LECTURE 14

$$\frac{d^2x}{dt^2} = \frac{x(t_0+h) - 2x(t_0) + x(t_0-h)}{(\quad)}$$

$t_0 = 0$

$$\frac{d^2x}{dt^2} = \frac{x(h) - 2x(0) + x(-h)}{(\quad)}$$

$x(0) = x_0$

$v(0) = v_0$

$$v = \frac{dx}{dt} = \frac{x(t_0) - x(t_0-h)}{(\quad)}$$

$$v_0 = \frac{x(0) - x(-h)}{(\quad)}.$$

Solve for $x(-h)$ and sub.

$$\frac{d^2u}{d\theta^2} + \cdot \quad \cdot$$

$$\left(\frac{du}{d\theta}\right)^2 \qquad \left(\frac{du}{d\theta}\right) = f(u, \theta)$$

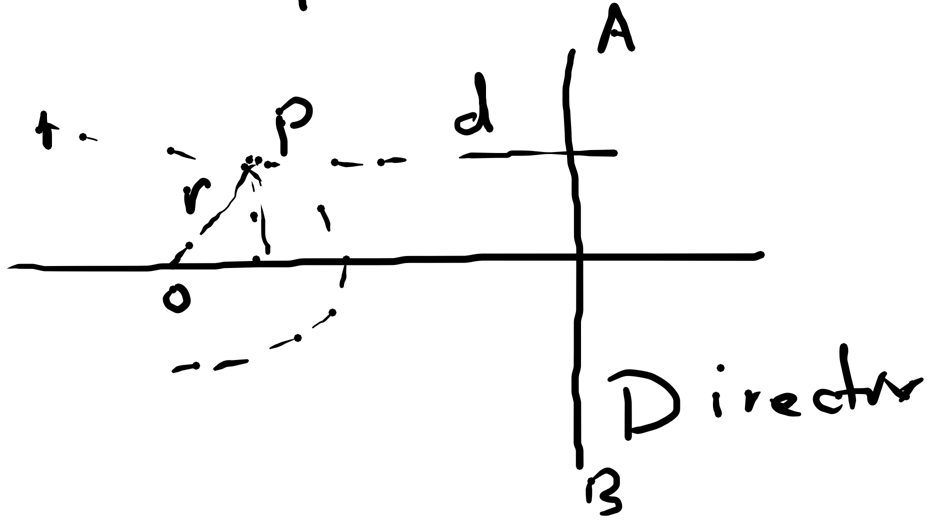
☑ <u>Recap</u>: $V(r) = -\alpha/r$ (gravitational potential)

$= -\alpha u$, where $u = 1/r$.

Used

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{L^2}(E - V)$$

to derive, $r = \dfrac{p}{1 + \epsilon \cos\theta}$ → polar eqn. for conic sections. which are defined as



$$\frac{r}{d} = \epsilon = \text{const}.$$

$\boxed{/\!/}$ $V(r) = \beta / r^2$

$$V_{eff}(r) = \left(\frac{L^2}{2mr^2} + \frac{\beta}{r^2} \right) = \frac{L^2}{2mr^2} \left(1 + \frac{2m\beta}{L^2} \right) = \frac{a^2 L^2}{2mr^2}$$

Eqn. for path :-

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{L^2} E - \frac{2m}{L^2} V \qquad = \frac{2mE}{L^2} - \frac{2m}{L^2} \beta u^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + a^2 u^2 = \frac{2mE}{L^2} \quad , \quad a^2 \equiv 1 + \frac{2m\beta}{L^2}$$

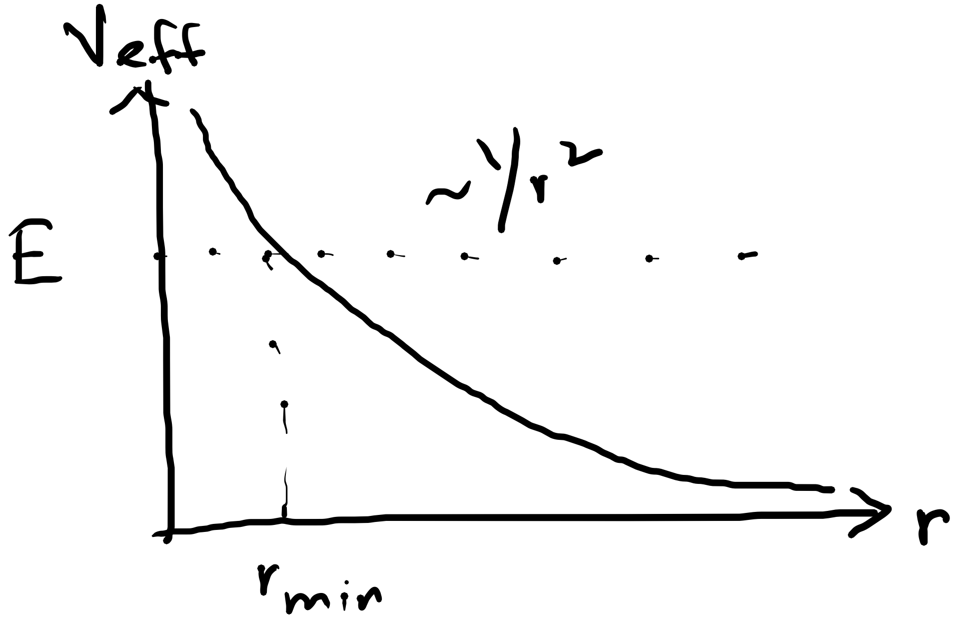
Case 1 :- $\quad a^2 > 0 \Rightarrow 1 + \frac{2m\beta}{L^2} > 0 \Rightarrow \beta > -\frac{L^2}{2m}$

$$\left(\frac{du}{d\theta}\right) = \sqrt{\frac{2mE}{L^2} - a^2 u^2}$$

$$\Rightarrow u = \frac{1}{a} \sqrt{\frac{2mE}{L^2}} \sin a\theta = \frac{1}{r}$$

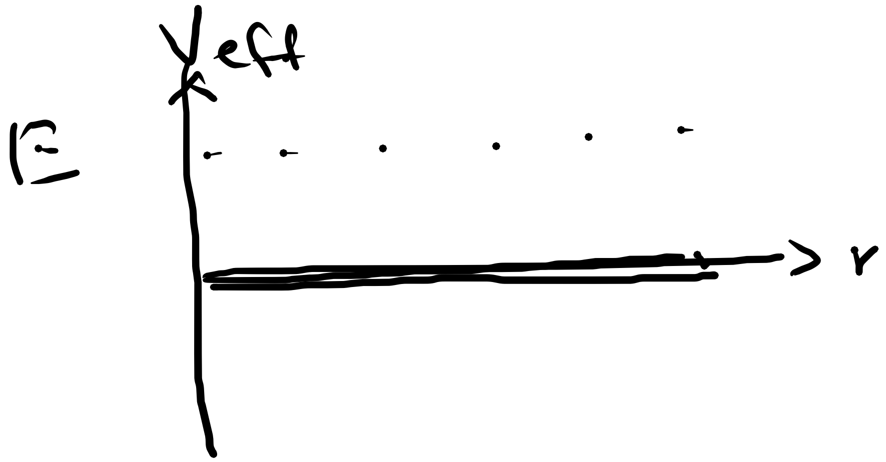
When $\theta = 0$, $r \to \infty = r_{max}$

$$r_{min} = a \sqrt{L^2 / (2mE)}$$



Case 2 :- $a = 0$

$$a^2 = 1 + \frac{2m\beta}{L^2} = 0 \Rightarrow \beta = -\frac{L^2}{2m}$$



$$V_{eff} = \frac{a^2 L^2}{2mr^2} = 0$$

$$\left(\frac{du}{d\theta}\right)^2 + \cancel{u^2} = \frac{2m}{L^2}\left(E - \beta/r^2\right)$$

$$= \frac{2m}{L^2}\left(E + \frac{L^2}{2m}u^2\right) = \frac{2mE}{L^2} + \cancel{u^2}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right) = \sqrt{\frac{2mE}{L^2}}$$

$$\Rightarrow \quad u = \theta \sqrt{\frac{2mE}{L^2}}$$

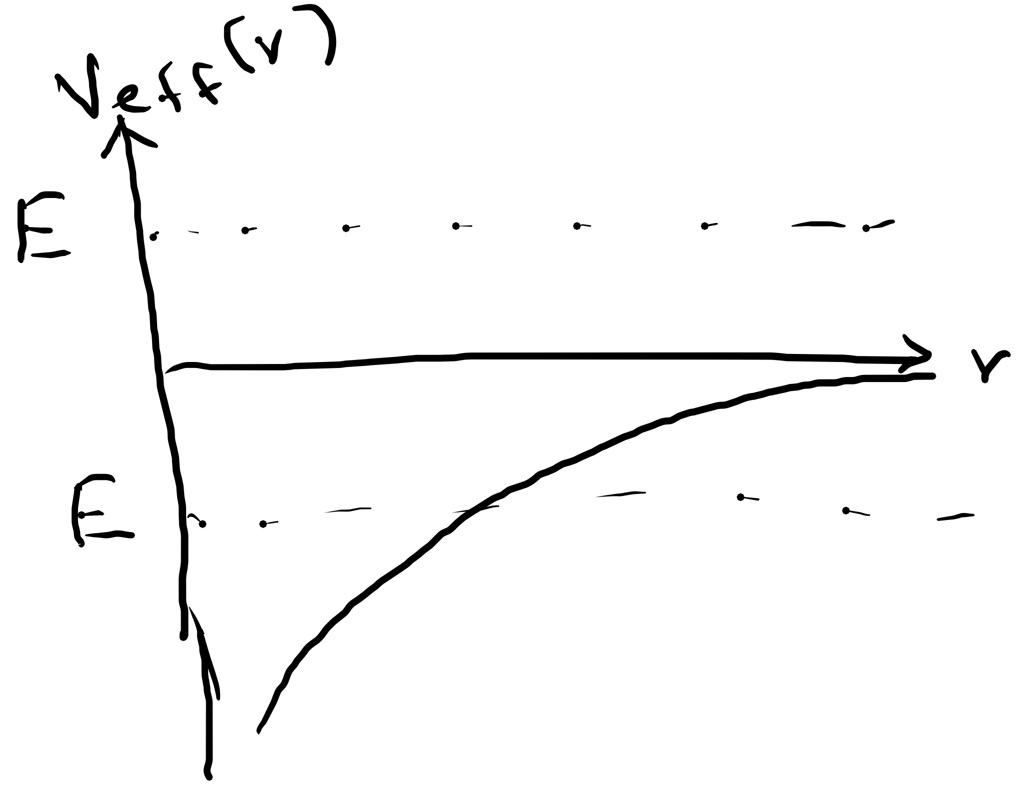
$$\Rightarrow \quad r = \frac{1}{\theta} \sqrt{\frac{L^2}{2mE}}$$

<u>Case 3</u> :- $a^2 < 0$.

$$1 + \frac{2m\beta}{L^2} < 0$$

$$\Rightarrow \quad \frac{2m\beta}{L^2} < -1$$

$$\Rightarrow \quad \beta < -\frac{L^2}{2m}$$



$$\left(\frac{du}{d\theta}\right)^2 + a^2u^2 = \frac{2mE}{L^2}$$

$$a^2 = -b^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 - b^2u^2 = \frac{2mE}{L^2}$$

Case 1 :- $\underline{E > 0}$

$$\left(\frac{du}{d\theta}\right) = \sqrt{\frac{2mE}{L^2} + b^2u^2}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{b}\sqrt{\frac{2mE}{L^2}}\ \underbrace{\sinh b\theta}_{\text{has no maximum}}$$

$$\sqrt{(\ ) - u^2}$$

$$\sin^2 x + \cos^2 x = 1.$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{1}{r} = \frac{1}{b}\sqrt{\frac{2mE}{L^2}}\ \sinh b\theta$$

$r \to 0$ at some finite θ $\Rightarrow$ particle hits the origin.

Remark: Adjust initial conditions to obtain an orbit of choice.

Case 2:- $E < 0$:- R.H.S. $= \frac{2mE}{L^2} = -\frac{2m|E|}{L^2}$

$$\left(\frac{du}{d\theta}\right)^2 - b^2u^2 = -\frac{2m|E|}{L^2}$$

$$\Rightarrow \quad b^2u^2 - \left(\frac{du}{d\theta}\right)^2 = \frac{2m|E|}{L^2}$$

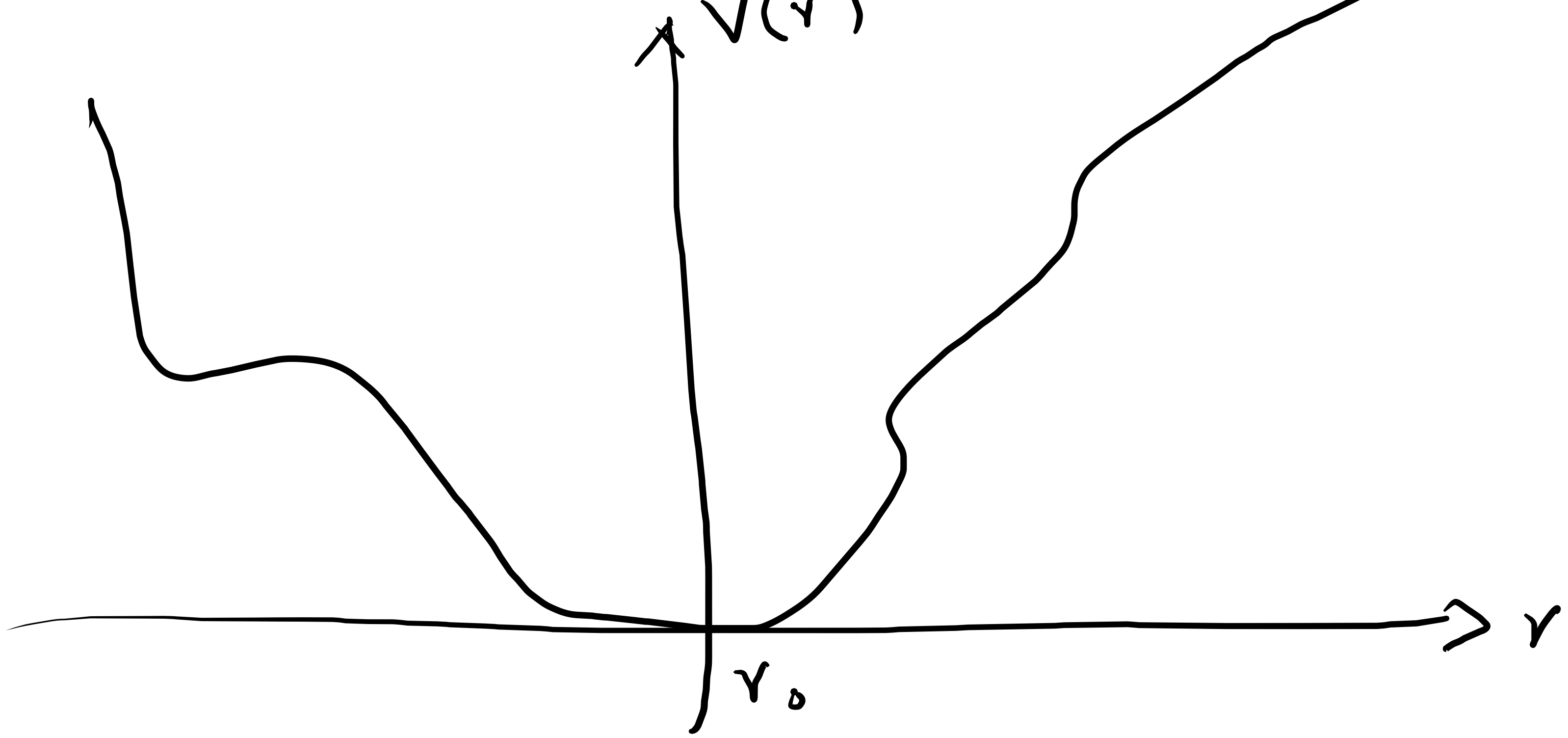
$$\Rightarrow \quad \frac{1}{r} = \frac{1}{b}\sqrt{\frac{2m|E|}{L^2}}\ \underbrace{\cosh(b\theta)}_{\text{has no maximum but has a finite minimum.}}$$

$$r_{max} = b\sqrt{\frac{L^2}{2m|E|}}$$

POSSIBLE TYPO, VERIFY

☒ $V(r) \approx r^2$ $\longrightarrow$ orbit is an ellipse.

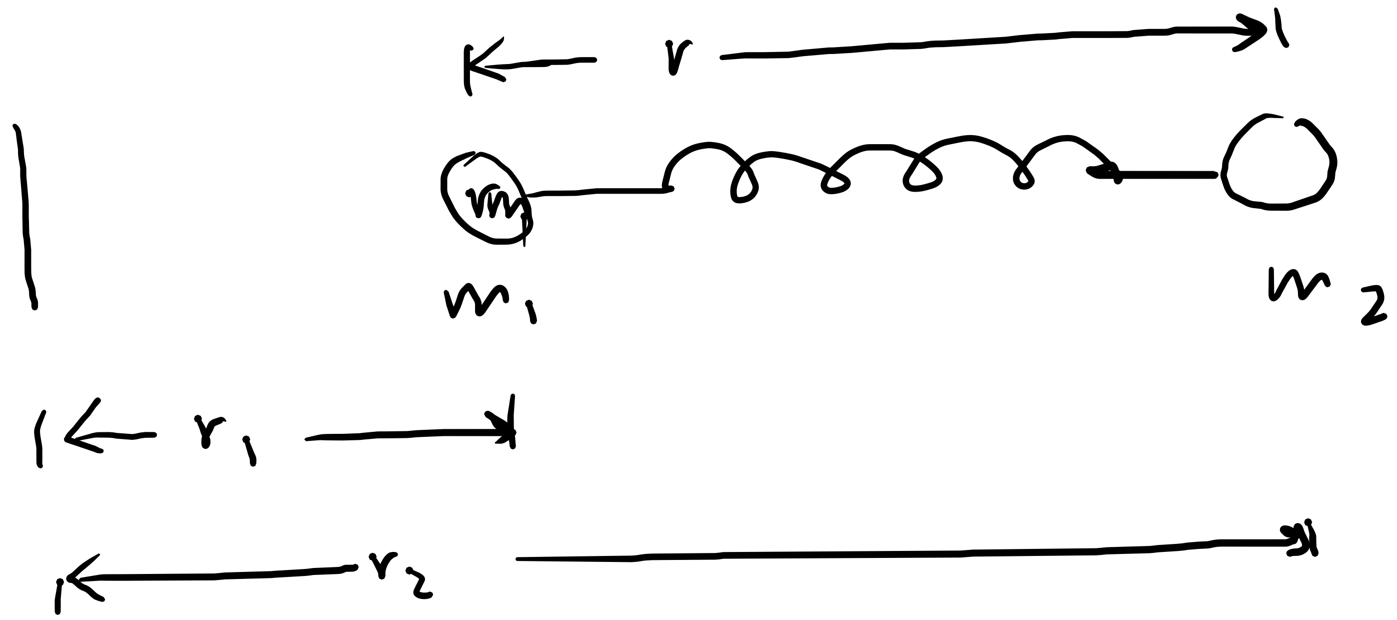
$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \left(\quad \right)$$



$$V(r) = V(r_0) + \cancel{(r-r_0)\frac{dV}{dr_0}\Big|_{r=r_0}} \xrightarrow{} 0$$
$$+ \frac{1}{2}(r-r_0)^2 \left(\frac{d^2V}{dr_0}\right)\Big|_{r=r_0}$$
$$+ \cdots$$
$$= V(r_0) + \frac{1}{2}(r-r_0)^2\left(\frac{d^2V}{dr^2}\right)\Big|_{r=r_0}$$

- Any potential with an extremum can be modeled near its extremum by a quadratic potential.

– Reduced mass.



$$r = r_2 - r_1$$

$$V(r) = \frac{1}{2} k (r - r_0)^2$$

$$F(r) \sim (r - r_0)$$

$$m_1 \ddot{r}_1 = k (r - r_0)$$

$$m_2 \ddot{r}_2 = -k (r - r_0) .$$

$$\ddot{r}_1 = \frac{k}{m_1} (r - r_0)$$

$$\ddot{r}_2 = -\frac{k}{m_2} (r - r_0)$$

~~Lagrangian~~

$$\ddot{r}_2 - \ddot{r}_1 = -k \left(\frac{1}{m_1} + \frac{1}{m_2} \right) (r - r_0)$$

$$\Rightarrow \ddot{r} = -\frac{k}{\mu} (r - r_0)$$

$$\boxed{\mu = \frac{1}{m_1} + \frac{1}{m_2}}$$