Example 1: Mortar Formula

Let $y_{11}, y_{12}, y_{13}, .. y_{1n1}$ be n_1 observations from the first factor level (Modified Mortar)

and $y_{21}, y_{22}, y_{23}, .. y_{2n1}$ be n_2 observations from the second factor level (UNmodified Mortar)

What is the hypothesis test?

A simple statistical model to describe the data is

$$y_{ij} = \mu_i + \epsilon_{ij} \begin{cases} i = 1, 2 \\ j = 1, 2, \ldots, n_i \end{cases}$$

where y_{ij} is the jth observation from factor level i, μ_i is the mean of the response at the ith factor level, and ϵ_{ij} is a normal random variable associated with the ijth observation.

Ref: Design and Analysis of Experiments, 8th Ed.

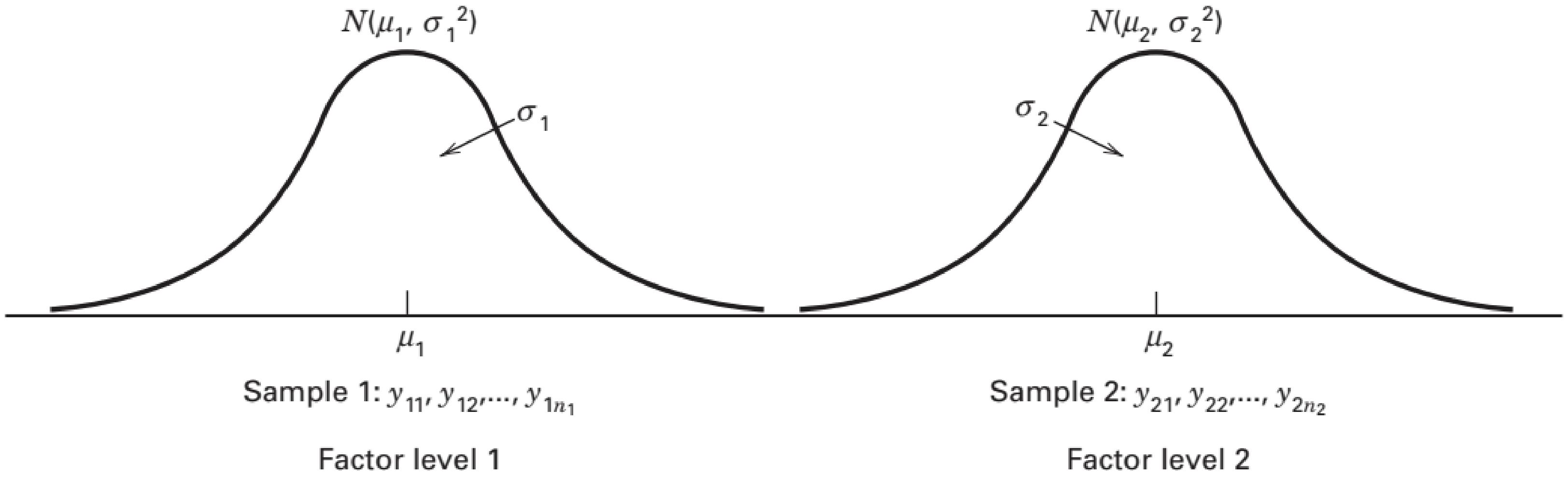
■ **TABLE 2.1**
Tension Bond Strength Data for the Portland Cement Formulation Experiment

j	Modified Mortar y_{1j}	Unmodified Mortar y_{2j}
1	16.85	16.62
2	16.40	16.75
3	17.21	17.37
4	16.35	17.12
5	16.52	16.98
6	17.04	16.87
7	16.96	17.34
8	17.15	17.02
9	16.59	17.08
10	16.57	17.27

Example 1: Mortar Formula

We assume that the random error components ϵ_{1j} and ϵ_{2j} are normally distributed with means 0 and variances σ_1^2 and σ_2^2

Which would follow that the y_{1j} and y_{2j} are normally distributed with means μ_1 *and* μ_2 and variances σ_1^2 and σ_2^2



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7	16.96	17.34
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9	16.59	17.08
10	16.57	17.27

Example 1: Mortar Formula

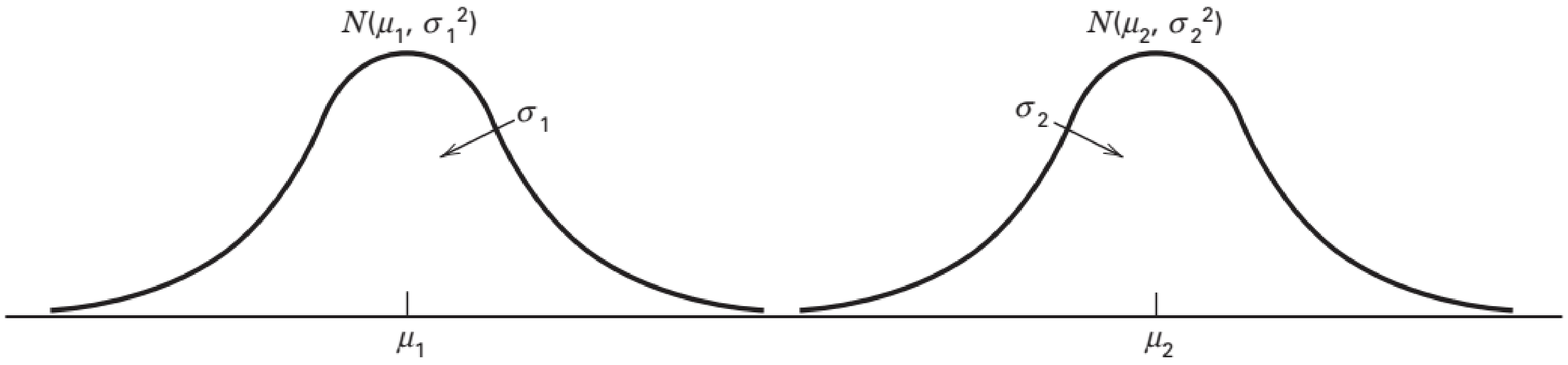
Now the question is whether μ_1 & μ_2 are statistically different

Hypothesis Testing

$H_0: \mu_1 = \mu_2$ Null Hypothesis

$H_1: \mu_1 \neq \mu_2$ Alternate Hypothesis (two-sided)

$\mu_1 < \mu_2$ or if $\mu_1 > \mu_2$.



Sample 1: $y_{11}, y_{12}, \ldots, y_{1n_1}$

Factor level 1

Sample 2: $y_{21}, y_{22}, \ldots, y_{2n_2}$

Factor level 2

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10	16.57	17.27

Two-Sample t-Test

Suppose that we could assume that the variances of tension bond strengths were identical for both mortar formulations. $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Then the **appropriate test statistic** to use for comparing two treatment means in the completely randomized design is

$\frac{SS}{V} = S^2$

$\sum y_{1j} - \bar{y}_1 = 0$

$\sum y_{2j} - \bar{y}_2 = 0$

$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{1/n_1 + 1/n_2}}$

Pooled Var $= \frac{SS}{V}$

$$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$= \frac{SS}{V}$

$= \frac{SS_1 + SS_2}{n_1 + n_2 - 2}$

$= \frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}$

Where

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

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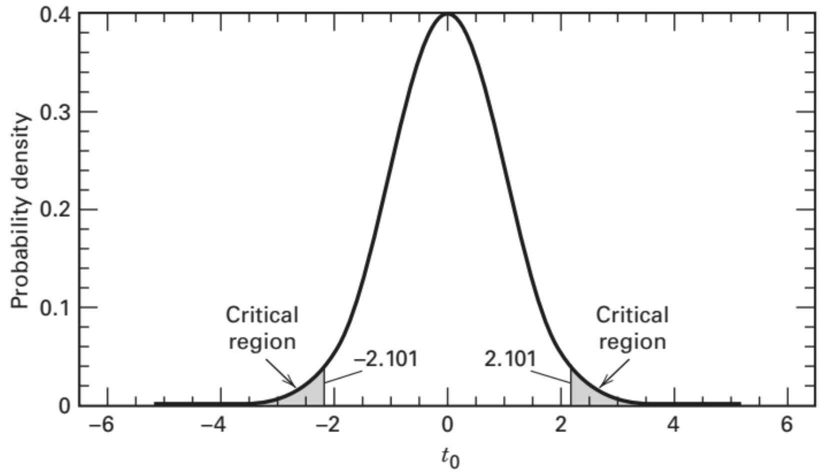
S_1^2 $\bar{y}_1$ S_2^2 $\bar{y}_2$

S_p^2 is an estimate of the common variance $\sigma_1^2 = \sigma_2^2 = \sigma^2$

t-Test

Two-Sample t-Test Procedure

$$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$



1) To determine whether to reject $H_0\colon \mu_1 = \mu_2$, we would compare t_0 to the t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

2) If $t_0 > t_{\frac{\alpha}{2}, n_1+n_2-2}$ OR $t_0 < -t_{\frac{\alpha}{2}, n_1+n_2-2}$, then we will reject $H_0\colon \mu_1 = \mu_2$

Justification of Two-Sample t-Test

If we were sampling from two independent normal distributions, then the distribution of $\overline{y_1} - \overline{y_2}$ will be a normal distribution with mean $\mu_1 - \mu_2$ and variance $\sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)$

If σ^2 were known, and if $H_0: \mu_1 = \mu_2$ were true, then the Z_0 distribution would be a normal distribution with mean 0 and variance 1

$$Z_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

But since we do NOT know σ^2, we use S_p^2

and the normal distribution changes to t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

Two-Sample t-Test

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{1/n_1 + 1/n_2}} = \frac{16.76 - 17.04}{\sqrt{0.081}\sqrt{\frac{2}{10}}} = -2.21$$

In this example

Modified Mortar	Unmodified Mortar
$\bar{y}_1 = 16.76$ kgf/cm^2	$\bar{y}_2 = 17.04$ kgf/cm^2
$S_1^2 = 0.100$	$S_2^2 = 0.061$
$S_1 = 0.316$	$S_2 = 0.248$
$n_1 = 10$	$n_2 = 10$

$$S_p^2 = \frac{S_1^2(n_1-1) + S_2^2(n_2-1)}{n_1+n_2-2} = \frac{0.1\times 9 + 0.061\times 9}{18}$$

$$= 0.161/2 = 0.0805 \rightarrow S_p =$$

■ **TABLE 2.1**
Tension Bond Strength Data for the Portland Cement Formulation Experiment

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7	16.96	17.34
8	17.15	17.02
9	16.59	17.08
10	16.57	17.27

Two-Sample t-Test

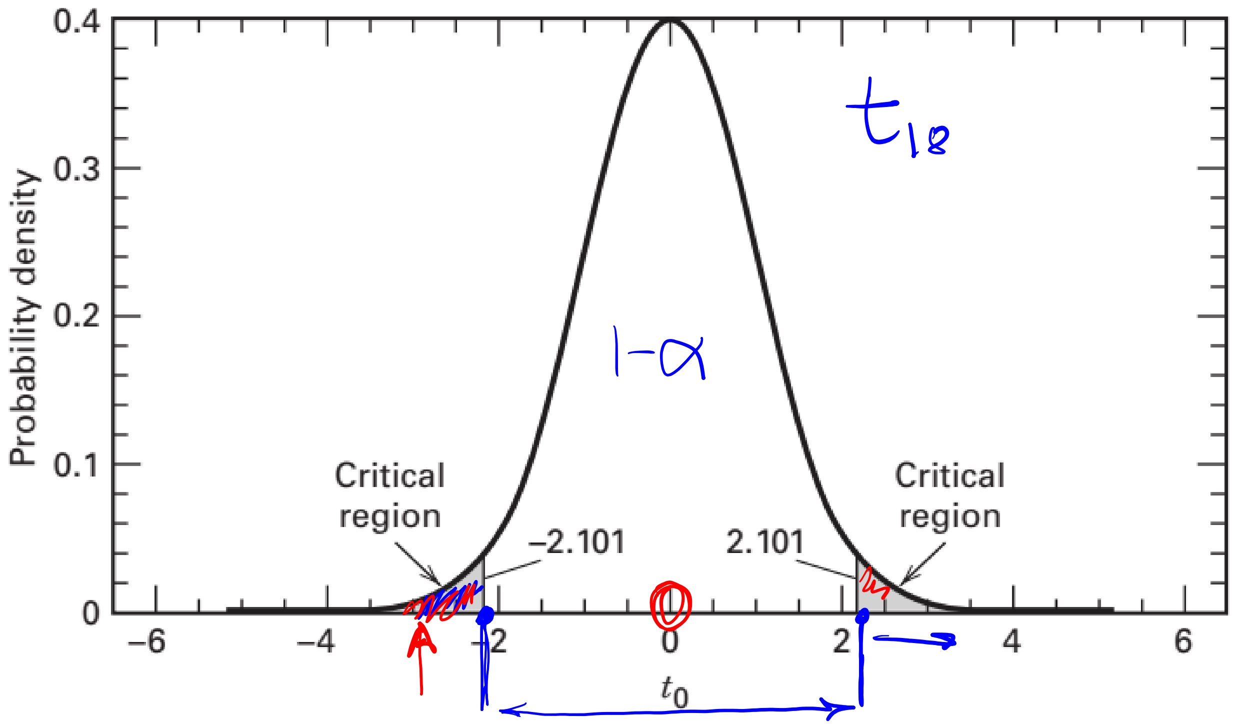
$\alpha = 5\%$

In this example

Modified Mortar	Unmodified Mortar
$\bar{y}_1 = 16.76\ \text{kgf/cm}^2$	$\bar{y}_2 = 17.04\ \text{kgf/cm}^2$
$S_1^2 = 0.100$	$S_2^2 = 0.061$
$S_1 = 0.316$	$S_2 = 0.248$
$n_1 = 10$	$n_2 = 10$

your $t_0 = -2.21$

H_0 is Rejected.



■ **FIGURE 2.10** **The *t* distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$**

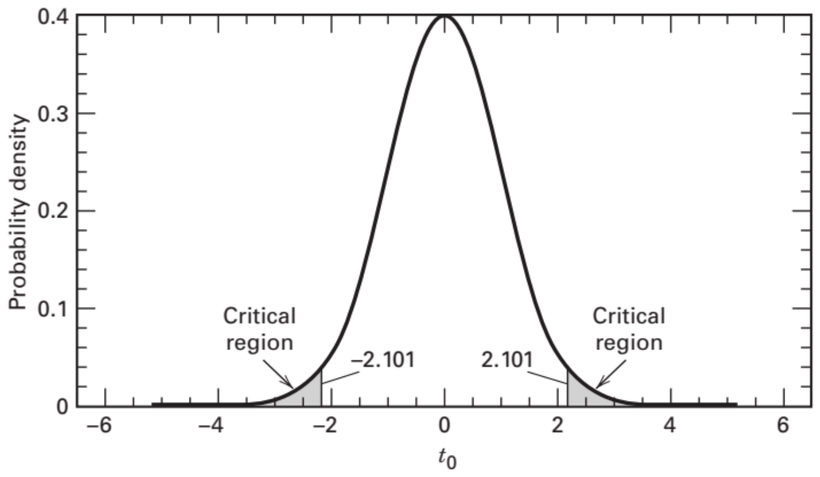
Furthermore, $n_1 + n_2 - 2 = 10 + 10 - 2 = 18$, and if we choose $\alpha = 0.05$, then we would reject $H_0\colon \mu_1 = \mu_2$ if the numerical value of the test statistic $t_0 > t_{0.025,18} = 2.101$, or if $t_0 < -t_{0.025,18} = -2.101$. These boundaries of the critical region are shown on the reference distribution (t with 18 degrees of freedom) in Figure 2.10.

t-Test Calculations

Two-Sample t-Test

In this example

Modified Mortar	Unmodified Mortar
$\bar{y}_1 = 16.76 \text{ kgf/cm}^2$	$\bar{y}_2 = 17.04 \text{ kgf/cm}^2$
$S_1^2 = 0.100$	$S_2^2 = 0.061$
$S_1 = 0.316$	$S_2 = 0.248$
$n_1 = 10$	$n_2 = 10$



■ **FIGURE 2.10** The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

$$= \frac{9(0.100) + 9(0.061)}{10 + 10 - 2} = 0.081$$

$$S_p = 0.284$$

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{16.76 - 17.04}{0.284\sqrt{\frac{1}{10} + \frac{1}{10}}}$$

$$= \frac{-0.28}{0.127} = -2.20$$

We Reject $H_0: \mu_1 = \mu_2$ at Significance level of 0.05

Two-Sample t-Test

In this example, we concluded that we Reject $H_0: \mu_1 = \mu_2$ at significance level of α = 0.05

Do you see any problem/limitation of this?

For example, what will be the conclusion if the significance level is 0.04 or 0.03 or 0.01?

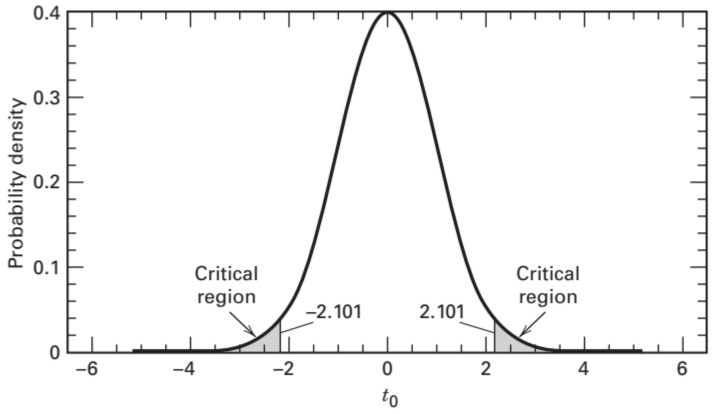
We do not know whether the test-statistic t0 lies just barely in the rejection region OR very far into the rejection region

Thus, we can specify P-value, which is the minimum significance value which will

Result in rejection of the null hypothesis

For example, in the mortar experiments, the null hypothesis will be rejected for

any level of significance > 0.0411



■ **FIGURE 2.10** The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

Concept of Confidence Interval

- Given a random sample of '*n*' observations from some process of interest and an estimate of the process mean, it is of interest to make some statement about the *"goodness" of that sample mean*, as an estimate of μ, i.e., the degree of belief or confidence that can be placed on it.
- One way of approaching this problem is through *the concept of the confidence interval.*

- **Remember:** Distribution of sample means is a normal distribution (CLT)
- *That means,* for random samples of size 'n' drawn from a population, we expect that 95% of all sample means will be within an interval of $\mu \pm 1.96$ standard deviations of the distribution of the sample mean, i.e., $\mu \pm \frac{1.96\sigma_x}{\sqrt{n}}$

 https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Concept of Confidence Interval

In other words, $\bar{y} \pm \frac{1.96\sigma_y}{\sqrt{n}}$ is called a 95% confidence interval for the true mean μ

$\mu \pm 1.96\,\sigma_{\bar{y}}$

$\bar{y}$

In general,

$\boldsymbol{\bar{y} \pm \left(z_{1-\frac{\alpha}{2}}\right)\frac{\sigma_y}{\sqrt{n}}}$ **is a 100*(1-α)% confidence interval for the true mean μ**

μ

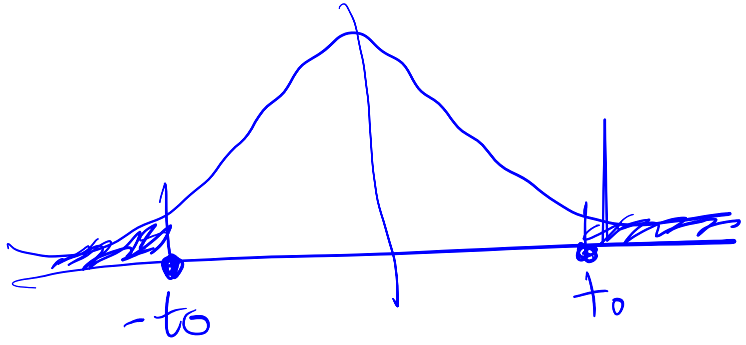
$\bar{y} \pm 1.96\,\sigma_{\bar{y}}$

95%

When sample size is small and σ_y is UNKNOWN,

the confidence interval is given by $\boldsymbol{\bar{y} \pm \left(t_{v,1-\frac{\alpha}{2}}\right)\frac{s}{\sqrt{n}}}$

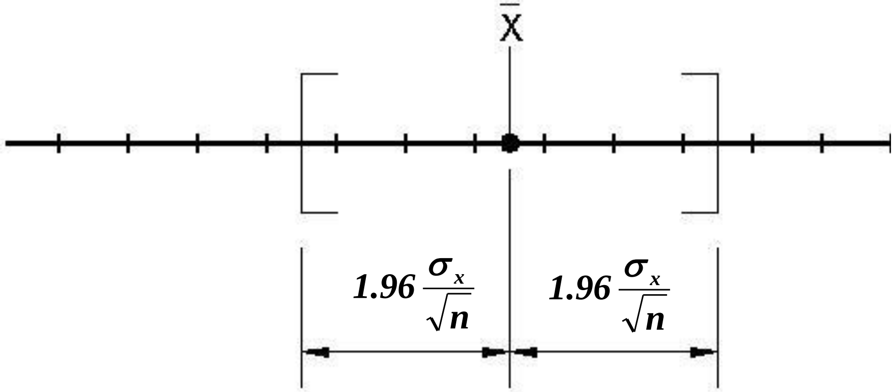
Where v = n-1 is the degree of freedom

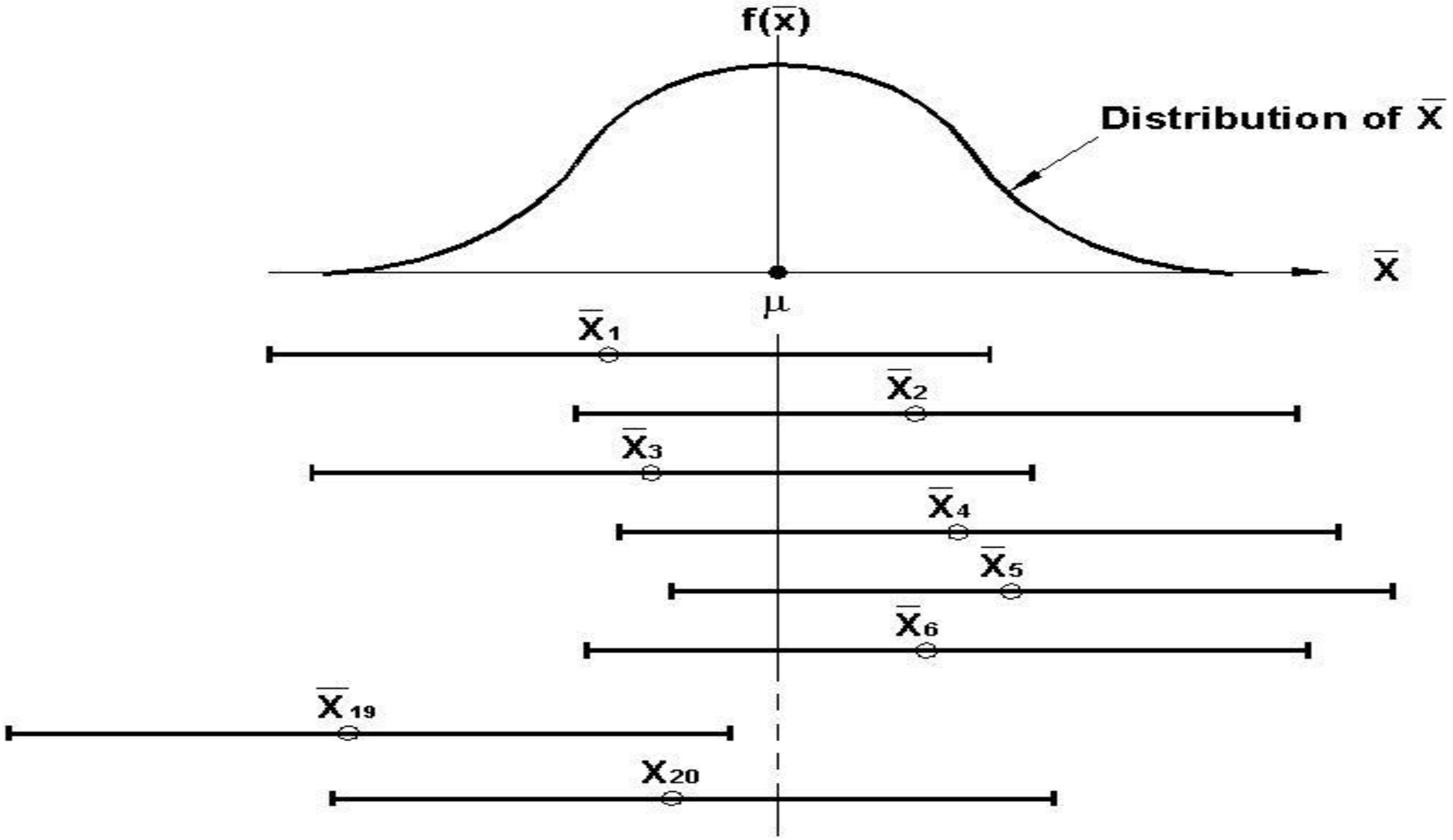


$t_{v,\alpha/2} = -t_{v,1-\alpha/2}$

https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Confidence Interval





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Confidence Interval Approach

CEP2022_Notebook (2.1.5)

To define a confidence interval, suppose that θ is an unknown parameter. To obtain an interval estimate of θ, we need to find two statistics L and U such that the probability statement

$$P(L \leqslant \theta \leqslant U) = 1 - \alpha \qquad \textbf{(2.27)}$$

is true. The interval

$$L \leqslant \theta \leqslant U \qquad \textbf{(2.28)}$$

is called a **100(1 − α) percent confidence interval** for the parameter θ. The interpretation of this interval is that if, in repeated random samplings, a large number of such intervals are constructed, $100(1 - \alpha)$ percent of them will contain the true value of θ. The statistics L and U are called the **lower** and **upper confidence limits**, respectively, and $1 - \alpha$ is called the **confidence coefficient**. If $\alpha = 0.05$, Equation 2.28 is called a 95 percent confidence interval for θ. Note that confidence intervals have a frequency interpretation; that is, we do not know if the statement is true for this specific sample, but we do know that the *method* used to produce the confidence interval yields correct statements $100(1 - \alpha)$ percent of the time.

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8	17.15	17.02
9	16.59	17.08
10	16.57	17.27

Suppose that we wish to find a $100(1 - \alpha)$ percent confidence interval on the true difference in means $\mu_1 - \mu_2$ for the Portland cement problem. The interval can be derived in the following way. The statistic

$$\frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

is distributed as $t_{n_1+n_2-2}$. Thus,

$$P\left(-t_{\alpha/2,n_1+n_2-2} \le \frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \le t_{\alpha/2,n_1+n_2-2}\right) = 1 - \alpha$$

or

$$P\left(\bar{y}_1 - \bar{y}_2 - t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \right.$$

$$\left. \le \bar{y}_1 - \bar{y}_2 + t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) = 1 - \alpha$$

Comparing Equations 2.29 and 2.27, we see that

$$\bar{y}_1 - \bar{y}_2 - t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2$$

$$\le \bar{y}_1 - \bar{y}_2 + t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

is a $100(1 - \alpha)$ percent confidence interval for $\mu_1 - \mu_2$.

$-t_{\alpha/2} = t_{1-\alpha/2}$

$\Delta y = y_1 - y_2$

$\overline{\Delta y} = \bar{y}_1 - \bar{y}_2$

$$\frac{\overline{\Delta y} - \Delta\mu}{S_p\sqrt{1/n_1 + 1/n_2}}$$

The actual 95 percent confidence interval estimate for the difference in mean tension bond strength for the formulations of Portland cement mortar is found by substituting in Equation 2.30 as follows:

$$16.76 - 17.04 - (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}} \leq \mu_1 - \mu_2$$

$$\leq 16.76 - 17.04 + (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}}$$

$$-0.28 - 0.27 \leq \mu_1 - \mu_2 \leq -0.28 + 0.27$$

$$-0.55 \leq \mu_1 - \mu_2 \leq -0.01$$

95% $\alpha = 5\%$

$\mu_1 \neq \mu_2$

$\mu_1 - \mu_2 \neq 0$

Note that because $\mu_1 - \mu_2 = 0$ is *not* included in this interval, the data do not support the hypothesis that $\mu_1 = \mu_2$ at the 5 percent level of significance (recall that the P-value for the two-sample t-test was 0.042, just slightly less than 0.05).

~~H_0~~

Example

Given that 9 bearings made by a certain process have an average diameter of 0.305 cm and the sample standard deviation of 0.003 cm, construct a 99 % confidence interval for the true mean diameter of bearings made by the process. What is the width of the confidence interval? $2 \times \frac{0.003}{3}\, 3.35$

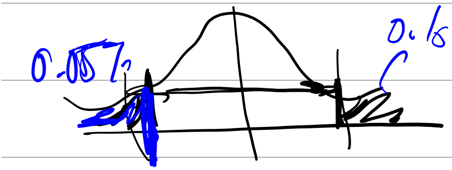
$n = 9$ $\quad \bar{y} = 0.305$ $\quad s = 0.003$

$\alpha = 1\%$

$\dfrac{\bar{y} - \mu}{s/\sqrt{n}}$ will follow t dist with doF = 8

$$-3.35 \le \frac{\bar{y} - \mu}{s/\sqrt{n}} \le 3.35$$

1%



$$\bar{y} - \frac{s}{\sqrt{n}}\, 3.35 \le \mu \le \bar{y} + \frac{s}{\sqrt{n}}\, 3.35$$

$$0.305 \pm \frac{0.003}{3}\, 3.35$$

Degrees of freedom (ν)	Amount of area in one tail (α)							
	0.0005	0.001	0.005	0.010	0.025	0.050	0.100	0.200
1	636.6192	318.3088	63.65674	31.82052	12.70620	6.313752	3.077684	1.376382
2	31.59905	22.32712	9.924843	6.964557	4.302653	2.919986	1.885618	1.060660
3	12.92398	10.21453	5.840909	4.540703	3.182446	2.353363	1.637744	0.978472
4	8.610302	7.173182	4.604095	3.746947	2.776445	2.131847	1.533206	0.940965
5	6.868827	5.893430	4.032143	3.364930	2.570582	2.015048	1.475884	0.919544
6	5.958816	5.207626	3.707428	3.142668	2.446912	1.943180	1.439756	0.905703
7	5.407883	4.785290	3.499483	2.997952	2.364624	1.894579	1.414924	0.896030
8	5.041305	4.500791	3.355387	2.896459	2.306004	1.859548	1.396815	0.888890
9	4.780913	4.296806	3.249836	2.821438	2.262157	1.833113	1.383029	0.883404
10	4.586894	4.143700	3.169273	2.763769	2.228139	1.812461	1.372184	0.879058
11	4.436979	4.024701	3.105807	2.718079	2.200985	1.795885	1.363430	0.875530
12	4.317791	3.929633	3.054540	2.680998	2.178813	1.782288	1.356217	0.872609
13	4.220832	3.851982	3.012276	2.650309	2.160369	1.770933	1.350171	0.870152
14	4.140454	3.787390	2.976843	2.624494	2.144787	1.761310	1.345030	0.868055
15	4.072765	3.732834	2.946713	2.602480	2.131450	1.753050	1.340606	0.866245
16	4.014996	3.686155	2.920782	2.583487	2.119905	1.745884	1.336757	0.864667
17	3.965126	3.645767	2.898231	2.566934	2.109816	1.739607	1.333379	0.863279

https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Choice of Sample Size

- Selection of appropriate sample size 'n' is critical in any experimental design
- In the previous example, have a look at the length of 100*(1-α)% confidence interval for difference in means ($\mu_1 - \mu_2$)
- It was determined by

$$t_{\alpha/2,\, n_1+n_2-2}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$\bar{y}_1 - \bar{y}_2 - t_{\alpha/2, n_1+n_2-2}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2$$

$$\leq \bar{y}_1 - \bar{y}_2 + t_{\alpha/2, n_1+n_2-2}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

is a $100(1-\alpha)$ percent confidence interval for $\mu_1 - \mu_2$.

- What is the effect of sample size on this width?
  - Say n1 = n2 = n, and α = 0.05, Sp could be anything (we don't have control over it)
  - So essentially, the width is a function of

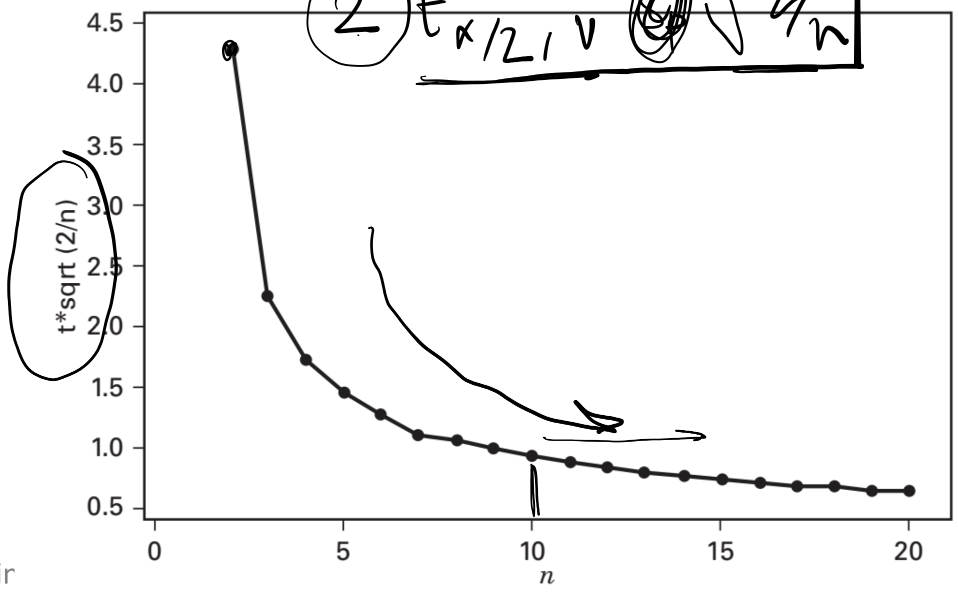
95%

μ

$\bar{y}$

$$t_{\alpha/2,\, 2n-2}\, S_p \sqrt{\frac{2}{n}}$$

$2\, t_{\alpha/2, \nu}\, \sigma \sqrt{2/n}$



if you n ↑ width of conf. int ↓