EE 110B Signals and Systems

Introduction to Discrete-time Signals and Systems

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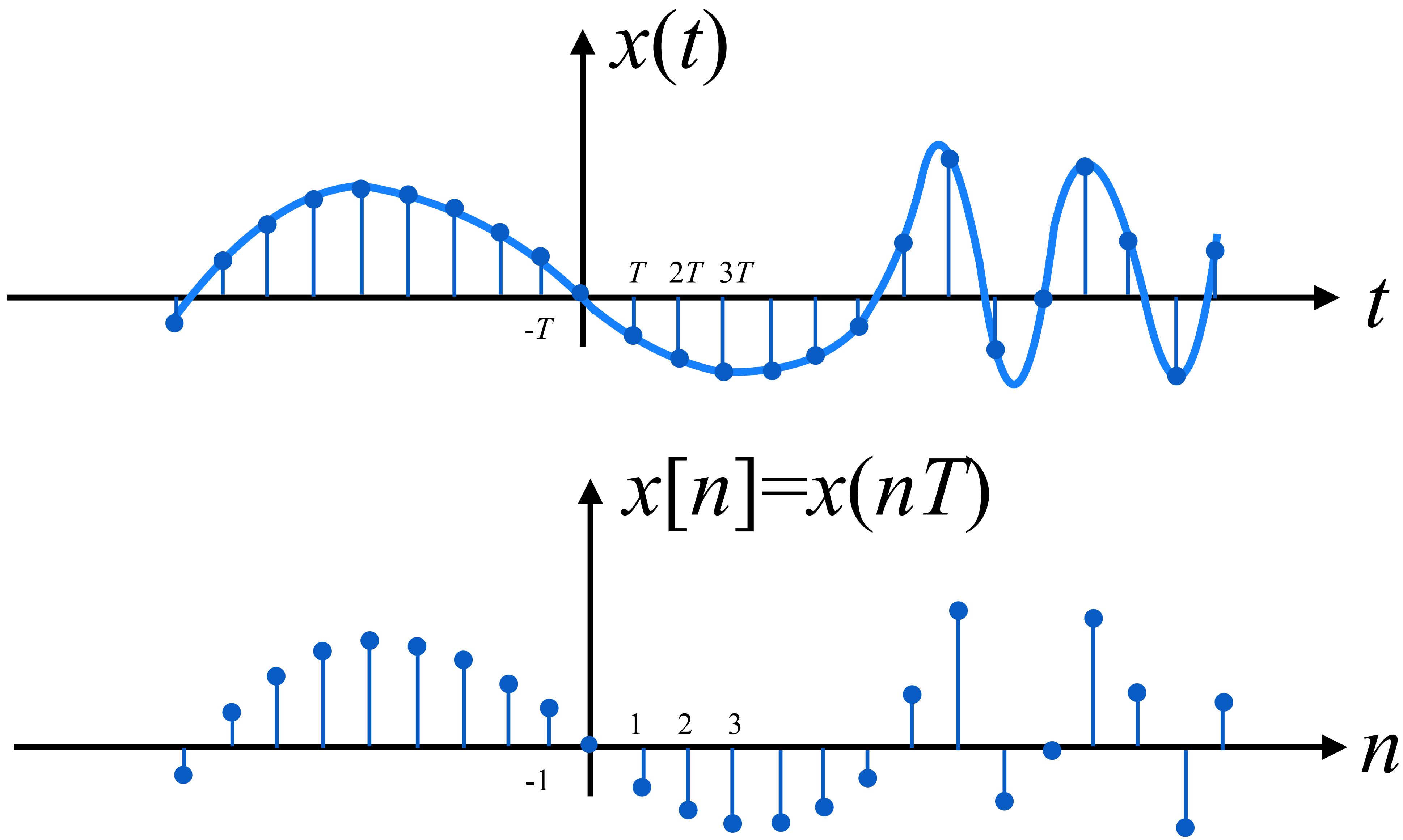
Discrete-time signals

- **Motivation**: We may have access to only periodic samples $x(nT)$ of a signal $x(t)$.



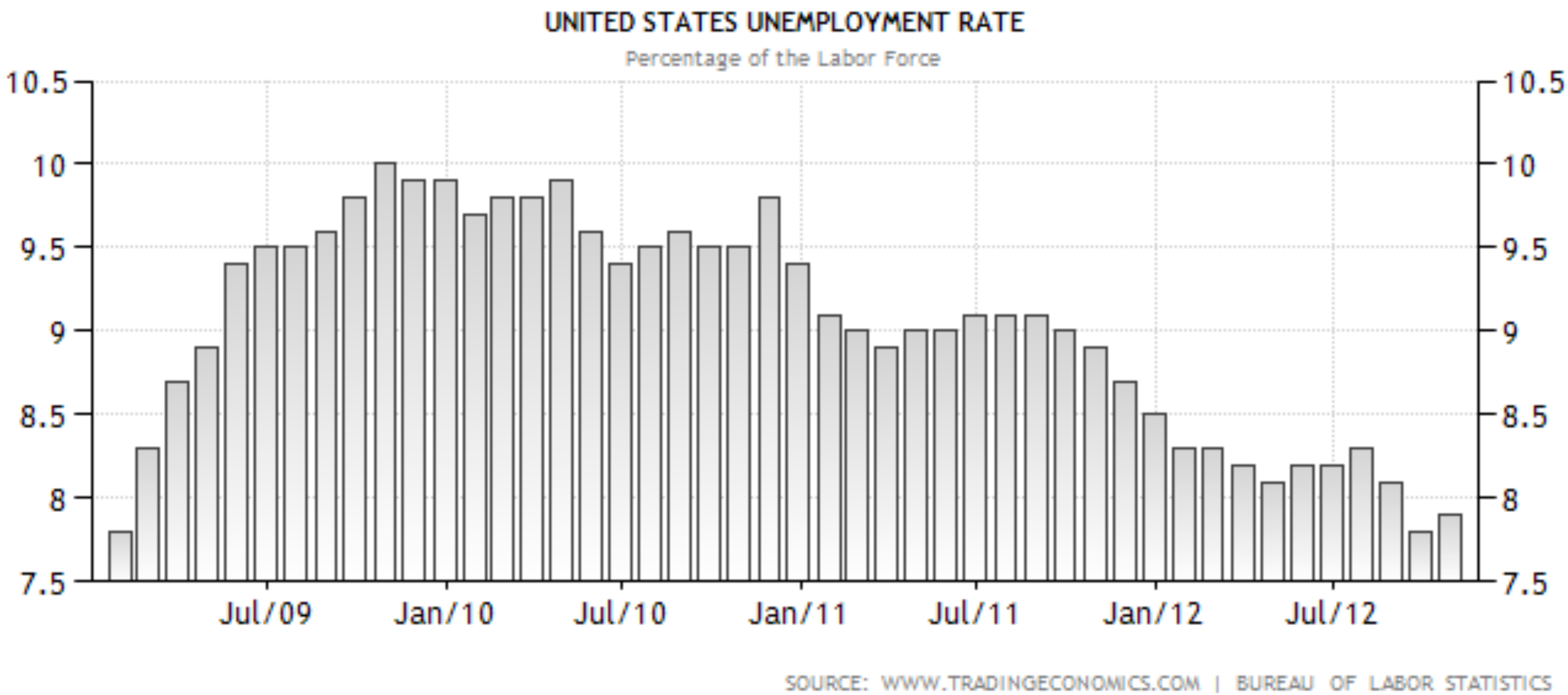
Discrete-time signals

- **Motivation**: We may want to process a signal $x(t)$ using digital means.

- **Motivation**: We may want to store a signal $x(t)$ digitally.

Discrete-time signals

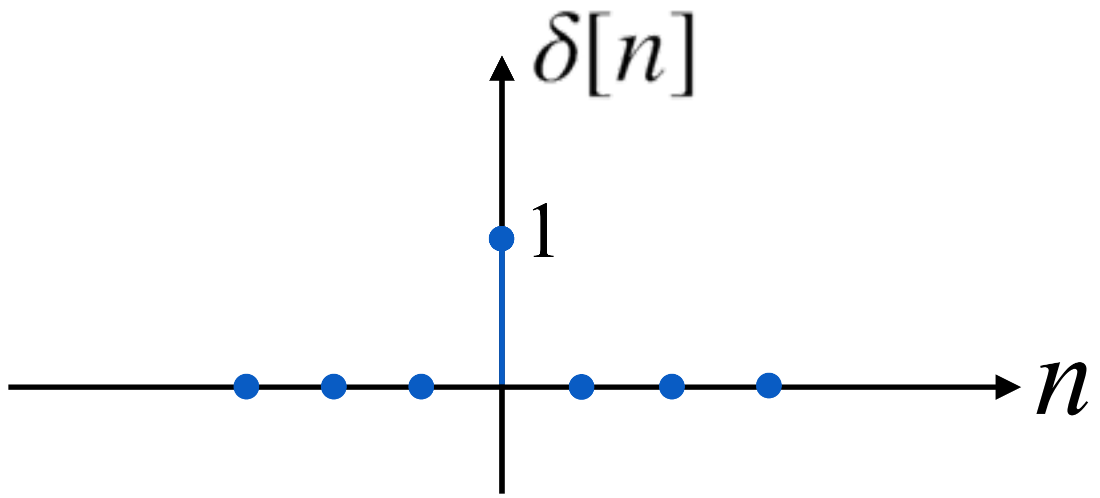
- **Motivation**: The signal might be discrete-time in its nature.



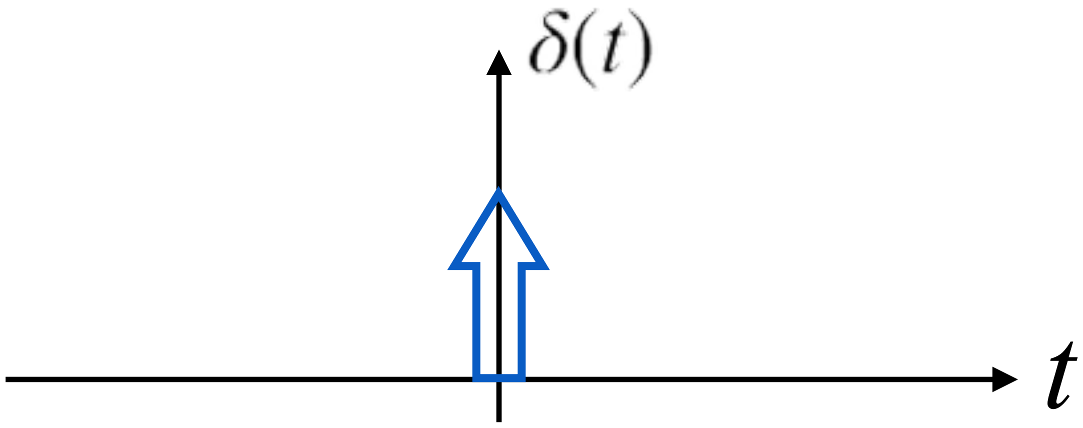
Some important signals

- **The impulse signal:**

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$



- Compare with the continuous-time version:

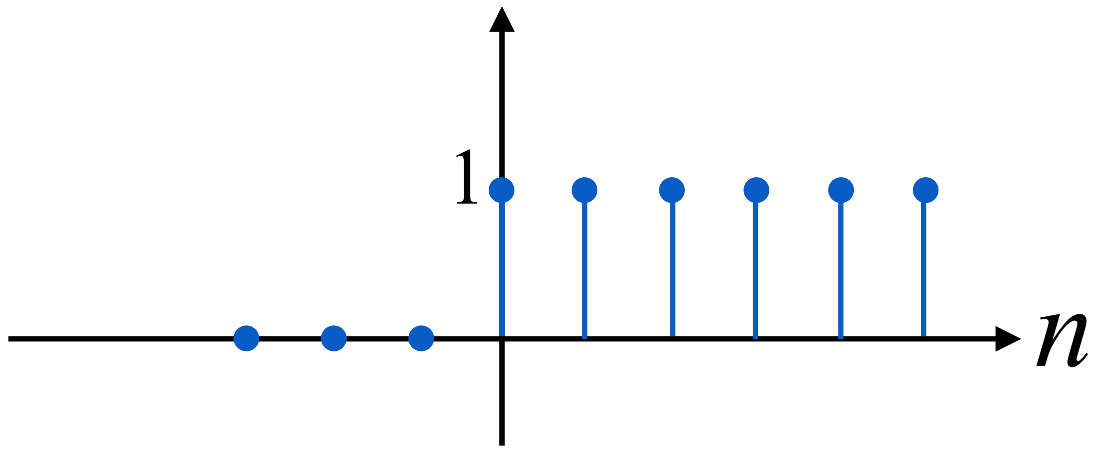


- For this case only, $\delta[n] \neq \delta(nT)$

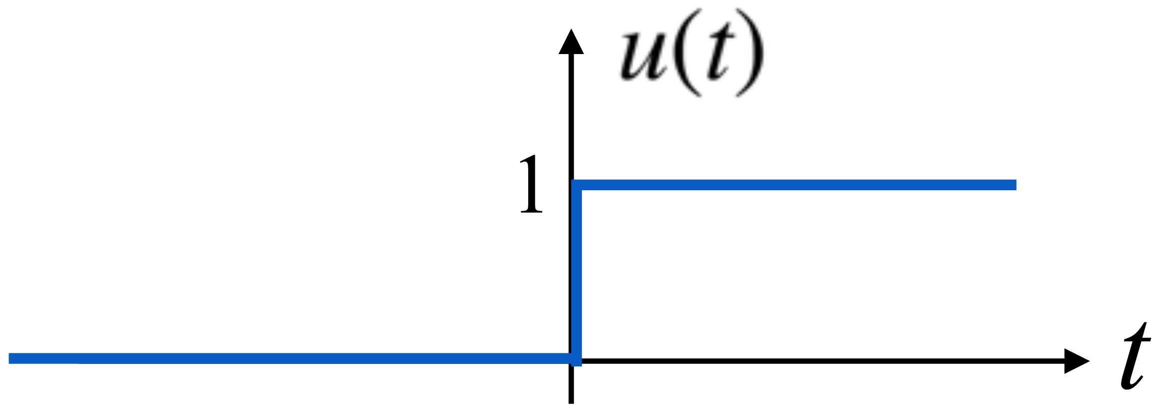
Some important signals

- **The unit step signal:**

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



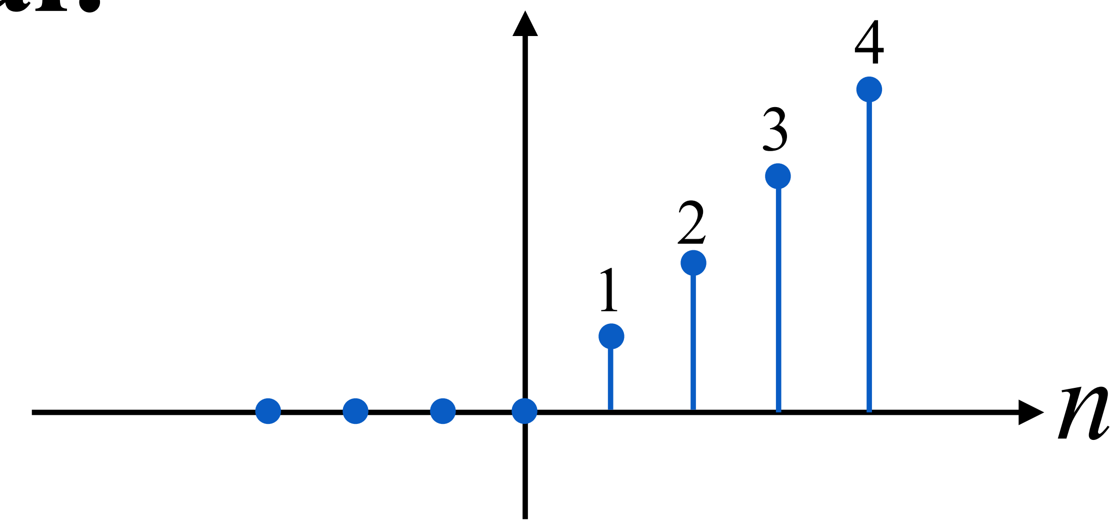
- Compare with the continuous-time version:



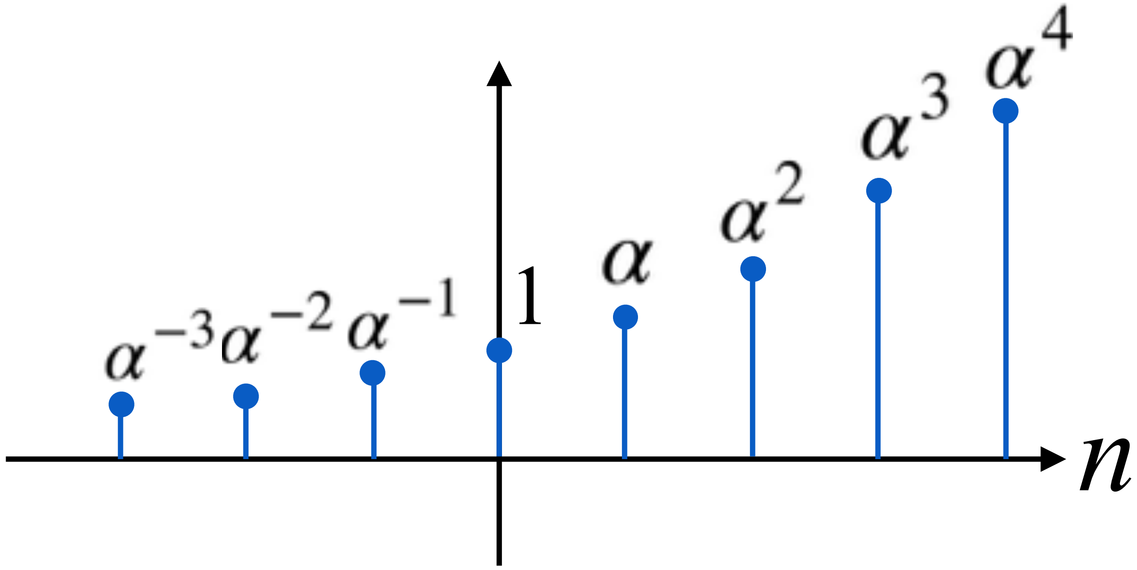
Some important signals

- **The unit ramp signal:**

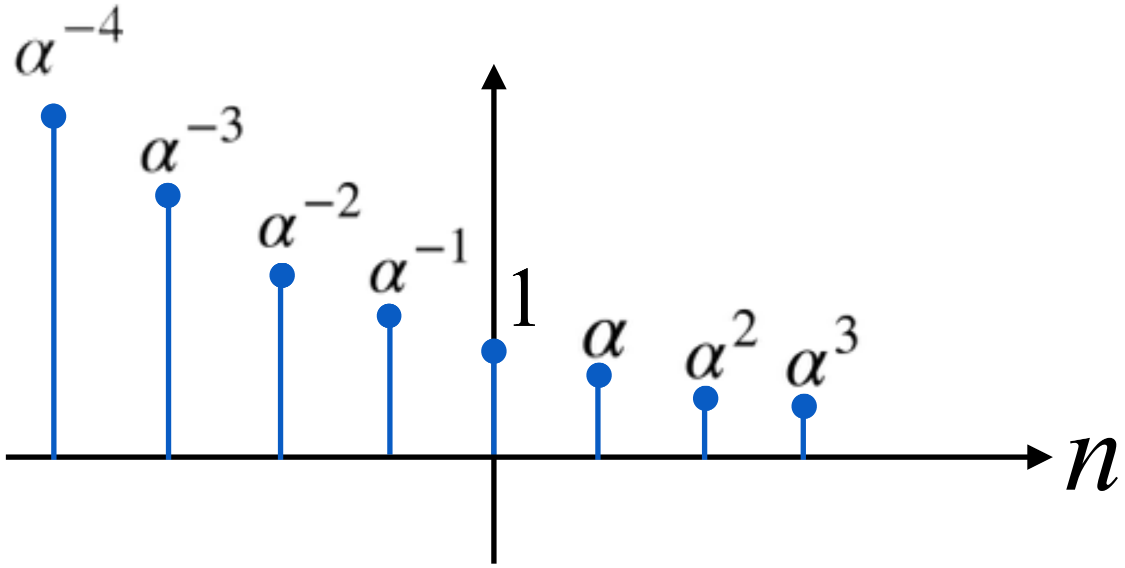
$$r[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- **The exponential signal:** $x[n] = \alpha^n$



$\alpha > 1$



$\alpha < 1$

Some important signals

- **Some relations between these signals:**
  - The impulse signal can be seen as the difference between consecutive samples of the unit step signal:

$$\delta[n] = u[n] - u[n-1]$$

  - Conversely, the unit step signal is the cumulative sum of all samples of the impulse signal up to time instant *n*:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Some important signals

- **Some relations between these signals:**
  - The unit step signal can be seen as the difference between consecutive samples of the unit ramp signal:

$$u[n] = r[n+1] - r[n]$$

  - Conversely, the unit ramp signal is the cumulative sum of all samples of the unit step signal up to time instant n-1:

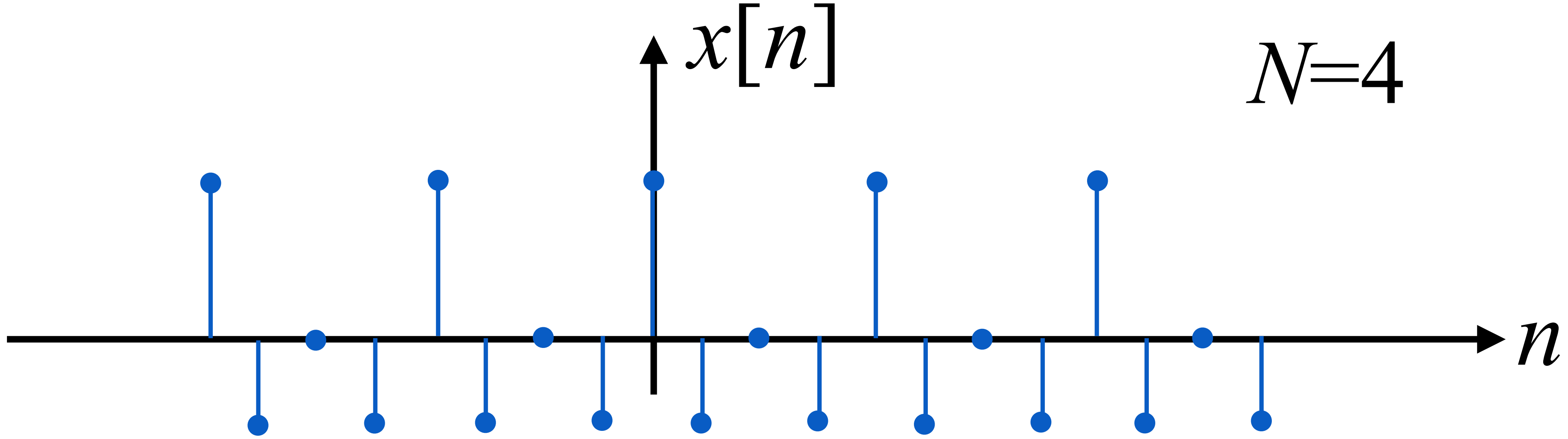
$$r[n] = \sum_{k=-\infty}^{n-1} u[k]$$

Periodic signals

- A signal is said to have a period N if

$$x[n] = x[n+N] \quad \forall n$$

- Example:



- If N is a period, so are $2N$, $3N$, $4N$, ...

Periodic signals

- Another class of examples is sinusoidals:

$$x[n] = \cos(\omega n)$$

- Note that this is periodic if and only if for some integers *N* and *k*,

$$\omega N = 2\pi k$$

- In other words, $\dfrac{\omega}{2\pi}$ must be a rational number.
- Recall that we did not have any such condition for the periodicity of $\cos(\omega t)$

Periodic signals

- Example: Find the smallest period of the signal

$$x[n] = \cos\left(\frac{4\pi}{5}n\right)$$

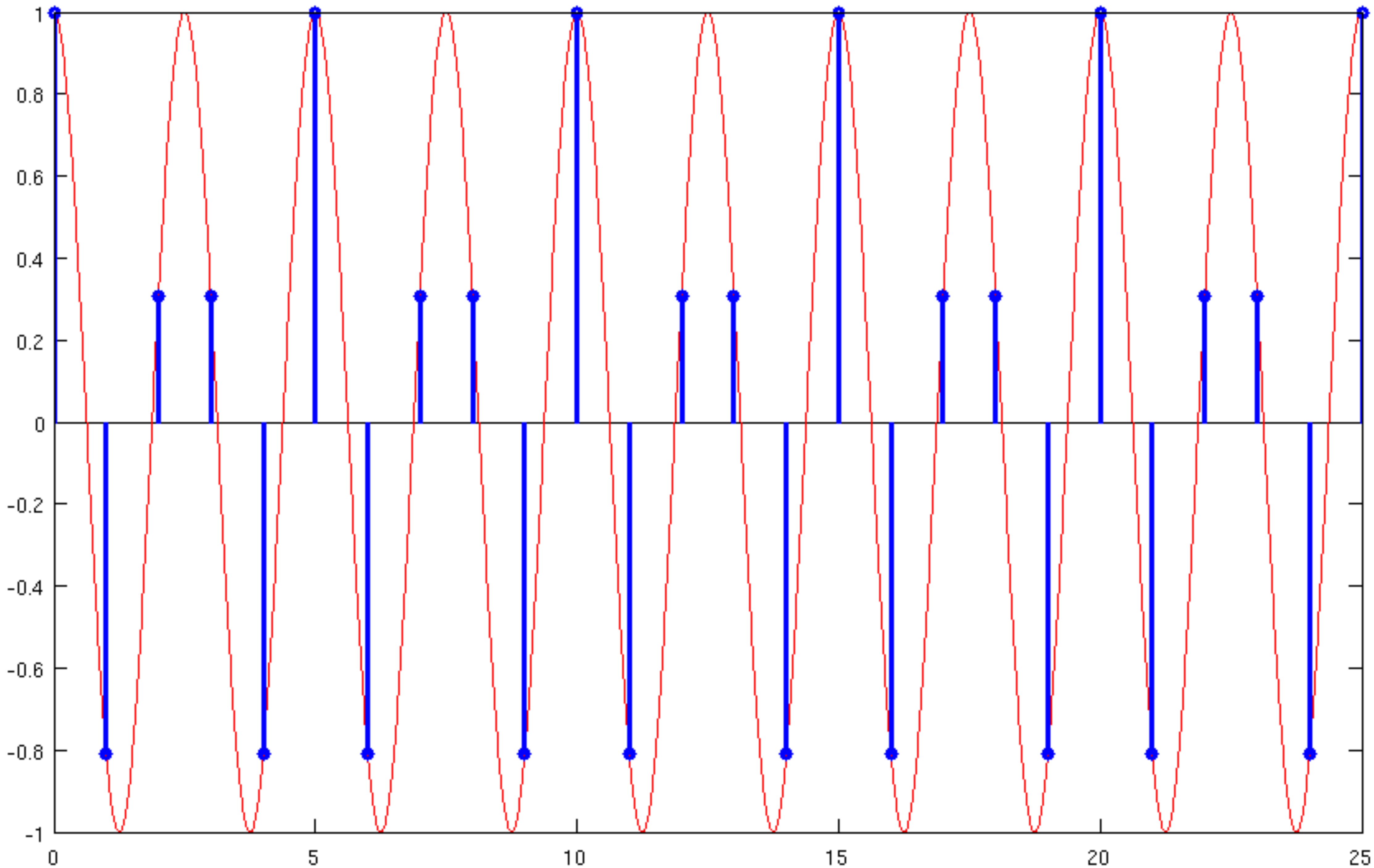
- Solution: We need $\frac{4\pi}{5}N = 2\pi k$, or equivalently,

$$2N = 5k$$

- What is the smallest N such that there exists an integer k satisfying this?

$$N = 5$$

Periodic signals



Periodic signals

- Now, $\frac{4\pi}{5} \approx 2.5133$. So let's see what happens when

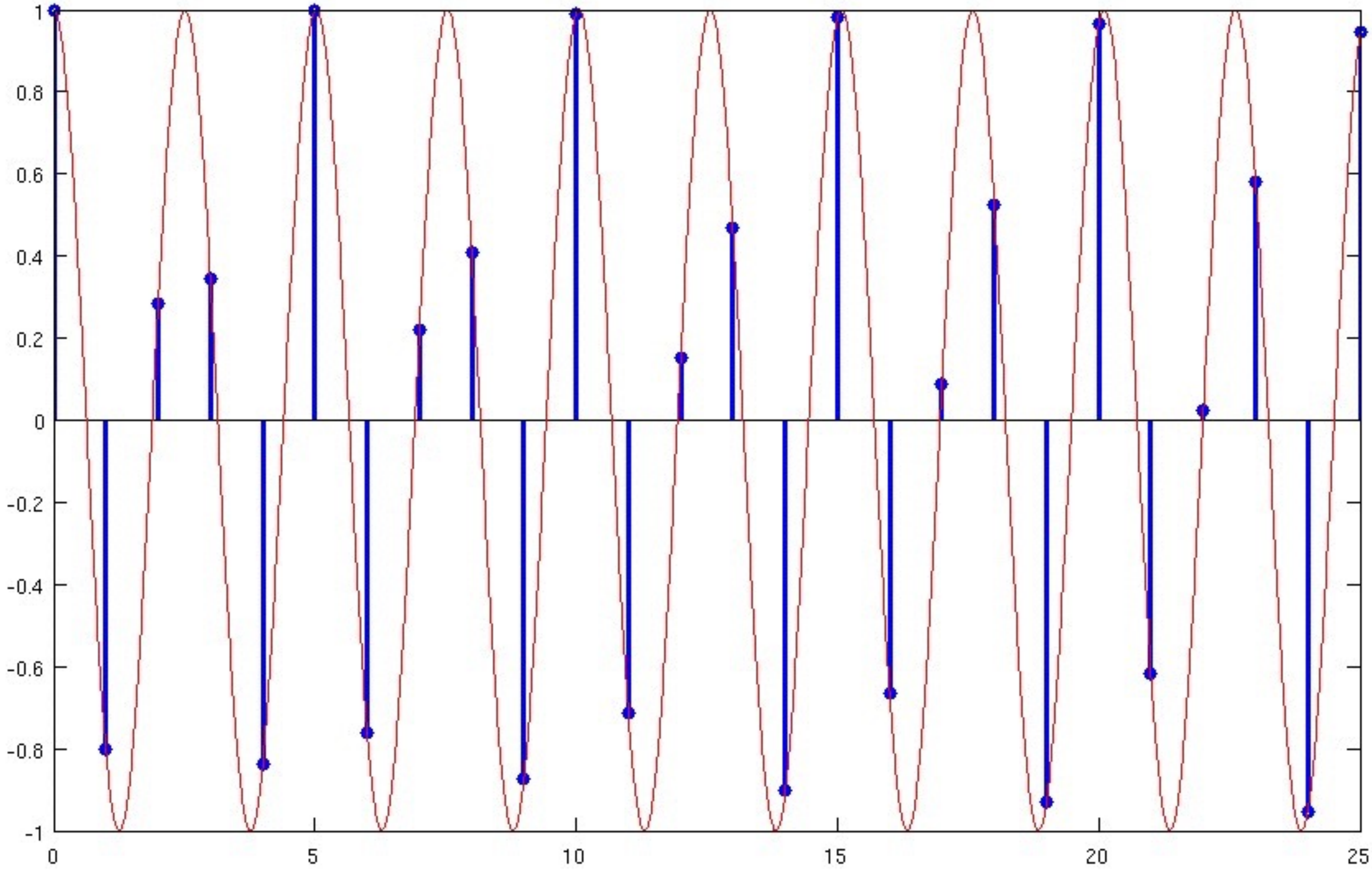
$$x[n] = \cos(2.5n)$$

- Clearly,

$$2.5N = 2\pi k$$

does not have a solution for integer N, k.

Periodic signals



Periodic signals

- Another way to understand this odd behavior is to look at complex exponentials

$$x[n] = e^{j\omega n}$$

- For $x[n]$ to have a period N,

$$e^{j\omega n} = e^{j\omega(n+N)} = e^{j\omega n} e^{j\omega N}$$

implying that

$$e^{j\omega N} = 1$$

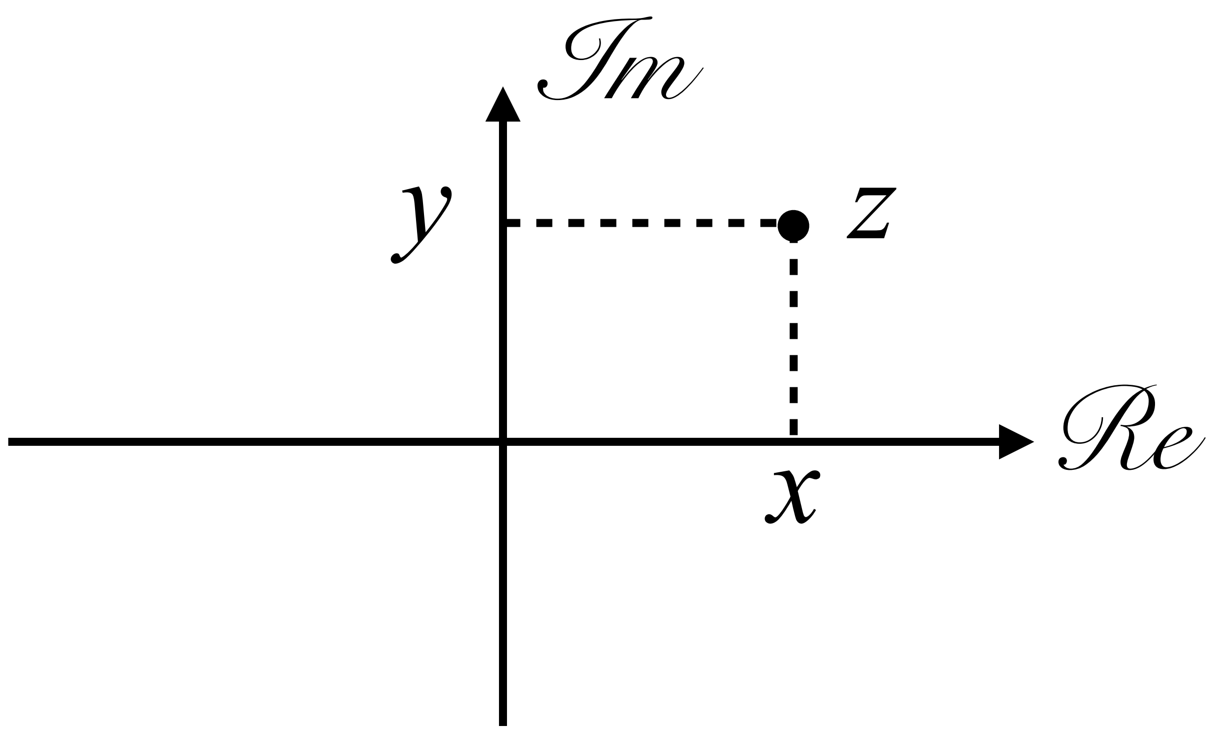
or that

$$\omega N = 2\pi k$$

Digression: Complex algebra

- In rectangular coordinates,

$$z = \underbrace{x}_{\text{Real}} + j\underbrace{y}_{\text{Imaginary}}$$



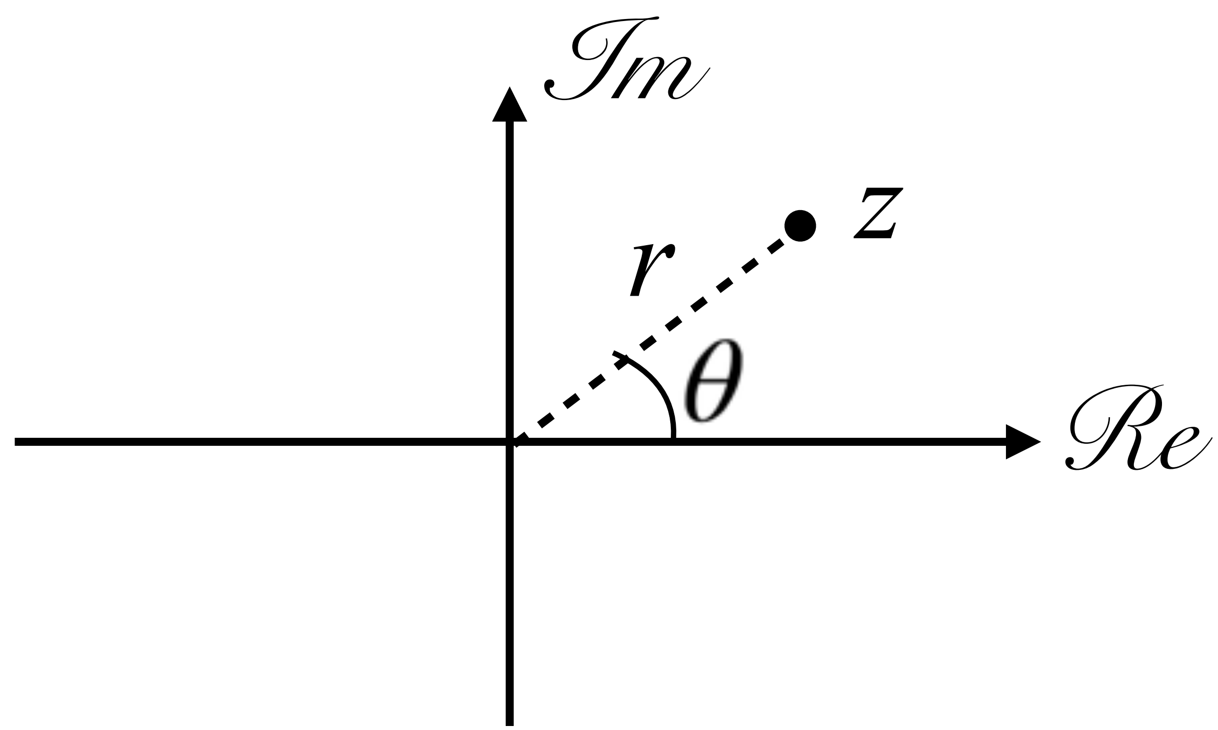
Digression: Complex algebra

- In polar coordinates,

$$z = re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

Angle

Magnitude



Digression: Complex algebra

- In rectangular coordinates,

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$

$$= (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 \times z_2 = (x_1 + jy_1) \times (x_2 + jy_2)$$

$$= x_1x_2 + j(x_1y_2 + x_2y_1) + j^2y_1y_2$$

$$= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$$

Digression: Complex algebra

- In polar coordinates,

$$z_1 \times z_2 = r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= r_1(\cos\theta_1 + j\sin\theta_1) \\ &\quad + r_2(\cos\theta_2 + j\sin\theta_2) \\ &= (r_1\cos\theta_1 + r_2\cos\theta_2) \\ &\quad + j(r_1\sin\theta_1 + r_2\sin\theta_2) \end{aligned}$$

Digression: Complex algebra

- Two important identities:

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos(\theta) \qquad \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin(\theta)$$

- <u>Proof</u>:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta)$$

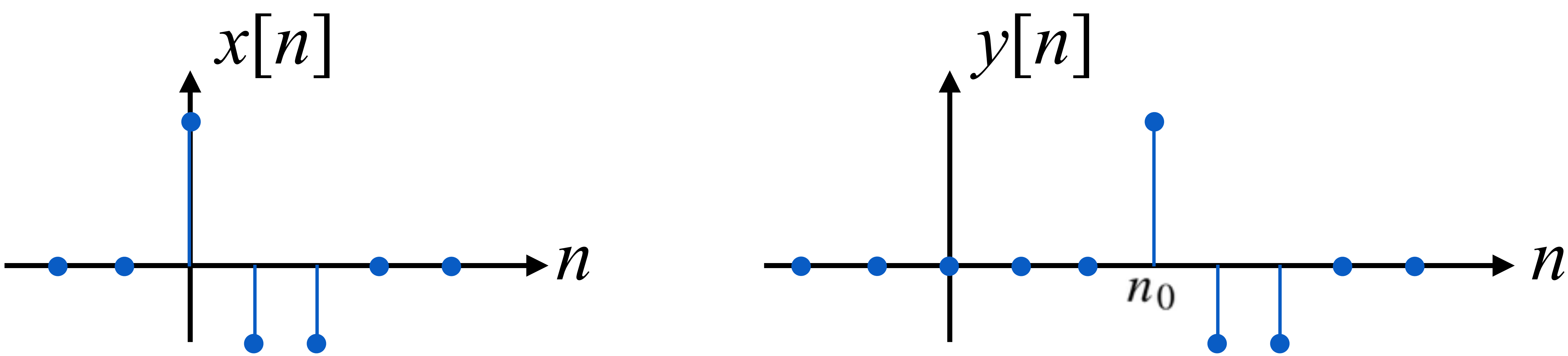
- Add and subtract the two to find the desired result.

Simple signal transformations

- **Time shift:** Let

$$y[n] = x[n - n_0]$$

for some $n_0 > 0$.

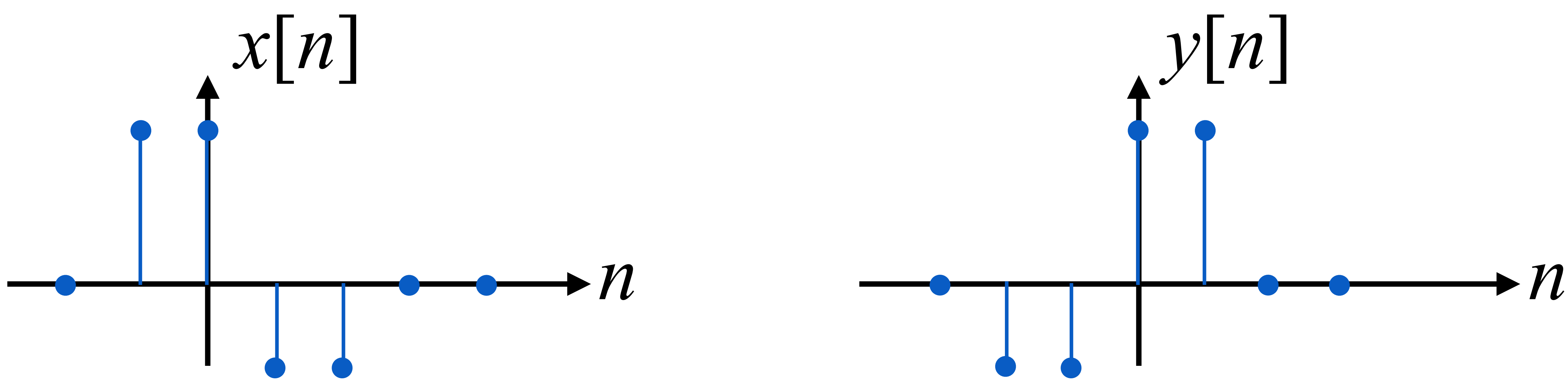


- Why does this cause a right shift?
- The key is to see that the signal *y* copies at time instant *n* the "old value" of *x* at $n - n_0$

Simple signal transformations

- **Time reversal:** Let

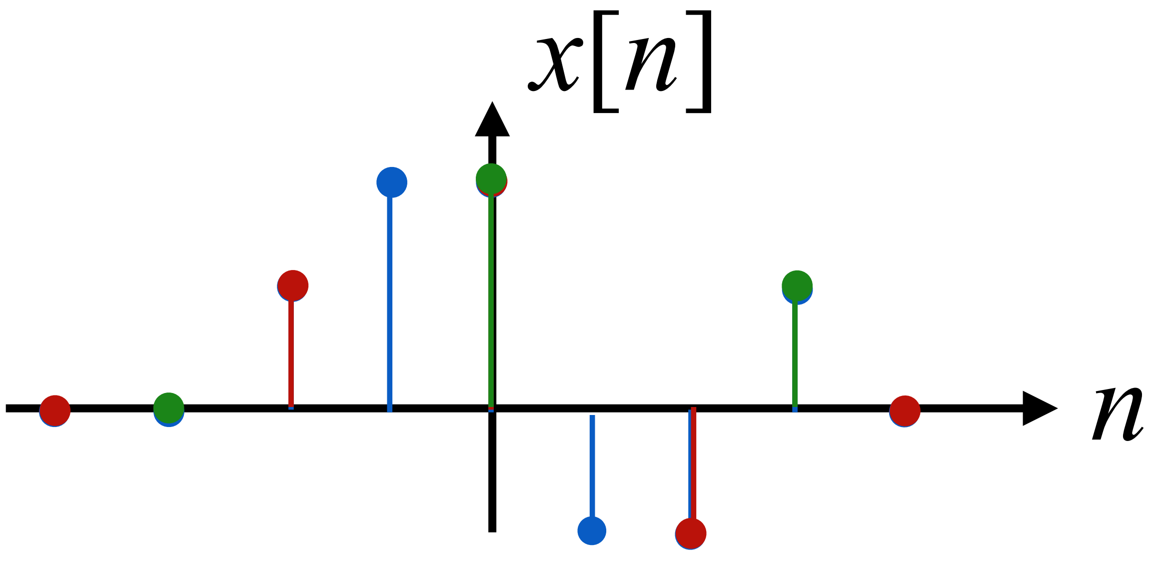
$$y[n] = x[-n]$$

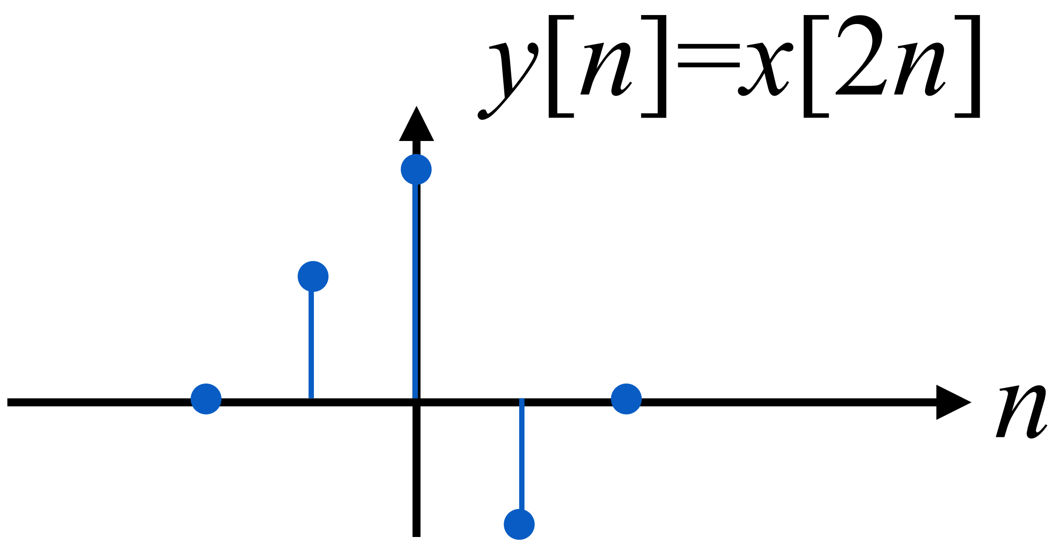


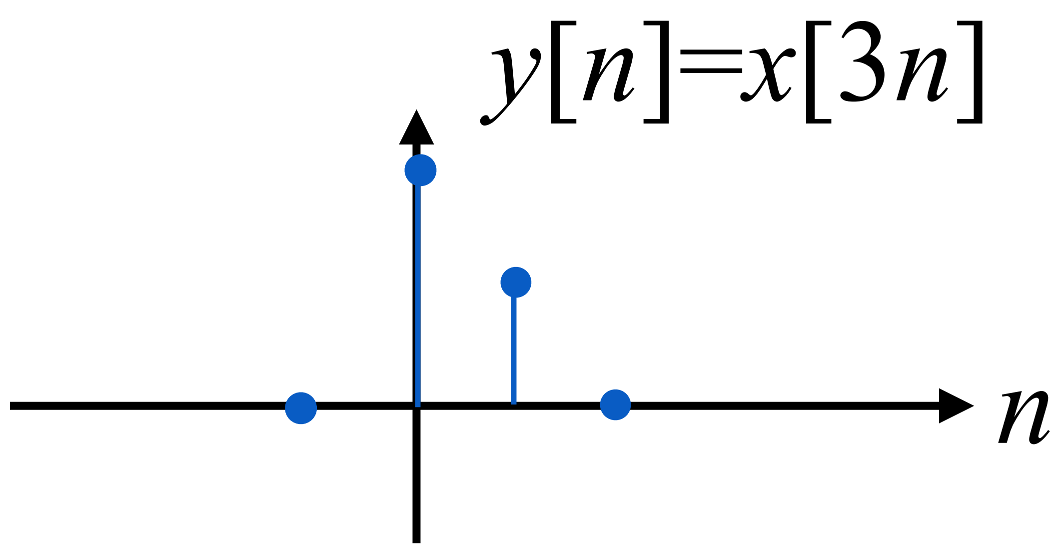
- So this is resulting in the "mirror image" of the signal around the y-axis.

Simple signal transformations

- **Subsampling:** $y[n] = x[Mn]$







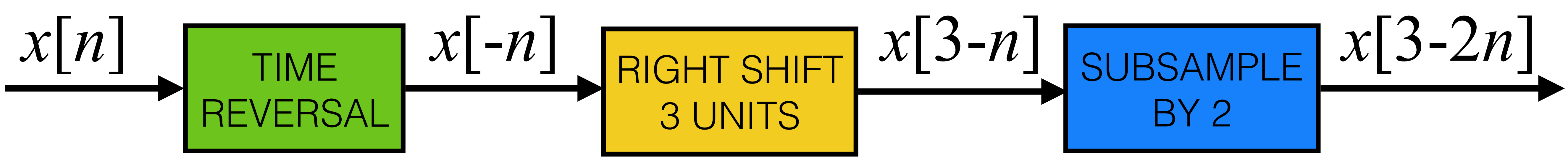
- Permanent information loss!!

Real world examples

- The original:
- Time-reversal:
- 2 times sub-sampling:
- 3 times sub-sampling:
- 5 times sub-sampling:

Transformation combos

- What if we have a transformation such as
$$y[n] = x[3 - 2n] \ \ ?$$
- Looks like a combo of time shift, time reversal, and subsampling. But with what order?
- Option 1:

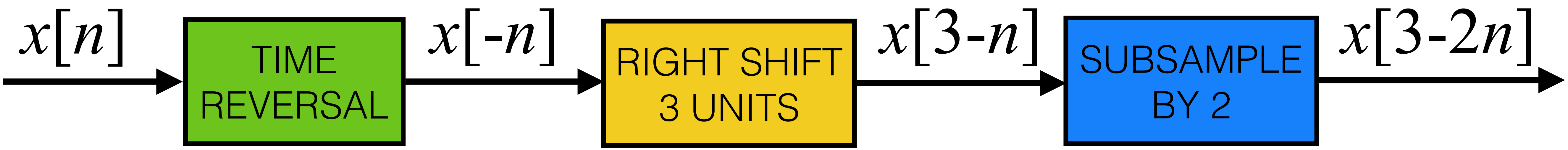


- Option 2:

Transformation combos

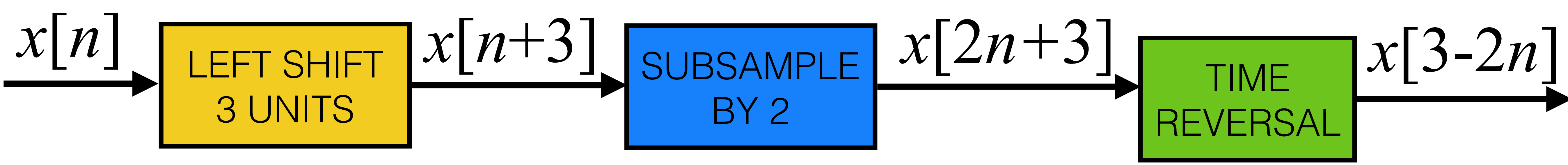
$$y[n] = x[3 - 2n]$$

- Option 1:

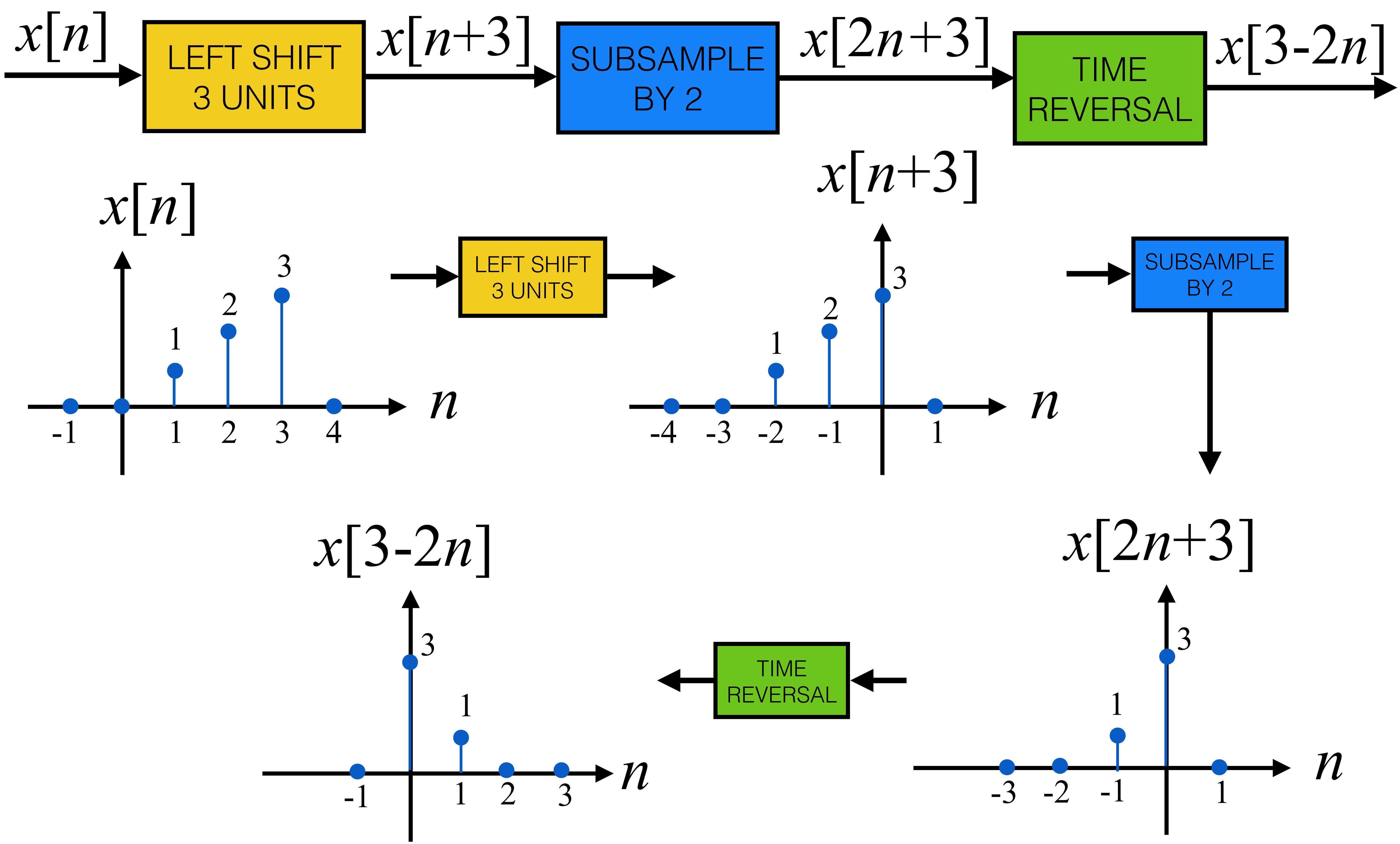


- Option 2:

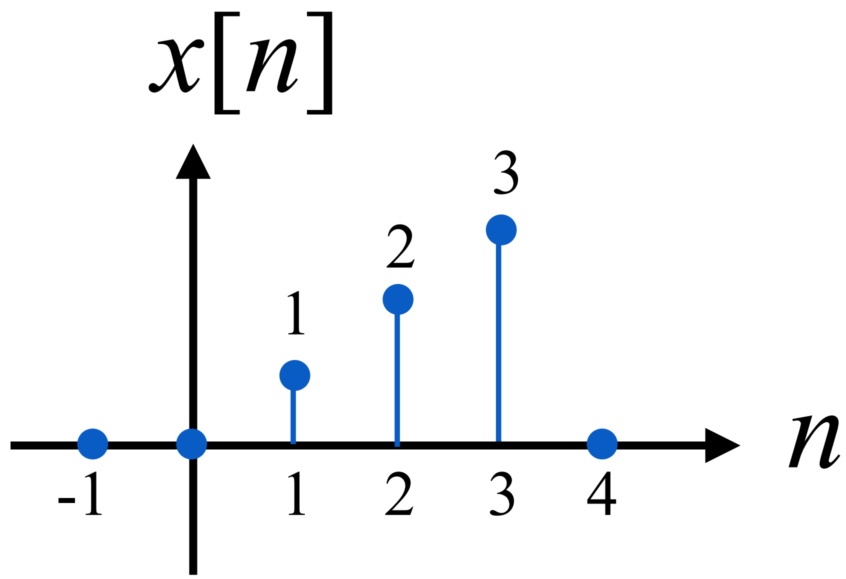
- Option 3:

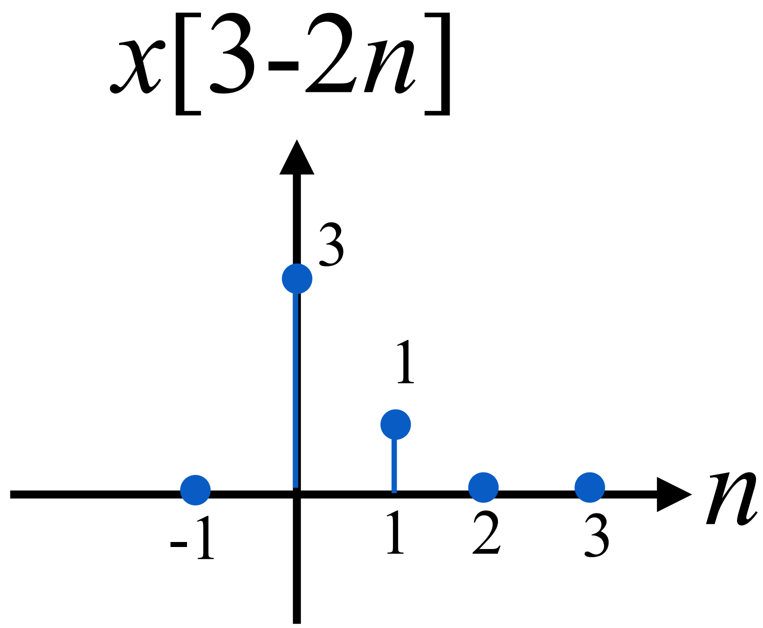


Transformation combos



Transformation combos



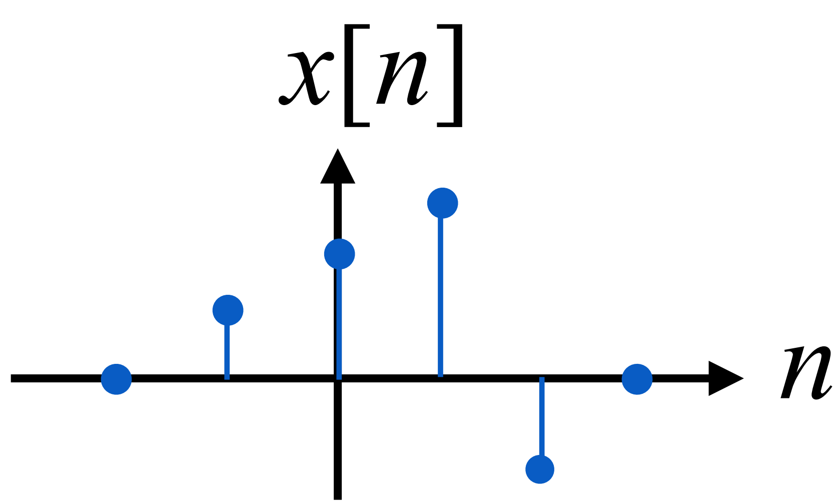


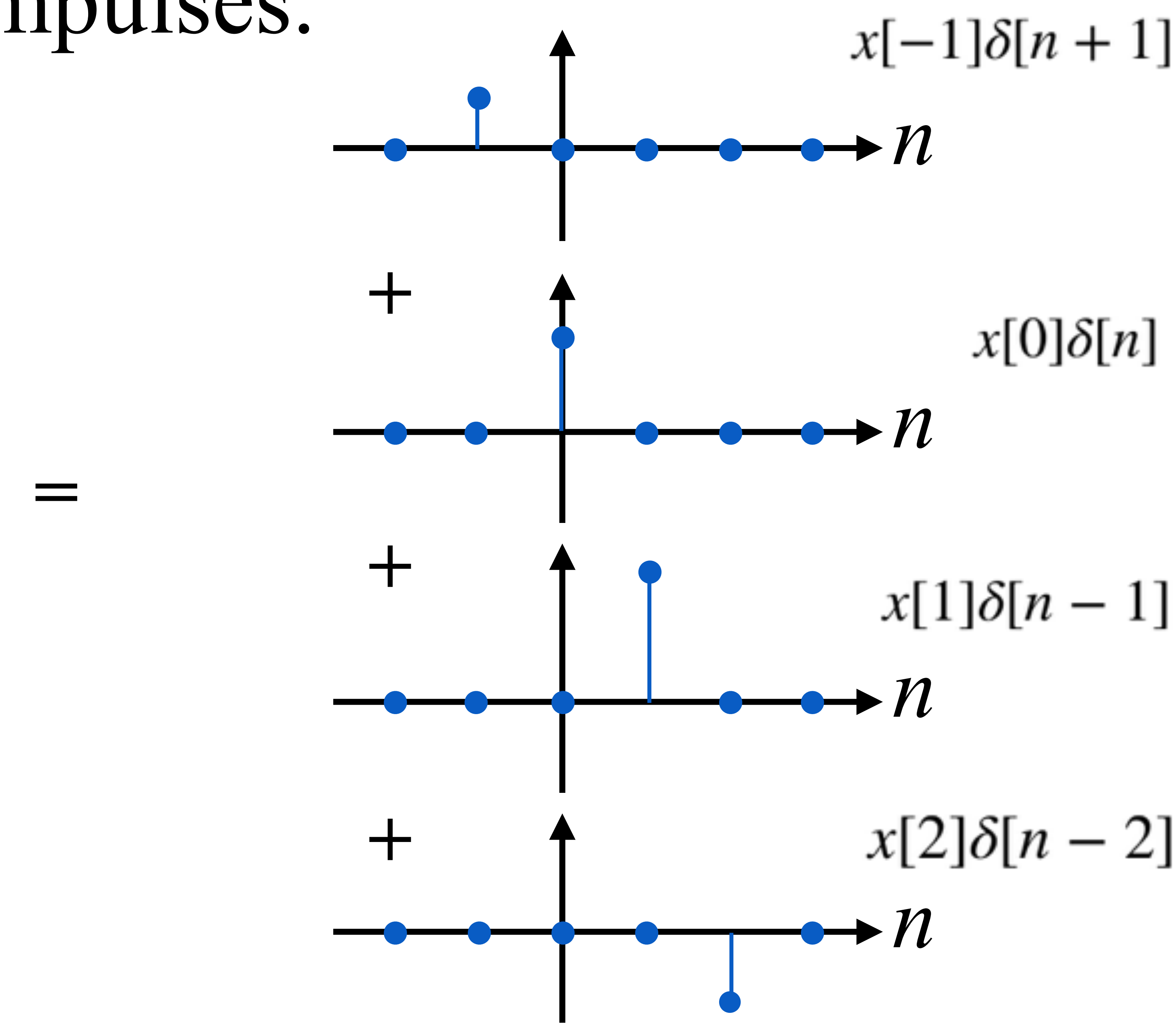
- Could we have computed this without going through the transformations one by one?
- We sure could have:

n	$3\text{-}2n$	$x[3\text{-}2n]$
-1	5	0
0	3	3
1	1	1
2	-1	0
3	-3	0

A simple decomposition

- We can decompose every signal into a sum of shifted impulses.
- Example:





A simple decomposition

- Generalizing, we obtain

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Two ways to interpret this formula:
  - As in the previous slide, a summation of infinitely many signals $\delta[n-k]$ each scaled with $x[k]$.
  - For every n, an infinite sum over the samples of the product signal $x[k]\delta[n-k]$.

Discrete-time systems

- A discrete-time system is a mapping from the input signal $x[n]$ to the output signal $y[n]$.
- We already saw the examples

$$y[n] = x[n - n_0]$$
$$y[n] = x[-n]$$
$$y[n] = x[Mn]$$

- But sky is the limit:

$$y[n] = x[n]^2 \qquad y[n] = x[n] + x[n-1]$$
$$y[n] = x[n^2] \qquad y[n] = y[n-1] + x[n]^3$$
$$y[n] = \cos(x[n]) \qquad y[n] = 4$$

Linarity

- A discrete-time system is **linear** if

$$x_1[n] \longrightarrow \boxed{\text{DISCRETE-TIME SYSTEM}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{DISCRETE-TIME SYSTEM}} \longrightarrow y_2[n]$$

implies

$$ax_1[n] + bx_2[n] \longrightarrow \boxed{\text{DISCRETE-TIME SYSTEM}} \longrightarrow ay_1[n] + by_2[n]$$

for any $x_1[n], x_2[n], a,$ and b

Linearity

- <u>Problem:</u> Is $y[n] = x[n]\cos(n)$ a linear system?

$$x_1[n] \longrightarrow y_1[n] = x_1[n]\cos(n)$$

$$x_2[n] \longrightarrow y_2[n] = x_2[n]\cos(n)$$

$$\begin{aligned} ax_1[n] + bx_2[n] \longrightarrow &\ (ax_1[n] + bx_2[n])\cos(n) \\ &= ax_1[n]\cos(n) \\ &\quad + bx_2[n]\cos(n) \\ &= ay_1[n] + by_2[n] \ \checkmark \end{aligned}$$

Linearity

- Problem: Is $y[n] = x[n]^2$ a linear system?

$$x_1[n] \longrightarrow y_1[n] = x_1[n]^2$$

$$x_2[n] \longrightarrow y_2[n] = x_2[n]^2$$

$$ax_1[n] + bx_2[n] \longrightarrow (ax_1[n] + bx_2[n])^2$$

$$\neq ax_1[n]^2 + bx_2[n]^2$$

- To see this, just take a=2, b=0.

Linearity

- Problem: Is $y[n] = 4$ a linear system?

$$x_1[n] \longrightarrow y_1[n] = 4$$

$$x_2[n] \longrightarrow y_2[n] = 4$$

$$ax_1[n] + bx_2[n] \longrightarrow 4 \neq a \cdot 4 + b \cdot 4$$

- To see this, just take any *a* and *b* other than $a+b = 1$.

Time invariance

- A discrete-time system is **time-invariant** if

$$x_1[n] \longrightarrow \boxed{\begin{array}{c}\text{DISCRETE-}\\ \text{TIME SYSTEM}\end{array}} \longrightarrow y_1[n]$$

implies

$$x_1[n-n_0] \longrightarrow \boxed{\begin{array}{c}\text{DISCRETE-}\\ \text{TIME SYSTEM}\end{array}} \longrightarrow y_1[n-n_0]$$

for any $x_1[n]$ and n_0 .

Time invariance

- <u>Problem:</u> Is $y[n] = \cos(x[n])$ a time-invariant system?

$$x_1[n] \longrightarrow y_1[n] = \cos(x_1[n])$$

$$x_1[n-n_0] \longrightarrow \cos(x_1[n-n_0])$$

$$= y_1[n-n_0] \checkmark$$

Time invariance

- Problem: Is $y[n] = n\,x[n]$ a time-invariant system?

$$x_1[n] \longrightarrow y_1[n] = n\,x_1[n]$$

$$x_1[n-n_0] \longrightarrow n\,x_1[n-n_0]$$

$$\neq y_1[n-n_0]$$

Time invariance

- Problem: Is $y[n] = x[n] + x[n-1]^2$ a time-invariant system?

$$x_1[n] \longrightarrow y_1[n] = x_1[n] + x_1[n-1]^2$$

$$x_1[n-n_0] \longrightarrow x_1[n-n_0] + x_1[n-n_0-1]^2$$

$$= y_1[n-n_0] \checkmark$$

Time invariance

- Problem: Is $y[n] = x[n^2]$ a time-invariant system?

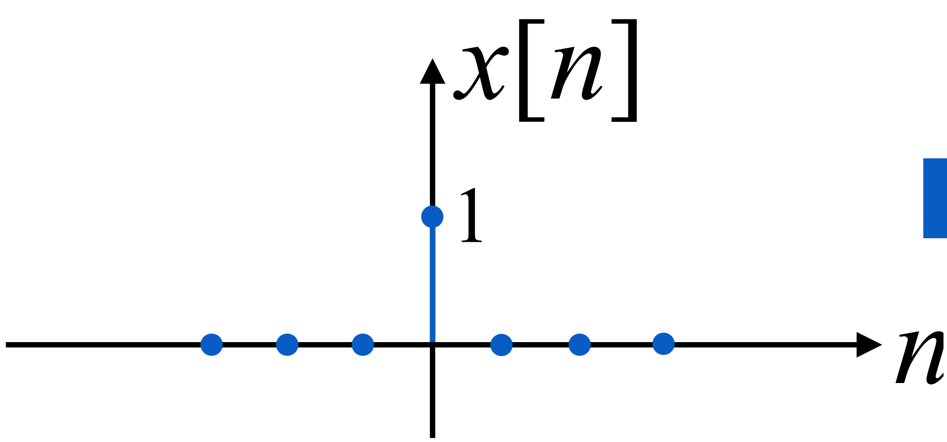
$$x_1[n] \longrightarrow y_1[n] = x_1[n^2]$$

$$x_1[n - n_0] \longrightarrow x_1[n^2 - n_0]$$

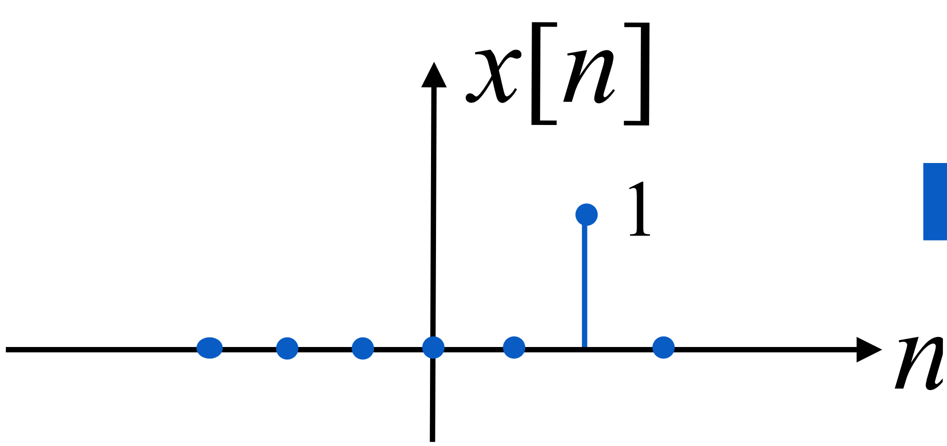
$$\neq y_1[n - n_0]$$

If in doubt, try this out

- You can try an example to prove **non-linearity** or **time-variance**
- For the last example, $y[n] = x[n^2]$, try the impulse signal as input:



n	n^2	$x[n^2]$
-2	4	0
-1	1	0
0	0	1
1	1	0
2	4	0



n	n^2	$x[n^2]$
-2	4	0
-1	1	0
0	0	0
1	1	0
2	4	0

Memory and Causality

- A discrete-time system is **memoryless** if at time instant n, the value of $y[n]$ depends only on the *current* value of $x[n]$, and not on any *past* or *future* value of it.
- A discrete-time system is **causal** if at time instant n, the value of $y[n]$ depends only on the *current* and *past* value of $x[n]$, and not on any *future* value of it.
- Obviously, memorylessness implies causality, but not vice versa.

Memory and Causality

- <u>Problem:</u> Is $y[n] = \cos(x[n])$ a memoryless system? If not memoryless, is it causal?

- <u>Solution:</u>

$$y[0] = \cos(x[0])$$

$$y[1] = \cos(x[1])$$

$$y[2] = \cos(x[2])$$

$$\vdots$$

MEMORYLESS ✓

CAUSAL ✓

Memory and Causality

- <u>Problem:</u> Is $y[n] = x[n] + x[n-1]^2$ a memoryless system? If not memoryless, is it causal?
- <u>Solution:</u>

$$y[0] = x[0] + x[-1]^2$$
$$y[1] = x[1] + x[0]^2$$
$$y[2] = x[2] + x[1]^2$$
$$\vdots$$

HAS MEMORY

CAUSAL ✓

Memory and Causality

- Problem: Is $y[n] = x[n^2]$ a memoryless system? If not memoryless, is it causal?

- Solution:

$$y[0] = x[0]$$

$$y[1] = x[1]$$

$$y[2] = x[4]$$

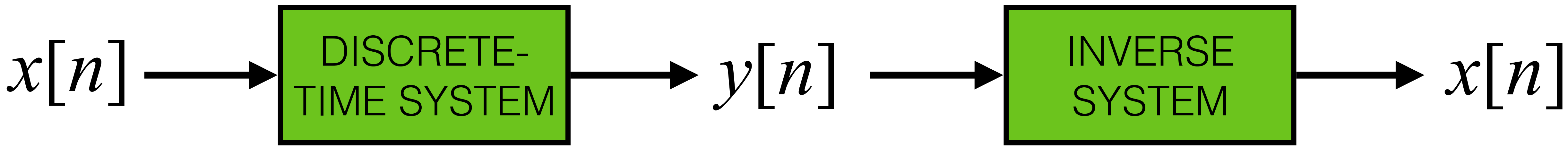
$$\vdots$$

HAS MEMORY

NON-CAUSAL

Invertibility

- A discrete-time system is **invertible** if there exists another system which outputs $x[n]$ when its input is $y[n]$.

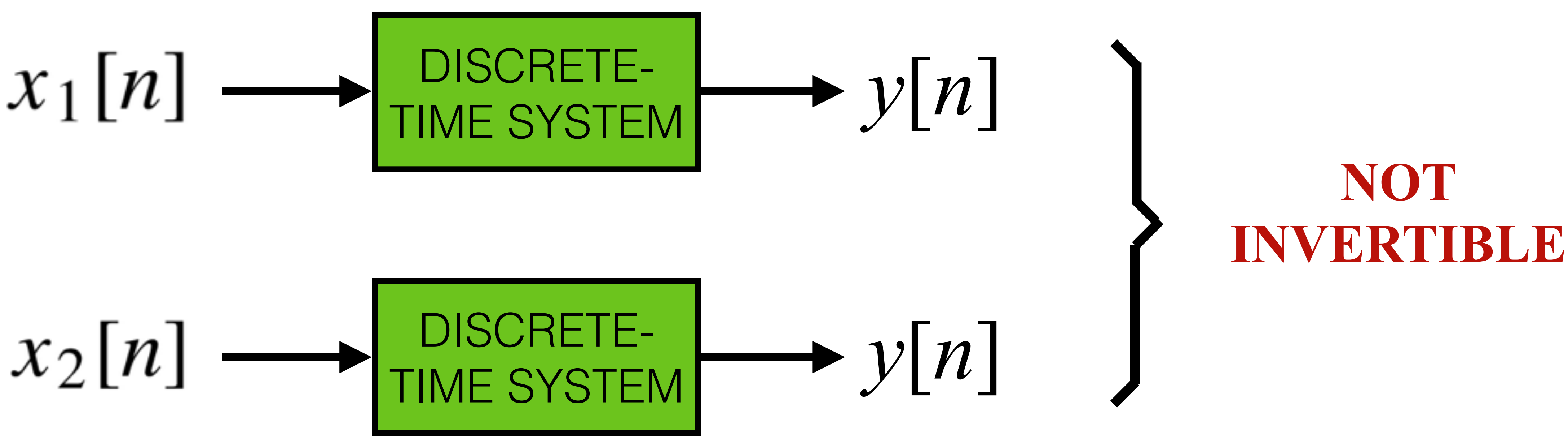


- This should be true for ALL $x[n]$.

Invertibility

- But this definition seems to require that you actually *find* the inverse system.

- <u>Alternative definition</u>: A discrete-time system is **invertible** if no two distinct input signals yield the same output.



Invertibility

- <u>Problem:</u> Is $y[n] = \cos(x[n])$ an invertible system?

$$x_1[n] = 0 \longrightarrow y[n] = 1$$

$$x_2[n] = 2\pi\delta[n] \longrightarrow y[n] = 1$$

NOT INVERTIBLE

Invertibility

- Problem: Is $y[n] = x[n^2]$ an invertible system?

$$x_1[n] = 0 \longrightarrow y[n] = 0$$

$$x_2[n] = \delta[n-2] \longrightarrow y[n] = 0$$

NOT INVERTIBLE

Invertibility

- <u>Problem:</u> Is $y[n] = \sum_{k=-\infty}^{n} x[k]$ an invertible system?
- <u>Solution:</u> In this case, we can come up with the inverse system:

INVERTIBLE

$$x[n] = y[n] - y[n-1]$$

- **Proof:**

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k]$$

$$= x[n] + \cancel{\sum_{k=-\infty}^{n-1} x[k]} - \cancel{\sum_{k=-\infty}^{n-1} x[k]}$$

Invertibility

- Alternatively, if the system were not invertible, there would exist two inputs $x_1[n]$ and $x_2[n]$ yielding the same output.

- But that would mean that for all *n*,

$$\sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x_2[k] \quad \text{and} \quad \sum_{k=-\infty}^{n-1} x_1[k] = \sum_{k=-\infty}^{n-1} x_2[k]$$

implying that $x_1[n] = x_2[n]$

Stability

- A discrete-time system is **stable** if bounded inputs yield bounded outputs.
- Mathematically speaking, a system is stable if

$$|x[n]| < B \quad \forall n$$

for some B implies

$$|y[n]| < C \quad \forall n$$

for some C.

Stability

- Problem: Is $y[n] = x[n^2]$ a stable system?
- Solution: If $|x[n]| < B \quad \forall n$, then certainly

$$|y[n]| = |x[n^2]| < B \quad \forall n$$

- Taking $C = B$ in the definition then leads to the conclusion that the system is...

STABLE ✓

Stability

- <u>Problem:</u> Is $y[n] = \sum_{k=-\infty}^{n} x[k]$ a stable system?

- <u>Solution:</u> Take $x[n] = 1$ for all n.

- It is certainly a bounded input:

$$x[n] < 1.00001 \quad \forall n$$

- But $y[n]$ is accumulating all these 1's, and therefore must diverge to infinity.

UNSTABLE

Stability

- Problem: Is $y[n] = \sum_{k=-\infty}^{n} 0.5^{n-k} x[k]$ a stable system?

- Solution: This time, the accumulation has a "forgetting factor" of 0.5.

- In fact, if $|x[n]| < B$ for all n, then

STABLE

$$|y[n]| = \left| \sum_{k=-\infty}^{n} 0.5^{n-k} x[k] \right| \le \sum_{k=-\infty}^{n} 0.5^{n-k} |x[k]|$$

$$< B \sum_{k=-\infty}^{n} 0.5^{n-k} \overset{(m=n-k)}{=} B \sum_{m=0}^{\infty} 0.5^{m} = B \cdot \frac{1}{1-0.5} \triangleq C$$