Problem 1 [30pts]: Let a system be given as

$$y(t) = x(r(t))$$

where $r(\cdot)$ is the ramp function, i.e. $r(t) = tu(t)$.

a) Determine whether the system is linear.

b) Determine whether the system is time-invariant.

c) Determine whether the system is memoryless.

d) Determine whether the system is causal.

e) Determine whether the system is stable.

f) Show that the system is <u>not</u> invertible.

Hint: It will be helpful to draw the output for some arbitrary input.

Solution:

A close look into this system reveals a simpler formula:

$$y(t) = \begin{cases} x(0) & t < 0 \\ x(t) & t \geq 0 \end{cases}$$

because $r(t) = 0$ for $t < 0$ and $r(t) = t$ for $t \geq 0$. Now using this, we analyze the 6 properties:

a) <u>Linearity:</u> If for arbitrary inputs $x_1(t)$ and $x_2(t)$, the outputs are given as

$$y_1(t) = \begin{cases} x_1(0) & t < 0 \\ x_1(t) & t \geq 0 \end{cases} \qquad y_2(t) = \begin{cases} x_2(0) & t < 0 \\ x_2(t) & t \geq 0 \end{cases}$$

then for the input $ax_1(t) + bx_2(t)$, the output would be

$$\begin{cases} ax_1(0) + bx_2(0) & t < 0 \\ ax_1(t) + bx_2(t) & t \geq 0 \end{cases}$$

which can then be written as $ay_1(t) + by_2(t)$. Therefore, the system is **linear**.

b) Time invariance: The system is **time-varying** because

$$x_1(t) = u(t+1) \longrightarrow y_1(t) = 1 \quad \text{for all } t$$

whereas setting $t_0 = 2$,

$$x_2(t) = x_1(t-2) = u(t-1) \longrightarrow y_2(t) = u(t-1) \neq y_1(t-2) \, .$$

c) Memory: The system **has memory** since $y(-1) = x(0)$.

d) Causality: The system is **not causal** since $y(-1) = x(0)$.

e) Stability: The system is **stable**, because if $|x(t)| < B$ for all t, then $|y(t)| < B$ for all t as well.

f) Invertability: The system is **not invertible** because all of $x(t)$ for $t < 0$ is forgotten. In other words, it is easy to create 2 separate inputs giving the same output. For example, $x_1(t) = 1$ and $x_2(t) = u(t+1)$ would yield the same output $y(t) = 1$.

Problem 2 [30pts]: Consider impulse response shown in Fig. 1 (a) below. Answer the following questions.

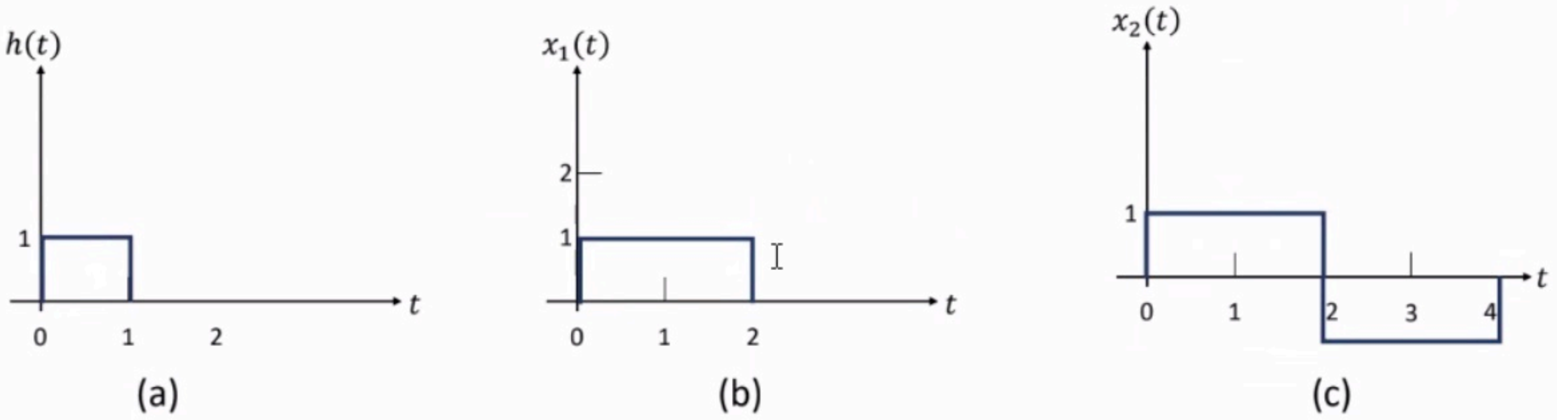

Figure 1: Plots for problem 3.

a) Evaluate and plot the convolution output $y_1(t) = x_1(t) * h(t)$, where $x_1(t)$ is shown in Fig. 1 (b).

b) Write down $x_2(t)$ in Fig. 1(c) in terms of $x_1(t)$.

c) Now evaluate and plot the convolution output $y_2(t) = x_2(t) * h(t)$, where $x_2(t)$ is shown in Fig. 1 (c).

Hint: Use the linearity and time invariance of convolution to simplify your solution to this problem.

Solution:

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a) Using flip and drag, we have five scenarios

$\underline{t < 0}$: $y_1(t) = 0$

$\underline{0 \le t < 1}$:

$$y_1(t) = \int_0^t 1dt$$
$$= t.$$

$\underline{1 \le t < 2}$:

$$y_1(t) = 1 \times 1 = 1.$$

$\underline{2 \le t < 3}$:

$$y_2(t) = \int_{t-1}^2 1dt$$
$$= 3 - t.$$

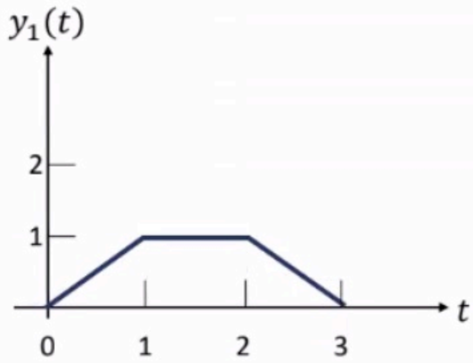

Figure 2: Solution for part (a).

$\underline{t \geq 3}$: $y_1(t) = 0$.

b) Notice that $x_2(t)$ can be constructed by combining $x_1(t)$ and its copy shifted right (advanced) by two units, and scaled by -1. By observation, $x_2(t) = x_1(t) - x_1(t-2)$.

c) Using LTI properties of convolution, $y2(t) = y_1(t) - y_1(t-2)$.

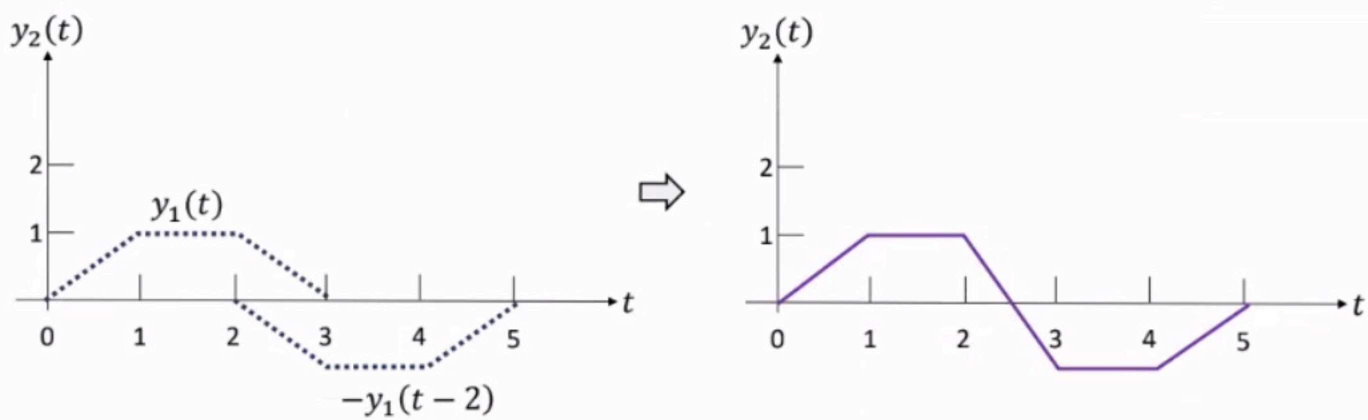

Figure 3: Solution for part (c).

Problem 3 [40pts]: Consider a causal LTI system given by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

Answer the following questions.

a) **[15 pts]** Draw the block diagram for this LTI system. Mark all parts clearly.

b) **[15 pts]** Evaluate the impulse response $h(t)$ of this system.

c) **[10 pts]** Draw the impulse response $h(t)$.

Solution:

a) We have,

$$y(t) = \frac{1}{2}x(t) - \frac{1}{2}\frac{dy(t)}{dt}.$$

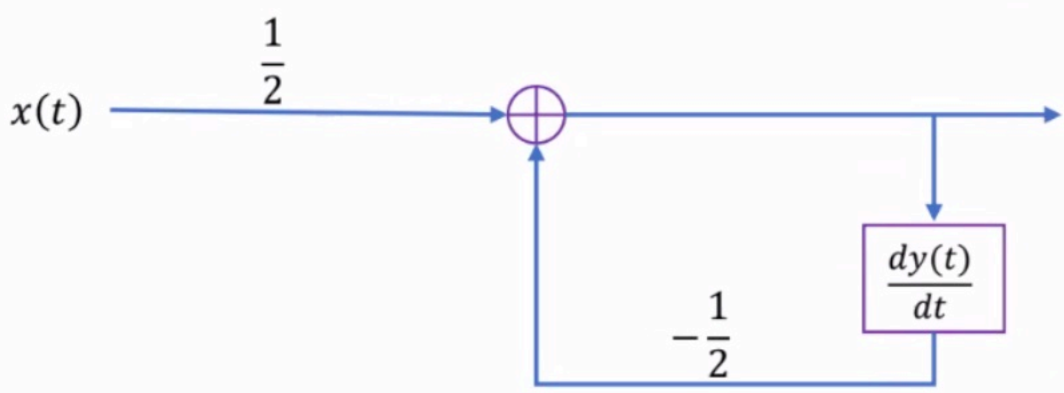

Figure 4: Solution for part (a).

Figure 4: Solution for part (a).

b) For impulse response, $x(t) = \delta(t)$ and $y(t) = h(t)$. Therefore,

$$\frac{dh(t)}{dt} + 2h(t) = \delta(t).$$

For some $t > 0$, we have,

$$\frac{dh(t)}{dt} + 2h(t) = 0.$$

Using the general solution, $h(t) = e^{\alpha t}$,

$$\alpha e^{\alpha t} + 2e^{\alpha t} = 0$$
$$\implies \alpha = -2.$$

Hence the solution is, $h(t) = c_0 e^{-2t}$. Since we have a $\delta(t)$ on the right hand side, $h(0^+)$ can be non-zero.

Integrating the equation, we get,

$$\int_{0^-}^{0^+} \frac{dh(t)}{dt} + 2\int_{0^-}^{0^+} h(t)dt = \int_{0^-}^{0^+} \delta(t)dt$$
$$\implies h(0^+) = 1$$
$$\implies c_0 = 1.$$

The complete solution is hence,

$$h(t) = e^{-2t}u(t).$$

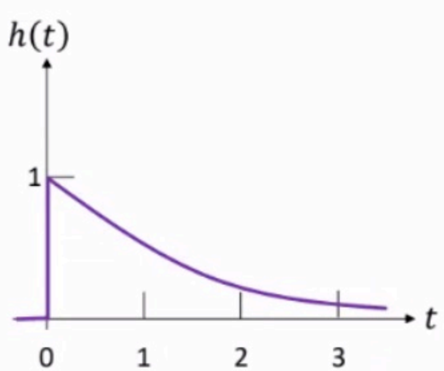

Figure 5: Solution for part (c).

c)