Homework 4 solutions

Problem 1 [10pts]: Consider the **causal LTI** system we saw in the previous homework, whose input-output relation is given by

$$y(t) + \frac{dy(t)}{dt} = x(t) \ .$$

a) Determine the impulse response of the system, $h(t)$.

b) Now determine the output of the system to the input $x(t) = e^{-3t}u(t)$ using convolution with $h(t)$ that you determined in the first part. How does it compare to the answer you obtained in your previous homework?

Solution:

a) The impulse response is the solution to

$$h(t) + \frac{dh(t)}{dt} = \delta(t) \ .$$

We can borrow the homogeneous solution from previous homework as ce^{-t}. The solution therefore is of the form

$$h(t) = ce^{-t}u(t) \ .$$

To find c, we need to have $h(0^+)$. That, in turn, can be obtained by integrating the differential equation from 0^- to 0^+:

$$\int_{0^-}^{0^+} \left[h(t) + \frac{dh}{dt}\right] dt = \int_{0^-}^{0^+} \delta(t)dt \ .$$

Since $h(t)$ cannot have an impulse (otherwise $\frac{dh}{dt}$ would have a doublet and that is not accounted for on the right-hand side), its integral from 0^- to 0^+ is just 0. Using this, the above equation can be simplified to

$$h(0^+) - h(0^-) = 1 \ .$$

But since the system is in initial rest, $h(0^-) = 0$, and therefore $h(0^+) = 1$. Substituting this piece of information in $h(t) = ce^{-t}u(t)$, we get that $c = 1$, and therefore the impulse response is given as

$$h(t) = e^{-t}u(t) \ .$$

b) We indeed have

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau \\ &= e^{-3t}\int_{-\infty}^{\infty} e^{2\tau}u(\tau)u(t-\tau)d\tau \\ &\overset{(a)}{=} e^{-3t}u(t)\int_{0}^{t} e^{2\tau}d\tau \\ &= e^{-3t}u(t)\frac{1}{2}[e^{2t}-1] \\ &= \frac{1}{2}[e^{-t}-e^{-3t}]u(t)\ . \end{aligned}$$

Here, (a) follows from the fact that

$$u(\tau)u(t-\tau) = 0$$

for all τ if $t < 0$, and

$$u(\tau)u(t-\tau) = \begin{cases} 1 & 0 \le \tau \le t \\ 0 & \text{otherwise.} \end{cases}$$

if $t \ge 0$.

Problem 2 [10pts]: Draw block diagram representations for causal LTI systems described by the following differential equations:

a) $y(t) + \frac{1}{2}\frac{dy(t)}{dt} = 4x(t)$

b) $25y(t) - 6\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = x(t) - \frac{dx(t)}{dt}$

Solution:

a) $y(t) = -\frac{1}{2}\frac{dy(t)}{dt} + 4x(t)$.

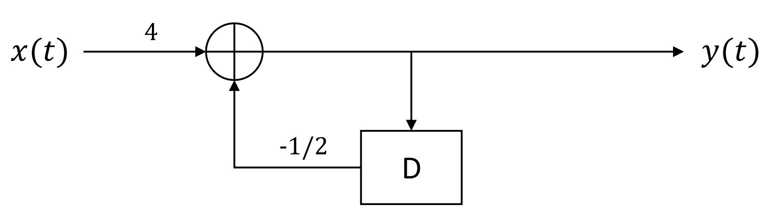

Figure 1: Solution for 2(a)

b) $y(t) = \frac{6}{25}\frac{dy(t)}{dt} - \frac{1}{25}\frac{d^2y(t)}{dt^2} + \frac{1}{25}x(t) - \frac{1}{25}\frac{dx(t)}{dt}$.

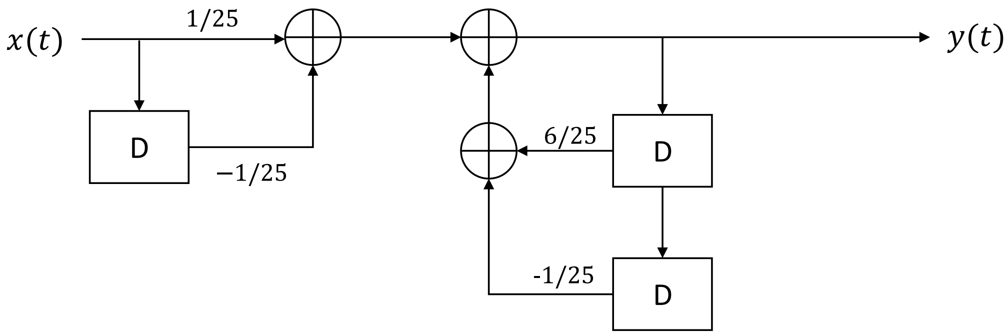

Figure 2: Solution for 2(b)