SC223 - Linear Algebra
Autumn 2023

Tutorial 7 | Basis and Dimension

1. Find the dimension of the vector spaces/subspaces U given below:
   (a) Let $A = xy^T, x, y \in \mathbb{R}^n$. $U = N(A)$.
   (b) Let $V = \{(x_i)_{i=0}^{\infty} \mid x_i \in \mathbb{R}\}$, i.e., the set of all real-valued sequence beginning at index 0. $U = \{(x_i)_{i=0}^{\infty} \in V \mid x_0 = a, x_1 = b, x_n = x_{n-1} + x_{n-2}, n \geq 2\}$.
   (c) $V = \mathbb{R}^{n \times n}, U = \{A \in V \mid A^T = A\}$.
   (d) Let $V = span(\{1, \sin t, \cos t\}), U = \{f \in V \mid \frac{d^2}{dt^2} f(t) + f(t) = 0, \forall t \in \mathbb{R}\}$.

2. Show that the vector space of real-valued continuous functions on $[0, 1] \subset \mathbb{R}$ is infinite dimensional.

3. Let $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 4}$. Find the basis and dimensions of $C(A), C(A^T), N(A), N(A^T)$.

4. Consider the finite field F of integers modulo 7, with operations of addition modulo 7 and multiplication modulo 7. We define the list of all sequences over F with addition of two sequence being term by term addition in the field, and multiplication by a scalar being term by term multiplication by that scalar.
   (a) Show that the set of such sequences form a vector space.
   (b) Is this vector space finite dimensional?
   (c) Consider the subset of all strings of the type
   $$s_{init}.s_{repeat}^{\omega}$$
   Here $|S_{repeat}| > 0$. These are called rational strings. Do the set of all rational strings form a subspace?
   (d) Give a non-trivial (not only the identity element) example of a finite dimensional subspace of this vector space.

5. If U and W are subspaces of a 6 dimensional vector space V, what can be said about the dimension of $U \cap W$.

6. Let $A \in \mathbb{R}^{100\times 10}, B \in \mathbb{R}^{10\times 100}$ be matrices such that $rank(A) = 10, rank(B) = 5$. Find $rank(AB)$.