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EE110B-SIGNALS AND SYSTEMS
HOMEWORK 6 SOLUTIONS

Problem 1:
a) $x[n] = 0.1^n u[n]$.
b) $x[n] = -0.1^n u[-n-1]$.
c) Multiplying both the numerator and the denominator by z^2, and applying partial fraction expansion, we get

$$X(z) = \frac{0.5z + 0.1}{(z-0.3)(z-0.4)} = \frac{A}{z-0.3} + \frac{B}{z-0.4} = \frac{A(z-0.4)+B(z-0.3)}{(z-0.3)(z-0.4)} = \frac{(A+B)z-(0.4A+0.3B)}{(z-0.3)(z-0.4)}$$

which means

$$\begin{aligned} A+B &= 0.5 \\ 0.4A+0.3B &= -0.1 \end{aligned}$$

and therefore, $A = -2.5$ and $B = 3$. Going back to the z^{-1} notation, we obtain

$$X(z) = \frac{-2.5z^{-1}}{1-0.3z^{-1}} + \frac{3z^{-1}}{1-0.4z^{-1}} \; .$$

Since the ROC is $\{z : 0.3 < |z| < 0.4\}$, the first part must be due to a right-sided sequence, and the second due to a left-sided one. Using this fact, and also observing the extra z^{-1} at the numerator for each term,

$$x[n] = -2.5(0.3)^{n-1}u[n-1] - 3(0.4)^{n-1}u[-n] \; .$$

d) The Taylor expansion of $\cos(z)$ is given by

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

for all z. Substituting z^{-1} instead, we obtain

$$\begin{aligned} \cos(z^{-1}) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n} \\ &= 1 - \frac{1}{2}z^{-2} + \frac{1}{4!}z^{-4} - \frac{1}{6!}z^{-6} + \ldots \end{aligned}$$

As with any power series, coefficients of z^{-n} correspond to $x[n]$. Therefore,

$$x[n] = \begin{cases} 0 & n < 0 \\ 0 & n \geq 0, n \text{ odd} \\ \frac{(-1)^{n/2}}{n!} & n \geq 0, n \text{ even} \end{cases}$$

Problem 2: We know that if $X(z)$ is the z-transform of $x[n]$, then

$$(1 - 0.4z^{-1})X(z)$$

is the z-transform of $x[n]-0.4x[n-1]$, because of the shifting rule and linearity of the z-transform. Further, from the convolution property, we obtain

$$\begin{aligned} W(z) &= (1-0.4z^{-1})X(z)Y(z) \\ &= (1-0.4z^{-1}) \cdot \frac{1}{1-0.5z^{-1}} \cdot \frac{1}{1-0.4z^{-1}} \\ &= \frac{1}{1-0.5z^{-1}} \, . \end{aligned}$$

Since $W(z)$ is the product of two z-transforms, its ROC must contain the intersection of the two ROCs. In other words,

$$\{z : |z| > 0.5\}$$

must be part of the resultant ROC. Since the pole at $z = 0.5$ is not cancelled, ROC is indeed as above.

Problem 3: Rewriting $X(z)$ as

$$X(z) = \frac{z^2}{z^2+z+1},$$

and performing long division, one gets

$$X(z) = 1 - z^{-1} + z^{-3} - z^{-4} + z^{-6} - z^{-7} + \ldots$$

Thus,

$$\begin{aligned} x[0] &= 1 \\ x[1] &= -1 \\ x[2] &= 0 \\ x[3] &= 1 \\ x[4] &= -1 \\ x[5] &= 0 \\ x[6] &= 1 \\ x[7] &= -1 \\ \vdots &= \vdots \end{aligned}$$

A better way of writing this is

$$x(n) = \begin{cases} 0 & n < 0 \\ 0 & n \geq 0, n = 2 \ (\mathrm{mod} 3) \\ 1 & n \geq 0, n = 0 \ (\mathrm{mod} 3) \\ -1 & n \geq 0, n = 1 \ (\mathrm{mod} 3) \end{cases}$$

Another way of writing it is

$$x[n] = \sum_{k=0}^{\infty} \delta[n-3k] - \sum_{k=0}^{\infty} \delta[n-3k-1] \, .$$

Sanity check: If we did not know how to do long division, we could still use partial fraction expansion to solve this problem. We have

$$X(z) = \frac{1}{1+z^{-1}+z^{-2}} = \frac{1}{(1-rz^{-1})(1-r^*z^{-1})}$$

where r and r^* are the roots of the polynomial

$$z^2 + z + 1 = 0 \; .$$

From the usual formula $\frac{-\frac{b}{2} \pm \sqrt{b^2-4ac}}{2a}$ for the roots of $az^2 + bz + c$, we obtain

$$r = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j\frac{2\pi}{3}} \; .$$

Using partial fraction expansion,

$$X(z) = \frac{A}{1 - rz^{-1}} + \frac{B}{1 - r^*z^{-1}} = \frac{A(1 - r^*z^{-1}) + B(1 - rz^{-1})}{(1 - rz^{-1})(1 - r^*z^{-1})}$$

and therefore

$$\begin{aligned} A + B &= 1 \\ Ar^* + Br &= 0 \end{aligned}$$

which means $A = \frac{r}{r-r^*}$ and $B = \frac{-r^*}{r-r^*}$. Inverting the z-transform, we obtain

$$x[n] = \frac{r}{r - r^*} r^n u[n] - \frac{r^*}{r - r^*} (r^*)^n u[n] \; .$$

Since

$$\left(\frac{r}{r - r^*} r^n \right)^* = -\frac{r^*}{r - r^*} (r^*)^n \; ,$$

we can succinctly write $x[n]$ as

$$\begin{aligned} x[n] &= 2\mathrm{Re}\left\{ \frac{r}{r - r^*} r^n \right\} u[n] \\ &= 2\mathrm{Re}\left\{ \frac{r}{2j\mathrm{Im}\{r\}} r^n \right\} u[n] \\ &= 2\mathrm{Re}\left\{ \frac{r}{j\sqrt{3}} r^n \right\} u[n] \\ &= \frac{2}{\sqrt{3}} \mathrm{Re}\left\{ -jr^{n+1} \right\} u[n] \\ &= \frac{2}{\sqrt{3}} \mathrm{Re}\left\{ e^{-j\frac{\pi}{2}} e^{j\frac{2\pi}{3}(n+1)} \right\} u[n] \\ &= \frac{2}{\sqrt{3}} \cos\left(\frac{2\pi n}{3} + \frac{2\pi}{3} - \frac{\pi}{2} \right) u[n] \\ &= \frac{2}{\sqrt{3}} \cos\left(\frac{2\pi n}{3} + \frac{\pi}{6} \right) u[n] \; . \end{aligned}$$

Now, this does not immediately look like our answer, but it is clear that $x[n]$ repeats itself with a period of 3 as before, so it suffices to check $x[0]$, $x[1]$, and $x[2]$. We have

$$x[0] = \frac{2}{\sqrt{3}} \cos\left(\frac{2\pi 0}{3} + \frac{\pi}{6} \right) = \frac{2}{\sqrt{3}} \cos\left(\frac{\pi}{6} \right) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = 1$$

which checks,

$$x[1] = \frac{2}{\sqrt{3}} \cos\left(\frac{2\pi 1}{3} + \frac{\pi}{6} \right) = \frac{2}{\sqrt{3}} \cos\left(\frac{5\pi}{6} \right) = \frac{2}{\sqrt{3}} \cdot \frac{-\sqrt{3}}{2} = -1$$

which also checks, and finally

$$x[2] = \frac{2}{\sqrt{3}} \cos\left(\frac{2\pi 2}{3} + \frac{\pi}{6} \right) = \frac{2}{\sqrt{3}} \cos\left(\frac{9\pi}{6} \right) = \frac{2}{\sqrt{3}} \cos\left(\frac{3\pi}{2} \right) = 0 \; .$$