Computational Numerical Methods

CS 374

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Number of iteration needed.

$$|r - c_n| \le \varepsilon.$$

This will be satisfied if.

$$\frac{1}{2^{n+1}}(b_0 - a_0) \le \varepsilon.$$

$$\frac{1}{2^{n+1}} \le \frac{\varepsilon}{b_0 - a_0}$$

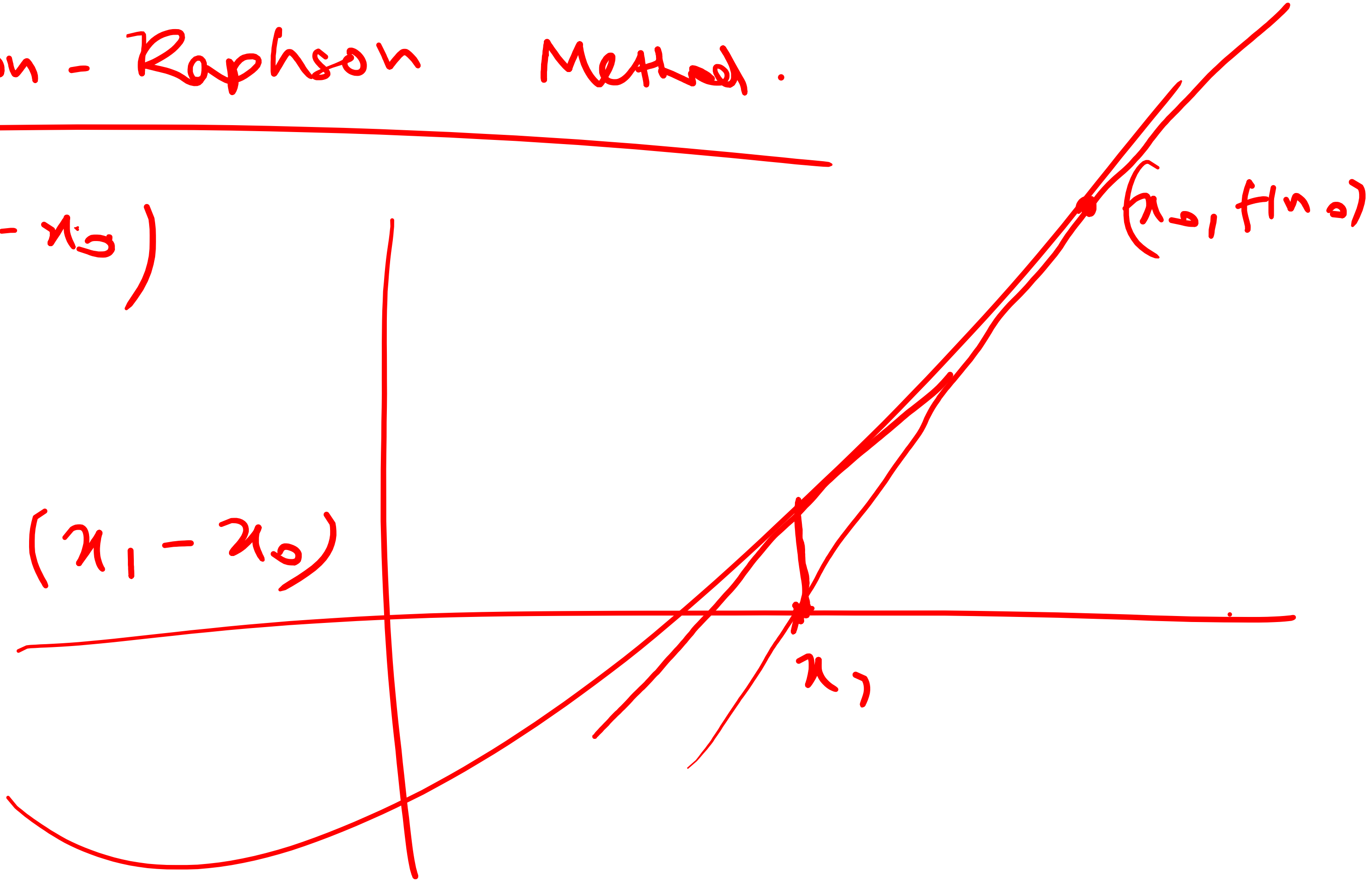
$$2^{n+1} \geq \frac{b_0 - a_0}{\varepsilon}.$$

$$n+1 \geq \log_2\left(\frac{b_0 - a_0}{\varepsilon}\right)$$

$$n \geq \log_2\left(\frac{b_0 - a_0}{\varepsilon}\right) - 1$$

Newton's Method / Newton-Raphson Method.

$$f(y) = f(x_0) + f'(x_0)(x_1 - x_0)$$



$$y_1 = f(x_0) + f'(x_0)(x_1 - x_0)$$

at $x = x_1$

$y_1 = 0$

$$\therefore \quad f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

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$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Ex

$f(x) = x^6 - x - 1$ $\quad f'(x) = 6x^5 - 1$

i.e.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$	$\alpha - x_{n-1}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	1.5	8.89	—	—	1.3049088
1	1.3049088	2.54	-2×10^{-1}	-3.65×10^{-1}	1.1814804
2	1.1814804	0.538	~~-119~~ -0.119	-1.66×10^{-1}	1.13945559
3	1.13945559	0.0492	-0.042	4.68×10^{-2}	1.13477763
4	1.13477763	0.00055	-0.00468	-4.72×10^{-3}	1.13472415
5	1.13472415	0.0000000711	-0.0000535	-5.35×10^{-5}	1.13472414
6	$\boxed{1.13472414}$	1.5×10^{-11}	-6.91×10^{-9}	-6.91×10^{-9}	—

Early days's computer arithmatic.

Suppose we need to find a/b.

Try to find out $\frac{1}{b}$ & then compute $a \cdot \frac{1}{b}$.

To solve $\frac{1}{b}$

Consider

$$f(x) = b - \frac{1}{x}.$$

Here we assume $b > 0$

$$f'(x) = \frac{1}{x^2}.$$

Use Newton's method.

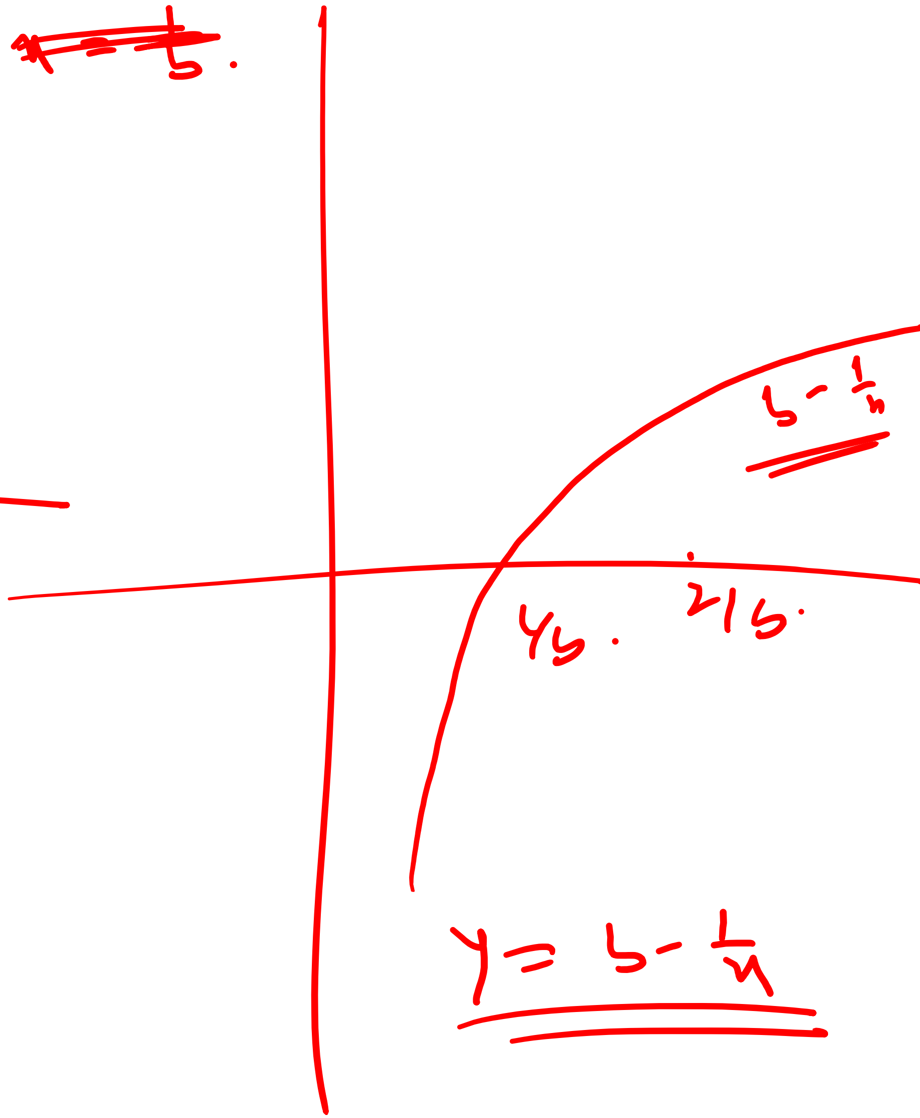
~~$x = \frac{1}{b}$~~.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{b - \frac{1}{x_n}}{1/x_n^2}.$$

$$\underline{x_{n+1} = x_n(2 - b x_n)}$$

assume $\underline{x_0 > 0}$



$\underline{y = b - \frac{1}{x}}$

Show that

$$Rel(x_{n+1}) = [Rel(x_n)]^2.$$

$$Rel(x_n) = \frac{\alpha - x_n}{\alpha}.$$

Try this

Relative error when considering x_n as the approximation of $\alpha = \frac{1}{b}$.

then we must have -

$$|Rel(x_0)| < 1$$

$$-1 < \operatorname{Re}(x_0) < 1$$

$$-1 < \frac{\frac{1}{5} - x_0}{1/5} < 1$$

$$\boxed{0 < x_0 < \frac{2}{5}.}$$

$$\cancel{5} \quad \underline{\underline{5 - \frac{1}{x} = t}}$$

Error Analysis

$y = f(x)$.

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \underline{\frac{1}{2}(\alpha - x_n)^2 f''(c_n)}$$

where c_n is an unknown point b/w $\underline{\underline{\alpha \,\&\, x_n}}$.

as α is a root

$$\therefore \quad 0 = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 \cdot f''(c_n)$$

$$\Rightarrow \quad 0 = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(c_n)}{f'(x_n)}$$

$$0 = \cancel{x_n} - x_{n+1} + (\alpha - \cancel{x_n}) + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(c_n)}{f'(x_n)}$$

$$\therefore \ \alpha - x_{n+1} = \underline{(\alpha - x_n)^2} \cdot \frac{-f''(c_n)}{2 f'(x_n)}$$

$\therefore$ The error in $n+1$ ~~th~~ th iteration. is nearly proportional to the square of the error in n th iteration.

Ex Consider $f(x) = x^6 - x - 1$

$f''(x) = 30x^4$.

$$-\frac{f''(c_n)}{2f'(x_n)} \approx \frac{-f''(\alpha)}{2f'(\alpha)} = \frac{-30\alpha^4}{2(6\alpha^5 - 1)} = -2.42$$

$\therefore \quad \alpha - x_{n+1} \approx -2.42(\alpha - x_n)^2$.