SC223 - Linear Algebra
Autumn 2023

Aditya Tatu

Theorem 2(4): The inverse of any invertible lower triangular matrix is also a lower triangular matrix.
Proof: Let $L \in \mathbb{R}^{n\times n}$ be an arbitrary lower triangular matrix as shown below:

$$L = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ l_{*1} & l_{*2} & \dots & l_{*n} \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

The inverse of L, when it exits, is a matrix $V \in \mathbb{R}^{n\times n}$ that satisfies the equation $LV = VL = I_n$, where I_n is the $n \times n$ identity matrix. This gives,

$$\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ l_{*1} & l_{*2} & \dots & l_{*n} \\ \vdots & \vdots & \dots & \vdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ v_{*1} & v_{*2} & \dots & v_{*n} \\ \vdots & \vdots & \dots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ e_1 & e_2 & \dots & e_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix},$$

where $e_k \in \mathbb{R}^n, k = 1, \dots, n$ denotes a column vector that contains 1 in the k^{th} row, while all other entries are zero. Multiplying the two matrices on the left of the above equation gives,

$$\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ Lv_{*1} & Lv_{*2} & \dots & Lv_{*n} \\ \vdots & \vdots & \dots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ e_1 & e_2 & \dots & e_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix},$$

which can be rewritten as $Lv_{*k} = e_k, k = 1, \dots, n$. Using the fact that multiplying a matrix to a column vector yields a linear combination of columns of the matrix with scalars from the column vector, we get

$$v_{1k}l_{*1} + v_{2k}l_{*2} + \dots + v_{nk}l_{*n} = e_k, k = 1, \dots, n.$$

We have column vectors containing n entries on both sides of the above equation. Thus, each of the n entries of the column vectors must be the same.
Let us consider the i^{th} entry of both vectors, for any $i < k$. We know that $e_{ik} = 0, i \neq k$. We thus get,

$$v_{1k}l_{i1} + v_{2k}l_{i2} + \dots + v_{nk}l_{in} = 0, i < k.$$

Given that L is lower triangular, $l_{ij} = 0, \forall j > i$. In particular $l_{i,i+1} = l_{i,i+2} = \dots = l_{in} = 0$, (comma is used in the subscripts to avoid confusion) which simplifies the above equation to:

$$v_{1k}l_{i1} + v_{2k}l_{i2} + \dots + v_{ik}l_{ii} = 0, i < k.$$

Since $l_{i1}, l_{i2}, \dots, l_{ii}$ can be any arbitrary numbers, the only way to ensure that the above equations are satisfied for all $i < k$ is $v_{1k} = v_{2k} = \dots = v_{k-1,k} = 0, k = 1, \dots, n$. This implies that the matrix V, the inverse of L, is lower triangular.