EE 110A Signals and Systems

Linear and Time-Invariant (LTI) Systems

Ertem Tuncel

Why LTI systems?

- Linear and time-invariant systems are especially easy to analyze and design.
- In a lot of cases, they are good enough to do the "signal processing" job.
- Amenable to frequency analysis in the Fourier domain.

The impulse response

- An LTI system's response to an impulse input is called its **impulse response**.

$$\delta(t) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow h(t)$$

- Because the system is LTI,

$$a\delta(t - t_0) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow ah(t - t_0)$$

The impulse response

- Not only that, but also

$$\begin{matrix} a_1\delta(t-t_1) \\ +a_2\delta(t-t_2) \end{matrix} \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \begin{matrix} a_1 h(t-t_1) \\ +a_2 h(t-t_2) \end{matrix}$$

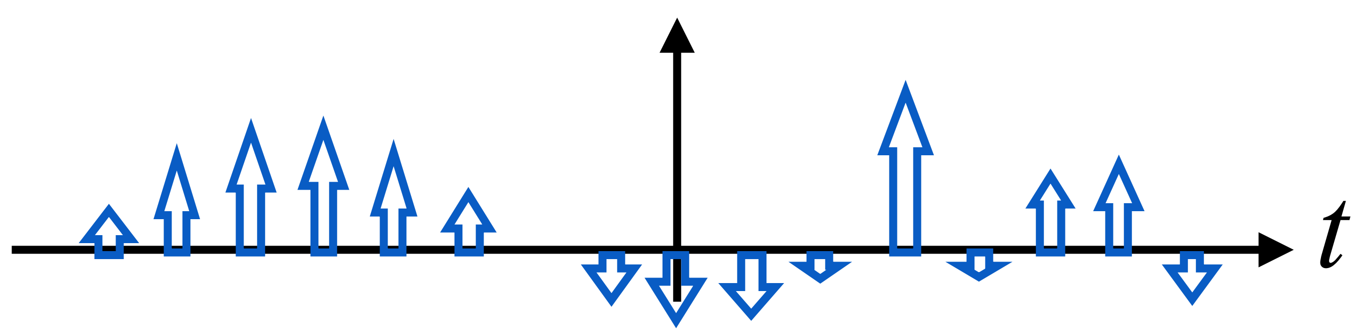
- Now, extending this all the way,

$$\sum_{k=-\infty}^{\infty} a_k\delta(t-t_k) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k h(t-t_k)$$

The impulse response

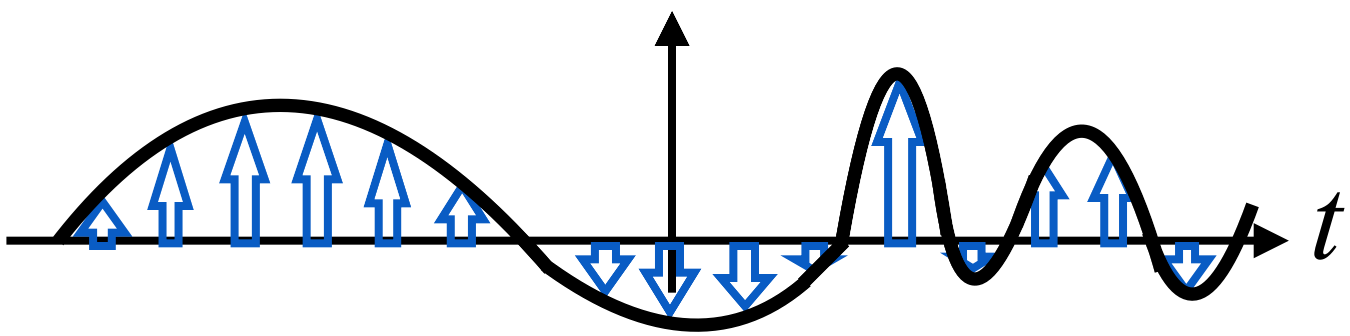
$$\sum_{k=-\infty}^{\infty} a_k \delta(t - t_k) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k h(t - t_k)$$

- If you place the impulses, densely and uniformly, across the real line, say at $t_k = k\Delta$



$$\sum_{k=-\infty}^{\infty} a_k \delta(t - k\Delta) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k h(t - k\Delta)$$

The impulse response



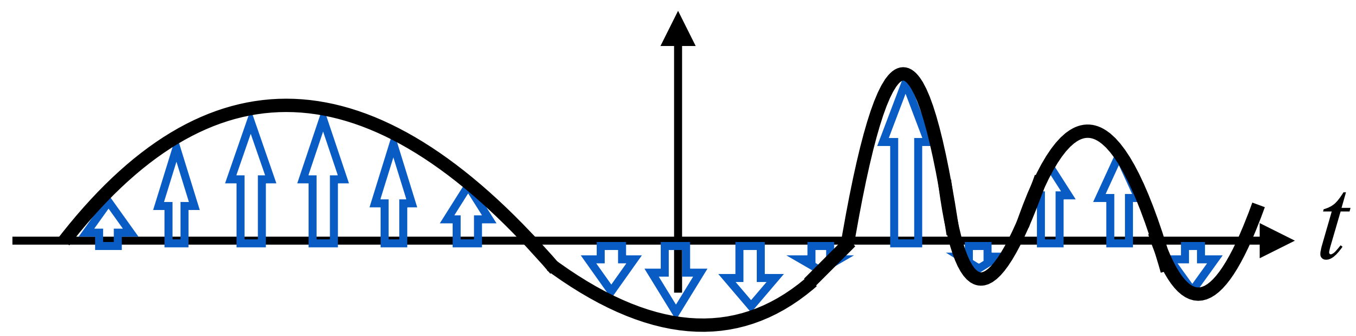
$$\sum_{k=-\infty}^{\infty} a_k \delta(t - k\Delta) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k h(t - k\Delta)$$

- Now think of a_k's as samples from a function $x(t)$ taken at time instants $t_k = k\Delta$, i.e.,

$$a_k = x(k\Delta)$$

$$\sum_{k=-\infty}^{\infty} x(k\Delta)\delta(t - k\Delta) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} x(k\Delta)h(t - k\Delta)$$

The impulse response



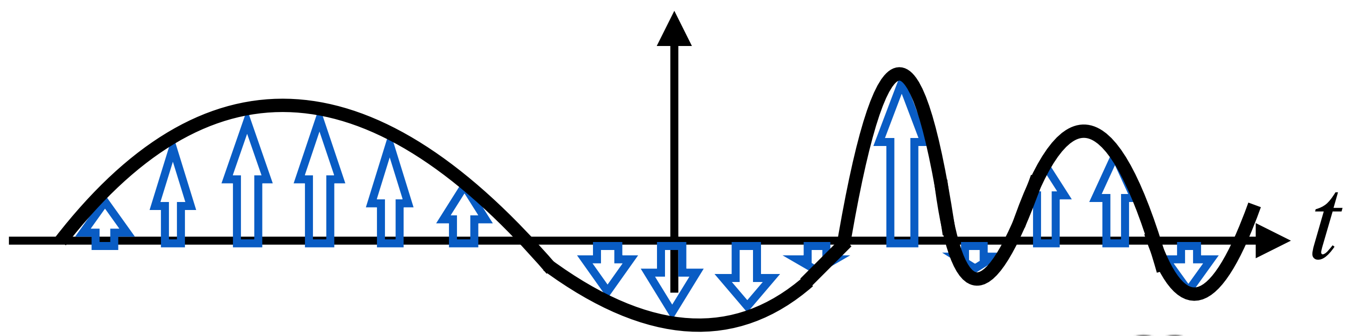
$$\sum_{k=-\infty}^{\infty} x(k\Delta)\delta(t-k\Delta) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} x(k\Delta)h(t-k\Delta)$$

- From linearity,

$$\sum_{k=-\infty}^{\infty} x(k\Delta)\delta(t-k\Delta)\Delta \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} x(k\Delta)h(t-k\Delta)\Delta$$

Riemann sums

The impulse response



$$\sum_{k=-\infty}^{\infty} x(k\Delta)\delta(t-k\Delta)\Delta \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} x(k\Delta)h(t-k\Delta)\Delta$$

- As $\Delta \rightarrow 0$, this becomes ...

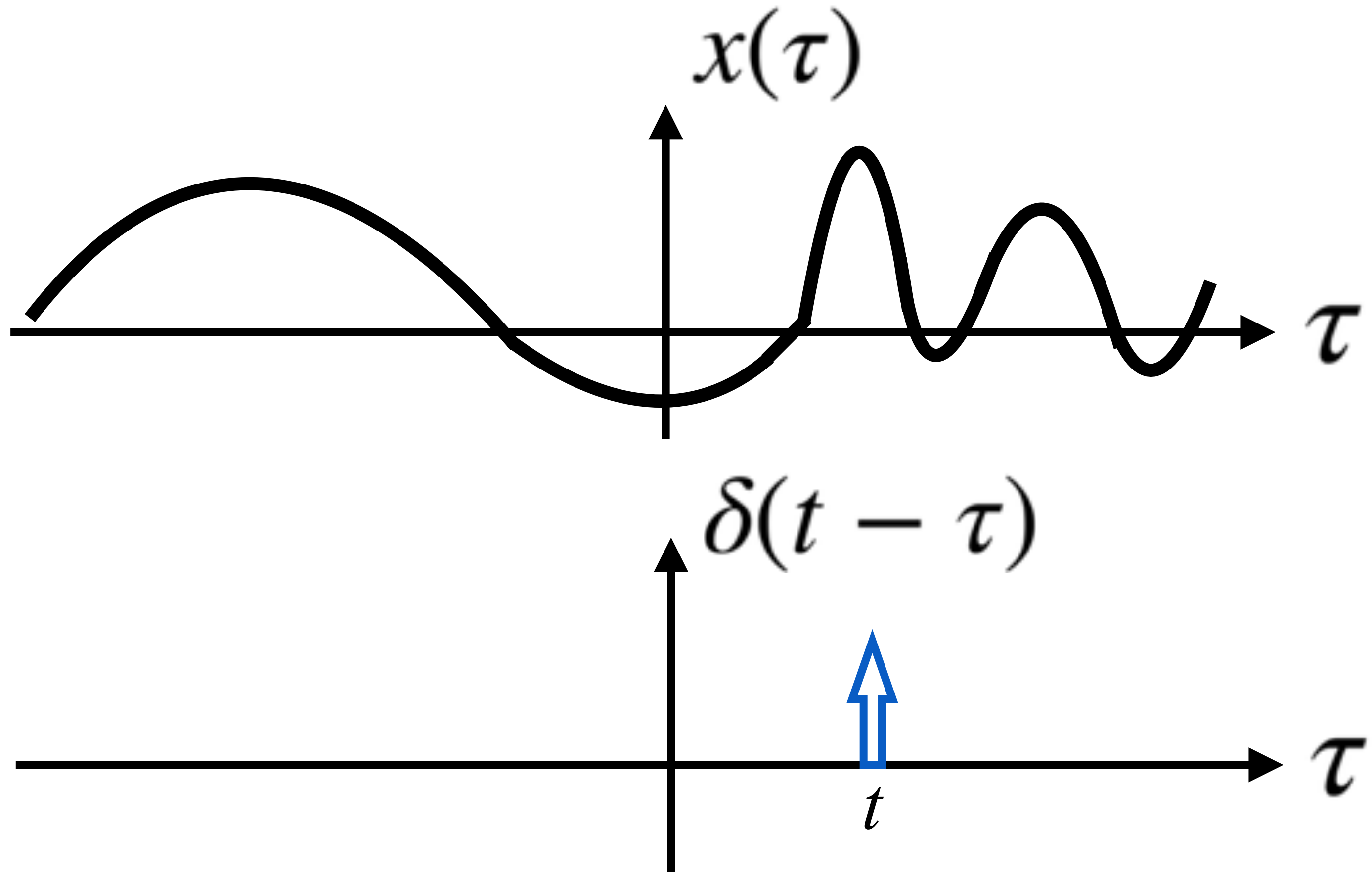
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- But of what use is this?

The impulse response

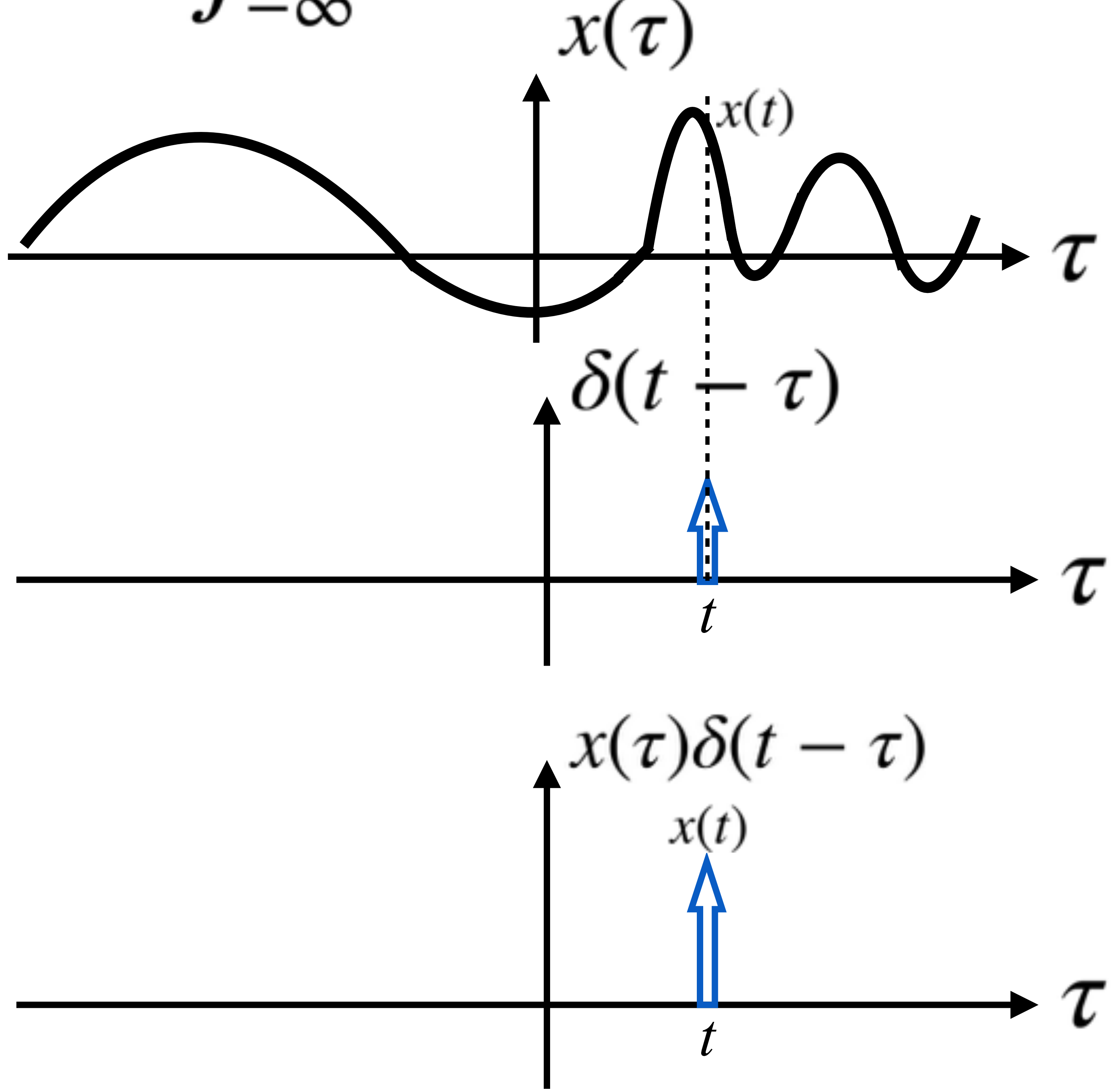
$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- Instead of looking at this as a collection of impulses located at each $t = \tau$, focus on the evaluation of the integral for each fixed t.



The impulse response

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$



The impulse response

- In other words,

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$$

- A.k.a. the **sampling** property of the impulse.
- This simplifies the integral greatly:

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau \\ &= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau \\ &= x(t)\end{aligned}$$

- A.k.a. the **sifting** property of the impulse.

The impulse response

- Thus,

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

is the same as

$$x(t) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- **Conclusion**: If you know how an LTI system responds to the impulse signal, you know how it responds to *any* input signal.

The convolution integral

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- This is known as the **convolution** integral
- **Procedure:**
  - Time reverse the signal $h(\tau)$ to obtain $h(-\tau)$
  - For each t,
    - Shift $h(-\tau)$ to the right by t units to obtain $h(t-\tau)$
    - Multiply $h(t-\tau)$ with $x(\tau)$
    - Integrate the product signal over the entire real line

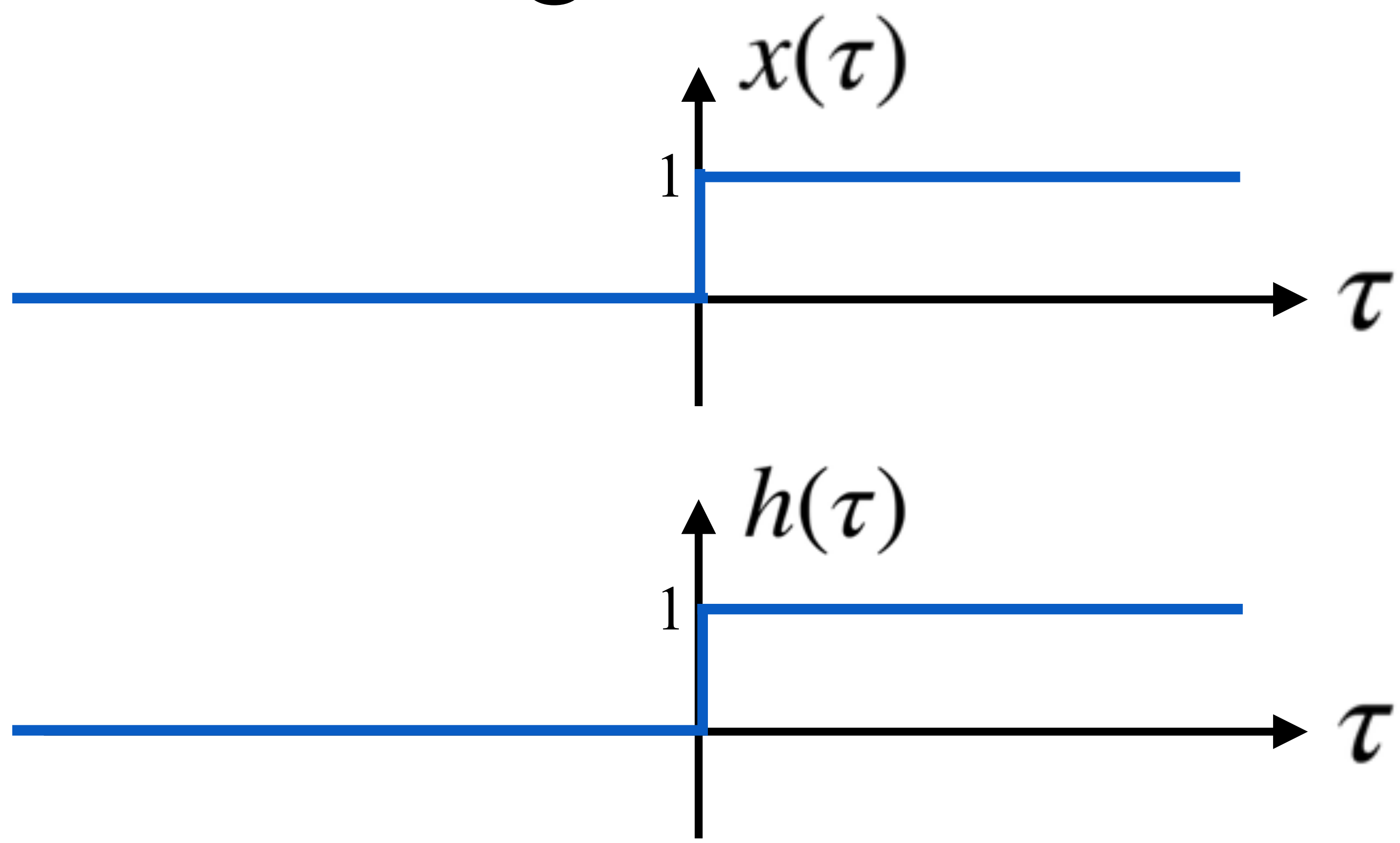
The convolution integral

- Example: Compute

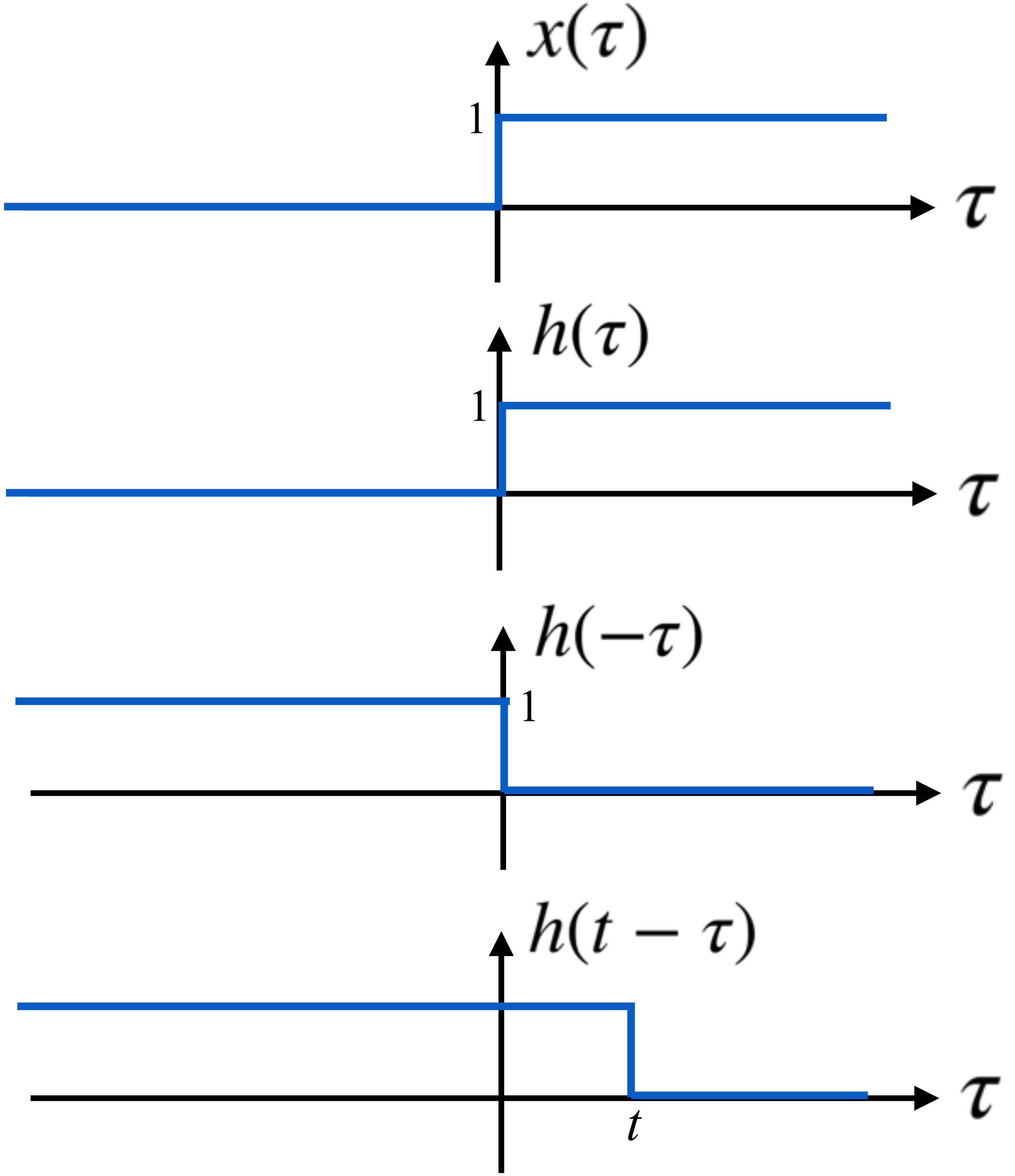
$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

for $x(t) = h(t) = u(t)$

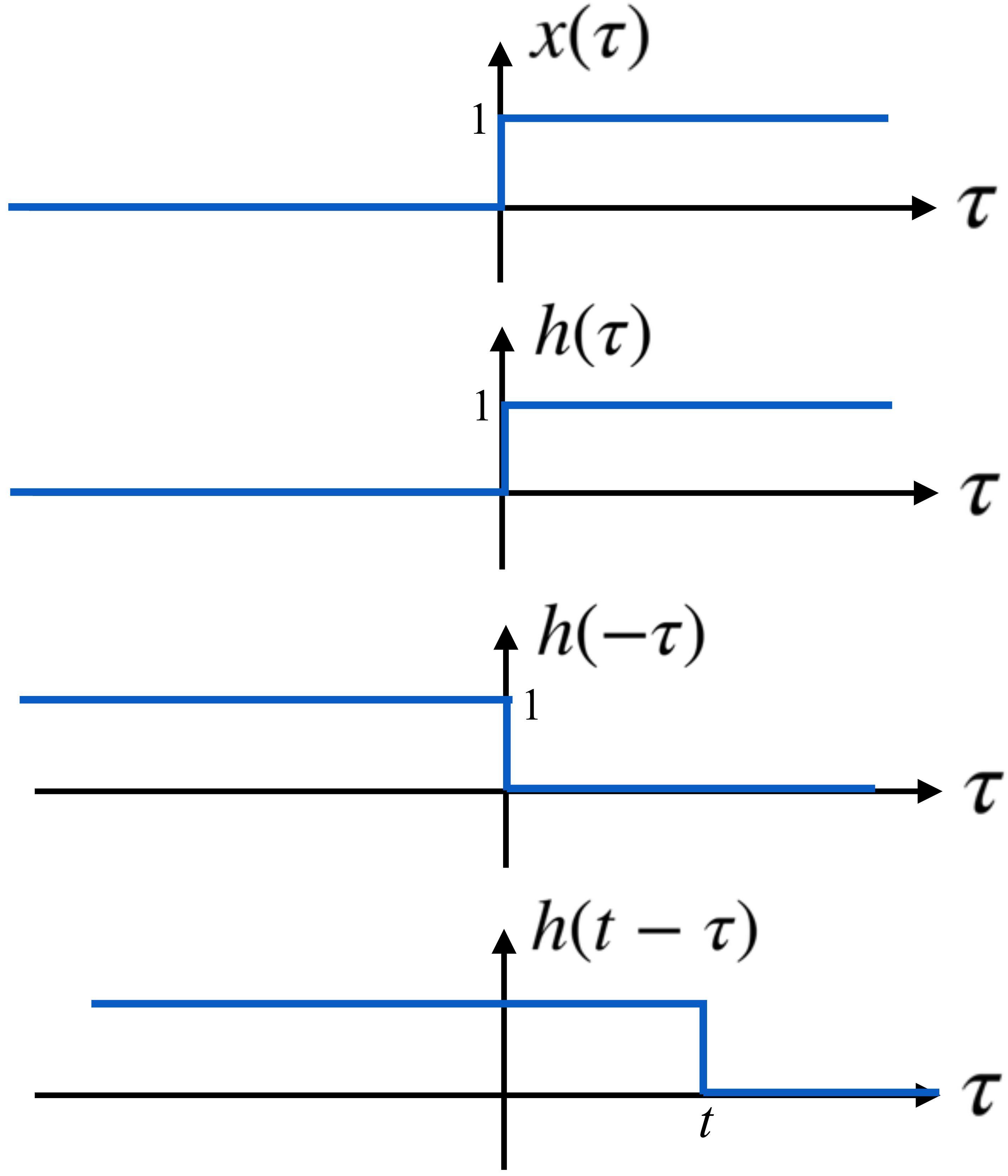
- Solution: Looking at the functions graphically,



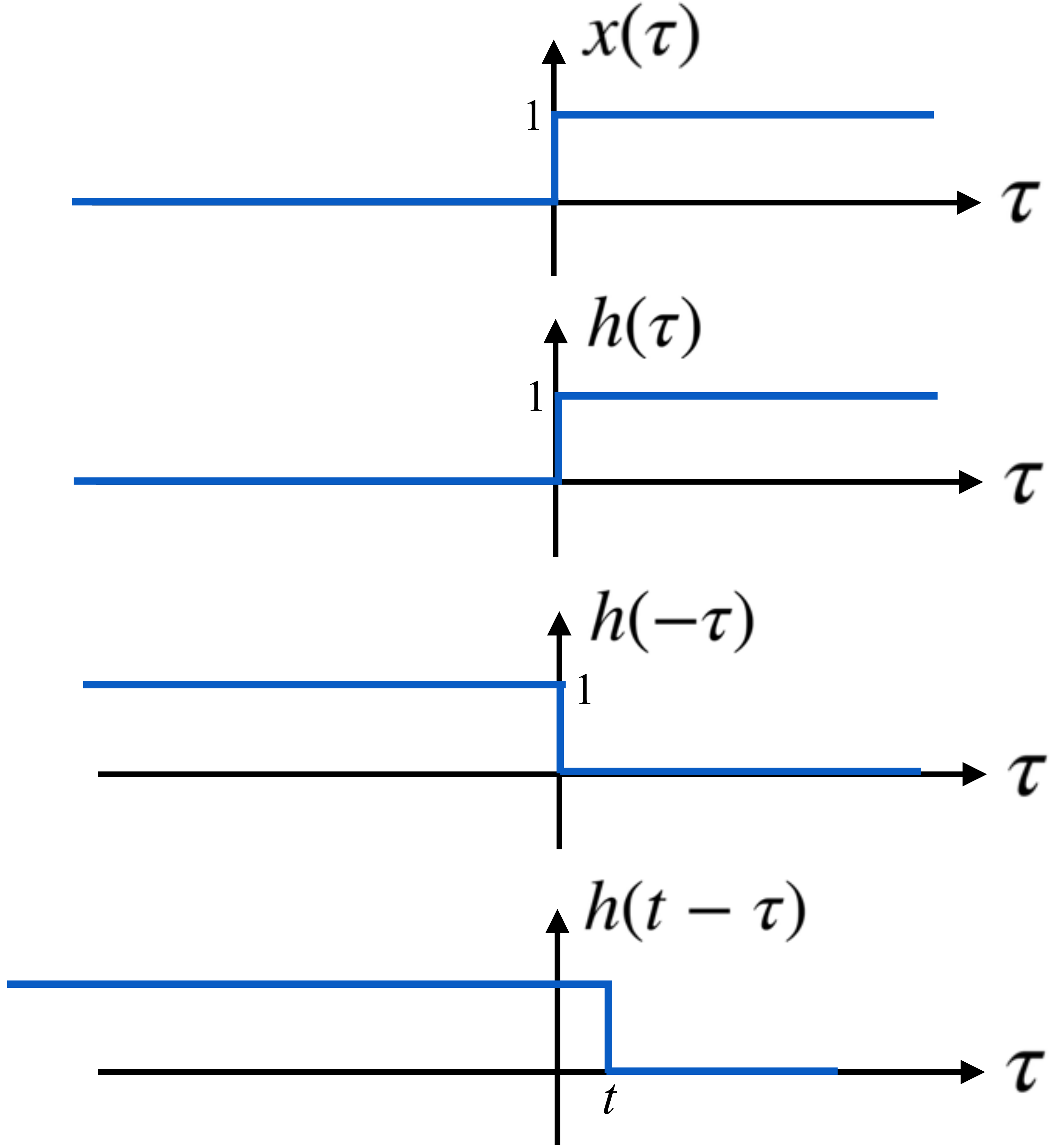
The convolution integral



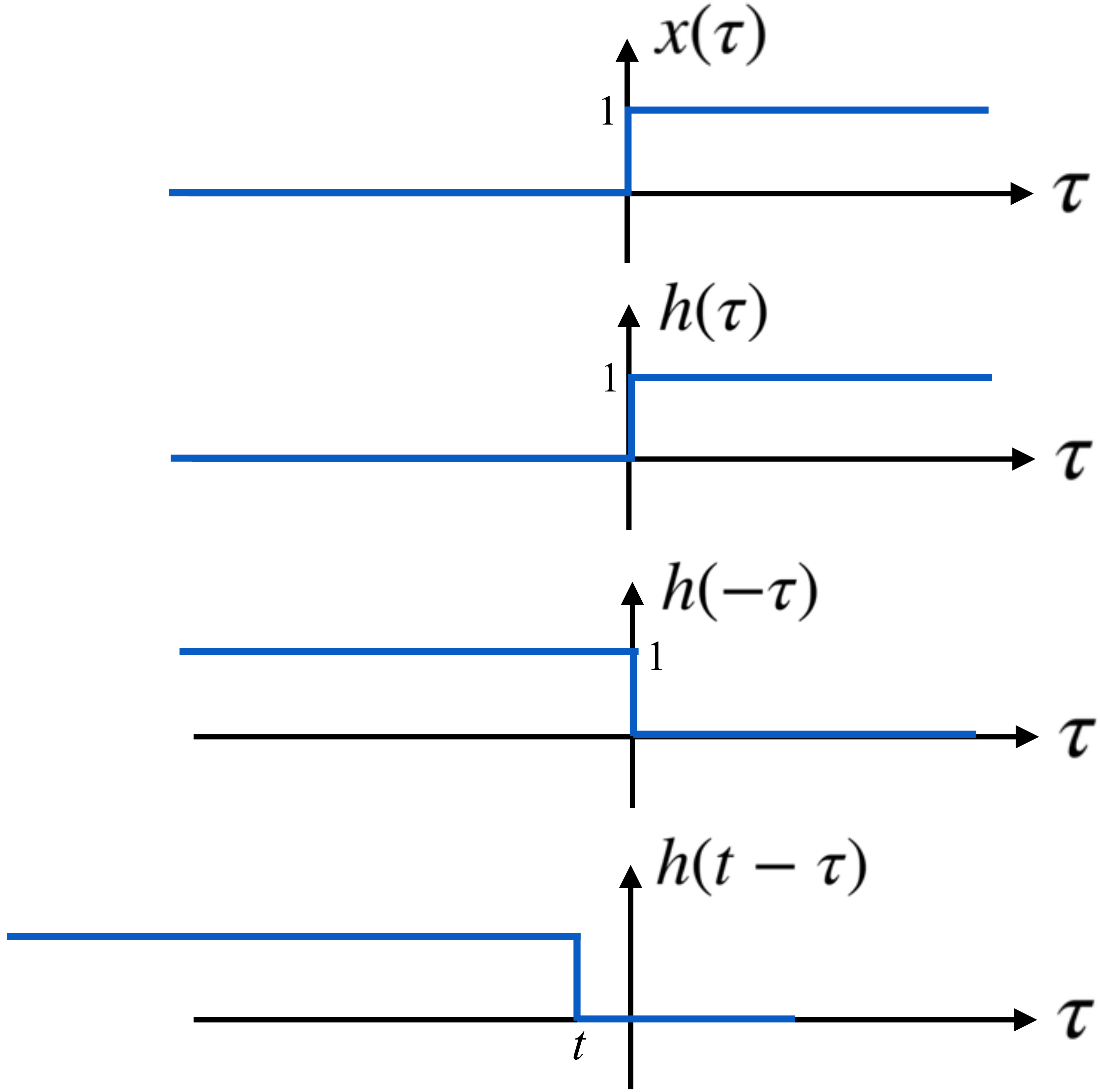
The convolution integral



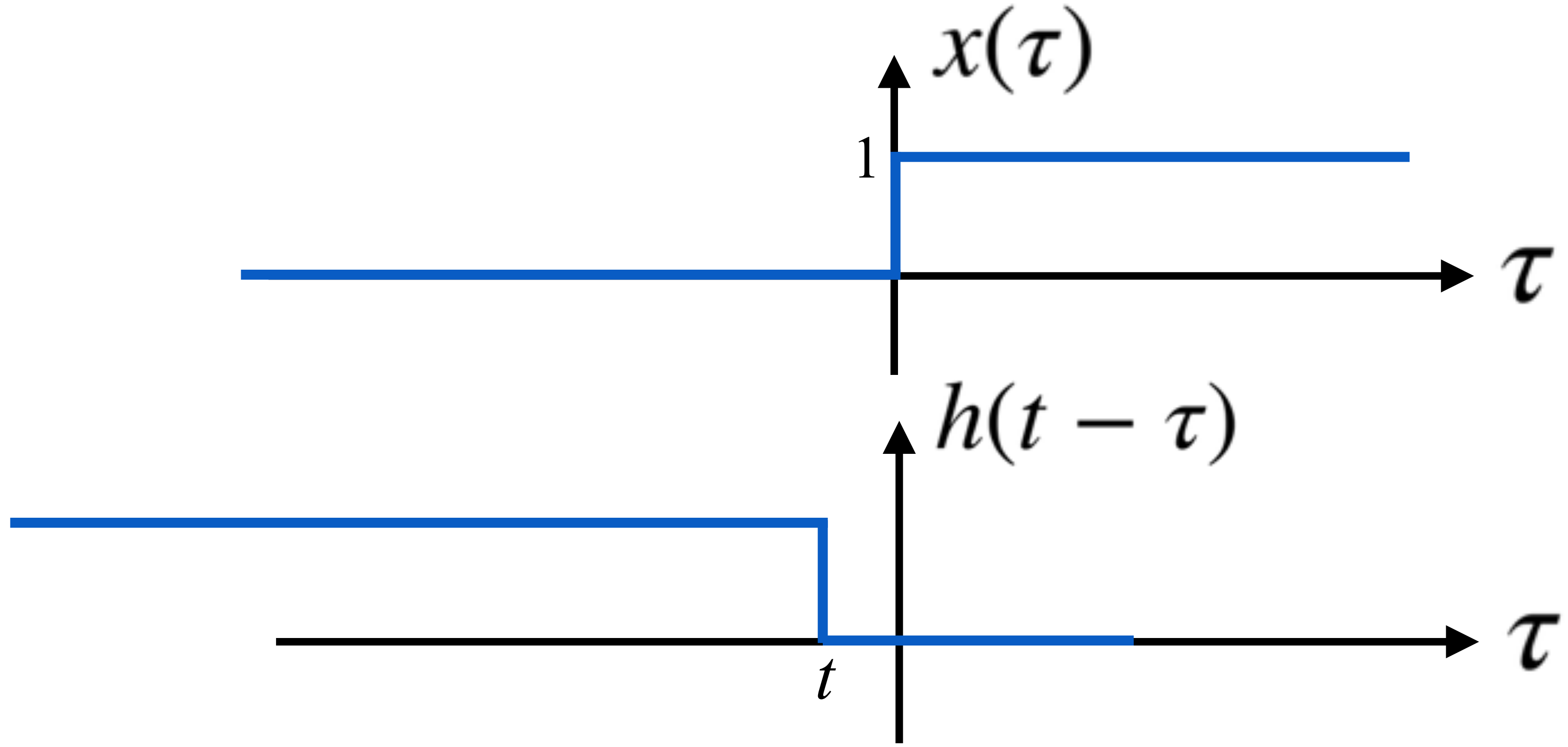
The convolution integral



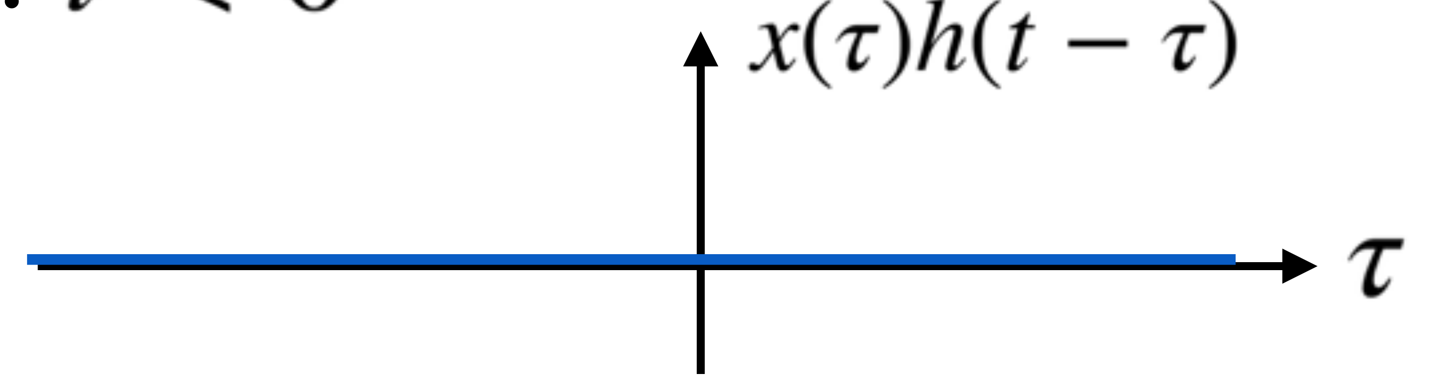
The convolution integral



The convolution integral

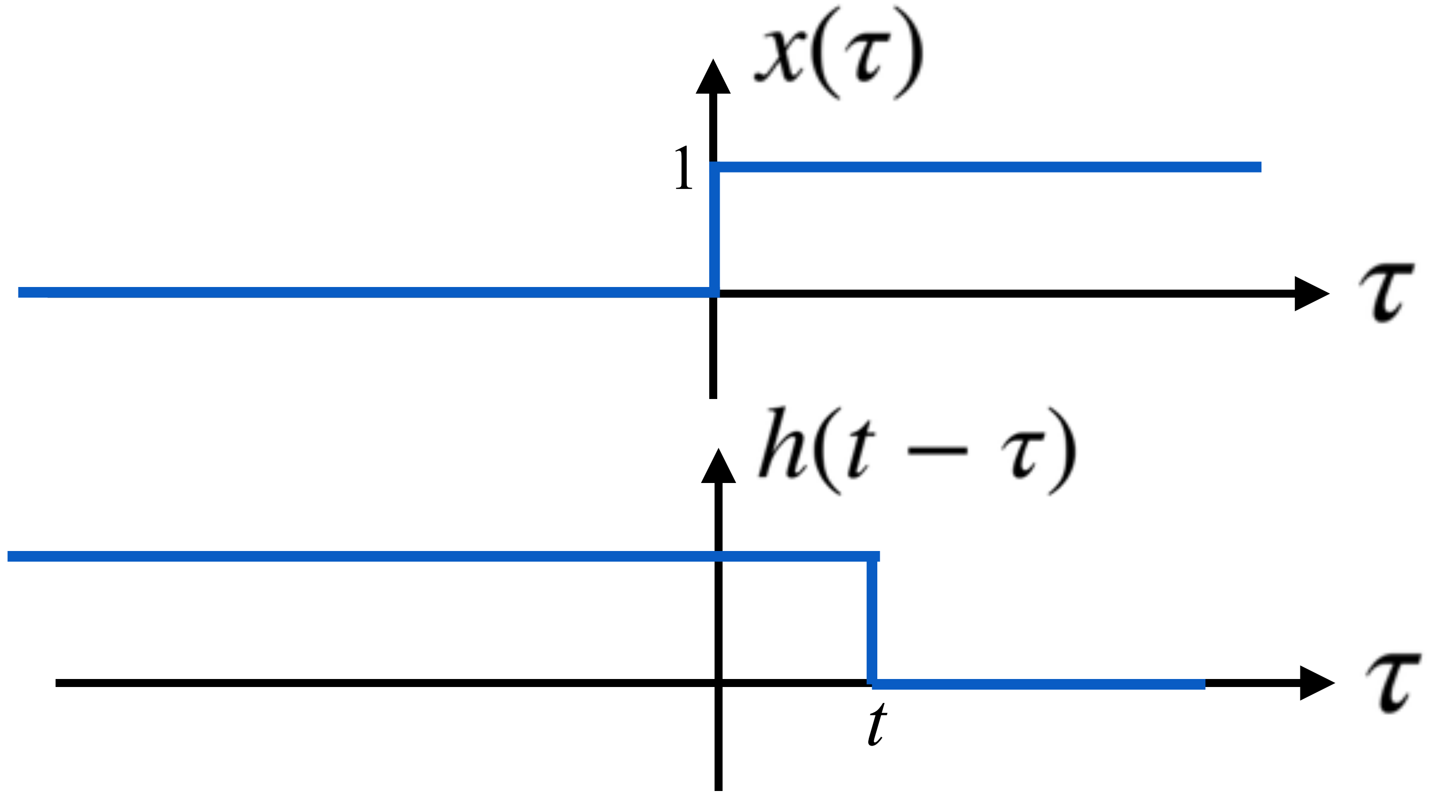


- **Case 1**: $t < 0$

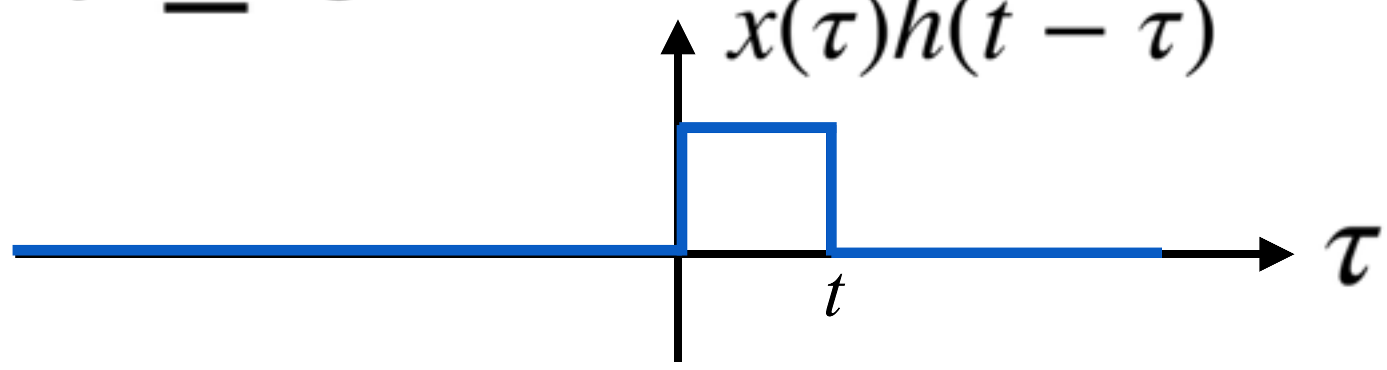


$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$

The convolution integral



- **Case 2:** $t \geq 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} 1 d\tau = t$$

The convolution integral

- Therefore,

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases} = r(t)$$

- Convolving the step function with itself resulted in the *integral* of the step function.
- Coincidence? Hardly.
- Take any $x(t)$ and $h(t) = u(t)$,

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$$

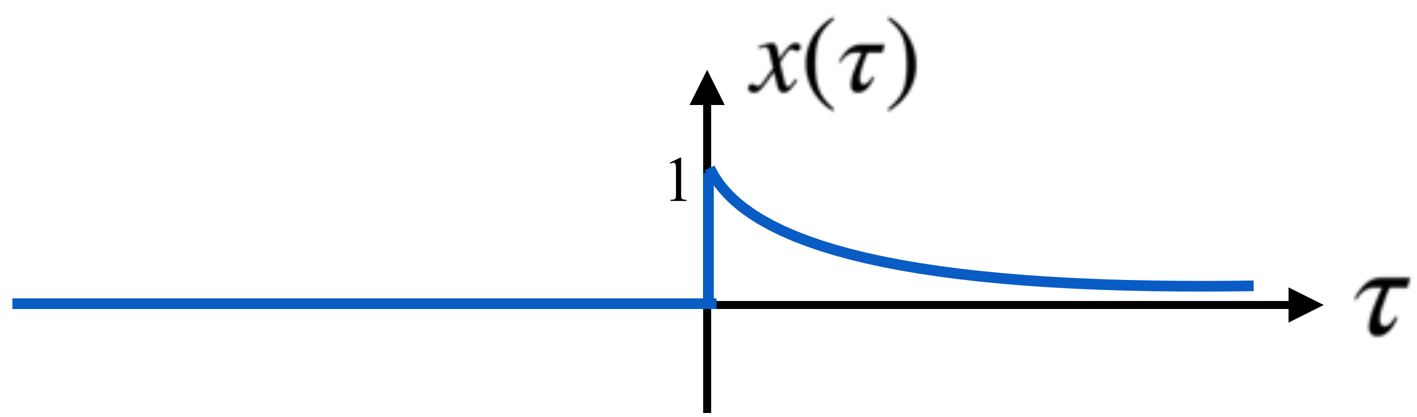
The convolution integral

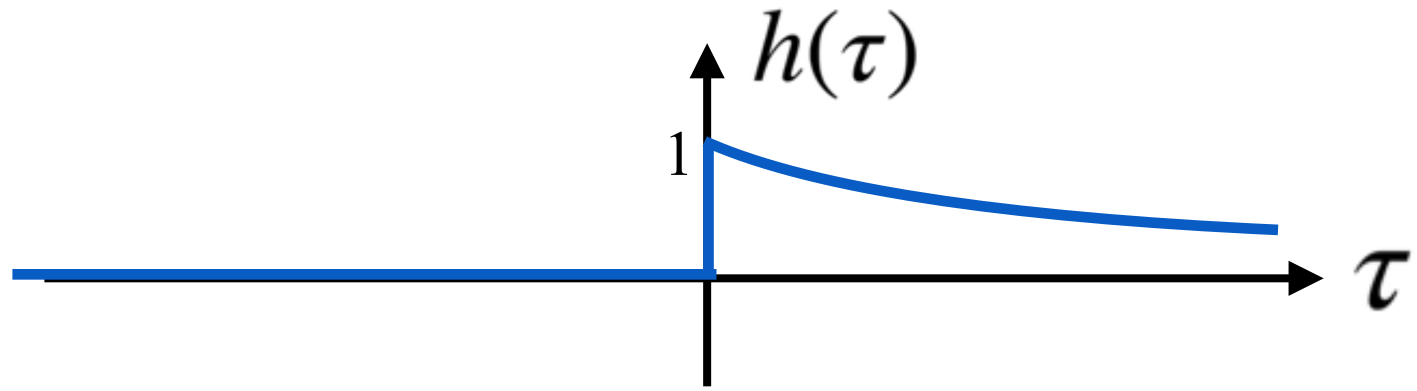
- <u>Example</u>: Compute

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

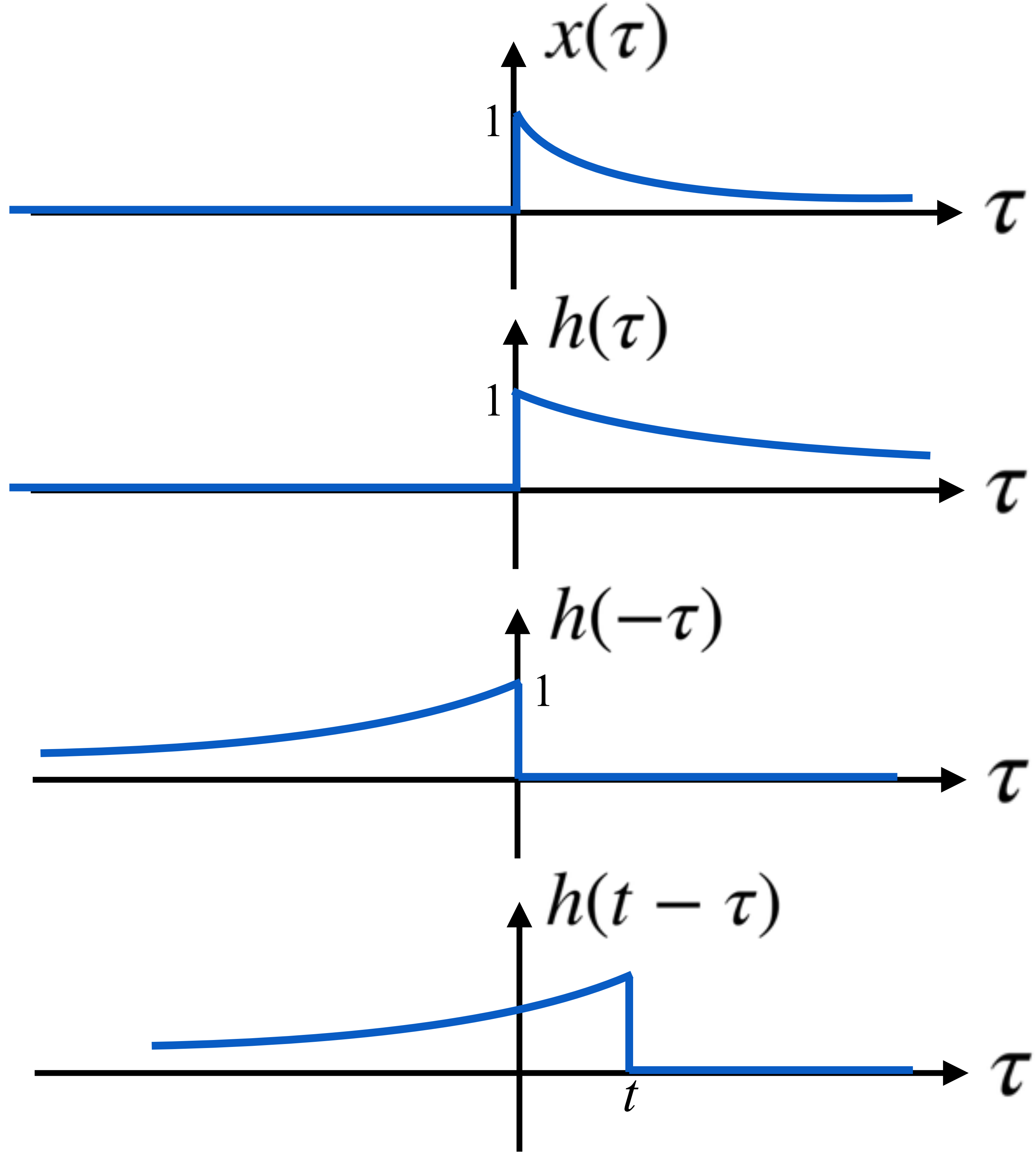
for $x(t) = e^{-2t}u(t)$ and $h(t) = e^{-t}u(t)$

- <u>Solution</u>:





The convolution integral



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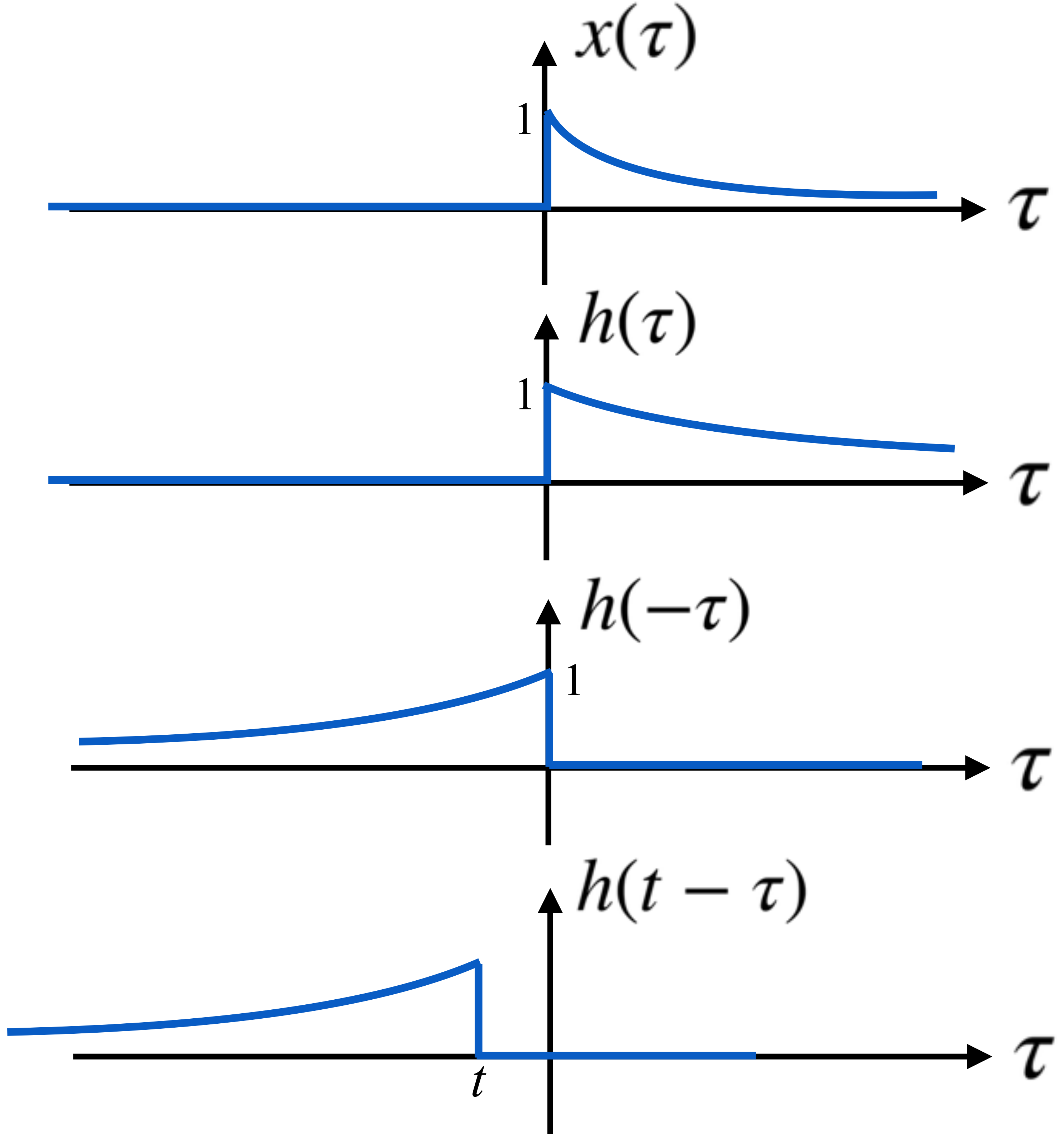
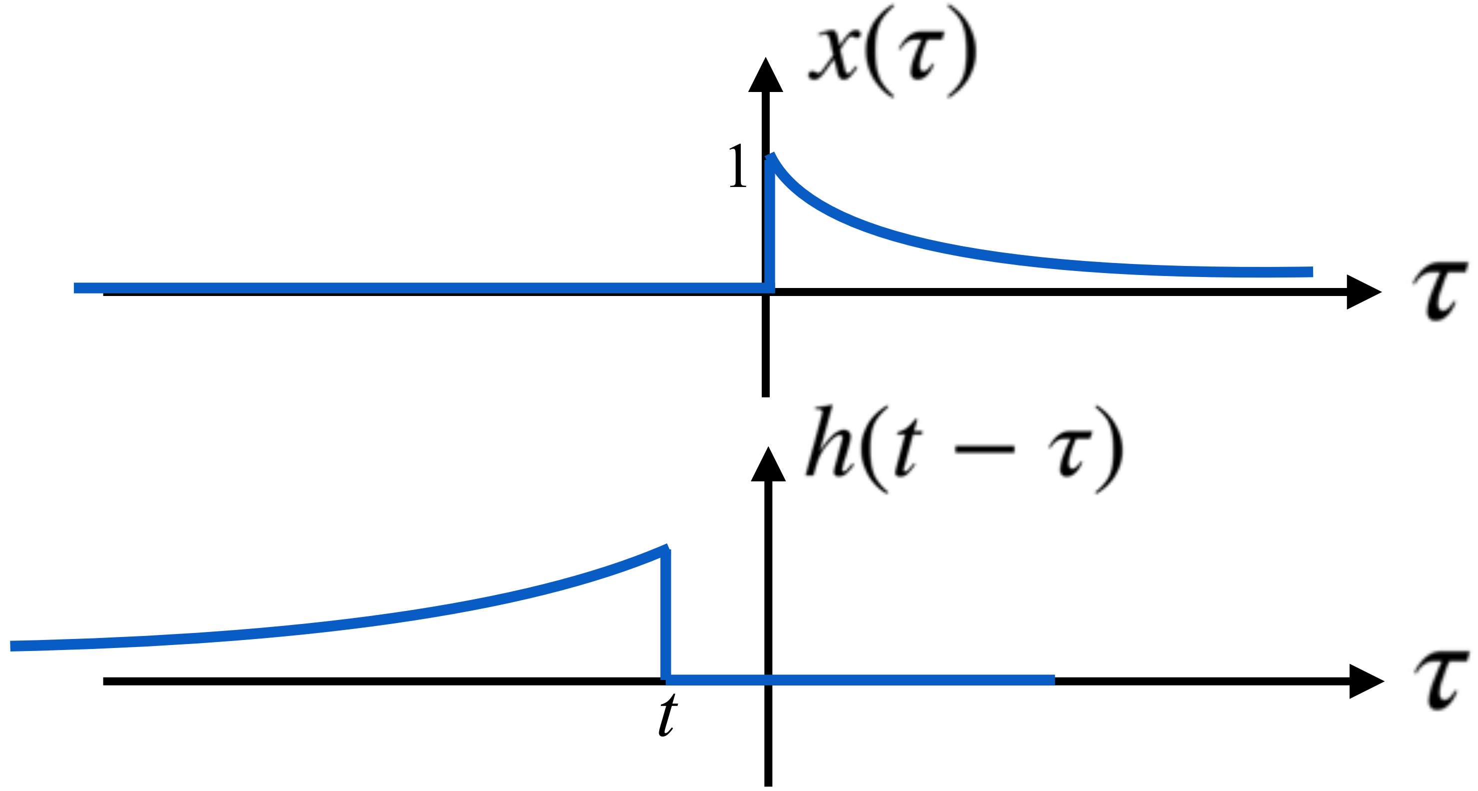
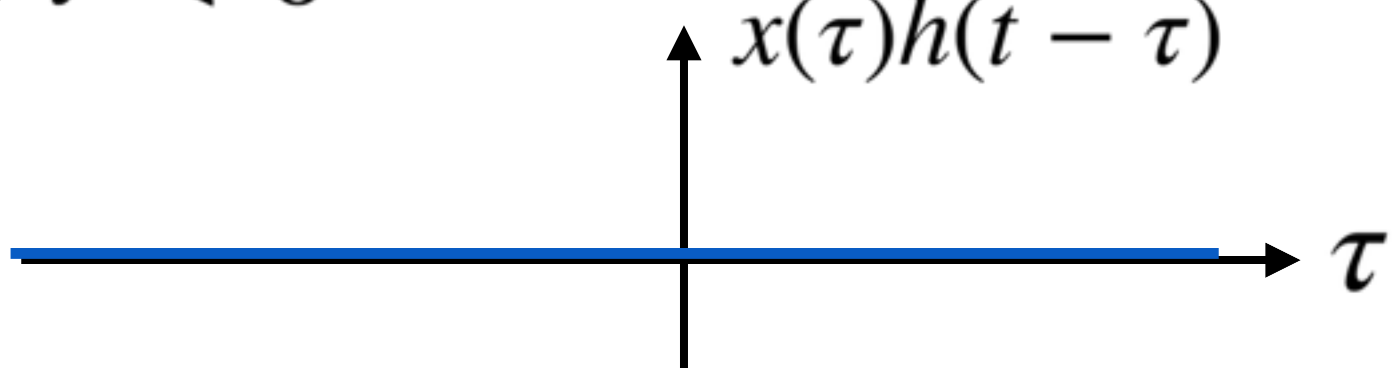


The convolution integral

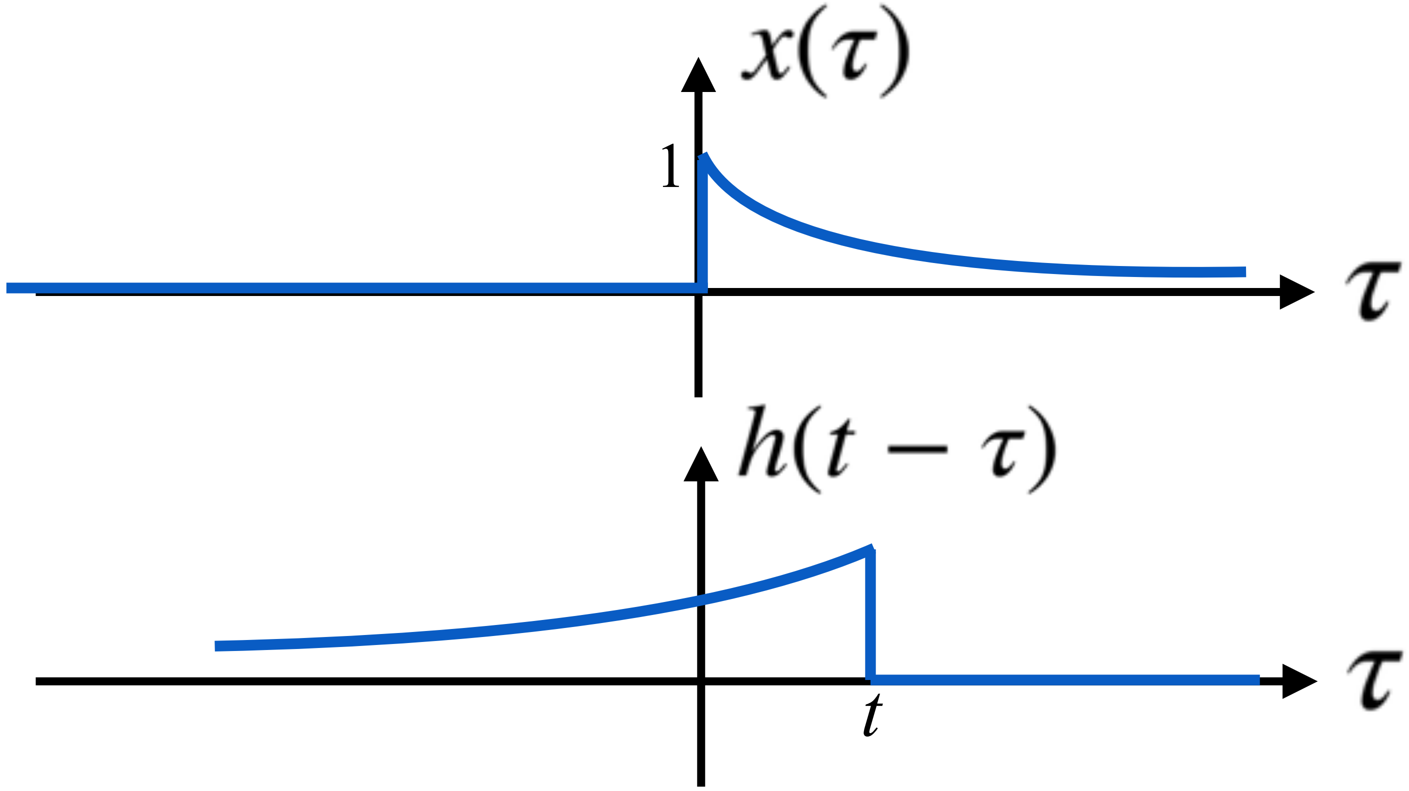


- **Case 1**: $t < 0$

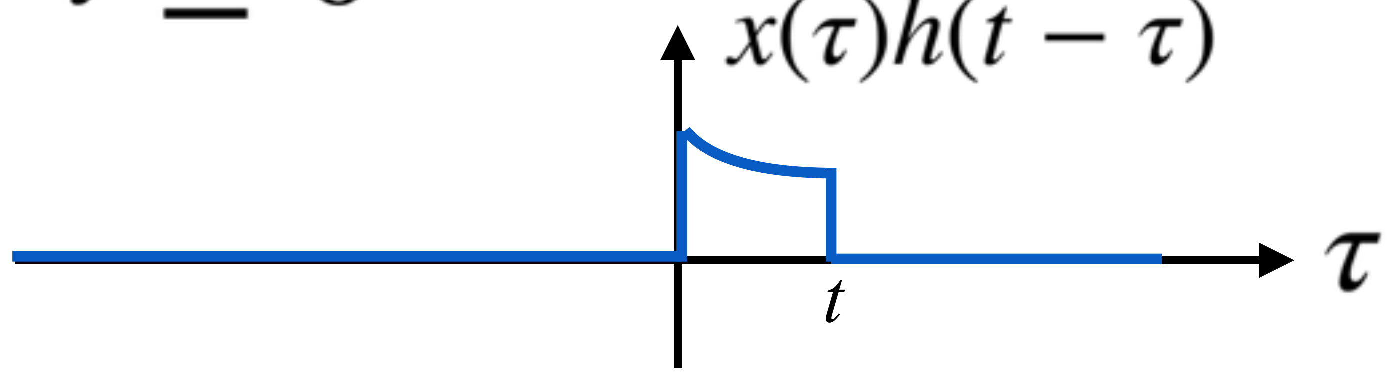


$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$

The convolution integral



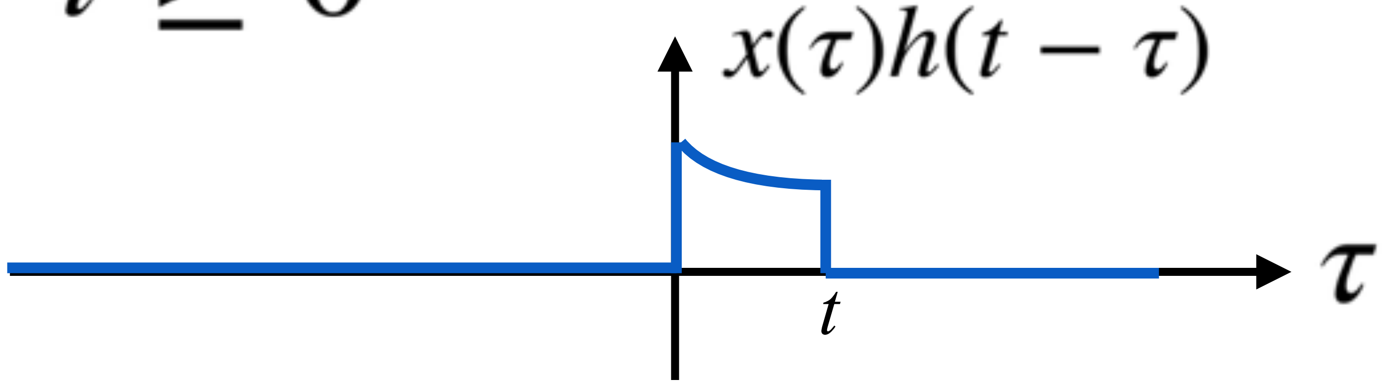
- **Case 2:** $t \geq 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \ = \int_{0}^{t} e^{-2\tau}e^{-(t-\tau)}d\tau \ = \int_{0}^{t} e^{-t-\tau}d\tau$$

The convolution integral

- **Case 2**: $t \geq 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^t e^{-2\tau}e^{-(t-\tau)}d\tau = \int_0^t e^{-t-\tau}d\tau$$

$$= e^{-t}\int_0^t e^{-\tau}d\tau$$

$$= e^{-t}(1-e^{-t})$$

The convolution integral

- Therefore,

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} 0 & t < 0 \\ e^{-t}(1-e^{-t}) & t \geq 0 \end{cases}$$

$$= e^{-t}(1-e^{-t})u(t)$$

Convolution as an operator

- It is very common to write $x(t) \star h(t)$ for

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Unlike the common operators $+, -, \times, \div$, the operator $\star$ requires knowledge on all time instants
- It still shares some nice properties with those common operators

Properties of convolution

- **Commutativity:**

$$x(t) \star h(t) = h(t) \star x(t)$$

- <u>Proof:</u>

$$x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$\overset{z=t-\tau}{=} \int_{\infty}^{-\infty} x(t-z)h(z)(-dz)$$

$$= \int_{-\infty}^{\infty} h(z)x(t-z)dz = h(t) \star x(t)$$

Properties of convolution

- **Associativity:**

$$x(t) \star [y(t) \star z(t)] = [x(t) \star y(t)] \star z(t)$$

- <u>Proof:</u>

$$x(t) \star [y(t) \star z(t)] = x(t) \star \int_{-\infty}^{\infty} y(\tau) z(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} x(\sigma) \int_{-\infty}^{\infty} y(\tau) z(t-\sigma-\tau) d\tau d\sigma$$

$$\overset{\rho=\sigma+\tau}{=} \int_{-\infty}^{\infty} x(\sigma) \int_{-\infty}^{\infty} y(\rho-\sigma) z(t-\rho) d\rho d\sigma$$

Properties of convolution

$$x(t) \star [y(t) \star z(t)] \overset{\rho=\sigma+\tau}{=} \int_{-\infty}^{\infty} x(\sigma) \int_{-\infty}^{\infty} y(\rho - \sigma) z(t - \rho) d\rho d\sigma$$

$$= \int_{-\infty}^{\infty} z(t - \rho) \int_{-\infty}^{\infty} x(\sigma) y(\rho - \sigma) d\sigma d\rho$$

$$= \int_{-\infty}^{\infty} z(t - \rho) \Big[x(\rho) \star y(\rho) \Big] d\rho$$

$$= \int_{-\infty}^{\infty} \Big[x(\rho) \star y(\rho) \Big] z(t - \rho) d\rho$$

$$= [x(t) \star y(t)] \star z(t)$$

Properties of convolution

- **Distribution:**

$$[ax_1(t) + bx_2(t)] \star h(t) = a[x_1(t) \star h(t)] + b[x_2(t) \star h(t)]$$

- <u>Proof</u>: Follows directly from the fact that convolution of the input with the impulse response yields the output for **linear** and time-invariant systems.

Properties of convolution

- Using the same logic with **time-invariance**, if

$$x(t) \star h(t) = y(t)$$

then

$$x(t - t_0) \star h(t) = y(t - t_0)$$

- Thanks to commutativity, we also have

$$x(t - t_1) \star h(t - t_2) = y(t - [t_1 + t_2])$$

Properties of convolution

- **Time-reversal:**

$$x(t) \star h(t) = y(t)$$

implies

$$x(-t) \star h(-t) = y(-t)$$

- <u>Proof:</u>

$$\begin{aligned} x(-t) \star h(-t) &= \int_{-\infty}^{\infty} x(-\tau)h(\tau - t)d\tau \\ &\overset{\sigma=-\tau}{=} \int_{-\infty}^{\infty} x(\sigma)h(-t - \sigma)d\sigma \\ &= y(-t) \end{aligned}$$

Properties of convolution

- **Identity element:**

$$x(t) \star \delta(t) = x(t)$$

- Proof:

$$x(t) \star \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

due to the sifting property of the impulse

System properties revisited

- For an LTI system, we can tell whether the system is **memoryless**, **causal**, **stable**, or **invertible** just by analyzing the impulse response.
- It may be more convenient to write the convolution integral as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- This better shows how $y(t)$ depends on various samples of $x(t)$.

System properties revisited

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- $h(\tau) \neq 0$ for any $\tau \neq 0$ shows that $y(t)$ depends on the input at some time instant other than t.
  - Then the system **has memory.**
- $h(\tau) \neq 0$ for any $\tau < 0$ shows that $y(t)$ depends on the input at some future time instant $t - \tau$.
  - Then the system is **non-causal.**

System properties revisited

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Conversely, if $h(\tau) = 0$ for all $\tau \neq 0$, the system is **memoryless**.
  - The only interesting example is $h(t) = a\delta(t)$
- Also conversely, if $h(\tau) = 0$ for all $\tau < 0$, the system is causal.
  - All causal systems have an impulse response of the form $h(t) = f(t)u(t)$ for some function $f(t)$.

System properties revisited

- For **stability**, let us analyze $|y(t)|$:

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right|$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)|\ |x(t-\tau)|\, d\tau$$

- Now, if $|x(t)|$ is bounded by B for all t,

$$|y(t)| \leq B \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

System properties revisited

$$|y(t)| \leq B \int_{-\infty}^{\infty} |h(\tau)| \, d\tau$$

- Therefore, a **sufficient** condition for stability is

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty}$$

System properties revisited

- Therefore, a **sufficient** condition for stability is

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty}$$

- Also **necessary** because otherwise, select the obviously bounded $x(t) = \mathrm{sign}[h(-t)]$ to obtain

$$y(0) = \int_{-\infty}^{\infty} h(\tau)x(-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\mathrm{sign}[h(\tau)]d\tau$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \infty$$

System properties revisited

- An LTI system is **invertible** if and only if no two distinct $x_1(t)$ and $x_2(t)$ exist such that

$$x_1(t) \star h(t) = x_2(t) \star h(t)$$

or equivalently,

$$[x_1(t) - x_2(t)] \star h(t) = 0$$

- In other words, the system is invertible if and only if no non-zero input signal $x(t)$ satisfies

$$x(t) \star h(t) = 0$$

Examples

- <u>Example</u>: Determine if the system is memoryless, causal, stable, or invertible if its impulse response is given by

$$h(t) = e^{-3t}u(t)$$

- **Memory:** $h(t)$ is not of the form $a\delta(t)$

HAS MEMORY

- **Causality:** $h(t)$ <u>is</u> of the form $f(t)u(t)$

CAUSAL ✓

- **Stability:**

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{0}^{\infty} e^{-3\tau} d\tau = -\frac{1}{3} e^{-3\tau}\Big|_{\tau=0}^{\tau=\infty} = \frac{1}{3}$$

STABLE ✓

Examples

- Example: Determine if the system is memoryless, causal, stable, or invertible if its impulse response is given by

$$h(t) = e^{-3t}u(t)$$

- **Invertibility:** We need to see whether or not there exists nonzero $x(t)$ such that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$

- That is the same as $\displaystyle\int_{-\infty}^{t} x(\tau)e^{-3(t-\tau)}d\tau = 0$

Examples

- **Invertibility:** We need to see whether or not there exists nonzero $x(t)$ such that

$$\int_{-\infty}^{t} x(\tau)e^{-3(t-\tau)}d\tau = 0$$

or

$$e^{-3t}\int_{-\infty}^{t} x(\tau)e^{3\tau}d\tau = 0$$

- Since $e^{-3t} \neq 0$ for any t, this is the same as

$$\int_{-\infty}^{t} x(\tau)e^{3\tau}d\tau = 0$$

Examples

- **Invertibility:** We need to see whether or not there exists nonzero $x(t)$ such that

$$\int_{-\infty}^{t} x(\tau)e^{3\tau}d\tau = 0$$

- Differentiating both sides w.r.t. t, we get

$$x(t)e^{3t} = 0$$

- But that is possible only if $x(t) = 0$

INVERTIBLE

✓

Examples

- <u>Example</u>: Determine if the system is memoryless, causal, stable, or invertible if its impulse response is given by

$$h(t) = \cos(2t)$$

- **Memory:** $h(t)$ is not of the form $a\delta(t)$

HAS MEMORY

- **Causality:** $h(t)$ is not of the form $f(t)u(t)$

NON-CAUSAL

- **Stability:**

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{-\infty}^{\infty} |\cos(2t)|\, d\tau \quad = \infty$$

UNSTABLE

Examples

- Example: Determine if the system is memoryless, causal, stable, or invertible if its impulse response is given by

$$h(t) = \cos(2t)$$

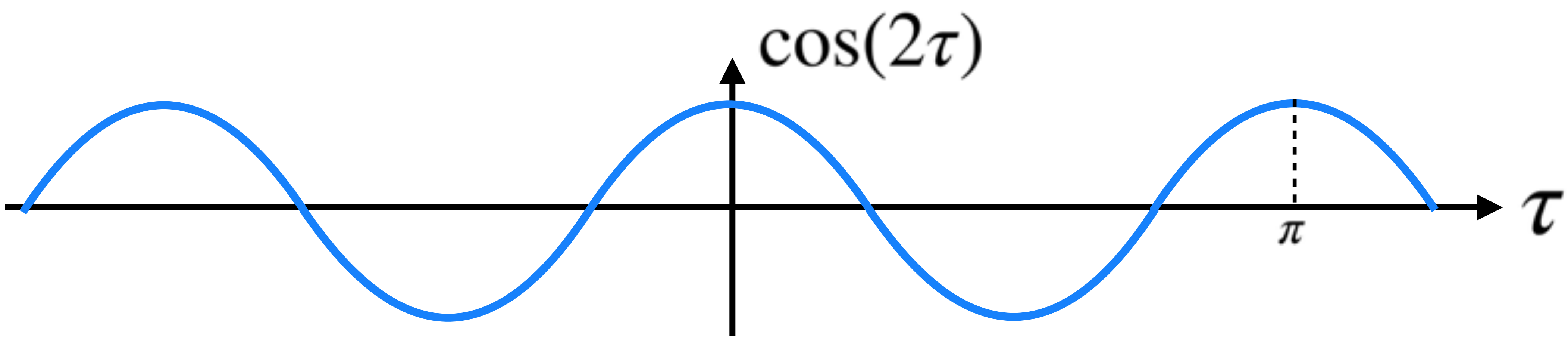
- **Invertibility:** Does there exist $x(t)$ with

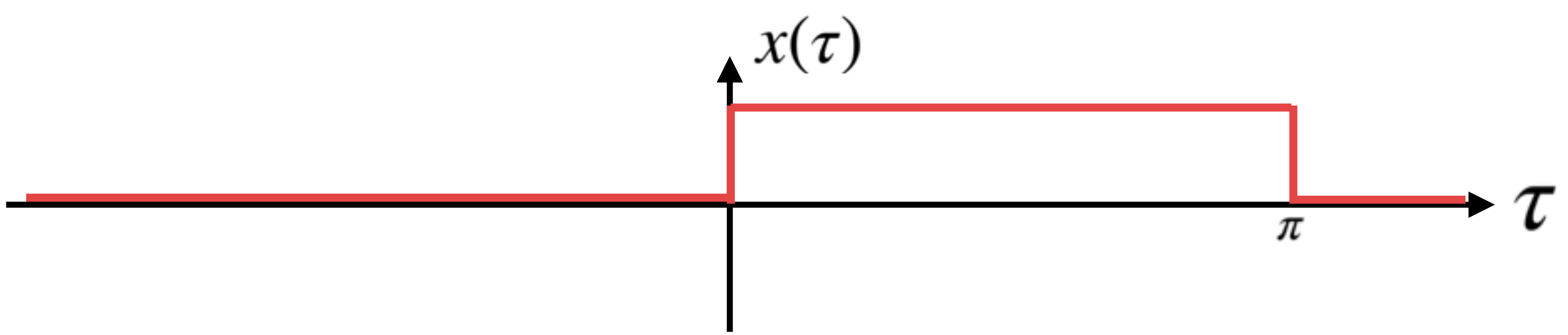
$$\int_{-\infty}^{\infty} x(t-\tau)\cos(2\tau)d\tau = 0 \quad ?$$

Examples

- **Invertibility:** Does there exist $x(t)$ with

$$\int_{-\infty}^{\infty} x(t-\tau)\cos(2\tau)d\tau = 0 \quad ?$$

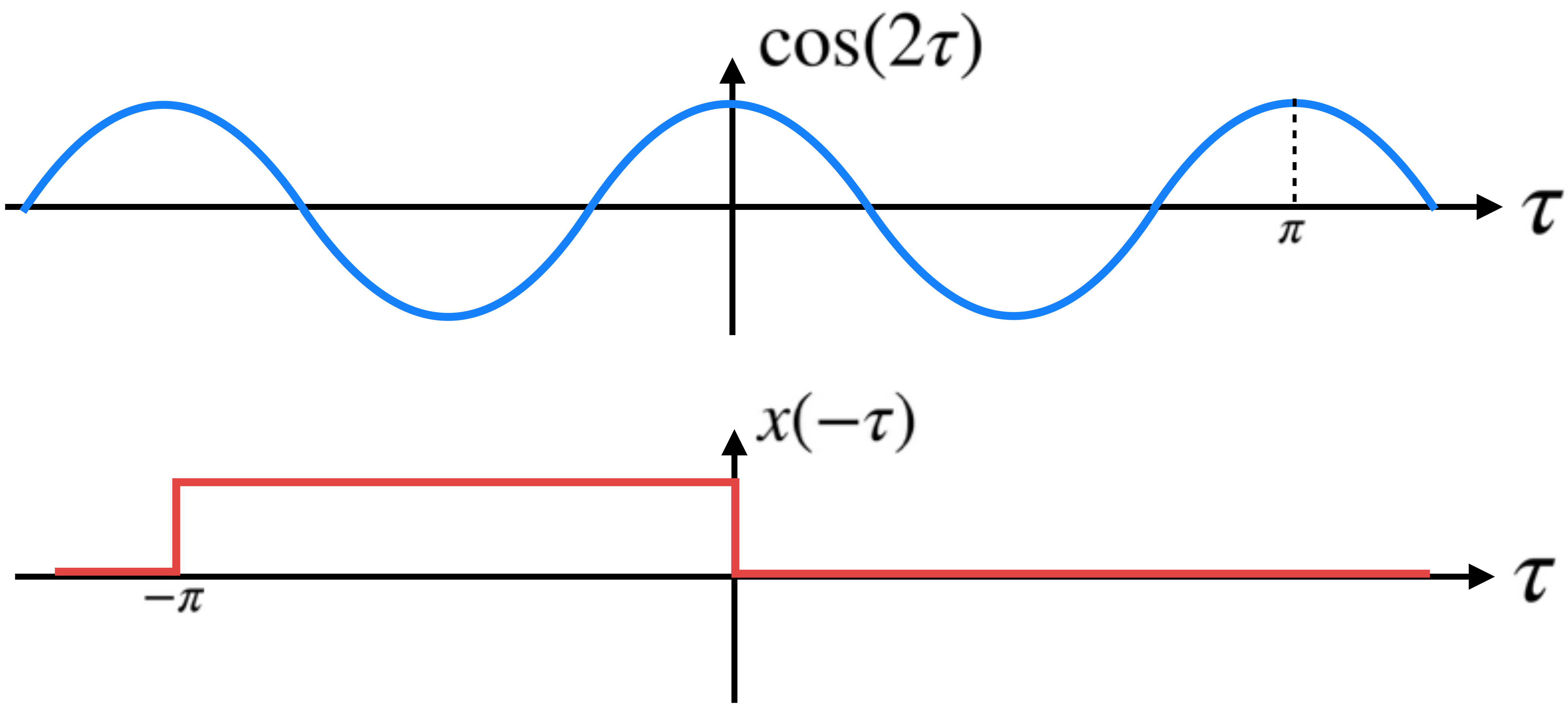




Examples

- **Invertibility:** Does there exist $x(t)$ with

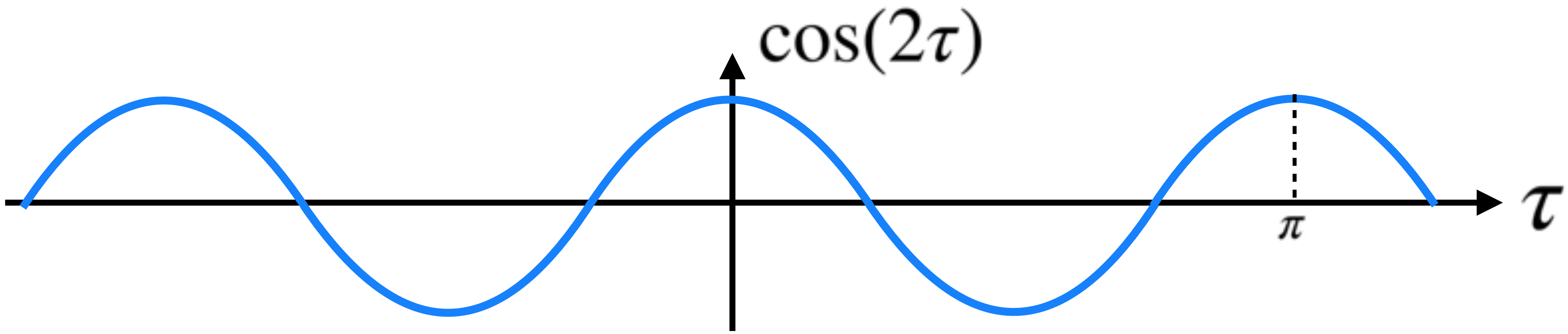
$$\int_{-\infty}^{\infty} x(t-\tau)\cos(2\tau)d\tau = 0 \quad ?$$

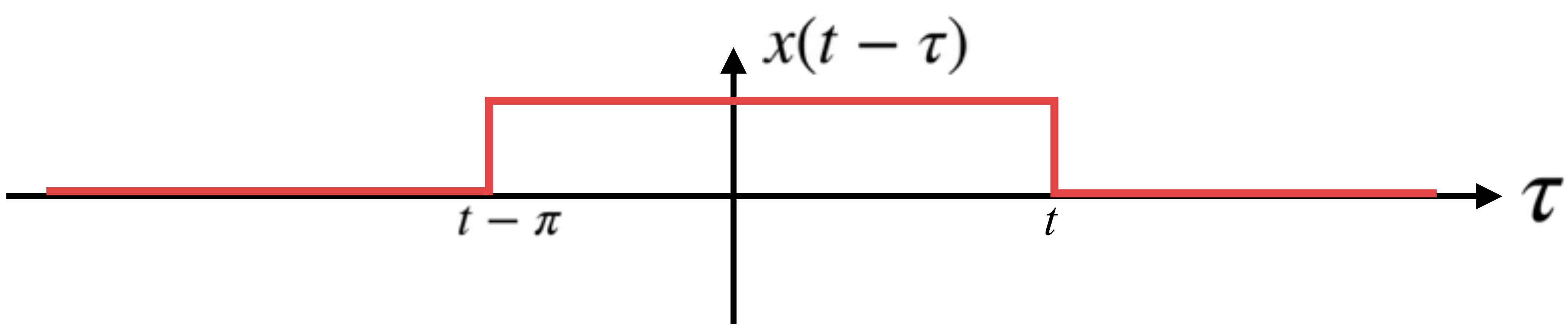


Examples

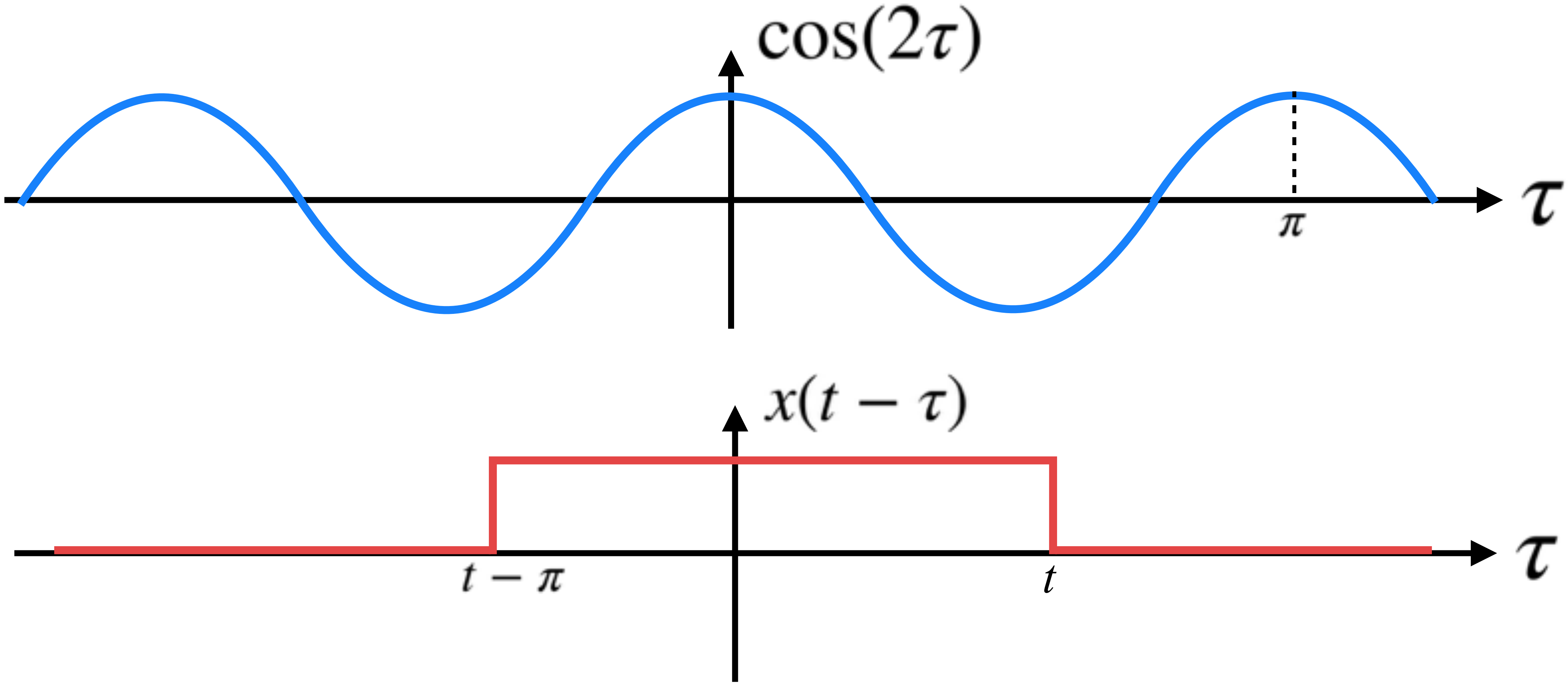
- **Invertibility:** Does there exist $x(t)$ with

$$\int_{-\infty}^{\infty} x(t-\tau)\cos(2\tau)d\tau = 0 \quad ?$$





Examples



$$\int_{-\infty}^{\infty} x(t-\tau)\cos(2\tau)d\tau = \int_{t-\pi}^{t} \cos(2\tau)d\tau \;= 0$$

NOT INVERTIBLE