III ALTERNATE TREATMENT OF NORMAL MODES

(TYPOS CORRECTED)

$M\ddot{x} = -Kx$.

$x = S\xi$.

$\ddot{x} = S\ddot{\xi}$

$\Rightarrow M S\ddot{\xi} = -KS\xi$

$\Rightarrow S\ddot{\xi} = -M^{-1}KS\xi$.

$\Rightarrow \ddot{\xi} = -S^{-1}(M^{-1}K)S\xi$.

$\boxed{\xi = S^{-1}x}$

Let $K' = M^{-1}K$.

$\ddot{\xi} = -S^{-1}K'S\xi$.

$\Rightarrow \ddot{\xi} = -K'_D\xi$.

EXAMPLE 1-

$$\left.\begin{aligned} 2\ddot{x} + \omega^2(5x-3y) &= 0 \\ 2\ddot{y} + \omega^2(-3x+5y) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 2\ddot{x} &= -\omega^2(5x-3y) \\ 2\ddot{y} &= -\omega^2(-3x+5y) \end{aligned}$$

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $M\ddot{X} = -KX$, where $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$K = \omega^2\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$

$X = S\xi$.

$\Rightarrow \ddot{X} = S\ddot{\xi}$

$K' = M^{-1}K$.

$= \omega^2\begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$

$\therefore M S\ddot{\xi} = -KS\xi$.

$\Rightarrow S^{-1}MS\ddot{\xi} = -S^{-1}KS\xi$

$\Rightarrow \ddot{\xi} = -S^{-1}(M^{-1}K)S\xi$.

$$K' = \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \omega^2 \rightarrow$$

Need to determine eigenvalues and eigenvectors of this matrix.

The eigenvectors are needed to construct the matrix S. which diagonalizes the matrix $M^{-1}K$ as $S^{-1}K'S = K_D'$

Eigenvalues:-

$|K' - \lambda I| = 0$ (have omitted overall ω^2 factor)

$\Rightarrow \left(\frac{5}{2} - \lambda\right)^2 - \frac{9}{4} = 0$

$\Rightarrow \frac{5}{2} - \lambda = \pm \frac{3}{2}$

$\frac{5}{2} - \lambda_1 = \frac{3}{2}$

$\Rightarrow \lambda_1 = 1 \times \omega^2$

$\frac{5}{2} - \lambda_2 = -\frac{3}{2}$

$\Rightarrow \lambda_2 = 4 \cdot \times \omega^2$

$$K_D' = \begin{pmatrix} \omega^2 & 0 \\ 0 & 4\omega^2 \end{pmatrix}$$

Eigenvectors:-

$$K' \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\Rightarrow \omega^2 \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \omega^2 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \frac{5}{2} e_1 - \frac{3}{2} e_2 &= e_1 \\ -\frac{3}{2} e_1 + \frac{5}{2} e_2 &= e_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{3}{2} e_1 - \frac{3}{2} e_2 &= 0 \\ -\frac{3}{2} e_1 + \frac{3}{2} e_2 &= 0 \end{aligned}\right\} \Rightarrow e_1 = e_2$$

Again,

$$\left.\begin{aligned} \frac{5}{2} e_1 - \frac{3}{2} e_2 &= 4 e_1 \\ -\frac{3}{2} e_1 + \frac{5}{2} e_2 &= 4 e_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} -\frac{3}{2} e_1 - \frac{3}{2} e_2 &= 0 \\ -\frac{3}{2} e_1 - \frac{3}{2} e_2 &= 0 \end{aligned}\right\} \Rightarrow e_1 = -e_2$$

So, eigenvectors are $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

corresponding to ~~eigenvect~~ eigenvalues ω^2 and $4\omega^2$ respectively.

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Verify:

$$S^{-1} K' S = \omega^2 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 \\ 0 & 4\omega^2 \end{pmatrix}.$$

$$\xi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} x_1 + \frac{1}{2} x_2 \\ \frac{1}{2} x_1 - \frac{1}{2} x_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \xi_1 &= \tfrac{1}{2}(x_1 + x_2) \\ \xi_2 &= \tfrac{1}{2}(x_1 - x_2) \end{aligned}\right\} \text{Normal coordinates.}$$

$$\ddot{\xi}_1 + \omega^2 \xi_1 = 0$$

$$\ddot{\xi}_2 + 4\omega^2 \xi_2 = 0.$$