SC223 - Linear Algebra
Autumn 2023

Tutorial 7 Linear Transformations

1. Which of the following functions are Linear transformations?
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (x,0,0), \forall (x,y,z) \in \mathbb{R}^3$
   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (5x,-x,10y), \forall (x,y,z) \in \mathbb{R}^3$
   (c) $T : \mathbb{R} \to \mathbb{R}, T(x) = ax + b, \forall x \in \mathbb{R}$, where a, b are some real-valued non-zero constants.
   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (x,y,z) + (1,2,-2), \forall (x,y,z) \in \mathbb{R}^3$
   (e) $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R}), T(p) = q \cdot p \forall p \in \mathcal{P}_3(\mathbb{R})$, where $p \cdot q$ denotes multiplication between polynomials, and $q = q_0 + q_1 x + q_2 x^2 + q_3 x^3$, with $q_0, q_1, q_2, q_3 \in \mathbb{R}$ are fixed constants.
   (f) Let V be a vectors space, $T : V \to V, T(u) = w, \forall u \in V$, where $w \in V$ is a fixed non-zero vector.
   (g) Consider the vector space $V = \{f : \mathbb{R} \to \mathbb{R}\}$ over $\mathbb{R}$. Let $T : V \to V, (T(f))(t) = \sin t \cdot f(t), \forall t \in \mathbb{R}, \forall f \in V$
   (h) Let $T : \mathbb{C} \to \mathbb{C}$ be such that $T(x) = \bar{x}, \forall x \in \mathbb{C}$.
2. Let $T : U \to V$ be a linear transformation between vector spaces U and V. Show that, if W is a subspace of U, then the image $T(W)$ will be a subspace of V.
3. Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined as $T(A) = BA - AB, \forall A \in \mathbb{R}^{n\times n}$, where $B \in \mathbb{R}^{n\times n}$ is a fixed invertible matrix. Is T a linear transformation? Is it an isomorphism?
4. Suppose V is a finite dimensional vector space over $\mathbb{R}$ and U is a non-trivial subspace of V. Corresponding to any vector v in the vector space V, we define the set $S_v(U) = \{v + u | u \in U\}$.
   (a) Show that any two such sets are either identical or disjoint.
   (b) Show that no such set is closed under vector addition, unless it is created by using an element of U.
   (c) Let S_1, S_2 be two such sets (possibly identical). Define $S_1 + S_2 = \{s_1 + s_2 | s_1 \in S_1, s_2 \in S_2\}$. Show that any such sum of two sets is also a set generated in this way. **[10]**