**UNIVERSITY OF CALIFORNIA, RIVERSIDE**
**Department of Electrical Engineering**

WINTER 2024
**EE 110B SIGNALS AND SYSTEMS**
**MIDTERM**

You have 50 minutes to complete the exam. Please fully justify your work.

**Question 1)** (40 points)
In this question, we will derive the famous formula

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

using difference equations. To see this, we will first define

$$y[n] = \sum_{k=0}^{n} k^2$$

for $n \geq 0$, and notice that

$$y[n] - y[n-1] = n^2 \ . \tag{2}$$

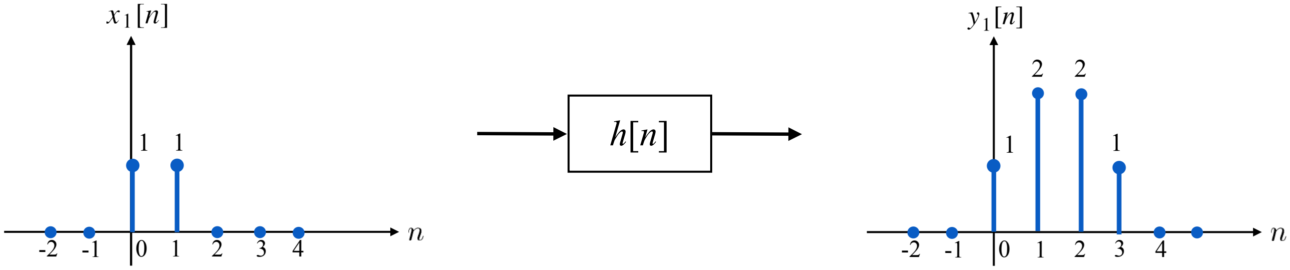
Now,
**a)** Find the homogeneous solution family to (2), and denote it by $y_h[n]$.
**b)** As always, we will add the particular solution $y_p[n]$. Towards that end, *guess* the solution

$$y_p[n] = an^3 + bn^2 + cn$$

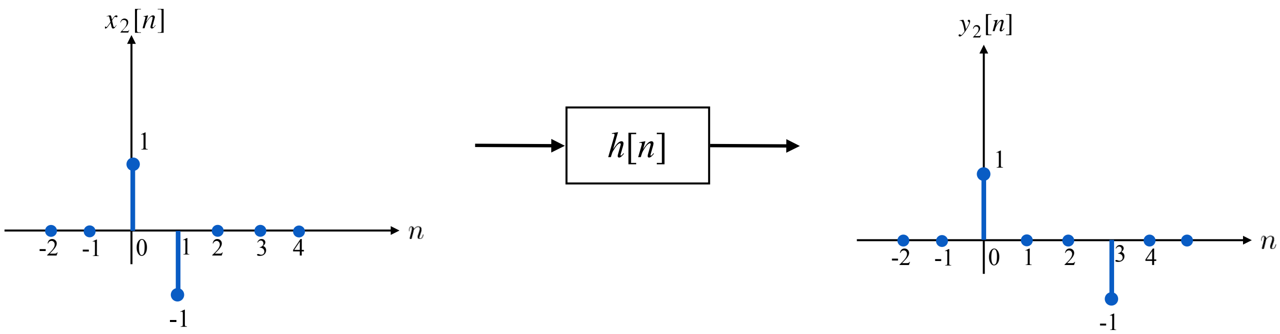
and solve for $a, b, c$ to satisfy (2) for all $n$.
**c)** Bring the two together and write the solution family as $y[n] = y_h[n] + y_p[n]$. Noting that $y[0] = 0$, use it as an initial condition to single out the unique solution to $y[n]$. Confirm that you have (1) as the solution.

**Answer:**

**Question 2)** (25 points)
A linear and time-invariant (LTI) system has been observed to output $y_1[n]$ when $x_1[n]$ is the input, as shown below:



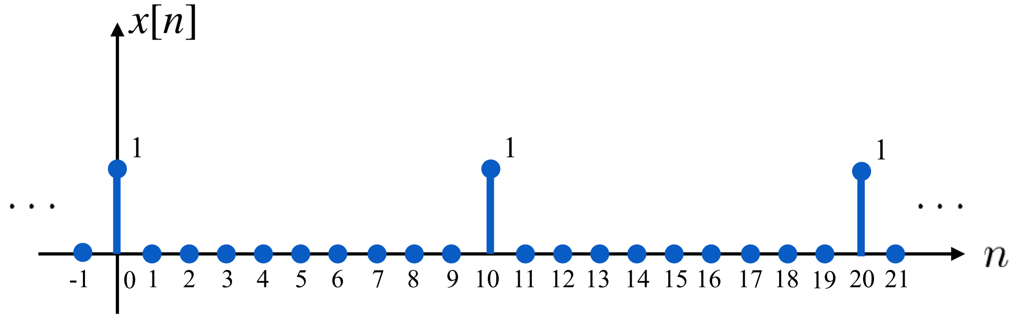

**The same LTI system** has also been observed to output $y_2[n]$ when $x_2[n]$ is the input:




Find the impulse response $h[n]$. You can plot $h[n]$ or describe it mathematically, whichever is easier.

**Answer:**

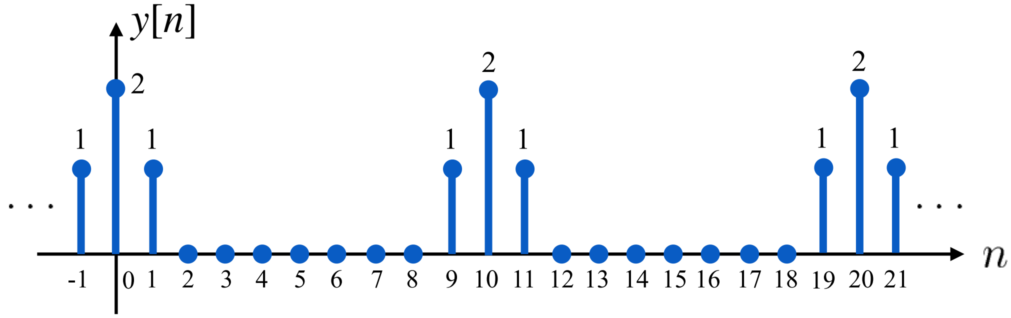
**Question 3)** (35 points)




In one of the homework problems, we have seen that the signal above has the DTFS coefficients given by

$$a_k = \frac{1}{10}$$

for all $k$. Using this result, together with the properties of the DTFS, find the DTFS coefficients $b_k$ of the signal below.




**Answer:**

## FORMULAS AND CONCEPTS YOU MIGHT FIND USEFUL

- **Complex numbers:** If $z$ is a complex number, then it can be expressed in one of two forms:
$$z = a + jb \qquad \text{or} \qquad z = re^{j\theta}$$
where
$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \ .$$
Values of $r$ and $\theta$ can be found from $a$ and $b$ using
$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ \theta &= \tan^{-1}\left(\frac{b}{a}\right) \ . \end{aligned}$$
The complex conjugate of $z$ is $z^* = a - jb$ or $z^* = re^{-j\theta}$. We also have the following relationships:
$$\begin{aligned} z + z^* &= 2Re\{z\} &= 2a \\ z - z^* &= 2Im\{z\} &= 2b \\ zz^* &= |z|^2 &= a^2 + b^2 \end{aligned}$$
- **Sine waves and complex exponentials:**
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \text{and} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$
- **Geometric series:** For any $\alpha \neq 1$ (complex or real),
$$\sum_{n=0}^{N} \alpha^n = \frac{1-\alpha^{N+1}}{1-\alpha} \ .$$
If $|\alpha| < 1$, in the limit $N \to \infty$, this becomes $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$. When $\alpha = 1$, we immediately have
$$\sum_{n=0}^{N} \alpha^n = N + 1 \ .$$
- **System properties:** A system is
  - **linear** if
$$\begin{matrix} x_1[n] \to y_1[n] \\ x_2[n] \to y_2[n] \end{matrix} \implies ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$$
  - **time-invariant** if
$$x[n] \to y[n] \implies x[n-n_0] \to y[n-n_0]$$
  - **memoryless** if $y[n]$ at time $n$ depends only on $x[n]$ on time $n$.
  - **causal** if $y[n]$ at time $n$ depends only on $x[k]$ on times $k \leq n$.
  - **stable** if $|x[n]| \leq B$ for some $M$ implies $|y[n]| \leq C$ for some $C$.
  - **invertible** if two distinct $x_1[n]$ and $x_2[n]$ does not result in the same $y[n]$.
- **LTI System properties:** An LTI system with impulse response $h[n]$ is
  - **memoryless** if $h[n] = 0$ for all $n \neq 0$.
  - **causal** if $h[n] = 0$ for all $n < 0$.
  - **stable** if
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$
  - **invertible** if
$$x[n] \star h[n] = 0$$
implies $x[n] = 0$. In other words, there is no nonzero signal whose convolution with $h[n]$ outputs the zero signal.
- **Difference equations:** For a $K$th order equation of the form
$$\sum_{k=0}^{K} \alpha_k y[n-k] = x[n]$$
you need to
  - find a *particular* solution $y_p[n]$, i.e., *any* $y[n]$ that satisfies the equation,
  - find a family of *homogeneous* solutions, $y_h[n]$, that satisfy the equation with $x[n] = 0$,
  - write the overall solution family as $y[n] = y_p[n] + y_h[n]$,
  - find the specific member of the family by using initial conditions $y[0], y[1], \ldots, y[K-1]$.

  If the system is said to be in *initial rest*, then you need to derive your own initial conditions using the fact that $y[-1] = y[-2] = \ldots = 0$.
- **DTFS:** For a signal with period $N$, and $\omega_0 = \frac{2\pi}{N}$
$$x[n] = \sum_{k \in \mathcal{N}} a_k e^{jk\omega_0 n}$$
where
$$a_k = \frac{1}{N} \sum_{n \in \mathcal{N}} x[n] e^{-jk\omega_0 n} \ .$$
By $\mathcal{N}$, we mean any interval of integers with length $N$, e.g., $\{0, 1, \ldots, N-1\}$.
- **Properties of DTFS:**
  - **Linearity:**
$$\begin{matrix} x[n] \to a_k \\ y[n] \to b_k \end{matrix} \implies \alpha x[n] + \beta y[n] \to \alpha a_k + \beta b_k$$
  - **Conjugation:**
$$x[n] \to a_k \implies x^*[n] \to a^*_{-k}$$
  - **Time reversal:**
$$x[n] \to a_k \implies x[-n] \to a_{-k}$$
  - **Time shifting:**
$$x[n] \to a_k \implies x[n-n_0] \to a_k e^{-jk\omega_0 n_0}$$
  - **Frequency shifting:**
$$x[n] \to a_k \implies x[n] e^{jk_0\omega_0 n} \to a_{k-k_0}$$
  - **Periodic convolution:**
$$\begin{matrix} x[n] \to a_k \\ y[n] \to b_k \end{matrix} \implies x[n] \tilde{*} y[n] \to N a_k b_k$$
where
$$x[n] \tilde{*} y[n] = \sum_{l \in \mathcal{N}} x[l] y[n-l]$$
  - **Multiplication:**
$$\begin{matrix} x[n] \to a_k \\ y[n] \to b_k \end{matrix} \implies x[n]y[n] \to a_k \tilde{*} b_k$$
where
$$a_k \tilde{*} b_k = \sum_{l \in \mathcal{N}} a_l b_{k-l}$$
  - **Parseval's relation:**
$$\frac{1}{N} \sum_{n \in \mathcal{N}} |x[n]|^2 = \sum_{k \in \mathcal{N}} |a_k|^2 \ .$$
  - **Real signals:** If $x[n]$ is real, $a_k = a^*_{-k}$, therefore in reconstructing $x[n]$, we can conveniently couple $a_k e^{jk\omega_0 n}$ and $a_{-k} e^{-jk\omega_0 n}$ to write
$$\begin{aligned} a_k e^{jk\omega_0 n} + a_{-k} e^{-jk\omega_0 n} &= a_k e^{jk\omega_0 n} + a^*_k e^{-jk\omega_0 n} \\ &= 2Re\{a_k e^{jk\omega_0 n}\} \\ &= 2|a_k| \cos(k\omega_0 n + \angle a_k) \end{aligned}$$