SC223 - Linear Algebra

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Lecture 33

October 27, 2023

Summary of Lecture 32

- To compute eigenvalues and eigenvectors of T, find roots of the characteristic polynomial: $c(x) = det([T]_\beta^\beta - xI_n)$. For any root λ, $u \neq \theta, u \in N(T - \lambda I)$ is an eigenvector.
- For $T \in \mathcal{L}(U), dim(U) = n$, if a diagonal matrix representation is preferred, we need an *eigenbasis* of the linear operator: $e = \{u_1, \ldots, u_n\}$.
- The process of similarity transformation on a matrix A, using eigenvectors as columns of a matrix, say E, to get a diagonal matrix Λ: $\Lambda = E^{-1}AE$ is called **matrix diagonalization**.
- **Algebraic mutiplicity of** λ: Multiplicity of λ as a root of $c(x)$. Denoted as $AM(\lambda)$
- **Geometric Multiplicity of** λ: $GM(\lambda) = dim(N(T - \lambda I))$.

Example

- Let $S_N = \{x : \mathbb{Z} \to \mathbb{C} \mid x[n+N] = x[n], \forall n \in \mathbb{Z}, x[n] \in \mathbb{C}, \forall n \in \mathbb{Z}\}$. $dim(S_N) = N$.
- Let $\beta = \{\delta_0, \ldots, \delta_{N-1}\}$ denote the basis of S^N, where $\forall k = 0, \ldots, N-1$.
- Let $x \in S_N, [x]_\beta \in \mathbb{C}^N, [x]_\beta = (x[0], \ldots, x[N-1])$.
- Delay/Shift operator: $D \in \mathcal{L}(S_N)$ be defined as $D(x)[n] = x[n-1], \forall n \in \mathbb{Z}$.

$[D(x)]_\beta [n]$

- Shift-invariant operator: $T \in \mathcal{L}(S_N)$ be a linear operator such that $T \cdot D = D \cdot T$.

$= x[n - 1 \bmod N]$

-

$$[T]_\beta^\beta = \begin{bmatrix} h[0] & h[N-1] & \ldots & h[1] \\ h[1] & h[0] & \ldots & h[2] \\ \vdots & \vdots & \vdots & \vdots \\ h[N-1] & h[N-2] & \ldots & h[0] \end{bmatrix}$$

- Such a matrix is called a *Circulant matrix*.

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- Let $f_p = \sum_{n=0}^{N-1} w^{np}\delta_n$ for a fixed p between 0 and $N-1$.

$$f_p[k] = f_p[k+N], \quad \forall k \in \mathbb{Z}.$$

$$f_p[k] = \sum_{n=0}^{N-1} \omega^{np} \delta_n[k]$$

$$f_p[k+N] = \sum_{n=0}^{N-1} \omega^{np} \delta_n[k+N] = \sum_{n=0}^{N-1} \omega^{np} \delta_n[k] = f_p.$$

$$p = 0, \ldots, N-1.$$

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- The corresponding output y for any $n = 0, \ldots, N-1$ is

$$y[n] = \sum_{k=0}^{N-1} f_p[(n-k) mod N]h[k]$$

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- Let $H(p) := \left(\sum_{k=0}^{N-1} h[k] w^{-kp}\right), p = 0, \ldots, N-1$.

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- Thus, $y = \left(\sum_{k=0}^{N-1} w^{-kp} h[k]\right) x$, for any $p = 0, \ldots, N-1$.
- Let $H(p) := \left(\sum_{k=0}^{N-1} h[k] w^{-kp}\right), p = 0, \ldots, N-1$.
- We have shown that $Tf_p = H(p) f_p, p = 0, \ldots, N-1$, and so $[T]_\beta^\beta [f_p]_\beta = H(p)[f_p]_\beta, p = 0, \ldots, N-1$.

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$$[T]_\beta^\beta \left[\begin{array}{cccc} | & | & \cdots & | \\ [f_0]_\beta & [f_1]_\beta & \cdots & [f_{N-1}]_\beta \\ | & | & \cdots & | \end{array} \right]$$

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freq. domain.

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$$[T]_\beta^\beta [x]_\beta = [y]_\beta$$

$$\underbrace{[N]_F^\beta [T]_F^F [N]_\beta^F}_{[T]_\beta^\beta} [x]_\beta = [y]_\beta$$

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- $H(p) = \sum_{n=0}^{N-1} h[n] w^{-np}, p = 0, \ldots, N-1$ is called the *Discrete Fourier Transform(DFT)* of the sequence h.
- Also, in the basis F,

$$[T]_F^F = \begin{bmatrix} H(0) & 0 & \ldots & 0 \\ 0 & H(1) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & H(N-1) \end{bmatrix}$$

Results on Eigenvectors and Eigenvalues

- **Proposition 20**: Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T \in \mathcal{L}(U)$. Then the eigenvectors $v_1, \ldots, v_m$ associated with these eigenvalues are linearly independent.

Assume $\{v_1, \ldots, v_m\}$ are LD.

smallest $k \leq m$ such that $\{v_1, \ldots, v_{k-1}\}$ is LI but $\{v_1, \ldots, v_k\}$ is LD.

$$\boxed{v_k = c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1}} \rightarrow (1)$$

$$\lambda_k v_k = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_{k-1} \lambda_{k-1} v_{k-1} \rightarrow (2)$$

$$\hat{0} = \underline{c_1(\lambda_1 - \lambda_k) v_1 + c_2(\lambda_2 - \lambda_k)} v_2 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1}$$

— END OF CLASS —

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- **Proposition 21**: For $T \in \mathcal{L}(U), dim(U) = n$, if $\sum_{i=1}^{m} AM(\lambda_i) = n$, and $GM(\lambda_i) = AM(\lambda_i), i = 1, \ldots, m$, then T is diagonalizable.

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- **Proposition 21**: For $T \in \mathcal{L}(U), dim(U) = n$, if $\sum_{i=1}^{m} AM(\lambda_i) = n$, and $GM(\lambda_i) = AM(\lambda_i), i = 1, \ldots, m$, then T is diagonalizable.
- **Proposition 22**: For $T \in \mathcal{L}(U), dim(U) = n$, for any eigenvalue λ of T, $GM(\lambda) \leq AM(\lambda)$

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Assume that the rate of change of predator population is a times the current predator population and b times that of the prey population. Similarly, the rate of change of prey population is c times the current predator population and d times the prey population, compute the predator-prey population at any time t, given some initial population.

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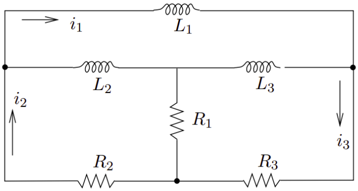

Figure: Source: Differential Equations and Linear Algebra, by GB Gustafson.

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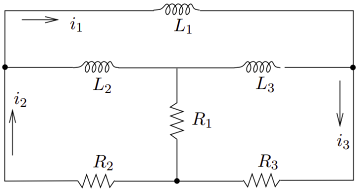

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$$
\begin{aligned}
i_1' &= - \left(\frac{R_2}{L_1}\right) i_2 - \left(\frac{R_3}{L_1}\right) i_3, \\
i_2' &= - \left(\frac{R_2}{L_2} + \frac{R_2}{L_1}\right) i_2 + \left(\frac{R_1}{L_2} - \frac{R_3}{L_1}\right) i_3, \\
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\end{aligned}
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- Epidemics Modeling:

▶ Population can be divided into: Susceptible/Healthy (S), Infected (I), and Dead (D) classes.

▶ Susceptible $\xrightarrow{-a}$ Infected $\xrightarrow{b}$ Susceptible: $\frac{d}{dt}S = -aS(t) + rI(t)$

▶ Infected $\xrightarrow{-d}$ Dead: $\frac{d}{dt}D = dI(t)$

▶ Infected: $\frac{d}{dt}I = aS(t) - dI(t) - rI(t)$

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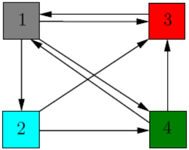

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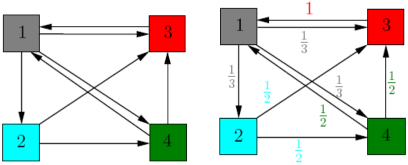

Figure: Model: If P_2 has links to P_3 and P_4, a surfer go to these pages with equal probability. Source: `pi.math.cornell.edu`

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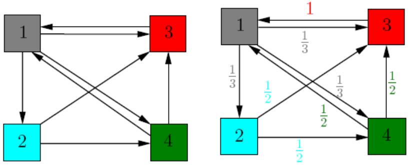

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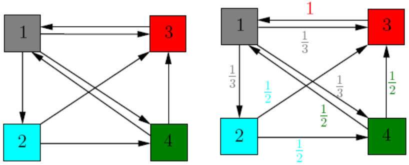

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- $\lim_{n\to\infty} A^n x_0 = y, Ay = y, y = [0.38, 0.12, 0.29, 0.19]^T$.