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Department of Electrical and Computer Engineering

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EE110B-SIGNALS AND SYSTEMS
HOMEWORK 5 SOLUTIONS

Problem 1:

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} -3je^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi} 3je^{j\omega n}d\omega \\
&= \frac{-3j}{2\pi}\left.\frac{e^{j\omega n}}{jn}\right|_{-\pi}^{0} + \frac{3j}{2\pi}\left.\frac{e^{j\omega n}}{jn}\right|_{0}^{\pi} \\
&= \frac{3}{2\pi n}\left(-e^{j0n} + e^{-j\pi n} + e^{j\pi n} - e^{j0n}\right) \\
&= \frac{3}{\pi n}\left(\cos(\pi n) - 1\right)
\end{aligned}$$

Problem 2:
From class notes, we know that

$$H(e^{j\omega}) = \begin{cases} 1 & -\frac{\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

and

$$X(e^{j\omega}) = \begin{cases} 1 & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & \text{otherwise .} \end{cases}$$

Using the convolution property, we obtain

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \begin{cases} 1 & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & \text{otherwise .} \end{cases}$$

That means $Y(e^{j\omega}) = X(e^{j\omega})$, and thus $y[n] = x[n]$, i.e., the filter $h[n]$ has absolutely no effect on $x[n]$.

Problem 3:
We can immediately state

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

with ROC given as $|z| > 0.5$. As for $X(z)$, the easiest way is to first find the z-transform of $u[n]$, which is $U(z) = \frac{1}{1-z^{-1}}$ with $|z| > 1$. Then using the time-reversal property, we can state

$$X(z) = U(z^{-1}) = \frac{1}{1 - z}$$

with ROC given as $|z| < 1$. Rewriting $X(z)$ as $\frac{-z^{-1}}{1-z^{-1}}$, we obtain

$$Y(z) = X(z)H(z) = \frac{-z^{-1}}{1 - z^{-1}} \cdot \frac{1}{1 - 0.5z^{-1}}$$

with the ROC $0.5 < |z| < 1$.

Using the partial fraction expansion technique, we can rewrite $Y(z)$ as

$$Y(z) = \frac{A}{1 - z^{-1}} + \frac{B}{1 - 0.5z^{-1}} = \frac{A(1 - 0.5z^{-1}) + B(1 - z^{-1})}{(1 - z^{-1})(1 - 0.5z^{-1})} .$$

We can then find A and B by solving

$$\begin{aligned} A + B &= 0 \\ 0.5A + B &= 1 \end{aligned}$$

as $A = -2$ and $B = 2$. Thus,

$$\begin{aligned} y[n] &= -A \cdot 1^n u[-n-1] + B \cdot 0.5^n u[n] \\ &= 2u[-n-1] + 2 \cdot 0.5^n u[n] . \end{aligned}$$

Note that we did not invert $\frac{A}{1-z^{-1}}$ as $A \cdot 1^n u[n]$, because the ROC dictates that $|z| < 1$, and therefore that we should invert it as a left-sided sequence $-A \cdot 1^n u[-n-1]$.

Problem 4:

a) Applying our knowledge of the z-transform of right-sided exponentials,

$$X(z) = \frac{1}{1 - (-1)z^{-1}} = \frac{1}{1 + z^{-1}}$$

The only pole is at $z = -1$, and the only zero is at $z = 0$. Since the sequence is right-sided,

$$\text{ROC} = \{z : |z| > 1\} .$$

Since the unit circle is not included in the ROC, the Fourier transform does not exist.

b) We can express $x[n]$ alternatively as

$$\begin{aligned} x[n] &= \frac{1}{2} 4^n \left(e^{j\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)} + e^{-j\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)} \right) u[-n-1] \\ &= \frac{1}{2} \left(4e^{j\frac{2\pi}{6}} \right)^n e^{j\frac{\pi}{4}} u[-n-1] + \frac{1}{2} \left(4e^{-j\frac{2\pi}{6}} \right)^n e^{-j\frac{\pi}{4}} u[-n-1] . \end{aligned}$$

Letting $a = 4e^{j\frac{2\pi}{6}}$ and $b = e^{j\frac{\pi}{4}}$, this is the same as

$$x[n] = \frac{1}{2} \left(a^n b + (a^*)^n b^* \right) u[-n-1]$$

resulting in

$$X(z) = -\frac{1}{2} \left[\frac{b}{1 - az^{-1}} + \frac{b^*}{1 - a^* z^{-1}} \right]$$

with an ROC$= \{z : |z| < 4\}$. We can simplify $X(z)$ as

$$\begin{aligned} X(z) &= -\frac{b(1 - a^* z^{-1}) + b^*(1 - az^{-1})}{2(1 - az^{-1})(1 - a^* z^{-1})} \\ &= -\frac{b + b^* - (a^* b + ab^*)z^{-1}}{2(1 - az^{-1})(1 - a^* z^{-1})} \\ &= -\frac{\text{Re}\{b\} - \text{Re}\{ab^*\} z^{-1}}{(1 - az^{-1})(1 - a^* z^{-1})} \\ &= -\frac{z[\text{Re}\{b\} z - \text{Re}\{ab^*\}]}{(z - a)(z - a^*)} . \end{aligned}$$

There are two poles, one at $z = a$ and one at $z = a^*$. The two zeros are at $z = 0$ and at

$$z = \frac{\mathrm{Re}\{ab^*\}}{\mathrm{Re}\{b\}} = \frac{4\cos(\pi/12)}{\cos(\pi/4)} .$$

Since $|z| = 1$ is inside the ROC, Fourier transform exists.

c) The infinite sum

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

boils down to

$$X(z) = z - z^{-1} = \frac{z^2 - 1}{z}$$

which means there are two poles: $z = 0$ and $z = \infty$. The zeros are at $z = 1$ and $z = -1$. The ROC is given as $\{z : 0 < |z| < \infty\}$. Since this includes $|z| = 1$, the Fourier transform exists.

d) The best way is to put $x[n]$ into a form where we recognize the z-transforms:

$$\begin{aligned} x[n] &= 2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1] \\ &= 2 \cdot 2^{n-1} u[-n] + \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u[n-1] \\ &= -2x_1[n-1] + \frac{1}{4} x_2[n-1] \end{aligned}$$

where

$$x_1[n] = -2^n u[-n-1]$$

and

$$x_2[n] = \left(\frac{1}{4}\right)^n u[n]$$

Now we know that

$$X_1(z) = \frac{1}{1 - 2z^{-1}}$$

with ROC= $\{z : |z| < 2\}$, and

$$X_2(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

with ROC= $\{z : |z| > \frac{1}{4}\}$. Thus,

$$\begin{aligned} X(z) &= \frac{-2z^{-1}}{1 - 2z^{-1}} + \frac{\frac{1}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} \\ &= \frac{z^{-1}\left[-2(1 - \frac{1}{4}z^{-1}) + \frac{1}{4}(1 - 2z^{-1})\right]}{(1 - 2z^{-1})(1 - \frac{1}{4}z^{-1})} \\ &= \frac{-\frac{7}{4} \cdot z^{-1}}{(1 - 2z^{-1})(1 - \frac{1}{4}z^{-1})} \\ &= \frac{-\frac{7}{4} \cdot z}{(z-2)(z - \frac{1}{4})} . \end{aligned}$$

The poles are at $z = 2$ and $z = \frac{1}{4}$, and the zeros are at $z = 0$ and $z = \infty$. Since there is no zero-pole cancellation, the ROC is the intersection of those of $X_1(z)$ and $X_2(z)$, i.e., ROC$= \{z : \frac{1}{4} < |z| < 2\}$, which includes the unit circle, so the Fourier transform exists.

e) We have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{9} a^n z^{-n} \\ &= \sum_{n=0}^{9} (az^{-1})^n \\ &= \frac{1-(az^{-1})^{10}}{1-az^{-1}} \\ &= \frac{z^{10}-a^{10}}{z^9(z-a)} \end{aligned}$$

Although it looks like $z = a$ would be a pole, it cancels with one of the zeros, so it won't determine the ROC. In fact, the only value of z for which $X(z)$ diverges is $z = 0$. Therefore

$$\text{ROC} = \{z : |z| > 0\}\ .$$