

AMS 553.430 - Introduction to Statistics

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Introduction

Math 553.430 is one of the most important courses that is required/recommended for the engineering-based majors at Johns Hopkins University.

These notes are being live-Texed, through I edot for Typos and add diagrams requiring the *TikZ* package separately. I am using Texpad on Mac OS X.

I would like to thank Zev Chonoles from The University of Chicago and Max Wang from Harvard University for providing me with the inspiration to start live-Texing my notes. They also provided me the starting template for this, which can be found on their personal websites.

Please email any corrections or suggestions to ksriniv4@jhu.edu.

Lecture 0 (2018-08-30)

Introduction to Probability (553.420) Review

Part 1 - Counting

① Multiplication rule (Basic Counting Principle)

② Combinations/Permutations

- Sampling with or without replacement. \Rightarrow Inclusion-Exclusion Principle

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad {}^nP_k = \frac{n!}{(n-k)!}$$

③ Birthday Problem

④ Matching Problem (inclusion-exclusion principle)

$$- P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$- P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

- etc...

⑤ n balls going into m boxes (all are distinguishable)

Example. n balls numbered $1, 2, \dots, n$. n boxes labelled $1, 2, \dots, n$. Distribute the balls into the boxes, one in each box. M_i = ball i is in box i

⑥ Multinomial Coefficients e.g. assign A, B, C, D, to different students \rightarrow anagram problem
- n distinct objects into r distinct groups

$$\frac{n!}{n_1!n_2!n_3!\dots n_r!} = \binom{n}{n_1, n_2, n_3, \dots, n_r}$$

⑦ Pairing Problem

$$2n \text{ people, paired up } \begin{cases} \text{ordered: } \binom{2n}{2, 2, \dots, 2} & \text{e.g. different courts for players} \\ \text{unordered: } \frac{\binom{2n}{2, 2, \dots, 2}}{n!} \end{cases}$$

⑧ Partition of integers $\rightarrow n$: sum of integer, r : number of partitions

$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

Basics of Probability

Axioms

- ① $0 \leq P(A) \leq 1, \forall A$
- ② $P(\Omega) = 1 \rightarrow$ where Ω is the sample space
- ③ Countable additivity
 - if A_1, \dots, A_n are mutually exclusive, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(A_1) + P(A_2) + \dots = \sum_{i=1}^{\infty} P(A_i)$$

$$\Rightarrow P(A) = 1 - P(A^c)$$

$$P(A) = \frac{|A|}{|\Omega|}$$

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Law of Total Probability

$$P(A) = \sum_j P(A|B_j)P(B_j) = \sum P(A \cap B_j) \quad \underbrace{\bigcup_j B_j = \Omega}_{\text{partition of } \Omega}$$

Bayes Rule

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)} \quad \underbrace{\bigcup_j B_j = \Omega}_{\text{partition of } \Omega}$$

Independent events

If we have events A_1, A_2, \dots, A_n , then

$$P(A_1 \cap A_2 \cap A_3 \dots A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \dots \cdot P(A_n)$$

Introduction to Discrete and Continuous Random Variables

Random Variable - a real valued function defined on the sample space of an experiment $X : \Omega \rightarrow \mathbb{R}, \forall \omega \in \Omega, X(\omega) \in \mathbb{R}$

| Function | Discrete | Continuous |
|--------------------------|------------------------------|------------------------------|
| Probability Function | PMF: $P(X = x)$ | PDF: $f_x(x)$ |
| Probability Distribution | $\sum_x P(X = x) = 1$ | $\int_x f_x(x)dx = 1$ |
| Expectation | $E[X] = \sum_x xP(X = x)$ | $E[X] = \int_x xf(x)dx$ |
| Variance | $Var[X] = E[X^2] - (E[X])^2$ | $Var[X] = E[X^2] - (E[X])^2$ |

Law of the Unconscious Statistician (LOTUS)

$$\text{1-dim} \quad E[g(x)] = \sum_x g(x)P(X = x) \quad \Bigg/ \quad E[g(x)] = \int_x g(x)f(x)dx$$

$$\text{2-dim} \quad E[g(X, Y)] = \sum_y \sum_x g(x, y)P(X = x, Y = y) \quad \Bigg/ \quad E[g(X, Y)] = \int_y \int_x g(x, y)f(x, y)dxdy$$

Discrete Distributions

- | | |
|--------------------------|--------------------------------|
| 1. Bernoulli(p) | 4. Geometric(p) |
| 2. Binomial(n, p) | 5. Negative Binomial(n, p) |
| 3. Poisson (λ) | 6. Hypergeometric(N, M, n) |

Bernoulli Distribution

X is a random variable with Bernoulli(p) distribution

$$X \sim \text{Bernoulli}(p)$$

$$P(X = x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

Binomial Distribution

A sum of i.i.d. (identical, independent distribution) Bernoulli(p) R.V.

$$X \sim \text{Binomial}(n, p)$$

Support : $x \in \{0, 1, \dots, n\}$

n : sample size p : probability of success

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{(n-k)}$$

$$E[X] = np \qquad \qquad \qquad Var(X) = np(1 - p)$$

- Approximation methods \Rightarrow

- if n is large, p very small and $np < 10$. \Rightarrow use Normal $(np, np(1-p))$
- $p \approx \frac{1}{2} \Rightarrow$ Use Poisson $(\lambda = np)$
- Mode:
 - if $(n+1)p$ integer, mode = $(n+1)p$ or $(n+1)p - 1$.
 - if $(n+1)p \notin \mathbb{Z}$ mode is $\lfloor (n+1)p \rfloor$
 - **Proof:** consider $\frac{P(X=x)}{P(X=x-1)}$ going below 1.

Poisson Distribution

$$\begin{aligned}
 X &\sim \text{Poisson}(\lambda) \\
 x &\in \{0, 1, \dots\} \\
 \lambda &: \text{parameter} \\
 P(X=x) &= \frac{e^{-\lambda} \lambda^x}{x!} \\
 E[X] &= \lambda & \text{Var}(X) &= \lambda
 \end{aligned}$$

- Approximations
 - if n is large \Rightarrow Normal (λ, λ)
- Sums of Poisson

Let $X \sim Po(\lambda)$ $Y \sim Po(\mu)$ \Rightarrow $X + Y \sim Po(\mu + \lambda)$

Negative Binomial

$$\begin{aligned}
 X &\sim NB(r, p) \\
 \text{Support : } x &= \{r, r+1, \dots\} \\
 r &= \text{the } r\text{th success} \\
 p &= \text{probability of success} \\
 P(X=k) &= \binom{k+r-1}{k} \cdot (1-p)^r \cdot p^k
 \end{aligned}$$

A sum of i.i.d Geometric(p) R.V.

■ a^{th} head before b^{th} tail

Example. A coin has probability p to land on a head, $q = 1 - p$ to land on a tail.

$P[5^{th} \text{ tail occurs before the } 10^{th} \text{ head}]?$

$$\left\{ \begin{array}{l} = P[5^{th} \text{ tail occurs before or on the } 14^{th} \text{ flip}] \\ = P[\text{Neg Binomial}(5, q) = 5, 6, 7, \dots, 14] \\ = \sum_{x=5}^{14} \binom{x-1}{4} q^5 p^{x-5} \end{array} \right. \quad (\text{or}) \quad \left\{ \begin{array}{l} = P[\text{at least } 5 \text{ tails in } 14 \text{ flips}] \\ = P[\text{binom}(14, q) = 5, 6, 7, \dots, 14] \\ = \sum_{x=5}^{14} \binom{14}{x} q^x p^{14-x} \end{array} \right.$$

Geometric Distribution

$$\begin{aligned} X &\sim \text{Geometric}(p) \\ \text{Support} : x &\in \{1, 2, \dots\} \\ p &: \text{probability of success} \\ P(X = r) &= (1 - p)^{(r-1)} \cdot p \\ \text{prob for 1st success on } r\text{th trial} \\ E[X] &= \frac{1}{p} \qquad \qquad \qquad \text{Var}(X) = \frac{1-p}{p^2} \end{aligned}$$

Example. ■ Coupon Question

Variation A: N different types of coupons $\rightarrow P(\text{get a specific type}) = \frac{1}{N}$

Question: $E[\text{draws to get } 10 \text{ different coupons}]?$

Answer:

$$X = X_1 + X_2 + \dots + X_{10} \qquad X_i = \# \text{ draws to get the } i\text{th distinct coupon type}$$

$$\boxed{X_i \sim \text{Geo}(p_i)} \qquad p_i : \text{prob to get a new coupon} \leftarrow \text{success, given that we have } i-1 \text{ types of coupons}$$

Hence, $E[X_1] = 1$

$$E[X_2] = \frac{1}{p_2} = \frac{1}{\frac{N-1}{N}} = \frac{N}{N-1}$$

$$E[X_3] = \frac{1}{p_3} = \frac{1}{\frac{N-2}{N}} = \frac{N}{N-2}$$

\vdots

$$E[X_{10}] = \frac{1}{p_{10}} = \frac{1}{\frac{N-9}{N}} = \frac{N}{N-9}$$

$$\text{So, } E[X] = E[X_1] + E[X_2] + \dots + E[X_{10}] = E\left[\sum_{i=1}^{10} X_i\right] = 1 + \frac{N}{N-1} + \frac{N}{N-2} + \dots + \frac{N}{N-9}$$

Variation B: Same setting, now you draw 10 times.

Question: $E[\# \text{ different types of coupons}]?$

Answer:

$$X = I_1 + I_2 + \dots + I_N$$

$$I_i \begin{cases} 1 & \text{if we have this type of coupon} \\ 0 & \text{o/w} \end{cases}$$

$$\begin{aligned}
E[I_i] &= P(\text{we draw coupon } i \text{ in } 10 \text{ draws}) \\
&= 1 - P(\text{we don't have coupon } i) \quad \text{we use binomial distribution where } 1 - P(N = 0) \\
&= 1 - \left(\frac{N-1}{N}\right)^{10}
\end{aligned}$$

$$E[X] = E\left[\sum_{i=1}^N I_i\right] = NE[I_i] = \boxed{N\left[1 - \left(\frac{N-1}{N}\right)^{10}\right]}$$

Hypergeometric Distribution

$$\begin{aligned}
X &\sim \text{Hyp}(N, M, n) \\
N &\in \{0, 1, 2, \dots\} \quad M \in \{0, 1, \dots, N\} \quad n \in \{0, 1, \dots, N\} \\
\text{Support : } k &\in \{\max(0, n + M - N), \min(n, M)\} \\
N &\text{ is the population size} \quad K \text{ is the no. of success states in the population} \\
n &\text{ is the no. of draws (i.e. quantity drawn in each trial)} \\
k &\text{ is the no. of observed successes} \\
P(X = k) &= \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}}
\end{aligned}$$

Continuous Distributions

Uniform Distribution

$$\begin{aligned}
X &\sim \text{Unif}(a, b) \\
f_X(x) &= \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{o/w} \end{cases} \\
E[X] &= \frac{a+b}{2} \qquad \qquad \qquad \text{Var}(X) = \frac{(b-a)^2}{12}
\end{aligned}$$

Normal Distribution

$$\begin{aligned} X \sim N(\mu, \sigma^2) &\Rightarrow Z = \frac{X - \mu}{\sigma} \sim N(0, 1) \text{ with CDF } P(Z \leq z) = \Phi(z) \\ \Phi(-x) &= 1 - \Phi(x) \\ \text{Support: } x &\in (-\infty, \infty) \\ f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ E[X] &= \mu \qquad \qquad \qquad \text{Var}(X) = \sigma^2 \end{aligned}$$

- Sums and differences of Normal R.V.

$$\begin{array}{cc} X_1 \sim N(\mu, \sigma^2) & X_2 \sim N(\mu, \sigma^2) \\ Y_1 = X_1 + X_2 & Y_2 = X_1 - X_2 \\ \underbrace{Y_1 \sim N(2\mu, 2\sigma^2)}_{\text{has } \mu} & \underbrace{Y_2 \sim N(0, 2\sigma^2)}_{\text{doesn't have } \mu} \end{array}$$

- The sum and difference of Normal R.V. are Normal R.V.
- Any Linear Combination of Independent Normal R.V. is a Normal R.V.
- Dependence
 - $Y_2 = X_1 - X_2$ density does not depend on μ . But density of $X_1 + X_2$ does.
 - Key idea is used in Data Reduction

Exponential distribution

$$\begin{aligned} X &\sim \text{Exp}(\lambda) \\ \text{Support: } x &\in [0, \infty) \\ f_X(x) &= \lambda e^{-\lambda x} \\ E[X] &= \frac{1}{\lambda} \qquad \qquad \qquad \text{Var}(X) = \frac{1}{\lambda^2} \end{aligned}$$

Lack of memory property: $P(X \geq s + t | X \geq t) = P(X \geq s)$

- $M = \min \text{ of } \text{exp}(\lambda) \text{ and } \text{exp}(\mu) \Rightarrow M \sim \text{exp}(\lambda + \mu)$
- $M = \min \text{ of } X_1, X_2, \dots, X_n, \text{ where } X_i \sim_{\text{i.i.d.}} \text{exp}(\lambda) \Rightarrow \text{exp}(n\lambda)$

Gamma Distribution

$$X \sim \text{Gamma}(\alpha, \beta)$$

$$\text{Support: } x \in [0, \infty)$$

$$F_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

$$E[X] = \frac{\alpha}{\beta}$$

$$\text{Var}(X) = \frac{\alpha}{\beta^2}$$

$$\textbf{Gamma Function: } \Gamma(z) = (z-1)! = \int_0^\infty x^{z-1} e^{-x} dx$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

- Sums of Gamma

$$- \underset{\text{ind}}{\text{Gamma}(s, \lambda)} + \text{Gamma}(s, \lambda) = \text{Gamma}(s+t, \lambda)$$

Beta Distribution

$$X \sim \text{Beta}(\alpha, \beta)$$

$$\text{Support: } x \in [0, 1]$$

$$f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

$$E[X] = \frac{\alpha}{\alpha + \beta}$$

$$\text{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

- Gamma to Beta

$$X \sim \text{Gamma}(\alpha_1, \beta) \quad Y \sim \text{Gamma}(\alpha_2, \beta)$$

$$\text{Then transformation } U = \frac{X}{X+Y} \sim \text{Beta}(\alpha_1, \alpha_2) \quad (\text{Use } X = UV, Y = V - UV)$$

Chi-Square

Chi-Square: χ_n^2 is Chi-square with degrees of Freedom n

$$\chi_n^2 = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \quad \text{where } Z_i \sim \text{standard normal. } Z_i \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\Rightarrow \chi_n^2 = n \text{ i.i.d. } Z_i \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

CDF in General

- $F_x(t) = P(X \leq t)$

$$= \sum_{x \leq t} P(X = x) \quad \text{discrete}$$

$$= \int_{-\infty}^t f(x) dx \quad \text{continuous}$$

- **Discrete:** "Left open, right closed" \Rightarrow if you flip the sign (from $<$ to \leq) in the left, you flip the sign of a (from a to a^-)

$$- P(a < x \leq b) = F(b) - F(a)$$

$$- P(a \leq x \leq b) = F(b) - F(a^-)$$

$$- P(a < x < b) = F(b^-) - F(a)$$

$$- P(a \leq x < b) = F(b^-) - F(a^-)$$

- **Continuous:** (because a point doesn't have a mass)

$$P(a \leq x \leq b) = \int_a^b f(x) dx = F(b) - F(a)$$

Integration by Recognition

$$1 = \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx \quad \sigma\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx \quad (\text{normal dist.})$$

Joint Distribution

Discrete

$$P_{X,Y}(x, y) = P(X = x, Y = y)$$

$$\text{Indep} \Rightarrow P_X(x)P_Y(y)$$

Continuous

$$F_{X,Y}(x, y) = F_X(x)f_Y(y)$$

$$= \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$$

- **Marginal Density/PMF:**

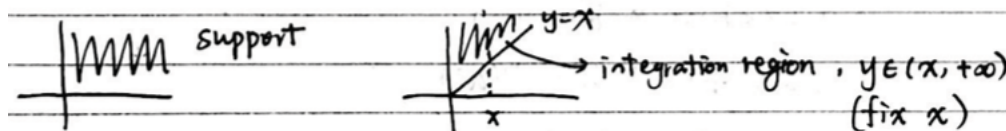
$$\text{Continuous:} \quad f_X(x) = \int_x f_{X,Y}(x, y) dy \quad \text{and} \quad f_Y(y) = \int_y f_{X,Y}(x, y) dx$$

** the bounds for y in the integration can depend on x , and vice versa*

$$\text{Discrete:} \quad P_X(x) = \sum_y P(X = x, Y = y) \quad \text{and} \quad P_Y(y) = \sum_x P(X = x, Y = y)$$

- Use joint pdf to compute probability

e.g. $P(X < Y) = \int_0^{\infty} \int_x^{\infty} f(x, y) dy dx$ assume $x > 0, y > 0$



- **Independence:** If X, Y are independent, then

Continuous: $f(x, y) = f_X(x)f_Y(y)$

Discrete: $P(X = x, Y = y) = P(X = x)P(Y = y)$

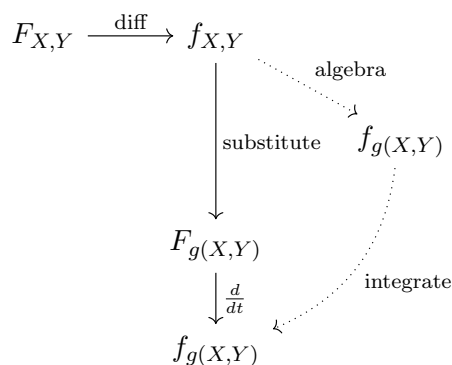
- **Convolution:** assume X, Y are independent

Discrete: $P_{X+Y}(a) = \sum_y P_X(a-y)P_Y(y) = \sum_x P_X(x)P_Y(a-x)$

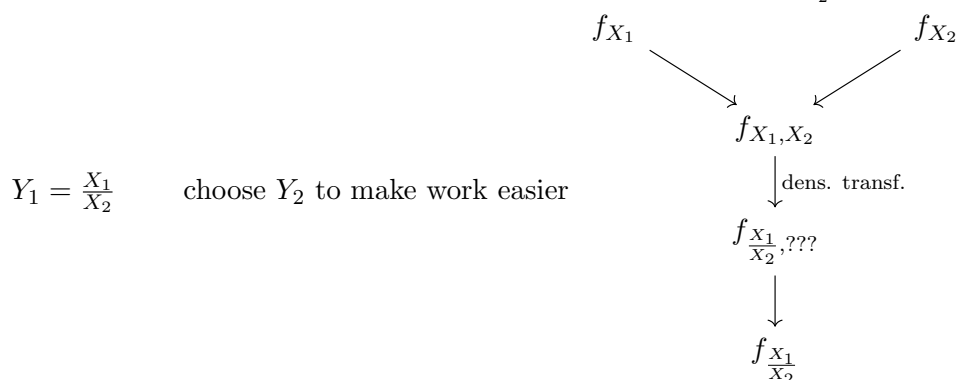
Continuous: $f_{X+Y}(a) = \int_y f_X(a-y)f_Y(y)dy = \int_y f_X(x)f_Y(a-x)dx$

MGF: we can use this $M_{X+Y}(t) = M_X(t)M_Y(t) \rightarrow$ then identify dist of $X+Y$ from mgf

- **Density Transformation:**



X_1 & X_2 are indep r.v. \Rightarrow want to find density of $\frac{X_1}{X_2}$



Density Transformation

For density transformation e.g. finding pdf of $U = X + Y$

- Convolution
- MGF
- Jacobian
- CDF Transformation

- Use CDF: Computer $P(Y \leq y) = P(g(x) = y)$

- **1-dim:** If Y is monotonically increasing or decreasing: $Y = g(x)$ $f_Y(y) = f_X(x(y)) \cdot |(x^{-1})'(y)|$

- **2-dim:** Joint Density:

$$(X, Y) \rightarrow (U, V) \quad U = h_1(X, Y) \quad V = h_2(X, Y)$$

$$f_{U,V}(u, v) = f_{X,Y}(x(u, v), y(u, v)) \cdot |J|$$

$$\text{where } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{determinant}$$

- if $Z = X + Y$ (2-dim \rightarrow 1-dim) use CDF. Compute $P(Z \leq z) = P(X + Y \leq z)$. Integrate $f(x, y)$ over this region.

Sterling's Formula

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

This is only really useful when n is large, when factorials are represented as ratios.

Conditional distribution

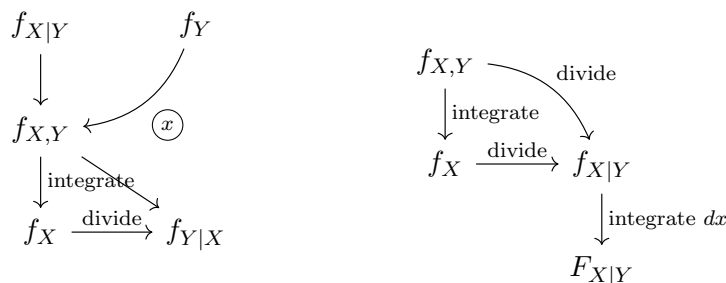
$$\text{Discrete} \quad P_{X|Y=y}(x|y) = \frac{P_{X,Y}(x, y)}{P_Y(y)} = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$\Rightarrow \sum_y P_{X,Y}(x, y) = \sum_y P_{X|Y=y}(x|y) \cdot P_Y(y)$$

$$\text{Continuous} \quad f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$\Rightarrow f_X(x) = \int_y f(x, y) dy = \int_y f_{X|Y=y}(x|y) \cdot f_Y(y) dy$$

$$F_{X|Y}(x|y) = \int_{-\infty}^x f_{X|Y}(x|y) dx$$



Conditional Expectation

$$E[X|Y = y] = \sum_x xP(X = x|Y = y)$$

$$E[X|Y = y] = \int_x xf(x|y)dx$$

$E[X|Y]$: compute $E[X|Y = y]$ first, replace y with Y

• Properties:

- $E[aU + bV|Y = y] = aE[U|Y = y] + bE[V|Y = y]$ *LOTUS*
- If $g(Y) = X$ then $E[X|Y = y] = X$
- If X and Y are independent, then $E[X|Y = y] = E[X]$

Conditional Variance

$$\boxed{Var(X|Y) = E[(X - E[X|Y])^2]} \quad (\text{conditional variance})$$

$$\boxed{Var(X|Y) = E[X^2|Y] - (E[X|Y])^2} \quad (\text{unconditional variance})$$

Ordered Statistics

Consider X_1, X_2, \dots, X_n $X_{(j)}$ = j-th smallest

$$F_{\max(X_i)}(t) = P(\max X_i \leq t) = P(X_1 \leq t) \cdot P(X_2 \leq t) \cdots P(X_n \leq t)$$

$$= [F_X(t)]^n \quad \boxed{f_{\max X_i}(t) = nF(t)^{n-1}f_X(t)}$$

$$F_{\min(X_i)}(t) = 1 - P(\min x_i \geq t) = 1 - P(X_1 \geq t) \cdot P(X_2 \geq t) \cdots P(X_n \geq t)$$

$$= 1 - [1 - F_X(t)]^n \quad \boxed{f_{\min X_i}(t) = n[1 - F(t)]^{n-1}f_X(t)}$$

General: j -th order statistic

$$f_{x(j)}(t) = \binom{n}{j-1, 1, n-j} F_X(t)^{j-1} \cdot f_X(t) \cdot [1 - F_X(t)]^{n-j}$$

As Beta distribution: Let $U_1, U_2, \dots, U_N \sim i.i.d.$ Uniform(0, 1) and let $1 \leq j \leq N$
 $U_{(j)}$ = jth smallest in $U_{(1)}, U_{(2)}, \dots, U_{(N)}$ (ordered statistics). Then,

$$U_{(j)} \sim \text{Beta}(j, N - j + 1)$$

$$E[U_{(j)}] = \frac{j}{N + 1}$$

Expectation and Variance

Law of Total Expectation:

$$E[X] = E[E[X|Y]]$$

Law of Total Variance:

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$$

Expectation

① linearity of expectation

② How to compute

(a) LOTUS or definition (use density to integrate)

(b) MGF: $M^{(n)}(0) = E[X^n]$ or by recognition

(c) $E[X^2] = Var[X] + E[X]^2$

(d) Tail probability X is non-neg R.V. ($x > 0$) then $E[X] = \sum_{t=0}^{\infty} P(X \geq t)$ or $= \int_0^{\infty} P(X \geq t) dt$

Variance

① $Var(X_1 + X_2 + \cdots + X_n) = \sum_{i=1}^n Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$

if X_i, X_j identical (not independent) $= nVar(X_i) + n(n-1)Cov(X_i, X_j)$ $i \neq j$

$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

② **Covariance:**

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

$$Cov(X, c) = 0 \quad c \text{ is a constant}$$

$$Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$$

$$Cov(cX, dZ) = cd \cdot Cov(X, Z)$$

$$Cov(aX + b, cY + d) = ac \cdot Cov(X, Y) \quad a, b, c, d \text{ are constants}$$

$$Cov(X, Y) = 0 \quad \text{If } X \perp Y \text{ (independent)}$$

③ **Correlation Coefficient:**

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = \frac{Cov(X, Y)}{\sigma_x \sigma_y}$$

MGFs

Let X be a random variable. Then

$$M_X(t) = E[e^{tX}]$$

it can also be written as:

$$\begin{aligned} &= E\left[\sum_{j=0}^{\infty} \frac{(tX)^j}{j!}\right] \\ &= E\left[\sum_{j=0}^{\infty} \left(\frac{X^j}{j!} \cdot t^j\right)\right] \\ &\boxed{M_X^{(n)}(0) = E[X^n]} \end{aligned}$$

If X and Y are independent, then

$$\begin{aligned} M_{X+Y}(t) &= E[E^{(X+Y)t}] \\ &= E[e^{tX}]E[e^{tY}] \\ &= M_X(t)M_Y(t) \end{aligned}$$

Limit Theorems

Markov's Inequality

For any non-negative random variable X

$$P(X \geq a) \leq \frac{E(X)}{a} \quad (\text{for } \underline{\text{any}} \ a > 0)$$

Proof. Let $X \geq 0$ a random variable and let $a > 0$. Define new random variable from X as Y_a

$$\begin{aligned} Y_a &= \begin{cases} 0 & \text{if } X < a \\ a & \text{if } X \geq a \end{cases} \\ 0 \leq Y_a \leq X &\implies \underbrace{E[Y_a]}_{a \cdot P(X \geq a)} \leq E[X] \\ E[Y_a] &= 0 \cdot P(Y_a < a) + a \cdot P(X \geq a) \\ E[Y_a] = a \cdot P(X \geq a) \leq E[X] &\implies \boxed{P(X \geq a) \leq \frac{E(X)}{a}} \end{aligned}$$

■

Chebyshev's Inequality

For any random variable Y with mean μ_y and variance σ_y^2

$$P(|Y - \mu| \geq c) \leq \frac{\sigma_y^2}{c^2} \quad (\text{for } \underline{\text{any}} \ c > 0)$$

Proof.

$$P(|Y - \mu_y| \geq c) = P(\underbrace{|Y - \mu_y|^2}_{=X} \geq c^2)$$

$$P(|Y - \mu_y|^2 \geq c^2) \leq \frac{E[|Y - \mu_y|^2]}{c^2} = \frac{\sigma_y^2}{c^2}$$

■

This is the same as

$$- P(|Y - \mu_y| \geq k\sigma_y) \leq \frac{1}{k^2}$$

$$- P(|Y - \mu_y| \leq k\sigma_y) \geq \underbrace{1 - \frac{1}{k^2}}_{\text{very conservative}}$$

Central Limit Theorem

$$\sum_{i=1}^n X_i \sim \mathcal{N}(n\mu_n, n\sigma_x^2)$$

$$\frac{1}{n} \sum_{i=1}^n X_i \sim \mathcal{N}\left(\mu_n, \frac{\sigma_x^2}{n}\right)$$

Weak Law of Large Numbers

If X_1, X_2, \dots are *i.i.d.* with a mean μ

$$\text{then } \lim_{n \rightarrow \infty} P(|\bar{X}_n - \mu| \geq \epsilon) = 0$$

Strong Law of Large Numbers

$$X \xrightarrow{p} \mu_X \quad \text{as } n \rightarrow \infty$$

$$Pr\left(\lim_{n \rightarrow \infty} \bar{X}_n = \mu\right) = 1$$

Lecture 1 (2018-08-30)

Survey Sampling

We have a population of objects under study (people, animals, places, etc.). We will consider a single numerical measurement associated to object i : x_i

Example. $N = 5000$, x_i = height of person i , Population size = N . We denote population measurements $\{x_1, x_2, \dots, x_N\}$

Compute population quantities:

- population total $\tau = \sum_{i=1}^N x_i$
- population mean $\mu = \frac{\tau}{N} = \frac{\sum_{i=1}^N x_i}{N}$

Note: τ and μ are population parameters, their computation depends on all the population data.

Question. How to estimate τ and μ based on a sample of observation from this population?

Classical Answer: Choose a "random" sample of objects and associated measurements denoted $\{x_1, x_2, \dots, x_n\}$. *Note:* capital X_i denote random variables.

Whiter "Random"? Two types of ways to sample:

– without replacement

– with replacement

Claim 1. If X_i are drawn without replacement, then the distribution of X_1 and X_2 are identical. Is this true? **In fact, it is** \Rightarrow They are **NOT** independent but they are identically distributed.

$$P(\text{Ace in Pos 1}) = P(\text{Ace in Pos 2}) = \frac{4}{52}$$

Combinatorial Approach

"well-shuffled deck" \leftrightarrow all $52!$ rearrangements of the card are equally likely. How many rearrangements have ace at pos 1? $4 \cdot 51!$

$$P(A_1) = \frac{4 \cdot 51!}{52!} = \frac{4}{52} = P(A_2) = P(A_{19}) = P(A_{36})$$

Question. If X_1 and X_2 are identically distributed, then how do they differ between corresponding draws with replacement?

Answer. Independence. We can have Random Variables that are identically distributed and not independent. Note if independent, $P(A_2|A_1) = P(A_2)$.

with replacement

$$P(A_1) = \frac{4}{52}, \quad P(A_2) = \frac{4}{52}$$
$$P(A_2|A_1) = \frac{4}{52}$$

without replacement

$$P(A_1) = \frac{4}{52}, \quad P(A_2) = \frac{4}{52}$$
$$P(A_2|A_1) = \frac{3}{51}$$

We can see from this that depending on sampling method, we gain or lose independence. In the finite population sampling method, we have $1, \dots, N$ objects we care about.

Loss of Independence when choosing sampling method is important.

Lecture 2 (2018-09-05)

Finite Population sampling – without replacement. Mean/expected value and variance of \bar{X}

Suppose our population is given by $\{x_1, \dots, x_N\} = \{1, 2, 2, 7, 8, 9\}$ where

$$N = 6, \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 2 \quad x_4 = 7 \quad x_5 = 8 \quad x_6 = 9$$

Could also describe it by counting.

| Distinct Value | frequency |
|-----------------|-----------|
| $\varphi_1 = 1$ | $n_1 = 1$ |
| $\varphi_2 = 2$ | $n_2 = 2$ |
| $\varphi_3 = 7$ | $n_3 = 1$ |
| $\varphi_4 = 8$ | $n_4 = 1$ |
| $\varphi_5 = 9$ | $n_5 = 1$ |

Possible sample of size $n = 6$, where we sample without replacement

$$X_1 = 7 \quad X_2 = 2 \quad X_3 = 8 \quad X_4 = 9 \quad X_5 = 1 \quad X_6 = 2$$

Sample here is the same as population as $\bigcirc n=N$

Same thing with replacement

$$X_1 = 9 \quad X_2 = 9 \quad X_3 = 9 \quad X_4 = 9 \quad X_5 = 9 \quad X_6 = 9$$

Typically N is large and $n \ll N$

Recall population parameters

$$\mu = \frac{\sum_{i=1}^N X_i}{N} \quad \tau = N\mu = \sum_{i=1}^N X_i$$

Next, σ^2 (population variance)

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2 \quad (\sigma^2 \text{ is pop. variance})$$

Alternatively, we can also express σ^2 as

$$\begin{aligned} \sigma^2 &= \frac{\sum_{i=1}^N (x_i - \mu)^2}{N} = \frac{\sum_{i=1}^N (x_i^2 - 2\mu x_i + \mu^2)}{N} \\ &= \frac{\sum_{i=1}^N x_i^2}{N} - \frac{2\mu}{N} \underbrace{\sum_{i=1}^N x_i}_{\mu} + \frac{N\mu^2}{N} \\ &= \frac{\sum_{i=1}^N x_i^2}{N} - 2\mu^2 + \mu^2 \end{aligned}$$

$$= \underbrace{\left(\frac{1}{N} \sum_{i=1}^N x_i^2 \right)}_{\text{2nd moment}} - \mu^2 = \mu^{(2)} - \mu^2$$

Define: $\mu^{(k)} = \frac{1}{N} \sum_{i=1}^N x_i^k$

Sample Mean \bar{X} as an estimator

A function of the sample data for the population μ .

Note: If the sample is random (X_1, \dots, X_n are R.Vs), then \bar{X} is **random!**

Questions:

- ① How is \bar{X} distributed? - in theory, if we know ①, then we know the answers ② & ③ too.
- ② What is $E[\bar{X}]$?
- ③ What is $Var(\bar{X})$?

Let's address ②

Consider $E[\underbrace{X_1}_{\text{first draw}}]$

possible values for $X_1 = \{x_1, \dots, x_N\}$

$$P(X_1 = x_k) = \frac{1}{\binom{N}{1}} = \frac{1}{N}$$

e.x. $\{\underbrace{1}_{x_1}, \underbrace{2}_{x_2}, \underbrace{2}_{x_3}, \underbrace{7}_{x_4}, \underbrace{7}_{x_5}, \underbrace{9}_{x_6}\}$ gives every separate entry a unique ticket even if they are the same

$$E[X_1] = \frac{1}{N} \sum_{k=1}^N x_k = \mu = E[X_2] \quad (\text{b/c } X_1 \text{ \& } X_2 \text{ are identically dist.})$$

In sampling without replacement X_i & X_j are still identically distributed, but they are not independent.

In sampling with replacement, X_i & X_j are *i.i.d.*

Note that whether or not X_1, \dots, X_n are independent,

$$E\left[\sum_{i=1}^N X_i\right] = \sum_{i=1}^N E[X_i]$$

Note: The sample mean is equal to expected population mean regardless of sampling with or without replacement.

$$\begin{aligned} E[\bar{X}] &= E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] \\ &= \frac{n\mu}{n} = \mu \end{aligned}$$

Since $E[\bar{X}] = \mu$, we say \bar{X} is an unbiased estimator for μ . **BUT** $\underbrace{\bar{X}}_{\text{R.V.}} \neq \underbrace{\mu}_{\text{constant}}$

Let's address ③

Sampling with replacement.

Theorem. *Sampling from finite population with replacement*

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

Proof. Here X_1, \dots, X_n are *i.i.d.*. In general, X_i 's are R.V. and a_i 's are constants

$$\text{Var}\left(\sum_i a_i X_i\right) = \sum_i \sum_j a_i a_j \text{cov}(X_i, X_j)$$

If X_1, \dots, X_N are independent, $\text{Cov}(X_i, X_j) = 0$! Hence

$$\begin{aligned} \text{Var}(\bar{X}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n \underbrace{\text{Var}(X_i)}_{\text{a constant}} \\ &\boxed{\text{Var}(\bar{X}) = \frac{\text{Var}(X_i)}{n} = \frac{\sigma^2}{n}} \end{aligned}$$

■

We need to compute $\text{Var}(X_i)$. Observe that $\text{Var}(X_i)$ are same for all: *Why?* because they are identical.

Also notice $\frac{\text{Var}(X_i)}{n}$ decreases with n .

Observe that for all finite n , $\text{Var}(\bar{X})$ is not 0 unless $\text{Var}(X_i) = 0$!

Note: $\text{Var}(X_i) = E[(X_i - E(X_i))^2] = E[(X_i - \mu)^2] = \frac{1}{N} \sum (x_i - \mu)^2 = \sigma^2$

So $\text{Var}(X_i) = 0$ **iff** all $X_i \equiv \mu$

Lemma. \bar{X} is consistent for μ , i.e. $\forall \delta > 0$, the $P(|\bar{X} - \mu| > \delta) \rightarrow 0$ as $n \rightarrow \infty$

For this Lemma, we need to Prove Chebyshev's Inequality, which is

$$P(|Z - E(Z)| > \delta) \leq \frac{\text{Var}(Z)}{\delta^2}$$

Use this identity!

$$\begin{aligned} E[\bar{X}] &= \mu, & \text{Var}(\bar{X}) &= \frac{\sigma^2}{n} \\ P(|\bar{X} - E(\bar{X})| > \delta) &\leq \frac{\text{Var}(\bar{X})}{\delta^2} = \frac{\sigma^2}{n\delta^2} \rightarrow 0 && \text{as } n \rightarrow \infty \end{aligned}$$

Lecture 3 (2018-09-10)

Sampling without replacement

$Var(\bar{X})$ = when sampling without replacement

Theorem. *Sampling from finite population without replacement*

$$Var(\bar{X}) = \frac{\sigma^2}{n} \underbrace{\left[\frac{N-n}{n-1} \right]}_{FPN} \quad (\text{finite population correction})$$

Points to Note - In sample without replacement,

- If $n = N$, $Var(\bar{X}) = 0$
- If $n = 1$, $Var(\bar{X}) = \frac{\sigma^2}{n} = \sigma^2$, same as with replacement
- Check: for $n > 1$, how does $\frac{N-n}{N-1}$ relate to 1? The $Var(\bar{X})$ is always less without replacement

Proof. Start

①

$$Var(\bar{X}) = Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_i \sum_j Cov(X_i, X_j)$$

(When sampling with replacement, $Cov(X_i, X_j) = 0$ if $i \neq j$)

In sampling without replacement, we cannot assert that $Cov(X_i, X_j) = 0$ and we'll compute it explicitly.

$$\text{Recall} \quad Cov(X_i, X_j) = E[X_i X_j] - \underbrace{E[X_i]E[X_j]}_{\mu^2}$$

$$\mu^2 \leftarrow \text{as identical but not independent} \quad = E[X_i X_j] - \mu^2$$

② To calculate $E[X_i X_j]$, let us list distinct values in population

Example. $\{\underbrace{5}_{x_1}, \underbrace{5}_{x_2}, \underbrace{8}_{x_3}, \underbrace{11}_{x_4}, \underbrace{8}_{x_5}, \underbrace{17}_{x_6}, \underbrace{9}_{x_7}\}$ Let $n_l = \#$ of times ζ_l appears in population.

| Distinct Value | frequency |
|----------------|-----------|
| $\zeta_1 = 5$ | $n_1 = 2$ |
| $\zeta_2 = 8$ | $n_2 = 2$ |
| $\zeta_3 = 11$ | $n_3 = 1$ |
| $\zeta_4 = 17$ | $n_4 = 1$ |
| $\zeta_5 = 9$ | $n_5 = 1$ |

$$P[X_i = 5] = \frac{2}{7} = \frac{n_1}{N} \quad (\text{i draws identical})$$

$$\Rightarrow P[X_i = \zeta_l] = \frac{n_l}{N}$$

$$n_1 + n_2 + \dots + n_m = \sum_{j=1}^m n_j = N$$

$$E[X_i X_j] = \sum_{k=1}^m \sum_{l=1}^m \zeta_k \zeta_l \underbrace{P[X_i = \zeta_k, X_j = \zeta_l]}_?$$

$$P[X_i = \zeta_k, X_j = \zeta_l] = \underbrace{P[X_j = \zeta_l | X_i = \zeta_k]}_{\textcircled{3}} \cdot \underbrace{P[X_i = \zeta_k]}_{= \frac{n_k}{N}}$$

③ Cases for Conditional probability

$$P[X_j = \zeta_l | X_i = \zeta_k] \stackrel{\text{cases}}{=} \begin{cases} \frac{n_l}{N-1} & l \neq k \rightarrow \text{numbers are diff.} \\ \frac{n_l - 1}{N-1} & l = k \rightarrow \text{numbers are same} \end{cases}$$

④ So we have

$$E[X_i X_j] = \sum_{k=1}^m \sum_{l=1}^m \zeta_k \zeta_l P[X_i = \zeta_k, X_j = \zeta_l]$$

$$E[X_i X_j] = \sum_{k=1}^m \sum_{l=1}^m \zeta_k \zeta_l P[X_j = \zeta_l | X_i = \zeta_k] \cdot P[X_i = \zeta_k]$$

$$= \sum_k \zeta_k P[X_i = \zeta_k] \zeta_k \left(\sum_l \zeta_l P[X_j = \zeta_l | X_i = \zeta_k] \right)$$

$$= \sum_k \zeta_k P[X_i = \zeta_k] \zeta_k \left(\sum_{l \neq k} \zeta_l P[X_j = \zeta_l | X_i = \zeta_k] + \zeta_k P[X_j = \zeta_k | X_i = \zeta_k] \right)$$

$$= \sum_k \zeta_k P[X_i = \zeta_k] \zeta_k \underbrace{\left(\sum_{l \neq k} \zeta_l \frac{n_l}{N-1} + \zeta_k \frac{n_k - 1}{N-1} \right)}_{\textcircled{5}}$$

⑤ When $l \neq k$ and we want to remove all l terms

$$\begin{aligned} \sum_{l \neq k} \zeta_l \frac{n_l}{N-1} &= \frac{1}{N-1} \sum_{l \neq k} \zeta_l n_l \\ \left(\sum_l \zeta_l n_l = \tau = n\mu \right) &\quad \text{population total} \\ &= \frac{1}{N-1} (\tau - \zeta_k n_k) \end{aligned}$$

⑥ Now Back

$$\begin{aligned}
 E[X_i X_j] &= \sum_k \zeta_k \frac{n_k}{N} \left(\frac{1}{N-1} (\tau - \zeta_k n_k) + \zeta_k \frac{n_k - 1}{N-1} \right) \\
 &= \frac{1}{N(N-1)} \sum_k \zeta_k n_k [(\tau - \cancel{\zeta_k n_k}) + \cancel{\zeta_k n_k} - \zeta_k] \\
 &= \frac{1}{N(N-1)} \sum_k \zeta_k n_k [\tau - \zeta_k] \\
 &= \frac{1}{N(N-1)} \left(\sum_k \zeta_k n_k \tau - \sum_k \zeta_k^2 n_k \right) \\
 &= \frac{1}{N(N-1)} \left[\tau^2 - \sum_k \zeta_k^2 n_k \right]
 \end{aligned}$$

⑦ What is $\sum_k (\zeta_k)^2 \frac{n_k}{N}$? Second moment $E[X_i^2]$ $E[X_i^2] = \sigma^2 + \mu^2$

$$\begin{aligned}
 E[X_i^2] &= \sigma^2 + \mu^2 & \frac{\tau^2}{N} &= N \mu^2 \text{ as } \mu = \frac{\tau}{N} \\
 E[X_i X_j] &\Rightarrow \frac{1}{N-1} \left[N \mu^2 - (\sigma^2 + \mu^2) \right] \\
 &= \frac{1}{N-1} [(N-1) \mu^2 - \sigma^2] = \mu^2 - \frac{\sigma^2}{N-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{So } \text{Cov}(X_i, X_j) &= \mu^2 - \frac{\sigma^2}{N-1} - \mu^2 \\
 &= -\frac{\sigma^2}{N-1}
 \end{aligned}
 \tag{Cov < 0}$$

$$\text{So } \text{Cov}(X_i, X_j) = \text{Var}(X_i) = \sigma^2$$

⑧ Putting it all together

$$\begin{aligned}
 \text{Var}(\bar{X}) &= \frac{1}{n^2} \left(\sum_{i \neq j} \text{Cov}(X_i, X_j) + \sum_{i=1}^n \text{Var}(X_i) \right) \\
 &= \frac{1}{n^2} \left(\sum_{i \neq j} -\frac{\sigma^2}{N-1} + n \sigma^2 \right) \\
 &= \frac{1}{n^2} \left(\frac{-n(n-1)\sigma^2}{N-1} + \frac{\sigma^2}{n} \right) \\
 &= \frac{\sigma^2}{n} \left(1 - \frac{n-1}{N-1} \right) \\
 &= \frac{\sigma^2}{n} \left(\frac{N-n}{N-1} \right)
 \end{aligned}$$

