

Problem sheet 0

The problem we are looking at today is the following: For each positive integer n , let S_n denote the set consisting of the first n natural numbers, that is

$$S_n = \{1, 2, 3, 4, \dots, n-1, n\}.$$

The Problem: For which values of n is it possible to express S_n as the union of two non-empty disjoint subsets so that the elements in the two subsets have equal sums? Can you *prove* why?

In other words, can you separate the numbers $1, 2, \dots, n-1, n$ into two separate lists (list A and list B, say), so that the sum of all the numbers in list A equals the sum of all the numbers in list B? You must use all the numbers, and each number can only appear *once*.

Extension Problem: For which values of n is it possible to express S_n as the union of three non-empty disjoint subsets so that the elements in the three subsets have equal sums? Can you *prove* why?

General strategies

1. *Simplify the problem if necessary* - for example, you would attempt the original before the extension problem, because there are fewer "combinations" for partitioning the numbers into two than three subsets.
2. *Look at a few numerical examples* - some easy to study examples would be $n = 1, 2, 3, 4, 5, 6, \dots$ up until you are comfortable enough that you understand what the problem is asking. Ideally, you want to have come across several values of n where you can partition, and several values of n where you can't partition.
3. *Record your observations and any patterns you notice* - always useful if you want to refer to your work later on.
4. *Make (reasonable) conjectures, and test them out* - this is how new mathematics is discovered, by following hunches/leads/educated guesses and seeing what happens. It can also stop you from following false leads, as you tend to remember any conjectures which are wrong.
5. *Don't worry about trying to immediately find a general formula/theorem* - too many students want a magic theorem which solves a problem. That's not how mathematics works. You need to draw different ideas together, and it is perfectly ok to have to carry out lots of little steps