Chapter 10 | Lecture 44

The Z-transform

Motivation :→

① The role of Laplace transform for CTS is played by Z-transform for DTS

LT → CTS

ZT → DTS

② Eigenanalysis of Discrete-time LTI [Chapter 3]

systems:→

DTS

$$x[n] = e^{j\omega n} \longrightarrow \boxed{\text{LTI} \quad h[n] \leftrightarrow H(e^{j\omega})} \longrightarrow y[n] = H(e^{j\omega}) \cdot e^{j\omega n}$$

eigenfunction

$$x[n] = z^{n} \longrightarrow y[n] = H(z) \cdot z^{n}$$

z = complex number

$z = x + jy = r e^{j\omega}$

where

$$Z\{h[n]\} = H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} \quad ①$$

In general, for sequence $x[n]$

$$\boxed{\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n] \cdot z^{-n} = X(z)}$$

③ Notational convinience

$$DTFT\{x[n]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

[Chapter 5]

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega})$$

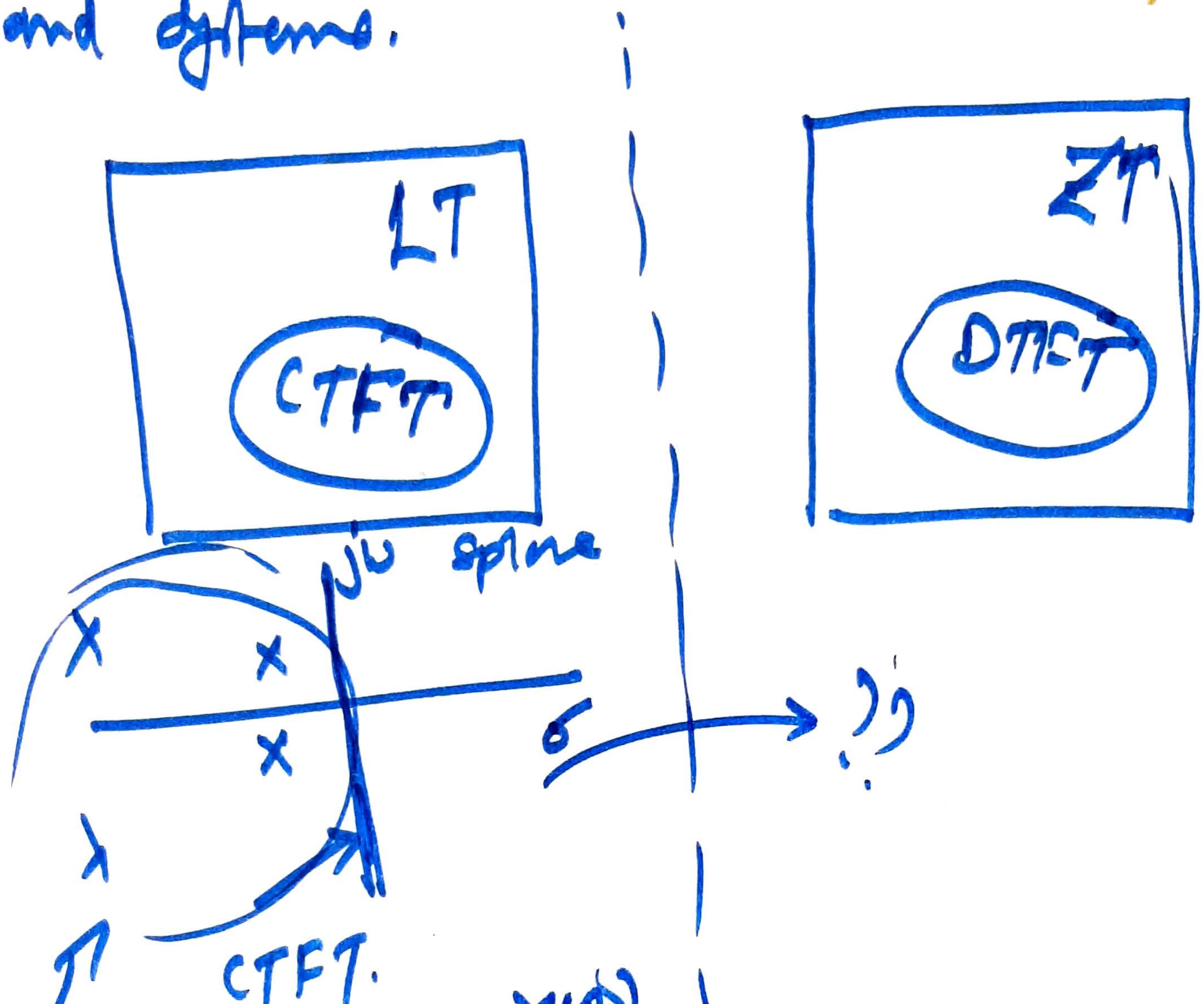
$X(z)$ has notational convinience than $X(e^{j\omega})$.

④ $x[n] = u[n] \longrightarrow DTFT$ ✗

However $\mathcal{Z}\{u[n]\}$ = exists.

$L\{u(t)\} = 1/s$

(5) Like LT, ZT is also broader spectrum for analysis of signals and systems.



$X(s)\big|_{s=j\omega} = X(j\omega)$

(6) ZT is useful for solution of difference equation

(7) ZT for analyses systems → stability

(8) Powerful results from complex variable theory, such as Cauchy Residue Theorem, analytic functions,

Tutorial

① Find the Z transform of

$x(n) = a^n \cdot u(n)$

Soln: → $Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$

$= \sum_{n=-\infty}^{+\infty} \left[a^n (u(n)) \right] \cancel{e^{-j\omega n}}\, z^{-n}$

$= \sum_{n=0}^{+\infty} (a z^{-1})^n$

G.P.: First term $= (a z^{-1})^0 = 1$

Common ratio $= a z^{-1} = r$

$= \dfrac{1}{1 - a z^{-1}}$; $|r| < 1$

$|a z^{-1}| < 1$

$\boxed{X(z) = \dfrac{z}{z-a}}$ ↳ $\boxed{|z| > |a|}$

$|z| > |a|$.

$z = x + jy$.

$|z| = \sqrt{x^2 + y^2}$

$\sqrt{x^2 + y^2} > |a|$

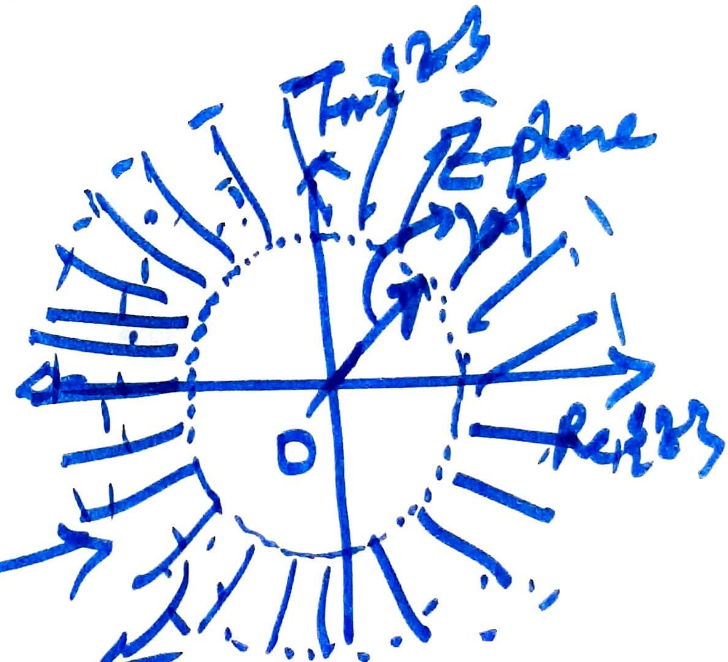
$\Longrightarrow \boxed{x^2 + y^2 > a^2}$

locus of points

$x(n) = a^n \cdot u(n)$

$\downarrow$

$X(z) = \dfrac{z}{z-a}$, $|z| > |a|$



Region of convergence (ROC)

Set of points of 'z' for which X(z) convergence.

Problem ② Find ZT of

$x(n) = -a^n \cdot u(-n-1)$

$$\therefore X(z) = \sum_{n=-\infty}^{+\infty} \left[-a^n u(-n-1)\right] z^{-n}$$

$$= \frac{z}{z-a} \; ; \quad \text{ROC } |z| < |a|$$

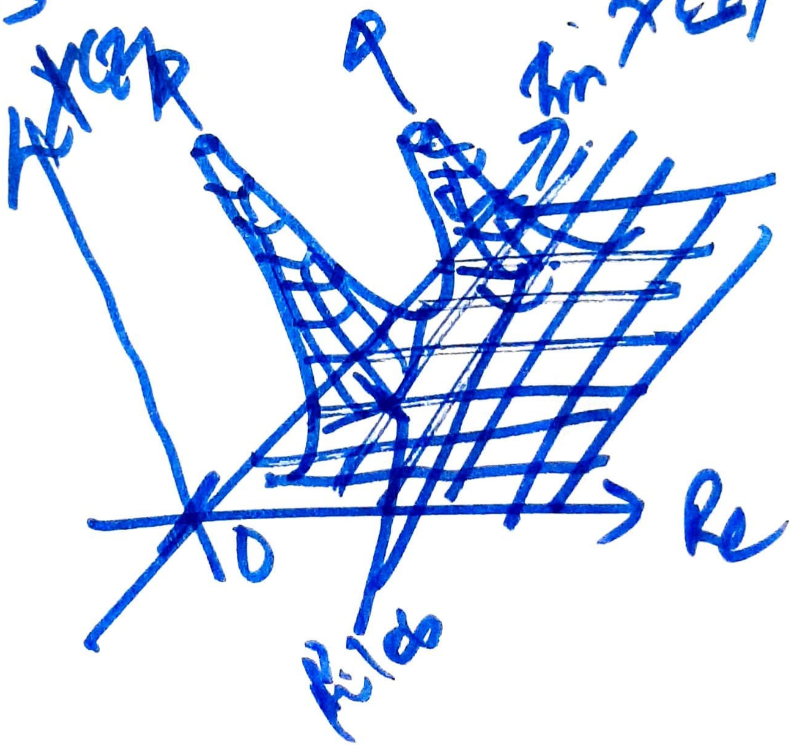
Like LT specification of ROC is important along with algebraic expression for ZT.

oncept of Poles and Zeros

$$x(n) \rightarrow X(z) = \frac{(z-1)}{(z-2)(z-3)}$$

$$X(z)\Big|_{z=1} = 0 \Rightarrow z=1 \Rightarrow \text{zero of } X(z)$$

$$X(z)\Big|_{z=2,3} = \infty \Rightarrow z=2,3 \Rightarrow \text{poles of } X(z)$$



Inverse Z transform

$$x(t) \rightarrow X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \cdot e^{st}\, ds$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \oint X(z) \cdot z^{n-1}\, dz$$

Phenomena

If $X(s)$, & $X(z)$ are rational function then contour integration is NOT required

$\Rightarrow$ rather we can use partial fraction expansion method.

Properties of Z-transform :→

① Linearity.

$$x(n) \xleftrightarrow{Z} X(Z).$$

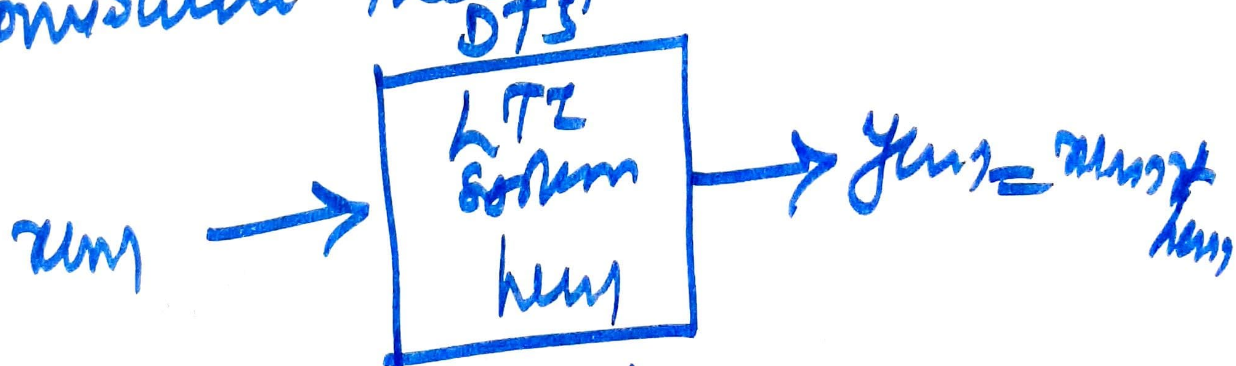
$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} a_1 X_1(Z) + a_2 X_2(Z)$$

ZT → Linear

② Time shifting.

$$\therefore\ x(n-k) \xleftrightarrow{Z(\cdot)} z^{-k} X(Z)$$

③ Convolution theorem



$$x(n) * h(n) \xleftrightarrow{Z(\cdot)} X(Z) \cdot H(Z).$$

$$\therefore\ Z\{y(n)\} = Z\{x(n) * h(n)\}$$

$$\boxed{Y(Z) = X(Z) \cdot H(Z)}$$

①

$$H(z) = \frac{Y(z)}{X(z)}$$ = system function or transfer function in z-domain

$$\therefore H(z) = \text{Transfer function} = \frac{\mathcal{Z}\{y(n)\}}{\mathcal{Z}\{x(n)\}}$$

with system initially at rest
(Zero initial conditions)

LTI

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(n)\end{matrix}} \longrightarrow y(n)$$

LCCDE

$$\mathcal{Z}\left\{\sum_{k=0}^{N} a_k\, y(n-k)\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k\, x(n-k)\right\}$$

linearity, time-shifting, convolution theorem

CTFT, LT

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$H(z) = \frac{N(z)}{D(z)}$ → roots of $N(z)$ → zeros of $H(z)$

roots of $D(z)$ ⇒ Poles of $H(z)$.

→ ZT

function

polynomial

Weierstrass-stone approach.

$$\boxed{H(s) = \frac{N(s)}{D(s)}} \to LT$$

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} \to CTFT$$

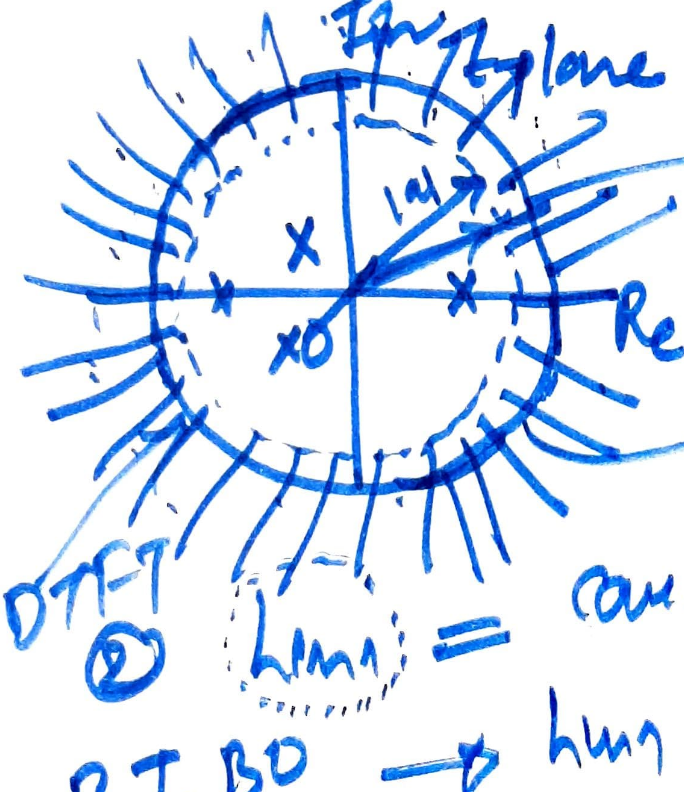
Analysis of LTI systems using Z-T.

(1) $h(n) \rightarrow$ causal, stable.

$h(n) = a^n \cdot u(n) \longrightarrow H(z) = \frac{z}{z-a}$

$|z| > |a| \quad H(z) = \frac{(z-b_1)(z-b_2)}{(z-a_1)(z-a_2)}$

$\Rightarrow$ For a causal signal $\rightarrow$ ROC is outside the circle, $|z| > |a|$



$\rightarrow z = e^{j\omega}$

$|z| = 1$

$\rightarrow$ unit circle

DTFT

(2) $h(n) =$ causal & stable

BIBO $\rightarrow h(n)$ absolutely summable.

(Chapter 2) Chapter 1

$\Rightarrow \sum_{\langle n \rangle} |h(n)| < +\infty$

$\Rightarrow$ Dirichlet condition is satisfied

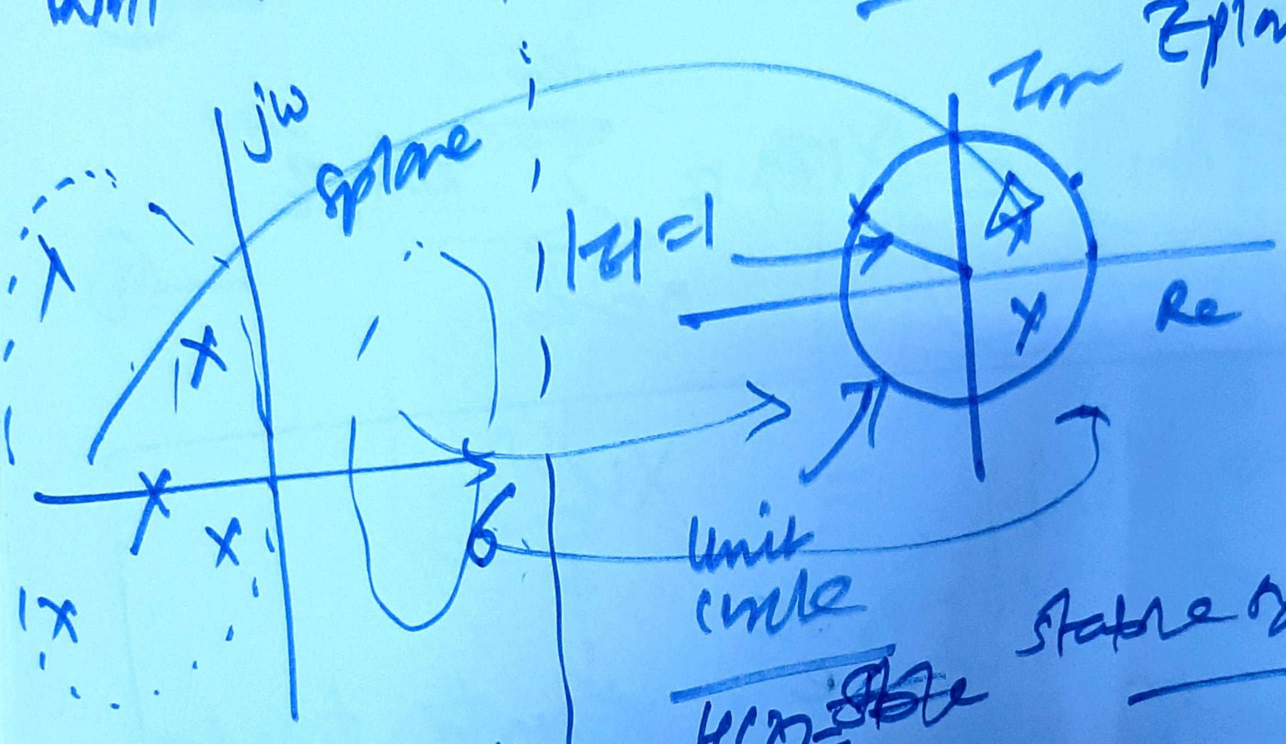
$\Rightarrow H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

exists & finite

(12)

⇒ ROC of causal & stable sequence includes the contour $z = e^{j\omega}$.

⇒ Poles of causal & stable $H(z)$ will lie inside the unit circle



Stable system $= H(z)$

$H(z) =$ ~~State~~ Stable system

Laplace Domain

Z domain

Stable s domain poles (in left hand side of s plane) are mapped to inside unit circle in Z plane.

Relationship betn ZT & DTFT

x[n]

↳ $X(e^{j\omega}) = DTFT\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$ —— (1)

$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$ —— (2)

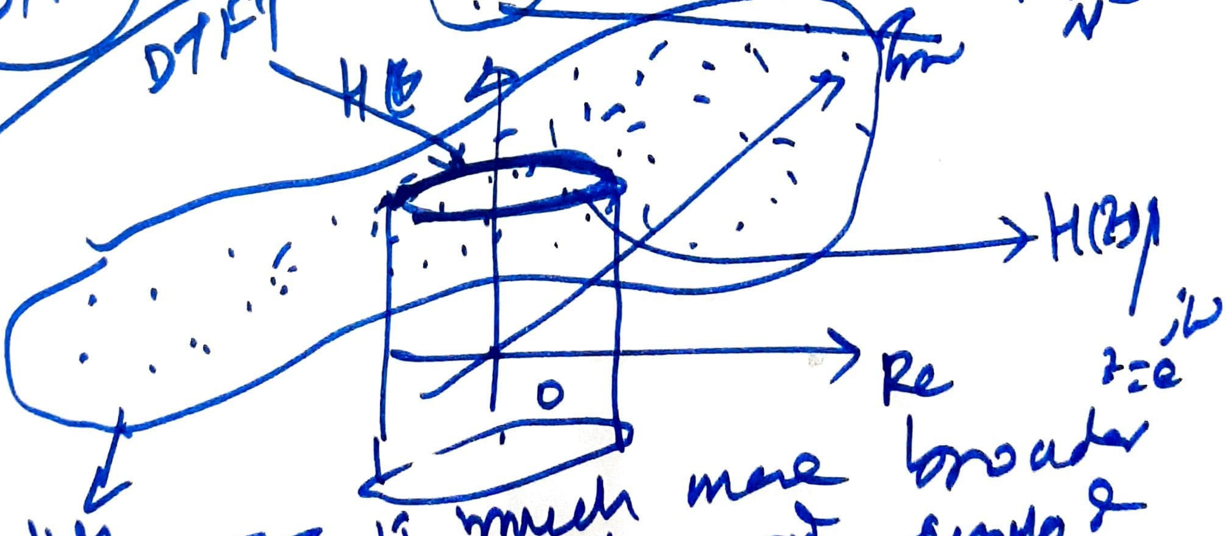
$$\boxed{X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega})}$$

Z transform — Fourier transform

ZT

DTFT

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

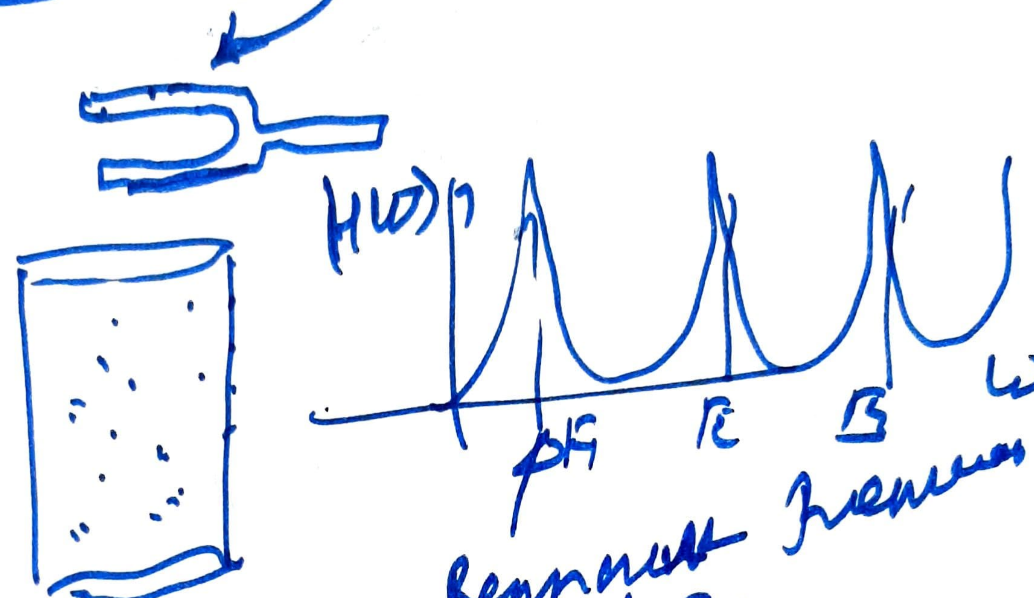


ZT is much more broader representation of signals & systems.

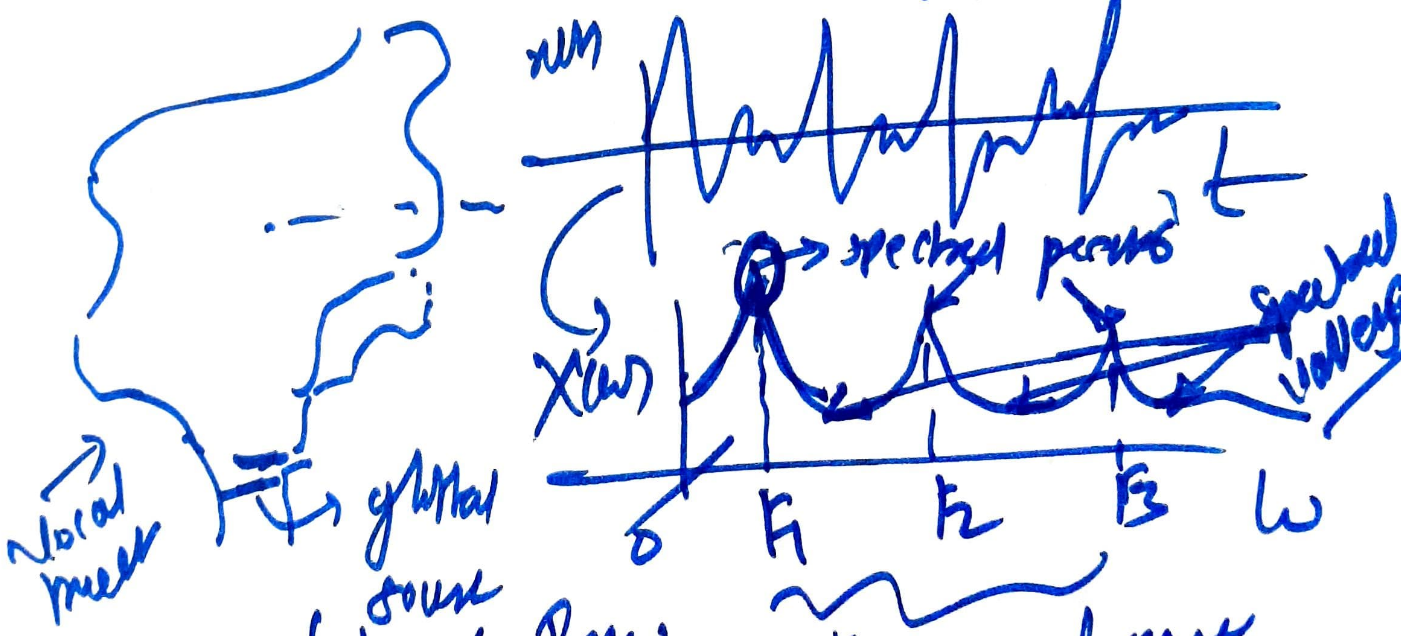
Application: Design of Digital 2nd order Resonators

Motivation:

Resonance: → Tuning fork (Physics).



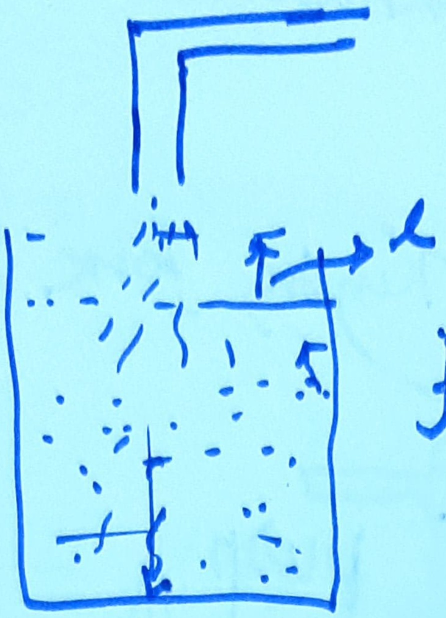
Resonance frequencies of pipe / vessel



Resonance frequency of human vocal tract

Spectral peaks are very important for speech production & speech feedback systems.

(3) Bucket (tank) filling with water



$$f \uparrow \propto \frac{1}{l \downarrow}$$

(4) Harmonic band fillers for Design & communication → Masses - 3/8 band

Manfred Schroeder: Human senses emit sounds and perceive sounds by emitting spectral peaks dominantly than spectral variety r 2nd order resonator

$M=0$, $N=2$, $a_0=1$

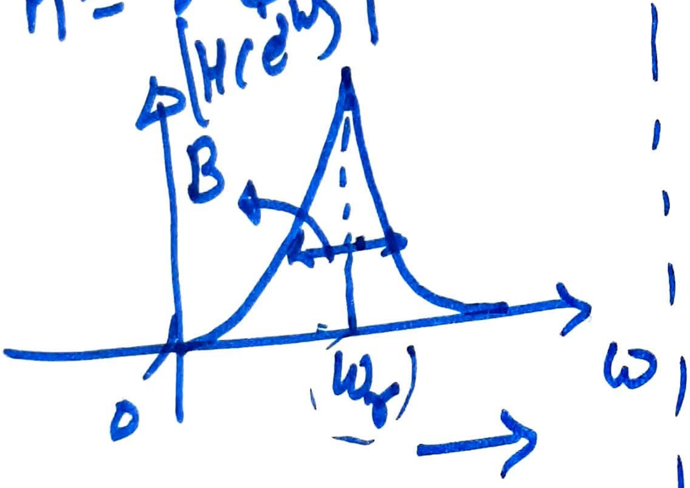
$$H(z) = \frac{b_0}{1+a_1 z^{-1} + a_2 z^{-2}} = \text{Second-order system}$$

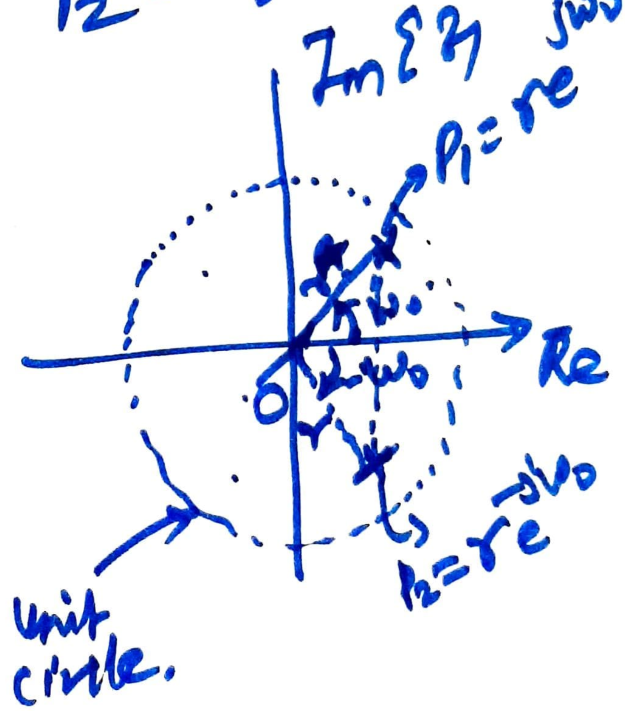
$$H(z) = \frac{b_0}{(1-P_1 z^{-1})(1-P_2 z^{-2})}$$

P_1 & P_2 = Poles of $H(z)$ in Z-plane.

$P_1 = P_2 = r e^{\pm j\omega_0}$

$P_1 = r e^{j\omega_0}$ & $P_2 = r e^{-j\omega_0}$





$Q \propto \frac{1}{B}$

ω_0 = pole angle.

r = pole radius.

Q.1) How $\omega_{r_1} = f(\omega_0)$ → $f = ?$
Resonant frequency of resonator

Q.2) How B (-3dB BW) $= g(r) \to g(r)?$
pole radius

Q.3) Analysis of impulse response of resonator?

Soln Q.3 $H(z) = \dfrac{b_0}{(1-P_1 z^{-1})(1-P_2 z^{-1})}$

Partial fraction expansion

$$H(z) = \frac{A_1}{1-P_1 z^{-1}} + \frac{A_2}{1-P_2 z^{-1}}$$

$$Z^{-1}\{H(z)\} = Z^{-1}\left\{\frac{A_1}{1-P_1 z^{-1}}\right\} + Z^{-1}\left\{\frac{A_2}{1-P_2 z^{-1}}\right\}$$

$$= A_1 Z^{-1}\left\{\frac{z}{z-P_1}\right\} + A_2 Z^{-1}\left\{\frac{z}{z-P_2}\right\}$$

$$a^n u(n) \xleftrightarrow{Z} \frac{z}{z-a}$$

$$h(n) = A_1 (P_1)^n u(n) + A_2 (P_2)^n u(n)$$

$$A_1 = \frac{b_0 P_1}{P_1 - P_2}, \quad A_2 = \frac{b_0 P_2}{P_2 - P_1}$$

$$h(n) = \frac{b_0}{P_1 - P_2}\left[(P_1)^{n+1} - (P_2)^{n+1}\right] \cdot u(n)$$

$$h(n) = \frac{b_0}{\frac{r e^{j\omega_0} - r e^{-j\omega_0}}{2j}} \left[\frac{(r e^{j\omega_0})^{n+1} - (r e^{-j\omega_0})^{n+1}}{2j}\right] u(n)$$

$$\boxed{h(n) = \frac{b_0 (r)^{n}}{\sin(\omega_0)} \sin\left[\omega_0 (n+1)\right] u(n)}$$

Impulse response of 2nd order digital resonator.

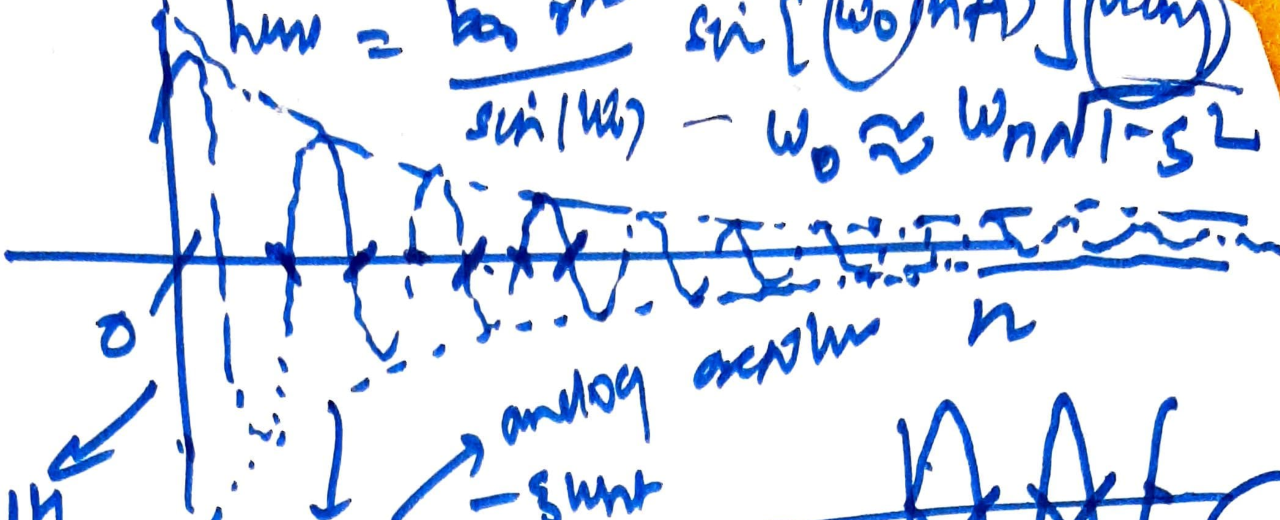
1) $h(n)$ = causal

2) For stable resonator $|r| < 1$ to ensure poles are inside unit circle

3) For $|r| < 1 \rightarrow h(n) \Rightarrow$ damped sinusoid.

$$h(n) = \frac{k_0 r^n}{\sin(\omega_0)} \sin[\omega_0(n+1)]\, u(n)$$

$\omega_0 \approx \omega_n\sqrt{1-\zeta^2}$

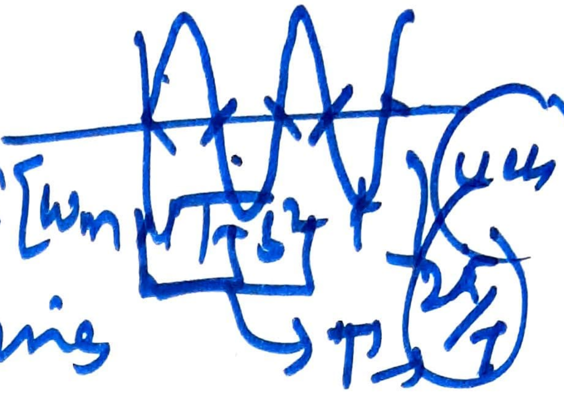


analog version

$h(t)$

characterize $h(n)$

$$h(t) = \frac{k\, e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin[\omega_n \sqrt{1-\zeta^2}\, t]\, u(t)$$



→ T, $2\pi/T$

→ r controls damping

→ Zero-crossings in $h(n)$ are controlled by frequency (resonant) of resonator.

Useful in analysis of far-field (or distant speech)

Preserves resonance of speech & important for speech perception.

...lun to 2.1

$$\omega_r = f(\omega_0)$$

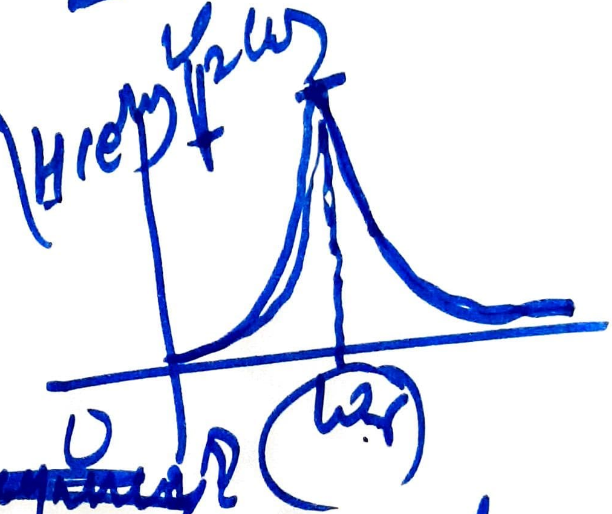
$$H(z) = \frac{b_0}{(1-P_1 z^{-1})(1-P_2 z^{-1})}$$

For frequency response,

$$H(e^{j\omega})\Big| = H(z)\Big|_{z=e^{j\omega}} = \frac{b_0}{(1-P_1 e^{-j\omega})(1-P_2 e^{-j\omega})}$$

$$H(e^{j\omega}) = \frac{b_0}{\underbrace{(1-re^{j\omega_0}e^{-j\omega})}_{V_1(\omega)}\underbrace{(1-re^{-j\omega_0}e^{-j\omega})}_{V_2(\omega)}}$$

$$|H(e^{j\omega})| = \frac{|b_0|}{|V_1(\omega)|\cdot|V_2(\omega)|}$$



What is resonance? ~~frequency~~?
↳ spectrum energy is concentrated at or around resonance frequency

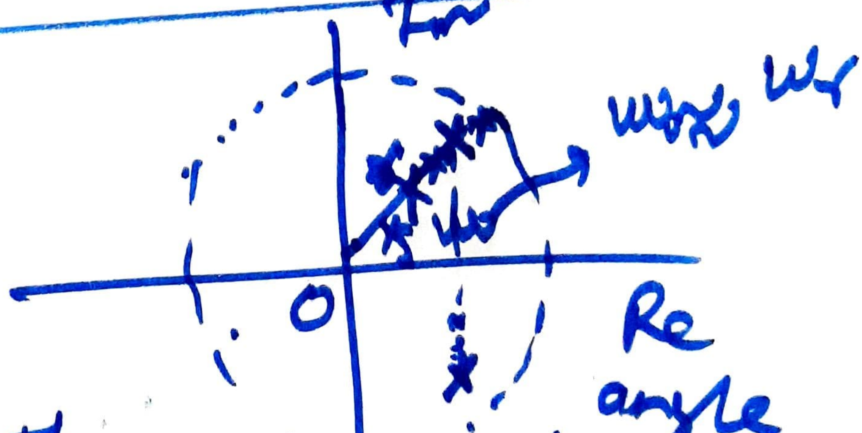
$$\frac{\partial |H(e^{j\omega})|}{\partial \omega} = 0$$

$$\Rightarrow \frac{\partial}{\partial \omega}\left[|U_1(\omega)| \cdot U_2(\omega)\right] = 0$$

$$\cos(\omega_r) = \frac{1+r^2}{2r}\cos(\omega_0)$$

$$\therefore \omega_r = \cos^{-1}\left[\frac{1+r^2}{2r}\cos(\omega_0)\right]$$

As poles radius $r \to 1$, $\omega_r \approx \omega_0$



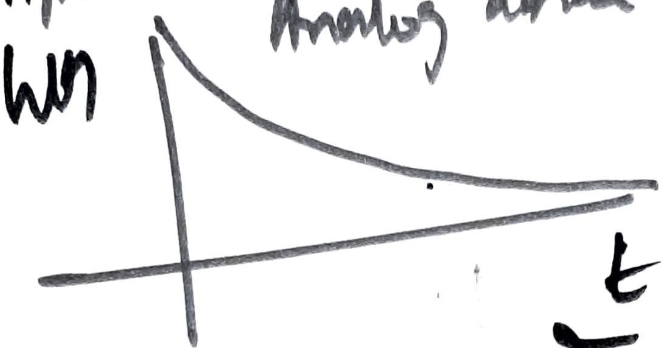
Soln 2.2 → The pole ~~radius~~ ω_0 approximate well to ~~peak~~ resonance for $r \to 1$

... to 2.2

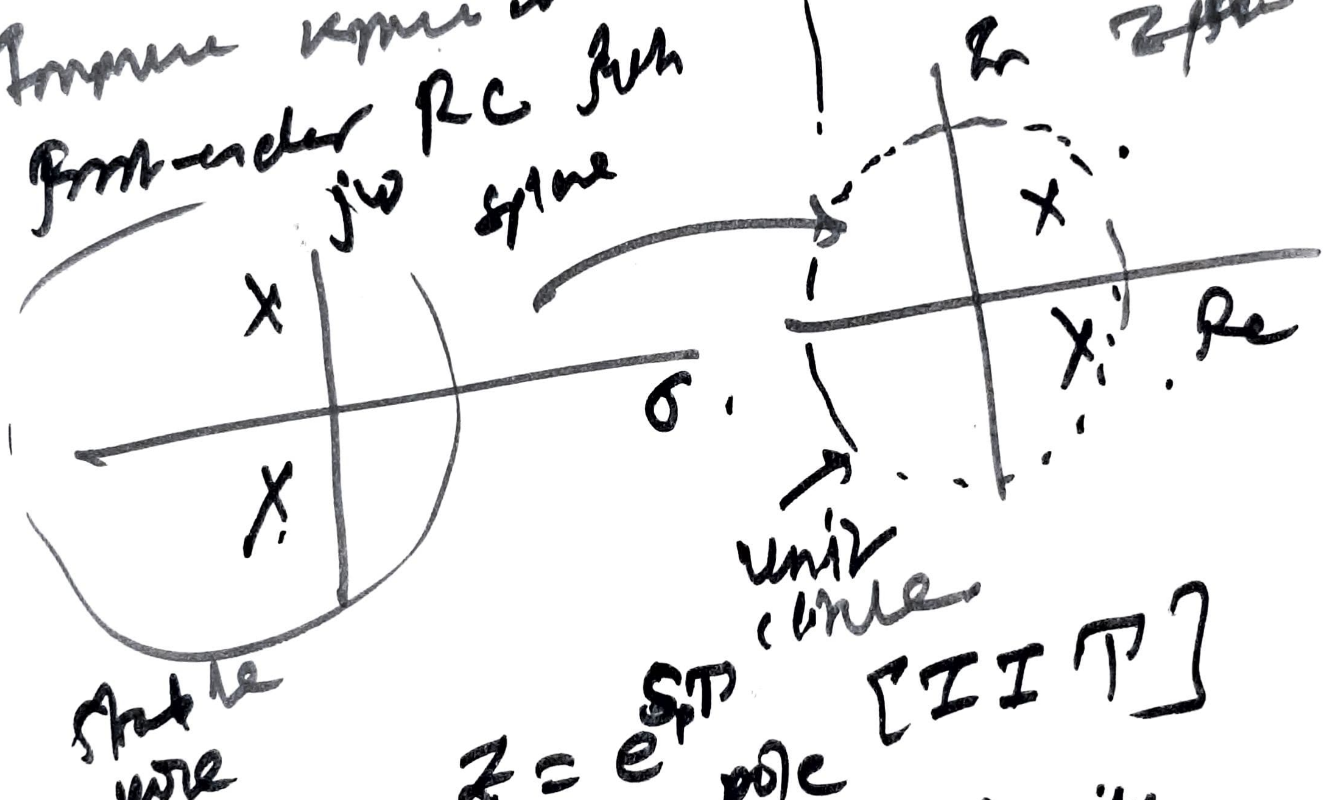
$B(-3dB \text{ width}) = g(r)?$

$g(\omega)?$

Impulse Invariance Transform (IIT)



Impulse response of first-order RC filter



$z_p = e^{s_p T}$ [IIT]

s_p = s-domain ~~pole~~ pole $= \sigma_p + j\omega_p$.

z_p = z-domain pole $= r e^{j\omega_p}$

$$r e^{j\omega_0} = e^{(\sigma_p + j\omega_p)T}$$

$$r e^{j\omega_0} = \left(e^{\sigma_p T}\right) \cdot \left(e^{j\omega_p T}\right)$$

$$r = e^{\sigma_p T}$$

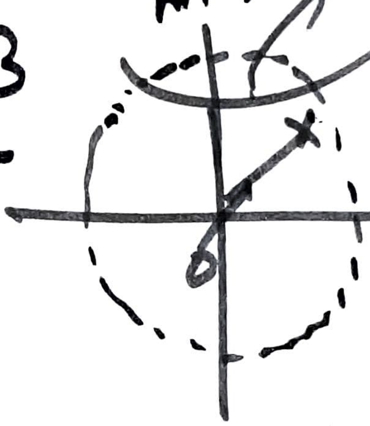
Laplace Transform → last topic

$$B\,(\text{in Hz}) = \frac{|\sigma_p|}{\pi_{\text{Radians}}}$$

$2\uparrow$

$$\therefore \quad \sigma_p = \pm \pi B$$

$$r = e^{\pm \pi B T}$$



Re(z)

$q(z) = ?$

$r = e^{\pi B T}$ ✓ ?

$r > 1$

Not realizable for stable resonator

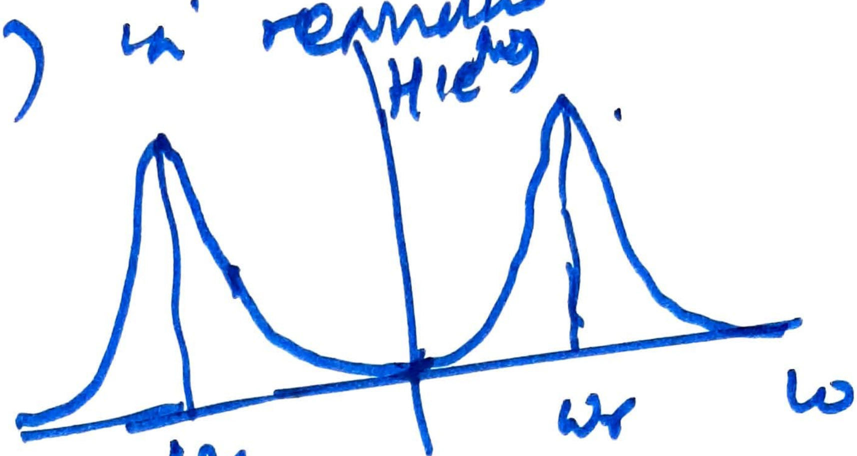
$r = e^{-\pi B T}$ ✓ ?

$$r = e^{-\pi B T}$$

$$\boxed{r\uparrow \propto \frac{1}{B\downarrow}}$$

$$2\uparrow \propto \frac{1}{B\downarrow}$$

…quency Response of Digital Resonator, if the pole radius of poles of digital resonator moves towards unit circle we tend to achieve high quality factor (Q) in resonator



$$h(n) = \underbrace{\frac{b_0 \cdot r^n}{\sin(w_0)}}_{K} \times \sin[w_0(n+1)]\,u(n)$$

$$h(n) = \underbrace{(r^n u(n))}_{x_1(n)} \times \underbrace{\sin(w_0 n)}_{x_2(n)}$$

$$F\{h(n)\} = F\{r^n u(n)\} * F\{\sin(w_0 n)\}$$

$Q \propto \frac{1}{B}$

B → 3dB Bandwidth

Q → Quality factor

$B \Rightarrow Q \propto \frac{1}{B}$