EE 110B Signals and Systems

Sampling and Reconstruction of Continuous-Time Signals

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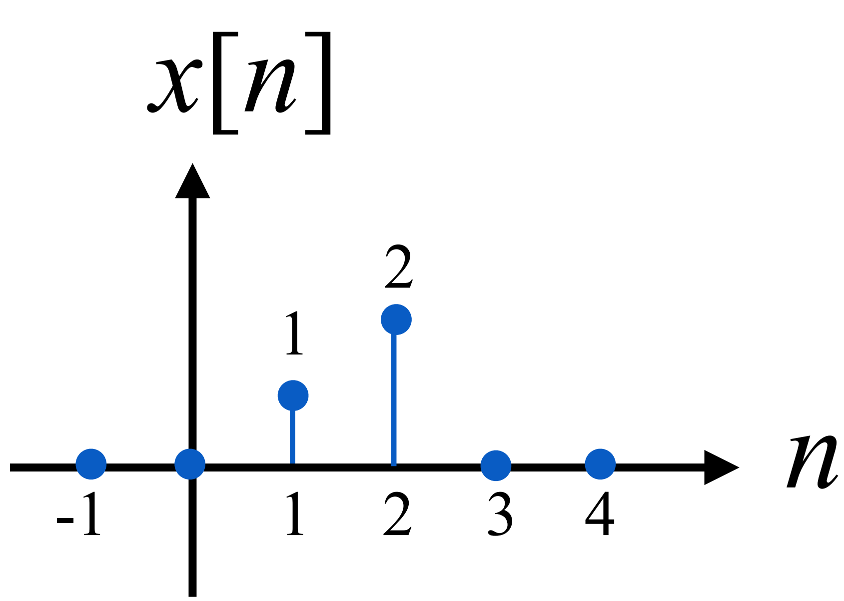
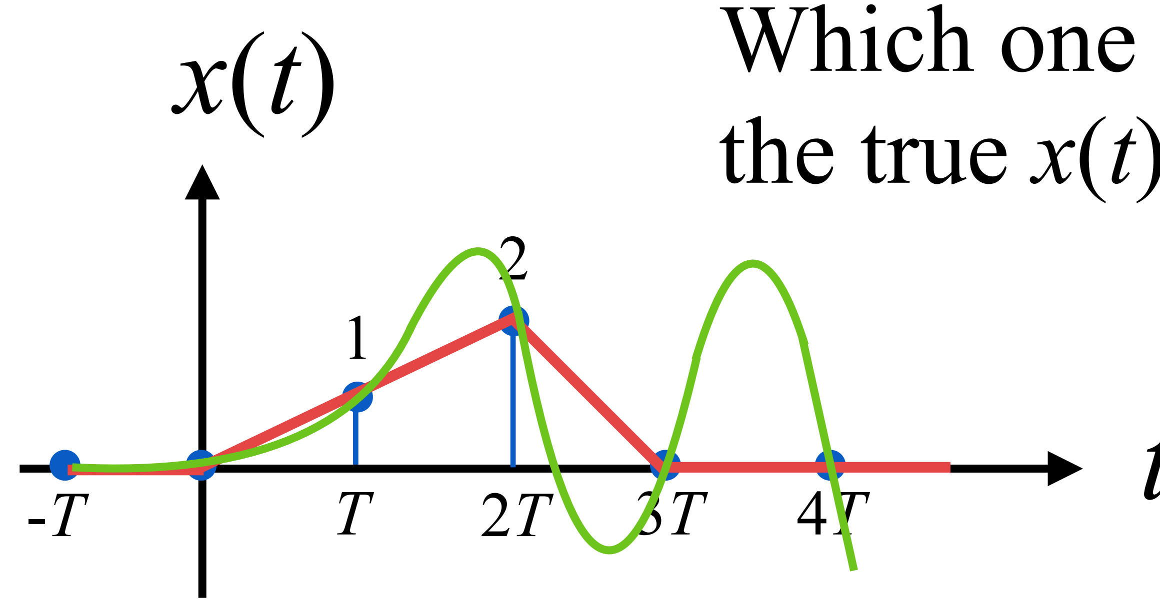
Sampling

- Recall that one of the motivations for studying discrete-time signals was the sampling scenario

$$x[n] = x(nT)$$

- <u>Fundamental question</u>: Can we reconstruct $x(t)$ only from $x[n]$?
- The answer seems to be no:

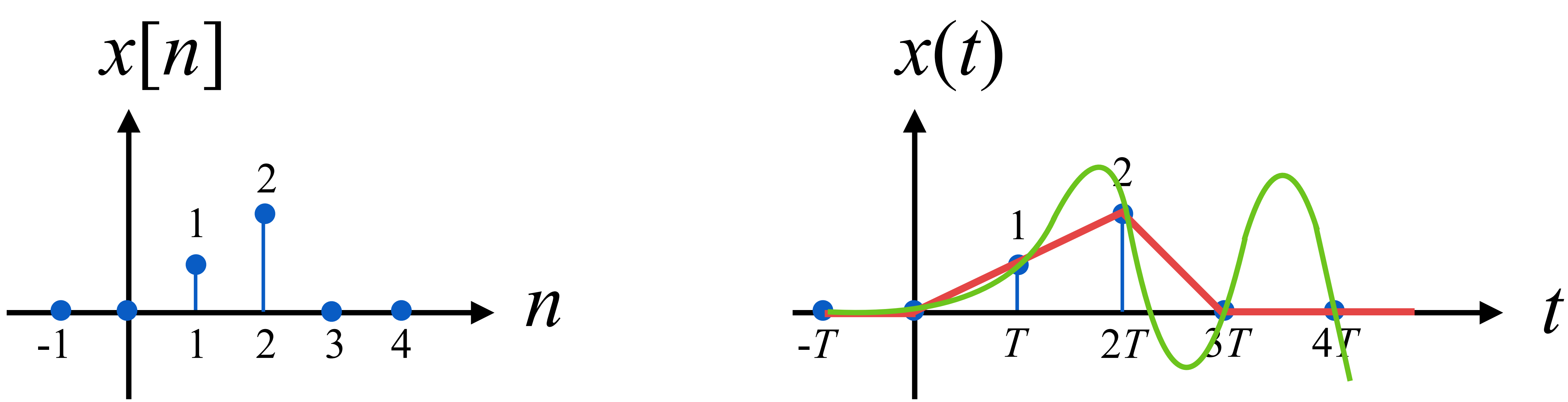




Which one is the true $x(t)$?

Sampling

- What if I told you that $x(t)$ is in a certain class?
  - Piece-wise linear signals.
  - Polynomials of degree M
  - ...
  - This lecture will be exclusively about the class of "band limited signals"



It's all about Fourier

- How is the DTFT of $x[n]$ related to the CTFT of $x(t)$?

$$X_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

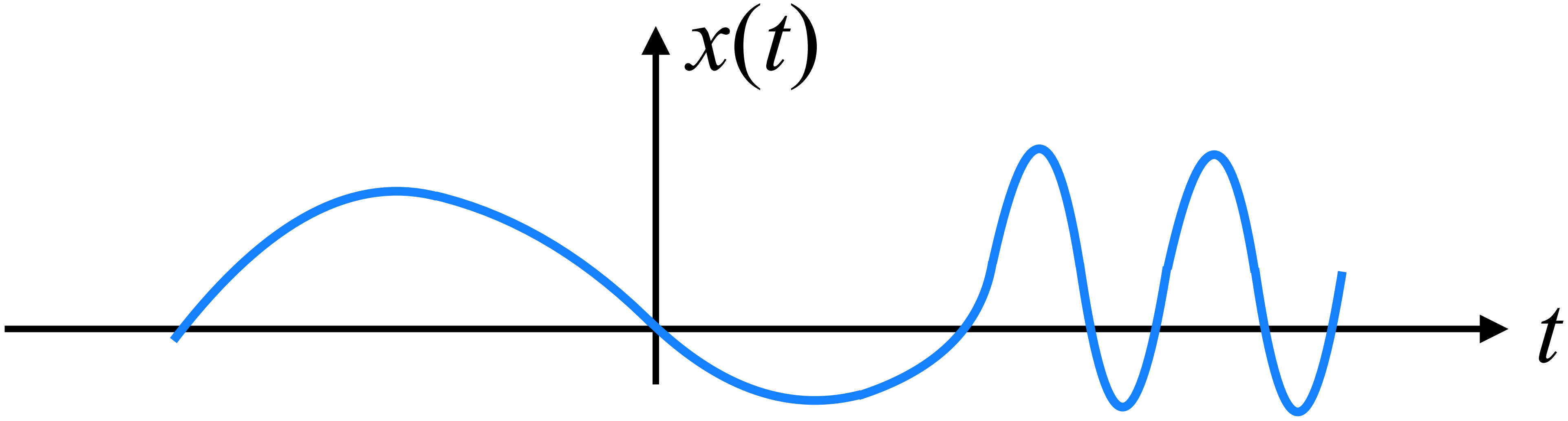
$$X_c(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

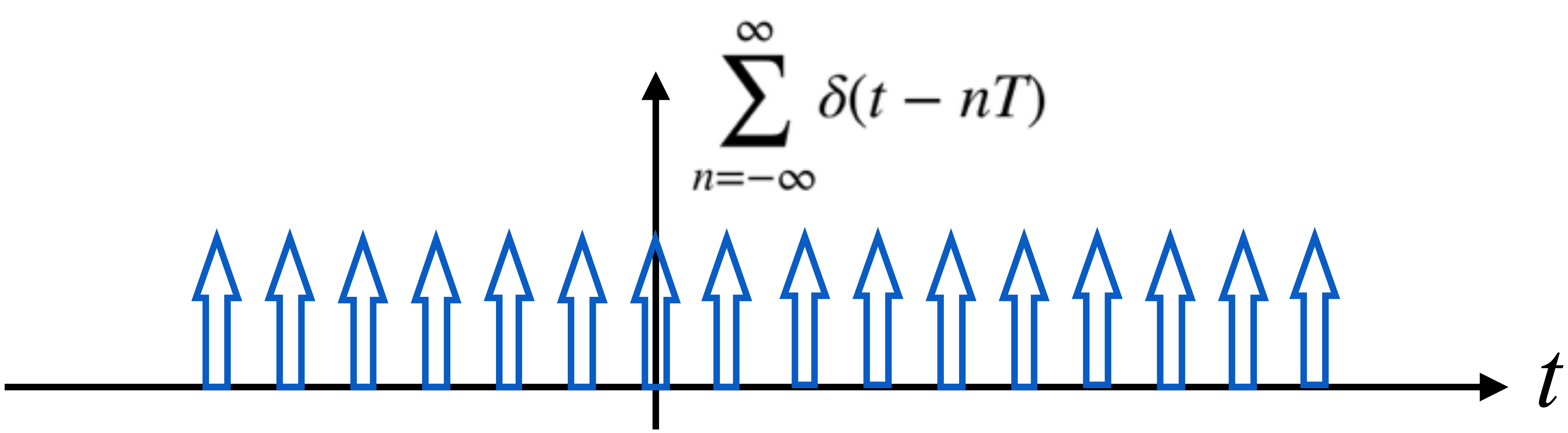
- They certainly look alike.

DTFT vs CTFT

- The first step is to define an intermediate signal

$$x_s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)$$





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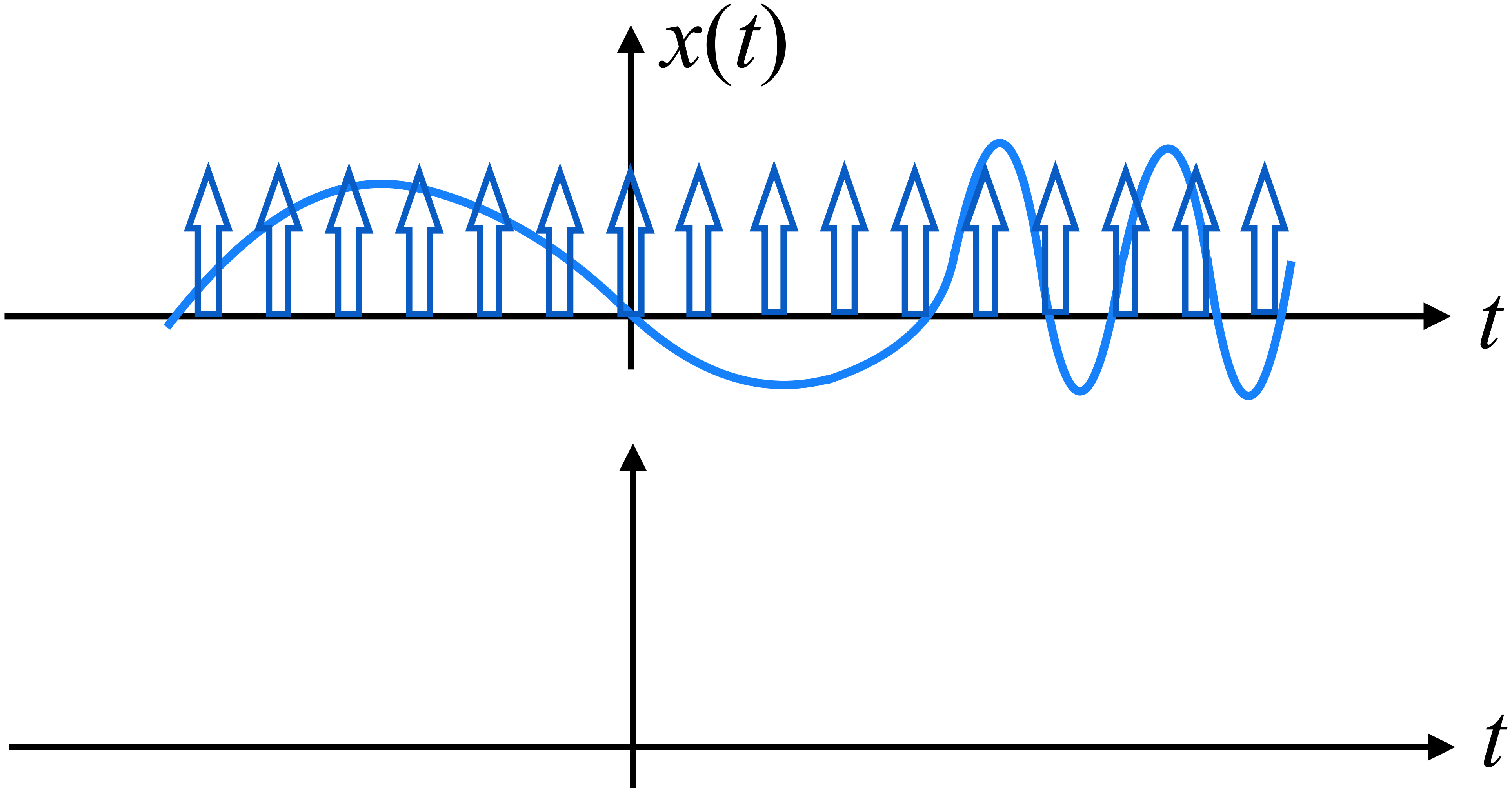

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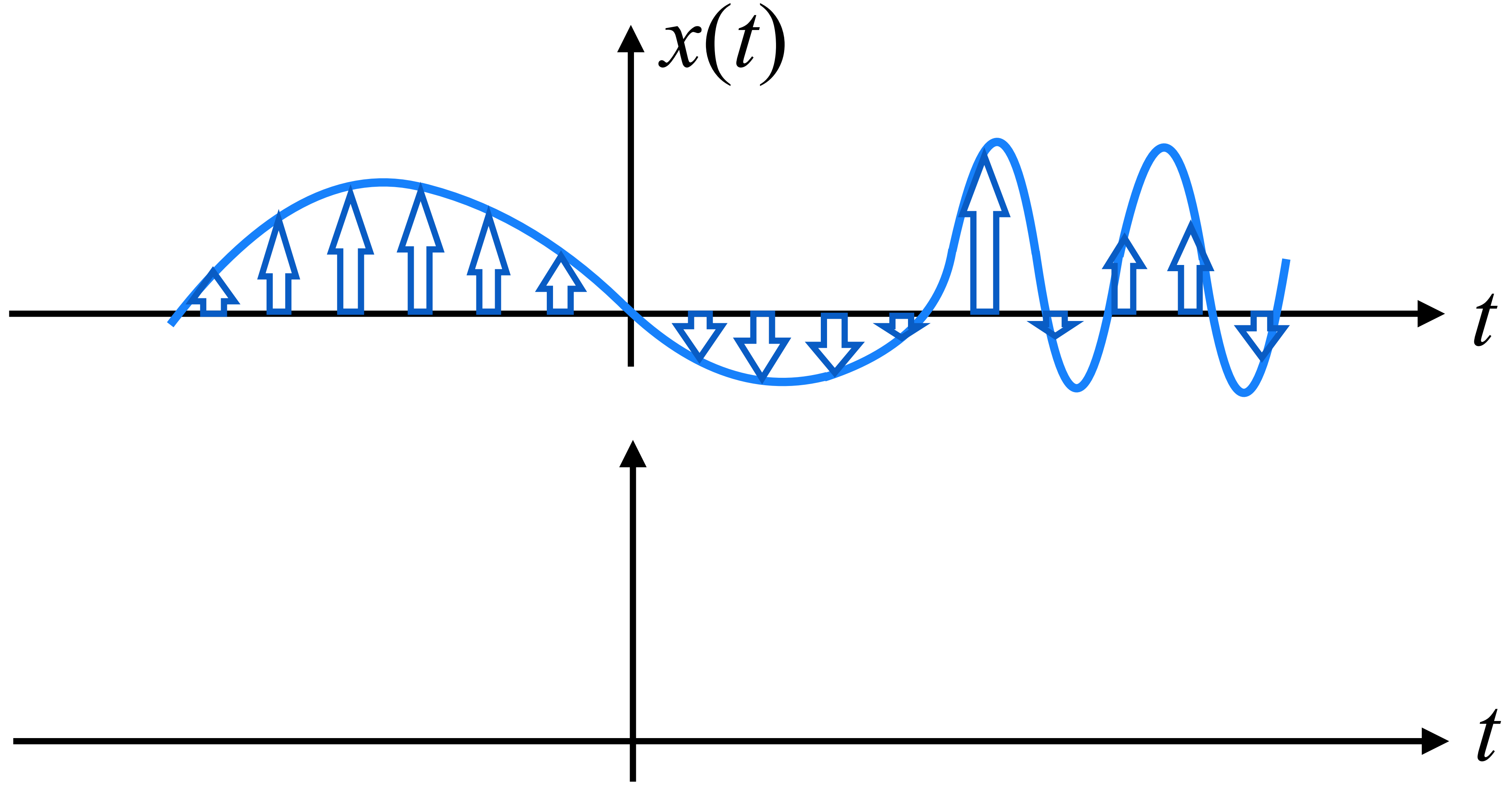

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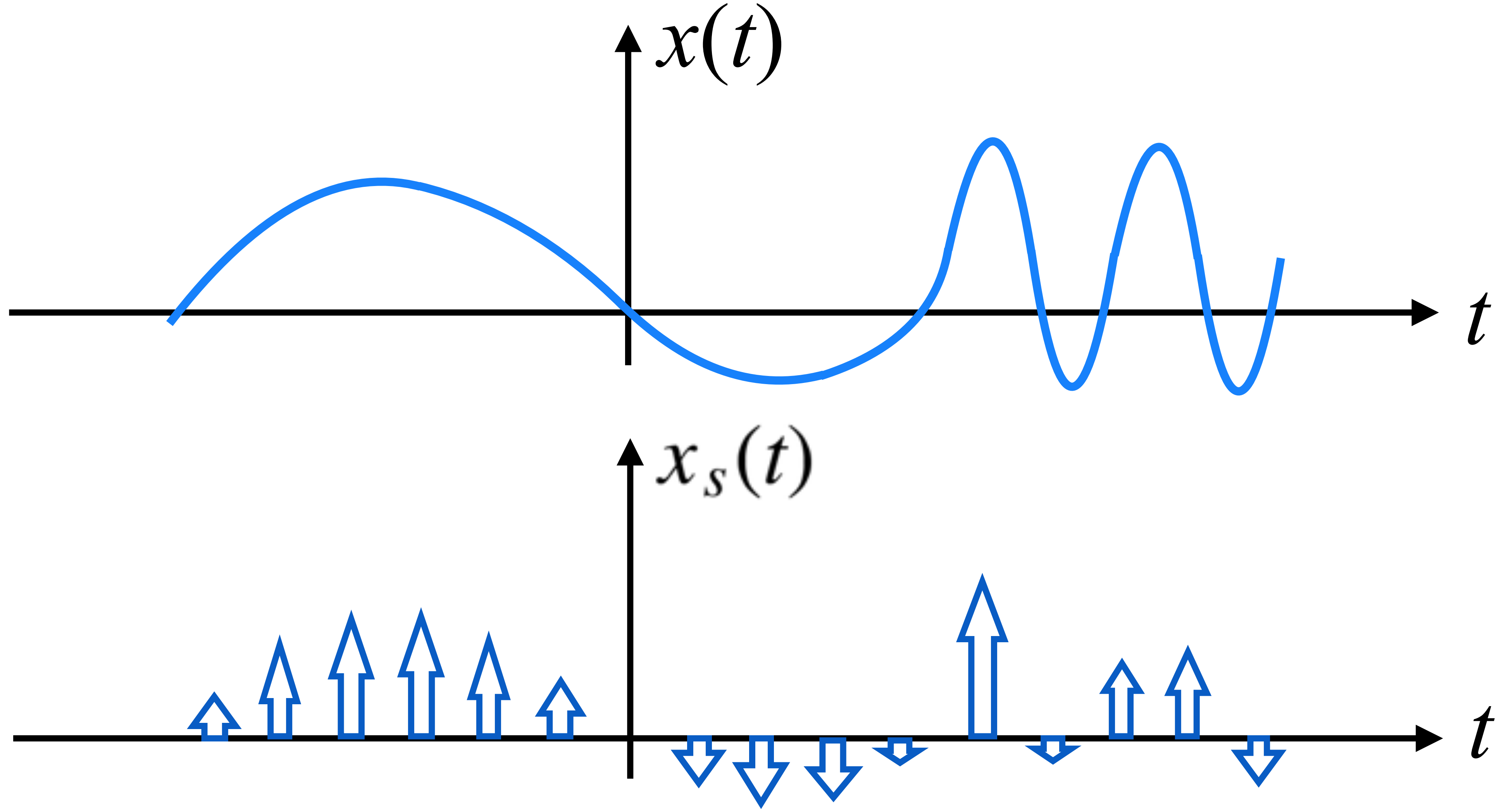
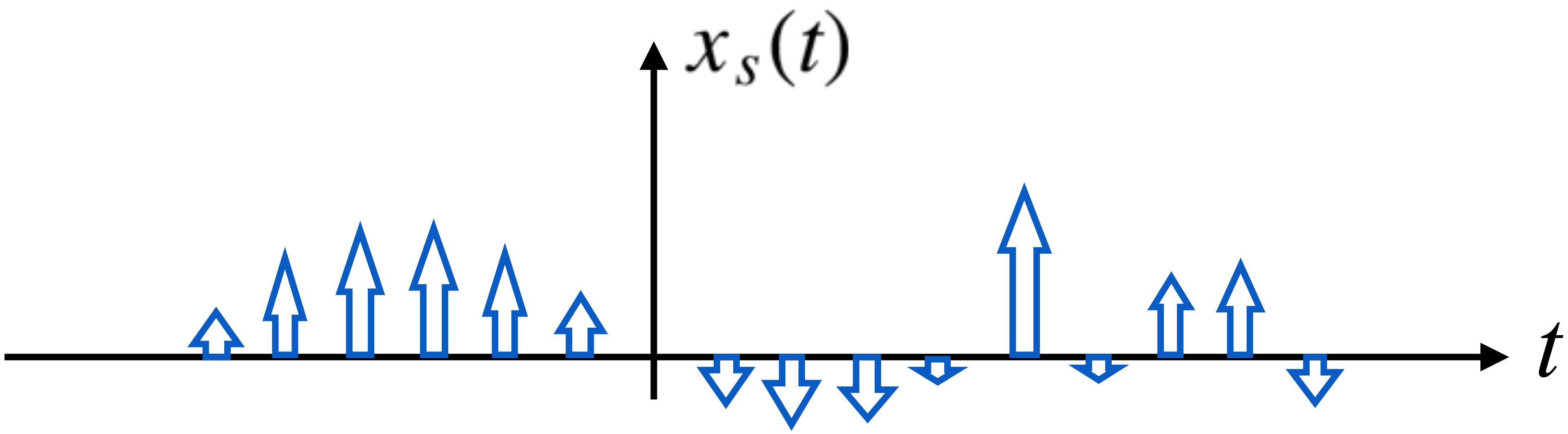


DTFT vs CTFT

- The first step is to define an intermediate signal

$$
\begin{aligned}
x_s(t) &= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)
\end{aligned}
$$



DTFT vs CTFT

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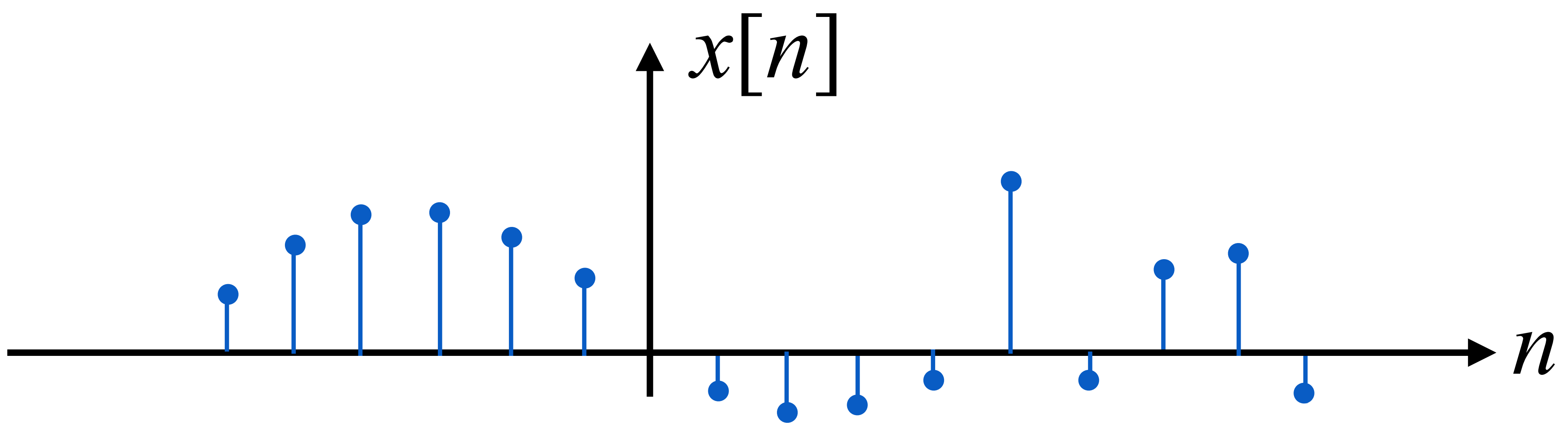
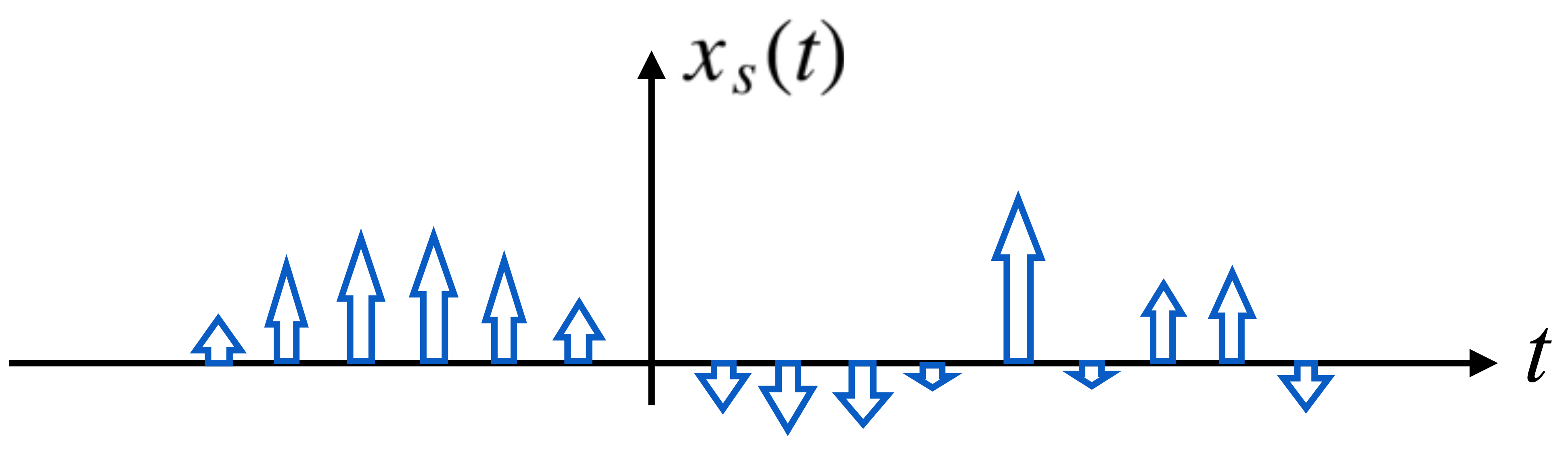




DTFT vs CTFT

- The first step is to define an intermediate signal

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

- Let us now relate $X_d(e^{j\omega})$ and $X_s(j\Omega)$:

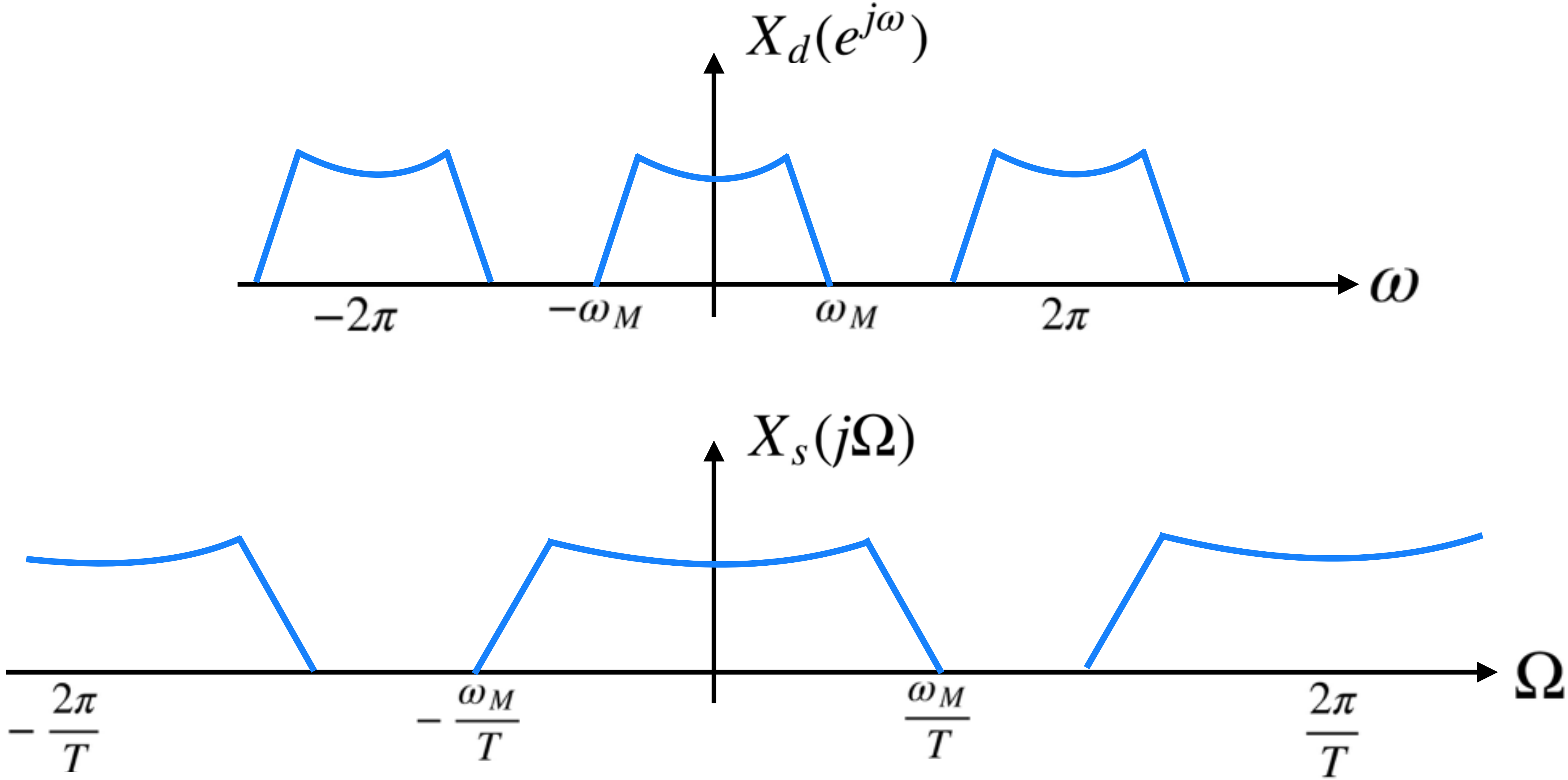
$$X_s(j\Omega) = \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)\right)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]\int_{-\infty}^{\infty}\delta(t - nT)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT} = X_d(e^{j\omega})\Big|_{\omega=\Omega T} = X_d(e^{j\Omega T})$$

DTFT vs CTFT

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \qquad X_s(j\Omega) = X_d(e^{j\Omega T})$$



DTFT vs CTFT

- Observation: The CTFT of $x_s(t)$ has to have a period of $2\pi/T$. But why?
- The answer lies in the relation

$$x_s(t) = x(t) \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{p(t)}$$

- Multiplication in the time domain implies convolution in the Fourier domain:

$$X_s(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\theta) P(j(\Omega - \theta)) d\theta$$

$$X_s(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\theta)P(j(\Omega-\theta))d\theta$$

- But what is $P(j\Omega)$?
- The best way to find it is to see $p(t)$ as a periodic signal and find its CTFS coefficients:

$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

with

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2}\Big[\sum_{n=-\infty}^{\infty}\delta(t-nT)\Big]e^{-jk\frac{2\pi}{T}t}dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\frac{2\pi}{T}t}dt \quad = \frac{1}{T}$$

$$X_s(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\theta)P(j(\Omega-\theta))d\theta$$

- So

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

implying that

$$P(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\Omega - k\frac{2\pi}{T}\right)$$

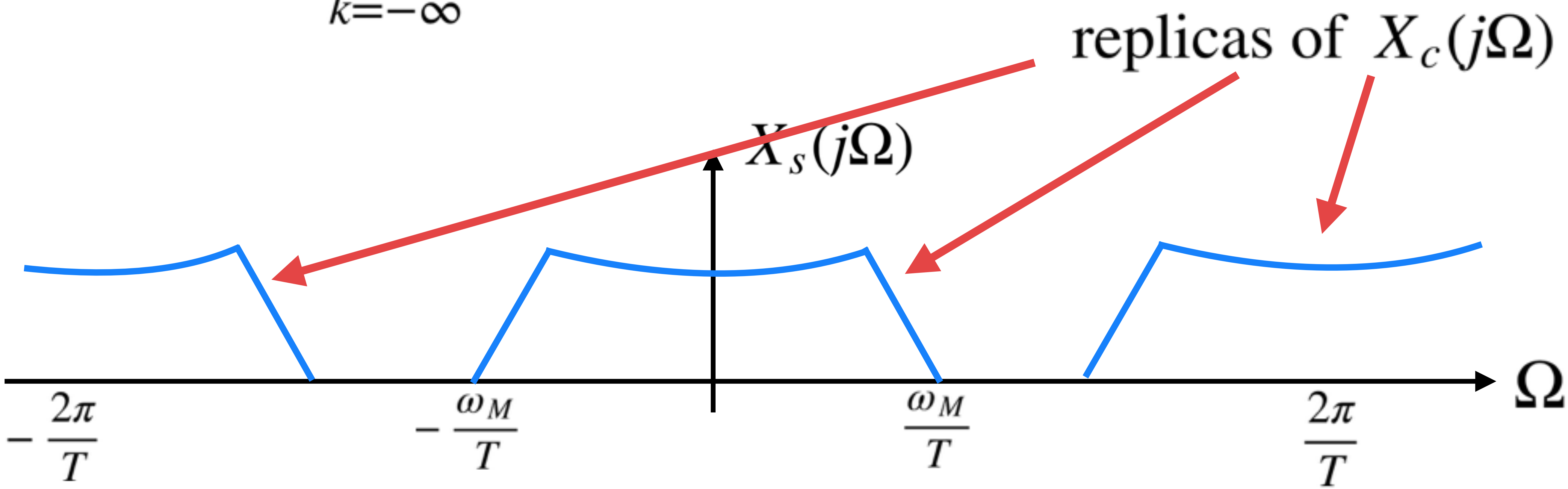
- Therefore,

$$X_s(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\theta)\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\Omega - \theta - k\frac{2\pi}{T}\right)\right]d\theta$$

$$X_s(j\Omega) = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} X_c(j\theta) \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} \cancel{2\pi} \delta\left(\Omega - \theta - k\,\frac{2\pi}{T} \right) \right] d\theta$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(j\theta) \delta\left(\Omega - \theta - k\,\frac{2\pi}{T} \right) d\theta$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\,\frac{2\pi}{T} \right)$$



Recap

- Using the relation

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

we found

$$X_s(j\Omega) = X_d(e^{j\Omega T})$$

- And using the relation

$$x_s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

we found

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$$

Recap

- This can only mean

$$X_d(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\,\frac{2\pi}{T}\right)$$

or equivalently,

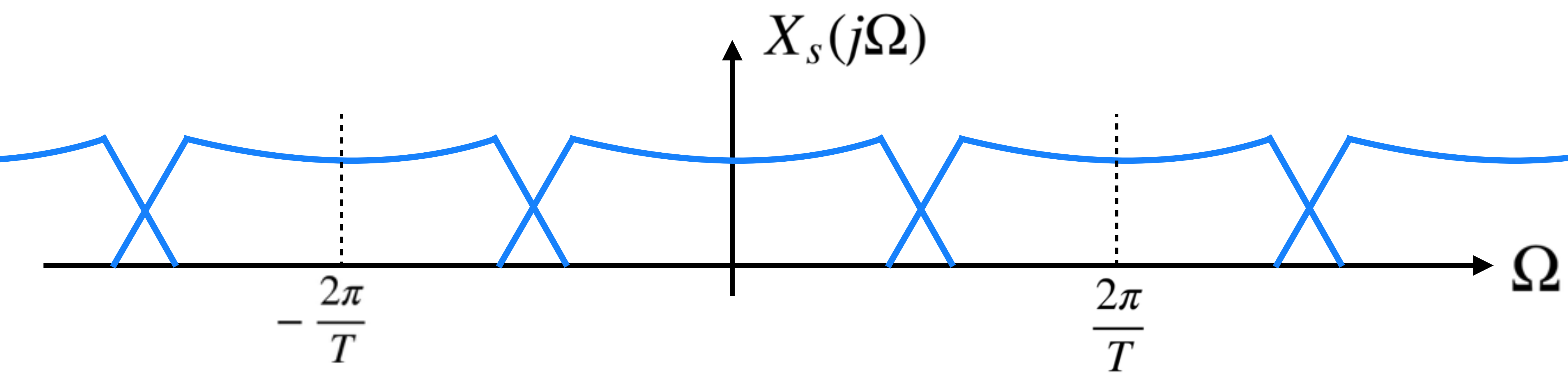
$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - k2\pi}{T}\right)$$

Choosing the sampling period

- Ideally, we prefer larger T (fewer samples/sec).
- But large T means small $\frac{2\pi}{T}$ in

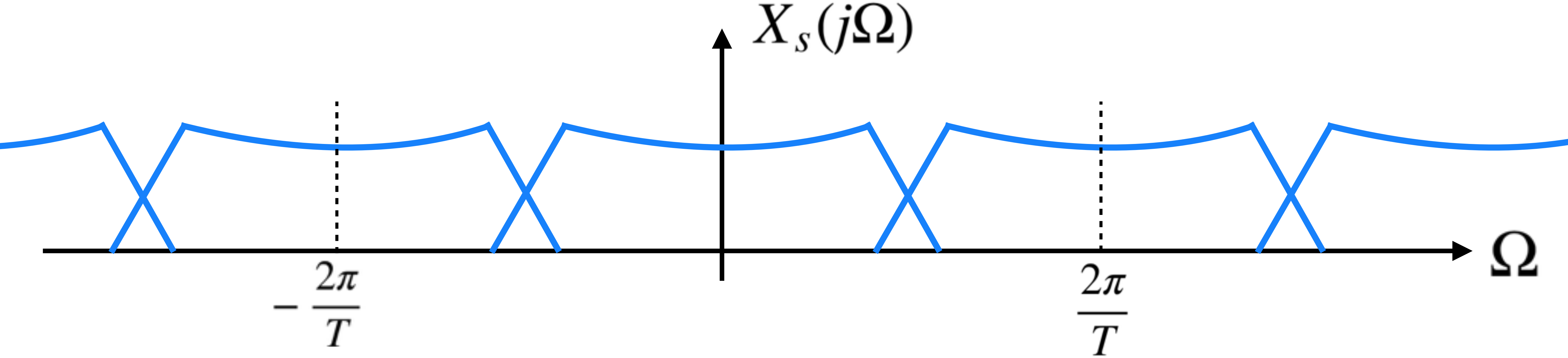
$$X_d(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$$

- That, in turn, creates a situation as below:



Choosing the sampling period

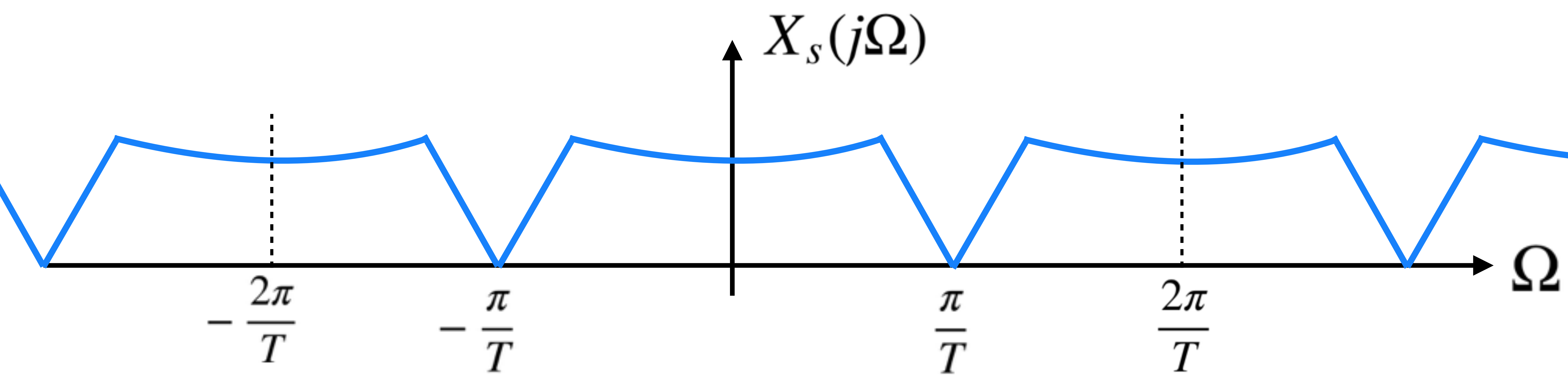
- Therefore, we want a T large, but not too large.



Choosing the sampling period

- Therefore, we want a T large, but not too large.
- To be precise, the largest T we can choose is given by

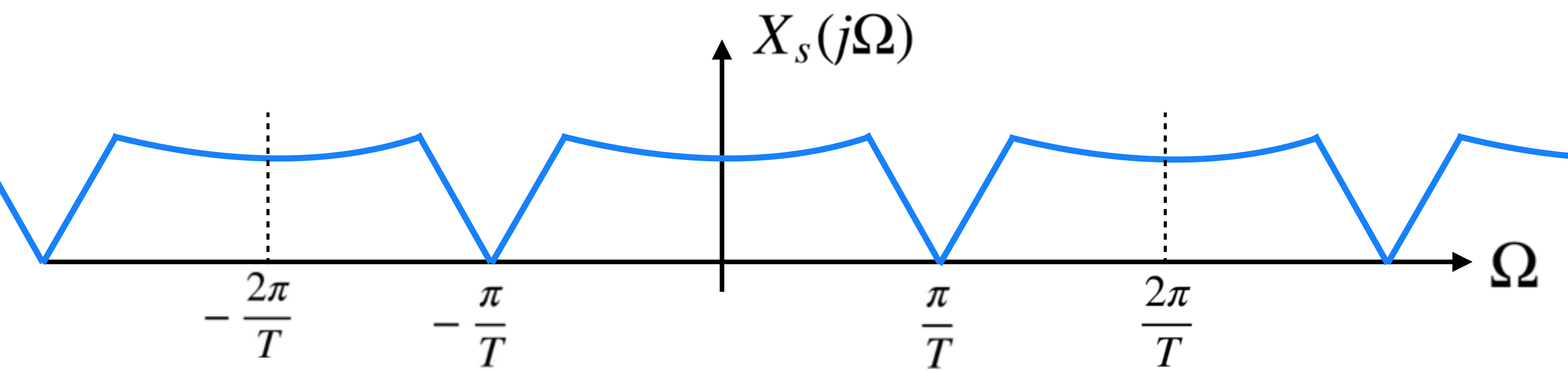
$$\frac{\pi}{T} = \Omega_M \text{ (bandwidth of } X_c(j\Omega)\text{)}$$



Choosing the sampling period

- Therefore, we want a T large, but not too large.
- To be precise, the largest T we can choose is given by

$$\cancel{\pi} f_S = \frac{\pi}{T} = \Omega_M = 2\cancel{\pi} f_M$$



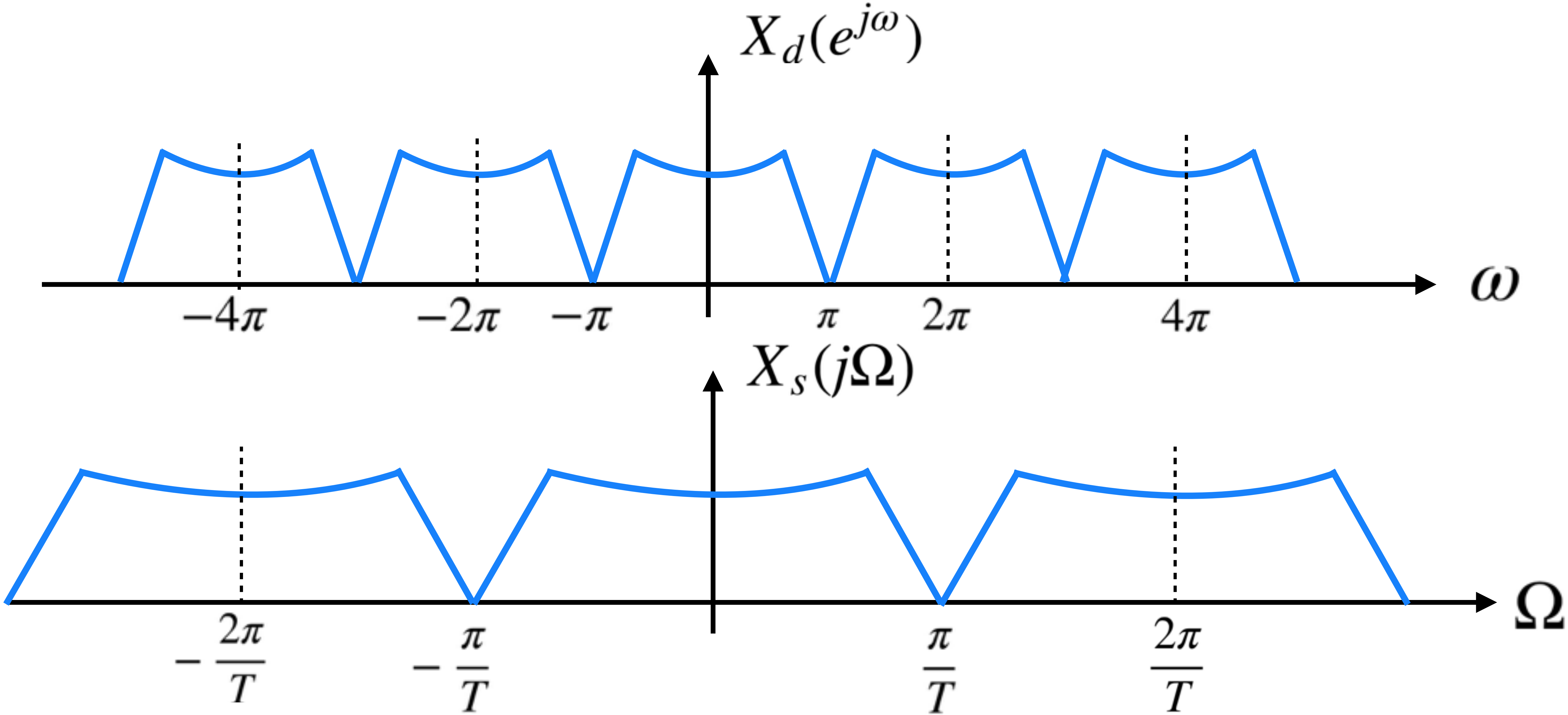
Choosing the sampling period

- This result is known as the Nyquist-Shannon sampling theorem.
- Applied in so many places:
  - Digital telephony ($f_S = 8\text{kHz}$)
    - The premise: 4kHz is close to the bandwidth for human speech
  - Audio CDs ($f_S = 44.1\text{kHz}$)
    - The premise: You can't hear higher than 22.05kHz
  - iPad retina display ($f_S = 264$ pixels/inch)
    - The premise: at the "comfortable viewing distance" of 18", you cannot detect a frequency of more than 132 oscillations per inch.

Reconstruction from samples

- How do we reconstruct $x(t)$ from $x[n]$?
- Reverse the sampling process:

$$x[n] \longrightarrow x_s(t) \longrightarrow x(t)$$



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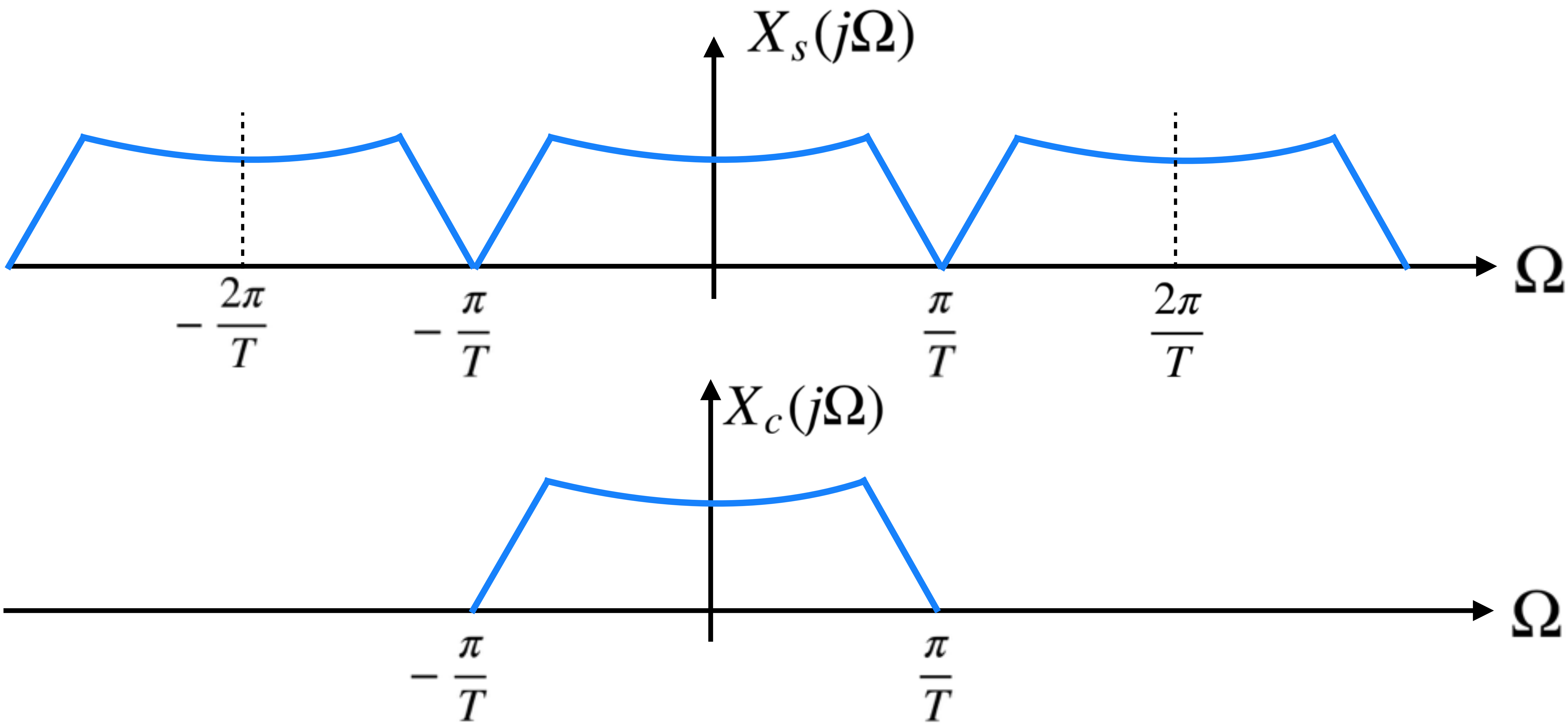
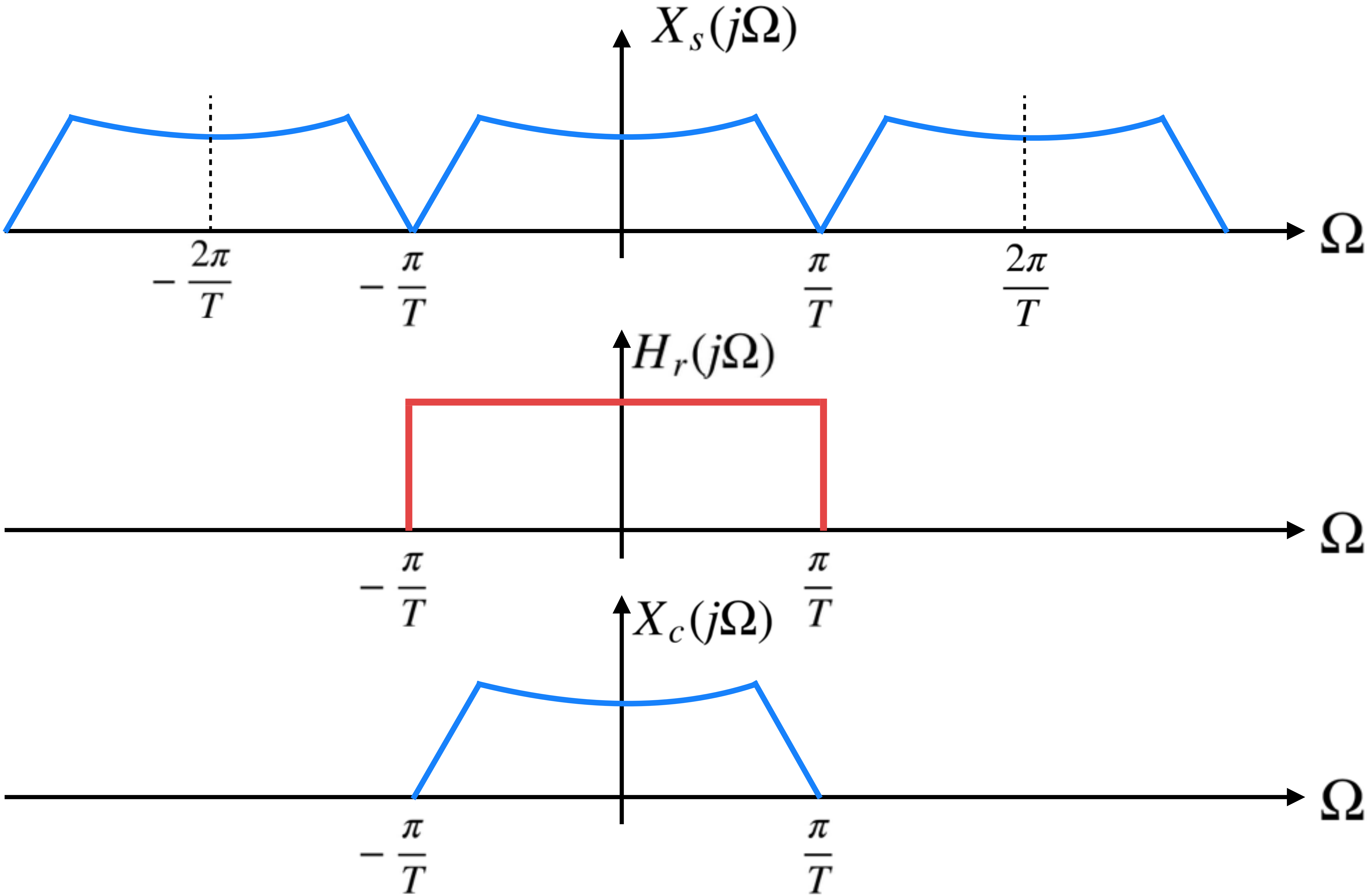


Reconstruction from samples

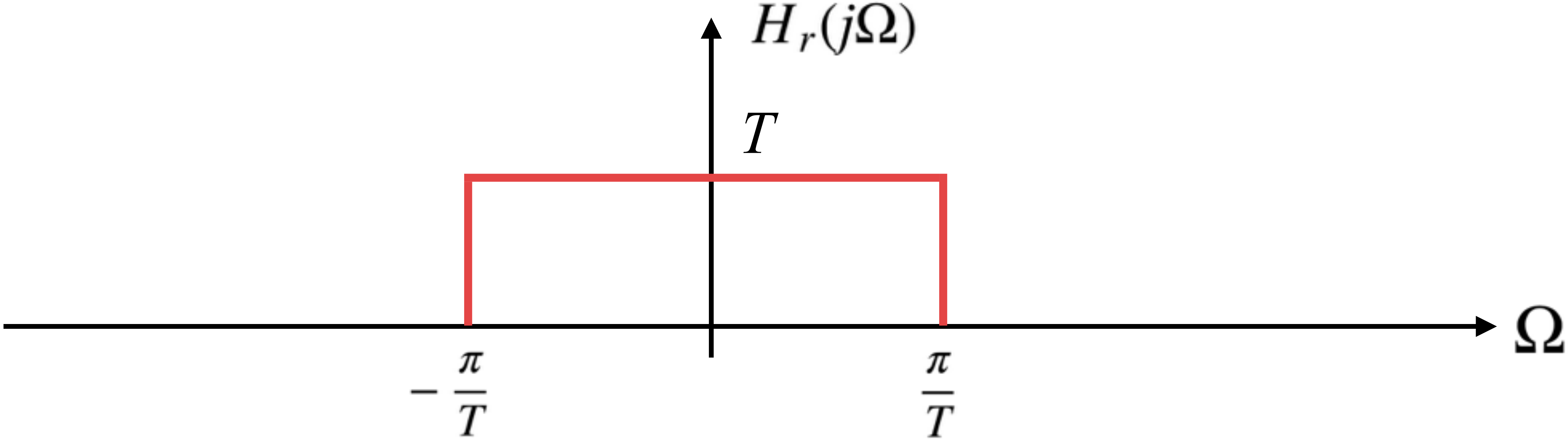


Reconstruction from samples

- A small detail: Since

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$$

the ideal reconstruction filter must have a magnitude of T.



Reconstruction from samples

- What does this low-pass filtering correspond to in the time domain?
- Convolution with the impulse response

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega \;= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} e^{j\Omega t}d\Omega$$

$$= \frac{T}{2\pi jt}\, e^{j\Omega t}\Big|_{-\pi/T}^{\pi/T} = \frac{T}{2\pi jt}\left(e^{j\frac{\pi t}{T}} - e^{-j\frac{\pi t}{T}}\right)$$

$$= \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} = \mathrm{sinc}\left(\frac{\pi t}{T}\right)$$

Reconstruction from samples

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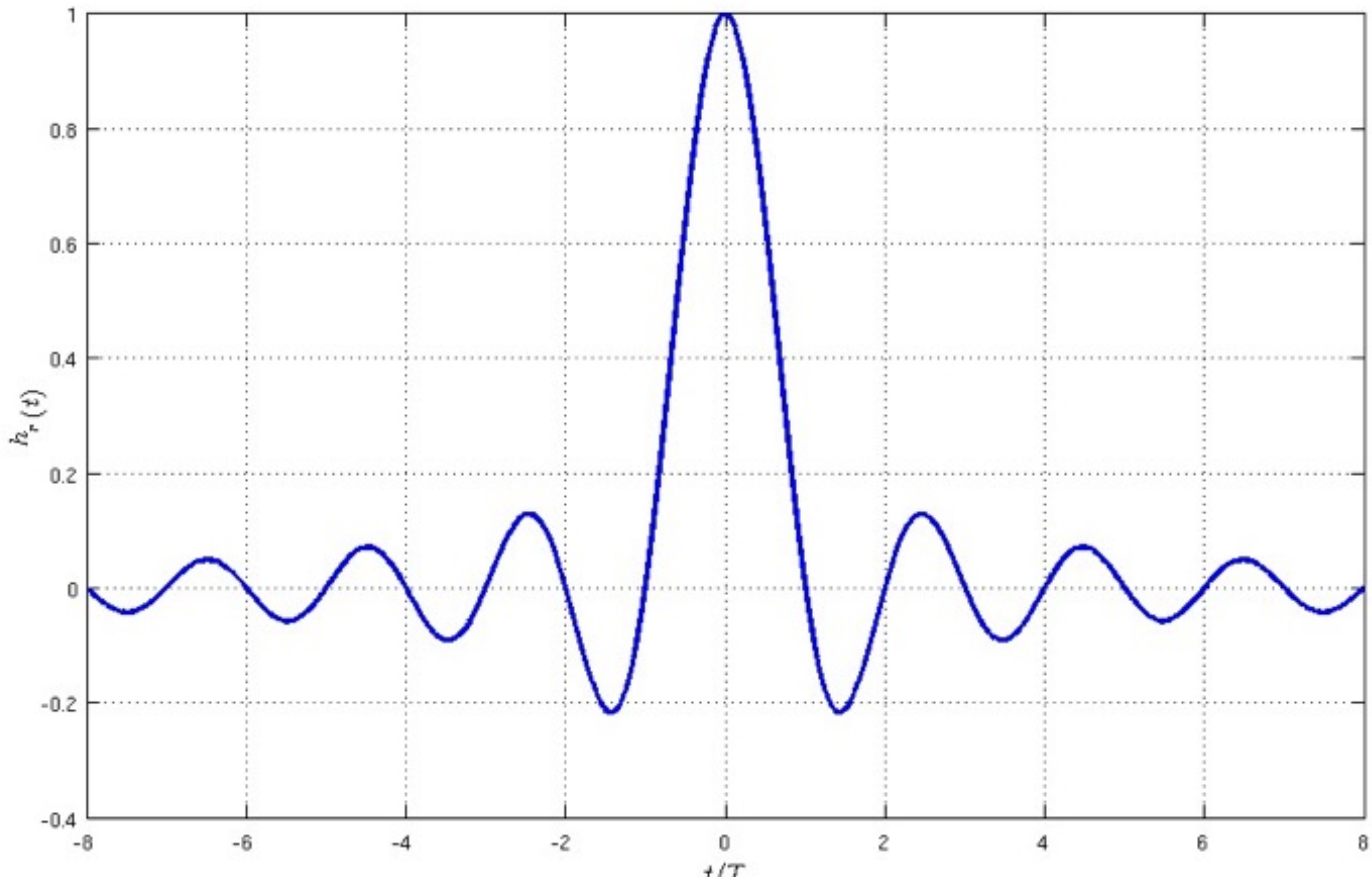


Reconstruction from samples

- Since

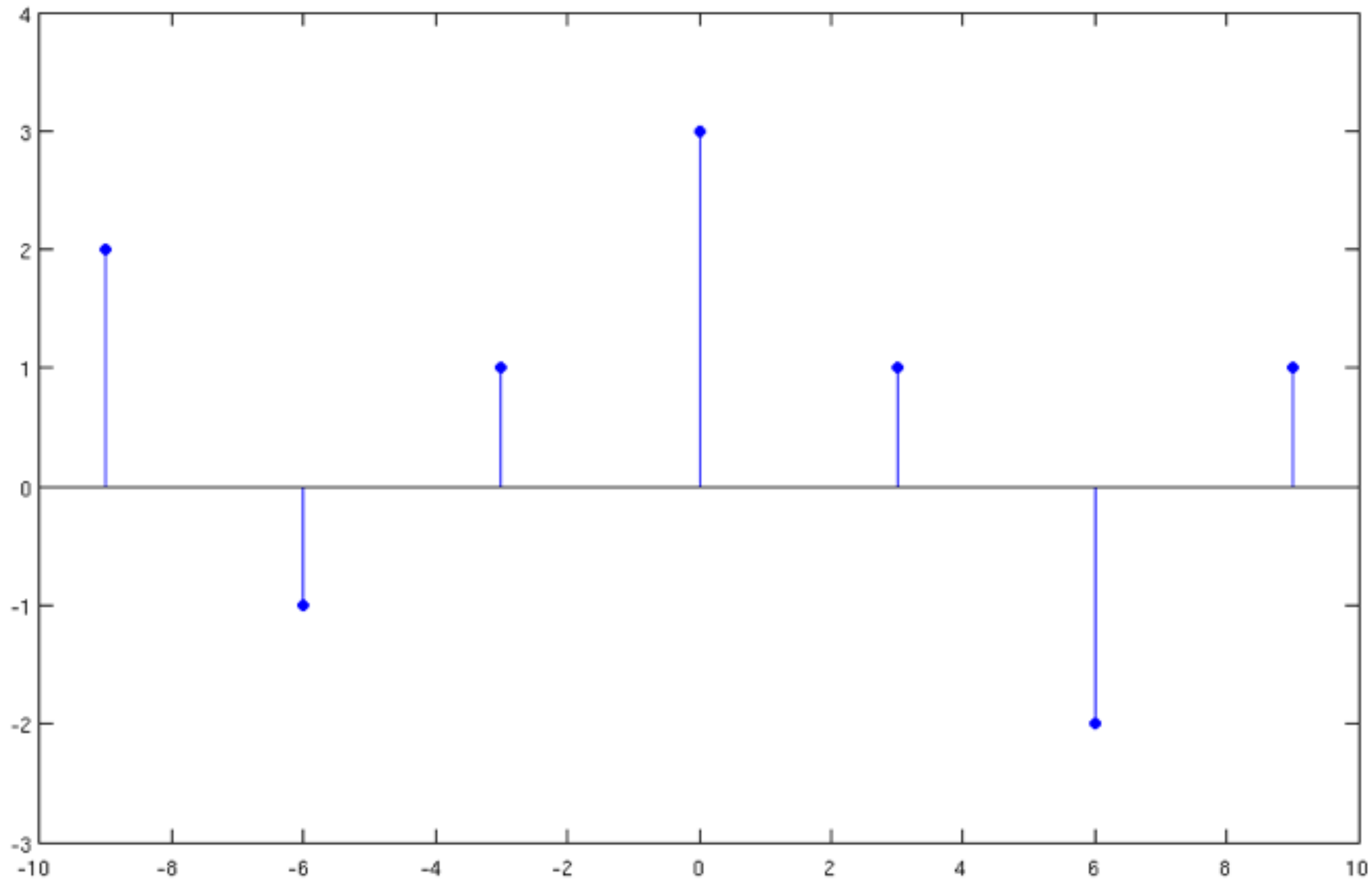
$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

we have

$$\begin{aligned} x_c(t) &= x_s(t) \star h_r(t) \\ &= \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}\left(\frac{\pi(t - nT)}{T} \right) \end{aligned}$$

Reconstruction from samples

- Pictorially,



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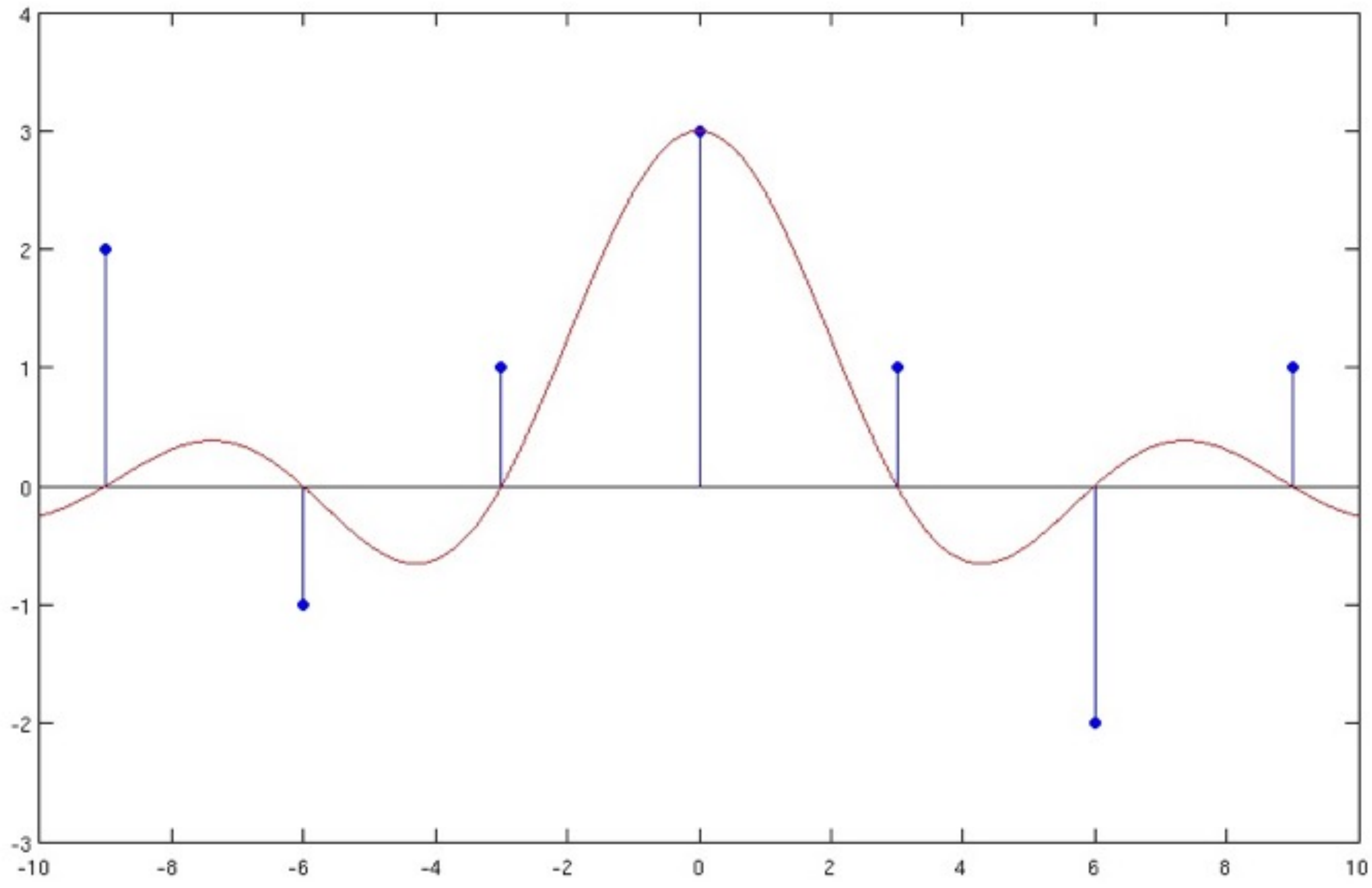

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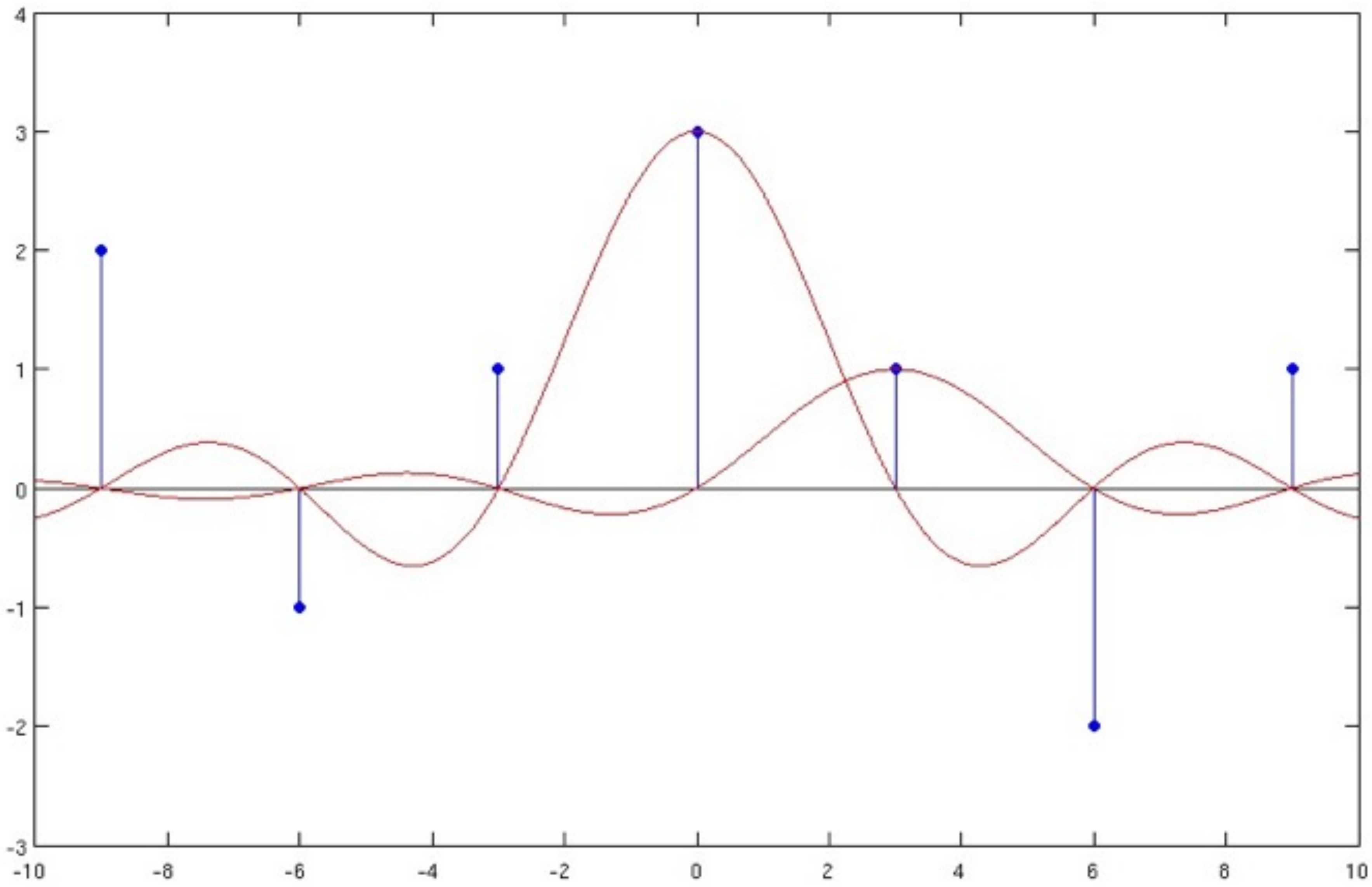

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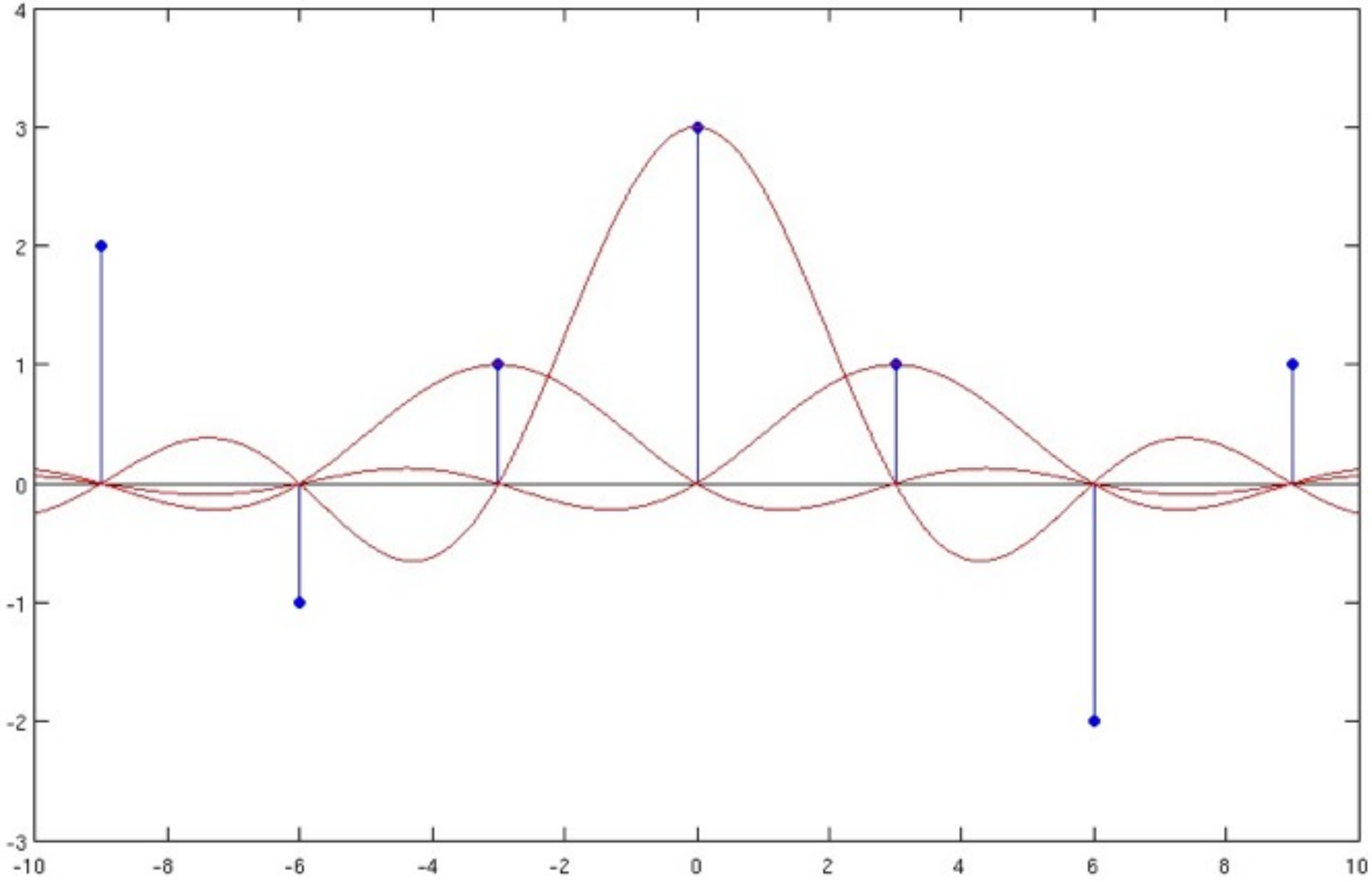

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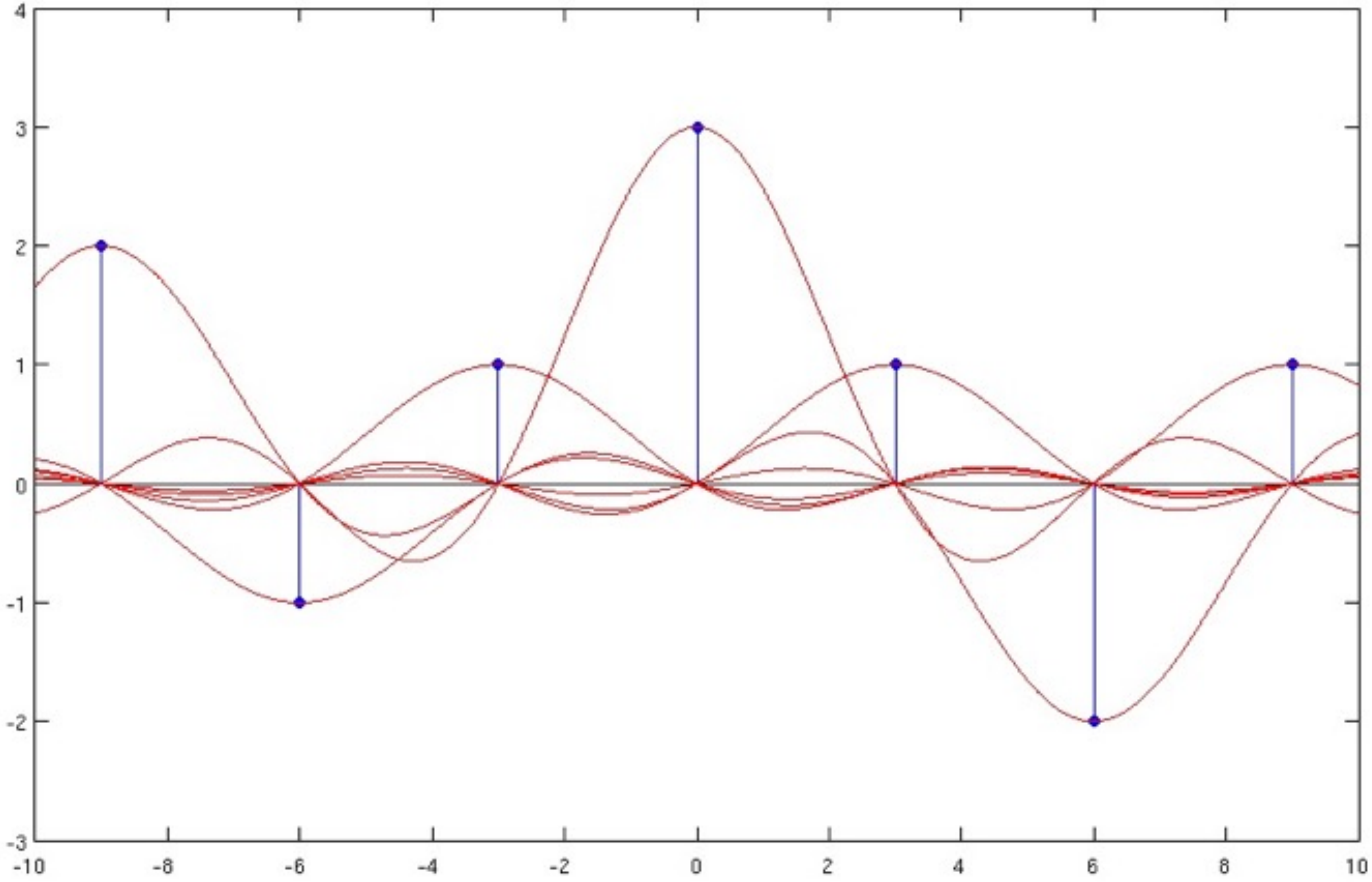

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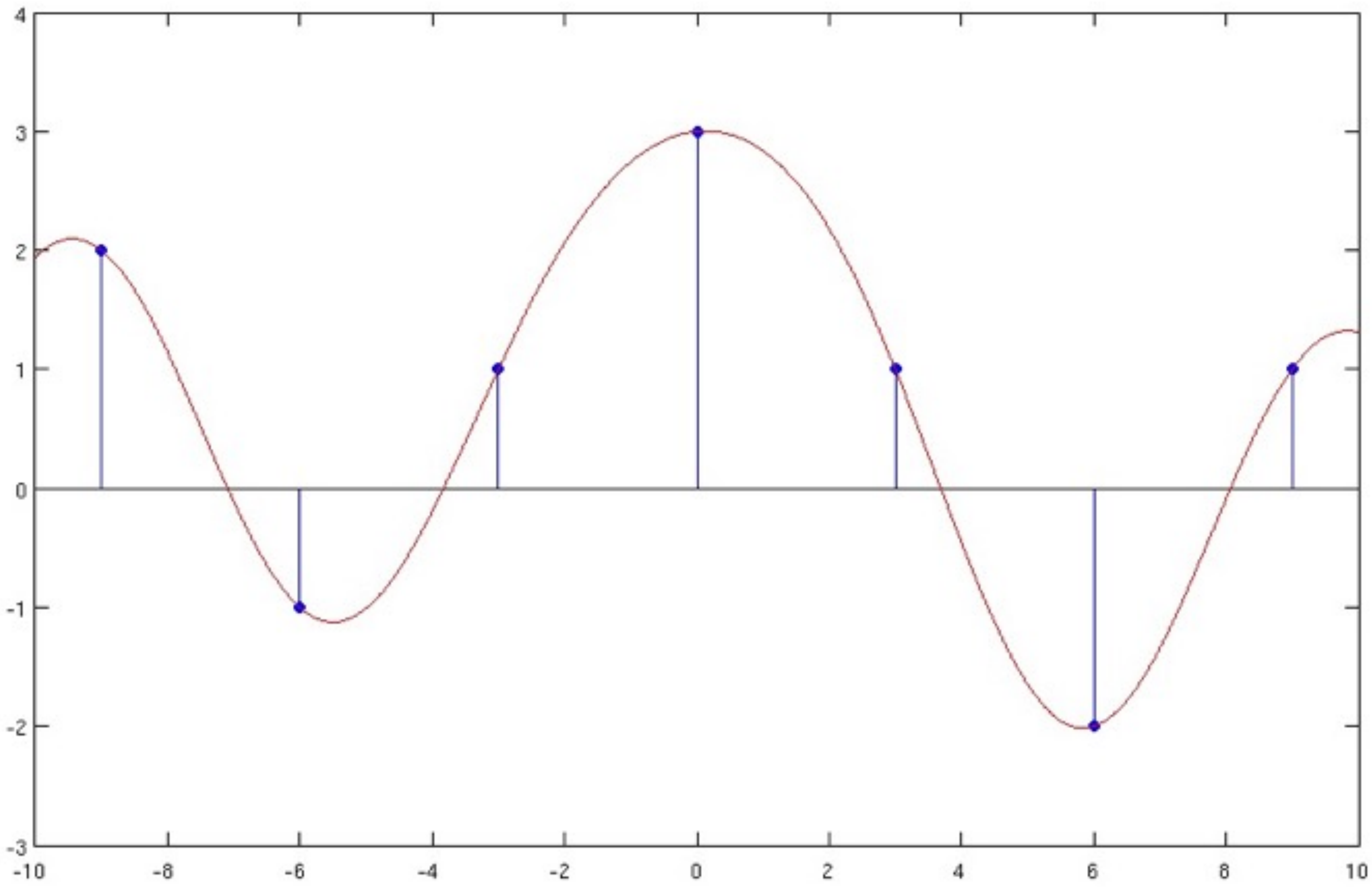
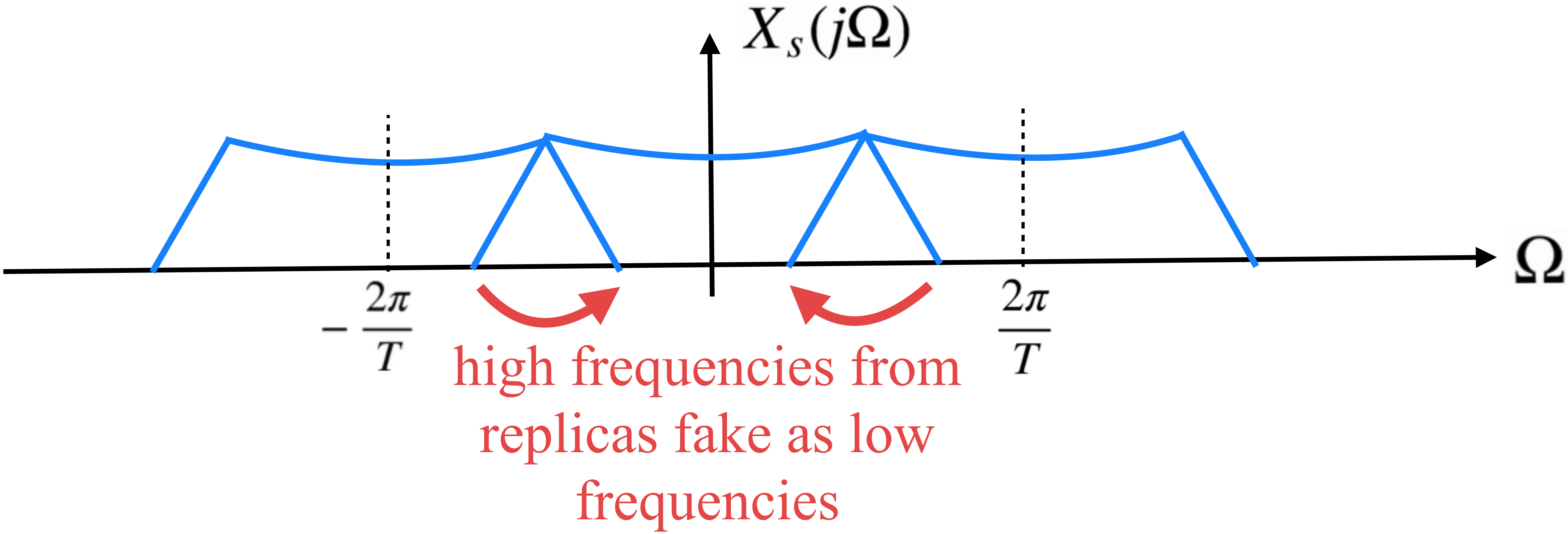


Aliasing

- What happens if we undersample?

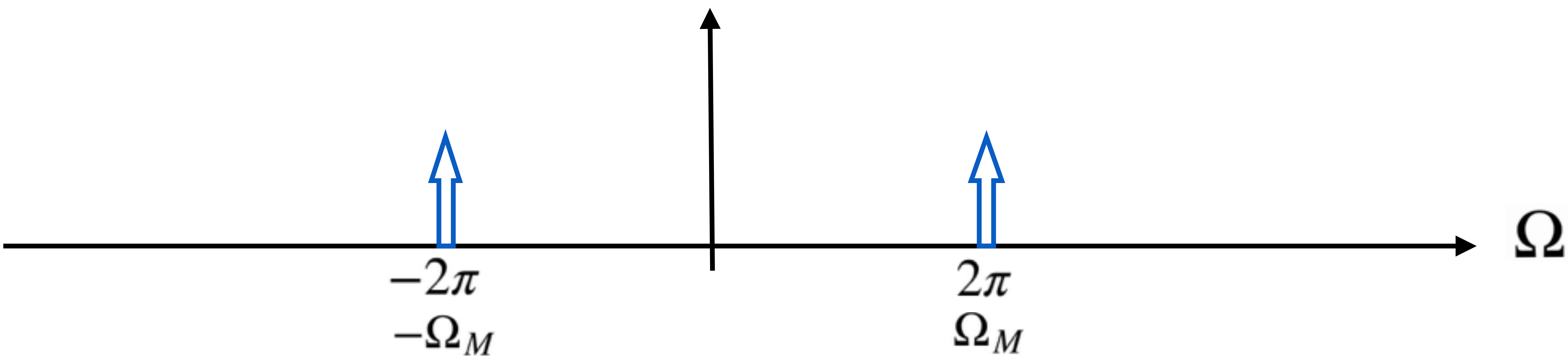


Aliasing

- A more dramatic example:

$$x(t) = \cos(2\pi t)$$

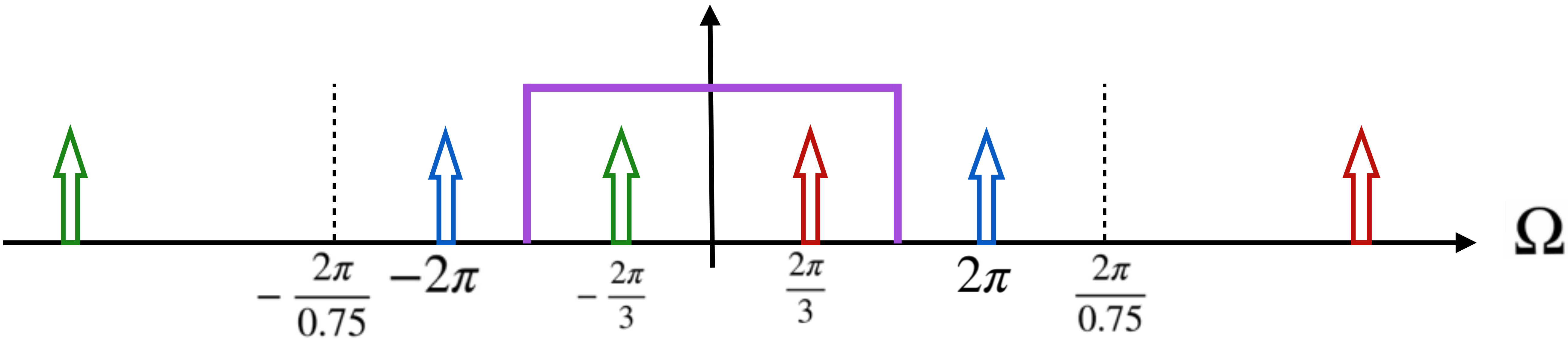
$$X_c(j\Omega) = \pi\delta(\Omega - 2\pi) + \pi\delta(\Omega + 2\pi)$$



- We need $\dfrac{\pi}{T} \geq \Omega_M$ to have perfect reconstruction
- So, the maximum allowed T is 0.5
- What if we choose $T = 0.75$?

$$x(t) = \cos(2\pi t)$$

$$X_c(j\Omega) = \pi\delta(\Omega - 2\pi) + \pi\delta(\Omega + 2\pi)$$



$$x_r(t) = \cos\left(\frac{2\pi}{3}t\right)$$

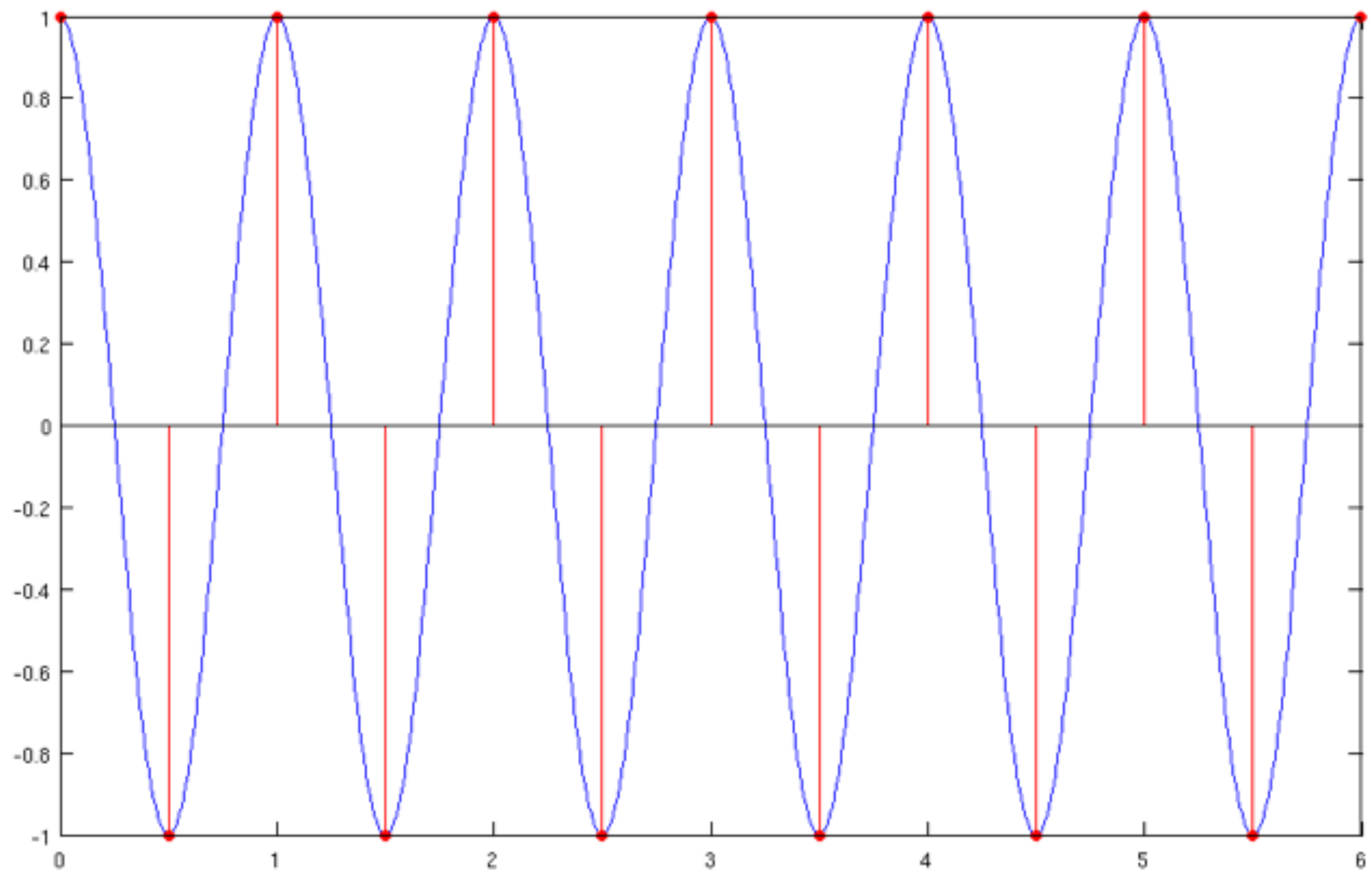
- The original frequency 2π is faking as frequency $2\pi/3$!

Aliasing

- That's what's happening in the frequency domain.
- What about the time domain?

$$T = 0.5$$

$$f_s = 2\text{Hz}$$

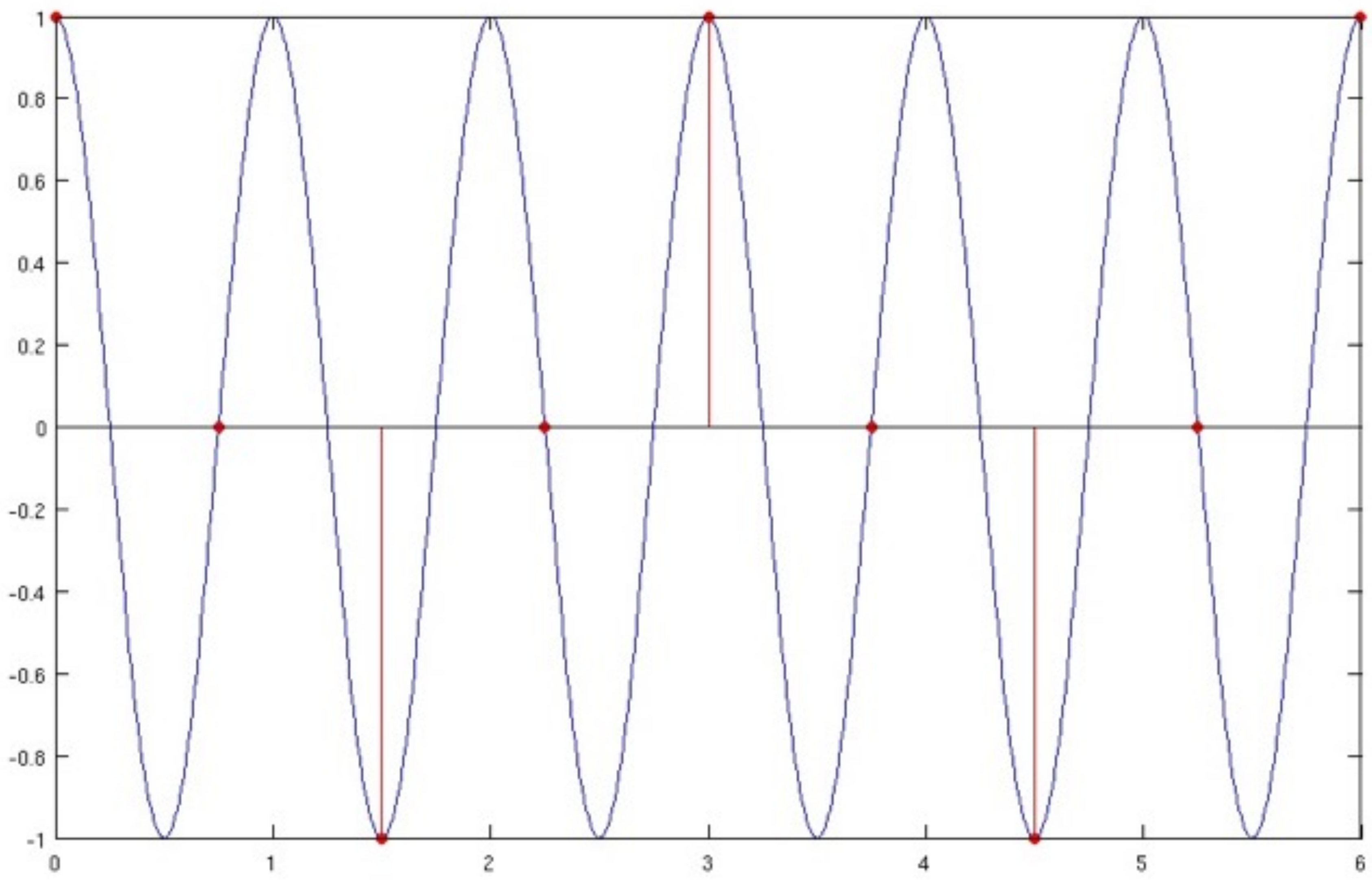


Aliasing

- That's what's happening in the frequency domain.
- What about the time domain?

$$T = 0.75$$

$$f_s \approx 1.33\text{Hz}$$



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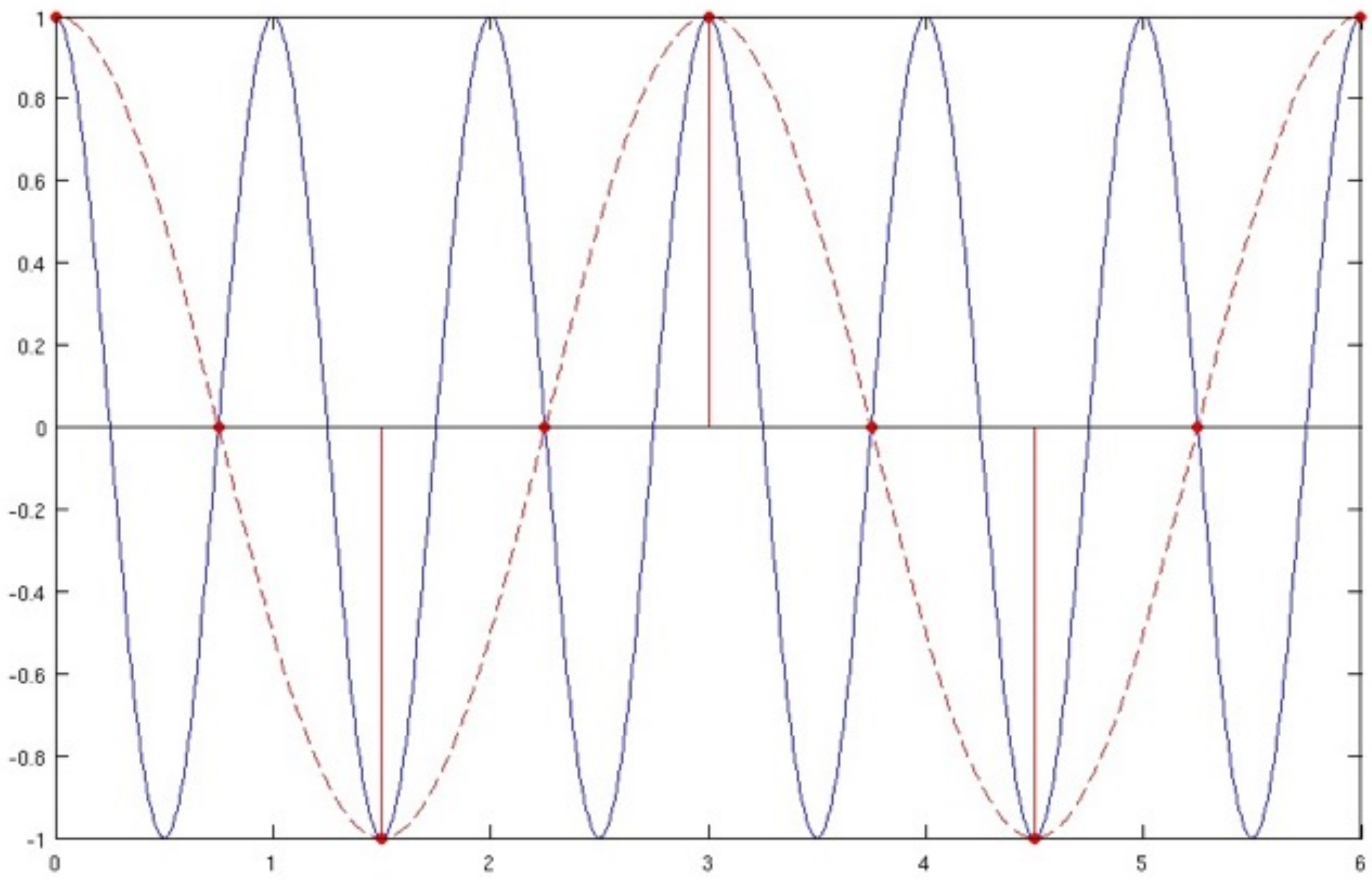
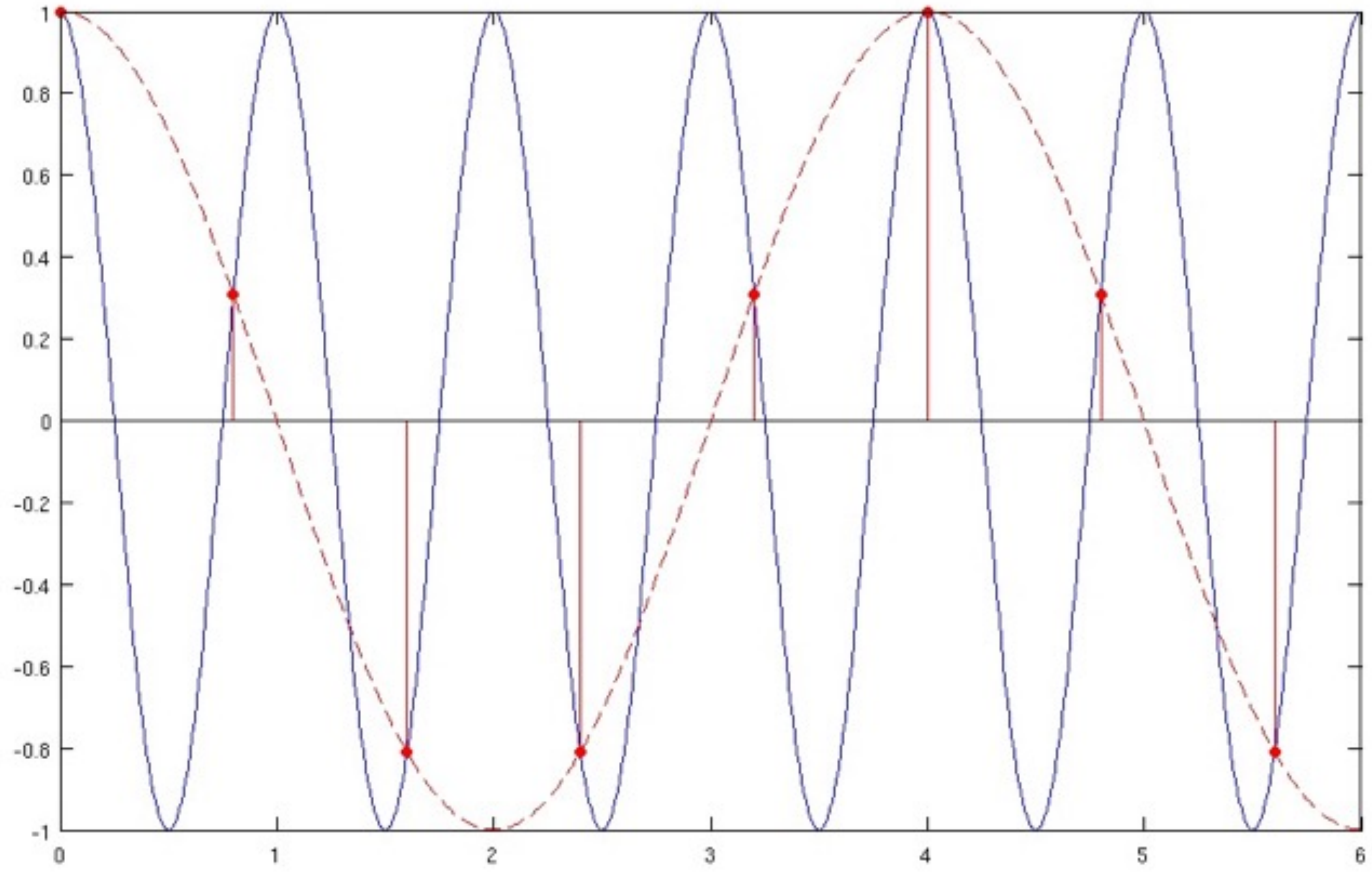


Aliasing

- That's what's happening in the frequency domain.
- What about the time domain?

$$T = 0.8$$

$$f_s \approx 1.25\text{Hz}$$



Let's hear aliasing

- Take a cosine wave with frequency 3,520Hz.
- Minimum (Nyquist) sampling rate: 7,040Hz.
- With a sampling rate of 20,000Hz:
  - Reconstruction has the correct frequency of 3,520 Hz.
- With a sampling rate of 5,000Hz:
  - Reconstruction poses as 5,000-3,520 = 1,480Hz.
- With a sampling rate of 4,000Hz:
  - Reconstruction poses as 4,000-3,520 = 480Hz.
- With a sampling rate of 3,800Hz:
  - Reconstruction poses as 3,800-3,520 = 280Hz.

Let's hear aliasing

- Let's hear that Beatles song again.
- Sampling rate = 22,050 Hz
- Sampling rate = 11,025 Hz
- Sampling rate = 5,512 Hz
- Sampling rate = 2,756 Hz
- Sampling rate = 1,378 Hz

Let's see aliasing

- In 2-D signal processing, we can use the same Fourier analysis.
- This time, we decompose onto sinusoidals with frequency and "direction":

Let's see aliasing

Original

Half the sampling rate

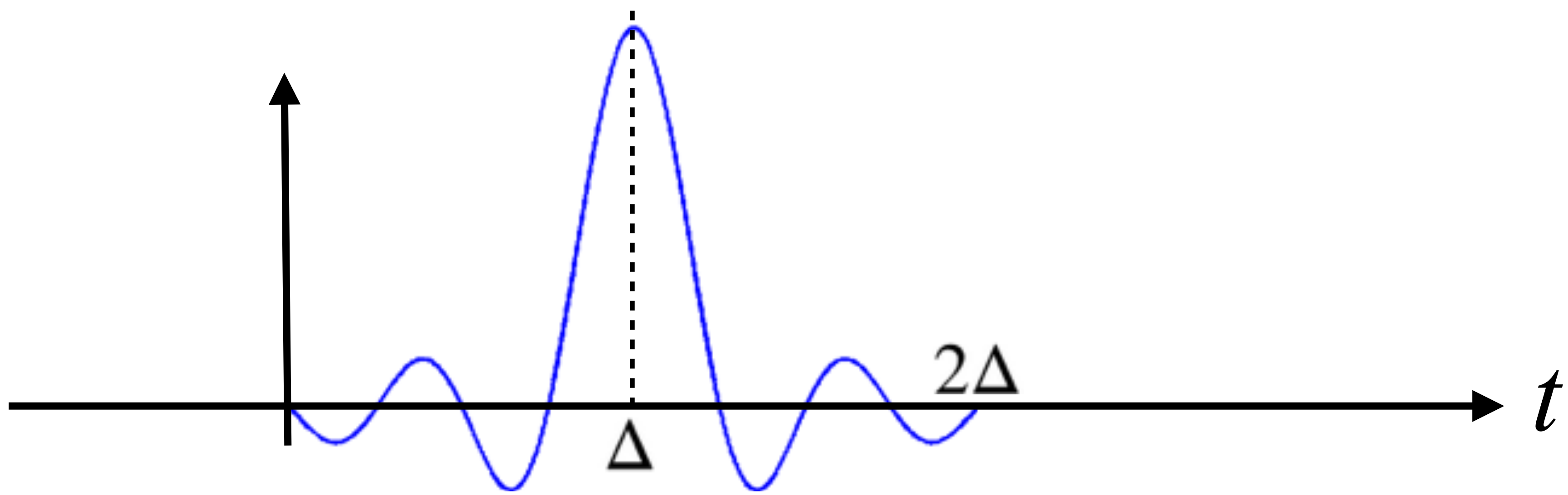
Quarter the sampling rate

Practical considerations

- Non-causality of the sinc function
  - Other "interpolators" and their effect on reconstruction
- Not all signals are bandlimited
  - Anti-aliasing pre-filters
- Difficulty of implementation of the ideal impulse train during reconstruction..
  - Impulses should be replaced with narrow pulses.
- The sample values need to be quantized.

Practical interpolation

- How about a truncated and shifted sinc?



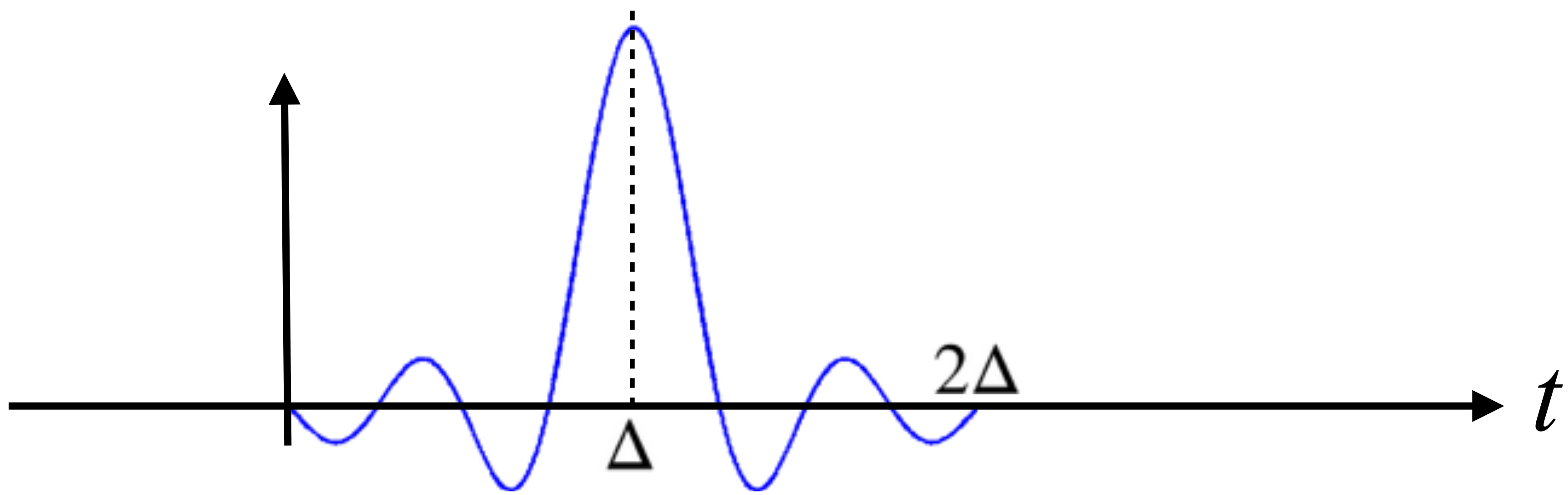
- Causes delay, but not a huge problem.

$$x_s(t) \star \mathrm{sinc}\left(\frac{\pi(t-\Delta)}{T}\right) = x(t-\Delta)$$

  - Example: digital telephony (8kHz) has a sampling period of $T = 0.125$ms. A delay of several T is practically not noticeable

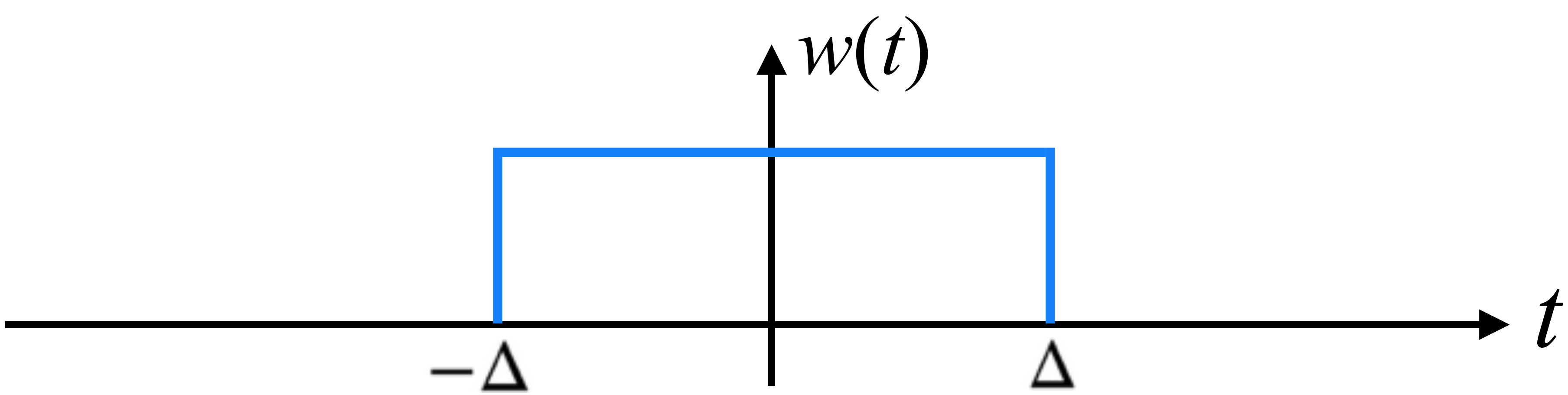
Practical interpolation

- How about a truncated and shifted sinc?

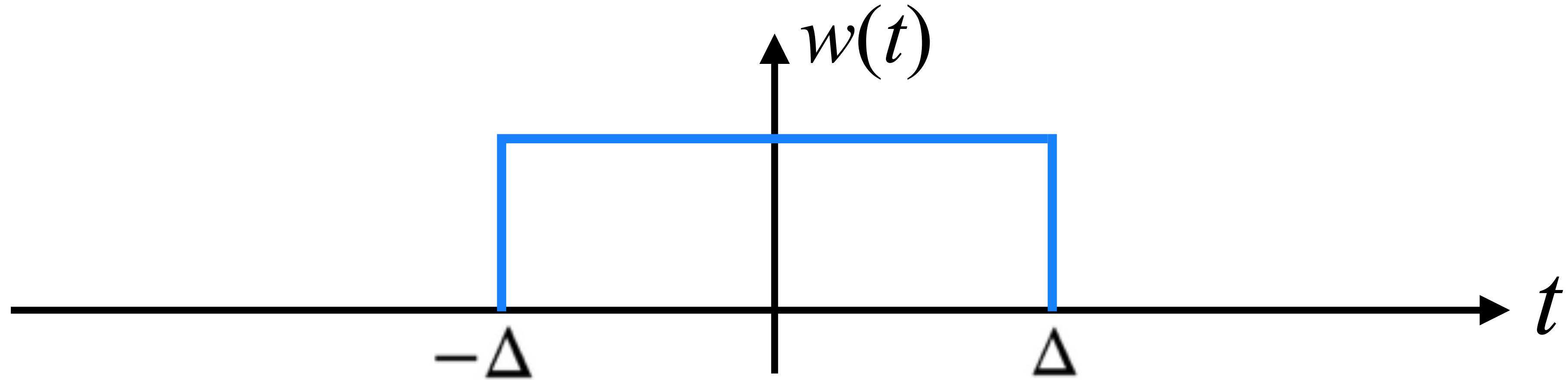


- Truncation is the bigger problem:

$$x_r(t) = x_s(t) \star \left[w(t)\,\mathrm{sinc}\left(\frac{\pi t}{T} \right) \right]$$



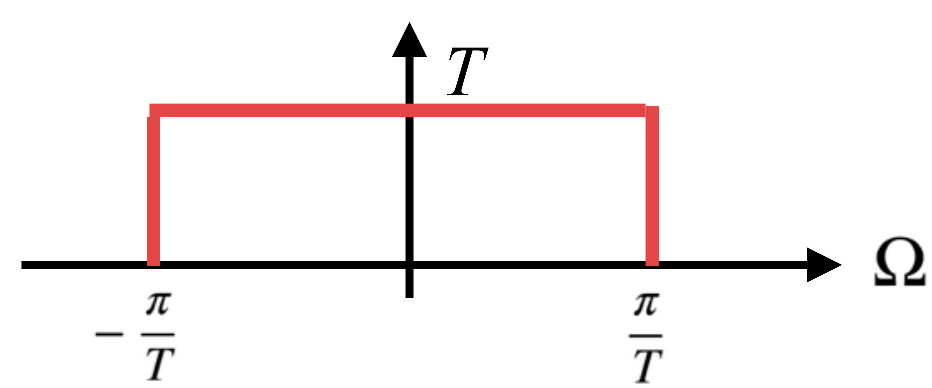
$$x_r(t) = x_s(t) \star \left[w(t)\,\text{sinc}\left(\frac{\pi t}{T} \right) \right]$$



$$W(j\Omega) = \int_{-\Delta}^{\Delta} e^{-j\Omega t} dt \quad = \frac{e^{j\Omega\Delta} - e^{-j\Omega\Delta}}{j\Omega} = \frac{2\sin(\Omega\Delta)}{\Omega}$$

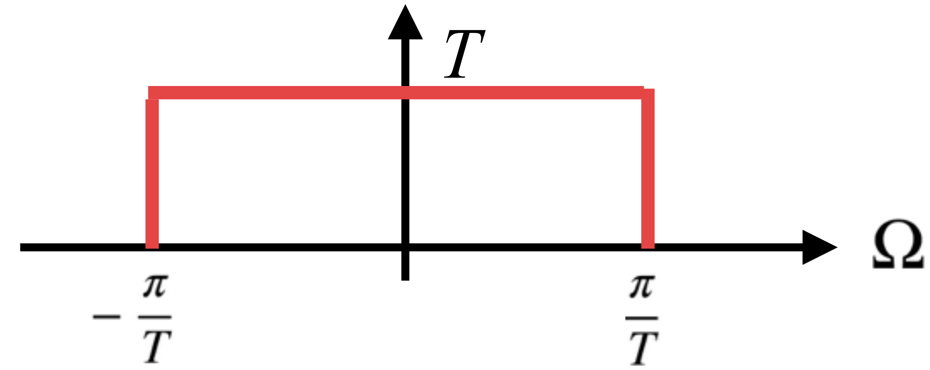
- Therefore,

$$w(t)\,\text{sinc}\left(\frac{\pi t}{T} \right) \xrightarrow{CTFT} \frac{1}{2\pi}\frac{2\sin(\Omega\Delta)}{\Omega} \star$$

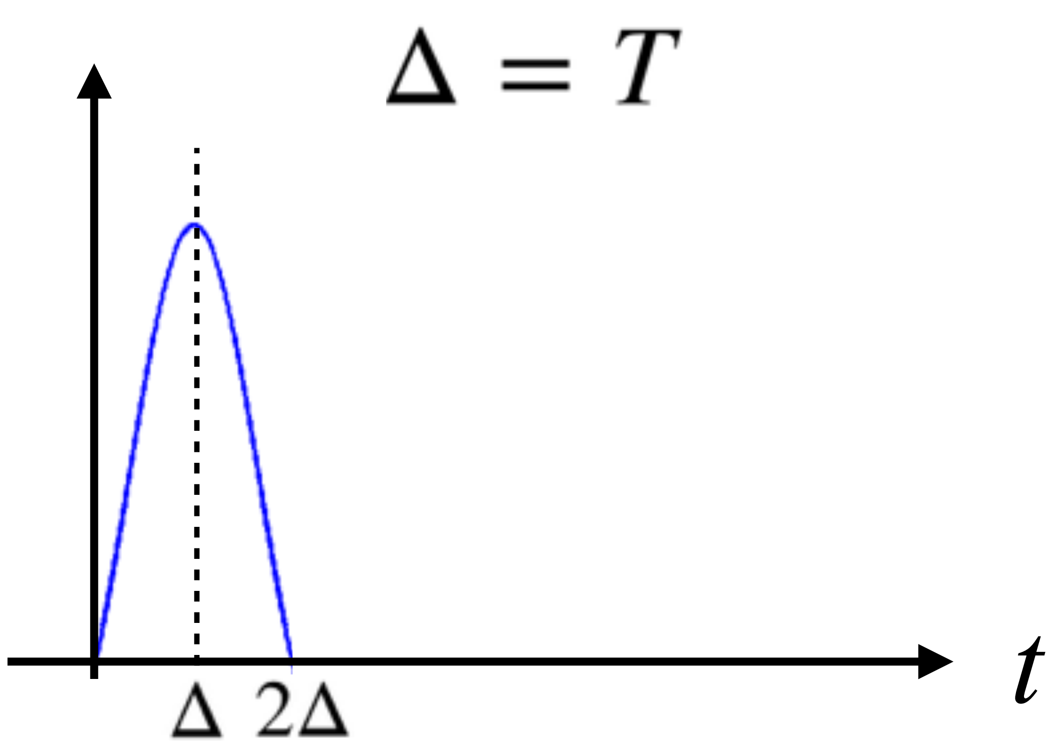


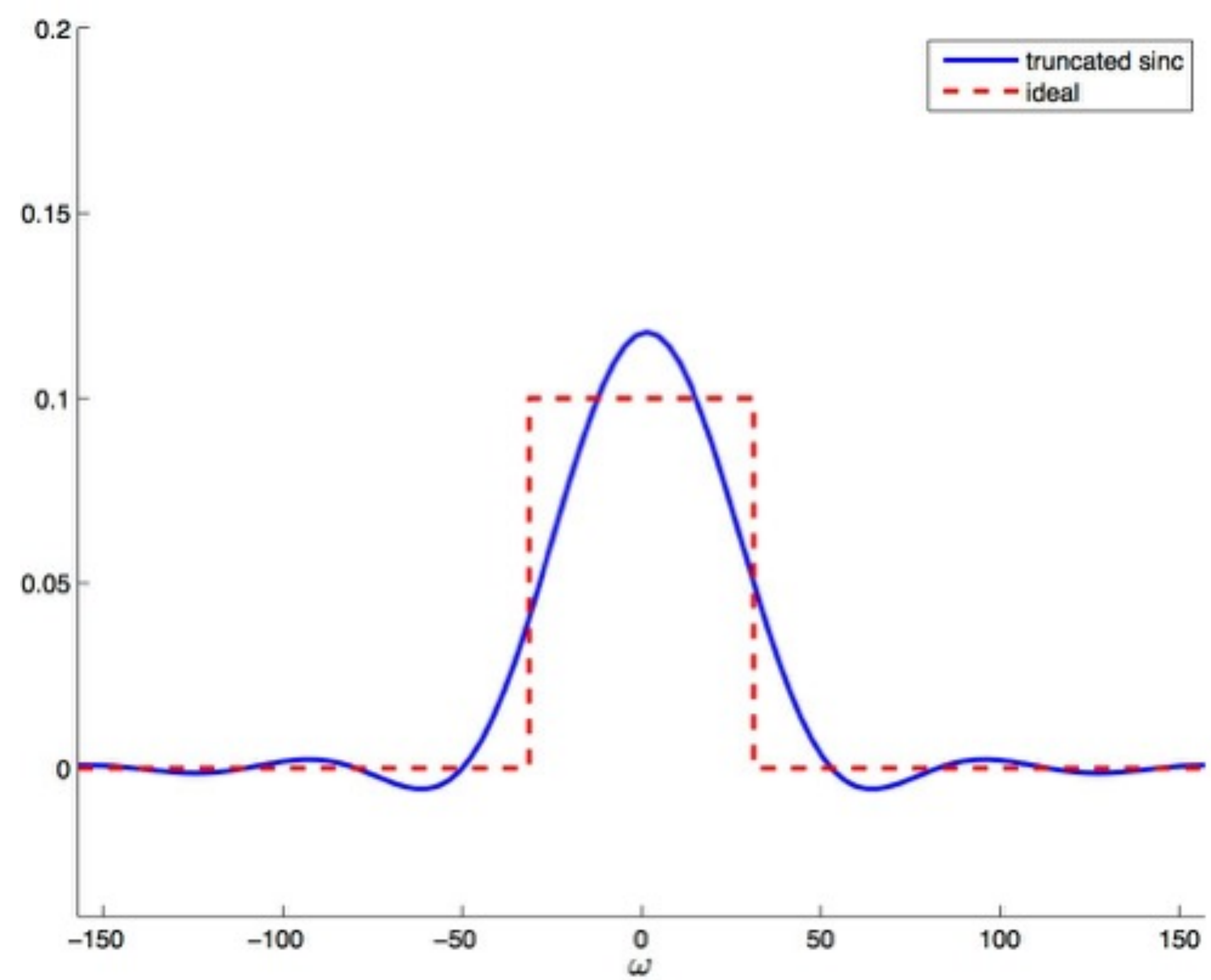
- Will this ruin the lowpass nature of the filter?

$$w(t)\,\mathrm{sinc}\left(\frac{\pi t}{T}\right) \xrightarrow{CTFT} \frac{1}{2\pi}\,\frac{2\sin(\Omega\Delta)}{\Omega} \star$$

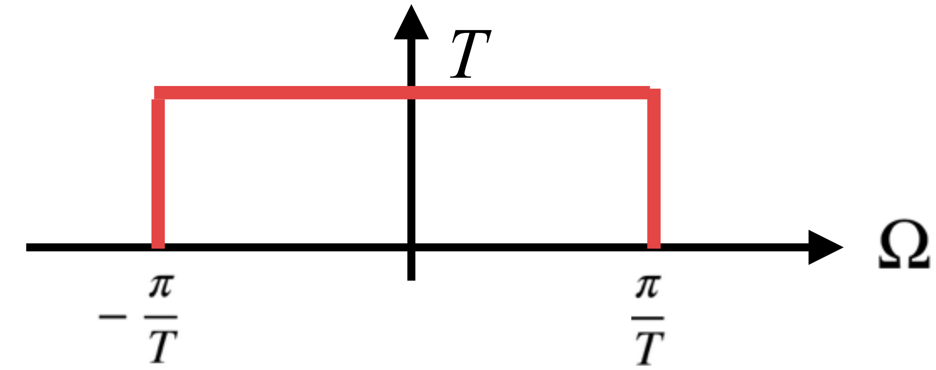


- Will this ruin the lowpass nature of the filter?
- Hardly.





$$w(t)\,\mathrm{sinc}\left(\frac{\pi t}{T}\right) \xrightarrow{CTFT} \frac{1}{2\pi}\,\frac{2\sin(\Omega\Delta)}{\Omega} \star$$



- Will this ruin the lowpass nature of the filter?
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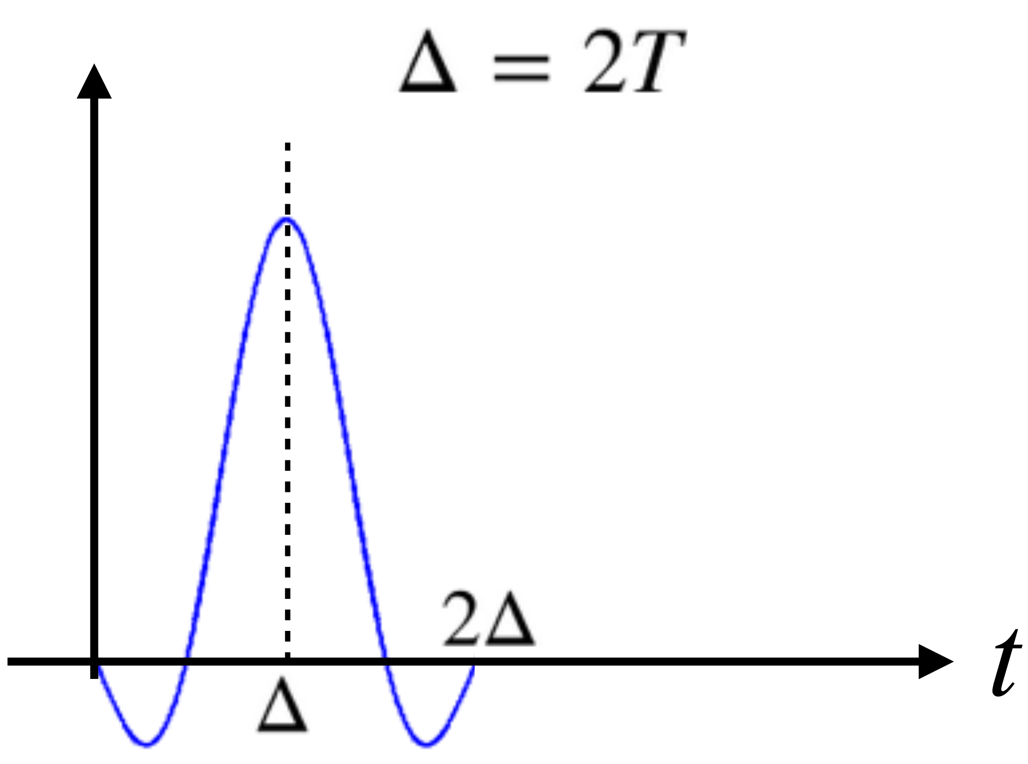
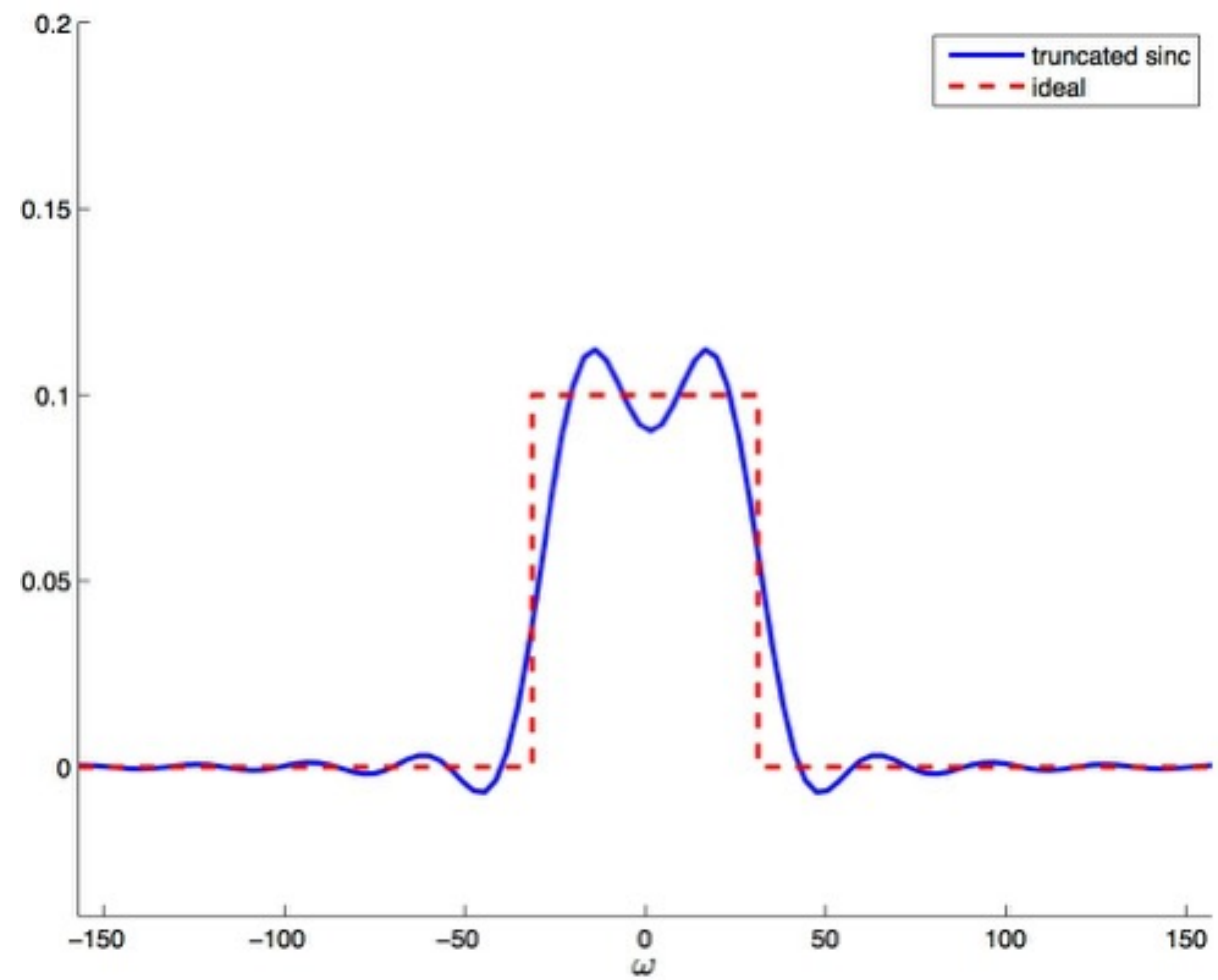
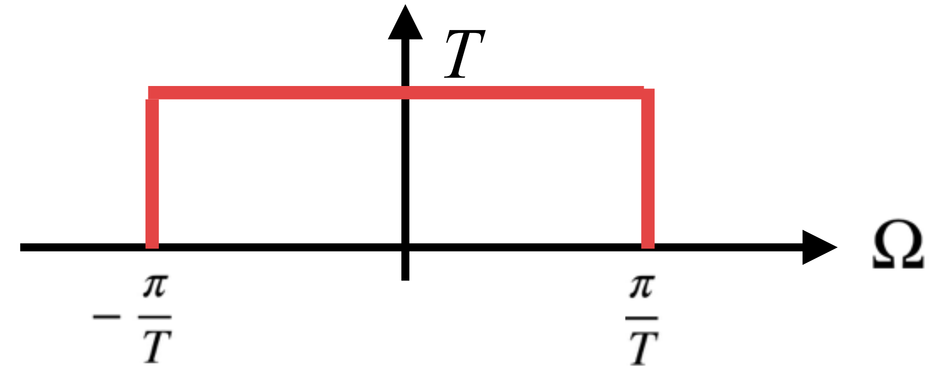
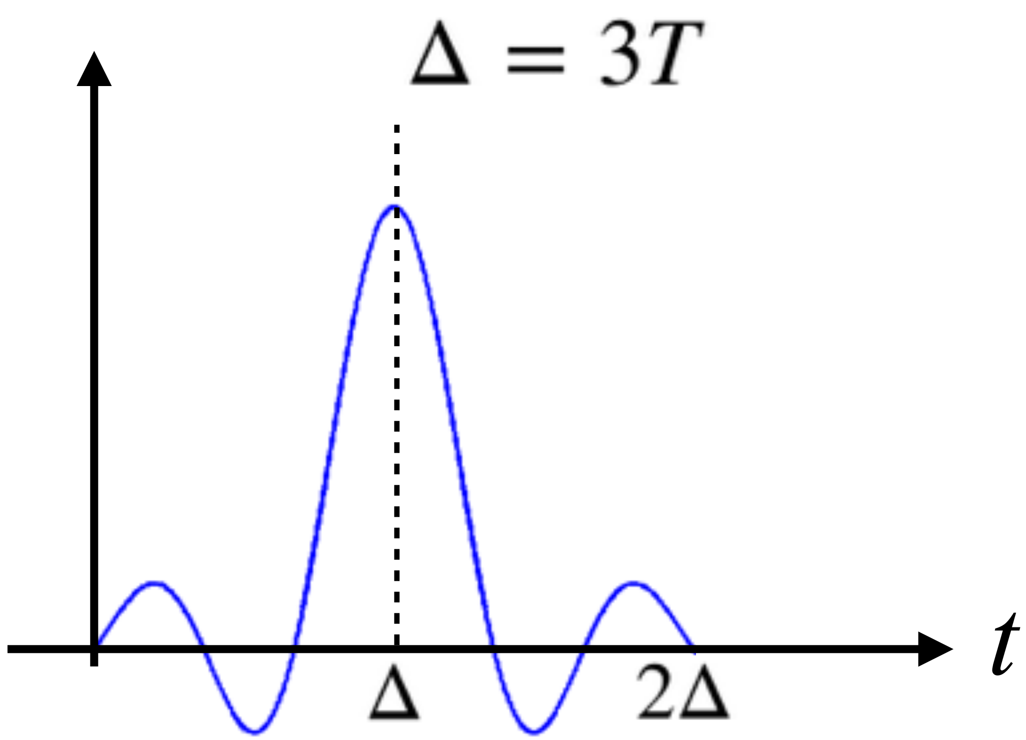


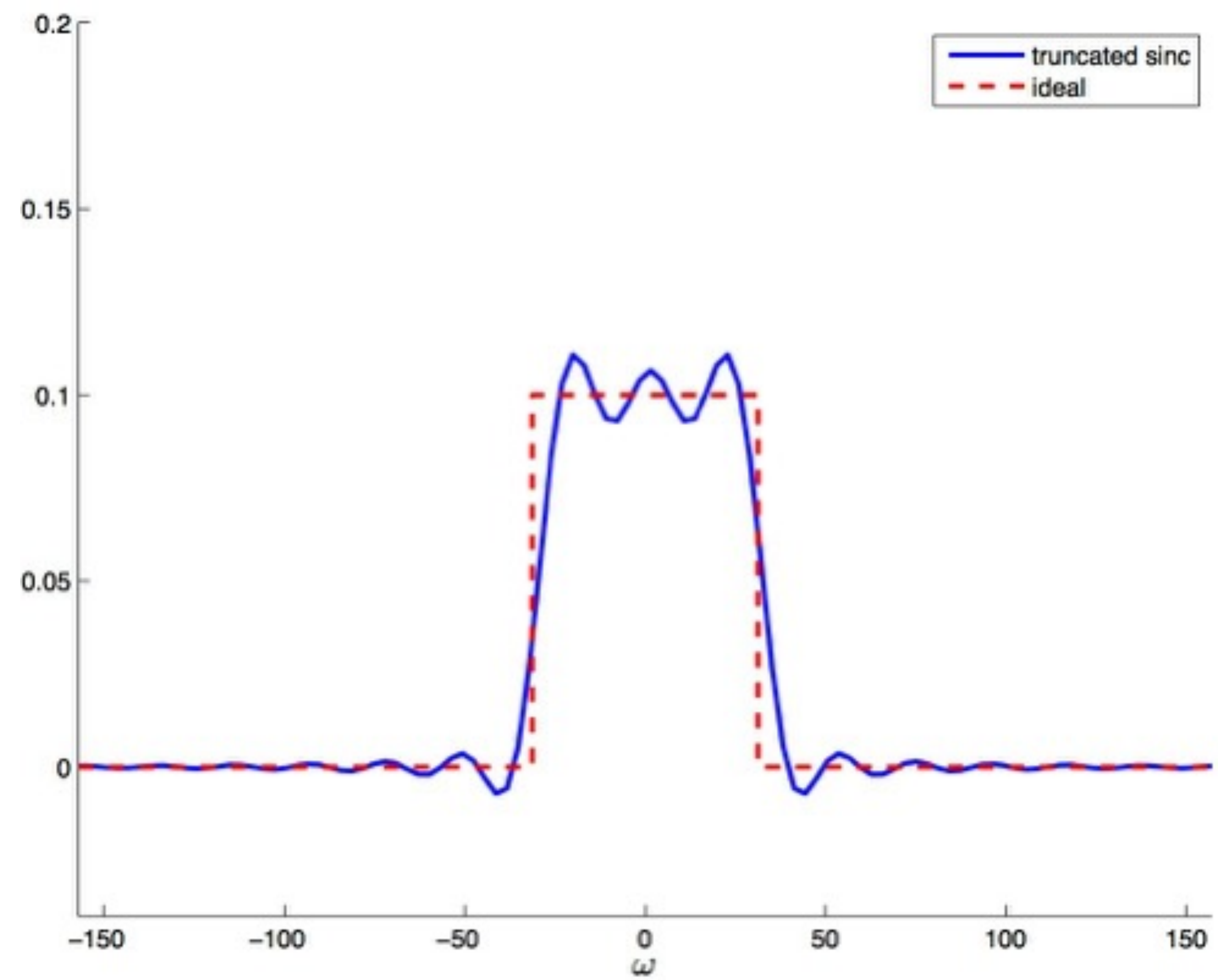


$$w(t)\,\mathrm{sinc}\left(\frac{\pi t}{T}\right) \xrightarrow{CTFT} \frac{1}{2\pi}\,\frac{2\sin(\Omega\Delta)}{\Omega} \star$$

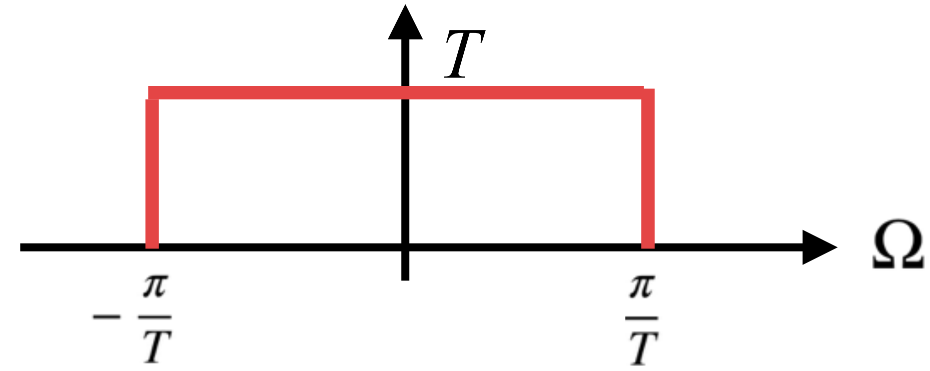


- Will this ruin the lowpass nature of the filter?
- Hardly.





$$w(t)\,\mathrm{sinc}\left(\frac{\pi t}{T}\right) \xrightarrow{CTFT} \frac{1}{2\pi}\frac{2\sin(\Omega\Delta)}{\Omega} \star$$



- Will this ruin the lowpass nature of the filter?
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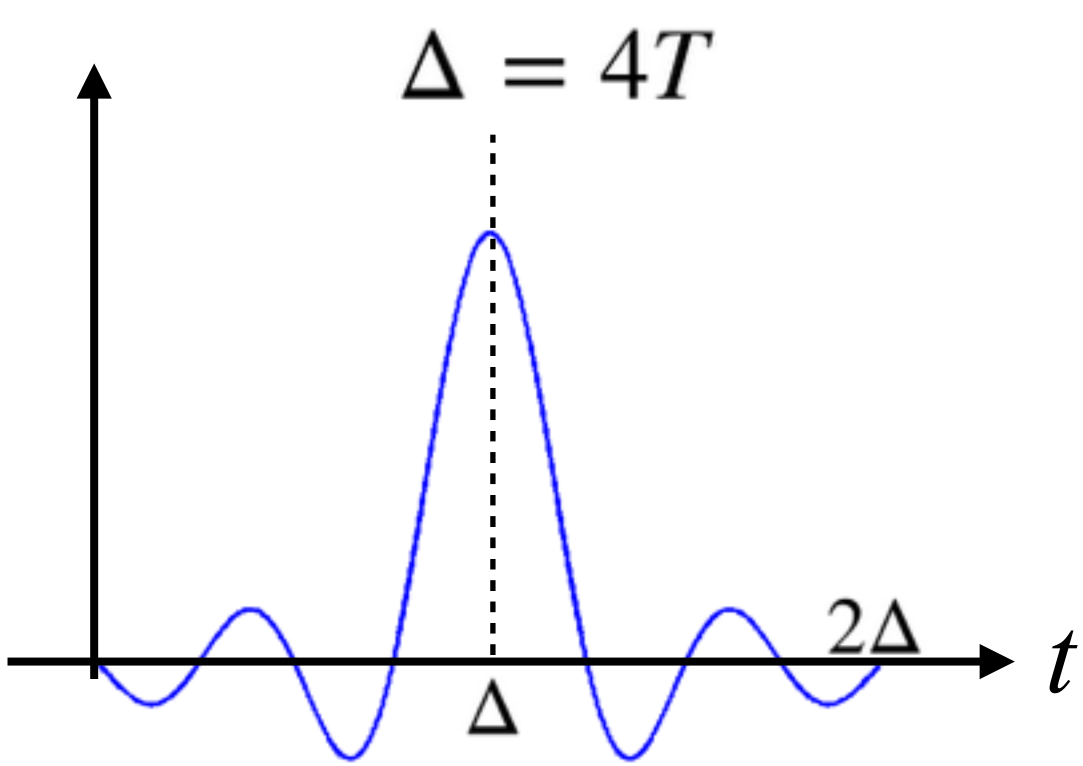
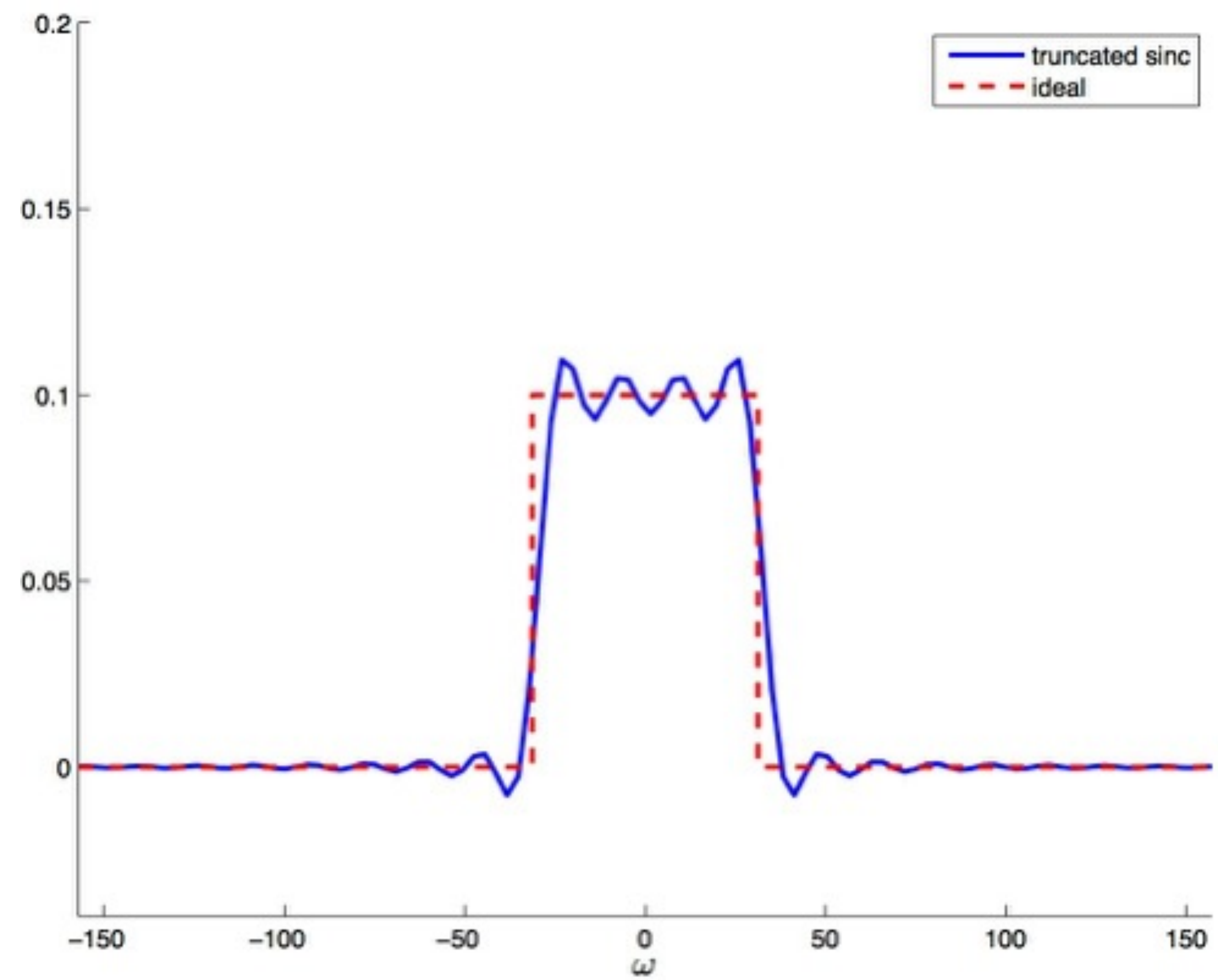
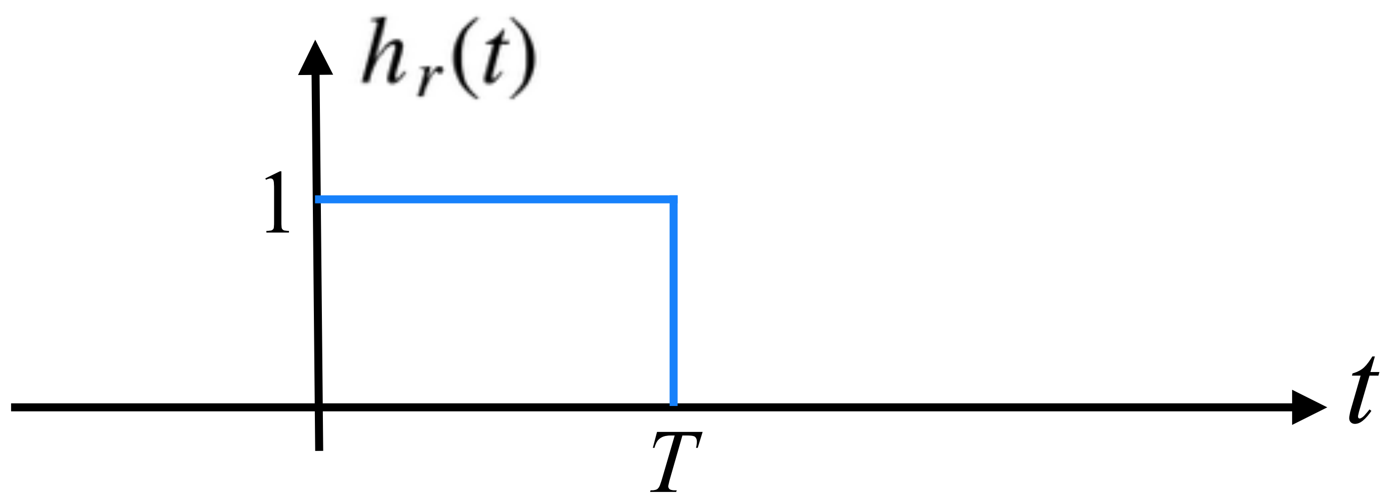
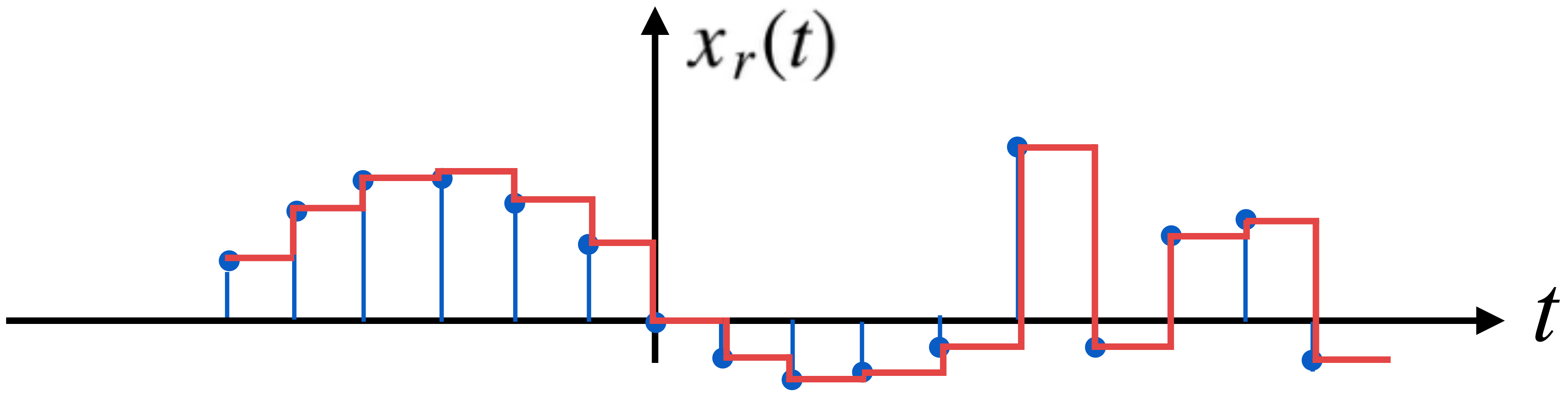


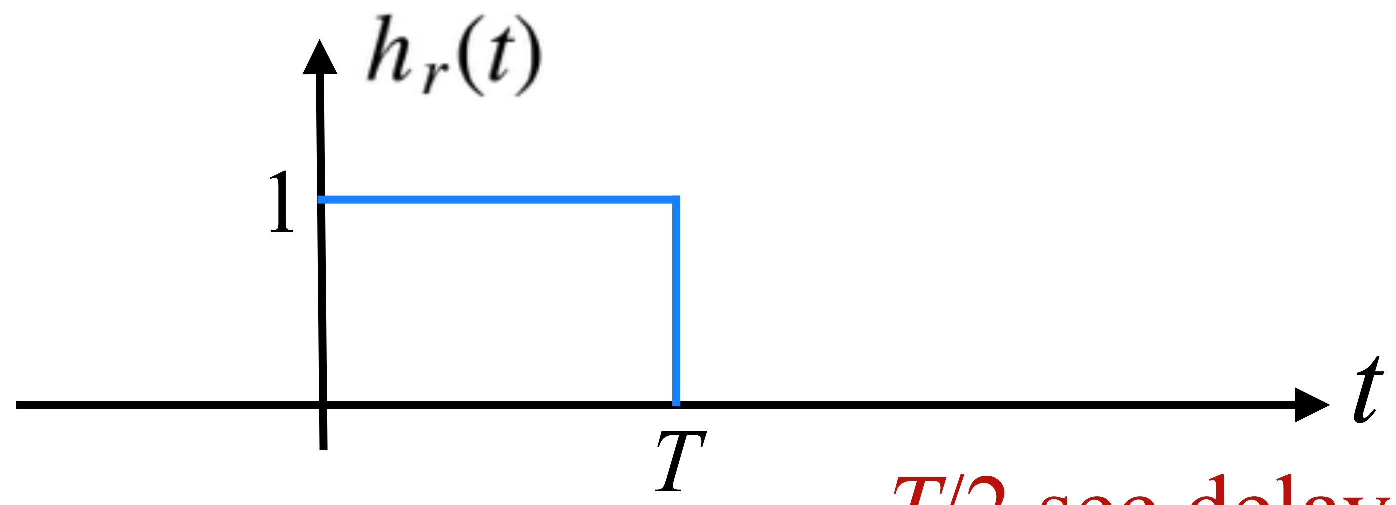


Practical interpolation

- Zero-order hold (ZOH)

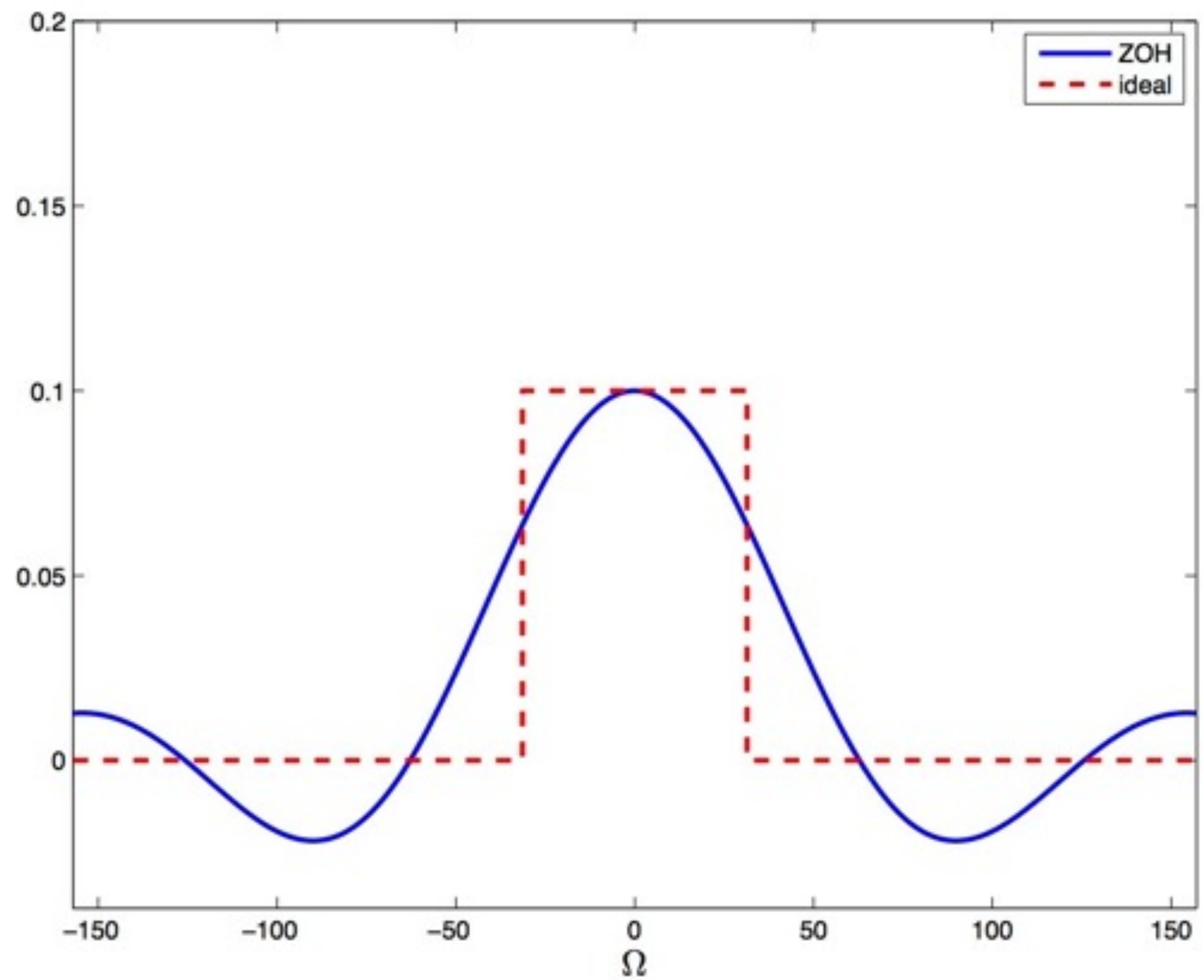






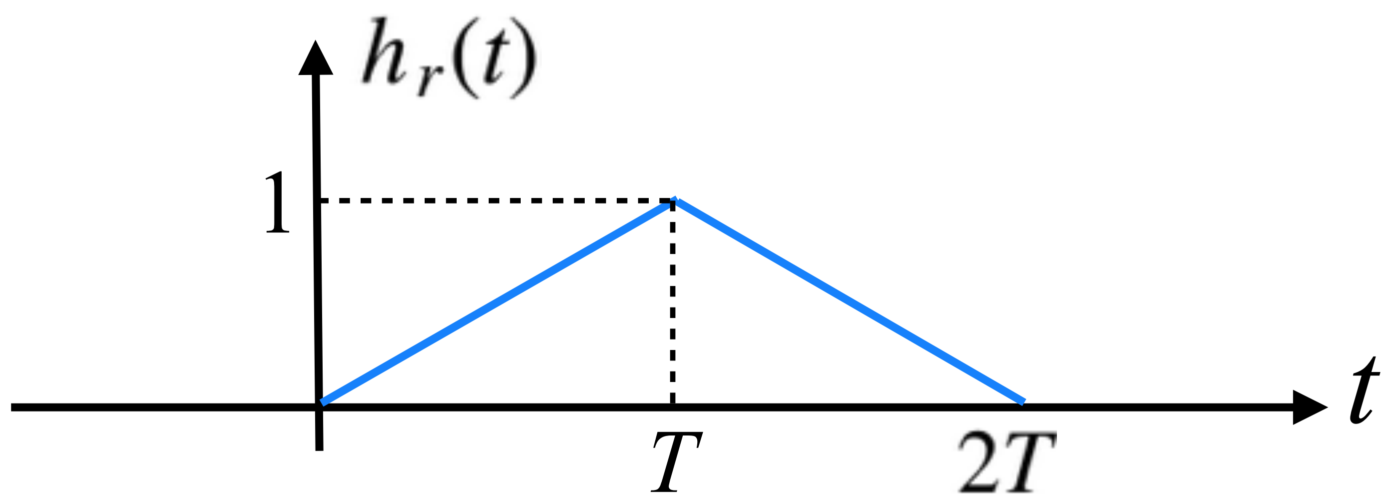
$T/2$ sec delay

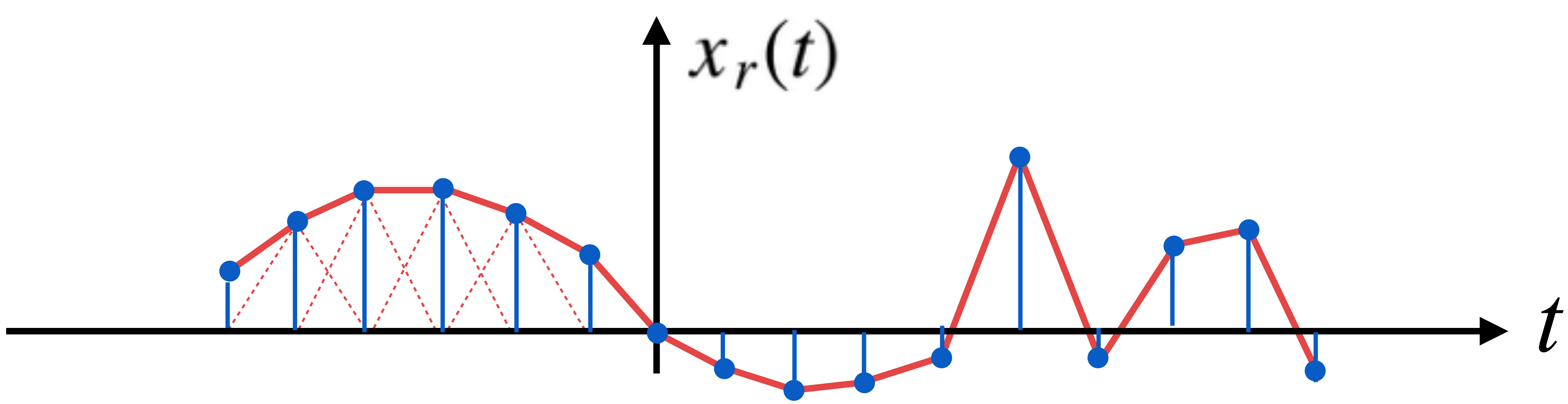
$$H_r(j\Omega) = \int_0^T e^{-j\Omega t}\,dt = \frac{1 - e^{-j\Omega T}}{j\Omega} = e^{-j\Omega T/2}\, T \,\mathrm{sinc}\left(\frac{\Omega T}{2}\right)$$

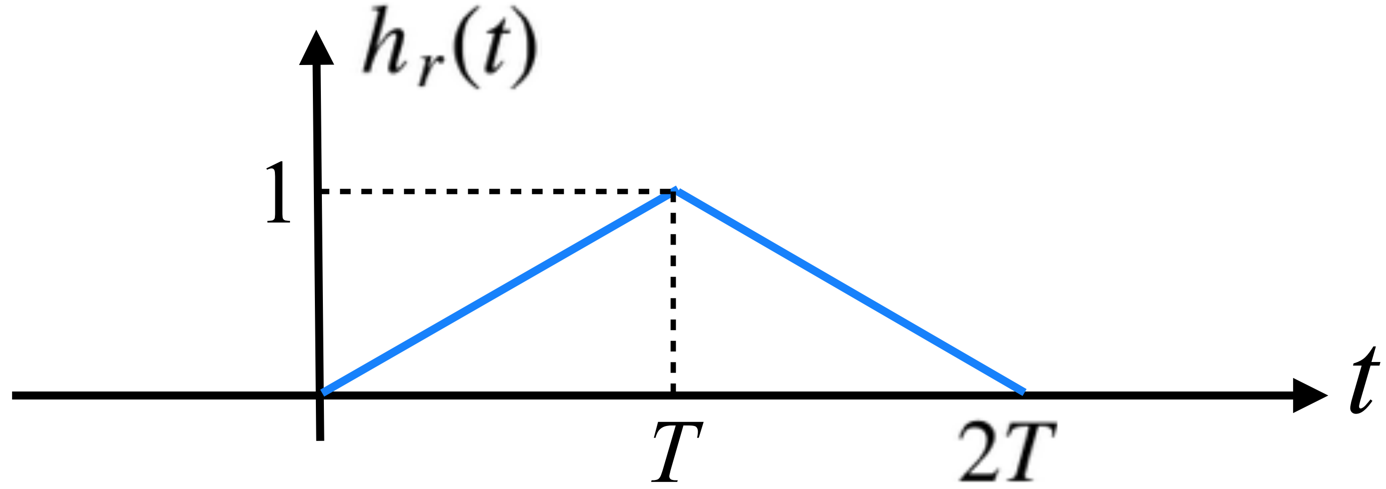


Practical interpolation

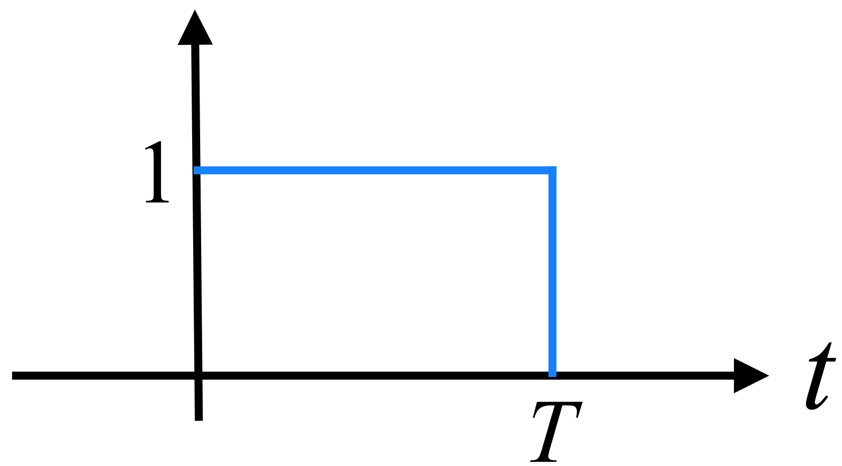
- First-order hold (FOH)



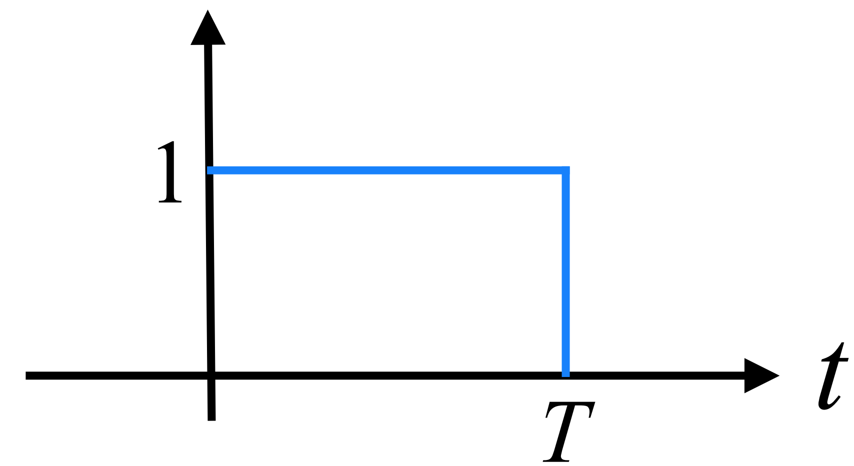




$$= \frac{1}{T}$$

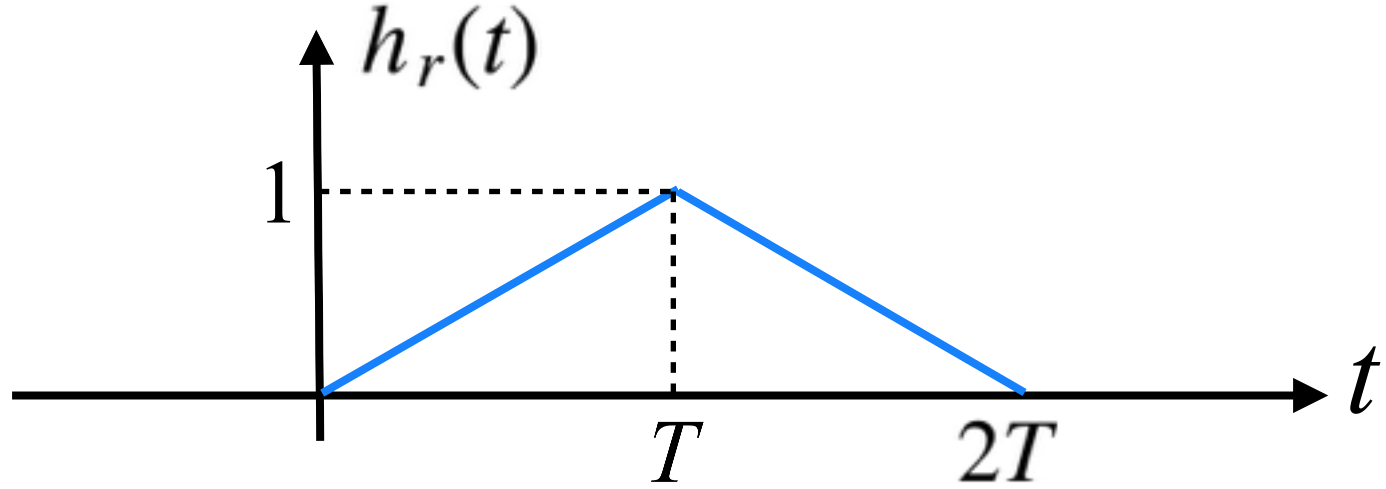


$\star$

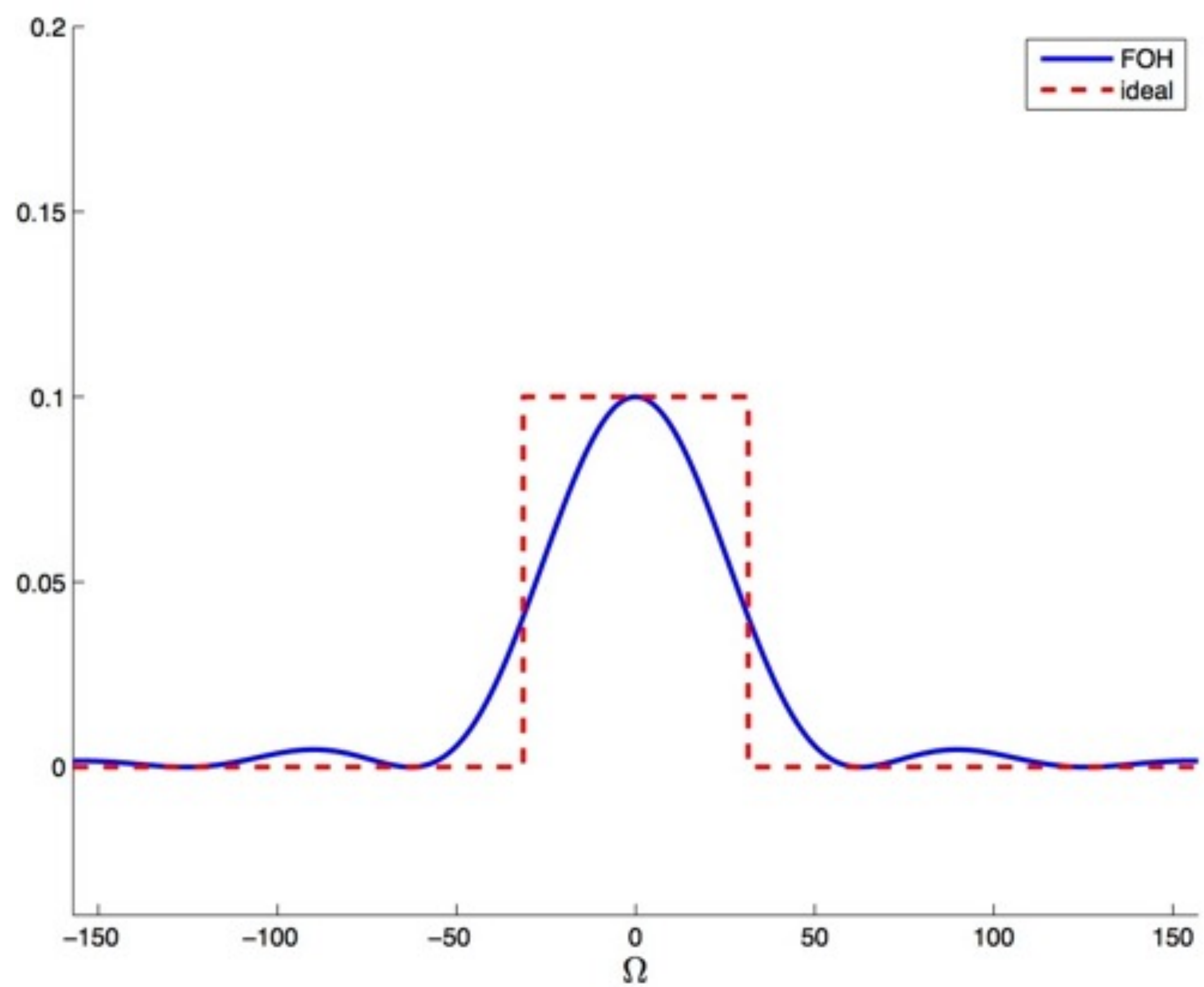


T sec delay

$$H_r(j\Omega) = \frac{1}{T}\left[e^{-j\Omega T/2}\, T\,\mathrm{sinc}\left(\frac{\Omega T}{2}\right)\right]^2 = e^{-j\Omega T}\, T\,\mathrm{sinc}^2\left(\frac{\Omega T}{2}\right)$$

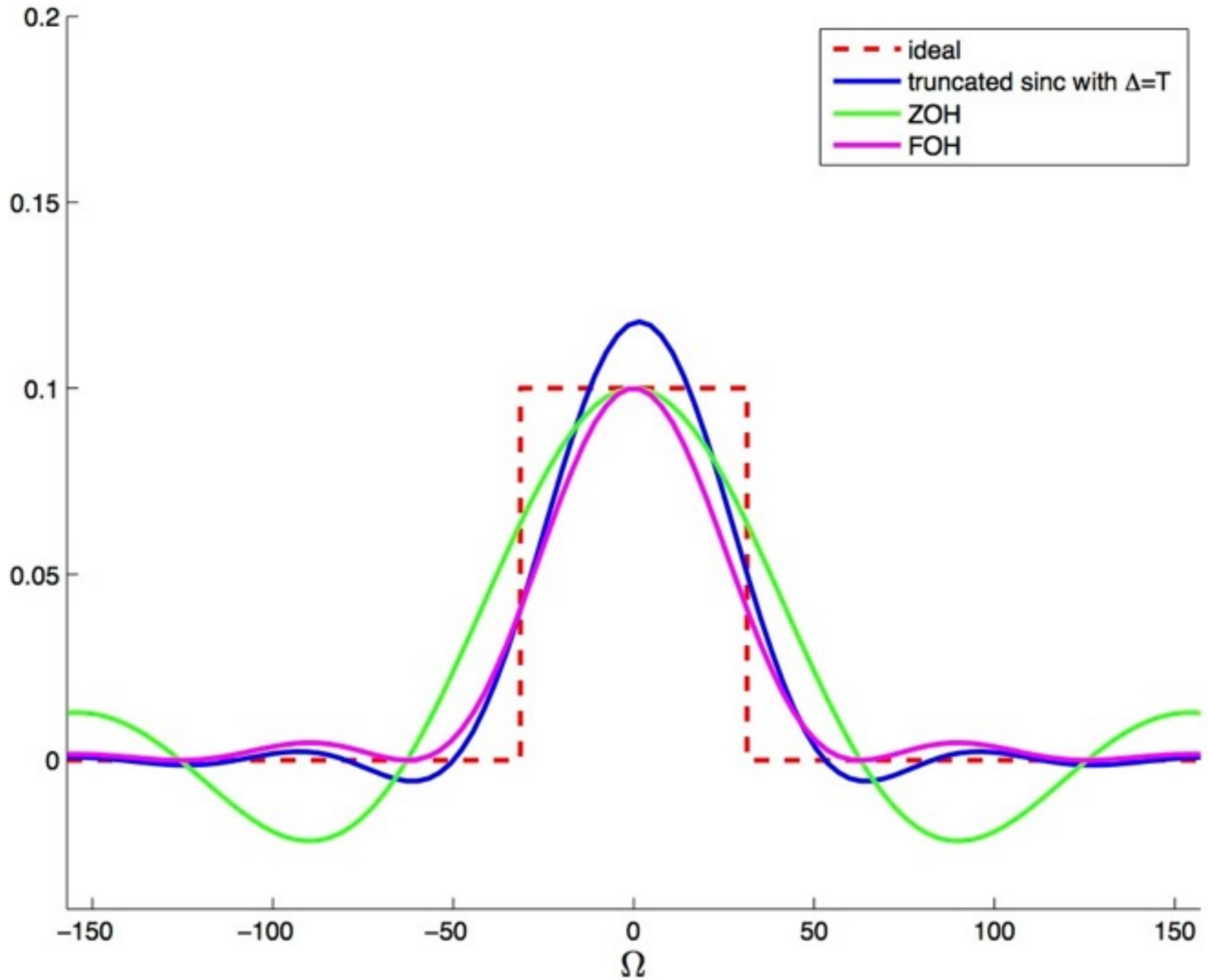


$$H_r(j\Omega) = \frac{1}{T}\left[e^{-j\Omega T/2}\, T\,\mathrm{sinc}\left(\frac{\Omega T}{2}\right)\right]^2 = e^{-j\Omega T}\, T\,\mathrm{sinc}^2\left(\frac{\Omega T}{2}\right)$$



Practical interpolation

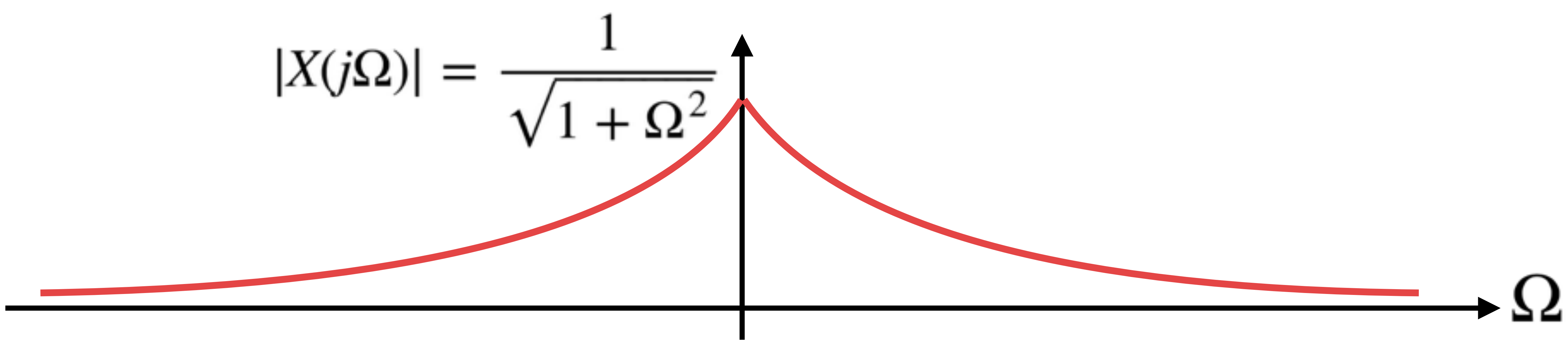
- Comparison of all three

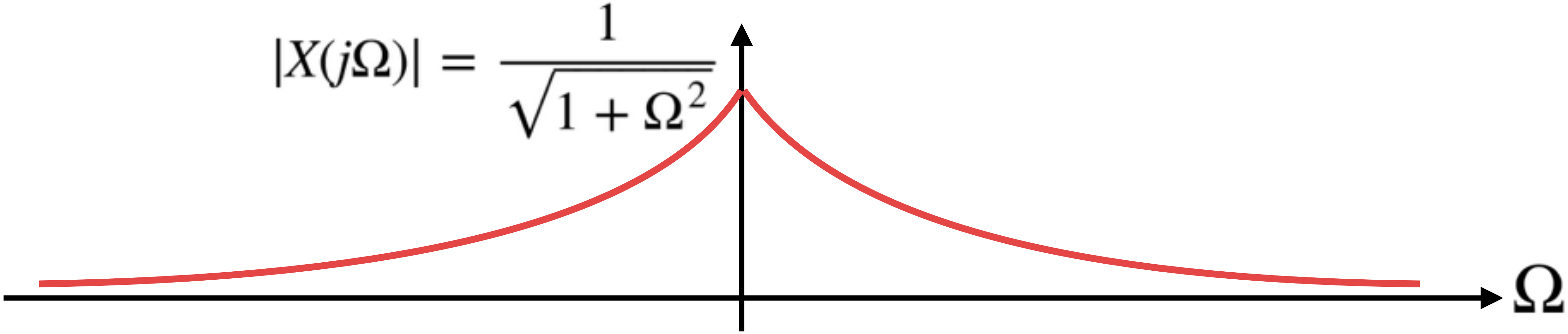


Infinite bandwidth signals

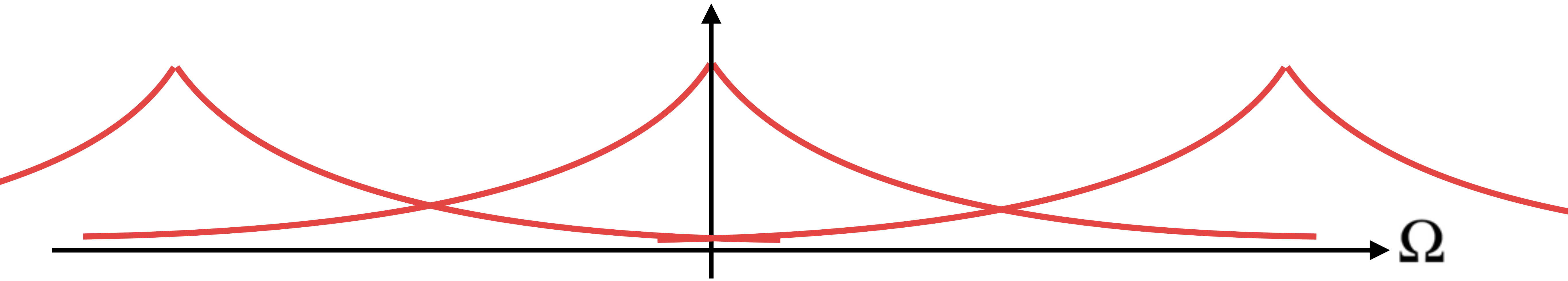
- Loss of information is unavoidable if the bandwidth is not finite.
- Example:

$$x(t) = e^{-t}u(t) \quad \Longrightarrow X(j\Omega) = \frac{1}{1 + j\Omega}$$

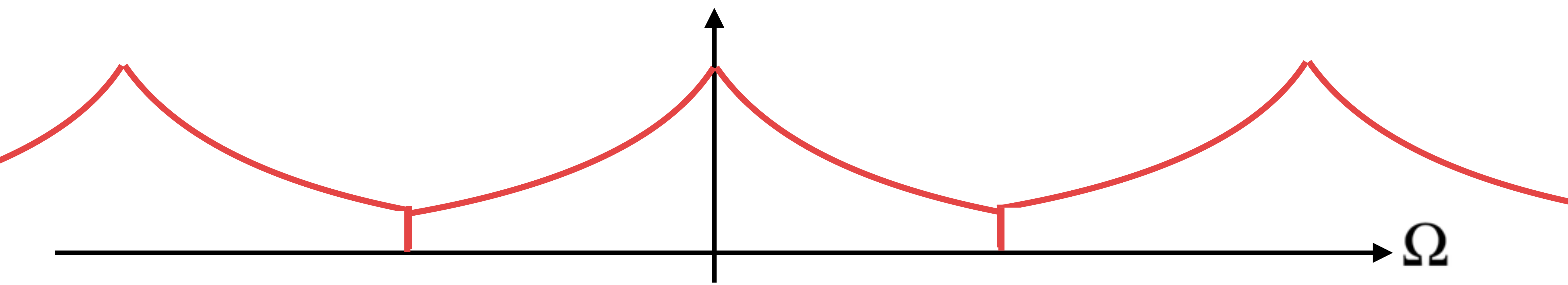




- Which option is better:



or



Infinite bandwidth signals

- So, the better option is to use an "anti-aliasing" filter before sampling.

Original

Half the sampling rate

Quarter the sampling rate

Let's hear anti-aliasing

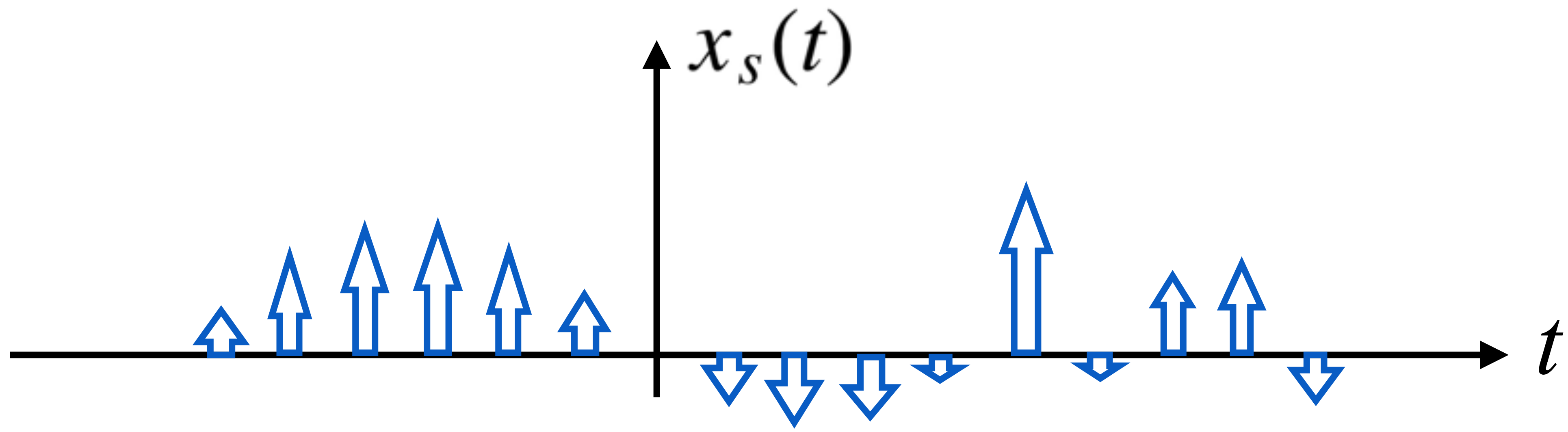
- Sampling rate = 2,756 Hz
  - With no filtering
  - After anti-aliasing filter
- Sampling rate = 1,378 Hz
  - With no filtering
  - After anti-aliasing filter

The ideal impulse train

- Recall that during reconstruction, we need

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

which is an impulse train

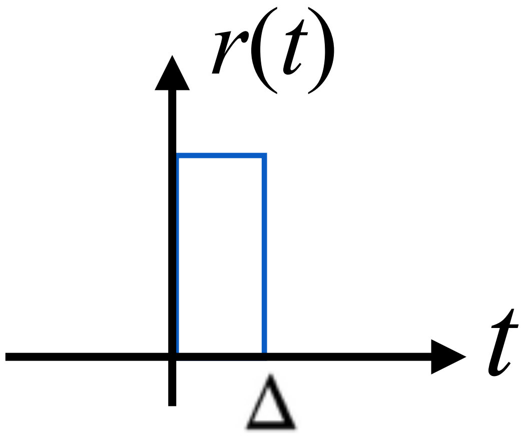


- Constructing a signal like this from the samples is not practical.

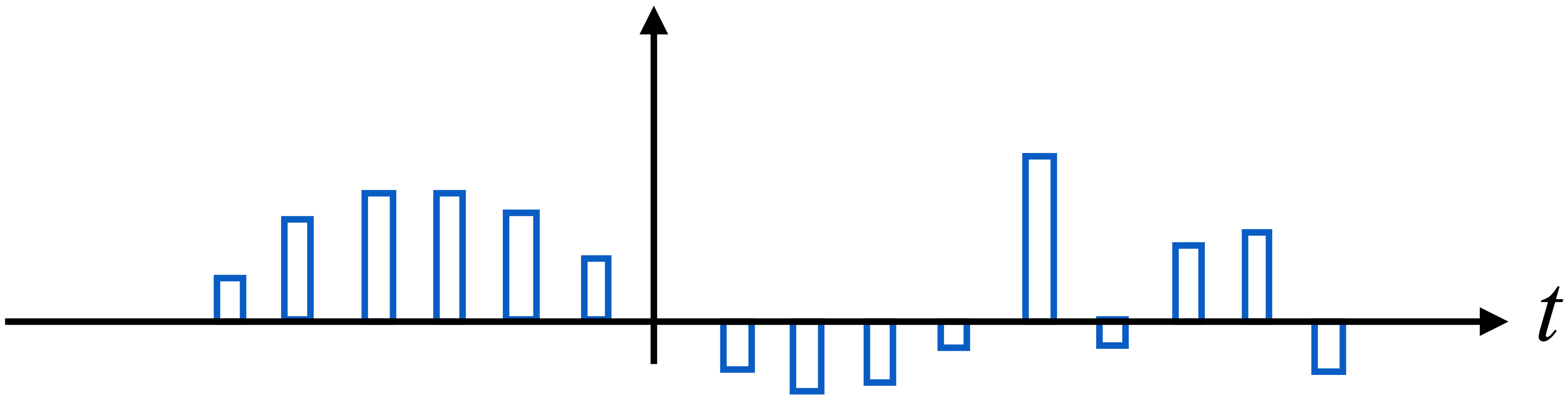
The pulse train

- We must instead use

$$\hat{x}_s(t) = \sum_{n=-\infty}^{\infty} x[n]\, r(t - nT)$$

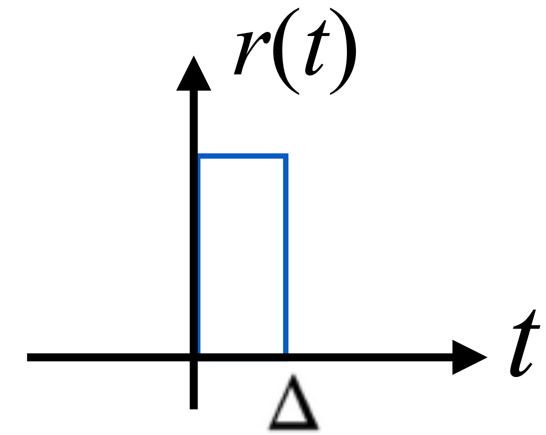


which is a "pulse" train



- How will this affect reconstruction?

$$\hat{x}_s(t) = \sum_{n=-\infty}^{\infty} x[n]\, r(t - nT)$$



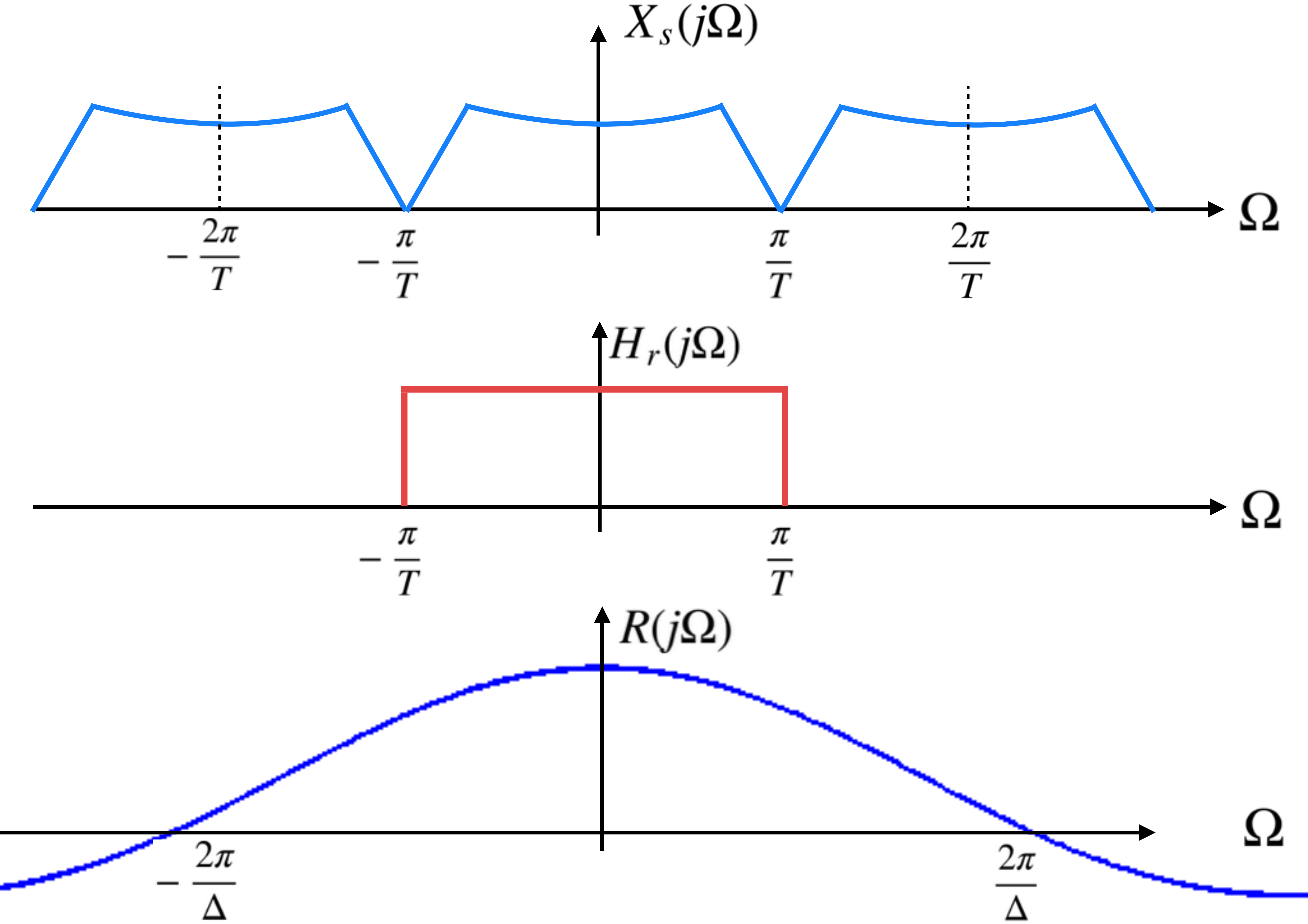
- How will this affect reconstruction?
- Observe that

$$\hat{x}_s(t) = \sum_{n=-\infty}^{\infty} x[n]\left[r(t) \star \delta(t - nT)\right] = r(t) \star \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

$$= r(t) \star x_s(t)$$

- Therefore, during reconstruction,

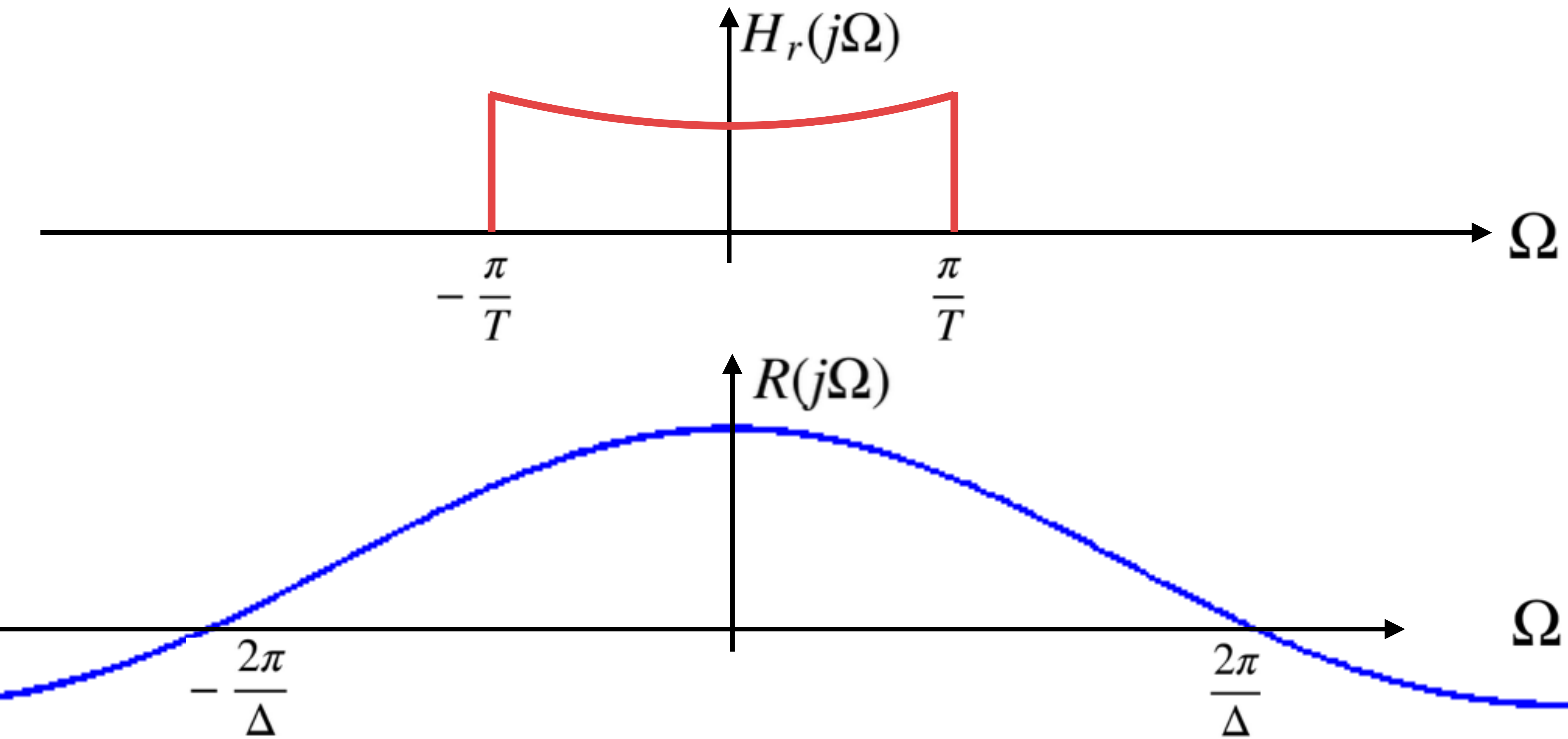
$$\hat{x}_r(t) = \hat{x}_s(t) \star h_r(t) = x_s(t) \star \underbrace{h_r(t) \star r(t)}_{\text{equivalent reconstruction filter}}$$



$X_s(j\Omega)$
Ω
$-\frac{2\pi}{T}$
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
$\frac{2\pi}{T}$
$H_r(j\Omega)$
Ω
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
$R(j\Omega)$
Ω
$-\frac{2\pi}{\Delta}$
$\frac{2\pi}{\Delta}$

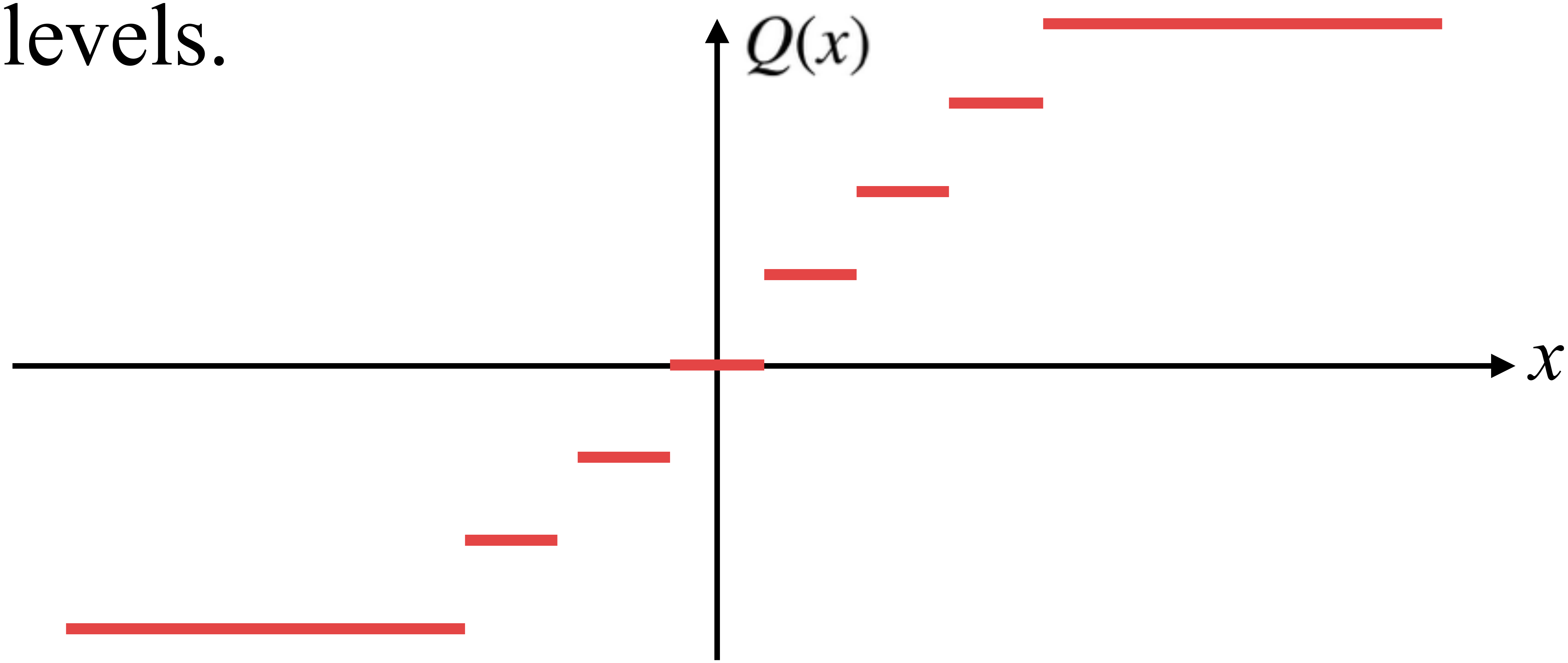
The pulse train

- To compensate, we need a new reconstruction filter as below:



Effects of quantization

- After sampling, discrete-time signals are stored or processed after quantization to 2^B levels.



- Each sound signal played so far was quantized to 256 levels, i.e., was represented with 8 bits.

Let's hear quantization

- What if we use less number of bits?
  - 6 bits:
  - 5 bits:
  - 4 bits:
- Are you hearing an ever increasing amount of "white noise?"
  - A very good model for quantization is that of "additive white noise"
  - The less #bits, the higher quantization error, and hence the louder the noise becomes.