Computational Numerical Methods

CS 374

Prosenjit Kundu

Error in the kth iteration $x^{(k)}$ compared to the exact solution.

$$e^{(k)} = x - x^{(k)}$$

$$x^{(k+1)} = B x^{(k)} + C.$$

$$e^{k+1} = x - x^{k+1}$$

$$= Bx + C - \cancel{B}\, Bx^{(k)} - C.$$

$$= B(x - x^{k}) = B.e^{(k)}.$$

$$e^{(k+1)} = B^{k+1}(e^{0})$$

$$x = Bx + C$$

$x \cancel{0} =$

$$ax + by = c$$

$$cx + dy = e.$$

$$x = \frac{1}{a}(c - by)$$

$$y = \frac{1}{d}(e - cx)$$

$$Ax = b.$$

$$(D - C)x = b$$

$$Dx - Cx = b.$$

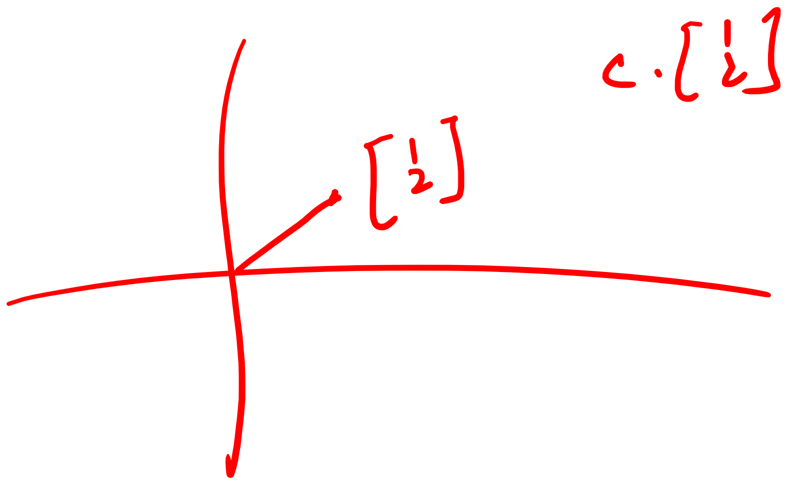
$$Dx = Cx + b.$$

$$x = D^{-1}Cx + D^{-1}b$$

Vector norm

$\|x - y\|$

A vector norm on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ having the following properties.



- $\|x\| \geq 0 \quad \forall\, x \in \mathbb{R}^n$
- $\|x\| = 0$ iff $x = 0$
- $\|\alpha x\| = |\alpha|\,\|x\|$ where $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y, \in \mathbb{R}^n$.

Eg. Euclidean norm

$$\|x\|_2 = \sqrt{\sum_i |x_i|^2}.$$

- L_∞ norm. also called maximum norm.

$$\|x\|_\infty = \max\{|x_1|, |x_2| \ldots |x_n|\}.$$

l_1 norm. on R^n

$$\|x\|_1 = |x_1| + |x_2| \cdots + |x_n|.$$

Matrix norm

$$\|A\| = \sqrt{\sum_i \sum_j |a_{ij}|^2}.$$

$$\|A\| = \max \{ |a_{ij}| ,\ 1 \le i, j \le n \}$$

$$\|A\| = \sum_i \sum_j |a_{ij}|$$

Important norms on matrices.

For a $n \times n$ matrix A with real entries.

Let $\lambda_1, \lambda_2 \ldots \lambda_n$ be the eigen values.

of $\underline{\underline{A^T A}}$

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} (\lambda_i)}$$

Matrix norm subordinate to a vector.

Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$.

The matrix norm subordinate to $\|\cdot\|$ is defined by

$$\|A\| = \sup \{ \|Ax\| : x \in \mathbb{R}^n \; \|x\| = 1 \}$$

Lemma

For any $A \in M_n(\mathbb{R})$ and a given vector norm $\|\cdot\|$ we have

$$\|A\| = \max_{z \neq 0} \frac{\|Az\|}{\|z\|} \qquad z \in \mathbb{R}^n$$

$$e^{(k+1)} = B^{k+1} e^{(0)}.$$

$$0 \le \underline{\| e^{(k+1)} \|} = \| B^{k+1} e^{(0)} \|$$

$$\le \underline{\| B^{k+1} \|} \; \| e^{(0)} \| \neq 0 \text{ for diff choicy.}$$

$$= \|B\|^{k+1}$$

for convergence $\underline{\underline{\| B^{k+1} \| \to 0.}}$

is possible only when $\|B\| < 1$.

Diagonally dominant

A matrix is said to be diagonally dominant

if $$\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| < |a_{ii}|$$

Th If the coefficient matrix A is diagonally dominant then, the Jacobi method

$$x^{(k+1)} = Bx^{(k)} + c \qquad k = 0, 1, 2 \ldots$$

converges.

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\, x_j^{(k)}\right)$$

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}\, x_j\right)$$

The iteration

$$e^{(k+1)} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_{ij}}{a_{ii}} \left(x_j^{(k)} - x_j \right)$$