# SC223 - Linear Algebra

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Lecture 14

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# Vector Spaces

$(\mathbb{R}, +, \times)$

- **Definition:** A Vector space is a set $V$ with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
  - $(V, +)$ is an **Abelian group**:
    - $\forall x, y \in V, x + y \in V$.
    - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$.
    - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote $y$ by $-x$.
    - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
    - $\forall x, y \in V, x + y = y + x$.
  - **Closure with respect to Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$.
  - **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.
  - **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.
  - **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (\underbrace{a \times b}_{\mathbb{F}}) \cdot u = a \cdot (b \cdot u)$.

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  - $(\mathbb{F} - \{0\}, \times)$ is an **Abelian group**. The mutiplicative identity will be denoted by 1.
  - **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c$.

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$$(a+bi) \times (c+di) =$$

$$\underbrace{(ac-bd) + (bc+ad)i}_{=0}$$

$$\forall (a+bi) \in \mathbb{C} - \{0\} \quad = 0$$

$$(a+bi) \times 1 = a+bi \Rightarrow \underline{1} \text{ is the } \times\text{-identity}$$

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- $(\mathbb{R}[x], +, \times)$, where $\mathbb{R}[x]$ is the set of all rational polynomials of the form $\frac{p(x)}{q(x)}$, with $q \neq 0$, and $p$ and $q$ are polynomials in one variable with real coefficients.

$$\frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)}, \quad q_1 \neq 0, \; q_2 \neq 0.$$

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- If the 3-tuple $(V, +, \cdot)$ with field $(\mathbb{F}, +_F, \times)$ satisfies all vector space axioms, we say that $(V, +, \cdot)$ **forms a vector space over** $\mathbb{F}$.

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- If the 3-tuple $(V, +, \cdot)$ with field $(\mathbb{F}, +_F, \times)$ satisfies all vector space axioms, we say that $(V, +, \cdot)$ **forms a vector space over** $\mathbb{F}$.

● Any element of the vector space $(V, +, \cdot)$ will be referred to as a **vector**, and any element $a \in \mathbb{F}$ will be referred to as a **scalar**.

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- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$.

$V = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$.

$\forall u, w \in V$, $u + w := \begin{pmatrix} u_1 + w_1 \\ \vdots \\ u_n + w_n \end{pmatrix}$

$\forall a \in \mathbb{R}, \forall u \in V$, $a \cdot u := \begin{pmatrix} a \times u_1 \\ a \times u_2 \\ \vdots \\ a \times u_n \end{pmatrix}$

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$(\mathbb{Z}_2^n, +, \circ)$ over $\mathbb{Z}_2$

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- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.

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- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.
- $(\mathbb{L}_2(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathbb{L}_2(\mathbb{R})$ denotes the set of all square-integrable functions $f : \mathbb{R} \to \mathbb{R}$.

$\forall f \in \mathbb{L}_2(\mathbb{R})$, $\underline{\int_{-\infty}^{\infty} (f(t))^2 dt} < \infty$.

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- **Proposition 4:** $\forall a \in \mathbb{F}, a \cdot \theta = \theta$.

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- **Proposition 4:** $\forall a \in \mathbb{F}, a \cdot \theta = \theta$.
- **Proposition 5:** $\forall v \in V, (-1) \cdot v = -v$.