EE 110B Signals and Systems

LTI Systems Defined by Difference Equations

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Difference equations

- The input/output relation of an LTI system can sometimes be expressed as a constant-coefficient **difference equation**.
- Analogous to constant-coefficient differential equations for continuous-time systems.
- Example:

$$y[n] + 2y[n-1] - 3y[n-2] = x[n] + 4x[n-1]$$

is very much like

$$y(t) + 2\frac{dy}{dt} - 3\frac{d^2y}{dt^2} = x(t) + 4\frac{dx}{dt}$$

Difference equations

- <u>Example</u>:

$$y[n] + 2y[n-1] - 3y[n-2] = x[n] + 4x[n-1]$$

- We can solve this recursively if two **initial conditions**, $y[0]$ and $y[1]$, are given to us:

$$\underbrace{y[2]}_{\text{CALCULATED}} + 2\underbrace{y[1]}_{\text{GIVEN}} - 3\underbrace{y[0]}_{\text{GIVEN}} = \underbrace{x[2]}_{\text{KNOWN}} + 4\underbrace{x[1]}_{\text{KNOWN}}$$

$$y[3] + 2y[2] - 3y[1] = x[3] + 4x[2]$$

$$y[4] + 2y[3] - 3y[2] = x[4] + 4x[3]$$

$$\vdots$$

Diff. equations in "real" life

- <u>Example</u>: You open a CD account with \$1,000. The account has an annual percentage rate (APR) of 6%. At the end of each month n, you are allowed to deposit an additional $x[n]$ dollars as cash.
- The money in the account at the end of month n is governed by the difference equation

$$y[n] = \left(1 + \frac{6}{100} \cdot \frac{1}{12}\right) y[n-1] + x[n]$$

with the initial condition $y[0]$=1,000.

Diff. equations in "real" life

- Example: You finance a \$20,000 car using a 60-month loan with 3% APR. You pay \$5,000 up front as down payment. What is your monthly payment?
- If you pay P dollars each month, the loan amount at the end of month n is governed by the difference equation

$$y[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y[n-1] - P$$

with the initial condition $y[0]=15{,}000$.

- P must be chosen to satisfy $y[60]=0$.

Non-recursive solutions

- It seems that we need a technique to solve difference equations non-recursively.
- Otherwise, it is virtually impossible to solve the last problem, for instance.
  - We might guess a P, solve the difference equation recursively, and see if $y[60]=0$. If $y[60]>0$, increase P and try again. If $y[60]<0$, decrease P and try again.
  - This is a very painstaking process, and car dealers certainly do not do this when they are offering you a loan.

How to solve

- Without the initial conditions, there will be an infinite family of solutions. Initial conditions are used to zero in on the unique solution.
- To find that infinite family of solutions, start with just "a" solution, called the **particular solution**.
- For the car loan example

$$y[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y[n-1] - P$$

try the solution $y[n] = K$.

How to solve

- For the car loan example

$$y[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y[n-1] - P$$

try the solution $y[n] = K$.

$$K = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) K - P$$

- This yields the particular solution $y_p[n] = 400P$
- Add onto this the **homogeneous solution,** i.e., the solution to

$$y_h[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y_h[n-1]$$

How to solve

$$y[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y[n-1] - P$$

- Why is $y_p[n] + y_h[n]$ a solution?
- Because

$$y_h[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y_h[n-1]$$

$$y_p[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y_p[n-1] - P$$

$+$

$$y_h[n] + y_p[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) (y_h[n-1] + y_p[n-1]) - P$$

How to solve

- Conversely, if $y[n]$ is a solution, $y[n] - y_p[n]$ must be a homogeneous solution:

$$y[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y[n-1] - P$$

$$-\quad y_p[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y_p[n-1] - P$$

$$y[n] - y_p[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) (y[n-1] - y_p[n-1])$$

- <u>Conclusion</u>: all solutions must be of the form

$$y_p[n] + y_h[n]$$

How to solve

- Back to the **homogeneous solution**

$$y_h[n] = \left(1 + \frac{3}{100} \cdot \frac{1}{12}\right) y_h[n-1]$$

- What function increases with a factor r when its argument increases by 1?
- No function other than cr^n for some arbitrary c.
- Therefore,

$$y_h[n] = c\left(1 + \frac{3}{100} \cdot \frac{1}{12}\right)^n$$

How to solve

- Therefore,

$$y_h[n] = c\left(1 + \frac{3}{100} \cdot \frac{1}{12}\right)^n$$

which makes

$$y[n] = c\left(1 + \frac{3}{100} \cdot \frac{1}{12}\right)^n + 400P$$

the entire family of solutions.

- Now find c using $y[0]$=15,000:

$$y[0] = 15000 = c + 400P$$

meaning $c = 15000 - 400P$.

How to solve

- Recall that the original problem asked for the monthly payment P in order for $y[60] = 0$.

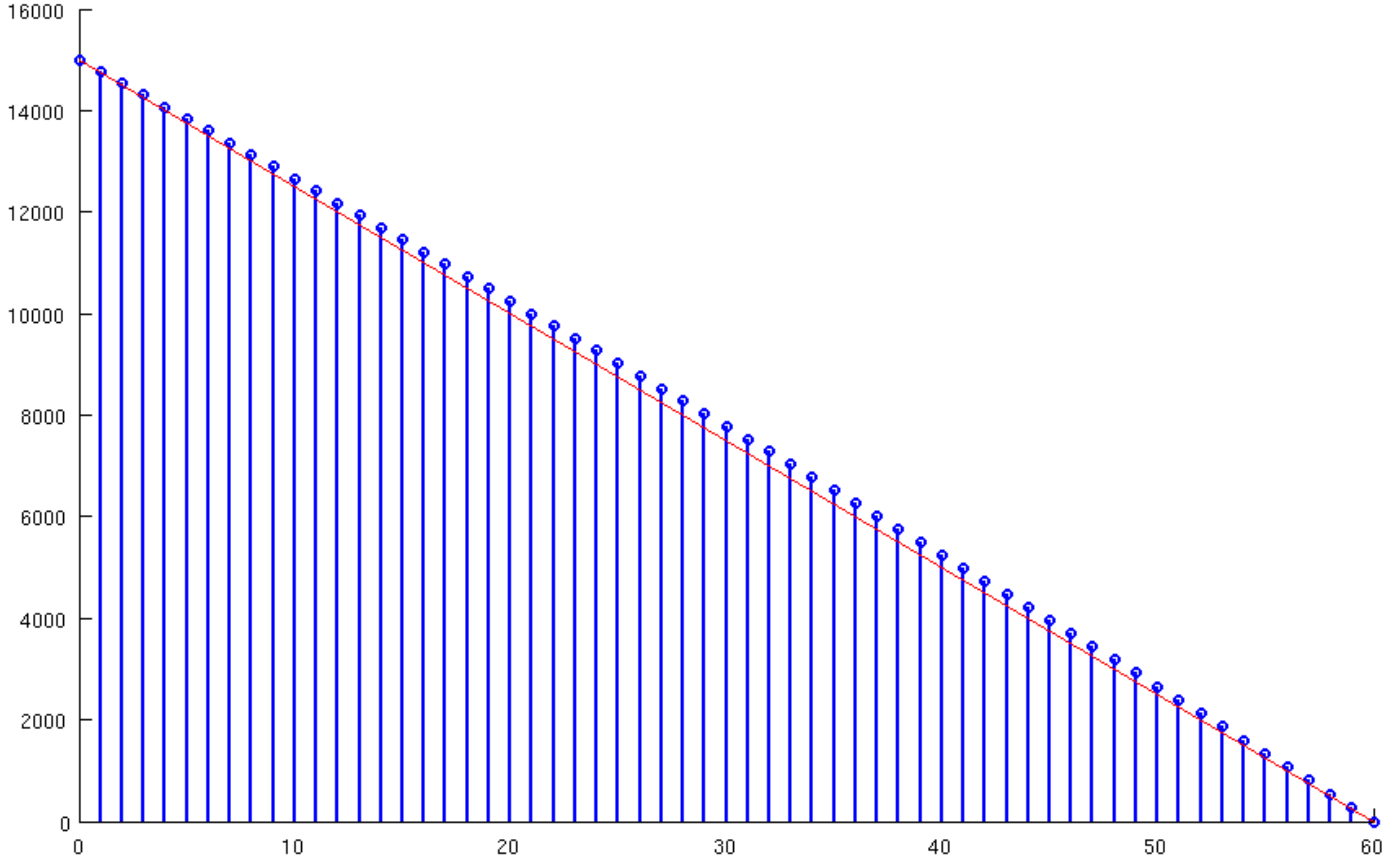
$$y[60] = (15000 - 400P)\left(1 + \frac{3}{100}\cdot\frac{1}{12}\right)^{60} + 400P$$

- Solving for P, we obtain

$$P = \frac{15000}{400}\cdot\frac{\left(1+\frac{3}{100}\cdot\frac{1}{12}\right)^{60}}{\left(1+\frac{3}{100}\cdot\frac{1}{12}\right)^{60} - 1}$$

$$\approx 269.53$$

Resultant $y[n]$



General car loan case

- Loan amount: L dollars.
- Interest: I % APR.
- Loan period: M months.
- Monthly payment: P dollars.
- The solution is given by

$$y[n] = \left(L - \frac{P}{r-1}\right) r^n + \frac{P}{r-1}$$

with $r = 1 + \frac{I}{100} \cdot \frac{1}{12}$.

- The monthly payment is $P = L \cdot \frac{r^M (r-1)}{r^M - 1}$

Causal LTI systems

- If the system is known to be causal, we do not need explicit initial conditions.
- The system would be in **initial rest**, i.e., $y[n] = 0$ for all n before the input comes.
- We could use this fact together with the recursive method to figure out a few $y[n]$'s after the input comes (as many as we need to solve the difference equation).

Causal LTI systems

- Under the initial rest regime, the system indeed becomes linear and time-invariant.
- If $y[n]$ is the solution to the difference equation

$$\sum_{k=0}^{K} \alpha_k y[n-k] = \sum_{m=0}^{M} \beta_m x[n-m]$$

with input $x[n]$, then

$$\sum_{k=0}^{K} \alpha_k y[n-n_0-k] = \sum_{m=0}^{M} \beta_m x[n-n_0-m]$$

- That is, $y[n-n_0]$ is a solution for input $x[n-n_0]$

TIME-INVARIANT ✓

Causal LTI systems

- Similarly, if $y_1[n]$ and $y_2[n]$ are solutions to the difference equation

$$\sum_{k=0}^{K} \alpha_k y[n-k] = \sum_{m=0}^{M} \beta_m x[n-m]$$

with inputs $x_1[n]$ and $x_2[n]$, respectively, then

$$\sum_{k=0}^{K} \alpha_k (a y_1[n-k] + b y_2[n-k]) = \sum_{m=0}^{M} \beta_m (a x_1[n-m] + b x_2[n-m])$$

- That is, $a y_1[n] + b y_2[n]$ is a solution for input $a x_1[n] + b x_2[n]$

LINEAR

Causal LTI systems

- Example: Consider the causal LTI system whose input-output relation is given by

$$y[n] - y[n-1] - 2y[n-2] = x[n]$$

Find $y[n]$ if $x[n] = u[n]$.

- Solution: Since the system is causal, we immediately have $y[n] = 0$ for $n < 0$.
- For $n \geq 0$, we have the equivalent equation

$$y[n] - y[n-1] - 2y[n-2] = 1$$

- For $n \geq 0$, we have the equivalent equation

$$y[n] - y[n-1] - 2y[n-2] = 1$$

- For particular solution, try $y_p[n] = K$:

$$K - K - 2K = 1$$

or $K = -0.5$.

- For homogeneous solution, try $y_h[n] = r^n$:

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

or

$$r^2 - r - 2 = 0$$

or

$$(r+1)(r-2) = 0$$

- So both $(-1)^n$ and 2^n are homogeneous solutions.

- For $n \geq 0$, we have the equivalent equation

$$y[n] - y[n-1] - 2y[n-2] = 1$$

- Combining these together,

$$y[n] = c_1(-1)^n + c_2 2^n - 0.5$$

becomes the family of all solutions.

- To find c_1 and c_2, it suffices to know $y[0]$ and $y[1]$.
- But since we know that the system is in initial rest,

$$\left.\begin{aligned} y[0] - \cancel{y[-1]} - 2\cancel{y[-2]} &= 1 \\ y[1] - y[0] - 2\cancel{y[-1]} &= 1 \end{aligned}\right\} \begin{aligned} y[0] &= 1 \\ y[1] &= 2 \end{aligned}$$

$$y[n] = c_1(-1)^n + c_2 2^n - 0.5$$

$$\left.\begin{aligned} y[0] &= 1 = c_1 + c_2 - 0.5 \\ y[1] &= 2 = -c_1 + 2c_2 - 0.5 \end{aligned}\right\} \begin{aligned} c_1 &= 1/6 \\ c_2 &= 4/3 \end{aligned}$$

- Recall that this was all for $n \geq 0$, so

$$y[n] = \left(\frac{1}{6}(-1)^n + \frac{4}{3}2^n - 0.5\right)u[n]$$

Causal LTI systems

- Example: Consider the causal LTI system whose input-output relation is given by

$$y[n] - y[n-1] + 0.5y[n-2] = x[n]$$

Find $y[n]$ if $x[n] = 0.5^n u[n]$.

- Solution: Since the system is causal, we immediately have $y[n] = 0$ for $n < 0$.
- For $n \geq 0$, we have the equivalent equation

$$y[n] - y[n-1] + 0.5y[n-2] = 0.5^n$$

- For $n \geq 0$, we have the equivalent equation

$$y[n] - y[n-1] + 0.5y[n-2] = 0.5^n$$

- For particular solution, try $y_p[n] = K\,0.5^n$:

$$K\,0.5^n - K\,0.5^{n-1} + 0.5K\,0.5^{n-2} = 0.5^n$$

or

$$K\,0.5^{n-1}\{0.5 - 1 + 1\} = 0.5^n$$

or $K = 1$.

- For homogeneous solution, try $y_h[n] = r^n$:

$$r^n - r^{n-1} + 0.5r^{n-2} = 0$$

or

$$r^2 - r + 0.5 = 0$$

or

$$(r - 0.5 - 0.5j)(r - 0.5 + 0.5j) = 0$$

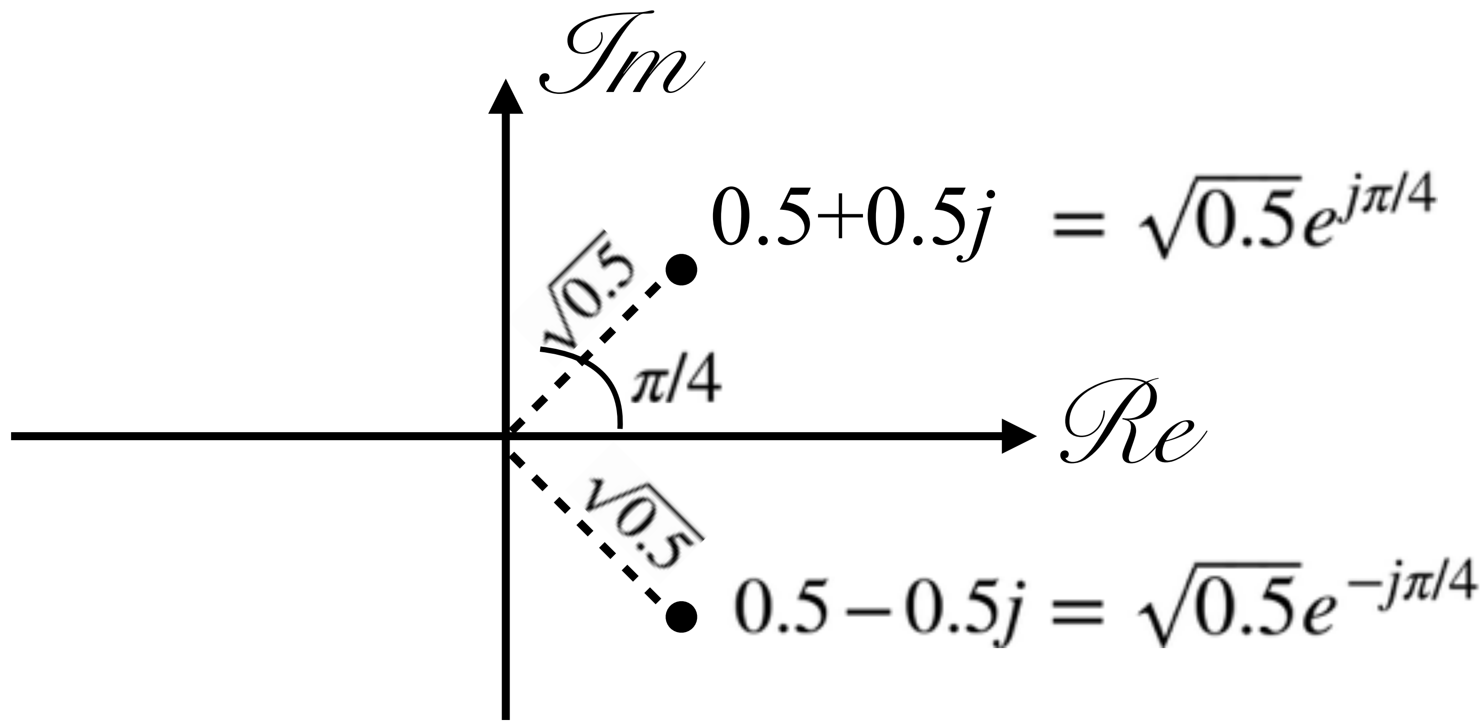
- For $n \geq 0$, we have the equivalent equation

$$y[n] - y[n-1] + 0.5y[n-2] = 0.5^n$$

- Combining these together,

$$y[n] = c_1(0.5 + 0.5j)^n + c_2(0.5 - 0.5j)^n + 0.5^n$$

- But,



- Therefore,

$$y[n] = \sqrt{0.5}^n \left(c_1 e^{j\pi n/4} + c_2 e^{-j\pi n/4}\right) + 0.5^n$$

$$y[n] = \sqrt{0.5}^n \left(c_1 e^{j\pi n/4} + c_2 e^{-j\pi n/4}\right) + 0.5^n$$

- Since we know that the system is in initial rest,

$$\left.\begin{aligned} y[0] - \cancel{y[-1]} + 0.5\cancel{y[-2]} &= 1 \\ y[1] - y[0] + 0.5\cancel{y[-1]} &= 0.5 \end{aligned}\right\} \begin{aligned} y[0] &= 1 \\ y[1] &= 1.5 \end{aligned}$$

- Use these to find the unknown coefficients:

$$y[0] = 1 = c_1 + c_2 + 1$$

$$y[1] = 1.5 = c_1(0.5 + 0.5j) + c_2(0.5 - 0.5j) + 0.5$$

- In other words,

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= \frac{1}{0.5j} = -2j \end{aligned}\right\} \begin{aligned} c_1 &= -j \\ c_2 &= j \end{aligned}$$

$$
\begin{aligned}
y[n] &= \sqrt{0.5}^n \left(c_1 e^{j\pi n/4} + c_2 e^{-j\pi n/4}\right) + 0.5^n \\
&= \sqrt{0.5}^n (-j)(e^{j\pi n/4} - e^{-j\pi n/4}) + 0.5^n \\
&= \sqrt{0.5}^n (-j)(2j) \sin(\pi n/4) + 0.5^n \\
&= 2\sqrt{0.5}^n \sin(\pi n/4) + 0.5^n
\end{aligned}
$$

- Recall that this was all for $n \geq 0$, so

$$
y[n] = \left(2\sqrt{0.5}^n \sin(\pi n/4) + 0.5^n\right) u[n]
$$

Finding the impulse response

- For every input, we need to guess another particular solution. This can be hard for some inputs.
- Since every difference equation with initial rest describes an LTI system, why not find the impulse response instead, and use the convolution sum to find the output for any input?

Finding the impulse response

- Example: Consider the first example with

$$y[n] - y[n-1] - 2y[n-2] = x[n]$$

Find the impulse response, and find the output for the input $x[n] = u[n]$ using convolution.

- Solution: In other words, we are asked to solve

$$h[n] - h[n-1] - 2h[n-2] = \delta[n]$$

- The important observation is that for $n > 0$, we got ourselves a homogeneous equation. So, no need to guess and try a particular solution.

Finding the impulse response

$$h[n] - h[n-1] - 2h[n-2] = \delta[n]$$

- Recall that we already found the homogeneous solution to be of the form $c_1(-1)^n + c_2 2^n$.
- As before, c_1 and c_2 are determined by $h[0]$ and $h[1]$.

$$\left.\begin{aligned} h[0] - \cancel{h[-1]} - 2\cancel{h[-2]} &= \delta[0] = 1 \\ h[1] - h[0] - 2\cancel{h[-1]} &= \delta[1] = 0 \end{aligned}\right\} \begin{aligned} h[0] &= 1 \\ h[1] &= 1 \end{aligned}$$

$$\left.\begin{aligned} h[0] &= 1 = c_1 + c_2 \\ h[1] &= 1 = -c_1 + 2c_2 \end{aligned}\right\} \begin{aligned} c_1 &= 1/3 \\ c_2 &= 2/3 \end{aligned}$$

Finding the impulse response

$$h[n] - h[n-1] - 2h[n-2] = \delta[n]$$

- Thus, the impulse response is given by

$$h[n] = \left(\frac{1}{3}(-1)^n + \frac{2}{3}2^n\right)u[n]$$

- Now, when the input is $x[n] = u[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty}\left(\frac{1}{3}(-1)^k + \frac{2}{3}2^k\right)u[k]u[n-k]$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{3}(-1)^k + \frac{2}{3}2^k\right)u[n-k]$$

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{\infty}\left(\frac{1}{3}(-1)^k + \frac{2}{3}2^k\right)u[n-k] \\
&= u[n]\sum_{k=0}^{n}\left(\frac{1}{3}(-1)^k + \frac{2}{3}2^k\right) \\
&= u[n]\left(\frac{1}{3}\sum_{k=0}^{n}(-1)^k + \frac{2}{3}\sum_{k=0}^{n}2^k\right) \\
&= u[n]\left(\frac{1}{3}\cdot\frac{(-1)^{n+1}-1}{(-1)-1} + \frac{2}{3}\cdot\frac{2^{n+1}-1}{2-1}\right) \\
&= u[n]\left(\frac{-1}{6}\left((-1)^{n+1}-1\right) + \frac{2}{3}\left(2^{n+1}-1\right)\right) \\
&= u[n]\left(\frac{1}{6}(-1)^n + \frac{4}{3}2^n + \frac{1}{6} - \frac{2}{3}\right)
\end{aligned}
$$

Finding the impulse response

- Example: Consider the causal LTI system with

$$y[n] - y[n-1] + 0.5y[n-2] = x[n]$$

Find the impulse response.

- Solution: As before, we will use the homogeneous solution

$$h[n] = \sqrt{0.5}^n \left(c_1 e^{j\pi n/4} + c_2 e^{-j\pi n/4}\right)$$

we derived earlier, together with the initial conditions $h[0]$ and $h[1]$ we will derive recursively.

Finding the impulse response

$$h[n] = \sqrt{0.5}^{\,n}\left(c_1 e^{j\pi n/4} + c_2 e^{-j\pi n/4}\right)$$

$$\left.\begin{aligned} h[0] - \cancel{h[-1]} + 0.5\cancel{h[-2]} &= \delta[0] = 1 \\ h[1] - h[0] + 0.5\cancel{h[-1]} &= \delta[1] = 0 \end{aligned}\right\} \begin{aligned} h[0] &= 1 \\ h[1] &= 1 \end{aligned}$$

$$\left.\begin{aligned} h[0] &= 1 = c_1 + c_2 \\ h[1] &= 1 = \sqrt{0.5}\left(c_1 e^{j\pi/4} + c_2 e^{-j\pi/4}\right) \end{aligned}\right\} \begin{aligned} c_1 &= \sqrt{0.5}e^{-j\pi/4} \\ c_2 &= \sqrt{0.5}e^{j\pi/4} \end{aligned}$$

- Therefore,

$$h[n] = \sqrt{0.5}^{\,n+1}\left(e^{j\pi(n-1)/4} + e^{-j\pi(n-1)/4}\right)u[n]$$

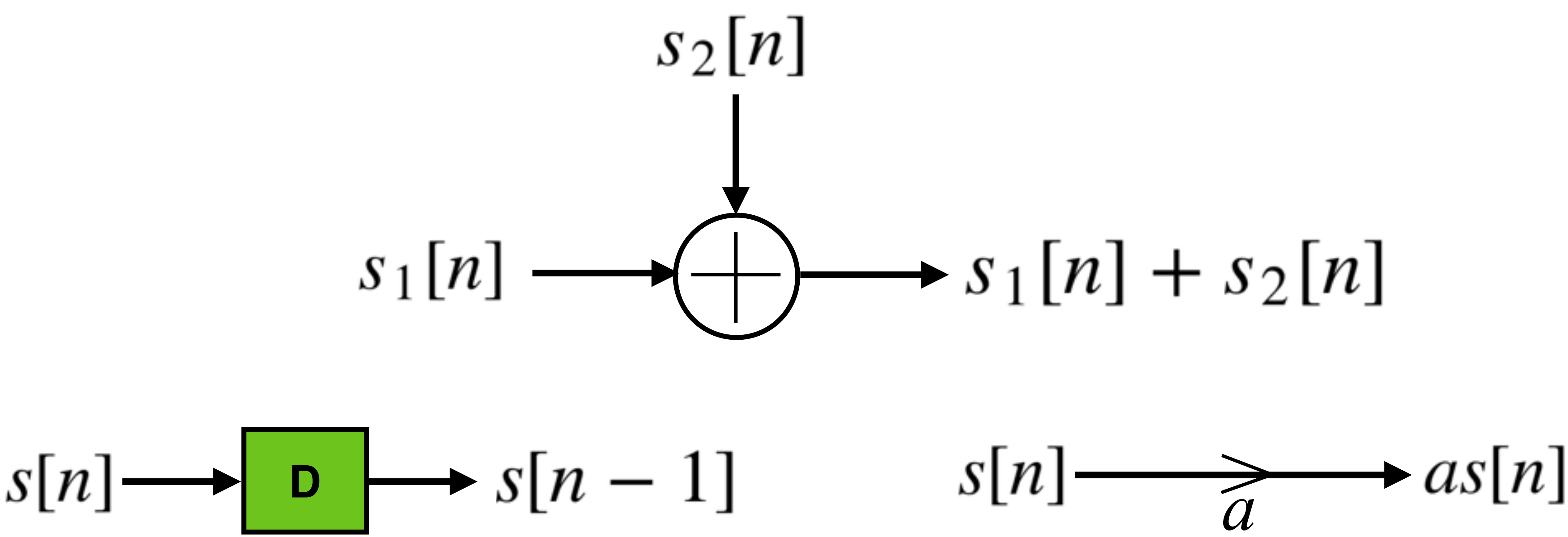
$$= 2\sqrt{0.5}^{\,n+1}\cos\left(\pi(n-1)/4\right)u[n]$$

Block diagrams

- We can also represent the LTI system for

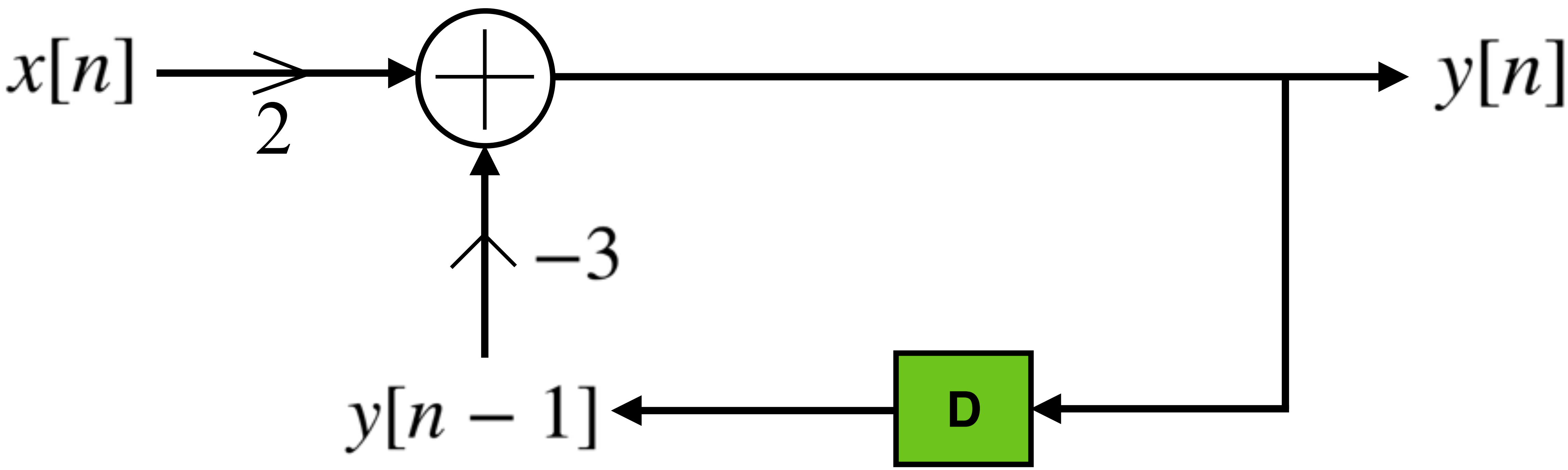
$$\sum_{k=0}^{K} \alpha_k y[n-k] = \sum_{m=0}^{M} \beta_m x[n-m]$$

with a block diagram using basic elements:



Block diagrams

- <u>Example</u>: $y[n] + 3y[n-1] = 2x[n]$
- This is the same as $y[n] = -3y[n-1] + 2x[n]$



Block diagrams

- <u>Example</u>:

$$y[n] - 2y[n-1] + 4y[n-2] = x[n] + 2x[n-1]$$