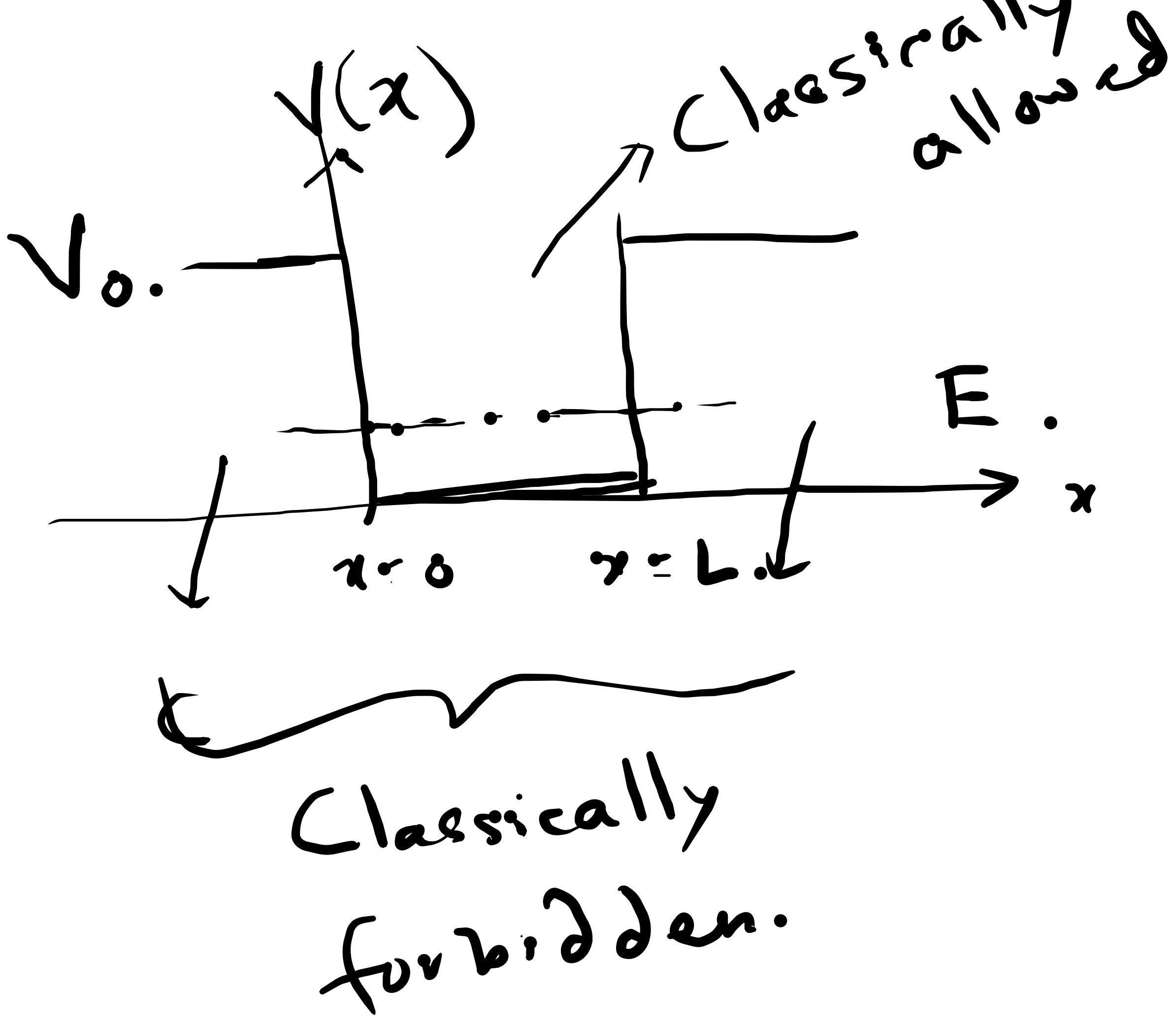
LECTURE 35

RECAP.

- Infinite potential well
- Finite potential well.



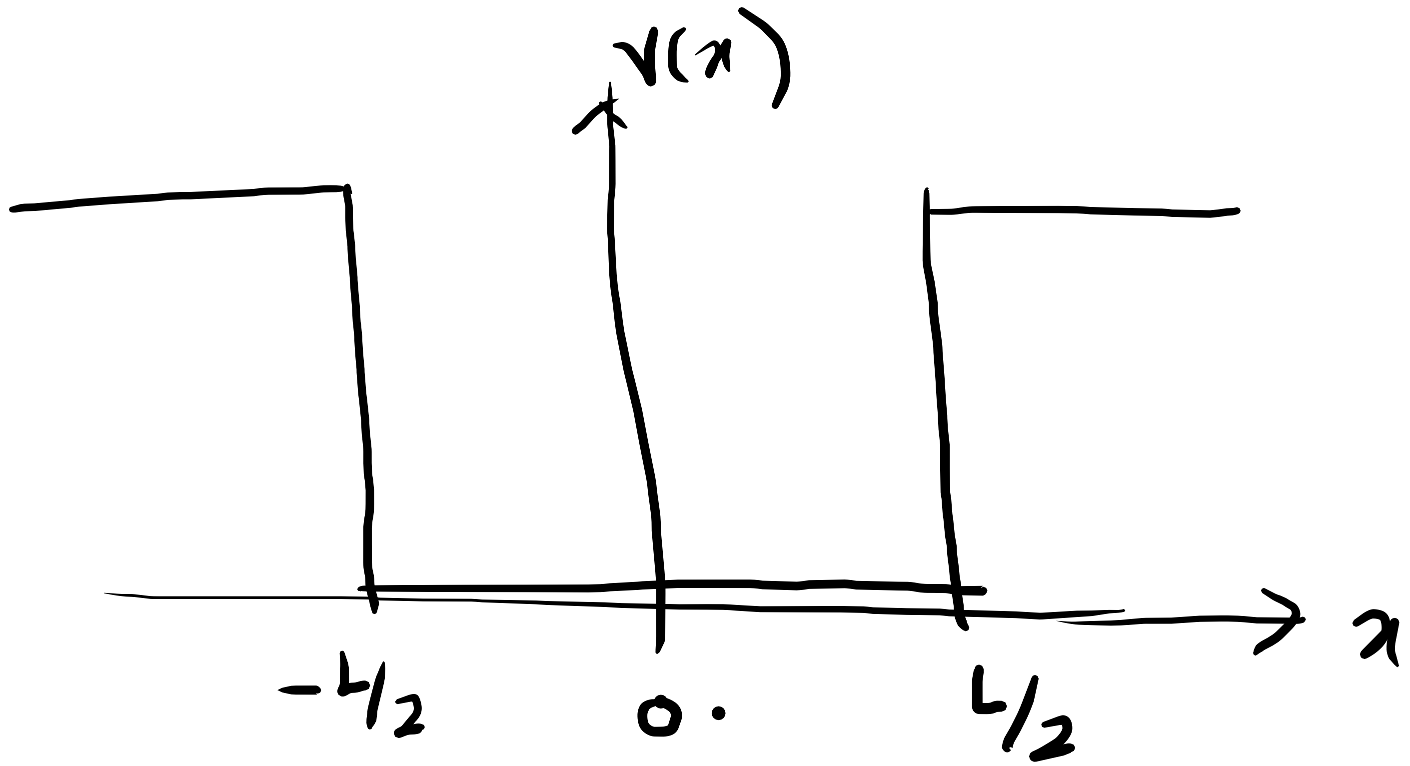
For classically forbidden regions,

$$\phi(x) = Ce^{k'x} + De^{-k'x},$$

where $k' = \sqrt{\dfrac{2m(V_0-E)}{\hbar^2}}$.

$$\phi(x) \sim e^{-k'x}, \quad x>0.$$

$|\phi(x)|^2$



$$x' = x - L/2 .$$

Old

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) .$$

$$x' = ax + b .$$

$$-\frac{L}{2} = b .$$

$$\frac{L}{2} = aL + b .$$

$$\Rightarrow \frac{L}{2} = aL - \frac{L}{2}$$

$$\Rightarrow L = aL$$

$$\Rightarrow a = 1$$

PROB: Prob. that a particle trapped in a box of length L can be found between x_1 and x_2.

$x_1 = 0.5L$, $x_2 = L$

Soln:- $\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

$$P = \int_{x_1}^{x_2} dx \, |\phi_n(x)|^2 = \frac{2}{L}\int_{x_1}^{x_2} dx \, \sin^2\left(\frac{n\pi x}{L}\right).$$

$$= \frac{1}{L}\int_{x_1}^{x_2} dx \left[1 - \cos\left(\frac{2\pi n x}{L}\right)\right] = \frac{(x_2 - x_1)}{L}.$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

$$E_{n+1} - E_n = \frac{\pi^2 \hbar^2}{2mL^2}\left[(n+1)^2 - n^2\right]$$

$$= \frac{\pi^2 \hbar^2}{2mL^2}\left[\cancel{n^2} + 2n + 1 - \cancel{n^2}\right]$$

$$= \frac{\pi^2 \hbar^2}{2mL^2}(2n+1).$$

PROB:- Given wavefn:

$$\phi(x) = 0, \qquad x < -L/2$$

$$= C\left(\frac{2x}{L}+1\right), \qquad -\frac{L}{2} < x < 0$$

$$= C\left(-\frac{2x}{L}+1\right), \qquad 0 < x < L/2$$

$$= 0, \qquad x > L/2.$$

$C = ?$

$$\int_{-L/2}^{+L/2} dx\,|\phi(x)|^2 = 1 \quad \Rightarrow \quad C^2 \int_{-L/2}^{0} dx \left(\frac{2x}{L}+1\right)^2 + C^2 \int_{0}^{L/2} dx \left(-\frac{2x}{L}+1\right)^2 = 1.$$

$$\rightarrow C^2 \int_{-L/2}^{0} dx \left(\frac{4x^2}{L^2} + \frac{4x}{L} + 1 \right) + C^2 \int_{0}^{L/2} dx \left(\frac{4x^2}{L^2} - \frac{4x}{L} + 1 \right) = 1 .$$

PROB: Particle in a box.

Given $\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$.

Verify Heisenberg's uncertainty relⁿ:

$\Delta x = ?$

$\Delta p = ?$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} .$$

$\langle x \rangle = \frac{L}{2}$

$\langle p \rangle = 0 .$

$$\langle x^2 \rangle = \int_0^L dx \; x^2 \, |\phi_n(x)|^2$$

$$= \frac{2}{L} \int_0^L dx \; x^2 \; \sin^2\left(\frac{n\pi x}{L}\right).$$

$$= \frac{1}{L} \int_0^L dx \; x^2 \left[1 - \cos\left(\frac{2\pi n x}{L}\right) \right]$$

$$= \frac{1}{L} \left. \frac{x^3}{3} \right|_0^L - \frac{1}{L} \int_0^L dx \; x^2 \cos\left(\frac{2\pi n x}{L}\right).$$

$$= \frac{L^2}{3} - \frac{1}{L} \int_0^L dx \; x^2 \cos\left(\frac{2\pi n x}{L}\right).$$

$$= \frac{L^2}{3} - \frac{1}{L}\int_0^L dx\, x^2 \cos\left(\frac{2\pi n x}{L}\right)$$

$$= \frac{L^2}{3} - \frac{1}{\cancel{L}} \underbrace{\left.\frac{x^2 \sin\left(\frac{2\pi n x}{L}\right)}{\frac{2\pi n}{\cancel{L}}}\right|_0^L}_{0} + \frac{\cancel{2}}{\cancel{L}}\int_0^L dx\, x\, \frac{\sin\left(\frac{2\pi n x}{L}\right)}{\frac{\cancel{2}\pi n}{\cancel{L}}}$$

$$= \frac{L^2}{3} + \frac{1}{n\pi}\int_0^L dx\, x \sin\left(\frac{2\pi n x}{L}\right).$$

$$= \frac{L^2}{3} + \frac{1}{n\pi}\left[\left. -x\frac{\cos\left(\frac{2\pi n x}{L}\right)}{\frac{2\pi n}{L}}\right|_0^L + \underbrace{\int_0^L dx\, \frac{\cos\left(\frac{2\pi n x}{L}\right)}{\frac{2\pi n}{L}}}_{0}\right].$$

$$= \frac{L^2}{3} + \frac{1}{n\pi}\left[-\frac{L^2}{2\pi n}\cos(2\pi n)\right] = \underline{\frac{L^2}{3} - \frac{L^2}{2\pi^2 n^2}}$$

$$\langle p^2 \rangle = \int dx\, \phi_n^*(x)\, p^2\, \phi_n(x)$$

$$= -\hbar^2 \int dx\, \phi_n^*(x) \frac{d^2\phi_n}{dx^2}.$$

$$= \hbar^2 \left(\frac{2}{L}\right) \frac{n^2\pi^2}{L^2} \int_0^L dx \sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{\hbar^2}{L} \frac{n^2\pi^2}{L^2} \int_0^L dx \left[1 - \cos\left(\frac{2\pi n x}{L}\right)\right]$$

$$= \frac{n^2\pi^2\hbar^2}{L^3} L = \frac{n^2\pi^2\hbar^2}{L^2}.$$

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}.$$

$$p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$\phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\Delta x \Delta p = \sqrt{\langle x^2\rangle - \langle x\rangle^2}\sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

$$= \sqrt{\frac{L^2}{3} - \frac{L^2}{2\pi^2 n^2} - \left(\frac{L}{2}\right)^2}\sqrt{\frac{n^2\pi^2\hbar^2}{L^2}}$$

$$= \frac{n\pi\hbar}{L}\sqrt{\frac{L^2}{12} - \frac{L^2}{2\pi^2 n^2}}$$

$$= \frac{\hbar}{2}\frac{2n\pi}{L}\sqrt{\frac{L^2}{12} - \frac{L^2}{2\pi^2 n^2}}$$

$$= \frac{\hbar}{2} 2n\pi\sqrt{\frac{1}{12} - \underbrace{\frac{1}{2\pi^2 n^2}}} \geq \frac{\hbar}{2}.$$

$$\Delta p = \sqrt{\frac{n^2\pi^2\hbar^2}{L^2}}$$

$$\Delta p_1 = \sqrt{\frac{\pi^2\hbar^2}{L^2}}$$

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2}$$

$$= \frac{\Delta p_1^2}{2m}$$

Zero-energy phenomenon.

$$\psi_n(x,t) = \phi_n(x)e^{-iE_nt/\hbar} .$$