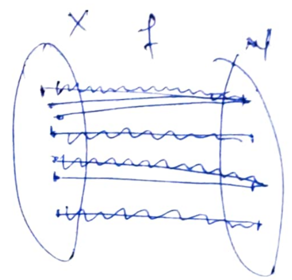
Observation: Suppose $f: X \to Y$ is a surjective/onto function, then $|Y| \leq |X|$, that is, there is exist an injective ~~fro~~ function from Y to X.

Proof:



Let $g: Y \to X$ be defined as follows,

for an arbitrary $y \in Y$, there must exist $x \in X$ s.t. $f(x) = y$.

We chose such an x and set $g(y) = x$.

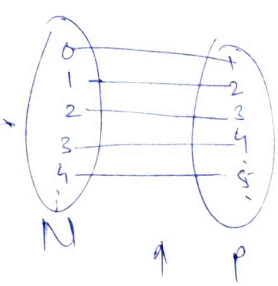
We need to show that g is injective

Question: If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$?

$\longrightarrow$ CSB Th^m

Def^n: A set is countable if either finite or there is a bijection/one to one correspondence bet^n X and the set of positive integers.

$$\boxed{\begin{aligned} &\mathbb{N} = \{0, 1, 2, 3, \dots\} \\ &P = \{1, 2, 3, \dots\} \\ &\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} \\ &\mathbb{Q} = \{ p/q \mid q \neq 0 \text{ \& } p, q \in \mathbb{Z} \} \end{aligned}}$$

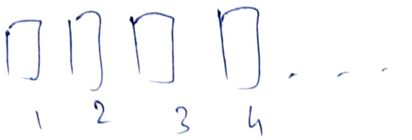


Bijection

$f: \mathbb{N} \to P$

Ex. ① any finite set
② P
③ Is $\mathbb{N}$ countable? (Yes)

Hilbert's grand hotel:



new guy } to accomodate shift n to $n+1$.

} to accomodate infinite guys shift n to $2n$.

(4) $\mathbb{Z}$ is countable. $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

Observation: An infinite set X is countable if and only if one can list all the elements of X in a sequence.

$x_1, x_2, x_3, x_4, \ldots$

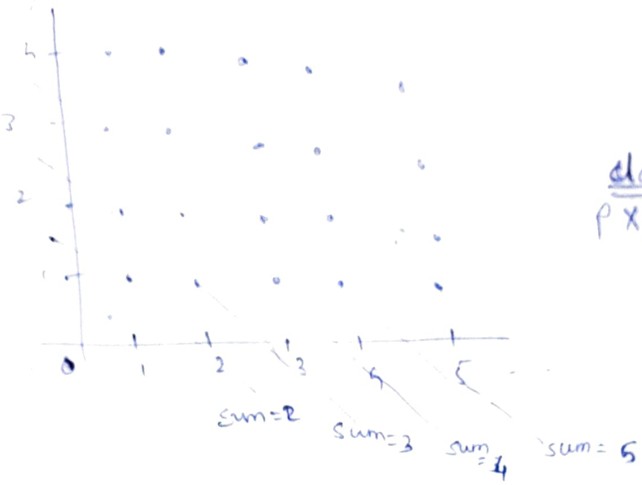
Proof: Let $f: X \longrightarrow \mathbb{P}$ is bijection.

bijective f^n's are invertible.

Lemma: If X & Y are countable, then $X \cup Y$ is also countable.
($X \times Y$ is also countable) } See book for proof

$\mathbb{Q}$ = the rationals

$\mathbb{Q}^+$ = positive rationals. = $\{ p/q \mid q \neq 0 \text{ \& } p/q > 0 \}$



claim:
$P \times P$ is countable.

To get the index of (m,n), we will encounter finitely many anti diagonals each on which contains finite points, hence we can ~~not~~ get index of (m,n).

$$|X| = |P| = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$$

* If X is countable, then so is every subset of X.

* A set X is called uncountable if it is not countable.

Cantor's theorem:

X is a set ; $P(X) = 2^X$ = power set of X

$$2^X = \{ Y \mid Y \in X \}$$

If X is finite ; $|2^X| = 2^{|X|}$ = finite

$2^{\mathbb{N}}$ is uncountable. $\Rightarrow$ $|\mathbb{N}| \neq |2^{\mathbb{N}}|$

Cantor's th^m : There is no bijection bet^n $\mathbb{N}$ and $2^{\mathbb{N}}$.

Proof: Suppose, for contradiction, that $2^{\mathbb{N}}$ is countable.

Let, $A_0, A_1, A_2, A_3, \ldots$ be an ennumeration / listing of $2^{\mathbb{N}}$.

Let An infinite matrix,

$$
\begin{array}{c|ccccccc|l}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \ldots & \\
\hline
A_0 \rightarrow & 1 & 0 & 1 & 1 & 1 & 0 & 1 \ldots & \text{row} \rightarrow \text{subsets in the ennumeration} \\
A_1 \rightarrow & 0 & 0 & 1 & 0 & 1 & 0 & 0 \ldots & i^{th} \text{ row} = A_i \\
A_2 \rightarrow & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ldots & \text{column} - \text{element of } \mathbb{N} \\
A_3 \rightarrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots & j^{th} \text{ column} \rightarrow j \in \mathbb{N} \\
\vdots & & & & & 1 & & & \\
\hline
\text{char}(B) & 0 & 1 & 1 & 1 & 0 & \ldots & &
\end{array}
$$

$(i,j)^{th}$ entry $= 1 \rightarrow$ if $j \in A_i$

$= 0 \rightarrow$ if $j \notin A_i$

Define $B \subseteq \mathbb{N}$ as follows,

$$B = \{ i \in \mathbb{N} \mid (i,i)^{th} \text{ entry} = \cancel{1}\ 0 \} = \{ i \in \mathbb{N} \mid i \notin A_i \}$$

For ~~a~~ assumed infinite matrix,

$$B = \{ 1, 2, 3, \ldots \}$$

claim: There is no $i \in \mathbb{N}$ such that ~~$B \notin A_i$~~ $B = A_i$

~~Proof~~: Proof for claim:

characteristic vector of $B = \{0, 1, 1, 1, 0, \ldots\}$

$char(B) \neq i^{th}$ row of matrix

$\therefore B \neq A_i$

But, $B \subseteq N$.

→ This provides a contradiction to the fact that $A_0, A_1, A_2, \ldots$ is an ennumeration of 2^N. (assumption)

$$\rightarrow |N| < |2^N| \overset{?}{<} |2^{(2^N)}|$$

Cantor's thm:

Let X be any set. Then $|X| < |2^X|$. — injection betn X & 2^X.

Proof: We shall show that $|X| \neq |2^X|$. As $|X| \leq |2^X|$, this will imply that $|X| < |2^X|$.

We will prove by contradiction that there is no bijection betn X & 2^X.

Suppose for contradiction, that there is a bijection $f: X \to 2^X$.

We define a subset $Y \subseteq X$ as follows:

$$Y = \{ x \in X \mid x \notin f(x) \}$$

claim: there is no $z \in X$ such that $f(z) = Y$

Proof of claim: Let $w \in X$ be arbitrary. consider the subset $f(w) \subseteq X$ and the following two exhaustive cases:

case 1 : $w \in f(w)$ $\therefore$ by defn of Y, $w \notin Y$

$\therefore \boxed{f(w) \neq Y}$

Case 2 : $w \notin f(w) \Rightarrow$ by def^n of Y, $w \in Y$

Hence, $f(w) \neq Y$

This concludes the proof of the claim.

So, this provides a contradiction to the fact assumption that f is a bijection from X to 2^X.

Therefore, there is no bijection from X to 2^X.

$|2^{\mathbb{N}}| = |\mathbb{R}|$ real numbers ? } Book.

Experiment-2

CS207

- → 3 quizzes (10% each) }
- — midsem (30%) } ±5%
- — endsem (50%) }

— Book: Kenneth Rosen, Applied discrete maths.

— formal proof to reason about discrete structures.

— Topics: (1) Sets, functions, relations, ...
(2) Counting techniques – inclusion-exclusion, recurrence relations, generating functions, bijective proofs.
(3) Partially ordered sets.
(4) Graph theory.
(5) Group theory. (counting modulo symmetries)

* Sets *

— A set is a "collection" of objects/things.

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

— X is a set, then $P(X) = 2^X = \{Y \mid Y \subseteq X\}$

e.g. $X = \{1, 2, 3\}$; $P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \ldots\}$

— Russell's paradox:

Consider the following set

$B = \{A \mid A \notin A\}$; is $B \overset{?}{\in} B$?

Ans: suppose $B \in B$, then by defn of B, $B \notin B$.
suppose $B \notin B$, then by defn of B, $B \in B$

- X is a set.

  $P(X) = 2^X = \{Y \mid Y \subseteq X\}$

- X, Y are sets,

  $X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$

* Functions *

- A function from X to Y ($f: X \to Y$) is a rule which assigns to every element of X a unique element of Y.
- $x \longmapsto f(x)$

Function composition:

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$

$f \circ g : X \to Z \quad ; \quad (f \circ g)(x) = g(f(x))$

Lemma: $(f \circ g) \circ h = f \circ (g \circ h)$ } Associativity

Proof: we will show that $\forall x \in X$,

$$((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$$

we know, $(f' \circ g')(x) = g'(f'(x))$

Fix $x \in X$,

$$h((f \circ g)(x)) = h(g(f(x)))$$

$$(f \circ (g \circ h))(x) = (g \circ h)(f(x)) = h(g(f(x)))$$

Experiment - 2

CS207

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- midsem (30%)
- endsem (50%)

} ±5%

- Book: Kenneth Rosen, Applied discrete maths.
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