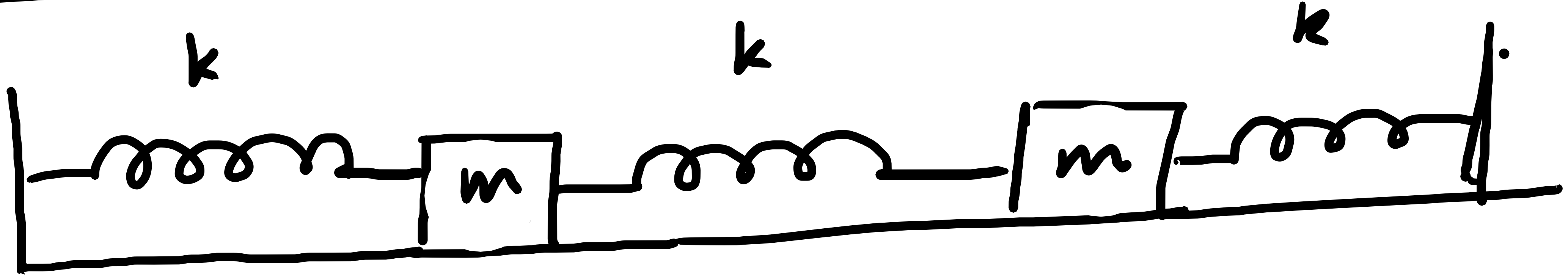
LECTURE 22



EOM:-

$$m_1 \ddot{x}_1 = -2k x_1 + k x_2$$

$$m_2 \ddot{x}_2 = k x_1 - 2k x_2 .$$

In matrix notation:- $M \ddot{x} = -K x$, where $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$

Trial sol^n:- $z_1(t) = a_1 e^{i\omega t} = \alpha_1 e^{i(\omega t - \delta)}$

$z_2(t) = a_2 e^{i\omega t} = \alpha_2 e^{i(\omega t - \delta)}$

Substituting the trial solutions,

$$\left(\bar{K} - \omega^2 \bar{M}\right) \bar{a} = 0 .$$

For existence of non-trivial solutions,

$$\left| \bar{K} - \omega^2 \bar{M} \right| = 0 . \longrightarrow \text{quadratic in } \omega .$$

Two solutions:- $\boxed{\omega_1 = \sqrt{\frac{k}{m}} , \quad \omega_2 = \sqrt{\frac{3k}{m}}}$

Two types of motion:- $\bar{z}(t) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_1 t - \delta_1)}$

$$\bar{z}(t) = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_2 t - \delta_2)} .$$

— Choosing real parts,

$$\bar{x}(t) = \mathrm{Re}\left[\bar{z}(t)\right]$$

$$x_1(t) = A\cos(\omega_1 t - \delta_1)$$
$$x_2(t) = A\cos(\omega_1 t - \delta_1)$$

$$x_1(t) = A\cos(\omega_2 t - \delta_2)$$
$$x_2(t) = -A\cos(\omega_2 t - \delta_2)$$

GENERAL SOLUTION IS:-

$$\bar{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2).$$

Original example:-

$$2\ddot{x} + \omega^2(5x - 3y) = 0$$
$$2\ddot{y} + \omega^2(5y - 3x) = 0.$$

$$\underline{(\ddot{x}+\ddot{y}) = -\omega^2(x+y)} \qquad \underline{(\ddot{x}-\ddot{y})} = -4\underline{\omega^2(x-y)}$$

Define:-
$$\xi_1 = \frac{1}{2}(x_1 + x_2)$$
$$\xi_2 = \frac{1}{2}(x_1 - x_2) \, .$$

<u>1st normal mode</u>:-
$$\xi_1(t) = A_1 \cos(\omega_1 t - \delta_1)$$
$$\xi_2(t) = 0 \, .$$

<u>2nd normal mode</u>:-
$$\xi_1(t) = 0$$
$$\xi_2(t) = A_2 \cos(\omega_2 t - \delta)$$

<u>Basic idea</u>:- Coupled eqns. of motion simplify for certain linear combinations of the original variables.

▨ HOW DO THESE LINEAR COMBINATIONS OCCUR.

$$\bar{M}\ddot{\bar{x}} = -\bar{K}\bar{x}$$

$$\bar{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$AX = \lambda X.$$

$$|A - \lambda I| = 0 \longrightarrow \lambda_1, \lambda_2$$

$$A_D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$$AX_1 = \lambda_1 X_1$$

$$AX_2 = \lambda_2 X_2.$$

$$S^T = (X_1, X_2).$$

$$\bar{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \rightarrow$$ Aim:- Write $\bar{K}$ in a basis where it is diagonal.

$$\underbrace{\bar{M}^{-1}\bar{K} = \frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}$$

diagonalise this matrix.

$$\bar{\xi} = \bar{O}\,\bar{x}.$$

$$\Rightarrow \quad x = O^T \xi.$$

$$\bar{M}\ddot{x} = -\bar{K}x$$

$$\Rightarrow \quad \bar{M}O^T\ddot{\xi} = -\bar{K}O^T\xi..$$

$$\Rightarrow \quad \bar{O}^T\bar{M}\ddot{\xi} = -KO^T\xi \qquad \left(\because M \text{ is } \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right).$$

$$\Rightarrow \quad O^T M \ddot{\xi} = -K O^T \xi$$

$$\Rightarrow \quad O O^T M \ddot{\xi} = -O K O^T \xi$$

$$\Rightarrow \quad M \ddot{\xi} = -\underbrace{O K O^T}_{} \xi.$$

$$\underbrace{= -K_D \xi}_{\text{uncoupled eqns.}}$$

$$K_D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$$M \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \end{pmatrix} = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$m \ddot{\xi}_1 = -\lambda_1 \xi_1$$

$$m \ddot{\xi}_2 = -\lambda_2 \xi_2$$

Coupled eqns. in x $\longleftrightarrow$ uncoupled eqns. in ξ.

— Summary.

$$\bar{M}\ddot{x} = -\bar{K}x, \quad \text{where} \quad \bar{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\bar{K} \rightarrow$ non-diagonal

Algorithm (i) Construct eigenvalues and eigenvectors of $\bar{M}^{-1}\bar{K}$.

Eigenvalues are :- $\left(\frac{3k}{m}, \frac{k}{m}\right)$.

Eigenvectors:- $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$O = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$\xi = \begin{pmatrix} -x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}.$$

$$= \begin{pmatrix} -(x_1 - x_2) \\ (x_1 + x_2) \end{pmatrix}.$$

$$2\ddot{x} + \omega^2(5x - 3y) = 0$$
$$2\ddot{y} + \omega^2(-3x + 5y) = 0.$$

$$\bar{K} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}\omega^2$$

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

1. Determine eigenvalues of $\bar{M}^{-1}\bar{K}$.

$$\omega_1 = \omega^2 \qquad \omega_2 = 4\omega^2.$$

$$\ddot{\xi}_1 + \omega_1^2 \xi_1 = 0.$$
$$\ddot{\xi}_2 + \omega_2^2 \xi_2 = 0.$$

2. Determine eigenvectors X_1 and X_2 to construct $O \doteq (X_1, X_2)$.

3. $\xi = O x$.