# Improper Integration (contd....)

## (Infinite Range of Integration)

**Def$^n$ ④:** Let us consider the integral $\int_a^\infty f(x)\,dx$ $(a \in \mathbb{R})$.

If for any real number $X > a$, $f$ is integrable over $[a, X]$ and if $\lim_{X \to \infty} \int_a^X f(x)\,dx$ exists finitely, then the improper integral $\int_a^\infty f(x)\,dx$ is convergent and $\int_a^\infty f(x)\,dx = \lim_{X \to \infty} \int_a^X f(x)\,dx$.

Otherwise, the improper integral is said to be divergent.

**Def$^n$ ⑤:** Let us consider the integral $\int_{-\infty}^a f(x)\,dx$ $(a \in \mathbb{R})$.

If for any real no. $X < a$, $f$ is integrable over $[X, a]$ and if $\lim_{X \to -\infty} \int_X^a f(x)\,dx$ exists finitely, then the improper integral $\int_{-\infty}^a f(x)\,dx$ is convergent and $\int_{-\infty}^a f(x)\,dx = \lim_{X \to -\infty} \int_X^a f(x)\,dx$.

Otherwise, the improper integral is said to be divergent.

**Def$^n$ ⑥:** The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is said to be convergent if $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ converge for any real no. $c$.

In this case $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$.

Ex: Examine the convergence of

(i) $\displaystyle\int_0^\infty \frac{dt}{\sqrt{t}\,(1+t)}$ , (ii) $\displaystyle\int_1^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$

(iii) $\displaystyle\int_0^\infty \frac{x^3+1}{x^4}\,dx$ , (iv) $\displaystyle\int_e^\infty \frac{dx}{x\,(\log x)^{3/2}}$ .

Sol. (i) Let us consider the integrals

$$I_1 = \int_0^1 \frac{dt}{\sqrt{t}\,(1+t)} \quad , \quad I_2 = \int_1^\infty \frac{dt}{\sqrt{t}\,(1+t)} .$$

$I_1 = \frac{\pi}{2}$ (solved in the class)

For $I_2$, the integrand $f(t) = \frac{1}{\sqrt{t}\,(1+t)}$ is continuous and hence integrable in $[1, X]$ for any $X > 1$.

$\displaystyle\int_1^X \frac{dt}{\sqrt{t}\,(1+t)}$ , substituting, $t = z^2$ i.e. $dt = 2z\,dz$

$$= \int_1^{\sqrt{X}} \frac{2z\,dz}{z\,(1+z^2)} = 2\int_1^{\sqrt{X}} \frac{dz}{1+z^2} = 2\left[\tan^{-1} z\right]_1^{\sqrt{X}}$$

$$= 2\left[\tan^{-1}\sqrt{X} - \frac{\pi}{4}\right] \longrightarrow \frac{\pi}{2}$$

as $X \longrightarrow \infty$

So $I_2$ is convergent and $I_2 = \frac{\pi}{2}$

Thus, $\displaystyle\int_0^\infty \frac{dt}{\sqrt{t}\,(1+t)} = \frac{\pi}{2} + \frac{\pi}{2} = \pi .$

(II) $I = \displaystyle\int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2}\,dx$ , Let $f(x) = \dfrac{\sqrt{x}}{(1+x)^2}$

$f$ is continuous and hence bounded and integrable over $[1, X]$ for any $X > 1$.

$$\int_1^X \frac{\sqrt{x}}{(1+x)^2}\,dx \quad , \quad \text{putting } x = \tan^2\theta$$
$$dx = 2\tan\theta \sec^2\theta \, d\theta$$

$$= \int_{\pi/4}^{\tan^{-1}\sqrt{X}} \frac{\tan\theta \; 2\tan\theta \; \sec^2\theta}{\sec^4\theta}\, d\theta$$

$$= 2\int_{\pi/4}^{\tan^{-1}\sqrt{X}} \sin^2\theta \, d\theta = \int_{\pi/4}^{\tan^{-1}\sqrt{X}} (1-\cos 2\theta)\, d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\tan^{-1}\sqrt{X}} = \tan^{-1}\sqrt{X} - \frac{1}{2}\sin\left(2\tan^{-1}\sqrt{X}\right) - \frac{\pi}{4} + \frac{1}{2}$$

$$= \underbrace{\tan^{-1}\sqrt{X}}_{\pi/2} - \frac{\sqrt{X}}{1+X} - \frac{\pi}{4} + \frac{1}{2} \longrightarrow \frac{\pi}{4} + \frac{1}{2} \quad \text{as } X \to \infty$$

(III) $I = \displaystyle\int_1^{\infty} \frac{x^3+1}{x^4}\,dx$ , Let, $f(x) = \dfrac{x^3+1}{x^4}$

The integrand is continuous and hence bounded and integrable over $[1, X]$ for any $X > 1$.

$$\int_1^X \frac{x^3+1}{x^4}\,dx = \int_1^X \left(\frac{1}{x} + \frac{1}{x^4}\right) dx$$

$$= \left[\log|x| + \frac{x^{-3}}{-3}\right]_1^X = \log X - \frac{1}{3X^3} + \frac{1}{3} \longrightarrow \infty \quad \text{as } X \to \infty$$

Hence $I$ is divergent.

## $\mu$-Test:

$$\int_a^\infty \frac{dx}{x^\mu}, \quad (a>0) \quad \text{Converges iff } \mu>1.$$

Pf: If $\mu \neq 1$. For $X>a$, $\displaystyle\int_a^X \frac{dx}{x^\mu} = \left[\frac{x^{-\mu+1}}{-\mu+1}\right]_a^X = \frac{1}{1-\mu}\left[\frac{1}{X^{\mu-1}} - \frac{1}{a^{\mu-1}}\right]$

$$\longrightarrow \begin{cases} \dfrac{1}{(\mu-1)\,a^{\mu-1}} & \text{if } \mu-1>0 \\ \infty & \text{if } \mu-1<0 \end{cases}$$

as $X \longrightarrow \infty$

If $\mu=1$. $\displaystyle\int_a^X \frac{dx}{x} = \log|X| - \log|a| \longrightarrow \infty$ as $X \longrightarrow \infty$

Thus $\displaystyle\int_a^\infty \frac{dx}{x^\mu}$, $(a>0)$ converges iff $\mu>1$.

## Comparison Test:

Let $0<f\leq g$ be defined and integrable over $[a,X]$ for any $X>a$. Then

(i) if $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.

(ii) if $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

## Ratio Test:

Let $f(x)\geq 0$, $g(x)\geq 0$ be defined $\forall x\geq a$ and $f, g$ be bdd. and integrable over $[a,X]$ $\forall X>a$. Moreover, $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$, a non-zero +ve number.

Then $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_a^\infty g(x)dx$ converge or diverge together.

## $\Gamma$ - function

Examine convergence of $\int_0^\infty e^{-x} x^{n-1}\,dx$.

Consider $I_1 = \int_0^1 e^{-x} x^{n-1}\,dx$, $I_2 = \int_1^\infty e^{-x} x^{n-1}\,dx$.

For $I_1$

If $n-1 \geq 0$, $I_1$ is proper integral.

If $n-1 < 0$, integrand of $I_1$ has infinite discontinuity only at 0.

Let $f(x) = e^{-x} x^{n-1}$, $g(x) = x^{n-1}$.

Both $f, g$ have infinite discontinuity at $x=0$, but are continuous and hence bounded and integrable in $[\varepsilon, 1]$ for any $\varepsilon \in (0,1)$. Also $f(x) > 0$, $g(x) > 0$ $\forall x \in (0,1]$.

Moreover, $\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} e^{-x} = 1$.

Thus using Ratio Test $\int_0^1 e^{-x} x^{n-1}\,dx$ and $\int_0^1 x^{n-1}\,dx$ converge or diverge together.

But $\int_0^1 x^{n-1}\,dx$ converges iff $1-n < 1$ or $n > 0$

Thus $I_1$ converges iff $n > 0$.

For $I_2$ here also we can take g(x) = 1/x^2

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \cdots \quad \forall x \in \mathbb{R}$$

$$\Rightarrow e^x > \frac{x^r}{r!} \text{ for } x > 0 \Rightarrow e^{-x} < \frac{r!}{x^r} \text{ for } x > 0$$

$$\Rightarrow e^{-x} x^{n-1} < \frac{r!}{x^{r-n+1}}$$

Now $\int_1^\infty \frac{r!}{x^{r-n+1}}\,dx$ converges iff $r-n+1 > 1$

i.e. iff $r > n$

So by Comparison Test, $\int_1^{\infty} e^{-x} x^{n-1}\,dx$ converges if $r > n$

Choice of $r$ always possible since $n$ is fixed.

Therefore $I_1$ and $I_2$ both exist for $n > 0$ and $\int_0^{\infty} e^{-x} x^{n-1}\,dx$ converges for $n > 0$.

Define:

$$\boxed{\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx, \quad n > 0}$$

Before we defined:

$$\boxed{B(m,n) = \int_0^{1} x^{m-1} (1-x)^{n-1}\,dx, \quad m,n > 0}$$

Note: $m, n$ in $B(m,n)$ or $n$ in $\Gamma(n)$ are not necessarily integers.

## Absolutely Convergent

**Def$^n$:** The improper integral $\int_a^b f\,dx$ is said to be absolutely convergent if $\int_a^b |f|\,dx$ is convergent.

**Th** Every absolutely convergent integral is convergent, i.e.

$$\int_a^b |f|\,dx \text{ exists} \Rightarrow \int_a^b f\,dx \text{ exists}.$$

**Ex:** Show that $\displaystyle\int_0^1 \frac{\sin 1/x}{x^p}\,dx$, $(p>0)$ converges absolutely for $p<1$.

**Sol.** Let $f(x) = \dfrac{\sin 1/x}{x^p}$, $p>0$

'0' is only point of infinite discontinuity.

$f$ does not keep same sign in any neighbourhood of '0'.

In $[0,1]$ ~~$(0,1]$~~, $|f(x)| = \left|\dfrac{\sin \frac{1}{x}}{x^p}\right| < \dfrac{1}{x^p}$

$\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges iff $p<1$.

By Comparison Test, $\displaystyle\int_0^1 \left|\frac{\sin 1/x}{x^p}\right| dx$ Converges iff $p<1$

$\Rightarrow \displaystyle\int_0^1 \frac{\sin 1/x}{x^p}\,dx$ converges absolutely for $p<1$.

# Properties of Beta and Gamma Functions

1. $\boxed{B(m,n) = B(n,m)}$ $\quad (m, n > 0)$

Pf: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ , Putting $x = 1-z$

$$= \int_0^1 z^{n-1}(1-z)^{m-1}\,dz$$

$$= B(n,m)$$

2. $\boxed{B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(n,m)}$ $\quad (m,n>0)$

Pf: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ , Putting $x = \frac{1}{z+1}$

$\Rightarrow z = \frac{1}{x} - 1$

$\Rightarrow dz = -\frac{1}{x^2}\,dx$

$$= -\int_\infty^0 \frac{1}{(z+1)^{m-1}}\left(1-\frac{1}{z+1}\right)^{n-1}\frac{dz}{(z+1)^2}$$

$$= \int_0^\infty \frac{z^{n-1}}{(1+z)^{m+n}}\,dz$$

3. $\boxed{B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta}$ $\quad (m,n>0)$

Pf: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ , putting $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta\,(\sin\theta\cos\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta .$$

④ $$\boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

Pf: $$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

putting $2m-1 = p$ , $2n-1 = q$ we get the result.

⑤ $$\boxed{B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \pi}$$

Pf: $$B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi .$$

⑥ $$\boxed{\Gamma(n+1) = n\Gamma(n)} , \quad n > 0$$

how????

Pf: $$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx = \left[-e^{-x} x^n\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x} \, dx$$

$$= n\Gamma(n) .$$

⑦ $$\boxed{\Gamma(1) = 1}$$

Pf: $$\Gamma(1) = \int_0^\infty e^{-x} \, dx = \left[-e^{-x}\right]_0^\infty = 1 .$$

⑧ $$\boxed{\Gamma(n+1) = n! \quad \text{if } n \text{ is a +ve integer}}$$

Pf: $$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots\cdots$$

$$= n(n-1)(n-2)\ldots\ldots 3.2.1\Gamma(1)$$

$$= n!$$

⑨ $$\boxed{\text{For } a>0, \quad \int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}, \quad n>0}$$

Pf: Put $ax = y$.

⑩ $$\boxed{\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx, \quad n>0}$$

Pf. Put $x = y^2$ in original $\Gamma(n)$.

(*) ⑪ $$\boxed{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}} \quad (m,n>0)$$ ???

Pf: Using double ~~derivative~~ integration

⑫ $$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

Pf: $B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \Rightarrow \pi = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

⑬ $$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}}, \quad \boxed{\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}}$$

~~⑭ $\Gamma(m)\Gamma(n-1) = \pi$~~

⑭ $$\boxed{\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi} \quad (0<m<1)}$$

Pf: Using double ~~derivative~~ integration.