Homework 2 solutions

Problem 1 [8pts]: Determine whether the following transformations are invertible. If they are, express $x(t)$ in terms of $y(t)$.

a) $y(t) = tx(t)$

b) $y(t) = (1 + t^2)x(t)$

c) $y(t) = \begin{cases} x(t) & t < 0 \\ 0 & 0 \leq t < 1 \\ x(t-1) & t \geq 1 \end{cases}$

d) $y(t) = \begin{cases} x(t) & t < 0 \\ 0 & 0 \leq t \leq 1 \\ x(t-1) & t > 1 \end{cases}$

Solution:

a) Not invertible, because $y(0) = 0$ regardless of what $x(0)$ is. That means the information about $x(0)$ is lost forever. One can exploit this to come up with examples where $x_1(t)$ and $x_2(t)$ yield the same $y(t)$. For instance,

$$x_1(t) = \begin{cases} 1 & t \neq 0 \\ 2 & t = 0 \end{cases}$$

and

$$x_2(t) = \begin{cases} 1 & t \neq 0 \\ 1,000 & t = 0 \end{cases}$$

both yield $y(t) = t$.

b) Invertible, because this time each $x(t)$ is multiplied by a non-zero entity. Therefore one can obtain $x(t)$ back from $y(t)$ simply by dividing it by the correct amount. More specifically,

$$x(t) = \frac{1}{1+t^2} y(t) \ .$$

c) Invertible, because one can write

$$x(t) = \begin{cases} y(t) & t < 0 \\ y(t+1) & t \geq 0 \end{cases} \ .$$

d) Not invertible, because this time, $y(t)$ does not copy the value $x(0)$ anywhere, thereby losing it forever. The same example of $x_1(t)$ and $x_2(t)$ in part a) can be used here as well. They will result in the same $y(t)$ given as

$$y(t) = \begin{cases} 1 & t < 0 \\ 0 & 0 \le t \le 1 \\ 1 & 1 < t \end{cases}$$

Problem 2 [8pts]: Let the impulse response of an LTI system be given by

$$h(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

Find the output $y(t)$ of this system if the input is given by

a) $x(t) = 1 + \cos(2\pi t)$

b) $x(t) = \cos(\pi t)$

c) $x(t) = |\sin(\pi t)|$

d) $x(t) = \delta(t-3) + \delta(t-5)$

Solution:

Writing $h(t-\tau)$ as

$$h(t-\tau) = \begin{cases} 0 & t-\tau < 0 \\ 1 & 0 \le t-\tau \le 1 \\ 0 & t-\tau > 1 \end{cases} = \begin{cases} 0 & \tau < t-1 \\ 1 & t-1 \le \tau \le t \\ 0 & \tau > t \end{cases}$$

One can simplify the convolution integral as below:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{t-1}^{t} x(\tau)d\tau \ . \end{aligned}$$

In light of this simplification, we answer the question as follows.

a)

$$\begin{aligned} y(t) &= \int_{t-1}^{t} [1+\cos(2\pi\tau)]d\tau \\ &= 1 + \int_{t-1}^{t} \cos(2\pi\tau)d\tau \\ &= 1 + \frac{1}{2\pi}\sin(2\pi\tau)\Big|_{t-1}^{t} \\ &= 1 + \frac{1}{2\pi}[\sin(2\pi t) - \sin(2\pi t - 2\pi)] \\ &= 1\ . \end{aligned}$$

Note that in general the convolution integral will output a function of t, for this input, the output is constant at all t.

b)

$$\begin{aligned} y(t) &= \int_{t-1}^{t} \cos(\pi\tau)d\tau \\ &= \frac{1}{\pi}\sin(\pi\tau)\Big|_{t-1}^{t} \\ &= \frac{1}{\pi}[\sin(\pi t) - \sin(\pi t - \pi)] \\ &= \frac{2}{\pi}\sin(\pi t)\ . \end{aligned}$$

c) We start by making the observation that $|\sin(\pi\tau)|$ has a period of 1. Therefore, an integral over it from $t-1$ to t will create a signal which has constant value. It then suffices to find only that value, i.e., $y(0)$. Proceeding,

$$\begin{aligned} y(0) &= \int_{0}^{1} |\sin(\pi\tau)|d\tau \\ &= \int_{0}^{1} \sin(\pi\tau)d\tau \\ &= \frac{-1}{\pi}\cos(\pi\tau)\Big|_{0}^{1} \\ &= \frac{-1}{\pi}\Big[\cos(\pi) - \cos(0)\Big] \\ &= \frac{-1}{\pi}\Big[-1-1\Big] \\ &= \frac{2}{\pi}\ . \end{aligned}$$

d)

$$y(t) = \int_{t-1}^{t} [\delta(\tau - 3) + \delta(\tau - 5)]\, d\tau = \int_{t-1}^{t} \delta(\tau - 3) d\tau + \int_{t-1}^{t} \delta(\tau - 5) d\tau$$

The two integrals above depend on whether $\tau = 3$ or $\tau = 5$ is within the interval $[t-1, t]$. In other words,

$$\int_{t-1}^{t} \delta(\tau - 3) d\tau = \begin{cases} 1 & t-1 < 3 < t \\ 0 & \text{otherwise} \end{cases}$$

and similarly,

$$\int_{t-1}^{t} \delta(\tau - 5) d\tau = \begin{cases} 1 & t-1 < 5 < t \\ 0 & \text{otherwise} \end{cases}$$

Bringing these together,

$$y(t) = \begin{cases} 0 & t \leq 3 \\ 1 & 3 < t < 4 \\ 0 & 4 \leq t \leq 5 \\ 1 & 5 < t < 6 \\ 0 & t \geq 6 \end{cases}$$

Problem 3 [4pts]: Consider an LTI system whose response to signal $x_1(t)$ in Fig.1(a) is the signal $y_1(t)$ illustrated in Fig. 1(b). Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted in Fig. 1(c).

Hint: Express $x_2(t)$ in terms of $x_1(t)$ and then use linearity property.

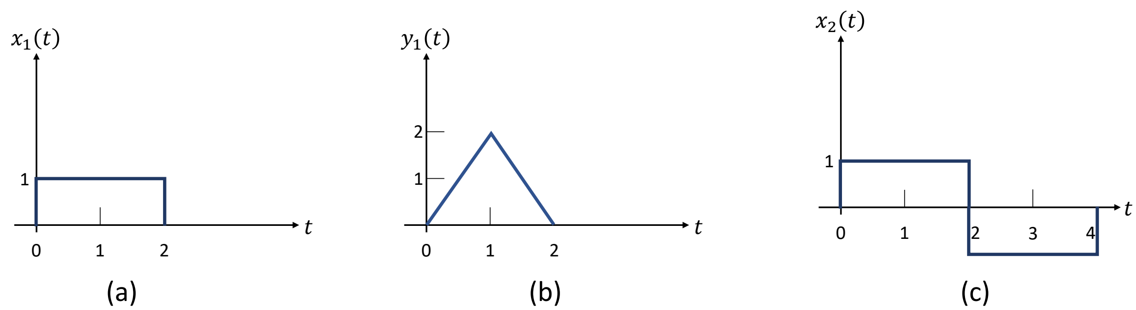

Figure 1: Figure for problem 3

Solution:

Observe that $x_2(t) = x_1(t) - x_1(t-2)$. Using linearity property, we get,

$$y_2(t) = y_1(t) - y_1(t-2). \tag{1}$$

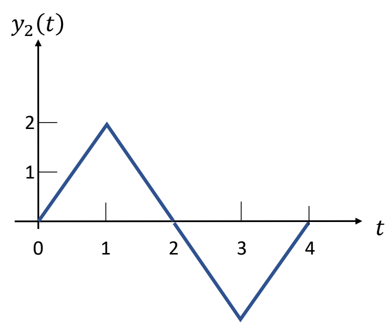

Figure 2: Solution for problem 3