SC223 - Linear Algebra

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Lecture 11

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Sets associated with a matrix $A \in \mathbb{R}^{m \times n}, A : \mathbb{R}^n \to \mathbb{R}^m$

- **Column Space:**

$$C(A) = \{Ax \mid \forall x \in \mathbb{R}^n\}, C(A) \subseteq \mathbb{R}^m$$

- **Nullspace:**

$$N(A) := \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}_m\}, N(A) \subseteq \mathbb{R}^n$$

- **Rowspace:**

$$C(A^T) = \{A^T y \mid \forall y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

- **Left Nullspace:**

$$N(A^T) := \{y \in \mathbb{R}^m \mid A^T y = \mathbf{0}_n\} \subseteq \mathbb{R}^m$$

● All the above sets $S \subseteq \mathbb{R}^k, k = m, n$ as appropriate, satisfy: (1) $\mathbf{0}_k \in S$, (2) $\forall p, q \in S, \forall k_1, k_2 \in \mathbb{R}, k_1 p + k_2 q \in S$.

● LD columns of $A \Leftrightarrow \exists z \neq 0$ such that $Az = \mathbf{0}_m$.

● **Rank of a Matrix:** The number of linearly independent rows or columns of a matrix is called the Rank of the matrix.

- Given that $A : \mathbb{R}^n \to \mathbb{R}^m$, note that $C(A^T) \subseteq \mathbb{R}^n, C(A) \subseteq \mathbb{R}^m$.
- Consider the restriction $A|_{C(A^T)} : C(A^T) \to C(A) \subseteq \mathbb{R}^m$.
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$\forall x \in \mathbb{R}^n$, $\underline{\underline{Ax}} \in C(A)$.

Does there exist $z \in C(A^T)$ such that $Az = Ax$

$\Rightarrow$ Is there a $y \in \mathbb{R}^m$ s.t $AA^Ty = Ax$?

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- Find a solution (i.e., $y \in \mathbb{R}^r$ for any given $x \in \mathbb{R}^n$) to $A_{LI}A_{LI}^T y = A_{LI}x$.

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- Given that $A_{LI} \in \mathbb{R}^{r\times n}$, what do we know about $A_{LI}A_{LI}^T$?

If $x \in N(A_{LI}A_{LI}^T)$,

$$A_{LI}A_{LI}^T x = \vec{0}$$

$$\underline{x^T A_{LI}} \, \underline{A_{LI}^T x} = 0$$

$$\Rightarrow A_{LI}^T x = \vec{0}$$

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- Thus, $A_{LI}A_{LI}^T$ is invertible! $\exists y \in \mathbb{R}^r$ and therefore an element $A_{LI}^T y \in C(A^T)$ such that $AA_{LI}^T y = Ax$ for a given $x \in \mathbb{R}^n$.

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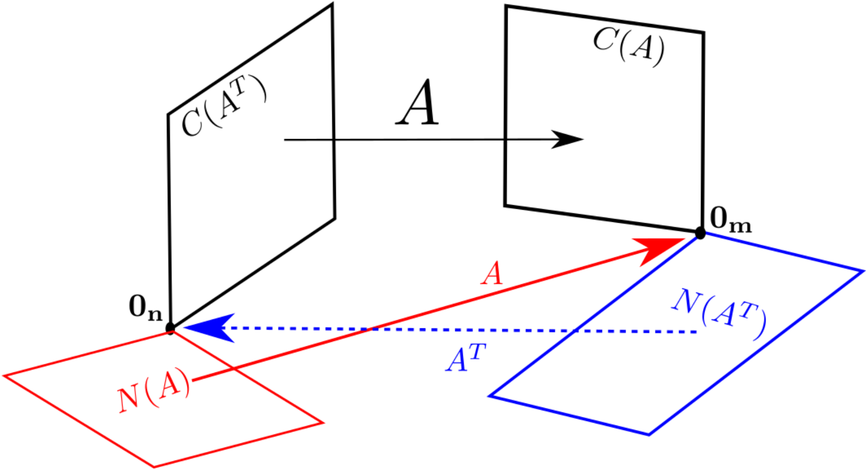
- Thus, $A|_{C(A^T)} : C(A^T) \to C(A)$ is **invertible**!

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$



Application

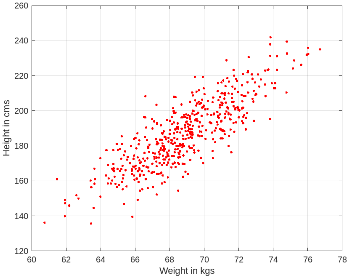


Figure: 500 height,weight tuples. How will you store this data?

- $(h_i, w_i), i = 1, \ldots, 500$. How much memory will be required?

Application

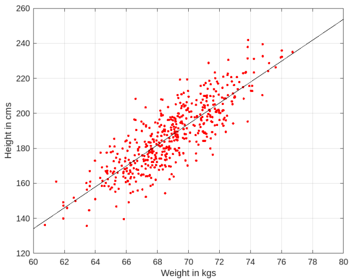
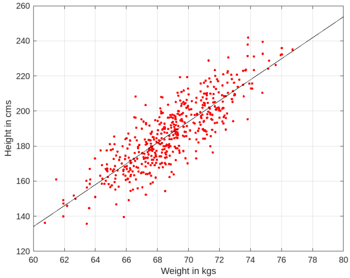


Figure: 500 height,weight tuples. How will you store this data?

- Since we know that $h_i \simeq a + bw_i, i = 1, \ldots, 500$, with $a = -225.08, b = 5.98$, $\begin{bmatrix} h_i \\ w_i \end{bmatrix} \simeq \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 1 \end{bmatrix} w_i$, store only $a, b, w_i, i = 1, \ldots, 500$.

Application



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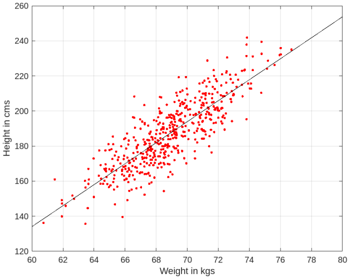

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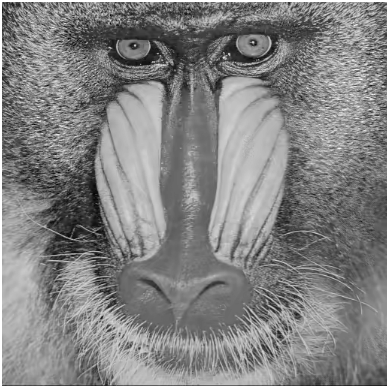
- Average error $= 0.47$ cm
- Compression ratio $= \dfrac{\text{Uncompressed size}}{\text{Compressed size}} \simeq 2$.



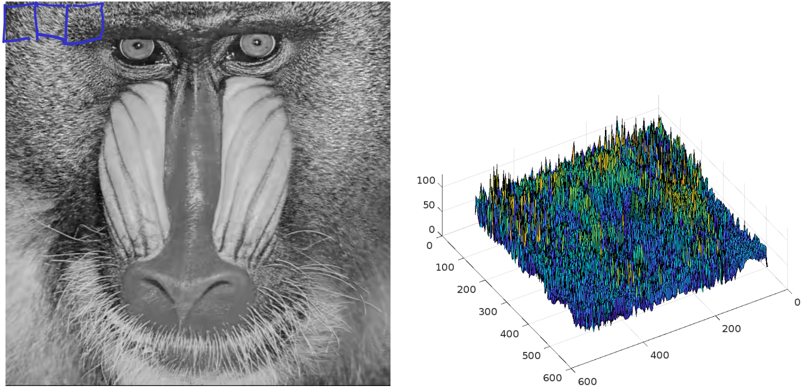


Figure: (left) Babboon image, size 512×512 pixels, (right) Each pixel is an integer between $0 - 255$, 0 encoding dark black, 255 encoding pure white.

- How much memory is required to store this image?

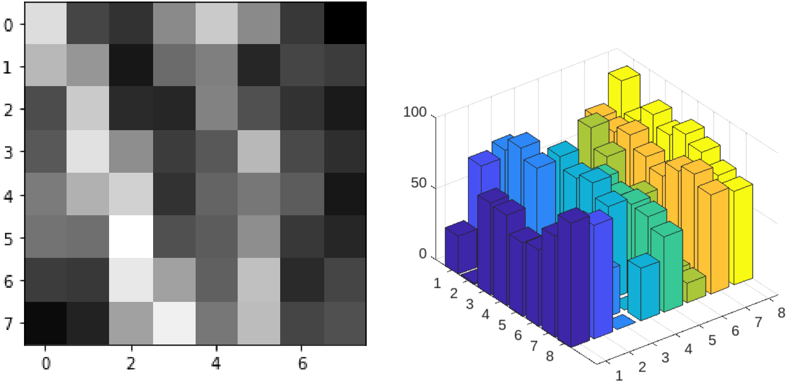


Figure: First 8 $\times$ 8 sub-matrix from top-left of the image shown as (top) an image, (bottom) bar graph.

- Consider a sub-matrix of size 8×8:

$$u = \begin{bmatrix} u_{11} & \dots & u_{18} \\ \vdots & \dots & \vdots \\ u_{81} & \dots & u_{88} \end{bmatrix},$$

which in a column-concatenated form can be considered as the vector

$$\bar{u} = \begin{bmatrix} u_{11} \\ \dots \\ u_{81} \\ u_{12} \\ \dots \\ u_{82} \\ \vdots \\ u_{88} \end{bmatrix} \in \mathbb{R}^{64},$$

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- How many LI vectors can there be in $\mathbb{R}^{64}$?

- Consider the following 64 LI vectors in $\mathbb{R}^{8\times 8}$:

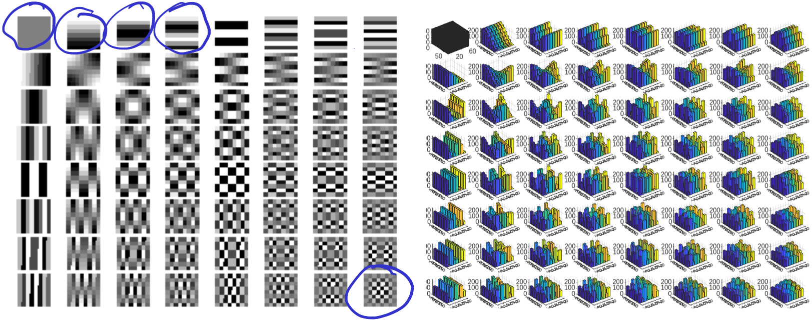
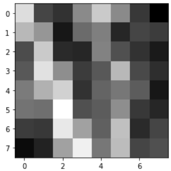
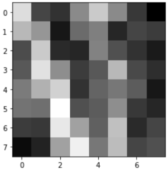
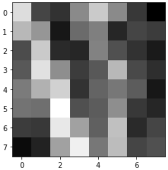


Figure: 64 LI vectors in $\mathbb{R}^{8\times 8}$ shown as (left) images, and as (right) bar graphs.


0
1
2
3
4
5
6
7
0
2
4
6



$$\begin{bmatrix} 634.25 & 88.05494 & -74.784996 & 32.74705 & -43.99998 & -24.357815 & 35.03592 & -7.2648544 \\ -26.507694 & 41.838562 & 67.079315 & 84.20723 & 54.4698 & -31.839682 & -82.9738 & 10.983028 \\ 2.8712888 & -37.569656 & -53.52906 & 16.196518 & 89.56473 & 51.585373 & 9.951899 & -21.102015 \\ 21.926683 & 34.701126 & 2.3982697 & 14.235403 & 2.4065735 & 30.57963 & 44.41404 & 43.47802 \\ 23.999998 & -9.934709 & -22.007797 & 18.674824 & -7.75 & 10.566483 & 3.703967 & -10.806338 \\ 8.985329 & -7.6490726 & -22.368402 & 13.29105 & -18.128761 & -10.898352 & 29.498426 & -14.009026 \\ -12.013248 & -1.3537698 & -14.548095 & 9.875553 & -3.2741735 & -9.035305 & -2.970933 & 9.557777 \\ -12.959625 & 0.6944461 & -7.2741323 & 5.049839 & 1.3122789 & -5.1217155 & -7.9491153 & 13.824383 \end{bmatrix}$$



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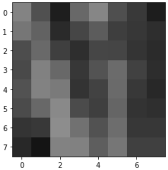
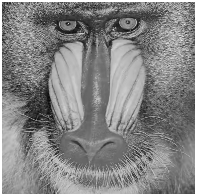


Figure: (top) Original 8×8 portion of the *babboon* image, (center) Coefficients of the LI vectors from previous slide, (bottom) Reconstructed image from 36 of the 64 coefficients.

- Average error per pixel $= 1.42$.



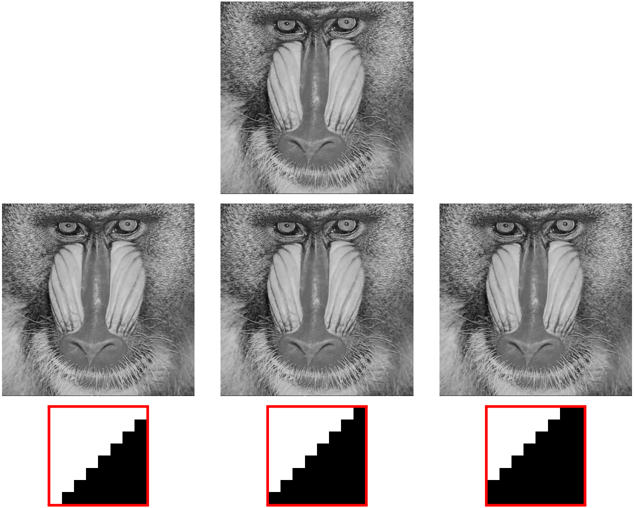


Figure: (top row) Original image - 512 $\times$ 512 pixels, (Center row): (left) Image reconstructed from 36 out of 64 (center) 28 out of 64, (right) 21 out of 64 coefficients. (Bottom row) Masks of size 8 $\times$ 8 pixels. White pixels shows the coefficients that are preserved, black pixel show coefficients that are not preserved for the corresponding image above. Average error per pixel: (left) 0.017, (center) 0.0215, (right) 0.026.

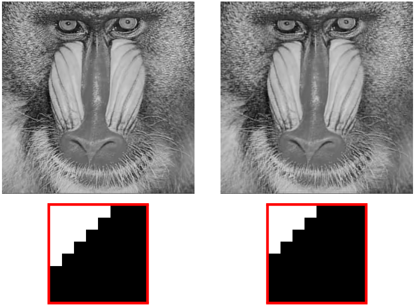


Figure: (top row): (left) Image reconstructed from 15 out of 64 (right) 10 out of 64, coefficients. (Bottom row) Masks. White pixels shows the coefficients that are preserved, black pixel show coefficients that are not preserved for the corresponding image above. Average error per pixel: (left) 0.03, (right) 0.0357.