SC224: Tutorial Sheet 2

Problems based on Random Variables

Pb 1) Consider a random experiment of two independent tosses of a coin so that the sample space is $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$. Let $P(.)$ be the probability function defined on Ω such that $P(\mathrm{HH}) = p^2$, $P(\mathrm{HT}) = P(\mathrm{TH}) = p(1-p)$ and $P(\mathrm{TT}) = (1-p)^2$, where $p \in (0,1)$. Define the function $X : \Omega \to \mathbb{R}$ by $X(\mathrm{HH}) = 2$, $X(\mathrm{HT}) = X(\mathrm{TH}) = 1$ and $X(\mathrm{TT}) = 0$, i.e., $X(w)$ denotes the number of heads in w. What is the type of r.v. X? Write the distribution function of X and discuss its nature using its plot?

Pb 2) Let F_X be the distribution function of X. Then, establish the following results:

(i) $P(\{X = x\}) = F_X(x) - F_X(x-)$.
(ii) $P(\{X \geq a\}) = 1 - F_X(a-)$.
(iii) $P(\{X > a\}) = 1 - F_X(a)$.
(iv) $P(\{a < X \leq b\}) = F_X(b) - F_X(a)$.
(v) $P(\{a \leq X \leq b\}) = F_X(b) - F_X(a-)$.
(vi) $P(\{a < X < b\}) = F_X(b-) - F_X(a)$.

Can we conclude that the d.f. F_X is continuous (discontinuous) at a point $x \in \mathbb{R}$ if and only if $P(\{X = x\}) = 0 (P(\{X = x\}) > 0)$?

Pb 3) In each of the following cases determine whether or not $F_X : \mathbb{R} \to \mathbb{R}$ is a distribution function of some r.v..

(i) $F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1, & \text{if } x > \frac{1}{2}. \end{cases}$
(ii) $F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \geq 0. \end{cases}$

Pb 4) Let X be a r.v. and F_X be the d.f. of X given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{x+2}{4}, & \text{if } -1 \leq x < 1 \\ 1, & \text{if } x \geq 1. \end{cases}$$

Sketch the graph of F_X and compute the probabilities $P\left(\left\{-\frac{1}{2} < X \leq \frac{1}{2}\right\}\right)$, $P(\{X = 0\})$, $P(\{-1 \leq X < 1\})$ and $P(\{-1 < X < 1\})$.

Pb 5) Let $F(.)$ and $G(.)$ be two distribution functions. Verify whether or not the following functions are distribution functions:

(i) $H(x) = F(x) + G(x)$.
(ii) $H(x) = \max\{F(x), G(x)\}$.
(iii) $H(x) = \min\{F(x), G(x)\}$.

Pb 6) Do the following functions define probability mass functions/probability density functions of some random variables of discrete/continuous type?

(a) $p_X(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \{-1, 0, 1, 2\} \\ 0, & \text{otherwise.} \end{cases}$ (b) $p_X(x) = \begin{cases} \frac{e^{-1}}{x!}, & \text{if } x \in \{0, 1, 2, ...\} \\ 0, & \text{otherwise.} \end{cases}$

(c) $f_X(x) = \begin{cases} \frac{(x^2+1)e^{-x}}{2}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$ (d) $f_X(x) = \begin{cases} \frac{2+\cos x}{2\pi}, & \text{if } 0 < x < \pi \\ 0, & \text{otherwise.} \end{cases}$

Pb 7) In each of the following, find the value of constant c so that $p_X(.)$(or $f_X(.)$) is a p.m.f.(or p.d.f.) of some r.v. (say X).

(a) $p_X(x) = \begin{cases} c(1-p)^x, & \text{if } x \in \{1, 2, 3, ...\} \\ 0, & \text{otherwise} \end{cases}$ (b) $p_X(x) = \begin{cases} \frac{c\lambda^x}{x!}, & \text{if } x \in \{1, 2, ...\} \\ 0, & \text{otherwise} \end{cases}$

(c) $f_X(x) = \begin{cases} cxe^{-x^2}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$ (d) $f_X(x) = \begin{cases} cxe^{-(x-2)}, & \text{if } x > 2 \\ 0, & \text{otherwise} \end{cases}$

where $p \in (0, 1)$ and $\lambda > 0$ are fixed constants. Also, for each of the above, find $P(\{X > 3\})$, $P(\{X \leq 3\})$, $P(\{3 < X < 4\})$ and $P(\{1 < X < 2\})$.

Problems based on Jointly Distributed Random Variables, Expectation, Uniform Random Variables, and Geometric Random Variables.

Pb 1) We say that X is a **geometric r.v.**, if it has the p.m.f.

$$p_X(k) = \begin{cases} p\,(1-p)^{k-1}, & \text{if } k \in \{1, 2, ...\} \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < p \leq 1$ and we write symbolically $X \sim \text{Geo}(p)$. Geometric r.v. implies that the occurrence of the first success requires k independent trials, each with success probability p. If a r.v. $X \sim \text{Geo}\left(\frac{1}{3}\right)$, then find the distribution function of $Y = \frac{X}{X+1}$ and hence determine its p.m.f.

Pb 2) Let X be a r.v. with following p.d.f., i.e., $X \sim \text{U}[0, 1]$,

$$f_X(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the p.d.f.s of the following random variables: (i) $Y_1 = \sqrt{X}$; (ii) $Y_2 = X^2$; (iii) $Y_3 = 2X + 3$; (iv) $Y_4 = -\ln X$; (v) $Y_5 = \sin X$; (vi) $Y_6 = 1/X$.

Pb 3) Let X be a r.v. with p.m.f.

$$p_X(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in \{1, 2, ..., n\} \\ 0, & \text{otherwise} \end{cases}$$

where $n(\geq 2)$ is an integer. Find the mean of X.

Pb 4) In three independent tosses of a fair coin, let X denotes the number of tails appearing. Let $Y = X^2$ and $Z = 2X^2 + 1$. Find the means of random variables Y and Z.

Pb 5) A communication system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function.

(a) For what values of p, a 5-component system more likely to operate effectively than a 3-component system?

(b) In general, when is a $2k+1-$compnent system better than a $2k-1-$component system?

Pb 6) Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

(a) less than 5 minutes for a bus.

(b) at least 12 minutes for a bus.

Pb 7) The joint p.d.f. of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{12}{5}x(2-x-y), & \text{if } 0<x<1, 0<y<1, \\ 0 & \text{otherwise} \end{cases}$$

Compute the density of X/Y and of $X+Y$.

Pb 8) A secretary has typed N letters along with their respective envelopes. The envelops get mixed up when they fall on the floor. If the letters are placed in the mixed-up envelops in a completely random manner (that is, each letter is equally likely to end up in any of the envelopes), what is the expected number of letters that are placed in the correct envelopes?

Pb 9) The joint PMF of a discrete random vector (X_1, X_2) is given by the following table

$x_2 \setminus x_1$	-1	0	1
0	1/9	2/9	1/9
1	1/9	2/9	1/9
2	0	1/9	0

a) Find the expectation of the random variables X_1, X_2, X_1+X_2 and X_1X_2.

b) Are X_1 and X_2 independent random variables? Justify your answer.

Pb 10) Consider the problem above. Define a new random vector

$$Y = (Y_1, Y_2) = g(X_1, X_2) = (X_1 + X_2, X_1).$$

Find the joint p.m.f. of random vector Y.

Pb 11) Let $X \sim \mathrm{U}[0,1]$ and $Y \sim \mathrm{U}[0,1]$ be two random variables. Define, $Z = X + Y$ and $W = X - Y$. Find the joint p.d.f. of Z and W. Suppose that X and Y are independent. What can we say about Z and W?