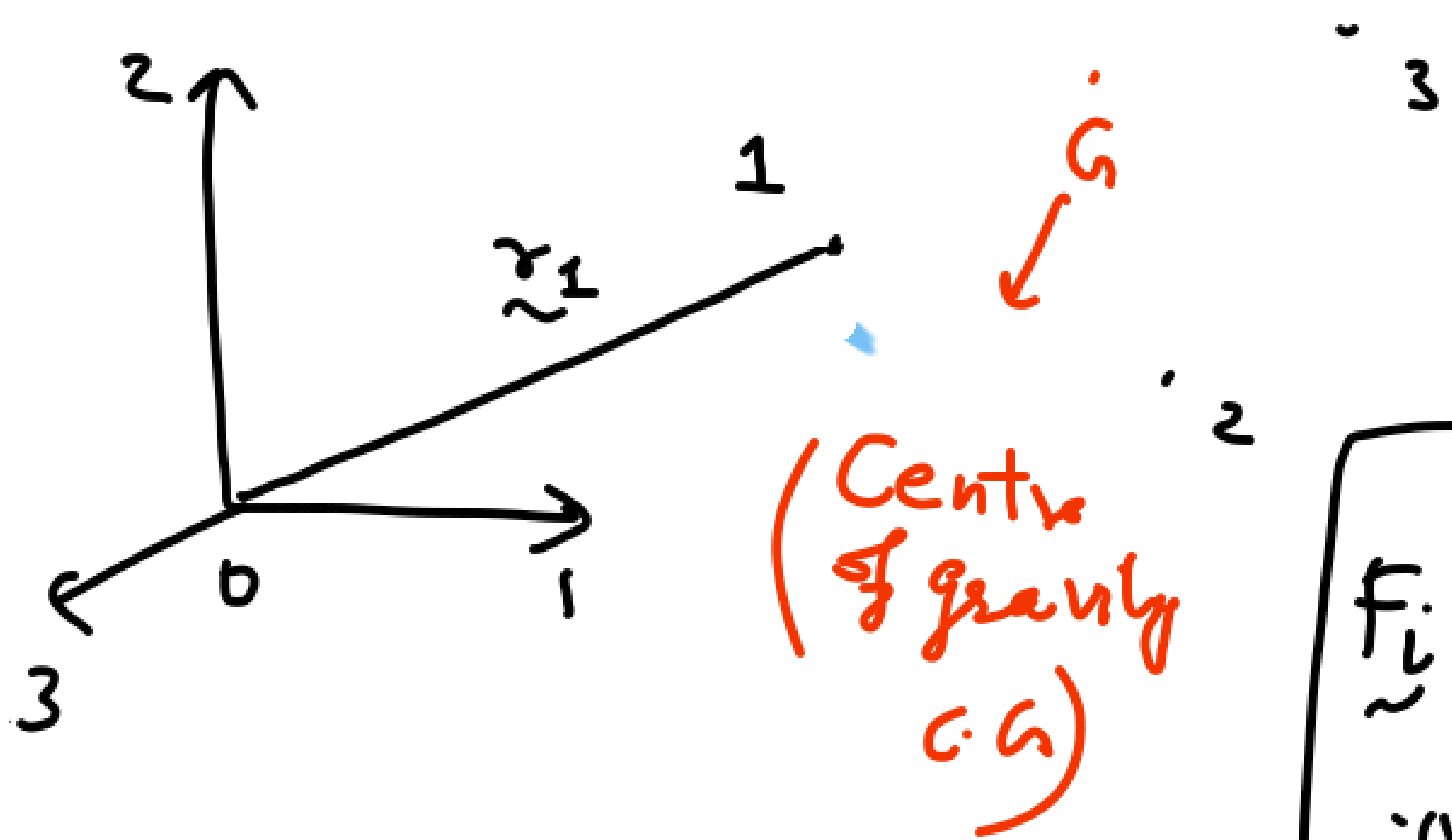
Kinetics of many particles

4.



$\underset{\sim}{r}_i$ = Position vector of i^{th} particle

m_i = mass of i^{th} particle

$\bar{\underset{\sim}{r}}$ = Position vector of Centre of gravity

$$= \frac{\sum m_i \underset{\sim}{r}_i}{\sum m_i} = \frac{\sum m_i \underset{\sim}{r}_i}{m}$$

$\underset{\sim}{F}_i$ = External force on i^{th} particle

$\underset{\sim}{f}_i$ = Force on i^{th} particle due to interaction with remaining particles.

For a particle:

$$\underset{\sim}{F}_i + \underset{\sim}{f}_i = m_i \underset{\sim}{a}_i$$

Summing for all particles:

$$\underbrace{\sum \underset{\sim}{F}_i}_{\underset{\sim}{F}} + \overset{0}{\cancel{\sum \underset{\sim}{f}_i}} = \sum m_i \underset{\sim}{a}_i$$

$$\boxed{\underset{\sim}{F} = m \ddot{\underset{\sim}{r}}}$$

or $\boxed{\underset{\sim}{F} = m \underset{\sim}{a}_G}$ — (2D/3D)

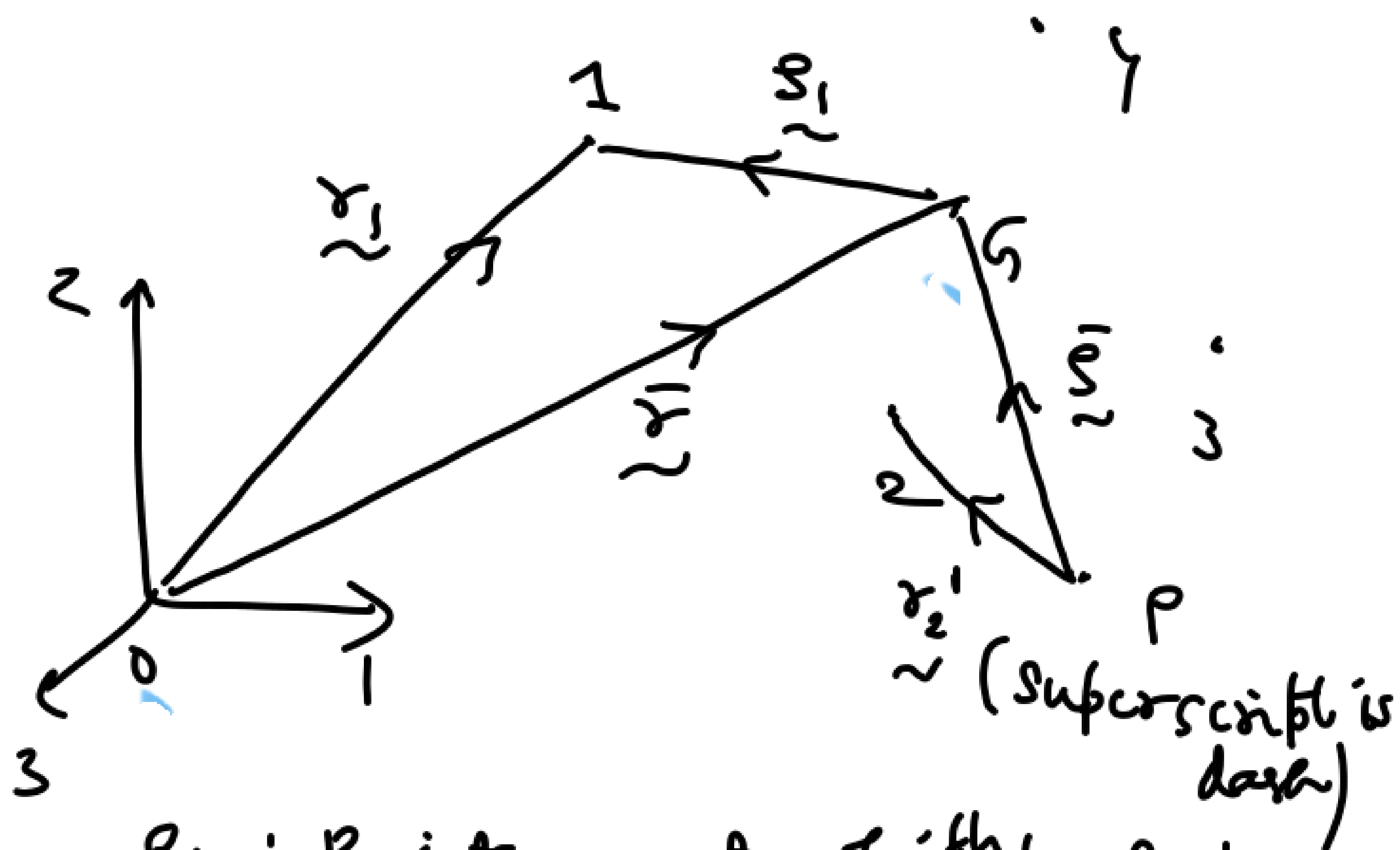
Linear momentum

$$\underset{\sim}{G}_i = m_i \underset{\sim}{v}_i$$

$$\underset{\sim}{G} = \sum \underset{\sim}{G}_i = \sum m_i \underset{\sim}{v}_i$$

$$\dot{\underset{\sim}{G}} = \sum m_i \dot{\underset{\sim}{v}}_i = \sum m_i \underset{\sim}{a}_i = \sum \underset{\sim}{F}_i + \underset{\sim}{f}_i = \underset{\sim}{F}$$

Angular momentum ($\underset{\sim}{H}$)



ρ_i : Position vector of i^{th} particle w.r.t G.

So using vector addition,

$$\boxed{r_i = \rho_i + \bar{r}}$$

$$H_O = \Sigma (H_O)_i = \Sigma r_i \times (m_i \dot{r}_i)$$

$$H_G = \Sigma (H_G)_i = \Sigma \rho_i \times (m_i \dot{r}_i)$$

$$\dot{H}_O = \frac{dH_O}{dt} = \Sigma \dot{r}_i \times \cancelto{0}{(m_i \dot{r}_i)} + \Sigma r_i \times \cancelto{0}{(\dot{m}_i \dot{r}_i)} + \Sigma r_i \times (m_i \ddot{r}_i)$$

$$\dot{H}_O = \Sigma r_i \times (F_i + f_i)$$

$$= \Sigma r_i \times F_i + \cancelto{0}{\Sigma r_i \times f_i}$$

Pairwise interaction ensures that

$$\Sigma r_i \times f_i = 0$$

R.H.S $= \sum \underset{\sim}{M}_0 \rightarrow$ moment due to external forces at the fixed point O

$$\therefore \boxed{\dot{\underset{\sim}{H}}_0 = \sum \underset{\sim}{M}_0} \Rightarrow$$

Rate of change of angular momentum w.r.t fixed point = moment due to external forces at the fixed point O.

$$\underset{\sim}{r}_i = \underset{\sim}{\bar{r}} + \underset{\sim}{\rho}_i$$

$$\dot{\underset{\sim}{\rho}}_i \times (m_i \dot{\underset{\sim}{r}}_i) = \dot{\underset{\sim}{\rho}}_i \times m_i \left(\dot{\underset{\sim}{\bar{r}}} + \dot{\underset{\sim}{\rho}}_i\right)$$

$$= m_i \dot{\rho}_i \times \dot{\underset{\sim}{\bar{r}}} + \cancel{m_i \dot{\underset{\sim}{\rho}}_i \times \dot{\underset{\sim}{\rho}}_i} \rightarrow 0$$

$$\underset{\sim}{\rho}_i \times m_i \ddot{\underset{\sim}{r}}_i = \underset{\sim}{\rho}_i \times m_i \left(\ddot{\underset{\sim}{\rho}}_i + \ddot{\underset{\sim}{\bar{r}}}\right)$$

$$= m_i \underset{\sim}{\rho}_i \times \ddot{\underset{\sim}{\rho}}_i + \left(m_i \underset{\sim}{\rho}_i \times \ddot{\underset{\sim}{\bar{r}}}\right)$$

$$\therefore \dot{\underset{\sim}{H}}_G = \cancel{\left(\sum m_i \dot{\underset{\sim}{\rho}}_i\right)} \rightarrow 0 \times \dot{\underset{\sim}{\bar{r}}} + \sum m_i \underset{\sim}{\rho}_i \times \ddot{\underset{\sim}{\rho}}_i + \cancel{\left(\sum m_i \underset{\sim}{\rho}_i\right)} \rightarrow 0 \times \ddot{\underset{\sim}{\bar{r}}}$$

<u>For point G:</u>

$$\underset{\sim}{H}_G = \sum (\underset{\sim}{H}_G)_i = \sum \underset{\sim}{\rho}_i \times (m_i \underset{\sim}{v}_i), \quad \underset{\sim}{v}_i \rightarrow \dot{\underset{\sim}{r}}_i$$

$$\dot{\underset{\sim}{H}}_G = \sum \dot{\underset{\sim}{\rho}}_i \times (m_i \dot{\underset{\sim}{r}}_i) + \sum \underset{\sim}{\rho}_i \times \cancel{\dot{m}_i} \underset{\sim}{r}_i \rightarrow 0 + \sum \underset{\sim}{\rho}_i \times m_i \ddot{\underset{\sim}{r}}_i$$

$$\underset{\sim}{\bar{r}} = \frac{\sum m_i \underset{\sim}{r_i}}{m} = \frac{\sum m_i (\underset{\sim}{\bar{r}} + \underset{\sim}{\rho_i})}{m}$$

$$\underset{\sim}{\bar{r}} = \frac{\sum m_i \bar{r} + \sum m_i \underset{\sim}{\rho_i}}{m}$$

$$= \frac{\cancel{(\sum m_i)} \underset{\sim}{\bar{r}}}{\cancel{m}} + \frac{\sum m_i \underset{\sim}{\rho_i}}{m}$$

$$\Rightarrow \boxed{\sum m_i \underset{\sim}{\rho_i} = \underset{\sim}{0}}$$

Derivative w.r.t time:

$$\sum m_i \dot{\underset{\sim}{\rho_i}} = \underset{\sim}{0}$$

$$\sum m_i \ddot{\underset{\sim}{\rho_i}} = \underset{\sim}{0}$$

$$\dot{\underset{\sim}{H_G}} = \sum m_i \underset{\sim}{\rho_i} \times \ddot{\underset{\sim}{\rho_i}}$$

$$= \sum m_i \underline{\rho_i} \times (\ddot{\underset{\sim}{r_i}} - \ddot{\underset{\sim}{\bar{r}}})$$

$$= \sum \underset{\sim}{\rho_i} \times (m_i \ddot{\underset{\sim}{r_i}})$$

$$- \overbrace{\cancel{\left(\sum m_i \underset{\sim}{\rho_i}\right)}}^{0} \times \ddot{\underset{\sim}{\bar{r}}}$$

$$= \sum \underset{\sim}{\rho_i} \times (\underset{\sim}{F_i} + \underset{\sim}{f_i})$$

$$= \sum \underset{\sim}{\rho_i} \times \underset{\sim}{F_i} + \overbrace{\cancel{\sum \underset{\sim}{\rho_i} \times \underset{\sim}{f_i}}}^{0} = \sum \underset{\sim}{\rho_i} \times \underset{\sim}{F_i}$$

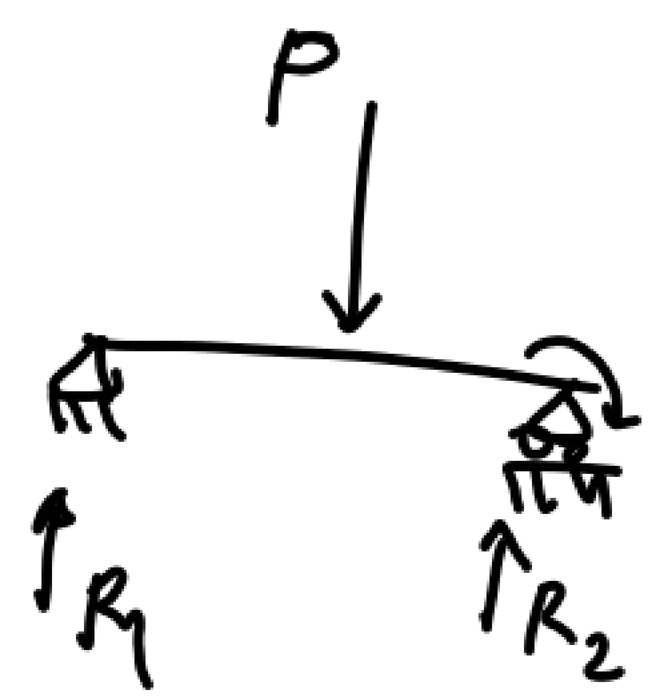
$\sum \underset{\sim}{M_G}$

Moment due to external force @ C.G i.e. G.

Same argument (reasoning) as $\sum \underset{\sim}{r_i} \times \underset{\sim}{f_i} = \underset{\sim}{0}$ for $\sum \underset{\sim}{\rho_i} \times \underline{f_i} = \underline{0}$

$$\therefore \boxed{\dot{\underset{\sim}{H}}_G = \Sigma \underset{\sim}{M}_G}$$

Rate of change of angular momentum w.r.t centre of mass = moment due to external forces w.r.t centre of mass.



How does the balance of angular momentum changes if we consider a point P different from origin O or centre of mass G?

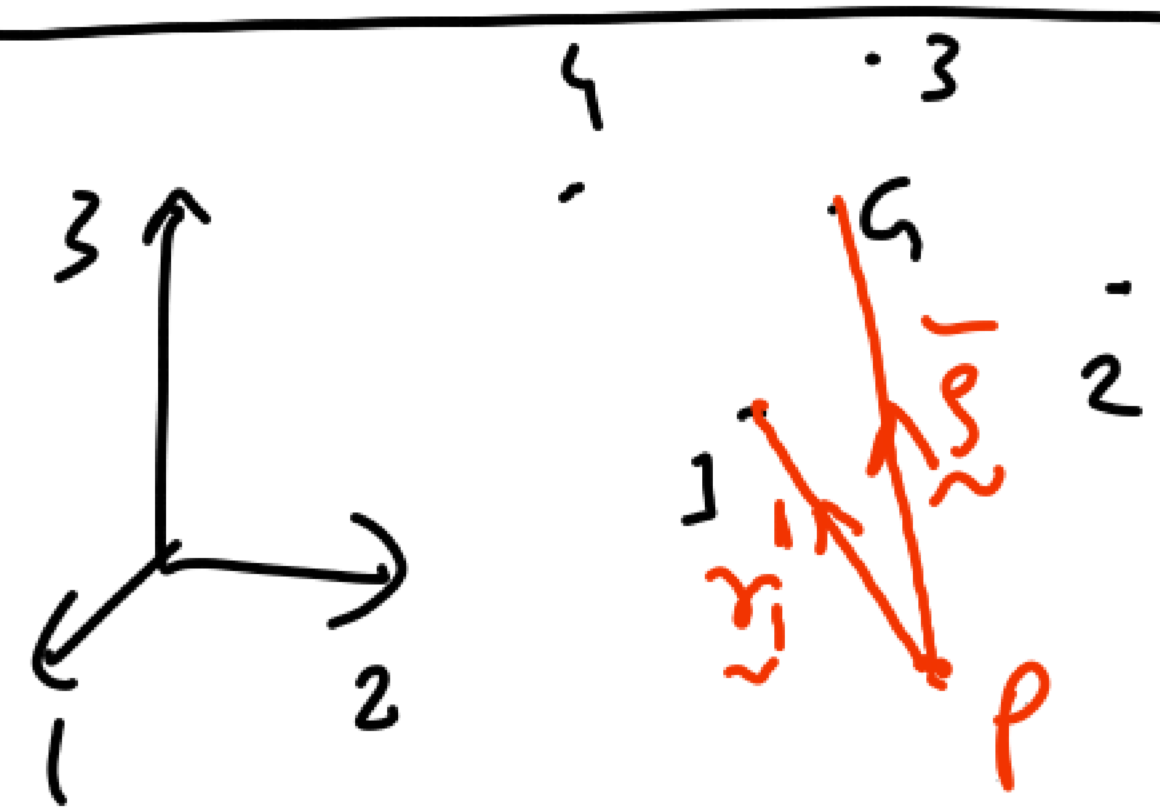
Angular momentum w.r.t point P:

$$\underset{\sim}{H}_P = \Sigma(\underset{\sim}{H}_P)_i = \Sigma\, \underset{\sim}{r}_i' \times (m_i \underset{\sim}{v}_i)$$

$$= \Sigma\, \underset{\sim}{r}_i' \times (m_i \dot{\underset{\sim}{r}}_i)$$

$$= \Sigma\, \underset{\sim}{r}_i' \times (m_i(\dot{\bar{\underset{\sim}{r}}} + \dot{\underset{\sim}{\rho}}_i))$$

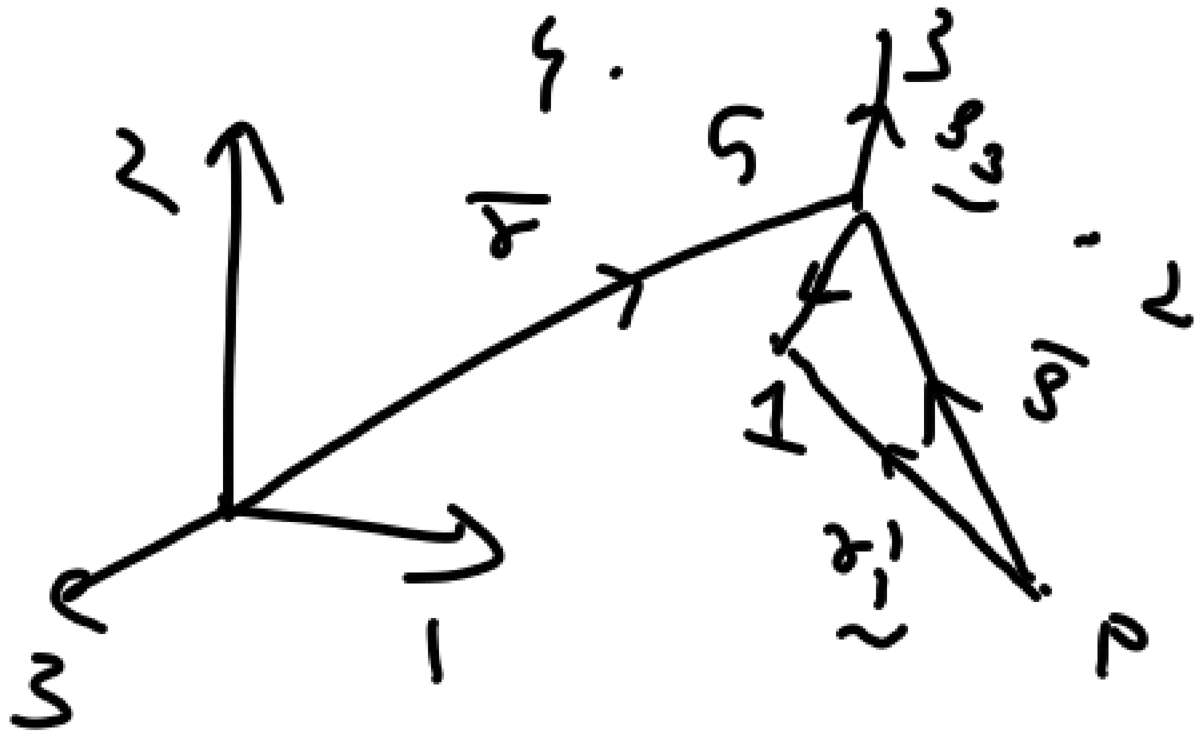
$$= \Sigma\, \underset{\sim}{r}_i' \times m_i \dot{\bar{\underset{\sim}{r}}} + \Sigma\, \underset{\sim}{r}_i' \times m_i \dot{\underset{\sim}{\rho}}_i$$



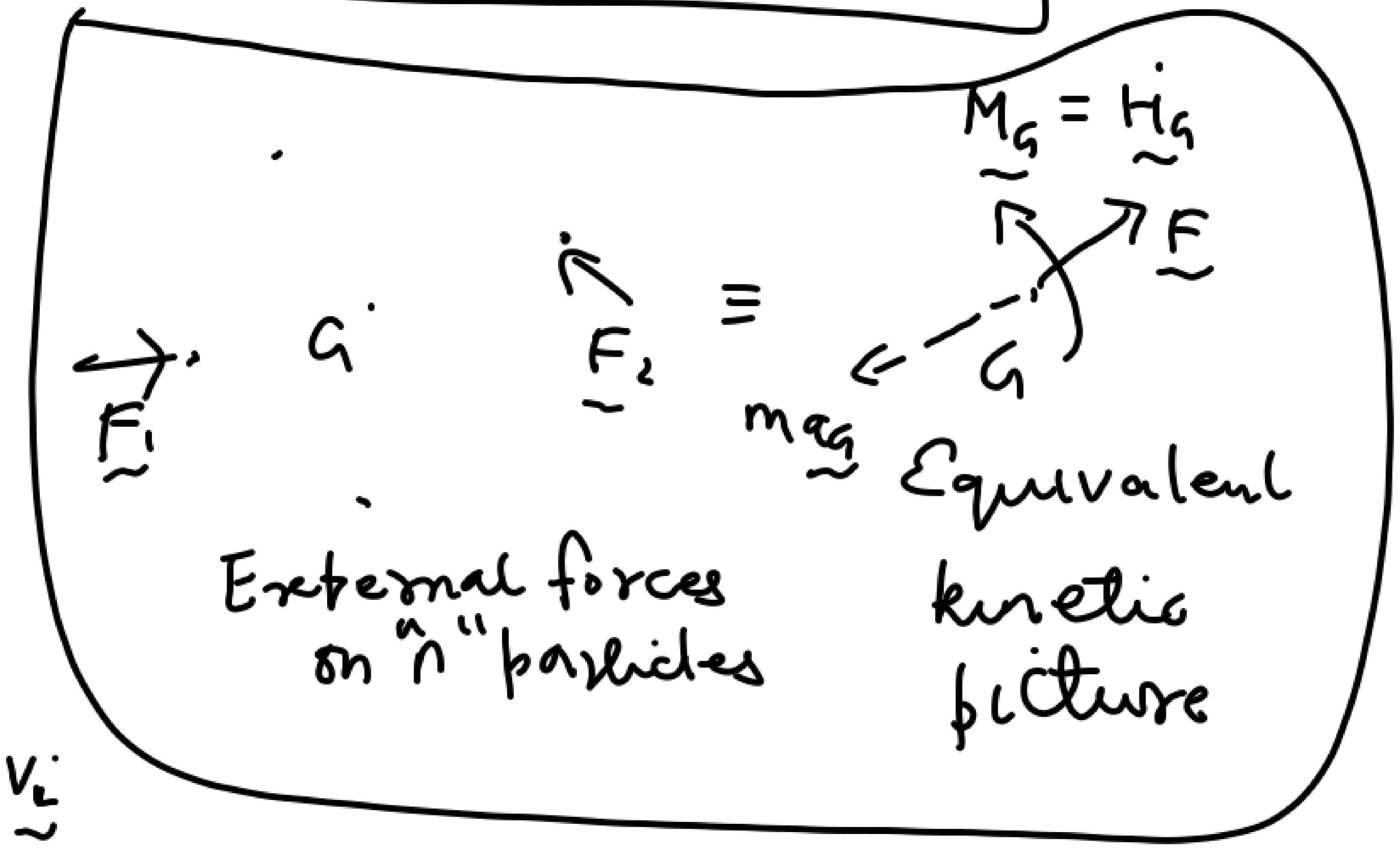
$\underset{\sim}{\bar{\rho}}$ is position vector joining P to G.

$\underset{\sim}{r}_i'$: Position vector of i^{th} particle w.r.t P.

superscript (Dash)



$$\boxed{H_p = \bar{\rho} \times G + H_G}$$



$$r_i' = \bar{\rho} + \rho_i$$

$$H_p = \sum (\bar{\rho} + \rho_i) \times (m_i v_i)$$

$$= \bar{\rho} \times \sum m_i v_i + \sum \rho_i \times m_i v_i$$

$$= \bar{\rho} \times G + \sum \rho_i \times m_i (\dot{r}_i)$$ → Angular momentum w.r.t G

$$\sum \rho_i \times m_i (\dot{\bar{r}} + \dot{\rho}_i)$$

|| (equal)

$$= \left(\cancel{\sum \rho_i m_i}\right)^{0} \times \dot{\bar{r}} + \boxed{\sum \rho_i \times (m_i \dot{\rho}_i)}$$ → Angular momentum w.r.t G using relative linear momentum