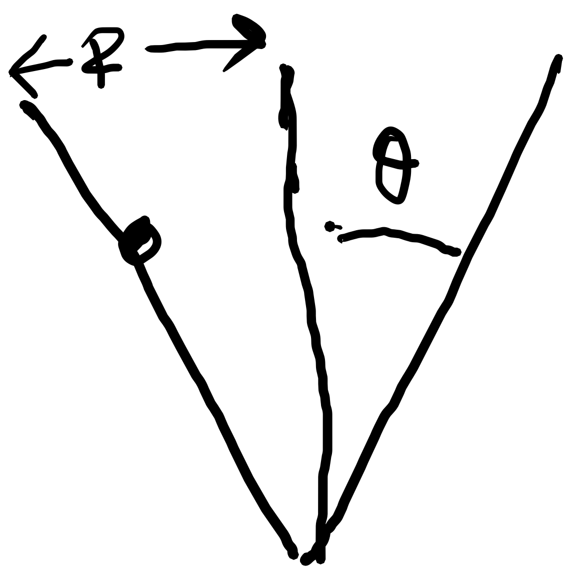
LECTURE 2

RECAP

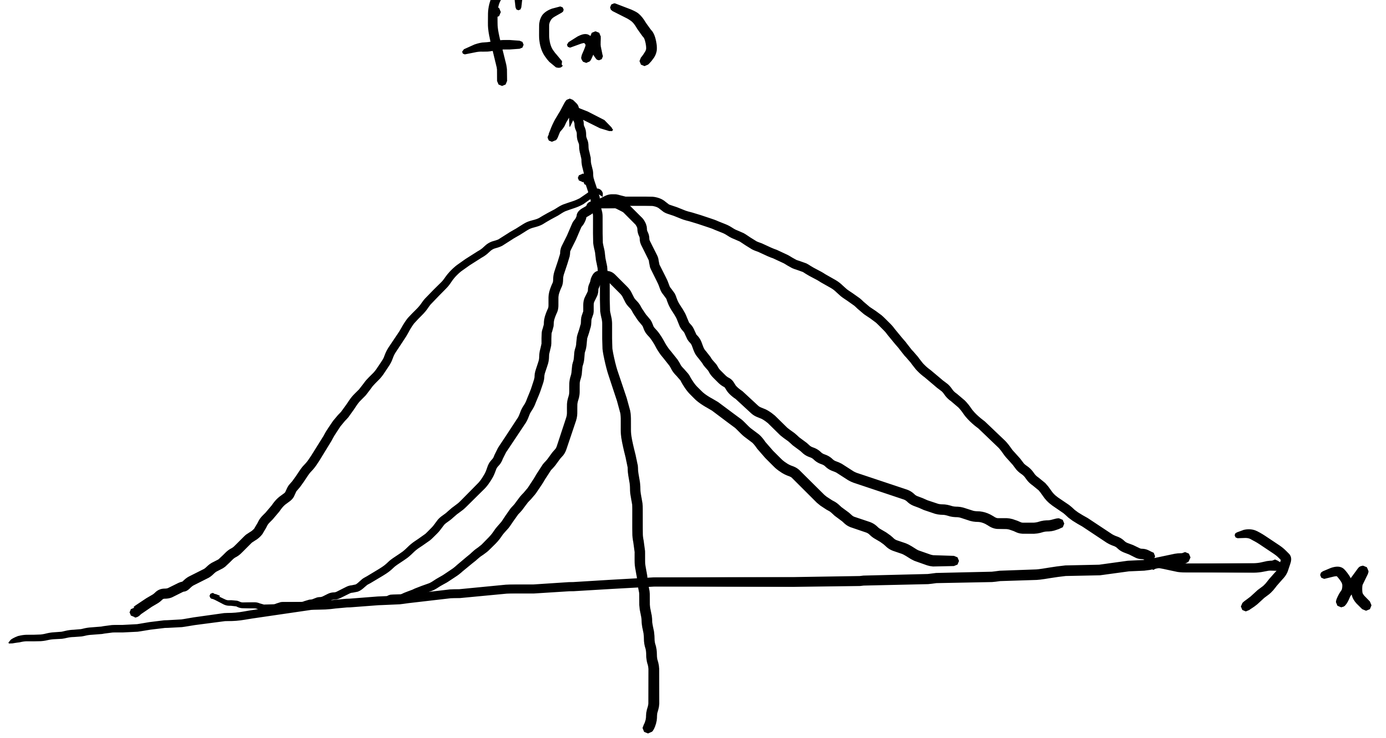
- Dimensional analysis.
  - how to derive qualitative formula from dimensional considerations.



- derived a formula for the time period.

Prescription :- Take a minimal set of parameters which should dictate the physics of the model and match dimension

$$t = f(g, R, \theta, m)$$
$$= h(\theta)\sqrt{\frac{R}{g}}$$



- Limiting cases:

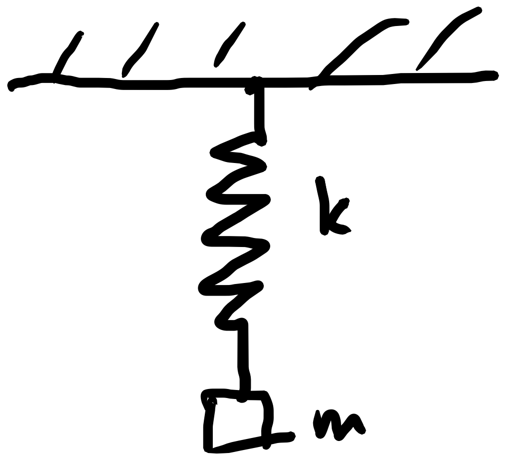
$$I(\alpha) = \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2} \propto \frac{1}{\sqrt{\alpha}}$$

Take the following limits: $\alpha \to 0 \Rightarrow I \to \infty$

$\alpha \to \infty \Rightarrow I \to 0$

☑ COMBINING DIMENSIONAL ANALYSIS AND LIMITING CASES.

- **Example:**

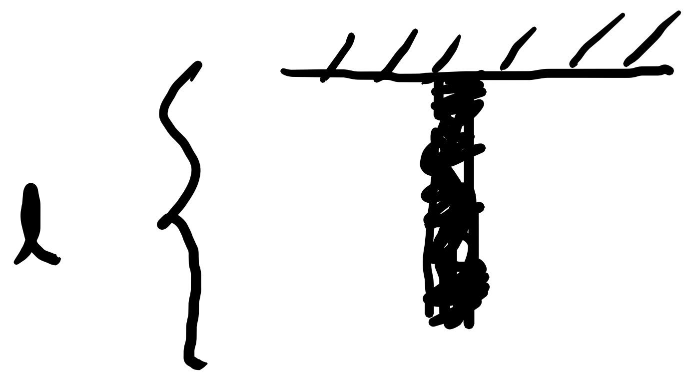


$$F = -kx.$$

$$[k] = \frac{[F]}{[x]}$$

idealised spring → no mass of its own.

- **Example:** Consider a spring / cable / rope with a non-zero mass of its own.



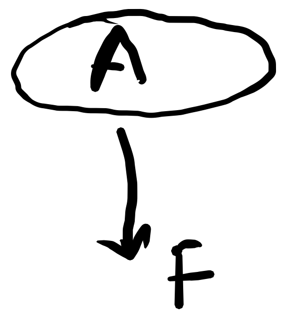
l = unstretched length of the cable

s = stretch.

λ = mass density of the cable.

K = Bulk modulus defined as $\Delta p = -K\left(\frac{\Delta V}{V}\right)$

Derive a formula for s as a fraction of l.



- Enumerate the variables/parameters of interest.

$$s = F(\lambda, g, K, l)$$

$$\Rightarrow s = l \, f(\lambda, g, K, l)$$

s	L
λ	ML^{-1}
l	L
g	LT^{-2}
K	$ML^{-1}T^{-2}$

$$[\lambda^{\alpha} g^{\beta} l^{\gamma} K^{\delta}]$$

$$= M^{\alpha} L^{-\alpha} L^{\beta} T^{-2\beta} L^{\gamma} M^{\delta} L^{-\delta} T^{-2\delta}$$

$$= M^{\alpha+\delta} L^{-\alpha+\beta+\gamma-\delta} T^{-2\beta-2\delta}$$

– Enumerate the variables/parameters of interest.

$$s = F(\lambda, g, K, \ell)$$

$$\Rightarrow \; s = \ell \, f(\lambda, g, K, \ell)$$

s	L
λ	ML^{-1}
ℓ	L
g	LT^{-2}
K	$ML^{-1}T^{-2}$

$$[\lambda^{\alpha} g^{\beta} \ell^{\gamma} K^{\delta}]$$

$$= M^{\alpha} L^{-\alpha} L^{\beta} T^{-2\beta} L^{\gamma} M^{\delta} L^{-\delta} T^{-2\delta}$$

$$= M^{\alpha+\delta} L^{-\alpha+\beta+\gamma-\delta} T^{-2\beta-2\delta}$$

$$\left.\begin{aligned} \alpha + \delta &= 0 \\ -\alpha + \beta + \gamma - \delta &= 0 \\ -2\beta - 2\delta &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha = 1, \quad \beta = 1 \\ \gamma = -1, \quad \delta = -1 \end{aligned}$$

$\lambda^\alpha g^\beta l^\gamma K^\delta$

$$s = l\, f\left(\frac{\lambda g}{l K}\right)$$

The form of f is unknown. $f \equiv f(x)$

$f(x)$ possible forms :- as $x \gg 1$ and $x \ll 1$. :- (i) f approaches a constant.

(ii) $f \sim C x^n$, $n > 1$

(iii) $f \sim \log x$, exp.

(i) + (ii) $\Rightarrow$ $f(x) = C x^n$, $n \geq 0$

$$\left.\begin{aligned} \alpha + \delta &= 0 \\ -\alpha + \beta + \gamma - \delta &= 0 \\ -2\beta - 2\delta &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &\alpha = 1, \quad \beta = 1 \\ &\gamma = -1, \quad \delta = -1 \end{aligned}$$

$\lambda^{\alpha} g^{\beta} l^{\gamma} K^{\delta}$

$$s = l\, f\left(\frac{\lambda g}{l K}\right)$$

The form of f is unknown. $f \equiv f(x)$

$f(x)$ possible forms:- as $x \gg 1$ and $x \ll 1$. :- (i) f approaches a constant.

(ii) $f \sim C x^n$, $n > 1$

(iii) $f \sim \log x$, exp.

(i) + (ii) $\Rightarrow$ $f(x) = C x^n$, $n \geq 0$

- External information/assumption is necessary.

  $f(x) = e^{x}$.

- A priori, there is no reason to believe that a small change in any one parameter is going to lead to a large change in s.

$$s \sim l\left(\frac{\lambda g}{lk}\right)^{n}, \quad n \geq 0$$

So, $n = 0$ is ruled out.

$$\boxed{0 < n < 1}$$

$$s \sim l\left(\frac{\lambda g}{lk}\right)^{n}, \text{ where } \boxed{0 < n < 1}.$$

VII APPROXIMATION OF FUNCTIONS

- **Motivation** :- Dynamical system $\equiv$ systems which changes with time.

  - changes are represented by derivatives. $\frac{dx}{dt}$, $\frac{dv}{dt}$ and so on.
    x and v are variables which specify the behaviour of the system.

  - Usually, thes are obtained by solving ordinary diff. eqns ODE's or PDE's

(i) $\dfrac{dy}{dx} = f(x)$ $\longrightarrow$ 1st order ODE

(ii) $\dfrac{d^2y}{dx^2} = f(x)$ $\longrightarrow$ 2nd order ODE

$$y(x) = \int dx\, f(x)$$

$$y_1 = \frac{dy}{dx} \rightarrow y(x) = \int dx\; y_1(x)$$

$$f(x) = \frac{dy_1}{dx} \longrightarrow y_1(x) = \int dx\, f(x)$$

- Finding the solⁿ depends on the evaluation of

$$\int dx\, f(x)$$

$$f(x) = \frac{e^x}{\cos x + x^5}$$

- For difficult integrals, it is convenient to have an approximation for $f(x)$.

- Taylor polynomial $p_n(x)$

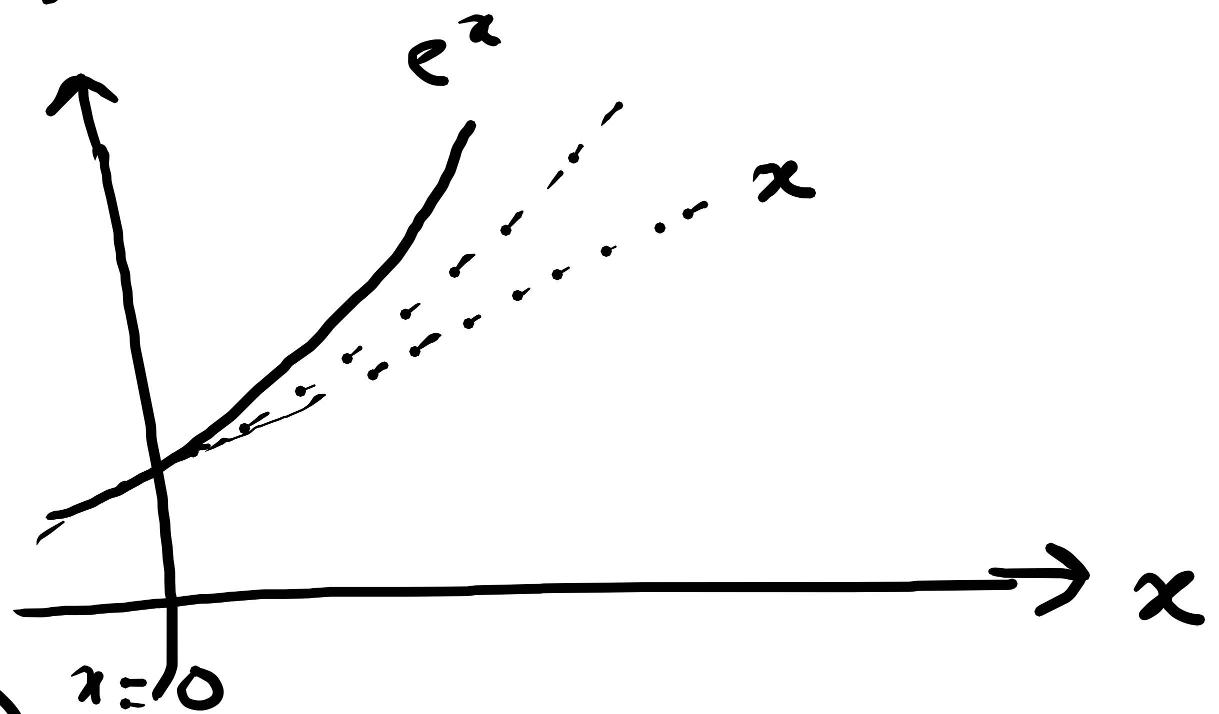
<u>Remarks</u>:- Approximations generally only useful in the neighbourhood of a specific point.

- $f(x) = e^x$.

Let's try to construct an approximation to $f(x) = e^x$ which is linear in x.

$p_n(x) \to$ Taylor polynomial of degree n.

$$\left.\begin{aligned} p_1(a) &= f(a) \\ p_2'(a) &= f'(a) \end{aligned}\right\} \Rightarrow p_1(x) = f(a) + (x-a) f'(a).$$



- $p_2(x) = b_0 + b_1 x + b_2 x^2$

$p_2(a) = f(a)$, $p_2'(a) = f'(a)$, $p_2''(a) = f''(a)$

- For practical purposes, we have to terminate the Taylor series at some chosen point.

- Systematic procedure to terminate Taylor series is by calculating :

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c), \qquad a<c<x.$$