SC223 - Linear Algebra

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Lecture 38

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Summary of Lecture 37 $A \in \mathbb{C}^{n\times n}$, $V = \mathbb{C}^n$, $\langle x,y\rangle = y^* x$

- **Definition:** (Inner Product Space) A vector space V with an inner product is called an **Inner Product space**(IPS) and is denoted by $(V, \langle\cdot,\cdot\rangle)$.
- Given an IPS $(V, \langle\cdot,\cdot\rangle), \forall x \in V, ||x|| = \sqrt{\langle x, x\rangle}$ is a valid norm, called the **induced norm**.
- Given an IPS $(V, \langle\cdot,\cdot\rangle)$, two vectors $x, y \in V$ are said to be **orthogonal** if $\langle x, y\rangle = 0$, and are said to be **orthonormal** if $\langle x, y\rangle = 0, \underline{||x||} = \underline{||y||} = 1$. $||x|| = \sqrt{\langle x,x\rangle}$
- A set of vectors $\{v_1, \ldots, v_n\}$ is said to be **orthogonal** if $\langle v_i, v_j\rangle = 0, \forall i \neq j$ and is said to be **orthonormal** if $\langle v_i, v_j\rangle = 0, \forall i \neq j$, $||v_i|| = 1, \forall i$.
- A set of orthonormal vectors that also forms a basis of the given vector space is called an **Orthonormal basis**.
- A matrix $A \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ is said to be an **orthogonal matrix** if all its n columns are orthonormal, i.e., $A^*A = I$, where A^* denotes the conjugate transpose of A. In this case, $A^{-1} = A^*$.

$A \in \mathbb{R}^{n\times n}$ $/(\mathbb{F} = \mathbb{R})$ $V = \mathbb{R}^n$, $\langle x,y\rangle = x^T y$.

Properties

- **Proposition 23:** In a IPS $(V, \langle \cdot, \cdot \rangle)$, a set of n non-zero orthogonal vectors $\{v_1, \dots, v_n\}$ is linearly independent.

$$\sum_{i=1}^{n} a_i v_i = \theta, \quad \text{not all } a_i\text{'s are } 0.$$

$$\left\langle \sum_{i=1}^{n} a_i v_i, v_1 \right\rangle = \langle \theta, v_1 \rangle = 0$$

$$\Rightarrow \sum_{i=1}^{n} a_i \langle v_i, v_1 \rangle = 0$$

$$\Rightarrow a_1 \|v_1\|^2 = 0 \Rightarrow a_1 = 0$$

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- **Proposition 24:** (Pythagoras Theorem): In an IPS $(V, \langle \cdot, \cdot \rangle)$, if $\langle x, y \rangle = 0$, $||x+y||^2 = ||x||^2 + ||y||^2$.

$$||x+y||^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle$$
$$+ \cancel{\langle x, y \rangle}^{0} + \cancel{\langle y, x \rangle}^{0}$$
$$= ||x||^2 + ||y||^2$$

$$||x_1 + \cdots + x_n||^2 = \sum_{i=1}^{n} ||x_i||^2$$

(If $\langle x_i, x_j \rangle = 0, \forall i \neq j$)

$$\left|\left| \sum c_n e^{jn\omega_0 t} \right|\right|^2 = \sum |c_n|^2$$

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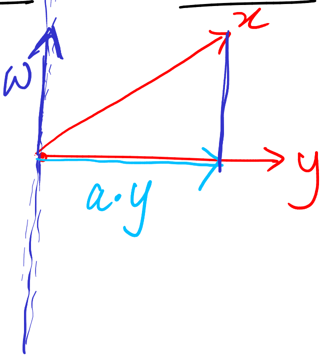
- **Orthogonal Decomposition:** Let $x, y \neq \theta \in V$. Find $w \in V$ such that $x = a \cdot y + w$, with $a \in \mathbb{F}$, $\langle w, y \rangle = 0$.

$w = x - a \cdot y$



$\langle x - a \cdot y, y \rangle = 0$

$\langle x, y \rangle - a\, ||y||^2 = 0$

$\Rightarrow \boxed{a = \dfrac{\langle x, y \rangle}{||y||^2}}$

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$$
\begin{aligned}
&\langle w, y\rangle = 0\\
&\langle x - a\cdot y, y\rangle = \langle x, y\rangle - a\langle y, y\rangle = 0\\
&a = \frac{\langle x, y\rangle}{\langle y, y\rangle}\\
&\text{Thus}, x = \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y + \left(x - \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y\right)
\end{aligned}
$$

Properties

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- Proof: If $x = \theta$, or $y = \theta$, both sides are equal to zero. So let us assume $x, y \neq \theta$.
- From previous proposition,

$x = \frac{\langle x,y\rangle}{\langle y,y\rangle} \cdot y + \left(x - \frac{\langle x,y\rangle}{\langle y,y\rangle} \cdot y\right) = \underline{\frac{\langle x,y\rangle}{\langle y,y\rangle} \cdot y + w}$, with $w \perp y$.

$$||x||^2 = \left|\left|\frac{\langle x,y\rangle}{\langle y,y\rangle} \cdot y + w\right|\right|^2 = \left|\left|\frac{\langle x,y\rangle}{\langle y,y\rangle} y\right|\right|^2 + ||w||^2$$

$$= \frac{|\langle x,y\rangle|^2}{|\langle y,y\rangle|^2} \cdot ||y||^2 + ||w||^2$$

$$||x||^2 = \frac{|\langle x,y\rangle|^2}{||y||^2} + ||w||^2 \geq \frac{|\langle x,y\rangle|^2}{||y||^2}$$

Properties

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Gram-Schmidt Procedure

- **Proposition 26:** (Gram-Schmidt Procedure): Let $\{v_1, \ldots, v_m\}$ be a list of linearly independent vectors. Then there exists a list of orthonormal vectors $\{e_1, \ldots, e_m\}$ such that $span(\{v_1, \ldots, v_j\}) = span(\{e_1, \ldots, e_j\}), \forall j = 1, \ldots, m$.

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- Let $e_1 = \frac{v_1}{||v_1||}$. Define $e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{||v_2 - \langle v_2, e_1 \rangle e_1||}$.

$$w = x - a \cdot y, \quad a = \frac{\langle x, y \rangle}{||y||^2}$$

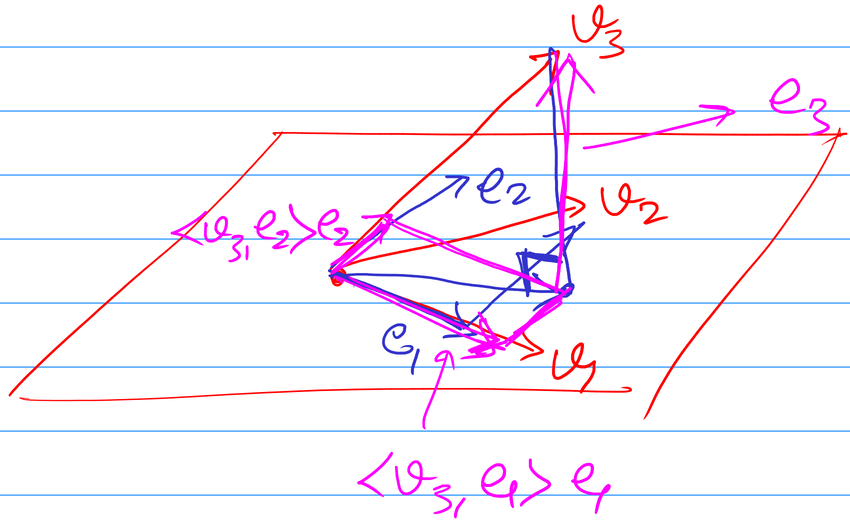
$x \to v_2$, $y \to e_1$

$$\Rightarrow a = \frac{\langle v_2, e_1 \rangle}{||e_1||^2} = \langle v_2, e_1 \rangle$$

$$e_3 = \underline{v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2}$$

$$v_1 = ||v_1|| \cdot e_1 + 0 \cdot e_2$$

$$v_2 = ||v_2 - \langle v_2, e_1 \rangle e_1|| \cdot e_2 + \langle v_2, e_1 \rangle e_1$$



v_3
e_3
e_2
v_2
$\langle v_3, e_2 \rangle e_2$
e_1
v_1
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- Similarly, $e_3 = \frac{v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)}{||v_3 - (\langle v_3, e_1 \rangle e_1 + \langle v_3, e_2 \rangle e_2)||}$, and $e_k = \frac{v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i}{||v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i||}$.

— End of Class. —

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- Observe that $span(\{v_1, \ldots, v_j\}) = span(\{e_1, \ldots, e_j\})$.

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- Observe that $span(\{v_1, \ldots, v_j\}) = span(\{e_1, \ldots, e_j\})$.
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- It is easy to see that $e_1 \perp e_2$. Assume that $\{e_1, \ldots, e_j\}$ are orthonormal.
- Then $\forall l = 1, \ldots j$, with $\tilde{e_{j+1}} = v_{j+1} - \sum_{i=1}^{j} \langle v_{j+1}, e_i \rangle e_i$

$$
\begin{aligned}
\langle e_{j+1}, e_l \rangle &= \frac{1}{||\tilde{e_{j+1}}||} \left(\langle v_{j+1}, e_l \rangle - \sum_{i=1}^{j} \langle v_{j+1}, e_i \rangle \langle e_i, e_l \rangle \right) \\
&= \frac{1}{||\tilde{e_{j+1}}||} \left(\langle v_{j+1}, e_l \rangle - \langle v_{j+1}, e_l \rangle \right) = 0
\end{aligned}
$$

Orthogonal Complement

- Let V be a FD IPS and let U be a subset of V. The **Orthogonal Complement** of U is defined as

$$U^{\perp} = \{v \in V \mid \langle v, u \rangle = 0, \forall u \in U\}$$

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- **Proposition 27:** Irrespective of whether U is a subspace of V or not, $U^{\perp}$ is a subspace.

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  1. $\text{Range}(P_U) = U$
  2. $\text{Null}(P_U) =$

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3. Idempotent: $(P_U)^2 =$

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4. (Conjugate) Symmetric: If $U = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T =$

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1. $\text{Range}(P_U) = U$
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3. Idempotent: $(P_U)^2 = P_U$
4. (Conjugate) Symmetric: If $U = \mathbb{R}^n$ (or $\mathbb{C}^n$), $P_U^T = P_U$ ($P_U^* = P_U$).
5. $\forall v \in V, P_U(v) = \arg\min_{u \in U} ||u - v||^2$.