CS 207M Discrete Structures: Quiz-I Solutions

1. [**4 marks**] Let $f : X \to Y$ be a function from a set X to a set Y. For $A \subseteq X$ and $B \subseteq Y$, we define

$$f(A) = \{f(a) | a \in A\} \text{ and } f^{-1}(B) = \{x \in X | f(x) \in B\}$$

For each of the following four statements, decide if it is true of false. You must justify your answer. In other words, either provide a small proof or provide a counter-example.

Answer:
Suppose $X = \{1, 2\}, Y = \{a, b\}$ and $f : X \to Y$ with $f(1) = f(2) = a$. This will be used as a counter-example to the first three statements which are false.

(a) $A = f^{-1}(f(A))$.
The statement is false. With $A = \{1\}$, $f(A) = \{a\}$ and $f^{-1}(f(A)) = \{1, 2\}$. Therefore, $A \neq f^{-1}(f(A))$.
We always have $A \subseteq f^{-1}(f(A))$.

(b) $B = f(f^{-1}(B))$.
The statement is false. With $B = \{a, b\}$, $f^{-1}(B) = \{1, 2\}$ and $f(f^{-1}(B)) = \{a\}$. Therefore, $B \neq f(f^{-1}(B))$.
We always have $B \supseteq f(f^{-1}(B))$.

(c) $f(A) = B$ iff $f^{-1}(B) = A$.
The statement is false. With $A = \{1\}$ and $B = \{a\}$, $f(A) = B$ but $\{1, 2\} = f^{-1}(B) \neq A$.
We always have $f(A) = B$ *implies* $A \subseteq f^{-1}(B)$ *and* $f^{-1}(B) = A$ *implies* $f(A) \subseteq B$.

(d) $f(A) \cap B \neq \emptyset$ iff $A \cap f^{-1}(B) \neq \emptyset$.
This statement is true and can be proved as follows.
Suppose $f(A) \cap B \neq \emptyset$. Then there exists $b \in B$ and $a \in A$ such that $f(a) = b$. Clearly, $a \in f^{-1}(B)$ and hence $A \cap f^{-1}(B) \neq \emptyset$.
Conversely, suppose $A \cap f^{-1}(B) \neq \emptyset$. Then there exists $a \in A$ such that $f(a) \in B$. Clearly, $f(a) \in f(A)$. As $f(a)$ also belongs to B, $f(A) \cap B \neq \emptyset$.

Marking Scheme: Correct answer with correct justification is awarded 1 mark. A correct true/false answer with no/incorrect justification will NOT be awarded any mark.

2. [**2 marks**] Let R be a relation on a set A. Show that R is transitive iff $R^2 \subseteq R$. Note that the relation R^2 is $R \circ R$, that is, composition of R with itself.

Answer:
($\Rightarrow$) Suppose R is transitive. Let $(a, b) \in R^2$. By definition of R^2, there exists $c \in A$ such that $(a, c), (c, b) \in R$. As R is transitive, $(a, b) \in R$. This shows that $R^2 \subseteq R$.

($\Leftarrow$) Now suppose $R^2 \subseteq R$. Let $a, b, c \in A$ be such that $(a, b), (b, c) \in R$. By definition of R^2, $(a, c) \in R^2$. As $R^2 \subseteq R$, $(a, c) \in R$. This shows that R is transitive.

Marking Scheme: A complete and correct proof of each (left-to-right and right-to-left implication) direction is awarded 1 mark.

3. [**4 marks**] Prove or disprove: The sets $(2^N)^N$ and $2^{(2^N)}$ have the same cardinality.

Answer:
There is a natural and canonical bijection between $(X^Y)^Z$ and $X^{Y \times Z}$. Given $f \in (X^Y)^Z$, that is $f : Z \to X^Y$, we define $g : Y \times Z \to X$ as: $g((y, z)) = (f(z))(y)$. Note that $f(z)$ is itself a function from Y to X. It is easily checked that $f \mapsto g$ is a bijection between $(X^Y)^Z$ and $X^{Y \times Z}$ by explicitly writing down its inverse.

Let $A = (2^N)^N$ and $B = 2^{(2^N)}$ be the two sets under consideration. By the above argument, we have a canonical bijection between A and $2^{N \times N}$. We have a bijection from $N \times N$ to N (done in the class). This leads to a direct and natural bijection between $2^{N \times N}$ and 2^N (tutorial problem: a bijection between X and Y gives a bijection between 2^X and 2^Y). Altogether, there is a bijection between A and 2^N.

By Cantor's diagonalization theorem (done in the class), there does not exist a bijection between 2^N and $2^{(2^N)} = B$. As a result, there does not exist a bijection between A and B and they do not have the same cardinality.

Marking Scheme: Three marks for the proof that $(2^N)^N$ has same cardinality as 2^N. Any correct argument which does that is fine. Application of Cantor's theorem, to conclude that the two sets under consideration have different cardinalities, will be awarded 1 mark.