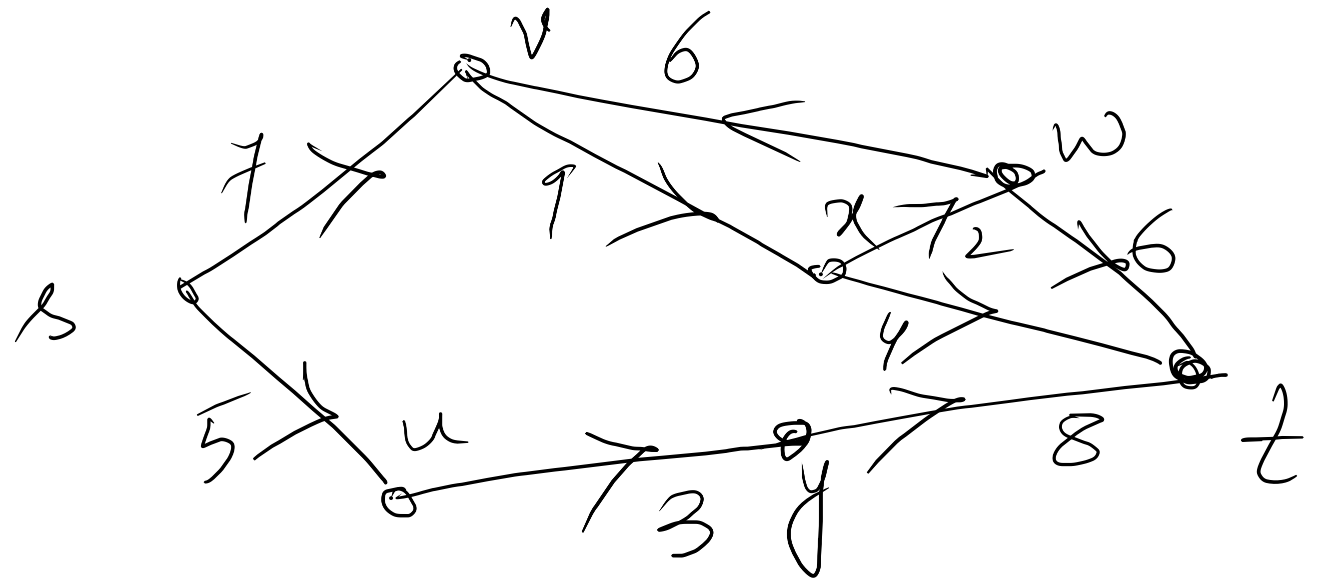
Network - Flow Problem

Flow network : Input: $G(V,E)$, s, t, $c: E \to \mathbb{R}^+$

A directed graph G

s : source t : sink

For each edge $e \in E$ $c(e) \geq 0$ called the capacity of the edge.



Maximum flow problem

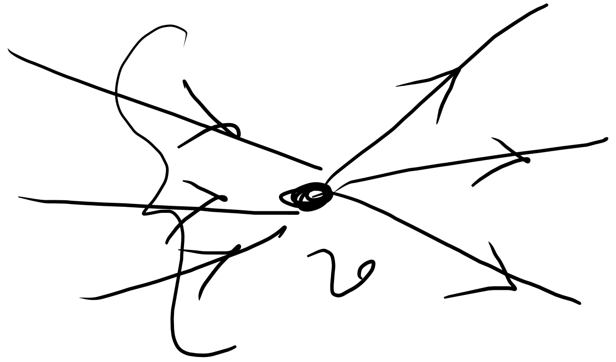
An s-t flow f that satifies the following two conditions/ constraints:

① **capacity constraint:**

for each edge $e \in E$, $0 \le f(e) \le c(e)$

② **flow conservation constraint**

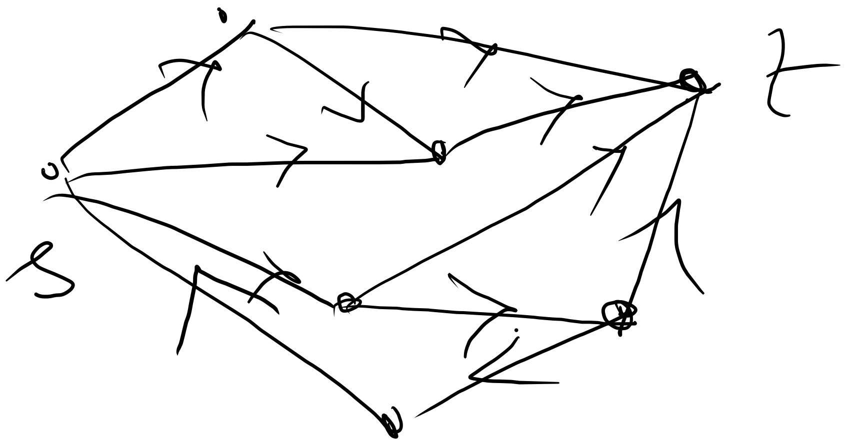
for each vertex $v \in V \setminus \{s,t\}$



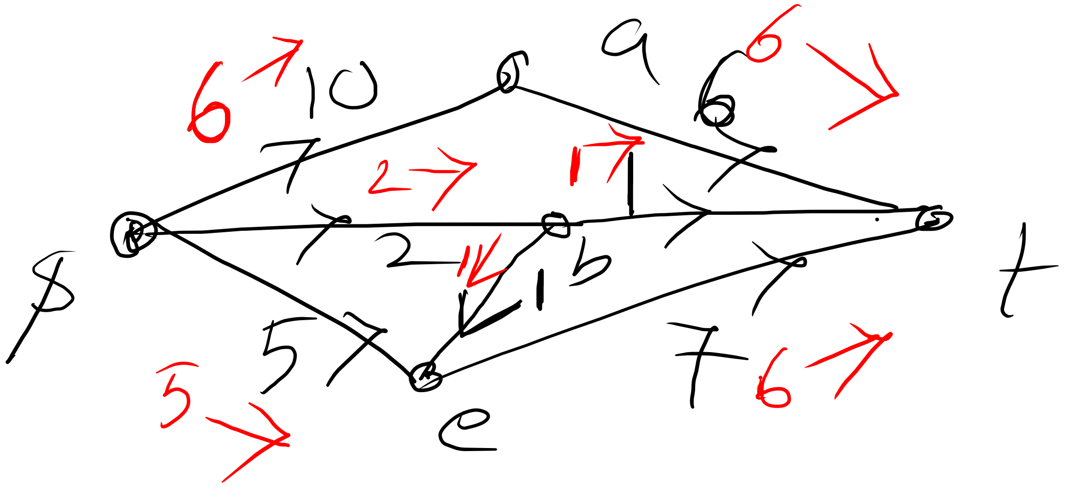
$$\sum_{\substack{e \text{ is incident} \\ \text{towards } v}} f(e) = \sum_{\substack{e \text{ is incident} \\ \text{outward } v}} f(e)$$

The value of the flow

$$val(f) = \sum_{e \text{ is going out of } s} f(s) - \sum_{e \text{ is going inside } s} f(s)$$

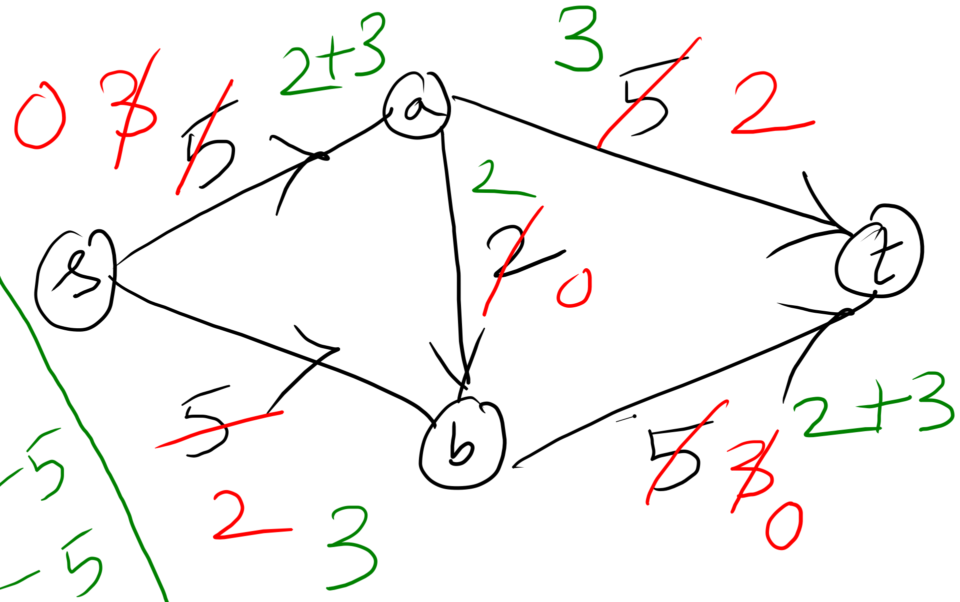


<u>Objective:</u> Find a flow of maximum value.



An algorithm

- for each edge flow is 0 ie $f(e) = 0$
- if there is a $s \rightsquigarrow t$ path where $f(e) < c(e)$
- augment flow along the path
- Repeat until no $s \rightsquigarrow t$ path exist.



$s \to a \to b \to t \to 2$

$s \to a \to t - 3$

$s \to b \to t - 3$

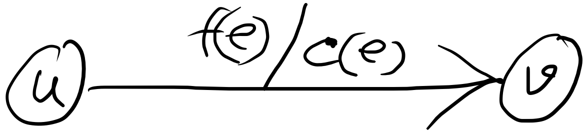
Total flow $= 8$

Optimum flow $= 10$

$s \to a \to t - 5$

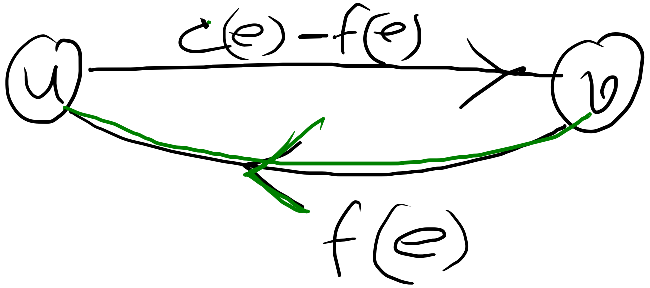
$s \to b \to t - 5$

Residual network



Residual capacity

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e^{reverse}) & \text{if } e^{reverse} \in E \end{cases}$$



Residual network: $G_f(V, E_f)$, s, t, c_f is residual capacity function.

$$E_f = \{ e : f(e) < c(e) \} \cup \{ e : f(e^{reverse}) > 0 \}$$

Ford-Fulkerson algorithm

Ford-Fulkerson(G)

for each edge $e \in E$

$f(e) = 0$

$G_f \leftarrow$ residual network of G w.r.t. f

while there is an $s \rightsquigarrow t$ path P in G_f

$f \leftarrow$ augment(G, P, c)

update G_f

return f.

augment (G, P, c)

$\delta \leftarrow$ bottleneck capacity in P

For each edge e in P

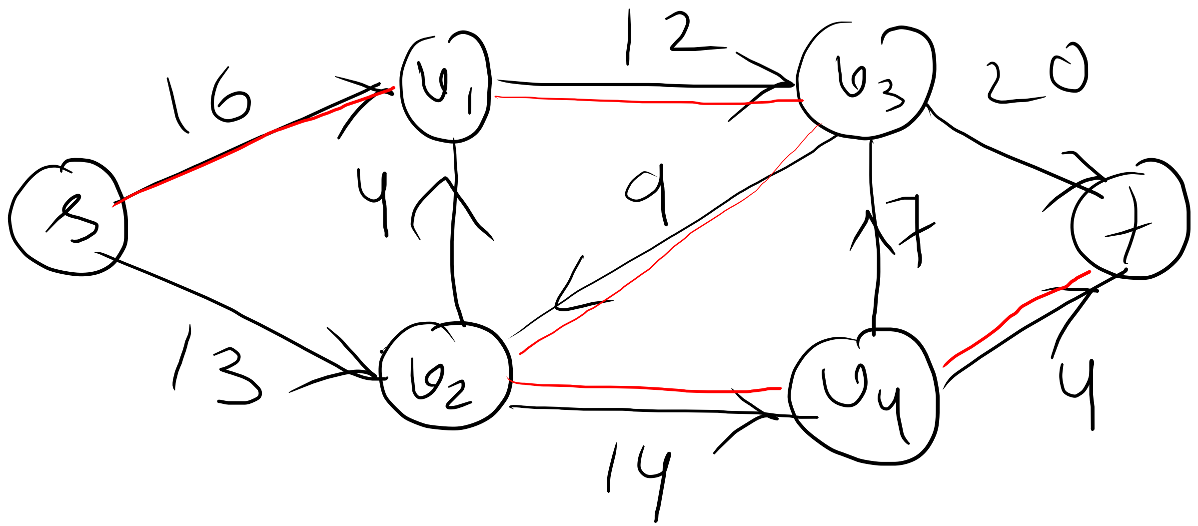
if $e \in E$

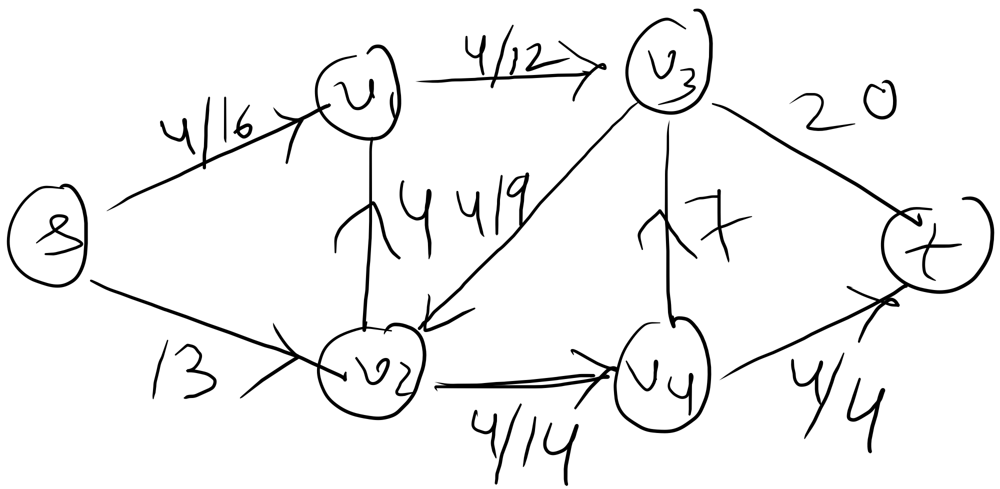
$f(e) \leftarrow f(e) + \delta$

else

$f(e^{reverse}) = f(e^{reverse}) - \delta$

Return f.





$s \to v_1 \to v_3 \to v_2 \to v_4 \to t$