UCR Department of Electrical and Computer Engineering
WINTER 2023
EE 110A SIGNALS AND SYSTEMS
MOCK FINAL EXAM

You have 180 minutes to complete the exam. Please fully justify your work.

Question 1) (40 points)
Calculate each convolution below by performing multiplication in the Laplace domain and inverting back to the time domain:

a) $e^{-t}u(t) \star e^{-2t}u(t) = ?$
b) $e^{-t}u(t) \star e^{-t}u(t) = ?$
c) $e^{-t}u(t) \star e^{t}u(-t) = ?$
d) $[\delta(t) - e^{-2t}u(t)] \star u(t) = ?$
e) $te^{-t}u(t) \star u(t-1) = ?$

Question 2) (30 points)
Consider the filter whose Laplace transform is given by

$$H(s) = \frac{s^2 + 9}{(s+3)^2}$$

a) State the ROC for the filter to be **causal** and **stable**.

b) Invert $H(s)$ to find the impulse response $h(t)$.

c) Using the pole-zero plot, argue that this is a **band-stop** filter which supresses frequencies around $\Omega = \pm 3$, but passes frequencies away from $\Omega = \pm 3$.

Hint: Find out $|H(j0)|$, $|H(\pm j\infty)|$, and of course $|H(\pm j3)|$.

Question 3) (30 points)
Find the impulse response $h(t)$ of the **causal** LTI system described by the differential equation:

$$-y(t) + \frac{d^4 y(t)}{dt^4} = x(t) \ .$$

Is this a stable system?

SOLUTIONS

Question 1)
a) We need to invert $\frac{1}{s+1} \cdot \frac{1}{s+2}$ with an ROC of $\{s : \mathrm{Re}\{s\} > -1\}$, which can be written as

$$\frac{1}{s+1} \cdot \frac{1}{s+2} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)} \ .$$

One can compute $A = 1$, $B = -1$, resulting in the output

$$e^{-t}u(t) - e^{-2t}u(t) \ .$$

b) This time, we have to invert $\frac{1}{(s+1)^2}$ with an ROC of $\{s : \mathrm{Re}\{s\} > -1\}$, directly yielding the output

$$te^{-t}u(t)$$

c) Since the Laplace transform of $e^t u(-t)$ is $-\frac{1}{s-1}$ with an ROC of $\{s : \mathrm{Re}\{s\} < 1\}$, the convolution will create a need to invert $\frac{1}{s+1} \cdot \frac{-1}{s-1}$ with ROC of $\{s : -1 < \mathrm{Re}\{s\} < 1\}$. Proceeding with the partial fraction expansion,

$$\frac{-1}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} = \frac{A(s-1) + B(s+1)}{(s+1)(s-1)}$$

yielding $A = \frac{1}{2}$ and $B = -\frac{1}{2}$, and therefore the output

$$\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

or alternatively

$$\frac{1}{2}e^{-|t|} .$$

d) This time, we need to invert

$$\left(1 - \frac{1}{s+2}\right) \cdot \frac{1}{s} = \frac{s+1}{(s+2)s}$$

with the ROC given by $\{s : \mathrm{Re}\{s\} > 0\}$. Proceeding with the partial fraction expansion,

$$\frac{s+1}{(s+2)s} = \frac{A}{s+2} + \frac{B}{s} = \frac{As + B(s+2)}{(s+2)s} .$$

One can find $A = B = \frac{1}{2}$, and thus the output would be given by

$$\frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t) .$$

e) We need to invert $\frac{1}{(s+1)^2} \cdot \frac{e^{-s}}{s}$. We can forget about the e^{-s} term and can account for it as the last step by shifting the time domain signal by one unit to the right. Now,

$$\frac{1}{(s+1)^2 s} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s} = \frac{As(s+1) + Bs + C(s+1)^2}{(s+1)^2 s} .$$

Substituting $s = 0$ in the numerator yields $C = 1$. Similarly, substituting $s = -1$ gives $B = -1$. To solve for A, we can rewrite the numerator as

$$1 = As(s+1) - s + (s+1)^2 = s^2(A+1) + s(A+1) + 1$$

which clearly points to $A = -1$ as well. Without the extra shift, we would therefore have gotten

$$-e^{-t}u(t) - te^{-t}u(t) + u(t) .$$

Accounting for the right shift then yields

$$-e^{-t+1}u(t-1) - (t-1)e^{-t+1}u(t-1) + u(t-1)$$

as the answer.

Question 2)

a) Since both roots are at $s = -1$, the ROC for causality and stability is given by $\text{Re}\{s\} > -3$.

b) Adding and subtracting $(s+3)^2$ on the numerator,

$$\begin{aligned} H(s) &= \frac{s^2+9}{(s+3)^2} \\ &= \frac{(s+3)^2 + s^2 + 9 - (s+3)^2}{(s+3)^2} \\ &= 1 + \frac{s^2+9-(s^2+6s+9)}{(s+3)^2} \\ &= 1 - \frac{6s}{(s+3)^2} \ . \end{aligned}$$

Applying partial fraction expansion on the second term,

$$\frac{6s}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2} = \frac{A(s+3)+B}{(s+3)^2}$$

from which it is pretty clear that $A = 6$ and $B = -18$. Inverting the overall expression with the right-sided ROC in mind, we obtain

$$h(t) = \delta(t) - 6e^{-3t}u(t) + 18te^{-3t}u(t) \ .$$

c) There are two repeated poles at $s = -3$ and two zeros at $s = j3$ and $s = -j3$. Following the hint, we obtain

$$|H(j0)| = \frac{\prod \text{distance to zeros}}{\prod \text{distance to poles}} = \frac{3 \times 3}{3 \times 3} = 1 \ .$$

So the low frequencies are preserved. Similarly, for very large Ω

$$|H(j\Omega)| = \frac{\prod \text{distance to zeros}}{\prod \text{distance to poles}} \approx \frac{\Omega \times \Omega}{\Omega \times \Omega} = 1 \ .$$

Note that this is an approximation that is increasingly accurate as $\Omega \to \infty$. If you really want the true value of $|H(j\Omega)|$ for $\Omega > 3$, it is given as

$$|H(j\Omega)| = \frac{(\Omega-3)(\Omega+3)}{\sqrt{\Omega^2+9}\sqrt{\Omega^2+9}} = \frac{\Omega^2-9}{\Omega^2+9} \ .$$

Finally, because there is a zero exactly at $s = j3$, we have

$$|H(\pm j3)| = 0 \ .$$

Question 3)

Going to the Laplace domain, the differential equation becomes

$$-Y(s) + s^4 Y(s) = X(s)$$

or in other words,

$$Y(s) = \frac{1}{s^4-1} X(s) \ .$$

Therefore, $H(s) = \frac{1}{s^4-1}$. Since the poles are at -1, 1, j, and $-j$, we have the causal ROC given by $\text{Re}\{s\} > 0$.

We can immediately see that the system is not stable, as the ROC does not include the imaginary axis. Inverting $H(s)$ into the time domain is best accomplished by the partial fraction expansion

$$\begin{aligned} H(s) &= \frac{1}{s^4-1} \\ &= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s-j} + \frac{D}{s+j} \\ &= \frac{A(s+1)(s^2+1) + B(s-1)(s^2+1) + C(s+j)(s^2-1) + D(s-j)(s^2-1)}{s^4-1} \ . \end{aligned}$$

Here we used the shortcuts $(s-j)(s+j) = s^2+1$ and $(s+1)(s-1) = s^2-1$.

Now, substituting $s = 1$, $s = -1$, $s = j$, and $s = -j$ will respectively yield

$$\begin{aligned} 4A &= 1 \\ -4B &= 1 \\ -j4C &= 1 \\ j4D &= 1 \end{aligned}$$

or equivalently, $A = \frac{1}{4}$, $B = \frac{-1}{4}$, $C = \frac{j}{4}$, and $D = \frac{-j}{4}$. Therefore,

$$\begin{aligned} h(t) &= \frac{1}{4}\left[e^t - e^{-t} + je^{jt} - je^{-jt}\right] u(t) \\ &= \frac{e^t - e^{-t}}{4} u(t) - \frac{e^{jt} - e^{-jt}}{4j} u(t) \\ &= \frac{e^t - e^{-t}}{4} u(t) - \frac{\sin(t)}{2} u(t) \ . \end{aligned}$$