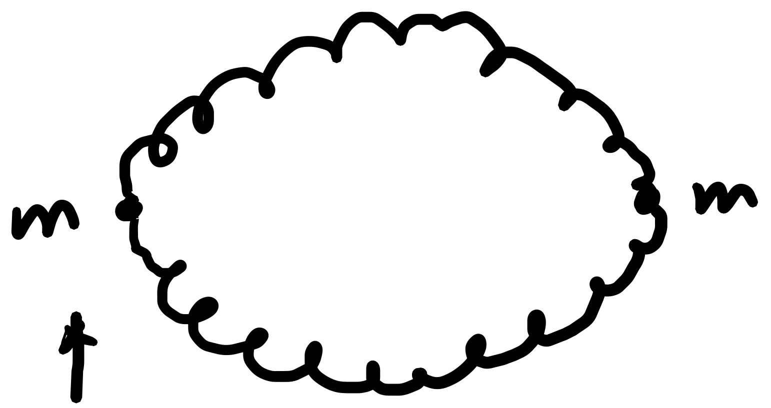
# LECTURE 24

**RECAP :**

Masses on a circle.



$F_d \cos\omega_d t$.

$$x_1(t) = -\frac{F_d\left(2\omega^2-\omega_d^2\right)}{\omega_d^2\left(4\omega^2-\omega_d^2\right)}\cos\omega_d t$$

$$x_2(t) = -\frac{2F_d\,\omega^2}{\omega_d^2\left(4\omega^2-\omega_d^2\right)}\cos\omega_d t.$$

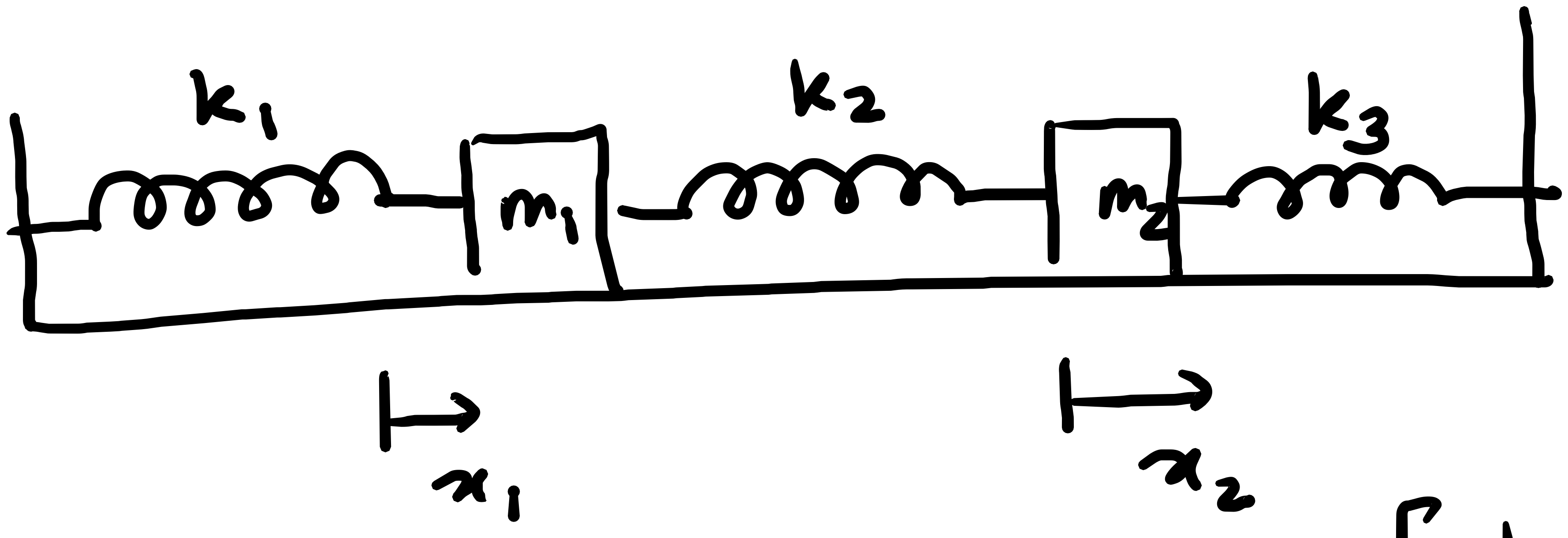
(i) $4\omega^2-\omega_d^2 = 0 \rightarrow$ resonance.

(ii) $2\omega^2-\omega_d^2 = 0 \rightarrow x_1(t) = 0$.

**EOM:** $m\ddot{x}_1 + 2k(x_1 - x_2) = F_d \cos\omega_1 t$

$m\ddot{x}_2 + 2k(x_2 - x_1) = F_d \cos\omega_2 t$ .

**PROB:**



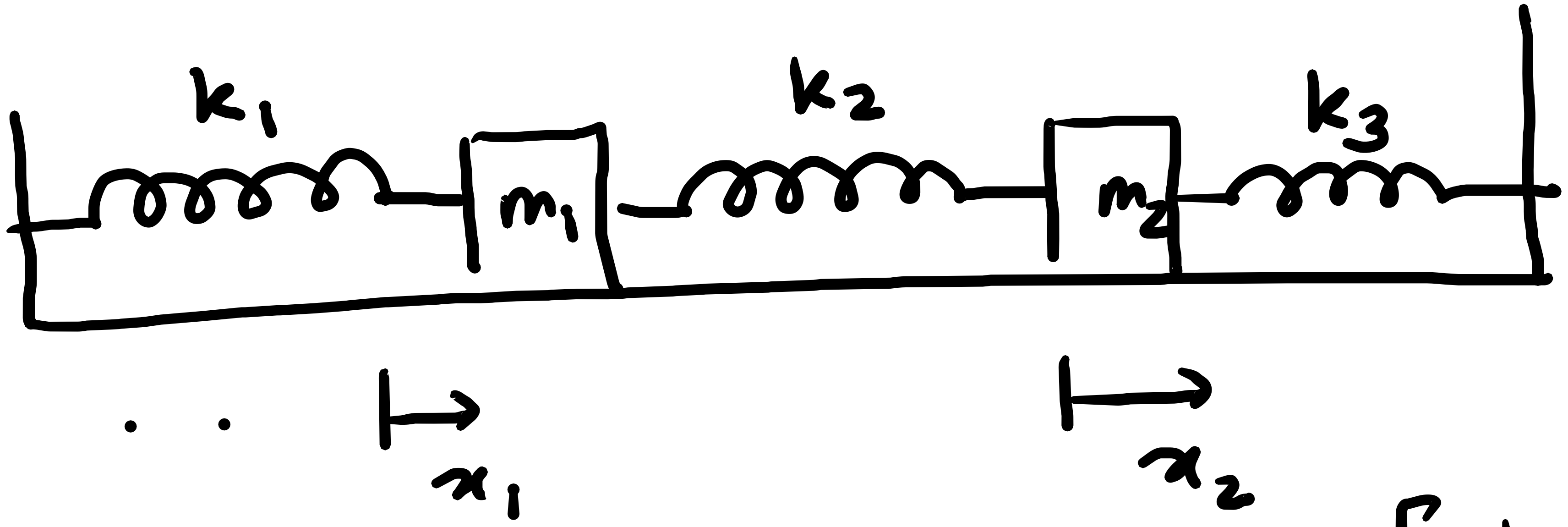

$$x(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2) .$$

**Simplification:** $m_1 = m_2 = m$, $\quad k_1 = k_2 = k$.

EOM :- $m\ddot{x}_1 + 2k(x_1 - x_2) = F_d \cos\omega_1 t$

$m\ddot{x}_2 + 2k(x_2 - x_1) = F_d \cos\omega_2 t$ .

PROB :-




EOM :

$m_1\ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2$

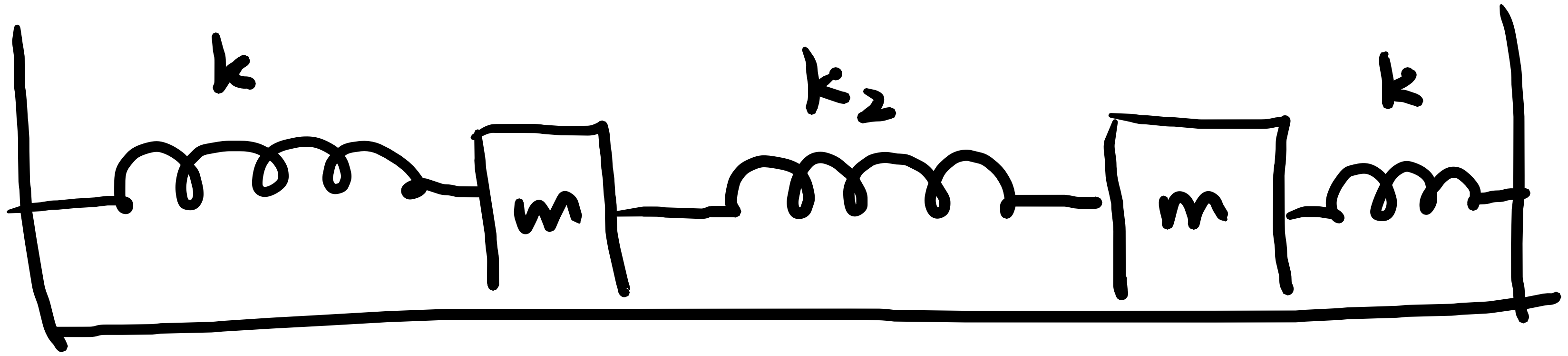
$m_2\ddot{x}_2 = k_2 x_1 - (k_2+k_3)x_2$

$z(t) = \ldots$

$$x(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2) .$$

Simplification :- $m_1 = m_2 = m$, $\quad k_1 = k_2 = k_3 = k$.

PROB :- $k_1 = k_3 = k$ , $k_2 \neq k$ . $m_1 = m_2$




$k_2 << k$ .

$\Downarrow$

Weakly coupled oscillators.

$$M\ddot{x} = -Kx .$$

Earlier prob.

$$K_{old} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

In this problem

$$K = \begin{bmatrix} k + k_2 & -k_2 \\ -k_2 & k + k_2 \end{bmatrix}$$

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

Freq. of the normal modes obtained from the eqn.

$$|K - \omega^2 M| = 0 .$$

$$(K - \omega^2 M) = \begin{bmatrix} k + k_2 - m\omega^2 & -k_2 \\ -k_2 & k + k_2 - m\omega^2 \end{bmatrix}$$

$$\begin{vmatrix} k + k_2 - m\omega^2 & -k_2 \\ -k_2 & k + k_2 - m\omega^2 \end{vmatrix} = 0 .$$

$$\Rightarrow (k + k_2 - m\omega^2)^2 - k_2^2 = 0 .$$

$$k + k_2 - m\omega^2 = \pm k_2 .$$

$$\Rightarrow \underbrace{\omega_1 = \sqrt{\frac{k}{m}}}_{\text{in-phase mode.}} \qquad \omega_2 = \sqrt{\frac{k + 2k_2}{m}} .$$

Approximation, $k_2 << k$.

Define, $\omega_0 = \frac{\omega_1 + \omega_2}{2}$

$\omega_1 = \omega_0 - \epsilon$.

$\omega_2 = \omega_0 + \epsilon$.

Solve for normal modes.

$$\bar{z}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_1 t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega_2 t}$$

$$= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 - \epsilon)t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0 + \epsilon)t}$$

$$= \underbrace{\left\{ C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t} \right\}}_{\text{since } \epsilon \text{ is small, this terms oscillates very slowly.}} e^{i\omega_0 t}.$$

— For convenience, let initial conditions be chosen such that,

$$C_1 = C_2 = \frac{A}{2}$$

$$\bar{Z}(t) = \frac{A}{2}\begin{bmatrix} e^{-i\epsilon t} + e^{i\epsilon t} \\ e^{-i\epsilon t} - e^{i\epsilon t} \end{bmatrix} e^{i\omega_0 t} .$$

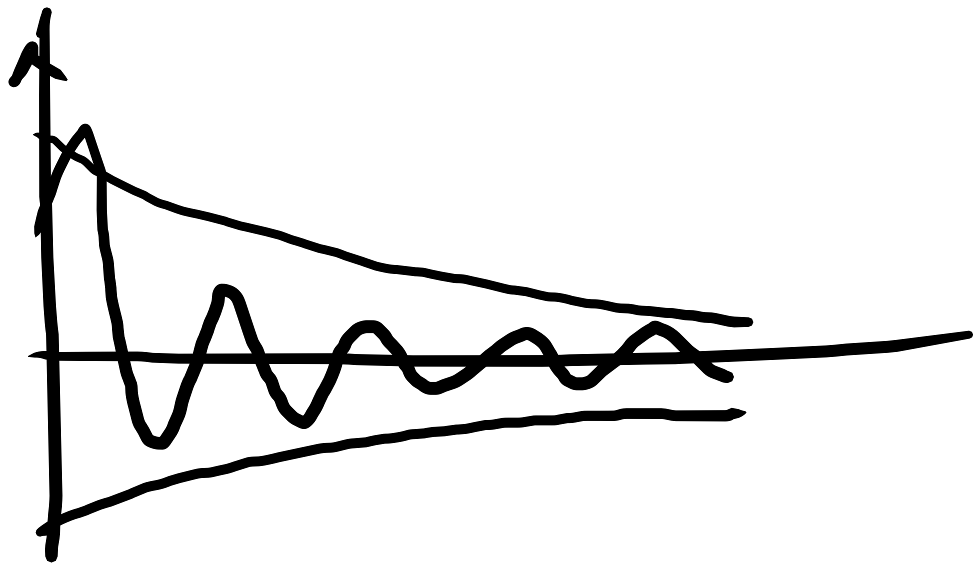
$$= A \begin{bmatrix} \cos\epsilon t \\ -i\sin\epsilon t \end{bmatrix} e^{i\omega_0 t} .$$

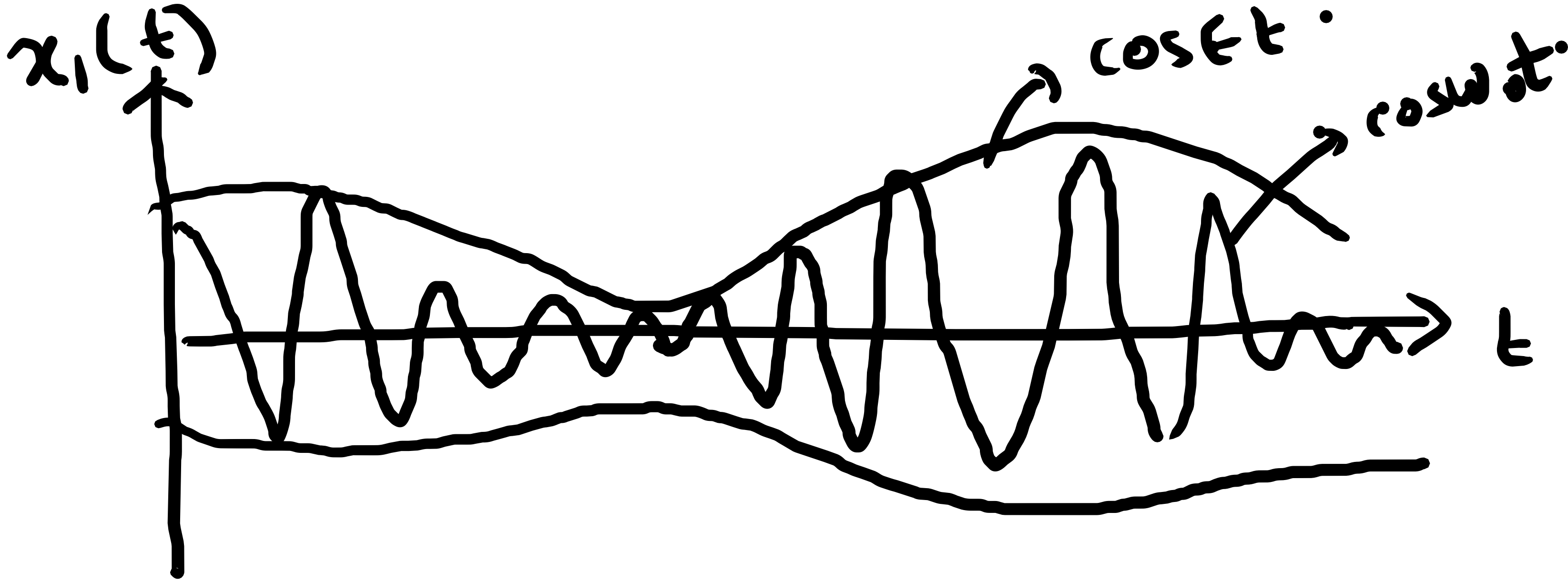
$$\bar{x}(t) = \mathrm{Re}\,\bar{Z}(t) \Rightarrow \begin{aligned} x_1(t) &= (A\cos\epsilon t)\cos\omega_0 t \\ x_2(t) &= (A\sin\epsilon t)\sin\omega_0 t . \end{aligned}$$

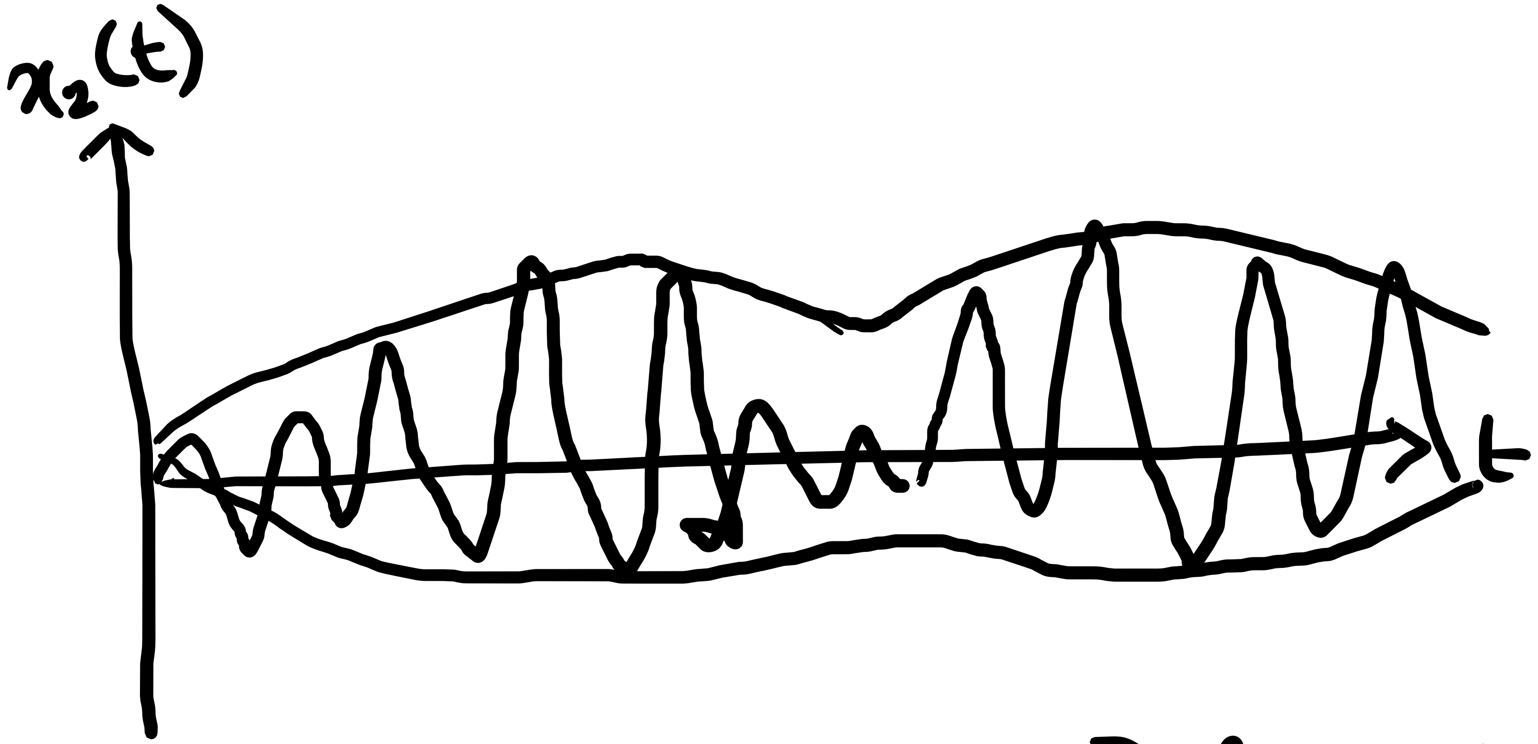
$$x_1(t) = A\cos\epsilon t \cos\omega_0 t$$

$$x_2(t) = A\sin\epsilon t \sin\omega_0 t$$

In case of damped oscillation






BEATING PATTERN

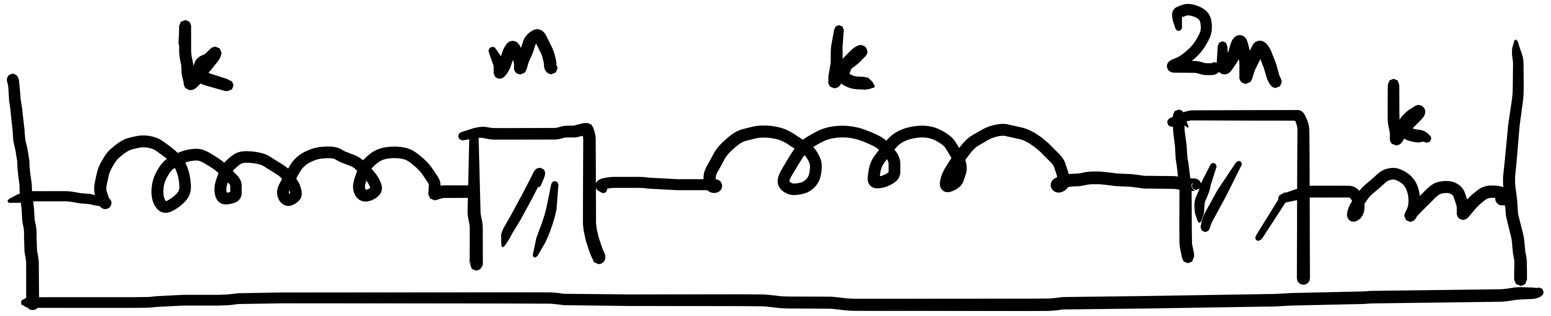
/ BEATS.

$$\left.\begin{aligned} \xi_1 &= \tfrac{1}{2}(x_1 + x_2) \\ \xi_2 &= \tfrac{1}{2}(x_1 - x_2) \end{aligned}\right\} \quad \text{Exercise:- prove this.}$$

$$\xi_1(t) = \frac{1}{2} A \cos(\omega_0 - \epsilon) t \quad = \frac{1}{2} A \cos \omega_1 t$$

$$\xi_2(t) = \frac{1}{2} A \cos(\omega_0 + \epsilon) t \quad = \frac{1}{2} A \cos \omega_2 t \ .$$

PROB :-




## Inhomogeneous ODEs

— Arise from (say). driving force applied to oscillator.

$$\frac{dx}{dt} + P(t)\,x(t) = \underbrace{Q(t)}_{\neq 0}.$$

$$\frac{dx}{dt} + \underline{P(t)\,x(t) = Q(t)}.$$

- Need to solve this eqn. analytically.

Remark:- So far have considered R.H.S. $= e^{i\omega t}$.

In this case, trial solⁿ can be guessed $x(t) = A e^{i\omega t}$.

$$\dot{x}(t) = i\omega A e^{i\omega t}$$

$$\ddot{x}(t) = -\omega^2 A \underbrace{e^{i\omega t}}$$

cancels from both L.H.S. and R.H.S.

For any general $Q(t)$, multiply this eqn. by $M(t)$.

$$M(t)\frac{dx}{dt} + M(t)P(t)\,x(t) = M(t)Q(t) \,. \longrightarrow$$

Demand. $L.H.S. = \dfrac{d}{dt}\big(M(t)\,x(t)\big)$.

$$= M(t)\frac{dx}{dt} + \frac{dM}{dt}x$$

Comparing, $\dfrac{dM}{dt}x = MPx$.

$$\Rightarrow \frac{dM}{dt} = MP \,. \qquad \Rightarrow \frac{dM}{M} = P\,dt \,.$$

$$\ln M = \int P\,dt + \text{const}.$$

$$\Rightarrow \quad M = e^{\int dt\, P(t)} .$$

$$\frac{d}{dt}(Mx) = QM .$$

$$\Rightarrow \quad Mx = \int dt\, MQ .$$

$$\Rightarrow \quad \boxed{x = M^{-1} \int dt\, MQ .}$$