Chapter 7 | Lecture 40

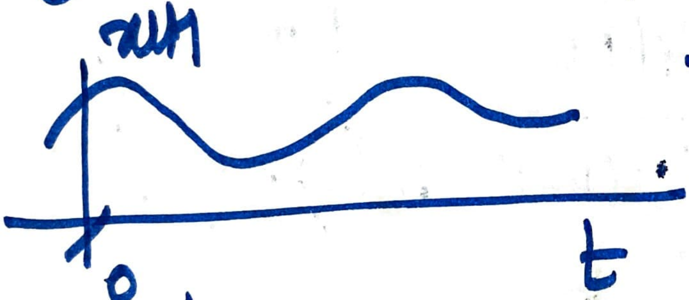
Sampling and Reconstruction

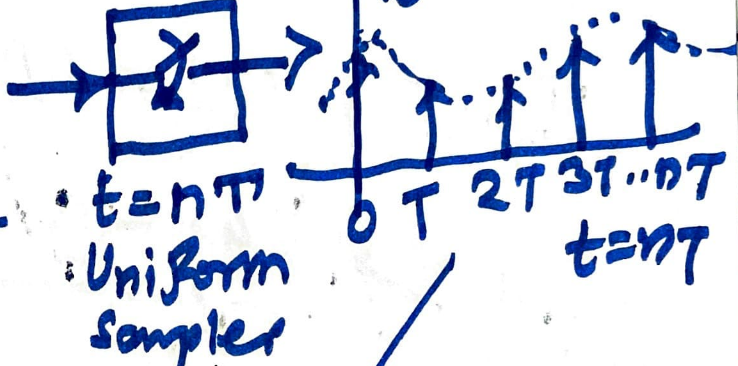
Digital Communication :→ B.Tech. core course in Sem V.

Motivation for Sampling

[Analog signal v.s. Digital signal]

① $x(t)$ Continuous-time signal | $x(n)$, $x(nT)$ Discrete-time signal





T = Sampling interval

$\{x(t)\}_{t \in \mathbb{R}}$ $\{x(nT)\}_{n \in \mathbb{Z}}$

speech, image, video

→ Ability to achieve narrowband (−3dB) filters in digital domain than analog domain

In chapter 3,

$\rightarrow$ $-3dB$ bandwidth

$\downarrow B(Hz) = \frac{2}{RC\uparrow}$

$RC\uparrow \Rightarrow$ higher cost

$\Rightarrow$ component tolerance

$\Rightarrow$ aging and losses (heat)

$Q\uparrow \propto \frac{1}{B\downarrow}$

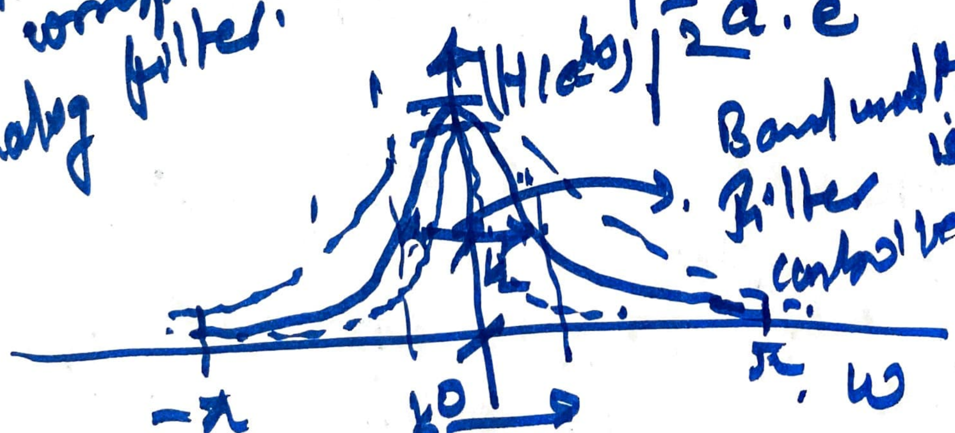
Quality factor

Digital Filter (Difference Equation)

$$y(n) - a\, y(n-1) = b\, x(n)$$

Digital representation of corresponding analog filter.

$$H(e^{jw}) = \frac{b}{1 - a \cdot e^{-jw}}$$



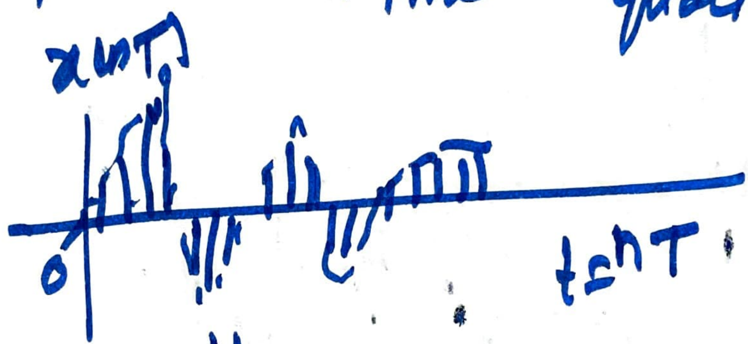
Bandwidth of digital filter is being controlled by |a|

4] Storage of data in analog vs. digital domain

Speech and audio recordings

Magnetic tape → Over a time → Deterioration in tape quality

Sampling →



Digital data is stored in 1's or 0's

$$\left\{ \underbrace{1, 0, 1, 1, 0, 1, \dots}_{8} \right\}$$

Digital memory (registers)

[3] Ability to delay the signal in discrete time domain

$y(n) \rightarrow y(n) \rightarrow$

$y(n) = b \cdot x(n)$

(No delay element)

$Y(e^{j\omega}) = b \cdot X(e^{j\omega})$

Convolution theorem

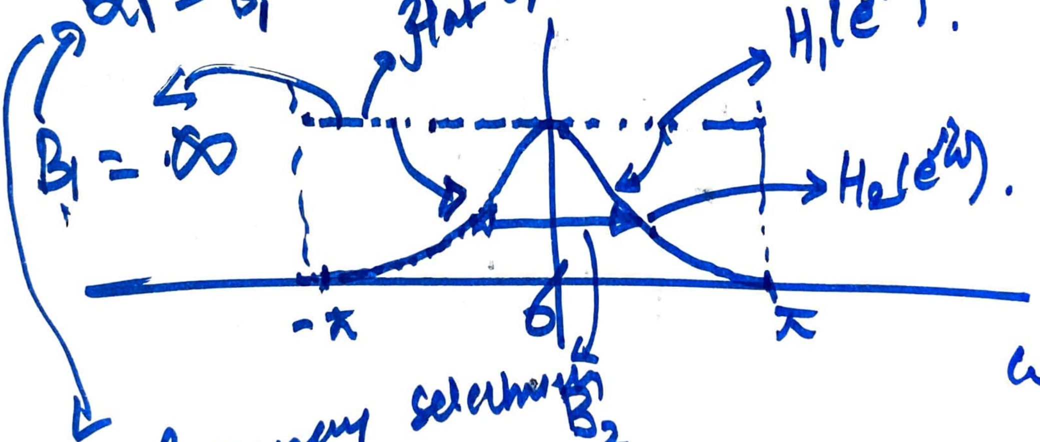
$H_1(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = b$

$Q_1 = \frac{0}{B_1} = 0$ (Bad quality)

Flat spectrum

$y(n) - \underbrace{a\, y(n-1)}_{\text{Delay } y(n) \text{ by 1 sample}} = b\, x(n)$

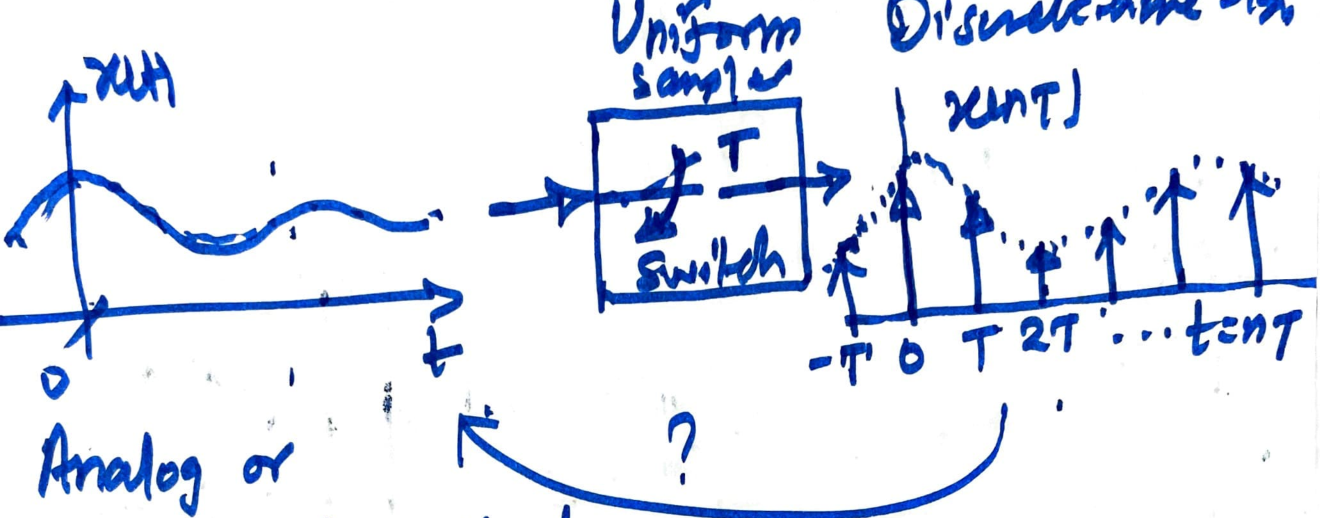
$H_2(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b}{1 - a e^{-j\omega}}$



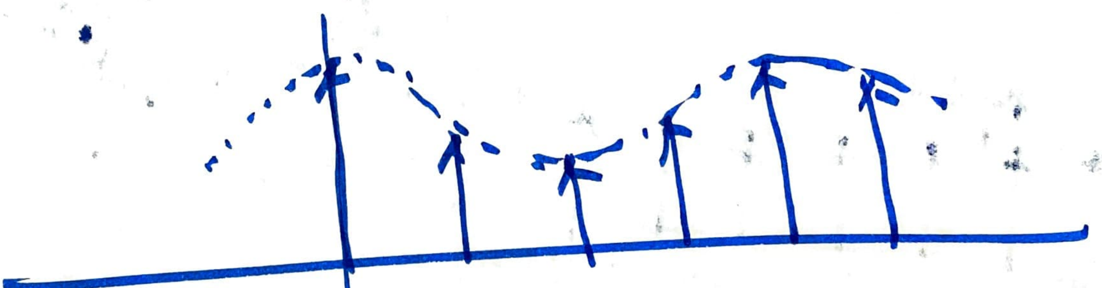
No frequency selectivity

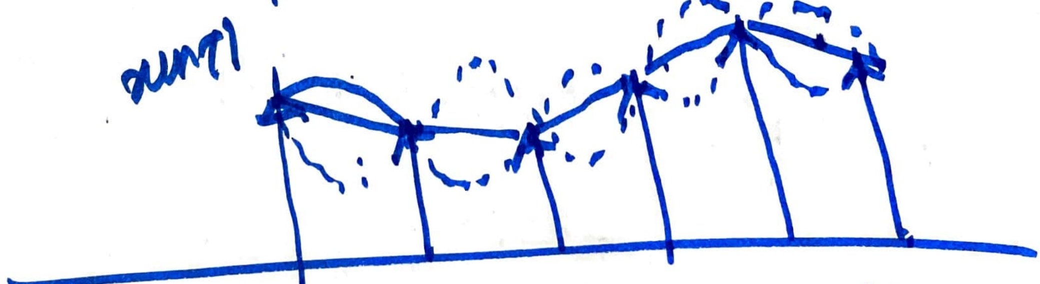
→ against design of frequency selective filters and amplifiers

Key Research Issue in Sampling: →



Analog or
Continuous-time signal





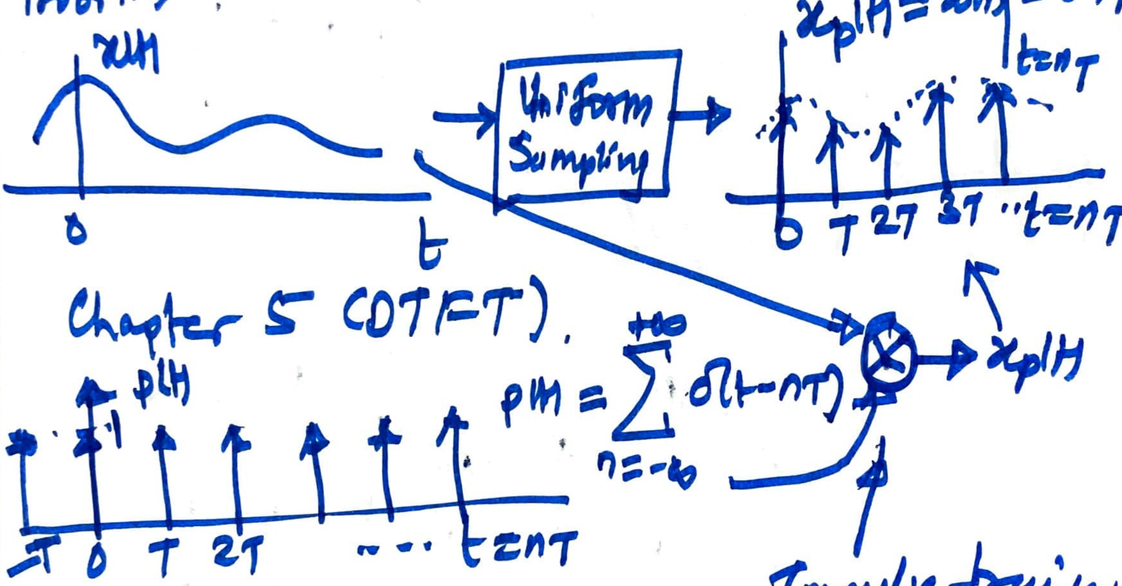
Issue: → What is the guarantee that we can reconstruct original analog signal $x(t)$ from its discrete-time samples $\{x(nT)\}_{n \in \mathbb{Z}}$ "uniquely".

$\implies$ Unique reconstruction of $x(t)$ from $x(nT)$ is the issue? 5

* Key Result in Digital Communication and Information Theory :→

[Shannon's Sampling Theorem]

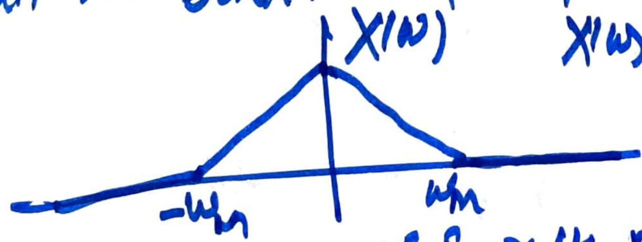
Proof :→



$\therefore \quad x_p(t) = x(t) \cdot p(t)$ —(1) Impulse train sampling.

Let $x(t)$ be ~~x(t)~~ bandlimited signal

$X(\omega) = 0, \ |\omega| \geq \omega_M$



$$F\{x_p(t)\} = F\{x(t) \cdot p(t)\}$$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * (P(\omega))$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \xrightarrow{F(\cdot)} P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

$$= \left(\frac{1}{2\pi}\right) X(\omega) * \left[\frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T} \cdot n\right)\right]$$

$$= \frac{1}{T} \left\{ X(\omega) * \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T} \cdot n\right) \right\}.$$

Distributive property of convolution.

$$x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

$$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(\omega) * \delta\left(\omega - \frac{2\pi}{T} \cdot n\right)$$

$$\boxed{X_p(\omega) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(\omega - \frac{2\pi}{T} n\right);}$$

T = sampling interval

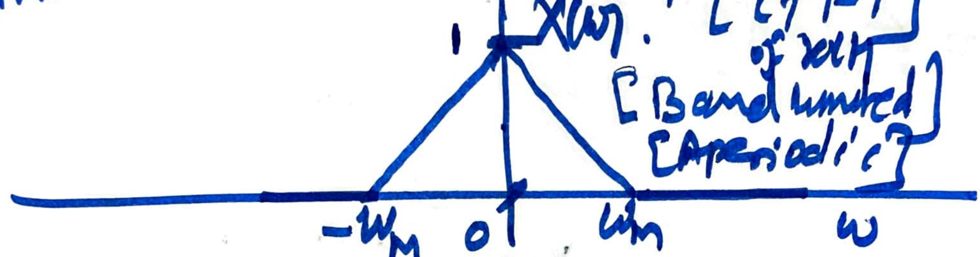
$\therefore \frac{2\pi}{T} = \omega_s$ sampling frequency

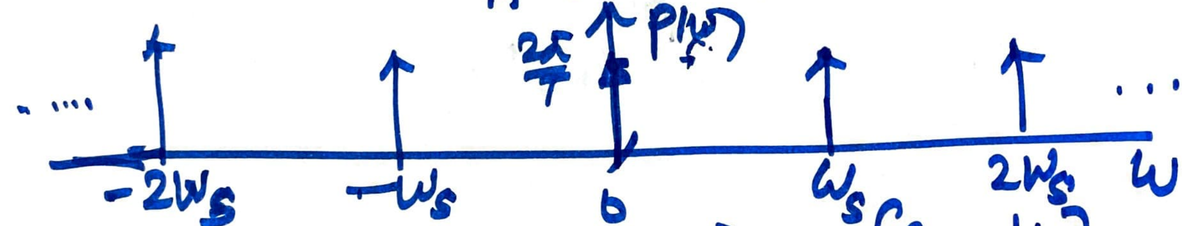
$$X_p(\omega) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(\omega - \omega_s \cdot n)$$

$$X_p(\omega) = \frac{1}{T} \left[\cdots + X(\omega + \omega_s) + X(\omega) + X(\omega - \omega_s) + \cdots \right]$$

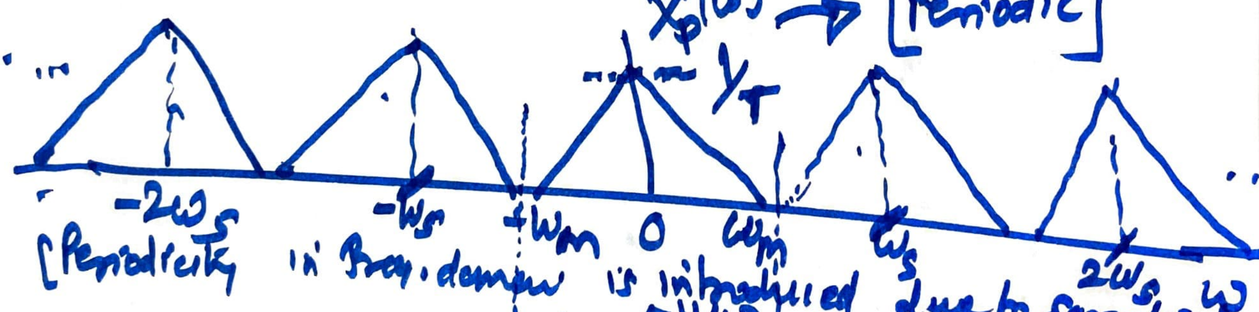
→ spectrum of discrete-time signal.

[CTFT] of x(t)
[Band limited]
[Aperiodic]





$X_p(\omega)$ → [Periodic]

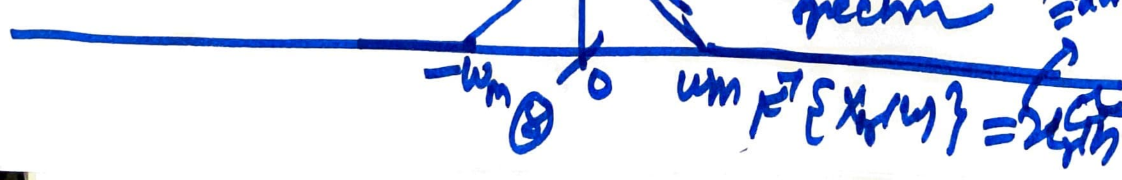


[Periodicity in freq. domain is introduced due to sampling]

Ideal Lowpass filter.

$X_r(\omega) = X_p(\omega) \cdot H(\omega)$

↳ Reconstructed spectrum



$F^{-1}\{X_r(\omega)\} = x(t)$

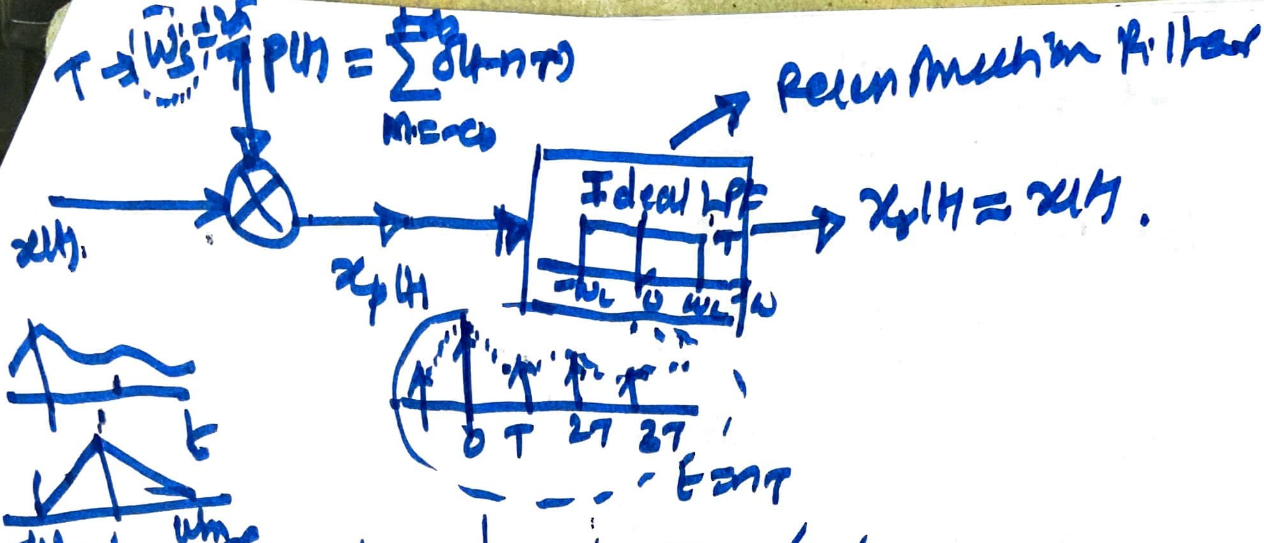
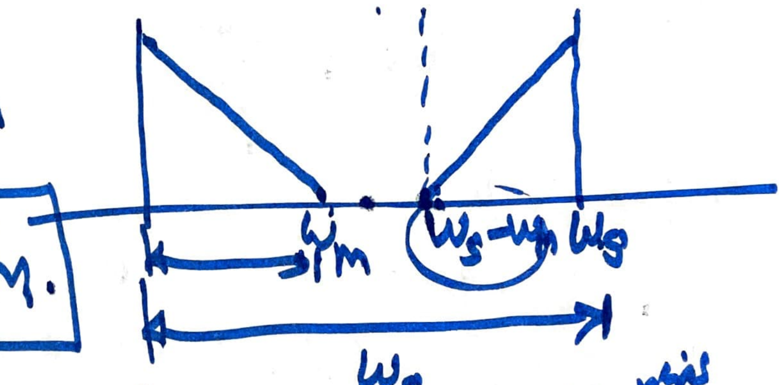


Fig. Impulse-train sampling

Relationship betn ω_m and ω_s ??

$\omega_s - \omega_m > \omega_m$

$$\boxed{\therefore \omega_s > 2\omega_M.}$$



aliasing in the frequency domain.

Shannon's sampling theorem

OR

Nyquist sampling theorem

if $\omega_s < 2\omega_m$

Sampling Theorem [Shannon's Sampling theorem]

Let $x(t)$ be a band-limited signal with $X(\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n \in \mathbb{Z}$, if

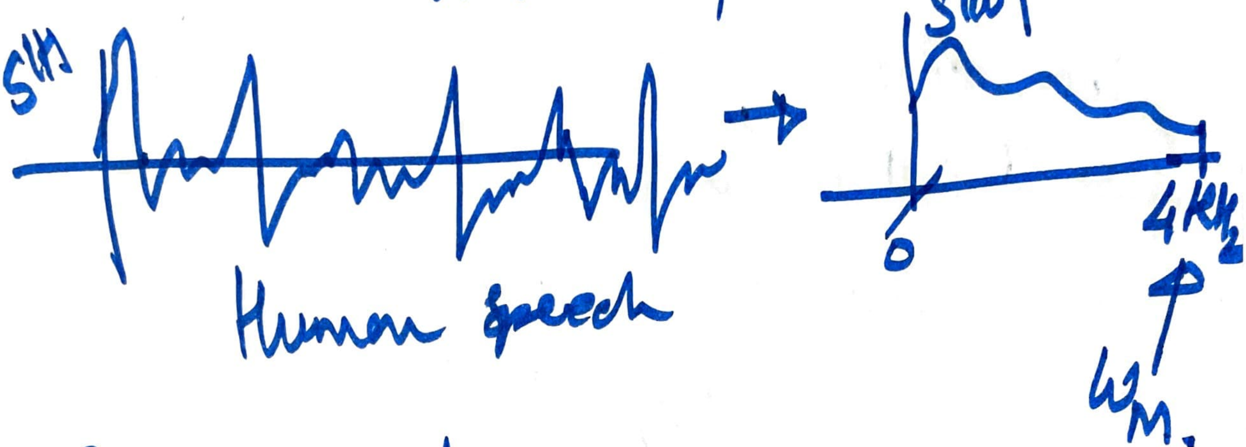
$$\omega_s > 2 \cdot \omega_M$$

where $\omega_s = \frac{2\pi}{T}$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output will exactly equal to $x(t)$

Application of Sampling

↳ Telephone Systems :→ Speech transmission over telephone (PCM)

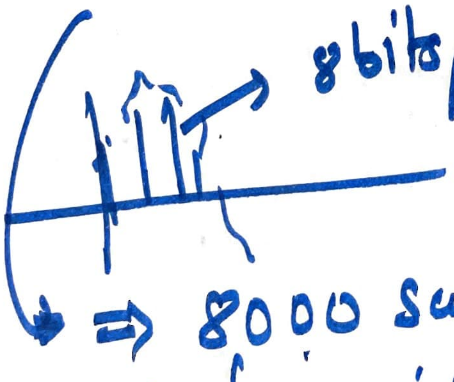


Human speech

Shannon's theorem

$$\boxed{\omega_s > 2\omega_m}$$

$\omega_s = \underline{8 KHz}$ → Sampling frequency used in telephone via PCM.



⇒ 8000 samples per second.

⇒ To transmit 1 second of speech we need to store 8000 speech samples

∴ 1 second of speech ⇒ (8000 samples/sec) × 8 bits

bit-rate of speech transmission (1) = 64 kbps

History of Sampling: →

① Prof. Claude E. Shannon

Communication in the presence of noise.

Proc. of IRE, 1949.

→ Digital Communication

→ Information Theory

→ Father of Digital Communication.

$\boxed{\omega_s > 2\omega_m}$ → Shannon's Sampling.

② H. Nyquist → Certain Topics in Telegraph Transmission Theory → 1928

⑧ Prof. Y. Eldar

③ J.M. Whittaker, "Interpolatory Function Theory" → 1936

④ D. Gabor 'Theory of Communication'

J. of IEE, 1946

⑤ EPFT → Prof. Michael Unser

→ Splines and Wavelets

⑦ Frame Theory

[Frame Research Institute NY 19??]

⑥ Compressive Sensing

Prof. Richard Baraniuk

⑫ Rice University