

# VIP Refresher: Linear Algebra and Calculus

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October 6, 2018

## General notations

□ **Vector** – We note  $x \in \mathbb{R}^n$  a vector with  $n$  entries, where  $x_i \in \mathbb{R}$  is the  $i^{th}$  entry:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

□ **Matrix** – We note  $A \in \mathbb{R}^{m \times n}$  a matrix with  $m$  rows and  $n$  columns, where  $A_{i,j} \in \mathbb{R}$  is the entry located in the  $i^{th}$  row and  $j^{th}$  column:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

*Remark: the vector  $x$  defined above can be viewed as a  $n \times 1$  matrix and is more particularly called a column-vector.*

□ **Identity matrix** – The identity matrix  $I \in \mathbb{R}^{n \times n}$  is a square matrix with ones in its diagonal and zero everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

*Remark: for all matrices  $A \in \mathbb{R}^{n \times n}$ , we have  $A \times I = I \times A = A$ .*

□ **Diagonal matrix** – A diagonal matrix  $D \in \mathbb{R}^{n \times n}$  is a square matrix with nonzero values in its diagonal and zero everywhere else:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & d_n \end{pmatrix}$$

*Remark: we also note  $D$  as  $\text{diag}(d_1, \dots, d_n)$ .*

## Matrix operations

□ **Vector-vector multiplication** – There are two types of vector-vector products:

- inner product: for  $x, y \in \mathbb{R}^n$ , we have:

$$x^T y = \sum_{i=1}^n x_i y_i \in \mathbb{R}$$

- outer product: for  $x \in \mathbb{R}^m, y \in \mathbb{R}^n$ , we have:

$$xy^T = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{pmatrix} \in \mathbb{R}^{m \times n}$$

□ **Matrix-vector multiplication** – The product of matrix  $A \in \mathbb{R}^{m \times n}$  and vector  $x \in \mathbb{R}^n$  is a vector of size  $\mathbb{R}^m$ , such that:

$$Ax = \begin{pmatrix} a_{r,1}^T x \\ \vdots \\ a_{r,m}^T x \end{pmatrix} = \sum_{i=1}^n a_{c,i} x_i \in \mathbb{R}^m$$

where  $a_{r,i}^T$  are the vector rows and  $a_{c,j}$  are the vector columns of  $A$ , and  $x_i$  are the entries of  $x$ .

□ **Matrix-matrix multiplication** – The product of matrices  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times p}$  is a matrix of size  $\mathbb{R}^{m \times p}$ , such that:

$$AB = \begin{pmatrix} a_{r,1}^T b_{c,1} & \cdots & a_{r,1}^T b_{c,p} \\ \vdots & & \vdots \\ a_{r,m}^T b_{c,1} & \cdots & a_{r,m}^T b_{c,p} \end{pmatrix} = \sum_{i=1}^n a_{c,i} b_{r,i}^T \in \mathbb{R}^{m \times p}$$

where  $a_{r,i}^T, b_{r,i}^T$  are the vector rows and  $a_{c,j}, b_{c,j}$  are the vector columns of  $A$  and  $B$  respectively.

□ **Transpose** – The transpose of a matrix  $A \in \mathbb{R}^{m \times n}$ , noted  $A^T$ , is such that its entries are flipped:

$$\forall i, j, \quad A_{i,j}^T = A_{j,i}$$

*Remark: for matrices  $A, B$ , we have  $(AB)^T = B^T A^T$ .*

□ **Inverse** – The inverse of an invertible square matrix  $A$  is noted  $A^{-1}$  and is the only matrix such that:

$$AA^{-1} = A^{-1}A = I$$

*Remark: not all square matrices are invertible. Also, for matrices  $A, B$ , we have  $(AB)^{-1} = B^{-1}A^{-1}$*

□ **Trace** – The trace of a square matrix  $A$ , noted  $\text{tr}(A)$ , is the sum of its diagonal entries:

$$\text{tr}(A) = \sum_{i=1}^n A_{i,i}$$

*Remark: for matrices  $A, B$ , we have  $\text{tr}(A^T) = \text{tr}(A)$  and  $\text{tr}(AB) = \text{tr}(BA)$*

□ **Determinant** – The determinant of a square matrix  $A \in \mathbb{R}^{n \times n}$ , noted  $|A|$  or  $\det(A)$  is expressed recursively in terms of  $A_{\setminus i, \setminus j}$ , which is the matrix  $A$  without its  $i^{th}$  row and  $j^{th}$  column, as follows:

$$\det(A) = |A| = \sum_{j=1}^n (-1)^{i+j} A_{i,j} |A_{\setminus i, \setminus j}|$$

*Remark:  $A$  is invertible if and only if  $|A| \neq 0$ . Also,  $|AB| = |A||B|$  and  $|A^T| = |A|$ .*

## Matrix properties

□ **Symmetric decomposition** – A given matrix  $A$  can be expressed in terms of its symmetric and antisymmetric parts as follows:

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{Symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{Antisymmetric}}$$

□ **Norm** – A norm is a function  $N : V \rightarrow [0, +\infty[$  where  $V$  is a vector space, and such that for all  $x, y \in V$ , we have:

- $N(x + y) \leq N(x) + N(y)$
- $N(ax) = |a|N(x)$  for  $a$  scalar
- if  $N(x) = 0$ , then  $x = 0$

For  $x \in V$ , the most commonly used norms are summed up in the table below:

Norm	Notation	Definition	Use case
Manhattan, $L^1$	$\ x\ _1$	$\sum_{i=1}^n  x_i $	LASSO regularization
Euclidean, $L^2$	$\ x\ _2$	$\sqrt{\sum_{i=1}^n x_i^2}$	Ridge regularization
$p$ -norm, $L^p$	$\ x\ _p$	$\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$	Hölder inequality
Infinity, $L^\infty$	$\ x\ _\infty$	$\max_i  x_i $	Uniform convergence

□ **Linearly dependence** – A set of vectors is said to be linearly dependent if one of the vectors in the set can be defined as a linear combination of the others.

*Remark: if no vector can be written this way, then the vectors are said to be linearly independent.*

□ **Matrix rank** – The rank of a given matrix  $A$  is noted  $\text{rank}(A)$  and is the dimension of the vector space generated by its columns. This is equivalent to the maximum number of linearly independent columns of  $A$ .

□ **Positive semi-definite matrix** – A matrix  $A \in \mathbb{R}^{n \times n}$  is positive semi-definite (PSD) and is noted  $A \succeq 0$  if we have:

$$A = A^T \quad \text{and} \quad \forall x \in \mathbb{R}^n, \quad x^T A x \geq 0$$

*Remark: similarly, a matrix  $A$  is said to be positive definite, and is noted  $A \succ 0$ , if it is a PSD matrix which satisfies for all non-zero vector  $x$ ,  $x^T A x > 0$ .*

□ **Eigenvalue, eigenvector** – Given a matrix  $A \in \mathbb{R}^{n \times n}$ ,  $\lambda$  is said to be an eigenvalue of  $A$  if there exists a vector  $z \in \mathbb{R}^n \setminus \{0\}$ , called eigenvector, such that we have:

$$Az = \lambda z$$

□ **Spectral theorem** – Let  $A \in \mathbb{R}^{n \times n}$ . If  $A$  is symmetric, then  $A$  is diagonalizable by a real orthogonal matrix  $U \in \mathbb{R}^{n \times n}$ . By noting  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ , we have:

$$\exists \Lambda \text{ diagonal, } A = U \Lambda U^T$$

□ **Singular-value decomposition** – For a given matrix  $A$  of dimensions  $m \times n$ , the singular-value decomposition (SVD) is a factorization technique that guarantees the existence of  $U \in \mathbb{R}^{m \times m}$  unitary,  $\Sigma \in \mathbb{R}^{m \times n}$  diagonal and  $V \in \mathbb{R}^{n \times n}$  unitary matrices, such that:

$$A = U \Sigma V^T$$

## Matrix calculus

□ **Gradient** – Let  $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  be a function and  $A \in \mathbb{R}^{m \times n}$  be a matrix. The gradient of  $f$  with respect to  $A$  is a  $m \times n$  matrix, noted  $\nabla_A f(A)$ , such that:

$$\left( \nabla_A f(A) \right)_{i,j} = \frac{\partial f(A)}{\partial A_{i,j}}$$

*Remark: the gradient of  $f$  is only defined when  $f$  is a function that returns a scalar.*

□ **Hessian** – Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function and  $x \in \mathbb{R}^n$  be a vector. The hessian of  $f$  with respect to  $x$  is a  $n \times n$  symmetric matrix, noted  $\nabla_x^2 f(x)$ , such that:

$$\left( \nabla_x^2 f(x) \right)_{i,j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

*Remark: the hessian of  $f$  is only defined when  $f$  is a function that returns a scalar.*

□ **Gradient operations** – For matrices  $A, B, C$ , the following gradient properties are worth having in mind:

$$\nabla_A \text{tr}(AB) = B^T$$

$$\nabla_{A^T} f(A) = (\nabla_A f(A))^T$$

$$\nabla_A \text{tr}(ABA^T C) = CAB + C^T A B^T$$

$$\nabla_A |A| = |A| (A^{-1})^T$$

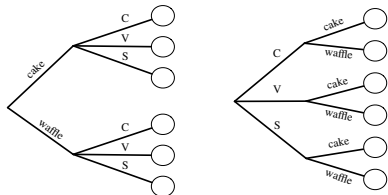
# Probability Cheatsheet v2.0

Compiled by William Chen (<http://wzchen.com>) and Joe Blitzstein, with contributions from Sebastian Chiu, Yuan Jiang, Yuqi Hou, and Jessy Hwang. Material based on Joe Blitzstein's (@stat110) lectures (<http://stat110.net>) and Blitzstein/Hwang's Introduction to Probability textbook (<http://bit.ly/introprobability>). Licensed under CC BY-NC-SA 4.0. Please share comments, suggestions, and errors at [http://github.com/wzchen/probability\\_cheatsheet](http://github.com/wzchen/probability_cheatsheet).

Last Updated September 4, 2015

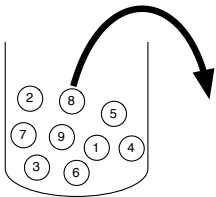
## Counting

### Multiplication Rule



Let's say we have a compound experiment (an experiment with multiple components). If the 1st component has  $n_1$  possible outcomes, the 2nd component has  $n_2$  possible outcomes, ..., and the  $r$ th component has  $n_r$  possible outcomes, then overall there are  $n_1 n_2 \dots n_r$  possibilities for the whole experiment.

### Sampling Table



The sampling table gives the number of possible samples of size  $k$  out of a population of size  $n$ , under various assumptions about how the sample is collected.

	Order Matters	Not Matter
With Replacement	$n^k$	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

### Naive Definition of Probability

If all outcomes are equally likely, the probability of an event  $A$  happening is:

$$P_{\text{naive}}(A) = \frac{\text{number of outcomes favorable to } A}{\text{number of outcomes}}$$

## Thinking Conditionally

### Independence

**Independent Events**  $A$  and  $B$  are independent if knowing whether  $A$  occurred gives no information about whether  $B$  occurred. More formally,  $A$  and  $B$  (which have nonzero probability) are independent if and only if one of the following equivalent statements holds:

$$\begin{aligned}P(A \cap B) &= P(A)P(B) \\ P(A|B) &= P(A) \\ P(B|A) &= P(B)\end{aligned}$$

**Conditional Independence**  $A$  and  $B$  are conditionally independent given  $C$  if  $P(A \cap B|C) = P(A|C)P(B|C)$ . Conditional independence does not imply independence, and independence does not imply conditional independence.

### Unions, Intersections, and Complements

**De Morgan's Laws** A useful identity that can make calculating probabilities of unions easier by relating them to intersections, and vice versa. Analogous results hold with more than two sets.

$$\begin{aligned}(A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c\end{aligned}$$

### Joint, Marginal, and Conditional

**Joint Probability**  $P(A \cap B)$  or  $P(A, B)$  – Probability of  $A$  and  $B$ .

**Marginal (Unconditional) Probability**  $P(A)$  – Probability of  $A$ .

**Conditional Probability**  $P(A|B) = P(A, B)/P(B)$  – Probability of  $A$ , given that  $B$  occurred.

**Conditional Probability is Probability**  $P(A|B)$  is a probability function for any fixed  $B$ . Any theorem that holds for probability also holds for conditional probability.

### Probability of an Intersection or Union

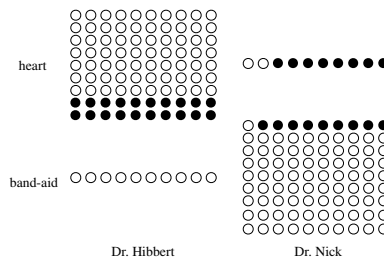
**Intersections via Conditioning**

$$\begin{aligned}P(A, B) &= P(A)P(B|A) \\ P(A, B, C) &= P(A)P(B|A)P(C|A, B)\end{aligned}$$

**Unions via Inclusion-Exclusion**

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C).\end{aligned}$$

### Simpson's Paradox



It is possible to have

$$\begin{aligned}P(A | B, C) &< P(A | B^c, C) \text{ and } P(A | B, C^c) < P(A | B^c, C^c) \\ \text{yet also } P(A | B) &> P(A | B^c).\end{aligned}$$

### Law of Total Probability (LOTP)

Let  $B_1, B_2, B_3, \dots, B_n$  be a *partition* of the sample space (i.e., they are disjoint and their union is the entire sample space).

$$\begin{aligned}P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n) \\ P(A) &= P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n)\end{aligned}$$

For **LOTP with extra conditioning**, just add in another event  $C$ !

$$\begin{aligned}P(A|C) &= P(A|B_1, C)P(B_1|C) + \dots + P(A|B_n, C)P(B_n|C) \\ P(A|C) &= P(A \cap B_1|C) + P(A \cap B_2|C) + \dots + P(A \cap B_n|C)\end{aligned}$$

Special case of LOTP with  $B$  and  $B^c$  as partition:

$$\begin{aligned}P(A) &= P(A|B)P(B) + P(A|B^c)P(B^c) \\ P(A) &= P(A \cap B) + P(A \cap B^c)\end{aligned}$$

### Bayes' Rule

**Bayes' Rule, and with extra conditioning (just add in  $C$ !)**

$$\begin{aligned}P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\ P(A|B, C) &= \frac{P(B|A, C)P(A|C)}{P(B|C)}\end{aligned}$$

We can also write

$$P(A|B, C) = \frac{P(A, B, C)}{P(B, C)} = \frac{P(B, C|A)P(A)}{P(B, C)}$$

**Odds Form of Bayes' Rule**

$$\frac{P(A|B)}{P(A^c|B)} = \frac{P(B|A)}{P(B|A^c)} \frac{P(A)}{P(A^c)}$$

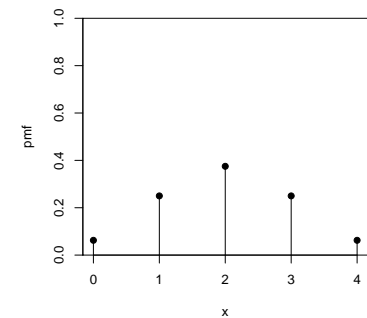
The *posterior odds* of  $A$  are the *likelihood ratio* times the *prior odds*.

## Random Variables and their Distributions

### PMF, CDF, and Independence

**Probability Mass Function (PMF)** Gives the probability that a *discrete* random variable takes on the value  $x$ .

$$p_X(x) = P(X = x)$$

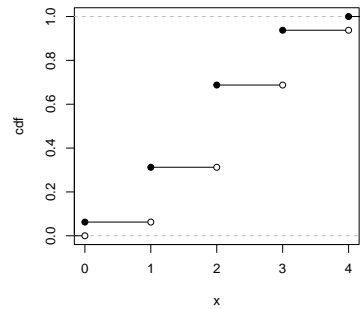


The PMF satisfies

$$p_X(x) \geq 0 \text{ and } \sum_x p_X(x) = 1$$

**Cumulative Distribution Function (CDF)** Gives the probability that a random variable is less than or equal to  $x$ .

$F_X(x) = P(X \leq x)$



The CDF is an increasing, right-continuous function with  $F_X(x) \rightarrow 0$  as  $x \rightarrow -\infty$  and  $F_X(x) \rightarrow 1$  as  $x \rightarrow \infty$

**Independence** Intuitively, two random variables are independent if knowing the value of one gives no information about the other. Discrete r.v.s  $X$  and  $Y$  are independent if for *all* values of  $x$  and  $y$

$P(X = x, Y = y) = P(X = x)P(Y = y)$

Expected Value and Indicators

Expected Value and Linearity

**Expected Value** (a.k.a. *mean*, *expectation*, or *average*) is a weighted average of the possible outcomes of our random variable. Mathematically, if  $x_1, x_2, x_3, \dots$  are all of the distinct possible values that  $X$  can take, the expected value of  $X$  is

$E(X) = \sum_i x_i P(X = x_i)$

$X$	$Y$	$X + Y$
3	4	7
2	2	4
6	8	14
10	23	33
1	-3	-2
1	0	1
5	9	14
4	1	5
...	...	...

$$\frac{1}{n} \sum_{i=1}^n x_i + \frac{1}{n} \sum_{i=1}^n y_i = \frac{1}{n} \sum_{i=1}^n (x_i + y_i)$$
$$E(X) + E(Y) = E(X + Y)$$

**Linearity** For any r.v.s  $X$  and  $Y$ , and constants  $a, b, c$ ,

$E(aX + bY + c) = aE(X) + bE(Y) + c$

**Same distribution implies same mean** If  $X$  and  $Y$  have the same distribution, then  $E(X) = E(Y)$  and, more generally,

$E(g(X)) = E(g(Y))$

**Conditional Expected Value** is defined like expectation, only conditioned on any event  $A$ .

$E(X|A) = \sum_x xP(X = x|A)$

Indicator Random Variables

**Indicator Random Variable** is a random variable that takes on the value 1 or 0. It is always an indicator of some event: if the event occurs, the indicator is 1; otherwise it is 0. They are useful for many problems about counting how many events of some kind occur. Write

$$I_A = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

Note that  $I_A^2 = I_A$ ,  $I_A I_B = I_{A \cap B}$ , and  $I_{A \cup B} = I_A + I_B - I_A I_B$ .

**Distribution**  $I_A \sim \text{Bern}(p)$  where  $p = P(A)$ .

**Fundamental Bridge** The expectation of the indicator for event  $A$  is the probability of event  $A$ :  $E(I_A) = P(A)$ .

Variance and Standard Deviation

$$\text{Var}(X) = E(X - E(X))^2 = E(X^2) - (E(X))^2$$
$$\text{SD}(X) = \sqrt{\text{Var}(X)}$$

Continuous RVs, LOTUS, UoU

Continuous Random Variables (CRVs)

**What's the probability that a CRV is in an interval?** Take the difference in CDF values (or use the PDF as described later).

$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a)$

For  $X \sim \mathcal{N}(\mu, \sigma^2)$ , this becomes

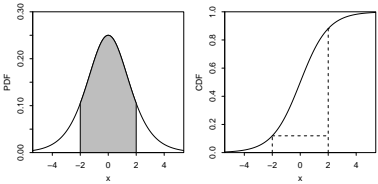
$$P(a \leq X \leq b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

**What is the Probability Density Function (PDF)?** The PDF  $f$  is the derivative of the CDF  $F$ .

$F'(x) = f(x)$

A PDF is nonnegative and integrates to 1. By the fundamental theorem of calculus, to get from PDF back to CDF we can integrate:

$$F(x) = \int_{-\infty}^x f(t)dt$$



To find the probability that a CRV takes on a value in an interval, integrate the PDF over that interval.

$$F(b) - F(a) = \int_a^b f(x)dx$$

**How do I find the expected value of a CRV?** Analogous to the discrete case, where you sum  $x$  times the PMF, for CRVs you integrate  $x$  times the PDF.

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

LOTUS

**Expected value of a function of an r.v.** The expected value of  $X$  is defined this way:

$$E(X) = \sum_x xP(X = x) \text{ (for discrete } X)$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx \text{ (for continuous } X)$$

The **Law of the Unconscious Statistician (LOTUS)** states that you can find the expected value of a *function of a random variable*,  $g(X)$ , in a similar way, by replacing the  $x$  in front of the PMF/PDF by  $g(x)$  but still working with the PMF/PDF of  $X$ :

$$E(g(X)) = \sum_x g(x)P(X = x) \text{ (for discrete } X)$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx \text{ (for continuous } X)$$

**What's a function of a random variable?** A function of a random variable is also a random variable. For example, if  $X$  is the number of bikes you see in an hour, then  $g(X) = 2X$  is the number of bike wheels you see in that hour and  $h(X) = \binom{X}{2} = \frac{X(X-1)}{2}$  is the number of *pairs* of bikes such that you see both of those bikes in that hour.

**What's the point?** You don't need to know the PMF/PDF of  $g(X)$  to find its expected value. All you need is the PMF/PDF of  $X$ .

Universality of Uniform (UoU)

When you plug any CRV into its own CDF, you get a Uniform(0,1) random variable. When you plug a Uniform(0,1) r.v. into an inverse CDF, you get an r.v. with that CDF. For example, let's say that a random variable  $X$  has CDF

$F(x) = 1 - e^{-x}$ , for  $x > 0$

By UoU, if we plug  $X$  into this function then we get a uniformly distributed random variable.

$F(X) = 1 - e^{-X} \sim \text{Unif}(0, 1)$

Similarly, if  $U \sim \text{Unif}(0, 1)$  then  $F^{-1}(U)$  has CDF  $F$ . The key point is that for any continuous random variable  $X$ , we can transform it into a Uniform random variable and back by using its CDF.

Moments and MGFs

Moments

Moments describe the shape of a distribution. Let  $X$  have mean  $\mu$  and standard deviation  $\sigma$ , and  $Z = (X - \mu)/\sigma$  be the *standardized* version of  $X$ . The  $k$ th moment of  $X$  is  $m_k = E(X^k)$  and the  $k$ th standardized moment of  $X$  is  $m_k = E(Z^k)$ . The mean, variance, skewness, and kurtosis are important summaries of the shape of a distribution.

**Mean**  $E(X) = \mu_1$

**Variance**  $\text{Var}(X) = \mu_2 - \mu_1^2$

**Skewness**  $\text{Skew}(X) = m_3$

**Kurtosis**  $\text{Kurt}(X) = m_4 - 3$

Moment Generating Functions

MGF For any random variable X, the function

M\_X(t) = E(e^{tX})

is the **moment generating function (MGF)** of X, if it exists for all t in some open interval containing 0. The variable t could just as well have been called u or v. It's a bookkeeping device that lets us work with the function M\_X rather than the sequence of moments.

**Why is it called the Moment Generating Function?** Because the kth derivative of the moment generating function, evaluated at 0, is the kth moment of X.

\mu\_k = E(X^k) = M\_X^{(k)}(0)

This is true by Taylor expansion of e^{tX} since

M\_X(t) = E(e^{tX}) = \sum\_{k=0}^\infty \frac{E(X^k)t^k}{k!} = \sum\_{k=0}^\infty \frac{\mu\_k t^k}{k!}

MGF of linear functions If we have Y = aX + b, then

M\_Y(t) = E(e^{t(aX+b)}) = e^{bt} E(e^{(at)X}) = e^{bt} M\_X(at)

**Uniqueness** If it exists, the MGF uniquely determines the distribution. This means that for any two random variables X and Y, they are distributed the same (their PMFs/PDFs are equal) if and only if their MGFs are equal.

**Summing Independent RVs by Multiplying MGFs.** If X and Y are independent, then

M\_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY}) = M\_X(t) \cdot M\_Y(t)

The MGF of the sum of two random variables is the product of the MGFs of those two random variables.

Joint PDFs and CDFs

Joint Distributions

The **joint CDF** of X and Y is

F(x, y) = P(X \leq x, Y \leq y)

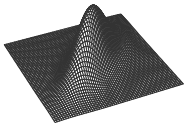
In the discrete case, X and Y have a **joint PMF**

p\_{X,Y}(x, y) = P(X = x, Y = y).

In the continuous case, they have a **joint PDF**

f\_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F\_{X,Y}(x, y).

The joint PMF/PDF must be nonnegative and sum/integrate to 1.



Conditional Distributions

Conditioning and Bayes' rule for discrete r.v.s

P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{P(X = x|Y = y)P(Y = y)}{P(X = x)}

Conditioning and Bayes' rule for continuous r.v.s

f\_{Y|X}(y|x) = \frac{f\_{X,Y}(x, y)}{f\_X(x)} = \frac{f\_{X|Y}(x|y)f\_Y(y)}{f\_X(x)}

Hybrid Bayes' rule

f\_X(x|A) = \frac{P(A|X = x)f\_X(x)}{P(A)}

Marginal Distributions

To find the distribution of one (or more) random variables from a joint PMF/PDF, sum/integrate over the unwanted random variables.

Marginal PMF from joint PMF

P(X = x) = \sum\_y P(X = x, Y = y)

Marginal PDF from joint PDF

f\_X(x) = \int\_{-\infty}^\infty f\_{X,Y}(x, y)dy

Independence of Random Variables

Random variables X and Y are independent if and only if any of the following conditions holds:

- Joint CDF is the product of the marginal CDFs
- Joint PMF/PDF is the product of the marginal PMFs/PDFs
- Conditional distribution of Y given X is the marginal distribution of Y

Write X \perp\!\!\!\perp Y to denote that X and Y are independent.

Multivariate LOTUS

LOTUS in more than one dimension is analogous to the 1D LOTUS. For discrete random variables:

E(g(X, Y)) = \sum\_x \sum\_y g(x, y)P(X = x, Y = y)

For continuous random variables:

E(g(X, Y)) = \int\_{-\infty}^\infty \int\_{-\infty}^\infty g(x, y)f\_{X,Y}(x, y)dxdy

Covariance and Transformations

Covariance and Correlation

**Covariance** is the analog of variance for two random variables.

Cov(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)

Note that

Cov(X, X) = E(X^2) - (E(X))^2 = Var(X)

**Correlation** is a standardized version of covariance that is always between -1 and 1.

Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}

**Covariance and Independence** If two random variables are independent, then they are uncorrelated. The converse is not necessarily true (e.g., consider X \sim \mathcal{N}(0, 1) and Y = X^2).

X \perp\!\!\!\perp Y \implies Cov(X, Y) = 0 \implies E(XY) = E(X)E(Y)

**Covariance and Variance** The variance of a sum can be found by

Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)

Var(X\_1 + X\_2 + \dots + X\_n) = \sum\_{i=1}^n Var(X\_i) + 2 \sum\_{i < j} Cov(X\_i, X\_j)

If X and Y are independent then they have covariance 0, so

X \perp\!\!\!\perp Y \implies Var(X + Y) = Var(X) + Var(Y)

If X\_1, X\_2, \dots, X\_n are identically distributed and have the same covariance relationships (often by **symmetry**), then

Var(X\_1 + X\_2 + \dots + X\_n) = nVar(X\_1) + 2\binom{n}{2}Cov(X\_1, X\_2)

**Covariance Properties** For random variables W, X, Y, Z and constants a, b:

Cov(X, Y) = Cov(Y, X)

Cov(X + a, Y + b) = Cov(X, Y)

Cov(aX, bY) = abCov(X, Y)

Cov(W + X, Y + Z) = Cov(W, Y) + Cov(W, Z) + Cov(X, Y) + Cov(X, Z)

**Correlation is location-invariant and scale-invariant** For any constants a, b, c, d with a and c nonzero,

Corr(aX + b, cY + d) = Corr(X, Y)

Transformations

**One Variable Transformations** Let's say that we have a random variable X with PDF f\_X(x), but we are also interested in some function of X. We call this function Y = g(X). Also let y = g(x). If g is differentiable and strictly increasing (or strictly decreasing), then the PDF of Y is

f\_Y(y) = f\_X(x) \left| \frac{dx}{dy} \right| = f\_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|

The derivative of the inverse transformation is called the **Jacobian**.

**Two Variable Transformations** Similarly, let's say we know the joint PDF of U and V but are also interested in the random vector (X, Y) defined by (X, Y) = g(U, V). Let

\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}

be the **Jacobian matrix**. If the entries in this matrix exist and are continuous, and the determinant of the matrix is never 0, then

f\_{X,Y}(x, y) = f\_{U,V}(u, v) \left| \frac{\partial(u, v)}{\partial(x, y)} \right|

The inner bars tells us to take the matrix's determinant, and the outer bars tell us to take the absolute value. In a 2 \times 2 matrix,

\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|

Convolutions

**Convolution Integral** If you want to find the PDF of the sum of two independent CRVs X and Y, you can do the following integral:

f\_{X+Y}(t) = \int\_{-\infty}^\infty f\_X(x)f\_Y(t - x)dx

**Example** Let X, Y \sim \mathcal{N}(0, 1) be i.i.d. Then for each fixed t,

f\_{X+Y}(t) = \int\_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \frac{1}{\sqrt{2\pi}}e^{-(t-x)^2/2}dx

By completing the square and using the fact that a Normal PDF integrates to 1, this works out to f\_{X+Y}(t) being the \mathcal{N}(0, 2) PDF.

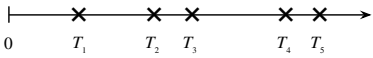


Poisson Process

**Definition** We have a **Poisson process** of rate  $\lambda$  arrivals per unit time if the following conditions hold:

- 1. The number of arrivals in a time interval of length  $t$  is  $\text{Pois}(\lambda t)$ .
- 2. Numbers of arrivals in disjoint time intervals are independent.

For example, the numbers of arrivals in the time intervals  $[0, 5]$ ,  $(5, 12]$ , and  $[13, 23]$  are independent with  $\text{Pois}(5\lambda)$ ,  $\text{Pois}(7\lambda)$ ,  $\text{Pois}(10\lambda)$  distributions, respectively.



**Count-Time Duality** Consider a Poisson process of emails arriving in an inbox at rate  $\lambda$  emails per hour. Let  $T_n$  be the time of arrival of the  $n$ th email (relative to some starting time 0) and  $N_t$  be the number of emails that arrive in  $[0, t]$ . Let's find the distribution of  $T_1$ . The event  $T_1 > t$ , the event that you have to wait more than  $t$  hours to get the first email, is the same as the event  $N_t = 0$ , which is the event that there are no emails in the first  $t$  hours. So

P(T1 > t) = P(Nt = 0) = e^{-λt} → P(T1 ≤ t) = 1 - e^{-λt}

Thus we have  $T_1 \sim \text{Expo}(\lambda)$ . By the memoryless property and similar reasoning, the interarrival times between emails are i.i.d.  $\text{Expo}(\lambda)$ , i.e., the differences  $T_n - T_{n-1}$  are i.i.d.  $\text{Expo}(\lambda)$ .

Order Statistics

**Definition** Let's say you have  $n$  i.i.d. r.v.s  $X_1, X_2, \dots, X_n$ . If you arrange them from smallest to largest, the  $i$ th element in that list is the  $i$ th order statistic, denoted  $X_{(i)}$ . So  $X_{(1)}$  is the smallest in the list and  $X_{(n)}$  is the largest in the list.

Note that the order statistics are *dependent*, e.g., learning  $X_{(4)} = 42$  gives us the information that  $X_{(1)}, X_{(2)}, X_{(3)}$  are  $\leq 42$  and  $X_{(5)}, X_{(6)}, \dots, X_{(n)}$  are  $\geq 42$ .

**Distribution** Taking  $n$  i.i.d. random variables  $X_1, X_2, \dots, X_n$  with CDF  $F(x)$  and PDF  $f(x)$ , the CDF and PDF of  $X_{(i)}$  are:

FX(i)(x) = P(X(i) ≤ x) = ∑\_{k=i}^n (n choose k) F(x)^k (1 - F(x))^{n-k}

fX(i)(x) = n (n-1 choose i-1) F(x)^{i-1} (1 - F(x))^{n-i} f(x)

**Uniform Order Statistics** The  $j$ th order statistic of i.i.d.  $U_1, \dots, U_n \sim \text{Unif}(0, 1)$  is  $U_{(j)} \sim \text{Beta}(j, n - j + 1)$ .

Conditional Expectation

**Conditioning on an Event** We can find  $E(Y|A)$ , the expected value of  $Y$  given that event  $A$  occurred. A very important case is when  $A$  is the event  $X = x$ . Note that  $E(Y|A)$  is a *number*. For example:

- The expected value of a fair die roll, given that it is prime, is  $\frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 5 = \frac{10}{3}$ .
- Let  $Y$  be the number of successes in 10 independent Bernoulli trials with probability  $p$  of success. Let  $A$  be the event that the first 3 trials are all successes. Then

E(Y|A) = 3 + 7p

since the number of successes among the last 7 trials is  $\text{Bin}(7, p)$ .

- Let  $T \sim \text{Expo}(1/10)$  be how long you have to wait until the shuttle comes. Given that you have already waited  $t$  minutes, the expected additional waiting time is 10 more minutes, by the memoryless property. That is,  $E(T|T > t) = t + 10$ .

Discrete $Y$	Continuous $Y$
$E(Y) = \sum_y yP(Y = y)$	$E(Y) = \int_{-\infty}^{\infty} yf_Y(y)dy$
$E(Y A) = \sum_y yP(Y = y A)$	$E(Y A) = \int_{-\infty}^{\infty} yf(y A)dy$

**Conditioning on a Random Variable** We can also find  $E(Y|X)$ , the expected value of  $Y$  given the random variable  $X$ . This is a *function of the random variable  $X$* . It is *not* a number except in certain special cases such as if  $X \perp\!\!\!\perp Y$ . To find  $E(Y|X)$ , find  $E(Y|X = x)$  and then plug in  $X$  for  $x$ . For example:

- If  $E(Y|X = x) = x^3 + 5x$ , then  $E(Y|X) = X^3 + 5X$ .
- Let  $Y$  be the number of successes in 10 independent Bernoulli trials with probability  $p$  of success and  $X$  be the number of successes among the first 3 trials. Then  $E(Y|X) = X + 7p$ .
- Let  $X \sim \mathcal{N}(0, 1)$  and  $Y = X^2$ . Then  $E(Y|X = x) = x^2$  since if we know  $X = x$  then we know  $Y = x^2$ . And  $E(X|Y = y) = 0$  since if we know  $Y = y$  then we know  $X = \pm\sqrt{y}$ , with equal probabilities (by symmetry). So  $E(Y|X) = X^2, E(X|Y) = 0$ .

Properties of Conditional Expectation

1.  $E(Y|X) = E(Y)$  if  $X \perp\!\!\!\perp Y$
2.  $E(h(X)W|X) = h(X)E(W|X)$  (**taking out what's known**)  
In particular,  $E(h(X)|X) = h(X)$ .
3.  $E(E(Y|X)) = E(Y)$  (**Adam's Law**, a.k.a. Law of Total Expectation)

**Adam's Law (a.k.a. Law of Total Expectation)** can also be written in a way that looks analogous to LOTP. For any events  $A_1, A_2, \dots, A_n$  that partition the sample space,

E(Y) = E(Y|A1)P(A1) + ... + E(Y|An)P(An)

For the special case where the partition is  $A, A^c$ , this says

E(Y) = E(Y|A)P(A) + E(Y|A^c)P(A^c)

Eve's Law (a.k.a. Law of Total Variance)

Var(Y) = E(Var(Y|X)) + Var(E(Y|X))

MVN, LLN, CLT

Law of Large Numbers (LLN)

Let  $X_1, X_2, X_3 \dots$  be i.i.d. with mean  $\mu$ . The **sample mean** is

̄Xn = (X1 + X2 + X3 + ... + Xn) / n

The **Law of Large Numbers** states that as  $n \rightarrow \infty, \bar{X}_n \rightarrow \mu$  with probability 1. For example, in flips of a coin with probability  $p$  of Heads, let  $X_j$  be the indicator of the  $j$ th flip being Heads. Then LLN says the proportion of Heads converges to  $p$  (with probability 1).

Central Limit Theorem (CLT)

Approximation using CLT

We use  $\sim$  to denote *is approximately distributed*. We can use the **Central Limit Theorem** to approximate the distribution of a random variable  $Y = X_1 + X_2 + \dots + X_n$  that is a sum of  $n$  i.i.d. random variables  $X_i$ . Let  $E(Y) = \mu_Y$  and  $\text{Var}(Y) = \sigma_Y^2$ . The CLT says

Y ~ N(μY, σY^2)

If the  $X_i$  are i.i.d. with mean  $\mu_X$  and variance  $\sigma_X^2$ , then  $\mu_Y = n\mu_X$  and  $\sigma_Y^2 = n\sigma_X^2$ . For the sample mean  $\bar{X}_n$ , the CLT says

̄Xn = 1/n (X1 + X2 + ... + Xn) ~ N(μX, σX^2/n)

Asymptotic Distributions using CLT

We use  $\xrightarrow{D}$  to denote *converges in distribution to* as  $n \rightarrow \infty$ . The CLT says that if we standardize the sum  $X_1 + \dots + X_n$  then the distribution of the sum converges to  $\mathcal{N}(0, 1)$  as  $n \rightarrow \infty$ :

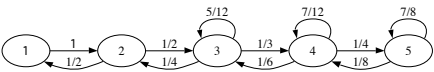
1/(σ√n) (X1 + ... + Xn - nμX) →D N(0, 1)

In other words, the CDF of the left-hand side goes to the standard Normal CDF,  $\Phi$ . In terms of the sample mean, the CLT says

√n(̄Xn - μX) / σX →D N(0, 1)

Markov Chains

Definition



A Markov chain is a random walk in a **state space**, which we will assume is finite, say  $\{1, 2, \dots, M\}$ . We let  $X_t$  denote which element of the state space the walk is visiting at time  $t$ . The Markov chain is the sequence of random variables tracking where the walk is at all points in time,  $X_0, X_1, X_2, \dots$ . By definition, a Markov chain must satisfy the **Markov property**, which says that if you want to predict where the chain will be at a future time, if we know the present state then the entire past history is irrelevant. *Given the present, the past and future are conditionally independent*. In symbols,

P(Xn+1 = j|X0 = i0, X1 = i1, ..., Xn = i) = P(Xn+1 = j|Xn = i)

State Properties

A state is either recurrent or transient.

- If you start at a **recurrent state**, then you will always return back to that state at some point in the future. ♡*You can check-out any time you like, but you can never leave.* ♡
- Otherwise you are at a **transient state**. There is some positive probability that once you leave you will never return. ♡*You don't have to go home, but you can't stay here.* ♡

A state is either periodic or aperiodic.

- If you start at a **periodic state** of period  $k$ , then the GCD of the possible numbers of steps it would take to return back is  $k > 1$ .
- Otherwise you are at an **aperiodic state**. The GCD of the possible numbers of steps it would take to return back is 1.

## Transition Matrix

Let the state space be  $\{1, 2, \dots, M\}$ . The transition matrix  $Q$  is the  $M \times M$  matrix where element  $q_{ij}$  is the probability that the chain goes from state  $i$  to state  $j$  in one step:

$$q_{ij} = P(X_{n+1} = j | X_n = i)$$

To find the probability that the chain goes from state  $i$  to state  $j$  in exactly  $m$  steps, take the  $(i, j)$  element of  $Q^m$ .

$$q_{ij}^{(m)} = P(X_{n+m} = j | X_n = i)$$

If  $X_0$  is distributed according to the row vector PMF  $\vec{p}$ , i.e.,  $p_j = P(X_0 = j)$ , then the PMF of  $X_n$  is  $\vec{p}Q^n$ .

## Chain Properties

A chain is **irreducible** if you can get from anywhere to anywhere. If a chain (on a finite state space) is irreducible, then all of its states are recurrent. A chain is **periodic** if any of its states are periodic, and is **aperiodic** if none of its states are periodic. In an irreducible chain, all states have the same period.

A chain is **reversible** with respect to  $\vec{s}$  if  $s_i q_{ij} = s_j q_{ji}$  for all  $i, j$ . Examples of reversible chains include any chain with  $q_{ij} = q_{ji}$ , with  $\vec{s} = (\frac{1}{M}, \frac{1}{M}, \dots, \frac{1}{M})$ , and random walk on an undirected network.

## Stationary Distribution

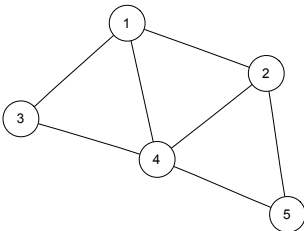
Let us say that the vector  $\vec{s} = (s_1, s_2, \dots, s_M)$  be a PMF (written as a row vector). We will call  $\vec{s}$  the **stationary distribution** for the chain if  $\vec{s}Q = \vec{s}$ . As a consequence, if  $X_t$  has the stationary distribution, then all future  $X_{t+1}, X_{t+2}, \dots$  also have the stationary distribution.

For irreducible, aperiodic chains, the stationary distribution exists, is unique, and  $s_i$  is the long-run probability of a chain being at state  $i$ . The expected number of steps to return to  $i$  starting from  $i$  is  $1/s_i$ .

To find the stationary distribution, you can solve the matrix equation  $(Q' - I)\vec{s}' = 0$ . The stationary distribution is uniform if the columns of  $Q$  sum to 1.

**Reversibility Condition Implies Stationarity** If you have a PMF  $\vec{s}$  and a Markov chain with transition matrix  $Q$ , then  $s_i q_{ij} = s_j q_{ji}$  for all states  $i, j$  implies that  $\vec{s}$  is stationary.

## Random Walk on an Undirected Network



If you have a collection of **nodes**, pairs of which can be connected by undirected **edges**, and a Markov chain is run by going from the current node to a uniformly random node that is connected to it by an edge, then this is a random walk on an undirected network. The stationary distribution of this chain is proportional to the **degree sequence** (this is the sequence of degrees, where the degree of a node is how many edges are attached to it). For example, the stationary distribution of random walk on the network shown above is proportional to  $(3, 3, 2, 4, 2)$ , so it's  $(\frac{3}{14}, \frac{3}{14}, \frac{2}{14}, \frac{4}{14}, \frac{2}{14})$ .

## Continuous Distributions

### Uniform Distribution

Let us say that  $U$  is distributed  $\text{Unif}(a, b)$ . We know the following:

**Properties of the Uniform** For a Uniform distribution, the probability of a draw from any interval within the support is proportional to the length of the interval. See *Universality of Uniform and Order Statistics* for other properties.

**Example** William throws darts really badly, so his darts are uniform over the whole room because they're equally likely to appear anywhere. William's darts have a Uniform distribution on the surface of the room. The Uniform is the only distribution where the probability of hitting in any specific region is proportional to the length/area/volume of that region, and where the density of occurrence in any one specific spot is constant throughout the whole support.

### Normal Distribution

Let us say that  $X$  is distributed  $\mathcal{N}(\mu, \sigma^2)$ . We know the following:

**Central Limit Theorem** The Normal distribution is ubiquitous because of the Central Limit Theorem, which states that the sample mean of i.i.d. r.v.s will approach a Normal distribution as the sample size grows, regardless of the initial distribution.

**Location-Scale Transformation** Every time we shift a Normal r.v. (by adding a constant) or rescale a Normal (by multiplying by a constant), we change it to another Normal r.v. For any Normal  $X \sim \mathcal{N}(\mu, \sigma^2)$ , we can transform it to the standard  $\mathcal{N}(0, 1)$  by the following transformation:

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

**Standard Normal** The Standard Normal,  $Z \sim \mathcal{N}(0, 1)$ , has mean 0 and variance 1. Its CDF is denoted by  $\Phi$ .

### Exponential Distribution

Let us say that  $X$  is distributed  $\text{Expo}(\lambda)$ . We know the following:

**Story** You're sitting on an open meadow right before the break of dawn, wishing that airplanes in the night sky were shooting stars, because you could really use a wish right now. You know that shooting stars come on average every 15 minutes, but a shooting star is not "due" to come just because you've waited so long. Your waiting time is memoryless; the additional time until the next shooting star comes does not depend on how long you've waited already.

**Example** The waiting time until the next shooting star is distributed  $\text{Expo}(4)$  hours. Here  $\lambda = 4$  is the **rate parameter**, since shooting stars arrive at a rate of 1 per  $1/4$  hour on average. The expected time until the next shooting star is  $1/\lambda = 1/4$  hour.

**Expos as a rescaled Expo(1)**

$$Y \sim \text{Expo}(\lambda) \rightarrow X = \lambda Y \sim \text{Expo}(1)$$

**Memorylessness** The Exponential Distribution is the only continuous memoryless distribution. The memoryless property says that for  $X \sim \text{Expo}(\lambda)$  and any positive numbers  $s$  and  $t$ ,

$$P(X > s + t | X > s) = P(X > t)$$

Equivalently,

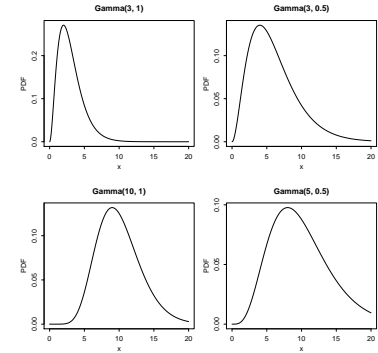
$$X - a | (X > a) \sim \text{Expo}(\lambda)$$

For example, a product with an  $\text{Expo}(\lambda)$  lifetime is always "as good as new" (it doesn't experience wear and tear). Given that the product has survived  $a$  years, the additional time that it will last is still  $\text{Expo}(\lambda)$ .

**Min of Expos** If we have independent  $X_i \sim \text{Expo}(\lambda_i)$ , then  $\min(X_1, \dots, X_k) \sim \text{Expo}(\lambda_1 + \lambda_2 + \dots + \lambda_k)$ .

**Max of Expos** If we have i.i.d.  $X_i \sim \text{Expo}(\lambda)$ , then  $\max(X_1, \dots, X_k)$  has the same distribution as  $Y_1 + Y_2 + \dots + Y_k$ , where  $Y_j \sim \text{Expo}(j\lambda)$  and the  $Y_j$  are independent.

## Gamma Distribution

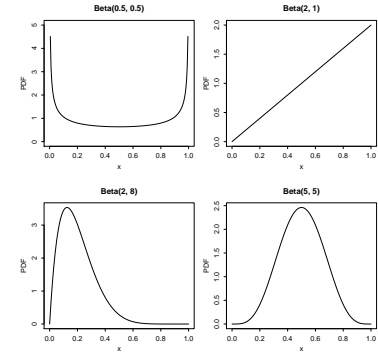


Let us say that  $X$  is distributed  $\text{Gamma}(a, \lambda)$ . We know the following:

**Story** You sit waiting for shooting stars, where the waiting time for a star is distributed  $\text{Expo}(\lambda)$ . You want to see  $n$  shooting stars before you go home. The total waiting time for the  $n$ th shooting star is  $\text{Gamma}(n, \lambda)$ .

**Example** You are at a bank, and there are 3 people ahead of you. The serving time for each person is Exponential with mean 2 minutes. Only one person at a time can be served. The distribution of your waiting time until it's your turn to be served is  $\text{Gamma}(3, \frac{1}{2})$ .

## Beta Distribution



**Conjugate Prior of the Binomial** In the Bayesian approach to statistics, parameters are viewed as random variables, to reflect our uncertainty. The *prior* for a parameter is its distribution before observing data. The *posterior* is the distribution for the parameter after observing data. Beta is the *conjugate* prior of the Binomial because if you have a Beta-distributed prior on  $p$  in a Binomial, then the posterior distribution on  $p$  given the Binomial data is also Beta-distributed. Consider the following two-level model:

$$X | p \sim \text{Bin}(n, p) \\ p \sim \text{Beta}(a, b)$$

Then after observing  $X = x$ , we get the posterior distribution

$$p | (X = x) \sim \text{Beta}(a + x, b + n - x)$$

**Order statistics of the Uniform** See *Order Statistics*.

**Beta-Gamma relationship** If  $X \sim \text{Gamma}(a, \lambda)$ ,  $Y \sim \text{Gamma}(b, \lambda)$ , with  $X \perp\!\!\!\perp Y$  then

- $\frac{X}{X+Y} \sim \text{Beta}(a, b)$
- $X + Y \perp\!\!\!\perp \frac{X}{X+Y}$

This is known as the **bank-post office result**.

### $\chi^2$ (Chi-Square) Distribution

Let us say that  $X$  is distributed  $\chi_n^2$ . We know the following:

**Story** A Chi-Square( $n$ ) is the sum of the squares of  $n$  independent standard Normal r.v.s.

#### Properties and Representations

$X$  is distributed as  $Z_1^2 + Z_2^2 + \cdots + Z_n^2$  for i.i.d.  $Z_i \sim \mathcal{N}(0, 1)$

$$X \sim \text{Gamma}(n/2, 1/2)$$

## Discrete Distributions

### Distributions for four sampling schemes

	Replace	No Replace
<b>Fixed # trials (<math>n</math>)</b>	Binomial (Bern if $n = 1$ )	HGeom
<b>Draw until <math>r</math> success</b>	NBin (Geom if $r = 1$ )	NHGeom

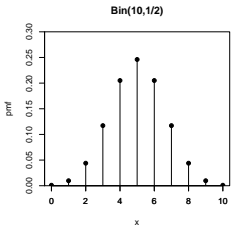
### Bernoulli Distribution

The Bernoulli distribution is the simplest case of the Binomial distribution, where we only have one trial ( $n = 1$ ). Let us say that  $X$  is distributed  $\text{Bern}(p)$ . We know the following:

**Story** A trial is performed with probability  $p$  of “success”, and  $X$  is the indicator of success: 1 means success, 0 means failure.

**Example** Let  $X$  be the indicator of Heads for a fair coin toss. Then  $X \sim \text{Bern}(\frac{1}{2})$ . Also,  $1 - X \sim \text{Bern}(\frac{1}{2})$  is the indicator of Tails.

### Binomial Distribution



Let us say that  $X$  is distributed  $\text{Bin}(n, p)$ . We know the following:

**Story**  $X$  is the number of “successes” that we will achieve in  $n$  independent trials, where each trial is either a success or a failure, each with the same probability  $p$  of success. We can also write  $X$  as a sum of multiple independent  $\text{Bern}(p)$  random variables. Let  $X \sim \text{Bin}(n, p)$  and  $X_j \sim \text{Bern}(p)$ , where all of the Bernoullis are independent. Then

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

**Example** If Jeremy Lin makes 10 free throws and each one independently has a  $\frac{3}{4}$  chance of getting in, then the number of free throws he makes is distributed  $\text{Bin}(10, \frac{3}{4})$ .

**Properties** Let  $X \sim \text{Bin}(n, p)$ ,  $Y \sim \text{Bin}(m, p)$  with  $X \perp\!\!\!\perp Y$ .

- **Redefine success**  $n - X \sim \text{Bin}(n, 1 - p)$
- **Sum**  $X + Y \sim \text{Bin}(n + m, p)$

- **Conditional**  $X|(X + Y = r) \sim \text{HGeom}(n, m, r)$
- **Binomial-Poisson Relationship**  $\text{Bin}(n, p)$  is approximately  $\text{Pois}(\lambda)$  if  $p$  is small.
- **Binomial-Normal Relationship**  $\text{Bin}(n, p)$  is approximately  $\mathcal{N}(np, np(1 - p))$  if  $n$  is large and  $p$  is not near 0 or 1.

### Geometric Distribution

Let us say that  $X$  is distributed  $\text{Geom}(p)$ . We know the following:

**Story**  $X$  is the number of “failures” that we will achieve before we achieve our first success. Our successes have probability  $p$ .

**Example** If each pokeball we throw has probability  $\frac{1}{10}$  to catch Mew, the number of failed pokeballs will be distributed  $\text{Geom}(\frac{1}{10})$ .

#### First Success Distribution

Equivalent to the Geometric distribution, except that it includes the first success in the count. This is 1 more than the number of failures. If  $X \sim \text{FS}(p)$  then  $E(X) = 1/p$ .

### Negative Binomial Distribution

Let us say that  $X$  is distributed  $\text{NBin}(r, p)$ . We know the following:

**Story**  $X$  is the number of “failures” that we will have before we achieve our  $r$ th success. Our successes have probability  $p$ .

**Example** Thundershock has 60% accuracy and can faint a wild Raticate in 3 hits. The number of misses before Pikachu faints Raticate with Thundershock is distributed  $\text{NBin}(3, 0.6)$ .

### Hypergeometric Distribution

Let us say that  $X$  is distributed  $\text{HGeom}(w, b, n)$ . We know the following:

**Story** In a population of  $w$  desired objects and  $b$  undesired objects,  $X$  is the number of “successes” we will have in a draw of  $n$  objects, without replacement. The draw of  $n$  objects is assumed to be a **simple random sample** (all sets of  $n$  objects are equally likely).

**Examples** Here are some HGeom examples.

- Let’s say that we have only  $b$  Weedles (failure) and  $w$  Pikachus (success) in Viridian Forest. We encounter  $n$  Pokemon in the forest, and  $X$  is the number of Pikachus in our encounters.
- The number of Aces in a 5 card hand.
- You have  $w$  white balls and  $b$  black balls, and you draw  $n$  balls. You will draw  $X$  white balls.
- You have  $w$  white balls and  $b$  black balls, and you draw  $n$  balls without replacement. The number of white balls in your sample is  $\text{HGeom}(w, b, n)$ ; the number of black balls is  $\text{HGeom}(b, w, n)$ .
- **Capture-recapture** A forest has  $N$  elk, you capture  $n$  of them, tag them, and release them. Then you recapture a new sample of size  $m$ . How many tagged elk are now in the new sample?  $\text{HGeom}(n, N - n, m)$

### Poisson Distribution

Let us say that  $X$  is distributed  $\text{Pois}(\lambda)$ . We know the following:

**Story** There are rare events (low probability events) that occur many different ways (high possibilities of occurrences) at an average rate of  $\lambda$  occurrences per unit space or time. The number of events that occur in that unit of space or time is  $X$ .

**Example** A certain busy intersection has an average of 2 accidents per month. Since an accident is a low probability event that can happen many different ways, it is reasonable to model the number of accidents in a month at that intersection as  $\text{Pois}(2)$ . Then the number of accidents that happen in two months at that intersection is distributed  $\text{Pois}(4)$ .

**Properties** Let  $X \sim \text{Pois}(\lambda_1)$  and  $Y \sim \text{Pois}(\lambda_2)$ , with  $X \perp\!\!\!\perp Y$ .

1. **Sum**  $X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$
2. **Conditional**  $X|(X + Y = n) \sim \text{Bin}\left(n, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$
3. **Chicken-egg** If there are  $Z \sim \text{Pois}(\lambda)$  items and we randomly and independently “accept” each item with probability  $p$ , then the number of accepted items  $Z_1 \sim \text{Pois}(\lambda p)$ , and the number of rejected items  $Z_2 \sim \text{Pois}(\lambda(1 - p))$ , and  $Z_1 \perp\!\!\!\perp Z_2$ .

## Multivariate Distributions

### Multinomial Distribution

Let us say that the vector  $\vec{X} = (X_1, X_2, X_3, \dots, X_k) \sim \text{Mult}_k(n, \vec{p})$  where  $\vec{p} = (p_1, p_2, \dots, p_k)$ .

**Story** We have  $n$  items, which can fall into any one of the  $k$  buckets independently with the probabilities  $\vec{p} = (p_1, p_2, \dots, p_k)$ .

**Example** Let us assume that every year, 100 students in the Harry Potter Universe are randomly and independently sorted into one of four houses with equal probability. The number of people in each of the houses is distributed  $\text{Mult}_4(100, \vec{p})$ , where  $\vec{p} = (0.25, 0.25, 0.25, 0.25)$ . Note that  $X_1 + X_2 + \cdots + X_4 = 100$ , and they are dependent.

**Joint PMF** For  $n = n_1 + n_2 + \cdots + n_k$ ,

$$P(\vec{X} = \vec{n}) = \frac{n!}{n_1!n_2! \dots n_k!} p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

**Marginal PMF, Lumping, and Conditionals** Marginally,  $X_i \sim \text{Bin}(n, p_i)$  since we can define “success” to mean category  $i$ . If you lump together multiple categories in a Multinomial, then it is still Multinomial. For example,  $X_i + X_j \sim \text{Bin}(n, p_i + p_j)$  for  $i \neq j$  since we can define “success” to mean being in category  $i$  or  $j$ . Similarly, if  $k = 6$  and we lump categories 1-2 and lump categories 3-5, then

$$(X_1 + X_2, X_3 + X_4 + X_5, X_6) \sim \text{Mult}_3(n, (p_1 + p_2, p_3 + p_4 + p_5, p_6))$$

Conditioning on some  $X_j$  also still gives a Multinomial:

$$X_1, \dots, X_{k-1} | X_k = n_k \sim \text{Mult}_{k-1}\left(n - n_k, \left(\frac{p_1}{1 - p_k}, \dots, \frac{p_{k-1}}{1 - p_k}\right)\right)$$

**Variances and Covariances** We have  $X_i \sim \text{Bin}(n, p_i)$  marginally, so  $\text{Var}(X_i) = np_i(1 - p_i)$ . Also,  $\text{Cov}(X_i, X_j) = -np_i p_j$  for  $i \neq j$ .

### Multivariate Uniform Distribution

See the univariate Uniform for stories and examples. For the 2D Uniform on some region, probability is proportional to area. Every point in the support has equal density, of value  $\frac{1}{\text{area of region}}$ . For the 3D Uniform, probability is proportional to volume.

### Multivariate Normal (MVN) Distribution

A vector  $\vec{X} = (X_1, X_2, \dots, X_k)$  is Multivariate Normal if every linear combination is Normally distributed, i.e.,  $t_1 X_1 + t_2 X_2 + \cdots + t_k X_k$  is Normal for any constants  $t_1, t_2, \dots, t_k$ . The parameters of the Multivariate Normal are the **mean vector**  $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$  and the **covariance matrix** where the  $(i, j)$  entry is  $\text{Cov}(X_i, X_j)$ .

**Properties** The Multivariate Normal has the following properties.

- Any subvector is also MVN.
- If any two elements within an MVN are uncorrelated, then they are independent.
- The joint PDF of a Bivariate Normal  $(X, Y)$  with  $\mathcal{N}(0, 1)$  marginal distributions and correlation  $\rho \in (-1, 1)$  is

$$f_{X,Y}(x, y) = \frac{1}{2\pi\tau} \exp\left(-\frac{1}{2\tau^2}(x^2 + y^2 - 2\rho xy)\right),$$

with  $\tau = \sqrt{1 - \rho^2}$ .



Distribution Properties

Important CDFs

Standard Normal  $\Phi$

Exponential( $\lambda$ )  $F(x) = 1 - e^{-\lambda x}$ , for  $x \in (0, \infty)$

Uniform(0,1)  $F(x) = x$ , for  $x \in (0, 1)$

Convolutions of Random Variables

A convolution of  $n$  random variables is simply their sum. For the following results, let  $X$  and  $Y$  be *independent*.

- $X \sim \text{Pois}(\lambda_1), Y \sim \text{Pois}(\lambda_2) \longrightarrow X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$
- $X \sim \text{Bin}(n_1, p), Y \sim \text{Bin}(n_2, p) \longrightarrow X + Y \sim \text{Bin}(n_1 + n_2, p)$ .   
  $\text{Bin}(n, p)$  can be thought of as a sum of i.i.d.  $\text{Bern}(p)$  r.v.s.
- $X \sim \text{Gamma}(a_1, \lambda), Y \sim \text{Gamma}(a_2, \lambda) \longrightarrow X + Y \sim \text{Gamma}(a_1 + a_2, \lambda)$ .   
  $\text{Gamma}(n, \lambda)$  with  $n$  an integer can be thought of as a sum of i.i.d.  $\text{Expo}(\lambda)$  r.v.s.
- $X \sim \text{NBin}(r_1, p), Y \sim \text{NBin}(r_2, p) \longrightarrow X + Y \sim \text{NBin}(r_1 + r_2, p)$ .   
  $\text{NBin}(r, p)$  can be thought of as a sum of i.i.d.  $\text{Geom}(p)$  r.v.s.
- $X \sim \mathcal{N}(\mu_1, \sigma_1^2), Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \longrightarrow X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Special Cases of Distributions

- $\text{Bin}(1, p) \sim \text{Bern}(p)$
- $\text{Beta}(1, 1) \sim \text{Unif}(0, 1)$
- $\text{Gamma}(1, \lambda) \sim \text{Expo}(\lambda)$
- $\chi_n^2 \sim \text{Gamma}(\frac{n}{2}, \frac{1}{2})$
- $\text{NBin}(1, p) \sim \text{Geom}(p)$

Inequalities

- Cauchy-Schwarz**  $|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$
- Markov**  $P(X \geq a) \leq \frac{E|X|}{a}$  for  $a > 0$
- Chebyshev**  $P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$  for  $E(X) = \mu, \text{Var}(X) = \sigma^2$
- Jensen**  $E(g(X)) \geq g(E(X))$  for  $g$  convex; reverse if  $g$  is concave

Formulas

Geometric Series

$$1 + r + r^2 + \dots + r^{n-1} = \sum_{k=0}^{n-1} r^k = \frac{1 - r^n}{1 - r}$$
$$1 + r + r^2 + \dots = \frac{1}{1 - r} \text{ if } |r| < 1$$

Exponential Function ( $e^x$ )

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

Gamma and Beta Integrals

You can sometimes solve complicated-looking integrals by pattern-matching to a gamma or beta integral:

$$\int_0^{\infty} x^{t-1} e^{-x} dx = \Gamma(t) \qquad \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Also,  $\Gamma(a + 1) = a\Gamma(a)$ , and  $\Gamma(n) = (n - 1)!$  if  $n$  is a positive integer.

Euler’s Approximation for Harmonic Sums

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \approx \log n + 0.577 \dots$$

Stirling’s Approximation for Factorials

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Miscellaneous Definitions

**Medians and Quantiles** Let  $X$  have CDF  $F$ . Then  $X$  has median  $m$  if  $F(m) \geq 0.5$  and  $P(X \geq m) \geq 0.5$ . For  $X$  continuous,  $m$  satisfies  $F(m) = 1/2$ . In general, the  $a$ th quantile of  $X$  is  $\min\{x : F(x) \geq a\}$ ; the median is the case  $a = 1/2$ .

**log** Statisticians generally use  $\log$  to refer to natural log (i.e., base  $e$ ).

**i.i.d r.v.s** Independent, identically-distributed random variables.

Example Problems

Contributions from Sebastian Chiu

Calculating Probability

A textbook has  $n$  typos, which are randomly scattered amongst its  $n$  pages, independently. You pick a random page. What is the probability that it has no typos? **Answer:** There is a  $(1 - \frac{1}{n})$  probability that any specific typo isn’t on your page, and thus a

$$\left(1 - \frac{1}{n}\right)^n$$
 probability that there are no typos on your page. For  $n$  large, this is approximately  $e^{-1} = 1/e$ .

Linearity and Indicators (1)

In a group of  $n$  people, what is the expected number of distinct birthdays (month and day)? What is the expected number of birthday matches? **Answer:** Let  $X$  be the number of distinct birthdays and  $I_j$  be the indicator for the  $j$ th day being represented.

$$E(I_j) = 1 - P(\text{no one born on day } j) = 1 - (364/365)^n$$

By linearity,  $E(X) = 365(1 - (364/365)^n)$ . Now let  $Y$  be the number of birthday matches and  $J_i$  be the indicator that the  $i$ th pair of people who leave have the same birthday. The probability that any two specific people share a birthday is  $1/365$ , so  $E(Y) = \binom{n}{2}/365$ .

Linearity and Indicators (2)

*This problem is commonly known as the **hat-matching problem**.* There are  $n$  people at a party, each with hat. At the end of the party, they each leave with a random hat. What is the expected number of people who leave with the right hat? **Answer:** Each hat has a  $1/n$  chance of going to the right person. By linearity, the average number of hats that go to their owners is  $n(1/n) = 1$ .

Linearity and First Success

*This problem is commonly known as the **coupon collector problem**.* There are  $n$  coupon types. At each draw, you get a uniformly random coupon type. What is the expected number of coupons needed until you have a complete set? **Answer:** Let  $N$  be the number of coupons needed; we want  $E(N)$ . Let  $N = N_1 + \dots + N_n$ , where  $N_1$  is the draws to get our first new coupon,  $N_2$  is the *additional* draws needed to draw our second new coupon and so on. By the story of the First Success,  $N_2 \sim \text{FS}((n - 1)/n)$  (after collecting first coupon type, there’s  $(n - 1)/n$  chance you’ll get something new). Similarly,  $N_3 \sim \text{FS}((n - 2)/n)$ , and  $N_j \sim \text{FS}((n - j + 1)/n)$ . By linearity,

$$E(N) = E(N_1) + \dots + E(N_n) = \frac{n}{n} + \frac{n}{n - 1} + \dots + \frac{n}{1} = n \sum_{j=1}^n \frac{1}{j}$$

This is approximately  $n(\log(n) + 0.577)$  by Euler’s approximation.

Orderings of i.i.d. random variables

I call 2 UberX’s and 3 Lyfts at the same time. If the time it takes for the rides to reach me are i.i.d., what is the probability that all the Lyfts will arrive first? **Answer:** Since the arrival times of the five cars are i.i.d., all  $5!$  orderings of the arrivals are equally likely. There are  $3!2!$  orderings that involve the Lyfts arriving first, so the probability

that the Lyfts arrive first is  $\frac{3!2!}{5!} = 1/10$ . Alternatively, there are  $\binom{5}{3}$

ways to choose 3 of the 5 slots for the Lyfts to occupy, where each of the choices are equally likely. One of these choices has all 3 of the

Lyfts arriving first, so the probability is  $1/\binom{5}{3} = 1/10$ .

Expectation of Negative Hypergeometric

What is the expected number of cards that you draw before you pick your first Ace in a shuffled deck (not counting the Ace)? **Answer:** Consider a non-Ace. Denote this to be card  $j$ . Let  $I_j$  be the indicator that card  $j$  will be drawn before the first Ace. Note that  $I_j = 1$  says that  $j$  is before all 4 of the Aces in the deck. The probability that this occurs is  $1/5$  by symmetry. Let  $X$  be the number of cards drawn before the first Ace. Then  $X = I_1 + I_2 + \dots + I_{48}$ , where each indicator corresponds to one of the 48 non-Aces. Thus,

$$E(X) = E(I_1) + E(I_2) + \dots + E(I_{48}) = 48/5 = \boxed{9.6}.$$

Minimum and Maximum of RVs

What is the CDF of the maximum of  $n$  independent  $\text{Unif}(0,1)$  random variables? **Answer:** Note that for r.v.s  $X_1, X_2, \dots, X_n$ ,

$$P(\min(X_1, X_2, \dots, X_n) \geq a) = P(X_1 \geq a, X_2 \geq a, \dots, X_n \geq a)$$

Similarly,

$$P(\max(X_1, X_2, \dots, X_n) \leq a) = P(X_1 \leq a, X_2 \leq a, \dots, X_n \leq a)$$

We will use this principle to find the CDF of  $U_{(n)}$ , where  $U_{(n)} = \max(U_1, U_2, \dots, U_n)$  and  $U_i \sim \text{Unif}(0, 1)$  are i.i.d.

$$\begin{aligned} P(\max(U_1, U_2, \dots, U_n) \leq a) &= P(U_1 \leq a, U_2 \leq a, \dots, U_n \leq a) \\ &= P(U_1 \leq a)P(U_2 \leq a) \dots P(U_n \leq a) \\ &= \boxed{a^n} \end{aligned}$$

for  $0 < a < 1$  (and the CDF is 0 for  $a \leq 0$  and 1 for  $a \geq 1$ ).

Pattern-matching with  $e^x$  Taylor series

For  $X \sim \text{Pois}(\lambda)$ , find  $E\left(\frac{1}{X + 1}\right)$ . **Answer:** By LOTUS,

$$E\left(\frac{1}{X + 1}\right) = \sum_{k=0}^{\infty} \frac{1}{k + 1} \frac{e^{-\lambda} \lambda^k}{k!} = \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k + 1)!} = \boxed{\frac{e^{-\lambda}}{\lambda} (e^{\lambda} - 1)}$$

Adam’s Law and Eve’s Law

William really likes speedsolving Rubik’s Cubes. But he’s pretty bad at it, so sometimes he fails. On any given day, William will attempt  $N \sim \text{Geom}(s)$  Rubik’s Cubes. Suppose each time, he has probability  $p$  of solving the cube, independently. Let  $T$  be the number of Rubik’s Cubes he solves during a day. Find the mean and variance of  $T$ . **Answer:** Note that  $T|N \sim \text{Bin}(N, p)$ . So by Adam’s Law,

$$E(T) = E(E(T|N)) = E(Np) = \frac{p(1-s)}{s}$$

Similarly, by Eve’s Law, we have that

$$\begin{aligned} \text{Var}(T) &= E(\text{Var}(T|N)) + \text{Var}(E(T|N)) = E(Np(1-p)) + \text{Var}(Np) \\ &= \frac{p(1-p)(1-s)}{s} + \frac{p^2(1-s)}{s^2} = \frac{p(1-s)(p+s(1-p))}{s^2} \end{aligned}$$

MGF – Finding Moments

Find  $E(X^3)$  for  $X \sim \text{Expo}(\lambda)$  using the MGF of  $X$ . **Answer:** The MGF of an  $\text{Expo}(\lambda)$  is  $M(t) = \frac{\lambda}{\lambda - t}$ . To get the third moment, we can take the third derivative of the MGF and evaluate at  $t = 0$ :

$$E(X^3) = \frac{6}{\lambda^3}$$

But a much nicer way to use the MGF here is via pattern recognition: note that  $M(t)$  looks like it came from a geometric series:

$$\frac{1}{1 - \frac{t}{\lambda}} = \sum_{n=0}^{\infty} \left(\frac{t}{\lambda}\right)^n = \sum_{n=0}^{\infty} \frac{n!}{\lambda^n} \frac{t^n}{n!}$$

The coefficient of  $\frac{t^n}{n!}$  here is the  $n$ th moment of  $X$ , so we have  $E(X^n) = \frac{n!}{\lambda^n}$  for all nonnegative integers  $n$ .

Markov chains (1)

Suppose  $X_n$  is a two-state Markov chain with transition matrix

$$Q = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \end{matrix}$$

Find the stationary distribution  $\vec{s} = (s_0, s_1)$  of  $X_n$  by solving  $\vec{s}Q = \vec{s}$ , and show that the chain is reversible with respect to  $\vec{s}$ . **Answer:** The equation  $\vec{s}Q = \vec{s}$  says that

$$s_0 = s_0(1 - \alpha) + s_1\beta \text{ and } s_1 = s_0(\alpha) + s_0(1 - \beta)$$

By solving this system of linear equations, we have

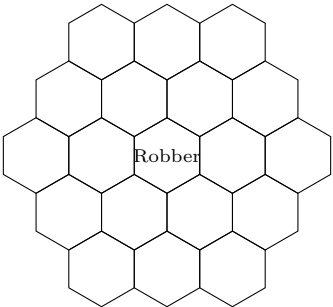
$$\vec{s} = \left( \frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta} \right)$$

To show that the chain is reversible with respect to  $\vec{s}$ , we must show  $s_i q_{ij} = s_j q_{ji}$  for all  $i, j$ . This is done if we can show  $s_0 q_{01} = s_1 q_{10}$ . And indeed,

$$s_0 q_{01} = \frac{\alpha\beta}{\alpha + \beta} = s_1 q_{10}$$

Markov chains (2)

William and Sebastian play a modified game of Settlers of Catan, where every turn they randomly move the robber (which starts on the center tile) to one of the adjacent hexagons.



- (a) Is this Markov chain irreducible? Is it aperiodic? **Answer:** Yes to both. The Markov chain is irreducible because it can get from anywhere to anywhere else. The Markov chain is aperiodic because the robber can return back to a square in 2, 3, 4, 5, . . . moves, and the GCD of those numbers is 1.
- (b) What is the stationary distribution of this Markov chain? **Answer:** Since this is a random walk on an undirected graph, the stationary distribution is proportional to the degree sequence. The degree for the corner pieces is 3, the degree for the edge pieces is 4, and the degree for the center pieces is 6. To normalize this degree sequence, we divide by its sum. The sum of the degrees is  $6(3) + 6(4) + 7(6) = 84$ . Thus the stationary probability of being on a corner is  $3/84 = 1/28$ , on an edge is  $4/84 = 1/21$ , and in the center is  $6/84 = 1/14$ .
- (c) What fraction of the time will the robber be in the center tile in this game, in the long run? **Answer:** By the above,  $1/14$ .
- (d) What is the expected amount of moves it will take for the robber to return to the center tile? **Answer:** Since this chain is irreducible and aperiodic, to get the expected time to return we can just invert the stationary probability. Thus on average it will take  $14$  turns for the robber to return to the center tile.

Problem-Solving Strategies

Contributions from Jessy Hwang, Yuan Jiang, Yuqi Hou

- 1. **Getting started.** Start by *defining relevant events and random variables*. (“Let  $A$  be the event that I pick the fair coin”; “Let  $X$  be the number of successes.”) Clear notion is important for clear thinking! Then decide what it is that you’re supposed to be finding, in terms of your notation (“I want to find  $P(X = 3|A)$ ”). Think about what type of object your answer should be (a number? A random variable? A PMF? A PDF?) and what it should be in terms of.  
*Try simple and extreme cases.* To make an abstract experiment more concrete, try *drawing a picture* or making up numbers that could have happened. Pattern recognition: does the structure of the problem resemble something we’ve seen before?
- 2. **Calculating probability of an event.** Use counting principles if the naive definition of probability applies. Is the probability of the complement easier to find? Look for symmetries. Look for something to condition on, then apply Bayes’ Rule or the Law of Total Probability.
- 3. **Finding the distribution of a random variable.** First make sure you need the full distribution not just the mean (see next item). Check the *support* of the random variable: what values can it take on? Use this to rule out distributions that don’t fit. Is there a *story* for one of the named distributions that fits the problem at hand? Can you write the random variable as a function of an r.v. with a known distribution, say  $Y = g(X)$ ?

- 4. **Calculating expectation.** If it has a named distribution, check out the table of distributions. If it’s a function of an r.v. with a named distribution, try LOTUS. If it’s a count of something, try breaking it up into indicator r.v.s. If you can condition on something natural, consider using Adam’s Law.
- 5. **Calculating variance.** Consider independence, named distributions, and LOTUS. If it’s a count of something, break it up into a sum of indicator r.v.s. If it’s a sum, use properties of covariance. If you can condition on something natural, consider using Eve’s Law.
- 6. **Calculating  $E(X^2)$ .** Do you already know  $E(X)$  or  $\text{Var}(X)$ ? Recall that  $\text{Var}(X) = E(X^2) - (E(X))^2$ . Otherwise try LOTUS.
- 7. **Calculating covariance.** Use the properties of covariance. If you’re trying to find the covariance between two components of a Multinomial distribution,  $X_i, X_j$ , then the covariance is  $-np_i p_j$  for  $i \neq j$ .
- 8. **Symmetry.** If  $X_1, \dots, X_n$  are i.i.d., consider using symmetry.
- 9. **Calculating probabilities of orderings.** Remember that all  $n!$  ordering of i.i.d. continuous random variables  $X_1, \dots, X_n$  are equally likely.
- 10. **Determining independence.** There are several equivalent definitions. Think about simple and extreme cases to see if you can find a counterexample.
- 11. **Do a painful integral.** If your integral looks painful, see if you can write your integral in terms of a known PDF (like Gamma or Beta), and use the fact that PDFs integrate to 1?
- 12. **Before moving on.** Check some simple and extreme cases, check whether the answer seems plausible, check for biohazards.

Biohazards

Contributions from Jessy Hwang

- 1. **Don’t misuse the naive definition of probability.** When answering “What is the probability that in a group of 3 people, no two have the same birth month?”, it is *not* correct to treat the people as indistinguishable balls being placed into 12 boxes, since that assumes the list of birth months {January, January, January} is just as likely as the list {January, April, June}, even though the latter is six times more likely.
- 2. **Don’t confuse unconditional, conditional, and joint probabilities.** In applying  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ , it is *not* correct to say “ $P(B) = 1$  because we know  $B$  happened”;  $P(B)$  is the *prior* probability of  $B$ . Don’t confuse  $P(A|B)$  with  $P(A, B)$ .
- 3. **Don’t assume independence without justification.** In the matching problem, the probability that card 1 is a match and card 2 is a match is not  $1/n^2$ . Binomial and Hypergeometric are often confused; the trials are independent in the Binomial story and dependent in the Hypergeometric story.
- 4. **Don’t forget to do sanity checks.** Probabilities must be between 0 and 1. Variances must be  $\geq 0$ . Supports must make sense. PMFs must sum to 1. PDFs must integrate to 1.
- 5. **Don’t confuse random variables, numbers, and events.** Let  $X$  be an r.v. Then  $g(X)$  is an r.v. for any function  $g$ . In particular,  $X^2$ ,  $|X|$ ,  $F(X)$ , and  $I_{X>3}$  are r.v.s.  $P(X^2 < X|X \geq 0)$ ,  $E(X)$ ,  $\text{Var}(X)$ , and  $g(E(X))$  are numbers.  $X = 2$  and  $F(X) \geq -1$  are events. It does not make sense to write  $\int_{-\infty}^{\infty} F(X)dx$ , because  $F(X)$  is a random variable. It does not make sense to write  $P(X)$ , because  $X$  is not an event.

6. **Don't confuse a random variable with its distribution.**  
To get the PDF of  $X^2$ , you can't just square the PDF of  $X$ . The right way is to use transformations. To get the PDF of  $X + Y$ , you can't just add the PDF of  $X$  and the PDF of  $Y$ . The right way is to compute the **convolution**.
7. **Don't pull non-linear functions out of expectations.**  
 $E(g(X))$  does not equal  $g(E(X))$  in general. The St. Petersburg paradox is an extreme example. See also Jensen's inequality. The right way to find  $E(g(X))$  is with **LOTUS**.

## Recommended Resources

- Introduction to Probability Book (<http://bit.ly/introprobability>)
- Stat 110 Online (<http://stat110.net>)
- Stat 110 Quora Blog (<https://stat110.quora.com/>)
- Quora Probability FAQ (<http://bit.ly/probabilityfaq>)
- R Studio (<https://www.rstudio.com>)
- LaTeX File ([github.com/wzchen/probability\\_cheatsheet](https://github.com/wzchen/probability_cheatsheet))

Please share this cheatsheet with friends!  
<http://wzchen.com/probability-cheatsheet>

## Distributions in R

Command	What it does
<code>help(distributions)</code>	shows documentation on distributions
<code>dbinom(k,n,p)</code>	PMF $P(X = k)$ for $X \sim \text{Bin}(n, p)$
<code>pbinom(x,n,p)</code>	CDF $P(X \leq x)$ for $X \sim \text{Bin}(n, p)$
<code>qbinom(a,n,p)</code>	ath quantile for $X \sim \text{Bin}(n, p)$
<code>rbinom(r,n,p)</code>	vector of $r$ i.i.d. $\text{Bin}(n, p)$ r.v.s
<code>dgeom(k,p)</code>	PMF $P(X = k)$ for $X \sim \text{Geom}(p)$
<code>dhyp(k,w,b,n)</code>	PMF $P(X = k)$ for $X \sim \text{HGeom}(w, b, n)$
<code>dnbinom(k,r,p)</code>	PMF $P(X = k)$ for $X \sim \text{NBin}(r, p)$
<code>dpois(k,r)</code>	PMF $P(X = k)$ for $X \sim \text{Pois}(r)$
<code>dbeta(x,a,b)</code>	PDF $f(x)$ for $X \sim \text{Beta}(a, b)$
<code>dchisq(x,n)</code>	PDF $f(x)$ for $X \sim \chi_n^2$
<code>dexp(x,b)</code>	PDF $f(x)$ for $X \sim \text{Expo}(b)$
<code>dgamma(x,a,r)</code>	PDF $f(x)$ for $X \sim \text{Gamma}(a, r)$
<code>dlnorm(x,m,s)</code>	PDF $f(x)$ for $X \sim \mathcal{LN}(m, s^2)$
<code>dnorm(x,m,s)</code>	PDF $f(x)$ for $X \sim \mathcal{N}(m, s^2)$
<code>dt(x,n)</code>	PDF $f(x)$ for $X \sim t_n$
<code>dunif(x,a,b)</code>	PDF $f(x)$ for $X \sim \text{Unif}(a, b)$

The table above gives R commands for working with various named distributions. Commands analogous to `pbinom`, `qbinom`, and `rbinom` work for the other distributions in the table. For example, `pnorm`, `qnorm`, and `rnorm` can be used to get the CDF, quantiles, and random generation for the Normal. For the Multinomial, `dmultinom` can be used for calculating the joint PMF and `rmultinom` can be used for generating random vectors. For the Multivariate Normal, after installing and loading the `mvtnorm` package `dmvnorm` can be used for calculating the joint PDF and `rmvnorm` can be used for generating random vectors.

## Table of Distributions

Distribution	PMF/PDF and Support	Expected Value	Variance	MGF
Bernoulli Bern( $p$ )	$P(X = 1) = p$ $P(X = 0) = q = 1 - p$	$p$	$pq$	$q + pe^t$
Binomial Bin( $n, p$ )	$P(X = k) = \binom{n}{k} p^k q^{n-k}$ $k \in \{0, 1, 2, \dots, n\}$	$np$	$npq$	$(q + pe^t)^n$
Geometric Geom( $p$ )	$P(X = k) = q^k p$ $k \in \{0, 1, 2, \dots\}$	$q/p$	$q/p^2$	$\frac{p}{1-qe^t}, qe^t < 1$
Negative Binomial NBin( $r, p$ )	$P(X = n) = \binom{r+n-1}{r-1} p^r q^n$ $n \in \{0, 1, 2, \dots\}$	$rq/p$	$rq/p^2$	$(\frac{p}{1-qe^t})^r, qe^t < 1$
Hypergeometric HGeom( $w, b, n$ )	$P(X = k) = \binom{w}{k} \binom{b}{n-k} / \binom{w+b}{n}$ $k \in \{0, 1, 2, \dots, n\}$	$\mu = \frac{nw}{b+w}$	$\left(\frac{w+b-n}{w+b-1}\right) n \frac{\mu}{n} (1 - \frac{\mu}{n})$	messy
Poisson Pois( $\lambda$ )	$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$ $k \in \{0, 1, 2, \dots\}$	$\lambda$	$\lambda$	$e^{\lambda(e^t-1)}$
Uniform Unif( $a, b$ )	$f(x) = \frac{1}{b-a}$ $x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$
Normal $\mathcal{N}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ $x \in (-\infty, \infty)$	$\mu$	$\sigma^2$	$e^{t\mu + \frac{\sigma^2 t^2}{2}}$
Exponential Expo( $\lambda$ )	$f(x) = \lambda e^{-\lambda x}$ $x \in (0, \infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}, t < \lambda$
Gamma Gamma( $a, \lambda$ )	$f(x) = \frac{1}{\Gamma(a)} (\lambda x)^a e^{-\lambda x} \frac{1}{x}$ $x \in (0, \infty)$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^a, t < \lambda$
Beta Beta( $a, b$ )	$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}$ $x \in (0, 1)$	$\mu = \frac{a}{a+b}$	$\frac{\mu(1-\mu)}{(a+b+1)}$	messy
Log-Normal $\mathcal{LN}(\mu, \sigma^2)$	$\frac{1}{x\sigma\sqrt{2\pi}} e^{-(\log x - \mu)^2/(2\sigma^2)}$ $x \in (0, \infty)$	$\theta = e^{\mu + \sigma^2/2}$	$\theta^2(e^{\sigma^2} - 1)$	doesn't exist
Chi-Square $\chi_n^2$	$\frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$ $x \in (0, \infty)$	$n$	$2n$	$(1-2t)^{-n/2}, t < 1/2$
Student- $t$ $t_n$	$\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} (1+x^2/n)^{-(n+1)/2}$ $x \in (-\infty, \infty)$	0 if $n > 1$	$\frac{n}{n-2}$ if $n > 2$	doesn't exist

## Limits Definitions

**Precise Definition :** We say  $\lim_{x \rightarrow a} f(x) = L$  if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that whenever  $0 < |x - a| < \delta$  then  $|f(x) - L| < \varepsilon$ .

**“Working” Definition :** We say  $\lim_{x \rightarrow a} f(x) = L$  if we can make  $f(x)$  as close to  $L$  as we want by taking  $x$  sufficiently close to  $a$  (on either side of  $a$ ) without letting  $x = a$ .

**Right hand limit :**  $\lim_{x \rightarrow a^+} f(x) = L$ . This has the same definition as the limit except it requires  $x > a$ .

**Left hand limit :**  $\lim_{x \rightarrow a^-} f(x) = L$ . This has the same definition as the limit except it requires  $x < a$ .

**Limit at Infinity :** We say  $\lim_{x \rightarrow \infty} f(x) = L$  if we can make  $f(x)$  as close to  $L$  as we want by taking  $x$  large enough and positive.

There is a similar definition for  $\lim_{x \rightarrow -\infty} f(x) = L$  except we require  $x$  large and negative.

**Infinite Limit :** We say  $\lim_{x \rightarrow a} f(x) = \infty$  if we can make  $f(x)$  arbitrarily large (and positive) by taking  $x$  sufficiently close to  $a$  (on either side of  $a$ ) without letting  $x = a$ .

There is a similar definition for  $\lim_{x \rightarrow a} f(x) = -\infty$  except we make  $f(x)$  arbitrarily large and negative.

### Relationship between the limit and one-sided limits

$$\lim_{x \rightarrow a} f(x) = L \Rightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \quad \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \Rightarrow \lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x) \Rightarrow \lim_{x \rightarrow a} f(x) \text{ Does Not Exist}$$

## Properties

Assume  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist and  $c$  is any number then,

1.  $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
2.  $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
3.  $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
4.  $\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  provided  $\lim_{x \rightarrow a} g(x) \neq 0$
5.  $\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n$
6.  $\lim_{x \rightarrow a} \left[ \sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

### Basic Limit Evaluations at $\pm\infty$

1.  $\lim_{x \rightarrow \infty} e^x = \infty$  &  $\lim_{x \rightarrow -\infty} e^x = 0$
  2.  $\lim_{x \rightarrow \infty} \ln(x) = \infty$  &  $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$
  3. If  $r > 0$  then  $\lim_{x \rightarrow \infty} \frac{b}{x^r} = 0$
  4. If  $r > 0$  and  $x^r$  is real for negative  $x$  then  $\lim_{x \rightarrow -\infty} \frac{b}{x^r} = 0$
  5.  $n$  even :  $\lim_{x \rightarrow \pm\infty} x^n = \infty$
  6.  $n$  odd :  $\lim_{x \rightarrow \infty} x^n = \infty$  &  $\lim_{x \rightarrow -\infty} x^n = -\infty$
  7.  $n$  even :  $\lim_{x \rightarrow \pm\infty} ax^n + \dots + bx + c = \text{sgn}(a)\infty$
  8.  $n$  odd :  $\lim_{x \rightarrow \infty} ax^n + \dots + bx + c = \text{sgn}(a)\infty$
  9.  $n$  odd :  $\lim_{x \rightarrow -\infty} ax^n + \dots + bx + c = -\text{sgn}(a)\infty$
- Note :  $\text{sgn}(a) = 1$  if  $a > 0$  and  $\text{sgn}(a) = -1$  if  $a < 0$ .



## Evaluation Techniques

### Continuous Functions

If  $f(x)$  is continuous at  $a$  then  $\lim_{x \rightarrow a} f(x) = f(a)$

### Continuous Functions and Composition

$f(x)$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$  then

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b)$$

### Factor and Cancel

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+6)}{x(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{x+6}{x} = \frac{8}{2} = 4\end{aligned}$$

### Rationalize Numerator/Denominator

$$\begin{aligned}\lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{x^2 - 81} &= \lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{x^2 - 81} \cdot \frac{3 + \sqrt{x}}{3 + \sqrt{x}} \\ &= \lim_{x \rightarrow 9} \frac{9 - x}{(x^2 - 81)(3 + \sqrt{x})} = \lim_{x \rightarrow 9} \frac{-1}{(x+9)(3 + \sqrt{x})} \\ &= \frac{-1}{(18)(6)} = -\frac{1}{108}\end{aligned}$$

### Combine Rational Expressions

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{x+h} - \frac{1}{x} \right) &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{x - (x+h)}{x(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{-h}{x(x+h)} \right) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}\end{aligned}$$

### L'Hospital's/L'Hôpital's Rule

If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$  or  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm \infty}{\pm \infty}$  then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \text{ } a \text{ is a number, } \infty \text{ or } -\infty$$

### Polynomials at Infinity

$p(x)$  and  $q(x)$  are polynomials. To compute

$\lim_{x \rightarrow \pm \infty} \frac{p(x)}{q(x)}$  factor largest power of  $x$  in  $q(x)$  out of

both  $p(x)$  and  $q(x)$  then compute limit.

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{3x^2 - 4}{5x - 2x^2} &= \lim_{x \rightarrow -\infty} \frac{x^2(3 - \frac{4}{x^2})}{x^2(\frac{5}{x} - 2)} \\ &= \lim_{x \rightarrow -\infty} \frac{3 - \frac{4}{x^2}}{\frac{5}{x} - 2} = -\frac{3}{2}\end{aligned}$$

### Piecewise Function

$$\lim_{x \rightarrow -2} g(x) \text{ where } g(x) = \begin{cases} x^2 + 5 & \text{if } x < -2 \\ 1 - 3x & \text{if } x \geq -2 \end{cases}$$

Compute two one sided limits,

$$\begin{aligned}\lim_{x \rightarrow -2^-} g(x) &= \lim_{x \rightarrow -2^-} x^2 + 5 = 9 \\ \lim_{x \rightarrow -2^+} g(x) &= \lim_{x \rightarrow -2^+} 1 - 3x = 7\end{aligned}$$

One sided limits are different so  $\lim_{x \rightarrow -2} g(x)$  doesn't exist. If the two one sided limits had been equal then  $\lim_{x \rightarrow -2} g(x)$  would have existed and had the same value.

## Some Continuous Functions

Partial list of continuous functions and the values of  $x$  for which they are continuous.

1. Polynomials for all  $x$ .
2. Rational function, except for  $x$ 's that give division by zero.
3.  $\sqrt[n]{x}$  ( $n$  odd) for all  $x$ .
4.  $\sqrt[n]{x}$  ( $n$  even) for all  $x \geq 0$ .
5.  $e^x$  for all  $x$ .
6.  $\ln(x)$  for  $x > 0$ .
7.  $\cos(x)$  and  $\sin(x)$  for all  $x$ .
8.  $\tan(x)$  and  $\sec(x)$  provided  $x \neq \dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
9.  $\cot(x)$  and  $\csc(x)$  provided  $x \neq \dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots$

## Intermediate Value Theorem

Suppose that  $f(x)$  is continuous on  $[a, b]$  and let  $M$  be any number between  $f(a)$  and  $f(b)$ . Then there exists a number  $c$  such that  $a < c < b$  and  $f(c) = M$ .

## Derivatives

### Definition and Notation

If  $y = f(x)$  then the derivative is defined to be  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ .

If  $y = f(x)$  then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = Df(x)$$

If  $y = f(x)$  all of the following are equivalent notations for derivative evaluated at  $x = a$ .

$$f'(a) = y'|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{dy}{dx} \right|_{x=a} = Df(a)$$

### Interpretation of the Derivative

If  $y = f(x)$  then,

1.  $m = f'(a)$  is the slope of the tangent line to  $y = f(x)$  at  $x = a$  and the equation of the tangent line at  $x = a$  is given by  $y = f(a) + f'(a)(x - a)$ .
2.  $f'(a)$  is the instantaneous rate of change of  $f(x)$  at  $x = a$ .
3. If  $f(t)$  is the position of an object at time  $t$  then  $f'(a)$  is the velocity of the object at  $t = a$ .

### Basic Properties and Formulas

If  $f(x)$  and  $g(x)$  are differentiable functions (the derivative exists),  $c$  and  $n$  are any real numbers,

1.  $\frac{d}{dx}(c) = 0$
2.  $\left(c f(x)\right)' = c f'(x)$
3.  $\frac{d}{dx}(x^n) = n x^{n-1}$  – **Power Rule**
4.  $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$
5.  $\left(f(x) g(x)\right)' = f'(x) g(x) + f(x) g'(x)$  – **Product Rule**
6.  $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) g(x) - f(x) g'(x)}{(g(x))^2}$  – **Quotient Rule**
7.  $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) g'(x)$  – **Chain Rule**

### Common Derivatives

$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(\csc(x)) = -\csc(x) \cot(x)$	$\frac{d}{dx}(a^x) = a^x \ln(a)$
$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad x > 0$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\ln x ) = \frac{1}{x}, \quad x \neq 0$
$\frac{d}{dx}(\sec(x)) = \sec(x) \tan(x)$	$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln(a)}, \quad x > 0$

**Chain Rule Variants**

The chain rule applied to some specific functions.

$$1. \frac{d}{dx} \left( [f(x)]^n \right) = n[f(x)]^{n-1} f'(x)$$

$$2. \frac{d}{dx} \left( e^{f(x)} \right) = f'(x) e^{f(x)}$$

$$3. \frac{d}{dx} \left( \ln [f(x)] \right) = \frac{f'(x)}{f(x)}$$

$$4. \frac{d}{dx} \left( \sin [f(x)] \right) = f'(x) \cos [f(x)]$$

$$5. \frac{d}{dx} \left( \cos [f(x)] \right) = -f'(x) \sin [f(x)]$$

$$6. \frac{d}{dx} \left( \tan [f(x)] \right) = f'(x) \sec^2 [f(x)]$$

$$7. \frac{d}{dx} \left( \sec [f(x)] \right) = f'(x) \sec [f(x)] \tan [f(x)]$$

$$8. \frac{d}{dx} \left( \tan^{-1} [f(x)] \right) = \frac{f'(x)}{1 + [f(x)]^2}$$

**Higher Order Derivatives**

The 2<sup>nd</sup> Derivative is denoted as

$$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2} \text{ and is defined as}$$

$f''(x) = (f'(x))'$ , i.e. the derivative of the first derivative,  $f'(x)$ .

The  $n^{\text{th}}$  Derivative is denoted as

$$f^{(n)}(x) = \frac{d^n f}{dx^n} \text{ and is defined as}$$

$f^{(n)}(x) = (f^{(n-1)}(x))'$ , i.e. the derivative of the  $(n-1)^{\text{st}}$  derivative,  $f^{(n-1)}(x)$ .

**Implicit Differentiation**

Find  $y'$  if  $e^{2x-9y} + x^3y^2 = \sin(y) + 11x$ . Remember  $y = y(x)$  here, so products/quotients of  $x$  and  $y$  will use the product/quotient rule and derivatives of  $y$  will use the chain rule. The “trick” is to differentiate as normal and every time you differentiate a  $y$  you tack on a  $y'$  (from the chain rule). Then solve for  $y'$ .

$$e^{2x-9y}(2 - 9y') + 3x^2y^2 + 2x^3y y' = \cos(y)y' + 11$$

$$2e^{2x-9y} - 9y'e^{2x-9y} + 3x^2y^2 + 2x^3y y' = \cos(y)y' + 11 \quad \Rightarrow \quad y' = \frac{11 - 2e^{2x-9y} - 3x^2y^2}{2x^3y - 9e^{2x-9y} - \cos(y)}$$

$$(2x^3y - 9e^{2x-9y} - \cos(y)) y' = 11 - 2e^{2x-9y} - 3x^2y^2$$

**Increasing/Decreasing – Concave Up/Concave Down****Critical Points**

$x = c$  is a critical point of  $f(x)$  provided either

1.  $f'(c) = 0$  or,
2.  $f'(c)$  doesn't exist.

**Increasing/Decreasing**

1. If  $f'(x) > 0$  for all  $x$  in an interval  $I$  then  $f(x)$  is increasing on the interval  $I$ .
2. If  $f'(x) < 0$  for all  $x$  in an interval  $I$  then  $f(x)$  is decreasing on the interval  $I$ .
3. If  $f'(x) = 0$  for all  $x$  in an interval  $I$  then  $f(x)$  is constant on the interval  $I$ .

**Concave Up/Concave Down**

1. If  $f''(x) > 0$  for all  $x$  in an interval  $I$  then  $f(x)$  is concave up on the interval  $I$ .
2. If  $f''(x) < 0$  for all  $x$  in an interval  $I$  then  $f(x)$  is concave down on the interval  $I$ .

**Inflection Points**

$x = c$  is a inflection point of  $f(x)$  if the concavity changes at  $x = c$ .

## Extrema

### Absolute Extrema

1.  $x = c$  is an absolute maximum of  $f(x)$  if  $f(c) \geq f(x)$  for all  $x$  in the domain.
2.  $x = c$  is an absolute minimum of  $f(x)$  if  $f(c) \leq f(x)$  for all  $x$  in the domain.

### Fermat's Theorem

If  $f(x)$  has a relative (or local) extrema at  $x = c$ , then  $x = c$  is a critical point of  $f(x)$ .

### Extreme Value Theorem

If  $f(x)$  is continuous on the closed interval  $[a, b]$  then there exist numbers  $c$  and  $d$  so that,

1.  $a \leq c, d \leq b$ ,
2.  $f(c)$  is the absolute maximum in  $[a, b]$ ,
3.  $f(d)$  is the absolute minimum in  $[a, b]$ .

### Finding Absolute Extrema

To find the absolute extrema of the continuous function  $f(x)$  on the interval  $[a, b]$  use the following process.

1. Find all critical points of  $f(x)$  in  $[a, b]$ .
2. Evaluate  $f(x)$  at all points found in Step 1.
3. Evaluate  $f(a)$  and  $f(b)$ .
4. Identify the absolute maximum (largest function value) and the absolute minimum (smallest function value) from the evaluations in Steps 2 & 3.

### Relative (local) Extrema

1.  $x = c$  is a relative (or local) maximum of  $f(x)$  if  $f(c) \geq f(x)$  for all  $x$  near  $c$ .
2.  $x = c$  is a relative (or local) minimum of  $f(x)$  if  $f(c) \leq f(x)$  for all  $x$  near  $c$ .

### 1<sup>st</sup> Derivative Test

If  $x = c$  is a critical point of  $f(x)$  then  $x = c$  is

1. a relative maximum of  $f(x)$  if  $f'(x) > 0$  to the left of  $x = c$  and  $f'(x) < 0$  to the right of  $x = c$ .
2. a relative minimum of  $f(x)$  if  $f'(x) < 0$  to the left of  $x = c$  and  $f'(x) > 0$  to the right of  $x = c$ .
3. not a relative extrema of  $f(x)$  if  $f'(x)$  is the same sign on both sides of  $x = c$ .

### 2<sup>nd</sup> Derivative Test

If  $x = c$  is a critical point of  $f(x)$  such that  $f'(c) = 0$  then  $x = c$

1. is a relative maximum of  $f(x)$  if  $f''(c) < 0$ .
2. is a relative minimum of  $f(x)$  if  $f''(c) > 0$ .
3. may be a relative maximum, relative minimum, or neither if  $f''(c) = 0$ .

### Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of  $f(x)$ .
2. Use the 1<sup>st</sup> derivative test or the 2<sup>nd</sup> derivative test on each critical point.

## Mean Value Theorem

If  $f(x)$  is continuous on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$  then there is a number  $a < c < b$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

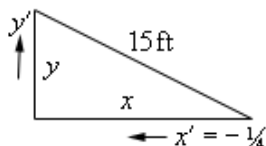
## Newton's Method

If  $x_n$  is the  $n^{th}$  guess for the root/solution of  $f(x) = 0$  then  $(n + 1)^{st}$  guess is  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$  provided  $f'(x_n)$  exists.

### Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to  $t$  using implicit differentiation (i.e. add on a derivative every time you differentiate a function of  $t$ ). Plug in known quantities and solve for the unknown quantity.

**Example** A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at  $\frac{1}{4}$  ft/sec. How fast is the top moving after 12 sec?



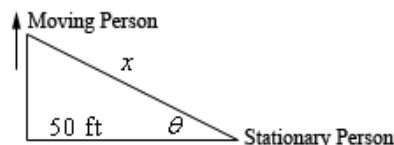
$x'$  is negative because  $x$  is decreasing. Using Pythagorean Theorem and differentiating,

$$x^2 + y^2 = 15^2 \Rightarrow 2x x' + 2y y' = 0$$

After 12 sec we have  $x = 10 - 12\left(\frac{1}{4}\right) = 7$  and so  $y = \sqrt{15^2 - 7^2} = \sqrt{176}$ . Plug in and solve for  $y'$ .

$$7\left(-\frac{1}{4}\right) + \sqrt{176} y' = 0 \Rightarrow y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

**Example** Two people are 50 ft apart when one starts walking north. The angle  $\theta$  changes at 0.01 rad/min. At what rate is the distance between them changing when  $\theta = 0.5$  rad?



We have  $\theta' = 0.01$  rad/min. and want to find  $x'$ . We can use various trig functions but easiest is,

$$\sec(\theta) = \frac{x}{50} \Rightarrow \sec(\theta) \tan(\theta) \theta' = \frac{x'}{50}$$

We know  $\theta = 0.5$  so plug in  $\theta'$  and solve.

$$\sec(0.5) \tan(0.5) (0.01) = \frac{x'}{50}$$

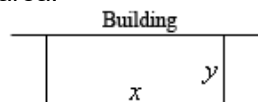
$$x' = 0.3112 \text{ ft/min}$$

Remember to have calculator in radians!

### Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

**Example** We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize  $A = xy$  subject to constraint  $x + 2y = 500$ .

Solve constraint for  $x$  and plug into area.

$$x = 500 - 2y \Rightarrow A = y(500 - 2y) = 500y - 2y^2$$

Differentiate and find critical point(s).

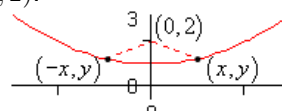
$$A' = 500 - 4y \Rightarrow y = 125$$

By 2<sup>nd</sup> derivative test this is a relative maximum and so is the answer we're after. Finally, find  $x$ .

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

**Example** Determine point(s) on  $y = x^2 + 1$  that are closest to  $(0, 2)$ .



Minimize  $f = d^2 = (x - 0)^2 + (y - 2)^2$  and the constraint is  $y = x^2 + 1$ . Solve constraint for  $x^2$  and plug into the function.

$$x^2 = y - 1 \Rightarrow f = x^2 + (y - 2)^2 = y - 1 + (y - 2)^2 = y^2 - 3y + 3$$

Differentiate and find critical point(s).

$$f' = 2y - 3 \Rightarrow y = \frac{3}{2}$$

By the 2<sup>nd</sup> derivative test this is a relative minimum and so all we need to do is find  $x$  value(s).

$$x^2 = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

The 2 points are then  $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$  and  $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ .



## Integrals Definitions

**Definite Integral :** Suppose  $f(x)$  is continuous on  $[a, b]$ . Divide  $[a, b]$  into  $n$  subintervals of width  $\Delta x$  and choose  $x_i^*$  from each interval. Then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

**Anti-Derivative :** An anti-derivative of  $f(x)$  is a function,  $F(x)$ , such that  $F'(x) = f(x)$ .

**Indefinite Integral :**  $\int f(x) dx = F(x) + c$  where  $F(x)$  is an anti-derivative of  $f(x)$ .

## Fundamental Theorem of Calculus

**Part I :** If  $f(x)$  is continuous on  $[a, b]$  then

$g(x) = \int_a^x f(t) dt$  is also continuous on  $[a, b]$  and

$$g'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

**Part II :**  $f(x)$  is continuous on  $[a, b]$ ,  $F(x)$  is an anti-derivative of  $f(x)$ , i.e.  $F(x) = \int f(x) dx$ , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

**Variants of Part I :**

$$\frac{d}{dx} \int_a^{u(x)} f(t) dt = u'(x) f[u(x)]$$

$$\frac{d}{dx} \int_{v(x)}^b f(t) dt = -v'(x) f[v(x)]$$

$$\frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt = u'(x) f[u(x)] - v'(x) f[v(x)]$$

## Properties

$$\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$$

$$\int c f(x) dx = c \int f(x) dx, c \text{ is a constant}$$

$$\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx, c \text{ is a constant}$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^b c dx = c(b - a), c \text{ is a constant}$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \text{ for any value } c.$$

$$\text{If } f(x) \geq g(x) \text{ on } a \leq x \leq b \text{ then } \int_a^b f(x) dx \geq \int_a^b g(x) dx$$

$$\text{If } f(x) \geq 0 \text{ on } a \leq x \leq b \text{ then } \int_a^b f(x) dx \geq 0$$

$$\text{If } m \leq f(x) \leq M \text{ on } a \leq x \leq b \text{ then } m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$$

**Common Integrals**

$$\begin{aligned}
 \int k \, dx &= kx + c & \int x^n \, dx &= \frac{1}{n+1} x^{n+1} + c, \, n \neq -1 & \int x^{-1} \, dx &= \int \frac{1}{x} \, dx = \ln|x| + c \\
 \int e^u \, du &= e^u + c & \int \frac{1}{ax+b} \, dx &= \frac{1}{a} \ln|ax+b| + c & \int \ln(u) \, du &= u \ln(u) - u + c \\
 \int \cos(u) \, du &= \sin(u) + c & \int \sec(u) \tan(u) \, du &= \sec(u) + c & \int \tan(u) \, du &= \ln|\sec(u)| + c \\
 \int \sin(u) \, du &= -\cos(u) + c & \int \csc(u) \cot(u) \, du &= -\csc(u) + c & \int \cot(u) \, du &= -\ln|\cos(u)| + c \\
 \int \sec^2(u) \, du &= \tan(u) + c & \int \sec(u) \, du &= \ln|\sec(u) + \tan(u)| + c & \int \frac{1}{a^2 + u^2} \, du &= \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c \\
 \int \csc^2(u) \, du &= -\cot(u) + c & \int \csc(u) \, du &= -\ln|\csc(u) + \cot(u)| + c & \int \frac{1}{\sqrt{a^2 - u^2}} \, du &= \sin^{-1}\left(\frac{u}{a}\right) + c
 \end{aligned}$$

**Standard Integration Techniques**

**u Substitution :**  $\int_a^b f(g(x)) g'(x) \, dx$  will convert the integral into  $\int_a^b f(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$  using the substitution  $u = g(x)$  where  $du = g'(x)dx$ . For indefinite integrals drop the limits of integration.

<p><b>Example</b> <math>\int_1^2 5x^2 \cos(x^3) \, dx</math></p> <p><math>u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du</math></p> <p><math>x = 1 \Rightarrow u = 1^3 = 1 \quad \therefore \quad x = 2 \Rightarrow u = 2^3 = 8</math></p>	$  \begin{aligned}  \int_1^2 5x^2 \cos(x^3) \, dx &= \int_1^8 \frac{5}{3} \cos(u) \, du \\  &= \frac{5}{3} \sin(u) \Big _1^8 = \frac{5}{3} (\sin(8) - \sin(1))  \end{aligned}  $
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**Products and (some) Quotients of Trig Functions**

For  $\int \sin^n(x) \cos^m(x) \, dx$  we have the following :

1. **n odd.** Strip 1 sine out and convert rest to cosines using  $\sin^2(x) = 1 - \cos^2(x)$ , then use the substitution  $u = \cos(x)$ .
2. **m odd.** Strip 1 cosine out and convert rest to sines using  $\cos^2(x) = 1 - \sin^2(x)$ , then use the substitution  $u = \sin(x)$ .
3. **n and m both odd.** Use either 1. or 2.
4. **n and m both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For  $\int \tan^n(x) \sec^m(x) \, dx$  we have the following :

1. **n odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using  $\tan^2(x) = \sec^2(x) - 1$ , then use the substitution  $u = \sec(x)$ .
2. **m even.** Strip 2 secants out and convert rest to tangents using  $\sec^2(x) = 1 + \tan^2(x)$ , then use the substitution  $u = \tan(x)$ .
3. **n odd and m even.** Use either 1. or 2.
4. **n even and m odd.** Each integral will be dealt with differently.

**Trig Formulas :**  $\sin(2x) = 2 \sin(x) \cos(x)$ ,  $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ ,  $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

<p><b>Example</b> <math>\int \tan^3(x) \sec^5(x) \, dx</math></p> $  \begin{aligned}  \int \tan^3 x \sec^5 x \, dx &= \int \tan^2 x \sec^4 x \tan x \sec x \, dx \\  &= \int (\sec^2(x) - 1) \sec^4(x) \tan(x) \sec(x) \, dx \\  &= \int (u^2 - 1) u^4 \, du \quad [u = \sec(x)] \\  &= \frac{1}{7} \sec^7(x) - \frac{1}{5} \sec^5(x) + c  \end{aligned}  $
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<p><b>Example</b> <math>\int \frac{\sin^5(x)}{\cos^3(x)} \, dx</math></p> $  \begin{aligned}  \int \frac{\sin^5 x}{\cos^3 x} \, dx &= \int \frac{\sin^4 x \sin x}{\cos^3 x} \, dx = \int \frac{(\sin^2 x)^2 \sin x}{\cos^3 x} \, dx \\  &= \int \frac{(1 - \cos^2(x))^2 \sin(x)}{\cos^3(x)} \, dx \quad [u = \cos(x)] \\  &= - \int \frac{(1 - u^2)^2}{u^3} \, du = - \int \frac{1 - 2u^2 + u^4}{u^3} \, du \\  &= \frac{1}{2} \sec^2(x) + 2 \ln \cos(x)  - \frac{1}{2} \cos^2(x) + c  \end{aligned}  $
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**Integration by Parts :**  $\int u dv = uv - \int v du$  and  $\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$ . Choose  $u$  and  $dv$  from integral and compute  $du$  by differentiating  $u$  and compute  $v$  using  $v = \int dv$ .

**Example**  $\int x e^{-x} dx$   
 $u = x \quad dv = e^{-x} \Rightarrow du = dx \quad v = -e^{-x}$   
 $\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx$   
 $= -x e^{-x} - e^{-x} + c$

**Example**  $\int_3^5 \ln(x) dx$   
 $u = \ln(x) \quad dv = dx \Rightarrow du = \frac{1}{x} dx \quad v = x$   
 $\int_3^5 \ln(x) dx = x \ln(x) \Big|_3^5 - \int_3^5 dx = (x \ln(x) - x) \Big|_3^5$   
 $= 5 \ln(5) - 3 \ln(3) - 2$

**Trig Substitutions :** If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$$\sqrt{a^2 - b^2 x^2} \Rightarrow x = \frac{a}{b} \sin(\theta) \quad \left\| \quad \sqrt{b^2 x^2 - a^2} \Rightarrow x = \frac{a}{b} \sec(\theta) \quad \left\| \quad \sqrt{a^2 + b^2 x^2} \Rightarrow x = \frac{a}{b} \tan(\theta) \right. \right.$$

$$\cos^2(\theta) = 1 - \sin^2(\theta) \quad \left\| \quad \tan^2(\theta) = \sec^2(\theta) - 1 \quad \left\| \quad \sec^2(\theta) = 1 + \tan^2(\theta) \right.$$

**Example**  $\int \frac{16}{x^2 \sqrt{4 - 9x^2}} dx$   
 $x = \frac{2}{3} \sin(\theta) \Rightarrow dx = \frac{2}{3} \cos(\theta) d\theta$   
 $\sqrt{4 - 9x^2} = \sqrt{4 - 4 \sin^2(\theta)} = \sqrt{4 \cos^2(\theta)} = 2 |\cos(\theta)|$   
 Recall  $\sqrt{x^2} = |x|$ . Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute  $\theta$ 's and remove absolute value bars based on that and,

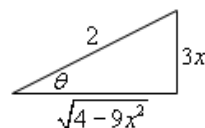
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this case we have  $\sqrt{4 - 9x^2} = 2 \cos(\theta)$ .

$$\int \frac{16}{\frac{4}{9} \sin^2(\theta) (2 \cos(\theta))} \left( \frac{2}{3} \cos(\theta) \right) d\theta = \int \frac{12}{\sin^2(\theta)} d\theta$$

$$= \int 12 \csc^2(\theta) d\theta = -12 \cot(\theta) + c$$

Use Right Triangle Trig to go back to  $x$ 's. From substitution we have  $\sin(\theta) = \frac{3x}{2}$  so,



From this we see that  $\cot(\theta) = \frac{\sqrt{4-9x^2}}{3x}$ . So,

$$\int \frac{16}{x^2 \sqrt{4 - 9x^2}} dx = -\frac{4\sqrt{4 - 9x^2}}{x} + c$$

**Partial Fractions :** If integrating a rational expression involving polynomials,  $\int \frac{P(x)}{Q(x)} dx$ , where the degree of  $P(x)$  is smaller than the degree of  $Q(x)$ . Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor of $Q(x)$	Term in P.F.D	Factor is $Q(x)$	Term in P.F.D
$ax + b$	$\frac{A}{ax + b}$	$(ax + b)^k$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_k}{(ax + b)^k}$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$	$(ax^2 + bx + c)^k$	$\frac{A_1 x + B_1}{ax^2 + bx + c} + \dots + \frac{A_k x + B_k}{(ax^2 + bx + c)^k}$

**Example**  $\int \frac{7x^2 + 13x}{(x-1)(x^2+4)} dx$   
 $\int \frac{7x^2 + 13x}{(x-1)(x^2+4)} dx = \int \frac{4}{x-1} + \frac{3x+16}{x^2+4} dx$   
 $= \int \frac{4}{x-1} + \frac{3x}{x^2+4} + \frac{16}{x^2+4} dx$   
 $= 4 \ln|x-1| + \frac{3}{2} \ln(x^2+4) + 8 \tan^{-1}\left(\frac{x}{2}\right)$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4) + (Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

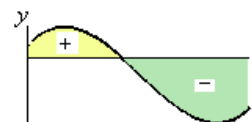
$$7x^2 + 13x = (A + B)x^2 + (C - B)x + 4A - C$$

Set coefficients equal to get a system and solve to get constants.

$$\begin{array}{rcl} A + B = 7 & C - B = 13 & 4A - C = 0 \\ A = 4 & B = 3 & C = 16 \end{array}$$

## Applications of Integrals

**Net Area :**  $\int_a^b f(x) dx$  represents the net area between  $f(x)$  and the  $x$ -axis with area above  $x$ -axis positive and area below  $x$ -axis negative.



**Area Between Curves :** The general formulas for the two main cases for each are,

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}] dx \quad \& \quad x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}] dy$$

If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.

$$A = \int_a^b f(x) - g(x) dx$$

$$A = \int_c^d f(y) - g(y) dy$$

$$A = \int_a^c f(x) - g(x) dx + \int_c^b g(x) - f(x) dx$$

**Volumes of Revolution :** The two main formulas are  $V = \int A(x) dx$  and  $V = \int A(y) dy$ . Here is some general information about each method of computing and some examples.

### Rings

$$A = \pi((\text{outer radius})^2 - (\text{inner radius})^2)$$

Limits:  $x/y$  of right/bot ring to  $x/y$  of left/top ring

Horz. Axis use  $f(x)$ , Vert. Axis use  $f(y)$ ,  
 $g(x)$ ,  $A(x)$  and  $dx$ .  $g(y)$ ,  $A(y)$  and  $dy$ .

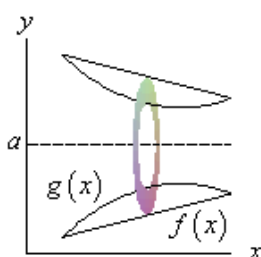
### Cylinders/Shells

$$A = 2\pi(\text{radius})(\text{width / height})$$

Limits :  $x/y$  of inner cyl. to  $x/y$  of outer cyl.

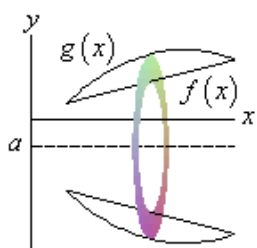
Horz. Axis use  $f(y)$ , Vert. Axis use  $f(x)$ ,  
 $g(y)$ ,  $A(y)$  and  $dy$ .  $g(x)$ ,  $A(x)$  and  $dx$ .

**Ex. Axis :**  $y = a > 0$



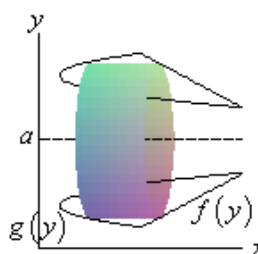
outer radius :  $a - f(x)$   
 inner radius :  $a - g(x)$

**Ex. Axis :**  $y = a \leq 0$



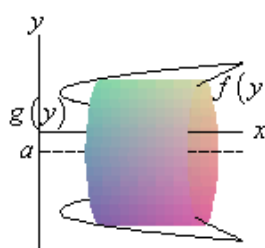
outer radius:  $|a| + g(x)$   
 inner radius:  $|a| + f(x)$

**Ex. Axis :**  $y = a > 0$



radius :  $a - y$   
 width :  $f(y) - g(y)$

**Ex. Axis :**  $y = a \leq 0$



radius :  $|a| + y$   
 width :  $f(y) - g(y)$

These are only a few cases for horizontal axis of rotation. If the axis of rotation is the  $x$ -axis use the  $y = a \leq 0$  case with  $a = 0$ . For vertical axis of rotation ( $x = a > 0$  and  $x = a \leq 0$ ) interchange  $x$  and  $y$  to get appropriate formulas.

**Work :** If a force of  $F(x)$  moves an object in  $a \leq x \leq b$ , the work done is  $W = \int_a^b F(x) dx$

**Average Function Value :** The average value of  $f(x)$  on  $a \leq x \leq b$  is  $f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$

**Arc Length & Surface Area :** The three basic formulas are,

$$L = \int_a^b ds \quad SA = \int_a^b 2\pi y \, ds \text{ (rotate about } x\text{-axis)} \quad SA = \int_a^b 2\pi x \, ds \text{ (rotate about } y\text{-axis)}$$

where  $ds$  is dependent upon the form of the function being worked with as follows.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ if } y = f(x), a \leq x \leq b \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \text{ if } x = f(t), y = g(t), a \leq t \leq b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \text{ if } x = f(y), a \leq y \leq b \quad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \text{ if } r = f(\theta), a \leq \theta \leq b$$

With surface area you may have to substitute in for the  $x$  or  $y$  depending on your choice of  $ds$  to match the differential in the  $ds$ . With parametric and polar you will always need to substitute.

### Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called **convergent** if the limit exists and has a finite value and **divergent** if the limit doesn't exist or has infinite value.

#### Infinite Limit

$$1. \int_a^\infty f(x) \, dx = \lim_{t \rightarrow \infty} \int_a^t f(x) \, dx \quad 2. \int_{-\infty}^b f(x) \, dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) \, dx$$

$$3. \int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^\infty f(x) \, dx \text{ provided both integrals are convergent.}$$

#### Discontinuous Integrand

$$1. \text{ Discontinuity at } a: \int_a^b f(x) \, dx = \lim_{t \rightarrow a^+} \int_t^b f(x) \, dx \quad 2. \text{ Discontinuity at } b: \int_a^b f(x) \, dx = \lim_{t \rightarrow b^-} \int_a^t f(x) \, dx$$

$$3. \text{ Discontinuity at } a < c < b: \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \text{ provided both are convergent.}$$

**Comparison Test for Improper Integrals :** If  $f(x) \geq g(x) \geq 0$  on  $[a, \infty)$  then,

1. If  $\int_a^\infty f(x) \, dx$  is convergent then  $\int_a^\infty g(x) \, dx$  is convergent (if larger converges so does the smaller).
2. If  $\int_a^\infty g(x) \, dx$  is divergent then  $\int_a^\infty f(x) \, dx$  is divergent (if smaller diverges so does the larger).

Useful fact : If  $a > 0$  then  $\int_a^\infty \frac{1}{x^p} \, dx$  converges if  $p > 1$  and diverges for  $p \leq 1$ .

### Approximating Definite Integrals

For given integral  $\int_a^b f(x) \, dx$  and  $n$  (must be even for Simpson's Rule) define  $\Delta x = \frac{b-a}{n}$  and divide  $[a, b]$  into  $n$  subintervals  $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$  with  $x_0 = a$  and  $x_n = b$  then,

**Midpoint Rule :**  $\int_a^b f(x) \, dx \approx \Delta x [f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)]$ ,  $x_i^*$  is midpoint  $[x_{i-1}, x_i]$

**Trapezoid Rule :**  $\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$

**Simpson's Rule :**  $\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$