UNIVERSITY OF CALIFORNIA, RIVERSIDE
Department of Electrical and Computer Engineering
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EE 110B SIGNALS AND SYSTEMS
FINAL EXAM

You have 180 minutes to complete the exam. Answers must be fully justified. Good luck!

Question 1) Recall that a great mechanism to implement continuous-time filters is to convert signals into discrete-time domain, process them there, and then convert back to continuous-time. This only works if the signal is bandlimited and the sampling rate T is chosen appropriately. Also recall that if the desired filter is $H_c(j\Omega)$, the discrete-time implementation should use the filter

$$H_d(e^{j\omega}) = H_c(j\tfrac{\omega}{T})$$

for $-\pi \leq \omega \leq \pi$, and then repeat with a period of 2π on the ω axis. To figure out $h[n]$, we either apply inverse DTFT to $H_d(e^{j\omega})$, or perform a "thought experiment" of inputing $x_c(t) = \mathrm{sinc}(\frac{\pi t}{T})$, tracking the resultant $y_c(t)$, and then using $h[n] = y_c(nT)$.

In class, we analyzed how a differentiator can be implemented using this technique. Now we will do the same for calculating the second derivative, i.e., implementing a system

$$y_c(t) = \frac{d^2 x_c(t)}{dt^2}$$

Assume that $X_c(j\Omega) = 0$ for $|\Omega| > \pi$ and take $T = 1$ for simplicity.

a) Find $H_c(j\Omega)$ such that $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$.
b) Find the corresponding discrete-time filter $H_d(e^{j\omega})$.
c) Find $h[n]$ either through $\mathrm{DTFT}^{-1}\{\mathrm{H_d(e^{j\omega})}\}$ or through the thought experiment. With either choice, pay special attention to $n = 0$.

Answer:

Question 2) This question will walk you through steps of showing

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \ .$$

using z-transforms. You may recall that the above formula was first derived by the famous mathematician Gauss in late 1700s when he was an elementary school student.

a) Letting

$$x[n] = u[n] \sum_{k=0}^{n} k \ ,$$

show that

$$x[n] = u[n] \star nu[n]$$

b) Using part a) and properties of the z-transform, compute $X(z)$. Do not forget to state the ROC.

c) Find an expression for $x[n]$ by inverting the z-transform using **long division**. If you did everything properly, the expression must be equivalent to $x[n] = \frac{n(n+1)}{2} u[n]$. In other words, $x[0] = 0$, $x[1] = 1$, $x[2] = 3$, $x[3] = 6$, $x[4] = 10, \ldots$

Answer:

Question 3) Consider the filter

$$G(z) = \frac{z^{-1} - a}{1 - az^{-1}}$$

with ROC $= \{z : |z| > a\}$ for some real number $0 < a < 1$.

a) Find and sketch $g[n]$.
Hint: Write $\frac{z^{-1}-a}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}} - a \cdot \frac{1}{1-az^{-1}}$, and use the delay property of the z-transform.

b) Show that this is an *all-pass* filter, i.e.,

$$|G(e^{j\omega})| = 1$$

for all ω.

c) Now consider the non-causal filter

$$h[n] = -2^n u[-n-1] \; .$$

Find $H(z)$ and state the ROC. Argue why the DTFT $H(e^{j\omega})$ exists.

d) Let's say we want to be able to implement a filter with the magnitude response $|H(e^{j\omega})|$. To achieve that goal, there is no way to use $h[n]$ in its current form, because $h[n]$ is a left-sided infinite sequence and therefore wants to look ahead in the infinite future. The solution is to implement

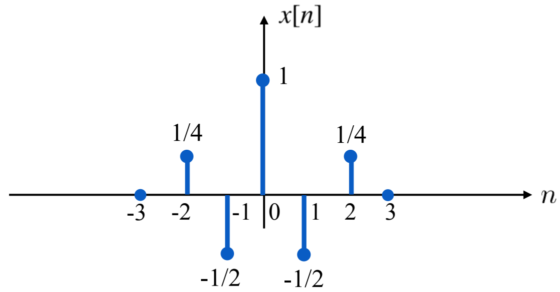
$$H(z)G(z)$$

instead. Since $|G(e^{j\omega})| = 1$, this modification will not change the magnitude response. Choose an a such that $H(z)G(z)$ becomes a causal and stable filter.

Answer:

Question 4)

Let $x[n]$ be as shown below.



a) Find the discrete-time Fourier transform (DTFT) of $x[n]$.
Hint: Write the sum

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

openly, and pair up suitable terms to simplify.

b) Sketch $|X(e^{j\omega})|$. What kind of signal is $x[n]$? Low-pass or high-pass?
Hint: You can have a rough idea when you evaluate $X(e^{j\omega})$ at $\omega = 0$, $\omega = \pm\frac{\pi}{2}$, and $\omega = \pm\pi$.

c) Calculate

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\sin(\omega) - \sin(2\omega)\Big)^2 d\omega$$

using the **differentiation property** of the DTFT, **Parseval's relation**, and your answer from part a.

Answer:

FORMULAS AND CONCEPTS YOU MIGHT FIND USEFUL

- **Sine waves and complex exponentials:**
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
and
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$
- **Geometric series:** For any $\alpha \neq 1$ (complex or real),
$$\sum_{n=0}^{N} \alpha^n = \frac{1-\alpha^{N+1}}{1-\alpha} .$$
If $|\alpha| < 1$, then
$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} .$$
- **LTI properties:** An LTI system with impulse response $h[n]$ is
  - **memoryless** if $h[n] = c\delta[n]$ for some constant c.
  - **causal** if $h[n] = 0$ for all $n < 0$.
  - **stable** if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
  - **invertible** if there is $g[n]$ such that $g[n] * h[n] = \delta[n]$.
- **DTFS:** For a signal with period N, and $\omega_0 = \frac{2\pi}{N}$
$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\omega_0 kn}$$
where
$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\omega_0 kn} .$$
- **DTFT:** For any signal,
$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) e^{j\omega n}$$
where
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} .$$
- **Properties of DTFT:**
  - **Linearity:**
$$\begin{matrix} x[n] \to X(e^{j\omega}) \\ y[n] \to Y(e^{j\omega}) \end{matrix} \Longrightarrow \alpha x[n] + \beta y[n] \to \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$
  - **Time reversal:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow x[-n] \to X(e^{-j\omega})$$
  - **Time shifting:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow x[n-n_0] \to X(e^{j\omega}) e^{-j\omega n_0}$$
  - **Frequency shifting:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow x[n] e^{j\omega_0 n} \to X(e^{j(\omega-\omega_0)})$$
  - **Convolution:**
$$\begin{matrix} x[n] \to X(e^{j\omega}) \\ y[n] \to Y(e^{j\omega}) \end{matrix} \Longrightarrow x[n] * y[n] \to X(e^{j\omega}) Y(e^{j\omega})$$
  - **Multiplication:**
$$\begin{matrix} x[n] \to X(e^{j\omega}) \\ y[n] \to Y(e^{j\omega}) \end{matrix} \Longrightarrow x[n] y[n] \to \frac{1}{2\pi} \left[X(e^{j\omega}) \tilde{*} Y(e^{j\omega}) \right]$$
  - **Differentiation in frequency domain:**
$$x[n] \to X(e^{j\omega}) \Longrightarrow nx[n] \to j \frac{dX(e^{j\omega})}{d\omega}$$
- **z-Transform:**
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} .$$
Region of convergence (ROC) is the set of z for which the above sum converges.
- **Properties of the z-Transform:**
  - If $X(z) = \frac{(z-z_0)(z-z_1)\cdots(z-z_M)}{(z-p_0)(z-p_1)\cdots(z-p_K)}$, then z_m are *zeros* and p_k are *poles*. In addition, there are $K-M$ zeros at ∞ if $K > M$, $M-K$ poles at ∞ if $M > K$.
  - The ROC can only take one of three possible forms:
    - $|z| < a$ when $x[n]$ is left-sided,
    - $a < |z| < b$ when $x[n]$ is two-sided,
    - $b < |z|$ when $x[n]$ is right-sided.

    In addition, the ROC cannot contain any poles.
  - An LTI system that has a transfer function $H(z)$ with ROC $\mathcal{R}$ is
    - **causal** if $\mathcal{R}$ is of the form $b < |z|$ and $H(z)$ has no pole at $z = \infty$.
    - **stable** if $\mathcal{R}$ contains the unit circle $|z| = 1$.
  - The DTFT exists (converges) if the ROC contains $|z| = 1$.
  - **Linearity:** If $x[n] \to X(z)$ with ROC $\mathcal{R}_1$ and $y[n] \to Y(z)$ with ROC $\mathcal{R}_2$, then
$$\alpha x[n] + \beta y[n] \to \alpha X(z) + \beta Y(z)$$
with an ROC containing $\mathcal{R}_1 \cap \mathcal{R}_2$. There might be a zero-pole cancellation resulting in a larger ROC.
  - **Time reversal:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$x[-n] \to X(z^{-1})$$
with ROC $= \mathcal{R}^{-1} \triangleq \{z : z^{-1} \in \mathcal{R}\}$
  - **Time shifting:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$x[n-n_0] \to X(z) z^{-n_0}$$
with ROC $= \mathcal{R}$ (possibly excluding $z = 0$ or $z = \infty$).
  - **Frequency shifting:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$x[n] z_0^n \to X(z/z_0)$$
with ROC $= |z_0|\mathcal{R} \triangleq \{z : z/z_0 \in \mathcal{R}\}$.
  - **Convolution:** If $x[n] \to X(z)$ with ROC $\mathcal{R}_1$ and $y[n] \to Y(z)$ with ROC $\mathcal{R}_2$, then
$$x[n] * y[n] \to X(z) Y(z)$$
with an ROC containing $\mathcal{R}_1 \cap \mathcal{R}_2$. There might be a zero-pole cancellation resulting in a larger ROC.
  - **Differentiation in z-domain:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
$$nx[n] \to -z \frac{dX(z)}{dz}$$
with ROC $= \mathcal{R}$ (possibly excluding $z = 0$ or $z = \infty$).

- **Some known signal/z-Transform pairs:**
  - If $x[n] = a^n u[n]$, then
$$X(z) = \frac{1}{1 - az^{-1}}$$
with ROC= $\{z : |z| > a\}$.
  - If $x[n] = -a^n u[-n-1]$, then
$$X(z) = \frac{1}{1 - az^{-1}}$$
with ROC= $\{z : |z| < a\}$.
  - If $x[n] = na^n u[n]$, then
$$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}$$
with ROC= $\{z : |z| > a\}$.
  - If $x[n] = -na^n u[-n-1]$, then
$$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}$$
with ROC= $\{z : |z| < a\}$.
  - If $x[n] = \delta[n - n_0]$, then
$$X(z) = z^{-n_0}$$
with ROC= $\{z : |z| > 0\}$ if $n_0 > 0$. If $n_0 < 0$, then ROC is the entire complex plane with the exception of ∞.
- **Continuous time Fourier Transform (CTFT):** For any signal $x(t)$,
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$$
where
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \ .$$
- **Properties of CTFT:**
  - **Linearity:**
$$\begin{array}{l} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{array} \Longrightarrow \alpha x(t) + \beta y(t) \to \alpha X(j\Omega) + \beta Y(j\Omega)$$
  - **Time reversal:**
$$x(t) \to X(j\Omega) \Longrightarrow x(-t) \to X(-j\Omega)$$
  - **Time shifting:**
$$x(t) \to X(j\Omega) \Longrightarrow x(t - t_0) \to X(j\Omega)e^{-j\Omega t_0}$$
  - **Frequency shifting:**
$$x(t) \to X(j\Omega) \Longrightarrow x(t)e^{j\Omega_0 t} \to X(j(\Omega - \Omega_0))$$
  - **Convolution:**
$$\begin{array}{l} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{array} \Longrightarrow x(t) * y(t) \to X(j\Omega)Y(j\Omega)$$
  - **Multiplication:**
$$\begin{array}{l} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{array} \Longrightarrow x(t)y(t) \to \frac{1}{2\pi}\left[X(j\Omega) * Y(j\Omega)\right]$$
  - **Differentiation in time domain:**
$$x(t) \to X(j\Omega) \Longrightarrow \frac{dx(t)}{dt} \to j\Omega X(j\Omega)$$
  - **Differentiation in frequency domain:**
$$x(t) \to X(j\Omega) \Longrightarrow -jtx(t) \to \frac{dX(j\Omega)}{d\Omega}$$
- **Some known signal-CTFT pairs:**
  - If $x(t) = \delta(t - t_0)$, then
$$X(j\Omega) = e^{-j\Omega t_0} \ .$$
  - If $x(t) = e^{j\Omega_0 t}$, then
$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0) \ .$$
  - If $x(t) = \frac{\sin(At)}{t}$, then
$$X(j\Omega) = \begin{cases} \pi & -A \le \Omega \le A \\ 0 & \text{otherwise} \end{cases}$$
  - If $x(t) = \begin{cases} 1/2 & -B \le t \le B \\ 0 & \text{otherwise} \end{cases}$, then
$$X(j\Omega) = \frac{\sin(B\Omega)}{\Omega} \ .$$
- **Sampling and reconstruction:** If $x_d[n] = x_c(nT)$, x_d is said to be the sampled version of the continuous-time signal x_c. The sampling period is T, and the sampling frequency is $\Omega_s = \frac{2\pi}{T}$.
  - Defining the intermediate signal $x_s(t)$ as
$$x_s(t) = x_c(t)\left(\sum_{k=-\infty}^{\infty} \delta(t - kT)\right) \ ,$$
we have the relation
$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \ .$$
  - The relation between $X_s(j\Omega)$ and $X_d(e^{j\omega})$ is
$$X_d(e^{j\omega}) = X_s\left(j\left(\frac{\omega}{T}\right)\right)$$
or
$$X_s(j\Omega) = X_d(e^{j\Omega T}) \ .$$
  - Therefore,
$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega - k2\pi}{T}\right)\right) \ .$$
  - These imply that in order for us to reconstruct $x_c(t)$ perfectly, we need
$$\Omega_s = \frac{2\pi}{T} \ge 2\Omega_M$$
where Ω_M is the bandwidth, i.e., maximum frequency where $X_c(j\Omega) \neq 0$.
  - Reconstruction is accomplished by eliminating the replicas of $X_c(j\Omega)$ from $X_s(j\Omega)$. Ideally, this is done by a filter
$$H(j\Omega) = \begin{cases} T & -\frac{\pi}{T} \le \Omega \le \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$
which, in the time domain becomes $h(t) = \frac{\sin(\frac{\pi}{T}t)}{\frac{\pi}{T}t}$.