Note: There are 6 problems with a total of 130 points. You are required to do all the problems. In the problems, the base of $\log n$ is 2.

1. ($7 \times 5 = 35$ points) Let $f(n)$ and $g(n)$ be two functions from N^+ to R^+. Prove or disprove the following assertions. To disprove, you only need to give a counter example for functions $f(n)$ and/or $g(n)$ which make the assertion false.

   (a) $O(O(f(n))) = O(f(n))$
   (b) $O(\Theta(f(n))) = O(f(n))$
   (c) $\Theta(O(f(n))) = \Theta(f(n))$
   (d) $\Omega(O(f(n))) = O(\Omega(f(n)))$
   (e) If $f(n) = \Theta(h(n))$ and $g(n) = \Theta(h(n))$, then $f(n) + g(n) = \Theta(h(n))$
   (f) If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$
   (g) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$

2. ($2 \times 5 = 10$ points) Use mathematical induction to prove the following.

   (a) $\sum_{i=1}^{n} i r^{i-1} = \frac{1-r^{n+1}-(n+1)(1-r)r^n}{(1-r)^2}$ for all $n \geq 1$, where $0 \leq r < 1$.
   (b) Every integer $n \geq 1$ can be represented as the sum of distinct Fibonacci numbers, no two of which are consecutive in the Fibonacci sequence.

3. ($4 \times 5 = 20$ points) Prove or disprove the following assertions.

   (a) $n! = O(n^n)$
   (b) $\sum_{i=1}^{n} i \log i = \Theta(n^2 \log n)$
   (c) If $n = 2^k$, then $\sum_{i=0}^{k} \log(n/2^i) = \Theta(\log^2 n)$
   (d) $n^n = O(2^n)$

4. (15 points) Rank the following functions in asymptotically increasing order based on O-notation and justify your ordering: $n!$, $(lgn)^{lg(lgn)}$, $[lg(lgn)]^{lgn}$, $2^{n^{0.001}}$, $n^{1/lgn}$, $lg^*(lgn)$, $2^{\sqrt{2lgn}}$, 2^{2^n}, n^5, $\sqrt{lgn}$.

5. ($8 \times 5 = 40$ points) Find a closed form for each $T(n)$. You may assume that $T(1) = 1$.

   (a) $T(n) = T(n-1) + 2^n$
   (b) $T(n) = 4T(n/3) + n^2$
   (c) $T(n) = 6T(n/7) + n$
   (d) $T(n) = T(\sqrt{n}) + logn$
   (e) $T(n) = 2 + \sum_{i=1}^{n-1} T(i)$
   (f) $T(n) = 3T(n/2) + n \log n$
   (g) $T(n) = 2T(n/2) + n/\log n$
   (h) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

6. (10 points) The sequence $< a_n >$ is defined for $n \geq 0$ by $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n > 1$. The first few elements of the sequence are 2, 5, 13, 35, 97. Find a closed form for a_n.

1) a) $O(O\ f(n)) = O(f(n))$

We know that, $O(g(n)) = \{ f(n)$: there exists constants C an[d]
n_0 such that $0 \le f(n) \le C\,g($[n)]
for all $n \ge n_0 \}$

This means that $O(g(n))$ implies upper bound on a function
which basically means, $f(n) = C.g(n)$ for some constant C
→ (

Therefore, $O(f(n)) = O(C\ g(n))$

$\le C.C_1\ g(n)$ ...... from ①

$\le K\ g(n)$ for some constant
$K = C.C_1$ — ②

From ① & ②, we get $O(f(n)) = f(n)$

$\therefore O(O(f(n))) = O(f(n))$.

b) $O(\theta f(n)) = O(f(n))$

let say $O(\theta f(n)) = h(n)$ & $\theta(f(n)) = g(n)$ — ①

By property of θ-notation, we get
$C_1 f(n) \le g(n) \le C_2\ f(n)$

lets consider, $g(n) \le C_2\ f(n)$, — ②

Now, $h(n) = O(\theta f(n))$
$= O(g(n))$ ...... from 1

$\therefore h(n) \le C.\ g(n)$ ③

From ② & ③ we get
$h(n) \le C.C_2\ f(n)$

Similarly, By property of O-notation

$h(n) = O(f(n))$ — (4)

From (1) & (4), $O(\Theta(f(n))) = O(f(n))$

c) $\Theta(O(f(n))) = \Theta(f(n))$

we assume the given statement to be true and $f(n) = n^3$

$\therefore \Theta(O(n^3)) = \Theta(n^3)$ — 1

By definition of Big O-notation

$h(n) \leq O(n^3)$

$\therefore h\,n = c_1 \cdot n^2$ for some constant $c_1 > 0$ & $n \geq n_0$ — 2

From (1) & (2), we get

$\Theta(c_1 \cdot n^2) = \Theta(n^3)$

By definition of Θ-notation, we get

L.H.S $= \Theta(c_1 \cdot n^2)$	R.H.S $= \Theta(n^3)$
$h_1(n) = \Theta(c_1 n^2)$	$h_2(n) = \Theta(n^3)$
$c_1 c_2 n^2 \leq h_1(n) \leq c_1 c_3 n^2$	$c_4 n^3 \leq h_2(n) \leq c_5 n^3$

As we can see that $\Theta(O(f(n)))$ is not equal to $\Theta(f(n))$ for all cases.

1) e) If $f(n) = \theta[h(n)]$ then $f(n) = c_1 h(n)$ where c_1 is constant.

Similarly, $g(n) = c_2 h(n)$ where c_2 is constant

then, $f(n) + g(n) = c_1 h(n) + c_2 h(n)$

$= (c_1 + c_2) h(n)$

$= k\, h(n)$ where $k = c_1 + c_2$ is a constant

From above we can say that, $f(n) + g(n) = \theta(h(n))$

f) let say $f(n) = 2n$ & $g(n) = n$

$\therefore$ If $f(n) = \theta(g(n))$

then, $n = \theta(2n)$

Now, L.H.S: $2^{f(n)} = 2^{2n}$ R.H.S: $\theta(2^n)$

$\therefore 4^n = \theta(2^n)$

By the definition of θ-notation,

$c_1 4^n \leq 2^n \leq c_2 4^n$.

But $c_1 4^n \nleq 2^n$.

$\therefore$ the given statement is not true

i.e. $2^{f(n)} \neq \theta(2^{g(n)})$ If $f(n) = \theta(g(n))$

g) $f(n) + g(n) = \theta(\min(f(n), g(n)))$

let say $f(n) = n$ & $g(n) = n^2$ for all $n \geq 1$

then $\min(f(n), g(n)) = n$ — ①

$L.H.S = f(n) + g(n)$

$= n + n^2$

$= \theta(n^2)$ — ②

From ① & ②, we get that

$$f(n) + g(n) \neq \theta(\min(f(n), g(n)))$$

i.e $n + n^2 \neq \theta(n)$ //

2) a) $\sum_{i=1}^{n} i x^{i-1} = \frac{1 - x^{n+1} - (n+1)(1-x)x^n}{(1-x)^2}$ for all $n \geq 1$

By using $n=1$ & $i=1$ we get

$L.H.S = (1)x^{(1-1)}$

$= 1$

$R.H.S = \frac{1 - x^{(1+1)} - (1+1)(1-x)x^1}{(1-x)^2}$

$= \frac{1 - x^2 - 2(x - x^2)}{(1-x)^2}$

$= \frac{1 - x^2 - 2x + 2x^2}{(1-x)^2}$

$= \frac{1 - 2x + x^2}{(1-x)^2} = \frac{(1-x)^2}{(1-x)^2}$

$= 1$

for $n=1$ $L.H.S = R.H.S$

By Induction method we know that,
$f(K)$ is true i.e, $f(K) = \sum_{i=1}^{K} i\gamma^{i-1}$

$$= \frac{1-\gamma^{K+1} - (K+1)(1-\gamma)\gamma^K}{(1-\gamma)^2} \quad -(1)$$

Simplifying the above equation, we get

$$\sum_{i=1}^{K} i\gamma^{i-1} = \frac{1-\gamma^{K+1} - (K - K\gamma + 1 - \gamma)\gamma^K}{(1-\gamma)^2}$$

$$= \frac{1-\gamma^{K+1} - (K\gamma^K - K\gamma^{K+1} + \gamma^K - \gamma^{K+1})}{(1-\gamma)^2}$$

$$= \frac{1 \cancel{-\gamma^{K+1}} - K\gamma^K + K\gamma^{K+1} - \gamma^K \cancel{+\gamma^{K+1}}}{(1-\gamma)^2}$$

$$= \frac{1 - \gamma^K - K\gamma^K + K\gamma^{K+1}}{(1-\gamma)^2} \quad -(2)$$

The eq (1) should also be true for $f(K+1)$

$$\therefore f(K+1) = \sum_{i=1}^{K+1} i\gamma^{i-1} = \frac{1-\gamma^{(K+1)+1} - (K+2)(1-\gamma)\gamma^{K+1}}{(1-\gamma)^2}$$

$$\therefore \sum_{i=1}^{K} i\gamma^{i-1} + (K+1)\gamma^K = \frac{1-\gamma^{K+2} - (K+2)(1-\gamma)\gamma^{K+1}}{(1-\gamma)^2}$$

$$\therefore \sum_{i=1}^{K} i\gamma^{i-1} = \frac{1-\gamma^{K+2} - (K - K\gamma + 2 - 2\gamma)\gamma^{K+1}}{(1-\gamma)^2} - (K+1)\gamma^K$$

$$\therefore \sum_{i=1}^{K} i\gamma^{i-1} = \frac{1-\gamma^{K+2} - (K\gamma^{K+1} - K\gamma^{K+2} + 2\gamma^{K+1} - 2\gamma^{K+2}) - (K+1)\gamma^K (1-\gamma)^2}{(1-\gamma)^2}$$

$$\therefore \sum_{i=1}^{k} i x^{i-1} = \frac{1-x^{k+2}-(kx^{k+1}-kx^{k+2}+2x^{k+1}-2x^{k+2})-(kx^{k}+x^{k})(1-2x+x^{2})}{(1-x)^{2}}$$

$$= \frac{1-x^{k+2}-kx^{k+1}+kx^{k+2}-2x^{k+1}+2x^{k+2}-(kx^{k}-2kx^{k+1}+kx^{k+2}+x^{k}-2x^{k+1}+x^{k+2})}{(1-x)^{2}}$$

$$= \frac{1-\cancel{x^{k+2}}-kx^{k+1}+kx^{k+2}-2x^{k+1}+\cancel{2x^{k+2}}-kx^{k}+2kx^{k+1}-kx^{k+2}-x^{k}+2x^{k+1}-\cancel{x^{k+2}}}{(1-x)^{2}}$$

$$= \frac{1-kx^{k+1}+\cancel{kx^{k+2}}-\cancel{2x^{k+1}}-kx^{k}+2kx^{k+1}-\cancel{kx^{k+2}}-x^{k}+\cancel{2x^{k+1}}}{(1-x)^{2}}$$

$$= \frac{1-kx^{k+1}-kx^{k}+2kx^{k+1}-x^{k}}{(1-x)^{2}}$$

$$\therefore \sum_{i=1}^{k} i x^{i-1} = \frac{1-x^{k}-kx^{k}+kx^{k+1}}{(1-x)^{2}} \quad - \ (3)$$

From (2) & (3), ~~f(k)~~ $f(k+1)$ also holds true.

$\therefore$ By mathematical induction, we prove that

$$\sum_{i=1}^{n} i x^{i-1} = \frac{1-x^{n+1}-(n+1)(1-x)x^{n}}{(1-x)^{2}} \quad \text{for all } n \geq 1$$

4) As per the question, let say we have $h(n)$ which is sum of two non consecutive fibonacci numbers

So $h(n) = F(p) + F(q)$ where p & q are non consecutive fibonacci num. — (1)

For base condition where $n = 1$

$h(n) = 1$ which itself is a fibonacci number

The equation holds true for $f(k) = h(k) = F(p) + F(q)$

By mathematical equation it should also be true for $f(k+1)$, $n \Rightarrow k+1$

$\therefore h(k+1) = F(p) + F(q) + F(r) + \ldots + 1$

$\therefore h(k+1) =$ sum of non consecutive fibonacci number $+ F(1)$ — (2)

$\therefore h(k+1)$ also holds true ... From (1) & (2)

In eq (2), $F(1) = F(2) = 1$ i.e first two numbers in a fibonacci series but to compute the sum we will never take $F(1)$ & $F(2)$ together

For example, for $n = 12$

$h(12) = 8 + 3 + 1 = F(6) + F(4) + F(2)$

where $F(6), F(4), F(2)$ are terms in fibonacci series 0 1 1 2 3 5 8 13 ...

Hence by mathematical ~~equation~~ induction the statement is proved

3) d) $n! = O(n^n)$

L'Hospital Rule does not work for this problem as we don't know how to take derivative of $n!$

According to sterling formula,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n+1} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}$$

which can be simplified to

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot \left(1 + \theta\left(\frac{1}{n}\right)\right)$$

$\therefore$ By sterling formula & appling limit test, we get

$T_1(n) = n!$ $\quad T_2(n) = n^n$

$$\therefore \lim_{n\to\infty} \frac{n!}{n^n} \geq \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{ne}\right)^n$$

$$\therefore \lim_{n\to\infty} \frac{n!}{n^n} \geq \lim_{n\to\infty} \sqrt{2\pi n} \cdot \lim_{n\to\infty} \left(\frac{1}{e}\right)^n$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

which means that $n!$ is smaller than n^n i.e. $n! \leq n^n$

$\therefore n! = o(n^n)$ — By definition of small o notation

Also small 'o' is a subset of big 'O' therefor it implies that $n! = O(n^n)$

//

v) $\sum_{i=1}^{n} i.\log i = \Theta(n^2 \log n)$

L.H.S = 1.log 1 + 2.log 2 + 3 log 3 + ... + n.log n.

therefore it is an increasing function

∴ By integration method

$$\int_{a-1}^{b} f(x)\,dx \le \sum_{i=a}^{b} f(i) \le \int_{a}^{b+1} f(x)\,dx$$

where a = 1, b = n.

$$\therefore \int_{0}^{n} x.\log(x)\,dx \le \sum_{i=1}^{n} f(i) \le \int_{1}^{n+1} x.\log(x)\,dx \quad -(1)$$

$\therefore \int_{a}^{b} x.\log x\,dx$ can be found by integration by parts, $u = \log x$ $du = \frac{1}{x}dx$.

$v = x\,dx$ $dv = \frac{1}{2}x^2$.

Then we have,

$$\int_{a}^{b} x.\log x\,dx = \int_{a}^{b} u\,dv$$

$$= uv - \int_{a}^{b} v\,du$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{2}\int_{a}^{b} x\,dx$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 \Big]_{a}^{b} \quad -(2)$$

From (1) & (2), we get

$$\left[\frac{1}{2}x^2 \log x - \frac{1}{4}x^2\right]_0^n \le \sum_{i=1}^{n} f(i) \le \left[\frac{1}{2}x^2 \log x - \frac{1}{4}x^2\right]_1^{n+1} \quad -(3)$$

In eq (3), we will find the dominant term in $\frac{1}{2}x^2 \log x - \frac{1}{4}x^2$

by limit test where $T_1(x) = \frac{1}{2}x^2 \log(x)$ & $T_2 = \frac{1}{4}x^2$

$$\therefore \lim_{x \to \infty} \frac{2x^2 \log x}{x^2} = \lim_{x \to \infty} 2 \cdot \log x = \infty$$

$\therefore$ we can say that, dominant term is $\frac{1}{2}x^2 \cdot \log x = \Theta(x^2 \log$

$-(4)$

from (3) & (4) we get,

$$c_1\Theta(x^2 \cdot \log x)\Big]_0^n \le \sum_{i=1}^{n} f(i) \le c_2\Theta(x^2 \cdot \log x)\Big]_1^{n+1}$$

$$\therefore c_1\Theta(n^2 \cdot \log n) \le \sum_{i=1}^{n} f(i) \le c_2\Theta((n+1)^2 \cdot \log n+1 - \log 1)$$

$$\therefore \cancel{c_1\Theta(n^2 \cdot \log n) \le \sum_{i=1}^{n} f(i) \le c_2\Theta(n^2 \cdot \log n)}$$

$$\therefore c_1(n^2 \cdot \log n) \le \sum_{i=1}^{n} f(i) \le c_2((n+1)^2 \cdot \log n+1)$$

By definition of Θ-notation, the $f(i)$ lies between its upper bound & lower bound.

$$\therefore \sum_{i=1}^{n} i \cdot \log i = \Theta(n^2 \cdot \log n) \quad //$$

c) we know that $n = 2^k$.

$$\therefore L.H.S = \sum_{i=0}^{k} \log\left(\frac{n}{2^i}\right) = \sum_{i=0}^{k} \log\left(\frac{2^k}{2^i}\right)$$

$$= \sum_{i=0}^{k} \log 2^{k-i}$$

$$= \sum_{i=0}^{k} k-i$$

By evaluating ~~te~~ the series, we get,

$$L.H.S = k + k-1 + k-2 + k-3 + \dots + 3+2+1$$

$$= \frac{k(k+1)}{2} \quad \text{--- (1)}$$

As per the summation formulas, $\frac{k(k+1)}{2} = \theta(k^2)$

$$\therefore L.H.S = \theta(k^2) \quad \text{--- (2)}$$

$$R.H.S = \theta(\log^2 n)$$

$$= \theta(\log \log 2^k)$$

$$= \theta(\log k \log 2)$$

$$= \theta(\log k) \quad \text{--- (3)}$$

From (2) & (3), we get that $L.H.S \neq R.H.S$

$$\therefore \sum_{i=0}^{k} \log\left(\frac{n}{2^i}\right) \neq \theta(\log^2 n)$$

d) $n^n = O(2^n)$

let $T_1(n) = n^n$ and $T_2(n) = 2^n$

By limit Test, $c = \lim_{n \to \infty} \frac{T_1(n)}{T_2(n)}$

$$= \lim_{n \to \infty} \frac{n^n}{2^n}$$

$$= \lim_{n \to \infty} \left(\frac{n}{2}\right)^n$$

$$= \infty$$

We know that is $c = \infty$ then $T_1(n) = \omega(T_2(n))$

$n^n = \omega(2^n)$

Hence the given statement is false i.e. $n^n \neq O(2^n)$

In the list of function given, there are exponential, polynomial and polylogarithmic function. The ordering is based on following properties and identities.

P1 → Exponential functions grow faster than polynomial functions

P2 → Polynomial functions grow faster than polylogarithmic functions

P2 → In asymptotical analysis the base of logarithmic functions doesn't matter, but the base of exponential and degree of polynomial do matter.

I1 → $n^{1/\log n} \Rightarrow O(1)$

I2 → $\log^*(\log n) = (\log^* n) - 1$

Proof 1 :

let say $T_1(n) = [\log(\log n)]^{\log n}$ & $T_2(n) = (\log n)^{\log \log n}$

lets compare the above term, say $\log n = x$.

$\therefore T_1(x) = (\log x)^{x}$ $T_2(x) = x^{\log x}$.

By limit test and L' Hospital rule

$$\lim_{x \to \infty} \frac{T_1(x)}{T_2(x)} = \frac{T_1'(x)}{T_2'(x)} = \lim_{x \to \infty} \frac{(\log x)^{x}}{x^{\log x}}$$

$$= 0.$$

This means that $x^{\log x}$ grows faster than ~~n~~ numerator

$\therefore [\log n]^{\log \log n}$ grows faster than $[\log(\log n)]^{\log n}$.

Proof 2 :

let say $T_1(n) = 2^{\sqrt{2 \log n}}$ $T_2(n) = (\log n)^{\log(\log n)}$

suppose $\log n = x$ and $n = 2^{x}$

$T_1(2^x) = 2^{\sqrt{2x}}$ $T_2(2^x) = x^{\log x}$.

comparing T_1 & T_2 by limit Test we get that

$$\lim_{x \to \infty} \frac{T_1}{T_2} = \lim_{x \to \infty} \frac{2^{\sqrt{2x}}}{x^{\log x}}$$

$$= \infty$$

$\therefore 2^{\sqrt{2x}}$ grows faster than $x^{\log x}$.

This means that $2^{\sqrt{2 \log n}}$ grows faster than $(\log n)^{\log(\log n)}$

Proof 3:

let $T_1(n) = 2^{\sqrt{2\log n}}$ $\qquad$ $T_2(n) = [\log \log n]^{\log n}$

suppose $\log n = x$ then $n = 2^x$

$\therefore T_1(2^x) = 2^{\sqrt{2x}}$ $\qquad$ $T_2(2^x) = (\log x)^x$

lets compare the above eqn by limit Test and we get.

$$\lim_{x \to \infty} \frac{T_1(2^x)}{T_2(2^x)} = \lim_{x \to \infty} \frac{2^{\sqrt{2x}}}{(\log x)^x}$$

$$= 0$$

$\therefore (\log x)^x$ grows faster than $2^{\sqrt{2x}}$

This means that $[\log \log n]^{\log n}$ grows faster than $2^{\sqrt{2\log n}}$

Proof 4:

let $T_1(n) = 2^{2^n}$ $\qquad$ $T_2(n) = n!$

By sterling formula, $n!$ can be evaluated as

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}$$

This can be simplified to

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n} \quad \text{where } \frac{1}{12n+1} < \alpha_n < \frac{1}{12n}.$$

lets compare $T_1(n)$ & $T_2(n)$ by limit test

$$\lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} = \lim_{n\to\infty} \frac{2^{2^n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha n}}$$

$$= \lim_{n\to\infty} \frac{2^{2^n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \times \lim_{n\to\infty} \frac{1}{e^{\alpha n}}$$

As $n \to \infty$, $\frac{1}{e^{\alpha n}} \to 1$

$$\therefore \lim_{n\to\infty} \frac{T_1(n)}{T_2'(n)} = \lim_{n\to\infty} \frac{2^{2^n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

let $y = 2^{2^n}$ and $z = 2^n$

Applying log, we get

$\log y = 2^n \log 2$ $\qquad\qquad$ $\log z = n \log 2$

$\therefore \log y = 2^n$ $\qquad\qquad$ $\log z = n$

$$\frac{1}{z} dz = dn$$

$$\frac{dz}{dz} = z = 2^n$$

Now, $T_1'(n) = \frac{1}{y} dy = 2^n$

$$T_1'(n) = dy = 2^n . y$$

$$T_1'(n) = 2^{2^n} . 2^n .$$

$$T_2'(n) = \frac{\sqrt{2\pi}\,(n + 1/2)\, n^{n-1/2}}{e^n}$$

By L'Hospital rule we get

$$\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{T_1'(n)}{T_2'(n)} = \frac{2^{2^n} \cdot 2^n}{\frac{\sqrt{2\pi}\left(n+\frac{1}{2}\right) n^{n-1/2}}{e^n}}$$

$$\therefore \lim_{n \to \infty} \frac{T_1'(n)}{T_2'(n)} = \frac{1}{\sqrt{2\pi}} \lim_{n \to \infty} \frac{2^{2^n} \cdot 2^n}{\left(n+\frac{1}{2}\right) n^{n-1/2}}$$

If we apply L'Hospital rule again & again then the denominator

$\left(n+\frac{1}{2}\right) n^{n-1/2}$ → constant value will the numerator

will tend to ∞

$\therefore$ we can say that 2^{2^n} will grow faster than $n!$

The ranking of functions in increasing order based on Θ-notation is:

$n^{1/\log n}$ .... I 1

$\log^* \log n$ .... P1 & P2

$\sqrt{\log n}$ .... P1 & P2

$(\log n)^{\log \log n}$ } Proof 2

$2^{\sqrt{2 \log n}}$ } Proof 3

$[\log \log n]^{\log n}$ } Proof 1

n^5 } P1 & P2

$2^{n^{0.001}}$

$n!$ } Proof 4

2^{2^n}

6) $a_0 = 2$, $a_1 = 5$; $a_n = 5a_{n-1} - 6a_{n-2}$.

let say $n = n+2$ then we get.

$$a_{n+2} = 5a_{n+1} - 6a_n.$$

The above eq^n is of the form linear recursive sequence
∴ by comparing with the characteristic eq^n

$$x^k = c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \dots + c_1x^1 + c_0$$

we get that $k = 2$, $c_1 = 5$, $c_0 = -6$.

$$\therefore x^2 = c_1x^1 + c_0$$
$$\therefore x^2 = 5x - 6$$

$$\therefore x^2 - 5x + 6 = 0.$$

$$\therefore (x-3)(x-2) = 0.$$

The roots of characteristic eq^n are $\alpha_1 = 3$; $\alpha_2 = 2$.

Since the roots are distinct, the solution has the for. form $t_n = a_1(\alpha_1)^n + a_2(\alpha_2)^n$.

$$\therefore t_n = a_1 3^n + a_2 2^n.$$

when $n = 0$, $t_0 = 2 = a_1 + a_2$. — ①

$n = 1$ $t_n = 5 = 3a_1 + 2a_2$. — ②

multiply eq^n ① by 2 and subtract from eq^n ② we get

$$\begin{array}{r} 3a_1 + 2a_2 = 5 \\ - \quad 2a_1 + 2a_2 = 4 \\ - \quad - \quad - \\ \hline a_1 = 1 \end{array}$$

From eqⁿ ② & ① we get.

$$a_1 = a_2 = 1.$$

$\therefore$ $t_n = 3^n + 2^n$

The closed form of $a_n = 5a_{n-1} - 6a_{n-2}$ is $3^n + 2^n$

a) $T(n) = T(n-1) + 2^n$. We know that $T(1) = 1$

$\therefore$ $T(2) = T(1) + 2^2 = 5$

for $n=3$, $T(3) = T(2) + 2^3 = 5 + 8 = 13$

$n=4$, $T(4) = T(3) + 2^4 = 13 + 16 = 29$

By observing the outputs, the eqⁿ $2^{n+1} - 3$ satisfies the condition of the recurrence.

lets test $t_n = 2^{n+1} - 3$ by mathematical induction.

$t_3 = 2^{3+1} - 3 = 16 - 3 = 13$

$t_4 = 2^{4+1} - 3 = 32 - 3 = 29.$

$\therefore$ The closed form for $T(n) = T(n-1) + 2^n$ is $2^{n+1} - 3$ //

b) $T(n) = 4T(n/3) + n^2$.

A simpler version of Master Theorem is given by

$$T(n) = \begin{cases} O(1) & \text{if } n \leq n_0 \\ aT(n/b) + \theta(n^k) & \text{if } n > n_0 \end{cases}$$

Therefore we get $a=4$, $b=3$, $k=2$.

$4 < 3^2$ i.e $a < b^k$.

As per case 3 of master theorem, if $a < b^k$ then $T(n) = \theta(n^k)$

$\therefore$ For $T(n) = 4T(n/3) + n^2 = \theta(n^2)$ //

c) $T(n) = 6T(n/7) + n$

Master theorem is given by

$$T(n) = \begin{cases} O(1) & \text{if } n \leq n_0 \\ aT(n/b) + \theta(n^k) & \text{if } n > n_0 \end{cases}$$

Therefore, we get $a = 6$; $b = 7$; $k = 1$

$6 < 7^1$ i.e $a < b^k$

As per case 3 of master theorem, if $a < b^k$ then $T(n) = \theta(n^k)$

$\therefore$ For $T(n) = 6T(n/7) + n = \theta(n)$ //

d) $T(n) = T(\sqrt{n}) + \log n$

let $m = \log n$ $\therefore$ $n = 2^m$

$$\therefore T(n) = T(2^m) = T(\sqrt{2^m}) + m.$$

$$\therefore T(2^m) = T(2^{m/2}) + m.$$

let say $H(m) = T(2^m)$

$$\therefore H(m) = H(m/2) + m.$$

By applying Masters theorem to the above equation, we get

$$a = 1, \; b = 2, \; k = 1$$

$$\therefore a < b^k \text{ i.e. } 1 < 2^1$$

As per case 3 of master theorem, $H(m) = \Theta(m^k) = \Theta(m)$

But $m = \log n$.

$$\therefore \cancel{H(n) = \Theta(\log n)} \; H(\log n) = \Theta(\log n)$$

But $H(\log n) = T(n)$

$$\therefore T(n) = \Theta(\log n) \quad //$$

e) $T(n) = 2 + \sum_{i=1}^{n-1} T(i)$ & $T(1) = 1$

For $n = 2$, $T(2) = 2 + \sum_{i=1}^{1} T(i) = 2 + T_1 = 3$

For $n = 3$, $T(3) = 2 + \sum_{i=1}^{2} T(i) = 2 + T(2) + T(1) = 2+3+1 = 6$

for $n=4$, $T(4) = 2 + \sum_{i=1}^{3} T(i) = 2 + T(3) + T(2) + T1 = 2+6+3+1 = 12$

By observing the terms, for $n_0 = 2$ & $n \geq n_0$ the series looks like a geometric progression, ar^{n-2} $\therefore n_0 = 2$.

where $a = 3$ & $r = \frac{T(3)}{T(2)} = \frac{6}{3} = 2$.

$$\therefore T(n) = 2 + \sum_{i=1}^{n-1} T(i) = 3 \cdot 2^{n-2}.$$

The above eqn holds true for $T(K) = 3 \cdot 2^{K-2}$ — ①

By mathematical induction we will prove that $T(K+1)$ also holds true.

$$T(K+1) = 2 + \sum_{i=1}^{K+1} T(i)$$

$$= 2 + \sum_{i=1}^{K} T(i) + T(K) \quad -②$$

$$= \cancel{3} \cdot T(K) + T(K) \quad \text{... From ① \& ②}$$

$$2\,(3 \cdot 2^{K-2})$$

$$= 3 \cdot 2^{K-1}$$

$$\therefore T(K+1) = 3 \cdot 2^{(K+1)-2}$$

$\therefore T(n) = 3 \cdot 2^{n-2}$ Holds true for all $n \geq 2$

f) $T(n) = 3T(n/2) + n\log n$

Master theorem is given by;

$$T(n) = \begin{cases} O(1) & \text{if } n \leq n_0 \\ aT(n/b) + \theta(n^k) & \text{if } n \geq n_0 \end{cases}$$

$\therefore$ $a = 3$; $b = 2$ $\quad f(n) = n\log n = O(n^{\log_b^a - \varepsilon})$ —(1)

$\therefore$ $\log_b^a = \log 3$ —(2)

From (1) & (2) & taking $\varepsilon = 0.1$ i.e $\varepsilon > 0$ we get

$T(n) = \theta(n^{\log_b^a})$ case-I of master theorem

$\therefore$ $T(n) = \theta(n^{\log 3})$ //

g) $T(n) = 2T(n/2) + n/\log n$

Rewriting $T(n) = 2T(n/2) + n(\log n)^{-1}$

where $a = 2$; $b = 2$; $f(n) = n(\log n)^{-1}$

$\log_b a = \log_2 2 = 1$ $\quad f(n) = n^1(\log n)^{-1}$

$= n^{\log_b^a}(\log n)^{-1}$

we know that, If $T(n) = aT(n/b) + f(n)$ where

$f(n) = \theta(n^{\log_b^a}(\log n)^k)$ then $T(n) = \theta(n^{\log_b^a}(\log n)^{k+1})$

Therefore, we get $T(n) = \theta(n^1(\log n)^{-1+1}) = \theta(n)$ //

n) $T(n) = \sqrt{n}\,T(\sqrt{n}) + n$.

let $n = 2^K$ ∴ $\sqrt{n} = 2^{K/2}$ and $K = \log n$ — ①

substituting back in the eqn we get

$$T(2^K) = 2^{K/2}\,T(2^{K/2}) + 2^K \quad -②$$

Dividing both sides by 2^K, we get

$$\frac{T(2^K)}{2^K} = \frac{T(2^{K/2})}{2^{K/2}} + 1 \quad -③$$

let say $h(K) = \frac{T(2^K)}{2^K}$ then eqn ③ becomes

$$h(K) = h(K/2) + 1 \quad -④$$

we can now apply Master theorem to above eqn, $T(n) = aT(n/b) + n^p$

where $a = 1$; $b = 2$; $p = 0$
$a = b^p$ i.e $a = b^p = 1$

∴ $h(K) = K^p \cdot \log K$ case ② of simplex master theorem

∴ $h(K) = \log K$ ⑤

From eqn ③ we get that

$T(2^K) = 2^K \cdot h(K)$

∴ $T(2^K) = 2^K \cdot \log K$.

But $n = 2^K$ & $K = \log n$.

∴ $T(n) = n \cdot \log\log n$

∴ $T(n) = \Theta(n \log\log n)$ //

d) $\Omega(O(f(n))) = O(\Omega(f(n)))$

let say $f(n) = n^2$

$\therefore L.H.S = \Omega(O(n^2))$

By definition of $O(n^2)$ O-notation

$O(n^2) = h(n)$

$h(n) \le C_1 n^2$ where C_1 is constant $C_1 > 0$ & $n \ge n_0$.

$\therefore L.H.S = \Omega(C_1 n^2)$

By definition of Ω-notation.

$\Omega(C_1 n^2) = h(n)$

$\therefore C_1 . C . n^2 \le h(n)$

$\therefore$ L.H.S = $h(n)$ which is a polynomial term greater than n^2.

$R.H.S = O(\Omega(n^2))$

By definition of Ω-notation

$\Omega(n^2) = h(n)$

$\therefore h(n) \ge C n^2$ where C is some constant

$\therefore R.H.S = O(C n^2)$

By definition of O-notation.

$O(C n^2) = h(n)$

$\therefore h(n) \leq c_1 c\, n^2$

$\therefore h(n) \leq K n^2$ where $K = c_1 \cdot c$ is some constant.

$\therefore$ R.H.S = $h(n)$ which is polynomial with lesser than n^2

$\therefore$ L.H.S $\neq$ R.H.S for most cases

$\therefore$ we can say that the given statement is false