LECTURE 35

Beiser.

- RECAP

- Time-dependent Schrödinger eqn.

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\, \psi(x,t).$$

- When $V(x,t) \equiv V(x)$

$$-\frac{\hbar^2}{2m} \frac{d^2\phi}{dx^2} + V(x)\,\phi(x) = E\,\phi(x).$$ → Time-independent Schrödinger eqn.

$$\left(\frac{p^2}{2m} + V(x)\right)\phi(x)$$ → total energy

L.H.S. = R.H.S.

- $\boxed{p \equiv \frac{\hbar}{i}\frac{\partial}{\partial x}} \longrightarrow$

- $|\phi(x)|^2 \rightarrow$ p.d.f.

$$\langle x \rangle = \int dx\, x|\phi(x)|^2 = \int dx\, \phi^*(x)\, x\, \phi(x)$$

$$\langle p \rangle = \int dx\, \phi^*(x)\, p\, \phi(x).$$

$$\cancel{\int dx\, |\phi(x)|^2 \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)}$$

$$\langle x^2 \rangle = \int dx\, \phi^*(x)\, x^2 \phi(x).$$

$$\sqrt{\langle x^2 \rangle - \langle x \rangle^2}.$$

– Particle in a box

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

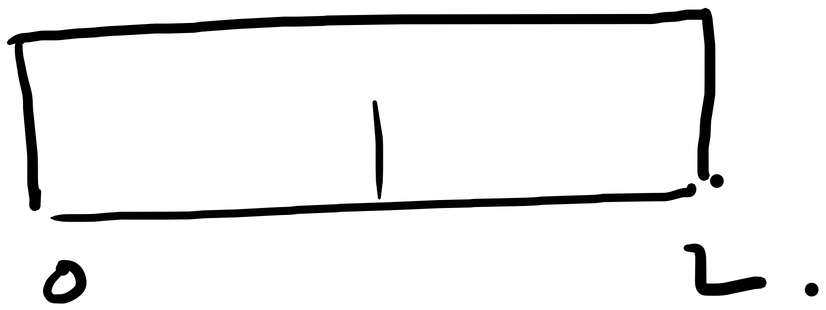
$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}.$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx \; x \, |\phi|^2 = \frac{2}{L} \int_0^L dx \; x \sin^2 \frac{n\pi x}{L}$$

$$= \frac{2}{L} \int_0^L dx \; \frac{x}{2} \left[1 - \cos\left(\frac{2\pi n x}{L} \right) \right]$$

$$= \underline{\frac{2}{L} \frac{1}{2} \int_0^L dx \; x} - \frac{1}{L} \int_0^L \cancel{d} dx \; x \cos\left(\frac{2\pi n x}{L} \right)$$

$$= \frac{1}{L}\cdot\frac{x^2}{2}\Bigg|_0^L = \frac{L}{2}.$$



$$\langle x\rangle = \frac{L}{2}$$

$$\langle p\rangle = \int_{-\infty}^{+\infty} dx\, \phi^*\, p\, \phi$$

$$= \left(\frac{\hbar}{i}\right)\int_0^L dx \left(\sqrt{\frac{2}{L}}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\frac{d}{dx}\left(\sin\frac{n\pi x}{L}\right).$$

$$= \left(\frac{\hbar}{i}\right)\left(\frac{2}{L}\right)\frac{\pi n}{L}\int_0^L dx\ \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right). \quad = 0 \ .$$

VII Simple Examples of Schrödinger eqn.

— Constant potential energy. $V(x) = V_0 = \text{const}.$

$$-\frac{\hbar^2}{2m}\phi''(x) + V_0\phi(x) = E\phi(x)$$

$$\Rightarrow \quad \phi''(x) + k^2\phi(x) = 0. \quad , \text{where } k^2 = \frac{2m}{\hbar^2}(E - V_0).$$

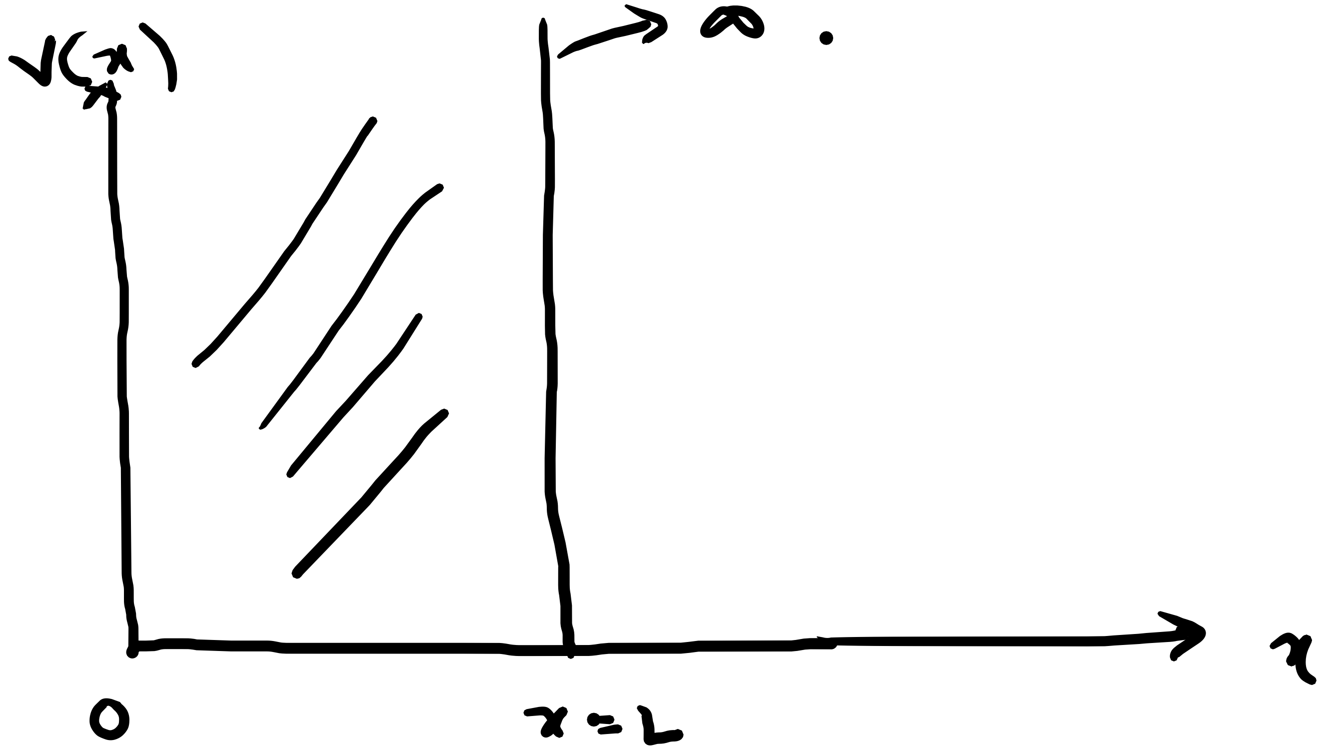
$$\phi(x) = A\cos kx + B\sin kx.$$

$E > V_0 . \rightarrow$ allowed region

Forbidden :- $V_0 > E$.

$$\phi(x) = Ae^{k'x} + Be^{-k'x}$$

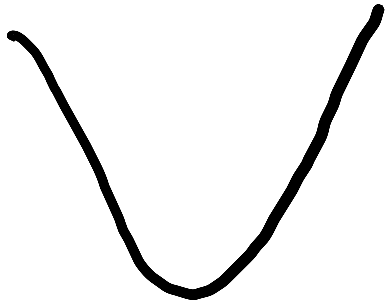
▨ PARTICLE IN A BOX.



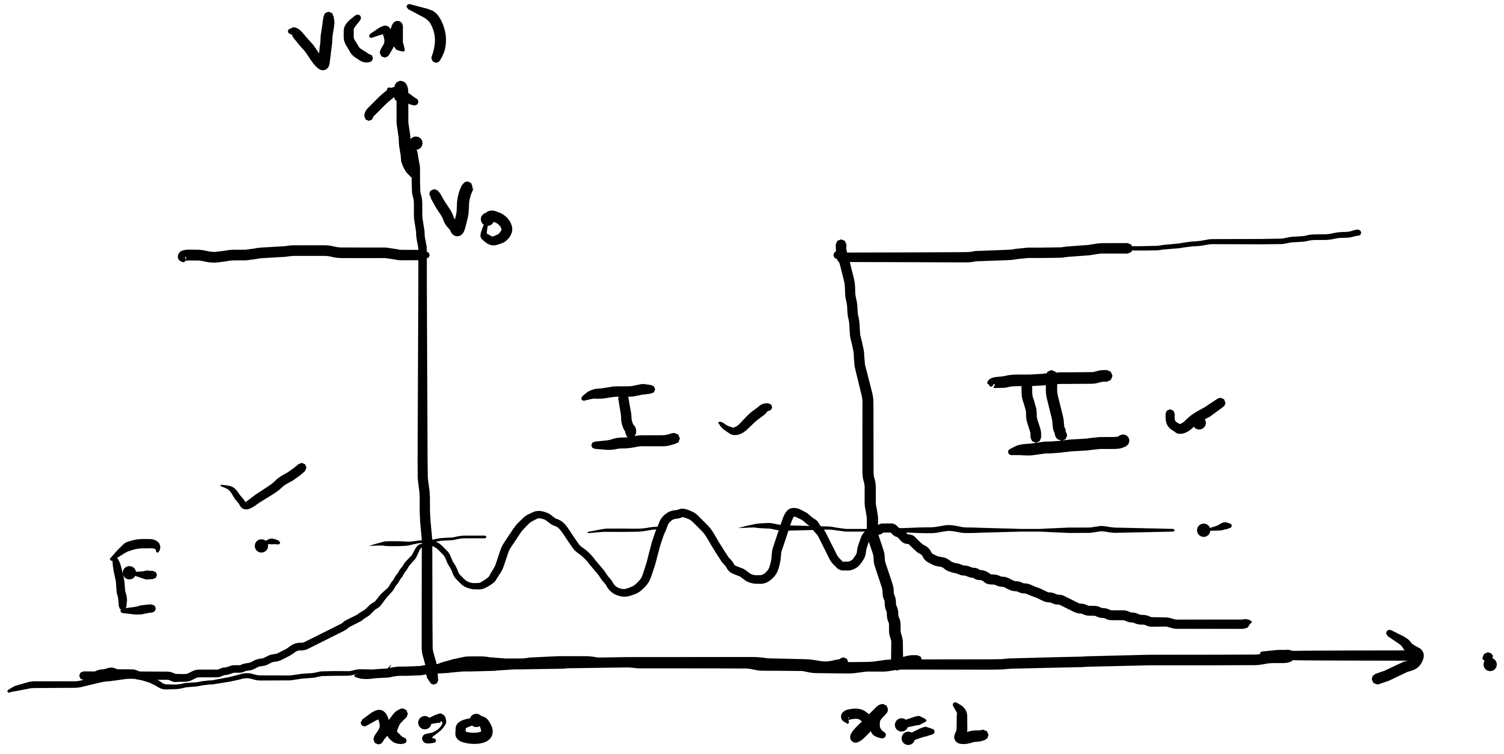
Infinite potential well.

▨ FINITE POTENTIAL WELL

$$V(x) = 0, \quad 0 \leq x \leq L$$



$$= V_0, \quad x < 0, \; x > L$$



Region I

$$-\frac{\hbar^2}{2m}\phi''(x) = E\,\phi(x).$$

$$\Rightarrow \phi(x) = A\cos kx + B\sin kx \quad , \quad 0 \le x \le L.$$

Region II :

$$-\frac{\hbar^2}{2m}\phi''(x) + V_0\,\phi(x) = E\,\phi(x)$$

$$\Rightarrow \phi''(x) - \frac{2m}{\hbar^2}V_0\,\phi = -\frac{2m}{\hbar^2}\phi$$

$$\Rightarrow \phi''(x) - \frac{2m}{\hbar^2}\underbrace{(V_0 - E)}_{k'^2}\phi = 0. \Rightarrow \phi(x) = Ce^{k'x} + De^{-k'x}.$$

$x < 0$

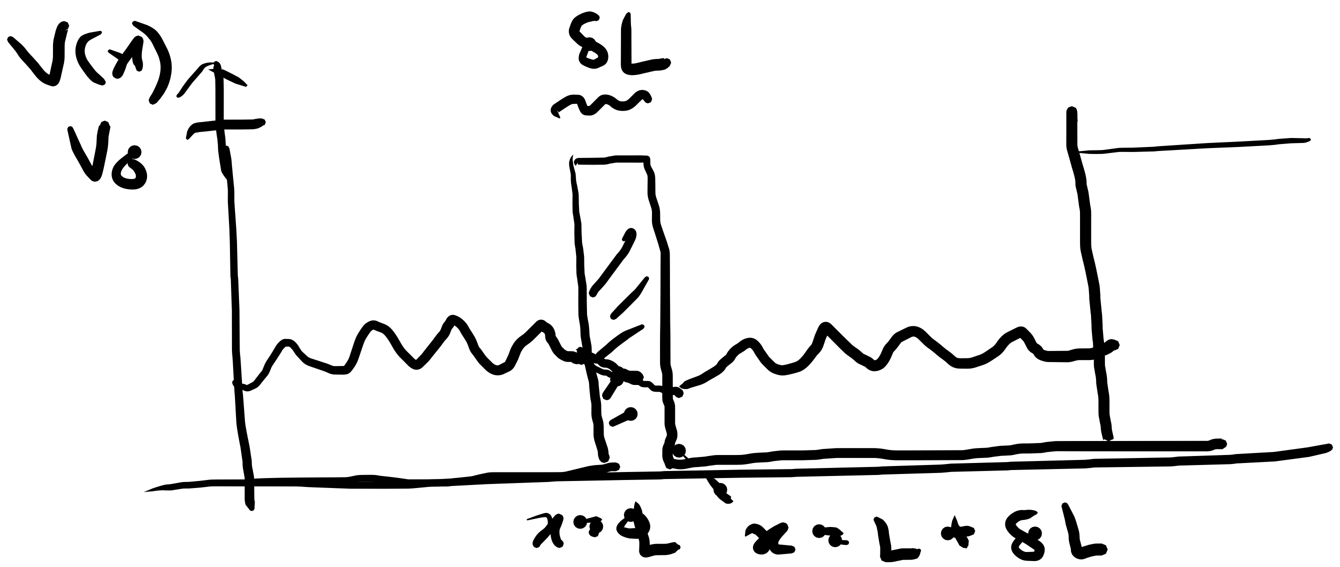
$x > L$

$$\phi(x) = Ce^{k'x} + De^{-k'x}, \quad x < 0$$
$$x > L.$$

$$\phi_{x>0} = De^{-k'x}$$

$$\phi_{x<0} = Ce^{k'x}.$$

$\Rightarrow$ Classically forbidden regions.



QUANTUM TUNNELLING

- Left to evaluate both (A, B, C, D)
- $\phi(x)$ and $\phi'(x)$ should be continuous at $x = 0$ and $x = L$.

For particle in a box,

$$E = \frac{p^2}{2m} \Rightarrow \frac{n^2 \hbar^2 \pi^2}{\cancel{2m} L^2} = \frac{p^2}{\cancel{2m}}$$

$$\Rightarrow p^2 = \frac{n^2 \pi^2 \hbar^2}{L^2}.$$

$$\Rightarrow p = \pm \frac{n \pi \hbar}{L}$$

$$p_{avg} = \frac{\frac{n\pi h}{L} - \frac{n\pi h}{L}}{2} = 0.$$

For a particle in a box,

$$\langle p^2 \rangle = -\hbar^2 \int_0^L dx\, \phi^*(x) \frac{d^2\phi(x)}{dx^2}$$

$$p = \frac{\hbar}{i} \frac{\partial}{\partial x}.$$

$$\langle x^2 \rangle = \int_0^L dx\, x\, |\phi(x)|^2 .$$

$$p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}.$$

$\Delta x \quad \Delta p$.

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}.$$