SC223 - Linear Algebra

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Lecture 15

August 29, 2023

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$.
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$.
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.
- **Closure with respect to Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$.
- **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.
- **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.
- **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

Field

- **Definition:**(Field). A field is a set $\mathbb{F}$ with two binary operations, addition $+_F$ and multiplication $\times$ that satisfy the following axioms:
  - $(\mathbb{F}, +_F)$ is an **Abelian group**. The additive identity will be denoted by 0.
  - $(\mathbb{F} - \{0\}, \times)$ is an **Abelian group**. The mutiplicative identity will be denoted by 1.
  - **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c$.

Vector Space

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Vector Space

- If the 3-tuple $(V, +, \cdot)$ with field $(\mathbb{F}, +_F, \times)$ satisfies all vector space axioms, we say that $(V, +, \cdot)$ **forms a vector space over** $\mathbb{F}$.
- Any element of the vector space $(V, +, \cdot)$ will be referred to as a **vector**, and any element $a \in \mathbb{F}$ will be referred to as a **scalar**.

Examples of Vector spaces

- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{C}^n, +, \cdot)$ over $\mathbb{C}$.
- $(\mathbb{F}^n, +, \cdot)$ over $\mathbb{F}$.
- $(\mathbb{R}^\infty, +, \cdot)$ over $\mathbb{R}$, where $\mathbb{R}^\infty$ is the set of all doubly-infinite sequences.
- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.
- $(\mathbb{L}_2(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathbb{L}_2(\mathbb{R})$ denotes the set of all square-integrable functions $f : \mathbb{R} \to \mathbb{R}$.

$$V = \mathbb{L}_2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{-\infty}^{\infty} f(t)^2 \, dt < \infty \right\}.$$

$$\mathbb{F} = \mathbb{R}$$

① $\forall f, g \in \mathbb{L}_2(\mathbb{R})$, $(f + g)(t) := f(t) + g(t)$, $\forall t \in \mathbb{R}$

② $\forall a \in \mathbb{R}$, $\forall f \in \mathbb{L}_2(\mathbb{R})$, $(a \cdot f)(t) := a \times f(t)$, $\forall t \in \mathbb{R}$.

Axioms:

1. Closure of $\mathbb{L}_2(\mathbb{R})$ w.r.t $+$.

$\forall f, g \in \mathbb{L}_2(\mathbb{R})$, $\underline{f+g \in \mathbb{L}_2(\mathbb{R})}$.

Show: $\int_{-\infty}^{\infty} (f+g)(t)^2 \, dt < \infty$.

$$= \int_{-\infty}^{\infty} (f(t) + g(t))^2 \, dt = \underbrace{\int_{-\infty}^{\infty} f(t)^2 \, dt}_{<\infty} + \underbrace{\int_{-\infty}^{\infty} g(t)^2 \, dt}_{<\infty} + 2 \underline{\int_{-\infty}^{\infty} f(t) g(t) \, dt}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f(x)g(y) - f(y)g(x))^2 \, dx \, dy \geq 0$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\underline{f(x)^2 g(y)^2} + f(y)^2 g(x)^2 - 2 f(x) f(y) g(x) g(y) \right) dx \, dy.$$

$$= 2\left(\int_{-\infty}^{\infty} f(x)^2\,dx\right)\left(\int_{-\infty}^{\infty} g(y)^2\,dy\right) - 2\left(\int_{-\infty}^{\infty} f(x)\,g(x)\,dx\right)\left(\int_{-\infty}^{\infty} f(y)\,g(y)\,dy\right)$$

$$\left[\left(\int_{-\infty}^{\infty} f(x)^2\,dx\right)\left(\int_{-\infty}^{\infty} g(x)^2\,dx\right) - \underline{\underline{\left(\int_{-\infty}^{\infty} f(x)\,g(x)\,dx\right)^2}}\right] \geq 0$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,g(x)\,dx \leq \sqrt{\int_{-\infty}^{\infty} f(x)^2\,dx}\sqrt{\int_{-\infty}^{\infty} g(x)^2\,dx}}$$

CAUCHY-SCHWARTZ INEQUALITY.

$$\int_{-\infty}^{\infty} (f+g)(t)^2\,dt \leq \int_{-\infty}^{\infty} f(t)^2\,dt + \int_{-\infty}^{\infty} g(t)^2\,dt + 2\sqrt{\int_{-\infty}^{\infty} f(t)^2\,dt}\sqrt{\int_{-\infty}^{\infty} g(t)^2\,dt}$$

$$< \infty.$$

$\therefore \forall f, g \in \mathbb{L}_2(\mathbb{R}),\quad f+g \in \mathbb{L}_2(\mathbb{R})$

(2) Identity w.r.t +.

$\theta \in \mathbb{L}_2(\mathbb{R})$, s.t $f + \theta = \theta + f = f$. $\forall f \in \mathbb{L}_2(\mathbb{R})$

$$(f+\theta)(t) = f(t) + \theta(t) = f(t), \ \forall t \in \mathbb{R}.$$

$$\Rightarrow \theta(t) = 0, \ \forall t \in \mathbb{R}.$$

$$\int_{-\infty}^{\infty} \theta(t)^2 \, dt = \int_{-\infty}^{\infty} 0 \, dt = 0 < \infty$$

③ <u>Inverse w.r.t +</u>

Let $f \in \mathbb{L}_2(\mathbb{R})$, find $g \in \mathbb{L}_2(\mathbb{R})$

s.t $\quad f+g = g+f = \theta$

$$\Rightarrow (f+g)(t) = \theta(t) = 0, \quad \forall t \in \mathbb{R}.$$

$$f(t) + g(t) = 0, \ \forall t \in \mathbb{R}$$

$$\Rightarrow g(t) = -f(t), \ \forall t \in \mathbb{R}.$$

Such an element $g \in \mathbb{L}_2(\mathbb{R})$ will be denoted by $-f$.

<u>Scalar Multiplication Identity</u>

$\forall f \in \mathbb{L}_2(\mathbb{R})$, $(1 \cdot f)(t) = 1 \times f(t), \ \forall t \in \mathbb{R}$

$$= f(t), \ \forall t \in \mathbb{R}.$$

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- **Proposition 1:** Every Vector space has a unique additive identity.
- **Proposition 2:** Every vector in a vector space has a unique additive inverse.
- **Proposition 3:** $\forall v \in V, 0 \cdot v = \theta$
- **Proposition 4:** $\forall a \in \mathbb{F}, a \cdot \theta = \theta$.
- **Proposition 5:** $\forall v \in V, (-1) \cdot v = -v$.