Equations Used

$$-EI\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}} = \frac{1}{2}\rho u^2 C_D \int_{s^*}^{l} \cos^2(\theta)\cos(\theta-\theta^*)ds + \frac{1}{2}\rho u^2 C_L \int_{s^*}^{l} \cos^2(\theta)\sin(\theta-\theta^*)ds \tag{1}$$

$$-\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}} = \frac{\rho u^2 l^3}{2EI}\left[C_D \int_{s^*}^{1} \cos^2(\theta)\cos(\theta-\theta^*)ds + C_L \int_{s^*}^{1} \cos^2(\theta)\sin(\theta-\theta^*)ds\right] \tag{2}$$

$$-\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}} = \frac{\rho u^2 l^3}{2EI}\left[\int_{s^*}^{1} \cos^2(\theta)\left(C_D\cos(\theta-\theta^*) + C_L\sin(\theta-\theta^*)\right)ds\right] \tag{3}$$

$$\begin{aligned}-\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}} = \frac{\rho u^2 l^3}{2EI}\sqrt{C_D^2+C_L^2}\Bigg[\int_{s^*}^{1} \cos^2(\theta)\Bigg(&\frac{C_D}{\sqrt{C_D^2+C_L^2}}\cos(\theta-\theta^*) \\ &+ \frac{C_L}{\sqrt{C_D^2+C_L^2}}\sin(\theta-\theta^*)\Bigg)ds\Bigg]\end{aligned} \tag{4}$$

$$-\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}} = \frac{\rho u^2 l^3}{2EI}\sqrt{C_D^2+C_L^2}\left[\int_{s^*}^{1} \cos^2(\theta)\sin\left(\theta-\theta^*+\sin^{-1}\left(\frac{C_D}{\sqrt{C_D^2+C_L^2}}\right)\right)ds\right] \tag{5}$$

$$-\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}} = \mathrm{Ca}\frac{\sqrt{C_D^2+C_L^2}}{C_D}\left[\int_{s^*}^{1} \cos^2(\theta)\sin\left(\theta-\theta^*+\phi\right)ds\right] \tag{6}$$

where,

$$\begin{aligned}\mathrm{Ca} &= \text{Cauchy Number} \\ &= \frac{\rho u^2 l^3 C_D}{2EI} \\ C_D &= 1.95 \\ C_L &= 0.178 \\ \phi &= \sin^{-1}\left(\frac{C_D}{\sqrt{C_D^2+C_L^2}}\right)\end{aligned}$$

And the boundary conditions being,

$$\theta^*(0) = 0 \tag{7}$$

$$\left.\frac{\mathrm{d}\theta^*}{\mathrm{d}s^*}\right|_{s^*=1} = 0 \tag{8}$$

$$\left.\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}}\right|_{s^*=1} = 0 \tag{9}$$

$$\tag{10}$$

Numerical Methods Used

Since the analytical solution to the equation (6) is very difficult, hence we are trying to find the solution numerically. We need the value of θ at various s^*, so to do that we divide the s^* space into $n+1$ nodes and the distance between between two adjacent nodes being $h = 1/n$.
Starting to solve the eqaution by applying *Backward Euler* difference equation to (9):

$$\frac{\mathrm{d}\theta^*}{\mathrm{d}s^*} \approx \frac{\theta_n - \theta_{n-1}}{h} = 0 \tag{11}$$
$$\implies \theta_n - \theta_{n-1} \tag{12}$$

Applying the *Backward Difference* equation to (10):

$$\frac{\mathrm{d}^2\theta^*}{\mathrm{d}s^{*2}} \approx \frac{\theta_n + \theta_{n-1} - 2\theta_{n-1}}{h} = 0 \tag{13}$$
$$\implies \theta_{n-2} = 2\theta_{n-1} - \theta_n \tag{14}$$
$$\implies \theta_{n-2} = \theta_n \tag{15}$$

Till now 2 out of the three boundary conditions have been used, and we will use equation (8) in the end. Now, we will focus on our main differential equation (6). Applying the *Backward Difference* equation to (6) at s^*_{n-1} we get:

$$\frac{\theta_{n-1} + \theta_{n-3} - 2\theta_{n-2}}{h^2} \approx \mathrm{Ca}\frac{\sqrt{C_D^2 + C_L^2}}{C_D}\left[\int_{1-h}^{1} \cos^2(\theta)\sin(\theta - \theta^* + \phi)\,ds\right] \tag{16}$$

The RHS of the above equation is computed using *Trapezoidal Method*. Hence applying *Trapezoidal Method* to the above equation:

$$\begin{aligned}\frac{\theta_{n-1} + \theta_{n-3} - 2\theta_{n-2}}{h^2} = \frac{h}{2}\mathrm{Ca}\frac{\sqrt{C_D^2 + C_L^2}}{C_D}\Bigg[&\cos^2(\theta_n)\sin(\theta_n - \theta_{n-1} + \phi) \\ &+ \cos^2(\theta)\sin(\theta_{n-1} - \theta_{n-1} + \phi)\Bigg]\end{aligned} \tag{17}$$

$$\begin{aligned}\theta_{n-3} = 2\theta_{n-2} - \theta_{n-1} + \frac{h^3}{2}\mathrm{Ca}\frac{\sqrt{C_D^2 + C_L^2}}{C_D}\Bigg[&\cos^2(\theta_n)\sin(\theta_n - \theta_{n-1} + \phi) \\ &+ \cos^2(\theta)\sin(\theta_{n-1} - \theta_{n-1} + \phi)\Bigg]\end{aligned} \tag{18}$$

As we can see clearly the RHS is known[1] and we can easily calculate θ_{n-3} by just plugging in the values. Similarly applying the same method at any internal node i we obtain:

$$\frac{\theta_i + \theta_{i-2} - 2\theta_{i-1}}{h^2} \approx \mathrm{Ca}\frac{\sqrt{C_D^2 + C_L^2}}{C_D}\left[\int_{s_i^*}^{1} \cos^2(\theta)\sin(\theta - \theta^* + \phi)\,ds\right] \tag{19}$$

[1] In terms of θ_n

And again applying *Trapezoidal Formula* to the above equation we get:

$$
\begin{aligned}
\frac{\theta_i + \theta_{i-2} - 2\theta_{i-1}}{h^2} = \frac{h}{2}\mathrm{Ca}\frac{\sqrt{C_D^2 + C_L^2}}{C_D}\Big[& \cos^2(\theta_n)\sin(\theta_n - \theta_i + \phi) \\
& + 2\cos^2(\theta_{n-1})\sin(\theta_{n-1} - \theta_i + \phi) \\
& + 2\cos^2(\theta_{n-2})\sin(\theta_{n-2} - \theta_i + \phi) \\
& \vdots \\
& + 2\cos^2(\theta_{i+1})\sin(\theta_{i+1} - \theta_i + \phi) \\
& + \cos^2(\theta_i)\sin(\theta_i - \theta_i + \phi)\Big]
\end{aligned} \tag{20}
$$

Hence the above equation can be simply represented as:

$$
\begin{aligned}
\theta_i = 2\theta_{i-1} - \theta_{i-2} + \frac{h^3}{2}\mathrm{Ca}\frac{\sqrt{C_D^2 + C_L^2}}{C_D}\Big[& \cos^2(\theta_n)\sin(\theta_n - \theta_i + \phi) \\
& + 2\sum_{j=i+1}^{n-1}\cos^2(\theta_j)\sin(\theta_j - \theta_i + \phi) \\
& + \cos^2(\theta_i)\sin(\theta_i - \theta_i + \phi)\Big]
\end{aligned} \tag{21}
$$

And, again we get an expression for θ_i in terms of θ_n. Following this procedure till $i = 2$ we get an expression for θ_0 in terms of θ_n. And from equation (8), we equate $\theta_0 = 0$, by using numerical algorithms to find the roots of non-linear equations[2]. Hence getting a value for θ_n, and once we get the value of θ_n we could easily calculate each θ_i from $i = 0$ to $n - 1$ from the above equations.

Now having calculated the θ array we can extract all data from it. For example the equivalent length can be given by the expression:

$$
\frac{L_e}{L} = \int_0^1 cos^3(\theta)d\hat{s} \tag{22}
$$

[2] In this case Newton-Raphson method was used