University of California at Riverside

Department of Electrical Engineering

EE 110A: Signals and Systems

Midterm Exam (Dec 13th, 2013)

Question	MAX	GRADE
1	16	
2	14	
3	25	
4	15	
5	15	
6	15	
TOTAL	100	

Surname **First name**

Student ID Number

Time: 150 Minutes

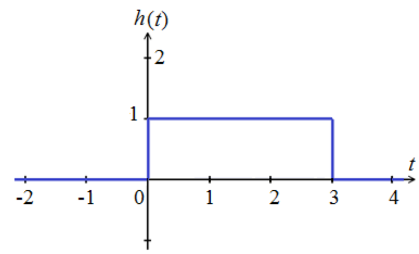
Instructions:

- **Attempt all questions.**
- **One page of <u>one-sided hand-written letter-size</u> information sheet is allowed.**
- **No other reference material is allowed.**
- **<u>No calculator of any kind</u> is allowed.**
- **All answers should be written in this booklet.**
- **In addition to final answers, include the necessary steps to show your derivations.**
- **Enjoy your exam, and *Good Luck!***

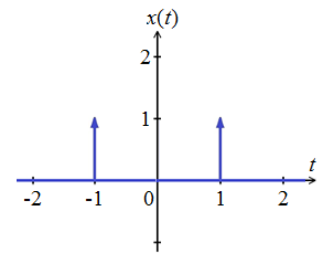
Instructor's Name: Dr. Hamed Mohsenian-Rad

<u>Question One</u>:

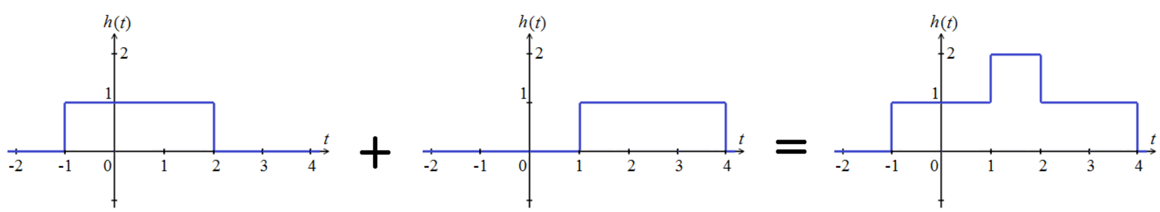
Part A) Consider an LTI system with unit impulse response $h(t)$



Plot the output of this system to the input signal $x(t)$ below.

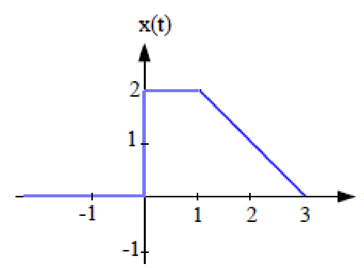


Using superposition and time-shift properties of LTI systems:

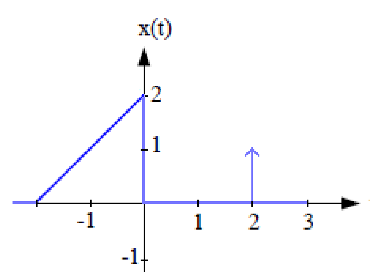


Part B) Given the following signal, sketch signal $y(t) = x(1-t)u(-t) + x(t)\delta(t-2)$.

$$x(t) = \begin{cases} 2 & 0 \le t < 1 \\ 3-t & 1 \le t < 3 \\ 0 & otherwise \end{cases} \Rightarrow x(1-t) = \begin{cases} 2 & 0 \le 1-t < 1 \\ 3-(1-t) & 1 \le 1-t < 3 \\ 0 & otherwise \end{cases} = \begin{cases} 2 & 0 \le t < 1 \\ 2+t & -2 \le t < 0 \\ 0 & otherwise \end{cases}$$



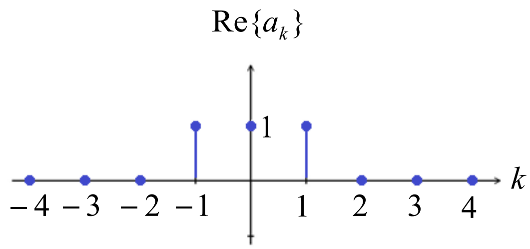
Therefore, we have $x(1-t)u(-t) = \begin{cases} 2+t & -2 \le t < 0 \\ 0 & otherwise \end{cases}$.

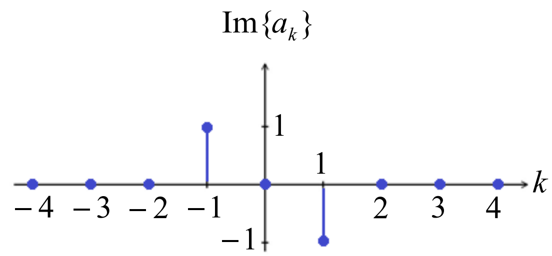


Also, using the sampling property of the delta function, we have $x(t)\delta(t-2) = x(2)\delta(t-2) = \delta(t-2)$.

<u>Question Two</u>:

Consider the Fourier series coefficients $\{a_k\}$ of a periodic signal $x(t)$ with fundamental period $T = 2\pi$:





Part A) Obtain a mathematical expression for signal $x(t)$ in <u>time</u> domain.

First, we note that based on the diagram above, we have $a_{-1} = 1 + j$, $a_0 = 1$, and $a_1 = 1 - j$.

From Slide #13 in Chapter 3, we have

$$
\begin{aligned}
x(t) &= a_{-1}e^{-jt} + a_0 + a_1 e^{jt} = (1+j)e^{-jt} + 1 + (1-j)e^{jt} \\
&= 1 - j\left(e^{jt} - e^{-jt}\right) + \left(e^{jt} + e^{-jt}\right) = 1 + \left(\frac{e^{jt} - e^{-jt}}{j}\right) + \left(e^{jt} + e^{-jt}\right) \\
&= 1 + 2\left(\frac{e^{jt} - e^{-jt}}{2j}\right) + 2\left(\frac{e^{jt} + e^{-jt}}{2}\right) = 1 + 2\sin(t) + 2\cos(t).
\end{aligned}
$$

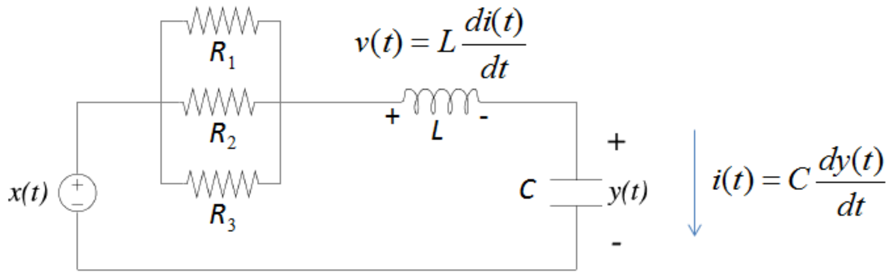
Part B) What is the average power P_∞ of periodic signal $x(t)$?

Using Parseval's Theorem on Slide # 24 in Chapter 3, we have:

$$
P_\infty = |a_{-1}|^2 + |a_0|^2 + |a_1|^2 = |1+j|^2 + |1|^2 + |1-j|^2 = 2 + 1 + 2 = 5
$$

k k

<u>Question Three</u>: Consider the following circuit with three resistors, one capacitor, and one inductor. The input signal is $x(t)$ and the output signal is $y(t)$. We have $R_1 = R_2 = R_3 = 3$, $L = 0.5$, $C = 0.5$.



Part A) Obtain the transfer function of the system.

Since the two resistors are parallel, we can replace them with $R = \dfrac{R_1R_2R_3}{R_1R_2 + R_1R_2 + R_2R_3} = 1$.

From Slide 22 in Chapter 9, we have

$$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)} = \frac{4}{s^2 + 2s + 4} = \frac{4}{(s+1+j\sqrt{3})(s+1-j\sqrt{3})}$$

Part B) Identify the zeros and poles of the transfer function.

$Zeros$: None $\qquad\qquad$ $Poles$: $-1 \pm j\sqrt{3}$

Part C) Is the system stable? Justify your answer.

Yes, the system is stable because the real parts of both poles are negative.

Part D) Assume that $h(t)$ denotes unit impulse response and $s(t)$ denote the unit step response for this system. Calculate the following signal values:

$$h(+\infty) = \lim_{s\to 0} sH(s) = \lim_{s\to 0} \frac{4s}{s^2 + 2s + 4} = 0 .$$

To calculate $s(+\infty)$, first we note that the Laplace transform of the unit step response for this system is

$$\frac{1}{s}H(s) = \frac{1}{s}\frac{4}{s^2 + 2s + 4} .$$

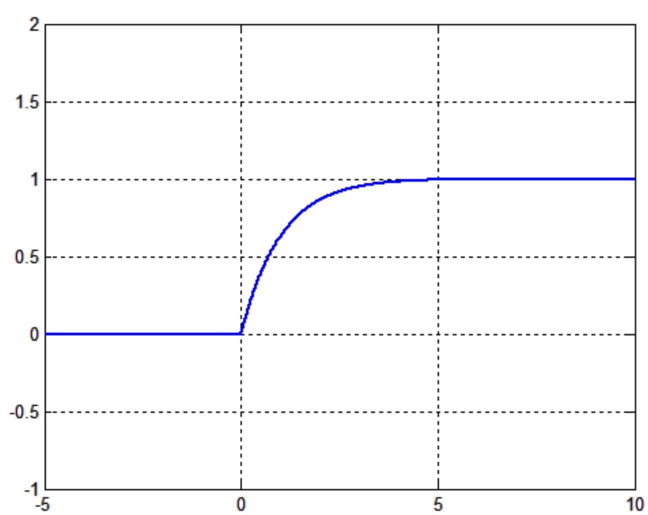
Therefore, we have

$$s(+\infty) = \lim_{s\to 0} s\frac{1}{s}H(s) = \lim_{s\to 0} \frac{4}{s^2 + 2s + 4} = 1 .$$

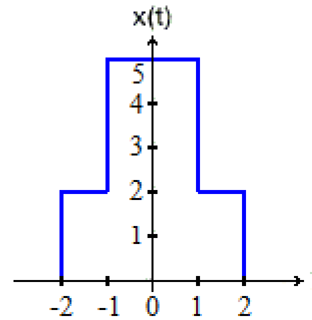
<u>Question Four</u>: Consider an LTI system with the following Transfer Function: $H(s) = \dfrac{1}{s+1}$.

Calculate and <u>plot</u> the *unit step response* of this system, i.e., the output when the input is $x(t) = u(t)$.

$$Y(s) = H(s)X(s) = \frac{1}{s+1} \times \frac{1}{s} = -\frac{1}{s+1} + \frac{1}{s} \qquad \Rightarrow \qquad y(t) = L^{-1}\{Y(s)\} = u(t) - e^{-t}u(t) = \left(1 - e^{-t}\right)u(t)$$



<u>Question Five</u>: Obtain the Fourier Transform $X(j\omega)$ of the following signal:



Signal $x(t)$ can be written as a weighted summation of two square pulses:

$$x(t) = 3x_1(t) + 2x_2(t),$$

where

$$x_1(t) = \begin{cases} 1, & if\ |t| \leq 1, \\ 0, & otherwise \end{cases} \quad \text{and} \quad x_2(t) = \begin{cases} 1, & if\ |t| \leq 2, \\ 0, & otherwise \end{cases}.$$

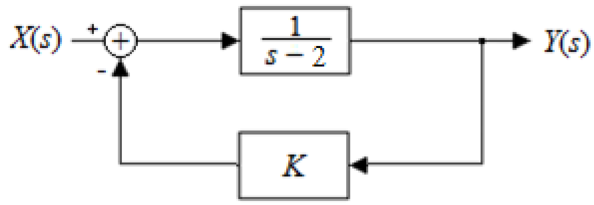
From Slide #12, we know that given a square pulse $y(t)$, where

$$y(t) = \begin{cases} 1, & if\ |t| \leq \Delta, \\ 0, & otherwise \end{cases},$$

the Fourier transform is $Y(j\omega) = \frac{2}{\omega}\sin(\omega\Delta)$. Following this equation, we have $X_1(j\omega) = \frac{2}{\omega}\sin(\omega)$ and $X_2(j\omega) = \frac{2}{\omega}\sin(2\omega)$. Thus, the Fourier transform of $x(t)$ is obtained as follows.

$$X(j\omega) = 3X_1(j\omega) + 2X_2(j\omega) = 3 \times \frac{2}{\omega}\sin(\omega) + 2 \times \frac{2}{\omega}\sin(2\omega) = \frac{6}{\omega}\sin(\omega) + \frac{4}{\omega}\sin(2\omega).$$

Question Six: Consider the following system with feedback:



For what values of the feedback gain K the above closed-loop system is stable?

The following relationship holds between the Laplace transforms of the input and output signals:

$$Y(s) = \left(X(s) - KY(s)\right)\frac{1}{s-2} \quad \Rightarrow \quad (s-2)Y(s) = X(s) - KY(s) \quad \Rightarrow \quad (s-2+K)Y(s) = X(s)$$

Therefore, the closed-loop transfer function can be obtained as follows:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s-2+K}.$$

The pole of the above closed-loop system is

$$p = 2 - K.$$

For the closed-loop system to be stable, the real-values of the above poles must be negative. That is,

$$K > 2.$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & T_1 < \lvert t \rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & \lvert t \rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt} x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j \frac{d}{d\omega} X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega) \rvert = \lvert X(-j\omega) \rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ \[$x(t)$ real\] $x_o(t) = \mathcal{O}d\{x(t)\}$ \[$x(t)$ real\]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\omega) \rvert^2 d\omega$		

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} < -\alpha$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$

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