SC223 - Linear Algebra

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Lecture 29

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Change of Basis (Similarity Transformation)

- Let $T \in \mathcal{L}(U, V)$.
- We have seen how to compute $[T]_{\beta_U}^{\beta_V}$, the matrix representation of T w.r.t the basis β_U and β_V.
- What happens if we choose a different basis, say α_U and α_V. Are $[T]_{\beta_U}^{\beta_V}$ and $[T]_{\alpha_U}^{\alpha_V}$ different?
- How are they related?

Change of Basis (Similarity Transformation)

- Let $T \in \mathcal{L}(U, V)$.
- Let $\beta_U = \{u_1, \ldots, u_n\}$ and $\beta_V = \{v_1, \ldots, v_m\}$ be basis of U and V, and let $[T]_{\beta_U}^{\beta_V}$ denote the matrix representation of T w.r.t β_U and β_V.

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- Now, $U \cong \mathbb{F}^n$, and $V \cong \mathbb{F}^m$.

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- Now, $U \cong \mathbb{F}^n$, and $V \cong \mathbb{F}^m$.
- Let $N_{\beta_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\beta_U}(u_1) = e_1^n, \dots, N_{\beta_U}(u_n) = e_n^n$,

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}_{n\times 1} \qquad \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}_{n\times 1}$$

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- Now, $U \cong \mathbb{F}^n$, and $V \cong \mathbb{F}^m$.
- Let $N_{\beta_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\beta_U}(u_1) = e_1^n, \ldots, N_{\beta_U}(u_n) = e_n^n$, and $M_{\beta_V} \in \mathcal{L}(V, \mathbb{F}^m)$ be defined as $\underline{M_{\beta_V}(v_1)} = \underline{e_1^m}, \ldots, M_{\beta_V}(v_n) = e_n^m$.

$\forall x \in U$, $N_{\beta_U}(x) = [x]_{\beta_U}$

$$e_1^m = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

$\forall y \in V$, $M_{\beta_V}(y) = [y]_{\beta_V}$

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- Now, $U \cong \mathbb{F}^n$, and $V \cong \mathbb{F}^m$.
- Let $N_{\beta_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\beta_U}(u_1) = e_1^n, \ldots, N_{\beta_U}(u_n) = e_n^n$, and $M_{\beta_V} \in \mathcal{L}(V, \mathbb{F}^m)$ be defined as $M_{\beta_V}(v_1) = e_1^m, \ldots, M_{\beta_V}(v_n) = e_n^m$.
- $x \in U, N_{\beta_U}(x) = [x]_{\beta_U}$, and $y \in V, M_{\beta_V}(y) = [y]_{\beta_V}$.

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- Now, $U \cong \mathbb{F}^n$, and $V \cong \mathbb{F}^m$.
- Let $N_{\beta_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\beta_U}(u_1) = e_1^n, \ldots, N_{\beta_U}(u_n) = e_n^n$, and $M_{\beta_V} \in \mathcal{L}(V, \mathbb{F}^m)$ be defined as $M_{\beta_V}(v_1) = e_1^m, \ldots, M_{\beta_V}(v_n) = e_n^m$.
- $x \in U$, $N_{\beta_U}(x) = [x]_{\beta_U}$, and $y \in V$, $M_{\beta_V}(y) = [y]_{\beta_V}$.
- Let $\alpha_U = \{p_1, \ldots, p_n\}, \alpha_V = \{q_1, \ldots, q_m\}$, $U \cong \mathbb{F}^n$, be different set of basis vector for U and V resp.

$U \to \mathbb{F}^n$ $\qquad$ $V \to \mathbb{F}^m$.

$N_{\alpha_U}(p_i) = e_i^n$, $i = 1, \ldots, n$ $\qquad$ $M_{\alpha_V}(q_i) = e_i^m$, $i = 1, \ldots, m$.

$N_{\alpha_U}(x) = [x]_{\alpha_U}$ $\qquad$ $M_{\alpha_V}(y) = [y]_{\alpha_V}$.

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- $x \in U, N_{\beta_U}(x) = [x]_{\beta_U}$, and $y \in V, M_{\beta_V}(y) = [y]_{\beta_V}$.
- Let $\alpha_U = \{p_1, \dots, p_n\}, \alpha_V = \{q_1, \dots, q_m\}$, $U \cong \mathbb{F}^n$, be different set of basis vector for U and V resp. Let $N_{\alpha_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\alpha_U}(p_1) = e_1^n, \dots, N_{\alpha_U}(p_n) = e_n^n$

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- $x \in U, N_{\beta_U}(x) = [x]_{\beta_U}$, and $y \in V, M_{\beta_V}(y) = [y]_{\beta_V}$.
- Let $\alpha_U = \{p_1, \ldots, p_n\}, \alpha_V = \{q_1, \ldots, q_m\}$, $U \cong \mathbb{F}^n$, be different set of basis vector for U and V resp. Let $N_{\alpha_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\alpha_U}(p_1) = e_1^n, \ldots, N_{\alpha_U}(p_n) = e_n^n$ and $M_{\alpha_V} \in \mathcal{L}(V, \mathbb{F}^m)$ be defined as $M_{\alpha_V}(q_1) = e_1^m, \ldots, M_{\alpha_V}(q_n) = e_n^m$.

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- $x \in U, N_{\beta_U}(x) = [x]_{\beta_U}$, and $y \in V, M_{\beta_V}(y) = [y]_{\beta_V}$.
- Let $\alpha_U = \{p_1, \ldots, p_n\}, \alpha_V = \{q_1, \ldots, q_m\}$, $U \cong \mathbb{F}^n$, be different set of basis vector for U and V resp. Let $N_{\alpha_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\alpha_U}(p_1) = e_1^n, \ldots, N_{\alpha_U}(p_n) = e_n^n$ and $M_{\alpha_V} \in \mathcal{L}(V, \mathbb{F}^m)$ be defined as $M_{\alpha_V}(q_1) = e_1^m, \ldots, M_{\alpha_V}(q_n) = e_n^m$.
- $x \in U, N_{\alpha_U}(x) = [x]_{\alpha_U}$, and $y \in V, M_{\alpha_V}(y) = [y]_{\alpha_V}$.
- Given $[T]_{\beta_U}^{\beta_V}$, how to compute $[T]_{\alpha_U}^{\alpha_V}$?

- Commutative Diagram

$$
\begin{tikzcd}
{[x]_{\beta_U} \in \mathbb{F}^n} \arrow[rr, "{[T]_{\beta_U}^{\beta_V}}"] & & \mathbb{F}^m \ni {[y = Tx]_{\beta_V}} \arrow[d, bend left, "(M_{\beta_V})^{-1}"] \\
x \in U \arrow[u, "N_{\beta_U}"] \arrow[d, "N_{\alpha_U}"'] \arrow[rr, "T"] & & V \arrow[u, "M_{\beta_V}"'] \arrow[d, "M_{\alpha_V}"] \\
{[x]_{\alpha_U} \in \mathbb{F}^n} \arrow[u, bend left, "(N_{\alpha_U})^{-1}"] \arrow[rr, "{[T]_{\alpha_U}^{\alpha_V}\,?}"] & & \mathbb{F}^m \ni {[y = Tx]_{\alpha_V}}
\end{tikzcd}
$$

?

X

$$[y]_{\alpha_V} = \underbrace{M_{\alpha_V} M_{\beta_V}^{-1} [T]_{\beta_U}^{\beta_V} N_{\beta_U} N_{\alpha_U}^{-1}} [x]_{\alpha_U}$$

$$[y]_{\alpha_V} = \underline{\underline{[T]_{\alpha_U}^{\alpha_V}}} [x]_{\alpha_U}$$

$$\underbrace{\left(N_{\beta_U} N_{\alpha_U}^{-1}\right)}_{N_{\alpha_U}^{\beta_U}} \ ?$$

- Commutative Diagram

$$
\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \\
U \arrow[u, "N_{\beta_U}"] \arrow[r, "T"] \arrow[d, "N_{\alpha_U}"'] & V \arrow[u, "M_{\beta_V}"'] \arrow[d, "M_{\alpha_V}"] \\
\mathbb{F}^n \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & \mathbb{F}^m
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \arrow[d, "\underbrace{M_{\alpha_V}(M_{\beta_V})^{-1}}_{M_{\beta_V}^{\alpha_V}}"] \\
\mathbb{F}^n \arrow[u, "\underbrace{N_{\beta_U}(N_{\alpha_U})^{-1}}_{N_{\alpha_U}^{\beta_U}}"] \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & \mathbb{F}^m
\end{tikzcd}
$$

$M_{\alpha_V} M_{\beta_V}^{-1}$

$M_{\beta_V}^{\alpha_V}$

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U \arrow[u, "N_{\beta_U}"] \arrow[r, "T"] \arrow[d, "N_{\alpha_U}"'] & V \arrow[u, "M_{\beta_V}"'] \arrow[d, "M_{\alpha_V}"] \\
\mathbb{F}^n \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & \mathbb{F}^m
\end{tikzcd}
$$

$$
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\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \arrow[d, "\underbrace{M_{\alpha_V}(M_{\beta_V})^{-1}}_{M_{\beta_V}^{\alpha_V}}"] \\
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\end{tikzcd}
$$

$$
\begin{tikzcd}
{[x]_{\beta_U} = N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}} \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & {[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V}[x]_{\beta_U}} \arrow[d, "M_{\beta_V}^{\alpha_V}"] \\
{[x]_{\alpha_U}} \arrow[u, "N_{\alpha_U}^{\beta_U}"] \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & {[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[y]_{\beta_V}}
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\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \arrow[d, "\underbrace{M_{\alpha_V}(M_{\beta_V})^{-1}}_{M_{\beta_V}^{\alpha_V}}"] \\
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{[x]_{\beta_U} = N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}} \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & {[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V}[x]_{\beta_U}} \arrow[d, "M_{\beta_V}^{\alpha_V}"] \\
{[x]_{\alpha_U}} \arrow[u, "N_{\alpha_U}^{\beta_U}"] \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & {[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[y]_{\beta_V}}
\end{tikzcd}
$$

- Thus,

$$[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}, \forall x \in U$$

- Commutative Diagram

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\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \\
U \arrow[u, "N_{\beta_U}"] \arrow[r, "T"] \arrow[d, "N_{\alpha_U}"'] & V \arrow[u, "M_{\beta_V}"'] \arrow[d, "M_{\alpha_V}"] \\
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\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \arrow[d, "\underbrace{M_{\alpha_V}(M_{\beta_V})^{-1}}_{M_{\beta_V}^{\alpha_V}}"] \\
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{[x]_{\beta_U} = N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}} \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & {[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V}[x]_{\beta_U}} \arrow[d, "M_{\beta_V}^{\alpha_V}"] \\
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- Thus,

$$[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}, \forall x \in U$$

$$[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}$$

- $[T]^{\alpha_V}_{\alpha_U} = M^{\alpha_V}_{\beta_V}[T]^{\beta_V}_{\beta_U}N^{\beta_U}_{\alpha_U}$

- $[T]^{\alpha_V}_{\alpha_U} = M^{\alpha_V}_{\beta_V}[T]^{\beta_V}_{\beta_U}N^{\beta_U}_{\alpha_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and $\alpha_U = \alpha_V = \alpha$.

$$[T]^{\alpha}_{\alpha} = M^{\alpha}_{\beta}[T]^{\beta}_{\beta}N^{\beta}_{\alpha}$$

$$N^{\alpha}_{\beta} = (N^{\beta}_{\alpha})^{-1}$$

?

- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V} N_{\alpha_U}^{\beta_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and

$\alpha_U = \alpha_V = \alpha$.
- In this case, $[T]_\alpha^\alpha = M_\beta^\alpha [T]_\beta^\beta N_\alpha^\beta$.

- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and

$\alpha_U = \alpha_V = \alpha$.
- In this case, $[T]_{\alpha}^{\alpha} = M_{\beta}^{\alpha}[T]_{\beta}^{\beta}N_{\alpha}^{\beta}$.
- Note that $M_{\beta}^{\alpha} = \left(N_{\alpha}^{\beta}\right)^{-1}$.

- $[T]^{\alpha_V}_{\alpha_U} = M^{\alpha_V}_{\beta_V}[T]^{\beta_V}_{\beta_U}N^{\beta_U}_{\alpha_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and $\alpha_U = \alpha_V = \alpha$.
- In this case, $[T]^{\alpha}_{\alpha} = M^{\alpha}_{\beta}[T]^{\beta}_{\beta}N^{\beta}_{\alpha}$.
- Note that $M^{\alpha}_{\beta} = \left(N^{\beta}_{\alpha}\right)^{-1}$. Denote M^{α}_{β} by S, which gives us $[T]^{\alpha}_{\alpha} = S[T]^{\beta}_{\beta}S^{-1}$.

- $[T]^{\alpha_V}_{\alpha_U} = M^{\alpha_V}_{\beta_V}[T]^{\beta_V}_{\beta_U}N^{\beta_U}_{\alpha_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and $\alpha_U = \alpha_V = \alpha$.
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- Note that $M^{\alpha}_{\beta} = \left(N^{\beta}_{\alpha}\right)^{-1}$. Denote M^{α}_{β} by S, which gives us $[T]^{\alpha}_{\alpha} = S[T]^{\beta}_{\beta}S^{-1}$.
- **Similar matrices and Similarity transformation**: We say two matrices A and B are similar if there exists an invertible matrix, say S such that $B = SAS^{-1}$. The transformation $A \mapsto SAS^{-1}$ is said to be a similarity transformation of A by S.

$$[T]^{time}_{time} = \left(N^{freq}_{time}\right)^{-1}[T]^{freq}_{freq}N^{freq}_{time}$$

$$T : U \to U$$

β is a basis of U.

$$[T]^{\beta}_{\beta} \to [T]^{\alpha}_{\alpha}$$

$$[T]_{\beta}^{\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\downarrow$$

$$\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$$

$$\underline{\begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & a \\ b & \lambda_2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{\lambda_3} \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cancel{\alpha_1} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \end{bmatrix}$$