EE 115 Lecture Note 2
Analog Communication Techniques

Y. Hua

I. Terminology and Notation

1) Message signal $m(t)$. It can be of finite duration or infinite duration. We will assume that $m(t)$ is real valued.
2) The spectrum $M(f)$ of $m(t)$ is the Fourier transform of $m(t)$, i.e., $M(f) = \int_{-\infty}^{\infty} m(t)e^{-j2\pi ft}dt$. Since $m(t)$ is real, we know that $M(f) = M^*(-f)$, which is said to be of conjugate symmetry.
3) The averaged power of $m(t)$ is
$$\overline{m^2} = \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} m^2(t)dt. \tag{1}$$
If $m(t)$ is of finite duration, the average power is zero. If the average power is zero, $m(t)$ may or may not be of finite duration.
4) The average (or DC value) of $m(t)$ is
$$\overline{m} = \lim_{T_0\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} m(t)dt. \tag{2}$$
If and only if the positive area of $m(t)$ is the same as the negative area of $m(t)$, $\overline{m}$ is zero. Typically, we consider the case of $\overline{m} = 0$.
5) Example: $m(t) = A_m \cos(2\pi f_m t)$. Here, $\overline{m} = 0$ and $\overline{m^2} = \frac{1}{2}A_m^2$. Also note that $M(f) = \frac{A_m}{2}\delta(f - f_m) + \frac{A_m}{2}\delta(f + f_m)$ where $\left(\frac{A_m}{2}\right)^2 + \left(\frac{A_m}{2}\right)^2 = \frac{1}{2}A_m^2 = \overline{m^2}$.
6) Note that for any periodic signal $m(t) = m(t + T)$ with the period T, there is the Fourier series expansion:
$$m(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T} t} \tag{3}$$
with $c_k = \frac{1}{T}\int_{-T/2}^{T/2} m(t)e^{-j2\pi\frac{k}{T}t}dt$ (coefficients). If $m(t)$ is also real, then $c_k = c_{-k}^*$ for all k. The Fourier transform of $m(t)$ is
$$M(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - \frac{k}{T}) \tag{4}$$
and the average power of $m(t)$ is
$$\overline{m^2} = \sum_{k=-\infty}^{\infty} |c_k|^2. \tag{5}$$

We can call $|c_k|^2$ as a function of k the power spectrum of $m(t)$.

7) The transmitted radio-frequency (RF) signal always has the form:

$$u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t) = a(t)\cos(2\pi f_c t + \theta(t)) \tag{6}$$

where

   a) f_c is the carrier frequency.
   b) $u_c(t)$ is called the in-phase component or I component.
   c) $u_s(t)$ is called the quadrature component or Q component.
   d) $a(t)$ is the envelope of the RF signal.
   e) $\theta(t)$ is the phase of the RF signal.
   f) $a(t)e^{j\theta(t)}$ is called the complex envelope of the RF signal. It can be shown that $a(t)e^{j\theta(t)} = u_c(t) + ju_s(t)$.

8) How a message $m(t)$ affects $u_c(t)$, $u_s(t)$, $a(t)$ and/or $\theta(t)$ depends on a specific modulation method.

II. Amplitude Modulation

A. Double-sideband (DSB) suppressed carrier (SC)

A DSB-SC signal is

$$u_{DSB}(t) = Am(t)\cos(2\pi f_c t) \tag{7}$$

where $m(t)$ is the message, f_c is the carrier frequency, and A controls the power. The Fourier transform (FT) of $u_{DSB}(t)$ is

$$U_{DSB}(f) = \frac{A}{2}M(f - f_c) + \frac{A}{2}M(f + f_c). \tag{8}$$

For example, we can let $m(t) = A_m\cos(2\pi f_m t)$. We can now sketch $u_{DSB}(t)$ and $U_{DSB}(f)$.

More generally, we can assume that the real $m(t)$ has a bandwidth B and $M(f) = Re(M(f)) + jIm(M(f))$ where $Re(M(f))$ is even and $Im(M(f))$ is odd. We can then sketch $Re(U_{DSB}(f))$ and $Im(U_{DSB}(f))$. The portion of $U_{DSB}(f)$ for $0 < f < f_c$ or $-f_c < f < 0$ is called the *lower-sideband* (LSB). The portion of $U_{DSB}(f)$ for $f > f_c$ or $f < -f_c$ is called the *upper-sideband* (USB).

Since $M(f) = M^*(-f)$ or equivalently $Re(M(f)) = Re(M(-f))$ and $Im(M(f)) = -Im(M(-f))$, there is a waste of bandwidth. We will discuss a more efficient modulation method later.

$u_{DSB}(t)$ is not only called DSB but also SC because there is no pure carrier component or equivalently $U_{DSB}(f)$ is not infinite at $f = \pm f_c$.

A DSB-SC modulator (a mixer) takes $m(t)$ and $\cos(2\pi f_c t)$ as inputs and produces $u_{DSB}(t)$ as output. But the factor A can be a combination of the factors from the mixer and an amplifier.

1) DSB-SC demodulator: A DSB-SC demodulator (a demixer) can be described as follows. Let the received signal at the destination be

$$y(t) = A_r A m(t-\tau)\cos(2\pi f_c(t-\tau)) = A_r A m(t-\tau)\cos(2\pi f_c t - \theta) \tag{9}$$

where A_r is a factor due to the channel power loss and the frontend power gain of the receiver, and $\theta = 2\pi f_c \tau$ is a phase due to the propagation delay τ.

A demixer requires a locally generated carrier signal $\cos(2\pi f_c t - \phi)$. Multiplying this carrier signal to $2y(t)$ yields

$$2y(t)\cos(2\pi f_c t - \phi) = A_r A m(t-\tau)\cos(\phi - \theta) + A_r A m(t-\tau)\cos(4\pi f_c t - \theta - \phi). \tag{10}$$

Since the second term above has a much higher frequency than the first term, we can use a lowpass filter (LPF) to filter out the second term. The output of the LPF is

$$A_r A m(t-\tau)\cos(\phi - \theta) \tag{11}$$

which differs from the original $m(t)$ by the delay τ and the overall factor $A_r A \cos(\phi - \theta)$.

To minimize the attenuation of the received message, we need the phase difference $\phi - \theta$ to be as small as possible.

If the receiver's carrier frequency differs from the transmitter's carrier frequency by Δf_c, then the receiver produces the carrier signal $\cos(2\pi(f_c + \Delta f_c)t - \phi) = \cos(2\pi f_c - \phi')$ where $\phi' = -2\pi \Delta f_c t + \phi$. Therefore, the signal from the demixer is now

$$A_r A m(t-\tau)\cos(\phi - 2\pi \Delta f_c t - \theta) \tag{12}$$

which is now a (not acceptably) distorted version of $m(t)$.

B. Conventional AM

The conventional amplitude-modulated (AM) signal is

$$u_{AM}(t) = A m(t)\cos(2\pi f_c t) + A_c \cos(2\pi f_c t) \tag{13}$$

where there is a pure carrier signal. The FT of this signal is

$$U_{AM}(t) = \frac{A}{2}\left(M(f-f_c) + M(f+f_c)\right) + \frac{A_c}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right). \tag{14}$$

Here we see spikes at $f = \pm f_c$.

For the conventional AM signal, there is a simple (low cost) demodulator as follows. Let the envelope of $u_{AM}(t)$ be

$$e(t) = |Am(t) + A_c|. \tag{15}$$

If $Am(t) + A_c$ is always positive, then $e(t) = Am(t) + A_c$ and hence $m(t)$ can be extracted from $e(t)$ by subtracting out its DC component.

Let $M_0 = |\min_t m(t)|$ and $a_{mod} = \frac{AM_0}{A_c}$ (*Modulation Index*). We can see that as long as $a_{mod} < 1$, $Am(t) + A_c$ is always positive and hence $e(t) = Am(t) + A_c$. Note that

$$u_{AM}(t) = A_c(1 + a_{mod}m_n(t))\cos(2\pi f_c t) \tag{16}$$

where $m_n(t) = \frac{1}{M_0}m(t)$ is the normalized message signal. Typically, $\max_t |m_n(t)| < 1$.

The envelope detector consists of a diode, a R-C circuit and a DC blocker. The R-C circuit has a time constant equal to RC (the product of resistance and capacitance). See page 96. The RC value should be such that

$$\frac{1}{f_c} \ll RC \ll \frac{1}{B} \tag{17}$$

where the left inequality makes the charging-phase of the R-C circuit highly responsive to the input RF signal, and the right inequality makes the discharging phase of the R-C circuit highly responsive to the change due to the message.

For example, if $f_c = 500$kHz and $B = 5$kHz, then we can choose RC to be

$$2\mu s \ll RC \ll 200\mu s. \tag{18}$$

If $R = 50\Omega$ (ohms) and $C = 400nF$ (nanofarads), then $RC = 20\mu s$.

Note that the envelope detector does not need a locally generated carrier signal or a mixer. The latter can be costly at high frequency.

1) Power efficiency: The total power of the AM signal is

$$\overline{u_{AM}^2(t)} = \overline{A_c^2(1 + a_{mod}m_n(t))^2\cos^2(2\pi f_c t)} = \frac{1}{2}A_c^2(1 + a_{mod}^2\overline{m_n(t)^2}). \tag{19}$$

while the power of the message carrying component in the AM signal is

$$\overline{A_c^2 a_{mod}^2 m_n^2(t)\cos^2(2\pi f_c t)} = \frac{1}{2}A_c^2 a_{mod}^2\overline{m_n^2(t)}. \tag{20}$$

Then the *power efficiency* is

$$\eta_{AM} = \frac{a_{mod}^2 \overline{m_n^2(t)}}{1 + a_{mod}^2 \overline{m_n^2(t)}}. \tag{21}$$

which is obviously less than one. Furthermore, since $a_{mod} < 1$ and $\overline{m_n^2(t)} < 1$, we see that $\eta_{AM} < 1/2$.

If $\overline{m_n^2(t)} < 1/2$, then $\eta_{AM} < \frac{0.5a_{mod}^2}{1+0.5a_{mod}^2} < 1/3$.

The larger is the "ratio of the peak power over the average power" (*peak-to-average power ratio*) of $m(t)$, the smaller is $\overline{m_n^2(t)}$ and hence so is η_{AM}.

The conventional AM signal is suitable for broadcast where the transmitter has a large power source and the receivers are of low cost.

C. Single-Sideband Modulation (SSB)

1) Hilbert Transformer: A filter with the following frequency response is called the Hilbert Transformer:

$$H(f) = -j \times sgn(f) = \begin{cases} -j, & f > 0 \\ j & f < 0 \end{cases} \tag{22}$$

Notice that $|H(f)| = 1$ and $\angle(H(f)) = -\pi/2$ for $f > 0$ and $\angle(H(f)) = \pi/2$ for $f < 0$.

It is clear that

$$H(f) = \lim_{\epsilon \to 0} H_\epsilon(f) \tag{23}$$

with

$$H_\epsilon(f) = \begin{cases} -je^{-\epsilon f}, & f > 0 \\ je^{\epsilon f}, & f < 0 \end{cases} \tag{24}$$

where $\epsilon > 0$.

The inverse Fourier transform of $H_\epsilon(f)$ is

$$\begin{aligned} h_\epsilon(t) = F^{-1}(H_\epsilon(f)) &= \int_{-\infty}^{\infty} H_\epsilon(f) e^{j2\pi f t} df \\ &= \int_{-\infty}^{0} je^{\epsilon f} e^{j2\pi f t} df + \int_{0}^{\infty} -je^{-\epsilon f} e^{j2\pi f t} df \\ &= \frac{j}{\epsilon + j2\pi t} + \frac{j}{-\epsilon + j2\pi t}. \end{aligned} \tag{25}$$

It follows that the impulse response of the Hilbert transformer is

$$h(t) = \lim_{\epsilon \to 0} h_\epsilon(t) = \frac{1}{\pi t}. \tag{26}$$

Let $\hat{m}(t)$ be the output of the Hilbert transformer driven by $m(t)$. Then

$$\hat{M}(f) = \begin{cases} -jM(f), & f > 0 \\ jM(f), & f < 0 \end{cases} . \tag{27}$$

Note that if $m(t) = \cos(2\pi f_c t)$ and $\hat{m}(t) = \sin(2\pi f_c t)$. Also note that $\hat{\hat{m}}(t) = -\cos(2\pi f_c t)$.

2) Upper-Sideband (USB) Signal: Consider

$$u_{USB}(t) = m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t) \tag{28}$$

whose Fourier transform is

$$U_{USB}(f) = \frac{1}{2}M(f - f_c) + \frac{1}{2}M(f + f_c) - \frac{1}{2j}\hat{M}(f - f_c) + \frac{1}{2j}\hat{M}(f + f_c). \tag{29}$$

It follows that for $f > f_c$,

$$U_{USB}(f) = \frac{1}{2}M(f - f_c) - \frac{1}{2j}(-j)M(f - f_c) = M(f - f_c), \tag{30}$$

and for $0 < f < f_c$,

$$U_{USB}(f) = \frac{1}{2}M(f - f_c) - \frac{1}{2j}jM(f - f_c) = 0. \tag{31}$$

Similarly, for $f < -f_c$,

$$U_{USB}(f) = \frac{1}{2}M(f + f_c) + \frac{1}{2j}jM(f + f_c) = M(f + f_c), \tag{32}$$

and for $f_c < f < 0$,

$$U_{USB}(f) = \frac{1}{2}M(f + f_c) + \frac{1}{2j}(-j)M(f + f_c) = 0. \tag{33}$$

The above analysis shows that $u_{USB}(t)$ is indeed a USB signal.

3) Lower-Sideband (LSB) Signal: We can show that the following is a LSB signal:

$$u_{LSB}(t) = m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t). \tag{34}$$

Notice the positive sign in front of the Q-component $\hat{m}(t)$.

Examples:

$$u_{USB}(t) = \cos(2\pi(f_c + f_m)t) = \cos(2\pi f_m t)\cos(2\pi f_c t) - \sin(2\pi f_m t)\sin(2\pi f_c t) \tag{35}$$

$$u_{LSB}(t) = \cos(2\pi(f_c - f_m)t) = \cos(2\pi f_m t)\cos(2\pi f_c t) + \sin(2\pi f_m t)\sin(2\pi f_c t) \tag{36}$$

4) Coherent SSB Demodulation: Assume that the received RF SSB signal is

$$y(t) = m(t)\cos(2\pi f_c t + \theta) - \hat{m}(t)\sin(2\pi f_c t + \theta) \tag{37}$$

where θ is the phase offset with respect to the locally generated carrier signal $\cos(2\pi f_c t)$.

(As shown later) The output of the I-and-Q demodulator based on $\cos(2\pi f_c t)$ is the complex envelope of $y(t)$ with respect to $\cos(2\pi f_c t)$, which is

$$\tilde{y}(t) = m(t)e^{j\theta} + j\hat{m}(t)e^{j\theta}. \tag{38}$$

The real part of the complex envelope is

$$\Re(\tilde{y}(t)) = m(t)\cos\theta - \hat{m}(t)\sin\theta \tag{39}$$

and the imaginary part of the complex envelope is

$$\Im(\tilde{y}(t)) = m(t)\sin\theta + \hat{m}(t)\cos\theta. \tag{40}$$

We see that as long as $\theta \neq 0$, the I component of the output of the (coherent) SSB demodulator is a distorted version of $m(t)$.

5) Noncoherent SSB Demodulator: If a carrier component is added to the SSB signal at the transmitter, then the received RF signal is

$$y(t) = (A + m(t))\cos(2\pi f_c t + \theta) \pm \hat{m}(t)\sin(2\pi f_c t + \theta) \tag{41}$$

whose real envelope is

$$e(t) = \sqrt{(A + m(t))^2 + \hat{m}^2(t)}. \tag{42}$$

If A is such that $(A + m(t))^2 \gg \hat{m}^2(t)$, then

$$e(t) \approx A + m(t) \tag{43}$$

from which $m(t)$ can be retrieved easily. In this case, we only need an envelope detector.

D. Vestigial-Sideband (VSB) Modulation

We have seen that SSB modulation is efficient in spectral usage. But it requires a good Hilbert transformer.

An alternative to SSB modulation is called vestigial-sideband (VSB) modulation as shown next. Let a DSB-SC signal be

$$u_{DSB}(t) = 2m(t)\cos(2\pi f_c t) \tag{44}$$

and then pass it through a bandpass filter with the frequency response $H_p(f)$ to partially remove one of the two sidebands. The output of $H_p(f)$ is

$$u_{VSB}(t) = u_{DBS}(t) * h_p(t) \tag{45}$$

whose spectrum is

$$U_{VSB}(f) = H_p(f)[M(f - f_c) + M(f + f_c)]. \tag{46}$$

To further specify $H_p(f)$, let us apply the coherent SSB demodulator to $u_{VSB}(t)$ as follows. The output of the mixer is

$$2u_{VSB}(t)\cos(2\pi f_c t) \tag{47}$$

whose spectrum is

$$U_{VSB}(f-f_c)+U_{VSB}(f+f_c) = H_p(f-f_c)[M(f-2f_c)+M(f)]+H_p(f+f_c)[M(f)+M(f+2f_c)]. \tag{48}$$

Using a LPF, we obtain a signal with the following spectrum:

$$H_p(f - f_c)M(f) + H_p(f + f_c)M(f) = [H_p(f - f_c) + H_p(f + f_c)]M(f). \tag{49}$$

To have a perfect $m(t)$, we need $H_p(f - f_c) + H_p(f + f_c) = \texttt{constant}$ within the band of $M(f)$. A filter with this property is called VSB filter. Typically, $M(f)$ is zero in a region around $f = 0$, which allows a relaxation of $H_p(f - f_c) + H_p(f + f_c) = \texttt{constant}$ in that region.

Since the coherent SSB demodulator extracts out the exact $m(t)$, we can conclude

$$u_{VSB}(t) = u_{DBS}(t) * h_p(t) = m(t)\cos(2\pi f_c t) - m'(t)\sin(2\pi f_c t). \tag{50}$$

If we also add a strong enough carrier component $A\cos(2\pi f_c t)$ to $u_{VSB}(t)$, then we can approximately extract out $m(t)$ using an envelope detector.

E. Quadrature Amplitude (QAM) Modulation

A QAM signal is

$$u_{QAM}(t) = m_c(t)\cos(2\pi f_c t) - m_s(t)\sin(2\pi f_c t) \tag{51}$$

where $m_c(t)$ and $m_s(t)$ are independent messages (different from SSB and VSB). The complex envelope of $u_{QAM}(t)$ with respect to $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ is

$$u(t) = m_c(t) + jm_s(t). \tag{52}$$

The complex envelope of $u_{QAM}(t)$ with respect to $\cos(2\pi f_c t - \theta)$ is

$$u_\theta(t) = m_c(t)e^{j\theta} + jm_s(t)e^{j\theta} = u(t)e^{j\theta}. \tag{53}$$