# AMS 553.361 - Introduction to Optimization

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## Introduction

Math 110.304 is one of the semi-important courses that is required/recommended for the engineering-based majors at Johns Hopkins University.

These notes are being live-TeXed, through I edot for Typos and add diagrams requiring the Ti*k*Z package separately. I am using Texpad on Mac OS X.

I would like to thank Zev Chonoles from The University of Chicago and Max Wang from Harvard University for providing me with the inspiration to start live-TeXing my notes. They also provided me the starting template for this, which can be found on their personal websites.

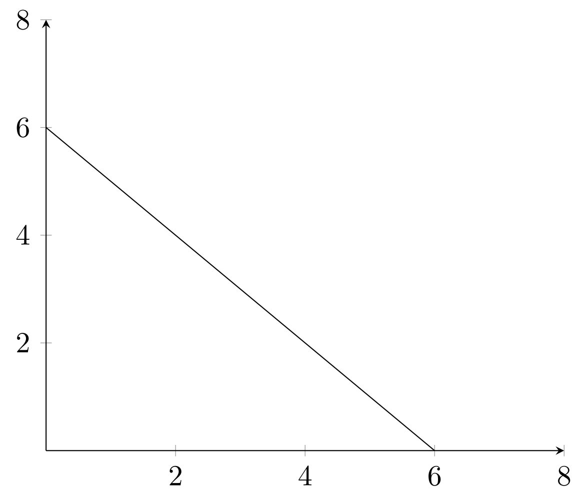
Please email any corrections or suggestions to ksriniv4@jhu.edu.

# Lecture 1 (2018-08-30)

## Introduction

**Example 1.** Up to 6 units of two nutrients can be added to solution and we require the number of units of nutrient 2 has to be atleast (natural) logarithm of # units of nutrient one. Goal: choose $x_1$ = number units of nutrient 1, choose $x_2$ = number units of nutrient 2. To maximize expected height of plant $1 + x_1^2(x_2-1)^3e^{-x_1-x_2}$

$$\begin{aligned} \text{Maximise} \quad & 1 + x_1^2(x_2-1)^3e^{-x_1-x_2} \\ & x_1 + x_2 \leq 6 \\ & x_2 > \log x_1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$




generic optimization problem: say $\subseteq \mathbb{R}$

$$f : \underbrace{\mathbb{S}}_{\text{feasible region}} \to \underbrace{\mathbb{R}}_{\text{objective function}}$$

$$x^* \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{optimal}$$

however optimal objective function value is1.2677

**Definition 1.** We say $x^*$ is an optimal solution if

- $x^* \in \mathbb{S}$
- For any $y \in \mathbb{S}$

**Example 2.** Find min $\log x$ s.t. $-\infty \leq x \leq 7 \rightarrow$ unbounded, has no solution.

**Example 3.** Find min $\log x$ s.t. $1 < x \leq 7 \rightarrow$ bounded, but is also no solution. (as we can go 1.000001)

**Example 4.** Find min $\log x$ s.t. $x > 1$, $x \leq 0.5 \rightarrow$ infeasible!!!

**Example 5.** minimize $3 + (x-1)^2$ s.t. $1 \leq x \leq 3 \rightarrow$ feasible as optimal solution $x^* = 2$ is an interior part of feasible region.

$$\begin{gathered} f(x) = 3 + (x-1)^2 \\ f'(x) = 0 \\ f''(x) > 0 \end{gathered}$$

**Example 6.** Minimize $3 + (x-2)^2$ s.t. $x \geq 10 \rightarrow$ optimal solution is $x^* = 10$. but $f'(x) \neq 0$. But is not an interior point - it is a boundary point of feasible region.

## Lecture 2 (2018-08-31)

**Definition 1.** $\forall x \in \mathbb{R}^n$ Euclidian length of $x$ is $||x|| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$

$\forall x, y \in \mathbb{R}^n$ Euclidian distance from $x$ to $y$ is $||x - y||$

**Definition 2.** $\forall S \subseteq \mathbb{R}^n$ point $x \in S$ is an interior point of S if $J$ a neighborhood of $x$ which is a subset of $S$ point $S \in \mathbb{R}^n$ is a boundary point of S if every neighborhood of $x$ contains a point in $S$ and a point not in $S$

- Set $S \subseteq \mathbb{R}^n$ is <u>open</u> if every point in $S$ is an interior point. example: open ball.
- Set $S \subseteq \mathbb{R}^n$ is <u>closed</u> if $S$ contains all boundary points of $S$.

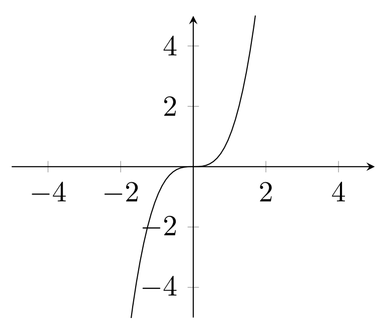
*Note:* $\forall S \subseteq \mathbb{R}^n, S$ is open **iff** $S^c$ is closed.
In $(P)$ min $f(x)$ s.t. $x \in S$, suppose $x^* \in S$:

- $x^*$ is a <u>global maximizer</u> if $\forall y \in S$, $f(x^*) \leq f(y)$
- $x^*$ is a <u>strict</u> global minimizer if $\forall y \in S$, s.t. $y \neq x^*$, $f(x^*) < f(y)$
- $x^*$ is a <u>local minimizer</u> if $\exists$ any neighborhood $N$ of $x$ s.t. $\forall y \in N \cap S$, $f(x^*) \leq f(y)$
- $x^*$ is a <u>strict</u> local minimizer if $\exists$ any neighborhood $N$ of $x$ s.t. $\forall y \in (S \cap N) \underbrace{\setminus x^*}_{\text{besides}}$, $f(x^*) < f(y)$

*Note:* If $S \subseteq \mathbb{R}^1$, $x^*$ is interior of $S$, $f$ is suitably differentiable at $x^*$.

If $x^*$ is a *local minimizer* $\Rightarrow f'(x^*) = 0$ $\qquad$ $x^*$ is a *local maximizer* $\Rightarrow f'(x^*) = 0$

$$\textbf{BUT } f'(x^*) = 0 \not\Rightarrow x^* \text{ local min/max}$$




**BUT**

$$f'(x^*) = 0 \& f''(x^*) > 0 \Longrightarrow x^* \text{ strict local max}$$
$$f'(x^*) = 0 \& f''(x^*) < 0 \Longrightarrow x^* \text{ strict local min}$$

If $S \subseteq \mathbb{R}^n$, $x^*$ interior part of $S$, $f$ strictly differentiable, $x^*$ local min/max $\Longrightarrow \nabla f(x^*) = \vec{0}$

$$\nabla f(x^*) = \vec{0} \quad \& \quad [?] \Rightarrow x^* \text{ strict local min/max}$$

## Lecture 3 (2018-09-05)

## Linear Programming

**Example 1.** *Diet Problem*: You will pick levels of four ingredients for chicken feed.

- $x_1$ = units of ingredient 1
- $x_2$ = units of ingredient 2
- $x_3$ = units of ingredient 3
- $x_4$ = units of ingredient 4

All are real numbers, so fractions allowed. Given minimum levels of 3 nutrients $\begin{bmatrix} 6.2 \\ 11.9 \\ 10 \end{bmatrix} = \begin{bmatrix} \text{nutrient 1} \\ \text{nutrient 2} \\ \text{nutrient 3} \end{bmatrix}$

Given how many units of nutrient per unit of ingredient

| nutrients \ ingredients | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 (protein) | 1.2 | 2.6 | 0 | 9.2 |
| 2 (carbs) | 3.9 | 1 | .8 | 2 |
| 3 (cholesterol) | 6 | 0 | 4 | 3.1 |

**Problem**: Find yand of ingredients that meet nutritional requirements cheaply as possible.

Minimum Cost of the ingredients $\begin{bmatrix} 6.2 \\ 2 \\ 1.6 \\ 3.2 \end{bmatrix}$

$$\text{minimize} \qquad 6.2x_1 + 2x_2 + 1.6x_3 + 3.2x_4$$

$$\begin{cases} 1.2x_1 + 2.6x_2 + 0x_3 + 9.2x_4 \geq 6.2 \\ 3.9x_1 + x_2 + 0.8x_3 + 2x_4 \geq 11.9 \\ 3.9x_1 + 1x_2 + 0.8x_3 + 2x_4 \geq 11.9 \\ 6x_1 + 0x_2 + 4x_3 + 3.1x_4 \geq 11.9 \end{cases} \quad \text{and} \quad x_1, x_2, x_3, x_4 \geq 0$$

What makes a ingredient list special?

1. proportionality
2. additivity
3. divisibility
   - fractions are allowed – referring to $x_1, x_2, x_3$

**(Linear Programming)** in <u>canonical form</u>, an example

$$\begin{gathered} \text{minimize} \qquad c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \\ \text{where} \qquad x_1, x_2, \cdots, x_n \geq 0 \end{gathered}$$

$$\Updownarrow$$

$$\min \quad c^T x \quad \text{s.t.} \quad A\vec{x} \geq \vec{b}, x \geq \vec{0}$$
$$A \in \mathbb{R}^{mxn}, b \in \mathbb{R}^m, C \in \mathbb{R}^n, x \in \mathbb{R}^n$$

**Example 2.** *Transportation problem*: There are 3 electricity generation plants $\alpha, \beta, \gamma$ and two cities $u, v$. $\alpha, \beta, \gamma$ respectively produce $65.2, 98.6, 32.5$ units of electricity. $u, v$ respectively use $86.2, 110.1$ units of electricity
**Question:** How to supply transportation as cheaply as possible?

| generator to city | cost variable | cost |
|---|---|---|
| $\alpha \to u$ | $x_1$ | 31.7 |
| $\alpha \to v$ | $x_2$ | 28.6 |
| $\beta \to u$ | $x_3$ | 17.6 |
| $\beta \to v$ | $x_4$ | 37.4 |
| $\gamma \to u$ | $x_5$ | 22.8 |
| $\gamma \to v$ | $x_6$ | 29.7 |

$$\min \qquad 31.7x_1 + 28.6x_2 + 17.6x_3 + 374x_4 + 22.8x_5 + 29.7x_6$$

$$\left.\begin{array}{ll} x_1 + x_2 \leq 65.2 & \alpha \\ x_3 + x_4 \leq 98.6 & \beta \\ x_5 + x_6 \leq 32.5 & \gamma \end{array}\right\} \text{electricity produced at } \alpha, \beta, \gamma$$
$$\left.\begin{array}{ll} u & x_1 + x_3 + x_5 \geq 86.2 \\ v & x_2 + x_4 + x_6 \geq 110.1 \end{array}\right\} \text{Need for electricity for } u, v$$

**(LP)** in standard form $\qquad$ minimize $c^T x$ $\qquad$ s.t. $Ax = b, \quad x \geq \vec{0}$
Here

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 65.2 \\ 98.6 \\ 32.5 \\ 86.2 \\ 110.1 \end{bmatrix}$$
$$c^T = \begin{bmatrix} 31.7 & 28.6 & 17.6 & 37.4 & 22.8 & 29.7 \end{bmatrix}$$

Converting one form of (LP) to another
$\max \to \min \qquad \min \to \max \quad \Longrightarrow \quad \max\ 6x_1 + 3x_2 - 4x_3 \leftrightarrow \min\ -6x_1 - 3x_2 + 4x_3$

$$Ax \geq b \to \leq$$
$$Ax \geq b \leftrightarrow \boxed{-A}x \leq \boxed{-b}$$

**Standard form $\to$ canonical form**

$$\left.\begin{array}{l} \min\ c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \min\ c^T x \\ \text{s.t. } Ax \geq b \\ Ax \leq b \\ x \geq \vec{0} \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \min\ c^T x \\ \text{s.t. } Ax \geq b \\ -Ax \geq -b \\ x \geq \vec{0} \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \min\ c^T x \text{ s.t.} \\ \begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \end{bmatrix} \\ x \geq \vec{0} \end{array}\right.$$

**Canoconical form → Standard form**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13} &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23} &\geq b_2 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13} + x_4 \qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23} \underbrace{\qquad +x_5}_{\text{slack variables}} &= b_2 \end{aligned}$$