EE 115 – Homework 2

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Problem 1

The passband signal is

$$x(t) = 3\big(\sin(100\pi t) + \sin(200\pi t)\big)\cos(1000\pi t) = m(t)\cos(1000\pi t),$$

where $m(t) = 3\big(\sin(100\pi t) + \sin(200\pi t)\big)$ carries baseband components at 50 Hz and 100 Hz. The channel $h(t)$ is an ideal lowpass filter with

$$H(f) = \mathrm{rect}\left(\frac{f}{220}\right) = \begin{cases} 1, & |f| < 110 \text{ Hz}, \\ 0, & |f| > 110 \text{ Hz}. \end{cases}$$

Multiplication by a sinusoid in time shifts the spectrum in frequency; the output of $h(t)$ therefore keeps only those shifted components that fall inside $|f| < 110$ Hz.

(a)

$$y_1(t) = h(t) * \big[x(t)\cos(1000\pi t)\big] = h(t) * \big[m(t)\cos^2(1000\pi t)\big].$$

Using $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, the term at $2 \cdot 500 = 1000$ Hz is rejected by $H(f)$, leaving

$$y_1(t) = \frac{1}{2}m(t) = \frac{3}{2}\big(\sin(100\pi t) + \sin(200\pi t)\big).$$

Work. Using the identity

$$\cos(1000\pi t)\cos(1000\pi t) = \tfrac{1}{2}(1 + \cos(2000\pi t)),$$

we write

$$x(t)\cos(1000\pi t) = m(t)\cos^2(1000\pi t) = \tfrac{1}{2}m(t) + \tfrac{1}{2}m(t)\cos(2000\pi t).$$

Multiplication by $\cos(2000\pi t)$ shifts the baseband spectrum of $m(t)$ to ± 1000 Hz, which lies outside the low-pass passband $|f| < 110$ Hz. Hence the filter removes that term and passes only $\frac{1}{2}m(t)$.

$$\boxed{y_1(t) = \tfrac{1}{2}m(t) = \tfrac{3}{2}\big(\sin(100\pi t) + \sin(200\pi t)\big)}$$

(b)

$$y_2(t) = h(t) * \big[m(t)\cos(1000\pi t)\cos(1000\pi t + \tfrac{\pi}{4})\big].$$

With the identity $\cos\alpha\cos\beta = \frac{1}{2}\big[\cos(\alpha-\beta) + \cos(\alpha+\beta)\big]$,

$$\cos(1000\pi t)\cos(1000\pi t + \tfrac{\pi}{4}) = \tfrac{1}{2}\big[\cos(\tfrac{\pi}{4}) + \cos(2000\pi t + \tfrac{\pi}{4})\big].$$

The component near 1000 Hz is filtered out, yielding

$$y_2(t) = \frac{\sqrt{2}}{4}m(t) = \frac{3\sqrt{2}}{4}\big(\sin(100\pi t) + \sin(200\pi t)\big).$$

Work. Apply

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]\,.$$

With $\alpha = 1000\pi t$ and $\beta = 1000\pi t + \frac{\pi}{4}$,

$$\cos(1000\pi t)\cos(1000\pi t + \tfrac{\pi}{4}) = \tfrac{1}{2}\big[\cos(\tfrac{\pi}{4}) + \cos(2000\pi t + \tfrac{\pi}{4})\big]\,.$$

The second term is centred at ± 1000 Hz and is rejected by $H(f)$; the constant factor $\frac{1}{2}\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{4}$ scales $m(t)$.

$$\boxed{y_2(t) = \tfrac{\sqrt{2}}{4}m(t) = \tfrac{3\sqrt{2}}{4}\big(\sin(100\pi t) + \sin(200\pi t)\big)}$$

(c)

$$y_3(t) = h(t) * \big[m(t)\cos(1000\pi t)\sin(1000\pi t)\big] = h(t) * \Big[\frac{m(t)}{2}\sin(2000\pi t)\Big].$$

All spectral content is centred at ± 1000 Hz, hence

$$y_3(t) = 0.$$

Work. First,

$$\cos(1000\pi t)\sin(1000\pi t) = \tfrac{1}{2}\sin(2000\pi t).$$

Therefore $m(t)\cos(1000\pi t)\sin(1000\pi t) = \frac{1}{2}m(t)\sin(2000\pi t)$, whose spectrum is the spectrum of $m(t)$ shifted to ± 1000 Hz. Since the ideal low-pass keeps only $|f| < 110$ Hz, the entire term is removed.

$$\boxed{y_3(t) = 0}$$

(d) The detuned local oscillator produces

$$\cos(1000\pi t)\cos(1010\pi t) = \tfrac{1}{2}\big[\cos(10\pi t) + \cos(2010\pi t)\big],$$

so the lowpass output retains only the term at 5 Hz:

$$y_4(t) = \frac{1}{2}m(t)\cos(10\pi t) = \frac{3}{2}\big(\sin(100\pi t) + \sin(200\pi t)\big)\cos(10\pi t).$$

Expanding the product highlights the new baseband tones at 45, 55, 95, and 105 Hz:

$$y_4(t) = \frac{3}{4}\Big[\sin(110\pi t) + \sin(90\pi t) + \sin(210\pi t) + \sin(190\pi t)\Big].$$

Work. The detuned LO gives

$$\cos(1000\pi t)\cos(1010\pi t) = \tfrac{1}{2}[\cos(10\pi t) + \cos(2010\pi t)]\,.$$

The low-pass passes only the 5 Hz factor $\frac{1}{2}\cos(10\pi t)$, so $y_4(t) = \frac{1}{2}m(t)\cos(10\pi t)$. Using $\sin A \cos B = \frac{1}{2}[\sin(A{+}B) + \sin(A{-}B)]$ with $A \in \{100\pi t,\, 200\pi t\}$ and $B = 10\pi t$ yields the four tones at $45, 55, 95, 105$ Hz.

$$\boxed{y_4(t) = \frac{3}{4}\Big[\sin(110\pi t) + \sin(90\pi t) + \sin(210\pi t) + \sin(190\pi t)\Big]}$$

Problem 1: Spectrum sketches

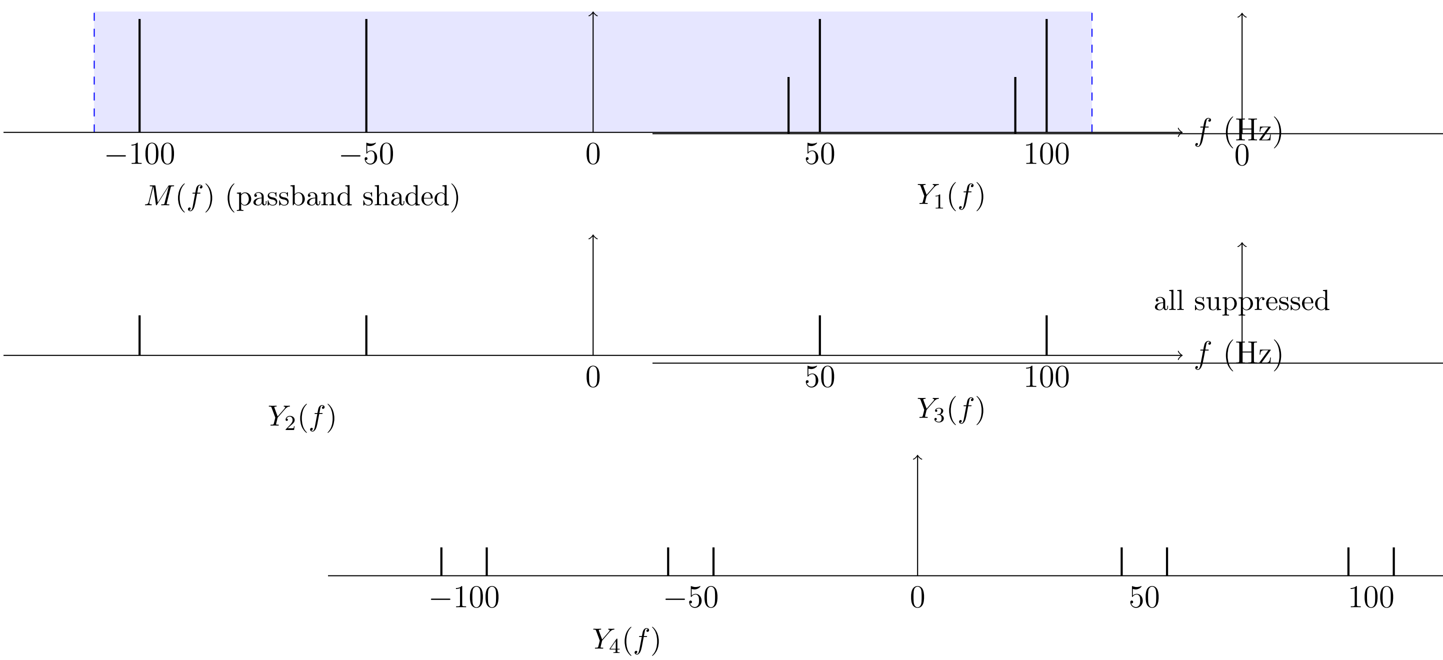

Figure 1: Problem 1: stick-spectrum sketches (heights proportional to amplitude-spectrum values; $|f| < 110$ Hz passband indicated on $M(f)$).

Problem 2

The message waveform is a bipolar pulse train:

$$m(t) = \sum_{k=-\infty}^{\infty} [\mathrm{rect}(t - 2k) - \mathrm{rect}(t - 2k - 1)]\,,$$

so $m(t) = +1$ on $(2k - \frac{1}{2}, 2k + \frac{1}{2})$ and $m(t) = -1$ on $(2k + \frac{1}{2}, 2k + \frac{3}{2})$. The AM input to the envelope detector is $x(t) = (m(t) + \alpha)\cos(2\pi f_c t)$, and an ideal detector outputs the instantaneous envelope,

$$y(t) = |m(t) + \alpha|.$$

Consequently, for any α

$$y(t) = \begin{cases} |\alpha + 1|, & t \in (2k - \frac{1}{2}, 2k + \frac{1}{2}), \\ |\alpha - 1|, & t \in (2k + \frac{1}{2}, 2k + \frac{3}{2}), \end{cases} \quad k \in \mathbb{Z}.$$

Work. The AM input is $x(t) = (m(t) + \alpha)\cos(2\pi f_c t)$. An ideal envelope detector outputs the instantaneous magnitude of the complex envelope, i.e., $|m(t) + \alpha|$. Over intervals where $m(t) = +1$ the level is $|\alpha + 1|$; where $m(t) = -1$ the level is $|\alpha - 1|$. Special cases: (i) $\alpha = 0$ $\Rightarrow$ constant output 1 (full-wave rectification of a suppressed-carrier AM); (ii) $\alpha = 1 \Rightarrow$ the envelope just touches zero (critical modulation); (iii) $\alpha > 1 \Rightarrow$ strictly positive envelope and no sign inversions.

$$\boxed{y(t) = |m(t) + \alpha|}$$

$$\boxed{\text{Levels: } |\alpha + 1| \text{ on } (2k - \tfrac{1}{2}, 2k + \tfrac{1}{2}), \quad |\alpha - 1| \text{ on } (2k + \tfrac{1}{2}, 2k + \tfrac{3}{2})}$$

α	$\alpha + 1$	$\alpha - 1$	Output levels	Comment
0	$+1$	-1	constant 1	Carrier suppressed; full-wave rectification.
0.5	$+1.5$	-0.5	1.5 and 0.5	Unequal positive plateaus.
1	$+2$	0	2 and 0	Critical modulation; envelope just touches zero.
1.5	$+2.5$	$+0.5$	2.5 and 0.5	Strong carrier, always positive.

A qualitative sketch for one period of $y(t)$ in each case is shown below (period $T = 2$).

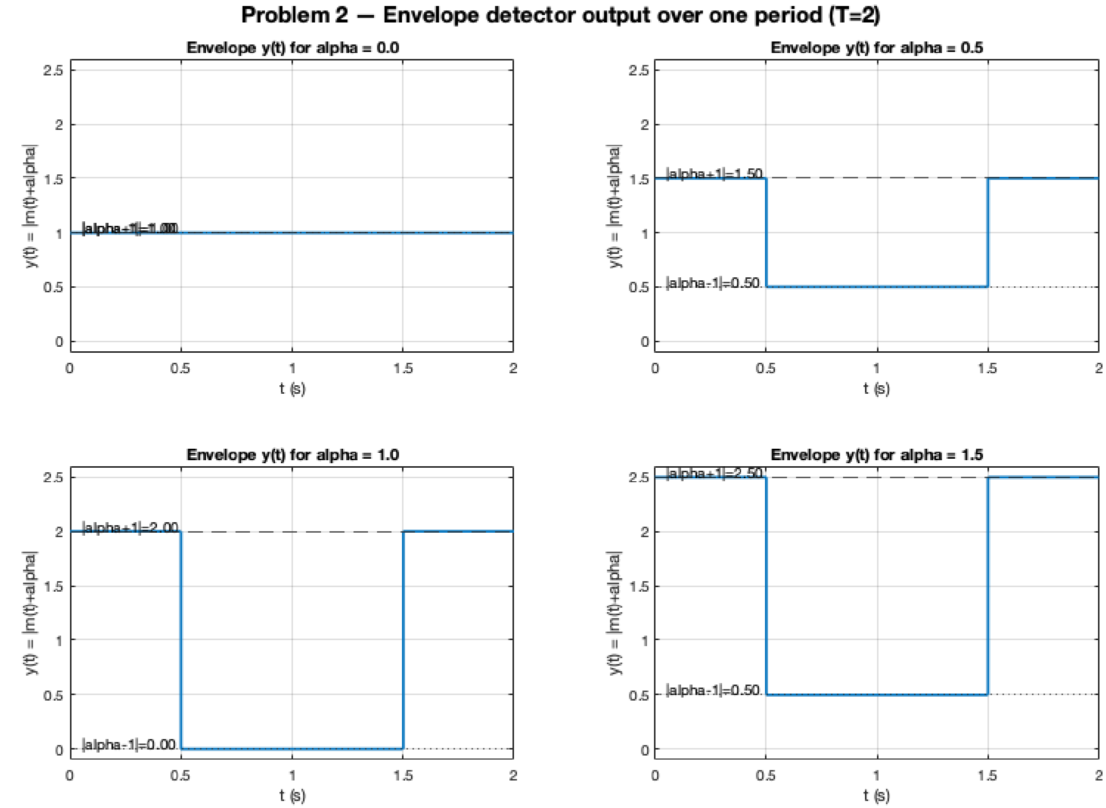

Figure 2: Envelope detector output $y(t)$ over one period ($T = 2$) for $\alpha \in \{0, 0.5, 1, 1.5\}$.

Problem 3

The AM waveform is

$$x_{\mathrm{AM}}(t) = 10\big(m(t) + 3\big)\cos(100\pi t), \quad m(t) = \sin(20\pi t) + 2\sin(30\pi t),$$

with carrier frequency $f_c = 50\,\mathrm{Hz}$.

(a) *Work.* Expand with product-to-sum:

$$\begin{aligned} 10(m{+}3)\cos(100\pi t) &= 30\cos(100\pi t) + 10\big[\sin(20\pi t) + 2\sin(30\pi t)\big]\cos(100\pi t) \\ &= 30\cos(100\pi t) \\ &\quad + 5[\sin(120\pi t) + \sin(80\pi t)] + 10[\sin(130\pi t) + \sin(70\pi t)]\,, \end{aligned}$$

where we used $\sin\omega_1 t\cos\omega_c t = \frac{1}{2}\left[\sin(\omega_c{+}\omega_1)t + \sin(\omega_c{-}\omega_1)t\right]$. The single-tone contributions are summarized below (Hz):

Term	Frequencies	Amplitude
$\sin(20\pi t)\cos(100\pi t)$	50 ± 10	5
$2\sin(30\pi t)\cos(100\pi t)$	50 ± 15	10
$\cos(100\pi t)$	50	30

(For a two-sided amplitude spectrum, a cosine of amplitude A gives impulses of height $A/2$ at $\pm f$.) Expanding $x_{\mathrm{AM}}(t)$ gives

$$x_{\mathrm{AM}}(t) = 30\cos(100\pi t) + 5\sin(120\pi t) - 5\sin(80\pi t) + 10\sin(130\pi t) - 10\sin(70\pi t).$$

Therefore the (two-sided) amplitude spectrum contains impulses at the carrier and at $f_c \pm 10\,\mathrm{Hz}$ and $f_c \pm 15\,\mathrm{Hz}$. Magnitudes of the spectral lines are 15 for the carrier, 2.5 for the $\pm 10\,\mathrm{Hz}$ offsets, and 5 for the $\pm 15\,\mathrm{Hz}$ offsets:

$$\begin{aligned} |X_{\mathrm{AM}}(f)| = 15\big[&\delta(f-50)+\delta(f+50)\big] \\ &+ 2.5\big[\delta(f-60)+\delta(f+60)+\delta(f-40)+\delta(f+40)\big] \\ &+ 5\big[\delta(f-65)+\delta(f+65)+\delta(f-35)+\delta(f+35)\big]. \end{aligned}$$

$$\boxed{\text{Carrier: 50 Hz@15; Sidebands: } 50 \pm 10 \text{ Hz@2.5},\ 50 \pm 15 \text{ Hz@5}}$$

A single-sided sketch is shown in Fig. 3; the negative-frequency components mirror the positives.

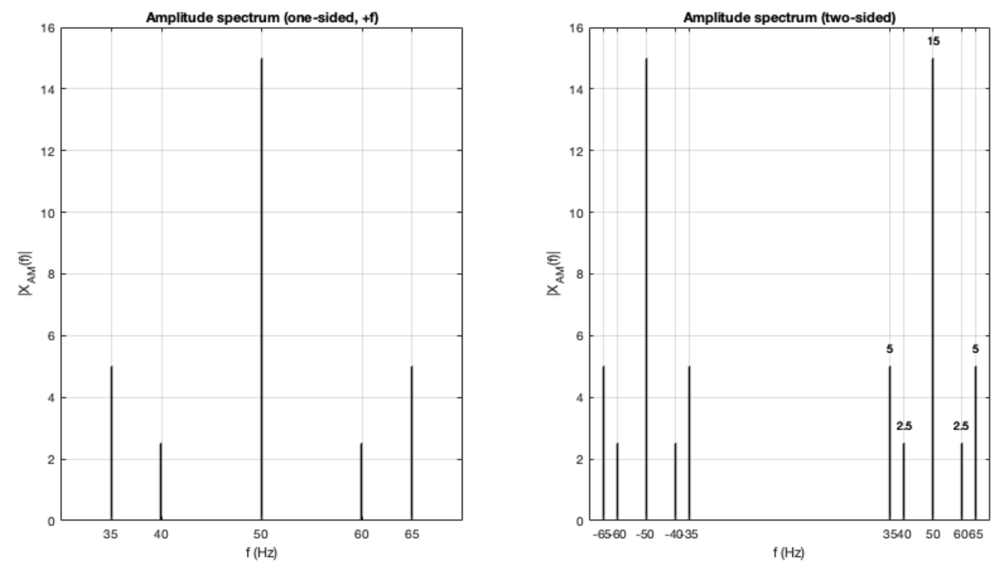

Figure 3: Amplitude spectrum of $x_{AM}(t)$ showing carrier and sidebands at 50 Hz, 50 ± 10 Hz, and 50 ± 15 Hz.

(b) *Work.* Write

$$x_{\mathrm{AM}}(t) = A_c\big[1 + k_a m(t)\big]\cos(2\pi f_c t), \qquad A_c = 30,\ k_a = \tfrac{1}{3},\ f_c = 50 \text{ Hz}.$$

Since $m(t) = \sin(20\pi t) + 2\sin(30\pi t)$ attains $\max|m(t)| = 3$ and $\min m(t) = -3$, the envelope $A_c\big|1 + k_a m(t)\big|$ just touches zero at the minimum, hence the modulation index

$$\mu = k_a \max|m(t)| = \tfrac{1}{3} \cdot 3 = 1.$$

$$\boxed{A_c = 30, \quad k_a = \tfrac{1}{3}, \quad \mu = 1}$$

(c) *Work.* For a real sinusoid $A\cos(\omega t)$ (or $A\sin(\omega t)$), the time-average power is $A^2/2$. Orthogonality of distinct sinusoids at different frequencies makes cross-terms average to zero, so the total power is the sum of each component's $A^2/2$. Thus the carrier contributes $30^2/2$ and the four sidebands contribute $5^2/2$, $5^2/2$, $10^2/2$, $10^2/2$, respectively. The average transmitted power equals the carrier power plus the sideband power. From the expansion above,

$$P_c = \frac{A_c^2}{2} = \frac{30^2}{2} = 450, \qquad P_{\text{sb}} = \frac{5^2}{2} + \frac{5^2}{2} + \frac{10^2}{2} + \frac{10^2}{2} = 125,$$

giving $P_{\text{total}} = 575$. Hence the power efficiency is

$$\eta = \frac{P_{\text{sb}}}{P_{\text{total}}} = \frac{125}{575} \approx 0.217\,(21.7\%).$$

$$\boxed{P_c = 450, \quad P_{\text{sb}} = 125, \quad P_{\text{total}} = 575, \quad \eta = \tfrac{125}{575} \approx 0.217}$$