Planar Kinetics ;

Work-Energy theorem

For a particle: $\Delta U = \Delta T$

$$\underbrace{\int_1^2 \mathbf{F}\cdot d\mathbf{r}}_{\text{Work done by external force}} = \underbrace{\frac{1}{2} m\left(v_2^2 - v_1^2\right)}_{\text{change in Kinetic energy}}$$

Work done by gravity and elastic spring was accounted separately through potential V^g and V^e respectively

$$\therefore \Delta U = \Delta U' - \Delta V^e - \Delta V^g$$

$$\therefore \boxed{\Delta U' = \Delta T + \Delta V^e + \Delta V^g}$$

$\Delta U'$: Work done by forces other than gravity and elastic spring.

Extension to number (tending to ∞) particles i.e. continuous body:

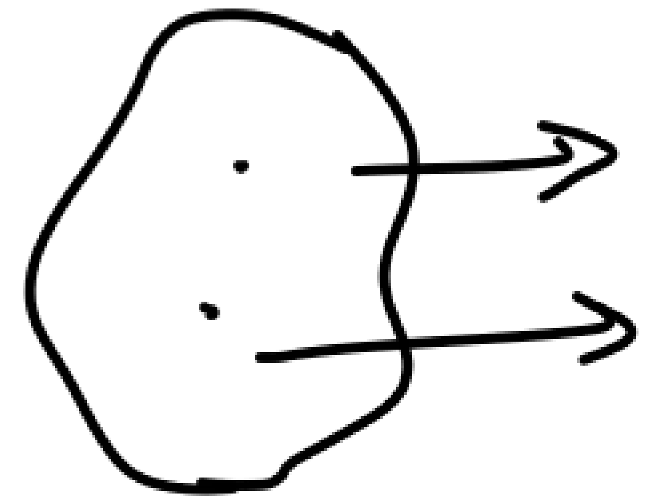
No change in the work done ΔU expressions.

$$T = \sum \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \int_m \frac{1}{2} dm (\mathbf{v}\cdot\mathbf{v})$$

Our focus is planar motion.

① <u>Translation</u> :

$$\underset{\sim}{v} = \underset{\sim}{\bar{v}}$$



$$T = \int_m \frac{1}{2}\, dm\, \underset{\sim}{v} \cdot \underset{\sim}{v} = \frac{m}{2}\left(\underset{\sim}{\bar{v}} \cdot \underset{\sim}{\bar{v}}\right) = \frac{m}{2}\left(\bar{v}\right)^2$$

② <u>Rotation</u> :

$OA = r$

$\underset{\sim}{r} = r\,\underset{\sim}{\hat{e}_r}$ $\quad \underset{\sim}{\hat{e}_\theta}$ $\quad \underset{\sim}{\hat{e}_r}$

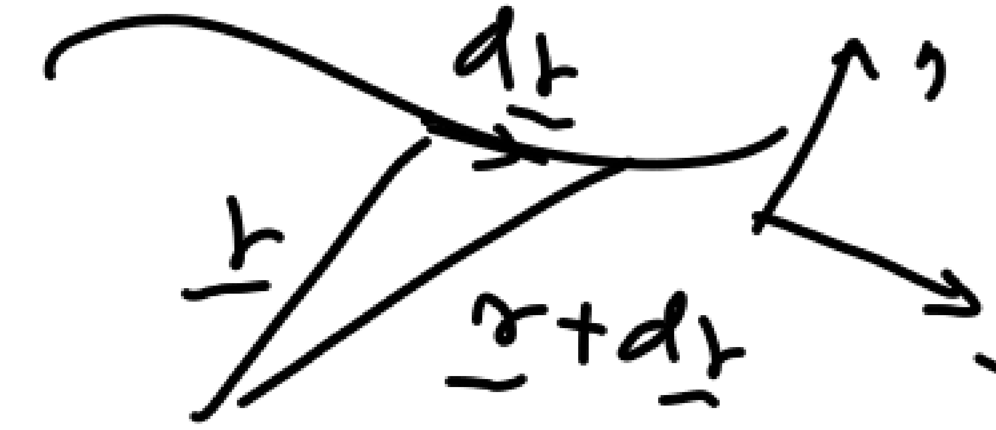
$\underset{\sim}{\omega} = \omega\, \underset{\sim}{\hat{e}_3}$

$$\underset{\sim}{v} = \underset{\sim}{\omega} \times \underset{\sim}{r}$$

$$\underset{\sim}{v} \cdot \underset{\sim}{v} = \left(\underset{\sim}{\omega} \times \underset{\sim}{r}\right) \cdot \left(\underset{\sim}{\omega} \times \underset{\sim}{r}\right)$$

$$= |\underset{\sim}{\omega}|^2 |\underset{\sim}{r}|^2 = \omega^2 r^2$$

$$\underset{\sim}{F} \cdot d\underset{\sim}{r}$$



$$= \left(F_t\, \underset{\sim}{\hat{t}} + F_n\, \underset{\sim}{\hat{n}}\right) \cdot (ds)\, \underset{\sim}{\hat{t}}$$

$$= \left(F_t\, ds\right) \qquad (F = ma)$$

$$= m\, a_t\, ds$$

$$= m \frac{dv}{dt}\, ds$$

$$= m \frac{dv}{dt} \frac{ds}{dt}\, dt$$

$$= m\, v \frac{dv}{dt}\, dt$$

$$= \int m\, v\, dv$$

Coming back to rotation

$$T = \int_m \frac{1}{2} dm\, \omega^2 r^2$$

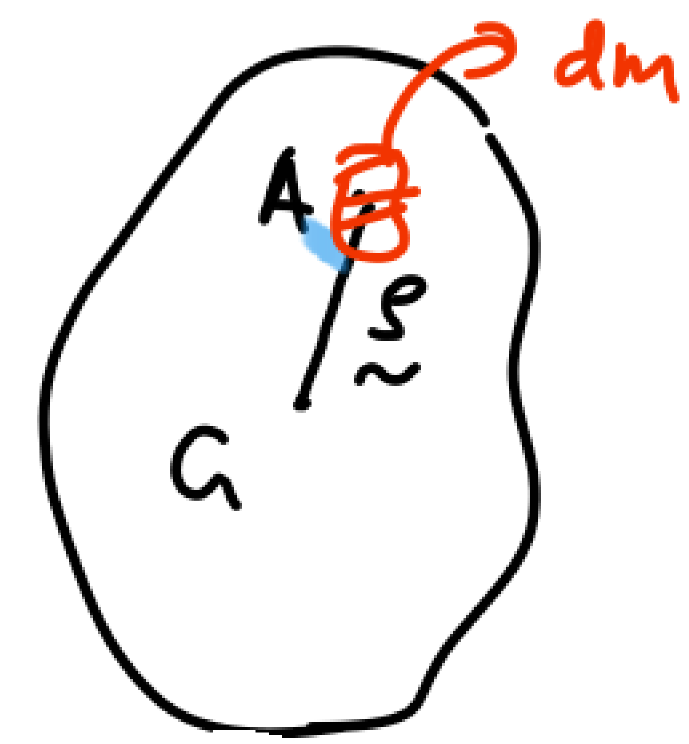
$$= \frac{\omega^2}{2} \int_m (dm\, r^2)$$

$$\boxed{T = \frac{I_o \omega^2}{2}}$$

where I_o } mass moment of inertia about axis $\perp$ to plane and passing through fixed point O.

③ General motion

motion = Translation + Rotation



$\underset{\sim}{\rho}$ } vector joining centre of mass G to a general point A

For point A : $\underset{\sim}{v} = \underset{\sim}{v}_G + \underset{\sim}{v}_{A/G}$

$$= \bar{\underset{\sim}{v}} + \underset{\sim}{\omega} \times \underset{\sim}{\rho}$$

$$\underset{\sim}{v} \cdot \underset{\sim}{v} = (\bar{\underset{\sim}{v}} + \underset{\sim}{\omega} \times \underset{\sim}{\rho}) \cdot (\bar{\underset{\sim}{v}} + \underset{\sim}{\omega} \times \underset{\sim}{\rho})$$

$$= \bar{\underset{\sim}{v}} \cdot \bar{\underset{\sim}{v}} + (\underset{\sim}{\omega} \times \underset{\sim}{\rho}) \cdot \bar{\underset{\sim}{v}} + \bar{\underset{\sim}{v}} \cdot (\underset{\sim}{\omega} \times \underset{\sim}{\rho})$$

$$+ (\underset{\sim}{\omega} \times \underset{\sim}{\rho}) \cdot (\underset{\sim}{\omega} \times \underset{\sim}{\rho})$$

$$\underset{\sim}{v}\cdot\underset{\sim}{v} = \bar{\underset{\sim}{v}}\cdot\bar{\underset{\sim}{v}} + 2(\underset{\sim}{\omega}\times\underset{\sim}{\rho})\cdot\bar{\underset{\sim}{v}} + (\underset{\sim}{\omega}\times\underset{\sim}{\rho})\cdot(\underset{\sim}{\omega}\times\underset{\sim}{\rho})$$

$$= (\bar{v})^2 + \omega^2\rho^2 + 2\,[\bar{\underset{\sim}{v}}, \underset{\sim}{\omega}, \underset{\sim}{\rho}]$$

$$[\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}] := \underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c})$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\underset{\sim}{c}, \underset{\sim}{a}, \underset{\sim}{b}] = [\underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{a}]$$

$$= (\bar{v})^2 + \omega^2\rho^2 + 2\,[\underset{\sim}{\rho}, \bar{\underset{\sim}{v}}, \underset{\sim}{\omega}]$$

cyclic property

$$\underset{\sim}{v}\cdot\underset{\sim}{v} = (\bar{v})^2 + \omega^2\rho^2 + 2\underset{\sim}{\rho}\cdot(\bar{\underset{\sim}{v}}\times\underset{\sim}{\omega})$$

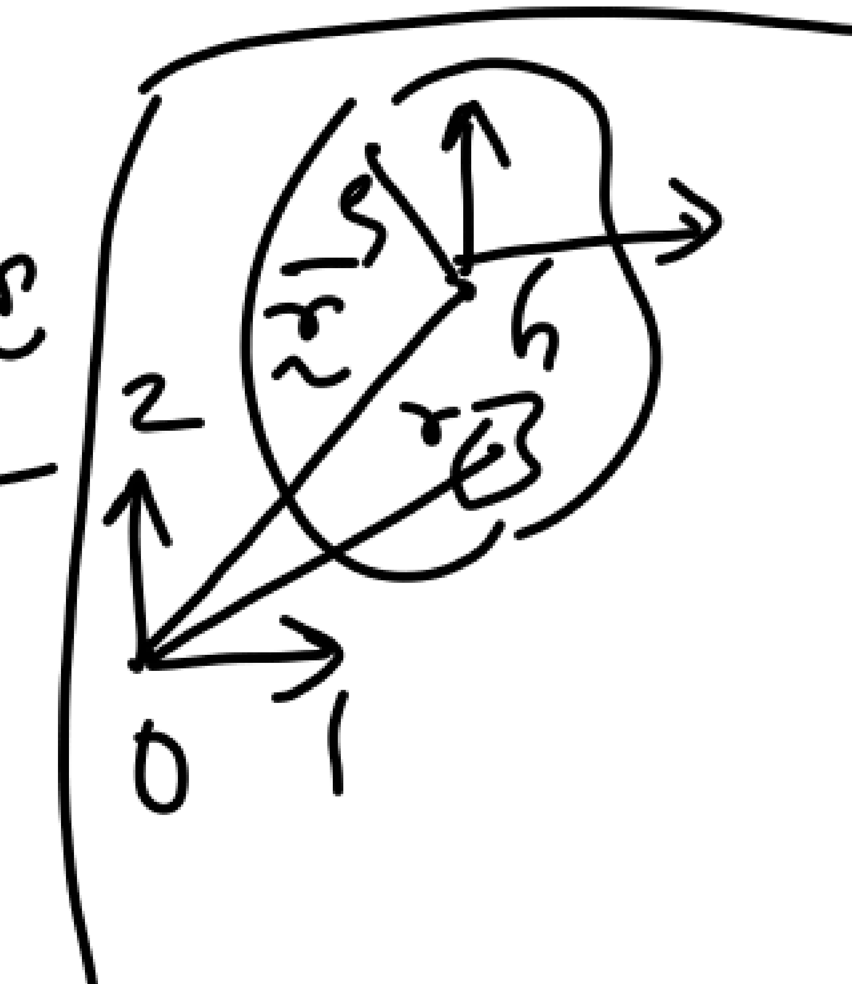
$$T = \int_m \frac{1}{2}\,dm\,(\underset{\sim}{v}\cdot\underset{\sim}{v})$$

$$= \int_m \frac{1}{2}\,dm\left[(\bar{v})^2 + \omega^2\rho^2 + 2\underset{\sim}{\rho}\cdot(\bar{\underset{\sim}{v}}\times\underset{\sim}{\omega})\right]$$

$$= \frac{1}{2}m(\bar{v})^2 + \frac{\omega^2}{2}\int_m \rho^2\,dm + \underbrace{\cancel{\left(\int_m \underset{\sim}{\rho}\,dm\right)}}_{\to\,\underset{\sim}{0}}\cdot(\bar{\underset{\sim}{v}}\times\underset{\sim}{\omega})$$

$$\bar{\underset{\sim}{r}} = \frac{\int_m dm\,\underset{\sim}{r}}{m}$$

$$\underset{\sim}{0} = \frac{\int_m \underset{\sim}{\rho}\,dm}{m}$$



$$\boxed{T = \frac{m(\bar{v})^2}{2} + \frac{1}{2} I_G \omega^2} \quad (*)$$

where $I_G = \int_m \frac{1}{2} \rho^2 \, dm$

is mass moment of inertia about an axis $\perp$ to the plane and passing through centre of mass G.

$$\boxed{T = \text{K.E of translation} + \text{K.E of rotation}}$$

(*) is the most general expression.

<u>For pure translation</u>

$$\omega = 0; \quad T = \frac{1}{2} m(\bar{v})^2$$

<u>For pure rotation:</u>

$$\underset{\sim}{\bar{v}} = \underset{\sim}{\omega} \times \underset{\sim}{\bar{r}}$$

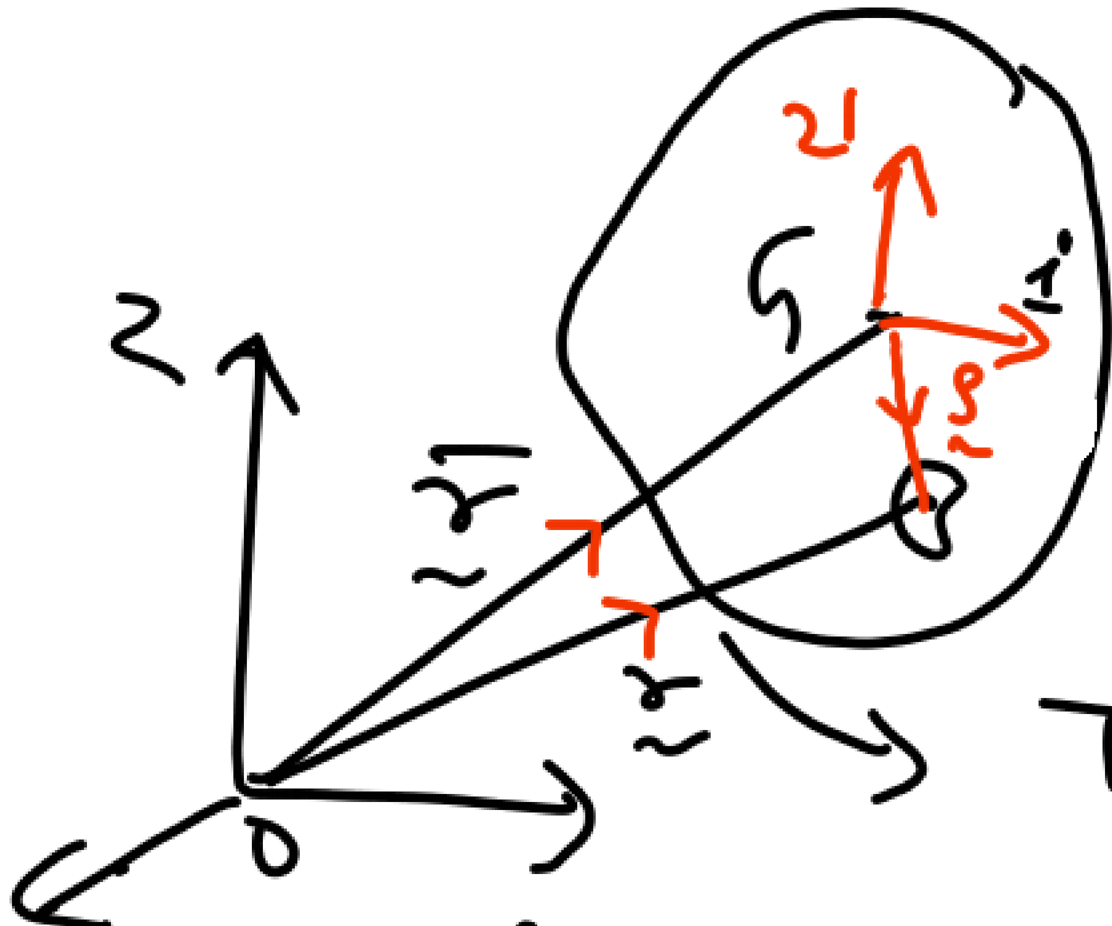
$$(\bar{v})^2 = \underset{\sim}{\bar{v}} \cdot \underset{\sim}{\bar{v}}$$

$$= \omega^2 (\bar{r})^2$$

$$T = \frac{1}{2} m \omega^2 (\bar{r})^2 + \frac{1}{2} I_G \omega^2$$

$$T = \frac{\omega^2}{2} \underbrace{\left[I_G + m(\bar{r})^2 \right]}_{\text{Parallel axis theorem} \; = I_O}$$

$$T = \frac{1}{2} I_O \omega^2$$



Triangle law

$$\bar{\underset{\sim}{r}} + \underset{\sim}{\rho} = \underset{\sim}{r}$$

$$\bar{\underset{\sim}{r}} = \frac{\int_m dm\, \underset{\sim}{r}}{m}$$

$$\bar{\underset{\sim}{r}} = \frac{\int_m dm\,(\bar{\underset{\sim}{r}} + \underset{\sim}{\rho})}{m}$$

$$= \frac{\int_m dm\, \bar{\underset{\sim}{r}} + \int_m \underset{\sim}{\rho}\, dm}{m}$$

$$= \frac{\cancel{m}\,\bar{\underset{\sim}{r}}}{\cancel{m}} + \frac{\int_m \underset{\sim}{\rho}\, dm}{m}$$

$$\bar{\underset{\sim}{r}} = \bar{\underset{\sim}{r}} + \frac{\int_m \underset{\sim}{\rho}\, dm}{m}$$

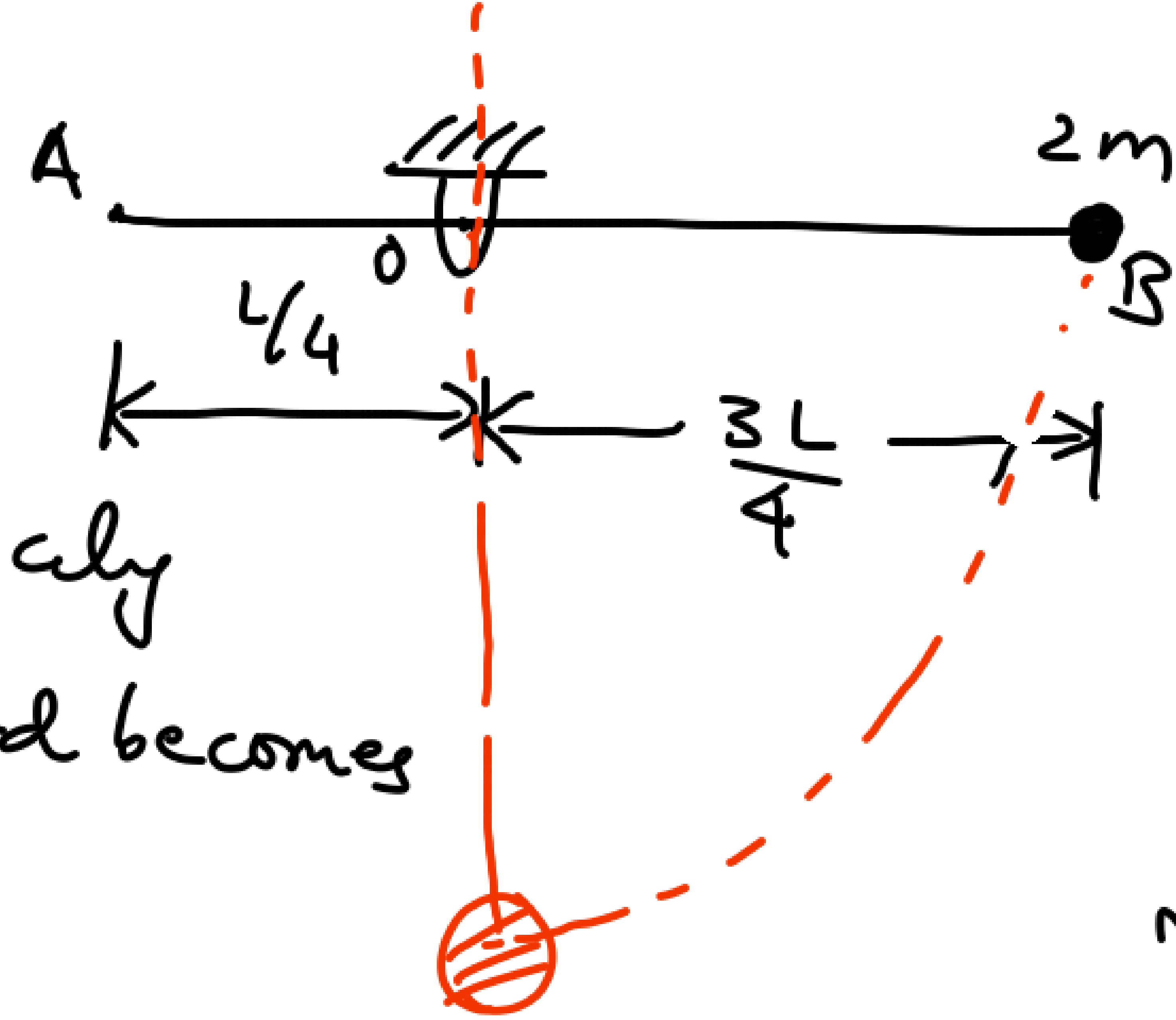
$$\Rightarrow \boxed{\int_m \underset{\sim}{\rho}\, dm = \underset{\sim}{0}}$$

If we differentiate w.r.t time;

$$\int_m \dot{\underset{\sim}{\rho}}\, dm = \underset{\sim}{0}$$

$$\int_m \ddot{\underset{\sim}{\rho}}\, dm = \underset{\sim}{0}$$

Example 1 :



Uniform rod of mass m and length L

Angular velocity when the rod becomes vertical

Horizontal is position 1

Vertical is position 2.

Work - Energy balance

$$\overset{0}{\cancel{\Delta V'}} = \Delta T + \cancel{\Delta V^e} + \Delta V^g$$

No external force

No elastic spring

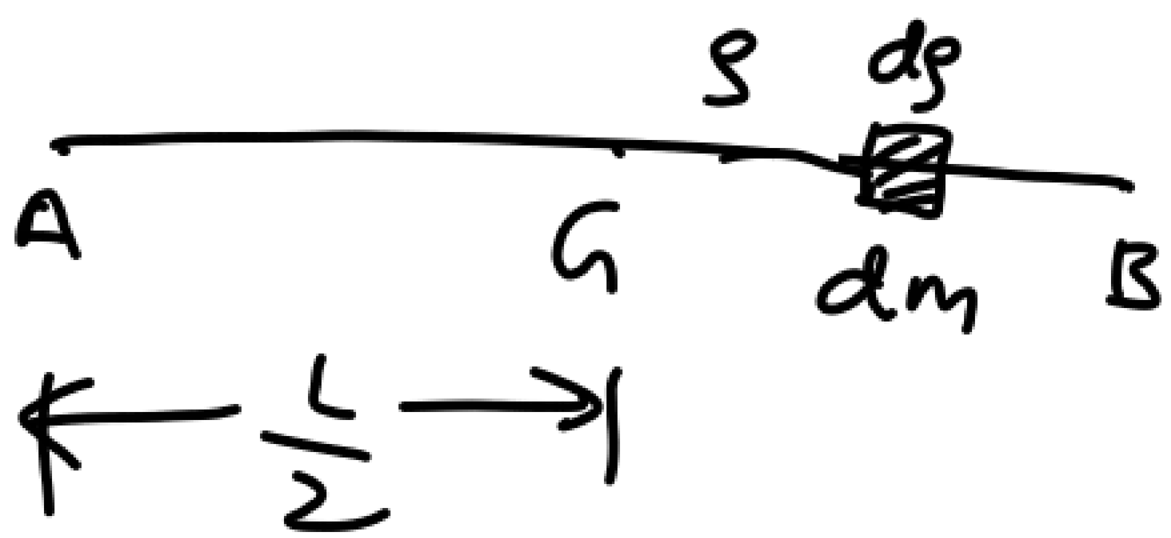
$$\Delta T = T_2 - \cancel{T_1}^{\,0} \text{ (Initial velocity = 0)}$$

$$= \frac{1}{2} I_0 \omega^2$$

$$I_0 = (I_0)_{rod} + (I_0)_{2m} \rightarrow \text{(Point mass)}$$

$$(I_0)_{2m} = 2m\left(\frac{3L}{4}\right)^2 = \frac{9mL^2}{8}$$

Uniform Rod



$I_G = \int_m dm\, \rho^2$ mass density

$$= \int_{-L/2}^{L/2} (d\rho)\, A\, \eta\, \rho^2$$

Rod cross-section

$$= A\eta \left(\frac{\rho^3}{3}\right)_{-L/2}^{L/2}$$

$$= A\eta \left(\frac{L^3}{8\times 3}\right)\times 2 = \frac{A\eta L^3}{12} = (A\eta L)\frac{L^2}{12}$$

$$= \frac{mL^2}{12}$$