#### SC223 - Linear Algebra

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Lecture 28



October 10, 2023

#### Summary of Lecture 27

- ullet For any two vector spaces U and V over the same field  $\mathbb{F}$ , and any linear transformation T from U to V, following subspaces exist:
- ▶ Nullspace of T (a.k.a. kernel of T):  $N(T) = \{x \in U \mid Tx = \theta_V\}$
- ▶ Range of T:  $R(T) = \{Tx \mid \forall x \in U\}$

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- ▶ Range of T:  $R(T) = \{Tx \mid \forall x \in U\}$
- Let U and V be n and m dimensional vector space over field  $\mathbb{F}$ , resp. If T is LT, then T can be **represented** as a matrix from  $\mathbb{F}^{m \times n}$ .
- If  $\beta_U$  and  $\beta_V$  are the chosen basis for U and V, we denote the matrix representation of T by  $[T]_{\beta_U}^{\beta_V}$ , and if y = Tx, then  $[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V}[x]_{\beta_U}$ .

3. Additive Identity Let OBIU, v) be defined as: OBIU, v) (u) = Ov, HuEU. Ozius E £ 10,V). Y T ∈ £ (0,V), T+ Ozius = Ozius + T = T. 

4. Additive Inverse Let TEB(0,0) Define (-T)(u) = -T(u), fue U.  $(7+(-7)) = O_{\mathcal{L}(0,V)}$ (T+(-T))(u) = T(u) + (-T)(u) = T(u) + - T(u)= Ov, tueU. · (T+(-T))(u) = Op(v,v)(u), +ueu. J Associative (+) 6- Commutative (+) 7. Scalar Mult Folentity: HTER(UN), (1.T) (U) = 1. T(U), HUEU 00 1.7 = T, HTEB(U,V). 8. Distributivity 9. Compatibility of SM with FM.  $(a \times b) \cdot T = a \cdot (b \cdot T) + T \in \mathcal{B}(v, V)$   $f_{a,b} \in F$ (L(U,V), , o) is a VS over F LL(L(U,V), L(U,V)) L(L(U,U), L(U,V))

END OF CLASS

- Let  $T \in \mathcal{L}(U, V)$ .
- ullet We have seen how to compute  $[T]_{\beta_U}^{\beta_V}$ , the matrix representation of T w.r.t the basis  $\beta_U$  and  $\beta_V$ .
- What happens if we choose a different basis, say  $\alpha_U$  and  $\alpha_V$ . Are  $[T]_{\beta_U}^{\beta_V}$  and  $[T]_{\alpha_U}^{\alpha_V}$  different?
- How are they related?

- Let  $T \in \mathcal{L}(U, V)$ .
- Let  $\beta_U = \{u_1, \dots, u_n\}$  and  $\beta_V = \{v_1, \dots, v_m\}$  be basis of U and V, and let  $[T]_{\beta_U}^{\beta_V}$  denote the matrix representation of T w.r.t  $\beta_U$  and  $\beta_V$ .

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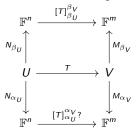
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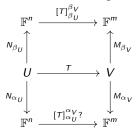
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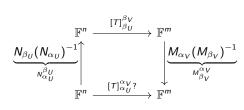
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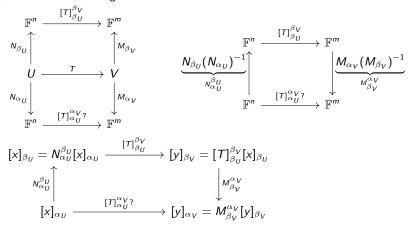
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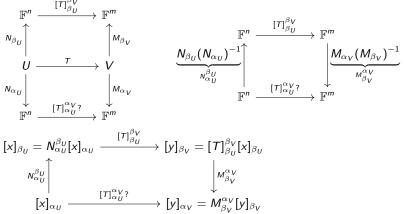
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- Given  $[T]_{\beta_U}^{\beta_V}$ , how to compute  $[T]_{\alpha_U}^{\alpha_V}$ ?





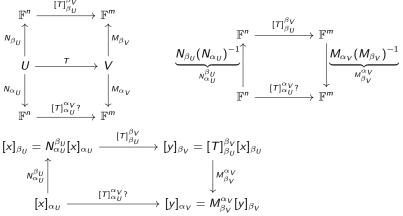






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- For a linear operator  $T: U \to U$ , assume  $\beta_U = \beta_V = \beta$  and  $\alpha_U = \alpha_V = \alpha$ .
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- **Similar matrices and Similarity transformation:** We say two matrices A and B are similar if there exists an invertible matrix, say S such that  $B = SAS^{-1}$ . The transformation  $A \mapsto SAS^{-1}$  is said to be a similarity transformation of A by S.