LECTURE 13

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{m^2L^2}$$

$$\left(\frac{du}{d\theta}\right)^2 + u = (E - V)$$

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$\dot{r}^2 + r^2\dot{\theta}^2 = \frac{2E}{m} - V(r) \qquad r = 1/u \, .$$

$$V(r) = -\alpha/r$$

$$V_{eff}(r) = \frac{L^2}{2mr} - \frac{\alpha}{r}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{L^2}(E - V)$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = -u^2 + \frac{2mE}{L^2} + \frac{2m\alpha}{L^2}u$$

Complete the square

$$\left(\frac{du}{d\theta}\right)^2 = -\underbrace{\left(u - \frac{m\alpha}{L^2}\right)}_{z}^2 + B^2,$$

$$\left(\frac{dz}{d\theta}\right)^2 = -z^2 + B^2 \quad \Rightarrow \quad \int \frac{dz}{\sqrt{B^2 - z^2}} = \int d\theta = \theta - \theta_0$$

$$\frac{1}{r} = \frac{m\alpha}{L^2}\left(1 + \epsilon\cos\theta\right), \qquad \epsilon = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$$

$$\epsilon = 0 \Rightarrow \quad r = \frac{L^2}{m\alpha} = \text{const.} \longrightarrow \text{circular orbit}$$

$$E = -\frac{m\alpha^2}{2L}$$

$$V_{eff}(r) = \frac{L^2}{2mr^2} - \frac{\alpha}{r}$$

$$\frac{dV_{eff}(r)}{dr} = -\frac{L^2}{mr^3} + \frac{\alpha}{r^2} = 0$$

$$\Rightarrow \quad r = \frac{L^2}{m\alpha} = r_*$$

$$V_{eff}(r_*) = \frac{L^2}{2mr_*^2} - \frac{\alpha}{r_*} = \frac{m\alpha^2}{2L^2} - \frac{m\alpha^2}{L^2} = -\frac{m\alpha^2}{2L^2}$$

$\epsilon > 1$

$$\frac{1}{r} = \frac{m\alpha}{L^2}\left(1 + \epsilon\cos\theta\right)$$

$$= \frac{m\alpha}{L^2}\,\epsilon\left(\cos\theta + \frac{1}{\epsilon}\right)$$

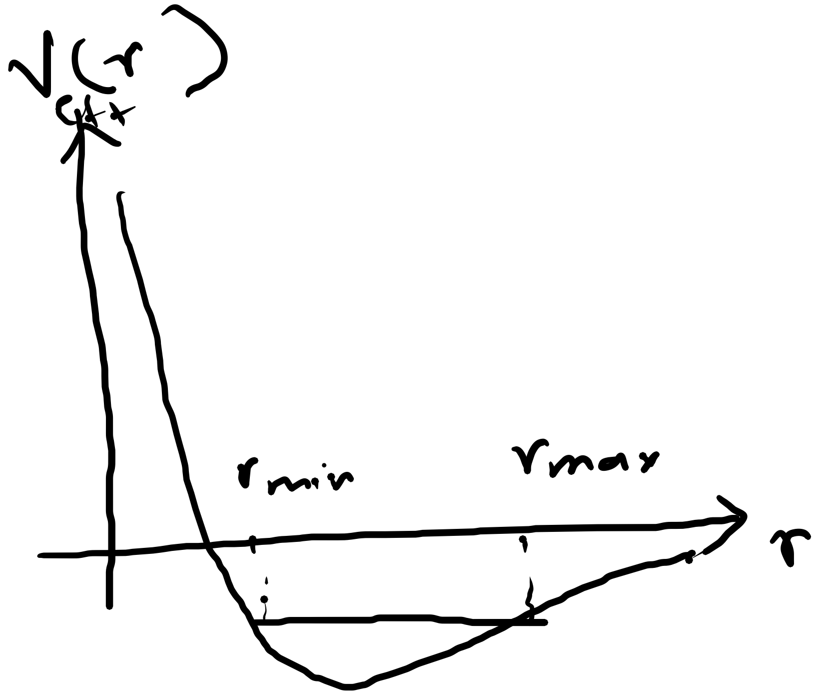
$$r_{max} = \infty .$$

$$r_{min} = \frac{L^2}{m\alpha(1+\epsilon)}$$

$0 < \epsilon < 1$

$$r_{max} = \frac{L^2}{m\alpha(1-\epsilon)}$$

$$r_{min} = \frac{L^2}{m\alpha(1+\epsilon)}$$



$0 < \epsilon < 1$:-

$$0 < 1 + \frac{2EL^2}{m\alpha^2} < 1$$

$$\Rightarrow \quad -1 < \frac{2EL^2}{m\alpha^2} < 0$$

$$\Rightarrow \quad -m\alpha^2/(2L^2) < E < 0$$

Geometry of path/orbit

$$\frac{1}{r} = \left(\frac{m\alpha}{L^2}\right)(1 + \epsilon \cos\theta)$$

$0 < \epsilon < 1$

$\epsilon = 1$

$\epsilon > 1$

To work out geometry of orbit, easier to use Cartesian coordinates. (x, y).

$$k \equiv \frac{L^2}{m\alpha}$$

$\cos\theta = x/r$

$$\frac{1}{r} = \frac{1}{k}(1 + \epsilon \cos\theta)$$

$$\Rightarrow \quad k = r(1 + \epsilon \cos\theta)$$

$$\Rightarrow \quad k = r(1 + \epsilon\, x/r) = r + \epsilon x$$

$$k = r + \epsilon x$$

Squaring,

$$r^2 = (k - \epsilon x)^2$$

$$\Rightarrow \quad r^2 = k^2 + \epsilon^2 x^2 - 2\epsilon k x$$

$$\Rightarrow \quad x^2 + y^2 = k^2 + \epsilon^2 x^2 - 2\epsilon k x$$

<u>Case :-</u> $0 < \epsilon < 1$

$$(1 - \epsilon^2) x^2 + 2\epsilon k x + y^2 = k^2$$

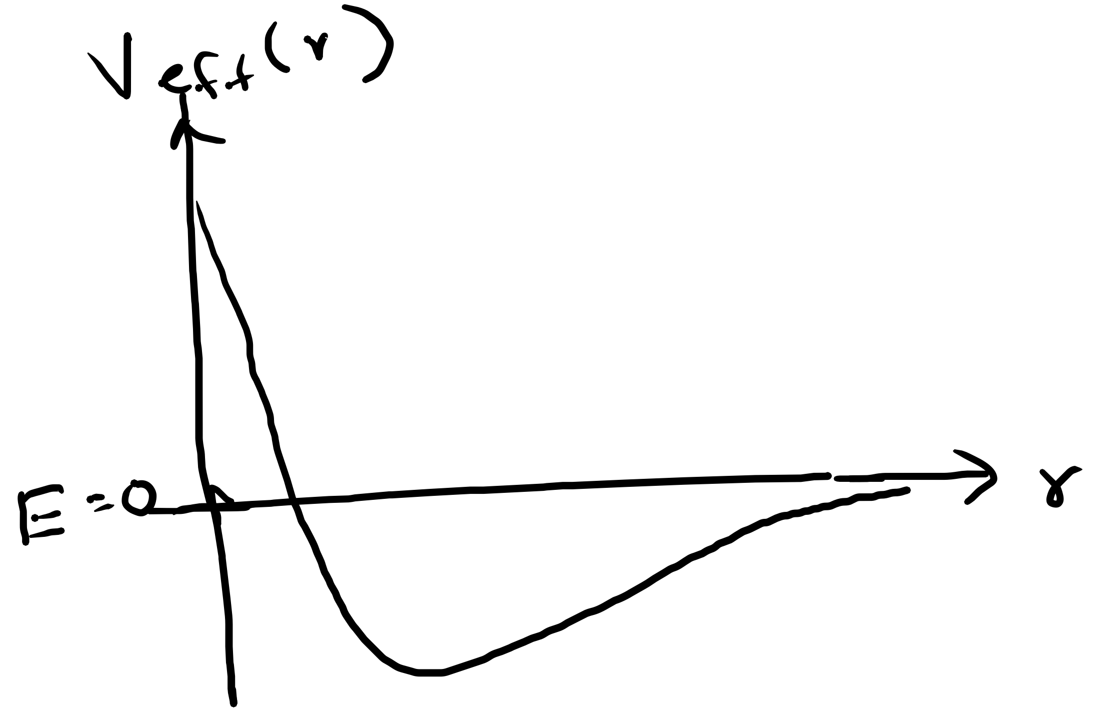
$$\Rightarrow \quad x^2 + 2x \frac{k\epsilon}{1-\epsilon^2} + \frac{y^2}{1-\epsilon^2} = \frac{k^2}{(1-\epsilon^2)}$$

$$\Rightarrow \quad x^2 + 2x \frac{k\epsilon}{1-\epsilon^2} + \left(\frac{k\epsilon}{1-\epsilon^2}\right)^2 - \left(\frac{k\epsilon}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{k^2}{1-\epsilon^2}$$

$\underline{\epsilon = 1}$:-

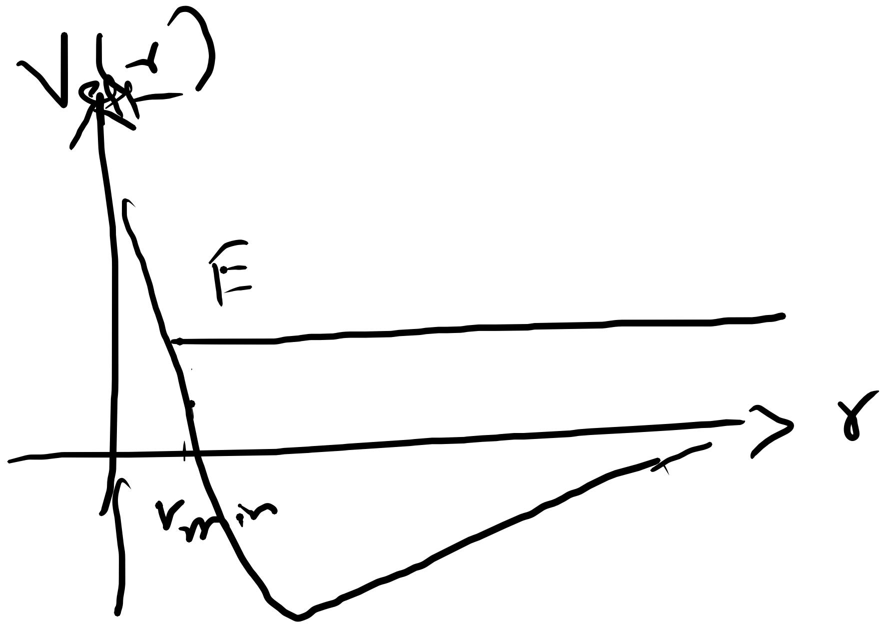
$$1 + \frac{2EL^2}{m\alpha^2} = 1$$

$$\Rightarrow \quad E = 0$$



and $V_{eff} \to 0$ for large r.

$$v\big|_{r\to\infty} = 0$$



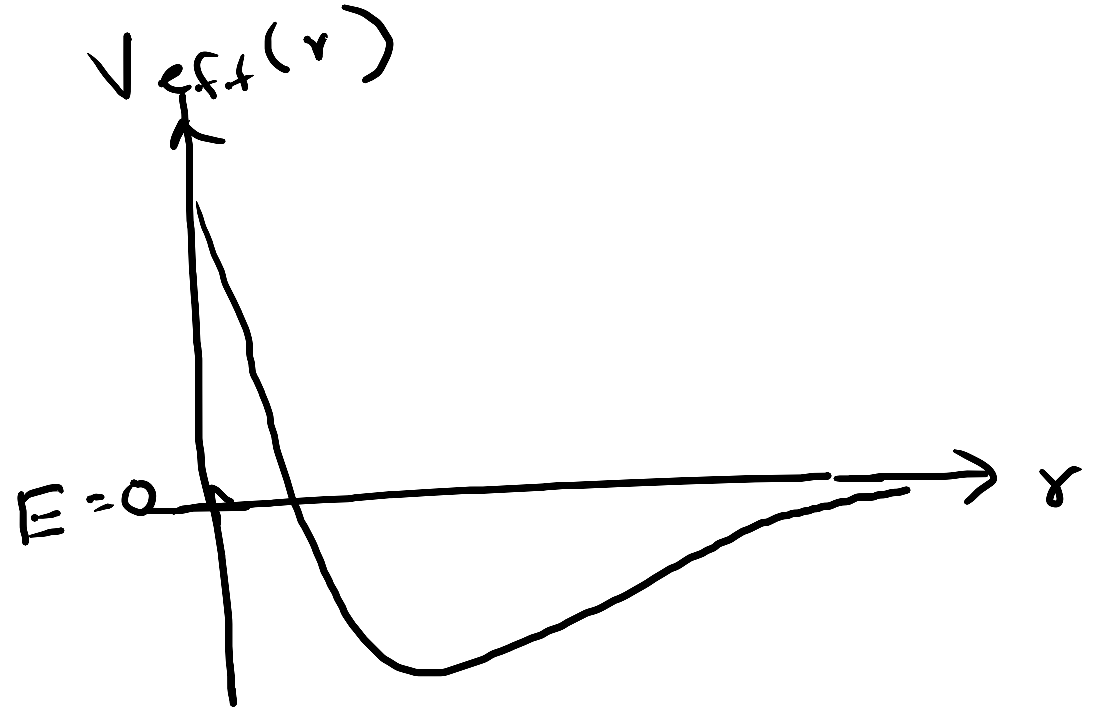
$\underline{\epsilon > 1}$:-

$$\frac{1}{2} m v_\infty^2 = E.$$

$$\Rightarrow \quad v_\infty = \sqrt{\frac{2E}{m}}$$

$\underline{\epsilon = 1}$:- $1 + \frac{2EL^2}{m\alpha^2} = 1$

$\Rightarrow \quad E = 0$



and $V_{eff} \to 0$ for large r.

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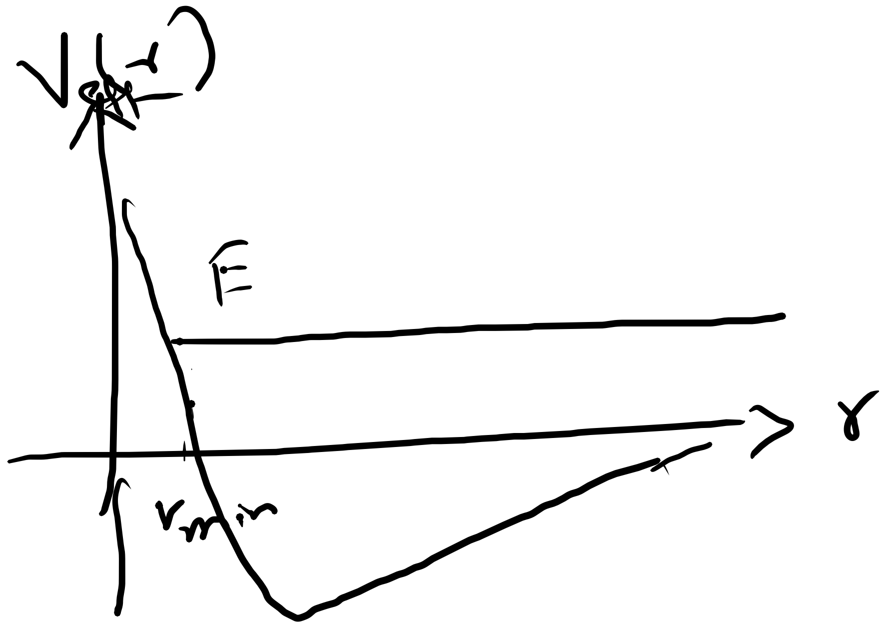
$\underline{\epsilon > 1}$:-



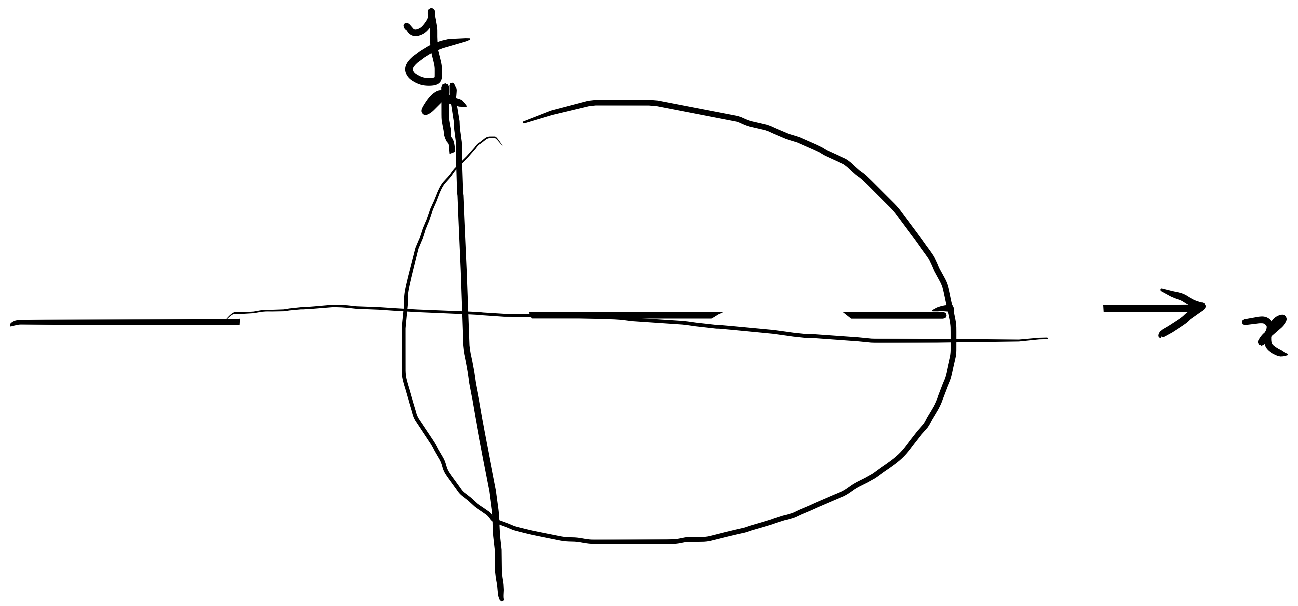
$\frac{1}{2} m v_\infty^2 = E$.

$\Rightarrow \quad v_\infty = \sqrt{\frac{2E}{m}}$

$$\Rightarrow \left(x + \frac{k\epsilon}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{k^2}{1-\epsilon^2}\left[1 + \frac{\epsilon^2}{1-\epsilon^2}\right] = \frac{k^2}{(1-\epsilon^2)^2}$$

$$\Rightarrow \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \frac{k}{1-\epsilon^2}$$



$$b = \frac{k}{\sqrt{1-\epsilon^2}}$$

$$a = \frac{b}{\sqrt{1-\epsilon^2}}$$

Case:- $\epsilon = 1$

$$\cancel{x^2} + y^2 = k^2 + \cancel{x^2} - 2kx$$

$\Rightarrow \quad y^2 = k^2 - 2kx \longrightarrow$ parabola.

Case:- $\epsilon > 1$

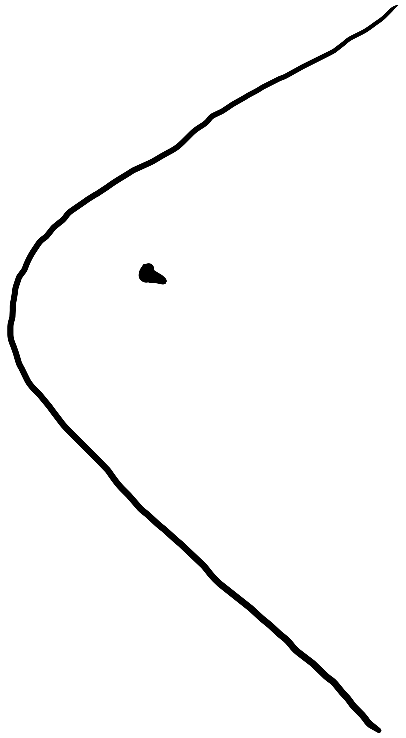
$\dfrac{\left(x - \frac{k\epsilon}{\epsilon^2 - 1}\right)^2}{a^2} - \dfrac{y^2}{b^2} = 1. \longrightarrow$ hyperbola.

$\boxed{V(r) = -\alpha/r}$ → Gravitational potential

$F = -\frac{dV}{dr} \propto \frac{1}{r^2}$

↕

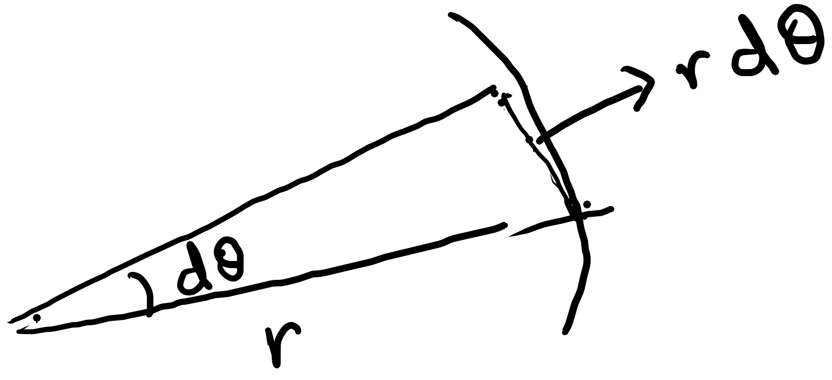
Kepler's law.



Kepler's law

(i) Planets move around sun in elliptic orbits.

(ii) Areal velocity of planets is const.



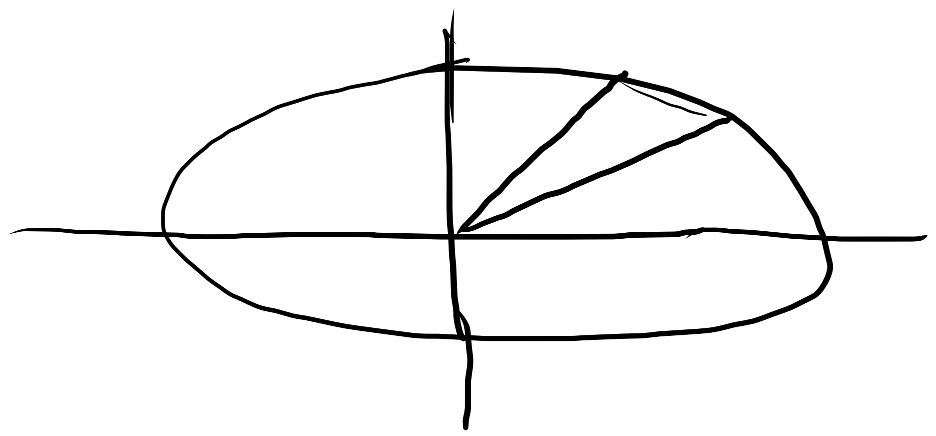
$$dA = \frac{1}{2} r (r\,d\theta)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m} = \text{constant} = \text{areal velocity}.$$

(iii) Time period T of planets around the sun,

$$\boxed{T^2 \propto a^3}$$

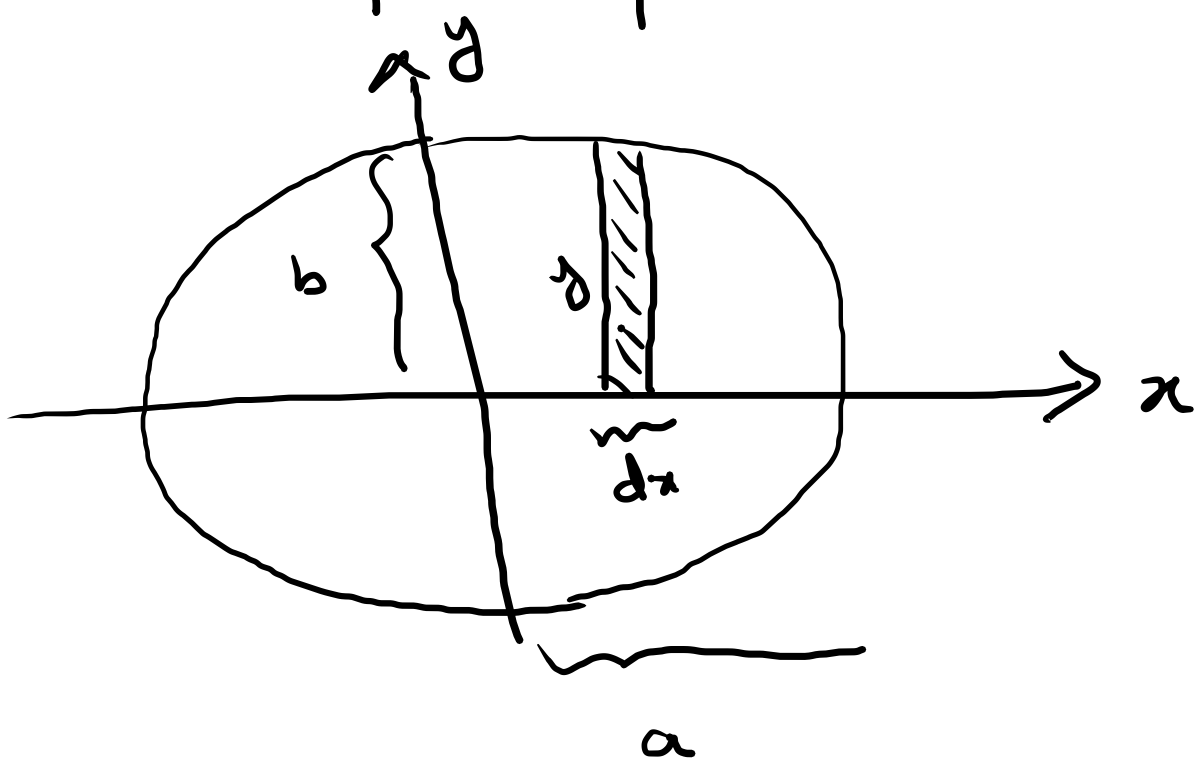
<u>Proof</u>:- Path/orbit is an area



$$T = \frac{A}{\text{areal velocity}}$$

$$= \frac{A}{\left(L/2m\right)}$$

— Area of ellipse :



Area of shaded region

$$= \int_0^a dx \; y(x)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{A}{4} = \int_0^a dx \; y(x) = ab \int_0^a dx \sqrt{a^2 - x^2}$$

Sub :- $x = a \sin \alpha$

$$\frac{A}{4} = ab \int_0^{\pi/2} d\alpha \cos^2 \alpha = \frac{ab\pi}{4} \Rightarrow A = \pi ab$$

$$\Rightarrow \quad \pi ab = \frac{LT}{2m} \qquad\qquad a = \frac{b}{\sqrt{1-\epsilon^2}}.$$

$$\Rightarrow \quad \pi^2 a^2 b^2 = \frac{L^2}{4m^2} T^2$$

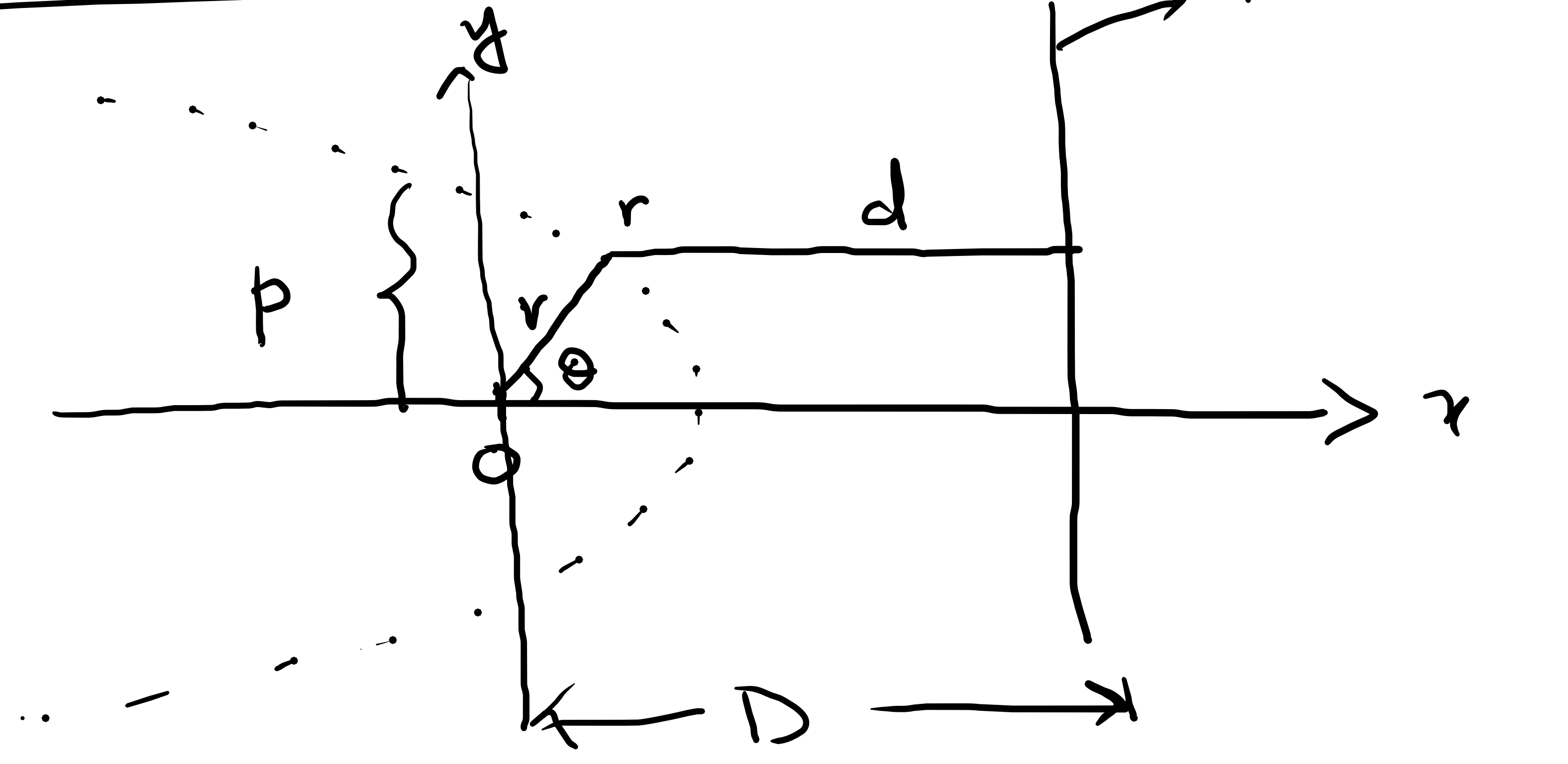
$$\Rightarrow \quad \pi^2 a^4 = \frac{m\alpha k}{m(1-\epsilon^2)} \frac{T^2}{4m}$$

$$\Rightarrow \quad \pi^2 a^4 \propto a T^2$$

$$\Rightarrow \quad T^2 \propto a^3$$

$$\Rightarrow \quad \boxed{T^2 \propto a^3}$$

Conic sections



O ≡ fixed pt.

$$\frac{r}{d} = \boxed{\epsilon = \frac{p}{D}} = \text{const}.$$

$$D = d + r\cos\theta$$

$$\Rightarrow \quad p = \epsilon(d + r\cos\theta) = \epsilon d + \epsilon r\cos\theta$$

$$= r + \epsilon r\cos\theta = r(1+\epsilon\cos\theta)$$

$$\Rightarrow \quad \boxed{r = \frac{p}{1+\epsilon\cos\theta}}$$ → general eqn. for conic section.