UNIVERSITY OF CALIFORNIA, RIVERSIDE
Department of Electrical and Computer Engineering
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EE 110B SIGNALS AND SYSTEMS
FINAL EXAM SOLUTIONS

Question 1)
a) We have

$$\begin{aligned}
\lim_{z\to\infty} X(z) &= \lim_{z\to\infty}\left(x[0]+x[1]z^{-1}+x[2]z^{-2}+x[3]z^{-3}+\dots\right)\\
&= x[0]+x[1]\cdot 0+x[2]\cdot 0+x[3]\cdot 0+\dots\\
&= x[0]\ .
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{z\to\infty} z\left(X(z)-x[0]\right) &= \lim_{z\to\infty} z\left(x[0]+x[1]z^{-1}+x[2]z^{-2}+x[3]z^{-3}+\dots-x[0]\right)\\
&= \lim_{z\to\infty} x[1]+x[2]z^{-1}+x[3]z^{-2}+\dots\\
&= x[1]+x[2]\cdot 0+x[3]\cdot 0+x[4]\cdot 0+\dots\\
&= x[1]
\end{aligned}$$

and similarly

$$\begin{aligned}
\lim_{z\to\infty} z^2\left(X(z)-x[0]-x[1]z^{-1}\right) &= \lim_{z\to\infty} z^2\left(x[0]+x[1]z^{-1}+x[2]z^{-2}+x[3]z^{-3}+\dots-x[0]-x[1]z^{-1}\right)\\
&= \lim_{z\to\infty} x[2]+x[3]z^{-1}+x[4]z^{-2}+\dots\\
&= x[2]+x[3]\cdot 0+x[4]\cdot 0+x[5]\cdot 0+\dots\\
&= x[2]\ .
\end{aligned}$$

c) In general, we can write

$$\begin{aligned}
\lim_{z\to\infty} z^n\left(X(z)-\sum_{k=0}^{n-1}x[k]z^{-k}\right) &= \lim_{z\to\infty} z^n\left(\sum_{k=0}^{\infty}x[k]z^{-k}-\sum_{k=0}^{n-1}x[k]z^{-k}\right)\\
&= \lim_{z\to\infty} z^n\sum_{k=n}^{\infty}x[k]z^{-k}\\
&= \lim_{z\to\infty} \sum_{k=n}^{\infty}x[k]z^{n-k}\\
&= \lim_{z\to\infty} x[n]+x[n+1]z^{-1}+x[n+2]z^{-2}+x[n+3]z^{-3}+\dots\\
&= x[n]+x[n+1]\cdot 0+x[n+2]\cdot 0+x[n+3]\cdot 0+\dots\\
&= x[n]\ .
\end{aligned}$$

d) We need to start from

$$x[0]=\lim_{z\to\infty}X(z)=\lim_{z\to\infty}\frac{1}{1-z^{-1}}=1\ .$$

Then,

$$\begin{aligned}
x[1] &= \lim_{z\to\infty} z\left(X(z) - x[0]\right) \\
&= \lim_{z\to\infty} z\left(\frac{1}{1-z^{-1}} - 1\right) \\
&= \lim_{z\to\infty} z\left(\frac{1}{1-z^{-1}} - \frac{1-z^{-1}}{1-z^{-1}}\right) \\
&= \lim_{z\to\infty} z\left(\frac{z^{-1}}{1-z^{-1}}\right) \\
&= \lim_{z\to\infty} \frac{1}{1-z^{-1}} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
x[2] &= \lim_{z\to\infty} z^2\left(X(z) - x[0] - x[1]z^{-1}\right) \\
&= \lim_{z\to\infty} z^2\left(\frac{1}{1-z^{-1}} - 1 - z^{-1}\right) \\
&= \lim_{z\to\infty} z^2\left(\frac{1}{1-z^{-1}} - \frac{1-z^{-1}}{1-z^{-1}} - \frac{z^{-1}(1-z^{-1})}{1-z^{-1}}\right) \\
&= \lim_{z\to\infty} z^2\left(\frac{z^{-2}}{1-z^{-1}}\right) \\
&= \lim_{z\to\infty} \frac{1}{1-z^{-1}} \\
&= 1\ .
\end{aligned}$$

We seem to have caught the trend here. In fact, we can show that if $x[0] = x[1] = x[2] = \ldots = x[n-1] = 1$, we must have $x[n] = 1$, as follows:

$$\begin{aligned}
x[n] &= \lim_{z\to\infty} z^n\left(X(z) - \sum_{k=0}^{n-1} x[k]z^{-k}\right) \\
&= \lim_{z\to\infty} z^n\left(\frac{1}{1-z^{-1}} - \left[1 + z^{-1} + z^{-2} + \ldots + z^{-(n-1)}\right]\right) \\
&= \lim_{z\to\infty} z^n\left(\frac{1}{1-z^{-1}} - \frac{(1-z^{-1})\left[1 + z^{-1} + z^{-2} + \ldots + z^{-(n-1)}\right]}{1-z^{-1}}\right) \\
&= \lim_{z\to\infty} z^n \cdot \frac{1 - \left[1 + z^{-1} + z^{-2} + \ldots + z^{-(n-1)}\right] + z^{-1}\left[1 + z^{-1} + z^{-2} + \ldots + z^{-(n-1)}\right]}{1-z^{-1}} \\
&= \lim_{z\to\infty} z^n \cdot \frac{1 - \left[1 + z^{-1} + z^{-2} + \ldots + z^{-(n-1)}\right] + \left[z^{-1} + z^{-2} + z^{-3} + \ldots + z^{-n}\right]}{1-z^{-1}} \\
&= \lim_{z\to\infty} z^n \cdot \frac{z^{-n}}{1-z^{-1}} \\
&= \lim_{z\to\infty} \frac{1}{1-z^{-1}} \\
&= 1\ .
\end{aligned}$$

The conclusion is that $x[n] = u[n]$, which is what was expected in the first place, because we could have recognized it by the form of the z-transform.

Question 2)
a) We know that

$$\sin\left(\frac{\pi}{2}n\right) = \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j} = \frac{j^n - (-j)^n}{2j} \ .$$

Therefore

$$\begin{aligned} X(z) &= \frac{1}{2j}\left[\frac{1}{1-jz^{-1}} - \frac{1}{1+jz^{-1}}\right] \\ &= \frac{1}{2j}\frac{1+jz^{-1}-(1-jz^{-1})}{(1-jz^{-1})(1+jz^{-1})} \\ &= \frac{z^{-1}}{1+z^{-2}} \\ &= \frac{z}{z^2+1} \end{aligned}$$

There are two poles, one at $z = j$ and another at $z = -j$, and a zero at $z = 0$. There is also a hidden zero at $z = \infty$. The ROC is given by $\{z : |z| > 1\}$.

b) We know that $x[-n]$ has a z-transform equal to $X(z^{-1})$. But

$$\begin{aligned} X(z^{-1}) &= \frac{z^{-1}}{z^{-2}+1} \\ &= \frac{z}{1+z^2} \\ &= X(z) \ . \end{aligned}$$

The ROC of $X(z)$ is the reciprocal of that of $X(z)$, i.e., $\{z : |z| < 1\}$.

The reason why poles and zeros of $X(z^{-1})$ are at the exact same spots as those of $X(z)$ is that for each zero z_i of $X(z)$, there is another zero at z_i^{-1}, and similarly, for each pole p_i, there is one at p_i^{-1}.

c) We must first find $H(z)$. But since $h[n] = 0$ at all but two locations, we can easily conclude

$$H(z) = 0.5 + 0.5z^{-2}$$

with an ROC$= \{z : |z| > 0\}$. Then,

$$Y(z) = \frac{z^{-1}}{1+z^{-2}} \cdot \frac{1+z^{-2}}{2} = \frac{z^{-1}}{2}$$

with also an ROC$= \{z : |z| > 0\}$, because the poles and zeros canceled each other. Clearly, this is the shifted version of half an impulse, i.e.,

$$y[n] = 0.5\delta[n-1] \ .$$

d) Since $X(z^{-1}) = X(z)$, we also have $X(z^{-1})H(z) = X(z)H(z)$ yielding $T(z) = Y(z) = \frac{z^{-1}}{2}$. So, despite the fact that $x[-n]$ is a left-sided sequence, and hence has a very different ROC, in the end it does not matter because of the pole-zero cancellation. The answer is the same as in part c, i.e.,

$$t[n] = y[n] \ .$$

Question 3)
Following the hint, we can write $x_1[n] = x[-n]$ yielding

$$X_1(e^{j\omega}) = X(e^{-j\omega}) \ .$$

Then, $x_2[n] = x_1[n-3]$ yielding

$$X_2(e^{j\omega}) = X_1(e^{j\omega})e^{-j3\omega} = X(e^{-j\omega})e^{-j3\omega} \ .$$

Next, $x_3[n] = nx_2[n]$, implying

$$\begin{aligned} X_3(e^{j\omega}) &= j\frac{dX_2(e^{j\omega})}{d\omega} \\ &= j\frac{d}{d\omega}\left[X(e^{-j\omega})e^{-j3\omega}\right] \\ &= j\frac{dX(e^{-j\omega})}{d\omega}e^{-j3\omega} + jX(e^{-j\omega})\frac{de^{-j3\omega}}{d\omega} \\ &= -jX'(e^{-j\omega})e^{-j3\omega} + jX(e^{-j\omega})e^{-j3\omega}(-j3) \\ &= -jX'(e^{-j\omega})e^{-j3\omega} + 3X(e^{-j\omega})e^{-j3\omega} \end{aligned}$$

where $X'(e^{j\omega})$ is the derivative of $X(e^{j\omega})$. Finally, $y[n] = nx_3[n]$, and therefore,

$$\begin{aligned} Y(e^{j\omega}) &= j\frac{dX_3(e^{j\omega})}{d\omega} \\ &= j\frac{d}{d\omega}\left[-jX'(e^{-j\omega})e^{-j3\omega} + 3X(e^{-j\omega})e^{-j3\omega}\right] \\ &= \frac{d}{d\omega}\left[X'(e^{-j\omega})e^{-j3\omega}\right] + 3j\frac{d}{d\omega}\left[X(e^{-j\omega})e^{-j3\omega}\right] \\ &= \frac{dX'(e^{-j\omega})}{d\omega}e^{-j3\omega} + X'(e^{-j\omega})\frac{de^{-j3\omega}}{d\omega} + 3j\frac{dX(e^{-j\omega})}{d\omega}e^{-j3\omega} + 3jX(e^{-j\omega})\frac{de^{-j3\omega}}{d\omega} \\ &= -X''(e^{-j\omega})e^{-j3\omega} + X'(e^{-j\omega})e^{-j3\omega}(-j3) - 3jX'(e^{-j\omega})e^{-j3\omega} + 3jX(e^{-j\omega})e^{-j3\omega}(-j3) \\ &= -X''(e^{-j\omega})e^{-j3\omega} - 6jX'(e^{-j\omega})e^{-j3\omega} + 9X(e^{-j\omega})e^{-j3\omega} \ . \end{aligned}$$

If you do not want to express $Y(e^{j\omega})$ in so much detail, you can leave the relation as

$$\begin{aligned} Y(e^{j\omega}) &= j\frac{dX_3(e^{j\omega})}{d\omega} \\ &= j\frac{d}{d\omega}\left[j\frac{dX_2(e^{j\omega})}{d\omega}\right] \\ &= -\frac{d^2X_2(e^{j\omega})}{d\omega^2} \\ &= -\frac{d^2}{d\omega^2}\left[X(e^{-j\omega})e^{-j3\omega}\right] \end{aligned}$$

which will be accepted as the correct solution.

Question 4)
a) Since

$$F_c(j\Omega) = \int_{-\infty}^{\infty} f(t)e^{-j\Omega t}dt$$

we immediately obtain

$$F_c(0) = \int_{-\infty}^{\infty} f(t)dt \ .$$

b) Similarly, since

$$F_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\omega n}$$

it follows that

$$F_d(e^{j0}) = \sum_{n=-\infty}^{\infty} f[n]$$

c) Now, if $f[n] = f(n\Delta)$, sampling theory tells us that

$$F_d(e^{j\omega}) = \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} F_c\left(j\left[\frac{\omega - k2\pi}{\Delta}\right]\right) \ .$$

Specializing to $\omega = 0$ yields

$$F_d(e^{j0}) = \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} F_c\left(j\left[\frac{-k2\pi}{\Delta}\right]\right) \ .$$

which can also be written as

$$F_d(e^{j0}) = \frac{1}{\Delta}\left[F_c(j0) + \sum_{k\neq 0} F_c\left(j\left[\frac{k2\pi}{\Delta}\right]\right)\right] \ .$$

d) Gathering results from a-c, we can rewrite the limit as

$$\lim_{\Delta\to 0}\left[F_c(j0) + \sum_{k\neq 0} F_c\left(j\left[\frac{k2\pi}{\Delta}\right]\right)\right] = F_c(j0)$$

or even more simply as

$$\lim_{\Delta\to 0} \sum_{k\neq 0} F_c\left(j\left[\frac{k2\pi}{\Delta}\right]\right) = 0 \ .$$

e) From part d, it can be seen that if the infinite sum on the left-hand side is zero for any $\Delta > 0$, then the Riemann approximation is *exact*, i.e.,

$$\sum_{n=-\infty}^{\infty} f(n\Delta)\Delta = \int_{-\infty}^{\infty} f(t)dt \ .$$

One such case is when the signal is band-limited. If $F_c(j\Omega) = 0$ for all $|\Omega| > \Omega_M$, then all we need is

$$\frac{2\pi}{\Delta} \geq \Omega_M \ ,$$

or in other words

$$\Delta \leq \frac{2\pi}{\Omega_M} = \Delta_{\max} \ .$$