Chapter 9

Lecture 42

The Laplace Transform

The relevance of LT to LTI system

Chapter 3 → Lecture 29

General representation of eigenfunction of an LTI system.

$$e^{j\omega t} \longrightarrow e^{st}, \quad \text{where } s = \sigma + j\omega t$$

complex variable.

$$x(t) = e^{st} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = H(s) \cdot e^{st}$$

where $\underline{\underline{H(s)}} = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt$

$$L\{h(t)\} = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt$$

In general, for a signal $x(t)$, Laplace transform is defined by

$$L\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st}\, dt = \text{Bilateral LT}$$

$$\int_{0}^{+\infty} x(t)\, e^{-st}\, dt = \text{Unilateral LT.}$$

* Motivation for Laplace Transform (LT)

① The function e^{st} represents generalized eigenfunction of an continuous-time LTI system.. If $x(t) = e^{st}$ then $y(t) = H(s) \cdot e^{st}$

② LT is a generalization of CTFT (Chapter 4)

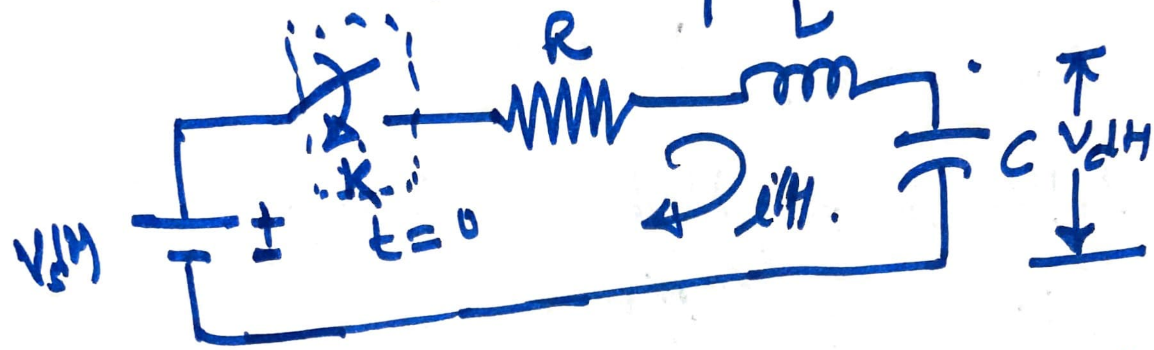
$$H(j\omega) = H(\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt = CTFT\{h(t)\}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt$$

$$s = \sigma + j\omega \Rightarrow \sigma = 0 \;\therefore\; s = j\omega.$$

$\Rightarrow$ LT is much general representation for LTI systems than CTFT.

LT is highly useful for analysis of electrical networks / circuits.



transient analysis of circuits is done using LT approach.

④ LT is useful for solution of linear constant coefficient differential equation

$$\boxed{\frac{dy(t)}{dt} + a \cdot y(t) = b \cdot x(t)}$$

⑤ $x(t) = u(t)$ [unit step signal]

$$\int |u(t)|\,dt \not< \infty$$

Dirichlet condition is NOT satisfied

∴ Formally CTFT do not exist

However $L\{u(t)\}$ = exists $= 1/s$

(7) LT is highly useful in DSP, Circuits and Network Analysis, Control Systems, Power Systems, Van Valkenberg, Network Analysis and Synthesis → LT

(8) We can employ powerful results from complex variable theory to analyse LTI systems in Laplace domain.

X Cauchy Residue Theorem

Relationship Laplace Transform and Fourier Transform

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$X(j\omega) = X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt \quad \text{--- (1)}$$

$$L\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st} dt \quad \text{--- (2)}$$

where $s = \sigma + j\omega$

$$= \int_{-\infty}^{+\infty} x(t) \cdot e^{-(\sigma + j\omega)t} dt$$

$$X(s)\Big|_{s=\sigma+j\omega} = \int_{-\infty}^{+\infty} \left(x(t)\, e^{-\sigma t}\right) \cdot e^{-j\omega t} dt \quad \text{--- (3)}$$

$$\therefore X(s)\Big|_{s=\sigma+j\omega} = CTFT\{x(t) \cdot e^{-\sigma t}\}$$

if $\sigma = 0$

$$X(s)\Big|_{s=j\omega} = X(j\omega) = CTFT \quad [\text{Chapter 4}] \quad (5)$$

Tutorials on LT: →

Problem: – Find the Laplace transform of

$$x(t) = e^{-at} \cdot u(t) \;;\; a > 0$$

Sol^n: → $X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$

$\therefore X(s) = \int_{-\infty}^{+\infty} \left[e^{-at} \cdot u(t) \right] \cdot e^{-st}\, dt.$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$= \int_{0}^{+\infty} e^{-(a+s)t}\, dt$$

$$\boxed{X(s) = \frac{1}{a+s}} = \frac{1}{a+\sigma+j\omega}$$

$$F\{e^{-at} \cdot u(t)\} = \frac{1}{a+j\omega}$$

$a > 0$

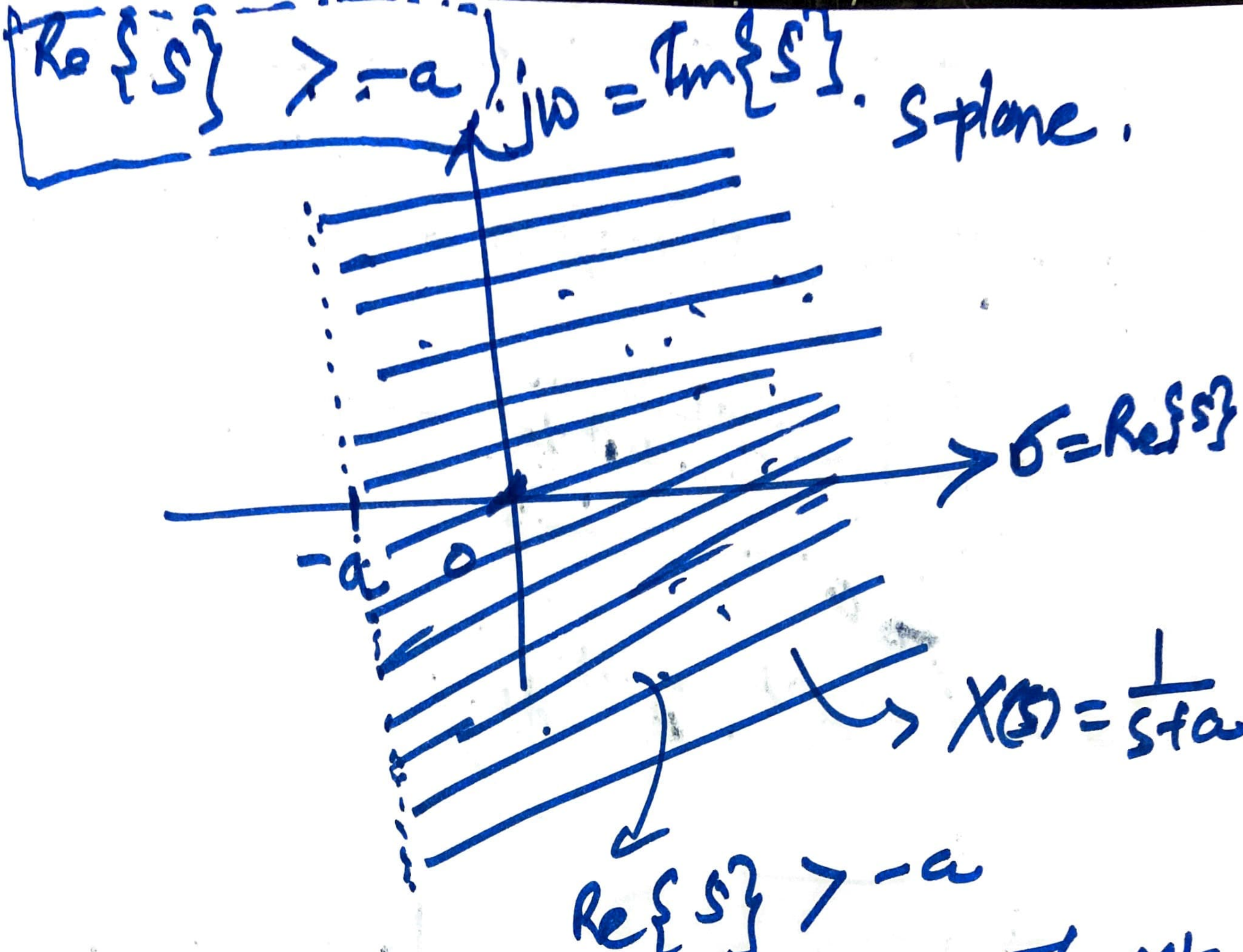
$s = \sigma + j\omega$

$\sigma = Re\{s\}$

$a + \sigma > 0$

$\therefore \sigma > -a$

$$\boxed{Re\{s\} > -a}$$



Region of Convergence (ROC) : → The set of points 's' for which Laplace transform exists, i.e., $X(s)$ converges.

Inference :

Problem ② Find LT of $x(t) = -e^{-at} \cdot u(-t)$

Soln : $LT\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st}\, dt$

$$X(s) = \int_{-\infty}^{+\infty} \left[-e^{-at} \cdot u(-t) \right] e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-at} \cdot e^{-st}\, dt$$

$$\boxed{X(s) = \frac{1}{s+a}} \rightarrow \text{ROC} :\ \boxed{Re\{s\} < -a}$$

↳ algebraic expression

Inference: → Two distinct / different signals can have exactly identical algebraic expression in Laplace-domain, however, they can be distinguished through ROC.

⟹ Specification of ROC is important in Laplace-domain

(3) Find the LT of

$x(t) = 3\, e^{-2t} \cdot u(t) - 2\, e^{-t} \cdot u(t).$

$$X(s) = 3\left[\frac{1}{s+2}\right] - 2\left[\frac{1}{s+1}\right]$$

ROC $= R_1 = \mathrm{Re}\{s\} > -2$

ROC $= R_2$: $\mathrm{Re}\{s\} > -1$

ROC $(R_1 \cap R_2)$

$$\boxed{X(s) = \frac{s-1}{(s+1)(s+2)}}$$

$R_1 \cap R_2 = \mathrm{Re}\{s\} > -1$

$$X(s) = \frac{N(s)}{D(s)} = \frac{\text{Numerator polynomial in } s}{\text{Denominator polynomial in } s}$$

→ Characterised by roots of $N(s)$ & $D(s)$

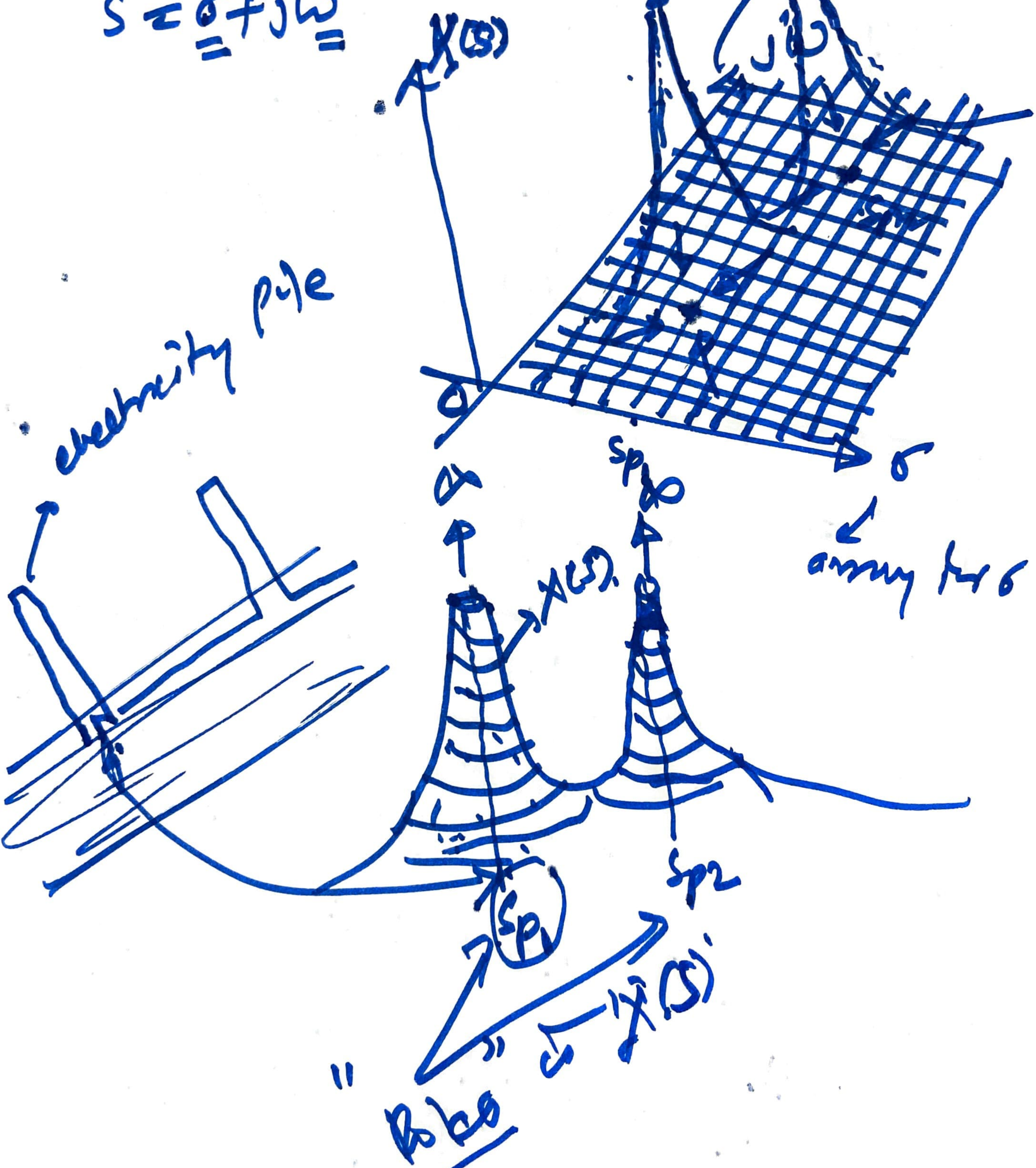
$X(s)\big|_{s=1} = 0 \Rightarrow s = 1$ is called zero of $X(s)$.

$X(s)\big|_{s=-1 \text{ or } -2} = \infty \Rightarrow s = -1,\ s = -2$ are called as poles of $X(s)$.

Logic behind terminology i Poles?
(Why to call poles???)

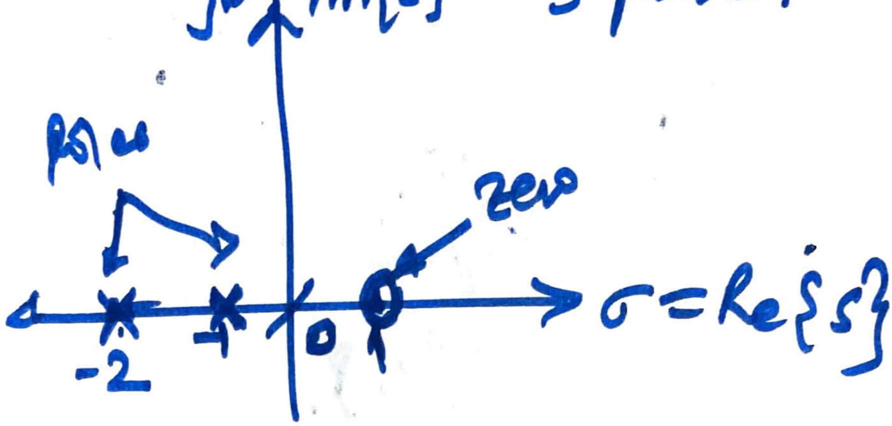
→ We visualise X(s) in 3-D

$s = \underline{\sigma} + j\underline{\omega}$



$$X(s) = \frac{(s-1)}{(s+1)\cdot(s+2)}$$

s-plane.



O → Zeros of X(s)

X → poles of X(s).

'X' is used to denote pole location in 's' plane because at poles $X(s) = \infty$ (undefined)!

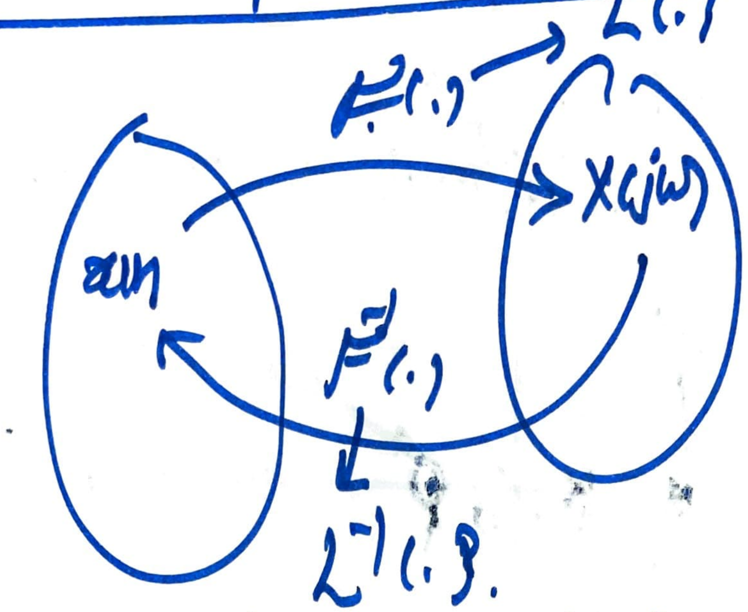
$\boxed{\text{Key Property of ROC}}$

ROC = X(s) converges (exists / finite) at each point of s in ROC.

Poles ⇒ X(s) is undefined, i.e., $X(s) = \infty$

For rational Laplace transform, the ROC does not contain any poles.

* Inverse Laplace Transform



Q. Whether $L^{-1}\{.\}$ for $X(s)$. exists??

Ans:

$$CTFT\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t}\, dt = X(j\omega)$$

$$x(t) = \left(\frac{1}{2\pi}\right) \cdot \int_{-\infty}^{+\infty} X(j\omega) \cdot e^{j\omega t}\, d\omega .$$

$$\therefore\ X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st}\, dt$$

$s = \sigma + j\omega$

$$\therefore\ X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-(\sigma + j\omega)t}\, dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} \left(x(t) e^{-\sigma t}\right) \cdot e^{-j\omega t}\, dt$$

$$= CTFT\{\underline{\underline{x(t) e^{-\sigma t}}}\}$$

$\therefore\ x(t) \cdot e^{-\sigma t} = F^{-1}\{X(\sigma + j\omega)\}$

$$x(t) \cdot e^{-\sigma t} = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=+\infty} X(\sigma + j\omega)\, e^{j\omega t}\, d\omega \quad [A]$$

Method of substitution, let $\underline{\sigma + j\omega = s}$

$\therefore\ \omega = -\infty \Rightarrow s = \sigma - j\infty$

$\therefore\ \omega = +\infty \Rightarrow s = \sigma + j\infty$

$\sigma + j\omega = ds$

$$\frac{ds}{d\omega} = \frac{d}{d\omega}[\sigma + j\omega] = j$$

$$\therefore\ \boxed{d\omega = ds/j}$$

multiply eqn [A] by $e^{\sigma t}$

$$\therefore\ x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} \underline{X(\sigma + j\omega)} \cdot \underline{\underline{e^{\sigma t} \cdot e^{j\omega t}}}\, d\omega$$

$$= \frac{1}{2\pi} \int_{s=\sigma-j\infty}^{s=\sigma+j\infty} X(s)\, e^{st} \times \frac{ds}{j}$$

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\infty}^{s=\sigma+j\infty} \underline{X(s)} \cdot e^{st} \cdot ds}$$

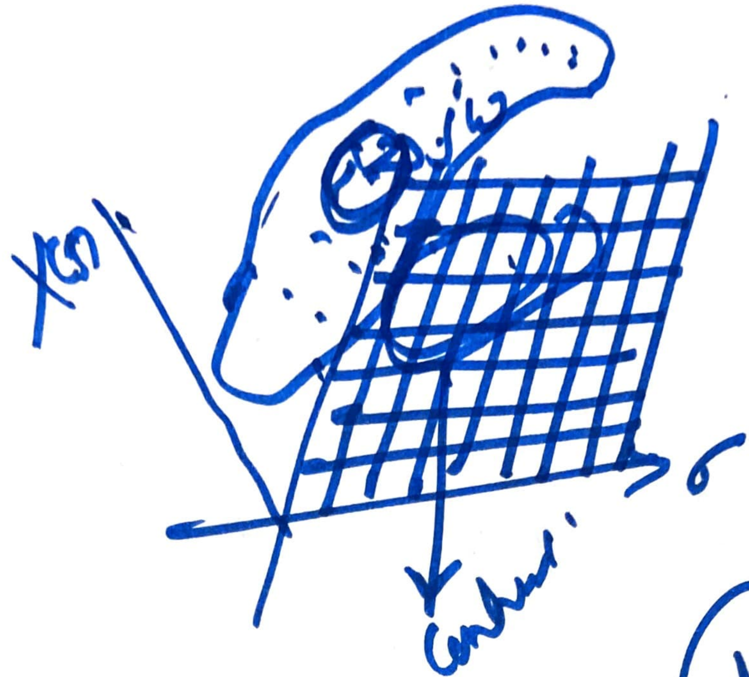
∵ S = complex variable (Not real)
so formal evaluation of above integral requires <u>contour integration</u>. However, for LTI system, $X(s)$ is in general rational, so we can use Partial Fraction Expansion & No need of formal <u>contour integration</u>

$$A = \int_{x=a}^{x=b} f(x)\,dx$$

there $x \in$ Real line

∴ The integral, $\int_{x=a}^{x=b} f(x)\,dx$ is called line integral.

Since s is a complex variable $s = \sigma + j\omega$
thus, we need to a grasp of 'points of s'



$$\int X(s)\, ds$$

Contour Integration

$s \in \mathbb{C}$

Technical : Find $\mathcal{L}\{x(t) = u(t)\}$

$$\Rightarrow \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st}\, dt$$

$$\therefore \mathcal{L}\{u(t)\} = \int_{-\infty}^{+\infty} 1 \cdot \underline{u(t)}\, e^{-st}\, dt$$

$$= \int_{0}^{+\infty} 1 \cdot e^{-st}\, dt = \frac{1}{s}.$$

$$\boxed{L\{u(t)\} = \frac{1}{s}}$$

② Find $L\{\delta(t)\} = ?$

$$\text{sol}^n \quad L\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)\, e^{-st}\, dt$$

$$L\{\delta(t)\} = 1$$

③ Find $L\{\cos(\omega t).u(t)\} =$

$$\text{let } x(t) = \cos(\omega t)\, u(t)$$

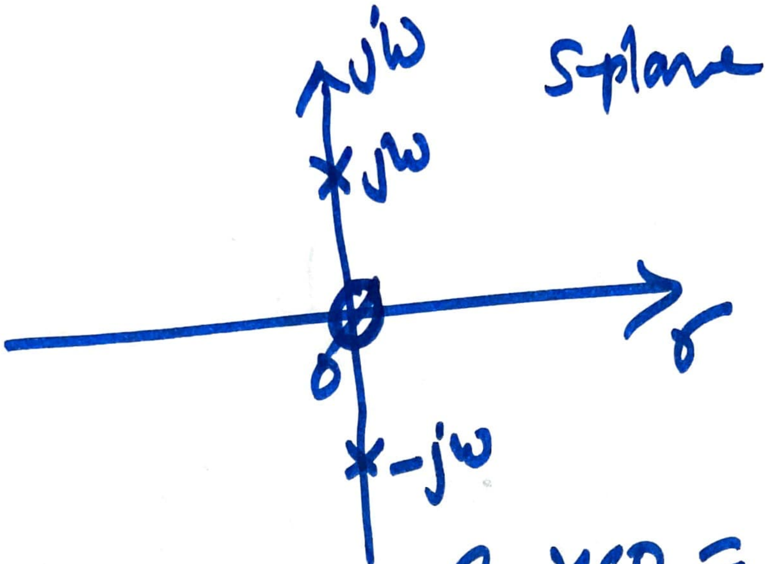
$$= \left[\frac{e^{+j\omega t} + e^{-j\omega t}}{2}\right] . u(t)$$

$$L\{x(t)\} = \frac{1}{2} L\{e^{+j\omega t} u(t)\} + \frac{1}{2} L\{e^{-j\omega t} u(t)\}$$

$$X(s) = \left(\frac{1}{2}\right) \cdot \left(\frac{1}{-j\omega + s}\right) + \left(\frac{1}{2}\right) \frac{1}{s + j\omega}$$

$$X(s) = \frac{s}{s^2 + \omega^2} = \frac{N(s)}{D(s)} = \frac{(s-0)}{(s-j\omega)(s+j\omega)}.$$

⑥



Pole-zero plot for $X(s) = \frac{s}{s^2+\omega^2}$

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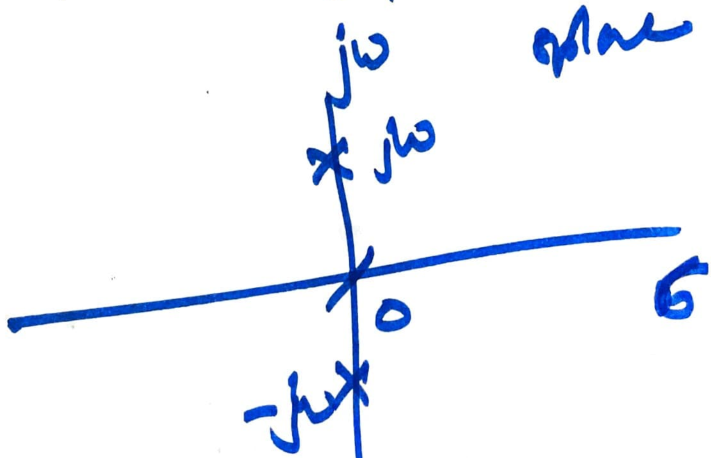
7) Find $\mathcal{L}\{\sin(\omega t)u(t)\}$ & also plot pole-zero plot

Soln: $X(s) = \mathcal{L}\{\sin(\omega t)u(t)\}$

$$X(s) = \frac{\omega}{s^2+\omega^2}$$ has zero $s=\infty$



Pole-zero plot

* Properties of Laplace Transform

[1] Linearity

$$x(t) \xleftrightarrow{L} X(s)$$

ROC : $R_1 \cap R_2$

$$a_1 x_1(t) + a_2(t) x_2(t) \xleftrightarrow{L} a_1 X_1(s) + a_2 X_2(s)$$

$$\sum_{\langle k \rangle} a_k x_k(t) \xleftrightarrow{L} \sum_{\langle k \rangle} a_k X_k(s)$$

[2] Time Shifting.

$$x(t) \xleftrightarrow{L} X(s)$$

$$x(\underline{t - t_0}) \xleftrightarrow{L} e^{-st_0} . X(s).$$

[3] Shifting in s-domain

$$x(t) \xleftrightarrow{L} X(s)$$

$$e^{s_0 t} . x(t) \longleftrightarrow X(s - s_0)$$

Tutorial $\mathcal{L}\{ e^{-at} \cos(\omega t) . u(t) \} = ??$

Here, let $x(t) = \cos(\omega t) u(t)$.

& $s_0 = -a$

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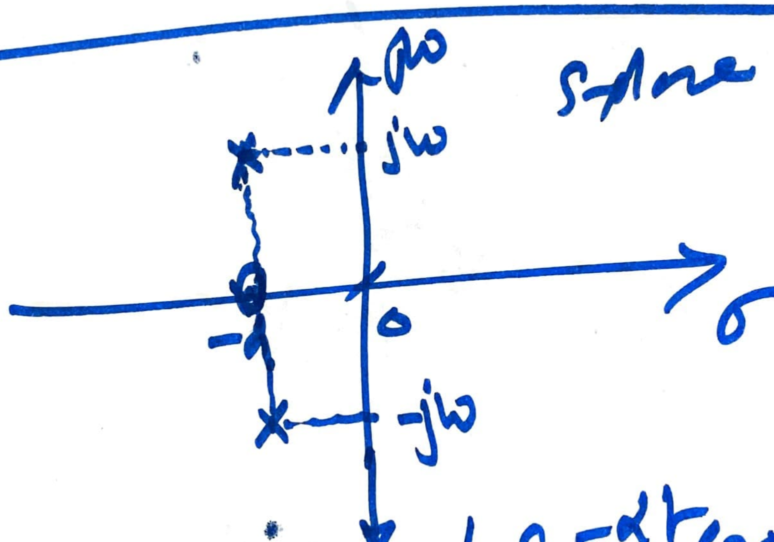
$$\therefore X(s) = \frac{s}{s^2+\omega^2}$$

Using "Shifting" in s-domain:

$$\therefore \mathcal{L}\{e^{-\alpha t}\cdot x(t)\} = X(s+\alpha)$$

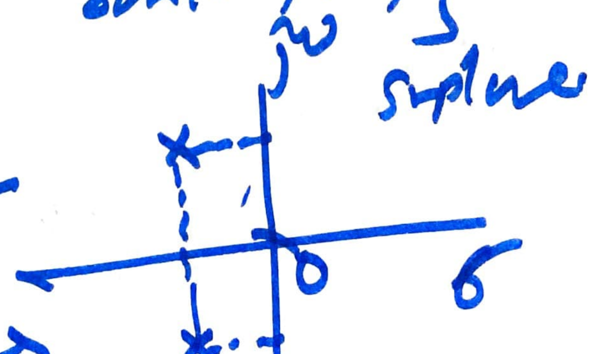
$$\boxed{\mathcal{L}\{e^{-\alpha t}\cos(\omega t)u(t)\} = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}}$$



Pole-zero plot of $\mathcal{L}\{e^{-\alpha t}\cos(\omega t)u(t)\}$

Problem: Find $\mathcal{L}\{e^{-\alpha t}\sin(\omega t)u(t)\}$

$$= \frac{\omega}{(s+\alpha)^2+\omega^2}$$



* Differentiation in time-domain

$$x(t) \xleftrightarrow{L} X(s)$$

$$\frac{d}{dt}[x(t)] \xleftrightarrow{L} s\,X(s)$$

Proof: $x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\omega} X(s)\, e^{st}\, ds$

Differentiate both sides w.r.t. t

$$\therefore \frac{d}{dt}[x(t)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, \underbrace{\frac{d}{dt}[e^{st}]}\, ds$$

$$\frac{d}{dt}[x(t)] = \left(\frac{1}{2\pi j}\right) \int_{\sigma-j\infty}^{\sigma+j\infty} [s\,X(s)]\, e^{st}\, ds$$

$$\Rightarrow L\left\{\frac{d}{dt}x(t)\right\} = s\,X(s)$$

$$\therefore L\left\{\frac{d^2}{dt^2}x(t)\right\} = s^2 X(s)$$

$$\vdots \qquad \qquad \vdots$$

$$\boxed{L\left\{\frac{d^n}{dt^n}x(t)\right\} = s^n X(s)}$$