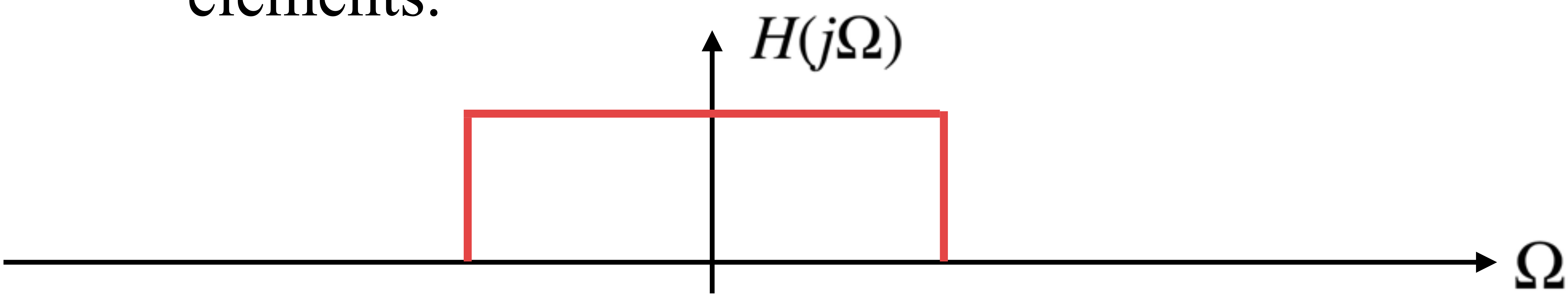
EE 110B Signals and Systems

Discrete-time Processing of Continuous-time Signals

Ertem Tuncel

Continuous-time filtering

- A very important question we avoided so far:
  - Given a continuous-time impulse response $h(t)$ or transfer function $H(j\Omega)$, how do we implement the corresponding LTI system?
  - For example, the ideal low-pass filter is not trivial to implement using classical linear circuit elements.

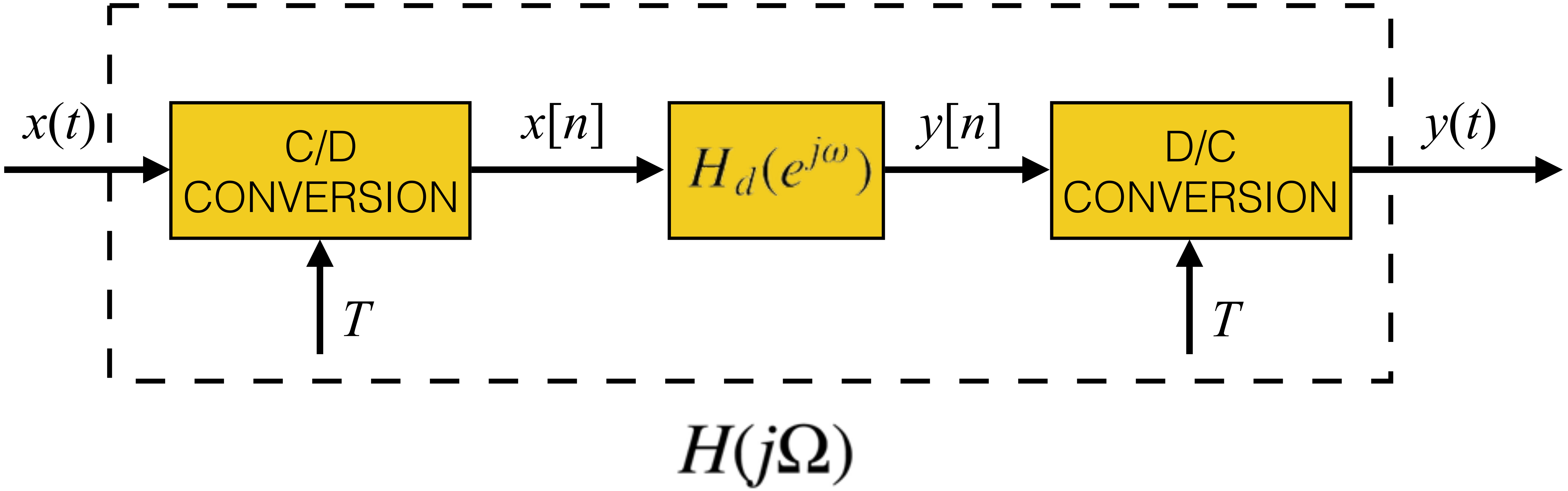


  - Non-causality is not your only problem!

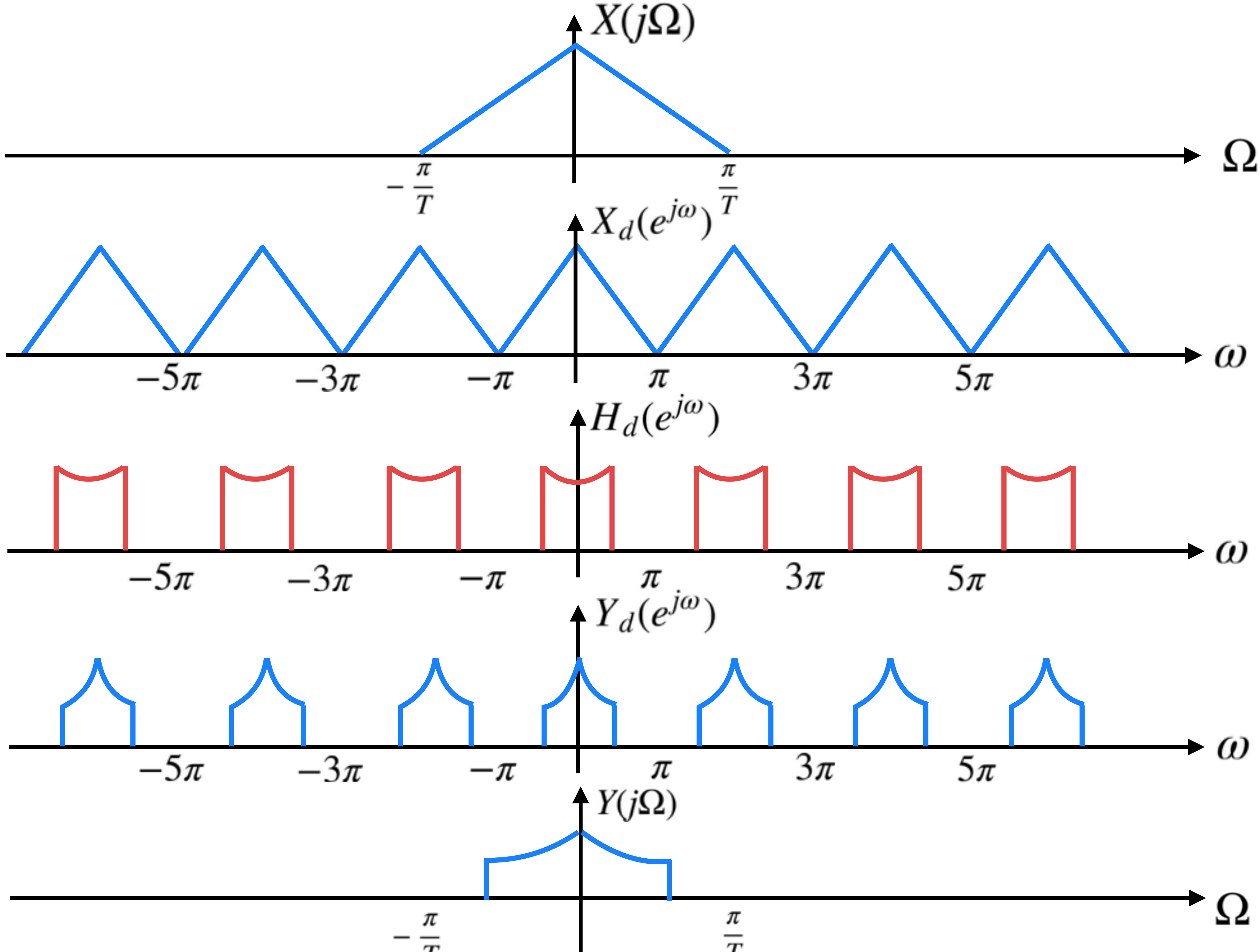
Discrete-time filtering

- For discrete-time processing, we already saw that any causal LTI system can be implemented using very simple digital circuit elements:
  - Multiplier, adder, and one-sample delay.
- Can we tap into this simplicity and implement continuous-time filters in the discrete-time world?
  - The answer is yes if the input signal is guaranteed to be band-limited.
  - Is it as simple as sampling $h(t)$ as $h[n] = h(nT)$?

Discrete-time filtering

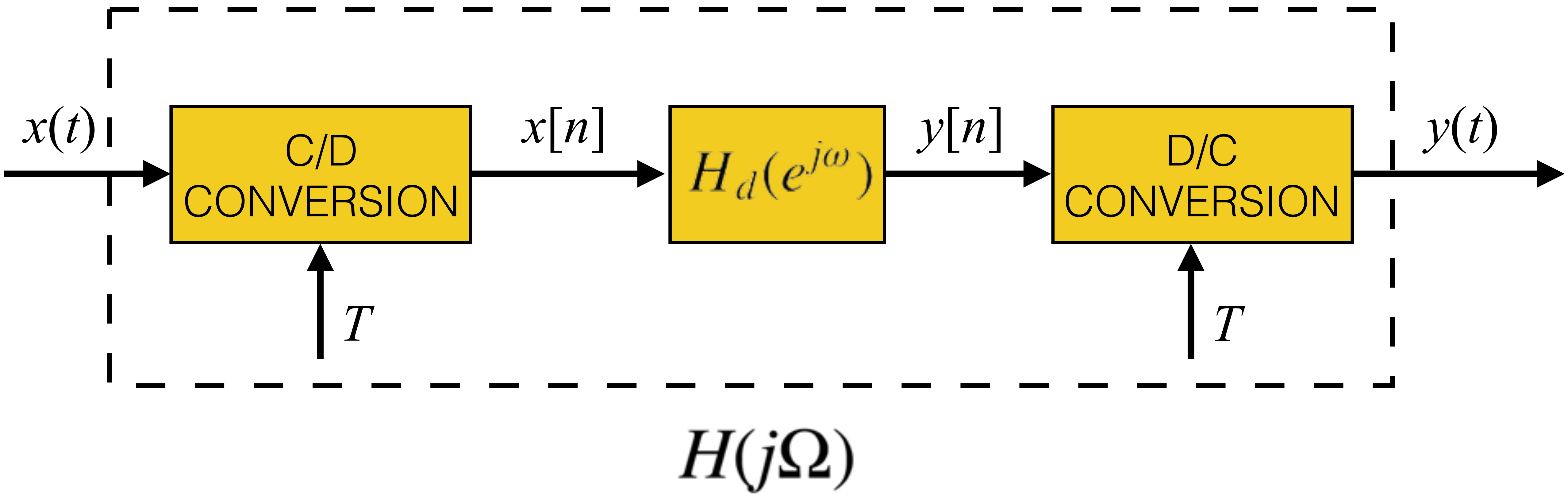


- How does $H(j\Omega)$ relate to $H_d(e^{j\omega})$?



$X(j\Omega)$
Ω
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
$X_d(e^{j\omega})$
ω
-5π
-3π
$-\pi$
π
3π
5π
$H_d(e^{j\omega})$
ω
-5π
-3π
$-\pi$
π
3π
5π
$Y_d(e^{j\omega})$
ω
-5π
-3π
$-\pi$
π
3π
5π
$Y(j\Omega)$
Ω
$-\frac{\pi}{T}$
$\frac{\pi}{T}$

Discrete-time filtering



- How does $H(j\Omega)$ relate to $H_d(e^{j\omega})$?

$$H(j\Omega) = \begin{cases} H_d(e^{j\Omega T}) & |\Omega| \leq \dfrac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

Relation between $h[n]$ and $h(t)$

- To find the discrete-time impulse response $h[n]$, there are two alternative techniques:

  1) From the desired $H(j\Omega)$, find $H_d(e^{j\omega})$ and invert.

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(j\frac{\omega}{T}\right)e^{j\omega n}\,d\omega$$

- Since $H(j\Omega)$ is bandlimited, this is the same as

$$h[n] = h(nT)$$

Relation between $h[n]$ and $h(t)$

2) If the relation between $x(t)$ and $y(t)$ is especially simple, first figure out $y(t)$ when

$$x(t) = \text{sinc}\left(\frac{\pi t}{T}\right)$$

- This implies

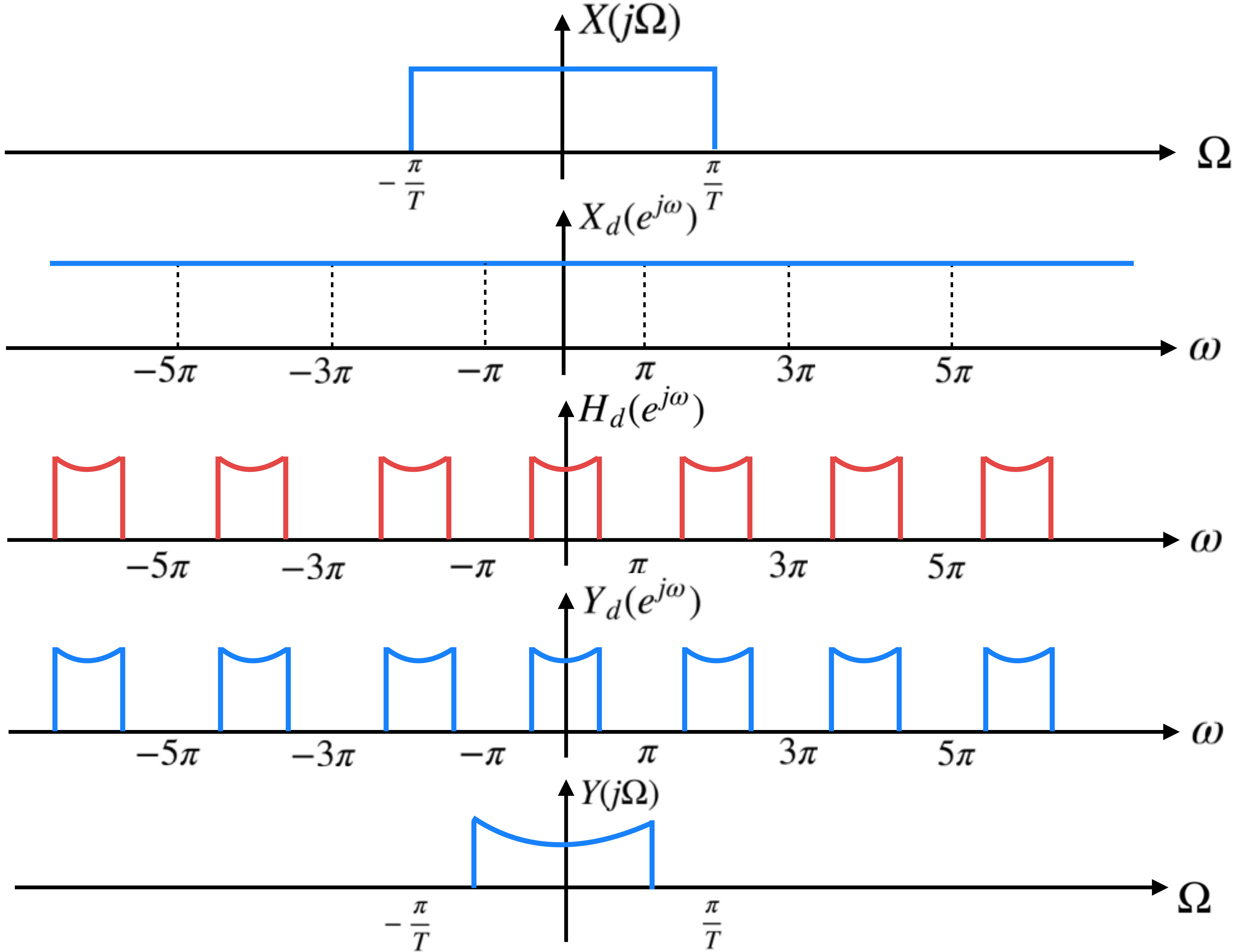
$$x[n] = x(nT) = \text{sinc}(\pi n) = \delta[n]$$

- Therefore,

$$y[n] = x[n] \star h[n] = \delta[n] \star h[n] = h[n]$$

- But it's also true that $y[n] = y(nT)$
- In conclusion,

$$h[n] = y(nT) \quad \text{when} \quad x(t) = \text{sinc}\left(\frac{\pi t}{T}\right)$$



$X(j\Omega)$
Ω
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
$X_d(e^{j\omega})$
ω
-5π -3π $-\pi$ π 3π 5π
$H_d(e^{j\omega})$
ω
-5π -3π $-\pi$ π 3π 5π
$Y_d(e^{j\omega})$
ω
-5π -3π $-\pi$ π 3π 5π
$Y(j\Omega)$
Ω
$-\frac{\pi}{T}$
$\frac{\pi}{T}$

Example

- **Differentiator**: $y(t) = \dfrac{dx(t)}{dt}$
- Since $Y(j\Omega) = j\Omega\, X(j\Omega)$, the desired filter seems to be

$$H(j\Omega) = j\Omega$$

- But this is not bandlimited!
- Since $x(t)$ itself is assumed to be bandlimited, no harm in implementing

$$H(j\Omega) = \begin{cases} j\Omega & |\Omega| \leq \dfrac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

instead.

$$H(j\Omega) = \begin{cases} j\Omega & |\Omega| \le \dfrac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

- The corresponding discrete-time impulse response is

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H\Big(j\,\frac{\omega}{T}\Big) e^{j\omega n}\, d\omega \;=\; \frac{j}{2\pi T} \int_{-\pi}^{\pi} \omega e^{j\omega n}\, d\omega$$

$$= \frac{j}{2\pi T} \left[\frac{\omega e^{j\omega n}}{jn} \bigg|_{-\pi}^{\pi} - \frac{1}{jn} \int_{-\pi}^{\pi} e^{j\omega n}\, d\omega \right]$$

$$= \frac{1}{2\pi T n} \left[\omega e^{j\omega n} \Big|_{-\pi}^{\pi} - \frac{e^{j\omega n}}{jn} \bigg|_{-\pi}^{\pi} \right]$$

$$= \frac{1}{2\pi T n} \left[\pi e^{j\pi n} - (-\pi) e^{-j\pi n} \right] \;=\; \frac{(-1)^n}{Tn}$$

- But differentiation is simple enough that if

$$x(t) = \mathrm{sinc}\left(\frac{\pi t}{T}\right)$$

then

$$y(t) = \frac{d}{dt}\,\frac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}} = \frac{\left(\frac{\pi t}{T}\right)\frac{d}{dt}\sin(\frac{\pi t}{T}) - \sin(\frac{\pi t}{T})\frac{d}{dt}\left(\frac{\pi t}{T}\right)}{\left(\frac{\pi t}{T}\right)^2}$$

$$= \frac{\left(\frac{\pi t}{T}\right)\frac{\pi}{T}\cos(\frac{\pi t}{T}) - \sin(\frac{\pi t}{T})\frac{\pi}{T}}{\left(\frac{\pi t}{T}\right)^2} = \frac{1}{t}\cos\left(\frac{\pi t}{T}\right) - \frac{T}{\pi t^2}\sin\left(\frac{\pi t}{T}\right)$$

- Thus,

$$h[n] = y(nT) = \frac{1}{nT}\cos(\pi n) - \frac{1}{\pi n^2 T}\sin(\pi n) = \frac{(-1)^n}{nT}$$

- The same result!

Example

- **Delay**: $y(t) = x(t - \Delta)$
- If $\Delta = kT$, this would be very easy to implement:

$$h[n] = \delta[n-k]$$

- To make it more interesting, take $\Delta \neq kT$
- Again, since $x(t)$ is bandlimited, let us implement

$$H(j\Omega) = \begin{cases} e^{-j\Omega\Delta} & |\Omega| \leq \dfrac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

which is the bandlimited version of the delay operator.

$$H(j\Omega) = \begin{cases} e^{-j\Omega\Delta} & |\Omega| \le \dfrac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

- The corresponding discrete-time impulse response is

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\frac{\omega\Delta}{T}} e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\frac{\Delta}{T})} d\omega$$

$$= \frac{1}{2\pi j(n-\frac{\Delta}{T})} e^{j\omega(n-\frac{\Delta}{T})}\Big|_{-\pi}^{\pi} = \frac{e^{j\pi(n-\frac{\Delta}{T})} - e^{-j\pi(n-\frac{\Delta}{T})}}{2\pi j(n-\frac{\Delta}{T})}$$

$$= \frac{\sin\left(\pi(n-\frac{\Delta}{T})\right)}{\pi(n-\frac{\Delta}{T})} = \operatorname{sinc}\left(\pi\left(n-\frac{\Delta}{T}\right)\right)$$

Example

- But time shift is simple enough that if

$$x(t) = \mathrm{sinc}\left(\frac{\pi t}{T}\right)$$

then

$$y(t) = \mathrm{sinc}\left(\frac{\pi(t-\Delta)}{T}\right)$$

- Thus,

$$h[n] = y(nT) = \mathrm{sinc}\left(\frac{\pi(nT-\Delta)}{T}\right) = \mathrm{sinc}\left(\pi\left(n - \frac{\Delta}{T}\right)\right)$$

- The same result!

Example

$$h[n] = \mathrm{sinc}\left(\pi\left(n - \frac{\Delta}{T}\right)\right)$$

$$\Delta = \frac{T}{3}$$