[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 6706 **D** **Your Roll No...............**

Unique Paper Code : 237501

Name of the Course : **B.Sc. (H) Statistics**

Name of the Paper : STHT – 501 : Statistical Inference I

Semester : V

Time : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **six** questions in all selecting **three** from each section.

SECTION – I

1. (a) Explain the concept of consistency and state its properties.

   State and prove sufficient conditions for consistency.

   (b) Let $X_1, X_2, \ldots \ldots X_n$ be a random sample from uniform population with

   $$f(x, \beta) = 1/\beta, \ 0 < X < \beta.$$

   Show that $T_1 = \frac{n+1}{n} Y_n$ (where Y_n is the largest order statistic in a random sample $X_1, X_2, \ldots \ldots X_n$) and $T_2 = 2\bar{X}$ are both unbiased estimators for β. Also compare the relative efficiencies of T_1 to T_2. (6,6½)

2. (a) State Cramer-Rao inequality and its regularity conditions. Under what conditions does the equality hold ? Verify that there exists an MVB estimator for the parameter θ in population $N(\mu, \theta)$ where μ is known.

(b) Let $f(x, \theta)$ be the probability mass function of X, where the range of X does not depend on θ. Then a necessary and sufficient condition for a statistic T to be sufficient for θ is, that the probability function should belong to exponential family. (6½,6)

3. (a) Let T_0 be MVU estimator, while T_1 is an unbiased estimator with efficiency e_θ. If ρ_θ is the correlation coefficient between T_0 and T_1 then show that $\rho_\theta = \sqrt{e_\theta}$.

(b) Let X be a Bernoulli random variable with parameter θ. Obtain an unbiased estimator for θ^2 for the random sample $X_1, X_2, \ldots \ldots X_n$. (7,5½)

4. (a) Let $X_1, X_2, \ldots \ldots X_n$ be a random sample from $N(\theta,\sigma^2)$, where σ^2 is known.

(i) Obtain sufficient statistic for θ.

(ii) Show that it is complete.

(iii) Hence obtain UMVUE for θ.

(b) A random sample $X_1, X_2, X_3, \ldots, X_5$ of size 5 is drawn from a normal population with unknown mean μ and known variance σ^2. Consider the following estimators to estimate μ :

$$T_1 = \frac{X_1+X_2+X_3+X_4+X_5}{5}, \quad T_2 = \frac{X_1+X_2}{2} + X_3 \text{ and } T_3 = \frac{2X_1 + X_2 + \lambda X_3}{3}$$

Find λ such that T_3 is unbiased for μ. Are T_1 and T_2 unbiased ? Find the best estimator among T_1, T_2 and T_3. (6½,6)

SECTION – II

5. (a) State and prove Rao-Blackwell Theorem and explain its significance in point estimation.

(b) If X_1, X_2 is a random sample of size 2 from a distribution having p.d.f

$$f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}, \; 0 < x < \infty .$$

Show that $Y_1 = X_1 + X_2$ is sufficient for θ and find the joint p.d.f of Y_1 and $Y_2 = X_2$. Show that Y_2 is unbiased for θ with variance θ^2. Find $E(Y_2|y_1)$ and compare its variance with that of Y_2. (6,6½)

6. (a) Find the maximum likelihood estimators of parameters α and λ (λ being large) of the distribution

$$f(x;\alpha,\lambda) = \frac{1}{\Gamma(\lambda)}\left(\frac{\lambda}{\alpha}\right)^{\lambda} e^{-(\lambda x/\alpha)} x^{\lambda-1}; \; 0 < x < \infty, \; \lambda > 0 .$$

You may use, that for large values of λ,

$$\Psi(\lambda) = \frac{\partial}{\partial\lambda}\log\Gamma(\lambda) = \log\lambda - \frac{1}{2\lambda} \quad \text{and} \quad \Psi'(\lambda) = \frac{1}{\lambda} + \frac{1}{2\lambda^2}.$$

(b) Obtain the most general form of the continuous distribution for which maximum likelihood estimator of parameter θ is the geometric mean of the sample values. (6½,6)

7. (a) Describe the method of moments. A random variable X takes the values 0, 1, 2 with respective probabilities

$$\frac{\theta}{4N} + \frac{1}{2}\left(1 - \frac{\theta}{N}\right), \; \frac{\theta}{2N} + \frac{\alpha}{2}\left(1 - \frac{\theta}{N}\right) \text{ and } \frac{\theta}{4N} + \frac{1-\alpha}{2}\left(1 - \frac{\theta}{N}\right),$$

where N is a known number. Estimate θ and α by the method of moments.

(b) Explain the method of maximum likelihood estimation and state its optimum properties. (7,5½)

8\. (a) Consider a random sample of size n from the distribution having p.d.f.

$$f(x, \theta) = \exp[-(x-\theta)];\ \theta \le x < \infty.$$

Show that $P\left[X_1 + \frac{1}{n}\log\alpha \le \theta \le X_1\right] = 1-\alpha$.

(b) Let $X_1, X_2, \ldots \ldots X_n$ be a random sample from Poisson distribution with parameter θ. Obtain $100(1-\alpha)$ % confidence interval for θ, when n is large. (6½,6)

[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1191 **E** **Your Roll No................**

Unique Paper Code : 237501

Name of the Course : **B.Sc. (H) Statistics**

Name of the Paper : STHT – 501 : Statistical Inference I

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **six** questions in all selecting **three** from each section.

SECTION I

1. (a) Define MVU estimator and show that it is unique.

   (b) Stating clearly the regularity conditions, show that in any regular case of estimation, the variance of an unbiased estimator T for $\gamma(\theta)$, which is supposed to be differentiable with respect to θ is given by

$$V_\theta(T) \geq -\frac{[\gamma'(\theta)]^2}{E\left[\frac{\partial^2 \log L(x,\theta)}{\partial\theta^2}\right]}$$

   Also explain the significance of the result. (5,7½)

2. (a) If X_1, X_2, ------, X_n is a random sample from a Cauchy distribution with p.d.f.

$$f(x,\theta) = \frac{1}{\pi}\frac{1}{\left[1+(x-\theta)^2\right]}; \quad -\infty < x < \infty.$$

Verify whether there exists an MVB estimator for the parameter θ. Also obtain the value of MVB.

(b) Let $\overline{X}_1$ be the mean of a random sample of size n from $N(\mu, \sigma_1^2)$ and $\overline{X}_2$ be the mean of another random sample of size n from $N(\mu, \sigma_2^2)$. Assuming that two samples are independent, show that

(i) $T = K\overline{X}_1 + (1-K)\overline{X}_2$ is unbiased for μ,

(ii) variance of T is minimum when $K = \dfrac{\sigma_2^2}{(\sigma_1^2+\sigma_2^2)}$. (6,6½)

3. (a) State and prove Fisher – Neyman criterion for the existence of a sufficient statistic.

(b) Let $X_1, X_2, \text{------}, X_n$ be a random sample of size n from $N(\theta_1, \theta_2)$. Show that $T_1 = \overline{X}$ and $T_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are jointly sufficient for (θ_1, θ_2).

(8,4½)

4. (a) Let T be an MVU estimator for $\gamma(\theta)$ and T_1 and T_2 be two other unbiased estimators of $\gamma(\theta)$, with efficiencies e_1 and e_2 respectively. If ρ_θ is the correlation coefficient between T_1 and T_2 then show that

$$(e_1e_2)^{1/2} - \{(1-e_1)(1-e_2)\}^{1/2} \leq \rho_\theta \leq (e_1e_2)^{1/2} + \{(1-e_1)(1-e_2)\}^{1/2}.$$

Further if T_1 and T_2 have the same variance then show that $\rho_\theta \geq 2e - 1$.

(b) Describe the method of minimum Chi-square. (8,4½)

5. (a) Show that the first order statistic of a random sample of size n from the distribution with p.d.f.

$$f(x, \theta) = e^{-(x-\theta)}, \quad \theta < x < \infty,$$

is a complete sufficient statistic for θ. Find a unique continuous function of this statistic which is the best statistic for θ.

(b) Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the order statistics of a random sample of size 5 from the uniform distribution $U(0, \theta)$. Show that $2Y_3$ is an unbiased statistic for θ. Determine the joint probability density function of Y_3 and the sufficient statistic for θ. Find the conditional expectation $E(2Y_3 | Y_5)$ and compare its variance with that of $2Y_3$. (6,6½)

6. (a) Let $X_1, X_2, \text{-----}, X_n$ be a random sample of size n from a distribution with p.d.f.

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_2}\exp\left[-\frac{(x-\theta_1)}{\theta_2}\right], \quad x \geq \theta_1, -\infty < \theta_1 < \infty, \theta_2 > 0$$

Obtain the maximum likelihood esimators of θ_1 and θ_2.

(b) If $X_1, X_2, \text{-----}, X_n$ is a random sample from the distribution with p.d.f.

$$f(x, \theta) = \frac{\theta^{k+1}x^k e^{-\theta x}}{\Gamma(k+1)}; \quad 0 \leq x < \infty, \theta > 0,$$

where k is a known constant, show that the maximum likelihood estimator $\hat{\theta}$ of θ is $\frac{k+1}{\bar{x}}$. Also show that the estimator is biased but consistent and that its asymptotic distribution for large n is $N\left(\theta, \frac{\theta^2}{n(k+1)}\right)$. (4,8½)

7. (a) Describe the method of moments. Discuss the case when the estimates obtained by the method of moments are identical with those of maximum likelihood estimates.

Estimate α and β by the method of moments for the distribution :

$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}, \quad 0 \leq x < \infty$$

(b) State optimum properties of maximum likelihood estimators. (8,4½)

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8. (a) Describe the pivotal quantity method for obtaining confidence intevals.

Let $X_1, X_2, \text{-----}, X_n$ be a random sample of size n from a rectangular distribution with p.d.f.

$$f(x,\theta) = \begin{cases} \dfrac{1}{\theta}, & 0<x<\theta,\ \theta>0, \\ 0, & \text{elsewhere}. \end{cases}$$

If $X_{(n)}$ is the largest observation from this sample then show that

$$P\left[X_{(n)} \le \theta \le X_{(n)}/\alpha^{1/n}\right] = 1-\alpha.$$

(b) Obtain 95% confidence limits, for the parameter θ of the distribution with p.d.f.

$$f(x, \theta) = \theta e^{-\theta x}, \quad 0 < x < \infty,$$

for large sample size. (8,4½)

[This question paper contains 3 printed pages.]

Sr. No. of Question Paper : 6235 F-5 Your Roll No...............

Unique Paper Code : 2371501

Name of the Paper : Statistical Inference-I

Name of the Course : Erstwhile FYUP B.Sc. (H) Statistics

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any five questions.

1. (a) In case of random sample of size n drawn from N (θ, σ^2) where σ^2 is known, obtain a consistent estimator for θ. Further, show that the estimator of the form $a\bar{X}$ for θ, has minimum mean square when

$$a = \frac{\theta^2}{\theta^2 + \sigma^2 / n}$$

(b) Let $X_1, X_2, X_3, \ldots\ldots X_n$, (n > 2) be a random sample of size n from the distribution having density function

$$f(x, \theta) = \theta x^{\theta - 1}, \ 0 < x < 1, \ \theta > 0$$

If $Z = -\Sigma_{i=1}^{n} \log X_i$, show that (n-1)/Z is an unbiased estimator for θ and its efficiency is (n-2)/n. (6, 9)

2. (a) Prove the following result:

$$\int_{-\infty}^{\infty} [T - \varphi(\theta)]^2 L(x,\theta) dx . \int_{-\infty}^{\infty} \left[\frac{\partial \log L}{\partial \theta}\right]^2 L(x,\theta) dx \geq \left[\frac{d\varphi(\theta)}{d\theta}\right]^2, \textit{where}$$

P.T.O.

$L(x, \theta)$ is the joint p.d.f. of $X_1, X_2, \ldots X_n$, and $T = T(X_1, X_2, \ldots X_n)$ is an unbiased estimate of $\varphi(\theta)$.

(b) If T_1 and T_2 are two unbiased estimator of $\gamma(\theta)$ with variances σ_1^2 and σ_2^2 and correlation coefficient ρ_θ, what is the best unbiased linear combination of T_1 and T_2 ? What is the variance of such a combination? (7, 8)

3. (a) If T_1 is an MVU estimator of θ and T_2 is any other unbiased estimator of θ with efficiency e_θ then prove that the correlation between T_1 and T_2 is $\sqrt{e_\theta}$. Further, if T_1 and T_2 have same variance then show that ρ_θ can not be smaller than $2e_\theta - 1$.

(b) In case of a random sample of size n drawn from binomial population $B(m, \theta)$ with parameter θ, $0 < \theta < 1$, Obtain complete sufficient statistic for θ. Hence obtain UMVU estimate of θ. (7,8)

4. (a) Let $f(x, \theta)$ be the probability function of θ where range of x does not depend on θ. Then show that a necessary and sufficient condition for a statistic T to be sufficient for θ is that the probability function should belong to exponential family.

(b) Let the random variable X and Y have the joint p.d.f.

$$f(x, y) = \frac{2}{\theta^2} e^{\frac{-(x+y)}{\theta}}, \quad 0 < x < \infty.$$

show that $E(Y|x) = x + \theta$. Obtain the expected value of $x+\theta$ and compare the variance of $x+\theta$ with that of y. (7, 8)

5. (a) Describe the method of minimum Chi-Square and modified minimum Chi-Square.

(b) Show that the most general form of a distribution for which the sample harmonic mean is the maximum likelihood estimator of θ has p.d.f. (or p.m.f.)

$$f(x,\theta) = \exp\left[\frac{1}{x}\{\theta A'(\theta) - A(\theta)\} - A'(\theta) + c(x)\right]$$

where $A(\theta)$ and $c(x)$ are arbitrary functions of θ and x respectively. (7,8)

6. (a) For the distribution

$$f(x,\theta)=\frac{1}{\theta^p\Gamma(p)}exp\left[-x/\theta\right]x^{p-1},0\le x<\infty,p>1,\theta>0$$

where p is known, obtain the maximum likelihood estimate of θ on the basis of a random sample of size n from the distribution. Find the asymptotic variance of estimate.

(b) Describe the method of moments. Under what conditions the estimates, obtained by the method of moments are identical with those of maximum likelihood estimates? Estimate θ by the method of moment for the distribution

$$f(x,\theta)=\begin{cases}(1+\theta)x\theta, & 0<x<1, \quad \theta>-1\\ 0 & otherwise\end{cases}$$ (7,8)

7. (a) Describe a general method for constructing confidence intervals. Consider a random sample of size n from the exponential distribution with p.d.f.

$$f(x,\theta)=\exp[-(x-\theta)], \quad \theta\le x<\infty, \quad -\infty<\theta<\infty.$$

Show that $100(1-\alpha)\%$ confidence interval for θ is

$$P\left[X_{(1)}+\frac{1}{n}\log\alpha \le\theta\le X_{(1)}\right]=1-\alpha$$

(b) Describe the method of maximum likelihood estimation. Mention all the properties of maximum likelihood estimators. (6,9)