EE 110B Signals and Systems

Fourier Analysis of Discrete-Time Signals

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Decomposition of periodic signals

- Recall that we can decompose any signal into shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Also remember how this was useful in understanding the response of an LTI system to any input:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Decomposition of periodic signals

- Is there any other decomposition that may be similarly useful?
- For periodic signals, we will find exactly that.
- <u>Claim</u>: For signals with period N, we can always write

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi k}{N}n}$$

- Setting $\omega_0 = 2\pi/N$, this is the same as

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

Decomposition of periodic signals

- Before proving this, let's see why it is even useful.
- For any LTI system with impulse response $h[n]$, if the input is a complex exponential signal $x[n] = e^{j\omega n}$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = e^{j\omega n} H(\omega)$$

Decomposition of periodic signals

- Pictorially,

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow e^{j\omega n} H(\omega)$$

- Therefore, if a periodic signal can indeed be decomposed as mentioned above,

$$\sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow \sum_{k=0}^{N-1} a_k H(k\omega_0) e^{jk\omega_0 n}$$

Back to the formula

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

- Multiply both sides by $e^{-jl\omega_0 n}$ for some integer l, and sum over n in one period:

$$\sum_{n=0}^{N-1} x[n] e^{-jl\omega_0 n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} e^{-jl\omega_0 n}$$

$$= \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j(k-l)\omega_0 n}$$

$$= \frac{e^{j(k-l)\omega_0 N} - 1}{e^{j(k-l)\omega_0} - 1} \quad \text{if } k \neq l$$

$$= N \quad \text{if } k = l$$

$$\sum_{n=0}^{N-1} x[n]e^{-jl\omega_0 n} = \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j(k-l)\omega_0 n} \quad \begin{cases} = \dfrac{e^{j(k-l)\omega_0 N} - 1}{e^{j(k-l)\omega_0} - 1} & \text{if } k \neq l \\ = N & \text{if } k = l \end{cases}$$

- On the other hand,

$$\frac{e^{j(k-l)\omega_0 N} - 1}{e^{j(k-l)\omega_0} - 1} = \frac{e^{j(k-l)2\pi} - 1}{e^{j(k-l)\omega_0} - 1} = \frac{1 - 1}{e^{j(k-l)\omega_0} - 1} = 0$$

- This simplifies the outer summation: For fixed l, a_k is multiplied by 0 for all k except at $k = l$ where it is multiplied by N.

- Therefore, $\displaystyle\sum_{n=0}^{N-1} x[n]e^{-jl\omega_0 n} = Na_l$

Back to the formula

- To recap, if

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad (*)$$

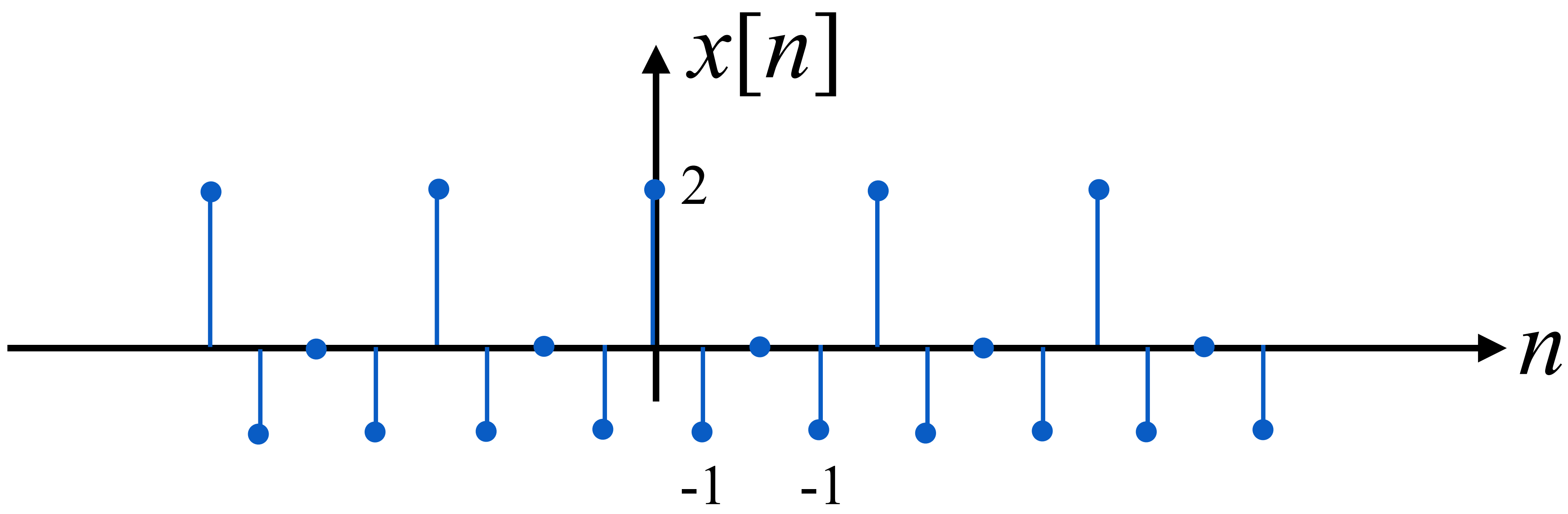
then

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n} \qquad (**)$$

- Conversely, for any $x[n]$, a_k calculated as in (**) satisfies (*).
- a_k are called the discrete-time Fourier series (DTFS) coefficients.

Examples

- Find the DTFS coefficients for the signal



- Solution:

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{kn}$$

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{kn}$$

$$= \frac{1}{4}\left[x[0](-j)^0 + x[1](-j)^k + x[2](-j)^{2k} + x[3](-j)^{3k}\right]$$

$$= \frac{1}{4}\left[2(-j)^0 + (-1)(-j)^k + 0(-j)^{2k} + (-1)(-j)^{3k}\right]$$

$$= \frac{1}{4}\left[2 - (-j)^k - (-j)^{3k}\right] = \frac{1}{4}\left[2 - (-j)^k - j^k\right]$$

$$a_0 = \frac{1}{4}\left[2 - 1 - 1\right] = 0 \qquad a_2 = \frac{1}{4}\left[2 - (-1) - (-1)\right] = 1$$

$$a_1 = \frac{1}{4}\left[2 - (-j) - j\right] = 0.5 \qquad a_3 = \frac{1}{4}\left[2 - j - (-j)\right] = 0.5$$

$$a_0 = \frac{1}{4}\left[2 - 1 - 1\right] = 0 \qquad a_2 = \frac{1}{4}\left[2 - (-1) - (-1)\right] = 1$$

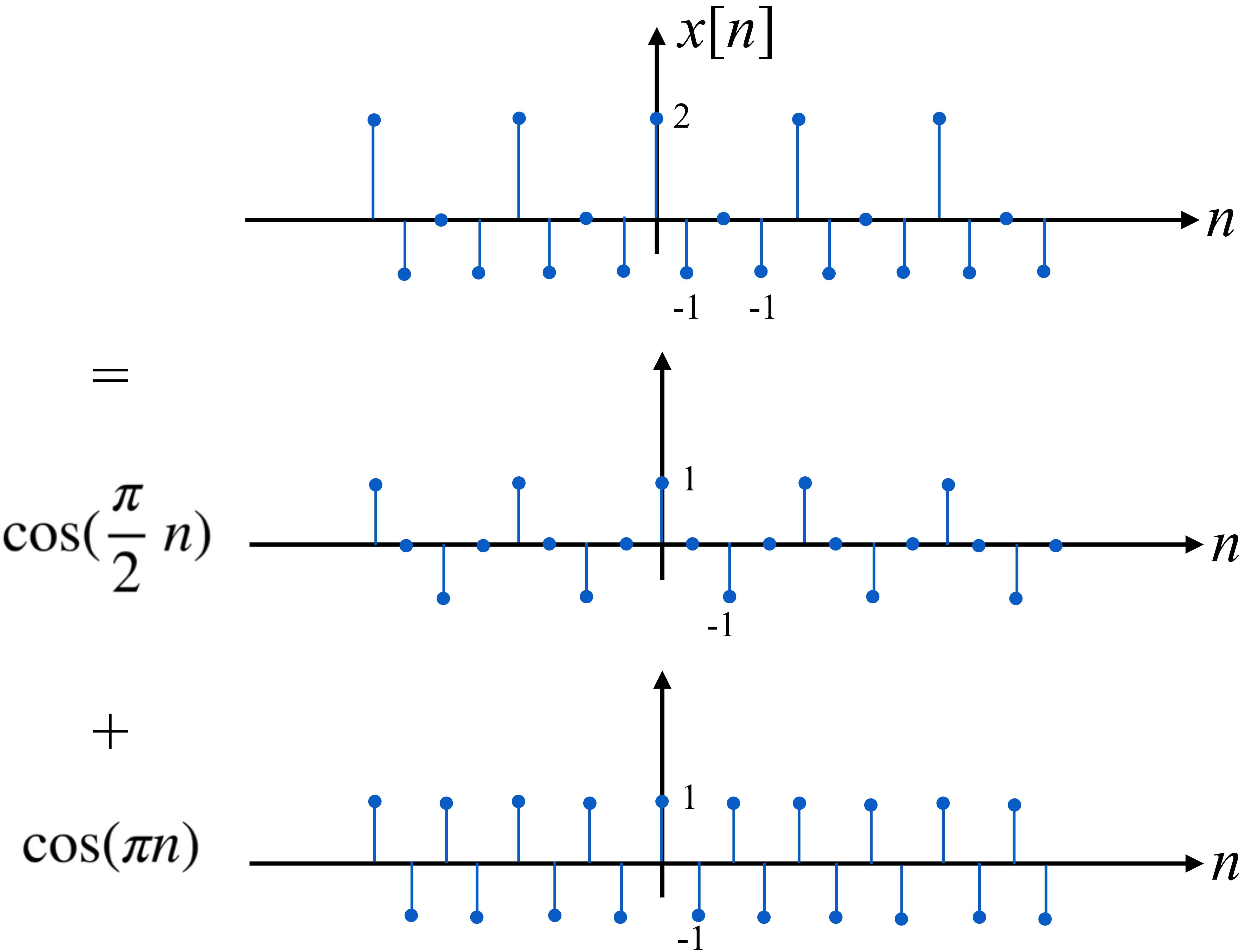
$$a_1 = \frac{1}{4}\left[2 - (-j) - j\right] = 0.5 \qquad a_3 = \frac{1}{4}\left[2 - j - (-j)\right] = 0.5$$

$$x[n] = \sum_{k=0}^{3} a_k e^{jk\frac{\pi}{2}n} = \sum_{k=0}^{3} a_k j^{kn}$$

$$= 0.5j^n + j^{2n} + 0.5j^{3n}$$

$$= 0.5j^n + \underbrace{(-1)^n}_{\cos(\pi n)} + 0.5(-j)^n$$

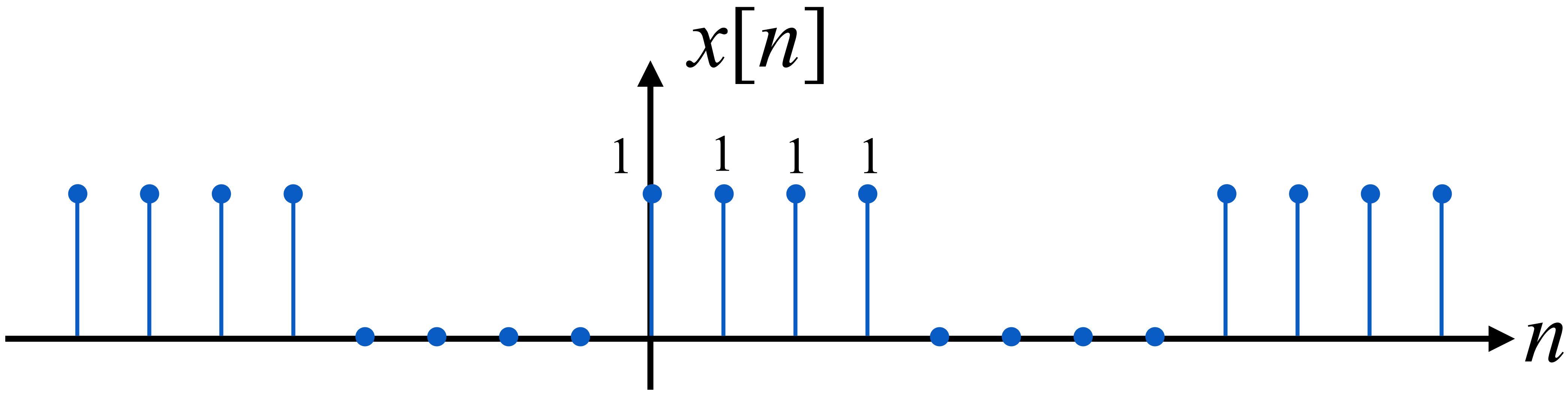
$$\cos(\frac{\pi}{2}n)$$



$x[n]$
2
-1
-1
n
$=$
$\cos(\frac{\pi}{2} n)$
1
-1
n
$+$
$\cos(\pi n)$
1
-1
n

Examples

- Find the DTFS coefficients for the signal



- <u>Solution</u>:

$$a_k = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-jk\frac{\pi}{4}n} = \frac{1}{8}\sum_{n=0}^{3} e^{-jk\frac{\pi}{4}n}$$

$$a_0 = 0.5 \qquad a_k = \frac{1}{8}\cdot\frac{e^{-jk\frac{\pi}{4}4}-1}{e^{-jk\frac{\pi}{4}}-1} \quad \text{for} \quad k \neq 0$$

$$a_0 = 0.5 \qquad a_k = \frac{1}{8} \cdot \frac{e^{-jk\frac{\pi}{4}4} - 1}{e^{-jk\frac{\pi}{4}} - 1} \quad \text{for} \quad k \neq 0$$

$$= \frac{1}{8} \cdot \frac{e^{-jk\frac{\pi}{2}}\left[e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right]}{e^{-jk\frac{\pi}{8}}\left[e^{-jk\frac{\pi}{8}} - e^{jk\frac{\pi}{8}}\right]}$$

$$= \frac{1}{8} \cdot \frac{\sin(k\,\frac{\pi}{2})}{\sin(k\,\frac{\pi}{8})} \cdot e^{-jk\frac{3\pi}{8}}$$

- <u>Observation</u>:

$$a_{8-k} = \frac{1}{8} \cdot \frac{\sin(4\pi - k\,\frac{\pi}{2})}{\sin(\pi - k\,\frac{\pi}{8})} \cdot e^{jk\frac{3\pi}{8}} e^{-j3\pi}$$

$$= \frac{1}{8} \cdot \frac{-\sin(k\,\frac{\pi}{2})}{\sin(k\,\frac{\pi}{8})} \cdot e^{jk\frac{3\pi}{8}}(-1) \quad = a_k^*$$

- Why is $a_{8-k} = a_k^*$ even useful?
- Because pairing up those terms in the reconstruction, we obtain

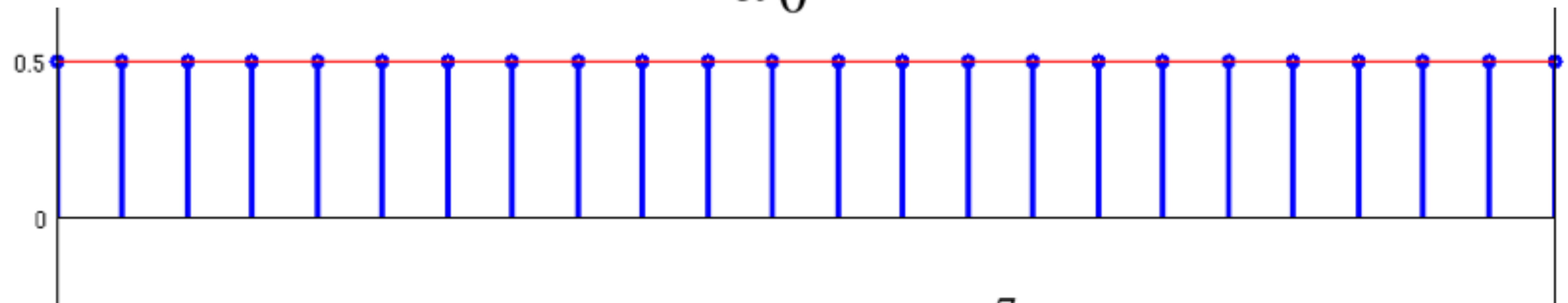
$$a_k e^{jk\frac{\pi}{4}n} + a_{8-k} e^{j(8-k)\frac{\pi}{4}n}$$

$$= a_k e^{jk\frac{\pi}{4}n} + a_k^* e^{-jk\frac{\pi}{4}n}$$

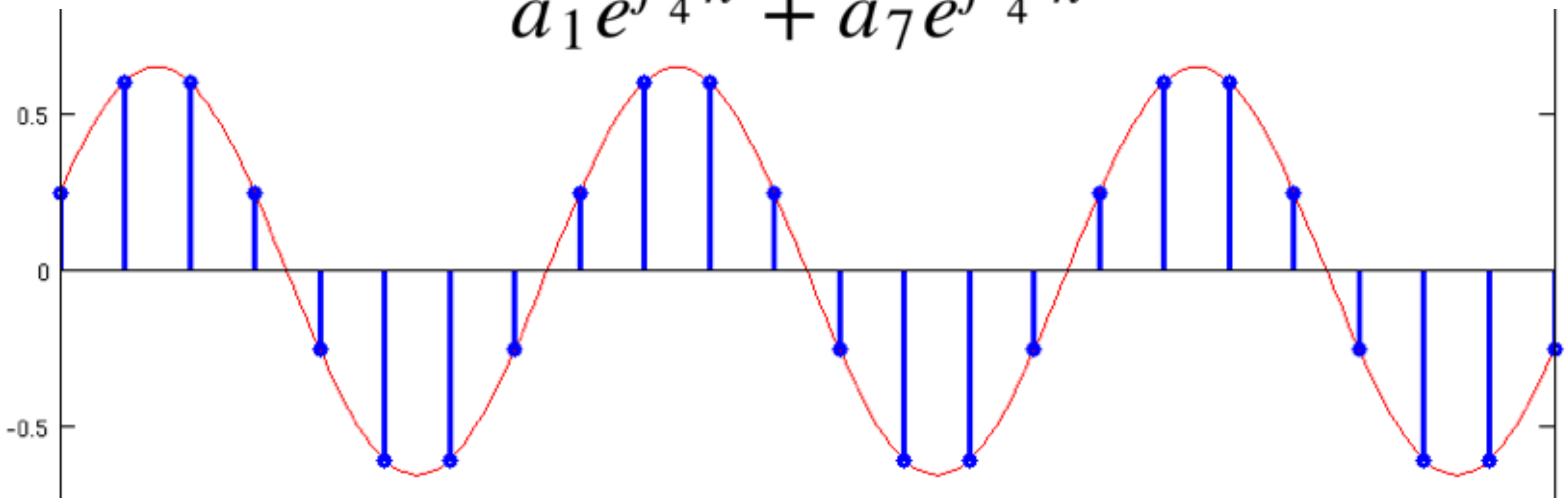
$$= |a_k| e^{j\angle a_k} e^{jk\frac{\pi}{4}n} + |a_k| e^{-j\angle a_k} e^{-jk\frac{\pi}{4}n}$$

$$= 2|a_k| \cos\left(\frac{kn\pi}{4} + \angle a_k\right)$$

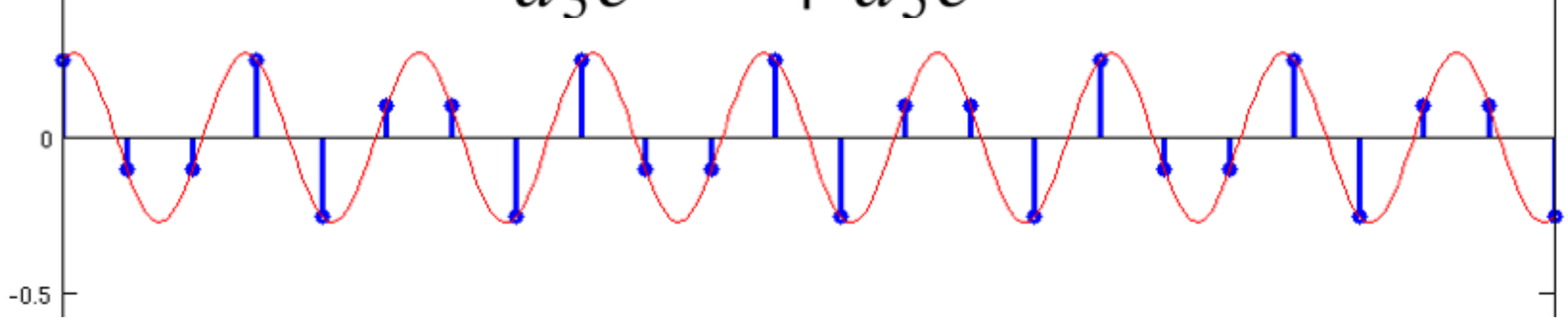
$$a_0$$



$$a_1 e^{j\frac{\pi}{4}n} + a_7 e^{j\frac{7\pi}{4}n}$$



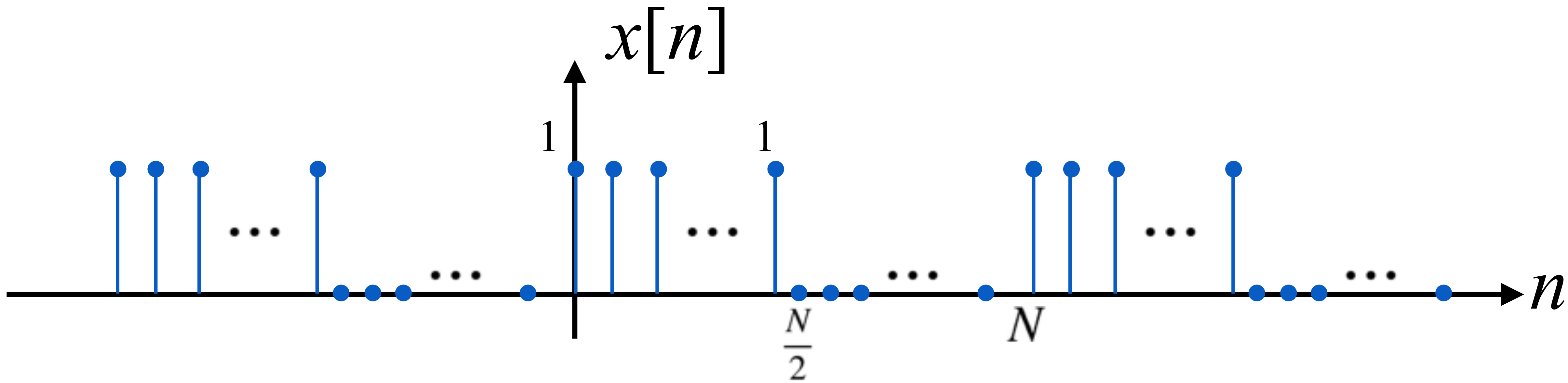
$$a_3 e^{j\frac{3\pi}{4}n} + a_5 e^{j\frac{5\pi}{4}n}$$



$$a_2 = a_4 = a_6 = 0$$

Examples

- In general, if

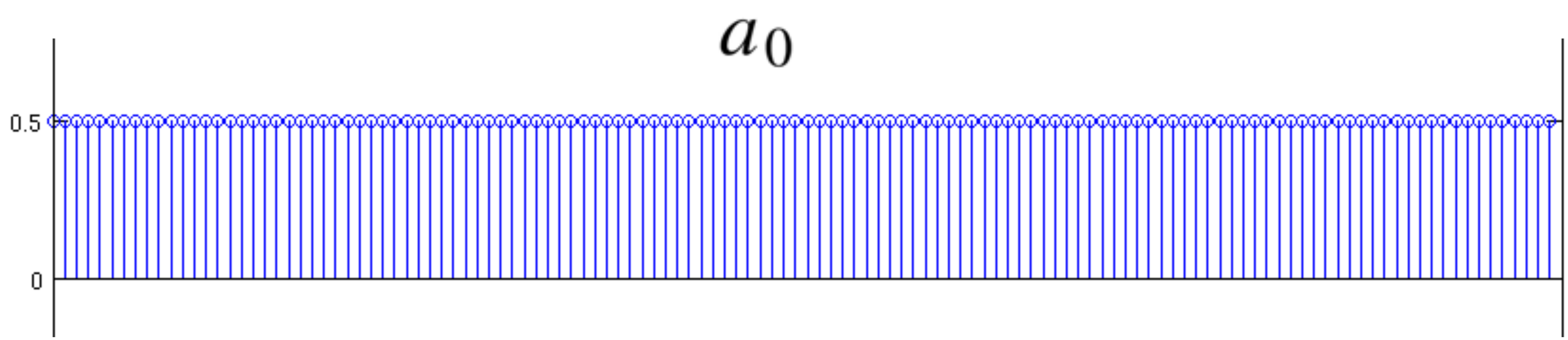


we can similarly compute

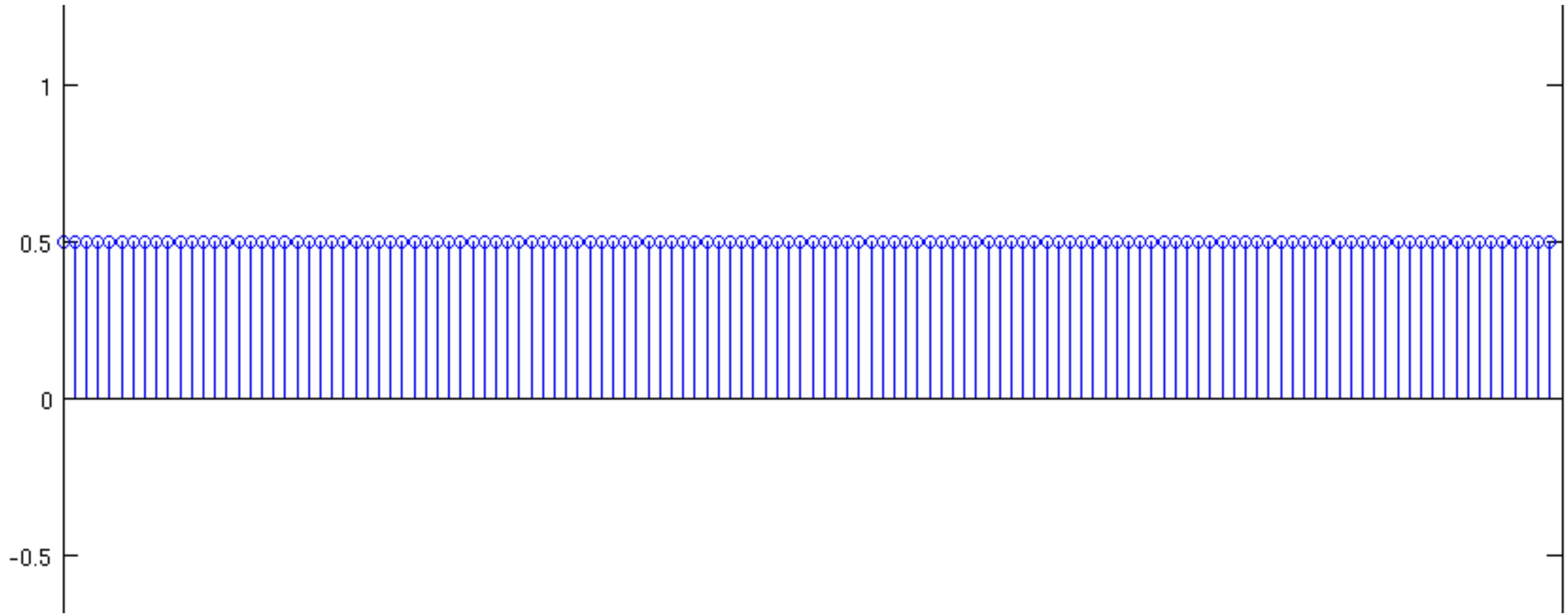
$$a_0 = 0.5 \quad a_k = \frac{1}{N} \cdot \frac{\sin(\frac{k\pi}{2})}{\sin(\frac{k\pi}{N})} e^{-jk(\frac{\pi}{2} - \frac{\pi}{N})} \quad k \neq 0$$

- Still true that $a_{N-k} = a_k^*$ and $a_k = 0$ for even k

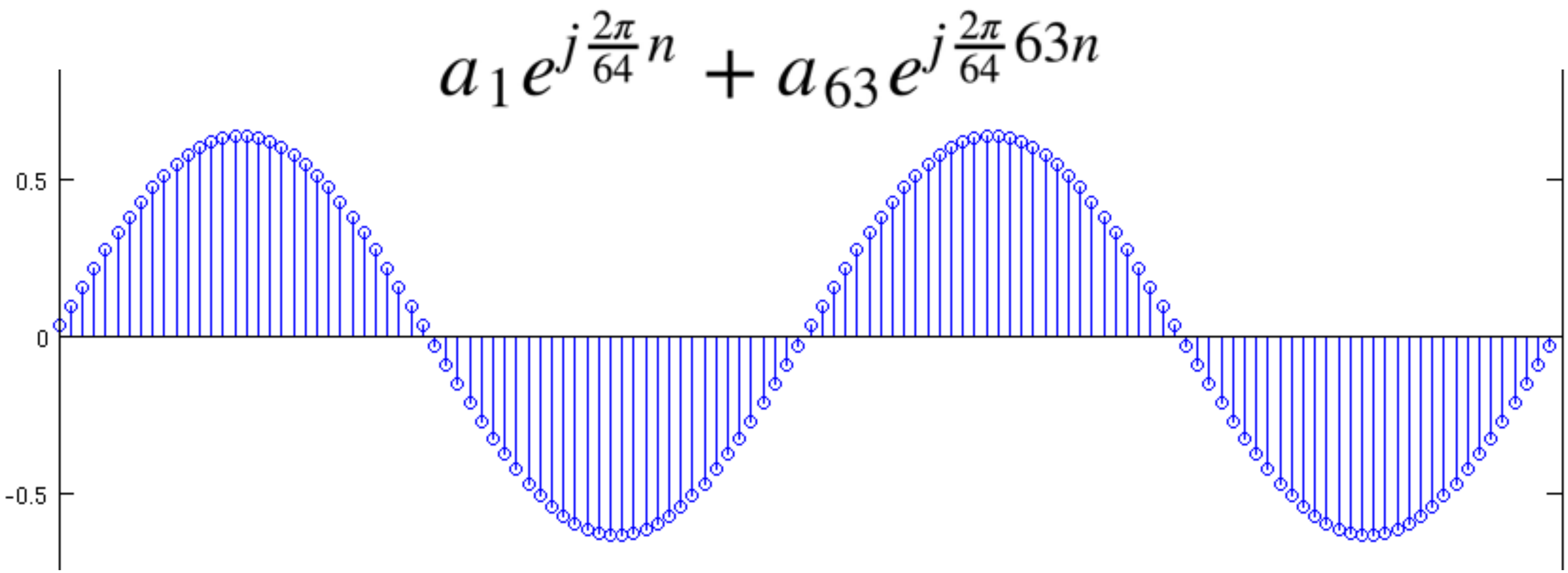
Let's keep adding



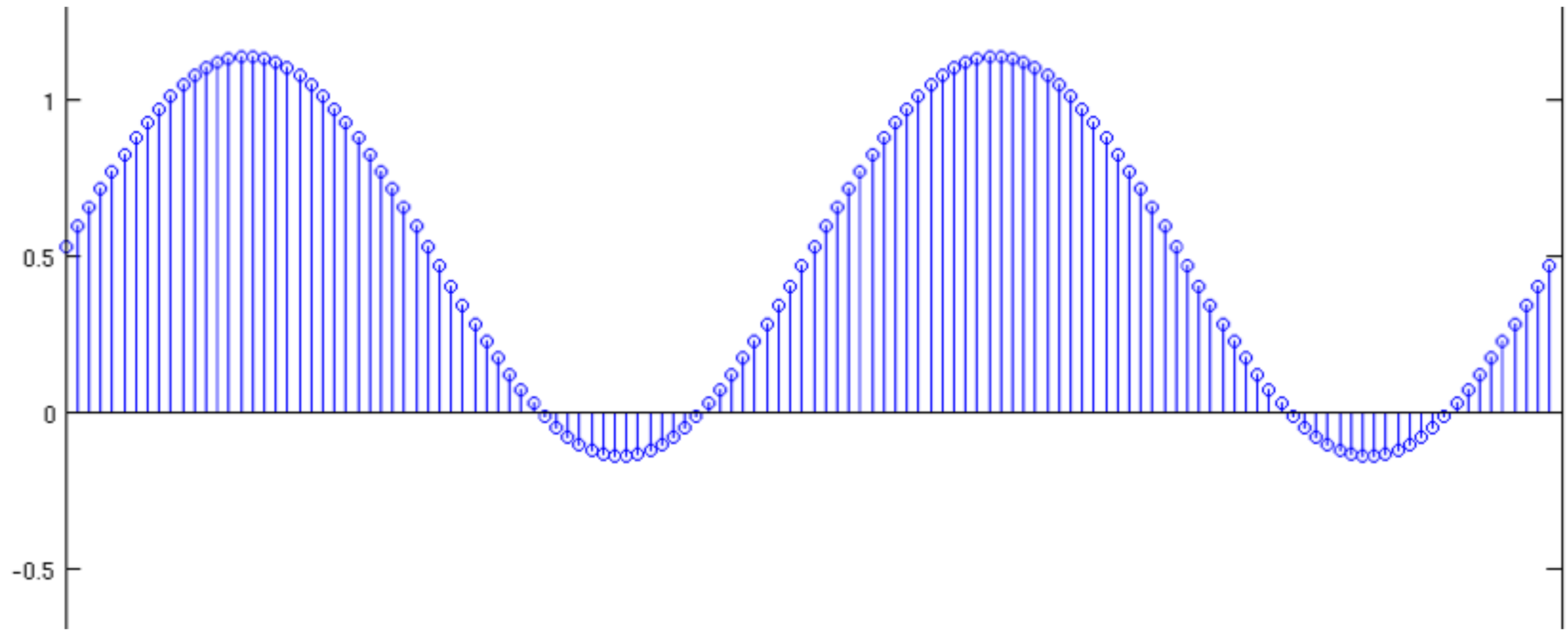
Components added: $k = 0$



Add

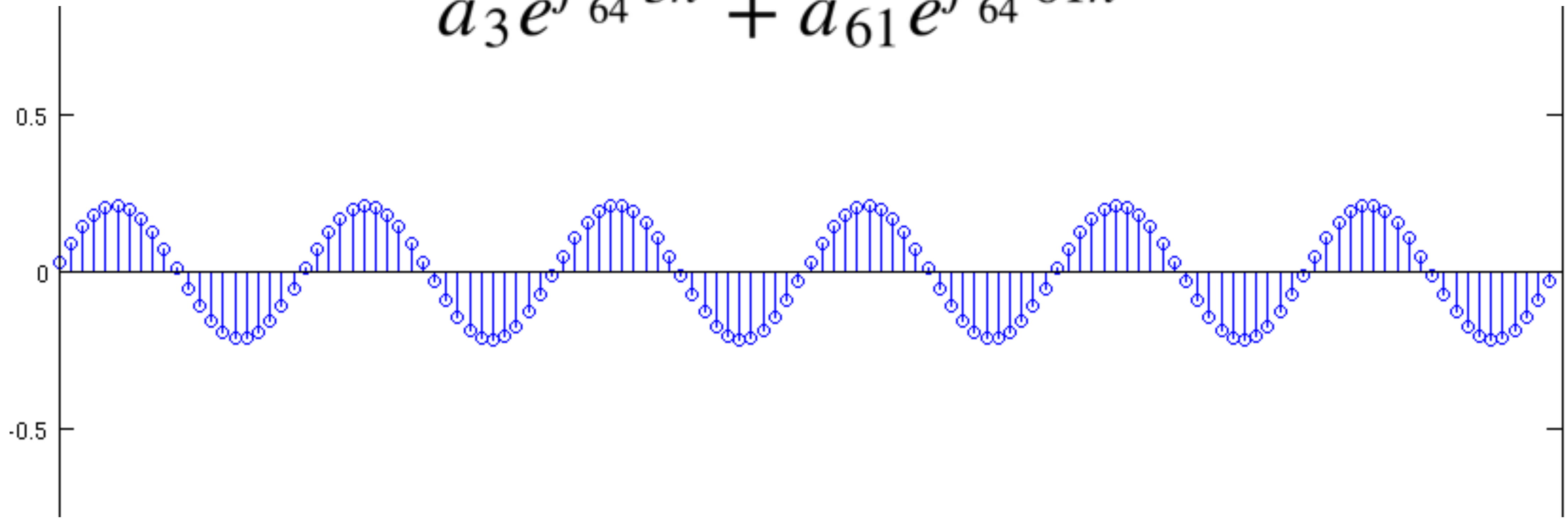


Components added: k =0, 1, 63

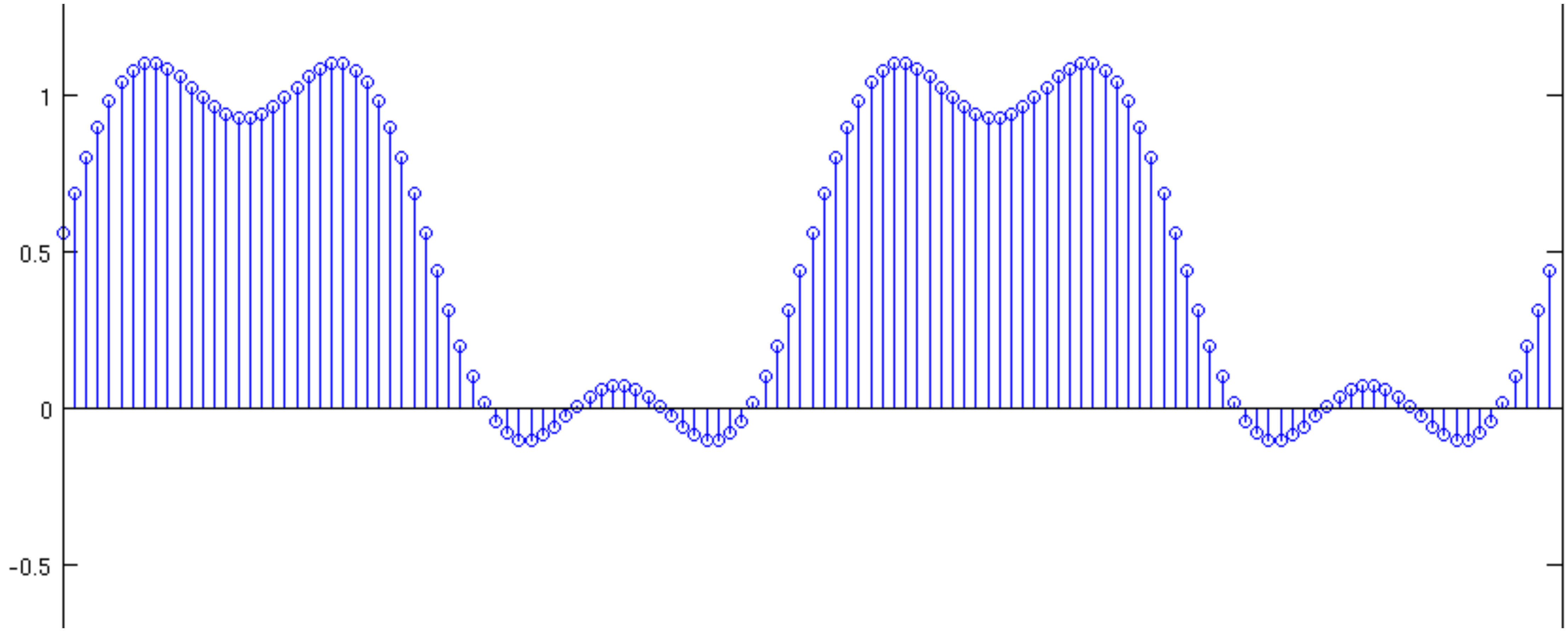


Add

$$a_3 e^{j\frac{2\pi}{64}3n} + a_{61} e^{j\frac{2\pi}{64}61n}$$

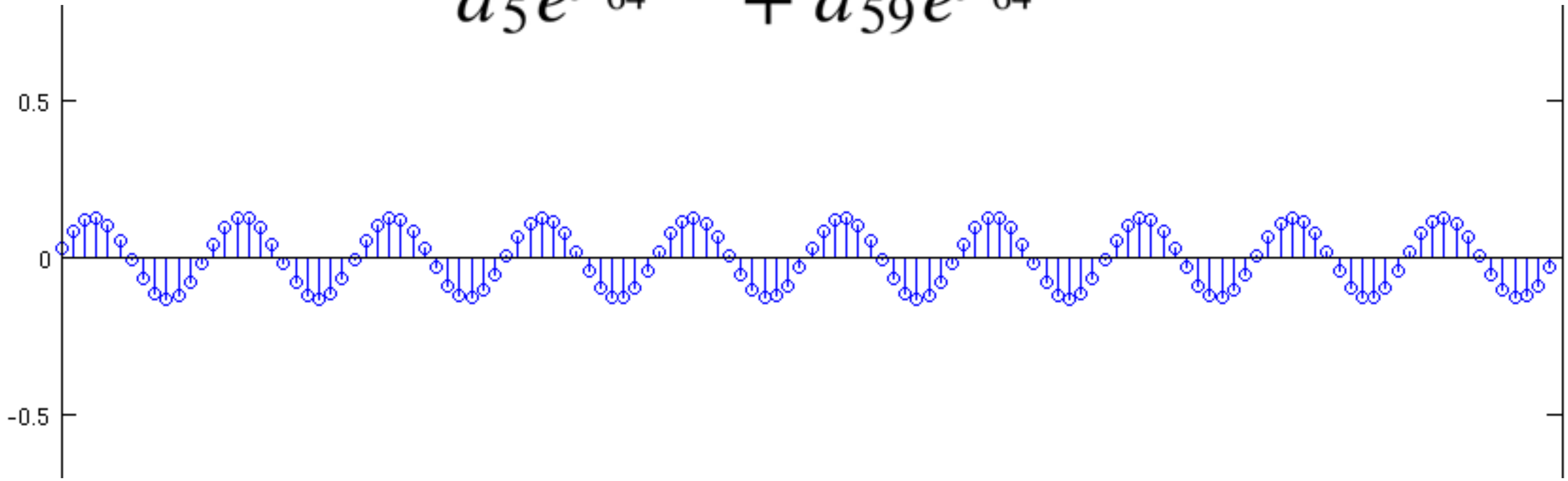


Components added: k =0, 1, 3, 61, 63

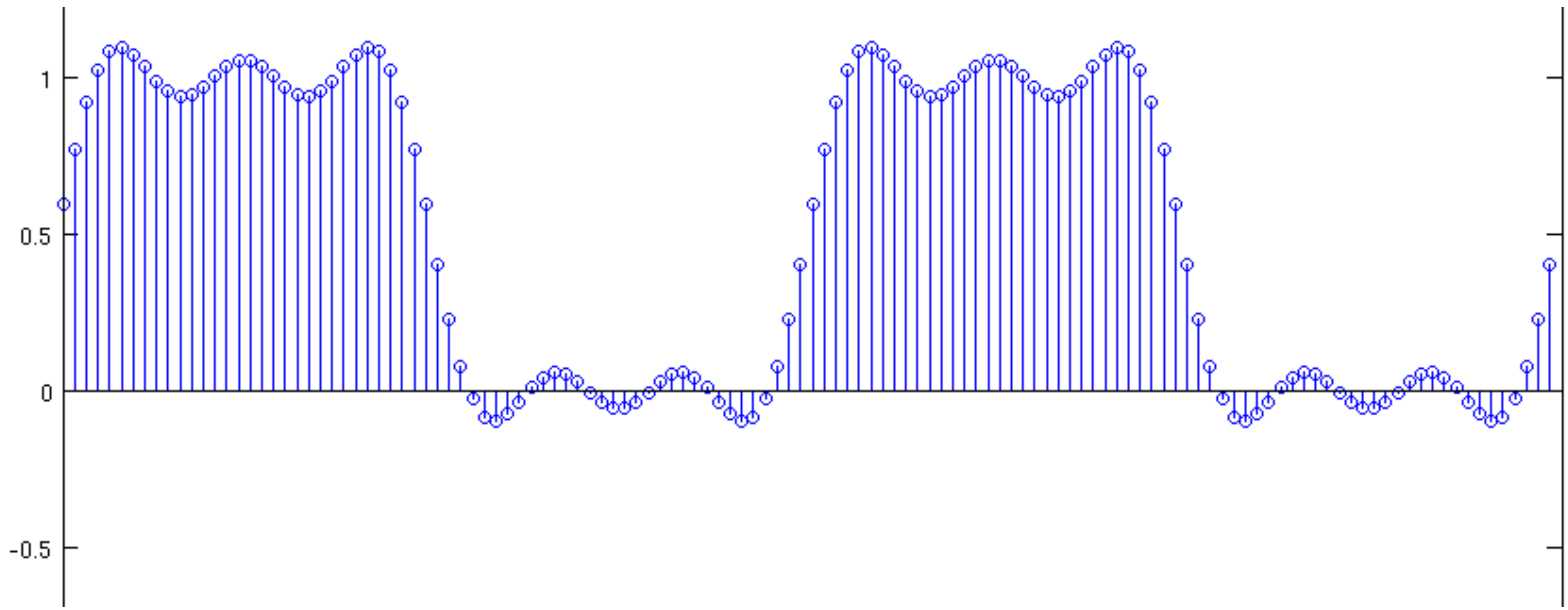


Add

$$a_5 e^{j\frac{2\pi}{64}5n} + a_{59} e^{j\frac{2\pi}{64}59n}$$

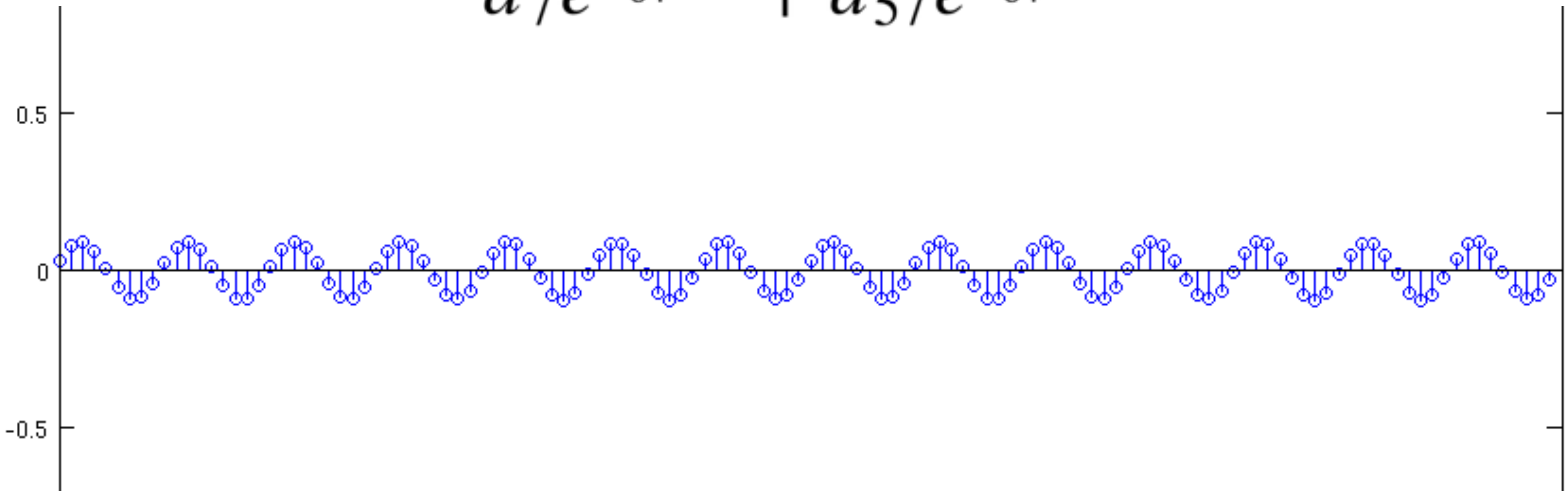


Components added: k =0, 1, 3, 5, 59, 61, 63

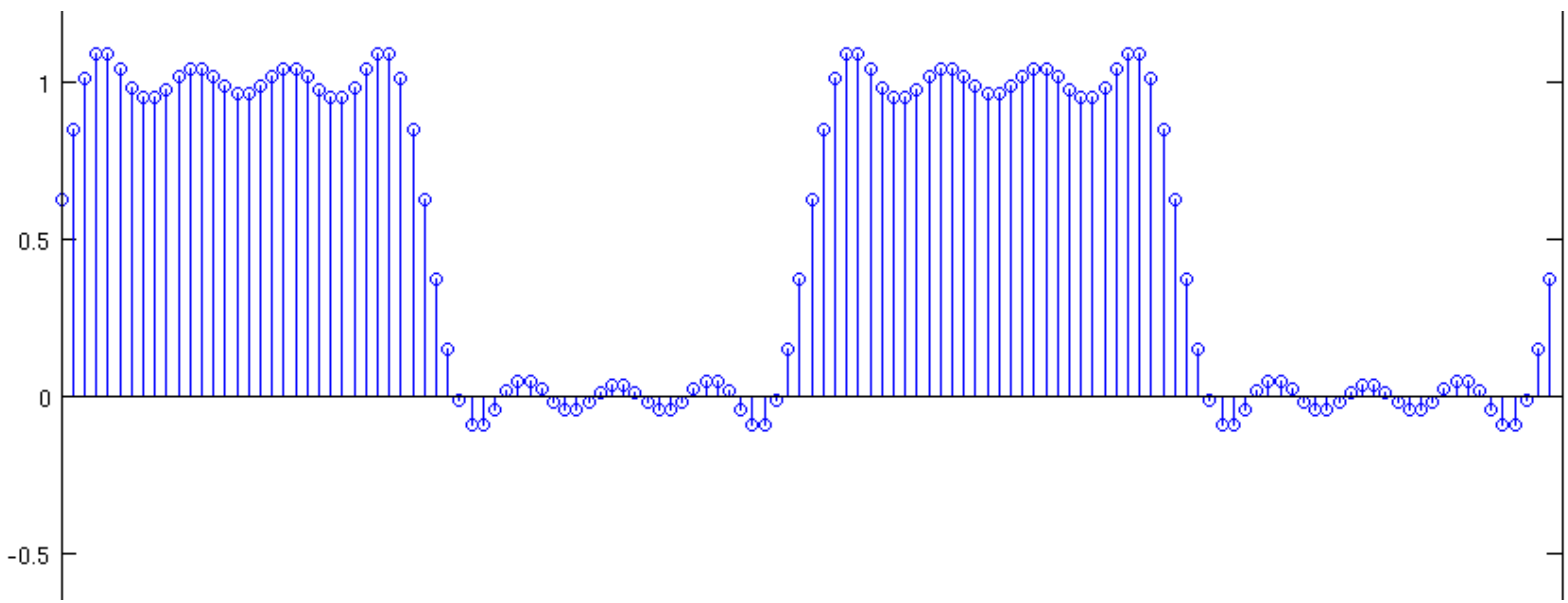


Add

$$a_7 e^{j\frac{2\pi}{64}7n} + a_{57} e^{j\frac{2\pi}{64}57n}$$



Components added: k =0, 1, 3, 5, 7, 57, 59, 61, 63



Let’s hear them

- Assume 44,100 samples per second (CD quality)
- Take $N = 100$.
- The square wave sounds like this:
- Components 1 and 99:
- Components 3 and 97:
- Components 5 and 95:
- Components 7 and 93:

- 0, 1, 3, 97, 99:
- 0, 1, 3, 5, 95, 97, 99:
- 0, 1, 3, 5, 7, 93, 95, 97, 99:

A neat observation

- Note that we are only interested in a_k's for

$$0 \le k \le N-1$$

- However, if the interval of interest is extended further, we observe that a_k also is a periodic sequence (of k) with period N.

$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(k+N)\omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}e^{-j\overset{2\pi}{N\omega_0}n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n} = a_k$$

A neat observation

- Moreover, the complex exponentials are also repetitive in the same manner:

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} e^{jN\omega_0 n} = e^{jk\omega_0 n}$$

- So the formulae can be updated as

$$a_k = \frac{1}{N} \sum_{n \text{ in one period}} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n \in \mathcal{N}} x[n] e^{-jk\omega_0 n}$$

$$x[n] = \sum_{k \text{ in one period}} a_k e^{jk\omega_0 n} = \sum_{k \in \mathcal{N}} a_k e^{jk\omega_0 n}$$

Properties

- **Linearity**:

$$x[n] \xrightarrow{DTFS} a_k$$

$$y[n] \xrightarrow{DTFS} b_k$$

implies

$$Ax[n] + By[n] \xrightarrow{DTFS} Aa_k + Bb_k$$

- Proof:

$$\frac{1}{N}\sum_{n=0}^{N-1}(Ax[n] + By[n])e^{-jk\omega_0 n} = \frac{A}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\omega_0 n} + \frac{B}{N}\sum_{n=0}^{N-1}y[n]e^{-jk\omega_0 n}$$

Properties

- **Time shifting**:

$$x[n] \xrightarrow{DTFS} a_k$$

implies

$$x[n-n_0] \xrightarrow{DTFS} a_k e^{-jk\omega_0 n_0}$$

- <u>Proof</u>:

$$\frac{1}{N}\sum_{n\in\mathcal{N}} x[n-n_0]e^{-jk\omega_0 n}$$

$$\overset{(m=n-n_0)}{=} \frac{1}{N}\sum_{m\in\mathcal{N}} x[m]e^{-jk\omega_0(m+n_0)} = \frac{e^{-jk\omega_0 n_0}}{N}\sum_{m\in\mathcal{N}} x[m]e^{-jk\omega_0 m}$$

Properties

- **Frequency shifting**:

$$x[n] \xrightarrow{DTFS} a_k$$

implies

$$x[n]e^{jk_0\omega_0 n} \xrightarrow{DTFS} a_{k-k_0}$$

- <u>Proof</u>:

$$\frac{1}{N}\sum_{n\in\mathcal{N}} x[n]e^{jk_0\omega_0 n}e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n\in\mathcal{N}} x[n]e^{-j(k-k_0)\omega_0 n}$$

$$= a_{k-k_0}$$

Properties

- **Conjugation**:

$$x[n] \xrightarrow{DTFS} a_k$$

implies

$$x[n]^* \xrightarrow{DTFS} a_{-k}^*$$

- <u>Proof</u>:

$$\frac{1}{N}\sum_{n\in\mathcal{N}} x[n]^* e^{-jk\omega_0 n} = \left\{\underbrace{\frac{1}{N}\sum_{n\in\mathcal{N}} x[n] e^{jk\omega_0 n}}_{a_{-k}}\right\}^*$$

Properties

- **Time reversal**:

$$x[n] \xrightarrow{DTFS} a_k$$

implies

$$x[-n] \xrightarrow{DTFS} a_{-k}$$

- <u>Proof</u>:

$$\frac{1}{N}\sum_{n\in\mathcal{N}} x[-n]e^{-jk\omega_0 n} \overset{(m=-n)}{=} \frac{1}{N}\sum_{m\in\mathcal{N}} x[m]e^{jk\omega_0 m}$$

$$= a_{-k}$$

Properties

- Implications of the last two properties:

$$x[n]^* \xrightarrow{DTFS} a^*_{-k} \qquad x[-n] \xrightarrow{DTFS} a_{-k}$$

- Real signals: $x[n] = x^*[n] \Longrightarrow a_k = a^*_{-k}$
- Even signals: $x[n] = x[-n] \Longrightarrow a_k = a_{-k}$
- Real and even signals:

$$x[n] = x^*[n] = x[-n] \Longrightarrow a_k = \boxed{a^*_{-k} = a_{-k}}$$

real coefficients

- DTFS coefficients are also real and even!!!

Properties

- **Periodic convolution**:

$$x[n] \xrightarrow{DTFS} a_k$$

$$y[n] \xrightarrow{DTFS} b_k$$

implies

$$\sum_{l \in \mathcal{N}} x[l] y[n-l] \xrightarrow{DTFS} N a_k b_k$$

periodic convolution

also shown as $x[n] \tilde{\star} y[n]$

Properties

- Proof:

$$\frac{1}{N}\sum_{n\in\mathcal{N}}\sum_{l\in\mathcal{N}}x[l]y[n-l]e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{l\in\mathcal{N}}x[l]\sum_{n\in\mathcal{N}}y[n-l]e^{-jk\omega_0 n}$$

$$\overset{(m=n-l)}{=} \frac{1}{N}\sum_{l\in\mathcal{N}}x[l]\sum_{m\in\mathcal{N}}y[m]e^{-jk\omega_0 (m+l)}$$

$$= \frac{1}{N}\sum_{l\in\mathcal{N}}x[l]e^{-jk\omega_0 l}\sum_{m\in\mathcal{N}}y[m]e^{-jk\omega_0 m}$$

$$= Na_k b_k$$

Properties

- **Multiplication**:

$$x[n] \xrightarrow{DTFS} a_k \qquad \text{implies} \qquad x[n]y[n] \xrightarrow{DTFS} a_k \tilde{\star}\, b_k$$
$$y[n] \xrightarrow{DTFS} b_k$$

- Proof: Whose DTFS is $a_k \tilde{\star}\, b_k$?

$$\sum_{k\in\mathcal{N}} (a_k \tilde{\star}\, b_k) e^{jk\omega_0 n} = \sum_{k\in\mathcal{N}} \sum_{l\in\mathcal{N}} a_l b_{k-l} e^{jk\omega_0 n} = \sum_{l\in\mathcal{N}} a_l \sum_{k\in\mathcal{N}} b_{k-l} e^{jk\omega_0 n}$$

$$\overset{(m=k-l)}{=} \sum_{l\in\mathcal{N}} a_l \sum_{m\in\mathcal{N}} b_m e^{j(m+l)\omega_0 n}$$

$$= \sum_{l\in\mathcal{N}} a_l e^{jl\omega_0 n} \sum_{m\in\mathcal{N}} b_m e^{jm\omega_0 n} = x[n]y[n]$$

Properties

- **Parseval's relation**:

$$x[n] \xrightarrow{DTFS} a_k$$

implies

$$\frac{1}{N} \sum_{n \in \mathcal{N}} \left| x[n] \right|^2 = \sum_{k \in \mathcal{N}} |a_k|^2$$

- <u>Proof:</u> First, let's first show

$$\sum_{n \in \mathcal{N}} \left| x[n] \right|^2 = \left(x[n] \,\tilde{\star}\, x[-n]^* \right) \Big|_{n=0}$$

$$\frac{1}{N}\sum_{n\in\mathcal{N}}\left|x[n]\right|^2 = \sum_{k\in\mathcal{N}}|a_k|^2$$

- Proof: First, let's first show

$$\sum_{n\in\mathcal{N}}\left|x[n]\right|^2 = \left(x[n] \mathbin{\tilde{\star}} x[-n]^*\right)\Big|_{n=0}$$

To see that, write

$$x[n] \mathbin{\tilde{\star}} x[-n]^* = \sum_{m\in\mathcal{N}} x[m]x[m-n]^*$$

and substitute $n = 0$ on the RHS:

$$\left(x[n] \mathbin{\tilde{\star}} x[-n]^*\right)\Big|_{n=0} = \sum_{m\in\mathcal{N}}\left|x[m]\right|^2 = \sum_{n\in\mathcal{N}}\left|x[n]\right|^2$$

$$\frac{1}{N}\sum_{n\in\mathcal{N}}\left|x[n]\right|^2=\sum_{k\in\mathcal{N}}|a_k|^2$$

$$\sum_{n\in\mathcal{N}}\left|x[n]\right|^2=\left(x[n]\tilde{\star}x[-n]^*\right)\Big|_{n=0}$$

But we know that

$$x[-n]\xrightarrow{DTFS}a_{-k}\quad\text{and}\quad x[n]^*\xrightarrow{DTFS}a_{-k}^*$$

Therefore,

$$x[-n]^*\xrightarrow{DTFS}a_k^*$$

Then using the convolution property,

$$x[n]\tilde{\star}x[-n]^*\xrightarrow{DTFS}Na_ka_k^*=N|a_k|^2$$

In other words, $x[n]\tilde{\star}x[-n]^*=\sum_{k\in\mathcal{N}}N|a_k|^2e^{jk\omega_0n}$

$$\frac{1}{N}\sum_{n\in\mathcal{N}}\left|x[n]\right|^2=\sum_{k\in\mathcal{N}}|a_k|^2$$

$$\sum_{n\in\mathcal{N}}\left|x[n]\right|^2=\left(x[n]\ \tilde{\star}\ x[-n]^*\right)\Big|_{n=0}$$

$$x[n]\ \tilde{\star}\ x[-n]^*=\sum_{k\in\mathcal{N}}N|a_k|^2e^{jk\omega_0 n}$$

The proof can be finished as

$$\frac{1}{N}\sum_{n\in\mathcal{N}}\left|x[n]\right|^2=\frac{1}{N}\left(x[n]\ \tilde{\star}\ x[-n]^*\right)\Big|_{n=0}$$

$$=\frac{1}{N}\sum_{k\in\mathcal{N}}N|a_k|^2e^{jk\omega_0 n}\Bigg|_{n=0}=\sum_{k\in\mathcal{N}}|a_k|^2$$

Example problems

- Problem: Find the sum $\sum_{n=0}^{63} \cos\left(\frac{5\pi n}{32}\right)^2$
- Solution: Think of this as $\sum_{n=0}^{63} |x[n]|^2$

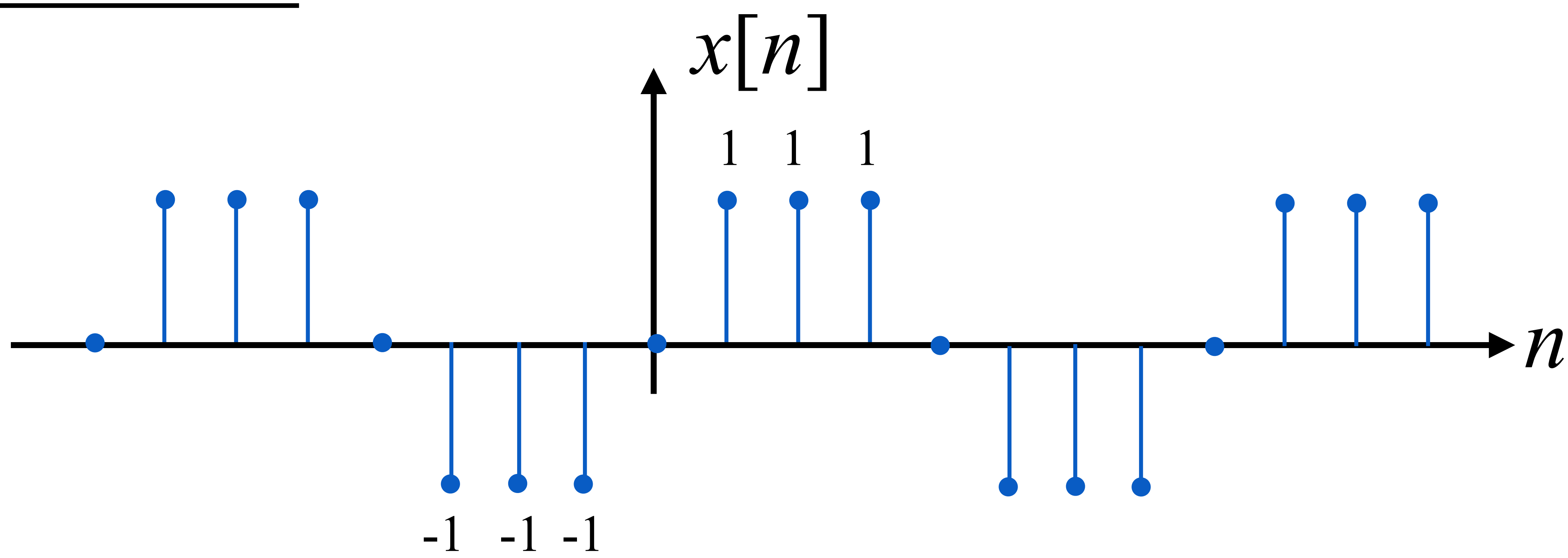
Thanks to Parseval's relation, all we need is a_k
But that is particularly easy, because

$$\cos\left(\frac{5\pi n}{32}\right) = \frac{e^{j\frac{10\pi n}{64}} + e^{-j\frac{10\pi n}{64}}}{2} \left.\begin{matrix} a_5 = 0.5 \\ a_{-5} = 0.5 \\ a_k = 0 \ \text{otherwise} \end{matrix}\right\}$$

So $\sum_{n=0}^{63} |x[n]|^2 = 64\left(|a_5|^2 + |a_{-5}|^2\right) = 32$

Example problems

- <u>Problem:</u> Find the DTFS coefficients for



- <u>Solution:</u> This signal can be written as

$$x[n] = y[n] - y[-n]$$

where $y[n]$ is the same square wave we analyzed before.

- Recall that

$$y[n] \xrightarrow{DTFS} \begin{cases} 0.5 & k = 0 \\ \frac{1}{8} \cdot \frac{\sin(\frac{k\pi}{2})}{\sin(\frac{k\pi}{8})} e^{-jk\frac{3\pi}{8}} & k \neq 0 \end{cases}$$

- Using the time reversal property,

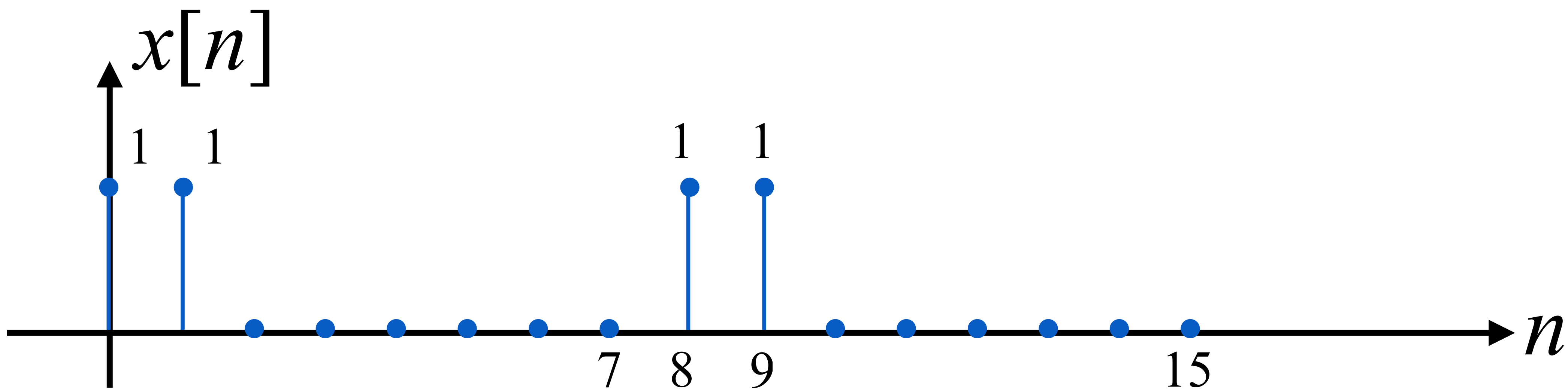
$$y[-n] \xrightarrow{DTFS} \begin{cases} 0.5 & k = 0 \\ \frac{1}{8} \cdot \frac{\sin(\frac{k\pi}{2})}{\sin(\frac{k\pi}{8})} e^{jk\frac{3\pi}{8}} & k \neq 0 \end{cases}$$

- Bringing the two together,

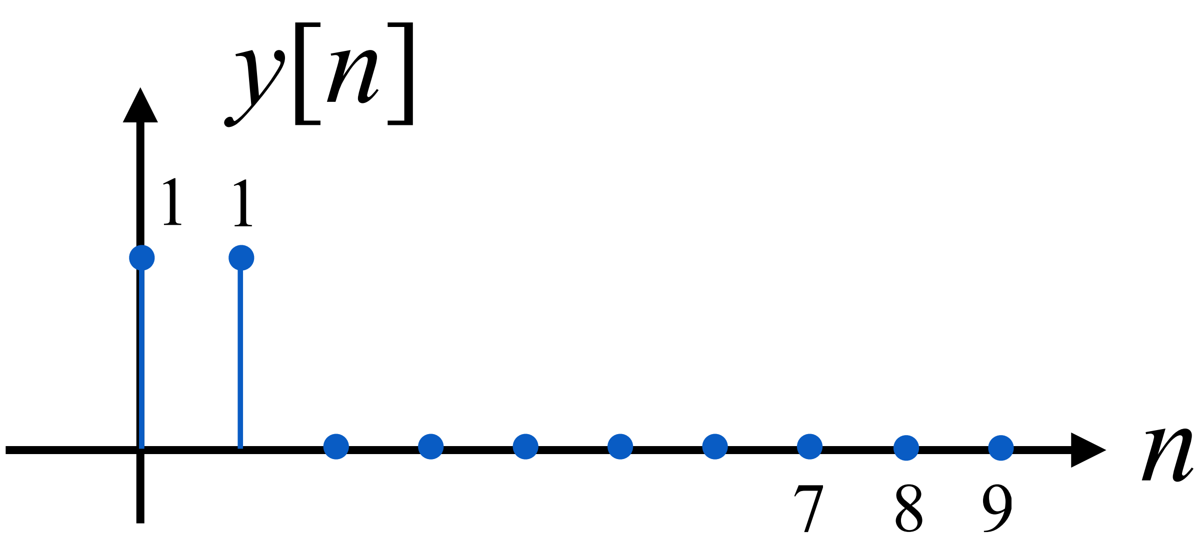
$$y[n] - y[-n] \xrightarrow{DTFS} \begin{cases} 0 & k = 0 \\ \frac{-j}{4} \cdot \frac{\sin(\frac{k\pi}{2})}{\sin(\frac{k\pi}{8})} \sin\left(\frac{3k\pi}{8}\right) & k \neq 0 \end{cases}$$

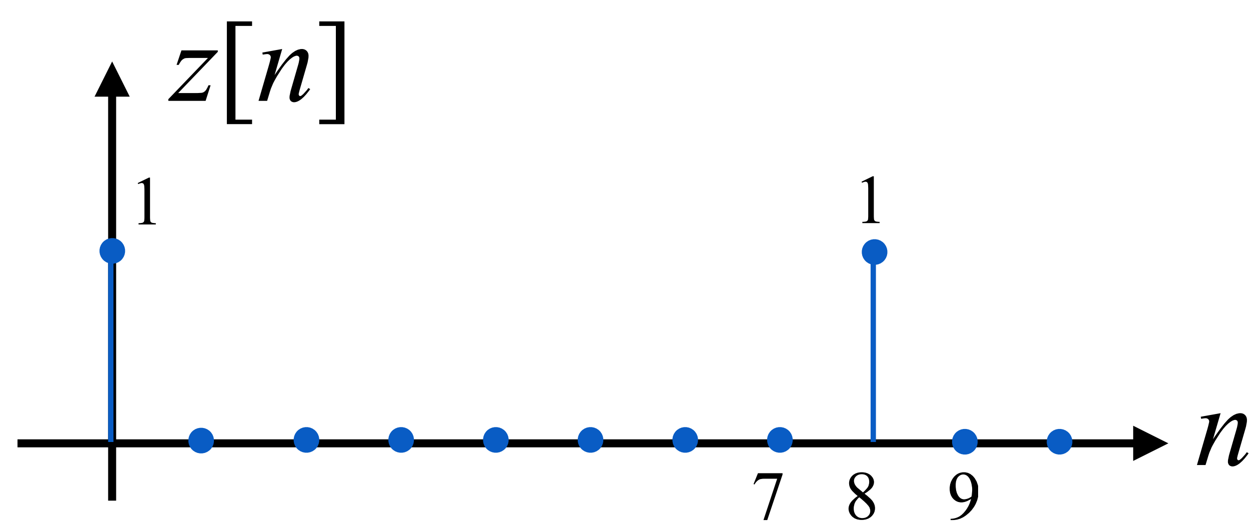
Example problems

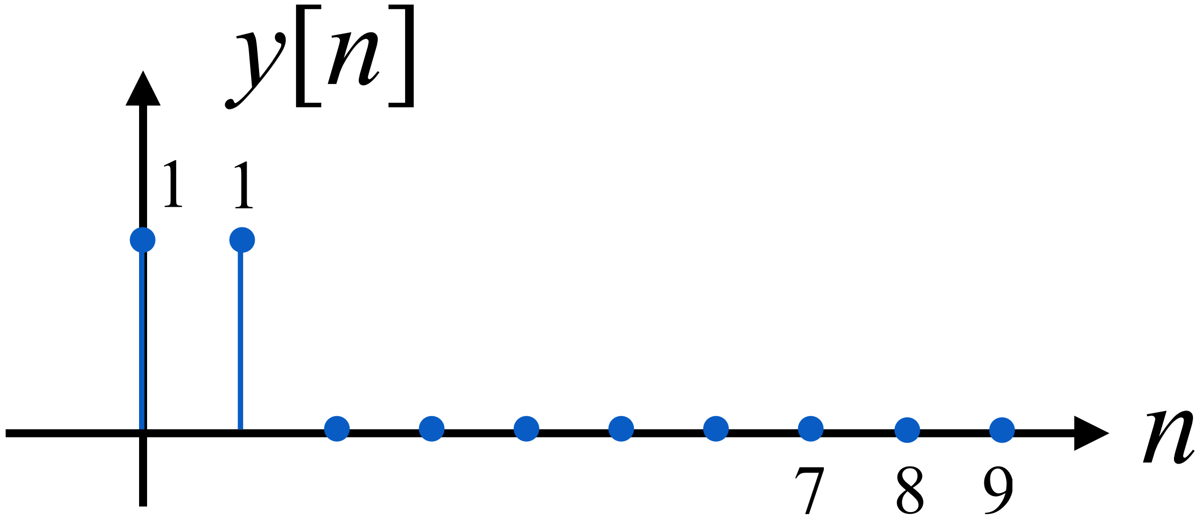
- Problem: Find the DTFS coefficients for the signal whose one period is shown below:

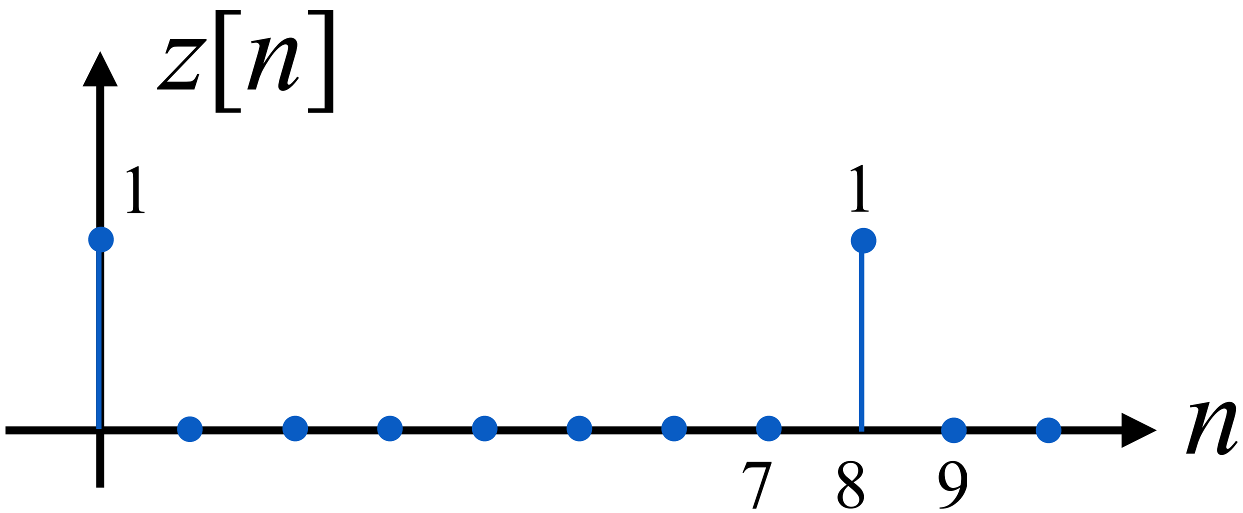


- Solution: It might be much easier to think of this as a periodic convolution of two signals:









$$b_k = \frac{1}{16}\sum_{n=0}^{15} y[n]e^{-jk\frac{2\pi}{16}n} \qquad\qquad c_k = \frac{1}{16}\sum_{n=0}^{15} z[n]e^{-jk\frac{2\pi}{16}n}$$

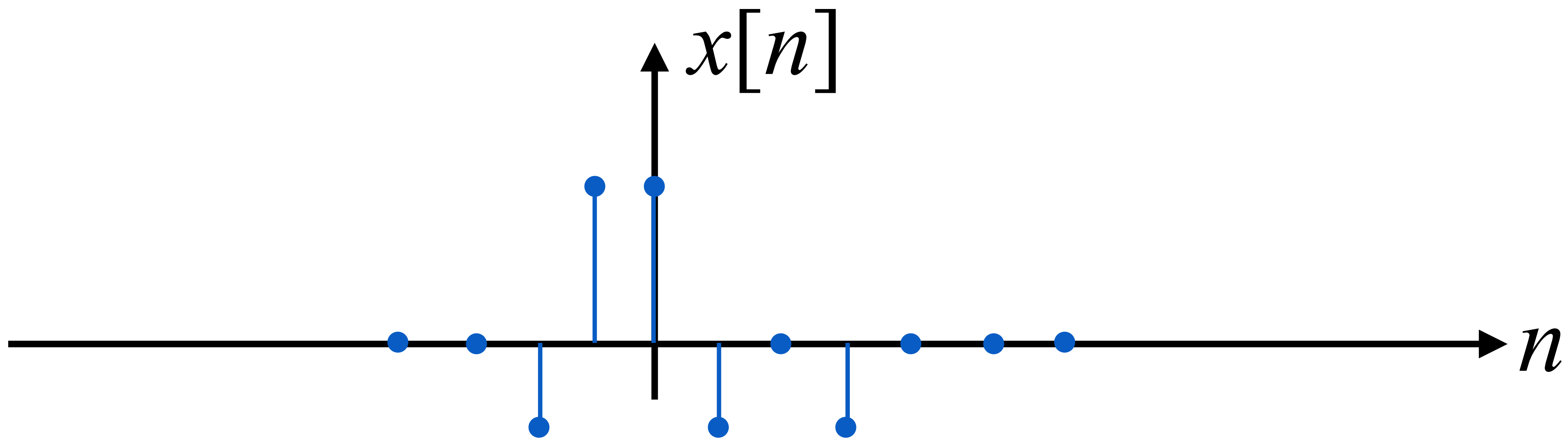
$$= \frac{1+e^{-jk\frac{\pi}{8}}}{16} \qquad\qquad = \frac{1+e^{-jk\pi}}{16}$$

$$x[n] = y[n] \tilde{\star} z[n] \Longrightarrow a_k = 16b_kc_k$$

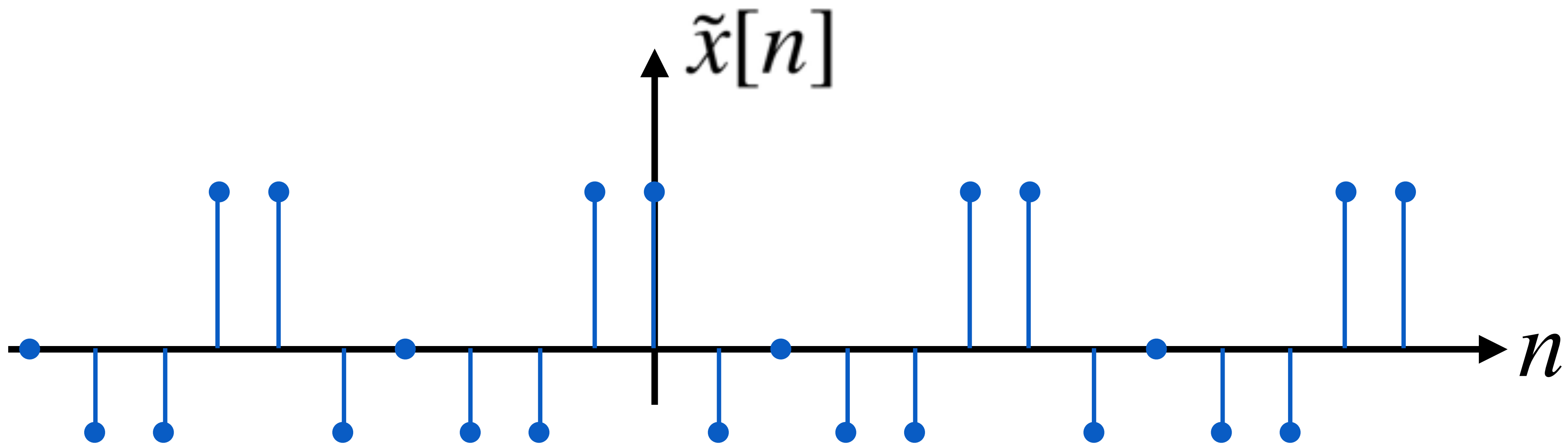
$$= \frac{(1+e^{-jk\frac{\pi}{8}})(1+e^{-jk\pi})}{16}$$

What about nonperiodic signals?

- If the signal has finite duration, everything is fine:



- Extend the signal into a periodic one and decompose it onto however many $e^{jk\omega_0 n}$ needed



What about nonperiodic signals?

- We can then apply the usual analysis/synthesis formulae:

$$a_k = \frac{1}{N} \sum_{n \in \mathcal{N}} \tilde{x}[n] e^{-jk\omega_0 n}$$

$$\tilde{x}[n] = \sum_{k \in \mathcal{N}} a_k e^{jk\omega_0 n}$$

- If the original signal was nonzero only in the interval $N_1 \le n \le N_2$, then the analysis formula can also be written as

$$a_k = \frac{1}{N} \sum_{n=N_1}^{N_2} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{N}\sum_{n=N_1}^{N_2} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk\omega_0 n}$$

- Now define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

so that

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

and

$$\tilde{x}[n] = \sum_{k\in\mathcal{N}} \frac{X(e^{jk\omega_0})}{N} e^{jk\omega_0 n} = \frac{1}{2\pi}\sum_{k\in\mathcal{N}} \omega_0 X(e^{jk\omega_0})e^{jk\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k \in \mathcal{N}} \omega_0 X(e^{jk\omega_0}) e^{jk\omega_0 n}$$

- Note that this is true for any $N \geq N_2 - N_1 + 1$
- What happens if $N \to \infty$?

$$\omega_0 = \frac{2\pi}{N} \to 0$$

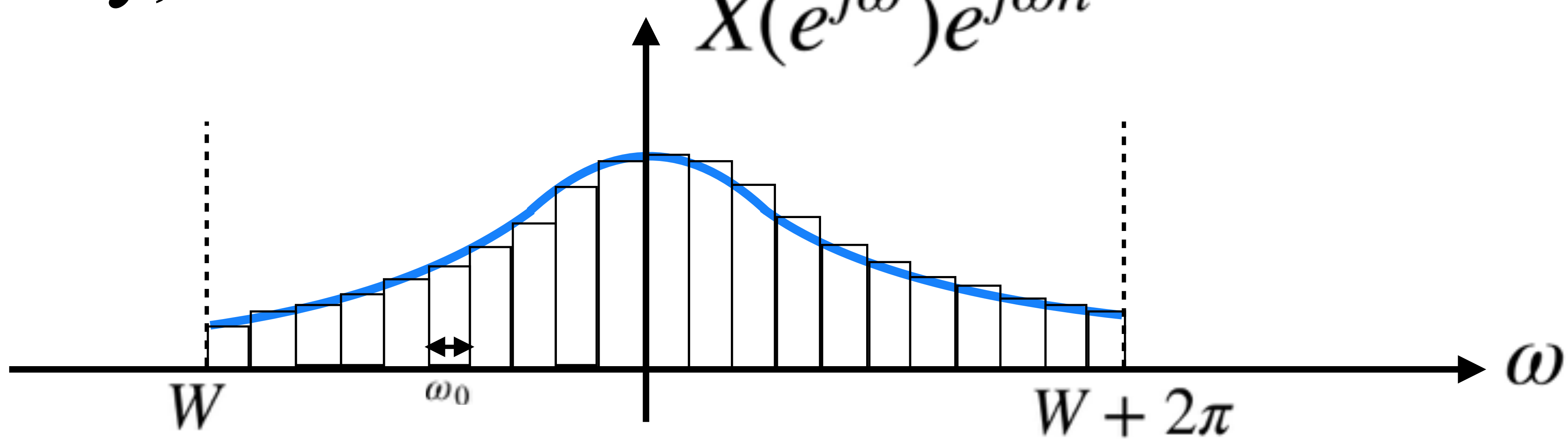
$k\omega_0$ spans an interval of length 2π

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k\in\mathcal{N}} \omega_0 X(e^{jk\omega_0})e^{jk\omega_0 n}$$

= total area of the rectangles

- Pictorially,



- As $\omega_0 \to 0$, where does the total area of the rectangles go?

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k\in\mathcal{N}} \omega_0 X(e^{jk\omega_0})e^{jk\omega_0 n}$$

$$\Big\downarrow \ \omega_0 \to 0$$

$$\tilde{x}[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$$

- But $N \to \infty$ also implies that $\tilde{x}[n] \to x[n]$
- Therefore,

$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$$

Discrete-time Fourier Transform

- This pair is known as the discrete-time Fourier transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$$

Properties

- Properties of DTFS carry over to DTFT:

$$x[n] \xrightarrow{DTFT} X(e^{j\omega}) \qquad y[n] \xrightarrow{DTFT} Y(e^{j\omega})$$

imply

- **Linearity:** $ax[n] + by[n] \xrightarrow{DTFT} aX(e^{j\omega}) + bY(e^{j\omega})$
- **Time shifting:** $x[n - n_0] \xrightarrow{DTFT} X(e^{j\omega})e^{-j\omega n_0}$
- **Frequency shifting:** $x[n]e^{j\omega_0 n} \xrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$
- **Time reversal:** $x[-n] \xrightarrow{DTFT} X(e^{-j\omega})$
- **Conjugation:** $x[n]^* \xrightarrow{DTFT} X(e^{-j\omega})^*$

- **Linearity:** $ax[n] + by[n] \xrightarrow{DTFT} aX(e^{j\omega}) + bY(e^{j\omega})$
- **Time shifting:** $x[n - n_0] \xrightarrow{DTFT} X(e^{j\omega})e^{-j\omega n_0}$
- **Frequency shifting:** $x[n]e^{j\omega_0 n} \xrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$
- **Time reversal:** $x[-n] \xrightarrow{DTFT} X(e^{-j\omega})$
- **Conjugation:** $x[n]^* \xrightarrow{DTFT} X(e^{-j\omega})^*$
- **Convolution:** $x[n] \star y[n] \xrightarrow{DTFT} X(e^{j\omega})Y(e^{j\omega})$
- **Multiplication:**

$$x[n]y[n] \xrightarrow{DTFT} \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

- **Parseval's:** $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{<2\pi>} |X(e^{j\omega})|^2 d\omega$

Properties

- But there is also one more:
  - **Differentiation in the frequency domain:**

$$nx[n] \xrightarrow{DTFT} j\frac{dX(e^{j\omega})}{d\omega}$$

- Proof:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}(-jn)$$

Rearranging finishes the proof.

Examples

- Find the DTFT of $x[n] = a^n u[n]$ for $|a| < 1$
- <u>Solution:</u>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \quad \text{since} \quad |a| < 1$$

- How do we plot this?
  - Real and imaginary parts separately
  - Magnitude and phase separately

$$
\begin{aligned}
X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} \\
&= \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} \\
&= \frac{1 - a\cos\omega - ja\sin\omega}{1 + a^2 - 2a\cos\omega}
\end{aligned}
$$

$$
\mathrm{Re}\{X(e^{j\omega})\} = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega}
$$

$$
\mathrm{Im}\{X(e^{j\omega})\} = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}
$$

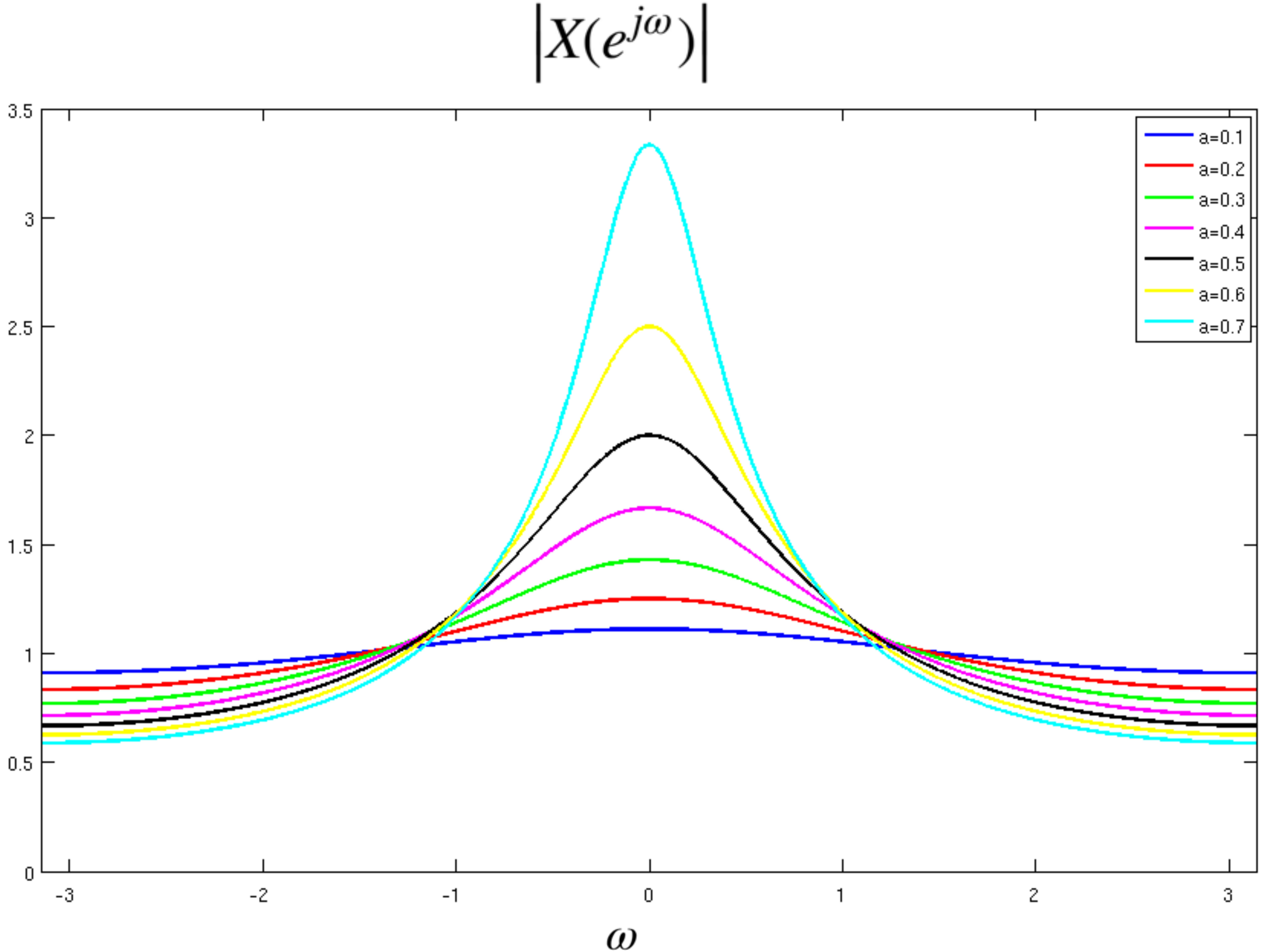
$$\mathrm{Re}\{X(e^{j\omega})\} = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega}$$

$$\mathrm{Im}\{X(e^{j\omega})\} = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}$$

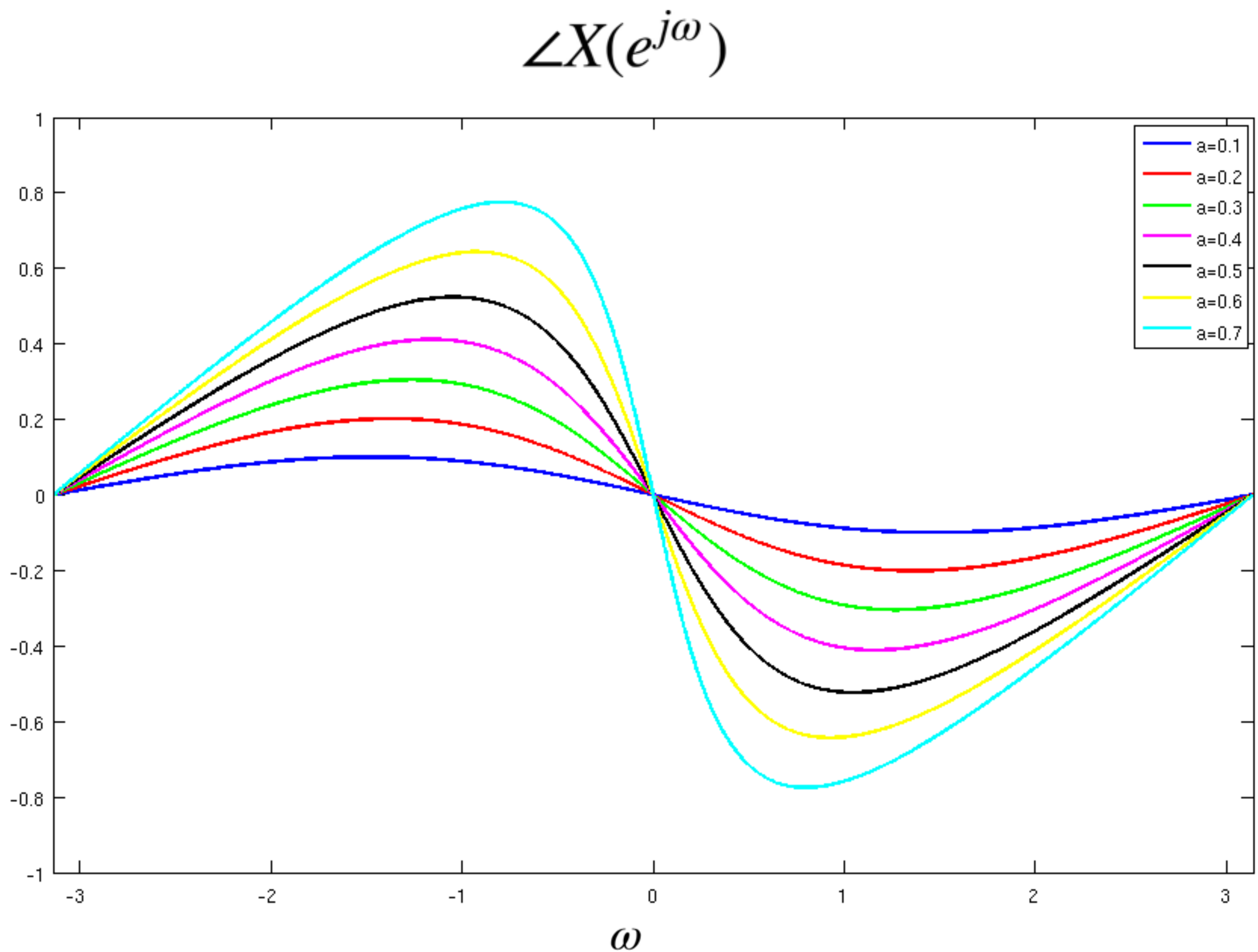
$$\left|X(e^{j\omega})\right| = \frac{\sqrt{(1 - a\cos\omega)^2 + a^2(\sin\omega)^2}}{1 + a^2 - 2a\cos\omega}$$

$$= \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$



$\left|X(e^{j\omega})\right|$
a=0.1
a=0.2
a=0.3
a=0.4
a=0.5
a=0.6
a=0.7
3.5
3
2.5
2
1.5
1
0.5
0
-3
-2
-1
0
1
2
3
ω


$\angle X(e^{j\omega})$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3
ω
a=0.1
a=0.2
a=0.3
a=0.4
a=0.5
a=0.6
a=0.7

Implementation as a filter

- Note that if this is the impulse response of a system, the system will suppress high frequencies and boost low frequencies.

$$x[n] \longrightarrow \boxed{h[n] = a^n u[n]} \longrightarrow y[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

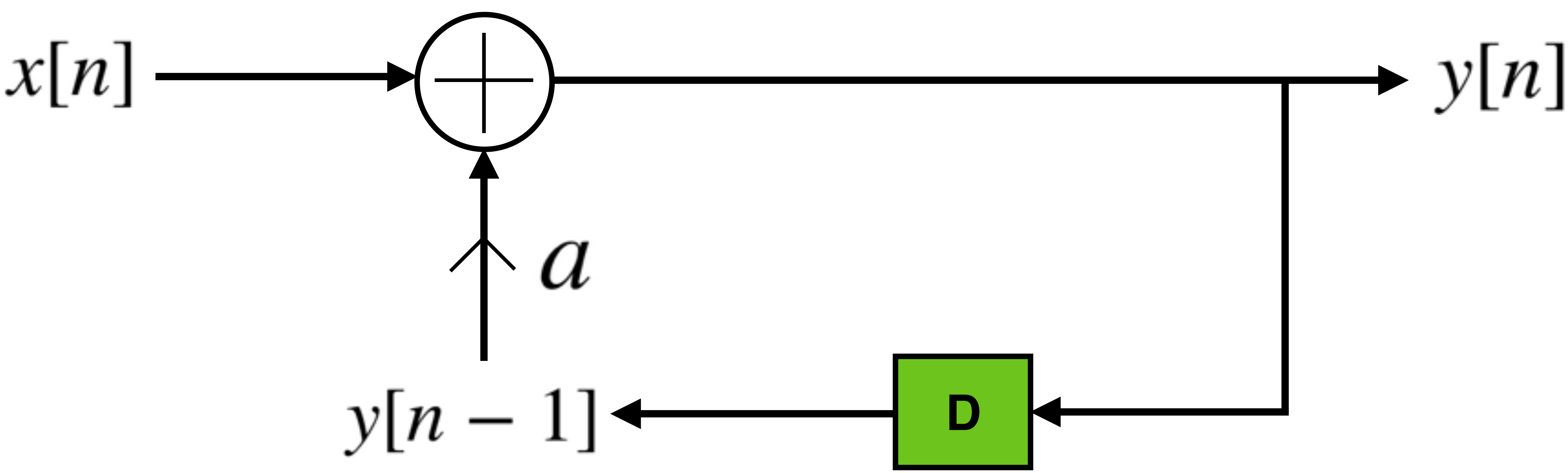
Implementation as a filter

- Let's hear the effect when $a = 0.95$
- Original:
- Filtered:

Implementation as a filter

- This filter is easy to implement once we figure out that $h[n]$ is the impulse response of the system with difference equation

$$y[n] - ay[n-1] = x[n]$$



Examples

- Find the DTFT of $x[n] = a^{|n|}$ for $|a| < 1$
- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{0} a^{-n} e^{-j\omega n} - 1$$

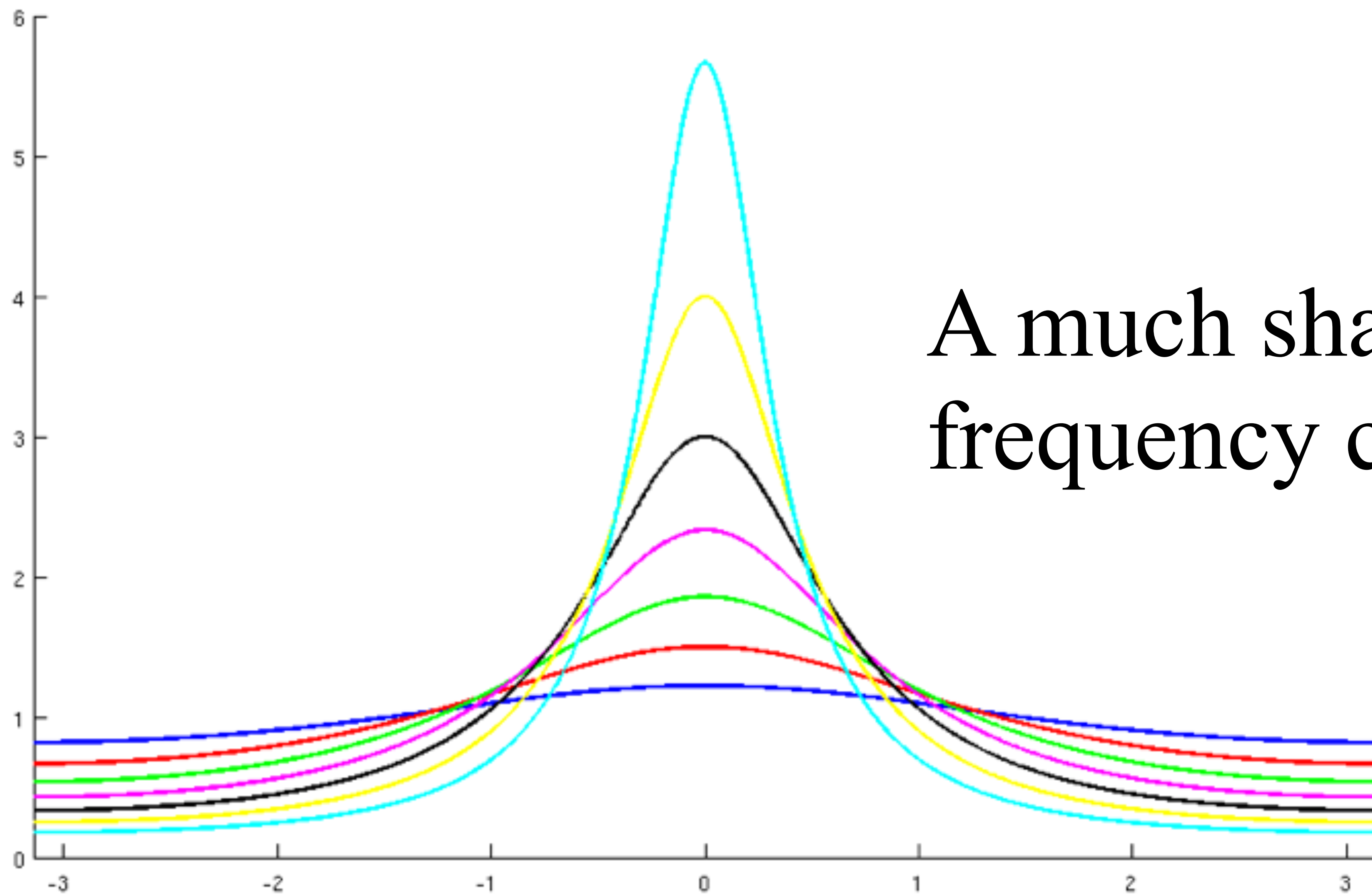
$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{m=0}^{\infty} a^m e^{j\omega m} - 1$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - ae^{j\omega}} - 1$$

$$= \frac{1 - ae^{j\omega} + 1 - ae^{-j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} - 1 = \frac{2(1 - a\cos\omega)}{1 + a^2 - 2a\cos\omega} - 1$$

$$X(e^{j\omega}) = \frac{2(1 - a\cos\omega)}{1 + a^2 - 2a\cos\omega} - 1$$

$$= \frac{1 - a^2}{1 + a^2 - 2a\cos\omega}$$



A much sharper frequency content!!

Examples

- Find the DTFT of $x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$
- <u>Solution:</u>

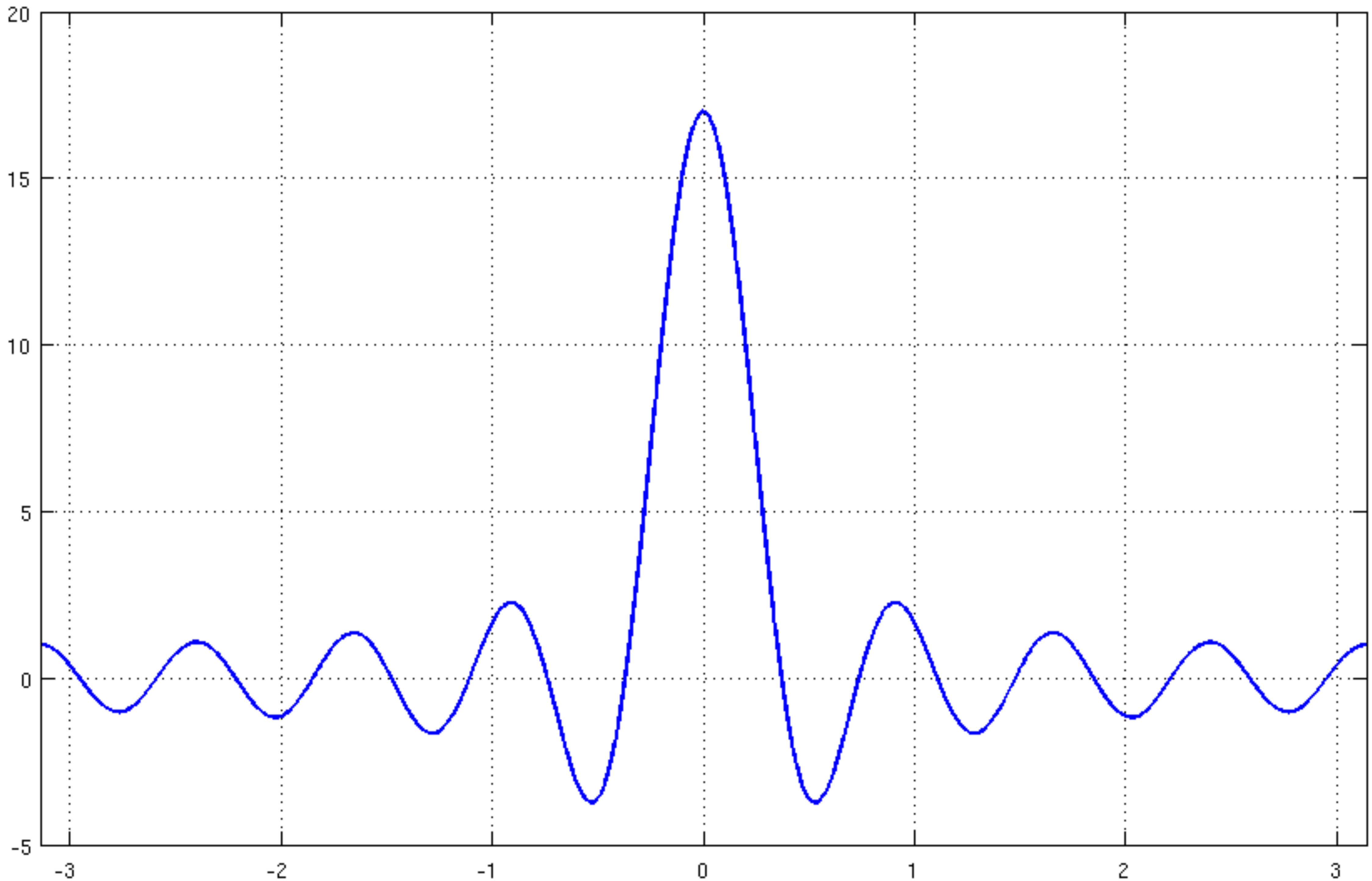
$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} \stackrel{(m=n+N_1)}{=} \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)}$$

$$= e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m} = e^{j\omega N_1} \frac{e^{-j\omega(2N_1+1)} - 1}{e^{-j\omega} - 1}$$

$$= \cancel{e^{j\omega N_1}} \frac{e^{-j\omega(\cancel{N_1}+\cancel{1/2})}}{\cancel{e^{-j\omega/2}}} \frac{e^{-j\omega(N_1+1/2)} - e^{j\omega(N_1+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}}$$

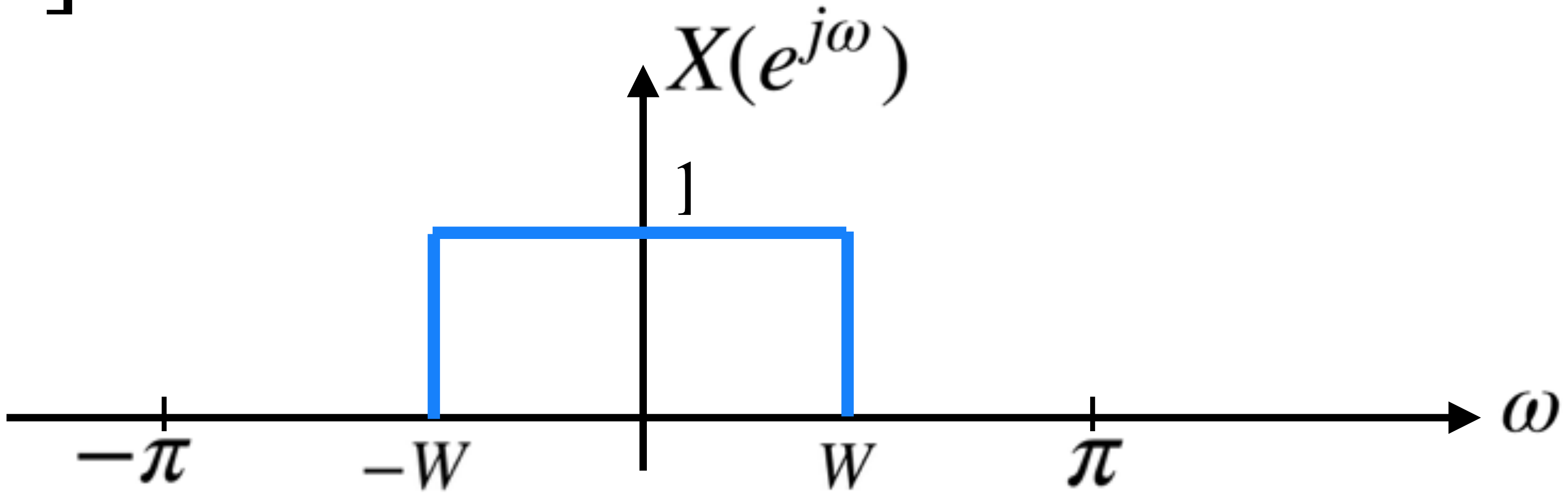
$$= \frac{\sin\left(\omega(N_1 + 1/2)\right)}{\sin(\omega/2)}$$

$$X(e^{j\omega}) = \frac{\sin\left(\omega(N_1 + 1/2)\right)}{\sin(\omega/2)}$$



Examples

- Find $x[n]$ whose DTFT is



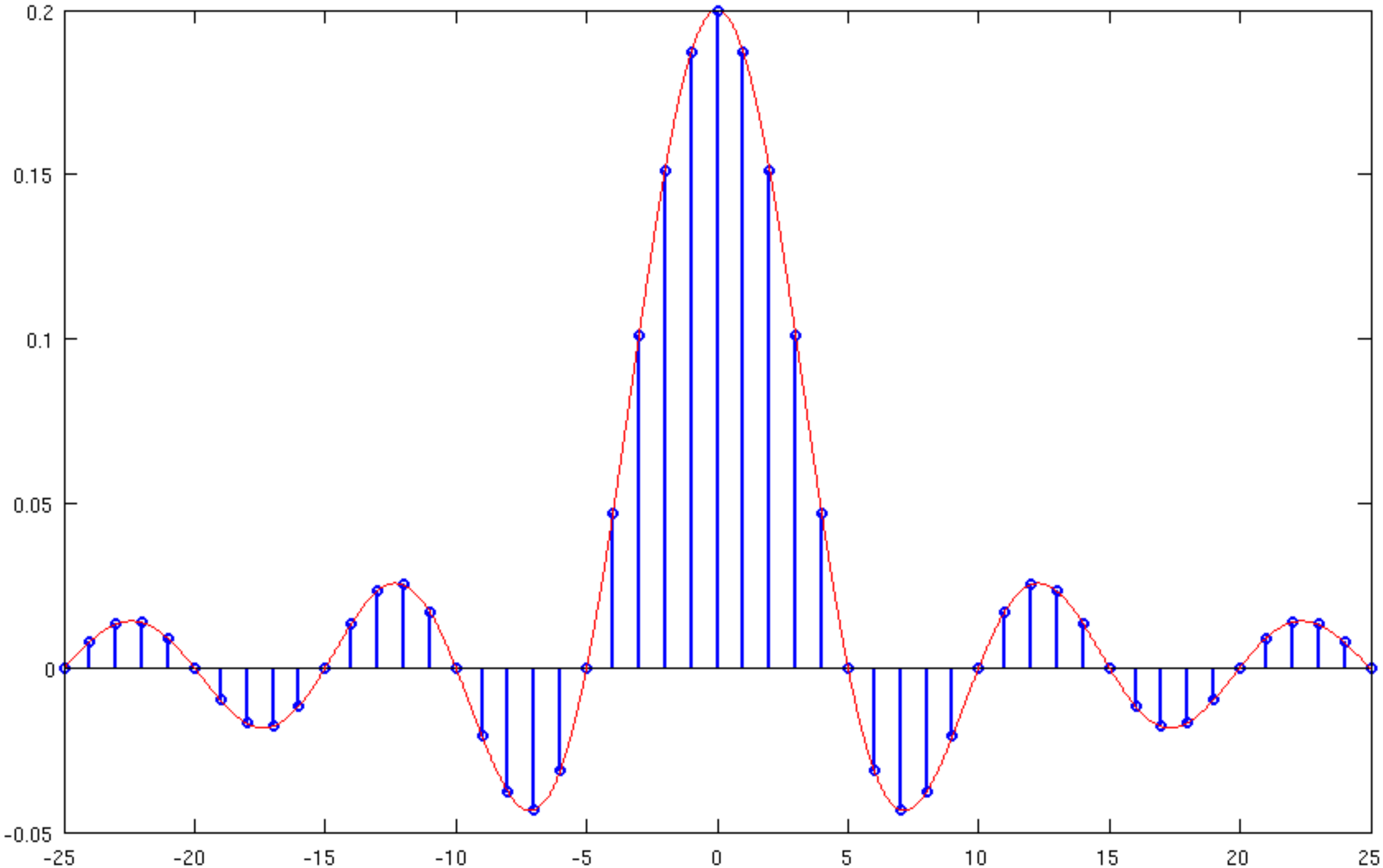
- <u>Solution:</u>

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \;= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}d\omega$$

$$n = 0 \Longrightarrow \quad = \frac{W}{\pi}$$

$$n \neq 0 \Longrightarrow \quad = \frac{e^{j\omega n}}{j2\pi n}\Bigg|_{-W}^{W} = \frac{e^{jWn} - e^{-jWn}}{j2\pi n} = \frac{\sin(Wn)}{\pi n}$$

For the case $W = \frac{\pi}{5}$



DTFT of periodic signals

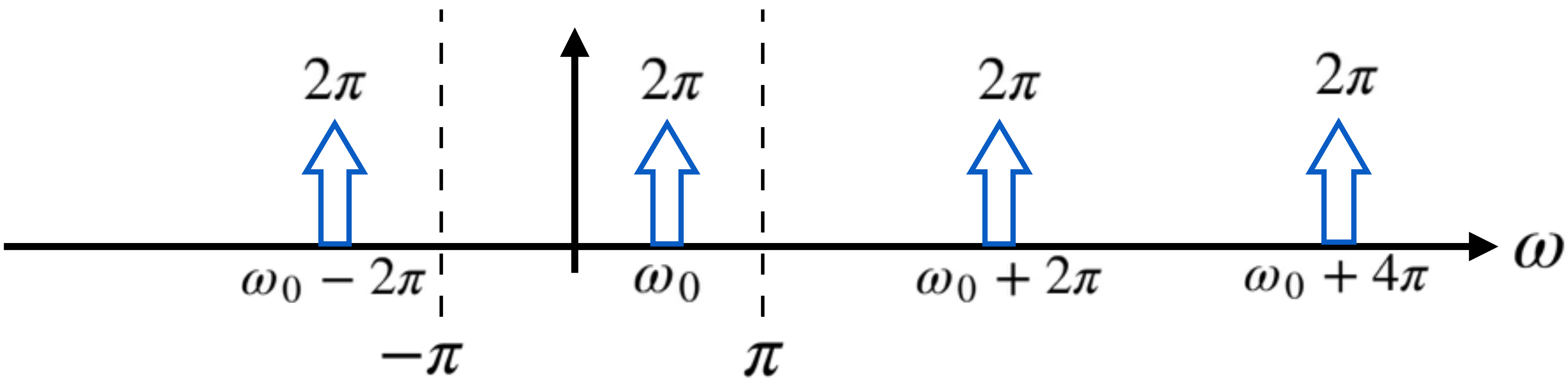
- Normally, DTFS suffices in decomposing onto complex exponentials.
- What if, just out of intellectual curiosity, we compute the DTFT of a periodic signal?
- Since

$$x[n] = \sum_{k \in \mathcal{N}} a_k e^{jk\omega_0 n}$$

and since DTFT is linear, it suffices to find the DTFT of $e^{jk\omega_0 n}$.

DTFT of periodic signals

- Claim: $e^{j\omega_0 n} \xrightarrow{DTFT} 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$



- Proof: Using the synthesis formula,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}\,d\omega \quad = e^{j\omega_0 n}$$

- Therefore,

$$x[n] = \sum_{k \in \mathcal{N}} a_k e^{jk\omega_0 n}$$

implies

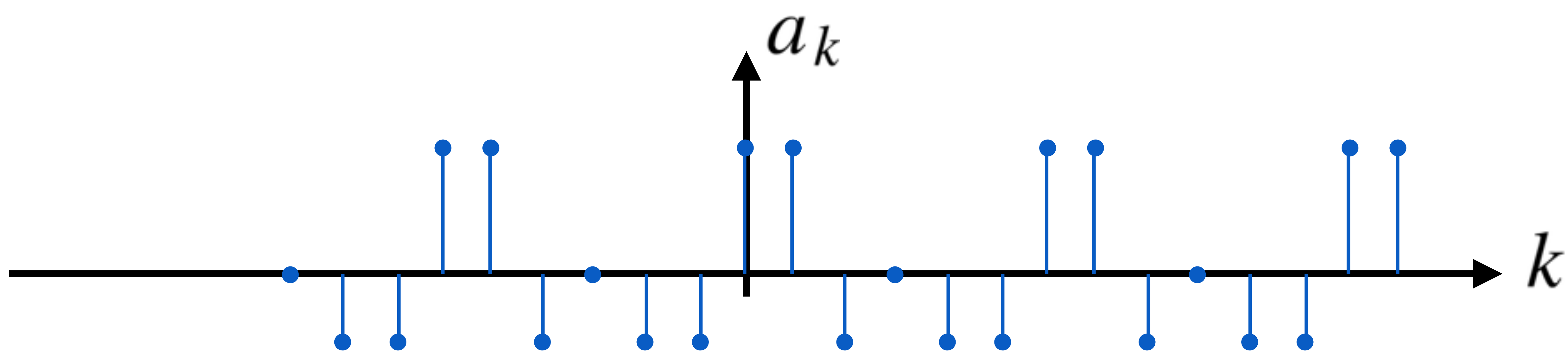
$$X(e^{j\omega}) = 2\pi \sum_{k \in \mathcal{N}} a_k \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$$

$$= 2\pi \sum_{k \in \mathcal{N}} a_k \sum_{m=-\infty}^{\infty} \delta\left(\omega - 2\pi\left(\frac{k}{N} + m\right)\right)$$

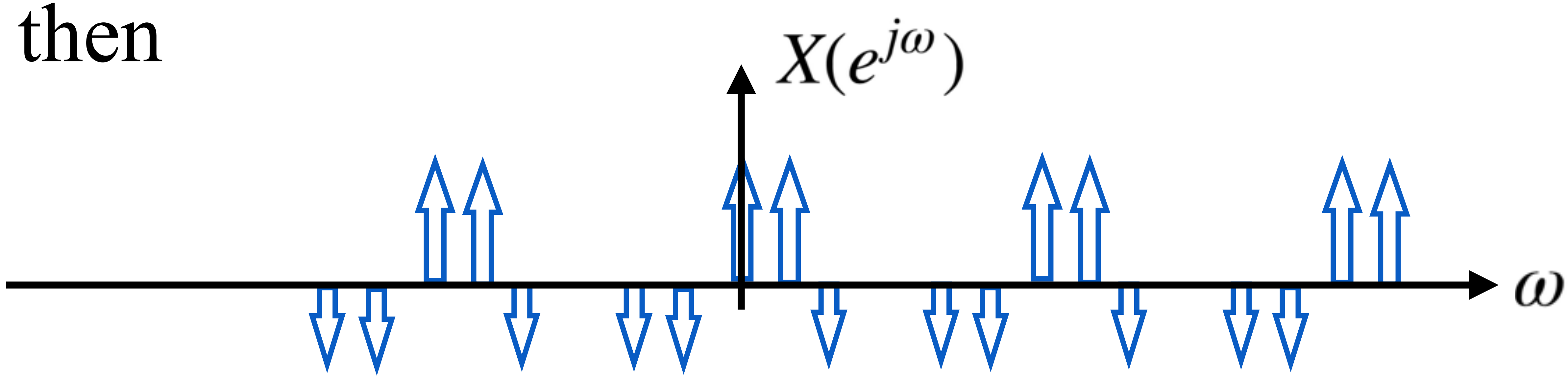
$$= 2\pi \sum_{l=-\infty}^{\infty} a_l \delta\left(\omega - \frac{2\pi l}{N}\right)$$

DTFT of periodic signals

- For example, if



then



Each impulse at $\omega = \dfrac{2\pi k}{N}$ is of amplitude $2\pi a_k$

DTFT of periodic signals

- Example: Find the DTFT of

$$x[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$

- Solution: Finding the DTFS first,

$$a_k = \frac{1}{N} \sum_{n \in \mathcal{N}} x[n] e^{-jk\omega_0 n} = \frac{1}{N}$$

we obtain

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{l=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi l}{N} \right)$$