SC223 - Linear Algebra

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Lecture 24

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Some beautiful results

- **Theorem 3:** (Rank-Nullity Theorem). Let $A \in \mathbb{R}^{m \times n}$, with $rank(A) = r$. Then $r + dim(N(A)) = n$.
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Proof:

Since $U \cap W$ is a FDVS, it has a basis.

Let $\beta_{U \cap W} = \{\underbrace{p_1, \ldots, p_k}\}$ be a basis of $U \cap W$ → (1)

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Let $\beta_U = \{p_1, \ldots, p_k, u_1, \ldots, u_n\}$ be a basis of U → (2)

Let $\beta_W = \{p_1, \ldots, p_k, w_1, \ldots, w_m\}$ " of W → (3)

$$\beta_{U+W} = \beta_U \cup \beta_W$$

$$u_1 = \sum_{i=2}^{n} a_i u_i + \sum_{i=1}^{k} b_i p_i + \sum_{i=1}^{m} c_i w_i \Rightarrow \begin{matrix} a_i = 0 \\ b_i = 0 \\ c_i = 0 \end{matrix} \quad \forall i$$

$$\left.\begin{matrix} \Rightarrow \beta_U \cup \beta_W \text{ is LI.} \\ \text{span}(\beta_U \cup \beta_W) = U+W \end{matrix}\right\} \Rightarrow \beta_{U+W} = \beta_U \cup \beta_W$$

$$\dim(U+W) = |\beta_U \cup \beta_W| = n+k+m$$

$$\dim(U) = n+k$$

$$\dim(W) = m+k$$

$$\dim(U \cap W) = k.$$

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$$span(\phi) = \{0\}.$$

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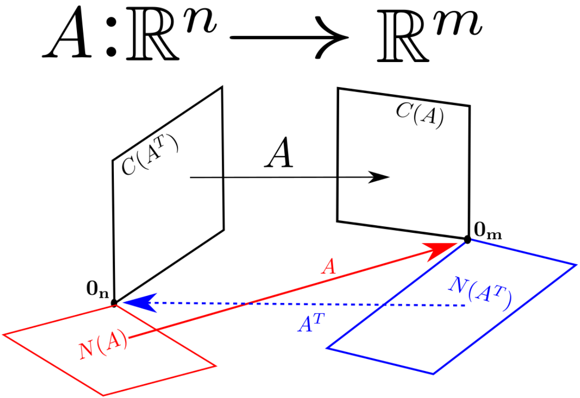
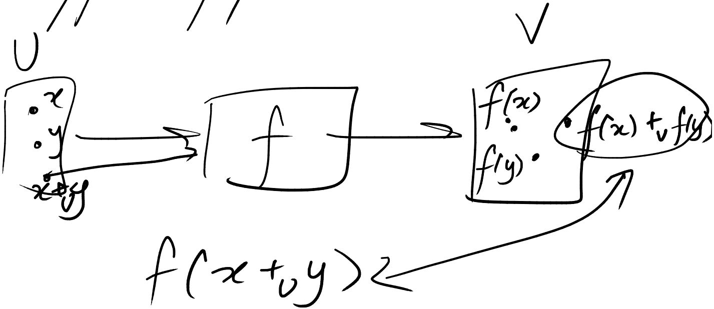


Linear Transformations

Linear transformations

- Let U and V be vector spaces over the same field $\mathbb{F}$. A function $f : U \to V$ is said to be **Linear transformation** from U to V if

$$\begin{aligned} &\textit{Additive} : \forall x, y \in U, f(x+y) = f(x) + f(y) \\ &\textit{Homogeneous} : \forall a \in \mathbb{F}. \forall x \in U, f(a \cdot x) = a \cdot f(x). \end{aligned}$$



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● Examples:

① U. $\quad I : U \to U$, $\forall x \in U, I(x) = x$. } Identity mapping

② $U = \mathcal{L}_2(\mathbb{R})$, Let $h \in \mathcal{L}_2(\mathbb{R})$ $h(x) = 0$ $|x| \geq a$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$

③ $U = \mathbb{R}^n$, $V = \mathbb{R}^m$.

$A \in \mathbb{R}^{m \times n}$, $\quad A : x \longrightarrow A \cdot x$.

④ $U = V = \mathbb{R}^2$.

$\theta \in [0, 2\pi)$.

$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

⑤ $U = P_n(\mathbb{R})$. $\quad \forall p \in P_n(\mathbb{R})$, $\frac{d}{dx}(p) \in P_n(\mathbb{R})$

Is $\frac{d}{dx}$ a L.T?

④ $U = P_n(\mathbb{R})$, $V = P_{n+1}(\mathbb{R})$

Is $\int : U \to V$ a L.T ?

$$\int (p+q)(x)\,dx = \int p(x)\,dx + \int q(x)\,dx$$

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- We say that two vector spaces over $\mathbb{F}$, U and V are **homomorphic** if there exists an linear transformation between them.
- We say that two vector spaces over $\mathbb{F}$, U and V are **isomorphic** if there exists an **invertible** linear transformation between them.
- If T is an invertible linear transformation between U and V, then we say that T is an **isomorphism** between U and V.
- **Proposition 19:** Show that two vector spaces U and V over $\mathbb{F}$ are isomorphic iff they have the same dimensions.

Representation of Linear Transformations between FDVS

- Let $T : U \to V$ be a LT, and let $\beta_U := \{u_1, \dots, u_n\}$ and $\beta_V = \{v_1, \dots, v_m\}$ denote the basis of U and V respectively.

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- Thus, $T(u_i), i = 1, \ldots, n$ is enough to allow us to compute $T(x), \forall x \in U$.
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- Also, since $y \in V, y = \sum_{j=1}^{m} b_j v_j$.
- Thus, $\sum_{j=1}^{m} b_j v_j = \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ji} a_i v_j$.

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- For $k \in \{1, \ldots, m\}$, $b_k = \sum_{i=1}^{n} c_{ki} a_i$, or,

$$\underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{[y]_{\beta_V}} = \underbrace{\begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \ldots & c_{mn} \end{bmatrix}}_{[T]_{\beta_U}^{\beta_V}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{[x]_{\beta_U}}$$

- The matrix $[T]_{\beta_U}^{\beta_V}$ is called the matrix representation of the linear tranformation T with respect to the basis β_U and β_V.

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- $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.
- Let $p \in \mathcal{P}_3(\mathbb{R})$ be such that $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$. Define $T_p : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ by $T_p(q) = p \cdot q, \forall q \in \mathcal{P}_3(\mathbb{R})$, where $\cdot$ represents multiplication between polynomials.