FORMULAS AND CONCEPTS YOU MIGHT FIND USEFUL

- **Sine waves and complex exponentials:**
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
and
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$
- **Geometric series:** For any $\alpha \neq 1$ (complex or real),
$$\sum_{n=0}^{N} \alpha^n = \frac{1 - \alpha^{N+1}}{1 - \alpha} \ .$$
If $|\alpha| < 1$, then
$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha} \ .$$
- **LTI properties:** An LTI system with impulse response $h[n]$ is
  - **memoryless** if $h[n] = c\delta[n]$ for some constant c.
  - **causal** if $h[n] = 0$ for all $n < 0$.
  - **stable** if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
  - **invertible** if there is $g[n]$ such that $g[n] * h[n] = \delta[n]$.
- **DTFS:** For a signal with period N, and $\omega_0 = \frac{2\pi}{N}$
$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\omega_0 kn}$$
where
$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\omega_0 kn} \ .$$
- **DTFT:** For any signal,
$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega n}$$
where
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \ .$$
- **Properties of DTFT:**
  - **Linearity:**
$$\begin{array}{l} x[n] \to X(e^{j\omega}) \\ y[n] \to Y(e^{j\omega}) \end{array} \Longrightarrow \alpha x[n] + \beta y[n] \to \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$
  - **Time reversal:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow x[-n] \to X(e^{-j\omega})$$
  - **Time shifting:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow x[n - n_0] \to X(e^{j\omega}) e^{-j\omega n_0}$$
  - **Frequency shifting:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow x[n] e^{j\omega_0 n} \to X(e^{j(\omega - \omega_0)})$$
  - **Convolution:**
$$\begin{array}{l} x[n] \to X(e^{j\omega}) \\ y[n] \to Y(e^{j\omega}) \end{array} \Longrightarrow x[n] * y[n] \to X(e^{j\omega}) Y(e^{j\omega})$$
  - **Multiplication:**
$$\begin{array}{l} x[n] \to X(e^{j\omega}) \\ y[n] \to Y(e^{j\omega}) \end{array} \Longrightarrow x[n] y[n] \to \frac{1}{2\pi} \left[X(e^{j\omega}) \tilde{*} Y(e^{j\omega}) \right]$$
  - **Differentiation in frequency domain:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow nx[n] \to j \frac{dX(e^{j\omega})}{d\omega}$$
- **z-Transform:**
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \ .$$
Region of convergence (ROC) is the set of z for which the above sum converges.
- **Properties of the z-Transform:**
  - If $X(z) = \frac{(z-z_0)(z-z_1)\cdots(z-z_M)}{(z-p_0)(z-p_1)\cdots(z-p_K)}$, then z_m are *zeros* and p_k are *poles*. In addition, there are $K-M$ zeros at ∞ if $K > M$, $M - K$ poles at ∞ if $M > K$.
  - The ROC can only take one of three possible forms:
    - $|z| < a$ when $x[n]$ is left-sided,
    - $a < |z| < b$ when $x[n]$ is two-sided,
    - $b < |z|$ when $x[n]$ is right-sided.

    In addition, the ROC cannot contain any poles.
  - An LTI system that has a transfer function $H(z)$ with ROC $\mathcal{R}$ is
    - **causal** if $\mathcal{R}$ is of the form $b < |z|$ and $H(z)$ has no pole at $z = \infty$.
    - **stable** if $\mathcal{R}$ contains the unit circle $|z| = 1$.
  - The DTFT exists (converges) if the ROC contains $|z| = 1$.
  - **Linearity:** If $x[n] \to X(z)$ with ROC $\mathcal{R}_1$ and $y[n] \to Y(z)$ with ROC $\mathcal{R}_2$, then
$$\alpha x[n] + \beta y[n] \to \alpha X(z) + \beta Y(z)$$
with an ROC containing $\mathcal{R}_1 \cap \mathcal{R}_2$. There might be a zero-pole cancellation resulting in a larger ROC.
  - **Time reversal:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$x[-n] \to X(z^{-1})$$
with ROC $= \mathcal{R}^{-1} \triangleq \{z : z^{-1} \in \mathcal{R}\}$
  - **Time shifting:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$x[n - n_0] \to X(z) z^{-n_0}$$
with ROC $= \mathcal{R}$ (possibly excluding $z = 0$ or $z = \infty$).
  - **Frequency shifting:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$x[n] z_0^n \to X(z/z_0)$$
with ROC $= |z_0| \mathcal{R} \triangleq \{z : z/z_0 \in \mathcal{R}\}$.
  - **Convolution:** If $x[n] \to X(z)$ with ROC $\mathcal{R}_1$ and $y[n] \to Y(z)$ with ROC $\mathcal{R}_2$, then
$$x[n] * y[n] \to X(z) Y(z)$$
with an ROC containing $\mathcal{R}_1 \cap \mathcal{R}_2$. There might be a zero-pole cancellation resulting in a larger ROC.
  - **Differentiation in z-domain:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$nx[n] \to -z \frac{dX(z)}{dz}$$
with ROC $= \mathcal{R}$ (possibly excluding $z = 0$ or $z = \infty$).

- **Some known signal/z-Transform pairs:**
  - If $x[n] = a^n u[n]$, then
  $$X(z) = \frac{1}{1 - az^{-1}}$$
  with ROC= $\{z : |z| > a\}$.
  - If $x[n] = -a^n u[-n-1]$, then
  $$X(z) = \frac{1}{1 - az^{-1}}$$
  with ROC= $\{z : |z| < a\}$.
  - If $x[n] = na^n u[n]$, then
  $$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}$$
  with ROC= $\{z : |z| > a\}$.
  - If $x[n] = -na^n u[-n-1]$, then
  $$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}$$
  with ROC= $\{z : |z| < a\}$.
  - If $x[n] = \delta[n - n_0]$, then
  $$X(z) = z^{-n_0}$$
  with ROC= $\{z : |z| > 0\}$ if $n_0 > 0$. If $n_0 < 0$, then ROC is the entire complex plane with the exception of ∞.
- **Continuous time Fourier Transform (CTFT):** For any signal $x(t)$,
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$
where
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \ .$$
- **Properties of CTFT:**
  - **Linearity:**
  $$\begin{array}{l} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{array} \Longrightarrow \alpha x(t) + \beta y(t) \to \alpha X(j\Omega) + \beta Y(j\Omega)$$
  - **Time reversal:**
  $$x(t) \to X(j\Omega) \Longrightarrow x(-t) \to X(-j\Omega)$$
  - **Time shifting:**
  $$x(t) \to X(j\Omega) \Longrightarrow x(t - t_0) \to X(j\Omega) e^{-j\Omega t_0}$$
  - **Frequency shifting:**
  $$x(t) \to X(j\Omega) \Longrightarrow x(t) e^{j\Omega_0 t} \to X(j(\Omega - \Omega_0))$$
  - **Convolution:**
  $$\begin{array}{l} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{array} \Longrightarrow x(t) * y(t) \to X(j\Omega) Y(j\Omega)$$
  - **Multiplication:**
  $$\begin{array}{l} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{array} \Longrightarrow x(t) y(t) \to \frac{1}{2\pi} [X(j\Omega) * Y(j\Omega)]$$
  - **Differentiation in time domain:**
  $$x(t) \to X(j\Omega) \Longrightarrow \frac{dx(t)}{dt} \to j\Omega X(j\Omega)$$
  - **Differentiation in frequency domain:**
  $$x(t) \to X(j\Omega) \Longrightarrow -jtx(t) \to \frac{dX(j\Omega)}{d\Omega}$$
- **Some known signal-CTFT pairs:**
  - If $x(t) = \delta(t - t_0)$, then
  $$X(j\Omega) = e^{-j\Omega t_0} \ .$$
  - If $x(t) = e^{j\Omega_0 t}$, then
  $$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0) \ .$$
  - If $x(t) = \frac{\sin(At)}{t}$, then
  $$X(j\Omega) = \begin{cases} \pi & -A \le \Omega \le A \\ 0 & \text{otherwise} \end{cases}$$
  - If $x(t) = \begin{cases} 1/2 & -B \le t \le B \\ 0 & \text{otherwise} \end{cases}$, then
  $$X(j\Omega) = \frac{\sin(B\Omega)}{\Omega} \ .$$
- **Sampling and reconstruction:** If $x_d[n] = x_c(nT)$, x_d is said to be the sampled version of the continuous-time signal x_c. The sampling period is T, and the sampling frequency is $\Omega_s = \frac{2\pi}{T}$.
  - Defining the intermediate signal $x_s(t)$ as
  $$x_s(t) = x_c(t) \left(\sum_{k=-\infty}^{\infty} \delta(t - kT) \right) \ ,$$
  we have the relation
  $$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \ .$$
  - The relation between $X_s(j\Omega)$ and $X_d(e^{j\omega})$ is
  $$X_d(e^{j\omega}) = X_s\left(j\left(\frac{\omega}{T}\right)\right)$$
  or
  $$X_s(j\Omega) = X_d(e^{j\Omega T}) \ .$$
  - Therefore,
  $$X_d(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega - k2\pi}{T}\right)\right) \ .$$
  - These imply that in order for us to reconstruct $x_c(t)$ perfectly, we need
  $$\Omega_s = \frac{2\pi}{T} \ge 2\Omega_M$$
  where Ω_M is the bandwidth, i.e., maximum frequency where $X_c(j\Omega) \neq 0$.
  - Reconstruction is accomplished by eliminating the replicas of $X_c(j\Omega)$ from $X_s(j\Omega)$. Ideally, this is done by a filter
  $$H(j\Omega) = \begin{cases} T & -\frac{\pi}{T} \le \Omega \le \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$
  which, in the time domain becomes $h(t) = \frac{\sin(\frac{\pi}{T}t)}{\frac{\pi}{T}t}$.