Properties of CTFT [Contd.]

Lecture 32

[2] Time Shifting

If $f(t) \xleftrightarrow{F(\cdot)} F(\omega)$

then $f(t-t_0) \xleftrightarrow{F(\cdot)} ?\ e^{-j\omega t_0} \cdot F(\omega)$

But: $F(\omega) = \underline{F[f(t)]} = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t}\, dt$

$\therefore F\{\underline{f(t-t_0)}\} = \int_{-\infty}^{+\infty} f(t-t_0)\, e^{-j\omega t}\, dt$

Using method substitution, let $t - t_0 = t_1$

$dt = dt_1$

$t = t_1 + t_0.$

$\therefore F\{f(t-t_0)\} = \int_{-\infty}^{+\infty} f(t_1) \cdot e^{-j\omega(t_1+t_0)} \cdot dt_1$

$= \left(e^{-j\omega t_0}\right) \cdot \left\{ \int_{-\infty}^{+\infty} f(t_1) \cdot e^{-j\omega t_1}\, dt_1 \right\}.$

$$\boxed{F\{f(t-t_0)\} = \left(e^{-j\omega t_0}\right) \cdot F(\omega)} = F_1(\omega)$$

$|F_1(\omega)| = |e^{-j\omega t_0} \cdot F(\omega)| = |e^{-j\omega t_0}| \cdot |F(\omega)|.$

$|e^{-j\omega t_0}| = 1$

(1)

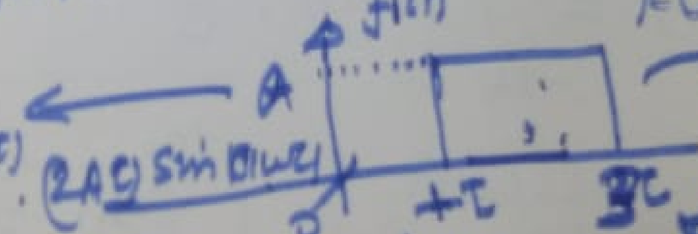
$$|F_1(\omega)| = |F(\omega)|$$

$$\angle F_1(\omega) = \angle e^{-j\omega t_0} + \angle F(\omega).$$

$$\boxed{\angle F_1(\omega) = -\omega t_0 + \angle F(\omega).}$$

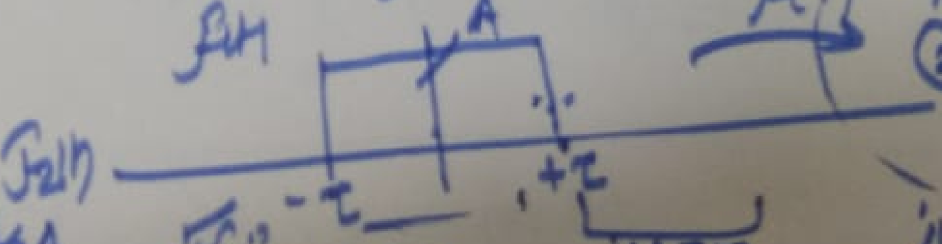
Inference: If we time shift (delay) a signal $f(t)$ in time-domain, then in frequency domain, magnitude spectrum of $f(t)$ and $f(t-t_0)$ remains unchanged whereas the phase spectrum of $f(t-t_0)$ = phase due to $-\omega t_0 + \angle F(\omega)$.

Problem: Find the Fourier transform of

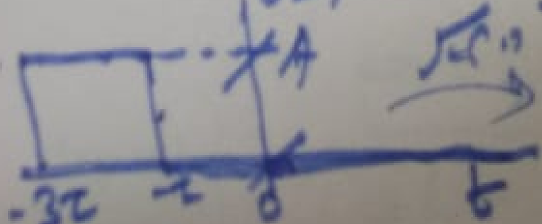


$f(t) \xrightarrow{F.T.} F_1(\omega) = e^{-j\omega(2\tau)} \times F(\omega).$

$F_1(\omega) = e^{-j\omega(2\tau)} \cdot (2A\tau)\,\text{sinc}(\omega\tau)$



$f(t) \xrightarrow{F.T.} F(\omega) = (2A\tau)\,\text{sinc}(\omega\tau)$



$f_2(t) \xrightarrow{F.T.} F_2(\omega) = e^{j\omega 2\tau} F(\omega) = e^{j\omega(2\tau)}(2A\tau)\,\text{sinc}(\omega\tau)$

[3] Conjugation and Symmetry

If $f(t) \xleftrightarrow{F(\cdot)} F(\omega)$

then $f^*(t) \xleftrightarrow{F(\cdot)}$? $F^*(-\omega)$.

Proof: $\rightarrow$ $F(\omega) = F\{f(t)\} = \int_{-\infty}^{+\infty} \underline{f(t)} \cdot \underline{e^{-j\omega t}}\, dt$

Take complex conjugation on both the sides.

$$\therefore F^*(\omega) = \left[\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt\right]^*$$

$$= \int_{-\infty}^{+\infty} \left[f(t) \cdot e^{-j\omega t}\right]^* dt \quad \text{[Tutorial 1]}$$

$$F^*(\omega) = \int_{-\infty}^{+\infty} \underbrace{f^*(t)} \cdot \underbrace{e^{j\omega t}}\, dt$$

$\omega \rightarrow -\omega$.

$$\therefore F^*(-\omega) = \underbrace{\int_{-\infty}^{+\infty} f^*(t) \cdot e^{-j\omega t}\, dt} = F\{f^*(t)\}$$

$\blacksquare$

Side Result :→ Let $f(t)$ be a real signal.

$$f^*(t) = f(t).$$

$$\therefore F\{f^*(t)\} = F\{f(t)\}.$$

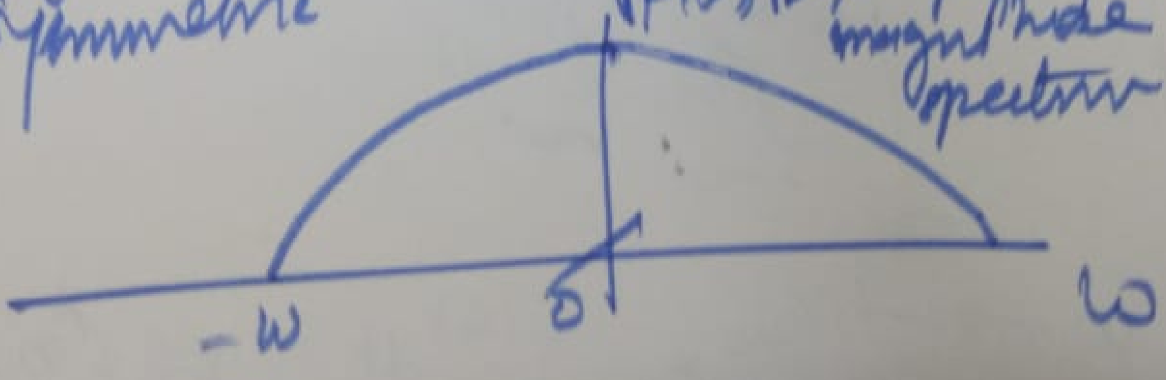
$$\boxed{F^*(-\omega) = F(\omega)} \rightarrow \text{if } f(t) = \text{real}.$$

$$|F^*(-\omega)| = |F(\omega)|.$$

$$|F(\omega)| = |F(-\omega)|.$$

⇒ If a signal $f(t)$ is real then its Fourier or magnitude spectrum is even function or symmetric

$|F(\omega)|$ → Symmetry in magnitude spectrum.



Side point ②.

$F(\omega)$ = complex quantity

$$F(\omega) = F_R(\omega) + j\, F_I(\omega) \quad \text{—— ①}$$

$\omega \rightarrow -\omega$

$$F(-\omega) = F_R(-\omega) + j\, F_I(-\omega).$$

Take complex conjugate,

$$F^*(-\omega) = \left[F_R(-\omega) + j\, F_I(-\omega)\right]^*$$

$$F^*(-\omega) = F_R(-\omega) - j\, F_I(-\omega) \quad \text{—— ②}$$

If $f(t)$ = real, then $F(\omega) = F^*(-\omega)$

$\therefore$ $F_R(\omega) = F_R(-\omega)$

$\Rightarrow$ $F_R(\omega)$ is even function of 'ω'

$F_I(\omega) = -F_I(-\omega)$

$\Rightarrow$ $F_I(\omega)$ is an odd function in of 'ω'.

$\boxed{\text{Proposition 4.1}}$ If a function/signal, $f(t)$ is real and even then its Fourier transform, $F(\omega)$, is also real and even.

Proof: $\rightarrow$ $F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$

$t \rightarrow \tau$

$$= \int_{-\infty}^{+\infty} f(\tau) e^{-j\omega\tau} d\tau$$

Using method of substitution, $\tau = -t_1$

$d\tau = -dt_1$

$\tau \rightarrow -\infty, \; t_1 \rightarrow +\infty$

$\tau \rightarrow +\infty, \; t_1 \rightarrow -\infty$

$$\therefore F(\omega) = \int_{+\infty}^{-\infty} f(-t_1) \, e^{+j\omega t_1} (-dt_1)$$

$\because f(t)$ is even

$f(t_1) = f(-t_1)$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t_1) \, e^{j\omega t_1} dt_1$$

6

$\omega \longrightarrow -\omega$.

$$\therefore F(-\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\boxed{F(-\omega) = F(\omega).} \quad \text{— (2)}$$

$\Longrightarrow F(\omega)$ is even function in 'ω'!

$\because f(t)$ = real function,

$$F^{*}(-\omega) = F(\omega). \quad \text{— (1)}$$

$$\therefore F^{*}(\omega) = F(\omega).$$

$\Longrightarrow F(\omega)$ is a real function ∎

Proposition 4.2 :→ If a function / signal $f(t)$ is real and odd then its Fourier transform will be purely imaginary and odd function.

$\Longrightarrow$ Proof : Homework.

(P)

Side Remarks: (3)

$$f(t) = f_e(t) + f_o(t) \quad \text{[Chapter 1]}$$

By linearity property of CTFT,

$$F\{f(t)\} = F\{f_e(t)\} + F\{f_o(t)\}.$$

$$F(\omega) = F_e(\omega) + F_o(\omega)$$

$$F(\omega) = F_R(\omega) + j\, F_I(\omega)$$

$$\boxed{\underbrace{F_e(\omega)}_{\text{even function}} = \underbrace{F_R(\omega)}_{\text{even function}}}$$

$$\underbrace{F_o(\omega)}_{\text{Odd function in } \omega} = j\, \underbrace{F_I(\omega)}$$

& purely imaginary

[4] Differentiation in time-domain

If $f(t) \xleftrightarrow{F(\cdot)} F(\omega)$

then $\frac{d}{dt}[f(t)] \xleftrightarrow{F(\cdot)} (j\omega\, F(\omega))$.

Proof:→ $$f(t) = \left(\frac{1}{2\pi}\right) \cdot \int_{\omega=-\infty}^{\omega=+\infty} F(\omega) \cdot e^{j\omega t}\, d\omega \quad \text{---} (1)$$

Differentiate w.r.t. t under integral sign

$$\frac{d}{dt} f(t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \frac{d}{dt}\left[F(\omega) \cdot e^{j\omega t}\right] d\omega$$

$$\therefore \frac{d f(t)}{dt} = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} F(\omega) \left[j\omega\, e^{j\omega t}\right] d\omega$$

$$\frac{d}{dt} f(t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \left[j\omega\, F(\omega)\right] \cdot e^{j\omega t}\, d\omega \quad \text{---} (2)$$

Similarly,

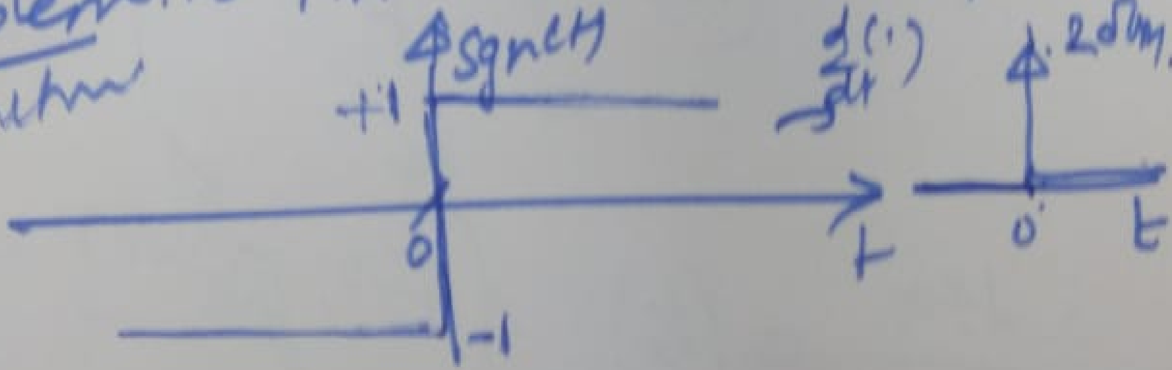
$$\frac{d^2 f(t)}{dt^2} \xleftrightarrow{F\{\cdot\}} (j\omega)^2 \cdot F(\omega)$$

$$\frac{d^3 f(t)}{dt^3} \xleftrightarrow{F} (j\omega)^3 \cdot F(\omega)$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d^n f(t)}{dt^n} \xleftrightarrow{F\{\cdot\}} (j\omega)^n \cdot F(\omega)$$

<u>Problem</u>:→ Find the CTFT of signum function



$$\frac{d}{dt}[sgn(t)] = 2\delta(t)$$

$$F\left\{\frac{d}{dt} sgn(t)\right\} = F\{2\delta(t)\}$$

Using differentiation property of CTFT,

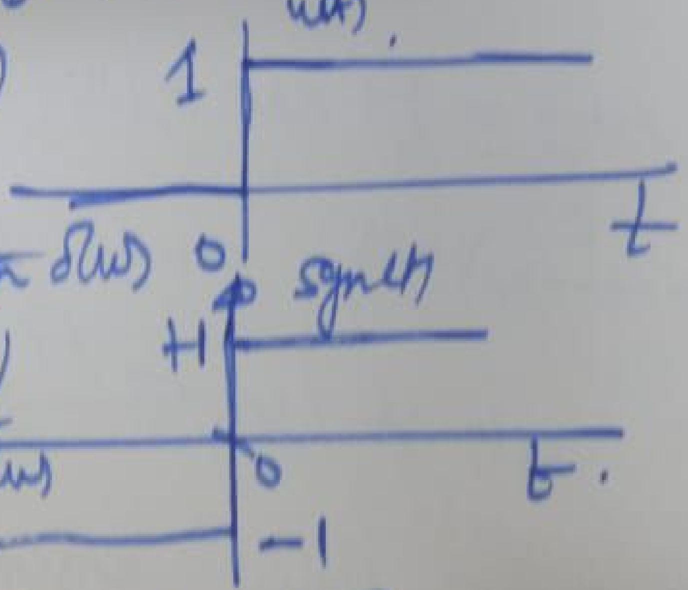
$$(j\omega)\, F\{sgn(t)\} = 2\, F\{\delta(t)\}$$

$$(j\omega)\, F\{sgn(t)\} = 2 \times 1$$

$$\therefore \boxed{F\{sgn(t)\} = \frac{2}{j\omega}}$$

Problem 2: Find the CTFT of u(t)

$f(t) = u(t)$



$$\int_{-\infty}^{\infty} e^{-j\omega t}\, dt = 2\pi\,\delta(\omega)$$

$$\therefore F\{1\} = 2\pi\,\delta(\omega)$$

$$u(t) = \frac{1}{2}\left[1 + sgn(t)\right]$$

$$F\{u(t)\} = \frac{1}{2}\left[F\{1\} + F\{sgn(t)\}\right]$$

$$= \frac{1}{2}\left[2\pi\,\delta(\omega) + \frac{2}{j\omega}\right]$$

$$\boxed{\mathcal{F}\{u(t)\} = U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}}$$

[5] Convolution Theorem

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t)$$

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\} = X(\omega) \cdot H(\omega)$$

If $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$ and $h(t) \xleftrightarrow{\mathcal{F}(\cdot)} H(\omega)$

then $x(t) * h(t) \xleftrightarrow{\mathcal{F}(\cdot)} X(\omega) \cdot H(\omega)$.

Proof: Method I Using eigenfunction theory of LTI operator:

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \underline{X(\omega)}\, e^{j\omega t}\, d\omega$$

fundamental theorem of integral calculus

$$= \left(\frac{1}{2\pi}\right) \left\{ \lim_{\omega_0 \to 0} \sum_{n=-\infty}^{+\infty} \left(X(n\omega_0)\, e^{jn\omega_0 t}\right) \cdot \omega_0 \right\}$$

$$x(t) = \frac{1}{2\pi} \lim_{w_0 \to 0} \sum_{\langle n \rangle} \underbrace{[X(w_n) \cdot w_0]}_{\text{constant wrt t.}} \cdot \overbrace{e^{jw_n t}}^{\text{eigenfunction}} \rightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \rightarrow$$

Using eigenfunction property

$$\therefore y(t) = \frac{1}{2\pi} \lim_{w_0 \to 0} \sum_{n=-\infty}^{+\infty} [X(w_n) \cdot w_0] \cdot H(w_n) \cdot e^{jw_n t}$$

$$y(t) = \frac{1}{2\pi} \left\{ \lim_{w_0 \to 0} \sum_{n=-\infty}^{+\infty} [X(w_n) \cdot H(w_n) e^{jw_n t}] w_0 \right\}$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(w) e^{jwt} dw.$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(w) \cdot H(w) e^{jwt} dw.$$

$$\therefore Y(w) = \mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\} = X(w) \cdot H(w)$$

Method II) Using method of substitution

$$y(t) = x(t) * h(t).$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$$\therefore F\{y(t)\} = Y(\omega) = \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t}\, dt$$

$$Y(\omega) = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} x(\tau)\, h(\underbrace{t-\tau}_{})\, d\tau \right\} e^{-j\omega t}\, dt.$$

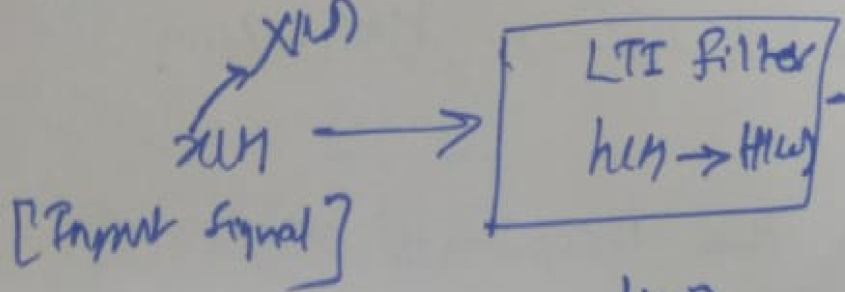
$$t - \tau = t_1$$

$$= \left\{ \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega \tau}\, d\tau \right\} \underbrace{\left\{ \int_{-\infty}^{+\infty} h(t_1)\, e^{-j\omega t_1}\, dt_1 \right\}}$$

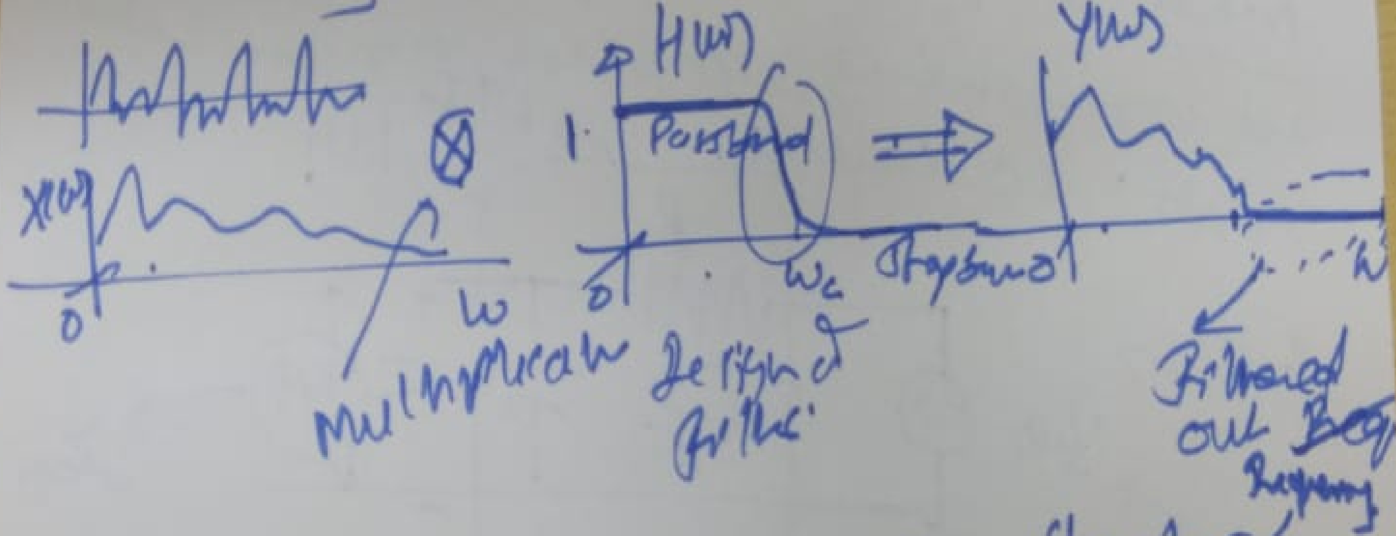
$$\boxed{Y(\omega) = X(\omega) \cdot H(\omega)}$$

Inference :→

To understand LTI filtering



$Y(\omega) = X(\omega) \cdot H(\omega)$



Chapter 2 ⟶ Chapter 4

$$y(n) = x(n) * h(n) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \xrightarrow{F(\cdot)} Y(\omega) = X(\omega) \cdot H(\omega)$$

No discussion of frequency or LTI filters

Convolution Theorem

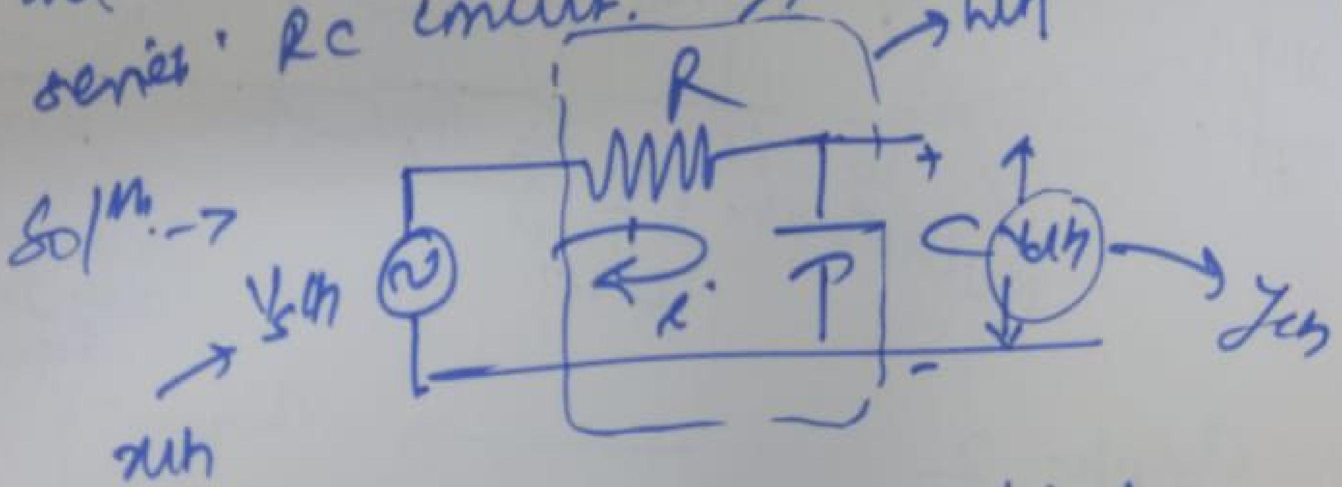
Discuss Filters

Key strength of CTFT: Convolution in time-domain is multiplication in frequency-domain

$$x(t) * h(t) \xleftrightarrow{F} X(\omega) \cdot H(\omega).$$

Problem ① Find the frequency response and corresponding impulse response of series RC circuit.

Sol^n →



$$\frac{dv_c(t)}{dt} + \left(\frac{1}{RC}\right) v_c(t) = \left(\frac{1}{RC}\right) v_s(t).$$

Take CTFT & use linearity,

$$F\left\{\frac{dv_c(t)}{dt}\right\} + \left(\frac{1}{RC}\right) F\{v_c(t)\} = \left(\frac{1}{RC}\right) F\{v_s(t)\}$$

$v_c(t)$ differentiation property

$$(j\omega) V_c(\omega) + \left(\frac{1}{RC}\right) V_c(\omega) = \left(\frac{1}{RC}\right) V_s(\omega).$$

$$\therefore \frac{V_c(\omega)}{V_s(\omega)} = \frac{(1/RC)}{(1/RC) + j\omega}$$

We have $RC \rightarrow$ LTI system
ckt.

$$\therefore v_c(t) = h(t) * \cancel{x(t)}\, v_s(t)$$

$$\therefore V_c(\omega) = H(\omega) \cdot \times V_s(\omega)$$

$$\therefore \frac{V_c(\omega)}{V_s(\omega)} = H(\omega)$$

$$\therefore \boxed{H(\omega) = \frac{1}{1+j\omega RC}}$$

frequency response of RC series circuit

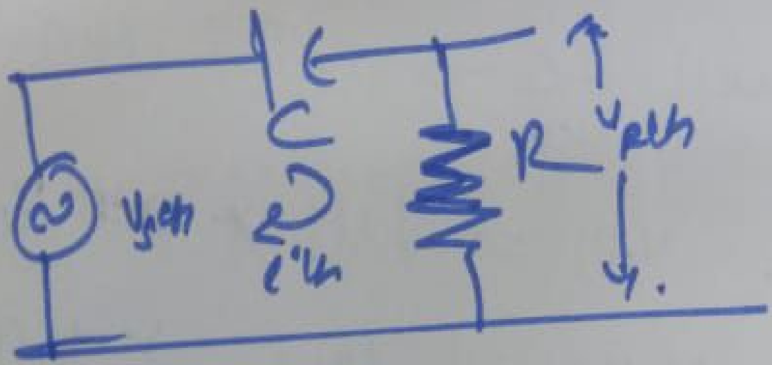
$$F^{-1}\{H(\omega)\} = h(t) = F^{-1}\left\{\frac{1}{1+j\omega RC}\right\} =$$

$$F\{e^{-at} \cdot u(t)\} = \frac{1}{a+j\omega}$$

$$\therefore F^{-1}\left\{\frac{(1/RC)}{1/RC + j\omega}\right\} = \boxed{(1/RC) \cdot e^{-t/RC} \cdot u(t) = h(t)}$$

(7)

Problem 2 Find the frequency response and corresponding impulse response of series RC highpass filter



Homework

Problem (3) For an LTI system shown below, find output $y(t)$?

$$x(t) = e^{-at}.u(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI} \\ h(t) = e^{-bt}u(t)\end{array}} \longrightarrow y(t) = ??$$

We have, $F\{x(t)\} = F\{e^{-at}.u(t)\} =$

$$X(\omega) = \frac{1}{a+j\omega}$$

similarly, $H(\omega) = \dfrac{1}{b+j\omega}$.

Using Convolution Theorem,

$$x(t) * h(t) \xleftrightarrow{F} \underbrace{X(\omega).H(\omega)}_{Y(\omega)}$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$Y(\omega) = \frac{1}{a+j\omega} \cdot \frac{1}{b+j\omega}$$

Using Method of Partial Fraction Expansion

CP

$$Y(\omega) = \left[\frac{1}{a+j\omega}\right] \cdot \left[\frac{1}{b+j\omega}\right] = \frac{A_1}{(a+j\omega)} + \frac{A_2}{b+j\omega}$$

$$1 = A_1(b+j\omega) + A_2(a+j\omega)$$

To find A_1, $a+j\omega = 0$

$$\cancel{1 = A_1(0) + A_2(}$$

$$1 = A_1(b+j\omega) + A_2(0).$$

$$A_1 = \frac{1}{b-a}, \quad A_2 = \frac{1}{a-b}$$

$$Y(\omega) = \left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]\frac{1}{b-a}$$

$$y(t) = F^{-1}\{Y(\omega)\} = \left(\frac{1}{b-a}\right)\left[F^{-1}\left\{\frac{1}{a+j\omega}\right\} - F^{-1}\left\{\frac{1}{b+j\omega}\right\}\right]$$

$$\boxed{y(t) = \left(\frac{1}{b-a}\right)\left[e^{-at} - e^{-bt}\right] u(t)}$$

Ans. Q.1