SC223 - Linear Algebra

Aditya Tatu

Lecture 35

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Applications

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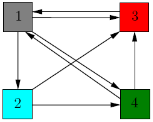

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

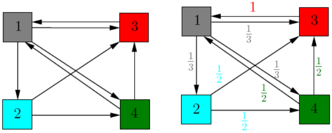

Figure: Source: `pi.math.cornell.edu`

$$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} x_1^0 \\ \vdots \\ x_4^0 \end{pmatrix}$$

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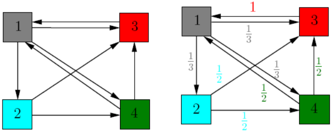

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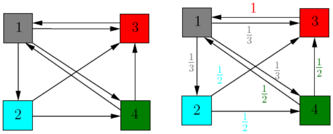

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- Let $x^0 \in \mathbb{R}^4$ denote initial estimate of importance, or probability that a person might visit these webpages.
- With the above model, after once click:

$$x^1 = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} x^0$$

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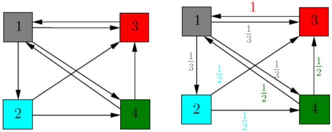
f is C

$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$



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$A(\lim_{n\to\infty} A^n x_0) = Ay$

$= \lim_{n\to\infty} A^{n+1} x_0 = y$

$\Rightarrow Ay = y$

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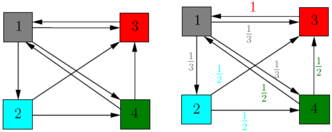

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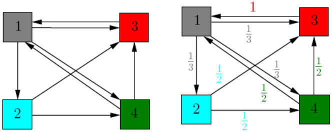

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- ► $\lim_{n\to\infty} A^n x_0 = y, Ay = y, y = [0.38, 0.12, 0.29, 0.19]^T$.

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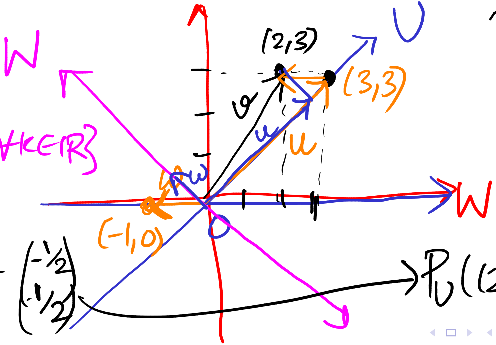
Example:

① $V = \mathbb{R}^2$, $U = \{k\begin{pmatrix}1\\1\end{pmatrix}, \forall k \in \mathbb{R}\}$. $U \oplus W = \mathbb{R}^2$

$W = \{k\begin{pmatrix}1\\0\end{pmatrix}, \forall k \in \mathbb{R}\}$

$P_U((2,3)) = (3,3)$

$W = \{k\begin{pmatrix}-1\\1\end{pmatrix}, \forall k \in \mathbb{R}\}$



$\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}5/2\\5/2\end{pmatrix} + \begin{pmatrix}-1/2\\1/2\end{pmatrix}$ $\rightarrow P_U((2,3)) = \begin{pmatrix}5/2\\5/2\end{pmatrix}$

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- Properties:

① $N(P_U) = W$, ② $R(P_U) = U$

$$V = R(P_U) \oplus N(P_U)$$

② Let $v \in V$, $v = u + w$, $u \in U$, $w \in W$.

$\forall v \in V$, $\quad P_U(v) = u$

$$P_U(P_U(v)) = P_U(u) = u$$

$$\Rightarrow \quad P_U^2 = P_U \Rightarrow \underline{\text{Idempotent op.}}$$

(3) Any Idempotent op. is diagonalizable

(4) Eigenvalues are 0 & 1.

(5) Let $T \in \mathcal{L}(V)$ such that $T^2 = T$.

then T is a projection. (H.W)

——— END of Class ———

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● Is this the only way to define length?

● What are the necessary conditions for a function on vector space for it be called *length*?

Normed Vector space

- **Definition:** (Normed Vector Space) A normed vector space (NVS) is a vector space $(V, +, \cdot)$ over either $\mathbb{R}$ or $\mathbb{C}$ with a **norm**, a function $|| \cdot || : V \to \mathbb{R}$ which satisfies the following properties:

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● Also note that given a NVS $(V, ||\cdot||)$, we can define distance between two vectors x and y as $d(x, y) := ||x - y||$. Such a distance or metric is called the **induced metric**.

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- L_2 norm on $\mathcal{P}_n([-1,1])$: $||x||_{L_2} = \sqrt{\int_{-1}^{1} (x(t))^2 \; dt}$.