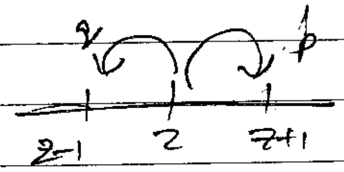
Expected duration of the game.

The prob distribution of the duration of the game will be obtained in this section.

However, its expected value can be obtained by adopting difference eqn techinque in a much simpler way.

Let us assume that the expected duration of the game be known as D_z and has a finite value. If the first trial results in a success the game continues as if the initial position has been $z+1$. The conditional expectation of the duration assuming a success at the first trial is thus $D_{z+1}+1$



Similarly, if the first trial results in a failure the game continues as if the initial position has been $z-1$. The conditional expectation of the duration assuming a failure at the first trial is thus $D_{z-1}+1$.

This argument shows that the expected duration D_z satisfies the following difference eqn

$$D_z = p\left[D_{z+1}+1\right] + q\left[D_{z-1}+1\right]$$

$$= (p+q) + pD_{z+1} + qD_{z-1}$$

$$= pD_{z+1} + qD_{z-1} + 1$$

for $0 < z < a$ $\quad D_0 = 0 \quad D_a = 0$ ——— (1)

Thus, the above is a non-homogenous second order difference eqn with boundary conditions $0 < z < a$ and $D_0 = 0$ and $D_a = 0$

A general solⁿ to (1) is obtained to get

$D_z = \lambda^z$ to get

$$\lambda^z = p\lambda^{z+1} + q\lambda^{z-1}$$

$$p\lambda^2 - \lambda + q = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{1 - 4pq}}{2p}$$

$$= \frac{1 \pm \sqrt{(p+q)^2 - 4pq}}{2p}$$

$$= \frac{1 \pm \sqrt{(p-q)^2}}{2p}$$

$$= \frac{1 \pm (p-q)}{2p}$$

$$\lambda = \begin{cases} 1 \\ \frac{q}{p} = r \quad \text{for } p \neq q \end{cases}$$

case (1) : $p \neq q$ the solⁿ to eqn (1) is of the form

$$D_z = \left(A + B\left(\frac{q}{p}\right)^z\right) + (\text{A particular sol}^n)$$

PAGE NO.
DATE

for finding a particular solⁿ, in this case we put in eqn ①

$D_z = \lambda z$ and get,

$$\lambda z = p\lambda(z+1) + q\lambda(z-1) + 1$$

$$\lambda z - p\lambda z - p\lambda - q\lambda z + q\lambda = 1$$

$$\cancel{\Rightarrow \lambda z - \lambda z(p+q) - \lambda(p+q) = 1}$$

$$\cancel{\Rightarrow \lambda z - \lambda z - \lambda = 1}$$

$$\cancel{\Rightarrow \lambda}$$

$$\lambda z - \lambda z(p+q) + \lambda(q-p) = 1$$

$$\Rightarrow \lambda z - \lambda z + \lambda(q-p) = 1$$

$$\Rightarrow \boxed{\lambda = \frac{1}{q-p}}$$

∴ When $p \neq q$, D_z is of the form

$$D_z = A + B\left(\frac{q}{p}\right)^z + \frac{z}{q-p}$$

with conditions $D_0 = 0 = D_a$

$$D_0 = 0$$

$$\Rightarrow D_0 = A + B\left(\frac{q}{p}\right)^0 = 0 \Rightarrow A + B = 0$$

and $D_a = A + B\left(\frac{q}{p}\right)^a + \frac{a}{q-p} = 0$

PAGE NO.
DATE

$$-B + B\left(\frac{q}{p}\right)^a + \frac{a}{q-p} = 0$$

$$-B + Br^a = -\frac{a}{q-p}$$

$$\Rightarrow B(r^a - 1) = -\frac{a}{q-p}$$

$$\Rightarrow B = \frac{-a}{(r^a-1)(q-p)} = \boxed{\frac{a}{q-p}(1-r^a)^{-1}}$$

$$A = \boxed{\frac{-a}{q-p}(1-r^a)^{-1}}$$

$$D_z = \frac{z}{q-p} + \frac{a}{(q-p)(1-r^a)}(1-r^z)$$

$$\boxed{D_z = \frac{z}{q-p} + \frac{a}{q-p}\left(\frac{1-r^z}{1-r^a}\right)} \quad (2)$$

In this case, the Particular solⁿ obtained above had

$$\lambda = \frac{1}{q-p} = \infty$$

Thus, we obtain the same by putting $D_z = \lambda z^2$ instead of $D_z = \lambda z$ in eqn (1)

$$\lambda z^2 = p\lambda(z+1)^2 + q\lambda(z-1)^2 + 1$$

$$\Rightarrow \lambda z^2 = p\lambda(z^2+1+2z) + q\lambda(z^2+1-2z) + 1$$

PAGE NO.
DATE

$$\lambda z^2 - p\lambda z^2 - q\lambda z^2 - p\lambda - q\lambda - 2p\lambda z + 2q\lambda z = 1$$

$$\Rightarrow \lambda z^2(1-(p+q)) - \lambda(p+q) + 2\lambda z(q-p) = 1$$

$$\Rightarrow \quad 2\lambda z(q-p) - \lambda \cancel{q} = 1$$

$$\Rightarrow \quad \lambda\left[2z(q-p) - 1\right] = 1$$

$$\Rightarrow \quad \lambda = \frac{1}{2z(q-p)-1}$$

$$\Rightarrow \lambda = -1 \quad \text{if } p = q = \frac{1}{2}$$

$\therefore$ particular solution $= \lambda z^2 = -z^2$ and the required solⁿ for D_z in eqn ① is of the form.

$$D_z = (A + Bz) + (\text{A particular sol}^n)$$

$D_z = A + Bz - z^2$
with initial condition

$$D_0 = A \quad \Rightarrow \boxed{A = 0}$$

$$D_a = A + Ba - a^2 = 0$$

$$\Rightarrow \quad Ba - a^2 = 0$$

$$\Rightarrow \quad Ba = a^2 \Rightarrow \boxed{B = a}$$

So, $\boxed{D_z = az - z^2 = z(a-z)}$ ③

conclusion.

for $p \neq q$

(i) If $p > q$, the game may go on forever and (ii) thus in such a case we cannot find the expected duration of the game (in case $a \to \infty$)

(ii) If $p < q$, the game is likely to terminate in the near future as the expected duration in this case is

$$\lim_{a \to \infty} D_z = \lim_{a \to \infty} \left[\frac{z}{q-p} + \frac{a}{q-p} \frac{(1-r^z)}{(1-r^a)} \right.$$

$$\because \quad r = \frac{q}{p} > 1$$

we have $r^a \to \infty$ faster than $a \to \infty$ itself

so that

$$\lim_{a \to \infty} D_z = \frac{z}{q-p}$$

(iii) If $p = q = \frac{1}{2}$

$$\lim_{a \to \infty} D_z = \lim_{a \to \infty} z(a-z) = \infty$$

PAGE NO.

DATE

Page 349 (Feller)

Example Duration is considerably ~~at~~ larger than what we thought

(i) If $z = \$500$, $a-z$ with prob $1/2$ where $a = \$1000$

$$D_z = D_{500} = 500(1000-500) = 250{,}000$$ trials.

(ii) If $z = \$1$ and $a-z = \$1000$

$$D_z = D_1 = 1(1000) = 1000$$ trials.

(iii) Consider, $p = 0.6$ $q = 0.4$

$$D_z = \frac{z}{q-p} + \frac{q}{q-p}\left(\frac{1-r^z}{1-r^a}\right)$$

$$D_z = \frac{500}{-0.2} + \frac{0.4}{-0.2}\left(1-(q/p)\right.$$