SC223 - Linear Algebra

.

Aditya Tatu

Lecture 1

July 25, 2023

Linear Equations

- Solve for x, y in:

$$\begin{aligned} 8x + 4y &= 2 \\ 5x + 3y &= 1 \end{aligned}$$

$x = 1/2$, $y = -1/2$.

Elimination

Linear Equations

- Solve for x, y in:

$$8x + 4y = 2$$
$$5x + 3y = 1$$

Or

$$\underbrace{\begin{bmatrix} 8 & 4 \\ 5 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{det(A)}$$

Linear Equations

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Or

$$\begin{bmatrix} 8 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or

$$\underbrace{\left[\begin{array}{cc|c} 8 & 4 & 2 \\ 5 & 3 & 1 \end{array}\right]}_{\textbf{Augmented Matrix}}$$

- Solve:

$$\left[\begin{array}{ccc|c} 8 & 1 & -1 & 2 \\ 5 & 3 & 1 & -2 \\ 2 & -1 & 1 & -2 \end{array}\right]$$

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- Solve:

$$\left[\begin{array}{cccc|c} 8 & 1 & -1 & 2 & 2 \\ 5 & 3 & 1 & -3 & -2 \\ 2 & -1 & 1 & 5 & 2 \\ 0 & -2 & 1 & 3 & 0 \end{array}\right]$$

- Solve:

$$\left[\begin{array}{rrr|r} 8 & 1 & -1 & 2 \\ 5 & 3 & 1 & -2 \\ 2 & -1 & 1 & -2 \end{array}\right]$$

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$$\begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \left[\begin{array}{rrrr|r} 8 & 1 & -1 & 2 & 2 \\ 5 & 3 & 1 & -3 & -2 \\ 2 & -1 & 1 & 5 & 2 \\ 0 & -2 & 1 & 3 & 0 \end{array}\right] \end{array}$$

- Solve:

$$\left[\begin{array}{rrrrr|r} 8 & 1 & -1 & 2 & 0 & 2 \\ 5 & 3 & 1 & -3 & 5 & -2 \\ 2 & -1 & 1 & 5 & -2 & 2 \\ 0 & -2 & 1 & 3 & 1 & 7 \\ -1 & 4 & -2 & 2 & 5 & 0 \end{array}\right]$$

- In general, solve for $x_1, \ldots, x_n$ in

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

- Augmented matrix $[A \mid b]$

Applications

- Circuits:

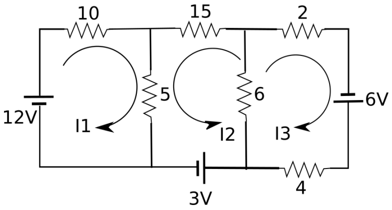

Figure: Solve for the loop currents

- Linear equations:

Applications

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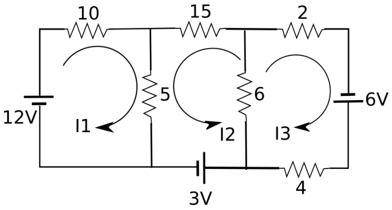


Figure: Solve for the loop currents

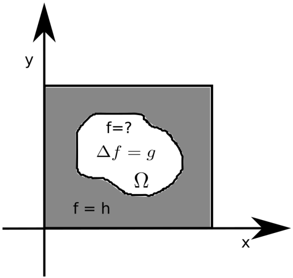
- Linear equations:

$$\begin{bmatrix} 15 & -5 & 0 \\ -5 & 26 & 6 \\ 0 & 6 & 12 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 6 \end{bmatrix}$$

Applications

- Numerical Solution to PDEs.
- Poisson's Equation:

$$\Delta f(x, y) = g(x, y), (x, y) \in \Omega \subset \mathbb{R}^2$$
$$f(x, y) = h(x, y), (x, y) \in \partial\Omega$$



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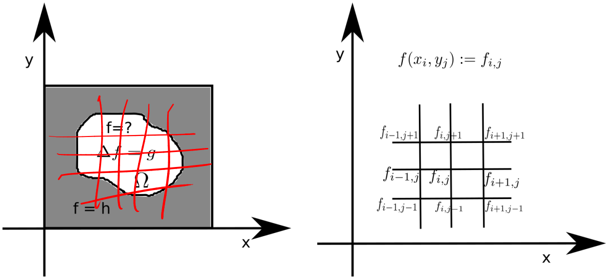

Figure: (left) Poisson equation on $\Omega \subset \mathbb{R}^2$ (right) Solving Poisson's equation on a grid in Ω

- FD Approximation: Let $x_i - x_{i-1} = y_j - y_{j-1} = h, \forall i, j$, then

$$\boxed{\Delta f(x_i, y_j) \simeq \frac{f(x_{i+1}, y_j) + f(x_{i-1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_{j-1}) - 4f(x_i, y_j)}{h^2}}$$

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y)$$

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- Discrete Poisson's equation:

$$\frac{f(x_{i+1}, y_j) + f(x_{i-1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_{j-1}) - 4f(x_i, y_j)}{h^2}$$
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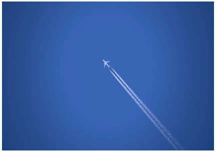

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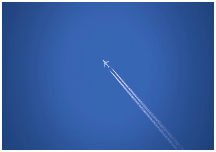
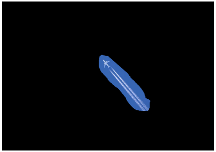

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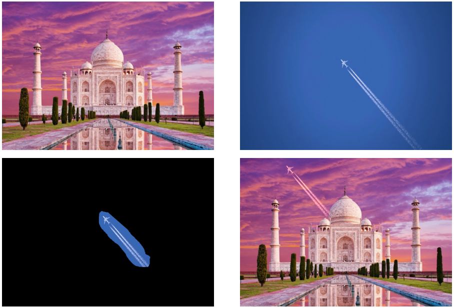

Figure: (top-left) Destination Image, (top-right) Source Image, (btm-left) Part of Source to be copied, (btm-right) Solution of the Poisson Image. Source: di Martino etal. IPOL 2016, P. Perez Poisson Image editing, ACM TOG, 2003

Rectangular matrices?

- Solve for x in $Ax = b$, where $\underline{A \in \mathbb{R}^{m \times n}}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, with $m > n$?

$A \in \mathbb{R}^{m \times n}$

$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m \times n}$

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$$ht_1 = p\, wt_1 + q.$$

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$$\overbrace{\begin{bmatrix} wt_1 & 1 \\ wt_2 & 1 \\ \vdots & \vdots \\ wt_m & 1 \end{bmatrix}_{m \times 2}}^{A} \overbrace{\begin{bmatrix} p \\ q \end{bmatrix}}^{x} = \overbrace{\begin{bmatrix} ht_1 \\ ht_2 \\ \vdots \\ ht_m \end{bmatrix}}^{b}$$

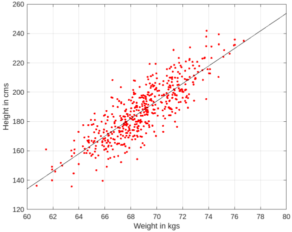

Figure: Black line depicts the prediction obtained after solving the above equations. $p \simeq 5.98, q \simeq -225$

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$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \end{bmatrix}_{3 \times 4} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

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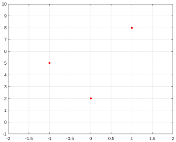

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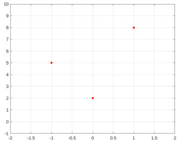

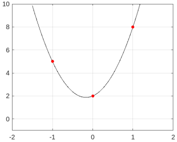

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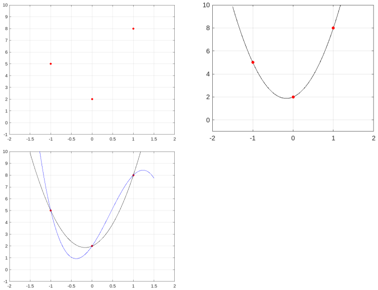

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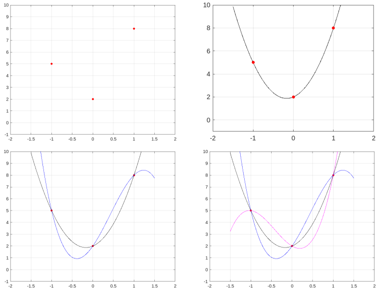

Figure: (top-left) 3 points to be interpolated, (top-rt) One soln., (btm-left) Two solutions, (btm-rt) Third soln.