LECTURE 25

☒ Recap:

— Solving inhomogeneous linear differential eqns

— Motivation:- in this context, obtaining solutions for oscillators subject to generic driving forces.

Ex. $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t) \;.= F_0 \cos\omega t$.

$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = F_0 e^{i\omega t}$.

Trial sol$\underline{n}$: $z = (\quad) e^{i\omega t}$.

$F(t) = t\cos\omega t$

$\boxed{/\!/}$ $\underline{\dfrac{dx}{dt} + P(t)x(t)} = Q(t)$. $\longrightarrow$ inhomogeneous eqn.

$$M(t)\frac{dx}{dt} + M P x = QM.$$

$$\Rightarrow \frac{d}{dt}(Mx) = QM \qquad \text{(demand)}.$$

$$\frac{dM}{dt} = MP.$$

$$\Rightarrow M = \exp\left[\int dt\, P(t)\right]$$

$$x(t) = M^{-1}\int dt\, M Q.$$

- IF method is applicable for 1st order ODEs
- How to solve higher order ODEs?

Ex:- $y'' + y' - 2y = e^x$.

$D \equiv \frac{d}{dx}$.

$\Rightarrow (D-1)(D+2)\,y = e^x$. $\longrightarrow$ $D^2 y + 2Dy - Dy - 2y = e^x$.

$\Rightarrow \frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = e^x$.

Let $u = (D+2)\,y$.

$P = -1$

$Q = e^x$

$M = e^{+\int dx\, P(x)} = e^{-x} + C_1$

$(D-1)\,u = e^x$.

$\Rightarrow u' - u = e^x$.

Now, reduced to 1st order ODE, so, IF method can be used now.

$$u = M^{-1} \int dx\, M Q .$$

$$\Rightarrow \quad u = x e^{x} + c_1 e^{x}$$

$$(D+2) y = x e^{x} + c_1 e^{x} .$$

$$\Rightarrow \quad y' + 2y = x e^{x} + c_1 e^{x} .$$

$$y = e^{-2x} \int dx\, e^{3x} (x + c_1) + c_2 .$$

Exercise :- Arrive at the same result, in the form of complementary functions and particular integrals.

- Motivation for using D-notation.

$$y'' + y - 2y = 0.$$
$$\Rightarrow (D+2)(D-1)y = 0.$$

$$y = e^{rx}.$$
$$r^2 + r - 2 = 0.$$
$$\Rightarrow (r+2)(r-1) = 0.$$

Ex:- $(D-1)(D+5)y = 7e^{2x}$.

Guess a trial soln: $y_p = Ce^{2x}$.

$$y_p'' + 4y_p' - 5y_p = C(4e^{2x} + 8e^{2x} - 5e^{2x}) = 7\underline{e}^{2x}.$$
$$\Rightarrow 7Ce^{2x} = e^{2x}$$
$$\Rightarrow C = 1.$$ (method of undetermined coefficients).

PROB: $\ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 = 0$

$$2\ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 = 0 .$$

Method 1:- Let $\left.\begin{aligned} x_1 &= A_1 e^{i\alpha t} \\ x_2 &= A_2 e^{i\alpha t} . \end{aligned}\right\}$ Trial sol^n.

Sub: solve for α.

Method 2:- $M\ddot{x} = -Kx$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Eigenvalues of $M^{-1}K$:

$$\underline{\underline{M^{-1}K}} = \begin{pmatrix} 2 & -1 \\ -1/2 & 1 \end{pmatrix}.$$

$$\lambda = \tfrac{1}{2}\left(3 \pm \sqrt{3}\right).$$

$$m\ddot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + \omega^2 x = 0.$$

Check::

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha_1 t + \delta_1) \longrightarrow \text{1st normal mode}$$

$$\begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha_2 t + \delta_2). \longrightarrow \text{2nd normal mode}$$

General solⁿ is:

$$x(t) = B_1 \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha_1 t + \delta_1) + B_2 \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha_2 t + \delta_2) .$$

$$\boxed{\begin{aligned} \ddot{\xi}_1 + \omega_1^2 \xi_1 &= 0 \\ \ddot{\xi}_2 + \omega_2^2 \xi_2 &= 0 . \end{aligned}}$$

- How to guess normal coord.s from above
- Recall, normal coords. exhibit "pure" freq.
- So, procedure should be to find linear combinations which show either no α_1-dependence or no α_2-dependence.

$$B_1(\sqrt{3}+1) + B_2(\sqrt{3}-1) = x_1$$
$$-B_1 + B_2 = x_2$$

solving,

$$x_1 + (\sqrt{3}+1)x_2$$
$$x_1 - (\sqrt{3}-1)x_2$$

— $\dot{M}^{-1}K = \begin{pmatrix} 2 & -1 \\ -1/2 & 1 \end{pmatrix}$.

$$O = \begin{pmatrix} -(1+\sqrt{3}) & -(1-\sqrt{3}) \\ 1 & 1 \end{pmatrix}$$

$\downarrow$ eig 1 $\qquad$ $\downarrow$ eig 2.