SC223 - Linear Algebra

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Lecture 27

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Summary of Lecture 26

- Let $T : U \to V$ be a LT, and let $\beta_U := \{u_1, \dots, u_n\}$ and $\beta_V = \{v_1, \dots, v_m\}$ denote the basis of U and V respectively.
- For an $x \in U, x = \sum_{i=1}^{n} a_i u_i$, $y = Tx = \sum_{j=1}^{m} b_j v_j$.
- For $k \in \{1, \dots, m\}$, $b_k = \sum_{i=1}^{n} c_{ki} a_i$, or,

$$\underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{[y]_{\beta_V}} = \underbrace{\begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \dots & c_{mn} \end{bmatrix}}_{[T]_{\beta_U}^{\beta_V}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{[x]_{\beta_U}}$$

- The matrix $[T]_{\beta_U}^{\beta_V}$ is called the matrix representation of the linear tranformation T with respect to the basis β_U and β_V.
- $T(u_i), i = 1, \dots, n$ is enough to allow us to compute $T(x), \forall x \in U$.

Examples

- $\frac{d}{dx} : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}), \beta_1 = \{1, x, x^2\}, \beta_2 = \{1, x^2, x\}$.

$$\left[\frac{d}{dx}\right]_{\beta_1}^{\beta_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \left[\frac{d}{dx}\right]_{\beta_2}^{\beta_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \left[\frac{d}{dx}\right]_{\beta_1}^{\beta_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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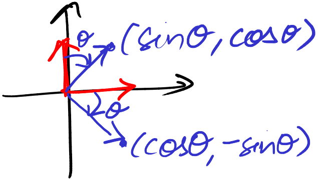
- $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.

$$\beta_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad [R_\theta]_{\beta_1}^{\beta_1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



$$[R_\theta]_{\beta_1}^{\beta_1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\beta_2 = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{u_2} \right\}. \qquad [R_\theta]_{\beta_2}^{\beta_2} = \begin{bmatrix} \; \end{bmatrix}$$

$$R_\theta u_1 = \left(\sqrt{2}\cos(\theta+\pi/4), \sqrt{2}\sin(\theta+\pi/4)\right) \qquad R_\theta u_2 = \left(\sqrt{2}\cos(\theta-\pi/4), \sqrt{2}\sin(\theta-\pi/4)\right)$$

$$\begin{bmatrix} \sqrt{2}\cos(\theta+\pi/4) \\ \sqrt{2}\sin(\theta+\pi/4) \end{bmatrix} = \frac{\cos(\theta+\pi/4)+\sin(\theta+\pi/4)}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\cos(\theta+\pi/4)-\sin(\theta+\pi/4)}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2}\cos(\theta-\pi/4) \\ \sqrt{2}\sin(\theta-\pi/4) \end{bmatrix} = \frac{\cos(\theta-\pi/4)+\sin(\theta-\pi/4)}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\cos(\theta-\pi/4)-\sin(\theta-\pi/4)}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$[R_\theta]_{\beta_2}^{\beta_2} = \begin{bmatrix} \dfrac{\cos(\theta+\pi/4)+\sin(\theta+\pi/4)}{\sqrt{2}} & \dfrac{\cos(\theta-\pi/4)+\sin(\theta-\pi/4)}{\sqrt{2}} \\[2ex] \dfrac{\cos(\theta+\pi/4)-\sin(\theta+\pi/4)}{\sqrt{2}} & \dfrac{\cos(\theta-\pi/4)-\sin(\theta-\pi/4)}{\sqrt{2}} \end{bmatrix}$$

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- $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.
- Let $p \in \mathcal{P}_3(\mathbb{R})$ be such that $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$. Define $T_p : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ by $T_p(q) = p \cdot q, \forall q \in \mathcal{P}_3(\mathbb{R})$, where $\cdot$ represents multiplication between polynomials.

$\beta_{P_3(\mathbb{R})} = \{1, x, x^2, x^3\}$, $\beta_{P_6(\mathbb{R})} = \{1, x, x^2, x^3, x^4, x^5, x^6\}$

$$[T_p]_{\beta_{P_3(\mathbb{R})}}^{\beta_{P_6(\mathbb{R})}} = ?$$

$$[T_p]_{\beta_{P_3(\mathbb{R})}}^{\beta_{P_6(\mathbb{R})}} = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 & p_0 \\ 0 & p_3 & p_2 & p_1 \\ 0 & 0 & p_3 & p_2 \\ 0 & 0 & 0 & p_3 \end{bmatrix}_{7\times 4} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}_{4\times 1}$$

$$q = q_0 + q_1 x + q_2 x^2 + q_3 x^3$$

$$[T_p(q)]_{\beta_{P_6(\mathbb{R})}} = \begin{bmatrix} p_0 q_0 \\ p_1 q_0 + p_0 q_1 \\ p_2 q_0 + p_1 q_1 + p_0 q_2 \\ p_3 q_0 + p_2 q_1 + p_1 q_2 + p_0 q_3 \\ p_3 q_1 + p_2 q_2 + p_1 q_3 \\ p_3 q_2 + p_2 q_3 \\ p_3 q_3 \end{bmatrix}$$

$$[T_p(q)]_{\beta_{P_6(\mathbb{R})}}(n) = \sum_{k=0}^{n} p(k)q(n-k), \quad n = 0, \cdots, 6.$$

Subspaces associated with a linear transformation

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Subspaces associated with a linear transformation

- Let $T : U \to V$ be a linear transformation between U and V.
- Nullspace of T (a.k.a. kernel of T): $N(T) = \{x \in U \mid Tx = \theta_V\}$
- Range of T: $R(T) = \{Tx \mid \forall x \in U\}$

$$
\begin{aligned}
T(\theta_U) = T(u - u) &= T(u) + T(-u) \\
&= T(u) + T(-1 \cdot u) \\
&= T(u) + (-1) \cdot T(u) \\
&= T(u) - T(u) = \theta_V .
\end{aligned}
$$

$\mathcal{L}(U, V)$

$$\mathcal{L}(\mathcal{L}(U,V), V)$$

End of class

Change of Basis (Similarity Transformation)

- Let $T \in \mathcal{L}(U, V)$.
- We have seen how to compute $[T]_{\beta_U}^{\beta_V}$, the matrix representation of T w.r.t the basis β_U and β_V.
- What happens if we choose a different basis, say α_U and α_V. Are $[T]_{\beta_U}^{\beta_V}$ and $[T]_{\alpha_U}^{\alpha_V}$ different?
- How are they related?

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- Now, $U \cong \mathbb{F}^n$, and $V \cong \mathbb{F}^m$.
- Let $N_{\beta_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\beta_U}(u_1) = e_1^n, \ldots, N_{\beta_U}(u_n) = e_n^n$,

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- $x \in U, N_{\alpha_U}(x) = [x]_{\alpha_U}$, and $y \in V, M_{\alpha_V}(y) = [y]_{\alpha_V}$.
- Given $[T]_{\beta_U}^{\beta_V}$, how to compute $[T]_{\alpha_U}^{\alpha_V}$?

- Commutative Diagram

$$
\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \\
U \arrow[u, "N_{\beta_U}"] \arrow[r, "T"] \arrow[d, "N_{\alpha_U}"'] & V \arrow[u, "M_{\beta_V}"'] \arrow[d, "M_{\alpha_V}"] \\
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\end{tikzcd}$$

- Thus,

$$[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}, \forall x \in U$$

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\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \arrow[d, "\underbrace{M_{\alpha_V}(M_{\beta_V})^{-1}}_{M_{\beta_V}^{\alpha_V}}"] \\
\mathbb{F}^n \arrow[u, "\underbrace{N_{\beta_U}(N_{\alpha_U})^{-1}}_{N_{\alpha_U}^{\beta_U}}"] \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & \mathbb{F}^m
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[x]_{\beta_U} = N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}} \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & {[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V}[x]_{\beta_U}} \arrow[d, "M_{\beta_V}^{\alpha_V}"] \\
{[x]_{\alpha_U}} \arrow[u, "N_{\alpha_U}^{\beta_U}"] \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & {[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[y]_{\beta_V}}
\end{tikzcd}
$$

- Thus,

$$[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V} N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}, \forall x \in U$$

$$[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V} N_{\alpha_U}^{\beta_U}$$

- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}$

- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V} [T]_{\beta_U}^{\beta_V} N_{\alpha_U}^{\beta_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and

$\alpha_U = \alpha_V = \alpha$.

- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V} N_{\alpha_U}^{\beta_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and

$\alpha_U = \alpha_V = \alpha$.

- In this case, $[T]_\alpha^\alpha = M_\beta^\alpha [T]_\beta^\beta N_\alpha^\beta$.

- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and $\alpha_U = \alpha_V = \alpha$.
- In this case, $[T]_\alpha^\alpha = M_\beta^\alpha[T]_\beta^\beta N_\alpha^\beta$.
- Note that $M_\beta^\alpha = \left(N_\alpha^\beta\right)^{-1}$.

- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V}N_{\alpha_U}^{\beta_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and $\alpha_U = \alpha_V = \alpha$.
- In this case, $[T]_{\alpha}^{\alpha} = M_{\beta}^{\alpha}[T]_{\beta}^{\beta}N_{\alpha}^{\beta}$.
- Note that $M_{\beta}^{\alpha} = \left(N_{\alpha}^{\beta}\right)^{-1}$. Denote M_{β}^{α} by S, which gives us $[T]_{\alpha}^{\alpha} = S[T]_{\beta}^{\beta}S^{-1}$.

- $[T]^{\alpha_V}_{\alpha_U} = M^{\alpha_V}_{\beta_V}[T]^{\beta_V}_{\beta_U} N^{\beta_U}_{\alpha_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and $\alpha_U = \alpha_V = \alpha$.
- In this case, $[T]^{\alpha}_{\alpha} = M^{\alpha}_{\beta}[T]^{\beta}_{\beta} N^{\beta}_{\alpha}$.
- Note that $M^{\alpha}_{\beta} = \left(N^{\beta}_{\alpha}\right)^{-1}$. Denote M^{α}_{β} by S, which gives us $[T]^{\alpha}_{\alpha} = S[T]^{\beta}_{\beta} S^{-1}$.
- **Similar matrices and Similarity transformation:** We say two matrices A and B are similar if there exists an invertible matrix, say S such that $B = SAS^{-1}$. The transformation $A \mapsto SAS^{-1}$ is said to be a similarity transformation of A by S.