SC223 - Linear Algebra

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Lecture 2

July 26, 2023

- In general, solve for $x_1, \dots, x_n$ in

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

Rectangular matrices?

- Solve for x in $Ax = b$, where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, b \in \mathbb{R}^m$, with $m > n$?
- Example: Assume $ht = p \cdot wt + q$. Given $(wt_i, ht_i), i = 1, \ldots, m$, find p, q.

$$\begin{bmatrix} wt_1 & 1 \\ wt_2 & 1 \\ \vdots & \vdots \\ wt_m & 1 \end{bmatrix}_{m \times 2} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} ht_1 \\ ht_2 \\ \vdots \\ ht_m \end{bmatrix}$$

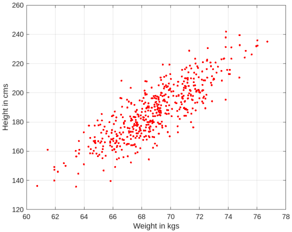

Figure: Height and Weight data for 500 Males, Source: Mustafa Ali, Kaggle.

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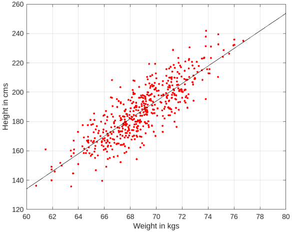

Figure: Black line depicts the prediction obtained after solving the above equations. $p \simeq 5.98, q \simeq -225$

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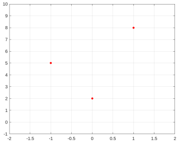

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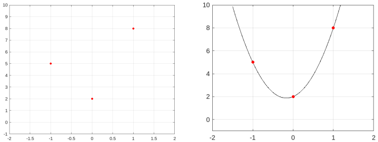

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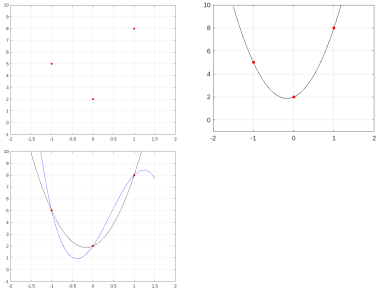

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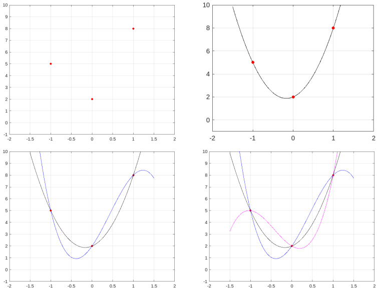

Figure: (top-left) 3 points to be interpolated, (top-rt) One soln., (btm-left) Two solutions, (btm-rt) Third soln.

Notations

- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{n2} & \dots & a_{mn} \end{bmatrix}$$

be any $m \times n$ matrix.

- Notation: $A = [a_{ij}]_{i=1,j=1}^{n,n}$ where i is the row index, and j is the column index. The elements $\{a_{11}, a_{22}, \dots, a_{kk}\}$, $k = min(m, n)$, are referred to as the elements on the **main diagonal** or **diagonal elements**.

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● If $m = 1, n > 1$ the matrix A is said to be a **row matrix/vector**, if $n = 1, m > 1$, it is referred to as a **column matrix/vector**. If $m = n$, the matrix is said to be a **Square** matrix.

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- A list of numbers $k_1, \dots, k_n$ can be represented as a
  - **column matrix/vector**: $k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$, or
  - **row matrix/vector**: $k = [k_1 \quad k_2 \quad \dots \quad k_n]$.

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be any $m \times n$ matrix.

● The p^{th} column of A will be denoted by a_{*p}, $1 \leq p \leq n$, and the q^{th} row of A represented as a column vector will be denoted by a_{q*}, $1 \leq q \leq m$.

$(a_{q*})^T$

Solving Linear Equations via Gaussian Elimination

- Solve using elimination

$[A \mid b]$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 2 & 0 & 3 & 2 & 4 \\ -2 & 3 & -2 & 1 & 6 \\ 3 & -4 & 2 & 1 & 1 \end{array}\right]$$

Using Row operations

1. Multiply $k \in \mathbb{R}$ to any row
2. Add a row to any other row
3. Exchange 2 rows.

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- Elimination sequence:
  - Eliminate x_1 in Eqns 4,3,2 using Eqn 1
  - Eliminate x_2 in Eqns 4,3 using Eqn 2
  - Eliminate x_3 in Eqns 4 using Eqn 3

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- The positions and entries used to eliminate variables from Eqns. below are referred to as **Pivot positions/elements**.

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 2 & 0 & 3 & 2 & 4 \\ -2 & 3 & -2 & 1 & 6 \\ 3 & -4 & 2 & 1 & 1 \end{array}\right]$$

① $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + 2R_1$, $R_4 \leftarrow R_4 - 3R_1$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 0 & 4 & 5 & 4 & 6 \\ 0 & -1 & -4 & -1 & 4 \\ 0 & 2 & 5 & 4 & 4 \end{array}\right]$$

② $R_3 \leftarrow R_3 + \frac{1}{4}R_2$

$R_4 \leftarrow R_4 - R_2/2$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 0 & 4 & 5 & 4 & 6 \\ 0 & 0 & -11/4 & 0 & 11/2 \\ 0 & 0 & 5/2 & 2 & 1 \end{array}\right]$$

③ $R_4 \leftarrow R_4 + \frac{10}{11}R_3$

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & -1 & -1 & -1 \\ 0 & \boxed{4} & 5 & 4 & 6 \\ 0 & 0 & \boxed{-11/4} & 0 & 11/2 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right]$$

- Equations after elimination:

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- Solution:

$$2x_4 = 6 \Rightarrow \boxed{x_4 = 3}$$

$$-11/4\, x_3 = 11/2 \Rightarrow \boxed{x_3 = -2}$$

$$4x_2 + 5x_3 + 4x_4 = 6$$

$$\boxed{x_2 = 1}$$

$$x_1 - 2x_2 - x_3 - x_4 = -1$$

$$\boxed{x_1 = 2}$$

$$x = (2, 1, -2, 3)$$

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- The process of solving the linear equations obtained after elimination is called **Back-Substitution**.
- In any row of the matrix, the first non-zero element from the left is said to be its leading entry. ● A matrix A is said to be in row-echelon form if
  - All non-zero rows are above any rows of all zeros
  - Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  - All entries in a column below a leading entry are zeros.

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- Observe U. All entries below the main diagonal are 0.

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- Observe U. All entries below the main diagonal are 0. Such a (square) matrix is called an **Upper Triangular matrix**.