UNIVERSITY OF CALIFORNIA, RIVERSIDE
Department of Electrical Engineering

WINTER 2025
EE 110B SIGNALS AND SYSTEMS
MIDTERM

You have 50 minutes to complete the exam. Please fully justify your work.

Question 1) (60 points)
In this question, you will derive the geometric sum formula

$$\sum_{k=0}^{n} \alpha^k = \frac{\alpha^{n+1} - 1}{\alpha - 1} \tag{1}$$

for any $\alpha \neq 1$ using difference equations. To see this, we will first define

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

for $n \geq 0$, and notice that $y[0] = 1$

Now,
a) Show that

$$y[n] - y[n-1] = \alpha^n \tag{2}$$

for all $n > 0$.

b) Treat (2) as a difference equation with the initial condition $y[0] = 1$, and solve it using the standard algorithm (homogeneous + particular). Does your solution agree with (1)?

c) Alternatively, show that

$$y[n] - \alpha y[n-1] = 1 \tag{3}$$

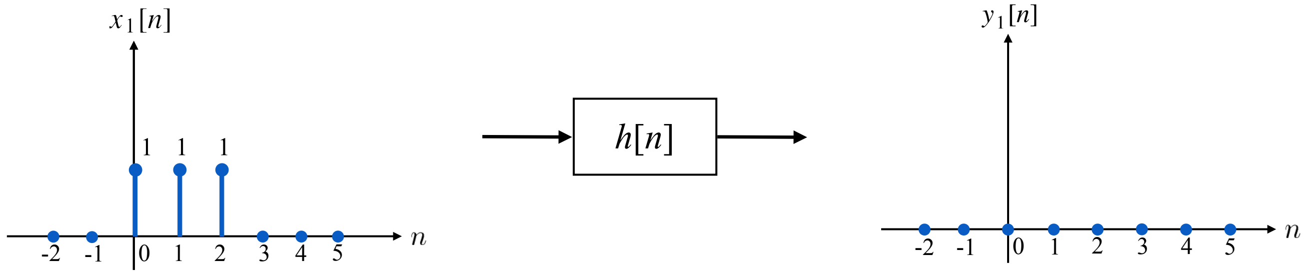
for all $n > 0$.

d) Solve this alternative difference equation (3) with the initial condition $y[0] = 1$ using the standard algorithm. Does your solution agree with (1)?

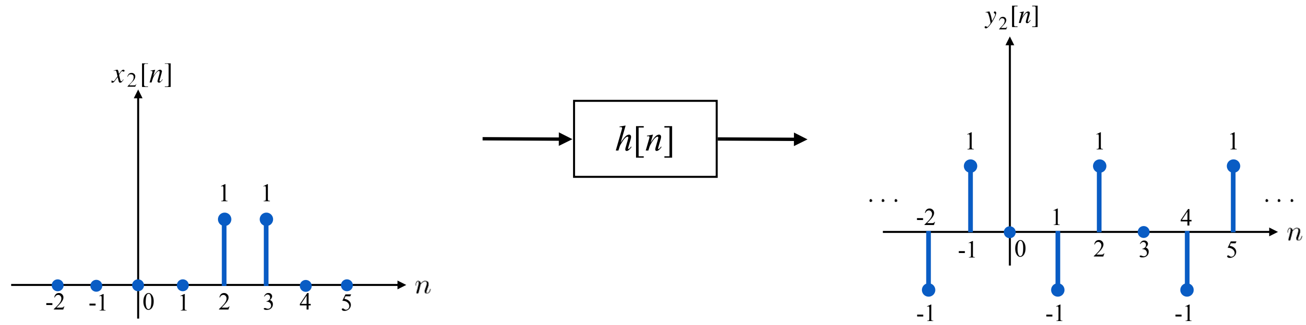
Answer:

Question 2) (40 points)

A linear and time-invariant (LTI) system has been observed to output $y_1[n]$ when $x_1[n]$ is the input, as shown below:



The same LTI system has also been observed to output $y_2[n]$ when $x_2[n]$ is the input:



a) Find the impulse response $h[n]$. You can plot $h[n]$ or describe it mathematically, whichever is easier.
b) Is this system causal? Is it stable? Finally, is it invertible?

Answer:

FORMULAS AND CONCEPTS YOU MIGHT FIND USEFUL

- **Quadratic equations:** The roots of the equation $as^2 + bs + c = 0$ are given by
$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
with the understanding that $\sqrt{-1} = j$.
- **Complex numbers:** If z is a complex number, then it can be expressed in one of two forms:
$$z = a + jb \qquad \text{or} \qquad z = re^{j\theta}$$
where
$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \ .$$
Values of r and θ can be found from a and b using
$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1}\left(\frac{b}{a}\right) \ . \end{aligned}$$
The polar form makes multiplication easy. That is, if $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then
$$z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)} \ .$$
The complex conjugate of z is $z^* = a - jb$ or $z^* = re^{-j\theta}$. We also have the following relationships:
$$\begin{aligned} z + z^* &= 2Re\{z\} &&= 2a \\ z - z^* &= 2Im\{z\} &&= 2b \\ zz^* &= |z|^2 &&= a^2 + b^2 \end{aligned}$$
It is helpful to know the following about the complex number j:
$$\begin{aligned} j^2 &= -1 \\ j^3 &= -j \\ j^4 &= 1 \\ 1/j &= -j \\ je^{j\theta} &= e^{j(\theta + \pi/2)} \end{aligned}$$
where the last equation can be read as "multiplication by j rotates a complex number by 90 degrees counter-clockwise."
- **Sine waves and complex exponentials:**
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$
- **Geometric series:** For any $\alpha \neq 1$ (complex or real),
$$\sum_{n=0}^{N} \alpha^n = \frac{1 - \alpha^{N+1}}{1 - \alpha} \ .$$
If $|\alpha| < 1$, in the limit $N \to \infty$, this becomes $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$. When $\alpha = 1$, we immediately have
$$\sum_{n=0}^{N} \alpha^n = N + 1 \ .$$
- **System properties:** A system is
  - **linear** if
$$\begin{matrix} x_1[n] \to y_1[n] \\ x_2[n] \to y_2[n] \end{matrix} \Longrightarrow ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$$
  - **time-invariant** if
$$x[n] \to y[n] \Longrightarrow x[n - n_0] \to y[n - n_0]$$
  - **memoryless** if $y[n]$ at time n depends only on $x[n]$ on time n.
  - **causal** if $y[n]$ at time n depends only on $x[k]$ on times $k \leq n$.
  - **stable** if $|x[n]| \leq B$ for some M implies $|y[n]| \leq C$ for some C.
  - **invertible** if two distinct $x_1[n]$ and $x_2[n]$ does not result in the same $y[n]$.
- **LTI Systems:** The input-output relationship in an LTI system with an impulse response $h[n]$ is given by the convolution sum
$$y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$
- **Convolution for right-sided signals:** If
$$x[n] = f[n]u[n]$$
and
$$h[n] = g[n]u[n] \ ,$$
for some $f[n]$ and $g[n]$, then then $y[n] = x[n] \star h[n]$ simplifies to
$$y[n] = u[n] \cdot \sum_{k=0}^{n} f[k]g[n-k]$$
- **Properties of convolution:**
  - **Commutativity**:
$$x[n] \star y[n] = y[n] \star x[n]$$
  - **Associativity**:
$$x[n] \star (y[n] \star z[n]) = (x[n] \star y[n]) \star z[n]$$
  - **Distribution**:
$$x[n] \star (ay[n] + bz[n]) = a(x[n] \star y[n]) + b(x[n] \star z[n])$$
  - **Time invariance**:
$$x[n] \star h[n] = y[n] \Longrightarrow x[n] \star h[n - n_0] = y[n - n_0]$$
  - **Identity**:
$$x[n] \star \delta[n] = x[n] \ .$$
- **LTI System properties:** An LTI system with impulse response $h[n]$ is
  - **memoryless** if $h[n] = 0$ for all $n \neq 0$.
  - **causal** if $h[n] = 0$ for all $n < 0$.
  - **stable** if
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$
  - **invertible** if
$$x[n] \star h[n] = 0$$
implies $x[n] = 0$. In other words, there is no nonzero signal whose convolution with $h[n]$ outputs the zero signal.
- **Difference equations:** For a Kth order equation of the form
$$\sum_{k=0}^{K} \alpha_k y[n-k] = x[n]$$
you need to
  - find a *particular* solution $y_p[n]$, i.e., *any* $y[n]$ that satisfies the equation,
  - find a family of *homogeneous* solutions, $y_h[n]$, that satisfy the equation with $x[n] = 0$,
  - write the overall solution family as $y[n] = y_p[n] + y_h[n]$,
  - find the specific member of the family by using initial conditions $y[0], y[1], \ldots, y[K-1]$.

If the system is said to be in *initial rest*, then you need to derive your own initial conditions using the fact that $y[-1] = y[-2] = \ldots = 0$.