LECTURE 20

Recap:- — Damped oscillator.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0.$$

- underdamped $(\beta < \omega_0)$
- overdamped $(\beta > \omega_0)$
- critically damped $(\beta = \omega_0)$.

— Driven damped oscillator.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t.$$ $\rightarrow$ example of inhomogeneous ODE.

$$x(t) = x_h(t) + x_p(t).$$

Corresponding homogeneous ODE :-

$$\ddot{x}_h + 2\beta\dot{x} + \omega_0^2 x_h = 0 \quad .$$

$x_h = e^{rt}$. Substitute and solve for r.

"Complexified" x : : $\underline{\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}}$.

Take the real part of the above eqr. to obtain the desired solution. since $\mathrm{Re}(e^{i\omega t}) = \cos\omega t$.

$$z_p(t) = C e^{i\omega t}.$$

Substitute, $$(-\omega^2 + 2\beta i \omega + \omega_0^2)\, C e^{i\omega t} = f_0 e^{i\omega t}.$$

$$\Rightarrow \quad C = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega}$$ $\longrightarrow$ in general, a complex quantity.

Can write, $$C = A e^{-i\delta}.$$

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right).$$

$$z_p(t) = A e^{i(\omega t - \delta)}.$$

$$A^2 = C^* C = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

The general solⁿ is :-

$$x(t) = \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}_{x_h} + \underbrace{A \cos(\omega t - \delta)}_{x_p}.$$

C_1 and C_2 are constants that should be determined from the initial conditions.

A is <u>NOT</u> determined from initial conditions.

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$$x(t) = A\cos(\omega t - \delta) + \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}_{\text{transient}}.$$

☒ Over longer time intervals,

$$x(t) = A\cos(\omega t - \delta).$$

$$A^2 = \underbrace{\frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}.$$

depends on ω_0 (natural freq) and ω (freq. of driving force).

A^2 is maximum when denominator $\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]$ is minimum.

Case i) :- Vary with ω fixed $\therefore$ denominator maximised when

$\omega_0 = \underline{\omega}$.

Case 12 :- Vary ω with ω_0 fixed.

denominator $= (\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$.

$$\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right] = 0.$$

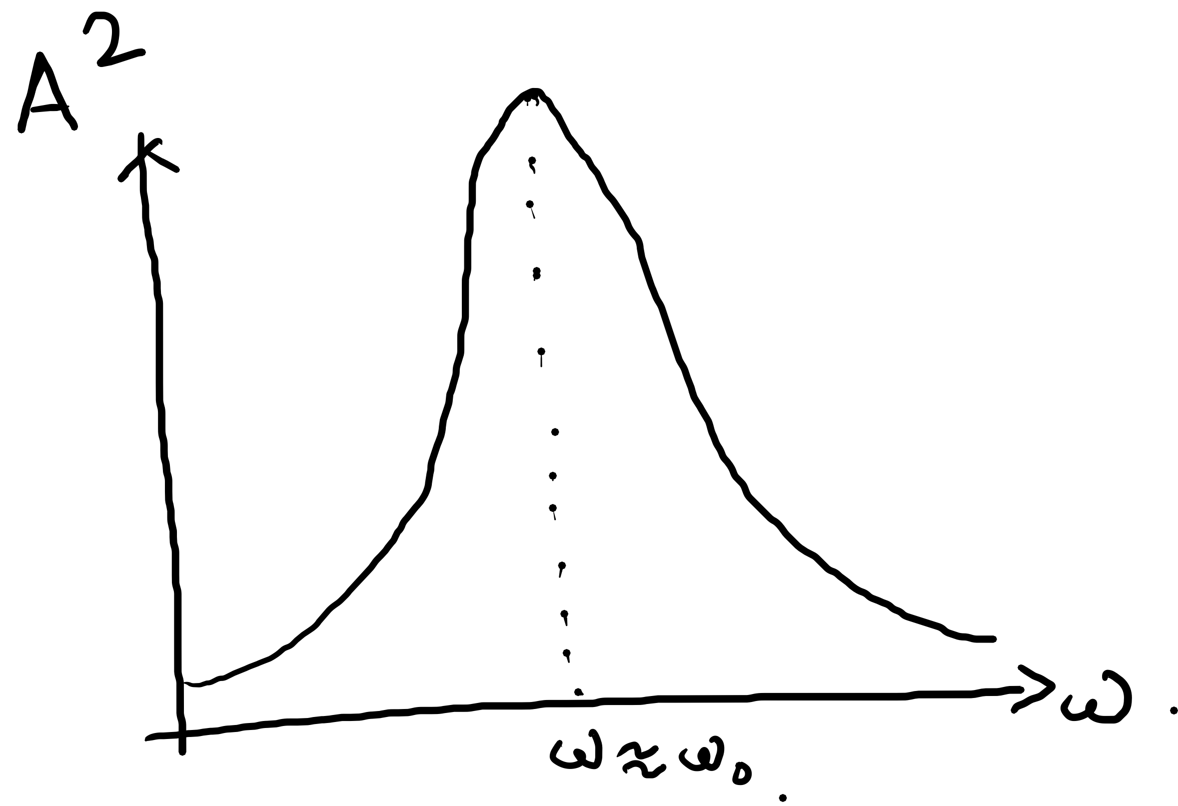
$$2(\omega_0^2 - \omega^2)(-2\omega) + 8\beta^2\omega = 0.$$

$$\Rightarrow \omega_0^2 - \omega^2 = 2\beta^2.$$

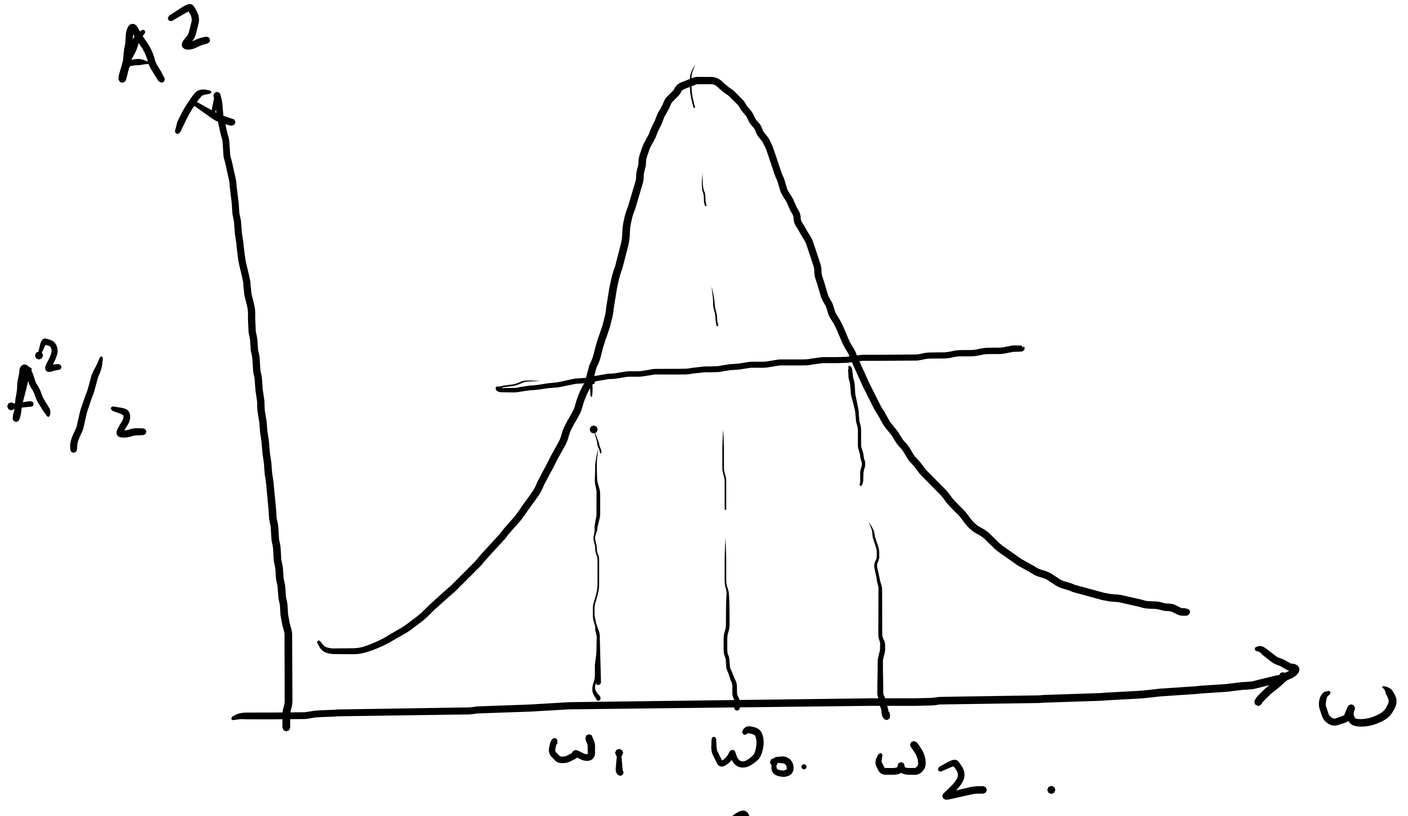
$$\Rightarrow \omega = \sqrt{\omega_0^2 - 2\beta^2}.$$

$$\Rightarrow$$

For low β, (underdamped) $\omega \approx \omega_0$.



RESONANCE.



— Width of resonance.

— Can be quantified by,

FWHM $\equiv$ Full width at half maximum.

$$\boxed{\begin{aligned} \omega_1 &= \omega_0 - \beta \\ \omega_2 &= \omega_1 + \beta \end{aligned}}$$

$$A_{max}^2 = \frac{f_0^2}{4\beta^2\omega_0^2}.$$

Half max is $A^2 = \dfrac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \approx \dfrac{f_0^2}{8\beta^2\omega_0^2}.$

$$\frac{f_0^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} = \frac{f_0^2}{8\beta^2\omega_0^2}.$$

$$\Rightarrow \quad 8\beta^2\omega_0^2 = (\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2.$$

$$\Rightarrow \quad \left[(\omega_0+\omega)(\omega_0-\omega)\right]^2 = 8\beta^2\omega_0^2 - 4\beta^2\omega^2.$$

$$\Rightarrow \quad \left[2\omega_0(\omega-\omega_0)\right]^2 \approx 4\beta^2\omega_0^2.$$

$$\Rightarrow \quad 4\omega_0^2(\omega-\omega_0)^2 = 4\beta^2\omega_0^2.$$

$$\Rightarrow \quad \omega = \omega_0 \pm \beta.$$

$$\text{FWHM} = 2\beta.$$ $\longrightarrow$ measures sharpness of the resonance peak.

Formally, define,

$$Q = \frac{\omega_0}{2\beta}. \longrightarrow \text{large } Q \text{ indicates narrow resonance.}$$

$$\text{small } \beta \Longrightarrow \text{smaller FWHM}.$$

At resonance, $\omega_0 \approx \omega.$,

$$\ddot{x} + \omega^2 x = f_0 \cos\omega t. \quad (\text{no damping}).$$

$$x_h = A\cos\omega t + B\sin\omega t.$$

$x_p = ?$ $\Rightarrow$ Seek a particular solution of the form,

$$x_p = t\,(c_1\cos\omega t + c_2\sin\omega t).$$

Substitute into inhomogeneous ODE.

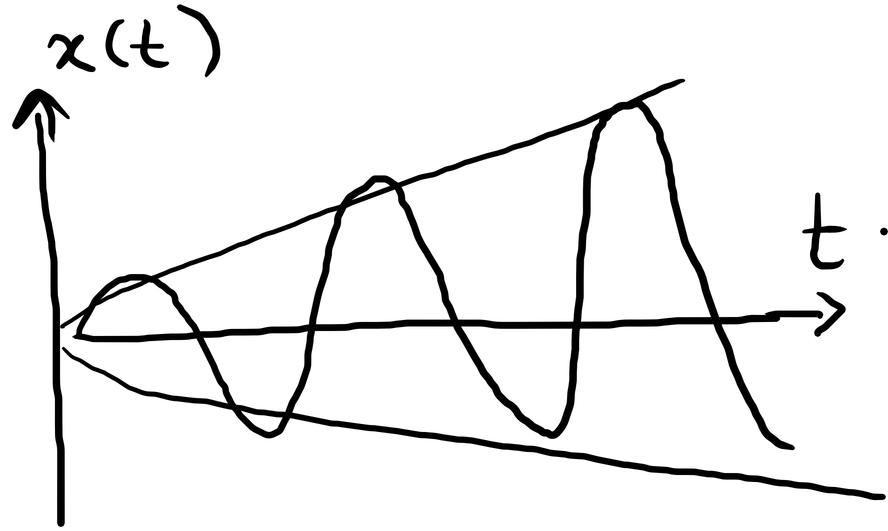
$$x_p(t) = t\,(c_1\cos\omega t + c_2\sin\omega t)$$

$$\ddot{x}_p(t) = 2\omega(-c_1\sin\omega t + c_2\cos\omega t) - \omega^2 \underbrace{t\,(c_1\cos\omega t + c_2\sin\omega t)}_{x_p(t)}.$$

$$\Rightarrow \quad \ddot{x}_p(t) + \omega^2 x_p(t) = 2\omega(-c_1\sin\omega t + c_2\cos\omega t) = f_0\cos\omega t.$$

$$c_1 = 0. \qquad 2\omega c_2 = f_0 \Rightarrow c_2 = \frac{f_0}{2\omega}.$$

$$x(t) = A\cos\omega t + B\sin\omega t + t\left(\frac{f_0}{2\omega}\right)\sin\omega t.$$



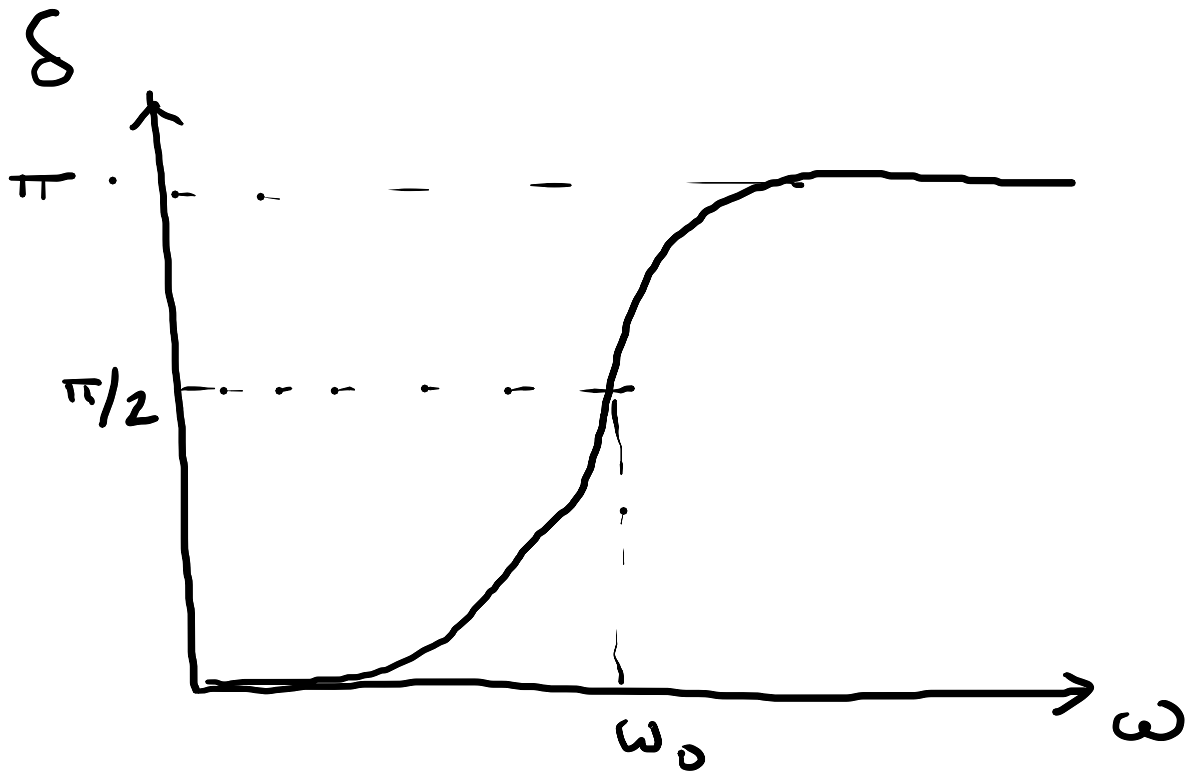
— Behaviour of δ.

$$x = A\cos(\omega t - \delta)$$

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right).$$

At small value of ω, $\delta \approx 0$.

At $\omega \approx \omega_0$, $\delta = \pi/2$.



Consider, $2\ddot{x} + \omega^2(5x-3y) = 0$ —— (1)

$2\ddot{y} + \omega^2(5y-3x) = 0$. —— (2)

(NORMAL MODES).

(1) + (2),

$2(\ddot{x}+\ddot{y}) + 2\omega^2(x+y) = 0$.

$\Rightarrow (\ddot{x}+\ddot{y}) + \omega^2(x+y) = 0$. —— (3)

(1) − (2),

$(\ddot{x}-\ddot{y}) + 4\omega^2(x-y) = 0$ —— (4).

$x+y = A_1\cos(\omega t + \phi_1)$

$x-y = A_2\cos(2\omega t + \phi_2)$.