1. For the following vector spaces V and their corresponding subsets U, find whether U is a subspace of V or not.

   (a) $V = \mathbb{R}, U = \mathbb{Q}$.

   (b) $V = \mathbb{C}$ over $\mathbb{R}$, $U = \mathbb{R}$.

   (c) $V = \mathbb{R}^{n\times n}, U = \{A \in V, A^T = A\}$.

   (d) $V = \mathbb{R}^n$. For a fixed $A \in \mathbb{R}^{n\times n}$, $U_\lambda = \{x \in V \mid Ax = \lambda x\}$, where $\lambda \in \mathbb{R}$.

   (e) $V = \{f : \mathbb{R} \to \mathbb{R}\}, U_{t_0} = \{f \in V \mid f(t_0) = 0\}$ for some $t_0 \in \mathbb{R}$.

   (f) Let $V = \{(x_i)_{i=0}^{\infty} \mid x_i \in \mathbb{R}\}$, i.e., the set of all real-valued sequence beginning at index 0. $U = \{(x_i)_{i=0}^{\infty} \in V \mid x_0 = a, x_1 = b, x_n = x_{n-1} + x_{n-2}, n \geq 2\}$.

2. For the following vector spaces V, and subspaces U, W, find $U + W$. Also find if the sum of U and W is a Direct sum or not.

   (a) $V = \mathbb{R}^{n\times n}, U = \{A \in V \mid A^T = A\}, W = \{A \in V \mid A^T = -A\}$

   (b) $V = \mathcal{P}(\mathbb{R}), U_{x_0} = \{p \in V, p(x_0) = 0\}, W_{x_1} = \{p \in V, p(x_1) = 0\}, x_0, x_1 \in \mathbb{R}, x_0 \neq x_1$.

   (c) Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 2 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}, V = \mathbb{R}^4, U = C(A), W = C(A^T)$.

   (d) For the above matrix A, $V = \mathbb{R}^4, U = C(A), W = N(A)$.

3. Let U_1, U_2, U_3 be three subspaces of the vector space V. If $U_i \cap U_j = \{\theta\}, 1 \leq i, j \leq 3, i \neq j$, is the sum of these three subspaces a Direct sum?

4. Let V be a vector space over $\mathbb{F}$. Let $S = \{W \mid W \text{ is a subspace of } V\}$ be equipped with the following binary operations: $+$ denoting addition of subspaces in V and $\cdot$ denoting scalar multiplication defined by: $\forall a \in \mathbb{F}, \forall W \in S, a \cdot W = \{a \cdot w \mid \forall w \in W\}$, where $a \cdot w$ denotes the scalar multiplication defined on V. Is $(S, +, \cdot)$ a vector space?