DA-IICT Gandhinagar
SC224: Tutorial Sheet-1

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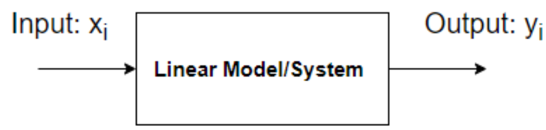
1. For the Linear model as shown in the Fig. 1



x_i's and y_i's are the input and observed output respectively and $\xi_i \sim \mathcal{N}(0, \sigma^2)$ denotes the random error where

$$y_i = (\beta_0 + \beta_1 x_i) + \xi_i; \qquad i = 1, 2, \cdots, n \tag{1}$$

Here, β_0 and β_1 represent the unknown parameters of the model. Meanwhile, $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ and $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$ are the LSE estimators of β_0 and β_1 respectively where $S_{xy} = \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})$ and $S_{xx} = \sum_{i=1}^{N} (x_i - \bar{x})^2$

(a) Obtain the covariance $Cov(\hat{\beta}_0, \hat{\beta}_1)$ between the estimators $\hat{\beta}_0$ and $\hat{\beta}_1$.

(b) Given that the input is $x_i = \frac{i-1}{n-1}$, show that the asymptotic correlation between $\hat{\beta}_0$ and $\hat{\beta}_1$ is $-\frac{\sqrt{3}}{2}$, i.e.,

$$\lim_{n \to \infty} corr(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\sqrt{3}}{2}$$

where $corr(\hat{\beta}_0, \hat{\beta}_1) = \frac{Cov(\hat{\beta}_0, \hat{\beta}_1)}{\sqrt{Var(\hat{\beta}_0) \cdot var(\hat{\beta}_1)}}$

(c) Given that the fitted Model is $\hat{y}_i = (\hat{\beta}_0 + \hat{\beta}_1 x_i)$ and residual error is $r_i = y_i - \hat{y}_i$. Here, $SS_{res} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$ is the squared sum of the residuals. Derive the statistics of the residual r_i, i.e., expected value $E\{r_i\}$ and variance $Var\{r_i\}$.

(d) The Linear model in Eq. (1) is modified to model a particular experiment/scenario where the observed output values are zero when there is no input. Obtain the estimator of the unknown parameter of the model using the principles of LSE.

(e) Obtain the statistics of the estimator of the modified model, i.e., obtain the expected value and the variance of the estimator.

(f) Illustrate whether the obtained estimator is a biased or unbiased estimator of the unknown parameter.