SC223 - Linear Algebra

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Lecture 26

1. If $V = W_1 \oplus W_2$

2. Is $W_1 + W_2$ a direct sum?

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$$U = \{ (x_i)_{i=0}^{\infty} \mid x_0 = a, \ x_1 = b, \ x_n = x_{n-1} + x_{n-2}, \ n \geq 2 \}.$$

$\dim(U) = 2$ $\qquad V = \mathbb{R}^2.$

$$s_1 = (1, 0, 1, 1, 2, 3, \ldots) \in U$$

$$s_2 = (0, 1, 1, 2, 3, 5, \ldots) \in U.$$

$T : U \to V$

$$T(s_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T(s_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Summary of Lecture 25

- We say that two vector spaces over $\mathbb{F}$, U and V are **homomorphic** if there exists an linear transformation between them.
- We say that two vector spaces over $\mathbb{F}$, U and V are **isomorphic** if there exists an **invertible** linear transformation between them.
- If T is an invertible linear transformation between U and V, then we say that T is an **isomorphism** between U and V.
- **Proposition 19:** Show that two vector spaces U and V over $\mathbb{F}$ are isomorphic iff they have the same dimensions.

Representation of Linear Transformations between FDVS

- Let $T : U \to V$ be a LT, and let $\beta_U := \{u_1, \ldots, u_n\}$ and $\beta_V = \{v_1, \ldots, v_m\}$ denote the basis of U and V respectively.

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- For an $x \in U, x = \sum_{i=1}^{n} a_i u_i$,

$$y = Tx = T\left(\sum a_i u_i\right) = \sum_{i=1}^{n} a_i T u_i$$

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- Thus, $T(u_i), i = 1, \ldots, n$ is enough to allow us to compute $T(x), \forall x \in U$.
- Now, $T(u_i) \in V$, thus $T(u_i) = \sum_{j=1}^{m} c_{ji} v_j$.

$$y = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} c_{ji} v_j \right)$$

$$y = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i c_{ji} v_j = \sum_{j=1}^{m} b_j v_j$$

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- Thus, $T(u_i), i = 1, \ldots, n$ is enough to allow us to compute $T(x), \forall x \in U$.
- Now, $T(u_i) \in V$, thus $T(u_i) = \sum_{j=1}^{m} c_{ji} v_j$.
- Then, $y = \sum_{i=1}^{n} a_i (\sum_{j=1}^{m} c_{ji} v_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i c_{ji} v_j$.

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- Also, since $y \in V, y = \sum_{j=1}^{m} b_j v_j$.

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- Also, since $y \in V, y = \sum_{j=1}^{m} b_j v_j$.
- Thus, $\sum_{j=1}^{m} b_j v_j = \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ji} a_i v_j$.

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- For an $x \in U, x = \sum_{i=1}^{n} a_i u_i$, $y = Tx = T(\sum_{i=1}^{n} a_i u_i) = \sum_{i=1}^{n} a_i T(u_i)$.
- Thus, $T(u_i), i = 1, \ldots, n$ is enough to allow us to compute $T(x), \forall x \in U$.
- Now, $T(u_i) \in V$, thus $T(u_i) = \sum_{j=1}^{m} c_{ji} v_j$.
- Then, $y = \sum_{i=1}^{n} a_i(\sum_{j=1}^{m} c_{ji} v_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i c_{ji} v_j$.
- Also, since $y \in V, y = \sum_{j=1}^{m} b_j v_j$.
- Thus, $\sum_{j=1}^{m} b_j v_j = \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ji} a_i v_j$.

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

- For $k \in \{1, \dots, m\}$, $b_k = \sum_{i=1}^{n} c_{ki} a_i$,

- For $k \in \{1, \ldots, m\}$, $b_k = \sum_{i=1}^{n} c_{ki} a_i$, or,

$$\underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{[y]_{\beta_V}} = \underbrace{\begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \ldots & c_{mn} \end{bmatrix}}_{[T]_{\beta_U}^{\beta_V}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{[x]_{\beta_U}}$$

co-domain

domain

- The matrix $[T]_{\beta_U}^{\beta_V}$ is called the matrix representation of the linear tranformation T with respect to the basis β_U and β_V.

$$[y]_{\beta_V} = [T]_{\beta_U}^{\beta_V} [x]_{\beta_U}$$

Examples

- $\frac{d}{dx} : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$.

$$\beta^1_{\mathcal{P}_2(\mathbb{R})} = \{1, x, x^2\}$$

$$\left[\frac{d}{dx}\right]^{\beta^1_{\mathcal{P}_2(\mathbb{R})}}_{\beta^1_{\mathcal{P}_2(\mathbb{R})}} = ?$$

$$\beta^2_{\mathcal{P}_3(\mathbb{R})} = \underline{\{1, x^2, x\}}.$$

$$\left[\frac{d}{dx}\right]^{\beta^2_{\mathcal{P}_3(\mathbb{R})}}_{\beta^2_{\mathcal{P}_3(\mathbb{R})}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$p(x) = x^2 + 2x + 1$$

$$[p]_{\beta^2_{\mathcal{P}_3(\mathbb{R})}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = [p']_{\beta^2_{\mathcal{P}_3(\mathbb{R})}}$$

$$u_1 = 1 \qquad \frac{d}{dx}(1) = 0$$

$$= \underline{0} \cdot 1 + \underline{0} \cdot x^2 + \underline{0} \cdot x$$

$$u_2 = x^2 \qquad \frac{d}{dx}(x^2) = 2x$$

$$= \underline{0} \cdot 1 + \underline{0} \cdot x^2 + \underline{2} \cdot x$$

$$u_3 = x. \qquad \frac{d}{dx}(x) = 1$$

$$= \underline{1} \cdot 1 + \underline{0} \cdot x^2 + \underline{0} \cdot x$$

Examples

- $\frac{d}{dx} : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$.
- $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.

End of class.

Examples

- $\frac{d}{dx} : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$.
- $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.
- Let $p \in \mathcal{P}_3(\mathbb{R})$ be such that $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$. Define $T_p : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ by $T_p(q) = p \cdot q, \forall q \in \mathcal{P}_3(\mathbb{R})$, where $\cdot$ represents multiplication between polynomials.

$\mathcal{L}(U, V)$