UNIVERSITY OF CALIFORNIA, RIVERSIDE
Department of Electrical and Computer Engineering
WINTER 2025
EE 110B SIGNALS AND SYSTEMS
FINAL EXAM SOLUTIONS

Question 1)

a) Using the differentiation property twice, we get

$$Y(j\Omega) = j\Omega \cdot j\Omega \cdot X(j\Omega) = -\Omega^2 X(j\Omega)$$

and therefore $H(j\Omega) = -\Omega^2$.

b) The corresponding discrete-time filter is given by

$$H_d(e^{j\omega}) = H(j\tfrac{\omega}{T}) = -\frac{\omega^2}{T^2} = -\omega^2$$

for $\pi \leq \omega \leq \pi$.

c) Using the inverse Fourier transform:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \omega^2 e^{j\omega n}d\omega \ . \end{aligned}$$

For $n = 0$, we have

$$h[0] = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\omega^2 d\omega = -\frac{\pi^2}{3} \ .$$

For all other n,

$$\begin{aligned} h[n] &= -\frac{1}{2\pi}\left[\left.\frac{\omega^2 e^{j\omega n}}{jn}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2\omega e^{j\omega n}}{jn}d\omega\right] \\ &= -\frac{1}{2\pi}\left[\frac{\pi^2(e^{j\pi n} - e^{-j\pi n})}{jn} - \int_{-\pi}^{\pi}\frac{2\omega e^{j\omega n}}{jn}d\omega\right] \\ &= -\frac{1}{2\pi}\left[\frac{2\pi^2\sin(\pi n)}{n} - \int_{-\pi}^{\pi}\frac{2\omega e^{j\omega n}}{jn}d\omega\right] \\ &= \frac{1}{\pi jn}\int_{-\pi}^{\pi}\omega e^{j\omega n}d\omega \end{aligned}$$

since $\sin(\pi n) = 0$ for all integer n. Continuing,

$$\begin{aligned} h[n] &= \frac{1}{\pi jn}\left[\left.\frac{\omega e^{j\omega n}}{jn}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{j\omega n}}{jn}d\omega\right] \\ &= \frac{1}{\pi jn}\left[\frac{\pi(e^{j\pi n} + e^{-j\pi n})}{jn} - \int_{-\pi}^{\pi}\frac{e^{j\omega n}}{jn}d\omega\right] \\ &= \frac{1}{\pi jn}\left[\frac{2\pi\cos(\pi n)}{jn} - \int_{-\pi}^{\pi}\frac{e^{j\omega n}}{jn}d\omega\right] \\ &= -\frac{2(-1)^n}{n^2} + \frac{1}{\pi n^2}\int_{-\pi}^{\pi}e^{j\omega n}d\omega \end{aligned}$$

since $\cos(\pi n)$ alternates between 1 and -1. Therefore,

$$\begin{aligned}
h[n] &= -\frac{2(-1)^n}{n^2} + \frac{e^{j\pi n} - e^{-j\pi n}}{j\pi n^3} \\
&= -\frac{2(-1)^n}{n^2} + \frac{2\sin(\pi n)}{\pi n^3} \\
&= -\frac{2(-1)^n}{n^2}\,.
\end{aligned}$$

<u>Using the thought experiment</u>: If $x_c(t) = \frac{\sin(\pi t)}{\pi t}$, then

$$\begin{aligned}
y_c(t) &= \frac{d}{dt}\frac{d}{dt}\left(\frac{\sin(\pi t)}{\pi t}\right) \\
&= \frac{d}{dt}\left(\frac{\pi^2 t\cos(\pi t) - \pi\sin(\pi t)}{\pi^2 t^2}\right) \\
&= \frac{d}{dt}\frac{\cos(\pi t)}{t} - \frac{d}{dt}\frac{\sin(\pi t)}{\pi t^2} \\
&= \frac{-\pi t\sin(\pi t) - \cos(\pi t)}{t^2} - \frac{\pi^2 t^2\cos(\pi t) - 2\pi t\sin(\pi t)}{\pi^2 t^4} \\
&= -\frac{\pi\sin(\pi t)}{t} - \frac{\cos(\pi t)}{t^2} - \frac{\cos(\pi t)}{t^2} + \frac{2\sin(\pi t)}{\pi t^3} \\
&= -\frac{\pi\sin(\pi t)}{t} - \frac{2\cos(\pi t)}{t^2} + \frac{2\sin(\pi t)}{\pi t^3}\,.
\end{aligned}$$

For any $n \neq 0$, this means

$$h[n] = y_c(nT) = y_c(n) = -\frac{\pi\sin(\pi n)}{n} - \frac{2\cos(\pi n)}{n^2} + \frac{2\sin(\pi n)}{\pi n^3} = -\frac{2(-1)^n}{n^2}$$

which is the same result as before. For $n = 0$, we need

$$\begin{aligned}
h[0] &= y_c(0) \\
&= \lim_{t\to 0}\left(-\frac{\pi\sin(\pi t)}{t} - \frac{2\cos(\pi t)}{t^2} + \frac{2\sin(\pi t)}{\pi t^3}\right) \\
&= -\lim_{t\to 0}\frac{\pi\sin(\pi t)}{t} - \frac{2}{\pi}\lim_{t\to 0}\frac{\pi t\cos(\pi t) - \sin(\pi t)}{t^3} \\
&= -\pi^2 - \frac{2}{\pi}\lim_{t\to 0}\frac{\pi\cos(\pi t) - \pi^2 t\sin(\pi t) - \pi\cos(\pi t)}{3t^2} \\
&= -\pi^2 + \frac{2\pi^2}{3}\lim_{t\to 0}\frac{\sin(\pi t)}{\pi t} \\
&= -\frac{\pi^2}{3}
\end{aligned}$$

which also agrees with what we found earlier.

Question 2)

a) We have

$$\begin{aligned}
u[n] \star nu[n] &= \sum_{k=-\infty}^{\infty} ku[k]u[n-k] \\
&= \sum_{k=0}^{\infty} ku[n-k] \\
&= \begin{cases} \sum_{k=0}^{n} k & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
&= u[n] \sum_{k=0}^{n} k \\
&= x[n] \; .
\end{aligned}$$

b) The z-transforms of $u[n]$ and $nu[n]$ can be found to be $\frac{1}{1-z^{-1}}$ and $\frac{z^{-1}}{(1-z^{-1})^2}$, respectively, from the cheat sheet. The ROC for either signal is $|z| > 1$. Then, using the convolution property,

$$X(z) = \frac{1}{1-z^{-1}} \cdot \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z^{-1}}{(1-z^{-1})^3}$$

with ROC $|z| > 1$.

c) $X(z)$ can also be written as

$$X(z) = \frac{z^{-1}}{1 - 3z^{-1} + 3z^{-2} - z^{-3}}$$

Performing the long division, we get

$$\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 6z^{-3} + 10z^{-4} + \ldots \\
\hline
1 - 3z^{-1} + 3z^{-2} - z^{-3} & z^{-1} \\
 & -\;\; z^{-1} - 3z^{-2} + 3z^{-3} - z^{-4} \\
\hline
 & 3z^{-2} - 3z^{-3} + z^{-4} \\
 & -\;\; 3z^{-2} - 9z^{-3} + 9z^{-4} - 3z^{-5} \\
\hline
 & 6z^{-3} - 8z^{-4} + 3z^{-5} \\
 & -\;\; 6z^{-3} - 18z^{-4} + 18z^{-5} - 6z^{-6} \\
\hline
 & 10z^{-4} - 15z^{-5} + 6z^{-6}
\end{array}$$

from which we can see the trend $x[0] = 0$, $x[1] = 1$, $x[2] = 3$, $x[3] = 6$, $x[4] = 10, \ldots$, and therefore,

$$x[n] = \frac{n(n+1)}{2} u[n] \; .$$

Question 3)

a) Since we know that $a^n u[n]$ yields a z-transform $\frac{1}{1-az^{-1}}$, we can rewrite $G(z)$ as

$$G(z) = \frac{z^{-1}}{1-az^{-1}} - \frac{a}{1-az^{-1}}$$

and invert the z-transform readily as

$$g[n] = a^{n-1}u[n-1] - a \cdot a^n u[n] = \begin{cases} -a & n = 0 \\ a^{n-1}(1-a^2) & n > 0 \end{cases} .$$

b) We have

$$\begin{aligned} |G(e^{j\omega})| &= \left| \frac{e^{-j\omega} - a}{1 - ae^{-j\omega}} \right| \\ &= \left| \frac{e^{-j\omega}(1 - ae^{j\omega})}{1 - ae^{-j\omega}} \right| \\ &= \left| e^{-j\omega} \right| \cdot \frac{\left|1 - ae^{j\omega}\right|}{\left|1 - ae^{-j\omega}\right|} . \end{aligned}$$

Now, since $\left|e^{-j\omega}\right| = 1$, it suffices to show $\left|1 - ae^{j\omega}\right| = \left|1 - ae^{-j\omega}\right|$. But that is true because $1 - ae^{-j\omega} = (1 - ae^{j\omega})^*$, and any z and z^* have the same magnitude.

c)We already know that, just like $2^n u[n]$,

$$h[n] = -2^n u[-n-1]$$

has the z-transform

$$H(z) = \frac{1}{1 - 2z^{-1}} .$$

Unlike $2^n u[n]$, however, its ROC is given as $|z| < 2$. The DTFT $H(e^{j\omega})$ exists because the ROC contains the unit circle.

d) If we choose $a = \frac{1}{2}$, we obtain

$$\begin{aligned} H(z)G(z) &= \frac{1}{1 - 2z^{-1}} \cdot \frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}} \\ &= \frac{1}{1 - 2z^{-1}} \cdot \frac{1}{2} \cdot \frac{2z^{-1} - 1}{1 - \frac{1}{2}z^{-1}} \\ &= -\frac{1}{2} \frac{1}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$

which is stable and causal.

Question 4)
a) We have

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= x[-2]e^{-j\omega(-2)} + x[-1]e^{-j\omega(-1)} + x[0]e^{-j\omega 0} + x[1]e^{-j\omega 1} + x[2]e^{-j\omega 2} \\ &= \frac{1}{4}e^{j\omega 2} - \frac{1}{2}e^{j\omega} + 1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j\omega 2} \\ &= \frac{1}{2}\left(\frac{e^{j\omega 2} + e^{-j\omega 2}}{2}\right) - \left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 1 \\ &= \frac{1}{2}\cos(2\omega) - \cos(\omega) + 1 \end{aligned}$$

b) Evaluating $X(e^{j\omega})$ at the suggested ω values yields

$$\begin{aligned} X(e^{j0}) &= 0.5 \\ X(e^{j(\pm\frac{\pi}{2})}) &= 0.5 \\ X(e^{j(\pm\pi)}) &= 2.5\ . \end{aligned}$$

This clearly indicates a high-pass signal.

c) Following the hint, we first have from the differentiation property that

$$nx[n] \stackrel{\text{DTFT}}{\longrightarrow} j\frac{d}{d\omega}X(e^{j\omega}) = j\Big[\sin(\omega) - \sin(2\omega)\Big]$$

and then from Parseval's relation that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|j\frac{d}{d\omega}X(e^{j\omega})\right|^2 d\omega = \sum_{n=-\infty}^{\infty} n^2 x[n]^2\ .$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\sin(\omega) - \sin(2\omega)\Big]^2 d\omega &= \sum_{n=-\infty}^{\infty} n^2 x[n]^2 \\ &= \sum_{n=-2}^{2} n^2 x[n]^2 \\ &= (-2)^2 \times \left(\frac{1}{4}\right)^2 + (-1)^2 \times \left(-\frac{1}{2}\right)^2 + 0^2 \times 1^2 + 1^2 \times \left(-\frac{1}{2}\right)^2 + 2^2\left(\frac{1}{4}\right)^2 \\ &= \frac{1}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{4} \\ &= 1\ . \end{aligned}$$