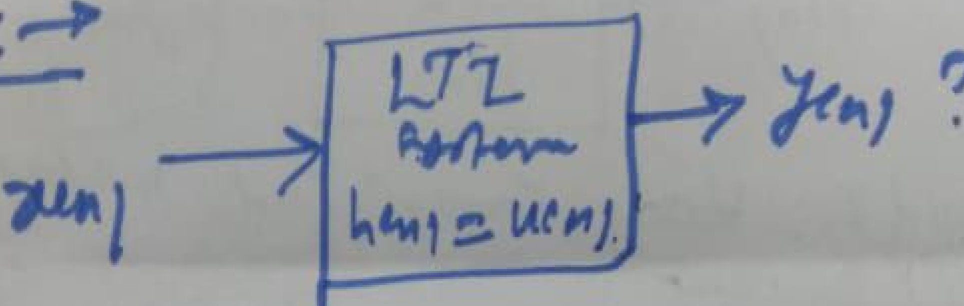
Lecture 16

Problem: → For any LTI system, impulse response is given by

$$h[n] = u[n]$$

Find input-output relationship.

Solution: →



$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k]$$

$$u[n-k] = \begin{cases} 1 \\ 0 \end{cases}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{n} x[k].}$$ → Accumulator system

Problems on Convolution Sum :→

(1) For an LTI system

$x(n) = \{-2, \underset{\uparrow n=0}{1}, 1\}$ and $h(n) = \{-1, \underset{\uparrow n=0}{1}, 2, -1\}$

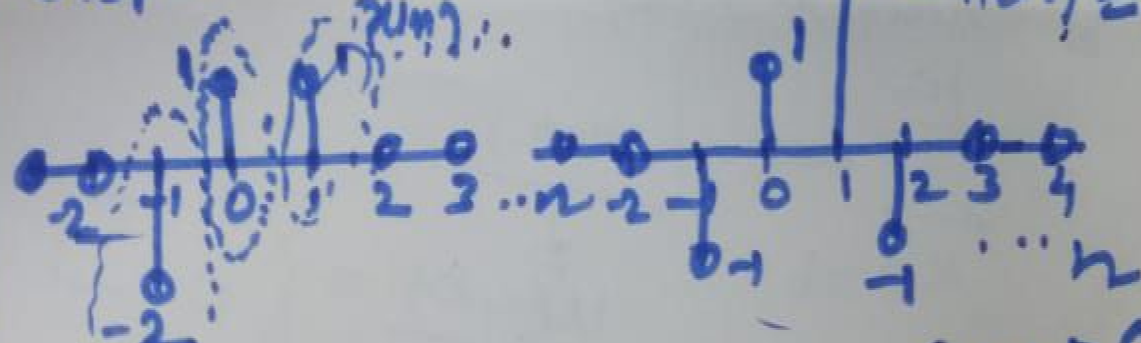
Find output response of system

$$x(n) \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n) = ?$$

Method I: ~~Using~~ Graphical method using (LTI) concept

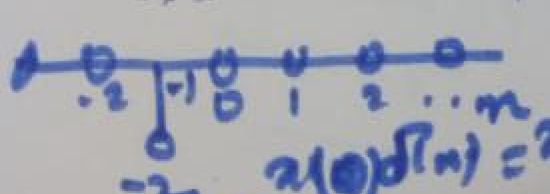
Step I Graph the signals $x(n)$ and $h(n)$.

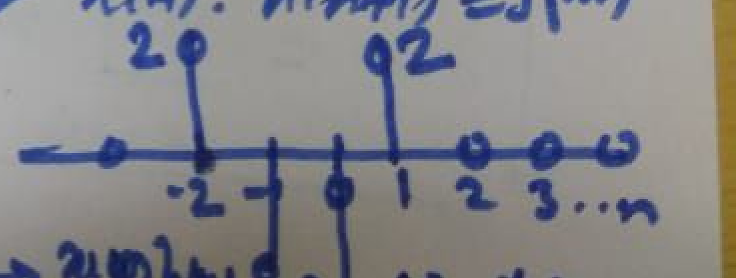
$h(n) = D\{\delta(n)\}$



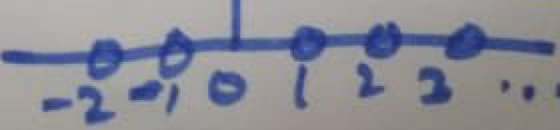
$x_1(n) = x(-1)\cdot\delta(n+1) \longrightarrow \{LTI\} \longrightarrow x(-1)\cdot h(n+1) = y_1(n)$

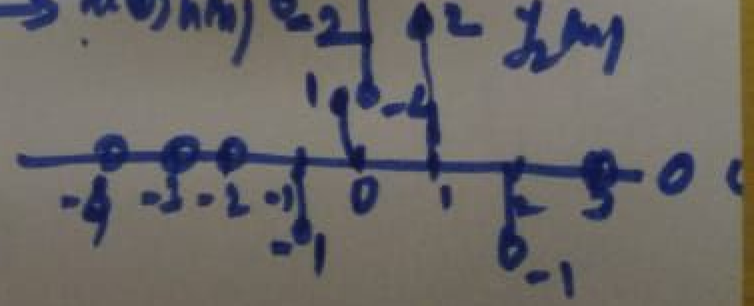
$x(-1)\, D\{\delta(n+1)\}$





$x(0)\delta(n) = x_2(n) \longrightarrow \{LTI\} \longrightarrow x(0)h(n)$ $y_2(n)$





$\therefore y(n) = D\{x(n)\}$

If $x(n) = \delta(n)$ then

$y(n) = h(n) = D\{\delta(n)\}$.

$$\underbrace{D}_{\text{LTI}}\{\underbrace{x(0)}_{\text{scalar}}.\delta(n)\} = x(0)\, D\{\delta(n)\}$$
$$= x(0).h(n).$$

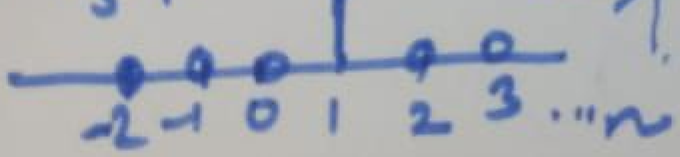
[Homogeneity]

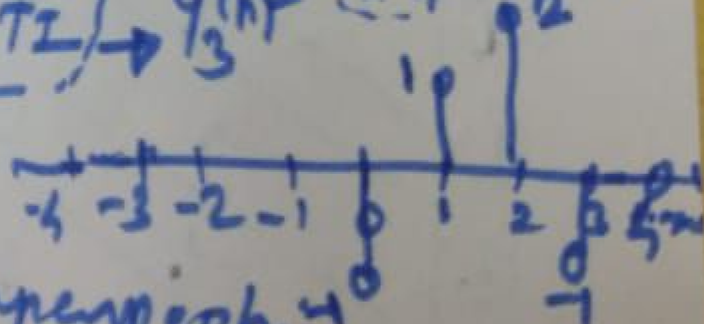
$$\therefore D\{x(-1).\delta(n+1)\} = x(-1).\, D\{\delta(n+1)\}$$
$$= x(-1)\, h(n+1)$$

(Time Invariance

$$\therefore D\{x(1).\delta(n-1)\} = x(1).\, h(n-1)$$

$x_3(n) = x(1)\,\delta(n-1) \rightarrow$ [LTI] $\rightarrow y_3(n) = x(1)\, h(n-1)$





Using Principle of Superposition

(Because operator $D\{\cdot\}$ is Linear)

$$\therefore D\{x_1(n) + x_2(n) + x_3(n)\} = D\{x_1(n)\} + D\{x_2(n)\} + D\{x_3(n)\}$$

$$\therefore D\{x(n)\} = y_1(n) + y_2(n) + y_3(n) = y(n)$$

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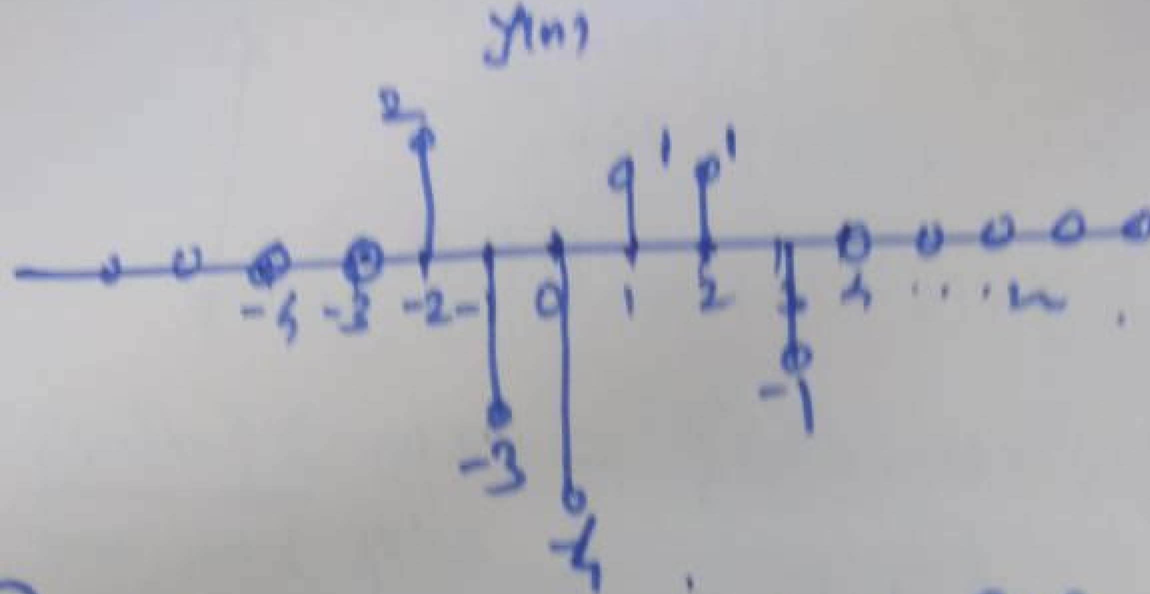

$y(n) = \{0, 2, -3, \underset{\uparrow}{-4}, 5, 1, -1, 0, 0, \ldots\}$

$n = 0$

output response.

Method II - Tabular Method.

$h(n) \downarrow$ / $x(n) \rightarrow$	-2	1	1
-1	2	-1	-1
1	-2	1	1
2	-4	2	2
-1	2	-1	-1

$y(n_x + n_h) = y(-2) = 2$

$y(n) = \{0, 0, 2, -3, \underset{\uparrow}{-4}, 5, 1, -1, 0, 0\}$

$n = 0$

Method III Analytical Method:

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n-k)$$

Let $n=0$

$$y(0) = \sum_{\langle k \rangle} x(k) \cdot \underline{h(-k)}$$

$$= \ldots + x(-2)\, h(2) + x(-1)\, h(1) + x(0)\, h(0) + x(1)\, h(-1) + x(4)\, h(-2) + x(5)\, h(3) + \ldots$$

$$= 0 \times (-1) + (-2) \times (2) + 1 \times (1) + 1 \times -1 + 0 \times 0 + 0 + 0.$$

$$\boxed{y(0) = -4}$$

For $y(1)$, put $n=1$,

$$\therefore y(1) = \sum_{\langle k \rangle} x(k)\, h(1-k)$$

$$= \cancel{x: + x(-2)\, h(3) + x:}$$

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$y(1) = \dots + x(-2)\, h(3) + x(-1)\, h(2)$
$+ x(0)\, h(1) + x(1)\, h(0) + x(2)\, h(-1) + \dots$

$y(1) = 0 + 0 \times 0 + (-2) \times (-1)$
$+ 1 \times 2 + 1 \times 1 + 0 + 0 + 0$

$y(1) = 5$

$y(-1) = \sum_{\langle k \rangle} x(k)\, h(-1-k) = ?? \; -3$

$y(-2) = \sum x(k)\, h(-2-k) = ?? \; 2$

$y(2) = \sum_{\langle k \rangle} x(k)\, h(2-k) = ?? \; 1$

$y(3) = \sum_{\langle k \rangle} x(k)\, h(3-k) = ?? \; -1$

$y(4) = 0$

$y(5) = 0$

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Method III: Graphical Method using convolution algorithm

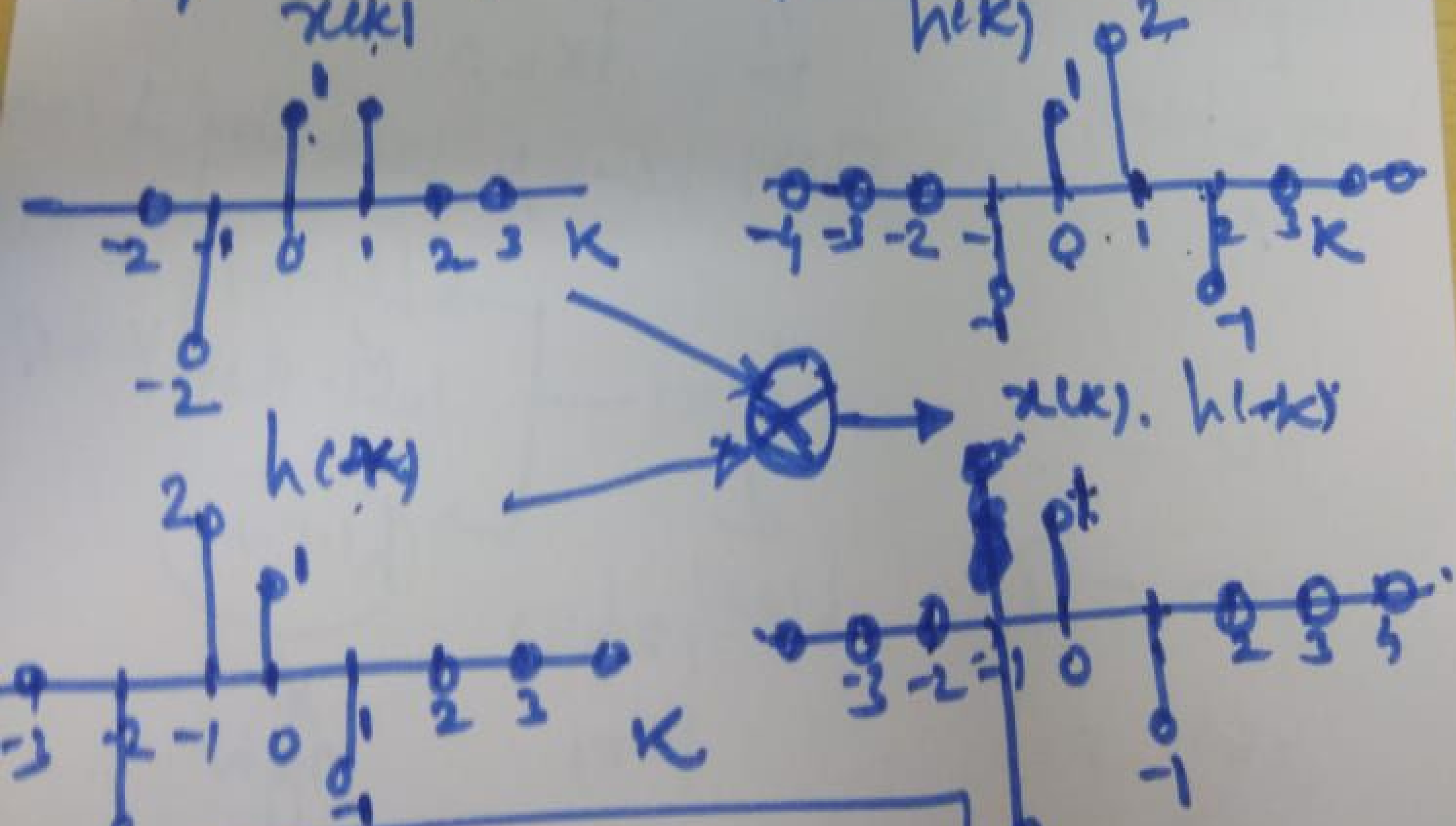
$$y(n) = x(n) * h(n)$$

$$\boxed{y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)}$$

Let $n=0$

$$y(0) = \sum_{k=-\infty}^{+\infty} x(k) \cdot h(-k)$$

Step I Graph the signals $x(k)$ & $h(k)$



$$\boxed{y(0) = \sum_{k} x(k) \cdot h(-k) = -4}$$

$$y(1) = \sum_{\langle k \rangle} x(k)\, h(1-k)$$

$$y(-1) = \sum_{\langle k \rangle} x(k)\, h(-1-k)$$

$h(-k)$ → $h(1-k)\Big|_{k=?\,2} = h(-1)$

Shift $h(-k)$ right by 1 sample

$h(-k)$ → $h(-1-k)\Big|_{k=0} = h(-1)$

Shift $h(-k)$ left by 1 sam

$k = 1$

$h(-k)\Big|_{k=1} = h(-1)$

$h(k)$ → $h(1-k)\Big|_{k=0} = h(1$

left shift

$h(k)$ → $h(1-k)$

right.

$h(k)\Big|_{k=1} = h(1)$

$y(4) = ?$

$y(3) = ?$

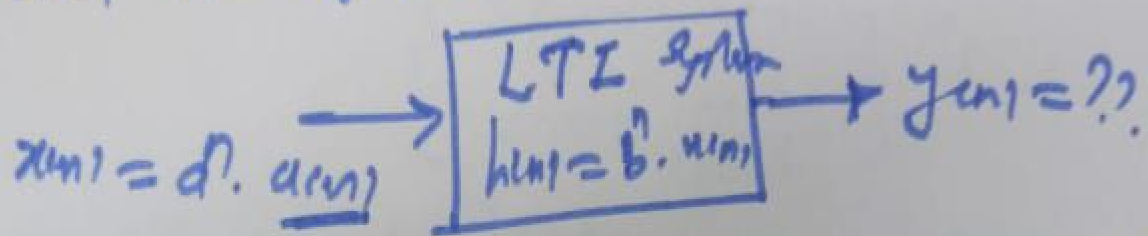
$y(-2) = ?$

$y(-3) = ?$

__Problem__ For an LTI system,

$x(n) = a^n \cdot u(n)$, $h(n) = b^n \cdot u(n)$

Find the system output.



__Solution:__ → Since the given system is LTI,

$\therefore\ y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{+\infty} x(k) \cdot h(n+k)$$

$$= \sum_{k=-\infty}^{+\infty} [a^k \cdot u(k)] \, [b^{n-k} \cdot u(n-k)]$$

$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$; $u(n-k) = \begin{cases} 1, & n-k \geq 0 \\ 0, & n-k < 0 \end{cases}$

$$y(n) = \cancel{\sum_{k=-\infty}^{-1} (\cdot 0)} + \sum_{k=0}^{n} (\cdot 1) + \cancel{\sum_{k=n+1}^{+\infty} (\cdot 0)}$$

$$y(n) = \sum_{k=0}^{n} a^k \times 1 \times b^{n-k} \times 1$$

$$= b^n \left\{ \sum_{k=0}^{n} (a/b)^k \right\}$$

G.P. , $a_1 = (a/b)^0 = 1$,

$r = (a/b)$.

$$= b^n \times 1 \left[\frac{1 - (a/b)^{n+1}}{1 - (a/b)} \right]$$

$$\boxed{y(n) = \frac{b^{n+1} - a^{n+1}}{b-a}}$$