Computational Numerical Methods

CS 374

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Eg Find the root of $x^4 - x - 10 = 0$ using F.P iteration

① $x = \dfrac{10}{x^3 - 1}$ $\qquad x_0 = 2$

② $x = (x+10)^{1/4}$ $\qquad x_0 = 1, 4.$

③ $x = (x+10)^{1/2}/x.$ $\qquad x_0 = 1.8.$

④ $x = x^4 - 10.$ $\qquad x_0 = 1$

(2) $x_0 = 1$

$i = 5$

$x_n = 1.85558$

$x_0 = 4$

$x_n = 1.85555$

(1) $x_5 = -9.97 \times 10^{-3}$

$x_6 = -10$

$x_7 = -9.99 \times 10^{-3}$

$x_8 = -10$

③

$x_0 = 1.8$

$x_5 = 1.89355$

$x_6 = 1.82129$

$\vdots$

$x_{98} = 1.85555$

④

$$\alpha - x_{n+1} = -\frac{f''(\xi)}{2f'(\xi)}(\alpha - x_n)(\alpha - x_{n-1})$$

Error in Secant method

Let $\varepsilon_n = x_n - \alpha$. abs error $\varepsilon_n = |x_n - \alpha|$

$$\varepsilon_{n+1} = x_{n+1} - \alpha.$$

$$= x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} - \alpha.$$

$$= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} - \alpha.$$

$$= \frac{(\alpha+\bar{\varepsilon}_{n-1}) f(x_n) - (\alpha+\bar{\varepsilon}_n) f(x_{n-1})}{f(x_n) - f(x_{n-1})} - \alpha.$$

$$= \frac{\bar{\varepsilon}_{n-1} f(x_n) - \bar{\varepsilon}_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Using Taylor's expansion.

$$f(x_n) = f(\alpha) + f'(\alpha)\bar{\varepsilon}_n + \frac{1}{2} f''(\xi)\bar{\varepsilon}_n^2$$

ξ lies b/w x_n & α.

$$= f'(\alpha)\bar{\varepsilon}_n + \frac{1}{2} f''(\underline{\xi})\bar{\varepsilon}_n^2$$

Using MVT.

$$f(x_n) - f(x_{n-1}) = f'(\xi)(x_n - x_{n-1})$$

where ξ lies b/w x_{n-1} & x_n.

Note that $\xi \approx \xi \approx \alpha$. as α lies in the interval x_n & x_{n+1}.

$$f(x_n) \approx f'(\alpha)\,\bar{\varepsilon}_n + \frac{1}{2} f''(\alpha)\,\bar{\varepsilon}_n^2$$

$$f(x_{n-1}) \approx f'(\alpha)\,\bar{\varepsilon}_{n-1} + \frac{1}{2} f''(\alpha)\,\bar{\varepsilon}_{n-1}^2.$$

$$f(x_n) - f(x_{n-1}) \approx f'(\alpha)(x_n - x_{n-1})$$

$$= f'(\alpha)(x_n - \alpha - x_{n-1} + \alpha)$$

$$\bar{\varepsilon}_{n+1} = \frac{\bar{\varepsilon}_{n-1} f(x_n) - \bar{\varepsilon}_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\approx \frac{\bar{\varepsilon}_{n-1}\left(f'(\alpha)\bar{\varepsilon}_n + \frac{1}{2} f''(\alpha)\bar{\varepsilon}_n^2\right) - \bar{\varepsilon}_n\left(f'(\alpha)\bar{\varepsilon}_{n-1} + \frac{1}{2} f''(\alpha)\bar{\varepsilon}_{n-1}^2\right)}{f'(\alpha)\cdot\left(\bar{\varepsilon}_n - \bar{\varepsilon}_{n-1}\right)}$$

$$= \frac{\frac{1}{2}\bar{\varepsilon}_{n-1}\bar{\varepsilon}_n \quad f''(\gamma)\left[\bar{\varepsilon}_n - \bar{\varepsilon}_{n-1}\right]}{f'(\alpha)\left(\bar{\varepsilon}_n - \bar{\varepsilon}_{n-1}\right)}$$

$$= \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\bar{\varepsilon}_{n-1}\bar{\varepsilon}_n$$

$$x_{n+1} - \alpha = \quad '' \quad (x_{n-1} - \alpha)(x_n - \alpha)$$

∴ From the expression the absolute error.

$$\varepsilon_{n+1} = \underline{K}\,\varepsilon_{n-1}\,\varepsilon_n .$$

Let $K\varepsilon_n = \delta_n$.

$$K\varepsilon_{n+1} = K\varepsilon_{n-1}\; K\varepsilon_n .$$

$$\delta_{n+1} = \delta_{n-1}\cdot\delta_n .$$

$$\delta_2 = \delta_1\,\delta_0$$

$$\delta_3 = \delta_2\cdot\delta_1 = \delta_1^2\,\delta_0$$

$$\delta_4 = \delta_3\,\delta_2 = \delta_1^3\,\delta_0^2$$

$$\delta_5 = \delta_4\,\delta_3 = \delta_1^5\,\delta_0^3 .$$

$$\therefore\ \delta_n = \delta_0^{\alpha_n}\, \delta_1^{\beta_n}.$$

$$\delta_0^{\alpha_{n+1}}\, \delta_1^{\beta_{n+1}} = \delta_0^{\alpha_{n-1}}\, \delta_1^{\beta_{n-1}}\, \delta_0^{\alpha_n}\, \delta_1^{\beta_n}$$

$$\alpha_{n+1} = \alpha_{n-1} + \alpha_n. \qquad \beta_{n+1} = \beta_{n-1} + \beta_n.$$

These are Fibonacci sequence which is $(0, 1, 1, 2, 3, 5, \ldots)$

$$\alpha_n = \frac{\phi^n}{\sqrt{5}} \quad \text{where } \phi = \frac{1+\sqrt{5}}{2} = 1.61803$$

for large n.

$$K\varepsilon_n = \delta_n \approx \delta_0^{\alpha_{n-1}} \cdot \delta_1^{\alpha_n}.$$

$$= \delta_0^{\frac{\phi^{n-1}}{\sqrt{5}}}\ \delta_1^{\phi^n/\sqrt{5}}.$$

$$= d^{\phi^n} \quad \text{where} \quad d = \delta_0^{1/\sqrt{5}\,\phi}\ \phi^n\ \delta_1^{\frac{1}{\sqrt{5}}\phi^n}$$

$$K\varepsilon_n = \cancel{d^{1.6^n}}.\ d^{1.618^n}.$$