SC223 - Linear Algebra

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Lecture 28

October 10, 2023

Summary of Lecture 27

● For any two vector spaces U and V over the same field $\mathbb{F}$, and any linear transformation T from U to V, following subspaces exist:

- Nullspace of T (a.k.a. kernel of T): $N(T) = \{x \in U \mid Tx = \theta_V\}$
- Range of T: $R(T) = \{Tx \mid \forall x \in U\}$

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● Let U and V be n and m dimensional vector space over field $\mathbb{F}$, resp. If T is LT, then T can be **represented** as a matrix from $\mathbb{F}^{m\times n}$.

● If β_U and β_V are the chosen basis for U and V, we denote the matrix representation of T by $[T]_{\beta_U}^{\beta_V}$, and if $y = Tx$, then $[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V}[x]_{\beta_U}$.

$\mathcal{L}(U, V)$

$\forall T_1, T_2 \in \mathcal{L}(U,V)$ $(T_1 + T_2)(u) := T_1(u) + T_2(u), \ \forall u \in U.$

$\forall T \in \mathcal{L}(U,V)$, $\forall a \in \mathbb{F}$, $(a \cdot T)(u) := a \cdot T(u) \ \forall u \in U.$

1. Closure w.r.t $+$
2. " " $\cdot$
3. Additive Identity Let $0_{\mathcal{L}(U,V)}$ be defined as: $0_{\mathcal{L}(U,V)}(u) = 0_V, \ \forall u \in U.$

$0_{\mathcal{L}(U,V)} \in \mathcal{L}(U,V).$

$\forall T \in \mathcal{L}(U,V)$, $T + 0_{\mathcal{L}(U,V)} = 0_{\mathcal{L}(U,V)} + T = T.$

4. Additive Inverse

Let $T \in \mathcal{L}(U,V)$

Define $(-T)(u) = -T(u), \forall u \in U$.

Show. $(T+(-T)) = \theta_{\mathcal{L}(U,V)}$

$$
\begin{aligned}
(T+(-T))(u) &= T(u) + (-T)(u) \\
&= T(u) + -T(u) \\
&= \theta_V, \quad \forall u \in U.
\end{aligned}
$$

$\therefore (T+(-T))(u) = \theta_{\mathcal{L}(U,V)}(u), \forall u \in U.$

$\Rightarrow T+(-T) = \theta_{\mathcal{L}(U,V)}$.

5. Associative (+)

6. Commutative (+)

7. Scalar Mult. Identity.

$\forall T \in \mathcal{L}(U,V), \ (1 \cdot T)(u) = 1 \cdot T(u), \ \forall u \in U$

$= T(u)$

$\therefore 1 \cdot T = T, \ \forall T \in \mathcal{L}(U,V)$.

8. Distributivity

9. Compatibility of SM with FM.

$(a \times b) \cdot T = a \cdot (b \cdot T) \quad \forall T \in \mathcal{L}(U,V)$
$\forall a, b \in \mathbb{F}$.

$(\mathcal{L}(U,V), +, \cdot)$ is a VS over $\mathbb{F}$

$\mathcal{L}(\mathcal{L}(\mathcal{L}(U,V), \mathcal{L}(U,V)), \mathcal{L}(\mathcal{L}(U,V), \mathcal{L}(U,V)))$

END OF CLASS

Change of Basis (Similarity Transformation)

- Let $T \in \mathcal{L}(U, V)$.
- We have seen how to compute $[T]_{\beta_U}^{\beta_V}$, the matrix representation of T w.r.t the basis β_U and β_V.
- What happens if we choose a different basis, say α_U and α_V. Are $[T]_{\beta_U}^{\beta_V}$ and $[T]_{\alpha_U}^{\alpha_V}$ different?
- How are they related?

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- Let $T \in \mathcal{L}(U, V)$.
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- Now, $U \cong \mathbb{F}^n$, and $V \cong \mathbb{F}^m$.
- Let $N_{\beta_U} \in \mathcal{L}(U, \mathbb{F}^n)$ be defined as $N_{\beta_U}(u_1) = e_1^n, \ldots, N_{\beta_U}(u_n) = e_n^n$,

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- $x \in U, N_{\beta_U}(x) = [x]_{\beta_U}$, and $y \in V, M_{\beta_V}(y) = [y]_{\beta_V}$.
- Let $\alpha_U = \{p_1, \ldots, p_n\}, \alpha_V = \{q_1, \ldots, q_m\}$, $U \cong \mathbb{F}^n$, be different set of basis vector for U and V resp.

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- $x \in U, N_{\alpha_U}(x) = [x]_{\alpha_U}$, and $y \in V, M_{\alpha_V}(y) = [y]_{\alpha_V}$.
- Given $[T]_{\beta_U}^{\beta_V}$, how to compute $[T]_{\alpha_U}^{\alpha_V}$?

- Commutative Diagram

$$
\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \\
U \arrow[u, "N_{\beta_U}"] \arrow[r, "T"] \arrow[d, "N_{\alpha_U}"'] & V \arrow[u, "M_{\beta_V}"'] \arrow[d, "M_{\alpha_V}"] \\
\mathbb{F}^n \arrow[r, "{[T]_{\alpha_U}^{\alpha_V}?}"] & \mathbb{F}^m
\end{tikzcd}
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\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & \mathbb{F}^m \arrow[d, "\underbrace{M_{\alpha_V}(M_{\beta_V})^{-1}}_{M_{\beta_V}^{\alpha_V}}"] \\
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{[x]_{\beta_U} = N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}} \arrow[r, "{[T]_{\beta_U}^{\beta_V}}"] & {[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V}[x]_{\beta_U}} \arrow[d, "M_{\beta_V}^{\alpha_V}"] \\
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- Thus,

$$[y]_{\alpha_V} = M_{\beta_V}^{\alpha_V}[T]_{\beta_U}^{\beta_V} N_{\alpha_U}^{\beta_U}[x]_{\alpha_U}, \forall x \in U$$

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- $[T]_{\alpha_U}^{\alpha_V} = M_{\beta_V}^{\alpha_V} [T]_{\beta_U}^{\beta_V} N_{\alpha_U}^{\beta_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and

$\alpha_U = \alpha_V = \alpha$.

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- In this case, $[T]_\alpha^\alpha = M_\beta^\alpha[T]_\beta^\beta N_\alpha^\beta$.
- Note that $M_\beta^\alpha = \left(N_\alpha^\beta\right)^{-1}$.

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- In this case, $[T]_\alpha^\alpha = M_\beta^\alpha[T]_\beta^\beta N_\alpha^\beta$.
- Note that $M_\beta^\alpha = \left(N_\alpha^\beta\right)^{-1}$. Denote M_β^α by S, which gives us $[T]_\alpha^\alpha = S[T]_\beta^\beta S^{-1}$.

- $[T]^{\alpha_V}_{\alpha_U} = M^{\alpha_V}_{\beta_V}[T]^{\beta_V}_{\beta_U} N^{\beta_U}_{\alpha_U}$
- For a linear operator $T : U \to U$, assume $\beta_U = \beta_V = \beta$ and $\alpha_U = \alpha_V = \alpha$.
- In this case, $[T]^{\alpha}_{\alpha} = M^{\alpha}_{\beta}[T]^{\beta}_{\beta} N^{\beta}_{\alpha}$.
- Note that $M^{\alpha}_{\beta} = \left(N^{\beta}_{\alpha}\right)^{-1}$. Denote M^{α}_{β} by S, which gives us $[T]^{\alpha}_{\alpha} = S[T]^{\beta}_{\beta} S^{-1}$.
- **Similar matrices and Similarity transformation:** We say two matrices A and B are similar if there exists an invertible matrix, say S such that $B = SAS^{-1}$. The transformation $A \mapsto SAS^{-1}$ is said to be a similarity transformation of A by S.