UNIVERSITY OF CALIFORNIA, RIVERSIDE
Department of Electrical Engineering

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EE110B-SIGNALS AND SYSTEMS
HOMEWORK 7 SOLUTIONS

Problem 1:
a) Define the signal $\mathrm{rect}_A(x)$ as

$$\mathrm{rect}_A(x) = \begin{cases} 1 & -A \le x \le A \\ 0 & \text{otherwise} \end{cases}$$

which is nothing but a rectangular pulse signal. Now let us look at the inverse Fourier transform of $\mathrm{rect}_A(\Omega)$:

$$\begin{aligned} \mathcal{F}^{-1}(\mathrm{rect}_A(\Omega)) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{rect}_A(\Omega) e^{j\Omega t} d\Omega \\ &= \frac{1}{2\pi}\int_{-A}^{A} e^{j\Omega t} d\Omega \\ &= \frac{1}{2\pi j t}\, e^{j\Omega t}\Big|_{-A}^{A} \\ &= \frac{e^{jAt} - e^{-jAt}}{2\pi j t} \\ &= \frac{\sin(At)}{\pi t} \end{aligned}$$

Defining this signal as $s_A(t)$, we observe that

$$x_c(t) = [\pi s_\pi(t)]^2 = \pi^2 s_\pi(t)^2$$

From the multiplication property, which states that if $z(t) = x(t)y(t)$ then $Z(j\Omega) = \frac{1}{2\pi}\left[X(j\Omega) * Y(j\Omega)\right]$, it follows that

$$X_c(j\Omega) = \pi^2 \cdot \frac{1}{2\pi}\left[\mathrm{rect}_\pi(\Omega) * \mathrm{rect}_\pi(\Omega)\right] = \frac{\pi}{2}\mathrm{triangle}_{2\pi}(\Omega)$$

where

$$\mathrm{triangle}_B(x) = \begin{cases} B + x & -B \le x \le 0 \\ B - x & 0 \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the bandwidth of $x_c(t)$ is $\Omega_M = 2\pi$.

b) We want

$$\Omega_s = \frac{2\pi}{T} \ge 2\Omega_M = 4\pi\ .$$

Therefore

$$T \le \frac{1}{2}\ .$$

We need the sampling period T to be at most 0.5 secs.

c) If $T = 1$, then the first replica is at $\Omega = 2\pi$, which is the bandwidth of the original signal. So, there will be huge amount of aliasing. Looking at

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left[\Omega - \frac{2\pi k}{T}\right]\right) = \sum_{k=-\infty}^{\infty} X_c\left(j\left[\Omega - 2\pi k\right]\right)$$

we see that it is summing a perfectly aligned sequence of triangles where the peak and the end point of two consecutive triangles are coinciding. Since the peak of each triangle is at π^2, this means

$$X_s(j\Omega) = \pi^2$$

for all $-\infty < \Omega < \infty$. If we filter this ideally with cutoff frequency $\frac{\pi}{T}$, we obtain

$$\hat{X}_c(j\Omega) = \pi^2 \mathrm{rect}_\pi(\Omega)$$

whose inverse Fourier transform is

$$\hat{x}_c(t) = \pi^2 \cdot \frac{\sin(\pi t)}{\pi t} = \pi \frac{\sin(\pi t)}{t} .$$

So, instead of sinc^2, we obtained sinc as the reconstruction.

Alternative solution: By just plugging in $t = nT$ in $x_c(t)$, we have the relation

$$x_d[n] = \left(\frac{\sin(\pi n T)}{nT}\right)^2 .$$

But $T = 1$ simplifies this into

$$x_d[n] = \left(\frac{\sin(\pi n)}{n}\right)^2 = \begin{cases} \pi^2 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

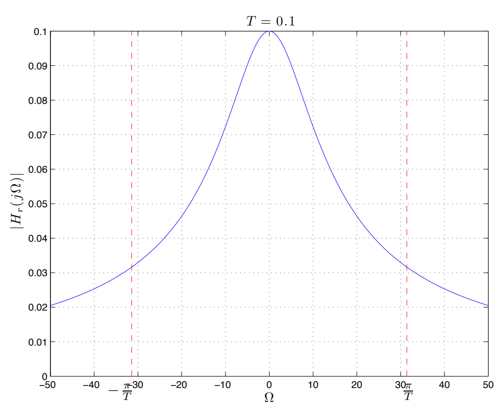
In other words, $x_d[n] = \pi^2 \delta[n]$, implying that

$$x_s(t) = \pi^2 \delta(t) .$$

The Fourier transform of this is also clearly $X_s(j\Omega) = \pi^2$ as in the above solution.

Problem 2:

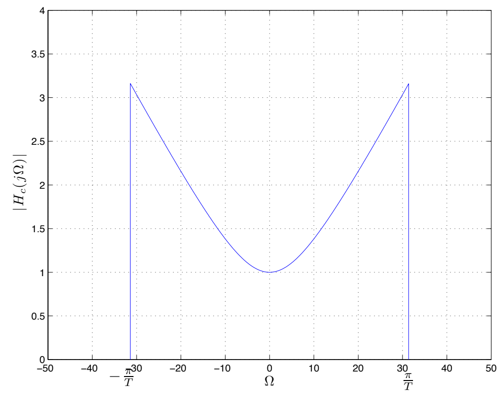
a) When we sketch the magnitude, it looks like below:



As can be seen, at the frequency $\Omega = \frac{\pi}{T}$, the magnitude falls to about 30% of the peak value (which is about 10dB drop in the dB domain.) This implies that the problem of lingering replicas won't be that severe.

b) To take care of the fact that $H_r(j\Omega)$ is not flat, we can design an overall filter $H_c(j\Omega)$ which counters the effect of $1 + j\frac{3T}{\pi}\Omega$ in the denominator. That is, it could be of the form

$$H_c(j\Omega) = \begin{cases} 1 + j\frac{3T}{\pi}\Omega & -\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T} \\ 0 & \text{otherwise.} \end{cases}$$



whose magnitude will look like this:

To implement this in the discrete-time world, all we have to do is to design $H_d(e^{j\omega})$ so that

$$H_d(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right) = 1 + j\frac{3}{\pi}\omega$$

for all $-\pi \leq \omega \leq \pi$.

c) The discrete-time impulse response $h[n]$ can be calculated as

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 + j\frac{3}{\pi}\omega\right)e^{j\omega n}d\omega\ . \end{aligned}$$

For $n = 0$, this simplifies to

$$h[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 + j\frac{3}{\pi}\omega\right)d\omega = 1\ .$$

For all other n, letting $u = 1 + j\frac{3}{\pi}\omega$ and $dv = e^{j\omega n}d\omega$, integration by parts gives us

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\left(1 + j\frac{3}{\pi}\omega\right)\frac{e^{j\omega n}}{jn}\bigg|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{j\omega n}}{jn}\frac{j3}{\pi}d\omega \\ &= \frac{1}{2jn\pi}\left[(1+j3)e^{j\pi n} - (1-j3)e^{-j\pi n}\right] - \frac{3}{2n\pi^2}\int_{-\pi}^{\pi}e^{j\omega n}d\omega \\ &= \frac{1}{n\pi}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} + \frac{3\left(e^{j\pi n} + e^{-j\pi n}\right)}{2}\right] - \frac{3}{n^2\pi^2}\frac{e^{j\pi n} - e^{-j\pi n}}{2j} \\ &= \frac{1}{n\pi}\left[\sin(\pi n) + 3\cos(\pi n)\right] - \frac{3}{n^2\pi^2}\sin(\pi n) \\ &= \frac{3(-1)^n}{n\pi} \end{aligned}$$

where the last step follows from $\sin(\pi n) = 0$ and $\cos(\pi n) = (-1)^n$ for all n.