110.405 - Real Analysis I

Kaushik Srinivasan

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Topic 1 — Preliminaries

Topic 1.1 — Quantifier Logic

A	B	¬A	¬B	$A \cup B$	$A \cap B$	$A \to B$	$A \leftrightarrow B$
T	T	F	F	T	T	T	T
T	F	F	T	T	F	F	F
F	T	T	F	T	F	T	F
F	F	T	T	F	F	T	T

- ($\forall$): For all - universal quantifier (Big AND) - reject by counterexample
- ($\exists$) there exists - universal existential quantifier (Big OR) - reject by showing false for all instances.

negation of conjuction (and) $\equiv$ disjunction (or) of negations

- $\exists y \in U$, s.t. $\forall x \in V$, $A(x, y) \equiv$ one y will make $A(x, y)$ true no matter what x is.
- $\forall x \in V, \exists y \in U, A(x, y) \equiv A(x, y)$ can be made true by choosing y depending on x.

The existential-universal implies the universal-existential, but not vice versa. The universal-existential is equivalent to asserting the existence of a function (domain of universally quantified variable to domain of existentially quantified variable).

Example. Goldbach's Conjecture: every even number greater than 2 is the sum of two primes.

$$\forall x \in E \text{ (evens)}, \exists p, q \in P \text{ (primes)}, \text{ s.t. } x = p + q$$

The negation becomes, in stages

$$(not)(\forall x \in E)(\exists p, q \in P)(x = p + q)$$
$$(\exists x \in E)(not)(\exists p, q \in P)(x = p + q)$$
$$(\exists x \in E)(\forall p, q \in P)(not)(x = p + q)$$

Asserts the existence of a counter example:

- **Direct Proof:** gives strategy of showing true directly
- **Indirect proof:** we only prove (by contraction) that such a strategy must exist

Topic 1.2 — Infinite Sets

Countable Sets

Definition 1.2.1. A set with the same cardinality as natural numbers is called a ***countable*** set. It can form a one-to-one correspondence with the natural numbers.

Lemma 1.2.2. *If there is a mapping of the natural numbers to set U (not necessarily one-to-one), then U is either finite or countable*

We can show that the union of any finite number of countable sets is countable. Suppose $A_1, A_2, A_3, \ldots$ are sets each countable with A_1 having elements $a_{11}, a_{12}, a_{13}, \ldots$. We can then write all the elements of the union $\bigcup_{k=1}^{\infty} A_k$ in infinite matrix

$$\begin{array}{ccccc} A_1 \to & a_{11} & a_{12} & a_{13} & \cdots \\ A_2 \to & a_{21} & a_{22} & a_{23} & \cdots \\ A_3 \to & a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We can enumerate sideways $a_{11} \to a_{21} \to a_{12} \to a_{31} \to a_{22} \to a_{11} \to a_{13} \to \cdots$ and this is countable. We can also show that the cartesian product of finitely many countable sets is countable.

Uncountable Sets

Claim 1.2.3. *Prove that $2^{\mathbb{N}}$ is uncountable, where $\mathbb{N}$ is the set of natural numbers*

Proof. Suppose $2^{\mathbb{N}}$ is counttable, then obtain contradiction. Let $u_1, u_2, u_3, \ldots$ be enumeration of $2^{\mathbb{N}}$ where u_k is a set of $\mathbb{N}$. Let us construct $V = \{k : k \text{ is not in } u_k\}$, V depends on enumeration $u_1, u_2, \ldots$. Construction of V guarentees that it doesn't appear in the list because it differes from u_k in matter of $\#k \to$ one contains k the other doesn't. Hence contradiction. ■

Definition 1.2.4. Set A has greater cardinality than set B if A cannot be put in one-to-one correspondence with B, but a proper subset of A can be put in one-to-one correspondence with B

Topic 1.3 — Proofs

Definition 1.3.1. A proof is a sequence of logical deductions from the hypotheses to the conclusion of the statement proved.

Sources of Error

- applying theorem withhout verifying hypothesis $\to$ incorrect/incomplete proof
- making extra assumptions
- proving for all instances specified

Writing up proofs

- order of discovery
- rearrange and polish argument

How to understand proofs

1. able to read tthrough proof and convinced reassoning is correct
2. grasping structure of proof
3. how every hypottheses enter proof
4. way theorem is used and how it can be generalized

Topic 1.4 — Rational Number System

Before we define rational numbers, let's define the set of integers ($\mathbb{Z}$)

Definition 1.4.1. (Integers). An *integer* is an expression of the form $a - b$ s.t. $a, b \in \mathbb{N}$. Two integers are equal $a - b = c - d$ **iff** $a + d = c + b$. Denote the set of integers as $\mathbb{Z}$

Now we can try to define the rational numbers ($\mathbb{Q}$)

Definition 1.4.2. A *rational number* is an expression in the form of p/q where $p, q \in \mathbb{Z}$ and $q \neq 0$. Two rational $p/q = p'/q'$ **iff** $pq' = qp'$

Definition 1.4.3. We define the *addition* and *multiplication* of rationals as

$$\frac{p}{q} + \frac{p'}{q'} = \frac{pq' + p'q}{qq'}, \qquad \frac{p}{q} \times \frac{p'}{q'} = \frac{p \cdot p'}{q \cdot q'}$$

Proposition 1.4.4. *The set of rationals ($\mathbb{Q}$) form a **field***

Let a, b, c be rational numbers, then the following properties hold.

1. *Addition* and *multiplication* are commutative and associative.
   - commutative: $a \cdot b = b \cdot a$, $a + b = b + a$
   - associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. Multiplication distributes over addition
3. 0 is the additive identity, 1 is the multiplicative identity.
4. Every rational number has a negative $a + (-a) = 0$. Every non-zero rational has a reciprocal.

Proposition 1.4.5. *The rational numbers follow the basic properties of **order**.*

Let x, y, z be rational numbers, then the following properties hold.

a) *(Order trichotomy)* Exactly one of the statements $x = y, x < y, x > y$ is true.
b) *(Order is anti-symmetric)* One has $x < y$ *iff* $y > x$
c) *(Order is transitive)* If $x < y$ and $y < z$ then $x < z$
d) *(Addition preserves order).* If $x < y$, then $x+z < y+z$
e) *(+ve multiplication preserves order)* If $x < y$, then $xz < yz$

Because of the two propositions above, we can assert that rational numbers form an ***ordered field***.

Definition 1.4.6. We define the ***absolute value*** of $a \in \mathbb{Q}$ as $|a|$ where $|a| = a$ if $a \geq 0$ and $-a$ if $a < 0$.

Corollary 1.4.7. *The **triangle inequality** follows from absolute value where $a, b \in \mathbb{Q}$ s.t.*

$$|a - b| \geq |a| - |b| \quad \textit{and} \quad |a + b| \leq |a| + |b|$$

Axiom of Archimedes: for every positive rational number $a > 0$, there exists an integer n such that $a > 1/n$. We can also take the reciprocal: where every positive rational is less than some integer - important as also true for real numbers.

Remark. Between any two distinct rationals, there is an infinite amount of other rationals. Hence there is no next largest rational.

Topic 2 — Construction of the Real Number System

Proposition 2.0.1. *There is no rational number $x \in \mathbb{Q}$ s.t. $x^2 = 2$*

Proof. Assume FSOC that $a, b \in \mathbb{Z}$, $x = \frac{a}{b}$ and $x^2 = 2$. This means that $\frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. By unique factorization, $a = 2^k \cdot a'$, $b = 2^l \cdot b'$ where a', b' are odd. Then $a^2 = 2^{2k} \cdot (a')^2 = 2 \cdot (2^{2l} \cdot (b')^2) = 2^{2l+1} \cdot (b')^2$. By unique factorization, $2k \neq 2l + 1 \Rightarrow\Leftarrow$. ∎

Topic 2.1 — Cauchy Sequences

Definition 2.1.1. (Sequences). For all natural numbers n, there exists a natural number m (dependent on n) s.t. $|x - x_k| < \frac{1}{n}$ for all $k \geq m$

Definition 2.1.2. (Cauchy Sequence). A sequence $x_1, x_2, \ldots \in \mathbb{Q}$ is ***Cauchy*** ***iff*** for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ s.t. for all $i, j > m$,

$$|x_i - x_j| < \frac{1}{n}$$

Proposition 2.1.3. *If $\{x_i\}$ converges to x, then $\{x_i\}$ is Cauchy*

Triangle Inequality

$$|x_i - x_j| = |x_i - x + x - x_j| \leq |x_i - x| + |x - x_j|$$

Definition 2.1.4. Sequences $\{x_n\}$, $\{y_n\}$ are equivalent, written $\{x_i\} \sim \{y_i\}$, if for all $\frac{1}{n}$, $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ s.t. for all $i > m$

$$|x_i - y_i| < \frac{1}{n}$$

Lemma 2.1.5. *The equivalence of Cauchy sequences is transitive.*

Proof. Given $\frac{1}{n}$, $n \in \mathbb{N}$, let $m_1, m_2 \in \mathbb{N}$ s.t. for all $i > m_1, j > m_2$

$$|x_i - y_i| < \frac{1}{2n} \qquad |y_j - z_j| < \frac{1}{2n}$$

Let $m = \max(m_1, m_2)$, then $\forall i > m$

$$\begin{aligned} |x_i - z_i| = |(x_i - y_i) + (y_i - z_i)| &\leq |x_i - y_i| + |y_i - z_i| \\ &< \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \end{aligned}$$

■

Definition 2.1.6. (Real Number System) We denote $\mathcal{C}$ to be the set of all Cauchy sequences $x_1, x_2, \ldots$ and let $\mathbb{R}$ denote the set of equivalence classes of elements of $\mathcal{C}$

$$\mathbb{R} = \{\text{Cauchy sequences of rationals}\}/\sim$$

Topic 2.2 — Reals as an Ordered Field

Arithmetic

Remark. When defining an operation on real numbers, we need to verify that it first preserves Cauchy sequences of rationals and then to verify that it respects equivalence classes.

Lemma 2.2.1. *(Sum of Cauchy seq. is Cauchy) Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be Cauchy sequence of Rationals. Then $x_1 + y_1, x_2 + y_2, \ldots$ is also a Cauchy Sequence of rationals.*

Proof. Given $n \in \mathbb{N}$, let there exist $m \in \mathbb{N}$ s.t. for $i > m$

$$|x_i - x_j| < \frac{1}{2n} \qquad |y_i - y_j| < \frac{1}{2n}$$

Then for $i, j > m$

$$|(x_i + y_i) - (x_j + y_j)| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

■

Lemma 2.2.2. *(Sum of equivalent Cauchy seq. are equivalent) If $\{x_i\} \sim \{x_i'\}$ and $\{y_i\} \sim \{y_i'\}$, then $\{x_i + y_i\} \sim \{x_i' + y_i'\}$*

Proof. Given $n \in \mathbb{N}$ then $m \in \mathbb{N}$ s.t. for all $i > m$

$$|x_i - x_i'| < \frac{1}{2n} \qquad |y_i - y_i'| < \frac{1}{2n}$$

Then for $i > m$

$$\begin{aligned} |(x_i + y_i) - (x_i' + y_i')| &= |(x_i - x_i') + (y_i - y_i')| \\ &\leq |x_i - x_i'| + |y_i - y_i'| \\ &< \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \end{aligned}$$

■

Definition 2.2.3. The real number $x + y$ is the *equivalence class* of the Cauchy sequence $x_1 + y_1, x_2 + y_2, \ldots$ where $x_1, x_2, \ldots$ represents x and $y_1, y_2, \ldots$ represents y

Lemma 2.2.4. *(Bounded) Every Cauchy sequence of rationals is bounded. There exists $N \in \mathbb{N}$ s.t. $\forall k \in \mathbb{N}$, $|x_k| < N$.*

Proof. By Cauchy criterion there exist m such that $|x_j - x_k| \leq 1$ for all $j, k \geq m$. If we choose N larger than $|x_m| + 1$, Then

$$\begin{aligned} |x_j| = |(x_j - x_m) + x_m| &\leq |x_j - x_m| + |x_m| \\ &\leq |x_m| + 1 \leq N \end{aligned}$$

for all $j \geq m$. If we also take N greater than $|x_1|, \ldots, |x_{m-1}|$, then we will trivially have $|x_j| \leq N$ for $j < m$. This imposes only a finite number of conditions on N, so such a number can be found. ■

Lemma 2.2.5. *(Multiplication is Cauchy) If $\{x_i\}, \{y_i\} \subset \mathbb{Q}$ are Cauchy, then so is $\{x_i y_i\}$*

Proof. For all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ for all $i, j \geq m$ s.t. $|x_i - x_k| \leq \frac{1}{n}$ and $|y_i - y_k| \leq \frac{1}{n}$. We also know from the previous Lemma that there exists N (fixed, doesn't depend on n) s.t. $|x_j| \leq N$ and $|y_j| \leq N$ for j.

$$\begin{aligned} |x_j y_j - x_k y_k| &= |y_j(x_j - x_k) + x_k(y_j - y_k)| \\ &\leq |y_j||(x_j - x_k)| + |x_k||(y_j - y_k)| \\ &\leq N \cdot \frac{1}{n} + N \cdot \frac{1}{n} = \frac{2N}{n} \end{aligned}$$

for all $j, k \geq m$. we can change m to m' by $2Nn$ rather than n. We have $|x_j y_j - x_k y_k| \leq \frac{1}{n}$ for all $j, k \geq m'$, proving $x_1 y_1, x_2 y_2, \ldots$ is Cauchy sequence. ■

Lemma 2.2.6. *(Multiplication is well defined) If $\{x_i\} \sim \{x_i'\}$ and $\{y_i\} \sim \{y_i'\}$ then $\{x_i y_i\} \sim \{x_i' y_i'\}$*

Proof. Let N be an upper bound for all four Cauchy sequences. For all $n \in \mathbb{N}$, we use equivalence of $\{x_i\}$ and $\{x_i'\}$ to find m_1 s.t. for $j \geq m_1$ $|x_j - x_j'| \leq \frac{1}{2Nn}$ and for $j > m_2$ s.t. $|y_j - y_j'| \leq \frac{1}{2Nn}$. We choose $m = \max(m_1, m_2)$ and for all $j \geq m$, we have

$$\begin{aligned}|x_j y_j - x_j' y_j'| &= |y_j(x_j - x_j') + x_k'(y_j - y_j')| \\ &\leq |y_j||x_j - x_j'| + |x_j'||y_j - y_j'| \\ &\leq N \cdot \frac{1}{2Nn} + N \cdot \frac{1}{2Nn} = \frac{1}{n}\end{aligned}$$

■

Definition 2.2.7. The real number $x \cdot y$ is the equivalence class of the Cauchy sequence $x_1y_1, x_2y_2, \ldots$, where $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ respectively represent x and y.

Field Axioms

Theorem 2.2.8. *$\mathbb{R}$ is an ordered field.*

1. Addition and multiplication are associative and commutative
2. $\exists 0, 1 \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$, $x + 0 = x$, $x \cdot 1 = x$
3. $\forall x \in \mathbb{R}$, $\exists - x \in \mathbb{R}$ s.t. $(x) + (-x) = 0$
4. $\forall x \in \mathbb{R}$, $x \neq 0$, $\exists x^{-1} \in \mathbb{R}$ s.t. $x \cdot x^{-1} = 1$
5. $\forall x, y, z \in \mathbb{R}$, $x \cdot (y + z) = xy + xz$

Most of the proofs for these are trivial, with the exception of 4).

Lemma 2.2.9. *Let $x \in \mathbb{R}, x \neq 0$. Then there exists $N \in \mathbb{N}$ s.t. for any sequence $\{x_i\}$ s.t. $[\{x_i\}] = X$, there exists $m \in \mathbb{N}$ s.t. $\forall i > m$, $|x_i| > \frac{1}{N}$*

Proof. The proof of this is in the textbook, and is used to prove axiom field 4). ■

Order

Theorem 2.2.10. *(Order of Reals) Each real number is either positive, negative or zero.*

Proof. 0 is not positive, as Cauchy Sequence 0, 0, 0, ... clearly shows. Since $-0 = 0$, it is not negative. By the previous Lemma, we know there exists N, m for all $j \geq m$ s.t. $|x_j| \geq 1/N$. However signs cannot keep changing, as changes produce a jump of $2/N$ between terms which violates the Cauchy criterion. We can increase m s.t. for all $j \geq m$, $x_j \geq 1/N$ or $x_j \leq -1/N$. First case we say x is positive, the second case $-x$ is positive. WE cannot have both be the case for the same x. ■

Lemma 2.2.11. *Given $\{x_i\}, \{y_i\}$ in $\mathbb{Q}$ if there exists $m \in \mathbb{N}$ for all $i > m$, $x_i \geq y_i$ then $[\{x_i\}] \geq [\{y_i\}]$*

Proof. Equivalently, Given $\{x_i\}$ s.t. $\exists m \in \mathbb{N}$ s.t. $\forall i > m, x_i \geq 0$, then $[\{x_i\}] \geq 0$
Contrapositive, if $[\{x_i\}] < 0$, then $\forall m \in \mathbb{N}$, $\exists i > m$ s.t. $x_i > 0$. ■

Theorem 2.2.12. *(Triangle inequality of Reals) $|x+y| \leq |x| + |y|$ for real numbers x and y*

Proof. Let $x = [\{x_i\}]$ and $y = [\{y_i\}]$. Then by triangle inequality for $\mathbb{Q}$, $|x_i + y_i| \leq |x_i| + |y_i|$ ■

Theorem 2.2.13. *(Axiom of Archimedes) For all $x \in \mathbb{R} > 0$, there exists $N \in \mathbb{N}$ s.t. $\frac{1}{N} < x$*

Proof. Let $x = [\{x_i\}]$ be a Cauchy sequence. We have shown that there exists an n s.t. $x_j > 1/n$ for all $j \geq m$. By the lemma (2.2.9), this implies $x \geq 1/n$ ■

Theorem 2.2.14. *(Density of rationals) For all $x \in \mathbb{R}$, for any $\frac{1}{n}, n \in \mathbb{N}$ there exists $y \in \mathbb{Q}$ s.t. $|y - x| < \frac{1}{n}$*

Proof. Let $x = [\{x_i\}]$ be a Cauchy sequence. Let $m \in \mathbb{N}$ s.t. $\forall i, j > m$ $|x_i - x_j| < \frac{1}{2n}$. Let $y = x_{m+1}$. For all $i > m$, $|x_i - y| < \frac{1}{2n}$ by Lemma 2.2.9. Then $|x - y| \leq \frac{1}{2n} < \frac{1}{n}$. ■

Topic 2.3 — Limits and Completeness

Remark. We want to show ***completeness***, that is that if we repeated the process with reals, we will not end up with anything new.

Proof of Completeness

Definition 2.3.1. (Limit) A sequence $\{x_i\}$ of $\mathbb{R}$ converges to $x \in \mathbb{R}$ if $\forall \frac{1}{n}$, $n \in \mathbb{N}$, $\exists m \in \mathbb{N}$ s.t. $\forall i > m$, $|x_i - x| < \frac{1}{n}$

$$\boxed{\lim_{i \to \infty} x_i = x}$$

Theorem 2.3.2. *(Completeness of Reals) A sequence $\{x_i\}$ in $\mathbb{R}$ converges (has a limit) **iff** it is a Cauchy Sequence.*

Proof. ($\Rightarrow$) Assume $\lim_{i\to\infty} x_i = x$. Given $\frac{1}{n}$, $n \in \mathbb{N}$ let $m \in \mathbb{N}$ s.t. $\forall i > m$, $|x_i - x| < \frac{1}{2n}$. Then if $i, j \geq m$

$$|x_i - x_j| \leq |x_i - x| + |x - x_j| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

($\Leftarrow$) Let $\{x_i\} \in \mathbb{R}$ be Cauchy. By density of $\mathbb{Q} \subset \mathbb{R}$, for every $i \in \mathbb{N}$, there exists $y_i \in \mathbb{Q}$ s.t. $|y_i - x_i| < 1/i$. Idea is we want a rational y_i close to the real x_i. Now we just need to show that $\{y_i\}$ is Cauchy.
Claim 1: $\{y_i\}$ is Cauchy. Given $\frac{1}{n}, n \in \mathbb{N}$. Let $m \in \mathbb{N}$,

$m < 3n$ s.t. $\forall i, j > m$, $|x_i - x_j| < \frac{1}{3n}$. Then for all $i, j > m$

$$\begin{aligned}|y_i - y_j| &\leq |y_i - x_i| + |x_i - x_j| + |x_j - y_j| \\ &< \frac{1}{i} + |x_i - x_j| + \frac{1}{j} < \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n}\end{aligned}$$

Claim 2: Need to show $\lim_{i\to\infty} x_i = y = [\{y_i\}]$. Given that $\frac{1}{n}$, $n \in \mathbb{N}$, let $m \in \mathbb{N}$, $m > 2n$, $\forall i > m$, $|y_i - y| < 1/2n$

$$|y - x_i| \leq |y - y_i| + |y_i - x_i| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

■

Theorem 2.3.3. *(Limit identities)*

a) if $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} y_i = y$, then $\lim_{i\to\infty}(x_i + y_i) = x + y$ and $\lim_{i\to\infty} x_i y_i = xy$

b) If $x_i \geq y_i$, for all $i > m$ for some $m \in \mathbb{N}$, then $lim_{i\to\infty} x_i \geq \lim_{i\to\infty} y_i$

Square Roots

Theorem 2.3.4. *Let $x \in \mathbb{R}_+$, there exists a unique $y \in \mathbb{R}_+$, s.t. $y^2 = x$*

Proof. **Uniqueness:** East to show. We assume $z^2 = x$. Then $y^2 - z^2 = 0$. But $y^2 - z^2 = (y + z)(y - z) = 0$. We now know $y+z = 0$ or $y-z = 0$. Since we know that reals is an ordered field, and $y, z > 0$, the sum of two positives are positive. Hence $y = z$. We have shown uniqueness.
Existence: *Divide and conquer.* We define $\{y_i\}, \{z_i\}$ s.t.

i) $y_i^2 \leq x \leq z_i^2$
ii) $|y_i - z_i| = (y_1 - z_1)/2^{i-1}$

If $x \leq 1 \to y_1 = x, z_1 = 1$. If $x > 1 \to y_1 = 1, z_1 = x$. We then define $\{y_i\}, \{z_i\}$ inductively, given by each value of y_i, z_i. Define $m = (y_i + z_i)/2$.
If $m^2 \leq x \to y_{i+1} = m, z_{i+1} = z_i$
If $m^2 > x \to y_{i+1} = y_i, z_{i+1} = m$
Claim 1: $\{y_i\}, \{z_i\}$ are Cauchy Sequences: For all $n \in \mathbb{N}$, let $m \in M$ for all $i, j > m$, $|y_1 - z_1|/2^{m-1} < \frac{1}{n}$.

$$y_m \leq y_i, y_j \leq z_m \Rightarrow |y_i - y_j| < \frac{1}{n} \Rightarrow \{y_i\} \text{ is Cauchy}$$

Claim 2: $\lim_{i\to\infty} |y_i - z_i| = 0$ i.e. $y = \lim_{i\to\infty} y_i = \lim_{i\to\infty} z_i = z$
Since $y_i^2 \leq x \leq z_i^2$, we have $y_2 \leq x \leq z^2$. As we have $y^2 = z^2 \Rightarrow y^2 = x$ ■

Topic 3 — Topology of a Real Line

Topic 3.1 — The Theory of Limits

Limit, Sups, Infs

Definition 3.1.1. $\lim x_i = \infty$ if $\forall N \in \mathbb{N}$, there exists $m \in \mathbb{N}$ s.t. for every $i > m$, $x_i > N$

Definition 3.1.2. (Upper Bound). Let $E \subset \mathbb{R}$. Then $y \in R$ is an ***upper bound*** of E if $\forall x \in E$, $x \leq y$

Example. If $A = \{x : 0 \leq x \leq 1\}$, then 1 is an upper bound of A, but so is 2. We need to define the *lowest upper bound*

Definition 3.1.3. (Least upper bound) Let $E \subset \mathbb{R}$ and is bounded above. Then there exists a unique real number $\sup E$ s.t.

i) $\sup E$ is an *upper bound* for E.

ii) if y is any upper bound for E, then $y \geq \sup E$

Definition 3.1.4. (Greatest lower bound) Let $E \subset \mathbb{R}$ and is bounded below. Then there exists a unique real number $\inf E$ s.t.

i) $\inf E$ is an *lower bound* for E.

ii) if y is any lower bound for E, then $y \leq \inf E$

Proposition 3.1.5. *(Uniqueness of Least Upper Bound). Let $E \subset \mathbb{R}$.. Then E can have at most one least upper bound.*

Proof. Let y_1 and y_2 be least upper bounds of E. Since y_1 is the least upper bound and y_2 is an upper bound, then $y_1 \leq y_2$ by definition. Since y_2 is a least upper bound and y_1 is an upper bound, $y_2 \leq y_1$ by definition. Here we get $y_1 = y_2$ — hence there is at most one upper bound. ■

Definition 3.1.6. A monotone increasing sequence that is bounded from above has a finite limit, and the limit equals the sup. $x_{j+1} \geq x_j$ for every j.

Limit Points

Definition 3.1.7. (Limit point) $x \in \mathbb{R}$ is a ***limit point*** of $\{x_i\}$ if for every $n \in \mathbb{N}$, there are infinitely many $j \in \mathbb{N}$ s.t. $|x - x_j| \leq \frac{1}{n}$

Definition 3.1.8. (Subsequences) $\{y_i\}$ is a ***subsequence*** of $\{x_i\}$ if there exists a strictly increasing function $m : \mathbb{N} \to \mathbb{N}$ s.t. $y_i = x_{m(i)}$

Theorem 3.1.9. *Let $\{x_i\}$ be a Cauchy sequence of real numbers. $x \in \mathbb{R}$ is a limit point of $\{x_i\}$ **iff** there exists a subsequence $\{y_i\}$ s.t. $\lim_{i\to\infty} y_i = x$*

Proof. ($\Rightarrow$) Let $x \in \mathbb{R}$. If $\{y_i\}$ is a subsequence and converges to x, given for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ s.t. $\forall i > m$, $|y_i - x| \le \frac{1}{n} \longrightarrow x$ is a limit point.
($\Leftarrow$) We assume that $\{x_i\}$ has a limit point x. We choose subsequence $\{y_i\}$ so that $|y_n - x| \le \frac{1}{n}$. This implies $\lim_{i\to\infty} y_i = x$. How do we choose y_n? By definition of limit-point there are infinitely many x_i satisfying $|x_i - x| \le \frac{1}{n}$; we choose $y_1, y_2, y_3, \ldots$ in order such that after choosing $y_1, \ldots, y_{n-1}$, we take for y_n some x_i beyond $y_1, \ldots, y_{n-1}$ in original sequence that satisfies $|x_i - x| < \frac{1}{n}$. This way $\{y_i\}$ is a subsequence of $\{x_i\}$ with limit x. ■

Definition 3.1.10. (lim inf and lim sup) The lim sup of a *sequence* is the extended real number

$$\limsup_{i\to\infty} x_i = \lim_{i\to\infty} \sup_{j>i} x_j = \inf_i \sup_{j>i} x_j$$

The lim inf definition is as follows:

$$\liminf_{i\to\infty} x_i = \lim_{i\to\infty} \inf_{j>i} x_j == \sup_i \inf_{j>i} x_j$$

Theorem 3.1.11. *The* lim sup *of a sequence is a limit-point of the sequence and is the* sup *of the set of limit points of the sequence.*

Theorem 3.1.12. *A bounded sequence is convergent* ***<u>iff</u>*** *the* lim sup ***equals*** *the* lim inf*, or equivalently,* ***<u>iff</u>*** *it has only one limit point.*

Proof. ($\Rightarrow$) easy
($\Leftarrow$) $\underbrace{\inf\{x_j |\ j \ge i\}}_{z_i} \le x_i \le \underbrace{\sup\{x_j |\ j \ge i\}}_{y_i}$. Define $m \in \mathbb{N}$ and $|y_i - y| < 1/3n$ and $|z_i - y| < 1/3n$, then for all $i > m$

$$\begin{aligned} |x_i - y| &\le |x_i - y_i| + |y_i - y| \\ &\le |z_i - y_i| + |y_i - y| \\ &\le |z_i - y| + |y - y_i| + |y_i - y| \\ &< \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n} \end{aligned}$$

■

Topic 3.2 — Open and Closed Sets

Open Sets

Definition 3.2.1. (Open) A set $E \subset \mathbb{R}$ is ***open*** if $\forall x \in E$, there exists an open interval $(a, b) \subset E$ containing x

Definition 3.2.2. (Neighborhood) A ***Neighborhood*** of x is any open set containing x.

Definition 3.2.3. (Convergence) Sequence $\{x_i\}$ converges to x if for any neighborhood U of x, there exists $m \in \mathbb{N}$ s.t. $\forall i > m$, $x_i \in U$

Proposition 3.2.4. *An open set $U \subset \mathbb{R}$ is a disjoint (doesn't intersect) countable union of open intervals.*

Proof. For $x \in R$, lett $I_x \subset E$ be an open interval containing x. Hence $E = \bigcup_{x\in E} I_x$ and is not disjoint.
If I_1 & I_2 are open intervals, then $I_1 \cup I_2$ is a union of disjoin open intervals. We can define $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ where $b_1 > a_2$. Then $I_1 \cup I_2 = (\min(a_1, a_2), \max(b_1, b_2)$
Define $\sim$ on E by $x \sim y$ if $[x, y]$ or $[y, x]$ is contained in E.
Claim 1: $[x]$ is an open interval
Claim 2: $[x] \ne [y] \Rightarrow [x] \cup [y] = \emptyset$. This means

$$\bigcup_{x\in E/\sim} [x] \leftarrow \text{ disjoint}$$

Claim 3: $E/\sim$ is countable. By density of rationals, every open interval contains a rational number. Countable $\to \mathbb{Q} \cup E \to E/\sim$. Hence $x \longrightarrow [x]$ ■

Theorem 3.2.5. *(Union and intersection of open sets)*

i) The union of any number of open sets is an open set.

ii) The intersection of a finite number of open sets is an open set.

Proof.

1) Let $U = \bigcup_{i\in\mathcal{I}} U_i$ where U_i is open. For $x \in U$, there exists $i \in \mathcal{I}$ s.t. $x \in U_i$. Since U_i is open, there exists an open interval $I \subset U_i$ s.t. $x \in I$. Hence $x \in I \subset U_i \subset U$
2) Let $U = \bigcap_{i=1}^n U_i$, where all U_i are open and i is finite. For $x \in U$, for every i, there exists an interval $(a_i, b_i) \subset U_i$ containing x. Then $(\max_i a_i, \min_i b_i) \subset U$ contains x. Hence U is open.

■

Closed Sets

Definition 3.2.6. (Closed) A set $E \subset \mathbb{R}$ is ***closed*** if its complement $E^c = \mathbb{R} - E$ is open

Definition 3.2.7. (Limit point for sets). A point $x \in \mathbb{R}$ is a limit point of $E \subset \mathbb{R}$ if every neighborhood of x contains points of E distinct from x. That is, x_0 is a ***limit point*** of $S \subset \mathbb{R}$ if for every $\epsilon > 0$,

$$(x_0 - \epsilon, x_0 + \epsilon) \cap S\backslash\{x_0\} \ne \emptyset$$

Remark. What's the difference between the limit point of sets and sequences? If a sequence has no repetition or a finite amount of repetitions, then the concepts of limit point coincide.

Definition 3.2.8. (Perfect set) A set E is a ***perfect set*** if it equals the set o its limit points.

Remark. A set with no limit points, such as the empty set is automatically closed.

Theorem 3.2.9. *A set is closed* ***iff*** *it contains all its limit points*

Proof. ($\Rightarrow$) Assume E is closed. Then $x \in E^c \Rightarrow \exists (a,b) \subset E^c$ s.t. $x \in (a,b)$.
Claim: x limit point of E if $\exists \{x_i\}$ with $x_i \in E, x_i \neq x, \lim_{i\to\infty} x_i = x$
($\Leftarrow$) Contrapositive: If E is not closed, then E does not contain all its limit points. Assume E is not closed, that is E^c is not open. Then there exists $x \in E^c$ s.t. $\forall n \in \mathbb{N}$, $(x - \frac{1}{n}, x + \frac{1}{n}) \notin E^c$, i.e. $\exists x_n \in E$ and $x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$. Therefore, $\lim_{i\to\infty} x_i = x$, so x is limit point of E but not contained in E ∎

Remark. Let E be closed and $\{x_i\}$ be a sequence in E. Then all limit points of $\{x_i\}$ belong to E.

Theorem 3.2.10. *(Union & Intersection of closed sets)*

i) The union of finite number of closed set is a closed set

ii) The intersection of any number of closed set is a closed set.

Proof. i) Let $B_1, \dots, B_n$ be closed sets and $B = \bigcup_{i=1}^n B_i$. To show B is closed, we need to show that it contains all its limit points. Let x be a limit point of B, then every neighborhood of x contains a point (not x) of B. So each neighborhood of x $(x - \frac{1}{k}, x + \frac{1}{k})$, there is a point y_k in the neighborhood of one of the B_i's. Since there are a finite number of sets and an infinite amount of points $y_1, y_2, \dots$, one set must contain an infinite number of points. Hence the points $y_1', y_2', \dots$ approximatte x arbitrarily closely in B_m where $1 \geq m \geq n$, then x is a limit point of B_m. As B_m is closed, x is in B.

ii) Let $\mathcal{B}$ be a collection of open sets and and we let $\bigcap_{B \in \mathcal{B}} B$ be the intersection. We need to show intersection is closed by showing it contains all its limit points - let x be a limit point. Since each $B \in \mathcal{B}$ contains the intersection, it follows thhat x is a limit point for each B. As each B is closed, $x \in B$ and hence is in the intersection.

∎

Definition 3.2.11. (Closure) The ***closure*** of $E \subset \mathbb{R}$ is $E \cup \{\text{limit points of } E\}$

Definition 3.2.12. (Dense). Let $A \subset B \subset \mathbb{R}$. Then A is ***dense*** in B if $A \subset B \subset \text{closure}(A)$

Remark. $\mathbb{Q}$ is *dense* in $\mathbb{R}$, because every $x \in \mathbb{R}$ can be constructed as a sequence of $x_1, x_2, \dots \in \mathbb{Q}$ (how we constructed the real number system).

Example. $\text{closure}\big((a,b)\big) = [a,b]$

Definition 3.2.13. (Interior) The ***interior*** of $E \subset \mathbb{R}$ is $\{x \in E |\text{ there exists a neighborhood of } x \text{ contained in } E\}$

Topic 3.3 — Compact Sets

Definition 3.3.1. (Compact) A set A of real numbers is said to be ***compact*** if it has the property that for every sequence $x_1, x_2, \dots$ of real numbers, that lies entirely in A has a (finite) limit-point in A.

Theorem 3.3.2. *A set of real numbers is compact* ***iff*** *it is closed and bounded.*

Proof.

$\Longrightarrow$ If A is not closed, then it must have a limit-point y which is not in A, but then by definition of limit point we construct $x_1, x_2, \dots$ of points in A that converge to y. Thus y would be the sole limit-point. Similarly, unbounded sets cannot be compact as if A is unbounded, then we can find a sequence $x_1, x_2, \dots$ s.t. $x_n > n$ or $x_n < -n$, and such sequence has no finite limit point. Hence only *closed* and *bounded* can be compact.

$\Longleftarrow$ We know that A is closed and bounded. Hence let $x_1, x_2, \dots$ be a sequence of points in A. Since A is bounded, we know that bounded sequences possess a limit-point. Additionally, since A is closed it contains its limit points. NTS limit-point of sequence is limit point of set. By definition for y to be a limit point of set, every neighborhood of y must contain points of A *not equal to* y. But if y appears in the sequence, then it already appears in the set A. If y *never* appears in the sequence, we conclude that y is a limit point of A and then y is in A since A is closed.

∎

Definition 3.3.3. (Open cover). An ***open cover*** of a set $E \subset \mathbb{R}$ is a collection of open sets $\{U_i\}_{i \in \mathcal{I}}$ s.t. $E \subset \bigcup_{i \in \mathcal{I}} U_i$ where $\mathcal{I}$ is a finite or infinite set.

Definition 3.3.4. (Subcover). A ***subcover*** of $\{U_i\}_{i \in \mathcal{I}}$ is $\{U_j\}_{j \in \mathcal{J}}$ where $\mathcal{J} \subset \mathcal{I}$ that's also a *cover*.

Remark. Every cover contains a finite subcover.

Theorem 3.3.5. *Every open cover of a compact set has a finite subcover.*

Proof. Let A be a compact set and $\mathcal{B}$ be an open cover. We want to show that $\mathcal{B}$ has a countable subcover. We do this by choosing rational endpoints (as rationals are countable and ordered). To do this, we start with point $x \in A$ and show it's covered. Since $\mathcal{B}$ is a cover, there exists $B \in \mathcal{B}$ s.t. $x \in B$. Since B is open, there exists interval $B \supset \mathcal{I} = (a, b) \ni x$. We can shrink $\mathcal{I}$ s.t. it has rational endpoints. Since $x \in \mathcal{I} \subset B$, x is covered.
Now we have countable subcover $B_1, B_2, \ldots$. We take n large enough s.t. $B_1, \ldots, B_n$ already covers A. We assume it does not cover A. Hence $\forall n$, there exists $x_n \in A$ that isn't covered by $B_1, \ldots, B_n$. Since A is compact, the sequence $x_1, x_2, \ldots$ has a limit-point $x \in A$. Since the infinite $B_1, B_2, \ldots$ covers A, there exists B_k that contains x. But $x_k, x_{k+1}, x_{k+2}, \ldots$ are not in B_k by choice of x_k. $\Rightarrow\Leftarrow$as if x is the limit point of the sequence, then the neighborhood of $x \in B_k$ must contain infinitely many of them. Hence $B_1, \ldots, B_n$ must cover A for some n. ■

Theorem 3.3.6. *(Heine-Borel Theorem) Set A is compact* ***iff*** *it is closed & bounded.*

Proof. Need to complete. ■

Note. The following are equivalent given a set A

1. A is closed and bounded
2. Every sequence of points in A has a limit-point in A
3. Every open cover of A has a finite subcover

Definition 3.3.7. (Nested sets). A sequence of sets $A_1, A_2, \ldots$ is ***nested*** if A_n contains A_{n+1} for every n. Hence $\bigcap_{i=1}^{n} A_i = A_n$

Definition 3.3.8. A nested sequence of non-empty compact sets has a non-empty intersection.

Proof. Take $x_i \in K$. Then let x be a limit point of $\{x_i\}$. Then $x \in \bigcap_{i=1}^{\infty} K_i$. ■

Remark.
$$\bigcap_{i=1}^{\infty} \left(0, \frac{1}{i}\right) = \emptyset$$

Topic 4 — Continuous Functions

Topic 4.1 — Concepts of Continuity

Definitions

Definition 4.1.1. A function consists of a domain D and range R, and correspondence $x \longrightarrow f(x)$ assigning point $f(x) \in R$ to each point $x \in D$. Image $f(D)$ is set of all values $f(x)$. Function is onto if the image equals the range and is one-to-one if $x \neq y$ implies $f(x_1) \neq f(x_2)$.

Definition 4.1.2. (Continuous at x_0). A function is *continuous at* $x_0 \in D$ if for every $\frac{1}{n}$, there exists $\frac{1}{m}$ such that $\forall x \in D$ with $|x - x_0| < \frac{1}{m}$, we have $|f(x) - f(x_0)| < \frac{1}{n}$

Example. Show that $f(x) = 1/x$, $x > 0$ is continuous. We fix x_0 in the domain. Given any error $\frac{1}{n}$, we need to find an error $\frac{1}{m}$ s.t. $|x - x_0| < \frac{1}{m}$ and $x > 0$ that implies $|\frac{1}{x} - \frac{1}{x_0}| < \frac{1}{n}$. We can show that $\frac{1}{x} - \frac{1}{x_0} = (x_0 - x)/xx_0$ - we need to bound this from above and xx_0 from below. Since x_0 does not vary, we need to keep x from getting close to 0. Thus we would like something like $|x - x_0| < 1/m < x_0/2$, which implies $x > x_0/2$. Then
$$\left|\frac{1}{x} - \frac{1}{x_0}\right| \leq \frac{|x - x_0|}{xx_0} \leq \frac{2}{x_0^2}|x - x_0|$$
$$\frac{2}{mx_0^2} \leq \frac{1}{m}$$

Definition 4.1.3. (Continuous) We say function $f : D \to R$ is continuous if for every $x_0 \in D$ and for every $\frac{1}{n}$, there exists $\frac{1}{m}$ (depending on x_0, $\frac{1}{n}$) s.t. $|f(x) - f(x_0)| < \frac{1}{n}$ for every $x \in D$ satisfying $|x - x_0| < \frac{1}{m}$

Definition 4.1.4. (Uniformly continuous). We say function $f : D \to R$ is *uniformly continuous* if for every $\frac{1}{n}$, there exists $\frac{1}{m}$ s.t. $|f(x) - f(x_0)| < \frac{1}{n}$ for all $x, x_0 \in D$ satisfying $|x - x_0| < \frac{1}{m}$

Definition 4.1.5. (Limit) Consider $f : D \to R$ and $x_0 \in \mathbb{R}$ (not be in D) a limit point of D. Then $\lim_{x \to x_0} f(x) = y$ if for every $\frac{1}{n}$, there exists $\frac{1}{m}$ s.t. $\forall x \in D$, with $x \neq x_0$ and $|x_0 - x| < \frac{1}{m}$ we have $|f(x) - y| < \frac{1}{n}$

Limits of functions and limits of sequences

Theorem 4.1.6. *For x_0 a limit point of domain D of a function f. Then $\lim_{x \to x_0} f(x)$ exists* ***iff*** *for every sequence $\{x_i\}$ in D s.t. $x_i \neq x$ for all i and $\lim_{i \to \infty} x_i = x$ and $\lim_{i \to \infty} f(x_i)$ exists.*

Proof.

$\Longrightarrow$ Suppose $\lim_{x \to x_0} f(x) = y$ exists and let $x_1, x_2, \ldots$ converge to x_0. Given any error $\frac{1}{n}$, we want to make $|f(x_i) - y| < 1/n$ by going far in the sequence. By existence of $\lim_{x \to x_0} f(x)$, we know we can make $|f(x) - y| < 1/n$ where $\frac{1}{n}$ depends on $\frac{1}{m}, x_0$. Given $\frac{1}{n}$, there exists k s.t. $|x_j - x_0| < \frac{1}{m}$ for all $j \geq k$ by convergence of $x_1, x_2, \ldots$ to x_0. We have assumed $x_j \neq x_0$. Hence $|x - x_0| < \frac{1}{m}$ implies $|f(x) - y| < \frac{1}{m}$, derived above to $x = x_j$ if $j \geq k$. We obtain $j \geq k$ implies $|f(x_j) - y| < \frac{1}{n}$. Hence we have shown $f(x_1), f(x_2), \ldots$ converges to y.

$\Longleftarrow$ **Claim 1:** Let $x_i \to x_0$ and $x_i' \to x_0$ sequences in D. Then $\lim_{i\to\infty} f(x_i) = \lim_{i\to\infty} f(x_i')$ This is equal to the limit we shall call $\lim_{x\to x_0} f(x)$

Claim 2: Suppose $\lim_{x\to x_0} f(x) = y$ false. Hence there exists $\frac{1}{n}$ s.t. for all $\frac{1}{m}$, there exists $x_m \in D$ $|x_m - x_0| < \frac{1}{m}$ and $|f(x_m) - y| > \frac{1}{n}$. If this were true, that means the domain converges but the range diverges - $\lim_{i\to\infty} x_i = x_0$ as $|x - x_0| < \frac{1}{m}$ but $\lim_{i\to\infty} f(x_i) \neq y$ as $|f(x_m)| > \frac{1}{n}$. $\Rightarrow\Leftarrow$.

■

Remark. Necessarily the image under f of a convergent sequence (converging to a point in the domain) is convergent in the range.

$$\lim_{x\to x_0} f(x) = \lim_{i\to\infty} f(x_i)$$

Theorem 4.1.7. *Let $f : D \to R$. Then f is continuous* ***iff*** *for every convergent sequence $\{x_i\} \in D$ with $\lim_{i\to\infty} x_i \in D$. Then $\lim_{i\to\infty} f(x_i)$ exists.*

Proof.

$\Longrightarrow$ Suppose f is continuous. Then if $\{x_i\}$ converges to x_0 we can show $f(x_1), f(x_2), \ldots$ converges to $f(x_0)$. Given $\frac{1}{n}$, we find a $\frac{1}{m}$ s.t. $|x - x_0| < \frac{1}{m}$ that implies $|f(x) - f(x_0)| < \frac{1}{n}$ by continuity of f and find k s.t. $\forall j \geq k$ implies $|x_j - x_0| < \frac{1}{m}$ by convergence of $\{x_i\}$ to x_0. Then $j \geq k$ implies $|f(x_j) - f(x_0)| < \frac{1}{n}$

$\Longleftarrow$ We use shuffling to show the limit of sequence $f(x_1), f(x_2), \ldots$ is the same for all sequences $\{x_i\}$ converging to x_0. We can see that $x_0, x_0, \ldots$ is such sequence with limit x_0 and the following $f(x_0), f(x_0), \ldots$ has limit $f(x_0)$, it follows that limit of all sequences if $f(x_0)$. By the previous theorem, if x_0 is a limit point of D, then $\lim_{x\to x_0} f(x)$ exists and $\lim_{x\to x_0} f(x) = f(x_0) = y$

■

Inverse Images of Open Sets

Theorem 4.1.8. *Let $f : D \to \mathbb{R}$ with D being open. Then f is continuous* ***iff*** *for every open $U \in \mathbb{R}$, $f^{-1}(U)$ is open.*

Proof.

$\Longrightarrow$ Let $x_0 \in f^{-1}(U)$ where U is in the range. Since U is open, then $\exists n$ s.t. $(f(x_0) - \frac{1}{n}, f(x_0) + \frac{1}{n}) \subset U$. By continuity of f, there exists $\frac{1}{m}$ s.t. $f(D \cap \underbrace{(x_0 - \frac{1}{m}, x_0 + \frac{1}{m})}_{\text{open interval in } f^{-1}(U)}) \subset f(f(x_0) - \frac{1}{n}, f(x_0) - \frac{1}{n}) \subset U$

$\Longleftarrow$ Suppose $f^{-1}(U)$ is open for every set U. To do so, we show that $\forall \frac{1}{n}$, $\exists \frac{1}{m}$ s.t. $|x - x_0| < \frac{1}{m}$ implies $|f(x) - f(x_0)| < \frac{1}{n}$. Since U is an open set, $\exists n$ s.t. $(f(x_0) - \frac{1}{n}, f(x_0)_{\frac{1}{n}}) \subset U$. By hypothesis $f^{-1}(U)$ is also open. Hence, $\exists m$ s.t. $(x_0 - \frac{1}{m}, x_0 + \frac{1}{m}) \subset f^{-1}(U)$ that is open. This implies that $|x - x_0| < \frac{1}{m}$ implies $f(x) \in U$

■

Related definitions

Definition 4.1.9. (Lipschitz condition). Function $f : D \to \mathbb{R}$ is ***Lipschitz*** if there exists $M \in \mathbb{R}$ s.t. $\forall x, y \in D$

$$|f(x) - f(y)| \leq M|x - y|$$

Example. Function satisfying Lipschitz is $f(x) = |x|$ where $M = 1$

Definition 4.1.10. (Hölder condition). Function $f : D \to \mathbb{R}$ is ***Hölder*** of order α, $0 < \alpha \leq 1$, if $\exists M \in \mathbb{R}$ s.t. $\forall x, y \in D$

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

Topic 4.2 — Continuous Functions Properties

Basic Properties

Theorem. Let f, g be two continuous functions. Then $f + g$, $f \cdot g$, $f - g$, f/g are also continuous

Theorem 4.2.1. $\max(f, g)(x)$ *and* $\min(f, g)(x)$ *are continuous sequences*

Proof follows from following lemma and theorem 4.1.7

Lemma 4.2.2. *Let $\{x_i\}$, $\{x_i'\}$ be convergent sequences. Then $\{\max(x_i, x_i')\}$ and $\{\min(x_i, x_i')\}$ are convergent sequence.*

Proof. **Case 1:** $\lim_{i\to\infty} x_i = \lim_{i\to\infty} x_i'$. Then sequence $\{\max x_i, x_i'\}$ is just the converging sequence $x_1, x_1', x_2, x_2', \ldots$ which converges to the same limit point. Similar argument for min

Case 2: $x = \lim_{i\to\infty} x_i < \lim_{i\to\infty} x_i' = x'$. Let n be s.t. $\forall i > m$, $|x - x_i| < \frac{x' - x}{2}$ and $|x' - x_i'| < \frac{x' - x}{2}$ (Sequences don't overlap.). Then

$$\begin{aligned} x_i' - x_i &= (x_i' - x') + (x' - x) + (x - x_i) \\ &> -|x_i' - x'| + (x' - x) - |x - x_i| > 0 \\ x_i' &> x_i \end{aligned}$$

Therefore $\forall i > m$, $x_i' > x_i \Rightarrow \max(x_i, x_i') = x_i'$. Therefore $\lim_{i\to\infty}(x_i, x_i') = x'$ ■

Proposition 4.2.3. *Let $A \subset \mathbb{R}$ be closed. Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ s.t. $f^{-1}(0) = A$.*

Remark. $f^{-1}(0)$ is always closed if f is continuous.

Proof. Let $f(x) = \min\{|x-a| : a \in A\}$. For $x_1, x_2 \in \mathbb{R}$, there exists $a \in A$ s.t. $f(x_1) = |x_1 - a|$ and $f(x_2) \leq |x_2 - a| \leq |x_2 - x_1| + |x_1 - a|$. Then

$$\begin{aligned} f(x_2) - f(x_1) \leq |x_2 - x_1| &\Rightarrow f(x_1) - f(x_2) \leq |x_1 - x_2| \\ &\Rightarrow |f(x_1) - f(x_2)| \leq |x_1 - x_2| \end{aligned}$$

■

Theorem 4.2.4. *(Intermediate Value Theorem) Let $f : [a,b] \to \mathbb{R}$ be continuous with $f(b) > f(a)$. Then for every y between $f(a) < y < f(b)$, $\exists x \in (a,b)$ s.t. $f(x) = y$*

Proof. Proof using divide and conquer method. We fix the value y, we consider sequences of intervals $[a_1, b_1], [a_2, b_2], \dots$ where $[a_1, b_1] = [a, b]$ and $[a_2, b_2] = [a, \frac{a+b}{2}]$ IF $f(\frac{a+b}{2}) > y$ ELSE $[\frac{a+b}{2}, b]$ chosen so $f(a_2) < y < f(b_2)$. Hence we will get a sequence $\{a_i\}$ and $\{b_i\}$ s.t. $f(a_k) < y < f(b_k)$ where $b_k - a_k = \frac{b-a}{2^{k-1}}$. This condition shows both sequences are Cauchy and converge to common limit x as $f(x) = \lim_{i\to\infty} f(a_i) = \lim_{i\to\infty} f(b_i)$. Since $f(a_i) < y < f(b_i)$, we get $f(x) \leq y \leq f(x)$. ■

Remark. This can be used to show that polynomials with odd degrees have a real root.

Cont. Functions on Compact Domains

Theorem 4.2.5. *With $f : D \to \mathbb{R}$ be continuous with D compact. Then $f(D)$ is also compact.*

Proof. Let $\{y_i\}$ be a sequence in $f(D)$. We choose $x_i \in D$ s.t. $f(x_i) = y_i$. Since D is compact, theres a subsequence $\{x_{k_i}\}$ which converges to limit. point $x \in D$. Then for each value of the subsequence, there's $f(x_{k_1}), f(x_{k_2}), \dots$ that converges to $f(x)$. Hence the image is compact. ■

Theorem 4.2.6. *Let $f : D \to \mathbb{R}$. and D is compact. Then f is bounded and there exists $x, y \in D$ (not necessarily unique) s.t. $f(y) = \sup\{f(x) : x \in D\}$ and $f(z) = \inf\{f(x) : x \in D\}$*

Proof. By the previous theorem, since D is compact, $f(D)$ is compact. We know that compact sets contain all its limit points, so $\inf(f(D)), \sup(f(D)) \subset f(D)$. Hence we can pick a $y, z \in D$ s.t. $f(y) = \inf(f(D))$ and $f(z) = \sup(f(D))$ ■

Theorem 4.2.7. *(Uniform Continuity Theorem). Let $f : D \to \mathbb{R}$ where D is compact and continuous. Then it is uniformly continuous.*

Proof. For f to be continuous, for any error $\frac{1}{n}$, there exists $\frac{1}{m}$ s.t. $|x - x_0| < \frac{1}{m}$ implies $|f(x) - f(x_0)| < \frac{1}{n}$. By negation for f to not be continuous, there exists $\frac{1}{n}$ s.t. for all $\frac{1}{m}$, there exists two points $x_n, y_n \in D$ s.t. $|x_n - y_n| < \frac{1}{m}$ but $|f(x_n) - f(y_n)| \geq \frac{1}{n}$. NTS this leads to contradiction and is not possible.
Since D is assumed compact, we have convergent sequences $\{x_i\}$, $\{y_i\}$ with the condition $|x_m - y_m| \leq \frac{1}{m}$ which implies both subsequences converge to same limit x_0. Let the subsequences be $\{x'_i\}$ and $\{y'_i\}$, we have $|f(x'_m) - f(y'_m)| \geq \frac{1}{n}$ but $f(x_0) = \lim_{m\to infty} f(x'_m) = \lim_{m\to infty} f(y'_m)$ by the continuity of f at the point x_0. $\Rightarrow\Leftarrow$as we cannot have $\lim_{n\to\infty}(f(x'_m) - f(y'_m)) = f(x_0) - f(x_0) = 0$ and $|f(x'_m) - f(y'_m)| \geq \frac{1}{n}$ for all m. ■

Monotone Functions

Theorem 4.2.8. *(Monotone Function Theorem). A monotone function on an interval has one-sided limits at all points of the domain, finite except prehaps the end points.*

Corollary 4.2.9. *A monottone function on an interval has at most a countable number of discontinuities, all of which are jump discontinuities.*

Topic 5 — Differential Calculus

Topic 5.1 — Concepts of the Derivative

Equivalent Definitions

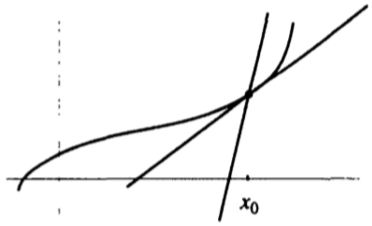
Definition 5.1.1. Let f be a function defined in the neighborhood of x_0. Then f is differentiable at x_0 with derivative $f'(x_0)$ if

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Similarly, for every $\frac{1}{m}$, there exists $\frac{1}{n}$ s.t. for all $x \neq x_0$, $|x - x_0| < \frac{1}{n}$ implies

$$\begin{aligned} \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| \leq \frac{1}{m} \Rightarrow &\left|f(x) - \underbrace{\left(f(x_0) - f'(x_0)(x - x_0)\right)}_{\text{affine function}}\right| \\ &\leq \frac{|x - x_0|}{m} \end{aligned}$$

Remark. We consider $f(x) - f(x_0) - f'(x_0)(x - x_0)$ as an *affine function* $ax + b$, where $a = f'(x_0)$, $b = f(x_0) - x_0 f'(x_0)$.



Remark. It's not just $f(x) - g(x)$ tends to 0 as $x \to x_0$, but it goes faster than any other choice of a. This is what condition, for all m, there exists n s.t. $|x - x_0| < \frac{1}{n}$ implies $|f(x) - g(x)| \leq |x - x_0|/m$ means it goes faster than $|x - x_0|$

Definition 5.1.2. (Big-O and Little-o) Let f, g be functions around x_0. We say $f(x) = O(g(x))$ as $x \to x_0$ if $\exists \frac{1}{n}$ and $c > 0$. If $|x - x_0| < \frac{1}{n}$, then $|f(x)| \leq c|g(x)|$. This means $\frac{f}{g}$ is bounded for $|x - x_0| < \frac{1}{n}$.
We say $f(x) = o(g(x))$ as $x \to x_0$ if for every $\frac{1}{m}$, there exists $\frac{1}{n}$ s.t. if $|x - x_0| < \frac{1}{n}$, then $|f(x)| < |g(x)|/m$ (or equivalently, $\lim_{x \to x_0} f(x)/g(x) = 0$).

Remark. $o(g(x))$ is a stronger statement $O(g(x))$. $o(g(x))$ implies $O(g(x))$, but not conversely.

Example. 1. $x^2 = O(5x^2)$ as $x \to 0$

2. $x^2 \neq o(5x^2)$ as $x \to 0$
3. $x^2 = o(5x)$
4. $|x|^i = O(|x|^j)$ as $x \to 0$ if $i \geq j$
5. $|x|^i = o(|x|^j)$ as $x \to 0$ if $i > j$
6. $f(x) = O(1)$ as $x \to x_0 \Leftrightarrow f$ is bounded in a neighborhood of x_0.

Proposition 5.1.3. *f is differentiable at x_0 if there exists an affine function $g(x)$ s.t.*

$$f(x) = g(x) + o(x - x_0) \text{ as } x \to x_0$$

Definition 5.1.4. (Differentiable on a set) f is differentiable on A if for every $x_0 \in A$, there exists constant $f'(x_0)$ s.t. for every $\frac{1}{m}$, there exists $\frac{1}{n}$ (depending on $\frac{1}{m}$ and x_0) s.t. if $|x - x_0| < \frac{1}{n}$, then

$$|f(x) - (f(x_0) + f'(x_0)(x - x_0))| \leq \frac{|x - x_0|}{m}$$

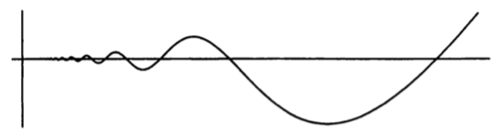
Continuity and Continuous Differentiability

Remark. Since differentiability at a point implies continuity at a point, differentiability on an open set implies continuity on that set.

Remark. Differentiability of a function implies continuity of a function, but it does *not* imply continuity of derivative. Take example $f(x) = |x|$. This is not differentiable at $x_0 = 0$, where discontinuity of the derivative occurs.

Example. (Discontinuities of the second kind). Consider a function that oscillates more rapidly as x approaches x_0, but size of oscillations are less rapid. Consider function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$



The differentiability is

$$f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} \quad \text{if} \quad x \neq 0$$

From this we can natually guess that $f'(0)$ does not exist, because of all the oscillations nearby. But it turns out the decay of factor x^2 is enough to overwhelm the oscillations to produce a 0 derivative at $x = 0$.

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin(1/x^2) - 0}{x - 0} = x \sin \frac{1}{x^2}$$

which has limit equal to 0 as $|\sin x^{-2}| \leq 1$. Thus f is differentiable at 0 and $f'(0) = 0$

Definition 5.1.5. A function whose derivative exists and is continuous is called *continuously differentiable* or *of class C^1*

Remark. There can be functions that are continuous and smooth, but the derivative can fail to exist. This is seen when $f(x) = \sqrt[3]{2}$, where as $x \to 0$, the derivative is $+\infty$, the tangent is verticle.

Topic 5.2 — Properties of the Derivative

Local Properties

Definition 5.2.1. Let f be a {strict} monotone { increasing } function on A. Then $f(x_1) < f(x_2)$ if $x_1 < x_2$. We define x_0 to be a { strict } local minima if $f(x_0) < f(x)$ for all $x \in N_\epsilon(x_0)$ for all $\epsilon > 0$.

Remark. Confusion: saying monotonically increasing at x_0 is *not* the same as saying monotonically at neighborhood of x_0.

Theorem 5.2.2. *Let f and f' be defined in a neighborhood of x_0.*

a) *If $f'(x_0) > 0$, then f is strictly increasing at x_0, and vice versa.*

b) *If f is monotone increasing at x_0, then $f'(x_0) > 0$, and vice versa*

c) *If f has a local minimum or maximum at x_0, then $f'(x_0) = 0$.*

Proof.

a) If $f'(x_0) > 0$ then there must be a neighborhood of x_0 where the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

is also strictly positive. Then the numerator is $f(x) - f(x_0) > 0$ if $x > x_0$ and $f(x) - f(x_0) < 0$ if $x < x_0$

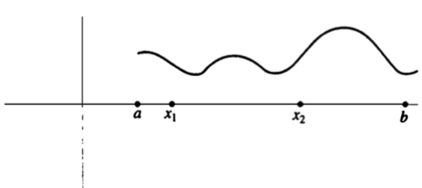
b) if f is monotonically increasing at x_0, then there exists a neighborhood where $\forall x_1 \in N_\epsilon(x_0)$, $\epsilon > 0$ and $x_1 > x_0$, then $f(x_1) > f(x_0)$. This means the difference quotient is > 0 and hence the derivative is also > 0.

c) Let's say f has a local maximum at x_0. Then the difference quotient will be ≥ 0 for all $x < x_0$ and ≤ 0 for all $x > x_0$. Hence we can take limit from either side, then the difference quotient is both $\leq$ and ≥ 0. Hence it is 0.

■

Remark. It **does not** say that if the function is strictly increasing the derivative is positive. Take $f(x) = x^3$ at $x = 0$. Additionally, we cannot draw conclusions if the derivative is 0 at a point

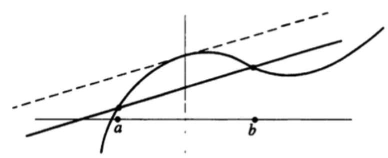
Theorem 5.2.3. *(Intermediate Value Theorem) Let $f(x)$ be differentiable on open interval (a, b). Then it's derivative has intermediate value property - if $x_1 < x_2$ are two points in the same interval, then $f'(x)$ assumes all values between $f'(x_1)$ and $f'(x_2)$ on interval (x_1, x_2)*



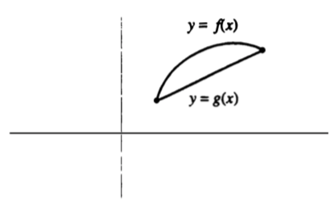
Proof. Lets prove for when $f'(x) = 0$ i.e. $f'(x_1) < 0 < f(x_2)$. This means to find $f'(x) = 0$, we only show there is a maximum or minimum in open interval (x_1, x_2). To find local minimum we look for point in f where it attains its inf on closed interval $[x_1, x_2]$. We know that differentiability implies continuity of f and continuous functions on compact sets attain their inf. If the inf is attained at an interior point x_0 of the interval, then x_0 is a local minimum as $(x_0 \in (x_1, x_2)$ and $f(x) \geq f(x_0)$, $\forall x \in (x_1, x_2)$. But if the inf occurs at the endpoint, we cannot assert we have a local minimum like $f(x) = x$. We use $f'(x_1) < 0 < f(x_2)$ to show that inf does not occur at either end point. It cannot occur at the left endpoint x_1 as function is strictly decreasing so $f(x) < f(x_1)$ for x near the interval near x_1 and it cannot occur on the right end point as $f(x) < f(x_2)$ for x near the interval (if we did $f'(x_1) < 0 < f(x_2)$, we show f can't attain it's sup). We have shown that it attains an intermediate value if that happens to be 0. We can show the general case by taking $f'(x_1) < y_0 < f(x_2)$. The function $g(x) = y_0 x$ has derivative everywhere equal y_0 (constant). Hence we solve $F'(x) = 0$ with $F(x) = f - g$ to get $F'(x_1) = f' - y_0 < 0$ and $F'(x_2) = f' - y_0 > 0$ ■

Theorem 5.2.4. *(Mean Value Theorem) Let f be a continuous function on a compact interval $[a, b]$ that is differentiable at every interior point. Then there exists a point x_0 where*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$



Proof. We can begin with $f(a) = f(b)$ where the difference quotient is zero. We want to find $f'(x) = 0$ for $x \in [a, b]$. We assume f is continuous on compactt $[a, b]$, so we know f attains its sup or inf. If either occurs on interior point, then we have an extrema. Thus we only consider special case when f has both inf and sup at endpoints (can occur as $f(x) = x$); but since we assumed $f(b) = f(a)$, this implies f must be constant on $[a, b]$ so $f'(x) = 0$ at any point in $[a, b]$



Now to reduce to the general case, let $g(x)$ be an affine function passing through $(a, f(a))$ and $(b, f(b))$. Hence $g(x)$ has a derivative equal to $(f(b) - f(a))/(b - a)$ at every point. By subtraction, we can get $F = f - g$. We reduce the problem of solving $f'(x_0) = \frac{f(b)-f(a)}{b-a}$ to $F'(x) = 0$. But F vanishes at both endpoints because f and g attain the same values at the endpoints. so $F(b) = F(a)$, so the previous argument applies. Hence the affine function is

$$g(x) = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)$$

and $g'(x) = \frac{f(b)-f(a)}{b-a}$ ■

Global Properties

Theorem 5.2.5. *Let f be differentiable on interval (a, b)*

*a) f is monotone increasing on (a, b) **iff** $f'(x) \geq 0$ for every x in the interval and vice versa*

b) If $f'(x) > 0$ for every x in interval, then f is strictly increasing on the interval, and vice versa.

c) If $f'(x) = 0$ for every x in the interval, then f is constant on the interval.

Proof.

a) $\Longleftarrow$ If f is monotone increasing on interval, then same argument for pointwise case. For every $x_1, x_2 \in (a, b)$, $x_1 < x_2$, $f(x_1) < f(x_2)$, the difference quotient is ≥ 0 because of the numerator. Hence shown.

$\Longrightarrow$ Suppose $f'(x) \geq 0$ on (a, b) and $x_1 < x_2$. We can apply the mean value theorem on compact set $[a, b]$. We know f is continuous as f is differentiable. MVT gives us $f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_2}$. Since $f'(x_0) \geq 0$, we get $f(x_2) - f(x_1) \geq 0$ hence f is monotonically increasing.

b) We use part *a)* to argue that by MVT, if $f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_2}$, then $f(x_2) - f(x_1) > 0$ and hence it is strictly increasing.

c) This time from $f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_2}$ and $f'(x_0) = 0$, we obtain $f(x_2) - f(x_1) = 0$ hence $f(x_2) = f(x_1)$

■

Topic 5.3 — Calculus of Derivatives

Product and Quotient Rules

Remark. If f is a function, we write $\Delta_h f(x) = f(x + h) - f(x)$. Hence we get

$$f'(x) = \lim_{h\to 0} \frac{\Delta_h f(x)}{h}$$

So how do the operators look on functions?

$$\begin{aligned}\Delta_h(f \pm g)(x) &= (f(x + h) \pm g(x + h)) - (f(x) \pm g(x)) \\ &= \Delta_h f(x) \pm \Delta_h g(x)\end{aligned}$$

$$\begin{aligned}\Delta_h(f \cdot g)(x) &= f(x + h)g(x + h) - f(x)g(x) \\ &= f(x + h)\big[g(x + h) - g(x)\big] \\ &\quad + g(x)\big[f(x + h) - f(x)\big] \\ &= f(x + h)\Delta_h g(x) + g(x)\Delta_h f(x)\end{aligned}$$

$$\begin{aligned}\Delta_h(f \cdot g)(x) &= f(x)\big[g(x + h) - g(x)\big] \\ &\quad + g(x + h)\big[f(x + h) - f(x)\big] \\ &= f(x)\Delta_h g(x) + g(x + h)\Delta_h f(x)\end{aligned}$$

$$\begin{aligned}\Delta_h\left(\frac{f}{g}\right)(x) &= \frac{f(x + h)}{g(x + h)} - \frac{f(x)}{g(x)} \\ &= \frac{f(x + h)g(x) - f(x)g(x + h)}{g(x + h)g(x)} \\ &= \frac{g(x)\big[f(x + h) - f(x)\big] - f(x)\big[g(x + h) - g(x)\big]}{g(x)g(x + h)} \\ &= \frac{g(x)\Delta_h hf(x) - f(x)\Delta_h g(x)}{g(x)g(x + h)}\end{aligned}$$

Theorem 5.3.1. *If f and g are differentiable at x_0, then $f \pm g$ and $f \cdot g$ are also differentiable at x_0 and $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$, $(f \cdot g)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$. In addition, if $g(x_0) \neq 0$, then f/g is differentiable at x_0, and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Proof. Assumptions about differentiability mean $\lim_{h\to 0} \Delta_h f(x_0)/h = f'(x_0)$ and $\lim_{h\to 0} \Delta_h g(x_0)/h = g'(x_0)$. We know g is differentiable at x_0 implies it is continuous at x_0, so $\lim_{h\to 0} g(x_0 + h) = g(x_0)$. We have

$$\lim_{h\to 0} \frac{\Delta_h(f/g)(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{g(x_0)\Delta_h f(x_0) - f(x_0)\Delta_h(x_0)}{hg(x_0 + h)g(x_0)}$$

$$\vdots$$

$$= \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

■

Chain Rule

Remark. For chain rule, we use notation $g \circ f(x) = g(f(x))$.

$$\begin{aligned}
\Delta_h g \circ f(x) &= g(f(x+h) - g(f(x))) \\
&= g(f(x) + (f(x+h) - f(x))) - g(f(x)) \\
&= (\Delta_{\Delta_h f(x)} g)(f(x))
\end{aligned}$$

Inverse Function Theorem