EE 110A Signals and Systems

Fourier Transform of Continuous-Time Signals

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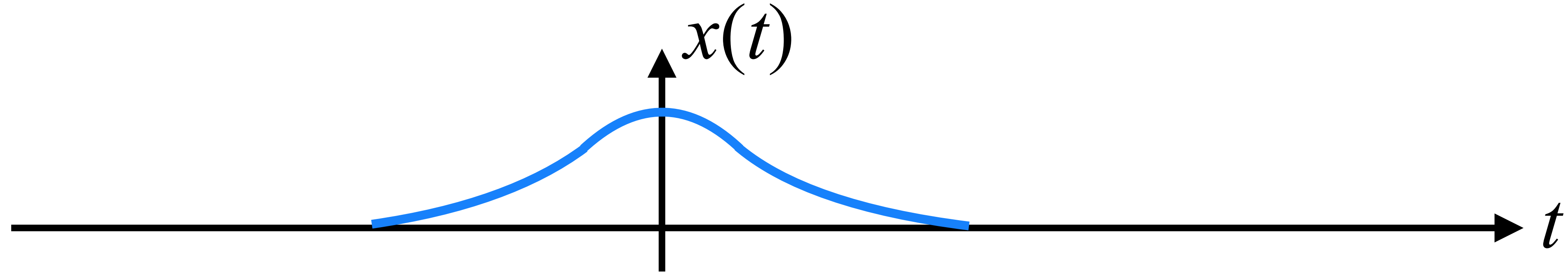
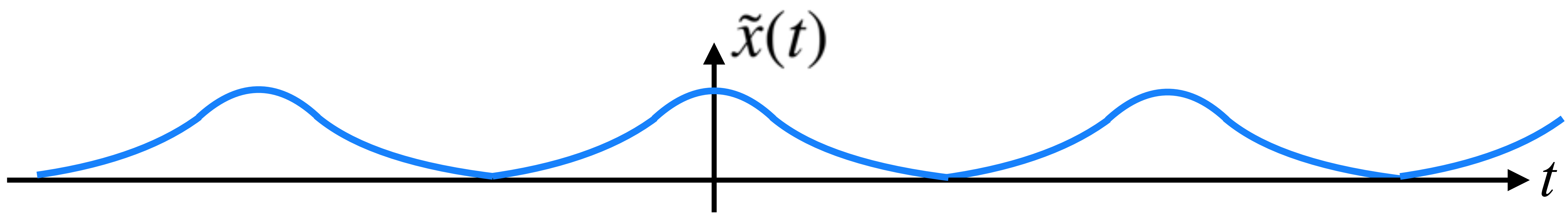
What about nonperiodic signals?

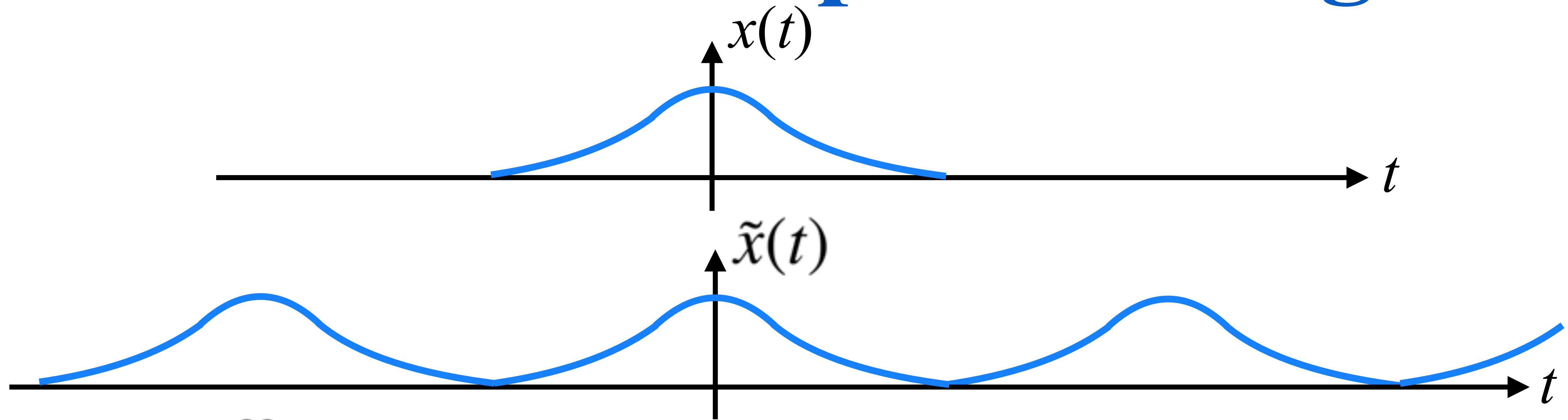


- If the signal has finite duration, everything is fine.
- Extend the signal into a periodic one and decompose onto $e^{jk\Omega_0 t}$.



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \quad \text{with} \quad a_k = \frac{1}{T}\int_{\mathcal{T}} \tilde{x}(t) e^{-jk\Omega_0 t} dt$$

What about nonperiodic signals?



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \quad \text{with} \quad a_k = \frac{1}{T}\int_{T} \tilde{x}(t) e^{-jk\Omega_0 t}\, dt$$

- Alternatively,

$$a_k = \frac{1}{T}\int_{T} x(t) e^{-jk\Omega_0 t}\, dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t}\, dt$$

What about nonperiodic signals?

$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\Omega_0 t}dt \;\; = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\Omega_0 t}dt$$

- Now define

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

so that

$$a_k = \frac{1}{T}X(jk\Omega_0)$$

and

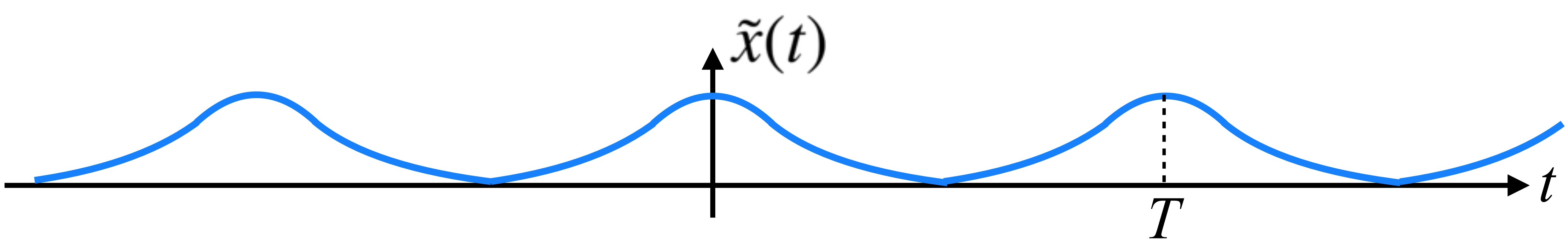
$$\tilde{x}(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(jk\Omega_0)e^{jk\Omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0)e^{jk\Omega_0 t}$$

What about nonperiodic signals?

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0)e^{jk\Omega_0 t}$$

- What happens if we set T to more than its minimum possible value?



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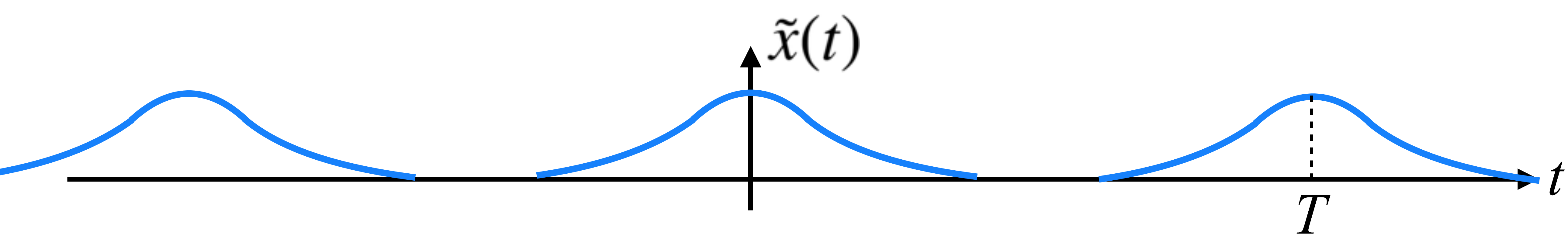
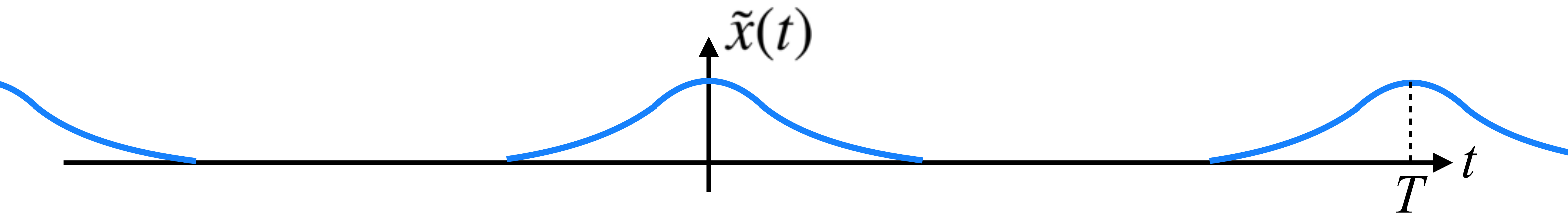


What about nonperiodic signals?

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}$$

- What happens if we set T to more than its minimum possible value?



- As $T \rightarrow \infty$, we have $\Omega_0 \rightarrow 0$ and $\tilde{x}(t) \longrightarrow x(t)$

What about nonperiodic signals?

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0)e^{jk\Omega_0 t}$$

- As $T \rightarrow \infty$, we have $\Omega_0 \rightarrow 0$ and $\tilde{x}(t) \longrightarrow x(t)$
- That implies

$$x(t) = \lim_{\Omega_0 \rightarrow 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0)e^{jk\Omega_0 t}$$

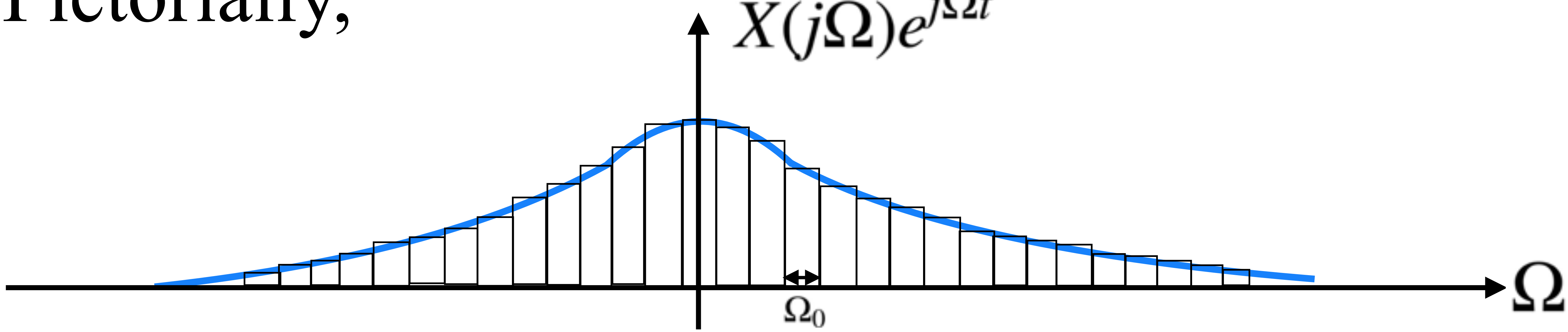
What about nonperiodic signals?

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$x(t) = \lim_{\Omega_0 \to 0} \frac{1}{2\pi} \boxed{\sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}}$$

= total area of the rectangles

- Pictorially,



- As $\Omega_0 \to 0$, where does the total area of the rectangles go?

What about nonperiodic signals?

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$x(t) = \lim_{\Omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0)e^{jk\Omega_0 t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$$

Continuous-time Fourier Transform

- This pair is known as the continuous-time Fourier transform (CTFT)

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,d\Omega$$

Properties

- Properties of CTFS carry over to CTFT:

$$x(t) \xrightarrow{\text{CTFT}} X(j\Omega) \qquad\qquad y(t) \xrightarrow{\text{CTFT}} Y(j\Omega)$$

imply

- **Linearity:** $ax(t) + by(t) \xrightarrow{\text{CTFT}} aX(j\Omega) + bY(j\Omega)$
- **Time shifting:** $x(t - t_0) \xrightarrow{\text{CTFT}} X(j\Omega)e^{-j\Omega t_0}$
- **Frequency shifting:** $x(t)e^{j\Omega_0 t} \xrightarrow{\text{CTFT}} X(j(\Omega - \Omega_0))$
- **Time reversal:** $x(-t) \xrightarrow{\text{CTFT}} X(-j\Omega)$
- **Conjugation:** $x(t)^* \xrightarrow{\text{CTFT}} X(-j\Omega)^*$

- **Linearity:** $ax(t) + by(t) \xrightarrow{\text{CTFT}} aX(j\Omega) + bY(j\Omega)$
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- **Time reversal:** $x(-t) \xrightarrow{\text{CTFT}} X(-j\Omega)$
- **Conjugation:** $x(t)^* \xrightarrow{\text{CTFT}} X(-j\Omega)^*$
- **Convolution:** $x(t) \star y(t) \xrightarrow{\text{CTFT}} X(j\Omega)Y(j\Omega)$
- **Multiplication:** $x(t)y(t) \xrightarrow{\text{CTFT}} \dfrac{1}{2\pi} X(j\Omega) \star Y(j\Omega)$
- **Parseval's:** $\displaystyle\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega$

Properties

- But there are also two more:
  - **Differentiation in the time domain:**

$$\frac{dx(t)}{dt} \xrightarrow{\text{CTFT}} j\Omega \cdot X(j\Omega)$$

- Proof:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) \frac{de^{j\Omega t}}{dt} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) j\Omega e^{j\Omega t} d\Omega$$

Properties

- **Differentiation in the frequency domain:**

$$tx(t) \xrightarrow{\text{CTFT}} j\,\frac{dX(j\Omega)}{d\Omega}$$

- <u>Proof:</u>

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

$$\frac{dX(j\Omega)}{d\Omega} = \int_{-\infty}^{\infty} x(t)\,\frac{de^{-j\Omega t}}{d\Omega}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\Omega t}\,dt$$

Rearranging finishes the proof.

Examples

- Find the CTFT of $x(t) = e^{-\alpha t}u(t)$ for $\alpha > 0$
- <u>Solution:</u>

$$X(j\Omega) = \int_{-\infty}^{\infty} e^{-\alpha t}u(t)e^{-j\Omega t}dt = \int_{0}^{\infty} e^{-(\alpha+j\Omega)t}dt$$

$$= \frac{1}{\alpha + j\Omega} \quad \text{since} \ \alpha > 0$$

- How do we plot this?
  - Real and imaginary parts separately
  - Magnitude and phase separately

$$X(j\Omega) = \frac{1}{\alpha + j\Omega}$$

$$= \frac{\alpha - j\Omega}{(\alpha + j\Omega)(\alpha - j\Omega)}$$

$$= \frac{\alpha - j\Omega}{\alpha^2 + \Omega^2}$$

$$\mathrm{Re}\{X(j\Omega)\} = \frac{\alpha}{\alpha^2 + \Omega^2} \quad \mathrm{Im}\{X(j\Omega)\} = \frac{-\Omega}{\alpha^2 + \Omega^2}$$

$$\mathrm{Re}\{X(j\Omega)\} = \frac{\alpha}{\alpha^2 + \Omega^2} \qquad \mathrm{Im}\{X(j\Omega)\} = \frac{-\Omega}{\alpha^2 + \Omega^2}$$

$$|X(j\Omega)| = \sqrt{\frac{\alpha^2}{\left(\alpha^2 + \Omega^2\right)^2} + \frac{\Omega^2}{\left(\alpha^2 + \Omega^2\right)^2}}$$

$$= \frac{1}{\sqrt{\alpha^2 + \Omega^2}}$$

$$\angle X(j\Omega) = \tan^{-1}\left(\frac{-\Omega}{\alpha}\right)$$

Examples

- If we normalize this family of functions as

$$x(t) = \alpha e^{-\alpha t} u(t) \quad \text{for} \quad \alpha > 0$$

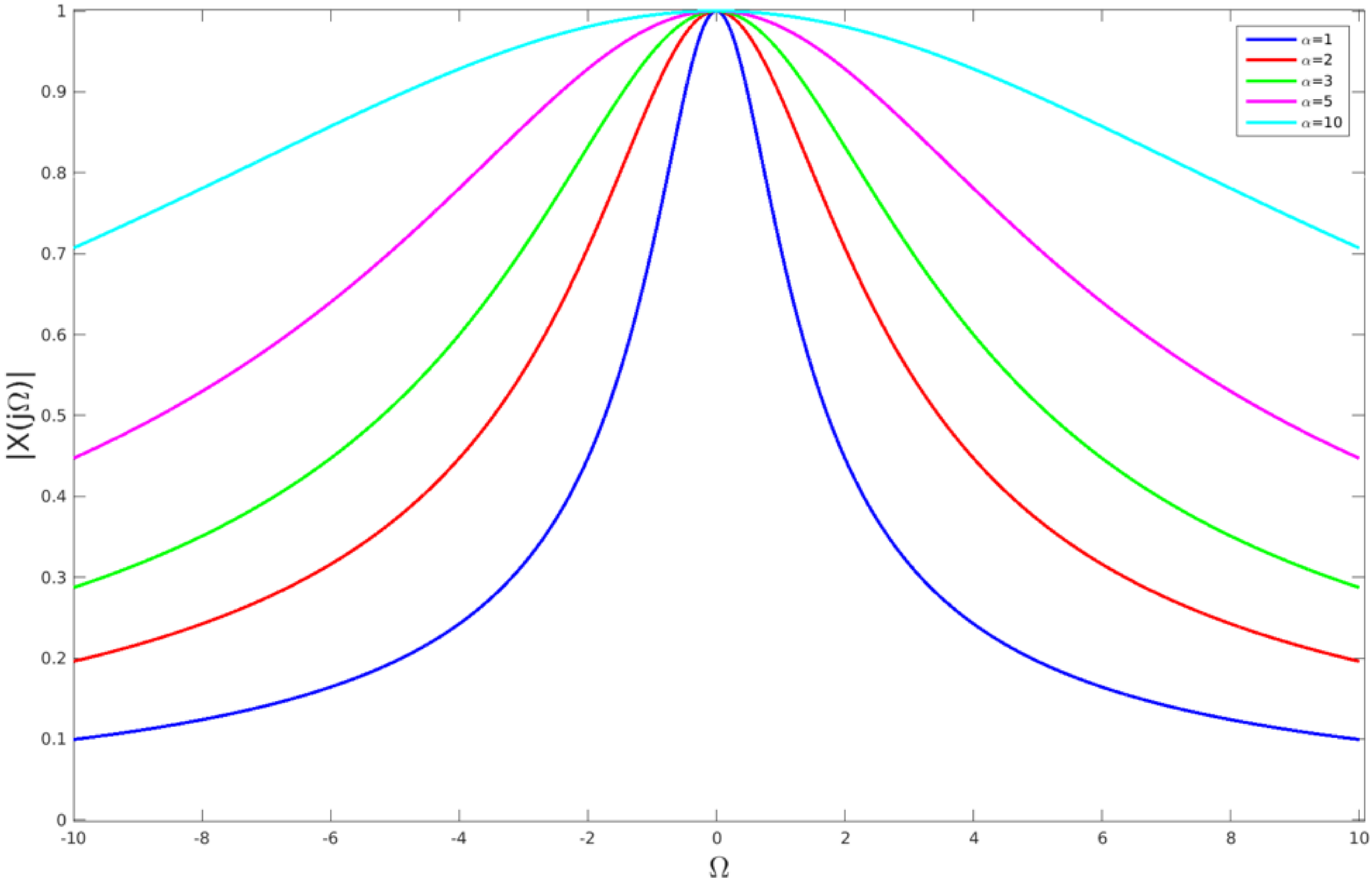
we obtain from linearity that

$$X(j\Omega) = \frac{\alpha}{\alpha + j\Omega}$$

and therefore that

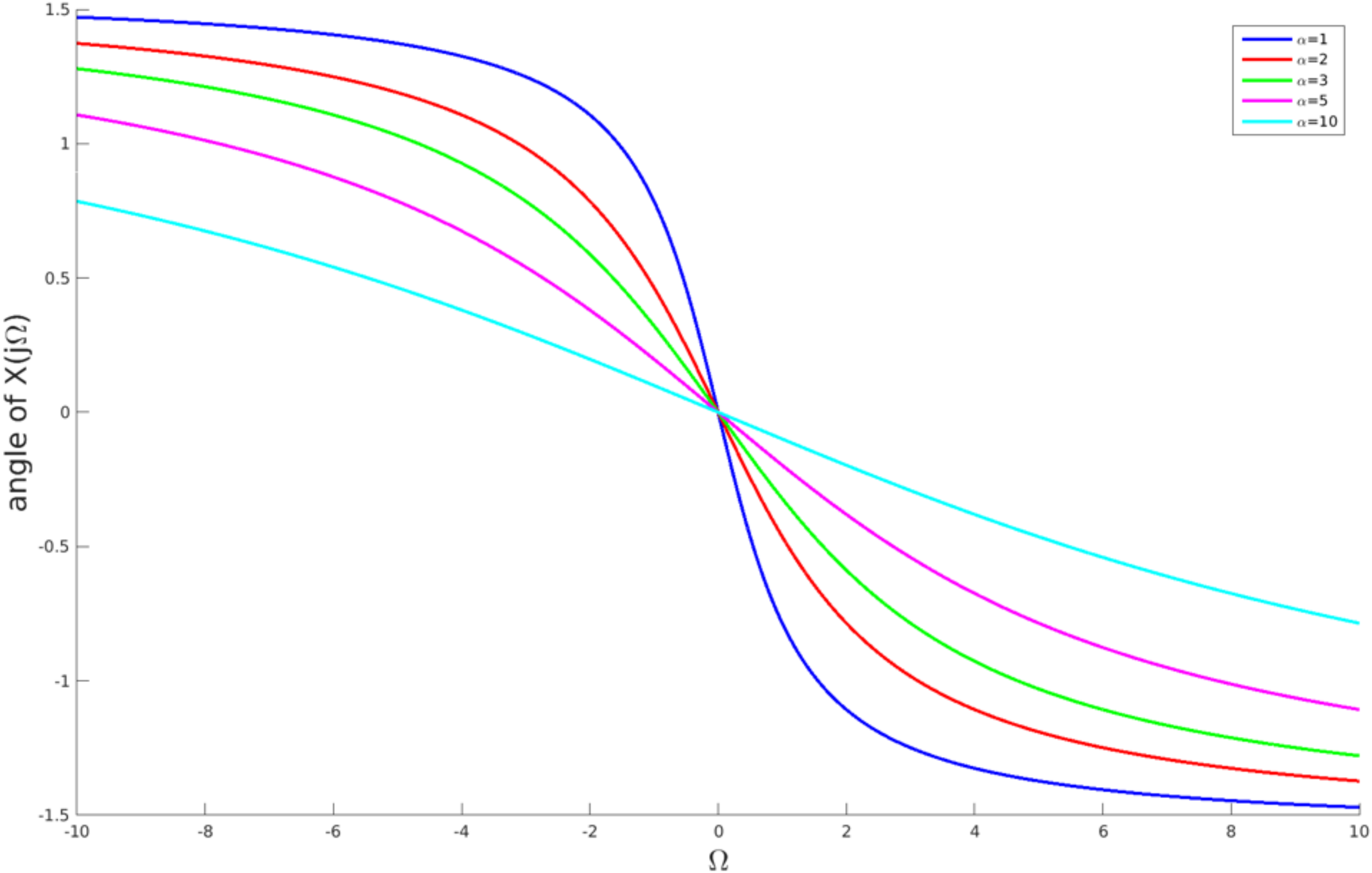
$$|X(j\Omega)| = \frac{\alpha}{\sqrt{\alpha^2 + \Omega^2}} \qquad \angle X(j\Omega) = \tan^{-1}\left(\frac{-\Omega}{\alpha}\right)$$

Examples



α=1
α=2
α=3
α=5
α=10
|X(jΩ)|
Ω
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-8
-6
-4
-2
0
2
4
6
8
10

Examples



Implementation as a filter

- Note that if this is the impulse response of a system, the system will suppress high frequencies and boost low frequencies.

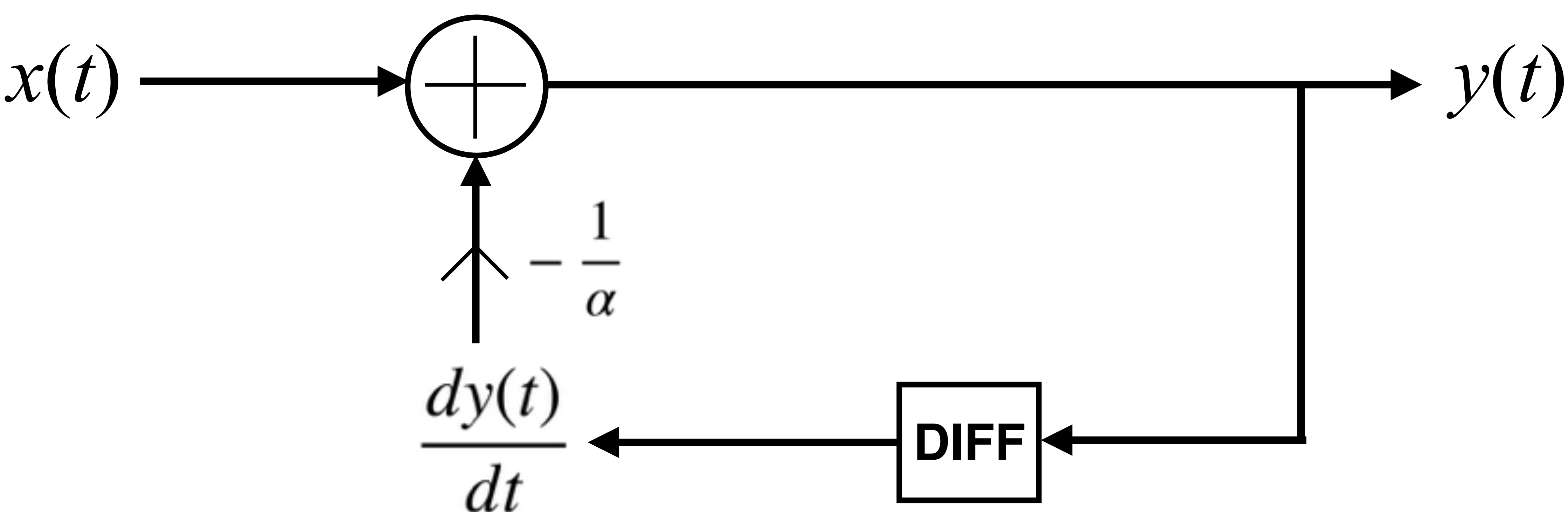
$$x(t) \longrightarrow \boxed{h(t) = \alpha e^{-\alpha t}u(t)} \longrightarrow y(t)$$

$$Y(j\Omega) = X(j\Omega)H(j\Omega)$$

Implementation as a filter

- This filter is easy to implement once we figure out that $h(t)$ is the impulse response of the system with differential equation

$$y(t) + \frac{1}{\alpha}\frac{dy(t)}{dt} = x(t)$$



Examples

- Find the CTFT of $x(t) = e^{-\alpha t}\cos(\beta t)u(t)$ for $\alpha > 0$
- Solution: It is best to rewrite $x(t)$ as

$$x(t) = 0.5e^{-\alpha t}e^{j\beta t}u(t) + 0.5e^{-\alpha t}e^{-j\beta t}u(t)$$

We already know that

$$e^{-\alpha t}u(t) \xrightarrow{\text{CTFT}} \frac{1}{\alpha + j\Omega}$$

Using the frequency shift property,

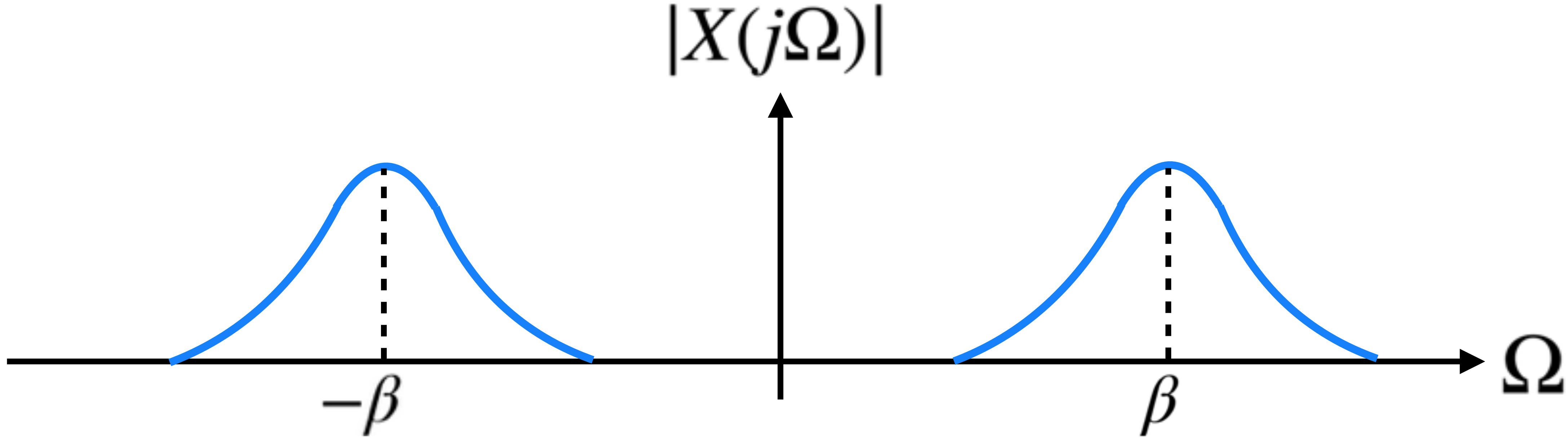
$$X(j\Omega) = \frac{0.5}{\alpha + j(\Omega - \beta)} + \frac{0.5}{\alpha + j(\Omega + \beta)}$$

Examples

- Find the CTFT of $x(t) = e^{-\alpha t}\cos(\beta t)u(t)$ for $\alpha > 0$

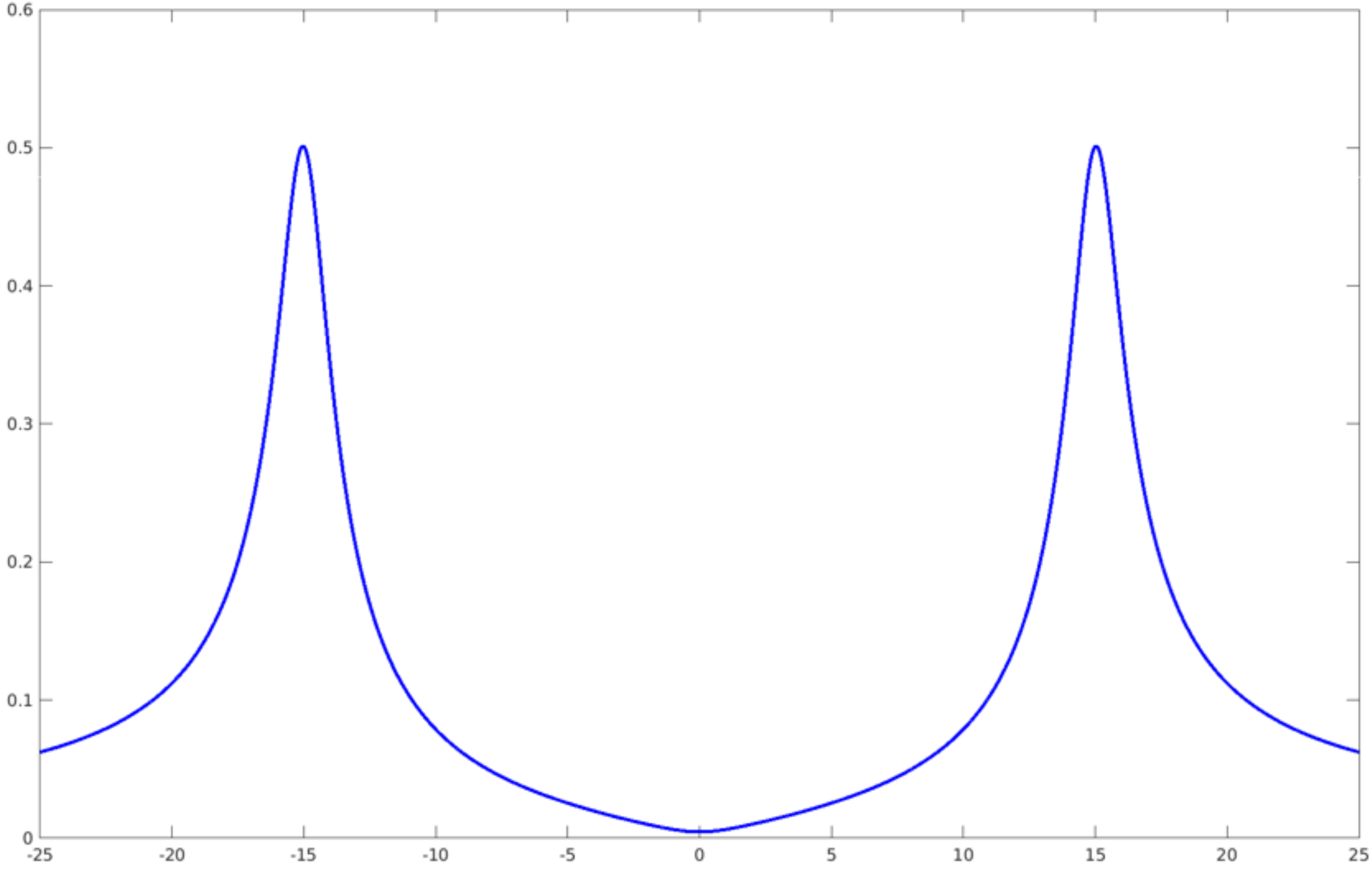
$$X(j\Omega) = \frac{0.5}{\alpha + j(\Omega - \beta)} + \frac{0.5}{\alpha + j(\Omega + \beta)}$$

- Intuitively, if α is small enough, and β is big enough,



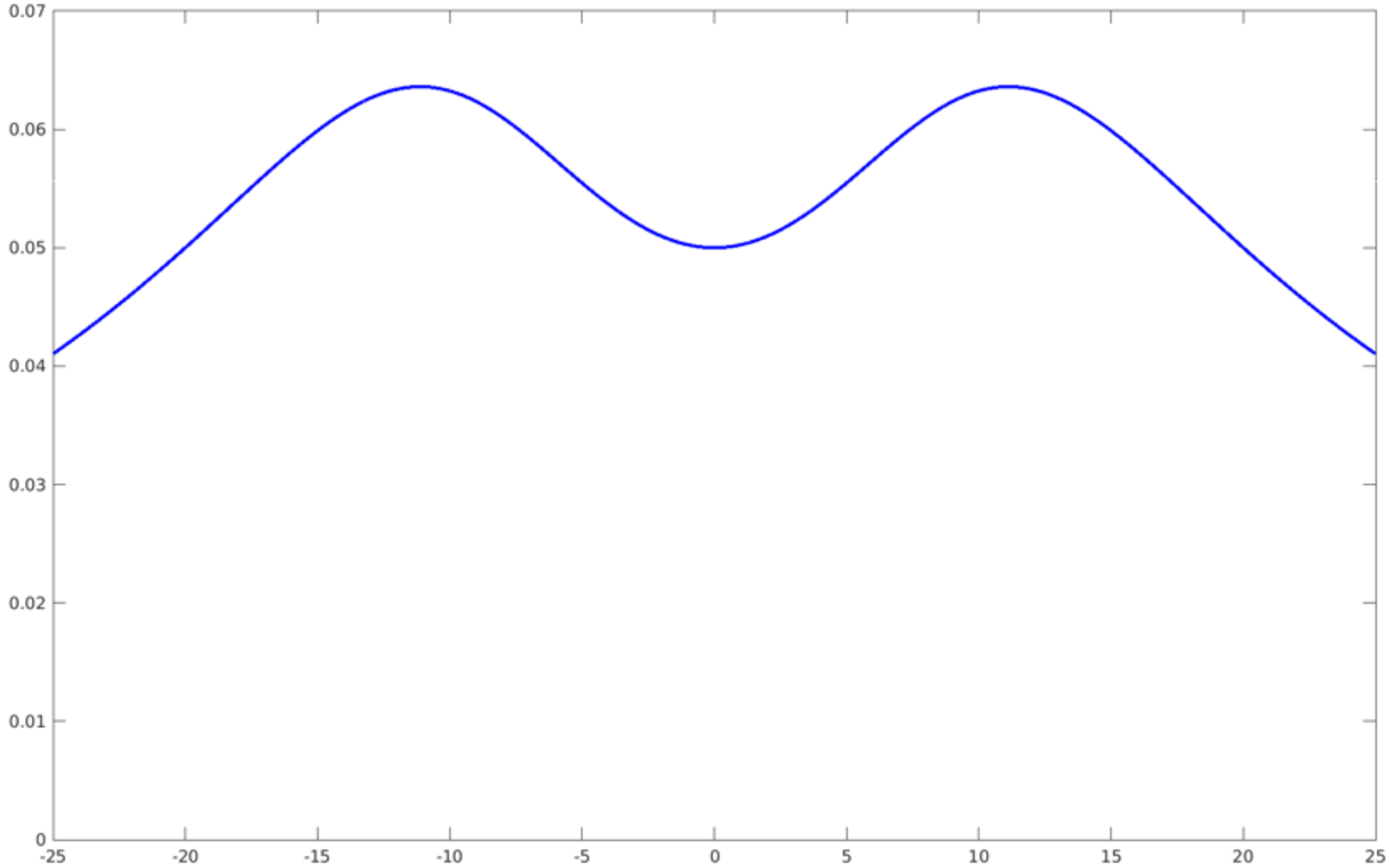
Examples

- Indeed, when $\alpha = 1$ and $\beta = 15$,



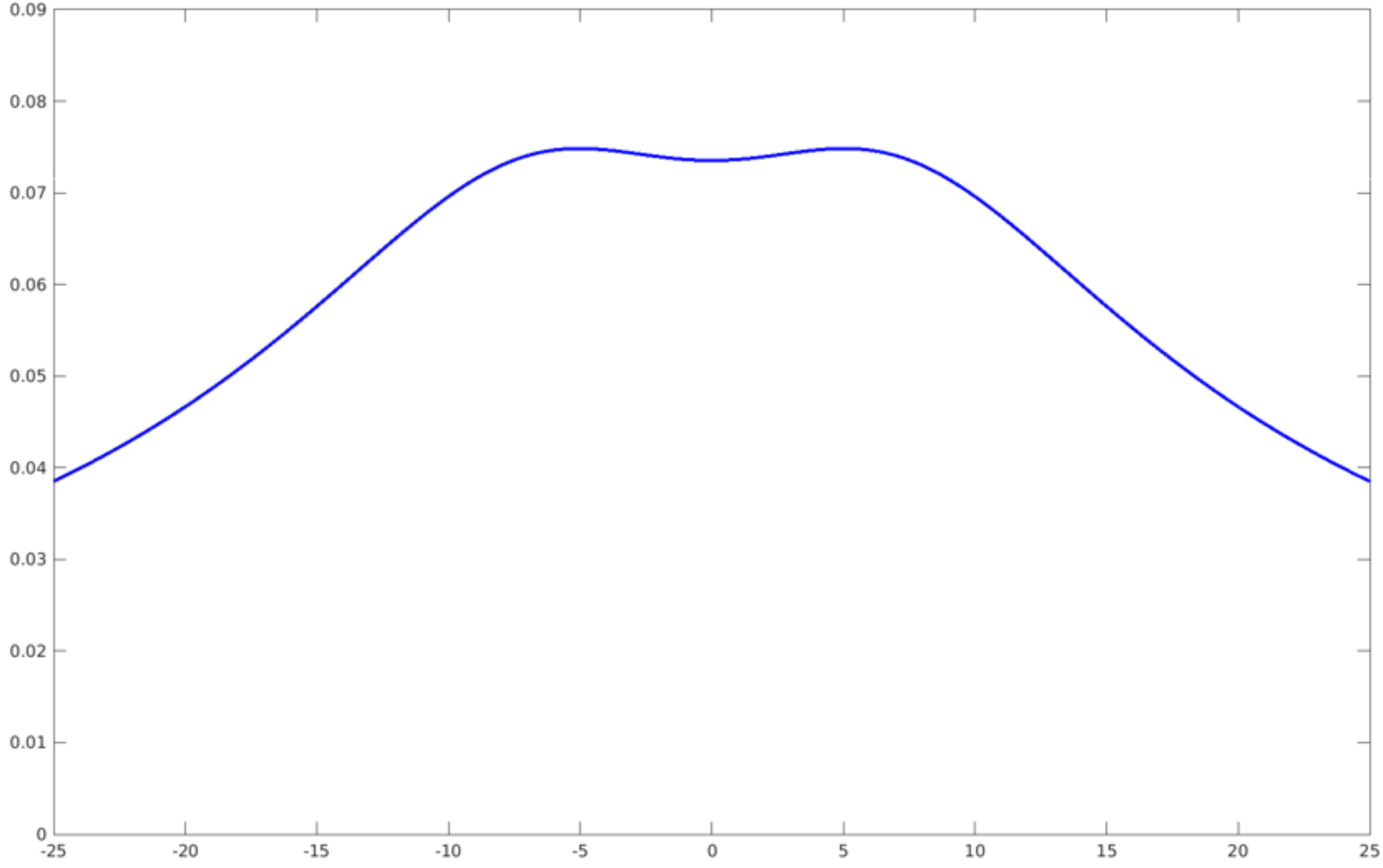
Examples

- However, when $\alpha = 10$ and $\beta = 10$,



Examples

- And when $\alpha = 10$ and $\beta = 6$,



Implementation as a filter

- What system has an impulse response

$$h(t) = e^{-\alpha t}\cos(\beta t)u(t) \quad ?$$

- The homogeneous solution must be in the form of $e^{-(\alpha - j\beta)t}$ and $e^{-(\alpha + j\beta)t}$
- Therefore, the homogeneous part of the differential equation must be

$$(\alpha^2 + \beta^2)h(t) + 2\alpha\frac{dh(t)}{dt} + \frac{d^2h(t)}{dt^2} = 0$$

- The initial conditions need to be

$$h(0^+) = 1 \qquad \frac{dh}{dt}(0^+) = -\alpha$$

Implementation as a filter

$$(\alpha^2 + \beta^2)h(t) + 2\alpha \frac{dh(t)}{dt} + \frac{d^2h(t)}{dt^2} = 0$$

$$h(0^+) = 1 \qquad \frac{dh}{dt}(0^+) = -\alpha$$

- Since $h(t)$ is not continuous at $t = 0$, the most general possibility is

$$(\alpha^2 + \beta^2)h(t) + 2\alpha \frac{dh(t)}{dt} + \frac{d^2h(t)}{dt^2} = a\delta(t) + b\delta'(t)$$

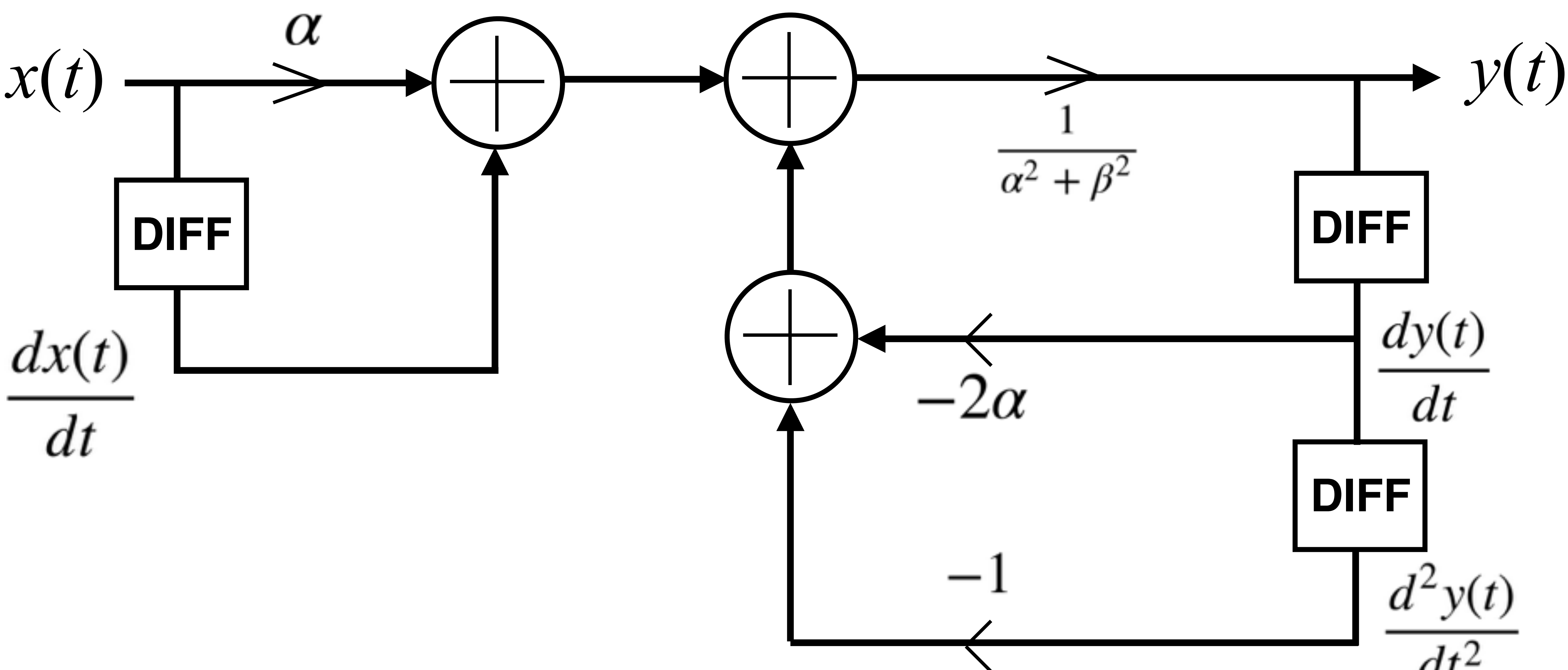
- Integrating both sides twice yields

$$a = \alpha, \quad b = 1$$

Implementation as a filter

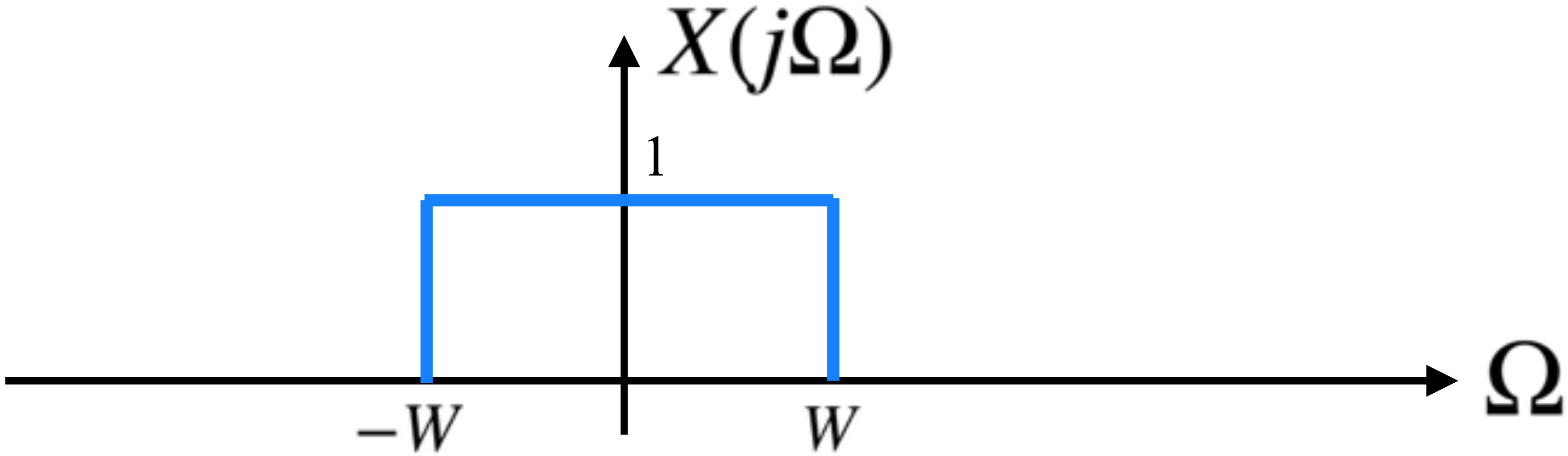
- The system therefore has a differential equation given by

$$(\alpha^2+\beta^2)y(t)+2\alpha\frac{dy(t)}{dt}+\frac{d^2y(t)}{dt^2}=\alpha x(t)+\frac{dx(t)}{dt}$$



Examples

- Find the signal $x(t)$ whose CTFT is as below:



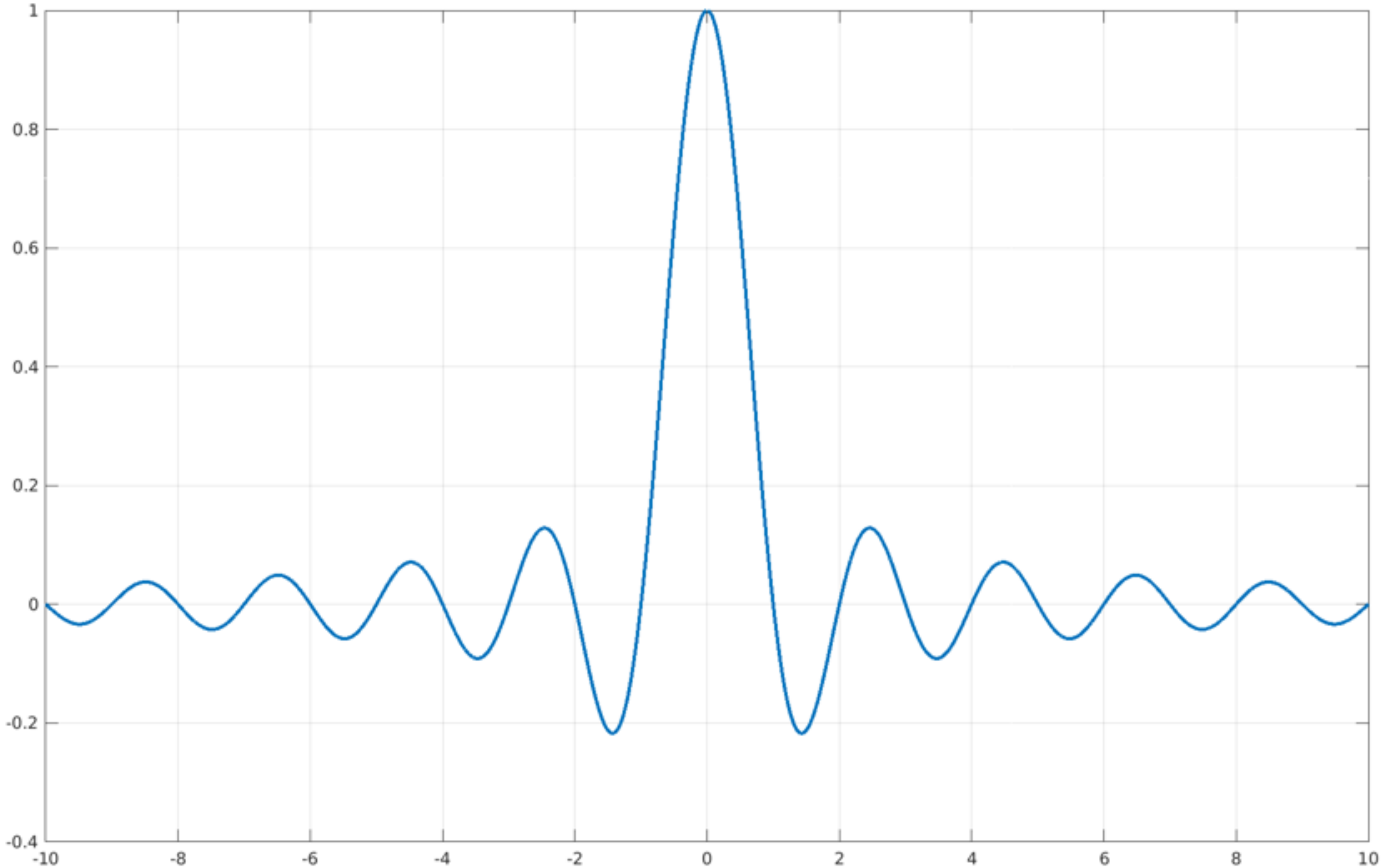
- <u>Solution:</u>

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega \;= \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega t}d\Omega$$

$$= \frac{1}{2\pi jt}\,e^{j\Omega t}\Big|_{-W}^{W} = \frac{e^{jWt} - e^{-jWt}}{2\pi jt} = \frac{\sin(Wt)}{\pi t}$$

Examples

- As an example, if $W = \pi$,



Examples

- Conversely, find the CTFT of

$$x(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

- <u>Solution:</u>

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \;= \int_{-T}^{T} e^{-j\Omega t}dt$$

$$= \frac{1}{-j\Omega}\, e^{-j\Omega t}\Big|_{-T}^{T} \;= \frac{e^{j\Omega T} - e^{-j\Omega T}}{j\Omega} \;= \frac{2\sin(\Omega T)}{\Omega}$$

CTFT of periodic signals

- Normally, CTFS suffices in decomposing onto complex exponentials.
- What if, just out of intellectual curiosity, we compute the CTFT of a periodic signal?
- Since

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

and since CTFT is linear, it suffices to find the CTFT of $e^{jk\Omega_0 t}$.

CTFT of periodic signals

- <u>Claim</u>:

$$e^{jk\Omega_0 t} \xrightarrow{\text{CTFT}} 2\pi\delta(\Omega - k\Omega_0)$$

- <u>Proof</u>: Using the synthesis formula, we see

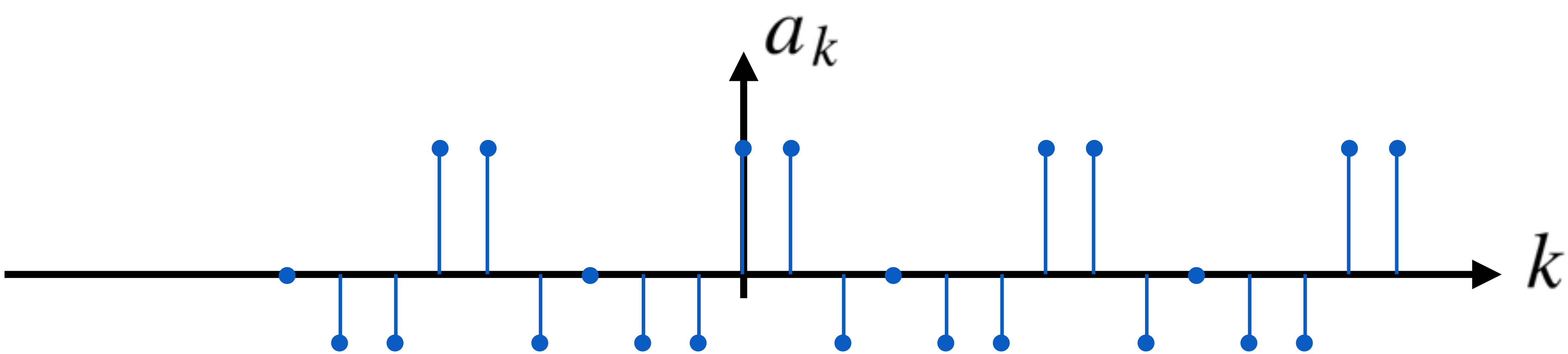
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega - k\Omega_0)e^{j\Omega t}d\Omega \quad = e^{jk\Omega_0 t}$$

- Therefore,

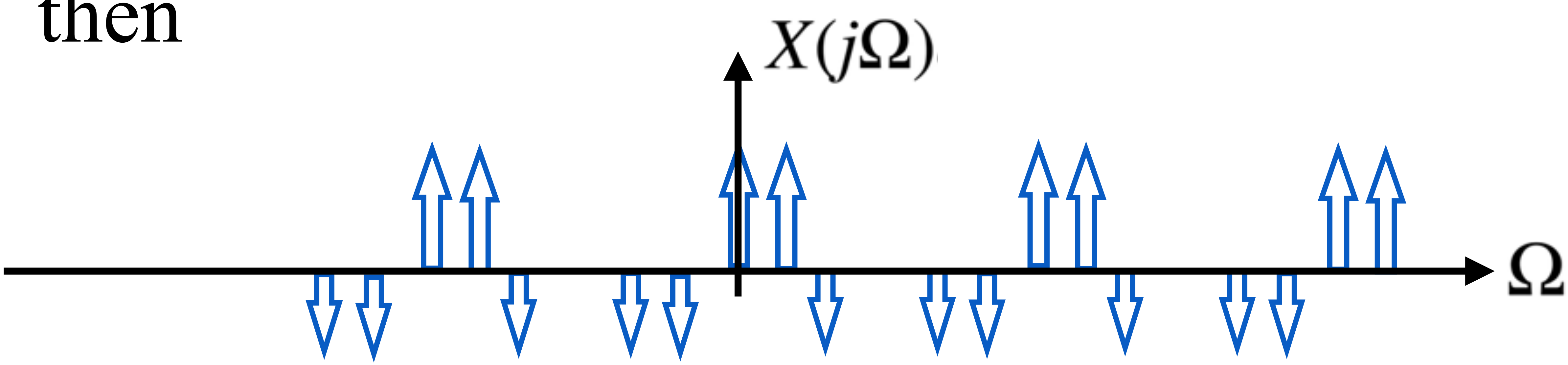
$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \xrightarrow{\text{CTFT}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$$

CTFT of periodic signals

- For example, if



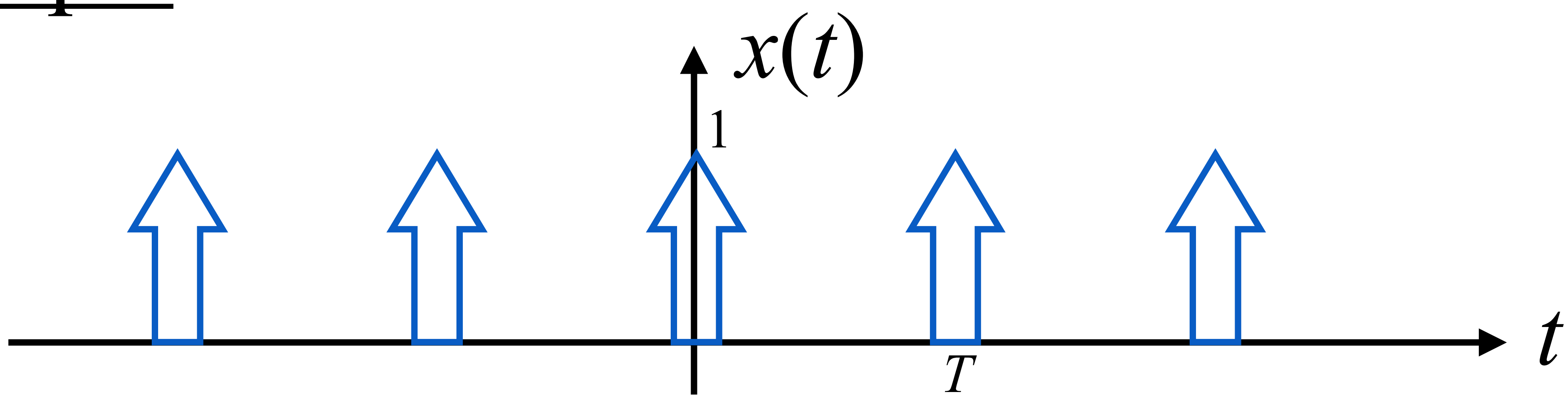
then



Each impulse at $\Omega = k\Omega_0$ is of amplitude $2\pi a_k$

CTFT of periodic signals

- Example: Find the CTFT of



- Solution: Recall that the CTFS coefficients were

$$a_k = \frac{1}{T}$$

Therefore,

$$X(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)$$

CTFT of periodic signals

- <u>Example</u>: Find the CTFT of