Ridge Systems

- Consider 2nd order model with canonical form $$\hat{y} = \hat{y}_s + \lambda_1 w_1^2 + \lambda_2 w_2^2 + \cdots + \lambda_k w_k^2$$
- Suppose the stationary point (x_s) is in the region of experimentation, and some λ_i are small, i.e., $\lambda_i \approx 0$
- Then, the response variable y is very insensitive to variables with small λ

Is small

$$\hat{y} = \hat{y}_s + \lambda_1 w_1^2 + \lambda_2 w_2^2$$

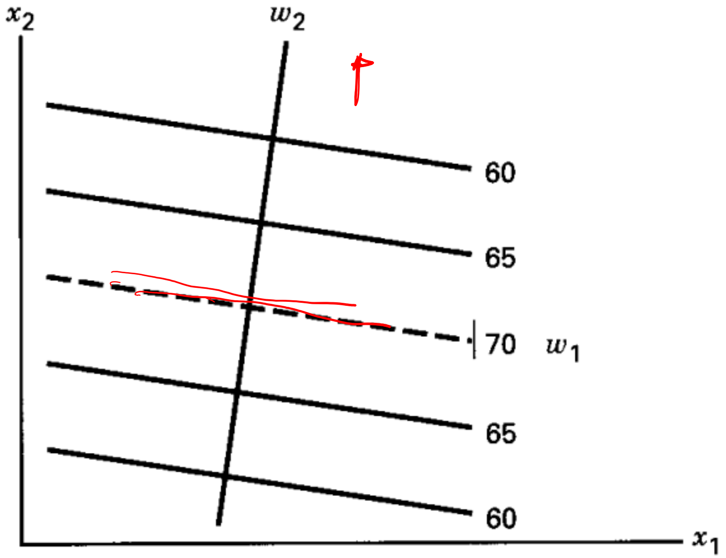


Figure 11-12 A contour plot of a stationary ridge system.

- Because of small λ_1, the optimum can be taken anywhere along the line of y = 70
- This type of response surface is called **'stationary ridge system'**

Ridge System

- If the stationary point (x_s) is far outside the region of experimentation, and some λ_i are small, i.e., $\lambda_i \approx 0$
- Then the response surface could be a 'rising ridge' or 'falling ridge'
- In such type of systems, we can NOT draw conclusions about the true surface or the stationary point
- BECAUSE the stationary point is far outside the region where we fitted the model
- In this example, further exploration is needed in the w1 direction

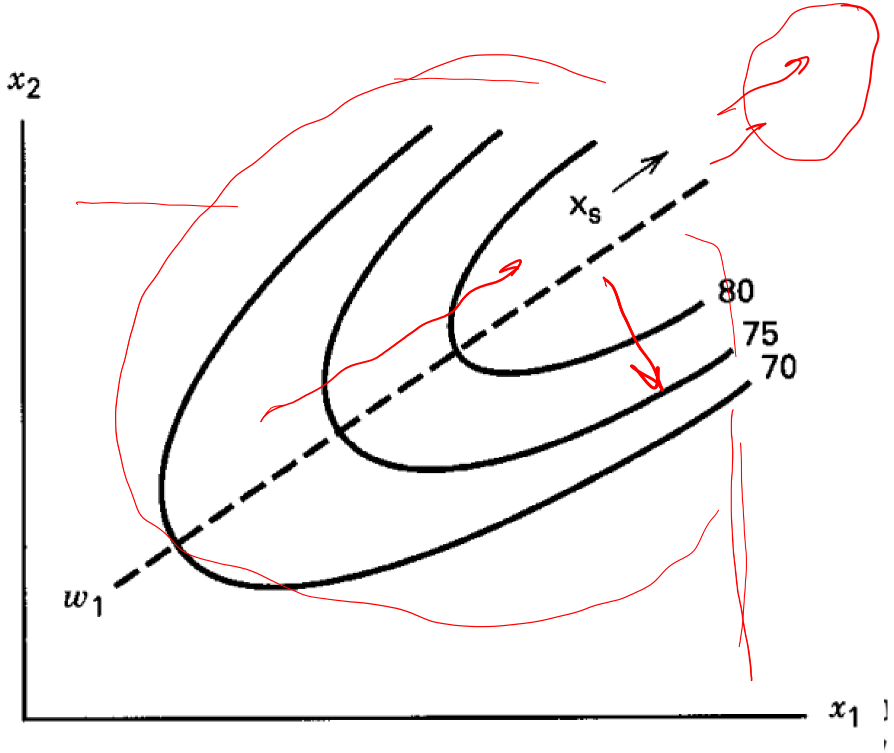


Figure 11-13 A contour plot of a rising ridge system.

Multiple Responses

- Consider the previous example
- Similar to yield, we can also obtain models for viscosity and molecular weight

We only found a response surface for ONE of the responses - Yield

Final Equation in Terms of Coded Factors:

yield =
+79.94
+0.99 * A
+0.52 * B
−1.38 * A²
−1.00 * B²
+0.25 * A * B

$$\hat{y}_2 = 70.00 - 0.16x_1 - 0.95x_2 - 0.69x_1^2 - 6.69x_2^2 - 1.25x_1x_2$$
$$\hat{y}_3 = 3386.2 + 205.1x_1 + 17.4x_2$$

In terms of the natural levels of time (ξ_1) and temperature (ξ_2), these models are

$$\hat{y}_2 = -9030.74 + 13.393\xi_1 + 97.708\xi_2 - 2.75 \times 10^{-2}\xi_1^2 - 0.26757\xi_2^2 - 5 \times 10^{-2}\xi_1\xi_2$$

and

$$\hat{y}_3 = -6308.8 + 41.025\xi_1 + 35.473\xi_2$$

Time, Temp

Natural Variables		Coded Variables		Responses		
ξ_1	ξ_2	x_1	x_2	y_1 (yield)	y_2 (viscosity)	y_3 (molecular weight)
80	170	−1	−1	76.5	62	2940
80	180	−1	1	77.0	60	3470
90	170	1	−1	78.0	66	3680
90	180	1	1	79.5	59	3890
85	175	0	0	79.9	72	3480
85	175	0	0	80.3	69	3200
85	175	0	0	80.0	68	3410
85	175	0	0	79.7	70	3290
85	175	0	0	79.8	71	3500
92.07	175	1.414	0	78.4	68	3360
77.93	175	−1.414	0	75.6	71	3020
85	182.07	0	1.414	78.5	58	3630
85	167.93	0	−1.414	77.0	57	3150

How would you optimize multiple responses?

A relatively straightforward approach to optimizing several responses that works well when there are only a few process variables is to **overlay the contour plots** for each response. Figure 11-16 (page 451) shows an overlay plot for the three responses in Example 11-2, with contours for which y_1 (yield) $\geqslant 78.5$, $62 \leqslant y_2$ (viscosity) $\leqslant 68$, and y_3 (molecular weight Mn) $\leqslant 3400$. If these boundaries represent important conditions that must be met by the process, then as the unshaded portion of Figure 11-16 shows, there are a number of combinations of time and temperature that will result in a satisfactory process. The experimenter can visually examine the contour plot to determine appropriate operating conditions. For example, it is likely that the experimenter would be most interested in the larger of the two feasible operating regions shown in Figure 11-16.

But what will you do if there are even more responses OR if there are more than two independent variables?

Graphical method won't work!

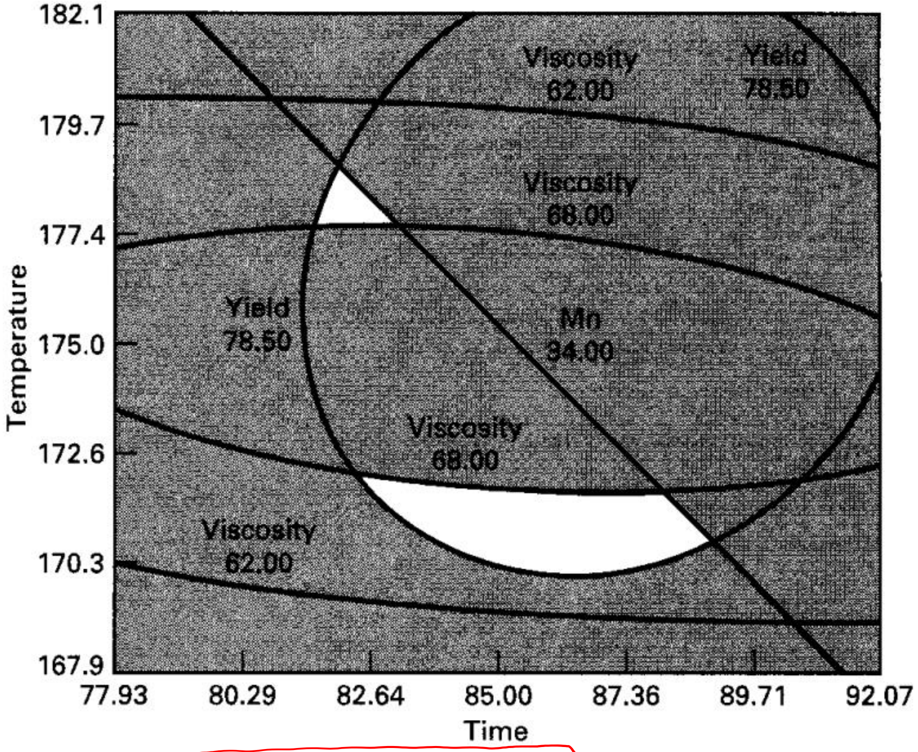


Figure 11-16 Region of the optimum found by overlaying yield, viscosity, and molecular weight response surfaces, Example 11-2.

Constrained Optimization

- Popular approach is to formulate the problem as constrained optimization problem
- For example,

$$\begin{gathered}\text{Max } y_1 \\ \text{subject to} \\ 62 \leqslant y_2 \leqslant 68 \\ y_3 \leqslant 3400\end{gathered}$$

- Then one can use numerical techniques to solve such a problem ('non-linear programming methods')

The two solutions found are

$$\text{time} = 83.5 \qquad \text{temp} = 177.1 \qquad \hat{y}_1 = 79.5$$

$$\text{time} = 86.6 \qquad \text{temp} = 172.25 \qquad \hat{y}_1 = 79.5$$

Desirability Functions

Another useful approach to optimization of multiple responses is to use the simultaneous optimization technique popularized by Derringer and Suich (1980). Their procedure makes use of **desirability functions**. The general approach is to first convert each response y_i into an individual desirability function d_i that varies over the range

$$0 \leqslant d_i \leqslant 1$$

where if the response y_i is at its goal or target, then $d_i = 1$, and if the response is outside an acceptable region, $d_i = 0$. Then the design variables are chosen to maximize the overall desirability

$$D = (d_1 \cdot d_2 \cdot \cdots \cdot d_m)^{1/m}$$

where there are m responses.

yield visc mol. wt

y_1 y_2 y_3

$y_1 = 80$

$d_1 = 1$

$y_1 \nless 70\%$

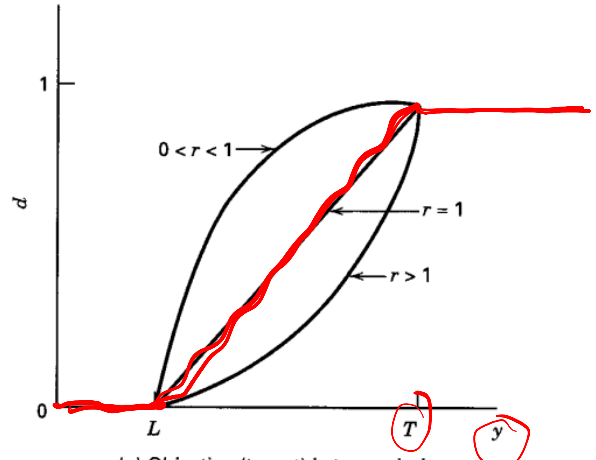
$y_1 \leq 70$

$d_1 = 0$

d_1

If the objective or target T for the response y is a maximum value,

$$d = \begin{cases} 0 & y < L \\ \left(\dfrac{y-L}{T-L}\right)^r & L \leqslant y \leqslant T \\ 1, & y > T \end{cases}$$

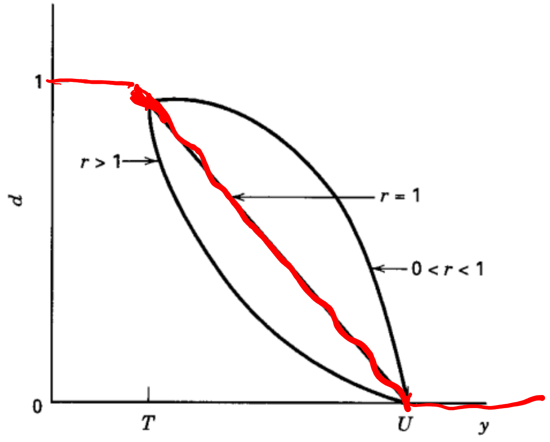


(a) Objective (target) is to maximize y

$y_1 \to d_1$

If the target for the response is a minimum value,

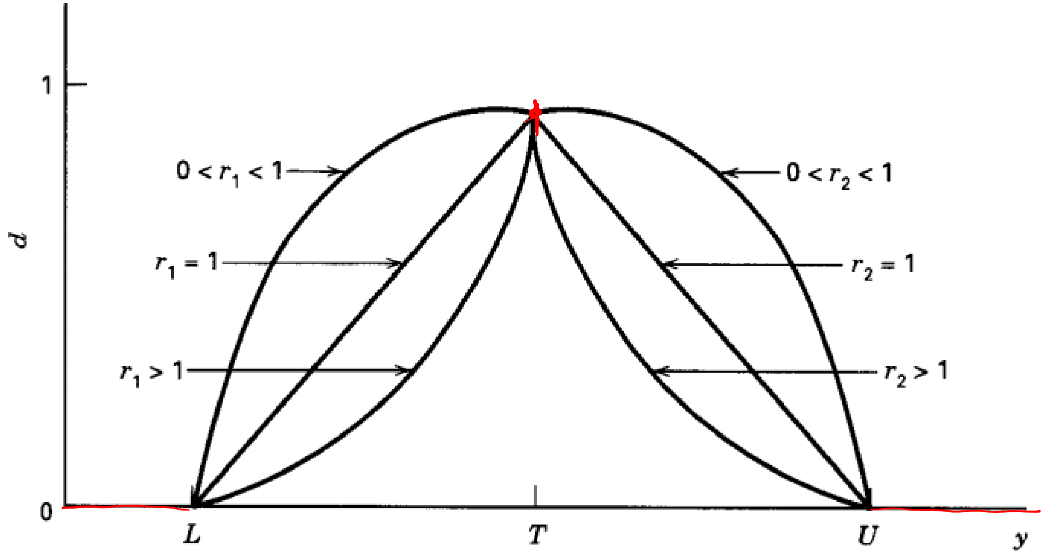
$$d = \begin{cases} 1 & y < T \\ \left(\dfrac{U-y}{U-T}\right)^r & T \leqslant y \leqslant U \\ 0 & y > U \end{cases}$$



(b) Objective (target) is to minimize y

The two-sided desirability function shown in Figure 11-17(c) assumes that the target is located between the lower (L) and upper (U) limits, and is defined as

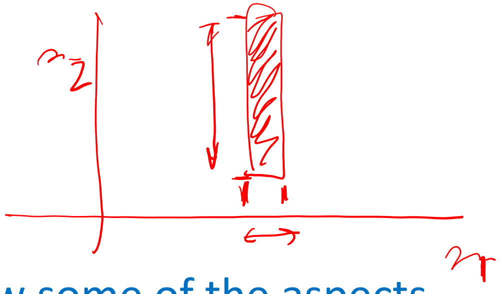
$$d = \begin{cases} 0 & y < L \\ \left(\dfrac{y-L}{T-L}\right)^{r_1} & L \leqslant y \leqslant T \\ \left(\dfrac{U-y}{U-T}\right)^{r_2} & T \leqslant y \leqslant U \\ 0 & y > U \end{cases} \qquad (11\text{-}13)$$



(c) Objective is for y to be as close as posible to the target

Design of Experiments for Fitting Response Surfaces

- Fitting and analysing a response surface can be made very effective by *proper choice of experimental design* to collect the data
- What is a ‘good experimental design’?



1. Provides a reasonable distribution of data points (and hence information) throughout the region of interest
2. Allows model adequacy, including lack of fit, to be investigated
3. Allows experiments to be performed in blocks
4. Allows designs of higher order to be built up sequentially
5. Provides an internal estimate of error
6. Provides precise estimates of the model coefficients
7. Provides a good profile of the prediction variance throughout the experimental region
8. Provides reasonable robustness against outliers or missing values
9. Does not require a large number of runs
10. Does not require too many levels of the independent variables
11. Ensures simplicity of calculation of the model parameters

Note how some of the aspects are conflicting -> We need to apply our judgement

DOE for fitting First-Order Model

- First order model with 'k' variables: $y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \epsilon$

- Using the experimental data, one would find the regression coefficients β_i
- There is a unique class of experimental designs that can minimize the variance of β_i ->

Orthogonal First-Order Designs

- A first-order design is orthogonal if the off-diagonal elements of (X'X) matrix are ALL ZERO
- That is the <cross> products of the columns of the X matrix add to ZERO

$$\sum x_{1i} x_{2i} = 0$$

2^2

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

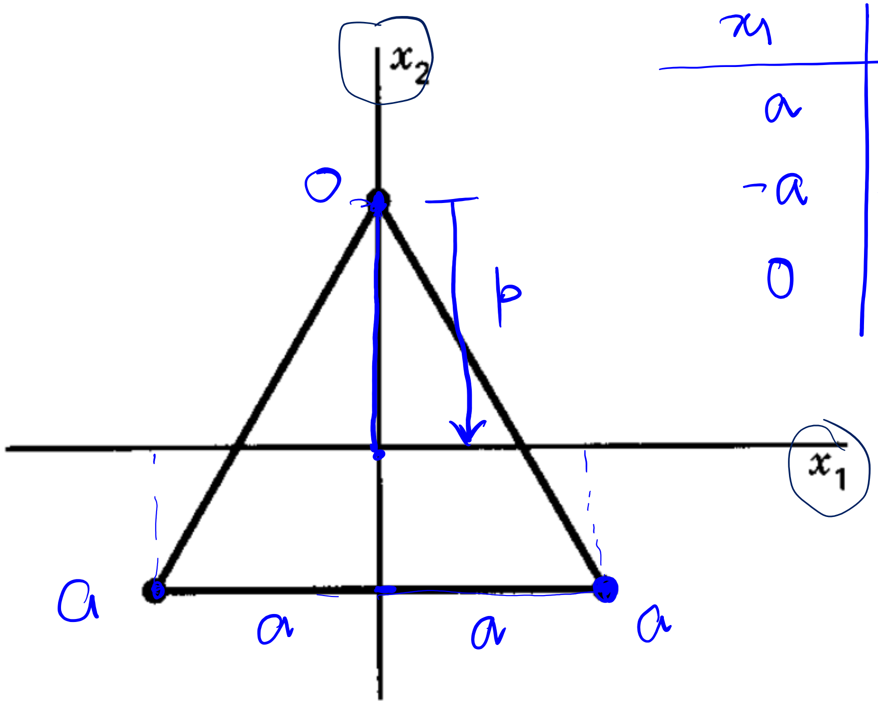
$$y_1 = \beta_0 - \beta_1 - \beta_2$$
$$y_2 = \beta_0 + \beta_1 - \beta_2$$
$$y_3 = \beta_0 - \beta_1 + \beta_2$$
$$y_4 = \beta_0 + \beta_1 + \beta_2$$

x_1	x_2	y
−1	−1	y_1
+1	−1	y_2
−1	+1	y_3
+1	+1	y_4

Orthogonal First-order Designs

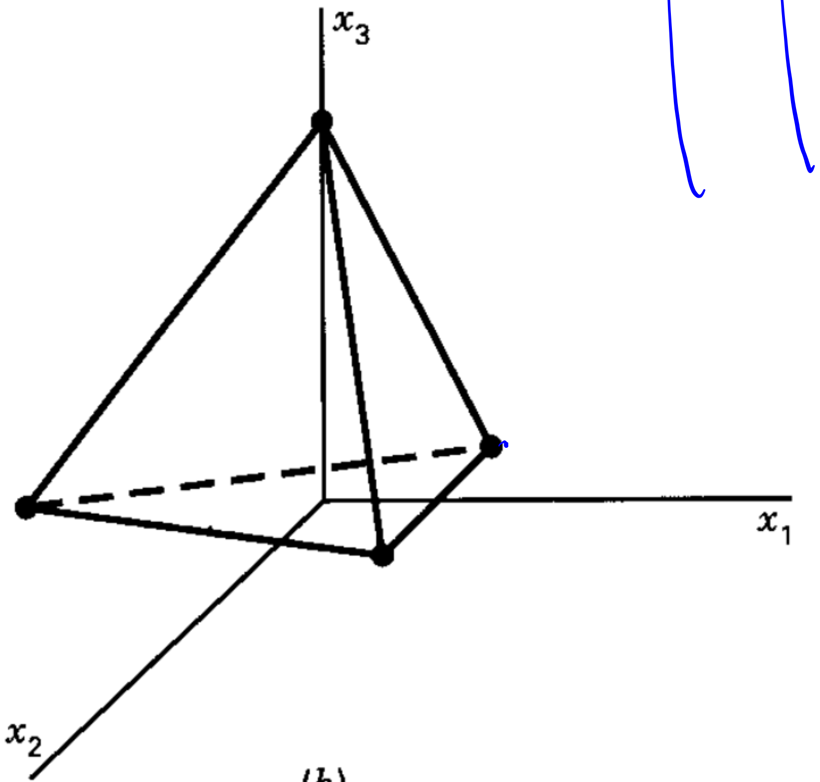
- Are 2^k factorial designs orthogonal?
- Yes – they are a part of orthogonal first-order designs
  - BUT, 2^k design does not afford an estimate of experimental error unless we do replication
  - A common method is to augment the 2^k design with several observations at the centre point
  - The addition of centre point doesn't influence the $\beta_1, \beta_2, \ldots$, BUT β_0 becomes a grand average of ALL the observations
  - Also, note how addition of centre point does not affect the orthogonality of the design

Another orthogonal first-order design is the **simplex**. The simplex is a regularly sided figure with $k + 1$ vertices in k dimensions. Thus, for $k = 2$ the simplex design is an equilateral triangle and for $k = 3$ it is a regular tetrahedron. Simplex designs in two and three dimensions are shown in Figure 11-19.



(a)

x_1	x_2
a	$-\sqrt{3}/2\, a$
$-a$	$-\sqrt{3}/2\, a$
0	0



(b)

x_1	x_2	x_3

Recap: How do you find regression coefficients?

First Order Model

Regression Analysis of a 2^3 Factorial Design

A chemical engineer is investigating the yield of a process. Three process variables are of interest: temperature, pressure, and catalyst concentration. Each variable can be run at a low and a high level, and the engineer decides to run a 2^3 design with four center points. The design and the resulting yields are shown in Figure 10-5, where we have shown both the natural levels of the design factor and the +1, −1 coded variable notation normally employed in 2^k factorial designs to represent the factor levels.

Suppose that the engineer decides to fit a main effects only model, say

$$y = \boxed{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3} + \epsilon$$

How many unknowns = ? 4

$$y_1 = \beta_0 + \beta_1(-1) + \beta_2(-1) + \beta_3(-1) + \varepsilon_1$$

$$y_2 = \beta_0 + \beta_1(1) + \beta_2(-1) + \beta_3(-1) + \varepsilon_2$$

12 eq → ~~4~~ 16 unknowns

$$\boxed{x_1 + x_2 = 1}$$

$$\cancel{4x_1 + x_2 = 3}$$

$$\cancel{x_1 + 4x_2 = 0}$$

$3x = 1 \Rightarrow x = 1/3$

$x_1 = 1/3$

$2^3 = 8$

Run	Temp (°C)	Pressure (psig)	Conc. (g/l)	x_1	x_2	x_3	Yield, y
	Process Variables			Coded Variables			
1	120	40	15	−1	−1	−1	32
2	160	40	15	1	−1	−1	46
3	120	80	15	−1	1	−1	57
4	160	80	15	1	1	−1	65
5	120	40	30	−1	−1	1	36
6	160	40	30	1	−1	1	48
7	120	80	30	−1	1	1	57
8	160	80	30	1	1	1	68
9	140	60	22.5	0	0	0	50
10	140	60	22.5	0	0	0	44
11	140	60	22.5	0	0	0	53
12	140	60	22.5	0	0	0	56

$$x_1 = \frac{\text{Temp} - 140}{20}, \quad x_2 = \frac{\text{Pressure} - 60}{20}, \quad x_3 = \frac{\text{Conc} - 22.5}{7.5}$$

Regression Analysis

First order approx → $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3$ 4 unknowns

	x_1	x_2	x_3	y	
1	-1	-1	-1	32	y_1
2	x_{1_2}				y_2
3					y_3
⋮					⋮
12	0	0	0	56	y_{12}

$32 = \hat{\beta}_0 + \hat{\beta}_1(-1) + \hat{\beta}_2(-1) + \beta_3(-1) + \epsilon_1$ ①

$y_2 = \hat{\beta}_0 + \hat{\beta}_1(x_{1_2}) + \hat{\beta}_2(x_{2_2}) + \beta_3(x_{3_2}) + \epsilon_2$ ②

⑫

12 eqn

"least square method"

want to find out $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ such that the error bet model prediction and exp data is minimum.

for row/run 1 model predicts → $\hat{y}_1 = \hat{\beta}_0 + \hat{\beta}_1 x_{1_1} + \hat{\beta}_2 x_{2_1} + \beta_3 x_{3_1}$

actual data $y = y_1$

$\epsilon_1 = y_1 - \hat{y}_1$

Regression Analysis

Least square function (i.e. sum of sq. of errors)

$$L = \sum_{i=1}^{n} \epsilon_i^2 = \sum \Big(\underset{\text{exp}}{y_i} - \underset{\text{model}}{\hat{y}_i} \Big)^2 = \sum \Big(y_i - \big(\beta_0 + \sum_{j=1}^{k} \beta_j x_j\big) \Big)^2$$

To minimize L, I need to minimize wrt each β

$$\left.\frac{\partial L}{\partial \beta_0}\right|_{\hat{\beta}_1, \hat{\beta}_2, \ldots} = \frac{\partial L}{\partial \beta_1} = \frac{\partial L}{\partial \beta_2} = 0$$

How many eq — $k+1$

regression coeff — $k+1$

$$\frac{\partial L}{\partial \beta_j} = 0 \quad \text{for all } j\text{s}$$

$$\frac{\partial L}{\partial \beta_0} = \frac{\partial L}{\partial \beta_1} = \frac{\partial L}{\partial \beta_2} = \frac{\partial L}{\partial \beta_3} = 0$$

Regression Analysis

In matrix format, first order model $\bar{\bar{y}}_{n\times 1} = \bar{\bar{X}}\bar{\bar{\beta}} + \bar{\bar{\varepsilon}}$ —— A

$$\bar{\bar{y}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}, \quad \bar{\bar{X}} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ & & & & \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}_{n\times(k+1)}, \quad \bar{\bar{\beta}} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}_{(k+1)\times 1}$$

$n = \#$ runs

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}$$

Least square function $= \sum \varepsilon_i^2 = \bar{\bar{\varepsilon}}^T \bar{\bar{\varepsilon}} = (\bar{\bar{y}} - \bar{\bar{X}}\bar{\bar{\beta}})^T(\bar{\bar{y}} - \bar{\bar{X}}\bar{\bar{\beta}})$

$$L = \bar{\bar{y}}^T\bar{\bar{y}} - 2\bar{\bar{\beta}}^T\bar{\bar{X}}^T y + \bar{\bar{\beta}}^T\bar{\bar{X}}^T\bar{\bar{X}}\bar{\bar{\beta}}$$

$\partial L/\partial \beta = 0$

Regression Analysis

$$\left.\frac{\partial L}{\partial \bar{\bar{\beta}}}\right|_{\beta} = -2\bar{\bar{X}}'y + 2\bar{\bar{X}}'\bar{\bar{X}}\bar{\beta} = 0$$

$$\hat{\beta} = (\bar{\bar{X}}'\bar{\bar{X}})^{-1}\bar{\bar{X}}'y$$

$$\hat{y} = \bar{\bar{X}}\hat{\bar{\beta}}$$

$$SS_E = \sum_{i=1}^{n \text{ runs}} (\underset{\text{exp}}{y_i} - \underset{\text{mod}}{\hat{y}_i})^2$$

$$= e'e = \bar{y}'\bar{y} - \hat{\bar{\beta}}\bar{\bar{X}}'\bar{y}$$

how many degrees of freedom?

$$n - (k+1)$$

Can be shown that $E(SS_E) = \sigma^2 (n-k-1)$

$$\hat{\sigma}^2 = \frac{SS_E}{n-k-1}$$

Example

First Order Model

DIY

Find regression coeff.

$$\hat{\beta} = (X'X)^{-1} X'y = \begin{bmatrix} 51 \\ 5.625 \\ 10.625 \\ 1.125 \end{bmatrix} \begin{matrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{matrix}$$

$4\times12 \quad 12\times1$

$$X_{12\times4} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{12\times4}$$

$$X'X = [\quad]_{4\times4}$$

$$= \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

ortho

$$X'y = [\quad]_{4\times1}$$

Regression Analysis of a 2^3 Factorial Design

A chemical engineer is investigating the yield of a process. Three process variables are of interest: temperature, pressure, and catalyst concentration. Each variable can be run at a low and a high level, and the engineer decides to run a 2^3 design with four center points. The design and the resulting yields are shown in Figure 10-5, where we have shown both the natural levels of the design factor and the +1, −1 coded variable notation normally employed in 2^k factorial designs to represent the factor levels.

Suppose that the engineer decides to fit a main effects only model, say

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon$$

	Process Variables			Coded Variables			
Run	Temp (°C)	Pressure (psig)	Conc. (g/l)	x_1	x_2	x_3	Yield, y
1	120	40	15	−1	−1	−1	32
2	160	40	15	1	−1	−1	46
3	120	80	15	−1	1	−1	57
4	160	80	15	1	1	−1	65
5	120	40	30	−1	−1	1	36
6	160	40	30	1	−1	1	48
7	120	80	30	−1	1	1	57
8	160	80	30	1	1	1	68
9	140	60	22.5	0	0	0	50
10	140	60	22.5	0	0	0	44
11	140	60	22.5	0	0	0	53
12	140	60	22.5	0	0	0	56

$$x_1 = \frac{\text{Temp} - 140}{20}, \quad x_2 = \frac{\text{Pressure} - 60}{20}, \quad x_3 = \frac{\text{Conc} - 22.5}{7.5}$$

Regression Analysis: Second-Order Model

$$y = \beta_0 + \sum \beta_i x_i + \sum \beta_{ij} x_i x_j + \sum \beta_{ii} x_i^2 + \epsilon$$

For example, if we have 2 variables $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \epsilon$

how many regression coeff $= 1 + k + {}^kC_2 + k$

$$= 1 + 2k + \frac{k(k-1)}{2} = \frac{k^2}{2} + \frac{3}{2}k + 1$$

$$= \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

$$= {}^{k+2}C_2$$

How to find 6 regression coeff.?

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \epsilon$$

$\beta_3 = \beta_{12}$ $\quad \beta_4 = \beta_{11}$ $\quad \beta_{22}$

$x_3 = x_1 x_2$ $\quad x_4 = x_1^2$ $\quad x_2^2$

			x_3	x_4	x_5
1	x_1	x_2	x_1x_2	x_1^2	x_2^2
1	1	1	1	1	1
1	1	-1	-1	1	1
1	-1	-1	1	1	1
1	-1	1	-1	1	1

X

$X_{n \times {}^{k+2}C_2}$

DOE for Fitting Second Order Model

Central Composite Design

- CCD consists of 2^k factorial design with n_F runs, 2k axial or star runs, and n_c center runs

total runs $= n_F + 2k + n_c$

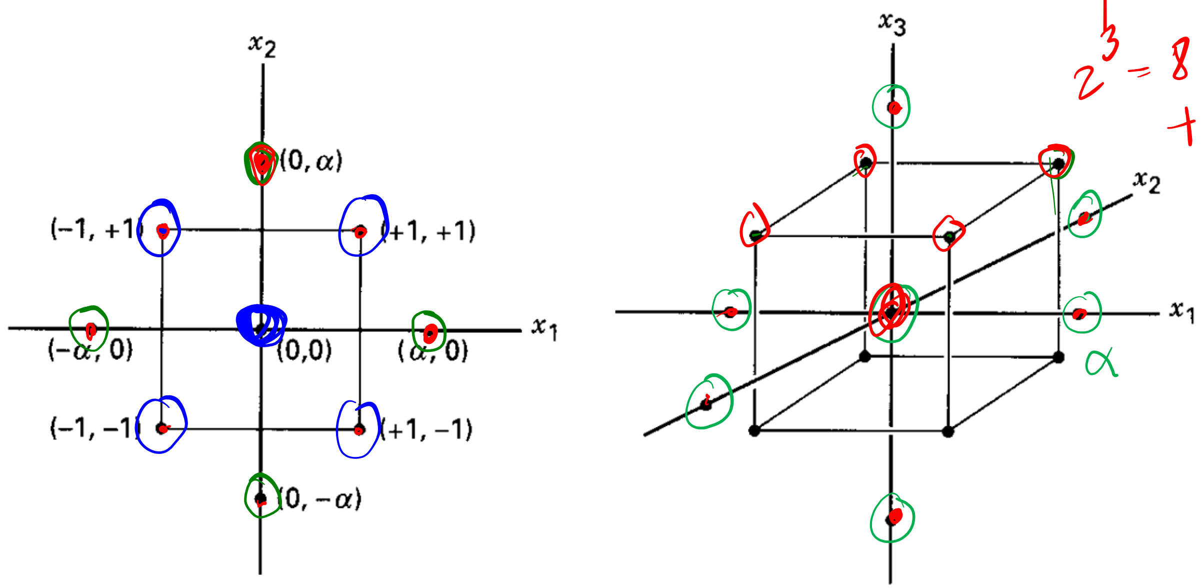


Figure 11-20 Central composite designs for $k = 2$ and $k = 3$.

$2^3 = 8$
$+ 2 \times 3$
$+ 3/4/5$

how do you choose α

Central Composite Design

- CCD consists of 2^k factorial design with n_F runs, 2k axial or star runs, and n_c center runs

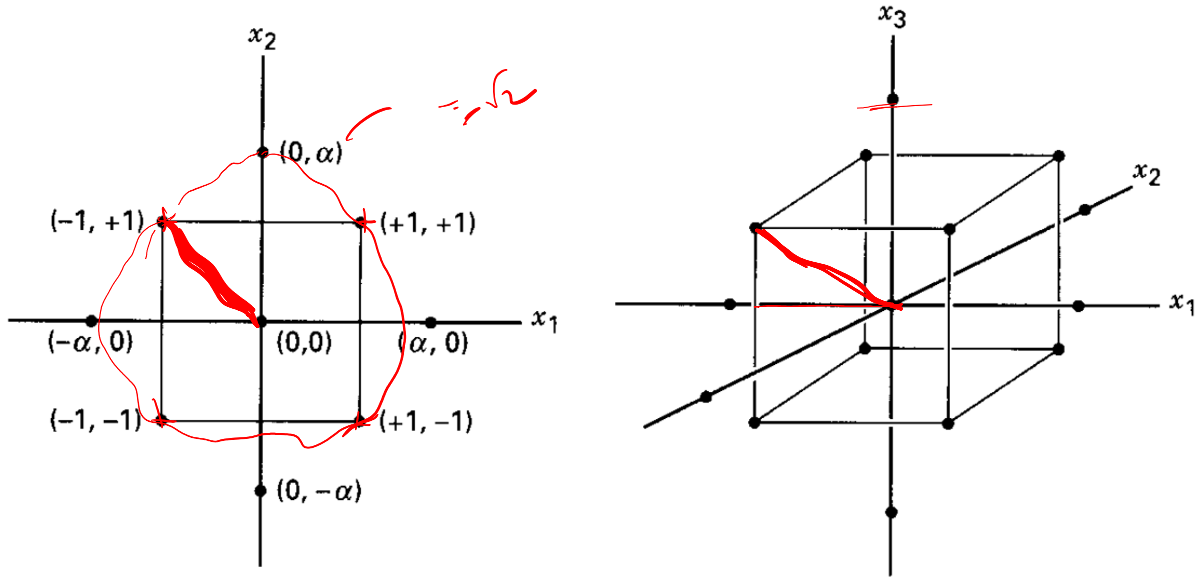


Figure 11-20 Central composite designs for $k = 2$ and $k = 3$.

Rotatibility

Rotatability

$E\left(\left(\hat{y}(x) - E(\hat{y}(x))\right)^2\right)$

It is important for the second-order model to provide good predictions throughout the region of interest. One way to define ''good'' is to require that the model have a reasonably consistent and stable variance of the predicted response at points of interest **x**. Recall Equation 10-40 that the variance of the predicted response at some point **x** is

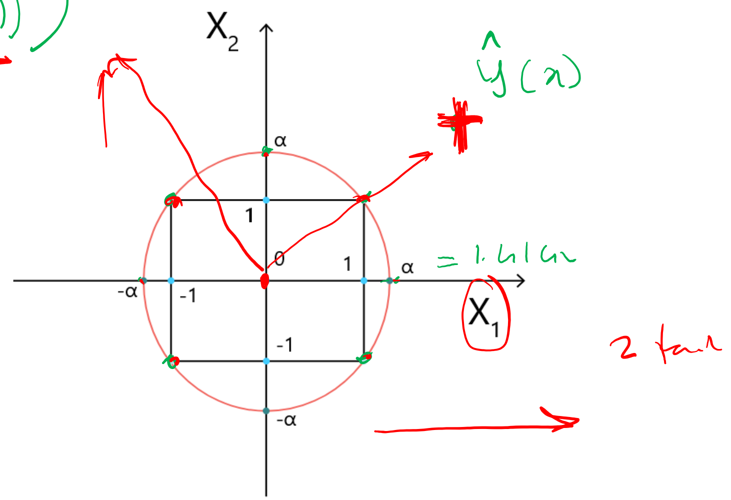
$$V[\hat{y}(\mathbf{x})] = \sigma^2\mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x} \qquad \hat{y}(x)$$

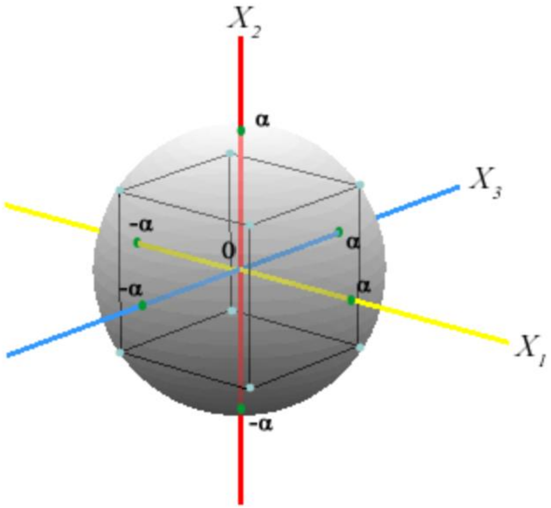
Box and Hunter (1957) suggested that a second-order response surface design should be **rotatable**. This means that the $V[\hat{y}(\mathbf{x})]$ is the same at all points **x** that are the same distance from the design center. That is, the variance of predicted response is constant on spheres.

Figure 11-21 (page 458) shows contours of constant $\sqrt{V[\hat{y}(\mathbf{x})]}$ for the second-order model fit using the CCD in Example 11-2. Notice that the contours of constant standard deviation of predicted response are concentric circles. A design with this property will leave the variance of $\hat{y}$ unchanged when the design is rotated about the center (0, 0, . . . , 0), hence, the name *rotatable* design.

Rotatability is a reasonable basis for the selection of a response surface design. Because the purpose of RSM is optimization and the location of the optimum is unknown prior to running the experiment, it makes sense to use a design that provides equal precision of estimation in all directions (it can be shown that any first-order orthogonal design is rotatable).

A central composite design is made rotatable by the choice of α. The value of α for rotatability depends on the number of points in the factorial portion of the design; in fact, $\alpha = (n_F)^{1/4}$ yields a rotatable central composite design where n_F is the number of points used in the factorial portion of the design.





Rotatibility

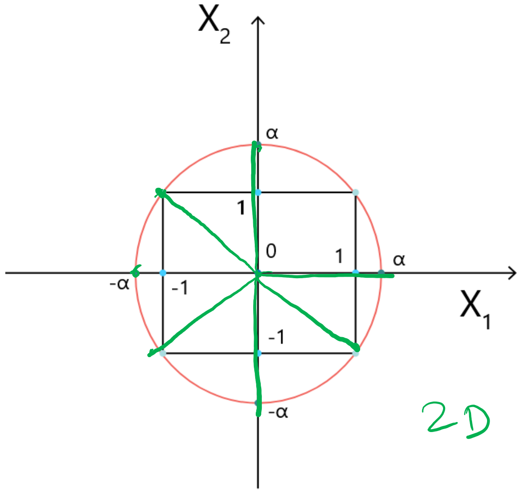
		k = 2	k = 3	k = 4	k = 5
Central Composite Designs	Factorial points 2^k	4	8	16	32
	Star points $2k$	4	6	8	10
	Center points n_c (varies)	5	5	6	6
	Total	13	19	30	48
3^k Designs		9	27	81	243
Choice of α	Spherical design ($\alpha = \sqrt{k}$)	1.4	1.73	2	2.24
	Rotatable design ($\alpha = (n_F)^{\frac{1}{4}}$)	1.4	1.68	2	2.38

n_F

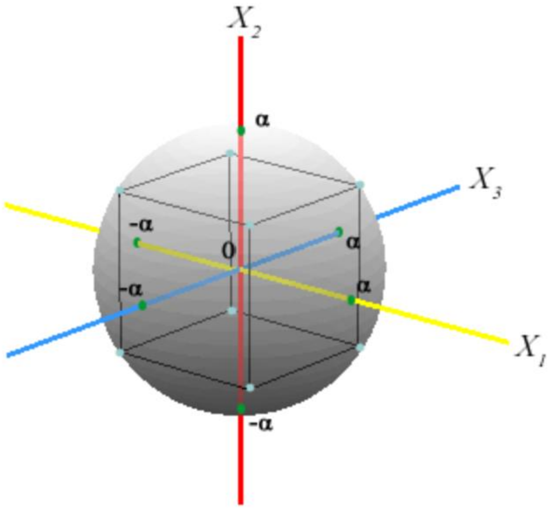
2^k

$+ 2k$ at

$+ n_c$



2D



3D

Center Runs in the CCD

The choice of α in the CCD is dictated primarily by the region of interest. When this region is a sphere, the design must include center runs to provide reasonably stable variance of predicted response. Generally, three to five center runs are recommended.

Box-Behnken Design

Table 11-8 A Three-Variable Box–Behnken Design

Run	x_1	x_2	x_3
1	−1	−1	0
2	−1	1	0
3	1	−1	0
4	1	1	0
5	−1	0	−1
6	−1	0	1
7	1	0	−1
8	1	0	1
9	0	−1	−1
10	0	−1	1
11	0	1	−1
12	0	1	1
13	0	0	0
14	0	0	0
15	0	0	0

Table 11-8 shows a three-variable Box–Behnken design. The design is also shown geometrically in Figure 11-22. Notice that the Box–Behnken design is a spherical design, with all points lying on a sphere of radius $\sqrt{2}$. Also, the Box–Behnken design does not contain any points at the vertices of the cubic region created by the upper and lower limits for each variable. This could be advantageous when the points on the corners of the cube represent factor-level combinations that are prohibitively expensive or impossible to test because of physical process constraints.

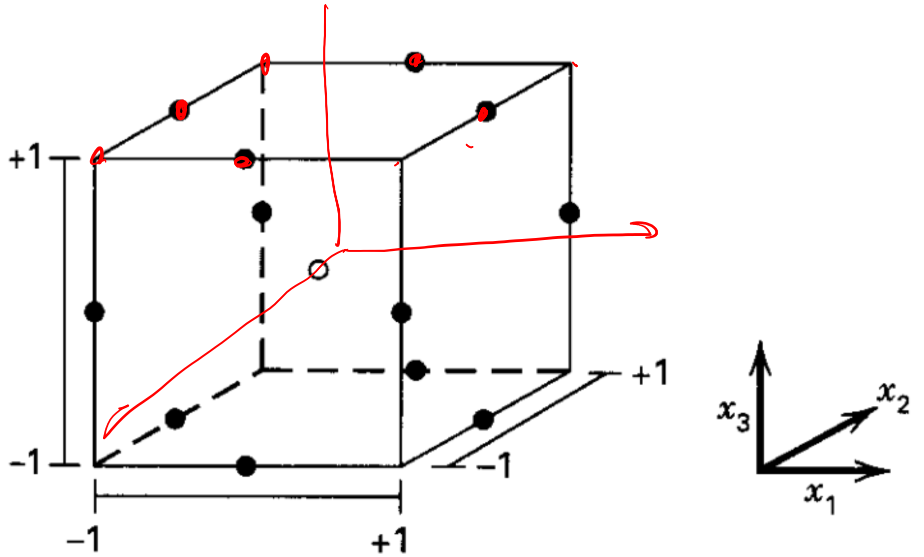


Figure 11-22 A Box–Behnken design for three factors.

Taguchi Orthogonal Designs

L4 Array

Row Number	Col 1	Col 2	Col 3
1	1	1	1
2	1	2	2
3	2	1	2
4	2	2	1

L8 Array

Test No	Col1	Col2	Col3	Col4	Col5	Col6	Col7
1	1	1	1	1	1	1	1
2	1	1	1	2	2	2	2
3	1	2	2	1	1	2	2
4	1	2	2	2	2	1	1
5	2	1	2	1	2	1	2
6	2	1	2	2	1	2	1
7	2	2	1	1	2	2	1
8	2	2	1	2	1	1	2

L9 Array

Row No	Col 1	Col 2	Col 3	Col 4
1	1	1	1	1
2	1	2	2	2
3	1	3	3	3
4	2	1	2	3
5	2	2	3	1
6	2	3	1	2
7	3	1	3	2
8	3	2	1	3
9	3	3	2	1

DIY:

(a) What is the effects model for these designs?

(b) How would you perform ANOVA of these designs?

Taguchi Orthogonal Designs

L18 Array

Row Number	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8
1	1	1	1	1	1	1	1	1
2	1	1	2	2	2	2	2	2
3	1	1	3	3	3	3	3	3
4	1	2	1	1	2	2	3	3
5	1	2	2	2	3	3	1	1
6	1	2	3	3	1	1	2	2
7	1	3	1	2	1	3	2	3
8	1	3	2	3	2	1	3	1
9	1	3	3	1	3	2	1	2
10	2	1	1	3	3	2	2	1
11	2	1	2	1	1	3	3	2
12	2	1	3	2	2	1	1	3
13	2	2	1	2	3	1	3	2
14	2	2	2	3	1	2	1	3
15	2	2	3	1	2	3	2	1
16	2	3	1	3	2	3	1	2
17	2	3	2	1	3	1	2	3
18	2	3	3	2	1	2	3	1

DIY:

(a) What is the effects model for these designs?

(b) How would you perform ANOVA of these designs?

ME 794

Statistical Design of Experiments

Instructors: Prof. Suhas Joshi, Prof. Soham Mujumdar (sohammujumdar@iitb.ac.in)

Robust Design Methodology

Acknowledgement: Design and Analysis of Experiments by Montegamory. Some of the course material has been adopted from similar courses taught previously by Prof. Shiv Kapoor (UofI), and Prof. Suhas Joshi (IITB).

Department of Mechanical Engineering
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The Quality Revolution

- In 1980, US lost their dominant position in manufacturing and electronics
  - 90% of semiconductors were being produced in Asia despite their advent in US
  - Large segment of US population were driving automobiles that were not produced in US
- What happened? Quality Revolution!
- What was quality revolution about?
  - Was it about the technological breakthroughs? ✖
  - Or development of new methods and approaches towards design and manufacturing? ✖
  - **It was a fundamental shift in the philosophical approach to quality** ✓
  - **Shifting away from '*product control mentality*' to '*process control mentality*'** ✓
  - **Shifting away from inspection as means to filter bad quality product to a system embracing robust design and process monitoring** ✓

The Quality Revolution: Change in Philosophy

- Moving away from a mentality which **tolerated and remained oblivious to the ills of scrap, defects, and waste** to one which embraces the never-ending pursuit of quality and productivity improvement.
- It was a departure from the paradigm which viewed quality as the **responsibility of one person or department** to one which views quality as the responsibility of everyone.
- It was essentially an embrace of "an ounce of prevention is worth a pound of cure"
- The revolution has sparked a new relationship between design and manufacturing.

Redefining Quality

- Traditional view of quality control that drove the field for more than half a century

  “Quality Control was the final filter between manufacturing and shipping established to ensure that the product met stated specifications before it went out to the customer” → **“Conformance to Specifications”**

- Today this narrow view of quality has been
  - **Pushed Upward:** Management’s role and responsibility to Quality
  - **Pushed Backwards:** Upstream into engineering design
  - **Spread Laterally:** No longer just a manufacturing issue
- “Continuous improvement in all aspects of the company”

What prevents continual improvement?

- Definition of quality as conformance to specifications promotes ‘product control’ and hence, significant inhibitor to principle of never-ending improvement through ‘process control’ approach.
- Some think it is important to calculate ‘cost of quality’ in some misdirected efforts towards ‘optimizing’ the system.
- Quality as an add-on function with associated cost:
  - Cost of inspection
  - Cost of implementing statistical process control (SPC) program
  - Cost of warranty and return
- We should NOT ask ‘What is the cost of quality?’
- We should ask ‘What is the cost of NOT having quality?’, i.e, what are the costs to the system for not continually striving for stable and predictable processes with low variability.

New Definition of Quality

“Loss imparted to society during product use as a result of functional variation”

Functional variation refers to the deviation of product performance from that intended by design, i.e., deviation of performance from the design target (nominal)

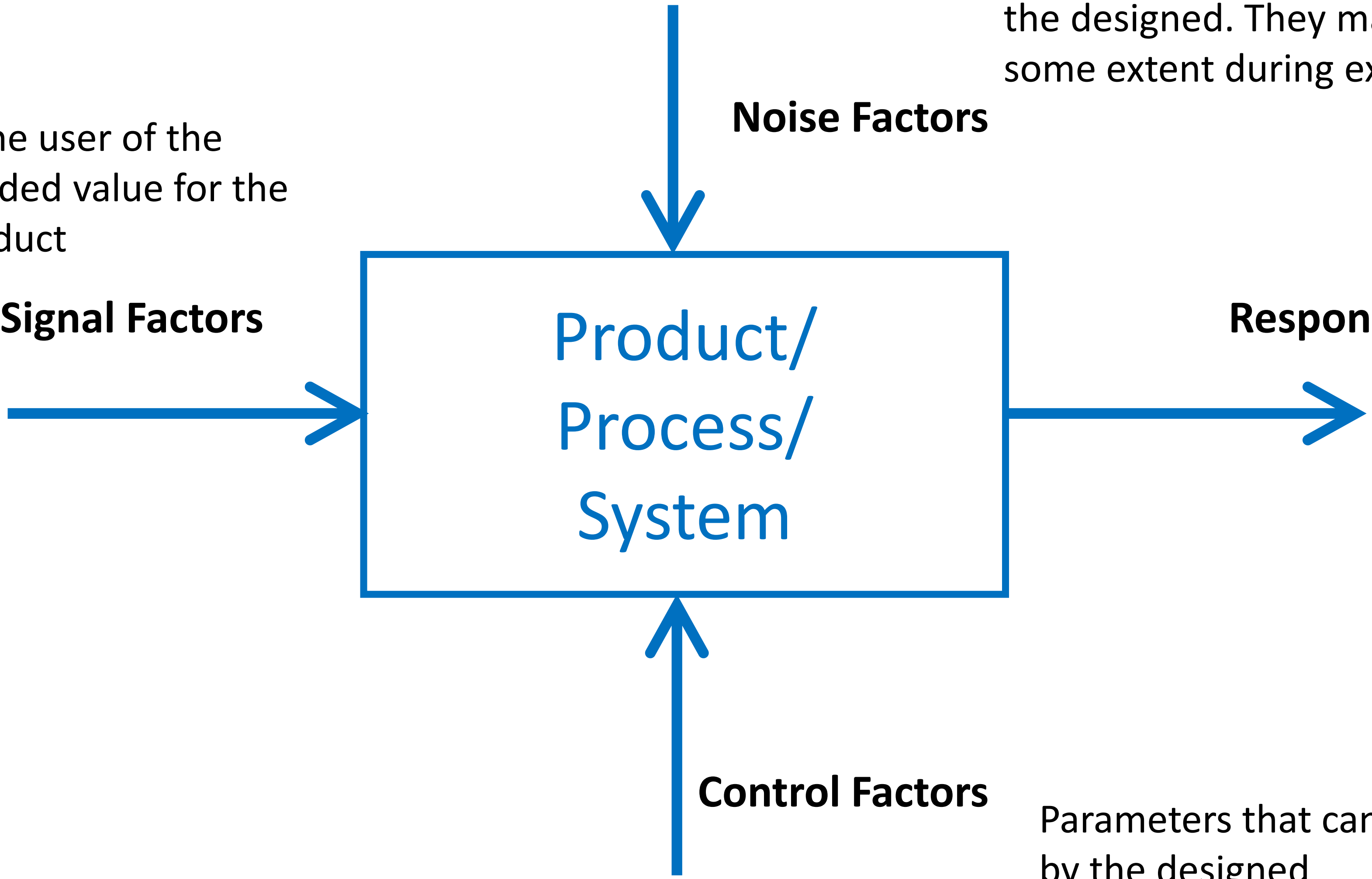
What is Variability?

- ALL manufacturing and measurement processes exhibit variation.
- For example, when we take sample data on the output of a process, such as critical dimensions, oxide thickness, or resistivity, we observe that all the values are NOT the same.
- This results in a collection of observed values distributed about some location value. This is what we call spread or variability.
- We represent variability numerically with the variance calculation and graphically with a histogram.

P Diagram

Parameters that cannot be controlled by the designed. They may be controllable to some extent during experiment

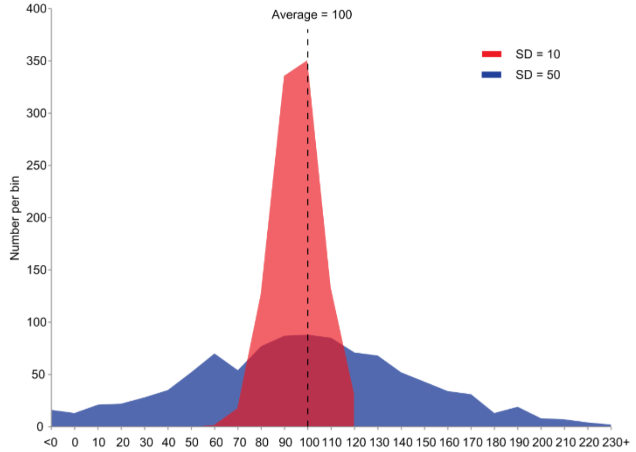
Parameters set by the user of the product to get intended value for the response of the product



Parameters that can be specified freely by the designed

Sources of Functional Variation

Taguchi suggests that variation in product and process function arises from three basic sources:



- **Outer Noise:** Sources of noise which influence performance as measured during field use under actual *operating conditions*, e.g., temp, humidity, supply voltage, vibration

- **Inner Noise:** Internal change in product characteristics such as drift from the nominal over time due to *deterioration*, e.g., mechanical wear, aging

- **Variational Noise:** Variation in the product parameters from one unit to another as a result of the manufacturing process, e.g., *manufacturing imperfection*

Example 1: Temperature Controlled Refrigerator

- **Outer Noise** [Operating Conditions]
    - The number of times the door is opened and closed
    - The amount of food kept and the initial temperature of the food
    - Variation in the ambient temperature
    - Supply voltage variation

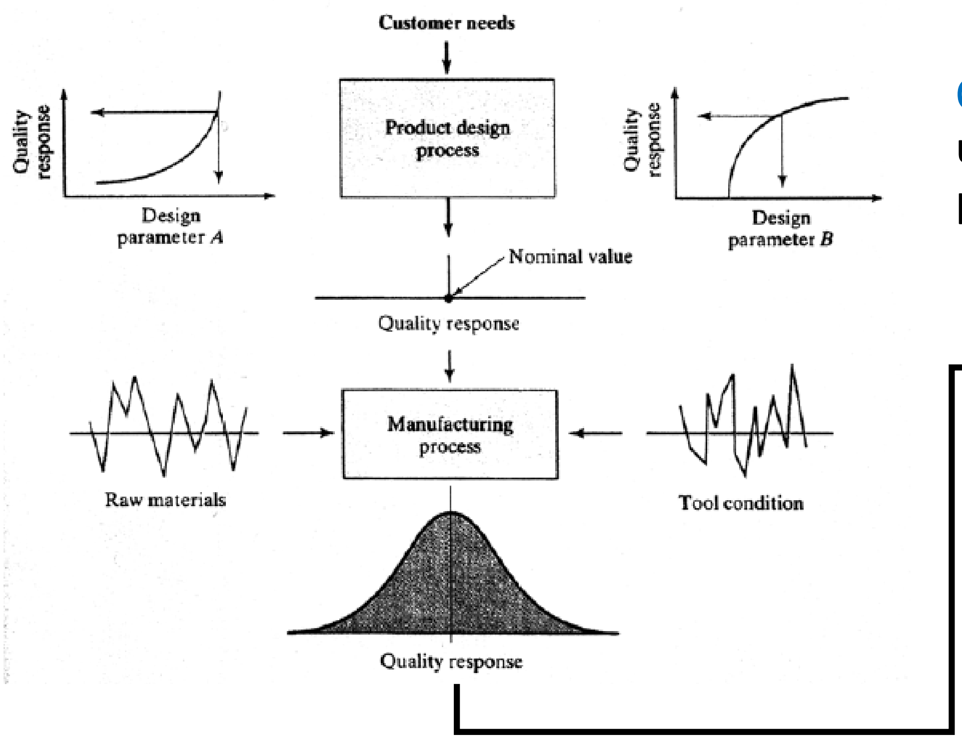
- **Inner Noise** [Deterioration]
    - The leakage of Refrigerant
    - Mechanical Wear of Compressor parts

- **Variational Noise** [Mfg/Use Imperfection]
    - The tightness of door closure
    - The amount of refrigerant used

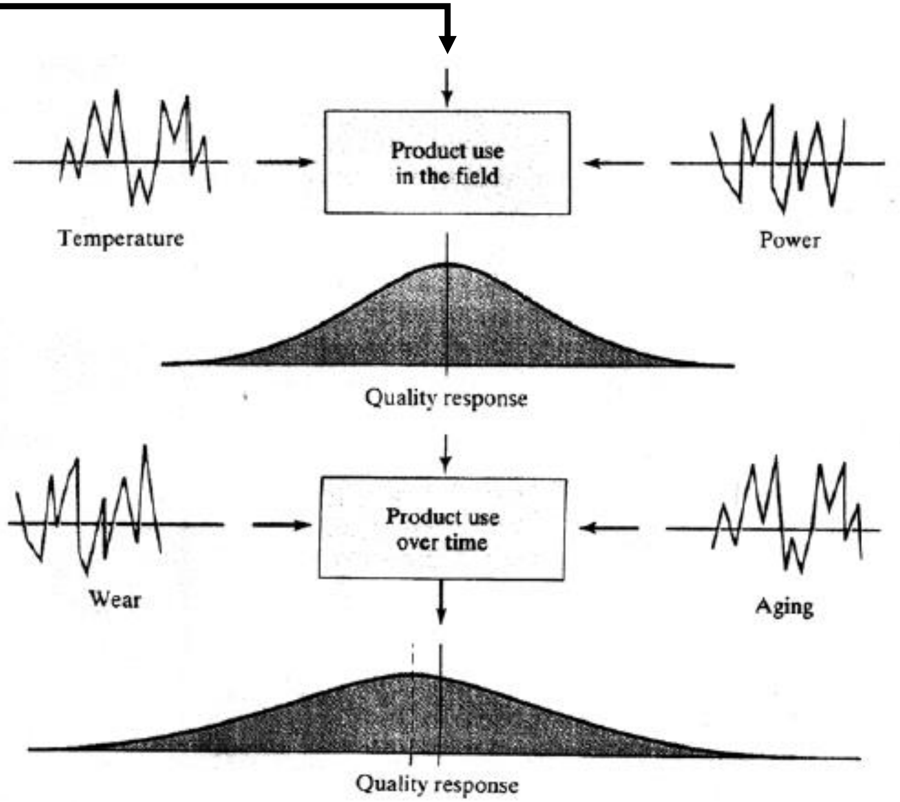
Example 2: Braking Distance of a Car

- Outer Noise [Operating Conditions]
  - Wet or dry road
  - Concrete or Asphalt pavement
  - Number of passengers in the car
- Inner Noise [Deterioration]
  - The leakage of brake fluid
  - Wear of brake drums and brake pads
- Variational Noise [Mfg/Use Imperfection]
  - Variation in friction coefficient of pads and drums
  - The amount of brake fluid

How do these sources affect Quality?



Outer Noise and Inner Noise can be dealt with upstream effectively only by the design process



Variational noise is a matter of manufacturing imperfection and can be dealt with *Statistical Process Control (SPC).*

Robust Parameter Design

- **Robust parameter design (RPD)** is an approach to product realization activities that focuses on choosing the levels of controllable factors (or parameters) in a process or a product to achieve two objectives:
  1. to ensure that the mean of the output response is at a desired level or target
  2. to ensure that the variability around this target value is as small as possible
- When an RPD study is conducted on a process, it is usually called a **process robustness study.**
- The general RPD problem was developed by a Japanese engineer, Genichi Taguchi, and introduced in the United States in the 1980s
- Taguchi proposed an approach to solving the RPD problem based on designed experiments and some novel methods for analysis of the resulting data.

Robust Parameter Design

In a robust design problem, the focus is usually on one or more of the following:

1. Designing systems that are insensitive to environmental factors that can affect performance once the system is deployed in the field. Example: Development of an exterior paint that should exhibit long life when exposed to a variety of weather conditions.

2. Designing products so that they are insensitive to variability transmitted by the components of the system. An example is designing an electronic amplifier so that the output voltage is as close as possible to the desired target regardless of the variability in the electrical parameters of the transistors, resistors, and power supplies that are the components of the system.

3. Designing processes so that the manufactured product will be as close as possible to the desired target specifications, even though some process variables (such as temperature) or raw material properties are impossible to control precisely.

4. Determining the operating conditions for a process so that the critical process characteristics are as close as possible to the desired target values and the variability around this target is minimized. Examples of this type of problem occur frequently. For example, in semiconductor manufacturing we want the oxide thickness on a wafer to be as close as possible to the target mean thickness, and we want the variability in thickness across the wafer (a measure of uniformity) to be as small as possible

Crossed Array Design

- The original Taguchi methodology for the RPD problem revolved around the use of a statistical design for the controllable variables and another statistical design for the noise variables.
- Then these two designs were "crossed"; that is, every treatment combination in the design for the controllable variables was run in combination with every treatment combination in the noise variable design. This type of experimental design was called a crossed array design.

Example: Leaf Spring Experiment

In this experiment, **five factors** were studied to determine their effect on the free height of a leaf spring used in an automotive application.

Out of five,

four are control variables, (a) furnace Temperature, (b) heating time, (c) transfer time, (d) hold don time

and one is noise variable (e) quench oil temperature

Crossed Array Design

Example: Leaf Spring Experiment

The design for the controllable factors is a 2^{4-1} fractional factorial design with generator D = ABC. This is called the inner array design.

The design for the single noise factor is a 2^1 design, and it is called the outer array design.

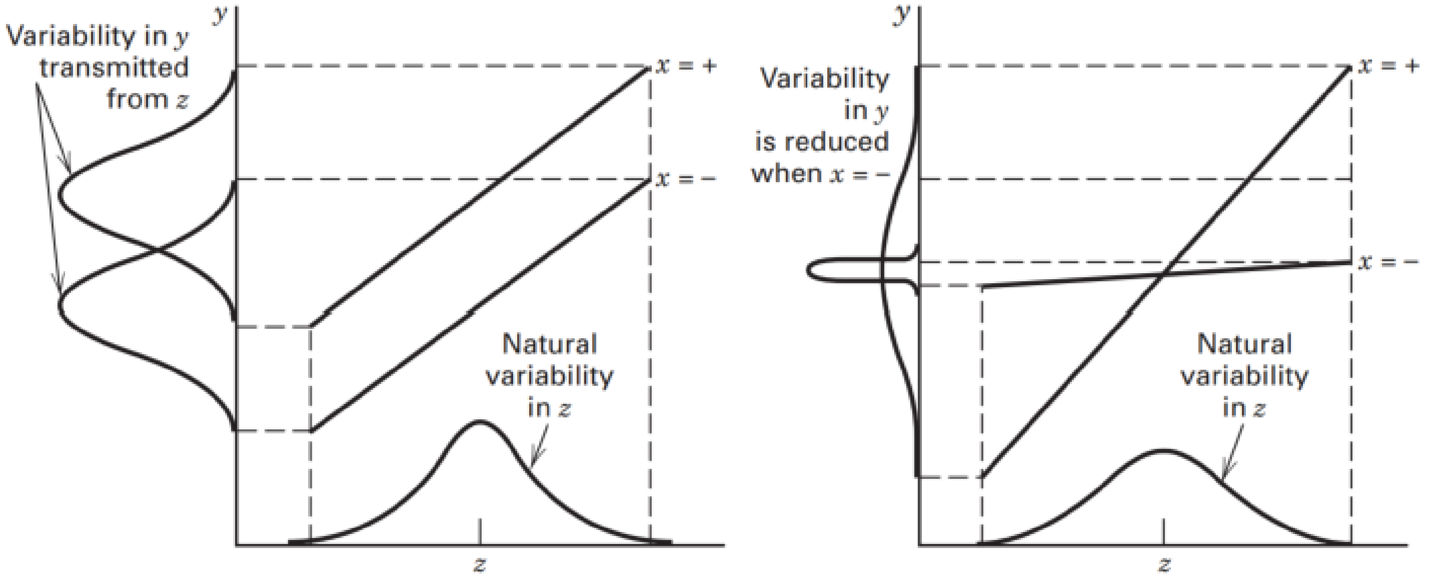
Notice how each run in the outer array is performed for all eight treatment combinations in the inner array, producing the crossed array structure. In the leaf spring experiment, each of the 16 distinct design points was replicated three times, resulting in 48 observations on free height.

The Leaf Spring Experiment

A	B	C	D	$E = -$	$E = +$	$\bar{y}$	s^2
−	−	−	−	7.78, 7.78, 7.81	7.50, 7.25, 7.12	7.54	0.090
+	−	−	+	8.15, 8.18, 7.88	7.88, 7.88, 7.44	7.90	0.071
−	+	−	+	7.50, 7.56, 7.50	7.50, 7.56, 7.50	7.52	0.001
+	+	−	−	7.59, 7.56, 7.75	7.63, 7.75, 7.56	7.64	0.008
−	−	+	+	7.54, 8.00, 7.88	7.32, 7.44, 7.44	7.60	0.074
+	−	+	−	7.69, 8.09, 8.06	7.56, 7.69, 7.62	7.79	0.053
−	+	+	−	7.56, 7.52, 7.44	7.18, 7.18, 7.25	7.36	0.030
+	+	+	+	7.56, 7.81, 7.69	7.81, 7.50, 7.59	7.66	0.017

Crossed Array Design

- An important point about the crossed array design is that it provides information about interactions between controllable factors and noise factors. These interactions are crucial to the solution of an RPD problem.
- For example, consider the two-factor interaction graph, where x is the controllable factor and z is the noise factor.
- In Figure A, there is no interaction between x and z; therefore, there is no setting for the controllable variable x that will affect the variability transmitted to the response by the variability in the noise factor z.
- However, in Figure B, there is a strong interaction between x and z. Note that when x is set to its low level, there is much less variability in the response variable than when x is at the high level.



(a) No control × noise interaction (b) Significant control × noise interaction

Robust Design Strategy

- **Stage 1: Parameter Design**
  - Step 1: Reduce sensitivity to noise (reduce variability)
    - Find mean ($\bar{x}$) and std. dev (s)
    - Define signa-to-noise ratio, $\eta = 10\log_{10}\left(\frac{\bar{x}^2}{s^2}\right)$
    - Minimize η
  - Step 2: Set the mean ($\bar{x}$) on target using parameters that affect the variability the least

- **Stage 2: Tolerance Design**
  - Select Tolerances based on Cost vs. Performance (not covered in this class)
  - This step may become unnecessary if design parameters reduce variance sufficiently
  - Systematically balance cost and performance improvement

Example

Anti Satellite Attack Missile

Response variable is distance to shoot

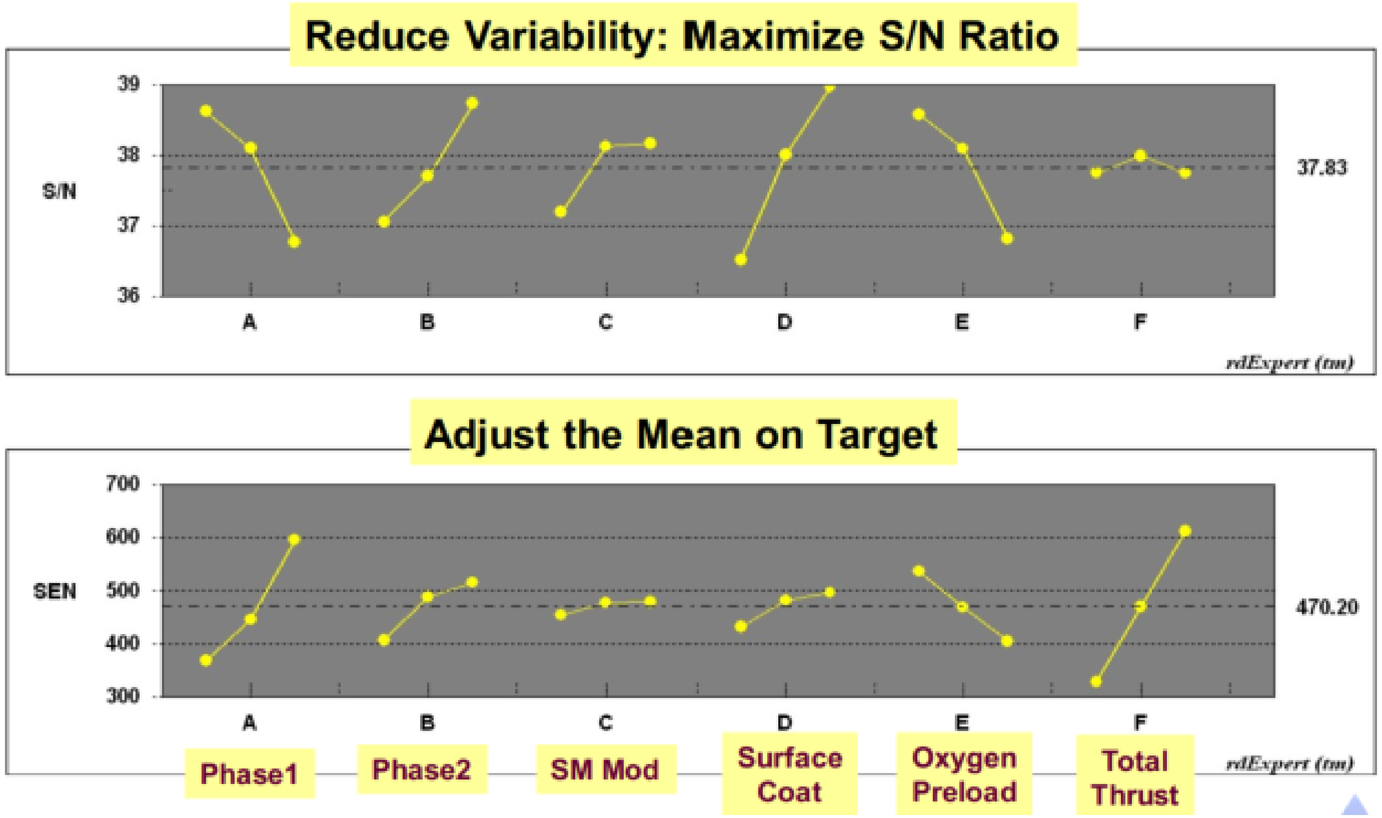


Image from Dr. Phadke's workshop