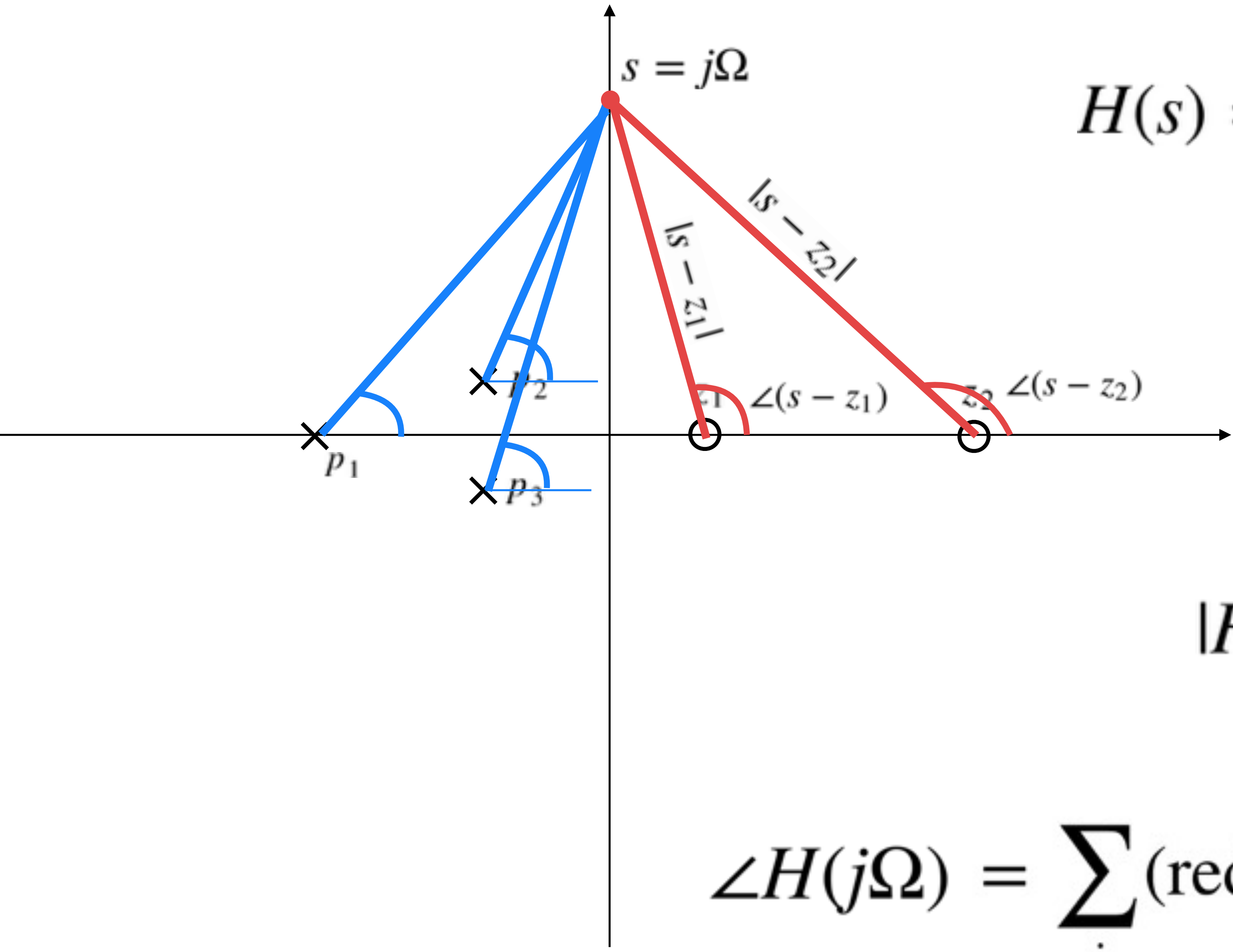
EE 110A Signals and Systems

The Laplace Transform
Part II

Ertem Tuncel

CTFT from poles and zeros

- The pole-zero plot is useful in understanding the frequency behavior of the signal (or filter).



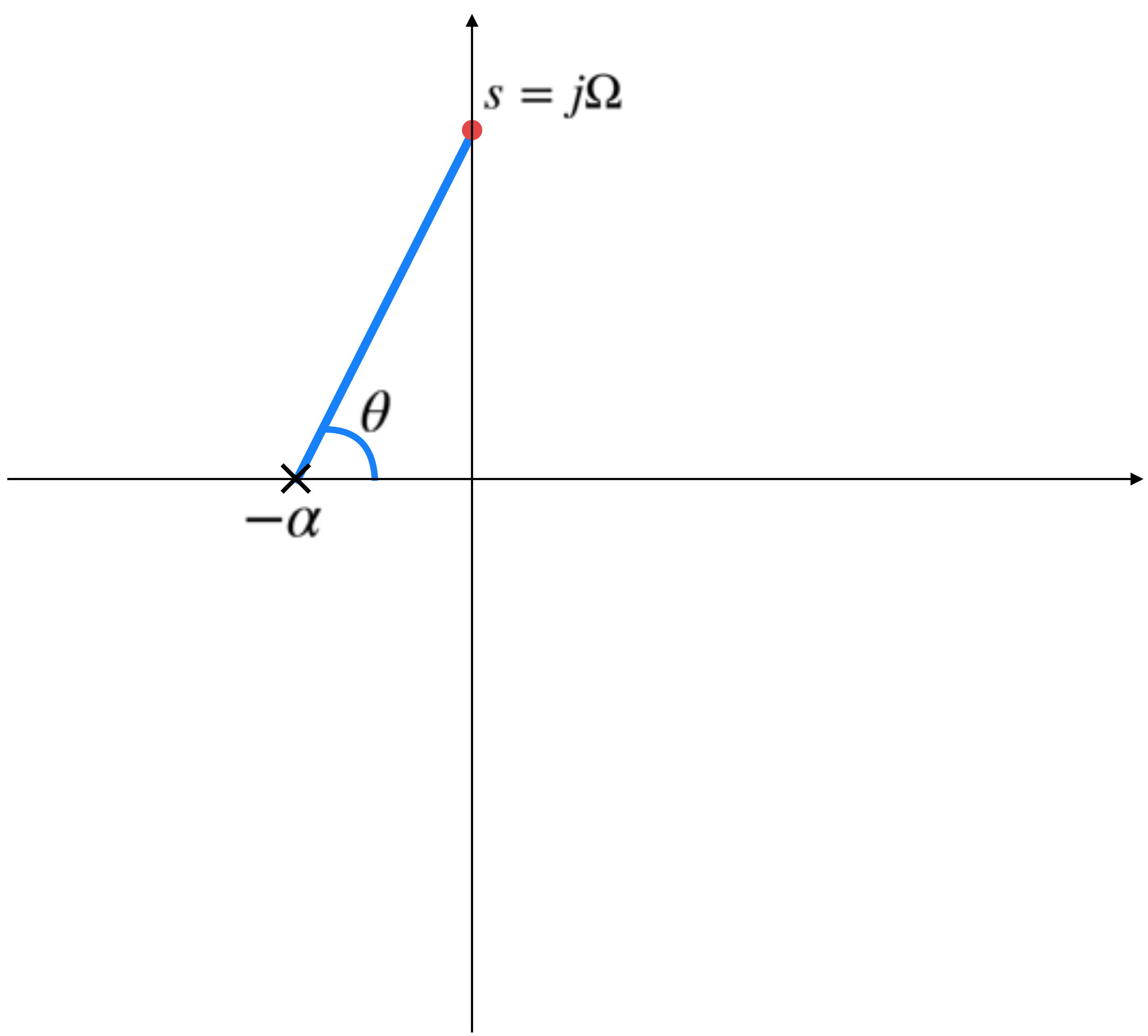
$$H(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)(s - p_3)}$$

$$|H(j\Omega)| = \frac{\prod_i (\text{red length})_i}{\prod_j (\text{blue length})_j}$$

$$\angle H(j\Omega) = \sum_i (\text{red angle})_i - \sum_j (\text{blue angle})_j$$

CTFT from poles and zeros

- Example: A first-order filter: $h(t) = e^{-\alpha t} u(t)$

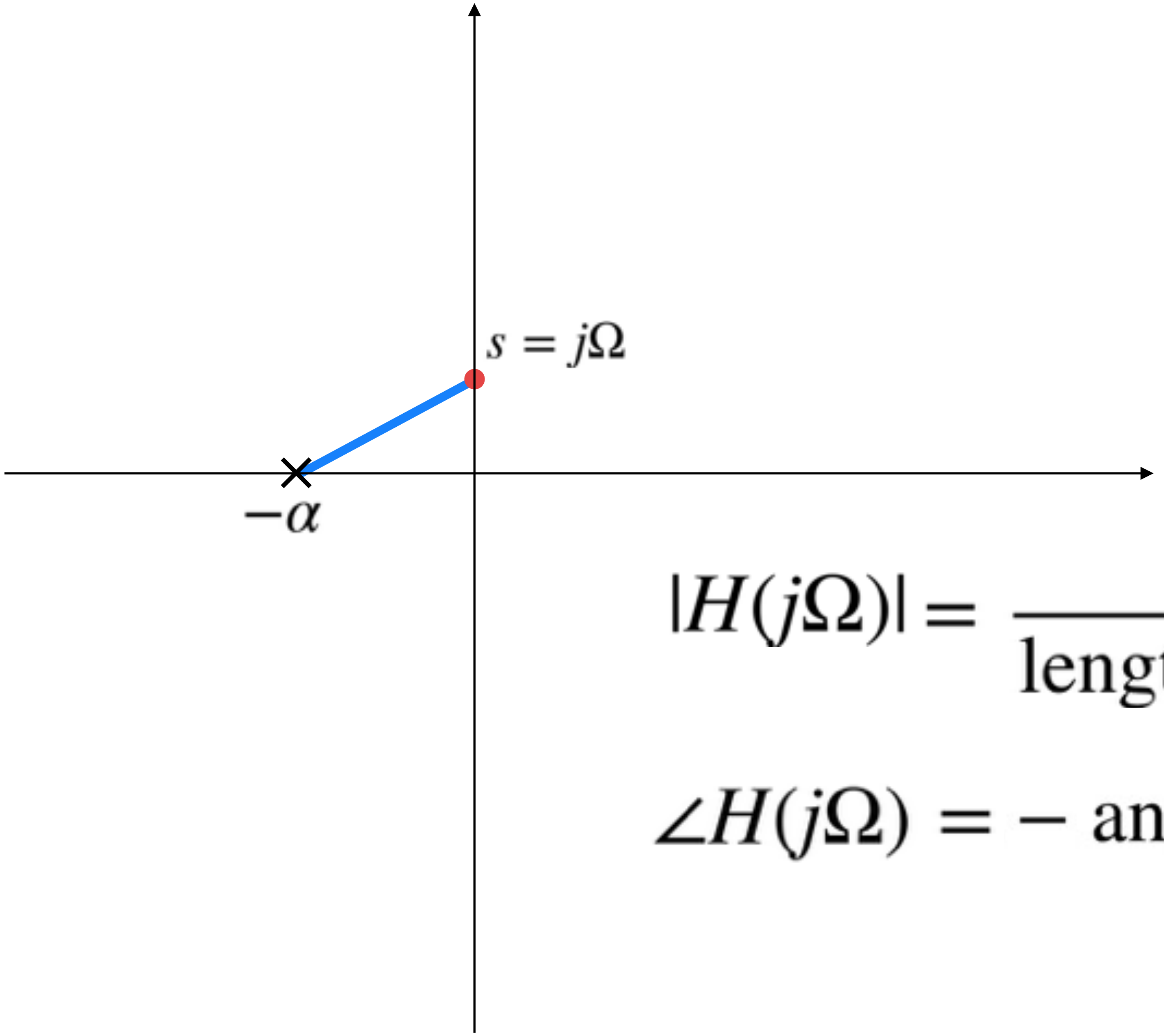


$$H(s) = \frac{1}{s + \alpha}$$

$$|H(j\Omega)| = \frac{1}{\text{length of blue}} = \frac{1}{\sqrt{\alpha^2 + \Omega^2}}$$

$$\angle H(j\Omega) = -\text{ angle of blue} = -\theta = -\tan^{-1}\left(\frac{\Omega}{\alpha}\right)$$

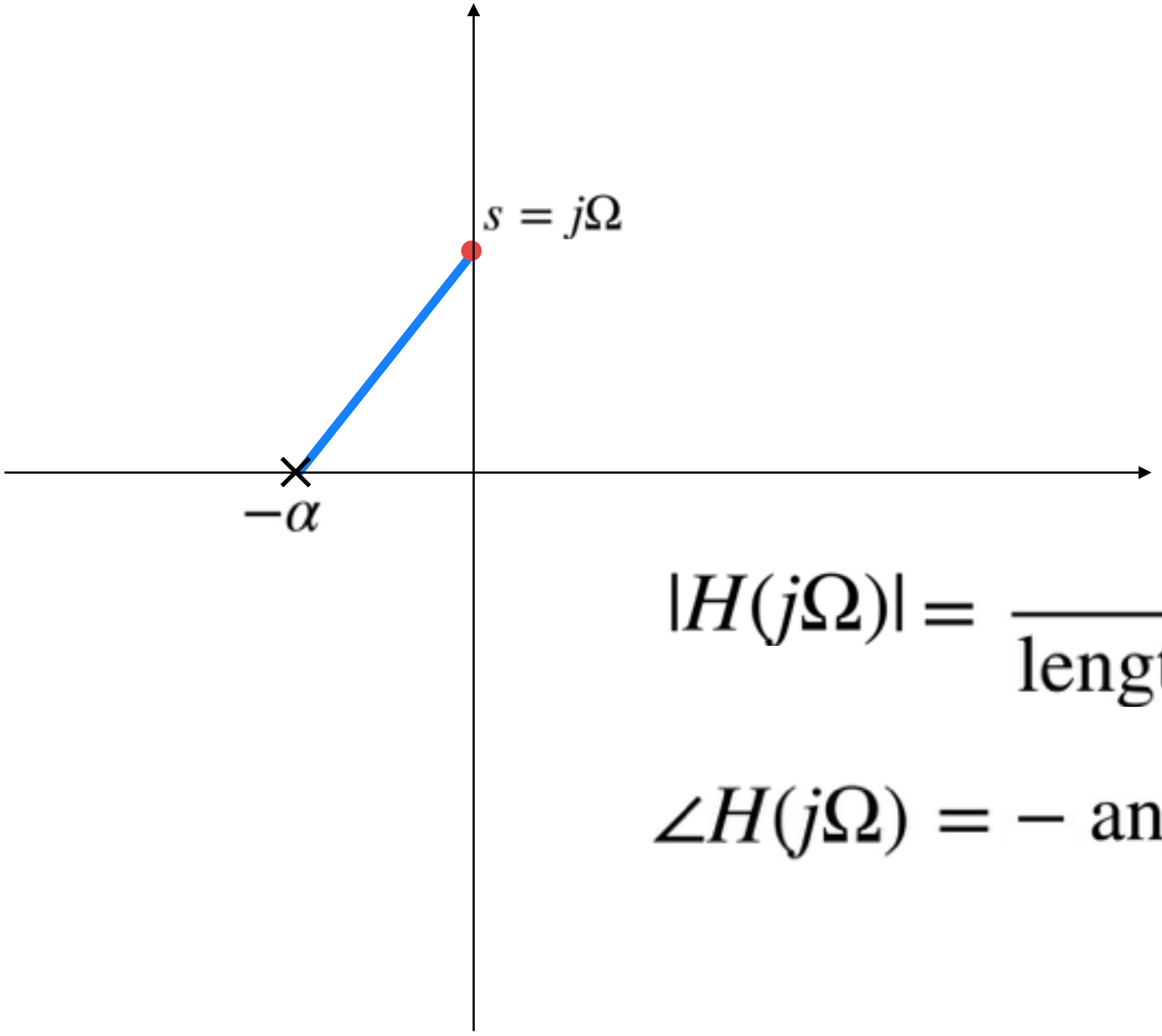
CTFT from poles and zeros



$$|H(j\Omega)| = \frac{1}{\text{length of blue}}$$ **LARGE**

$$\angle H(j\Omega) = -\text{ angle of blue}$$ **SMALL**

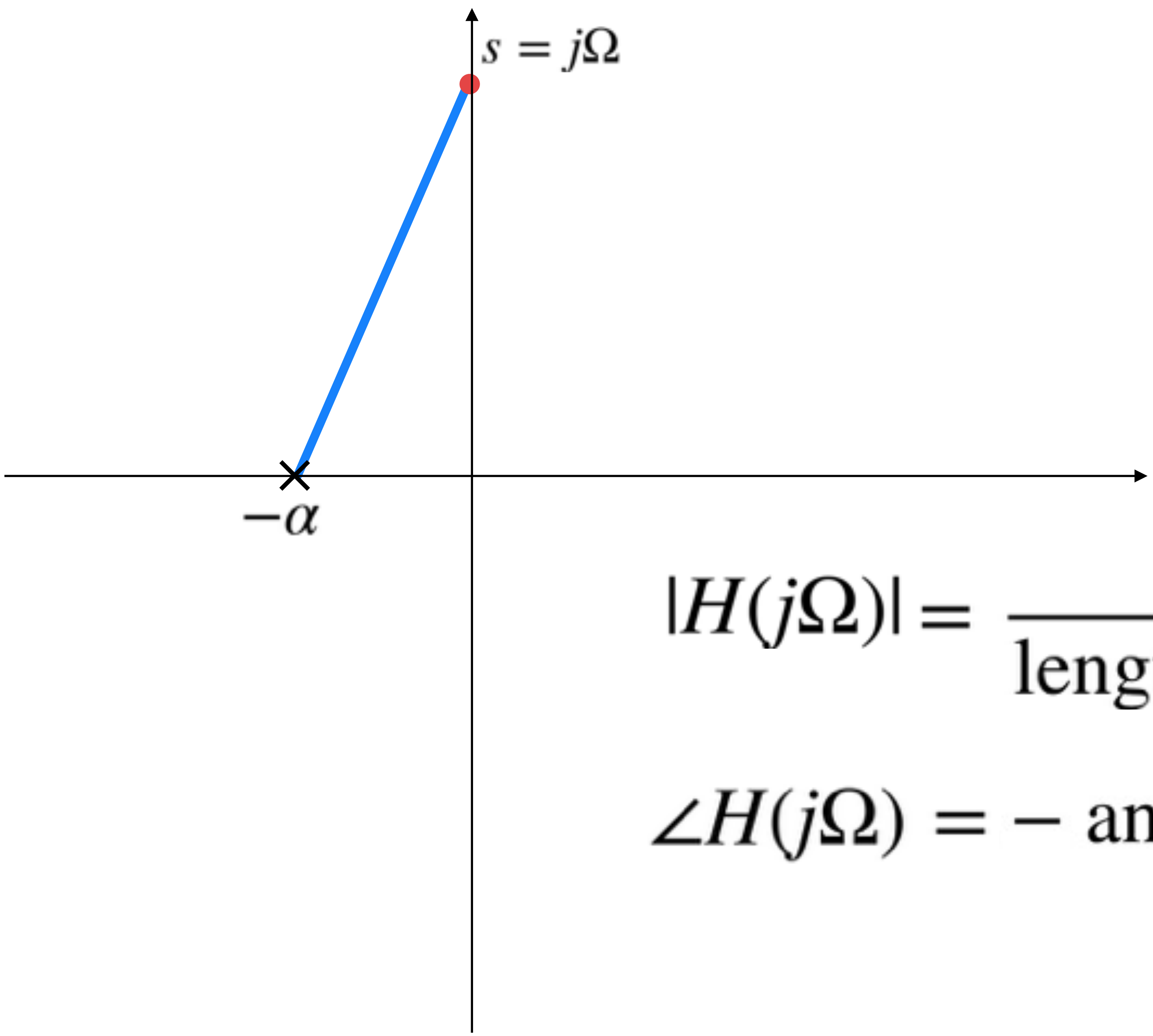
CTFT from poles and zeros



$$|H(j\Omega)| = \frac{1}{\text{length of blue}}$$ **MEDIUM**

$$\angle H(j\Omega) = -\text{ angle of blue}$$ **MEDIUM**

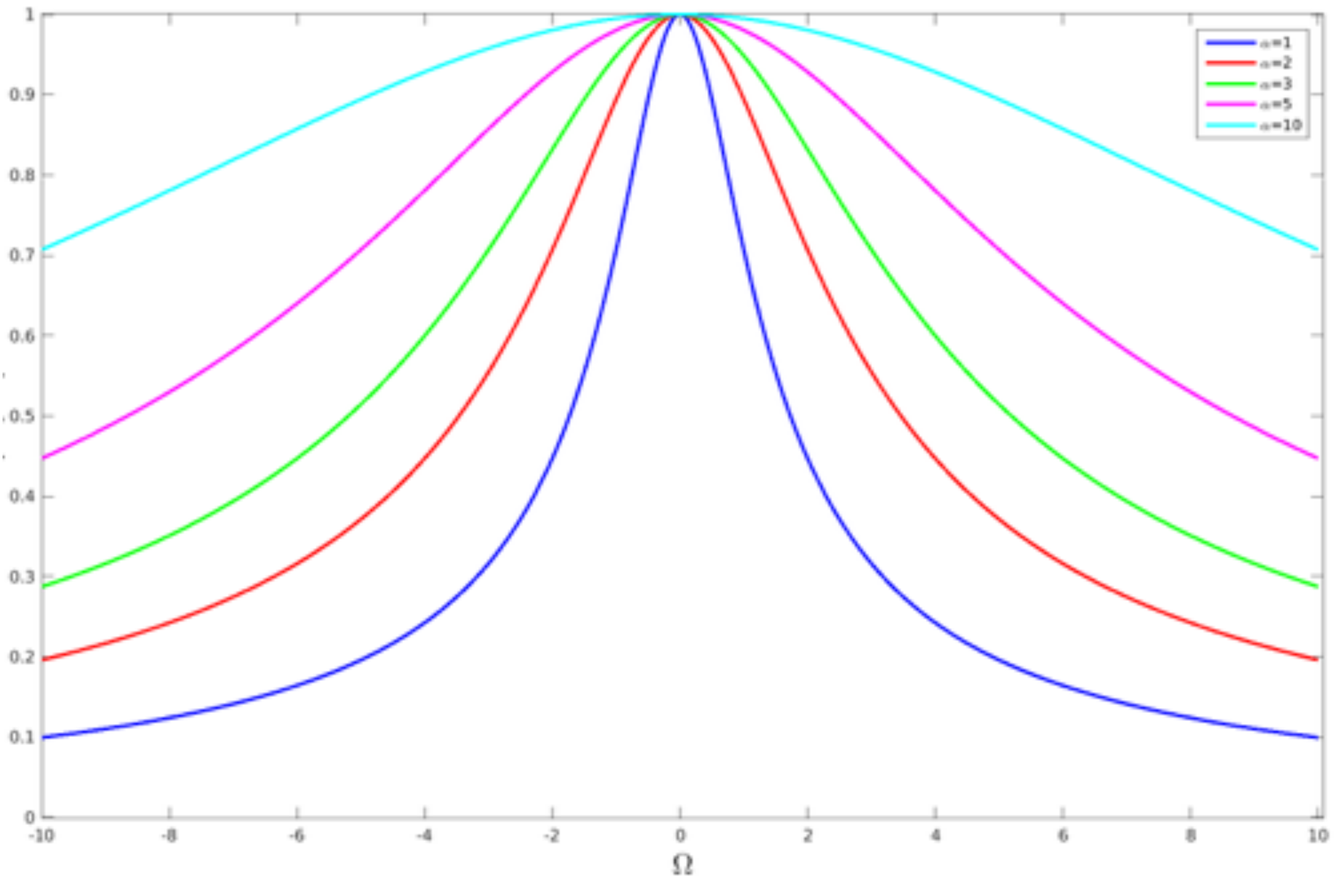
CTFT from poles and zeros



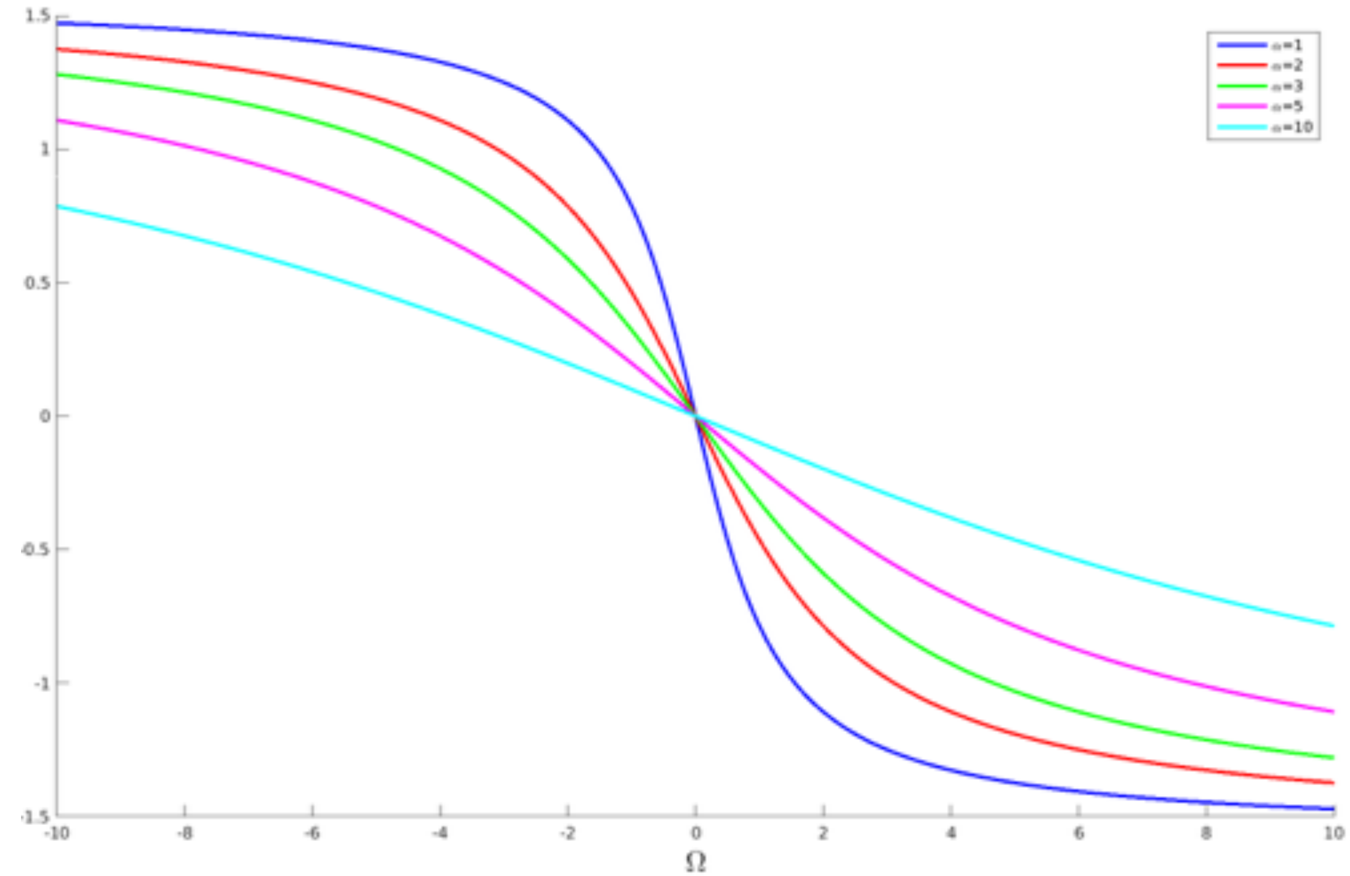
$$|H(j\Omega)| = \frac{1}{\text{length of blue}}$$ **SMALL**

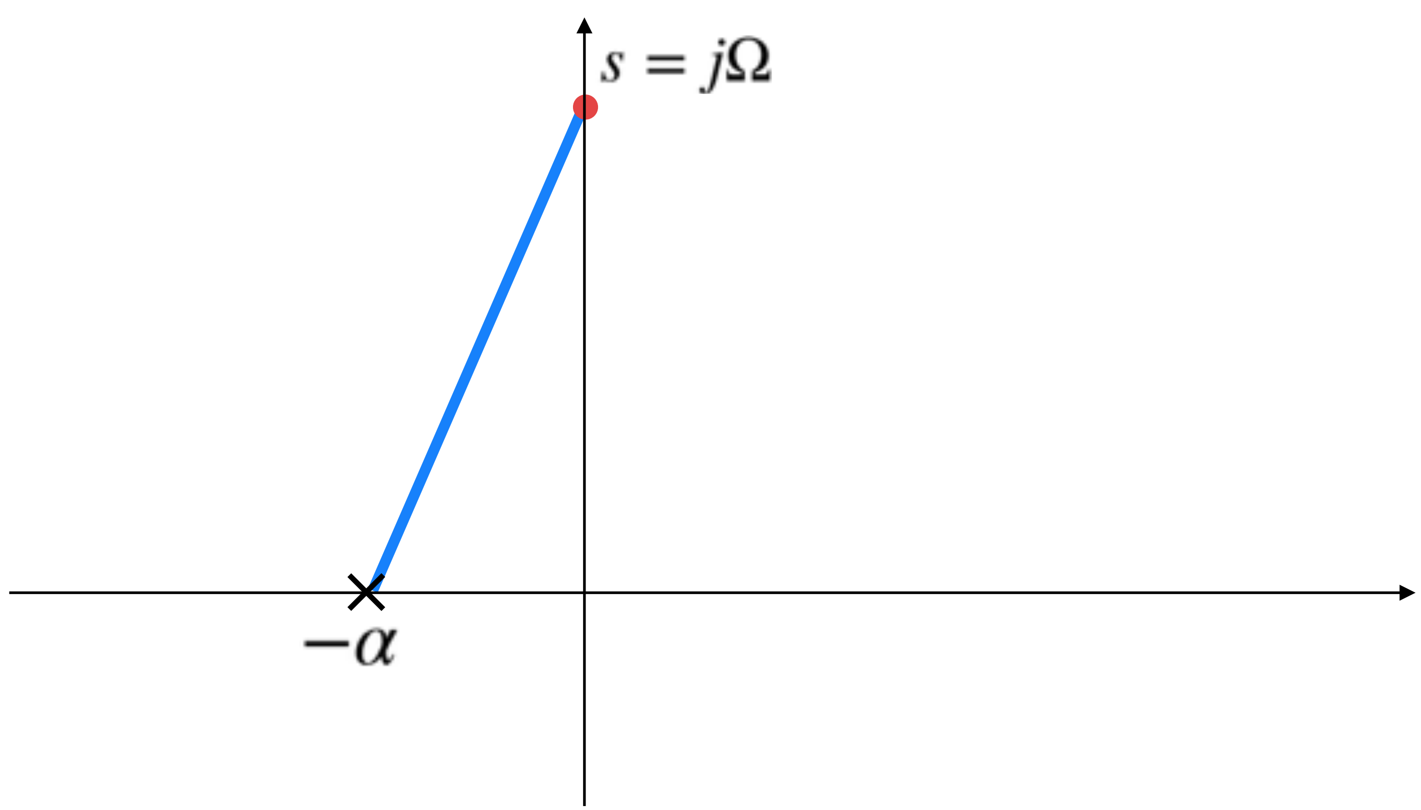
$$\angle H(j\Omega) = -\text{ angle of blue}$$ **LARGE**

$|H(j\Omega)|$



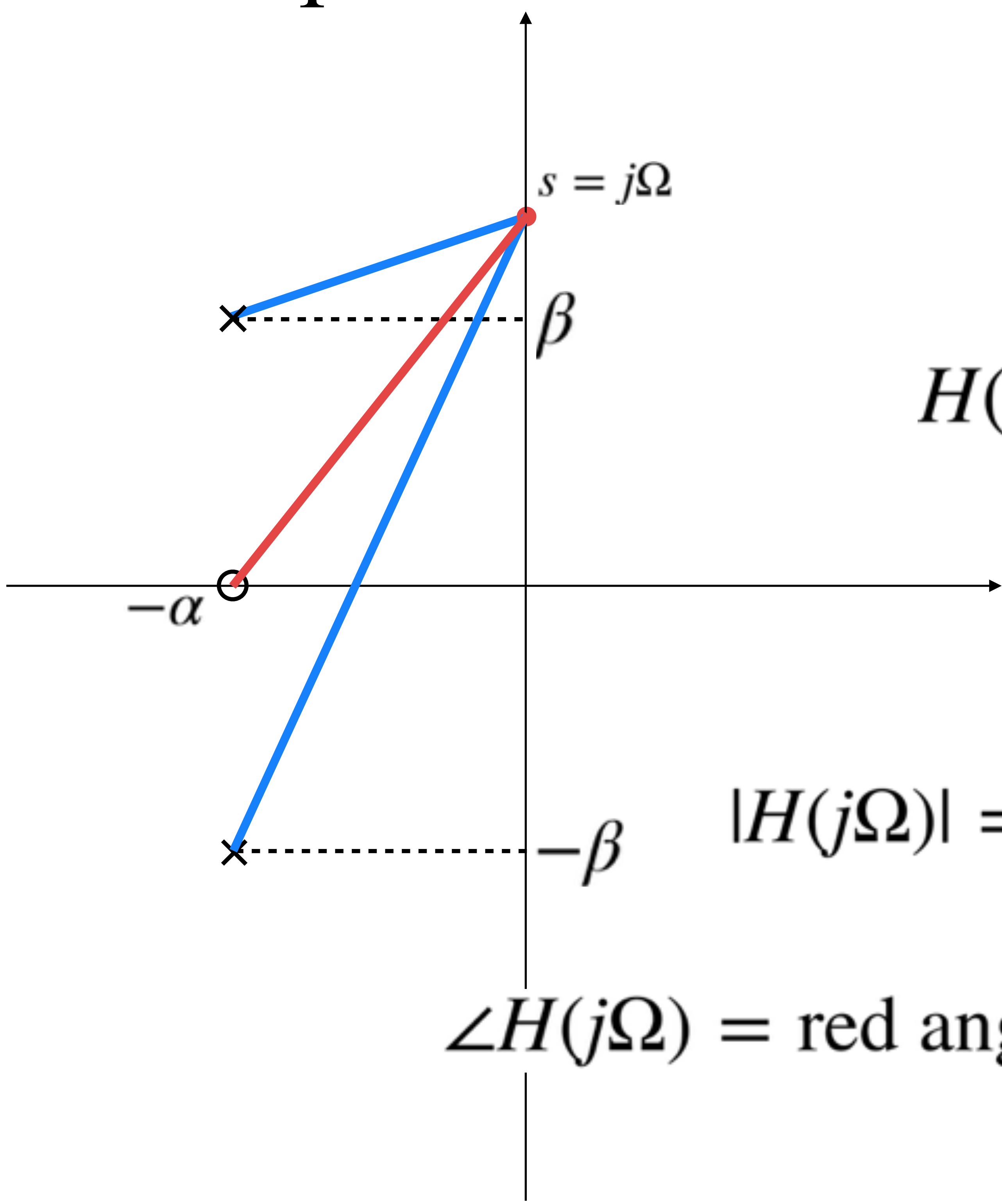
$\angle H(j\Omega)$





CTFT from poles and zeros

- <u>Example</u>: A 2nd-order filter: $h(t) = e^{-\alpha t}\cos(\beta t)u(t)$

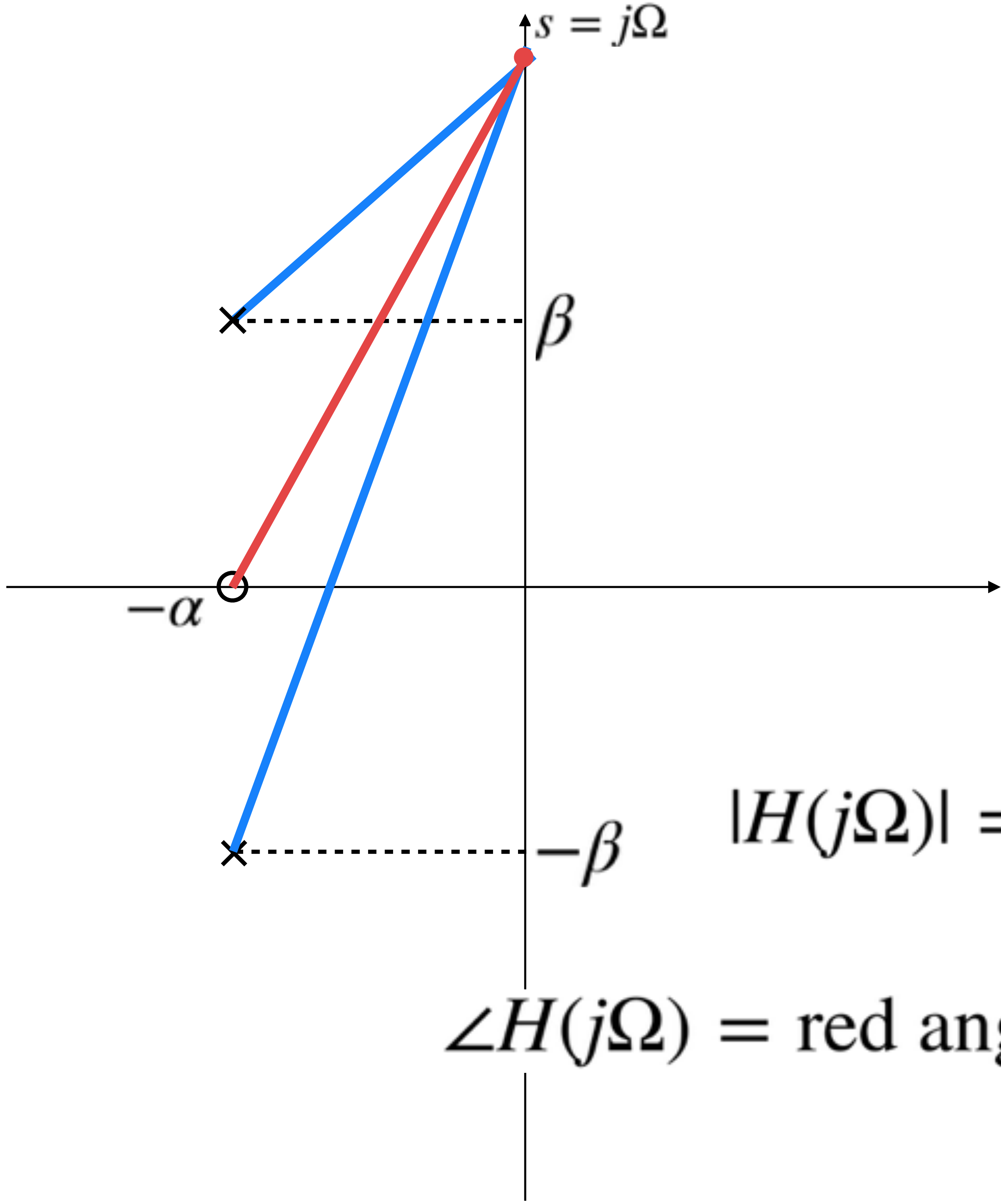


$$H(s) = \frac{s+\alpha}{(s+\alpha-j\beta)(s+\alpha+j\beta)}$$

$$|H(j\Omega)| = \frac{\text{red length}}{\text{blue length 1} \times \text{blue length 2}}$$

$$\angle H(j\Omega) = \text{red angle} - \text{blue angle 1} - \text{blue angle 2}$$

CTFT from poles and zeros

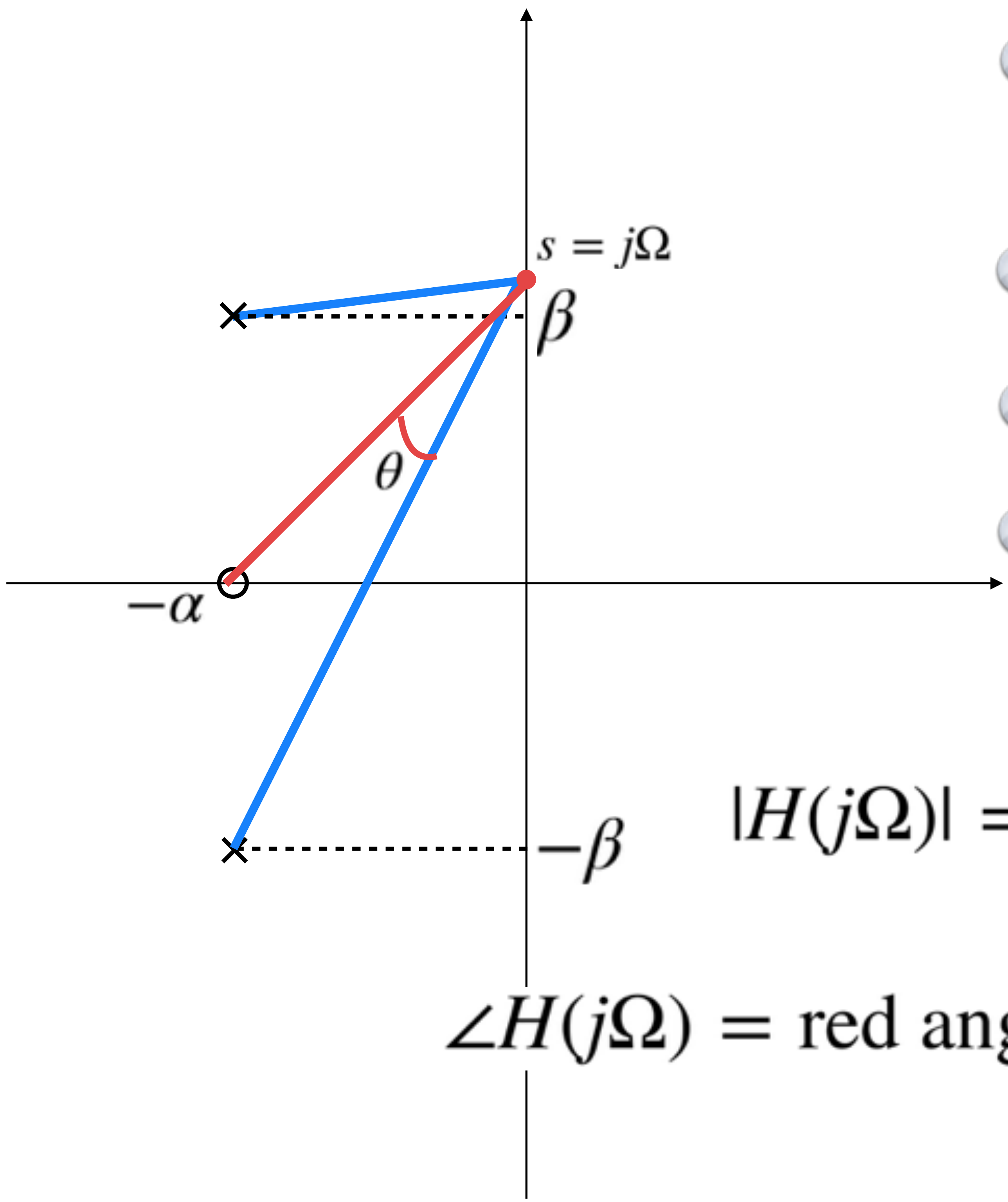


- As $\Omega \to \infty$,
  - All the lengths become equal and $\to \infty$
  - $|H(j\Omega)| \to 0$
  - All the angles become equal and $\to \pi/2$
  - $\angle H(j\Omega) \to -\pi/2$

$$|H(j\Omega)| = \frac{\text{red length}}{\text{blue length 1} \times \text{blue length 2}}$$

$$\angle H(j\Omega) = \text{red angle} - \text{blue angle 1} - \text{blue angle 2}$$

CTFT from poles and zeros



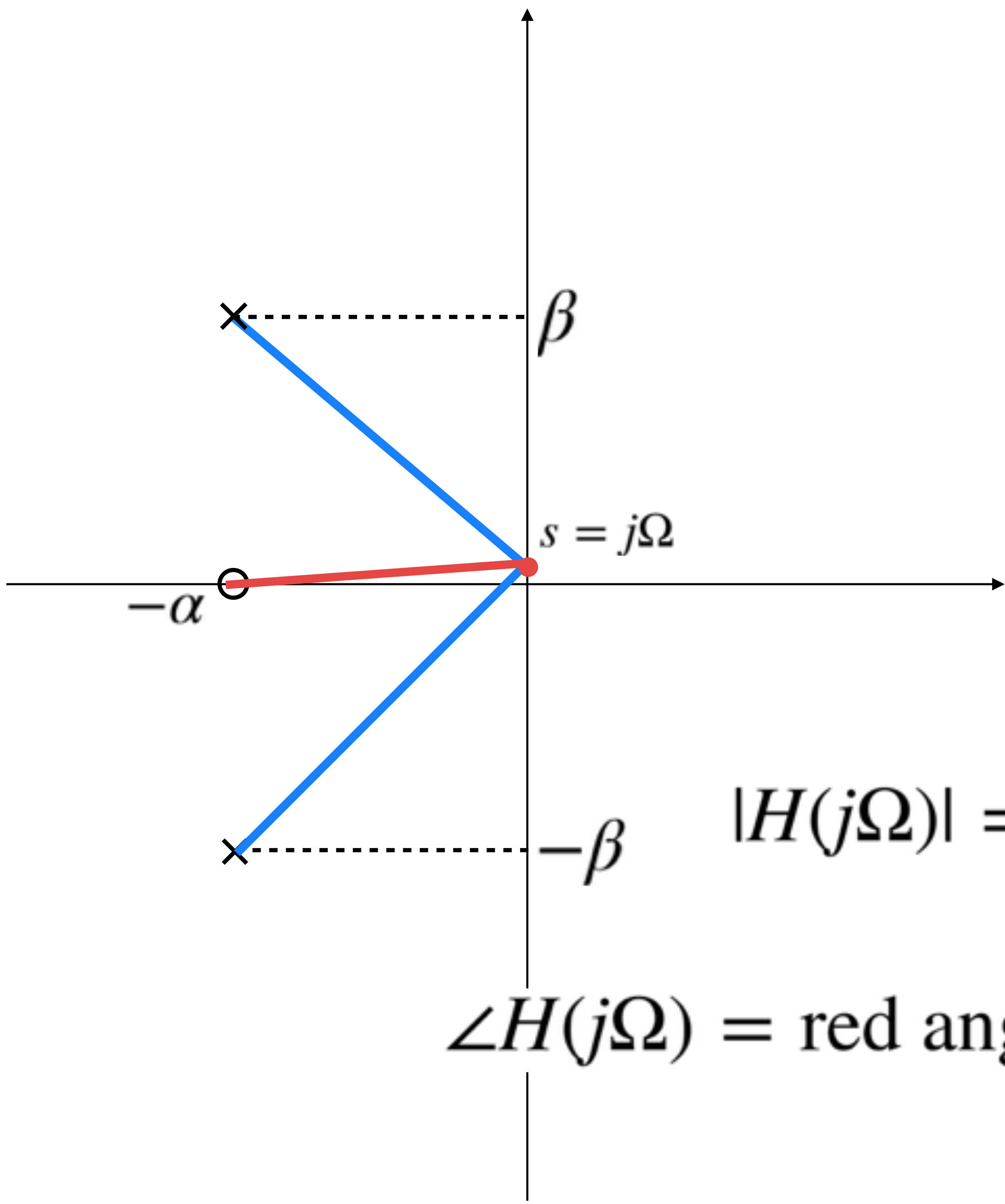
- As $\Omega \approx \beta$,
  - Blue length 1 becomes the smallest
  - $|H(j\Omega)| \approx 1/2\alpha$ for small α
  - $|H(j\Omega)| \approx 1/\alpha$ for large α
  - $\angle H(j\Omega) \approx -\theta$

$$|H(j\Omega)| = \frac{\text{red length}}{\text{blue length 1} \times \text{blue length 2}}$$

$$\angle H(j\Omega) = \text{red angle} - \text{blue angle 1} - \text{blue angle 2}$$

CTFT from poles and zeros

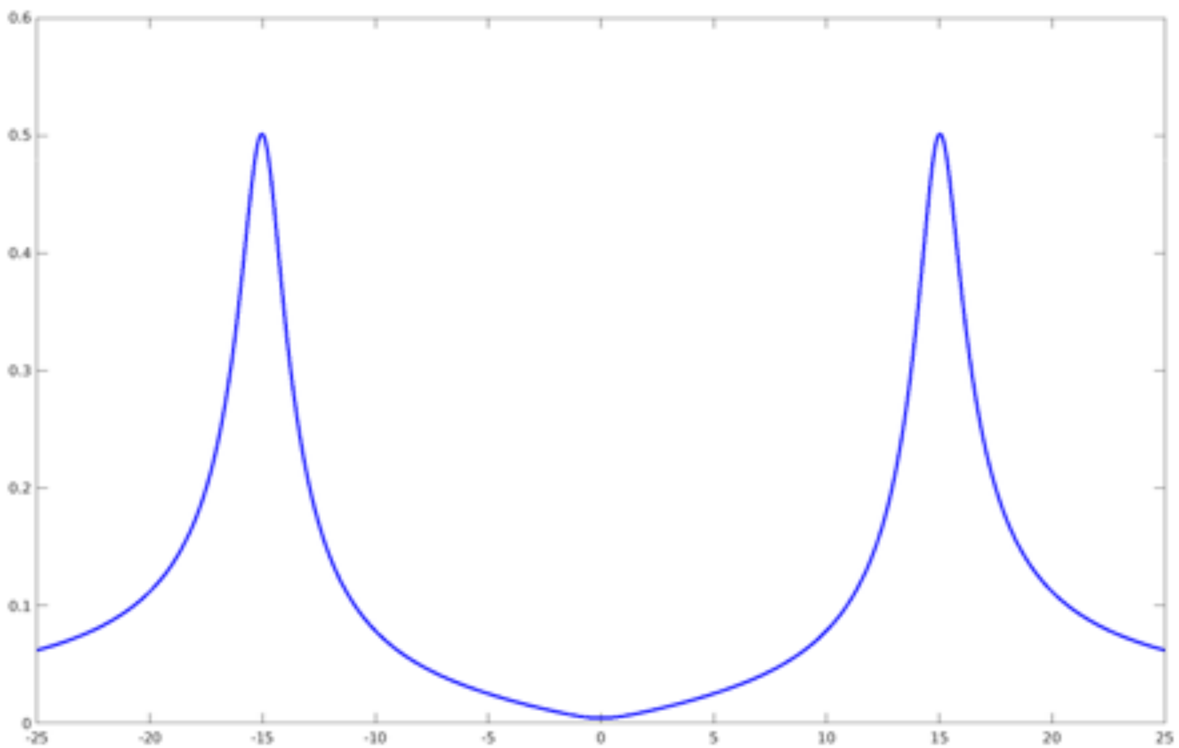
- As $\Omega \approx 0$,
  - $|H(j\Omega)| \approx \alpha/\beta^2$ for small α
  - $|H(j\Omega)| \approx 1/\alpha$ for large α
  - $\angle H(j\Omega) \approx 0$



$$|H(j\Omega)| = \frac{\text{red length}}{\text{blue length 1} \times \text{blue length 2}}$$

$$\angle H(j\Omega) = \text{red angle} - \text{blue angle 1} - \text{blue angle 2}$$

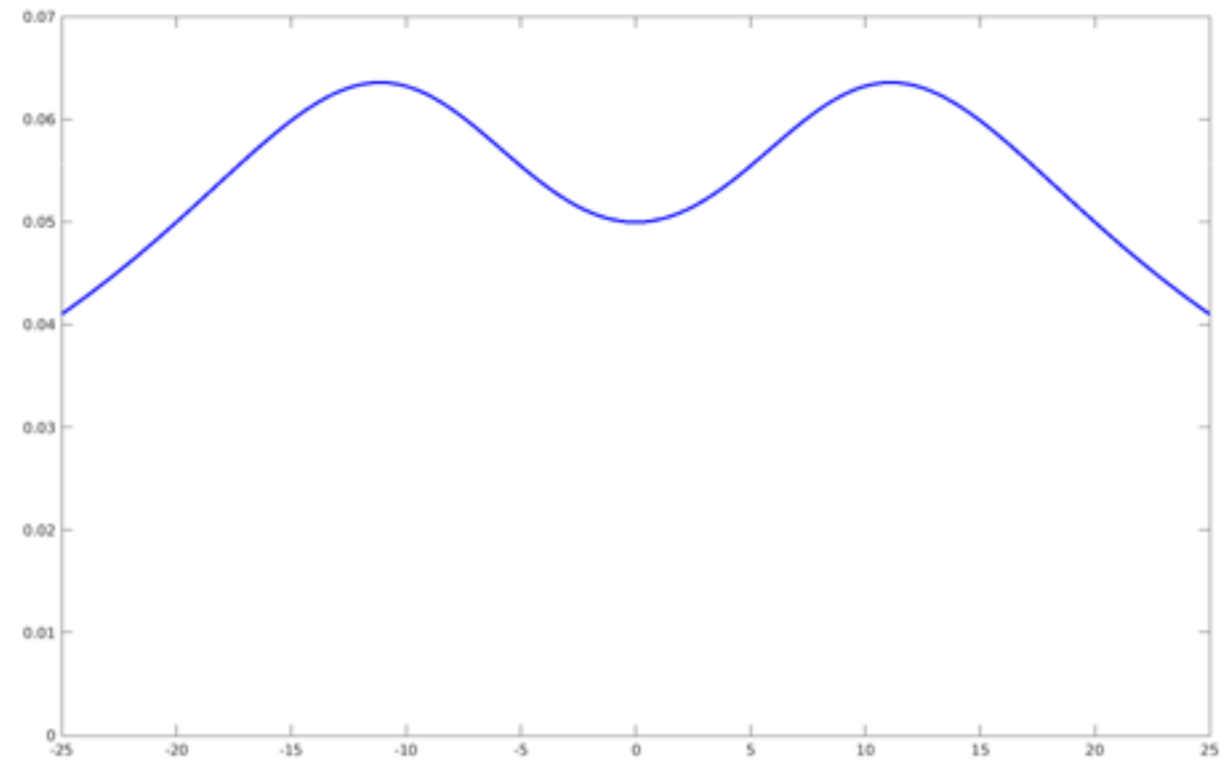
$|H(j\Omega)|$ with $\alpha = 1$ and $\beta = 15$



$$|H(j\beta)| \approx 1/2\alpha = 0.5$$

$$|H(j0)| \approx \alpha/\beta^2 \approx 0$$

$|H(j\Omega)|$ with $\alpha = 10$ and $\beta = 10$



$$|H(j\beta)| \approx 1/\alpha = 0.1$$

$$|H(j0)| \approx 1/\alpha = 0.1$$

System properties re-revisited

- We can tell from the Laplace transform and its ROC whether the LTI system is causal, stable, and even invertible.
  - **Causality**: We already saw that the ROC has to be of the form $Re\{s\} > \alpha$ and must include $s = \infty$.
  - **Stability**: Recall that if the system is stable,

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty$$

System properties re-revisited

- **Stability**: Recall that if the system is stable,

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

But we also have

$$\left|\int_{-\infty}^{\infty} h(t)e^{-j\Omega t} dt\right| \leq \int_{-\infty}^{\infty} \left|h(t)e^{-j\Omega t}\right| dt = \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Hence, stability implies the existence of CTFT

Stability $\Longrightarrow$ ROC includes the imaginary axis

It can be shown that $\Longleftarrow$ is also true.

System properties re-revisited

- Example: If $H(s) = \dfrac{s-1}{(s+2)(s-3)}$,

  there are three possible ROCs:

  - $Re\{s\} < -2$: Non-causal (in fact, anti-causal) and unstable.
  - $-2 < Re\{s\} < 3$: Non-causal and stable.
  - $Re\{s\} > 3$: Causal and unstable.
  - So we can never have both causality and stability with this set of poles.

System properties re-revisited

- So, what does it take to have both causality and stability?
  - All poles must be on the left side of the imaginary axis (including the hidden ones)

- Example: The following is causal and stable.

$$H(s) = \frac{s-1}{(s+2)(s+3)} \quad \text{with ROC: } Re\{s\} > -2$$

- Example: The following is stable but not causal.

$$H(s) = \frac{(s-1)s^2}{(s+1)(s+5)} \quad \text{with ROC: } Re\{s\} > -1$$

System properties re-revisited

- **Invertibility**: At first, it looks like every LTI system with non-empty ROC is invertible.
  - The inverse is simply $G(s) = \dfrac{1}{H(s)}$
  - However, for practicality, we need both the system and its inverse to be causal and stable.
  - This implies that not only all poles, but also all zeros must be on the left side of the imaginary axis (including the hidden ones).

System properties re-revisited

- Example:

$$H(s) = \frac{s-3}{(s+1)(s+5)} \quad \text{with ROC: } Re\{s\} > -1$$

- The inverse $G(s) = \dfrac{(s+1)(s+5)}{s-3}$ cannot be implemented as both causal and stable
- Example:

$$H(s) = \frac{s+3}{(s+1)(s+5)} \quad \text{with ROC: } Re\{s\} > -1$$

- There is still a hidden zero at $s = \infty$ so the inverse won't be causal.

System properties re-revisited

$$H(s) = \frac{s+3}{(s+1)(s+5)} \quad \text{with ROC: } Re\{s\} > -1$$

- There is still a hidden zero at $s = \infty$ so the inverse won't be causal.

- Example:

$$H(s) = \frac{(s+3)(s+2)}{(s+1)(s+5)} \quad \text{with ROC: } Re\{s\} > -1$$

- This one has a causal and stable inverse.

Differential equations again!

- We can actually solve a differential equation using the Laplace transform.

$$\sum_{k=0}^{K} a_k \frac{d^k y}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x}{dt^m}$$

becomes

$$\sum_{k=0}^{K} a_k s^k Y(s) = \sum_{m=0}^{M} b_m s^m X(s)$$

or equivalently,

$H(s)$

$$Y(s) = \boxed{\frac{\sum_{m=0}^{M} b_m s^m}{\sum_{k=0}^{K} a_k s^k}} X(s)$$

Differential equations again!

- What about the ROC?
- Choose the one that will give you a causal $H(s)$
- <u>Example</u>: $y(t) - \frac{dy(t)}{dt} - 2\,\frac{d^2y(t)}{dt^2} = x(t)$
- Recall that we had found

$$h(t) = \frac{1}{3}\,\left[e^{-t} - e^{0.5t}\right]u(t)$$

- <u>Solution:</u> Start with going to the s-domain:

$$Y(s) - sY(s) - 2s^2Y(s) = X(s)$$

- Example: $y(t) - \frac{dy(t)}{dt} - 2\,\frac{d^2y(t)}{dt^2} = x(t)$
- Recall that we had found

$$h(t) = \frac{1}{3}\,\left[e^{-t} - e^{0.5t}\right] u(t)$$

- Solution: Start with going to the s-domain:

$$Y(s) - sY(s) - 2s^2 Y(s) = X(s)$$

or equivalently,

$$Y(s) = \frac{1}{1 - s - 2s^2} X(s) = \underbrace{\boxed{\frac{1}{-2(s+1)(s-0.5)}}}_{H(s)} X(s)$$

$$H(s) = \frac{A}{s+1} + \frac{B}{s-0.5} = \frac{A(s-0.5) + B(s+1)}{(s+1)(s-0.5)}$$

- Example: $y(t) - \dfrac{dy(t)}{dt} - 2\,\dfrac{d^2y(t)}{dt^2} = x(t)$
- Recall that we had found

$$h(t) = \frac{1}{3}\,\left[e^{-t} - e^{0.5t}\right] u(t)$$

$$Y(s) = \frac{1}{1 - s - 2s^2}\,X(s) = \frac{1}{-2(s+1)(s-0.5)}\,X(s)$$

$$H(s) = \frac{A}{s+1} + \frac{B}{s-0.5} = \frac{A(s-0.5) + B(s+1)}{(s+1)(s-0.5)}$$

- Substitute $s = 0.5$ to get $1.5B = -0.5 \Rightarrow B = -\dfrac{1}{3}$
- Substitute $s =$ -1 to get $-1.5A = -0.5 \Rightarrow A = \dfrac{1}{3}$

- <u>Example</u>: $y(t) - \frac{dy(t)}{dt} - 2\,\frac{d^2y(t)}{dt^2} = x(t)$
- Recall that we had found

$$h(t) = \frac{1}{3}\,\left[e^{-t} - e^{0.5t}\right]u(t)$$

$$Y(s) = \frac{1}{1 - s - 2s^2}X(s) = \frac{1}{-2(s+1)(s-0.5)}X(s)$$

$$H(s) = \frac{A}{s+1} + \frac{B}{s-0.5} = \frac{A(s-0.5) + B(s+1)}{(s+1)(s-0.5)}$$

$$= \frac{1/3}{s+1} - \frac{1/3}{s-0.5}$$

- Choose the causal ROC: $Re\{s\} > 0.5$

Differential equations again!

- Example: Find $h(t)$ if

$$25y(t) - 6\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = x(t) - \frac{dx(t)}{dt}$$

- Recall that we had found

$$h(t) = 2re^{3t}\cos(4t + \theta)u(t)$$

with $re^{j\theta} = -0.5 + 0.25j$

- Solution:

$$25Y(s) - 6sY(s) + s^2Y(s) = X(s) - sX(s)$$

or $$Y(s) = \underbrace{\boxed{\frac{1-s}{25-6s+s^2}}}_{H(s)} X(s)$$

- Example: Find $h(t)$ if

$$25y(t) - 6\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = x(t) - \frac{dx(t)}{dt}$$

- Recall that we had found

$$h(t) = 2re^{3t}\cos(4t + \theta)u(t)$$

with $re^{j\theta} = -0.5 + 0.25j$

- Solution:

$$H(s) = \frac{1 - s}{25 - 6s + s^2} = \frac{A}{s - (3 + 4j)} + \frac{B}{s - (3 - 4j)}$$

$$= \frac{A[s - (3 - 4j)] + B[s - (3 + 4j)]}{[s - (3 + 4j)][s - (3 - 4j)]}$$

- Recall that we had found

$$h(t) = 2re^{3t}\cos(4t + \theta)u(t)$$

with $re^{j\theta} = -0.5 + 0.25j$

- Solution:

$$H(s) = \frac{1-s}{25 - 6s + s^2} = \frac{A}{s - (3+4j)} + \frac{B}{s - (3-4j)}$$

$$= \frac{A[s - (3-4j)] + B[s - (3+4j)]}{[s - (3+4j)][s - (3-4j)]}$$

- Substitute $s = 3 - 4j$ to get $(-8j)B = -2 + 4j$

$$\Rightarrow B = -0.5 - 0.25j$$

- Substitute $s = 3 + 4j$ to get $(8j)A = -2 - 4j$

$$\Rightarrow A = -0.5 + 0.25j$$

- Recall that we had found

$$h(t) = 2re^{3t}\cos(4t + \theta)u(t)$$

with $re^{j\theta} = -0.5 + 0.25j$

- Solution:

$$H(s) = \frac{1 - s}{25 - 6s + s^2} = \frac{A}{s - (3 + 4j)} + \frac{B}{s - (3 - 4j)}$$

$$\Rightarrow A = -0.5 + 0.25j$$

$$\Rightarrow B = -0.5 - 0.25j$$

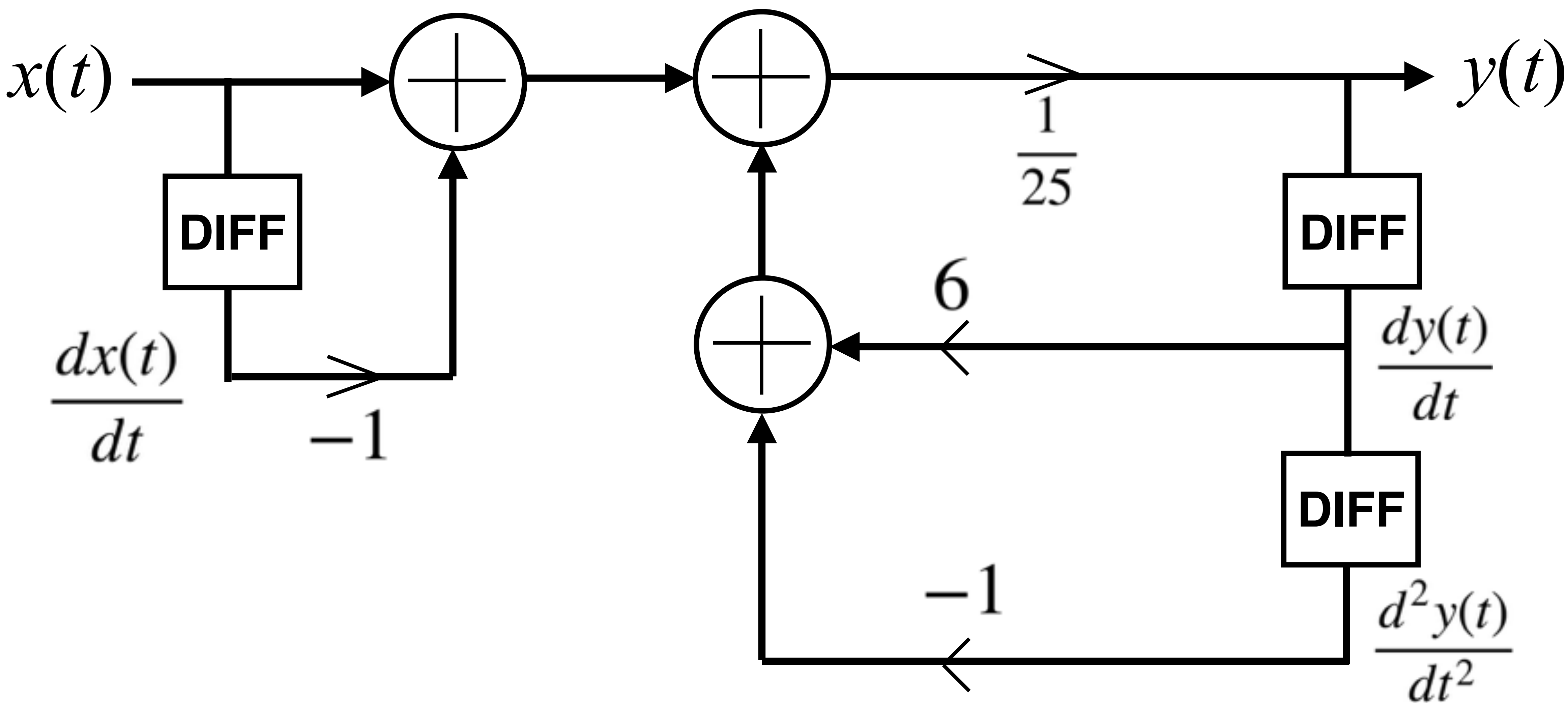
- Choose the causal ROC: $Re\{s\} > 3$

$$h(t) = re^{j\theta}e^{3t}e^{4jt}u(t) + re^{-j\theta}e^{3t}e^{-4jt}u(t)$$

Block diagrams

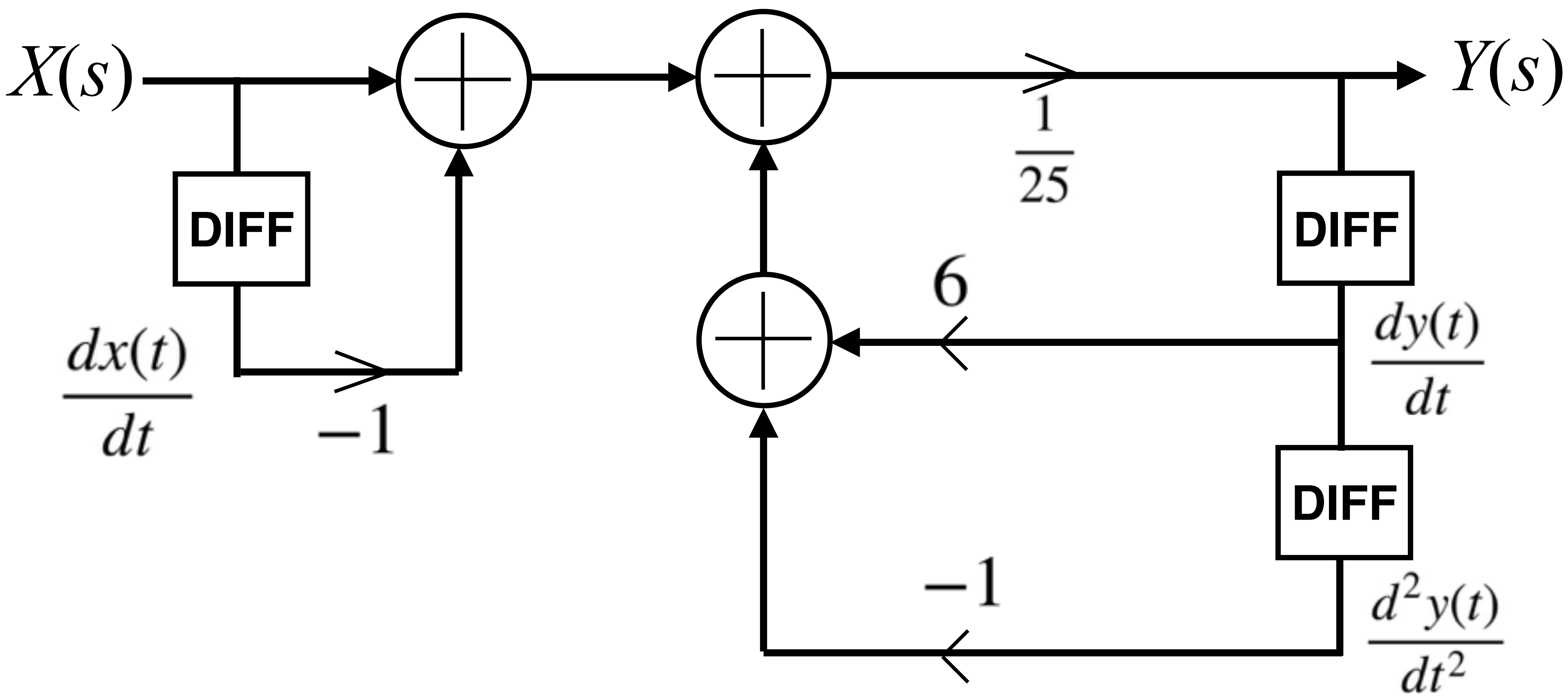
- Recall that we can convert differential equations to block diagrams easily:

$$25y(t) - 6\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = x(t) - \frac{dx(t)}{dt}$$



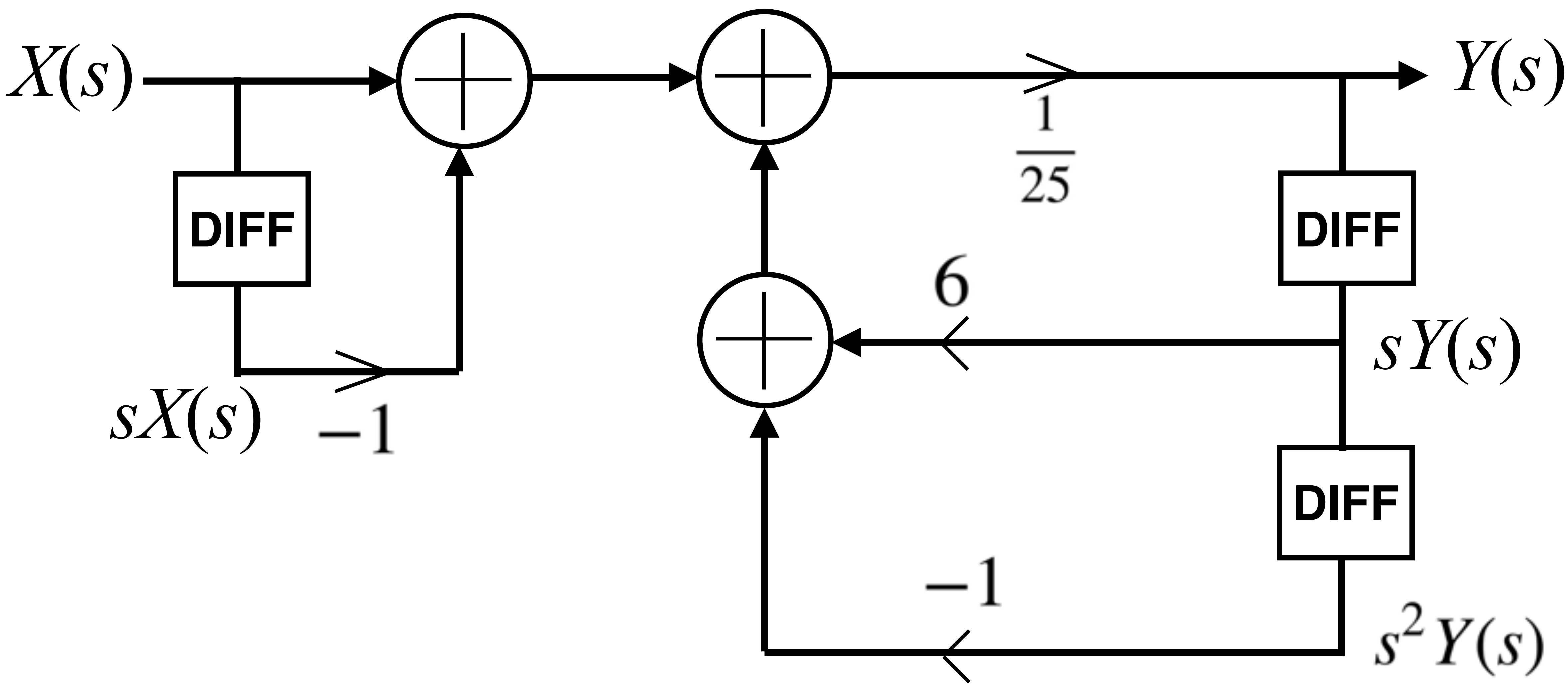
Block diagrams

- Re-think this in the s-domain:



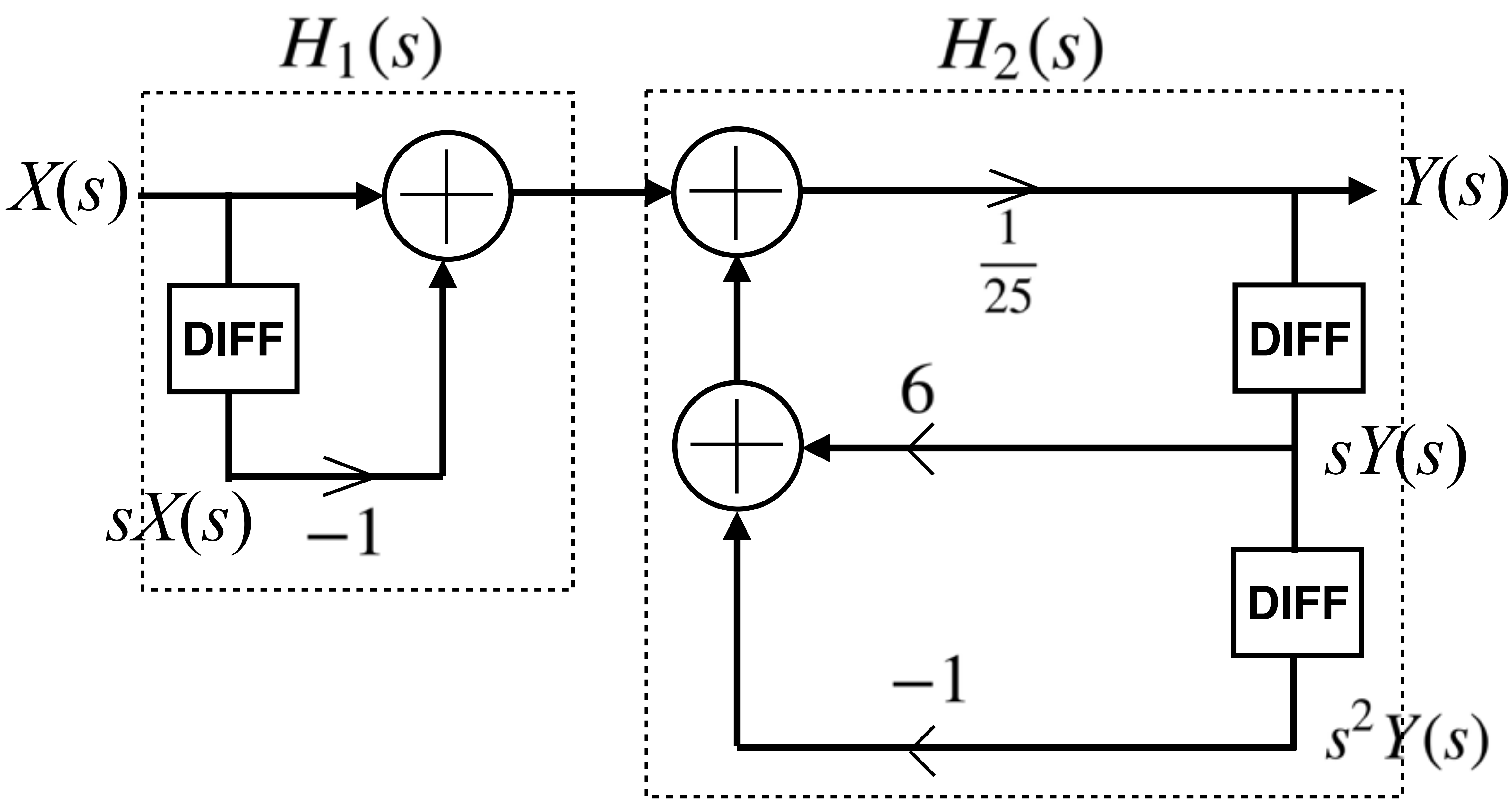
Block diagrams

- Re-think this in the s-domain:



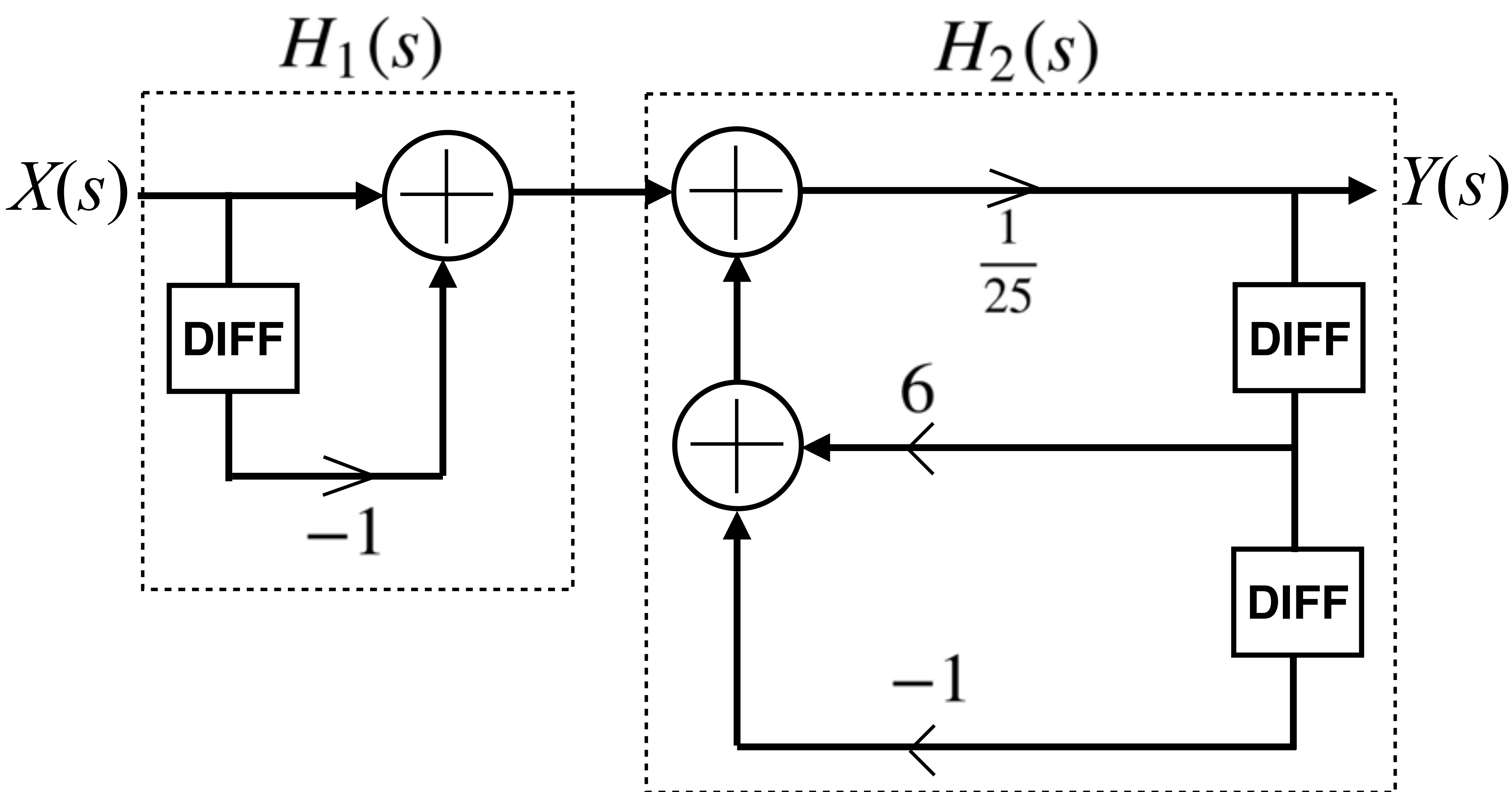
Block diagrams

- You can also think of this as a cascade of two LTI systems:



Block diagrams

- ... which means you can actually swap these two sub-systems:



Block diagrams

- ... which means you can actually swap these two sub-systems: