# SC223 - Linear Algebra

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Lecture 20

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# Direct sum of Subspaces

- **Definition:** (Direct Sum of Subspaces) In a VS $V$ with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, $w$ is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.
- Direct sum notation: $W = U_1 \oplus U_2 \oplus \ldots \oplus U_n$.
- **Proposition 8:** Let $U_1, \ldots, U_n$ be subspaces of $V$. Then $V = U_1 \oplus \ldots \oplus U_n$ if and only if: (1) $V = U_1 + \ldots + U_n$, and (2) The only decomposition of $\theta \in V$ is $(\theta, \ldots, \theta)$.
- **Proposition 9:** Let $V$ be a VS with subspaces $U_1, U_2$. Then $V = U_1 \oplus U_2$ *iff* $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\theta\}$.

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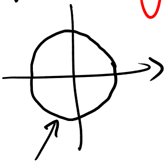
- If $|U| = \infty$? $\Rightarrow$ $span(U) :=$ Set of all possible LC of all finite subsets of $U$.

$$y = \sum_{i=1}^{\infty} x_i, \quad x_i \in \mathbb{R}.$$

$V = \mathbb{R}^2$



$U$

$$S_N = \sum_{i=1}^{N} x_i \qquad \lim_{N \to \infty} S_N = y.$$

$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$

$$|S_n - y| < \epsilon$$

2. $V = P(\mathbb{R})$

$U = \{1, x, x^2, x^3, x^4, \ldots\}$

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- If $|U| = \infty$?
- Is $span(U)$ a subspace of $V$?

Any $w_1 \in span(U)$

$$w_1 = \sum_{i=1}^{n} a_i u_i \ , \ u_i \in U.$$

Not necessarily the same

$$w_2 = \sum_{i=1}^{m} b_i u_i \ , \ u_i \in U$$

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- If $|U| = \infty$?
- Is $span(U)$ a subspace of $V$?
- **Proposition 10:** Let $U \subseteq V$. Then $span(U)$ is a subspace of $V$.

● Let $V$ be a VS, and let $W \subset V$. If $span(W) = V$, we say that $W$ is a **spanning set** of $V$, or $W$ **spans** $V$.

Ex: $V = \mathbb{R}^{n \times n}$

$$W = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & 0 \\ \vdots & & & & \\ 0 & \cdots & & & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \right\}$$

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$$a_1 v_1 + \ldots + a_n v_n = \theta \Rightarrow a_i = 0, i = 1, \ldots, n$$

$V = \{f : \mathbb{R} \to \mathbb{R}\}.$

$W = \{\mathbb{1}, \sin t, \cos t\}.$

$\mathbb{1}(t) = 1, \forall t \in \mathbb{R}.$

$a_1 \cdot \mathbb{1} + a_2 \cdot \sin t + a_3 \cos t = 0, \forall t \in \mathbb{R}.$

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● What if $|W| = \infty$.

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- Is it possible that $m > n$?
- If so, after $n$ iterations, we will reach a contradiction: $span(\{w_1, w_2, \ldots, w_n\}) = V$

# Basis of a Vector space

- **Definition:** (Hamel Basis) Let $V$ be a finite dimensional vector space. An ordered set $\beta := \{v_1, \ldots, v_n\}$ is said to be a **(Hamel) basis** of $V$ if (1) $span(\beta) = V$, and (2) $\beta$ is a set of linearly independent vectors.

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- Examples:

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- **Proposition 13:** Every FDVS has a basis.
- **Proposition 14:** Any set of basis vectors of a VS contains the same number of elements.
- **Dimension of a Vector Space:** Let $V$ be a FDVS. For any set of basis vectors $\beta$ of $V$, we define the dimension of $V$ as $dim(V) := |\beta|$.