1. Find the dimension of the vector spaces/subspaces U given below:

   (a) Let $A = xy^T, x, y \in \mathbb{R}^n$. $U = N(A)$.
   Solution:
   Here, the rank of A is either 0, when all its entries are 0, Otherwise the rank is 1. This is because each row of A is a scalar multiple of y^T. Thus U has dimension either $n-1$ or n.

   (b) Let $V = \{(x_i)_{i=0}^{\infty} \mid x_i \in \mathbb{R}\}$, i.e., the set of all real-valued sequence beginning at index 0. $U = \{(x_i)_{i=0}^{\infty} \in V \mid x_0 = a, x_1 = b, x_n = x_{n-1} + x_{n-2}, n \geq 2\}$.
   Solution:
   Here V is the set of all real valued sequences, which we have seen in the lectures, to have dimension ∞.

   The subspace has dimension 2. We can use the basis vectors $b_1 = 1, 0, 1, 1, 2, 3, \ldots$ and $b_2 = 0, 1, 1, 2, 3, \ldots$

   (c) $V = \mathbb{R}^{n \times n}, U = \{A \in V \mid A^T = A\}$.
   Solution:
   The dimension of V is n^2, since we cannot impact any matrix index using values at other indices. The basis vectors are the matrices which have one entry 1 and remaining entries 0. The subspace of symmetric matrices has a smaller dimension. We need only specify all entries on or above the diagonal, as the rest are implied by symmetry. However symmetry places no correlation between values all on or above the diagonal. Thus, this number of elements is the dimension of this subspace. This works out as $\frac{n^2+n}{2}$.

   (d) Let $V = span(\{1, \sin t, \cos t\}), U = \{f \in V \mid \frac{d^2}{dt^2} f(t) + f(t) = 0, \forall t \in \mathbb{R}\}$.
   Solution:
   There is no solution to $a \cdot 1 + b \cdot \sin t + c \cdot \cos t = 0, \forall t$, other than $a = b = c = 0$. Thus these three are independent and the vector has dimension 3.

   Solving $\frac{d^2}{dt^2} f(t) + f(t) = 0$ on $f(t) = a + b \cdot \sin t + c \cdot \cos t$ yields
   $-b \cdot \sin t - c \cdot \cos t + a + b \cdot \sin t + c \cdot \cos t = 0$.

   Thus $a = 0$, and this gives us a two dimensional subspace.

2. Show that the vector space of real-valued continuous functions on $[0, 1] \subset \mathbb{R}$ is infinite dimensional.
   Solution:
   We know that every polynomial is a real valued function, and continues to be so when restricted to $[0, 1] \subset \mathbb{R}$. We also know that this subset of functions forms an infinite dimensional vector space. Being a subspace of all real valued functions, and having infinite dimension implies that the vector space of all real valued functions is also infinite dimensional.

3. Let $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{3 \times 4}$. Find the basis and dimensions of $C(A), C(A^T), N(A), N(A^T)$.
   Solution:
   It can be checked by guassian elimination that the rank of the given matrix is 3.

Basis of $C(A)$ consists of $(2,0,1),(1,2,1),(0,-2,-1)$.

Dimension of $C(A)$ is 3.

Basis of $C(A^T)$ consists of $(2,1,0,-1),(0,2,-2,3),(1,1,-1,2)$.

Dimension of $C(A^T)$ is 3.

Basis of $N(A)$ is obtained by solving the null space equation $Ax = 0$. Applying guassian elimination to A, we get

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & -1/2 & 7/4 \end{bmatrix}$$

The general solution is:

$$\begin{bmatrix} -\frac{1}{2}x_4 \\ 2x_4 \\ \frac{7}{2}x_4 \\ x_4 \end{bmatrix}$$

Plugging in $x_4 = 2$, we get a basis of $N(A)$ as:

$\begin{bmatrix} -1 \\ 4 \\ 7 \\ 2 \end{bmatrix}$ Dimension of $N(A)$ is 1.

Basis of $N(A^T)$ is θ.

Dimension of $N(A^T)$ is 0.

4. Consider the finite field F of integers modulo 7, with operations of addition modulo 7 and multiplication modulo 7. We define the list of all sequences over F with addition of two sequence being term by term addition in the field, and multiplication by a scalar being term by term multiplication by that scalar.

   (a) Show that the set of such sequences form a vector space.
   Solution:

   - additive closure:
     Since it is term by term addition and the entries are from a field, the closure follows.
   - Identity:
     Since sequence equality is term by term equality, we must retain the same term at each position upon adding the identity. It follows that the identity is the all 0 sequence.
   - Inverse:
     At each position, we must get 0 upon adding the inverse vector, which means the corresponding position must have the inverse of the element in the field.
   - associativity:
     Since the sequence value is the aggregate of its entries at each position and the entry at each position is from a field, the associativity axiom extends.
   - Commutativity:
     This follows from the same line as the previous axiom.
   - Scalar multiplicative identity is the number 1. This is because it is the identity element of the underlying field and the sequences have entries from the same field and scalar multiplication is term by term.

- the two distributivity laws follow from the fact that whether we consider two scalars and a vector or two vectors and a scalar, when considered pointwise it is three scalars from the underlying field and the distributivity laws hold there.
- Scalar multiplication closure follows from the fact that the scalar and the value of the vector/sequence at each point are from the underlying field.
- compatibility:
  This follows because at each point the sequence value is a scalar from the field, and so in effect this axiom corresponds to the associativity of the underlying field.

Thus, this is a vector space.

(b) Is this vector space finite dimensional?
Solution:
This is infinite dimensional. To establish this, we will use the well known diagonalisation technique. Assume the dimension is a finite number n. Let the basis vectors be $b_1, \ldots, b_n$. Each basis vector an only generate 7 distinct vectors, corresponding to the 7 scalars in the field. If we try to make linear combinations of the n basis vectors, we can get at most 7^n and this gives a finite list of vectors. However, we can always generate a vetor outside this finite list, by a new sequence that differs from vector number i at position i.

(c) Consider the subset of all strings of the type

$$s_{init}.s_{repeat}^{\omega}$$

Here $|S_{repeat}| > 0$. These are called rational strings. Do the set of all rational strings form a subspace?
Solution:
Yes, they form a subspace. θ is a rational sequence as the repetition begins from the first position itself.

If we add two rational sequences which have periods p_1 and p_2 will begin the repetitive phase within the larger index where repetition begins $+lcm(p_1, p_2)$.

(d) Give a non-trivial (not only the identity element) example of a finite dimensional subspace of this vector space.
Solution:
Take any finite set of vectors and the span of them is a finite dimensional vector space/subspace.

5. If U and W are subspaces of a 6 dimensional vector space V, what can be said about the dimension of $U \cap W$.
As we know, a vector space is completely determined by specifying any of its bases. The same therefore holds for subspaces. We can thus view the subspaces U and W in terms of their bases (any bases will do). We can always recalculate the basis of one of the two subspaces to ensure maximum overlap possible. That is, if some vector of W belongs to the chosen basis of U, then it can be made a part of the basis of W, by introducing it and removing some other vector. in effect the question becomes on on set theory, and venn diagrams. we know the standard formula

$$|A \cap B| = |A| + |B| - |A \cup B|$$

Thus the possible dimension of $U \cap W$ is at least $dim(U) + dim(W) - 6$ and at most $\min\{dim(U), dim(W)\}$.

6. Let $A \in \mathbb{R}^{100\times 10}, B \in \mathbb{R}^{10\times 100}$ be matrices such that $rank(A) = 10, rank(B) = 5$. Find $rank(AB)$.

Solution:

AB is a matrix representing applying a transform corresponding to B followed by a transform corresponding to A. B is a map from $\mathbb{R}^{100} \to \mathbb{R}^{10}$, where the range space has dimension 5. This is because B has rank 5. The action AB, in its second phase, operates on the range space corresponding to a 5 dimensional subspace of $\mathbb{R}^{10}$ and produces a vector in $\mathbb{R}^{100}$.

We know that A maps $\mathbb{R}^{10}$ to a 10 dimensional subspace of $\mathbb{R}^{100}$. Let b_1, b_2, b_3, b_4, b_5 be a basis of this subspace. We can extend this basis by adding five other vectors $b_6, b_7, b_8, b_9, b_{10}$ to span all of $\mathbb{R}^{10}$.

We claim the action of A on the 5 dimensional subspace maps to a 5 dimensional subspace. Consider $\alpha_1 Ab_1 + \alpha_2 Ab_2 + \alpha_3 Ab_3 + \alpha_4 Ab_4 + \alpha_5 Ab_5 = \theta$, indicating linear dependence unless the only solution is that all multipliers are 0. However, that would imply

$A(\alpha_1 \cdot b_1 + \alpha_2 \cdot b_2 + \alpha_3 \cdot b_3 + \alpha_4 \cdot b_4 + \alpha_5 \cdot b_5) = \theta$. This is impossible as the null space of this matrix has dimension 0, and thus cannot map a non-zero vector to the zero vector.