SC223 - Linear Algebra

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Lecture 31

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Summary of Lecture 30

- **Definition:** Let $T \in \mathcal{L}(U)$. If V is a subspace of U such that $\forall u \in V, Tu \in V$, then V is said to be an *invariant subspace* of T, or *T-invariant* subspace.
- If $U = V_1 \oplus V_2$, where V_1, V_2 are $T-$invariant, then with $B = B_{V_1} \cup B_{V_2}$ as the basis of U, $[T]_B^B$ becomes block-diagonal.
- If V is a 1-dimensional $T-$ invariant subspace, $\forall u \in V, u \neq \theta, Tu = \lambda u$. λ is the eigenvalue associated with eigenvector u.

$V = \{ku \mid \forall k \in \mathbb{F}\}, u \neq \theta.$

$\forall x \neq \theta, x \in V, \quad x = a_1 \cdot u, a_1 \neq 0.$

$Tx = a_2 \cdot u$

$T(k \cdot u) = \lambda (k \cdot u) \quad T(a_1 \cdot u) = a_2 \cdot u$

eigenvectors: (eigenspace). $\Rightarrow Tu = \frac{a_2}{a_1} u = \lambda u.$

1 dimensional Invariant Subspaces

- If $U = \oplus_{i=1}^{n} V_i$, where V_i are 1-dimensional T-invariant subspaces, then with basis $\boxed{e = \{u_i, i = 1, \ldots, n \mid u_i \in V_i, u_i \neq \theta\}}$, $[T]_e^e = diag(\lambda_1, \ldots, \lambda_n)$.

$\dim(U) = n$.

$V_1, \ldots, V_n$.

$\downarrow \qquad \downarrow$

$u_1 \qquad u_n$

$\neq \theta \qquad \neq \theta$

$e = \{u_1, \ldots, u_n\}$.

$u_i \in V_i$

$Tu_i = \lambda_i u_i$

$$[T]_e^e = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & \lambda_n \end{bmatrix}_{n \times n}$$

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- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is not invertible, and $det([T]_\beta^\beta - \lambda I_n) = 0$.

$\exists u \neq \theta$, $(T - \lambda I)u = \theta$

$[T - \lambda I]_B^B [u]_B = \vec{0} = [\theta]_B$

$Ax_1 = y$

$Ax_2 = y$

$A(x_1 - x_2) = \vec{0}$

$det([T - \lambda I]_B^B) = 0$

$\boxed{det([T]_B^B - \lambda I_n) = 0}$

1 dimensional Invariant Subspaces

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- Note that if $Tu = \lambda u$, then $(T - \lambda I)u = \theta$.
- Thus $(T - \lambda I)$ is not invertible, and $det([T]_\beta^\beta - \lambda I_n) = 0$.
- **Characteristic polynomial** of T: $c(\lambda) = det([T]_\beta^\beta - \lambda I_n)$.

Eigenvalues of T are roots of the polynomial $c(\lambda)$, and the eigenvector is a non-zero vector belonging to $\mathcal{N}(T - \lambda I)$.

$$c(\lambda) = det(\lambda I_n - [T]_\beta^\beta)$$

$$c_1(\lambda) = det([T]_\beta^\beta - \lambda I_n)$$

$$c_2(\lambda) = det([T]_\alpha^\alpha - \lambda I_n).$$

$$det(ABC) = det(CAB) = det(BCA)$$

$$\det([T]_\alpha^\alpha - \lambda I_n) = \det(S^{-1}[T]_\beta^\beta S - \lambda I_n)$$

$$= \det(S^{-1}[T]_\beta^\beta - \lambda S^{-1} I_n S)$$

$$= \det(S^{-1}([T]_\beta^\beta - \lambda I_n) S)$$

$$= \det(S \cdot S^{-1}([T]_\beta^\beta - \lambda I_n))$$

$$= \det([T]_\beta^\beta - \lambda I_n)$$

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$$[T]_\beta^\beta \begin{bmatrix} | & | & & | \\ [v_1]_\beta & [v_2]_\beta & \cdots & [v_n]_\beta \\ | & | & & | \end{bmatrix} = ?$$

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$$= \underbrace{\begin{bmatrix} | & & | \\ [v_1]_\beta & \cdots & [v_n]_\beta \\ | & & | \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{n\times n}$$

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$$[T]_\beta^\beta E = E\Lambda \Rightarrow \Lambda = E^{-1}[T]_\beta^\beta E$$

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$$[T]_\beta^\beta E = E\Lambda \Rightarrow \Lambda = [T]_e^e = E^{-1}[T]_\beta^\beta E$$

- Eigenvalues of T are roots of $det([T]_\beta^\beta - \lambda\ I_n)$, and corresponding eigenvectors are linearly independent vectors in $\mathcal{N}(T - \lambda\ I)$.

END OF CLASS.

Examples

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

Results on Eigenvectors and Eigenvalues

- **Proposition 20**: Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T \in \mathcal{L}(U)$. Then the eigenvectors $v_1, \ldots, v_m$ associated with these eigenvalues are linearly independent.