EE 110B Signals and Systems

Linear and Time-Invariant (LTI) Systems

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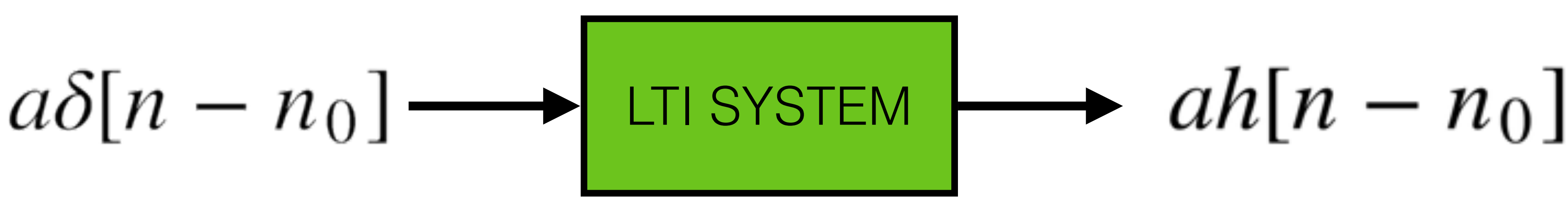
Why LTI systems?

- Linear and time-invariant systems are especially easy to analyze and design.
- In a lot of cases, they are good enough to do the "signal processing" job.
- Amenable to frequency analysis in the Fourier domain.

The impulse response

- An LTI system's response to an impulse input is called its **impulse response**.

- Because the system is LTI,



The impulse response

- Not only that, but also

$$\begin{array}{c} a_1\delta[n-n_1] \\ +a_2\delta[n-n_2] \end{array} \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \begin{array}{c} a_1h[n-n_1] \\ +a_2h[n-n_2] \end{array}$$

- Now, extending this all the way,

$$\sum_{k=-\infty}^{\infty} a_k\delta[n-k] \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k h[n-k]$$

The impulse response

- If only all signals came in the form

$$\sum_{k=-\infty}^{\infty} a_k \delta[n-k]$$

- Then we would figure out the output for any input in terms of the impulse response.
- But they DO come in that form!!! Recall that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The impulse response

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

which implies

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

- This sum is known as the **convolution sum**.
- Summary: If you know the impulse response of an LTI system, you know everything there is to know!

The convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

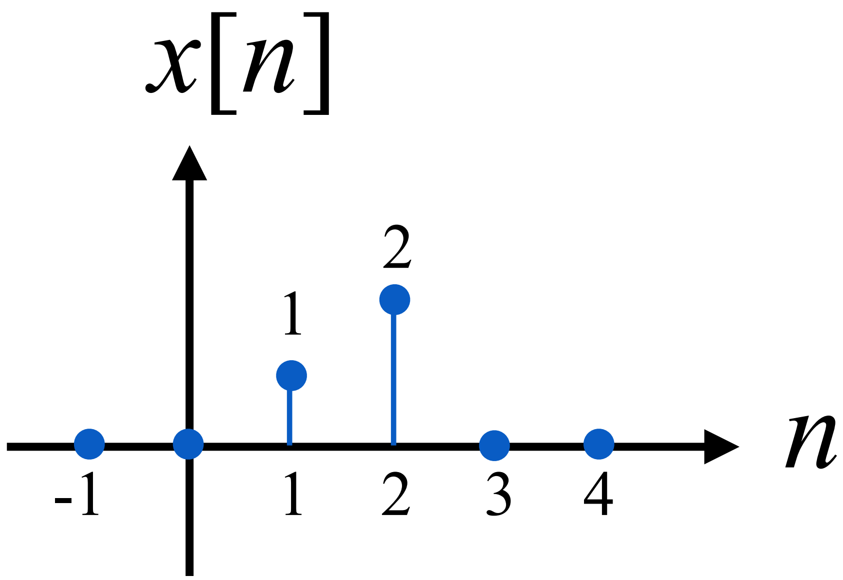
- This operation is also denoted as

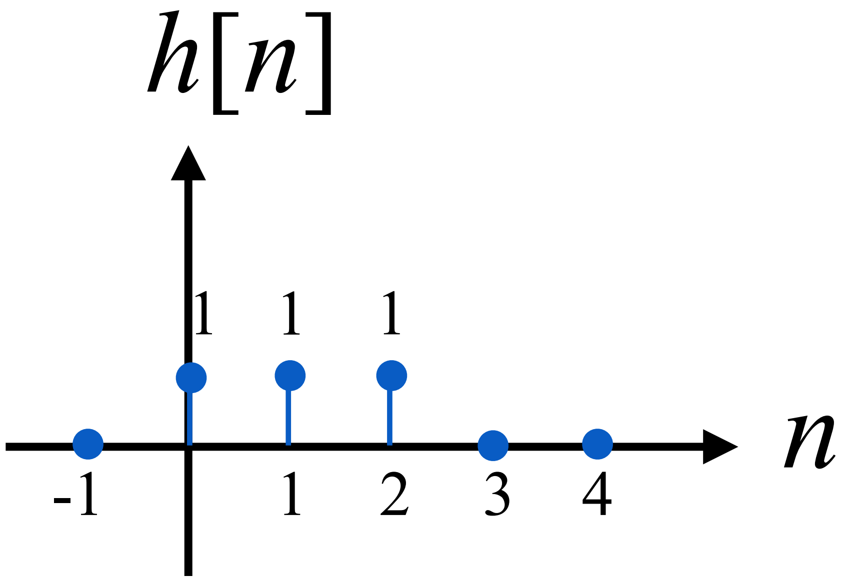
$$y[n] = x[n] \star h[n]$$

- As before, two ways to interpret this formula:
  - An infinite summation of shifted impulse responses $h[n\text{-}k]$ each scaled with $x[k]$.
  - For every n, an infinite sum of the samples of the product signal $x[k]\ h[n\text{-}k]$.

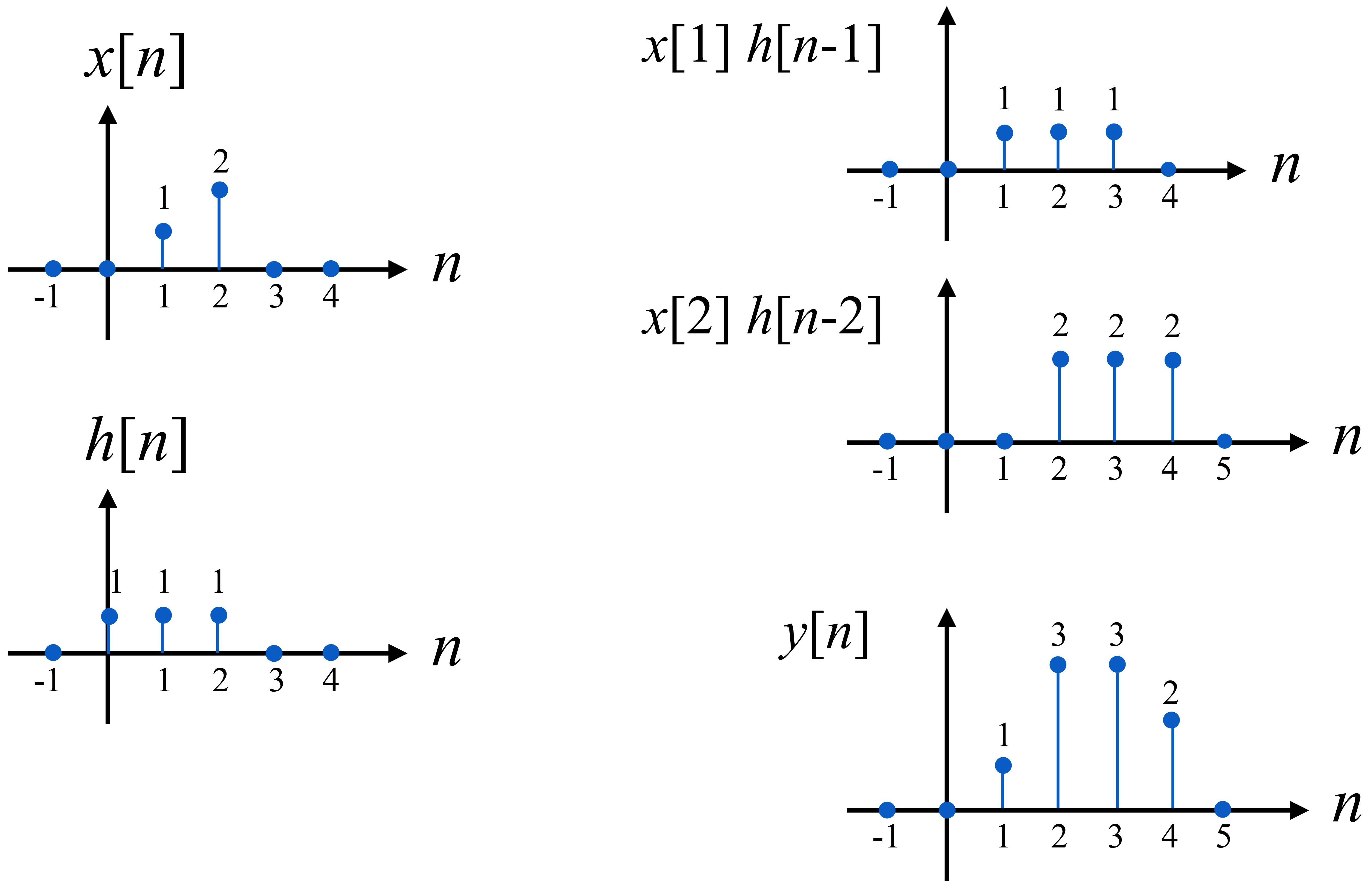
The convolution sum

- <u>Example</u>: Find $y[n] = x[n] \star h[n]$ if

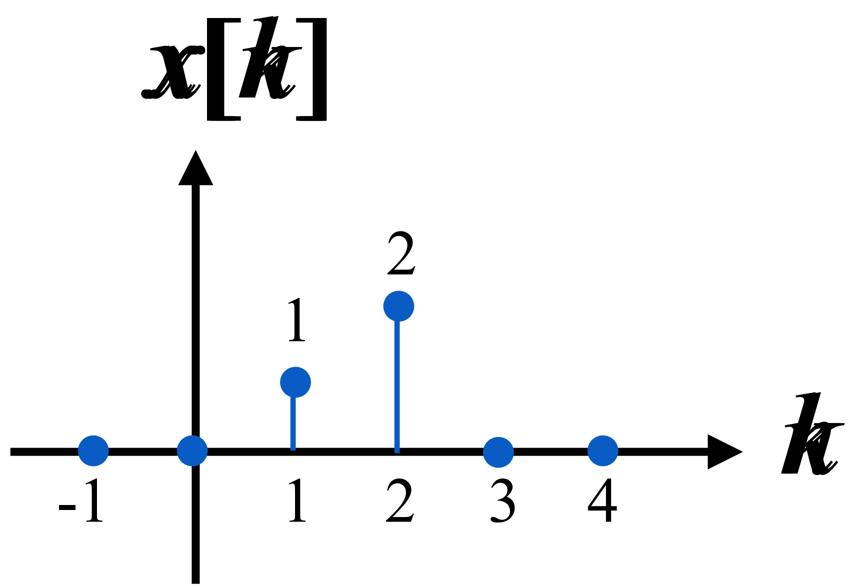


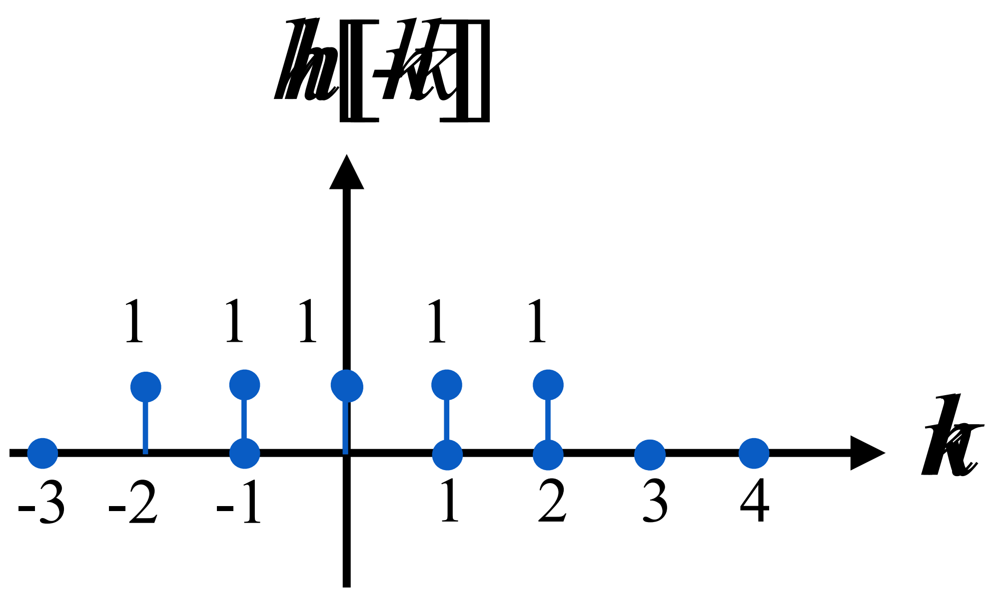


- <u>Method 1</u>: Accumulate $x[k]\ h[n-k]$'s.

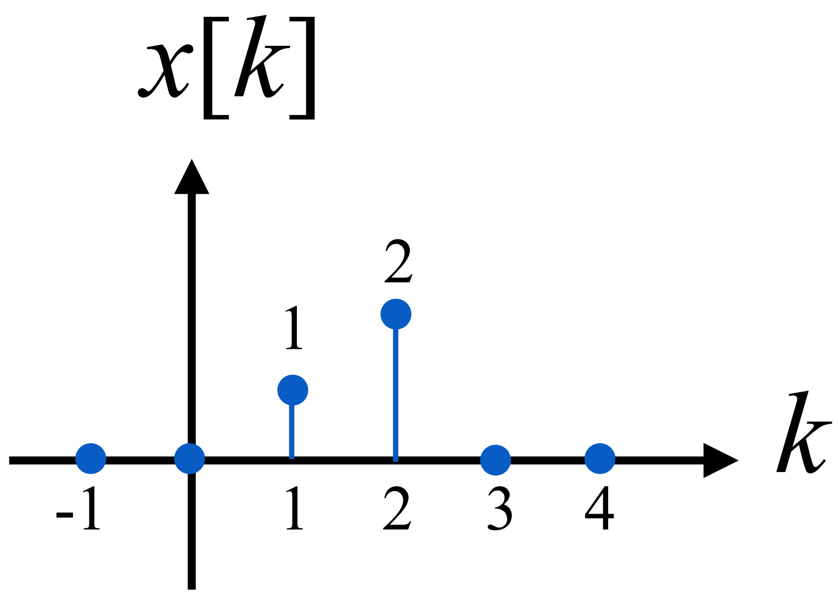


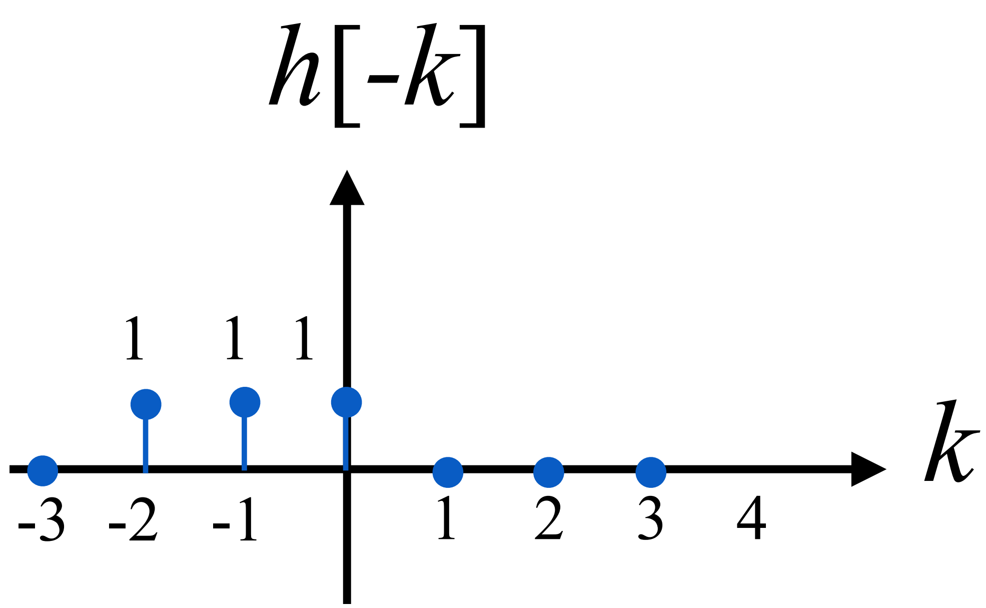
- <u>Method 2</u>: Calculate sum for each n.

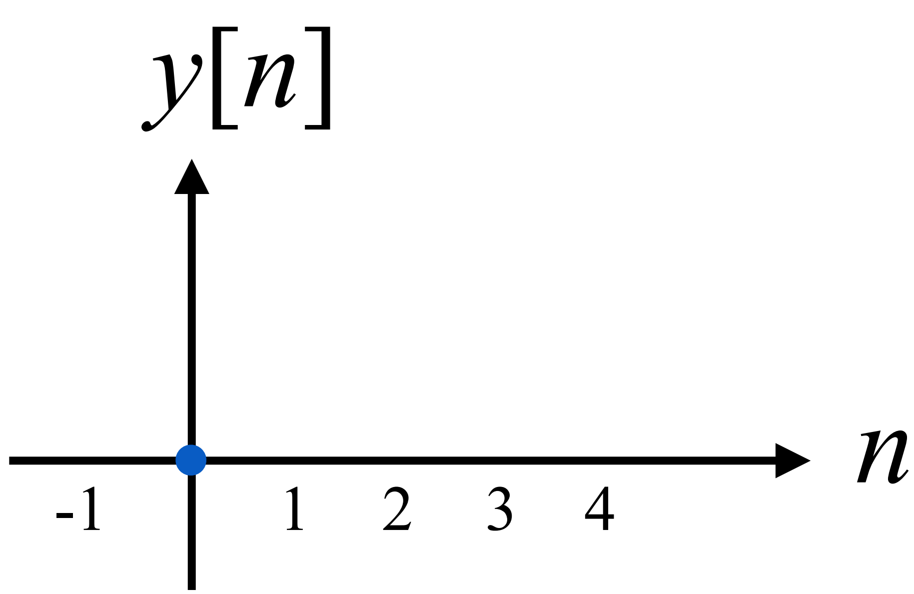




- Change the running variable to k
- To plot $h[-k]$, flip $h[k]$ around the y-axis.
- To plot $h[n-k]$, shift $h[-k]$ to the right by n units.
- For each n, sum up all samples of the product signal $x[k]h[n-k]$.



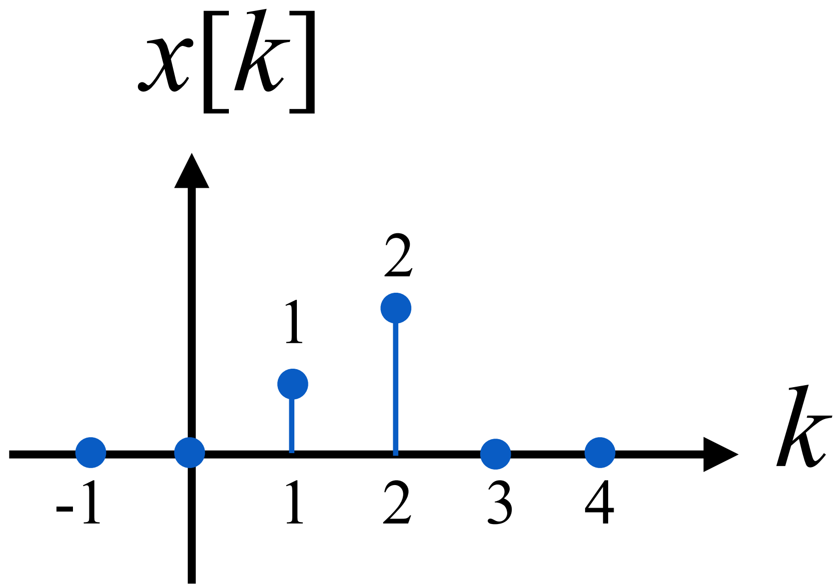


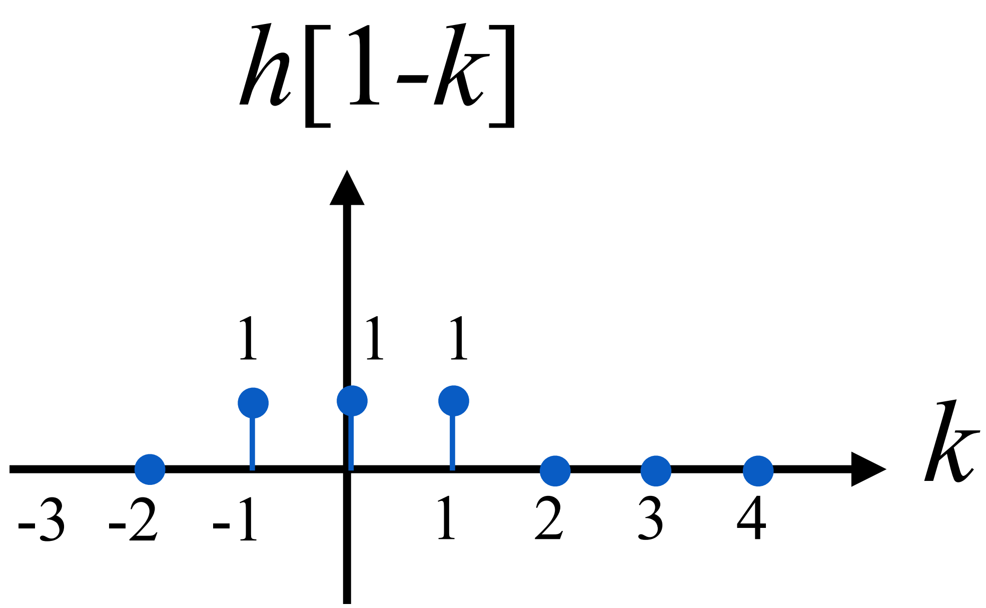


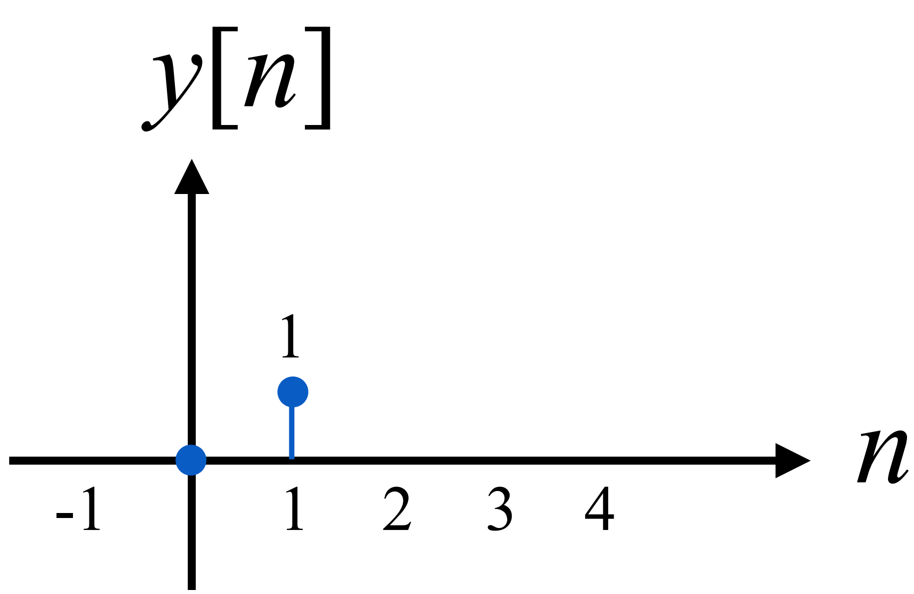
- To plot $h[n\text{-}k]$, shift $h[\text{-}k]$ to the right by n units.
- For each n, sum up all samples of the product signal $x[k]h[n\text{-}k]$.
- For $n = 0$, the sum yields

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$

$$= 0$$





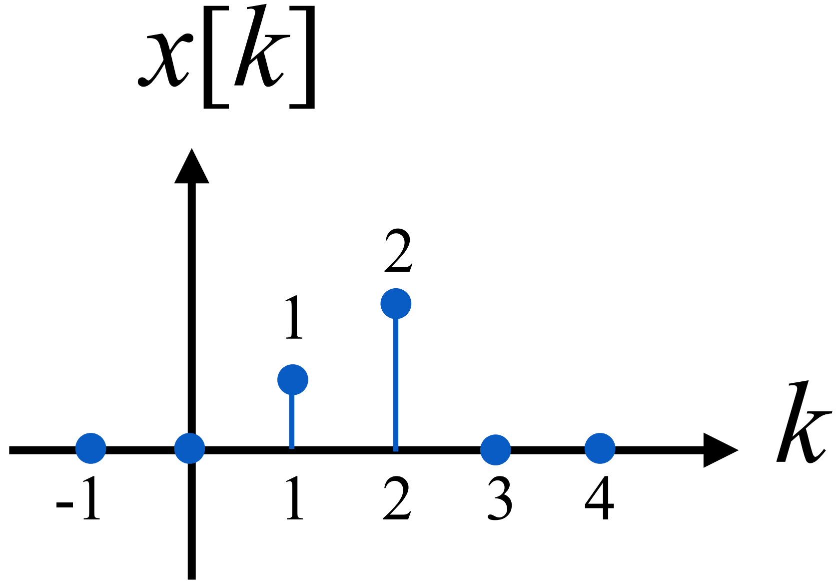


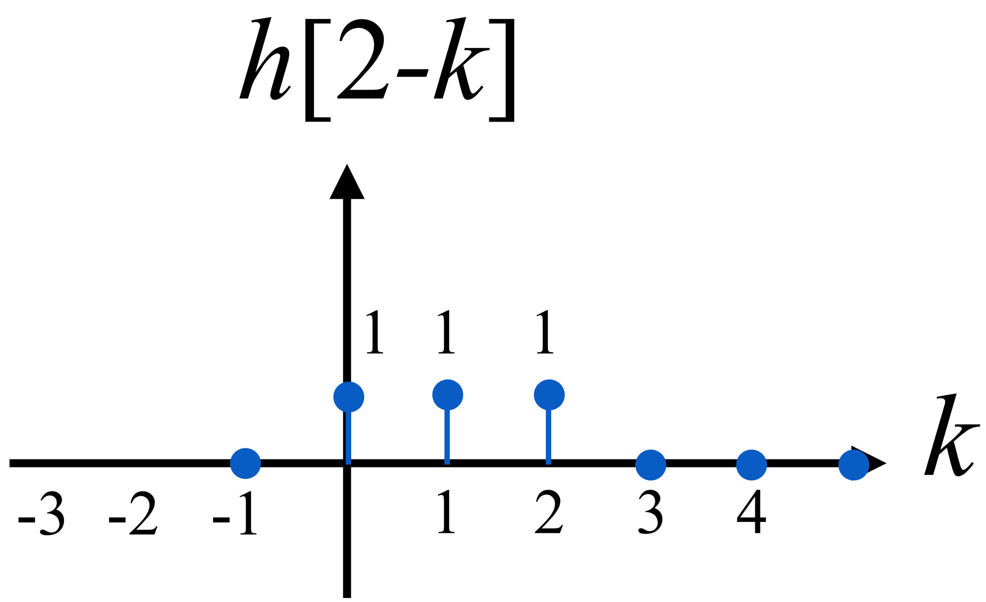
- To plot $h[n\text{-}k]$, shift $h[\text{-}k]$ to the right by n units.
- For each n, sum up all samples of the product signal $x[k]h[n\text{-}k]$.
- For $n = 1$, the sum yields

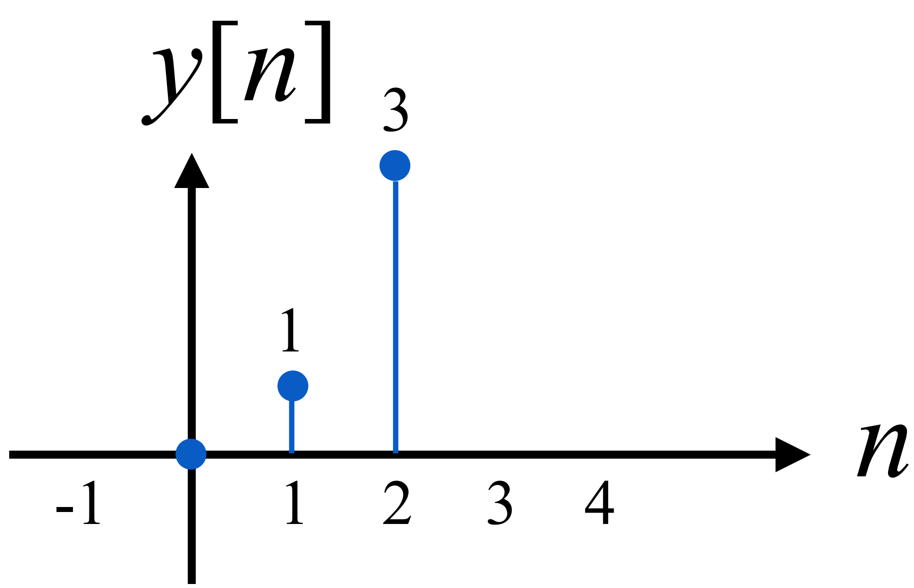
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$

$$= 1 \times 1$$

$$= 1$$





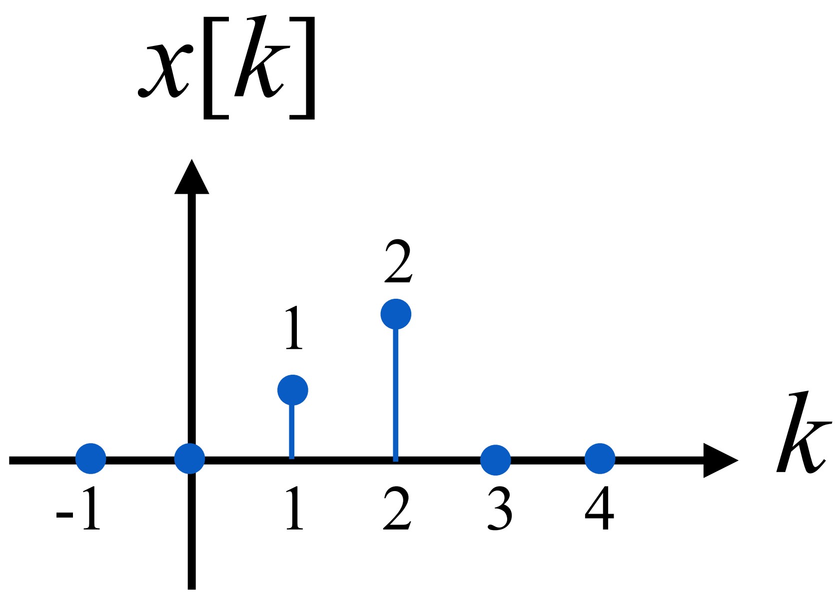


- To plot $h[n\text{-}k]$, shift $h[\text{-}k]$ to the right by n units.
- For each n, sum up all samples of the product signal $x[k]h[n\text{-}k]$.
- For $n = 2$, the sum yields

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$

$$= 1 \times 1 + 2 \times 1$$

$$= 3$$

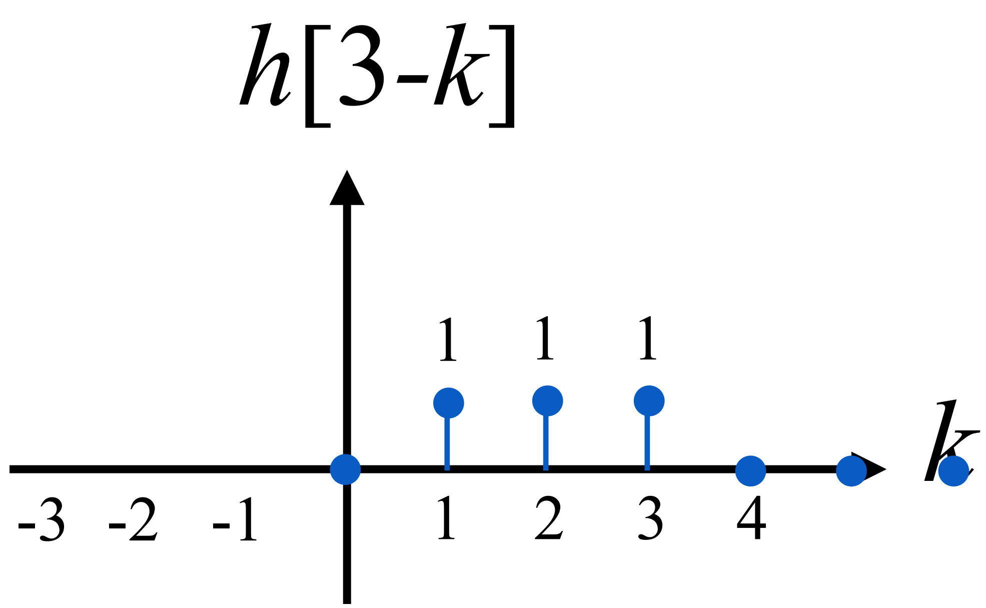


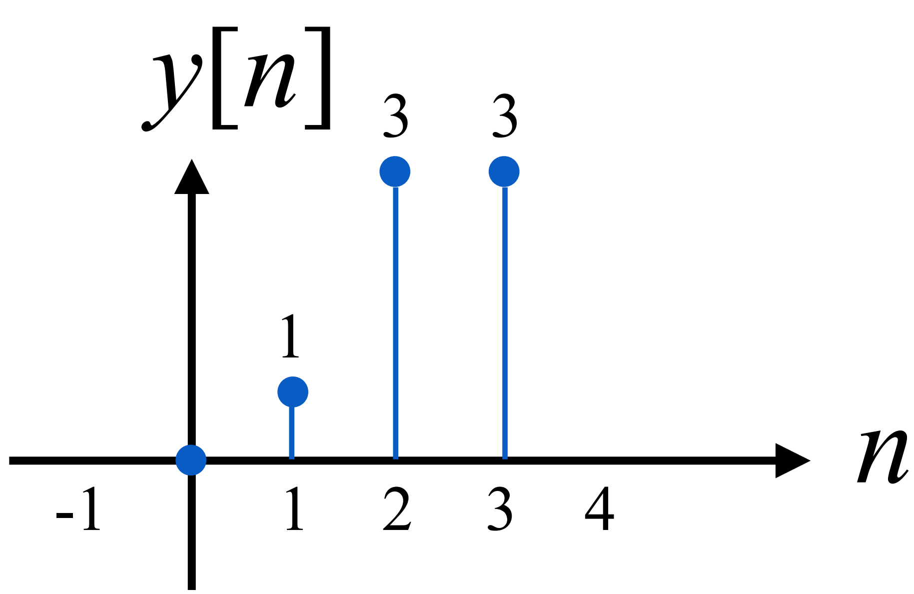
- To plot $h[n-k]$, shift $h[-k]$ to the right by n units.
- For each n, sum up all samples of the product signal $x[k]h[n-k]$.
- For $n = 3$, the sum yields

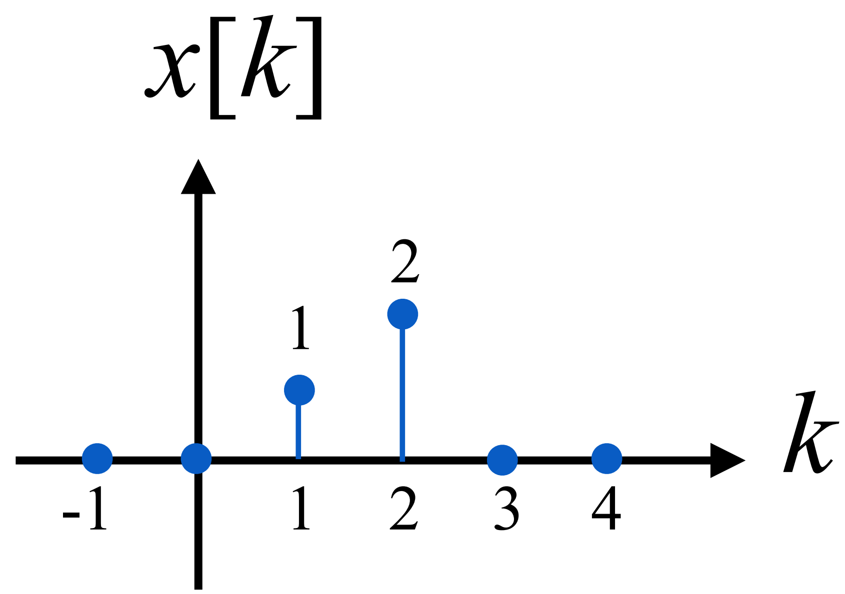
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$

$$= 1 \times 1 + 2 \times 1$$

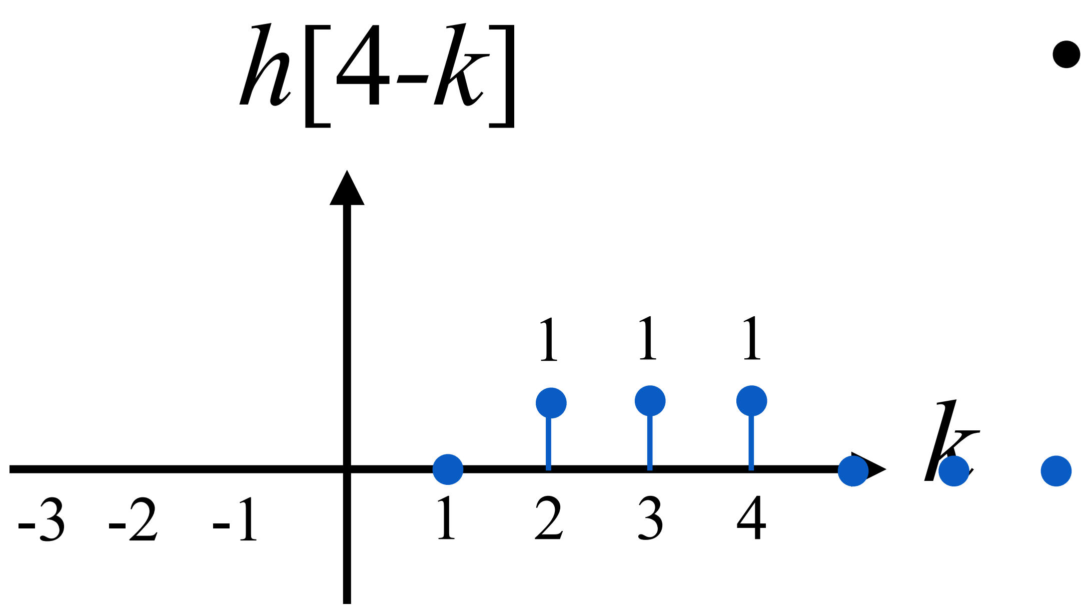
$$= 3$$







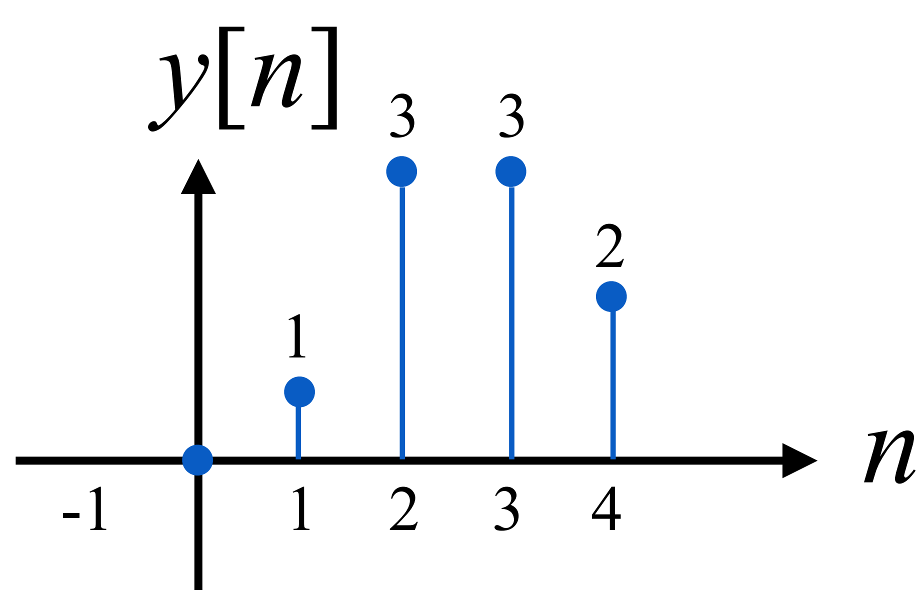
- To plot $h[n\text{-}k]$, shift $h[\text{-}k]$ to the right by n units.
- For each n, sum up all samples of the product signal $x[k]h[n\text{-}k]$.
- For $n = 4$, the sum yields

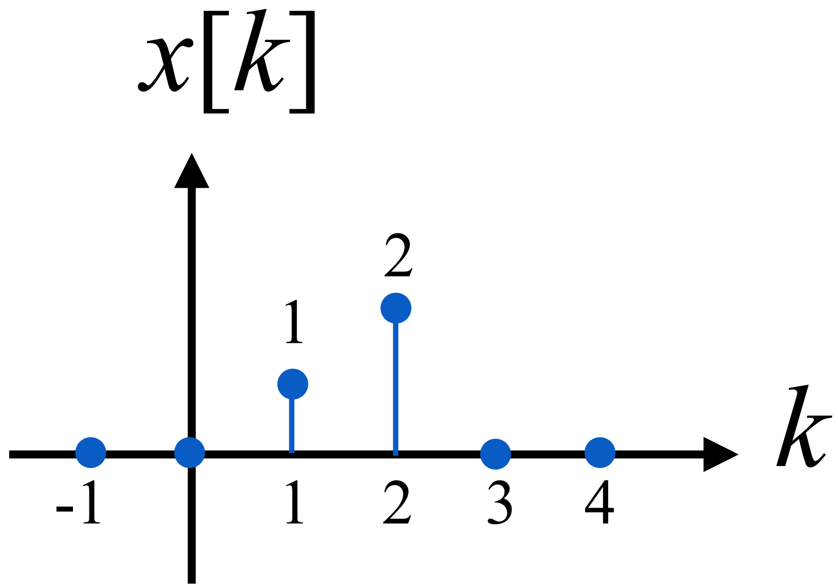


$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

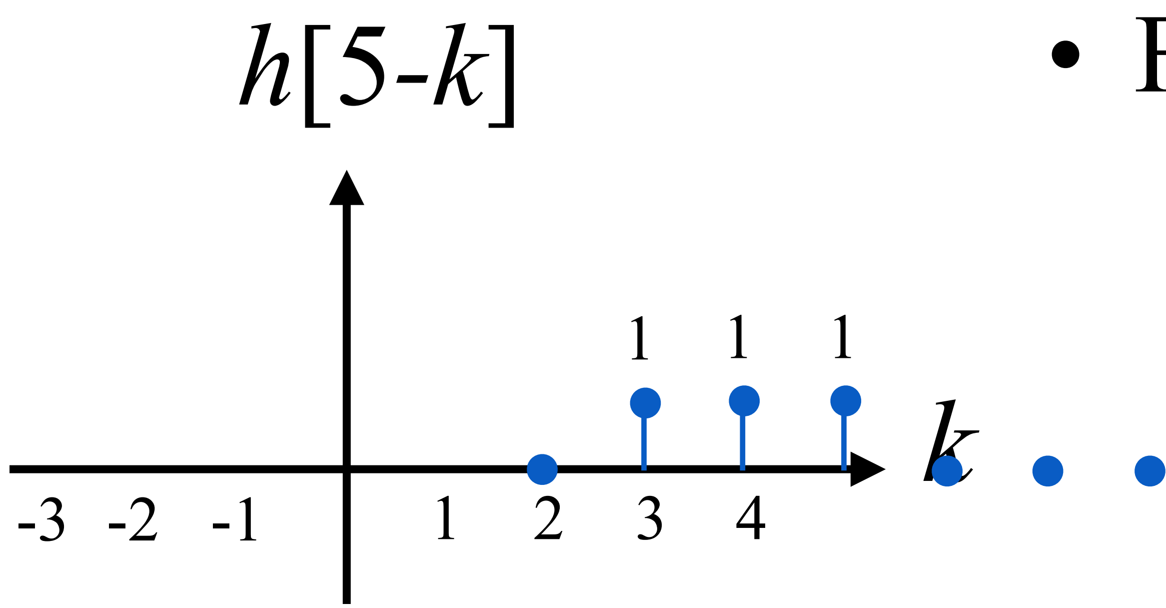
$$= 2 \times 1$$

$$= 2$$



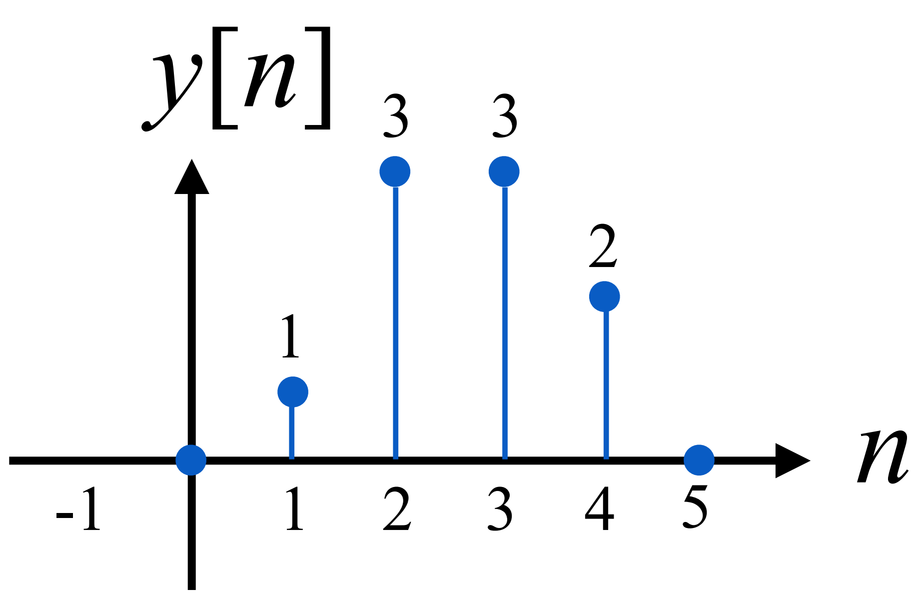


- To plot $h[n\text{-}k]$, shift $h[\text{-}k]$ to the right by n units.
- For each n, sum up all samples of the product signal $x[k]h[n\text{-}k]$.
- For $n = 5$, the sum yields



$$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$$

$$= 0$$



The convolution sum

- Example: Find $y[n] = x[n] \star h[n]$ if

$$x[n] = u[n]$$
$$h[n] = 0.5^n u[n]$$

- Now, since $x[n]$ has infinitely many non-zero samples, Method 1 won't work.
- For Method 2, the infinite sum becomes

$$y[n] = \sum_{k=-\infty}^{\infty} 0.5^{n-k} u[k] u[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} 0.5^{n-k} u[k] u[n-k]$$

$$= \sum_{k=0}^{\infty} 0.5^{n-k} u[n-k]$$

$$= 0.5^{n} \sum_{k=0}^{\infty} 0.5^{-k} u[n-k]$$

- If $n < 0$, the above sum is zero since it contains $u[n]$, $u[n\text{-}1]$, $u[n\text{-}2]$, ..., all of which is zero.
- Otherwise, $u[n\text{-}k] = 0$ only when $k > n$. Thus,

$$y[n] = 0.5^{n} u[n] \sum_{k=0}^{n} 0.5^{-k}$$

Digression: Power sums

- How do we compute $S = \sum_{k=0}^{n} \alpha^k$?

- Here is the trick:

$$S = 1 + \alpha + \alpha^2 + \cdots + \alpha^n$$

$$\alpha S = \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n+1}$$

$$\alpha S + 1 - \alpha^{n+1} = 1 + \alpha + \alpha^2 + \cdots + \alpha^n$$

$$\alpha S + 1 - \alpha^{n+1} = S$$

$\alpha S + 1 - \alpha^{n+1} = S$

- In other words,

$$S = \frac{\alpha^{n+1} - 1}{\alpha - 1}$$

- What about the infinite sum $S = \sum_{k=0}^{\infty} \alpha^k$?

- Converges only if $|\alpha| < 1$, and to

$$S = \frac{-1}{\alpha - 1} = \frac{1}{1 - \alpha}$$

Back to the example

- Example: Find $y[n] = x[n] \star h[n]$ if

$$x[n] = u[n]$$

$$h[n] = 0.5^n u[n]$$

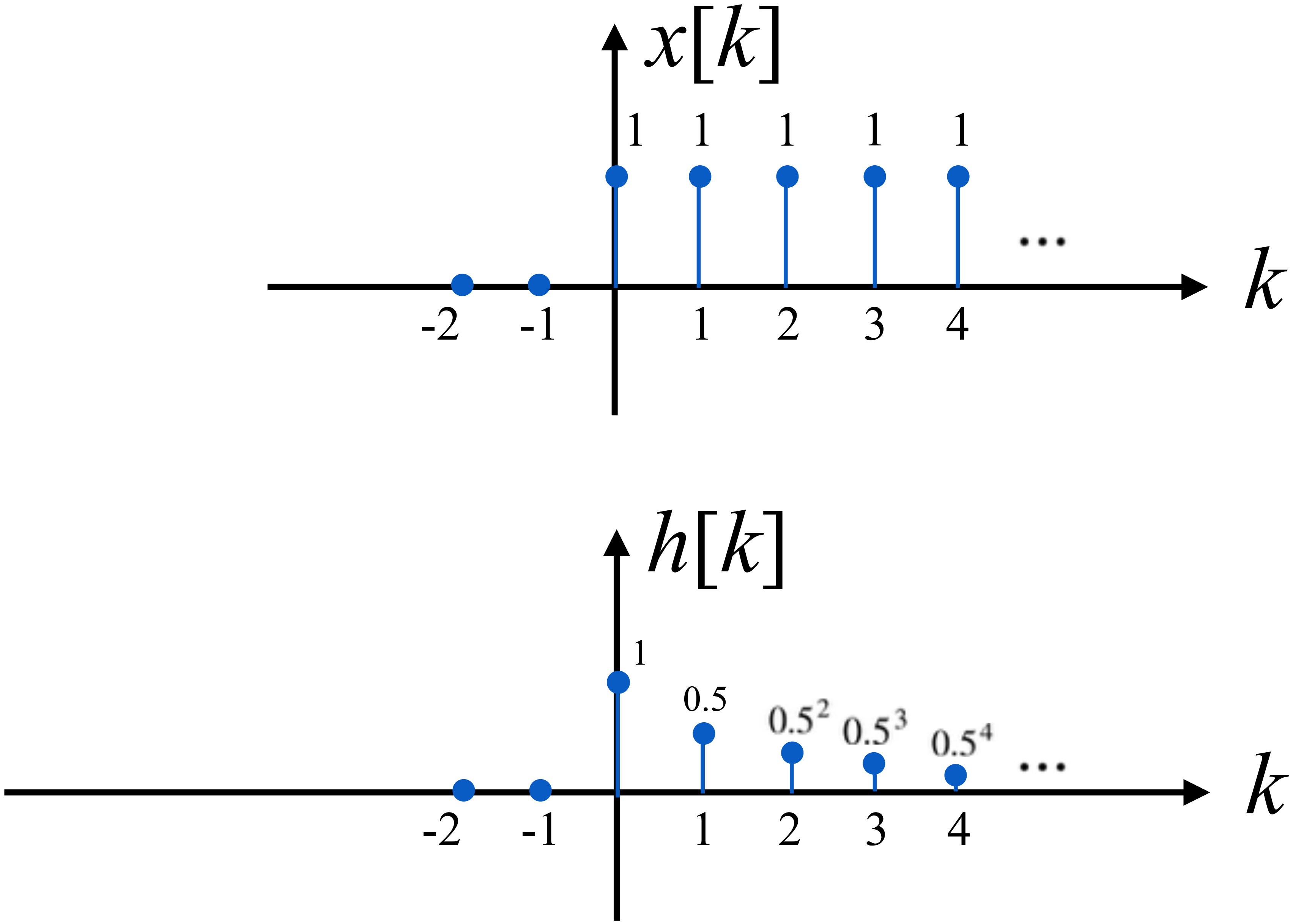
- We had found

$$y[n] = 0.5^n u[n] \sum_{k=0}^{n} 0.5^{-k} = 0.5^n u[n] \frac{2^{n+1} - 1}{2 - 1}$$

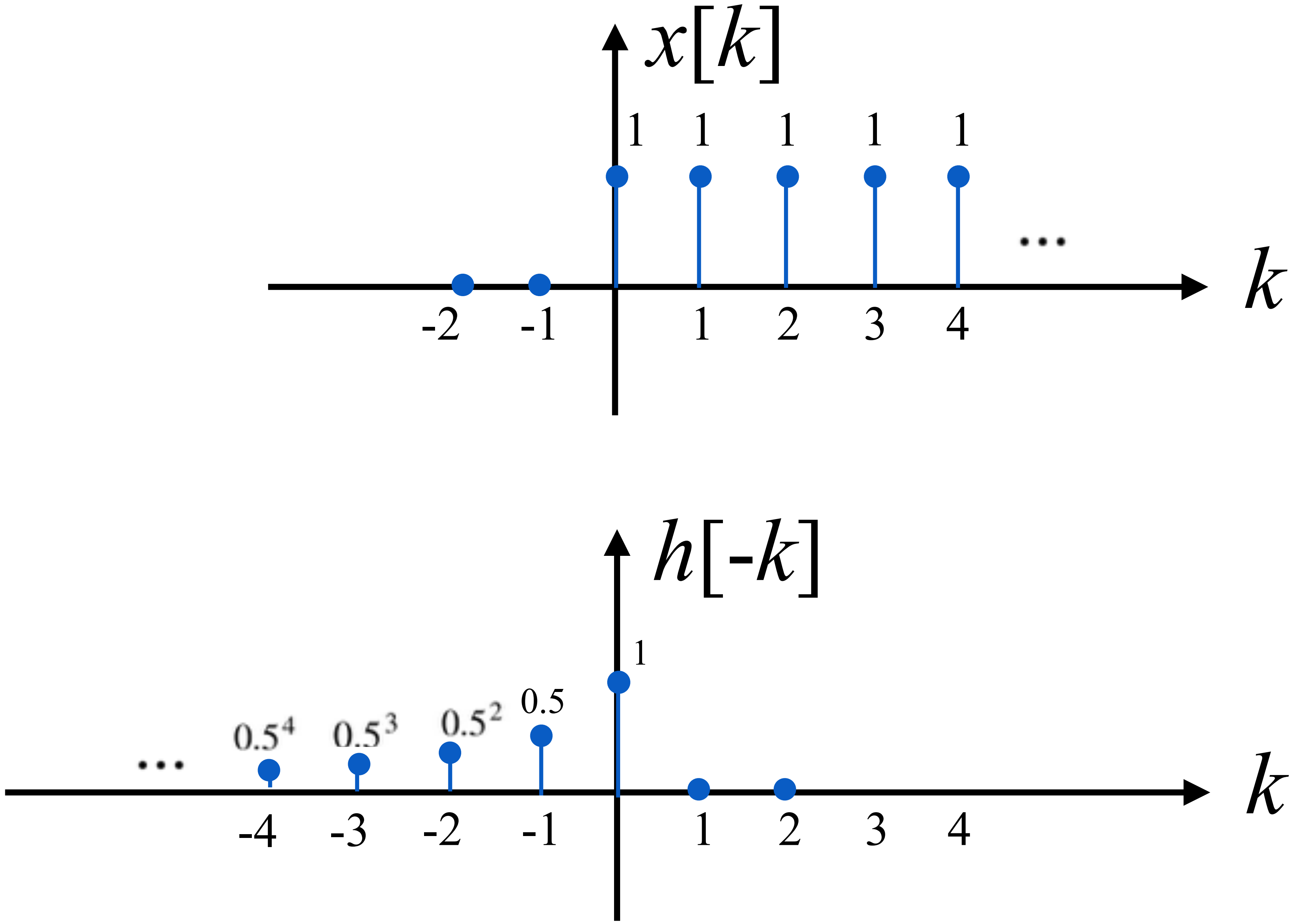
$$= u[n]\left(2^{n+1} 0.5^n - 0.5^n\right)$$

$$= \left(2 - 0.5^n\right) u[n]$$

A more visual solution

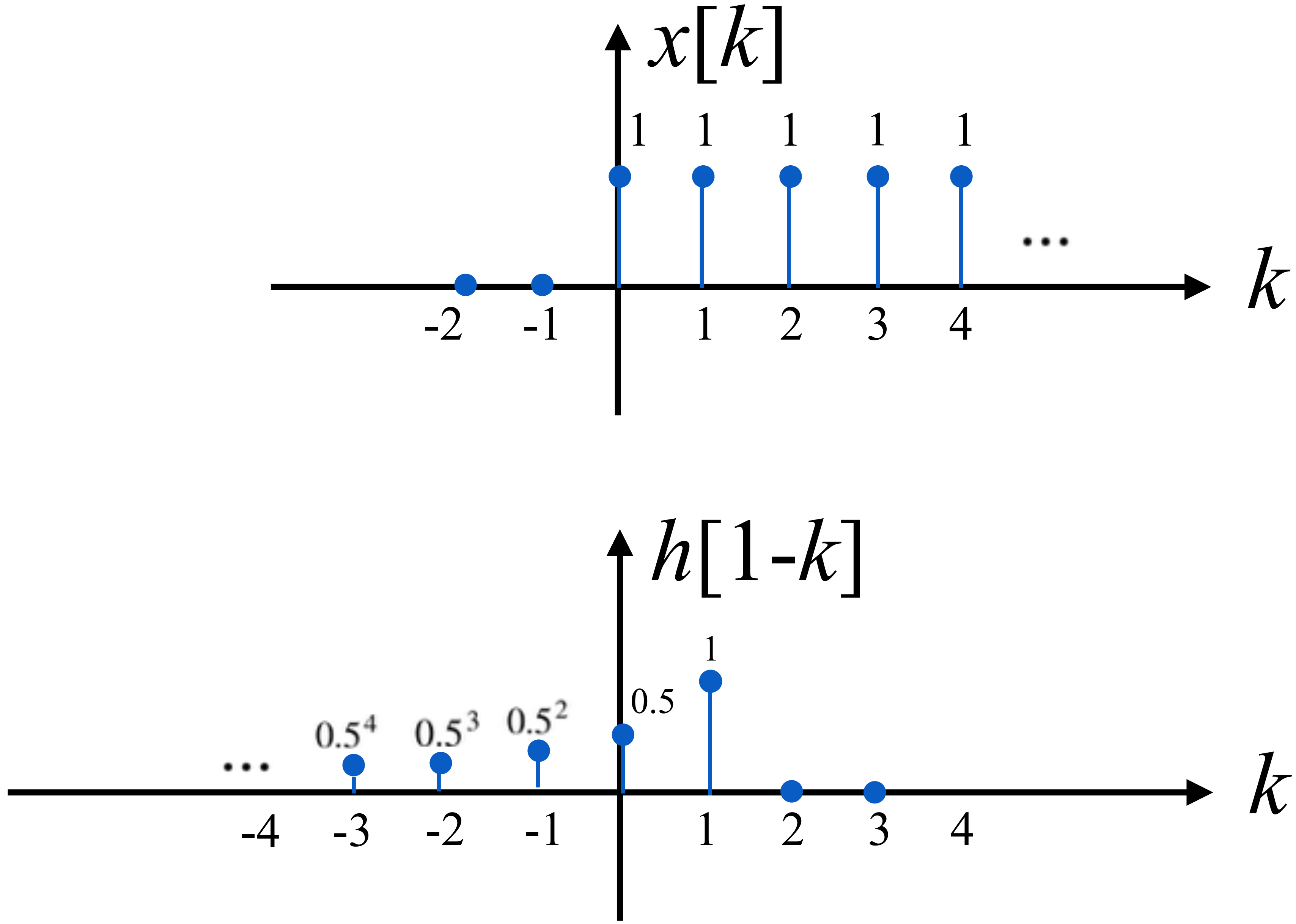


A more visual solution



$$y[0] = 1$$

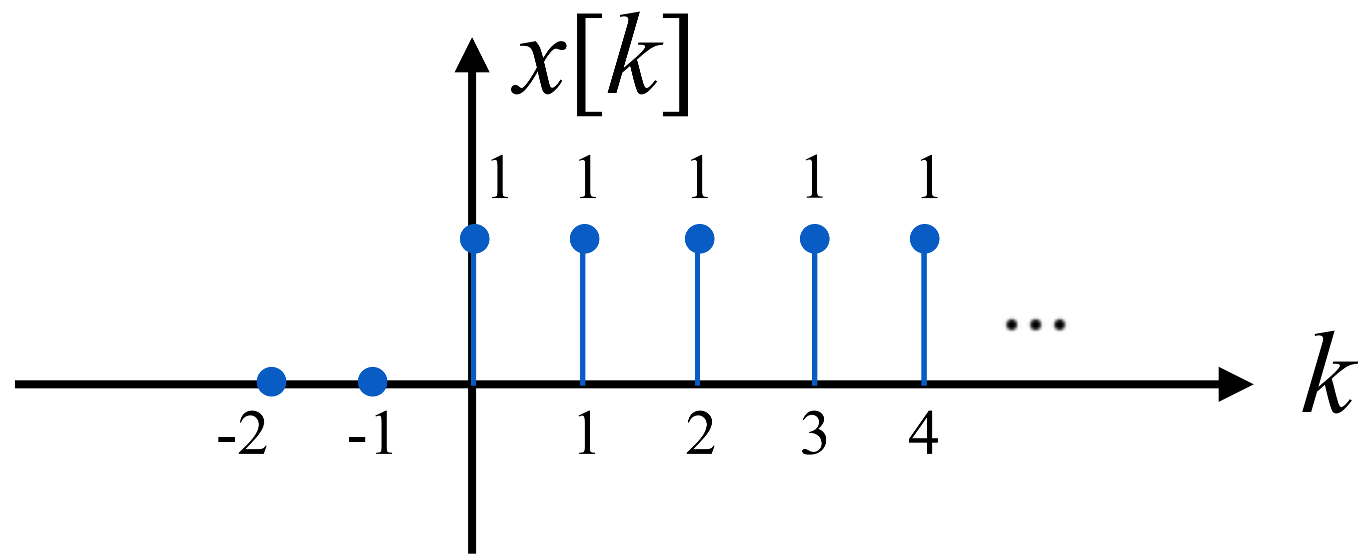
A more visual solution



$$y[0] = 1$$

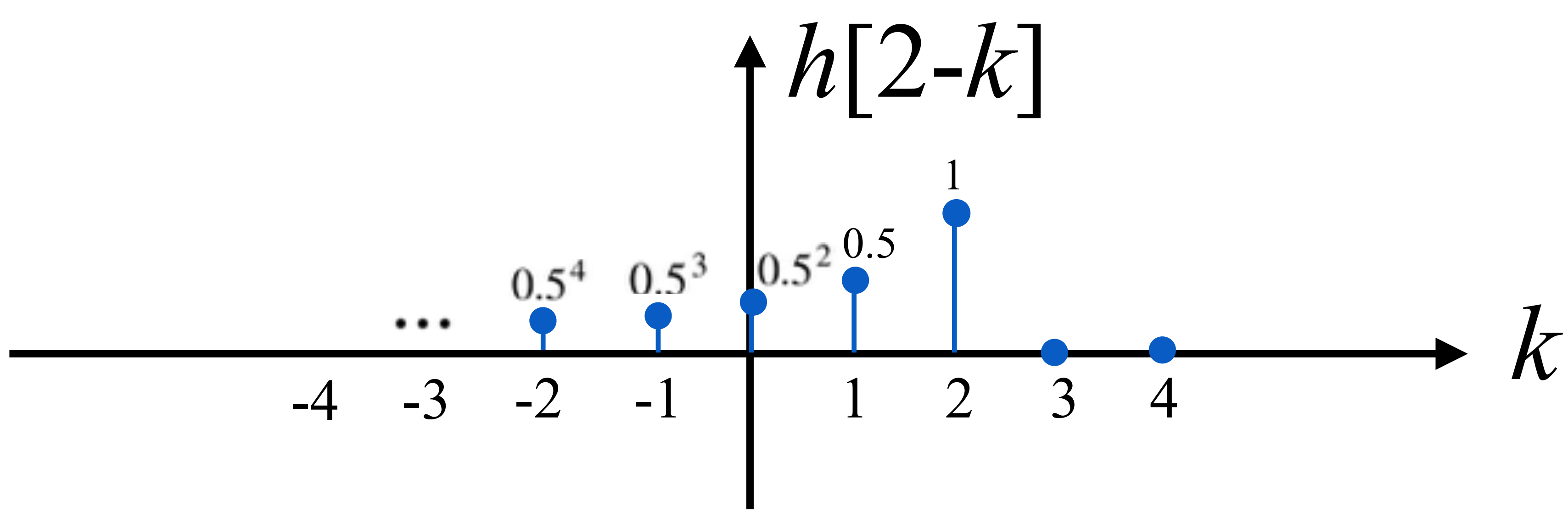
$$y[1] = 1 + 0.5 = 1.5$$

A more visual solution



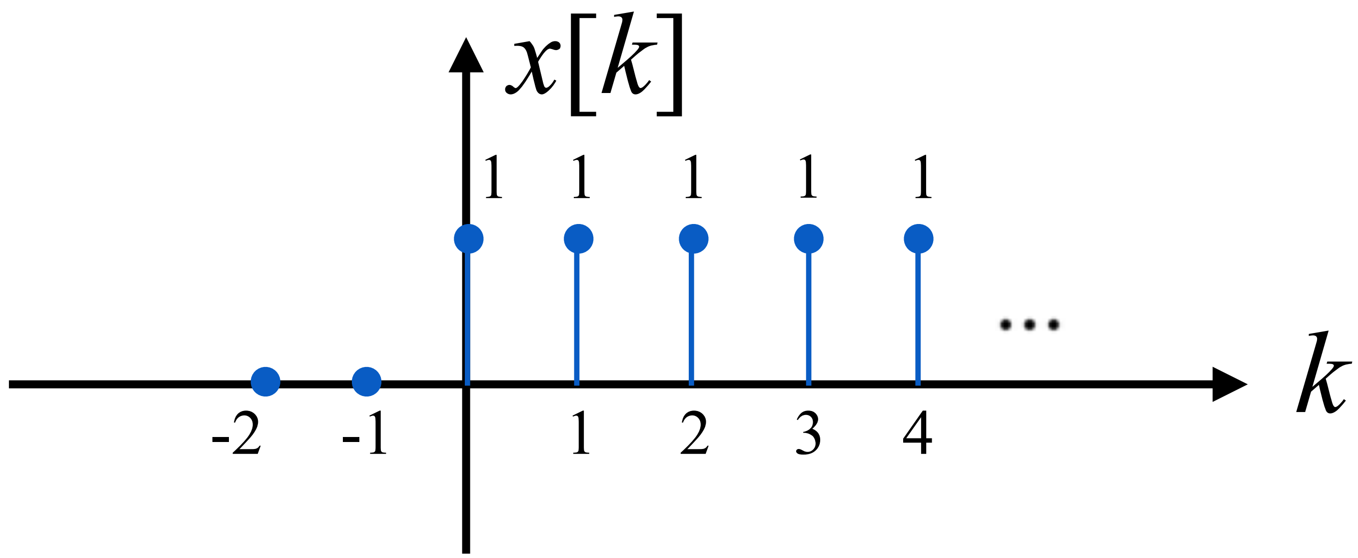
$$y[0] = 1$$

$$y[1] = 1.5$$

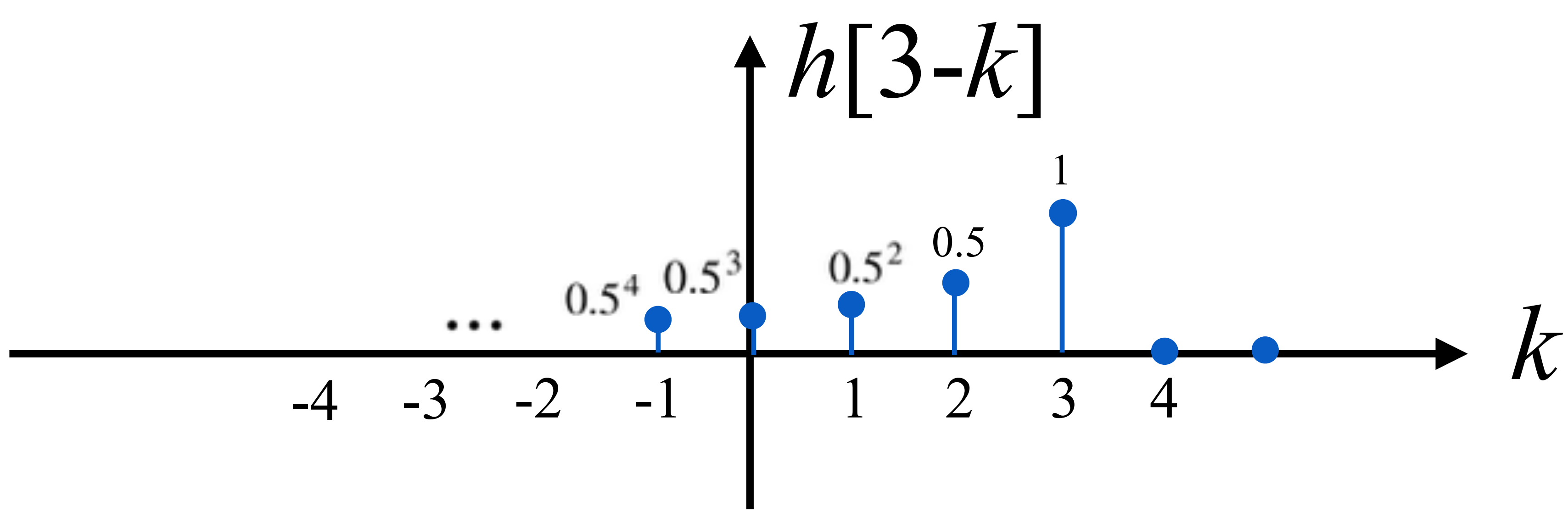


$$y[2] = 1 + 0.5 + 0.5^2 = 1.75$$

A more visual solution



$$y[0] = 1$$
$$y[1] = 1.5$$
$$y[2] = 1.75$$



$$y[3] = 1 + 0.5 + 0.5^2 + 0.5^3 = 1.875$$

Properties of convolution

- **Commutativity:**

$$x[n] \star h[n] = h[n] \star x[n]$$

- <u>Proof:</u>

$$x[n] \star h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$\overset{(m=n-k)}{=} \sum_{m=\infty}^{-\infty} x[n-m]h[m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] \star x[n]$$

Properties of convolution

- **Associativity:**

$$x[n] \star (y[n] \star z[n]) = (x[n] \star y[n]) \star z[n]$$

- <u>Proof:</u>

$$x[n] \star (y[n] \star z[n]) = x[n] \star \sum_{k=-\infty}^{\infty} y[k]z[n-k]$$

$$= \sum_{l=-\infty}^{\infty} x[l] \sum_{k=-\infty}^{\infty} y[k]z[n-l-k]$$

$$\overset{(m=k+l)}{=} \sum_{l=-\infty}^{\infty} x[l] \sum_{m=-\infty}^{\infty} y[m-l]z[n-m]$$

$$
\begin{aligned}
x[n] \star (y[n] \star z[n]) &\overset{(m=k+l)}{=} \sum_{l=-\infty}^{\infty} x[l] \sum_{m=-\infty}^{\infty} y[m-l] z[n-m] \\
&= \sum_{m=-\infty}^{\infty} z[n-m] \sum_{l=-\infty}^{\infty} x[l] y[m-l] \\
&= \sum_{m=-\infty}^{\infty} z[n-m] (x[m] \star y[m]) \\
&= (x[n] \star y[n]) \star z[n]
\end{aligned}
$$

Properties of convolution

- **Linearity:**

$$x_1[n] \star h[n] = y_1[n]$$

$$x_2[n] \star h[n] = y_2[n]$$

implies

$$(ax_1[n] + bx_2[n]) \star h[n] = ay_1[n] + by_2[n]$$

- <u>Proof</u>: Follows from the fact that convolution of the input with the impulse response yields the output for **linear** and time-invariant systems.

Properties of convolution

- The same logic leads to **Time-invariance:**

$$x[n] \star h[n] = y[n]$$

implies

$$x[n - n_0] \star h[n] = y[n - n_0]$$

- Thanks to commutativity, we also have

$$x[n - n_1] \star h[n - n_2] = y[n - (n_1 + n_2)]$$

Properties of convolution

- **Time-reversal:**

$$x[n] \star h[n] = y[n]$$

implies

$$x[-n] \star h[-n] = y[-n]$$

- <u>Proof:</u>

$$x[-n] \star h[-n] = \sum_{k=-\infty}^{\infty} x[-k]h[k-n]$$

$$\overset{(l=-k)}{=} \sum_{l=\infty}^{-\infty} x[l]h[-l-n] \ = y[-n]$$

Properties of convolution

- **Identity element:**

$$x[n] \star \delta[n] = x[n]$$

- <u>Proof:</u>

$$x[n] \star \delta[n] = \sum_{k=-\infty}^{\infty} \delta[k]x[n-k]$$

- All the terms in the above sum is zero, except at $k = 0$, where it is equal to $x[n]$.

System properties revisited

- For an LTI system, we can tell whether the system is **memoryless**, **causal**, **stable**, or **invertible** just by analyzing the impulse response.
- It may be more convenient to write the convolution sum as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- In general, this indicates that $y[n]$ depends on all samples of $x[n]$.

System properties revisited

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Write this more openly as

$$\begin{aligned} y[n] = \ldots + h[2]x[n-2] & \\ +h[1]x[n-1] & \quad \textbf{PAST} \\ \hline +h[0]x[n] & \quad \textbf{PRESENT} \\ \hline +h[-1]x[n+1] & \quad \textbf{FUTURE} \\ +h[-2]x[n+2] + \ldots & \end{aligned}$$

System properties revisited

- For the system to be **memoryless**, the present value of $y[n]$ must depend only on the **present** value of $x[n]$.
- That is the same as

$$h[n] = 0 \quad \forall n \neq 0$$

- In other words, the impulse response must be of the form $h[n] = c\delta[n]$ for some c.

System properties revisited

- For the system to be **causal**, the present value of $y[n]$ must depend only on the **present** and **past** values of $x[n]$.
- That is the same as

$$h[n] = 0 \quad \forall n < 0$$

- In other words, the impulse response must be of the form $h[n] = g[n]u[n]$ for some $g[n]$.

System properties revisited

- For **stability**, let us analyze $|y[n]|$:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} \left|h[k]\right| \cdot \left|x[n-k]\right|$$

- Now, if $|x[n]|$ is bounded by B for all n,

$$|y[n]| < B \sum_{k=-\infty}^{\infty} \left|h[k]\right|$$

System properties revisited

$$|y[n]| < B \sum_{k=-\infty}^{\infty} |h[k]|$$

- Therefore, a **sufficient** condition for stability is

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty}$$

- It is also **necessary** because otherwise, we could just select $x[n] = \mathrm{sign}(h[-n])$ to obtain

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} h[k]\mathrm{sign}(h[k]) = \sum_{k=-\infty}^{\infty} |h[k]|$$

$$= \infty$$

System properties revisited

- The system has an LTI **inverse** if and only if there exists a signal $g[n]$ such that
$$x[n] \star h[n] \star g[n] = x[n]$$
for all $x[n]$.
- This is equivalent to
$$h[n] \star g[n] = \delta[n]$$
- If such $g[n]$ exists, it is the impulse response of the inverse system.

Examples

- <u>Example</u>: Determine if the system is memoryless, causal, stable, or invertible if its impulse response is given by

$$h[n] = 0.5^n u[n]$$

- **Memory:** $h[n]$ is not of the form $c\delta[n]$ **HAS MEMORY**
- **Causality:** $h[n]$ is of the form $g[n]u[n]$ **CAUSAL** ✓
- **Stability:**

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} 0.5^k = \frac{1}{1-0.5} = 2$$

STABLE ✓

Examples

- Example: Determine if the system is memoryless, causal, stable, or invertible if its impulse response is given by

$$h[n] = 0.5^n u[n]$$

- **Invertibility:** Observe that

$$\begin{aligned} h[n] - 0.5h[n-1] &= 0.5^n u[n] - 0.5 \cdot 0.5^{n-1} u[n-1] \\ &= 0.5^n \left(u[n] - u[n-1] \right) \\ &= 0.5^n \delta[n] \\ &= \delta[n] \end{aligned}$$

Examples

- **Invertibility:** Observe that

$$h[n] - 0.5h[n-1] = \delta[n]$$

- Now, can we rewrite this as $h[n] \star g[n] = \delta[n]$ for some $g[n]$?

- Yes. Take $g[n] = \delta[n] - 0.5\delta[n-1]$:

$$\begin{aligned} h[n] \star (\delta[n] - 0.5\delta[n-1]) \\ &= h[n] \star \delta[n] - 0.5h[n] \star \delta[n-1] \\ &= h[n] - 0.5h[n-1] \end{aligned}$$

INVERTIBLE ✓

Examples

- Example: Determine if the system is memoryless, causal, stable, or invertible if its impulse response is given by

$$h[n] = (-1)^n$$

- **Memory:** $h[n]$ is not of the form $c\delta[n]$ **HAS MEMORY**
- **Causality:** $h[n]$ is not of the form $g[n]u[n]$ **NON-CAUSAL**
- **Stability:**

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \left|(-1)^k\right| = \sum_{k=-\infty}^{\infty} 1 = \infty$$ **UNSTABLE**

Examples

- **Invertibility:** If $g[n]$ exists such that

$$h[n] \star g[n] = \delta[n]$$

what would be the result of $h[n-2] \star g[n]$?

- Due to time invariance, it must be $\delta[n-2]$
- Due to the fact that $h[n-2] = h[n]$, it must be $\delta[n]$
- Contradiction!!!!
- No such $g[n]$ can exist.

NOT INVERTIBLE