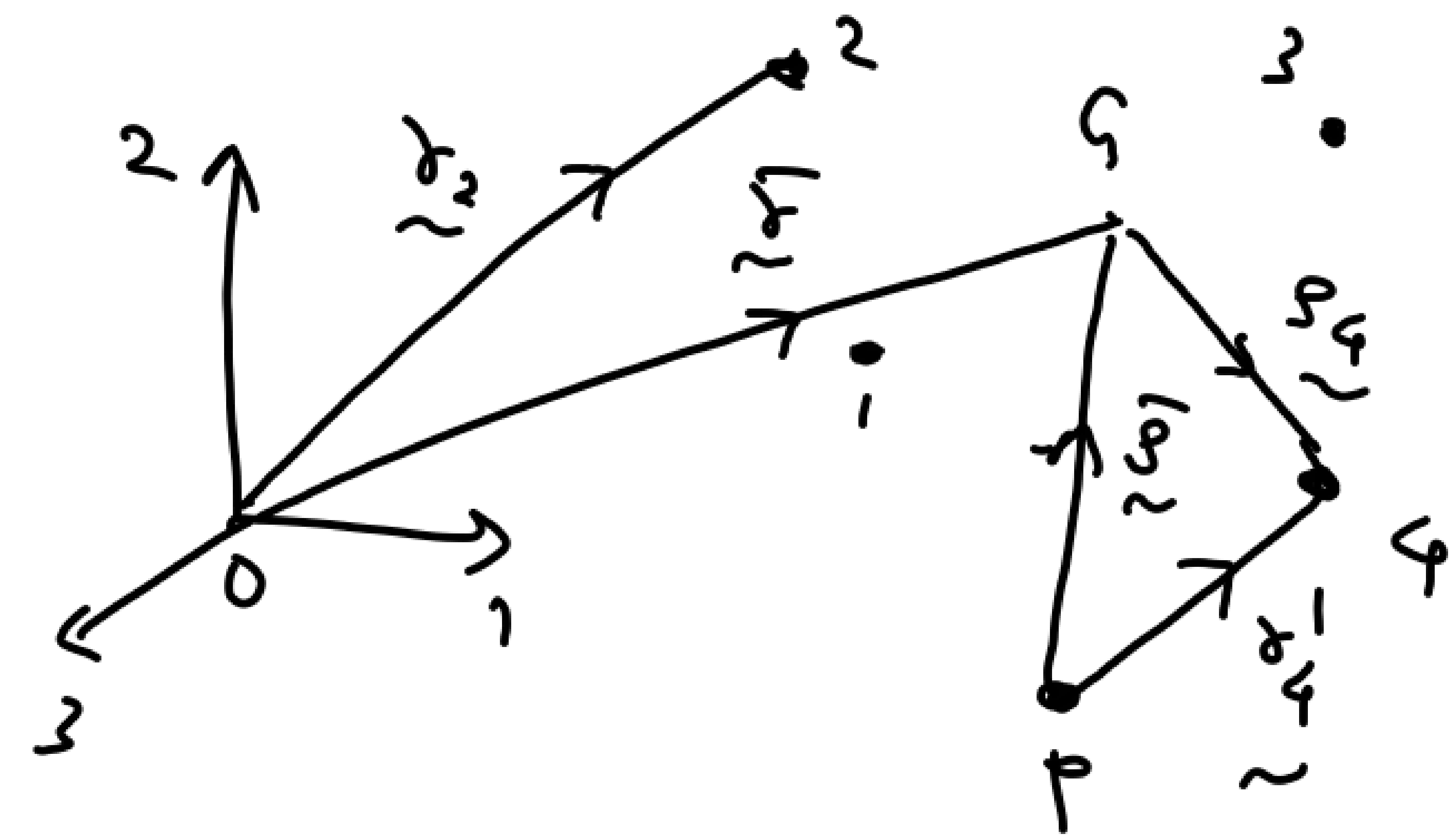
Balance of angular momentum :

$$\left.\begin{aligned} &① \quad \dot{\underset{\sim}{H}}_O = \Sigma \underset{\sim}{M}_O \\ &② \quad \dot{\underset{\sim}{H}}_G = \Sigma \underset{\sim}{M}_G \end{aligned}\right\} \text{moment due to external forces}$$

Rate of change of angular momentum

$\boxed{\begin{array}{l} O : \text{Fixed point} \\ G : \text{Centre of mass} \end{array}}$

What is the form of above eq^n at any other point P ?



$\underset{\sim}{r}_i'$: Position vector of i^{th} particle w.r.t P

$$\underset{\sim}{r}_i' = \underset{\sim}{\bar{\rho}} + \underset{\sim}{\rho}_i$$

$$\underset{\sim}{H}_P = \Sigma\, \underset{\sim}{r}_i' \times (m_i \underset{\sim}{v}_i) = \Sigma\, \underset{\sim}{r}_i' \times (m_i \dot{\underset{\sim}{r}}_i)$$

$$= \Sigma (\underset{\sim}{\bar{\rho}} + \underset{\sim}{\rho}_i) \times (m_i \dot{\underset{\sim}{r}}_i)$$

$$= \underset{\sim}{\bar{\rho}} \times \Sigma m_i \dot{\underset{\sim}{r}}_i + \Sigma\, \underset{\sim}{\rho}_i \times m_i \dot{\underset{\sim}{r}}_i$$

$$\boxed{\mathbf{H}_P = \bar{\boldsymbol{\rho}} \times \underbrace{(m\bar{\mathbf{v}})}_{\text{(Lin. momentum)}} + \mathbf{H}_G = \bar{\boldsymbol{\rho}} \times \mathbf{G} + \mathbf{H}_G}$$

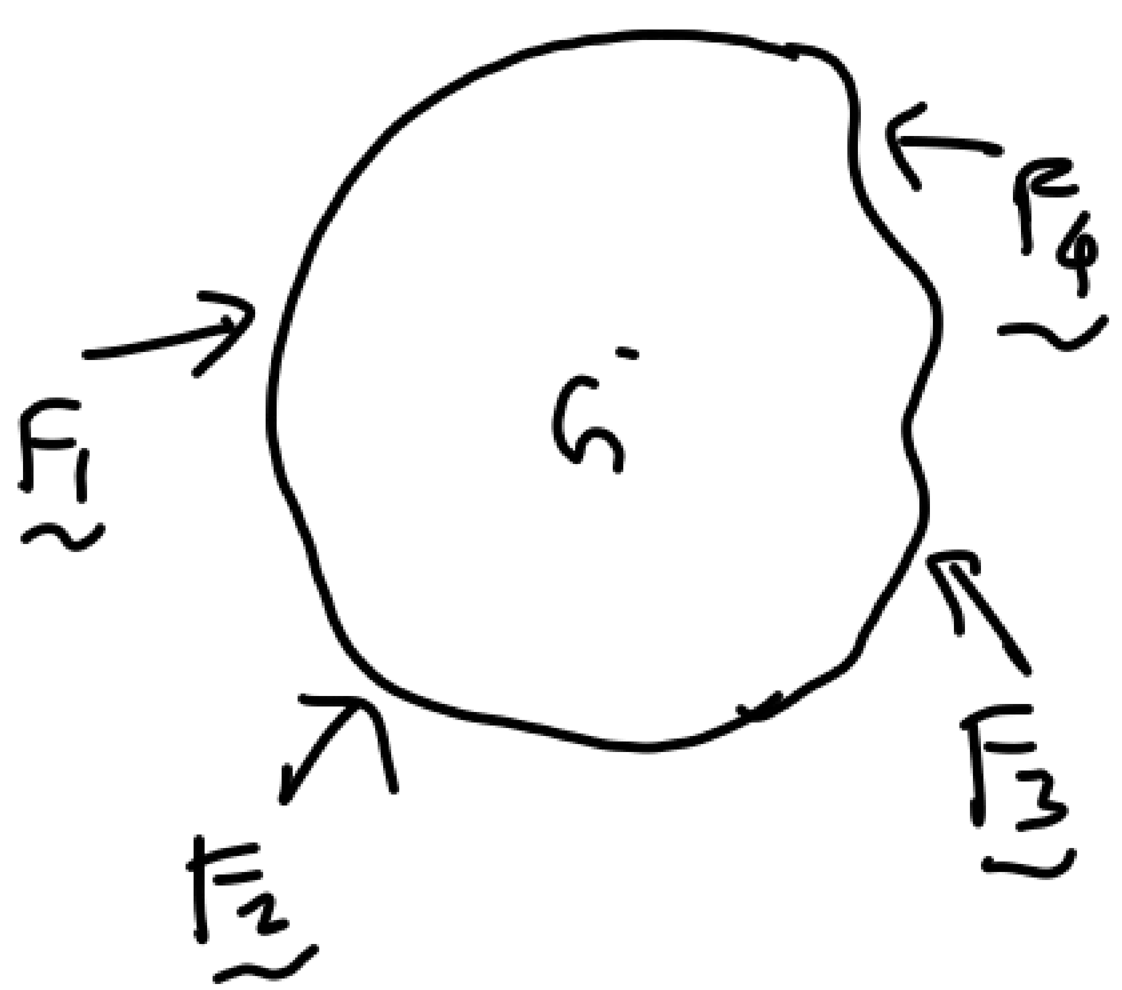
Def'n of angular momentum at point other than $\mathbf{G}$

Moment @ P

$$\Rightarrow \mathbf{M}_P = \mathbf{M}_G + \bar{\boldsymbol{\rho}} \times \mathbf{F}$$

$$\boxed{\mathbf{M}_P = \dot{\mathbf{H}}_G + \bar{\boldsymbol{\rho}} \times m\mathbf{a}}$$

<u>Moment @ P in terms of moment @ G</u>



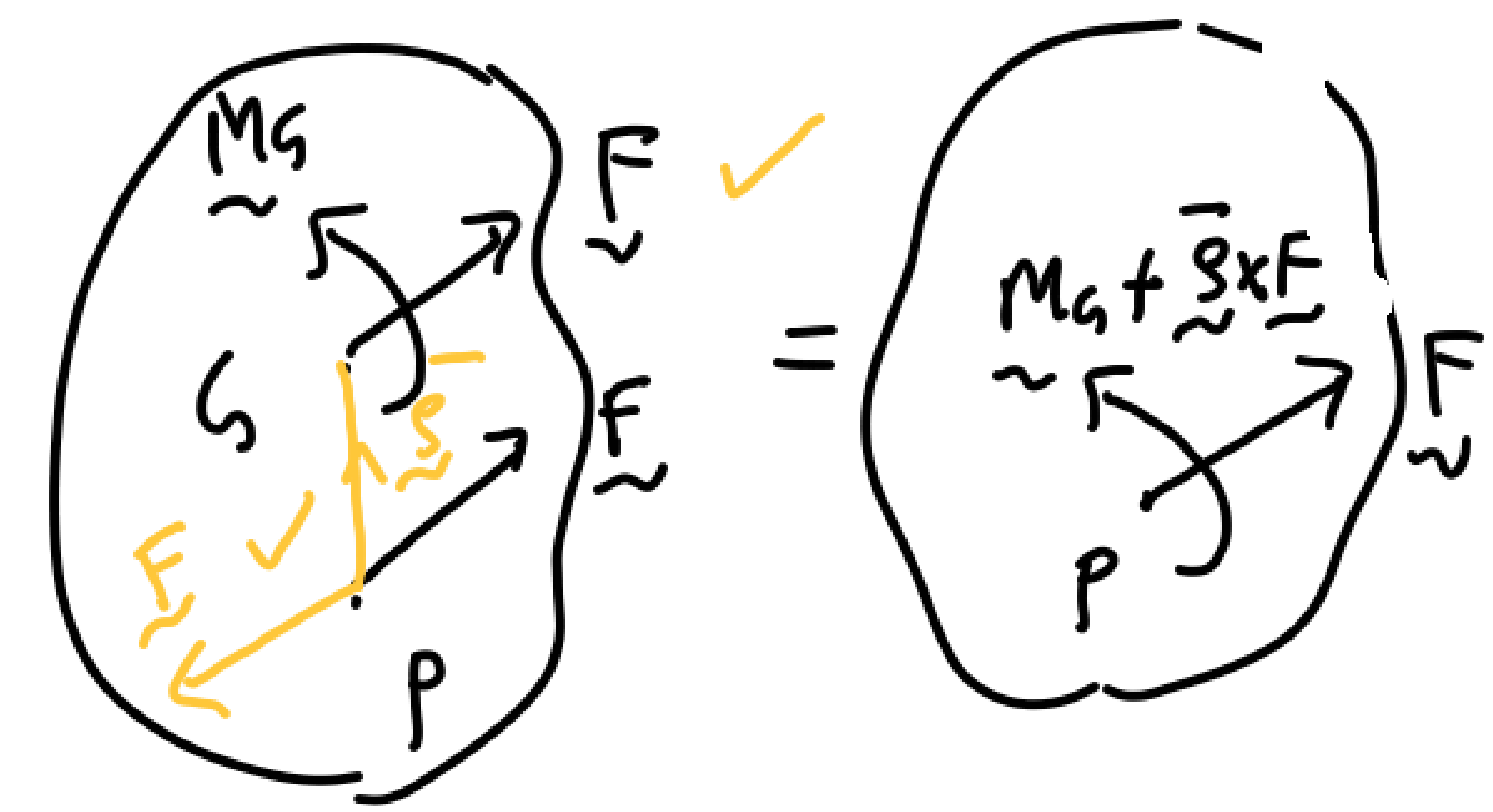
$\equiv$

$$\mathbf{F} = \sum_i \mathbf{F}_i$$

$$\mathbf{M}_G = \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i$$

Moment @ P in terms of moment @ G

$=$



Rate of change of $\underset{\sim}{H}_P$:

$$\underset{\sim}{H}_P^{rel} = \sum_i \underset{\sim}{r}_i' \times \left(m_i \dot{\underset{\sim}{r}}_i'\right)$$

(Relative angular momentum)

Relative linear momentum

$$\dot{\underset{\sim}{H}}_P^{rel} = \sum_i \dot{\underset{\sim}{r}}_i' \times \cancelto{0}{\left(m_i \dot{\underset{\sim}{r}}_i'\right)} + \sum_i \underset{\sim}{r}_i' \times \cancelto{0}{\left(\dot{m}_i \dot{\underset{\sim}{r}}_i'\right)} + \sum \underset{\sim}{r}_i' \times \left(m_i \ddot{\underset{\sim}{r}}_i'\right)$$

$$\dot{\underset{\sim}{H}}_P^{rel} = \sum \left(\bar{\underset{\sim}{\rho}} + \underset{\sim}{\rho}_i\right) \times m_i \left(\ddot{\bar{\underset{\sim}{\rho}}} + \ddot{\underset{\sim}{\rho}}_i\right)$$

$$= \sum \bar{\underset{\sim}{\rho}} \times m_i \ddot{\bar{\underset{\sim}{\rho}}} + \sum \bar{\underset{\sim}{\rho}} \times m_i \ddot{\underset{\sim}{\rho}}_i + \sum \underset{\sim}{\rho}_i \times m_i \ddot{\bar{\underset{\sim}{\rho}}} + \sum \underset{\sim}{\rho}_i \times m_i \ddot{\underset{\sim}{\rho}}_i$$

$$= \bar{\underset{\sim}{\rho}} \times \left(\sum m_i\right) \ddot{\bar{\underset{\sim}{\rho}}} + \bar{\underset{\sim}{\rho}} \times \cancelto{0}{\left(\sum m_i \ddot{\underset{\sim}{\rho}}_i\right)} + \cancelto{0}{\left(\sum m_i \underset{\sim}{\rho}_i\right)} \times \ddot{\bar{\underset{\sim}{\rho}}} + \sum \underset{\sim}{\rho}_i \times m_i \ddot{\underset{\sim}{\rho}}_i$$

$$= \bar{\underset{\sim}{\rho}} \times m \ddot{\bar{\underset{\sim}{\rho}}} + \sum \underset{\sim}{\rho}_i \times m_i \ddot{\underset{\sim}{\rho}}_i$$

$$= \bar{\underset{\sim}{\rho}} \times m \ddot{\bar{\underset{\sim}{\rho}}} + \underbrace{\dot{\underset{\sim}{H}}_G}_{\underset{\sim}{M}_G}$$

$$\boxed{\underset{\sim}{r}_i = \bar{\underset{\sim}{r}} + \underset{\sim}{\rho}_i}$$

$$\underset{\sim}{H}_G^{rel} = \sum \underset{\sim}{\rho}_i \times m_i \dot{\underset{\sim}{\rho}}_i$$

$$\underset{\sim}{H}_G = \sum \underset{\sim}{\rho}_i \times m_i \dot{\underset{\sim}{r}}_i$$

$$\boxed{\underset{\sim}{H}_G = \underset{\sim}{H}_G^{rel}}$$

$$\underset{\sim}{H}_G = \sum \underset{\sim}{\rho}_i \times (m_i \underset{\sim}{v}_i)$$

$$= \sum \underset{\sim}{\rho}_i \times (m_i \dot{\underset{\sim}{r}}_i)$$

$$= \sum \underset{\sim}{\rho}_i \times m_i (\dot{\bar{\underset{\sim}{r}}} + \dot{\underset{\sim}{\rho}}_i)$$

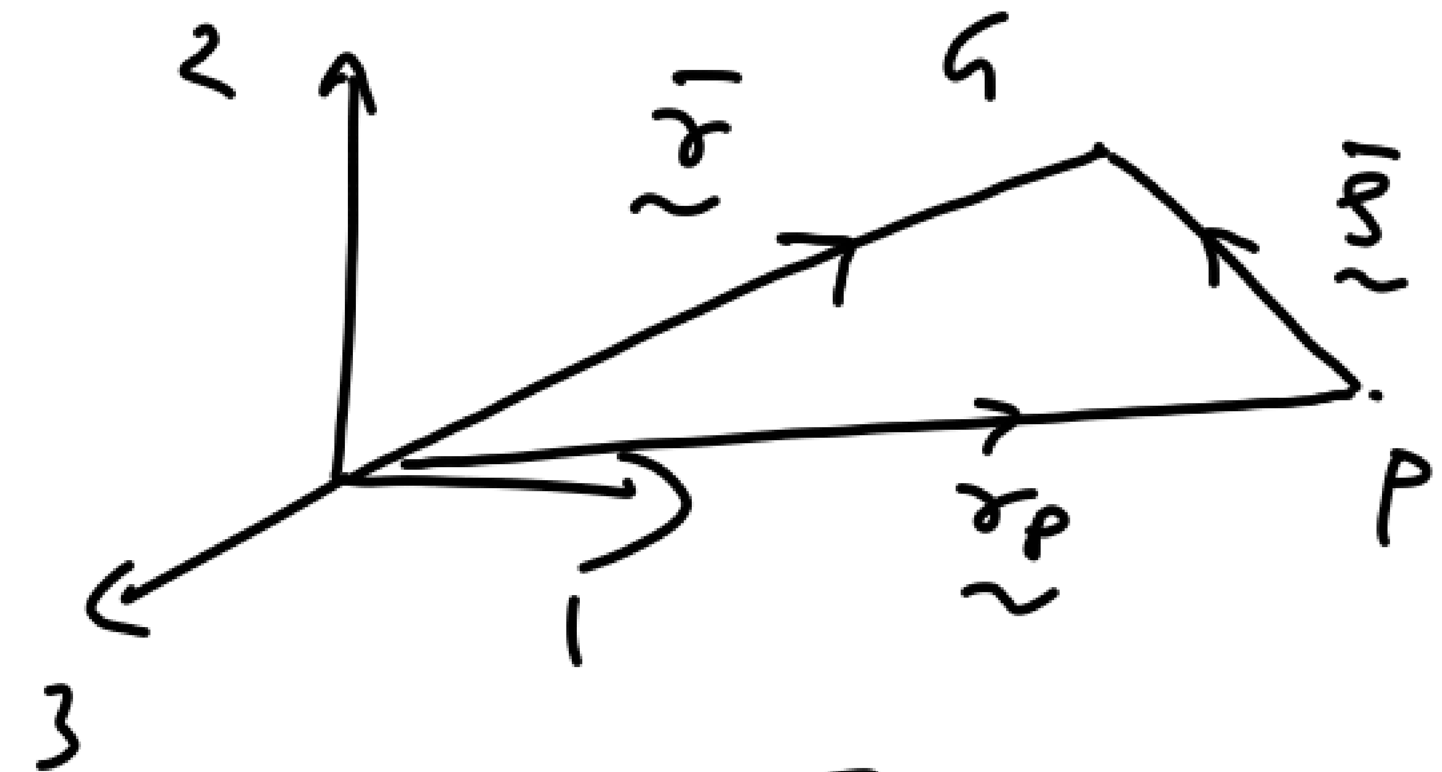
$$= \sum (\underset{\sim}{\rho}_i \times m_i \dot{\underset{\sim}{\rho}}_i)$$

$$\dot{\underset{\sim}{H}}{}_P^{rel} = \bar{\underset{\sim}{\rho}} \times m \ddot{\bar{\underset{\sim}{\rho}}} + \underset{\sim}{M}_G$$

$$\underset{\sim}{M}_P = \underset{\sim}{M}_G + \bar{\underset{\sim}{\rho}} \times m \bar{\underset{\sim}{a}}$$

$$\dot{\underset{\sim}{H}}{}_P^{rel} = \bar{\underset{\sim}{\rho}} \times m \ddot{\bar{\underset{\sim}{\rho}}} + \underset{\sim}{M}_P - \bar{\underset{\sim}{\rho}} \times m \bar{\underset{\sim}{a}}$$

$$\underset{\sim}{M}_P = \dot{\underset{\sim}{H}}{}_P^{rel} + \bar{\underset{\sim}{\rho}} \times m (\bar{\underset{\sim}{a}} - \ddot{\bar{\underset{\sim}{\rho}}})$$



$$\bar{\underset{\sim}{r}} = \bar{\underset{\sim}{\rho}} + \underset{\sim}{r}_p$$

$$\underset{\sim}{r}_p = \bar{\underset{\sim}{r}} - \bar{\underset{\sim}{\rho}}$$

$$\therefore \ddot{\underset{\sim}{r}}_p = \underset{\sim}{a}_p = \ddot{\bar{\underset{\sim}{r}}} - \ddot{\bar{\underset{\sim}{\rho}}}$$

$$= \bar{\underset{\sim}{a}} - \ddot{\bar{\underset{\sim}{\rho}}}$$

$$\therefore \underset{\sim}{M}_P = \dot{\underset{\sim}{H}}{}_P^{rel} + \bar{\underset{\sim}{\rho}} \times m \underset{\sim}{a}_p$$

Balance of angular momentum w.r.t P

(*)

Most general form.

If $P = G$

$$\bar{\underline{\rho}} = \underline{0}$$

$$\underline{M}_G = \dot{\underline{H}}_G^{rel} = \dot{\underline{H}}_G$$

If $P = O$

$$\underline{a}_P = \underline{0}$$

$$\underline{M}_O = \dot{\underline{H}}_O^{rel}$$

$$\underline{M}_O = \dot{\underline{H}}_O$$

Summary

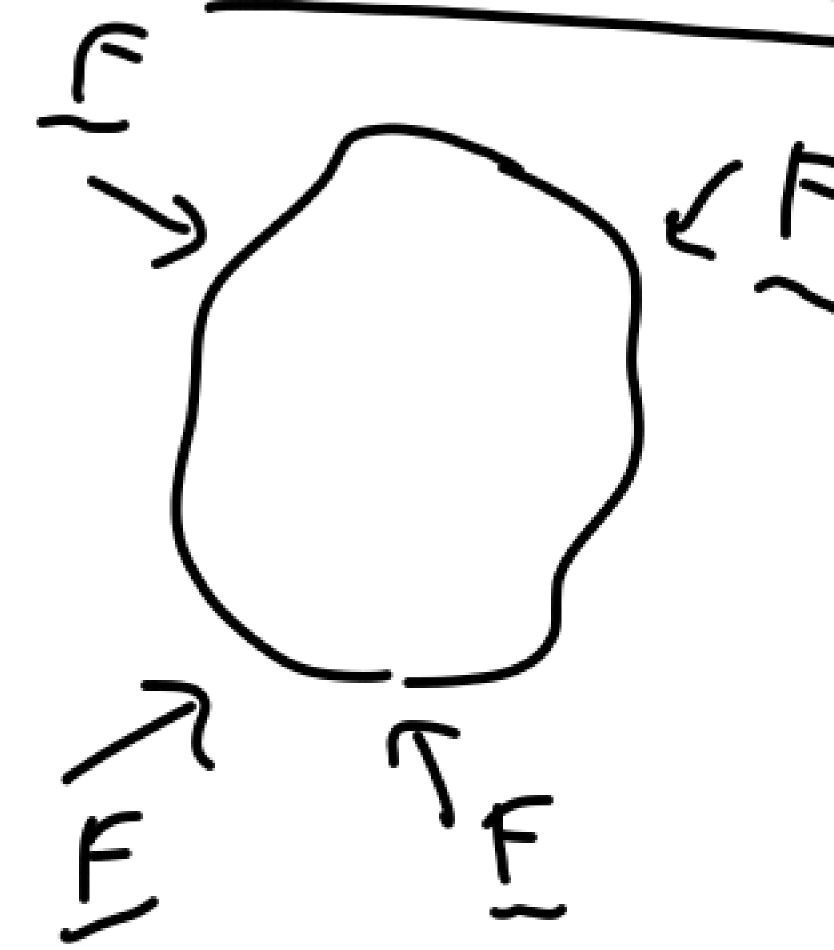
1. $\underline{F} = m\bar{\underline{a}}$
2. $\Sigma \underline{M}_O = \dot{\underline{H}}_O$

$$\Sigma \underline{M}_G = \dot{\underline{H}}_G$$

$$\Sigma \underline{M}_P = \dot{\underline{H}}_P^{rel} + \bar{\underline{\rho}} \times m\underline{a}_P$$

These laws carry forward to system of ∞ particles i.e. a continuous body

Planar kinetics



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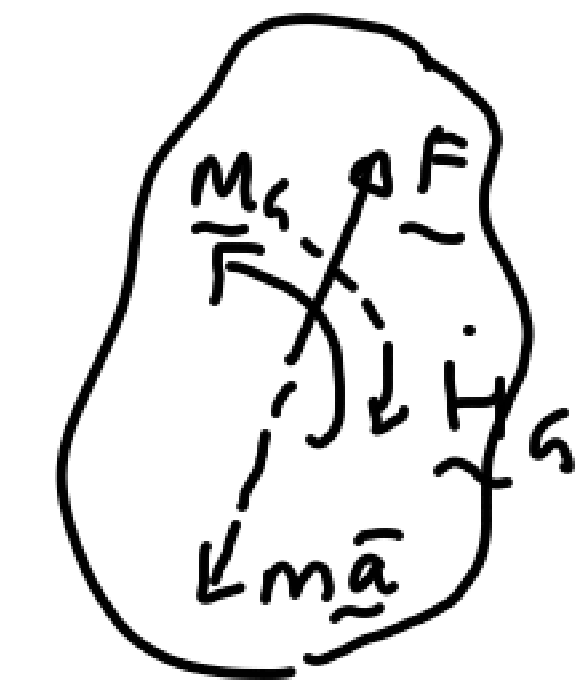


Free body diagram

Kinetic diagram

Angular momentum

$$\underline{H}_G = \sum_i \underline{\rho}_i \times (m_i \dot{\underline{\rho}}_i)$$

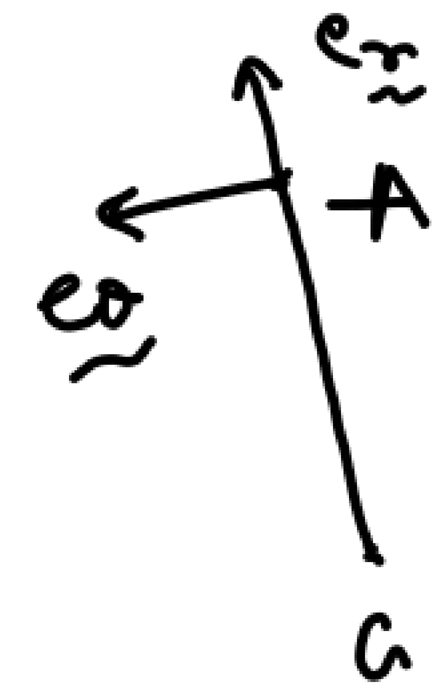
$$= \int_m \underline{\rho} \times (dm\, \dot{\underline{\rho}})$$

$\underset{\sim}{\rho}$ } Position vector of "dm" mass w.r.t G i.e. centre of mass

$\dot{\underset{\sim}{\rho}}$ = Velocity of "dm" mass w.r.t G

=

$\to$ dm (small infinitesimal mass surrounding A)



$\underset{\sim}{e_3} = \underset{\sim}{e_r} \times \underset{\sim}{e_\theta}$

$$\dot{\underset{\sim}{\rho}} = \underset{\sim}{V}_{A/G} = \underset{\sim}{\omega} \times \underset{\sim}{\rho}$$

$$\underset{\sim}{H}_G = \int_m \underset{\sim}{\rho} \times \left(dm\left(\underset{\sim}{\omega} \times \underset{\sim}{\rho}\right)\right)$$

For planar motion

$$\underset{\sim}{\rho} = \rho\, \underset{\sim}{e_r}$$

$$\underset{\sim}{\omega} = \omega\, \underset{\sim}{e_3}$$

$$\underset{\sim}{\omega} \times \underset{\sim}{\rho} = \omega\, \underset{\sim}{e_3} \times (\rho\, \underset{\sim}{e_r})$$

$$= \omega \rho\, \underset{\sim}{e_\theta}$$

$$\underset{\sim}{\rho} \times (\underset{\sim}{\omega} \times \underset{\sim}{\rho}) = \rho\, \underset{\sim}{e_r} \times (\omega\rho)\, \underset{\sim}{e_\theta}$$

$$= \omega \rho^2\, \underset{\sim}{e_3}$$

$$\underset{\sim}{H}_G = \left(\int_m \omega \rho^2\, dm\right) \underset{\sim}{e_3}$$

$$= \omega\left(\int_m \rho^2\, dm\right) \underset{\sim}{e_3}$$

$$I_G = \underbrace{\int \rho^2 \, dm}$$

is the mass moment of inertia

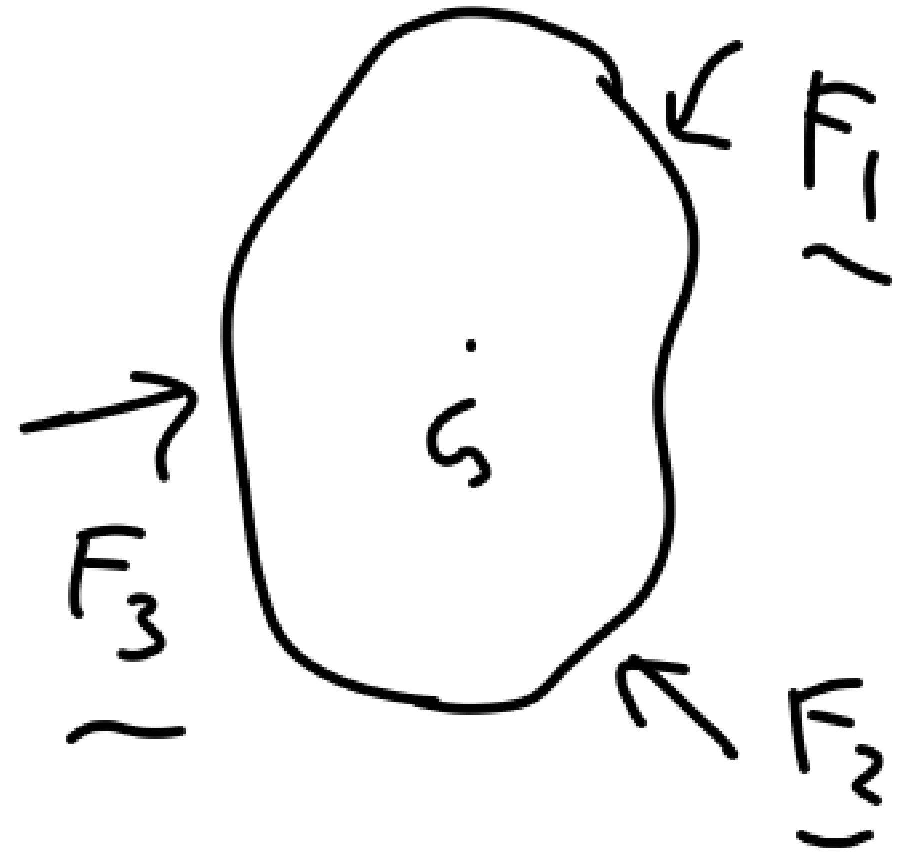
$$\underset{\sim}{H_G} = (I_G \omega)\, \underset{\sim}{e_3}$$

↳ so $\underset{\sim}{H_G}$ has only the out-of-plane component.

$$H_G = I_G \omega$$

$$\dot{H}_G = I_G \dot{\omega}$$

$$\boxed{\dot{H}_G = I_G \alpha}$$



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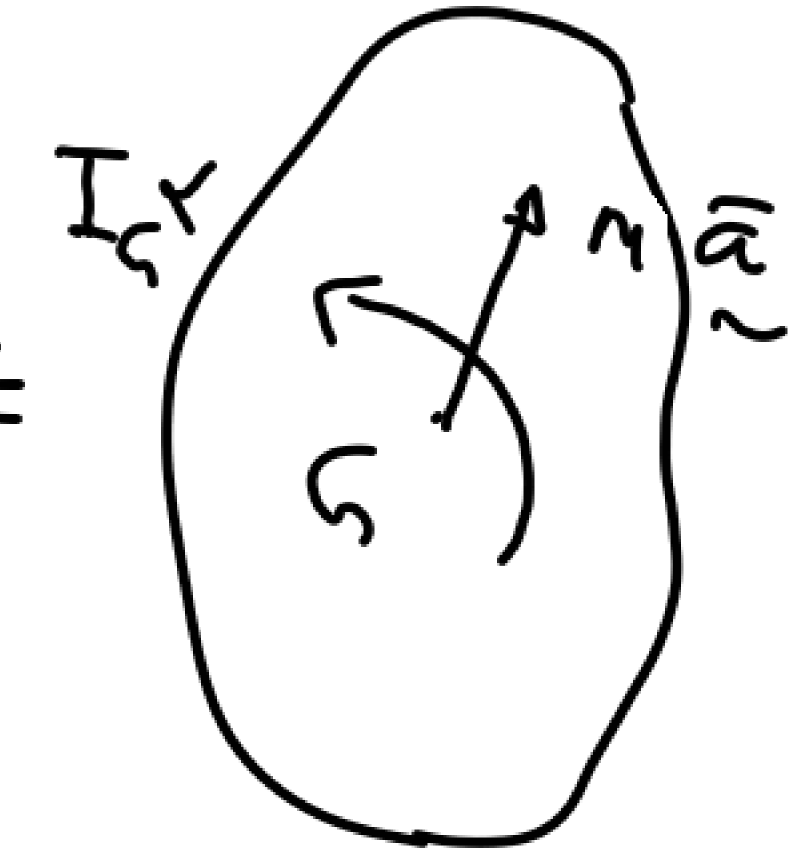
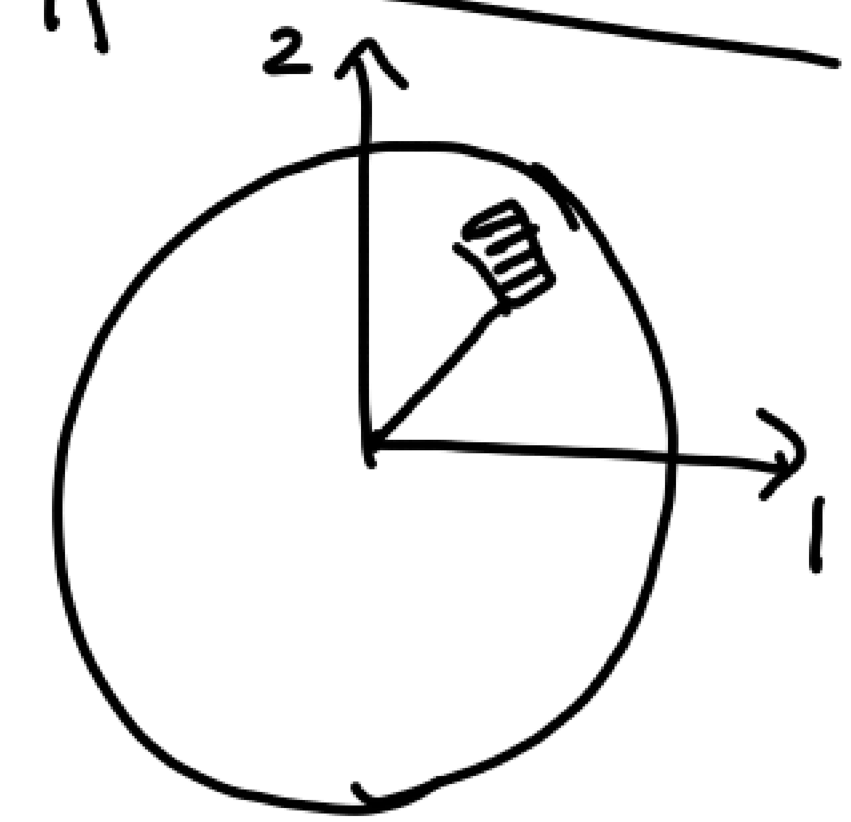


Kinetic diagram

If the geometry is a uniform circular disc of radius "a":



$$I_G = \int \rho^2 \, dm$$

$$dm = \eta \, dV$$

↳ mass density

$$= \eta \, (\rho \, d\rho \, d\theta)\, t \rightarrow \text{out-of-plane thickness}$$

$$I_G = \int_0^a \int_0^{2\pi} (\eta \rho \, d\rho \, d\theta)\, \rho^2 t$$

$$= (2\pi \eta t) \int_0^a \rho^3 \, d\rho$$

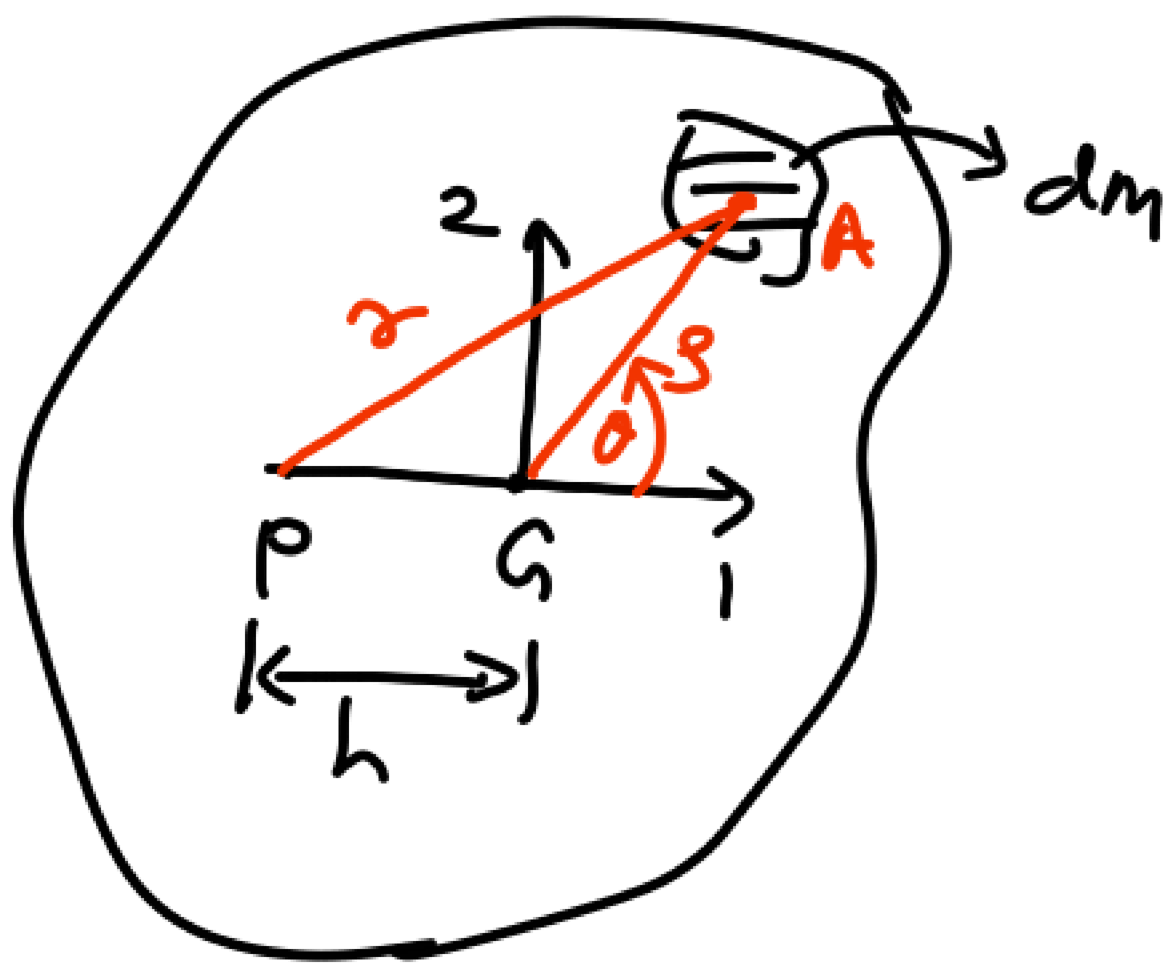
$$= 2\pi \eta t \left(\frac{\rho^4}{4}\right)_0^a$$

$$= \cancel{2}\pi \eta t \frac{a^4}{\cancel{4}\,2}$$

$$= \left[(\pi a^2) t\right] \eta \frac{a^2}{2}$$

$$= \frac{m a^2}{2}$$

For any other point



$$I_P = \int r^2 dm$$

$$I_G = \int \rho^2 dm$$

In $\triangle PGA$:

$$\cos \angle PGA = \frac{h^2+\rho^2-r^2}{2h\rho}$$

$$\angle PGA = \pi - \theta$$

$$\cos(\pi-\theta) = \frac{h^2+\rho^2-r^2}{2h\rho}$$

$$r^2 = h^2+\rho^2-2h\rho\cos(\pi-\theta)$$

$$r^2 = h^2+\rho^2+2h\rho\cos\theta$$

$$\therefore I_P = \int_m r^2 dm$$

$$= \int_m h^2 dm + \int_m \rho^2 dm + \int_m 2h\rho\cos\theta\, dm$$

$$= mh^2 + I_G + 2h\underbrace{\left(\int_m x_1\, dm\right)}_{0}$$

$$\therefore \boxed{I_P = I_G + mh^2}$$

Parallel axis theorem

Forms of angular momentum

$$\dot{\underset{\sim}{H}}_G = \underset{\sim}{M}_G \Rightarrow M_G = I_G \alpha$$

$$\dot{\underset{\sim}{H}}_O = \underset{\sim}{M}_O \Rightarrow M_O = I_O \alpha$$

$$\underset{\sim}{M}_P = \dot{\underset{\sim}{H}}{}_P^{rel} + \underset{\sim}{\bar{\rho}} \times m \underset{\sim}{a}_P$$

↳ This translates to

$$M_P = I_P \alpha + m a_P \bar{\rho}_\perp$$

$$\underline{a_P \rho_\perp} = \underset{\sim}{e}_3 \cdot (\underset{\sim}{\bar{\rho}} \times \underset{\sim}{a}_P)$$

$$\underset{\sim}{H}{}_P^{rel} = \sum \underset{\sim}{r}_i' \times (m_i \dot{\underset{\sim}{r}}{}_i')$$

$$= \int \underset{\sim}{r}' \times (dm\, \dot{\underset{\sim}{r}}')$$

$\underset{\sim}{r}'$] Position vector of "dm" mass w.r.t P

$$\dot{\underset{\sim}{r}}' = \underset{\sim}{\omega} \times \underset{\sim}{r}'$$

$$\underset{\sim}{H}{}_P^{rel} = \int_m \underset{\sim}{r}' \times (\underset{\sim}{\omega} \times \underset{\sim}{r}')\, dm$$

For planar case:

$$\underset{\sim}{r}' \times (\underset{\sim}{\omega} \times \underset{\sim}{r}') = \omega (r')^2 \underset{\sim}{e}_3$$