EE 115 — Homework 1: Solutions

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Fourier series/transform, complex envelopes, DSB–SC demod

Problem 1

A real, periodic signal $m(t) = m(t+T)$ is defined over one symmetric period ($|t| < T/2$) by

$$m(t) = \begin{cases} -3, & -\frac{T}{3} \le t < 0, \\ 3, & 0 \le t < \frac{T}{3}, \\ 0, & \text{otherwise on } (-\frac{T}{2}, \frac{T}{2}), \end{cases}$$

and extended periodically.

(a) Mean, energy in $|t| < T/2$, and average power

Mean. Over any period, areas cancel by odd symmetry:

$$\bar{m} = \frac{1}{T}\int_{-T/2}^{T/2} m(t)\,dt = \frac{1}{T}\left[3 \cdot \tfrac{T}{3} + (-3)\cdot \tfrac{T}{3}\right] = 0.$$

Energy within $|t| < T/2$. On $(-T/2, T/2)$ the signal is ± 3 on a total length $2(T/3)$ and 0 elsewhere, so

$$E_{|t|<T/2} = \int_{-T/2}^{T/2} m^2(t)\,dt = 9 \cdot \frac{2T}{3} = 6T.$$

Average power. For a periodic signal, $P_{\text{avg}} = \dfrac{1}{T}\int_{-T/2}^{T/2} m^2(t)\,dt = \dfrac{6T}{T} = 6.$

(b) Exponential Fourier series coefficients

Let $\omega_0 = \frac{2\pi}{T}$ and $k \in \mathbb{Z}$. With

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} m(t)\,e^{-jk\omega_0 t}\,dt,$$

only the two nonzero subintervals contribute:

$$
\begin{aligned}
c_k &= \frac{1}{T}\left(\int_0^{T/3} 3\,e^{-jk\omega_0 t}\,dt + \int_{-T/3}^{0} (-3)\,e^{-jk\omega_0 t}\,dt\right) \\
&= \frac{3}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^{T/3} - \frac{3}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T/3}^{0} \\
&= \frac{3}{-jk\omega_0 T}\left(e^{-j\frac{2\pi k}{3}} - 1\right) + \frac{3}{jk\omega_0 T}\left(1 - e^{j\frac{2\pi k}{3}}\right) \\
&= \frac{6}{jk\omega_0 T}\left(1 - \cos\tfrac{2\pi k}{3}\right) = -j\,\frac{6}{2\pi k}\left(1 - \cos\tfrac{2\pi k}{3}\right).
\end{aligned}
$$

Thus for $k \neq 0$,

$$
\boxed{c_k = -j\,\frac{3}{\pi k}\left(1 - \cos\tfrac{2\pi k}{3}\right) = -j\,\frac{6}{\pi k}\,\sin^2\!\left(\frac{\pi k}{3}\right)}
$$

and since $m(t)$ is odd and real, $\boxed{c_0 = 0}$, each c_k is *purely imaginary* (no cosine terms), and $c_{-k} = -c_k$ (odd in k).
A convenient simplification uses the 3-periodicity of $\cos(2\pi k/3)$:

$$
\cos\!\left(\tfrac{2\pi k}{3}\right) = \begin{cases} 1, & 3 \mid k, \\ -1/2, & 3 \nmid k, \end{cases} \qquad \Rightarrow \qquad \boxed{c_k = \begin{cases} 0, & 3 \mid k, \\ -j\,\dfrac{9}{2\pi k}, & 3 \nmid k\,. \end{cases}}
$$

(c) Real and imaginary parts of $M(f)$ for $|f| < \frac{6}{T}$

For a periodic signal, the Fourier transform is a line spectrum:

$$
M(f) = \sum_{k=-\infty}^{\infty} c_k\,\delta\!\left(f - \frac{k}{T}\right).
$$

Here c_k are purely imaginary and vanish for k divisible by 3. Therefore

$$
\mathrm{Re}\ M(f) \equiv 0, \qquad \mathrm{Im}\ M(f) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathrm{Im}\{c_k\}\,\delta\!\left(f - \frac{k}{T}\right),
$$

with $\mathrm{Im}\{c_k\} = -\frac{9}{2\pi k}$ for $k \equiv \pm1, \pm2 \pmod 3$, and 0 for $k \equiv 0 \pmod 3$. The first few impulses ("sketch data") within $|f| < 6/T$:

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f = \frac{k}{T}$	$-\frac{5}{T}$	$-\frac{4}{T}$	$-\frac{3}{T}$	$-\frac{2}{T}$	$-\frac{1}{T}$	0	$\frac{1}{T}$	$\frac{2}{T}$	$\frac{3}{T}$	$\frac{4}{T}$	$\frac{5}{T}$
$\mathrm{Im}\{c_k\}$	$\frac{9}{10\pi}$	$\frac{9}{8\pi}$	0	$\frac{9}{4\pi}$	$\frac{9}{2\pi}$	0	$-\frac{9}{2\pi}$	$-\frac{9}{4\pi}$	0	$-\frac{9}{8\pi}$	$-\frac{9}{10\pi}$

(Real part is identically zero.)

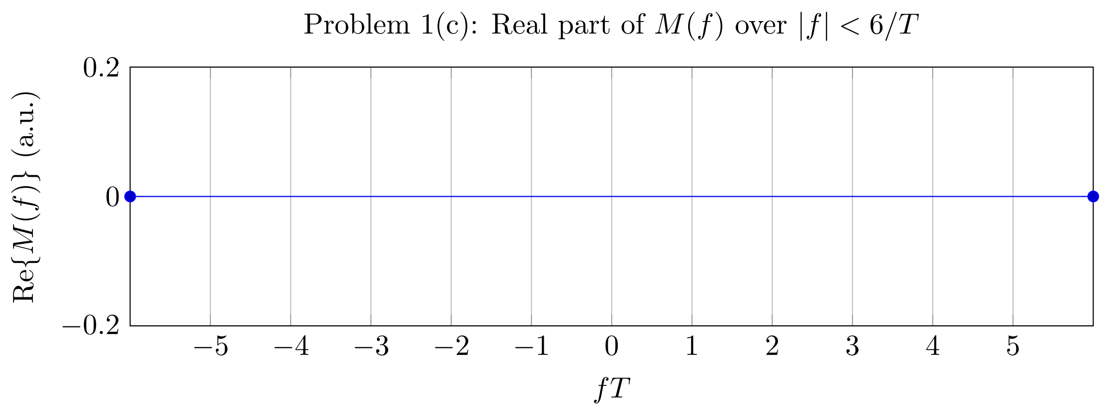

Figure 1: Real part $\mathrm{Re}\{M(f)\}$ is identically zero because $m(t)$ is odd.

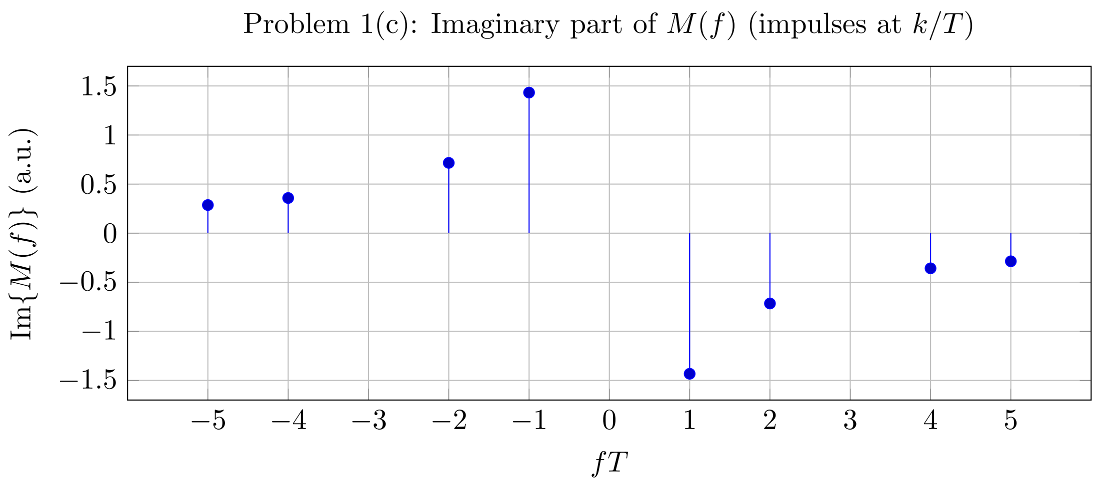

Figure 2: Imaginary part $\mathrm{Im}\{M(f)\}$ showing nonzero spectral lines at k/T for $k \equiv \pm 1, \pm 2 \pmod 3$. Values use $\mathrm{Im}\{c_k\} = -9/(2\pi k)$.

Problem 2

For a real passband $x(t)$, the complex envelope $g(t)$ *with respect to* $\cos(2\pi f_c t)$ is the baseband signal satisfying

$$x(t) = \mathrm{Re}\{\, g(t)\, e^{j2\pi f_c t}\,\}.$$

(a) $x(t) = a(t)\cos(2\pi f_c t + \theta(t)) + b(t)\cos(2\pi f_c t + \phi(t))$

Each cosine term has envelope equal to its complex phasor:

$$\boxed{g(t) = a(t)\, e^{j\theta(t)} + b(t)\, e^{j\phi(t)}}$$

since $\mathrm{Re}\{a e^{j\theta} e^{j2\pi f_c t}\} = a\cos(2\pi f_c t + \theta)$ (and similarly for the b term).

(b) $x(t) = a(t)\cos(2\pi f_c t) - b(t)\sin(2\pi f_c t) + c(t)\sin(2\pi f_c t + \phi(t))$

Use the standard I/Q identity $\mathrm{Re}\{(I + jQ)e^{j\omega_c t}\} = I\cos\omega_c t - Q\sin\omega_c t$ and $\sin(\omega_c t + \phi) = \sin\omega_c t\cos\phi + \cos\omega_c t\sin\phi$. Grouping coefficients:

$$I(t) = a(t) + c(t)\sin\phi(t), \qquad Q(t) = b(t) - c(t)\cos\phi(t),$$

so

$$\boxed{g(t) = I(t) + jQ(t) = a(t) + j\,b(t) + c(t)\,e^{j(\phi(t) - \frac{\pi}{2})}}$$

(equivalently $a + jb + c\sin\phi + j\,[\,b - c\cos\phi\,]$).

(c) $x(t) = a(t)\sin\big(2\pi(f_c + \Delta)t + \theta(t)\big)$

Write $\sin(\cdot) = \mathrm{Re}\{e^{j(\cdot - \pi/2)}\}$:

$$x(t) = \mathrm{Re}\left\{\, a(t)\,e^{j(\theta(t) - \frac{\pi}{2})}\,e^{j2\pi(f_c+\Delta)t}\right\} = \mathrm{Re}\Big\{\,\underbrace{a(t)\,e^{j(\theta(t) - \frac{\pi}{2})}\,e^{j2\pi\Delta t}}_{g(t)}\;e^{j2\pi f_c t}\Big\}.$$

Hence

$$\boxed{g(t) = a(t)\,e^{j(\theta(t) - \frac{\pi}{2})}\,e^{j2\pi\Delta t}}$$

i.e., a baseband phasor $a\,e^{j(\theta - \pi/2)}$ with a residual frequency shift Δ.

Problem 3

Given a DSB–SC signal $u(t) = m(t)\cos(500\pi t)$ with

$$m(t) = \sin(10\pi t) + 2\cos(20\pi t) \quad \Rightarrow \quad f_c = \frac{500\pi}{2\pi} = 250\,\mathrm{Hz}, \quad f_1 = 5\,\mathrm{Hz}, \quad f_2 = 10\,\mathrm{Hz}.$$

(a) $M(f)$ **and** $U(f)$

Using the Fourier transform convention $\mathcal{F}\{x(t)\} = \int x(t)e^{-j2\pi f t}dt$,

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \tfrac{1}{2}\big[\delta(f - f_0) + \delta(f + f_0)\big], \qquad \mathcal{F}\{\sin(2\pi f_0 t)\} = \tfrac{1}{2j}\big[\delta(f - f_0) - \delta(f + f_0)\big].$$

Therefore

$$\boxed{M(f) = \frac{1}{2j}\big[\delta(f - 5) - \delta(f + 5)\big] + \big[\delta(f - 10) + \delta(f + 10)\big].}$$

Multiplication by $\cos(2\pi f_c t)$ produces a *half-sum of shifts*:

$$\mathcal{F}\{x(t)\cos(2\pi f_c t)\} = \tfrac{1}{2}[X(f - f_c) + X(f + f_c)]\,.$$

Hence

$$\boxed{U(f) = \frac{1}{2}\Big(M(f - 250) + M(f + 250)\Big).}$$

Explicitly, impulses occur at

$$f = \pm(250 \pm 5), \quad \pm(250 \pm 10),$$

with magnitudes:

$$\begin{aligned} &\text{from } \sin(10\pi t): && \pm\frac{1}{4j} \text{ at } 250 \pm 5,\ -250 \pm 5, \\ &\text{from } 2\cos(20\pi t): && \frac{1}{2} \text{ (real, +) at } 250 \pm 10,\ -250 \pm 10. \end{aligned}$$

(b) Sketches of $|U(f)|$, $\mathrm{Re}\, U(f)$, and $\mathrm{Im}\, U(f)$

- $\mathrm{Re}\, U(f)$: nonzero only at $f = \pm(250 \pm 10)$, with real amplitude $+1/2$ at each delta; zero elsewhere.
- $\mathrm{Im}\, U(f)$: nonzero only at $f = \pm(250 \pm 5)$, with purely imaginary weights $\pm 1/(4j)$ (odd signs so that $U(-f) = U^*(f)$).
- $|U(f)|$: line spectrum with four real lines (at $\pm(250 \pm 10)$) of height $1/2$ and four imaginary lines (at $\pm(250 \pm 5)$) of height $1/4$.

Tabular "sketch data":

Location f (Hz)	$\mathrm{Re}\, U(f)$	$\mathrm{Im}\, U(f)$
$250 \pm 10,\ -250 \pm 10$	$+\frac{1}{2}$	0
$250 \pm 5,\ -250 \pm 5$	0	$\pm\left(\mp \frac{1}{4}\right)$ (purely imaginary, Hermitian)

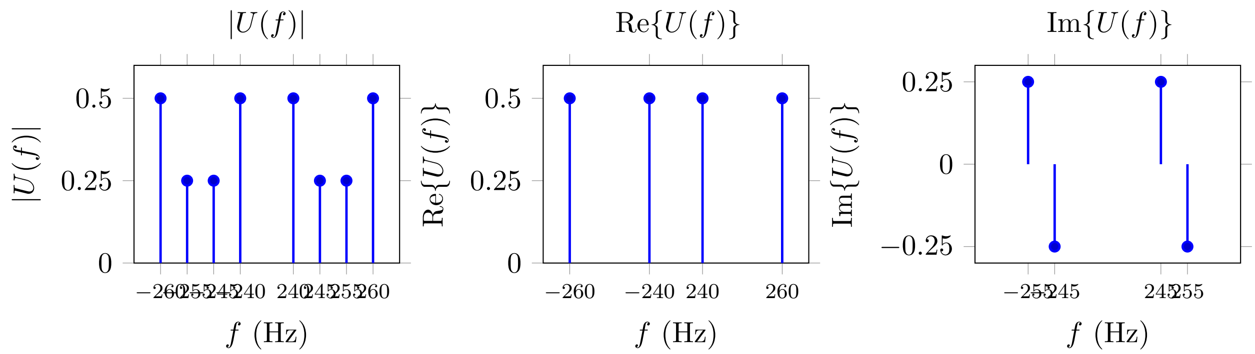

Figure 3: Problem 3(b) sketches. Left: magnitude $|U(f)|$ with impulses at $\pm(240, 245, 255, 260)$ Hz. Middle: real part (only at $\pm 240, \pm 260$) with weight $1/2$. Right: imaginary part (only at $\pm 245, \pm 255$) with weights $\pm 1/4$ arranged so $U(-f) = U^*(f)$.

(c) Lowpass filter for coherent demod

The mixer output is

$$y(t) = u(t)\cos(500\pi t) = m(t)\cos^2(500\pi t) = \frac{1}{2}m(t) + \frac{1}{2}m(t)\cos(1000\pi t).$$

The baseband term is $\frac{1}{2}m(t)$ (support $|f| \le 10\,\mathrm{Hz}$); the second term sits around $\pm 500\,\mathrm{Hz}$ (with ± 5 and $\pm 10\,\mathrm{Hz}$ offsets). An *ideal* LPF that passes baseband and rejects the high-frequency term, while restoring unity gain for $m(t)$, is:

$$\boxed{H(f) = \begin{cases} 2, & |f| \le 10\,\mathrm{Hz}, \\ 0, & |f| \ge 240\,\mathrm{Hz}, \end{cases} \quad \text{(any practical transition band between 10 Hz and 240 Hz is fine).}}$$

(The passband gain 2 compensates the $\frac{1}{2}$ factor.)