Kinematics

Monday, 8 January 2024 10:33 AM

⇒ Kinematics of Particles

Displacement $= \vec{r}_2 - \vec{r}_1$, where $\vec{r}$: position vector

Inertial Frame : Fixed coordinate System (fixed to the Earth)

Velocity $= \dfrac{d\vec{r}}{dt}$ Acceleration $= \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$

Rectilinear motion : Motion along a straight line.

In cartesian coordinate system, $\dfrac{d\vec{e}_i}{dt} = 0$ ($\vec{e}_i$: unit coordinate vector)

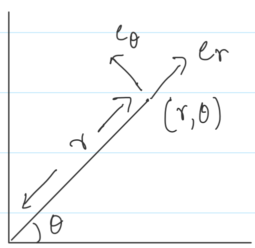
$\vec{r} = (x, y, z)$ $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$, $\vec{a} = (\ddot{x}, \ddot{y}, \ddot{z})$

Curvilinear motion : Along a curve (dimension > 1)

Similarly, $\vec{v}(t) = \int_0^t \vec{a}(\tau)\, d\tau + \vec{v}(0)$

$$\vec{x}(t) = \int_0^t \vec{v}(t)\, dt + \vec{x}(0)$$

→ Polar coordinates :



$x = r\cos\theta$ $y = r\sin\theta$

(e_r, e_θ) → Curvilinear coordinate System

$\hat{e}_r = \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2$ $\hat{e}_\theta = -\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_2$

$$\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = 0$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = -\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_2 = \hat{e}_\theta = \frac{d\hat{e}_r}{d\theta}$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\theta\,\hat{e}_1 - \sin\theta\,\hat{e}_2 = -\hat{e}_r = \frac{d\hat{e}_\theta}{d\theta}$$

$$\boxed{\vec{r}(t) = r(t)\, e_r}$$

$$\vec{v}(t) = \frac{d(r(t)\,e_r)}{dt} = \dot{r}(t)\,e_r + r(t)\cdot\frac{d(e_r)}{dt}$$

$$\boxed{\vec{v}(t) = (\dot{r}(t))e_r + (r(t)\,\dot{\theta})e_\theta}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot{r}\,e_r + \dot{r}\dot{\theta}\,e_\theta + \dot{r}\dot{\theta}\,e_\theta + r\ddot{\theta}\,e_\theta - r(\dot{\theta})^2 e_r$$

$$\boxed{\vec{a} = (\ddot{r} - r(\dot{\theta})^2)\,e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,e_\theta}$$

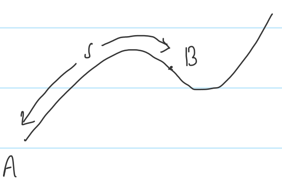
Circular Motion : $\dot{r} = 0$

$\Rightarrow$ $\vec{v} = r\dot{\theta}\,e_\theta = (r\omega)\,e_\theta$

$\Rightarrow$ $\vec{a} = -r(\dot{\theta})^2 e_r + r\ddot{\theta}\,e_\theta = -(\omega^2 r)\,e_r + (r\alpha)\,e_\theta$

ω: angular velocity α: angular acceleration

$\Rightarrow$ Tangential - Normal coordinate system :



s: Arc Length

$$\vec{r}(t) = \vec{r}(s(t))$$

$\vec{v}(t) = \frac{d\vec{r}}{ds}\cdot\frac{ds}{dt}$ $e_t = \frac{d\vec{r}}{ds}$ = unit tangent vector (its magnitude = 1 $\because |d\vec{r}| = |ds|$)

$$\vec{a}(t) = \frac{d^2\vec{r}}{ds^2}\cdot\left(\frac{ds}{dt}\right)^2 + \frac{d\vec{r}}{ds}\cdot\frac{d^2 s}{dt^2}$$

Example: Helix $-$ $\vec{r} = a\cos t\,\hat{e}_1 + a\sin t\,\hat{e}_2 + ct\,\hat{e}_3$

Arc length $\quad ds = \sqrt{d\vec{r}\cdot d\vec{r}}$

$$d\vec{r} = -a\sin t\,dt\,\hat{e}_1 + a\cos t\,dt\,\hat{e}_2 + c\hat{e}_3$$
$$d\vec{r}\cdot d\vec{r} = (a^2+c^2)(dt)^2$$

$$\therefore\ ds = \sqrt{a^2+c^2}\,dt \quad \rightarrow \quad s = \left(\sqrt{a^2+c^2}\right)t \quad \rightarrow \quad t = \frac{s}{\sqrt{a^2+c^2}}$$

$$\vec{r}(s) = a\cos\left(s/(a^2+c^2)^{1/2}\right)\hat{e}_1 + a\sin\left(s/(a^2+c^2)^{1/2}\right)\hat{e}_2 + \frac{cs}{(a^2+c^2)^{1/2}}\hat{e}_3$$

$$\vec{v} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \left[-\frac{a}{(a^2+c^2)^{1/2}}\cdot\sin\left(\frac{s}{(a^2+c^2)^{1/2}}\right)\hat{e}_1 + \frac{a}{(a^2+c^2)^{1/2}}\cos\left(\frac{s}{(a^2+c^2)^{1/2}}\right)\hat{e}_2 + \frac{c}{(a^2+c^2)^{1/2}}\right]($$

$$\vec{v} = -a\sin\left(s/(a^2+c^2)^{1/2}\right)\hat{e}_1 + a\cos\left(s/(a^2+c^2)^{1/2}\right)\hat{e}_2 + c\hat{e}_3$$

$$\hat{e}_t = \frac{d\vec{r}}{ds} \qquad \hat{e}_t\cdot\hat{e}_t = 1$$

Differentiate wrt s.

$$\frac{d\hat{e}_t}{ds}\cdot\hat{e}_t = 0 \qquad \text{i.e.}\quad k\hat{e}_n = \frac{d\hat{e}_t}{ds} \Rightarrow$$

k: curvature

$\hat{n}$: normal vector

Summary: Serret - Ferret frame ($e_t - e_n$ frame)

$$\vec{r}(t) = \vec{r}(s(t)), \quad \vec{v} = \dot{s}e_t, \quad \vec{a} = \ddot{s}e_t + \frac{(\dot{s})^2}{\rho}e_n$$

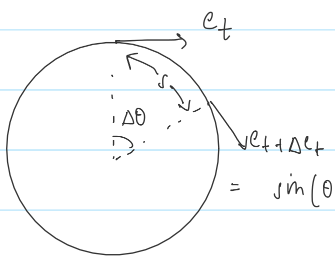
$e_t = \dfrac{d\vec{r}}{ds} \Rightarrow$ unit tangent vector

$\dfrac{de_t}{ds} = ke_n \Rightarrow$ k : curvature $= 1/R$ $\quad$ R: Radius of curvature

e_n: normal vector $\quad$ Note that e_n points towards centre of the circle.

The out of plane normal is called binormal.

$b = \hat{e}_t \times \hat{e}_n$



$$\frac{de_t}{ds} = \lim_{\Delta s \to 0} \frac{\Delta e_t}{\Delta s}$$

$$= \sin(\theta - \Delta\theta) e_1 - \cos(\theta - \Delta\theta)\, e_2$$

$$\Delta e_t = \left(\sin(\theta - \Delta\theta) - \sin\theta\right) e_1 + \left(\cos(\theta) - \cos(\theta - \Delta\theta)\right) e_2$$

$$= 2\cos\left(\frac{2\theta - \Delta\theta}{2}\right) \sin\left(\frac{-\Delta\theta}{2}\right) e_1 - 2\sin\left(\frac{2\theta - \Delta\theta}{2}\right) \sin\left(\frac{\Delta\theta}{2}\right) e_2$$

$$= -\cos\theta \cdot \Delta\theta\, e_1 - \sin\theta\, \Delta\theta\, e_2$$

$$= -\Delta\theta \cdot \left(\cos\theta\, e_1 + \sin\theta\, e_2\right)$$

$$\Delta\theta = \frac{\Delta s}{R}$$

$$\boxed{\therefore \lim_{\Delta s \to 0} \frac{\Delta e_t}{\Delta s} = \frac{1}{R} \cdot e_n}$$

i.e. $k = \frac{1}{R}$ $\quad e_n$ = inward unit normal

→ **Relative Motion :**

$r_A = r_B + r_{A/B}$

$v_A = v_B + v_{A/B}$

$a_A = a_B + a_{A/B}$

If B is stationary or undergoing translation at uniform speed, $a_A = a_{A/B}$

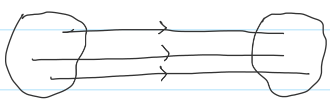
↳ Inertial frame

⇒ **Rigid Body**

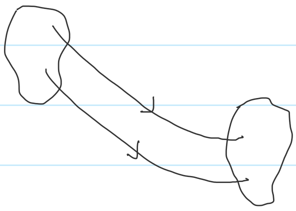
Collection of infinite particles. Distance between any two particles remains constant.

The motion of any rigid body is a combination of translation and rotation.

a) Translation :



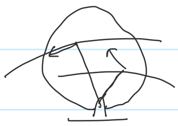
Rectilinear translation



Curvilinear translation

All trajectories are parallel to each other.
Knowledge of only one point is enough.

b) (i) About a fixed axis :



All points trace circular paths.
Centre is the fixed path. Radius is distance from the fixed point.
Angular speed, acceleration is the same.

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\boxed{\vec{V} = r\dot{\theta}\, e_\theta = \vec{\omega} \times \vec{r}} \qquad (\vec{\omega} = \omega e_3 \text{ where } e_3 = e_r \times e_\theta)$$

In general case, $\omega = \dot{\theta} c^*$ & $\vec{V} = \vec{\omega} \times \vec{r}$

where c^* is the axis of rotation.

Observe that $\frac{de_r}{dt} = \vec{\omega} \times e_r$ i.e. Rate of change of a vector = Cross product of its angular velocity vector and the vector itself.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\vec{\omega}\times\vec{r})}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

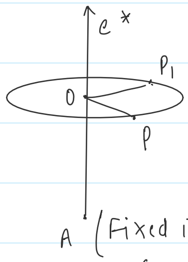
$$= \frac{d(\dot{\theta} e_3)}{dt} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \ddot{\theta} e_3 \times \vec{r} + \dot{\theta} \frac{de_3}{dt} \times \vec{r} + \omega(\vec{r}\cdot\vec{\omega}) - \vec{r}(\vec{\omega}\cdot\vec{\omega})$$

$$= \vec{\alpha} \times \vec{r} - |\omega|^2 \vec{r}$$

$$\boxed{\vec{a} = \vec{\alpha} \times \vec{r} - |w|^2 \vec{r}}$$

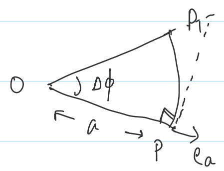
(ii) Rotation about a fixed point:



At the given instant let e^* be the axis of rotation at time t.

(Fixed in space) $\angle POP_1 = \phi$ $\quad P_1 = P(t+\Delta t)$

$$V_p = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$



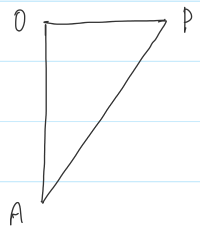
$\Delta r = a\,\Delta\phi$

Direction: $e^* \times e_a$

$$V_p = \lim_{\Delta t \to 0} \frac{a\,\Delta\phi\,(e^* \times e_a)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(\Delta\phi\, e^*) \times (a e_a)}{\Delta t}$$

$$= \vec{w} \times \vec{r}_{P/O}$$



$$\vec{r}_{PA} = \vec{r}_{OA} + \vec{r}_{PO}$$

$$\vec{r}_{PO} = \vec{r}_{PA} - \vec{r}_{OA}$$

$$\vec{V}_p = \vec{w} \times (\vec{r}_{PA} - \vec{r}_{OA})$$

$$\boxed{\vec{V}_p = \vec{w} \times \vec{r}_{PA}} \qquad (\because \vec{w} \parallel \vec{r}_{OA})$$

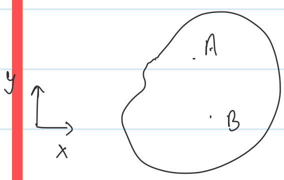
$$\vec{a} = \frac{d}{dt}\left(\vec{w} \times \vec{r}_{PA}\right)$$

$$= \frac{d\vec{w}}{dt} \times \vec{r}_{PA} + \vec{w} \times \frac{d\vec{r}_{PA}}{dt}$$

$$= \vec{\alpha} \times \vec{r}_{PA} + \vec{w} \times (\vec{w} \times \vec{r})$$

$$\boxed{\vec{a} = \vec{\alpha} \times \vec{r}_{PA} - |\vec{w}|^2 \vec{r}}$$

c) Combination of rotation & translation:



$$r_A = r_B + r_{A/B}$$
$$v_A = v_B + v_{A/B}$$
$$a_A = a_B + a_{A/B}$$

$\because |r_{A/B}| = \text{const.}$, relative motion of A wrt B is purely rotational.

$\therefore v_{A/B} = \omega \times r_{A/B}$

$\therefore \boxed{v_A = v_B + \omega \times r_{A/B}}$ $\Rightarrow$ 2 point formula for velocity

Differentiating w.r.t. time,

$$a_A = a_B + \frac{d\omega}{dt} \times r_{A/B} + \omega \times \frac{dr_{A/B}}{dt}$$

$$= a_B + \alpha \times r_{A/B} + \omega \times (\omega \times r_{A/B})$$

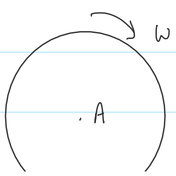
$$\boxed{a_A = a_B + \alpha \times r_{A/B} - \omega^2 r_{A/B}}$$

$\Rightarrow$ For any point B, if $v_B = 0$,

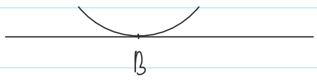
$v_A = \omega \times r_{A/B}$ i.e. body undergoes pure rotation w.r.t. B.

Point B is called as the instantaneous center of rotation and the axis of rotation through B is called the instantaneous axis of rotation.

eg.

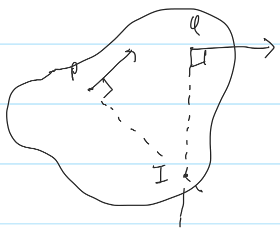


If no slippage at B,
$v_B = 0$



i.e. B is the instantaneous center of rotation.

⇒ <u>Finding the instantaneous center of rotation</u>

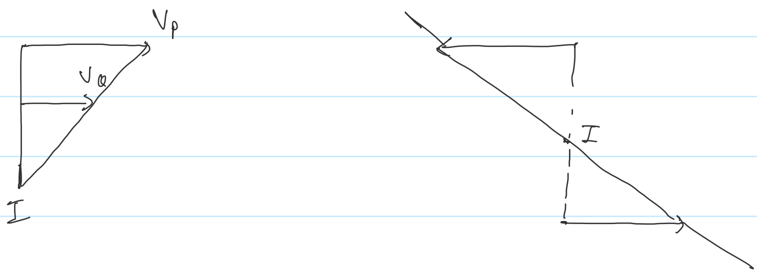


Let I be the instantaneous center of rotation.

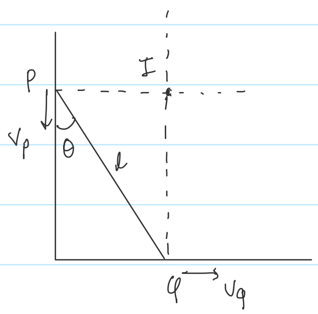
$v_P = \cancel{v_I}^{\,0} + \omega \times r_{P/I}$ $\quad$ $v_P \perp r_{P/I}$

$v_Q = \cancel{v_I}^{\,0} + \omega \times r_{Q/I}$ $\quad$ $v_Q \perp r_{Q/I}$

Special Case:



eg.

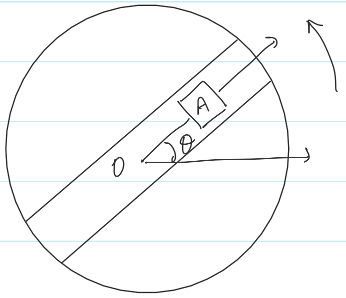


Rod PQ is supported by the walls.

$x_I = l\sin\theta \qquad y_I = l\cos\theta$

$\dfrac{x_I^2}{l^2} + \dfrac{y_I^2}{l^2} = 1 \rightarrow$ locus of I

eg. turntable with a slot



At any given time, point P on the table is coincident with the point mass A.

i.e. $r_{A/O} = r_{P/O}$

$$V_P = V_O + \omega \times r_{P/O}$$

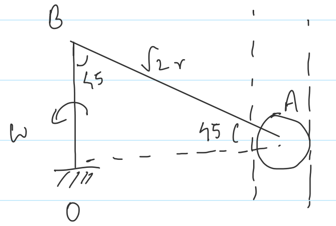
There will be a relative motion between A & P.

$$V_A = V_P + V_{rel}$$

$$V_A = V_O + \omega \times r_{P/O} + V_{rel}$$

$$\boxed{V_A = V_O + \omega \times r_{A/O} + V_{rel}}$$

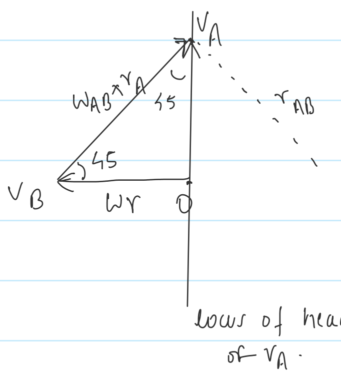
Q.



What is α_{AB}?

Graphical Method:

$$V_A = V_B + \omega_{AB} \times r_A$$



$\therefore V_A = \omega r$

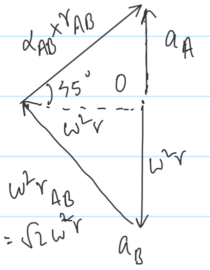
$\omega_{AB} \times r_A = \omega r \sqrt{2}$

$\omega_{AB} \cdot \sqrt{2} r = \omega r \sqrt{2}$

$\omega_{AB} = \omega$

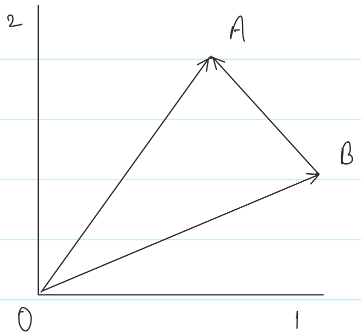
Acceleration analysis:

$$a_A = a_B + \alpha_{AB} \times r_{AB} - \omega_{AB}^2 r_{AB}$$



$$\boxed{\therefore a_A = \omega^2 r \; \hat{j}}$$

⇒ Rotating Frame of reference :

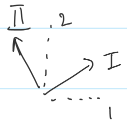


$$r_A = r_B + r_{A/B}$$

$$v_A = v_B + v_{A/B}$$

$$a_A = a_B + a_{A/B}$$

Consider a frame rotating with respect to the inertial frame 1-2.



$$\vec{r}_{AB} = x_I e_I + x_{II} e_{II}$$

$$\vec{r}_A = \vec{r}_B + x_I e_I + x_{II} e_{II}$$

$$\vec{V}_A = \vec{V}_B + \dot{x}_I e_I + x_I \dot{e}_I + \dot{x}_{II} e_{II} + x_{II} \dot{e}_{II}$$

$$= \vec{V}_B + \dot{x}_I e_I + x_I (\omega e_{II}) + \dot{x}_{II} e_{II} + x_{II} (-\omega e_I)$$

$$\vec{V}_A = \vec{V}_B + (\dot{x}_I e_I + \dot{x}_{II} e_{II}) + \omega \times (x_I e_I + x_{II} e_{II})$$

$$= \vec{V}_B + (\dot{x}_I e_I + \dot{x}_{II} e_{II}) + \omega \times (\vec{r}_{A/B})$$

$$\boxed{\vec{V}_A = \vec{V}_B + \omega \times \vec{r}_{A/B} + \vec{V}_{rel}}$$ → 3 point formula

where $\vec{V}_{rel} = \dot{x}_I e_I + \dot{x}_{II} e_{II}$

Acceleration:

$$\vec{a}_A = \vec{a}_B + (\ddot{x}_I e_I + \ddot{x}_{II} e_{II}) + \dot{x}_I (\vec{\omega} \times \vec{e}_I) + \dot{x}_{II} (\vec{\omega} \times e_{II}) + \vec{\omega} \times (\vec{V}_{rel} + \vec{\omega} \times \vec{r}_{A/B}) + \dot{\vec{\omega}} \times \vec{r}_{A/B}$$

$$= \vec{a}_B + \vec{\omega} \times \vec{V}_{rel} + \vec{\omega} \times \vec{V}_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/B}) + (\ddot{x}_I e_I + \ddot{x}_{II} e_{II}) + \vec{\alpha} \times \vec{r}_{A/B}$$

$$\boxed{\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2 \vec{r}_{A/B} + 2\vec{\omega} \times \vec{V}_{rel} + \vec{a}_{rel}}$$

where $\vec{a}_{rel} = \ddot{x}_I e_I + \ddot{x}_{II} e_{II}$ (acc. in the rot. frame)

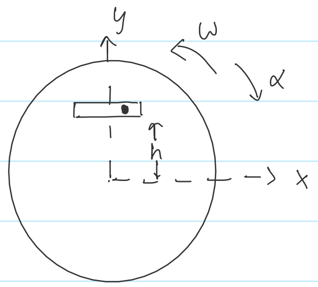
For the turntable,

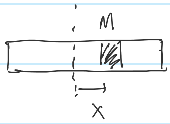
$$a_P = a_O + \omega \times (\omega \times r_{P/O}) + \vec{\alpha} \times r_{P/O}$$

$$a_{A/P} = \underbrace{2\vec{\omega} \times \vec{v}_{A/P}}_{\text{coriolis acceleration}} + \vec{a}_{rel}$$

It is the difference between the accelerations of point A wrt in inertial frame $(a_{A/P})$ and rotating frame $(\vec{a}_{rel})$.

eg.



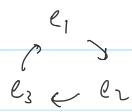


Given $\omega, \alpha, h, x, dx/dt, \frac{d^2x}{dt^2}$.

Find (i) Absolute velocity and acceleration of M.

Using 3 point formula for velocity:

$$\underline{v}_A = \underline{v}_O + \underline{\omega} \times \underline{r}_{A/O} + \underline{v}_{rel}$$



$v_O = 0$ $\quad \underline{\omega} \times \underline{r}_{A/O} = (\omega e_3) \times (x e_1 + h e_2)$

$$= \omega(x e_2 - h e_1)$$

$$v_{rel} = \frac{dx}{dt} e_1$$

$$\therefore \underline{v}_A = \left(\frac{dx}{dt} - \omega h\right) e_1 + (\omega x) e_2$$

Using 5-point acceleration formula:

$$\underline{a}_A = \underline{a}_0 + \underline{\alpha} \times \underline{r}_{A/0} - \omega^2 \underline{r}_{A/0} + 2\underline{\omega} \times \underline{v}_{rel} + \underline{a}_{rel}$$

$\underline{a}_0 = 0$

$$\underline{\alpha} \times \underline{r}_{A/0} = -\alpha e_3 \times (x e_1 + h e_2)$$
$$= \alpha(-x e_2 + h e_1)$$

$$-\omega^2 \underline{r}_{A/0} = -\omega^2 (x e_1 + h e_2)$$

$$2\underline{\omega} \times \underline{v}_{rel} = 2\omega e_3 \times \frac{dx}{dt} e_1$$

$$= 2\omega \frac{dx}{dt} e_2$$

$$a_{rel} = \frac{d^2x}{dt^2} e_1$$

$$\underline{a}_A = \left(\alpha h - \omega^2 x + \frac{d^2 x}{dt^2}\right) e_1 + \left(-\alpha x - \omega^2 h + 2\omega \frac{dx}{dt}\right) e_2$$