RANDOM WALK & RUIN PROBLEM

Classical Ruin Problem

Consider a game of gambling in which player A ~~who~~ wins or loses a dollar with prob p and q $(p+q=1)$ respectively. Let his initial capital ~~z~~ be $z \equiv$ initial capital and let him play against an adversary with initial capital $a-z$ so that combined capital for the game is a.

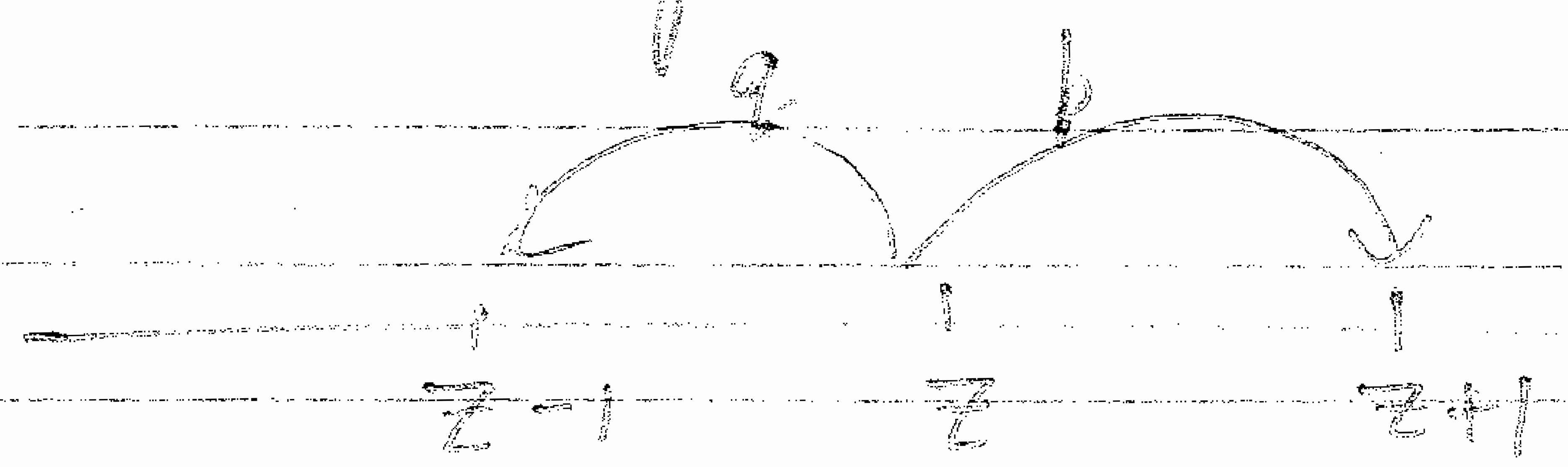
The game continues until the gamblers capital either reduces to 0 or increases to a i.e., one of the two players is ruined.

At each trial with prob p, the player wins a dollar from his adversary. Our objective is to find the prob of the gamblers ruin and the expected duration of the game. This is the classical ruin problem.

$z \equiv$ initial capital of the player

$a-z =$ initial capital of his adversary

$a \equiv$ combined capital available for the game.



Interpretation in terms of random walk.

- A classical ruin problem can also be given random walk interpretation by considering a particle to be moving along the x-axis with unit steps. It starts from the initial position z and moves to $z+1$ or $z-1$ with prob p and q respectively.

- The position of the particle after n steps represents the Gamblers capital at the conclusion of the n^{th} step. The trials terminate on the particle's reaching 0 or a for the first time. Thus we have an one dimensional random walk with barriers at 0 and a.

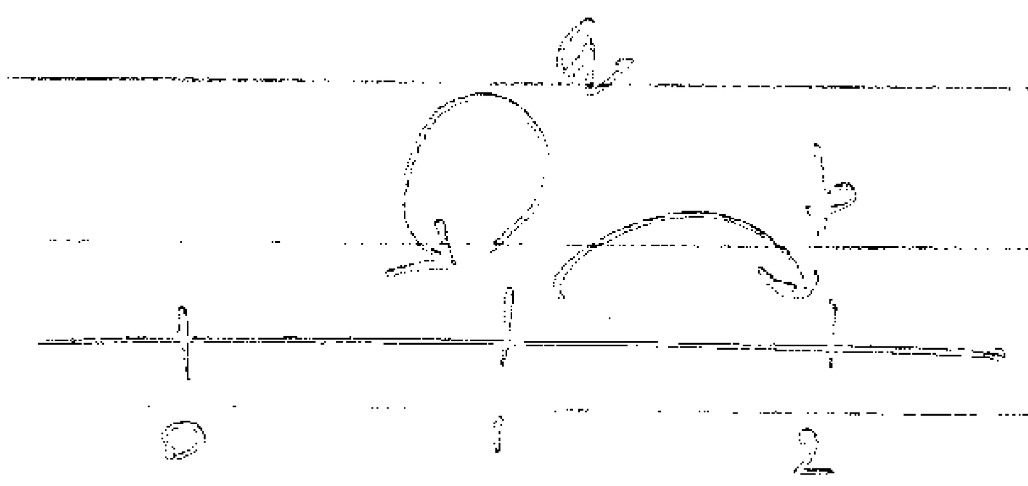
- In the absence of absorbing barriers we say that we have an unrestricted random walk. This is the formulation of the original classical ruin problem as a random walk problem.

→ If $p > q$, a drift to the right is more likely.

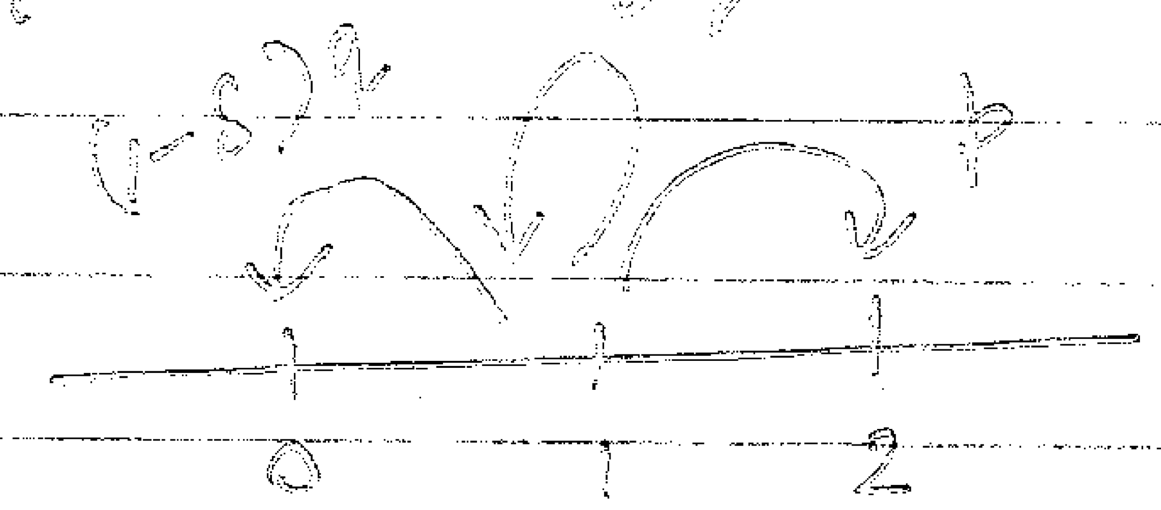
→ If $p = q = \frac{1}{2}$, we have a symmetric random walk.

Reflecting and Elastic Barriers

Consider a random walk b/w 0 to a. In reflecting barriers we take a restriction that whenever the particle is at '1' it has prob p to move to '2' and q to stay at '1'.



We define elastic barrier at the origin by the rule that from '1' the particle moves to '2' with prob p and δq stays at '1' with



prob δq, $(\delta > 0)$ and to 0 with $(1-\delta)q$, where it is absorbed (at '1').

Particular cases.

Absorbing & reflecting barriers are particular case of elastic barriers.

(i) $\delta = 0$ absorbing

(ii) $\delta = 1$ reflecting

(iii) $\delta \in (0,1)$ we have intermediate cases.

Notations

q_z = Pr [of gambler's ultimate ruin in starting with capital z]

p_z = Pr [- - - - - - - win - - -]

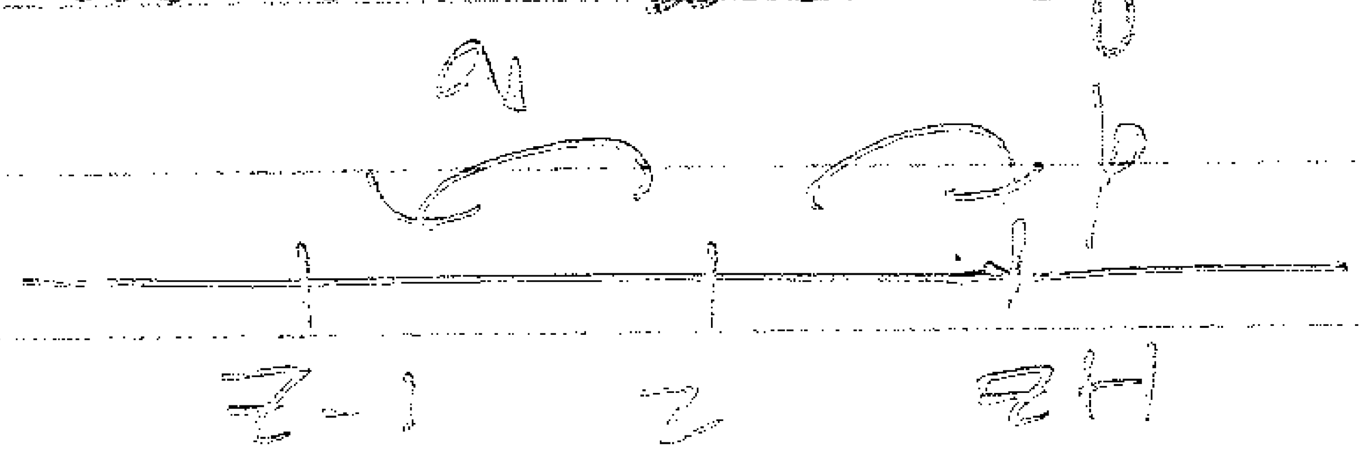
In random walk terminology we define above as

q_z = Pr [the particle starting at 'z' will be absorbed at '0']

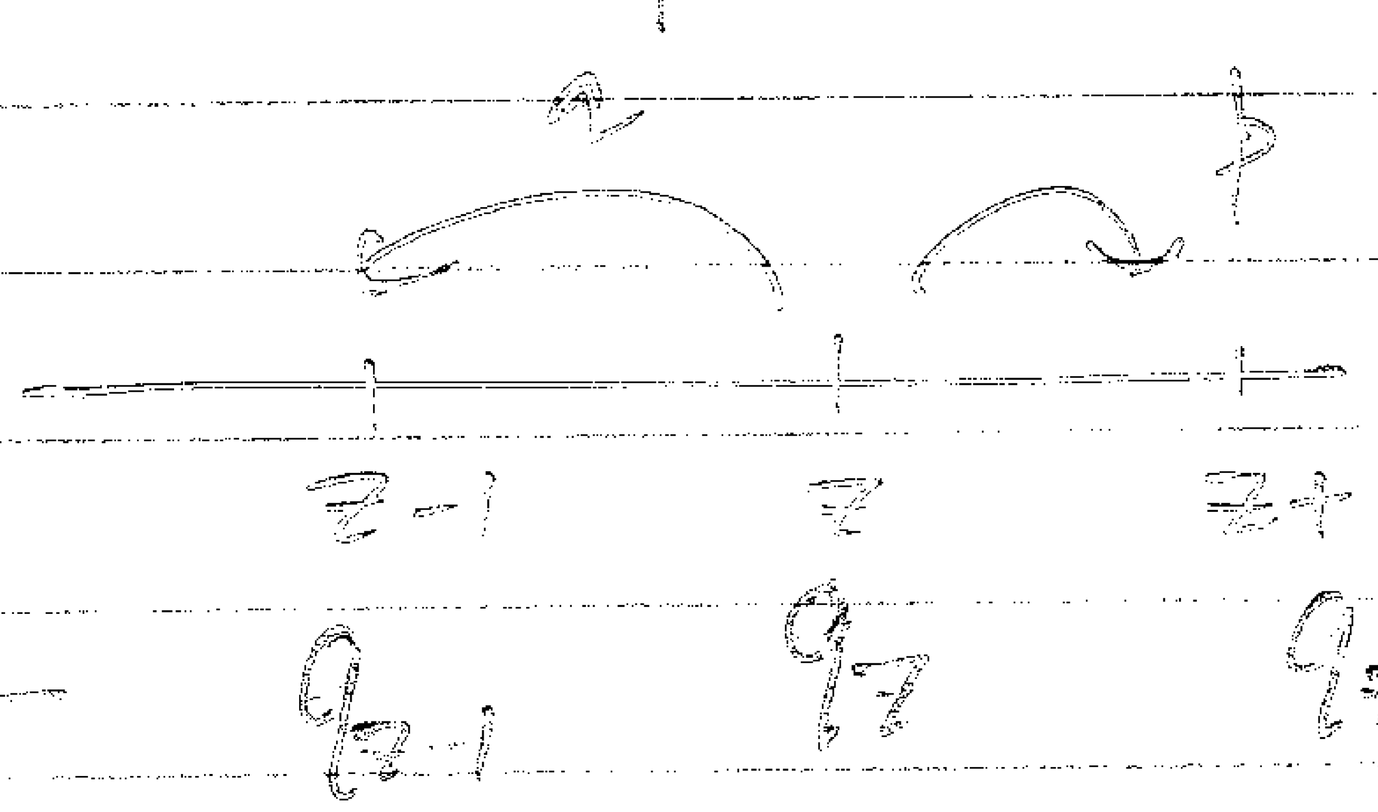
p_z = Pr [. . . . . 'z' . . . . at 'a']

such that $p_z + q_z = 1$.

~~At~~ Starting with an initial capital 'z', the following are two mutually exclusive outcomes at the end of next trial, so that



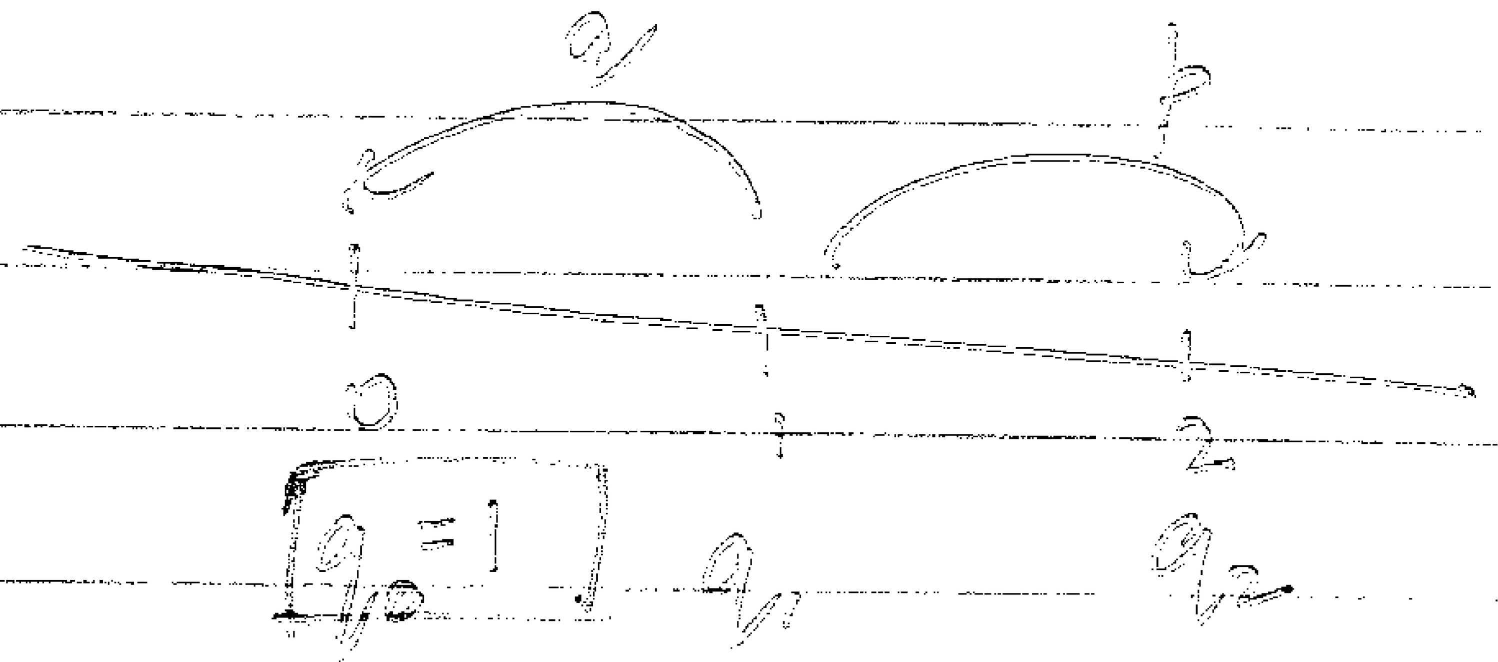
the respective ruin probabilities are



respective ruin probabilities.

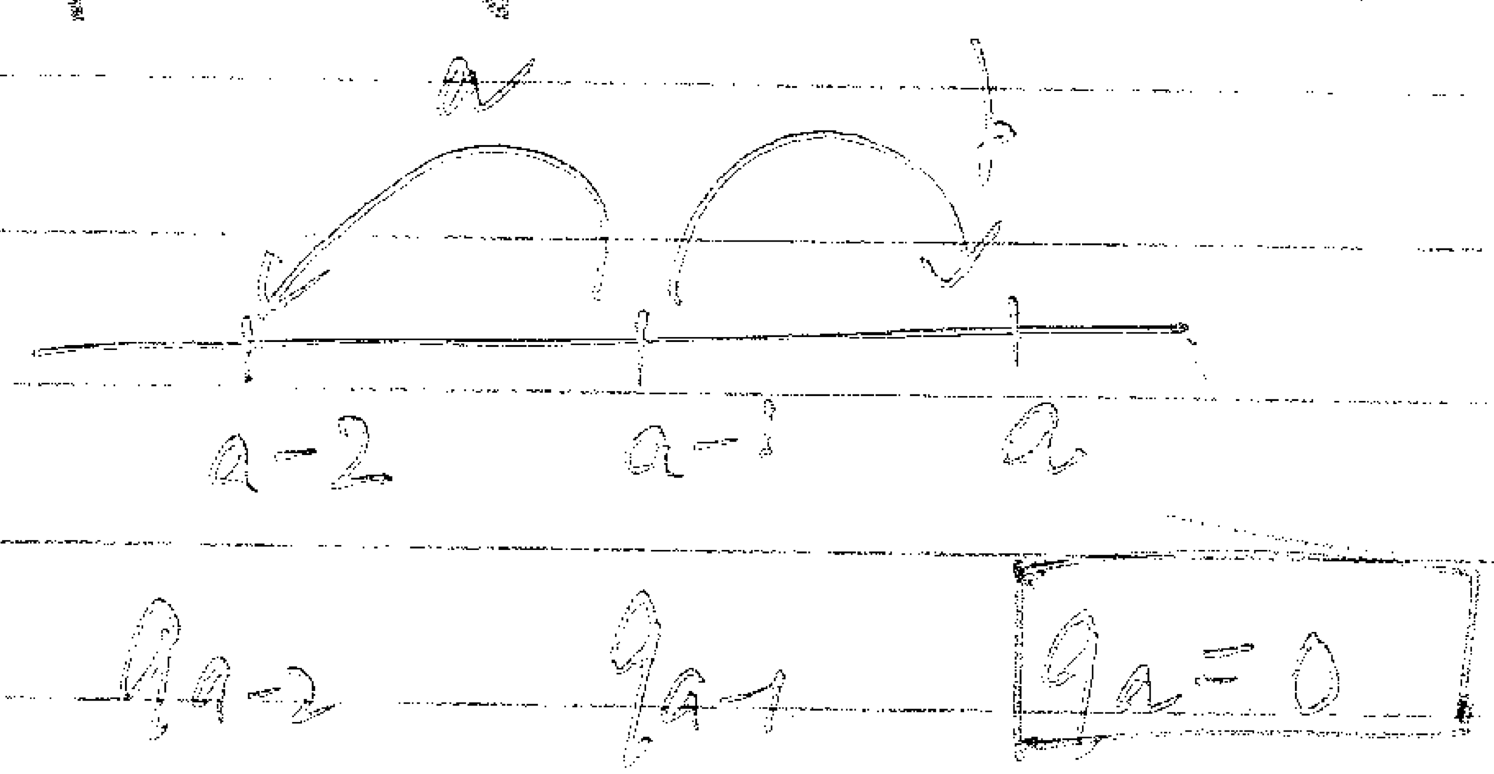
$\therefore \quad q_z = p \cdot q_{z+1} + q \cdot q_{z-1}$; $1 < z < a-1$ (A)

(2) If $z=1$, at the end of 1^{st} trial, the following may result



and hence, $q_1 = pq_2 + q1 = pq_2 + q$; $q_0 = 1$ (B)

(3) Similarly, if $z = a-1$, at the end of 1^{st} trial the following may result



with $q_{a-1} = pq_a + q\, q_{a-2}$

$$q_{a-1} = q\, q_{a-2} \; ; \; q_a = 0$$ (C)

~~In oth~~ (D)

With this convention ($q_0 = 1, q_a = 0$), the prob q_z of ruin satisfies eqn (A) for $z = 1, 2, 3, \ldots, a-1$

It is our objective to solve the difference eqn in (A) for the conditions in (B) and (C)

Recall : $U_{x+n} + A_1 U_{x+n-1} + A_2 U_{x+n-2} + \cdots + A_n U_x = 0$

Auxillary eqn $f(a) = 0$

$$a^n + A_1 a^{n-1} + A_2 a^{n-2} + \cdots + A_n = 0$$

case I : If $a_1, a_2, \cdots, a_n$ are distinct roots then general solⁿ :

$$U_x = C_1 a_1^x + C_2 a_2^x + \cdots + C_n a_n^x$$

case II : If a_1 is repeated k times and other roots are $a_{k+1}, a_{k+2}, \cdots, a_n$

g. solⁿ $U_x = (C_1 + C_2 x + \cdots + C_k x^{k-1}) a_1^x + C_{k+1} a_{k+1}^x + \cdots + C_n a_n^x$

The solⁿ :

put $q_z = \lambda^z$ in (A)

$$\lambda^z = p\lambda^{z+1} + q\lambda^{z-1}$$

or, $p\lambda^2 - \lambda + q = 0$

$$\lambda = \frac{1 \pm \sqrt{1-4pq}}{2p}$$

$$= \frac{1 \pm \sqrt{(p+q)^2 - 4pq}}{2p}$$

$$= \frac{1 \pm \sqrt{(p-q)^2}}{2p} = \frac{(p+q) \pm |p-q|}{2p}$$

If $p > q$

$$\frac{p+q+p-q}{2p} = \frac{2p}{2p} = 1$$

If. $p < q$

$$\frac{p+q-p+q}{2p} = \frac{2q}{2p} = \frac{q}{p}.$$

$\boxed{\text{Case I}: (p \neq q)}$

Then, the particular sol^{n}s are

$$q_z = (1)^z \quad \text{and} \quad q_z = \left(\frac{q}{p}\right)^z = r^z \text{ (say)}$$

Thus, the general sol^n of the eqn is

$$\boxed{q_z = A + B r^z} \quad \text{———— (E)}$$

Where, A and B are obtained using the condition in (D)

$$q_0 = 1 \Rightarrow A + B = 1$$

$$q_a = 0 \Rightarrow A + B r^a = 0$$

$$\text{—(B)} \quad A = 1 - B$$

$$1 - B + B r^a = 0 \Rightarrow B = (1 - r^a)^{-1}$$

$$A = \frac{1 - 1}{1-r^a} = \frac{-r^a}{1-r^a}$$

$$\therefore \boxed{q_z = \frac{r^z - r^a}{1-r^a}} \quad \text{———} \quad (F)$$

Now, in order to prove that eqn (F) is the req. prob of ruin it remains to show that the solⁿ is unique i.e., all solutions of eqn (A) are of the form in eqn (E).

Given, an arbitrary solⁿ of eqn (A), the two constants A and B can be chosen so that eqn (E) will agree with it for $z=0$, $z=1$. From these two values (0 and 1) all other values can be obtained by using eqn (A) recursively. Therefore, two solutions of the eqn which agree for $z = 0, 1$ are identical and hence every solⁿ to the eqn is of the form in eqn (E).

$$\boxed{\text{case II} : \quad p = q = \frac{1}{2}}$$

The above argument breaks down if $p=q=\frac{1}{2}$ for in that case eqn (F) is meaningless.

In this case, we obtain the solⁿ directly as below we have,

$$q_z = \frac{1}{2} q_{z+1} + \frac{1}{2} q_{z-1}$$

or $\quad 2q_z = q_{z+1} + q_{z-1}$

put $\quad q_z = \lambda^z$ to get $\quad f(\lambda) = 0$ as

$$2\lambda^z = \lambda^{z+1} + \lambda^{z-1}$$

$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)(\lambda - 1) = 0$$
$$\lambda = 1, 1$$

then general sol^n is of the form:

$$q_z = (A + Bz)(1)^z$$

$$q_z = A + Bz$$

with $\quad q_0 = 1 \Rightarrow A = 1$

$$q_a = 0 \Rightarrow A + Ba = 0 \Rightarrow \boxed{B = \frac{-1}{a}}$$

$$\therefore \quad \boxed{q_z = 1 - \frac{z}{a}} \quad \text{———} \quad (G)$$

the same can also be obtained

$p \to \frac{1}{2}, q \to \frac{1}{2}$ or $(\lambda \to 1)$ in eqn (F)

$$\lim_{r \to 1} q_z = \lim_{r \to 1} \frac{r^z - r^a}{1 - r^a}$$

Applying l'Hospitals rule,

$$= \lim_{r \to 1} \frac{z r^{z-1} - a r^{a-1}}{-a r^{a-1}}$$

$$= \lim \frac{z - a}{-a} = \frac{a - z}{a}$$

$$\boxed{q_z = 1 - \frac{z}{a}}$$

Prob ~~for~~ of ruin for the Adversary

prob p_z of the gambler's ~~winning~~ winning the game equals the prob his adversary ruins the game. And is thus obtained from eqn (F) and (G) on replacing p, q, z by $q, p, a-z$.

case 1 : $(p \neq q)$

$r = \frac{q}{p}$, is replaced by $r^* = \frac{p}{q} = \frac{1}{r}$

$$\therefore \quad p_z = \frac{\left(\frac{1}{r}\right)^a - \left(\frac{1}{r}\right)^{a-z}}{\left(\frac{1}{r}\right)^a - 1}$$

$$= \frac{r^{a-z} - r^{a}}{\cancel{r^{a}} \times r^{a-z}} \frac{\times \cancel{r^{a}}}{(1-r^{a})}$$

$$= \boxed{\frac{r^{z}-1}{r^{a}-1} = p_z}$$

case 2 : $p = q = \frac{1}{2}$

$$p_z = 1 - \frac{a-z}{a}$$

$$\Rightarrow \boxed{p_z = \frac{z}{a}}$$

verification :

→ for $p \neq q$

$$p_z + q_z = \underbrace{\left(\frac{r^{z} - r^{a}}{1-r^{a}}\right)}_{q_z} + \underbrace{\left(\frac{r^{z}-1}{r^{a}-1}\right)}_{p_z}$$

$$= \frac{\cancel{r^{z}} - r^{a} \cancel{- r^{z}} + 1}{1-r^{a}}$$

$$= \frac{1-r^{a}}{1-r^{a}} = \boxed{1}$$

→ for $p = q = 1$

$$p_z + q_z = \left(\frac{z}{a}\right) + \left(1 - \frac{z}{a}\right) = \boxed{1}$$

Reformulation of the game problem

Let the gambler A has initial capital z. He is playing against an infinitely rich adversary (gambler B) who is always willing to play although A has the previlege to stop playing any further. Gambler A adopts a strategy of playing until either he loses his capital or increases into 'a' with the net game of $-z$ or $a-z$. Then, q_z and $1-q_z$ are the respective probabilities. Under these assumptions let G denote the gamblers ultimate ~~am~~ gain or loss, then, we have

$$G = \begin{cases} a - z & \text{with prob } 1 - q_z \\ -z & \text{with prob } q_z \end{cases}$$

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Therefore, the gamblers expected gain is

$$E(G) = -z q_z + (a-z)(1-q_z)$$

$$= a(1-q_z) - z.$$

We note that the game is termed as fair if

(i) $p = q = \frac{1}{2}$

(ii) $E(G) = 0$. i.e., none of the players win or loose in the game.

When $p = q = \frac{1}{2}$, we have, $q_z = 1 - \frac{z}{a}$

then, $E(G) = a\left(1 - \left(1 - \frac{z}{a}\right)\right) - z = 0.$

Further, we also note that if $E(G) = 0$ implies

$\Rightarrow a(1 - q_z) - z = 0$

$\Rightarrow q_z = 1 - \frac{z}{a}$, which is the probability for ultimate ruin of the gambler in the case $p = q = \frac{1}{2}$.

Conclusion: From above we conclude that

$E(G) = 0$ if and only if $p = q = \frac{1}{2}$. i.e. $p = q = \frac{1}{2}$ is the necessary condition and sufficient cond for expected gain ~~is zero~~ to be zero.

In other words, A fair game remains fair and no unfair game can be changed into a fair one.

NOTE:

When $p = q = \frac{1}{2}$, $q_z = 1 - \frac{z}{a}$.

A player with initial capital $z = \$999$ has the probability 0.999 to win a dollar before losing his capital with $q = 0.6$, $p = 0.4$

the game is unfavourable for the player indeed, but still the probability of winning a dollar before losing his capital can be obtained from

$$q_z = \frac{r^z - r^a}{1 - r^a}$$

$z = 999$, $a = 1000$, $r = \frac{q}{p} = \frac{0.6}{0.4} = \frac{3}{2} = 1.5$

$$q_{z=999} = \frac{\left(\frac{3}{2}\right)^{999} - \left(\frac{3}{2}\right)^{1000}}{1 - \left(\frac{3}{2}\right)^{1000}} = 0.3333$$

$$p_{z=999} = 1 - 0.3333 = 0.6667 \simeq \frac{2}{3}$$

Thus, we notice that even though the game is unfavourable to the player ($q > p$), the probability of his winning a dollar before loosing his capital is very high at $\frac{2}{3}$ (approx).

So, the gambler with relatively large capital z has a reasonable chance to win a small amount $a-z$ before being ruined.

Effect of changing stakes

Changing the unit at stake from a dollar to half a dollar is equivalent to doubling the initial capital. In other words if gambler A had initial capital z and B had $a-z$ units, after the unit at stake being changed from a dollar to half a dollar. They become $2z$, $2(a-z)$ respectively, then the probability of ruin q_z^* becomes

$$q_z^* = \frac{r^{2a} - r^{2z}}{r^{2a} - 1}$$

$$= \frac{(r^a + r^z)(r^a - r^z)}{(r^a+1)(r^a-1)}$$

$$= \left(\frac{r^a + r^z}{r^a + 1}\right) \cdot q_z$$

$$z,\ a-z \longrightarrow 2z,\ 2(a-z)$$

$$q_z \qquad\qquad q_z^*$$

for, $q > p$, $r > 1 \Rightarrow \left(\frac{r^a + r^z}{r^a+1}\right) > 1$

$\therefore$ for $q > p$

$$q_z^* > q_z$$

conclusion

(1) If the stakes are double while the initial capital remains the same (unchanged), the prob of ruin decreases for the player whose prob of success is $p < \frac{1}{2}$, and increases for the adversary.

(for whom the game is advantageous).

In short even if the game is disadvantageous to the player by changing the unit at stake the probability of ruin can be decreased.

Limiting case

In case $a=\infty$, let gambler A has initial capital z and is playing against adversary B such that $a=\infty$. In this case, the prob of ultimate ruin for $q \neq p$ is

$$\lim_{a\to\infty} q_z = \lim_{a\to\infty} \frac{r^z - r^a}{1-r^a} \qquad r = \frac{q}{p}$$

<u>case(1)</u> : $q > p$

$$\lim_{a\to\infty} \frac{z r^{z-1} - a r^{a-1}}{-a r^{a-1}}$$

$$= \lim_{a\to\infty} 1 - \frac{z r^{z-1}}{a r^{a-1}} = \boxed{1}$$

case(2) ; $q < p$

$$\lim_{a\to\infty} \frac{r^z - 0}{1-0} = \boxed{r^z}$$

case (3) $q = p \Rightarrow r = 1$

Apply L Hospitals rule

$$\lim_{a\to\infty} 1 - \frac{z}{a} = \boxed{1}$$

$$\lim_{a\to\infty} \frac{\left(\frac{q}{p}\right)^z - \left(\frac{q}{p}\right)^a}{1-r^a} = \lim_{a\to\infty} \frac{z r^{z-1} - a r^{a-1}}{-a r^{a-1}} = \lim_{a\to\infty} 1 - \frac{z}{a}$$

(*) cont. $\therefore \lim_{a \to \infty} q_z = \begin{cases} 1 & \text{for } q \geq p \\ \left(\frac{q}{p}\right)^z & \text{for } q < p \end{cases}$

we interpret q_z as the prob of ultimate ruin of Gambler A playing against an infinitely rich adversary.

In random walk terminology q_z is the prob that a particle starting at $z > 0$ will ever reach the origin. This can also be restated as

(1) In a random walk starting at origin, the prob of ever reaching the position $z > 0$ is 1 for $q \leq p$ and $\left(\frac{p}{q}\right)^z$ for $q \geq p$.

1 for $q \leq p$. $\left(\frac{p}{q}\right)^z$ for $q > p$.

~~(2) Expected duration of the game.~~

~~the prob duration of the~~

(2) The above expression can also be treated as the probability for the particle's first passage through zero.

(*) (3) cont. For $p > q$, It will be beneficial for the player to start with a larger initial capital z. in which case the prob of ruin will be more.