Computational Numerical Methods

CS 374

Prosenjit Kundu

$$e^{(k+1)} = B^{k+1} e^{(0)}.$$

$$0 \le \| e^{(k+1)} \| = \| B^{k+1} e^{(0)} \|$$

$$\le \| B^{k+1} \| \, \| e^{(0)} \| \quad \neq 0 \text{ for diff choices.}$$

$$= \| B \|^{k+1}$$

for convergence. $\| B^{k+1} \| \to 0$.

is possible only when $\| B \| < 1$.

Diagonally dominant

A matrix is said to be diagonally dominant

if $\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| < |a_{ii}|$

Th If the coefficient matrix A is diagonally dominant then, the Jacobi method.

$$x^{(k+1)} = B x^{(k)} + C \qquad k = 0, 1, 2 \ldots$$

converges.

$$x_i^{k+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)} \right)$$

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j \right)$$

The error

$$e^{(k+1)} = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{a_{ij}}{a_{ii}} \left(x_j^{(k)} - x_j \right)$$

$$\|e^{(k+1)}\| \le \sum \left|\frac{a_{ij}}{a_{ii}}\right| \|e^{(k)}\|$$

$$e_\infty^{(k)} = \max . |e_j^{(k)}|$$

$$\|e^{(k+1)}\|_\infty \le \sum \left|\frac{a_{ij}}{a_{ii}}\right| \|e^{(k)}\|_\infty$$

$$\le \mu \|e^{(k)}\|_\infty$$

$$\underline{\mu = \max} \sum_{\substack{j=1\\ i\neq j}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|$$

$$\|e^{(k+1)}\|_\infty \le \mu^2 \|e^{(k-1)}\|_\infty \le \mu^{(k+1)} \|e^0\|_\infty$$

✓

$$\|e^{(k+1)}\|_\infty \leq \underline{\underline{\mu^{k+1}}} \|e^{(k)}\|_\infty .$$

The method will converge. or.

$\|e^{(k+1)}\|_\infty \to 0$. when.

$\mu^{k+1} \to 0$ as $k \to \infty$.

$\therefore \quad \underline{\underline{\mu < 1}}$

$\Rightarrow$ The coefficient matrix is diagonally dominant.

Gauss - Seidel method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right)$$

$$x_2^{k+1} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right)$$

$$x_3^{k+1} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right)$$

Ex $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix}$ $AX = 0$. find the solⁿ using Jacobi & Gauss Seidel's method.

Jacobi $x^{(0)} = (1, 1, 1)$

		error
1	$x^{(1)} = (-1, 1, 1)$	$\lVert e^{(1)} \rVert_2 = 1.7320$
	$x^{(10)} = (0.77778, 0.38272, -0.19342)$	$\lVert e^{(10)} \rVert_2 = 0.88816$.
	$x^{(50)} = (-0.07015, 0.00385, 0.5541)$	$e_2^{(50)} = 0.08946$.
	$x^{(100)} = (-0.00161, -0.00472, -0.00361)$	$e_2^{(100)} = 0.00583$

Gauss - Seidel $(1, 1, 1)$

$x^{(1)} = (-1, +1, +1)$

$x^{(2)} = (1, 1, 1)$

1.7320508.

Solve using Gauss Seidel

$$4x_1 + 2x_2 - x_3 = 5$$

$$x_1 + 4x_2 + x_3 = 12$$

$$2x_1 - x_2 + 4x_3 = 12.$$

Use $x^{(0)} = (0, 0, 0)$

$$x_1^{k+1} = \frac{1}{4}\left(5 - 2x_2^{k} + x_3^{k}\right)$$

$$x_2^{k+1} = \frac{1}{4}\left(12 - x_1^{k+1} - x_3\right)$$

$$x_3^{k+1} = \frac{1}{4}\left(12 - 2x_1^{k+1} + x_2^{k+1}\right)$$

it		Error (l_∞)
1	$x^{(1)} = (1.25, 2.6875, 3.0468)$	3.0468.
2.	$x^{(2)} = (0.6679, 2.07138, 3.1838)$	0.6161
3.	$x^{(3)} = (1.0102, 1.9515, 2.9827)$	0.3423
4.	$x^{(4)} = (1.0199, 1.9993, 2.9898)$	0.0478.