SC223 - Linear Algebra

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Lecture 5

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Elementary row transformations

- Can we encode ERO by matrices?

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -1 \\ 2 & 0 & 3 & 2 & 4 \\ -2 & 3 & -2 & 1 & 6 \\ 3 & -4 & 2 & 1 & 1 \end{array}\right]$$

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- **Elementary Row Transformations(ERT)** - Matrix representation of ERO

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{bmatrix},$$

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Structure of E_k?

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- How about $E = E_3 E_2 E_1$?

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3/2 & 1/4 & 1 & 0 \\ -7/11 & -3/11 & 10/11 & 1 \end{bmatrix}$$

- **Theorem 2:**
  1. Using the convention for ERO, any ERT will be a lower triangular matrix.

$$E = \begin{array}{c} \\ \\ q \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & k & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ q \end{array}$$

$$R_p \leftarrow R_p + kR_q$$

$\big\downarrow$ Inverse

$$R_p \leftarrow R_p - kR_q$$

$$R_2 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- **Theorem 2:**
  1. Using the convention for ERO, any ERT will be a lower triangular matrix. (except row exchange)
  2. The product of any two lower triangular matrices is a lower triangular matrix.

$A, B \to$ lower Δ . $n \times n$

$C = AB$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

For $i < j$, $A_{ik} = 0$ for $k > i$

$B_{kj} = 0$ for $k < j$

Both A_{ik} & B_{kj} are nonzeros for $k \le i$, $k \ge j$

But $i < j$, so $C_{ij} = 0$

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    1. Using the convention for ERO, any ERT will be a lower triangular matrix. (except row exchange)
    2. The product of any two lower triangular matrices is a lower triangular matrix.
    3. Any ERT is an invertible matrix.
    4. The inverse of any invertible lower triangular matrix is also a lower triangular matrix.

LU Decomposition

$$A = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}$$

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- By Theorem 2, E^{-1} is a lower triangular matrix. Define $L := E^{-1}$.

Thus $A = LU$, known as the **LU decomposition**.

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- For this example:

$$\underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & 0 & 3 & 2 \\ -2 & 3 & -2 & 1 \\ 3 & -4 & 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1/4 & 1 & 0 \\ 3 & 1/2 & -10/11 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 4 & 5 & 4 \\ 0 & 0 & -11/4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U}$$