University of California Riverside, Department of ECE
EE110A: Signals and Systems
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Homework 3 solutions

Problem 1 [8pts]: Determine if the LTI system is memoryless, causal, stable, and invertible if its impulse response is given by

a) $h(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$

b) $h(t) = \delta(t-1)$

c) $h(t) = \delta(t-2) + \delta(t-4)$

d) $h(t) = \cos(t)$

Solution:

a)
Not memoryless: $h(t)$ is not of the form $a\delta(t)$.
Causal: $h(t) = 0$ for $t < 0$.
Stable: $\int_{-\infty}^{\infty} |h(\tau)| d\tau = 1 < \infty$.
Not invertible: The input $x(t) = \cos(2\pi t)$ results in $y(t) = 0$ as evidenced by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{t} \cos(2\pi\tau)d\tau = \frac{\sin(2\pi\tau)}{2\pi}\Bigg|_{t-1}^{t} = \frac{\sin(2\pi t) - \sin(2\pi t - 2\pi)}{2\pi} = 0$$

b)
Not memoryless: $h(t)$ is not of the form $a\delta(t)$.
Causal: $h(t) = 0$ for $t < 0$.
Stable: $\int_{-\infty}^{\infty} |h(\tau)| d\tau = 1 < \infty$.
Invertible: The inverse should be obvious: A system with impulse response $\delta(t+1)$, i.e., a left shifter (by 1 unit). But if it is not as clear, one can still use the same tool we have: If there exists an $x(t)$ that drives the output to 0, we must have

$$0 = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} \delta(\tau-1)x(t-\tau)d\tau = x(t-1)$$

directly showing that $x(t) = 0$ must be true (the last step being a result of the sifting property of the impulse).

c)
Not memoryless: $h(t)$ is not of the form $a\delta(t)$.
Causal: $h(t) = 0$ for $t < 0$.
Stable: $\int_{-\infty}^{\infty} |h(\tau)| d\tau = 2 < \infty$.
Not invertible: Using the integral tool we have, if there exists an $x(t)$ that drives the output to 0, we must have

$$0 = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} [\delta(\tau-2) + \delta(\tau-4)]x(t-\tau)d\tau = x(t-2) + x(t-4) .$$

But this is indeed possible by choosing any *periodic* x(t) with period 4 and the second half of each period being the negative of the first half. One such example is $x(t) = \sin(\frac{\pi t}{2})$.

d)
Not memoryless: $h(t)$ is not of the form $a\delta(t)$.
Non-causal: $h(t) \neq 0$ for some $t < 0$.
Unstable: $\int_{-\infty}^{\infty} |h(\tau)| d\tau = \infty$.
Not invertible: As evidenced by the choice $x(t)$ being a pulse of width 2π. The shifted version of it always coincides with exactly one period of $h(t)$, thereby setting $\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = 0$ for all t.

Problem 2 [6pts]: Let the LTI system have an impulse response given by $h(t) = e^{-|t|}$. In this question, we will prove that this system is invertible. Recall that one way of proving invertibility is to show that if

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$

then $x(t) = 0$ for all t.

a) Show that the above integral can be written as $A(t) + B(t)$, where

$$A(t) = e^{-t} \int_{-\infty}^{t} x(\tau)e^{\tau} d\tau$$

and

$$B(t) = e^{t} \int_{t}^{\infty} x(\tau)e^{-\tau} d\tau .$$

b) Show that if $A(t) + B(t) = 0$ then $A(t) = B(t) = 0$. This could be accomplished by differentiating both sides of $A(t) + B(t) = 0$.

c) Now, show that if $A(t) = 0$, then we must also have $x(t) = 0$, again by differentiating both sides of $A(t) = 0$.

Solution:

a) We have

$$\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-\infty}^{\infty} x(\tau)e^{-|t-\tau|}d\tau \\
&= \int_{-\infty}^{t} x(\tau)e^{-(t-\tau)}d\tau + \int_{t}^{\infty} x(\tau)e^{-(\tau-t)}d\tau \\
&= e^{-t}\int_{-\infty}^{t} x(\tau)e^{\tau}d\tau + e^{t}\int_{t}^{\infty} x(\tau)e^{-\tau}d\tau \\
&\triangleq A(t)+B(t)
\end{aligned}$$

with

$$A(t) = e^{-t}\int_{-\infty}^{t} x(\tau)e^{\tau}d\tau$$

and

$$B(t) = e^{t}\int_{t}^{\infty} x(\tau)e^{-\tau}d\tau \ .$$

b) We will need below the property that

$$\frac{d}{dt}\int_{-\infty}^{t} f(\tau)d\tau = f(t)$$

and

$$\frac{d}{dt}\int_{t}^{\infty} f(\tau)d\tau = -f(t)$$

for any function $f(t)$.

Now, if $A(t)+B(t)=0$, taking the derivative of both sides with respect to t, we obtain

$$\begin{aligned}
0 &= e^{-t}x(t)e^{t} - e^{-t}\int_{-\infty}^{t} x(\tau)e^{\tau}d\tau - e^{t}x(t)e^{-t} + e^{t}\int_{t}^{\infty} x(\tau)e^{-\tau}d\tau \\
&= B(t)-A(t)
\end{aligned}$$

But $A(t)+B(t)=0$ and $A(t)-B(t)=0$ are simultaneously satisfied if and only if

$$A(t)=B(t)=0 \ .$$

c) $A(t)=0$ implies

$$\int_{-\infty}^{t} x(\tau)e^{\tau}d\tau = 0 \ .$$

Differentiating both sides one more time yields $x(t)e^t = 0$, which in turn, means $x(t) = 0$.

Problem 3 [6pts]: Consider the **causal LTI** system whose input-output relation is given by

$$y(t) + \frac{dy(t)}{dt} = x(t) \ .$$

Find the output if $x(t) = e^{-3t}u(t)$ by evaluating the particular and homogeneous solutions to this input.

Solution:

1) We argue that $y(0^+) = y(0^-)$ because if $y(t)$ is not continuous at $t = 0$, that would cause an impulse in $\frac{dy}{dt}$. But that is not accounted for on the right-hand side. Therefore, $y(0^+) = 0$.
2) For $t > 0$, the right-hand side reduces to e^{-3t}. Therefore, the differential equation becomes

$$y(t) + \frac{dy(t)}{dt} = e^{-3t} \ .$$

3) We can substitute $y_p(t) = Ke^{-3t}$ in the equation and write

$$Ke^{-3t} - 3Ke^{-3t} = e^{-3t} \ .$$

It then follows that $K = -\frac{1}{2}$.
4) As always, the homogeneous equation must be of the form $y_h(t) = ce^{\alpha t}$. Substituting, we obtain

$$ce^{\alpha t} + c\alpha e^{\alpha t} = 0$$

which is the same as

$$1 + \alpha = 0 \ .$$

Therefore, $y_h(t) = ce^{-t}$.
5) The complete solution family is given by

$$y(t) = y_p(t) + y_h(t) = -\frac{1}{2}e^{-3t} + ce^{-t}$$

6) Substituting $t = 0^+$, we obtain

$$0 = y(0^+) = -\frac{1}{2} + c$$

and hence $c = \frac{1}{2}$.
7) Finally, multiplying our solution with $u(t)$ yields

$$y(t) = \frac{1}{2}[e^{-t} - e^{-3t}]u(t)$$