MATH 110.302 - Differential Equations and Applications

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Introduction

Math 110.304 is one of the semi-important courses that is required/recommended for the engineering-based majors at Johns Hopkins University.

These notes are being live-TeXed, through I edot for Typos and add diagrams requiring the TikZ package separately. I am using Texpad on Mac OS X.

I would like to thank Zev Chonoles from The University of Chicago and Max Wang from Harvard University for providing me with the inspiration to start live-TeXing my notes. They also provided me the starting template for this, which can be found on their personal websites.

Please email any corrections or suggestions to ksriniv4@jhu.edu.

Lecture 1 (2018-06-14)

Start with $y = f(x)$, sine unknown functional relation between 2 variables, where

- x - independent variable
- y - dependent variable

Such an equation (or sets of them are) called a mathematical model when the variables represent measuarble quantities in some application *(usually set up to study some unknown entity of based on its relationship to something controllable x).*

If $y = f(x)$ is known, then we can simply study its properties (using calculus).

Often, though, we do not know $y = f(x)$, but we do have information about some properties, like derivatives, for example.

Examples:

1. $\frac{dx}{dy} = ky$
2. $F = ma$ (Newton's 2nd Law of motion)
3. $f'(x) = x - e^{x/2}$ (restraint of: Find $f(x - e^{x/2})dx$)
4. $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta \rightarrow$ **The Pendulum**

The above are mathematical models whhere the actual function is only known implicitly...

Ordinary Differential Equations

Definition 1. An Ordinary Differential Equation **(ODE)** is an equation involving an unknown function between two entities and some of its derivatives
Note: "Ordinary" means that the unknown function is a function of one independent variable

Example 1. The Heat Equation (in 3-space)

$$\frac{\partial u}{\partial t} = \alpha(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

is a partial differential equation since $'u'$ is a function of more than 1-independent variable.

Definition 2. The order of an ODE is the same as the order of the highest derivative that appears in the equation:

Definition 3. The general form of an nth order ODE is

$$\boxed{F(x, y, y', y'', ..., y^{(n)}) = 0}$$

x-independent variable
y-dependent variable $y = f(x)$
$y^{(i)}$ is the ith derivative of $y = f(x)$
F is some expression in $x, y, y', y'', ..., y^{(n)}$

Note: Sometimes, we can solve for the highest derivative

$$\boxed{y^{(n)} = G(x, y, y', y'', ..., y^{(n-1)})}$$

Definition 4. A function $f(\underbrace{x_1, x_2, ..., x_n}_{\vec{x}}, y_1, ..., y_n)$ is linear in the variables $y_1, ..., y_n$ if

$f(x_1, x_2, ..., x_n, y_1, ..., y_n) = G_0(\vec{x}) + \sum_{i=1}^{m} G_i(\vec{x})y_i$ where the $G_i(x), i = 0, ..., m$ are arbitrary.

Notes: 1 For an ODE to be linear, it must be linear in $y, y', ..., y^{(n)}$, and can be written as:

$$\boxed{G_n \cdot y^{(n)} + \cdots + G_1(\vec{x}) \cdot y' + G_0(\vec{x}) \cdot y = g(x)}$$

Examples:

1 $(sinx)y' + (lnx)y = tan(e^x)$ is a linear first order ODE.

2 $y'' + xy' + siny = 0$ is NOT linear.

3 $y''y + y' = 0$ is NOT linear.

Suppose $y' = f(t, y)$ is a first order linear ODE. Then thehere exist functions $p(t), q(t)$ so that $f(t, q) = -p(t)y + q(t)$, and the ODE can be written as

$$\boxed{y' \pm p(t)y = q(t)} \qquad (\star)$$

This form will be very important to understanding how to study this type of ODE.

Example 2. Given $y' \sin x + y \ln x = \tan e^x$, identify $p(t)$

Solution: Divide by $sinx$ to set,

$$y' + \frac{\ln x}{\sin x}y = \frac{\tan e^x}{\sin x}. \quad \boxed{p(t) = \frac{\ln x}{\sin x}}$$

Definition 5. A solution to $(\star)$ on $I = (a, b) \subset \mathbb{R}$ is any function $y = f(x)$ that satisfies that equation $(\star)$. Sometimes a solution is only known implicitly

Let's play this out and see just how the integrating factor is helpful.

Lecture 2 (2018-06-15)

Compare the following:

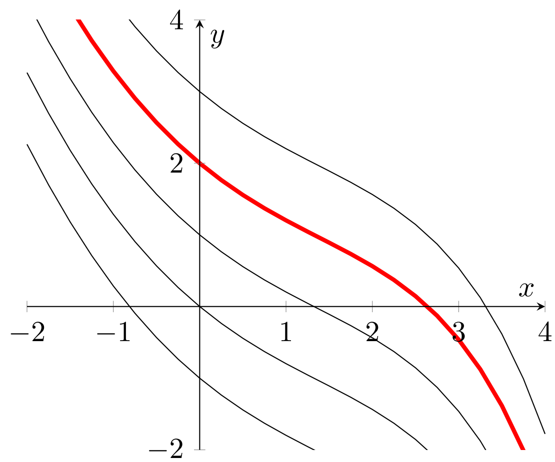
$$\text{(a)} \quad \frac{dy}{dx} = x - e^{x/2} \qquad \text{(b)} \quad \frac{dp}{dy} = \frac{p}{2} - 450$$

(1) Both are First Order ODEs.

(2) Both are linear.

(3) (a) us if the form $y' = f(x)$ and is simply a calculus problem (think finding $\int f(x)dx$)
Here the RHS is only a function of the independent variable

- Integrate both sides with respect to x to set $y(x) = \frac{x^2}{2} - 2e^{x/2} + c$
- This formula is called a general solution to (a) as it is an expression that specifies all possible solutions.
- A particular solution to (a) involves choosing a value for the constant $'c'$
- Graphs of solutions are called integral curves. The red line is when $c = 4$



- We also call the general solution to (a) as 1-parameter family of solutions

With the addition of a single pt in the xy-plane (ex. $y(0) = 2$), called an initial value, we can "choose" a particular solution from the family.
With this point, there is now only 1 solution to the problem:

$$y(x) = \frac{x^2}{2} - 2e^{x/2} + c$$

$$y(0) = \frac{x^2}{2} - 2e^{x/2} + c = 2 = 0 - 2 + c = 2 \Rightarrow \text{c} = 4$$

Here the solution to the Initial Value Problem **(IVP)** (an ODE with initial values)

$$\underbrace{y' = x - e^{x/2}, \quad y(0) = 2}_{\textbf{IVP}} \text{ is } y(x) = \frac{x^2}{2} - 2e^{x/2} + 4$$

4 b is not of the form $y' = f(x)$. Rather it is of the form $y' = f(y)$, where dependent variable on RHS.
Note: This is harder to solve but easier to study!

Definition 1. if in an ODE the independent variable is NOT explicitly present, the ODE is called autonomous.

Let's solve b: there are many ways. We choose something like what we will eventually call "separation of variables"

First recall from Calculus I:

I For any diff. $p(t)$, the function $p(t)$ and $p(t) - 900$ have the same derivative: $p'(t)$

II $\frac{d}{dt}[ln|f(t)|] = \frac{f'(t)}{f(t)}$ for $f(t)$ differentiable and $f(t) \neq 0$ *(Chain Rule)*

Hence we can rewrite $p' = \frac{p}{2} - 450 = \frac{p-900}{2} \Rightarrow \frac{p'}{p-900} = \frac{1}{2}$ *Why is this useful?*

It is useful since the LHS of $\frac{p'}{p-900} = \frac{1}{2}$ looks like the derivative:

$$\frac{d}{dt}\left[ln|p(t) - 900|\right] = \frac{1}{2}$$

Integrate Both sides or functions of $'t'$ to set

$$ln|p(t) - 900| = \frac{t}{2} + c$$

Exponentiate to set

$$p(t) - 900 = e^{\frac{t}{2}}e^{c}$$

Hence, we can "solve" this for $p(t)$ we are done since we would have an expression for the $p(t)$ that solves b

Question: Given $|p(t) - 900| = e^{\frac{t}{2}}e^{c}$, can $p(t) = 900$? Why or why not?
Answer: The ODE b has 3 types of solutions:

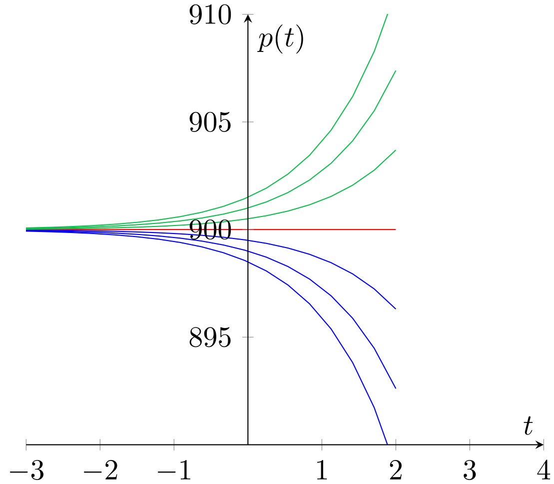
1 $p(t) > 900$ always $\Rightarrow p(t) - 900 = ke^{t/2}$ or $p(t) = 900 + ke^{t/2}$, where $k = e^{c} > 0$

2 $p(t) < 900$ always $\Rightarrow -(p(t) - 900) = ke^{t/2}$ or $p(t) = 900 + \underline{(-k)}e^{t/2}$ where $k = e^{c} > 0$

3 $p(t) = 900$ for all $t \in \mathbb{R}$. It works in the ODE and also be written as $p(t) = 900 + ke^{t/2}$, $k = 0$

Definition 2. This last solution is called a singular solution, which sometimes become hidden due to the method employed to find the other solutions.
Note: Our method included a divide by p-sov term which implied we discounted that possibility. We need to account for it.

Conclusion: $p(t) = 900 + ke^{t/2}$, $k \in \mathbb{R}$ is the general solution to (b)



Definition 3. Here the singular solution $p(t) = 900$, $\forall t$ is called an equilibrium solution (or a steady-state solution.) Equilibrium solutions will be important to us in the future.

Notice in both (a) and (b) above

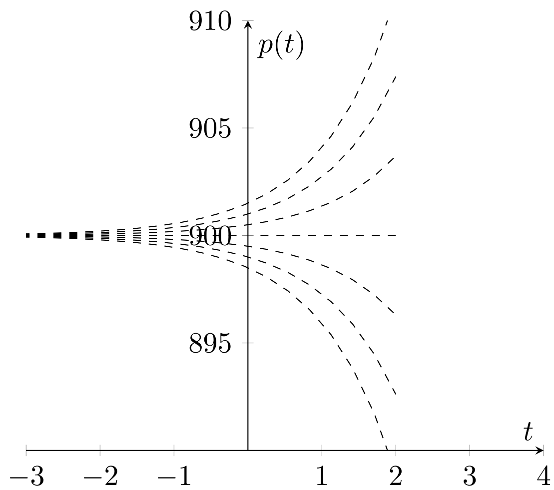
$$\frac{dy}{dx} = \text{something involving } x, y \qquad \frac{dp}{dt} = \text{something involving } p, t$$

This is useful since solutions "live" in the x,y-plane or the t,p-plane, only solution curve will hhave its tangent line at (x, y) with slope given by simply evaluating the RHS at that point.

Example: for $y' = x - e^{\frac{x}{2}}$, choose $(x_0, y_0) = (2, 1)$

Then $\left.\dfrac{dy}{dx}\right|_{x=2,y=1} = 2 - e^{2/2} \approx -0.718$

The solution curve passing through $(2, 1)$ will have a slope -0.718 there.
Without solving thhe ODE, we can take a grid of points in the xy-plane and evaluate these little slope lines. The result is called a slope field.



Lecture 3 (2018-06-16)

Very generally, a first-order ODE of the form

$$\frac{dy}{dt} = f(t, y)$$

will have a function of both t and y and will not be solvable. However, with some additional structure to f, thhere are methods to solve: In chapter 2, we explore some of these.

First Order Differential Equations

Linear Equations

Suppose $f(t, y) = -p(t)y + q(t)$ for some function $p(t), q(t)$. Then $(\star)$ can be rewritten

$$\boxed{y' = -p(t)y + q(t) \quad \textbf{or} \quad y' + p(t)y = q(t)} \tag{$\star\star$}$$

This new form exposes a structure that facilitates calculation: The LHS is almost the total derivative of a function. To make of so, we multiply the ODE by an expression called the integrating factor.

Definition 1. An integrating factor is a term that when multiplied to an expression renders the expression integrable.

To understand what we are looking for, look at the patterns here:
Let y be a differentiable function of t. Then, for any other diff. function of t, $f(t)$, we have

$$\frac{d}{dt}\Big[f(t)y\Big] = f(t)y' + f'(t)y \qquad \textit{(by Product Rule)}$$

And also,

$$\begin{aligned}\frac{d}{dt}\Big[e^{f(t)}y\Big] &= e^{f(t)}y' + e^{f(t)}f'(t)y \\ &= e^{f(t)}\Big[y' + f'(t)y\Big]\end{aligned}$$

We do this just to look for patterns. In this case, we see an important one: Inside the brackets, $[y' + f'(t)y]$ looks very close to the LHS of $y' + p(t)y = q(t)$
In fact, they are precisely the same when $f'(t) = p(t)$, or $f(t) = \displaystyle\int p(t)dt$

So we do more calculation for a pattern:

$$\begin{aligned}\frac{d}{dt}\Big[e^{\int p(t)dt}y\Big] &= e^{\int p(t)dt}y' + \frac{d}{dt}\Big[e^{\int p(t)dt}\Big]y \\ &= e^{\int p(t)dt}y' + e^{\int p(t)dt}p(t)y \\ &= e^{\int p(t)dt}\Big[\underbrace{y' + p(t)y}_{\text{precisely the LHS of } (\star\star)}\Big]\end{aligned}$$

This is useful because, if we take $y' + p(t)y = q(t)$ and multiply it the entire equation by $e^{\int p(t)dt}$, then the LHS becomes easily integrable.

We call $e^{\int p(t)dt}$ the integrating factor of $y' + p(t)y = q(t)$.

Solve $y' + p(t)y = q(t)$

Step 1: Multiply the entire equation by $e^{\int p(t)dt}$.

$$e^{\int p(t)dt}\Big[y' + p(t)y = q(t)\Big]$$

$$\underbrace{e^{\int p(t)dt}y' + e^{\int p(t)dt}p(t)y} = e^{\int p(t)dt}q(t)$$

$$\frac{d}{dt}\Big[e^{\int p(t)dt}y\Big] = e^{\int p(t)dt}q(t)$$

Step 2: Integrate with respect to (w.r.t.) t.

$$\int \frac{d}{dt}\big[e^{\int p(t)dt}y\big]dt = \int e^{\int p(t)dt}q(t)dt$$

$$e^{\int p(t)dt}y = \int e^{\int p(t)dt}q(t)dt + c$$

Step 3: Solve for y.

$$y(t) = e^{-\int p(t)dt}\Big[\int e^{\int p(t)dt}q(t)dt + c\Big]$$

Notes:

1. Theoretically, we can always do this. Practically, the integrating factor $e^{\int p(t)dt}$ is pretty easy to calculate usually.
2. You do not need to memorise anything of the form of step 3. Just remember the steps.
3. Any antiderivative of p(t) will do since a they all only differ by a constant and b You are multiplying the entire equation by the factor.

Example 1. Suppose $p(t) = 2t$. Then $e^{\int p(t)dt} = e^{\int 2tdt} = e^{t^2+c}$. Then $e^{t^2+c} = e^{t^2}e^c = e^{t^2}K$, for $K \in \mathbb{R}$ and constant.
Then $Ke^{t^2}\big[y' + p(t)y = q(t)\big]$ is same as $e^{t^2}\big[y' + p(t)y = q(t)\big]$ as far as solutions are concerned.

Some examples:

(I) Solve $ty' - 2y = t^3 e^{-2t}$

Strategy: This is linear so we use the integrating factor $e^{\int p(t)dt}$ to solve using the 3 steps above.
Solution: Place the ODE in standard form.

$$y' - \frac{2}{t}y = t^2 e^{-2t}$$

This gives us $p(t) = \frac{-2}{t}$, so the int. factor is $e^{\int p(t)dt} = e^{-2\int \frac{1}{t}dt} = e^{-2\ln|t|} = e^{\ln t^{-2}} = \underline{t^{-2}}$

Step 1: Multiply ODE by integrating factor.

$$= t^{-2}\left[y' - \frac{2}{t}y = t^2 e^{-2t}\right]$$

$$= \underbrace{t^{-2}y' - \frac{2}{t^3}y} = e^{-2t}$$

$$= \frac{d}{dt}\left[t^{-2}y\right] = e^{-2t}$$

Step 2: Integrate wrt t

$$= \int \frac{d}{dt}\left[t^{-2}y\right]dt = \int e^{-2t}dt$$

$$\Rightarrow t^{-2}y = \frac{-e^{-2t}}{2} + K$$

Step 3: Solve for $y(t)$

$$\boxed{y(t) = \frac{t^2 e^{-2t}}{2} + Kt^2} \qquad \text{(this solves the ODE)}$$

(II) $x' + 2tx = t^3$. Solve this.

Strategy: Use the integrating factor on this linear ODE to integrate through to an expression for $x(t)$
Solution: This ODE is linear, with $p(t) = 2t$. Thus the int. factor is $e^{\int p(t)dt} = e^{\int 2tdt} = \underline{e^{t^2}}$

Step 1: Multiply ODE by integrating factor.

$$e^{t^2}\left[x' + 2tx = t^3\right]$$

$$\underbrace{e^{t^2}x' + 2txe^{t^2}} = t^3 e^{t^2}$$

$$\frac{d}{dt}\left[e^{t^2}x\right] = t^3 e^{t^2}$$

Step 2: Integrate w.r.t. t

$$\int \frac{d}{dt}\left[e^{t^2}x\right]dt = e^{t^2}x + c_1 = \underbrace{\int t^3 e^{t^2}dt}$$

Expand using by parts and substitution. $s = t^2$, $ds = 2tdt$

$$\begin{aligned}\int t^3 e^{t^2}dt \longrightarrow & \frac{1}{2}\int se^s ds \\ =& \frac{1}{2}e^s(s-1) + c_2 \\ =& \frac{1}{2}e^{t^2}(t^2-1) + c_2\end{aligned}$$

Combine constants to set:

$$e^{t^2}x = \frac{1}{2}e^{t^2}(t^2-1) + K$$

Step 3: Solve for x(t) to get the general solution.

$$\boxed{x(t) = \frac{1}{2}t^2 - \frac{1}{2} + Ke^{-t^2}}$$

III Solve $\frac{dx}{ds} = \frac{x}{s} - s^2$, for $s > 0$

Here the ODE is again linear (note s is the independent variable), and $p(s) = -\frac{1}{s}$, the integrating factor is then...

$$e^{\int p(t)dt} = e^{-\int(\frac{1}{s})ds} = e^{-\ln s} = e^{\ln s^{-1}} = \underline{s^{-1}}$$

Step 1: Multiply through standard form ODE to set

$$\frac{1}{s}\left[\frac{dx}{ds} - \frac{x}{s} = -s^2\right] \Rightarrow \underbrace{\frac{1}{s}\frac{dx}{ds} - \frac{x}{s^2}}_{\frac{d}{ds}\left[\frac{1}{s}\cdot x\right]} = -s$$

$$\frac{d}{ds}\left[\frac{1}{s}\cdot x\right] = -s$$

Step 2: Integrate w.r.t. s to set

$$\frac{1}{s}\cdot x = \int(-s)ds + c = \frac{-s^2}{2} + c$$

Step 3: Solve for x(s):

$$\boxed{x(s) = \frac{-s^3}{2} + cs}$$

Lecture 4 (2018-06-17)

Separable Equations

Definition 1. Suppose for $y' = f(t, y)$ that $f(t, y) = g(t)h(y)$ for 2 functions $g(t)$, and $h(y)$. Then we say the ODE is separable. (we can separate RHS into the product of 2 functions; one of t alone and the other of y alone).

Then the ODE is $y' = g(t)h(y)$, and we can write...

$$\frac{1}{h(y)} \cdot \frac{dy}{dt} = g(t)$$

Since y is a function of t, both sides are functions of t and we can integrate wrt t.

$$\underbrace{\int \left[\frac{1}{h(y)} \cdot \frac{dy}{dt}\right] dt}_{\text{the antiderivative of} \frac{1}{h(y)} \text{as a function of t}} \qquad = \qquad \underbrace{\int g(t)dt}_{\text{the antiderivative of } g(t)} \qquad (\star)$$

The general solution to this kind of ODE is then found by integration alone.

Example 1. Find the general solution to $\frac{dy}{dx} = xy^2$. (Note: This ODE is separable, but NOT linear!)

Solution: Separate the variables: $\frac{1}{y^2}\frac{dy}{dx} = x$. Then integrate both sides wrt x:

$$\int \frac{1}{y^2}\frac{dy}{dx} = \int xdx$$

$$-\frac{1}{y} = \frac{x^2}{2} + c = \frac{x^2 + K}{2}$$

$$\Rightarrow \boxed{y = \frac{-2}{x^2 + K}}$$

Note: While this is the general solution, particular solutions will require more than just a choice of K!

This function works since for $y = \frac{-2}{x^2 + K}$

$$y' = \frac{2}{(x^2 + K)^2} \cdot 2x = x \cdot (\frac{2}{x^2 + K})^2 = x \cdot (\frac{-2}{x^2 + K})^2 = xy^2$$

Notes

1. The LHS of $(\star)$ is interesting:

$\int \left[\frac{1}{h(y)} \cdot \frac{dy}{dt}\right] dt$ is the antiderivative of $\frac{1}{h(y)}$ as a function of t; in the example,

$$\int \left[\frac{1}{y^2} \cdot \frac{dy}{\cancel{dx}}\right] \cancel{dx} = \int \frac{1}{y^2} dy = \frac{-1}{y} + c$$

To see this, rewrite LHS ($\star$) using u as the dependent variable:

$\int \left[\frac{1}{h(u(x))} \cdot \frac{du}{dx}\right] dx$ looks just like the integrand one would find a substitution problem:
Let $y = u(x), dy = u'(x)dx = \frac{du}{dx}dx$
Then $\int \left[\frac{1}{h(u(x))} \cdot \frac{du}{dx}\right] dx = \int \frac{1}{h(y)} dy$ and we can integrate w.r.t. y directly!
Strictly speaking one <u>does NOT</u> simply cross the dx's. But it does look that way.

2 The book uses a slightly different formula:
Any $y' = \frac{dy}{dx} = f(t, y)$ can be written as:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

(think $M = -f$ and $N = 1$, but this may sometimes not be the only way).
Then if $M(x, y) = M(x)$ and $N(x, y) = N(y)$ we set

$$M(x) + N(y)\frac{dy}{dx} = 0 \text{ or } N(y)\frac{dy}{dx} = M(x)$$

and ODE is separable.

3 In differential form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

may be presented as

$$M(x)dx + N(y)dy = 0 \text{ or } N(y)dy = -M(x)dx.$$

Integrating the differentials yields.

$$-\int M(x)dx = \int N(y)dy$$

Example 2. We may sometimes see $y' = xy^2$ as $\frac{dy}{dx} = xy^2$ or $\frac{dy}{dx} - xy^2 = 0$, or $\frac{dy}{y^2} = xdx$
The notation is different, but the ODE is the same.

4 Sometimes, a solution is known only implicitly:

<u>Example in book:</u> $\frac{dy}{dx} = \frac{x^2}{1 - y^2}$

Solution is $\frac{-x^3}{3} + y - \frac{y^3}{3} = K$

Question: How does one use this information to find an explicit solution?
Question: What is the domain of each solution?
Question: How do we know which piece to pick?

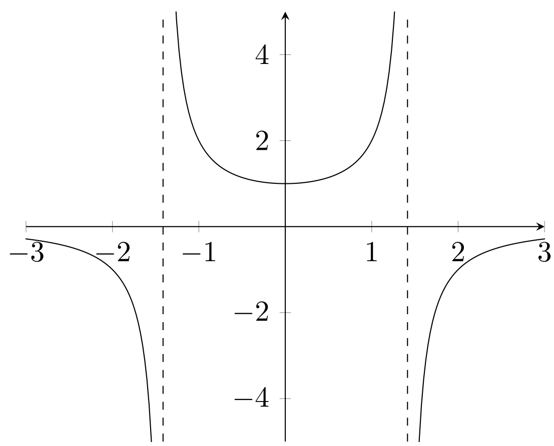
For $y' = \frac{x^2}{1-y^2}$, $y(0) = 0$, the solution is $y(x)$ where $\frac{-x^3}{3} + y - \frac{y^3}{3} = 0$, but the function $y(x)$ is only defined up to the vertical tangent lines: These are where $y^2 = 1$ on $y = \pm 1$.

Here is where $\underline{y=1}$, $\frac{-x^3}{3}+1-\frac{1}{3} \Rightarrow x=\sqrt[3]{2}$ and $\underline{y=-1} \Rightarrow x=-\sqrt[3]{2}$
Caution: A solution to an ODE is a function (even when defined implicitly) that <u>includes its **domain!**</u>

5 Back to $y = xy^2$ with its general solution, $y(x)=\frac{-2}{x^2+K}$. This is fine as a general solution. But for an IVP, we will also need a domain for which the solution is continuous!

$$\textbf{IVP: } y' = xy^2, y(0) = 1$$

here the particular solution has K-value $K=-2$.
But $y(x)=\frac{-2}{x^2-2}$ does not solve $y'=xy^2, y(0)=1$.
Only the <u>continuous piece</u> that contains the initial value <u>is</u> the solution.



The solution to $y'=xy^2$, $y(0)=1$ is $y(x)=\frac{-2}{x^2-2}$ on $(-\sqrt{2},\sqrt{2})$ only.
The solution to $y'=xy^2$, $y(2)=-1$ is $y(x)=\frac{-2}{x^2-2}$ on $(-\sqrt{2},0)$ only.

The proper domain is absolutely necessary to specifying a solution.

Example 3. One last example:

$$\text{Solve the IVP } ty' + 2y = 4t^2$$

$$\text{for (a) } y(-1)=1, \quad \text{(b) } y(-1)=2, \quad \text{(c) } y(0)=0, \quad \text{(d) } y(0)=1$$

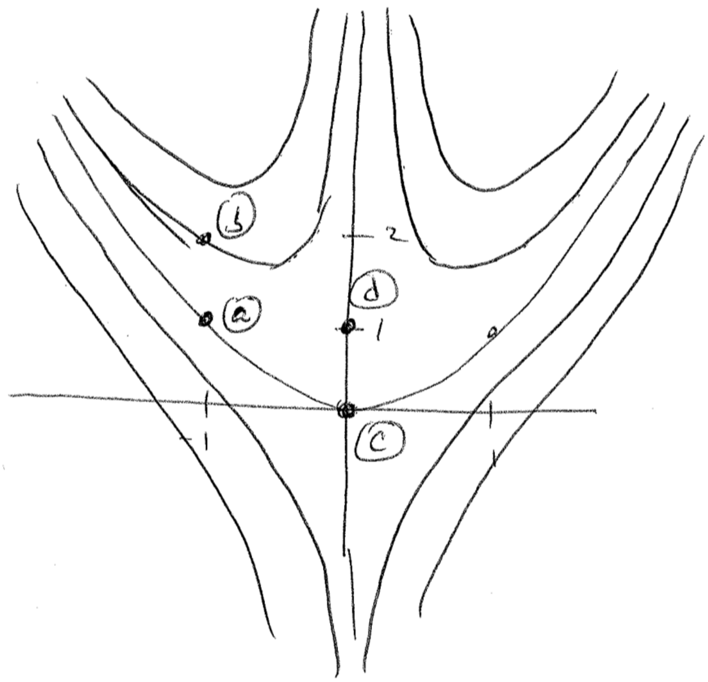
Solution: Here method of integration factors gives us... $y(t)=t^2+\frac{c}{t^2}$ as the general solution

(a) $y(-1)=1=(-1)^2+\frac{c}{(-1)^2} \Rightarrow c=0 \qquad y(t)=t^2$ for $t\in(-\infty,\infty)$.

(b) $y(-1)=2=(-1)^2+\frac{c}{(-1)^2} \Rightarrow c=1 \qquad y(t)=t^2+\frac{1}{t^2}$ for $t\in(-\infty,0)$.

(c) Connect plus in 0. But the point $(0,0)$ is on an integral curve of IVP. It is on the curve $y=t^2$ on $(-\infty,\infty)$.

(d) The point $t=0$, $y=1$ is not on any integral curve. The IVP $ty'+2y=4t^2$, $y(0)=1$ has <u>no solution.</u> $\longrightarrow$ <u>What gives??</u>



- the domain of the IVP solution in a is $(-\infty, \infty)$
- the domain of the IVP solution in b is $(-\infty, 0)$

(This is only one piece of $ty(t) = t^2 + \frac{1}{t^2}$ the one that includes the point.) $\Rightarrow$ Solution to the IVP $ty' + 2y = 4t^2$, $y(0) = 1$ is

$$\boxed{y(t) = t^2 + \frac{1}{t^2} \quad \text{on } (-\infty, 0)}$$

<u>Careful here.</u>

Lecture 5 (2018-06-19)

Modeling with First Order Equations

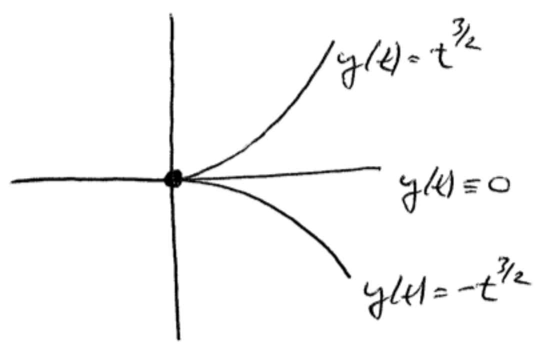
This section deals with applications and the details of how to model a process to construct on ODE. This, in general, is a difficult process and quite ad hoc. Also it cannot be mastered in such a short time. Read this section.

Non-Linear Equations

In all examples shown, there has always been a solution. And with initial data, there has always been an unique solution. Is this always the case?

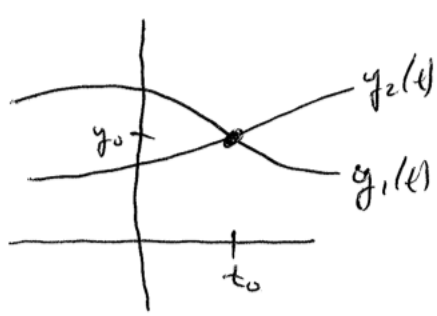
Example 1. Solve $(y')^2 + 1 = 0$
(Find a function whose derivative is -1?)

Example 2. Solve $y' = \frac{3}{2}y^{\frac{1}{3}}$, $y(0) = 0$
Note that here that a $y(t) \equiv 0$ solves this. But so does b $y(t) = t^{\frac{3}{2}}$ c $y(t) = -t^{\frac{3}{2}}$



Since all three of those distinct functions solve the IVP, we say solutions here (at $y(0) = 0$) are NOT unique!

Question: 16 Solutions to an ODE are not unique at a point $y(t_0) = y_0$, then 2 or more solutions that are distinct pass through the point. What does this say about the predictive power of your model?



There are criteria for ensuring an IVP has solutions (sols. exist) and whether they are unique or not (unique is good!)

Let $(\star)$ $\quad y'(t) = f(t,y),\ y(t_0) = y_0$ be an IVP

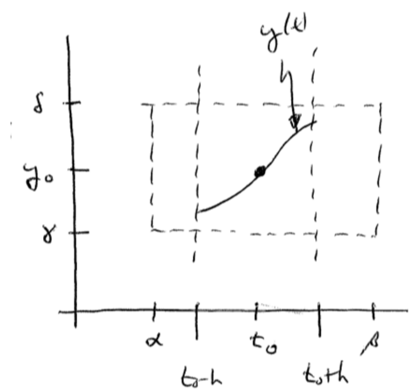
Theorem 1. *If $f(t,y)$ and $\dfrac{df}{dy}(t,y)$ are continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ inside $\alpha < t < \beta$, there exists a unique solution $y(t)$ to $(\star)$*

Many comments

(a) $f(t,y)$ and $\dfrac{df}{dy}(t,y)$ are functions of 2 variables. Continuity here is bigger than simply continuous in each variable while holding the other variable constant.

(b) $\dfrac{df}{dy}(t,y)$ is the derivative of f with respect to y while pretending t is a constant.. It is called a partial derivative.

(c) Geometrically, a solution passing though (t_0, y_0) is an integral curve in the *$ty-$plane.*



(d) $f(t,y)$ is continuous near (t_0, y_0), then $y'(t)$ is continuous. But then by Calc I, $y(t)$ is differentiable. Hence solution exists through (t_0, y_0)
By Fundamental Theorem of Calculus, $y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))ds$.
Since $f(t,y)$ is continuous near (t_0, y_0), then this integral will exist. Note:

$$y' = \frac{d}{dt}\Big[y(t)\Big] = \frac{d}{dt}\left[\int_{t_0}^{t} f(s,y(s))ds\right] = f(t,y)$$

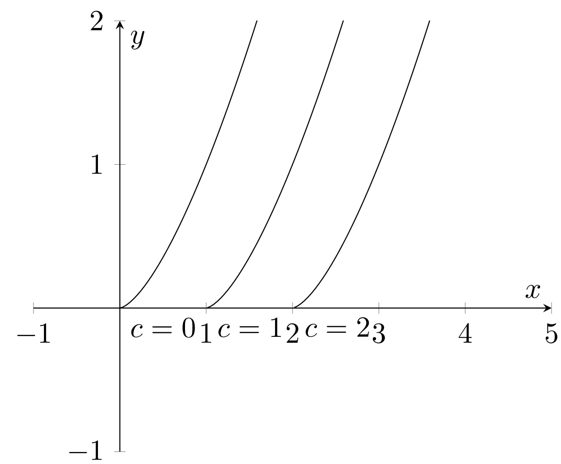
(e) if $\dfrac{df}{dy}(t,y)$ is continuous near (t_0, y_0), then solutions very nicely in the y-direction. This is enough to ensure solutions are unique.
$\Rightarrow$ Solution curves never touch or cross when uniquely defined.

(f) Example: Given $ty' + 2y = 4t^2$, if we place it in the form $y' = f(t,y)$, we set $y' = -\frac{2}{t}y + 4t$
Here, as long as our initial data does not include $t_0 = 0$, solutions will exist
($-\frac{2}{t}y + 4t$ is cont when $t \neq 0$) and unique when ($\dfrac{df}{dy}(t,y) = -\frac{2}{t}$ is continuous when $t \neq 0$)

Caution: it may be possible for solutions to exist and/or be unique when $t = 0$. But it is not assured!

Example 3. Given $y' = \frac{3}{2}y^{\frac{1}{3}}$, $f(t, y) = \frac{3}{2}y^{\frac{1}{3}}$.
Here $f(t, y)$ is defined and continuous everywhere (and for all $t \in \mathbb{R}$, and $y \in \mathbb{R}$). Hence by Thm, solutions are guaranteed to exist everywhere.
But $\dfrac{df}{dy}(t, y) = \frac{1}{2}y^{-\frac{2}{3}}$ is not continuous along the $y = 0$ line (in the $ty - plane$). Hence solutions exist for starting values like $y(t_0) = 0$, but may not be unique (everywhere else they are unique!)

For any value $c \geq 0$, the curve $y(t) = 0, (t < c)$ and $(t - c)^{\frac{3}{2}} when, (t \geq c)$.
Solve the IVP $y' = \frac{3}{2}y^{\frac{1}{3}}$, $y(0) = 0$



(g) For linear ODE, existence and uniqueness is easier. In standard form,

$$y' + p(t)y = q(t)$$

and in $y' = f(t, y)$ form

$$y' = \underbrace{-p(t)y + q(t)}_{f(t,y)}$$

Theorem 2. *As long as $p(t)$ and $q(t)$ are continuous at t_0, then solutions exist and are unique for $y' = -p(t)y + q(t)$, $y(t_0) = y_0$*

Lecture 6 (2018-06-20)

New Question: Suppose $y' = f(t, y)$, f is ONLY a function of y, but $\frac{1}{f(y)}$ is hard to integrate.

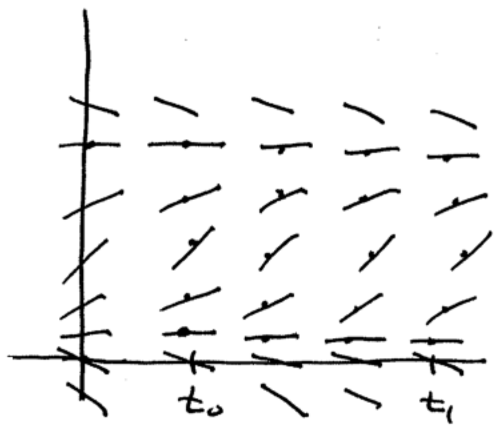
How to study? $\boxed{\frac{1}{f(y)}\frac{dy}{dt} = 1}$

Here $y' = f(y)$ is called autonomous (t is not explicit in equation.)

Example 1.

$$\dot{x} = x^{\frac{2}{3}}, \quad y' = Ky, \quad \frac{dz}{dt} = z(1-z)$$

Then the solution curve will not depend on the starting time, only on the elapsed time. In the slope field, every vertical slice looks the same.
Every horizontal slice is an isocline: a curve along which the slumps of the solution curves are the same.



Slopes along every vertical slice look like every other vertical slice. Slopes along every horizontal slice not very.

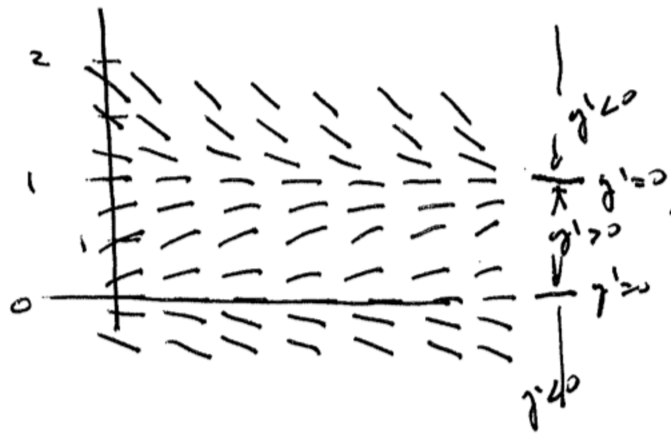
I Existence and uniqueness is determined by continuit of f in g and cont of $\frac{df}{dy}$ (not a partial here).

II At any place, y_0 where $f(y_0) = 0$, then $y'(t) = 0$ and $y(t) = y_0$ is a constant solution. Called an equilibrium solution, its graph is horizontal and is an isocline.

$$\frac{dz}{dt} = z(1-z). \quad \frac{dz}{dt} = 0 \quad \text{when } z = 0, 1$$

Hence $z(t) = 0$ is a solution

Hence $z(t) = 1$ is a solution



(III) Outside of the equilibria solutions, of the "sign" of $\dfrac{dz}{dt}$ does not change, hence solutions stock between equilibria will have slopes always negative or always positive.

<u>*Note:*</u> *Without solving, you know pretty much everything. In fact, the slope field is too much information!*

(IV) One vertical slice through the slope field gives you all relevant info. For $z' = f(z) = z(1-z)$

$$< \quad 0 \quad > \quad 1 \quad <$$

Sign of z' between equilibria

This is called a <u>phase line</u> (think of the y-axis slice in the ty-plane. It is a schematic that gives long term behavior of an autonomous 1st order ODE).

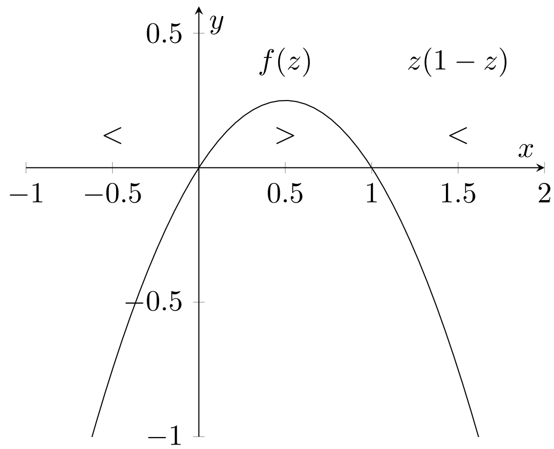
Example 2. Consider $z' = z(1-z), z(0) = \frac{3}{4}$.
Without solving, we know by phase line that $\lim_{t\to\infty} z(t) = 1$

In fact, we know the long term behavior of all solutions.

$$\lim_{t\to\infty} z(t) = \begin{cases} -\infty & z(0) < 0 \\ 0 & z(0) = 0 \\ 1 & z(0) > 0 \end{cases}$$

Without a slope field, one can simply graph $f(z)$.

Here on $(-\infty, 0)$, $\quad f(z) < 0, \quad z' < 0$ for solutions.
Here on $(0, 1)$, $\quad f(z) > 0, \quad z' > 0$ for solutions.
Here on $(1, \infty)$, $\quad f(z) > 0, \quad z' < 0$ for solutions.



Ⓥ We can use the the seaword derivative test to say more:
$y' = f(y)$ for a solution, then $y'' = f'(y)y'$ helps to determine concavity of infection.

<u>Det:</u> For $y' = f(y)$, the set $\{y \in \mathbb{R} | f(y) = 0\}$ is the set of critical points for the ODE (equilibrium solutions).
Classification of critical points.

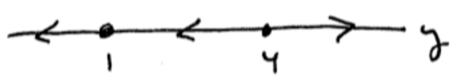
$$\text{let } y_\star \text{ be critical for } y' = f(y) \text{ and call } N_\varepsilon(y_\star) = \{y \in \mathbb{R} | |y - y_\star| < \varepsilon\}$$

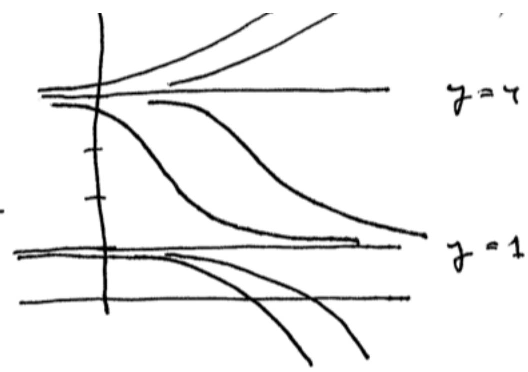
ⓐ If $\exists \varepsilon > 0$ where $\forall y \in N_\varepsilon(y_\varepsilon)$ and $\lim_{t\to\infty} y(t) = y_\star \Rightarrow y_\star$ is asymptotically stable.

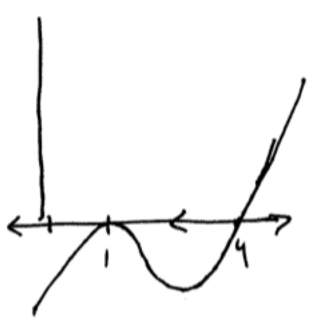
ⓑ If $\exists \varepsilon > 0$ where $\forall y \in N_\varepsilon(y_\varepsilon)$ and $\lim_{t\to-\infty} y(t) = y_\star \Rightarrow y_\star$ is unstable.

ⓒ If stable on one side and unstable on other, semi-stable.

ex. $y' = (1-y)^2(y-4)$







$f(y) = (1-y)^2(y-4)$

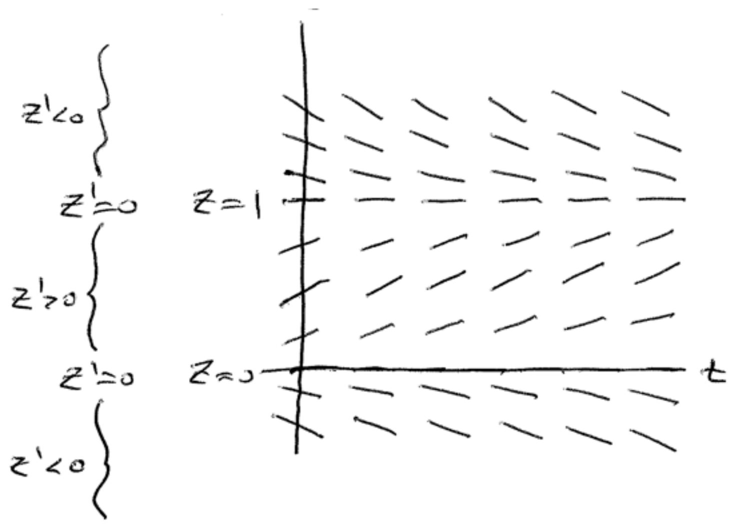
$y(t) = 4$ is unstable
$y(t) = 1$ is <u>semi</u>stable.

Lecture 7 (2018-06-28)

Autonomous Equations and Population Dynamics

Equilibrium Solutions
At any place y_0 where $f(y_0) = 0$, then $y'(t) = 0$ here, and thus $y(t) = y_0$ is a constant solution (or equilibrium, or steady-state solution). Its graph is a horizontal line is an isocline.

$$\frac{dz}{dt} = z(1-z)$$



Here $Z(t) = 0$ and $z(t) = 1$ are both equilibrium solutions.
And between the equilibria, the sign of z' does not choose. Hence solutions

(A) are trapped between equilibria, and

(B) always travel in the same direction.

In the example, we can say the following without solving:

(I) Solutions exist and are unique everywhere ($f(z)$ and $f'(z)$ are polynomials)

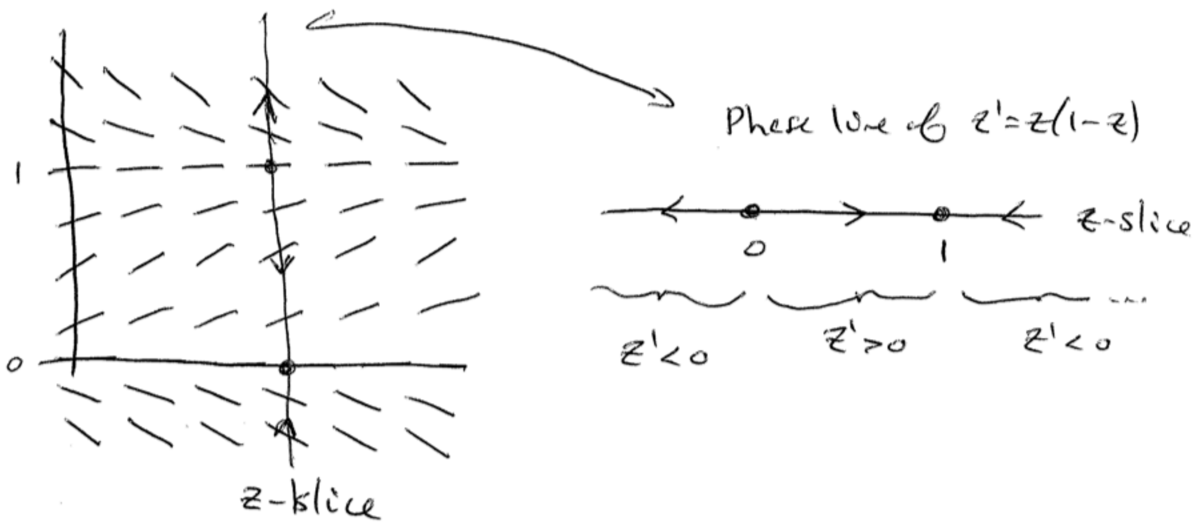
(II) Equilibria only at $z = 0$ and $z = 1$.

(III) Any solution that pushes through $0 < Z_0 < 1$ will find toward the equilibria $z(t) = 1$.
Any solution that starts at $Z_0 < 0$ will tend to $-\infty$
Any solution that starts at $Z_0 > 1$ will tend to $z(t) = 1$
Hence we can say

$$\lim_{t\to\infty} z(t) = \begin{cases} -\infty & Z_0 < 0 \\ 0 & Z_0 = 0 \\ 1 & Z_0 > 0 \end{cases}$$

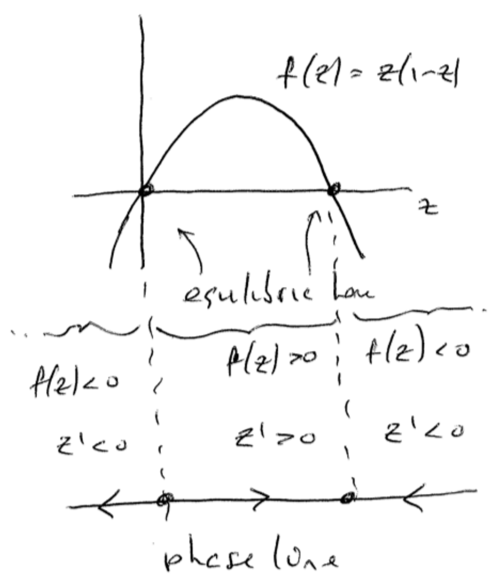
Phase Line

Any vertical slice through the slope field gives you all information about long term behavior solution:



Here, the phase line is a schematic that determines all long term behavior of all autonomous $z' = f(z)$

Without a slope field still easy to see phase line: Graph $f(z)$: $z' = z(1-z) = f(z)$



Definition 1. For $y' = f(y)$, the set $\{y \in \mathbb{R} \big| f(y) = 0\}$ is the set of critical points for the ODE. (equilibrium solutions.)

Critical points (equilibrium solutions) can be classified by how solutions behave around them:
Let $y_\star$ be a critical point for $y' = f(y)$ and let $N_\varepsilon(y_\star) = \{y \in \mathbb{R} \big| |y - y_\star| < \varepsilon\}$ be an $\varepsilon-$neighborhood of $y_\star$.

(a) If there is a $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_\star)$

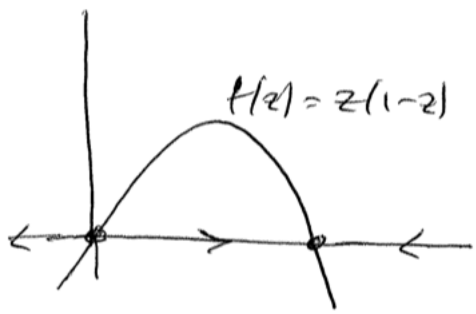
$$\lim_{t\to\infty} y(t) = y_\star \Rightarrow \text{ is aymptotically stable}$$

(b) If there is an $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_\star)$

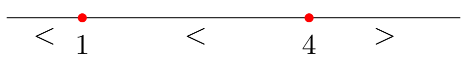
$$\lim_{t \to -\infty} y(t) = y_\star \Rightarrow \text{ is \underline{unstable}}$$

(c) If asymptotically stable on one side and unstable on the other, then $y_\star$ is semi-stable.

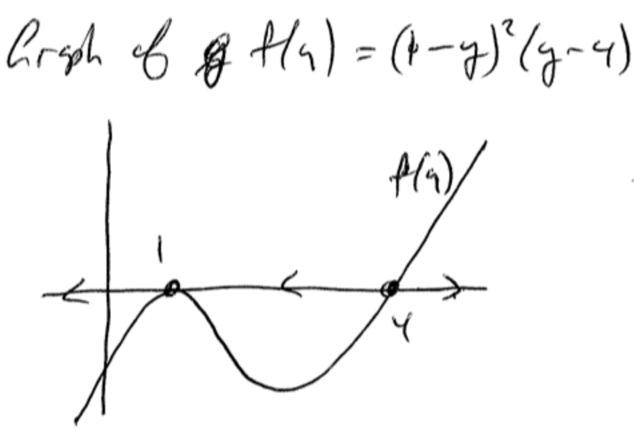
Example 1. $z' = z(1-z)$, Here critical points are $z = 0, 1$. And here $z(t) = 1$ is asymptotically stable and $z(t) = 0$ is unstable.

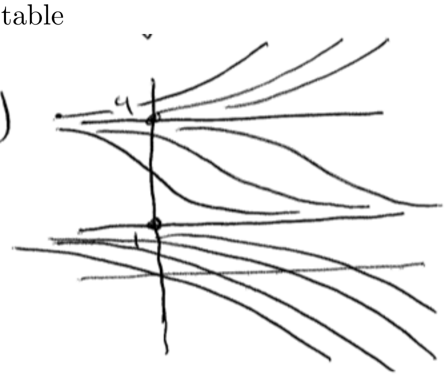


Example 2. $y' = (1-y)^2(y-4)$
Here, critical points at $y = 1, 4$. The phase line is...



$y(t) = 4$ is unstable
$y(t) = 1$ is semistable





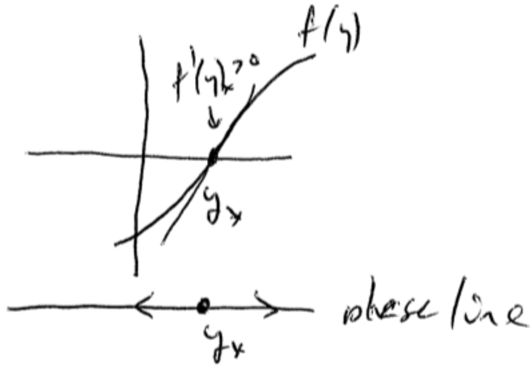
$$\lim_{t \to \infty} y(t) = \begin{cases} \infty & y_0 > 4 \\ 4 & y_0 = 4 \\ 1 & 1 \le y_0 < 4 \\ -\infty & y_0 < 1 \end{cases}$$

When graphing the $f(y)$ in $y' = f(y)$ and is constructing phrase lines, some patterns develop:
Let $y_\star$ be an equilibrium for $y' = f(y)$ (thus $f(y_\star) = 0$).

For y "near" y_0, $y' = f(y) \cong \underbrace{\cancelto{0}{f(y_\star)} + f'(y_\star)(y - y_\star)}_{\text{1st Taylor approx. to f(y) at } y_\star}$

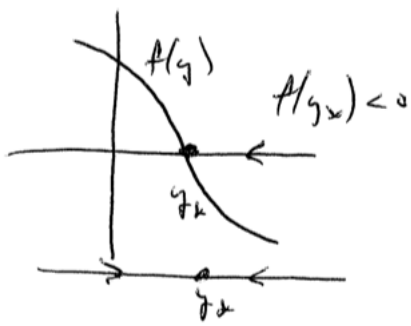
Case 1: Suppose $f'(y_\star) > 0$

$$\Rightarrow \text{for } y > y_\star, \quad y' > 0 \text{ and } y < y_\star, \quad y' < 0$$



All nearby solutions move away $\Rightarrow y_\star$ is on unstable node or source
Case 2: Suppose for $f'(y_\star) < 0$

$$\Rightarrow \text{ for } y > y_\star, \quad y' < 0 \text{ and } y < y_\star, \quad y' > 0$$



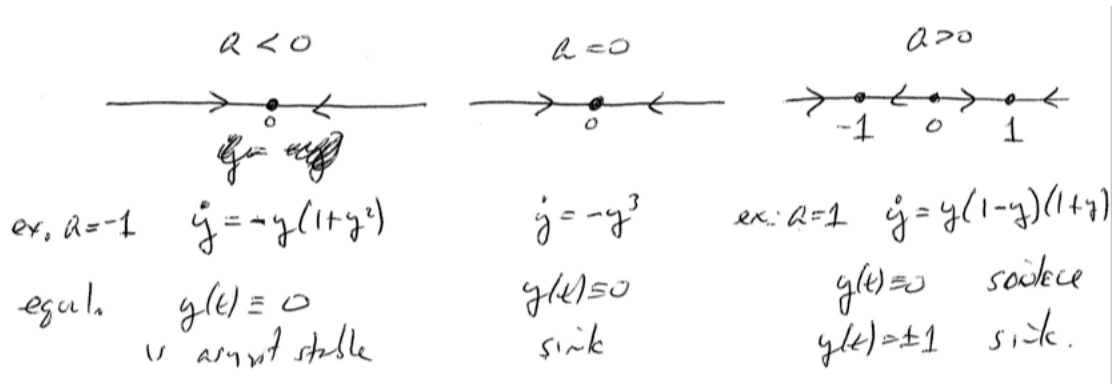
All nearby solutions converge to $y_\star \Rightarrow$ asymptotically stable on <u>sink.</u>
Case 3: $f(y_\star) = 0$. Need more information.

Lecture 8 (2018-06-30)

Bifurcations

Consider the autonomous $y' = f(a, y)$ where "a" is a parameter. (an uniform constant). Equilibrium may depend on the value of a (in location # and stability type).

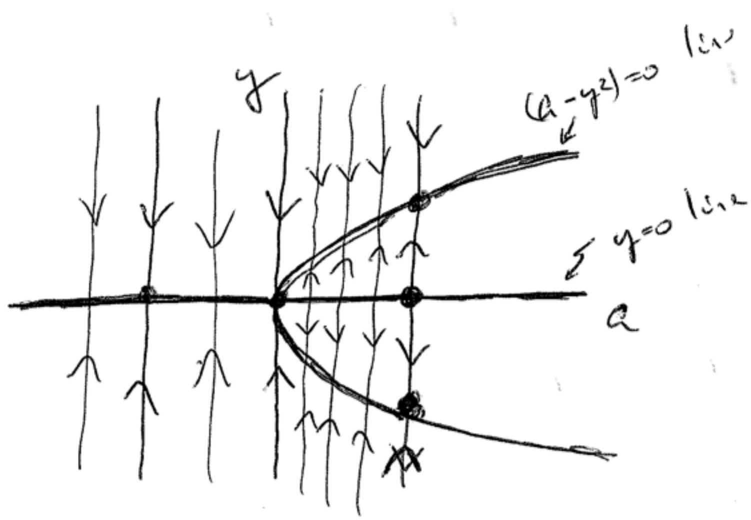
Example 1. $\dot{y} = ay - y^3 = y(a - y^2)$
Here, for $a < 0$, and $a > 0$, the number of equilibria are different, also the type.



We can study how "a" affects equilibrium via a <u>bifurcation diagram</u>

Definition 1. A bifurcation diagram is graph of equilibrium (and stability) in relation to parameter value.

<u>Properties</u>



- Each vertical slice is the phase line for a value of a.
- As a varies, equilibrium trace out curves of fixed points. Found by solving $f(a, y) = 0$
- Curves of fixed points can be found by solving $f(a, y) = 0$

$$f(a,y) = y(a-y^2) = 0, \quad \text{when } \underbrace{y=0}_{\text{a-axis}} \qquad \underbrace{y^2=a}_{\text{sideways parabola}} \quad \Rightarrow y = \sqrt{a},\ y = -\sqrt{a}$$

- Here the only bifurcation value is $a = 0$
- Values of a in which the stability and or # of equilibria change are called <u>bifurcation values</u> of a.
- $\boxed{\text{These are rare!}}$ - stability cannot change outside of these!

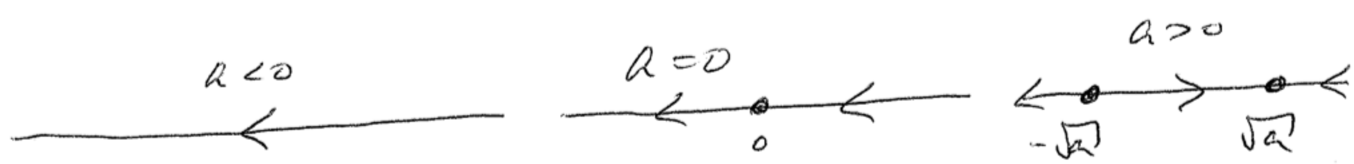
<u>Notes</u>

I Stable lines are used as phase lines to denote stability.

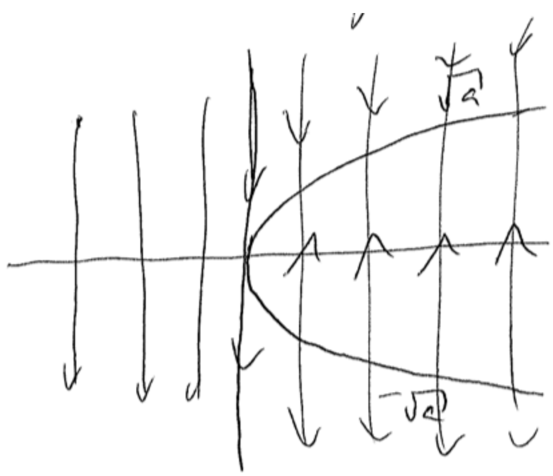
II Stability cannot change outside of bifurcation points, and cannot change far away.

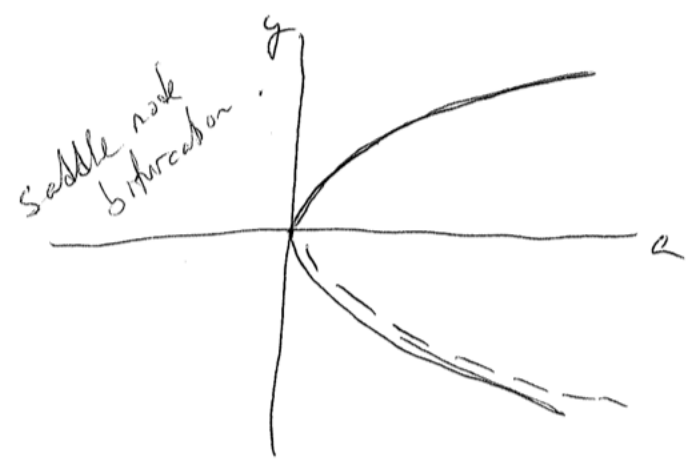
III $a = 0$ is called a pitchfork bifurcation for $\dot{y} = ay - y^3$.

Example 2. $\dot{y} = a - y^2$



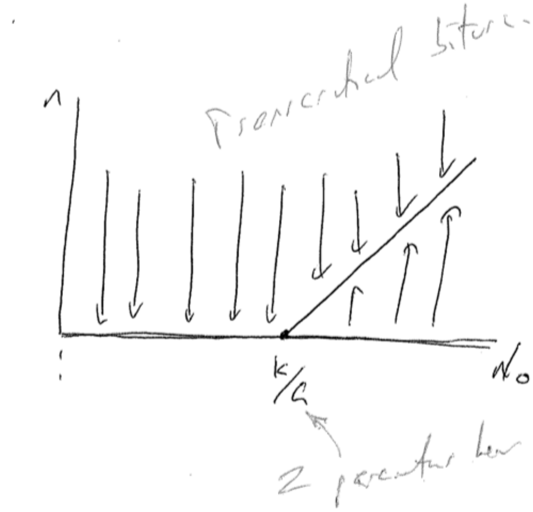
Lines of equil can only be $a - y^2 = 0 \Rightarrow a = y^2$ (sideways par).





Example 3. $\dot{n} = (GN_0 - k)n - Gn^2$
is an equation involving laser physics where G, N_0, k are positive constants. $n(t)$ is the # of photons along (always $n \geq 0$).
Equilibrium solutions are $n(GN_0 - k - Gn) = 0$, where $n = 0$, $n = N_0 - \frac{k}{a}$.

<u>Bifurcation diagram.</u>



(I) when $N_0 < \frac{k}{a}$
$\Rightarrow GN_0 - k < 0$
$\Rightarrow \dot{n} < 0$
$\Rightarrow n = 0$ is sink

(II) When $N_0 > \frac{k}{a}$, $GN_0 - k > 0$
$\Rightarrow GN_0 - k - Gn$
$N_0 - \frac{k}{a} - n > 0$ for small n.
$\Rightarrow n = 0$ is a source
if $N_0 - \frac{k}{a} - n < 0$ for $n > N_0 - \frac{k}{a}$
$\Rightarrow N_0 - \frac{k}{a}$ is a sink

Change track - Recall for any equation involving x, y,

- We can bring all terms to one side of the equation and create an equivalent equation $\varphi(x, y) = 0$. For $\varphi(x, y)$ a function of 2 variables. Then the curve in the xy-plane satisfying this equation is called the 0-level set of φ.

  Example. $y^2 = 1 - x^2$. We view this equation as the 0-level set of the function $\varphi(x, y) = x^2 + y^2 - 1 = 0$

- We can view y as a implicit function of x.
  In either case, the graph of the original equation (or the $\varphi(x, y) = 0$) is a curve in xy-plane that in general will not look like a function.
  We can calculate the tangent lines to this graph via differentiation in either interpretation.

Example. $x^2 + xy^2 = 4$, or $\varphi(x, y) = 0$, $\varphi(x, y) = x^2 + xy^2 - 4$

$$\text{Implicit diff} \quad \frac{d}{dx}(x^2 + xy^2 = 4) \Rightarrow \underbrace{2x + y^2 + 2xy\frac{dy}{dx}} = 0$$

$$\text{Calc III} \quad \frac{d\varphi}{dx}(x, y) = \frac{d\varphi}{dx} + \frac{d\varphi}{dy} \cdot \frac{dy}{dx} = \underbrace{\frac{d\varphi}{dx} + \frac{d\varphi}{dy} \cdot y'}_{(\star)}$$

when we think of y as an implicit fraction of x.

Lecture 9 (2018-07-08)

Exact Equations and Integrating Factors

Suppose a first ODE has the form

$$\boxed{M(x,y) + N(x,y)y' = 0} \tag{$*$}$$

Then $(\star)$ and $(*)$ are the same under the condition thaht there exists a function.

$$\varphi(x,y), where \quad ① \frac{d\varphi}{dx}(x,y) = M(x,y) \tag{1}$$

$$② \frac{d\varphi}{dx}(x,y) = N(x,y) \tag{2}$$

$$\text{So that} \quad M(x,y) + N(x,y)y' = 0 = \frac{d\varphi}{dx} = \frac{d\varphi}{dx} + \frac{d\varphi}{dy}y'$$

If this is the case, then the ODE $(*)$ can be rewritten as $\dfrac{d\varphi}{dx} = 0$, or $\varphi(x,y) = C$, a constant.

Example. Notice that $2x + y^2 + 2xyy' = 0$ is of the form $M(x,y) + N(x,y)y' = 0$ with

$$M(x,y) = 2x + y^2$$
$$N(x,y) = 2xy$$

But we also can see that the function $\varphi(x,y) = x^2 + xy^2$ has the partials.

$$\frac{d\varphi}{dx}(x,y) = 2x + y^2 \quad \frac{d\varphi}{dy}(x,y) = 2xy$$

Hence $2x + y^2 + 2xyy' = 0$ can be written $\dfrac{d\varphi}{dx} = 0 = \dfrac{d}{dx}(x^2 + 2xy^2)$. If we assume that y is an implicit function of x.
here we can (assuming y is an implicit function of x) integrate $\dfrac{d\varphi}{dx}(x,y) = 0$ w.r.t. x to set

$$\int \frac{d\varphi}{dx}(x,y)dx = \int 0dx$$

$$\varphi(x,y) = x^2 + xy^2 = C$$

This is the implicit solution to

$$2x + y^2 + 2xyy' = 0$$

Question: How do we know such a $\varphi(x,y)$ may exist and if so how to find it?

<u>Calc III Thm</u> Let $\varphi(x,y)$ have continuous partial derivatives in some open region. Then

$$\frac{d}{dx}\left(\frac{d\varphi}{dx}\right) = \frac{d}{dy}\left(\frac{d\varphi}{dx}\right) \quad \textit{i.e. Mixed 2nd partials are equal.}$$

Now if we had the ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

and knew there was a function $\varphi(x,y)$ when $\frac{d\varphi}{dx} = M$, $\frac{d\varphi}{dy} = N$. then the last criteria is

$$\frac{d}{dx}(\mathbf{N}) = \frac{d}{dy}(\mathbf{M}), \text{ or } \quad \boxed{N_{\mathbf{X}} = M_{\mathbf{Y}}}$$

Definition 1. The ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is called <u>exact</u> on a region.

$$R = \left\{ (x,y) \in \mathbb{R}^2 \middle| \begin{matrix} \alpha < x < \beta \\ \gamma < y < \delta \end{matrix} \right\}$$

If 1 M, N, M_y, N_x are C^0 on R, and 2 $M_y = N_x$ on R

Theorem 1. *Let $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ be exact o some open region R. Then $\exists$ a functions $\varphi(x,y)$ which is diff on R where*

$$1\ \frac{d\varphi}{dx} = M \quad 2\ \frac{d\varphi}{dy} = N$$

and $\varphi(x,y) = C$ is there general implicit solution to the ODE on R

Example 1. Solve $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$, $y(1) = 0$
<u>Strategy</u> First, we verify the exactness. Then we integrate to find the function whose level sets comprise solutions to the ODE.
<u>Solution</u> Here $M(x,y) = 3x^2 - 2xy + 2$, $N(x,y) = 6y^2 - x^2 + 3$ and since $M_y = \frac{dM}{dy} = -2x. = \frac{dN}{dx} = N_x$ the ODE is exact.
And since M, N, M_y, N_x are all C^0 on $\mathbb{R}^2$, by <u>Thm</u> $\varphi(x,y)$ will exist near $(1,0) \in \mathbb{R}^2$. To find $\varphi(x,y)$, note that $\frac{d\varphi}{dx} = M$, $\frac{d\varphi}{dy} = N$.
Integrate M <u>wrt</u> x.

$$\int M dx = \int \frac{d\varphi}{dx} dx = \int (3x^2 - 2xy + 2) dx = x^3 - x^2 y + 2x + \underbrace{h(y)}_{\text{why?}}$$

Hence $\varphi(x,y) = x^3 - x^2 y + 2x + h(y)$ for some unknown function $h(y)$. To find $h(y)$, note that $\frac{d\varphi}{dy} = N$:

$$\begin{aligned} \frac{d\varphi}{dy}(x,y) &= \frac{d}{dx}(x^3 - x^2 y + 2x + h(y)) \\ &= -x^2 + h'(y) \\ &= N(x,y) = 6y^2 - x^2 + 3 \end{aligned}$$

Hence $h'(y) = 6y^2 + 3$, or $h(y) = 2y^3 + 3y +$ *constant*. Thus our general implicit solution is

$$\varphi(x,y) = x^3 - x^2 y + 2x + 3y + 2y^2 = C$$

Our particular solution is $\varphi(1,0) = 1 - 0 + 2 + 0 + 0 = C$ or $c = 3$, so $\varphi(x,y) = 3 = x^3 - x^y + 2x + 3y + 2y^3$. Determining a valid interval is difficult here.

Example 2. Solve $2x + y^2 + 2xyy' = 0$, $y(1) = 1$
<u>Strategy</u> First, we verify the exactness. Then we integrate to find the function whose level sets comprise solutions to the ODE.
<u>Solution</u> Here $M(x,y) = 2x + y^2$, $N(x,y) = 2xy$ and since $M_y = \dfrac{dM}{dy} = 2y. = \dfrac{dN}{dx} = N_x$ the ODE is exact.
And since M, N, M_y, N_x are all C^0 on $\mathbb{R}^2$, by <u>Thm</u> $\varphi(x,y)$ will exist near $(1,1) \in \mathbb{R}^2$. To find $\varphi(x,y)$, note that $\dfrac{d\varphi}{dx} = M$, $\dfrac{d\varphi}{dy} = N$.
Integrate M <u>wrt</u> x.

$$\int M dx = \int \frac{d\varphi}{dx} dx = \int (2x + y^2) dx = x^2 + xy^2 + \underbrace{h(y)}_{\text{why?}}$$

Hence $\varphi(x,y) = x^2 + xy^2 + h(y)$ for some unknown function $h(y)$. To find $h(y)$, note that $\dfrac{d\varphi}{dy} = N$:

$$\begin{aligned}
\frac{d\varphi}{dy}(x,y) &= \frac{d}{dy}(x^2 + xy^2 + h(y)) \\
&= 2xy + h'(y) \\
&= N(x,y) = 2xy
\end{aligned}$$

Hence $h'(y) = 0$, or $h(y) = \textit{constant}$. Thus $\varphi(x,y) = x^2 + xy^2 + \textit{constant}$, and $\varphi(x,y) = x^2 + xy^2 = C$ is our general implicit solution The solution to $2x + y^2 + 2xyy' = 0$, $y(1) = 1$. $x^2 + xy^2 = 2$, variable on $x \in (0, \sqrt{2})$

<u>Notes</u>

I Sometimes, the ODE is in differential form:

Example. $(ye^{2xy} + x)dx + xe^{2xy}dy = 0$ is exact, since $M(x,y) = ye^{2xy} + x$, $N(x,y) = xe^{2xy}$ and

$$\textit{are equal} \begin{cases} M_y = e^{2xy} + 2xye^{2xy} \\ M_x = e^{2xy} + 2xye^{2xy} \end{cases}$$

II Caution: Sometimes a <u>non-exact</u> 1st order ODE can be made exact via an integration factor

Example. $dx + (\frac{x}{y} - \sin y)dy = 0$ is not exact $M_y = 0 \neq \frac{1}{y} = N_x$

$$y[dx + (\frac{x}{y} - \sin y)dy = 0]$$

$$ydx + (x - y\sin y)dy = 0 \text{ is exact since now}$$

$$M_y = 1 = \frac{d}{dx}[x - y\sin y] = 1 = N_x$$

We won't focus on this last technique.

Lecture 10 (2018-07-08)

I will relegate the discussion of Section of 2.8 to a worksheet posted. The theory can be deep (through very interesting). But the main takeaway is its usefulness in...

(I) seeing where a first order ODE is "nice"

(II) Understanding the intricacies of the theory even at this early stage.

Second Order Linear Equations

Homogeneous Equations with Constant Coefficients

A general form of a 2nd order ODE is, for some function f

$$\boxed{y'' = f(t, y, y')} \tag{$**$}$$

Definition 1. A 2nd order ODE is called <u>linear</u> if it can be written

$$y'' + p(t)y' + q(t)y = g(t)$$

$$\text{(so that } f(t, y, y') = g(t) - p(t)y' - q(t) \text{ in } (*))$$
$$(**) \text{ (actually } P(t)y'' + Q(t)y' + R(t)y = G(t)...)$$

<u>Notes</u>

(I) f needs to be linear in both y and y'

(II) If ODE is not linear, it is called non-linear.

Definition 2. If $g(t) \equiv 0$, then a linear ODE is called <u>homogeneous</u>.

Definition 3. An IVP with a 2nd order ODE continuous 2 pieces of initial data, usually $y(t_0) = y_0$, $y'(t_0) = y'_0$

<u>Question:</u> Why?
In General, it is heard or impossible to solve a 2nd order ODE. Even linear is very difficult in general!

One type that is solvable: Constant coefficients. Let $(**)$ here $P(t) \equiv a$, $Q(t) \equiv b$, $R(t) \equiv c$, and suppose $G(t) \equiv 0$ (homogenous).

$$\text{Then ODE is} \quad \boxed{ay'' + by' + cy = 0} \tag{$\star$}$$

Q: First think, what kinds of functions would possibly be solutions to their kind of ODE?

- Polynomials? Prove Functions?
- Trig functions?
- Exponentials?
- Logarithms?

Example 1. Suppose $a = 1$, $b = 0$, $c = -1$, Then $(**)$ is $y'' - y = 0$, or $y'' = y$
Solutions?
Here $y(t) = e^t$ and $y(t) = e^{-t}$ both solve $y'' - y = 0$ How about $e^t + e^{-t}$? $2e^t - 3e^{-t}$? Here $y(t) = c_1 e^t + c_2 e^{-t}$ is a solution for any choice of $c_1, c_2 \in \mathbb{R}$
What if IVP was $y'' - y = 0$, $y(0) = 3$, $y(0) = 4$?

$$\underbrace{\begin{aligned} y(0) &= 3 = c_1 e^0 + c_2 e^{-0} = c_1 + c_2 \\ y(0) &= 4 = c_1 e^0 - c_2 e^{-0} = c_1 - c_2 \end{aligned}}$$

$$\begin{aligned} c_1 &= \frac{7}{2} \\ c_2 &= -\frac{1}{2} \end{aligned} \quad \begin{cases} 3 = c_1 + c_2 \\ 4 = c_1 - c_2 \end{cases}$$

And the particular solution to IVP is

$$y(t) = \frac{7}{2} e^t - \frac{1}{2} e^{-t}$$

Example 2. $2y'' + 8y' - 10y = 0$
Here $y(t) = e^t$ and $y(t) = e^{-5t}$ both work! Check this... AND so does $y(t) = c_1 e^t + c_2 e^{-5t}$.
Is there a pattern?

For $ay'' + by' + cy = 0$, assume the solution is exponential (is this a good idea?) and looks like $y(t) = e^{\Gamma t}$, where Γ is an unknown parameter.
Then substituting $y(t)$ and its derivatives into the ODE, we set

$$a\Gamma^2 e^{\Gamma t} + b\Gamma e^{\Gamma t} + c e^{\Gamma t} = 0 \quad or \quad a\Gamma^2 + b\Gamma + c = 0$$

Any valid values for Γ must satisfy $\Gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Recognize this?

For a 2nd order homogenous ODE with constant coefficients $(\star)$ is called the characteristic equation
Important: When the two roots $\sqrt{1}$, $\sqrt{2}$ of the char. eqn. one real and distinct, then the general solution to the ODE is

$$\boxed{y(t) = c_1 e^{\Gamma_1 t} + c_2 e^{\Gamma_2 t}}$$

(we will need to make sure this is the case later)

Example. $y'' - y = 0$. Here $a = 1$, $b = 0$, $c = 1$, and characteristic equation is $\Gamma^2 - 1 = 0$, with roots $\Gamma = -1, 1$. General solution is

$$y(t) = c_1 e^t + c_2 e^{-t}$$

Example. Characteristic equation is $2\Gamma^2 + 8\Gamma - 10 = 0$ which factors to $(2\Gamma - 2)(\Gamma + 5) = 0$, so $\Gamma = 1, -5$.

$$y(t) = c_1 e^t + c_2 e^{-5t}$$

Lecture 11 (2018-07-09)

Before talking about the cases where roots of the characteristic equation are the same or not real, lets return to the more general linear 2nd order homogenous ODE

$$y'' + p(t)y' + q(t)y = 0$$

To study this, form the operator (an operator is a function whose domain and range are functions).

$$L[\gamma] = \gamma'' + p(t)\gamma' + q(t)\gamma$$

This operator is defined for all c^2 functions $y(t)$ on an interval like $\alpha < t < \beta$, where α may be a number or $-\infty$, and β may be a number or ∞.

Notes

I Can also write

$$L = \frac{d^2}{dt^2} + p\frac{d}{dt} + q$$

II An operator $L[\gamma]$ is linear if $L[c_1\gamma_1 + c_2\gamma_2] = c_1L[\gamma_1] + c_2L[\gamma_2]$

Claim. $L[\gamma] = \gamma'' + p(t)\gamma' + q(t)$ is linear as an operator

Proof.

$$\begin{aligned} L[c_1\gamma_1 + c_2\gamma_2] &= \frac{d^2}{dt^2}[c_1\gamma_1 + c_2\gamma_2] + p(t)\frac{d}{dt}[c_1\gamma_1 + c_2\gamma_2] + q(t)(c_1\gamma_1 + c_2\gamma_2) \\ &= c_1\gamma_1'' + c_2\gamma_2'' + p(t)(c_1\gamma_1' + c_2\gamma_2') + q(t)(c_1\gamma_1'') \\ &= c_1(\gamma_1'' + p(t)\gamma_1' + q(t)\gamma_1) + c2(\gamma_2'' + p(t)\gamma_2' + q(t)\gamma_2) \\ &= c_1L[\gamma_1] + c_2L[\gamma_2] \end{aligned}$$

■

Fact: The homogenous 2nd order linear ODE

$$y'' + p(t)y' + q(t)y = 0$$

is solved by any function $y(t)$, where $L[y(t)] = 0$

2 theorems on linear, 2^{nd} order ODE

I Existence of Uniqueness

Theorem. *The IVP $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_0'$, where p, q, and g are continuous on an open interval I containing t_0, has a unique solution $y(t)$ defined and twice differentiable on I.*

Note: Here, I can be taken to be the largest interval containing t_0, where p, q, and g are all simultaneously continuous.

II Superposition

Theorem. *If $y_1(t)$, $y_2(t)$ are 2 solutions to $L[y] = 0$, then so is $c_1\gamma_1 + c_2\gamma_2$ for all $c_1, c_2 \in \mathbb{R}$*

Lecture 12 (2018-07-09)

Solutions of Linear Homogenous Equations

New Questions: If you found 2 solutions y_1, y_2 to $L[y] = 0$, do all solutions look like $c_1y_1 + c_2y_2$? Can there be others?
To study this, let's "solve" the IVP.

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

using the idea of a "general" solution

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

$$\begin{aligned} \overbrace{c_1y_1(t_0)}^{\text{real \#}} + \overbrace{c_2y_2(t_0)}^{\text{real \#}} &= y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0' \end{aligned} \tag{$\star\star$}$$

Solve the system for c_1, c_2 (2 eqns., 2 unknowns)

$$c_1 = \frac{y_0 \cdot y_2'(t_0) - y_0' \cdot y_2(t_0)}{y_1(t_0) \cdot y_2'(t_0) - y_1'(t_0) \cdot y_2(t_0)} \qquad c_2 = \frac{y_0 \cdot y_1'(t_0) - y_0' \cdot y_1(t_0)}{y_1(t_0) \cdot y_2'(t_0) - y_1'(t_0) \cdot y_2(t_0)}$$

Note: Solutions to $(\star\star)$

1 solution	lines cross
0 solutions	lines parallel
∞ solutions	lines the same

Note: the numerators are different for c_1, c_2 but the <u>denominators</u> are <u>the same!</u>
Rewrite the denominator as the determinant of a $2x2$ matrix whose entries are the coefficients of $(\star\star)$:

$$y_1(t_0) \cdot y_2'(t_0) - y_1'(t_0) \cdot y_2(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

This comes from writing $(\star\star)$ as a matrix equation:

$$\underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

Given this matrix equation, if $\det A \neq 0$ there is a unique solution (c_1, c_2)

In our case, in the 2 expressions for c_1, c_2

I If denominator non-zero, then unique solution.

II if <u>ONLY</u> denominator zero, then no solutions

III If both numerator, denominator zero, tons of solutions

$$\text{Call} \quad W = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

The **Wronskian determinant** of y_1, y_2 at t_0.

- Tells you about the solutions to the IVP - $L[y] = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0'$

Theorem 1. *Suppose y_1, y_2 are 2 solutions to $l[y] = 0$ and at the initial values $y(t_0) = y_0$, $y_0'(t_0) = y_0'$ and $W(y_1, y_2)(t_0) \neq 0$*

$$\Rightarrow \exists c_1, c_2 \in \mathbb{R} \text{ so that } y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t) \text{ solves the IVP.}$$

Note: This ensures that y_1 and y_2 are fundamentally "different" solutions (read: independent)

Question: What does this mean?

Theorem 2. *If y_1, y_2 both solve $L[y] = 0$, and if $\exists t_0$ where $W = W(y_1, y_2)(t_0)$, $\Rightarrow y(t) = c_1 y_1(t) + c_2 y_2(t)$ includes every solution!*

Proof. Let γ be any solution to the IVP near t_0, where $W = W(y_1, y_2)(t_0)$. Then by theorem 1, $c_1 y_1(t) + c_2 y_2(t)$ solves the IVP for some choice of $c_1, c_2 \in \mathbb{R}$. But by uniqueness $\gamma(t) = c_1 y_1(t) + c_2 y_2(t)$ ∎

Here, given $L[y] = 0$, if you find any 2 solutions y_1, y_2 where Wronskian is somewhere non-zero, then $y(t) = c_1 y_1(t) + c_2 y_2(t)$ includes all solutions on the entire interval where Wronskin is non-non-zero. Called the general solution or the fundamental set of solutions to $L[y] = 0$

Example 1. Suppose $y_1 = e^{\Gamma_1 t}$ and $y_2 = e^{\Gamma_2 t}$ both solve $L[y] = 0$. The Wronskin is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{\Gamma_1 t} & e^{\Gamma_2 t} \\ \Gamma_1 e^{\Gamma_1 t} & \Gamma_2 e^{\Gamma_2 t} \end{vmatrix} = (\Gamma_1 + \Gamma_2) e^{(\Gamma_1 + \Gamma_2) t}$$

Here, as long as $\Gamma_1 \neq \Gamma_2$, $W(y_1, y_2)(t) \neq 0$ anywhere on $\mathbb{R}$, and $y(t) = c_1 e^{\Gamma_1 t} + c_2 e^{\Gamma_2 t}$ is a fundamental set of solutions to $L[y'] = 0$

Example $y_1 = \sin t$, $y_2 = \cos t$ and $W(y_1, y_2)(t) \equiv 1$
Example $y_1 = \sin t$, $y_2 = \cos(t - \frac{\pi}{2})$ and $W(y_1, y_2)(t) \equiv 0$
Where $W(y_1, y_2)(t) \neq 0$, we can say y_1, y_2 are linearly independent as functions.

Definition. Two functions $f(x)$, $g(x)$ are called linearly dependent **(LD)** on some open interval I if there exists 2 constants k_1, k_2 are not both 0, where

$$k_1 f(x) + k_2 g(x) = 0$$

$\forall x \in I$. Otherwise called linearly independent or **(LI)**

Note: If there exists one point $x \in I$ where 2 functions are **LI**, then the functions are **LI** on I.
Extra The Wronskin only really depends on the ODE in a fundamental way:

Theorem. *Given any 2 solutions to $L[y] = 0$, where $p(t)$, $q(t)$ are continuous on an open interval I, then*

$$W(y_1, y_2) = ce^{-\int p(t) dt}$$

where c depends on y_1, y_2 but not on t.

Notes

(I) If y_1, y_2 are **LD**, then $c = 0$

(II) If y_1, y_2 are **LI**, then $W \neq 0$ on all of I

(III) Proof is quire interesting!

(IV) LI and non Wronskin are the same thing for ODEs.

Proof. Since y_1, y_2 solve the ODE

(a) $y_1'' + p(t)y_1' + q(t)y_1 = 0$

(b) $y_2'' + p(t)y_2' + q(t)y_2 = 0$

Must (a) by $-y_2$ and (b) by y_1 and add (eliminating $q(t)$)

$$\underbrace{\underbrace{y_1y_2'' - y_2y_1''}_{W'(y_1,y_2)} + p(t)\underbrace{y_1y_2' - y_2y_1'}_{W(y_1,y_2)}}_{W'+p(t)W=0} = 0$$

is a 1st order ODE in the Wronskin det as a function of t. By separation of variables:

$$\frac{W'}{W} = -p(t) \Rightarrow \ln|W| = -\int p(t)\,\det K$$
$$\Rightarrow W = ce^{-\int p(t)dt}$$

Eitehr $W = 0$ on all of I or $W \neq 0$ on all of I ∎

Lecture 13 (2018-07-09)

Complex Roots of the Characteristic Equation

Back to the constant coefficients case:

$$ay'' + by = +cy = 0 \tag{$\star$}$$

Let $a = 1 = c$, $b = 0$. Here $y'' + y = 0$ has characteristic equation $\Gamma^2 + 1 = 0$ (no real solutions). But we know $y_1(t) = \cos t$, $y_2(t) = \sin t$. Solve $y'' + y = 0$

$$\boxed{y(t) = c_1 \cos t + c_2 \sin t}$$

Question: How can we get that from the characteristic equation?

First, $r^2 + 1 = 0$ does have 2 solutions: $r = \pm i$
Sticking to the exponential there, $y_1(t) = e^{it}$ and $y_2(t) = e^{-it}$ are the two solutions.

Recall Euler's Formula

$$e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$$

Question: Can we construct real solutions from there?

Suppose an ODE has characteristics equation.

$$a\Gamma^2 + b\Gamma + c = 0, \quad \Gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \lambda \pm \mu i, \quad \mu \neq 0$$

Writing 2 exponential solutions.

$$\begin{aligned} y_1(t) &= e^{(\lambda+\mu i)t} \\ &= e^{\lambda t}(\cos \mu t + i \sin \mu t) \end{aligned} \qquad \begin{aligned} y_1(t) &= e^{(\lambda-\mu i)t} \\ &= e^{\lambda t}(\cos \mu t - i \sin \mu t) \end{aligned}$$

We see they are not real. But real ODE must be real solutions!
The <u>superposition</u>, any linear combination of y_1, y_2 is also a solution:

$$\text{Hence} \quad \begin{aligned} \frac{1}{2}(y_1(t) + y_2(t)) &= \frac{1}{2}(e^{\lambda t} \cos \mu t + i \sin \mu t) + e^{\lambda t} \cos \mu t + i \sin \mu t \\ &= e^{\lambda t} \cos \mu t \quad \text{is a solution} \end{aligned}$$

$$\text{Hence} \quad \begin{aligned} \frac{1}{2i}(y_1(t) - y_2(t)) &= \frac{1}{2i}(e^{\lambda t} \cos \mu t + i \sin \mu t) - e^{\lambda t} \cos \mu t + i \sin \mu t \\ &= e^{\lambda t} \sin \mu t \quad \text{is a solution} \end{aligned}$$

lets call then $u(t) = e^{\lambda t} \cos \mu t, \quad v(t) = e^{\lambda t} \sin \mu t$
There a 2 real solutions (the real and imaginary parts of the original operator solutions.)
<u>And</u> since $W(u, v) = \text{calculate this} = \mu e^{2\lambda t} \neq 0$ anywhere as long as $\mu \neq 0$, the fund set of solutions is.

$$\begin{aligned} y(t) &= c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \\ &= e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) \end{aligned}$$

Example 1. $y'' + y = 0$. Roots of $\Gamma^2 + 1 = 0$ are $\Gamma = \pm i$. Here $\lambda = 0$, $\mu = 1$.

$$y(t) = e^{0t}(c_1 \cos 1t + c_2 \sin 1t) = \boxed{c_1 \cos t + c_2 \sin t}$$

Example 2. Solve the IVP $y'' + 4y' + 13y = 0$, $y(0) = 2$, $y'(0) = 7$
Solution: The characteristic equation is $\Gamma^2 + 4\Gamma + 13 = 0$, and $\Gamma. = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$

Here, find set of solutions is

$$y(t) = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$$

For a particular solution

$$y(0) = e^{-2(0)}(c_1 \cos 3(0) + c_2 \sin 3(0)) = \boxed{2 = c_1}$$

$$y'(0) = -2e^{-2t}(c_1 \cos 3t + c_2 \sin 3t) + e^{-2t}(-6 \sin 3t + 3c_2 cos3t)\Big|_{t=0}$$

$$= -4 + 3c_2 = 7 \quad c_2 = \frac{11}{3}$$

$$\boxed{y(t) = e^{-2t}(2 \cos 3t + \frac{11}{3} \sin 3t)}$$

Facts Given $ay'' + by' + cy = 0$ and its characteristic equation $a\Gamma^2 + b\Gamma + c = 0$ with roots Γ_1, Γ_2:

I $\Gamma_1 \neq \Gamma_2$, where real roots

$$y(t) = c_1 e^{\Gamma_1 t} + c_2 e^{\Gamma_2 t} \qquad \text{(real roots)}$$

II $\Gamma_1 = \lambda + i\mu \neq \lambda - i\mu = \Gamma_2$, complex roots

$$y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) \qquad \text{(complex roots)}$$

III $\Gamma_1 = \Gamma_2 = \Gamma$, equal roots

$$y(t) = c_1 e^{\Gamma t} + c_2 e^{\Gamma t} = (c_1 + c_2)e^{\Gamma t} = Ke^{\Gamma t} \qquad \text{(equal roots)}$$

Is only 1 solution. We will need another linear independent one to construct a fund set of solutions.

Lecture 14 (2018-07-10)

Repeated Roots; Reduction of Order

Reduction of order - Using one solution to an nth-order ODE to create on $(n-1)$th order ODE.
Suppose $(\star)$ $y''+p(t)y'+q(t)y=0$ has $y_1(t)$ as a non-zero solution. If you can find an independent second solution, you are done.
Guess: Assume second solution has the form $y_2(t)=v(t)y_1(t)$ for some function $v(t)$.
Why? you will see.
Good: Try to solve the $v(t)$.
Now if $y_2(t)$ solves the ODE, then

$$\begin{aligned} y_2'(t) &= \frac{d}{dt}[v(t)y_1(t)] = v'(t)y_1(t)+v(t)y_1(t) \\ y_2''(t) &= v''(t)y_1(t)+\underbrace{v'(t)y_1'(t)+v'(t)y_1'(t)}_{2v'(t)y_1'(t)}+v(t)y_1''(t) \end{aligned}$$

Substitute this back into the original ODE:

$$\underbrace{(v''y_1+2v'y_1'+vy'')}_{y_2''}+p\underbrace{(v'y_1+vy_1')}_{y_2'}+q\underbrace{vy_1}_{y_2}=0$$

Recall in terms of v and derivatives:

$$y_1v''+(2y_1'+py_1)v'+\underbrace{(y_1''+py_1'+qy_1)}_{=0}v=0 \qquad (\text{we guess } y_2=vy_1)$$

We left with:

$$y_1v''+(2y_1'+py)v'=0 \qquad (\star)$$

This is a 2nd order ODE in $v(t)$. But this is a 1st order ODE in $v'(t)$!!

$$\boxed{\text{Solve for } v'(t). \text{ Then integrate to set } v(t).}$$

Notes:

I Thinking of $(\star)$ is a 1st order ODE in $v'(t)$ is called reducing the order.

II If the coefficient of $v(t)$ weren't 0, then we cannot do this! Since the lowest order derivative is v', we can.

III For independence,

$$\begin{aligned} W(y_1,vy_1) &= \begin{vmatrix} y_1 & vy_1 \\ y_1' & v'y_1+vy_1' \end{vmatrix} \\ &= v_1y_1^2+\underbrace{vy_1y_1'-vy_1y_1'}_{\text{cancel}}=v_1y_1^2 \end{aligned}$$

As long as $v_1'\neq 0$, (so $v_1(t)$ is not a constant) $y_2=v_1y_1$ will be independent of y.

Example 1. $y_1(t) = \frac{1}{t}$ solves $t^2y'' + cty' + y = 0$ on the interval $t > 0$. Find the set of fundamental solutions.

Strategy: Use reduction of order.

Solution: Both $p(t) = \frac{3}{t}$, $q(t) = \frac{1}{t^2}$, are C^0 on $(0, \infty)$. Assume $y_2(t) = v(t)y_1(t) = \frac{v(t)}{t}$
Then $v(t)$ solves

$$\begin{aligned} y_1 v'' + (2y_1' + py_1)v' = 0, \text{ or} \\ \frac{1}{t}v'' + (2(-\frac{1}{t^2}) + (\frac{3}{t})(\frac{1}{t}))v' = 0, \text{ or} \\ \frac{v''}{t} + \frac{v'}{t^2} = 0 \Rightarrow tv'' + v = \frac{d}{dt}[tv'] = 0 \end{aligned}$$

which implies $tv' = c$, a constant, or $v'(t) = \frac{c}{t}$, or $v(t) = c_1 \ln t + c_2$ on $t = 0$

$$\text{Hence} \qquad y_2(t) = \frac{v(t)}{t} = \frac{c_1 \ln t + c_2}{t}$$

Questions to ask

I Does $y_2(t)$ actually solve the original ODE?

$$y_2(t) = \frac{c_1 \ln t + c_2}{t}, \quad y_2'(t) = c_1\left(\frac{1 - \ln t}{t^2}\right) - \frac{c_2}{t^2}, \quad y_2''(t) = c_1\left(\frac{2\ln t - 3}{t^3}\right) + \frac{2c_2}{t^3}$$

Here

$$\begin{aligned} t^2y'' + 3ty' + y = 0 &= t\left(c_1\left(\frac{2\ln t - 3}{t^3}\right) - \frac{2c_2}{t^3}\right) + 3t\left(c_1(\frac{1 - \ln t}{t^2} - \frac{c_2}{t^2})\right) + \frac{c_1 \ln t + c_2}{t} \\ &= c_1\left(\frac{2\ln t - 3}{t}\right) - \frac{2c_2}{t} + 3c_1(\frac{1 - \ln t}{t}) - \frac{3c_2}{t} + c_1\frac{\ln t}{t} + \frac{c_2}{t} = 0 \end{aligned}$$

II Notice that $y_1 = \frac{1}{t}$ appears as a summed in $y_2 = c_1\frac{\ln t}{t} + \frac{c_2}{t}$. By Superposition, it is not really needed. Check independence of y_1, y_2.

Hence fundamental set of solutions is

$$\begin{aligned} y(t) &= (\text{constant})y_1 + (\text{constant})y_2 \\ &= (\text{constant})\frac{1}{t} + (\text{constant})\left(c_1\frac{\ln t}{t} + c_2(\frac{1}{t})\right) \end{aligned}$$

Combine constants to get

$$\boxed{y(t) = \frac{K_1}{t} + K_2\frac{\ln t}{t}} \qquad \text{as the general solution.}$$

Application Given $ay'' + by' + cy = 0$.
Suppose characteristic equation has only 1 real solution.

$$\text{Then} \quad \Gamma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$$

Here $y_1(t) = e^{-\frac{b}{2a}t}$ solves the ODE, but this is the only exponential function that does. To find another function reduce the order:

Assume $y_2 = v(t)e^{-\frac{b}{2a}t}$, where $v(t)$ solves

$$v''y + (2y_1' + py_1)v_1 = 0, \quad \text{or} \quad e^{-\frac{b}{2a}t}v'' + \underbrace{\left(2(\frac{b}{2a}te^{-\frac{b}{2a}t}) + \frac{b}{a}\frac{b}{2a}t\right)}_{0} v' = 0$$

$$\Rightarrow e^{-\frac{b}{2a}t}v'' = 0 \Rightarrow v'' = 0 \Rightarrow v(t) = K_1t + K_2$$

$$\text{So } y_2(t) = (K_1t + K_2)e^{-\frac{b}{2a}t}$$

Exercise. Calculate $W(y_1, y_2)$ here!
Hence $y_1(t), y_2(t)$ form a fundamental set of solutions,

$$y(t) = (\text{constant})e^{-\frac{b}{2a}t} + (\text{constant})(K_1 + K_2)e^{-\frac{b}{2a}t}$$

$$\boxed{y(t) = C_1e^{-\frac{b}{2a}t} + C_2e^{-\frac{b}{2a}t}}$$

Example 2. Solve $25y'' - 20y' + 4y = 0, \quad y(0) = 5, y'(0) = \frac{3}{2}$
Solution: Here discriminant $b^2 - 4ac = 400 - 400 = 0$

Hence $\Gamma_1 = \Gamma_2 = \Gamma = -\frac{b}{2a} = \frac{2}{5}$
Hence fundamental set of solutions is:

$$y(t) = C_1e^{\frac{2}{5}t} + C_2e^{\frac{2}{5}t}$$

As for the particular solution:

$$y(0) = C_1e^0 + C_2e^0 = 5 = C_1$$

$$\text{So } y(t) = C_1e^{\frac{2}{5}t} + C_2e^{\frac{2}{5}t}$$

$$\text{And } y(t) = 5(\tfrac{2}{5})e^{\frac{2}{5}t} + C_2e^{\frac{2}{5}t} + \tfrac{2C_2}{5}te^{\frac{2}{5}t}\Big|_{t=0}$$

$$= 2 + C_2 = \tfrac{3}{2} \Rightarrow C_2 = \tfrac{-1}{2}$$

$$\text{Solution is} \quad \boxed{y(t) = 5e^{\frac{2}{5}t} - \frac{t}{2}e^{\frac{2}{5}t}}$$

Lecture 15 (2018-08-14)

Nonhomogeneous Equations; Method of Undetermined Coefficients

Let's go back to the original linear

$$L[y] = y'' + p(t)y' + q(t) = g(t) \tag{+}$$

where p, q, and g are continuous on some I and $g(t) \neq 0$ (The non-homogenous case)
Note: (+) is linear, but superposition only holds for the LHS!

Theorem. *Suppose $\bar{Y}_1(t)$ solves $L[y] = g_1(t)$ and $\bar{Y}_2(t)$ solves $L[y] = g_2(t)$*

Proof. $L[\bar{Y}_1 + \bar{Y}_2] = L[\bar{Y}_1] + L[\bar{Y}_2] = g_1(t) + g_2(t)$ ■

Corollary. *Suppose $\bar{Y}_1(t)$ and $\bar{Y}_2(t)$ both solve $L[y] = g(t)$. Then $\bar{Y}_2(t) - \bar{Y}_1(t)$ solve $L[y] = 0$.*

Let $L[y] = g(t)$ be non-homogenous, and $\bar{Y}_1(t)$ and $\bar{Y}_2(t)$ be 2 solutions.
Let $c_1y_1(t) + c_2y_2(t)$ be a fundamental set of solutions to the homogenous $L[y] = 0$.

Since $\bar{Y}_2(t) - \bar{Y}_1(t)$ solves $L[y] = 0$ also, we set:

I $\bar{Y}_2(t) - \bar{Y}_1(t) = c_1y_1(t) + c_2y_2(t)$ *for some values of* c_1, c_2

II $$\underbrace{\bar{Y}_2}_{\text{any other soln. to } L[y]=g(t)} = \underbrace{c_1y_1(t) + c_2y_2(t)}_{\text{fund. set of solns to L[y] = 0}} + \underbrace{\bar{Y}_1(t)}_{\text{a soln. to } L[y]=g(t)}$$

We use this to set

Theorem. *The general solution to $L[y] = g(t)$ is $y(t) = c_1y_1(t) + c_2y_2(t) + \bar{Y}(t)$ where y_1, y_2 form a fund. set of solns. to $L[y] = 0$, and $\bar{Y}(t)$ is ANY particular solution to $L[y] = g(t)$*

This gives us a method for solving $L[y] = g(t)$:

I First, solve $L[y] = 0$

II Find any soln to $L[y] = g(t)$

III Put them together to get general solution

So the new question appears: Find a particular solution to $L[y] = g(t)$. In general, this is hard, But there are ways. [Assume the form of a solution and solve.]

Undetermined coefficients

Say ODE part has:

I homogenous part with constant coefficients

II $g(t)$ is a sum of products of

a exponent

b sins and cosines

ⓒ polynomials

Then you can assume the solution is also of a similar type. Write it out with some unknown coefficients, sub in and solve for the coefficients.

Example 1. Solve $y'' - 2y' - 2y = 3e^{2t}$
Homogenous fundamental solution set is $c_1e^{3t} + c_2e^{-t}$.
Assume soln. $\bar{Y}(t) = Ae^{2t} \Rightarrow 4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t} \Rightarrow A = -1$

$$\text{General solution is} \quad y(t) = c_1e^{3t} + c_2e^{-t} - e^{2t}$$

Example 2. Solve $y'' - 2y' - 3y = 3\sin 3t$
Again, assume $\bar{Y}(t) = A\sin 3t + B\cos 3t \rightarrowtail$ **need to have both!!**

$$\begin{aligned}
\Rightarrow \quad & -9A\sin 3t - 9B\cos 3t - 6A\cos 3t + 6B\sin 3t - 3A\sin 3t - 3B\cos 3t = 3\sin 3t \\
\Rightarrow \quad & -9A + 6B - 3A = 3 \quad \text{(sins)} \\
& -9B - 6A - 3B = 0 \quad \text{(cos)}
\end{aligned}$$

$$\boxed{A = -\frac{1}{5} \quad B = \frac{1}{10}}$$

<u>Fundamental set of solutions is</u>

$$\boxed{y(t) = c_1e^{3t} + c_2e^{-t} - \frac{1}{8}\sin 3t + \frac{1}{10}\cos 3t}$$

Lecture 16 (2018-08-14)

Variation of parameter

Example 1. Solve $y'' - 2y' - 3y = 4e^{-t}$
Fundamental set of solutions: $c_1e^{3t}+c_2e^{-t}$. Since -1 is a root of chosen equation of the homogeneous part, we cannot assume $\bar{Y}(t) = Ae^{-t}$ as it is already part of $c_1e^{3t} + c_2e^{-t}$. We fix this by setting $S = 1$ and $\bar{Y}(t) = Ate^{-t}$

Example 2. Solve $y'' - 4y' + 4 = 12e^{2t}$
Here $\Gamma = 2$ is the only solution. Fundamental solution is: $c_1e^{2t} + c_2te^{2t}$. Here $g(t) = 12e^{2t}$ so assume $\bar{Y}(t) = At^2e^{2t}$ ($S = 2$ since $\Gamma = 2$ is a double root to $\Gamma^2 - 4\Gamma + 4$).

Example 3. Solve $y'' - 4y' + 4y = 3t^3e^{-2t}$
Here $\bar{Y}(t) = t^2(At^3 + Bt^2 + Ct + D)e^{2t}$

This method is useful but limited in scope:

1. LHS must have constant coefficients.
2. RHS must be **nice**.

Here is a more general idea: Variation of parameters.

1. Is a form of reduction of order.
2. Works for any second order linear non-homogenous ODE
3. Relies on two assumptions.

Given $y''+p(t)y'+q(t)y = g(t)$, suppose $c_1y_1(t)+c_2y_2(t)$ is a fundamental set of solutions to $L[y] = 0$.

Assumption 1 Assume $\boxed{\bar{Y}(t) = u_1(t)y_1(t) + u_2(t)y_2(t)}$ solves $L[y] = g(t)$, for u_1, u_2 unknown functions. (compare to reduction of order technique).

$$\text{Then } \bar{Y}'(t) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Note: This is messy, but a good assumption. We can make this easier to handle.

Assumption 2: Assume $\boxed{u_1'y_1 + u_2'y_2 = 0}$

$$\text{Then } \bar{Y}'(t) = u_1y_1' + u_2y_2' \text{ and } \bar{Y}''(t) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Substitute these into $L[y] = g(t)$ and get

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = g(t)$$

Rearrange to get

$$u_1\underbrace{(y_1'' + py_1' + qy_1)}_{0} + u_2\underbrace{(y_2'' + py_2' + qy_2)}_{0} + u_1'y_1 + u_2'y_2' = g(t)$$

$$\boxed{u_1'y_1' + u_2'y_2' = g(t)}$$

Here, Assumption 2 is a good one since

(a) First assumption allows a lot of freedom since 2 unknowns are present.

(b) Second assumption allows for no second derivatives of u_1, u_2 in ODE.

Both assumptions within ODE yield the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t) \end{aligned}$$

Solve this for u_1' and u_2', integrate each to find $u_1(t)$ and $u_2(t)$. Are there solutions? Solving, we get:

$$\begin{aligned} u_1' &= \frac{-y_2 g}{y_1y_2' - y_2y_1'} = \frac{-y_2g}{W(y_1, y_2)} \\ u_2' &= \frac{y_1 g}{y_1y_2' - y_2y_1'} = \frac{y_1g}{W(y_1, y_2)} \end{aligned}$$

Hence

$$u_1 = \int \frac{-y_2g}{W(y_1, y_2)}dt, \qquad u_2 = \int \frac{y_1g}{W(y_1, y_2)}dt$$

With these, $\bar{Y}(t) = u_1y_1 + u_2y_2$ is one particular solution, and

$$y(t) = c_1y_1(t) + c_2y_2(t) + \bar{Y}(t)$$

is the general solution to $y'' + py' + qy = g$

Example 4. Knowing $y_1(t) = t$, and $y_2(t) = te^t$ both solve $t^2y'' - t(t+2)y' + (t+2)y = 0$ or $t > 0$, find the general solution to $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$.

Strategy: We use the Variation of parameters method with $\bar{Y}(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1t + u_2te^t$

Note: Here, $g(t) = 2t$, not $2t^3$

Solution: Given this assumption for $\bar{Y}(t)$, we obtain the system $u_1'y)_1 + u_2'y_2 = 0$, $u_1'y_1' + u_2'y_2' = g(t)$,

$$\left.\begin{aligned} u_1't + u_2te^t &= 0 \\ u_1 + u_2'(e^t + te^t) &= 2t \end{aligned}\right\} (-t) * \text{ eqn 2 and add } \begin{cases} u_1't + u_2'te^t = 0 \\ -u_1t - u_2't(e^t + te^t) = 2t^2 \end{cases}$$

Add equation 1 to equation 2 to get $-u_2't^2e^t = -2t^2$ or $u_2' = 2e^{-t}$, so $\boxed{u_2(t) = -2e^{-t}}$

Then $\bar{Y}(t) = -2t(t) + (-2e^{-t})te^t = -2t^2 - 2t$. So general solution is $y(t) = c_1t + c_2te^t - 2t^2 - 2t$ or $\boxed{y(t) = K_1t + c_2te^t - 2t^2}$

Note: We could append directly to the general form for

$$u_1, u_2 : W(t, te^t) = \begin{vmatrix} t & te^t \\ 1 & e^t + te^t \end{vmatrix} = t^2e^t \text{ on } t > 0$$

$$\begin{aligned} u_1(t) &= \int \frac{-y_2g}{W(y_1, y_2)}dt = \int \frac{-(te^t)2t}{t^2e^t}dt = -\int 2dt = -2t \\ u_2(t) &= \int \frac{-y_1g}{W(y_1, y_2)}dt = \int \frac{t(2t)}{t^2e^t}dt = 2\int e^{-t}dt = -2e^{-t} \end{aligned}$$

Lecture 17 (2018-08-15)

Higher Order Linear Equations

General Theory of nth Order Linear Equation

The n-th order version. of a linear ODE is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y^{(1)} + a_n(0)y = G(t)$$

which can also be written like an operator

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y^{(1)} + p_n(t)y = g(t) \qquad (\star)$$

where we divide each of the coefficients in the top description by $a_n(t)$ (example $p(t) = \frac{a_{n-1}(t)}{a_n(t)}$)

The theory generalizes in the obvious ways:

I If the ODE is an IVP, then we will need n pieces of information to completely determine a solution (think n integration to get solution creating an n-parameter family of functions): $y(t_0) = y_0$, $y^1(t_0) = y_0^1$, $\cdots$, $y^{(n-2)}(t_0) = y_0^{(n-2)}$, $y^{(n-1)}(t_0) = y_0^{(n-1)}$

II **Theorem.** *(Existence and Uniqueness)*
In $(\star)$ if $p_1, \cdots, p_n$ gave all continuous on some common interval I, then there exists a unique solution to $(\star)$ passing through any set of initial values $t_0 \in I$.

III Superposition Holds: if $y_1(t)$ and $y_2(t)$ both solve $L[y] = 0$ (homogenous) where $L[y]$ corresponds to the nth order ODE $(\star)$ then $c_1y_1(t) + c_2y_2(t)$ is also a solution.

IV Given n solutions to $y_1, \cdots, y_n$ to an nth order homogenous $L[y] = 0$, if

$$W(y_1, \cdots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1^1(t) & y_2^1(t) & \cdots & y_n^1(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

is nonzero at $y \in I$, then every solution to the ODE is a linear combination of y_1, $\cdots$, y_n. In this case, a fundamental set of solutions then is $y(t) = c_1y_1(t) + \cdots + c_ny_n(t)$

V In fact, it can be shown that for any choice of y_1, $\cdots$, y_n solutions to $(\star)$,

$$W(y_1, \cdots, y_n) = ce^{-\int p_1(t)dt}$$

and is either always 0 on I where $p_1(t)$ is continuous or never 0 on I.

VI If in $(\star)$, $g(t) \neq 0$, then a general solution is the same as thjat of a 2nd order non-homogenous ODE:

$$y(t) = \underbrace{c_1y_1(t) + \cdots + c_ny_n(t)}_{\text{fund. set of the solns. of} L[y]=0} + \overbrace{\bar{Y}(t)}^{\text{any particular soln. to } L[y]=g(t)}$$

Lecture 18 (2018-08-15)

How to solve the n-th order linear ODE?

Homogeneous Equations with Constant Coefficients

$$\underbrace{a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y^1 + a_0(t)y}_{L[y]} = G(t)$$

$$\overbrace{y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y} = g(t)$$

Answer: Same as before is the short answer:
The homogenous part ($L[y] = 0$), if thhe coefficients are constants, can be solved by exponentials: Assume $L[e^{\Gamma t}] = 0$ to construct the characteristics equation:

$$a_n\Gamma^n + a_{n-1}\Gamma^{n-1} + \cdots + a_1\Gamma + a_0 = 0 \qquad (\star)$$

The roots of $(\star)$ correspond to solutions $y(t) = e^{\Gamma t}$ which are solutions to $L[y] = 0$.

The rest of the theory also:

1. If roots of $(\star)$ can be found all of them, counting multiplicity and complex conjugates, one can construct an n-parameter family of solutions the fundamental set of solutions:

   Example 1. Suppose $(\star)$ has all red distinct roots $\Gamma_1 \neq \Gamma_1 \neq \cdots \neq \Gamma_n$: Then $y(t) = c_1e^{\Gamma_1 t} + \cdots + c_ne^{\Gamma_n t}$ is the general solution.

   Example 2. For repeated roots, the pattern is similar to the 2nd order version.

   (a) Suppose characteristics equation of a 5th order ODE where $(\Gamma - 2)(\Gamma + 1)^3(\Gamma - 5) = 0$ then $\Gamma_1 = 2$, $\Gamma_2 = \Gamma_3 = \Gamma_4 = -1$, $\Gamma_5 = 5$, and

   $$y(t) = c_1e^{2t} + c_2e^{-t} + c_3te^{-t} + c_4t^2e^{-t} + c_5e^{5t}$$

   (b) Suppose $(\Gamma^2 - 6)(\Gamma^2 - 4\Gamma + 13)^2 = 0$

   $$\begin{aligned} \Rightarrow \Gamma_1 &= \sqrt{6} \\ \Gamma_2 &= -\sqrt{6} \\ \Gamma_3 = \Gamma_5 &= 2 + 3i \\ \Gamma_4 = \Gamma_6 &= 2 - 3i \end{aligned}$$

   and $y(t) = c_1e^{\sqrt{6}t} + c_2e^{-\sqrt{6}t} + e^{2t}(c_3\cos 3t + c_4\sin 3t) + te^{2t}(c_5\cos 3t + c_6\sin 3t)$

   Solution methods for non-homogenous linear nth order ODEs are the same:

   (i) Undetermined Coefficients - exactly the same as the 2nd order version

   (ii) Variation of Parameters

2. Variation of Parameters
   Assume $\bar{Y}(t) = u_1 y_1 + \cdots + u_n y_n$ for $y_1, \cdots, y_n$ solutions to the homogeneous version. Playing the same game by taking derivatives, making assumptions (to simplicity) and plugging into the ODE, one obtaining a set of n equations.

$$\begin{aligned} u_1' y_1 + \cdots + u_n' y_n &= 0 \\ u_1' y_1' + \cdots + u_n' y_n' &= 0 \\ u_1' y_1'' + \cdots + u_n' y_n'' &= 0 \\ \vdots \quad \vdots \quad &\vdots \\ u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{n-1} &= g(t) \end{aligned}$$

   This set of n-equations in n-unknowns (the derivatives of the u_i's) can be solved and the solution will be unique if $W(y_1, \cdots, y_n) \neq 0$

And lastly, Reduction of Order methods work perfectly well:
Assume $y_1(t)$ solves an nth order linear homogeneous ODE. Then $y_2(0) = v(t)y_1(t)$ leads to a $(n-1)$th order ODE in v'. Not necessarily easier, but perhaps $(n-1)$th order ODE may have obvious solutions?

Lecture 19 (2018-08-20)

Systems of First Order Linear Equations

Introduction

Consider the "system" of 2 1st order ODEs in 2 variables:

$$\left.\begin{aligned} \tfrac{dx}{dt} &= a_1x - b_1xy \\ \tfrac{dy}{dt} &= -a_2y - b_2xy \end{aligned}\right\} a_1, a_2, b_1, b_2 > 0 \text{ constants.}$$

Here, both $x(t)$, $y(t)$ are functions of time, and these evolution (derivatives) are intertwined (coupled).
Many applications appear this way. These are called the Lotka-Volterra equations: model the population size of 2 species in a closed environment (predator and prey). A solution is a set of expressions for $x(t)$ and $y(t)$ that satisfy both equations.
Question: Say $x(t)$ and $y(t)$ represent rabbits and foxes (not necessarily respectively). Can you tell from the ODE system which is which? How?
Why study systems of coupled equations? @ reasons:

1. Many apps appear this way. There are many measurable quartiles all depending on a single independent variable (not like vector calculus). In general, this looks like

$$\left.\begin{aligned} \dot{x_1} &= F_1(t_1x_1, \cdots, x_n) \\ \dot{x_1} &= F_2(t_1x_1, \cdots, x_n) \\ &\vdots \\ \dot{x_n} &= F_n(t_1x_1, \cdots, x_n) \end{aligned}\right\} \text{1st order system of ODEs}$$

where $x_1, \cdots, x_n$ are the set of n dependent variables and time t is the independent variable

2. <u>Any</u> higher ODE can be transformed (rewritten) as a system of 1st order ODEs:

$$\text{Let} \quad y^{(n)} = F(t_1y_1y', \cdots, y^{(n-1)})$$

Given the new vars:

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \cdots, x_n = y^{(n-1)}$$

$$\text{we get} \quad \dot{x_1} = x_2, \quad \dot{x_2} = x_3, \quad \dot{x_3} = x_4, \quad \cdots \dot{x_{n-1}} = x_n, \quad \dot{x_n} = F(t_1x_1, \cdots, x_n)$$

<u>*and*</u>

$$\begin{aligned} \dot{x_1} &= \dot{y} = y' = x_2 \\ \dot{x_2} &= (y')' = y'' = x_3 \\ &\vdots \\ \dot{x_n} &= (y^{n-1})' = y^{(n)} = F \end{aligned}$$

A solution to $(\star)$ is a set of functions

$$x_1(t), x_2(t), \cdots, x_n(t)$$

If initial values are specified, we would need

$$\left.\begin{array}{l} \text{1 date at each } x_i(t), i = 1, \cdots, n \\ x_1(t_0) = x_1^0, \cdots, x_n(t_0) = x_n^0 \\ \text{This is identical to} \\ y(t_0) = y_0, \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)} \end{array}\right\} \text{n-bits of initial data}$$

<u>Existence and uniqueness for 1st Order systems</u> (similar to that for a single eqn).

Theorem 1. *In ($\star$), let $F_1, \cdots, F_n$ <u>and</u> all of*

$$\frac{dF_1}{dx_1}, \cdots, \frac{dF_1}{dx_n}, \quad \frac{dF_2}{dx_1}, \cdots, \frac{dF_2}{dx_n}, \quad \cdots, \quad \frac{dF_n}{dx_1}, \cdots, \frac{dF_n}{dx_n},$$

(all partials w.r.t. dep. vars x_i but not t) be continuous in a region R of the $(n+1)$-dimiension $t_x, \cdots, x_n$-space defined by $\alpha < t < \beta, \quad \alpha_1 < x_1 < \beta_1, \quad \cdots, \alpha_n < x_n, \beta_n$

$\Rightarrow$ on an interval $|t - t_0| < h$, there is a unique solution to ($\star$) defined on the interval and passing through p.

Note: The proof is similar to that of the 1-dimension case

Definition. ($\star$) is linear if each F_i is linear in all of the x_i's $i = 1, \quad, n$. Indeed if

$$(*) \begin{cases} x_1' = p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ \qquad \vdots \\ x_n' = p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

and <u>homogenous</u> if each $g_i(t) \equiv 0$.

Note: Solutions exist <u>and</u> one unique for a linear system of IDEs (like ($*$) on an interval I it <u>all</u> $p_{ij}(t)$ and $g_i(t)$ and continuous on I).

Lecture 20 (2018-08-20)

Matrices Review

A linear system of equations looks like

$$\underline{neqns.}\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ \underbrace{a_{n1}x_1 + \cdots + a_{nn}x_n = b_n}_{\text{n-unknowns}} \end{cases}$$

We can write this as a single (matrix) equation by collecting up the constituent parts into arrays.

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A_{nxn}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{X}_{nx1}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\vec{b}_{nx1}}$$

Here $b_2 =$ (row 2 of A) $\cdot\vec{x}$ where the determinant is matrix multiplication.

Some facts about matrices and matrix equations

1. If $\vec{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $A\vec{x} = \vec{b}$, the equation is called homogenous.

2. A solution to $A\vec{x} = \vec{b}$ is a choice of $\vec{x}$ which satisfies the equation.

3. If $\det A \neq 0$, the system has a unique solution.

4. If $A\vec{x} = \vec{0}$ and $\det A \neq 0$, then $\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is the only solution.

   If $\det A = 0$, then tons of solutions ($A\vec{x} = \vec{0}$ is never inconsistent)

5. If $A\vec{x} \neq \vec{0}$, then the inverse of A, A^{-1} exists and can be used to "solve" $A\vec{x} = \vec{b}$: $A\vec{x} = \vec{b} \Rightarrow$ $\vec{x} = A^{-1}\vec{b}$. There is also an identity matrix $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ = n-dim Identity matrix

6. The idea of solving a system of equations involves adding multiples of equations to other equations in order to produce new simpler equations.
   In matrices, these are the elementary row operations one performs to A to reduce the number of non-zero entries. But what one does to A, one must also do to $\vec{I}$. We need to use an augmentation matrix.

7 All vectors, by convention, are considered column vectors. To talk about a row vector

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{x}^T = \begin{bmatrix} x_1, \cdots, x_n \end{bmatrix}$$

are should either specify "row vector", or take the transpose of a column vector.

Definition. A set of vectors $\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}$ (careful of the notation) of the same size are said to be linearly dependent (on each other) if $\exists$ a real numbers $c_1, \cdots, c_n \in \mathbb{R}$, not all at 0, where

$$c_1\vec{x}^{(1)} + \cdots + c_n\vec{x}^{(n)} = 0$$

Otherwise they are linearly independent
Note: The columns of A_{nxn} are linearly independent iff $\det A \neq 0$

8 For $A\vec{x} = \vec{b}$, think of $\vec{x}, \vec{b} \in \mathbb{R}^n$ where $\mathbb{R}^n$ = the set of all n-vectors.
Then an nxn matrix A_{nxn} can be considered a linear transformation of $\mathbb{R}^n$ (a function taking $\mathbb{R}^n$ to $\mathbb{R}^n$) taking $\vec{x}$ to $\vec{b} = A\vec{x}$:

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{x} \overset{A}{\longmapsto} \vec{b} = A\vec{x}$$

A takes n-vectors to n-vectors, where $\vec{b}$ is the image of $\vec{x}$ under A.

9 There is a special equation in linear algebra:

$$\boxed{A\vec{x} = \lambda\vec{x}} \qquad \begin{aligned} A_{nxn} &\text{ - matrix} \\ \vec{x} &\text{ - n-vector} \\ \lambda &\text{ - scalar} \end{aligned}$$

A choice of $\vec{x}$ and λ which satisfy this equation indicate a direction (of $\vec{x}$) unchanged via multiplication by A, and expanded or contracted by a factor λ.
Here $\vec{x}$ is called an eigenvector of A, and λ is its corresponding eigenvalue.

How to find eigenvalues?

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$A\vec{x} - \lambda I_n\vec{x} = \vec{0}$$
$$(A - \lambda I_n)\vec{x} = \vec{0}$$

The only way non-trivial solutions exist is if $\det(A - \lambda I_n) = 0$. But this equation only has λ in it!

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$. Then

$$\det(A - \lambda I_2) = 0 = \det(\left[\begin{smallmatrix} 1 & 1 \\ 6 & 0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 1-\lambda & 1 \\ 6 & -\lambda \end{smallmatrix}\right]$$
$$= (1-\lambda)(-\lambda) - 6 = 0 = (\lambda - 3)(\lambda + 2)$$
$$\boxed{\lambda = 3, \lambda = -2}$$

Lecture 21 (2018-08-21)

We have already seen that $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of the eigenvalue 3 for $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$
What is an eigenvector for $\lambda = -2$?
Back to $A\vec{x} = \lambda\vec{x}$:

$$\begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + x_2 &= -2x_1 \\ 6x_1 \qquad &= -2x_2 \end{aligned}\right\} \Rightarrow \begin{aligned} 3x_1 &= -x_2 \\ 6x_1 &= -2x_2 \end{aligned}$$

Notice how this system is degenerate (has lots of solutions!) It always is...

Back to ODEs:

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a vector of variables (functions of time), then $\vec{x}' = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$ is its derivative, and

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

is a linear system with matrix form...

$$\vec{x}' = \mathbb{P}(t)\vec{x} + g(\vec{t})$$

A solution is a vector of functions $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and a set of solutions (if more than one is)

$$\vec{x}^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{bmatrix}, \cdots, \vec{x}^{(b)}(t) = \begin{bmatrix} x_1^{(b)}(t) \\ \vdots \\ x_n^{(b)}(t) \end{bmatrix}.$$

Note the notational confusion, We will not deal with nth order systems, so the context is nice.

Some facts - Let $\vec{x}' = \mathbb{P}(t)\vec{x}$ be a hhomogenous linear system ($g(\vec{t}) = \vec{0}$)

1. Superposition holds if $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ both solve $\vec{x}' = \mathbb{P}(t)\vec{x}$, then so does

$$c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$$

for any choice of $c_1, c_2 \in \mathbb{R}$ (Any linear combination of solutions is a solution!)

Example 1. $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$. Here $\mathbb{P}(t) = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$ are <u>constants</u>. Hence verify that:

1. $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$ is a solution

2. $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix}$ is also a solution

Hence so is $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$.

$$\text{For } 2, \quad \vec{x}' = \frac{d}{dt} \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} - 3e^{-2t} \\ 6e^{-2t} + 0(-)e^{-2t} \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 6x_1 \end{bmatrix}$$

2 In $\mathbb{R}^n$, there can be at most n linearly independent vectors: For example:

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

It is reasonable to conclude that there can be up to n-linearly independent solutions to $\vec{x}' = \mathbb{P}(t)\vec{x}$

$$\vec{x}^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{bmatrix}, \cdots, \quad \vec{x}^{(n)}(t) = \begin{bmatrix} x_1^{(n)}(t) \\ \vdots \\ x_n^{(n)}(t) \end{bmatrix}$$

They will be independent on some interval I if for $t \in I$,

$$c_1\vec{x}^{(1)}(t) + \cdots + c_n\vec{x}^{(n)}(t) = \vec{0}$$

Can only be solved by $c_1 = c_2 = \cdots = c_n = 0$
Combine all of these vector solutions as columns in a single nxn matrix

$$\underset{\sim}{X}(t) = \left[\vec{x}^{(1)}, \cdots, \vec{x}^{(n)}\right] = \begin{bmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{bmatrix}$$

Then $\det \underset{\sim}{X} \neq 0$ iff <u>all columns are independent</u>.

Definition. $\det \underset{\sim}{X}(t)$ is called the <u>Wronskian (determinant)</u> of the solution set, and denoted $W(\vec{X}^{(1)}, \cdots, \vec{X}^{(n)})$

Theorem. *If $\vec{X}^{(1)}, \cdots, \vec{X}^{(n)}$ are all solutions to $\vec{x}' = \mathbb{P}(t)\vec{x}$ on $I = (\alpha, \beta) \in \mathbb{R}$, then for all $t \in I$, either $W(\vec{X}^{(1)}, \cdots, \vec{X}^{(n)})$ is identically 0 or is never 0.*

Definition. If $\vec{X}^{(1)}, \cdots, \vec{X}^{(n)}$ one all solutions to the n-dimension $\vec{x}' = \mathbb{P}(t)\vec{x}$ on I and $W(\vec{X}^{(1)}, \cdots, \vec{X}^{(n)}) \neq 0$ on I then a $\underset{\sim}{X}(t)$ is called <u>fundamental set of solutions</u>. and b $\vec{Y(t)} = c_1\vec{x}^{(1)} + \cdots + c_n\vec{x}^{(n)}$ is the <u>general solution</u>.

Example 2. Given $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$, with solutions

$$\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix}$$

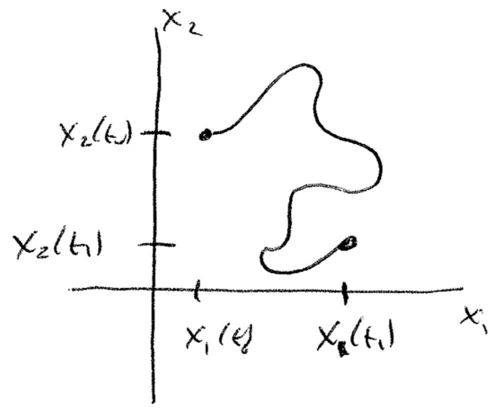
$$\text{Since } W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{vmatrix} = -3e^{3t}e^{-2t} - 2e^{3t}e^{-2t} = -5e^{t} \neq 0 \text{ on } \mathbb{R}$$

$$\underbrace{\bar{\underline{X}}(t) = \begin{vmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{vmatrix}}_{\underline{\text{Fundamental set of solutions}}} \quad \underline{\text{and}} \quad \begin{aligned} \vec{\varphi(t)} &= c_1 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} \\ &\underbrace{= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}}_{\text{as the } \underline{\text{general solution}}} \end{aligned}$$

Lecture 22 (2018-08-21)

Homogeneous with constant coefficients

Let $\vec{x}' = A_{nxn}\vec{x}$, $\quad A_{nxn}$ - matrix of constants. (**Note:** $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$ is an example.) For $n = 2$, solutions look like $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. For each t, $\vec{x}(t)$ is a point in $\mathbb{R}^2$. As t evolves, $\vec{x}(t)$ will trace out a parameterized curve.



We call the x_1, x_2-plane the phase plane for the system, noting that

(a) the independent variable t is implicit to the graph (not an axis, but on the curve)

(b) For one equation $y' = f(y)$, the solution $y(t)$ lived in the ty-plane, but we could also track its evolution in the phase line

(c) We could graph $\vec{x}(t)$ in the tx_1x_2-space but this is hard to see.

Some ideas for study

(I) $\vec{x}'(t) = A\vec{x}(t)$, by simply choosing points $\vec{x} \in \mathbb{R}^2$, we can plot tangent lines and make a slope field in the phase plane.

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$. Compute

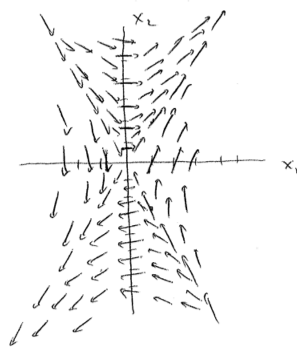
(a) $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

Notes:

(1) looks very similar to example 1

(2) Use JODE 2D calculator on website

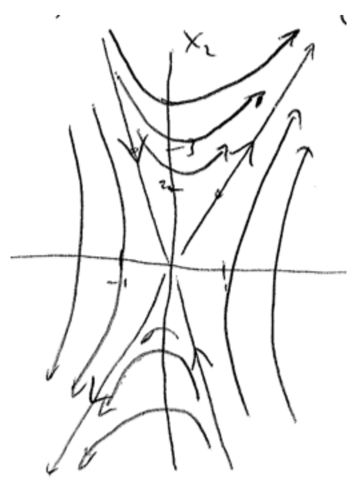


(II) The solution curves will be integral curves of this slope field:

(a) Given a value of $c_1, c_2 \in \mathbb{R}$, the curve $\vec{x(t)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$ will be one of these curves.

(b) Straight line motion occurs when c_1 or $c_2 = 0$

Choose $c_1 = -2, c_2 = 0$. then $\vec{x}(t) = -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} e^{3t}$



Lecture 23 (2018-08-22)

For $\vec{x}' = A_{2x2}\vec{x}$,

(I) One can construct a slope field in $\mathbb{R}^2$ via matrix multiplication: For $\vec{x} \in \mathbb{R}^2$, the tangent to the solution curve passing through is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is $A\vec{x}$

(a) $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

(c) $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

JODE 2D calculator or similar is helpful here.

(II) Solution curves are integral curves of the slope field:

(a) Given $c_1, c_2 \in \mathbb{R}$, the curve $x\vec{(t)} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$ is one of these curves.

(b) Straight line motion only occurs when $c_1 = -2, c_2 = 0$. Then $\vec{x}(t) = -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} e^{3t}$

(c) A copy of $\mathbb{R}^2$ with enough representative curve on it to give a sound sense of solutions is called a <u>phase portrait</u>.

(d) Solutions are called <u>trajectories</u> or <u>orbits</u>.

(e) General long term behaviour of trajectories can be read off easily from a phase portrait:

$$\text{Given} \quad \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t},$$

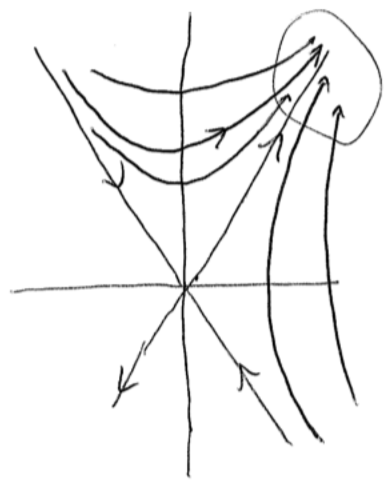
- If $c_1 = 0$, $c_2 \neq 0$, then $\vec{x}(t) = c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$ and $\vec{x}(t) = c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$ and $\lim\limits_{t\to\infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ How about $\lim\limits_{t\to-\infty} \vec{x}(t)$? We say it is <u>unbounded</u>.

  Note: Solutions never touch nor cross here. So since $\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in equilibrium solution, no other solution actually reader $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- If $c_1 \neq 0, c_2 = 0$?
  $\lim\limits_{t\to\infty} \vec{x}(t)$ DNE on trig is unbounded
  $\lim\limits_{t\to-\infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- If $c_1 \neq 0$ and $c_2 \neq 0$? These are the curved lines in the portrait. For these, what can we say about the forward trajectory ($\lim_{t\to\infty} \vec{x}(t)$), or the backward one ($\lim_{t\to-\infty} \vec{x}(t)$)?

  One Answer? Given $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$ and $c_1 \neq 0$, $c_2 \neq 0$, then as t gets large ($t \to \infty$), $\vec{x}(t)$ looks more and more like $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$, and less and less like $c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$. How about in backward time?



III Back to solution building via properties of A: For $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$,

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

the eigenvalues of A ($\Gamma_1 = 3, \Gamma_2 = -2$), and representive eigenvectors ($\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$) are explicitly part of the solution. Why?
Here, the eigenvalues and eigenvectors of a matrix A satisfy $A\vec{v} = \lambda\vec{v}$, or $(A - \lambda I)\vec{v} = \vec{0}$
For this to have non trivial solutions for $\vec{v}$,

- λ must be an eigenvalue, and
- $\det(A - \lambda I)\vec{v} = \vec{0}$

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$
$= \lambda^2 - (a + d)\lambda + (ad - bc) = 0$
This is called the characteristic equation of A, and solutions can be

1. real, distinct
2. real and repeated
3. complex conjugates

How to relate to solutions of $\vec{x}' = A\vec{x}$?

- Recall $\dot{x} = ax$ is solved by $x(t) = ce^{at}$
- Assume $\dot{\vec{x}} = A\vec{x}$ is also solved by exponentials. For $n = 2$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Assume $x_1(t) = c_1 e^{\Gamma t}$, $x_2(t) = c_2 e^{\Gamma t}$
  $\Rightarrow \vec{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\Gamma t}$, and $\vec{x}' = \Gamma \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\Gamma t}$ and then $\vec{x}' = A\vec{x}$ is $\Gamma \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\Gamma t} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\Gamma t}$ or $\Gamma \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, is $\Gamma\vec{v} = A\vec{v}$.
  And what do solutions to this look like?
  - Γ is an eigenvalue
  - $\vec{v}$ is an eigenvector of Γ.

Notes:

1. This works equally well for $n > 2$.
2. In the case where A_{nxn} is real and symmetric (i.e. when $a_{ij} = a_{ji}$)
   $\Rightarrow$ all eigenvalues one real and even if repeated, there is a full set of eigenvectors.

We have the following:
For $\dot{\vec{x}} = A_{nxn}\vec{x}$, when all eigenvalues of A are real and distinct, then

$$\vec{x}(t) = c_1 \vec{v_1} e^{\Gamma_1 t} + \cdots + c_n \vec{v_n} e^{\Gamma_n t}$$

is the general solution, where $\vec{v_i}$ is on eigenvector of Γ_i for A.

Lecture 24 (2018-08-22)

Back to $\vec{x}' = A_{2x2}\vec{x}$ in the case where the 2 eigenvalues of A, Γ_1, Γ_2 are real and distinct: $\Gamma_1 \neq \Gamma_2$

Some Notes:

1. This works fine when 0 is an eigenvalue

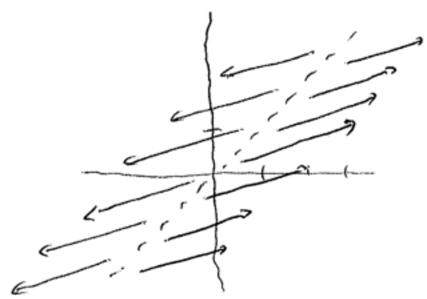
   Example 1. $\vec{x}' = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{x}$. Here characteristic equation is $\Gamma^2 - 2\Gamma = 0$, w/ solutions $\Gamma_1 = 0$, $\Gamma_2 = 2$.

   For $\Gamma_1 = 0$, $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a choice of eigenvector

   For $\Gamma_2 = 2$, $\vec{v_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is one choice

   Then $\vec{x}(t) = c_1\vec{v_1}e^{\Gamma_1 t} + c_2\vec{v_2}e^{\Gamma_2 t} = \boxed{c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}}$ is general solution.

   Question: So what does the phase portrait look like here?



2. This formulation of the general solution still works even for repeated eigenvalues <u>if</u> there are enough eigenvectors.

   Example 2. $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$. Here characteristic equation is $\Gamma^2 - 2\Gamma + 1 = 0 = (\Gamma - 1)^2$. Hence $\Gamma_1 = \Gamma_2 = 1$.

   The eigenvector equation $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

   One choice is $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   $$\text{Hence } \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \text{ is the general solution}$$

   Example 3. $\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$

   Here $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and the characteristic equation after $\det(A - \Gamma I)$ is $\boxed{(\Gamma + 1)^2(\Gamma - 2)}$.

   The eigenvalues are $\Gamma_1 = 2$, $\Gamma_2 = \Gamma_3 = -1$

The respective eigenvectors are $\vec{v_1} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{v_2} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\vec{v_3} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$

So the general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix}1\\1\\1\end{bmatrix} e^{2t} + c_2 \begin{bmatrix}1\\0\\-1\end{bmatrix} e^{-t} + c_3 \begin{bmatrix}0\\1\\-1\end{bmatrix} e^{-t}$$

3. This formulation for finding solution does not work when there are not enough independent eigenvectors for repeated eigenvalues

   Example 4. $\vec{x}' = \begin{bmatrix}-2 & 1\\0 & -2\end{bmatrix}\vec{x}$. Here eigenvalues are $\Gamma_1 = \Gamma_2 = -2$. But the eigenvector equation $A\vec{v} = -2\vec{v}$, or

$$\begin{aligned}-2x_1 + x_2 &= -2x_2\\ -2x_2 &= -2x_2\end{aligned}$$

   is only solved by $x_2 = 0$, $x_2 \neq 0$. Then only independent choice would be $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$

   Here, we will need to come up with another method to find another independent solution.

Properties of phase portraits

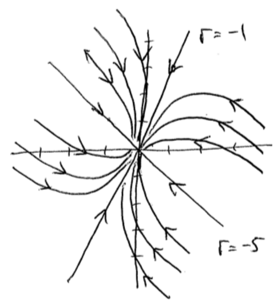
let $\vec{x}' = A\vec{x}$. We have

1. For any A_{2x2}, the origin is an equilibrium solution ($\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$ is always a solution to $A\vec{x} = \vec{0}$).
2. If $\det A \neq 0$, then $\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$ is the ONLY equilibrium solution (stationary point, or fixed point), since $A\vec{x} = \vec{0}$ has $\begin{bmatrix}0\\0\end{bmatrix}$ as the only solution.
3. The eigenvectors of A correspond to lines through the origin where the solutions exhibit straight line motion
   Note: These straight lines contain many distinct solutions.
4. The sign of each eigenvalue determines direction along lines (towered origin if < 0 outward if > 0).
5. If eigenvalues are real and distinct then there are the only straight lines (why?)
6. Signs of eigenvalues determine "stability" of equilibrium at origin. (How solutions behave "near" the origin. Do they stay nearby, converse to, or diverse from...)
7. Below, origin is called a "saddle point" . Would you consider it saddle?

Example 5. $\vec{x} = \begin{bmatrix} -4 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}$ Here $\Gamma_1 = -5, \Gamma_2 = -1$, with $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

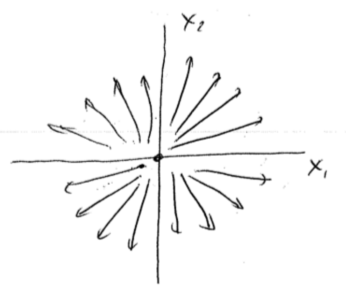
General solution is $\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$. Phase portrait is similar to above but different: how?

Here origin is a "sink" and asymptotically stable.



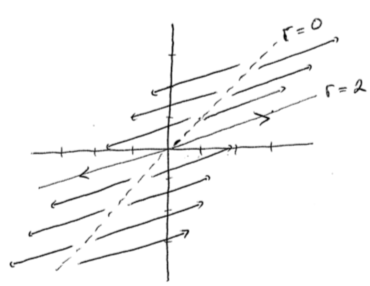
Example 6. $\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$ Here, as we have seen $\Gamma_1 = \Gamma_2 = 1$, $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and General solution is $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t = \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^t$.

This is another sources at the origin hhere called a <u>star node</u>



Example 7. $\vec{x} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{x}$ Here $\Gamma_1 = 0, \Gamma_2 = 2$, $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and General solution is $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$

This one is special: There is a line of equilibrium solution (determinant is 0). Off the dotted line, motion is straight along lines parallel to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and moving out from dotted line.



Lecture 25 (2018-08-29)

Complex Eigenvalues

New Question: What is the eigenvalues (solutions to the characteristic equation of A_{2x2} in $\vec{x}' = A\vec{x}$) are not real?
Then They are complex (the discriminant $b^2 - 4ac$ of the quadratic formula used to solve the characteristic equation is ¡ 0).
They must be complex conjugates (why?) How to use them? Let's play the same game for constructing solutions to $\vec{x}' = A\vec{x}$ using eigenvalue eigenvector pairs:
For $\vec{x}' = A_{2x2}\vec{x}$, suppose $\Gamma_1 \neq \Gamma_2$ are two distinct solutions to characteristic equation of A and we calculate eigenvectors $\vec{v_1}, \vec{v_2}$ respectively to Γ_1, Γ_2. Then general solution is

$$\boxed{\vec{x}(t) = c_1\vec{v_1}e^{\Gamma_1 t} + c_2\vec{v_2}e^{\Gamma_2 t}}$$

We try this with complex Γ:

Example 1. $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{x}$ Characteristic equation is $\Gamma^2 + 2\Gamma + 2 = 0$, solved by $\Gamma = -1 \pm i$
Leave them as $\Gamma_1 = -1 + i$, $\Gamma_2 = -1 - i$ and solve for $\vec{v_1}$: $A\vec{v} = \Gamma\vec{v}$

$$\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (-1+i) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$v_2 = -v_1 + iv_1$$
$$-2v_1 - 2v_2 = -v_2 + iv_2$$

We can substitute (1) into (2) and simplify to get $-2iv_1 = -2iv_1$, solved by any choice of v. For example, choose $v_1 = 1$. Then $v_1 = (-1+i)$ and

$$\Gamma_1 - 1 + i, \quad \vec{v_1} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} \qquad \text{forms an "eigenvalue/eigenvector" pair}$$

Notes:

1. This is not quite accurate since the definition of eigenvector is that of a vector whose direction does not change upon multiplication by a matrix. Without real eigenvalues, there are no real eigenvectors! But the term "complex eigenvector" is a commonly used one.

2. The other eigenvalue/eigenvector pair is $\Gamma_2 = -1 - i$, $\vec{v_1} = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$

3. Rewrite $\vec{v_1} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i = \vec{a} + i\vec{b}$. Then along with $\Gamma_1 = -1 + i = \lambda + i\mu$ we can attempt to form solutions.

General idea for a Method for constructing solutions?

Given:

- $\Gamma_1 = \lambda + i\mu$, $\vec{v_1} = \vec{a} + i\vec{b}$

- $\Gamma_1 = \lambda - i\mu$, $\vec{v_1} = \vec{a} - i\vec{b}$

Create a "complex" solution in the normal way:

$$\begin{aligned}\vec{x}(t) = \vec{v}_1 e^{\Gamma_1 t} &= (\vec{a} + i\vec{b})\underbrace{e^{\lambda t}(\cos\mu t) + i\sin\mu t}_{\text{Euler formula for } e^{\Gamma_1 t}} \\ &= e^{\lambda t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t) + ie^{\lambda t}(\vec{a}\sin\mu t + \vec{b}\cos t)\end{aligned}$$

Hence we can write $\vec{x}(t) = \vec{a}(t) + i\vec{w}(t)$, where

$$\begin{aligned}\vec{u}(t) &= e^{\lambda t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t) \\ \vec{u}(t) &= e^{\lambda t}(\vec{a}\sin\mu t - \vec{b}\cos\mu t)\end{aligned}$$

Notes:

1. These are 2 real-valued functions which each solve the ODE $\vec{x}' = A\vec{x}$ (check this!)
2. These are independent (check the Wronskian)
3. The general solution to $\vec{x}' = A\vec{x}$ when eigenvalues $\Gamma = \lambda \pm i\mu$ are complex and with eigenvectors $\vec{v} = \vec{a} \pm i\vec{b}$ is
$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{w}(t)$$

Back to example: $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\vec{x}$.

Here $\Gamma_1 = \lambda + i\mu = -1 + i$, $\vec{v_1} = \vec{a} + i\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So

$$\vec{x}(t) = c_1 e^{-t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + c_2 e^{-t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\sin t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t\right)$$

Lecture 26 (2018-08-29)