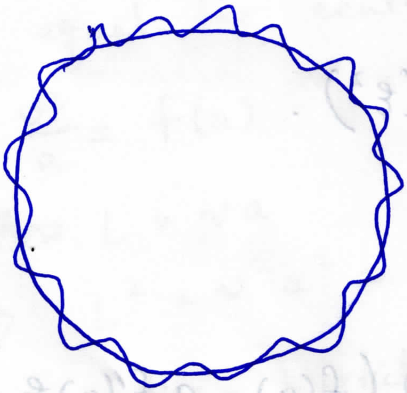
Qualitatively, $\xi = e^{m\theta} + e^{-m\theta}$

ξ grows with time, and as a result, the orbit does not remain circular. and is unstable

Case 2:- $\Omega^2 = 3 + \frac{a f'(a)}{f(a)} > 0$.

$\xi = (A) \sin \Omega\theta + (B) \cos \Omega\theta$.

stable circular orbit.



Condition for closure of orbit:-

$\Omega \rightarrow$ +ve integer.

Consider a power law,

$f(r) = k r^{\nu}$.

~~Assuming the condition for stability to be satisfied,~~

$f(r) = k r^{\nu}$

$$\Omega^2 = 3 + \frac{a f'(a)}{f(a)} = 3 + \frac{a\, \nu k a^{\nu-1}}{k a^{\nu}} = \nu + 3 > 0$$

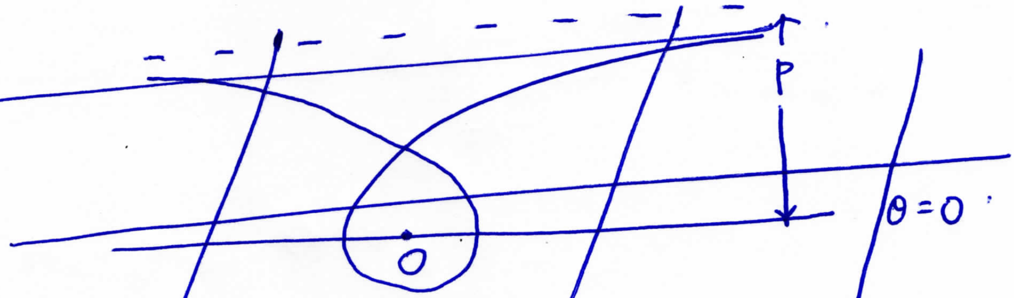
$$\Rightarrow \nu > -3.$$

$$\nu + 3 = m^2$$

$$\Rightarrow \nu = m^2 - 3,$$

Reminder:- the above results are only true for linearised perturbations.

PROB:-



~~Body at the centre~~ origin O ~~exerts a force~~ / potential. ~~$\vec{F} = \frac{\gamma}{r^3}\hat{r}$.~~

$$\boxed{V_{eff}(r) = \frac{L^2}{2mr^2} + k r^{\nu}} \rightarrow \text{object under consideration}$$

Radius of circular orbit $\Rightarrow \left.\frac{dV_{eff}}{dr}\right|_{r=r_*} = 0$

$$\cancel{\Rightarrow r_* = \left(\frac{L^2}{\beta \nu}\right)^{1/(\nu+2)}}$$

$$\cancel{V''_{eff}(r_0) = \frac{L^2}{r_0^4}}$$