EE 110B Signals and Systems

The Z-Transform

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The z-transform

- Our motivation for the DTFT was that it is easy to characterize the output of an LTI for exponential inputs:

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow e^{j\omega n} H(e^{j\omega})$$

- What about the input $r^n e^{j\omega n}$?

$$r^n e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow \; ?$$

The z-transform

- Let's find out:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] r^{n-k} e^{j\omega(n-k)}$$

$$= \underbrace{r^n e^{j\omega n}}_{\text{Input}} \underbrace{\sum_{k=-\infty}^{\infty} h[k] r^{-k} e^{-j\omega k}}_{\text{DTFT of } h[n]r^{-n}}$$

The z-transform

- Alternatively, we can think of the input as

$$r^n e^{j\omega n} = (re^{j\omega})^n \stackrel{\Delta}{=} z^n$$

- Then

$$y[n] = r^n e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k] r^{-k} e^{-j\omega k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

Let's call this $H(z)$

The z-transform

- The z-transform is then defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- For any z, we have

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow z^n H(z)$$

- If we specialize this to $z = e^{j\omega}$, we get back DTFT:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Examples

- Problem: Find the z-transform of $h[n] = a^n u[n]$

- Solution:

$$H(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} \quad \text{provided } |az^{-1}| < 1$$

$$\text{or } |z| > |a|$$

Examples

- Problem: What about for $h[n] = -a^n u[-n-1]$?
- Solution:

$$H(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$\stackrel{(m=-n-1)}{=} -\sum_{m=0}^{\infty} (az^{-1})^{-m-1} = -\sum_{m=0}^{\infty} (a^{-1}z)^{m+1}$$

$$= -a^{-1}z \sum_{m=0}^{\infty} (a^{-1}z)^m = \frac{-a^{-1}z}{1-a^{-1}z} \quad \text{for } |z| < |a|$$

$$= \frac{1}{1-az^{-1}}$$

Same as before!!!

What?

- How can

$$h[n] = a^n u[n]$$

  and

$$h[n] = -a^n u[-n-1]$$

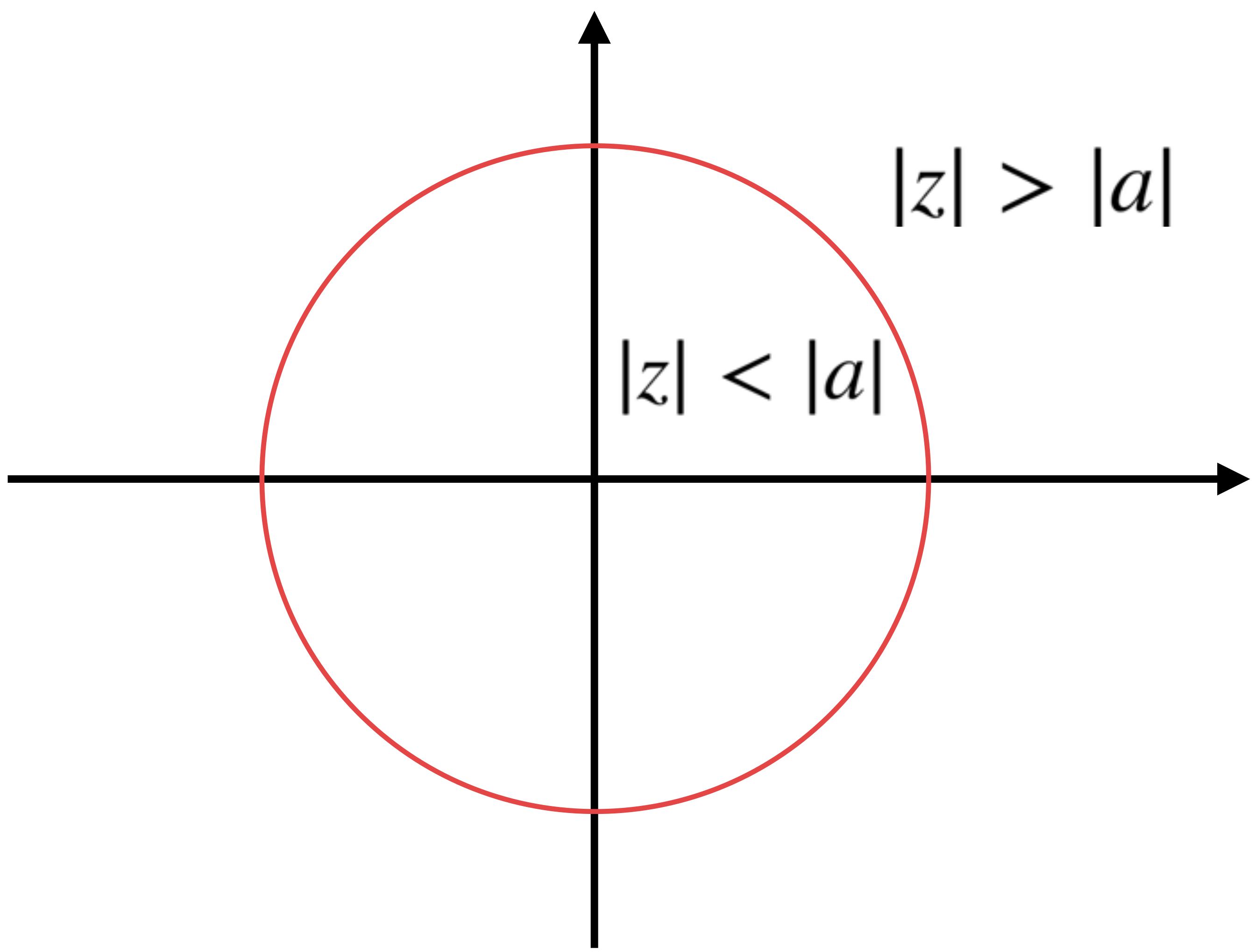
  have the same z-transform $H(z) = \dfrac{1}{1 - az^{-1}}$?
- Well, they don't.
- The former requires $|z| > |a|$ and the latter requires $|z| < |a|$.
- The z-transform is not complete without the specification of the region of convergence (ROC).

The region of convergence

- Defined as the region in the z-plane in which

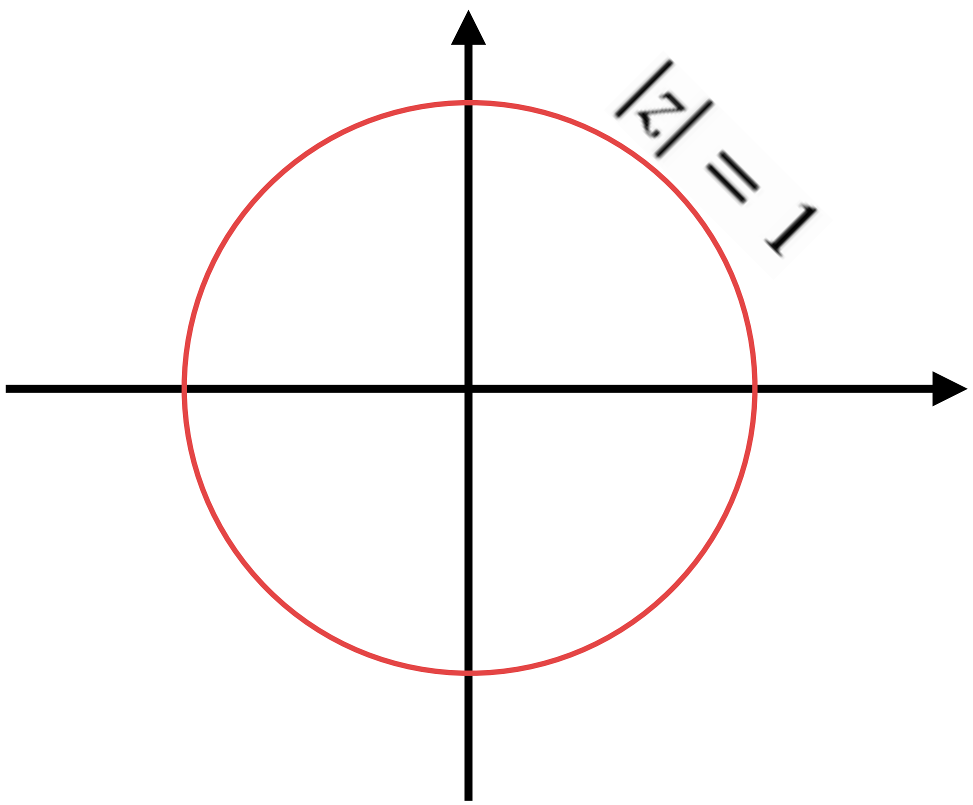
$$\sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

converges.



The region of convergence

- If the ROC includes $|z| = 1$, then the DTFT converges and hence is well-defined.



- For $h[n] = a^n u[n]$, this means $|a| < 1$.
- For $h[n] = -a^n u[-n-1]$, this means $|a| > 1$.

Examples continued

- Problem: Find the z-transform of

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

- Solution:

$$X(z) = 7\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n} - 6\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}$$

$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

more restrictive

assuming $|z| > \frac{1}{3}$ assuming $|z| > \frac{1}{2}$

ROC:

$|z| > \frac{1}{2}$

Zeros and poles

- The z-transform usually ends up being a rational function

$$X(z) = \frac{N(z)}{D(z)}$$

with polynomial $N(z)$ and $D(z)$.

- Zeros: Points on the z-plane where $X(z) = 0$
- Poles: Points on the z-plane where $X(z) = \infty$

Zeros and poles

- Example: $X(z) = \dfrac{1}{1 - az^{-1}} = \dfrac{z}{z - a}$

Zero: $z = 0$ **Pole**: $z = a$

- Example:

$$X(z) = \frac{7}{1 - \frac{1}{3}z^{-1}} - \frac{6}{1 - \frac{1}{2}z^{-1}} = \frac{7z}{z - \frac{1}{3}} - \frac{6z}{z - \frac{1}{2}}$$

$$= \frac{7z(z - \frac{1}{2}) - 6z(z - \frac{1}{3})}{(z - \frac{1}{3})(z - \frac{1}{2})} = \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}$$

Zeros: $z = 0$ and $z = 3/2$ **Poles**: $z = 1/3$ and $z = 1/2$

Examples continued

- Problem: Find the z-transform of

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi n}{4}\right) u[n]$$

- Solution:

$$x[n] = \left(\frac{1}{3}\right)^n \frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j} u[n]$$

$$= \frac{1}{2j}\left(\frac{e^{j\frac{\pi}{4}}}{3}\right)^n u[n] - \frac{1}{2j}\left(\frac{e^{-j\frac{\pi}{4}}}{3}\right)^n u[n]$$

$$X(z) = \frac{1}{2j} \cdot \frac{1}{1 - \frac{e^{j\frac{\pi}{4}}}{3} z^{-1}} - \frac{1}{2j} \cdot \frac{1}{1 - \frac{e^{-j\frac{\pi}{4}}}{3} z^{-1}}$$

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi n}{4}\right) u[n]$$

$$X(z) = \frac{1}{2j} \cdot \frac{1}{1 - \frac{e^{j\frac{\pi}{4}}}{3} z^{-1}} - \frac{1}{2j} \cdot \frac{1}{1 - \frac{e^{-j\frac{\pi}{4}}}{3} z^{-1}}$$

ROC: $|z| > \left|\frac{e^{j\frac{\pi}{4}}}{3}\right| = \frac{1}{3}$

- Simplifying,

$$X(z) = \frac{1}{2j} \cdot \frac{z}{z - \frac{e^{j\frac{\pi}{4}}}{3}} - \frac{1}{2j} \cdot \frac{z}{z - \frac{e^{-j\frac{\pi}{4}}}{3}}$$

$$= \frac{z}{2j} \cdot \frac{\left(z - \frac{e^{-j\frac{\pi}{4}}}{3}\right) - \left(z - \frac{e^{j\frac{\pi}{4}}}{3}\right)}{\left(z - \frac{e^{j\frac{\pi}{4}}}{3}\right)\left(z - \frac{e^{-j\frac{\pi}{4}}}{3}\right)}$$

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi n}{4}\right) u[n]$$

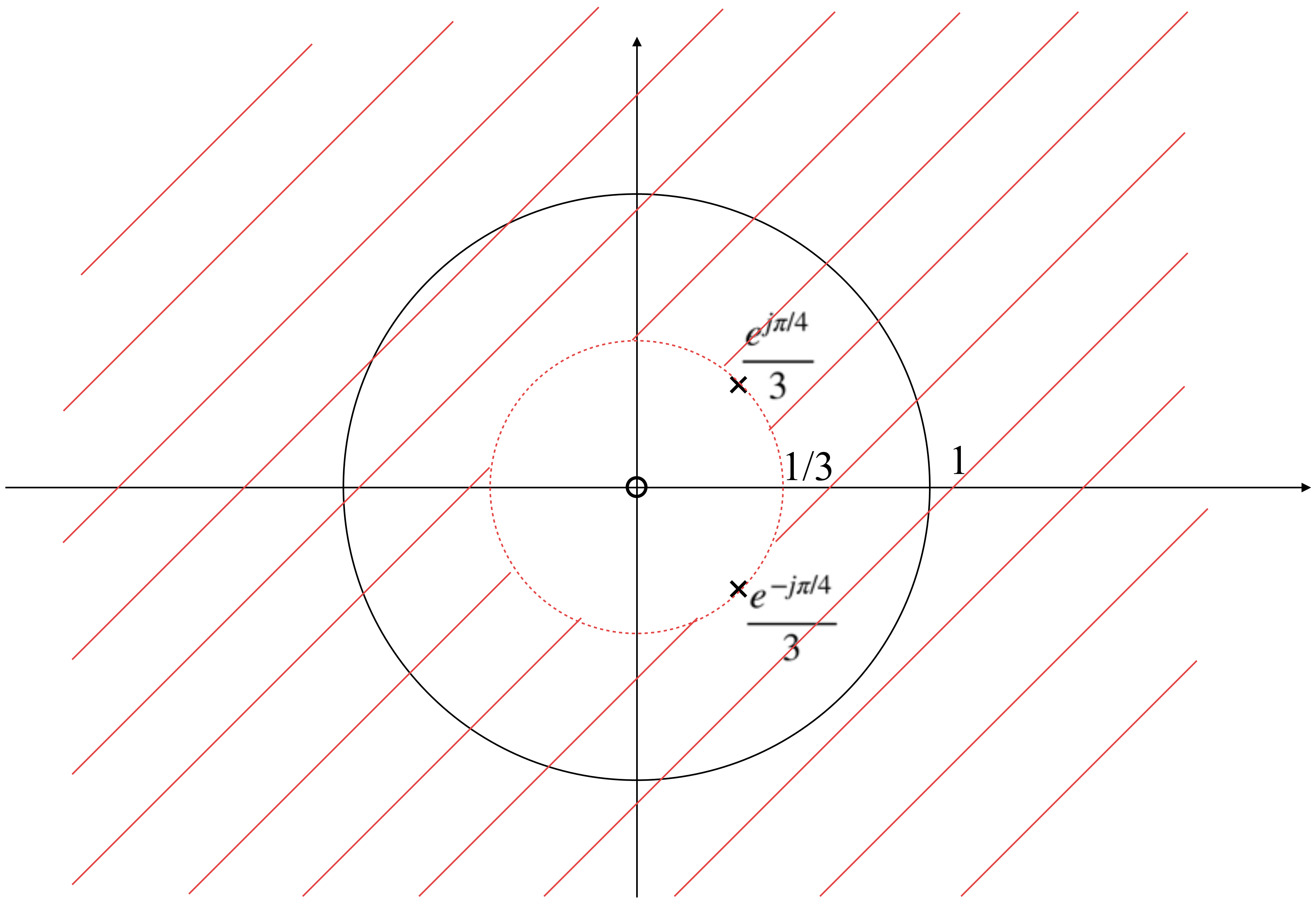
$$X(z) = \frac{z}{2j} \cdot \frac{\left(z - \frac{e^{-j\frac{\pi}{4}}}{3}\right) - \left(z - \frac{e^{j\frac{\pi}{4}}}{3}\right)}{\left(z - \frac{e^{j\frac{\pi}{4}}}{3}\right)\left(z - \frac{e^{-j\frac{\pi}{4}}}{3}\right)}$$

$$= \frac{z}{2j} \cdot \frac{\frac{e^{j\frac{\pi}{4}}}{3} - \frac{e^{-j\frac{\pi}{4}}}{3}}{\left(z - \frac{e^{j\frac{\pi}{4}}}{3}\right)\left(z - \frac{e^{-j\frac{\pi}{4}}}{3}\right)}$$

$$= \frac{\sin(\frac{\pi}{4})}{3} \cdot \frac{z}{\left(z - \frac{e^{j\frac{\pi}{4}}}{3}\right)\left(z - \frac{e^{-j\frac{\pi}{4}}}{3}\right)}$$

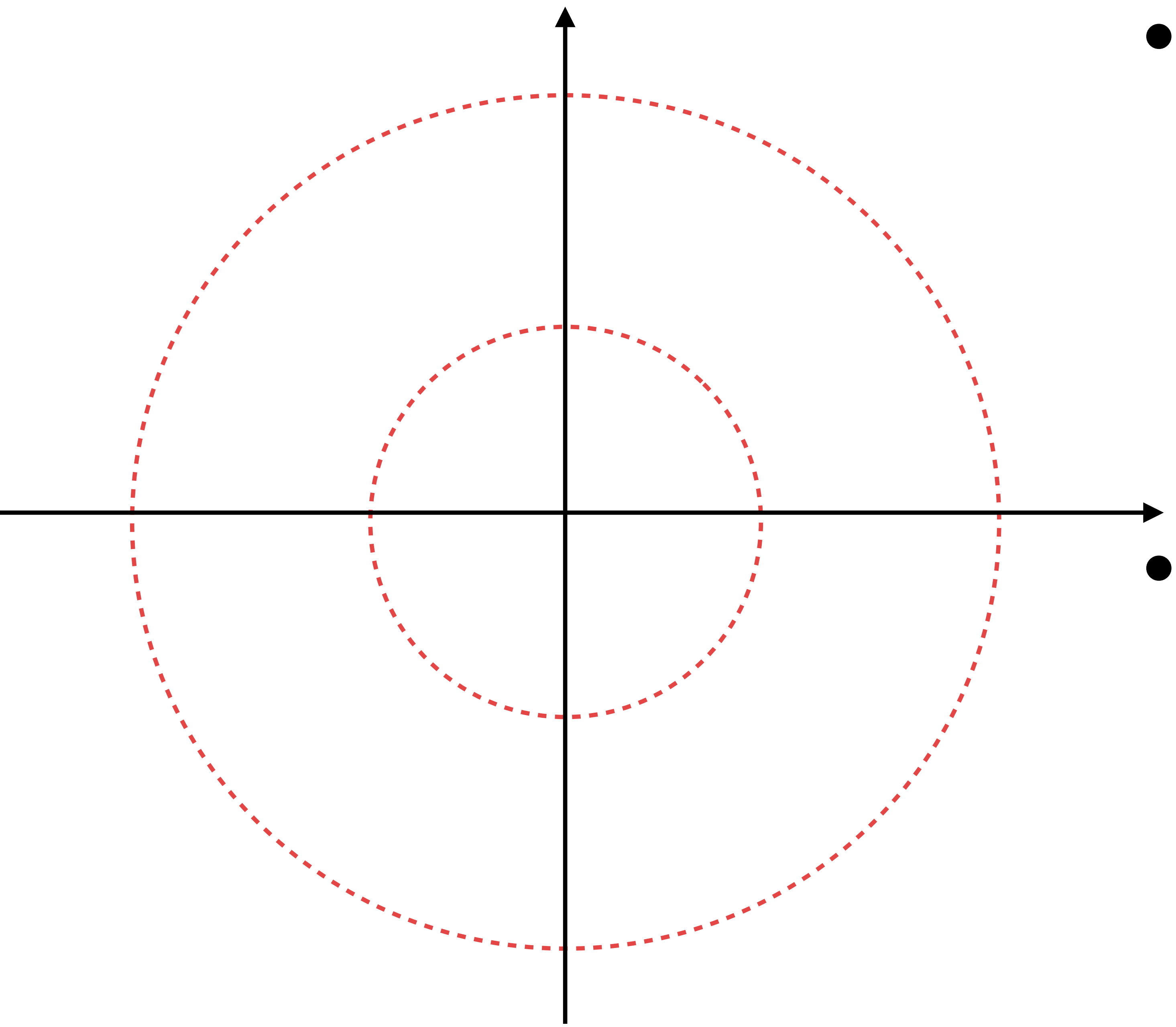
Poles: $z = \frac{e^{j\pi/4}}{3}$ and $z = \frac{e^{-j\pi/4}}{3}$ **Zeros**: $z = 0$ and $z = \infty$

The pole-zero plot and ROC



Properties

1) The ROC is always a ring centered around the origin.



- The inner circle may degenerate into the origin or disappear altogether
- The outer circle may degenerate into infinity or disappear altogether

Properties

2) The ROC can never contain any poles.

3) If $x[n]$ is of finite duration, the ROC is the entire plane except possibly at zero or infinity.

- <u>Example</u>: $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

ROC: All z

- <u>Example</u>: $x[n] = \delta[n] + \delta[n-1]$

$$X(z) = \sum_{n=0}^{1} z^{-n} = 1 + z^{-1} = \frac{z+1}{z}$$

ROC:
All z except
$z = 0$

Properties

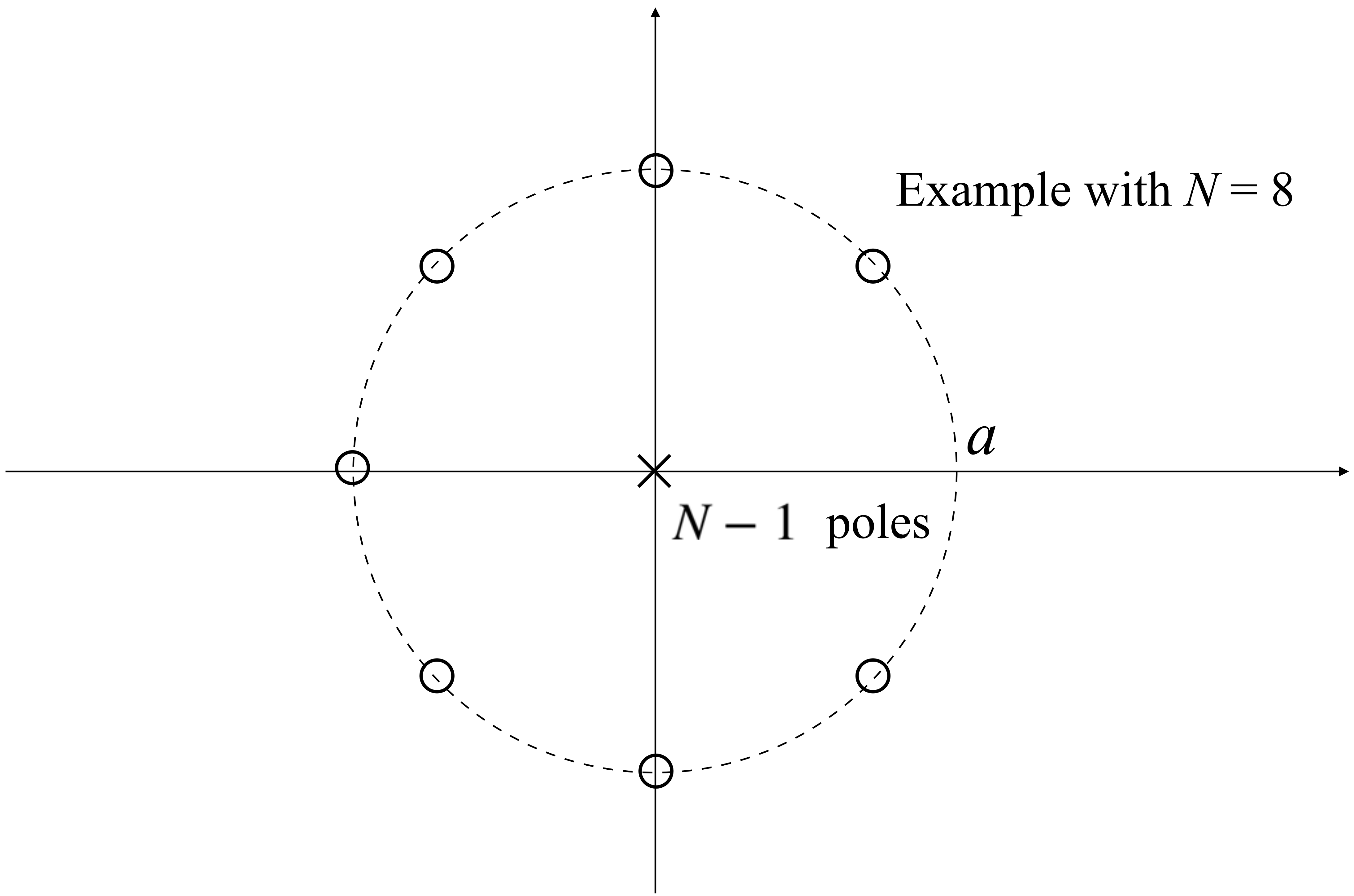
- <u>Example</u>: $x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}}$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)}$$

- **Poles**: $z = 0$ (repeated $N-1$ times)
- No pole at $z = a$ because it is cancelled with a zero.
- **Zeros**: Nth roots of a^N (except for the obvious root $z = a$)
- **ROC**: $|z| > 0$ (Yes, even $z = a$ is included)

Properties



Properties

4) If $x[n]$ is right-sided, then the ROC will be of the form $|z| > r_0$, possibly excluding infinity.

- Proof: If $x[n] = 0$ for $n < M$ for some M, and if $|z| = r$ is included in the ROC, then

$$\sum_{n=M}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

converges.

But then, so does $\sum_{n=M}^{\infty} x[n] R^{-n} e^{-j\omega n}$ for any $R > r$

Properties

- For right-sided sequences, $z = \infty$ will be excluded from the ROC if $x[n]$ is not causal.
  - Causal: $x[n] = 0$ for all $n < 0$
  - If $x[n]$ is not causal, then

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

will contain a positive power of z, which diverges when $z \to \infty$.

Properties

5) Similarly, if $x[n]$ is left-sided, then the ROC will be of the form $|z| < r_0$, possibly excluding zero.

- $z = 0$ will be excluded from the ROC if $x[n]$ is not anti-causal.
  - Anti-causal: $x[n] = 0$ for all $n > 0$
  - If $x[n]$ is not anti-causal, then $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ will contain a negative power of z, which diverges when $z \to 0$.

Properties

6) Finally, if $x[n]$ is two-sided, then the ROC will be of the form $r_0 < |z| < R_0$.

- Example: $x[n] = a^{|n|}$ with some $|a| < 1$

$$X(z) = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=-\infty}^{-1} (a^{-1}z^{-1})^n$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=1}^{\infty} (az)^n = \frac{1}{1 - az^{-1}} + \frac{1}{1 - az} - 1$$

$|z| > |a|$ $\qquad$ $|z| < \dfrac{1}{|a|}$

ROC : $|a| < |z| < \dfrac{1}{|a|}$

- Example: $x[n] = a^{|n|}$ with some $|a| < 1$

$$
\begin{aligned}
X(z) &= \frac{1}{1 - az^{-1}} + \frac{1}{1 - az} - 1 \\
&= \frac{z}{z - a} - \frac{1/a}{z - 1/a} - 1 \\
&= \frac{z(z - 1/a) - (z - a)/a - (z - a)(z - 1/a)}{(z - a)(z - 1/a)} \\
&= \frac{\cancel{z^2} - z/a - \cancel{z/a} + \cancel{1} - \cancel{z^2} + az + \cancel{z/a} - \cancel{1}}{(z - a)(z - 1/a)} \\
&= \frac{z(a - 1/a)}{(z - a)(z - 1/a)}
\end{aligned}
$$

- **Poles**: $z = a$ and $z = 1/a$
- **Zeros**: $z = 0$ and $z = \infty$

Properties

7) Linearity:

$$x_1[n] \longrightarrow X_1(z) \quad \text{with ROC} = \mathcal{R}_1$$

$$x_2[n] \longrightarrow X_2(z) \quad \text{with ROC} = \mathcal{R}_2$$

implies

$$ax_1[n] + bx_2[n] \longrightarrow aX_1(z) + bX_2(z)$$

with ROC $\supset (\mathcal{R}_1 \cap \mathcal{R}_2)$

- The actual ROC might be larger than $(\mathcal{R}_1 \cap \mathcal{R}_2)$ because zeros may cancel poles.

Properties

- Example: Find the z-transform of $x_1[n] + x_2[n]$ if

$$x_1[n] = 0.5^n u[n] \qquad x_2[n] = -0.5^n u[n-1]$$

- Solution: We already know that

$$X_1(z) = \frac{1}{1 - 0.5z^{-1}} \quad \text{with ROC } |z| > 0.5$$

Also,

$$X_2(z) = -\sum_{n=-\infty}^{\infty} 0.5^n u[n-1] z^{-n} = -\sum_{n=1}^{\infty} (0.5z^{-1})^n$$

$$= -\left(\frac{1}{1 - 0.5z^{-1}} - 1\right) = \frac{-0.5z^{-1}}{1 - 0.5z^{-1}} \quad \text{with ROC } |z| > 0.5$$

- Example: Find the z-transform of $x_1[n] + x_2[n]$ if

$$x_1[n] = 0.5^n u[n] \qquad x_2[n] = -0.5^n u[n-1]$$

$$X_1(z) = \frac{1}{1-0.5z^{-1}} \quad \text{with ROC } |z| > 0.5$$

$$X_2(z) = \frac{-0.5z^{-1}}{1-0.5z^{-1}} \quad \text{with ROC } |z| > 0.5$$

Therefore,

$$X_1(z) + X_2(z) = \frac{1-0.5z^{-1}}{1-0.5z^{-1}} = 1 \quad \text{with ROC } = \text{ all } z$$

Properties

8) Time shifting:

$$x_1[n] \longrightarrow X_1(z) \quad \text{with ROC} = \mathcal{R}_1$$

implies

$$x_1[n-n_0] \longrightarrow X_1(z)z^{-n_0} \quad \text{with ROC } = \mathcal{R}_1$$

(exclusion/inclusion of $z = 0$ or $z = \infty$ possible)

9) Time reversal:

$$x_1[n] \longrightarrow X_1(z) \quad \text{with ROC} = \mathcal{R}_1$$

implies

$$x_1[-n] \longrightarrow X_1(z^{-1}) \quad \text{with ROC } = 1/\mathcal{R}_1$$

Properties

10) Scaling in the z-domain:

$$x_1[n] \longrightarrow X_1(z) \quad \text{with ROC} = \mathcal{R}_1$$

implies

$$z_0^n x_1[n] \longrightarrow X_1(z/z_0) \quad \text{with ROC } = |z_0|\mathcal{R}_1$$

- <u>Special case</u>: $z_0 = e^{j\omega_0}$

$$e^{j\omega_0 n} x_1[n] \longrightarrow X_1(ze^{-j\omega_0}) \quad \text{with ROC } = \mathcal{R}_1$$

- This is nothing but a counter-clockwise rotation in the z-plane.

Properties

11) Convolution:

$$x_1[n] \longrightarrow X_1(z) \quad \text{with ROC} = \mathcal{R}_1$$

$$x_2[n] \longrightarrow X_2(z) \quad \text{with ROC} = \mathcal{R}_2$$

implies

$$x_1[n] \star x_2[n] \longrightarrow X_1(z)X_2(z)$$

with ROC $\supset (\mathcal{R}_1 \cap \mathcal{R}_2)$

- The actual ROC might be larger than $(\mathcal{R}_1 \cap \mathcal{R}_2)$ because zeros may cancel poles.

Inverting the z-transform

- Since for any radius r inside the ROC we have

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n} = \mathrm{DTFT}\left\{x[n]r^{-n}\right\}$$

we can write

$$x[n]r^{-n} = \mathrm{DTFT}^{-1}\left\{X(re^{j\omega})\right\}$$

or equivalently

$$x[n] = r^n \frac{1}{2\pi}\int_{<2\pi>} X(re^{j\omega})e^{j\omega n}\,d\omega$$

Partial fraction expansion

- In case this direct formula is not useful, we can use the partial fraction expansion technique.
- Example: Find $x[n]$ if

$$X(z) = \frac{z^2}{(z-1)(z-2)} \quad \text{with ROC: } |z| > 2$$

- Solution:

$$X(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-2z^{-1}}$$

$$= \frac{A(1-2z^{-1}) + B(1-z^{-1})}{(1-z^{-1})(1-2z^{-1})}$$

$$A + B = 1$$

$$2A + B = 0$$

$$A = -1, \;\; B = 2$$

Partial fraction expansion

- <u>Example</u>: Find $x[n]$ if

$$X(z) = \frac{z^2}{(z-1)(z-2)} \quad \text{with ROC: } |z| > 2$$

- <u>Solution</u>:

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-2z^{-1}} \qquad A = -1, \;\; B = 2$$

Therefore,

$$x[n] = -1 \cdot 1^n u[n] + 2 \cdot 2^n u[n] \;\; = (2^{n+1} - 1)u[n]$$

Partial fraction expansion

- Example: Find $x[n]$ if

$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})} \quad \text{with ROC: } \frac{1}{4} < |z| < \frac{1}{3}$$

- Solution:

$$X(z) = \frac{A}{1 - \frac{1}{4} z^{-1}} + \frac{B}{1 - \frac{1}{3} z^{-1}}$$

$$= \frac{A(1 - \frac{1}{3} z^{-1}) + B(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}$$

$$A + B = 3$$

$$\frac{A}{3} + \frac{B}{4} = \frac{5}{6}$$

$$A = 1, \quad B = 2$$

Partial fraction expansion

- Example: Find $x[n]$ if

$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})} \quad \text{with ROC: } \frac{1}{4} < |z| < \frac{1}{3}$$

- Solution:

$$X(z) = \frac{A}{1 - \frac{1}{4} z^{-1}} + \frac{B}{1 - \frac{1}{3} z^{-1}} \qquad A = 1, \ \ B = 2$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

Power series expansion

- Another technique is power series expansion.
- Example: Find $x[n]$ if

$$X(z) = 4z^2 + 2 + 3z^{-1} \quad \text{with ROC: } 0 < |z| < \infty$$

- Solution: Expanding the z-transform,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \ldots + \underset{\downarrow\, 4}{x[-2]}z^2 + \underset{\downarrow\, 0}{x[-1]}z + \underset{\downarrow\, 2}{x[0]} + \underset{\downarrow\, 3}{x[1]}z^{-1} + \underset{\downarrow\, 0}{x[2]}z^{-2} + \ldots$$

Power series expansion

- We may need to do Taylor series expansion
- Example: Find $x[n]$ if

$$X(z) = e^{z} \quad \text{with ROC: } |z| < \infty$$

- Solution: The Taylor series expansion gives

$$X(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \left.\frac{d^k e^z}{dz^k}\right|_{z=0} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{n=-\infty}^{0} \frac{z^{-n}}{(-n)!}$$

- Comparison with $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ yields

$$x[n] = \frac{1}{(-n)!} u[-n]$$

Power series expansion

- We may need to do long division.
- Example: Find $x[n]$ if

$$X(z) = \frac{1}{1 - 2z^{-1} + z^{-2}} \quad \text{with ROC: } |z| > 1$$

- Solution:

$$\begin{array}{r l}
 & 1 + 2z^{-1} \\ \hline
1 - 2z^{-1} + z^{-2} \,\big) & 1 \\
 & -\ (1 - 2z^{-1} + z^{-2}) \\ \hline
 & 2z^{-1} - z^{-2} \\
 & -\ (2z^{-1} - 4z^{-2} + 2z^{-3}) \\ \hline
 & 3z^{-2} - 2z^{-3}
\end{array}$$

- <u>Example</u>: Find $x[n]$ if

$$X(z) = \frac{1}{1 - 2z^{-1} + z^{-2}} \quad \text{with ROC: } |z| > 1$$

- <u>Solution</u>:

$$
\begin{array}{r l}
 & 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \cdots \\
1 - 2z^{-1} + z^{-2} \,\big) & 1 \\
 & -\; 1 - 2z^{-1} + z^{-2} \\ \hline
 & 2z^{-1} - z^{-2} \\
 & -\; 2z^{-1} - 4z^{-2} + 2z^{-3} \\ \hline
 & 3z^{-2} - 2z^{-3} \\
 & -\; 3z^{-2} - 6z^{-3} + 3z^{-4} \\ \hline
 & 4z^{-3} - 3z^{-4}
\end{array}
$$

$$x[n] = (n + 1)u[n]$$

Power series expansion

- <u>Example</u>: Find $x[n]$ if

$$X(z) = \frac{1}{1 - 2z^{-1} + z^{-2}} \quad \text{with ROC: } |z| < 1$$

- <u>Solution</u>: Note that we need a left-sided sequence now. Therefore, we perform

$$
\begin{array}{r l}
 & z^2 + 2z^3 \\
\cline{2-2}
z^{-2} - 2z^{-1} + 1 \,\big) & 1 \\
 & -\; 1 - 2z + z^2 \\
\cline{2-2}
 & \qquad 2z - z^2 \\
 & \qquad -\; 2z - 4z^2 + 2z^3 \\
\cline{2-2}
 & \qquad\qquad 3z^2 - 2z^3
\end{array}
$$

- Example: Find $x[n]$ if

$$X(z) = \frac{1}{1 - 2z^{-1} + z^{-2}} \quad \text{with ROC: } |z| < 1$$

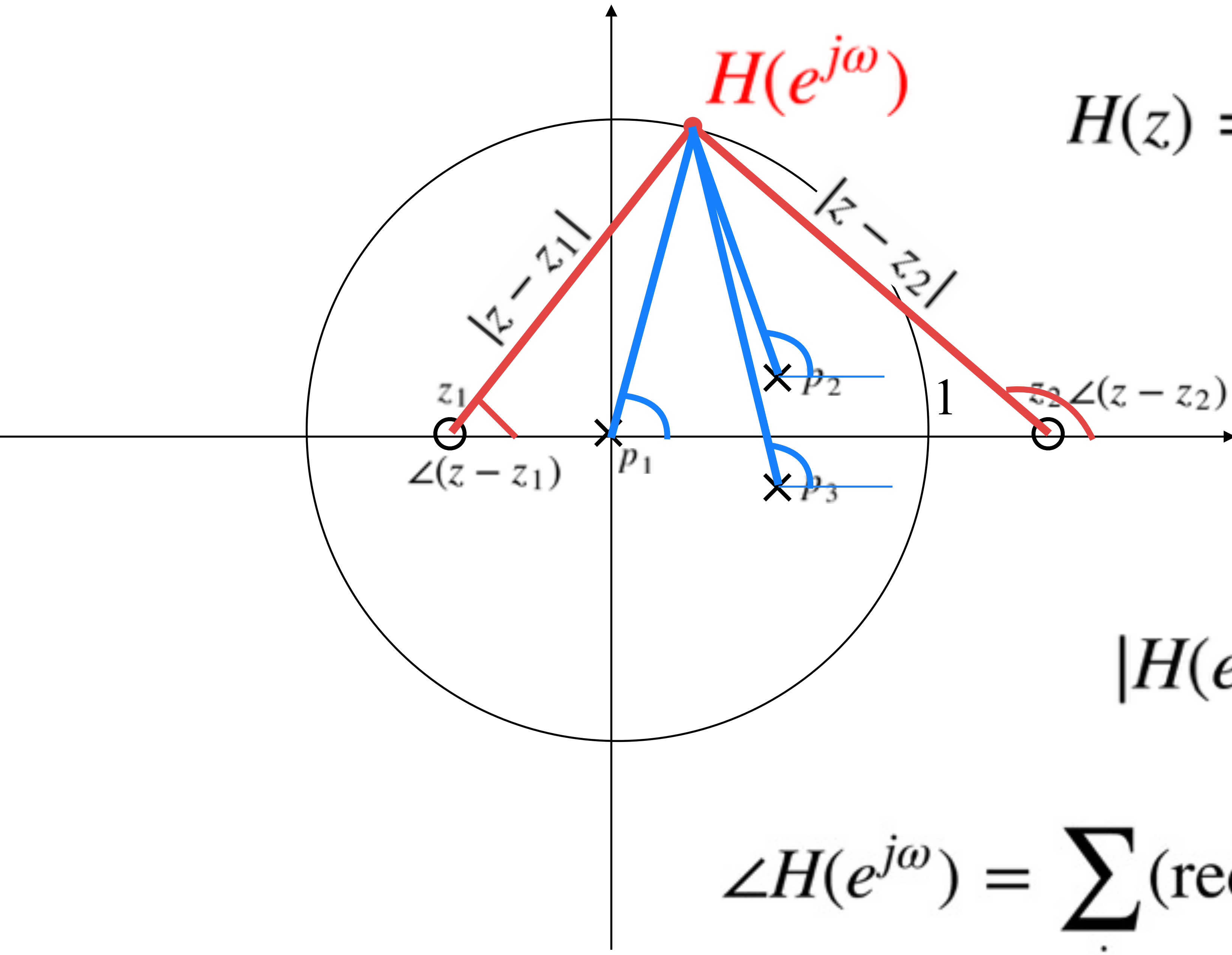
- Solution:

$$
\begin{array}{r|l}
 & z^2 + 2z^3 + 3z^4 + 4z^5 + \cdots \\
\hline
z^{-2} - 2z^{-1} + 1 & 1 \\
 & -\ 1 - 2z + z^2 \\
\cline{2-2}
 & \quad 2z - z^2 \\
 & \quad -\ 2z - 4z^2 + 2z^3 \\
\cline{2-2}
 & \qquad 3z^2 - 2z^3 \\
 & \qquad -\ 3z^2 - 6z^3 + 3z^4 \\
\cline{2-2}
 & \qquad\qquad 4z^3 - 3z^4
\end{array}
$$

$$x[n] = (-n - 1)u[-n - 2]$$

DTFT from poles and zeros

- The pole-zero plot is useful in understanding the frequency behavior of the signal (or filter).



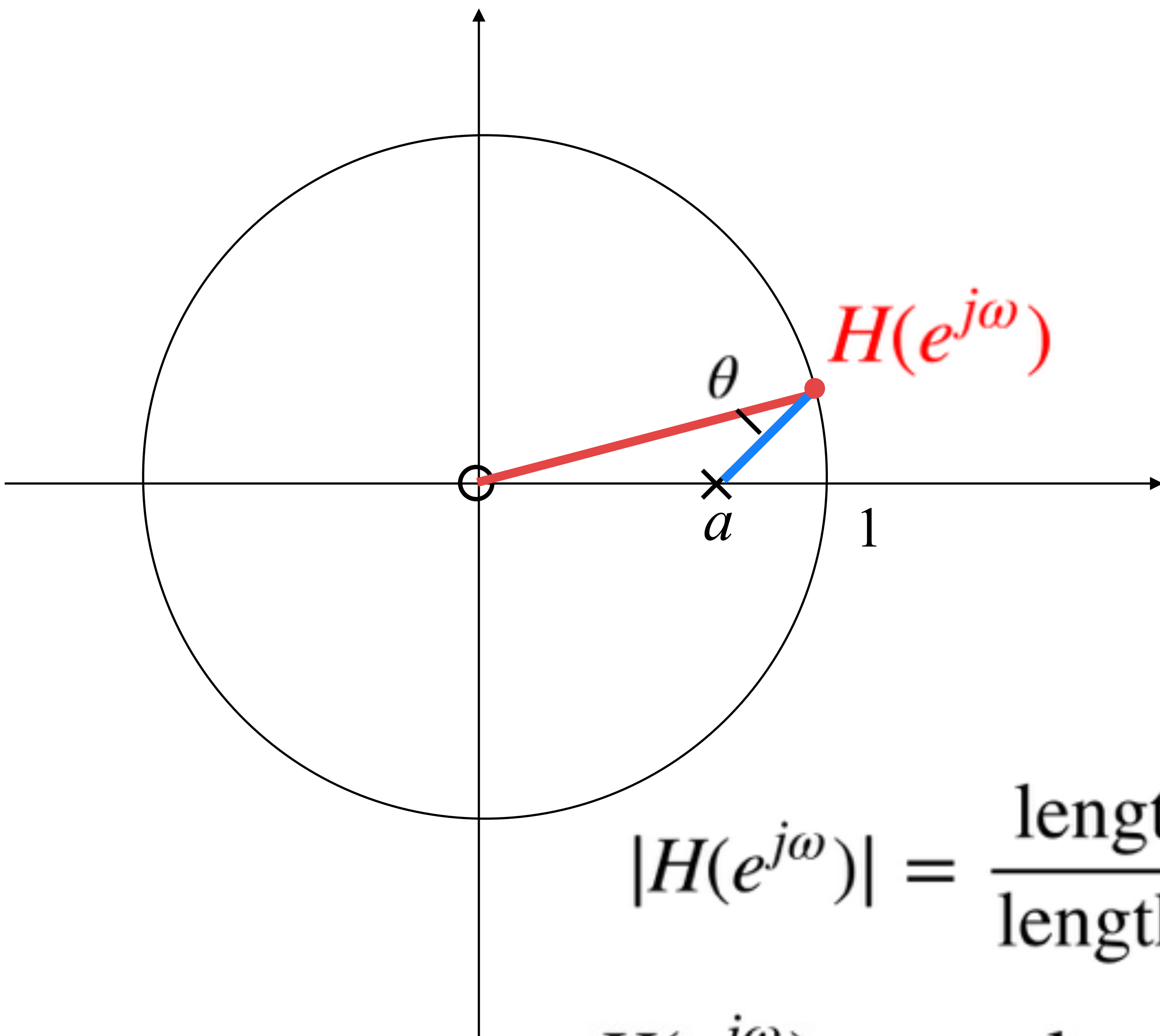
$$H(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)(z - p_3)}$$

$$|H(e^{j\omega})| = \frac{\prod_i (\text{red length})_i}{\prod_j (\text{blue length})_j}$$

$$\angle H(e^{j\omega}) = \sum_i (\text{red angle})_i - \sum_i (\text{blue angle})_j$$

DTFT from poles and zeros

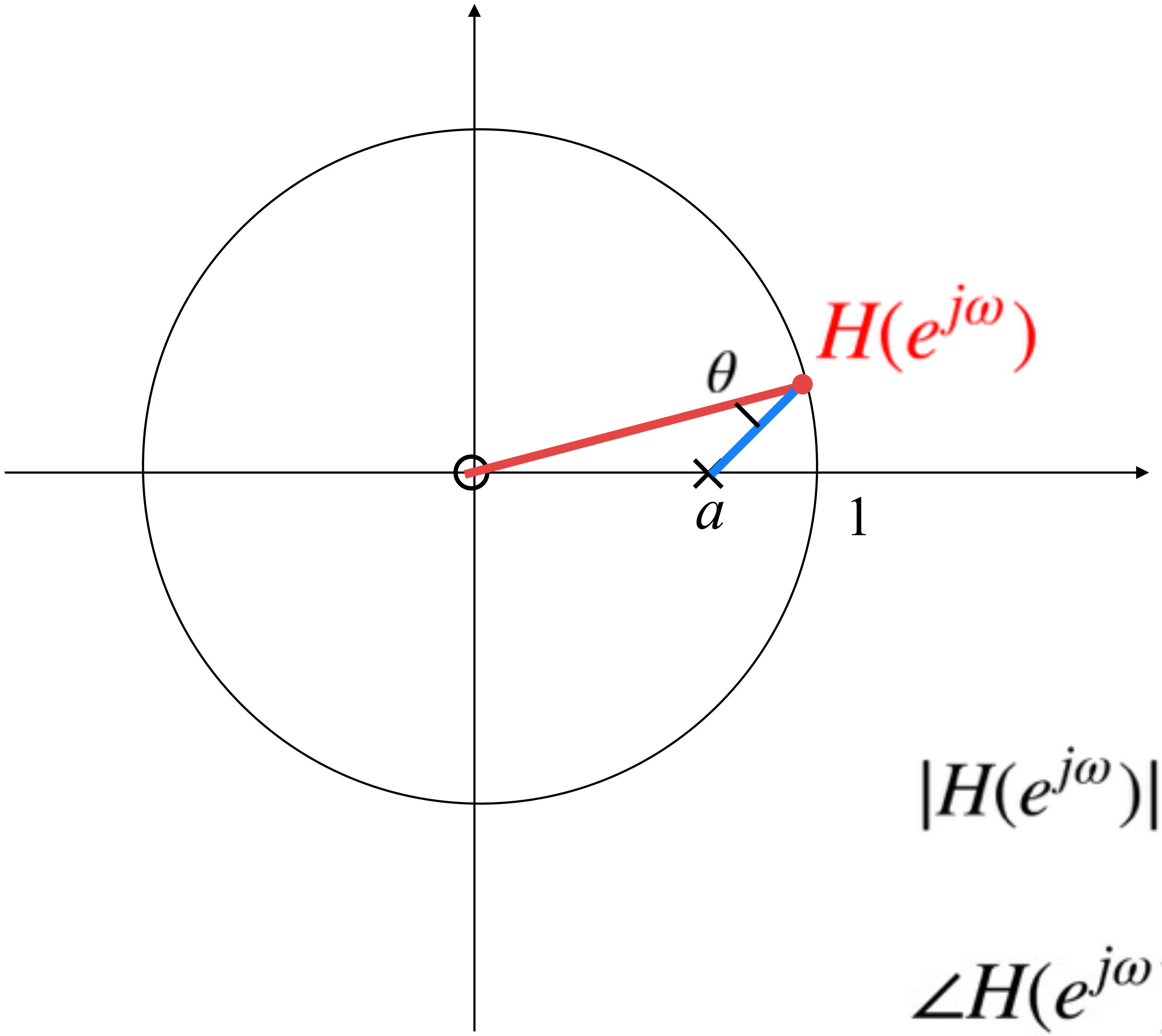
- <u>Example</u>: A first-order filter.



$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

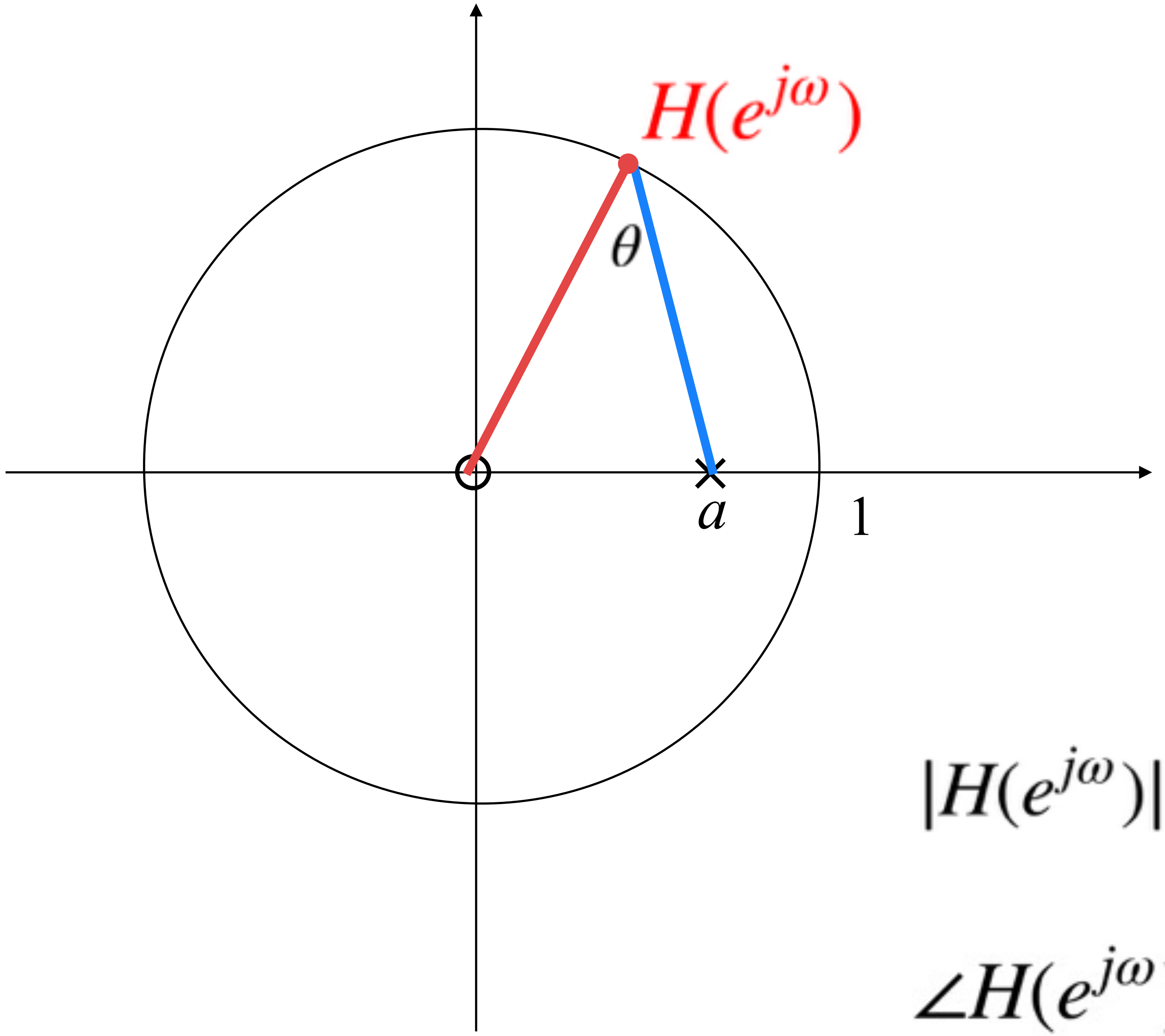
$$|H(e^{j\omega})| = \frac{\text{length of red}}{\text{length of blue}} = \frac{1}{\text{length of blue}}$$

$$\angle H(e^{j\omega}) = \text{angle of red} - \text{angle of blue} = -\theta$$



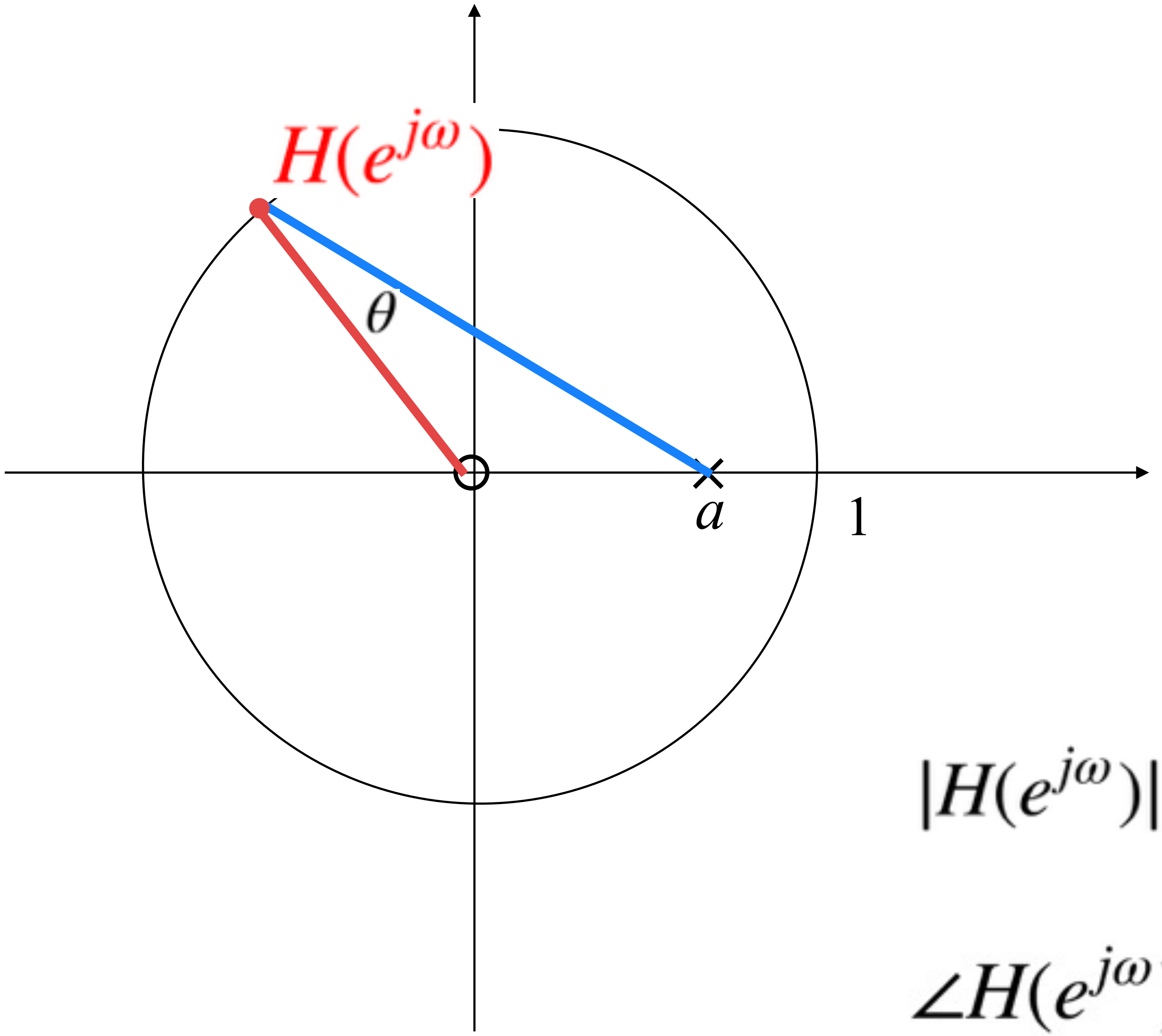
$$|H(e^{j\omega})| = \frac{1}{\text{length of blue}}$$ **LARGE**

$$\angle H(e^{j\omega}) = -\theta$$ **SMALL**



$$|H(e^{j\omega})| = \frac{1}{\text{length of blue}}$$ **MEDIUM**

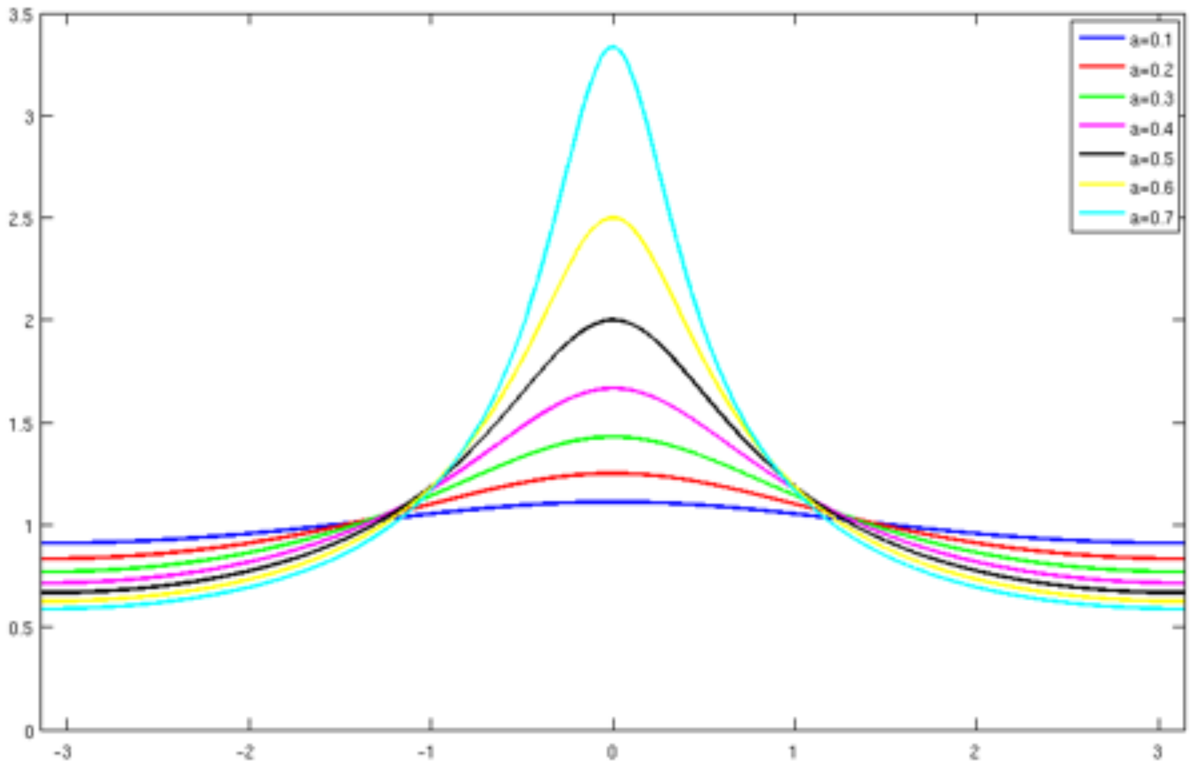
$$\angle H(e^{j\omega}) = -\theta$$ **MEDIUM**



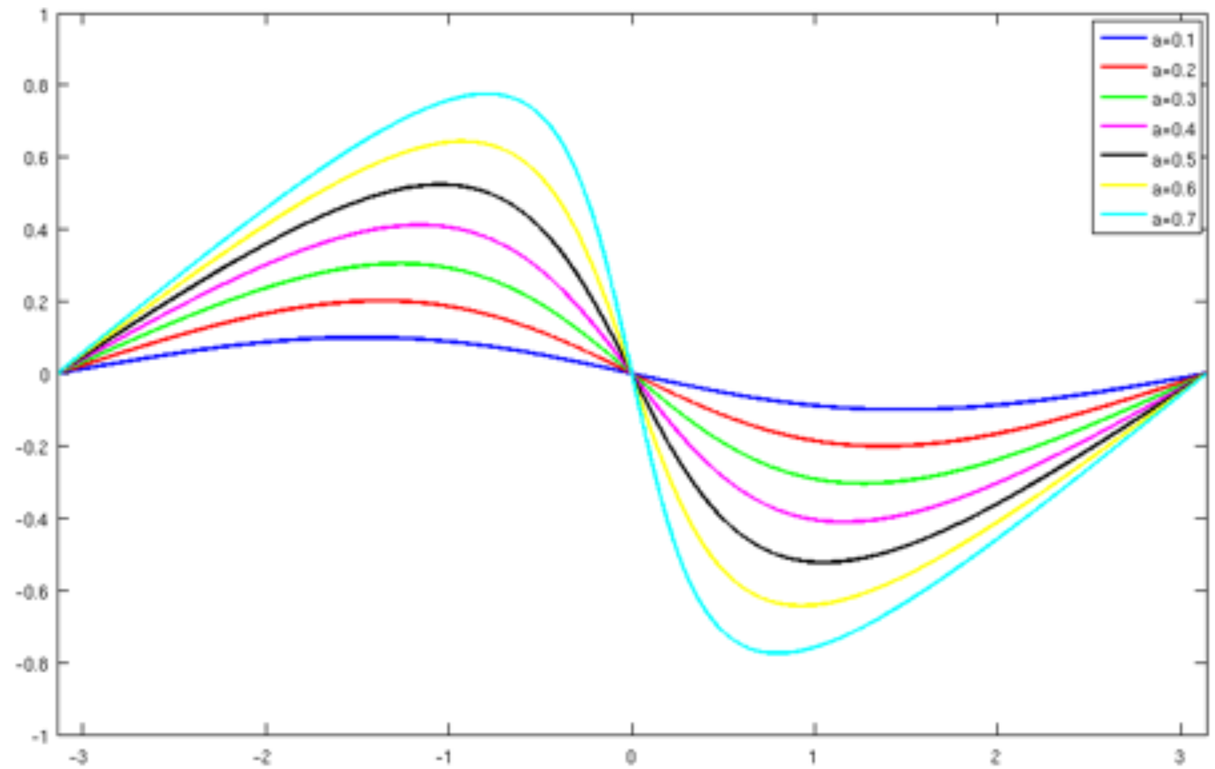
$$|H(e^{j\omega})| = \frac{1}{\text{length of blue}}$$ **SMALL**

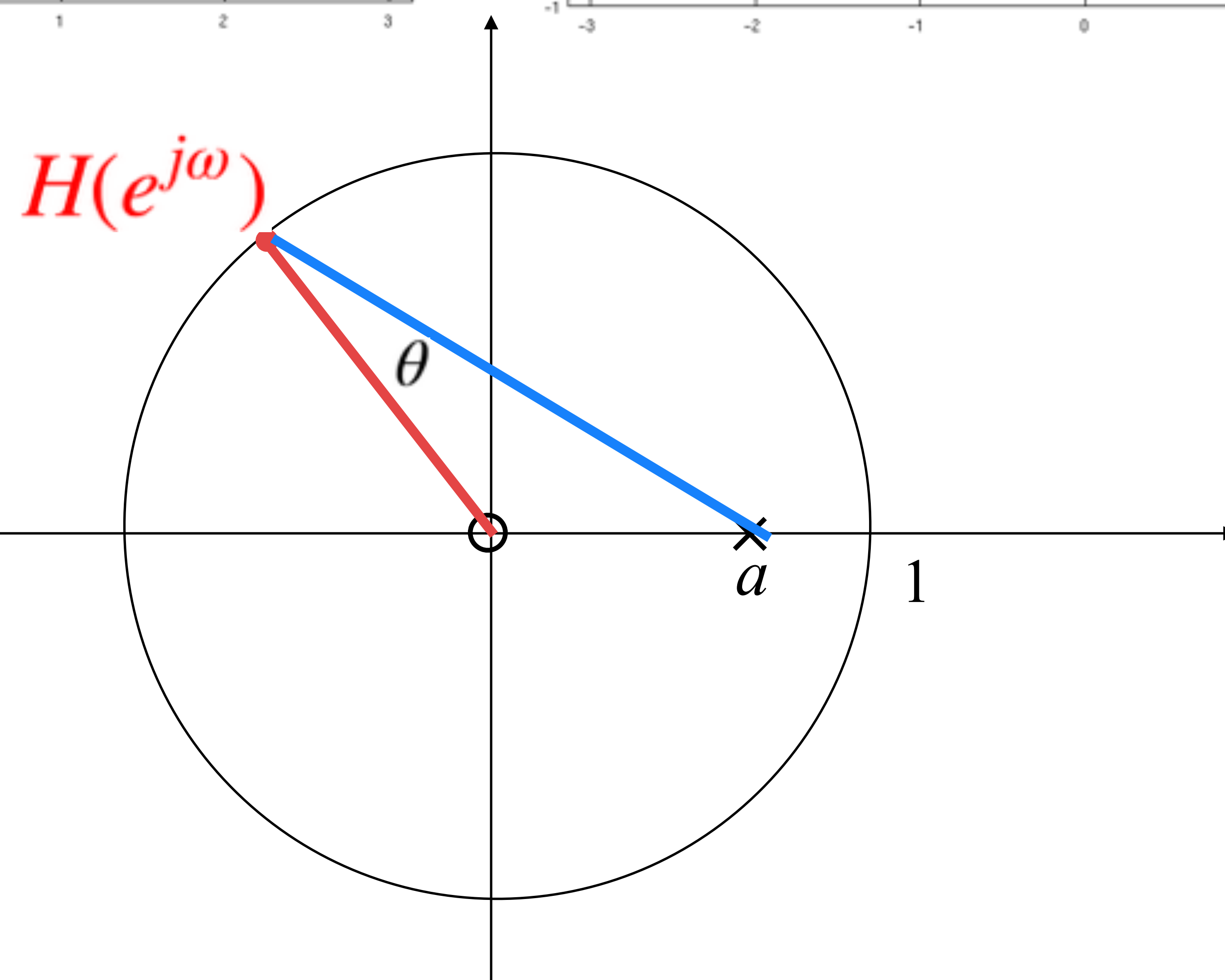
$$\angle H(e^{j\omega}) = -\theta$$ **SMALL**

$|H(e^{j\omega})|$



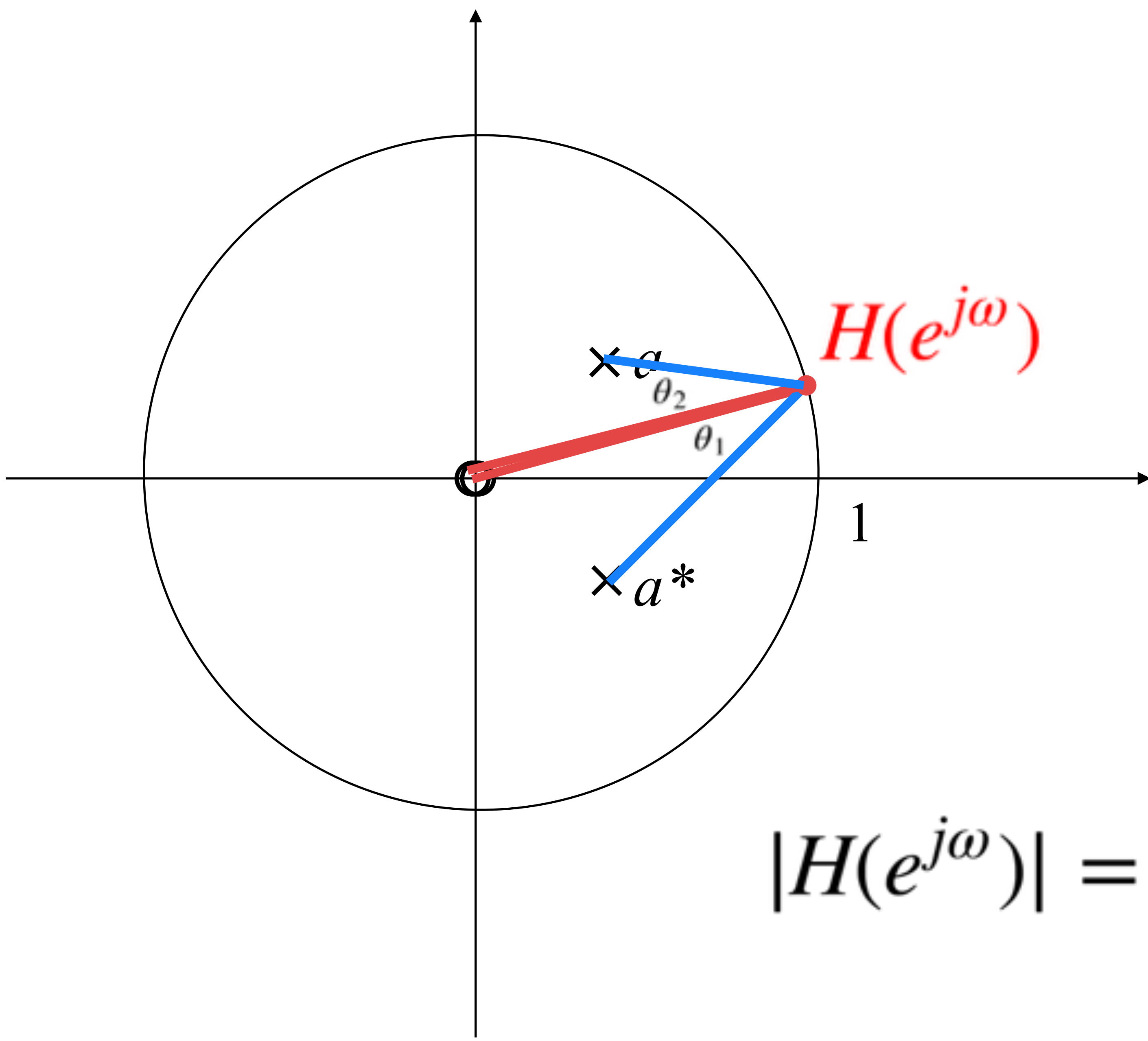
$\angle H(e^{j\omega})$





DTFT from poles and zeros

- Example: A second-order filter.



$$H(z) = \frac{z^2}{(z-a)(z-a^*)}$$

$$|H(e^{j\omega})| = \frac{1}{\text{blue}_1 \times \text{blue}_2}$$

$$\angle H(e^{j\omega}) = \theta_2 - \theta_1$$

System properties re-revisited

- We can tell from the z-transform and its ROC whether the LTI system is causal, stable, and even invertible.
  - **Causality**: We already saw that the ROC has to be of the form $|z| > r$ and must include $z = \infty$.
  - **Stability**: Recall that if the system is stable,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- **Stability**: Recall that if the system is stable,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

But we also have

$$\left|\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |h[n]e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |h[n]|$$

Hence, stability implies the existence of DTFT

<u>Translation</u>: stability $\Longrightarrow$ ROC includes $|z| = 1$

It can be shown that $\Longleftarrow$ is also true.

System properties re-revisited

- Example: If $H(z) = \dfrac{z^2}{(z-0.5)(z-2)}$,

  there are three possible ROCs:

  - $|z| < 0.5$: Non-causal (in fact, anti-causal) and unstable.
  - $0.5 < |z| < 2$: Non-causal and stable.
  - $2 < |z|$: Causal and unstable.
  - So we can never have both causality and stability with this set of poles.

System properties re-revisited

- So, what does it take to have both causality and stability?
  - All poles must be inside the unit circle (including the hidden ones)
- Example: The following is causal and stable.

$$H(z) = \frac{z^2}{(z-0.5)(z-0.7)} \quad \text{with ROC: } |z| > 0.7$$

- Example: The following is stable but not causal.

$$H(z) = \frac{z^3}{(z-0.5)(z-0.7)} \quad \text{with ROC: } 0.7 < |z| < \infty$$

System properties re-revisited

- **Invertibility**: At first, it looks like every LTI system with non-empty ROC is invertible.
  - The inverse is simply $G(z) = \dfrac{1}{H(z)}$
  - However, for practicality, we need both the system and its inverse to be causal and stable.
  - This implies that not only all poles, but also all zeros must be inside the unit circle (including the hidden ones).

Difference equations strike again!

- We can actually solve a difference equation using the z-transform.

$$\sum_{k=0}^{K} \alpha_k y[n-k] = \sum_{m=0}^{M} \beta_m x[n-m]$$

- Taking the z-transform of both sides,

$$\sum_{k=0}^{K} \alpha_k Y(z) z^{-k} = \sum_{m=0}^{M} \beta_m X(z) z^{-m}$$

or equivalently,

$$Y(z) = X(z) \boxed{\frac{\sum_{m=0}^{M} \beta_m z^{-m}}{\sum_{k=0}^{K} \alpha_k z^{-k}}} \rightarrow H(z)$$

Difference equations strike again!

- What about the ROC?
- Choose the one that will give you a causal $H(z)$
- Example: Find $y[n]$ if $x[n] = u[n]$ and

$$y[n] - y[n-1] - 2y[n-2] = x[n]$$

- Recall that we had found

$$y[n] = \left(\frac{1}{6}(-1)^n + \frac{4}{3}2^n - 0.5\right)u[n]$$

- Solution: Start with going to the z-domain:

$$Y(z) - Y(z)z^{-1} - 2Y(z)z^{-2} = X(z)$$

- Example: Find $y[n]$ if $x[n] = u[n]$ and

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- Solution: Start with going to the z-domain:

$$Y(z) - Y(z)z^{-1} - 2Y(z)z^{-2} = X(z)$$

or equivalently,

$$Y(z) = X(z) \cdot \frac{1}{1 - z^{-1} - 2z^{-2}}$$

ROC: $|z| > 2$

$$= \frac{1}{1 - z^{-1}} \cdot \frac{1}{(1 + z^{-1})(1 - 2z^{-1})}$$

- Example: Find $y[n]$ if $x[n] = u[n]$ and

$$y[n] - y[n-1] - 2y[n-2] = x[n]$$

- Recall that we had found

$$y[n] = \left(\frac{1}{6}(-1)^n + \frac{4}{3}2^n - 0.5\right)u[n]$$

$$\begin{aligned} Y(z) &= \frac{1}{1-z^{-1}} \cdot \frac{1}{(1+z^{-1})(1-2z^{-1})} \\ &= \frac{A}{1-z^{-1}} + \frac{B}{1+z^{-1}} + \frac{C}{1-2z^{-1}} \end{aligned}$$

Thus, we need to solve

$$A(1+z^{-1})(1-2z^{-1}) + B(1-z^{-1})(1-2z^{-1}) \\ +C(1-z^{-1})(1+z^{-1}) = 1$$

- <u>Example</u>: Find $y[n]$ if $x[n] = u[n]$ and

$$y[n] - y[n-1] - 2y[n-2] = x[n]$$

- Recall that we had found

$$y[n] = \left(\frac{1}{6}(-1)^n + \frac{4}{3}2^n - 0.5\right)u[n]$$

$$Y(z) = \frac{A}{1 - z^{-1}} + \frac{B}{1 + z^{-1}} + \frac{C}{1 - 2z^{-1}}$$

$$A(1 + z^{-1})(1 - 2z^{-1}) + B(1 - z^{-1})(1 - 2z^{-1})$$
$$+C(1 - z^{-1})(1 + z^{-1}) = 1$$

Substitute $z^{-1} = 1$ to get $-2A = 1 \Longrightarrow A = -0.5$

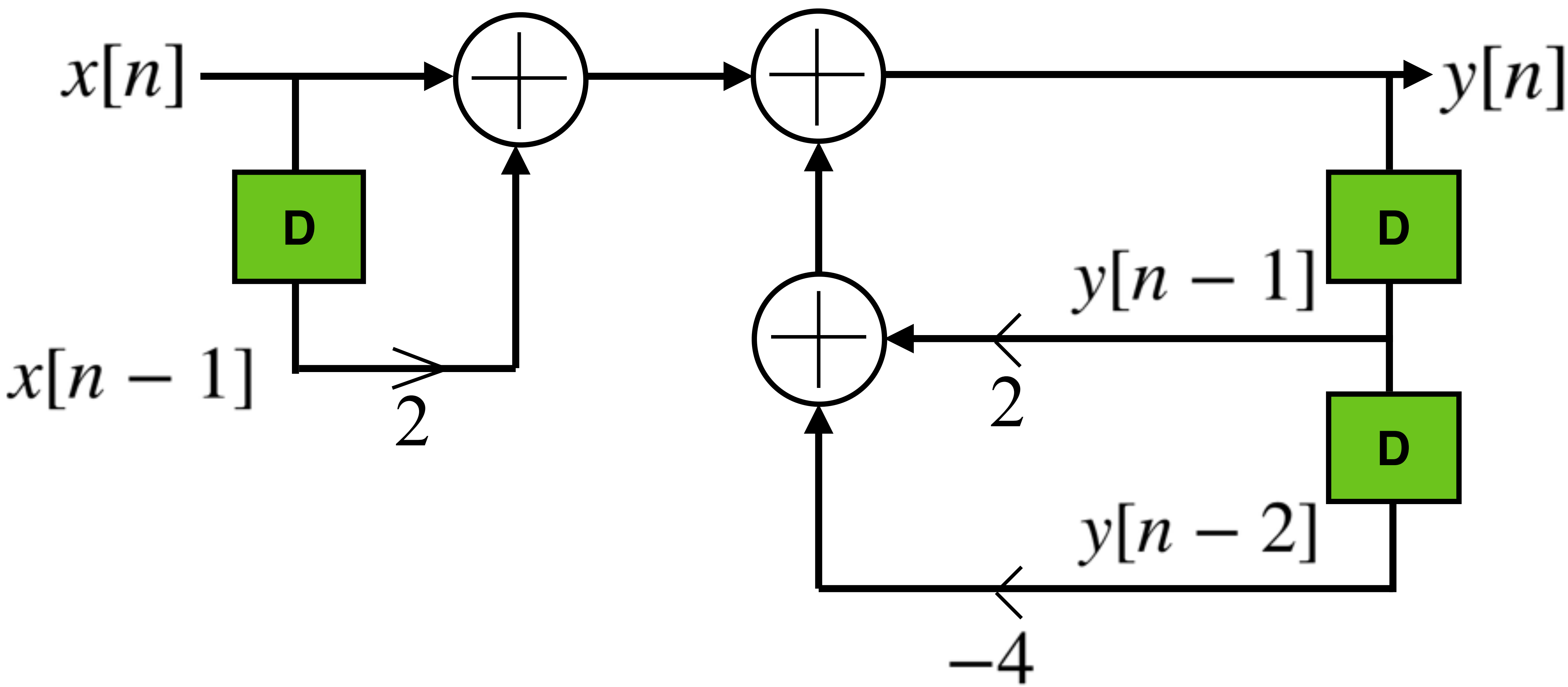
$z^{-1} = -1$ to get $6B = 1 \Longrightarrow B = 1/6$

$z^{-1} = 1/2$ to get $3C/4 = 1 \Longrightarrow C = 4/3$

Block diagrams

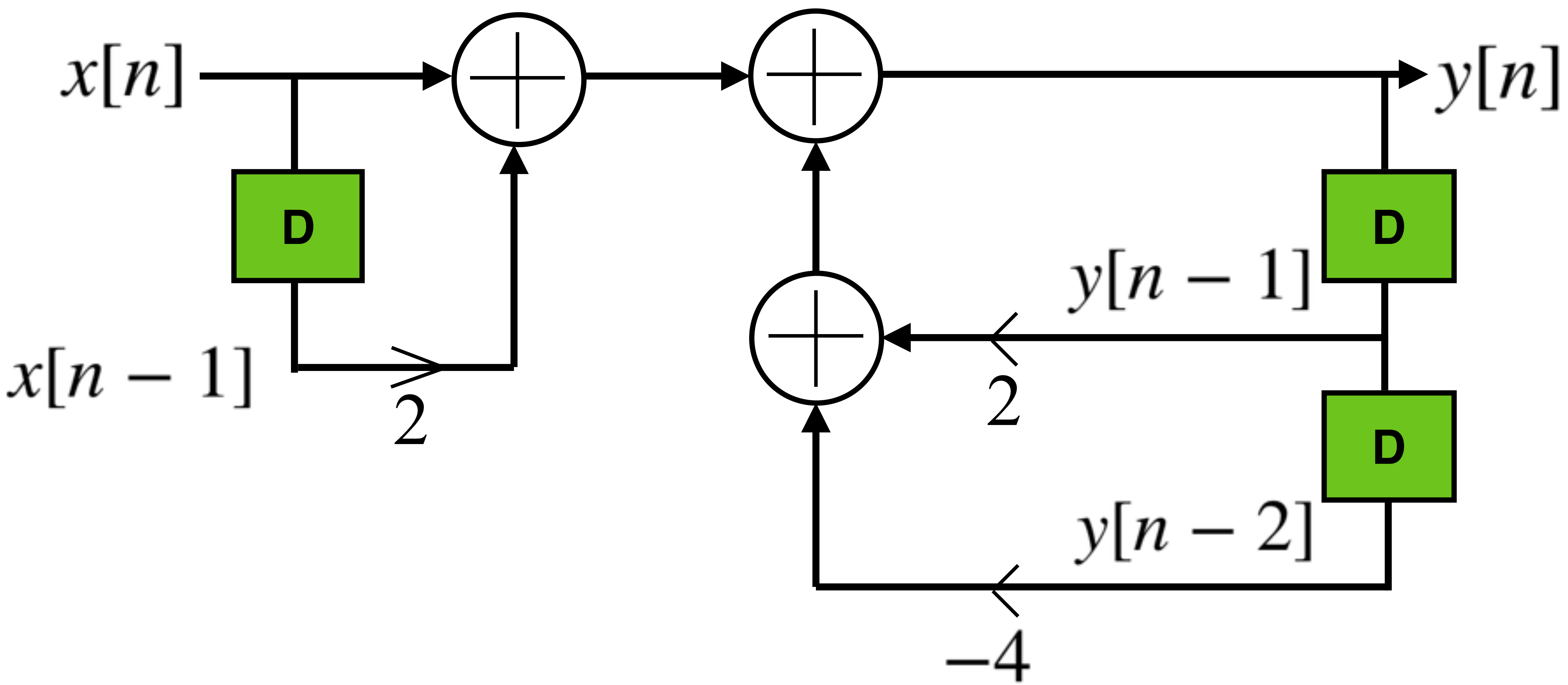
- Recall that we can convert difference equations to block diagrams easily:

$$y[n] - 2y[n-1] + 4y[n-2] = x[n] + 2x[n-1]$$



Block diagrams

- Re-think this in the z-domain:



Block diagrams

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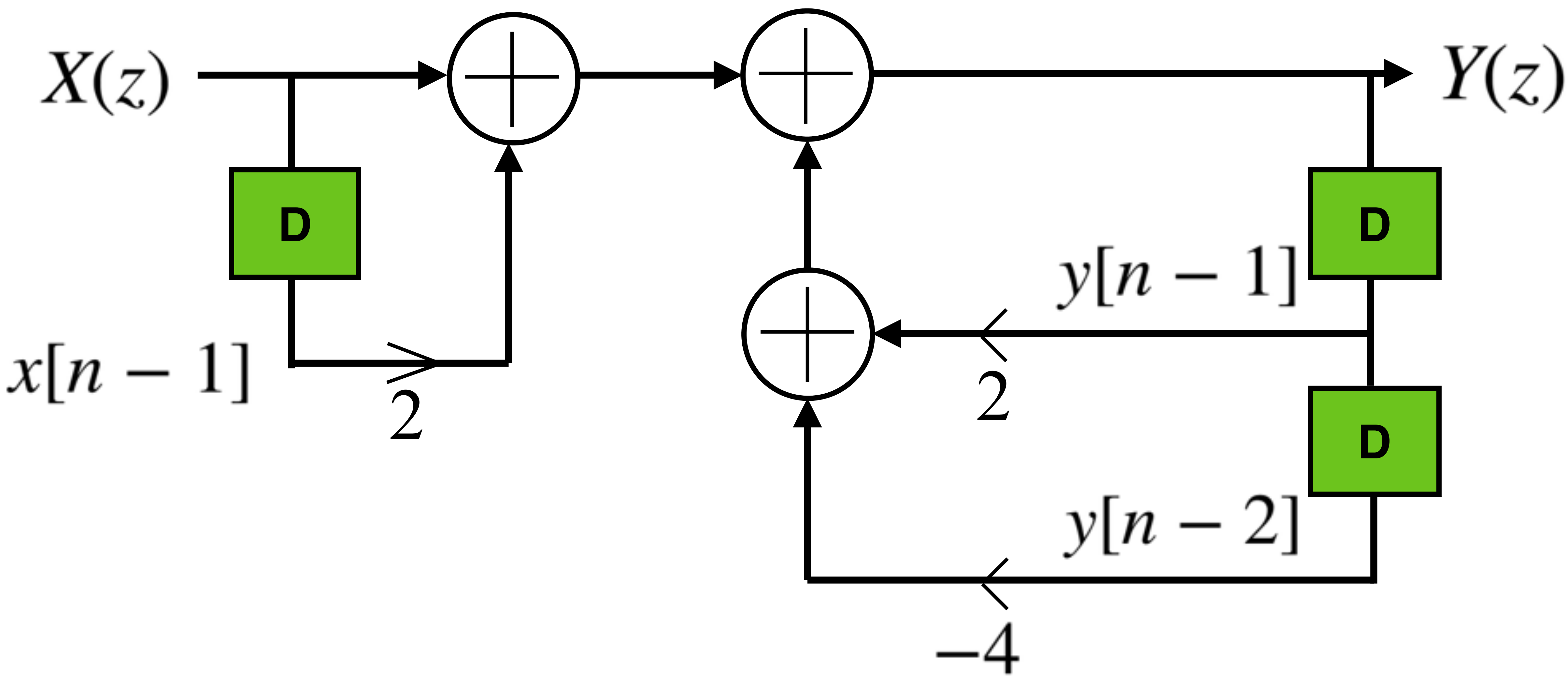

Block diagrams

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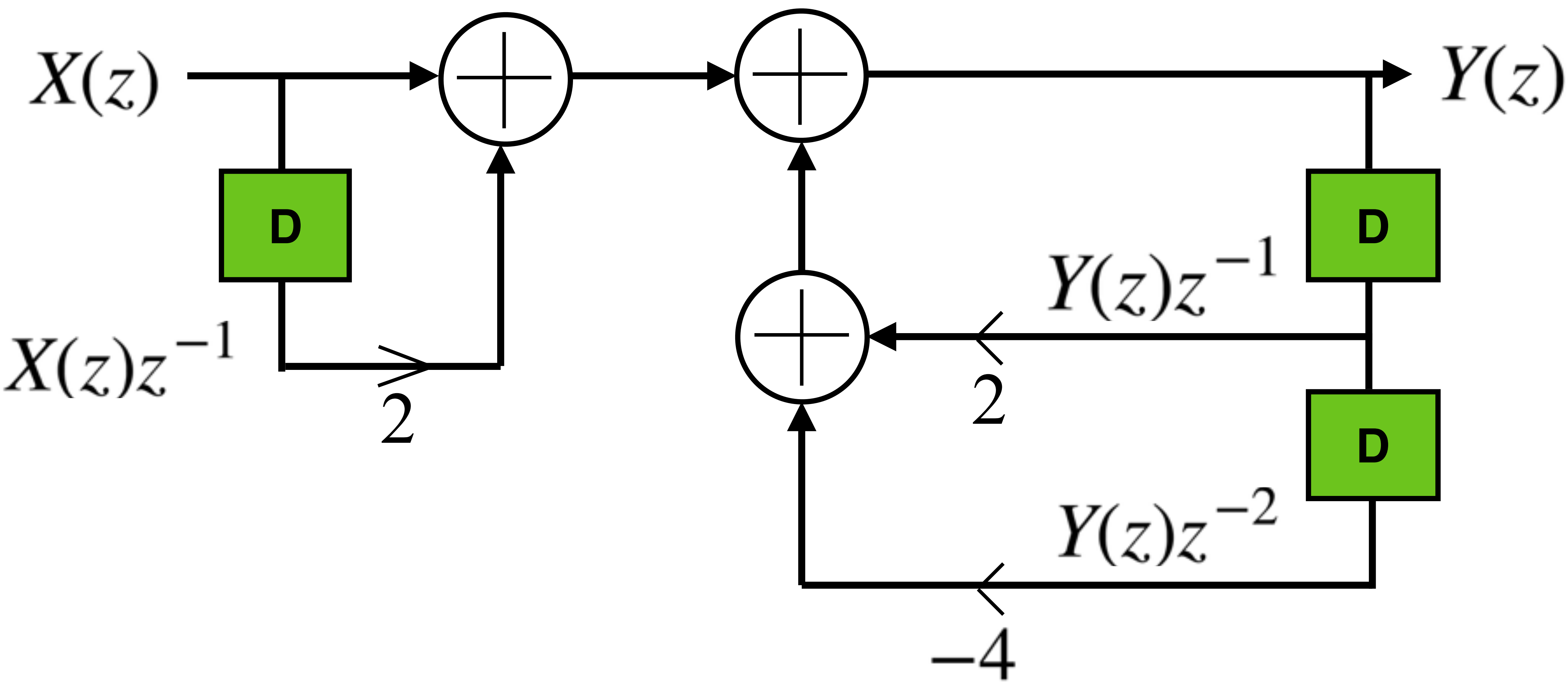

Block diagrams

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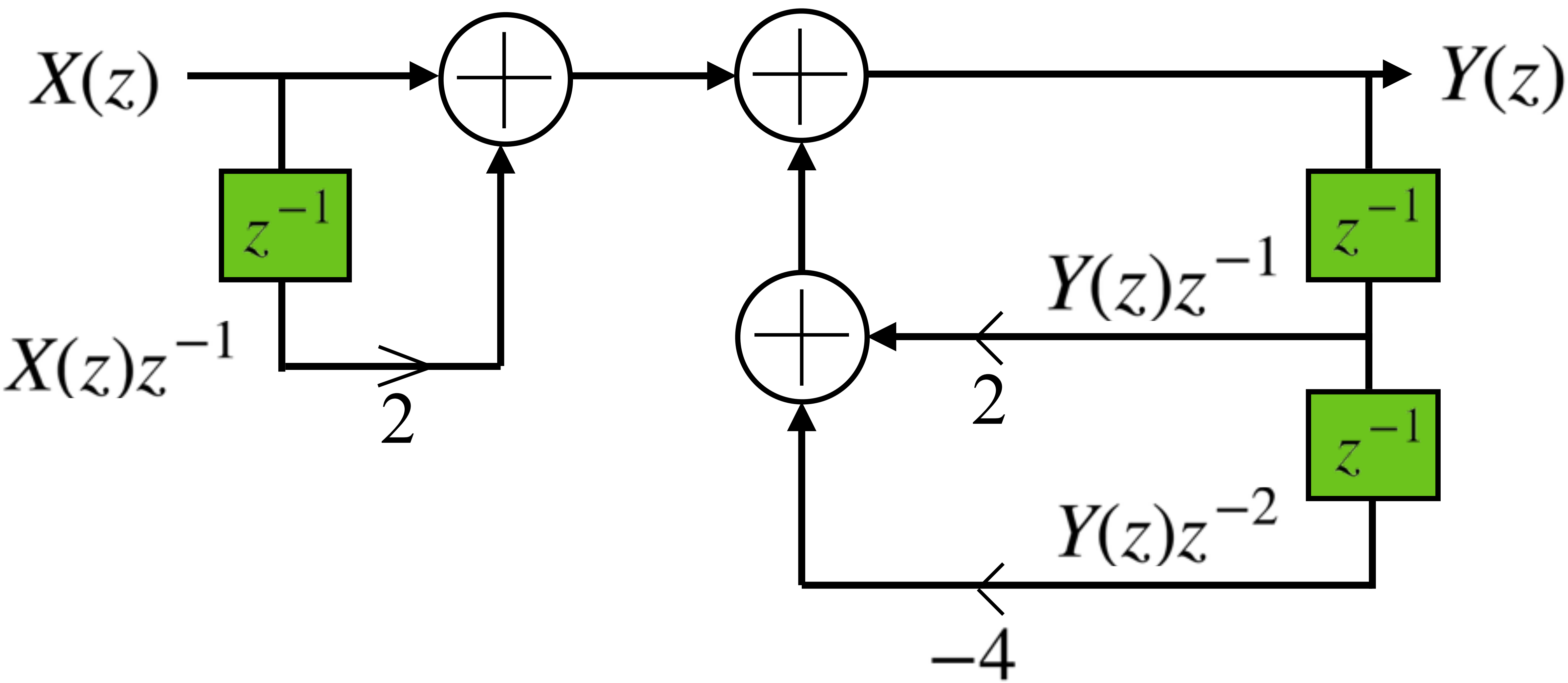
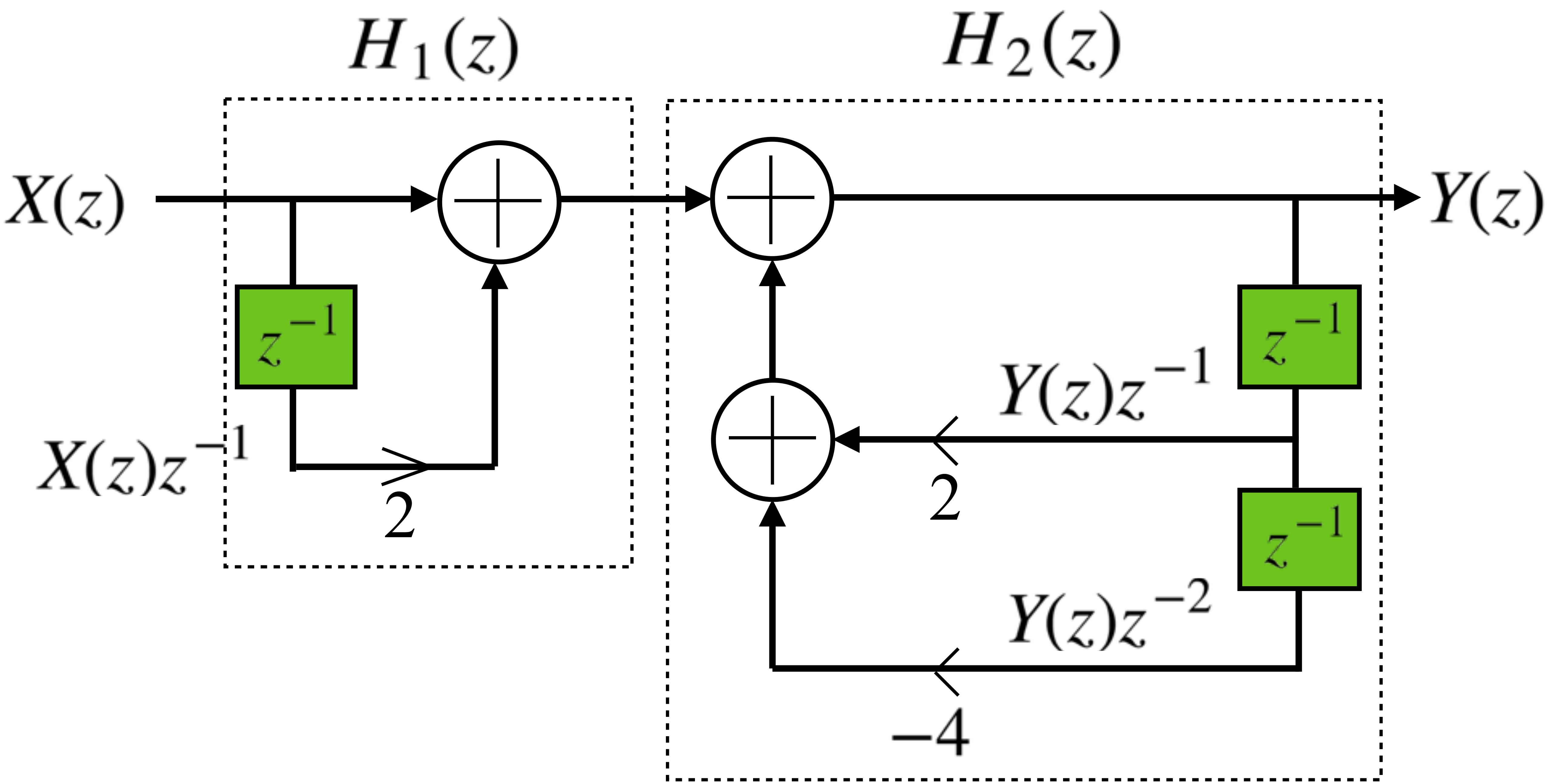


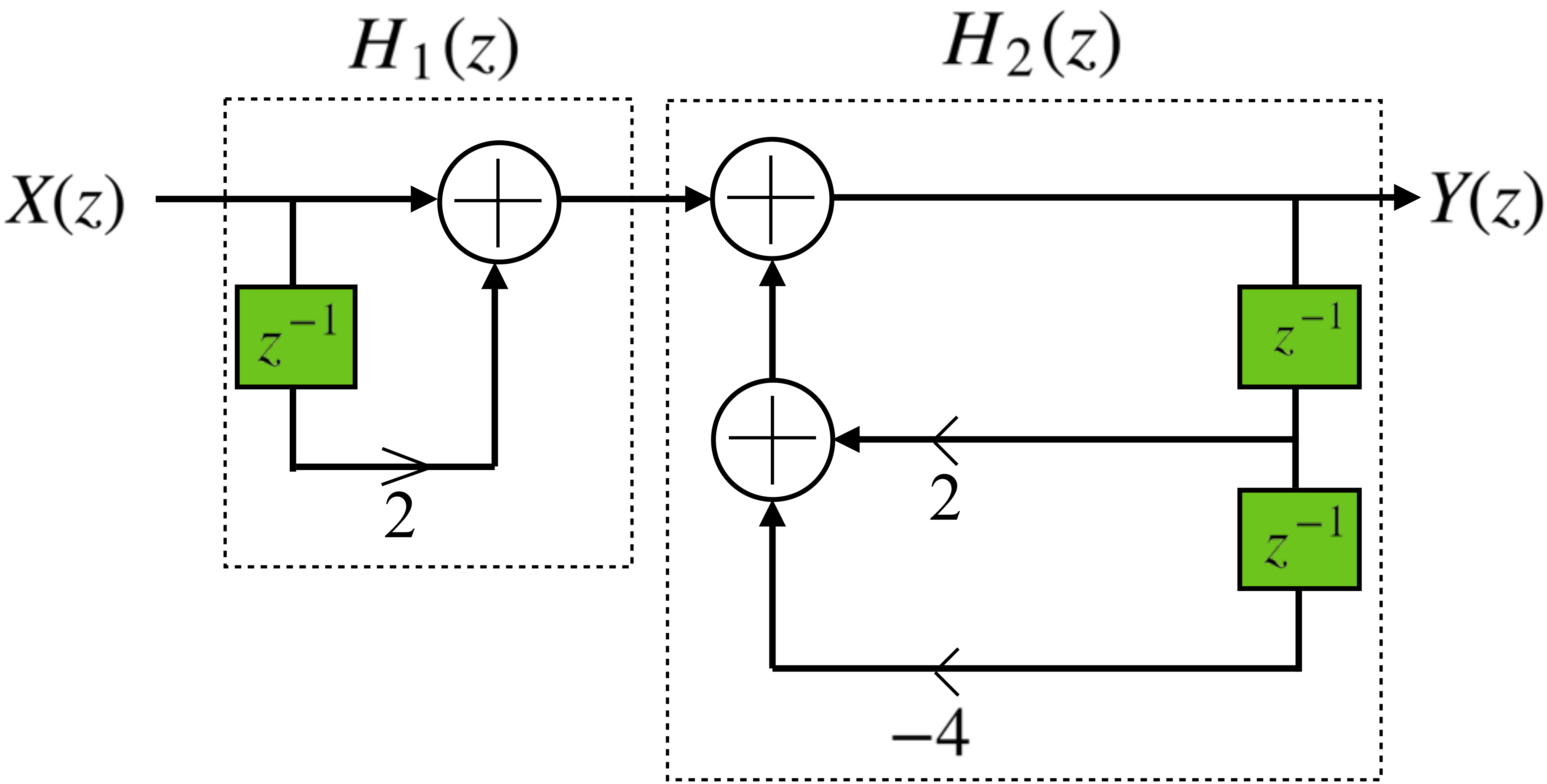
Block diagrams

- You can also think of this as a cascade of two LTI systems:



Block diagrams

- ... which means you can actually swap these two sub-systems:



Block diagrams

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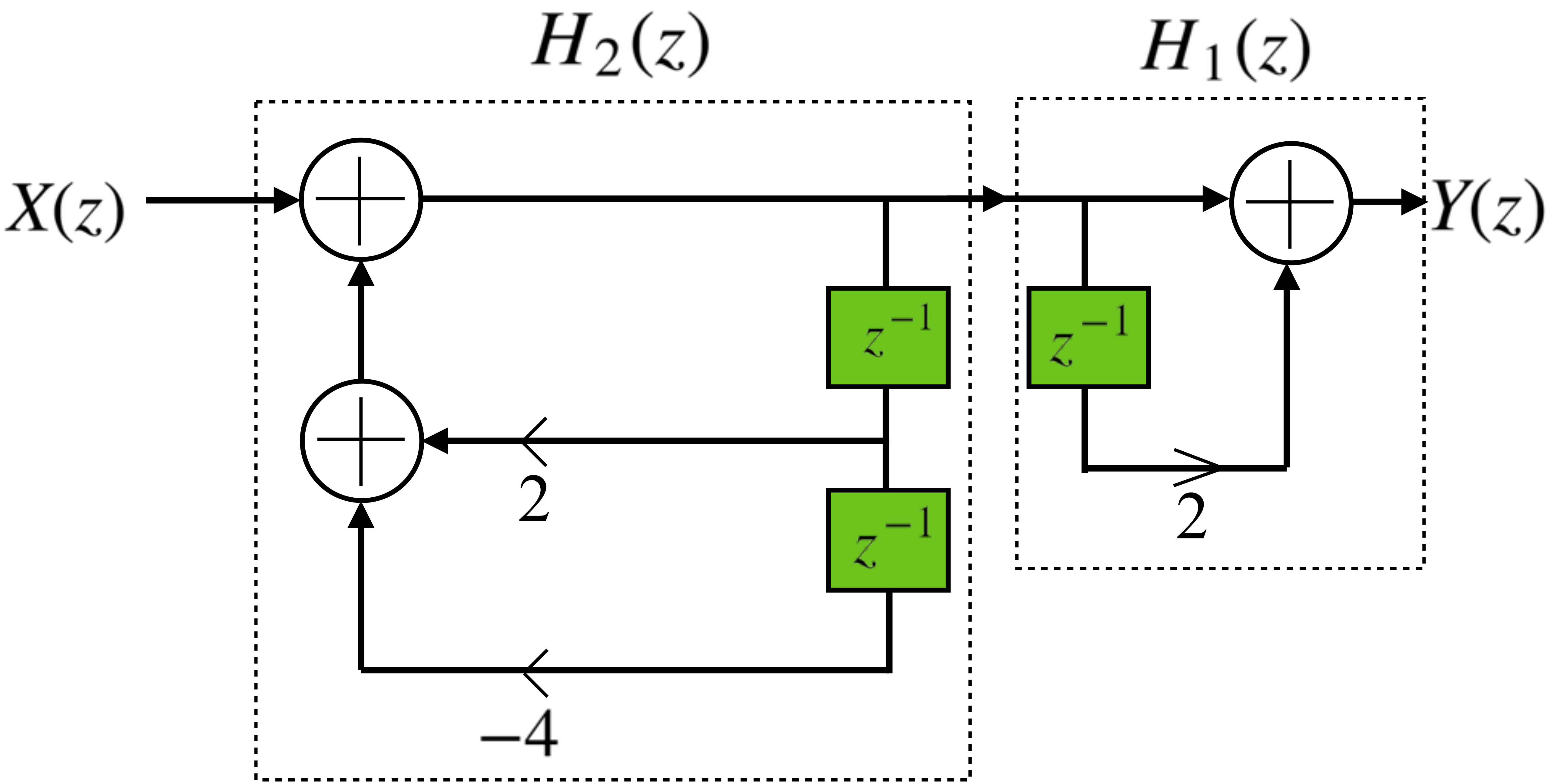

Block diagrams

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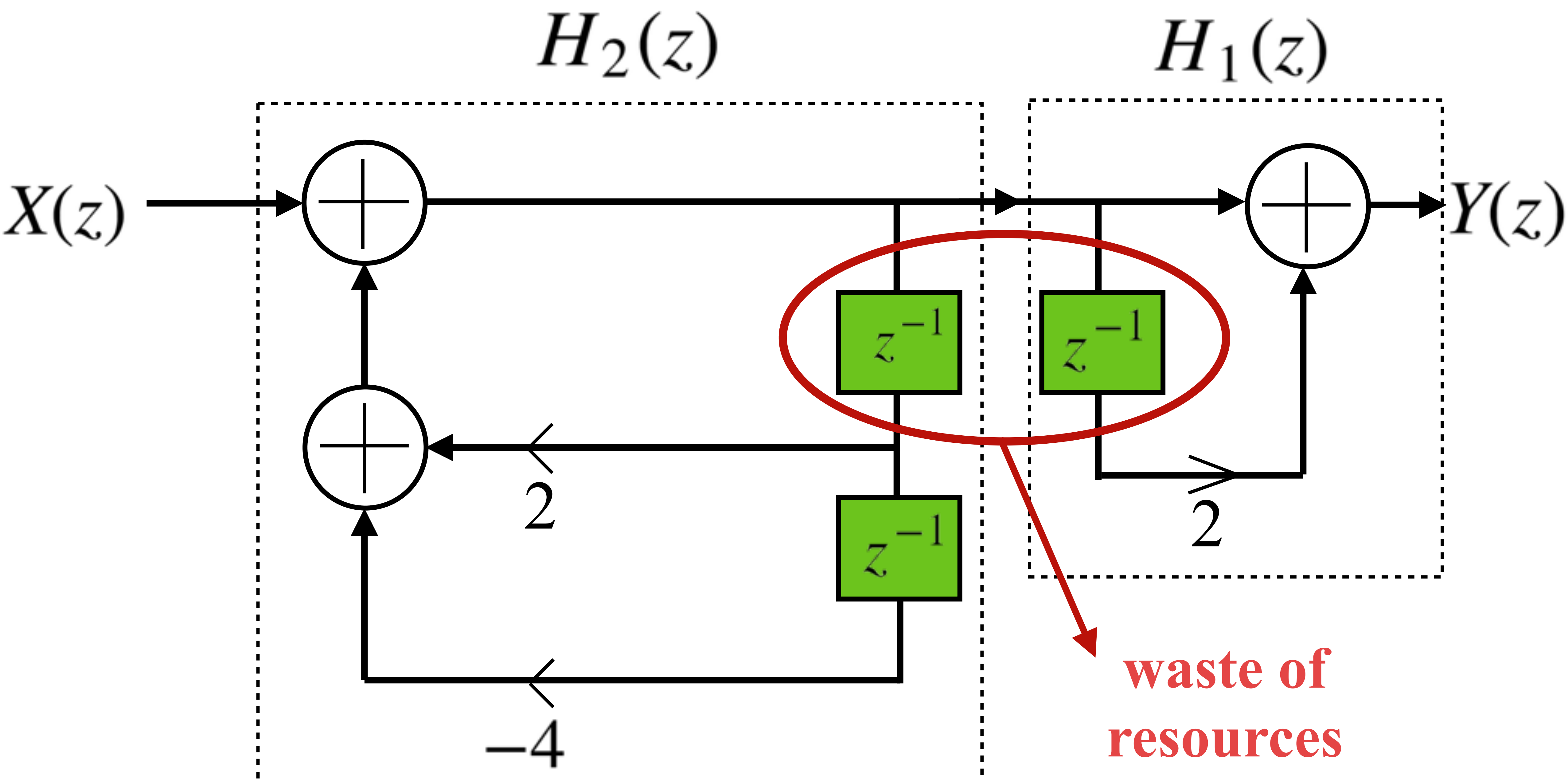

Block diagrams

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