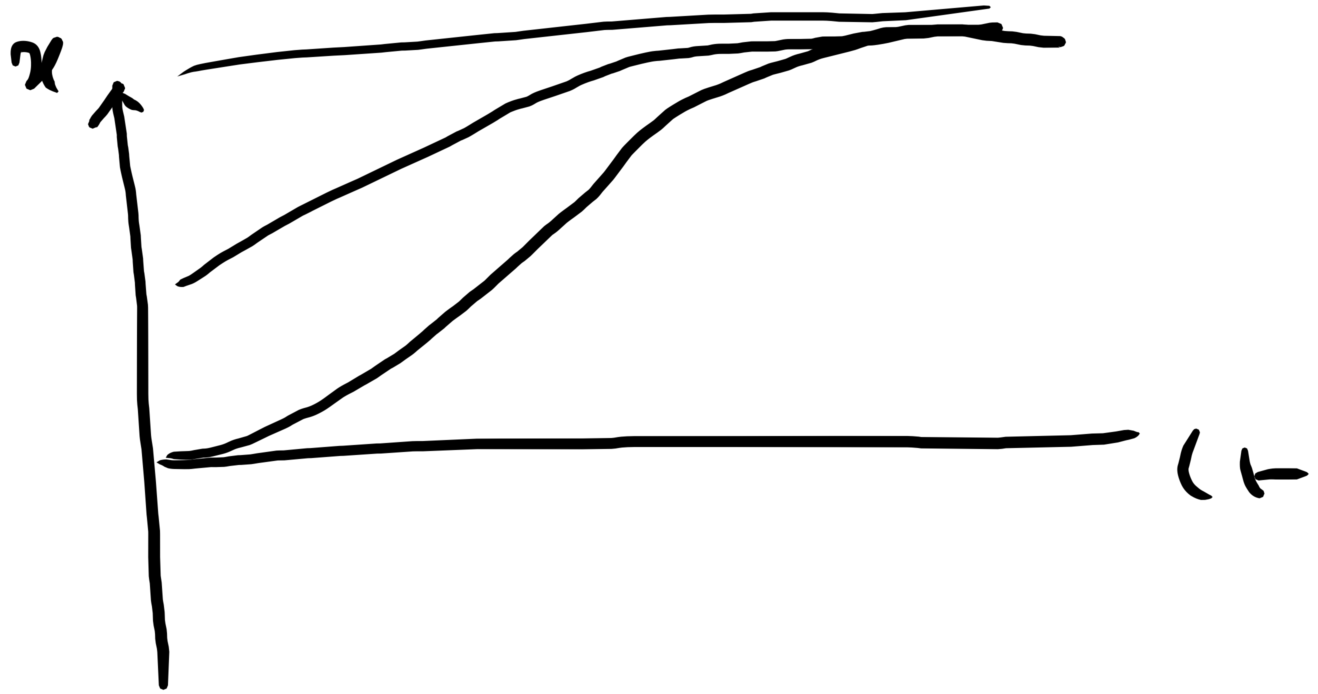
LECTURE 29

$\dot{x} = \sin x$.



- Not possible for first order autonomous systems to show oscillatory behaviour.

$\dot{x} = a - bx^2$. $\longrightarrow$ Fixed points $\dot{x} = 0 \Rightarrow x^* = \pm\sqrt{\frac{a}{b}}$.

$$\dot{x} = a - bx^2 , \quad a, b > 0 .$$

$$= a\left(1 - \frac{x^2}{a/b}\right) .$$

$$\Rightarrow \frac{dx}{dt} = a\left(1 - \frac{x^2}{a/b}\right) .$$

$$\Rightarrow \frac{dx}{d(at)} = 1 - \frac{x^2}{a/b} = (1 - X^2) .$$

$$\Rightarrow \sqrt{\frac{a}{b}} \frac{dX}{d(at)} = 1 - X^2$$

$$\Rightarrow \frac{dX}{d(\sqrt{ab}\,t)} = 1 - X^2$$

$$X^2 = \frac{x^2}{(a/b)}$$

$$\Rightarrow X = \frac{x}{\sqrt{a/b}} .$$

$$\Rightarrow dX = \frac{dx}{\sqrt{a/b}}$$

$$\Rightarrow dx = \sqrt{a/b}\, dX .$$

Exercise

Solve this example explicitly

$$T = \sqrt{ab}\, t .$$

Characteristic timescale $\Rightarrow T \simeq 1$

$$\Rightarrow t \simeq \frac{1}{\sqrt{ab}}$$

LINEAR STABILITY ANALYSIS.

$\dot{x} = f(x)$.

Let x^* be a fixed point.

Let $\eta(t) = x(t) - x^*$ be some small deviation from fixed point x^*.

$$\begin{aligned}
\dot{\eta}(t) &= f(\eta + x^*). \\
&= f(x^*) + \eta f'(x^*) + O(\eta^2). \\
&= \eta f'(x^*) + O(\eta^2).
\end{aligned}$$

$\dot{\eta} \simeq f'(x_*)\,\eta$. (Linearized).

If $f'(x^*) > 0$, $\eta(t)$ grows exponentially
If $f'(x^*) < 0$, $\eta(t)$ decays " .

$f'(x^*) < 0 \rightarrow x^*$ is a stable fixed point

$f'(x^*) > 0$, $\rightarrow x^*$ is unstable " " .

Ex:- $\dot{x} = f(x) = ax - bx^2$. (logistic eqn)

$$f(x^*) = 0$$

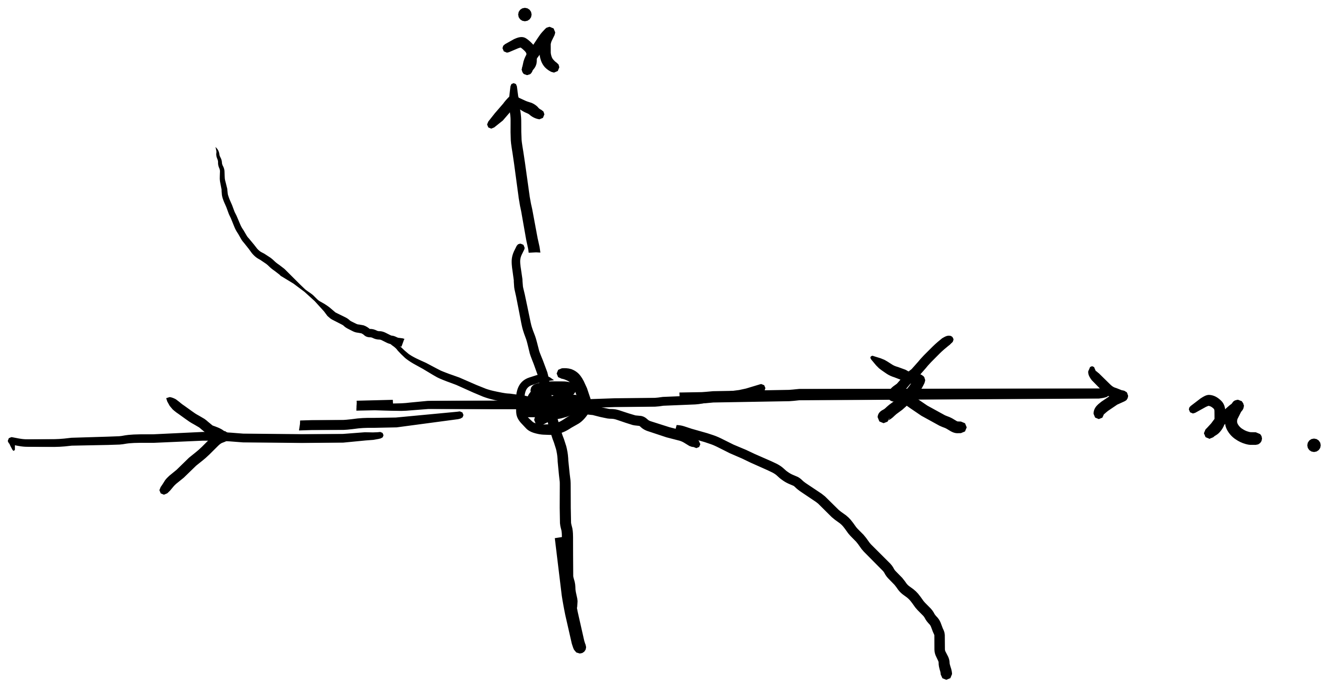
$$\Rightarrow \quad x^*(a - bx^*) = 0 \quad \Rightarrow \quad x^* = 0, (a/b).$$

$$f'(x) = a - 2bx$$

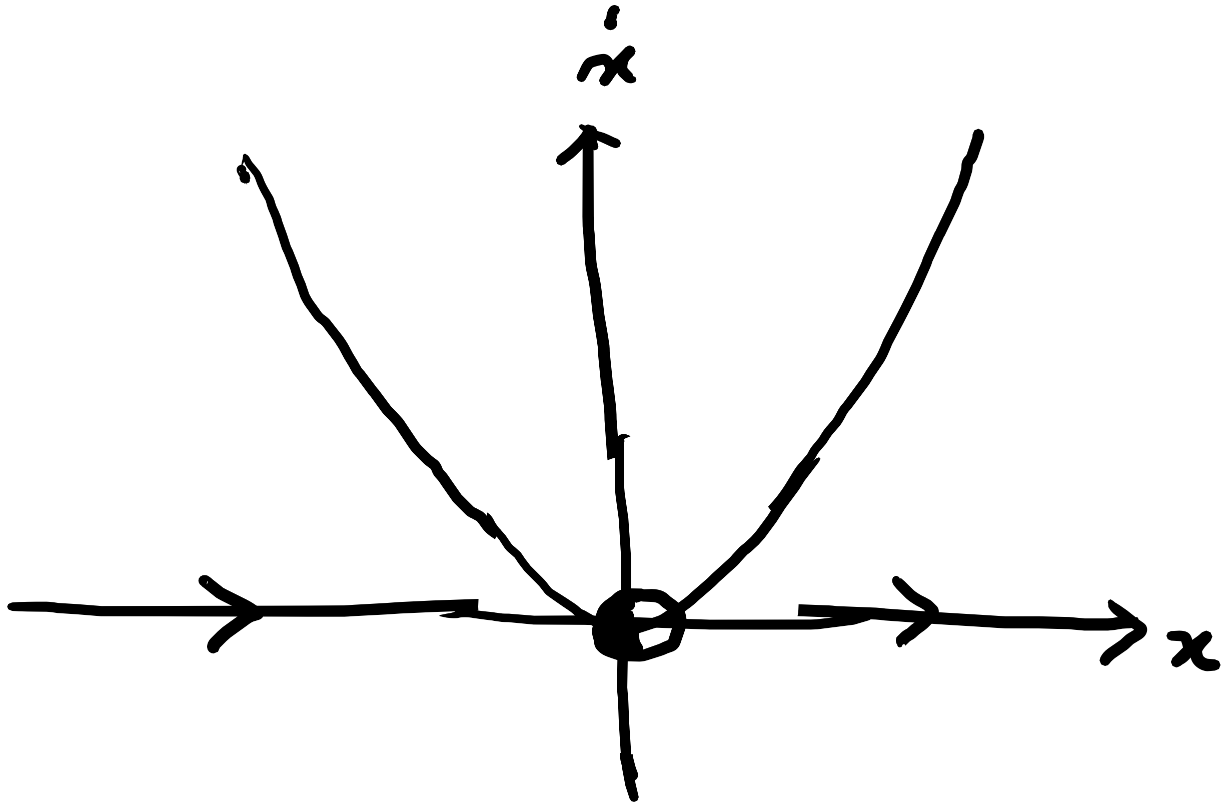
$$\left.\begin{aligned} f'(x^* = 0) &= a > 0. \\ f'(x^* = a/b) &= a - 2b \cdot \frac{a}{b} \\ &= -a < 0. \end{aligned}\right\} \Rightarrow \begin{array}{l} x^* = 0 \text{ is unstable} \\ x^* = (a/b) \text{ is stable.} \end{array}$$

Example :- (i) $\dot{x} = -x^3$, (ii), $\dot{x} = x^3$, (iii) $\dot{x} = x^2$.

(i)



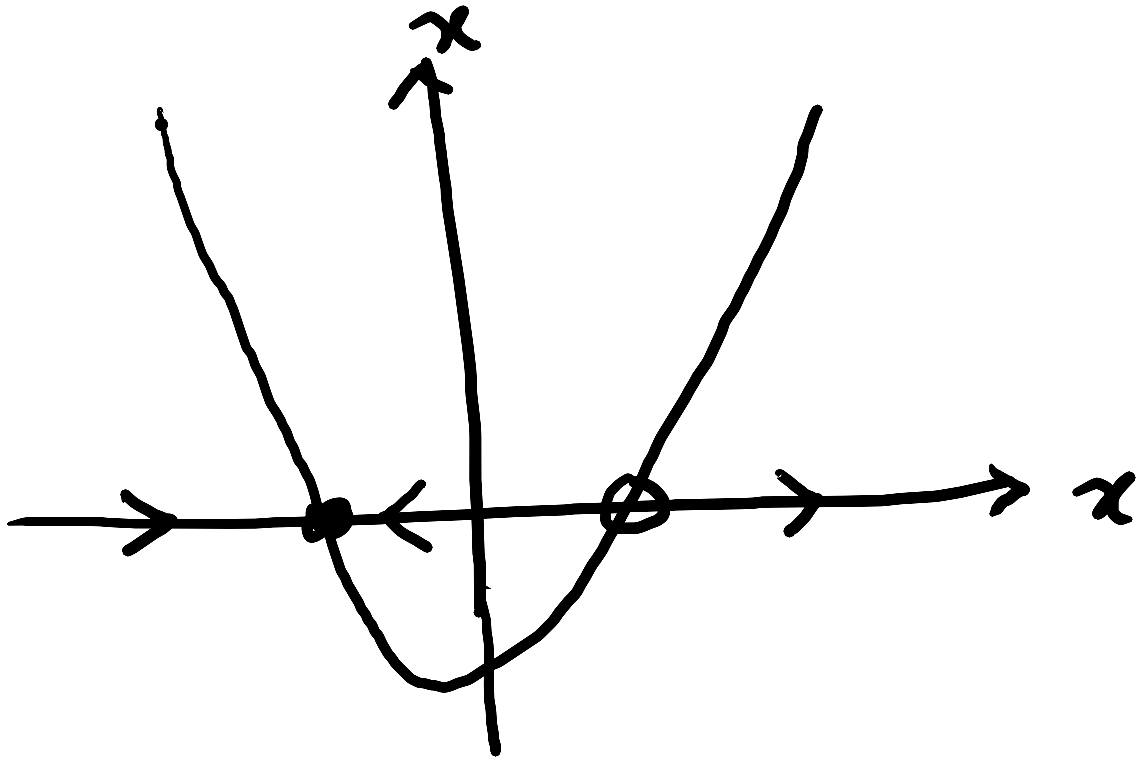
(iii)

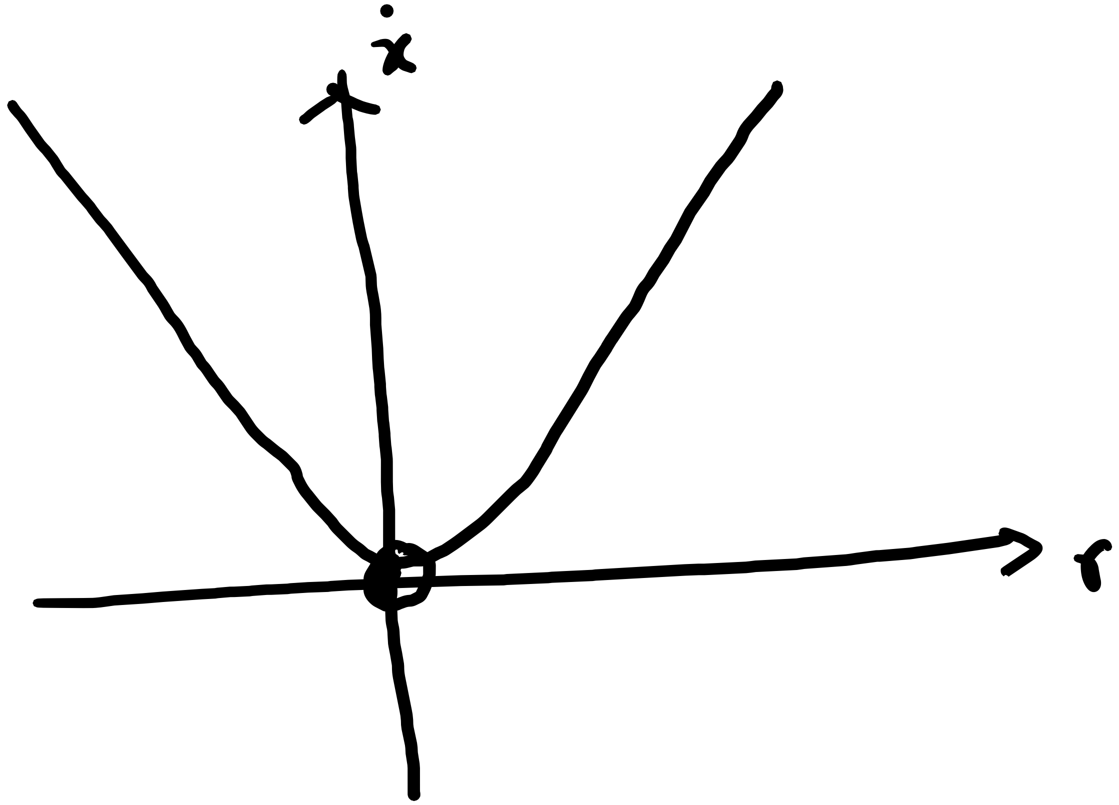


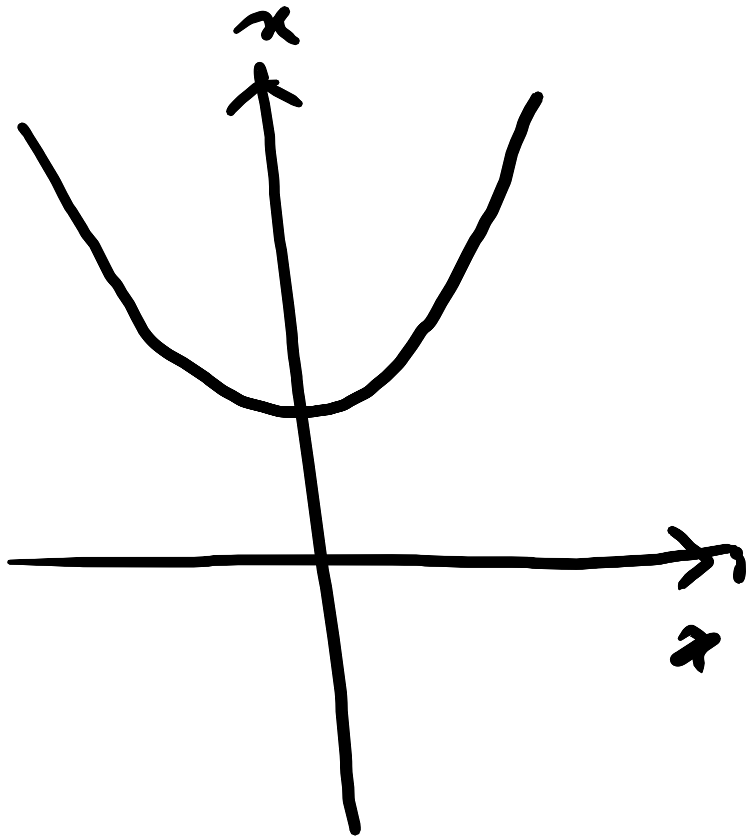
new class of fixed point
"half attractive/repulsive"

VIII BIFURCATION

$\boxed{\dot{x} = r + x^2}$, r allowed to be $\pm$ve, or 0 .







$r < 0$.

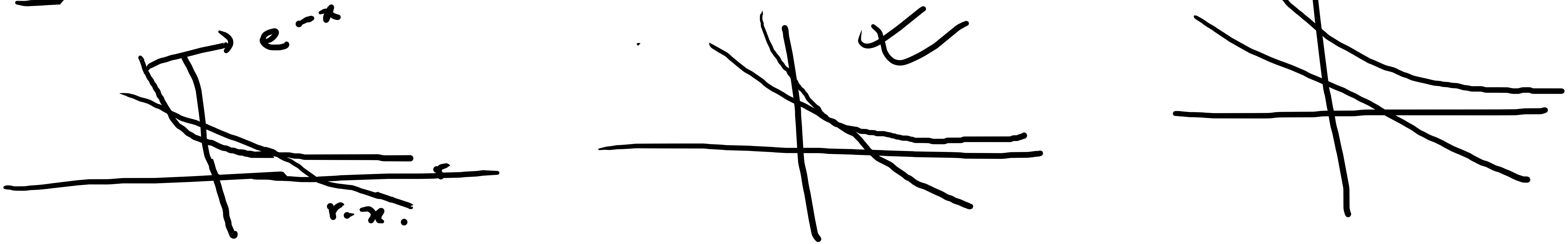
— Situation changes qualitatively as r is varied.

— Defn. of bifurcation.

— Saddle-node bifurcation.

$\dot{x} = r - x^2$. (Exercise)

<u>Ex</u> :- $\dot{x} = r - x - e^{-x} = f(x)$.



To find bifurcation point $r = r_c$

$$e^{-x} = r - x$$

$$\frac{d}{dx}\left(e^{-x}\right) = \frac{d}{dx}(r - x).$$

$\Rightarrow \quad e^{-x} = 1$

$\Rightarrow \quad x = 0.$

Sub:- $r_c = 1.$

Expand $\dot{x} = r - x - e^{-x}$ near $x = 0.$

$$= (r - x) - \left[1 - x - \frac{x^2}{2} + \dots\right]$$

$$\approx (r - 1) - \frac{x^2}{2}.$$

$\dot{x} = r + x^2.$

resembles normal form for saddle-node bifurcation.