Lecture 29

Problem: Find the DTFS (Discrete-time Fourier series) representation for $x(n) = \sin(\omega_0 n)$

$x(t) = \sin(\omega_0 t)$??

Sol^n :→

$$a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{1}{2j}.$$

$N = 5$ period of discrete-time signal, $x(n)$.

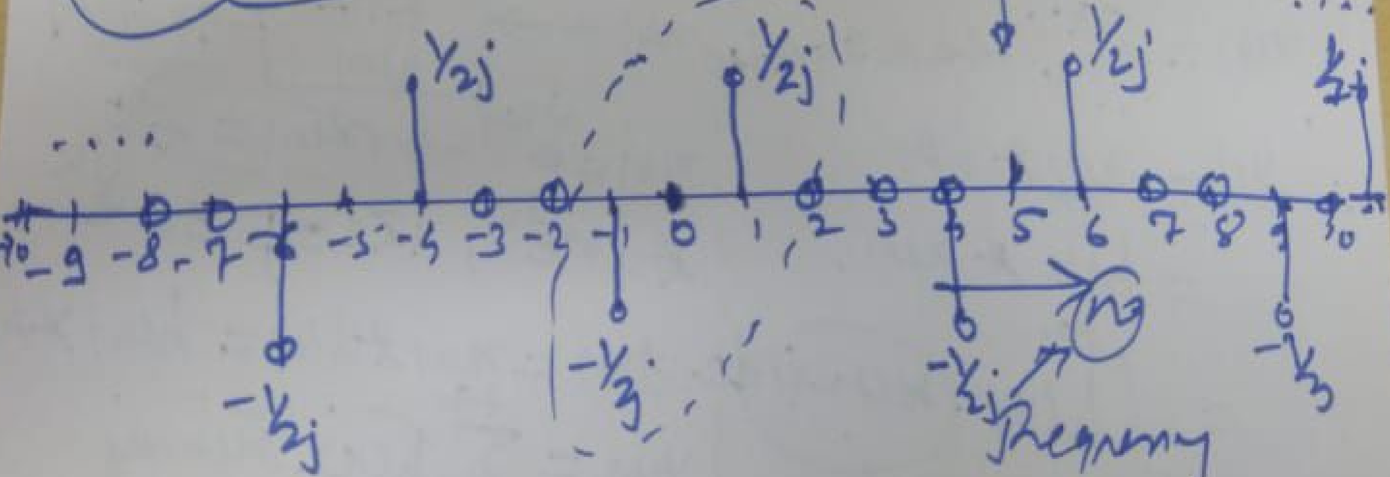

Fig. Fourier series coefficients (DTFS) for $x(n) = \sin(2\pi/5) \cdot n = \sin(\omega_0 n)$

$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{5}$$

Homework: 3.11, pp. 216–217
3.12 : pp. 218 ;

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Properties of DTFS → Homework.

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* Fourier Series and LTI Systems :→

Generalization of $x(t) = e^{j\omega t}$ or $x(n) = e^{j\omega n}$

$x(t) = e^{j\omega t}$

$\hookrightarrow x(t) = e^{st}$

$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$

A) CTS : LTI system

$x(t) = e^{st}$, where $s = \sigma + j\omega$

$\therefore y(t) = x(t) * h(t)$

$= h(t) * x(t).$

$$= \int_{\tau=-\infty}^{+\infty} h(\tau) \cdot x(t-\tau) \, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) \cdot e^{s(t-\tau)} \, d\tau$$

o/p

$$\boxed{y(t) = \underbrace{H(s)}_{\text{system function in } s\text{-domain}} \cdot e^{st}} \rightarrow \text{eigenfunction}$$

where $H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau = L\{h(t)\}$ ← Laplace Transform

(2)

B) DTS : LTI

$x(n) \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n)$

$x(n) = e^{j\omega n} \rightarrow x(n) = z^n$

$z = re^{j\omega}, \ |z| = 1 \quad z = e^{j\omega}$

$y(n) = x(n) * h(n) = h(n) * x(n)$

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k) \cdot x(n-k)$$

$$y(n) = \sum_{k=-\infty} h(k) \cdot z^{n-k}$$

o/p

$$y(n) = \underbrace{H(z)}_{\text{system function in } z\text{-domain}} \cdot z^n \rightarrow \text{eigenfunction}$$

where $H(z) = \sum_{k=-\infty}^{+\infty} h(k) \cdot z^{-k}$

$Z\{h(n)\} = Z$-transform of $h(n)$

Eigenfunction

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \rightarrow \boxed{\text{LTI System } h(t) \text{ LTI filter}} \rightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$$

Periodic signal

Linear combination (LC) of complex Exponentials

a_k ⇒ coeff. of LC

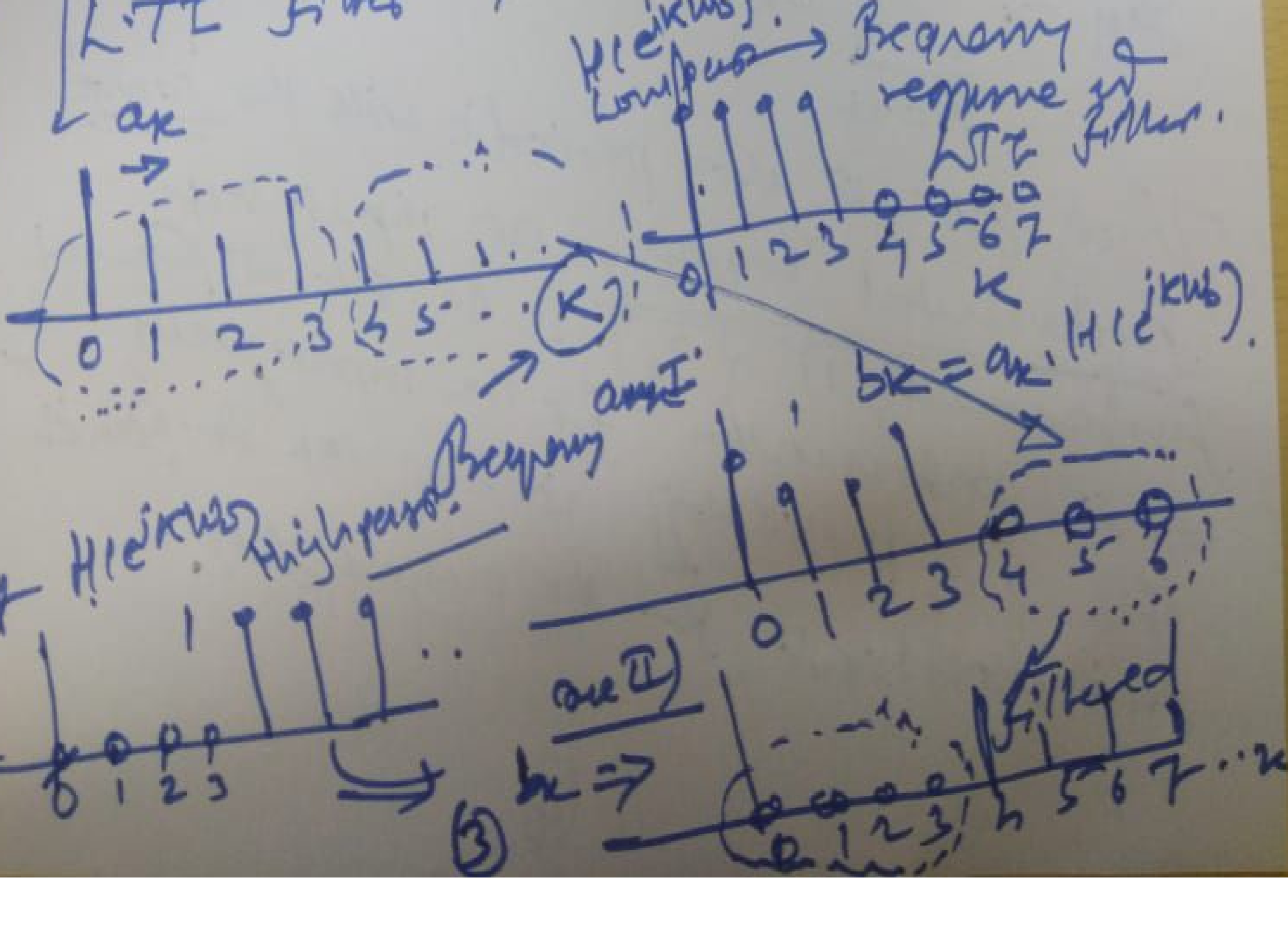
→ Fourier series coefficients

LC of complex Expon.

Fourier series coeff. of $y(t)$

$$b_k = a_k H(e^{jk\omega_0})$$

LTI Filter → $h(t)$ → $H(e^{jk\omega_0})$



<u>Inference</u>: → Let $x(t)$ be a periodic signal with a Fourier series representation given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Suppose that we apply this signal as the input to an LTI system with impulse response $h(t)$. Then, using eigenfunction theory for LTI operator, the output response of an LTI operator will be given by

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}.$$

Thus, $y(t)$ is also periodic with the <u>same</u> fundamental frequency as input, $x(t)$.

Furthermore, if $\{a_k\}$ is the set of Fourier series coefficients for the input, $x(t)$, then $\{a_k H(e^{jk\omega_0})\}$ is the set of Fourier series coefficients for the output, $y(t)$.

Thus, the effect of the LTI system is to modify individually each of the Fourier series coefficients of the input through multiplication by value of the frequency response at the corresponding frequency

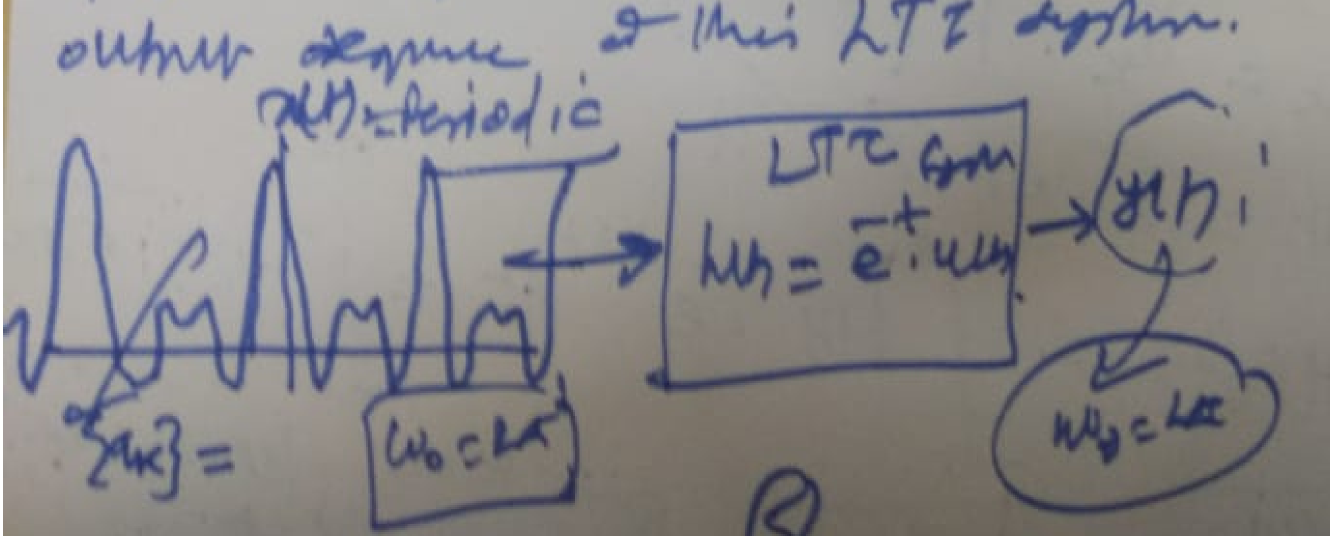
Most important!!..

Tutorial Problem :→

Suppose that the periodic signal $x(t)$ shown below is given as input to an LTI system with impulse response

$$h(t) = e^{-t}.u(t)$$

Find the Fourier series coefficients of output response of this LTI system.



We know that the Fourier series coefficients of given 'periodic' input signal (See solved problems in class)

$\{a_k\}$ ...

$a_0 = 1,$

$a_1 = a_{-1} = \frac{1}{4}$

$a_2 = a_{-2} = \frac{1}{2}$

$a_3 = a_{-3} = \frac{1}{3}$

...

Using the eigenfunction and LTI system,

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot H(e^{jk\omega_0}) \, e^{jk\omega_0 t}$$

∴ The Fourier series coefficients for the output signal i.e., b_k are given by

$$b_k = a_k \cdot H(e^{jk\omega_0})$$

??

$h(t) = e^{-t} \cdot u(t)$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{+\infty} e^{-t} u(t) e^{-j\omega t} dt$$

$$\boxed{H(j\omega) = \frac{1}{1+j\omega}} \rightarrow (5)$$

$\omega = k\omega_0$, let $\omega_0 = 2\pi$ ...

$$H(e^{jk\omega_0}) = \frac{1}{1+jk2\pi}.$$

$$b_k = a_k \cdot H(e^{jk\omega_0}) = (a_k) \cdot \frac{1}{1+jk2\pi}.$$

$$b_0 = a_0 \cdot \frac{1}{1+j0} = a_0 = 1$$

$$b_1 = (a_1) \times \frac{1}{1+j\cdot 1\, 2\pi} = \left(\frac{1}{4}\right) \times \frac{1}{1+j2\pi}$$

$b_{-1} = ?$

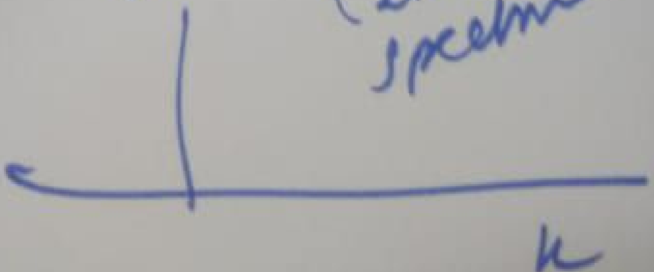
$b_2 = ?$

$b_{-2} = ?$

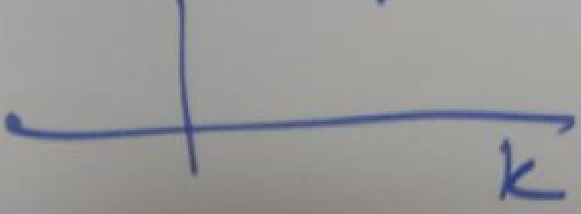
$b_3 = ?$

$b_{-3} = ?$

$|b_k|$ Magnitude (line) spectrum



$\measuredangle b_k$ phase spectrum.



Case II) for DTS: LTI

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(n)\end{matrix}} \longrightarrow y(n)$$

Periodic DTFS

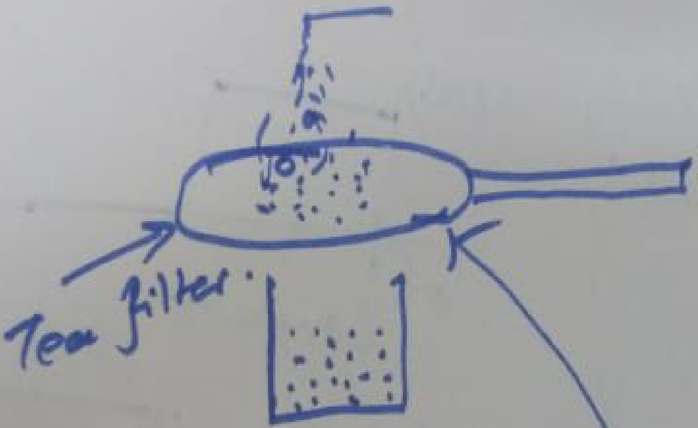
$$y(n) = \sum_{k=\langle N\rangle} \underbrace{a_k H(e^{j(2\pi/N)k})}_{b_k} e^{jk(2\pi/N)n}$$

Exam 3.17 → pp. 230 – 231

* LTI filtering :

Filtering :→

Example : Tea filter.



⇒ Tea particles are removed ⇒ filtered out.

Similarly

$$x(n) \longrightarrow \boxed{\text{LTI filter}} \longrightarrow y(n)$$

($y(n)$: output signal)

LTI filter will either change / modify OR remove certain frequency components in the input signal

LTI filtering

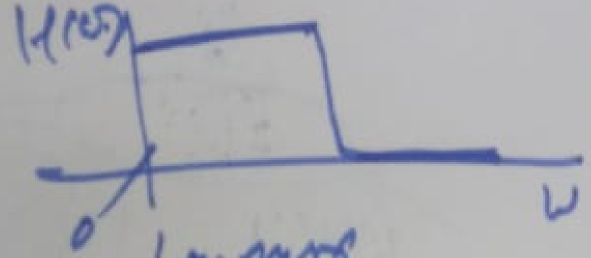
Frequency Shaping filters

e.g. {bass, treble} → audio

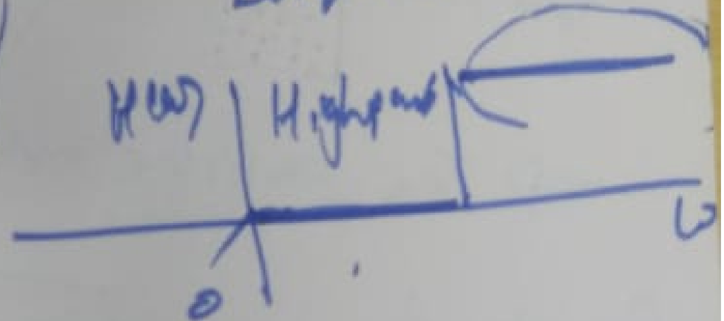
Lowpass Highpass

→ equalization filter

Frequency "selective" filters



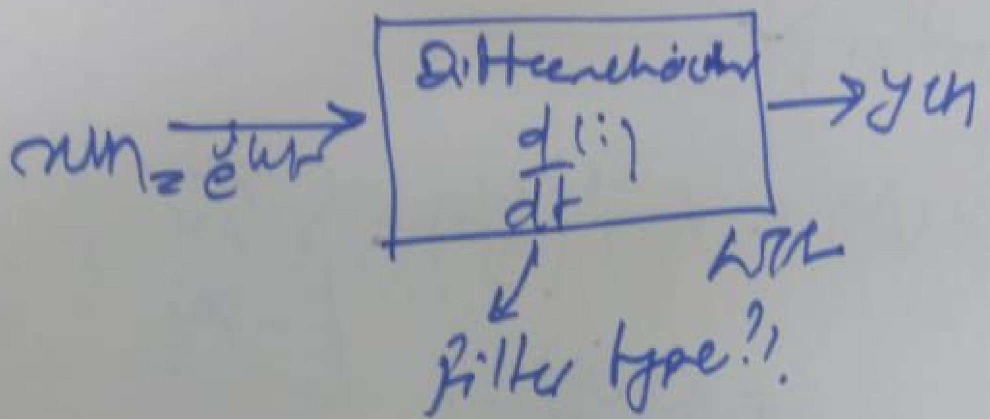
Lowpass



Problems on Frequency ~~Selective~~ Shaping Filters

$\boxed{\text{Problem 1}}$ Find the frequency response of differentiator system. What kind of filter it is??

Soln:



$$y(t) = \frac{d}{dt}[x(t)]$$

Let input $x(t) = e^{j\omega t}$ [eigenfunction]

$$\therefore \quad y(t) = \frac{d}{dt}\left[e^{j\omega t}\right] = (j\omega)\cdot e^{j\omega t} \quad \text{—— (1)}$$

$\therefore$ Using eigenfunction theory for LTI system,

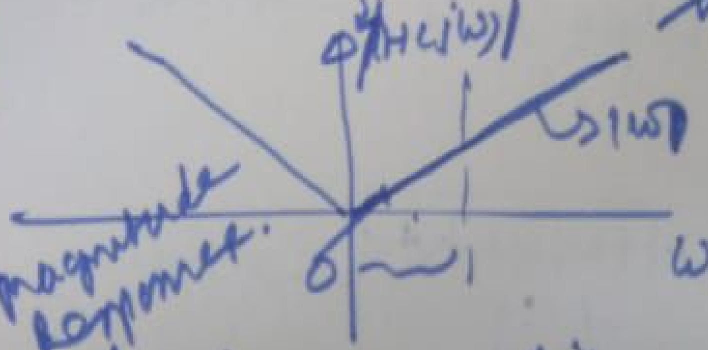
$$y(t) = H(j\omega)\cdot e^{j\omega t} \quad \text{—— (2)}$$

$$\therefore \quad \boxed{H(j\omega) = j\omega} \rightarrow \text{frequency response of given system}$$

$$|H(j\omega)| = |j\omega| = |j|\,|\omega|$$

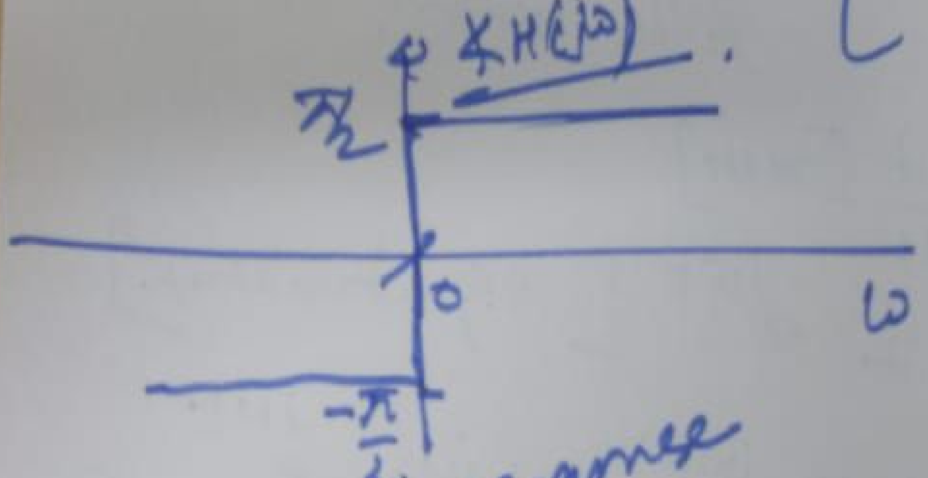
Highpass.

$$H(j\omega) = j\omega$$



magnitude response.

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = \begin{cases} \pi/2, & \omega > 0 \\ -\pi/2, & \omega < 0 \end{cases}$$



Phase response

Inference for $\omega \ll \Rightarrow |H(j\omega)| \downarrow$

for $\omega \gg$ $|H(j\omega)| \uparrow$

$\Rightarrow H(j\omega)$ is a highpass filter.

Let $x(n) = e^{j\omega n}$

$\therefore y(n) = x(n) - x(n-1)$

$$= e^{j\omega n} - e^{j\omega(n-1)}$$

$$= \left[1 - e^{-j\omega}\right] \cdot e^{j\omega n}$$

$$= e^{-j\omega/2} \cdot 2\left[\frac{e^{j\omega/2} - e^{-j\omega/2}}{2}\right] e^{j\omega n}$$

$$y(n) = \left[2\sin(\omega/2)\, e^{-j\omega/2}\right] e^{j\omega n} \quad \text{— (1)}$$

Using eigenfunction theory for LTI system

$$y(n) = H(e^{j\omega}) \cdot e^{j\omega n} \quad \text{— (2)}$$

$$H(e^{j\omega}) = 2\sin(\omega/2)\, e^{-j\omega/2}$$

$$\therefore \left|H(e^{j\omega})\right| = \left|2\sin(\omega/2)\right| \cdot \underbrace{\left|e^{-j\omega/2}\right|}_{1}$$

(3)

② Integrator

$$x(t) \longrightarrow \boxed{\int .\, dt} \longrightarrow y(t).$$

$$H(j\omega) = \frac{1}{j\omega}$$

$$|H(j\omega)| = \frac{1}{|j\omega|} = \frac{1}{|\omega|}$$

lowpass.

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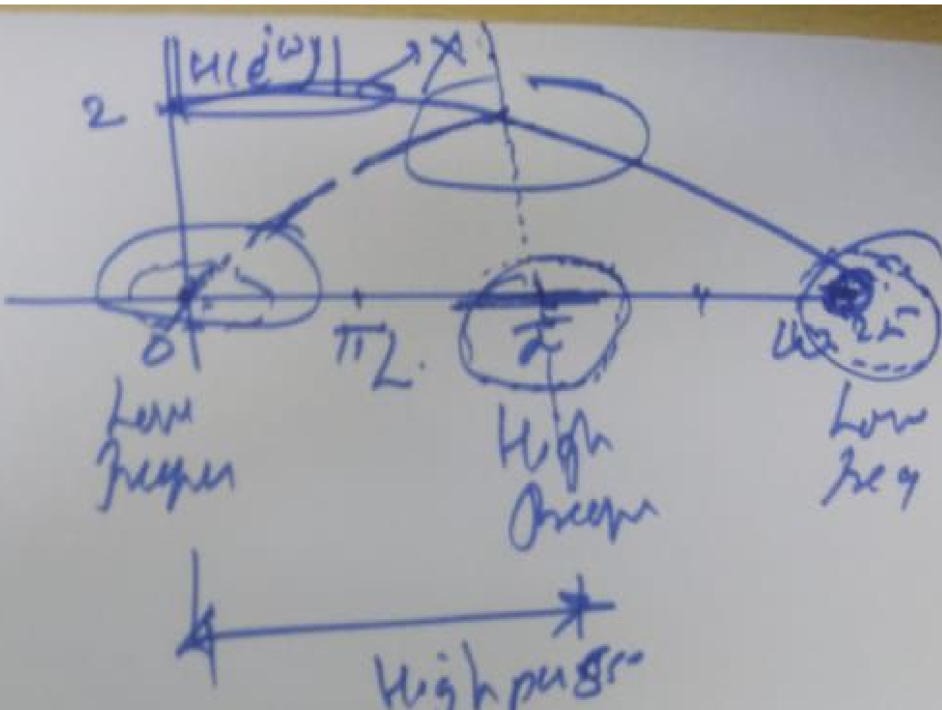
③ Discrete-time version of differentiator

$$y[n] = \frac{d}{dt}[x[n]] \longrightarrow y[n] = x[n] - x[n-1]$$

find the frequency response & filter type

$$x[n] \longrightarrow \boxed{\text{Finite long difference filter}} \longrightarrow y[n] = x[n] - x[n-1]$$

$$H(e^{j\omega}) = ??$$



$\therefore$ Finite difference system is a crude version of highpass filter.

$\boxed{\text{Problem 4}}$ Two-point moving average filter

$$y(n) = \frac{1}{2}\left[x(n) + x(n-1)\right]$$

$$H(e^{j\omega}) = \cos(\omega/2) \cdot e^{-j\omega/2}$$

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$$|H(e^{j\omega})| = |\cos(\omega/2)|$$

crude versions of lowpass filter

$\Rightarrow$ Sines and cosines in frequency domain are highpass and lowpass filters, respectively.