EE 110A Signals and Systems

The Laplace Transform
Part I

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The Laplace transform

- Our motivation for the CTFT was that it is easy to characterize the output of an LTI for complex exponential inputs:

$$e^{j\Omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\Omega)e^{j\Omega t}$$

- What about the input $e^{(\alpha+j\Omega)t}$?

$$e^{(\alpha+j\Omega)t} \longrightarrow \boxed{h(t)} \longrightarrow ?$$

The Laplace transform

- Let's find out:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{(\alpha+j\Omega)(t-\tau)} d\tau$$

$$= e^{(\alpha+j\Omega)t} \int_{-\infty}^{\infty} h(\tau) e^{-(\alpha+j\Omega)\tau} d\tau$$

Input

CTFT of $h(t)e^{-\alpha t}$

The Laplace transform

- Alternatively, we can think of the input as

$$e^{(\alpha+j\Omega)t} \triangleq e^{st}$$

- Then

$$y(t) = e^{(\alpha+j\Omega)t} \int_{-\infty}^{\infty} h(\tau) e^{-(\alpha+j\Omega)\tau} d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

Let's call this $H(s)$

The Laplace transform

- The Laplace transform is then defined as

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} \, dt$$

- For any s, we have

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

- If we specialize this to $s = j\Omega$, we get back CTFT:

$$H(j\Omega) = \int_{-\infty}^{\infty} h(t) e^{-j\Omega t} \, dt$$

Examples

- Problem: Find the Laplace transform of

$$x(t) = e^{\alpha t} u(t)$$

- Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad = \int_{0}^{\infty} e^{\alpha t} e^{-st} dt$$

$$= \frac{1}{\alpha - s} e^{(\alpha - s)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{s - \alpha} \quad \text{provided} \quad Re\{s\} > Re\{\alpha\}$$

Examples

- <u>Problem</u>: What about for $x(t) = -e^{\alpha t}u(-t)$?
- <u>Solution:</u>

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad = -\int_{-\infty}^{0} e^{\alpha t}e^{-st}dt$$

$$= \frac{1}{s-\alpha} e^{(\alpha - s)t}\Big|_{-\infty}^{0}$$

$$\boxed{= \frac{1}{s-\alpha}} \qquad \text{provided } Re\{s\} < Re\{\alpha\}$$

Same as in the
previous example!!!

What?

- How can

$$x(t) = e^{\alpha t}u(t)$$

  and

$$x(t) = -e^{\alpha t}u(-t)$$

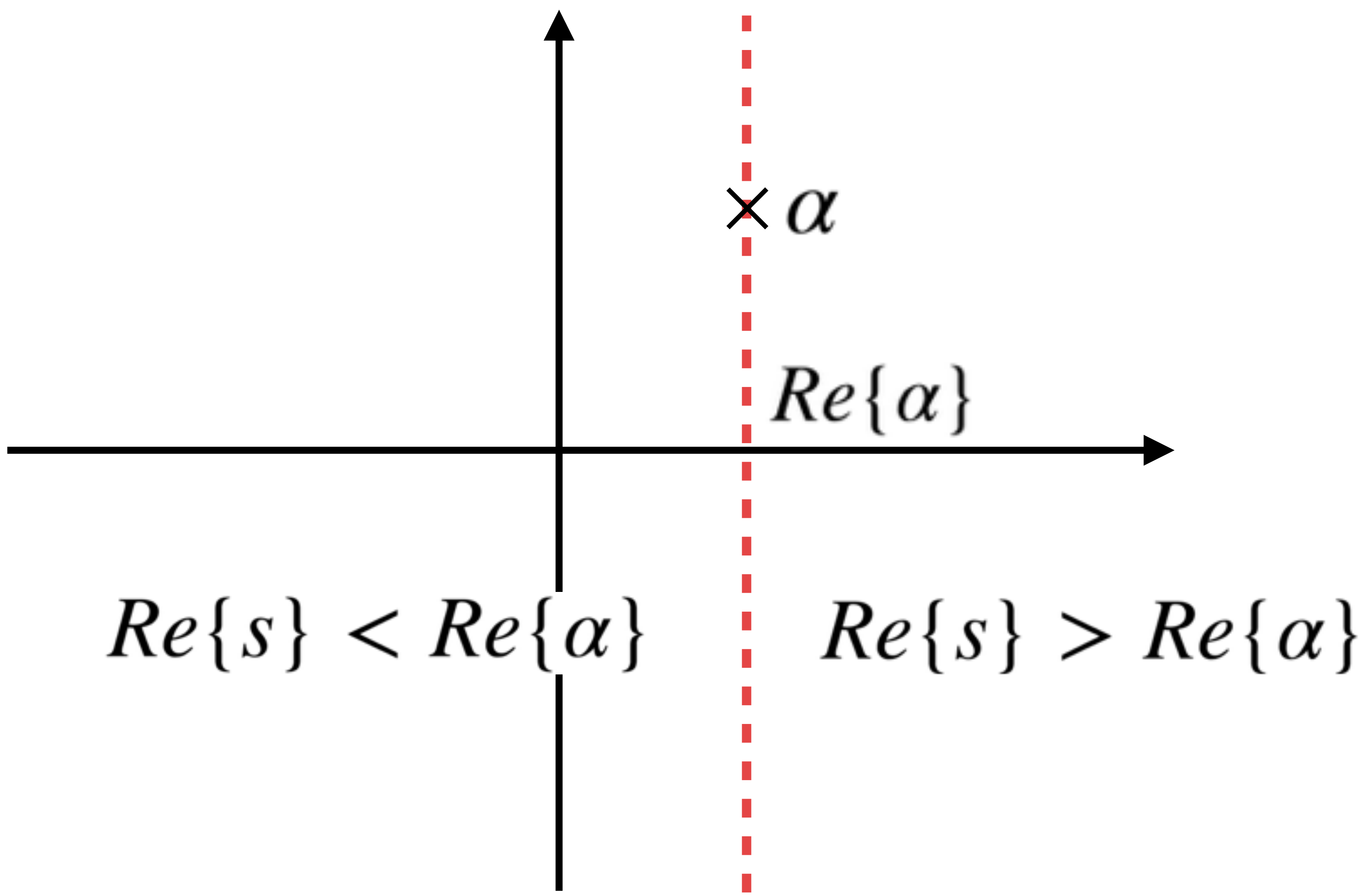
  have the same Laplace transform $X(s) = \frac{1}{s-\alpha}$?
- Well, they don't.
- The former requires $Re\{s\} > Re\{\alpha\}$ and the latter requires $Re\{s\} < Re\{\alpha\}$.
- The Laplace transform is not complete without the specification of the region of convergence (ROC).

The region of convergence

- Defined as the region in the *s*-plane in which

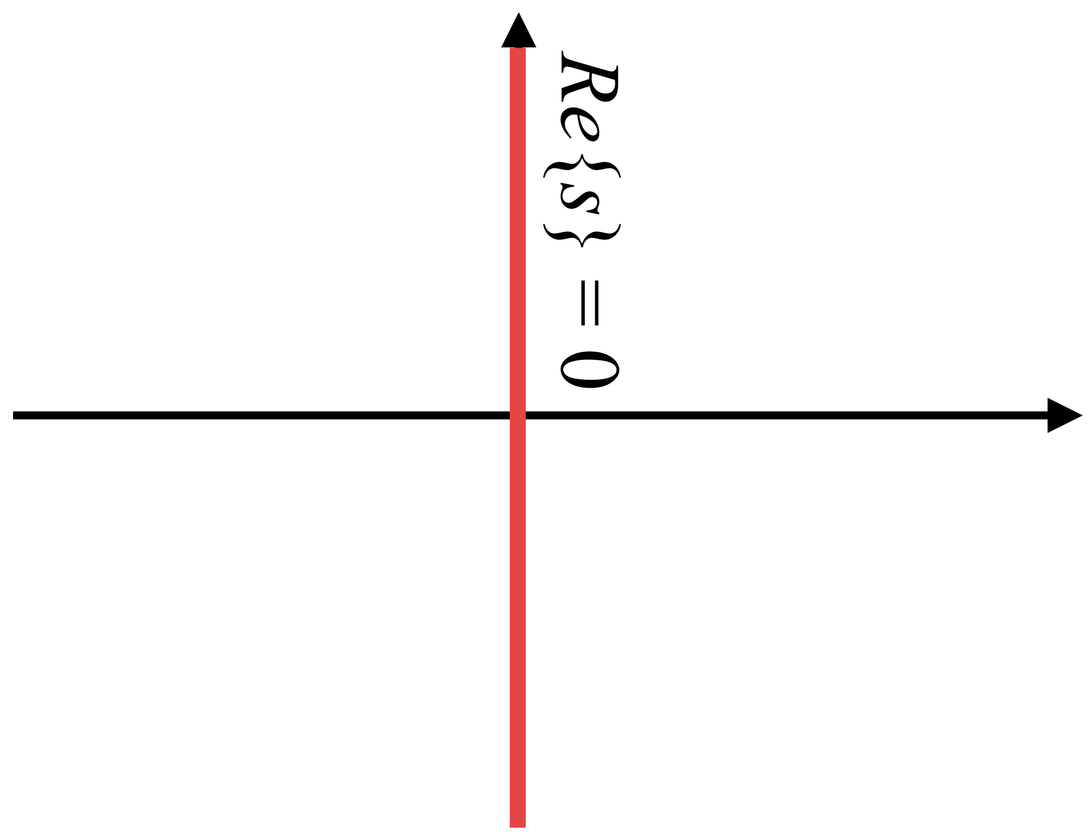
$$\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

converges.



The region of convergence

- If the ROC includes $Re\{s\} = 0$, then the CTFT converges and hence is well-defined.



  - For $x(t) = e^{\alpha t}u(t)$, this means $Re\{\alpha\} < 0$.
  - For $x(t) = -e^{\alpha t}u(-t)$, this means $Re\{\alpha\} > 0$.

Examples continued

- Problem: Find the Laplace transform of

$$x(t) = 3e^{-2t}u(t) - 2e^{-3t}u(t)$$

- Solution:

$$X(s) = 3\int_0^\infty e^{-2t}e^{-st}\,dt - 2\int_0^\infty e^{-3t}e^{-st}\,dt$$

$$= \frac{3}{s+2} - \frac{2}{s+3}$$

assuming $Re\{s\} > -2$

assuming $Re\{s\} > -3$

more restrictive

ROC: $Re\{s\} > -2$

Examples continued

- <u>Problem</u>: Find the Laplace transform of

$$x(t) = 3e^{-2t}u(t) - 2e^{-3t}u(t)$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+3}$$

ROC:

$Re\{s\} > -2$

- Since the ROC includes $Re\{s\} = 0$, the CTFT is well-defined and is given by

$$X(j\Omega) = \frac{3}{2+j\Omega} - \frac{2}{3+j\Omega}$$

Examples continued

- Problem: Find the Laplace transform of

$$x(t) = 3e^{-2t}u(t) - 2e^{3t}u(-t)$$

- Solution:

$$X(s) = \boxed{\frac{3}{s+2}} + \boxed{\frac{2}{s-3}}$$

assuming $Re\{s\} > -2$ 	assuming $Re\{s\} < 3$

ROC:
$-2 < Re\{s\} < 3$

- Since the ROC includes $Re\{s\} = 0$,

$$X(j\Omega) = \frac{3}{2 + j\Omega} + \frac{2}{-3 + j\Omega}$$

Zeros and poles

- The Laplace transform usually ends up being a rational function

$$X(s) = \frac{N(s)}{D(s)}$$

  with polynomial $N(s)$ and $D(s)$.

- Zeros: Points on the s-plane where $X(s) = 0$
- Poles: Points on the s-plane where $X(s) = \infty$

Zeros and poles

- <u>Problem</u>: For the same example

$$x(t) = 3e^{-2t}u(t) - 2e^{3t}u(-t)$$

with the Laplace transform

$$X(s) = \frac{3}{s+2} + \frac{2}{s-3}$$

find the poles and zeros.

- <u>Solution:</u>

$$X(s) = \frac{3(s-3) + 2(s+2)}{(s+2)(s-3)} = \frac{5s-5}{(s+2)(s-3)}$$

Poles: s = -2 and $s = 3$ **Zeros:** $s = 1$

Hidden zeros and poles

- If the degree of the polynomials $N(s)$ and $D(s)$ are different, there are hidden zeros or poles at infinity.
- <u>Example:</u>

$$X(s) = \frac{5s - 5}{(s+2)(s-3)}$$

Poles: $s = -2$ and $s = 3$ **Zeros:** $s = 1$

$$\lim_{s \to \infty} X(s) = 0$$

- Therefore, $s = \infty$ is a hidden zero.

Examples continued

- Problem: Find the Laplace transform of

$$x(t) = e^{-\alpha t}\cos(\beta t)u(t)$$

with $\alpha, \beta \geq 0$

- Solution:

$$x(t) = 0.5e^{-\alpha t}e^{j\beta t}u(t) + 0.5e^{-\alpha t}e^{-j\beta t}u(t)$$

$$X(s) = \frac{0.5}{s+\alpha-j\beta} + \frac{0.5}{s+\alpha+j\beta}$$

ROC: $Re\{s\} > -\alpha$

Examples continued

$$x(t) = e^{-\alpha t}\cos(\beta t)u(t)$$

$$X(s) = \frac{0.5}{s+\alpha-j\beta} + \frac{0.5}{s+\alpha+j\beta}$$

ROC: $Re\{s\} > -\alpha$

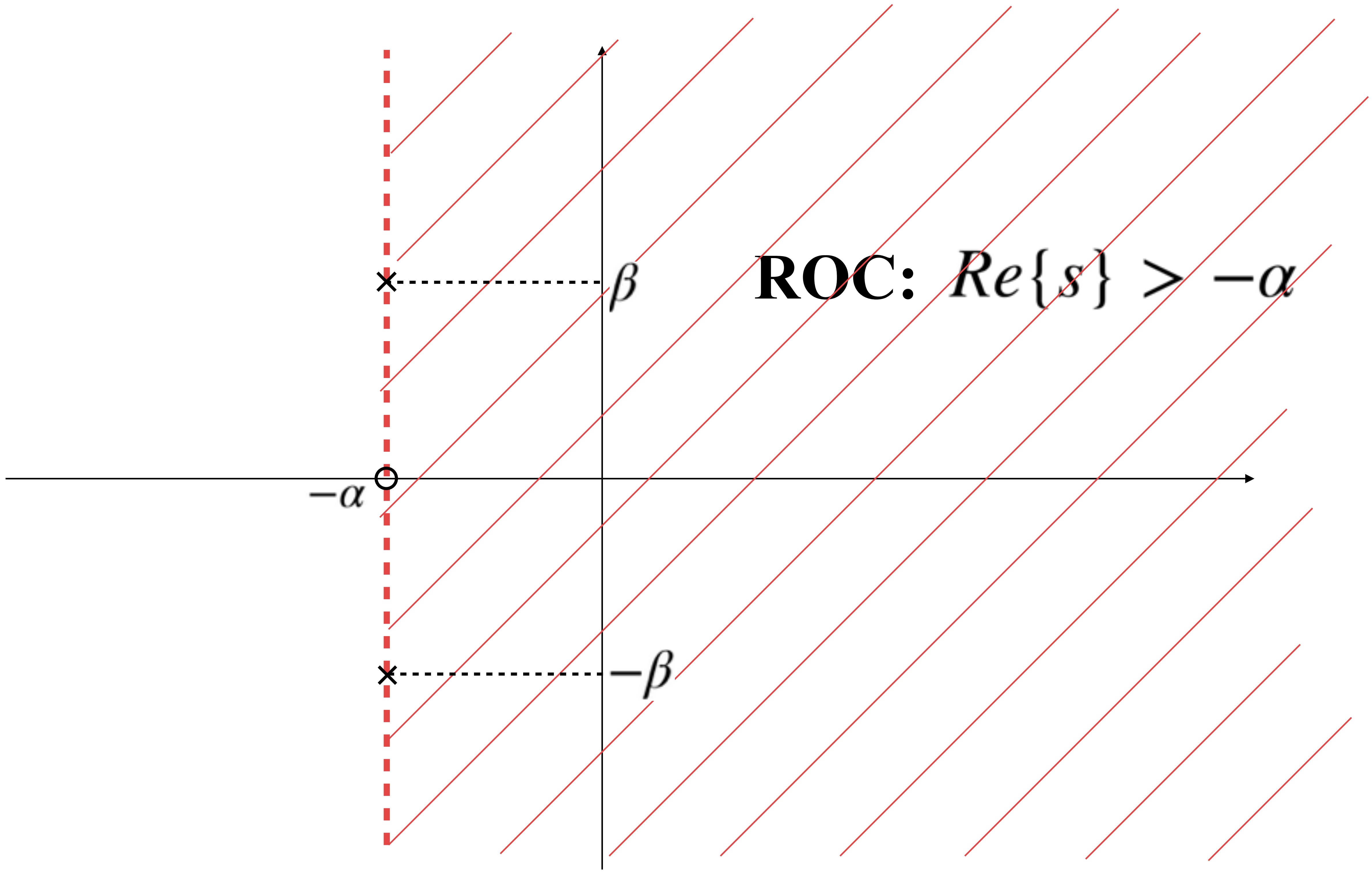
- Simplifying,

$$X(s) = \frac{0.5(s+\alpha+\cancel{j\beta}) + 0.5(s+\alpha\cancel{-j\beta})}{(s+\alpha-j\beta)(s+\alpha+j\beta)}$$

$$= \frac{s+\alpha}{(s+\alpha-j\beta)(s+\alpha+j\beta)}$$

Poles: $s=-\alpha+j\beta$ and $s=-\alpha-j\beta$ **Zeros:** $s=-\alpha$ and $s=\infty$

The pole-zero plot and ROC



Examples continued

- Had the signal been left-sided, i.e.,

$$x(t) = -e^{-\alpha t}\cos(\beta t)u(-t)$$

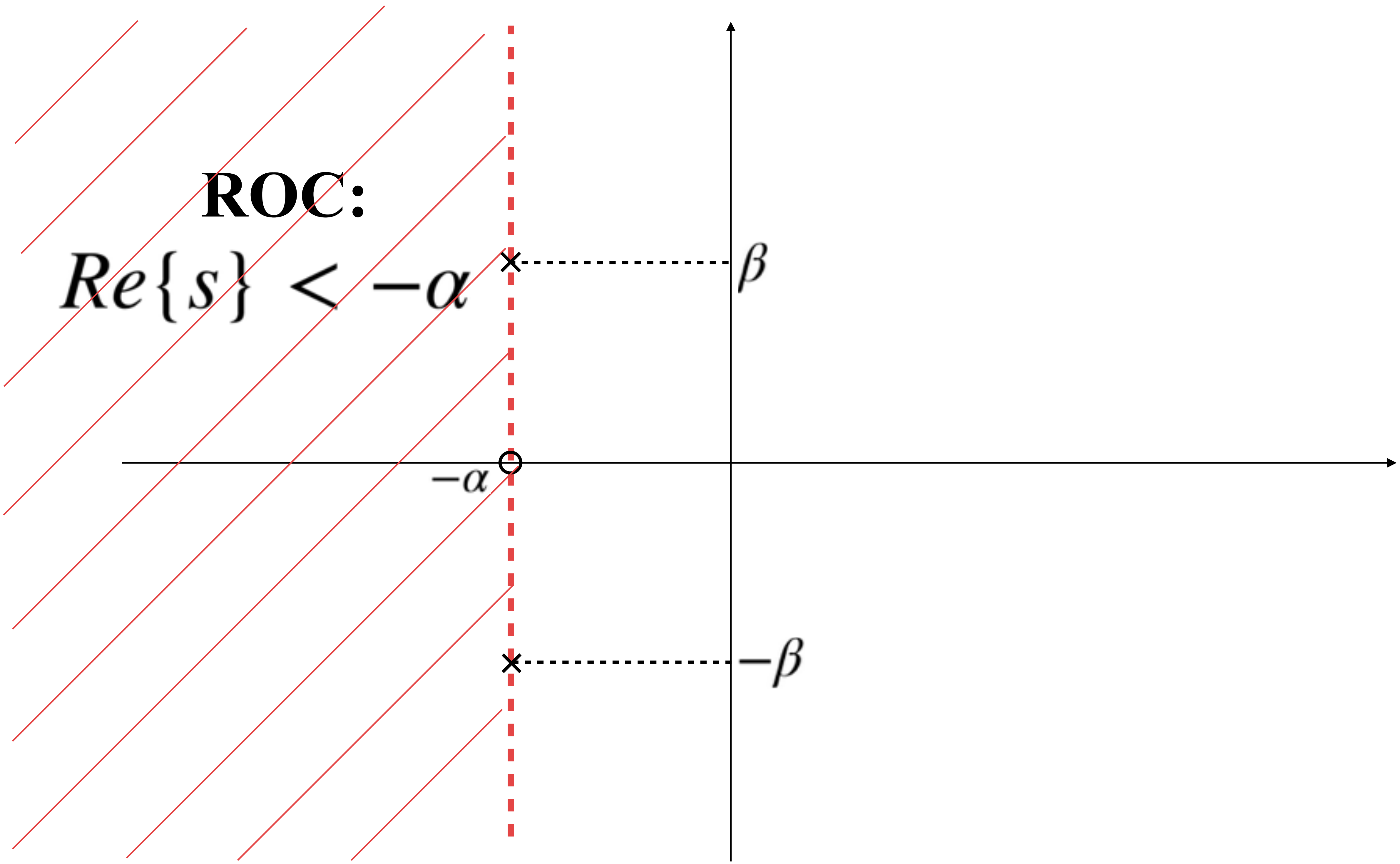
we would have had the same $X(s)$:

$$X(s) = \frac{s+\alpha}{(s+\alpha-j\beta)(s+\alpha+j\beta)}$$

with the opposite ROC:

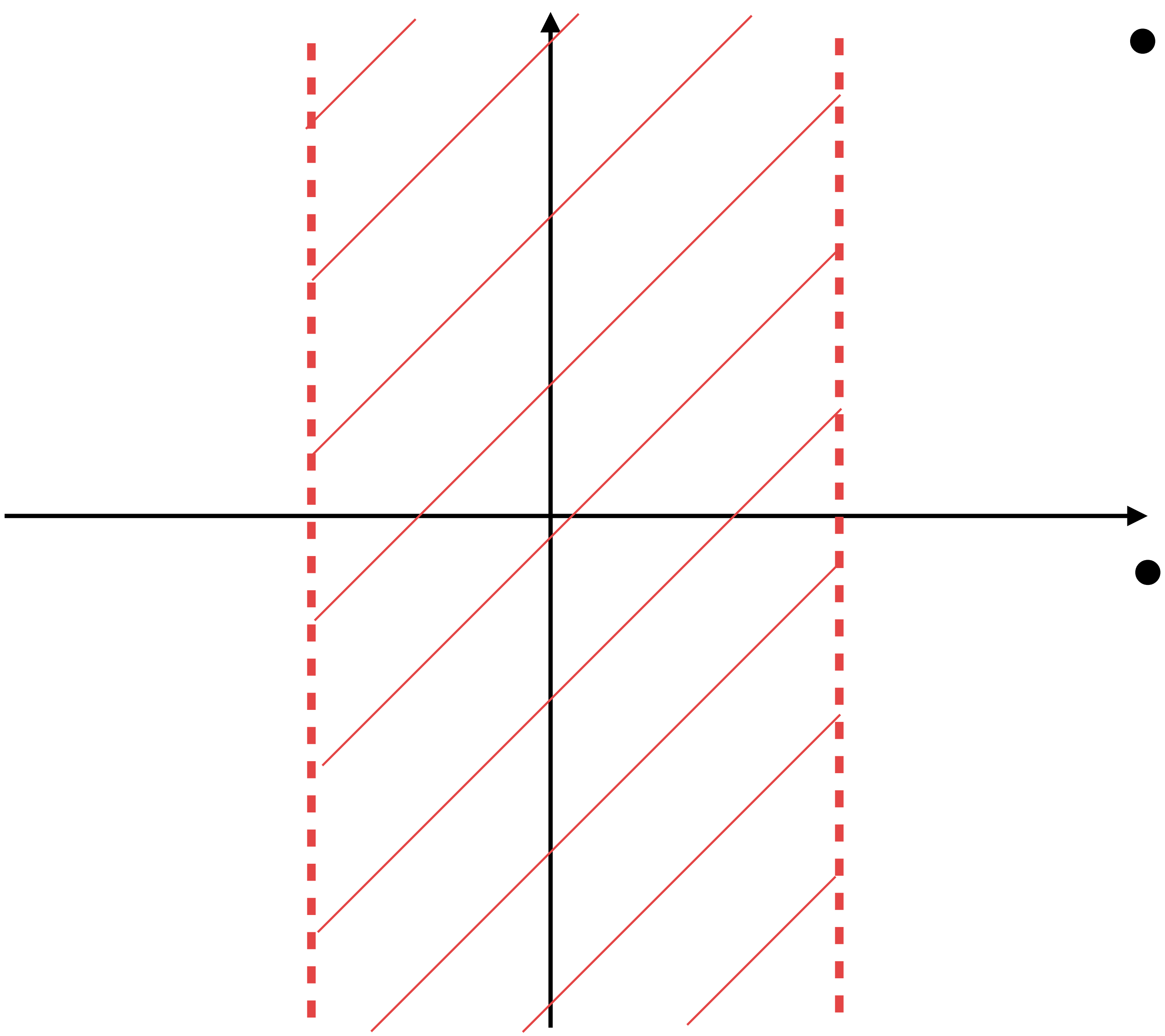
$$Re\{s\} < -\alpha$$

The pole-zero plot and ROC



Properties

1) The ROC is always an infinite vertical strip.



- The left boundary may degenerate into negative infinity or disappear altogether
- The right boundary may degenerate into positive infinity or disappear altogether

Properties

2) The ROC can never contain any poles.

3) If $x(t)$ is of finite duration, the ROC is the entire plane except possibly at $\pm\infty$.

- <u>Example</u>: $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

ROC: all s

Properties

4) If $x(t)$ is right-sided, then the ROC will be of the form $Re\{s\} > \alpha$, possibly excluding infinity.

- Proof: If $x(t) = 0$ for $t < T$ for some T, and if $Re\{s\} = a$ is included in the ROC, then

$$\int_T^{\infty} x(t)e^{-at}e^{-j\Omega t}dt$$

converges.

But then, so does $\int_T^{\infty} x(t)e^{-bt}e^{-j\Omega t}dt$ for any $b > a$

Properties

- For right-sided sequences, $s = \infty$ will be excluded from the ROC if $x(t)$ is not causal.
  - Causal: $x(t) = 0 \quad \text{for all} \quad t < 0$
  - If $x(t)$ is not causal, then

$$\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{0} x(t)\boxed{e^{-st}}dt + \int_{0}^{\infty} x(t)e^{-st}dt$$

Blows up as
$s \to \infty$

Properties

5) Similarly, if $x(t)$ is left-sided, then the ROC will be of the form $Re\{s\} < \alpha$, possibly excluding $s = -\infty$.

- $s = -\infty$ will be excluded from the ROC if $x(t)$ is not anti-causal.
  - Anti-causal: $x(t) = 0 \quad \text{for all} \quad t > 0$
  - If $x(t)$ is not anti-causal, then

$$\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{0} x(t)e^{-st}dt + \int_{0}^{\infty} x(t)\underbrace{e^{-st}}_{\text{Blows up as } s \to -\infty}dt$$

Properties

6) Finally, if $x(t)$ is two-sided, then the ROC will be of the form $\alpha_1 < Re\{s\} < \alpha_2$

- Example: $x(t) = e^{-\alpha|t|}$ for some $\alpha > 0$

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-st}\, dt \\
&= \int_{-\infty}^{0} e^{\alpha t} e^{-st}\, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-st}\, dt \\
&= \frac{1}{\alpha - s} e^{(\alpha - s)t} \Big|_{-\infty}^{0} - \frac{1}{\alpha + s} e^{-(\alpha + s)t} \Big|_{0}^{\infty}
\end{aligned}
$$

Properties

- Example: $x(t) = e^{-\alpha|t|}$ for some $\alpha > 0$

$$X(s) = \frac{1}{\alpha - s} e^{(\alpha - s)t}\Big|_{-\infty}^{0} - \frac{1}{\alpha + s} e^{-(\alpha + s)t}\Big|_{0}^{\infty}$$

$$= \underbrace{\frac{1}{\alpha - s}}_{\substack{\text{assuming} \\ Re\{s\} < \alpha}} + \underbrace{\frac{1}{\alpha + s}}_{\substack{\text{assuming} \\ Re\{s\} > -\alpha}} = \frac{2\alpha}{(\alpha - s)(\alpha + s)} = \frac{-2\alpha}{s^2 - \alpha^2}$$

ROC: $-\alpha < Re\{s\} < \alpha$

Properties

7) Linearity:

$$x_1(t) \rightarrow X_1(s) \text{ with ROC } = \mathcal{R}_1$$

$$x_2(t) \rightarrow X_2(s) \text{ with ROC } = \mathcal{R}_2$$

implies

$$ax_1(t) + bx_2(t) \rightarrow aX_1(s) + bX_2(s)$$

with ROC $\supset (\mathcal{R}_1 \cap \mathcal{R}_2)$

- The actual ROC might be larger than $(\mathcal{R}_1 \cap \mathcal{R}_2)$ because zeros may cancel poles.

Properties

8) Time shifting:

$$x_1(t) \rightarrow X_1(s) \text{ with ROC } = \mathcal{R}_1$$

implies

$$x_1(t - t_0) \rightarrow X_1(s)e^{-st_0} \text{ with ROC } = \mathcal{R}_1$$

(exclusion/inclusion of $s = -\infty$ or $s = \infty$ possible)

9) Time reversal:

$$x_1(t) \rightarrow X_1(s) \text{ with ROC } = \mathcal{R}_1$$

implies

$$x_1(-t) \rightarrow X_1(-s) \text{ with ROC } = -\mathcal{R}_1$$

Properties

10) Shifting in the s-domain:

$$x_1(t) \rightarrow X_1(s) \text{ with ROC } = \mathcal{R}_1$$

implies

$$x_1(t)e^{s_0 t} \rightarrow X_1(s - s_0) \text{ with ROC } = \mathcal{R}_1 + Re\{s_0\}$$

- <u>Special case</u>: $s_0 = j\Omega_0$

$$x_1(t)e^{j\Omega_0 t} \rightarrow X_1(s - j\Omega_0) \text{ with ROC } = \mathcal{R}_1$$

- This is nothing but a upward shift in the s-plane.

Properties

11) Convolution:

$$x_1(t) \rightarrow X_1(s) \text{ with ROC } = \mathcal{R}_1$$

$$x_2(t) \rightarrow X_2(s) \text{ with ROC } = \mathcal{R}_2$$

implies

$$x_1(t) \star x_2(t) \rightarrow X_1(s)X_2(s)$$

with ROC $\supset (\mathcal{R}_1 \cap \mathcal{R}_2)$

- The actual ROC might be larger than $(\mathcal{R}_1 \cap \mathcal{R}_2)$ because zeros may cancel poles.

Properties

12) Differentiation in time:

$$x_1(t) \to X_1(s) \text{ with ROC } = \mathcal{R}_1$$

implies

$$\frac{dx_1(t)}{dt} \to sX_1(s)$$

with ROC $\supset \mathcal{R}_1$

- The actual ROC might be larger than $\mathcal{R}_1$ because a pole at $s = 0$ may be canceled.

Properties

13) Differentiation in the s-Domain:

$$x_1(t) \rightarrow X_1(s) \text{ with ROC } = \mathcal{R}_1$$

implies

$$t x_1(t) \rightarrow -\frac{dX_1(s)}{ds}$$

with ROC $= \mathcal{R}_1$

Inverting the Laplace transform

- For any $s = \alpha + j\Omega$ inside the ROC we have

$$X(\alpha + j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-\alpha t}e^{-j\Omega t}dt \;\; = \mathrm{CTFT}\left\{x(t)e^{-\alpha t}\right\}$$

Thus, we can write

$$x(t)e^{-\alpha t} = \mathrm{CTFT}^{-1}\left\{X(\alpha + j\Omega)\right\}$$

or equivalently

$$x(t) = e^{\alpha t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha + j\Omega)e^{j\Omega t}d\Omega$$

Partial fraction expansion

- In case this direct formula is not useful, we can use the partial fraction expansion technique.
- Example: Find $x(t)$ if

$$X(s) = \frac{s}{(s+1)(s+2)} \quad \text{with ROC} \;: Re\{s\} > -1$$

- Solution:

$$X(s) = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

$$A + B = 1$$

$$2A + B = 0$$

$$A = -1, \;\; B = 2$$

Partial fraction expansion

- Example: Find $x(t)$ if

$$X(s) = \frac{s}{(s+1)(s+2)} \quad \text{with ROC} \ : Re\{s\} > -1$$

- Solution:

$$X(s) = \frac{A}{s+1} + \frac{B}{s+2} \qquad\qquad A = -1, \ \ B = 2$$

- Therefore, the solution must be given by

$$x(t) = -e^{-t}u(t) + 2e^{-2t}u(t)$$

Partial fraction expansion

- Example: Find $x(t)$ if

$$X(s) = \frac{s^2}{s^2+1} \quad \text{with ROC: } Re\{s\} > 0$$

- Solution: We need the numerator to be of lower degree than the denominator

$$X(s) = 1 - \frac{1}{s^2+1} = 1 - \left(\frac{A}{s-j} + \frac{B}{s+j}\right)$$

$$= 1 - \frac{A(s+j) + B(s-j)}{(s-j)(s+j)}$$

$$A + B = 0$$

$$jA - jB = 1$$

$$A = -0.5j \quad B = 0.5j$$

Partial fraction expansion

- Example: Find $x(t)$ if

$$X(s) = \frac{s^2}{s^2+1} \quad \text{with ROC: } Re\{s\} > 0$$

$$X(s) = 1 - \left(\frac{A}{s-j} + \frac{B}{s+j} \right) \qquad A = -0.5j \quad B = 0.5j$$

- Therefore, the solution must be given by

$$x(t) = \delta(t) + 0.5je^{jt}u(t) - 0.5je^{-jt}u(t)$$

$$= \delta(t) - \frac{e^{jt} - e^{-jt}}{2j} u(t) = \delta(t) - \sin(t)u(t)$$

Partial fraction expansion

- What if there is a repeated pole?
- Example: Find $x(t)$ if

$$X(s) = \frac{s-3}{(s+1)^2} \quad \text{with ROC: } Re\{s\} > -1$$

- Solution: Use the following expansion:

$$X(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} = \frac{A(s+1)+B}{(s+1)^2}$$

$$A = 1, B = -4$$

$$x(t) = e^{-t}u(t) + ?$$

Partial fraction expansion

$$X(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} = \frac{A(s+1)+B}{(s+1)^2}$$

$$A = 1, B = -4$$

$$x(t) = e^{-t}u(t) \ + ?$$

- Noting that $-\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$, we use

$$tx_1(t) \rightarrow -\frac{dX_1(s)}{ds}$$

to conclude $te^{-t}u(t) \rightarrow \dfrac{1}{(s+1)^2}$

Partial fraction expansion

$$X(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} = \frac{A(s+1)+B}{(s+1)^2}$$

$$A = 1, B = -4$$

$$x(t) = e^{-t}u(t) - 4te^{-t}u(t)$$

$$= (1 - 4t)e^{-t}u(t)$$

Partial fraction expansion

- Example: Find $x(t)$ if

$$X(s) = \frac{1}{(s^2+1)^2} \text{ with ROC: } Re\{s\} > 0$$

- Solution:

$$X(s) = \frac{1}{(s+j)^2(s-j)^2}$$

$$= \frac{A}{s+j} + \frac{B}{(s+j)^2} + \frac{C}{s-j} + \frac{D}{(s-j)^2}$$

$$= \frac{A(s+j)(s-j)^2 + B(s-j)^2 + C(s-j)(s+j)^2 + D(s+j)^2}{(s+j)^2(s-j)^2}$$

- Solution:

$$X(s) = \frac{1}{(s+j)^2(s-j)^2}$$

$$= \frac{A}{s+j} + \frac{B}{(s+j)^2} + \frac{C}{s-j} + \frac{D}{(s-j)^2}$$

$$= \frac{A(s+j)(s-j)^2 + B(s-j)^2 + C(s-j)(s+j)^2 + D(s+j)^2}{(s+j)^2(s-j)^2}$$

- The two numerators should be equal for all s.
- Substitute $s = j$ to get $1 = D(2j)^2$ $\Longrightarrow D = -1/4$
- Substitute $s = -j$ to get $1 = B(-2j)^2$ $\Longrightarrow B = -1/4$
- Substitute $s = j$ in the derivative of the numerator: $0 = C(2j)^2 + 2D(2j)$ $\Longrightarrow C = -j/4$

- Solution:

$$X(s) = \frac{1}{(s+j)^2(s-j)^2}$$

$$= \frac{A}{s+j} + \frac{B}{(s+j)^2} + \frac{C}{s-j} + \frac{D}{(s-j)^2}$$

$$= \frac{A(s+j)(s-j)^2 + B(s-j)^2 + C(s-j)(s+j)^2 + D(s+j)^2}{(s+j)^2(s-j)^2}$$

- Substitute $s = j$ to get $1 = D(2j)^2$ $\Longrightarrow D = -1/4$
- Substitute $s = -j$ to get $1 = B(-2j)^2$ $\Longrightarrow B = -1/4$
- Substitute $s = j$ in the derivative of the numerator: $0 = C(2j)^2 + 2D(2j)$ $\Longrightarrow C = -j/4$
- Substitute $s = -j$ in the derivative of the numerator: $0 = A(-2j)^2 + 2B(-2j)$ $\Longrightarrow A = j/4$

- Solution:

$$X(s) = \frac{1}{(s+j)^2(s-j)^2}$$

$$= \frac{A}{s+j} + \frac{B}{(s+j)^2} + \frac{C}{s-j} + \frac{D}{(s-j)^2}$$

$A = j/4 \qquad B = -1/4 \qquad C = -j/4 \qquad D = -1/4$

$$x(t) = \frac{j}{4}e^{-jt}u(t) - \frac{1}{4}te^{-jt}u(t) - \frac{j}{4}e^{jt}u(t) - \frac{1}{4}te^{jt}u(t)$$

$$= -\frac{j}{4}(e^{jt} - e^{-jt})u(t) - \frac{1}{4}(e^{jt} + e^{-jt})tu(t)$$

$$= \frac{1}{2}\sin(t)u(t) - \frac{1}{2}t\cos(t)u(t)$$