Tutorial class on DTFT

Lecture 36

Properties of DTFT :→

① Notation:

$$X(e^{j\omega}) = F\{x(n)\}$$

$$x(n) = F^{-1}\{X(e^{j\omega})\}$$

$$x(n) \overset{F}{\longleftrightarrow} X(e^{j\omega})$$

① Periodicity of DTFT → DTFT is periodic in frequency domain with period 2π

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

whereas CTFT is aperiodic in frequency domain

Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$

$\omega \to \omega + 2\pi$

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j(\omega+2\pi)\cdot n}$$

$$= \sum_{n=-\infty}^{+\infty} (x(n) \cdot e^{-j\omega n}) \times (e^{-j2\pi n})$$

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega})} \to \text{Proved}$$

①

② Linearity of DTFT

$x_1(n) \xleftrightarrow{F} X_1(e^{j\omega})$,

$x_2(n) \xleftrightarrow{F} X_2(e^{j\omega})$.

$\therefore \underline{a_1 x_1(n) + a_2 x_2(n)} \xleftrightarrow{F} a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$.

③ Time Shifting and Frequency Shifting.

eg: $x(n) \xleftrightarrow{F} \underline{X(e^{j\omega})}$

a) $\underline{x(n-n_0)} \xleftrightarrow{F} e^{-j\omega n_0} \cdot X(e^{j\omega})$

Soln: $\rightarrow F\{x(n-n_0)\} = \sum_{n=-\infty}^{+\infty} [x(n-n_0)] e^{-j\omega n}$

b) $e^{j\omega_0 n} \cdot \cancel{x(n)} \xrightarrow{F} X(e^{j(\omega-\omega_0)})$.

Proof: $\rightarrow F\{e^{j\omega_0 n} \cdot x(n)\} = \sum_{n=-\infty}^{+\infty} [e^{j\omega_0 n} \cdot x(n)] \cdot e^{-j\omega n}$

②

$$= \underbrace{\sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-j(\omega-\omega_0)n}}$$

$$= X(e^{j(\omega-\omega_0)})$$. Hence proved.

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④ Conjugation and Symmetry.

Let $x[n] \xleftrightarrow{F} X(e^{j\omega})$.

then $x^*[n] \xleftrightarrow{F} ?? \; X^*(e^{-j\omega})$.

If $x[n]$ is real-valued

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{ real}]$$

Proof: $\rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

$$X^*(e^{j\omega}) = \left(\sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}\right)^*$$

$$X^*(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x^*[n] \cdot e^{j\omega n}$$

Replace $\omega \rightarrow -\omega$ ③

$$\therefore X^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x^*(n)\, e^{-j\omega n}$$

$$= \text{DTFT}\{x^*(n)\}$$

$$x^*(n) \xleftrightarrow{F} X^*(e^{-j\omega}).$$

If $x(n) = \text{real}$, $x^*(n) = x(n)$.

$$\therefore X^*(e^{-j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$\boxed{X^*(e^{-j\omega}) = X(e^{j\omega})} \quad (x(n) = \text{real}).$$

$X(e^{j\omega})$ = complex quantity.

$$X(e^{j\omega}) = X_R(e^{j\omega}) + j\, X_I(e^{j\omega}) \quad \text{— (A)}$$

$$X^*(e^{j\omega}) = X_R(e^{j\omega}) - j\, X_I(e^{j\omega})$$

$$\omega \rightarrow -\omega$$

$$X^*(e^{-j\omega}) = X_R(e^{-j\omega}) - j X_I(e^{-j\omega}) \quad \text{(B)}$$

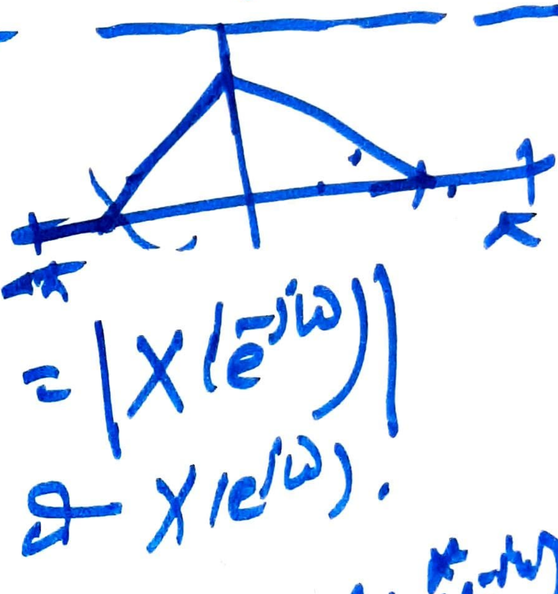
(x(n) = real), $X(e^{j\omega}) = X^*(e^{-j\omega})$

(A) & (B), we get

$$\boxed{X_R(e^{j\omega}) = X_R(e^{-j\omega})} \quad \& \quad \boxed{X_I(e^{j\omega}) = -X_I(e^{-j\omega})}$$

$\Downarrow$ Re$\{X(e^{j\omega})\}$, i.e., $X_R(e^{j\omega})$ is an even function

$\Downarrow$ Im$\{X(e^{j\omega})\}$, i.e., $X_I(e^{j\omega})$ is an odd function.



In addition,

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$\Rightarrow |X(e^{j\omega})| = |X^*(e^{-j\omega})| = |X(e^{-j\omega})|$$

$\Rightarrow$ Magnitude spectrum of $X(e^{j\omega})$ is even.

$$\Rightarrow |X(e^{j\omega})| \cdot e^{j\angle X(e^{j\omega})} = |X^*(e^{-j\omega})| \, e^{j\angle X^*(e^{-j\omega})}$$

$$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega}).$$

Phase spectrum is an odd function. (5)

$$x(n) = \underline{Ev\{x(n)\}} + Od\{x(n)\}.$$

$$\therefore \; Ev\{x(n)\} \xleftrightarrow{FT} Re\{X(e^{j\omega})\}$$

$$Od\{x(n)\} \xleftrightarrow{FT} j\,Im\{X(e^{j\omega})\}.$$

Application of conjugation and symmetry property :

Development of Discrete-Time Hilbert Transform
(Case II)

<u>Motivation</u>

Generation of <u>analytic</u> <u>signal</u> :→

Let $y(n)$ = analytic signal.

Fourier transform of $y(n)$ is zero for negative frequency.

$Y(e^{j\omega}) = 0$ for $\omega \in [-\pi, 0]$

→ Analytic signal, $y[n]$, is causal in frequency-domain.

∵ Spectrum of $y[n]$ is one-sided using conjugate and symmetric property, $y[n]$ has to be complex.

Let $y[n] = x_R[n] + j\, x_I[n]$

$= \tilde{x}[n] + j\, \hat{x}[n]$

where $\hat{x}[n] = Im\{y[n]\}$

$x[n] = Re\{y[n]\}$

$F\{y[n]\} = F\{x[n]\} + j\, F\{\hat{x}[n]\}$ [Linearity]

$\downarrow$

$Y(e^{j\omega}) = X(e^{j\omega}) + j\, \hat{X}(e^{j\omega})$. —— [A]

$(u+v)^* = u^* + v^*$

$\therefore Y^*(e^{j\omega}) = X^*(e^{j\omega}) - j\, \hat{X}^*(e^{j\omega})$.

$\omega \to -\omega$.

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$\therefore Y^*(e^{-j\omega}) = X^*(e^{-j\omega}) - j\hat{X}^*(e^{-j\omega})$

$\because x[n]$ is real

$\Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

$Y^*(e^{-j\omega}) = X(e^{j\omega}) - j\hat{X}(e^{+j\omega})$ — B

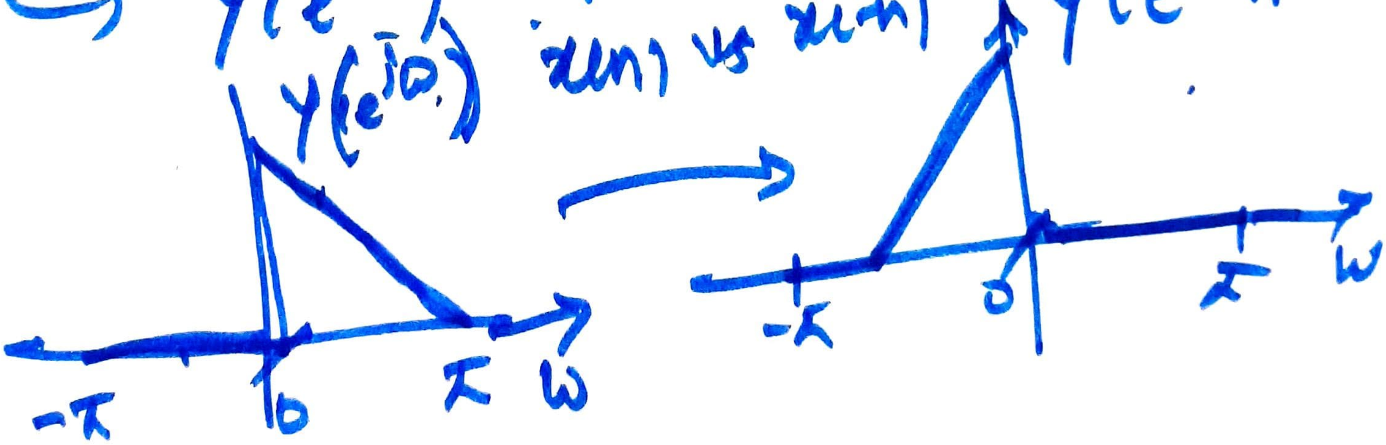
$Y(e^{j\omega}) = X(e^{j\omega}) + j\hat{X}(e^{j\omega})$ — A

A + B

$\hookrightarrow Y(e^{j\omega}) + Y^*(e^{-j\omega}) = \underline{2X(e^{j\omega})}$

A − B

$\hookrightarrow Y(e^{j\omega}) - Y^*(e^{-j\omega}) = 2j\hat{X}(e^{j\omega})$



Case I) $\omega \in [0, \pi]$, $Y^*(e^{j\omega}) = 0$

$$2X(e^{j\omega}) = 2j\hat{X}(e^{j\omega})$$

$$\therefore \hat{X}(e^{j\omega}) = -jX(e^{j\omega}) \quad \text{— (A1)}$$

Case II $\omega \in [-\pi, 0]$, $Y(e^{j\omega}) = 0$

$$\therefore 2X(e^{j\omega}) = -2j\hat{X}(e^{j\omega})$$

$$\boxed{\hat{X}(e^{j\omega}) = jX(e^{j\omega})} \quad \text{— (B1)}$$

$$\hat{X}(e^{j\omega}) = \begin{cases} -jX(e^{j\omega}), & \omega \in [0, \pi] \\ +jX(e^{j\omega}) & \omega \in [-\pi, 0] \end{cases}$$

LTI

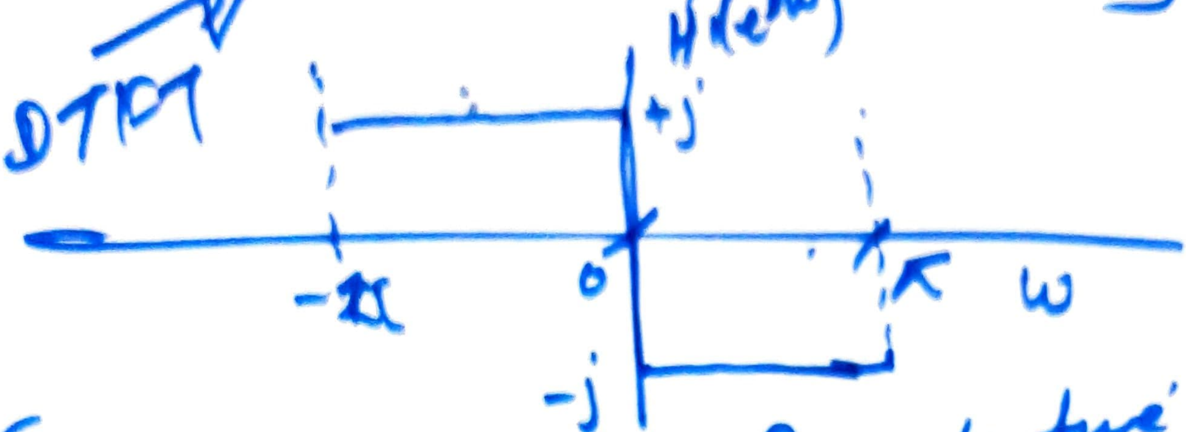
$$x(n) \longrightarrow \boxed{\begin{array}{c}\text{Hilbert Transform} \\ h(n) \rightarrow H(e^{j\omega})\end{array}} \longrightarrow \hat{x}(n) = \text{Hilbert}\{x(n)\}.$$

$$\hat{x}(n) = x(n) * h(n)$$

$$\hat{X}(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}).$$

$$H(e^{j\omega}) = \begin{cases} -j, & \omega \in [0, \pi] \\ +j, & \omega \in [-\pi, 0] \end{cases}$$

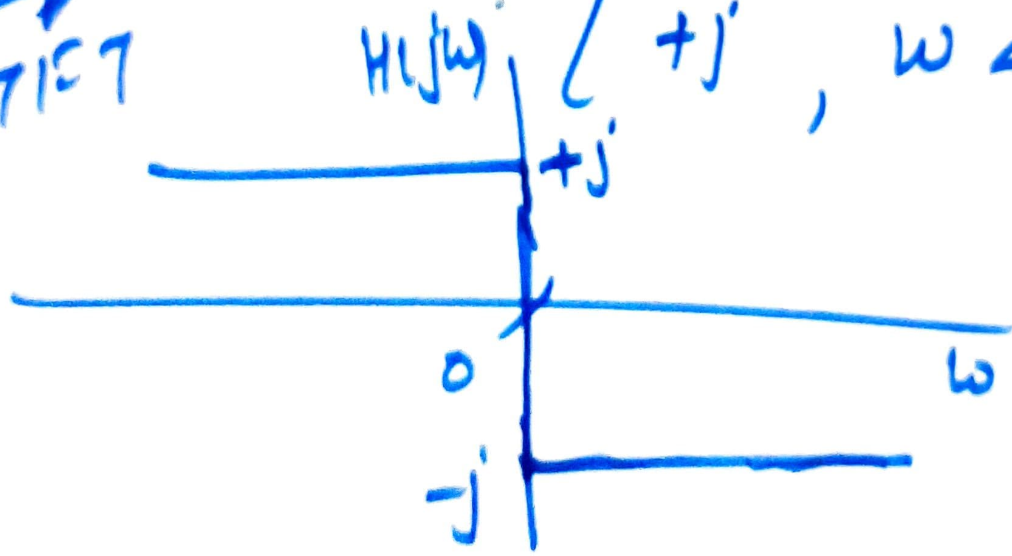
DTFT



Frequency Response of Discrete-time Hilbert Transformer

$$H(\omega) = H(j\omega) = \begin{cases} -j, & \omega > 0 \\ +j, & \omega < 0 \end{cases}$$

CTFT



Problem: Find the impulse response of discrete-time Hilbert transformer.

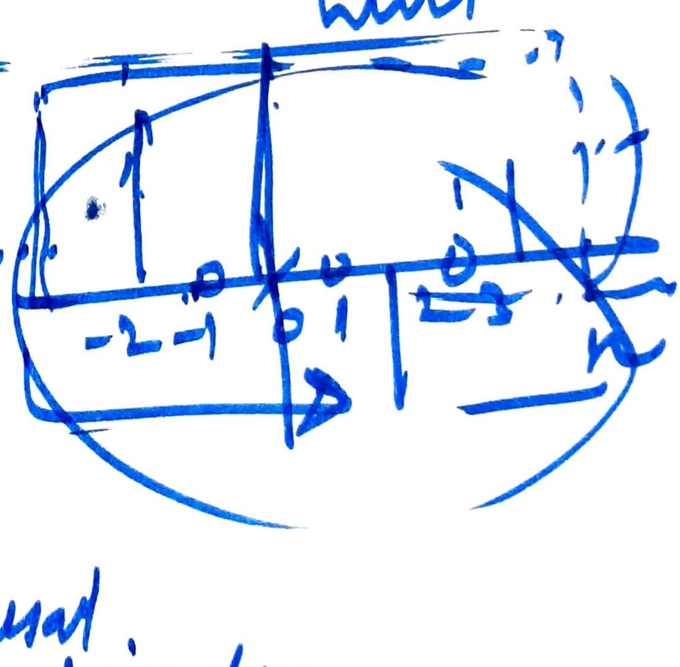
Sol^n: → $h[n] = F^{-1}\{H(e^{j\omega})\}$.

$$= \frac{1}{2\pi}\int_{\omega=-\pi}^{\omega=+\pi} H(e^{j\omega})\cdot e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\left\{\int_{-\pi}^{0} H(e^{j\omega})\cdot e^{j\omega n}\, d\omega + \int_{0}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{-\pi}^{0} (j)\, e^{j\omega n}\, d\omega + \int_{0}^{\pi} (-j)\, e^{j\omega n}\, d\omega\right\}$$

$h[n]$



$$h[n] = \frac{1 - \cos(\pi n)}{\pi n}$$

1) $h[n]$ is IIR.

2) $h[n]$ is non-causal

3) In practice, we [illegible] windowing & finite filters

Property of DTFT.

(5). Differencing and Accumulation

$$x[n] - x[n-1] \xleftrightarrow{F(\cdot)} [1-e^{-j\omega}]\cdot X(e^{j\omega}).$$

Proof. $y[n] = x[n] - x[n-1]$.

$$F\{y[n]\} = F\{x[n]\} - F\{x[n-1]\}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega}) = [1-e^{-j\omega}]\cdot X(e^{j\omega})$$

Inference

$$\frac{d}{dt}[x(t)] \xleftrightarrow{CTFT} (j\omega)\, X(\omega)$$

$$j\omega \Longleftrightarrow (1-e^{-j\omega})$$

Highpass filter

CTFT

DTFT.

$e^{-j\omega/2} \times \sin(\omega/2)$

Highpass

$$\sum_{m=-\infty}^{n} x(m) \xleftrightarrow{F.T.} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega-2\pi k)$$

Accumulation

DTFT is periodic

Let $y(n) = \sum_{m=-\infty}^{n} x(m)$

$\therefore x(n) = y(n) - y(n-1)$

$\therefore X(e^{j\omega}) = Y(e^{j\omega}) [1-e^{-j\omega}]$

$\therefore Y(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} X(e^{j\omega})$

$$\int_{\tau=-\infty}^{t} f(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(\omega) + \pi X(0)\cdot\delta(\omega)$$

Property 6 Differentiation in Frequency do..

If $x(n) \xleftrightarrow{F} X(e^{j\omega})$.

$$(-jn)\, x(n) \longleftrightarrow \frac{d}{d\omega}\left[X(e^{j\omega})\right]$$

Proof:- $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$

Diffn. w.r.t 'ω'

$$\frac{d}{d\omega} X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, \frac{d}{d\omega}\left(e^{-j\omega n}\right)$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, (-jn)\, e^{-j\omega n}$$

$$\boxed{\frac{d}{d\omega} X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left[(-jn)\, x(n)\right] \cdot e^{-j\omega n}}$$

...perty (7) Time Reversal.

If $x(n) \xleftrightarrow{D} X(e^{j\omega})$

$x(-n) \xleftrightarrow{F} ?\ X(e^{-j\omega})$.

Proof: Let $y(n) = x(-n)$

$Y(e^{j\omega}) = \sum_{n=-\infty} x(-n)\, e^{-j\omega n}$

$-n = m$

$\boxed{Y(e^{j\omega}) = X(e^{-j\omega})}$

$F\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$a = -1$

$\therefore F\{x(-t)\} = \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right)$

$= X(-\omega)$

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Property 8) Convolution Theorem

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n] \xleftrightarrow{F} X(e^{j\omega}) \cdot H(e^{j\omega})$$

Proof: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$

$$\therefore F\{y[n]\} = \sum_{n=-\infty}^{+\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \left\{ \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k] \right\} e^{-j\omega n}$$

$$= \sum_{n} \underbrace{\left\{ \sum_{k} x[k] \cdot e^{-j\omega k} \right\}}_{X(e^{j\omega})} \underline{h[n-k]\, e^{+j\omega k}\, e^{-j\omega n}}$$

$$= X(e^{j\omega}) \sum_{k=-\infty}^{+\infty} h[n-k]\, e^{-j(n-k)\omega} \quad \cdot H(e^{j\omega})$$

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$$\therefore \boxed{Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})}$$

$$\therefore \boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}}$$

Frequency response of an LTI system in DTFT-domain.

Systems Characterised by Linear Constant Coefficient Difference equation

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(n)\end{matrix}} \longrightarrow y(n)$$

LCCDE

$$F\left\{ \sum_{k=0}^{N} a_k \cdot y(n-k) \right\} = F\left\{ \sum_{k=0}^{M} b_k \, x(n-k) \right\}$$

Linearity & time-shifting

$$\sum_{k=0}^{N} a_k \, F\{y(n-k)\} = \sum_{k=0}^{M} b_k \, F\{x(n-k)\}$$

$$\sum_{k=0}^{N} a_k \cdot e^{-j\omega k} \, Y(e^{j\omega}) = \sum_{k=0}^{M} b_k \, e^{-j\omega k} \, X(e^{j\omega})$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k \, e^{-j\omega k}}{\sum_{k=0}^{N} a_k \, e^{-j\omega k}} = \underline{\underline{H(e^{j\omega})}}$$

Multiplication Property

Duality Property. → Homework

Time Expansion