LECTURE 10

Morin

Recap

- Central forces $\vec{F} = f(r)\hat{r} = f(r)\dfrac{\vec{r}}{r}$
- Motion takes place in a plane $\Rightarrow$ only TWO coordinates can be used to describe the motion.

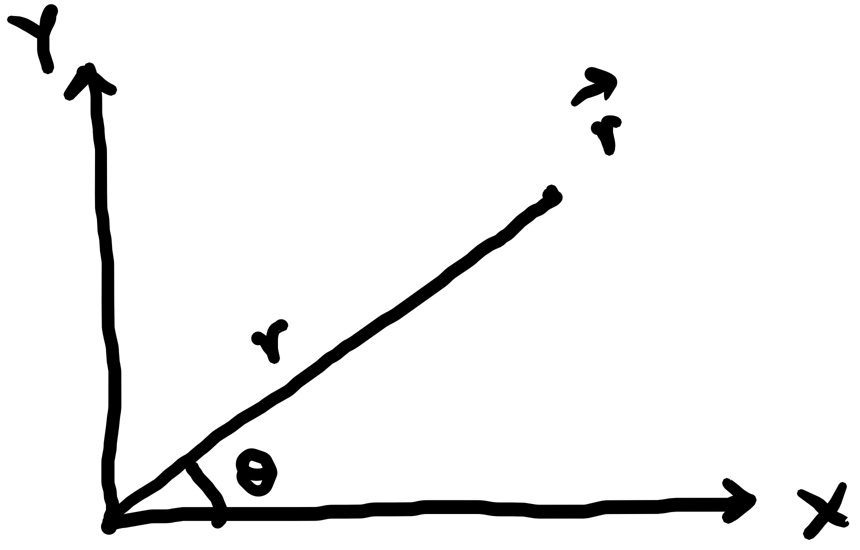
$$\Downarrow$$

$$\vec{h} = \vec{r} \times \vec{v} = \text{const.}$$

$$\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) = m\vec{h} = \text{const.}$$

- Central force is conservative $\Rightarrow$ total energy is const.

— Coordinate system.



$$\vec{F} = \frac{f(r)\,\vec{r}}{r}$$

$$(\hat{x}, \hat{y}) \longleftrightarrow (\hat{r}, \hat{\theta})$$

$$\vec{r} = r\,\hat{r}$$

$$\vec{r} = x\,\hat{x} + y\,\hat{y}$$

$$= r\,(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$

$$\hat{r} = \frac{\partial \vec{r}}{\partial r} \Big/ \left| \frac{\partial \vec{r}}{\partial r} \right|$$

$$\hat{\theta} = \frac{\partial \vec{r}}{\partial \theta} \Big/ \left| \frac{\partial \vec{r}}{\partial \theta} \right|$$

$$\boxed{\begin{aligned} \hat{r} &= \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \hat{\theta} &= -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \end{aligned}}$$

$$\hat{r}, \hat{\theta} = 0$$

CLAIM : $\vec{F}(r)$ is conservative.

$$\vec{F} = -\vec{\nabla} V.$$

Let, if possible, $\vec{F}$ be conservative.

$$\vec{F}\cdot d\vec{r} = \frac{f(r)\,\vec{r}}{r}\cdot d\vec{r} = \frac{f(r)\,r\,dr}{r} = f(r)\,dr.$$

$$\vec{F} = -\vec{\nabla} V = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$$

$$\vec{F}\cdot d\vec{r} = -\vec{\nabla} V\cdot d\vec{r} = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) = -dV$$

$$-dV = f(r)\,dr$$

$$\Rightarrow \quad V = -\int dr\, f(r) \longrightarrow \text{integration (in principle) can always be performed.}$$

— Need to evaluate $\vec{v} = \dfrac{d\vec{r}}{dt}$, $\vec{a} = \dfrac{d\vec{v}}{dt}$.

$$\vec{v} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}.$$

$$\frac{d\hat{r}}{dt} = \frac{\partial \hat{r}}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{r}}{\partial \theta}\frac{d\theta}{dt}.$$

$$= (0) + \underline{(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})}\,\dot{\theta}$$

$$= \dot{\theta}\,\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = \frac{\partial \hat{\theta}}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{\theta}}{\partial \theta}\frac{d\theta}{dt} = 0 + (-\cos\theta\,\hat{x} - \sin\theta\,\hat{y})\dot{\theta}$$

$$= -\dot{\theta}\,\hat{r}$$

$-$

$$\boxed{\begin{aligned} \frac{d\hat{r}}{dt} &= \dot{\theta}\,\hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= -\dot{\theta}\,\hat{r} \end{aligned}}$$

$$\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}.$$

$$= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta}\right)$$

$$= \ddot{r}\,\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\,\frac{d\hat{\theta}}{dt}.$$

$= \ddot{r}\,\hat{r} + 2\dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} - r\dot{\theta}^2\,\hat{r}$ (Fill in the missing step).

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}}$$

☑ EOM for central force.

$$\vec{F} = m\vec{a}$$

$$\Rightarrow f(r)\,\hat{r} = m(\ddot{r} - r\dot{\theta}^2)\,\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

Comparing both sides,

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \Rightarrow r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

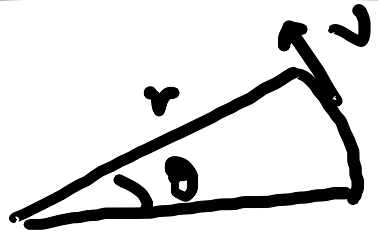
$$\Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$r^2\dot{\theta} = \text{const.} \qquad \vec{L} = \vec{r} \times \vec{p} = (mr)(r\dot{\theta})$$

$$= m r^2 \dot{\theta}$$

$$r^2\dot{\theta} = h = \text{constant.} \qquad = m\vec{h}$$

Integrating the EOM.



$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \quad \text{———} \quad (1)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad \text{———} \quad (2).$$

$s = r\theta$

$v = r\dot{\theta}$

$$(1) \times \dot{r} + (2) \times r\dot{\theta} . \Rightarrow \quad m(\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta}) = \dot{r} f(r)$$

$$\Rightarrow \quad \frac{1}{2} m \frac{d}{dt}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{d}{dt}\int dr\, f(r)$$

(Fundamental theorem of calculus).

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \int dr\, f(r)\right] = 0$$

$$\Rightarrow \quad \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \int dr\, f(r) = \text{const.}$$

$$V = -\int dr\, f(r)$$

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V = \text{const.}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \cdot (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$T + V = \text{constant} = E.$$

VII OBSERVATION

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int dr\, f(r) = E.$$

$$h = r^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{h}{r^2}$$

Substituting for $\dot{\theta}$ in the 1st EOM,

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\frac{h^2}{r^4}\right) - \int dr\, f(r) = E.$$

$$\Rightarrow \frac{1}{2}m\left[\dot{r}^2 + \frac{h^2}{r^2}\right] - \int dr\, f(r) = E.$$

Observe that due to substitution of $\dot{\theta}$ in the first equ, there is no explicit θ-dependence left.

Remarks:- Another form of the EOM :-

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$h = r^2 \dot{\theta}$$

$$m\left(\ddot{r} - r\frac{h^2}{r^4}\right) = f(r)$$

$$\Rightarrow \quad m\ddot{r} = f(r) + \frac{mh^2}{r^3}$$

Effectively, this Newton's eqn for a 1d problem, with inclusion of a new term on the RHS.