Chapter 6 | Lecture 37

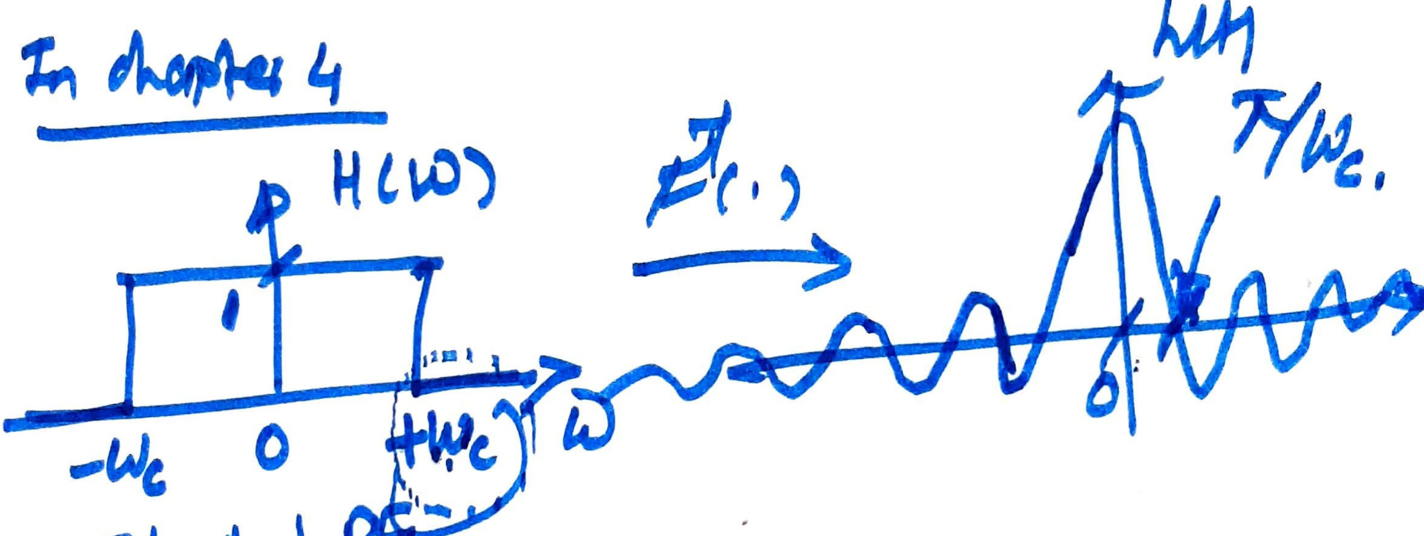
Time and Frequency-Domain Characteristics of Signals and Systems

OR

Significance of Phase of the Fourier Transform

Motivation: → Design of filters.

In chapter 4



Ideal LPF

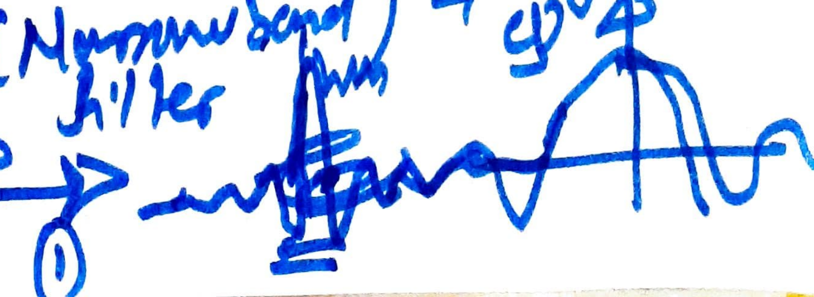
Observations.

① h(t) → Impulse response of ideal LPF is non-causal

② h(t) → IIR

③ As $\omega_c \downarrow$ ⇒ (Narrowband filter) ⇒ High quality

⇒ $\frac{\pi}{\omega_c} \uparrow$ ⟶



Key objective in chapter 6 is to investigate time and frequency-domain trade-offs during filter design.

⇒ Parameter tuning of filter in frequency domain ⇒ Parameter tuning for impulse response of filter in time-domain

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$\boxed{6.1}$ The magnitude-phase representation of the Fourier transform.

$x(t)$ = Aperiodic signal ; $x[n]$ ⇒ aperiodic signal

$$X(\omega) = |X(\omega)| \cdot e^{j\measuredangle X(\omega)}$$

(chapter 4)

$$X(e^{j\omega}) = |X(e^{j\omega})| \cdot e^{j\measuredangle X(e^{j\omega})}$$

(chapter 5)

$|X(\omega)|$ = Magnitude Spectrum

$\measuredangle X(\omega)$ = Phase spectrum

$\dfrac{|X(\omega)|^2}{\pi}$ = Energy Spectral Density (ESD).

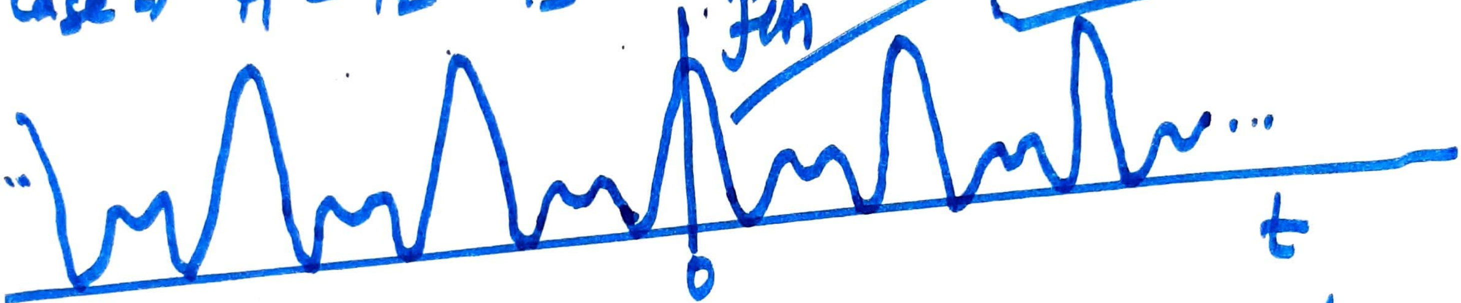
Significance of Fourier transform phase

[1] Synthetic signal

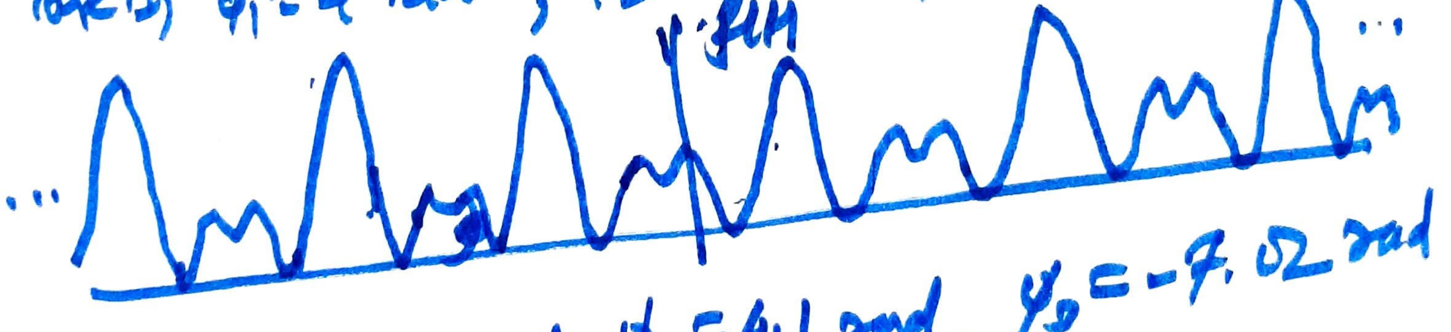
Chapter 3 $\longrightarrow$ Understanding Fourier series

$$f(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \left(\frac{2}{3}\right)\cos(6\pi t + \phi_3)$$

Case I) $\phi_1 = \phi_2 = \phi_3 = 0$ $\boxed{\text{chapter 2}}$



Case II) $\phi_1 = 4$ rad , $\phi_2 = 8$ rad , $\phi_3 = 12$ rad



Case III) $\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad

→ Linear phase changes in complex exponentials leads to change in "time shift" of reconstructed signal without affecting overall shape of signal during reconstruction.

→ However, nonlinear phase changes in complex exponentials leads to distortion in overall shape of reconstructed signal

⇒ Phase play a key role in determine shape of signal.

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[2] Speech, Audio and Music.

Ex. Experiment by an acoustician, Ohm "Human ear is phase deaf"

⇒ Human ear is insensitive to phase

⇒ However, there are recent studies reported in last 20 years, which gives significance of phase in speech.

Let x(t) → normal lay

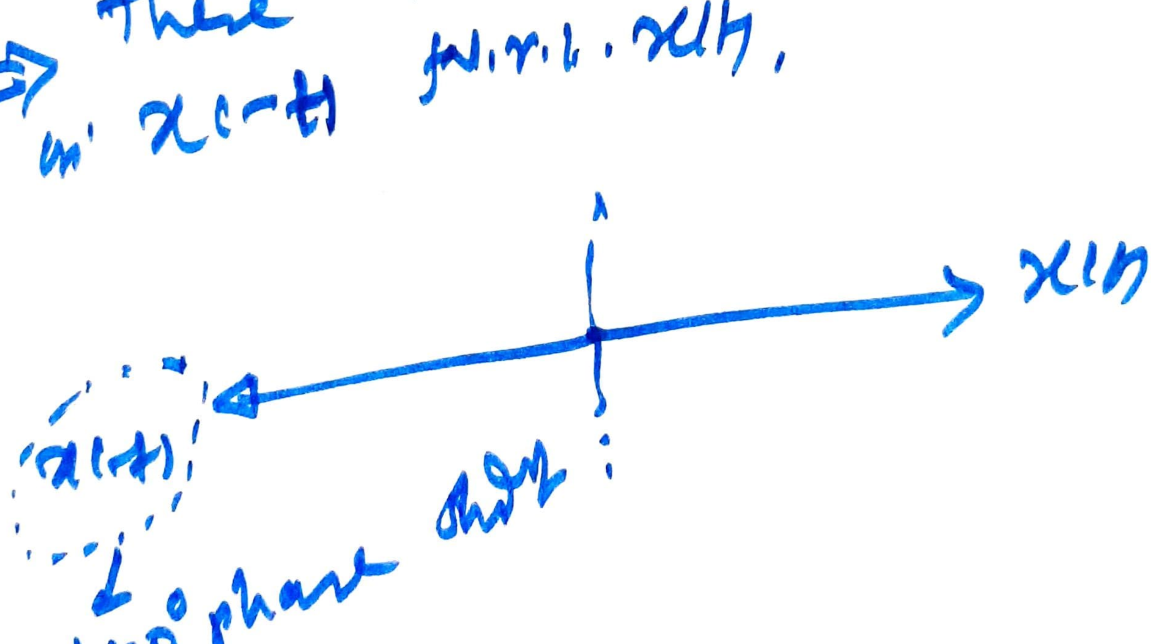
x(-t) → flipped / time-reversed

$$\mathcal{F}\{x(-t)\} = X(-\omega) = |X(\omega)| \cdot e^{-j\angle X(\omega)}$$

$$\mathcal{F}\{x(t)\} = X(\omega)$$

⇒ Time reversal corresponds 180° phase shift.

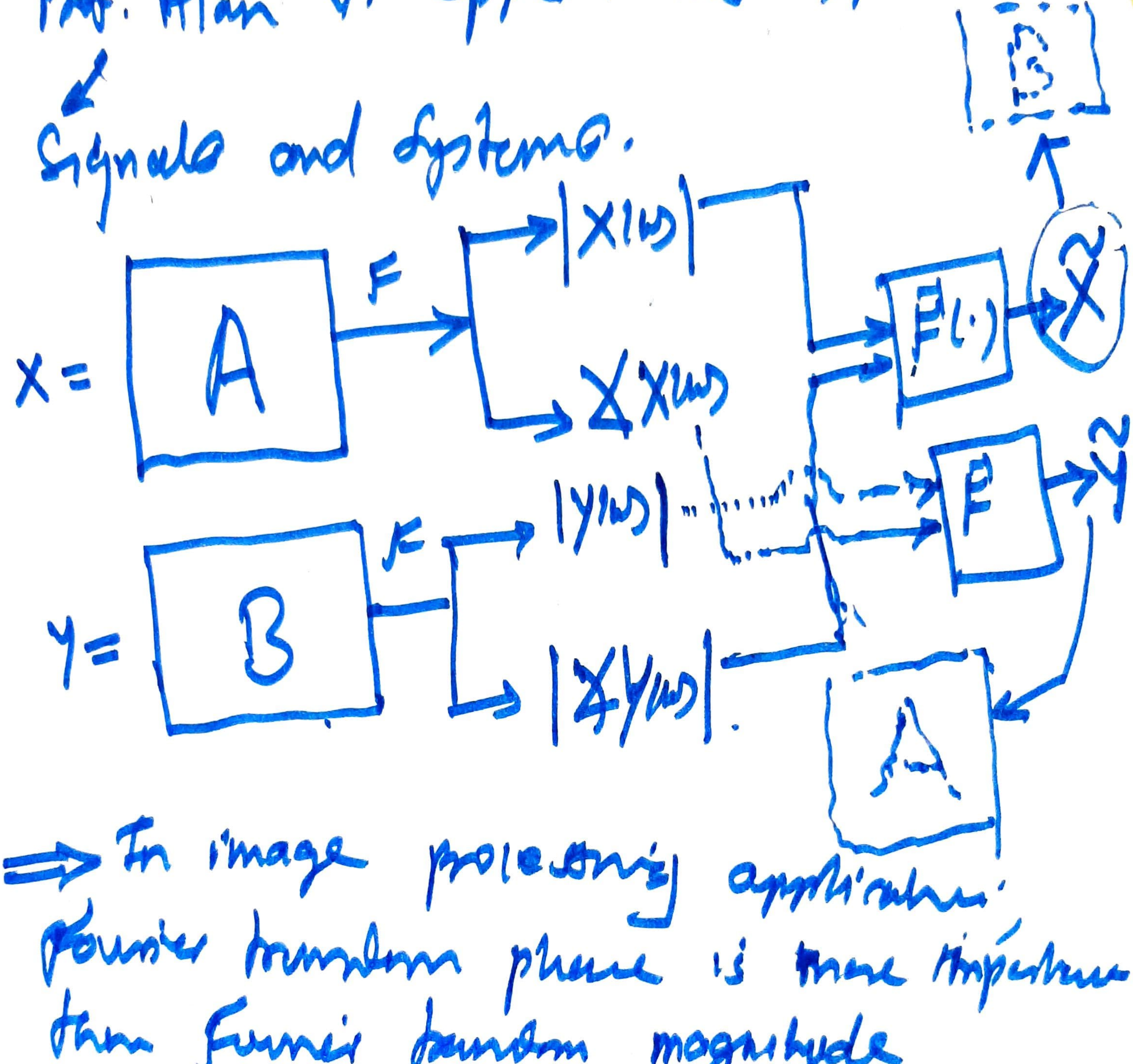
⇒ There will significant distortion in x(-t) w.r.t. x(t).



(3) Image Processing and Computer Vision

Prof. Alan V. Oppenheim's experiment.

Signals and Systems.



⇒ In image processing application: Fourier transform phase is more important than Fourier transform magnitude

4] Ocean Acoustics

in-phase

out-of phase

Why phase is significant in signal reconstruction??

Case I) $f(t)$ = periodic signal

$\therefore f(t)$ = Exponential Fourier representation.

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{jn\omega_0 t}$$

Fourier series coefficient

$\because F_n$'s in general are complex,

$\therefore F_n = |F_n| \cdot e^{j\angle F_n}$

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$$f(t) = \sum_{n=-\infty}^{+\infty} |F_n| \cdot e^{j\measuredangle F_n} \cdot e^{jn\omega_0 t}$$

$$\underset{\text{Reconstructed signal}}{f(t)} = \sum_{n=-\infty}^{+\infty} \underbrace{|F_n|}_{\text{Magnitude spectrum}} \cdot e^{j(\underbrace{\measuredangle F_n}_{\text{Phase spectrum}} + n\omega_0 t)}$$

Chapter 3 → Understanding Fourier series.

Inference:

① $|F_n|$ provides us with the information about the relative magnitudes of the complex exponentials, whereas

② $\measuredangle F_n$ → provides information concerns the relative phases of complex exponentials that make up the signal $f(t)$.

Case III) $f(t)$ = aperiodic

$\therefore f(t)$ represented via Fourier transform

$$f(t) = F^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=+\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Using fundamental theorem of integral calculus,

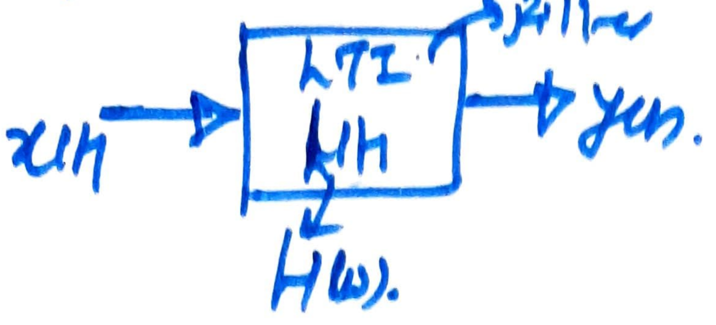
$$\therefore f(t) = \frac{1}{2\pi} \lim_{\omega_0 \to 0} \sum_{n=-\infty}^{+\infty} F(\omega_n) \cdot e^{j\omega_n t} \cdot \omega_0$$

$$F(\omega_n) = |F(\omega_n)| \cdot e^{j\angle F(\omega_n)}$$

$$\therefore f(t) = \frac{1}{2\pi} \lim_{\omega_0 \to 0} \sum_{n=-\infty}^{+\infty} |F(\omega_n)| \cdot e^{j(\angle F(\omega_n) + \omega_n t)} \times \omega_0$$

6.2 The Magnitude-Phase Representation of the Frequency Response of LTI Systems

Case I Continuous-time LTI filter



$$y(t) = x(t) * h(t)$$

Convolution theorem

$$Y(\omega) = X(\omega)\cdot H(\omega)$$

$$\therefore |Y(\omega)|\cdot e^{j\measuredangle Y(\omega)} = |X(\omega)|\cdot e^{j\measuredangle X(\omega)} \times |H(\omega)|\cdot e^{j\measuredangle H(\omega)}$$

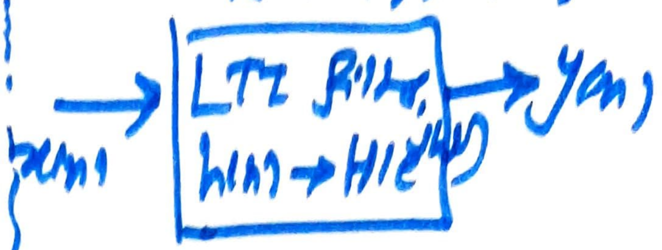
$$\Rightarrow |Y(\omega)| = |X(\omega)|\cdot|H(\omega)| \quad \text{—(A)}$$

gain of the filter

$$\measuredangle Y(\omega) = \measuredangle X(\omega) + \measuredangle H(\omega) \quad \text{—(B)}$$

C.T.F.T. (chapter 4)

Case II Discrete-time LTI system.



$$y[n] = x[n] * h[n]$$

Convolution theorem

$$Y(e^{j\omega}) = X(e^{j\omega})\cdot H(e^{j\omega}).$$

$$|Y(e^{j\omega})| = |X(e^{j\omega})|\cdot|H(e^{j\omega})|.$$

$$\measuredangle Y(e^{j\omega}) = \measuredangle X(e^{j\omega}) + \measuredangle H(e^{j\omega}).$$

D.T.F.T. (chapter 5)

Inference:

From eqn (A): → Since $|X(\omega)|$ (i.e., input signal's spectrum) gets scaled by magnitude response of LTI filter ($h(t) \rightarrow H(\omega)$), $H(\omega)$ is called as "gain" of the LTI filter.

$$\text{From eq}^n \text{ (B)}: \quad \angle Y(\omega) = \angle X(\omega) + \angle H(\omega);$$

From eqn (B): Since the phase spectrum of input signal, i.e. $\angle X(\omega)$ gets shifted by an amount $\angle H(\omega)$, the phase response of LTI filter, i.e. $\angle H(\omega)$ is called as phase shift of the LTI filter.

Design of Filters

(Digital Signal Processing, Network Theory,)

↓

LTI filters.

↓

Since impulse response of an LTI system is characterised completely via their impulse response, we design filter by appropriate design of impulse response.

$\Rightarrow$ Design of filters $\Rightarrow$ Design of $h[n]$'s.

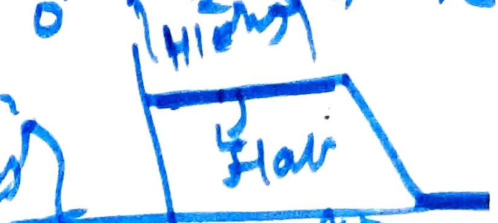
↓

(DSP)

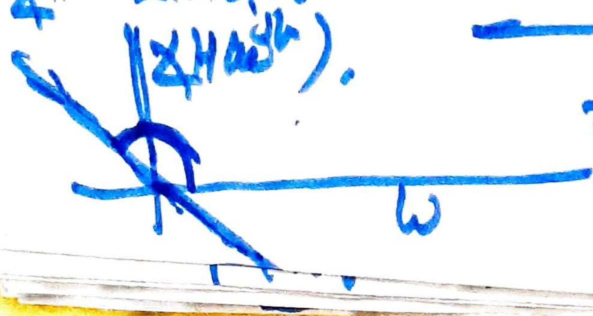
FIR filter | IIR filter.

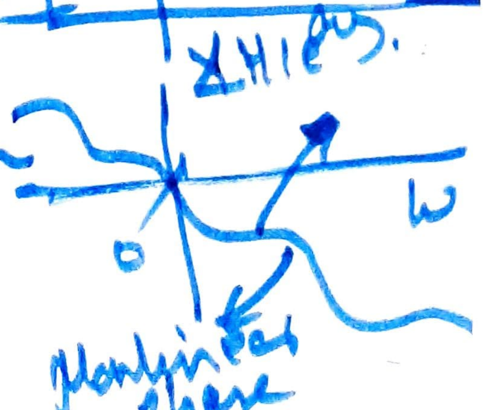
$h[n] = \begin{cases} \frac{1}{3}, & -1 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$

$h[n] = a^n \cdot u[n]$, $|a| < 1$



certain conditions on $h[n]$ {symmetric / antisymmetric} $\Rightarrow$ Linear phase ($\angle H(e^{j\omega})$).





Linear vs. Nonlinear Phase.

* Effect of "Linear Phase" on the Filtered Signal: →

$$\underset{\text{(Input signal)}}{x(n)} \longrightarrow \boxed{\begin{array}{c}\text{Linear Phase Filter}\\ h(n) \rightarrow H(e^{j\omega})\end{array}} \longrightarrow \underset{\text{Filter Output}}{y(n)}$$

Let $h(n) \rightarrow H(e^{j\omega})$ → $|H(e^{j\omega})| = 1$

→ $\angle H(e^{j\omega}) = -k\omega$

$$\therefore H(e^{j\omega}) = |H(e^{j\omega})| \cdot e^{j\angle H(e^{j\omega})}$$

$$\boxed{H(e^{j\omega}) = 1 \cdot e^{-jk\omega}}$$

$\because$ Given filter is LTI, $y(n) = x(n) * h(n)$.

Using convolution theorem.

$$\therefore Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$\therefore Y(e^{j\omega}) = X(e^{j\omega}) \cdot e^{-jk\omega}$$

$$Y(e^{j\omega}) = e^{-jk\omega} \cdot X(e^{j\omega})$$

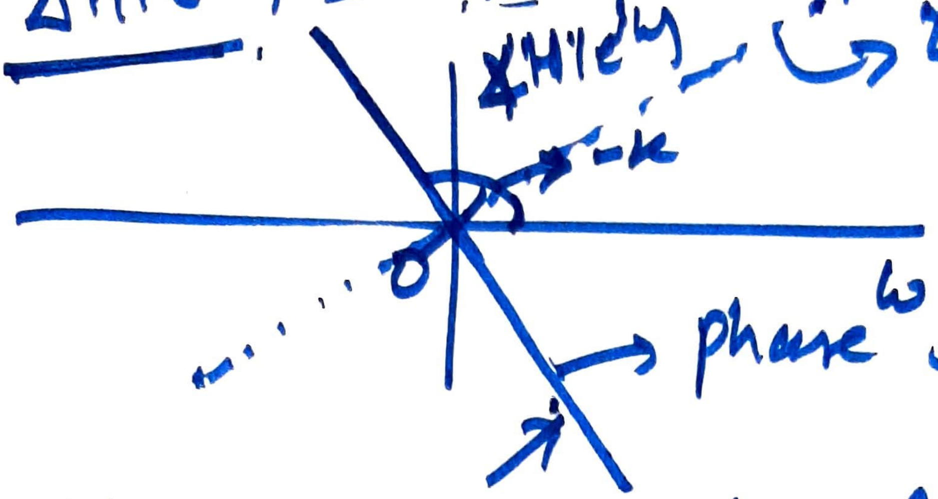
$$\therefore y(n) = F^{-1}\{Y(e^{j\omega})\} = F^{-1}\{e^{-jk\omega} \cdot X(e^{j\omega})\}$$

$$\boxed{y(n) = x(n-k)}$$

$y(n) = x(n+k)$

$\measuredangle H(e^{j\omega}) = -k\omega$

$\measuredangle H(e^{j\omega}) = k\omega$



phase slope function

$y = mx$ phase slope $= -k$

$\therefore m = -k$

Inference:

① If we pass an input signal $x(n)$ through an LTI filter having linear phase characteristic (with phase slope $= -k$ samples), then the output of filter $y(n)$ will be delayed version of $x(n)$. The amount of delay is equal phase slope of $\measuredangle H(e^{j\omega})$.

② Why phase slope $= -k$

How to quantify linear vs. nonlinear phase characteristics ??

$\angle H(e^{j\omega}) = -k\omega$.

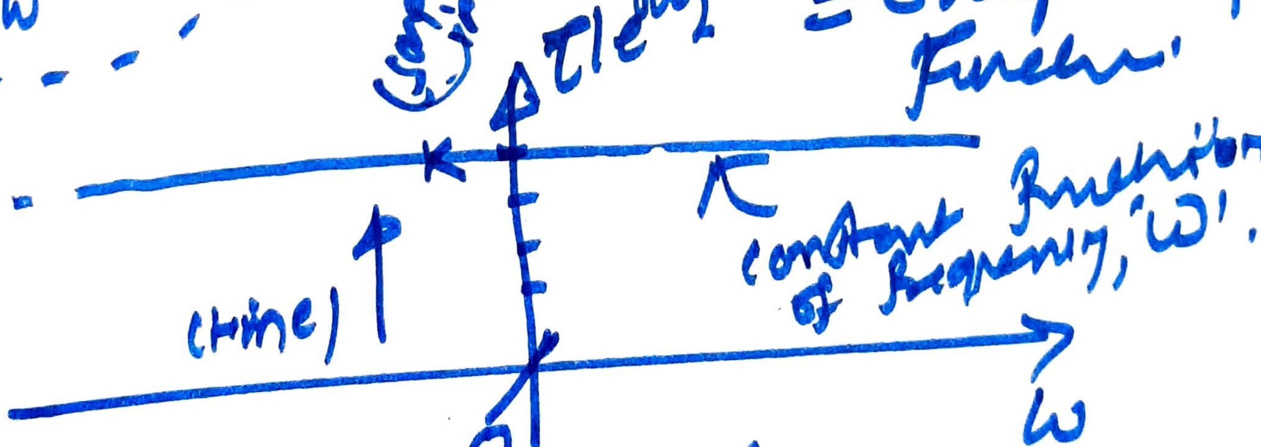
Differentiate w.r.t. ω:

$\therefore \frac{d}{d\omega} \angle H(e^{j\omega}) = \frac{d}{d\omega}(-k\omega)$

$-\frac{d}{d\omega} \angle H(e^{j\omega}) = -(-k) \times 1 = k$ samples

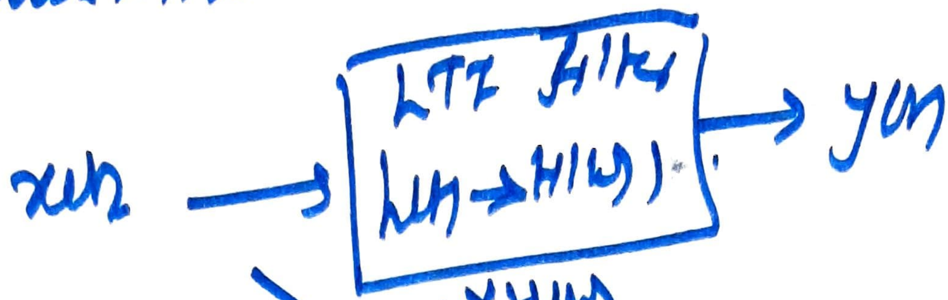
$-\frac{d}{d\omega} \angle H(e^{j\omega}) = k$ samples. $= \tau(e^{j\omega})$.

$\tau(e^{j\omega})$ = Group Delay Function.

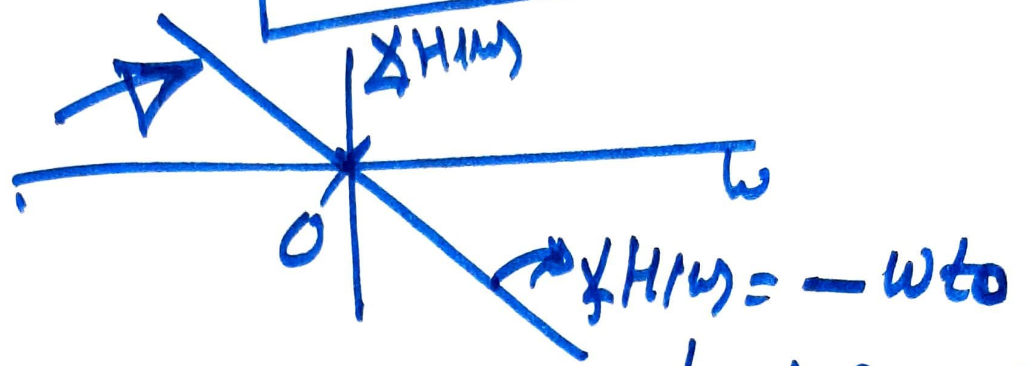


Similarly, for CTFT.

Continuous-time LTI filter.



Linear phase:



$\therefore y(t) = x(t - t_0)$.

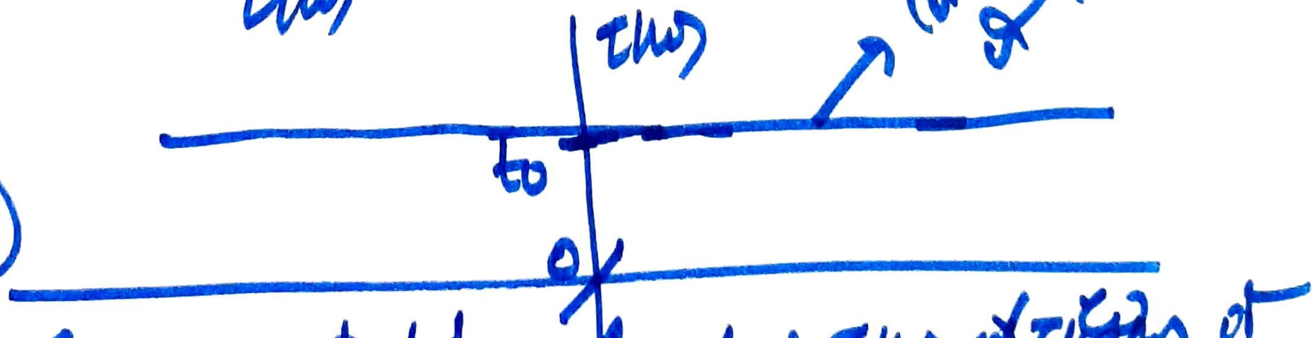
$\measuredangle H(\omega) = -\omega t_0$

Diff. w.r.t. ω

$$-\frac{d}{d\omega}\measuredangle H(\omega) = -\frac{d}{d\omega}(-\omega t_0)$$

$$\downarrow$$

$$\tau(\omega) = t_0 \text{ (seconds)}$$



(16)

∴ The group delay function $\tau(\omega)$ of a linear phase LTI filter, is constant function of frequency.