EE 110A Signals and Systems

Fourier Series Expansion of Continuous-Time Signals

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Decomposition of periodic signals

- Recall that we can decompose any signal into shifted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- Also remember how this was instrumental in understanding the response of an LTI system to any input:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Decomposition of periodic signals

- Is there any other decomposition that may be similarly useful?
- For periodic signals, we will find exactly that.
- Claim: For signals with period T, we can write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k}{T}t}$$

- Setting $\Omega_0 = 2\pi/T$, this is the same as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

Decomposition of periodic signals

- See why it is useful before proving the claim
- For any LTI system with impulse response $h(t)$, if the input is a complex exponential signal $x(t) = e^{j\Omega t}$,

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{j\Omega(t-\tau)}d\tau \\
&= e^{j\Omega t} \boxed{\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega\tau}d\tau}
\end{aligned}
$$

Depends only on $h(t)$

Function of Ω

Call it $H(\Omega)$

Decomposition of periodic signals

- Pictorially,

$$e^{j\Omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(\Omega)e^{j\Omega t}$$

- Therefore, if a periodic signal can indeed be decomposed as mentioned above,

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow \sum_{k=-\infty}^{\infty} a_k H(k\Omega_0) e^{jk\Omega_0 t}$$

Back to the formula

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

- Multiply both sides by $e^{-jl\Omega_0 t}$ for some integer l, and integrate over t in one period:

$$\int_0^T x(t) e^{-jl\Omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} e^{-jl\Omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \boxed{\int_0^T e^{j(k-l)\Omega_0 t} dt} \quad \begin{aligned} &= T \quad \text{if } k = l \\ &= 0 \quad \text{if } k \neq l \end{aligned}$$

$$= a_l T$$

Back to the formula

- Therefore, if

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \qquad (*)$$

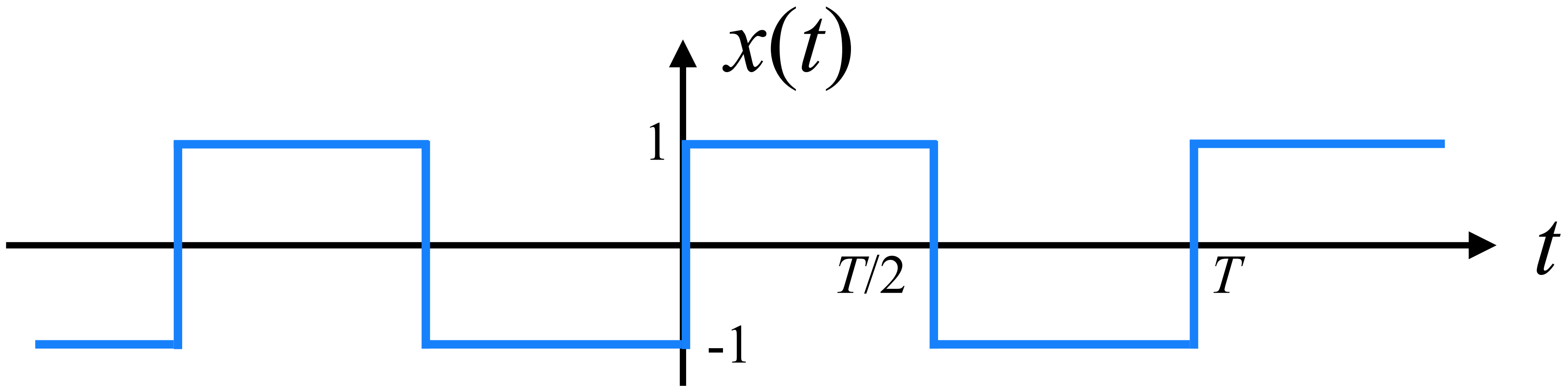
then

$$a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\Omega_0 t}\, dt \qquad (**)$$

- Conversely, for any $x(t)$, a_k calculated as in (**) satisfies (*).
- a_k are called the continuous-time Fourier series (CTFS) coefficients.

Example: Square wave

- Find the CTFS coefficients for the signal

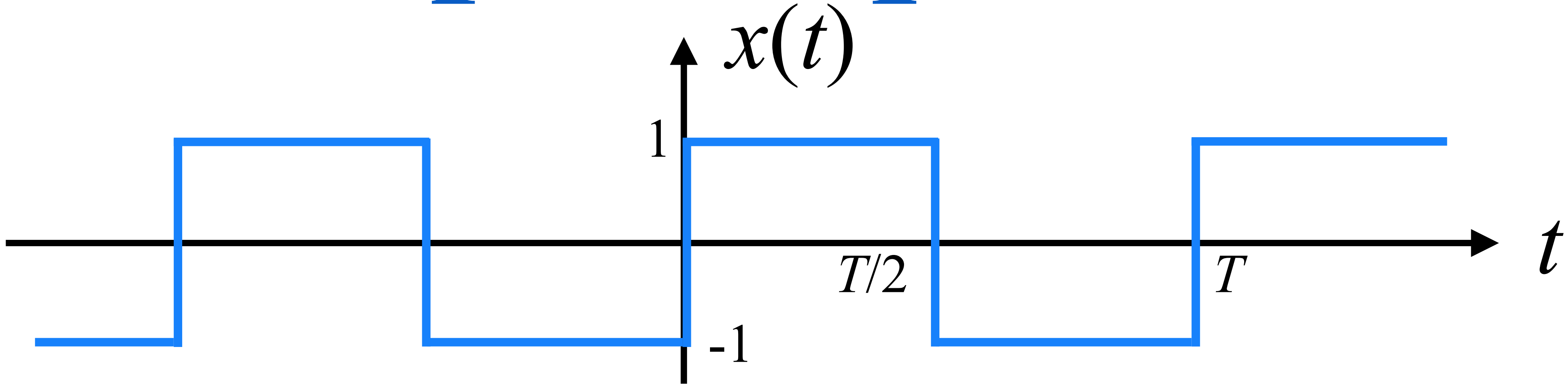


- Solution: Letting $\Omega_0 = 2\pi/T$,

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\Omega_0 t}dt \quad = 0 \;\text{ when }\; k = 0$$

$$= \frac{1}{T}\left[\int_0^{T/2} e^{-jk\Omega_0 t}dt - \int_{T/2}^{T} e^{-jk\Omega_0 t}dt\right]$$

Example: Square wave

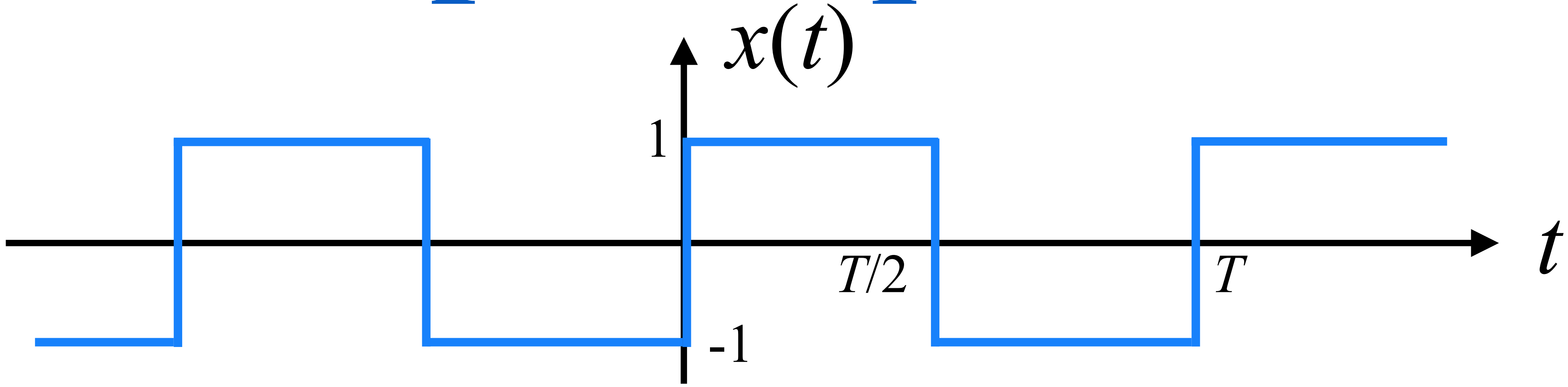


$$a_k = \frac{1}{T}\left[\int_0^{T/2} e^{-jk\Omega_0 t}dt - \int_{T/2}^{T} e^{-jk\Omega_0 t}dt\right]$$

$$= \frac{1}{T}\left[\left.\frac{e^{-jk\Omega_0 t}}{-jk\Omega_0}\right|_0^{T/2} - \left.\frac{e^{-jk\Omega_0 t}}{-jk\Omega_0}\right|_{T/2}^{T}\right]$$

$$= \frac{1}{-jk\Omega_0 T}\left[e^{-jk\Omega_0 T/2} - 1 - e^{-jk\Omega_0 T} + e^{-jk\Omega_0 T/2}\right]$$

Example: Square wave

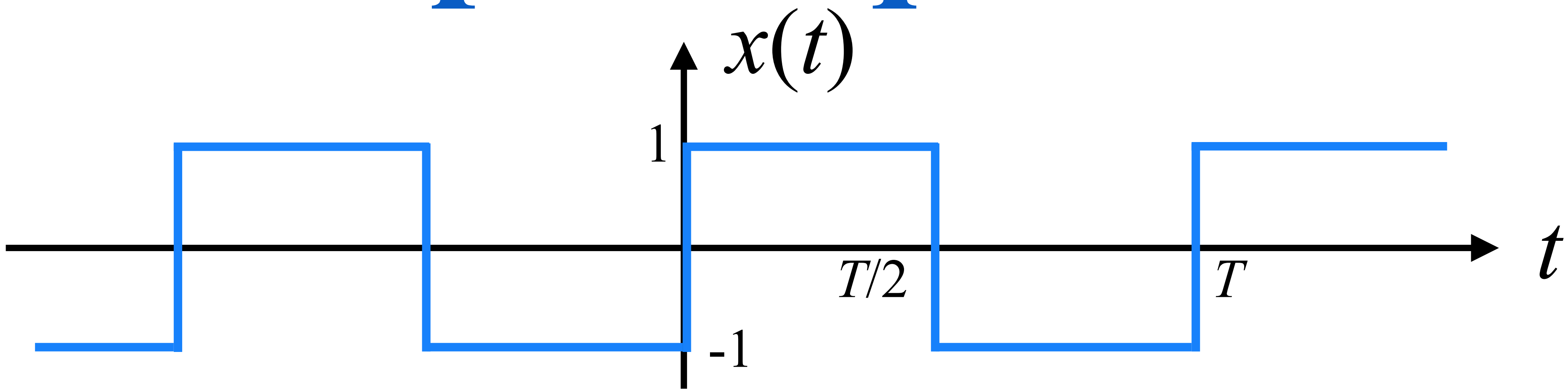


$$a_k = \frac{1}{-jk\Omega_0 T}\left[e^{-jk\Omega_0 T/2} - 1 - e^{-jk\Omega_0 T} + e^{-jk\Omega_0 T/2}\right]$$

$$= \frac{1}{-jk2\pi}\left[e^{-jk\pi} - 1 - e^{-jk2\pi} + e^{-jk\pi}\right]$$

$$= \frac{1}{-jk\pi}\left[(-1)^k - 1\right] = \begin{cases} 0 & \text{even } k \\ \dfrac{2}{jk\pi} & \text{odd } k \end{cases}$$

Example: Square wave

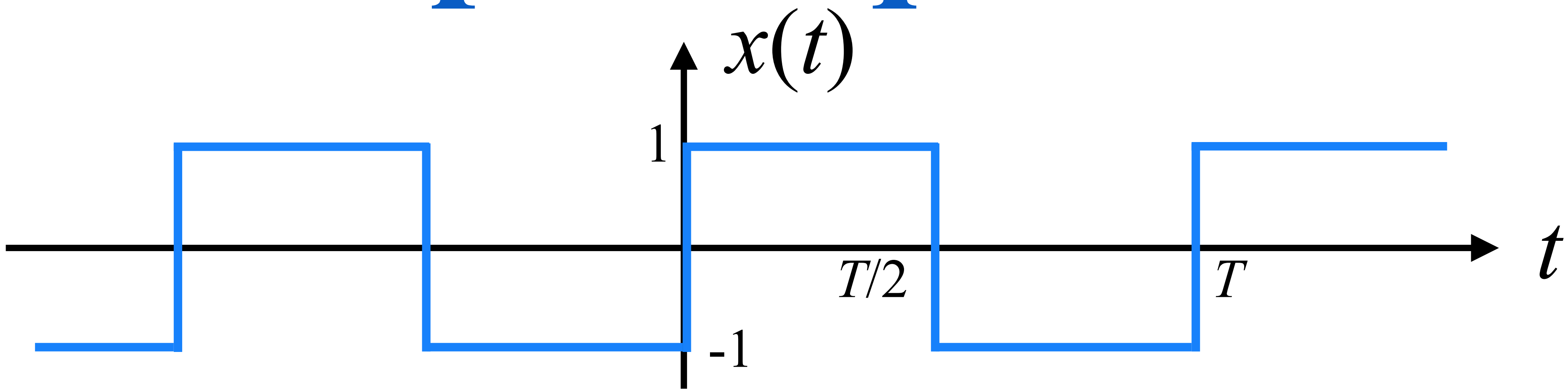


$$a_k = \begin{cases} 0 & \text{even } k \\ \dfrac{2}{jk\pi} & \text{odd } k \end{cases}$$

$$x(t) = \sum_{\text{odd } k} \frac{2}{jk\pi} e^{jk\Omega_0 t}$$

$$= \frac{2\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)}{j\pi} + \frac{2\left(e^{j3\Omega_0 t} - e^{-j3\Omega_0 t}\right)}{j3\pi} + \ldots$$

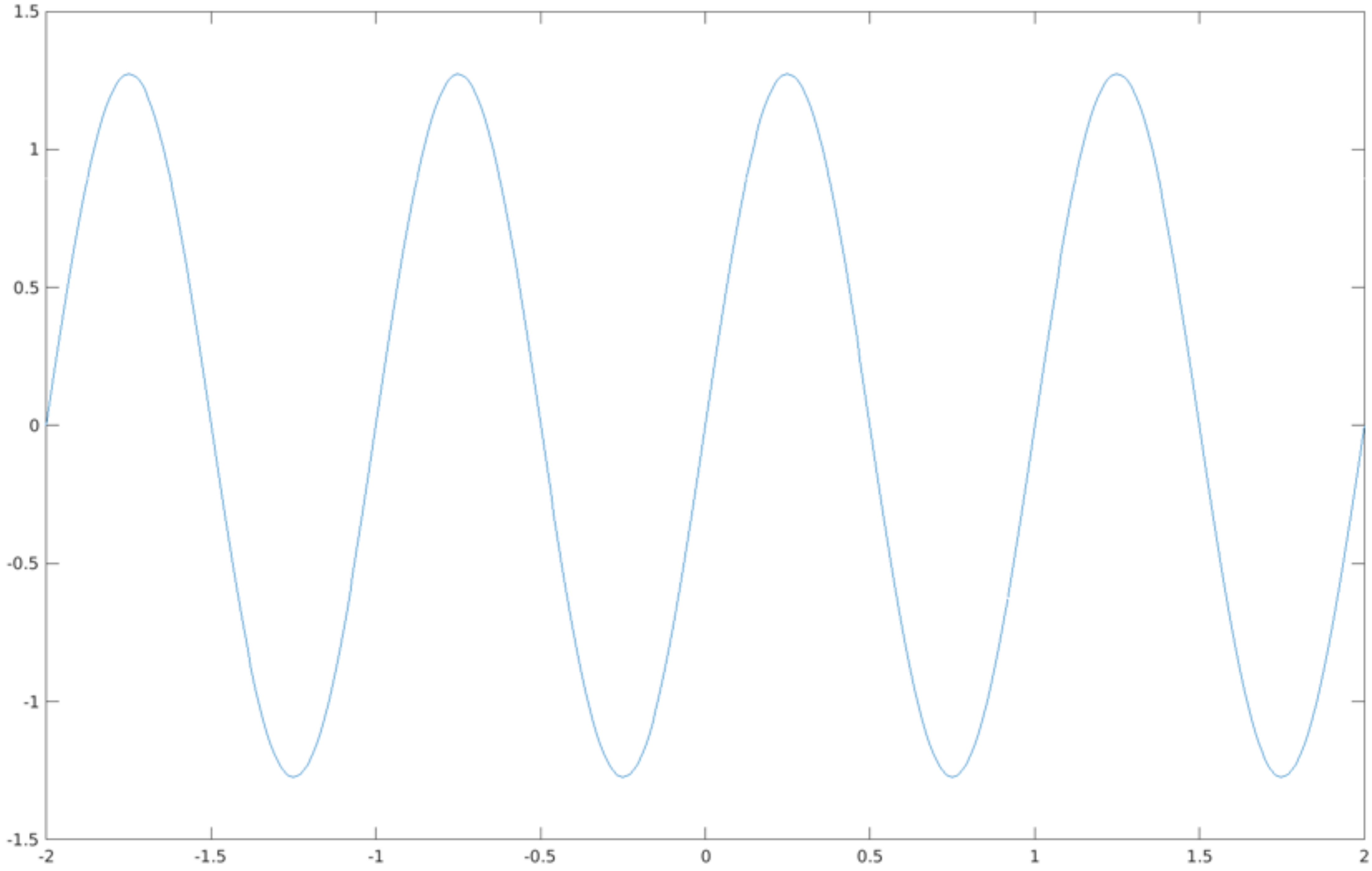
Example: Square wave



$$
\begin{aligned}
x(t) &= \frac{2\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)}{j\pi} + \frac{2\left(e^{j3\Omega_0 t} - e^{-j3\Omega_0 t}\right)}{j3\pi} + \ldots \\
&= \frac{4}{\pi}\left[\sin(\Omega_0 t) + \frac{\sin(3\Omega_0 t)}{3} + \frac{\sin(5\Omega_0 t)}{5} + \ldots\right] \\
&= \frac{4}{\pi}\sum_{\text{odd } k} \frac{\sin(k\Omega_0 t)}{k}
\end{aligned}
$$

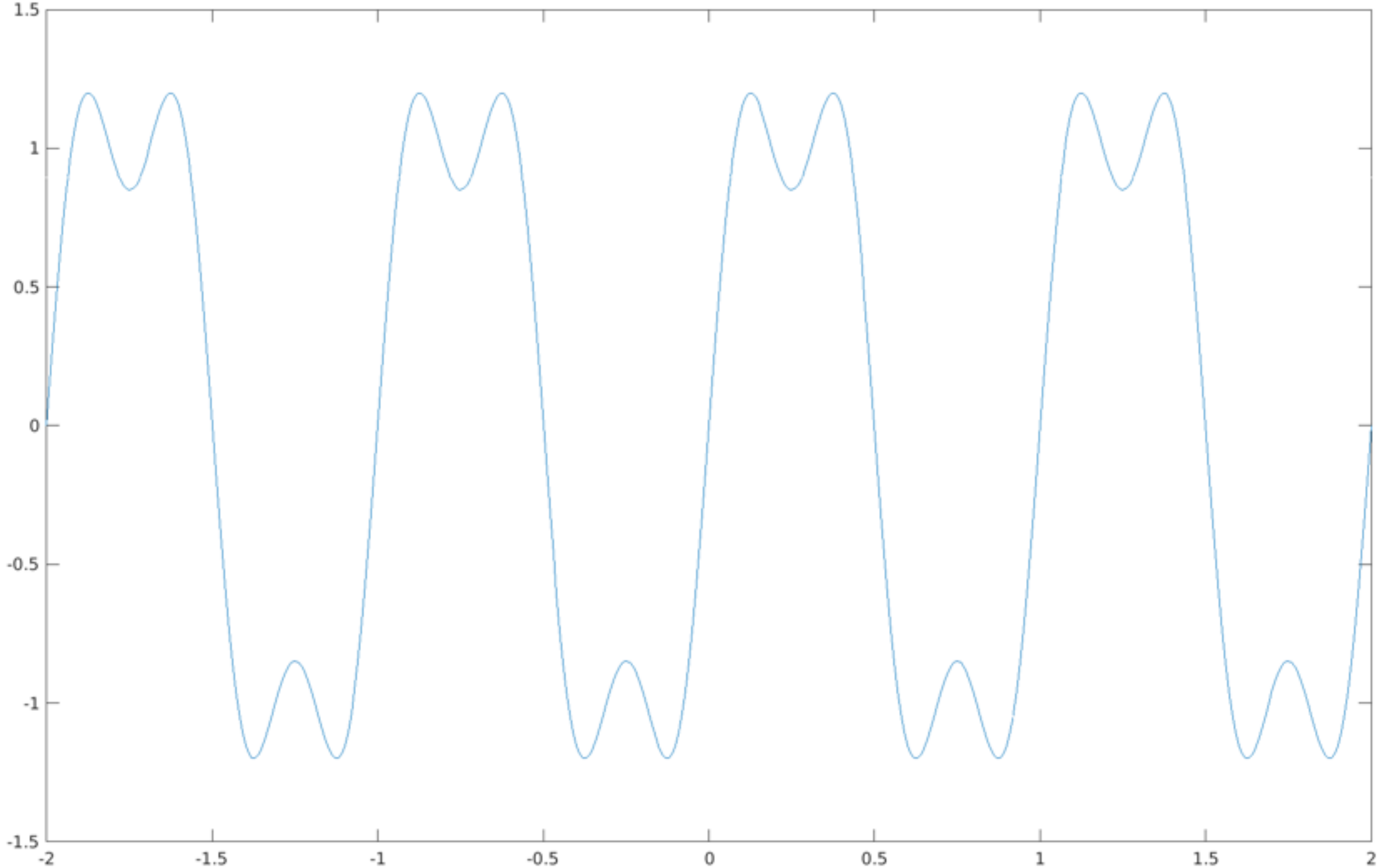
Example: Square wave

- The first sine wave



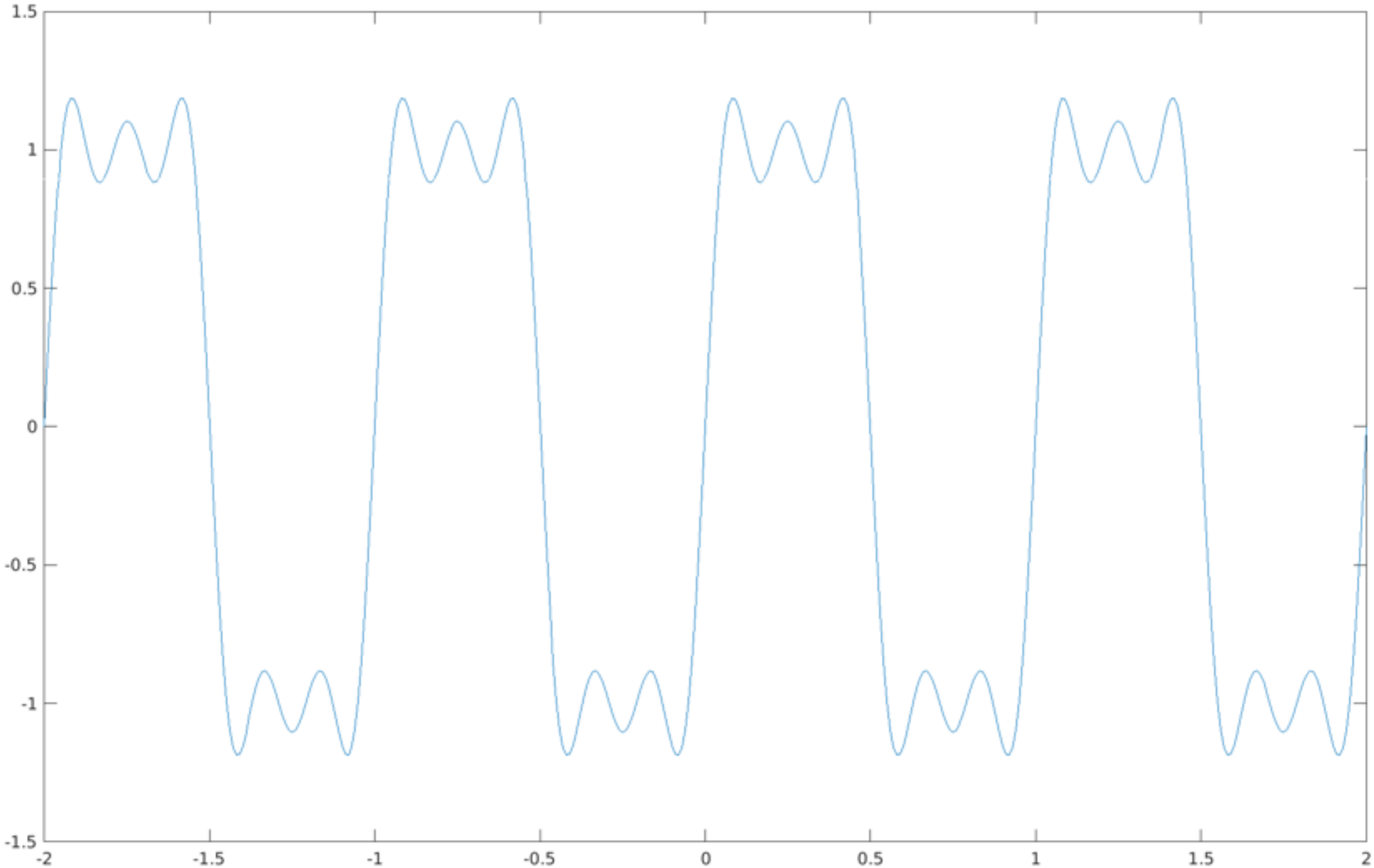
Example: Square wave

- The first 2 sine waves summed



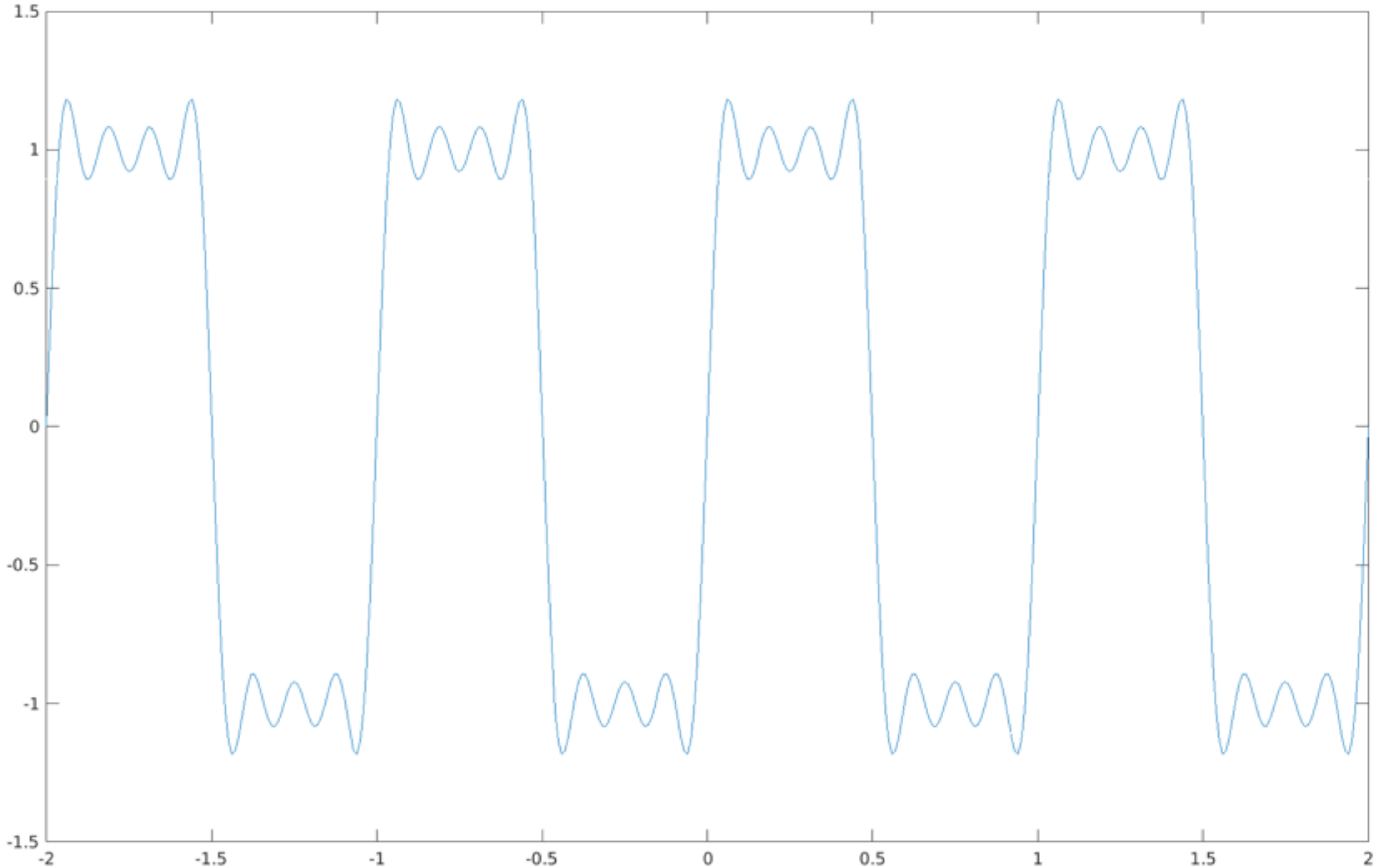
Example: Square wave

- The first 3 sine waves summed



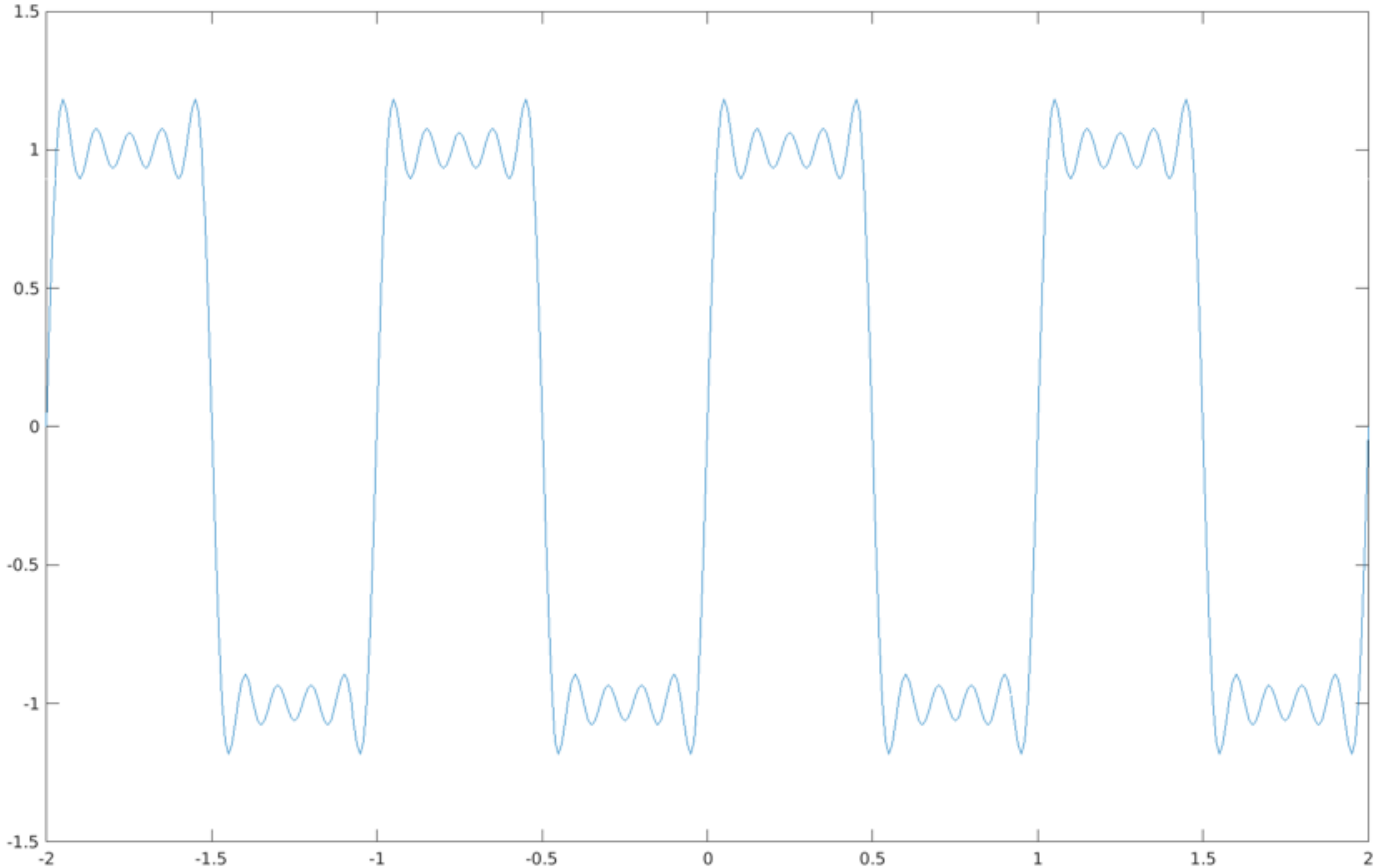
Example: Square wave

- The first 4 sine waves summed



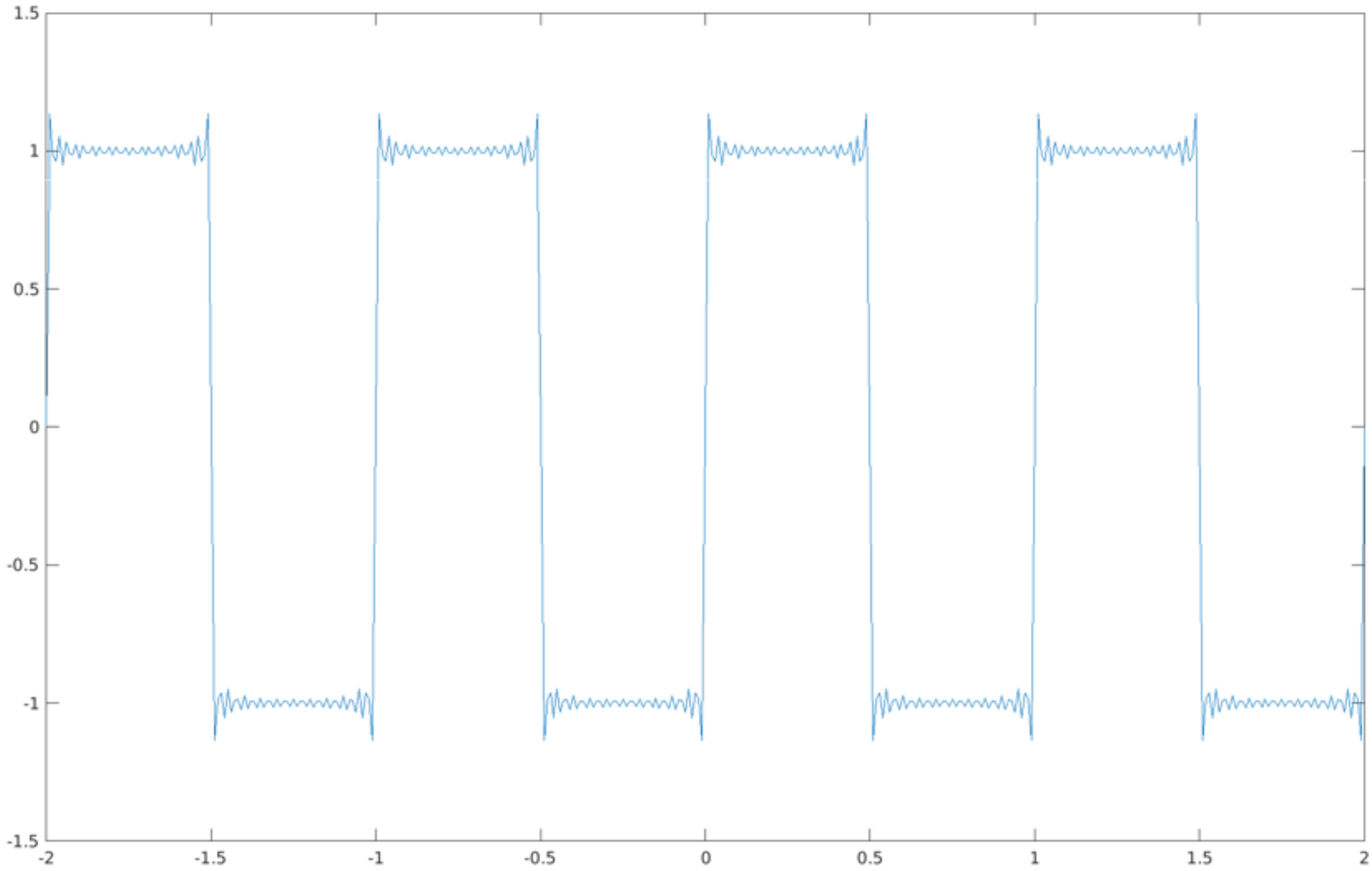
Example: Square wave

- The first 5 sine waves summed



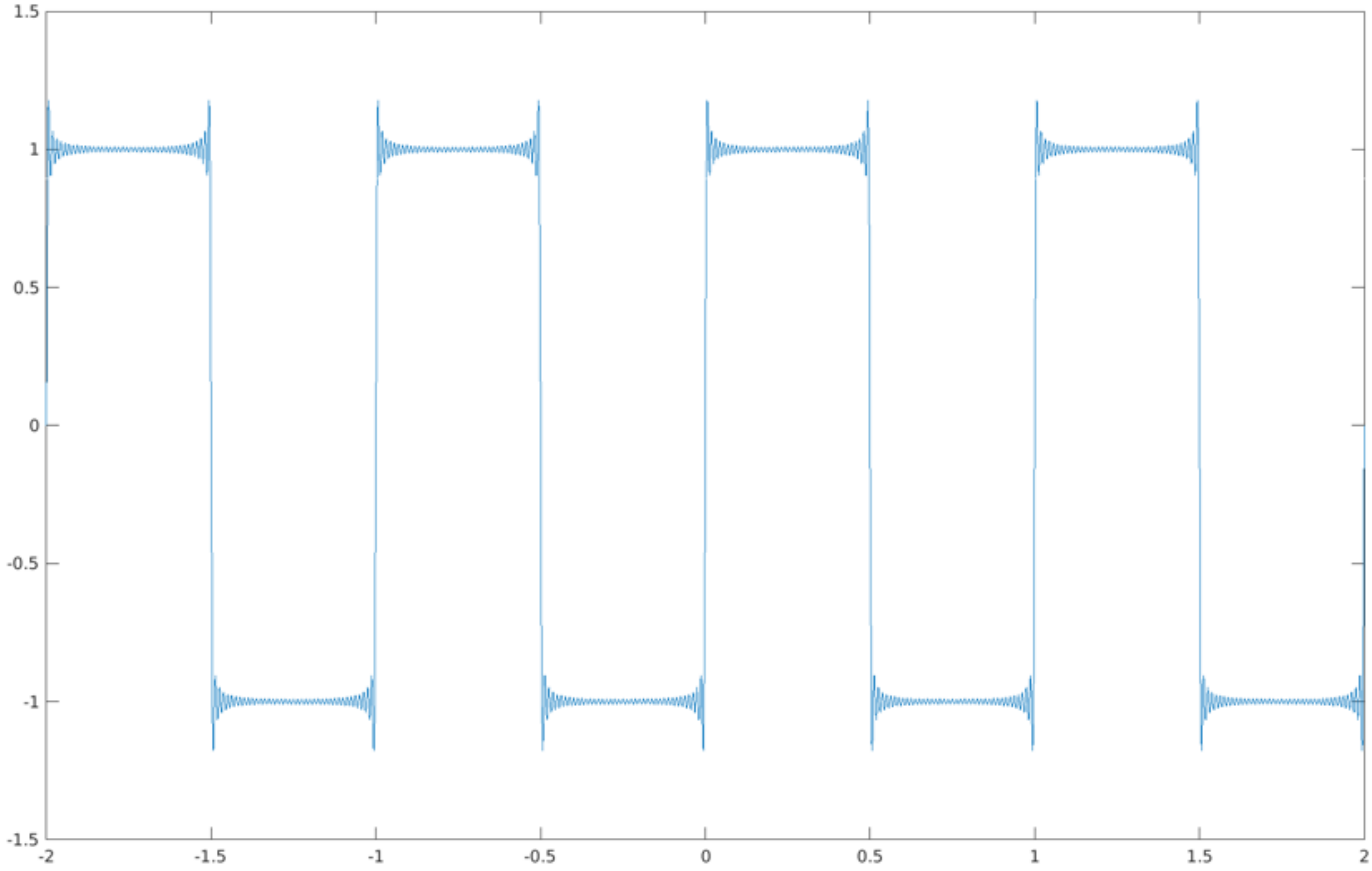
Example: Square wave

- The first 20 sine waves summed



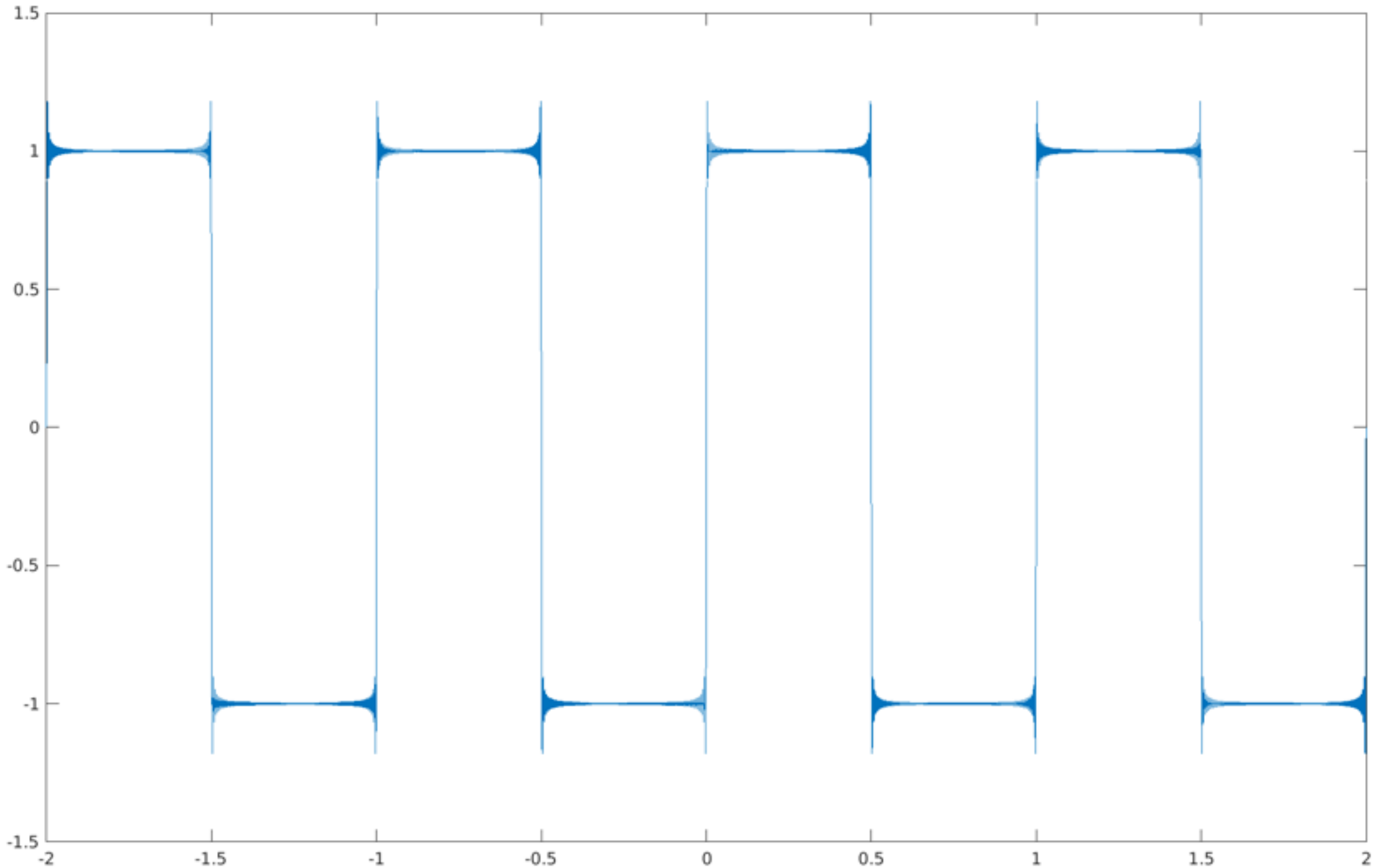
Example: Square wave

- The first 40 sine waves summed

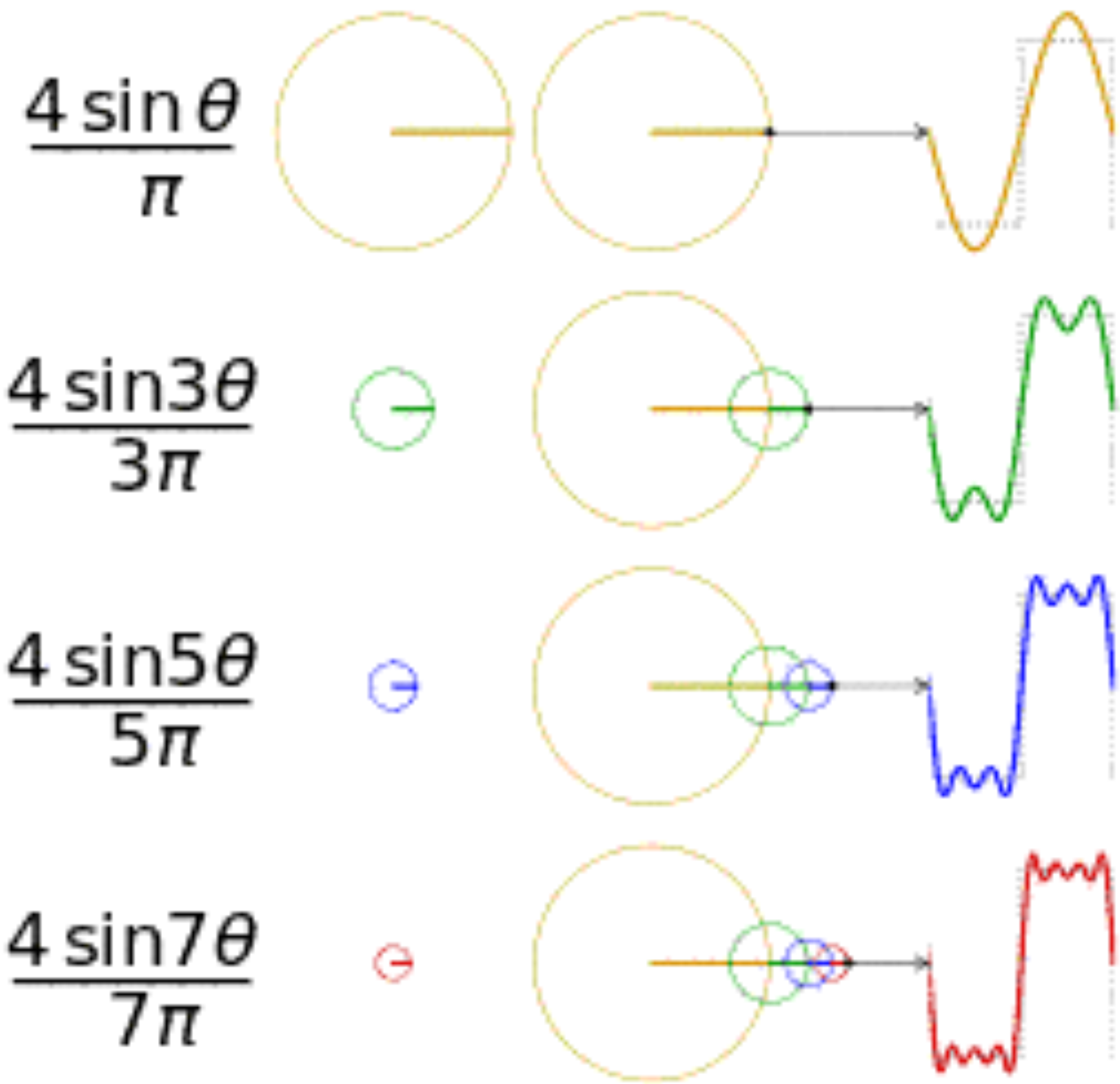


Example: Square wave

- The first 100 sine waves summed

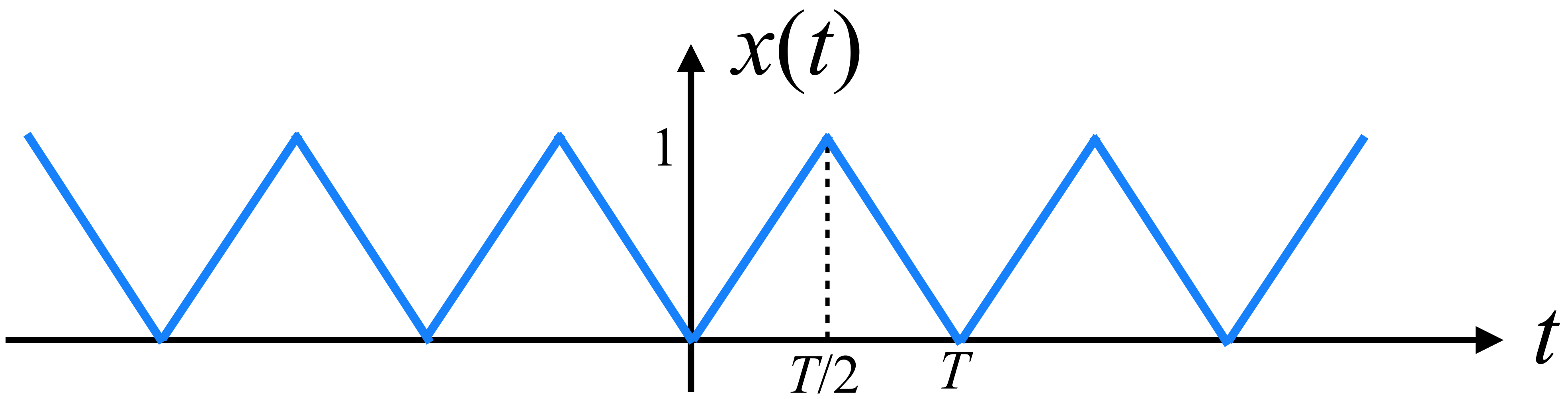


A beautiful illustration



Example: Triangular wave

- Find the CTFS coefficients for the signal



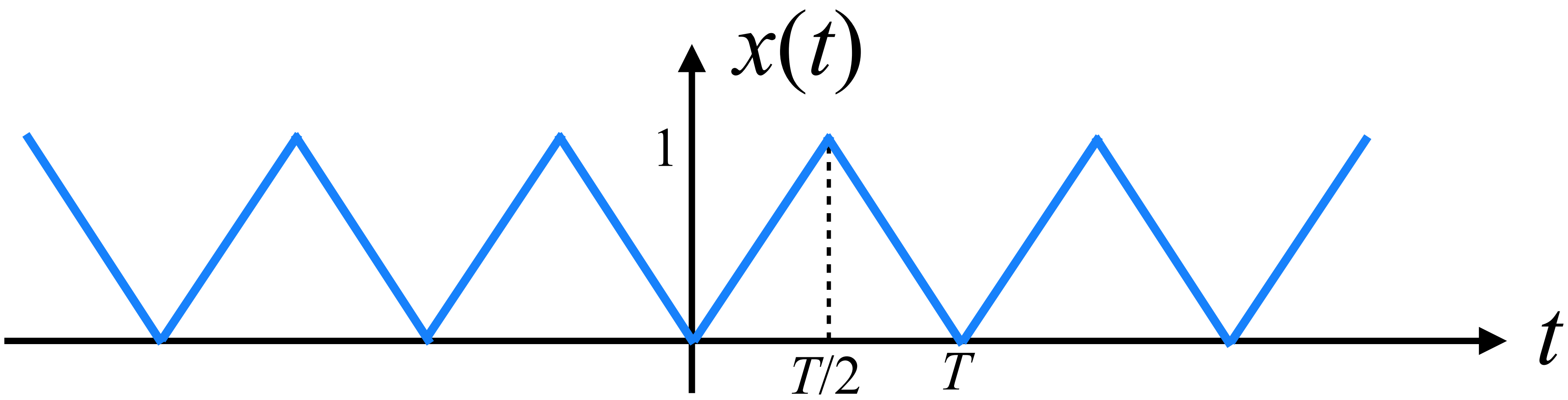
- <u>Solution</u>: Letting $\Omega_0 = 2\pi/T$,

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\Omega_0 t}dt$$

- For $k = 0$, we get $a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\frac{T}{2} = \frac{1}{2}$

Example: Triangular wave

- Find the CTFS coefficients for the signal

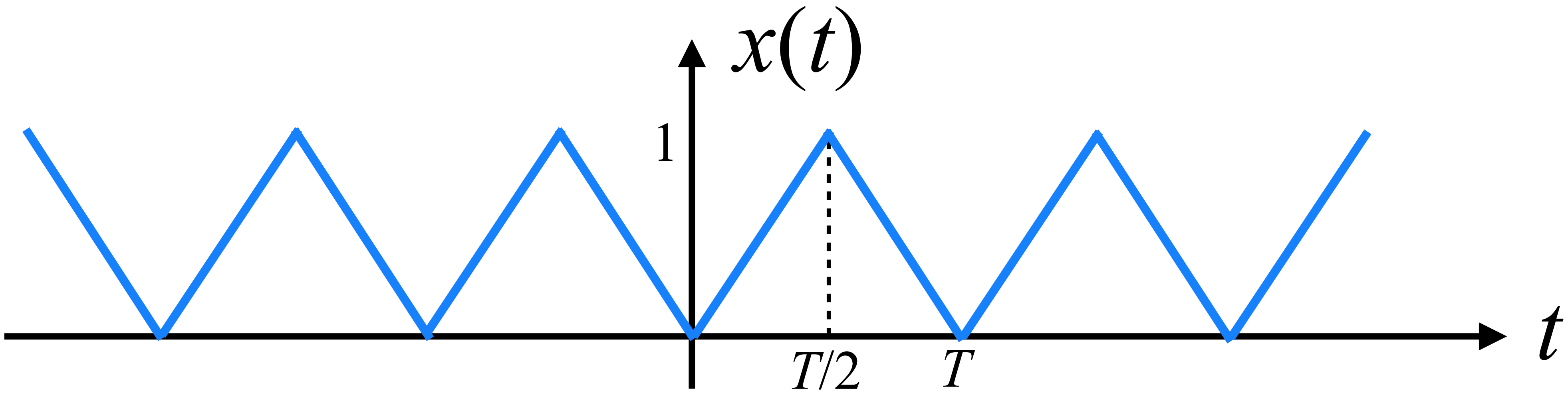


- <u>Solution</u>: Letting $\Omega_0 = 2\pi/T$,

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\Omega_0 t}dt$$

$$= \frac{1}{T}\left[\int_0^{T/2}\frac{2t}{T}e^{-jk\Omega_0 t}dt + \int_{T/2}^{T}\left(2 - \frac{2t}{T}\right)e^{-jk\Omega_0 t}dt\right]$$

Example: Triangular wave



$$a_k = \frac{1}{T}\left[\int_0^{T/2} \underbrace{\frac{2t}{T}}_{u}\,\underbrace{e^{-jk\Omega_0 t}\,dt}_{dv} + \int_{T/2}^{T} \underbrace{\left(2-\frac{2t}{T}\right)}_{u}\underbrace{e^{-jk\Omega_0 t}\,dt}_{dv}\right]$$

$$= \frac{1}{T}\left[\frac{2t}{-jk\Omega_0 T}\,e^{-jk\Omega_0 t}\Big|_0^{T/2} - \int_0^{T/2}\frac{2}{-jk\Omega_0 T}\,e^{-jk\Omega_0 t}\,dt\right.$$

$$\left. + \left(2-\frac{2t}{T}\right)\frac{e^{-jk\Omega_0 t}}{-jk\Omega_0}\Big|_{T/2}^{T} + \int_{T/2}^{T}\frac{2}{-jk\Omega_0 T}\,e^{-jk\Omega_0 t}\,dt\right]$$

Example: Triangular wave

$$a_k = \frac{1}{T}\left[\frac{2t}{-jk\Omega_0 T}e^{-jk\Omega_0 t}\bigg|_0^{T/2} - \int_0^{T/2}\frac{2}{-jk\Omega_0 T}e^{-jk\Omega_0 t}dt\right.$$

$$\left. + \left(2-\frac{2t}{T}\right)\frac{e^{-jk\Omega_0 t}}{-jk\Omega_0}\bigg|_{T/2}^{T} + \int_{T/2}^{T}\frac{2}{-jk\Omega_0 T}e^{-jk\Omega_0 t}dt\right]$$

$$= \frac{1}{-jk\Omega_0 T}\left[\frac{2t}{T}e^{-jk\Omega_0 t}\bigg|_0^{T/2} - \frac{2}{T}\int_0^{T/2}e^{-jk\Omega_0 t}dt\right.$$

$$\left. + \left(2-\frac{2t}{T}\right)e^{-jk\Omega_0 t}\bigg|_{T/2}^{T} + \frac{2}{T}\int_{T/2}^{T}e^{-jk\Omega_0 t}dt\right]$$

Example: Triangular wave

$$a_k = \frac{1}{-jk\Omega_0 T}\left[\frac{2t}{T}e^{-jk\Omega_0 t}\Big|_0^{T/2} - \frac{2}{T}\int_0^{T/2} e^{-jk\Omega_0 t}dt + \left(2-\frac{2t}{T}\right)e^{-jk\Omega_0 t}\Big|_{T/2}^{T} + \frac{2}{T}\int_{T/2}^{T} e^{-jk\Omega_0 t}dt\right]$$

- Using $\Omega_0 T = 2\pi$,

$$a_k = \frac{1}{-jk2\pi}\left[\cancel{e^{-jk\pi}} - \frac{2}{T}\frac{e^{-jk\Omega_0 t}}{-jk\Omega_0}\Big|_0^{T/2} - \cancel{e^{-jk\pi}} + \frac{2}{T}\frac{e^{-jk\Omega_0 t}}{-jk\Omega_0}\Big|_{T/2}^{T}\right]$$

$$= \frac{2}{(-jk2\pi)^2}\left[-e^{-jk\Omega_0 t}\Big|_0^{T/2} + e^{-jk\Omega_0 t}\Big|_{T/2}^{T}\right]$$

Example: Triangular wave

$$a_k = \frac{2}{(-jk2\pi)^2}\left[-e^{-jk\Omega_0 t}\Big|_0^{T/2} + e^{-jk\Omega_0 t}\Big|_{T/2}^{T}\right]$$

$$= \frac{-1}{2k^2\pi^2}\left[1 - e^{-jk\pi} + 1 - e^{-jk\pi}\right]$$

$$= \frac{-1}{k^2\pi^2}\left[1-(-1)^k\right] = \begin{cases} 0 & \text{even } k \\ \dfrac{-2}{k^2\pi^2} & \text{odd } k \end{cases}$$

- Remembering $a_0 = \dfrac{1}{2}$, this means

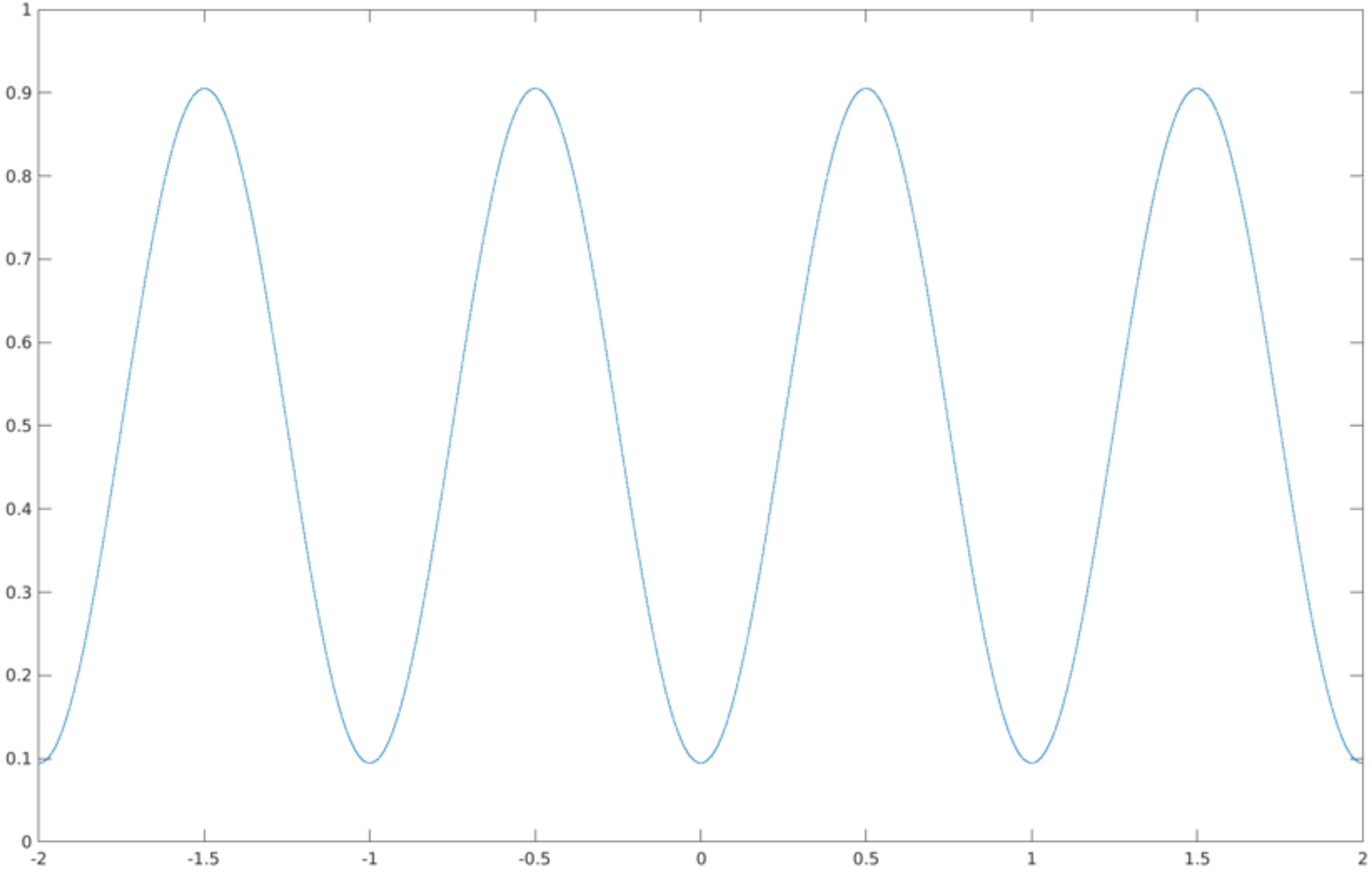
$$x(t) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{\text{odd } k}\frac{1}{k^2}\, e^{jk\Omega_0 t}$$

Example: Triangular wave

$$
\begin{aligned}
x(t) &= \frac{1}{2} - \frac{2}{\pi^2} \sum_{\text{odd } k} \frac{1}{k^2}\, e^{jk\Omega_0 t} \\
&= \frac{1}{2} - \frac{2}{\pi^2} \left[\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{1} + \frac{e^{j3\Omega_0 t} + e^{-j3\Omega_0 t}}{9} + \ldots \right] \\
&= \frac{1}{2} - \frac{4}{\pi^2} \left[\cos(\Omega_0 t) + \frac{1}{9}\cos(3\Omega_0 t) + \frac{1}{25}\cos(5\Omega_0 t) + \ldots \right] \\
&= \frac{1}{2} - \frac{4}{\pi^2} \sum_{\text{odd } k} \frac{1}{k^2} \cos(k\Omega_0 t)
\end{aligned}
$$

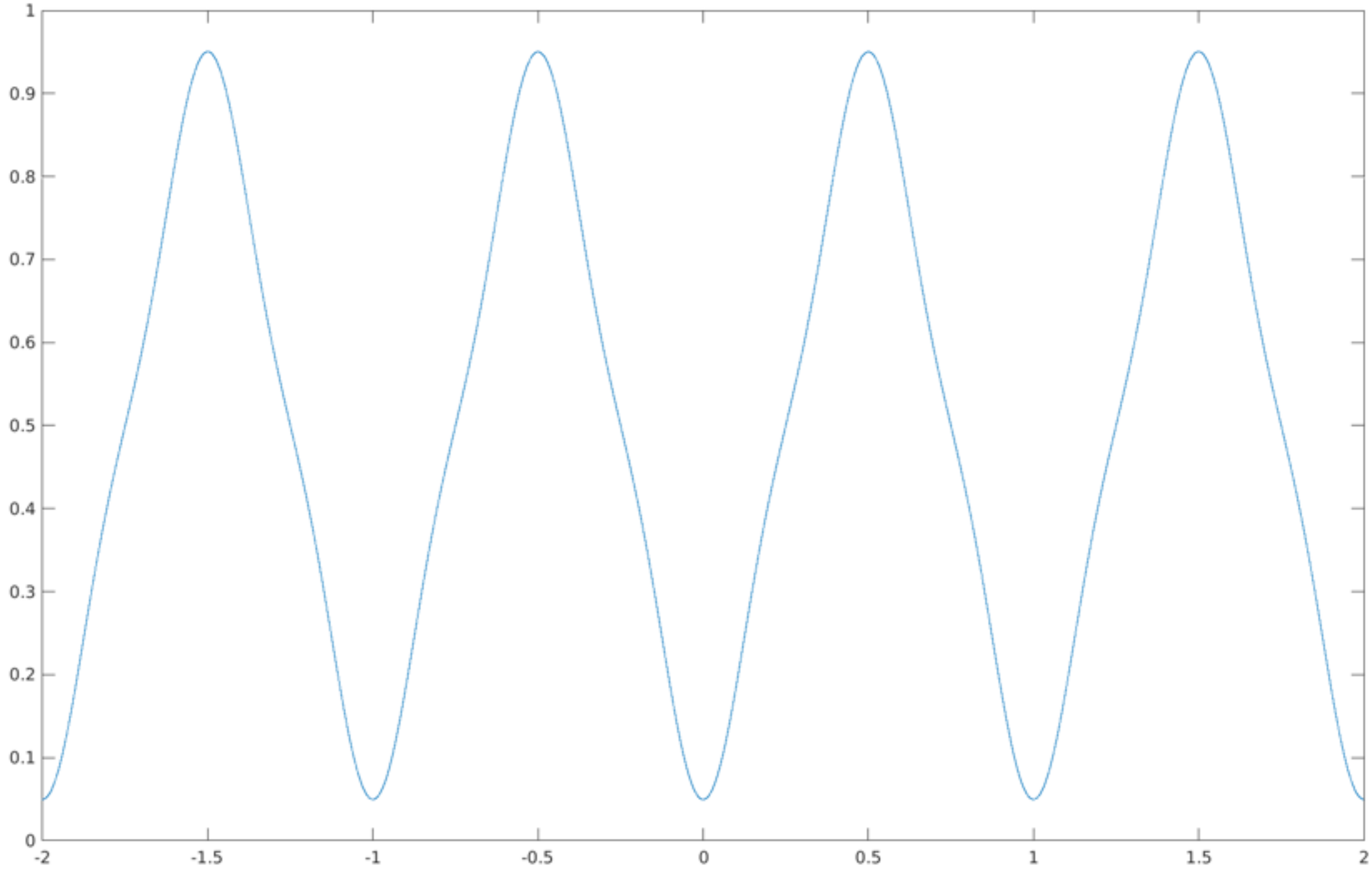
Example: Triangular wave

- The first cosine wave



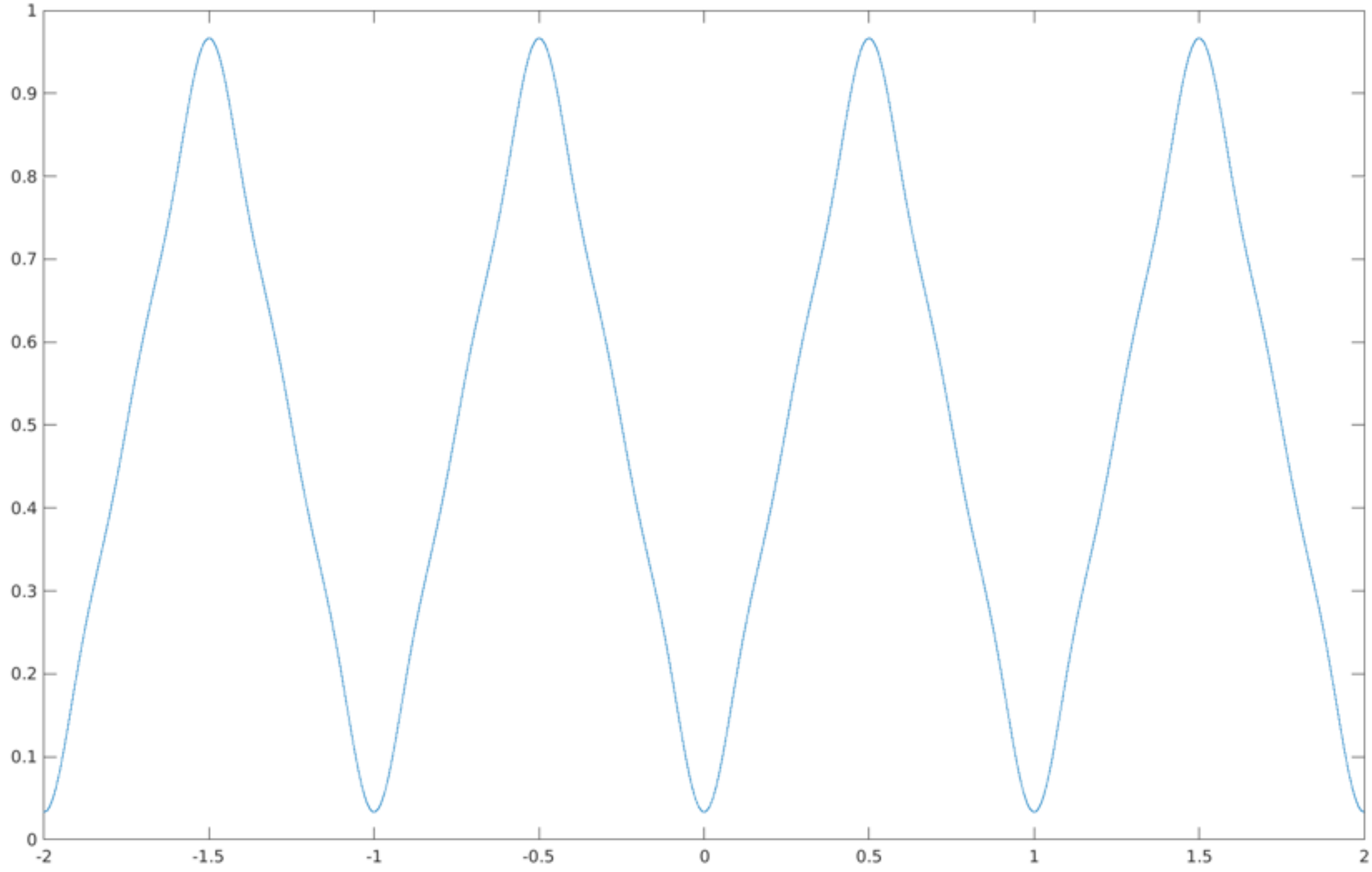
Example: Triangular wave

- The first 2 cosine waves



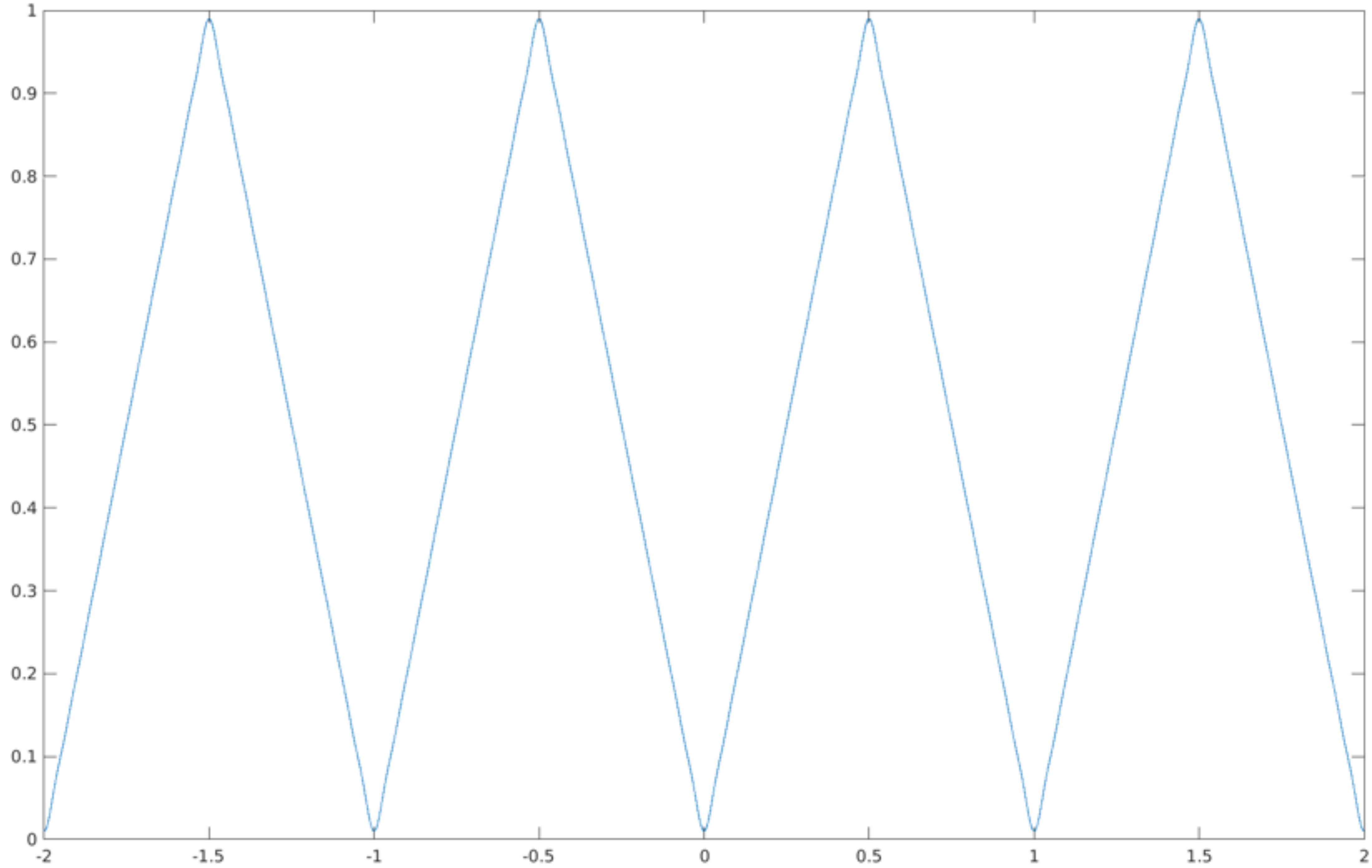
Example: Triangular wave

- The first 3 cosine waves



Example: Triangular wave

- The first 10 cosine waves



A neat observation

- Note that we write

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\Omega_0 t}\,dt$$

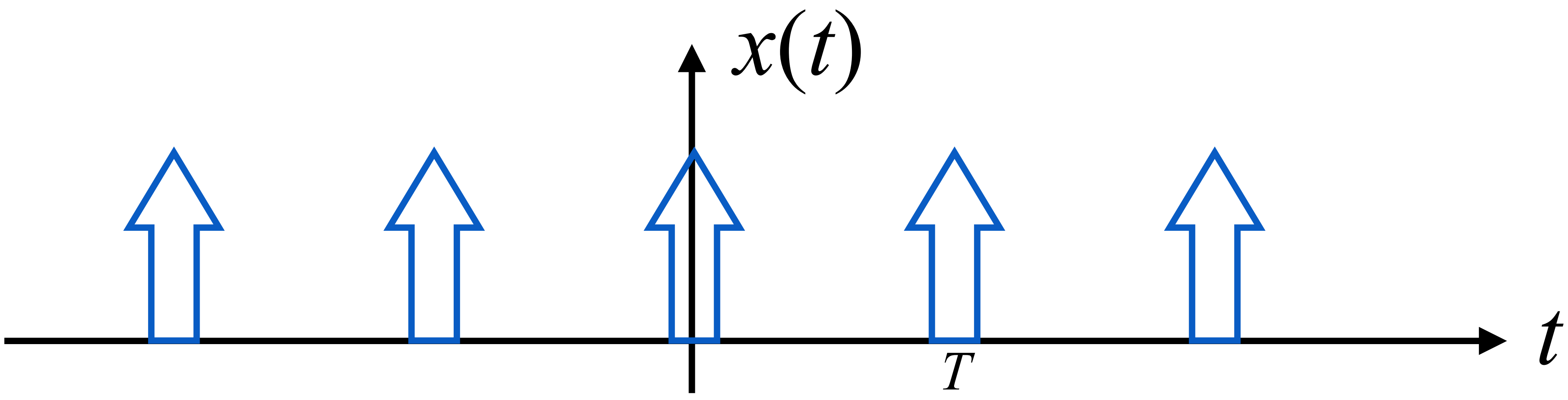
- However, since both $x(t)$ and $e^{-jk\Omega_0 t}$ have period T, the integral won't change if we write it as

$$a_k = \frac{1}{T}\int_c^{T+c} x(t)e^{-jk\Omega_0 t}\,dt$$

for any c.

Example: Impulse train

- Find the CTFS coefficients for the signal



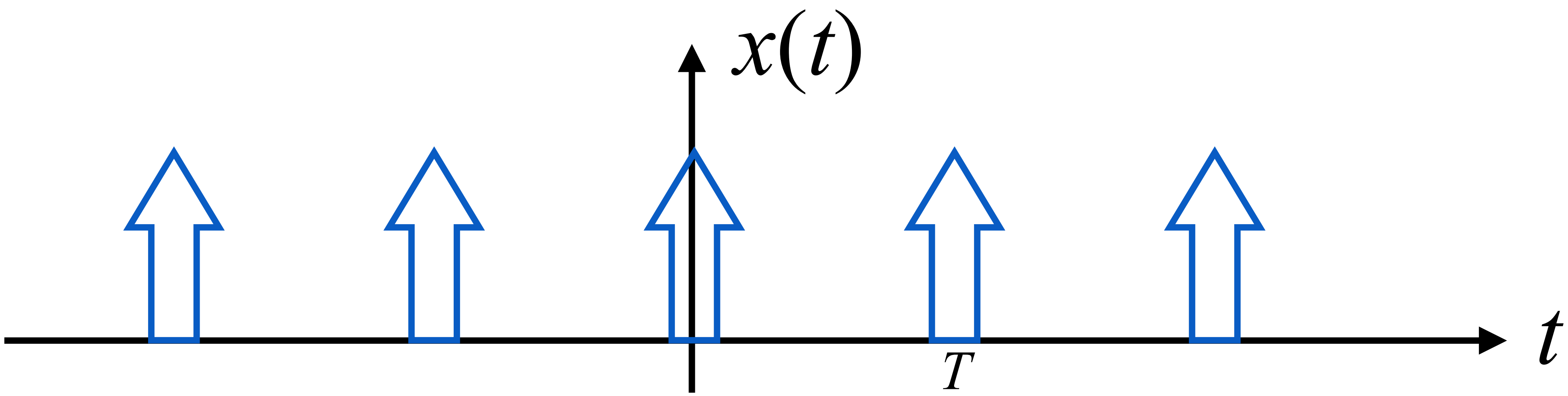
- <u>Solution</u>: Letting $\Omega_0 = 2\pi/T$,

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t}dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\Omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt = \frac{1}{T}$$

Example: Impulse train

- Find the CTFS coefficients for the signal



- That implies

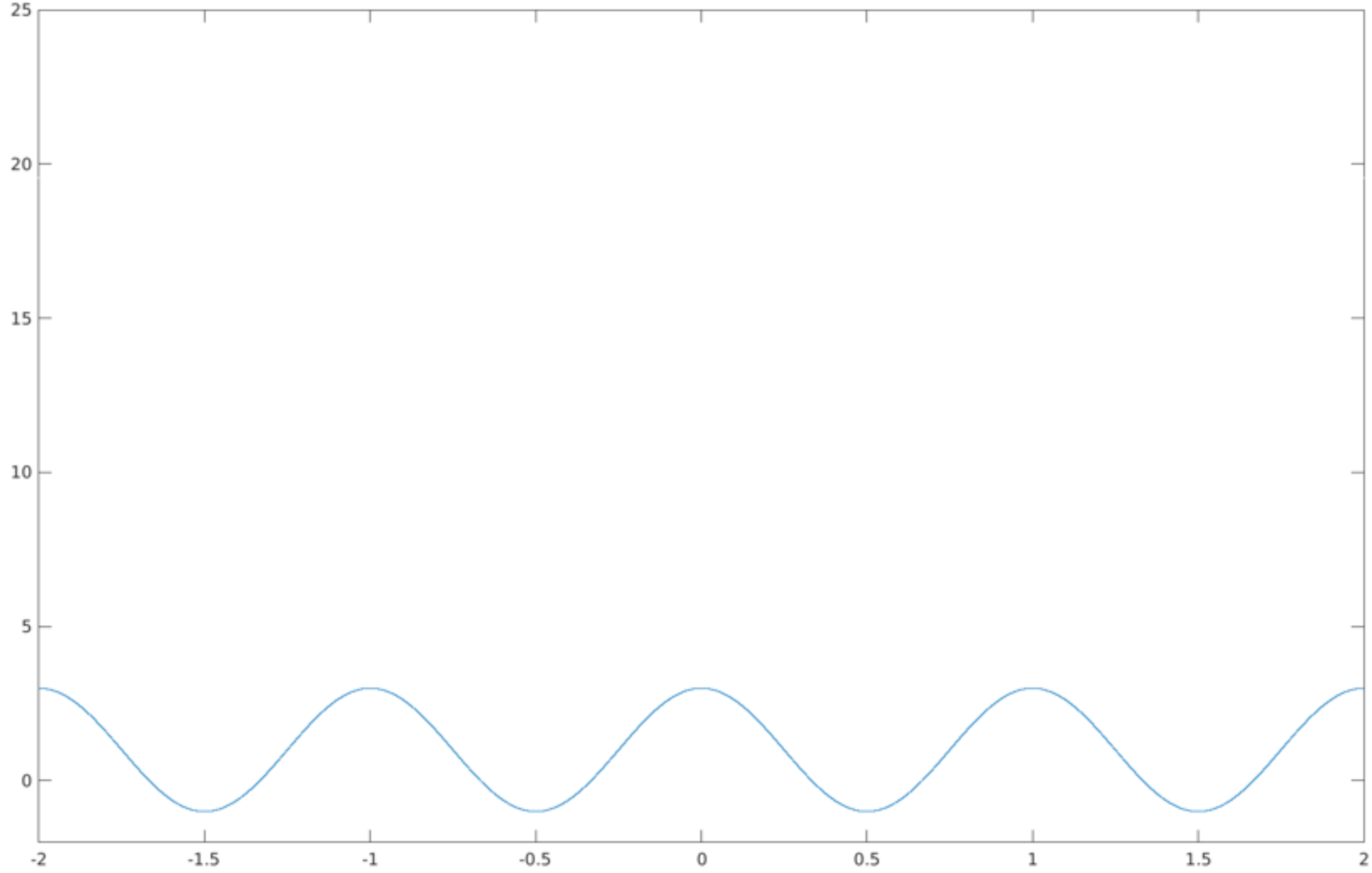
$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t}$$

$$= \frac{1}{T} \left(1 + e^{j\Omega_0 t} + e^{-j\Omega_0 t} + e^{j2\Omega_0 t} + e^{-j2\Omega_0 t} + \ldots\right)$$

$$= \frac{1}{T} \left(1 + 2\cos(\Omega_0 t) + 2\cos(2\Omega_0 t) + \ldots\right)$$

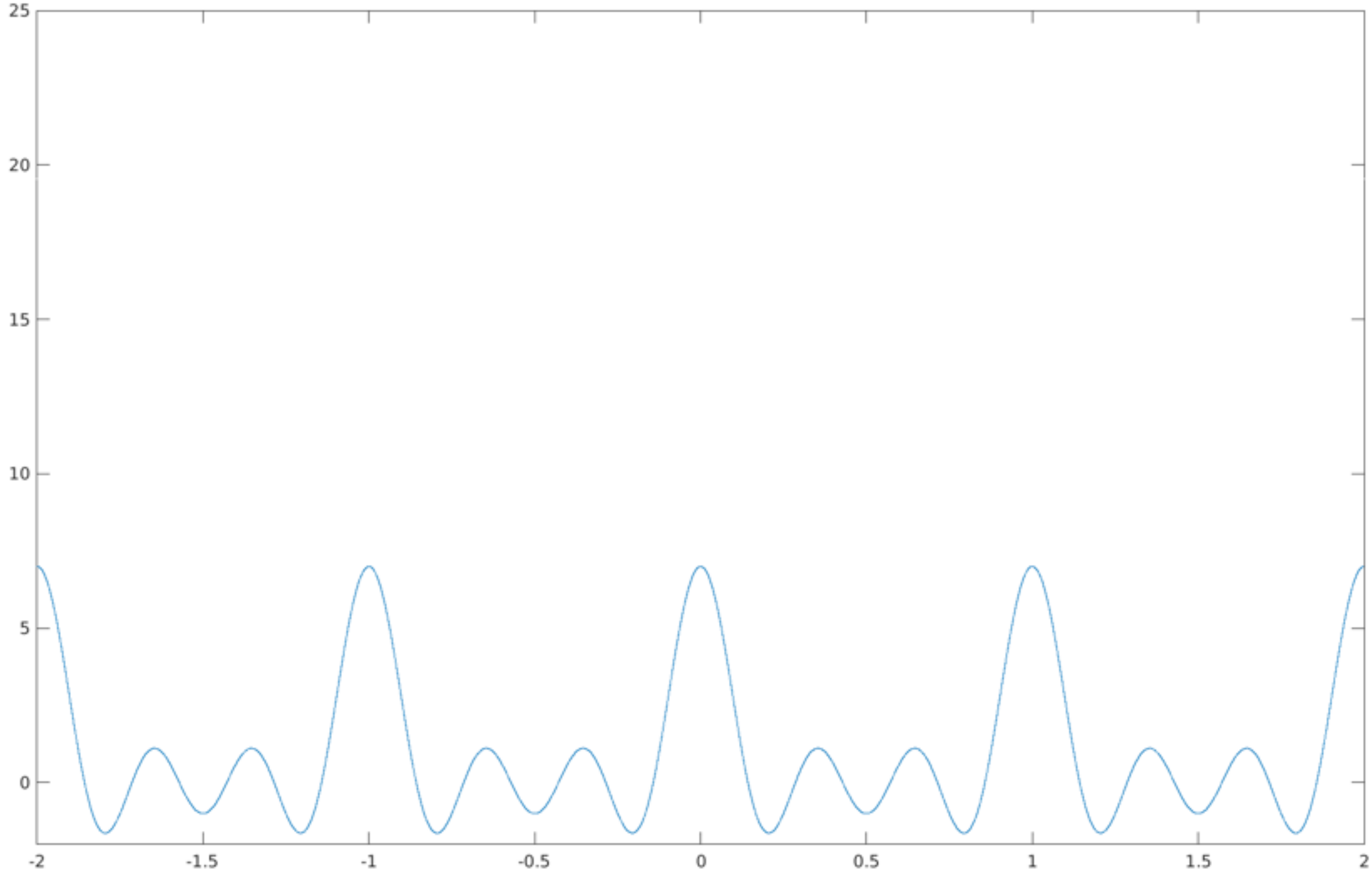
Example: Impulse train

- The first cosine wave



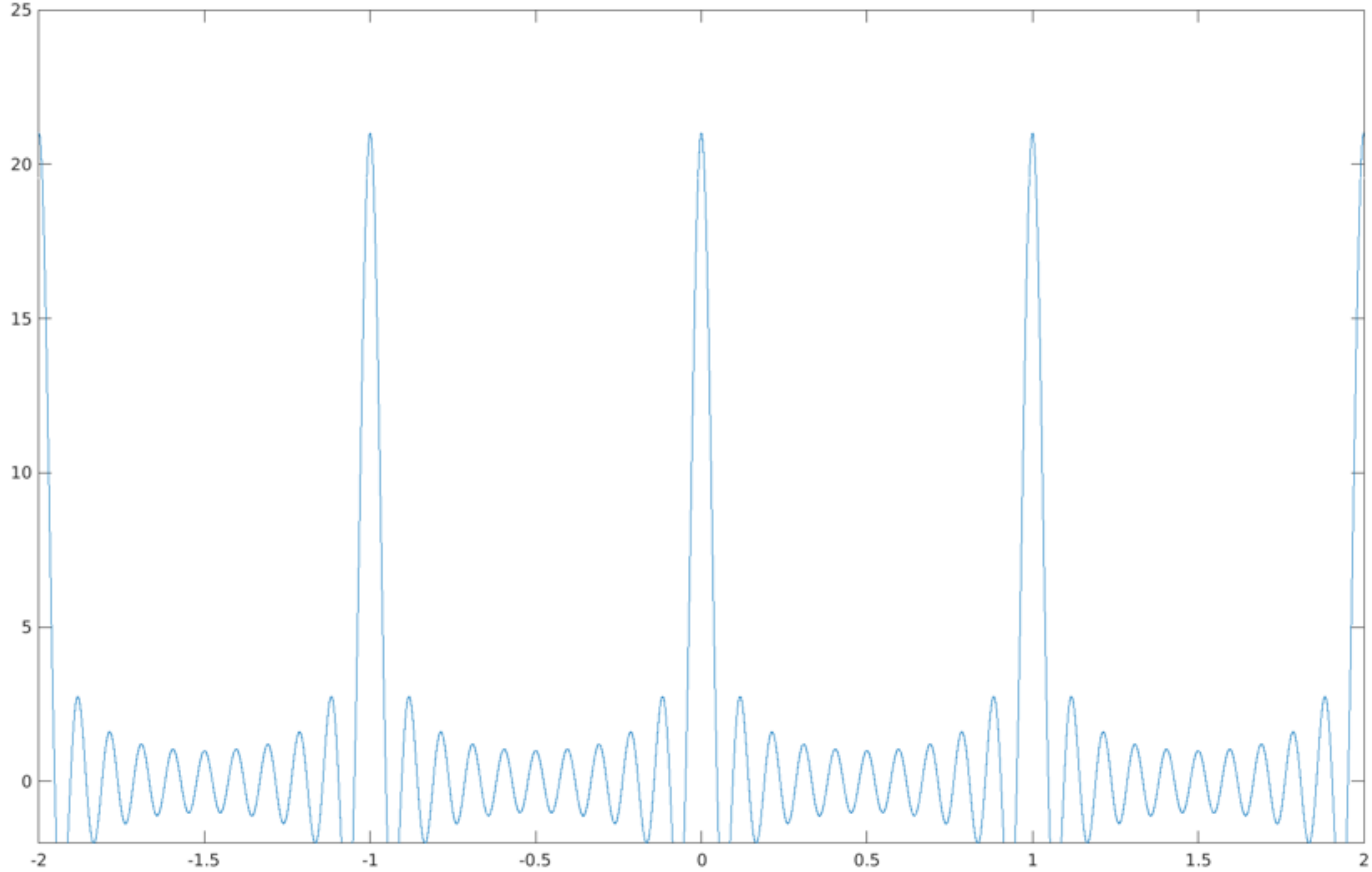
Example: Impulse train

- The first 3 cosine waves



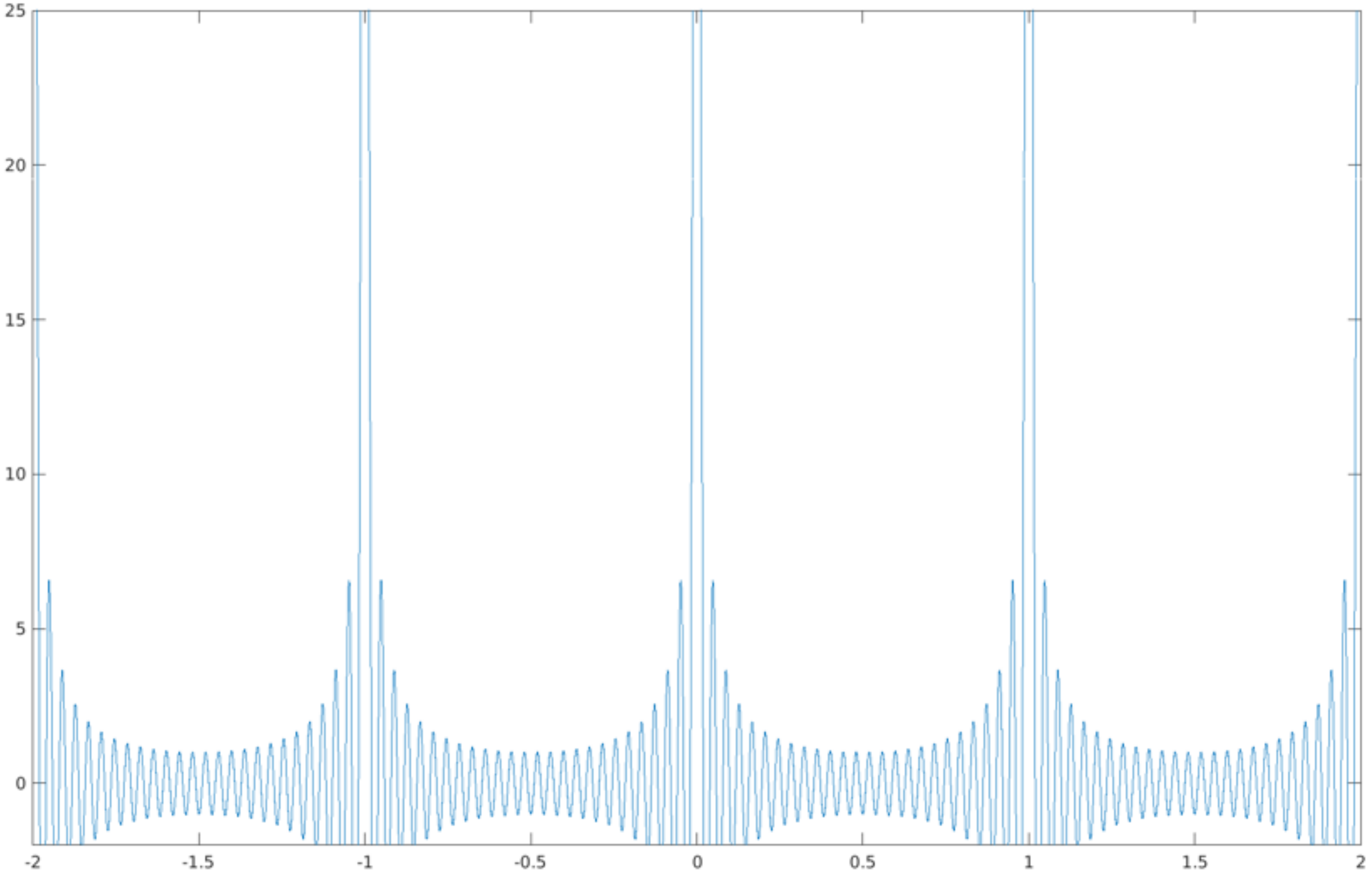
Example: Impulse train

- The first 10 cosine waves



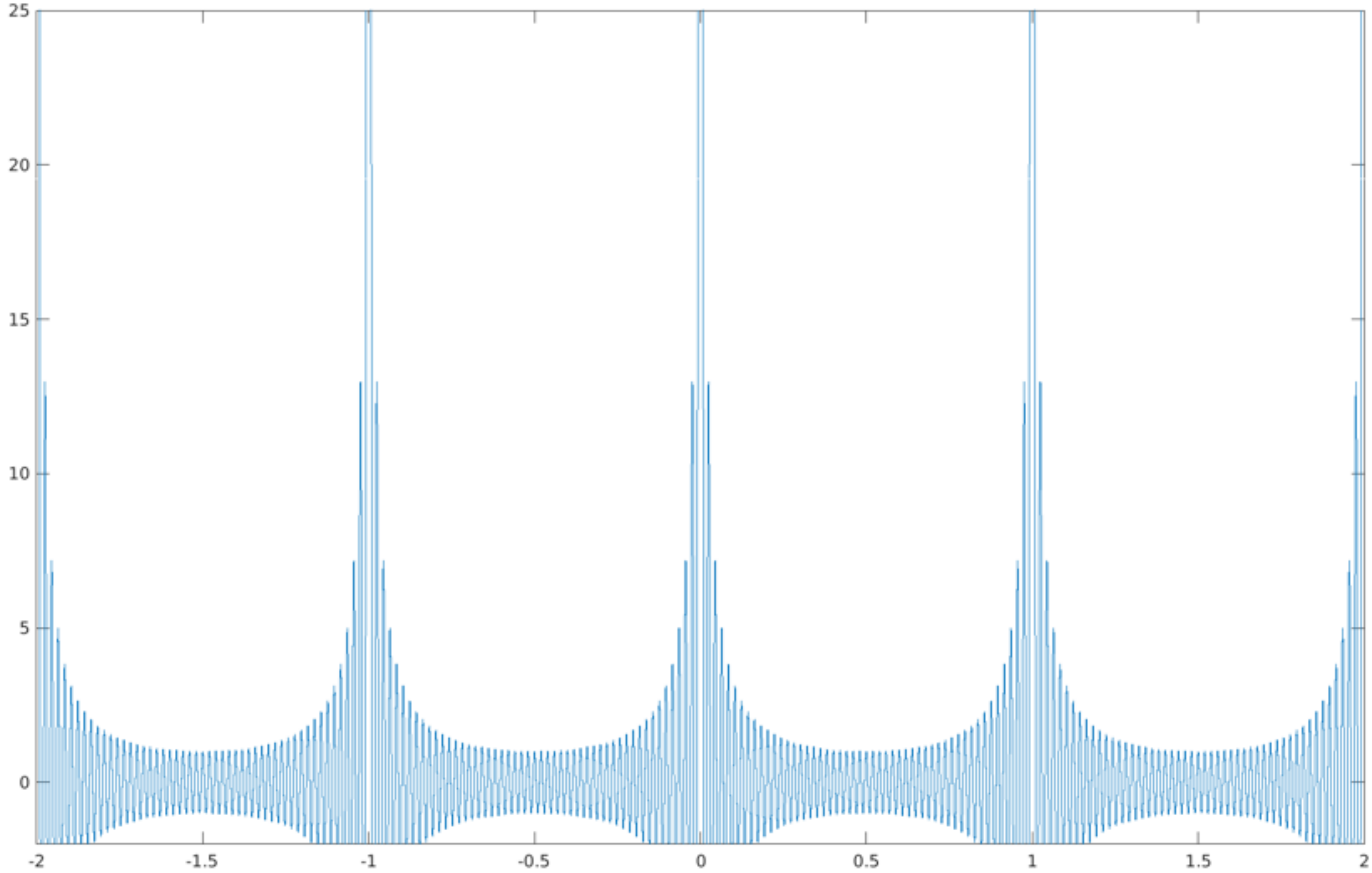
Example: Impulse train

- The first 25 cosine waves



Example: Impulse train

- The first 50 cosine waves



Properties

- **Linearity**:

$$x(t) \xrightarrow{\text{CTFS}} a_k$$

$$y(t) \xrightarrow{\text{CTFS}} b_k$$

implies

$$Ax(t) + By(t) \xrightarrow{\text{CTFS}} Aa_k + Bb_k$$

- Proof:

$$\frac{1}{T}\int_0^T [Ax(t) + By(t)]e^{-jk\Omega_0 t}dt$$

$$= \frac{A}{T}\int_0^T x(t)e^{-jk\Omega_0 t}dt + \frac{B}{T}\int_0^T y(t)e^{-jk\Omega_0 t}dt$$

Properties

- **Time shifting**:

$$x(t) \xrightarrow{\text{CTFS}} a_k$$

implies

$$x(t - t_0) \xrightarrow{\text{CTFS}} a_k e^{-jk\Omega_0 t_0}$$

- <u>Proof</u>:

$$\frac{1}{T}\int_0^T x(t - t_0)e^{-jk\Omega_0 t}\,dt \overset{(\tau=t-t_0)}{=} \frac{1}{T}\int_{-t_0}^{T-t_0} x(\tau)e^{-jk\Omega_0(\tau+t_0)}\,d\tau$$

$$= \frac{e^{-jk\Omega_0 t_0}}{T}\int_{-t_0}^{T-t_0} x(\tau)e^{-jk\Omega_0 \tau}\,d\tau$$

Properties

- **Frequency shifting**:

$$x(t) \xrightarrow{\text{CTFS}} a_k$$

implies

$$x(t)e^{jk_0\Omega_0 t} \xrightarrow{\text{CTFS}} a_{k-k_0}$$

- <u>Proof</u>:

$$\frac{1}{T}\int_0^T x(t)e^{jk_0\Omega_0 t}e^{-jk\Omega_0 t}dt = \frac{1}{T}\int_0^T x(t)e^{-j(k-k_0)\Omega_0 t}dt$$

$$= a_{k-k_0}$$

Properties

- **Conjugation**:

$$x(t) \xrightarrow{\text{CTFS}} a_k$$

implies

$$x(t)^* \xrightarrow{\text{CTFS}} a^*_{-k}$$

- Proof:

$$\frac{1}{T}\int_0^T x(t)^* e^{-jk\Omega_0 t}\,dt = \Bigg\{\underbrace{\frac{1}{T}\int_0^T x(t) e^{jk\Omega_0 t}\,dt}_{a_{-k}}\Bigg\}^*$$

Properties

- **Time reversal**:

$$x(t) \xrightarrow{\text{CTFS}} a_k$$

implies

$$x(-t) \xrightarrow{\text{CTFS}} a_{-k}$$

- <u>Proof</u>:

$$\frac{1}{T}\int_0^T x(-t)e^{-jk\Omega_0 t}\,dt \overset{(\tau=-t)}{=} \frac{1}{T}\int_{-T}^{0} x(\tau)e^{jk\Omega_0 \tau}\,d\tau$$

$$= a_{-k}$$

Properties

- Implications of the last two properties:

$$x(t)^* \xrightarrow{\text{CTFS}} a^*_{-k} \qquad x(-t) \xrightarrow{\text{CTFS}} a_{-k}$$

- Real signals: $x(t) = x(t)^* \Longrightarrow a_k = a^*_{-k}$
- Even signals: $x(t) = x(-t) \Longrightarrow a_k = a_{-k}$
- Real and even signals:

$$x(t) = x(t)^* = x(-t) \Longrightarrow a_k = \underbrace{a^*_{-k} = a_{-k}}_{\text{real coefficients}}$$

- CTFS coefficients are also real and even!!!

Properties

- Real signals: $x(t) = x(t)^* \implies a_k = a^*_{-k}$
- Rewriting a_k as $r_k e^{j\theta_k}$, this implies

$$r_k = r_{-k}$$

$$\theta_k = -\theta_{-k} \qquad \text{In particular, } \theta_0 = 0$$

- Therefore,

$$x(t) = \sum_{k=-\infty}^{\infty} r_k e^{j\theta_k} e^{jk\Omega_0 t}$$

$$= r_0 + \sum_{k=1}^{\infty} r_k e^{j\theta_k} e^{jk\Omega_0 t} + \sum_{k=-\infty}^{-1} r_k e^{j\theta_k} e^{jk\Omega_0 t}$$

Properties

$$r_k = r_{-k}$$
$$\theta_k = -\theta_{-k}$$

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} r_k e^{j\theta_k} e^{jk\Omega_0 t} \\
&= r_0 + \sum_{k=1}^{\infty} r_k e^{j\theta_k} e^{jk\Omega_0 t} + \sum_{k=-\infty}^{-1} r_k e^{j\theta_k} e^{jk\Omega_0 t} \\
&= r_0 + \sum_{k=1}^{\infty} r_k e^{j\theta_k} e^{jk\Omega_0 t} + \sum_{k=-\infty}^{-1} r_{-k} e^{-j\theta_{-k}} e^{jk\Omega_0 t} \\
&= r_0 + \sum_{k=1}^{\infty} r_k e^{j\theta_k} e^{jk\Omega_0 t} + \sum_{k=1}^{\infty} r_k e^{-j\theta_k} e^{-jk\Omega_0 t} \\
&= r_0 + \sum_{k=1}^{\infty} r_k \left[e^{j(k\Omega_0 t + \theta_k)} + e^{-j(k\Omega_0 t + \theta_k)} \right]
\end{aligned}
$$

Properties

$$x(t) = \sum_{k=-\infty}^{\infty} r_k e^{j\theta_k} e^{jk\Omega_0 t}$$

$$r_k = r_{-k}$$

$$\theta_k = -\theta_{-k}$$

$$= r_0 + \sum_{k=1}^{\infty} r_k \left[e^{j(k\Omega_0 t + \theta_k)} + e^{-j(k\Omega_0 t + \theta_k)} \right]$$

$$= r_0 + 2 \sum_{k=1}^{\infty} r_k \cos(k\Omega_0 t + \theta_k)$$

Properties

- **Periodic convolution**:

$$x(t) \xrightarrow{\text{CTFS}} a_k$$

$$y(t) \xrightarrow{\text{CTFS}} b_k$$

implies

$$\int_0^T x(\tau)y(t-\tau)d\tau \xrightarrow{\text{CTFS}} Ta_k b_k$$

periodic convolution

also shown as $x(t) \tilde{\star} y(t)$

Properties

- Proof:

$$\frac{1}{T}\int_0^T\int_0^T x(\tau)y(t-\tau)e^{-jk\Omega_0 t}\,d\tau dt$$

$$=\frac{1}{T}\int_0^T x(\tau)\int_0^T y(t-\tau)e^{-jk\Omega_0 t}\,dt d\tau$$

Time shift property: $b_k e^{-jk\Omega_0 \tau}$

$$=\int_0^T x(\tau)b_k e^{-jk\Omega_0 \tau}\,d\tau$$

$$=b_k\int_0^T x(\tau)e^{-jk\Omega_0 \tau}\,d\tau \quad = Ta_k b_k$$

Properties

- **Multiplication**:

$$x(t) \xrightarrow{\text{CTFS}} a_k \qquad \text{implies} \qquad x(t)y(t) \xrightarrow{\text{CTFS}} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$
$$y(t) \xrightarrow{\text{CTFS}} b_k$$

- <u>Proof:</u> Whose CTFS is $\sum_{l=-\infty}^{\infty} a_l b_{k-l}$?

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_l b_{k-l} e^{jk\Omega_0 t} = \sum_{l=-\infty}^{\infty} a_l \sum_{k=-\infty}^{\infty} b_{k-l} e^{jk\Omega_0 t}$$

$$= \sum_{l=-\infty}^{\infty} a_l \sum_{k=-\infty}^{\infty} b_{k-l} e^{j(k-l)\Omega_0 t} e^{jl\Omega_0 t} = \sum_{l=-\infty}^{\infty} a_l e^{jl\Omega_0 t} \sum_{k=-\infty}^{\infty} b_{k-l} e^{j(k-l)\Omega_0 t}$$

$$\overset{(m=k-l)}{=} \sum_{l=-\infty}^{\infty} a_l e^{jl\Omega_0 t} \sum_{m=-\infty}^{\infty} b_m e^{jm\Omega_0 t} = x(t)y(t)$$

Properties

- **Parseval's relation**:

$$x(t) \xrightarrow{\text{CTFS}} a_k$$

implies

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Proof: First, let's show

$$\int_0^T |x(t)|^2 dt = x(t) \,\tilde{\star}\, x(-t)^* \Big|_{t=0}$$

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- <u>Proof:</u> First, let's show

$$\int_0^T |x(t)|^2 dt = x(t) \tilde{\star} x(-t)^* \Big|_{t=0}$$

To see that, write

$$x(t) \tilde{\star} x(-t)^* = \int_0^T x(\tau)x(\tau - t)^* d\tau$$

and substitute $t = 0$ on the RHS.

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\int_0^T |x(t)|^2 dt = x(t) \,\tilde{\star}\, x(-t)^* \Big|_{t=0}$$

But we know that

$$x(-t) \xrightarrow{\text{CTFS}} a_{-k} \quad \text{and} \quad x(t)^* \xrightarrow{\text{CTFS}} a^*_{-k}$$

Therefore,

$$x(-t)^* \xrightarrow{\text{CTFS}} a_k^*$$

Then using the convolution property,

$$x(t) \,\tilde{\star}\, x(-t)^* \xrightarrow{\text{CTFS}} T a_k a_k^* = T|a_k|^2$$

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\int_0^T |x(t)|^2 dt = x(t) \,\tilde{\star}\, x(-t)^* \Big|_{t=0}$$

$$x(t) \,\tilde{\star}\, x(-t)^* \xrightarrow{\text{CTFS}} T a_k a_k^* = T|a_k|^2$$

In other words,

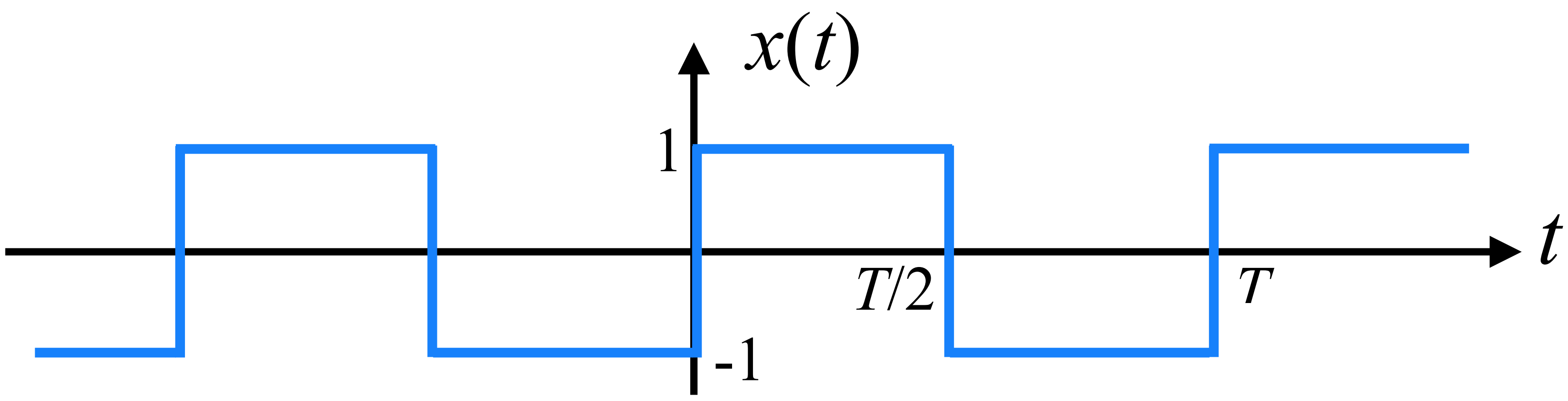
$$x(t) \,\tilde{\star}\, x(-t)^* = \sum_{k=-\infty}^{\infty} T|a_k|^2 e^{jk\Omega_0 t}$$

In particular,

$$x(t) \,\tilde{\star}\, x(-t)^* \Big|_{t=0} = \sum_{k=-\infty}^{\infty} T|a_k|^2$$

An example of its use

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$



$$a_k = \begin{cases} 0 & \text{even } k \\ \dfrac{2}{jk\pi} & \text{odd } k \end{cases}$$

- Parseval's relation implies

$$1 = \frac{4}{\pi^2} \sum_{\text{odd } k} \frac{1}{k^2}$$