SC224: Tutorial Sheet 3

Problems based on Jointly Distributed Random Variables, Expectation, Variance, Covariance and the Weak Law of Large Numbers.

Pb 1) The joint PMF of a discrete random vector (X_1, X_2) is given by the following table

$x_2 \setminus x_1$	-1	0	1
0	1/9	2/9	1/9
1	1/9	2/9	1/9
2	0	1/9	0

a) Determine the covariance of X_1 and X_2. (Ans: 15/81).

b) Calculate the correlation coefficient $\rho_{X_1,X_2} = \frac{\mathrm{Cov}(X_1,X_2)}{\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}}$ of X_1 and X_2. (Ans: 15/36).

c) Are X_1 and X_2 independent random variables? Justify your answer.

Pb 2) The moment generating function of a random variable X is a function $M_X(t)$ of a free parameter t, defined by $M_X(t) = E[e^{tX}]$ (if it exists).

(i) Compute the moment generating functions for the following distributions.

a) Geometric distribution with parameter p.

b) Uniform distribution over the interval $[a, b]$.

(ii) If the moment generating function exists for a random variable X, then show that the n^{th} moment about the origin (or $E[X^n]$) can be found by evaluating the n^{th} derivative of the moment generating function at $t = 0$.

Pb 3) Suppose that we have a resistance R. We know that the value of R follows a uniform law between 900 and 1100 Ω. What is the density of the corresponding conductance $G = 1/R$?

Pb 4) **(Universality of the uniform distribution)** Let X be a real valued random variable and let $U \sim U([0,1])$. Since $F_X : \mathbb{R} \to [0,1]$ is not always one-to-one, therefore, we define F_X^{-1} as

$$F_X^{-1}(u) = \sup\{x \in \mathbb{R} : F_X(x) \leq u\}.$$

Show that $F_X^{-1}(U)$ and X has the same distribution.

Pb 5) If X and Y are two independent random variables, then so are $g(X)$ and $h(Y)$.

Pb 6) Two random variables X and Y are said to be uncorrelated if their covariance is 0. Suppose X and Y are independent uniformly distributed random variables over the common interval $[0,1]$. Define $Z = X + Y$ and $W = X - Y$. Show that Z and W are not independent, but uncorrelated random variables.

Pb 7) Let the joint PDF of random variables X and Y be defined as

$$f_{X,Y}(x,y) = k\cos(x+y) \quad \text{for } 0 \le x \le \frac{\pi}{4}, 0 \le y \le \frac{\pi}{4}.$$

Determine the constant k and the marginal probability density functions ($f_X(x)$ and $f_Y(y)$) of X and Y. Are the random variables X and Y are orthogonal? Justify. (The random variables X and Y are said to be **orthogonal** if the mathematical expectation $E[XY] = 0$.)

Pb 8) Let X and Y be linearly dependent real valued random variables. Show that X and Y are not independent (in the probability sense.)

Pb 9) (**Multinomial Distribution**) Let Ω be a sample space associated with a random experiment E, and let $B_1, B_2, .., B_n$ be a partition of Ω. Assume that we perform m independent repetitions of the experiment E and that the probability $p_k = P[B_k]$ is constant from one repetition to another. If X_k denotes the number of times that the event B_k has occurred among the m repetitions, for $k = 1, 2, .., n$, then, determine the joint PMF of the random vector $(X_1, X_2, ..., X_n)$ and $\mathrm{Cov}(X_i, X_j)$. Also, calculate the expectation and variance of the random variable $\overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_n$.

Pb 10) Show that if $X \ge 0$ and $E(X) = \mu$ then $P(X \ge \sqrt{\mu}) \le \sqrt{\mu}$.

Pb 11) Let X have variance σ_X^2 and Y have variance σ_Y^2. Show that $-1 \le \rho_{X,Y} \le 1$. Further, argue that, if $\rho_{X,Y} = 1$ or -1, then X and Y are related by $Y = a + bX$, where $b > 0$ if $\rho_{X,Y} = 1$ and $b < 0$ if $\rho_{X,Y} = -1$.

Pb 12) Consider n independent trials, each of which results in any of the outcomes i, $i = 1, 2, 3$, with respective probabilities p_1, p_2, p_3, $\sum_{i=1}^{3} p_i = 1$. Let N_i denote the number of trials that result in outcome i, and show that $\mathrm{Cov}(N_1, N_2) = -np_1p_2$. Also explain why it is intuitive that this covariance is negative.

Pb 13) Suppose that X is a random variable with mean and variance both equal to 20. What can be said about $P[0 \le X \le 40]$?.

Pb 14) From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

(a) Give an upper bound to the probability that a student's test score will exceed 85.

(b) Suppose in addition the professor knows that the variance of a student's test score is equal to 25. What can be said about the probability that a student will score between 65 and 85?

Problems based on Special Discrete Random Variables.

Pb 1) The moment generating function of a random variable X is a function $M_X(t)$ of a free parameter t, defined by $M_X(t) = E[e^{tX}]$ (if it exists).

(i) Compute the moment generating functions for the following distributions.

   a) Bernoulli distribution with probability of success p.

   b) Binomial distribution with parameters n and p.

   c) Poisson distribution with parameter $\lambda > 0$.

(ii) Using the moment generating function, find the mean and variance of above mentioned distributions. Further, argue that sum of independent Binomial (Poisson) random variables follows Binomial (Poisson) distribution.

Pb 2) An urn contains n balls numbered 1 through n. If you withdraw m balls randomly in sequence, each time replacing the ball selected previously, find $P[X = k]$, $k = 1, ..., m$, where X is the maximum of the m chosen numbers.

Pb 3) If X is a binomial random variable with expected value 6 and variance 2.4, find $P[X = 5]$.

Pb 4) On average, 5.2 hurricanes hit a certain region in a year. What is the probability that there will be 3 or fewer hurricanes hitting this year?

Pb 5) The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter λ. However, such a random variable can be observed only if it is positive, since if it is 0 then we cannot know that such an insect was on the leaf. If we let Y denote the observed number of eggs, then

$$P[Y = i] = P[X = i | X > 0]$$

where X is Poisson with parameter λ. Find $E[Y]$.

Pb 6) Suppose that

$$P[X = a] = p \qquad P[X = b] = 1 - p.$$

Show that $\frac{X-b}{a-b}$ is a Bernoulli random variable. Find $\mathrm{Var}(X)$.

Pb 7) Each game you play is a win with probability p. You plan to play 5 games, but if you win the fifth game, then you will keep on playing until you lose. Find the expected number of games that you lose.

Pb 8) Ten balls are to be distributed among 5 urns, with each ball going into urn i with probability $p_i, \sum_{i=1}^{5} p_i = 1$. Let X_i denote the number of balls that go into urn i. Assume that events corresponding to the locations of different balls are independent.

   a) What type of random variable is X_i? Be as specific as possible.

   b) For $i \neq j$, what type of random variable is $X_i + X_j$?

   c) Find $P[X_1 + X_2 + X_3 = 7]$.