AMS 553.361 - Introduction to Optimization

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Introduction

Math 110.304 is one of the semi-important courses that is required/recommended for the engineering-based majors at Johns Hopkins University.

These notes are being live-TeXed, through I edot for Typos and add diagrams requiring the TikZ package separately. I am using Texpad on Mac OS X.

I would like to thank Zev Chonoles from The University of Chicago and Max Wang from Harvard University for providing me with the inspiration to start live-TeXing my notes. They also provided me the starting template for this, which can be found on their personal websites.

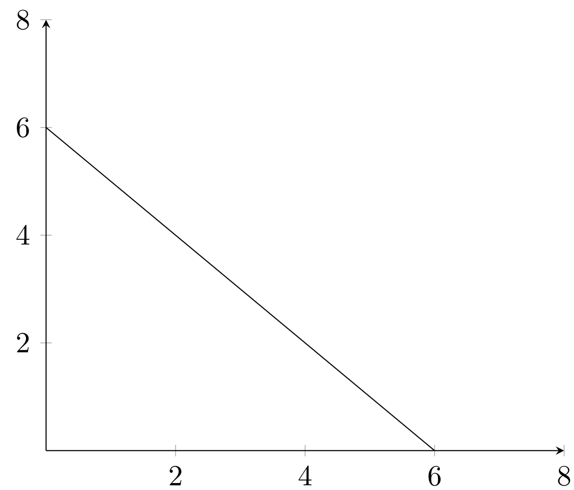
Please email any corrections or suggestions to ksriniv4@jhu.edu.

Lecture 1 (2018-08-30)

Introduction

Example 1. Up to 6 units of two nutrients can be added to solution and we require the number of units of nutrient 2 has to be atleast (natural) logarithm of # units of nutrient one. Goal: choose x_1 = number units of nutrient 1, choose x_2 = number units of nutrient 2. To maximize expected height of plant $1 + x_1^2(x_2-1)^3e^{-x_1-x_2}$

$$\begin{aligned} \text{Maximise} \quad & 1 + x_1^2(x_2-1)^3e^{-x_1-x_2} \\ & x_1 + x_2 \leq 6 \\ & x_2 > \log x_1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$



generic optimization problem: say $\subseteq \mathbb{R}$

$$f : \underbrace{\mathbb{S}}_{\text{feasible region}} \rightarrow \underbrace{\mathbb{R}}_{\text{objective function}}$$

$$x^* \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{optimal}$$

however optimal objective function value is1.2677

Definition 1. We say x^* is an optimal solution if

- $x^* \in \mathbb{S}$
- For any $y \in \mathbb{S}$

Example 2. Find min $\log x$ s.t. $-\infty \leq x \leq 7 \rightarrow$ unbounded, has no solution.

Example 3. Find min $\log x$ s.t. $1 < x \leq 7 \rightarrow$ bounded, but is also no solution. (as we can go 1.000001)

Example 4. Find min $\log x$ s.t. $x > 1$, $x \leq 0.5 \to$ infeasible!!!

Example 5. minimize $3 + (x-1)^2$ s.t. $1 \leq x \leq 3 \to$ feasible as optimal solution $x^* = 2$ is an interior part of feasible region.

$$\begin{aligned} f(x) &= 3 + (x-1)^2 \\ f'(x) &= 0 \\ f''(x) &> 0 \end{aligned}$$

Example 6. Minimize $3 + (x-2)^2$ s.t. $x \geq 10 \to$ optimal solution is $x^* = 10$. but $f'(x) \neq 0$. But is not an interior point - it is a boundary point of feasible region.

Lecture 2 (2018-08-31)

Definition 1. $\forall x \in \mathbb{R}^n$ Euclidian length of x is $||x|| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$
$\forall x, y \in \mathbb{R}^n$ Euclidian distance from x to y is $||x - y||$

Definition 2. $\forall S \subseteq \mathbb{R}^n$ point $x \in S$ is an interior point of S if J a neighborhood of x which is a subset of S point $S \in \mathbb{R}^n$ is a boundary point of S if every neighborhood of x contains a point in S and a point not in S

- Set $S \subseteq \mathbb{R}^n$ is <u>open</u> if every point in S is an interior point. example: open ball.
- Set $S \subseteq \mathbb{R}^n$ is <u>closed</u> if S contains all boundary points of S.

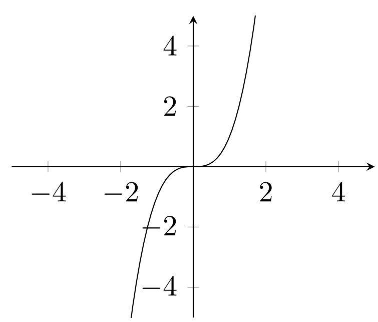
Note: $\forall S \subseteq \mathbb{R}^n, S$ is open **iff** S^c is closed.
In (P) min $f(x)$ s.t. $x \in S$, suppose $x^* \in S$:

- x^* is a <u>global maximizer</u> if $\forall y \in S$, $f(x^*) \leq f(y)$
- x^* is a <u>strict</u> global minimizer if $\forall y \in S$, s.t. $y \neq x^*$, $f(x^*) < f(y)$
- x^* is a <u>local minimizer</u> if $\exists$ any neighborhood N of x s.t. $\forall y \in N \cap S$, $f(x^*) \leq f(y)$
- x^* is a <u>strict</u> local minimizer if $\exists$ any neighborhood N of x s.t. $\forall y \in (S \cap N) \underbrace{\setminus x^*}_{\text{besides}}$, $f(x^*) < f(y)$

Note: If $S \subseteq \mathbb{R}^1$, x^* is interior of S, f is suitably differentiable at x^*.

If x^* is a *local minimizer* $\Rightarrow f'(x^*) = 0$ $\qquad$ x^* is a *local maximizer* $\Rightarrow f'(x^*) = 0$

$$\textbf{BUT } f'(x^*) = 0 \not\Longrightarrow x^* \text{ local min/max}$$



BUT

$$f'(x^*) = 0 \& f''(x^*) > 0 \Longrightarrow x^* \text{ strict local max}$$
$$f'(x^*) = 0 \& f''(x^*) < 0 \Longrightarrow x^* \text{ strict local min}$$

If $S \subseteq \mathbb{R}^n$, x^* interior part of S, f strictly differentiable, x^* local min/max $\Longrightarrow \nabla f(x^*) = \vec{0}$

$$\nabla f(x^*) = \vec{0} \quad \& \quad [?] \Rightarrow x^* \text{ strict local min/max}$$

Lecture 3 (2018-09-05)

Linear Programming

Example 1. *Diet Problem*: You will pick levels of four ingredients for chicken feed.

- x_1 = units of ingredient 1
- x_2 = units of ingredient 2
- x_3 = units of ingredient 3
- x_4 = units of ingredient 4

All are real numbers, so fractions allowed. Given minimum levels of 3 nutrients $\begin{bmatrix} 6.2 \\ 11.9 \\ 10 \end{bmatrix} = \begin{bmatrix} \text{nutrient 1} \\ \text{nutrient 2} \\ \text{nutrient 3} \end{bmatrix}$

Given how many units of nutrient per unit of ingredient

nutrients \ ingredients	1	2	3	4
1 (protein)	1.2	2.6	0	9.2
2 (carbs)	3.9	1	.8	2
3 (cholesterol)	6	0	4	3.1

Problem: Find yand of ingredients that meet nutritional requirements cheaply as possible.

Minimum Cost of the ingredients $\begin{bmatrix} 6.2 \\ 2 \\ 1.6 \\ 3.2 \end{bmatrix}$

$$\text{minimize} \qquad 6.2x_1 + 2x_2 + 1.6x_3 + 3.2x_4$$

$$\begin{cases} 1.2x_1 + 2.6x_2 + 0x_3 + 9.2x_4 \geq 6.2 \\ 3.9x_1 + x_2 + 0.8x_3 + 2x_4 \geq 11.9 \\ 3.9x_1 + 1x_2 + 0.8x_3 + 2x_4 \geq 11.9 \\ 6x_1 + 0x_2 + 4x_3 + 3.1x_4 \geq 11.9 \end{cases} \quad \text{and} \quad x_1, x_2, x_3, x_4 \geq 0$$

What makes a ingredient list special?

1. proportionality
2. additivity
3. divisibility
   - fractions are allowed – referring to x_1, x_2, x_3

(Linear Programming) in canonical form, an example

$$\begin{gathered} \text{minimize} \qquad c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{gathered}$$

$$\text{where} \qquad x_1, x_2, \cdots, x_n \geq 0$$
$$\Updownarrow$$
$$\min \quad c^T x \quad \text{s.t.} \quad A\vec{x} \geq \vec{b}, x \geq \vec{0}$$
$$A \in \mathbb{R}^{mxn}, b \in \mathbb{R}^m, C \in \mathbb{R}^n, x \in \mathbb{R}^n$$

Example 2. *Transportation problem*: There are 3 electricity generation plants α, β, γ and two cities u, v. α, β, γ respectively produce $65.2, 98.6, 32.5$ units of electricity. u, v respectively use $86.2, 110.1$ units of electricity

Question: How to supply transportation as cheaply as possible?

generator to city	cost variable	cost
$\alpha \to u$	x_1	31.7
$\alpha \to v$	x_2	28.6
$\beta \to u$	x_3	17.6
$\beta \to v$	x_4	37.4
$\gamma \to u$	x_5	22.8
$\gamma \to v$	x_6	29.7

$$\min \qquad 31.7x_1 + 28.6x_2 + 17.6x_3 + 374x_4 + 22.8x_5 + 29.7x_6$$

$$\left.\begin{array}{ll} x_1 + x_2 \leq 65.2 & \alpha \\ x_3 + x_4 \leq 98.6 & \beta \\ x_5 + x_6 \leq 32.5 & \gamma \end{array}\right\} \text{electricity produced at } \alpha, \beta, \gamma$$
$$\begin{array}{ll} u & x_1 + x_3 + x_5 \geq 86.2 \\ v & x_2 + x_4 + x_6 \geq 110.1 \end{array}\Bigg\} \text{Need for electricity for } u, v$$

Converting one form of (LP) to another

1. **(LP)** in standard form $\qquad$ minimize $c^T x$ $\qquad$ s.t. $Ax = b, \quad x \geq \vec{0}$
   Here

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 65.2 \\ 98.6 \\ 32.5 \\ 86.2 \\ 110.1 \end{bmatrix}$$
$$c^T = \begin{bmatrix} 31.7 & 28.6 & 17.6 & 37.4 & 22.8 & 29.7 \end{bmatrix}$$

2. $\max \to \min \quad$ or $\quad \min \to \max \quad \Longrightarrow \quad \max\ 6x_1 + 3x_2 - 4x_3 \leftrightarrow \min\ -6x_1 - 3x_2 + 4x_3$

$$Ax \geq b \to \leq$$
$$Ax \geq b \leftrightarrow \boxed{-A}x \leq \boxed{-b}$$

3. **Standard form → canonical form**

$$\left.\begin{array}{l}\min\ c^T x\\ \text{s.t.}\ \ Ax = b\\ x \geq 0\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\min\ c^T x\\ \text{s.t.}\ \ Ax \geq b\\ Ax \leq b\\ x \geq \vec{0}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\min\ c^T x\\ \ \text{s.t.}\ \ Ax \geq b\\ -Ax \geq -b\\ x \geq \vec{0}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\min\ c^T x \text{ s.t.}\\ \begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \end{bmatrix}\\ x \geq \vec{0}\end{array}\right.$$

4. **Canoconical form → Standard form**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13} &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23} &\geq b_2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \iff \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13} + x_4 \qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23} \underbrace{\qquad +x_5}_{\text{slack variables}} &= b_2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Lecture 4 (2018-09-07)

5. **non-negativity $\rightarrow$ include in matrix**

$$\left.\begin{matrix} Ax \geq b \\ x \geq \vec{0} \end{matrix}\right\} \longrightarrow \begin{bmatrix} A \\ I \end{bmatrix} x \geq \begin{bmatrix} \vec{b} \\ \vec{0} \end{bmatrix}$$

Simplex algorithm only works with non-negative elements.

6. **unconstrained sign $\rightarrow$ non-negativity** by <u>substitution</u>

$$\underbrace{Z}_{\text{uncons}} := \underbrace{Z_1}_{\geq 0} - \underbrace{Z_2}_{\geq 0}$$

$$\left.\begin{matrix} min & 5x_1 + 6x_2 & \\ s.t. & 2x_1 - 3x_2 \geq 9 & \\ & x_1 + x_2 \geq -8 & \\ & x_1 \geq 0 & x_2\text{unconst} \end{matrix}\right\} \rightarrow \left\{ X_2 = \underbrace{X_2'}_{\geq 0} - \underbrace{X_2''}_{\geq 0} \right\} \rightarrow \begin{cases} min & 5x_1 + 6x_2' - 6x_2'' \\ s.t. & 2x_1 - 3x_2' + 3x_2'' \geq 9 \\ & x_1 + x_2' - x_2'' \geq -8 \\ & x_1, x_2', x_2'' \geq 0 \end{cases}$$

7. **Have constraints** $Ax = b$

$$\text{illustration} \quad x_1, x_2, x_3 \begin{cases} 2x_1 - 2x_2 + 6x_3 = 8 \\ 3x_1 + x_2 - x_3 = 2 \end{cases}$$

Unchanged by row opeartions

- swap rows
- multiply by non-zero constant
- add one row to another

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 3 & 1 & -1 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} & -\frac{5}{2} \end{array}\right]$$

If row reduce $\left[\, A \,\vdots\, b \,\right] \rightarrow \left[\, A' \,\vdots\, b' \,\right]$ then $\exists$ invertible matrix C

$$C \left[\, A \,\vdots\, b \,\right] \rightarrow \left[\, A' \,\vdots\, b' \,\right]$$

Converse: if D is invertible, then $\left[\, A' \,\vdots\, b' \,\right]$ can be reduced to $D \left[\, A \,\vdots\, b \,\right]$

8. Changing the order of columns in $Ax = b$, it doesn't matter as long as you change the order of the variables conformally

$$\left.\begin{matrix} min & \begin{bmatrix} 17 \\ 83 \\ -11 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ s.t. & \begin{bmatrix} 2 & -2 & 6 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} \end{matrix}\right\} \longrightarrow \begin{cases} min & \begin{bmatrix} 83 \\ -11 \\ 17 \end{bmatrix}^T \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} \\ s.t. & \begin{bmatrix} -2 & 6 & 3 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} \end{cases}$$

Lecture 5 (2018-09-12)

Half spaces, Hyperplanes, Polyhedral sets

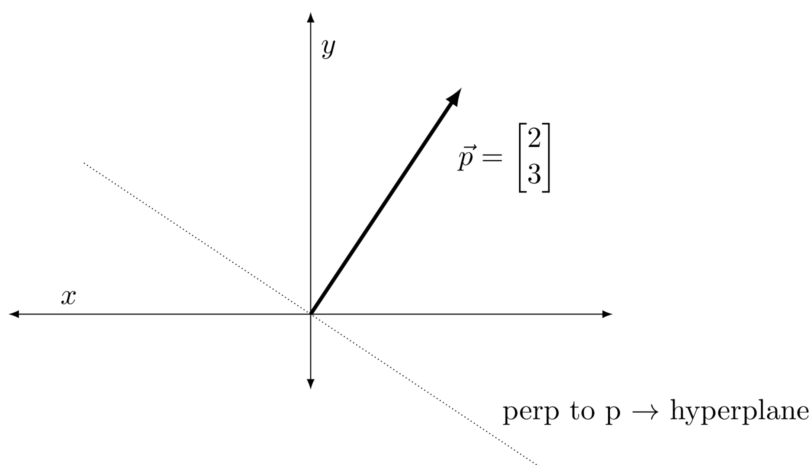
Recall: Suppose $p, x \in \mathbb{R}^n$. The inner product of p, x

$$p^T x = ||p||\,||x|| \cos\theta$$

$$\text{thus } p^T x \begin{cases} > 0 & \text{if } \theta \text{ acute} \\ = 0 & \text{if } p \perp x \text{ (perpendicular)} \\ < 0 & \text{if } \theta \text{ obtuse} \end{cases}$$

Definition 1. Suppose $p \in \mathbb{R}^n$ non-zero. $\{\vec{x} \in \mathbb{R}^n : p^T x = 0\}$ is a hyperplane through origin with normal vector p. $\{\vec{x} \in \mathbb{R}^n : p^T x \leq 0\}$ is (associated) and (closed) half space.

Example



Everything on either side of the hyperplane is the *half spaces*.

Definition 2. In general *hyperplanes* in $\mathbb{R}^n$ are sets $\{x \in \mathbb{R}^n, p^T x = \alpha\}$ for non-zero $p \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$

Definition 3. *half spaces* in $\mathbb{R}^n$ are sets $\{x \in \mathbb{R}^n, p^T x \geq \alpha\}$ for non-zero $p \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$

Example 1.

(2D)	$2x_1 + 5x_2 = 6$	hyperplane
	$2x_1 + 5x_2 \geq 6$	half space
(3D)	$3x_1 + 2x_2 - 7x_3 = 8$	hyperplane
	$3x_1 + 2x_2 - 7x_3 \geq 8$	half space

Definition 4. A *polyhedron* (polyhedral space) is the intersection of finely may half spaces.

Example 2. $x \cdot Ax \geq b$ for $A \in \mathbb{R}^{mxn}$, $\quad b \in \mathbb{B}^m$

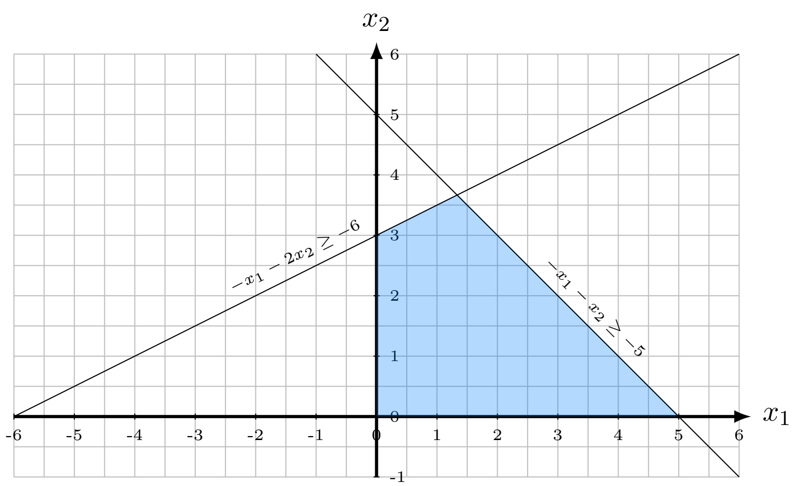
$$\begin{array}{c} \vec{p_1} \rightarrow \\ \vec{p_2} \rightarrow \\ \vdots \\ \vec{p_m} \rightarrow \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\vec{x}$ satisfies $Ax \geq b$ <u>iff</u> $\vec{x}$ satisfies $p_1{}^T x \geq b_1$, and $\vec{x}$ satisfies $p_2{}^T x \geq b_2$, …, $p_m{}^T x \geq b_m$

Example 3.

$$\begin{bmatrix} 1 & -2 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} -6 \\ -5 \\ 0 \\ 0 \end{bmatrix}$$

1. $x_1 - 2x_2 \geq -6$
2. $-x_1 - x_2 \geq 5$
3. $x_1 \geq 0$
4. $x_2 \geq 0$



Solving (LP) geometrically

$$\begin{aligned} \min \quad & \begin{bmatrix} 1 \\ -3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} \geq \begin{bmatrix} -6 \\ -5 \end{bmatrix} \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

polyhedral sets:

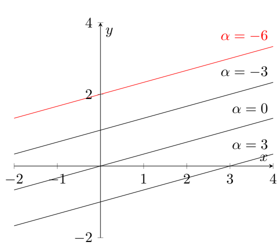
$$x : Ax \leq b \equiv \quad [(-A)x \geq (-b)]$$

$$x : Ax = b \equiv \quad \begin{bmatrix} Ax \geq b \\ Ax \leq b \end{bmatrix} \quad \text{i.e.} \begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \end{bmatrix}$$

$$x : \underbrace{Ax = b}_{x \geq 0} \equiv \quad \begin{bmatrix} A \\ -A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \\ \vec{0} \end{bmatrix}$$

Definition. <u>α-level set</u> is $\{x : f(x) = \alpha\}$

Here α-level set is $x : x_1 - 3x_2 = \alpha$ rewrite: $x_2 = \underbrace{\frac{1}{3}}_{\text{slope}} x - \underbrace{\frac{\alpha}{3}}_{\text{y-int}}$



The solution is the level set with the least α. In this case, $\alpha = -6$

Lecture 6 (2018-09-14)

<u>Hyperplane</u>:- $\quad x : p^T x = \alpha$ $\qquad\qquad$ <u>Halfspace</u>:- $\quad x : p^T x \geq \alpha$

<u>Polyhedral</u> set - intersection of halfspaces. The example $Ax \geq b$

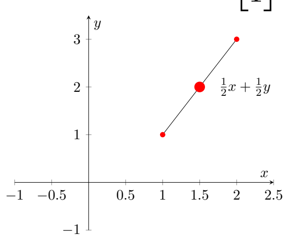
Example 1.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Level set $\quad c^T x = \alpha$

Definition 1. Let's say $x, y \in \mathbb{R}^n$. A <u>complex combination</u> of xy is $\lambda x + (1-\lambda)y \in \mathbb{R}^n$ where $\lambda \in [0,1]$

Example 2. $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad$ let $\lambda = \frac{1}{2}$



Note. In general, a convex combination of x, y are points on the line segment from x to y as $\lambda x + (1-\lambda)y = y + \lambda(x - y)$

Definition 2. $\mathcal{S} \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall x, y \in \mathcal{S}, \forall x \in [0,1] \quad \lambda x + (1-\lambda)y \in \mathcal{S}$

Example 3. Every polyhedron $\mathcal{S} := \{x \in \mathbb{R}^n : Ax \geq b\}$ is convex since if $y, z \in \mathcal{S}$ [i.e. $Ay \geq b$, $Ax \geq b$] and $\lambda \in [0,1]$, then

$$\begin{gathered} A(\lambda y + (1-\lambda)z) = \lambda Ay + (1-\lambda)Az \geq \lambda b + (1-\lambda)b = b \\ \lambda y + (1-\lambda)z \in \mathcal{S} \end{gathered}$$

Definition 3. Let $\mathcal{S}$ be a convex set. X is an <u>extreme point of $\mathcal{S}$</u> if $X = \lambda y + (1-\lambda)z$ for, $y, z \in \mathcal{S}$, $\lambda \in (0,1) \Rightarrow x = y = z$

Example 4. Consider

$$\begin{aligned} \min \quad & 10x_1 + 10x_2 + 0x_3 - 3x_4 - 5x_5 - 3x_6 \\ \text{s.t.} \quad & -x_1 + 2x_2 + 3x_3 + 6x_4 + 9x_5 + 8x_6 = 26 \\ & -2x_1 + 3x_2 + x_3 + x_4 + 6x_5 + 8x_6 = 17 \\ & x_1 + x_2 - x_3 + x_4 + x_5 + 3x_6 = 1 \\ & x_1, x_2, x_3, \ldots, x_6 \geq 0 \end{aligned}$$

What are feasible x? What if <u>**JUST FOR NOW**</u> we set $x_4, x_5, x_6 = 0$

$$\underbrace{\begin{bmatrix} -1 & 2 & 3 \\ -2 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{B \to \text{"Basic"}} \quad \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x_B \to \text{basic variable}} = \underbrace{\begin{bmatrix} 26 \\ 17 \\ 1 \end{bmatrix}}_{b}$$

Lucky # 1 $\rightarrow B$ is invertible

$$B^{-1} = \frac{1}{13}\begin{bmatrix} 4 & -5 & -1 \\ 1 & 2 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\text{so} \quad x_B = B^{-1}b = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\text{Hence the basic feasible solutions are } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$