SC223 - Linear Algebra

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Lecture 21

September 21, 2023

Summary of Lecture 20

- Let V be a VS, and $U = \{v_1, \ldots, v_n\}$.
- **Definition:** (Span of a set) We define the **Span of U** as

$$span(U) := \{a_1 v_1 + \ldots + a_n v_n \mid \forall a_1, \ldots, a_n \in \mathbb{F}\},$$

i.e., the set of all possible linear combinations of elements from U.
- If $|U| = \infty$, $span(U)$ is the set of **all** possible linear combinations of **all** possible finite subsets of U.
- **Proposition 10:** Let $U \subseteq V$. Then $span(U)$ is a subspace of V.
- Let V be a VS, and let $W \subset V$. If $span(W) = V$, we say that W is a **spanning set** of V, or W **spans** V.
- We say that a vector space V is **finite dimensional** if there exists a finite spanning set. Notation: FDVS.
- **Linearly independent set**: Let V be a vector space and let $W = \{v_1, \ldots, v_n\} \subset V$. We say that the set W is a set of linear independent vectors, if

$$a_1 v_1 + \ldots + a_n v_n = \theta \Rightarrow a_i = 0, i = 1, \ldots, n$$

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- What if $|W| = \infty$?

Let $W_1 = \{v_1, \dots, v_5\}$.

$$v_1 = \sum_{i=2}^{5} v_i$$

$W_1 \to LD$.

$W_2 = W_1 \cup \{v_6, \dots, v_{10}\}$.

$W_2 \to LD$

$W_3 = W_1 \cup \{p_i, i \in \mathbb{Z}\}$.

$\hookrightarrow LD$

W, $|W| = \infty$

W is said to be LD if there is a finite subset of W which is LD.

- Example: $V = \mathcal{P}(\mathbb{R}), U = \{x^i, i = 0, 1, \ldots\}$.

$$W = \{ x^{k_i}, \ k_i \in \mathbb{N}, \ i = 1, \ldots, m \}.$$

$$p(x) = \underline{a_1 x^{k_1} + a_2 x^{k_2} + \cdots + a_m x^{k_m} = 0} \to 1$$

$$p(x_1) = a_1 x_1^{k_1} + \cdots + a_m x_1^{k_m} = 0$$

$$\to \boxed{a_1 = a_2 = \cdots = a_m = 0}$$

Assume $k_j = \max_i \{k_i\}$.

$\to \deg(p) = k_j \to$ # of roots of $p = k_j$

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$span(\{w_1, u_1, \ldots, u_n\}) = V$

$u_j \in span(\{w_1, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n\})$

$w_1 = \sum_{i=1}^{n} a_i u_i$

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- Is it possible that $m > n$?
- If so, after n iterations, we will reach a contradiction:
$span(\{w_1, w_2, \ldots, w_n\}) = V$

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