FORMULAS AND CONCEPTS YOU MIGHT FIND USEFUL

- **Quadratic equations:** The roots of the equation $as^2 + bs + c = 0$ are given by

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the understanding that $\sqrt{-1} = j$.

- **Complex numbers:** If z is a complex number, then it can be expressed in one of two forms:

$$z = a + jb \qquad \text{or} \qquad z = re^{j\theta}$$

where

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \ .$$

Values of r and θ can be found from a and b using

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1}\left(\frac{b}{a}\right) \ . \end{aligned}$$

The polar form makes multiplication easy. That is, if $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)} \ .$$

The complex conjugate of z is $z^* = a - jb$ or $z^* = re^{-j\theta}$. We also have the following relationships:

$$\begin{aligned} z + z^* &= 2\text{Re}\{z\} &= 2a \\ z - z^* &= 2\text{Im}\{z\} &= 2b \\ zz^* &= |z|^2 &= a^2 + b^2 \end{aligned}$$

It is helpful to know the following about the complex number j:

$$\begin{aligned} j^2 &= -1 \\ j^3 &= -j \\ j^4 &= 1 \\ 1/j &= -j \\ je^{j\theta} &= e^{j(\theta + \pi/2)} \end{aligned}$$

where the last equation can be read as "multiplication by j rotates a complex number by 90 degrees counter-clockwise."

- **Sinusoidal waves and complex exponentials:**

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- **The impulse function:** $\delta(t)$ has the following properties:
  - **Integration:**

$$\int_{t_0^-}^{t_0^+} \delta(\tau - t_0) d\tau = 1$$

  - **Sampling:**

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$$

  - **Sifting:**

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

- **Relation between the impulse, step, and ramp functions:** We have

$$\frac{dr(t)}{dt} = u(t) \qquad \frac{du(t)}{dt} = \delta(t)$$

and

$$\int_{-\infty}^{t} u(\tau) d\tau = r(t) \qquad \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

- **System properties:** A system is
  - **linear** if

$$\begin{matrix} x_1(t) \to y_1(t) \\ x_2(t) \to y_2(t) \end{matrix} \Longrightarrow ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

  - **time-invariant** if

$$x(t) \to y(t) \Longrightarrow x(t - t_0) \to y(t - t_0)$$

  - **memoryless** if $y(t)$ at time t depends only on $x(t)$ on time t.
  - **causal** if $y(t)$ at time t depends only on $x(\tau)$ on times $\tau \leq t$.
  - **stable** if $|x(t)| \leq B$ for some B implies $|y(t)| \leq C$ for some C.
  - **invertible** if two distinct $x_1(t)$ and $x_2(t)$ can never result in the same $y(t)$.

- **The convolution integral:** For an LTI system with impulse response $h(t)$, the output $y(t)$ for any input $x(t)$ can be found as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau \ .$$

The convolution can also be seen as an operator, and be denoted as

$$y(t) = x(t) \star h(t) \ .$$

- **Convolution for right-sided signals:** If

$$x(t) = x_0(t)u(t)$$

and

$$h(t) = h_0(t)u(t) \ ,$$

then $y(t) = x(t) \star h(t)$ simplifies to

$$y(t) = u(t) \cdot \int_0^t x_0(\tau)h_0(t - \tau) d\tau \ .$$

- **Properties of the convolution operator:**
  - **Commutativity**:

$$x(t) \star y(t) = y(t) \star x(t)$$

  - **Associativity**:

$$x(t) \star [y(t) \star z(t)] = [x(t) \star y(t)] \star z(t)$$

  - **Distribution**:

$$x(t) \star [ay(t) + bz(t)] = a[x(t) \star y(t)] + b[x(t) \star z(t)]$$

  - **Time invariance**:

$$x(t) \star h(t) = y(t) \Longrightarrow x(t) \star h(t - t_0) = y(t - t_0)$$

  - **Identity**:

$$x(t) \star \delta(t) = x(t) \ .$$

- **System properties using the impulse response:** An LTI system with impulse response $h(t)$ is
  - **memoryless** if and only if $h(t) = a\delta(t)$.
  - **causal** if and only if $h(t) = 0$ for $t < 0$.
  - **stable** if and only if

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

  - **invertible** if and only if

$$\int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau = 0$$

  implies $x(t) = 0$.

- **Differential equations:** For a causal LTI system, the solution can be found as follows:
  1) Use initial rest assumption, continuity/discontinuity arguments, and integration to find $y(0^+)$, $\frac{dy}{dt}(0^+)$, $\frac{d^2y}{dt^2}(0^+)$, ... (however many initial conditions needed, i.e., as many as the degree of the differential equation).
  2) Focus only on $t > 0$, and simplify the right-hand side.
  3) Find a particular solution.
  4) Find the family of homogenous solutions.
  5) Add the particular and homogenous solutions to find the solution family.
  6) Use the initial conditions derived to find the unique solution in the solution family.
  7) Multiply your solution with $u(t)$.
- **Impulse response through differential equations:** Set $x(t) = \delta(t)$ and $y(t) = h(t)$ in the differential equation. Then follow the exact same steps as above, except you can skip Step 3, because the particular solution will always be zero.
- **Continuous-time Fourier Series (CTFS):** For a signal with period T, and $\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

Periodic Signals

where

$$a_k = \frac{1}{T}\int_{t\in\mathcal{T}} x(t)e^{-jk\Omega_0 t} \ .$$

By $\mathcal{T}$, we mean any interval with length T, e.g., from a to $a + T$.
- **Properties of CTFS:**
  - **Linearity:**

$$\begin{matrix} x(t) \to a_k \\ y(t) \to b_k \end{matrix} \implies \alpha x(t) + \beta y(t) \to \alpha a_k + \beta b_k$$

  - **Time shifting:**

$$x(t) \to a_k \implies x(t - t_0) \to a_k e^{-jk\Omega_0 t_0}$$

  - **Frequency shifting:**

$$x(t) \to a_k \implies x(t)e^{jk_0\Omega_0 t} \to a_{k-k_0}$$

  - **Time reversal:**

$$x(t) \to a_k \implies x(-t) \to a_{-k}$$

  - **Conjugation:**

$$x(t) \to a_k \implies x^*(t) \to a^*_{-k}$$

  - **Periodic convolution:**

$$\begin{matrix} x(t) \to a_k \\ y(t) \to b_k \end{matrix} \implies x(t) \tilde{\star} y(t) \to T a_k b_k$$

    where

$$x(t) \tilde{\star} y(t) = \int_0^T x(\tau)y(t - \tau)d\tau \ .$$

  - **Multiplication:**

$$\begin{matrix} x(t) \to a_k \\ y(t) \to b_k \end{matrix} \implies x(t)y(t) \to \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

  - **Parseval's relation:**

$$\frac{1}{T}\int_{t\in\mathcal{T}} |x(t)|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 \ .$$

- **Continuous-time Fourier transform (CTFT):** For any signal $x(t)$,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$$

inverse Fourier Transform

where

how much Frequency Ω is present in $x(t)$.

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \ .$$

time domain → frequency domain

- **Properties of CTFT:**
  - **Linearity:**

$$\begin{matrix} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{matrix} \implies \alpha x(t) + \beta y(t) \to \alpha X(j\Omega) + \beta Y(j\Omega)$$

  - **Time shifting:**

$$x(t) \to X(j\Omega) \implies x(t - t_0) \to X(j\Omega)e^{-j\Omega t_0}$$

  - **Frequency shifting:**

$$x(t) \to X(j\Omega) \implies x(t)e^{j\Omega_0 t} \to X(j(\Omega - \Omega_0))$$

  - **Time reversal:**

$$x(t) \to X(j\Omega) \implies x(-t) \to X(-j\Omega)$$

  - **Conjugation:**

$$x(t) \to X(j\Omega) \implies x^*(t) \to X^*(-j\Omega)$$

  - **Convolution:**

$$\begin{matrix} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{matrix} \implies x(t) \star y(t) \to X(j\Omega)Y(j\Omega)$$

  - **Multiplication:**

$$\begin{matrix} x(t) \to X(j\Omega) \\ y(t) \to Y(j\Omega) \end{matrix} \implies x(t)y(t) \to \frac{1}{2\pi}X(j\Omega) \star Y(j\Omega)$$

  - **Differentiation in time domain:**

$$x(t) \to X(j\Omega) \implies \frac{dx(t)}{dt} \to j\Omega X(j\Omega)$$

  - **Differentiation in frequency domain:**

$$x(t) \to X(j\Omega) \implies -jtx(t) \to \frac{dX(j\Omega)}{d\Omega}$$

  - **Parseval's relation:**

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega \ .$$

- **CTFT of periodic signals:** If $x(t)$ has a period T, and therefore has a CTFS expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \ ,$$

then

$$X(j\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0) \ .$$

- **Laplace transform:** The Laplace transform of any signal $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \ .$$

Region of convergence (ROC) is the set of s on the complex plane for which the above integral converges.

- **Properties of the Laplace transform:**
  - If $X(s) = \frac{(s-z_0)(s-z_1)\cdots(s-z_M)}{(s-p_0)(s-p_1)\cdots(s-p_K)}$, then z_m are *zeros* and p_k are *poles*. In addition, there are $K-M$ zeros at ∞ if $K > M$, $M-K$ poles at ∞ if $M > K$.
  - The ROC can only take one of four possible forms:
    * $Re\{s\} < a$ when $x(t)$ is left-sided,
    * $a < Re\{s\} < b$ when $x(t)$ is two-sided,
    * $Re\{s\} > b$ when $x(t)$ is right-sided,
    * all s when $x(t)$ is of finite duration

    with possible exception of $s = \pm\infty$. In addition, the ROC cannot contain any poles.
  - An LTI system that has a transfer function $H(s)$ with ROC $\mathcal{R}$ is
    * **causal** if $\mathcal{R}$ is of the form $Re\{s\} > b$ and $H(s)$ has no pole at $s = \infty$.
    * **stable** if $\mathcal{R}$ contains the imaginary axis $Re\{s\} = 0$.
  - The CTFT exists (converges) if the ROC contains $Re\{s\} = 0$.
  - **Linearity:** If $x(t) \to X(s)$ with ROC $\mathcal{R}_1$ and $y(t) \to Y(s)$ with ROC $\mathcal{R}_2$, then
    $$\alpha x(t) + \beta y(t) \to \alpha X(s) + \beta Y(s)$$
    with an ROC containing $\mathcal{R}_1 \cap \mathcal{R}_2$. There might be a zero-pole cancellation resulting in a larger ROC.
  - **Time reversal:** If $x(t) \to X(s)$ with ROC $\mathcal{R}$, then
    $$x(-t) \to X(-s)$$
    with ROC $= -\mathcal{R} \triangleq \{s : -s \in \mathcal{R}\}$
  - **Time shifting:** If $x[n] \to X(z)$ with ROC $\mathcal{R}$, then
    $$x(t-t_0) \to X(s)e^{-st_0}$$
    with ROC $= \mathcal{R}$ (possibly excluding $s = \pm\infty$).
  - **Frequency shifting:** If $x(t) \to X(s)$ with ROC $\mathcal{R}$, then
    $$x(t)e^{s_0 t} \to X(s-s_0)$$
    with ROC $= \mathcal{R} + Re\{s_0\} \triangleq \{s : s - s_0 \in \mathcal{R}\}$.
  - **Convolution:** If $x(t) \to X(s)$ with ROC $\mathcal{R}_1$ and $y(t) \to Y(s)$ with ROC $\mathcal{R}_2$, then
    $$x(t) \star y(t) \to X(s)Y(s)$$
    with an ROC containing $\mathcal{R}_1 \cap \mathcal{R}_2$. There might be a zero-pole cancellation resulting in a larger ROC.
  - **Differentiation in the s-domain:** If $x(t) \to X(s)$ with ROC $\mathcal{R}$, then
    $$tx(t) \to -\frac{dX(s)}{ds}$$
    with ROC $= \mathcal{R}$ (possibly excluding $s = \pm\infty$).
- **Some known signal/Laplace transform pairs:**
  - If $x(t) = e^{at}u(t)$, then
    $$X(s) = \frac{1}{s-a}$$
    with ROC: $Re\{s\} > a$.
  - If $x(t) = -e^{at}u(-t)$, then
    $$X(s) = \frac{1}{s-a}$$
    with ROC: $Re\{s\} < a$.
  - If $x(t) = te^{at}u(t)$, then
    $$X(s) = \frac{1}{(s-a)^2}$$
    with ROC: $Re\{s\} > a$.
  - If $x(t) = -te^{at}u(-t)$, then
    $$X(s) = \frac{1}{(s-a)^2}$$
    with ROC: $Re\{s\} < a$.
  - If $x(t) = \delta(t-t_0)$, then
    $$X(s) = e^{-st_0}$$
    with ROC being the entire plane except $s = -\infty$ or $s = \infty$ depending on whether $t_0 > 0$ or $t_0 < 0$, respectively.
  - If $x(t) = \sin(at)u(t)$, then
    $$X(s) = \frac{a}{s^2 + a^2}$$
    with ROC: $Re\{s\} > 0$.
  - If $x(t) = \cos(at)u(t)$, then
    $$X(s) = \frac{s}{s^2 + a^2}$$
    with ROC: $Re\{s\} > 0$.
- **System properties using the Laplace transform** An LTI system whose impulse response is $h(t)$, with Laplace transform $H(s)$, is
  - **memoryless** if and only if $H(s)$ has no zeros or poles.
  - **causal** if and only if the ROC of $H(s)$ of the form $Re\{s\} > a$ for some a, and $H(s)$ has no hidden poles at infinity.
  - **stable** if and only if the ROC of $H(s)$ includes the imaginary axis.

  In addition, the inverse system $G(s) = \frac{1}{H(s)}$ always exists, but to be able to implement it in practice, we need all poles *and* zeroes of $H(s)$ to the left of the imaginary axis.