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Linear Algebra

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Preface

Linear algebra is one of the most important subjects in the study of science and engineering because of its widespread applications in social or natural science, computer science, physics, or economics. As one of the most useful courses in undergraduate mathematics, it has provided essential tools for industrial scientists. The basic concepts of linear algebra are vector spaces, linear transformations, matrices and determinants, and they serve as an abstract language for stating ideas and solving problems.

This book is based on the lectures delivered several years in a sophomore-level linear algebra course designed for science and engineering students. The primary purpose of this book is to give a careful presentation of the basic concepts of linear algebra as a coherent part of mathematics, and to illustrate its power and usefulness through applications to other disciplines. We have tried to emphasize the computational skills along with the mathematical abstractions, which have also an integrity and beauty of their own. The book includes a variety of interesting applications with many examples not only to help students understand new concepts but also to practice wide applications of the subject to such areas as differential equations, statistics, geometry, and physics. Some of those applications may not be central to the mathematical development and may be omitted or selected in a syllabus at the discretion of the instructor. Most basic concepts and introductory motivations begin with examples in Euclidean space or solving a system of linear equations, and are gradually examined from different points of views to derive general principles.

For those students who have completed a year of calculus, linear algebra may be the first course in which the subject is developed in an abstract way, and we often find that many students struggle with the abstraction and miss the applications. Our experience is that, to understand the material, students should practice with many problems, which are sometimes omitted because of a lack of time. To encourage the students to do repeated practice, we placed in the middle of the text not only many examples but also some carefully selected problems, with answers or helpful hints. We have tried to make this book as easily accessible and clear as possible, but certainly there may be some awkward expressions in several ways. Any criticism or comment from the readers will be appreciated.

We are very grateful to many colleagues in Korea, especially to the faculty members in the mathematics department at Pohang University of Science and Technology (POSTECH), who helped us over the years with various aspects of this book. For their valuable suggestions and comments, we would like to thank the students at POSTECH, who have used photocopied versions of the text over the past several years. We would also like to acknowledge the invaluable assistance we have received from the teaching assistants who have checked and added some answers or hints for the problems and exercises in this book. Our thanks also go to Mrs. Kathleen Roush who made this book much more legible with her grammatical corrections in the final manuscript. Our thanks finally go to the editing staff of Birkhäuser for gladly accepting our book for publication.

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April 1997, in Pohang, Korea

"Linear algebra is the mathematics of our modern technological world of complex multivariable systems and computers"

– Alan Tucker –

"We (Halmos and Kaplansky) share a love of linear algebra. I think it is our conviction that we'll never understand infinite-dimensional operators properly until we have a decent mastery of finite matrices. And we share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury"

– Irving Kaplansky –

Contents

Linear Algebra

Chapter 1

Linear Equations and Matrices

1.1 Introduction

One of the central motivations for linear algebra is solving systems of linear equations. We thus begin with the problem of finding the solutions of a system of m linear equations in n unknowns of the following form:

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}\right.$$

$= 0 \Rightarrow$ homogeneous

where $x_1,\ x_2,\ \ldots,\ x_n$ are the unknowns and a_{ij}'s and b_i's denote constant (real or complex) numbers.

A sequence of numbers $(s_1,\ s_2,\ \ldots,\ s_n)$ is called a **solution** of the system if $x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n$ satisfy each equation in the system simultaneously. When $b_1 = b_2 = \cdots = b_m = 0$, we say that the system is **homogeneous**.

The central topic of this chapter is to examine whether or not a given system has a solution, and to find a solution if it has one. For instance, any homogeneous system always has at least one solution $x_1 = x_2 = \cdots = x_n = 0$, called the **trivial solution**. A natural question is whether such a homogeneous system has a nontrivial solution. If so, we would like to have a systematic method of finding all the solutions. A system of linear equations is said to be **consistent** if it has at least one solution, and **inconsistent** if it has no solution. The following example gives us an idea how to answer the above questions.

Example 1.1 When $m = n = 2$, the system reduces to two equations in two unknowns x and y:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2. \end{cases}$$

Geometrically, each equation in the system represents a straight line when we interpret x and y as coordinates in the xy-plane. Therefore, a point $P = (x, y)$ is a solution if and only if the point P lies on both lines. Hence there are three possible types of solution set:

(1) the empty set if the lines are parallel,

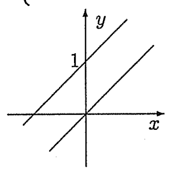
(2) only one point if they intersect,

(3) a straight line: *i.e.*, infinitely many solutions, if they coincide.

The following examples and diagrams illustrate the three types:

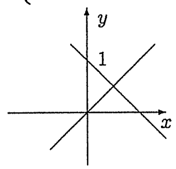
Case (1)

$$\begin{cases} x - y = -1 \\ x - y = 0 \end{cases}$$



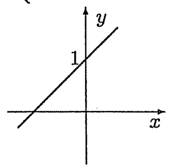
Case (2)

$$\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$$



Case (3)

$$\begin{cases} x - y = -1 \\ 2x - 2y = -2 \end{cases}$$



To decide whether the given system has a solution and to find a general method of solving the system when it has a solution, we repeat here a well-known elementary method of *elimination* and *substitution*.

Suppose first that the system consists of only one equation $ax + by = c$. Then the system has either infinitely many solutions (*i.e.*, points on the straight line $x = -\frac{b}{a}y + \frac{c}{a}$ or $y = -\frac{a}{b}x + \frac{c}{b}$ depending on whether $a \neq 0$ or $b \neq 0$) or no solutions when $a = b = 0$ and $c \neq 0$.

We now assume that the system has two equations representing two lines in the plane. Then clearly the two lines are parallel with the same slopes if and only if $a_2 = \lambda a_1$ and $b_2 = \lambda b_1$ for some $\lambda \neq 0$, or $a_1b_2 - a_2b_1 = 0$. Furthermore, the two lines either coincide (infinitely many solutions) or are distinct and parallel (no solutions) according to whether $c_2 = \lambda c_1$ holds or not.

Suppose now that the lines are not parallel, or $a_1b_2 - a_2b_1 \neq 0$. In this case, the two lines cross at a point, and hence there is exactly one solution: For instance, if the system is homogeneous, then the lines cross at the origin, so $(0,0)$ is the only solution. For a nonhomogeneous system, we may find the solution as follows: Express x in terms of y from the first equation, and then substitute it into the second equation (*i.e.*, eliminate the variable x from the second equation) to get

$$(b_2 - \frac{a_2}{a_1}b_1)y = c_2 - \frac{a_2}{a_1}c_1.$$

Since $a_1b_2 - a_2b_1 \neq 0$, this can be solved as

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

which is in turn substituted into one of the equations to find x and give a complete solution of the system. In detail, the process can be summarized as follows:

(1) Without loss of generality, we may assume $a_1 \neq 0$ since otherwise we can interchange the two equations. Then the variable x can be eliminated from the second equation by adding $-\frac{a_2}{a_1}$ times the first equation to the second, to get

$$\begin{cases} a_1x \; + \; b_1y = c_1 \\ (b_2 - \frac{a_2}{a_1}b_1)y = c_2 - \frac{a_2}{a_1}c_1. \end{cases}$$

(2) Since $a_1b_2 - a_2b_1 \neq 0$, y can be found by multiplying the second equation by a nonzero number $\frac{a_1}{a_1b_2 - a_2b_1}$ to get

$$\begin{cases} a_1x \; + \; b_1y = c_1 \\ y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \end{cases}$$

(3) Now, x is solved by substituting the value of y into the first equation, and we obtain the solution to the problem:

$$\begin{cases} x & = & \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \\ y & = & \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \end{cases}$$

Note that the condition $a_1b_2 - a_2b_1 \neq 0$ is necessary for the system to have only one solution. □

In this example, we have changed the original system of equations into a simpler one using certain operations, from which we can get the solution of the given system. That is, if (x, y) satisfies the original system of equations, then x and y must satisfy the above simpler system in (3), and vice versa.

It is suggested that the readers examine a system of three equations in three unknowns, each equation representing a plane in the 3-dimensional space $\mathbb{R}^3$, and consider the various possible cases in a similar way.

Problem 1.1 For a system of three equations in three unknowns

$$\begin{cases} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3, \end{cases}$$

describe all the possible types of the solution set in $\mathbb{R}^3$.

1.2 Gaussian elimination

As we have seen in Example 1.1, a basic idea for solving a system of linear equations is to change the given system into a simpler system, keeping the solutions unchanged; the example showed how to change a general system to a simpler one. In fact, the main operations used in Example 1.1 are the following three operations, called **elementary operations**:

(1) multiply a nonzero constant throughout an equation,

(2) interchange two equations,

(3) change an equation by adding a constant multiple of another equation.

After applying a finite sequence of these elementary operations to the given system, one can obtain a simpler system from which the solution can be derived directly.

Note also that each of the three elementary operations has its *inverse* operation which is also an elementary operation:

(1)′ divide the equation with the same nonzero constant,

(2)′ interchange two equations again,

(3)′ change the equation by subtracting the same constant multiple of the same equation.

By applying these inverse operations in reverse order to the simpler system, one can recover the original system. This means that a solution of the original system must also be a solution of the simpler one, and *vice versa.*

These arguments can be formalized in mathematical language. Observe that in performing any of these basic operations, only the coefficients of the variables are involved in the calculations and the variables $x_1, \ldots, x_n$ and the equal sign "=" are simply repeated. Thus, keeping the order of the variables and "=" in mind, we just extract the coefficients only from the equations in the given system and make a rectangular array of numbers:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

This matrix is called the **augmented matrix** for the system. The term *matrix* means just any rectangular array of numbers, and the numbers in this array are called the *entries* of the matrix. To explain the above operations in terms of matrices, we first introduce some terminology even though in the following sections we shall study matrices in more detail.

Within a matrix, the horizontal and vertical subarrays

$$[a_{i1}\ a_{i2}\ \cdots\ a_{in}\ b_i] \quad \text{and} \quad \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

are called the i-th *row* (matrix) and the j-th *column* (matrix) of the augmented matrix, respectively. Note that the entries in the j-th column are just the coefficients of j-th variable x_j, so there is a correspondence between columns of the matrix and variables of the system.

Since each row of the augmented matrix contains all the information of the corresponding equation of the system, we may deal with this augmented matrix instead of handling the whole system of linear equations.

The elementary operations to a system of linear equations are rephrased as the **elementary row operations** for the augmented matrix, as follows:

(1) multiply a nonzero constant throughout a row,

(2) interchange two rows,

(3) change a row by adding a constant multiple of another row.

The *inverse* operations are

(1)$'$ divide the row by the same constant,

(2)$'$ interchange two rows again,

(3)$'$ change the row by subtracting the same constant multiple of the other row.

Definition 1.1 Two augmented matrices (or systems of linear equations) are said to be **row-equivalent** if one can be transformed to the other by a finite sequence of elementary row operations.

If a matrix B can be obtained from a matrix A in this way, then we can obviously recover A from B by applying the inverse elementary row operations in reverse order. Note again that an elementary row operation does not alter the solution of the system, and we can formalize the above argument in the following theorem:

Theorem 1.1 *If two systems of linear equations are row-equivalent, then they have the same set of solutions.*

The general procedure for finding the solutions will be illustrated in the following example:

Example 1.2 Solve the system of linear equations:

$$\left\{\begin{array}{rcrcrcr} & & 2y & + & 4z & = & 2 \\ x & + & 2y & + & 2z & = & 3 \\ 3x & + & 4y & + & 6z & = & -1. \end{array}\right.$$

Solution: We could work with the augmented matrix alone. However, to compare the operations on systems of linear equations with those on the augmented matrix, we work on the system and the augmented matrix in parallel. Note that the associated augmented matrix of the system is

$$\begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{bmatrix}.$$

(1) Since the coefficient of x in the first equation is zero while that in the second equation is not zero, we interchange these two equations:

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ & & 2y & + & 4z & = & 2 \\ 3x & + & 4y & + & 6z & = & -1 \end{array}\right. \qquad \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 4 & 6 & -1 \end{bmatrix}.$$

(2) Add -3 times the first equation to the third equation:

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ & & 2y & + & 4z & = & 2 \\ & - & 2y & & & = & -10 \end{array}\right. \qquad \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 0 & -10 \end{bmatrix}.$$

The coefficient 1 of the first unknown x in the first equation (row) is called the **pivot** in this first elimination step.

Now the second and the third equations involve only the two unknowns y and z. Leave the first equation (row) alone, and the same elimination procedure can be applied to the second and the third equations (rows): The pivot for this step is the coefficient 2 of y in the second equation (row). To eliminate y from the last equation,

(3) Add 1 times the second equation (row) to the third equation (row):

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ & & 2y & + & 4z & = & 2 \\ & & & & 4z & = & -8 \end{array}\right. \qquad \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -8 \end{bmatrix}.$$

The elimination process done so far to obtain this result is called a **forward elimination**: *i.e.*, elimination of x from the last two equations (rows) and then elimination of y from the last equation (row).

Now the pivots of the second and third rows are 2 and 4, respectively. To make these entries 1,

(4) Divide each row by the pivot of the row:

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 3 \\ & & y & + & 2z & = & 1 \\ & & & & z & = & -2 \end{array}\right. \qquad \left[\begin{array}{cccr} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

The resulting matrix on the right side is called a **row-echelon form** of the matrix, and the 1's at the leftmost entries in each row are called the **leading** 1's. The process so far is called a **Gaussian elimination.**

We now want to eliminate numbers above the leading 1's;

(5) Add -2 times the third row to the second and the first rows,

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & & & = & 7 \\ & & y & & & = & 5 \\ & & & & z & = & -2 \end{array}\right. \qquad \left[\begin{array}{cccr} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

(6) Add -2 times the second row to the first row:

$$\left\{\begin{array}{rrrcr} x & & & = & -3 \\ & y & & = & 5 \\ & & z & = & -2 \end{array}\right. \qquad \left[\begin{array}{cccr} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

This matrix is called the **reduced row-echelon form.** The procedure to get this reduced row-echelon form from a row-echelon form is called the **back substitution.** The whole process to obtain the reduced row-echelon form is called a **Gauss-Jordan elimination.**

Notice that the corresponding system to this reduced row-echelon form is row-equivalent to the original one and is essentially a solved form: *i.e.*, the solution is $x = -3,\ y = 5,\ z = -2$. □

In general, a matrix of **row-echelon form** satisfies the following properties.

(1) The first nonzero entry of each row is 1, called a leading 1.

(2) A row containing only 0's should come after all rows with some nonzero entries.

(3) The leading 1's appear from left to the right in successive rows. That is, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Moreover, the matrix of the **reduced row-echelon form** satisfies

(4) Each column that contains a leading 1 has zeros everywhere else, in addition to the above three properties.

Note that an augmented matrix has only one reduced row-echelon form while it may have many row-echelon forms. In any case, the number of nonzero rows containing leading 1's is equal to the number of columns containing leading 1's. The variables in the system corresponding to columns with the leading 1's in a row-echelon form are called the **basic variables**. In general, the reduced row-echelon form U may have columns that do not contain leading 1's. The variables in the system corresponding to the columns without leading 1's are called **free variables**. Thus the sum of the number of basic variables and that of free variables is precisely the total number of variables.

For example, the first two matrices below are in reduced row-echelon form, and the last two just in row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Notice that in an augmented matrix $[A\ \mathbf{b}]$, the last column $\mathbf{b}$ does not correspond to any variable. Hence, if we consider the four matrices above as augmented matrices for some systems, then the systems corresponding to the first and the last two augmented matrices have only basic variables but no free variables. In the system corresponding to the second augmented matrix, the second and the forth variables, x_2 and x_4, are basic, and the first and the third variables, x_1 and x_3, are free variables. These ideas will be used in later chapters.

In summary, by applying a finite sequence of elementary row operations, the augmented matrix for a system of linear equations can be changed to its reduced row-echelon form which is row-equivalent to the original one. From the reduced row-echelon form, we can decide whether the system has a solution, and find the solution of the given system if it has one.

Example 1.3 Solve the following system of linear equations by Gauss-Jordan elimination.

$$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & = & 3 \\ 2x_1 & + & 6x_2 & - & 2x_3 & + & 4x_4 & = & 18 \\ & & x_2 & + & x_3 & + & 3x_4 & = & 10. \end{array} \right.$$

Solution: The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}.$$

The Gaussian elimination begins with:

(1) Adding -2 times the first row to the second produces

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}.$$

(2) Note that the coefficient of x_2 in the second equation is zero and that in the third equation is not. Thus, interchanging the second and the third rows produces

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix}.$$

(3) The pivot in the third row is 2. Thus, dividing the third row by 2 produces a row-echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}.$$

This is a row-echelon form, and we now continue the back-substitution:

(4) Adding -1 times the third row to the second, and 2 times the third row to the first produces

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 15 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}.$$

(5) Finally, adding -3 times the second row to the first produces the reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}.$$

The corresponding system of equations is

$$\left\{\begin{array}{ccccccc} x_1 & & & + & x_4 & = & 3 \\ & x_2 & & + & x_4 & = & 4 \\ & & x_3 & + & 2x_4 & = & 6. \end{array}\right.$$

Since x_1, x_2, and x_3 correspond to the columns containing leading 1's, they are the basic variables, and x_4 is the free variable. Thus by solving this system for the basic variables in terms of the free variable x_4, we have the system of equations in a solved form:

$$\left\{\begin{array}{ccccc} x_1 & = & 3 & - & x_4 \\ x_2 & = & 4 & - & x_4 \\ x_3 & = & 6 & - & 2x_4. \end{array}\right.$$

By assigning an arbitrary value t to the free variable x_4, the solutions can be written as

$$(x_1,\ x_2,\ x_3,\ x_4) = (3-t,\ 4-t,\ 6-2t,\ t),$$

for any $t \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. □

Remark: Consider a homogeneous system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0, \end{array}\right.$$

with the number of unknowns greater than the number of equations: that is, $m < n$. Since the number of basic variables cannot exceed the number of rows, a free variable always exists as in Example 1.3, so by assigning an arbitrary value to each free variable we can always find infinitely many nontrivial solutions.

Problem 1.2 Suppose that the augmented matrix for a system of linear equations has been reduced to the reduced row-echelon form below by elementary row operations. Solve the systems:

$$(1) \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad (2) \begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}.$$

We note that if a row-echelon form of an augmented matrix has a row of the type $[\,0\ 0\ \cdots\ 0\ b\,]$ with $b \neq 0$, then it represents an equation of the form $0x_1 + 0x_2 + \cdots + 0x_n = b$ with $b \neq 0$. In this case, the system has no solution. If $b = 0$, then it has a row containing only 0's that can be neglected. Hence, when we deal with a row-echelon form, we may assume that the zero rows are deleted. Note also that, as in Example 1.3, if there exists at least one free variable in the row-echelon form, then the system has infinitely many solutions. On the other hand, if the system has no free variable, the system has a unique solution.

To study systems of linear equations in terms of matrices systematically, we will develop some general theories of matrices in the following sections.

Problem 1.3 Solve the following systems of equations by Gaussian elimination. What are the pivots?

$$(1)\ \left\{\begin{array}{rcrcrcl} -x & + & y & + & 2z & = & 0 \\ 3x & + & 4y & + & z & = & 0 \\ 2x & + & 5y & + & 3z & = & 0. \end{array}\right. \qquad (2)\ \left\{\begin{array}{rcrcrcl} & & 2y & - & z & = & 1 \\ 4x & - & 10y & + & 3z & = & 5 \\ 3x & - & 3y & & & = & 6. \end{array}\right.$$

$$(3)\ \left\{\begin{array}{rcrcrcrcl} w & + & x & + & y & & & = & 3 \\ -3w & - & 17x & + & y & + & 2z & = & 1 \\ 4w & - & 17x & + & 8y & - & 5z & = & 1 \\ & - & 5x & - & 2y & + & z & = & 1. \end{array}\right.$$

Problem 1.4 Determine the condition on b_i so that the following system has a solution.

$$(1)\ \left\{\begin{array}{rcrcrcl} x & + & 2y & + & 6z & = & b_1 \\ 2x & - & 3y & - & 2z & = & b_2 \\ 3x & - & y & + & 4z & = & b_3. \end{array}\right. \qquad (2)\ \left\{\begin{array}{rcrcrcl} x & + & 3y & - & 2z & = & b_1 \\ 2x & - & y & + & 3z & = & b_2 \\ 4x & + & 2y & + & z & = & b_3. \end{array}\right.$$

1.3 Matrices

Rectangular arrays of real numbers arise in many real-world problems. Historically, it was the English mathematician A. Cayley who first introduced the word "matrix" in the year 1858. The meaning of the word is "that within which something originates," and he used matrices simply as a source for rows and columns to form squares.

In this section we are interested only in very basic properties of such matrices.

Definition 1.2 An m by n (written $m \times n$) **matrix** is a rectangular array of numbers arranged into m (horizontal) rows and n (vertical) columns. The

size of a matrix is specified by the number m of rows and the number n of columns.

In general, a matrix is written in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n},$$

or just $A = [a_{ij}]$ if the size of the matrix is clear from the context. The number a_{ij} is called the (i, j)-**entry** of the matrix A, and can be also written as $a_{ij} = [A]_{ij}$.

An $m\times 1$ matrix is called a **column (matrix)** or sometimes a **column vector**, and a $1\times n$ matrix is called a **row (matrix)**, or a **row vector**. These special cases are important, as we will see throughout the book. We will generally use capital letters like A, B, C for matrices and small boldface letters like $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ for columns or row vectors.

Definition 1.3 Let $A = [a_{ij}]$ be an $m\times n$ matrix. The **transpose** of A is the $n\times m$ matrix, denoted by A^T, whose j-th column is taken from the j-th row of A: That is, $A^T = [b_{ij}]$ with $b_{ij} = a_{ji}$.

For example, if $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

In particular, the transpose of a column vector is a row vector and vice versa. For example, for an $n \times 1$ column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

its transpose $\mathbf{x}^T = [x_1 \ x_2 \ \cdots \ x_n]$ is a row vector.

Definition 1.4 Let $A = [a_{ij}]$ be an $m \times n$ matrix.

(1) A is called a **square matrix of order** n if $m = n$.

In the following, we assume that A is a square matrix of order n.

(2) The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **diagonal entries** of A.

(3) A is called a **diagonal** matrix if all the entries except for the diagonal entries are zero.

(4) A is called an **upper (lower) triangular** matrix if all the entries below (above, respectively) the diagonal are zero.

The following matrices U and L are the general forms of the upper triangular and lower triangular matrices, respectively:

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Note that a matrix which is both upper and lower triangular must be a diagonal matrix, and the transpose of an upper (lower) triangular matrix is lower (upper, respectively) triangular matrix.

Definition 1.5 Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal**, written $A = B$, if they have the same size and corresponding entries are equal: *i.e.*, $a_{ij} = b_{ij}$ for all i and j.

This definition allows us to write matrix equations. A simple example is $(A^T)^T = A$ by definition.

Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with entries of real numbers. Among the elements of $M_{m\times n}(\mathbb{R})$, we can define two operations, called scalar multiplication and the sum of matrices, as follows:

Scalar multiplication: Given an $m \times n$ matrix $A = [a_{ij}]$ and a *scalar* k (which is simply a real number), the scalar multiplication kA of k and A is defined to be the matrix $kA = [ka_{ij}]$: *i.e.*, in an expanded form:

$$k\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{bmatrix}.$$

Sum of matrices: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same size, then the **sum** $A + B$ is defined to be the matrix $A + B = [a_{ij} + b_{ij}]$:

i.e., in an expanded form:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}.$$

Note that matrices of different sizes cannot be added. It is quite clear that $A+A=2A$, and $A+(A+A)=(A+A)+A=3A$. Thus, inductively we define $nA=(n-1)A+A$ for any positive integer n. If B is any matrix, then $-B$ is by definition the multiplication $(-1)B$. Moreover, if A and B are two matrices of the same size, then the difference $A-B$ is by definition the sum $A+(-1)B=A+(-B)$. A matrix whose entries are all zero is called a **zero matrix**, denoted by the symbol $\mathbf{0}$ (or $\mathbf{0}_{m\times n}$ when we emphasize the number of rows and columns).

Clearly, matrix addition has the same properties as the addition of real numbers. The real numbers in the context here are traditionally called **scalars** even though "numbers" is a perfectly good name and "scalar" sounds more technical. The following theorem lists some basic rules of these operations.

Theorem 1.2 *Suppose that the sizes of A, B and C are the same. Then the following rules of matrix arithmetic are valid:*

(1) $(A+B)+C=A+(B+C)$, (*written as* $A+B+C$) (Associativity),
(2) $A+\mathbf{0}=\mathbf{0}+A=A$,
(3) $A+(-A)=(-A)+A=\mathbf{0}$,
(4) $A+B=B+A$, (Commutativity),
(5) $k(A+B)=kA+kB$,
(6) $(k+\ell)A=kA+\ell A$,
(7) $(k\ell)A=k(\ell A)$.

Proof: We prove only (5) and the remaining are left for exercises. For any (i,j),

$$[k(A+B)]_{ij}=k[A+B]_{ij}=k([A]_{ij}+[B]_{ij})=[kA]_{ij}+[kB]_{ij}.$$

Consequently, $k(A+B)=kA+kB$. □

Definition 1.6 A square matrix A is said to be **symmetric** if $A^T = A$, or **skew-symmetric** if $A^T = -A$.

For example, the matrices A and B below

$$A = \begin{bmatrix} 1 & a & b \\ a & 3 & c \\ b & c & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

are symmetric and skew-symmetric, respectively. Notice here that all the diagonal entries of a skew-symmetric matrix must be zero, since $a_{ii} = -a_{ii}$.

By a direct computation, one can easily verify the following rules of the transpose of matrices:

Theorem 1.3 *Let A and B be $m \times n$ matrices. Then*

$$(kA)^T = kA^T, \quad \textit{and} \quad (A+B)^T = A^T + B^T.$$

Problem 1.5 Prove the remaining parts of Theorem 1.2.

Problem 1.6 Find a matrix B such that $A + B^T = (A - B)^T$, where

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 4 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}.$$

Problem 1.7 Find a, b, c and d such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2\begin{bmatrix} a & 3 \\ 2 & a+c \end{bmatrix} + \begin{bmatrix} 2+b & a+9 \\ c+d & b \end{bmatrix}.$$

1.4 Products of matrices

We introduced the operations sum and scalar multiplication of matrices in Section 1.3. In this section, we introduce the product of matrices. Unlike the sum of two matrices, the product of matrices is a little bit more complicated, in the sense that it is defined for two matrices of different sizes or for square matrices of the same order. We define the product of matrices in three steps:

(1) For a $1 \times n$ row matrix $\mathbf{a} = [a_1 \ \cdots \ a_n]$ and an $n \times 1$ column matrix $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$, the **product ax** is a $1{\times}1$ matrix (*i.e.*, just a number) defined by the rule

$$\mathbf{a}\mathbf{x} = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1x_1 + a_2x_2 + \cdots + a_nx_n] = \left[\sum_{i=1}^{n} a_ix_i\right].$$

Note that the number of columns of the first matrix must be equal to the number of rows of the second matrix to have entrywise multiplications of the entries.

(2) For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix},$$

where $\mathbf{a}_i$'s denote the row vectors, and for an $n \times 1$ column matrix $\mathbf{x} = [x_1 \ \cdots \ x_n]^T$, the **product** $A\mathbf{x}$ is by definition an $m \times 1$ matrix defined by the rule:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1\mathbf{x} \\ \mathbf{a}_2\mathbf{x} \\ \vdots \\ \mathbf{a}_m\mathbf{x} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \sum_{i=1}^{n} a_{2i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}x_i \end{bmatrix},$$

or in an expanded form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix},$$

which is just an $m \times 1$ column matrix of the form $[b_1 \, b_2 \, \cdots \, b_m]^T$.

Therefore, for a system of m linear equations in n unknowns, by writing the n unknowns as an $n \times 1$ column matrix $\mathbf{x}$ and the coefficients as an $m \times n$ matrix A the system may be expressed as a matrix equation $A\mathbf{x} = \mathbf{b}$. Notice that this looks just like the usual linear equation in one variable: $ax = b$.

(3) Product of matrices: Let A be an $m \times n$ matrix and B an $n \times r$ matrix. The **product** AB is defined to be an $m \times r$ matrix whose columns are the products of A and the columns of B in corresponding order.

Thus if A is $m \times n$ and B is $n \times r$, then B has r columns and each column of B is an $n \times 1$ matrix. If we denote them by $\mathbf{b}^1, \ldots, \mathbf{b}^r$, or $B = [\mathbf{b}^1 \cdots \mathbf{b}^r]$, then

$$\begin{aligned} AB &= \begin{bmatrix} A\mathbf{b}^1 & A\mathbf{b}^2 & \cdots & A\mathbf{b}^r \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1\mathbf{b}^1 & \mathbf{a}_1\mathbf{b}^2 & \cdots & \mathbf{a}_1\mathbf{b}^r \\ \mathbf{a}_2\mathbf{b}^1 & \mathbf{a}_2\mathbf{b}^2 & \cdots & \mathbf{a}_2\mathbf{b}^r \\ \vdots & & \ddots & \vdots \\ \mathbf{a}_m\mathbf{b}^1 & \mathbf{a}_m\mathbf{b}^2 & \cdots & \mathbf{a}_m\mathbf{b}^r \end{bmatrix}, \end{aligned}$$

which is an $m \times r$ matrix. Therefore, the (i,j)-entry $[AB]_{ij}$ of AB is the i-th entry of the j-th column matrix

$$A\mathbf{b}^j = \begin{bmatrix} \mathbf{a}_1\mathbf{b}^j \\ \mathbf{a}_2\mathbf{b}^j \\ \vdots \\ \mathbf{a}_m\mathbf{b}^j \end{bmatrix},$$

i.e., for $i = 1, \ldots, m$ and $j = 1, \ldots, r$, it is the product of i-th row and j-th column of A:

$$[AB]_{ij} = \mathbf{a}_i\mathbf{b}^j = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Example 1.4 Consider the matrices

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix}.$$

The columns of AB are the product of A and each column of B:

$$\begin{aligned} \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} &= \begin{bmatrix} 2\cdot 1 + 3\cdot 5 \\ 4\cdot 1 + 0\cdot 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \end{bmatrix}, \\ \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} &= \begin{bmatrix} 2\cdot 2 + 3\cdot(-1) \\ 4\cdot 2 + 0\cdot(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \\ \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2\cdot 0 + 3\cdot 0 \\ 4\cdot 0 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore, AB is

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 0 \\ 4 & 8 & 0 \end{bmatrix}.$$

Since A is a 2×2 matrix and B is a 2×3 matrix, the product AB is a 2×3 matrix. If we concentrate, for example, on the $(2,1)$-entry of AB, we single out the second row from A and the first column from B, and then we multiply corresponding entries together and add them up, *i.e.*, $4 \cdot 1 + 0 \cdot 5 = 4$. □

Note that the product AB of A and B is not defined if the number of columns of A and the number of rows of B are not equal.

Remark: In step (2), we could have defined for a $1 \times n$ row matrix A and an $n \times r$ matrix B using the same rule defined in step (1). And then in step (3) an appropriate modification produces the same definition of the product of matrices. We suggest the readers verify this (see Example 1.6).

The **identity matrix** of order n, denoted by I_n (or I if the order is clear from the context), is a diagonal matrix whose diagonal entries are all 1, *i.e.*,

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

By a direct computation, one can easily see that $AI_n = A = I_nA$ for any $n \times n$ matrix A.

Many, but not all, of the rules of arithmetic for real or complex numbers also hold for matrices with the operations of scalar multiplication, the sum and the product of matrices. The matrix $\mathbf{0}_{m \times n}$ plays the role of the number 0, and I_n that of the number 1 in the set of real numbers.

The rule that does not hold for matrices in general is the commutativity $AB = BA$ of the product, while the commutativity of the matrix sum $A + B = B + A$ does hold in general. The following example illustrates the noncommutativity of the product of matrices.

Example 1.5 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus the matrices A and B in this example satisfy $AB \neq BA$. □

The following theorem lists some rules of ordinary arithmetic that do hold for matrix operations.

Theorem 1.4 *Let A, B, C be arbitrary matrices for which the matrix operations below are defined, and let k be an arbitrary scalar. Then*
(1) $A(BC) = (AB)C$, (*written as* ABC) (Associativity),
(2) $A(B+C) = AB + AC$, *and* $(A+B)C = AC + BC$, (Distributivity),
(3) $IA = A = AI$,
(4) $k(BC) = (kB)C = B(kC)$,
(5) $(AB)^T = B^T A^T$.

Proof: Each equality can be shown by direct calculations of each entry of both sides of the equalities. We illustrate this by proving (1) only, and leave the others to the readers.

Assume that $A = [a_{ij}]$ is an $m \times n$ matrix, $B = [b_{k\ell}]$ is an $n \times p$ matrix, and $C = [c_{st}]$ is a $p \times r$ matrix. We now compute the (i,j)-entry of each side of the equation. Note that BC is an $n \times r$ matrix whose (i,j)-entry is $[BC]_{ij} = \sum_{\lambda=1}^{p} b_{i\lambda}c_{\lambda j}$. Thus

$$[A(BC)]_{ij} = \sum_{\mu=1}^{n} a_{i\mu}[BC]_{\mu j} = \sum_{\mu=1}^{n} a_{i\mu} \sum_{\lambda=1}^{p} b_{\mu\lambda}c_{\lambda j} = \sum_{\mu=1}^{n}\sum_{\lambda=1}^{p} a_{i\mu}b_{\mu\lambda}c_{\lambda j}.$$

Similarly, AB is an $m \times p$ matrix with the (i,j)-entry $[AB]_{ij} = \sum_{\mu=1}^{n} a_{i\mu}b_{\mu j}$, and

$$[(AB)C]_{ij} = \sum_{\lambda=1}^{p}[AB]_{i\lambda}c_{\lambda j} = \sum_{\lambda=1}^{p}\sum_{\mu=1}^{n} a_{i\mu}b_{\mu\lambda}c_{\lambda j} = \sum_{\mu=1}^{n}\sum_{\lambda=1}^{p} a_{i\mu}b_{\mu\lambda}c_{\lambda j}.$$

This clearly shows that $[A(BC)]_{ij} = [(AB)C]_{ij}$ for all i, j, and consequently $A(BC) = (AB)C$ as desired. □

Problem 1.8 Prove or disprove: If A is not a zero matrix and $AB = AC$, then $B = C$.

Problem 1.9 Show that any triangular matrix A satisfying $AA^T = A^TA$ is a diagonal matrix.

Problem 1.10 For a square matrix A, show that

(1) AA^T and $A + A^T$ are symmetric,

(2) $A - A^T$ is skew-symmetric, and

(3) A can be expressed as the sum of symmetric part $B = \frac{1}{2}(A + A^T)$ and skew-symmetric part $C = \frac{1}{2}(A - A^T)$, so that $A = B + C$.

As an application of our results on matrix operations, we shall prove the following important theorem:

Theorem 1.5 *Any system of linear equations has either no solution, exactly one solution, or infinitely many solutions.*

Proof: We have already seen that a system of linear equations may be written as $A\mathbf{x} = \mathbf{b}$, which may have no solution or exactly one solution. Now assume that the system $A\mathbf{x} = \mathbf{b}$ of linear equations has more than one solution and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two different solutions so that $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$. Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$. Since $A\mathbf{x}$ is just a particular case of a matrix product, Theorem 1.4 gives us

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + kA\mathbf{x}_0 = \mathbf{b} + k(A\mathbf{x}_1 - A\mathbf{x}_2) = \mathbf{b},$$

for any real number k. This says that $\mathbf{x}_1 + k\mathbf{x}_0$ is also a solution of $A\mathbf{x} = \mathbf{b}$ for any k. Since there are infinitely many choices for k, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. $\square$

Problem 1.11 For which values of a does each of the following systems have no solution, exactly one solution, or infinitely many solutions?

$$(1) \left\{ \begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2 - 14)z & = & a + 2. \end{array} \right.$$

$$(2) \left\{ \begin{array}{rcrcrcl} x & - & y & + & z & = & 1 \\ x & + & 3y & + & az & = & 2 \\ 2x & + & ay & + & 3z & = & 3. \end{array} \right.$$

1.5 Block matrices

In this section we introduce some techniques that will often be very helpful in manipulating matrices. A **submatrix** of a matrix A is a matrix obtained from A by deleting certain rows and/or columns of A. Using a system of horizontal and vertical lines, we can partition a matrix A into submatrices, called **blocks**, of A as follows: Consider a matrix

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right],$$

divided up into four blocks by the dotted lines shown. Now, if we write

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} a_{34} \end{bmatrix},$$

then A can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

called a **block matrix**.

The product of matrices partitioned into blocks also follows the matrix product formula, as if the A_{ij} were numbers:

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}; \\ AB &= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}, \end{aligned}$$

provided that the number of columns in A_{ik} is equal to the number of rows in B_{kj}. This will be true only if the columns of A are partitioned in the same way as the rows of B.

It is not hard to see that the matrix product by blocks is correct. Suppose, for example, that we have a 3×3 matrix A and partition it as

$$A = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

and a 3×2 matrix B which we partition as

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \hline b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}.$$

Then the entries of $C = [c_{ij}] = AB$ are

$$c_{ij} = (a_{i1}b_{1j} + a_{i2}b_{2j}) + a_{i3}b_{3j}.$$

The quantity $a_{i1}b_{1j} + a_{i2}b_{2j}$ is simply the $(i\,j)$-entry of $A_{11}B_{11}$ if $i \le 2$, and the $(i\,j)$-entry of $A_{21}B_{11}$ if $i = 3$. Similarly, $a_{i3}b_{3j}$ is the $(i\,j)$-entry of $A_{12}B_{21}$ if $i \le 2$, and of $A_{22}B_{21}$ if $i = 3$. Thus AB can be written as

$$AB = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}.$$

In particular, if an $m \times n$ matrix A is partitioned into blocks of column vectors: *i.e.*, $A = [\mathbf{a}^1\ \mathbf{a}^2\ \cdots\ \mathbf{a}^n]$, where each block $\mathbf{a}^j$ is the j-th column, then the product $A\mathbf{x}$ with $\mathbf{x} = [x_1\ \cdots\ x_n]^T$ is the sum of the block matrices (or column vectors) with coefficients x_j's:

$$A\mathbf{x} = [\mathbf{a}^1\ \mathbf{a}^2\ \cdots\ \mathbf{a}^n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}^1 + x_2\mathbf{a}^2 + \cdots + x_n\mathbf{a}^n,$$

where $x_j\mathbf{a}^j = x_j[a_{1j}\ a_{2j}\ \cdots\ a_{nj}]^T$.

Example 1.6 Let A be an $m \times n$ matrix partitioned into the row vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ as its blocks, and let B be an $n \times r$ matrix so that their product AB is well-defined. By considering the matrix B as a block, the product AB can be written

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}^1 & \mathbf{a}_1\mathbf{b}^2 & \cdots & \mathbf{a}_1\mathbf{b}^r \\ \mathbf{a}_2\mathbf{b}^1 & \mathbf{a}_2\mathbf{b}^2 & \cdots & \mathbf{a}_2\mathbf{b}^r \\ \vdots & & \ddots & \vdots \\ \mathbf{a}_m\mathbf{b}^1 & \mathbf{a}_m\mathbf{b}^2 & \cdots & \mathbf{a}_m\mathbf{b}^r \end{bmatrix},$$

where $\mathbf{b}^1, \mathbf{b}^2, \cdots, \mathbf{b}^r$ denote the columns of B. Hence, the row vectors of AB are the products of the row vectors of A and B.

Problem 1.12 Compute AB using block multiplication, where

$$A = \left[\begin{array}{rr|rr} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \\ \hline 0 & 0 & 2 & -1 \end{array}\right], \quad B = \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 2 & 3 & 4 \\ 3 & -2 & 1 \end{array}\right].$$

1.6 Inverse matrices

As we saw in Section 1.4, a system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$ in matrix form. This form resembles one of the simplest linear equation in one variable $ax = b$ whose solution is simply $x = a^{-1}b$ when $a \neq 0$. Thus it is tempting to write the solution of the system as $\mathbf{x} = A^{-1}\mathbf{b}$. However, in the case of matrices we first have to have a precise meaning of A^{-1}. To discuss this we begin with the following definition.

Definition 1.7 For an $m \times n$ matrix A, an $n \times m$ matrix B is called a **left inverse** of A if $BA = I_n$, and an $n \times m$ matrix C is called a **right inverse** of A if $AC = I_m$.

Example 1.7 From a direct calculation for two matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -3 \\ -1 & 5 \\ -2 & 7 \end{bmatrix},$$

we have $AB = I_2$, and $BA = \begin{bmatrix} -5 & 2 & -4 \\ 9 & -2 & 6 \\ 12 & -4 & 9 \end{bmatrix} \neq I_3$.

Thus, the matrix B is a right inverse but not a left inverse of A, while A is a left inverse but not a right inverse of B. Since $(AB)^T = B^TA^T$ and $I^T = I$, a matrix A has a right inverse if and only if A^T has a left inverse. □

However, if A is a square matrix and has a left inverse, then we prove later (Theorem 1.8) that it has also a right inverse, and vice versa. Moreover, the following lemma shows that the left inverses and the right inverses of a square matrix are all equal. (This is not true for nonsquare matrices, of course).

Lemma 1.6 *If an $n \times n$ square matrix A has a left inverse B and a right inverse C, then B and C are equal, i.e., $B = C$.*

Proof: A direct calculation shows that

$$B = BI = B(AC) = (BA)C = IC = C.$$

Now any two left inverses must be both equal to a right inverse C, and hence to each other, and any two right inverses must be both equal to a left inverse B, and hence to each other. So there exist only one left and only one right inverse for a square matrix A ifit is known that A has both left and right inverses. Furthermore, the left and right inverses are equal. $\square$

This theorem says that if a matrix A has both a right inverse and a left inverse, then they must be the same. However, we shall see in Chapter 3 that any $m\times n$ matrix A with $m \neq n$ cannot have both a right inverse and a left inverse: that is, a nonsquare matrix may have only a left inverse or only a right inverse. In this case, the matrix may have many left inverses or many right inverses.

Example 1.8 A nonsquare matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ can have more than one left inverse. In fact, for any x, $y \in \mathbb{R}$, one can easily check that the matrix $B = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}$ is a left inverse of A. $\square$

Definition 1.8 An $n \times n$ square matrix A is said to be **invertible** (or **nonsingular**) if there exists a square matrix B of the same size such that

$$AB = I = BA.$$

Such a matrix B is called the **inverse** of A, and is denoted by A^{-1}. A matrix A is said to be **singular** if it is not invertible.

Note that Lemma 1.6 shows that if a square matrix A has both left and right inverses, then it must be unique. That is why we call B "the" inverse of A. For instance, consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then it is easy to verify that

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix},$$

since $AA^{-1} = I_2 = A^{-1}A$. (Check this product of matrices for practice!) Note that any zero matrix is singular.

Problem 1.13 Let A be an invertible matrix and k any nonzero scalar. Show that
(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$;
(2) the matrix kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$;
(3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 1.7 *The product of invertible matrices is also invertible, whose inverse is the product of the individual inverses in reverse order:*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof: Suppose that A and B are invertible matrices of the same size. Then $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$, and similarly $(B^{-1}A^{-1})(AB) = I$. Thus AB has the inverse $B^{-1}A^{-1}$. □

We have written the inverse of A as "A to the power -1", so we can give the meaning of A^k for any integer k: Let A be a square matrix. Define $A^0 = I$. Then, for any positive integer k, we define the power A^k of A inductively as

$$A^k = A(A^{k-1}).$$

Moreover, if A is invertible, then the negative integer power is defined as

$$A^{-k} = (A^{-1})^k \quad \text{for } k > 0.$$

It is easy to check that with these rules we have $A^{k+\ell} = A^kA^\ell$ whenever the right hand side is defined. (If A is not invertible, $A^{3+(-1)}$ is defined but A^{-1} is not.)

Problem 1.14 Prove:
(1) If A has a zero row, so does AB.
(2) If B has a zero column, so does AB.
(3) Any matrix with a zero row or a zero column cannot be invertible.

Problem 1.15 Let A be an invertible matrix. Is it true that $(A^k)^T = (A^T)^k$ for any integer k? Justify your answer.

1.7 Elementary matrices

We now return to the system of linear equations $A\mathbf{x} = \mathbf{b}$. If A has a right inverse B such that $AB = I_m$, then $\mathbf{x} = B\mathbf{b}$ is a solution of the system since

$$A\mathbf{x} = A(B\mathbf{b}) = (AB)\mathbf{b} = \mathbf{b}.$$

In particular, if A is an invertible square matrix, then it has only one inverse A^{-1} by Lemma 1.6, and $\mathbf{x} = A^{-1}\mathbf{b}$ is the only solution of the system. In this section, we discuss how to compute A^{-1} when A is invertible.

Recall that Gaussian elimination is a process in which the augmented matrix is transformed into its row-echelon form by a finite number of elementary row operations. In the following, we will show that each elementary row operation can be expressed as a nonsingular matrix, called an *elementary matrix*, and hence the process of Gaussian elimination is simply multiplying a finite sequence of corresponding elementary matrices to the augmented matrix.

Definition 1.9 A matrix E obtained from the identity matrix I_n by executing only one elementary row operation is called an **elementary matrix**.

For example, the following matrices are three elementary matrices corresponding to each type of the three elementary row operations.

(1) $\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$: multiply the second row of I_2 by -5;

(2) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$: interchange the second and the fourth rows of I_4;

(3) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: add 3 times the third row to the first row of I_3.

It is an interesting fact that, if E is an elementary matrix obtained by executing a certain elementary row operation on the identity matrix I_m, then for any $m \times n$ matrix A, *the product EA is exactly the matrix that is obtained when the same elementary row operation in E is executed on A.* The following example illustrates this argument. (Note that AE is not what we want. For this, see Problem 1.17).

Example 1.9 For simplicity, we work on a 3×1 column matrix $\mathbf{b}$. Suppose that we want to do the operation "adding $(-2) \times$ the first row to the second row" on matrix $\mathbf{b}$. Then, we execute this operation on the identity matrix I first to get an elementary matrix E:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying the elementary matrix E to $\mathbf{b}$ on the left produces the desired result:

$$E\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix},$$

Similarly, the operation "interchanging the first and third rows" on the matrix $\mathbf{b}$ can be achieved by multiplying a *permutation matrix* P, which is an elementary matrix obtained from I_3 by interchanging two rows, to $\mathbf{b}$ on the left:

$$P\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}.$$

□

Recall that each elementary row operation has an inverse operation, which is also an elementary operation, that brings the matrix back to the original one. Thus, suppose that E denotes an elementary matrix corresponding to an elementary row operation, and let E' be the elementary matrix corresponding to its "inverse" elementary row operation in E. Then,

(1) if E multiplies a row by $c \neq 0$, then E' multiplies the same row by $\frac{1}{c}$;

(2) if E interchanges two rows, then E' interchanges them again;

(3) if E adds a multiple of one row to another, then E' subtracts it back from the same row.

Thus, for any $m \times n$ matrix A, $E'EA = A$, and $E'E = I = EE'$. That is, *every elementary matrix is invertible so that $E^{-1} = E'$, which is also an elementary matrix.*

For instance, if

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \; E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 1.10 A **permutation matrix** is a square matrix obtained from the identity matrix by permuting the rows.

Problem 1.16 Prove:

(1) A permutation matrix is the product of a finite number of elementary matrices each of which is corresponding to the "row-interchanging" elementary row operation.
(2) Any permutation matrix P is invertible and $P^{-1} = P^T$.
(3) The product of any two permutation matrices is a permutation matrix.
(4) The transpose of a permutation matrix is also a permutation matrix.

Problem 1.17 Define the **elementary column operations** for a matrix by just replacing "row" by "column" in the definition of the elementary row operations. Show that if A is an $m \times n$ matrix and if E is an elementary matrix obtained by executing an elementary column operation on I_n, then AE is exactly the matrix that is obtained from A when the same column operation is executed on A.

The next theorem establishes some fundamental relationships between $n \times n$ square matrices and systems of n linear equations in n unknowns.

Theorem 1.8 *Let A be an $n \times n$ matrix. The following are equivalent:*

(1) *A has a left inverse;*

(2) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$;*

(3) *A is row-equivalent to I_n;*

(4) *A is a product of elementary matrices;*

(5) *A is invertible;*

(6) *A has a right inverse.*

Proof: **(1)** $\Rightarrow$ **(2)** : Let $\mathbf{x}$ be a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$, and let B be a left inverse of A. Then

$$\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = BA\mathbf{x} = B\mathbf{0} = \mathbf{0}.$$

(2) $\Rightarrow$ (3) : Suppose that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$:

$$\left\{\begin{array}{cccccc} x_1 & & & & = & 0 \\ & x_2 & & & = & 0 \\ & & \ddots & & & \\ & & & x_n & = & 0. \end{array}\right.$$

This means that the augmented matrix $[A\ \mathbf{0}]$ of the system $A\mathbf{x} = \mathbf{0}$ is reduced to the system $[I_n\ \mathbf{0}]$ by Gauss-Jordan elimination. Hence, A is row-equivalent to I_n.

(3) $\Rightarrow$ (4) : Assume A is row-equivalent to I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. Thus, we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n.$$

Since $E_1, E_2, \ldots, E_k$ are invertible, by multiplying both sides of this equation on the left successively by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$, we obtain

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

which expresses A as the product of elementary matrices.

(4) $\Rightarrow$ (5) is trivial, because any elementary matrix is invertible. In fact, $A^{-1} = E_k \cdots E_2 E_1$.

(5) $\Rightarrow$ (1) and **(5) $\Rightarrow$ (6)** are trivial.

(6) $\Rightarrow$ (5) : If B is a right inverse of A, then A is a left inverse of B and we can apply (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) to B and conclude that B is invertible, with A as its unique inverse. That is, B is the inverse of A and so A is invertible. □

This theorem shows that a square matrix is invertible if it has a one-side inverse. In particular, if a square matrix A is invertible, then $\mathbf{x} = A^{-1}\mathbf{b}$ is a unique solution to the system $A\mathbf{x} = \mathbf{b}$.

Problem 1.18 Find the inverse of the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As an application of the preceding theorem, we give a practical method for finding the inverse A^{-1} of an invertible $n \times n$ matrix A. If A is invertible, there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n.$$

Hence,

$$A^{-1} = E_k \cdots E_2 E_1 = E_k \cdots E_2 E_1 I_n.$$

It follows that *the sequence of row operations that reduces an invertible matrix A to I_n will resolve I_n to A^{-1}*. In other words, let $[A \mid I]$ be the augmented matrix with the columns of A on the left half, the columns of I on the right half. A Gaussian elimination, applied to both sides, by some elementary row operations reduces the augmented matrix $[A \mid I]$ to $[U \mid K]$, where U is a row-echelon form of A. Next, the back substitution process by another series of elementary row operations reduces $[U \mid K]$ to $[I \mid A^{-1}]$:

$$\begin{aligned} [A \mid I] &\rightarrow [E_\ell \cdots E_1 A \mid E_\ell \cdots E_1 I] = [U \mid K] \\ &\rightarrow [F_k \cdots F_1 U \mid F_k \cdots F_1 K] = [I \mid A^{-1}], \end{aligned}$$

where $E_\ell \cdots E_1$ represents a Gaussian elimination and $F_k \cdots F_1$ represents the back substitution. The following example illustrates the computation of an inverse matrix.

Example 1.10 Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

We apply Gauss-Jordan elimination to

$$\begin{aligned} [A \mid I] &= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (-2)\text{row } 1 + \text{row } 2 \\ (-1)\text{row } 1 + \text{row } 3 \end{array} \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] (-1)\text{row } 2 \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] (2)\text{row } 2 + \text{row } 3 \end{aligned}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right].$$

This is $[U \mid K]$ obtained by Gaussian elimination. Now continue the back substitution to reduce $[U \mid K]$ to $[I \mid A^{-1}]$

$$\begin{array}{rcll} [U \mid K] & = & \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] & \begin{array}{l} (-1)\text{row } 3 + \text{row } 2 \\ (-3)\text{row } 3 + \text{row } 1 \end{array} \\ & \to & \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -8 & 6 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] & (-2)\text{row } 2 + \text{row } 1 \\ & \to & \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] & = \left[I \mid A^{-1}\right]. \end{array}$$

Thus, we get

$$A^{-1} = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}.$$

(The reader should verify that $AA^{-1} = I = A^{-1}A$.) □

Note that if A is not invertible, then, at some step in Gaussian elimination, a zero row will show up on the left side in $[U \mid K]$. For example, the matrix $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 0 \end{bmatrix}$ which is a noninvertible matrix.

Problem 1.19 Write A^{-1} as a product of elementary matrices for A in Example 1.10.

of A by using Gaussian elimination.

Problem 1.20 Find the inverse of each of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 2 \\ -6 & 4 & 11 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 2 & 4 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix} \quad (k \neq 0).$$

Problem 1.21 When is a diagonal matrix $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ nonsingular, and what is D^{-1}?

From Theorem 1.8, a square matrix A is nonsingular if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. That is, a square matrix A is singular if and only if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, say $\mathbf{x}_0$. Now, for any column vector $\mathbf{b} = [b_1 \ \cdots \ b_n]^T$, if $\mathbf{x}_1$ is a solution of $A\mathbf{x} = \mathbf{b}$ for a singular matrix A, then so is $k\mathbf{x}_0 + \mathbf{x}_1$ for any k:

$$A(k\mathbf{x}_0 + \mathbf{x}_1) = k(A\mathbf{x}_0) + A\mathbf{x}_1 = k\mathbf{0} + \mathbf{b} = \mathbf{b}.$$

This argument strengthens Theorem 1.5 as follows when A is a square matrix:

Theorem 1.9 *If A is an invertible $n \times n$ matrix, then for any column vector $\mathbf{b} = [b_1 \ \cdots \ b_n]^T$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$. If A is not invertible, then the system has either no solution or infinitely many solutions according to whether or not the system is inconsistent.* □

Problem 1.22 Write the system of linear equations

$$\begin{cases} x + 2y + 2z = 10 \\ 2x - 2y + 3z = 1 \\ 4x - 3y + 5z = 4 \end{cases}$$

in matrix form $A\mathbf{x} = \mathbf{b}$ and solve it by finding $A^{-1}\mathbf{b}$.

1.8 *LDU* factorization

Recall that a basic method of solving a linear system $A\mathbf{x} = \mathbf{b}$ is by Gauss-Jordan elimination. For a fixed matrix A, if we want to solve more than one system $A\mathbf{x} = \mathbf{b}$ for various values of $\mathbf{b}$, then the same Gaussian elimination on A has to be repeated over and over again. However, this repetition may be avoided by expressing Gaussian elimination as an invertible matrix which is a product of elementary matrices.

We first assume that no permutations of rows are necessary throughout the whole process of Gaussian elimination on $[A \ \mathbf{b}]$. Then the forward elimination is just to multiply finitely many elementary matrices $E_k, \ \ldots, \ E_1$ to the augmented matrix $[A \ \mathbf{b}]$: that is,

$$[E_k \cdots E_1 A \quad E_k \cdots E_1 \mathbf{b}] = [U \ \mathbf{c}],$$

where each E_i is a lower triangular elementary matrix whose diagonal entries are all 1's and $[U\ \mathbf{c}]$ is the augmented matrix of the system obtained after forward elimination on $A\mathbf{x} = \mathbf{b}$ (Note that U need not be an upper triangular matrix if A is not a square matrix). Therefore, if we set $L = (E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1}$, then $A = LU$ and

$$\mathbf{c} = U\mathbf{x} = E_k \cdots E_1 A\mathbf{x} = E_k \cdots E_1 \mathbf{b} = L^{-1}\mathbf{b}.$$

Note that L is a lower triangular matrix whose diagonal entries are all 1's (see Problem 1.24). Now, for any column matrix $\mathbf{b}$, the system $A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$ can be solved in two steps: first compute $\mathbf{c} = L^{-1}\mathbf{b}$ which is a forward elimination, and then solve $U\mathbf{x} = \mathbf{c}$ by the back substitution.

This means that, to solve the ℓ-systems $A\mathbf{x} = \mathbf{b}_i$ for $i = 1, \ldots, \ell$, we first find the matrices L and U such that $A = LU$ by performing forward elimination on A, and then compute $\mathbf{c}_i = L^{-1}\mathbf{b}_i$ for $i = 1, \ldots, \ell$. The solutions of $A\mathbf{x} = \mathbf{b}_i$ are now those of $U\mathbf{x} = \mathbf{c}_i$.

Example 1.11 Consider the system of linear equations

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = \mathbf{b}.$$

The elementary matrices for Gaussian elimination of A are easily found to be

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix},$$

so that

$$E_3E_2E_1A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} = U.$$

Note that U is the matrix obtained from A after forward elimination, and $A = LU$ with

$$L = E_1^{-1}\, E_2^{-1}\, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix},$$

which is a lower triangular matrix with 1's on the diagonal. Now, the system

$$L\mathbf{c} = \mathbf{b}: \quad \left\{ \begin{array}{rcrcrcr} c_1 & & & & & = & 1 \\ 2c_1 & + & c_2 & & & = & -2 \\ -\,c_1 & - & 3c_2 & + & c_3 & = & 7 \end{array} \right.$$

resolves to $\mathbf{c} = (1, -4, -4)$ and the system

$$U\mathbf{x} = \mathbf{c} : \left\{ \begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & - & x_2 & - & 2x_3 & + & x_4 & = & -4 \\ & & & - & 4x_3 & + & 4x_4 & = & -4 \end{array} \right.$$

resolves to

$$\mathbf{x} = \begin{bmatrix} -1 + t \\ 2 + 3t \\ 1 - t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix},$$

for $t \in \mathbb{R}$. It is suggested that the readers find the solutions for various values of $\mathbf{b}$. □

Problem 1.23 Determine an LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

and then find solutions of $A\mathbf{x} = \mathbf{b}$ for (1) $\mathbf{b} = [1\ 1\ 1]^T$ and (2) $\mathbf{b} = [2\ 0\ -1]^T$.

Problem 1.24 Let A, B be two lower triangular matrices. Prove that
(1) their product is also a lower triangular matrix;
(2) if A is invertible, then its inverse is also a lower triangular matrix;
(3) if the diagonal entries are all 1's, then the same holds for their product and their inverses.
Note that the same holds for upper triangular matrices, and for the product of more than two matrices.

Now suppose that A is a *nonsingular square* matrix with $A = LU$ in which no row interchanges were necessary. Then the pivots on the diagonal of U are all nonzero, and the diagonal of L are all 1's. Thus, by dividing each i-th row of U by the nonzero pivot d_i, the matrix U is factorized into a diagonal matrix D whose diagonals are just the pivots $d_1, d_2, \ldots, d_n$ and a new upper triangular matrix, denoted again by U, whose diagonals are all 1's so that $A = LDU$. For example,

$$\begin{bmatrix} d_1 & r & \cdots & s \\ 0 & d_2 & & t \\ \vdots & & \ddots & u \\ 0 & & \cdots & d_n \end{bmatrix} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & & \cdots & d_n \end{bmatrix} \begin{bmatrix} 1 & r/d_1 & & s/d_1 \\ 0 & 1 & & t/d_2 \\ \vdots & & \ddots & u/d_{n-1} \\ 0 & & \cdots & 1 \end{bmatrix}.$$

This decomposition of A is called the LDU **factorization** of A. Note that, in this factorization, U is just a row-echelon form of A (with leading 1's on the diagonal) after Gaussian elimination and before back substitution.

In Example 1.11, we found a factorization of A as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix}.$$

This can be further factored as $A = LDU$ by taking

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = DU.$$

Suppose now that during forward elimination row interchanges are necessary. In this case, we can first do all the row interchanges before doing any other type of elementary row operations, since the interchange of rows can be done at any time, before or after the other operations, with the same effect on the solution. Those "row-interchanging" elementary matrices altogether form a permutation matrix P so that no more row interchanges are needed during Gaussian elimination of PA. So PA has an LDU factorization.

Example 1.12 Consider a square matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. For Gaussian elimination, it is clearly necessary to interchange the first row with the third row, that is, we need to multiply the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ to A so that

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = LU.$$

□

Of course, if we choose a different permutation P', then the LDU factorization of $P'A$ may be different from that of PA, even if there is another permutation matrix P'' that changes $P'A$ to PA. However, if we fix a permutation matrix P when it is necessary, the uniqueness of the LDU factorization of A can be proved.

Theorem 1.10 *For an invertible matrix A, the LDU factorization of A is unique up to a permutation: that is, for a fixed P the expression $PA = LDU$ is unique.*

Proof: Suppose that $A = L_1D_1U_1 = L_2D_2U_2$, where the L's are lower triangular, the U's are upper triangular, all with 1's on the diagonal, and the D's are diagonal matrices with no zeros on the diagonal. We need to show $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$.

Note that the inverse of a lower (upper) triangular matrix is also a lower (upper) triangular matrix. And the inverse of a diagonal matrix is also diagonal. Therefore, by multiplying $(L_1D_1)^{-1} = D_1^{-1}L_1^{-1}$ on the left and U_2^{-1} on the right, our equation $L_1D_1U_1 = L_2D_2U_2$ becomes

$$U_1U_2^{-1} = D_1^{-1}L_1^{-1}L_2D_2.$$

The left side is an upper triangular matrix, while the right side is a lower triangular matrix. Hence, both sides must be diagonal. However, since the diagonal entries of the upper triangular matrix $U_1U_2^{-1}$ are all 1's, it must be the identity matrix I (see Problem 1.24). Thus $U_1U_2^{-1} = I$, *i.e.*, $U_1 = U_2$. Similarly, $L_1^{-1}L_2 = D_1D_2^{-1}$ implies that $L_1 = L_2$ and $D_1 = D_2$. □

In particular, if A is symmetric (*i.e.*, $A = A^T$), and if it can be factored into $A = LDU$ without row interchanges, then we have

$$LDU = A = A^T = (LDU)^T = U^TD^TL^T = U^TDL^T,$$

and thus, by the uniqueness of factorizations, we have $U = L^T$ and $A = LDL^T$.

Problem 1.25 Find the factors L, D, and U for $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

What is the solution to $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = [1\ 0\ -1]^T$?

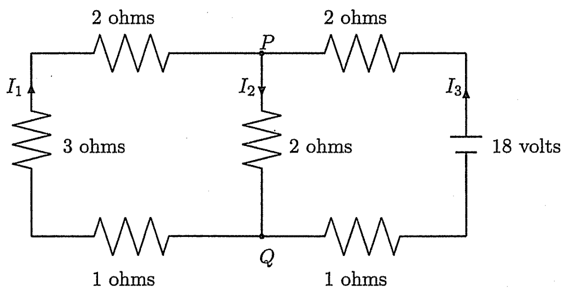
Problem 1.26 For all possible permutation matrices P, find the LDU factorization of PA for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

1.9 Application: Linear models

(1) In an electrical network, a simple current flow may be illustrated by a diagram like the one below. Such a network involves only voltage sources, like batteries, and resistors, like bulbs, motors, or refrigerators. The voltage is measured in *volts*, the resistance in *ohms*, and the current flow in amperes (*amps*, in short). For such an electrical network, current flow is governed by the following three laws:

- **Ohm's Law:** The voltage drop V across a resistor is the product of the current I and the resistance R: $V = IR$.
- **Kirchhoff's Current Law (KCL):** The current flow into a node equals the current flow out of the node.
- **Kirchhoff's Voltage Law (KVL):** The algebraic sum of the voltage drops around a closed loop equals the total voltage sources in the loop.

Example 1.13 Determine the currents in the network given in the above figure.



Solution: By applying KCL to nodes P and Q, we get equations

$$\begin{aligned} I_1 + I_3 &= I_2 \text{ at } P, \\ I_2 = I_1 &+ I_3 \text{ at } Q. \end{aligned}$$

Observe that both equations are the same, and one of them is redundant. By applying KVL to each of the loops in the network clockwise direction, we get

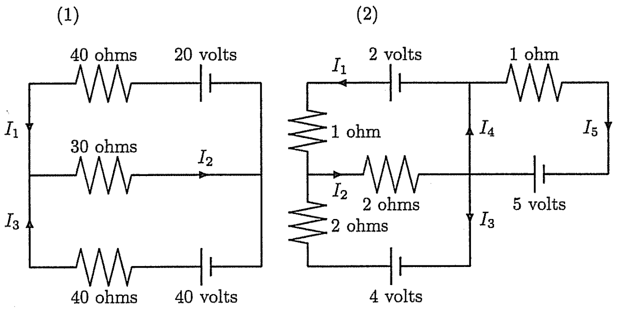
$$\begin{aligned} 6I_1 + 2I_2 &= 0 \text{ from the left loop,} \\ 2I_2 + 3I_3 &= 18 \text{ from the right loop.} \end{aligned}$$

Collecting all the equations, we get a system of linear equations:

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 6I_1 + 2I_2 = 0 \\ 2I_2 + 3I_3 = 18. \end{cases}$$

By solving it, the currents are $I_1 = -1$ amp, $I_2 = 3$ amps and $I_3 = 4$ amps. The negative sign for I_1 means that the current I_1 flows in the direction opposite to that shown in the figure. □

Problem 1.27 Determine the currents in the following networks.



(2) Cryptography is the study of sending messages in disguised form (secret codes) so that only the intended recipients can remove the disguise and read the message; modern cryptography uses advanced mathematics. As another application of invertible matrices, we introduce a simple coding. Suppose we associate a prescribed number with every letter in the alphabet; for example,

$$\begin{array}{ccccccccccc} A & B & C & D & \cdots & X & Y & Z & \text{Blank} & ? & ! \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 1 & 2 & 3 & \cdots & 23 & 24 & 25 & 26 & 27 & 28. \end{array}$$

Suppose that we want to send the message "GOOD LUCK". Replace this message by

$$6,\ 14,\ 14,\ 3,\ 26,\ 11,\ 20,\ 2,\ 10$$

according to the preceding substitution scheme. A code of this type could be cracked without difficulty by a number of techniques of statistical methods, like the analysis of frequency of letters. To make it difficult to crack the code, we first break the message into six vectors in $\mathbb{R}^3$, each with 3 components (optional), by adding extra blanks if necessary:

$$\begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 26 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 2 \\ 10 \end{bmatrix}.$$

Next, choose a nonsingular 3×3 matrix A, say

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which is supposed to be known to *both* sender and receiver. Then as a linear transformation A translates our message into

$$A \begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix} = \begin{bmatrix} 6 \\ 26 \\ 34 \end{bmatrix}, \quad A \begin{bmatrix} 3 \\ 26 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 32 \\ 40 \end{bmatrix}, \quad A \begin{bmatrix} 20 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 42 \\ 32 \end{bmatrix}.$$

By putting the components of the resulting vectors consecutively, we transmit

$$6,\ 26,\ 34,\ 3,\ 32,\ 40,\ 20,\ 42,\ 32.$$

To decode this message, the receiver may follow the following process. Suppose that we received the following reply from our correspondent:

$$19,\ 45,\ 26,\ 13,\ 36,\ 41.$$

To decode it, first break the message into two vectors in $\mathbb{R}^3$ as before:

$$\begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix}, \quad \begin{bmatrix} 13 \\ 36 \\ 41 \end{bmatrix}.$$

We want to find two vectors $\mathbf{x}_1$, $\mathbf{x}_2$ such that $A\mathbf{x}_i$ is the i-th vector of the above two vectors: *i.e.*,

$$A\mathbf{x}_1 = \begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix}, \quad A\mathbf{x}_2 = \begin{bmatrix} 13 \\ 36 \\ 41 \end{bmatrix}.$$

Since A is invertible, the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ can be found by multiplying the inverse of A to the two vectors given in the message. By an easy computation, one can find

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 19 \\ 45 \\ 26 \end{bmatrix} = \begin{bmatrix} 19 \\ 7 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 13 \\ 10 \\ 18 \end{bmatrix}.$$

The numbers one obtains are

$$19,\ 7,\ 0,\ 13,\ 10,\ 18.$$

Using our correspondence between letters and numbers, the message we have received is "THANKS".

Problem 1.28 Encode "TAKE UFO " using the same matrix A used in the above example.

(3) Another significant application of linear algebra is to a mathematical model in economics. In most nations, an economic society may be divided into many sectors that produce goods or services, such as the automobile industry, oil industry, steel industry, communication industry, and so on. Then a fundamental problem in economics is to find the *equilibrium* of the supply and the demand in the economy.

There are two kinds of demands for goods: the *intermediate demand* from the industries themselves (or the sectors) that are needed as inputs for their own production, and the *extra demand* from the consumer, the governmental use, surplus production, or exports. Practically, the interrelation between the sectors is very complicated, and the connection between the extra demand and the production is unclear. A natural question is *whether there is a production level such that the total amounts produced (or supply) will exactly balance the total demand for the production*, so that the equality

$$\begin{aligned}\{Total\ output\} &= \{Total\ demand\} \\ &= \{Intermediate\ demand\} + \{Extra\ demand\}\end{aligned}$$

holds. This problem can be described by a system of linear equations, which is called the *Leontief Input-Output Model.* To illustrate this, we show a simple example.

Suppose that a nation's economy consists of three sectors: I_1 = automobile industry, I_2 = steel industry, and I_3 = oil industry.

Let $\mathbf{x} = [x_1\, x_2\, x_3]^T$ denote the production vector (or production level) in $\mathbb{R}^3$, where each entry x_i denotes the total amount (in a common unit such as "dollars" rather than quantities such as "tons" or "gallons") of the output that the industry I_i produces per year.

The intermediate demand may be explained as follows. Suppose that, for the total output x_2 units of the steel industry I_2, 20% is contributed by the output of I_1, 40% by that of I_2 and 20% by that of I_3. Then we can write this as a column vector, called a *unit consumption vector* of I_2:

$$\mathbf{c}_2 = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.2 \end{bmatrix}.$$

For example, if I_2 decides to produce 100 units per year, then it will order (or demand) 20 units from I_1, 40 units from I_2, and 20 units from I_3: *i.e.*, the consumption vector of I_2 for the production $x_2 = 100$ units can be written as a column vector: $100\mathbf{c}_2 = [20\ 40\ 20]^T$. From the concept of the consumption vector, it is clear that the sum of decimal fractions in the column $\mathbf{c}_2$ must be ≤ 1.

In our example, suppose that the demands (inputs) of the outputs are given by the following matrix, called an *input-output matrix*:

$$A = \text{input} \begin{array}{c} \\ \\ I_1 \\ I_2 \\ I_3 \\ \\ \\ \end{array} \begin{array}{c} \text{output} \\ \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \left[\begin{array}{ccc} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{array} \end{array}.$$

In this matrix, an industry looks down a column to see how much it needs from where to produce its total output, and it looks across a row to see how much of its output goes to where. For example, the second row says that, out of the total output x_2 units of the steel industry I_2, as the intermediate demand, the automobile industry I_1 demands 10% of the output x_1, the steel industry I_2 demands 40% of the output x_2 and the oil industry I_3 demands 10% of the output x_3. Therefore, it is now easy to see that the intermediate demand of the economy can be written as

$$A\mathbf{x} = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_1 + 0.2x_2 + 0.3x_3 \\ 0.1x_1 + 0.4x_2 + 0.1x_3 \\ 0.3x_1 + 0.2x_2 + 0.3x_3 \end{bmatrix}.$$

Suppose that the extra demand in our example is given by $\mathbf{d} = [d_1, d_2, d_3]^T = [30, 20, 10]^T$. Then the problem for this economy is to find the production vector $\mathbf{x}$ satisfying the following equation:

$$\mathbf{x} = A\mathbf{x} + \mathbf{d}.$$

Another form of the equation is $(I - A)\mathbf{x} = \mathbf{d}$, where the matrix $I - A$ is called the *Leontief matrix*. If $I - A$ is not invertible, then the equation may have no solution or infinitely many solutions depending on what $\mathbf{d}$ is. If $I - A$ is invertible, then the equation has the unique solution $\mathbf{x} = (I - A)^{-1}\mathbf{d}$. Now, our example can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}.$$

In this example, it turns out that the matrix $I - A$ is invertible and

$$(I - A)^{-1} = \begin{bmatrix} 2.0 & 1.0 & 1.0 \\ 0.5 & 2.0 & 0.5 \\ 1.0 & 1.0 & 2.0 \end{bmatrix}.$$

Therefore,

$$\mathbf{x} = (I - A)^{-1}\mathbf{d} = \begin{bmatrix} 90 \\ 60 \\ 70 \end{bmatrix},$$

which gives the total amount of product x_i of the industry I_i for one year to meet the required demand.

Remark: (1) Under the usual circumstances, the sum of the entries in a column of the consumption matrix A is less than one because a sector should require less than one unit's worth of inputs to produce one unit of output. This actually implies that $I - A$ is invertible and the production vector $\mathbf{x}$ is feasible in the sense that the entries in $\mathbf{x}$ are all nonnegative as the following argument shows.

(2) In general, by using induction one can easily verify that for any $k = 1, 2, \ldots,$

$$(I - A)(I + A + \cdots + A^{k-1}) = I - A^k.$$

If the sums of column entries of A are all strictly less than one, then $\lim_{k\to\infty} A^k = \mathbf{0}$ (see Section 6.6 for the limit of a sequence of matrices). Thus, we get $(I - A)(I + A + \cdots + A^k + \cdots) = I$, that is,

$$(I - A)^{-1} = I + A + \cdots + A^k + \cdots.$$

This also shows a practical way of computing $(I - A)^{-1}$ since by taking k sufficiently large the right side may be made very close to $(I - A)^{-1}$. In Chapter 6, an easier method of computing A^k will be shown.

In summary, if A and $\mathbf{d}$ have nonnegative entries and if the sum of the entries of each column of A is less than one, then $I - A$ is invertible and the inverse is given as the above formula. Moreover, as the formula shows the entries of the inverse are all nonnegative, and so are those of the production vector $\mathbf{x} = (I - A)^{-1}\mathbf{d}$.

Problem 1.29 Determine the total demand for industries I_1, I_2 and I_3 for the input-output matrix A and the extra demand vector $\mathbf{d}$ given below:

$$A = \begin{bmatrix} 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \\ 0.4 & 0.2 & 0.2 \end{bmatrix} \text{ with } \mathbf{d} = \mathbf{0}.$$

Problem 1.30 Suppose that an economy is divided into three sectors: I_1 = services, I_2 = manufacturing industries, and I_3 = agriculture. For each unit of output, I_1 demands no services from I_1, 0.4 units from I_2, and 0.5 units from I_3. For each unit of output, I_2 requires 0.1 units from sector I_1 of services, 0.7 units from other parts in sector I_2, and no product from sector I_3. For each unit of output, I_3 demands 0.8 units of services I_1, 0.1 units of manufacturing products from I_2, and 0.1 units of its own output from I_3. Determine the production level to balance the economy when 90 units of services, 10 units of manufacturing, and 30 units of agriculture are required as the extra demand.

1.10 Exercises

1.1. Which of the following matrices are in row-echelon form or in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & -2 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1.2. Find a row-echelon form of each matrix.

$$(1) \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \\ 3 & -9 & 10 & 2 & 9 \\ 2 & -6 & 4 & 2 & 4 \\ 2 & -6 & 8 & 1 & 7 \end{bmatrix} \quad (2) \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

1.3. Find the reduced row-echelon form of the matrices in Exercise **1.2.**

1.4. Solve the systems of equations by Gauss-Jordan elimination.

$$(1) \begin{cases} x_1 + x_2 + x_3 - x_4 = -2 \\ 2x_1 - x_2 + x_3 + x_4 = 0 \\ 3x_1 + 2x_2 - x_3 - x_4 = 1 \\ x_1 + x_2 + 3x_3 - 3x_4 = -8\,. \end{cases}$$

$$(2) \begin{cases} 2x - 3y = 8 \\ 4x - 5y + z = 15 \\ 2x + 4z = 1\,. \end{cases}$$

What are the pivots in each elimination step?

1.5. Which of the following systems has a nontrivial solution?

$$(1) \begin{cases} x + 2y + 3z = 0 \\ 2y + 2z = 0 \\ x + 2y + 3z = 0. \end{cases} \quad (2) \begin{cases} 2x + y - z = 0 \\ x - 2y - 3z = 0 \\ 3x + y - 2z = 0. \end{cases}$$

1.6. Determine all values of the b_i that make the following system consistent:

$$\begin{cases} x + y - z = b_1 \\ 2y + z = b_2 \\ y - z = b_3\,. \end{cases}$$

1.7. Determine the condition on b_i so that the following system has no solution:

$$\begin{cases} 2x + y + 7z = b_1 \\ 6x - 2y + 11z = b_2 \\ 2x - y + 3z = b_3 . \end{cases}$$

1.8. Let A and B be matrices of the same size.

(1) Show that, if $A\mathbf{x} = 0$ for all $\mathbf{x}$, then A is the zero matrix.

(2) Show that, if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$, then $A = B$.

1.9. Compute ABC and CAB, for

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

1.10. Prove that if A is a 3×3 matrix such that $AB = BA$ for every 3×3 matrix B, then $A = kI_3$ for some constant k.

1.11. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. Find A^k for all integers k.

1.12. Compute $(2A - B)C$ and CC^T for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

1.13. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. For any square matrix A, a *matrix polynomial* $f(A)$ is defined as $f(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$. For $f(x) = 3x^3 + x^2 - 2x + 3$, find $f(A)$ for

$$(1)\ A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

1.14. Find the symmetric part and the skew-symmetric part of each of the following matrices.

$$(1)\ A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 5 & 9 \\ -1 & 3 & 2 \end{bmatrix} \qquad (2)\ A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

1.15. Find AA^T and A^TA for the matrix $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 3 & 1 \\ 2 & 8 & 4 & 0 \end{bmatrix}$.

1.16. Let $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$.

(1) Find a matrix B such that $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$.

(2) Find a matrix C such that $AC = A^2 + A$.

1.17. Find all possible choices of a, b and c so that $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ has an inverse matrix such that $A^{-1} = A$.

1.18. Decide whether or not each of the following matrices is invertible. Find the inverses for invertible ones.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

1.19. Suppose A is a 2×1 matrix and B is a 1×2 matrix. Prove that the product AB is not invertible.

1.20. Find three matrices which are row equivalent to $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 5 & 2 & -3 & 4 \end{bmatrix}$.

1.21. Write the following systems of equations as matrix equations $A\mathbf{x} = \mathbf{b}$ and solve them by computing $A^{-1}\mathbf{b}$:

$$(1) \left\{ \begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 3x_3 & = & 2 \\ & & x_2 & - & 4x_3 & = & 5 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 7, \end{array} \right. \quad (2) \left\{ \begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 5 \\ x_1 & + & x_2 & - & x_3 & = & -1 \\ 4x_1 & - & 3x_2 & + & 2x_3 & = & -3\,. \end{array} \right.$$

1.22. Find the LDU factorization for each of the following matrices:

(1) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$, (2) $A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$.

1.23. Find the LDL^T factorization of the following symmetric matrices:

(1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 3 & 8 & 10 \end{bmatrix}$, (2) $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

1.24. Solve $A\mathbf{x} = \mathbf{b}$ with $A = LU$, where L and U are given as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

Forward elimination is the same as $L\mathbf{c} = \mathbf{b}$, and back-substitution is $U\mathbf{x} = \mathbf{c}$.

1.25. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 5 \\ 1 & 4 & 7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

(1) Solve $A\mathbf{x} = \mathbf{b}$ by Gauss-Jordan elimination.
(2) Find the LDU factorization of A.
(3) Write A as a product of elementary matrices.
(4) Find the inverse of A.

1.26. A square matrix A is said to be *nilpotent* if $A^k = \mathbf{0}$ for a positive integer k.

(1) Show that an invertible matrix is not nilpotent.
(2) Show that any triangular matrix with zero diagonal is nilpotent.
(3) Show that if A is a nilpotent with $A^k = \mathbf{0}$, then $I - A$ is invertible with its inverse $I + A + \cdots + A^{k-1}$.

1.27. A square matrix A is said to be *idempotent* if $A^2 = A$.

(1) Find an example of an idempotent matrix other than $\mathbf{0}$ or I.
(2) Show that, if a matrix A is both idempotent and invertible, then $A = I$.

1.28. Determine whether the following statements are true or false, in general, and justify your answers.

(1) Let A and B be row-equivalent square matrices. Then A is invertible if and only if B is invertible.
(2) Let A be a square matrix such that $AA = A$. Then A is the identity.
(3) If A and B are invertible matrices such that $A^2 = I$ and $B^2 = I$, then $(AB)^{-1} = BA$.
(4) If A and B are invertible matrices, $A + B$ is also invertible.
(5) If A, B and AB are symmetric, then $AB = BA$.
(6) If A and B are symmetric and the same size, then AB is also symmetric.
(7) Let $AB^T = I$. Then A is invertible if and only if B is invertible.
(8) If a square matrix A is not invertible, then neither is AB for any B.
(9) If E_1 and E_2 are elementary matrices, then $E_1E_2 = E_2E_1$.
(10) The inverse of an invertible upper triangular matrix is upper triangular.
(11) Any invertible matrix A can be written as $A = LU$, where L is lower triangular and U is upper triangular.
(12) If A is invertible and symmetric, then A^{-1} is also symmetric.

Chapter 2

Determinants

2.1 Basic properties of determinant

Our primary interest in Chapter 1 was in the solvability or solutions of a system $A\mathbf{x} = \mathbf{b}$ of linear equations. For an invertible matrix A, Theorem 1.8 shows that the system has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$.

Now the question is how to decide whether or not a square matrix A is invertible. In this section, we introduce the notion of *determinant* as a real-valued function of square matrices that satisfies certain axiomatic rules, and then show that a square matrix A is invertible if and only if the determinant of A is not zero. In fact, we saw in Chapter 1 that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. This number is called the determinant of A, and is defined formally as follows:

Definition 2.1 For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$, the **determinant** of A is defined as $\det A = ad - bc$.

In fact, it turns out that geometrically the determinant of a 2×2 matrix A represents, up to sign, the area of a parallelogram in the xy-plane whose edges are constructed by the row vectors of A (see Theorem 2.9), so it will be very nice if we can have the same idea of determinant for higher order matrices. However, the formula itself in Definition 2.1 does not provide any clue of how to extend this idea of determinant to higher order matrices. Hence, we first examine some fundamental properties of the determinant function defined in Definition 2.1.

By a direct computation, one can easily verify that the function det in Definition 2.1 satisfies the following three fundamental properties:

(1) $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$

(2) $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = bc - ad = -(ad - bc) = -\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$

(3) $$\begin{aligned} \det \begin{bmatrix} ka + \ell a' & kb + \ell b' \\ c & d \end{bmatrix} &= (ka + \ell a')d - (kb + \ell b')c \\ &= k(ad - bc) + \ell(a'd - b'c) \\ &= k \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \ell \det \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}. \end{aligned}$$

Actually all the important properties of the determinant function can be derived from these three properties. We will show in Lemma 2.3 that if a function $f : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ satisfies the properties (1), (2) and (3) above, then it must be of the form $f(A) = ad - bc$. An advantage of looking at these properties of the determinant rather than looking at the explicit formula given in Definition 2.1 is that these three properties enable us to define the determinant function for any $n \times n$ square matrices.

Definition 2.2 A real-valued function $f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ of all $n \times n$ square matrices is called a **determinant** if it satisfies the following three rules:

($\mathbf{R}_1$) the value of f of the identity matrix is 1, *i.e.*, $f(I_n) = 1$;

($\mathbf{R}_2$) the value of f changes sign if any two rows are interchanged;

($\mathbf{R}_3$) f is linear in the first row: that is, by definition,

$$f\left(\begin{bmatrix} k\mathbf{r}_1 + \ell\mathbf{r}'_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix}\right) = kf\left(\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix}\right) + \ell f\left(\begin{bmatrix} \mathbf{r}'_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix}\right),$$

where $\mathbf{r}_i$'s denote the row vectors of a matrix.

It is already shown that the det on 2×2 matrices satisfies these rules. We will show later that for each positive integer n there always exists such a function $f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ satisfying the three rules in the definition, and, moreover, it is unique. Therefore, we say "the" determinant and designate it as "det" in any order.

Let us first derive some direct consequences of the three rules in the definition (the readers are suggested to verify that det of 2×2 matrices also satisfies the following properties):

Theorem 2.1 *The determinant satisfies the following properties.*

(1) *The determinant is linear in each row, i.e., for each row the rule* $(\mathbf{R}_3)$ *also holds.*

(2) *If A has either a zero row or two identical rows, then* $\det A = 0$.

(3) *The elementary row operation that adds a constant multiple of one row to another row leaves the determinant unchanged.*

Proof: **(1)** Any row can be placed in the first row with a change of sign in the determinant by rule $(\mathbf{R}_2)$, and then use rules $(\mathbf{R}_3)$ and $(\mathbf{R}_2)$.

(2) If A has a zero row, then the row is zero times the zero row. If A has two identical rows, then interchanging these identical rows changes only the sign of the determinant, but not A itself. Thus we get $\det A = -\det A$.

(3) By a direct computation using (1), we get

$$f\left(\begin{bmatrix} \vdots \\ \mathbf{r}_i + k\mathbf{r}_j \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{bmatrix}\right) = f\left(\begin{bmatrix} \vdots \\ \mathbf{r}_i \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{bmatrix}\right) + kf\left(\begin{bmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{bmatrix}\right),$$

in which the second term on the right side is zero by (2). □

It is now easy to see the effect of elementary row operations on evaluations of the determinant. The first elementary row operation that "multiplies a constant k to a row" changes the determinant to k times the determinant by (1) of Theorem 2.1. The rule $(\mathbf{R}_2)$ in the definition explains the effect of the elementary row operation that "interchanges two rows". The last elementary row operation that "adds a constant multiple of a row to another" is explained in (3) of Theorem 2.1.

Example 2.1 Consider a matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & b+a \end{bmatrix}.$$

If we add the second row to the third, then the third row becomes

$$[a+b+c \quad a+b+c \quad a+b+c],$$

which is a scalar multiple of the first row. Thus, $\det A = 0$. □

Problem 2.1 Show that, for an $n \times n$ matrix A and $k \in \mathbb{R}$, $\det(kA) = k^n \det A$.

Problem 2.2 Explain why $\det A = 0$ for

$$(1)\, A = \begin{bmatrix} a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \\ a+3 & a+6 & a+9 \end{bmatrix}, \qquad (2)\, A = \begin{bmatrix} a & a^4 & a^7 \\ a^2 & a^5 & a^8 \\ a^3 & a^6 & a^9 \end{bmatrix}.$$

Recall that any square matrix can be transformed to an upper triangular matrix by forward eliminations. Further properties of the determinant are obtained in the following theorem.

Theorem 2.2 *The determinant satisfies the following properties.*

(1) *The determinant of a triangular matrix is the product of the diagonal entries.*

(2) *The matrix A is invertible if and only if* $\det A \neq 0$.

(3) *For any two $n \times n$ matrices A and B,* $\det(AB) = \det A \det B$.

(4) $\det A^T = \det A$.

Proof: **(1)** If A is a diagonal matrix, then it is clear that $\det A = a_{11} \cdots a_{nn}$ by (1) of Theorem 2.1 and rule $(\mathbf{R}_1)$. Suppose that A is a lower triangular matrix. Then a forward elimination, which does not change the determinant, produces a zero row if A has a zero diagonal entry, or makes A row equivalent to the diagonal matrix D whose diagonal entries are exactly those of A if the diagonal entries are all nonzero. Thus, in the former case, $\det A = 0$ and the product of the diagonal entries is also zero. In the latter case, $\det A = \det D = a_{11} \cdots a_{nn}$. Similar arguments apply when A is an upper triangular matrix.

(2) Note again that a forward elimination reduces a square matrix A to an upper triangular matrix, which has a zero row if A is singular and has no zero row if A is nonsingular (see Theorem 1.8).

(3) If A is not invertible, then AB is not invertible, and so $\det(AB) = 0 = \det A \det B$. By the properties of the elementary matrices, it is clear that for any elementary matrix E, $\det(EB) = \det E \det B$. If A is invertible, it can be written as a product of elementary matrices, say $A = E_1E_2\cdots E_k$. Then by induction on k, we get

$$\begin{aligned}\det(AB) &= \det(E_1E_2\cdots E_kB)\\ &= \det E_1 \ \det E_2 \ \cdots \ \det E_k \ \det B\\ &= \det(E_1E_2\cdots E_k)\det B\\ &= \det A \det B.\end{aligned}$$

(4) Clearly, A is not invertible if and only if A^T is not. Thus for a singular matrix A we have $\det A^T = 0 = \det A$. If A is invertible, then there is a factorization $PA = LDU$ for a permutation matrix P. By (3), we get

$$\det P \ \det A = \det L \ \det D \ \det U.$$

Note that the transpose of $PA = LDU$ is $A^TP^T = U^TD^TL^T$ and that for any triangular matrix B, $\det B = \det B^T$ by (1). In particular, since L, U, L^T, and U^T are triangular with 1's on the diagonal, their determinants are all equal to 1. Therefore, we have

$$\begin{aligned}\det A^T \det P^T &= \det U^T \det D^T \det L^T\\ &= \det L \ \det D \ \det U \ = \ \det A \ \det P.\end{aligned}$$

By the definition, a permutation matrix P is obtained from the identity matrix by a sequence of row interchanges: that is, $P = E_k\cdots E_1I_n$ for some k, where each E_i is an elementary matrix obtained from the identity matrix by interchanging two rows. Thus, $\det E_i = -1$ for each $i = 1,\ldots,k$, and clearly $E_i^T = E_i = E_i^{-1}$. Therefore, $\det P = (-1)^k = \det P^T$ by (3), so $\det A = \det A^T$. □

Remark: From the equality $\det A = \det A^T$, we could define the determinant in terms of columns instead of rows in Definition 2.2, and Theorem 2.1 is also true with "columns" instead of "rows".

Example 2.2 Evaluate the determinant of the following matrix A:

$$A = \begin{bmatrix} 2 & -4 & 0 & 0\\ 1 & -3 & 0 & 1\\ 1 & 0 & -1 & 2\\ 3 & -4 & 3 & -1\end{bmatrix}.$$

Solution: By using forward elimination, A can be transformed to an upper triangular matrix U. Since the forward elimination does not change the determinant, the determinant of A is simply the product of the diagonal entries of U:

$$\begin{aligned} \det A = \det U &= \det \begin{bmatrix} 2 & -4 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 13 \end{bmatrix} \\ &= 2 \cdot (-1)^2 \cdot 13 = 26. \end{aligned}$$

□

Problem 2.3 Prove that if A is invertible, then $\det A^{-1} = 1/\det A$.

Problem 2.4 Evaluate the determinant of each of the following matrices:

$$(1) \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ -2 & 2 & 3 \end{bmatrix}, \quad (2) \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & x & x^2 & x^3 \\ x^3 & 1 & x & x^2 \\ x^2 & x^3 & 1 & x \\ x & x^2 & x^3 & 1 \end{bmatrix}.$$

2.2 Existence and uniqueness

Recall that $\det A = ad - bc$ defined in the previous section satisfies the three rules of Definition 2.2. Conversely, the following lemma shows that any function of $M_{2\times2}(\mathbb{R})$ into $\mathbb{R}$ satisfying the three rules $(\mathbf{R}_1)$ - $(\mathbf{R}_3)$ of Definition 2.2 must be det, which implies the uniqueness of the determinant function on $M_{2\times2}(\mathbb{R})$.

Lemma 2.3 *If $f : M_{2\times2}(\mathbb{R}) \to \mathbb{R}$ satisfies the three rules in* Definition 2.2, *then $f(A) = ad - bc$.*

Proof: First, note that $f\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$ by the rules $(\mathbf{R}_1)$ and $(\mathbf{R}_2)$.

$$\begin{aligned} f(A) &= f\begin{bmatrix} a & b \\ c & d \end{bmatrix} = f\begin{bmatrix} a+0 & 0+b \\ c & d \end{bmatrix} \\ &= f\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + f\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \\ &= f\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + f\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + f\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} + f\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \\ &= ad + 0 + 0 - bc. \end{aligned}$$

□

Therefore, when $n = 2$ there is only one function f on $M_{2\times 2}(\mathbb{R})$ which satisfies the three rules: *i.e.*, $f = \det$.

Now for $n = 3$, the same calculation as in the case of $n = 2$ can be applied. That is, by repeated use of the three rules $(\mathbf{R}_1)$ - $(\mathbf{R}_3)$ as in the proof of Lemma 2.3, we can obtain the explicit formula for the determinant function on $M_{3\times 3}(\mathbb{R})$ as follows:

$$\begin{aligned}
&\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
= &\det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \\
&+ \det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \\
= &\ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.
\end{aligned}$$

This expression of $\det A$ for a matrix $A \in M_{3\times 3}(\mathbb{R})$ satisfies the three rules. Therefore, for $n = 3$, it shows both the uniqueness and the existence of the determinant function on $M_{3\times 3}(\mathbb{R})$.

Problem 2.5 Show that the above formula of the determinant for 3×3 matrices satisfies the three rules in Definition 2.2.

To get the formula of the determinant for matrices of order $n > 3$, the same computational process can be repeated using the three rules again, but the computation is going to be more complicated as the order gets higher. To derive the explicit formula for $\det A$ of order $n > 3$, we examine the above case in detail. In the process of deriving the explicit formula for $\det A$ of a 3×3 matrix A, we can observe the following three steps:

(1st) By using the linearity of the determinant function in each row, $\det A$ of a 3×3 matrix A is expanded as the sum of the determinants of $3^3 = 27$ matrices. Except for exactly six matrices, all of them have zero columns so that their determinants are zero (see the proof of Lemma 2.3).

(2nd) In each of these remaining six matrices, all entries are zero except for exactly three entries that came from the given matrix A. Indeed, no two of the three entries came from the same column or from the same row of A.

In other words, in each row there is only one entry that came from A and at the same time in each column there is only one entry that came from A.

Actually, in each of the six matrices, the three entries from A, say a_{ij}, $a_{k\ell}$, and a_{pq}, are chosen as follows: If the first entry a_{ij} is chosen from the first row and the third column of A, say a_{13}, then the other entries $a_{k\ell}$ and a_{pq} in the product should be chosen from the second or the third row and the first or the second column. Thus, if the second entry $a_{k\ell}$ is taken from the second row, the column it belongs to must be either the first or the second, *i.e.*, either a_{21} or a_{22}. If a_{21} is taken, then the third entry a_{pq} must be, without option, a_{32}. Thus, the entries from A in the chosen matrix are a_{13}, a_{21}, and a_{32}. Therefore, the three entries in each of the six remaining matrices are determined as follows: when the row indices (the first indices i of a_{ij}) are arranged in the order 1, 2, 3, the assignment of the column indices 1, 2, 3 (the second indices j of a_{ij}) to each of the row indices is simply a re-arrangement of 1, 2, 3 without repetitions or omissions. In this way, one can recognize that the number $6 = 3!$ is simply the number of ways in which the three column indices 1, 2, 3 are rearranged.

(3rd) The determinant of each of the six matrices may be computed by converting it into a diagonal matrix using suitable "column interchanges" (see Theorem 2.2 (1)), so its determinant becomes $\pm a_{ij}a_{k\ell}a_{pq}$, where the sign depends on the number of column interchanges.

For example, for the matrix having entries a_{13}, a_{22}, and a_{31} from A, one can convert this matrix into a diagonal matrix in a couple of ways: for instance, one can take just one interchanging of the first and the third columns or take three interchanges: the first and the second, and then the second and the third, and then the first and the second. In any case,

$$\det \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} a_{13} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{31} \end{bmatrix} = -a_{13}a_{22}a_{31}.$$

Note that an interchange of two columns is the same as an interchange of two corresponding column indices. As mentioned above, there may be several ways of column interchanges to convert the given matrix to a diagonal matrix. However, it is very interesting that, whatever ways of column interchanges we take, the parity of the number of column interchanges remains the same all the time.

In this example, the given arrangement of the column is expressed in the arrangement of column indices, which is 3, 2, 1. Thus, to arrive at the order 1, 2, 3, which represents the diagonal matrix, we can take either just one interchanging of 3 and 1, or three interchanges: 3 and 2, 3 and 1, and then 2 and 1. In either case, the parity is odd so that the "$-$" sign in the computation of determinant came from $(-1)^1 = (-1)^3$, where the exponents mean the numbers of interchanges of the column indices.

In summary, in the expansion of $\det A$ for $A \in M_{3\times 3}(\mathbb{R})$, the number $6 = 3!$ of the determinants which contribute to the computation of $\det A$ is simply the number of ways in which the three numbers 1, 2, 3 are rearranged without repetitions or omissions. Moreover, the sign of each of the six determinants is determined by the parity (even or odd) of the number of column interchanges required to arrive at the order of 1, 2, 3 from the given arrangement of the column indices.

These observations can be used to derive the explicit formula of the determinant for matrices of order $n > 3$. We begin with the following definition.

Definition 2.3 A **permutation** of the set of integers $N_n = \{1, 2, \ldots, n\}$ is a one-to-one function from N_n onto itself.

Therefore, a permutation σ of N_n assigns a number $\sigma(i)$ in N_n to each number i in N_n, and this permutation σ is commonly denoted by

$$\sigma = \langle \sigma(1), \sigma(2), \ldots, \sigma(n) \rangle = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Here, the first row is the usual lay-out of N_n as the domain set, and the second row is just an arrangement in a certain order without repetitions or omissions of the numbers in N_n as the image set. A permutation that interchanges only two numbers in N_n, leaving the rest of the numbers fixed, such as $\sigma = \langle 3, 2, 1, \ldots, n \rangle$, is called a **transposition**. Note that the composition of any two permutations is also a permutation. Moreover, the composition of a transposition to a permutation σ produces an interchanging of two numbers in the permutation σ. In particular, the composition of a transposition with itself always produces the identity permutation.

It is not hard to see that if S_n denotes the set of all permutations of N_n, then S_n has exactly $n!$ permutations.

Once we have listed all the permutations, the next step is to determine the sign of each permutation. A permutation $\sigma = \langle j_1, j_2, \ldots, j_n \rangle$ is said to have an **inversion** if $j_s > j_t$ for $s < t$ (*i.e.*, a larger number precedes a smaller number). For example, the permutation $\sigma = \langle 3, 1, 2 \rangle$ has two inversions since 3 precedes 1 and 2.

An inversion in a permutation can be eliminated by composing it with a suitable transposition: for example, if $\sigma = \langle 3, 2, 1\rangle$ with three inversions, then by multiplying a transposition $\langle 2, 1, 3\rangle$ to it, we get $\langle 2, 3, 1\rangle$ with two inversions, which is the same as interchanging the first two numbers 3, 2 in σ. Therefore, given a permutation $\sigma = \langle \sigma(1), \sigma(2), \ldots, \sigma(n)\rangle$ in S_n, one can convert it to the identity permutation $\langle 1, 2, \ldots, n\rangle$, which is the only one with no inversions, by composing it with certain number of transpositions. For example, by composing the three (which is the number of inversions in σ) transpositions $\langle 2, 1, 3\rangle$, $\langle 1, 3, 2\rangle$ and $\langle 2, 1, 3\rangle$ with $\sigma = \langle 3, 2, 1\rangle$, we get the identity permutation. However, the number of necessary transpositions to convert the given permutation into the identity permutation need not be unique as we have seen in the third step. Notice that even if the number of necessary transpositions is not unique the parity (even or odd) is always consistent with the number of inversions.

Recall that all we need in the computation of the determinant is just the parity (even or odd) of the number of column interchanges, which is the same as that of the number of inversions in the permutation of the column indices.

A permutation is said to be **even** if it has an even number of inversions, and it is said to be **odd** if it has an odd number of inversions. For example, when $n = 3$, the permutations $\langle 1, 2, 3\rangle$, $\langle 2, 3, 1\rangle$ and $\langle 3, 1, 2\rangle$ are even, while the permutations $\langle 1, 3, 2\rangle$, $\langle 2, 1, 3\rangle$ and $\langle 3, 2, 1\rangle$ are odd. In general, for a permutation σ in S_n, the **sign** of σ is defined as

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

It is not hard to see that the number of even permutations is equal to that of odd permutations, so it is $\frac{n!}{2}$. In the case $n = 3$, one can notice that there are 3 terms with $+$ sign and 3 terms with $-$ sign.

Problem 2.6 Show that the number of even permutations and the number of odd permutations in S_n are equal.

Now, we repeat the three steps to get an explicit formula for $\det A$ of a square matrix $A = [a_{ij}]$ of order n. First, the determinant $\det A$ can be expressed as the sum of determinants of $n!$ matrices, each of which has zero entries except the n entries $a_{1\sigma(1)}, a_{2\sigma(2)}, \cdots, a_{n\sigma(n)}$ taken from A, where σ is a permutation of the set $\{1, 2, \ldots, n\}$ of column indices. The n entries $a_{1\sigma(1)}, a_{2\sigma(2)}, \cdots, a_{n\sigma(n)}$ are chosen from A in such a way that no two of them come from the same row or the same column. Such a matrix can be converted to a diagonal matrix. Hence, its determinant is equal to $\pm a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$, where the sign $\pm$ is determined by the parity of the number of column interchanges to convert the matrix to a diagonal matrix, which is equal to that of inversions in σ: $\mathrm{sgn}(\sigma)$. Therefore, the determinant of the matrix whose entries are all zero except for $a_{i\sigma(i)}$'s is equal to

$$\mathrm{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)},$$

which is called a **signed elementary product** of A. Now, our discussions can be summarized as follows:

Theorem 2.4 *For an $n \times n$ matrix A,*

$$\det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}.$$

That is, $\det A$ is the sum of all signed elementary products of A.

It is not difficult to see that this explicit formula for $\det A$ satisfies the three rules in the definition of the determinant. Therefore, we have both *existence* and *uniqueness* for the determinant function of square matrices of any order $n \geq 1$.

Example 2.3 Consider a permutation $\sigma = \langle 3,4,2,5,1\rangle \in S_5$: *i.e.*, $\sigma(1) = 3$, $\sigma(2) = 4, \ldots, \sigma(5) = 1$. Then σ has total $2 + 4 = 6$ inversions: two inversions caused by the position of $\sigma(1) = 3$, which precedes 1 and 2, and four inversions in the permutation $\tau = \langle 4,2,5,1\rangle$, which is a permutation of the set $\{1,2,4,5\}$. Thus,

$$\mathrm{sgn}(\sigma) = (-1)^{2+4} = (-1)^2\mathrm{sgn}(\tau).$$

Note that the permutation τ can be considered as a permutation of N_4 by replacing the numbers 4 and 5 by 3 and 4, respectively.

Moreover, $\sigma = \langle 3,4,2,5,1\rangle$ can be converted to $\langle 1,3,4,2,5\rangle$ by shifting the number 1 by four transpositions, and then $\langle 1,3,4,2,5\rangle$ can be converted to the identity permutation $\langle 1,2,3,4,5\rangle$ by two transpositions. Hence, σ can be converted to the identity permutation by six transpositions. $\square$

In general, for a fixed j, $1 \leq j \leq n$, there are $(n-1)!$ permutations σ's in S_n such that $\sigma(1) = j$. Each σ of those permutations has $j - 1$ inversions (j

precedes $j-1$ smaller numbers) and as many inversions as in the permutation $\tau = \langle\sigma(2), \ldots, \sigma(n)\rangle$. Therefore,

$$\mathrm{sgn}(\sigma) = (-1)^{j-1}\mathrm{sgn}(\tau).$$

Also, the permutation $\tau = \langle\sigma(2), \ldots, \sigma(n)\rangle$ can be considered as a permutation of N_{n-1} by replacing $\{j+1, \ldots, n\}$ by $\{j, \ldots, n-1\}$. Thus we have the following lemma.

Lemma 2.5 *For any permutation σ in S_n, if $\sigma(1) = j$, then*

$$\mathrm{sgn}(\sigma) = (-1)^{j-1}\mathrm{sgn}(\langle\sigma(2), \ldots, \sigma(n)\rangle),$$

where $\langle\sigma(2), \ldots, \sigma(n)\rangle$ is a permutation of $n-1$ numbers $N_n - \{j = \sigma(1)\}$.

Problem 2.7 Let $A = [\mathbf{c}_1 \cdots \mathbf{c}_n]$ be an $n \times n$ matrix with the column vectors $\mathbf{c}_j$'s. Show that $\det[\mathbf{c}_j\, \mathbf{c}_1 \cdots \mathbf{c}_{j-1}\, \mathbf{c}_{j+1} \cdots \mathbf{c}_n] = (-1)^{j-1} \det[\mathbf{c}_1 \cdots \mathbf{c}_j \cdots \mathbf{c}_n]$. Note that the same kind of equality holds when A is written in row vectors.

Problem 2.8 Compute the determinant of the matrix $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 1 \\ -2 & 2 & 3 \end{bmatrix}$.

2.3 Cofactor expansion

Recall that the determinant of an $n \times n$ matrix A is the sum of all signed elementary products of A, and

$$\det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

The first factor $a_{1\sigma(1)}$ in each term can be any one of $a_{11}, a_{12}, \ldots, a_{1n}$ in the first row of A. Among the $n!$ terms in this sum, there are precisely $(n-1)!$ permutations such that $a_{1\sigma(1)} = a_{11}$, *i.e.*, $\sigma(1) = 1$. The sum of those terms such that $\sigma(1) = 1$ can be written as $a_{11}A_{11}$, where

$$A_{11} = (-1)^0 \sum_{\tau} \mathrm{sgn}(\tau) a_{2\tau(2)} \cdots a_{n\tau(n)},$$

summing over all permutations τ of the numbers $\{2, 3, \ldots, n\}$. The term $(-1)^0$ means that there is no extra inversion other than that of τ if $\sigma(1) = 1$ is at the first place. Note that each term in A_{11} contains no entries from the first row or from the first column of A. Hence, all the terms of the sum in A_{11} are the signed elementary products of the submatrix M_{11} of A obtained by deleting the first row and the first column of A. Thus $A_{11} = (-1)^0 \det M_{11}$.

Similarly, if $a_{1\sigma(1)}$ is chosen to be a_{1j} with $1 \le j \le n$, then all $(n-1)!$ terms such that $\sigma(1) = j$ in the expression of $\det A$ add up to $a_{1j}A_{1j}$ with

$$A_{1j} = \sum_{\sigma \in S_n,\ \sigma(1)=j} \operatorname{sgn}(\sigma) a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)} = (-1)^{j-1} \det M_{1j},$$

where M_{1j} is the submatrix of A obtained by deleting the row and the column containing a_{1j}, and the sign $(-1)^{j-1}$ means the extra inversion numbers caused by placing $\sigma(1) = j$ at the first place as shown in Lemma 2.5.

By grouping $a_{1j}A_{1j}$ for all $j = 1, \ldots, n$ in the expression of $\det A$, we can get an expansion of $\det A$ with respect to the first row:

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n},$$

where $A_{1j} = (-1)^{j-1} \det M_{1j}$ and M_{1j} is the submatrix of A obtained by deleting the row and the column containing a_{1j}.

There is a similar expansion with respect to any other row, say the i-th row. To show this, first construct a new matrix $\bar{A}$ by using the i-th row of A as the first row and then shifting each of the preceding $i-1$ rows one row down. Then it is easy to see that $\det A = (-1)^{i-1} \det \bar{A}$ by Lemma 2.5. Now, with the expansion of $\det \bar{A}$ with respect to the first row $[a_{i1} \ \cdots \ a_{in}]$, we get

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in},$$

where $A_{ij} = (-1)^{i+j} \det M_{ij}$ and M_{ij} is the submatrix of A obtained by deleting the row and the column containing a_{ij}.

Also, we can do the same with the column vectors because $\det A^T = \det A$. This gives the following theorem:

Theorem 2.6 *Let A be an $n \times n$ matrix. Then,*

(1) *for each $1 \le i \le n$,*

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in},$$

called the **cofactor expansion** *of $\det A$ along the i-th row.*

(2) *For each $1 \le j \le n$,*

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj},$$

called the **cofactor expansion** *of $\det A$ along the j-th column.*

COFACTOR EXPANSION

The submatrix M_{ij} is called the **minor** of the entry a_{ij} and the number $A_{ij} = (-1)^{i+j} \det M_{ij}$ is called the **cofactor** of the entry a_{ij}. Therefore, the determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any one row by their cofactors and adding the resulting products. As a matter of fact, the determinant could be defined inductively by these explicit formulas.

Example 2.4 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Then the cofactors of a_{11}, a_{12} and a_{13} are

$$\begin{aligned} A_{11} &= (-1)^{1+1} \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} &= 5 \cdot 9 - 8 \cdot 6 = -3, \\ A_{12} &= (-1)^{1+2} \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} &= (-1)(4 \cdot 9 - 7 \cdot 6) = 6, \\ A_{13} &= (-1)^{1+3} \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} &= 4 \cdot 8 - 7 \cdot 5 = -3, \end{aligned}$$

respectively. Hence the expansion of $\det A$ along the first column is

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1 \cdot (-3) + 2 \cdot 6 + 3 \cdot (-3) = 0. \qquad \square$$

For a 3×3 matrix A, the cofactor expansion of A along the second column has the following form:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

$$= -a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{22} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{32} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}.$$

As this formula suggests, in the cofactor expansion of $\det A$ along a row or a column, the evaluation of A_{ij} can be avoided whenever $a_{ij} = 0$, because the product $a_{ij}A_{ij}$ is zero regardless of the value A_{ij}. Therefore it is beneficial to make the cofactor expansion along a row or a column that contains as many zero entries as possible. Moreover, by using the elementary operations, a matrix A may be simplified into another one having more zero entries in a row or in a column. This kind of simplification can be done by the elementary row (or column) operations, and generally gives the most efficient way to evaluate the determinant of a matrix. The next examples illustrate this method for an evaluation of the determinant.

Example 2.5 Evaluate the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{bmatrix}.$$

Solution: Apply the elementary operations:

$$\begin{array}{rcl} 3 \times \text{ row } 1 & + & \text{row } 2, \\ (-2) \times \text{ row } 1 & + & \text{row } 3, \\ 2 \times \text{ row } 1 & + & \text{row } 4 \end{array}$$

to A. Then

$$\det A = \det \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 7 & -4 \\ 0 & -3 & -7 & 10 \\ 0 & 4 & 0 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 7 & -4 \\ -3 & -7 & 10 \\ 4 & 0 & -1 \end{bmatrix}.$$

Now apply the operation: row 1 + row 2, to the matrix on the right side, then

$$\begin{aligned} \det \begin{bmatrix} 1 & 7 & -4 \\ -3 & -7 & 10 \\ 4 & 0 & -1 \end{bmatrix} &= \det \begin{bmatrix} 1 & 7 & -4 \\ -2 & 0 & 6 \\ 4 & 0 & -1 \end{bmatrix} \\ &= (-1)^{1+2} \cdot 7 \cdot \det \begin{bmatrix} -2 & 6 \\ 4 & -1 \end{bmatrix} \; ? \\ &= -7(2-24) = 154. \end{aligned}$$

Thus $\det A = 154$. □

Problem 2.9 Use cofactor expansions along a row or a column to evaluate the determinants of the following matrices:

$$(1)\; A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{bmatrix}, \quad (2)\; A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 5 & 0 \end{bmatrix}.$$

Example 2.6 Show that $\det A = (x-y)(x-z)(x-w)(y-z)(y-w)(z-w)$ for

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{bmatrix}.$$

Solution: Use Gaussian elimination. To begin with, add $(-1)\times$ row 1 to rows 2, 3, and 4 of A:

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & w-x & w^2-x^2 & w^3-x^3 \end{bmatrix} \\
&= \det \begin{bmatrix} y-x & y^2-x^2 & y^3-x^3 \\ z-x & z^2-x^2 & z^3-x^3 \\ w-x & w^2-x^2 & w^3-x^3 \end{bmatrix} \\
&= (y-x)(z-x)(w-x) \det \begin{bmatrix} 1 & y+x & y^2+xy+x^2 \\ 1 & z+x & z^2+xz+x^2 \\ 1 & w+x & w^2+xw+x^2 \end{bmatrix} \\
&= (x-y)(x-z)(w-x) \det \begin{bmatrix} 1 & y+x & y^2+xy+x^2 \\ 0 & z-y & (z-y)(z+y+x) \\ 0 & w-y & (w-y)(w+y+x) \end{bmatrix} \\
&= (x-y)(x-z)(w-x) \det \begin{bmatrix} z-y & (z-y)(z+y+x) \\ w-y & (w-y)(w+y+x) \end{bmatrix} \\
&= (x-y)(x-z)(x-w)(y-z)(w-y) \det \begin{bmatrix} 1 & z+y+x \\ 1 & w+y+x \end{bmatrix} \\
&= (x-y)(x-z)(x-w)(y-z)(y-w)(z-w).
\end{aligned}$$

□

Problem 2.10 Let A be the Vandermonde matrix of order n:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

Show that

$$\det A = \prod_{1\le i<j\le n} (x_j - x_i).$$

2.4 Cramer's rule

The cofactor expansion of the determinant gives a method for computing the inverse of an invertible matrix A. For $i \neq j$, let A^* be the matrix A with the j-th row replaced by the i-th row. Then the determinant of A^* must be zero, because the entries of the i-th and j-th rows are the same. Moreover, the cofactors of A^* with respect to the j-th row are the same as those of A: that is, $A^*_{jk} = A_{jk}$ for all $k = 1, \ldots, n$. Therefore, we have

$$\begin{aligned} 0 = \det A^* &= a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn} \\ &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}. \end{aligned}$$

This proves the following lemma.

Lemma 2.7

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det A & \textit{if } i = j \\ 0 & \textit{if } i \neq j. \end{cases}$$

Definition 2.4 If A is an $n \times n$ matrix and A_{ij} is the cofactor of a_{ij}, then the new matrix

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

is called the **matrix of cofactors** of A. The transpose of this matrix is called the **adjoint** of A and is denoted by adj$A$.

It follows from Lemma 2.7 that

$$A \cdot \mathrm{adj}A = \begin{bmatrix} \det A & 0 & \cdots & 0 \\ 0 & \det A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det A \end{bmatrix} = (\det A)I.$$

If A is invertible, then $\det A \neq 0$ and we may write $A\left(\frac{1}{\det A}\,\mathrm{adj}A\right) = I$. Thus

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}A, \quad \text{and} \quad A = (\det A)\,\mathrm{adj}(A^{-1})$$

by replacing A with A^{-1}.

Example 2.7 For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathrm{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, and if $\det A = ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Problem 2.11 Compute $\mathrm{adj}A$ and A^{-1} for $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

Problem 2.12 Show that A is invertible if and only if $\mathrm{adj}A$ is invertible, and that if A is invertible, then

$$(\mathrm{adj}A)^{-1} = \frac{A}{\det A} = \mathrm{adj}(A^{-1}).$$

Problem 2.13 Let A be an $n \times n$ matrix with $n > 1$. Show that
(1) $\det(\mathrm{adj}A) = (\det A)^{n-1}$;
(2) $\mathrm{adj}(\mathrm{adj}A) = (\det A)^{n-2}A$, if A is invertible.

The next theorem establishes a formula for the solution of a system of n equations in n unknowns. It is not useful as a practical method but can be used to study properties of the solution without solving the system.

Theorem 2.8 (Cramer's rule) *Let $A\mathbf{x} = \mathbf{b}$ be a system of n linear equations in n unknowns such that $\det A \neq 0$. Then the system has the unique solution given by*

$$x_j = \frac{\det C_j}{\det A}, \qquad j = 1,\ 2,\ \ldots,\ n,$$

where C_j is the matrix obtained from A by replacing the j-th column with the column matrix $\mathbf{b} = [b_1\ b_2\ \cdots\ b_n]^T$.

Proof: If $\det A \neq 0$, then A is invertible and $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. Since

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det A}(\mathrm{adj}A)\mathbf{b},$$

it follows that

$$x_j = \frac{b_1A_{1j} + b_2A_{2j} + \cdots + b_nA_{nj}}{\det A} = \frac{\det C_j}{\det A}. \qquad \square$$

Example 2.8 Use Cramer's rule to solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 50 \\ 2x_1 + 2x_2 + x_3 = 60 \\ x_1 + 2x_2 + 3x_3 = 90. \end{cases}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 50 & 2 & 1 \\ 60 & 2 & 1 \\ 90 & 2 & 3 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 50 & 1 \\ 2 & 60 & 1 \\ 1 & 90 & 3 \end{bmatrix}, \qquad C_3 = \begin{bmatrix} 1 & 2 & 50 \\ 2 & 2 & 60 \\ 1 & 2 & 90 \end{bmatrix}.$$

Therefore,

$$x_1 = \frac{\det C_1}{\det A} = 10, \quad x_2 = \frac{\det C_2}{\det A} = 10, \quad x_3 = \frac{\det C_3}{\det A} = 20. \qquad \square$$

Cramer's rule provides a convenient method for writing down the solution of a system of n linear equations in n unknowns in terms of determinants. To find the solution, however, one must evaluate $n+1$ determinants of order n. Evaluating even two of these determinants generally involves more computations than solving the system by using Gauss-Jordan elimination.

Problem 2.14 Use Cramer's rule to solve the systems

$$(1) \quad \begin{cases} \phantom{3x_1 +{}} 4x_2 + 3x_3 = -2 \\ 3x_1 + 4x_2 + 5x_3 = 6 \\ -2x_1 + 5x_2 - 2x_3 = 1. \end{cases}$$

$$(2) \quad \begin{cases} \dfrac{2}{x} - \dfrac{3}{y} + \dfrac{5}{z} = 3 \\ -\dfrac{4}{x} + \dfrac{7}{y} + \dfrac{2}{z} = 0 \\ \dfrac{2}{y} - \dfrac{1}{z} = 2. \end{cases}$$

Problem 2.15 Let A be the matrix obtained from the identity matrix I_n with i-th column replaced by the column vector $\mathbf{x} = [x_1 \cdots x_n]^T$. Compute $\det A$.

Problem 2.16 Prove that if A_{ij} is the cofactor of a_{ij} in $A = [a_{ij}]$, and if $n > 1$, then

$$\det \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n \\ x_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ x_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = -\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_ix_j.$$

2.5 Application: Area and Volume

In this section, we restrict our attention to the case of $n = 2$ or 3 in order to visualize the geometric figures conveniently, even if the same argument can be applied for $n > 3$.

For an $n \times n$ square matrix A, the row vectors $\mathbf{r}_i = [a_{i1} \ \cdots \ a_{in}]$, $i = 1, \ldots, n$, of A can be considered as elements in

$$\mathbb{R}^n = \{(a_1, \ldots, a_n) \ : \ a_i \in \mathbb{R}, i = 1, \ldots, n\}.$$

The set

$$\mathcal{P}(A) = \left\{ \sum_{i=1}^{n} t_i \mathbf{r}_i \ : \ 0 \le t_i \le 1, \ i = 1, \ldots, n \right\}$$

is called a **parallelogram** if $n = 2$, or a **parallelepiped** if $n \ge 3$. Note that the row vectors of A form the edges of $\mathcal{P}(A)$, and a different order of the row vectors does not alter the shape of $\mathcal{P}(A)$.

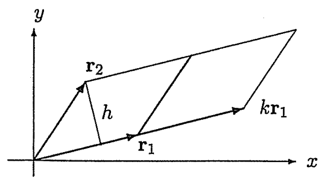
A geometrical meaning of the determinant is that it represents the volume (or area for $n = 2$) of the parallelepiped $\mathcal{P}(A)$.

Theorem 2.9 *The determinant* $\det A$ *of an* $n \times n$ *matrix* A *is the volume of* $\mathcal{P}(A)$ *up to sign. In fact, the volume of* $\mathcal{P}(A)$ *is equal to* $|\det A\,|$.

Proof: We present here a geometrical sketch since this way seems more intuitive and more convincing. We give only the proof of the case $n = 2$, and leave the case $n = 3$ to the readers. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix},$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ are the row vectors of A. Let $Area(A) = Area(\mathbf{r}_1, \ \mathbf{r}_2)$ denote the area of the parallelogram $\mathcal{P}(\mathbf{r}_1, \ \mathbf{r}_2)$ (see the figure below).



(1) It is quite clear that if $A = I_2$, then $Area \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$.

(2) Since the shape of the parallelogram $\mathcal{P}(A)$ does not depend on the order of placing the row vectors: *i.e.*, $\mathcal{P}(\mathbf{r}_1,\ \mathbf{r}_2) = \mathcal{P}(\mathbf{r}_2,\ \mathbf{r}_1)$, we have $Area(\mathbf{r}_1, \mathbf{r}_2) = Area(\mathbf{r}_2, \mathbf{r}_1)$. On the other hand, $\det(\mathbf{r}_1, \mathbf{r}_2) = -\det(\mathbf{r}_2, \mathbf{r}_1)$. Thus

$$\det(\mathbf{r}_1,\ \mathbf{r}_2) = \pm Area(\mathbf{r}_1,\ \mathbf{r}_2),$$

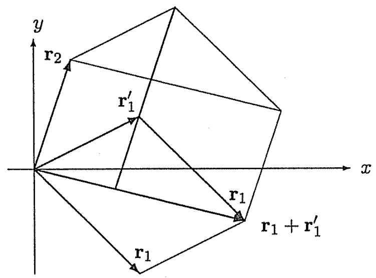
which explains why we say "up to sign".

(3) From the figure above, if we replace $\mathbf{r}_1$ by $k\mathbf{r}_1$ in A, then the bottom edge $\mathbf{r}_1$ of $\mathcal{P}(A)$ is elongated by $|k|$ while the height h remains unchanged. Thus

$$Area(k\mathbf{r}_1,\ \mathbf{r}_2) = |k| Area(\mathbf{r}_1,\ \mathbf{r}_2).$$

(4) The additivity in the first row is a trivial consequence of examining the following figure: That is, if we replace $\mathbf{r}_1$ by $\mathbf{r}_1 + \mathbf{r}_1'$ while fixing $\mathbf{r}_2$, then, as the following figure shows, we have

$$Area(\mathbf{r}_1 + \mathbf{r}_1',\ \mathbf{r}_2) = Area(\mathbf{r}_1,\ \mathbf{r}_2) + Area(\mathbf{r}_1',\ \mathbf{r}_2).$$



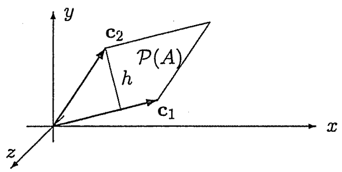
(5) Thus the area function $Area$ on $M_{2\times 2}(\mathbb{R})$ satisfies the rules $(\mathbf{R}_1)$ and $(\mathbf{R}_3)$ of the determinant except for the rule $(\mathbf{R}_2)$. Therefore, by uniqueness, $\det = \pm Area$. □

Remark: (1) Note that if we have constructed the parallelepiped $\mathcal{P}(A)$ using the column vectors of A, then the shape of the parallelepiped is totally different from the one constructed using the row vectors. However, $\det A = \det A^T$ means their volumes are the same, which is a totally nontrivial fact.

(2) For $n \geq 3$, the volume of $\mathcal{P}(A)$ can be defined by induction on n, and exactly the same argument in the proof can be applied to show that the volume is the determinant. However, there is another way of looking at this fact. Let $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be n column vectors of an $m \times n$ matrix A. They constitute an n-dimensional parallelepiped in $\mathbb{R}^m$ such that

$$\mathcal{P}(A) = \left\{ \sum_{i=1}^{n} t_i \mathbf{c}_i \ : \ 0 \leq t_i \leq 1, \ i = 1, \ldots, n \right\}.$$

A formula for the volume of this parallelepiped may be expressed as follows: We first consider a two-dimensional parallelepiped (a parallelogram) determined by two column vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ of $A = [\mathbf{c}_1 \ \mathbf{c}_2]$ in $\mathbb{R}^3$.



The area of this parallelogram is simply $Area(\mathcal{P}(A)) = \|\mathbf{c}_1\| h$, where $h = \|\mathbf{c}_2\| \sin\theta$ and θ is the angle between $\mathbf{c}_1$ and $\mathbf{c}_2$. Therefore, we have

$$\begin{aligned}
Area(\mathcal{P}(A))^2 &= \|\mathbf{c}_1\|^2 \|\mathbf{c}_2\|^2 \sin^2\theta \ = \ \|\mathbf{c}_1\|^2 \|\mathbf{c}_2\|^2 (1 - \cos^2\theta) \\
&= (\mathbf{c}_1 \cdot \mathbf{c}_1)(\mathbf{c}_2 \cdot \mathbf{c}_2) \left(1 - \frac{(\mathbf{c}_1 \cdot \mathbf{c}_2)^2}{(\mathbf{c}_1 \cdot \mathbf{c}_1)(\mathbf{c}_2 \cdot \mathbf{c}_2)} \right) \\
&= (\mathbf{c}_1 \cdot \mathbf{c}_1)(\mathbf{c}_2 \cdot \mathbf{c}_2) - (\mathbf{c}_1 \cdot \mathbf{r}_2)^2 \\
&= \det \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{c}_1 & \mathbf{c}_1 \cdot \mathbf{c}_2 \\ \mathbf{c}_2 \cdot \mathbf{c}_1 & \mathbf{c}_2 \cdot \mathbf{c}_2 \end{bmatrix} \\
&= \det \left(\begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \right) \ = \ \det(A^T A).
\end{aligned}$$

In general, let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be n column vectors of an $m \times n$ matrix A. Then one can show (for a proof see Exercise **5.16**) that the volume of the n-dimensional parallelepiped $\mathcal{P}(A)$ determined by those n column vectors $\mathbf{c}_j$'s in $\mathbb{R}^m$ is

$$\mathrm{vol}(\mathcal{P}(A)) = \sqrt{\det(A^T A)}\,.$$

In particular, if A is an $m \times m$ square matrix, then

$$\operatorname{vol}(\mathcal{P}(A)) = \sqrt{\det(A^T A)} = \sqrt{\det(A^T)\det(A)} = |\det(A)|,$$

as expected.

Problem 2.17 Show that the area of a triangle ABC in the plane $\mathbb{R}^2$, where $A = (x_1,\ y_1),\ B = (x_2,\ y_2),\ C = (x_3,\ y_3)$, is equal to the absolute value of

$$\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

2.6 Exercises

2.1. Determine the values of k for which $\det\begin{bmatrix} k & k \\ 4 & 2k \end{bmatrix} = 0$.

2.2. Evaluate $\det(A^2BA^{-1})$ and $\det(B^{-1}A^3)$ for the following matrices:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{bmatrix}.$$

2.3. Evaluate the determinant of

$$A = \begin{bmatrix} 3 & -2 & -5 & 4 \\ -5 & 2 & 8 & -5 \\ -3 & 4 & 7 & -3 \\ 2 & -3 & -5 & 8 \end{bmatrix}.$$

2.4. Evaluate $\det A$ for an $n \times n$ matrix $A = [a_{ij}]$ when

$$(1)\ a_{ij} = \begin{cases} 1 & i \neq j \\ 0 & i = j, \end{cases} \quad \text{or (2) } a_{ij} = i + j.$$

-A)$isapolynomialinxofc_1x + c_0$.

2.5. Find all solutions of the equation $\det(AB) = 0$ for

$$A = \begin{bmatrix} x+2 & 3x \\ 3 & x+2 \end{bmatrix}, \quad B = \begin{bmatrix} x & 0 \\ 5 & x+2 \end{bmatrix}.$$

2.6. Prove that if A is an $n \times n$ skew-symmetric matrix and n is odd, then $\det A = 0$. Give an example of 4×4 skew-symmetric matrix A with $\det A \neq 0$.

2.7. Use the determinant function to find

(1) the area of the parallelogram with edges determined by $(4,\ 3)$ and $(7,\ 5)$,

(2) the volume of the parallelepiped with edges determined by the vectors $(1,\ 0,\ 4)$, $(0,\ -2,\ 2)$ and $(3,\ 1,\ -1)$.

2.8. Use Cramer's rule to solve each system.

$$(1)\ \begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = -1\,. \end{cases} \qquad (2)\ \begin{cases} x_1 + x_2 + x_3 = 2 \\ x_1 + 2x_2 + x_3 = 2 \\ x_1 + 3x_2 - x_3 = -4\,. \end{cases}$$

$$(3)\ \begin{cases} \quad\ - x_2 \qquad\quad + x_4 = -1 \\ x_1 \qquad\ + x_3 \qquad\ = 3 \\ x_1 - x_2 - x_3 - x_4 = 2 \\ x_1 + x_2 + x_3 + x_4 = 0\,. \end{cases}$$

2.9. Use Cramer's rule to solve the given system:

$$(1)\ \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad (2)\ \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

2.10. Find a constant k so that the system of linear equations

$$\begin{cases} kx - 2y - z = 0 \\ \qquad (k+1)y + 4z = 0 \\ \qquad\qquad (k-1)z = 0\,. \end{cases}$$

has more than one solution. (Is it possible to apply Cramer's rule here?)

2.11. Solve the following system of linear equations by using Cramer's rule and by using Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

2.12. Solve the following system of equations by using Cramer's rule:

$$\begin{cases} 3x + 2y = 3z + 1 \\ 3x + 2z = 8 - 5y \\ 3z - 1 = x - 2y\,. \end{cases}$$

2.13. Calculate the cofactors A_{11}, A_{12}, A_{13} and A_{33} for the matrix A:

$$(1)\ A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix},\ (2)\ A = \begin{bmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix},\ (3)\ A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

2.14. Let A be the $n \times n$ matrix whose entries are all 1. Show that

(1) $\det(A - nI_n) = 0$.

(2) $(A - nI_n)_{ij} = (-1)^{n-1}n^{n-2}$ for all $i,\ j$, where $(A - nI_n)_{ij}$ denotes the cofactor of the $(i,\ j)$-entry of $A - nI_n$.

2.15. Show that if A is symmetric, then so is $\mathrm{adj}A$. Moreover, if it is invertible, then the inverse of A is also symmetric.

2.16. Use the adjoint formula to compute the inverse of the each of the following matrices:

$$A = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

2.17. Compute $\mathrm{adj}A$, $\det A$, $\det(\mathrm{adj}A)$, A^{-1}, and verify $A \cdot \mathrm{adj}A = (\det A)I$ for

$$(1)\ A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}, \quad (2)\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}.$$

2.18. Let A, B be invertible matrices. Show that $\mathrm{adj}(AB) = \mathrm{adj}B\ \mathrm{adj}A$. (The reader may also try to prove this equality for noninvertible matrices.)

2.19. For an $m \times n$ matrix A and $n \times m$ matrix B, show that

$$\det \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det(AB).$$

2.20. Find the area of the triangle with vertices at $(0,0)$, $(1,3)$ and $(3,1)$ in $\mathbb{R}^2$.

2.21. Find the area of the triangle with vertices at $(0,0,0)$, $(1,1,2)$ and $(2,2,1)$ in $\mathbb{R}^3$.

2.22. For $A, B, C, D \in M_{n\times n}(\mathbb{R})$, show that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \det D$. But, in general, $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq \det A \det D - \det B \det C$.

2.23. Determine whether or not the following statements are true in general, and justify your answers.

(1) For any square matrices A and B of the same size, $\det(A+B) = \det A + \det B$.
(2) For any square matrices A and B of the same size, $\det(AB) = \det(BA)$.
(3) If A is an $n \times n$ square matrix, then for any scalar c, $\det(cI_n - A) = c^n - \det A$.
(4) If A is an $n \times n$ square matrix, then for any scalar c, $\det(cI_n - A^T) = \det(cI_n - A)$.
(5) If E is an elementary matrix, then $\det E = \pm 1$.
(6) There is no matrix A of order 3 such that $A^2 = -I_3$.
(7) Let A be a nilpotent matrix, *i.e.*, $A^k = \mathbf{0}$ for some natural number k. Then $\det A = 0$.
(8) $\det(kA) = k \det A$ for any square matrix A.
(9) Any system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\det A \neq 0$.
(10) For any $n \times 1$, $n \geq 2$, column vectors $\mathbf{u}$ and $\mathbf{v}$, $\det(\mathbf{u}\mathbf{v}^T) = 0$.
(11) If A is a square matrix with $\det A = 1$, then $\mathrm{adj}(\mathrm{adj}A) = A$.

(12) If the entries of A are all integers and $\det A = 1$ or -1, then the entries of A^{-1} are also integers.
(13) If the entries of A are 0's or 1's, then $\det A = 1$, 0, or -1.
(14) Every system of n linear equations in n unknowns can be solved by Cramer's rule.
(15) If A is a permutation matrix, then $A^T = A$.

Chapter 3

Vector Spaces

3.1 Vector spaces and subspaces

We discussed how to solve a system $A\mathbf{x} = \mathbf{b}$ of linear equations, and we saw that the basic questions of the existence or uniqueness of the solution were much easier to answer after Gaussian-elimination. In this chapter, we introduce the notion of a vector space, which is an abstraction of the usual algebraic structures of the 3-space $\mathbb{R}^3$ and then elaborate our study of a system of linear equations to this framework.

Usually, many physical quantities, such as length, area, mass, temperature are described by real numbers as magnitudes. Other physical quantities like force or velocity have directions as well as magnitudes. Such quantities with direction are called **vectors**, while the numbers are called **scalars**. For instance, an element (or a point) $\mathbf{x}$ in the 3-space $\mathbb{R}^3$ is usually represented as a triple of real numbers:

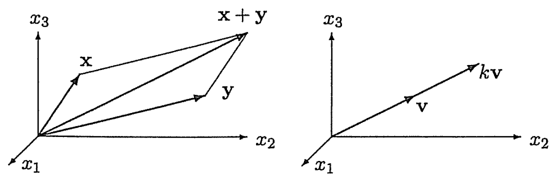
$$\mathbf{x} = (x_1,\ x_2,\ x_3),$$

where $x_i \in \mathbb{R}$, $i = 1,\ 2,\ 3$, are called the **coordinates** of $\mathbf{x}$. This expression provides a rectangular coordinate system in a natural way. On the other hand, pictorially such a point in the 3-space $\mathbb{R}^3$ can also be represented by an arrow from the origin to $\mathbf{x}$. In this way, a point in the 3-space $\mathbb{R}^3$ can be understood as a vector. The direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude.

In order to have a more general definition of vectors, we extract the most basic properties of those arrows in $\mathbb{R}^3$. Note that for all vectors (or points) in $\mathbb{R}^3$, there are two algebraic operations: the addition of any two vectors and scalar multiplication of a vector by a scalar. That is, for two vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ and k a scalar, we define

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (x_1+y_1,\ x_2+y_2,\ x_3+y_3), \\ k\mathbf{x} &= (kx_1,\ kx_2,\ kx_3). \end{aligned}$$

The addition of vectors and scalar multiplication of a vector in the 3-space $\mathbb{R}^3$ are illustrated as follows:



Even though our geometric visualization of vectors does not go beyond the 3-space $\mathbb{R}^3$, it is possible to extend the above algebraic operations of vectors in the 3-space $\mathbb{R}^3$ to the general **n-space** $\mathbb{R}^n$ for any positive integer n. It is defined to be the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ of real numbers, called *vectors*: *i.e.*,

$$\mathbb{R}^n = \{(a_1,\ a_2,\ \ldots,\ a_n)\ :\ a_i \in \mathbb{R},\ i = 1,\ 2,\ \ldots,\ n\}.$$

For any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in the n-space $\mathbb{R}^n$, and a scalar k, the *sum* $\mathbf{x}+\mathbf{y}$ and the *scalar multiplication* $k\mathbf{x}$ of them are vectors in $\mathbb{R}^n$ defined by

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (x_1+y_1,\ x_2+y_2,\ \ldots,\ x_n+y_n), \\ k\mathbf{x} &= (kx_1,\ kx_2,\ \ldots,\ kx_n). \end{aligned}$$

It is easy to verify the following list of arithmetical rules of the operations:

Theorem 3.1 *For any scalars k and ℓ, and vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, and $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ in the n-space $\mathbb{R}^n$, the following rules hold:*

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$,

(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$,

(3) $\mathbf{x}+\mathbf{0} = \mathbf{x} = \mathbf{0}+\mathbf{x}$,

(4) $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$,

(5) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$,

(6) $(k + \ell)\mathbf{x} = k\mathbf{x} + \ell\mathbf{x}$,

(7) $k(\ell\mathbf{x}) = (k\ell)\mathbf{x}$,

(8) $1\mathbf{x} = \mathbf{x}$,

where $\mathbf{0} = (0,\ 0,\ \ldots,\ 0)$ *is the* **zero** *vector.*

We usually write a vector $(a_1,\ a_2,\ \ldots,\ a_n)$ in the n-space $\mathbb{R}^n$ as an $n \times 1$ column matrix

$$(a_1,\ a_2,\ \ldots,\ a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [a_1\ a_2\ \cdots\ a_n]^T,$$

also called a *column vector*. Then the two operations of the matrix sum and the scalar multiplication of column matrices coincide with those of vectors in $\mathbb{R}^n$, and the above theorem is just Theorem 1.2.

These rules of arithmetic of vectors are the most important ones because they are the only rules that we need to manipulate vectors in the n-space $\mathbb{R}^n$. Hence, an (abstract) vector space can be defined with respect to these rules of operations of vectors in the n-space $\mathbb{R}^n$ so that $\mathbb{R}^n$ itself becomes a vector space. In general, a *vector space* is defined to be a set with two operations: an addition and a scalar multiplication which satisfy the above rules of operations in $\mathbb{R}^n$.

Definition 3.1 A (real) **vector space** is a nonempty set V of elements, called **vectors**, with two algebraic operations that satisfy the following rules.

(A) There is an operation called *vector addition* that associates to every pair $\mathbf{x}$ and $\mathbf{y}$ of vectors in V a unique vector $\mathbf{x} + \mathbf{y}$ in V, called the the **sum** of $\mathbf{x}$ and $\mathbf{y}$, so that the following rules hold for all vectors $\mathbf{x},\ \mathbf{y},\ \mathbf{z}$ in V:

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutativity in addition),

(2) $\mathbf{x} + (\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y}) + \mathbf{z} (= \mathbf{x} + \mathbf{y} + \mathbf{z})$ (associativity in addition),

(3) there is a unique vector $\mathbf{0}$ in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x} = \mathbf{0} + \mathbf{x}$ for all $\mathbf{x} \in V$ (it is called the **zero vector**),

(4) for any $\mathbf{x} \in V$, there is a vector $-\mathbf{x} \in V$, called the **negative** of $\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$.

(B) There is an operation called **scalar multiplication** that associates to each vector $\mathbf{x}$ in V and each scalar k a unique vector $k\mathbf{x}$ in V called the **multiplication** of $\mathbf{x}$ by a (real) scalar k, so that the following rules hold for all vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in V and all scalars k, ℓ:

(5) $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ (distributivity with respect to vector addition),

(6) $(k+\ell)\mathbf{x} = k\mathbf{x} + \ell\mathbf{x}$ (distributivity with respect to scalar addition),

(7) $k(\ell\mathbf{x}) = (k\ell)\mathbf{x}$ (associativity in scalar multiplication),

(8) $1\mathbf{x} = \mathbf{x}$.

Clearly, the n-space $\mathbb{R}^n$ is a vector space by Theorem 3.1. A **complex vector space** is obtained if, instead of real numbers, we take complex numbers for scalars. For example, the set $\mathbb{C}^n$ of all ordered n-tuples of complex numbers is a complex vector space. In Chapter 7 we shall discuss complex vector spaces, but until then we will discuss only real vector spaces unless otherwise stated.

Example 3.1 **(1)** For any positive integer m and n, the set $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices forms a vector space under the matrix sum and scalar multiplication defined in Section 1.3. The zero vector in this space is the zero matrix $\mathbf{0}_{m\times n}$, and $-A$ is the negative of a matrix A.

(2) Let $C(\mathbb{R})$ denote the set of real-valued continuous functions defined on the real line $\mathbb{R}$. For two functions $f(x)$ and $g(x)$, and a real number k, the sum $f+g$ and the scalar multiplication kf of them are defined by

$$\begin{aligned}(f+g)(x) &= f(x)+g(x),\\ (kf)(x) &= kf(x).\end{aligned}$$

Then one can easily verify, as an exercise, that the set $C(\mathbb{R})$ is a vector space under these operations. The zero vector in this space is the constant function whose value at each point is zero.

(3) Let A be an $m \times n$ matrix. Then it is easy to show that the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space (under the sum and scalar multiplication of matrices).

Theorem 3.2 *Let V be a vector space and let $\mathbf{x}$, $\mathbf{y}$ be vectors in V. Then*

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}$ *implies* $\mathbf{x} = \mathbf{0}$,

(2) $0\mathbf{x} = \mathbf{0}$,

(3) $k\mathbf{0} = \mathbf{0}$ *for any* $k \in \mathbb{R}$,

(4) $-\mathbf{x}$ *is unique and* $-\mathbf{x} = (-1)\mathbf{x}$,

(5) *if* $k\mathbf{x} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{x} = \mathbf{0}$.

Proof: (1) By adding $-\mathbf{y}$ to both sides of $\mathbf{x} + \mathbf{y} = \mathbf{y}$, we have

$$\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x} + \mathbf{y} + (-\mathbf{y}) = \mathbf{y} + (-\mathbf{y}) = \mathbf{0}.$$

(2) $0\mathbf{x} = (0+0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$ implies $0\mathbf{x} = \mathbf{0}$ by (1).

(3) This is an easy exercise.

(4) The uniqueness of the negative $-\mathbf{x}$ of $\mathbf{x}$ can be shown by a simple modification of Lemma 1.6. In fact, if $\bar{\mathbf{x}}$ is another negative of $\mathbf{x}$ such that $\mathbf{x} + \bar{\mathbf{x}} = \mathbf{0}$, then

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \bar{\mathbf{x}}) = (-\mathbf{x} + \mathbf{x}) + \bar{\mathbf{x}} = \mathbf{0} + \bar{\mathbf{x}} = \bar{\mathbf{x}}.$$

On the other hand, the equation

$$\mathbf{x} + (-1)\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = (1-1)\mathbf{x} = 0\mathbf{x} = \mathbf{0}$$

shows that $(-1)\mathbf{x}$ is another negative of $\mathbf{x}$, and hence $-\mathbf{x} = (-1)\mathbf{x}$ by the uniqueness of $-\mathbf{x}$.

(5) Suppose $k\mathbf{x} = \mathbf{0}$ and $k \neq 0$. Then $\mathbf{x} = 1\mathbf{x} = \frac{1}{k}(k\mathbf{x}) = \frac{1}{k}\mathbf{0} = \mathbf{0}$. □

Problem 3.1 Let V be the set of all pairs (x, y) of real numbers. Suppose that an addition and scalar multiplication of pairs are defined by

$$(x, y) + (u, v) = (x + 2u, y + 2v), \quad k(x, y) = (kx, ky).$$

Is the set V a vector space under those operations? Justify your answer.

A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined in V. Usually, in order to show that a subset W is a subspace, it is not necessary to verify all the rules of the definition of a vector space, because certain rules satisfied in the larger space V are automatically satisfied in every subset, if vector addition and scalar multiplication are closed in subset.

Theorem 3.3 *A nonempty subset W of a vector space V is a subspace if and only if $\mathbf{x} + \mathbf{y}$ and $k\mathbf{x}$ are contained in W (or equivalently, $\mathbf{x} + k\mathbf{y} \in W$) for any vectors $\mathbf{x}$ and $\mathbf{y}$ in W and any scalar $k \in \mathbb{R}$.*

Proof: We need only to prove the sufficiency. Assume both conditions hold and let $\mathbf{x}$ be any vector in W. Since W is closed under scalar multiplication, $\mathbf{0} = 0\mathbf{x}$ and $-\mathbf{x} = (-1)\mathbf{x}$ are in W, so rules (3) and (4) for a vector space hold. All the other rules for a vector space are clear. □

A vector space V itself and the zero vector $\{\mathbf{0}\}$ are trivially subspaces. Some nontrivial subspaces are given in the following examples.

Example 3.2 Let $W = \{(x,\ y,\ z) \in \mathbb{R}^3\ :\ ax + by + cz = 0\}$, where a, b, c are constants. If $\mathbf{x} = (x_1,\ x_2,\ x_3)$, $\mathbf{y} = (y_1,\ y_2,\ y_3)$ are points in W, then clearly $\mathbf{x}+\mathbf{y} = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$ is also a point in W, because it satisfies the equation in W. Similarly, $k\mathbf{x}$ also lies in W for any scalar k. Hence, W is a subspace of $\mathbb{R}^3$ and is a plane passing through the origin in $\mathbb{R}^3$.

Example 3.3 Let A be an $m \times n$ matrix. Then, as we have seen in Example 3.1 (3), the set

$$W = \{\mathbf{x} \in \mathbb{R}^n\ :\ A\mathbf{x} = \mathbf{0}\}$$

of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space. Moreover, since the operations in W and in $\mathbb{R}^n$ coincide, W is a subspace of $\mathbb{R}^n$.

Example 3.4 For a nonnegative integer n, let $P_n(\mathbb{R})$ denote the set of all real polynomials in x with degree $\leq n$. Then $P_n(\mathbb{R})$ is a subspace of the vector space $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$.

Example 3.5 Let W be the set of all $n \times n$ real symmetric matrices. Then W is a subspace of the vector space $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices, because the sum of two symmetric matrices is symmetric and a scalar multiplication of a symmetric matrix is also symmetric. Similarly, the set of all $n \times n$ skew-symmetric matrices is also a subspace of $M_{n\times n}(\mathbb{R})$.

Problem 3.2 Which of the following sets are subspaces of the 3-space $\mathbb{R}^3$? Justify your answer.

(1) $W = \{(x,\ y,\ z) \in \mathbb{R}^3\ :\ xyz = 0\}$,
(2) $W = \{(2t,\ 3t,\ 4t) \in \mathbb{R}^3\ :\ t \in \mathbb{R}\}$,
(3) $W = \{(x,\ y,\ z) \in \mathbb{R}^3\ :\ x^2 + y^2 - z^2 = 0\}$,
(4) $W = \{\mathbf{x} \in \mathbb{R}^3\ :\ \mathbf{x}^T\mathbf{u} = 0 = \mathbf{x}^T\mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are any two fixed nonzero vectors in $\mathbb{R}^3$.

Can you describe all subspaces of the 3-space $\mathbb{R}^3$?

Problem 3.3 Let $V = C(\mathbb{R})$ be the vector space of all continuous functions on $\mathbb{R}$. Which of the following sets W are subspaces of V? Justify your answer.

(1) W is the set of all differentiable functions on $\mathbb{R}$.

(2) W is the set of all bounded continuous functions on $\mathbb{R}$.

(3) W is the set of all continuous nonnegative-valued functions on $\mathbb{R}$, *i.e.*, $f(x) \geq 0$ for any $x \in \mathbb{R}$.

(4) W is the set of all continuous odd functions on $\mathbb{R}$, *i.e.*, $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.

(5) W is the set of all polynomials with integer coefficients.

3.2 Bases

Recall that any vector in the 3-space $\mathbb{R}^3$ is of the form $(x_1,\ x_2,\ x_3)$ which can alsobe written as

$$(x_1,\ x_2,\ x_3) = x_1(1,\ 0,\ 0) + x_2(0,\ 1,\ 0) + x_3(0,\ 0,\ 1).$$

That is, any vector in $\mathbb{R}^3$ can be expressed as the sum of scalar multiples of $\mathbf{e}_1 = (1,\ 0,\ 0)$, $\mathbf{e}_2 = (0,\ 1,\ 0)$ and $\mathbf{e}_3 = (0,\ 0,\ 1)$, which are also denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. The following definition gives a name to such expressions.

Definition 3.2 Let V be a vector space, and let $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m\}$ be a set of vectors in V. Then a vector $\mathbf{y}$ in V of the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m,$$

where $a_1,\ \ldots,\ a_m$ are scalars, is called a **linear combination** of the vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$.

The next theorem shows that the set of all linear combinations of a finite set of vectors in a vector space forms a subspace.

Theorem 3.4 *Let $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ be vectors in a vector space V. Then the set $W = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m\ :\ a_i \in \mathbb{R}\}$ of all linear combinations of $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ is a subspace of V called the* **subspace of V spanned** *by $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$.*

Proof: We want to show that W is closed under addition and scalar multiplication. Let $\mathbf{u}$ and $\mathbf{w}$ be any two vectors in W. Then

$$\begin{array}{rcl} \mathbf{u} & = & a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m, \\ \mathbf{w} & = & b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_m\mathbf{x}_m \end{array}$$

for some scalars a_i's and b_i's. Therefore,

$$\mathbf{u}+\mathbf{w} = (a_1+b_1)\mathbf{x}_1 + (a_2+b_2)\mathbf{x}_2 + \cdots + (a_m+b_m)\mathbf{x}_m$$

and, for any scalar k,

$$k\mathbf{u} = (ka_1)\mathbf{x}_1 + (ka_2)\mathbf{x}_2 + \cdots + (ka_m)\mathbf{x}_m.$$

Thus, $\mathbf{u}+\mathbf{w}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ and consequently contained in W. Therefore, W is a subspace of V. □

Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is any set of m vectors in a vector space V. If any vector in V can be written as a linear combination of these vector $\mathbf{x}_i$'s, we say that it is a **spanning set** of V.

Example 3.6 **(1)** For a nonzero vector $\mathbf{v}$ in a vector space V, linear combinations of $\mathbf{v}$ are simply scalar multiples of $\mathbf{v}$. Thus the subspace W of V spanned by $\mathbf{v}$ is $W = \{k\mathbf{v} : k \in \mathbb{R}\}$.

(2) Consider three vectors $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,-1,1)$ and $\mathbf{v}_3 = (1,0,1)$ in $\mathbb{R}^3$. The subspace W_1 spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is written as

$$W_1 = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = (a_1+a_2, a_1-a_2, a_1+a_2) : a_i \in \mathbb{R}\},$$

and the subspace W_2 spanned by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ is written as

$$W_2 = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (a_1+a_2+a_3, a_1-a_2, a_1+a_2+a_3) : a_i \in \mathbb{R}\}.$$

Then $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 0\mathbf{v}_3$ implies $W_1 \subseteq W_2$. On the other hand, any vector in W_2 is of the form $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. But, since $\mathbf{v}_3 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$, this can be rewritten as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This means that $W_2 \subseteq W_1$, thus $W_1 = W_2$ which is a plane in $\mathbb{R}^3$ containing the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In general, a subspace in a vector space can have many different spanning sets. □

Example 3.7 Let

$$\begin{aligned} \mathbf{e}_1 &= (1,\ 0,\ 0,\ \ldots,\ 0), \\ \mathbf{e}_2 &= (0,\ 1,\ 0,\ \ldots,\ 0), \\ &\vdots \\ \mathbf{e}_n &= (0,\ 0,\ 0,\ \ldots,\ 1) \end{aligned}$$

be n vectors in the n-space $\mathbb{R}^n$ $(n \geq 3)$. Then a linear combination of $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ is of the form

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0).$$

Hence, the set

$$W = \{(a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0) \in \mathbb{R}^n\ :\ a_1,\ a_2,\ a_3 \in \mathbb{R}\}$$

is the subspace of the n-space $\mathbb{R}^n$ spanned by the vectors $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$. Note that the subspace W can be identified with the 3-space $\mathbb{R}^3$ through the identification

$$(a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0) \equiv (a_1,\ a_2,\ a_3)$$

with $a_i \in \mathbb{R}$. In general, for $m \leq n$, the m-space $\mathbb{R}^m$ can be identified as a subspace of the n-space $\mathbb{R}^n$. □

Example 3.8 Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ be an $m \times n$ matrix. Then the column vectors $\mathbf{a}_i$'s are in $\mathbb{R}^m$, and the matrix product $A\mathbf{x}$ represents the linear combination of the column vector $\mathbf{a}_i$'s whose coefficients are the components of $\mathbf{x} \in \mathbb{R}^n$, *i.e.*, $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$. Therefore, the set

$$W = \{A\mathbf{x} \in \mathbb{R}^m\ :\ \mathbf{x} \in \mathbb{R}^n\}$$

of all linear combinations of the column vectors of A is a subspace of $\mathbb{R}^m$ called the **column space** of A. Therefore, $A\mathbf{x} = \mathbf{b}$ *has a solution* $(x_1,\ \cdots,\ x_n)$ *in* $\mathbb{R}^n$ *if and only if the vector* $\mathbf{b}$ *belongs to the subspace* W *spanned by the column vectors of* A. □

Problem 3.4 Let $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ be vectors in a vector space V and let W be the subspace spanned by $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$. Show that W is the smallest subspace of V containing $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$, *i.e.*, if U is a subspace of V containing $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$, then $W \subseteq U$.

Problem 3.5 Show that the set of all matrices of the form $AB - BA$ does not span the vector space $M_{n\times n}(\mathbb{R})$.

As we saw above, any nonempty subset of a vector space V spans a subspace through the linear combinations of the vectors, and a subspace can have many spanning sets with a different number of vectors. This means that a vector can be written as linear combinations in various ways. If one can find a finite number of vectors in V such that any vector in V can be expressed in a unique way as a linear combination of them, then the study of the vector space V might be easier and the computations of vectors may be simplified. Thus, for a fixed set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ in a vector space V, we look at their linear combinations $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m$ and see whether any vector in V can be written in this form in exactly one way. This problem can be rephrased as to whether or not a nontrivial linear combination produces the zero vector, while the trivial combination, with all scalars $c_i = 0$, obviously produces the zero vector.

Definition 3.3 A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in a vector space V is said to be **linearly independent** if the vector equation, called the **linear dependence** of $\mathbf{x}_i$'s,

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$. Otherwise, it is said to be **linearly dependent**.

Therefore, a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly dependent if and only if there is a linear dependence

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

with a nontrivial solution $(c_1, c_2, \ldots, c_m)$. In this case, we may assume that $c_m \neq 0$. Then the equation can be rewritten as

$$\mathbf{x}_m = -\frac{c_1}{c_m}\mathbf{x}_1 - \frac{c_2}{c_m}\mathbf{x}_2 - \cdots - \frac{c_{m-1}}{c_m}\mathbf{x}_{m-1}.$$

That is, *a set of vectors is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the others.*

Example 3.9 Let $\mathbf{x} = (1,2,3)$ and $\mathbf{y} = (3,2,1)$ be two vectors in the 3-space $\mathbb{R}^3$. Then clearly $\mathbf{y} \neq \lambda\mathbf{x}$ for any $\lambda \in \mathbb{R}$ (or $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$ only when $a = b = 0$). This means that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent in $\mathbb{R}^3$. If $\mathbf{w} = (3,6,9)$, then $\{\mathbf{x}, \mathbf{w}\}$ is linearly dependent since $\mathbf{w} - 3\mathbf{x} = \mathbf{0}$. In general, if $\mathbf{x}, \mathbf{y}$ are noncollinear vectors in the 3-space $\mathbb{R}^3$, the set of all linear combinations of $\mathbf{x}$ and $\mathbf{y}$ determines a plane W through the origin in $\mathbb{R}^3$, *i.e.*, $W = \{a\mathbf{x} + b\mathbf{y} : a, b \in \mathbb{R}\}$. Let $\mathbf{z}$ be another nonzero vector in the 3-space $\mathbb{R}^3$. If $\mathbf{z} \in W$, then there are some numbers $a, b \in \mathbb{R}$, not all of them are zero, such that $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$, that is, the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent. If $\mathbf{z} \notin W$, then $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ is possible only when $a = b = c = 0$ (prove it). Therefore, the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent if and only if $\mathbf{z}$ does not lie in W. □

By abuse of language, it is sometimes convenient to say that "the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent," although this is really a property of a set.

Example 3.10 The columns of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 4 & 2 & 6 & 8 \\ 2 & -1 & 1 & 3 \end{bmatrix}$$

are linearly dependent in the 3-space $\mathbb{R}^3$, since the third column is the sum of the first and the second.

As this example shows, the concept of linear dependence can be applied to the row or column vectors of any matrix.

Example 3.11 Consider an upper triangular matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix}.$$

The linear dependence of the column vectors of A may be written as

$$c_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which, in matrix notation, may be written as a homogeneous system:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the third row, $c_3 = 0$, from the second row $c_2 = 0$, and substitution of them into the first row forces $c_1 = 0$, *i.e.*, the homogeneous system has only the trivial solution, so that the column vectors are linearly independent. □

The following theorem can be proven by the same argument.

Theorem 3.5 *The nonzero rows of a matrix of a row-echelon form are linearly independent, and so are the columns that contain leading 1's.*

In particular, the rows of any triangular matrix with nonzero diagonals are linearly independent, and so are the columns.

In general, if $V = \mathbb{R}^m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are n vectors in $\mathbb{R}^m$, then they form an $m \times n$ matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$. On the other hand, Example 3.8 shows that the linear dependence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ of $\mathbf{v}_i$'s is nothing but the homogeneous equation $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (c_1, c_2, \cdots, c_n)$. Thus, *the column vectors $\mathbf{v}_i$'s of A are linearly independent in $\mathbb{R}^m$ if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution,* and *they are linearly dependent if and only if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*

If U is the reduced row-echelon form of A, then we know that $A\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{0}$ have the same set of solutions. Moreover, a homogeneous system $A\mathbf{x} = \mathbf{0}$ with unknowns more than the number of equations always has a nontrivial solution (see the remark on page 11). This proves the following lemma.

Lemma 3.6 **(1)** *Any set of n vectors in the m-space $\mathbb{R}^m$ is linearly dependent if $n > m$.*

(2) *If U is the reduced row-echelon form of A, then the columns of U are linearly independent if and only if the columns of A are linearly independent.*

Example 3.12 Consider the vectors $\mathbf{e}_1 = (1,\ 0,\ 0)$, $\mathbf{e}_2 = (0,\ 1,\ 0)$ and $\mathbf{e}_3 = (0,\ 0,\ 1)$ in the 3-space $\mathbb{R}^3$. The vector equation $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = \mathbf{0}$ becomes

$$c_1(1,\ 0,\ 0) + c_2(0,\ 1,\ 0) + c_3(0,\ 0,\ 1) = (0,\ 0,\ 0)$$

or, equivalently, $(c_1,\ c_2,\ c_3) = (0,\ 0,\ 0)$. Thus, $c_1 = c_2 = c_3 = 0$, so the set of vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ is linearly independent and also spans $\mathbb{R}^3$.

Example 3.13 In general, it is quite clear that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are linearly independent. Moreover, they span the n-space $\mathbb{R}^n$: In fact, when we write a vector $\mathbf{x} \in \mathbb{R}^n$ as $(x_1,\ \ldots,\ x_n)$, it means just the linear combination of the vector $\mathbf{e}_i$'s:

$$\mathbf{x} = (x_1,\ \ldots,\ x_n) = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

However, if any one of the $\mathbf{e}_i$'s is missed, then they cannot span $\mathbb{R}^n$. Thus, this kind of vector plays a special role in the vector space.

Definition 3.4 Let V be a vector space. A **basis** for V is a set of linearly independent vectors that spans V.

For example, as we saw in Example 3.13, the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ forms a basis, called the **standard basis** for the n-space $\mathbb{R}^n$. Of course, there are many other bases for $\mathbb{R}^n$.

Example 3.14 **(1)** The set of vectors $(1, 1, 0)$, $(0, -1, 1)$, and $(1, 0, 1)$ is not a basis for the 3-space $\mathbb{R}^3$, since this set is linearly dependent (the third is the sum of the first two vectors), and cannot span $\mathbb{R}^3$. (The vector $(1, 0, 0)$ cannot be obtained as a linear combination of them (prove it).) This set does not have enough vectors spanning $\mathbb{R}^3$.

(2) The set of vectors $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ and $(0, 1, 0)$ is not a basis either, since they are not linearly independent (the sum of the first two minus the third makes the fourth) even though they span $\mathbb{R}^3$. This set of vectors has some redundant vectors spanning $\mathbb{R}^3$.

(3) The set of vectors $(1, 1, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$ is linearly independent and also spans $\mathbb{R}^3$. That is, it is a basis for $\mathbb{R}^3$, different from the standard basis. This set has the proper number of vectors spanning $\mathbb{R}^3$, since the set cannot be reduced to a smaller set nor does it need any additional vector spanning $\mathbb{R}^3$. $\square$

By definition, in order to show that a set of vectors in a vector space is a basis, one needs to show two things: *it is linearly independent, and it spans the whole space.* The following theorem shows that a basis for a vector space represents a coordinate system just like the rectangular coordinate system by the standard basis for $\mathbb{R}^n$.

Theorem 3.7 *Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Then each vector $\mathbf{x}$ in V can be uniquely expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, i.e., there are unique scalars a_i's, $i = 1, \ldots, n$, such that*

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

In this case, the column vector $[a_1\ a_2\ \cdots\ a_n]^T$ is called the **coordinate vector** of $\mathbf{x}$ with respect to the basis α, and it is denoted $[\mathbf{x}]_\alpha$.

Proof: If $\mathbf{x}$ can be also expressed as $\mathbf{x} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$, then we have $\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n$. By the linear independence of $\mathbf{x}_i$'s, $a_i = b_i$ for all $i = 1, \ldots, n$. $\square$

Example 3.15 Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1) = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_2 = (0,1,1) = \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_3 = (0,0,1) = \mathbf{e}_3$. Then

$$[\mathbf{v}_1]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad [\mathbf{v}_2]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [\mathbf{v}_3]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

while $[\mathbf{v}_1]_\beta = [1\ 0\ 0]^T$, $[\mathbf{v}_2]_\beta = [0\ 1\ 0]^T$, $[\mathbf{v}_3]_\beta = [0\ 0\ 1]^T$.

Problem 3.6 Show that the vectors $\mathbf{v}_1 = (1,\ 2,\ 1)$, $\mathbf{v}_2 = (2,\ 9,\ 0)$ and $\mathbf{v}_3 = (3,\ 3,\ 4)$ in the 3-space $\mathbb{R}^3$ form a basis.

Problem 3.7 Show that the set $\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ is a basis for $P_n(\mathbb{R})$, the vector space of all polynomials of degree $\le n$ with real coefficients.

Problem 3.8 In the n-space $\mathbb{R}^n$, determine whether or not the set

$$\{\mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{e}_2 - \mathbf{e}_3,\ \ldots,\ \mathbf{e}_{n-1} - \mathbf{e}_n,\ \mathbf{e}_n - \mathbf{e}_1\}$$

is linearly dependent.

Problem 3.9 Let $\mathbf{x}_k$ denote the vector in $\mathbb{R}^n$ whose first $k-1$ coordinates are zero and whose last $n-k+1$ coordinates are 1. Show that the set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$.

3.3 Dimensions

We often say that the line $\mathbb{R}^1$ is one-dimensional, the plane $\mathbb{R}^2$ is two-dimensional and the space $\mathbb{R}^3$ is three-dimensional, etc. This is mostly due to the fact that the freedom in choosing coordinates for each element in the space is 1, 2 or 3, respectively. This means that the concept of *dimension* is closely related to the concept of bases. Note that for a vector space in general there is no unique way to choose a basis. However, there is something common to all bases, and this is related to the notion of dimension. We first need the following important lemma from which one can define the dimension of a vector space.

Lemma 3.8 *Let V be a vector space and let $\alpha = \{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m\}$ be a set of m-vectors in V.*

(1) *If α spans V, then every set of vectors with more than m vectors cannot be linearly independent.*

(2) *If α is linearly independent, then any set of vectors with fewer than m vectors cannot span V.*

Proof: Since (2) follows from (1) directly, we prove only (1). Let $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a set of n-vectors in V with $n > m$. We will show that β is linearly dependent. Indeed, since each vector $\mathbf{y}_j$ is a linear combination of the vectors in the spanning set α, *i.e.*, for $j = 1, \ldots, n$,

$$\mathbf{y}_j = a_{1j}\mathbf{x}_1 + a_{2j}\mathbf{x}_2 + \cdots + a_{mj}\mathbf{x}_m = \sum_{i=1}^{m} a_{ij}\mathbf{x}_i,$$

we have

$$\begin{aligned}
c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n &= c_1(a_{11}\mathbf{x}_1 + a_{21}\mathbf{x}_2 + \cdots + a_{m1}\mathbf{x}_m) \\
&\quad + c_2(a_{12}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{m2}\mathbf{x}_m) \\
&\qquad \vdots \\
&\quad + c_n(a_{1n}\mathbf{x}_1 + a_{2n}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_m) \\
&= (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n)\mathbf{x}_1 \\
&\quad + (a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n)\mathbf{x}_2 \\
&\qquad \vdots \\
&\quad + (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n)\mathbf{x}_m.
\end{aligned}$$

Thus, β is linearly dependent if and only if the system of linear equations

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{0}$$

has a nontrivial solution $(c_1, c_2, \ldots, c_n) \neq (0, 0, \cdots, 0)$. This is true if all the coefficients of $\mathbf{x}_i$'s are zero but not all of c_i's are zero. This means that the homogeneous system of linear equations in c_i's,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

has a nontrivial solution. This is guaranteed by Lemma 3.6, since A is an $m \times n$ matrix with $m < n$. □

It is clear by Lemma 3.8 that if a set $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of n vectors is a basis for a vector space V, then no other set $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ of r vectors can be a basis for V if $r \neq n$. This means that all bases for a vector space V have the same number of vectors, even if there are many different bases for a vector space. Therefore, we obtain the following important result:

Theorem 3.9 *If a basis for a vector space V consists of n vectors, then so does every other basis.*

Definition 3.5 The **dimension** of a vector space V is the number, say n, of vectors in a basis for V, denoted by $\dim V = n$. When V has a basis of a finite number of vectors, V is said to be **finite dimensional**.

Example 3.16 The following can be easily verified:

(1) If V has only the zero vector: $V = \{\mathbf{0}\}$, then $\dim V = 0$.

(2) If $V = \mathbb{R}^n$, then $\dim \mathbb{R}^n = n$, since V has the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

(3) If $V = P_n(\mathbb{R})$ of all polynomials of degree less than or equal to n, then $\dim P_n(\mathbb{R}) = n + 1$ since $\{1, x, x^2, \ldots, x^n\}$ is a basis for V.

(4) If $V = M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices, then $\dim M_{m\times n}(\mathbb{R}) = mn$ since $\{E_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a basis for V, where E_{ij} is the $m \times n$ matrix whose (i, j)-th entry is 1 and all others are zero. □

If $V = C(\mathbb{R})$ of all real-valued continuous functions defined on the real line, then one can show that V is not finite dimensional. A vector space V is **infinite dimensional** if it is not finite dimensional. In this book, we are concerned only with finite dimensional vector spaces unless otherwise stated.

Theorem 3.10 *Let V be a finite dimensional vector space.*

(1) *Any linearly independent set in V can be extended to a basis by adding more vectors if necessary.*

(2) *Any set of vectors that spans V can be reduced to a basis by discarding vectors if necessary.*

Proof: We prove **(1)** only and leave **(2)** as an exercise. Let $\alpha = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a linearly independent set in V. If α spans V, then α is a basis. If α does not span V, then there exists a vector, say $\mathbf{x}_{k+1}$, in V that is not contained in the subspace spanned by the vectors in α. Now $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ is linearly independent (check why). If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ spans V, then this is a basis for V. If it does not span V, then the same procedure can be repeated, yielding a linearly independent set that spans V, *i.e.*, a basis for V. This procedure must stop in a finite step because of Lemma 3.8 for a finite dimensional vector space V. □

Theorem 3.10 shows that a basis for a vector space V is a set of vectors in V which is *maximally* independent and *minimally* spanning in the above sense. In particular, if W is a subspace of V, then any basis for W is linearly independent also in V, and can be extended to a basis for V. Thus $\dim W \le \dim V$.

Corollary 3.11 *Let V be a vector space of dimension n. Then*
(1) *any set of n vectors that spans V is a basis for V, and*
(2) *any set of n linearly independent vectors is a basis for V.*

Proof: Again we prove **(1)** only. If a spanning set of n vectors were not linearly independent, then the set would be reduced to a basis that has a smaller number of vectors than n vectors. □

Corollary 3.11 means that if it is known that $\dim V = n$ and if a set of n vectors either is linearly independent or spans V, then it is already a basis for the space V.

Example 3.17 Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{x}_1 = (1,\ -2,\ 5,\ -3),\ \ \mathbf{x}_2 = (0,\ 1,\ 1,\ 4),\ \ \mathbf{x}_3 = (1,\ 0,\ 1,\ 0).$$

Find a basis for W and extend it to a basis for $\mathbb{R}^4$.

Solution: Note that $\dim W \le 3$ since W is spanned by three vectors $\mathbf{x}_i$'s. Let A be the 3×4 matrix whose rows are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Reduce A to a row-echelon form:

$$U = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & \frac{5}{6} \end{bmatrix}.$$

The three nonzero row vectors of U are clearly linearly independent, and they also span W because the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ can be expressed as a linear combination of these three nonzero row vectors of U. Hence, U provides a basis for W. (Note that this implies $\dim W = 3$ and hence $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is also a basis for W by Corollary 3.11. The linear independence of $\mathbf{x}_i$'s is a by-product of this fact).

To extend this basis, just add any nonzero vector of the form $\mathbf{x}_4 = (0, 0, 0, t)$ to the rows of U to get a basis for the space $\mathbb{R}^4$. □

Problem 3.10 Let W be a subspace of a vector space V. Show that if $\dim W = \dim V$, then $W = V$.

Problem 3.11 Find a basis and the dimension of each of the following subspaces of $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices:

(1) the space of all $n \times n$ diagonal matrices whose traces are zero;

(2) the space of all $n \times n$ symmetric matrices;

(3) the space of all $n \times n$ skew-symmetric matrices.

Now consider two subspaces U and W of a vector space V. The **sum** of these subspaces U and W is defined by

$$U + W = \{\mathbf{u} + \mathbf{w} \ : \ \mathbf{u} \in U, \ \mathbf{w} \in W\}.$$

It is not hard to see that this is a subspace of V.

Problem 3.12 Let U and W be subspaces of a vector space V.

(1) Show that $U + W$ is the smallest subspace of V containing U and W.

(2) Prove that $U \cap W$ is also a subspace of V. Is $U \cup W$ a subspace of V? Justify your answer.

Definition 3.6 A vector space V is called the **direct sum** of two subspaces U and W, written $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{\mathbf{0}\}$.

For example, one can easily show that $\mathbb{R}^3 = \mathbb{R}^1 \oplus \mathbb{R}^2 = \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1$.

Theorem 3.12 *A vector space V is the direct sum of subspaces U and W, i.e., $V = U \oplus W$, if and only if for any $\mathbf{v} \in V$ there exist unique $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.*

Proof: Suppose that $V = U \oplus W$. Then, for any $\mathbf{v} \in V$, there exist vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, since $V = U + W$. To show the uniqueness, suppose that $\mathbf{v}$ is also expressed as a sum $\mathbf{u}' + \mathbf{w}'$ for $\mathbf{u}' \in U$ and $\mathbf{w}' \in W$. Then $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$ implies

$$\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w} \in U \cap W = \{\mathbf{0}\}.$$

Hence, $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$.

Conversely, if there exists a nonzero vector $\mathbf{v}$ in $U \cap W$, then $\mathbf{v}$ can be written as sum of vectors in U and W in many different ways:

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{v} = \frac{1}{3}\mathbf{v} + \frac{2}{3}\mathbf{v} \in U + W. \qquad \square$$

Example 3.18 Consider the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in $\mathbb{R}^3$. Let $U = \{a_1\mathbf{e}_1 + b_3\mathbf{e}_3 \ : \ a_1, \ b_3 \in \mathbb{R}\}$ be the subspace spanned by $\mathbf{e}_1$ and $\mathbf{e}_3$ (xz-plane), and let $W = \{a_2\mathbf{e}_2 + c_2\mathbf{e}_3 \ : \ a_2, c_3 \in \mathbb{R}\}$ be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$ (yz-plane). Then a vector in $U + W$ is of the form

$$(a_1\mathbf{e}_1 + b_3\mathbf{e}_3) + (a_2\mathbf{e}_2 + c_2\mathbf{e}_3) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + (b_3 + c_2)\mathbf{e}_3 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

where $a_3 = b_3 + c_3$ and a_1, a_2, a_3 are arbitrary numbers. Thus $U + W = \mathbb{R}^3$. However, $\mathbb{R}^3 \neq U \oplus W$ since clearly $\mathbf{e}_3 \in U \cap W \neq \{\mathbf{0}\}$. In fact, the vector $\mathbf{e}_3 \in \mathbb{R}^3$ can be written as many linear combinations of vectors in U and W:

$$\mathbf{e}_3 = \frac{1}{2}\mathbf{e}_3 + \frac{1}{2}\mathbf{e}_3 = \frac{1}{3}\mathbf{e}_3 + \frac{2}{3}\mathbf{e}_3 \in U + W.$$

Note that if we had taken W to be the subspace spanned by $\mathbf{e}_2$ alone, then it would be easy to see that $\mathbb{R}^3 = U \oplus W$. Note also that there are many choices for W. $\square$

As a direct consequence of Theorem 3.10 and the definition of the direct sum, one can show the following.

Corollary 3.13 *If U is a subspace of V, then there is a subspace W in V such that $V = U \oplus W$.*

Proof: Choose a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for U, and extend it to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ for V. Then the subspace W spanned by $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ satisfies the requirement. $\square$

Problem 3.13 Let U and W be the subspaces of the vector space $M_{n\times n}(\mathbb{R})$ consisting of all symmetric matrices and all skew-symmetric matrices, respectively. Show that $M_{n\times n}(\mathbb{R}) = U \oplus W$. Therefore, the decomposition of a square matrix A given in (3) of Problem 1.10 is unique.

Problem 3.14 Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and let $W_i = \{r\mathbf{v}_i : r \in \mathbb{R}\}$ be the subspace of V spanned by $\mathbf{v}_i$. Show that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

3.4 Row and column spaces

In this section, we go back to systems of linear equations and study them in terms of the concepts introduced in the previous sections. Note that an $m \times n$ matrix A can be abbreviated by the row vectors or column vectors as follows:

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}, \end{aligned}$$

where the $\mathbf{r}_i$'s are the row vectors of A that are in $\mathbb{R}^n$, and the $\mathbf{c}_j$'s are the column vectors of A that are in $\mathbb{R}^m$.

Definition 3.7 Let A be an $m \times n$ matrix with row vectors $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ and column vectors $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.

(1) The **row space** of A is the subspace in $\mathbb{R}^n$ spanned by the row vectors $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$, denoted by $\mathcal{R}(A)$.

(2) The **column space** of A is the subspace in $\mathbb{R}^m$ spanned by the column vectors $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$, denoted by $\mathcal{C}(A)$.

(3) The solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is called the **null space** of A, denoted by $\mathcal{N}(A)$.

Note that the null space $\mathcal{N}(A)$ is a subspace of the n-space $\mathbb{R}^n$, whose dimension is called the **nullity** of A. Since the row vectors of A are just the column vectors of its transpose A^T, and the column vectors of A are the row vectors of A^T, the row space of A is the column space of A^T; that is,

$$\mathcal{R}(A) = \mathcal{C}(A^T) \quad \text{and} \quad \mathcal{C}(A) = \mathcal{R}(A^T).$$

Since $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \ldots x_n\mathbf{c}_n$ for any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we get

$$\mathcal{C}(A) = \{A\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^n\}.$$

Thus, for a vector $\mathbf{b} \in \mathbb{R}^m$, *the system* $A\mathbf{x} = \mathbf{b}$ *has a solution if and only if* $\mathbf{b} \in \mathcal{C}(A) \subseteq \mathbb{R}^m$. Thus, the column space $\mathcal{C}(A)$ is the set of vectors $\mathbf{b} \in \mathbb{R}^m$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

It is quite natural to ask what the dimensions of those subspaces are, and how one can find bases for them. This will help us to understand the structure of all the solutions of the equation $A\mathbf{x} = \mathbf{b}$. Since the set of the row vectors and the set of the column vectors of A are spanning sets for the row space and the column space, respectively, a minimally spanning subset of each of them will be a basis for each of them.

This is not difficult for a matrix of a (reduced) row-echelon form.

Example 3.19 Let U be in a reduced row-echelon form given as

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the first three nonzero row vectors containing leading 1's are linearly independent and they form a basis for the row space $\mathcal{R}(U)$, so that $\dim \mathcal{R}(U) = 3$. On the other hand, note that the first three columns containing leading 1's are linearly independent (see Theorem 3.5), and that the last two column vectors can be expressed as linear combinations of them. Hence, they form a basis for $\mathcal{C}(U)$, and $\dim \mathcal{C}(U) = 3$. To find a basis for the null space $\mathcal{N}(U)$, we first solve the system $U\mathbf{x} = \mathbf{0}$ with arbitrary values s and t for the free variables x_4 and x_5, and get the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 2t \\ s - 3t \\ -4s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = s\mathbf{n}_s + t\mathbf{n}_t,$$

where $\mathbf{n}_s = (-2,\ 1,\ -4,\ 1,\ 0)$, $\mathbf{n}_t = (-2,\ -3,\ 1,\ 0,\ 1)$. It shows that these two vectors $\mathbf{n}_s$ and $\mathbf{n}_t$ span the null space $\mathcal{N}(U)$, and they are clearly linearly independent. Hence, the set $\{\mathbf{n}_s,\ \mathbf{n}_t\}$ is a basis for the null space $\mathcal{N}(U)$. □

In the following, the row, the column or the null space of a matrix A will be discussed in relation to the corresponding space of its (reduced) row-echelon form. We first investigate the row space $\mathcal{R}(A)$ and the null space $\mathcal{N}(A)$ of A by comparing them with those of the reduced row-echelon form U of A. Since $A\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{0}$ have the same solution set by Theorem 1.1, we have $\mathcal{N}(A) = \mathcal{N}(U)$.

Let $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$ be an $m \times n$ matrix, where $\mathbf{r}_i$'s are the row vectors of A. The three elementary row operations change A into the following three types:

$$A_1 = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ k\mathbf{r}_i \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \text{ for } k \neq 0, \; A_2 = \begin{bmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_i \\ \vdots \end{bmatrix} \text{ for } i < j, \; A_3 = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i + k\mathbf{r}_j \\ \vdots \\ \mathbf{r}_m \end{bmatrix}.$$

It is clear that the row vectors of the three matrices A_1, A_2 and A_3 are linear combinations of the row vectors of A. On the other hand, by the inverse elementary row operations, these matrices can be changed into A. Thus, the row vectors of A can also be written as linear combinations of those of A_i's. This means that if matrices A and B are row equivalent, then their row spaces must be equal, *i.e.*, $\mathcal{R}(A) = \mathcal{R}(B)$.

Now the nonzero row vectors in the reduced row-echelon form U are always linearly independent and span the row space of U (see Theorem 3.5). Thus they form a basis for the row space $\mathcal{R}(A)$ of A. We have the following theorem.

Theorem 3.14 *Let U be a (reduced) row-echelon form of a matrix A. Then*

$$\mathcal{R}(A) = \mathcal{R}(U) \quad \text{and} \quad \mathcal{N}(A) = \mathcal{N}(U).$$

Moreover, if U has r nonzero row vectors containing leading 1's, then they form a basis for the row space $\mathcal{R}(A)$, so that the dimension of $\mathcal{R}(A)$ is r.

The following example shows how to find bases for the row and the null spaces, and at the same time how to find a basis for the column space $\mathcal{C}(A)$.

Example 3.20 Let A be a matrix given as

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{bmatrix}.$$

Find bases for the row space $\mathcal{R}(A)$, the null space $\mathcal{N}(A)$, and the column space $\mathcal{C}(A)$ of A.

Solution: (1) Find a basis for $\mathcal{R}(A)$: By Gauss-Jordan elimination on A, we get the reduced row-echelon form U:

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the three nonzero row vectors

$$\begin{aligned} \mathbf{v}_1 &= (1, \ 0, \ \ 2, \ 0, \ 1), \\ \mathbf{v}_2 &= (0, \ 1, \ -1, \ 0, \ 1), \\ \mathbf{v}_3 &= (0, \ 0, \ \ 0, \ 1, \ 1) \end{aligned}$$

of U are linearly independent, they form a basis for the row space $\mathcal{R}(U) = \mathcal{R}(A)$, so $\dim \mathcal{R}(A) = 3$. (Note that in the process of Gaussian elimination, we did not use a permutation matrix. This means that the three nonzero rows of U were obtained from the first three row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ of A and the fourth row $\mathbf{r}_4$ of A turned out to be a linear combination of them. Thus the first three row vectors of A also form a basis for the row space.)

(2) Find a basis for $\mathcal{N}(A)$. It is enough to solve the homogeneous system $U\mathbf{x} = \mathbf{0}$, since $\mathcal{N}(A) = \mathcal{N}(U)$. That is, neglecting the fourth zero equation, the equation $U\mathbf{x} = \mathbf{0}$ takes the following system of equations:

$$\left\{ \begin{array}{rcrcrcrcrcl} x_1 & & & + & 2x_3 & & & + & x_5 & = & 0 \\ & & x_2 & - & x_3 & & & + & x_5 & = & 0 \\ & & & & & & x_4 & + & x_5 & = & 0. \end{array} \right.$$

Since the first, the second and the fourth columns of U contain the leading 1's, we see that the basic variables are x_1, x_2, x_4, and the free variables are

x_3, x_5. By assigning arbitrary values s and t to the free variables x_3 and x_5, we find the solution $\mathbf{x}$ of $U\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - t \\ s - t \\ s \\ - t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = s\mathbf{n}_s + t\mathbf{n}_t,$$

where $\mathbf{n}_s = (-2,\ 1,\ 1,\ 0,\ 0)$ and $\mathbf{n}_t = (-1,\ -1,\ 0,\ -1,\ 1)$. In fact, the two vectors $\mathbf{n}_s$ and $\mathbf{n}_t$ are the solutions when the values of $(x_3, x_5) = (s, t)$ are $(1, 0)$ and those of $(x_3, x_5) = (s, t)$ are $(0, 1)$, respectively. They must be linearly independent, since $(1,\ 0)$ and $(0,\ 1)$, as the (x_3, x_5)-coordinates of $\mathbf{n}_s$ and $\mathbf{n}_t$ respectively, are clearly linearly independent. Since any solution of $U\mathbf{x} = \mathbf{0}$ is a linear combination of them, the set $\{\mathbf{n}_s,\ \mathbf{n}_t\}$ is a basis for the null space $\mathcal{N}(U) = \mathcal{N}(A)$. Thus $\dim \mathcal{N}(A) = 2 =$ *the number of free variables in* $U\mathbf{x} = \mathbf{0}$.

(3) Find a basis for $\mathcal{C}(A)$. Let $\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3,\ \mathbf{c}_4,\ \mathbf{c}_5$ denote the column vectors of A in the given order. Since these column vectors of A span $\mathcal{C}(A)$, we only need to discard some of the columns that can be expressed as linear combinations of other column vectors. But, the linear dependence

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + x_4\mathbf{c}_4 + x_5\mathbf{c}_5 = \mathbf{0}, \quad \textit{i.e.}, \ A\mathbf{x} = \mathbf{0},$$

holds if and only if $\mathbf{x} = (x_1,\ \cdots,\ x_5) \in \mathcal{N}(A)$. By taking $\mathbf{x} = \mathbf{n}_s = (-2,\ 1,\ 1,\ 0,\ 0)$ or $\mathbf{x} = \mathbf{n}_t = (-1,\ -1,\ 0,\ -1,\ 1)$, the basis vectors of $\mathcal{N}(A)$ given in (2), we obtain two nontrivial linear dependencies of $\mathbf{c}_i$'s:

$$\begin{aligned} -2\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 &= \mathbf{0}, \\ -\mathbf{c}_1 - \mathbf{c}_2 - \mathbf{c}_4 + \mathbf{c}_5 &= \mathbf{0}, \end{aligned}$$

respectively. Hence, the column vectors $\mathbf{c}_3$ and $\mathbf{c}_5$ corresponding to the free variables in $A\mathbf{x} = \mathbf{0}$ can be written as

$$\begin{aligned} \mathbf{c}_3 &= 2\mathbf{c}_1 - \mathbf{c}_2, \\ \mathbf{c}_5 &= \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_4. \end{aligned}$$

That is, the column vectors $\mathbf{c}_3, \mathbf{c}_5$ of A are linear combinations of the column vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4$, which correspond to the basic variables in $A\mathbf{x} = \mathbf{0}$. Hence, $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_4\}$ spans the column space $\mathcal{C}(A)$.

We claim that $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is linearly independent. Let $\tilde{A} = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_4]$ and $\tilde{U} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_4]$ be submatrices of A and U, respectively, where $\mathbf{u}_j$ is the j-th column vector of the reduced row-echelon form U of A obtained in (1). Then clearly $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$ so that $\mathcal{N}(\tilde{A}) = \mathcal{N}(\tilde{U})$. Since the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_4$ are just the columns of U containing leading 1's, they are linearly independent, by Theorem 3.5, and $\tilde{U}\mathbf{x} = \mathbf{0}$ has only a trivial solution. This means that $\tilde{A}\mathbf{x} = \mathbf{0}$ has also only a trivial solution, so $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is linearly independent. Therefore, it is a basis for the column space $\mathcal{C}(A)$ and $\dim \mathcal{C}(A) = 3 =$ *the number of basic variables*. That is, *the column vectors of A corresponding to the basic variables in $U\mathbf{x} = \mathbf{0}$ form a basis for the column space $\mathcal{C}(A)$.* □

In summary, given a matrix A, we first find the (reduced) row-echelon form U of A by Gauss-Jordan elimination. Then a basis for $\mathcal{R}(A) = \mathcal{R}(U)$ is the set of nonzero rows vectors of U, and a basis for $\mathcal{N}(A) = \mathcal{N}(U)$ can be found by solving $U\mathbf{x} = \mathbf{0}$, which is easy. On the other hand, one has to be careful for $\mathcal{C}(U) \neq \mathcal{C}(A)$ in general, since the column space of A is not preserved by Gauss-Jordan elimination. However, we have $\dim \mathcal{C}(A) = \dim \mathcal{C}(U)$, and a basis for $\mathcal{C}(A)$ can be selected from the column vectors in A, not in U, as those corresponding to the basic variables (or the leading 1's in U). To show that those column vectors indeed form a basis for $\mathcal{C}(A)$, we used a basis for the null space $\mathcal{N}(A)$ to eliminate the redundant columns.

Note that a basis for the column space $\mathcal{C}(A)$ can be also found with the elementary column operations, which is the same as finding a basis for the row space $\mathcal{R}(A^T)$ of A^T.

Problem 3.15 Let A be the matrix given in Example 3.20. Find a relation of a, b, c, d so that the vector $\mathbf{x} = (a, b, c, d)$ belongs to $\mathcal{C}(A)$.

Problem 3.16 Find bases for $\mathcal{R}(A)$ and $\mathcal{N}(A)$ of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}.$$

Also find a basis for $\mathcal{C}(A)$ by finding a basis for $\mathcal{R}(A^T)$.

Problem 3.17 Let A and B be two $n \times n$ matrices. Show that $AB = \mathbf{0}$ if and only if the column space of B is a subspace of the nullspace of A.

Problem 3.18 Find an example of a matrix A and its row-echelon form U such that $\mathcal{C}(A) \neq \mathcal{C}(U)$.

3.5 Rank and nullity

The argument in Example 3.20 is so general that it can be used to prove the following theorem, which is one of the most fundamental results in linear algebra. The proof given here is just a repetition of the argument in Example 3.20 in a general form, and so may be skipped at the reader's discretion.

Theorem 3.15 (The first fundamental theorem) *For any $m \times n$ matrix A, the row space and the column space of A have the same dimension; that is,* $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.

Proof: Let $\dim \mathcal{R}(A) = r$ and let U be the reduced row-echelon form of A. Then r is the number of the nonzero row (or column) vectors of U containing leading 1's, which is equal to the number of basic variables in $U\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \mathbf{0}$. We shall prove that the r columns of A corresponding to the r leading 1's (or basic variables) form a basis for $\mathcal{C}(A)$, so that $\dim \mathcal{C}(A) = r = \dim \mathcal{R}(A)$.

(1) They are linearly independent: Let $\tilde{A}$ denote the submatrix of A whose columns are those of A corresponding to the r basic variables (or leading 1's) in U, and let $\tilde{U}$ denote the submatrix of U containing r leading 1's. Then, it is quite clear that $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$, so that $\tilde{A}\mathbf{x} = \mathbf{0}$ if and only if $\tilde{U}\mathbf{x} = \mathbf{0}$. However, $\tilde{U}\mathbf{x} = \mathbf{0}$ has only a trivial solution since the columns of U containing the leading 1's are linearly independent by Theorem 3.5. Therefore, $\tilde{A}\mathbf{x} = \mathbf{0}$ also has only the trivial solution, so the columns of $\tilde{A}$ are linearly independent.

(2) They span $\mathcal{C}(A)$: Note that the columns A corresponding to the free variables are not contained in $\tilde{A}$, and each of these column vector of A can be written as a linear combination of the column vectors of $\tilde{A}$ (see Example 3.20). In fact, if $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is the set of free variables whose corresponding columns are not in $\tilde{A}$, then, for an assignment of value 1 to x_{i_j} and 0 to all the other free variables, one can always find a nontrivial solution of

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \ldots + x_n\mathbf{c}_n = \mathbf{0}.$$

When the solution is substituted into this equation, one can see that the column $\mathbf{c}_{i_j}$ of A corresponding to $x_{i_j} = 1$ is written as a linear combination of the columns of $\tilde{A}$. This can be done for each $j = 1, \ldots, k$, so the columns of A corresponding to those free variables are redundant in the spanning set of $\mathcal{C}(A)$. □

Remark: In the proof of Theorem 3.15, once we have shown that the columns in $\tilde{A}$ are linearly independent as in (1), we may replace step (2) by the following argument: One can easily see that $\dim \mathcal{C}(A) \geq \dim \mathcal{R}(A)$ by Theorem 3.10. On the other hand, since this inequality holds for arbitrary matrices, in particular for A^T, we get $\dim \mathcal{C}(A^T) \geq \dim \mathcal{R}(A^T)$. Moreover, $\mathcal{C}(A^T) = \mathcal{R}(A)$ and $\mathcal{R}(A^T) = \mathcal{C}(A)$ implies $\dim \mathcal{C}(A) \leq \dim \mathcal{R}(A)$, which means $\dim \mathcal{C}(A) = \dim \mathcal{R}(A)$. This also means that the column vectors of $\tilde{A}$ span $\mathcal{C}(A)$, and so form a basis.

In summary, the following equalities are now clear from Theorem 3.14 and 3.15:

$$
\begin{aligned}
\dim \mathcal{R}(A) &= \dim \mathcal{R}(U) \\
&= \text{the number of nonzero row vectors of } U \\
&= \text{the maximal number of linearly independent} \\
&\quad\ \text{row vectors of } A \\
&= \text{the number of basic variables in } U\mathbf{x} = \mathbf{0}. \\
&= \text{the maximal number of linearly independent} \\
&\quad\ \text{column vectors of } A \\
&= \dim \mathcal{C}(A).
\end{aligned}
$$

$$
\begin{aligned}
\dim \mathcal{N}(A) &= \dim \mathcal{N}(U) \\
&= \text{the number of free variables in } U\mathbf{x} = \mathbf{0}.
\end{aligned}
$$

Definition 3.8 For an $m \times n$ matrix A, the **rank** of A is defined to be the dimension of the row space (or the column space), denoted by rank A.

Clearly, rank $I_n = n$ and rank $A =$ rank A^T. And for an $m \times n$ matrix A, rank $A = \dim \mathcal{R}(A) = \dim \mathcal{C}(A)$. Since $\dim \mathcal{R}(A) \leq m$ and $\dim \mathcal{C}(A) \leq n$, we have the following corollary:

Corollary 3.16 *If A is an $m \times n$ matrix, then* rank $A \leq \min\{m, n\}$.

Since $\dim \mathcal{R}(A) = \dim \mathcal{C}(A) =$ rank A is the number of basic variables in $A\mathbf{x} = \mathbf{0}$, and $\dim \mathcal{N}(A) =$ nullity of A is the number of free variables $A\mathbf{x} = \mathbf{0}$, we have the following corollary.

Corollary 3.17 *For any $m \times n$ matrix A,*

$$
\begin{aligned}
\dim \mathcal{R}(A) + \dim \mathcal{N}(A) &= \text{rank } A + \text{nullity of } A &= n, \\
\dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) &= \text{rank } A + \text{nullity of } A^T &= m.
\end{aligned}
$$

If $\dim \mathcal{N}(A) = 0$ (or $\mathcal{N}(A) = \{\mathbf{0}\}$), then $\dim \mathcal{R}(A) = n$ (or $\mathcal{R}(A) = \mathbb{R}^n$), which means that A has exactly n linearly independent rows and n linearly independent columns. In particular, if A is a square matrix of order n, then the row vectors are linearly independent if and only if the column vectors are linearly independent. Therefore, by Theorem 1.8, we get the following corollary.

Corollary 3.18 *Let A be an $n \times n$ square matrix. Then A is invertible if and only if* rank $A = n$.

Example 3.21 For a 4×5 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix},$$

by Gaussian elimination, we get

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first three nonzero rows containing leading 1's in U form a basis for $\mathcal{R}(U) = \mathcal{R}(A)$. Note that x_1, x_3 and x_5 are the basic variables in $U\mathbf{x} = \mathbf{0}$, since the first, third and fifth columns of U contain leading 1's. Thus the three columns $\mathbf{c}_1 = (1, -1, 1, 1)$, $\mathbf{c}_3 = (0, 1, -3, -2)$ and $\mathbf{c}_5 = (1, 0, 2, 3)$ of A, not the three columns in U, corresponding to those basic variables x_1, x_3 and x_5 form a basis for $\mathcal{C}(A)$. Therefore, rank $A = \dim \mathcal{R}(A) = \dim \mathcal{C}(A) = 3$, the nullity of $A = \dim \mathcal{N}(A) = 2$, and $\dim \mathcal{N}(A^T) = 1$. □

Problem 3.19 Find the nullity and the rank of each of the following matrices:

$$(1)\, A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}, \quad (2)\, A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}.$$

For each of the matrices, show that $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$ directly by finding their bases.

Problem 3.20 Show that a system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if rank A = rank $[A\ \mathbf{b}]$, where $[A\ \mathbf{b}]$ denotes the augmented matrix of $A\mathbf{x} = \mathbf{b}$.

Theorem 3.19 *For any two matrices A and B for which AB can be defined,*
(1) $\mathcal{N}(AB) \supseteq \mathcal{N}(B)$,
(2) $\mathcal{N}((AB)^T) \supseteq \mathcal{N}(A^T)$,
(3) $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$,
(4) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$.

Proof: (1) and (2) are clear, since $B\mathbf{x} = \mathbf{0}$ implies $(AB)\mathbf{x} = A(B\mathbf{x}) = \mathbf{0}$.
(3) For an $m \times n$ matrix A and an $n \times p$ matrix B,

$$\begin{aligned} \mathcal{C}(AB) &= \{AB\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^p\} \\ &\subseteq \{A\mathbf{y} \ : \ \mathbf{y} \in \mathbb{R}^n\} = \mathcal{C}(A), \end{aligned}$$

because $B\mathbf{x} \in \mathbb{R}^n$ for any $\mathbf{x} \in \mathbb{R}^p$.
(4) $\mathcal{R}(AB) = \mathcal{C}((AB)^T) = \mathcal{C}(B^TA^T) \subseteq \mathcal{C}(B^T) = \mathcal{R}(B)$. □

Corollary 3.20 $\text{rank}(AB) \leq \min\{\text{rank } A, \ \text{rank } B\}$.

In some particular cases, the equality holds. In fact, it will be shown later in Theorem 5.23 that for any square matrix A, $\text{rank}(A^TA) = \text{rank } A = \text{rank}(AA^T)$. The following problem illustrates another such case.

Problem 3.21 Let A be an invertible square matrix. Show that, for any matrix B, $\text{rank}(AB) = \text{rank } B = \text{rank}(BA)$.

Theorem 3.21 *Let A be an $m \times n$ matrix of rank r. Then*
(1) *for every submatrix C of A,* $\text{rank } C \leq r$, *and*
(2) *the matrix A has at least one $r \times r$ submatrix of rank r, that is, A has an invertible submatrix of order r.*

Proof: (1) We consider an intermediate matrix B which is obtained from A by removing the rows that are not wanted in C. Then clearly $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and hence $\text{rank } B \leq \text{rank } A$. Moreover, since the columns of C are taken from those of B, $\mathcal{C}(C) \subseteq \mathcal{C}(B)$ and $\text{rank } C \leq \text{rank } B$.
(2) Note that we can find r linearly independent row vectors of A, which form a basis for the row space of A. Let B be the matrix whose row vectors consist of these vectors. Then $\text{rank } B = r$ and the column space of B must be of dimension r. By taking r linearly independent column vectors of B, one can find an $r \times r$ submatrix C of A with rank r. □

Problem 3.22 Prove that the rank of a matrix is equal to the largest order of its invertible submatrices.

Problem 3.23 For each of the matrices given in Problem 3.19, find an invertible submatrix of the largest order.

3.6 Bases for subspaces

In this section, we discuss how to find bases for $V+W$ and $V\cap W$ of two subspaces V and W of the n-space $\mathbb{R}^n$, and then derive an important relationship between the dimensions of those subspaces in terms of the dimensions of V and W.

Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ be bases for V and W, respectively. Let Q be the $n \times (k+\ell)$ matrix whose columns are those bases vectors:

$$Q = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_\ell]_{n\times(k+\ell)}.$$

Then it is quite clear that $\mathcal{C}(Q) = V+W$, so that a basis for $\mathcal{C}(Q)$ is a basis for $V+W$. On the other hand, one can show that $\mathcal{N}(Q)$ can be identified with $V\cap W$.

In fact, if $\mathbf{x} = (a_1, \ldots, a_k, b_1, \ldots, b_\ell) \in \mathcal{N}(Q) \subseteq \mathbb{R}^{k+\ell}$, then

$$Q\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell = \mathbf{0}.$$

This means that corresponding to $\mathbf{x}$ there is a vector

$$\mathbf{y} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = -(b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell)$$

that belongs to $V\cap W$, since the middle part is in V as a linear combination of the basis vectors in α and the right side is in W as a linear combination of the basis vectors in β. On the other hand, if $\mathbf{y} \in V\cap W$, $\mathbf{y}$ can be written as linear combinations of both bases for V and W:

$$\begin{aligned} \mathbf{y} &= a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \\ &= b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell, \end{aligned}$$

for some $a_1, \ldots, a_k$ and $b_1, \ldots, b_\ell$. Let $\mathbf{x} = (a_1, \ldots, a_k, -b_1, \ldots, -b_\ell)$. Then it is quite clear that $Q\mathbf{x} = \mathbf{0}$, *i.e.*, $\mathbf{x} \in \mathcal{N}(Q)$. That is, for each $\mathbf{x} \in \mathcal{N}(Q)$, there corresponds a vector $\mathbf{y} \in V\cap W$, and *vice versa*. Moreover, if $\mathbf{x}_i$, $i = 1, 2,$

correspond to $\mathbf{y}_i$, then one can easily check that $\mathbf{x}_1 + \mathbf{x}_2$ corresponds to $\mathbf{y}_1 + \mathbf{y}_2$, and $k\mathbf{x}_1$ corresponds to $k\mathbf{y}_1$. This means that the two vector spaces $\mathcal{N}(Q)$ and $V \cap W$ can be identified as vector spaces. In particular, for a basis for $\mathcal{N}(Q)$, the corresponding set in $V \cap W$ is also a basis, that is, if the set of vectors

$$\begin{cases} \mathbf{x}_1 &= (a_{11}, \ldots, a_{1k}, b_{11}, \ldots, b_{1\ell}), \\ \quad\vdots & \\ \mathbf{x}_s &= (a_{s1}, \ldots, a_{sk}, b_{s1}, \ldots, b_{s\ell}), \end{cases}$$

is a basis for $\mathcal{N}(Q)$, then the set

$$\begin{cases} \mathbf{y}_1 &= a_{11}\mathbf{v}_1 + \cdots + a_{1k}\mathbf{v}_k, \\ \quad\vdots & \\ \mathbf{y}_s &= a_{s1}\mathbf{v}_1 + \cdots + a_{sk}\mathbf{v}_k, \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{y}_1 &= -(b_{11}\mathbf{w}_1 + \cdots + b_{1\ell}\mathbf{w}_\ell), \\ \quad\vdots & \\ \mathbf{y}_s &= -(b_{s1}\mathbf{w}_1 + \cdots + b_{s\ell}\mathbf{w}_\ell) \end{cases}$$

is also a basis for $V \cap W$, and *vice versa*. This means that

$$\dim \mathcal{N}(Q) = \dim V \cap W.$$

Note that $\dim(V + W) \neq \dim V + \dim W$, in general. The following corollary gives a relation of them.

Corollary 3.22 *For any subspaces V and W of the n-space $\mathbb{R}^n$,*

$$\dim(V + W) + \dim(V \cap W) = \dim V + \dim W.$$

Proof: Let $\dim V = k$ and $\dim W = \ell$. Recall that rank $A+$ nullity $A =$ the number of the columns of a matrix A. Thus, for the matrix Q above, we have

$$\dim \mathcal{C}(Q) + \dim \mathcal{N}(Q) = k + \ell.$$

However, we have $\dim \mathcal{C}(Q) = \dim(V + W)$, $\dim \mathcal{N}(Q) = \dim(V \cap W)$. $\dim V = k$ and $\dim W = \ell$. □

Example 3.22 Let V and W be two subspaces of $\mathbb{R}^5$ with bases

$$\begin{cases} \mathbf{v}_1 &= (1, & 3, & -2, & 2, & 3), \\ \mathbf{v}_2 &= (1, & 4, & -3, & 4, & 2), \\ \mathbf{v}_3 &= (1, & 3, & 0, & 2, & 3), \end{cases} \qquad \begin{cases} \mathbf{w}_1 &= (2, & 3, & -1, & -2, & 9), \\ \mathbf{w}_2 &= (1, & 5, & -6, & 6, & 1), \\ \mathbf{w}_3 &= (2, & 4, & 4, & 2, & 8), \end{cases}$$

respectively. Then the matrix Q takes the following form:

$$Q = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{w}_1\ \mathbf{w}_2\ \mathbf{w}_3\ ] = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 3 & 5 & 4 \\ -2 & -3 & 0 & -1 & -6 & 4 \\ 2 & 4 & 2 & -2 & 6 & 2 \\ 3 & 2 & 3 & 9 & 1 & 8 \end{bmatrix}.$$

After Gauss-Jordan elimination, we get

$$U = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From this, one can directly see that $\dim(V + W) = 4$. The columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_3$ corresponding to the basic variables in $Q\mathbf{x} = \mathbf{0}$ form a basis for $\mathcal{C}(Q) = V + W$. Moreover, $\dim \mathcal{N}(Q) = \dim(V \cap W) = 2$, since there are two free variables x_4 and x_5 in $Q\mathbf{x} = \mathbf{0}$.

To find a basis for $V \cap W$, we solve $U\mathbf{x} = \mathbf{0}$ for $(x_1, x_2, x_3, 1, 0, x_5)$ and $(x_1, x_2, x_3, 0, 1, x_5)$. After a simple computation, we obtain a basis for $\mathcal{N}(Q)$:

$$\mathbf{x}_1 = (-5, 3, 0, 1, 0, 0) \text{ and } \mathbf{x}_2 = (0, -2, 1, 0, 1, 0).$$

From $Q\mathbf{x}_i = \mathbf{0}$, we obtain two equations:

$$\begin{array}{rcrcrcl} -5\mathbf{v}_1 & + & 3\mathbf{v}_2 & + & \mathbf{w}_1 & = & \mathbf{0}, \\ -2\mathbf{v}_2 & + & \mathbf{v}_3 & + & \mathbf{w}_2 & = & \mathbf{0}. \end{array}$$

Therefore, $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a basis for $V \cap W$, where

$$\mathbf{y}_1 = 5\mathbf{v}_1 - 3\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -2 \\ 9 \end{bmatrix} = \mathbf{w}_1, \quad \mathbf{y}_2 = 2\mathbf{v}_2 - \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ -6 \\ 6 \\ 1 \end{bmatrix} = \mathbf{w}_2.$$

Clearly, the equality

$$\dim(V + W) + \dim(V \cap W) = 4 + 2 = 3 + 3 = \dim V + \dim W$$

holds in this example. □

Remark: In Example 3.22, we showed a method for finding bases for $V+W$ and $V \cap W$ for given subspaces V and W of $\mathbb{R}^n$ by constructing a matrix Q whose columns are basis vectors for V and basis vectors for W. There is another method for finding their bases by constructing a matrix Q whose rows are basis vectors for V and basis vectors for W.

If Q is the matrix whose row vectors are basis vectors for V and basis vectors for W in order, then clearly $V + W = \mathcal{R}(Q)$. By finding a basis for the row space $\mathcal{R}(Q)$, we can get a basis for $V + W$.

On the other hand, a basis for $V \cap W$ can be found as follows: Let A be the $k \times n$ matrix whose rows are basis vectors for V, and B the $\ell \times n$ matrix whose rows are basis vectors for W. Then, $V = \mathcal{R}(A)$ and $W = \mathcal{R}(B)$. Let $\bar{A}$ denote the matrix A with an unknown vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ attached at the bottom row, *i.e.*,

$$\bar{A} = \begin{bmatrix} A \\ \mathbf{x} \end{bmatrix},$$

and the matrix $\bar{B}$ is defined similarly. Then it is clear that $\mathcal{R}(A) = \mathcal{R}(\bar{A})$ and $\mathcal{R}(B) = \mathcal{R}(\bar{B})$ if and only if $\mathbf{x} \in V \cap W = \mathcal{R}(A) \cap \mathcal{R}(B)$. This means that the row-echelon form of A and that of $\bar{A}$ should be the same via the same Gaussian elimination. Thus, by comparing the row vectors of the row-echelon form of A with those of $\bar{A}$, we can obtain a system of linear equations for $\mathbf{x} = (x_1, \ldots, x_n)$. By the same argument applied to B and $\bar{B}$, we get another system of linear equations for the same $\mathbf{x} = (x_1, \ldots, x_n)$. Solutions to these two systems together will provide us with a basis for $V \cap W$.

The following example illustrates how one can apply this argument to find bases for $V + W$ and $V \cap W$.

Example 3.23 Let V be the subspace of $\mathbb{R}^5$ spanned by

$$\begin{aligned} \mathbf{v}_1 &= (1,\ 3,\ -2,\ \ 2,\ 3), \\ \mathbf{v}_2 &= (1,\ 4,\ -3,\ \ 4,\ 2), \\ \mathbf{v}_3 &= (2,\ 3,\ -1,\ -2,\ 10), \end{aligned}$$

and W the subspace spanned by

$$\begin{aligned} \mathbf{w}_1 &= (1,\ 3,\ \ 0,\ 2,\ 1), \\ \mathbf{w}_2 &= (1,\ 5,\ -6,\ 6,\ 3), \\ \mathbf{w}_3 &= (2,\ 5,\ \ 3,\ 2,\ 1). \end{aligned}$$

Find a basis for $V + W$ and for $V \cap W$.

Solution: Note that the matrix A whose row vectors are $\mathbf{v}_i$'s is reduced to a row-echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

so that $\dim V = 3$. Similarly, the matrix B whose row vectors are $\mathbf{w}_j$'s is reduced to a row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that $\dim W = 2$.

Now, if Q denotes the 6×5 matrix whose row vectors are $\mathbf{v}_i$'s and $\mathbf{w}_j$'s, then $V+W = \mathcal{R}(Q)$. By Gaussian elimination, Q is reduced to a row-echelon form, excluding zero rows:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the four nonzero row vectors

$$(1,\ 3,\ -2,\ 2,\ 3),\ (0,\ 1,\ -1,\ 2,\ -1),\ (0,\ 0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 0,\ 1)$$

form a basis for $V + W$, so that $\dim(V + W) = 4$.

We now find a basis for $V \cap W$. A vector $\mathbf{x} = (x_1,\ x_2,\ x_3,\ x_4,\ x_5) \in \mathbb{R}^5$ is contained in $V \cap W$ if and only if $\mathbf{x}$ is contained in both the row space of A and that of B.

Let $\bar{A}$ be A with $\mathbf{x}$ attached at the last row:

$$\bar{A} = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 10 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}.$$

Then by the same Gaussian elimination $\bar{A}$ is reduced to

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -x_1+x_2+x_3 & 4x_1-2x_2+x_4 & 0 \end{bmatrix}.$$

Therefore, $\mathbf{x} \in \mathcal{R}(A) = V$ if and only if $\mathcal{R}(A) = \mathcal{R}(\bar{A})$. By comparing the row vectors of the row-echelon form of A with those of $\bar{A}$, it gives that $\mathbf{x} \in \mathcal{R}(A)$ if and only if the last row vector of the row-echelon form of $\bar{A}$ is the zero vector, that is, $\mathbf{x}$ is a solution of the homogeneous system of equations

$$\begin{cases} -x_1 + x_2 + x_3 = 0 \\ 4x_1 - 2x_2 + x_4 = 0. \end{cases}$$

We do the same calculation with $\bar{B}$, and obtain another homogeneous system of linear equations for $\mathbf{x}$:

$$\begin{cases} -9x_1 + 3x_2 + x_3 = 0 \\ 4x_1 - 2x_2 + x_4 = 0 \\ 2x_1 - x_2 + x_5 = 0. \end{cases}$$

Solving these two homogeneous systems together yields

$$V \cap W = \{t(1,\ 4,\ -3,\ 4,\ 2) : t \in \mathbb{R}\}.$$

Hence, $\{(1,\ 4,\ -3,\ 4,\ 2)\}$ is a basis for $V \cap W$ and $\dim(V \cap W) = 1$. □

Problem 3.24 Let V and W be the subspaces of the vector space $P_3(\mathbb{R})$ spanned by

$$\begin{cases} v_1(x) = 3 - x + 4x^2 + x^3, \\ v_2(x) = 5 + 5x^2 + x^3, \\ v_3(x) = 5 - 5x + 10x^2 + 3x^3, \end{cases}$$

and

$$\begin{cases} w_1(x) = 9 - 3x + 3x^2 + 2x^3, \\ w_2(x) = 5 - x + 2x^2 + x^3, \\ w_3(x) = 6 + 4x^2 + x^3, \end{cases}$$

respectively. Find the dimensions and bases for $V + W$ and $V \cap W$.

Problem 3.25 Let

$$\begin{aligned} V &= \left\{(x,\ y,\ z,\ u) \in \mathbb{R}^4 : y + z + u = 0\right\}, \\ W &= \left\{(x,\ y,\ z,\ u) \in \mathbb{R}^4 : x + y = 0,\ z = 2u\right\} \end{aligned}$$

be two subspaces of $\mathbb{R}^4$. Find bases for V, W, $V + W$, and $V \cap W$.

3.7 Invertibility

We now can have the following existence and uniqueness theorems for a solution of a system of linear equations $A\mathbf{x} = \mathbf{b}$ for an $m\times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^m$.

Theorem 3.23 (Existence) *Let A be an $m\times n$ matrix. Then the following statements are equivalent.*

(1) *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at least *one solution $\mathbf{x}$ in $\mathbb{R}^n$.*

(2) *The column vectors of A span $\mathbb{R}^m$, i.e., $\mathcal{C}(A) = \mathbb{R}^m$.*

(3) rank $A = m$, *and hence $m \leq n$.*

(4) *There exists an $n \times m$ right inverse B of A such that $AB = I_m$.*

Proof: **(1)** $\Leftrightarrow$ **(2)**: Note that $\mathcal{C}(A) \subseteq \mathbb{R}^m$ in general. For any $\mathbf{b} \in \mathbb{R}^m$, there is a solution $\mathbf{x} \in \mathbb{R}^n$ of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ is a linear combination of the column vectors of A. This is equivalent to saying that $\mathbb{R}^m = \mathcal{C}(A)$.

(2) $\Leftrightarrow$ **(3)**: Since $\dim \mathcal{C}(A) = \text{rank } A = \dim \mathcal{R}(A) \leq \min\{m,\ n\}$, $\mathcal{C}(A) = \mathbb{R}^m$ if and only if $\dim \mathcal{C}(A) = m \leq n$ (see Problem 3.10).

(1) $\Rightarrow$ **(4)** : Let $\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_m$ be the standard basis for $\mathbb{R}^m$. Then for each $i = 1,\ 2,\ \ldots,\ m$ we can find at least one solution $\mathbf{x}_i \in \mathbb{R}^n$ such that $A\mathbf{x}_i = \mathbf{e}_i$ by the condition. If B is the $n\times m$ matrix whose columns are these solutions, *i.e.*, $B = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m]$, then it follows by matrix multiplication that

$$AB = A\,[\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m] = [\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_m] = I_m.$$

Hence, the matrix B is a required right inverse.

(4) $\Rightarrow$ **(1)** : If B is a right inverse of A, then for any $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} = B\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. $\square$

Condition (2) means that A has m linearly independent column vectors, and condition (3) implies that there exist m linearly independent row vectors of A, since rank $A = m = \dim \mathcal{R}(A)$.

Note that if $\mathcal{C}(A) \subsetneq \mathbb{R}^m$, then $A\mathbf{x} = \mathbf{b}$ has no solution for $\mathbf{b} \notin \mathcal{C}(A)$.

Theorem 3.24 (Uniqueness) *Let A be an $m\times n$ matrix. Then the following statements are equivalent.*

(1) *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at most *one solution $\mathbf{x}$ in $\mathbb{R}^n$.*

(2) *The column vectors of A are linearly independent.*

(3) $\dim \mathcal{C}(A) = \text{rank } A = n$, *and hence* $n \le m$.

(4) $\mathcal{R}(A) = \mathbb{R}^n$.

(5) $\mathcal{N}(A) = \{\mathbf{0}\}$.

(6) *There exists an* $n \times m$ *left inverse* C *of* A *such that* $CA = I_n$.

Proof: **(1)** $\Rightarrow$ **(2)** : Note that the column vectors of A are linearly independent if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only a trivial solution. However, $A\mathbf{x} = \mathbf{0}$ has always a trivial solution $\mathbf{x} = \mathbf{0}$ and (1) means that it is the only one.

(2) $\Leftrightarrow$ **(3)** : Clear, because all the column vectors are linearly independent if and only if they form a basis for $\mathcal{C}(A)$, or $\dim \mathcal{C}(A) = n \le m$.

(3) $\Leftrightarrow$ **(4)**: Clear, because $\dim \mathcal{R}(A) = \text{rank } A = \dim \mathcal{C}(A) = n$ if and only if $\mathcal{R}(A) = \mathbb{R}^n$ (see Problem 3.10).

(4) $\Leftrightarrow$ **(5)**: Clear, since $\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n$.

(2) $\Rightarrow$ **(6)** : Suppose that the columns of A are linearly independent so that rank $A = n$. Extend these column vectors of A to a basis for $\mathbb{R}^m$ by adding $m - n$ more independent vectors to them. Construct an $m \times m$ matrix S with those vectors in columns. Then the matrix S has rank m and is hence invertible. Let C be the $n \times m$ matrix obtained from S^{-1} by throwing away the last $m - n$ rows. Since the first n columns of S constitute the matrix A, we have $CA = I_n$.

(6) $\Rightarrow$ **(1)** : Let C be a left inverse of A. If $A\mathbf{x} = \mathbf{b}$ has no solution, then we are done. Suppose that $A\mathbf{x} = \mathbf{b}$ has two solutions, say $\mathbf{x}_1$ and $\mathbf{x}_2$. Then

$$\mathbf{x}_1 = CA\mathbf{x}_1 = C\mathbf{b} = CA\mathbf{x}_2 = \mathbf{x}_2.$$

Hence, the system can have at most one solution. □

Remark: (1) We have proved that an $m \times n$ matrix A has a right inverse if and only if rank $A = m$, and A has a left inverse if and only if rank $A = n$. In the first case $A\mathbf{x} = \mathbf{b}$ always has a solution, and in the second case the solution (if it exists) is unique. Therefore, if $m \ne n$, A cannot have both left and right inverses.

(2) For a practical way of finding a right or a left inverse of an $m \times n$ matrix A, we will show later (see Corollary 5.24) that if rank $A = m$, then $(AA^T)^{-1}$ exists and $A^T(AA^T)^{-1}$ is a right inverse of A, and if rank $A = n$, then $(A^TA)^{-1}$ exists and $(A^TA)^{-1}A^T$ is a left inverse of A.

(3) Note that if $m = n$ so that A is a square matrix, then A has a right inverse (and a left inverse) if and only if rank $A = m = n$. Moreover, in this case the inverses are the same (see Theorem 1.8). Therefore, a square matrix A has rank n if and only if A is invertible. This means that for a square matrix "Existence = Uniqueness", and the ten conditions in the above two theorems are all equivalent. In particular, for the invertibility of a square matrix it is enough to show the existence of a one-side inverse.

Problem 3.26 For each of the following matrices, discuss the number of possible solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$:

$$(1)\ A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 2 & 3 \\ 3 & -7 \\ -6 & 1 \end{bmatrix},$$

$$(3)\ A = \begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}, \qquad (4)\ A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 1 & 2 & -2 \end{bmatrix}.$$

The following theorem is a collection of the results proved in Theorems 1.8, 3.23, 3.24, and the Remark before Definition 4.3.

Theorem 3.25 *For a square matrix A of order n, the following statements are equivalent.*

(1) *A is invertible.*

(2) $\det A \neq 0$.

(3) *A is row equivalent to I_n.*

(4) *A is a product of elementary matrices.*

(5) *Elimination can be completed: $PA = LDU$, with all $d_i \neq 0$.*

(6) *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$.*

(7) *$A\mathbf{x} = \mathbf{0}$ has only a trivial solution, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$.*

(8) *The columns of A are linearly independent.*

(9) *The columns of A span $\mathbb{R}^n$, i.e., $\mathcal{C}(A) = \mathbb{R}^n$.*

(10) *A has a left inverse.*

(11) rank $A = n$.

(12) *The rows of A are linearly independent.*

(13) *The rows of A span $\mathbb{R}^n$,* i.e., *$\mathcal{R}(A) = \mathbb{R}^n$.*

(14) *A has a right inverse.*

(15)* *The linear transformation* $A : \mathbb{R}^n \to \mathbb{R}^n$ *via* $A(\mathbf{x}) = A\mathbf{x}$ *is injective.*

(16)* *The linear transformation* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is surjective.*

(17)* *Zero is not an eigenvalue of* A.

Proof: Exercise: where have we proved which claim? Prove any not covered. The numbers with asterisks will be explained in the following places: (15) and (16) in the Remark on page 141 and (17) in Theorem 6.1. □

3.8 Application: Interpolation

In many scientific experiments, a scientist wants to find the precise functional relationship between input data and output data. That is, in his experiment, he puts various input values into his experimental device and obtains output values corresponding to those input values. After his experiment, what he has is a table of inputs and outputs. The precise functional relationship might be very complicated, and sometimes it might be very hard or almost impossible to find the precise function. In this case, one thing he can do is to find a polynomial whose graph passes through each of the data points and comes very close to the function he wanted to find. That is, he is looking for a polynomial that approximates the precise function. Such a polynomial is called an **interpolating polynomial**. This problem is closely related to systems of linear equations.

Let us begin with a set of given data: Suppose that for $n+1$ distinct experimental input values $x_0, x_1, \ldots, x_n$, we obtained $n+1$ output values $y_0 = f(x_0)$, $y_1 = f(x_1)$, ..., $y_n = f(x_n)$. The output values are supposed to be related to the inputs by a certain function f. We wish to construct a polynomial $p(x)$ of degree less than or equal to n which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$: *i.e.*, $p(x_i) = y_i = f(x_i)$ for $i = 0, 1, \ldots, n$.

Note that if there is such a polynomial, it must be unique. Indeed, if $q(x)$ is another such polynomial, then $h(x) = p(x) - q(x)$ is also a polynomial of degree less than or equal to n vanishing at $n+1$ distinct points $x_0, x_1, \ldots, x_n$. Hence $h(x)$ must be the identically zero polynomial so that $p(x) = q(x)$ for all $x \in \mathbb{R}$.

In fact, the unique polynomial $p(x)$ can be found by solving a system of linear equations: If we write $p(x) = a_0 + a_1x + \cdots + a_nx^n$, then we are supposed to determine the coefficients a_i's. The set of equations

$$p(x_i) = a_0 + a_1 x_i + \cdots + a_n x_i^n = y_i = f(x_i),$$

for $i = 0, 1, \ldots, n$, constitutes a system of $n+1$ linear equations in $n+1$ unknowns a_i's:

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

The coefficient matrix A is a square matrix of order $n+1$, known as **Vandermonde's matrix** (see Problem 2.10), whose determinant is

$$\det A = \prod_{0 \le i < j \le n} (x_j - x_i).$$

Since the x_i's are all distinct, $\det A \neq 0$. It follows that A is nonsingular, and hence $A\mathbf{x} = \mathbf{b}$ always has a unique solution, which determines the unique polynomial $p(x)$ of degree $\le n$ passing through the given $n+1$ points (x_0, y_0), (x_1, y_1), $\cdots$, (x_n, y_n) in the plane $\mathbb{R}^2$.

Example 3.24 Given four points

$$(0,\ 3),\ (1,\ 0),\ (-1,\ 2),\ (3,\ 6)$$

in the plane $\mathbb{R}^2$, let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be the polynomial passing through the given four points. Then, we have a system of equations

$$\begin{cases} a_0 & & & & & & & = & 3 \\ a_0 & + & a_1 & + & a_2 & + & a_3 & = & 0 \\ a_0 & - & a_1 & + & a_2 & - & a_3 & = & 2 \\ a_0 & + & 3a_1 & + & 9a_2 & + & 27a_3 & = & 6. \end{cases}$$

Solving this system, we find that $a_0 = 3$, $a_1 = -2$, $a_2 = -2$, $a_3 = 1$ is the unique solution, and the unique polynomial is $p(x) = 3 - 2x - 2x^2 + x^3$. □

Problem 3.27 Let $f(x) = \sin x$. Then at $x = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \pi$, the values of f are $y = 0, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 0$. Find the polynomial $p(x)$ of degree ≤ 4 that passes through these five points. (One may need to use a computer due to messy computation).

Problem 3.28 Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1$, $p'(0) = 2$, $p(1) = 4$, $p'(1) = 4$.

Problem 3.29 Find the equation of a circle that passes through the three points $(2, -2)$, $(3, 5)$, and $(-4, 6)$ in the plane $\mathbb{R}^2$.

Remark: (1) It is suggested that the readers think about the differences between this interpolation and the Taylor polynomial approximation to a differentiable function.

(2) Note again that the interpolating polynomial $p(x)$ of degree $\le n$ is uniquely determined when we have the correct data, *i.e.*, when we are given precisely $n+1$ values of y at precisely $n+1$ distinct points $x_0, x_1, \ldots, x_n$.

However, if we are given fewer data, then the polynomial is under-determined: *i.e.*, if we have m values of y with $m < n+1$ at m distinct points $x_1, x_2, \ldots, x_m$, then there are as many interpolating polynomials as the null space of A since in this case A is an $m \times (n+1)$ matrix with $m < n+1$.

On the other hand, if we are given more than $n+1$ data, then the polynomial is over-determined: *i.e.*, if we have m values of y with $m > n+1$ at m distinct points $x_1, x_2, \ldots, x_m$, then there need not be any interpolating polynomial since the system could be inconsistent. In this case, the best we can do is to find a polynomial of degree $\le n$ to which the data is closest. We will review this statement again in Section 5.8.

3.9 Application: The Wronskian

Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ be n vectors in an m-dimensional vector space V. To check the independence of the vectors $\mathbf{y}_i$'s, consider its linear dependence:

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{0}.$$

Let $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a basis for V. By expressing each $\mathbf{y}_i$ as a linear combination of the basis vectors $\mathbf{x}_i$'s, the linear dependence of $\mathbf{y}_i$'s can be written as a linear combination of the basis vectors $\mathbf{x}_i$'s, so that all of the coefficients (which are also linear combinations of c_i's) must be zero. It gives a homogeneous system of linear equations in c_i's, say $A\mathbf{c} = \mathbf{0}$ with an $m \times n$ matrix A, as in the proof of Lemma 3.8. Recall that the vectors $\mathbf{y}_i$'s are linearly independent if and only if the system $A\mathbf{c} = \mathbf{0}$ has only a trivial solution. Hence, the linear independence of a set of vectors in a finite dimensional vector space can be tested by solving a homogeneous system of linear equations. But, if V is not finite dimensional, this test for the linear independence of a set of vectors cannot be applied.

In this section, we introduce a test for the linear independence of a set of functions. For our purpose, let V be the vector space of all functions on $\mathbb{R}$ which are differentiable infinitely many times. Then one can easily see that V is not finite dimensional.

Let $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$ be n functions in V. The n functions are linearly independent in V if the linear equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in \mathbb{R}$ implies that all $c_i = 0$. By taking the differentiation $n-1$ times, we obtain n equations:

$$c_1 f_1^{(i)}(x) + c_2 f_2^{(i)}(x) + \cdots + c_n f_n^{(i)}(x) = 0, \quad 0 \le i \le n-1,$$

for all $x \in \mathbb{R}$. Or, in a matrix form:

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix is called the **Wronskian** for $\{f_1(x), f_2(x), \cdots, f_n(x)\}$ and denoted by $W(x)$. Therefore, if there is a point $x_0 \in \mathbb{R}$ such that $W(x) \ne 0$, then the coefficient matrix is nonsingular at $x = x_0$, and so all $c_i = 0$. Therefore, if the Wronskian is nonzero at a point in $\mathbb{R}$, then $\{f_1(x), f_2(x), \cdots, f_n(x)\}$ are linearly independent.

Example 3.25 For the sets of functions $F_1 = \{x, \cos x, \sin x\}$ and $F_2 = \{x, e^x, e^{-x}\}$, the Wronskians are

$$W_1(x) = \det \begin{bmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} = x$$

and

$$W_2(x) = \det \begin{bmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{bmatrix} = 2x.$$

Since $W_i(x) \ne 0$ for $x \ne 0$, both F_i are linearly independent. □

Problem 3.30 Show that $1, x, x^2, \cdots, x^n$ are linearly independent in the vector space $C(\mathbb{R})$ of continuous functions.

3.10 Exercises

3.1. Let V be the set of all pairs $(x,\ y)$ of real numbers. Define

$$\begin{aligned}(x,\ y)+(x_1,\ y_1) &= (x+x_1,\ y+y_1)\\ c(x,\ y) &= (cx,\ y).\end{aligned}$$

Is V a vector space with these operations?

3.2. For $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ and $k \in \mathbb{R}$, define two operations as

$$\mathbf{x} \oplus \mathbf{y} = \mathbf{x} - \mathbf{y}, \qquad k \cdot \mathbf{x} = -k\mathbf{x}.$$

The operations on the right sides are the usual ones. Which of the rules in the definition of a vector space are satisfied for $(\mathbb{R}^n,\ \oplus,\ \cdot)$?

3.3. Determine whether the given set is a vector space with the usual addition and scalar multiplication of functions.

(1) The set of all functions f defined on the interval $[-1,\ 1]$ such that $f(0) = 0$.
(2) The set of all functions f defined on $\mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$.
(3) The set of all twice differentiable functions f defined on $\mathbb{R}$ such that $f''(x) + f(x) = 0$.

3.4. Let $C^2[-1,\ 1]$ be the vector space of all functions with continuous second derivatives on the domain $[-1,\ 1]$. Which of the following subsets is a subspace of $C^2[-1,\ 1]$?

(1) $W = \{f(x) \in C^2[-1,\ 1] : f''(x) + f(x) = 0,\ -1 \le x \le 1\}$.
(2) $W = \{f(x) \in C^2[-1,\ 1] : f''(x) + f(x) = x^2,\ -1 \le x \le 1\}$.

3.5. Which of the following subsets of $C[-1,\ 1]$ is a subspace of the vector space $C[-1,\ 1]$ of continuous functions on $[-1,\ 1]$?

(1) $W = \{f(x) \in C[-1,\ 1] : f(-1) = -f(1)\}$.
(2) $W = \{f(x) \in C[-1,\ 1] : f(x) \ge 0$ for all x in $[-1,\ 1]\}$.
(3) $W = \{f(x) \in C[-1,\ 1] : f(-1) = -2$ and $f(1) = 2\}$.
(4) $W = \{f(x) \in C[-1,\ 1] : f(\frac{1}{2}) = 0\}$.

3.6. Does the vector $(3,\ -1,\ 0,\ -1)$ belong to the subspace of $\mathbb{R}^4$ spanned by the vectors $(2,\ -1,\ 3,\ 2)$, $(-1,\ 1,\ 1,\ -3)$ and $(1,\ 1,\ 9,\ -5)$?

3.7. Express the given function as a linear combination of functions in the given set Q.

(1) $p(x) = -1 - 3x + 3x^2$ and $Q = \{p_1(x),\ p_2(x),\ p_3(x)\}$, where $p_1(x) = 1 + 2x + x^2$, $p_2(x) = 2 + 5x$, $p_3(x) = 3 + 8x - 2x^2$.
(2) $p(x) = -2 - 4x + x^2$ and $Q = \{p_1(x),\ p_2(x),\ p_3(x),\ p_4(x)\}$, where $p_1(x) = 1 + 2x^2 + x^3$, $p_2(x) = 1 + x + 2x^3$, $p_3(x) = -1 - 3x - 4x^3$, $p_4(x) = 1 + 2x - x^2 + x^3$.

3.8. Is $\{\cos^2 x,\ \sin^2 x,\ 1,\ e^x\}$ linearly independent in the vector space $C(\mathbb{R})$?

3.9. Show that the given sets of functions are linearly independent in the vector space $C[-\pi,\ \pi]$.
(1) $\{1,\ x,\ x^2,\ x^3,\ x^4\}$
(2) $\{1,\ e^x,\ e^{2x},\ e^{3x}\}$
(3) $\{1,\ \sin x,\ \cos x,\ \ldots,\ \sin kx,\ \cos kx\}$

3.10. Are the vectors

$$\begin{array}{ll} \mathbf{v}_1 = (1,\ 1,\ 2,\ 4), & \mathbf{v}_2 = (2,\ -1,\ -5,\ 2), \\ \mathbf{v}_3 = (1,\ -1,\ -4,\ 0), & \mathbf{v}_4 = (2,\ 1,\ 1,\ 6) \end{array}$$

linearly independent in the 4-space $\mathbb{R}^4$?

3.11. In the 3-space $\mathbb{R}^3$, let W be the set of all vectors $(x_1,\ x_2,\ x_3)$ that satisfy the equation $x_1 - x_2 - x_3 = 0$. Prove that W is a subspace of $\mathbb{R}^3$. Find a basis for the subspace W.

3.12. With respect to the basis $\alpha = \{1,\ x,\ x^2\}$ for the vector space $P_2(\mathbb{R})$, find the coordinate vector of the following polynomials:

$$(1)\ f(x) = x^2 - x + 1, \quad (2)\ f(x) = x^2 + 4x - 1, \quad (3)\ f(x) = 2x + 5.$$

3.13. Let W be the subspace of $C[-\pi,\ \pi]$ consisting of functions of the form $f(x) = a\sin x + b\cos x$. Determine the dimension of W.

3.14. Let V denote the set of all infinite sequences of real numbers:

$$V = \{\mathbf{x} : \mathbf{x} = \{x_i\}_{i=1}^{\infty}, x_i \in \mathbb{R}\}.$$

If $\mathbf{x} = \{x_i\}$ and $\mathbf{y} = \{y_i\}$ are in V, then $\mathbf{x} + \mathbf{y}$ is the sequence $\{x_i + y_i\}_{i=1}^{\infty}$. If c is a real number, then $c\mathbf{x}$ is the sequence $\{cx_i\}_{i=1}^{\infty}$.
(1) Prove that V is a vector space.
(2) Prove that V is not finite dimensional.

3.15. For two matrices A and B for which AB can be defined, prove the following statements:
(1) If both A and B have linearly independent column vectors, then the column vectors of AB are also linearly independent.
(2) If both A and B have linearly independent row vectors, then the row vectors of AB are also linearly independent.
(3) If the column vectors of B are linearly dependent, then the column vectors of AB are also linearly dependent.
(4) If the row vectors of A are linearly dependent, then the row vectors of AB are also linearly dependent.

3.16. Let $U = \{(x,\ y,\ z)\ :\ 2x + 3y + z = 0\}$ and $V = \{(x,\ y,\ z)\ :\ x + 2y - z = 0\}$ be subspaces of $\mathbb{R}^3$.

(1) Find a basis for $U \cap V$.
(2) Determine the dimension of $U + V$.
(3) Describe U, V, $U \cap V$ and $U + V$ geometrically.

3.17. How many 5×5 permutation matrices are there? Are they linearly independent? Do they span the vector space $M_{5\times5}(\mathbb{R})$?

3.18. Find bases for the row space, the column space, and the null space for each of the following matrices.

$$(1)\ A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}, \quad (2)\ B = \begin{bmatrix} 0 & 2 & 1 & -5 \\ 1 & 1 & -2 & 2 \\ 1 & 5 & 0 & 0 \end{bmatrix},$$

$$(3)\ C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{bmatrix}, \quad (4)\ D = \begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{bmatrix}.$$

3.19. Find the rank of A as a function of x: $A = \begin{bmatrix} 2 & 2 & -6 & 8 \\ 3 & 3 & -9 & 8 \\ 1 & 1 & x & 4 \end{bmatrix}$.

3.20. Find the rank and the largest invertible submatrix of each of the following matrices.

$$(1) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2) \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 4 \end{bmatrix}, \quad (3) \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 4 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

3.21. For any nonzero column vectors $\mathbf{u}$, $\mathbf{v}$, show that the matrix $A = \mathbf{u}\mathbf{v}^T$ has rank 1. Conversely, every matrix of rank 1 can be written as $\mathbf{u}\mathbf{v}^T$ for some $\mathbf{u}$, $\mathbf{v}$.

3.22. Determine whether the following statements are true or false, and justify your answers.

(1) The set of all $n \times n$ matrices A such that $A^T = A^{-1}$ is a subspace of the vector space $M_{n\times n}(\mathbb{R})$.
(2) If α and β are linearly independent subsets of a vector space V, then so is their union $\alpha \cup \beta$.
(3) If U and W are subspaces of a vector space V with bases α and β respectively, then the intersection $\alpha \cap \beta$ is a basis for $U \cap W$.
(4) Let U be the row-echelon form of a square matrix A. If the first r columns of U are linearly independent, then so are the first r columns of A.
(5) Any two row-equivalent matrices have the same column space.

(6) Let A be an $m \times n$ matrix with rank m. Then the column vectors of A span $\mathbb{R}^m$.

(7) Let A be an $m \times n$ matrix with rank n. Then $A\mathbf{x} = \mathbf{b}$ has at most one solution.

(8) If U is a subspace of V and $\mathbf{x}$, $\mathbf{y}$ are vectors in V such that $\mathbf{x} + \mathbf{y}$ is contained in U, then $\mathbf{x} \in U$ and $\mathbf{y} \in U$.

(9) Let U and V are vector spaces. Then U is a subspace of V if and only if $\dim U \leq \dim V$.

(10) For any $m \times n$ matrix A, $\dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) = m$.

Chapter 4

Linear Transformations

4.1 Introduction

As we saw in Chapter 3, there are many vector spaces. Naturally, one can ask whether or not two vector spaces are the same. To say two vector spaces are the same or not, one has to compare them first as sets, and then see whether or not their arithmetical rules are preserved. A usual way of comparing two sets is defining a **function** between them. Recall that a function from a set X into another set Y is a rule which assigns a unique element y in Y to each element x in X. Such a function is denoted as $f : X \to Y$ and sometimes referred to as a **transformation** or a **mapping**. We say that f transforms (or maps) X into Y. When given sets are vector spaces, one can compare their arithmetical rules also by a transformation f if f preserves the arithmetical rules, that is, $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and $f(k\mathbf{x}) = kf(\mathbf{x})$ for any vectors $\mathbf{x}, \mathbf{y}$ and any scalar k. In this chapter, we discuss this kind of transformations between vector spaces via the linear equation $A\mathbf{x} = \mathbf{b}$.

For an $m \times n$ matrix A, the equation $A\mathbf{x} = \mathbf{b}$ means that to every vector $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]^T$ in $\mathbb{R}^n$ the matrix multiplication $A\mathbf{x}$ assigns a vector $\mathbf{b}$ $(= A\mathbf{x})$ in $\mathbb{R}^m$. That is, the matrix A transforms every vector $\mathbf{x}$ in $\mathbb{R}^n$ into a vector $\mathbf{b}$ in $\mathbb{R}^m$ by the matrix multiplication $A\mathbf{x} = \mathbf{b}$. Moreover, the distributive law $A(\mathbf{x} + k\mathbf{y}) = A\mathbf{x} + kA\mathbf{y}$, for $k \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, of matrix multiplication means that A preserves the sum of vectors and scalar multiplication.

Definition 4.1 Let V and W be vector spaces. A function $T : V \to W$ is called a **linear transformation** from V to W if for all $\mathbf{x}$, $\mathbf{y} \in V$ and scalar k the following conditions hold:

(1) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$,
(2) $T(k\mathbf{x}) = kT(\mathbf{x})$.

We often call T simply **linear**. It is not hard to see that the two conditions for a linear transformation can be combined into a single requirement

$$T(\mathbf{x} + k\mathbf{y}) = T(\mathbf{x}) + kT(\mathbf{y}).$$

Geometrically, this is just the requirement for a straight line to be transformed into a straight line, since $\mathbf{x} + k\mathbf{y}$ represents a straight line through $\mathbf{x}$ in the direction $\mathbf{y}$ in V, and its image $T(\mathbf{x})+kT(\mathbf{y})$ also represents a straight line through $T(\mathbf{x})$ in the direction of $T(\mathbf{y})$ in W. The following theorem is a direct consequence of the definition, and the proof is left for an exercise.

Theorem 4.1 *Let $T : V \to W$ be a linear transformation. Then*
(1) $T(\mathbf{0}) = \mathbf{0}$.
(2) *For any* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in V$ *and scalars* $k_1, k_2, \ldots, k_n$,
$$T(k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n) = k_1T(\mathbf{x}_1) + k_2T(\mathbf{x}_2) + \cdots + k_nT(\mathbf{x}_n).$$

Example 4.1 Consider the following functions:
(1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$;
(2) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 - x$;
(3) $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x, y) = (x - y, 2x)$;
(4) $k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $k(x, y) = (xy, x^2 + 1)$.
One can easily see that g and k are not linear, while f and h are linear.

Example 4.2 **(1)** For an $m \times n$ matrix A, the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by the matrix multiplication

$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation by the distributive law $A(\mathbf{x}+k\mathbf{y}) = A\mathbf{x}+kA\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and for any scalar $k \in \mathbb{R}$. Therefore, a matrix A, identified with T, may be considered to be a linear transformation of $\mathbb{R}^n$ to $\mathbb{R}^m$.

(2) For a vector space V, the **identity transformation** $Id : V \to V$ is defined by $Id(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$. If W is another vector space, the **zero transformation** $T_0 : V \to W$ is defined by $T_0(\mathbf{x}) = \mathbf{0}$ (the zero vector) for all $\mathbf{x} \in V$. Clearly, both transformations are linear. □

Nontrivial important examples of linear transformations are the rotations, reflections, and projections in geometry defined in the following example.

Example 4.3 **(1)** Let θ denote the angle between the x-axis and a fixed vector in $\mathbb{R}^2$. Then the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

defines a linear transformation on $\mathbb{R}^2$ that rotates any vector in $\mathbb{R}^2$ through the angle θ about the origin, and is called a **rotation** by the angle θ.

(2) The **projection** on the x-axis is the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by, for $\mathbf{x} = (x, y) \in \mathbb{R}^2$,

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

(3) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by, for $\mathbf{x} = (x, y)$,

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

is called the **reflection** about the x-axis. □

Problem 4.1 Find the matrix of reflection about the line $y = x$ in the plane $\mathbb{R}^2$.

Example 4.4 The transformation $\mathrm{tr} : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined as the sum of diagonal entries

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii},$$

for $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$, is called the **trace**. It is easy to show that

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B) \quad \text{and} \quad \mathrm{tr}(kA) = k\ \mathrm{tr}(A)$$

for any matrices A and B in $M_{n\times n}(\mathbb{R})$, which means that "tr" is a linear transformation. In particular, one can easily show that the set of all $n\times n$ matrices with trace 0 is a subspace of $M_{n\times n}(\mathbb{R})$. □

Problem 4.2 Let $W = \{A \in M_{n\times n}(\mathbb{R}) : \mathrm{tr}(A) = 0\}$. Show that W is a subspace, and then find a basis for W.

Problem 4.3 Show that, for any matrices A and B in $M_{n\times n}(\mathbb{R})$, tr $(AB) =$ tr (BA).

Example 4.5 From the calculus, it is well known that two transformations

$$D : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R}), \quad \mathcal{I} : P_{n-1}(\mathbb{R}) \to P_n(\mathbb{R})$$

defined by differentiation and integration,

$$D(f)(x) = f'(x), \quad \mathcal{I}(f)(x) = \int_0^x f(t)dt,$$

satisfy linearity, and so they are linear transformations. Many problems related with differential and integral equations may be reformulated in terms of linear transformations. □

Definition 4.2 Let V and W be two vector spaces, and let $T : V \to W$ be a linear transformation from V into W.

(1) $\text{Ker}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\} \subseteq V$ is called the **kernel** of T.

(2) $\text{Im}(T) = \{T(\mathbf{v}) \in W : \mathbf{v} \in V\} = T(V) \subseteq W$ is called the **image** of T.

Example 4.6 Let V and W be vector spaces and let $Id : V \to V$ and $T_0 : V \to W$ be the identity and the zero transformations, respectively. Then it is easy to see that $\text{Ker}(Id) = \{\mathbf{0}\}$, $\text{Im}(Id) = V$, $\text{Ker}(T_0) = V$, and $\text{Im}(T_0) = \{\mathbf{0}\}$. □

Theorem 4.2 *Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Then the kernel* $\text{Ker}(T)$ *and the image* $\text{Im}(T)$ *are subspaces of V and W, respectively.*

Proof: Since $T(\mathbf{0}) = \mathbf{0}$, each of $\text{Ker}(T)$ and $\text{Im}(T)$ is nonempty having $\mathbf{0}$. multiplication.

(1) For any $\mathbf{x}$, $\mathbf{y} \in \text{Ker}(T)$ and for any scalar k,

$$T(\mathbf{x} + k\mathbf{y}) = T(\mathbf{x}) + kT(\mathbf{y}) = \mathbf{0} + k\mathbf{0} = \mathbf{0}.$$

Hence $\mathbf{x} + k\mathbf{y} \in \text{Ker}(T)$ so that $\text{Ker}(T)$ is a subspace of V.

(2) If $\mathbf{v}$, $\mathbf{w} \in \text{Im}(T)$, then there exist $\mathbf{x}$ and $\mathbf{y}$ in V such that $T(\mathbf{x}) = \mathbf{v}$ and $T(\mathbf{y}) = \mathbf{w}$. Thus, for any scalar k,

$$\mathbf{v} + k\mathbf{w} = T(\mathbf{x}) + kT(\mathbf{y}) = T(\mathbf{x} + k\mathbf{y}).$$

Thus $\mathbf{v} + k\mathbf{w} \in \text{Im}(T)$, so that $\text{Im}(T)$ is a subspace of W. □

Example 4.7 Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by an $m \times n$ matrix A as in Example 4.2 (1). The kernel $\mathrm{Ker}(A)$ of A consists of all solution vectors $\mathbf{x}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Therefore, the kernel $\mathrm{Ker}(A)$ of A is nothing but the null space $\mathcal{N}(A)$ of the matrix A, and the image $\mathrm{Im}(A)$ of A is just the column space $\mathcal{C}(A) = \mathrm{Im}(A) = A(\mathbb{R}^n) \subseteq \mathbb{R}^m$ of the matrix A. Recall that $A\mathbf{x}$ is a linear combination of the column vectors of A. $\square$

One of the most important properties of linear transformations is that they are completely determined by their values on a basis.

Theorem 4.3 *Let V and W be vector spaces. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V and let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be any vectors (possibly repeated) in W. Then there exists a unique linear transformation $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \ldots, n$.*

Proof: Let $\mathbf{x} \in V$. Then it has a unique expression: $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{v}_i$ for some scalars $a_1, \ldots, a_n$. Define

$$T : V \to W \quad \text{by} \quad T(\mathbf{x}) = \sum_{i=1}^n a_i\mathbf{w}_i.$$

In particular, $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$.

Linearity: For $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{v}_i$, $\mathbf{y} = \sum_{i=1}^n b_i\mathbf{v}_i \in V$ and k a scalar, we have $\mathbf{x} + k\mathbf{y} = \sum_{i=1}^n (a_i + kb_i)\mathbf{v}_i$. Then

$$T(\mathbf{x} + k\mathbf{y}) = \sum_{i=1}^n (a_i + kb_i)\mathbf{w}_i = \sum_{i=1}^n a_i\mathbf{w}_i + k\sum_{i=1}^n b_i\mathbf{w}_i = T(\mathbf{x}) + kT(\mathbf{y}).$$

Uniqueness: Suppose that $S : V \to W$ is linear and $S(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \ldots, n$. Then for any $\mathbf{x} \in V$ with $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{v}_i$, we have

$$S(\mathbf{x}) = \sum_{i=1}^n a_i S(\mathbf{v}_i) = \sum_{i=1}^n a_i\mathbf{w}_i = T(\mathbf{x}).$$

Hence, we have $S = T$. $\square$

Therefore, from an assignment $T(\mathbf{v}_i) = \mathbf{w}_i$ of an arbitrary vector in $W$to each vector $\mathbf{v}_i$ in a basis for V, one can extend it uniquely to a linear transformation T from a vector space V into W. The uniqueness in the above theorem may be rephrased as the following corollary.

Corollary 4.4 *Let V and W be vector spaces, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. If S, $T : V \to W$ are linear transformations and $S(\mathbf{v}_i) = T(\mathbf{v}_i)$ for $i = 1, \ldots, n$, then $S = T$, i.e., $S(\mathbf{x}) = T(\mathbf{x})$ for all $\mathbf{x} \in V$.*

Example 4.8 Let $\mathbf{w}_1 = (1, 0)$, $\mathbf{w}_2 = (2, -1)$, $\mathbf{w}_3 = (4, 3)$ be three vectors in $\mathbb{R}^2$.

(1) Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for the 3-space $\mathbb{R}^3$, and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(\mathbf{e}_1) = \mathbf{w}_1, \qquad T(\mathbf{e}_2) = \mathbf{w}_2, \qquad T(\mathbf{e}_3) = \mathbf{w}_3.$$

Find a formula for $T(x_1, x_2, x_3)$, and then use it to compute $T(2, -3, 5)$.

(2) Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be another basis for $\mathbb{R}^3$, where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (1, 0, 0)$, and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(\mathbf{v}_1) = \mathbf{w}_1, \qquad T(\mathbf{v}_2) = \mathbf{w}_2, \qquad T(\mathbf{v}_3) = \mathbf{w}_3.$$

Find a formula for $T(x_1, x_2, x_3)$, and then use it to compute $T(2, -3, 5)$.

Solution: **(1)** For $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \mathbb{R}^3$,

$$\begin{aligned} T(\mathbf{x}) &= \sum_{i=1}^{3} x_i T(\mathbf{e}_i) = \sum_{i=1}^{3} x_i \mathbf{w}_i \\ &= x_1(1, 0) + x_2(2, -1) + x_3(4, 3) \\ &= (x_1 + 2x_2 + 4x_3, -x_2 + 3x_3). \end{aligned}$$

Thus, $T(2, -3, 5) = (16, 18)$. In matrix notation, this can be written as

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ -x_2 + 3x_3 \end{bmatrix}.$$

(2) In this case, we need to express $\mathbf{x} = (x_1, x_2, x_3)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, *i.e.*,

$$\begin{aligned} (x_1, x_2, x_3) = \sum_{i=1}^{3} k_i \mathbf{v}_i &= k_1(1, 1, 1) + k_2(1, 1, 0) + k_3(1, 0, 0) \\ &= (k_1 + k_2 + k_3)\mathbf{e}_1 + (k_1 + k_2)\mathbf{e}_2 + k_1\mathbf{e}_3. \end{aligned}$$

By equating corresponding components we obtain a system of equations

$$\begin{cases} k_1 + k_2 + k_3 = x_1 \\ k_1 + k_2 = x_2 \\ k_1 = x_3 . \end{cases}$$

The solution is $k_1 = x_3$, $k_2 = x_2 - x_3$, $k_3 = x_1 - x_2$. Therefore,

$$\begin{aligned} (x_1, x_2, x_3) &= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3, \quad \text{and} \\ T(x_1, x_2, x_3) &= x_3T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\ &= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \\ &= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3). \end{aligned}$$

From this formula we obtain $T(2, -3, 5) = (9, 23)$. In matrix notation, it can be written as

$$\begin{bmatrix} 4 & -2 & -1 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 - 2x_2 - x_3 \\ 3x_1 - 4x_2 + x_3 \end{bmatrix}. \qquad \square$$

Problem 4.4 Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(3, 1, 0) = (1, 1)$ and $T(-6, -2, 0) = (2, 1)$? If yes, can you find an expression of $T(\mathbf{x})$ for $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$?

Problem 4.5 Let V and W be vector spaces and $T : V \to W$ be linear. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be a linearly independent subset of the image $\mathrm{Im}(T) \subseteq W$. Suppose that $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is chosen so that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \ldots, k$. Prove that α is linearly independent.

4.2 Invertible linear transformations

Note that a function f from a set X to a set Y is said to be **invertible** if there is a function g, which is called the **inverse** function of f and denoted by $g = f^{-1}$, from Y to X such that their compositions satisfy $g \circ f = Id$ and $f \circ g = Id$. One can notice that if there exists an invertible function from a set X into another set Y, then it gives a one-to-one correspondence between these two sets so that they can be identified as sets. A useful criterion for a function between two given sets to be invertible is that it is one-to-one and onto. Recall that a function $f : X \to Y$ is said to be **one-to-one** (or

injective) if $f(u) = f(v)$ in Y implies $u = v$ in X, and said to be **onto** (or **surjective**) if for each element y in Y there is an element x in X such that $f(x) = y$. A function is said to be **bijective** if it is both one-to-one and onto, that is, if for each element y in Y there is a *unique* element x in X such that $f(x) = y$.

Lemma 4.5 *A function $f : X \to Y$ is invertible if and only if it is bijective (or one-to-one and onto).*

Proof: Suppose $f : X \to Y$ is invertible, and let $g : Y \to X$ be its inverse. If $f(u) = f(v)$, then $u = g(f(u)) = g(f(v)) = v$. Thus f is one-to-one. For each $y \in Y$, $g(y) = x \in X$. Then $f(x) = f(g(y)) = y$. Thus it is onto.

Conversely, suppose f is bijective. Then, for each $y \in Y$, there is unique $x \in X$ such that $f(x) = y$. Now for each $y \in Y$ define $g(y) = x$. Then one can easily check that $g : Y \to X$ is a well-defined function such that $f \circ g = Id$ and $g \circ f = Id$, *i.e.*, g is the inverse of f. □

The following lemma shows that if a given function is an *invertible linear* transformation from a *vector space* into another, then the linearity is also preserved by the inversion.

Lemma 4.6 *Let V and W be vector spaces. If $T : V \to W$ is an invertible linear transformation, then its inverse $T^{-1} : W \to V$ is also linear.*

Proof: Let $\mathbf{w}_1$, $\mathbf{w}_2 \in W$, and let k be any scalar. Since T is invertible, it is one-to-one and onto, so there exist unique vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Then

$$\begin{aligned}
T^{-1}(\mathbf{w}_1 + k\mathbf{w}_2) &= T^{-1}(T(\mathbf{v}_1) + kT(\mathbf{v}_2)) \\
&= T^{-1}(T(\mathbf{v}_1 + k\mathbf{v}_2)) \\
&= \mathbf{v}_1 + k\mathbf{v}_2 \\
&= T^{-1}(\mathbf{w}_1) + kT^{-1}(\mathbf{w}_2).
\end{aligned}$$ □

Definition 4.3 A linear transformation $T : V \to W$ from a vector space V to a vector space W is called an **isomorphism** if it is invertible (or one-to-one and onto). In this case, we say V and W are **isomorphic** to each other.

Lemma 4.6 shows that if T is an isomorphism, then its inverse T^{-1} is also an isomorphism with $(T^{-1})^{-1} = T$. Therefore, if V and W are isomorphic to each other, then it means that they look the same as vector spaces.

If $T : V \to W$ and $S : W \to Z$ are linear transformations, then it is quite easy to show that their composition $(S \circ T)(\mathbf{v}) = S(T(\mathbf{v}))$ is also a linear transformation from V to Z. In particular, if two linear transformations are given by matrices $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^m \to \mathbb{R}^k$, then their composition is nothing but the matrix multiplication BA of them, *i.e.*, $(B \circ A)(\mathbf{x}) = B(A\mathbf{x}) = (BA)\mathbf{x}$. Hence, if a linear transformation is given by an invertible $n \times n$ square matrix $A : \mathbb{R}^n \to \mathbb{R}^n$, then the inverse matrix A^{-1} plays the inverse linear transformation, so that it is an isomorphism of $\mathbb{R}^n$. That is, a linear transformation given by an $n \times n$ square matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism if and only if rank $A = n$.

Problem 4.6 Suppose that S and T are linear transformations whose composition $S \circ T$ is well-defined. Prove that

(1) if $S \circ T$ is one-to-one, so is T,
(2) if $S \circ T$ is onto, so is S,
(3) if S and T are isomorphisms, then so is $S \circ T$,
(4) if A and B are two $n \times n$ matrices of rank n, then so is AB.

Theorem 4.7 *Two vector spaces V and W are isomorphic if and only if* $\dim V = \dim W$.

Proof: Let $T : V \to W$ be an isomorphism, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Then we show that the set $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis for W so that $\dim W = n = \dim V$.

(1) *It is linearly independent:* Since T is one-to-one, the equation

$$\mathbf{0} = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n) = T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n)$$

implies that $\mathbf{0} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Since the $\mathbf{v}_i$'s are linearly independent, we have $c_i = 0$ for all $i = 1, \ldots, n$.

(2) *It spans W:* Since T is onto, for any $\mathbf{y} \in W$ there exists an $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{y}$. Write $\mathbf{x} = \sum_{i=1}^{n} a_i\mathbf{v}_i$. Then

$$\mathbf{y} = T(\mathbf{x}) = T(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1T(\mathbf{v}_1) + \cdots + a_nT(\mathbf{v}_n),$$

i.e., $\mathbf{y}$ is a linear combination of $T(\mathbf{v}_1), \cdots, T(\mathbf{v}_n)$.

Conversely, suppose that $\dim V = \dim W$. Then one can choose any bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for V and W, respectively. By Theorem 4.3 there exists a linear transformation $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \cdots, n$. It is not hard to show that T is invertible so that T is an isomorphism. Hence V and W are isomorphic. $\square$

Problem 4.7 Let $T : V \to W$ be a linear transformation. Prove that
(1) T is one-to-one if and only if $\mathrm{Ker}(T) = \{\mathbf{0}\}$,
(2) if $V = W$, then T is one-to-one if and only if T is onto.

Corollary 4.8 *Any n-dimensional vector space V is isomorphic to the n-space $\mathbb{R}^n$.*

An **ordered basis** for a vector space is a basis endowed with a specific order. Let V be a vector space of dimension n with an ordered basis $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $\beta = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$ in this order. Then clearly the linear transformation Φ defined by $\Phi(\mathbf{v}_i) = \mathbf{e}_i$ is an isomorphism from V to $\mathbb{R}^n$, called the **natural isomorphism** with respect to the basis α. Now for any $\mathbf{x} = \sum_{i=1}^n a_i\mathbf{v}_i \in V$, the image of $\mathbf{x}$ under this natural isomorphism is written as

$$\Phi(\mathbf{x}) = \sum_{i=1}^n a_i\Phi(\mathbf{v}_i) = \sum_{i=1}^n a_i\mathbf{e}_i = (a_1, \ldots, a_n) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n,$$

which is called the **coordinate vector** of $\mathbf{x}$ with respect to the basis α, and is denoted by $[\mathbf{x}]_\alpha (= \Phi(\mathbf{x}))$. Clearly $[\mathbf{v}_i]_\alpha = \mathbf{e}_i$.

Example 4.9 Recall that, from Example 4.3, the rotation by the angle θ of $\mathbb{R}^2$ is given by the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Clearly, it is invertible and hence an isomorphism of $\mathbb{R}^2$. In fact, one can easily check that the inverse R_θ^{-1} is simply $R_{-\theta}$.

Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_i = R_\theta\mathbf{e}_i$, $i = 1, 2$. Then β is also a basis for $\mathbb{R}^2$. The coordinate vectors of $\mathbf{v}_i$ with respect to α are themselves

$$[\mathbf{v}_1]_\alpha = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad [\mathbf{v}_2]_\alpha = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},$$

while

$$[\mathbf{v}_1]_\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_2]_\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

□

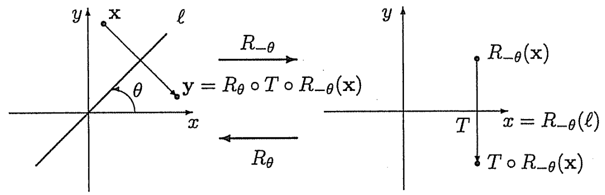
Example 4.10 In Problem 4.1, one can notice that the reflection about the line $y = x$ may be obtained by the compositions of rotation by $-\frac{\pi}{4}$ of the plane, reflection about the x-axis, and rotation by $\frac{\pi}{4}$. Actually, it is multiplication of the matrices given in (1) and (3) of Example 4.3 with $\theta = \frac{\pi}{4}$: that is, if we denote rotation by $\frac{\pi}{4}$ by

$$R_{\frac{\pi}{4}} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

and reflection about the x-axis by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $R_{-\frac{\pi}{4}} = R_{\frac{\pi}{4}}^{-1}$, and the matrix we want is

$$R_{\frac{\pi}{4}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{\frac{\pi}{4}}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The reflection about any line ℓ in the plane can be obtained in this way:



where T is the reflection about the x-axis. □

Problem 4.8 Find the matrix of reflection about the line $y = \sqrt{3}x$ in $\mathbb{R}^2$.

Problem 4.9 Find the coordinate vector of $5 + 2x + 3x^2$ with respect to the given ordered basis α for $P_2(\mathbb{R})$:
(1) $\alpha = \{1,\ x,\ x^2\}$; (2) $\alpha = \{1 + x,\ 1 + x^2,\ x + x^2\}$.

Example 4.11 Let A be an $n \times n$ matrix. It is a linear transformation on the n-space $\mathbb{R}^n$ defined by the matrix multiplication $A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. Suppose that $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are linearly independent vectors in $\mathbb{R}^n$ constituting a parallelepiped (see Remark (2) on page 70). Then A transforms this parallelepiped into another parallelepiped determined by $A\mathbf{r}_1, \ldots, A\mathbf{r}_n$. Hence, if we denote the $n \times n$ matrix whose j-th column is $\mathbf{r}_j$ by B, and the $n \times n$ matrix whose j-th column is $A\mathbf{r}_j$ by C, then clearly, $C = AB$, so

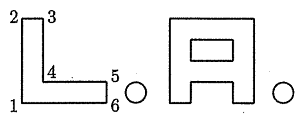
$$\mathrm{vol}(\mathcal{P}(C)) = |\det(AB)| = |\det\ A||\det\ B| = |\det\ A|\mathrm{vol}(\mathcal{P}(B)).$$

This means that, for a square matrix A considered as a linear transformation, the absolute value of the determinant of A is the ratio between the volumes of a parallelepiped $\mathcal{P}(B)$ and its image parallelepiped $\mathcal{P}(C)$ under the transformation by A. If $\det A = 0$, then the image $\mathcal{P}(C)$ is a parallelepiped in a subspace of dimension less than n. □

Problem 4.10 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(x, y, z) = (x + y, y + z, x + z)$. Let C denote the unit cube determined by the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Find the volume of the image parallelepiped $T(C)$ of C under T.

4.3 Application: Computer graphics

One of the simple applications of a linear transformation is to animations or graphical display of pictures on a computer screen. For a simple display of the idea, let us consider a picture in 2-plane $\mathbb{R}^2$. Note that a picture or an image on a screen usually consists of a number of points, lines or curves connecting some of them, and information about how to fill the regions bounded by the lines and curves. Assuming that the computer has information about how to connect the points and curves, a figure can be defined by a list of points. For example, consider the capital letters "LA" below:



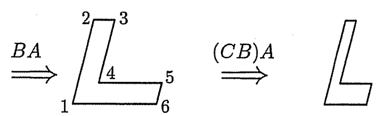
They can be represented by a matrix with coordinates of the vertices. For the sake of brevity we write it just for "L" as follows: The coordinates of the 6 vertices form a matrix:

$$\begin{array}{l} \textit{vertices} \\ \text{x}-\textit{coordinate} \\ \text{y}-\textit{coordinate} \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0.5 & 0.5 & 2.0 & 2.0 \\ 0 & 2.0 & 2.0 & 0.5 & 0.5 & 0.0 \end{array}\right] \end{array} = A.$$

Of course, we assume that the computer knows which vertices are connected to which by lines via some other algorithm. We know that line segments are transformed to other line segments by a matrix, considered as a linear transformation. Thus, by multiplying a matrix to A, the vertices are transformed to the other set of vertices, and the line segments connecting the vertices are preserved. For example, the matrix $B = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$ transforms the matrix A to the following form, which represents new coordinates of the vertices:

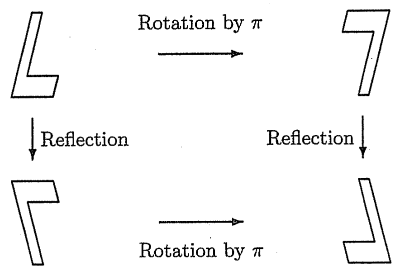
$$\begin{array}{l} \textit{vertices} \\ BA \ = \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0.5 & 1.0 & 0.625 & 2.125 & 2.0 \\ 0 & 2.0 & 2.0 & 0.5 & 0.5 & 0.0 \end{array}\right] \end{array}.$$

Now, the computer connects these vertices properly by lines according to the given algorithm and displays on the screen the changed figure as the left side of the following:



The multiplication of the matrix $C = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$ to BA shrinks the width of BA by half, the right side of the above figure. Thus, changes in the shape of a figure may be obtained by compositions of appropriate linear transformations. Now, it is suggested that the readers try to find various matrices such as reflections, rotations, or any other linear transformations, and multiply them to A to see how the shape of the figure changes.

Remark: Incidentally, one can see that the composition of a rotation by π followed by a reflection about an axis is the same as the composition of the reflection followed by the rotation. In general, a rotation and a reflection are not commutative, neither are a reflection and another reflection.



The above argument generally applies to figures in any dimension. For instance, a 3×3 matrix may be used to convert a figure in $\mathbb{R}^3$ since each point has 3 components.

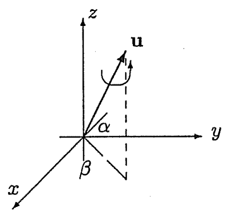
Example 4.12 It is easy to see that the matrices

$$R_{(x,\alpha)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \quad R_{(y,\beta)} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$R_{(z,\gamma)} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the rotations about the x, y, z-axes by the angles α, β and γ, respectively.

In general, the matrix that rotates $\mathbb{R}^3$ with respect to a given axis is useful in many applications. One can easily express such a general rotation as a composition of basic rotations such as $R_{(x,\alpha)}$, $R_{(y,\beta)}$ and $R_{(z,\gamma)}$.



Suppose that the axis of a rotation is the line determined by the vector $\mathbf{u} = (\cos\alpha\cos\beta,\ \cos\alpha\sin\beta,\ \sin\alpha)$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, $0 \le \beta \le 2\pi$, in spherical coordinates, and we want to find the matrix $R_{(\mathbf{u},\theta)}$ of the rotation about the $\mathbf{u}$-axis by θ: For this, we first rotate the $\mathbf{u}$-axis about the z-axis into the xz-plane by $R_{(z,-\beta)}$ and then about the y-axis into the x-axis by $R_{(y,-\alpha)}$. The rotation about the $\mathbf{u}$-axis is the same as the rotation about the x-axis, *i.e.*, one can use the rotation $R_{(x,\theta)}$ about the x-axis. After this, we get back to the rotation about the $\mathbf{u}$-axis via $R_{(y,\alpha)}$ and $R_{(z,\beta)}$. In summary,

$$R_{(\mathbf{u},\theta)} = R_{(z,\beta)}R_{(y,\alpha)}R_{(x,\theta)}R_{(y,-\alpha)}R_{(z,-\beta)}. \qquad \square$$

Problem 4.11 Find the matrix $R_{(\mathbf{u},\frac{\pi}{4})}$ for the rotation about the line determined by $\mathbf{u} = (1, 1, 1)$ by $\dfrac{\pi}{4}$.

4.4 Matrices of linear transformations

We saw that multiplication of an $m\times n$ matrix A with an $n\times 1$ column matrix $\mathbf{x}$ gives rise to a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this section, we show that for any vector spaces V and W (not necessarily the n-spaces), a linear transformation $T : V \to W$ can be represented by a matrix.

Recall that, for any n-dimensional vector space V with an ordered basis, there is a natural isomorphism from V to the n-space $\mathbb{R}^n$, which depends on the choice of a basis for V. Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Take ordered bases $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ for W, and fix them in the following discussion. Then each vector $T(\mathbf{v}_j)$ in W is expressed uniquely as a linear combination of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ in the basis β for W, say

$$\left\{\begin{array}{rcccccccc} T(\mathbf{v}_1) & = & a_{11}\mathbf{w}_1 & + & a_{21}\mathbf{w}_2 & + & \cdots & + & a_{m1}\mathbf{w}_m \\ T(\mathbf{v}_2) & = & a_{12}\mathbf{w}_1 & + & a_{22}\mathbf{w}_2 & + & \cdots & + & a_{m2}\mathbf{w}_m \\ & \vdots & & & & & & \vdots & \\ T(\mathbf{v}_n) & = & a_{1n}\mathbf{w}_1 & + & a_{2n}\mathbf{w}_2 & + & \cdots & + & a_{mn}\mathbf{w}_m, \end{array}\right.$$

or, in a short form,

$$T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i \quad \text{for } 1 \le j \le n,$$

for some scalars a_{ij} ($i = 1, \ldots, m$; $j = 1, \ldots, n$). Notice the indexing order of a_{ij} in this expression: The coordinate vector $[T(\mathbf{v}_j)]_\beta$ of $T(\mathbf{v}_j)$ with respect to the basis β can be written as a column vector

$$[T(\mathbf{v}_j)]_\beta = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Now for any vector $\mathbf{x} = \sum_{j=1}^n x_j \mathbf{v}_j \in V$,

$$\begin{aligned} T(\mathbf{x}) &= \sum_{j=1}^n x_j T(\mathbf{v}_j) = \sum_{j=1}^n x_j \sum_{i=1}^m a_{ij}\mathbf{w}_i \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n x_j a_{ij}\right)\mathbf{w}_i = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j\right)\mathbf{w}_i. \end{aligned}$$

Therefore, the coordinate vector of $T(\mathbf{x})$ with respect to the basis β is

$$[T(\mathbf{x})]_\beta = \begin{bmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A[\mathbf{x}]_\alpha,$$

where $[\mathbf{x}]_\alpha = [x_1 \ \cdots \ x_n]^T$ is the coordinate vector of $\mathbf{x}$ with respect to the basis α in V. In this sense, we say that matrix multiplication by A represents the transformation T. *Note that $A = [a_{ij}]$ is the matrix whose column vectors are just the coordinate vectors $[T(\mathbf{v}_j)]_\beta$ of $T(\mathbf{v}_j)$ with respect to the basis β.* Moreover, for the fixed bases α for V and β for W, the matrix A associated with the linear transformation T with respect to these bases is unique, because the coordinate expression of a vector with respect to a basis is unique. Thus, the assignment of the matrix A to a linear transformation T is well-defined.

Definition 4.4 The matrix A is called the **associated matrix** for T (or **matrix representation** of T) with respect to the bases α and β, and denoted by $A = [T]_\alpha^\beta$.

Now, the above argument can be summarized in the following theorem.

Theorem 4.9 *Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. For fixed* *ordered bases α for V and β for W, the coordinate vector $[T(\mathbf{x})]_\beta$ of $T(\mathbf{x})$ with respect to β is given as a matrix product of the associated matrix $[T]_\alpha^\beta$ of T and $[\mathbf{x}]_\alpha$, i.e.,*

$$[T(\mathbf{x})]_\beta = [T]_\alpha^\beta [\mathbf{x}]_\alpha.$$

The associated matrix $[T]_\alpha^\beta$ is given as

$$[T]_\alpha^\beta = [\,[T(\mathbf{v}_1)]_\beta \; [T(\mathbf{v}_2)]_\beta \; \cdots \; [T(\mathbf{v}_n)]_\beta\,].$$

This situation can be incorporated in the following commutative diagram:

$$\begin{array}{ccccc}
V & & \xrightarrow{\quad T \quad} & & W \\
 & \mathbf{x} & \longmapsto & T(\mathbf{x}) & \\
\Phi \downarrow & \downarrow & & \downarrow & \downarrow \Psi \\
 & [\mathbf{x}]_\alpha & \longmapsto & [T(\mathbf{x})]_\beta & \\
\mathbb{R}^n & & \xrightarrow[A = [T]_\alpha^\beta]{} & & \mathbb{R}^m,
\end{array}$$

where Φ and Ψ denote the natural isomorphisms, defined in Section 4.2, from V to $\mathbb{R}^n$ with respect to α, and from W to $\mathbb{R}^m$ with respect to β, respectively. Note that the commutativity of the above diagram means that $A \circ \Phi = \Psi \circ T$. When $V = W$ and $\alpha = \beta$, we simply write $[T]_\alpha$ for $[T]_\alpha^\alpha$.

Remark: (1) Note that an $m \times n$ matrix A is the matrix representation of A itself with respect to the standard bases α for $\mathbb{R}^n$ and γ for $\mathbb{R}^m$, *i.e.*, $A = [A]_\alpha^\gamma$. In particular, if A is an invertible $n \times n$ square matrix, then the column vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$ form another basis β for $\mathbb{R}^n$. Thus, A is simply the linear transformation on $\mathbb{R}^n$ that takes the standard basis α to β, in fact,

$$A\mathbf{e}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = \mathbf{c}_j = \sum_{i=1}^{n} a_{ij}\mathbf{e}_i,$$

the j-th column of A.

(2) Let V and W be vector spaces with bases α and β, respectively, and let $T : V \to W$ be a linear transformation with the matrix representation $[T]_\alpha^\beta = A$. Then it is quite clear that $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are isomorphic to $\mathcal{N}(A)$ and $\mathcal{C}(A)$, respectively, via the natural isomorphisms. In particular, if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ with the standard bases, then $\mathrm{Ker}(T) = \mathcal{N}(A)$, and $\mathrm{Im}(T) = \mathcal{C}(A)$. Therefore, from Corollary 3.17, we have

$$\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = \dim V.$$

The following examples illustrate the computation of matrices associated with linear transformations.

Example 4.13 Let $Id : V \to V$ be the identity transformation on a vector space V. Then for any ordered basis α for V, the matrix $[Id]_\alpha = I$, the identity matrix.

Example 4.14 Let $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$(T(p))(x) = xp(x).$$

Then, with the bases $\alpha = \{1,\ x\}$ and $\beta = \{1,\ x,\ x^2\}$ for $P_1(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively, the associated matrix for T is $[T]_\alpha^\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. □

Example 4.15 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(x,\ y) = (x + 2y,\ 0,\ 2x + 3y)$ with respect to the standard bases α and β for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then

$$\begin{aligned} T(\mathbf{e}_1) &= T(1,\ 0) = (1,\ 0,\ 2) = 1\mathbf{e}_1 + 0\mathbf{e}_2 + 2\mathbf{e}_3, \\ T(\mathbf{e}_2) &= T(0,\ 1) = (2,\ 0,\ 3) = 2\mathbf{e}_1 + 0\mathbf{e}_2 + 3\mathbf{e}_3. \end{aligned}$$

Hence, $[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}$. If $\beta' = \{\mathbf{e}_3,\ \mathbf{e}_2,\ \mathbf{e}_1\}$, then $[T]_\alpha^{\beta'} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$. □

Example 4.16 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(1,\ 1) = (0,\ 1)$ and $T(-1,\ 1) = (2,\ 3)$. Find the matrix representation $[T]_\alpha$ of T with respect to the standard basis $\alpha = \{\mathbf{e}_1,\ \mathbf{e}_2\}$.

Solution: Note that $(a,\ b) = a\mathbf{e}_1 + b\mathbf{e}_2$ for any $(a,\ b) \in \mathbb{R}^2$. Thus the definition of T shows

$$\begin{aligned} T(\mathbf{e}_1) + T(\mathbf{e}_2) &= T(\mathbf{e}_1 + \mathbf{e}_2) = T(1,\ 1) = (0,\ 1) = \mathbf{e}_2, \\ -T(\mathbf{e}_1) + T(\mathbf{e}_2) &= T(-\mathbf{e}_1 + \mathbf{e}_2) = T(-1,\ 1) = (2,\ 3) = 2\mathbf{e}_1 + 3\mathbf{e}_2. \end{aligned}$$

By solving these equations, we obtain

$$\begin{aligned} T(\mathbf{e}_1) &= -\mathbf{e}_1 - \mathbf{e}_2, \\ T(\mathbf{e}_2) &= \mathbf{e}_1 + 2\mathbf{e}_2. \end{aligned}$$

Therefore, $[T]_\alpha = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$. □

Example 4.17 Let T be the linear transformation in Example 4.16. Find $[T]_\beta$ for a basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = (0, 1)$ and $\mathbf{v}_2 = (2, 3)$.

Solution: From Example 4.16,

$$\begin{aligned} T(\mathbf{v}_1) &= \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [T(\mathbf{v}_1)]_\alpha, \\ T(\mathbf{v}_2) &= \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = [T(\mathbf{v}_2)]_\alpha. \end{aligned}$$

Writing these vectors with respect to β, we get

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2 = \begin{bmatrix} 2b \\ a+3b \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix} = c\mathbf{v}_1 + d\mathbf{v}_2 = \begin{bmatrix} 2d \\ c+3d \end{bmatrix}.$$

Solving for a, b, c and d, we obtain

$$[T(\mathbf{v}_1)]_\beta = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } [T(\mathbf{v}_2)]_\beta = \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Therefore, $[T]_\beta = \frac{1}{2}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$. □

Problem 4.12 Find the matrix representation of each of the following linear transformations T of $\mathbb{R}^3$ with respect to the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and $\beta = \{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1\}$:
(1) $T(x, y, z) = (2x - 3y + 4z, 5x - y + 2z, 4x + 7y)$,
(2) $T(x, y, z) = (2y + z, x - 4y, 3x)$.

Problem 4.13 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z, u) = (x + 2y, x - 3z + u, 2y + 3z + 4u).$$

Let α and β be the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Find $[T]_\alpha^\beta$.

Problem 4.14 Let $Id : \mathbb{R}^n \to \mathbb{R}^n$ be the identity transformation. Let $\mathbf{x}_k$ denote the vector in $\mathbb{R}^n$ whose first $k-1$ coordinates are zero and the last $n-k+1$ coordinates are 1. Then clearly $\beta = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$ (see Problem 3.9). Let $\alpha = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Find the matrix representations $[Id]_\alpha^\beta$ and $[Id]_\beta^\alpha$.

4.5 Vector spaces of linear transformations

Let V and W be two vector spaces. Let $\mathcal{L}(V;W)$ denote the set of all linear transformations from V to W, *i.e.*,

$$\mathcal{L}(V;W) = \{T \ : \ T \text{ is a linear transformation from } V \text{ into } W\}.$$

For $S, T \in \mathcal{L}(V;W)$ and $\lambda \in \mathbb{R}$, define the **sum** $S+T$ and the **scalar multiplication** λS by

$$(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v}), \quad \text{and} \quad (\lambda S)(\mathbf{v}) = \lambda(S(\mathbf{v}))$$

for any $\mathbf{v} \in V$. Then clearly $S+T$ and λS belong to $\mathcal{L}(V;W)$, so that $\mathcal{L}(V;W)$ becomes a vector space. In particular, if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then the set $M_{m\times n}(\mathbb{R})$ is precisely the vector space of the linear transformations of $\mathbb{R}^n$ into $\mathbb{R}^m$ with respect to the standard bases. Hence, by fixing the standard bases, we have identified $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m) = M_{m\times n}(\mathbb{R})$ via the matrix representation.

In general, for any vector spaces V and W of dimensions n and m with ordered bases α and β, respectively, there is a one-to-one correspondence between $\mathcal{L}(V;W)$ and $M_{m\times n}(\mathbb{R})$ via the matrix representation.

Let us first define a transformation $\phi : \mathcal{L}(V;W) \to M_{m\times n}(\mathbb{R})$ as

$$\phi(T) = [T]_\alpha^\beta \in M_{m\times n}(\mathbb{R})$$

for any $T \in \mathcal{L}(V;W)$ (see Section 4.4). If $[S]_\alpha^\beta = [T]_\alpha^\beta$ for S and $T \in \mathcal{L}(V;W)$, then we have $S = T$ by Corollary 4.4. This means that ϕ is one-to-one.

On the other hand, an $m \times n$ matrix A, considered as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, gives rise to a linear transformation T from V to W via the composition of A with the natural isomorphisms Φ and Ψ, *i.e.*, $T = \Psi^{-1} \circ A \circ \Phi$, which satisfies $[T]_\alpha^\beta = A$. This means that ϕ is onto.

Therefore, ϕ gives an one-to-one correspondence between $\mathcal{L}(V;W)$ and $M_{m\times n}(\mathbb{R})$. Furthermore, the following theorem shows that ϕ is linear, so that it is in fact an isomorphism from $\mathcal{L}(V;W)$ to $M_{m\times n}(\mathbb{R})$.

Theorem 4.10 *Let V and W be vector spaces with ordered bases α and β, respectively, and let S, $T : V \to W$ be linear. Then we have*

$$[S+T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta \quad \textit{and} \quad [kS]_\alpha^\beta = k[S]_\alpha^\beta.$$

Proof: Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Then we have unique expressions $S(\mathbf{v}_j) = \sum_{i=1}^m a_{ij}\mathbf{w}_i$ and $T(\mathbf{v}_j) = \sum_{i=1}^m b_{ij}\mathbf{w}_i$ for each $1 \le j \le n$. Hence

$$(S+T)(\mathbf{v}_j) = \sum_{i=1}^m a_{ij}\mathbf{w}_i + \sum_{i=1}^m b_{ij}\mathbf{w}_i = \sum_{i=1}^m (a_{ij} + b_{ij})\mathbf{w}_i.$$

Thus

$$[S+T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta.$$

The proof of the second equality $[kS]_\alpha^\beta = k[S]_\alpha^\beta$ is left as an exercise. □

In summary, for vector spaces V of dimension n and W of dimension m with fixed ordered bases α and β respectively, the vector space $\mathcal{L}(V;W)$ of all linear transformations from V to W can be identified with the vector space $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices so that

$$\dim \mathcal{L}(V;W) = \dim M_{m\times n}(\mathbb{R}) = mn = \dim V \dim W.$$

Remark: (1) Let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations for an $m\times n$ matrix A. By considering the coefficient matrix A as a linear transformation, one can have other equivalent conditions to those in Theorems 3.23 and 3.24: The conditions in Theorem 3.23 (*e.g.*, rank $A = m$) are equivalent to the condition that A is *surjective*, and those in Theorem 3.24 (*e.g.*, rank $A = n$) are equivalent to the condition that A is *one-to-one*. This observation gives the proof of (15)-(16) in Theorem 3.25.

(2) With the identification of vector spaces $\mathcal{L}(V;W)$ and $M_{m\times n}(\mathbb{R})$ as above, we can have, by Theorem 3.25, the following equivalent conditions for a linear transformation T on a vector space V:

(i) T *is an isomorphism,*

(ii) T *is one-to-one,*

(iii) T *is surjective.*

(One can also prove them directly by using the definition of a basis for V.)

The next theorem shows that the one-to-one correspondence between $\mathcal{L}(V;W)$ and $M_{m\times n}(\mathbb{R})$ preserves not only the linear structure but also the compositions of linear transformations. Let V, W and Z be vector spaces. Suppose that $S : V \to W$ and $T : W \to Z$ are linear transformations. Then clearly the composition $T \circ S : V \to Z$ is also linear. Often we refer this composition to the product operation of linear transformations.

Theorem 4.11 *Let V, W and Z be vector spaces with ordered bases α, β, and γ, respectively. Suppose that $S : V \to W$ and $T : W \to Z$ are linear transformations. Then*

$$[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta.$$

Proof: Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ and $\gamma = \{\mathbf{z}_1, \ldots, \mathbf{z}_\ell\}$. Let $[T]_\beta^\gamma = [a_{ij}]$ and $[S]_\alpha^\beta = [b_{pq}]$. Then, for $1 \le i \le n$

$$\begin{aligned} (T \circ S)(\mathbf{v}_i) &= T(S(\mathbf{v}_i)) = T\left(\sum_{k=1}^{m} b_{ki}\mathbf{w}_k\right) = \sum_{k=1}^{m} b_{ki} T(\mathbf{w}_k) \\ &= \sum_{k=1}^{m} b_{ki}\left(\sum_{j=1}^{\ell} a_{jk}\mathbf{z}_j\right) = \sum_{j=1}^{\ell}\left(\sum_{k=1}^{m} a_{jk}b_{ki}\right)\mathbf{z}_j. \end{aligned}$$

It shows that $[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$. □

Problem 4.15 Let α be the standard basis for $\mathbb{R}^3$, and let S, $T : \mathbb{R}^3 \to \mathbb{R}^3$ be two linear transformations given by

$$S(\mathbf{e}_1) = (2,\ 2,\ 1),\quad S(\mathbf{e}_2) = (0,\ 1,\ 2),\quad S(\mathbf{e}_3) = (-1,\ 2,\ 1),$$
$$T(\mathbf{e}_1) = (1,\ 0,\ 1),\quad T(\mathbf{e}_2) = (0,\ 1,\ 1),\quad T(\mathbf{e}_3) = (1,\ 1,\ 2).$$

Compute $[S+T]_\alpha$, $[2T - S]_\alpha$ and $[T \circ S]_\alpha$.

Problem 4.16 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f) = (3+x)f' + 2f$, and $S : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $S(a+bx+cx^2) = (a-b,\ a+b,\ c)$. For a basis $\alpha = \{1,\ x,\ x^2\}$ for $P_2(\mathbb{R})$ and the standard basis $\beta = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ for $\mathbb{R}^3$, compute $[S]_\alpha^\beta$, $[T]_\alpha$, and $[S \circ T]_\alpha^\beta$.

Theorem 4.12 *Let V and W be vector spaces with ordered bases α and β, respectively, and let $T : V \to W$ be an isomorphism. Then*

$$[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}.$$

Proof: Since T is invertible, $\dim V = \dim W$, and the matrices $[T]_\alpha^\beta$ and $[T^{-1}]_\beta^\alpha$ are square and of the same size. Thus,

$$[T]_\alpha^\beta [T^{-1}]_\beta^\alpha = [T \circ T^{-1}]_\beta = [Id]_\beta$$

is the identity matrix. Hence, $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$. □

In particular, if a linear transformation $T : V \to W$ is an isomorphism, then $[T]_\alpha^\beta$ is an invertible matrix for any bases α for V and β for W.

Problem 4.17 For the vector spaces $P_1(\mathbb{R})$ and $\mathbb{R}^2$, choose the bases $\alpha = \{1,\ x\}$ for $P_1(\mathbb{R})$ and $\beta = \{\mathbf{e}_1,\ \mathbf{e}_2\}$ for $\mathbb{R}^2$, respectively. Let $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined by $T(a + bx) = (a,\ a + b)$.

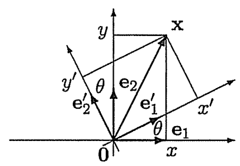
(1) Show that T is invertible. (2) Find $[T]_\alpha^\beta$ and $[T^{-1}]_\beta^\alpha$.

4.6 Change of bases

In Section 4.2, we saw that any vector space V of dimension n with an ordered basis α is isomorphic to the n-space $\mathbb{R}^n$ via the natural isomorphism Φ. It assigns the coordinate vector in $\mathbb{R}^n$ to each $\mathbf{x} \in V$, *i.e.*, $\Phi(\mathbf{x}) = [\mathbf{x}]_\alpha$. Of course, we can get a different isomorphism if we take another basis β instead of α: That is, the coordinate expression $[\mathbf{x}]_\beta$ of $\mathbf{x}$ with respect to β may be different from $[\mathbf{x}]_\alpha$. Thus, one may naturally ask what the relation between $[\mathbf{x}]_\alpha$ and $[\mathbf{x}]_\beta$ is for the two different bases. In this section, we discuss this question. One of the fundamental problems in linear algebra is to find bases for which the matrix representation of a linear transformation is as simple as possible.

Let us begin with an example in the plane $\mathbb{R}^2$. The coordinate expression of $\mathbf{x} = (x,\ y) \in \mathbb{R}^2$ with respect to the standard basis $\alpha = \{\mathbf{e}_1,\ \mathbf{e}_2\}$ is $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$, so that $[\mathbf{x}]_\alpha = \begin{bmatrix} x \\ y \end{bmatrix}$.

Now let $\beta = \{\mathbf{e}'_1,\ \mathbf{e}'_2\}$ be another basis for $\mathbb{R}^2$ obtained by rotating α counterclockwise through an angle θ.



Then the coordinate expression of $\mathbf{x} \in \mathbb{R}^2$ with respect to β is written as $\mathbf{x} = x'\mathbf{e}'_1 + y'\mathbf{e}'_2$, or $[\mathbf{x}]_\beta = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

In particular, the expression of the vectors in β with respect to α are

$$\begin{array}{rcrcl} \mathbf{e}_1' &=& Id(\mathbf{e}_1') &=& \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \\ \mathbf{e}_2' &=& Id(\mathbf{e}_2') &=& -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2, \end{array}$$

so

$$[\mathbf{e}_1']_\alpha = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad [\mathbf{e}_2']_\alpha = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{x} &= x'\mathbf{e}_1' + y'\mathbf{e}_2' = (x'\cos\theta - y'\sin\theta)\mathbf{e}_1 + (x'\sin\theta + y'\cos\theta)\mathbf{e}_2 \\ &= x\mathbf{e}_1 + y\mathbf{e}_2. \end{aligned}$$

This can be written as the following matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \text{or} \quad [\mathbf{x}]_\alpha = [Id]_\beta^\alpha [\mathbf{x}]_\beta,$$

where

$$[Id]_\beta^\alpha = [[\mathbf{e}_1']_\alpha \ [\mathbf{e}_2']_\alpha] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that $[Id]_\alpha^\beta = ([Id]_\beta^\alpha)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ by Theorem 4.12.

In general, let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be two ordered bases for V. Then any vector $\mathbf{x} \in V$ has two expressions:

$$\mathbf{x} = \sum_{i=1}^n x_i\mathbf{v}_i = \sum_{j=1}^n y_j\mathbf{w}_j.$$

Now, each vector in β is expressed as a linear combination of the vectors in α: $\mathbf{w}_j = Id(\mathbf{w}_j) = \sum_{i=1}^n q_{ij}\mathbf{v}_i$ for $j = 1, 2, \ldots, n$, so that

$$[\mathbf{w}_j]_\alpha = [Id(\mathbf{w}_j)]_\alpha = \begin{bmatrix} q_{1j} \\ \vdots \\ q_{nj} \end{bmatrix}.$$

Then for any $\mathbf{x} \in V$,

$$\mathbf{x} = \sum_{i=1}^n x_i\mathbf{v}_i = \sum_{j=1}^n y_j\mathbf{w}_j = \sum_{j=1}^n y_j \sum_{i=1}^n q_{ij}\mathbf{v}_i = \sum_{i=1}^n \left(\sum_{j=1}^n q_{ij}y_j \right) \mathbf{v}_i.$$

This is equivalent to the following matrix equation:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ & \ddots & \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

or

$$[\mathbf{x}]_\alpha = [Id]^\alpha_\beta [\mathbf{x}]_\beta$$

$$\begin{array}{ccccc} V & & \xrightarrow{\quad Id \quad} & & V \\ & \mathbf{x} & \longmapsto & \mathbf{x} & \\ \Phi' \Big\downarrow & \bar{\downarrow} & & \bar{\downarrow} & \Big\downarrow \Phi \\ & [\mathbf{x}]_\beta & \longmapsto & [\mathbf{x}]_\alpha & \\ \mathbb{R}^n & & \xrightarrow[Q = [Id]^\alpha_\beta]{} & & \mathbb{R}^n, \end{array}$$

where

$$[Id]^\alpha_\beta = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ & \ddots & \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} = [\ [\mathbf{w}_1]_\alpha \cdots [\mathbf{w}_n]_\alpha].$$

Definition 4.5 The matrix representation $[Id]^\alpha_\beta$ of the identity transformation $Id : V \to V$ with respect to any two bases α and β is called the **transition matrix** or the **coordinate change matrix** from β to α.

Since the identity transformation $Id : V \to V$ is invertible, the transition matrix $Q = [Id]^\alpha_\beta$ is also invertible by Theorem 4.12. If we had taken the expressions of the vectors in the basis α with respect to the basis β: $\mathbf{v}_j = Id(\mathbf{v}_j) = \sum_{i=1}^n p_{ij}\mathbf{w}_i$ for $j = 1, 2, \ldots, n$, then we would have $[p_{ij}] = [Id]^\beta_\alpha = Q^{-1}$ and

$$[\mathbf{x}]_\beta = [Id]^\beta_\alpha [\mathbf{x}]_\alpha = Q^{-1}[\mathbf{x}]_\alpha.$$

Example 4.18 Let the 3-space $\mathbb{R}^3$ be equipped with the standard xyz-coordinate system, *i.e.*, with the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Take a new $x'y'z'$-coordinate system by rotating the xyz-system around its z-axis counterclockwise through an angle θ, *i.e.*, we take a new basis $\beta = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ by rotating the basis α about z axis through θ. Then we get

$$[\mathbf{e}'_1]_\alpha = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad [\mathbf{e}'_2]_\alpha = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad [\mathbf{e}'_3]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the transition matrix from β to α is

$$Q = [Id]_\beta^\alpha = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so

$$[\mathbf{x}]_\alpha = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = Q[\mathbf{x}]_\beta.$$

Moreover, $Q = [Id]_\beta^\alpha$ is invertible and the transition matrix from α to β is

$$Q^{-1} = [Id]_\alpha^\beta = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

□

Problem 4.18 Find the transition matrix from a basis α to another basis β for the 3-space $\mathbb{R}^3$, where

$$\alpha = \{(1, 0, 1),\ (1, 1, 0),\ (0, 1, 1)\}, \quad \beta = \{(2, 3, 1),\ (1, 2, 0),\ (2, 0, 3)\}.$$

4.7 Similarity

The coordinate expression of a vector in a vector space V depends on the choice of an ordered basis. Hence, the matrix representation of a linear transformation is also dependent on the choice of bases.

Let V and W be two vector spaces of dimensions n and m with two ordered bases α and β, respectively, and let $T : V \to W$ be a linear transformation. In Section 4.4, we discussed how to find $[T]_\alpha^\beta$. If we have different bases α' and β' for V and W, respectively, then we get another matrix representation $[T]_{\alpha'}^{\beta'}$ of T. We, in fact, have two different expressions

$$\begin{array}{ll} [\mathbf{x}]_\alpha \text{ and } [\mathbf{x}]_{\alpha'} \text{ in } \mathbb{R}^n & \text{for each } \mathbf{x} \in V, \\ [T(\mathbf{x})]_\beta \text{ and } [T(\mathbf{x})]_{\beta'} \text{ in } \mathbb{R}^m & \text{for } T(\mathbf{x}) \in W. \end{array}$$

They are related by the transition matrices in the following equations:

$$[\mathbf{x}]_{\alpha'} = [Id_V]_{\alpha}^{\alpha'}[\mathbf{x}]_{\alpha}, \quad \text{and} \quad [T(\mathbf{x})]_{\beta'} = [Id_W]_{\beta}^{\beta'}[T(\mathbf{x})]_{\beta}.$$

On the other hand, by Theorem 4.9, we have

$$[T(\mathbf{x})]_{\beta} = [T]_{\alpha}^{\beta}[\mathbf{x}]_{\alpha}, \quad \text{and} \quad [T(\mathbf{x})]_{\beta'} = [T]_{\alpha'}^{\beta'}[\mathbf{x}]_{\alpha'}.$$

Therefore, we get

$$\begin{aligned} [T]_{\alpha'}^{\beta'}[\mathbf{x}]_{\alpha'} &= [T(\mathbf{x})]_{\beta'} = [Id_W]_{\beta}^{\beta'}[T(\mathbf{x})]_{\beta} = [Id_W]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[\mathbf{x}]_{\alpha} \\ &= [Id_W]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[Id_V]_{\alpha'}^{\alpha}[\mathbf{x}]_{\alpha'}. \end{aligned}$$

Actually, from Theorem 4.11, this relation can be obtained directly as

$$[T]_{\alpha'}^{\beta'} = [Id_W \circ T \circ Id_V]_{\alpha'}^{\beta'} = [Id_W]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[Id_V]_{\alpha'}^{\alpha},$$

since $T = Id_W \circ T \circ Id_V$. Note that $[T]_{\alpha}^{\beta}$ and $[T]_{\alpha'}^{\beta'}$ are $m \times n$ matrices, $[Id_V]_{\alpha'}^{\alpha}$ is an $n \times n$ matrix and $[Id_W]_{\beta}^{\beta'}$ is an $m \times m$ matrix.

The relation can also be incorporated in the following commutative diagrams:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "{[T]_{\alpha'}^{\beta'}}"] \arrow[ddd, "{[Id_V]_{\alpha'}^{\alpha}}"'] & & & \mathbb{R}^m \arrow[ddd, "{[Id_W]_{\beta'}^{\beta}}"] \\
& (V,\alpha') \arrow[ul, "\Phi'"'] \arrow[r, "T"] \arrow[d, "Id_V"'] & (W,\beta') \arrow[ur, "\Psi'"] \arrow[d, "Id_W"] & \\
& (V,\alpha) \arrow[dl, "\Phi"] \arrow[r, "T"] & (W,\beta) \arrow[dr, "\Psi"'] & \\
\mathbb{R}^n \arrow[rrr, "{[T]_{\alpha}^{\beta}}"] & & & \mathbb{R}^m
\end{tikzcd}$$

The following theorem summarizes the above argument.

Theorem 4.13 *Let $T : V \to W$ be a linear transformation on a vector space V with bases α and α' to another vector space W with bases β and β'. Then*

$$[T]_{\alpha'}^{\beta'} = P^{-1}[T]_{\alpha}^{\beta}Q,$$

where $Q = [Id_V]_{\alpha'}^{\alpha}$ and $P = [Id_W]_{\beta'}^{\beta}$ are the transition matrices.

In particular, if we take $W = V$, $\alpha = \beta$ and $\alpha' = \beta'$, then $P = Q$ and we get to the following corollary.

Corollary 4.14 *Let $T : V \to V$ be a linear transformation on a vector space V, and let α and β be ordered bases for V. Let $Q = [Id]_{\beta}^{\alpha}$ be the transition matrix from β to α. Then*

(1) *Q is invertible, and $Q^{-1} = [Id]_{\alpha}^{\beta}$.*

(2) *For any $\mathbf{x} \in V$, $[\mathbf{x}]_{\alpha} = Q[\mathbf{x}]_{\beta}$.*

(3) *$[T]_{\beta} = Q^{-1}[T]_{\alpha}Q$.*

Relation (3) of $[T]_{\beta}$ and $[T]_{\alpha}$ in Corollary 4.14 is called a *similarity*. In general, we have the following definition.

Definition 4.6 For any square matrices A and B, A is said to be **similar** to B if there exists a nonsingular matrix Q such that $B = Q^{-1}AQ$.

Note that if A is similar to B, then B is also similar to A. Thus we simply say that A and B are similar matrices. We saw in Theorem 4.14 that if A and B are $n \times n$ matrices representing the same linear transformation T, then A and B are similar.

Example 4.19 Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$ consisting of $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (1, 0, 1)$ and $\mathbf{v}_3 = (0, 1, 1)$. Let T be the linear transformation on $\mathbb{R}^3$ given by the matrix

$$[T]_{\beta} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}.$$

Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis. Find the transition matrix $[Id]_{\alpha}^{\beta}$ and $[T]_{\alpha}$.

Solution: Since $\mathbf{v}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{v}_3 = \mathbf{e}_2 + \mathbf{e}_3$, we have

$$[Id]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad [Id]_\alpha^\beta = ([Id]_\beta^\alpha)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Therefore,

$$[T]_\alpha = [Id]_\beta^\alpha [T]_\beta [Id]_\alpha^\beta = \frac{1}{2}\begin{bmatrix} 4 & 2 & 2 \\ 3 & -1 & 1 \\ -1 & 1 & 7 \end{bmatrix}.$$

□

Example 4.20 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1,\ x_2,\ x_3) = (2x_1 + x_2,\ x_1 + x_2 + 3x_3,\ -x_2).$$

Let $\alpha = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ be the standard ordered basis. Then we clearly have

$$[T]_\alpha = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{bmatrix}.$$

Let $\beta = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ be another ordered basis for $\mathbb{R}^3$ consisting of $\mathbf{v}_1 = (-1,\ 0,\ 0)$, $\mathbf{v}_2 = (2,\ 1,\ 0)$, and $\mathbf{v}_3 = (1,\ 1,\ 1)$. Let $Q = [Id]_\beta^\alpha$ be the transition matrix from β to α. Since α is the standard ordered basis for $\mathbb{R}^3$, the columns of Q are simply the vectors in β written in the same order, with an easily calculated inverse. Thus

$$Q = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Q^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

A straightforward multiplication shows that

$$[T]_\beta = Q^{-1}[T]_\alpha Q = \begin{bmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{bmatrix}.$$

To show that this is the correct matrix, we can verify that the image under T of the j-th vector of β is the linear combination of the vectors of β with the entries of the j-th column of $[T]_\beta$ as its coefficients. For example, for $j = 2$

we have $T(\mathbf{v}_2) = T(2,1,0) = (5,3,-1)$. On the other hand, the coefficients of $[T(\mathbf{v}_2)]_\beta$ are just the entries of the second column of $[T]_\beta$. Therefore,

$$\begin{aligned} T(\mathbf{v}_2) &= 2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3 \\ &= 12\mathbf{e}_1 + 4(2\mathbf{e}_1 + \mathbf{e}_2) - (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ &= 5\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3 \;=\; (5,3,-1), \end{aligned}$$

as expected. □

The next theorem shows that two similar matrices are matrix representations of the same linear transformation.

Theorem 4.15 *Suppose that A represents a linear transformation $T : V \to V$ on a vector space V with respect to an ordered basis $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, i.e., $[T]_\alpha = A$. If $B = Q^{-1}AQ$ for some nonsingular matrix Q, then there exists a basis β for V such that $B = [T]_\beta$, and $Q = [Id]^\alpha_\beta$.*

Proof: Let $Q = [q_{ij}]$ and let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be the vectors in V defined by

$$\left\{ \begin{aligned} \mathbf{w}_1 &= q_{11}\mathbf{v}_1 + q_{21}\mathbf{v}_2 + \cdots + q_{n1}\mathbf{v}_n \\ \mathbf{w}_2 &= q_{12}\mathbf{v}_1 + q_{22}\mathbf{v}_2 + \cdots + q_{n2}\mathbf{v}_n \\ &\;\vdots \\ \mathbf{w}_n &= q_{1n}\mathbf{v}_1 + q_{2n}\mathbf{v}_2 + \cdots + q_{nn}\mathbf{v}_n. \end{aligned} \right.$$

Then the nonsingularity of $Q = [q_{ij}]$ implies that $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an ordered basis for V, and Theorem 4.14 (3) shows that $[T]_\beta = Q^{-1}[T]_\alpha Q = Q^{-1}AQ = B$ with $Q = [Id]^\alpha_\beta$. □

Example 4.21 Let D be the differential operator on the vector space $P_2(\mathbb{R})$. Given two ordered bases $\alpha = \{1,\ x,\ x^2\}$ and $\beta = \{1,\ 2x,\ 4x^2 - 2\}$ for $P_2(\mathbb{R})$, we first note that

$$\begin{aligned} D(1) &= 0\cdot 1 + 0\cdot x + 0\cdot x^2 \\ D(x) &= 1\cdot 1 + 0\cdot x + 0\cdot x^2 \\ D(x^2) &= 0\cdot 1 + 2\cdot x + 0\cdot x^2. \end{aligned}$$

Hence, the matrix representation of D with respect to α is given by

$$[D]_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Applying D to 1, $2x$ and $4x^2 - 2$, one obtains

$$\begin{aligned} D(1) &= 0 \cdot 1 + 0 \cdot 2x + 0 \cdot (4x^2 - 2) \\ D(2x) &= 2 \cdot 1 + 0 \cdot 2x + 0 \cdot (4x^2 - 2) \\ D(4x^2 - 2) &= 0 \cdot 1 + 4 \cdot 2x + 0 \cdot (4x^2 - 2). \end{aligned}$$

Thus,

$$[D]_\beta = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

The transition matrix Q from $\beta = \{1,\ 2x,\ 4x^2 - 2\}$ to $\alpha = \{1,\ x,\ x^2\}$ and its inverse are easily calculated as

$$Q = [Id]_\beta^\alpha = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad Q^{-1} = [Id]_\alpha^\beta = \frac{1}{4}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A simple computation shows that $[D]_\beta = Q^{-1}[D]_\alpha Q$. □

Problem 4.19 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1,\ x_2,\ x_3) = (x_1 + 2x_2 + x_3,\ -x_2,\ x_1 + 4x_3).$$

Let α be the standard basis, and let $\beta = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ be another ordered basis consisting of $\mathbf{v}_1 = (1,\ 0,\ 0)$, $\mathbf{v}_2 = (1,\ 1,\ 0)$, and $\mathbf{v}_3 = (1,\ 1,\ 1)$ for $\mathbb{R}^3$. Find the associated matrix of T with respect to α and the associated matrix of T with respect to β. Are they similar?

Problem 4.20 Suppose that A and B are similar $n \times n$ matrices. Show that

(1) $\det A = \det B$,

(2) $\mathrm{tr}\ A = \mathrm{tr}\ B$,

(3) $\mathrm{rank}\ A = \mathrm{rank}\ B$.

Problem 4.21 Let A and B be $n \times n$ matrices. Show that if A is similar to B, then A^2 is similar to B^2.

4.8 Dual spaces

In this section, we are concerned exclusively with linear transformations from a vector space V to the one-dimensional vector space $\mathbb{R}^1$. Such a linear transformation is called a **linear functional** of V. The definite integrals of continuous functions is one of the most important examples of linear functionals in mathematics.

For a matrix A regarded as a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, we saw that the transpose A^T of A is another linear transformation $A^T : \mathbb{R}^m \to \mathbb{R}^n$. For a linear transformation $T : V \to W$ on a vector space V to W, one can naturally ask what its *transpose* is and what the definition is. This section will answer those questions.

Example 4.22 Let $C[a, b]$ be the vector space of all continuous real-valued functions on the interval $[a, b]$. The definite integral $\mathcal{I} : C[a, b] \to \mathbb{R}$ defined by

$$\mathcal{I}(f) = \int_a^b f(t)dt$$

is a linear functional of $C[a, b]$. In particular, if the interval is $[0, 2\pi]$ and n is an integer, then

$$\mathcal{F}_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$$

is a linear functional, called the n-th **Fourier coefficient** of f.

Example 4.23 The trace function $\mathrm{tr} : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a linear functional of $M_{n\times n}(\mathbb{R})$.

Note that as we saw in Section 4.5, the set of all linear functionals of V is the vector space $\mathcal{L}(V; \mathbb{R}^1)$ whose dimension equals the dimension of V (see page 141).

Definition 4.7 For a vector space V, the vector space of all linear functionals of V is called the **dual space** of V and denoted by V^*.

Recall that such a linear transformation $T : V \to \mathbb{R}$ is completely determined by the values on a basis for V. Thus if $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then the functions $\mathbf{v}_i^* : V \to \mathbb{R}$ defined by $\mathbf{v}_i^*(\mathbf{v}_j) = \delta_{ij}$ for each $i, j = 1, \ldots, n$ are clearly linear functionals of V, called the i-th **coordinate function** with respect to the basis α. In particular, for any $\mathbf{x} = \sum a_i\mathbf{v}_i \in V$, $\mathbf{v}_i^*(\mathbf{x}) = a_i$, the i-th coordinate of $\mathbf{x}$ with respect to α.

Theorem 4.16 *The set $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_n^*\}$ forms a basis for the dual space V^*, and for any $T \in V^*$ we have*

$$T = \sum_{i=1}^{n} T(\mathbf{v}_i)\mathbf{v}_i^*.$$

Proof: Clearly, the set $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^*\}$ is linearly independent, since $\mathbf{0} = \sum_{i=1}^n c_i\mathbf{v}_i^*$ implies $0 = \sum_{i=1}^n c_i\mathbf{v}_i^*(\mathbf{v}_j) = c_j$ for each $j = 1, \ldots, n$. Moreover, the set α^* spans V^*; for any $T \in V^*$ and any $\mathbf{v}_j \in \alpha$, we have

$$\left(\sum_{i=1}^{n} T(\mathbf{v}_i)\mathbf{v}_i^*\right)(\mathbf{v}_j) = \sum_{i=1}^{n} T(\mathbf{v}_i)(\mathbf{v}_i^*(\mathbf{v}_j)) = T(\mathbf{v}_j).$$

Hence, by Corollary 4.4, we get $T = \sum_{i=1}^n T(\mathbf{v}_i)\mathbf{v}_i^*$. □

Definition 4.8 For a basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for a vector space V, the basis α^* for V^* is called the **dual basis** of α.

This theorem says that, for a fixed basis $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V, the transformation $* : V \to V^*$ given by $*(\mathbf{v}_i) = \mathbf{v}_i^*$ is an isomorphism between V and V^*. Therefore, we have the following corollary.

Corollary 4.17 *Any finite-dimensional vector space is isomorphic to its dual space.*

Example 4.24 Let $\alpha = \{(1, 2), (1, 3)\}$ be a basis for $\mathbb{R}^2$. To determine the dual basis $\alpha^* = \{f, g\}$ of α, we consider the equations

$$\begin{aligned} 1 &= f(1, 2) = f(\mathbf{e}_1) + 2f(\mathbf{e}_2) \\ 0 &= f(1, 3) = f(\mathbf{e}_1) + 3f(\mathbf{e}_2). \end{aligned}$$

Solving these equations, we obtain that $f(\mathbf{e}_1) = 3$ and $f(\mathbf{e}_2) = -1$, and $f(x, y) = 3x - y$. Similarly, it can be shown that $g(x, y) = -2x + y$. □

Example 4.25 Consider $V = \mathbb{R}^n$ with the standard basis $\alpha = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, and its dual basis $\alpha^* = \{\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*\}$ for $\mathbb{R}^{n*}$. Then for a vector $\mathbf{a} = (a_1, \ldots, a_n) = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \in \mathbb{R}^n$, we have $\mathbf{e}_i^*(\mathbf{a}) = \mathbf{e}_i^*(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = a_i$. That is,

$$\mathbf{a} = (a_1, \ldots, a_n) = (\mathbf{e}_1^*(\mathbf{a}), \ldots, \mathbf{e}_n^*(\mathbf{a})) = (\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*)(\mathbf{a}).$$

On the other hand, when we write a vector in $\mathbb{R}^n$ as $\mathbf{x} = (x_1, \ldots, x_n)$ in coordinate functions (or unknowns) x_i, it means that given a point $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ each x_i gives us the i-th coordinate of $\mathbf{a}$, that is,

$$(x_1, \ldots, x_n)(\mathbf{a}) = (x_1(\mathbf{a}), \ldots, x_n(\mathbf{a})) = (a_1, \ldots, a_n).$$

In this way, we have identified $\mathbf{e}_i^* = x_i$ for $i = 1, \ldots, n$, *i.e.*, $\mathbb{R}^{n*} = \mathbb{R}^n$. Thus, the actual meaning of the usual coordinate expression $(x_1, \ldots, x_n)$ of $\mathbf{x}$ is just a vector in $\mathbb{R}^{n*}$ such that $(x_1, \ldots, x_n)(\mathbf{a}) = (a_1, \ldots, a_n)$ for a point $\mathbf{a} \in \mathbb{R}^n$. □

Now, consider two vector spaces V and W with fixed bases α and β, respectively. Let $S : V \to W$ be a linear transformation from V to W. Then for any linear functional $g \in W^*$, *i.e.*, $g : W \to \mathbb{R}$, it is easy to see that the composition $g \circ S(\mathbf{x}) = g(S(\mathbf{x}))$ for $\mathbf{x} \in V$ defines a linear functional on V, *i.e.*, $g \circ S \in V^*$. Thus, we have a transformation $S^* : W^* \to V^*$ defined by $S^*(g) = g \circ S$ for $g \in W^*$.

Theorem 4.18 *The mapping $S^* : W^* \to V^*$ defined by $S^*(g) = g \circ S$ for $g \in W^*$ is a linear transformation and $[S^*]_{\beta^*}^{\alpha^*} = \left([S]_\alpha^\beta\right)^T$.*

Proof: The mapping S^* is clearly linear by the definition of a composition of functions. Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be bases for V and W with their dual bases $\alpha^* = \{\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*\}$ and $\beta^* = \{\mathbf{w}_1^*, \ldots, \mathbf{w}_m^*\}$, respectively. Let $[S]_\alpha^\beta = [a_{ij}]$ and $[S^*]_{\beta^*}^{\alpha^*} = [b_{k\ell}]$. Then,

$$S(\mathbf{v}_i) = \sum_{k=1}^m a_{ki}\mathbf{w}_k \quad \text{and} \quad S^*(\mathbf{w}_j^*) = \sum_{i=1}^n b_{ij}\mathbf{v}_i^*,$$

for $1 \le i \le n$ and $1 \le j \le m$. Thus,

$$\begin{aligned} b_{ij} &= S^*(\mathbf{w}_j^*)(\mathbf{v}_i) = (\mathbf{w}_j^* \circ S)(\mathbf{v}_i) \\ &= \mathbf{w}_j^*(S(\mathbf{v}_i)) = \mathbf{w}_j^*\left(\sum_{k=1}^m a_{ki}\mathbf{w}_k\right) = \sum_{k=1}^m a_{ki}\mathbf{w}_j^*(\mathbf{w}_k) = a_{ji}. \end{aligned}$$

Hence, we get $[S^*]_{\beta^*}^{\alpha^*} = \left([S]_\alpha^\beta\right)^T$. □

Remark: Theorem 4.18 shows that the matrix representation of S^* is just the transpose of that of S. And hence, the linear transformation S^* is called the **transpose** (or *adjoint*) of S, denoted also by S^T.

Example 4.26 With the identification $\mathbb{R}^{n*} = \mathbb{R}^n$ in Example 4.25, the transpose A^T of a matrix A is actually A^*:

$$A^T = A^* : \mathbb{R}^{m*} \to \mathbb{R}^{n*}. \qquad \square$$

For two linear transformations $S : U \to V$ and $T : V \to W$, it is quite easy to show (the readers may try to) that

$$(T \circ S)^* = S^* \circ T^*.$$

Thus, if $S : V \to W$ is an isomorphism, then so is its transpose $S^* : W^* \to V^*$. In particular, since $* : V \to V^*$ is an isomorphism, so is its transpose $** : V^* \to V^{**}$. Note that even though the isomorphism $* : V \to V^*$ depends on a choice of a basis for V, there is an isomorphism between V and V^{**} that does not depend on a choice of bases for the two vector spaces: We first define, for each $\mathbf{x} \in V$, $\tilde{\mathbf{x}} : V^* \to \mathbb{R}$ by $\tilde{\mathbf{x}}(f) = f(\mathbf{x})$ for every $f \in V^*$. It is easy to verify that $\tilde{\mathbf{x}}$ is a linear functional on V^*, so $\tilde{\mathbf{x}} \in V^{**}$. We will show below that the mapping $\Phi : V \to V^{**}$ defined by $\Phi(\mathbf{x}) = \tilde{\mathbf{x}}$ is the desired isomorphism between V and V^{**}.

Lemma 4.19 *If $\tilde{\mathbf{x}}(f) = \mathbf{0}$ for all $f \in V^*$, i.e., $\tilde{\mathbf{x}} = \mathbf{0}$ in V^{**}, then $\mathbf{x} = \mathbf{0}$.*

Proof: Suppose that $\mathbf{x} \neq \mathbf{0}$. Choose a basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V with $\mathbf{v}_1 = \mathbf{x}$. Let $\alpha^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_n^*\}$ be the dual basis of α. Then

$$\tilde{\mathbf{x}}(\mathbf{v}_1^*) = \mathbf{v}_1^*(\mathbf{x}) = \mathbf{v}_1^*(\mathbf{v}_1) = 1,$$

which contradicts the hypothesis. $\square$

Theorem 4.20 *The mapping $\Phi : V \to V^{**}$ defined by $\Phi(\mathbf{x}) = \tilde{\mathbf{x}}$ is an isomorphism from V to V^{**}.*

Proof: To show the linearity of Φ, let $\mathbf{x},\ \mathbf{y} \in V$ and k a scalar. Then, for any $f \in V^*$,

$$\begin{aligned} \Phi(\mathbf{x} + k\mathbf{y})(f) &= (\widetilde{\mathbf{x} + k\mathbf{y}})(f) = f(\mathbf{x} + k\mathbf{y}) \\ &= f(\mathbf{x}) + kf(\mathbf{y}) = \tilde{\mathbf{x}}(f) + k\tilde{\mathbf{y}}(f) \\ &= (\tilde{\mathbf{x}} + k\tilde{\mathbf{y}})(f) = (\Phi(\mathbf{x}) + k\Phi(\mathbf{y}))\,(f). \end{aligned}$$

Hence, $\Phi(\mathbf{x} + k\mathbf{y}) = \Phi(\mathbf{x}) + k\Phi(\mathbf{y})$. The injectivity of Φ comes from Lemma 4.19. Since $\dim V = \dim V^{**}$, Φ is an isomorphism. $\square$

Problem 4.22 Let $\alpha = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$ be a basis for $\mathbb{R}^3$. Find the dual basis α^*.

Problem 4.23 Let $V = \mathbb{R}^3$ and define $f_i \in V^*$ as follows:

$$f_1(x, y, z) = x - 2y, \quad f_2(x, y, z) = x + y + z, \quad f_3(x, y, z) = y - 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual.

4.9 Exercises

4.1. Which of the following functions T are linear transformations?
(1) $T(x, y) = (x^2 - y^2, x^2 + y^2)$.
(2) $T(x, y, z) = (x + y, 0, 2x + 4z)$.
(3) $T(x, y) = (\sin x, y)$.
(4) $T(x, y) = (x + 1, 2y, x + y)$.
(5) $T(x, y, z) = (|x|, 0)$.

4.2. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation such that $T(1) = 1$; $T(x) = x^2$, and $T(x^2) = x^3 + x$. Find $T(ax^2 + bx + c)$.

4.3. Find $S \circ T$ and/or $T \circ S$ whenever it is defined.
(1) $T(x, y, z) = (x - y + z, x + z)$, $S(x, y) = (x, x - y, y)$;
(2) $T(x, y) = (x, 3y + x, 2x - 4y, y)$, $S(x, y, z) = (2x, y)$;

4.4. Let $S : C(\mathbb{R}) \to C(\mathbb{R})$ be the function on the vector space $C(\mathbb{R})$ defined by, for $f \in C(\mathbb{R})$,

$$S(f)(x) = f(x) - \int_1^x uf(u)du.$$

Show that S is a linear transformation on the vector space $C(\mathbb{R})$.

4.5. Let T be a linear transformation on a vector space V such that $T^2 = Id$ and $T \neq Id$. Let $U = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{v}\}$ and $W = \{\mathbf{v} \in V : T(\mathbf{v}) = -\mathbf{v}\}$. Show that
(1) at least one of U and W is a nonzero subspace of V;
(2) $U \cap W = \{\mathbf{0}\}$;
(3) $V = U + W$.

4.6. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (2x - z, 3x - 2y, x - 2y + z)$,
(1) determine the null space $\mathcal{N}(T)$ of T,
(2) determine whether T is one-to-one,
(3) find a basis for $\mathcal{N}(T)$.

4.7. Show that each of the following linear transformations T on $\mathbb{R}^3$ is invertible, and find a formula for T^{-1}:

(1) $T(x,\ y,\ z) = (3x,\ x - y,\ 2x + y + z)$.

(2) $T(x,\ y,\ z) = (2x,\ 4x - y,\ 2x + 3y - z)$.

4.8. Let $S, T : V \to V$ be linear transformations of a vector space V.

(1) Show that if $T \circ S$ is one-to-one, then T is an isomorphism.

(2) Show that if $T \circ S$ is onto, then T is an isomorphism.

(3) Show that if T^k is an isomorphism for some positive k, then T is an isomorphism.

4.9. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$, and let S be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. Prove that the composition $S \circ T$ is not invertible.

4.10. Let T be a linear transformation on a vector space V satisfying $T - T^2 = Id$. Show that T is invertible.

4.11. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$Tf(x) = f''(x) - 4f'(x) + f(x).$$

Find the matrix $[T]_\alpha$ for the basis $\alpha = \{x,\ 1 + x,\ x + x^2,\ x^3\}$.

4.12. Let T be the linear transformation on $\mathbb{R}^2$ defined by $T(x,\ y) = (-y,\ x)$.

(1) What is the matrix of T with respect to an ordered basis $\alpha = \{\mathbf{v}_1,\ \mathbf{v}_2\}$, where $\mathbf{v}_1 = (1,\ 2)$, $\mathbf{v}_2 = (1,\ -1)$?

(2) Show that for every real number c the linear transformation $T - c\,Id$ is invertible.

4.13. Find the matrix representation of each of the following linear transformations T on $\mathbb{R}^2$ with respect to the standard basis $\{\mathbf{e}_1,\ \mathbf{e}_2\}$.

(1) $T(x,\ y) = (2y,\ 3x - y)$.

(2) $T(x,\ y) = (3x - 4y,\ x + 5y)$.

4.14. Let $M = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

(1) Find the unique linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ so that M is the matrix of T with respect to the bases

$$\alpha_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \alpha_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(2) Find $T(x,\ y,\ z)$.

4.15. Find the matrix representation of each of the following linear transformations T on $P_2(\mathbb{R})$ with respect to the basis $\{1,\ x,\ x^2\}$.

(1) $T : p(x) \to p(x + 1)$.

(2) $T : p(x) \to p'(x)$.
(3) $T : p(x) \to p(0)x$.
(4) $T : p(x) \to \dfrac{p(x) - p(0)}{x}$.

4.16. Consider the following ordered bases of $\mathbb{R}^3$: $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the standard basis and $\beta = \{\mathbf{u}_1 = (1, 1, 1), \mathbf{u}_2 = (1, 1, 0), \mathbf{u}_3 = (1, 0, 0)\}$.
(1) Find the transition matrix P from α to β.
(2) Find the transition matrix Q from β to α.
(3) Verify that $Q = P^{-1}$.
(4) Show that $[\mathbf{v}]_\beta = P[\mathbf{v}]_\alpha$ for any vector $\mathbf{v} \in \mathbb{R}^3$.
(5) Show that $[T]_\beta = Q^{-1}[T]_\alpha Q$ for the linear transformation T defined by $T(x, y, z) = (2y + x, x - 4y, 3x)$.

4.17. There are no matrices A and B in $M_{n\times n}(\mathbb{R})$ such that $AB - BA = I_n$.

4.18. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by
$T(x, y, x) = (3x + 2y - 4z, x - 5y + 3z)$,
and let $\alpha = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ and $\beta = \{(1, 3), (2, 5)\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.
(1) Find the associated matrix $[T]_\alpha^\beta$ for T.
(2) Verify $[T]_\alpha^\beta[\mathbf{v}]_\alpha = [T(\mathbf{v})]_\beta$ for any $\mathbf{v} \in \mathbb{R}^3$.

4.19. Find the transition matrix $[Id]_\alpha^\beta$ from α to β, when
(1) $\alpha = \{(2,3), (0,1)\}$, $\beta = \{(6,4), (4,8)\}$;
(2) $\alpha = \{(5,1), (1,2)\}$, $\beta = \{(1,0), (0,1)\}$;
(3) $\alpha = \{(1,1,1), (1,1,0), (1,0,0)\}$, $\beta = \{(2,0,3), (-1,4,1), (3,2,5)\}$;
(4) $\alpha = \{t, 1, t^2\}$, $\beta = \{3 + 2t + t^2, t^2 - 4, 2 + t\}$.

4.20. Show that all matrices of the form $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ are similar.

4.21. Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ cannot be similar to a diagonal matrix.

4.22. Are the matrices $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ similar?

4.23. For a linear transformation T on a vector space V, show that T is one-to-one if and only if its transpose T^* is one-to-one.

4.24. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T(x, y, z) = (2y + z, -x + 4y + z, x + z)$.
Compute $[T]_\alpha$ and $[T^*]_{\alpha^*}$ for the standard basis $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

4.25. Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by
$T(x_1,\ x_2,\ x_3) = (x_1 + x_2,\ 2x_3 - x_1)$.
(1) For the standard ordered bases α and β for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, find the associated matrix for T with respect to the bases α and β.
(2) Let $\alpha = \{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3\}$ and $\beta = \{\mathbf{y}_1,\ \mathbf{y}_2\}$, where $\mathbf{x}_1 = (1, 0, -1)$, $\mathbf{x}_2 = (1, 1, 1)$, $\mathbf{x}_3 = (1, 0, 0)$, and $\mathbf{y}_1 = (0, 1)$, $\mathbf{y}_2 = (1, 0)$. Find the associated matrices $[T]_\alpha^\beta$ and $[T^*]_{\beta^*}^{\alpha^*}$.

4.26. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by
$T(x,\ y,\ z) = (2x + y + 4z,\ x + y + 2z,\ y + 2z,\ x + y + 3z)$.
Find the range and the kernel of T. What is the dimension of $\mathcal{C}(T)$? Find $[T]_\alpha^\beta$ and $[T^*]_{\beta^*}^{\alpha^*}$, where
$\alpha = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$
$\beta = \{(1, 0, 0, 0),\ (1, 1, 0, 0),\ (1, 1, 1, 0),\ (1, 1, 1, 1)\}$.

4.27. Let T be the linear transformation on $V = \mathbb{R}^3$, for which the associated matrix with respect to the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Find the bases for the range and the null space of the transpose T^* on V^*.

4.28. Define three linear functionals on the vector space $P_2(\mathbb{R})$ by
$f_1(p) = \int_0^1 p(x)dx,\quad f_2(p) = \int_0^2 p(x)dx,\quad f_3(p) = \int_0^{-1} p(x)dx.$
Show that $\{f_1,\ f_2,\ f_3\}$ is a basis for V^* by finding its dual basis for V.

4.29. Determine whether or not the following statements are true in general, and justify your answers.
(1) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\operatorname{Ker}(T) = \{\mathbf{0}\}$ if $m > n$.
(2) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\operatorname{Ker}(T) \neq \{\mathbf{0}\}$ if $m < n$.
(3) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if the nullspace of $[T]_\alpha^\beta$ is $\{\mathbf{0}\}$, for any bases α and β of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.
(4) For a linear transformation T on $\mathbb{R}^n$, the dimension of the image of T is equal to that of the row space of $[T]_\alpha$ for any basis α for $\mathbb{R}^n$.
(5) Any polynomial $p(x)$ is linear if and only if the degree of $p(x)$ is 1.
(6) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a function given as $T(\mathbf{x}) = (T_1(\mathbf{x}),\ T_2(\mathbf{x}))$ for any $\mathbf{x} \in \mathbb{R}^3$. Then T is linear if and only if their coordinate functions T_i, $i = 1,\ 2$, are linear.
(7) For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, if $[T]_\alpha^\beta = I_n$ for some bases α and β of $\mathbb{R}^n$, then T must be the identity transformation.
(8) If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, then any matrix representation of T is nonsingular.
(9) Any $m \times n$ matrix A can be a matrix representation of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

Chapter 5

Inner Product Spaces

5.1 Inner products

In order to study the geometry of a vector space, we go back to the case of the Euclidean 3-space $\mathbb{R}^3$. Recall that the **dot** (or **Euclidean inner**) **product** of two vectors $\mathbf{x} = (x_1,\ x_2,\ x_3)$ and $\mathbf{y} = (y_1,\ y_2,\ y_3)$ in $\mathbb{R}^3$ is defined by the formula

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 = \mathbf{x}^T\mathbf{y},$$

where $\mathbf{x}^T\mathbf{y}$ is the matrix product of $\mathbf{x}^T$ and $\mathbf{y}$. Using the dot product, the **length** (or **magnitude**) of a vector $\mathbf{x} = (x_1,\ x_2,\ x_3)$ is defined by

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

and the **distance** of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

In this way, the dot product can be considered to be a ruler for measuring the length of a line segment in $\mathbb{R}^3$. Furthermore, it can also be used to measure the angle between two vectors: in fact, the **angle** θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ is measured by the formula involving the dot product

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|},\quad 0 \le \theta \le \pi,$$

since the dot product satisfies the formula

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta.$$

In particular, two vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal (*i.e.*, they form a right angle $\theta = \pi/2$) if and only if the Pythagorean theorem holds:

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x}+\mathbf{y}\|^2.$$

By rewriting this formula in terms of the dot product, we obtain another equivalent condition:

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

In fact, this dot product is one of the most important structures with which $\mathbb{R}^3$ is equipped. Euclidean geometry begins with the vector space $\mathbb{R}^3$ together with the dot product, because the Euclidean distance can be defined by this dot product.

The dot product has a direct extension to the n-space $\mathbb{R}^n$ for any positive integer n, *i.e.*, for vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the dot product, also called the **Euclidean inner product**, and the **length** (or **magnitude**) of a vector are defined similarly as

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= x_1y_1 + \cdots + x_ny_n = \mathbf{x}^T\mathbf{y}, \\ \|\mathbf{x}\| &= (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \sqrt{x_1^2 + \cdots + x_n^2}. \end{aligned}$$

In order to extend this notion of dot product to vector spaces in general, we extract the most essential properties that the dot product in $\mathbb{R}^n$ satisfies and take these properties as axioms for an inner product of a vector space V. First of all, we note that it is a rule that assigns a real number $\mathbf{x} \cdot \mathbf{y}$ to each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, and the essential rules it satisfies are those in the following definition.

Definition 5.1 An **inner product** on a real vector space V is a function that associates a real number $\langle \mathbf{x}, \mathbf{y} \rangle$ to each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V in such a way that the following rules are satisfied for all vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in V and all scalars k in $\mathbb{R}$:

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry),

(2) $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (additivity),

(3) $\langle k\mathbf{x}, \mathbf{y} \rangle = k\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity),

(4) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$ (positive definiteness).

A pair $(V, \langle \, , \rangle)$ of a (real) vector space V and an inner product $\langle \, , \rangle$ is called a **(real) inner product space**. In particular, the pair $(\mathbb{R}^n, \cdot)$ is called the **Euclidean n-space**.

Note that by symmetry (1), additivity (2) and homogeneity (3) also hold for the second variable: *i.e.*,

(2′) $\langle \mathbf{x}, \mathbf{y}+\mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{x}, \mathbf{z}\rangle$,

(3′) $\langle \mathbf{x}, k\mathbf{y}\rangle = k\langle \mathbf{x}, \mathbf{y}\rangle$.

Now it is easy to show that $\langle \mathbf{0}, \mathbf{y}\rangle = 0\langle \mathbf{0}, \mathbf{y}\rangle = 0$, and $\langle \mathbf{x}, \mathbf{0}\rangle = 0$.

Example 5.1 For vectors $\mathbf{x} = (x_1,\ x_2)$ and $\mathbf{y} = (y_1,\ y_2)$ in $\mathbb{R}^2$, define

$$\langle \mathbf{x}, \mathbf{y}\rangle = ax_1y_1 + cx_1y_2 + cx_2y_1 + bx_2y_2,$$

where a, b and c are arbitrary real numbers. Then this function $\langle\ ,\ \rangle$ clearly satisfies the first three rules of the inner product. Moreover, if $a > 0$ and $ab - c^2 > 0$ hold, then it also satisfies rule (4), the positive definiteness of the inner product. (Hint: The equation $\langle \mathbf{x}, \mathbf{x}\rangle = ax_1^2 + 2cx_1x_2 + bx_2^2 \geq 0$ if and only if either $x_2 = 0$ or the discriminant of $\langle \mathbf{x}, \mathbf{x}\rangle/x_2^2$ is nonpositive.) Note that the equation can be written as matrix products:

$$\langle \mathbf{x}, \mathbf{y}\rangle = [x_1\ x_2]\begin{bmatrix} a & c \\ c & b\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} = \mathbf{x}^T A\mathbf{y}.$$

In the case of $c = 0$, this reduces to $\langle \mathbf{x}, \mathbf{y}\rangle = ax_1y_1 + bx_2y_2$. Notice also that $a = \langle \mathbf{e}_1, \mathbf{e}_1\rangle$, $b = \langle \mathbf{e}_2, \mathbf{e}_2\rangle$ and $c = \langle \mathbf{e}_1, \mathbf{e}_2\rangle = \langle \mathbf{e}_2, \mathbf{e}_1\rangle$. □

Example 5.2 Let $V = C[0,\ 1]$ be the vector space of all real-valued continuous functions on $[0,\ 1]$. For any two functions $f(x)$ and $g(x)$ in V, define

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx\,.$$

Then $\langle\ ,\ \rangle$ is an inner product on V (verify this). Let

$$f(x) = \begin{cases} 1-2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $f \neq \mathbf{0} \neq g$, but $\langle f, g\rangle = 0$. □

By a subspace W of an inner product space V, we mean a subspace of the vector space V together with the inner product that is the restriction of the inner product of V to W.

Example 5.3 The set $W = D^1[0,\ 1]$ of all real-valued *differentiable* functions on $[0,\ 1]$ is a subspace of $V = C[0,\ 1]$. The restriction to W of the inner product on V defined in Example 5.2 makes W an inner product subspace of V. However, suppose we define another inner product on W by the following formula: For any two functions $f(x)$ and $g(x)$ in W,

$$\langle\langle f, g\rangle\rangle = \int_0^1 f(x)g(x)dx + \int_0^1 f'(x)g'(x)dx\,.$$

Then $\langle\langle\ ,\ \rangle\rangle$ is also an inner product on W but is not defined on V. This means that this inner product is quite different from the restriction of the inner product of V to W, and hence W with this new inner product is not a subspace of the space V as an inner product space. □

5.2 The lengths and angles of vectors

The following inequality will enable us to define an angle between two vectors in an inner product space V.

Theorem 5.1 (Cauchy-Schwarz inequality) *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in an inner product space* V*, then*

$$\langle \mathbf{x}, \mathbf{y}\rangle^2 \le \langle \mathbf{x}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{y}\rangle.$$

Proof: If $\mathbf{x} = \mathbf{0}$, it is clear. Assume $\mathbf{x} \neq \mathbf{0}$. For any scalar t, we have

$$0 \le \langle t\mathbf{x}+\mathbf{y}, t\mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle t^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle t + \langle \mathbf{y}, \mathbf{y}\rangle.$$

This inequality implies that the polynomial $\langle \mathbf{x}, \mathbf{x}\rangle t^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle t + \langle \mathbf{y}, \mathbf{y}\rangle$ in t has either no real roots or a repeated real root. Therefore, its discriminant must be nonpositive:

$$\langle \mathbf{x}, \mathbf{y}\rangle^2 - \langle \mathbf{x}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{y}\rangle \le 0,$$

which implies the inequality. □

Problem 5.1 Prove that equality in the Cauchy-Schwarz inequality holds if and only if the vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

The lengths and angles of vectors in an inner product space are defined similarly to the case of the Euclidean n-space.

Definition 5.2 Let V be an inner product space. Then the **magnitude** or the **length** of a vector $\mathbf{x}$, denoted by $\|\mathbf{x}\|$, is defined by

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

The **distance** between two vectors $\mathbf{x}$ and $\mathbf{y}$, denoted by $d(\mathbf{x}, \mathbf{y})$, is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

From the Cauchy-Schwarz inequality, we have

$$-1 \leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1.$$

Hence, there is a unique number $\theta \in [0, \pi]$ such that $\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}$.

Definition 5.3 The real number θ in the interval $[0,\ \pi]$ that satisfies

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}, \quad \text{or} \quad \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta,$$

is called the **angle** between $\mathbf{x}$ and $\mathbf{y}$.

Example 5.4 In $\mathbb{R}^2$ equipped with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 3x_2y_2$, the angle between $\mathbf{x} = (1, 2)$ and $\mathbf{y} = (1, 0)$ is computed as

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{2}{\sqrt{14 \cdot 2}}.$$

Thus $\theta = \cos^{-1}(\frac{1}{\sqrt{7}})$. □

Problem 5.2 Prove the following properties of length in an inner product space V: For any vectors $\mathbf{x},\ \mathbf{y} \in V$,

(1) $\|\mathbf{x}\| \geq 0$,
(2) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,
(3) $\|k\mathbf{x}\| = |k|\,\|\mathbf{x}\|$,
(4) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangular inequality).

Problem 5.3 Let V be an inner product space. Show that for any vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in V,

(1) $d(\mathbf{x}, \mathbf{y}) \geq 0$,
(2) $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$,
(3) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$,
(4) $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ (triangular inequality).

Therefore, an inner product in the 3-space $\mathbb{R}^3$ may play the roles of a ruler and a protractor in our physical world.

Definition 5.4 Two vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space are said to be **orthogonal** (or **perpendicular**) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Note that for nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ if and only if $\theta = \pi/2$.

Lemma 5.2 *Let V be an inner product space and let $\mathbf{x} \in V$. Then the vector $\mathbf{x}$ is orthogonal to every vector $\mathbf{y}$ in V (i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y}$ in V) if and only if $\mathbf{x} = \mathbf{0}$.*

Proof: If $\mathbf{x} = \mathbf{0}$, clearly $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y}$ in V. Suppose that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y}$ in V. Then $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ in particular. The positive definiteness of the inner product implies that $\mathbf{x} = \mathbf{0}$. □

Corollary 5.3 *Let V be an inner product space, and let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Then a vector $\mathbf{x}$ in V is orthogonal to every basis vector $\mathbf{v}_i$ in α if and only if $\mathbf{x} = \mathbf{0}$.*

Proof: If $\langle \mathbf{x}, \mathbf{v}_i \rangle = 0$ for $i = 1, \ldots, n$, then $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} y_i \langle \mathbf{x}, \mathbf{v}_i \rangle = 0$ for any $\mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{v}_i \in V$. □

Example 5.5 (Pythagorean theorem) Let V be an inner product space, and let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors in V with the angle θ. Then, $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta$ gives the equality

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| \cos\theta.$$

Moreover, it deduces the Pythagorean theorem: $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ for any orthogonal vector $\mathbf{x}$ and $\mathbf{y}$. □

Theorem 5.4 *If* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ *nonzero vectors in an inner product space* V *are mutually orthogonal (i.e., each vector is orthogonal to every other vector), then they are linearly independent.*

Proof: Suppose $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$. Then for each $i = 1, \ldots, k$,

$$\begin{aligned} 0 &= \langle \mathbf{0}, \mathbf{x}_i \rangle = \langle c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k, \mathbf{x}_i \rangle \\ &= c_1\langle \mathbf{x}_1, \mathbf{x}_i \rangle + \cdots + c_i\langle \mathbf{x}_i, \mathbf{x}_i \rangle + \cdots + c_k\langle \mathbf{x}_k, \mathbf{x}_i \rangle \\ &= c_i\|\mathbf{x}_i\|^2, \end{aligned}$$

because $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are mutually orthogonal. Since each $\mathbf{x}_i$ is not the zero vector, $\|\mathbf{x}_i\| \neq 0$; so $c_i = 0$ for $i = 1, \ldots, k$. $\square$

Problem 5.4 Let $f(x)$ and $g(x)$ be continuous real-valued functions on $[0, 1]$. Prove

(1) $\left[\int_0^1 f(x)g(x)dx\right]^2 \leq \left[\int_0^1 f^2(x)dx\right]\left[\int_0^1 g^2(x)dx\right]$,

(2) $\left[\int_0^1 (f(x)+g(x))^2 dx\right]^{\frac{1}{2}} \leq \left[\int_0^1 f^2(x)dx\right]^{\frac{1}{2}} + \left[\int_0^1 g^2(x)dx\right]^{\frac{1}{2}}$.

5.3 Matrix representations of inner products

As we saw at the end of Example 5.1, the inner product on an inner product space $(V, \langle,\rangle)$ can be expressed in terms of a symmetric matrix. In fact, let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a fixed ordered basis for V. Then for any $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{v}_i$ and $\mathbf{y} = \sum_{j=1}^n y_j\mathbf{v}_j$ in V,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$

holds. If we set $a_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$ for $i, j = 1, \ldots, n$, then these numbers constitute a symmetric matrix $A = [a_{ij}]$, since $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_j, \mathbf{v}_i \rangle$. Thus, in matrix notation, the inner product may be written as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j a_{ij} = [\mathbf{x}]_\alpha^T A [\mathbf{y}]_\alpha.$$

The matrix A is called the **matrix representation** of the inner product with respect to α.

Example 5.6 **(1)** With respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of the Euclidean n-space $\mathbb{R}^n$, the matrix representation of the dot product is the identity matrix, since $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Thus for $\mathbf{x} = \sum x_i\mathbf{e}_i$, $\mathbf{y} = \sum y_j\mathbf{e}_j \in \mathbb{R}^n$ the dot product is the matrix product $\mathbf{x}^T\mathbf{y}$:

$$\mathbf{x}\cdot\mathbf{y} = [x_1 \cdots x_n]\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [x_1 \cdots x_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \mathbf{x}^T\mathbf{y}.$$

(2) Let $V = P_2(\mathbb{R})$, and define an inner product of V as

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

Then for a basis $\alpha = \{f_1(x) = 1,\ f_2(x) = x,\ f_3(x) = x^2\}$ for V, one can easily find $A = [a_{ij}]$: for instance,

$$a_{23} = \langle f_2, f_3\rangle = \int_0^1 f_2(x)f_3(x)dx = \int_0^1 x\cdot x^2dx = \frac{1}{4}. \qquad \square$$

The expression of the dot product as a matrix product is very useful in stating and proving theorems in the Euclidean space.

On the other hand, for any symmetric matrix A and for a fixed basis α, the formula $\langle\mathbf{x}, \mathbf{y}\rangle = [\mathbf{x}]_\alpha^T A[\mathbf{y}]_\alpha$ seems to give rise to an inner product on V. In fact, the formula clearly satisfies the first three rules in the definition of the inner product, but not necessarily the fourth rule, positive definiteness. The following theorem gives a necessary condition for a symmetric matrix A to give rise to an inner product. Some necessary and sufficient conditions will be discussed in Chapter 8.

Theorem 5.5 *The matrix representation A of an inner product (with respect to any basis) on a vector space V is invertible.*

Proof: It is enough to show that the column vectors of A are linearly independent. Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for an inner product space V. We denote the column vectors of $A = [a_{ij}] = [\langle\mathbf{v}_i, \mathbf{v}_j\rangle]$ by $\mathbf{a}_j$ for $j = 1, \cdots, n$. Consider the linear dependence of the column vectors of A: for $c_1, \cdots, c_n \in \mathbb{R}$,

$$\mathbf{0} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n,$$

Let $\mathbf{c} = \sum_{i=1}^n c_i\mathbf{v}_i \in V$ so that $[\mathbf{c}]_\alpha = [c_1 \ \cdots \ c_n]^T$. Then this equation becomes a homogeneous system $\mathbf{0} = A[\mathbf{c}]_\alpha$ of n linear equations in n unknowns:

$$\begin{cases} 0 = a_{11}c_1 + \cdots + a_{1n}c_n = \sum_{j=1}^n a_{1j}c_j = [\mathbf{v}_1]_\alpha^T A[\mathbf{c}]_\alpha = \langle \mathbf{v}_1, \mathbf{c}\rangle, \\ \quad \vdots \\ 0 = a_{n1}c_1 + \cdots + a_{nn}c_n = \sum_{j=1}^n a_{nj}c_j = [\mathbf{v}_n]_\alpha^T A[\mathbf{c}]_\alpha = \langle \mathbf{v}_n, \mathbf{c}\rangle, \end{cases}$$

where we used $[\mathbf{v}_i]_\alpha = \mathbf{e}_i$. Thus, by Corollary 5.3, we get $\mathbf{c} = \sum_{i=1}^n c_i\mathbf{v}_i = \mathbf{0}$, and the columns of A are linearly independent. □

Recall that the conditions $a > 0$ and $ab - c^2 > 0$ in (2) of Example 5.1 are sufficient for A to give rise to an inner product on $\mathbb{R}^2$.

The standard basis of the Euclidean n-space $\mathbb{R}^n$ has a special property: The basis vectors are mutually orthogonal and are of length 1. In this sense, it is called the **rectangular coordinate system** for $\mathbb{R}^n$. In an inner product space, a vector with length 1 is called a **unit vector**. If $\mathbf{x}$ is a nonzero vector in an inner product space V, the vector $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a unit vector. The process of obtaining a unit vector from a nonzero vector by multiplying by the inverse of its length is called a **normalization**. Thus, if there is a set of vectors (or a basis) in an inner product space consisting of mutually orthogonal vectors, then the vectors can be converted to unit vectors by normalizing them without losing their mutual orthogonality.

Problem 5.5 Normalize each of the following vectors in the Euclidean space $\mathbb{R}^3$:
(1) $\mathbf{u} = (2,\ 1,\ -1)$, (2) $\mathbf{v} = (1/2,\ 1/3,\ -1/4)$.

Definition 5.5 A set of vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_k$ in an inner product space V is said to be **orthonormal** if

$$\langle \mathbf{x}_i, \mathbf{x}_j\rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonality)}, \\ 1 & \text{if } i = j \quad \text{(normality)}. \end{cases}$$

A set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n\}$ of vectors is called an **orthonormal basis** for V if it is a basis and orthonormal.

It will be shown later that any inner product space has an orthonormal basis, just like the standard basis for the Euclidean n-space $\mathbb{R}^n$.

Problem 5.6 Determine whether each of the following sets of vectors in $\mathbb{R}^2$ is orthogonal, orthonormal, or neither with respect to the Euclidean inner product.

$$(1)\ \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}\right\} \quad (2)\ \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}\right\}$$

$$(3)\ \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\} \quad (4)\ \left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right\}$$

The next theorem shows a simple expression of a vector in terms of an orthonormal basis.

Theorem 5.6 *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is an orthonormal basis for an inner product space* V *and* $\mathbf{x}$ *is any vector in* V*, then*

$$\mathbf{x} = \langle\mathbf{x}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle\mathbf{x}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots + \langle\mathbf{x}, \mathbf{v}_n\rangle\mathbf{v}_n.$$

Proof: For any vector $\mathbf{x} \in V$, we can write $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$, as a linear combination of basis vectors. However, for each $i = 1, \ldots, n$,

$$\begin{aligned}
\langle\mathbf{x}, \mathbf{v}_i\rangle &= \langle x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n,\ \mathbf{v}_i\rangle \\
&= x_1\langle\mathbf{v}_1, \mathbf{v}_i\rangle + \cdots + x_i\langle\mathbf{v}_i, \mathbf{v}_i\rangle + \cdots + x_n\langle\mathbf{v}_n, \mathbf{v}_i\rangle \\
&= x_i,
\end{aligned}$$

because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is orthonormal. □

In an inner product space, the coordinate expression of a vector depends on the choice of an ordered basis, and the inner product is just a matrix product of the coordinate vectors with respect to an ordered basis involving some symmetric matrix between them, as we have seen already.

Actually, we will show in Theorem 5.12 in the following section that every inner product space V has an orthonormal basis, say $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then the matrix representation $A = [a_{ij}]$ of the inner product with respect to the orthonormal basis α is the identity matrix, since $a_{ij} = \langle\mathbf{v}_i, \mathbf{v}_j\rangle = \delta_{ij}$. Thus for any vector $\mathbf{x} = \sum x_i\mathbf{v}_i$ and $\mathbf{y} = \sum y_i\mathbf{v}_i$ in V,

$$\langle\mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij}x_iy_j = \sum_{i=1}^{n} x_iy_i.$$

This expression looks like the dot product in the Euclidean space $\mathbb{R}^n$. Thus *any inner product on* V *can be written just like the dot product in* $\mathbb{R}^n$*, if* V *is equipped with an orthonormal basis.*

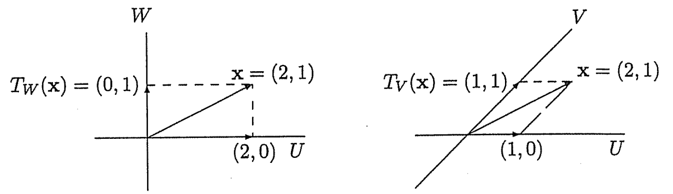
5.4 Orthogonal projections

Let U be a subspace of a vector space V. Then by Corollary 3.13 there is another subspace W of V such that $V = U \oplus W$, so that any $\mathbf{x} \in V$ has a unique expression as $\mathbf{x} = \mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$ and $\mathbf{w} \in W$. As an easy exercise, one can show that a function $T : V \to V$ defined by $T(\mathbf{x}) = T(\mathbf{u}+\mathbf{w}) = \mathbf{u}$ is a linear transformation, whose image $\mathrm{Im}(T) = T(V)$ is the subspace U and kernel $\mathrm{Ker}(T)$ is the subspace W.

Definition 5.6 Let U and W be subspaces of a vector space V. A linear transformation $T : V \to V$ is called the **projection** of V onto the subspace U along W if $V = U \oplus W$ and $T(\mathbf{x}) = \mathbf{u}$ for $\mathbf{x} = \mathbf{u} + \mathbf{w} \in U \oplus W$.

Note that for a given subspace U of V, there exist many projections T depending on the choice of a complementary subspace W of U. However, if we fix a complementary subspace W of U, then a projection T onto U is uniquely determined and by definition $T(\mathbf{u}) = \mathbf{u}$ for any $\mathbf{u} \in U$ and for any choice of W. In other words, $T \circ T = T$ for any projection T of V.

Example 5.7 Let U, V and W be the 1-dimensional subspaces of the Euclidean 2-space $\mathbb{R}^2$ spanned by the vectors $\mathbf{u} = \mathbf{e}_1$, $\mathbf{w} = \mathbf{e}_2$, and $\mathbf{v} = (1, 1)$, respectively.



Since the pairs $\{\mathbf{u},\ \mathbf{w}\}$ and $\{\mathbf{u},\ \mathbf{v}\}$ are linearly independent, the space $\mathbb{R}^2$ can be expressed as the direct sum in two ways: $\mathbb{R}^2 = U \oplus W = U \oplus V$. Thus a vector $\mathbf{x} = (2, 1) \in \mathbb{R}^2$ may be written in two ways:

$$\mathbf{x} = (2,1) = \begin{cases} 2(1,0) + (0,1) \in U \oplus W = \mathbb{R}^2, & \text{or} \\ (1,0) + (1,1) \in U \oplus V = \mathbb{R}^2. \end{cases}$$

Let T_W and T_V denote the projections of $\mathbb{R}^2$ onto W and V along U, respectively. Then

$$T_W(\mathbf{x}) = (0,1) \in V, \quad \text{and} \quad T_V(\mathbf{x}) = (1,1) \in W.$$

It also shows that a projection of $\mathbb{R}^2$ onto the subspace U (= the x-axis) depends on a choice of complementary subspace of U. □

The following shows an algebraic characterization of a projection.

Theorem 5.7 *A linear transformation $T : V \to V$ is a projection onto a subspace U if and only if $T = T^2$* ($= T \circ T$, by definition).

Proof: The necessity is clear, because $T \circ T = T$ for any projection T.

For e sufficiency, let $T^2 = T$. We want to show $V = \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$ and $T(\mathbf{u}+\mathbf{w}) = \mathbf{u}$ for $\mathbf{u}+\mathbf{w} \in \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$. For the first one, we need to prove $\mathrm{Im}(T) \cap \mathrm{Ker}(T) = \{\mathbf{0}\}$ and $V = \mathrm{Im}(T) + \mathrm{Ker}(T)$. Indeed, if $\mathbf{u} \in \mathrm{Im}(T) \cap \mathrm{Ker}(T)$, then there is $\mathbf{x} \in V$ such that $T(\mathbf{x}) = \mathbf{u}$ and $T(\mathbf{u}) = \mathbf{0}$. But

$$\mathbf{u} = T(\mathbf{x}) = T^2(\mathbf{x}) = T(T(\mathbf{x})) = T(\mathbf{u}) = \mathbf{0}$$

proves $\mathrm{Im}(T) \cap \mathrm{Ker}(T) = \{\mathbf{0}\}$. Note that this also shows $T(\mathbf{u}) = \mathbf{u}$ for $\mathbf{u} \in \mathrm{Im}(T)$. Then, $\dim V = \dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T))$ (see Remark (2) in page 138) implies $V = \mathrm{Im}(T) + \mathrm{Ker}(T)$. Now, note that $T(\mathbf{u}+\mathbf{w}) = T(\mathbf{u}) = \mathbf{u}$ for any $\mathbf{u}+\mathbf{w} \in \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$. □

Let $T : V \to V$ be a projection of V, so that $V = \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$. It is not difficult to show that $\mathrm{Im}(Id_V - T) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(Id_V - T) = \mathrm{Im}(T)$.

Corollary 5.8 *A linear transformation $T : V \to V$ is a projection if and only if $Id_V - T$ is a projection. Moreover, if T is the projection of V onto a subspace U along W, then $Id_V - T$ is the projection of V onto W along U.*

Proof: It is enough to show that $(Id_V - T) \circ (Id_V - T) = Id_V - T$. But

$$(Id_V - T) \circ (Id_V - T) = (Id_V - T) - (T - T^2) = Id_V - T. \qquad \square$$

Problem 5.7 Let $V = U \oplus W$. Let T_U denote the projection of V onto U along W, and T_W denote the projection of V onto W along U. Prove the following.

(1) For any $\mathbf{x} \in V$, $\mathbf{x} = T_U(\mathbf{x}) + T_W(\mathbf{x})$.
(2) $T_U \circ (Id_V - T_U) = \mathbf{0}$.
(3) $T_U \circ T_W = T_W \circ T_U = \mathbf{0}$.
(4) For any projection $T : V \to V$, $\mathrm{Im}(Id_V - T) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(Id_V - T) = \mathrm{Im}(T)$.

Now, let V be an inner product space and let U be a subspace of V. Recall that there exist many kinds of projections of V onto U depending on the choice of complementary subspace W of U. However, in an inner product space V, there is a particular choice of W, called the *orthogonal complement* of U, along which the projection onto U is called the *orthogonal projection.* Almost all projections used in linear algebra are orthogonal projections.

In an inner product space V, the orthogonality of two vectors can be extended to subspaces of V.

Definition 5.7 Let U and W be subspaces of an inner product space V.

(1) Two subspaces U and W are said to be **orthogonal**, written by $U \perp W$, if $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ for each $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

(2) The set of all vectors in V that are orthogonal to every vector in U is called the **orthogonal complement** of U, denoted by $U^\perp$, *i.e.*,

$$U^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{u} \in U\}.$$

One can easily show that $U^\perp$ is a subspace of V, and $\mathbf{v} \in U^\perp$ if and only if $\langle \mathbf{v}, \mathbf{u} \rangle = 0$ for every $\mathbf{u} \in \beta$, where β is a basis for U. Therefore, clearly $W \perp U$ if and only if $W \subseteq U^\perp$.

Problem 5.8 Show: (1) If $U \perp W$, $U \cap W = \{\mathbf{0}\}$. (2) $U \subseteq W$ if and only if $W^\perp \subseteq U^\perp$.

Theorem 5.9 *Let U be a subspace of an inner product space V. Then*

(1) $\dim U + \dim U^\perp = \dim V$.

(2) $(U^\perp)^\perp = U$.

(3) $V = U \oplus U^\perp$*: that is, for each* $\mathbf{x} \in V$*, there are unique vectors* $\mathbf{x}_U \in U$ *and* $\mathbf{x}_{U^\perp} \in U^\perp$ *such that* $\mathbf{x} = \mathbf{x}_U + \mathbf{x}_{U^\perp}$*. This is called the* **orthogonal decomposition** *of V (or of $\mathbf{x}$) by U.*

Proof: **(1)** Suppose that $\dim U = k$. Choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for U, and then extend it to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for V, where $n = \dim V$. Then $\mathbf{x} = \sum_{j=1}^n x_j \mathbf{v}_j \in U^\perp$ if and only if $0 = \langle \mathbf{x}, \mathbf{v}_i \rangle = \sum_{j=1}^n a_{ij} x_j$ for $1 \le i \le k$, where $a_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. The latter equations form a homogeneous system of k linear equations in n unknowns, that is, $U^\perp$ is precisely the null space of the $k \times n$ coefficient matrix $B = [a_{ij}]$, which is a submatrix of the matrix representation A of the inner product. Thus, by Theorem 5.5 the rows of B are linearly independent, so B is of rank k. Therefore, the null space has dimension $n - k$, or $\dim U^\perp = n - k = n - \dim U$.

(2) By definition, every vector in U is orthogonal to $U^\perp$, *i.e.*, $U \subseteq (U^\perp)^\perp$. On the other hand, by (1), $\dim (U^\perp)^\perp = n - \dim U^\perp = \dim U$. This proves that $(U^\perp)^\perp = U$.

(3) For a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for U, take any basis $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for $U^\perp$. Since $U \cap U^\perp = \{\mathbf{0}\}$, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is linearly independent, so it is a basis for V. Therefore, every vector $\mathbf{x} \in V$ has a unique expression

$$\mathbf{x} = \sum_{i=1}^k a_i \mathbf{v}_i + \sum_{j=k+1}^n b_j \mathbf{v}_j.$$

Now take $\mathbf{x}_U = \sum_1^k a_i \mathbf{v}_i \in U$ and $\mathbf{x}_{U^\perp} = \sum_{k+1}^n b_j \mathbf{v}_j \in U^\perp$. To show uniqueness, let $\mathbf{x} = \mathbf{u} + \mathbf{w}$ be another expression with $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$. Then $\mathbf{x}_U - \mathbf{u} = \mathbf{w} - \mathbf{x}_{U^\perp} \in U \cap U^\perp = \{\mathbf{0}\}$. So, $\mathbf{x}_U = \mathbf{u}$ and $\mathbf{x}_{U^\perp} = \mathbf{w}$. □

Definition 5.8 Let V be an inner product space, and let U be a subspace of V so that $V = U \oplus U^\perp$. Then the projection of V onto U along $U^\perp$ is called the **orthogonal projection** of V onto U, denoted Proj_U. For $\mathbf{x} \in V$, the component vector $\mathrm{Proj}_U(\mathbf{x}) \in U$ is called the **orthogonal projection** of $\mathbf{x}$ into U.

Example 5.8 As in Example 5.7, let U, V and W be subspaces of the Euclidean 2-space $\mathbb{R}^2$ generated by the vectors $\mathbf{u} = \mathbf{e}_1, \mathbf{v} = (1, 1)$, and $\mathbf{w} = \mathbf{e}_2$, respectively. Then clearly $W = U^\perp$ and $V \neq U^\perp$. Hence, for the projections T_V and T_W of $\mathbb{R}^2$ given in Example 5.7, the projection T_W is the orthogonal projection, but the projection T_V is not, so that $T_W = \mathrm{Proj}_W$ and $T_V \neq \mathrm{Proj}_V$. □

Theorem 5.10 *Let U be a subspace of an inner product space V, and let $\mathbf{x} \in V$. Then, the orthogonal projection* $\mathrm{Proj}_U(\mathbf{x})$ *of* $\mathbf{x}$ *satisfies*

$$\|\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})\| \le \|\mathbf{x} - \mathbf{y}\|$$

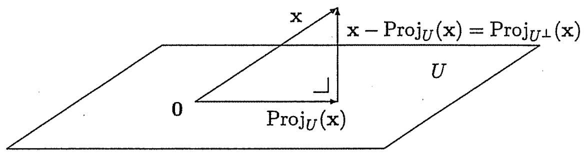
for all $\mathbf{y} \in U$*. The equality holds if and only if* $\mathbf{y} = \mathrm{Proj}_U(\mathbf{x})$.

Proof: Since $\mathbf{x} = \mathrm{Proj}_U(\mathbf{x}) + \mathrm{Proj}_{U^\perp}(\mathbf{x})$ for any vector $\mathbf{x} \in V$, $\mathbf{x} - \mathrm{Proj}_U(\mathbf{x}) = \mathrm{Proj}_{U^\perp}(\mathbf{x}) \in U^\perp$. Thus, for all $\mathbf{y} \in U$,

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|^2 &= \|(\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})) + (\mathrm{Proj}_U(\mathbf{x}) - \mathbf{y})\|^2 \\ &= \|\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})\|^2 + \|\mathrm{Proj}_U(\mathbf{x}) - \mathbf{y}\|^2 \\ &\ge \|\mathbf{x} - \mathrm{Proj}_U(\mathbf{x})\|^2, \end{aligned}$$

where the second equality comes from the Pythagorean theorem for $\mathbf{x} - \mathrm{Proj}_U(\mathbf{x}) \perp \mathrm{Proj}_U(\mathbf{x}) - \mathbf{y}$. $\square$

The theorem means that *the orthogonal projection* $\mathrm{Proj}_U(\mathbf{x})$ *of* $\mathbf{x}$ *is the unique vector in* U *that is closest to* $\mathbf{x}$ in the sense that it minimizes the distance to $\mathbf{x}$ from the vectors in U. Geometrically, the following picture depicts the vector $\mathrm{Proj}_U(\mathbf{x})$:



Problem 5.9 Let U and W be subspaces of an inner product space V. Show that
(1) $(U + W)^\perp = U^\perp \cap W^\perp$. (2) $(U \cap W)^\perp = U^\perp + W^\perp$.

Problem 5.10 Let $U \subset \mathbb{R}^4$ with the Euclidean inner product be the subspace spanned by (1, 1, 0, 0) and (1, 0, 1, 0), and $W \subset \mathbb{R}^4$ the subspace spanned by (0, 1, 0, 1) and (0, 0, 1, 1). Find a basis for and the dimension of each of the following subspaces:
(1) $U + W$, (2) $U^\perp$, (3) $U^\perp + W^\perp$, (4) $U \cap W$.

Lemma 5.11 *Let U be a subspace of an inner product space V, and let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be an orthonormal basis for U. Then, for any $\mathbf{x} \in V$, the orthogonal projection* $\mathrm{Proj}_U(\mathbf{x})$ *of* $\mathbf{x}$ *into* U *is*

$$\mathrm{Proj}_U(\mathbf{x}) = \langle \mathbf{x}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{x}, \mathbf{u}_m \rangle \mathbf{u}_m.$$

Proof: Let $\mathbf{z} = \langle \mathbf{x}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{x}, \mathbf{u}_m \rangle \mathbf{u}_m$. It is enough to show that $\mathbf{y} = \mathbf{x} - \mathbf{z}$ is orthogonal to U, because if $\mathbf{y} = \mathbf{x} - \mathbf{z} \in U^\perp$, then $\mathbf{x} = \mathbf{z} + \mathbf{y} \in U \oplus U^\perp$, so the uniqueness of this orthogonal decomposition gives $\mathbf{z} = \mathrm{Proj}_U(\mathbf{x})$. However, for each $j = 1, \dots, m$,

$$\langle \mathbf{x} - \mathbf{z}, \mathbf{u}_j \rangle = \langle \mathbf{x}, \mathbf{u}_j \rangle - \langle \mathbf{z}, \mathbf{u}_j \rangle = \langle \mathbf{x}, \mathbf{u}_j \rangle - \langle \mathbf{x}, \mathbf{u}_j \rangle = 0,$$

since $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an orthonormal basis for U. That is, the vector $\mathbf{x} - \mathbf{z} = \mathbf{x} - \sum_1^m \langle \mathbf{x}, \mathbf{u}_i \rangle \mathbf{u}_i$ is orthogonal to U. □

In particular, if $U = V$ in Lemma 5.11, then $\mathrm{Proj}_U(\mathbf{x}) = \mathbf{x}$, and we get Theorem 5.6.

A unit vector $\mathbf{u}$ in an inner product space V determines a 1-dimensional subspace $U = \{r\mathbf{u} : r \in \mathbb{R}\}$. Then, for a vector $\mathbf{x}$ in V, the orthogonal projection of $\mathbf{x}$ into U is simply

$$\mathrm{Proj}_U(\mathbf{x}) = \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u},$$

where $\langle \mathbf{x}, \mathbf{u} \rangle = \|\mathbf{x}\| \cos\theta$. On the other hand, it is quite clear that $\mathbf{y} = \mathbf{x} - \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u}$ is a vector orthogonal to $\mathbf{u}$. Thus

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u} + \mathbf{y} \in U \oplus U^\perp$$

so that $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 + |\langle \mathbf{x}, \mathbf{u} \rangle|^2$, which is just the Pythagorean theorem. In particular, if $V = \mathbb{R}^n$ the Euclidean space with the dot product, then

$$\mathrm{Proj}_U(\mathbf{x}) = (\mathbf{u} \cdot \mathbf{x})\mathbf{u} = (\mathbf{u}^T\mathbf{x})\mathbf{u} = \mathbf{u}(\mathbf{u}^T\mathbf{x}) = (\mathbf{u}\mathbf{u}^T)\mathbf{x}.$$

(Here the third equality comes from the matrix products). This equation shows that the matrix $\mathbf{u}\mathbf{u}^T$ is the matrix representation of the orthogonal projection Proj_U with respect to the standard basis for $\mathbb{R}^n$. Further discussions about matrix representations of the orthogonal projections will be given in Section 5.10.

Example 5.9 Let $P(x_0, y_0)$ be a point and $ax + by + c = 0$ a line in the $\mathbb{R}^2$ plane. One might know already from calculus that the nonzero vector $\mathbf{n} = (a, b)$ is perpendicular to the line $ax + by + c = 0$. In fact, for any two points $Q(x_1, y_1)$ and $R(x_2, y_2)$ on the line, the dot product $\overrightarrow{QR} \cdot \mathbf{n} = a(x_2 - x_1) + b(y_2 - y_1) = 0$, that is, $\overrightarrow{QR} \perp \mathbf{n}$.

For any point $P(x_0, y_0)$ in the plane $\mathbb{R}^2$, the distance d between the point $P(x_0, y_0)$ and the line $ax + by + c = 0$ is simply the length of the orthogonal projection of $\overrightarrow{QP}$ into $\mathbf{n}$, for any point $Q(x_1, y_1)$ in the line. Thus,

$$\begin{aligned} d &= \|\mathrm{Proj}_{\mathbf{n}}(\overrightarrow{QP})\| \\ &= \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|a(x_0 - x_1) + b(y_0 - y_1)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}. \end{aligned}$$

Note that the last equality is due to the fact that the point Q is on the line (*i.e.*, $ax_1 + by_1 + c = 0$). □

Problem 5.11 Let $V = P_3(\mathbb{R})$, the vector space of polynomials of degree ≤ 3 equipped with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx \quad \text{for any } f \text{ and } g \text{ in } V.$$

Let W be the subspace of V spanned by $\{1, x\}$, and define $f(x) = x^2$. Find the orthogonal projection $\mathrm{Proj}_W(f)$ of f on W.

5.5 The Gram-Schmidt orthogonalization

The construction of the orthogonal projection onto a subspace described in Section 5.4 can be used to find an orthonormal basis from any given basis, as the following example shows.

Example 5.10 Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find an orthonormal basis for the column space $\mathcal{C}(A)$ of A.

Solution: Let $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ be the column vectors of A in order from left to right. It is easily verified that they are linearly independent, so they form a basis for the 3-dimensional subspace $\mathcal{C}(A)$ of the Euclidean space $\mathbb{R}^4$, *i.e.*, the column space, but this basis is not orthonormal. To make an orthonormal basis, set

$$\mathbf{v}_1 = \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|} = \frac{\mathbf{c}_1}{2} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

which is a unit vector. Clearly, $\mathbf{v}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ span the column space $\mathcal{C}(A)$. Let W_1 denote the subspace spanned by $\mathbf{v}_1$. Then

$$\mathrm{Proj}_{W_1}(\mathbf{c}_2) = \langle \mathbf{c}_2, \mathbf{v}_1 \rangle \mathbf{v}_1 = 2\mathbf{v}_1,$$

and $\mathbf{c}_2 - \mathrm{Proj}_{W_1}(\mathbf{c}_2) = \mathbf{c}_2 - 2\mathbf{v}_1 = (0,\ 1,\ -1,\ 0)$ is a nonzero vector orthogonal to $\mathbf{v}_1$. To convert it to a unit vector, we set

$$\mathbf{v}_2 = \frac{\mathbf{c}_2 - 2\mathbf{v}_1}{\|\mathbf{c}_2 - 2\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}(0,\ 1,\ -1,\ 0) = \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right).$$

Since $\mathbf{c}_2 = 2\mathbf{v}_1 + \sqrt{2}\,\mathbf{v}_2$, we still have a spanning set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{c}_3\}$ of the column space $\mathcal{C}(A)$ and thus a basis. Let W_2 denote the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Then

$$\mathrm{Proj}_{W_2}(\mathbf{c}_3) = \langle \mathbf{c}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{c}_3, \mathbf{v}_2 \rangle \mathbf{v}_2 = 4\mathbf{v}_1 - \sqrt{2}\mathbf{v}_2,$$

so $\mathbf{c}_3 - \mathrm{Proj}_{W_2}(\mathbf{c}_3) = \mathbf{c}_3 - 4\mathbf{v}_1 + \sqrt{2}\mathbf{v}_2 = (0,\ 1,\ 1,\ -2)$ is a nonzero vector orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. In fact,

$$\begin{aligned}
\langle \mathbf{c}_3 - 4\mathbf{v}_1 + \sqrt{2}\,\mathbf{v}_2,\ \mathbf{v}_1 \rangle &= \langle \mathbf{c}_3, \mathbf{v}_1 \rangle - 4\langle \mathbf{v}_1, \mathbf{v}_1 \rangle + \sqrt{2}\,\langle \mathbf{v}_2, \mathbf{v}_1 \rangle = 0, \\
\langle \mathbf{c}_3 - 4\mathbf{v}_1 + \sqrt{2}\,\mathbf{v}_2,\ \mathbf{v}_2 \rangle &= \langle \mathbf{c}_3, \mathbf{v}_2 \rangle - 4\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \sqrt{2}\,\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 0,
\end{aligned}$$

since $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Thus we can normalize the vector $\mathbf{c}_3 - \mathrm{Proj}_{W_2}(\mathbf{c}_3)$ and set

$$\mathbf{v}_3 = \frac{\mathbf{c}_3 - \langle \mathbf{c}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{c}_3, \mathbf{v}_2 \rangle \mathbf{v}_2}{\|\mathbf{c}_3 - \langle \mathbf{c}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{c}_3, \mathbf{v}_2 \rangle \mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0,\ 1,\ 1,\ -2).$$

Then one can easily show that the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ still spans $\mathcal{C}(A)$ and forms an orthonormal basis for it. □

In fact, the orthonormalization process in Example 5.10 indicates how to prove the following general version, called the **Gram-Schmidt orthogonalization.**

Theorem 5.12 *Every inner product space has an orthonormal basis.*

Proof: [Gram-Schmidt orthogonalization process] Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for an n-dimensional inner product space V. Let

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}, \qquad \mathbf{v}_2 = \frac{\mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{v}_1\rangle \mathbf{v}_1}{\|\mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{v}_1\rangle \mathbf{v}_1\|}.$$

Of course, $\mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{v}_1\rangle \mathbf{v}_1 \neq \mathbf{0}$, because $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. Generally, we define by induction on $k = 1, 2, \ldots, n$

$$\mathbf{v}_k = \frac{\mathbf{x}_k - \langle \mathbf{x}_k, \mathbf{v}_1\rangle \mathbf{v}_1 - \langle \mathbf{x}_k, \mathbf{v}_2\rangle \mathbf{v}_2 - \cdots - \langle \mathbf{x}_k, \mathbf{v}_{k-1}\rangle \mathbf{v}_{k-1}}{\|\mathbf{x}_k - \langle \mathbf{x}_k, \mathbf{v}_1\rangle \mathbf{v}_1 - \langle \mathbf{x}_k, \mathbf{v}_2\rangle \mathbf{v}_2 - \cdots - \langle \mathbf{x}_k, \mathbf{v}_{k-1}\rangle \mathbf{v}_{k-1}\|}.$$

Thus, $\mathbf{v}_k$ is the normalized vector of $\mathbf{x}_k - \mathrm{Proj}_{W_{k-1}}(\mathbf{x}_k)$, where W_{k-1} is the subspace of V spanned by $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}\}$ (or equivalently, by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$). Then, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthonormal in the n-dimensional vector space V. Since every orthonormal set is linearly independent, it is an orthonormal basis for V. □

Here is a simpler proof of Theorem 5.9. Suppose that U is a subspace of an inner product space V. Then clearly we have $U \perp U^\perp$, by definition. To show $(U^\perp)^\perp = U$, take an orthonormal basis, say $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, for U by the Gram-Schmidt orthonormalization, and then extend it to an orthonormal basis for V, say $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$, which is always possible. Then clearly $\gamma = \{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ forms an (orthonormal) basis for $U^\perp$, which means that $(U^\perp)^\perp = U$ and $V = U \oplus U^\perp$.

Problem 5.12 Find an orthonormal basis for the subspace of the Euclidean space $\mathbb{R}^3$ given by $x + 2y - 3z = 0$, which is the orthogonal complement of the vector $(1, 2, -3)$ in $\mathbb{R}^3$.

Problem 5.13 Let $V = C[0, 1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \quad \text{for any } f \text{ and } g \text{ in } V.$$

Find an orthonormal basis for the subspace spanned by 1, x and x^2.

We can now identify an n-dimensional inner product space V with the Euclidean n-space $\mathbb{R}^n$ via the Gram-Schmidt orthogonalization. In fact, if $(V, \langle\ ,\ \rangle)$ is an inner product space, then by the Gram-Schmidt orthogonalization we can choose an orthonormal basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V. With this orthonormal basis α, the natural isomorphism $\Phi : V \to \mathbb{R}^n$ given by $\Phi(\mathbf{v}_i) = [\mathbf{v}_i]_\alpha = \mathbf{e}_i$, $i = 1, \ldots, n$ (see the last remark of Section 4.4) preserves the inner product of vectors: Every vector $\mathbf{x} \in V$ has a unique expression $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{v}_i$ with $x_i = \langle \mathbf{x}, \mathbf{v}_i\rangle$. Thus the coordinate vector of $\mathbf{x}$ with respect to α is a column matrix

$$[\mathbf{x}]_\alpha = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

which is a vector in $\mathbb{R}^n$. Moreover, if $\mathbf{y} = \sum_{i=1}^n y_i\mathbf{v}_i$ is another vector in V, then

$$\langle \mathbf{x}, \mathbf{y}\rangle = \left\langle \sum_{i=1}^n x_i\mathbf{v}_i, \sum_{j=1}^n y_j\mathbf{v}_j \right\rangle = \sum_{i=1}^n x_i y_i = [\mathbf{x}]_\alpha^T[\mathbf{y}]_\alpha.$$

The right side of this equation is just the dot product of vectors in the Euclidean space $\mathbb{R}^n$. That is,

$$\langle \mathbf{x}, \mathbf{y}\rangle = [\mathbf{x}]_\alpha^T[\mathbf{y}]_\alpha = \Phi(\mathbf{x}) \cdot \Phi(\mathbf{y})$$

for any $\mathbf{x}$, $\mathbf{y} \in V$. Hence, *the natural isomorphism Φ preserves the inner product,* and *identifies the inner product on V with the dot product on $\mathbb{R}^n$.* In this sense, we may restrict our study of an inner product space to the case of the Euclidean n-space $\mathbb{R}^n$ with the dot product.

A special kind of linear transformation that preserves the inner product such as the natural isomorphism from V to $\mathbb{R}^n$ plays an important role in linear algebra, and we will study this kind of transformation in Section 5.6.

Problem 5.14 Use the Gram-Schmidt orthogonalization on the Euclidean space $\mathbb{R}^4$ to transform the basis

$$\{(0,\ 1,\ 1,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (1,\ 2,\ 0,\ -1),\ (-1,\ 0,\ 0,\ -1)\}$$

into an orthonormal basis.

Problem 5.15 Find the point on the plane $x - y - z = 0$ that is closest to $\mathbf{p} = (1,\ 2,\ 0)$.

5.6 Orthogonal matrices and transformations

In Chapter 4, we saw that a linear transformation can be associated with a matrix, and vice versa. In this section, we are mainly interested in those linear transformations (or matrices) that preserve the lengths of vectors in an inner product space.

Let $A = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_n]$ be an $n \times n$ square matrix, where $\mathbf{c}_1, \ \ldots, \ \mathbf{c}_n \in \mathbb{R}^n$ are the column vectors of A. Then a simple computation shows that

$$A^TA = \begin{bmatrix} -- & \mathbf{c}_1^T & -- \\ & \vdots & \\ -- & \mathbf{c}_n^T & -- \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{c}_1 & \cdots & \mathbf{c}_n \\ | & & | \end{bmatrix} = [\mathbf{c}_i^T\mathbf{c}_j] = [\mathbf{c}_i \cdot \mathbf{c}_j].$$

Hence, if the column vectors are orthonormal, $\mathbf{c}_i^T\mathbf{c}_j = \delta_{ij}$, then $A^TA = I_n$, that is, A^T is a left inverse of A, and vice versa. Since A is a square matrix, this left inverse must be the right inverse of A, *i.e.*, $AA^T = I_n$. Equivalently, the row vectors of A are also orthonormal. This argument can now be summarized as follows.

Lemma 5.13 *Let A be an $n \times n$ matrix. The following are equivalent.*

(1) *The column vectors of A are orthonormal.*

(2) $A^TA = I_n$.

(3) $A^T = A^{-1}$.

(4) $AA^T = I_n$.

(5) *The row vectors of A are orthonormal.*

Definition 5.9 A square matrix A is called an **orthogonal** matrix if A satisfies one (and hence all) of the statements in Lemma 5.13.

Therefore, A is orthogonal if and only if A^T is orthogonal.

Example 5.11 It is easy to see that the matrices

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

are orthogonal, and satisfy

$$A^{-1} = A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad B^{-1} = B^T = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Note that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a rotation through the angle θ, while $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(\mathbf{x}) = B\mathbf{x}$ is the reflection about the line passing through the origin that forms an angle $\theta/2$ with the positive x-axis. □

Example 5.12 Show that every 2×2 orthogonal matrix must be one of the following forms

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Solution: Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an orthogonal matrix, so that $AA^T = I_2 = A^TA$. From the first equality, we get $a^2 + b^2 = 1$, $ac + bd = 0$, and $c^2 + d^2 = 1$. From the second equality, we get $a^2 + c^2 = 1$, $ab + cd = 0$, and $b^2 + d^2 = 1$. Thus, $b = \pm c$. If $b = -c$, then we get $a = d$. If $b = c$, then we get $a = -d$. Now, choose θ so that $a = \cos\theta$ and $b = \sin\theta$. □

Problem 5.16 Find the inverse of each of the following matrices.

$$(1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad (2) \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What are they as linear transformations on $\mathbb{R}^3$: rotations, reflections, or other?

Intuitively, any rotation or reflection on the Euclidean space $\mathbb{R}^n$ preserves both the lengths of vectors and the angle of two vectors. In general, any orthogonal matrix A preserves the lengths of vectors:

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2.$$

Definition 5.10 Let V and W be two inner product spaces. A linear transformation $T : V \to W$ is called an **isometry**, or an **orthogonal transformation**, if it preserves the lengths of vectors, that is, for every vector $\mathbf{x} \in V$

$$\|T(\mathbf{x})\| = \|\mathbf{x}\|.$$

Clearly, any orthogonal matrix is an isometry as a linear transformation. If $T : V \to W$ is an isometry, then T is a one-to-one, since the kernel of T is trivial: $T(\mathbf{x}) = \mathbf{0}$ implies $\|\mathbf{x}\| = \|T(\mathbf{x})\| = 0$. Thus, if $\dim V = \dim W$, then an isometry is also an isomorphism.

The following is an interesting characterization of an isometry.

Theorem 5.14 *Let $T : V \to W$ be a linear transformation on an inner product space V to W. Then T is an isometry if and only if T preserves inner products, that is,*

$$\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$$

for any vectors $\mathbf{x}$, $\mathbf{y}$ in V.

Proof: Let T be an isometry. Then $\|T(\mathbf{x})\|^2 = \|\mathbf{x}\|^2$ for any $\mathbf{x} \in V$. Hence,

$$\langle T(\mathbf{x}+\mathbf{y}), T(\mathbf{x}+\mathbf{y})\rangle = \|T(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle$$

for any $\mathbf{x}, \mathbf{y} \in V$. On the other hand,

$$\begin{aligned}
\langle T(\mathbf{x}+\mathbf{y}), T(\mathbf{x}+\mathbf{y})\rangle &= \langle T(\mathbf{x}), T(\mathbf{x})\rangle + 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle + \langle T(\mathbf{y}), T(\mathbf{y})\rangle, \\
\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle &= \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle,
\end{aligned}$$

from which we get $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$.

The converse is quite clear by choosing $\mathbf{y} = \mathbf{x}$. □

Corollary 5.15 *Let A be an $n \times n$ matrix. Then, A is an orthogonal matrix if and only if $A : \mathbb{R}^n \to \mathbb{R}^n$, as a linear transformation, preserves the dot product. That is, for any vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,*

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Proof: One way is clear. Suppose that A preserves the dot product. Then for any vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x}^T A^T A\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Take $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_j$. Then this equation is just $[A^T A]_{ij} = \delta_{ij}$. □

Since $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ for any $\mathbf{x}$ and $\mathbf{y}$ in V, one can easily derive the following corollary.

Corollary 5.16 *A linear transformation $T : V \to W$ is an isometry if and only if*

$$d(T(\mathbf{x}), T(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$$

for any $\mathbf{x}$ and $\mathbf{y}$ in V.

Recall that if θ is the angle between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V, then for any isometry $T : V \to V$,

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{\langle T\mathbf{x}, T\mathbf{y}\rangle}{\|T\mathbf{x}\|\|T\mathbf{y}\|}.$$

Hence, we have

Corollary 5.17 *An isometry preserves the angle.*

The following problem shows that the converse of Corollary 5.17 is not true in general.

Problem 5.17 Find an example of a linear transformation on the Euclidean space $\mathbb{R}^n$ that preserves the angles but not the lengths of vectors (*i.e.*, not an isometry). Such a linear transformation is called a **dilation**.

We have seen that any orthogonal matrix is an isometry as the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. The following theorem says that the converse is also true, that is, the matrix representation of an isometry with respect to an orthonormal basis is an orthogonal matrix.

Theorem 5.18 *Let $T : V \to W$ be an isometry of an inner product space V to W of the same dimension. Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be orthonormal bases for V and W, respectively. Then the matrix $[T]_\alpha^\beta$ for T with respect to the basis α and β is an orthogonal matrix.*

Proof: Note that the k-th column vector of the matrix $[T]_\alpha^\beta$ is just $[T(\mathbf{v}_k)]_\beta$. Since T preserves inner products and α is orthonormal, we get

$$[T(\mathbf{v}_k)]_\beta^T[T(\mathbf{v}_\ell)]_\beta = \langle T(\mathbf{v}_k), T(\mathbf{v}_\ell)\rangle = \langle \mathbf{v}_k, \mathbf{v}_\ell\rangle = \delta_{k\ell},$$

which shows that the column vectors of $[T]_\alpha^\beta$ are orthonormal. □

Therefore, a linear transformation $T : V \to V$ is an isometry if and only if $[T]_\alpha$ is an orthogonal matrix for an orthonormal basis α. Moreover, a square matrix A preserves the dot product if and only if it preserves the lengths of vectors.

Problem 5.18 Find values $r > 0$, $s > 0$, a, b and c such that matrix Q is orthogonal.

$$(1)\ Q = \begin{bmatrix} r & s & a \\ 0 & 2s & b \\ r & -s & c \end{bmatrix}, \quad (2)\ Q = \begin{bmatrix} r & -s & a \\ r & 3s & b \\ r & -2s & c \end{bmatrix}.$$

Problem 5.19 (Bessel's Inequality) Let V be an inner product space, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a set of orthonormal vectors in V (not necessarily a basis for V). Prove that for any $\mathbf{x}$ in V, $\|\mathbf{x}\|^2 \geq \sum_{i=1}^{m} |\langle \mathbf{x}, \mathbf{v}_i \rangle|^2$.

Problem 5.20 Determine whether the following linear transformations on Euclidean space $\mathbb{R}^3$ are orthogonal.
(1) $T(x, y, z) = (z, \frac{\sqrt{3}}{2}x + \frac{1}{2}y, \frac{x}{2} - \frac{\sqrt{3}}{2}y)$.
(2) $T(x, y, z) = (\frac{5}{13}x + \frac{11}{13}z, \frac{12}{13}y - \frac{5}{13}z, x)$.

5.7 Relations of fundamental subspaces

We now go back to the study of the system $A\mathbf{x} = \mathbf{b}$ of linear equations. One of the most important applications of the orthogonal projection of vectors onto a subspace is to study the relations or structures of the four fundamental subspaces $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{C}(A)$, and $\mathcal{N}(A^T)$ of an $m \times n$ matrix A.

Lemma 5.19 *For any $m \times n$ matrix A, the null space $\mathcal{N}(A)$ and the row space $\mathcal{R}(A)$ are orthogonal in $\mathbb{R}^n$. Similarly, the null space $\mathcal{N}(A^T)$ of A^T and the column space $\mathcal{C}(A) = \mathcal{R}(A^T)$ are orthogonal in $\mathbb{R}^m$.*

Proof: Note that $\mathbf{w} \in \mathcal{N}(A)$ if and only if $A\mathbf{w} = \mathbf{0}$, *i.e.*, for every row vector $\mathbf{r}$ in A, $\mathbf{r} \cdot \mathbf{w} = 0$. For the second statement, do the same with A^T. □

This theorem shows that $\mathcal{N}(A) \perp \mathcal{R}(A)$ and $\mathcal{C}(A) \perp \mathcal{N}(A^T)$, hence $\mathcal{N}(A) \subseteq \mathcal{R}(A)^\perp$ (or $\mathcal{R}(A) \subseteq \mathcal{N}(A)^\perp$) and $\mathcal{N}(A^T) \subseteq \mathcal{C}(A)^\perp$ (or $\mathcal{C}(A) \subseteq \mathcal{N}(A^T)^\perp$), but the equalities between them do not follow immediately. The next theorem shows that we have equalities in both inclusions, that is, the row space $\mathcal{R}(A)$ and the null space $\mathcal{N}(A)$ are orthogonal complements of each other, and the column space $\mathcal{C}(A)$ and the null space $\mathcal{N}(A^T)$ of A^T are orthogonal complements of each other. Note that the above theorem also shows that $\mathcal{N}(A) \cap \mathcal{R}(A) = \{\mathbf{0}\}$ and $\mathcal{C}(A) \cap \mathcal{N}(A^T) = \{\mathbf{0}\}$.

Theorem 5.20 (The second fundamental theorem) *For any $m\times n$ matrix A,*

(1) $\mathcal{N}(A) \oplus \mathcal{R}(A) = \mathbb{R}^n$,
(2) $\mathcal{N}(A^T) \oplus \mathcal{C}(A) = \mathbb{R}^m$.

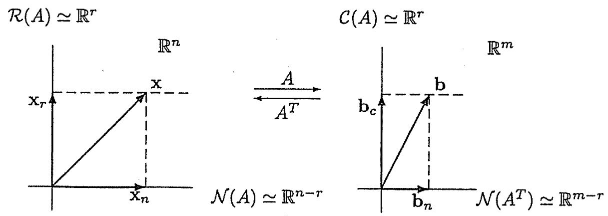
Proof: (1) Since both the row space $\mathcal{R}(A)$ and the null space $\mathcal{N}(A)$ of A are subspaces of $\mathbb{R}^n$, we have $\mathcal{N}(A)+\mathcal{R}(A) \subseteq \mathbb{R}^n$ in general. However,

$$\begin{aligned}
\dim(\mathcal{N}(A)+\mathcal{R}(A)) &= \dim\mathcal{N}(A)+\dim\mathcal{R}(A)-\dim(\mathcal{N}(A)\cap\mathcal{R}(A)) \\
&= \dim\mathcal{N}(A)+\dim\mathcal{R}(A) \\
&= \dim\mathcal{N}(A)+\operatorname{rank} A \\
&= n = \dim\mathbb{R}^n,
\end{aligned}$$

since dim(row space) + dim(null space) $= n =$ number of columns. This means that $\mathcal{N}(A)+\mathcal{R}(A)=\mathbb{R}^n$. Actually we have $\mathcal{N}(A)\oplus\mathcal{R}(A)=\mathbb{R}^n$ since $\mathcal{N}(A)\cap\mathcal{R}(A)=\{\mathbf{0}\}$. A similar argument applies to A^T to get (2). □

Corollary 5.21 (1) $\mathcal{N}(A)=\mathcal{R}(A)^{\perp}$, *and hence* $\mathcal{R}(A)=\mathcal{N}(A)^{\perp}$.
(2) $\mathcal{N}(A^T)=\mathcal{C}(A)^{\perp}$, *and hence* $\mathcal{C}(A)=\mathcal{N}(A^T)^{\perp}$.

For an $m\times n$ matrix A considered as a linear transformation $A:\mathbb{R}^n \to \mathbb{R}^m$, the decompositions $\mathbb{R}^n=\mathcal{R}(A)\oplus\mathcal{N}(A)$ and $\mathbb{R}^m=\mathcal{C}(A)\oplus\mathcal{N}(A^T)$ given in Theorem 5.20 depict the following figure with $r=\operatorname{rank} A$.



Note that if rank $A=r$, then $\dim\mathcal{R}(A)=r=\dim\mathcal{C}(A)$, $\dim\mathcal{N}(A)=n-r$ and $\dim\mathcal{N}(A^T)=m-r$. The figure shows that for any $\mathbf{b}_c$ in the column space $\mathcal{C}(A)$, which is the range of A, there is an $\mathbf{x}\in\mathbb{R}^n$ such that $A\mathbf{x}=\mathbf{b}_c$. Now there exist unique $\mathbf{x}_r\in\mathcal{R}(A)$ and $\mathbf{x}_n\in\mathcal{N}(A)$ such that $\mathbf{x}=\mathbf{x}_r+\mathbf{x}_n$. Thus $\mathbf{b}_c=A\mathbf{x}=A(\mathbf{x}_r+\mathbf{x}_n)=A\mathbf{x}_r$. Moreover, for any $\mathbf{x}'\in\mathcal{N}(A)$, $A(\mathbf{x}_r+\mathbf{x}')=A\mathbf{x}_r=\mathbf{b}_c$, since $A\mathbf{x}'=\mathbf{0}$. Therefore, the set of all solutions to $A\mathbf{x}=\mathbf{b}_c$ is precisely $\mathbf{x}_r+\mathcal{N}(A)$, which is the $n-r$ dimensional plane parallel to the null space $\mathcal{N}(A)$ and passing through $\mathbf{x}_r$.

In particular, if rank $A = m$, then $\mathcal{N}(A^T) = \{\mathbf{0}\}$ and hence $\mathcal{C}(A) = \mathbb{R}^m$. Thus for any $\mathbf{b} \in \mathbb{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ has solutions of the form $\mathbf{x}_r + \mathbf{x}_n$, where $\mathbf{x}_n \in \mathcal{N}(A)$ is arbitrary and $\mathbf{x}_r \in \mathcal{R}(A)$ is unique (this is the case in the existence Theorem 3.23).

On the other hand, if rank $A = n \leq m$, then $\mathcal{N}(A) = \{\mathbf{0}\}$ and hence $\mathcal{R}(A) = \mathbb{R}^n$. Therefore, the system $A\mathbf{x} = \mathbf{b}$ has at most one solution, that is, it has a unique solution $\mathbf{x}_r$ in the row space if $\mathbf{b} \in \mathcal{C}(A)$, and has no solution (that is, the system is inconsistent) if $\mathbf{b} \notin \mathcal{C}(A)$ (this is the case in the uniqueness Theorem 3.24). The latter case occurs when $m > r =$ rank A: that is, $\mathcal{N}(A^T)$ is a nontrivial subspace of $\mathbb{R}^m$.

Problem 5.21 Show that

(1) if $A\mathbf{x} = \mathbf{b}$ and $A^T\mathbf{y} = \mathbf{0}$, then $\mathbf{y}^T\mathbf{b} = 0$, and

(2) if $A\mathbf{x} = \mathbf{0}$ and $A^T\mathbf{y} = \mathbf{c}$, then $\mathbf{x}^T\mathbf{c} = 0$.

Problem 5.22 Given two vectors (1, 2, 1, 2) and (0, −1, −1, 1), find all vectors in $\mathbb{R}^4$ that are perpendicular to them.

Problem 5.23 Find a basis for the orthogonal complement of the row space of A:

$$(1)\ A = \begin{bmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ 3 & 0 & 6 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

5.8 Least square solutions

We consider again a system $A\mathbf{x} = \mathbf{b}$ of linear equations. Recall that the system $A\mathbf{x} = \mathbf{b}$ has at least one solution if and only if $\mathbf{b}$ belongs to the column space $\mathcal{C}(A)$ of A. In this case, such a solution is unique if and only if the null space $\mathcal{N}(A)$ of A is trivial.

Now the problem is *"what happens if* $\mathbf{b} \notin \mathcal{C}(A) \subseteq \mathbb{R}^m$ *so that* $A\mathbf{x} = \mathbf{b}$ *is inconsistent?"* Note that for any $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} \in \mathcal{C}(A)$. Thus the best we can do is to find a vector $\mathbf{x}_0 \in \mathbb{R}^n$ such that $A\mathbf{x}_0$ is *closest* to the given vector $\mathbf{b}$ in $\mathbb{R}^m$, *i.e.*, $\|A\mathbf{x}_0 - \mathbf{b}\|$ is as small as possible. Such a solution vector $\mathbf{x}_0$ gives the best approximation $A\mathbf{x}_0$ to $\mathbf{b}$, and is called a **least square solution** of $A\mathbf{x} = \mathbf{b}$. However, since we have the orthogonal decomposition $\mathbb{R}^m = \mathcal{C}(A) \oplus \mathcal{N}(A^T)$, we know that for any $\mathbf{b} \in \mathbb{R}^m$, $\mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b}) = \mathbf{b}_c \in \mathcal{C}(A)$

is the closest vector to $\mathbf{b}$ among the vectors in $\mathcal{C}(A)$. Therefore, a least square solution $\mathbf{x}_0 \in \mathbb{R}^n$ satisfies the following:

$$\begin{aligned} A\mathbf{x}_0 &= \mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b}), \\ \|A\mathbf{x}_0 - \mathbf{b}\| &\le \|A\mathbf{x} - \mathbf{b}\| \end{aligned}$$

for any vector $\mathbf{x}$ in $\mathbb{R}^n$. Since $\mathbf{b}_c \in \mathcal{C}(A)$, there always exists a least square solution $\mathbf{x}_0 \in \mathbb{R}^n$ such that $A\mathbf{x}_0 = \mathbf{b}_c$. It is quite easy to show that all other least square solutions are the vectors in $\mathbf{x}_0 + \mathcal{N}(A)$.

In summary, a least square solution of $A\mathbf{x} = \mathbf{b}$, when $\mathbf{b} \notin \mathcal{C}(A)$, is simply a solution of $A\mathbf{x} = \mathbf{b}_c$, where $\mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b}) \in \mathcal{C}(A)$ in the unique orthogonal decomposition of

$$\mathbf{b} = \mathbf{b}_c + \mathbf{b}_n \in \mathcal{C}(A) \oplus \mathcal{N}(A^T) = \mathbb{R}^m,$$

with $\mathbf{b}_n = \mathbf{b} - \mathbf{b}_c \in \mathcal{N}(A^T)$. That is, to find such a least square solution, we first have to find $\mathbf{b}_c$ and then solve $A\mathbf{x}_0 = \mathbf{b}_c$.

Practically, the computation of $\mathbf{b}_c$ from $\mathbf{b}$ could be quite complicated, since we first have to find an orthonormal basis for $\mathcal{C}(A)$ by using the Gram-Schmidt orthogonalization (whose computation is cumbersome) and then express $\mathbf{b}_c$ with respect to this orthonormal basis for a given $\mathbf{b}$.

To find an easier method, let us examine a least square solution in a little more detail. If $\mathbf{x}_0 \in \mathbb{R}^n$ is a least square solution of $A\mathbf{x} = \mathbf{b}$, *i.e.*, a solution of $A\mathbf{x}_0 = \mathbf{b}_c$, then $A\mathbf{x}_0 - \mathbf{b} = A\mathbf{x}_0 - (\mathbf{b}_c + \mathbf{b}_n) = -\mathbf{b}_n \in \mathcal{N}(A^T)$ holds. Thus, by applying A^T to the equation, we get $A^TA\mathbf{x}_0 = A^T\mathbf{b}$, *i.e.*, $\mathbf{x}_0$ is a solution of the equation

$$A^TA\mathbf{x} = A^T\mathbf{b}.$$

This equation is very interesting because it also gives a sufficient condition of a least square solution as the following theorem shows, and so is defined to be the **normal equation** of $A\mathbf{x} = \mathbf{b}$.

Theorem 5.22 *Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$ be any vector. Then a vector $\mathbf{x}_0 \in \mathbb{R}^n$ is a least square solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}_0$ is a solution of the normal equation $A^TA\mathbf{x} = A^T\mathbf{b}$.*

Proof: We only need to show the sufficiency of the normal equation: If $\mathbf{x}_0$ is a solution of the equation $A^TA\mathbf{x} = A^T\mathbf{b}$, then, $A^T(A\mathbf{x}_0 - \mathbf{b}) = \mathbf{0}$, so $A\mathbf{x}_0 - \mathbf{b} = A\mathbf{x}_0 - (\mathbf{b}_c + \mathbf{b}_n) \in \mathcal{N}(A^T)$. This means that, as a vector in $\mathcal{C}(A)$, $A\mathbf{x}_0 - \mathbf{b}_c \in \mathcal{N}(A^T) \cap \mathcal{C}(A) = \{\mathbf{0}\}$. Therefore $A\mathbf{x}_0 = \mathbf{b}_c = \mathrm{Proj}_{\mathcal{C}(A)}(\mathbf{b})$, *i.e.*,

$\mathbf{x}_0$ is a least square solution of $A\mathbf{x} = \mathbf{b}$. □

Note that if the rows of A are linearly independent, then $\operatorname{rank} A = m$ and $\mathcal{C}(A) = \mathbb{R}^m$ (or $\mathcal{N}(A^T) = \mathbf{0}$). Thus, a least square solution of $A\mathbf{x} = \mathbf{b}$ is simply a usual solution.

Example 5.13 Find all the least square solutions to $A\mathbf{x} = \mathbf{b}$, and then determine the orthogonal projection $\mathbf{b}_c$ of $\mathbf{b}$ into the column space $\mathcal{C}(A)$ of A, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solution:

$$A^T A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix},$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$

From the normal equation, a least square solution of $A\mathbf{x} = \mathbf{b}$ is a solution of $A^T A\mathbf{x} = A^T\mathbf{b}$, *i.e.*,

$$\begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$

By solving this system of equations (left for an exercise), we obtain all the least square solutions desired:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -8 \\ -5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

for any number t. Now

$$\mathbf{b}_c = A\mathbf{x} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} \in \mathcal{C}(A).$$

Note that the vector $\mathbf{x} = \frac{1}{3}\begin{bmatrix} -8 \\ -5 \\ 0 \end{bmatrix}$ is not in $\mathcal{R}(A)$. One needs to do a little more computation to find a least square solution $\mathbf{x} \in \mathcal{R}(A)$. □

Problem 5.24 Find all least square solutions $\mathbf{x}$ in $\mathbb{R}^3$ of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -3 \end{bmatrix}.$$

Practically, finding the solutions of the normal equation depends very much on A^TA. In the most fortunate case, if the square matrix A^TA is the identity matrix, then the normal equation $A^TA\mathbf{x} = A^T\mathbf{b}$ of the system $A\mathbf{x} = \mathbf{b}$ reduces to $\mathbf{x} = A^T\mathbf{b}$, which is simply a least square solution. Even if A^TA is not the identity matrix, we may still have several simple cases.

Remark: Let us now discuss the solvability of this normal equation. Observe that $A^T : \mathbb{R}^m \to \mathbb{R}^n$ and the row space $\mathcal{R}(A)$ of A and the column space $\mathcal{C}(A^T)$ of A^T are the same. Thus, for any $\mathbf{x}_r \in \mathcal{C}(A^T) = \mathcal{R}(A)$ there exists a vector $\mathbf{b} \in \mathbb{R}^m$ such that $A^T\mathbf{b} = \mathbf{x}_r$. If we write $\mathbf{b} = \mathbf{b}_c + \mathbf{b}_n$ for unique $\mathbf{b}_c \in \mathcal{C}(A)$ and $\mathbf{b}_n \in \mathcal{N}(A^T)$, then $\mathbf{x}_r = A^T\mathbf{b} = A^T\mathbf{b}_c$. Therefore, the restrictions

$$\begin{aligned} \bar{A} &= A|_{\mathcal{R}(A)} : \mathcal{R}(A) \subseteq \mathbb{R}^n \to \mathcal{C}(A) \subseteq \mathbb{R}^m \quad \text{and} \\ \bar{A^T} &= A^T|_{\mathcal{C}(A)} : \mathcal{C}(A) \subseteq \mathbb{R}^m \to \mathcal{R}(A) \subseteq \mathbb{R}^n \end{aligned}$$

are one-to-one and onto transformations, that is, they are invertible. However, even in this case we do not have $\bar{A}\bar{A^T} = I_r$ nor $\bar{A^T}\bar{A} = I_r$ in general.

The transpose A^T of a matrix A satisfies the following equation: For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, $A\mathbf{x} \in \mathbb{R}^m$, so

$$A\mathbf{x} \cdot \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^TA^T\mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y}.$$

The following theorem gives a condition for A^TA to be invertible.

Theorem 5.23 *For any $m \times n$ matrix A, $A^T A$ is a symmetric $n \times n$ square matrix and* $\operatorname{rank}(A^T A) = \operatorname{rank} A$.

Proof: Clearly, $A^T A$ is square and symmetric: $(A^T A)^T = A^T (A^T)^T = A^T A$. Since the number of columns of A and $A^T A$ are both n, we have

$$\operatorname{rank} A + \dim \mathcal{N}(A) = n = \operatorname{rank}\,(A^T A) + \dim \mathcal{N}(A^T A).$$

Hence, it suffices to show that A and $A^T A$ have exactly the same null space so that $\dim \mathcal{N}(A) = \dim \mathcal{N}(A^T A)$. If $\mathbf{x} \in \mathcal{N}(A)$, then $A\mathbf{x} = \mathbf{0}$ and also $A^T A\mathbf{x} = A^T \mathbf{0} = \mathbf{0}$, so that $\mathbf{x} \in \mathcal{N}(A^T A)$. Conversely, suppose that $A^T A\mathbf{x} = \mathbf{0}$. Then

$$A\mathbf{x} \cdot A\mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \mathbf{x}^T (A^T A\mathbf{x}) = \mathbf{x} \cdot A^T A\mathbf{x} = \mathbf{x} \cdot \mathbf{0} = 0.$$

Hence $A\mathbf{x} = \mathbf{0}$, *i.e.*, $\mathbf{x} \in \mathcal{N}(A)$. □

In the following discussion, we assume that the columns of A are linearly independent, *i.e.*, rank $A = n$, so that $\mathcal{N}(A) = \{\mathbf{0}\}$, or A is one-to-one. Hence the system $A\mathbf{x} = \mathbf{b}_c$ has a unique solution $\mathbf{x}$ in $\mathcal{R}(A) = \mathbb{R}^n$. Moreover, by Theorem 5.23, the square matrix $A^T A$ is also of rank n and it is invertible. In this case, from the normal equation, a least square solution is

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}.$$

Corollary 5.24 *If the columns of A are linearly independent, then*

(1) *$A^T A$ is invertible so that $(A^T A)^{-1} A^T$ is a left inverse of A,*

(2) *the vector $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$ is the unique least square solution of a system $A\mathbf{x} = \mathbf{b}$, and*

(3) *$A\mathbf{x} = A(A^T A)^{-1} A^T \mathbf{b}$ is the projection $\mathbf{b}_c$ of $\mathbf{b}$ into the column space $\mathcal{C}(A)$.*

By applying Corollary 5.24 to A^T, we can say that, if rank $A = m$ for an $m \times n$ matrix, then AA^T is invertible and $A^T (AA^T)^{-1}$ is a right inverse of A (cf. Remark after Theorem 3.24). Moreover, by using Theorem 5.23, we can show that for a matrix A, $A^T A$ is invertible if and only if the columns of A are linearly independent, and AA^T is invertible if and only if the rows of A are linearly independent.

Example 5.14 Consider the following system of linear equations:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix} = \mathbf{b}.$$

Clearly, the two columns of A are linearly independent and $\mathcal{C}(A)$ is the xy-plane. Thus $\mathbf{b} \notin \mathcal{C}(A)$. Note that

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 7 & 29 \end{bmatrix},$$

which is invertible. By a simple computation one can obtain

$$(A^T A)^{-1} = \frac{1}{9} \begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix}.$$

Hence,

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \frac{1}{9} \begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 23 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 42 \\ -3 \end{bmatrix} = \begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix}$$

is a least square solution, and the orthogonal projection of $\mathbf{b}$ in $\mathcal{C}(A)$ is

$$\mathbf{b}_c = A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}.$$

□

Problem 5.25 Let W be the subspace of the Euclidean space $\mathbb{R}^3$ spanned by the vectors $\mathbf{v}_1 = (1,\ 1,\ 2)$ and $\mathbf{v}_2 = (1,\ 1,\ -1)$. Find $\mathrm{Proj}_W(\mathbf{b})$ for $\mathbf{b} = (1,\ 3,\ -2)$.

5.9 Application: Polynomial approximations

In this section, one can find a reason for the name of the "*least square*" solutions, and the following example illustrates an application of the least square solution to the determination of the spring constants in physics.

Example 5.15 *Hooke's law* for springs in physics says that for a uniform spring, *the length stretched or compressed is a linear function of the force applied*, that is, the force F applied to the spring is related to the length x stretched or compressed by the equation

$$F = a + kx,$$

where a and k are some constants determined by the spring.

Suppose now that, given a spring of length 6.1 inches, we want to determine the constants a and k under the experimental data: The lengths are found to be 7.6, 8.7 and 10.4 inches when forces of 2, 4 and 6 kilograms, respectively, are applied to the spring. However, by plotting these data

$$(x, F) = (6.1,\ 0),\ (7.6,\ 2),\ (8.7,\ 4),\ (10.4,\ 6),$$

in the xF-plane, one can easily recognize that they are not on a straight line of the form $F = a + kx$ in the xF-plane, which may be caused by experimental errors. This means that the system of linear equations:

$$\begin{cases} F_1 &= a + 6.1k &= 0, \\ F_2 &= a + 7.6k &= 2, \\ F_3 &= a + 8.7k &= 4, \\ F_4 &= a + 10.4k &= 6 \end{cases}$$

is inconsistent (*i.e.*, has no solutions so the second equality in each equation may not be a true equality). Thus, the best thing one can do is to determine the straight line that "fits" the data, that is, the line that minimizes the sum of the squares of the vertical distances from the line to the data: *i.e.*, one needs to minimize

$$(0 - F_1)^2 + (2 - F_2)^2 + (4 - F_3)^2 + (6 - F_4)^2.$$

This quantity is simply the square of the distance between the vector $\mathbf{b} = (0, 2, 4, 6)$ in $\mathbb{R}^4$ and the vectors (F_1, F_2, F_3, F_4) in the column space $C(A)$ of the 4×2 matrix

$$A = \begin{bmatrix} 1 & 6.1 \\ 1 & 7.6 \\ 1 & 8.7 \\ 1 & 10.4 \end{bmatrix},$$

since the matrix form of the system of linear equations is

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 1 & 6.1 \\ 1 & 7.6 \\ 1 & 8.7 \\ 1 & 10.4 \end{bmatrix} \begin{bmatrix} a \\ k \end{bmatrix} \in \mathcal{C}(A).$$

The minimum of the sum of squares is obtained when (F_1, F_2, F_3, F_4) is the projection of the vector $\mathbf{b} = (0, 2, 4, 6)$ into the column space $\mathcal{C}(A)$, that is, what we are looking for is the least square solution of the system, which is now easily computed as

$$\begin{bmatrix} a \\ k \end{bmatrix} = \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} -8.6 \\ 1.4 \end{bmatrix}.$$

It gives $F = -8.6 + 1.4x$. □

In general, a common problem in experimental work is to obtain a polynomial $y = f(x)$ in two variables x and y that "fits" the data of various values of y determined experimentally for inputs x, say

$$(x_1,\ y_1),\ (x_2,\ y_2),\ \ldots,\ (x_n,\ y_n),$$

plotted in the xy-plane. Some possibilities are (1) by a straight line: $y = a+bx$, (2) by a quadratic polynomial: $y = a+bx+cx^2$, or (3) by a polynomial of degree k: $y = a_0 + a_1x + \cdots + a_kx^k$, etc.

As a general case, suppose that we are looking for a polynomial $y = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ of degree k that passes through the given data, then we obtain a system of linear equations,

$$\begin{cases} f(x_1) = a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_kx_1^k & = & y_1 \\ f(x_2) = a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_kx_2^k & = & y_2 \\ \qquad\vdots & & \\ f(x_n) = a_0 + a_1x_n + a_2x_n^2 + \cdots + a_kx_n^k & = & y_n, \end{cases}$$

or, in matrix form, the system may be written as $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The left side $A\mathbf{x}$ represents the values of the polynomial at x_i's and the right side represents the data obtained from the inputs x_i's in the experiment.

If $n \le k+1$, then the cases have already been discussed in Section 3.8. In general, this kind of system may be inconsistent (*i.e.*, it may have no solution) if $n > k+1$. This means that there may be no polynomial of degree $k < n-1$ whose graph passes through the n data (x_i, y_i) in the xy-plane. Practically, it is due to the fact that the experimental data usually have some errors.

Thus, the best thing we can do is to find the polynomial $f(x)$ that minimizes the sum of the squares of the vertical distances between the graph of the polynomial and the data. In matrix and vector space language, an inconsistency of the system means that the vector $\mathbf{b} \in \mathbb{R}^n$ representing the data is not in the column space $\mathcal{C}(A)$ of the coefficient matrix A. And minimizing the sum of the squares of the vertical distances between the graph of the polynomial and the data means looking for the least square solution of the system, because for any $\mathbf{c} \in \mathcal{C}(A)$ of the form

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} a_0 + a_1x_1 + \cdots + a_kx_1^k \\ a_0 + a_1x_2 + \cdots + a_kx_2^k \\ \vdots \\ a_0 + a_1x_n + \cdots + a_kx_n^k \end{bmatrix} = \mathbf{c},$$

we have

$$\begin{aligned} \|\mathbf{b} - \mathbf{c}\|^2 = \; & (y_1 - a_0 - a_1x_1 - \cdots - a_kx_1^k)^2 + \cdots \\ & + (y_n - a_0 - a_1x_n - \cdots - a_kx_n^k)^2. \end{aligned}$$

The previous theory says that the orthogonal projection $\mathbf{b}_c$ of $\mathbf{b}$ into the column space of A minimizes this quantity and shows how to find $\mathbf{b}_c$ and a least square solution $\mathbf{x}_0$.

Example 5.16 Find a straight line $y = a + bx$ that fits the given experimental data, (1, 0), (2, 3), (3, 4) and (4, 4), that is, a line $y = a + bx$ that minimizes the sum of squares of the vertical distances $|y_i - a - bx_i|$'s from the line $y = a + bx$ to the data $(x_i,\ y_i)$. By adapting matrix notation

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 4 \end{bmatrix},$$

we have $A\mathbf{x} = \mathbf{b}$ and want to find a least square solution of $A\mathbf{x} = \mathbf{b}$. But the columns of A are linearly independent, and the least square solution is $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$. Now,

$$A^TA = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}, \; (A^TA)^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} \end{bmatrix}, \; A^T\mathbf{b} = \begin{bmatrix} 11 \\ 34 \end{bmatrix}.$$

Hence, we have

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} -\frac{1}{2} \\ \frac{13}{10} \end{bmatrix},$$

and $y = -\frac{1}{2} + \frac{13}{10}x$ is the desired line. □

Problem 5.26 From Newton's second law of motion, a body near the surface of the earth falls vertically downward according to the equation

$$s(t) = s_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2,$$

where $s(t)$ is the distance that the body traveled in time t, and s_0, $\mathbf{v}_0$ are the initial displacement and velocity, respectively, of the body, and $\mathbf{g}$ is the gravitational acceleration at the earth's surface. Suppose a weight is released, and the distances that the body has fallen from some reference point were measured to be $s = -0.18,\ 0.31,\ 1.03,\ 2.48,\ 3.73$ feet at times $t = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ seconds, respectively. Determine approximate values of s_0, $\mathbf{v}_0$, $\mathbf{g}$ using these data.

5.10 Orthogonal projection matrices

In Section 5.8, we have seen that the orthogonal projection $\mathrm{Proj}_{\mathcal{C}(A)}$ of $\mathbb{R}^m$ on the column space $\mathcal{C}(A)$ of an $m \times n$ matrix A plays an important role in finding a least square solution of $A\mathbf{x} = \mathbf{b}$. Note that any subspace W of $\mathbb{R}^m$ is the column space of such a matrix A, whose columns are the vectors in a basis for W. Therefore, in this section, we only consider the orthogonal projection Proj_W of an inner product space V onto a subspace W, and aim to find its associated matrix, called an **orthogonal projection matrix**, for the projection Proj_W. This will give us a practical way of computing a given orthogonal projection.

First of all, if a subspace W of the Euclidean space $\mathbb{R}^m$ has an orthonormal basis $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, then for any $\mathbf{x} \in \mathbb{R}^m$,

$$\begin{aligned}
\mathrm{Proj}_W(\mathbf{x}) &= (\mathbf{u}_1 \cdot \mathbf{x})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{x})\mathbf{u}_2 + \cdots + (\mathbf{u}_n \cdot \mathbf{x})\mathbf{u}_n \\
&= \mathbf{u}_1(\mathbf{u}_1^T\mathbf{x}) + \mathbf{u}_2(\mathbf{u}_2^T\mathbf{x}) + \cdots + \mathbf{u}_n(\mathbf{u}_n^T\mathbf{x}) \\
&= (\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T)\mathbf{x},
\end{aligned}$$

by Lemma 5.11. Note that in this equation, Proj_W is a linear transformation, but the right side is the usual matrix product of vectors. It implies that if an orthonormal basis for a subspace W is given, the matrix representation (projection matrix) of the orthogonal projection Proj_W with respect to the standard basis α for $\mathbb{R}^m$ is given as

$$[\mathrm{Proj}_W]_\alpha = \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T.$$

Note that if we denote by $\mathrm{Proj}_{\mathbf{u}_i}$ the orthogonal projection of $\mathbb{R}^m$ on the subspace spanned by the basis vector $\mathbf{u}_i$ for each i, then matrix representation is $\mathbf{u}_i\mathbf{u}_i^T$ (see page 176). Moreover, by using the matrix representations, it can be shown that

$$\mathrm{Proj}_W = \mathrm{Proj}_{\mathbf{u}_1} + \mathrm{Proj}_{\mathbf{u}_2} + \cdots + \mathrm{Proj}_{\mathbf{u}_n}$$

and

$$\mathrm{Proj}_{\mathbf{u}_j} \circ \mathrm{Proj}_{\mathbf{u}_i} = \begin{cases} \mathbf{0} & \text{if } i \neq j, \\ \mathrm{Proj}_{\mathbf{u}_j} & \text{if } i = j. \end{cases}$$

Problem 5.27 Let $\mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ be a vector in $\mathbb{R}^2$ which determines 1-dimensional subspace $U = \{a\mathbf{u} = (\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}) \ : \ a \in \mathbb{R}\}$. Show that the matrix

$$A = \mathbf{u}\mathbf{u}^T = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}[1\,1] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

considered as a linear transformation on $\mathbb{R}^2$, is an orthogonal projection onto the subspace U.

Problem 5.28 Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then $\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T + \cdots + \mathbf{v}_m\mathbf{v}_m^T = I_m$.

Definition 5.11 Let W be a subspace of the Euclidean m-space $\mathbb{R}^m$. An $m \times m$ matrix P is called the **(orthogonal) projection matrix** on a subspace W if $\mathrm{Proj}_W(\mathbf{x}) = P\mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^m$. Equivalently, P is the matrix representation of the orthogonal projection Proj_W of $\mathbb{R}^m$ onto W with respect to the standard basis for $\mathbb{R}^m$.

It has already been shown that $\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_k\mathbf{u}_k^T$ is a projection matrix for any orthonormal set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ in $\mathbb{R}^m$. Such an expression of the projection matrix on a subspace W can be obtained only when an orthonormal basis for W is known.

Now, let W be an n-dimensional subspace of the Euclidean space $\mathbb{R}^m$, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a (not necessarily orthonormal) basis for W. If we find an orthonormal basis for W by the Gram-Schmidt orthogonalization, then we can get the projection matrix of the previous form. But the Gram-Schmidt orthogonalization process could be cumbersome and tedious. Sometimes, one can avoid this cumbersome process. Let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ be the $m \times n$ matrix having the basis vectors $\mathbf{v}_i$'s as columns. Clearly, we have $W = \mathcal{C}(A)$. For any vector $\mathbf{b} \in \mathbb{R}^m$, the projection vector $\mathrm{Proj}_W(\mathbf{b})$ is simply the vector $A\mathbf{x}_0$ for a least square solution $\mathbf{x}_0$ of $A\mathbf{x} = \mathbf{b}$ that is a solution of the normal equation $A^TA\mathbf{x} = A^T\mathbf{b}$.

On the other hand, since the columns of A are linearly independent, A^TA is invertible, so $\mathbf{x}_0 = (A^TA)^{-1}A^T\mathbf{b}$, and, furthermore,

$$\mathrm{Proj}_W(\mathbf{b}) = A\mathbf{x}_0 = A(A^TA)^{-1}A^T\mathbf{b},$$

by Corollary 5.24. This means that $A(A^TA)^{-1}A^T$ is the projection matrix on the subspace $W = \mathcal{C}(A)$. Note that this projection matrix is independent of the choice of basis for W due to the uniqueness of the matrix representation of a linear transformation with respect to a fixed basis. Some possible simple computations for the matrix $A(A^TA)^{-1}A^T$ will follow later. This argument proves the following theorem.

Theorem 5.25 *For any subspace W of $\mathbb{R}^m$, the projection matrix P on W can be written as*

$$P = [\mathrm{Proj}_W]_\alpha = A(A^TA)^{-1}A^T$$

for a matrix A whose columns form a basis for W.

Example 5.17 Find the projection matrix P on the plane $2x - y - 3z = 0$ in $\mathbb{R}^3$ and calculate $P\mathbf{b}$ for $\mathbf{b} = (1,\ 0,\ 1)$.

Solution: Choose any basis for the plane $2x - y - 3z = 0$, say,

$$\mathbf{v}_1 = (0,\ 3,\ -1) \quad \text{and} \quad \mathbf{v}_2 = (1,\ 2,\ 0).$$

Let $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$ be the matrix with $\mathbf{v}_1$ and $\mathbf{v}_2$ as columns. Then

$$(A^T A)^{-1} = \begin{bmatrix} 10 & 6 \\ 6 & 5 \end{bmatrix}^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}.$$

The projection matrix is

$$\begin{aligned} P &= A(A^T A)^{-1} A^T \\ &= \frac{1}{14}\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}\begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}, \end{aligned}$$

and

$$P\mathbf{b} = \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 16 \\ -1 \\ 11 \end{bmatrix}. \qquad \square$$

Remark: In particular, if the columns of A consist of an orthonormal basis $\alpha = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for W, then

$$A^T A = \begin{bmatrix} -- & \mathbf{u}_1^T & -- \\ & \vdots & \\ -- & \mathbf{u}_n^T & -- \end{bmatrix}\begin{bmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I_n.$$

since $\mathbf{u}_i^T \mathbf{u}_j = \delta_{ij}$. Hence, the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ becomes

$$\mathbf{x} = A^T\mathbf{b} = \begin{bmatrix} -- & \mathbf{u}_1^T & -- \\ & \vdots & \\ -- & \mathbf{u}_n^T & -- \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{b} \rangle \\ \vdots \\ \langle \mathbf{u}_n, \mathbf{b} \rangle \end{bmatrix},$$

which is just the expression of $\mathrm{Proj}_W(\mathbf{b})$ with respect to the orthonormal basis α for W that are the columns of A.

Corollary 5.26 *Suppose that the column vectors* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *of* A *form an orthonormal basis for* W *in* $\mathbb{R}^m$. *Then we get*

$$P = A(A^TA)^{-1}A^T = AA^T = \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T.$$

In particular, if A *is an* $m \times m$ *orthogonal matrix, then, for all* $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ *has the unique solution* $\mathbf{x} = A^{-1}\mathbf{b} = A^T\mathbf{b}$.

Proof: For any $\mathbf{x} \in \mathbb{R}^m$,

$$\begin{aligned} P\mathbf{x} = AA^T\mathbf{x} &= \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} -- & \mathbf{u}_1^T & -- \\ & \vdots & \\ -- & \mathbf{u}_n^T & -- \end{bmatrix} \mathbf{x} \\ &= \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T\mathbf{x} \\ \vdots \\ \mathbf{u}_n^T\mathbf{x} \end{bmatrix} \\ &= (\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T)\mathbf{x}, \end{aligned}$$

where each $\mathbf{u}_i^T\mathbf{x}$ is a scalar as the inner product of $\mathbf{u}_i$ and $\mathbf{x}$. □

Example 5.18 If $A = [\mathbf{c}_1\ \mathbf{c}_2]$, where $\mathbf{c}_1 = (1,\ 0,\ 0)$, $\mathbf{c}_2 = (0,\ 1,\ 0)$, then the column vectors of A are orthonormal, $\mathcal{C}(A)$ is the xy-plane, and the projection of $\mathbf{b} = (x,\ y,\ z) \in \mathbb{R}^3$ onto $\mathcal{C}(A)$ is $\mathbf{b}_c = (x,\ y,\ 0)$. In fact,

$$P = AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}.$$ □

Before discussing the computation of $P = [\mathrm{Proj}_W]_\alpha$ with a general basis for W, we exhibit a criterion for a square matrix to be a projection matrix.

Theorem 5.27 *A square matrix* P *is a projection matrix if and only if it is symmetric and idempotent, i.e.,* $P^T = P$ *and* $P^2 = P$.

Proof: Let P be a projection matrix. Then, by Theorem 5.25, the matrix P can be written as $P = A(A^TA)^{-1}A^T$ for some matrix A whose columns are linearly independent. A simple expansion of $P = A(A^TA)^{-1}A^T$ gives

$$\begin{aligned} P^T &= \left(A(A^TA)^{-1}A^T\right)^T = A(A^TA)^{-1^T}A^T = A(A^TA)^{-1}A^T = P, \\ P^2 &= PP = \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right) = A(A^TA)^{-1}A^T = P. \end{aligned}$$

We have already shown the second equation in Theorem 5.7.

For the converse, we have the orthogonal decomposition $\mathbb{R}^m = \mathcal{C}(P) \oplus \mathcal{N}(P^T)$ by Theorem 5.20. But $\mathcal{N}(P^T) = \mathcal{N}(P)$ since $P^T = P$. Note that $P^2 = P$ implies $P\mathbf{u} = \mathbf{u}$ for $\mathbf{u} \in \mathcal{C}(P)$ (see Theorem 5.7). $\square$

From Corollary 5.8, if P is a projection matrix on $\mathcal{C}(P)$, then $I - P$ is also a projection matrix on the null space $\mathcal{N}(P)$ $(= \mathcal{C}(I - P))$, which is orthogonal to $\mathcal{C}(P)$ $(= \mathcal{N}(I - P))$.

Example 5.19 Let $P_i : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$P_i(x_1, \ldots, x_m) = (0, \ldots, 0, x_i, 0, \ldots, 0),$$

for $i = 1, \ldots, m$. Then each P_i is the projection of $\mathbb{R}^m$ onto the i-th axis, whose matrix form looks like

$$P_i = \begin{bmatrix} \ddots & & & & 0 \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \end{bmatrix}, \qquad I - P_i = \begin{bmatrix} \ddots & & & & 0 \\ & 1 & & & \\ & & 0 & & \\ & & & 1 & \\ 0 & & & & \ddots \end{bmatrix}.$$

When we restrict the image to $\mathbb{R}$, P_i is an element in the dual space $\mathbb{R}^{n*}$, and usually denoted by x_i as the i-th coordinate function (see Example 4.25).

Problem 5.29 Show that any square matrix P that satisfies $P^T P = P$ is a projection matrix.

In general, if $A = [\mathbf{c}_1 \cdots \mathbf{c}_n]$ is an $m \times n$ matrix with linearly independent column vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$, then rank $A = n \leq m$ and $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ form a basis for the column space $W = \mathcal{C}(A)$ of dim n in $\mathbb{R}^m$. By using the Gram-Schmidt orthogonalization, one can obtain an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $\mathcal{C}(A)$ from this basis, so that the matrix $Q = [\mathbf{u}_1 \cdots \mathbf{u}_n]$ and A have the same column space W. Then, by the Remark on page 199, $\mathrm{Proj}_W = QQ^T$. The computation of the Gram-Schmidt orthogonalization might be messy, but these cases occur frequently in applied science and engineering problems, so we show the process in detail in the following.

From the Gram-Schmidt orthogonalization,

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{c}_1 \\
\mathbf{q}_2 &= \mathbf{c}_2 - \frac{\langle \mathbf{c}_2, \mathbf{q}_1\rangle}{\langle \mathbf{q}_1, \mathbf{q}_1\rangle}\,\mathbf{q}_1 \\
&\vdots \\
\mathbf{q}_n &= \mathbf{c}_n - \frac{\langle \mathbf{c}_n, \mathbf{q}_{n-1}\rangle}{\langle \mathbf{q}_{n-1}, \mathbf{q}_{n-1}\rangle}\,\mathbf{q}_{n-1} - \cdots - \frac{\langle \mathbf{c}_n, \mathbf{q}_1\rangle}{\langle \mathbf{q}_1, \mathbf{q}_1\rangle}\,\mathbf{q}_1,
\end{aligned}$$

gives an orthogonal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ for $\mathcal{C}(A)$. By taking normalization of these vectors, we obtain an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $\mathcal{C}(A)$, where $\mathbf{u}_i = \mathbf{q}_i/\|\mathbf{q}_i\|$. Rewriting these equations gives us

$$\begin{aligned}
\mathbf{c}_1 &= \mathbf{q}_1 &&= \|\mathbf{q}_1\|\mathbf{u}_1 \\
\mathbf{c}_2 &= a_{21}\mathbf{q}_1 + \mathbf{q}_2 &&= b_{21}\mathbf{u}_1 + b_{22}\mathbf{u}_2 \\
&\vdots \\
\mathbf{c}_n &= a_{n1}\mathbf{q}_1 + \cdots + a_{nn-1}\mathbf{q}_{n-1} + \mathbf{q}_n &&= b_{n1}\mathbf{u}_1 + \cdots + b_{nn}\mathbf{u}_n,
\end{aligned}$$

where $a_{ij} = \frac{\langle \mathbf{c}_i, \mathbf{q}_j\rangle}{\langle \mathbf{q}_j, \mathbf{q}_j\rangle}$ for $i > j$, $a_{ii} = 1$, and $b_{ij} = a_{ij}\|\mathbf{q}_j\|$ for $i \geq j$. Hence,

$$A = [\mathbf{c}_1 \cdots \mathbf{c}_n] = [\mathbf{u}_1 \cdots \mathbf{u}_n]\begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ 0 & b_{22} & \cdots & b_{n2} \\ & & \ddots & \\ 0 & & 0 & b_{nn} \end{bmatrix} = QR.$$

The matrix $Q = [\mathbf{u}_1 \cdots \mathbf{u}_n]$ is an $m \times n$ matrix with orthonormal columns, called the **orthogonal part** of A, and

$$R = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ 0 & b_{22} & \cdots & b_{n2} \\ & & \ddots & \\ 0 & & 0 & b_{nn} \end{bmatrix}$$

is an invertible upper triangular matrix, called the **upper triangular part** of A (note that all the diagonal $b_{ii} \neq 0$). Such an $A = QR$ is called the QR **factorization** of an $m \times n$ matrix A, when rank $A = n$. With this decomposition of A, the projection matrix can now be calculated easily as

$$P = A(A^TA)^{-1}A^T = QR(R^TQ^TQR)^{-1}R^TQ^T = QQ^T,$$

and $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = R^{-1}Q^T\mathbf{b}$.

Example 5.20 Let us find the projection matrix for

$$A = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: We first find the decomposition of A into Q and R, the orthogonal part and the upper triangular part:

$$\begin{aligned}
\mathbf{q}_1 &= \mathbf{c}_1 = (1,\ 1,\ 0,\ 0) \\
\mathbf{q}_2 &= \mathbf{c}_2 - \frac{\langle \mathbf{c}_2, \mathbf{q}_1\rangle}{\langle \mathbf{q}_1, \mathbf{q}_1\rangle}\mathbf{q}_1 = \left(\frac{1}{2},\ -\frac{1}{2},\ 1,\ 0\right) \\
\mathbf{q}_3 &= \mathbf{c}_3 - \frac{\langle \mathbf{c}_3, \mathbf{q}_2\rangle}{\langle \mathbf{q}_2, \mathbf{q}_2\rangle}\mathbf{q}_2 - \frac{\langle \mathbf{c}_3, \mathbf{q}_1\rangle}{\langle \mathbf{q}_1, \mathbf{q}_1\rangle}\mathbf{q}_1 = \left(-\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3},\ 1\right),
\end{aligned}$$

and $\|\mathbf{q}_1\| = \sqrt{2}$, $\|\mathbf{q}_2\| = \sqrt{3/2}$, $\|\mathbf{q}_3\| = \sqrt{7/3}$. Hence,

$$\begin{aligned}
\mathbf{u}_1 &= \frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) \\
\mathbf{u}_2 &= \frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} = \left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{\sqrt{2}}{\sqrt{3}},\ 0\right) \\
\mathbf{u}_3 &= \frac{\mathbf{q}_3}{\|\mathbf{q}_3\|} = \left(-\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{\sqrt{3}}{\sqrt{7}}\right).
\end{aligned}$$

Then $\mathbf{c}_1 = \sqrt{2}\mathbf{u}_1$, $\mathbf{c}_2 = \frac{1}{\sqrt{2}}\mathbf{u}_1 + \sqrt{\frac{3}{2}}\mathbf{u}_2$, $\mathbf{c}_3 = \frac{1}{\sqrt{2}}\mathbf{u}_1 + \frac{1}{\sqrt{6}}\mathbf{u}_2 + \sqrt{\frac{7}{3}}\mathbf{u}_3$. Therefore,

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{7}/\sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 0 & \sqrt{2}/\sqrt{3} & 2/\sqrt{21} \\ 0 & 0 & \sqrt{3}/\sqrt{7} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{7}/\sqrt{3} \end{bmatrix} = QR,
\end{aligned}$$

and

$$P = QQ^T = \begin{bmatrix} 6/7 & 1/7 & 1/7 & -2/7 \\ 1/7 & 6/7 & -1/7 & 2/7 \\ -1/7 & -1/7 & 6/7 & 2/7 \\ -2/7 & 2/7 & 2/7 & 3/7 \end{bmatrix}.$$

□

Problem 5.30 Find the 2×2 matrix P that projects the xy-plane onto the line $y = x$.

Problem 5.31 Find the projection matrix P of $\mathbb{R}^3$ onto the column space $C(A)$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem 5.32 Find the matrix for orthogonal projection from $\mathbb{R}^3$ to the plane spanned by the vectors $(1,\ 1,\ 1)$ and $(1,\ 0,\ 2)$.

Problem 5.33 Find the projection matrix P on the x_1, x_2, x_4 coordinate subspace of $\mathbb{R}^4$.

Problem 5.34 Find the QR factorization of the matrix $\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & 0 \end{bmatrix}$.

5.11 Exercises

5.1. Decide which of the following functions on $\mathbb{R}^2$ are inner products and which are not. For $\mathbf{x} = (x_1,\ x_2)$, $\mathbf{y} = (y_1,\ y_2)$,

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 x_2 y_2$,
(2) $\langle \mathbf{x}, \mathbf{y} \rangle = 4x_1 y_1 + 4x_2 y_2 - x_1 y_2 - x_2 y_1$,
(3) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 - x_2 y_1$,
(4) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2$,
(5) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$.

5.2. Show that the function $\langle A, B \rangle = \operatorname{tr}(A^T B)$ for $A, B \in M_{n\times n}(\mathbb{R})$ defines an inner product on $M_{n\times n}(\mathbb{R})$.

5.3. Find the angle between the vectors $(4,\ 7,\ 9,\ 1,\ 3)$ and $(2,\ 1,\ 1,\ 6,\ 8)$ in $\mathbb{R}^5$.

5.4. Determine the values of k so that the given vectors are orthogonal with respect to the Euclidean inner product in $\mathbb{R}^4$.

$$(1)\ \left\{ \begin{bmatrix} 2 \\ 3 \\ k \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ 3 \\ -5 \end{bmatrix} \right\}, \quad (2)\ \left\{ \begin{bmatrix} 2 \\ 8 \\ 4 \\ k \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 2 \\ k \end{bmatrix} \right\}.$$

5.5. Consider the space $C[0,\ 1]$ with the inner product defined by

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx.$$

Compute the length of each vector and the cosine of the angle between each pair of vectors in each of the following:

(1) $f(x) = 1,\ g(x) = x$;
(2) $f(x) = x^m,\ g(x) = x^n$, where $m,\ n$ are nonnegative integers;
(3) $f(x) = \sin \pi m x,\ g(x) = \sin \pi n x$, where $m,\ n$ are integers.

5.6. Prove that

$$(a_1 + \cdots + a_n)^2 \le n(a_1^2 + \cdots + a_n^2)$$

for any real numbers $a_1,\ a_2,\ \ldots,\ a_n$. When does equality hold?

5.7. Let $V = P_2([0,\ 1])$, the space of polynomials of degree ≤ 2 on $[0,\ 1]$. Equip V with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

(1) Compute $\langle f, g \rangle$ and $\|f\|$ for $f(x) = x + 2$ and $g(x) = x^2 - 2x - 3$.
(2) Find the orthogonal complement of the subspace of scalar polynomials.

5.8. Find an orthonormal basis for $\mathbb{R}^3$ with the Euclidean inner product by applying the Gram-Schmidt orthogonalization to the vectors $\mathbf{x} = (1,\ 0,\ 1)$, $\mathbf{x}_2 = (1,\ 0,\ -1)$, $\mathbf{x}_3 = (0,\ 3,\ 4)$.

5.9. Show that if $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then every scalar multiple of $\mathbf{u}$ is also orthogonal to $\mathbf{v}$. Find a unit vector orthogonal to $\mathbf{v}_1 = (1,\ 1,\ 2)$ and $\mathbf{v}_2 = (0,\ 1,\ 3)$ in $\mathbb{R}^3$.

5.10. Determine the orthogonal projection of $\mathbf{v}_1$ onto $\mathbf{v}_2$ for the following vectors in the n-space $\mathbb{R}^n$ with the Euclidean inner product.

(1) $\mathbf{v}_1 = (1,\ 2,\ 3)$, $\mathbf{v}_2 = (1,\ 1,\ 2)$,
(2) $\mathbf{v}_1 = (1,\ 2,\ 1)$, $\mathbf{v}_2 = (2,\ 1,\ -1)$,
(3) $\mathbf{v}_1 = (1,\ 0,\ 1,\ 0)$, $\mathbf{v}_2 = (0,\ 2,\ 2,\ 0)$.

5.11. Let $S = \{\mathbf{v}_i\}$, where $\mathbf{v}_i$'s are given below. For each S, find a basis for $S^\perp$ with respect to the Euclidean inner product on $\mathbb{R}^n$.

(1) $\mathbf{v}_1 = (0,\ 1,\ 0)$, $\mathbf{v}_2 = (0,\ 0,\ 1)$,
(2) $\mathbf{v}_1 = (1,\ 1,\ 0)$, $\mathbf{v}_2 = (1,\ 1,\ 1)$,
(3) $\mathbf{v}_1 = (1,\ 0,\ 1,\ 2)$, $\mathbf{v}_2 = (1,\ 1,\ 1,\ 1)$, $\mathbf{v}_3 = (2,\ 2,\ 0,\ 1)$.

5.12. Which of the following matrices are orthogonal?

$$(1) \begin{bmatrix} 1/2 & -1/3 \\ -1/2 & 1/3 \end{bmatrix}, \qquad (2) \begin{bmatrix} 4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix},$$

$$(3) \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}, \quad (4) \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}.$$

5.13. Consider $\mathbb{R}^4$ with the Euclidean inner product. Let W be the subspace of $\mathbb{R}^4$ consisting of all vectors that are orthogonal to both $\mathbf{x} = (1,\ 0,\ -1,\ 1)$ and $\mathbf{y} = (2,\ 3,\ -1,\ 2)$. Find a basis for W.

5.14. Let V be an inner product space. For vectors $\mathbf{x}$ and $\mathbf{y}$ in V, establish the following identities:

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x}-\mathbf{y}\|^2$ (polarization identity),

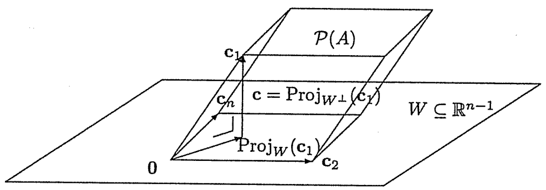
(2) $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\right)$ (Polarization identity),

(3) $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$ (parallelogram equality).

5.15. Show that $\mathbf{x}+\mathbf{y}$ is perpendicular to $\mathbf{x}-\mathbf{y}$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

5.16. Let A be the $m \times n$ matrix whose columns are $\mathbf{c}_1, \ldots, \mathbf{c}_n$ in $\mathbb{R}^m$. Prove that the volume of the n-dimensional parallelepiped $\mathcal{P}(A)$ determined by those vectors $\mathbf{c}_j$'s in $\mathbb{R}^m$ is given by

$$\mathrm{vol}(A) = \sqrt{\det(A^T A)}\,.$$



(Note that the volume of the n-dimensional parallelepiped determined by $\mathbf{c}_1$, ..., $\mathbf{c}_n$ in $\mathbb{R}^m$ is by definition the multiplication of the volume of the $(n-1)$-dimensional parallelepiped (base) determined by $\mathbf{c}_2, \ldots, \mathbf{c}_n$ and the height of $\mathbf{c}_1$ from the plane W which is spanned by $\mathbf{c}_2, \ldots, \mathbf{c}_n$. Here, the height is the length of the vector $\mathbf{c} = \mathbf{c}_1 - \mathrm{Proj}_W(\mathbf{c}_1)$, which is orthogonal to W. If the vectors are linearly dependent, then the parallelepiped is degenerate, *i.e.*, it is contained in a subspace of dimension less than n.)

5.17. Find the volume of the three-dimensional tetrahedron in $\mathbb{R}^4$ whose vertices are at $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,2,2)$ and $(0,0,1,2)$.

5.18. For an orthogonal matrix A, show that $\det A = \pm 1$. Give an example of an orthogonal matrix A for which $\det A = -1$.

5.19. Find orthonormal bases for the row space and the null space of each of the following matrices.

$$(1)\ \begin{bmatrix} 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix},\quad (2)\ \begin{bmatrix} 1 & 4 & 0 \\ -2 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix},\quad (3)\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

5.20. Let A be an $m \times n$ matrix of rank r. Find a relation of m, n and r so that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every $\mathbf{b} \in \mathbb{R}^m$.

5.21. Find the equation of the straight line that best fits the data of the four points $(0,\ 1)$, $(1,\ 3)$, $(2,\ 4)$, and $(3,\ 4)$.

5.22. Find the cubic polynomial that best fits the data of the five points $(-1,\ -14)$, $(0,\ -5)$, $(1,\ -4)$, $(2,\ 1)$, and $(3,\ 22)$.

5.23. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_i$'s given in each of the following problems. Find the projection matrix P for the subspace W and the null space $\mathcal{N}(P)$ of P. Compute $P\mathbf{b}$ for $\mathbf{b}$ given in each problem.

(1) $\mathbf{x}_1 = (1,\ 1,\ 1,\ 1)$, $\mathbf{x}_2 = (1,\ -1,\ 1,\ -1)$, $\mathbf{x}_3 = (-1,\ 1,\ 1,\ 0)$, and $\mathbf{b} = (1,\ 2,\ 1,\ 1)$.
(2) $\mathbf{x}_1 = (0,\ -2,\ 2,\ 1)$, $\mathbf{x}_2 = (2,\ 0,\ -1,\ 2)$, and $\mathbf{b} = (1,\ 1,\ 1,\ 1)$.
(3) $\mathbf{x}_1 = (2,\ 0,\ 3,\ -6)$, $\mathbf{x}_2 = (-3,\ 6,\ 8,\ 0)$, and $\mathbf{b} = (-1,\ 2,\ -1,\ 1)$.

5.24. Find the projection matrix for the row space and the null space of each of the following matrices:

$$(1)\ \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix},\quad (2)\ \begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix},\quad (3)\ \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 2 \\ 2 & 3 & -1 \end{bmatrix}.$$

5.25. Consider the space $C[-1,\ 1]$ with the inner product defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

A function $f \in C[-1,\ 1]$ is *even* if $f(-x) = f(x)$, or *odd* if $f(-x) = -f(x)$. Let U and V be the sets of all even functions and odd functions in $C[-1,\ 1]$, respectively.

(1) Prove that U and V are subspaces and $C[-1,\ 1] = U + V$.
(2) Prove that $U \perp V$.
(3) Prove that for any $f \in C[-1,\ 1]$, $\|f\|^2 = \|h\|^2 + \|g\|^2$ where $f = h + g \in U \oplus V$.

5.26. Determine whether the following statements are true or false, in general, and justify your answers.

(1) Two vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space are linearly independent if and only if the angle between $\mathbf{x}$ and $\mathbf{y}$ is not zero.
(2) If V is perpendicular to W, then $V^\perp$ is perpendicular to $W^\perp$.
(3) Every permutation matrix is an orthogonal matrix.
(4) The projection of $\mathbb{R}^m$ on a subspace W is a linear transformation of $\mathbb{R}^m$ into itself.
(5) Two different subspaces of $\mathbb{R}^m$ may have the same projection matrix.
(6) An $n \times n$ symmetric matrix A is a projection matrix if and only if $A^2 = I$.
(7) For any $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$, $A^T A\mathbf{x} = A^T\mathbf{b}$ always has a solution.
(8) An inner product can be defined on every vector space.
(9) Let V be an inner product space. Then $\|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|$ for any vectors $\mathbf{x}$ and $\mathbf{y}$ in V.
(10) The least square solution of $A\mathbf{x} = \mathbf{b}$ is unique for any symmetric matrix A.
(11) Every system of linear equations has a least square solution.
(12) The least square solution of $A\mathbf{x} = \mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ on the column space A.

Chapter 6

Eigenvectors and Eigenvalues

6.1 Introduction

Gaussian elimination plays a fundamental role in solving a system $A\mathbf{x} = \mathbf{b}$ of linear equations. In order to solve a system of linear equations, Gaussian elimination reduces the augmented matrix to a (reduced) row-echelon form by using elementary row operations that preserve row and null spaces.

In this chapter, as another method of simplifying a square matrix, we examine which matrices can be similar to diagonal matrices, and what the transition matrices are in this case. The tools are *eigenvalues* and *eigenvectors*. In fact, they play important roles in their own right in mathematics and have far-reaching applications not only in mathematics, but also other fields of science and engineering. Some specific applications with a square matrix A are **(1)** solving systems $A\mathbf{x} = \mathbf{b}$ of linear equations, **(2)** checking the invertibility of A or estimation of $\det A$, **(3)** calculating a power A^n or the limit of a matrix series $\sum_{n=1}^{\infty} A^n$, **(4)** solving systems of linear differential equations or difference equations, **(5)** finding a simple form of the matrix representation of a linear transformation, etc. One might notice that some of the problems listed above are easy if A is diagonal.

We begin by introducing eigenvalues and eigenvectors of a square matrix A. For an $n \times n$ square matrix A, there may exist a nonzero vector that is transformed by A into a scalar multiple of itself.

Definition 6.1 Let A be an $n \times n$ square matrix. A nonzero vector $\mathbf{x}$ in the n-space $\mathbb{R}^n$ is called an **eigenvector** (or **characteristic vector**) of A

if there is a scalar λ in $\mathbb{R}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The scalar λ is called an **eigenvalue** (or **characteristic value**) of A, and we say $\mathbf{x}$ **belongs** to λ.

Geometrically, an eigenvector of a matrix A is a nonzero vector $\mathbf{x}$ in the n-space $\mathbb{R}^n$ to which $A\mathbf{x}$ is parallel. Algebraically, an eigenvector $\mathbf{x}$ is a nontrivial solution of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ of linear equations, that is, an eigenvector $\mathbf{x}$ is a nonzero vector in the null space $\mathcal{N}(\lambda I - A)$. There are two unknowns in this equation: an eigenvalue λ and an eigenvector $\mathbf{x}$. To find those unknowns, first we should find an eigenvalue λ by using the fact that the equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$ if and only if λ satisfies the equation

$$\det(\lambda I - A) = 0.$$

Note that the left side is a polynomial of degree n in λ, called the **characteristic polynomial** of A. Thus the eigenvalues are simply the roots of the equation $\det(\lambda I - A) = 0$.

Thus, to find eigenvectors of A, first find the roots (or eigenvalues of A) of the equation $\det(\lambda I - A) = 0$, and then solve the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ for each eigenvalue λ. In summary, by referring to Theorem 3.25 we have the following theorem.

Theorem 6.1 *If A is an $n \times n$ matrix, then the following are equivalent:*

(1) *λ is an eigenvalue of A;*

(2) $\det(\lambda I - A) = 0$ (*or* $\det(A - \lambda I) = 0$);

(3) *$\lambda I - A$ is singular;*

(4) *the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*

Hence, the eigenvectors of A belonging to an eigenvalue λ are just the nonzero vectors $\mathbf{x}$ in the null space $\mathcal{N}(\lambda I - A)$. We call this null space the **eigenspace** of A belonging to λ, and denote it $E(\lambda)$.

Example 6.1 Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.$$