Beta and Gamma Function

The name gamma function and its symbol were introduced by Adrien-marie Legendre in 1811. It is found that some specific definite integrals can be conveniently used as Beta and Gamma function. The gamma and beta functions have wide applications in the area of quantum physics, Fluid dynamics, Engineering and statistics.

1. Beta Function

If $m > 0, n > 0$, then Beta function is defined by the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ and is denoted by $\beta(m,n)$.

$$\textbf{i.e. } \boldsymbol{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx}$$

Properties

- Beta function is a symmetric function. i.e. $B(m,n) = B(n,m)$, where $m > 0, n > 0$
- $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$
- $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

2. Gamma Function

If $n > 0$, then Gamma function is defined by the integral $\int_0^\infty e^{-x}x^{n-1}dx$ and is denoted by $\lceil n$.

$$\textbf{i.e. } \boldsymbol{\lceil n = \int_0^\infty e^{-x}x^{n-1}dx}$$

Properties

- Reduction formula for Gamma Function $\lceil(n+1) = n\lceil n$; where $n > 0$.
- If n is a positive integer, then $\lceil(n+1) = n!$
- Second Form of Gamma Function $\int_0^\infty e^{-x^2}x^{2m-1}dx = \frac{1}{2}\lceil m$
- Relation Between Beta and Gamma Function, $B(m,n) = \frac{\lceil m \lceil n}{\lceil(m+n)}$
- $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\frac{\lceil\left(\frac{p+1}{2}\right)\lceil\left(\frac{q+1}{2}\right)}{\lceil\left(\frac{p+q+2}{2}\right)}$
- $\left\lceil\frac{1}{2}\right. = \sqrt{\pi}$
- $\left\lceil\frac{n+1}{2}\right. = \frac{(2n)!\sqrt{\pi}}{n!4^n}$ for $n = 0,1,2,3,\ldots$

Special cases

For $n = 0, \left\lceil\frac{1}{2}\right. = \sqrt{\pi}$ For $n = 1, \left\lceil\frac{3}{2}\right. = \frac{\sqrt{\pi}}{2}$ For $n = 2, \left\lceil\frac{5}{2}\right. = \frac{3\sqrt{\pi}}{4}$

3. Error Function and Complementary Error Function

The error function of x is defined by the integral $\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, where x may be real or complex variable and is denoted by erf(x).

$$\text{i.e. } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

The complementary error function is denoted by $\operatorname{erf}_c(x)$ and defined as

$$\operatorname{erf}_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

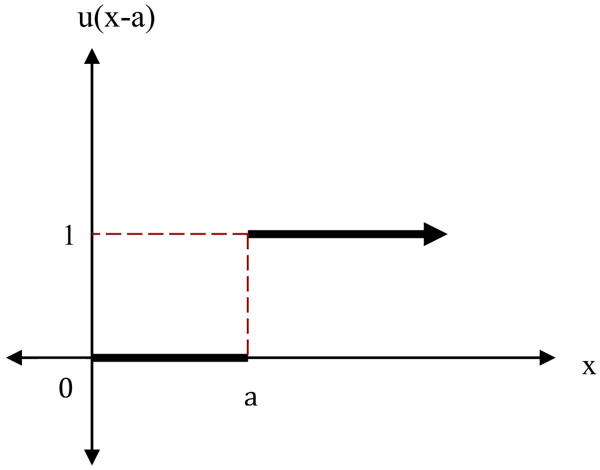
Properties

- $\operatorname{erf}(0) = 0$
- $\operatorname{erf}(\infty) = 1$
- $\operatorname{erf}(x) + \operatorname{erf}_c(x) = 1$
- $\operatorname{erf}(-x) = -\operatorname{erf}(x)$

4. Unit Step Function

The Unit Step Function is defined by

$$u(x-a) = \begin{cases} 1, & \text{for } x \geq a \\ 0, & \text{for } x < a \end{cases}, \text{ where } a \geq 0.$$



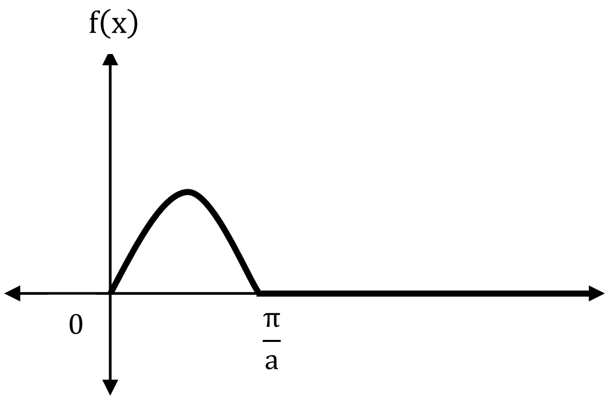
5. Pulse of unit Height

The pulse of unit height of duration T is defined by $f(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq T \\ 0, & \text{for } T < x \end{cases}$.

6. Sinusoidal Pulse Function

The sinusoidal pulse function is defined by

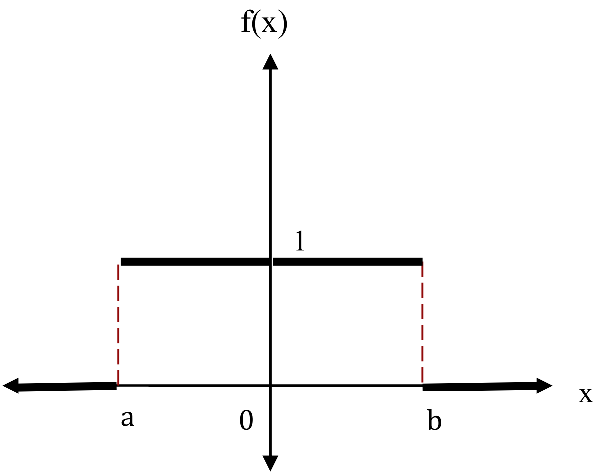
$$f(x) = \begin{cases} \sin ax, & \text{for } 0 \le x \le \frac{\pi}{a} \\ 0, & \text{for } x > \frac{\pi}{a} \end{cases}.$$



7. Rectangle Function

A Rectangular function f(x) defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} 1, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}.$$



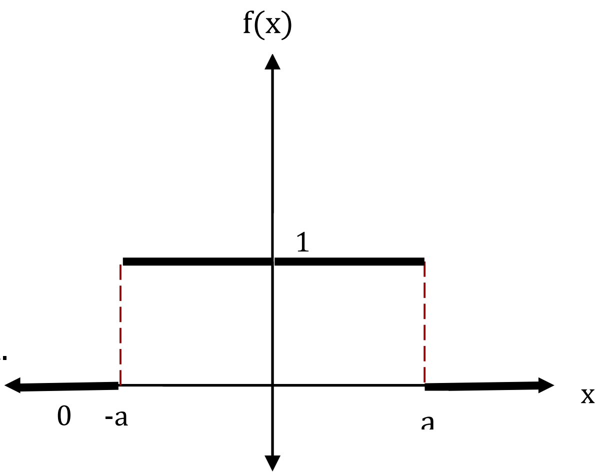
8. Gate Function

A Gate function $f_a(x)$ defined on $\mathbb{R}$ as

$$f_a(x) = \begin{cases} 1, & \text{for } |x| \le a \\ 0, & \text{for } |x| > a \end{cases}.$$

Note that gate function is symmetric about axis

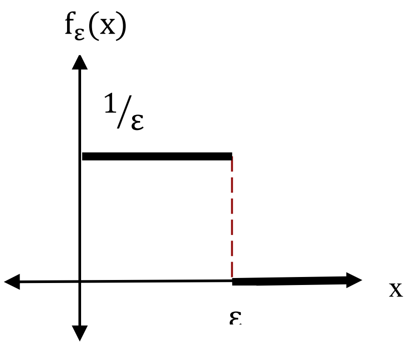
of co-domain. Gate function is also a rectangle function.



9. Dirac Delta Function

A Dirac delta Function $f_\varepsilon(x)$ defined of $\mathbb{R}$ as

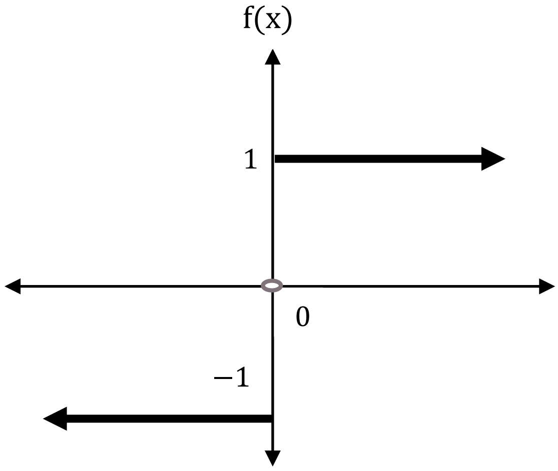
$$f_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & \text{for } 0 \le x \le \varepsilon \\ 0, & \text{for } x > \varepsilon \end{cases}.$$



10. Signum Function

The Signum function is defined by

$$f(x) = \begin{cases} 1, & \text{for } x > 0 \\ -1, & \text{for } x < 0 \end{cases}.$$



11. Periodic Function

A function f is said to be periodic, if $f(x + p) = f(x)$ for all x, If smallest positive number of set of all such p exists, then that number is called the Fundamental period of $f(x)$.

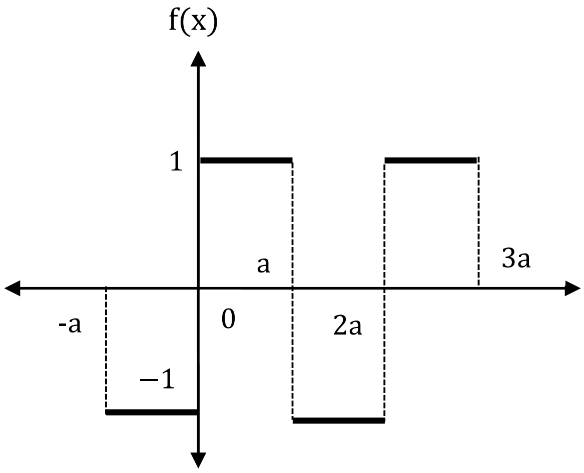
Note

- Constant function is periodic without Fundamental period.
- Sine and Cosine are Periodic functions with Fundamental period 2π.

12. Square Wave Function

A square wave function $f(x)$ of period 2a is defined by

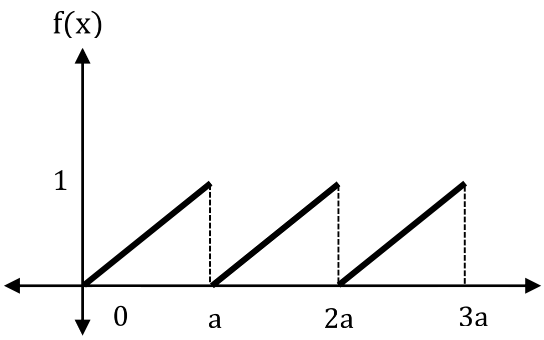
$$f(x) = \begin{cases} 1, & \text{for } 0 < x < a \\ -1, & \text{for } a < x < 2a \end{cases}.$$



13. Saw Tooth Wave Function

A saw tooth wave function $f(x)$ with period a is defined as

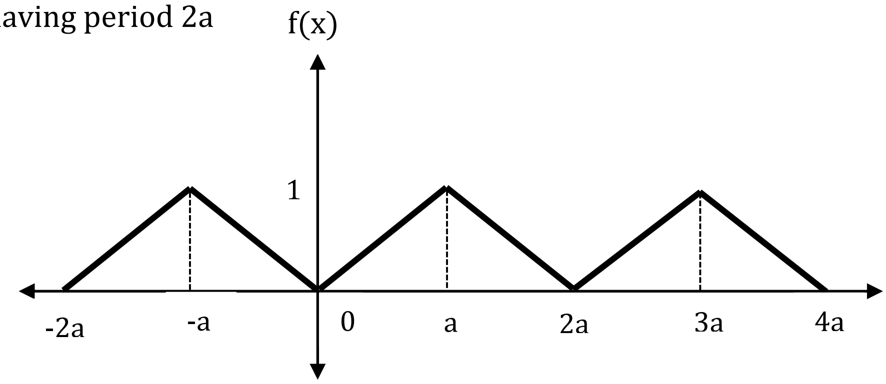
$f(x) = x\,;\, 0 \le x < a.$



14. Triangular Wave Function

A Triangular wave function f(x) having period 2a is defined by

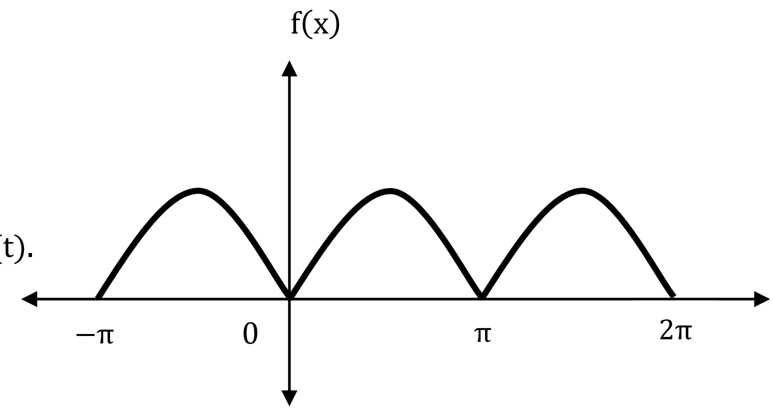
$$f(x) = \begin{cases} x & ; \quad 0 \le x < a \\ 2a - x & ; \quad a \le x < 2a \end{cases}.$$



15. Full Rectified Sine Wave Function

A full rectified sine wave function with period π is defined as

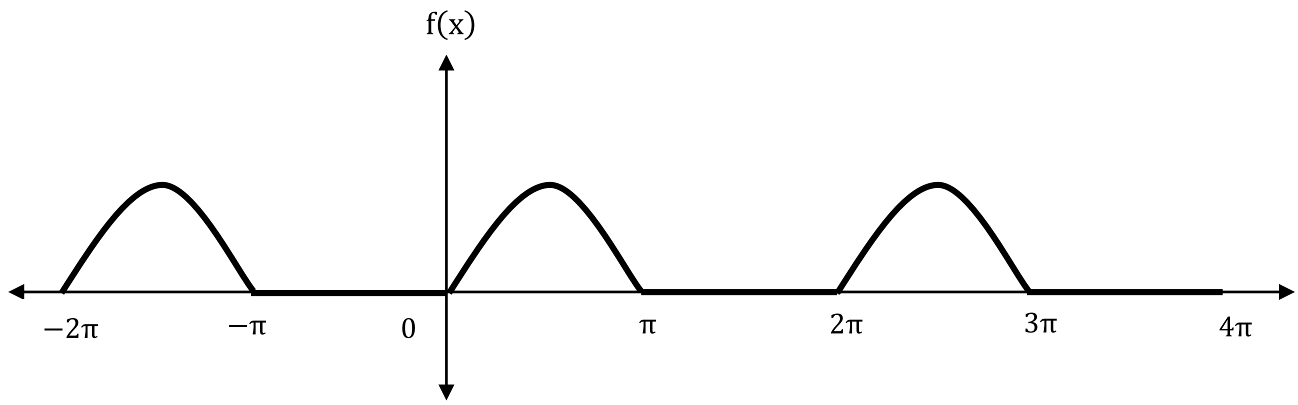
$f(t) = \sin t \;;\; 0 \le t < \pi$ and $f(t+\pi) = f(t)$.



16. Half Rectified Sine Wave Function

A half wave rectified sinusoidal function with period 2π is defined as

$$f(x) = \begin{cases} \sin x, & \text{for } 0 \le x < \pi \\ 0, & \text{for } \pi \le x < 2\pi \end{cases}.$$



17. Bessel's Function

A Bessel's function of order n is defined by

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2(n+2)}+\frac{x^4}{2\cdot 4(2n+2)(2n+4)}-\cdots\right]=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

Properties

- $J_{-n}(x)=(-1)^n J_n(x)$,If n is a positive integer.
- $J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$
- $\frac{d}{dx}\left(x^n J_n(x)\right)=x^n J_{n-1}(x)$

Fourier Series in the interval $(c, c + 2l)$

The Fourier series for the function f(x) in the interval (0,2l) is defined by

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]}$$

Where the constants a_0, a_n and b_n are given by

- $\mathbf{a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx}$
- $\mathbf{a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx}$
- $\mathbf{b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx}$

Note

At a point of discontinuity the sum of the series is equal to average of left and right hand limits of f(x) at the point of discontinuity, say x_0.

$$\text{i.e. } f(x_0) = \frac{f(x_0 - 0) + f(x_0 + 0)}{2}$$

FORMULAE

1. **Leibnitz's Formula**

$$\int u\ v\, dx = u\, v_1 - u'\, v_2 + u''\, v_3 - u'''\, v_4 + \cdots$$

Where, $u', u'', \ldots$ are successive derivatives of u and $v_1, v_2, \ldots$ are successive integrals of v.

Choice of u and v is as per LIATE order.

Where,

A means Algebraic Function

L means Logarithmic Function

T means Trigonometric Function

I means Invertible Function

E means Exponential Function

2. $\mathbf{\int e^{ax} \sin bx\ dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx] + c}$
3. $\mathbf{\int e^{ax} \cos bx\ dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx] + c}$

Fourier series in the interval $(0, 2l)$

The Fourier series for the function f(x) in the interval (0,2l) is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

Where the constants a_0, a_n and b_n are given by

- $a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx$
- $a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$
- $b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

Exercise-1

Fourier Series In Arbitrary Period $[0, 2l]$			
H	1.	Find the Fourier series for $f(x) = x^2$ in (0,1).	
	Ans.	$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi^2} \cos(2n\pi x) - \frac{1}{n\pi} \sin(2n\pi x) \right]$	
C	2.	Find the Fourier series to represent $f(x) = 2x - x^2$ in (0,3).	May-12
	Ans.	$f(x) = \sum_{n=1}^{\infty} \left[-\frac{9}{n^2\pi^2} \cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi} \sin\left(\frac{2n\pi x}{3}\right) \right]$	
T	3.	Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2$.	
	Ans.	$f(x) = \frac{(1-e^{-2})}{2} + \sum_{n=1}^{\infty} \left[\frac{(1-e^{-2})}{n^2\pi^2 + 1} \cos(n\pi x) + \frac{n\pi(1-e^{-2})}{n^2\pi^2 + 1} \sin(n\pi x) \right]$	
T	4.	Find the Fourier series of the periodic function $f(x) = \pi \sin \pi x$ Where $0 < x < 1$, $p = 2l = 1$.	Dec-10
	Ans.	$f(x) = 2 + \sum_{n=1}^{\infty} \frac{4}{1 - 4n^2} \cos(n\pi x)$	
H	5.	Develop f(x) in a Fourier series in the interval (0,2) if $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$	Dec-13
	Ans.	$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi^2} \cos(n\pi x) + \frac{(-1)^{n+1}}{n\pi} \sin(n\pi x) \right]$	

C	6.	Find the Fourier series for periodic function with period 2 of $f(x) = \begin{cases} \pi, & 0 \le x \le 1 \\ \pi\,(2-x), & 1 \le x \le 2. \end{cases}$	Jun-13
	Ans.	$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(n\pi x) + \frac{1}{n}\sin(n\pi x)\right]$	
T	7.	Find the Fourier series for periodic function with period 2 of $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi\,(2-x), & 1 \le x \le 2. \end{cases}$	Jun-15
	Ans.	$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2[(-1)^n - 1]}{\pi n^2}\cos(n\pi x)$	

Fourier series in the interval $(0, 2\pi)$

The Fourier series for the function f(x) in the interval $(0,2\pi)$ is defined by

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

Where the constants a_0, a_n and b_n are given by

- $\mathbf{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx}$
- $\mathbf{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx}$
- $\mathbf{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx}$

Exercise-2

Fourier Series In $[0, 2\pi]$			
H	1.	Find Fourier Series for $f(x) = x^2$; where $0 \le x \le 2\pi$.	Jan-13
	Ans.	$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right]$	
T	2.	Express $f(x) = \frac{\pi - x}{2}$ in a Fourier series in interval $0 < x < 2\pi$.	Dec-13
	Ans.	$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$	
C	3.	Show that, $\pi - x = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{\sin 2nx}{n}$, when $0 < x < \pi$.	
C	4.	Obtain the Fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 < x < 2\pi$. Hence prove that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.	Jan-15 Jun-15

	Ans.	$f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$	
C	5.	Find Fourier Series for $f(x)=e^{-x}$ where $0<x<2\pi$.	Jun-14
	Ans.	$f(x)=\frac{1-e^{-2\pi}}{2\pi}+\sum_{n=1}^{\infty}\left[\frac{1-e^{-2\pi}}{\pi(n^2+1)}\cos nx+\frac{n(1-e^{-2\pi})}{\pi(n^2+1)}\sin nx\right]$	
C	6.	Find the Fourier series of $f(x)=\begin{cases} x^2 & ;0<x<\pi \\ 0 & ;\pi<x<2\pi. \end{cases}$	Jan-13
	Ans.	$f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx+\frac{1}{\pi}\left\{-\frac{\pi^2(-1)^n}{n}+\frac{2(-1)^n}{n^3}-\frac{2}{n^3}\right\}\sin nx\right]$	

Fourier series in the interval $(-l,l)$

The Fourier series for the function f(x) in the interval $(-l,l)$ is defined by

$$\mathbf{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)}$$

Where the constants a_0, a_n and b_n are given by

- $\mathbf{a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx}$
- $\mathbf{a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx}$
- $\mathbf{b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$

Definition: Odd Function & Even Function

A function is said to be Odd Function if $\mathbf{f(-x)=-f(x)}$.

A function is said to be Even Function if $\mathbf{f(-x)=f(x)}$.

Fourier Series For Odd & Even Function

Let, f(x) be a periodic function defined in $(-\mathbf{l},\mathbf{l})$

- f(x) is even, $b_n=0$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)$$

- f(x) is odd, $a_n=0; n=0,1,2,3,\ldots$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$$

Exercise-3

Fourier Series In Arbitrary Period $[-l, l]$			
H	1.	Expand $f(x) = x$ in $-l < x < l$ the Fourier series.	
	Ans.	$f(x) = \sum_{n=1}^{\infty} \frac{2\,l\,(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{l}\right)$	
H	2.	Find the Fourier series of the periodic function $f(x) = 2x$ Where $-1 < x < 1, p = 2l = 2$.	Dec-09
	Ans.	$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi x)$	
T	3.	Find the Fourier series for $f(x) = x^2$ in $-2 < x < 2$.	Dec-11
	Ans.	$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$	
H	4.	Find the Fourier series of (i) $f(x) = x;\ -\pi < x < \pi,\ f(x) = f(x + 2\pi)$. (ii) $f(x) = x^2; -l < x < l$.	Jun-14
	Ans.	(i)$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$ (ii)$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} \cos nx$	
H	5.	Expand $f(x) = x^2 - 2$ in $-2 < x < 2$ the Fourier series.	Jan-15
	Ans.	$f(x) = -\frac{2}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{2}\right)$	
T	6.	Find the Fourier series of $f(x) = x^2 + x$ Where $-2 < x < 2$.	Jan-13
	Ans.	$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \left[\frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right) + \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{l}\right)\right]$	
H	7.	Find the Fourier expansion for function $f(x) = x - x^3$ in $-1 < x < 1$.	
	Ans.	$f(x) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3} \sin n\pi x$	
C	8.	Find the Fourier expansion for function $f(x) = x - x^2$ in $-1 < x < 1$.	Jun-15

	Ans.	$f(x) = -\frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2\pi^2}\cos(n\pi x) + \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)\right]$	
T	9.	Find the Fourier series for periodic function with period 2, which is given below $f(x) = \begin{cases} 0, -1 < x < 0 \\ x, \ 0 < x < 1 \end{cases}$.	Jun-13
	Ans.	$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos nx + \frac{(-1)^{n+1}}{n\pi}\sin nx\right]$	
H	10.	Find the Fourier series of $f(x) = \begin{cases} x, -1 < x < 0 \\ 2, \ 0 < x < 1 \end{cases}$.	Jan-13
	Ans.	$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{1 - (-1)^n}{n^2\pi^2}\cos nx + \frac{2 - 3(-1)^n}{n\pi}\sin nx\right]$	
C	11.	Find the Fourier series for periodic function f(x) with period 2 Where $f(x) = \begin{cases} -1, -1 < x < 0 \\ 1, \ 0 < x < 1 \end{cases}$.	Jan-13
	Ans.	$f(x) = \sum_{n=1}^{\infty}\frac{2 - 2(-1)^n}{\pi n}\sin nx$	

Fourier series in the interval $(-\pi, \pi)$

The Fourier series for the function f(x) in the interval $(-\pi, \pi)$ is defined by

$$\mathbf{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

- $\mathbf{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx}$
- $\mathbf{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx}$
- $\mathbf{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx}$

Exercise-4

Fourier Series In $[-\pi, \pi]$			
H	1.	Find the Fourier series expansion of $f(x) = x; -\pi < x < \pi$.	Dec-11 Jun-14
	Ans.	$f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$	
T	2.	Find the Fourier series expansion of $f(x) = \lvert x\rvert; -\pi < x < \pi$.	Jun-15
	Ans.	$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2\,[(-1)^n - 1]}{\pi n^2}\cos nx$	

C	3.	Obtain the Fourier series for $f(x) = x^2$ in the interval $-\pi < x < \pi$ and hence deduce that (i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$. (iii) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.	Dec-09 Jan-15
	Ans.	$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$	
H	4.	Find the Fourier series of $f(x) = \frac{x^2}{2}$; $-\pi < x < \pi$.	Mar-10
	Ans.	$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos nx$	
H	5.	Find the Fourier series of $f(x) = x^3$; $x \in (-\pi, \pi)$.	Jun-13
	Ans.	$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \left[\pi^2 - \frac{6}{n^2}\right] \sin nx$	
C	6.	Sketch the function $f(x) = x + \pi$; $-\pi < x < \pi$. Where $f(x) = f(x + 2\pi)$ and Find the Fourier series.	Mar-10
	Ans.	$f(x) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$	
T	7.	Find the Fourier series of $f(x) = x - x^2$; $-\pi < x < \pi$. Deduce that: $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.	Jun-14
	Ans.	$f(x) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n^2} \cos nx + \frac{2(-1)^{n+1}}{n} \sin nx\right]$	
H	8.	Find the Fourier series of $f(x) = x + x^2$; $-\pi < x < \pi$.	Jun-15
	Ans.	$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2} \cos nx + \frac{2(-1)^{n+1}}{n} \sin nx\right]$	
T	9.	Find the Fourier series of $f(x) = x + \|x\|$; $-\pi < x < \pi$.	Mar-10 Jun-13 Jan-15 Jan-15*
	Ans.	$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2[(-1)^n - 1]}{\pi n^2} \cos nx + \frac{2(-1)^{n+1}}{n} \sin nx\right]$	

T	10.	Find the Fourier series to representation e^x in the the interval$(-\pi, \pi)$.	Dec-13
	Ans.	$f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{[e^{\pi} - e^{-\pi}]\,(-1)^n}{\pi(n^2+1)}\cos nx + \frac{n\,[e^{-\pi} - e^{\pi}]\,(-1)^n}{\pi(n^2+1)}\sin nx\right]$	
C	11.	Find the Fourier series for $f(x) = \lvert\sin x\rvert$ in$-\pi < x < \pi$.	May-11
	Ans.	$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2\,[(-1)^n + 1]}{\pi(1-n^2)}\cos nx\ ;\ a_1 = 0$	
T	12.	Find the Fourier series expansion of $f(x) = \sqrt{1-\cos x}$ in the interval, (i)$-\pi < x < \pi$. (ii) $0 \le x \le 2\pi$.	
	Ans.	$f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(1-4n^2)}\cos nx$ (same answer in both intervals)	
H	13.	Find the Fourier series of the periodic function f(x) with period 2π defined as follows $f(x) = \begin{cases} 0\ ;\ -\pi < x < 0 \\ x\ ;\ 0 < x < \pi. \end{cases}$	Jun-13
	Ans.	$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$	
H	14.	Obtain the Fourier Series for the function f(x) given by $f(x) = \begin{cases} 0 & ; -\pi \le x \le 0 \\ x^2 & ;\ \ 0 \le x \le \pi \end{cases}$ Hence prove $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}$.	
	Ans.	$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \frac{1}{\pi}\left\{-\frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right\}\sin nx\right]$	
H	15.	Find the Fourier series expansion of the function $f(x) = \begin{cases} -\pi; & -\pi \le x < 0 \\ x; & 0 < x \le \pi \end{cases}$ Deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.	May-12 Jun-14
	Ans.	$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{1 - 2(-1)^n}{n}\sin nx\right]$	
C	16.	Find Fourier series for 2π periodic function $f(x) = \begin{cases} -k; \text{ if } -\pi < x < 0 \\ \ \ k; \text{ if } \ 0 < x < \pi \end{cases}$ Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.	Jan-15

	Ans.	$f(x) = \sum_{n=1} \frac{2k[1-(-1)^n]}{n\pi} \sin nx = \sum_{n=odd} \frac{4k}{n\pi} \sin nx$	
T	17.	If $f(x) = \begin{cases} \pi + x, -\pi < x < 0 \\ \pi - x, \ 0 < x < \pi \end{cases}$ $f(x) = f(x+2\pi)$,for all x then expand f(x) in a Fourier series.	**Jan-13** **Jun-15**
	Ans.	$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [1-(-1)^n] \cos nx$	
H	18.	Find the Fourier Series for the function f(x) given by $f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x - \pi, \ 0 < x < \pi \end{cases}$	**Jun-15**
	Ans.	$f(x) = -\frac{3\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right]$	
C	19.	Find the Fourier Series for the function f(x) given by $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & ; -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & ; \ 0 \le x \le \pi \end{cases}$ Hence prove $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.	**May-11** **Jan-15**
	Ans.	$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} [1-(-1)^n] \cos nx$	

Half Range Series

If a function f(x) is defined only on a half interval $(0, \pi)$ instead of $(0, 2\pi)$, then it is possible to obtain a Fourier cosine or Fourier sine series.

Half Range Cosine Series In The Interval $(0, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

- $\mathbf{a_0 = \frac{2}{l} \int_0^l f(x)\, dx}$
- $\mathbf{a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx}$

Exercise-5

Half Range Cosine Series			
C	1.	Find Fourier cosine series of the periodic function $f(x) = x$, $(0 < x < L), p = 2L$. also sketch $f(x)$ and its periodic extension.	Dec-10
	Ans.	$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L[(-1)^n - 1]}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)$	
H	2.	Find the Half range cosine series for $f(x) = x, 0 < x < 3$.	Jan-13
	Ans.	$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6[(-1)^n - 1]}{n^2\pi^2} \cos\left(\frac{n\pi x}{3}\right)$	
H	3.	Find Fourier cosine series for $f(x) = x^2;\ 0 < x \le c$. Also sketch $f(x)$.	Jun-13
	Ans.	$f(x) = \frac{c^2}{3} + \sum_{n=1}^{\infty} \frac{4c^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{c}\right)$	
H	4.	Find Half-range cosine series for $f(x) = x^2$ in $0 < x < \pi$.	Jun-15
	Ans.	$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$	
T	5.	Find Half-range cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$.	Jun-15
	Ans.	$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos(n\pi x)$	
T	6.	Find a cosine series for $f(x) = e^x$ in $0 < x < L$.	Mar-10 Jan-15
	Ans.	$f(x) = \frac{e^L - 1}{L} + \sum_{n=1}^{\infty} \frac{2L[e^L(-1)^n - 1]}{n^2\pi^2 + L^2} \cos\left(\frac{n\pi x}{L}\right)$	
H	7.	Find Half-range cosine series for $f(x) = e^x$ in $(0,1)$.	May-12
	Ans.	$f(x) = (e-1) + \sum_{n=1}^{\infty} \frac{2[e(-1)^n - 1]}{n^2\pi^2 + 1} \cos(n\pi x)$	

H	8.	Find Half-range cosine series for $f(x) = e^x$ in $0 < x < 2$.	Jun-14
	Ans.	$f(x) = \frac{e^2 - 1}{2} + \sum_{n=1}^{\infty} \frac{4[e^2(-1)^n - 1]}{n^2\pi^2 + 4} \cos\left(\frac{n\pi x}{2}\right)$	
H	9.	Find Half-range cosine series for $f(x) = e^x$ in $0 < x < \pi$.	Jun-15
	Ans.	$f(x) = \frac{e^\pi - 1}{\pi} + \sum_{n=1}^{\infty} \frac{2(e^\pi(-1)^n - 1)}{\pi(1 + n^2)} \cos nx$	
C	10.	Fine Half range cosine series for $\sin x$ in $(0, \pi)$ and show that $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$.	May-12 Jan-15
	Ans.	$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} -\frac{4}{\pi(n^2 - 1)} \cos nx, a_1 = 0$	

Half Range Sine Series In The Interval $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

- $\mathbf{b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx}$

Exercise-6

Half Range Sine Series			
H	1.	Express $f(x) = x$ as a Half range sine series in $0 < x < 2$ Half range cosine series in $0 < x < 2$.	Jun-14
	Ans.	(i)$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$ (ii)$f(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right)^2 [(-1)^n - 1] \cos\left(\frac{n\pi x}{2}\right)$	
H	2.	Find the Half range sine series for $f(x) = 2x, 0 < x < 1$.	Jun-15

	Ans.	$f(x)=\sum_{n=1}^{\infty}\frac{4}{n\pi}(-1)^{n+1}\sin n\pi x$	
H	3.	Find the Half range sine series for $f(x)=\pi-x, 0<x<\pi$.	Jan-13
	Ans.	$f(x)=\sum_{n=1}^{\infty}\frac{2}{n}\sin nx$	
T	4.	Find Half-range sine series for $f(x)=e^{x}$ in $0<x<\pi$.	Jun-15
	Ans.	$f(x)=\sum_{n=1}^{\infty}\frac{2n[e^{\pi}(-1)^{n+1}+1]}{\pi(1+n^2)}\sin nx$	
C	5.	Expand $\pi x-x^2$ in a half-range sine series in the interval $(0,\pi)$ up to first three terms.	Jan-15 Jun-15
	Ans.	$f(x)=\frac{8}{\pi}\left(\sin x+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+\cdots\right)$	
C	6.	Find the sine series $f(x)=\begin{cases}x & ; 0<x<\frac{\pi}{2}\\ \pi-x & ; \frac{\pi}{2}<x<\pi\end{cases}$.	May-11
	Ans.	$f(x)=\sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)\sin nx$	
T	7.	If $f(x)=\begin{cases}mx & ; 0<x<\frac{\pi}{2}\\ m(\pi-x) & ; \frac{\pi}{2}<x<\pi\end{cases}$ then, show that $f(x)=\frac{4m}{\pi}\left\{\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}-\frac{\sin 5x}{5^2}-\cdots\right\}$	May-12

Some Useful Integrals And Formulae

- Consider m and n be positive integers or zero.

1. $\int_0^{2\pi} \sin nx \, dx = 0; n \neq 0$
2. $\int_0^{2\pi} \cos nx \, dx = 0; n \neq 0$
3. $\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 0 \,; m \neq n \\ \pi \,; m = n \neq 0 \end{cases}$
4. $\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 \,; m \neq n \\ \pi \,; m = n \neq 0 \end{cases}$
5. $\int_0^{2\pi} \sin mx \cos nx \, dx = 0, \forall m, n$

- Let $m, n \in \mathbb{Z}$.

1. $\sin n\pi = 0 \,; \ \forall n$
2. $\cos n\pi = (-1)^n \,; \ \forall n$
3. $\sin(2n+1)\frac{\pi}{2} = (-1)^n$
4. $\cos(2n+1)\frac{\pi}{2} = 0$
5. $\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + c$
6. $\int \frac{[f(x)]^n}{f'(x)} dx = \frac{[f(x)]^{n+1}}{n+1} + c$

- If f(x) is an odd function defined in $(-a, a)$, then $\int_{-a}^{a} f(x) \, dx = 0$.
- If f(x) is a even function defined in $(-a, a)$, then $\int_{-a}^{a} f(x) \, dx = 2\int_0^a f(x) \, dx$.

Fourier Integrals

Fourier Integral of f(x) is given by

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

Where, $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$ & $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$

Fourier Cosine Integral

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega\;;\ \text{Where, } A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\,dv$$

Fourier Sine Integral

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega\;;\ \text{Where, } B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv$$

Exercise-6

FOURIER INTEGRAL			
C	1.	Using Fourier integral prove that $\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\,d\omega = \begin{cases} 0 & ; x<0 \\ \pi/2 & ; x=0 \\ \pi e^{-x} & ; x>0. \end{cases}$	Dec-10 Jan-15 Jun-15
C	2.	Find the Fourier integral representation of $f(x) = \begin{cases} 1 & ; \lvert x\rvert<1 \\ 0 & ; \lvert x\rvert>1 \end{cases}$. Hence calculate the followings. a) $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ b) $\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega$	Dec-13 Jan-15 Jun-15
	Ans.	$f(x) = \int_0^\infty \frac{2\sin\omega}{\pi\omega}\cos\omega x\,d\omega$ a) $\begin{cases} \frac{\pi}{2} & ; \lvert x\rvert<1 \\ 0 & ; \lvert x\rvert>1 \end{cases}$ b) $\frac{\pi}{2}$	
T	3.	Find the Fourier integral representation of $f(x) = \begin{cases} 2 & ; \lvert x\rvert<2 \\ 0 & ; \lvert x\rvert>2. \end{cases}$	Jan-13 Jun-14
	Ans.	$f(x) = \int_0^\infty \frac{4\sin 2\omega\cos\omega x}{\pi\omega}\,d\omega$	

H	4.	Find the Fourier cosine integral of $f(x) = e^{-kx}$ $(x > 0, k > 0)$.	Mar-10
	Ans.	$f(x) = \int_0^\infty \frac{2k}{\pi(k^2+\omega^2)} \cos \omega x \, d\omega$	
H	5.	Find Fourier cosine integral of $f(x) = \begin{cases} x\,; & 0 < x < a \\ 0 & ;\ x > a. \end{cases}$	Jun-13
	Ans.	$f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{a \sin \omega a}{\omega} + \frac{\cos \omega a}{\omega^2} - \frac{1}{\omega^2}\right] \cos \omega x \, d\omega$	
H	6.	Using Fourier integral prove that $\int_0^\infty \frac{1-\cos \omega\pi}{\omega} \sin \omega x \, d\omega = \begin{cases} \frac{\pi}{2} & ; 0 < x < \pi \\ 0 & ; x > \pi. \end{cases}$	Mar-10
	7.	Find Fourier cosine and sine integral of $f(x) = \begin{cases} 0 & ; 0 \le x < 1 \\ -1 & ; 1 < x < 2 \\ 0 & ; 2 < x < \infty \end{cases}$	
T	Ans.	a) $f(x) = \frac{2}{\pi\omega}\int_0^\infty (\sin \omega - \sin 2\omega) \cos \omega x \, d\omega$ b) $f(x) = \frac{2}{\pi\omega}\int_0^\infty (\cos 2\omega - \cos \omega) \sin \omega x \, d\omega$	
T	8.	Find Fourier cosine and sine integral of $f(x) = \begin{cases} \sin x & ; 0 \le x \le \pi \\ 0 & ; x > \pi. \end{cases}$	
	Ans.	a) $f(x) = \int_0^\infty \frac{2(1+\cos \omega\pi)}{\pi(1-\omega^2)} \cos \omega x \, d\omega$; $A(1) = 0$ b) $f(x) = \int_0^\infty \frac{2 \sin \omega\pi}{\pi(1-\omega^2)} \sin \omega x \, d\omega$; $B(1) = 1$	

Definition: Differential Equation

An eqn which involves differential co-efficient is called Differential Equation.

e.g. $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$

Definition: Ordinary Differential Equation

An eqn which involves function of single variable and ordinary derivatives of that function then it is called Ordinary Differential Equation.

e.g. $\frac{dy}{dx} + y = 0$

Definition: Partial Differential Equation

An eqn which involves function of two or more variable and partial derivatives of that function then it is called Partial Differential Equation.

e.g. $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0$

Definition: Order Of Differential Equation

The order of highest derivative which appeared in differential equation is "Order of D.E".

e.g. $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0$ Has order 1.

Definition: Degree Of Differential Equation

When a D.E. is in a polynomial form of derivatives, the highest power of highest order derivative occurring in D.E. is called a "Degree of D.E.".

e.g. $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0$ Has degree 2.

Exercise-1

Order And Degree Differential Equation			
C	1.	$\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{4}}$. **[2, 4]**	May-11
T	2.	$\left[\frac{dy}{dx} + y\right]^{\frac{1}{2}} = \sin x$. **[1, 1]**	May-11
C	3.	$y = x\frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$. **[1, 2]**	
C	4.	$\left(\frac{d^2y}{dx^2}\right)^3 = \left[x + \sin\left(\frac{dy}{dx}\right)\right]^2$. **[2, Undefined]**	
H	5.	$\frac{d^2y}{dx^2} = \ln\left(\frac{dy}{dx}\right) + y$. **[2, Undefined]**	

T	6.	Define order and degree of the differential equation. Find order and degree of differential equation $\sqrt{x^2\frac{d^2y}{dx^2}+2y}=\frac{d^3y}{dx^3}$. $[\mathbf{3},\mathbf{2}]$	**Jan-15**

Solution Of A Differential Equation

A solution or integral or primitive of a differential equation is a relation between the variables which does not involve any derivative(s) and satisfies the given differential equation.

1. **General Solution (G.S.)**
   A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation, is called the General solution or complete integral or complete primitive.
2. **Particular Solution**
   The solution obtained from the general solution by giving a particular value to the arbitrary constants is called a particular solution.
3. **Singular solution**
   A solution which cannot be obtained from a general solution is called a singular solution.

Definition: Linear Differential Equation

A differential equation is called “LINEAR DIFFERENTIAL EQUATION” if

1. dependent variable and its all derivative are of first degree
2. dependent variable and its derivative are not multiplied together

If one of above condition is not satisfy, then it is called “NON-LINEAR DIFFERENTIAL EQUATION”.

e.g.

1. $\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+y=0$ Is linear.
2. $\frac{d^2y}{dx^2}+y\frac{dy}{dx}+y=0$ Is non-linear.
3. $\frac{d^2y}{dx^2}+x^2\left(\frac{dy}{dx}\right)^2+y=0$ Is non-linear.

A Linear Differential Equation of first order is known as **Leibnitz’s linear Differential Equation**

i.e. $\frac{dy}{dx}+P(x)y=Q(x)+c$ **OR** $\frac{dx}{dy}+P(y)x=Q(y)+c$

Type Of First Order Differential Equation

I. Variable Separable Equation
II. Homogeneous Differential Equation
III. Exact Differential Equation
IV. Linear(Leibnitz's) Differential Equation
V. Bernoulli's Equation

Variable Separable Equation

If differential equation of type $\frac{dy}{dx} = f(x,y)$ can convert into $M(x)dx = N(y)dy$, then it is known as Variable Separable Equation.

The general solution of Variable Separable Equation is

$$\int \mathbf{M(x)dx} = \int \mathbf{N(y)dy + c}$$

Where, c is a arbitrary constant.

Homogeneous Differential Equation

A differential equation of the form $\mathbf{M(x,y)dx + N(x,y)dy = 0}$ is said to be Homogeneous Differential Equation if $M(x,y)$ & $N(x,y)$ are homogeneous function of same degree.

Such differential equation is solved by the substitution $\mathbf{y = vx}$.

Exercise-2

Separable method				
C	1.	$9yy' + 4x = 0.$	$[\mathbf{9y^2 + 4x^2 = c}]$	Dec-11
H	2.	$e^x dx - e^y dy = 0$	$[\mathbf{e^x = e^y + c}]$	
C	3.	$\frac{dy}{dx} = e^{2x+3y}$	$\left[\mathbf{\frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c}\right]$	Jun-14
C	4.	$y' = e^{x-y} + xe^{-y}$	$\left[\mathbf{e^y = e^x + \frac{x^2}{2} + c}\right]$	Jun-15
C	5.	$xy' + y = 0\,; y(2) = -2.$	$[\mathbf{xy = -4}]$	Dec-11
C	6.	$L\frac{dI}{dt} + RI = 0, I(0) = I_0.$	$\left[\mathbf{I = I_0 e^{-\frac{R}{L}t}}\right]$	Dec-10
H	7.	$(1+x)ydx + (1-y)xdy = 0.$	$[\mathbf{\log(xy) + x - y = c}]$	
T	8.	$e^x \tan y dx + (1-e^x)\sec^2 y dy = 0.$	$[\mathbf{(1-e^x)^{-1}\tan y = c}]$	Jun-13
C	9.	$xy' = y^2 + y.$	$\left[\mathbf{\frac{y}{y+1} = xc}\right]$	Dec-10

H	10.	$xy\frac{dy}{dx} = 1 + x + y + xy.$ $\quad$ $[\mathbf{y - \log(1+y) = \log x + x + c}]$	May-12
C	11.	$\tan y\frac{dy}{dx} = \sin(x+y) + \sin(x-y).$ $\quad$ $[\mathbf{\sec y = -2\cos x + c}]$	
C	12.	$1 + \frac{dy}{dx} = e^{x+y}.$ $\quad$ $[\mathbf{-(e^{-x-y}) = x + c}]$	
H	13.	$\frac{dy}{dx} = \cos x \cos y - \sin x \sin y.$ $\quad$ $\left[\mathbf{\tan\left(\frac{x+y}{2}\right) = x + c}\right]$	Jan- 13
T	14.	$(x+y)^2\frac{dy}{dx} = a^2.$ $\quad$ $\left[\mathbf{y - a\tan^{-1}\frac{x+y}{a} = c}\right]$	
C	15.	$x\frac{dy}{dx} = y + xe^{\frac{y}{x}}.$ $\quad$ $\left[\mathbf{-\left(e^{-\frac{y}{x}}\right) = \log x + c}\right]$	
T	16.	$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}.$ $\quad$ $[\mathbf{\sin\frac{y}{x} = xc}]$	Jan-15
H	17.	$(x+y)dx + (y-x)dy = 0.$ $\quad$ $\left[\mathbf{\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right) + c}\right]$	Jun-15
C	18.	$\left[1 + e^{\frac{x}{y}}\right]dx + e^{\frac{x}{y}}\left[1 - \frac{x}{y}\right]dy = 0.$ $\quad$ $\left[\mathbf{x + ye^{\frac{x}{y}} = c}\right]$	Jun-15
C	19.	$(x+y)^2\left[x\frac{dy}{dx} + y\right] = xy\left[1 + \frac{dy}{dx}\right].$ $\quad$ $\left[\mathbf{\log xy = -\frac{1}{x+y} + c}\right]$	May-12

Leibnitz's Linear Differential Equation

A differential equation of the form $\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{P(x)y = Q(x)}$ "OR" $\frac{\mathbf{dx}}{\mathbf{dy}} + \mathbf{P(y)x = Q(y)}$ is known as Linear Differential Equation.

The general solution of Linear Differential Equation is

$\mathbf{y(I.F.) = \int Q(x)(I.F.)dx + c}$ "OR" $\mathbf{x(I.F.) = \int Q(y)(I.F.)dy + c}$

Where, $\mathbf{I.F. = e^{\int p\,dx}}$ "OR" $\mathbf{I.F. = e^{\int p\,dy}}$

Exercise-3

Linear Differential Equation			
C	1.	$\frac{dy}{dx} - y = e^{2x}.$ $\quad$ $[\mathbf{ye^{-x} = e^x + c}]$	Dec-09
H	2.	$\frac{dy}{dx} + 2xy = e^{-x^2}.$ $\quad$ $\left[\mathbf{ye^{x^2} = x + c}\right]$	
C	3.	$y' + y\sin x = e^{\cos x}.$ $\quad$ $[\mathbf{ye^{-\cos x} = x + c}]$	Dec-11
H	4.	$\frac{dy}{dx} + \frac{1}{x^2}y = 6e^{\frac{1}{x}}.$ $\quad$ $\left[\mathbf{ye^{\frac{-1}{x}} = 6x + c}\right]$	Jan-13 Jun-15

T	5.	$y' + 6x^2y = \frac{e^{-2x^3}}{x^2}, y(1) = 0.$ $\left[\mathbf{ye^{2x^3} = \left(1 - \frac{1}{x}\right)}\right]$	May-10 Mar-10
C	6.	$(x+1)\frac{dy}{dx} - y = (x+1)^2e^{3x}.$ $\left[\mathbf{\frac{y}{x+1} = \frac{e^{3x}}{3} + c}\right]$	Jun-14
C	7.	$\frac{dy}{dx} + \frac{2y}{x} = \sin x.$ $[\mathbf{yx^2 = -x^2\cos x + 2x\sin x + 2\cos x + c}]$	
H	8.	$\frac{dy}{dx} + y = x.$ $[\mathbf{ye^x = e^x x - e^x + c}]$	Dec-09
T	9.	$x\frac{dy}{dx} + (1+x)y = x^3.$ $[\mathbf{xye^x = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c}]$	Jun-13
C	10.	$\frac{dy}{dx} + (\tan x)y = \cos x; y(0) = 2.$ $[\mathbf{y\sec x = x + 2}]$	Jan-15
H	11.	$\frac{dy}{dx} + 2\,y\tan x = \sin x.$ $[\mathbf{y\sec^2 x = \sec x + c}]$	Jan-15
C	12.	$\frac{dy}{dx} + y\cot x = 2\cos x.$ $\left[\mathbf{y\sin x = -\frac{\cos 2x}{2} + c}\right]$	
C	13.	$\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^3}.$ $[\mathbf{y(x^2+1)^2 = \tan^{-1}x + c}]$	Dec-13
C	14.	$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x.$ $\left[\mathbf{xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c}\right]$	Jun-13
T	15.	$ydx - xdy + (\log x)dx = 0.$ $[\mathbf{y + \log x + 1 = cx}]$	
T	16.	$y' - (1 + 3x^{-1})y = x + 2, \ y(1) = e - 1.$ $[\mathbf{y = -x + x^3e^x}]$	Dec-10
H	17.	$y' + \frac{1}{3}y = \frac{1}{3}(1-2x)x^4.$ $\left[\mathbf{ye^{\frac{x}{3}} = \frac{1}{3}e^{\frac{x}{3}}\{-6x^5 + 93x^4 - 1116x^3 + 10044x^2 - 60264x + 180792 + c\}}\right]$	Dec-10

Bernoulli's Differential Equation

A differential equation of the form $\frac{dy}{dx} + \mathbf{P(x)y = Q(x)y^n}$ OR $\frac{dx}{dy} + \mathbf{P(y)x = Q(y)x^n}$ is known as Bernoulli's Differential Equation. Where n is real number (except for $n = 0$ & 1)

Such differential equation can be converted into linear differential equation and accordingly can be solved.

Equation Reducible To Linear Differential Equation Form

- **CASE 1** : A differential equation of the form $\frac{dy}{dx} + \mathbf{P(x)y = Q(x)y^n}$ ____**(1)**

Dividing both sides of equation (1) by $\mathbf{y^n}$, We get, $\mathbf{y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)}$____(2)

Let $\mathbf{y^{1-n} = v}$

- $\mathbf{(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}}$

Equation (2) becomes $\mathbf{\frac{1}{(1-n)}\frac{dv}{dx} + P(x)v = Q(x) \Rightarrow \frac{dv}{dx} + P(x)(1-n)v = Q(x)(1-n)}$

Which is Linear Differential equation and accordingly can be solved.

- **CASE 2** : A differential of form $\mathbf{\frac{dy}{dx} + P(x)f(y) = Q(x)g(y)}$ **......(3)**

Dividing both sides of equation (3) by "y" ,

We get, $\mathbf{\frac{1}{g(y)}\frac{dy}{dx} + P(x)\frac{f(y)}{g(y)} = Q(x)}$ **......(4)**

Let $\mathbf{\frac{f(y)}{g(y)} = v}$

Differentiate with respect to x both the side,

Equation (4) becomes Linear Differential equation and accordingly can be solved.

<u>Exercise-4</u>

Bernoulli's Differential Equation				
C	1.	$\frac{dy}{dx} + y = -\frac{x}{y}.$	$\left[y^2e^{2x} = -xe^{2x} + \frac{e^{2x}}{2} + c\right]$	Dec-09
C	2.	$\frac{dy}{dx} + \frac{y}{x} = x^2y^6$	$\left[y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c\right]$	
H	3.	$\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}.$	$\left[\frac{e^{-y}}{x} = \frac{1}{2x^2} + c\right]$	May-11 Jun-15
H	4.	$x\frac{dy}{dx} + y\log y = x\,y\,e^x.$	$[x\log y = e^x x - e^x + c]$	
C	5.	$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y.$	$\left[\frac{\sin y}{1+x} = e^x + c\right]$	
T	6.	$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$	$\left[e^{x^2}\tan y = \frac{(x^2-1)e^{x^2}}{2} + c\right]$	Jan-15
C	7.	$\frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$	$\left[\frac{sex^2x}{y} = -\frac{\tan^3 x}{3} + c\right]$	
C	8.	$(x^3y^2 + xy)dx = dy.$	$\left[\frac{e^{\frac{x^2}{2}}}{y} = (2-x^2)e^{\frac{x^2}{2}} + c\right]$	

Exact Differential Equation

A differential equation of the form $\mathbf{M(x, y)dx + N(x, y)dy = 0}$ is said to be Exact Differential Equation if it can be derived from its primitive by direct differential without any further transformation such as elimination etc.

- The necessary and sufficient condition for differential equation to be exact i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The general solution of Exact Differential Equation is

$$\int_{y=\text{constant}} \mathbf{M(x)dx} + \int \mathbf{(terms\ of\ N\ not\ containing\ x)dy = c}$$

Where, c is an arbitrary constant

Exercise-5

Exact Differential Equation				
C	1.	$(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$ $\left[\frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c\right]$	Jan-15
H	2.	$(x^2 + y^2)dx + 2xydy = 0.$ $\left[\frac{x^3}{3} + y^2x = c\right]$	
H	3.	$2xydx + x^2dy = 0.$ $[x^2y = c]$	Dec-09
T	4.	$ye^xdx + (2y + e^x)dy = 0;\ y(0) = -1.$ $[ye^x + y^2 = 0]$	Mar-10 Jun-15
T	5.	$(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0.$ $[(e^y + 1)\sin x = c]$	
C	6.	Test for exactness and solve : $[(x + 1)e^x - e^y]dx - xe^ydy = 0, y(1) = 0.$ $[x(e^x - e^y) = e - 1]$	Jun-14 Dec -11
C	7.	$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$ $[y\sin x + x\sin y + xy = c]$	Jun-13

Definition: Non-Exact Differential Equation

A differential equation which is not exact differential equation is known as Non-Exact Differential Equation. i.e. if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then given equation is Non-Exact Differential Equation.

Definition: Integrating Factor

A differential equation which is not exact be made by multiplying it by a suitable function of x and y. Such a function is known as Integrating Factor.

Some Standard Rules for Finding I.F.

1. If $Mx + Ny \neq 0$ and the given equation is Homogeneous, then $\text{I. F.} = \frac{1}{Mx+Ny}$.
2. If $Mx - Ny \neq 0$ and the given equation is of the form $f(x, y)\, y\, dx + g(x, y)\, x\, dy = 0$ (OR Non-Homogeneous), then $\text{I. F.} = \frac{1}{Mx-Ny}$.
3. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ (i.e. function of only x), then $\text{I. F.} = e^{\int f(x)\, dx}$
4. If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ (i.e. function of only y), then $\text{I. F.} = e^{\int f(y)\, dy}$

   Then find, $M^* = M(\text{I. F.})$ & $N^* = N(\text{I. F.})$ and solution is

$$\int_{y=\text{constant}} \mathbf{M^* dx} + \int (\textbf{terms of } \mathbf{N^*} \textbf{ not containing } \mathbf{x})\mathbf{dy} = \mathbf{c};$$

Where, c is an arbitrary constant.

Exercise-6

Non-Exact Differential Equation				
C	1.	State the necessary & sufficient condition to be exact differential equation. And using it Solve $x^2y\, dx - (x^3 + xy^2)dy = 0$. $\left[-\frac{x^2}{2y^2} + \log y = \mathbf{c}\right]$	**Jan-13 Jan-15**
C	2.	$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ $\left[\frac{\mathbf{x}}{\mathbf{y}} - \mathbf{2\log x + 3\log y = c}\right]$	**Jan-15**
H	3.	$(xy - 2y^2)dx - (x^2 - 3xy)dy = 0$. $\left[\frac{\mathbf{x}}{\mathbf{y}} - \mathbf{2\log x + 3\log y = c}\right]$	
T	4.	$(x^3+y^3)dx - xy^2dy = 0$. $\left[\mathbf{\log x - \frac{y^3}{3x^3} = c}\right]$	
C	5.	$(x^2+y^2)dx - 2xydy = 0$. $[\mathbf{x^2 - y^2 = cx}]$	
C	6.	$(x^2y^2 + 2)ydx + (2 - x^2y^2)xdy = 0$. $\left[\mathbf{\log\frac{x}{y} - \frac{1}{x^2y^2} = c}\right]$	**May-12 Jan-15**
C	7.	$y(1 + xy)dx + x(1 + xy + x^2y^2)dy = 0$. $\left[\mathbf{\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c}\right]$	
H	8.	$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)xdy = 0$ $\left[\mathbf{-\frac{1}{xy} + \log\frac{x^2}{y} = c}\right]$	
C	9.	$(x^2+y^2 + x)dx + xydy = 0$. $[\mathbf{3x^4 + 6x^2y^2 + 4x^3 = 12c}]$	

Definition: Orthogonal Trajectory

A curve which cuts every member of a given family at right angles is a called an Orthogonal Trajectory.

1. **Methods of finding orthogonal trajectory of $\mathbf{f(x, y, c) = 0}$**

   I. Differentiate $f(x, y, c) = 0 \dots (1)$ w.r.t. x.

   II. Eliminate c by using eqn ...(1) and its derivative

   III. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. This will give you differential equation of the orthogonal trajectories.

   IV. Solve the differential equation to get the equation of the orthogonal trajectories.

2. **Methods of finding orthogonal trajectory of $\mathbf{f(r, \theta, c) = 0}$**

   I. Differentiate $f(r, \theta, c) = 0 \dots (1)$ w.r.t. θ.

   II. Eliminate c by using eqn ...(1) and its derivative

   III. Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. This will give you differential eqn of the orthogonal trajectories.

   IV. Solve the differential equation to get the equation of the orthogonal trajectories.

Exercise-7

Orthogonal Trajectory				
C	1.	$y = x^2 + c.$	$\left[\mathbf{y = -\frac{1}{2}\log x + \frac{c}{2}}\right]$	**Dec-10**
H	2.	$y^2 + (x - a)^2 = a^2$	$[\mathbf{(y - b)^2 + x^2 = b^2}]$	
T	3.	$x^2 = 4b(y + b).$	$[\mathbf{x^2 = 4b(y + b)}]$	

Higher Order Linear Differential Equation

A linear differential equation with more than one order is known as Higher Order Linear Differential Equation.

A general linear differential equation of the nth order is of the form

$$P_0\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_n y=f(x)\ \ldots\ldots\ldots(A)$$

Where, $P_0, P_1, P_2, \ldots$ are functions of x.

Higher Order Linear Differential Equation with constant co-efficient

The n^{th} order linear differential equation with constant co-efficient is

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_n y=f(x)\ \ldots\ldots\ldots(B)$$

Where, $a_0, a_1, a_2, \ldots$ are constants.

Notations

Eq. (B) can be written in operator form as below,

$$a_0D^n y+a_1D^{n-1}y+a_2D^{n-2}y+\cdots+a_n y=f(x)\ \ldots\ldots\ldots(C)$$

OR

$$[g(D)]y=f(x)\ \ldots\ldots\ldots(D)$$

Note

A n^{th} order linear differential equation has n linear independent solution.

Auxiliary Equation

The auxiliary equation for nth order linear differential equation

$$a_0D^n y+a_1D^{n-1}y+a_2D^{n-2}y+\cdots+a_n y=f(x)$$

is derived by replacing D by m and equating with 0.

$$\text{i.e. } a_0m^n y+a_1m^{n-1}y+a_2m^{n-2}y+\cdots+a_n y=0$$

Complimentary Function (C.F.) i.e.(y_c)

A general solution of $[g(D)]y=0$ is called complimentary function of $[g(D)]y=f(x)$.

Particular Integral (P.I.) i.e.(y_P)

A particular integral of $[g(D)]y=f(x)$ is $y=\frac{1}{g(D)}f(x)$.

General Solution [y (x)] Of Higher Order Linear Differential Equation

$$\text{G.S.} = \text{P.I.} + \text{C.F. i.e. } y(x)=y_p+y_c$$

Note

In case of higher order homogeneous differential equation, complimentary function is same as general solution.

Method For Finding C.F. Of Higher Order Differential Equation

Consider,

$$a_0D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots + a_ny = f(x)$$

The Auxiliary equation is

$$a_0m^ny + a_1m^{n-1}y + a_2m^{n-2}y + \cdots + a_ny = 0$$

Let, $m_1, m_2, m_3, \ldots$ be the roots of auxiliary equation.

Case	Nature of the “n” roots	L.I. solutions	General Solutions
1.	$m_1 \neq m_2 \neq m_3 \neq m_4 \neq \cdots$	$e^{m_1x}, e^{m_2x}, e^{m_3x}, \ldots$	$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots$
2.	$m_1 = m_2 = m$ $m_3 \neq m_4 \neq \cdots$	$e^{mx}, xe^{mx}, e^{m_3x}, e^{m_4x}, \ldots$	$y = (c_1 + c_2x)e^{mx} + c_3e^{m_3x} + c_4e^{m_4x} + \cdots$
3.	$m_1 = m_2 = m_3 = m$ $m_4 \neq m_5, \ldots$	$e^{mx}, xe^{mx}, x^2e^{mx}$, $e^{m_4x}, e^{m_5x}, \ldots$	$y = (c_1 + c_2x + c_3x^2)e^{mx} + c_4e^{m_4x} + c_5e^{m_5x} + \cdots$
4.	$m_1 = p + iq$ $m_2 = p - iq$ $m_3 \neq m_4, \ldots$	$e^{px}\cos qx, e^{px}\sin qx$, $e^{m_3x}, e^{m_4x}, \ldots$	$y = e^{px}(c_1\cos qx + c_2\sin qx) + c_2e^{m_3x} + c_3e^{m_4x} + \cdots$
5.	$m_1 = m_2 = p + iq$ $m_3 = m_4 = p - iq$ $m_5 \neq m_6, \ldots$	$e^{px}\cos qx, xe^{px}\cos qx$, $e^{px}\sin qx, xe^{px}\sin qx$, $e^{m_5x}, e^{m_6x}, \ldots$	$y = e^{px}[(c_1 + c_2x)\cos qx$ $+(c_3 + c_4x)\sin qx]$ $+c_5e^{m_5x} + c_6e^{m_6x} + \cdots$

Exercise-1

Solution Of Homogeneous Differential Equation			
C	1.	$y'' + y' - 2y = 0$ $\quad [c_1e^{-2x} + c_2e^{x}]$	Dec-09
H	2.	$y'' + 7y' - 18y = 0$ $\quad [c_1e^{-9x} + c_2e^{2x}]$	Jan-15
C	3.	$y'' + y' - 2y = 0,\ y(0) = 4,\ y'(0) = -5.$ $\quad [3e^{-2x} + e^{x}]$	Dec-09 Jan-15
T	4.	$y'' - 9y = 0\ ;\ \ y(0) = 2, y'(0) = -1.$ $\quad \left[\frac{5}{6}e^{3x} + \frac{7}{6}e^{-3x}\right]$	Jan-15
T	5.	$y'' - 5y' + 6y = 0; y(1) = e^2, y'(1) = 3e^2$ $\quad [e^{3x-1}]$	May-12
H	6.	$y'' - 2\sqrt{2}y' + 2y = 0.$ $\quad \left[(c_1 + c_2x)e^{\sqrt{2}x}\right]$	Jan-15
T	7.	$y'' + 4y' + 4y = 0; y(0) = 1, y'(0) = 1\ .$ $\quad [(1 + 3x)e^{-2x}]$	Dec-11
T	8.	$y'' - 4y' + 4y = 0; y(0) = 3, y'(0) = 1$ $\quad [(3 - 5x)e^{2x}]$	Jan-15
C	9.	$\frac{d^4y}{dx^4} - 18\frac{d^2y}{dx^2} + 81y = 0\ .$ $\quad [(c_1 + c_2x)e^{3x} + (c_3 + c_4x)e^{-3x}]$	May-12
H	10.	$(D^2 + 1)y = 0\ .$ $\quad [(c_1\cos x + c_2\sin x)]$	Dec-11
T	11.	$16y'' - 8y' + 5y = 0.$ $\quad \left[e^{\frac{x}{4}}\left(c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}\right)\right]$	Dec-11
C	12.	$(D^4 - 1)y = 0.$ $\quad [c_1e^{-x} + c_2e^{x} + (c_3\cos x + c_4\sin x)]$	Jun-14
H	13.	$y''' - y = 0.$ $\quad \left[c_1e^{x} + e^{-\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)\right]$	Jun-13 Jun-15
C	14.	$y''' - y'' + 100y' - 100y = 0; y(0) = 4, y'(0) = 11, y''(0) = -299$ $\quad [e^{x} + \sin 10x + 3\cos 10x]$	Dec-11

Method Of Finding The Particular Integral

There are many methods of finding the particular integral $\frac{1}{f(D)}X$, We shall discuss following four main methods,

- **A.** General Methods
- **B.** Short-cut Methods involving operators
- **C.** Method of Undetermined Co-efficient
- **D.** Method of Variation of parameters

A. General Methods

Consider the differential equation

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y = X$$

It may be written as $f(D)y = X$

$\therefore$ Particular Integral $= \dfrac{1}{f(D)} X$

Particular Integral may be obtained by following two ways:

1. Method Of Factors

The operator $\frac{1}{f(D)}$ X may be factorized into n linear factors; then the particular integral will be

$$P.I. = \frac{1}{f(D)} X = \frac{1}{(D-m_1)(D-m_2)\ldots\ldots\ldots\ldots(D-m_n)} X$$

Now, we know that,

$$\frac{1}{D-m_n} X = e^{m_n x} \int X e^{-m_n x} dx$$

On opening with the first symbolic factor, beginning at the right, the particular integral will have form

$$P.I. = \frac{1}{(D-m_1)(D-m_2)\ldots\ldots\ldots\ldots(D-m_{n-1})} e^{m_n x} \int X e^{-m_n x} dx$$

Then, on operating with the second and remaining factors in succession, taking them from right to left, one can find the desired particular integral.

2. Method of Partial Fractions

The operator $\frac{1}{f(D)}$ X may be factorized into n linear factors; then the particular integral will be

$$P.I. = \frac{1}{f(D)} X = \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}\right) X$$

$$= A_1 \frac{1}{D-m_1} X + A_2 \frac{1}{D-m_2} X + \cdots + A_n \frac{1}{D-m_n} X$$

Using $\dfrac{1}{D-m_n} X = e^{m_n x} \int X e^{-m_n x} dx$, we get

$$P.I. = A_1 e^{m_1 x} \int X e^{-m_1 x} dx + A_2 e^{m_2 x} \int X e^{-m_2 x} dx + \cdots + A_n e^{m_n x} \int X e^{-m_n x} dx$$

Out of these two methods, this method is generally preferred.

B. Shortcut Method

1. $F(x) = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{if } f(a) \neq 0$$

If $f(a) = 0$,

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{x}{f'(a)} e^{ax}, \text{if } f'(a) \neq 0$$

In general, If $f^{n-1}(a) = 0$,

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{x^n}{f^n(a)} e^{ax}, \text{if } f^n(a) \neq 0$$

2. $F(x) = \sin(ax + b)$

$$P.I. = \frac{1}{f(D^2)} \sin(ax + b) = \frac{1}{f(-a^2)} \sin(ax + b), \text{if } f(-a^2) \neq 0$$

If $f(-a^2) = 0$,

$$P.I. = \frac{1}{f(D^2)} \sin(ax + b) = \frac{x}{f'(-a^2)} \sin(ax + b), \text{if } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$,

$$P.I. = \frac{1}{f(D^2)} \sin(ax + b) = \frac{x^2}{f''(-a^2)} \sin(ax + b), \text{if } f''(-a^2) \neq 0 \quad \text{and so on} \ldots$$

3. $F(x) = \cos(ax + b)$

$$P.I. = \frac{1}{f(D^2)} \cos(ax + b) = \frac{1}{f(-a^2)} \cos(ax + b), \text{if } f(-a^2) \neq 0$$

If $f(-a^2) = 0$,

$$P.I. = \frac{1}{f(D^2)} \cos(ax + b) = \frac{x}{f'(-a^2)} \cos(ax + b), \text{if } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$,

$$P.I. = \frac{1}{f(D^2)} \cos(ax + b) = \frac{x^2}{f''(-a^2)} \cos(ax + b), \text{if } f''(-a^2) \neq 0 \quad \text{and so on} \ldots$$

4. $F(x) = x^m; m > 0$

In this case convert f (D) in the form of $1 + \phi(D)$ or $1 - \phi(D)$ form so that we get

$$P.I. = \frac{1}{f(D)} x^m = \frac{1}{1 + \phi(D)} x^m = \{1 - \phi(D) + [\phi(D)]^2 - \ldots\} x^m$$

(Using Binomial Theorem)

5. $F(x) = e^{ax} V(X)$,Where $V(X)$ is a function of x.

$$P.I. = \frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D + a)} V(x)$$

Exercise-2

Solution Of Non-Homogeneous Differential Equation			
C	1.	$\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=e^{6x}$ $\left[(c_1e^{-4x}+c_2e^{3x})+\frac{1}{30}e^{6x}\right]$	Jan-13
H	2.	$(D^2+5D+6)y=e^x$ $\left[c_1e^{-3x}+c_2e^{-2x}+\frac{e^x}{12}\right]$	Jun-14
H	3.	$y''-5y'+6y=3e^{-2x}$ $\left[(c_1e^{2x}+c_2e^{3x})+\frac{3}{20}e^{-2x}\right]$	Jan-15
H	4.	$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{4x}$ $\left[c_1e^{2x}+c_2e^{3x}+\frac{e^{4x}}{2}\right]$	Dec-13
H	5.	$y''-3y'+2y=e^x$ $[c_1e^x+c_2e^{2x}-xe^x]$	Dec-11
H	6.	$(D^2-3D+2)y=2e^x$. $[c_1e^x+c_2e^{2x}-2xe^x]$	May-12
H	7.	$(D^3-7D+6)y=e^{2x}$. $\left[c_1e^x+c_2e^{2x}+c_3e^{-3x}+\frac{x}{5}e^{2x}\right]$	Jun-15
C	8.	$(D^2-2D+1)y=10e^x$. $[(c_1+c_2x)e^x+5x^2e^x]$	Jun-15
T	9.	$y'''-3y''+3y'-y=4e^t$ $\left[(c_1+c_2t+c_3t^2)e^t+\frac{2}{3}t^3e^t\right]$	Jan-15
T	10.	$y'''-3y''+3y'-y=e^{-x}$ $\left[(c_1+c_2x+c_3x^2)e^x-\frac{1}{8}e^{-x}\right]$	
H	11.	$(D^2-49)y=\sinh 3x$ $\left[c_1e^{-7x}+c_2e^{7x}-\frac{1}{40}\sinh 3x\right]$	Jun-15
C	12.	$y''+2y'+2y=\sinh x$ $\left[e^{-x}(c_1\cos x+c_2\sin x)+\frac{1}{10}(e^x-5e^{-x})\right]$	
C	13.	$(D^2-25)y=\cos 5x$ $\left[c_1e^{-5x}+c_2e^{5x}-\frac{1}{50}\cos 5x\right]$	Jun-14
T	14.	Find the steady state oscillation of the mass-spring system governed by the equation: $y''+3y'+2y=20\cos 2t$. $[3\sin 2t-\cos 2t]$	Dec-09
H	15.	$(D^2+9)y=\cos 2x+\sin 2x$ $\left[c_1\cos 3x+c_2\sin 3x+\frac{1}{5}\cos 2x+\frac{1}{5}\sin 2x\right]$	Jun-15
T	16.	$(D^2-4D+3)y=\sin 3x\cos 2x$. $\left[c_1e^x+c_2e^{3x}+\frac{\sin x+2\cos x}{20}-\frac{10\cos 5x-11\sin 5x}{884}\right]$	Dec-13
T	17.	$(D^4+2a^2D^2+a^4)y=\cos ax$. $\left[(c_1+c_2x)\cos ax+(c_3+c_4x)\sin ax-\frac{x^2}{8a^2}\cos ax\right]$	May-11

T	18.	$(D^2+D-6)y=e^{2x}\sin 3x$ $\left[c_1e^{-3x}+c_2e^{2x}-\frac{e^{2x}}{306}(15\cos 3x+9\sin 3x)\right]$	Jan-13
C	19.	$(D^3-D^2+3D+5)y=e^x\cos 3x.$ $\left[c_1e^{-x}+e^x(c_2\cos 2x+c_3\sin 2x)-\frac{e^x}{65}(3\sin 3x+2\cos 3x)\right]$	May-12
T	20.	$y''+2y'+3y=2x^2$ $\left[e^{-x}\left(c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x\right)+\left(\frac{2}{3}x^2-\frac{8}{9}x-\frac{28}{27}\right)\right]$	Jan-15
H	21.	$y''+2y'+10y=25x^2+3$ $\left[e^{-x}(c_1\cos 3x+c_2\sin 3x)+\left(\frac{5}{2}x^2-x\right)\right]$	Dec-10
C	22.	$(D^3-D^2-6D)y=x^2+1$ $\left[c_1+c_2e^{3x}-\frac{x^3}{18}+\frac{x^2}{36}-\frac{25x}{108}\right]$	Jan-13
T	23.	$\frac{d^4y}{dt^4}-2\frac{d^2y}{dt^2}+y=\cos t+e^{2t}+e^t$ $\left[(c_1+c_2t)e^{-t}+(c_3+c_4t)e^t+\left(\frac{\cos t}{4}+\frac{e^{2t}}{9}+\frac{t^2e^t}{8}\right)\right]$	May-12
H	24.	$y''-4y=e^{-2x}-2x, y(0)=0, y'(0)=0$ $\left[-\frac{1}{8}\sinh 2x+\frac{1}{2}x-\frac{1}{4}xe^{-2x}\right]$	
C	25.	$(D^2+16)y=x^4+e^{3x}+\cos 3x$ $\left[(c_1\cos 4x+c_2\sin 4x)+\frac{1}{16}\left[x^4-\frac{3}{4}x^2+\frac{3}{32}\right]+\frac{1}{25}e^{3x}+\frac{1}{7}\cos 3x\right]$	Jan-15
H	26.	$y''+4y=8e^{-2x}+4x^2+2; y(0)=2, y'(0)=2.$ $[\cos 2x+2\sin 2x+e^{-2x}+x^2]$	Mar-10

C. Method of Undetermined Co-efficient

This method determines P.I. of $f(D)y=X$. In this method we will assume a trial solution containing unknown constants, which will be obtained by substitution in $f(D)y=X$. The trial solution depends upon X (the RHS of the given equation$f(D)y=X$).

Let the given equation be $f(D)y=X$ ...... (A)

$\therefore$ The general solution of (A) is $Y=Y_C+Y_P$

Here we guess the form of Y_P depending on X as per the following table.

	RHS of $f(D)y = X$	Form of Trial Solution
1.	$X = e^{ax}$	$Y_P = Ae^{ax}$
2.	$X = \sin ax$ $X = \cos ax$	$Y_P = A\sin ax + B\cos ax$
3.	$X = a + bx + cx^2 + dx^3$ $X = ax^2 + bx$ $X = ax + b$ $X = c$	$Y_P = A + Bx + Cx^2 + Dx^3$ $Y_P = A + Bx + Cx^2$ $Y_P = A + Bx$ $Y_P = A$
4.	$X = e^{ax}\sin bx$ $X = e^{ax}\cos bx$	$Y_P = e^{ax}(A\sin bx + B\cos bx)$
5.	$X = xe^{ax}$ $X = x^2e^{ax}$	$Y_P = e^{ax}(A + Bx)$ $Y_P = e^{ax}(A + Bx + Cx^2)$
6.	$X = x\sin ax$ $X = x^2\cos ax$	$Y_P = \sin ax\,(A + Bx) + \cos ax\,(C + Dx)$ $Y_P = \sin ax\,(A + Bx + Cx^2) + \cos ax\,(D + Ex + Fx^2)$
7.	$X = e^{2x}$ $X = e^{2x} - 3e^{-x}$	$Y_P = Ae^{2x}$ $Y_P = Ae^{2x} + Be^{-x}$
8.	$X = \cos 3x$ $X = 2\sin(4x - 5)$	$Y_P = A\sin 3x + B\cos 3x$ $Y_P = A\sin(4x - 5) + B\cos(4x - 5)$

Exercise-3

Solution By Method Of Undetermined Co-Efficient			
C	1.	$y'' + 4y = 2\sin 3x$ $\left[\mathbf{c_1\cos 2x + c_2\sin 2x - \frac{2}{5}\sin 3x}\right]$	**Mar-10** **Jun-14**
H	2.	Find particular Integral of Differential Equation $y'' + 9y = \cos 5x$ $\left[\mathbf{-\frac{1}{16}\cos 5x}\right]$	**Jun-15**
T	3.	$y'' + 4y = 8x^2$. $[\mathbf{c_1\cos 2x + c_2\sin 2x + 2x^2 - 1}]$	**Dec-09**
T	4.	$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 6x + 3x^2 - 6x^3$. $[\mathbf{c_1e^{-3x} + c_2e^{2x} + x^3}]$	**Jan-13**
C	5.	$y'' + 1.5y' - y = 12x^2 + 6x^3 - x^4$, $y(0) = 4, y'(0) = -8$ $[\mathbf{4e^{-2x} + x^4}]$	
T	6.	$y''' + 3y'' + 3y' + y = 30e^{-x}; y(0) = 3, y'(0) = -3, y''(0) = -47$. $[\mathbf{(3 - 25x^2 + 5x^3)e^{-x}}]$	**May-11**
H	7.	$y'' + 1.2y' + 0.36y = 4e^{-0.6x}, y(0) = 0, y'(0) = 1$ $[\mathbf{(x + 2x^2)e^{-0.6x}}]$	
C	8.	$y'' - 2y' + y = e^x + x$ $\left[\mathbf{(c_1 + xc_2)e^x + \left[\frac{x^2e^x}{2} + (x + 2)\right]}\right]$	

Definition: Wronskian

Wronskian of the n function $y_1, y_2, \dots, y_n$ is defined and denoted by the determinant

$$W(y_1, y_2, \dots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem: Let $y_1, y_2, \dots y_n$ be differentiable functions defined on some interval I. Then

1. $y_1, y_2, \dots y_n$ Are linearly independent on I if and only if $W(y_1, y_2, \dots y_n) \neq 0$ for all $x \in I$.
2. $y_1, y_2, \dots y_n$ Are linearly dependent on I then $W(y_1, y_2, \dots y_n) = 0$ for all $x \in I$.

Exercise-4

Check Whether L.D. Or L.I.			
T	1.	$x, \log x, x(\log x)^2$ **[L. I]**	May-11
C	2.	e^x, e^{-x} **[L. I]**	Dec-09

D. Method Of Variation Of Parameters

The process of replacing the parameters of an analytic expression by functions is called variation of parameters.

Consider, $y'' + p(x)y' + q(x)y = X$. Where, q and X are the functions of x.

The general solution of second order differential equation by the method of variation of parameters is

$$y(x) = y_c + y_p$$

Where $y_c = c_1y_1 + c_2y_2$ and $y_p(x) = -y_1 \int y_2 \frac{X}{W} dx + y_2 \int y_1 \frac{X}{W} dx$

Where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$, and $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y_2y'_1 \neq 0$.

The general solution of third order differential equation by the method of variation of parameters is

$$y(x) = y_c + y_p$$

Where, $y_c = c_1y_1 + c_2y_2 + c_3y_3$ and $y_p = P(X)y_1 + Q(X)y_2 + R(X)y_3$.

Where,

$$P(X) = \int (y_2y_3' - y_3y_2')\ \frac{X}{w}\ dx \quad Q(X) = \int (y_3y_1' - y_1y_3')\ \frac{X}{w}\ dx \quad R(X) = \int (y_1y_2' - y_2y_1')\ \frac{X}{w}\ dx$$

$$\text{Where, } w = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0.$$

Exercise-5

Solution By Variation OF Parameter			
C	1.	$(D^2 - 4D + 4)y = \frac{e^{2x}}{x^5}$ $\left[(c_1 + c_2x)e^{2x} + \frac{1}{12}\frac{e^{2x}}{x^3}\right]$	**May-12**
T	2.	$(D^2 + 4D + 4)y = \frac{e^{-2x}}{x^2}$ $\left[\left(c_1 + c_2x\ - (1 + \log(x))\right)e^{-2x}\right]$	**Mar-10**
H	3.	$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2}$ $[(c_1 + c_2x)e^x - e^x \log x - e^x]$	**Jan-13**
T	4.	$(D^2 - 2D + 1)y = 3x^{\frac{3}{2}}e^x$ $\left[\left(c_1 + c_2x + \frac{12}{35}x^{\frac{7}{2}}\right)e^x\right]$	**Dec-10** **Jun-13**
T	5.	$y'' + 2y' + y = e^{-x}\cos x$ $[(c_1 + c_2x - \cos x)e^{-x}]$	
T	6.	$(D^2 - 4D + 4)y = \frac{e^{2x}}{1 + x^2}$ $\left[(c_1 + c_2x)e^{2x} - e^{2x}\frac{1}{2}\log(1 + x^2)\ + xe^{2x}(\tan^{-1}x)\right]$	**May-12**
C	7.	$(D^2 - 3D + 2)y = \frac{e^x}{1 + e^x}$ $\left[c_1e^x + c_2e^{2x} + e^x[(1 + e^x)\log(1 + e^{-x}) - 1]\right]$	**Jan-13**
H	8.	$y'' + y = \sec x\,.$ $[c_1\cos x + c_2\sin x + x\sin x + \cos x\log(\cos x)]$	**Dec-09**
H	9.	$y'' + 9y = \sec 3x.$ $\left[y = c_1\cos 3x + c_2\sin 3x + \frac{1}{3}x\ \sin 3x + \frac{1}{9}\cos 3x\log\cos 3x\right]$	**Mar-10** **Jan-15** **Jun-15**
C	10.	$y'' + 4y = \tan 2x.$ $\left[c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)\right]$	**Jun-14**
H	11.	$y'' + y = \cot x\,.$ $[c_1\cos x + c_2\sin x + \sin x\log(\operatorname{cosec}x - \cot x)]$	**Jun-15**

H	12.	$(D^2 + a^2)y = \operatorname{cosec} ax$ $\left[\mathbf{c_1 cosax + c_2 sinax} + \left(\frac{-x\cos ax}{a} + \frac{1}{a^2}\sin ax \log\lvert\sin x\rvert\right)\right]$	**May-11**
C	13.	$\frac{d^3y}{dx^3} + \frac{dy}{dx} = \operatorname{cosec}x$ $\mathbf{y = [\log(\operatorname{cosec} x - \cot x) + \cos x\,(-\log \sin x) - x \sin x]}$	**Jan-13** **May-12**

Cauchy – Euler Equation

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-2}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_n y = X$$

Where $a_1, a_2, \ldots, a_n$ are constants and X is a function of x, is called Cauchy's homogeneous linear equation.

Steps To Convert Cauchy-Euler Eq. To Linear Differential Eq.

To reduce the above Cauchy – Euler Equation into a linear equation with constant coefficients, we use the transformation $x = e^z$ so that $z = logx$.

$$Now, z = log\, x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$Now, \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = Dy, where\ D = \frac{d}{dz}$$

$$Similarly, x^2\frac{d^2y}{dx^2} = D(D-1)y\ \&\ x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

Using this transformation, the given equation reduces to

$$[D(D-1)(D-2)\ldots(D-n+1) + a_1 D(D-1)\ldots(D-n+2) + \cdots + a_{n-1}D + a_n]y = f(e^z)$$

This is a linear equation with constant coefficients, which can be solved by the methods discussed earlier.

Exercise-6

Cauchy – Euler Equation			
H	1.	$(x^2D^2 - 3xD + 4)y = 0; y(1) = 0, y'(1) = 3$ $[\boldsymbol{3x^2 \log x}]$	**Dec-11**
T	2.	$x^2y'' - 4xy' + 6y = 21x^{-4}$. $\left[\boldsymbol{c_1x^2 + c_2x^3 + \frac{1}{2}x^{-4}}\right]$	**May-11**

C	3.	$(x^2D^2 - 3xD + 4)y = x^2; y(1) = 1, y'(1) = 0$ $\left[(1 - 2\log x)x^2 + \frac{1}{2}x^2(\log x)^2\right]$	May-12
C	4.	$x^3y''' + 2x^2y'' + 2y = 10\left(x + \frac{1}{x}\right).$ $[c_1 x^{-1} + x\{c_2 \cos(\log x) + c_3 \sin(\log x)\} + 5x + 2x^{-1}\log x]$	May-11 Jan-13 Jun-14
H	5.	$(x^2D^2 - 3xD + 3)y = 3\ln x - 4$ $[c_1x + c_2x^3 + \ln x]$	Mar-10
T	6.	$x^2D^2y - 3xDy + 5y = x^2 \sin(\log x)$ $[x^2\{c_1 \cos(\log x) + c_2 \sin(\log x)\} - x^2 \log x \cos(\log x)]$	Jun-15

Solution Of Differential Equation By One Of Its Solution

Step-1 Convert given D.E. into $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$ and find $P(x)$ & $Q(x)$.

Step-2 Find U.

$$U = \frac{1}{y_1^2} e^{-\int P\,dx}$$

Step-3 Find V.

$$V = \int U\ dx$$

Step-4 Second solution $y_2 = V \cdot y_1$

Finally, General solution is $y = c_1\, y_1 + c_2\, y_2$

Exercise-7

To Find Another Solution Of Differential Equation			
C	1.	$x^2y'' - xy' + y = 0; y_1 = x.$ $[y_2 = x\log x]$	Dec-10
T	2.	$x^2y'' - 4xy' + 6y = 0$ is $y_1 = x^2; x > 0.$ $[y_2 = x^3]$	Jan-13
H	3.	$xy'' + 2y' + xy = 0, y_1 = \frac{\sin x}{x}.$ $\left[y_2 = -\frac{\cos x}{x}\right]$	May-11
H	4.	$y'' + 4y' + 4y = 0, y_1 = e^{-2x}.$ $[y_2 = xe^{-2x}]$	Jun-15

Definition: Power Series

An infinite series of the form

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

is called a power series in $(x-x_0)$.

Definition: Analytic Function

A function is said to be analytic at a point x_0 if it can be expressed in a power series near x_0.

Definition: Ordinary and Singular Point

Let $P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$ be the given differential equation with variable co-efficient.

Dividing by $P_0(x)$,

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

Let, $P(x) = \frac{P_1(x)}{P_0(x)}$ & $Q(x) = \frac{P_2(x)}{P_0(x)}$

$$\therefore \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

A point x_0 is called an ordinary point of the differential equation if the functions $P(x)$ and $Q(x)$ both are analytic at x_0.

If at least one of the functions $P(x)$ or $Q(x)$ is not analytic at x_0 then x_0 is called a singular point.

Definition: Regular Singular Point And Irregular Singular Point

A singular point is further classified into regular singular point and irregular singular point as follows.

A singular point x_0 is called regular singular point if both $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at x_0 otherwise it is called an irregular singular point.

Exercise - 1

Classify The Singularities Of Following Differential Equation			
C	1.	Define Ordinary Point of the differential equation $y'' + P(x)y' + Q(x)y = 0$.	**Jun-14 Jun-15**
C	2.	$y'' + (x^2+1)y' + (x^3+2x^2+3x)y = 0$.	
	Ans.	No Singular Points.	

H	3.	$y'' + e^x y' + \sin(x^2)y = 0.$	
	Ans.	No Singular Points.	
T	4.	$x^3y'' + 5xy' + 3y = 0$	**Jan-15**
	Ans.	$x = 0$ is Irregular Singular Point.	
T	5.	$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0.$	**Dec-11**
	Ans.	$x = 1 \& -1$ are Regular Singular Point.	
C	6.	$(x^2 + 1)y'' + xy' - xy = 0.$	
	Ans.	$x = i, -i$ are Regular Singular Point.	
H	7.	$2x(x - 2)^2y'' + 3xy' + (x - 2)y = 0.$	**May-12**
	Ans.	$x = 0$ is Regular Singular Point & $x = 2$ is Irregular Singular Point.	
H	8.	$2x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + (x + 3)y = 0.$	**Dec-12**
	Ans.	$x = 0$ is Regular Singular Point.	
C	9.	$x(x + 1)^2y'' + (2x - 1)y' + x^2y = 0.$	**Jun-15**
	Ans.	$x = 0$ is Regular Singular Point & $x = -1$ is Irregular Singular Point.	
T	10.	Discuss singularities of $x^3(x - 1)y'' - 3(x - 1)y' + xy = 0.$	**Jun-13**
	Ans.	$x = 1$ is Regular Singular Point & $x = 0$ is Irregular Singular Point.	

Power Series Solution at an Ordinary Point

A power series solution of a differential equation $P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$

at an ordinary point x_0 can be obtained using the following steps.

- ✓ **STEP-1:** Assume that $y = \sum_{k=0}^{\infty} a_k(x - x_0)^k = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$ is the solution of the given differential equation.

  Differentiating with respect to y we get,

  $$y' = \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots\infty$$

  $$y'' = \frac{dy}{dx} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots\infty$$

- ✓ **STEP-2:** Substitute the expressions of $y, \frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equation.
- ✓ **STEP-3:** Equate to zero the co-efficient of various powers of x and find a_2, a_3, a_4 ... etc. in terms of a_0 and a_1.
- ✓ **STEP-4:** Substitute the expressions of $a_2, a_3, a_4, \ldots$ in

  $y = a_0 + a_1x + a_2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$ which will be the required solution.

Exercise - 2

Solution By Power Series Method			
C	1.	$y' + 2xy = 0.$	**Dec-11**
	Ans.	$y = a_0 - a_0\, x^2 + \frac{1}{2} a_0 x^4 - \frac{1}{6} a_0\, x^6 + \cdots$	
H	2.	$y' = 2xy.$	**May-11** **Jun-15**
	Ans.	$y = a_0 + a_0\, x^2 + \frac{1}{2} a_0 x^4 + \frac{1}{6} a_0\, x^6 + \cdots$	
T	3.	$y'' + y = 0.$	**Dec 09,11,12,13** **Jan-13** **Jun-14,15**
	Ans.	$y = a_0 + a_1\, x - \frac{1}{2}\, a_0\, x^2 - \frac{1}{6}\, a_1\, x^3 + \frac{1}{24}\, a_0\, x^4 + \frac{1}{120}\, a_1\, x^5 + \cdots$	
C	4.	$y'' + xy = 0$ in powers of x.	**Jun-14** **May-13**
	Ans.	$y = a_0 + a_1\, x - \frac{1}{6} a_0\, x^3 - \frac{1}{12} a_1\, x^4 + \frac{1}{180} a_0\, x^6 + \cdots$	
H	5.	$y'' + x^2 y = 0.$	**Dec-13** **Jan-15**
	Ans.	$y = a_0 + a_1\, x - \frac{1}{12} a_0\, x^4 - \frac{1}{20} a_1\, x^5 + \frac{1}{672} a_0\, x^8 + \frac{1}{1440} a_1 x^9 + \cdots$	
H	6.	$y'' = y'.$	**Mar-10**
	Ans.	$y = a_0 + a_1\, x + \frac{1}{2}\, a_1\, x^2 + \frac{1}{6}\, a_1\, x^3 + \frac{1}{24}\, a_1\, x^4 + \frac{1}{120}\, a_1\, x^5 + \cdots$	
H	7.	$y'' = 2y'$ in powers of x.	**Jan-13**
	Ans.	$y = a_0 + a_1\, x + a_1\, x^2 + \frac{2}{3} a_1\, x^3 + \frac{1}{3}\, a_1\, x^4 + \frac{2}{15} a_1\, x^5 + \cdots$	
C	8.	$y'' - 2xy' + 2py = 0.$	
	Ans.	$y = a_0 + a_1\, x - p\, a_0\, x^2 + \frac{(1-p)}{3} a_1 x^3 - \frac{p\,(2-p)}{6} a_0\, x^4 + \frac{(1-p)\,(3-p)}{30} a_1 x^5 + \cdots$	
C	9.	$(1 - x^2)y'' - 2xy' + 2y = 0.$	**Jun-13** **Dec-13** **Jan-15**
	Ans.	$y = a_0 + a_1\, x - a_0\, x^2 - \frac{1}{3} a_0\, x^4 - \frac{1}{5} a_0 x^6 + \cdots$	

T	10.	$\frac{d^2y}{dx^2}(1-x^2)-x\frac{dy}{dx}+py=0.$	**May-11**
	Ans.	$y=a_0+a_1x-\frac{p}{2}a_0x^2+\frac{(1-p)}{6}a_1x^3-\frac{p(4-p)}{24}a_0x^4+\frac{(9-p)(1-p)}{120}a_1x^5+\cdots$	
H	11.	$(1+x^2)y''+xy'-9y=0.$	**May-12** **Jun-15**
	Ans.	$y=a_0+a_1x+\frac{9}{2}a_0x^2+\frac{4}{3}a_1x^3+\frac{15}{8}a_0x^4-\frac{7}{16}a_0x^6+\cdots$	
C	12.	$(x^2+1)y''+xy'-xy=0$ near $x=0$.	**May-11,13** **Dec-12** **Jan-13** **Jun-13**
	Ans.	$y=a_0+a_1x+a_0\frac{x^3}{6}-a_1\frac{x^3}{6}+\left(\frac{a_1}{12}\right)x^4-\left(\frac{3}{40}\right)a_0x^5+..$	

Frobenius Method

Frobenius Method is used to find a series solution of a differential equation near regular singular point.

The following steps are useful.

- ✓ **STEP-1:** If x_0 is a regular singular point, we assume that the solution is

$$y=\sum_{k=0}^{\infty}a_k(x-x_0)^{m+k}$$

Differentiating with respect to x ,we get

$$\frac{dy}{dx}=\sum_{k=1}^{\infty}(m+k)a_k(x-x_0)^{m+k-1}\ \&\ \frac{d^2y}{dx^2}=\sum_{k=2}^{\infty}(m+k)(m+k-1)a_k(x-x_0)^{m+k-2}$$

- ✓ **STEP-2:** Substitute the expressions of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equation.
- ✓ **STEP-3:** Equate to zero the co-efficient of least power term in $(x-x_0)$, which gives a quadratic equation in m, called Indicial equation.

  The format of the series solution depends on the type of roots of the indicial equation.

Here we have the following three cases:

CASE-I Distinct roots not differing by an integer.

When $m_1-m_2\notin Z$, i.e. difference of $m_1$and m_2 is not a positive or negative integer.In this case, the series solution is obtained corresponding to both values of m. Let the solutions be $y=y_1$and $y=y_2$, then the general solution is $y=c_1y_1+c_2y_2$.

CASE-II Equal roots

In this case, we will have only one series solution. i.e. $y = y_1 = \sum_{k=0}^{\infty} a_k (x - x_0)^{m+k}$

In terms of a_0 and the variable m. The general solution is $y = c_1 (y_1)_m + c_2 \left(\frac{dy_1}{dm}\right)_m$

CASE-III Distinct roots differing by an integer

When $m_1 - m_2 \in Z$, i.e. difference of m_1 & m_2 is a positive or negative integer.Let the roots of the indicial equation be m_1 & m_2 with $m_1 < m_2$. In this case, the solutions corresponding to the values m_1 & m_2 may or may not be linearly independent.Smaller root must be taken as m_1.

Here we have the following two possibilities.

1. One of the co-efficient of the series becomes indeterminate for the smaller root m_1 and hence the solution for m_1 contains two arbitrary constants. In this case, we will not find solution corresponding to m_2.
2. Some of the co-efficient of the series becomes infinite for the smaller root m_1 ,then it is required to modify the series by replacing a_0 by $a_0(m + m_1)$. The two linearly independent solutions are obtained by substituting $m = m_1$ in the modified form of the series for y and in $\frac{dy}{dm}$ obtained from this modified form.

<u>**Exercise - 3**</u>

Solution By Frobenius Method			
C	1.	$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$	**Jan-15**
H	2.	$(x^2 - x)y'' - xy' + y = 0.$	**Mar-10**
C	3.	$xy'' + y' + xy = 0.$	**May-11**
H	4.	$xy'' + 2y' + xy = 0.$	**Mar-10**
C	5.	$x^2y'' + xy' + (x^2 - 1)y = 0.$	
T	6.	$2x(1 - x)y'' + (1 - x)y' + 3y = 0$	**Jun-13**
C	7.	$2x(x - 1)y'' + (1 + x)y' + y = 0 ; x = 0$	**Jun-15**
T	8.	$xy'' + y' - y = 0.$	**May-12**
C	9.	$x(x - 1)y'' + (3x - 1)y' + y = 0.$	**Dec-11**
T	10.	$x^2y'' + x^3y' + (x^2 - 2)y = 0.$	**Dec- 10**

Definition: Laplace Transform

Let f(t) be a given function defined for all $t \geq 0$, then the Laplace transform of f(t) is denoted by $\mathcal{L}\{f(t)\}$ or $\bar{f}(s)$ or $F(S)$, and is defined as

$$\boldsymbol{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt}$$

provided the integral exist.

Properties of Laplace Transforms

1. $\mathcal{L}\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot \mathcal{L}\{f(t)\} + \beta \cdot \mathcal{L}\{g(t)\}$

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Now, $\mathcal{L}\{\alpha \cdot f(t) + \beta \cdot g(t)\}$

$$= \int_0^\infty e^{-st} [\alpha \cdot f(t) + \beta \cdot g(t)]\, dt$$

$$= \int_0^\infty e^{-st} \cdot \alpha \cdot f(t)\, dt + \int_0^\infty e^{-st} \cdot \beta \cdot g(t)\, dt$$

$$= \alpha \cdot \int_0^\infty e^{-st} f(t)\, dt + \beta \cdot \int_0^\infty e^{-st} g(t)\, dt$$

$$= \alpha \cdot \mathcal{L}\{f(t)\} + \beta \cdot \mathcal{L}\{g(t)\}$$

Laplace Transform of some Standard Function

1. $\boldsymbol{\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} \overline{)n+1}; n > -1.}$ **Jun-13 ; Dec-13**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n\, dt$$

Let, $st = x \Longrightarrow sdt = dx$

When $t \longrightarrow 0 \Rightarrow x \to 0$ and $t \longrightarrow \infty \Rightarrow x \to \infty$

$$\Longrightarrow \mathcal{L}\{t^n\} = \int_0^\infty e^{-x} \frac{x^n}{s^n} \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$= \frac{1}{s^{n+1}} \overline{)n+1}$ (By definition of Gamma function $\overline{)n} = \int_0^\infty e^{-x} x^{n-1} \, dx$)

If n is a positive integer, then $n! = \overline{)n+1}$

$$\Rightarrow \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

2. $\mathcal{L}\{1\} = \frac{1}{s}$. **Dec-12 ; Jun-14 ; Jan-15**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{0-1}{-s} = \frac{1}{s}$$

2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$ **Jun-15**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

[When $t \longrightarrow \infty \Longrightarrow e^{-(s-a)t} \longrightarrow 0$ ($\because s > a \Rightarrow s - a > 0$)]

$$= \left[\frac{0-1}{-(s-a)}\right]$$

$$\Rightarrow \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

3. $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$, $s > -a$ Jun-13

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

[When $t \longrightarrow \infty \Rightarrow e^{-(s+a)t} \longrightarrow 0 (\because s > -a \Rightarrow s + a > 0)$]

$$= \left[\frac{0-1}{-(s+a)}\right] = \frac{1}{s+a}$$

$$\Rightarrow \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

4. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$ **and a is a constant.**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

[When $t \longrightarrow \infty \Rightarrow e^{-st} \longrightarrow 0 (\because s > 0)$]

$$= 0 - \frac{1}{s^2+a^2}(-a)$$

$$\Rightarrow \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

5. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$ **and a is a constant.**

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\ dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

[When $t \longrightarrow \infty \Rightarrow e^{-st} \longrightarrow 0\ (\because s > 0)$]

$$= 0 - \frac{1}{s^2+a^2}(-s)$$

$$\Rightarrow \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

6. $\boldsymbol{\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}\ , s^2 > a^2 (s > |a|)}$ **Dec-11;Dec-12 ; Jun-14 ; Jan-15 ; Jun-15**

Proof: $\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$$= \frac{1}{2}[\mathcal{L}\{e^{at}\} - \mathcal{L}\{e^{-at}\}]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$\Rightarrow \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

7. $\boldsymbol{\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}\ , s^2 > a^2 (s > |a|)}$ **Dec-13**

Proof: $\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}[\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$\Rightarrow \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

Exercise-1

Laplace Transform			
C	1.	Find the Laplace transform of $f(t) = \begin{cases} 0, 0 < t < \pi \\ \sin t, t > \pi \end{cases}$. $\left[\frac{-e^{-\pi s}}{s^2+1}\right]$	May-12 Jun-15
H	2.	Find the Laplace transform of $(t) = \begin{cases} 0, & 0 \le t < 2 \\ 3, \text{when } t \ge 2 \end{cases}$. $\left[\frac{3e^{-2s}}{s}\right]$	Dec-11
H	3.	Find the Laplace transform of $(t) = \begin{cases} 0, & 0 \le t < 3 \\ 4, \text{when } t \ge 3 \end{cases}$. $\left[\frac{4e^{-3s}}{s}\right]$	Jan-15
T	4.	Given that $f(t) = \begin{cases} t+1, 0 \le t \le 2 \\ 3, t \ge 2 \end{cases}$. Find $\mathcal{L}\{f(t)\}$. $\left[\frac{-e^{-2s}}{s^2} + \frac{1}{s} + \frac{1}{s^2}\right]$	Dec-13
T	5.	Find the Laplace transform of $t^3 + e^{-3t} + t^{\frac{1}{2}}$. $\left[\frac{3!}{s^4} + \frac{1}{s+3} + \frac{\sqrt{\pi}}{2s^{3/2}}\right]$	Jun-13
C	6.	Find the Laplace transform of $2t^3 + e^{-2t} + t^{\frac{4}{3}}$. $\left[\frac{12}{s^4} + \frac{1}{s+2} + \frac{4\overline{)1/3}}{9s^{7/3}}\right]$	Dec-13
H	7.	Find the Laplace transform of $t^5 + e^{-100t} + \cos 5t$. $\left[\frac{5!}{s^6} + \frac{1}{s+100} + \frac{s}{s^2+25}\right]$	Jun-14
C	8.	Find the Laplace transform of $100^t + 2t^{10} + \sin 10t$. $\left[\frac{1}{s - \log_e 100} + \frac{2 \cdot 10\,!}{s^{11}} + \frac{10}{s^2+100}\right]$	Jun-15
C	9.	Find $\mathcal{L}\{\sin 2t \cos 2t\}$. $\left[\frac{2}{s^2+16}\right]$	Dec-09 Jan-15
T	10.	Find $\mathcal{L}\{\sin 2t \sin 3t\}$. $\left[\frac{12s}{(s^2+25)(s^2+1)}\right]$	Jun-14
H	11.	Find the Laplace transform of $\cos^2 2t$. $\left[\frac{s^2+8}{s(s^2+16)}\right]$	Dec-12

H	12.	Find Laplace transform of $\cos^2(at)$, where a is a constant. $\left[\frac{s^2+2a^2}{s(s^2+4a^2)}\right]$	Dec-11
C	13.	Find the Laplace transform of following functions: (i) $\cos^3 t$. (ii) $\sin^2 t$. $\left[\text{(i)} \frac{s(s^2+7)}{(s^2+1)(s^2+9)}, \text{(ii)} \frac{2}{s(s^2+4)}\right]$	Jun-14
H	14.	Find the Laplace transform of following functions: (i) $\sin^3 2t$. (ii) $\sin^2 2t$. $\left[\text{(i)} \frac{48}{(s^2+4)(s^2+36)}, \text{(ii)} \frac{8}{s(s^2+16)}\right]$	
T	15.	Find the Laplace transform of (i) $\cos^3 2t$ (ii) $\sin^2 3t$. $\left[\text{(i)} \frac{s^3+28s}{(s^2+4)(s^2+36)}, \text{(ii)} \frac{18}{s(s^2+36)}\right]$	Jan-15

Theorem. First Shifting Theorem

Statement: If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Now, $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at} f(t)\, dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

Since, s and a are constants. $s-a$ is also a constant.

Thus, $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Corollary: If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$.

Exercise-2

First Shifting Theorem			
H	1.	By using first shifting theorem, obtain the value of $L\{(t+1)^2e^t\}$. $\left[\frac{2}{(s-1)^3}+\frac{2}{(s-1)^2}+\frac{1}{s-1}\right]$	Dec-09
T	2.	Find Laplace transform of $e^{-2t}\sin^2 2t$, where a is a constant. $\left[\frac{8}{(s+2)[(s+2)^2+16]}\right]$	Jun-13
C	3.	Find Laplace transform of $e^{-2t}(\sin^2 4t + t^2)$. $\left[\left(\frac{32}{(s+2)(s^2+4s+68)}+\frac{2}{(s+2)^3}\right)\right]$	Jan-15
H	4.	Find Laplace transform of $e^{-3t}(2\cos 5t - 3\sin 5t)$. $\left[\frac{2s-9}{(s+3)^2+25}\right]$	Jun-14

T	5.	Find Laplace transform of $e^{4t}(\sin 2t \cos t)$. $\frac{1}{2}\left[\frac{3}{(s^2-8s+25)}+\frac{1}{(s^2-8s+17)}\right]$	Jun-15

Theorem. Differentiation of Laplace Transform

Statement: If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),\ n = 1,2,3,\ldots$

Proof: By definition, $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$\Rightarrow \frac{d^n}{ds^n} F(s) = \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \left[\frac{\partial^n}{\partial s^n} e^{-st}\right] f(t)\, dt$$

$$= \int_0^\infty (-1)(t)\frac{\partial^{n-1}}{\partial s^{n-1}} e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-1)^2 (t)^2 \frac{\partial^{n-2}}{\partial s^{n-2}} e^{-st} f(t)\, dt$$

Continuing in this way, we have

$$= (-1)^n \int_0^\infty e^{-st}(t)^n f(t)\, dt$$

$$= (-1)^n \mathcal{L}\{t^n f(t)\}$$

Thus, $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),\ n = 1,2,3,\ldots$

Exercise-3

Differentiation Of Laplace Transform (Multiplication by t^n)			
T	1.	Find the value of $\mathcal{L}\{t \sin \omega t\}$. $\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right]$	Dec-09
H	2.	Find the value of $\mathcal{L}\{t \sin t\}$ & $\mathcal{L}\{t \sin 2t\}$ $\left[\frac{2s}{(s^2+1^2)^2}, \frac{4s}{(s^2+4)^2}\right]$	Jun-15
H	3.	Find the Laplace transform of $t^2 \sin \pi t$ & $t^2 \sin 2t$. $\left[\frac{2\pi(3s^2-\pi^2)}{(\pi^2+s^2)^3}, \frac{4(3s^2-4)}{(s^2+4)^3}\right]$	Dec-10 Jan-15 Jun-13

H	4.	If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$ use this result to obtained $\mathcal{L}\{e^{at}t\sin at\}$. $\left[\frac{\mathbf{2a(s-a)}}{\mathbf{[(s-a)^2+a^2]^2}}\right]$	**Dec-13**
C	5.	Find the value of $\mathcal{L}\{t\cosh t\}$. $\left[\frac{\mathbf{1+s^2}}{\mathbf{(s^2-1)^2}}\right]$	**Jun-14**
C	6.	Find the Laplace transform of $f(t) = t^2\sinh at$. $\left[\frac{\mathbf{1}}{\mathbf{(s-a)^3}} - \frac{\mathbf{1}}{\mathbf{(s+a)^3}}\right]$	**May-12**
H	7.	Find the Laplace transform of $f(t) = t^2\cos h\,\pi t$ & $f(t) = t^2\cos h\,3t$ $\left[\frac{\mathbf{1}}{\mathbf{(s-\pi)^3}} + \frac{\mathbf{1}}{\mathbf{(s+\pi)^3}}, \frac{\mathbf{1}}{\mathbf{(s-3)^3}} + \frac{\mathbf{1}}{\mathbf{(s+3)^3}}\right]$	**Jan-15** **Jun-15**
H	8.	Find the Laplace transform of $t^3\cosh 2t$. $\left[\frac{\mathbf{3}}{\mathbf{(s-2)^4}} + \frac{\mathbf{3}}{\mathbf{(s+2)^4}}\right]$	**Dec-12**

Theorem. Integration of Laplace Transform

Statement: If $\mathcal{L}\{f(t)\} = F(s)$ and if Laplace transform of $\frac{f(t)}{t}$ exists, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$.

Proof: By definition, $F(s) = \int_0^\infty e^{-st}f(t)\,dt$

Integrating both sides with respect to "s" in the range s to ∞.

$$\int_s^\infty F(s)ds = \int_0^\infty e^{-st}\left(\int_s^\infty e^{-st}ds\right)f(t)\,dt$$

$$= \int_0^\infty \left(\frac{e^{-st}}{-t}\right)_s^\infty f(t)\,dt = \int_0^\infty \left(\frac{0 - e^{-st}}{-t}\right)f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]\,dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Thus, $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$.

Exercise-4

Integration of Laplace Transform (Division by t)			
H	1.	Find $\mathcal{L}\left\{\frac{\sin 2t}{t}\right\}$. $\left[\frac{\boldsymbol{\pi}}{\mathbf{2}} - \mathbf{tan^{-1}}\left(\frac{\mathbf{s}}{\mathbf{2}}\right)\right]$	**Dec-12** **Jun-14**

C	2.	Find $\mathcal{L}\left\{\frac{1-e^t}{t}\right\}$ $\left[\log\left(\frac{s-1}{s}\right)\right]$	Dec-13
T	3.	Find the Laplace transform of $\frac{1-\cos t}{t}$. $\left[\log\left(\frac{\sqrt{s^2+1}}{s}\right)\right]$	May-12
H	4.	Find the Laplace transform of $\left[\frac{\sin wt}{t}\right]$. $\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{\omega}\right)\right]$	May-11

Theorem. Laplace Transform of integration of a function

Statement: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\int_0^t f(t)\ dt\right\} = \frac{F(s)}{s}$.

Proof: By definition $\mathcal{L}\left(\int_0^t f(u)\ du\right) = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\ du\right\}\ dt,$

Suppose that, $U = \int_0^t f(u)\ du$

$$= \int_0^\infty e^{-st}\ U\ dt = \left[U\cdot\left(\frac{e^{-st}}{-s}\right) - \int\left\{\frac{dU}{dt}\cdot\left(\frac{e^{-st}}{-s}\right)\right\}dt\right]_0^\infty$$

By Integration by parts,

$$= \left[U\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty\left\{\frac{dU}{dt}\left(\frac{e^{-st}}{-s}\right)\right\}dt$$

In first step,

When $t \to 0 \Longrightarrow U = \int_0^t f(u)\ du \to 0$ & $t \to \infty, \Longrightarrow e^{-st} \to 0$.

In second step,

By Fundamental theorem of calculus, $\frac{dU}{dt} = \frac{d}{dt}\left\{\int_0^t f(u)\ du\right\} = f(t)$.

$$\Longrightarrow \mathcal{L}\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}\int_0^\infty e^{-st} f(t)\ \ dt$$

Thus, $\mathcal{L}\left\{\int_0^t f(t)\ dt\right\} = \frac{1}{s}F(s)$.

Exercise-5

Laplace Transform of integration of a function			
C	1.	Find $\mathcal{L}\left\{\int_0^t e^{-x}\cos x\,dx\right\}$. $\left[\dfrac{\mathbf{s+1}}{\mathbf{s[(s+1)^2+1]}}\right]$	**Jun-13 May-12**
C	2.	Find $\mathcal{L}\left\{\int_0^t\int_0^t \sin au\,du\,du\right\}$. $\left[\dfrac{\mathbf{1}}{\mathbf{s^2}}\dfrac{\mathbf{a}}{\mathbf{(s^2+a^2)}}\right]$	**Jun-13**

Theorem. Laplace Transform of Periodic Function

Statement: The Laplace transform of a piecewise continuous periodic function f(t) having period "p" is $F(s) = \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$, where $s > 0$.

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt1$

$$= \int_0^p e^{-st}f(t)\,dt + \int_p^\infty e^{-st}f(t)\,dt \ldots\ (A)$$

Now, $\int_p^\infty e^{-st}f(t)\,dt$

Let, $t = u + p \Rightarrow dt = du$

When $t \longrightarrow p \Rightarrow u \longrightarrow 0$ and $t \longrightarrow \infty \Rightarrow u \longrightarrow \infty$.

$$\Rightarrow \int_p^\infty e^{-st}f(t)\,dt = \int_0^\infty e^{-s(u+p)}f(u+p)\,du$$

Since f(u) is Periodic. i. e. $f(u) = f(u+p)$

$$= \int_0^\infty e^{-s(u+p)}f(u)\,du$$

$$= e^{-sp}\int_0^\infty e^{-su}f(u)\,du = e^{-sp}F(s)$$

By eqn. …(A)

$$\mathcal{L}\{f(t)\} = \int_0^p e^{-st}f(t)\,dt + \int_p^\infty e^{-st}f(t)\,dt$$

$$\Rightarrow F(s) = \int_0^p e^{-st} f(t)\, dt + e^{-sp} F(s)$$

$$\Rightarrow (1 - e^{-sp}) F(s) = \int_0^p e^{-st} f(t)\, dt$$

$$\Rightarrow F(s) = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$

Thus, $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)dt$, where $s > 0$ is a Laplace Transform of periodic function $f(t)$ of period p.

Exercise-6

Laplace Transform of Periodic Function			
T	1.	Find the Laplace transform of the half wave rectifier $f(t) = \begin{cases} \sin \omega t, 0 < t < \frac{\pi}{\omega} \\ 0, \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ and $f(t) = f\left(t + \frac{2\pi}{\omega}\right)$. $\left[\frac{\boldsymbol{\omega}}{\left((\mathbf{s^2} + \mathbf{w^2})\left(\mathbf{1} - \mathbf{e}^{-\pi s/\omega}\right)\right)}\right]$	**Dec-12 Mar-10**
C	2.	Find the Laplace transform of $f(t) = \lvert\sin wt\rvert, t \geq 0$. $\left[\frac{\boldsymbol{\omega}\left(\mathbf{1}+\mathbf{e}^{-\pi s/\omega}\right)}{(\mathbf{s^2}+\mathbf{w^2})\left(\mathbf{1}-\mathbf{e}^{-\pi s/\omega}\right)}\right]$	**May-11**

Laplace Transform of Unit Step Function

Show that $\boldsymbol{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Now,

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

We know that, $u(t-a) = \begin{cases} 0, & 0 < x < a \\ 1, & x \geq a \end{cases}$

$$= \int_a^\infty e^{-st}\, dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty$$

$$= \left[-\frac{0 - e^{-sa}}{s}\right] = \frac{e^{-sa}}{s}$$

Thus, $\mathcal{L}\{u(t-a)\} = \frac{e^{-sa}}{s}$

Note: Instead of $u(t-a)$, we can also write $H(t-a)$,which will be Heaviside's unit step function.

Theorem. Second Shifting Theorem: Jun-13

Statement: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\, dt$

Now, $\mathcal{L}\{f(t-a)u(t-a)\}$

$$= \int_0^\infty e^{-st}f(t-a)u(t-a)\, dt$$

$$= \int_0^a e^{-st}f(t-a)u(t-a)\, dt + \int_a^\infty e^{-st}f(t-a)u(t-a)\, dt$$

We know that, $u(t-a) = \begin{cases} 0, & 0 < x < a \\ 1, & x \geq a \end{cases}$

$$= \int_a^\infty e^{-st}f(t-a)\, dt$$

Let, $t - a = u \Rightarrow dt = du$. When $t \longrightarrow a \Rightarrow u \longrightarrow 0$ and $t \longrightarrow \infty \Rightarrow u \longrightarrow \infty$.

$$= \int_0^\infty e^{-s(a+u)}f(u)\, du$$

$$= e^{-sa}\int_0^\infty e^{-su}f(u)\, du = e^{-as}F(s)$$

Hence, $\mathcal{L}\{f(t-a) \cdot u(t-a)\} = e^{-as}F(s)$

NOTE: $\mathcal{L}\{f(t) \cdot u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

Exercise-7

Second Shifting Theorem				
H	1.	Find the Laplace transform of $e^{-3t}u(t-2)$. $\left[\frac{e^{-2s}e^{-6}}{s+3}\right]$	May-12
H	2.	Find the Laplace transform of $e^{t}u(t-2)$. $\left[\frac{e^{-2s+2}}{s-1}\right]$	Dec-10
C	3.	Find the Laplace transform of $t^2u(t-2)$. $\left[e^{-2s}\left(\frac{2!}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)\right]$	Jan-15
T	4.	Find the Laplace transform of $(t-1)^2u(t-1)$. $\left[\frac{2e^{-s}}{s^3}\right]$	Jun-15
T	5.	Find the Laplace transform of $cost\ u(t-\pi)$. $\left[\frac{-se^{-\pi s}}{s^2+1}\right]$	

Use of partial fractions

Case 1 : If the denominator has non-repeated linear factors $(s-a),(s-b),(s-c)$, then

Partial fractions $=\frac{A}{(s-a)}+\frac{B}{(s-b)}+\frac{C}{(s-c)}$

Case 2 : If the denominator has repeated linear factors $(s-a)$, (n times), then

Partial fractions $=\frac{A_1}{(s-a)}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\cdots+\frac{A_n}{(s-a)^n}$

Case 3 : If the denominator has non-repeated quadratic factors (s^2+as+b),(s^2+cs+d), then

Partial fractions $=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+cs+d)}$

Case 4 : If the denominator has repeated quadratic factors (s^2+as+b), (n times), then

Partial fractions $=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+as+b)^2}+\cdots$(n times)

Exercise-8

Inverse Laplace Transform			
H	1.	Find $\mathcal{L}^{-1}\left\{\frac{6s}{s^2-16}\right\}$. $[6cosh4t]$	Dec-12
C	2.	Find $\mathcal{L}^{-1}\left\{\frac{3\left(s^2-1\right)^2}{2s^5}\right\}$. $\left[\frac{3}{2}\left[1-t^2+\frac{t^4}{4!}\right]\right]$	Dec-13
T	3.	Find $\mathcal{L}^{-1}\left\{\frac{s^3+2s^2+2}{s^3(s^2+1)}\right\}$. $[t^2+sint]$	Mar-10 Jun-13
T	4.	Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$. $[1-e^{-t}]$	Jan-15
H	5.	Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\}$. $[2e^{2t}-e^{t}]$	Jun-14
C	6.	Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}\right\}$. $\left[\frac{e^{\sqrt{3}t}-e^{-\sqrt{2}t}}{\sqrt{3}+\sqrt{2}}\right]$	Dec-10 Dec-09 Jun-15
H	7.	Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\}$. $\frac{1}{5}[e^{2t}-e^{-3t}]$	Jun-15
C	8.	Find $\mathcal{L}^{-1}\left\{\frac{5s^2+3s-16}{(s-1)(s+3)(s-2)}\right\}$. $[2e^{t}+e^{-3t}+2e^{2t}]$	Dec-12 Dec-11
T	9.	Find $\mathcal{L}^{-1}\left\{\frac{3s^2+2}{(s+1)(s+2)(s+3)}\right\}$. $\left[\frac{5}{2}e^{-t}-14e^{-2t}+\frac{29}{2}e^{-3t}\right]$	Jun-13 Jan-15
C	10.	Find $\mathcal{L}^{-1}\left\{\frac{s^3}{s^4-81}\right\}$. $\left[\frac{cos3t+cosh3t}{2}\right]$	Dec-12 Mar-10
H	11.	Find $\mathcal{L}^{-1}\left\{-\frac{s+10}{s^2-s-2}\right\}$. $[-4e^{2t}+3e^{-t}]$	Mar-10

Exercise-9

First Shifting Theorem

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

H	1.	Find $\mathcal{L}^{-1}\left\{\frac{10}{(s-2)^4}\right\}$. $\left[\frac{5e^{2t}t^3}{3}\right]$	Dec-12
C	2.	Evaluate $\mathcal{L}^{-1}\left\{\frac{3}{s^2+6s+18}\right\}$. $[e^{-3t}sin3t]$	Dec-09
T	3.	Find the inverse Laplace transform of $\frac{6+s}{s^2+6s+13}$, use shifting theorem. $\left[e^{-3t}cos2t + \frac{3}{2}e^{-3t}\sin 2t\right]$	Dec-11
T	4.	Find $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$. $\left[\frac{1}{4a^2}[e^{at}\sin at - e^{-at}\sin at]\right]$	Dec-13
C	5.	Find the inverse Laplace transform of $\frac{5s+3}{(s^2+2s+5)(s-1)}$. $\left[-e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t + e^t\right]$	May-12 Jun-14 Jan-15

Exercise-10

Second Shifting Theorem

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$

T	1.	Find the inverse Laplace transform of $\frac{se^{-2s}}{s^2+\pi^2}$. $[\cos\pi(t-2)\cdot u(t-2)]$	Dec-10 Dec-09
C	2.	Find the inverse Laplace transform of $\frac{e^{-4s}(s+2)}{s^2+4s+5}$. $[e^{-2(t-4)}\cos(t-4)\cdot u(t-4)]$	Jun-13

Exercise-11

Inverse Laplace Transform Of Derivatives

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$

T	1.	Find $\mathcal{L}^{-1}\left\{log\frac{s+a}{s+b}\right\}$. $\left[\frac{e^{-bt}-e^{-at}}{t}\right]$	Jun-13 Dec-10
H	2.	Find $\mathcal{L}^{-1}\left\{log\frac{s+1}{s-1}\right\}$. $\left[\frac{e^{t}-e^{-t}}{t}\right]$	Jun-13

H	3.	Obtain $\mathcal{L}^{-1}\left\{log\frac{1}{s}\right\}$. $\left[\frac{\mathbf{1}}{\boldsymbol{t}}\right]$	**May-11**
C	4.	Find the inverse transform of the function $ln\left(1+\frac{w^2}{s^2}\right)$. $\left[\frac{\mathbf{2}}{\boldsymbol{t}}(\mathbf{1}-\boldsymbol{cos\,wt})\right]$	**Mar-10** **Jun-14**

Definition: Convolution Product

The convolution of f and g is denoted by $f * g$ and is defined as

$$f * g = \int_0^t f(u)g(t-u)du$$

Exercise-12

Convolution Product			
T	1.	Find the value of $1 * 1$. Where " $*$ " denote convolution product. $[\boldsymbol{t}]$	**Dec-09** **Jun-15**
C	2.	Find the convolution of t and e^t. $[\boldsymbol{e^t - t - 1}]$	**Dec-10** **Jun-15**

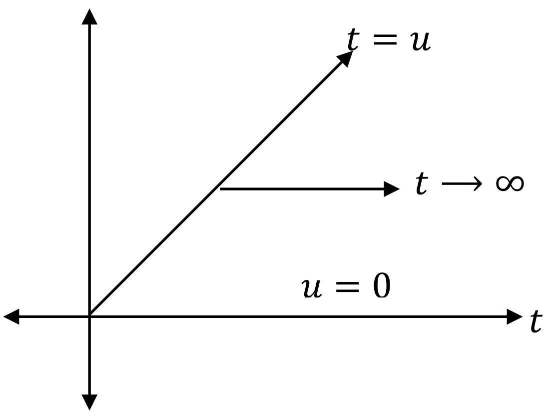
Theorem. Convolution Theorem

Statement: If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$,

Then $\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(u)g(t-u)du = f * g$

Proof: Let us suppose $F(t) = \int_0^t f(u)g(t-u)du$

Now, $\mathcal{L}\big(F(t)\big) = \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)du\right)dt = \int_0^\infty \int_0^t f(u)g(t-u)\,e^{-st}du\,dt$



Here, region of integration is entire area lying between the lines $u = 0$ and $u = t$ which is part of the first quadrant .

Changing the order of integration, we have

$$\mathcal{L}(F(t)) = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u)\, dt\, du$$

$$= \left(\int_0^\infty e^{-su} f(u) du\right)\left(\int_u^\infty e^{-s(t-u)} g(t-u)\, dt\right)$$

Let, $t - u = v \Rightarrow dt = dv$. When $t \longrightarrow u \Rightarrow v \longrightarrow 0andt \longrightarrow \infty \Rightarrow v \longrightarrow \infty$.

$$= \left(\int_0^\infty e^{-su} f(\mathrm{u}) du\right)\left(\int_0^\infty e^{-sv} g(v)\, dv\right)$$

$$= F(s) \cdot G(s)$$

Thus, $\mathcal{L}(F(t)) = F(s) \cdot G(s) \Rightarrow F(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s)\}$

$$\Rightarrow \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u) g(t-u) du$$

Hence, $f * g = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u) g(t-u) du$ is convolution product of function f & g.

<u>Exercise-13</u>

Convolution Theorem			
H	1.	Using convolution theorem, find the inverse Laplace transform of $\frac{1}{(s^2+a^2)^2}$. $\left[\frac{1}{2a^2}\left(\frac{\sin at}{a} - t\cos at\right)\right]$	May-11 Dec-10
T	2.	$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$ $\left[\frac{1}{2}\, t \sin t\right]$	Jan-15
T	3.	$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)\,(s-1)^2}\right\}$ $[\frac{1}{2} t\, e^t + \frac{e^t}{4} - \frac{e^{-t}}{4}]$	Jan-15
H	4.	State convolution theorem and using it find $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+3)}\right\}$. $\left[\frac{e^{-t} - e^{-3t}}{2}\right]$	Dec-13

H	5.	State convolution theorem and using it find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$. $\left[\frac{1}{5}(3\sin 3t - 2\sin 2t)\right]$	Jun-14
C	6.	State convolution theorem and using it find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$. $\left[\frac{1}{(a^2-b^2)}(a\sin at - b\sin bt)\right]$	Jun-15
T	7.	Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$. $\left[\frac{e^{-2t}\, t\sin t}{2}\right]$	Jun-13
H	8.	State the convolution theorem on Laplace transform and using it find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$. $\left[\frac{1-\cos 2t}{4}\right]$	Dec-12 Dec-09 Jan-15 Jun-15
H	9.	Apply convolution theorem to Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$. $\left[\frac{t\sin at}{2a}\right]$	Jun-14 Jan-15
T	10.	Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+a)^3}\right\}$. $\left[\frac{1}{2}\left[-\frac{t^2e^{-at}}{a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2}{a^3}\right]\right]$	Dec-13
C	11.	State convolution theorem and use to evaluate Laplace inverse of $\frac{a}{s^2(s^2+a^2)}$ $\left[\frac{at-\sin at}{a^2}\right]$	Mar-10

Theorem. Derivative Of Laplace Transform

Statement: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

Proof: By definition, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$\Rightarrow \mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \left[e^{-st}\int f'(t)dt - \int\left\{\left(\frac{d}{dt}e^{-st}\right)\cdot \int f'(t)dt\right\}dt\right]_0^\infty$$

$$= \left[e^{-st}f(t) - \int(-s)e^{-st}f(t)dt\right]_0^\infty$$

$$= 0 - f(0) + s\int_0^\infty e^{-st}f(t)\, dt = sF(s) - f(0)$$

Thus, $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

In general, $\mathcal{L}\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'^{(0)} - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$.

<u>Exercise-14</u>

		Application to Laplace Transform	
H	1.	Solve by Laplace transform $y'' + 6y = 1, y(0) = 2, y'(0) = 0$. $\left[\frac{11}{6}\cos\sqrt{6}t + \frac{1}{6}\right]$	Dec-12 Jun-15
T	2.	Using Laplace transform solve the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t, where\ x(0) = 0,\ x'(0) = 1$. $\left[\frac{e^{-t}(\sin t + \sin 2t)}{3}\right]$	May-12
C	3.	Solve the differential equation $\frac{d^2y}{dt^2} + 4y = f(t), y(0) = 0,\ y'(0) = 1$ by Laplace transform where (i) $f(t) = \begin{cases} 1, 0 < t < 1 \\ 0, t > 1 \end{cases}$, (ii) $f(t) = H(t-2)$. $\left[\begin{array}{c} (i)y = \frac{\sin 2t}{2} + \frac{1 - \cos 2t}{4} - \frac{[1 - \cos 2(t-1)]H(t-1)}{4} \\ (ii)y = \frac{\sin 2t}{2} + \frac{H(t-2) - \cos 2(t-2)\,H(t-2)}{4} \end{array}\right]$	May-12
C	4.	Solve IVP using Laplace transform $y'' + 4y = 0, y(0) = 1, y'(0) = 6$. $[\cos 2t + 3\sin 2t]$	Dec-11
H	5.	Using Laplace transform solve the IVP $y'' + y = \sin 2t, y(0) = 2, y'(0) = 1$. $\left[y = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + 2\cos t\right]$	Dec-10 Jan-15
C	6.	By Laplace transform solve, $y'' + a^2y = K\sin at$. $\left[y = \left(\frac{k}{2a^2} + \frac{B}{a}\right)\sin a\,t + \left(A - \frac{k}{2a}t\right)\cos at\right]$	Mar-10 Jun-14
C	7.	By using the method of Laplace transform solve the IVP : $y'' + 2y' + y = e^{-t}, y(0) = -1\ and\ y'(0) = 1$. $\left[\frac{e^{-t}\,t^2}{2!} - e^{-t}\right]$	Dec-09
H	8.	By using the method of Laplace transform solve the IVP : $y'' + 5y' + 6y = e^{-t}, y(0) = 0\ and\ y'(0) = -1$. $\left[\frac{1}{2}e^{-t} - 2e^{-2t} + \frac{3}{2}e^{-3t}\right]$	Jun-14

T	9.	By using the method of Laplace transform solve the IVP : $y'' + 4y' + 3y = e^{-t}, y(0) = 1 \ and \ y'(0) = 1$. $\left[\left(\frac{t}{2}+\frac{7}{4}\right)e^{-t}-\frac{3}{4}e^{-3t}\right]$	**Dec-13**
H	10.	By using the method of Laplace transform solve the IVP : $y'' + 3y' + 2y = e^{t}, y(0) = 1 \ and \ y'(0) = 0$. $\left[\frac{1}{6}e^{t}+\frac{3}{2}e^{-t}-\frac{2}{3}e^{-2t}\right]$	**Jun-15**
H	11.	Solve using Laplace transforms, $y''' + 2y'' - y' - 2y = 0$; where, $y(0) = 1$, $y'(0) = 2, y''(0) = 2$ $\left[\frac{1}{3}[5e^{t}+e^{-2t}]-e^{-t}\right]$	**Jan-15**
T	12.	By using the method of Laplace transform solve the IVP : $y'' - 4y' + 3y = 6t - 8, y(0) = 0 \ and \ y'(0) = 0$. $[y(t) = 2t + e^{t} - e^{3t}]$	**Jun-15**

Laplace Transform Of Some Standard Functions

1. $\mathcal{L}\{1\} = \frac{1}{s}$
2. $\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}}\overline{)\boldsymbol{n+1}}\ \boldsymbol{OR}\ \frac{n!}{s^{n+1}}$
3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
4. $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$
5. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$
6. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$
7. $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$
8. $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$
9. $\mathcal{L}\{e^{at}t^n\} = \frac{\overline{)\boldsymbol{n+1}}}{(s-a)^{n+1}}\ \boldsymbol{OR}\ \frac{n!}{(s-a)^{n+1}}$
10. $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$
11. $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$
12. $\mathcal{L}\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$
13. $\mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-a^2}$
14. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$

1. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
2. $\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\overline{)\boldsymbol{n}}}\ \boldsymbol{OR}\ \frac{t^{n-1}}{(n-1)!}$
3. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
4. $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$
5. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$
6. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
7. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$
8. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
9. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}\,t^{n-1}}{\overline{)\boldsymbol{n}}}\ \boldsymbol{OR}\ \frac{e^{at}\,t^{n-1}}{(n-1)!}$
10. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{a}e^{at}\sin bt$
11. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$
12. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b}e^{at}\sinh bt$
13. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$
14. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$

Definition: Partial Differential Equation

An equation which involves function of two or more variable and partial derivatives of that function then it is called Partial Differential Equation.

e.g. $\frac{\partial y}{\partial x}+\frac{\partial y}{\partial t}=0$

Definition: Order Of Differential Equation

The order of highest derivative which appeared in differential equation is "Order of D.E".

e.g. $\left(\frac{\partial y}{\partial x}\right)^2+\frac{\partial y}{\partial x}+5y=0$ has order 1.

Definition: Degree Of Differential Equation

When a D.E. is in a polynomial form of derivatives, the highest power of highest order derivative occurring in D.E. is called a "Degree Of D.E.".

e.g. $\left(\frac{\partial y}{\partial x}\right)^2+\frac{\partial y}{\partial x}+5y=0$ has degree 2.

Notation

Suppose $z=f(x,y)$. For that , we shall use $\frac{\partial z}{\partial x}=p$, $\frac{\partial z}{\partial y}=q$, $\frac{\partial^2 z}{\partial x^2}=r$, $\frac{\partial^2 z}{\partial x\,\partial y}=s$, $\frac{\partial^2 z}{\partial y^2}=t$.

Formation Of Partial Differential equation

1. **By Eliminating Arbitrary Constants**

Consider the function $f(x,y,z,a,b)=0$. Where, a & b are independent arbitrary constants.

- ✓ **Step 1:** $f(x,y,z,a,b)=0$. ……(1)
- ✓ **Step 2:** $f_x(x,y,z,a,b)=0$. ……(2) and $f_y(x,y,z,a,b)=0$. ……(3)
- ✓ **Step 3:** Eliminating a & b from eq. (1), eq. (2) & eq. (3).

We get partial differential equation of the form $F(x,y,z,p,q)=0$

2. **By Eliminating Arbitrary Functions**

Type 1: Consider, the function $\mathbf{f(u,v)=0}$

- ✓ **Step 1:** Let, $u=F(v)$.
- ✓ **Step 2:** Find u_x & u_y.
- ✓ **Step 3:** Eliminate the function **F** from u_x & u_y.

Note: In such case, for elimination of function, substitution method is used.

Type 2: Consider, the function $\mathbf{z=f(x,y)}$

- ✓ **Step 1:** Find z_x & z_y.
- ✓ **Step 2:** Eliminate the function **f** from z_x & z_y.

Note: In such case, for elimination of function, division of $\mathbf{z_x}$ & $\mathbf{z_y}$ is used.

Exercise-1

Formation Of Partial Differential Equation			
C	1.	Form the partial differential equation $z = (x-2)^2 + (y-3)^2$. $[\mathbf{4z = p^2 + q^2}]$	Jan-13 Jun-14
C	2.	Form the partial differential equation $z = (x+a)(y+b)$. $[\mathbf{z = pq}]$	Jun-15
T	3.	Eliminate the function f from the relation $f(xy + z^2, x + y + z) = 0$. $\left[\frac{\mathbf{p+1}}{\mathbf{q+1}} = \frac{\mathbf{y+2zp}}{\mathbf{x+2zq}}\right]$	Jun-13
C	4.	Form the partial differential equation of $f(x+y+z, x^2+y^2+z^2) = 0$. $\left[\frac{\mathbf{p+1}}{\mathbf{q+1}} = \frac{\mathbf{x+zp}}{\mathbf{y+zq}}\right]$	Jan-13 Jun-14 Jun-15
T	5.	Form the partial differential equation $f(x^2 - y^2, xyz) = 0$. $-\left[\frac{\mathbf{yz+xyp}}{\mathbf{xz+xyq}} = -\frac{\mathbf{x}}{\mathbf{y}}\right]$	Jan-15
H	6.	Form partial differential equation by eliminating the arbitrary function from $xyz = \Phi(x+y+z)$. $\left[\frac{\mathbf{p+1}}{\mathbf{q+1}} = \frac{\mathbf{yz+xyp}}{\mathbf{xz+xyq}}\right]$	Dec-13
H	7.	Form the partial differential equation by eliminating the arbitrary function from $z = f(x^2 - y^2)$. $\left[\frac{\mathbf{p}}{\mathbf{q}} = -\frac{\mathbf{x}}{\mathbf{y}}\right]$	Dec-13
C	8.	Form the partial differential equation of $z = f\left(\frac{x}{y}\right)$. $\left[\frac{\mathbf{p}}{\mathbf{q}} = -\frac{\mathbf{y}}{\mathbf{x}}\right]$	Jan-13
T	9.	Form the partial differential equation of $z = xy + f(x^2 + y^2)$. $\left[\frac{\mathbf{p-y}}{\mathbf{q-x}} = \frac{\mathbf{x}}{\mathbf{y}}\right]$	Jan-15
C	10.	Form the partial differential equation of $y = f(x - at) + F(x + at)$. $\frac{\partial^2 \mathbf{y}}{\partial \mathbf{t}^2} = \mathbf{a}^2 \frac{\partial^2 \mathbf{y}}{\partial \mathbf{x}^2}$	Jun-15

Exercise-2

Solution Of Partial Differential Equation			
C	1.	Solve $\frac{\partial^2 u}{\partial x\,\partial y} = x^3+y^3$. $\left[\mathbf{u(x,y) = \frac{x^4 y}{4} + \frac{x y^4}{4} + F(y) + g(x)}\right]$	Jun-14
C	2.	Solve $\frac{\partial^2 u}{\partial x\,\partial t} = e^{-t}\cos x$. $[\mathbf{u(x,y) = -e^{-t}\sin x + F(t) + g(x)}]$	Dec-13
H	3.	Solve $\frac{\partial^3 u}{\partial x^2\,\partial y} = \cos(2x+3y)$. $\left[\mathbf{u(x,y) = -\frac{\sin(2x+3y)}{12} + xF(y) + G(y) + h(x)}\right]$	Jan-15
T	4.	Solve $\frac{\partial^2 z}{\partial x\,\partial y} = \sin x \sin y$, given that $\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$ & $z = 0$, when y is an odd multiple of $\frac{\pi}{2}$. $[\mathbf{z(x,y) = \cos x\ \cos y + \cos y}]$	Jun-13
C	5.	Solve $\frac{\partial^2 z}{\partial x^2} = z$. $[\mathbf{u(x,y) = f(y)e^{x} + g(y)e^{-x}}]$	Jan-13 Jan-15

Lagrange's Differential Equation

A partial differential equation of the form $\mathbf{Pp + Qq = R}$ where P, Q and R are functions of x, y, z, or constant is called lagrange linear equation of the first order.

1. **Method obtaining general solution of $\mathbf{Pp + Qq = R}$**
   - ✓ **Step-1:** From the A.E. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
   - ✓ **Step-2:** Solve this A.E. by the method of grouping or by the method of multiples or both to get two independent solution $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$.
   - ✓ **Step-3:** The form $F(u,v) = 0$ or $u = f(v)$ and $v = f(u)$ is the general solution $\mathbf{Pp + Qq = R}$.

Following two methods will be used to solve Langrage's linear Equation

2. **Grouping Method**

This method is applicable only if the third variable z is absent in $\frac{dx}{P} = \frac{dy}{Q}$ or it is possible to eliminate z from $\frac{dx}{P} = \frac{dy}{Q}$.

Similarly, if the variable x is absent in last two fractions or it is possible to eliminate x from last two fractions $\frac{dx}{Q} = \frac{dy}{R}$, then we can apply grouping method.

3. Multipliers Method

In this method, we require two sets of multiplier l, m, n and l', m', n'.

By appropriate selection multiplier l, m, n (either constants or functions of x, y, z) we may write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

Such that, $lP + mQ + nR = 0$.

This implies $ldx + mdy + ndz = 0$

Solving it we get $u(x, y, z) = c_1 \quad \dots (1)$

Again we may find another set of multipliers l', m', n'

So that, $l'P + m'Q + n'R = 0$

This gives, $l'dx + m'dy + n'dz = 0$

Solving it we get $v(x, y, z) = c_2 \quad \dots (2)$

From (1) and (2), we get the general solution as $F(u, v) = 0$.

<u>Exercise-3</u>

Solution Of Lagrange's Differential Equation			
C	1.	Solve $(z - y)p + (x - z)q = y - x$. $\left[\mathbf{F(x + y + z, x^2 + y^2 + z^2) = 0}\right]$	Jun-15
C	2.	Solve $x(y - z)p + y(z - x)q = z(x - y)$. $\left[\mathbf{F(x + y + z, xyz) = 0}\right]$	Jun-13
T	3.	Solve$(x^2 - y^2 - z^2)p + 2xyq = 2xz$. $\left[\mathbf{F\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0}\right]$	Jun-14
T	4.	Solve $(y + z)p + (x + z)q = x + y$. $\left[\mathbf{F\left(\frac{x - y}{y - z}, (x + y + z)(x - y)^2\right) = 0}\right]$	Jan-15

Non Linear Partial Differential Equation Of First Order

A partial differential equation in which p & q occur in more than one order is known as Non Linear Partial Differential Equation.

Type 1: Equation Of the form $\mathbf{f(p,q) = 0}$.

- ✓ **Step 1:** Substitute $p = a$ & $q = b$.
- ✓ **Step 2:** Convert $b = g(a)$.
- ✓ **Step 3:** Complete Solution : $z = ax + by + c \Longrightarrow z = ax + g(a) + c$

Type 2: Equation Of the form $\mathbf{f(x,p) = g(y,q)}$.

- ✓ **Step 1:** $f(x,p) = g(y,q) = a$
- ✓ **Step 2:** Solving equations for p & q. Assume $p = F(x)$ & $q = G(y)$.
- ✓ **Step 3:** Complete Solution : $z = \int F(x)\, dx + \int G(y)\, dy + b$.

Type 3: Equation Of the form $\mathbf{z = px + qy + f(p,q)}$

- ✓ **Step 1:** Find value of p & q.
- ✓ **Step 2:** Complete Solution : $z = ax + by + f(a,b)$.

Type 4: Equation Of the form $\mathbf{f(z,p,q) = 0}$.

- ✓ **Step 1:** Assume $q = ap$
- ✓ **Step 2:** Solve the Equation in $dz = p\, dx + q\, dy$

Charpit's Method

Consider $f(x,y,z,p,q) = 0$

- ✓ **Step 1:** Find value of p & q by using the relation
$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} \text{ (lagrange } - \text{ Charpit eq}^{n} \text{)}$$
- ✓ **Step 2:** Find value of p & q.
- ✓ **Step 3:** Complete Solution : $z = \int p\, dx + \int q\, dy + c$.

<u>**Exercise-4**</u>

Non Linear Equation Of First Order			
C	1.	Solve $p + q^2 = 1$. $\left[\mathbf{z = ax \pm \left(\sqrt{1-a}\right)y + c}\right]$	Jan-13
H	2.	Solve $\sqrt{p} + \sqrt{q} = 1$ $\left[\mathbf{z = ax + \left(1 - \sqrt{a}\right)^2 y + c}\right]$	Jan-15
T	3.	Solve $p^2 + q^2 = npq$. $\left[\mathbf{z = ax + \frac{na \pm a\sqrt{n^2-4}}{2} y + c}\right]$	Jan-15
C	4.	Solve $p^2 + q^2 = x + y$. $\left[\mathbf{z = \frac{2}{3}(a+x)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b}\right]$	Jan-14

H	5.	Solve $p^2 - q^2 = x - y$. $\left[\mathbf{z = \frac{2}{3}(a+x)^{\frac{3}{2}} + \frac{2}{3}(a+y)^{\frac{3}{2}} + b}\right]$	**Jan-15**
T	6.	Solve $p - x^2 = q + y^2$. $\left[\mathbf{z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + b}\right]$	**Jun-15**
T	7.	Solve $z = px + qy + p^2q^2$. $[\mathbf{z = ax + by + a^2b^2}]$	**Jun-13**
C	8.	Solve $z = px + qy - 2\sqrt{pq}$. $[\mathbf{z = ax + by - 2\sqrt{ab}}]$	**Dec-13**
H	9.	Solve $qz = p(1+q)$. $[\mathbf{bz - 1 = c\,e^{x+ay}}]$	**Jun-14**
C	10.	Solve $pq = 4z$. $[\mathbf{4bz = (2x + 2by + b)^2}]$	**Jun-15**

Method Of Separation Of Variables

- ✓ **Step 1:** Let $u(x,y) = X(x) \cdot Y(y)$
- ✓ **Step 2:** Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$ as requirement and substitute in given Partial Differential Eq^n.
- ✓ **Step 3:** Convert it into Separable Variable equation and equate with constant say k individually.
- ✓ **Step 4:** Solve each Ordinary Differential Equation.
- ✓ **Step 5:** Put value of $X(x)$ & $Y(y)$ in equation $u(x,y) = X(x) \cdot Y(y)$.

Exercise-5

Method Of Separation Of Variables			
C	1.	Solve the equation by method of separation of variables $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, where $u(0,y) = 8e^{-3y}$. $[\mathbf{u(x,y) = 8\,e^{-12x-3y}}]$	**Dec-13**
C	2.	Solve $2\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u$ subject to the condition $u(x,0) = 4e^{-3x}$ $[\mathbf{u(x,t) = 4\,e^{-3x-7t}}]$	**Jan-13**
H	3.	Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ subject to the condition $u(x,0) = 6e^{-3x}$ $[\mathbf{u(x,t) = 6\,e^{-3x-2t}}]$	**Jun-15**

C	4.	Using method of separation of variables solve $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2(x+y)u$. $\left[\mathbf{u(x,y)=e^{x^2+y^2+kx-ky+c}}\right]$	Jan-15 Jun-15
T	5.	Using the method of separation variables, solve the partial differential equation $u_{xx}=16u_y$. $\left[\mathbf{u(x,y)=\left(c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}\right)\ c_3\ e^{\frac{ky}{16}}}\right]$	Dec-10
H	6.	Using method of separation of variables solve $\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial y}+2u$. $\left[\mathbf{u(x,y)=\left(c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}\right)\ c_3\ e^{(k-2)y}}\right]$	Jun-13
C	7.	Solve $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$ by the method of separation variables. $\left[\mathbf{z(x,y)=\left(c_1e^{(1+\sqrt{1+k})x}+c_2e^{(1-\sqrt{1+k})x}\right)\ c_3\ e^{-ky}}\right]$	May-12 Jun-14
C	8.	Solve two dimensional Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$,using the method separation of variables. $\left[\mathbf{u(x,y)=\left(c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}\right)\ \left(c_3\cos\sqrt{k}y+c_4\sin\sqrt{k}y\right)}\right]$	Jan-13 Dec-09
T	9.	Using the method of separation of variables, solve the partial differential equation $\frac{\partial^2 u}{\partial x^2}=16\frac{\partial^2 u}{\partial y^2}$. $\left[\mathbf{u(x,y)=\left(c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}\right)\left(c_3e^{\frac{\sqrt{k}}{4}y}+c_4e^{-\frac{\sqrt{k}}{4}y}\right)}\right]$	Jan-15
C	10.	Solve $x\frac{\partial u}{\partial x}-2y\frac{\partial u}{\partial y}=0$ using method of separation variables. $\left[\mathbf{u(x,y)=c_1\ c_2\ x^k\ y^{\frac{k}{2}}}\right]$	Jun-13