6.5 First-order and Second order Continuous-time Systems

Lecture 39

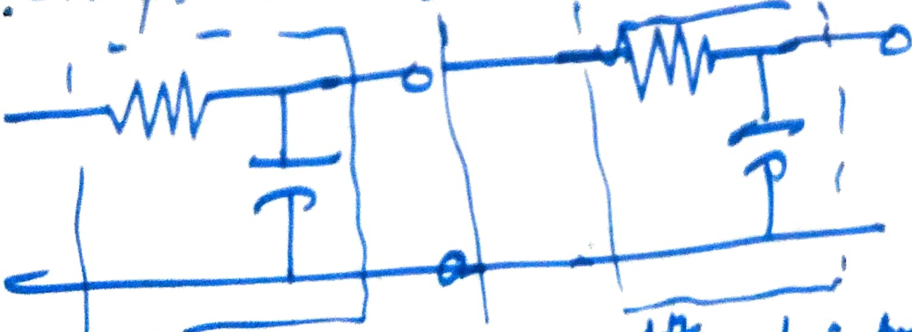
Why 1st and 2nd order?

⇒ Most of the higher order systems (including in control systems) can be modeled in terms of 1st and 2nd order primitives.

Case I) First order systems

Order of system = Order Differential Equation describing dynamics on underlying system.

A. Simple RC series circuit:



1st order system

1st order system.

Fig. 2nd order as a cascade of two 1st order system

In chapter 1, Differential Equation describes dynamics of RC ckt.

$$(RC)\frac{dy(t)}{dt} + y(t) = x(t)$$

Let $RC = \tau$

$\therefore$ Generalised mathematical model for first-order system is

$$\boxed{\tau \cdot \frac{dy(t)}{dt} + y(t) = x(t)}$$

$\boxed{\text{Chapter 4}}$ $\Rightarrow$ Frequency response of RC ckt is given by

$$\tau\, F\left\{\frac{dy(t)}{dt}\right\} + F\{y(t)\} = F\{x(t)\}$$

Convolution theorem

$$\therefore \boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1+j\omega\tau}}$$

$$\therefore h(t) = F^{-1}\{H(\omega)\} = F^{-1}\left\{\frac{1}{1+j\omega\tau}\right\}$$

$$h(t) = \left(\frac{1}{\tau}\right) F^{-1}\left\{\frac{1}{j\omega + 1/\tau}\right\}$$

$$\boxed{h(t) = \left(\frac{1}{\tau}\right) \cdot e^{-t/\tau} \cdot u(t)}$$

$$\tau h(t) = e^{-t/\tau} u(t) \longrightarrow f(t/\tau)$$

$$H(\omega) = \frac{1}{1+j(\omega\tau)} \longrightarrow f(\omega\tau)$$

⇒ inverse relationship betn time & frequency domain → Heisenberg's uncertainty.

∴ step response of first order system is given by

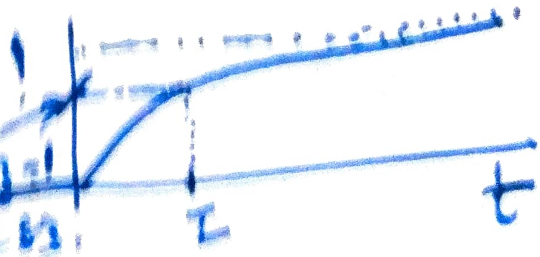
$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \int_{-\infty}^{t} \left[\frac{1}{\tau} e^{-t/\tau} \cdot u(t)\right] dt$$

$$\boxed{s(t) = \left[1 - e^{-t/\tau}\right] u(t)}$$

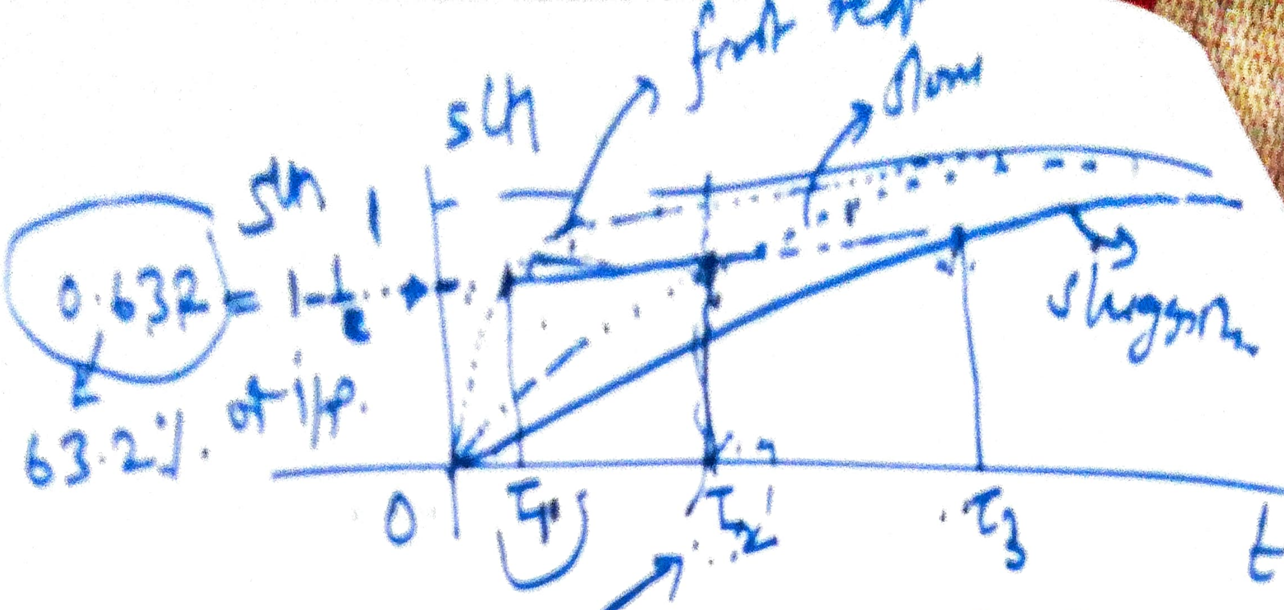
At $t = \tau$,

$$s(t)\Big|_{t=\tau} = \left[1 - e^{-\tau/\tau}\right] \cdot u(\tau)$$

$$= \left[1 - \frac{1}{e}\right] \cdot (u(\tau))$$



$$s(\tau) = \frac{e-1}{e} = \frac{2.718281\ldots - 1}{2.718281\ldots} \approx 0.632$$



Time constant of the first order system.

→ Time constant (τ) indicates speed of response (fast, slow, sluggish..

$$\tau_1 < \tau_2 < \tau_3$$

$$H(\omega) = \frac{1}{1+j(\omega\tau)}$$

$$|H(\omega)|^2 = \frac{|Y(\omega)|^2}{|X(\omega)|^2} = \frac{|1|^2}{|1+j\omega\tau|^2} = \frac{1}{1+\omega^2\tau^2}$$

$$10\log_{10}[|H(\omega)|^2] = 10\log_{10}\left(\frac{1}{1+\omega^2\tau^2}\right)$$

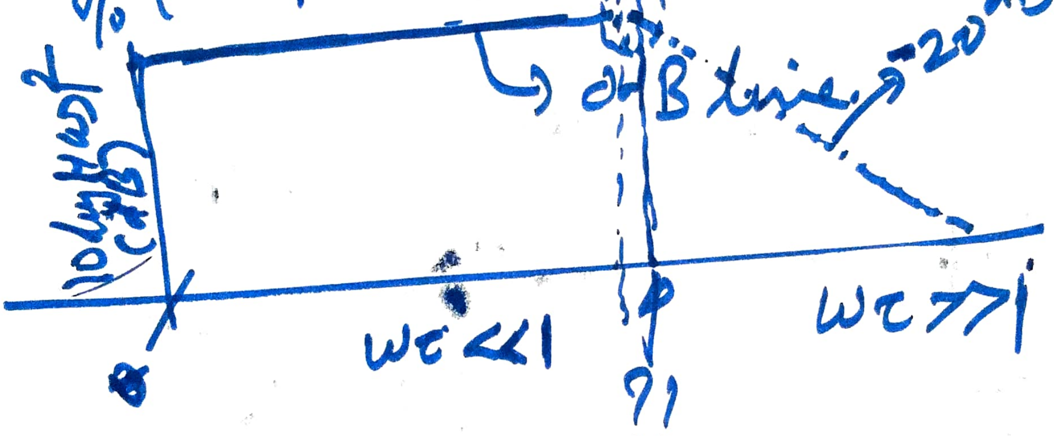
$$= \underbrace{10\log_{10}(1)}_{\to 0} - 10\log_{10}(1+\omega^2\tau^2).$$

$$10\log_{10}|H(\omega)|^2 = -10\log_{10}\left[1+(\omega\tau)^2\right]$$

Case I) $\omega\tau << 1 \Rightarrow (\omega\tau)^2 <<< 1$

$$\therefore 1+(\omega\tau)^2 \approx \cancel{(\omega\tau)^2}\ 1$$

$$10\log_{10}|H(\omega)|^2 \simeq 0\ \text{dB}$$



Case II) $\omega\tau >> 1$

$$1+(\omega\tau)^2 \approx (\omega\tau)^2$$

$$\therefore \underbrace{10\log_{10}|H(\omega)|^2}_{(\text{dB})} \simeq -10\log(\omega\tau)^2$$

$$\simeq -20\log_{10}(\omega) - 20\log(\tau)$$

$$y = mx + c$$

where $m = -20$ dB / decade

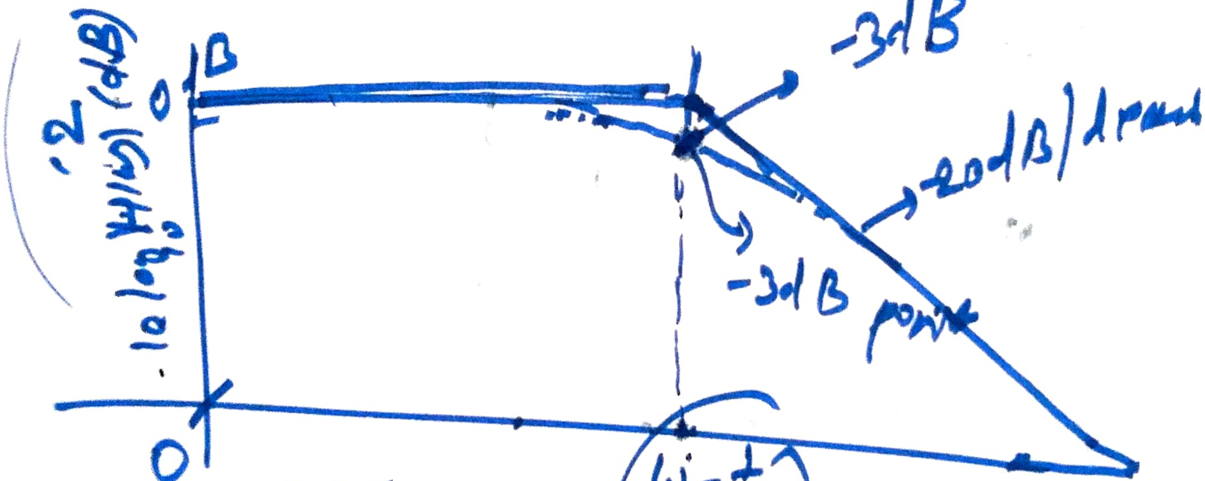
Let $\omega = \frac{1}{\tau}$

$$\therefore 10 \log_{10} |H(\omega)|^2 = -10 \log_{10} \left[1 + (\omega\tau)^2\right]$$

$$= -10 \log_{10} \left[1 + \left(\frac{1}{\tau} \times \tau\right)^2\right]$$

$$= -10 \log_{10} (2)$$

$$\doteq -3\,dB$$



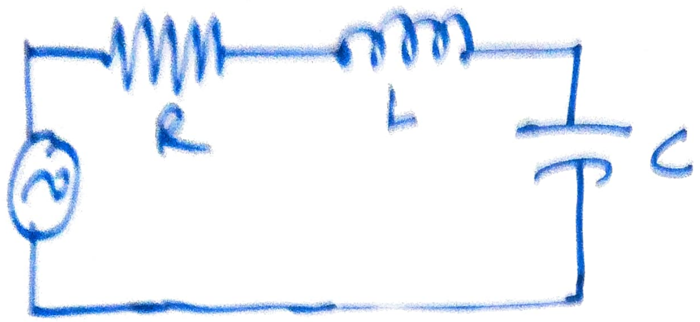
Bode plot for first order system $\rightarrow$ Break frequency. Magnitude

$\rightarrow$ Study phase plot $\rightarrow$ Home work.

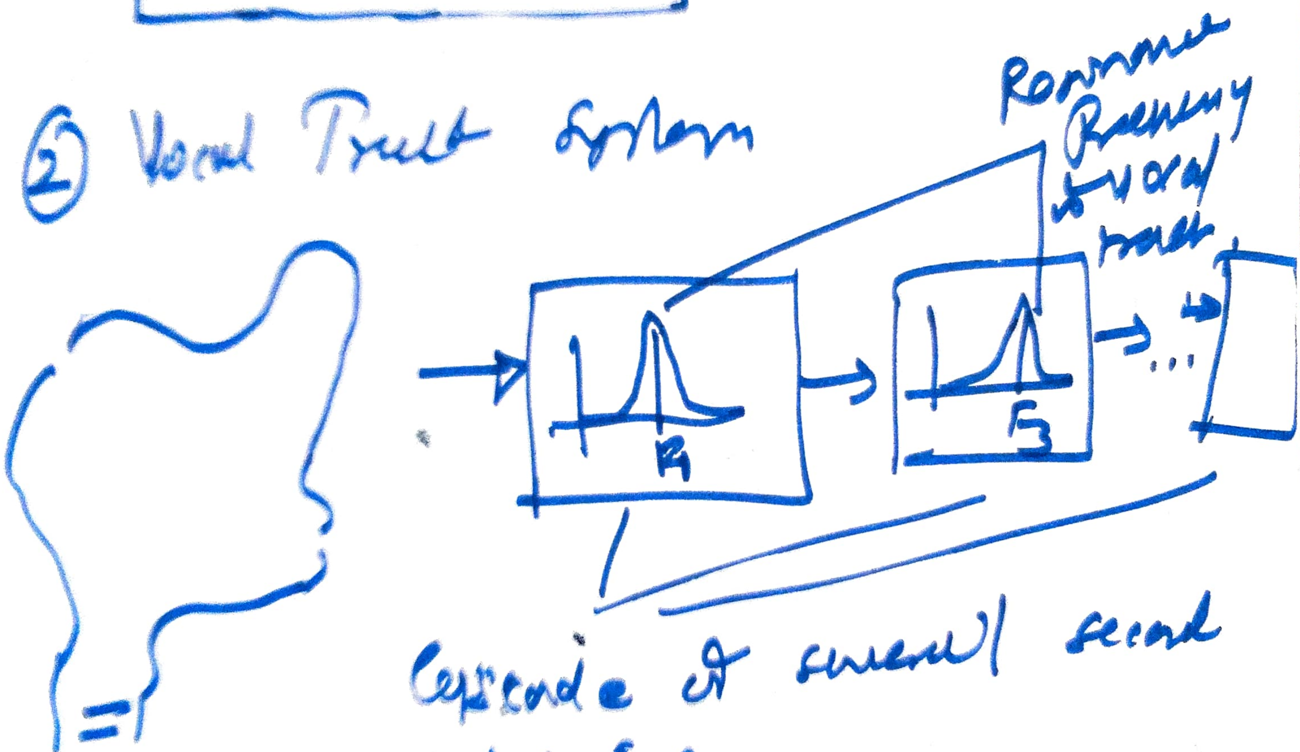
Case II Second order systems.

Examples :→

① Series RLC circuit.



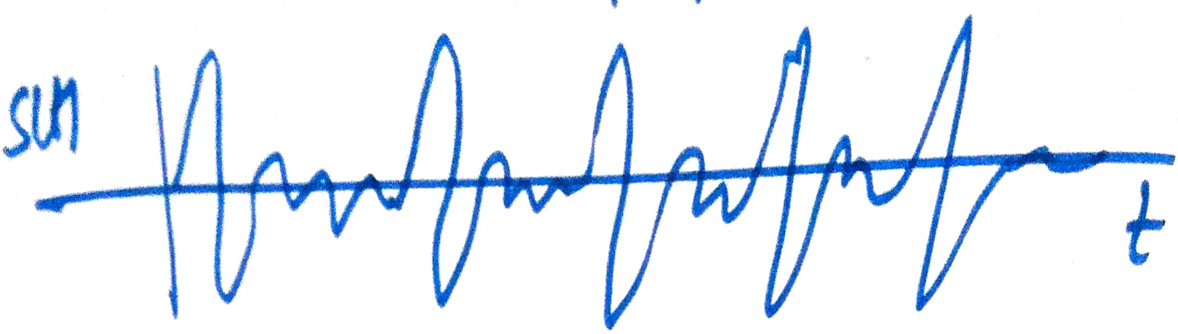
② Vocal Tract system



Cascade of several second order sys

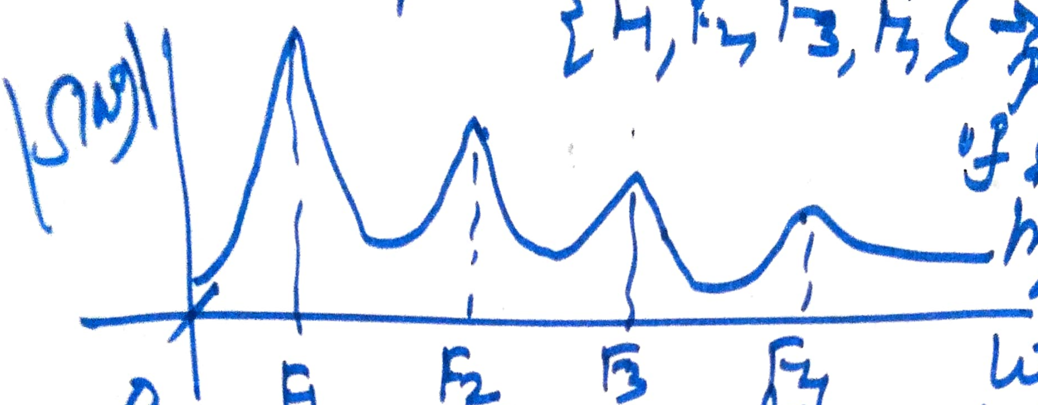
Manfred Schroeder (Bell Labs): Human beings produce and perceive sounds by spectral peaks than spectral valley.

Vowel /a/



Human speech

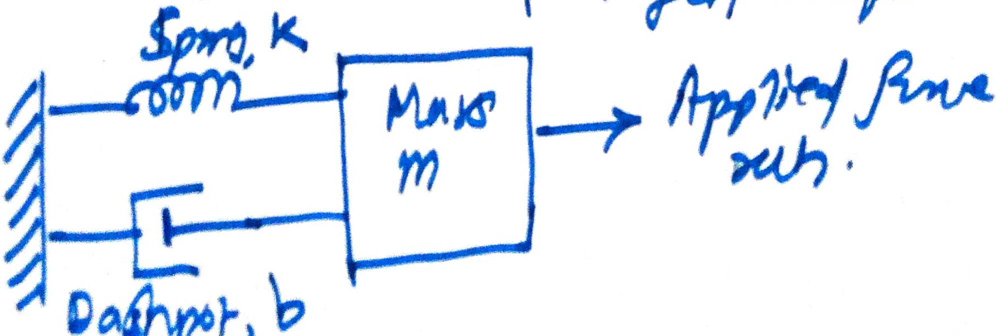
$\{F_1, F_2, F_3, F_4\}$ → Resonant frequencies of vocal tract



Magnitude spectrum of speech segment

→ Each of F_i's → corresponds to a 2nd order system.

(3) Mass-spring-damper system → gets x(t) displacement.



Ideal SHM → (No dashpot), $b=0$.

$\vec{F} = m \cdot a$

(Net force)

$$m \frac{d^2 y(t)}{dt^2} = x(t) - K \cdot y(t) - b \frac{dy(t)}{dt}.$$

$$\boxed{\frac{d^2 y(t)}{dt^2} + (b/m) \frac{dy(t)}{dt} + (K/m) \cdot y(t) = (1/m) \cdot x(t)} \rightarrow [A]$$

2nd order Differential Equation.

$$\boxed{\frac{d^2 y(t)}{dt^2} + 2\xi \cdot \omega_n \frac{dy(t)}{dt} + \omega_n^2 \cdot y(t) = \omega_n^2 \cdot x(t)} \rightarrow [B]$$

$$2\xi \omega_n = b/m.$$

$$\omega_n^2 = K/m \rightarrow \omega_n = \sqrt{K/m}$$

$$\therefore 2\xi \left(\sqrt{K/m}\right) = b/m$$

$$\therefore \xi = \frac{b}{2\sqrt{K \cdot m}} = \text{damping ratio} = f(b, K, m)$$

Objective is to find Bode Plot. For 2^{nd} order system

→ Use CTFT, convolution here

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2 \Rightarrow ax^2 + bx + c$$

where $x = j\omega$.

$$c_1, c_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2\xi\omega_n \pm \sqrt{(-2\xi\omega_n)^2 - 4 \times 1 \times \omega_n^2}}{2 \times 1}$$

$$c_1, c_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$\boxed{c_1, c_2 = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}}$$

complex roots.

$$H(\omega) = \frac{A_1}{\cancel{C}} \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

Using method of Partial Fraction Expansion.

$$= \frac{A_1}{j\omega - c_1} + \frac{A_2}{j\omega - c_2}.$$

$$A_2 = -A_1$$

$$H(\omega) = \frac{A_1}{j\omega - c_1} - \frac{A_1}{j\omega - c_2}$$

$$F^{-1}\{H(\omega)\} = A_1 F^{-1}\left\{\frac{1}{j\omega - c_1}\right\} - A_1 F^{-1}\left\{\frac{1}{j\omega - c_2}\right\}$$

$$h(t) = A_1 \left[e^{c_1 t} \cdot u(t)\right] - A_1 \left[e^{c_2 t} \cdot u(t)\right]$$

$$= A_1 \left[e^{c_1 t} - e^{c_2 t}\right] \cdot u(t).$$

$$= A \left[e^{(-\xi\omega_n + j\omega_n\sqrt{1-\xi^2})t} - e^{(-\xi\omega_n - j\omega_n\sqrt{1-\xi^2})t}\right] u(t)$$

$$= \frac{e^{-\xi\omega_n t}}{\cancel{2j}} A(2j) \left(\frac{e^{+j\omega_n\sqrt{1-\xi^2}\, t} - e^{-j\omega_n\sqrt{1-\xi^2}\, t}}{2j}\right) u(t)$$

(1)

$$\boxed{h(t) = \left(\frac{\omega_n}{\sqrt{1-\xi^2}}\right) \left(e^{-\xi\omega_n t}\right) \times \sin\left(\omega_n \sqrt{1-\xi^2}\, t\right) u(t)}$$

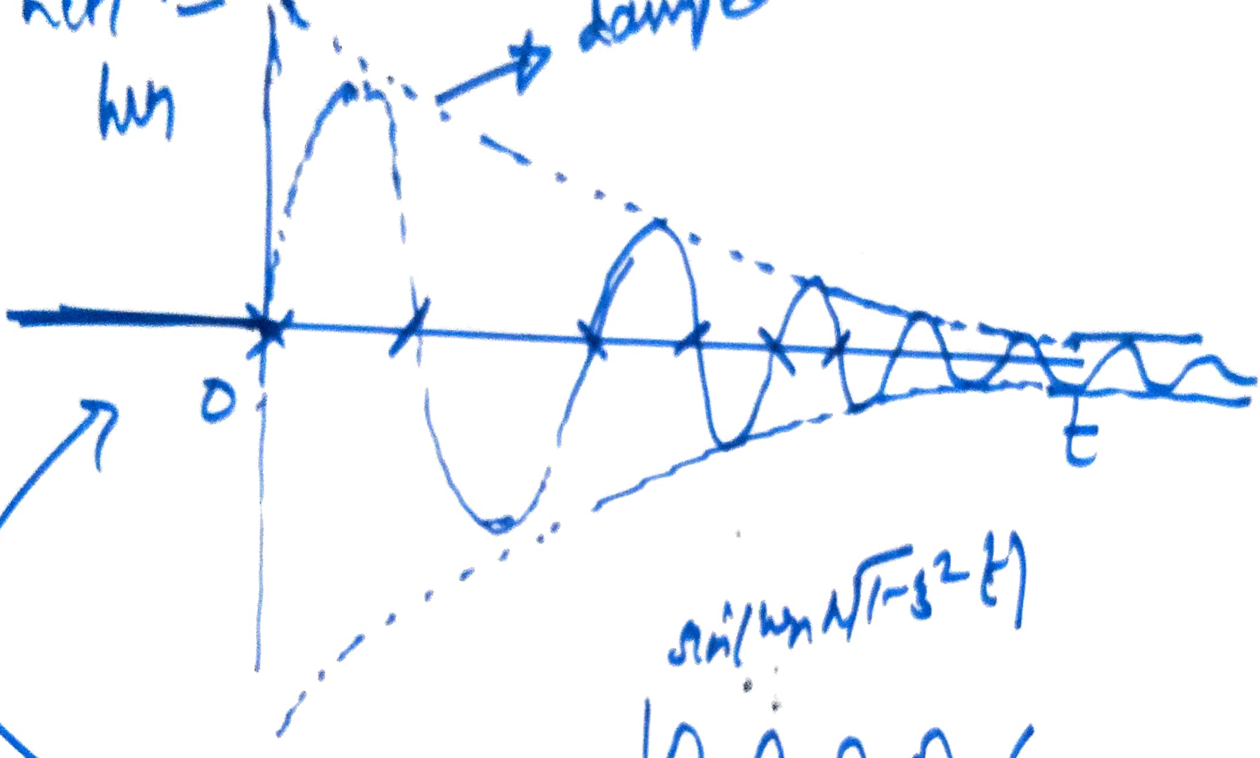
constant

(2j) · A1

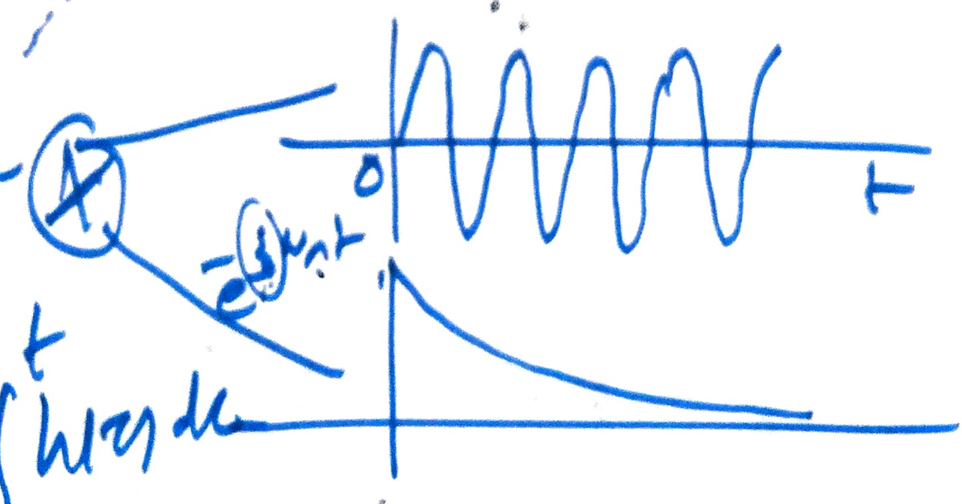
h(t) = causal

h(t)

damped curve.



$\sin(\omega_n \sqrt{1-\xi^2}\, t)$



$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

~~frequency response of 2nd order systems~~

(1) h(t) causal

(2) h(t) damped sinusoid.

(3) Zero crossings in h(t) are due to sinusoid or because frequency of sinusoid

$$\omega = \omega_n\sqrt{1-\xi^2}$$

$\sin(\omega t) = \sin[\omega_n\sqrt{1-\xi^2}\cdot t]$

If $\xi = 0$, → i.e., No damping

$\omega = \omega_n\sqrt{1-(0)^2} \Rightarrow$

$\omega = \omega_n \equiv$ undamped natural frequency of system

Bode Plot

$$H(\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n j\omega + \omega_n^2}$$

$$|H(\omega)|^2 = \frac{|\omega_n^2|^2}{\left|(j\omega)^2 + 2\xi\omega_n j\omega + \omega_n^2\right|^2}$$

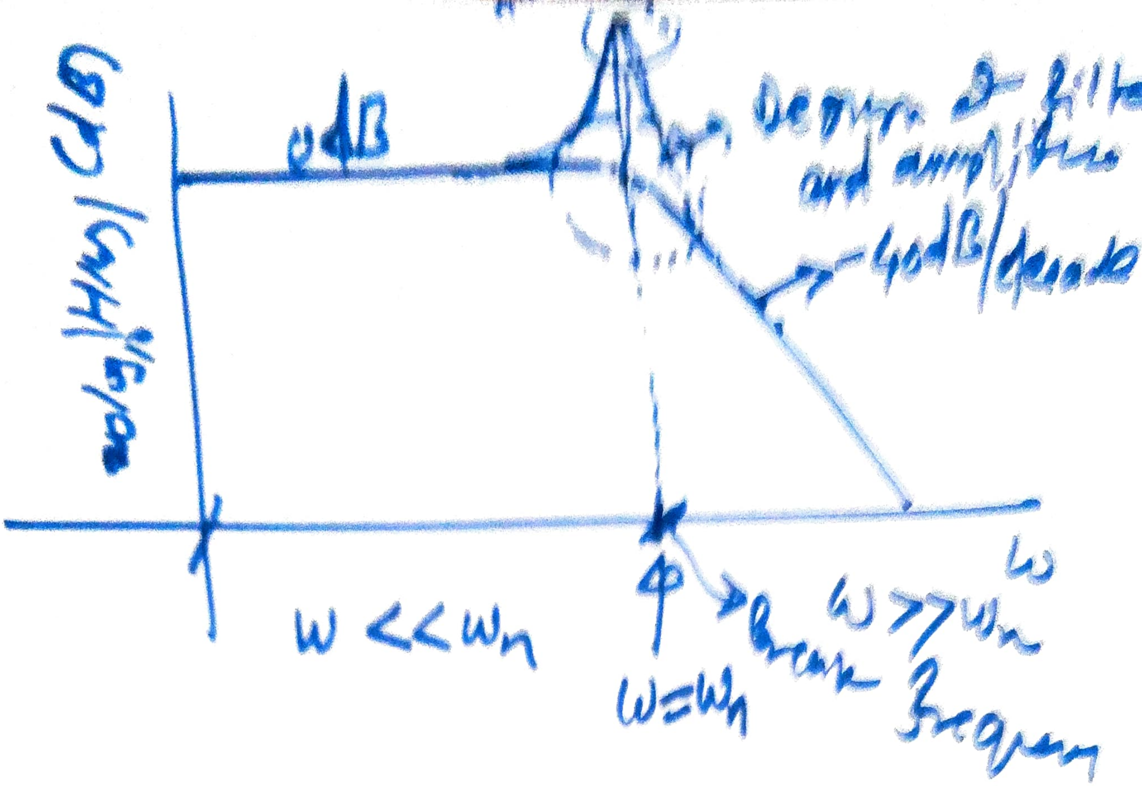
$$10\log_{10}|H(\omega)|^2 = 10\log_{10}\left[\ \cdots\ \right]$$

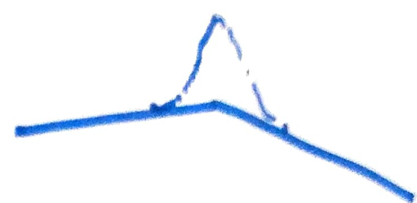
$$= \frac{1}{\left|\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\,2\xi\left(\frac{\omega}{\omega_n}\right)\right|^2}$$

$$= 0 - 10\log_{10}\left[\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2\right]$$

$$(dB) = \begin{cases} 0\ dB & \omega << \omega_n \\ -40\log\omega + 40\log_{10}(\omega_n) & \omega >> \omega_n \end{cases}$$

$\boxed{y = mx + c}$ $\quad$ (14) $\rightarrow$ $m = -40\,dB/decade$





Imp Note: The fact that a peak in Bode plot for 2nd order system is observed at $\omega < \omega_n$. This fact is used to design filters and frequency selective amplifiers.

$$\frac{20\log_{10}|H(j\omega)|\,(dB)}{d f(\omega)} = -10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$

$$\frac{\partial \log f(\omega)}{\partial \omega} = \frac{1}{f(\omega)} \times f'(\omega) = 0$$

$$\frac{\partial f(\omega)}{\partial \omega} = 0$$

$$\frac{\partial}{\partial \omega}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2\right\} = 0$$

$$2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\times\left(0-\frac{2\omega}{\omega_n^2}\right)$$

$$+4\xi^2 \times \frac{2\omega}{\omega_n^2} = 0$$

$$\boxed{\omega_{max} = \omega_n\sqrt{1-2\xi^2}}$$

$0<\xi<1 \rightarrow$ then the 2^{nd} order system has damped oscillatory behaviour.

$\rightarrow$ underdamped

$\xi=1 \rightarrow c_1 = c_2 \Rightarrow$ critically damped

$\xi>1 \rightarrow$ overdamped

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