EE 110A Signals and Systems

Introduction to Signals and Systems

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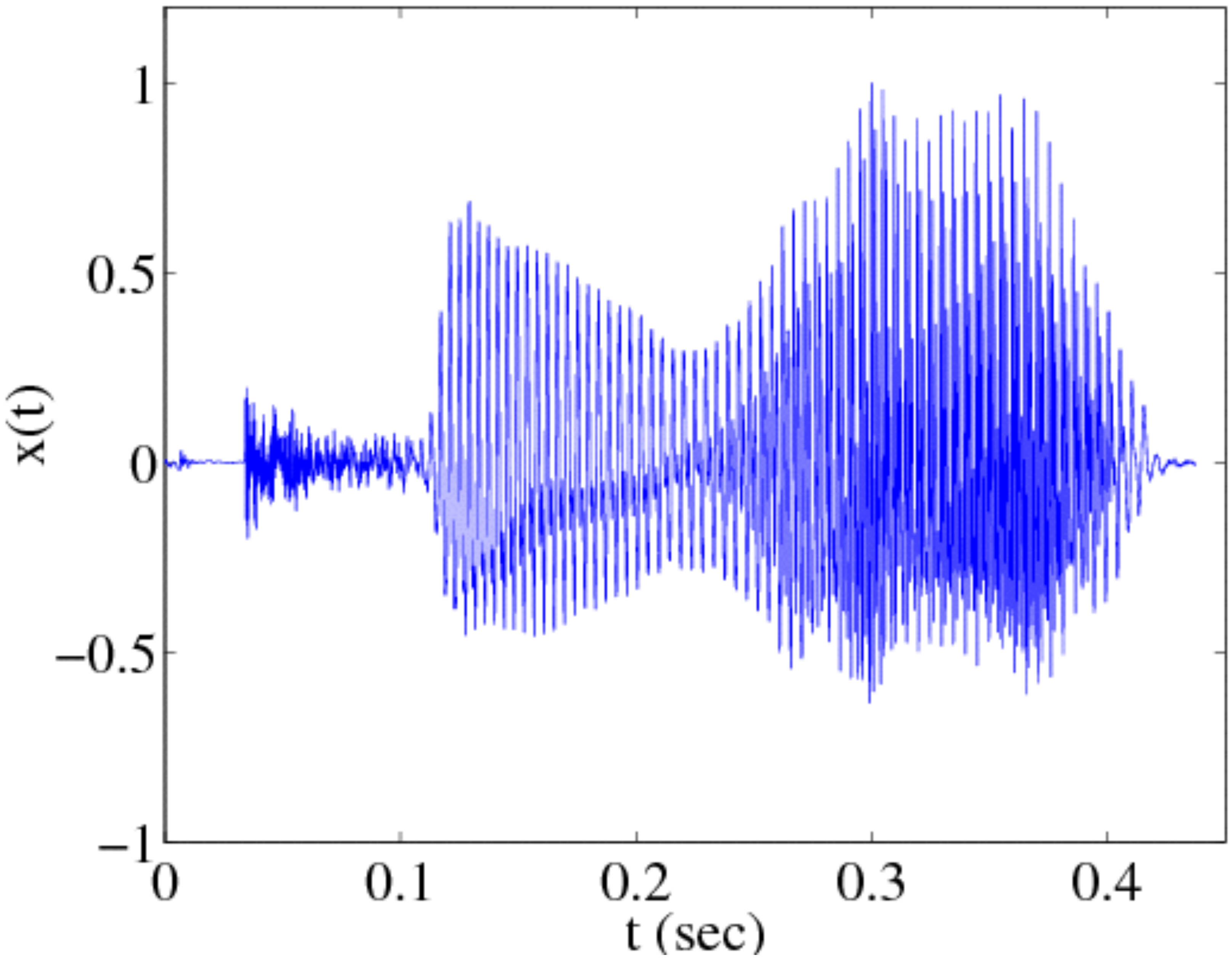
What is a signal?

- **Definition**: A signal is a function of one or more *independent variables*.
  - time (t)
  - space (x) or (x,y)
  - spatiotemporal (x,t) or (x,y,t)

Examples

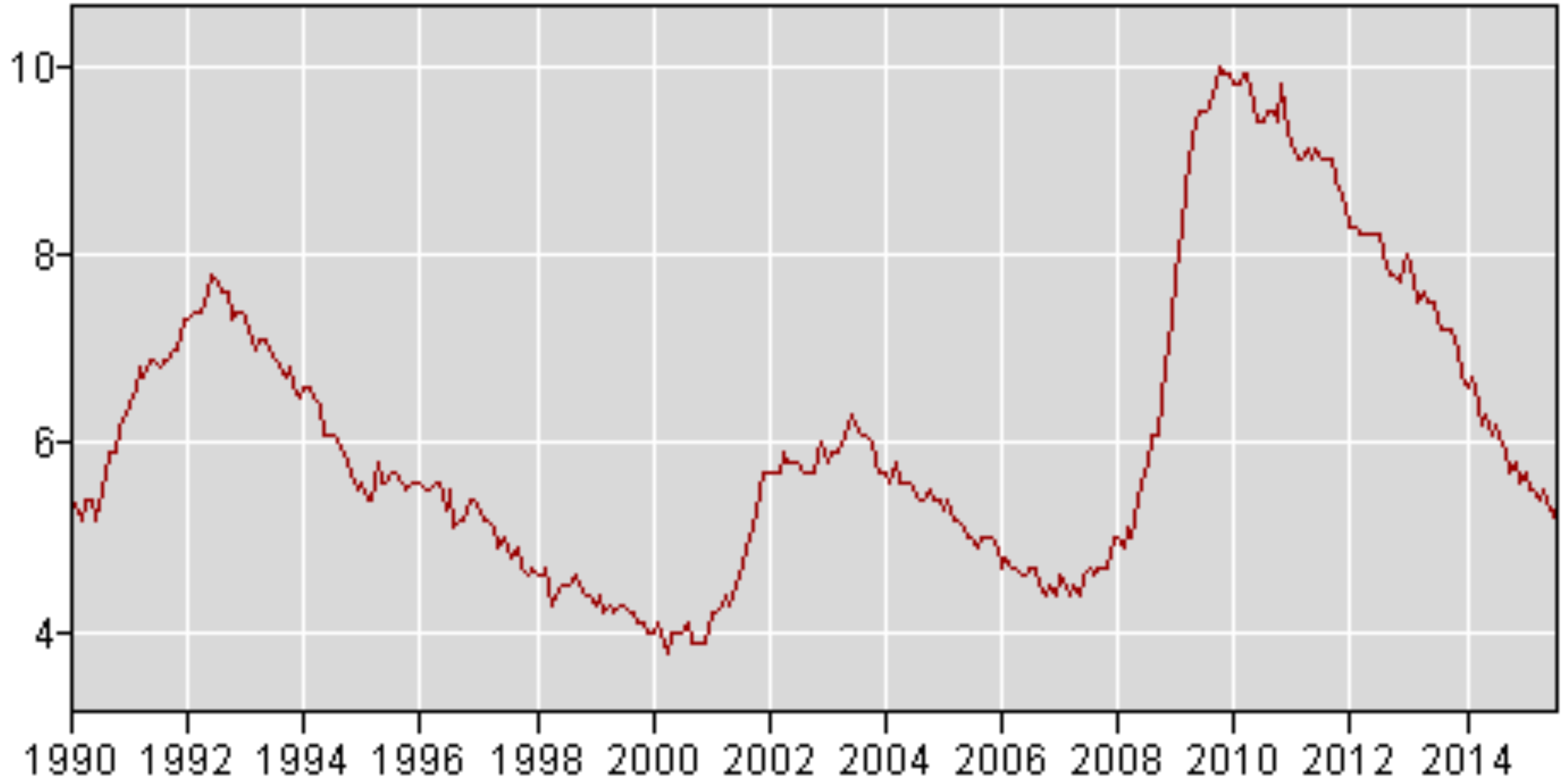
- **Speech signal.** Time is the independent variable



- Amplitude = acoustic pressure

Examples

- **Unemployment rate.** Also a time signal.



- Amplitude = % of unemployment among people over 16 years old.

Examples

- **Image signal.** Space variables (x,y)

- Amplitude = brightness

Examples

- **Video signal.** Space variables (x,y) **and** time variable t

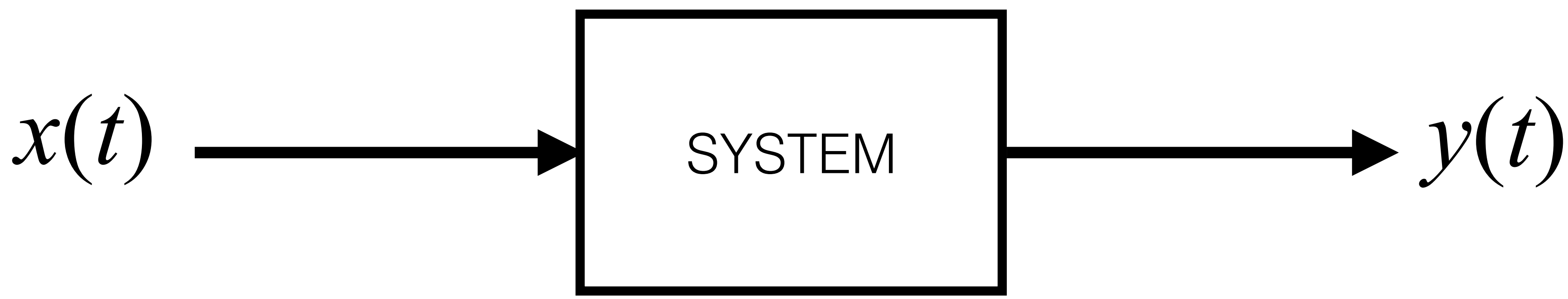
- Amplitude = brightness of RGB components

In this course

- We will focus on one-dimensional signals.
- Time will be the independent variable
  - Continuous-time signals (110A)
    - audio signals
    - voltage/current in a circuit with AC power
  - Discrete-time signals (110B)
    - unemployment rate
    - stock market data
    - sampled signals

What is a system?

- **Definition**: A system is a relationship between its input signal, typically $x(t)$, and its output signal, typically $y(t)$.

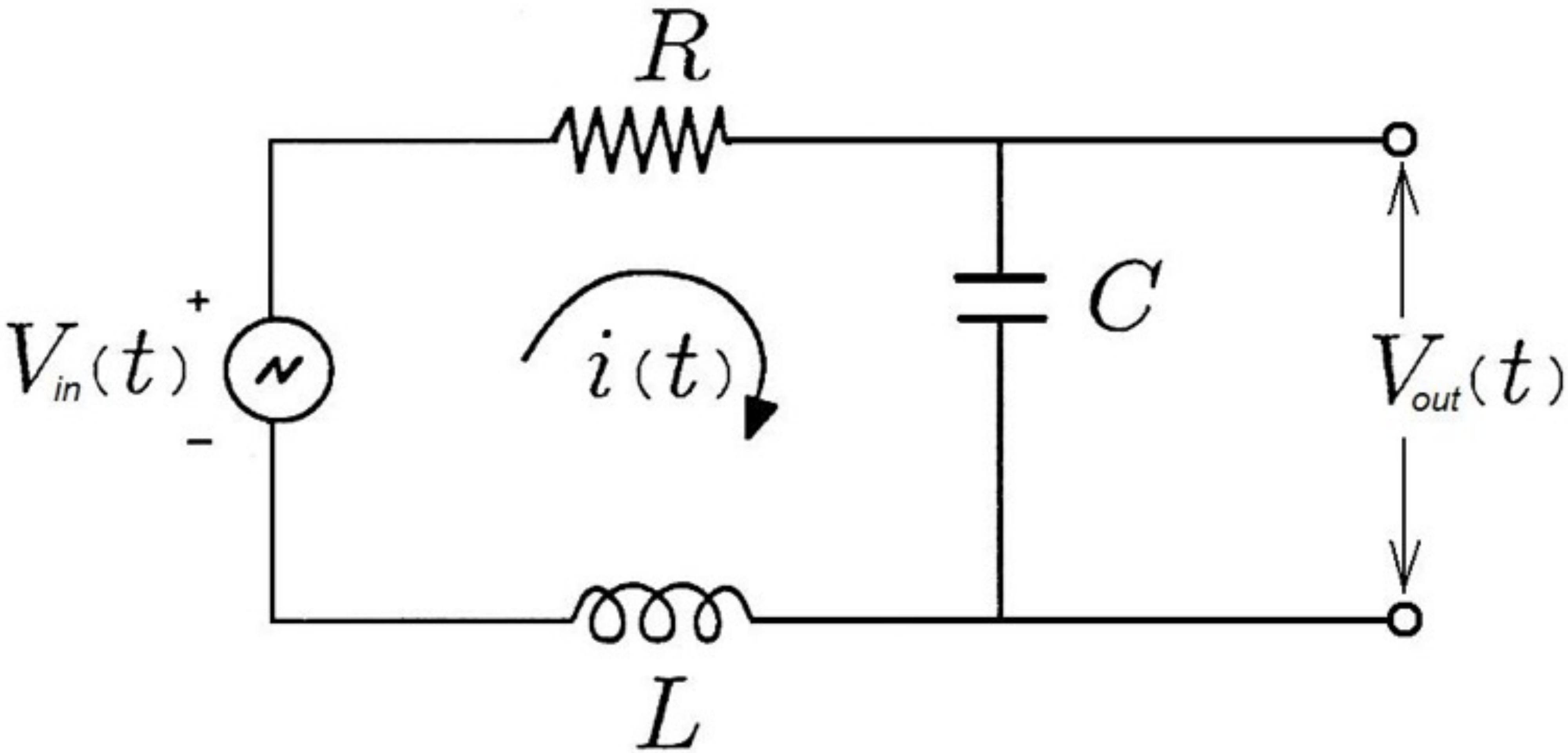


- Any legitimate relation between $x(t)$ and $y(t)$ forms a system.

Examples

- $x(t)$ = force applied on the car at time t
- $y(t)$ = displacement of the car at time t

Examples



- $V_{in}(t)$ = voltage applied on the circuit at time t
- $V_{out}(t)$ = voltage on the capacitor at time t

Examples

- $x(t)$ = solar radiation at time t
- $y(t)$ = temperature at a location at time t

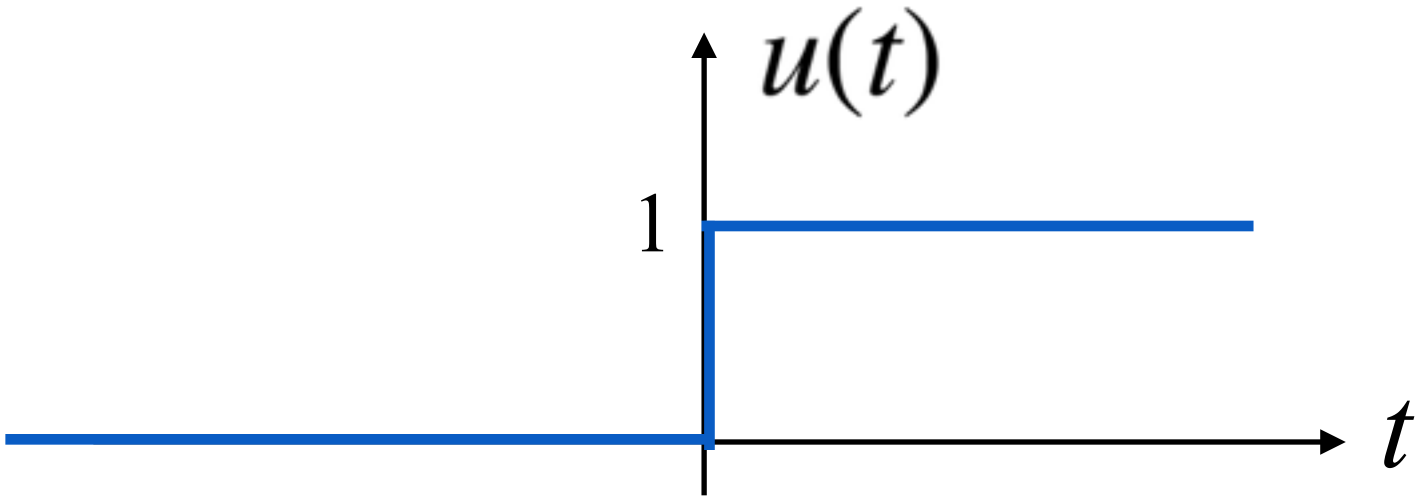
In this course

- We will study important properties of systems.
  - Memory
  - Causality
  - Linearity
  - Time-invariance
  - Invertibility
  - Stability
- The focus will be on linear time-invariant (LTI) systems
  - Continuous-time input and output in EE110A
  - Discrete-time input and output in EE110B

Some important signals

- **The unit step signal:**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



A very simple signal.

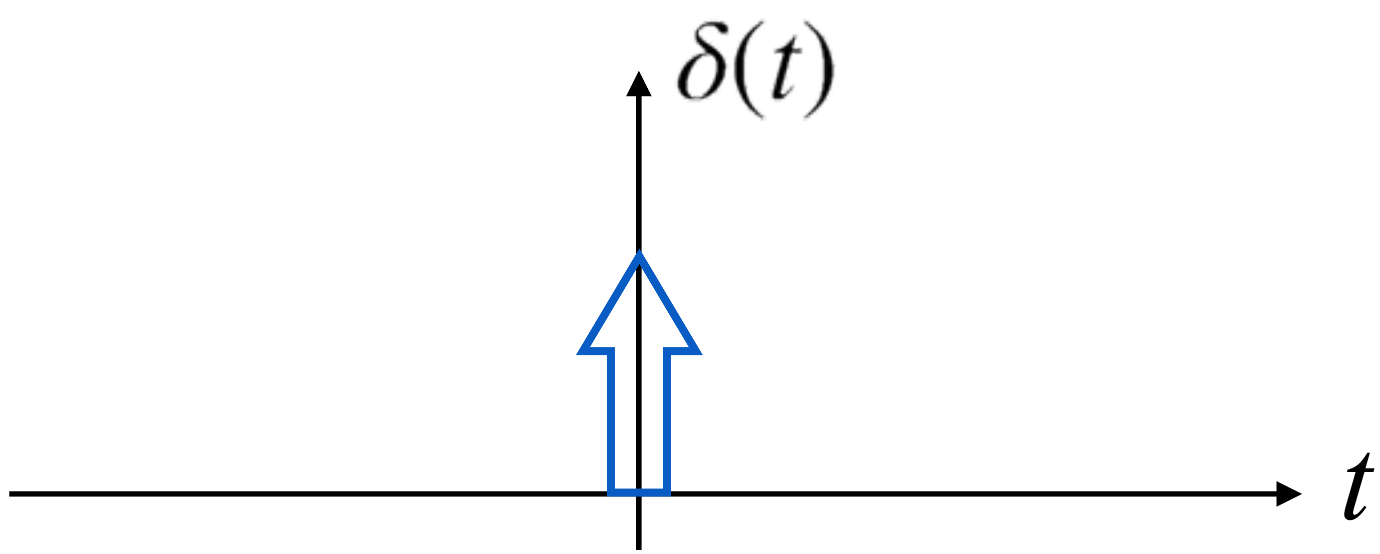
Some important signals

- **The impulse signal:**

What if we took the derivative of $u(t)$?

$$\delta(t) = \frac{du(t)}{dt} = \begin{cases} 0 & t > 0 \\ 0 & t < 0 \\ ? & t = 0 \end{cases} = \begin{cases} 0 & t > 0 \\ 0 & t < 0 \\ \infty & t = 0 \end{cases}$$

At $t = 0$, it seems that $u(t)$ has infinite slope.



Some important signals

- **The impulse signal:**

  For consistency, though, we must remember

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

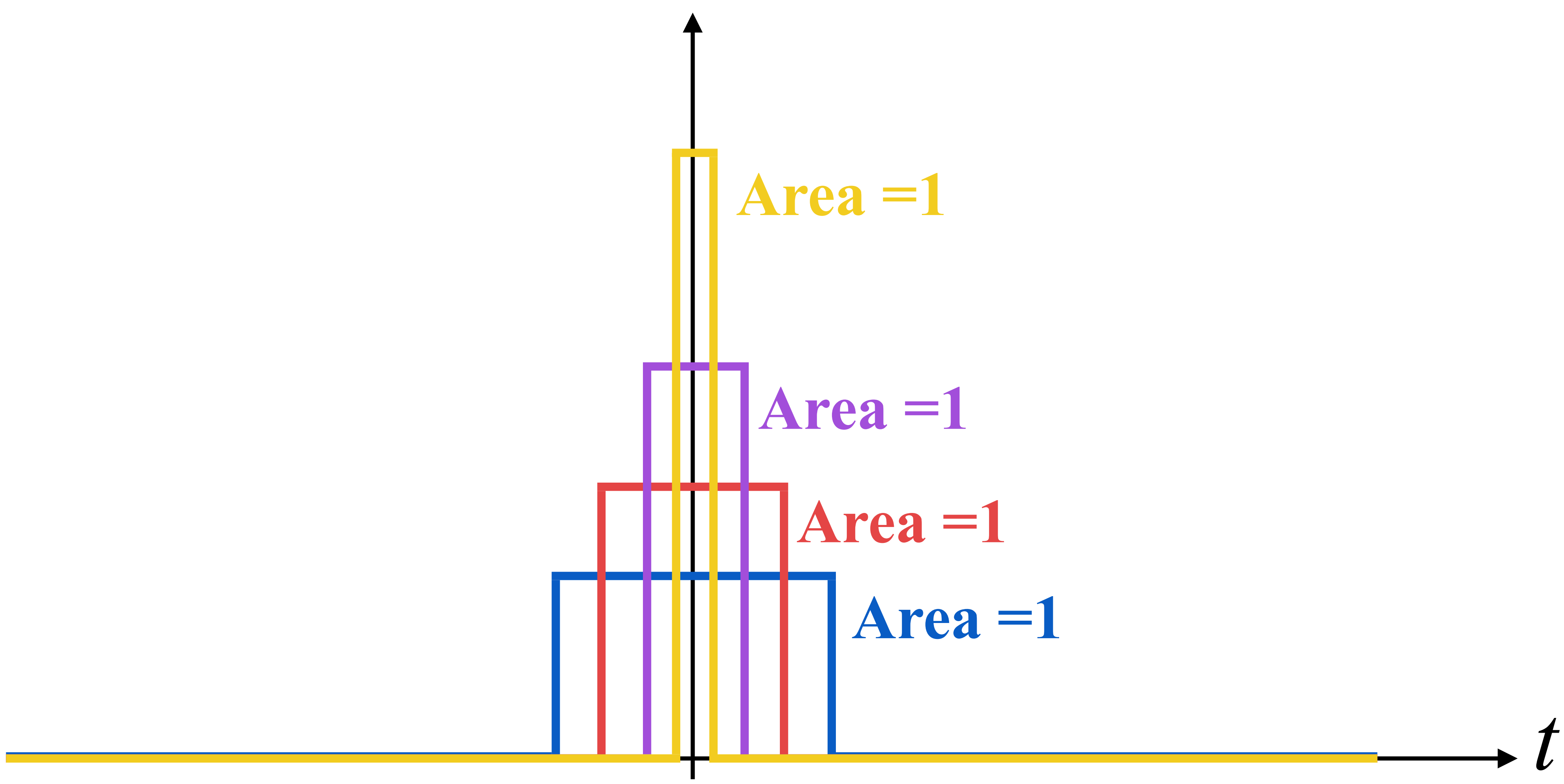
  What a strange function!!

  - integrate it from $-\infty$ to -0.000000001 and you get zero
  - integrate it from $-\infty$ to $+0.000000001$ and you get one

Some important signals

- **The impulse signal:**

  To keep this consistency, one can think of the impulse function as a limit.

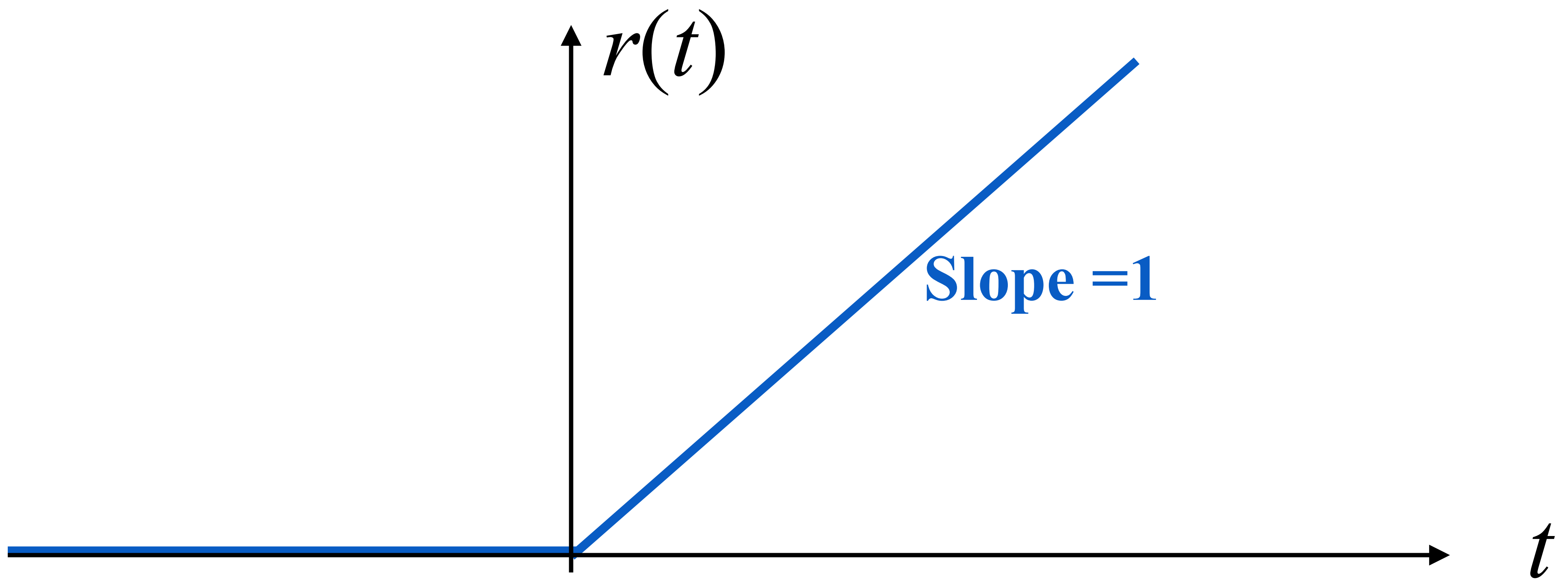


Some important signals

- **The unit ramp signal:**

  What if we integrate $u(t)$?

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$



Some important signals

- **The unit ramp signal:**

  Conversely, we also have

$$u(t) = \frac{dr(t)}{dt}$$

- **Summary:**

$$\delta(t) \underset{\text{differentiate}}{\overset{\text{integrate}}{\rightleftarrows}} u(t) \underset{\text{differentiate}}{\overset{\text{integrate}}{\rightleftarrows}} r(t)$$

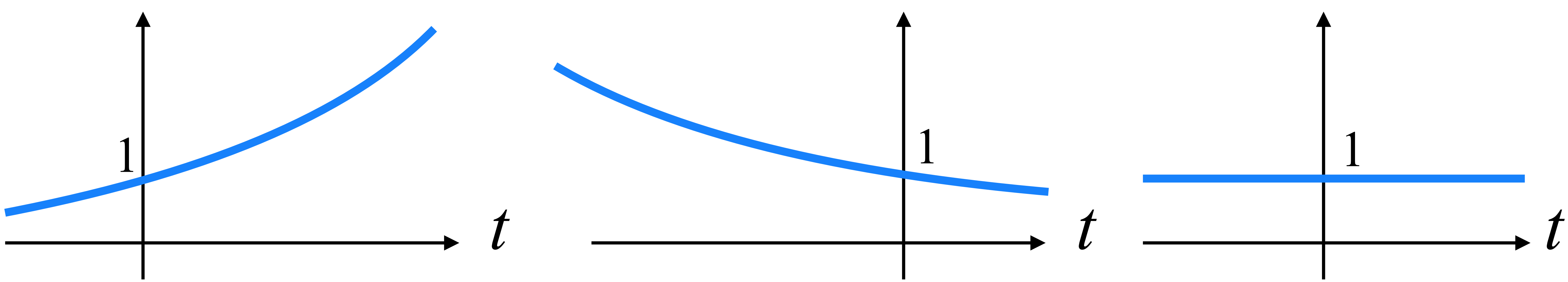
Some important signals

- **The exponential signal:**

$$x(t) = e^{\alpha t}$$

- Three cases:

$\alpha > 0$ $\alpha < 0$ $\alpha = 0$

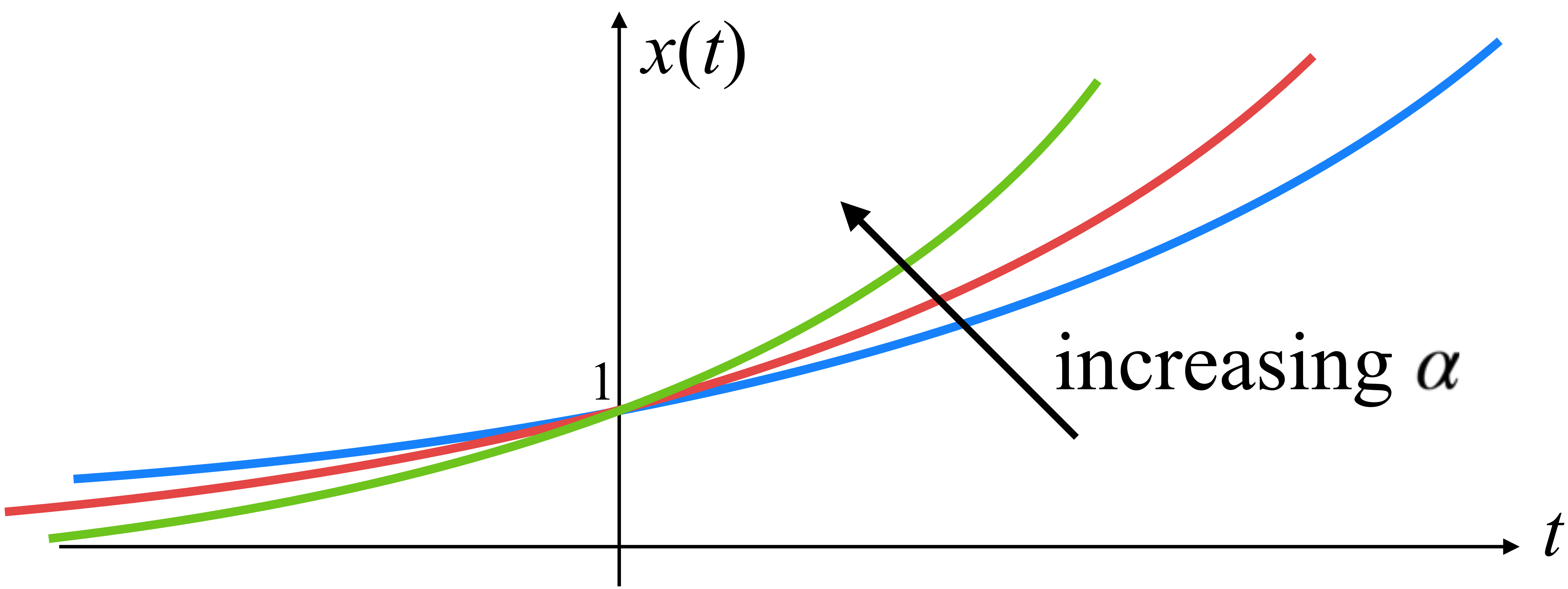


Some important signals

- **The exponential signal:**

$$x(t) = e^{\alpha t}$$

- As (positive) α changes,

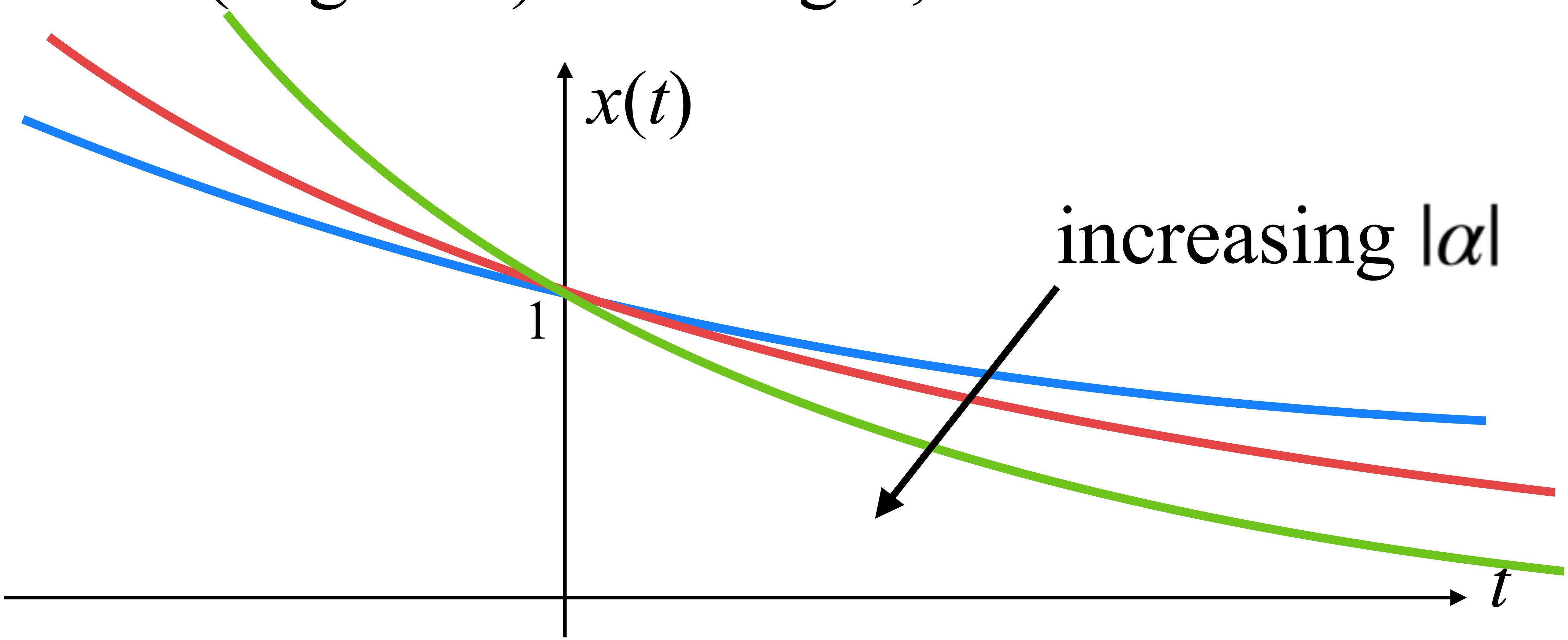


Some important signals

- **The exponential signal:**

$$x(t) = e^{\alpha t}$$

- As (negative) α changes,

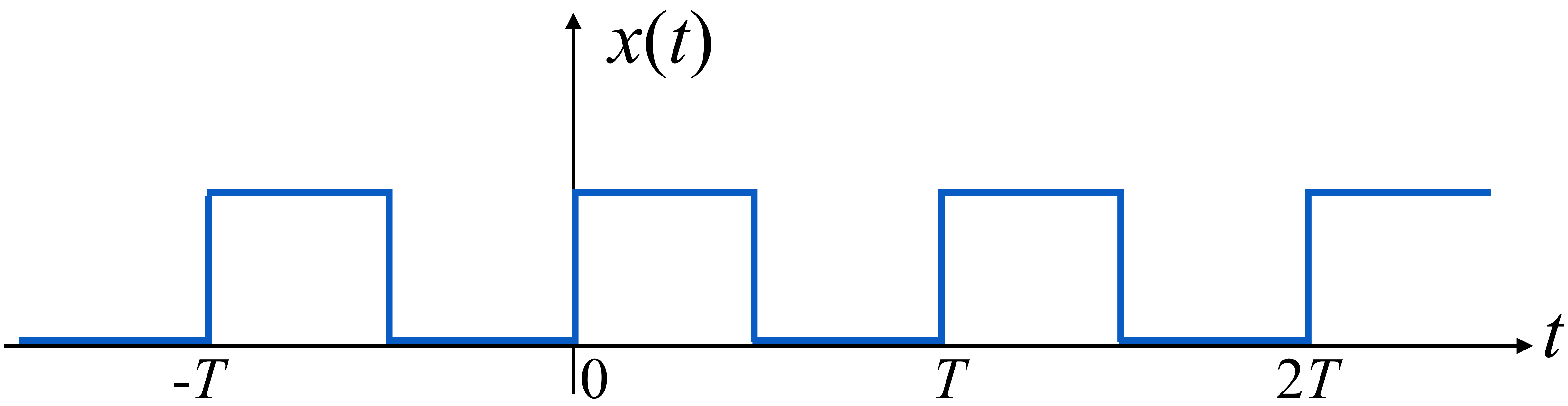


Periodic signals

- A signal is said to have a period T if

$$x(t+T) = x(t) \qquad \forall t$$

- Example: square wave



- If T is a period, so are $2T$, $3T$, $4T$, ...

Periodic signals

- An important class of examples is *sinusoidals*:

$$x(t) = A\cos(\Omega t + \phi)$$

  - A is called the *amplitude*
  - Ω is called the *frequency*
  - ϕ is called the *phase*

- Convention: $A > 0, \quad \Omega \geq 0, \quad -\pi \leq \phi \leq \pi$

Periodic signals

$$x(t) = A\cos(\Omega t + \phi)$$

- To find T, we need to solve

$$A\cos(\Omega t + \phi) = A\cos(\Omega(t + T) + \phi)$$

- Towards that end, we need to use the identity

$$\cos(\theta) = \cos(\theta + 2\pi k)$$

which is true for any integer k.

- In other words, T must satisfy $\Omega T = 2\pi k$ for some k.
- Smallest such T: $\boxed{T = \dfrac{2\pi}{\Omega}}$

Periodic signals

- Another way to understand this behavior is to look at complex exponentials

$$x(t) = Ae^{j(\Omega t+\phi)}$$

- For this signal to have a period T, we need

$$Ae^{j(\Omega t+\phi)} = Ae^{j(\Omega(t+T)+\phi)} = Ae^{j(\Omega t+\phi)}e^{j\Omega T}$$

implying that

$$e^{j\Omega T} = 1$$

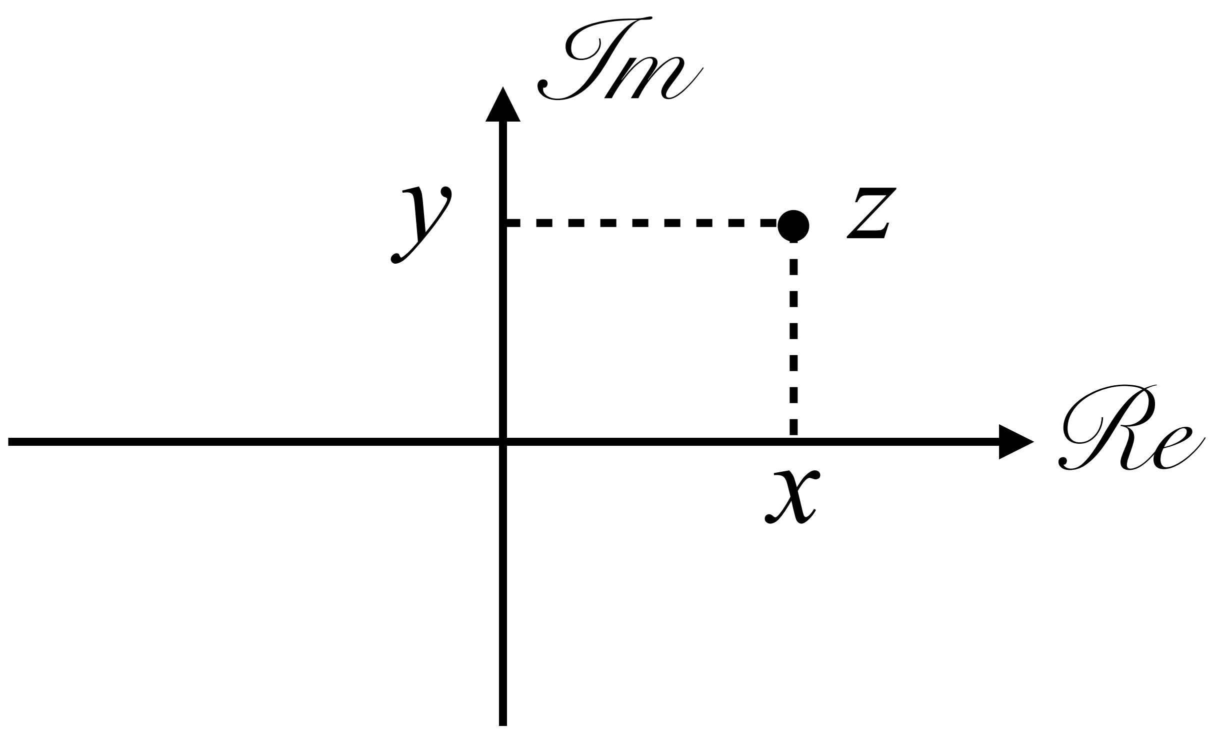
or that

$$T = \frac{2\pi k}{\Omega}$$

Digression: Complex algebra

- In rectangular coordinates,

$$z = \underbrace{x}_{\text{Real}} + j\underbrace{y}_{\text{Imaginary}}$$



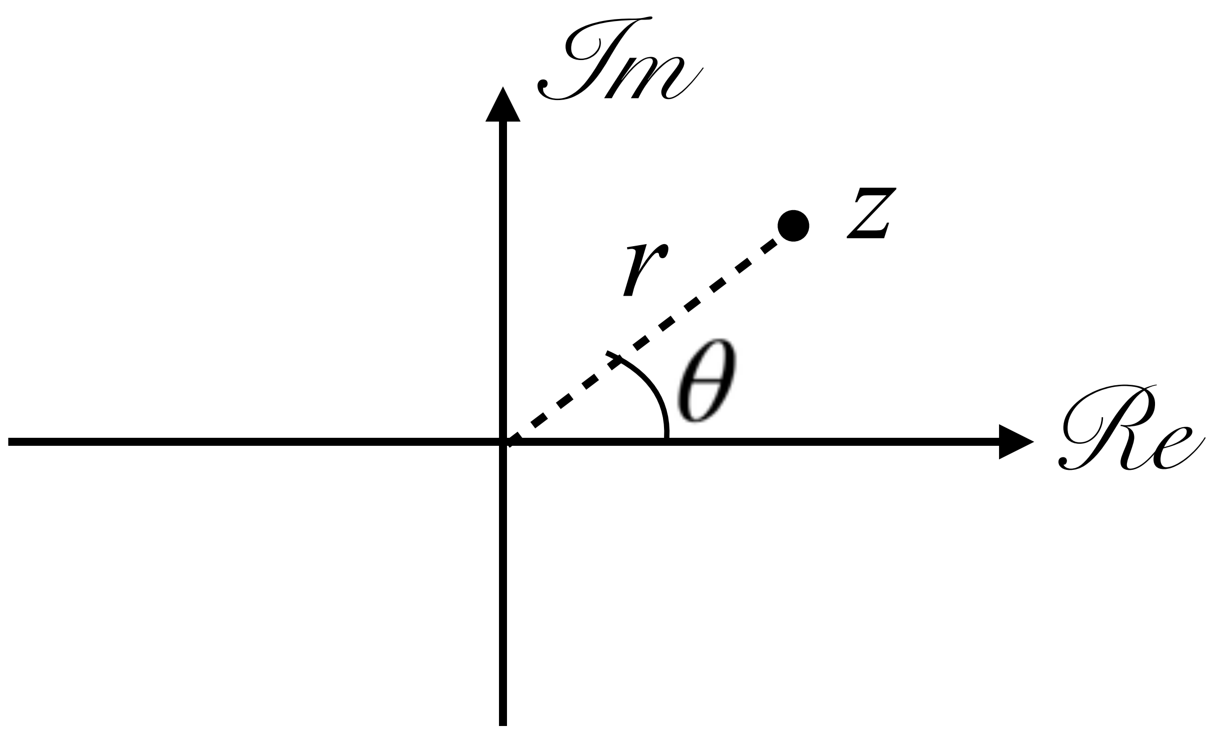
Digression: Complex algebra

- In polar coordinates,

$$z = re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

Angle

Magnitude



Digression: Complex algebra

- In rectangular coordinates,

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$

$$= (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 \times z_2 = (x_1 + jy_1) \times (x_2 + jy_2)$$

$$= x_1 x_2 + j(x_1 y_2 + x_2 y_1) + j^2 y_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

Digression: Complex algebra

- In polar coordinates,

$$z_1 \times z_2 = r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= r_1(\cos\theta_1 + j\sin\theta_1) \\ &\quad + r_2(\cos\theta_2 + j\sin\theta_2) \\ &= (r_1\cos\theta_1 + r_2\cos\theta_2) \\ &\quad + j(r_1\sin\theta_1 + r_2\sin\theta_2) \end{aligned}$$

Digression: Complex algebra

- Two important identities:

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos(\theta) \qquad \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin(\theta)$$

- Proof:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta)$$

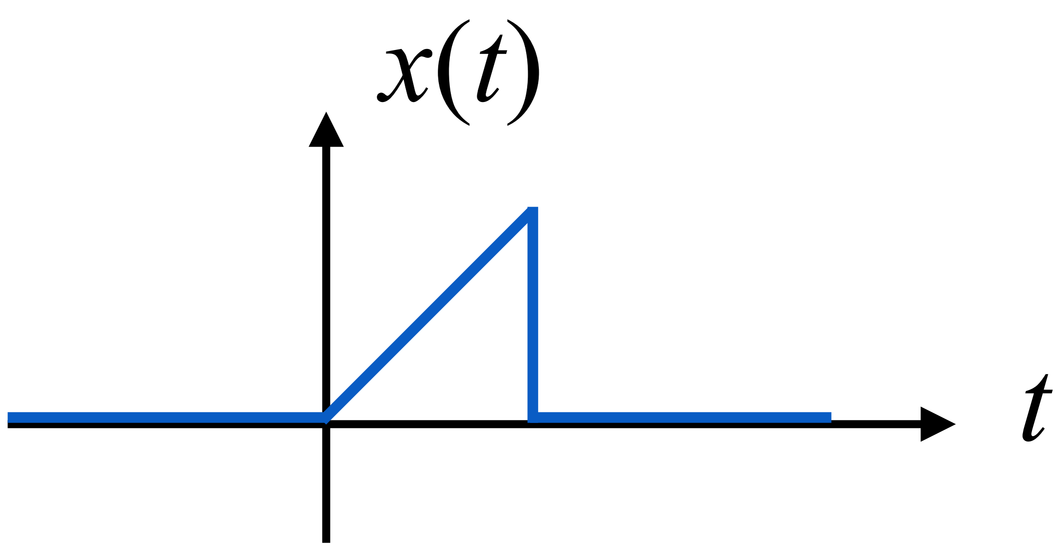
- Add and subtract the two to find the desired result.

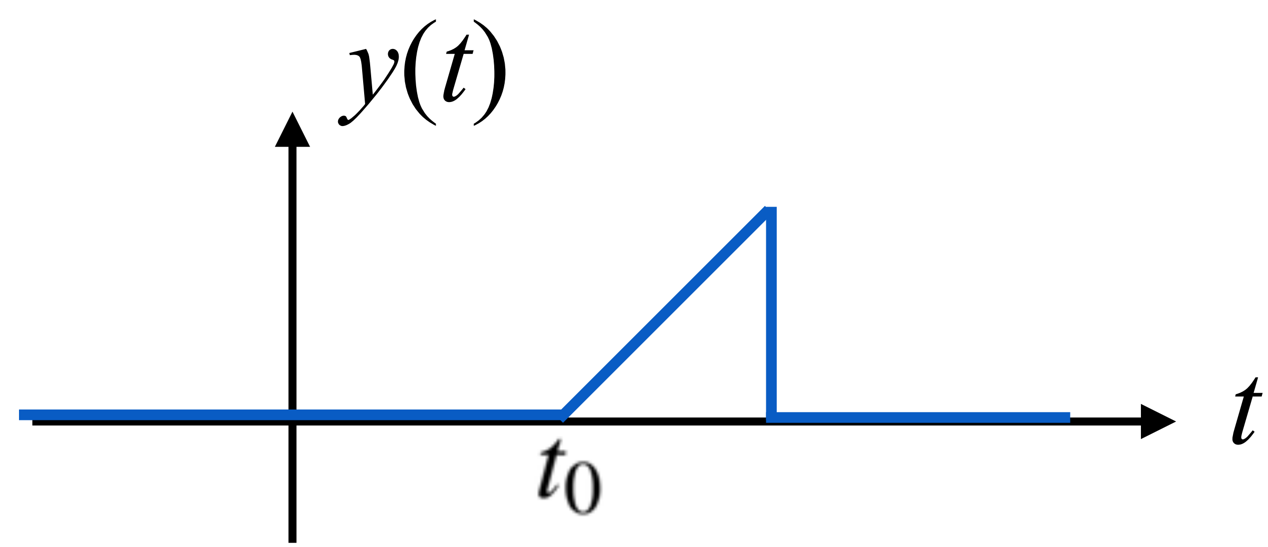
Simple signal transformations

- **Time shift:** Let

$$y(t) = x(t - t_0)$$

for some $t_0 > 0$.



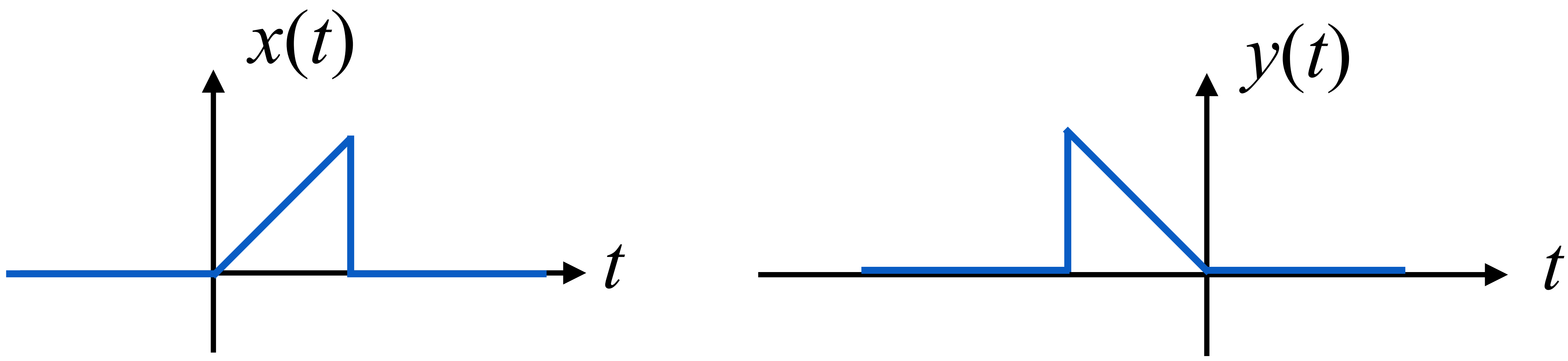


- Why does this cause a right shift?
- The key is to see that the signal y copies at time instant t the "old value" of x at $t - t_0$

Simple signal transformations

- **Time reversal:** Let

$$y(t) = x(-t)$$



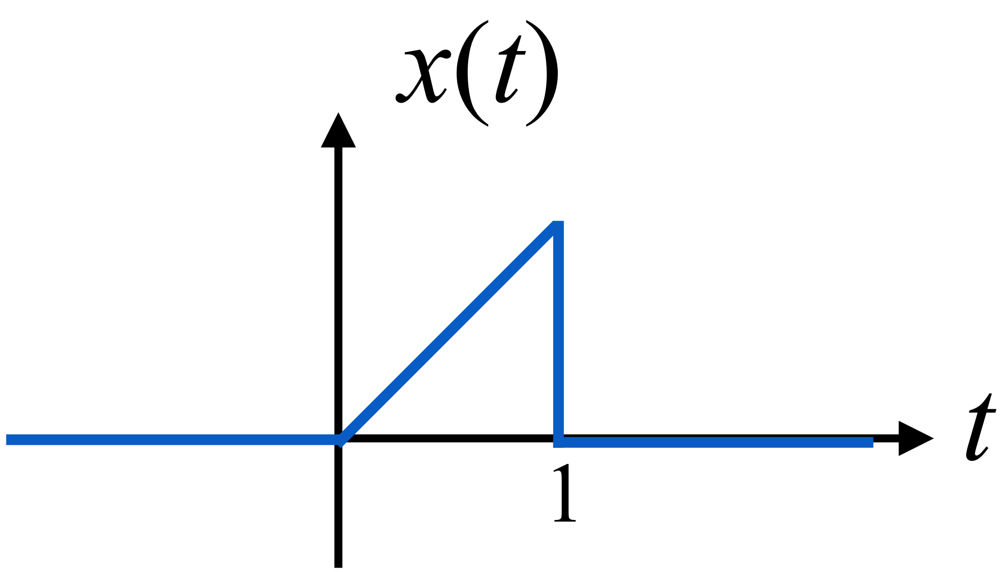
- So this is resulting in the "mirror image" of the signal around the y-axis.

Simple signal transformations

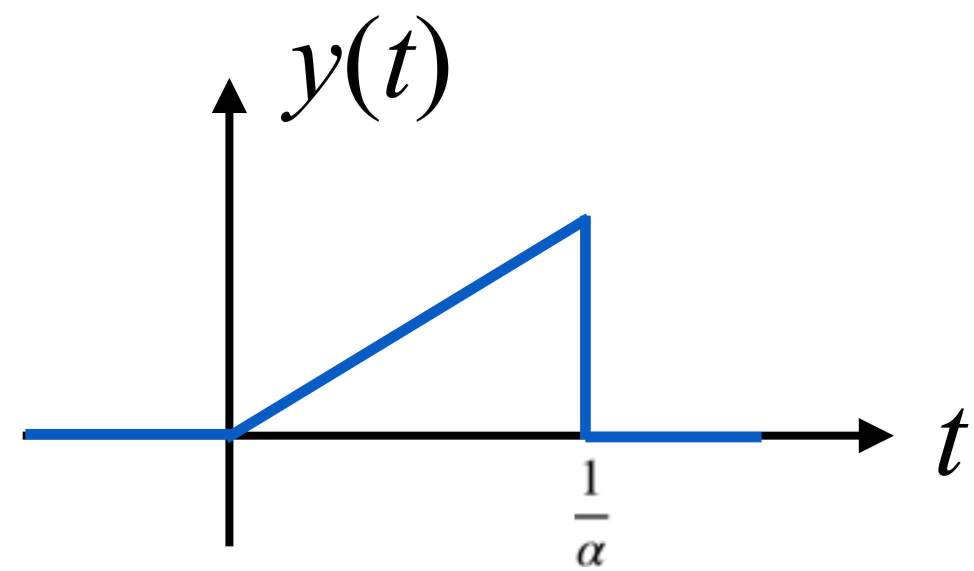
- **Scaling:**

$$y(t) = x(\alpha t)$$

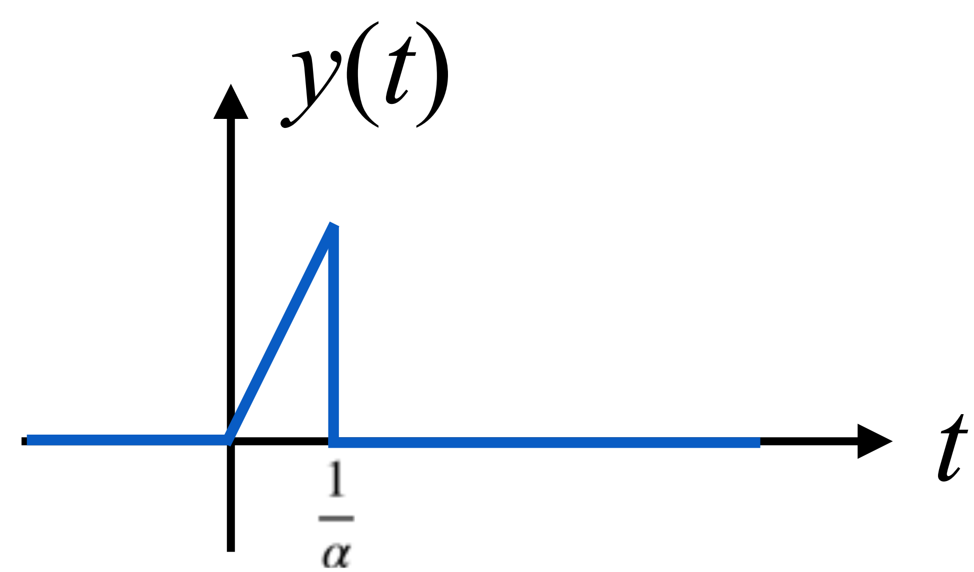
for some $\alpha > 0$.



$\alpha < 1$



$\alpha > 1$

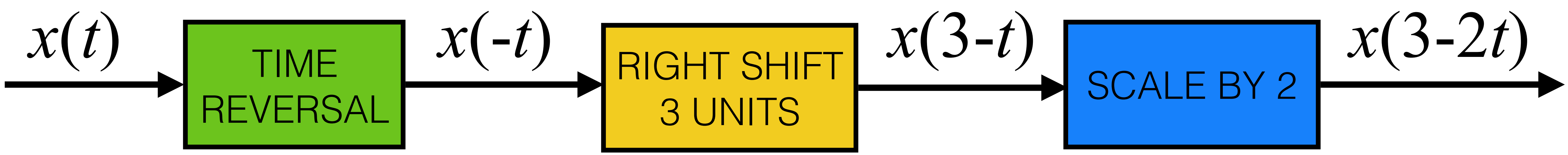


Transformation combos

- What if we have a transformation such as

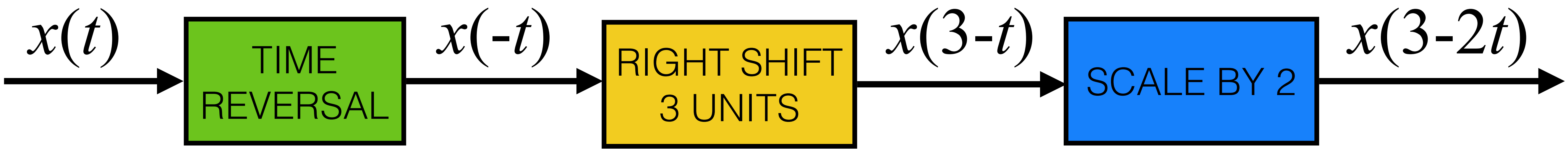
$$y(t) = x(3 - 2t) \ ?$$

- Looks like a combo of time shift, time reversal, and scaling. But with what order?
- Option 1:



- Option 2:

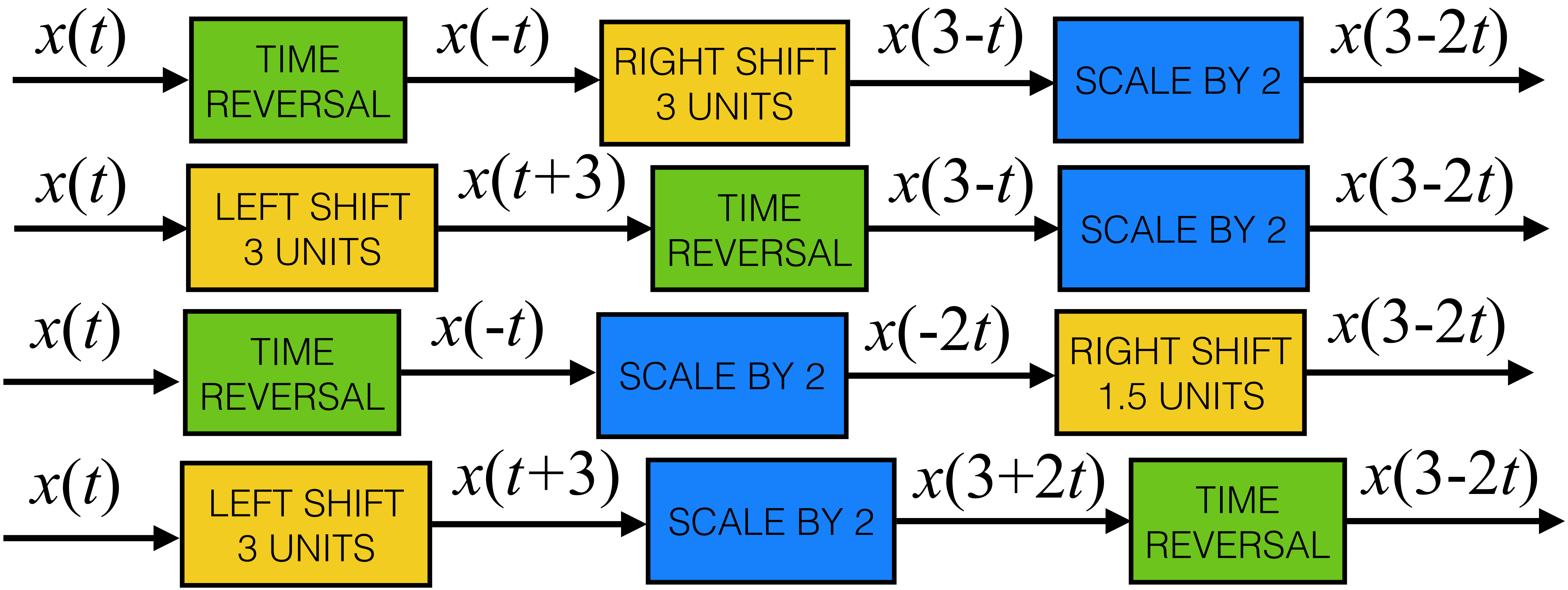
Transformation combos



- Option 3:

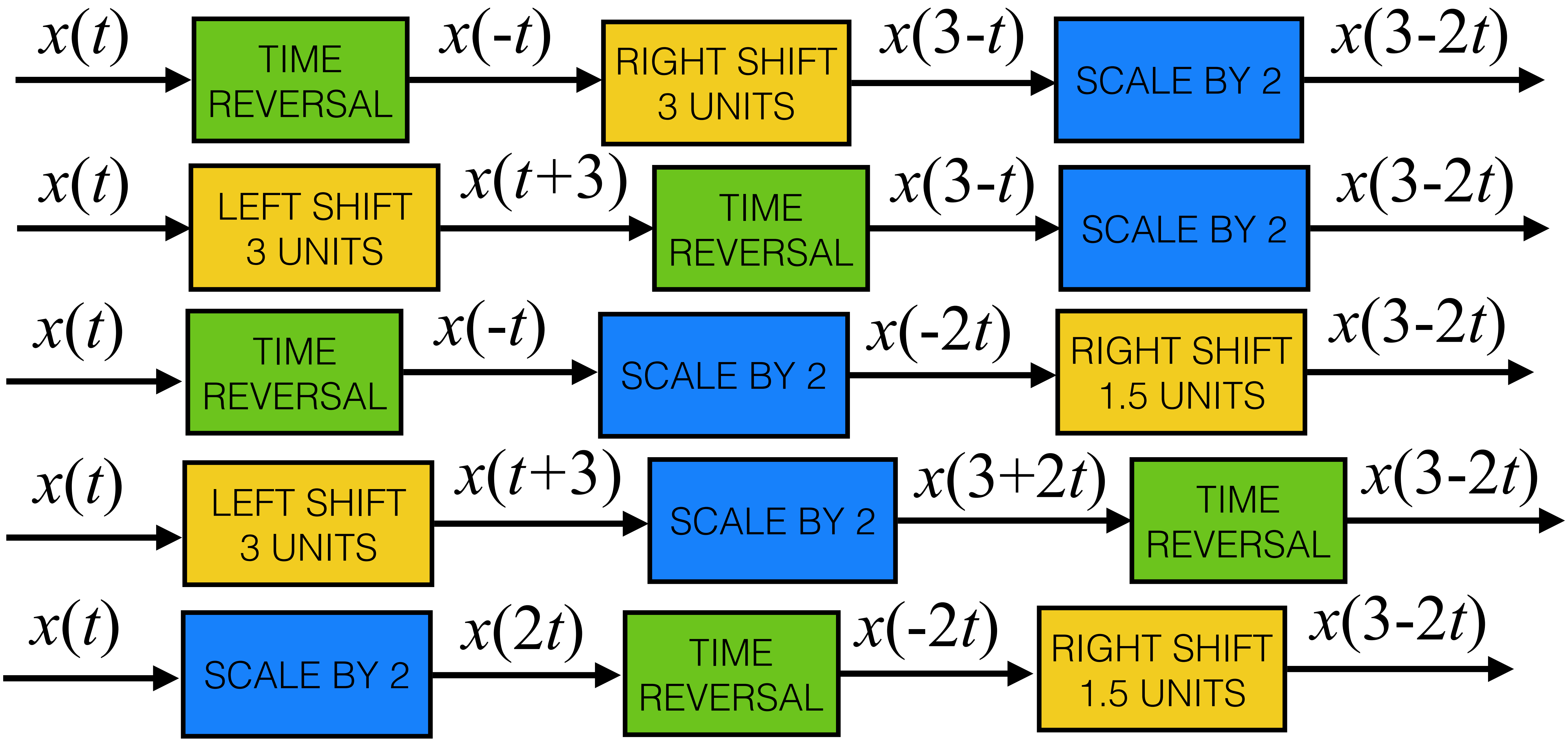
- Option 4:

Transformation combos

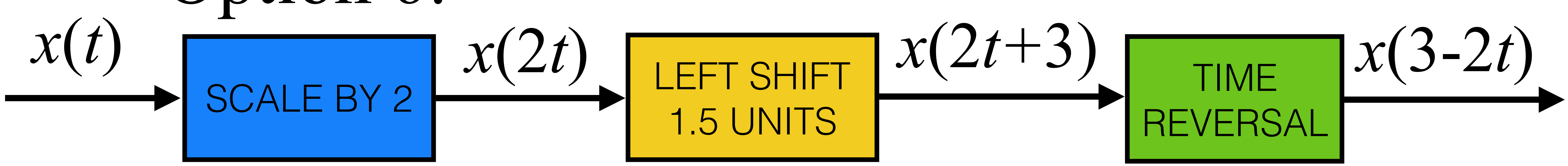


- Option 5:

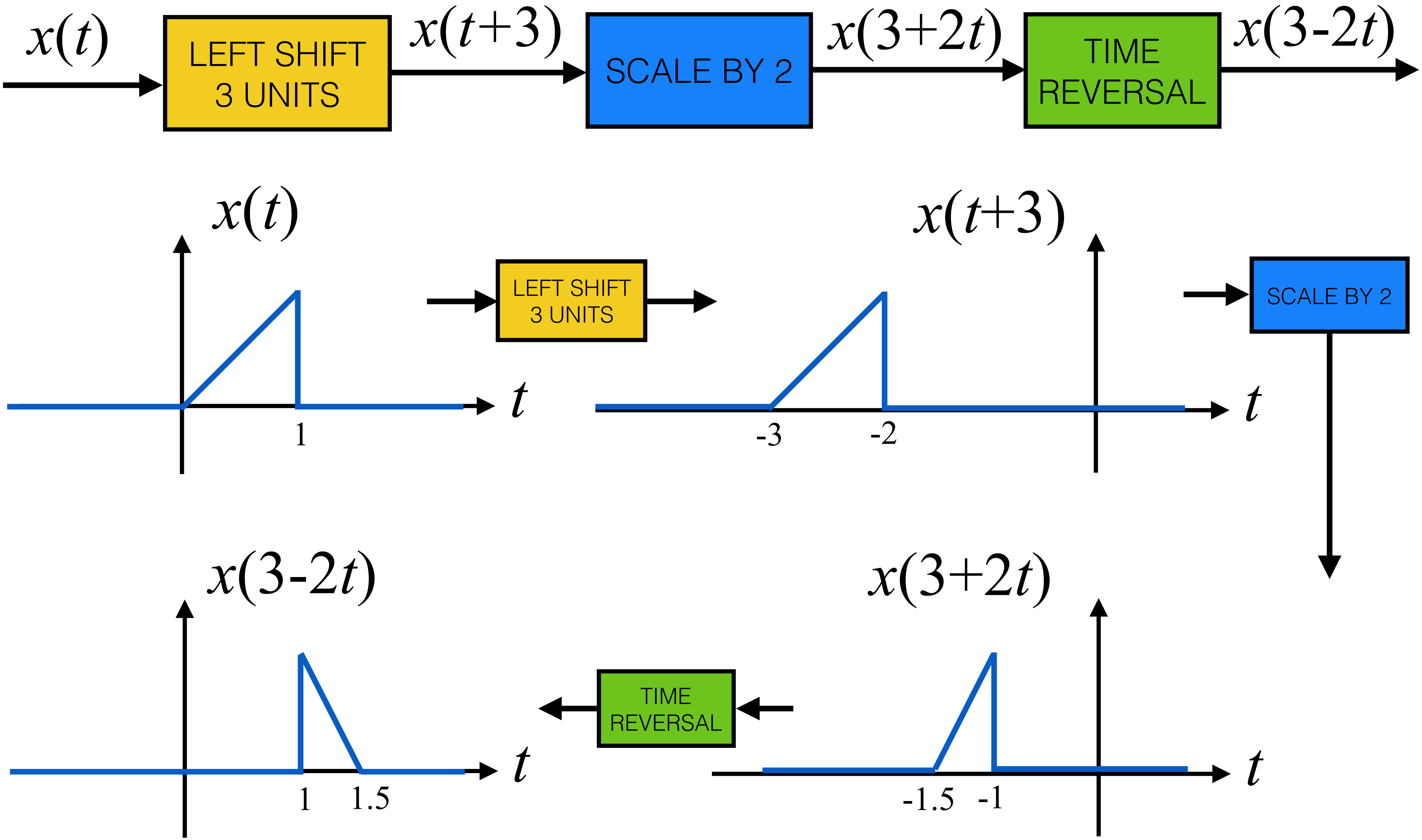
Transformation combos



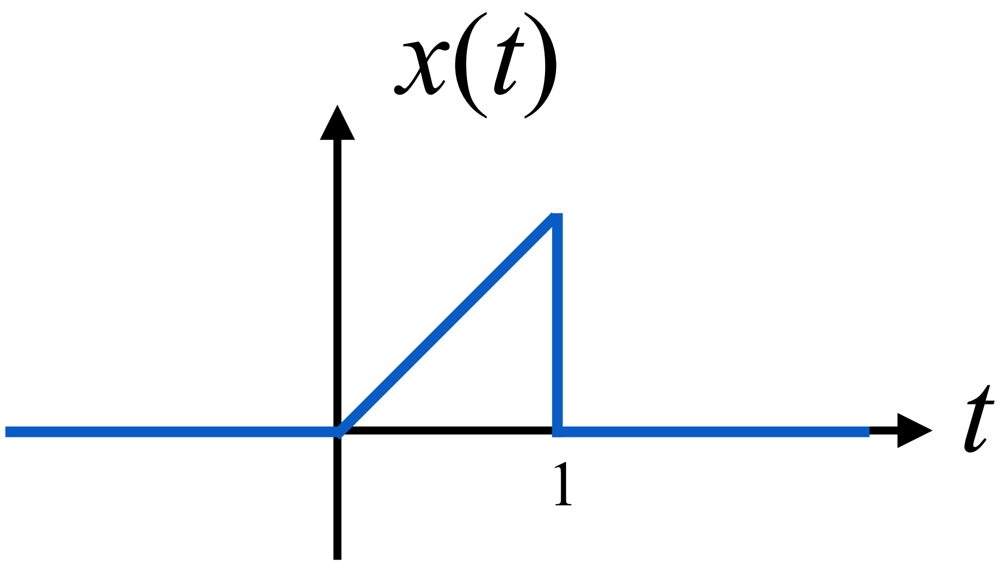
- Option 6:

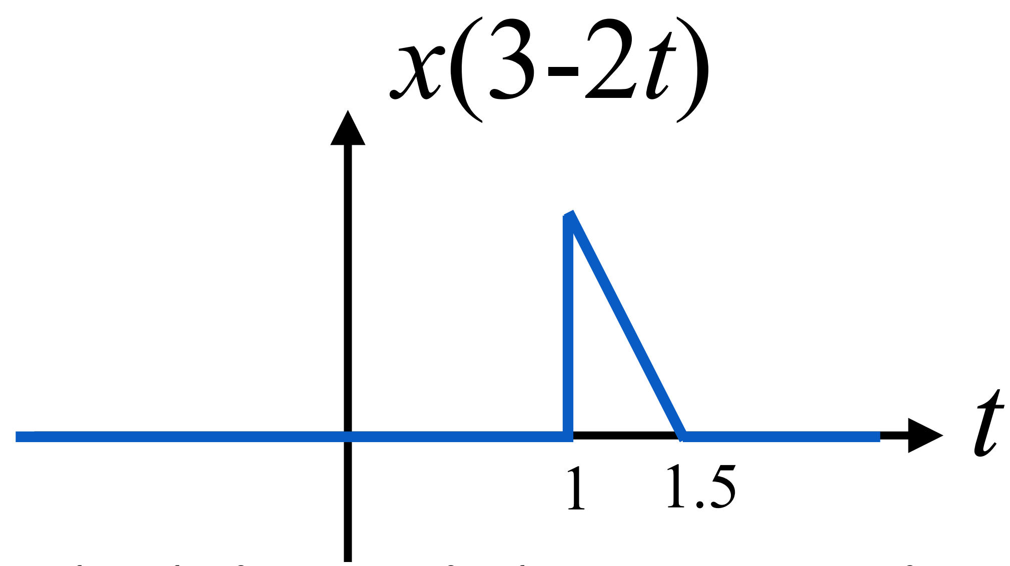


Transformation combos



Transformation combos





- Could we have computed this without going through the transformations one by one?
- Yes. Pick important time instants in the original signal, and find out what t needs to be for $3\text{-}2t$ to correspond to those instants.

$3\text{-}2t$	t
0	1.5
1	1

Linarity

- A system is **linear** if

$$x_1(t) \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_1(t)$$

and

$$x_2(t) \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_2(t)$$

implies

$$ax_1(t) + bx_2(t) \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow ay_1(t) + by_2(t)$$

for any $x_1(t), x_2(t), a$, and b

Linearity

- Problem: Is $y(t) = t^2 e^{-t} x(t)$ a linear system?

$$x_1(t) \longrightarrow y_1(t) = t^2 e^{-t} x_1(t)$$

$$x_2(t) \longrightarrow y_2(t) = t^2 e^{-t} x_2(t)$$

$$\begin{aligned} ax_1(t) + bx_2(t) \longrightarrow\ & t^2 e^{-t}(ax_1(t) + bx_2(t)) \\ &= at^2 e^{-t} x_1(t) \\ &\quad + bt^2 e^{-t} x_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Linearity

- Problem: Is $y(t) = x(3-2t)$ a linear system?

$$x_1(t) \longrightarrow y_1(t) = x_1(3-2t)$$

$$x_2(t) \longrightarrow y_2(t) = x_2(3-2t)$$

$$ax_1(t) + bx_2(t) \longrightarrow ax_1(3-2t) + bx_2(3-2t)$$

$$= ay_1(t) + by_2(t) \checkmark$$

Linearity

- <u>Problem:</u> Is $y(t) = \dfrac{dx(t)}{dt}$ a linear system?

$$x_1(t) \longrightarrow y_1(t) = \frac{dx_1(t)}{dt}$$

$$x_2(t) \longrightarrow y_2(t) = \frac{dx_2(t)}{dt}$$

$$ax_1(t) + bx_2(t) \longrightarrow \frac{d[ax_1(t) + bx_2(t)]}{dt}$$

$$= a\frac{dx_1(t)}{dt} + b\frac{dx_2(t)}{dt}$$

$$= ay_1(t) + by_2(t)$$

Linearity

- Problem: Is $y(t) = x(t)^2$ a linear system?

$$x_1(t) \longrightarrow y_1(t) = x_1(t)^2$$

$$x_2(t) \longrightarrow y_2(t) = x_2(t)^2$$

$$ax_1(t) + bx_2(t) \longrightarrow [ax_1(t) + bx_2(t)]^2$$

NONLINEAR $\quad \neq ay_1(t) + by_2(t)$

- But what about the case $a = 1$ and $b = 0$?
- Remember that the condition needs to be satisfied FOR ANY $x_1(t), x_2(t), a$, and b

Linearity

- Problem: Is $y(t) = x(t) + 4$ a linear system?

$$x_1(t) \longrightarrow y_1(t) = x_1(t) + 4$$

$$x_2(t) \longrightarrow y_2(t) = x_2(t) + 4$$

$$ax_1(t) + bx_2(t) \longrightarrow ax_1(t) + bx_2(t) + 4$$

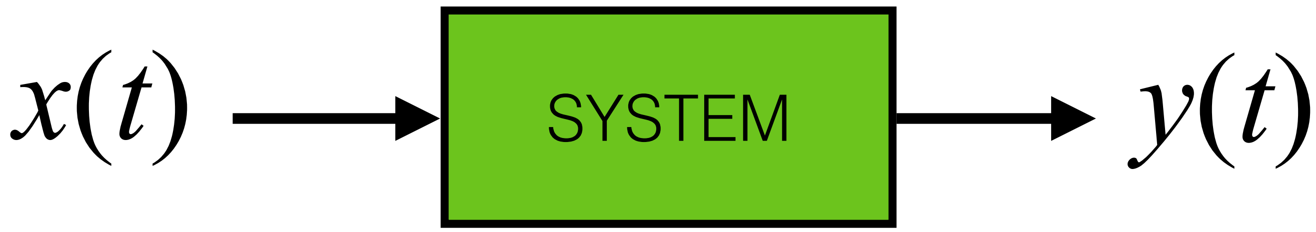
NONLINEAR

$$\neq ay_1(t) + by_2(t)$$

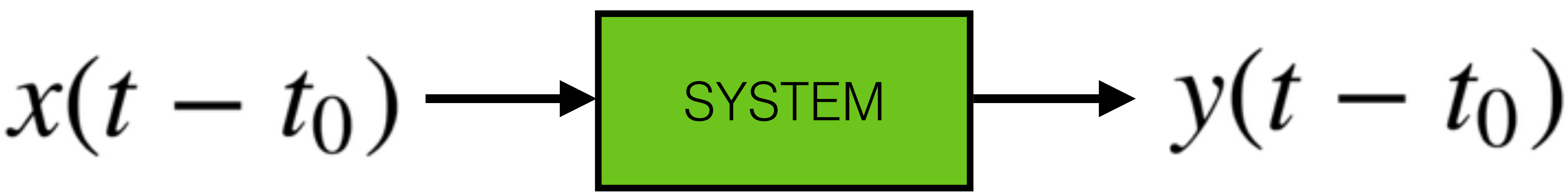
- To see this, just take any *a* and *b* NOT satisfying $a+b = 1$.

Time invariance

- A system is **time-invariant** if



implies



for any $x(t)$ and t_0.

Time invariance

- Problem: Is $y(t) = x(t)^2$ a time-invariant system?

$$x(t) \longrightarrow y(t) = x(t)^2$$

$$x(t - t_0) \longrightarrow x(t - t_0)^2$$

$$= y(t - t_0)$$

Time invariance

- Problem: Is $y(t) = t^2 e^{-t} x(t)$ a time-invariant system?

$$x(t) \longrightarrow y(t) = t^2 e^{-t} x(t)$$

$$x(t - t_0) \longrightarrow t^2 e^{-t} x(t - t_0)$$

TIME VARIANT $\neq y(t - t_0)$

Time invariance

- Problem: Is $y(t) = x(t) - 3x(t-1)^2$ a time-invariant system?

$$x(t) \longrightarrow y(t) = x(t) - 3x(t-1)^2$$

$$x(t-t_0) \longrightarrow x(t-t_0) - 3x(t-t_0-1)^2$$

$$= y(t-t_0)$$

Time invariance

- Problem: Is $y(t) = x(2t)$ a time-invariant system?

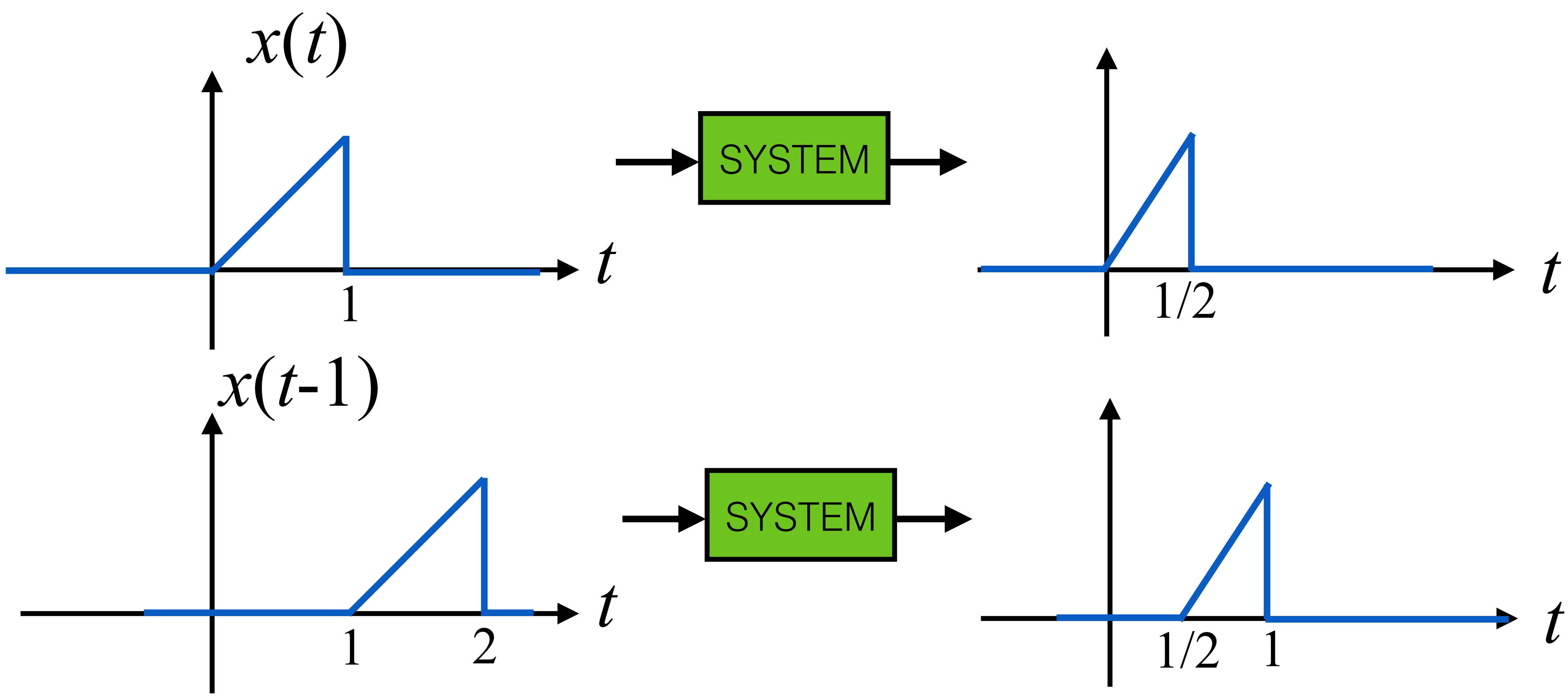
$$x(t) \longrightarrow y(t) = x(2t)$$

$$x(t - t_0) \longrightarrow x(2t - t_0)$$

TIME VARIANT $\neq y(t - t_0)$

If in doubt, try this out

- You can try to find an example to prove **non-linearity** or **time-variance**
- For the last example, $y(t) = x(2t)$, try this:



Memory and Causality

- A system is **memoryless** if at time instant t, the value of $y(t)$ depends only on the *current* value of $x(t)$, and not on any *past* or *future* value of it.
- A system is **causal** if at time instant t, the value of $y(t)$ depends only on the *current* and *past* value of $x(t)$, and not on any *future* value of it.
- Obviously, memorylessness implies causality, but not vice versa.

Memory and Causality

- Problem: Is $y(t) = x(t)^2$ a memoryless system? If not memoryless, is it causal?

- Solution:

$$y(0) = x(0)^2$$

$$y(1) = x(1)^2$$

$$y(1000.23) = x(1000.23)^2$$

$$\vdots$$

MEMORYLESS ✓

CAUSAL ✓

Memory and Causality

- <u>Problem:</u> Is $y(t) = x(t) - 3x(t-1)^2$ a memoryless system? If not memoryless, is it causal?
- <u>Solution:</u>

$$y(0) = x(0) - 3x(-1)^2$$

HAS MEMORY

$$y(1) = x(1) - 3x(0)^2$$

$$y(5.234) = x(5.234) - 3x(4.234)^2$$

$$\vdots$$

CAUSAL ✓

Memory and Causality

- Problem: Is $y(t) = x(2t)$ a memoryless system? If not memoryless, is it causal?

- Solution:

$$y(0) = x(0)$$

$$y(-1) = x(-2)$$

HAS MEMORY

$$y(2) = x(4)$$

NON-CAUSAL

Memory and Causality

- Problem: Is

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

a memoryless system? If not memoryless, is it causal?

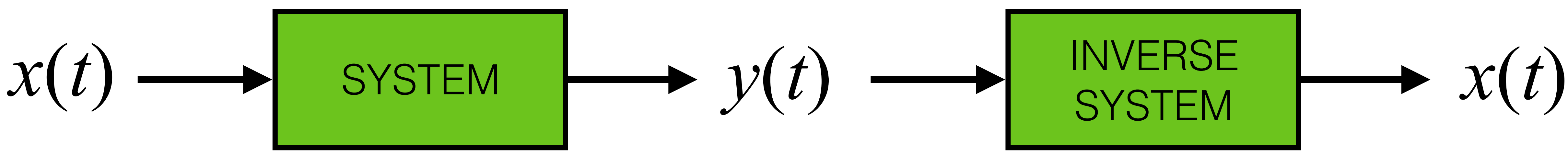
- Solution: $y(t)$ clearly depends on all values of $x()$ between the time instants $-\infty$ and t.

HAS MEMORY

CAUSAL ✓

Invertibility

- A system is **invertible** if there exists another system which outputs $x(t)$ when its input is $y(t)$.

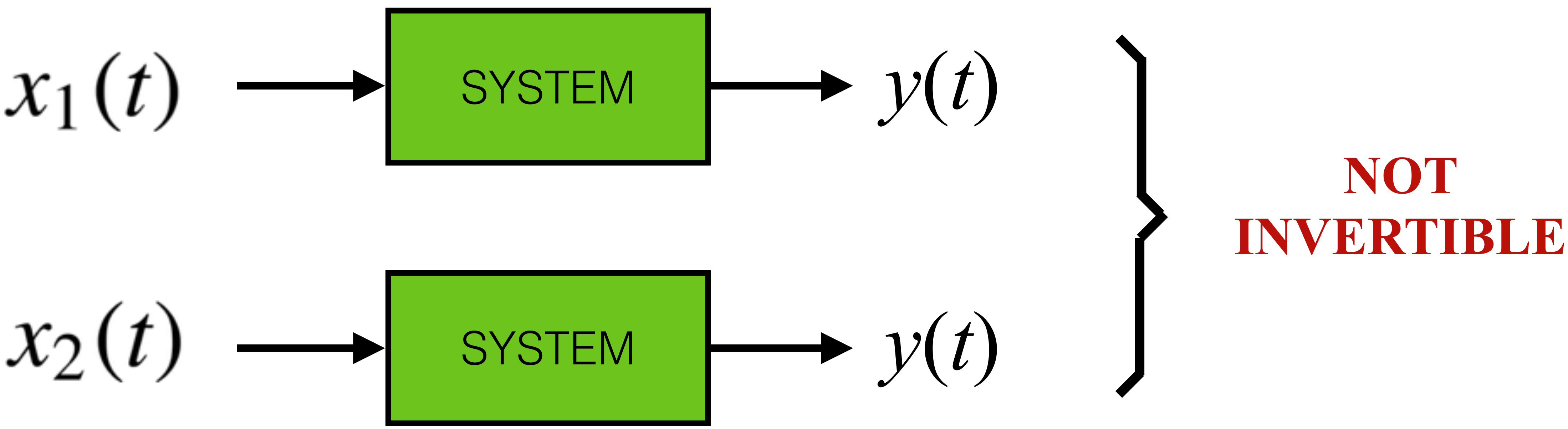


- This should be true for ALL $x(t)$.

Invertibility

- But this definition seems to require that you actually *find* the inverse system.

- <u>Alternative definition</u>: A system is **invertible** if no two distinct input signals yield the same output.



Invertibility

- Problem: Is $y(t) = x(t)^2$ an invertible system?

$$x_1(t) = u(t) \longrightarrow y(t) = u(t)^2 = u(t)$$

$$x_2(t) = -u(t) \longrightarrow y(t) = u(t)^2 = u(t)$$

NOT INVERTIBLE

Invertibility

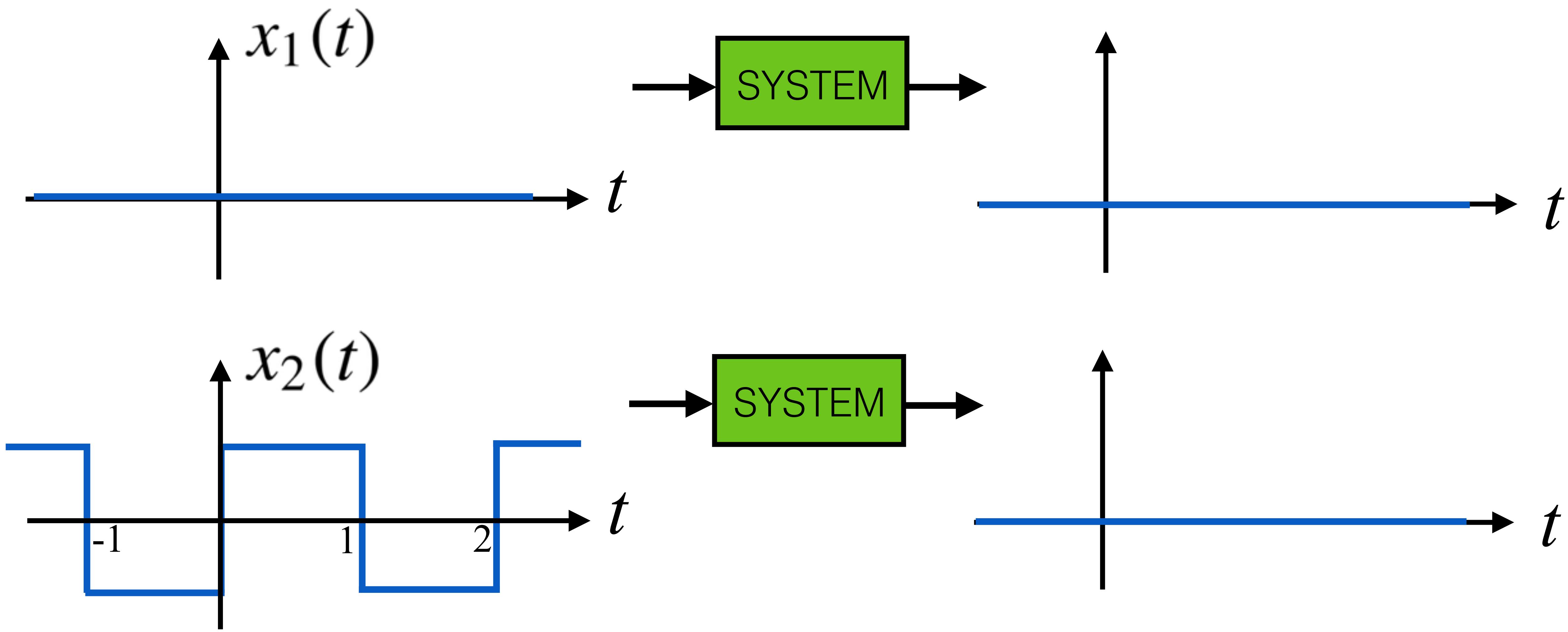
- <u>Problem:</u> Is $y(t) = x(2t)$ an invertible system?
- <u>Solution:</u> Yes, because you can reverse the operation by

$$x(t) = y(0.5t)$$

INVERTIBLE ✓

Invertibility

- Problem: Is $y(t) = x(t) + x(t-1)$ an invertible system?



NOT INVERTIBLE

Invertibility

- Problem: Is

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

an invertible system?

- Solution: Yes, because you can reverse the operation by

$$x(t) = \frac{dy(t)}{dt}$$

INVERTIBLE ✓

Invertibility

- Alternatively, if the system were not invertible, there would exist two inputs $x_1(t)$ and $x_2(t)$ yielding the same output.
- But that would mean that for all t and $t - \Delta t$,

$$\int_{-\infty}^{t} x_1(\tau)d\tau = \int_{-\infty}^{t} x_2(\tau)d\tau$$

and

$$\int_{-\infty}^{t-\Delta t} x_1(\tau)d\tau = \int_{-\infty}^{t-\Delta t} x_2(\tau)d\tau$$

Invertibility

$$\int_{-\infty}^{t} x_1(\tau)d\tau = \int_{-\infty}^{t} x_2(\tau)d\tau$$

$$-\quad \int_{-\infty}^{t-\Delta t} x_1(\tau)d\tau = \int_{-\infty}^{t-\Delta t} x_2(\tau)d\tau$$

$$\int_{t-\Delta t}^{t} x_1(\tau)d\tau = \int_{t-\Delta t}^{t} x_2(\tau)d\tau$$

- Letting $\Delta t \rightarrow 0$, this is the same as

$$x_1(t)\cancel{\Delta t} = x_2(t)\cancel{\Delta t}$$

INVERTIBLE ✓

- Contradiction! No such $x_1(t), x_2(t)$ can exist.

Stability

- A system is **stable** if bounded inputs yield bounded outputs.
- Mathematically speaking, a system is stable if

$$|x(t)| \leq B \quad \forall t$$

for some B implies

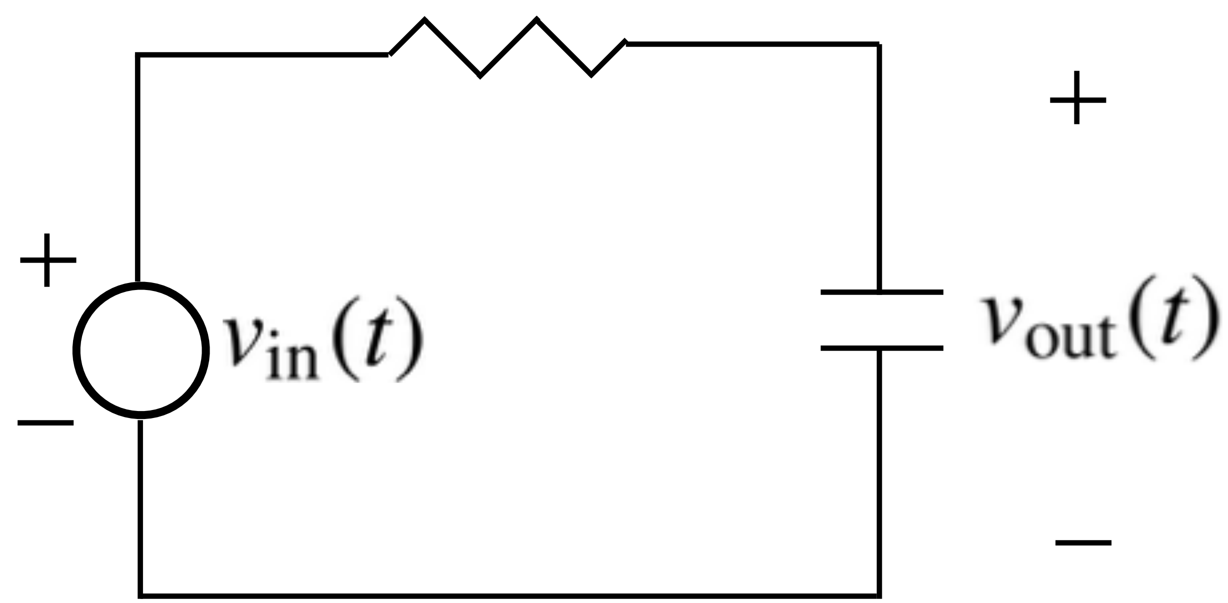
$$|y(t)| \leq C \quad \forall t$$

for some C.

- This should be true for ALL $x(t)$.

Stability

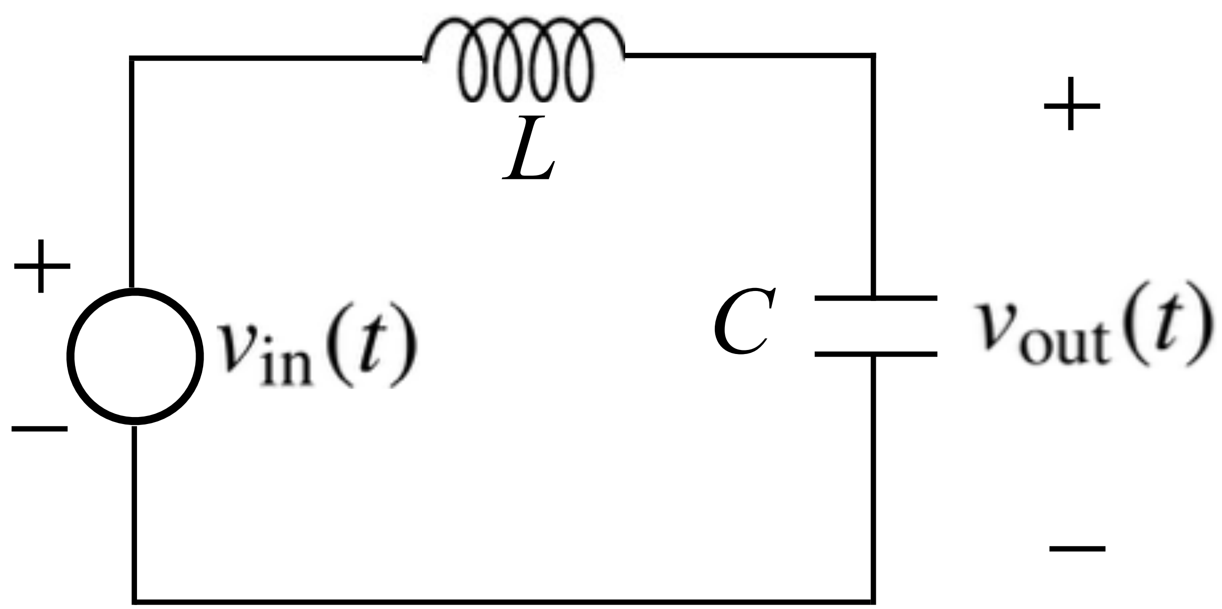
- A familiar example of a **stable** system:



- Obviously no bounded source can create an infinite voltage on the capacitor.

Stability

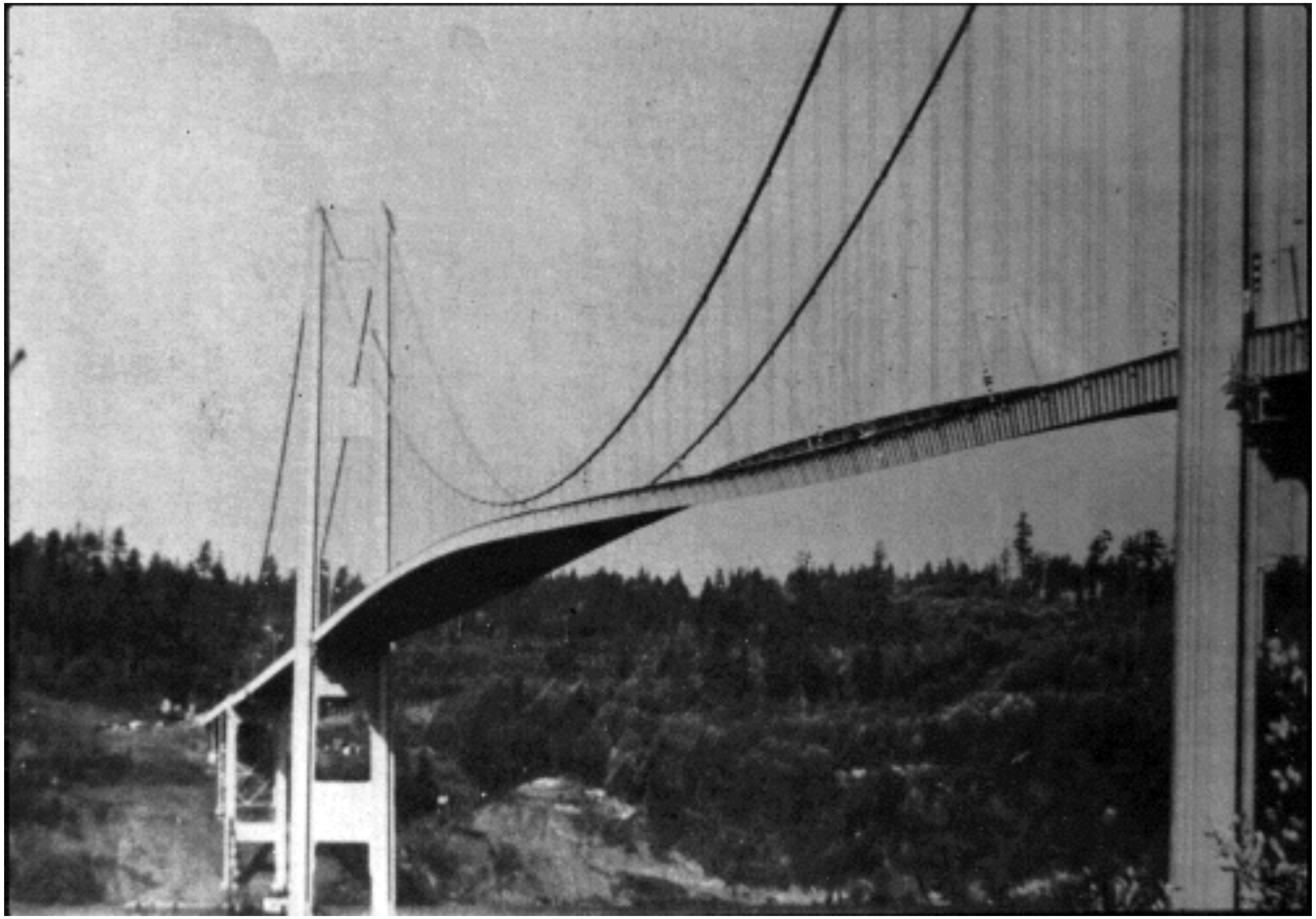
- A familiar example of an **unstable** system:



- If $v_{\text{in}}(t) = \cos(\omega t)$ with $\omega = \dfrac{1}{\sqrt{LC}}$, the output will blow up.

Stability

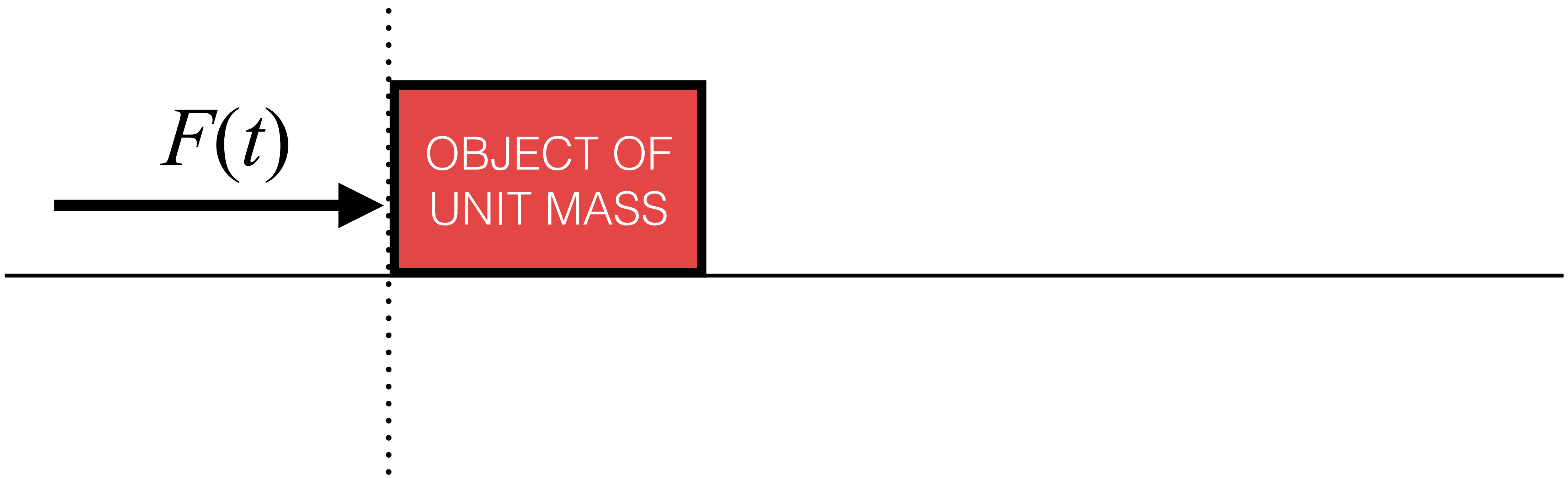
- A *frightening* example of an **unstable** system:



http://youtu.be/j-zczJXSxnw

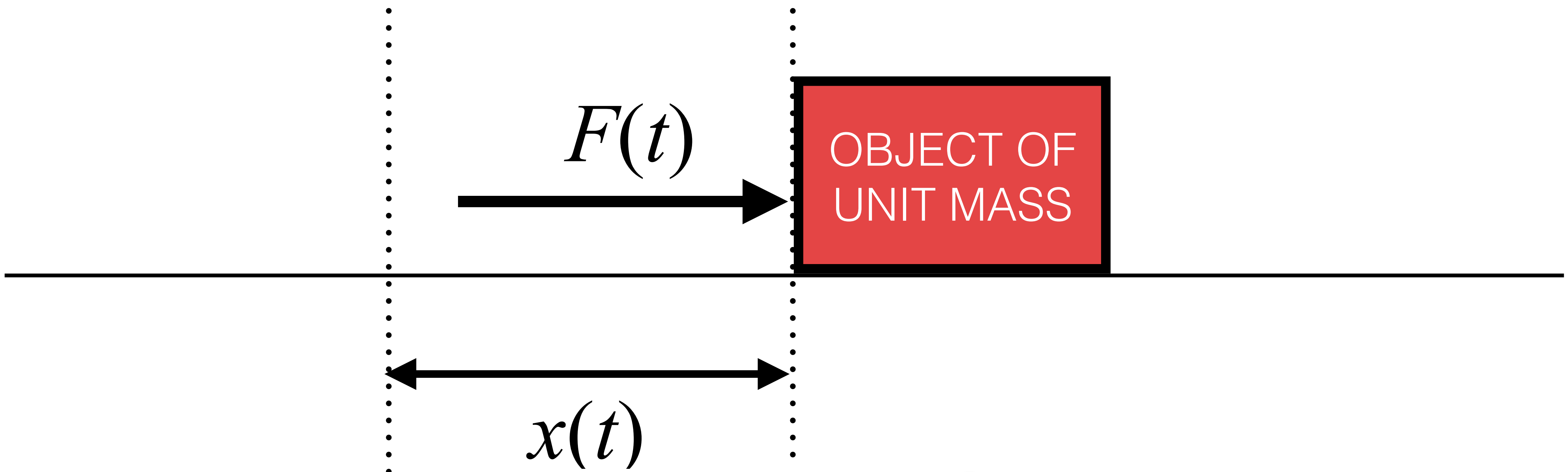
Stability

- Here is another **unstable** system:



Stability

- Here is another **unstable** system:



- Newton's Law: $F(t) = \dfrac{d^2x(t)}{dt^2}$
- This implies that even with a very small force, you can obtain infinite displacement.

Stability

- Problem: Is $y(t) = x(2t)$ a stable system?
- Solution: If $|x(t)| \leq B \quad \forall t$, then certainly

$$|y(t)| = |x(2t)| \leq B \quad \forall t$$

- Taking $C = B$ in the definition then leads to the conclusion that the system is...

STABLE

Stability

- Problem: Is $y(t) = x(t) - 3x(t-1)^2$ a stable system?
- Solution: If $|x(t)| \leq B \quad \forall t$, then

$$|y(t)| = \left|x(t) - 3x(t-1)^2\right|$$
$$\leq |x(t)| + \left|3x(t-1)^2\right| \leq B + 3B^2$$

- Take $C = B + 3B^2$

STABLE ✓