$\boxed{\text{Lecture} \cdot 25}$

$$x(n) = e^{j\omega n} \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n) = H(e^{j\omega}) \cdot e^{j\omega n}$$

$$x_1(n) = a_1 e^{j\omega_1 n} \longrightarrow \boxed{LTI} \longrightarrow y_1(n) = a_1 H(e^{j\omega_1}) e^{j\omega_1 n}$$

$$x_2(n) = a_2 e^{j\omega_2 n} \qquad y_2(n) = a_2 H(e^{j\omega_2})$$

$$x_3(n) = a_3 e^{j\omega_3 n} \longrightarrow \boxed{LTI} \longrightarrow y_3(n) = a_3 H(e^{j\omega_3}) \cdot e^{j\omega_3 n}$$

$$\underbrace{x_1(n) + x_2(n) + x_3(n)}_{x(n) =} \longrightarrow \boxed{LTI} \longrightarrow y(n) = y_1(n) + y_2(n) + y_3(n)$$

$$y(n) = \sum_{k=1}^{3} a_k H(e^{j\omega_k}) \cdot e^{j\omega_k n}$$

$$x(n) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k n} \longrightarrow \boxed{LTI} \longrightarrow y(n) = \sum_{k=-\infty}^{\infty} a_k H(e^{j\omega_k}) e^{j\omega_k n}$$

I/p linear combination (LC) of complex exponential

Coeff of LC = a_k

→ O/p is also LC of complex exponential

→ Coeff of LC = $a_k \cdot H(e^{j\omega_k})$.

(1)

Conclusion: If the input to an LTI system is represented ~~by~~ as a linear combination of complex exponentials,

i.e., $x[n] = \sum_{\langle k \rangle} a_k e^{j\omega_k n}$, then the output can also be represented as a linear combination of the 'same' complex exponentials signals, ~~However~~ i.e.,

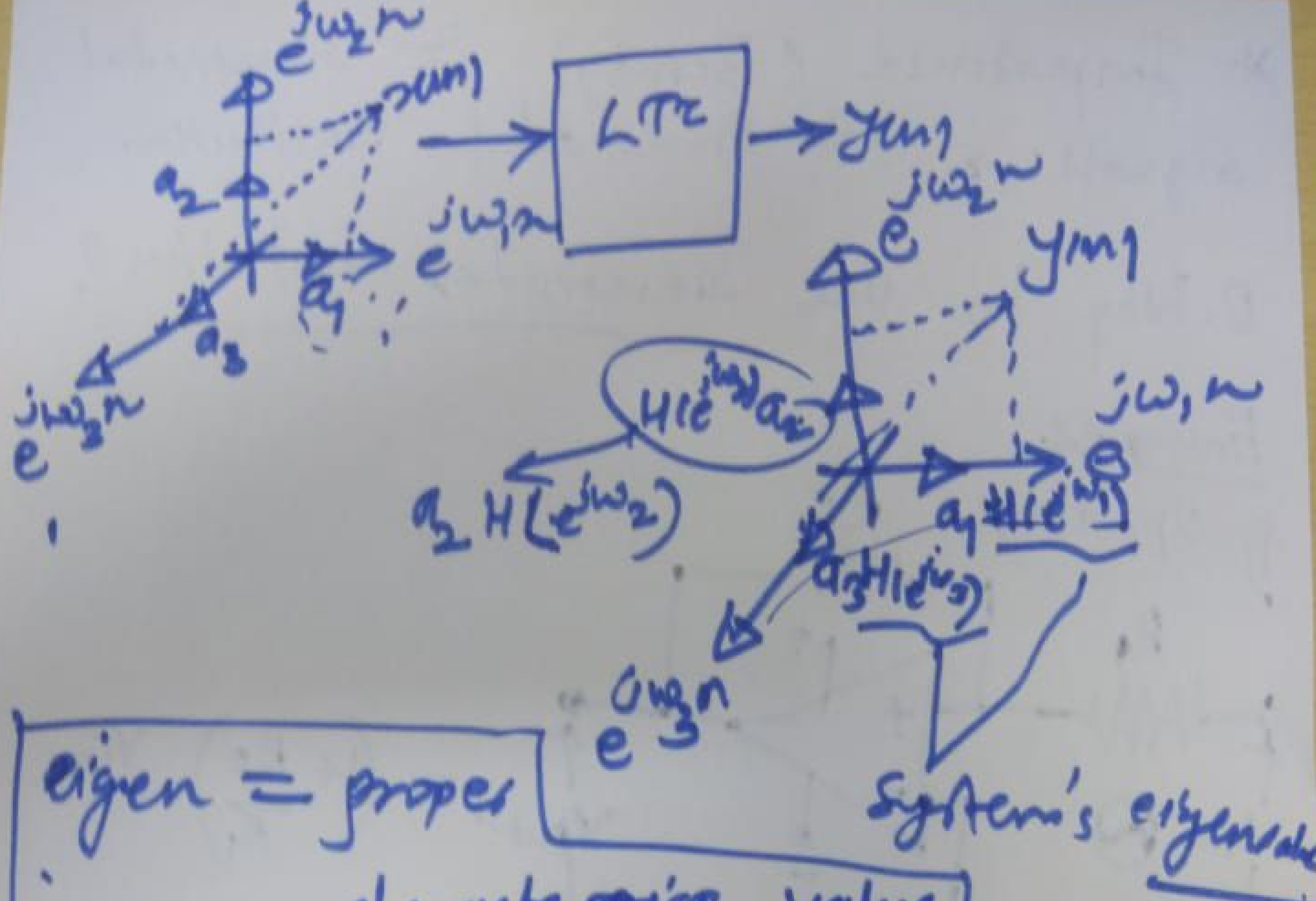
$$y[n] = \sum_{\langle k \rangle} a_k H(e^{j\omega_k}) \cdot e^{j\omega_k n}.$$

However, coefficients of LC of the output $=$ Coeffi. of LC of i/p $\times$ LTI system's eigenvalue, $H(e^{j\omega_k})$

$$\therefore \; = a_k \cdot H(e^{j\omega_k}).$$

Euler, Gauss, Fourier,

→ Problem of vibrating string



eigen = proper

= characteristics value.

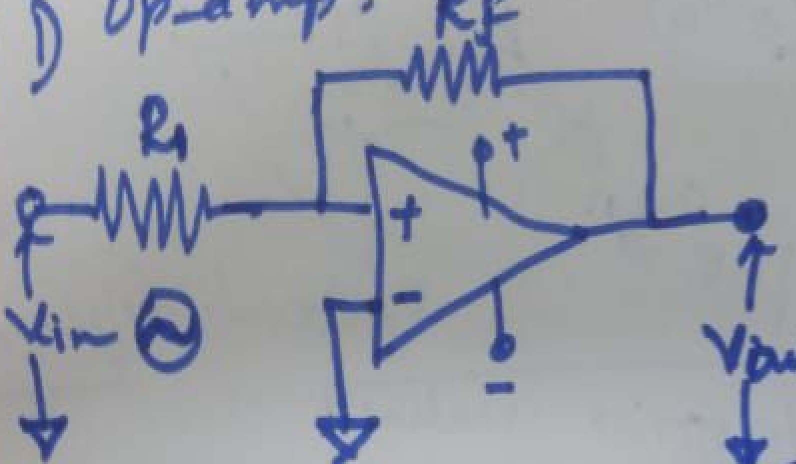
③ ④

* Importance (Beauty) of Sinusoidal Signals :→ OR Sinusoidal Excitation

Q. Why to use sinusoidal excitation?

Examples.

1) Op-amp.

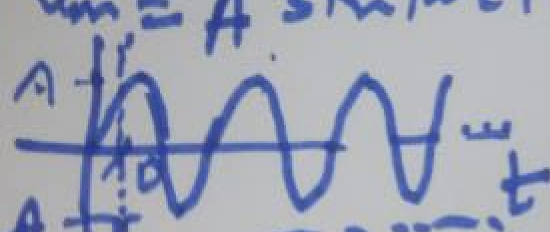


$$V_{out} = \left(1 + \frac{R_f}{R_1}\right) \cdot V_{in}$$

$$V_{in} = A \sin(\omega t + \phi)$$

$$\boxed{V_{out} = \left(1 + \frac{R_f}{R_1}\right) \cdot A \sin(\omega t + \phi)}$$

(A)



RC LPF

$$V_{out} = \int V(t)\, dt + C$$

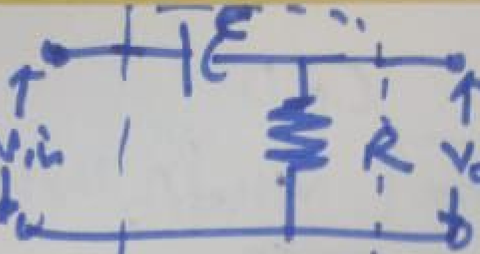
$$= \int A \sin(\omega t + \phi)\, dt$$

$$= -\frac{A}{\omega} \cos(\omega t + \phi)$$

(B)

(4)

$$V_{out} = \left(\frac{A}{\omega}\right) \cdot \sin\left[\omega t + \phi + \frac{\pi}{2}\right]$$



RC HPF

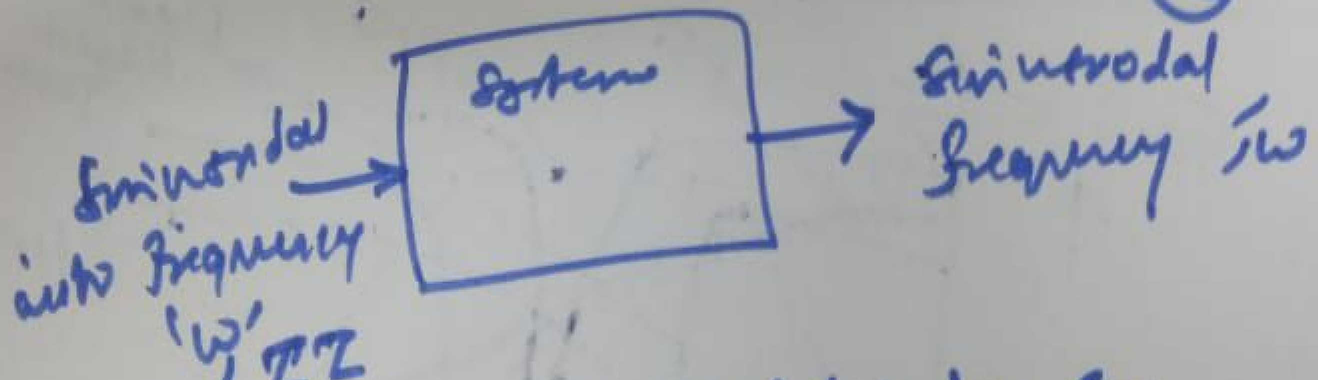
$$V_{out} = \frac{d}{dt}[V_{in}(t)]$$

$$= \frac{d}{dt}[A \sin(\omega t + \phi)]$$

$$= A\omega \cos(\omega t + \phi)$$

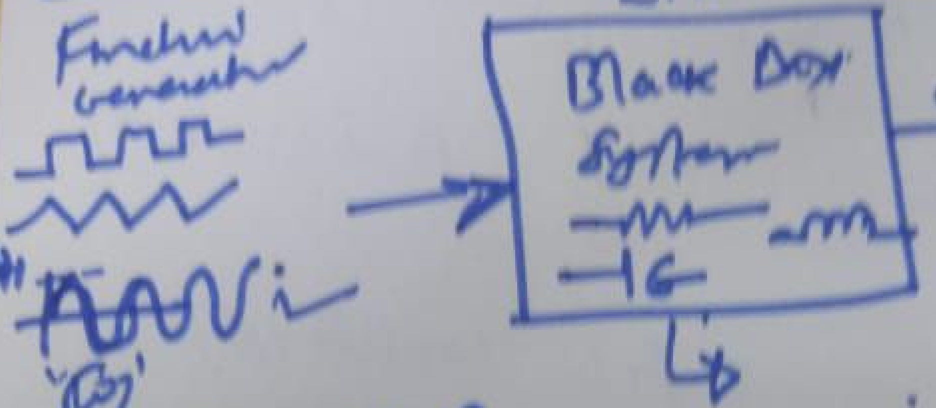
$$\boxed{V_{out} = (A\omega) \sin[\omega t + \phi + \pi/2]}$$

(C).



If a LTI system is excited by a sinusoidal function/signal of frequency 'ω', then the output of the system is also sinusoidal and with the same frequency. However, there is a change in amplitude and phase of sinusoid. This change is because of individual system characteristics.

System Identification



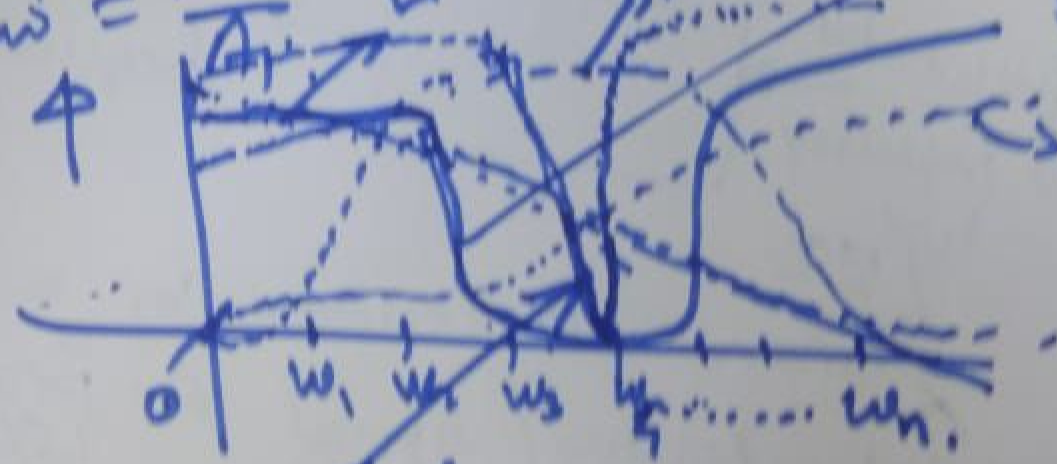
$A_1 \sin(wt + \phi_1)$

$$\text{Gain} = \frac{A_2}{A_1}$$

Both are known

$\{w_1, w_2, w_3, \ldots, w_n\}$

$$\text{Gain} = \frac{A_2}{A_1}$$



Frequency →

Ideal filter

6

* Concept of Inner Product :→

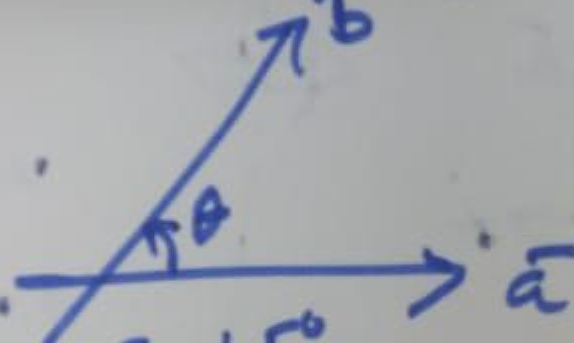
Motivation :→ To generalize the notion of dot product of two vectors.

Let $\bar{a} = a_1 i + a_2 j + a_3 k$

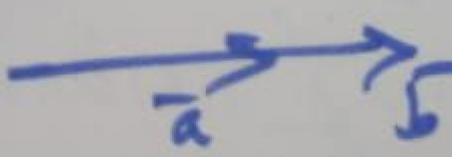
$\bar{b} = b_1 i + b_2 j + b_3 k$

$\therefore \bar{a}\cdot\bar{b} = |\bar{a}|\,|\bar{b}|\cos(\theta)$

$\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$



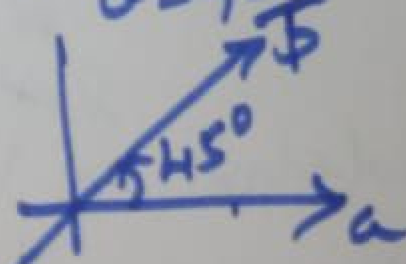
$\theta = 0^\circ$



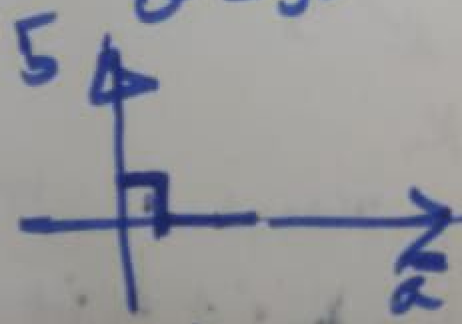
Collinear vectors.

$\bar{a}.\bar{b} = $ Max.

$\theta = 45^\circ$



$\theta = 90^\circ$



; Orthogonal vectors

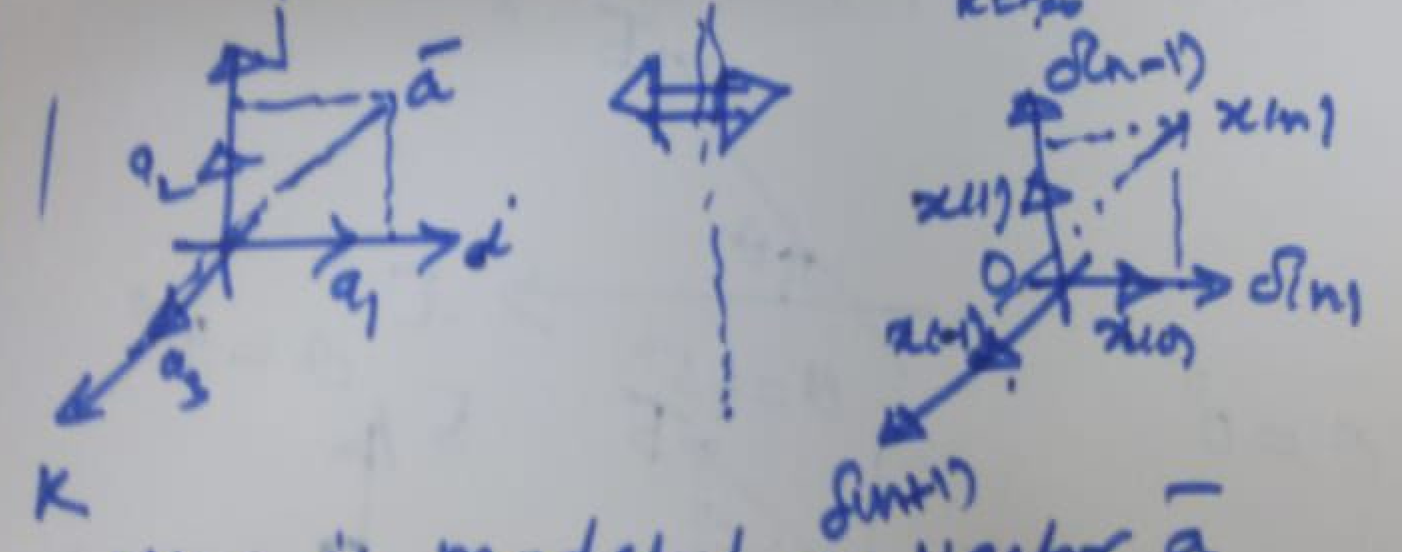
$\bar{a}.\bar{b} = 0$

Question: How to generalize $\bar{a}\cdot\bar{b}$ should $\bar{a}$ and $\bar{b}$ are signals?

Hint: In signals and systems we model signals as vector! (Why?)

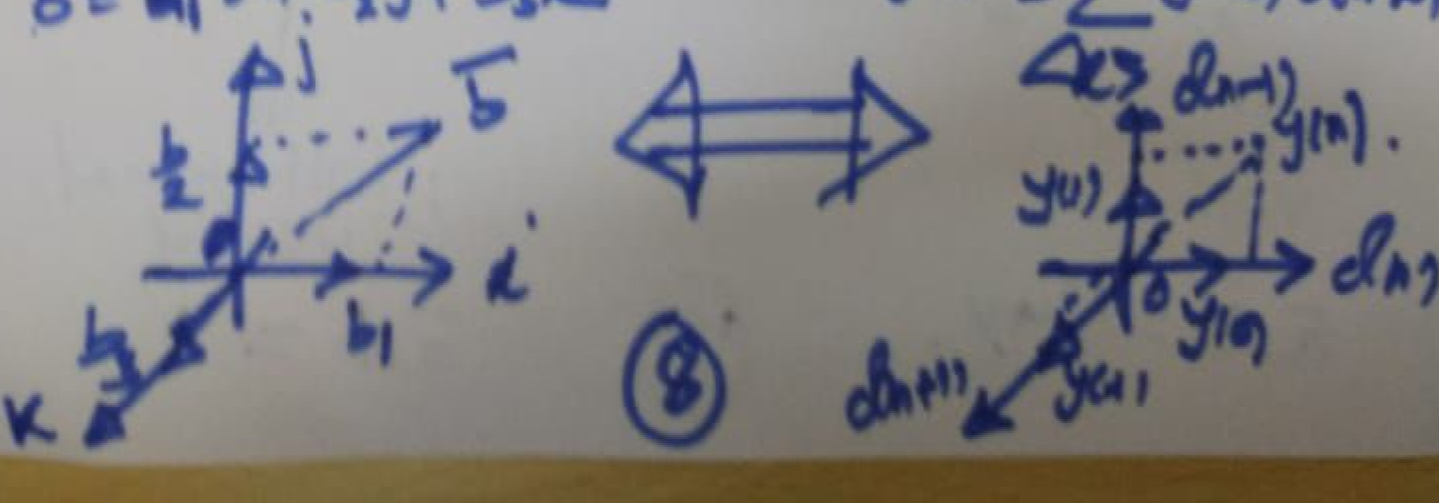
[Sifting property of Discrete Signal].

$\bar{a} = a_1 i + a_2 j + a_3 k$; $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$



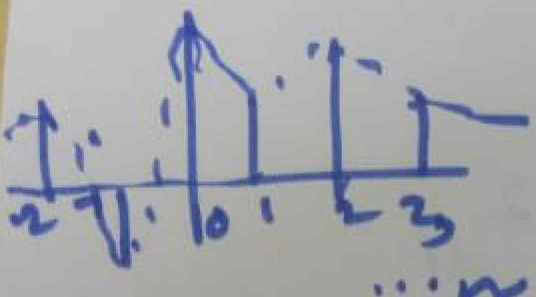
$\therefore$ $x(n)$ is modeled as vector $\bar{a}$

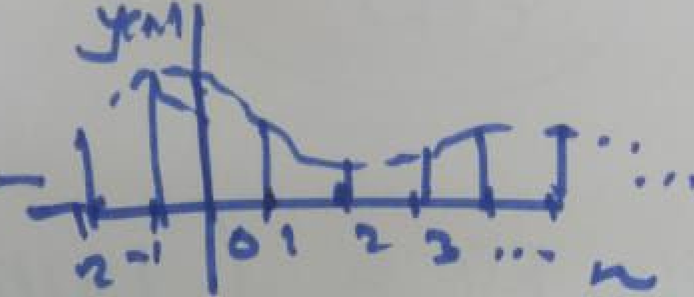
$\bar{b} = b_1 i + b_2 j + b_3 k$; $y(n) = \sum y(k)\delta(n-k)$



$$\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n$$

$$\overline{x[n]}\cdot\overline{y[n]} = \cdots + x(-1)y(-1) + x(0)y(0) + x(1)y(1) + \cdots$$





$$\therefore \overline{x[n]}\cdot\overline{y[n]} = \sum_{k=-\infty}^{+\infty} x[k]\cdot y[k]$$

If $x[n] \rightarrow x(t)$, $y[n] \rightarrow y(t)$

$$\therefore \overline{x(t)}\cdot\overline{y(t)} = \int_{t=-\infty}^{t=+\infty} x(t)\cdot y(t)\,dt$$

Let $y(t) = x(t)\cdot t\infty$

$$\therefore \overline{x(t)}\cdot\overline{x(t)} = \int_{-\infty} x^2(t)\,dt.$$

9

If $x(t)$ is complex signal

$$\int_{t\in\mathbb{R}} x^2(t)\, dt \rightarrow \text{Energy of a signal } x(t).$$

$$\bar{x}(t) \cdot \bar{x}(t) = \int_{t\in\mathbb{R}} |x(t)|^2\, dt$$

$$\bar{x}(t) \cdot \overline{x(t)} = \int_{t\in\mathbb{R}} x(t) \cdot x^*(t)\, dt$$

Let $x(t) = y(t)$

Inner Product of signal $x(t)$ with signal $y(t)$.

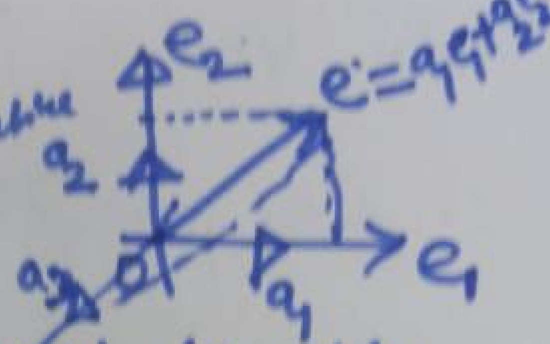
$$\boxed{\therefore \langle x(t), y(t) \rangle = \int_{t\in\mathbb{R}} x(t) \cdot y^*(t)\, dt}$$

For discrete-time signal (10) $\Rightarrow$

$$\langle x(n), y(n) \rangle = \sum_{n=-\infty}^{+\infty} x(n) \cdot y^*(n)$$

Concept of linear combination in terms of Inner Product:-

Let $\{e_1, e_2\}$ = set of basis vectors / representative vectors.



a_1 and a_2 are scalars along direction vector e_1 and e_2, e_3 respectively,

$\therefore$ An arbitrary vector, e can be represented as a linear combination

$$\therefore e = a_1 e_1 + a_2 e_2 + a_3 e_3 \text{ --- } (1)$$

Take inner product on both sides of eqn (1) w.r.t. vector e_1,

$$\langle e, e_1 \rangle = \langle a_1 e_1 + a_2 e_2, e_1 \rangle$$

$$\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$$

(ii) $$\bar{a} \cdot (k\bar{b}) = k(\bar{a} \cdot \bar{b})$$

$$\langle e, e_1\rangle = a_1 \langle e_1, e_1\rangle + a_2 \langle e_2, e_1\rangle$$

If vector e_1 $\perp$ar to e_2

$$\therefore \langle e_1, e_2\rangle = \langle e_2, e_1\rangle = 0$$

$$\therefore \langle e, e_1\rangle = a_1 \langle e_1, e_1\rangle$$

$$\therefore \boxed{a_1 = \frac{\langle e, e_1\rangle}{\langle e_1, e_1\rangle}}$$

for a_2 Inner product of eqn (1), w.r.t. e_2

$$\boxed{a_2 = \frac{\langle e, e_2\rangle}{\langle e_2, e_2\rangle}}$$

For a_3, inner product eqn (1), w.r.t. e_3

$$a_3 = \frac{\langle e, e_3\rangle}{\langle e_3, e_3\rangle}$$

12

If vector e is represented in n-dimension using the set $\{e_i\}_{i\in[1,n]}$

$$\therefore e = a_1 e_1 + a_2 e_2 + a_3 e_3 + \dots + a_n e_n$$

$\therefore$ For a_n inner product of eqn ①, w.r.t. e_n

$$\therefore a_n = \frac{\langle e, e_n\rangle}{\langle e_n, e_n\rangle}$$

$$\therefore e = \frac{\langle e, e_1\rangle}{\langle e_1, e_1\rangle}\cdot e_1 + \frac{\langle e, e_2\rangle}{\langle e_2, e_2\rangle}\cdot e_2 + \dots$$

$$+ \dots + \frac{\langle e, e_n\rangle}{\langle e_n, e_n\rangle}\cdot e_n$$

signal/vector

coeff of linear combination

$$e = \sum_{K=-\infty}^{+\infty} \frac{\langle e, e_K\rangle}{\langle e_K, e_K\rangle} e_K \qquad e = \sum_{K=-\infty}^{+\infty} a_K \cdot e_K$$

(13)

coeff of LE

Representative Vector

In Vector Algebra,

$\{i, j, k\}$ = Set of unit vector

$|i| = |j| = |k| = 1$

of

$\|e_1\| = \|e_2\| = \|e_3\| = \cdots = \|e_n\| = 1$

$\|\cdot\| \to$ Norm of a vector

$\downarrow$

Generalization of length of a vector from the origin.

$\therefore \langle e_1, e_2 \rangle = \underbrace{\|e_1\|}_{1} \cdot \underbrace{\|e_2\|}_{1} \cos(\theta)$

$\therefore \langle e_1, e_1 \rangle = \|e_1\| \, \|e_1\| \cos(0)$

$= 1 \cdot 1 \cdot 1 = 1$

$\therefore \langle e_2, e_2 \rangle = 1.$

$\vdots$

$\langle e_n, e_n \rangle = 1$

14

$$\therefore e = \sum_{k=-\infty}^{+\infty} \frac{\langle e, e_k \rangle}{1} \cdot e_k$$

$$e = \sum_{k=-\infty}^{+\infty} \langle e, e_k \rangle \cdot e_k$$

Coeffs. of linear combination

1 inner product operation is a multiply-add structure.

$\therefore \langle e_k, e_k \rangle \Rightarrow$ 'N' times then 'N' multiply-add structure operation will be saved.

$\Rightarrow \{e_n\}_{n \in \mathbb{Z}}$ is mutually orthogonal plus $\|e_n\| = 1 \; \forall n \in \mathbb{Z}$.

(1.5) Normalized orthogonal = ortho normal

If we represent a vector 'e'
in orthonormal system representation
then we have computational
advantage for computation and creation
of linear combination.