Degrees of Freedom (DOF)

If y is a random variable with variance σ^2 ,

and sum of squares SS = $\sum(y_i - \bar{y})^2$ has 'v' degrees of freedom, then $E\left(\frac{SS}{v}\right) = \sigma^2$

The number of **degrees of freedom of a sum of squares** is equal to the number of independent elements in that sum of squares.

For example, SS = $\sum(y_i - \bar{y})^2$ is a sum of squares of 'n' elements, i.e., $y_1 - \bar{y}$, $y_2 - \bar{y}$, …, $y_n - \bar{y}$

Note that these 'n' elements are not all independent because $\sum(y_i - \bar{y}) = 0$

Therefore, only n-1 of them are independent, implying that SS has (n-1) degrees of freedom.

$$E\left(\frac{SS}{n-1}\right) = \sigma^2$$

Consequence of CLT

If $y_1, y_2, y_3 \ldots, y_n$ is a sequence of 'n' independent and identically distributed random variables

with $E(y_i) = \mu$ and $\mathrm{V}(y_i) = \sigma^2$ (both finite)

If we define, $x = y_1 + y_2 + y_3 + \cdots + y_n$ Then what is the distribution of 'x' as 'n' becomes sufficiently large?

$$= n\,\bar{y}$$

In other words, $Z_n = \frac{x - n\mu}{\sqrt{n\sigma^2}}$ is a standard normal distribution as $n \to \infty$

IMP!

'sum of n independent and identically distributed random variables is approximately normally distributed'

Frequently, we think of the error in an experiment as arising in an additive manner from several independent sources; consequently, the normal distribution becomes a reasonable model for the combined experimental error.

z-Distribution (Std. Normal PDF)

If $y_1, y_2, \ldots, y_n$ is a random sample from the **ANY** distribution, then

$$z = \frac{\bar{y} - \mu}{\sigma / \sqrt{n}}$$

is distributed as **Standard Normal Distribution, i.e., NPDF (0, 1)**

Standard Normal Probabilities

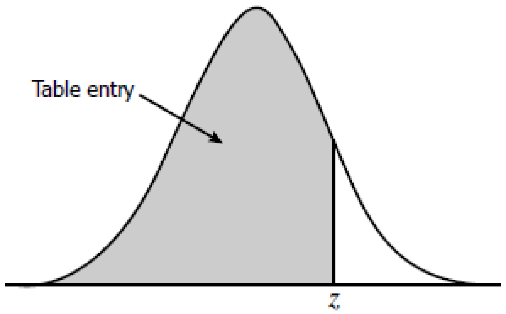


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879

DIY Read Chapter 2 Pages 32- 52 from Design and Analysis of Experiments, 8th Ed.

t-Distribution

If $y_1, y_2, \ldots, y_n$ is a random sample from the $N(\mu, \sigma^2)$ distribution, then

$$t = \frac{\bar{y} - \mu}{S/\sqrt{n}}$$

is distributed as t with $n - 1$ degrees of freedom.

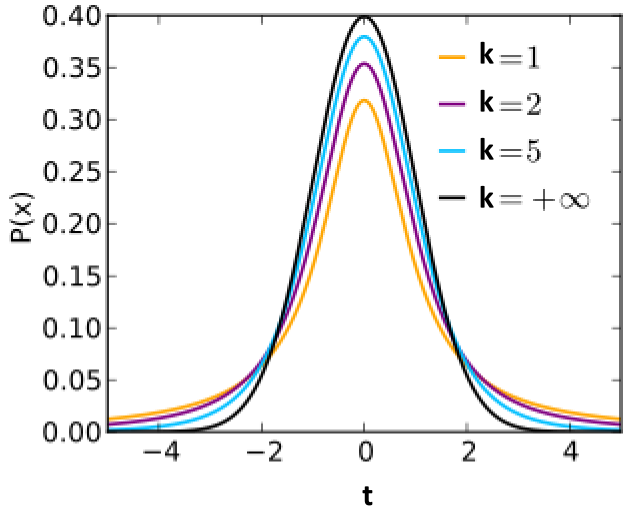
Side Notes:

What happens when $n \to N$?

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad \Re(z) > 0$$

***t* distribution with *k* degrees of freedom**, denoted t_k. The density function of t is

$$f(t) = \frac{\Gamma[(k+1)/2]}{\sqrt{k\pi}\,\Gamma(k/2)} \frac{1}{[(t^2/k) + 1]^{(k+1)/2}} \qquad -\infty < t < \infty$$



Note: The increased spread reflects the added uncertainty *due to unknown σ_y* that gets estimated by s, which itself is prone to sampling errors. **What will happen as k increases?**

DIY Read Chapter 2 Pages 32- 52 from Design and Analysis of Experiments, 8th Ed.

Chi-Square (χ^2) Distribution

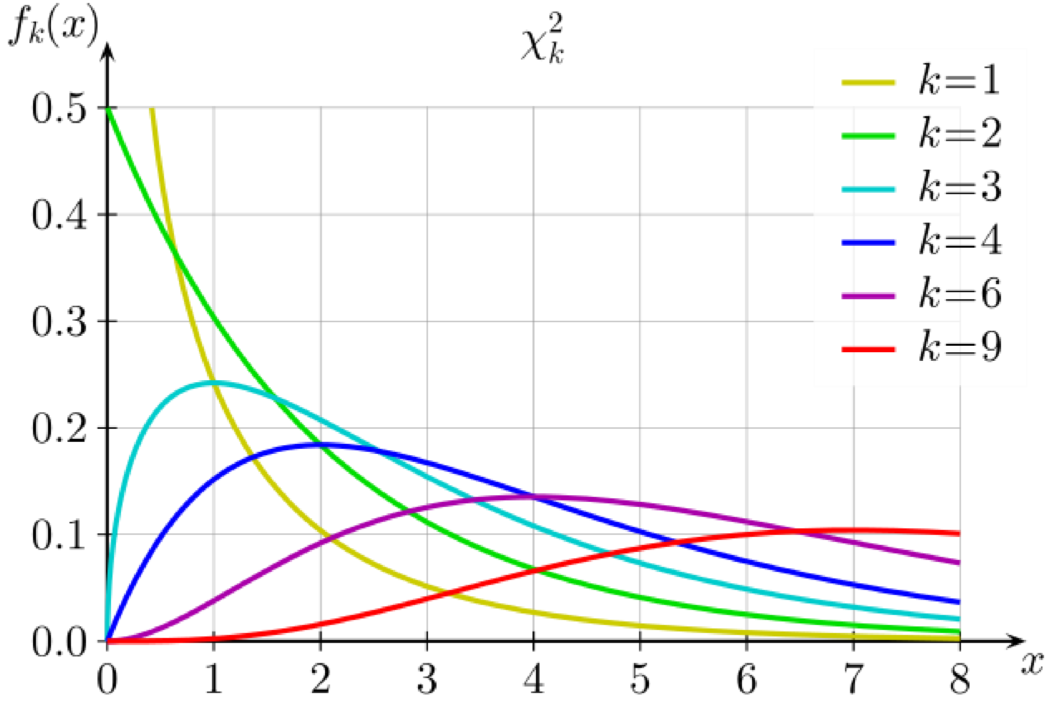
If $z_1, z_2, z_3 \ldots, z_k$ are **normally and independently distributed** random variables with mean 0 and variance 1 [NID (0,1)]

And if we define, $x = z_1^2 + z_2^2 + \cdots + z_k^2$

Then 'x' follows the **chi-square distribution with k degrees of freedom**

$$f(x;k) = \begin{cases} \dfrac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)}, & x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

- **The distribution is asymmetric or skewed**
- **Mean, $\mu = k$ and Variance, $\sigma^2 = 2k$**



Chi-Square (χ^2) Distribution

- **What would be an example of chi-square distribution?**
- Remember

$SS = \sum_{i=1}^{n}(y_i - \bar{y})^2$ is the **corrected sum of squares** of the observations y_i.

dOF = n-1

$$\boxed{\Sigma y_i - \bar{y} = 0}$$

$$E(S^2) = \frac{1}{n-1}E(SS) = \sigma^2$$

and we see that S^2 is an unbiased estimator of σ^2.

$$\Sigma \frac{(y_i - \bar{y})^2}{\sigma^2}$$

$$\frac{(y_1 - \bar{y})}{\sigma}, \frac{y_2 - \bar{y}}{\sigma} + \ldots \frac{y_n - \bar{y}}{\sigma}$$

$$\frac{SS}{\sigma^2} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sigma^2} \sim \chi^2_{n-1}$$

$SS = \sigma^2 \chi^2_{n-1}$

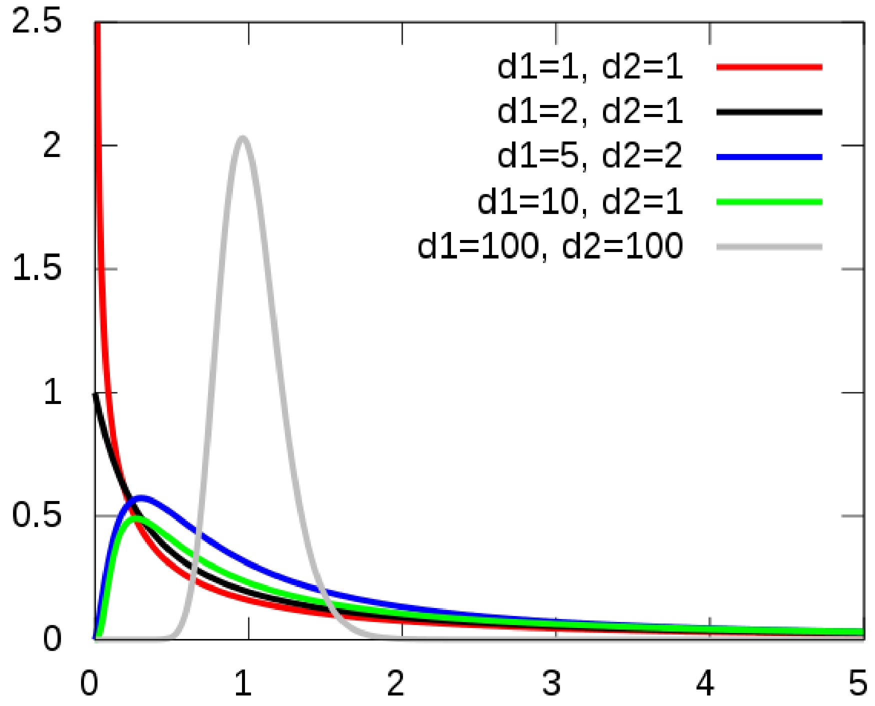
Sample variance, $S^2 = \frac{SS}{n-1}$

Therefore, if the observations in the sample are NID (μ, σ^2), then the distribution of S^2 *is* $\left(\frac{\sigma^2}{n-1}\right)\chi^2_{n-1}$

Thus, the sampling distribution of the sample variance is a constant times the chi-square distribution if the population is normally distributed.

If χ_u^2 and χ_v^2 are two independent chi-square random variables with u and v degrees of freedom, respectively, then, the ratio $\boldsymbol{F_{u,v} = \frac{\chi_u^2/u}{\chi_v^2/v}}$ follows a F-distribution with u degrees of freedom of numerator and v degrees of freedom of denominator

$$F(x) = \frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{u}{v}\right)^{u/2} x^{(u/2)-1}}{\Gamma\left(\frac{u}{x}\right)\Gamma\left(\frac{v}{2}\right)\left[\left(\frac{u}{v}\right)x+1\right]^{(u+v)/2}} \qquad 0 < x < \infty$$



Example of F-distribution

Suppose we have *two independent normal populations* with common variance σ^2.

If $y_{11}, y_{12}, y_{13}, .., y_{1n_1}$ is a random sample of n_1 observations from the first population and $y_{21}, y_{22}, y_{23}, .., y_{2n_2}$ is a random sample of n_2 observations from the second population, **Then,** $\boldsymbol{\frac{S_1^2}{S_2^2} \sim F_{n_1-1,n_2-1}}$

Chapter 2

Classical Design of Experiments

Importance of Experiments (Recap)

Why experimentation?

- To understand cause-and-effect relationships in a system/process.

Goal of experimentation:

- to determine which input variables are responsible for the observed changes in the response.
- to develop a model relating the response to the important input variables
- to use this model for process or system improvement or other decision-making.

Test: Each experimental run is a test

Experimentation plays an important role in technology commercialization and product realization activities.

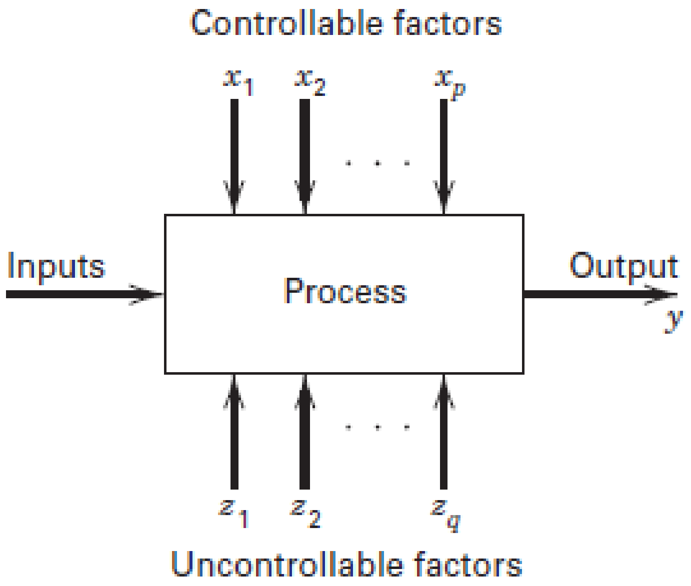
Models: i) physics based: mechanistic model, ii) experiment based: empirical model

Objectives of Experiments

In general, experiments are used to study the performance of processes and systems.

The objectives of the experiment:

- Determining which variables are **most influential** on the response y
- Determining **where to set** the influential x's so that y is almost always near the desired nominal value
- Determining where to set the influential x's so that **variability in y is small**
- Determining where to set the influential x's so that the **effects of the uncontrollable** variables $z_1, z_2, \ldots, z_q$ are minimized.



Applications (Recap)

Experimental design is a critically important tool in the scientific and engineering world for improving the product realization process.

Process development

- Improved process yields
- Reduced variability and closer conformance to nominal or target requirements
- Reduced development time
- Reduced overall costs.

Design activities

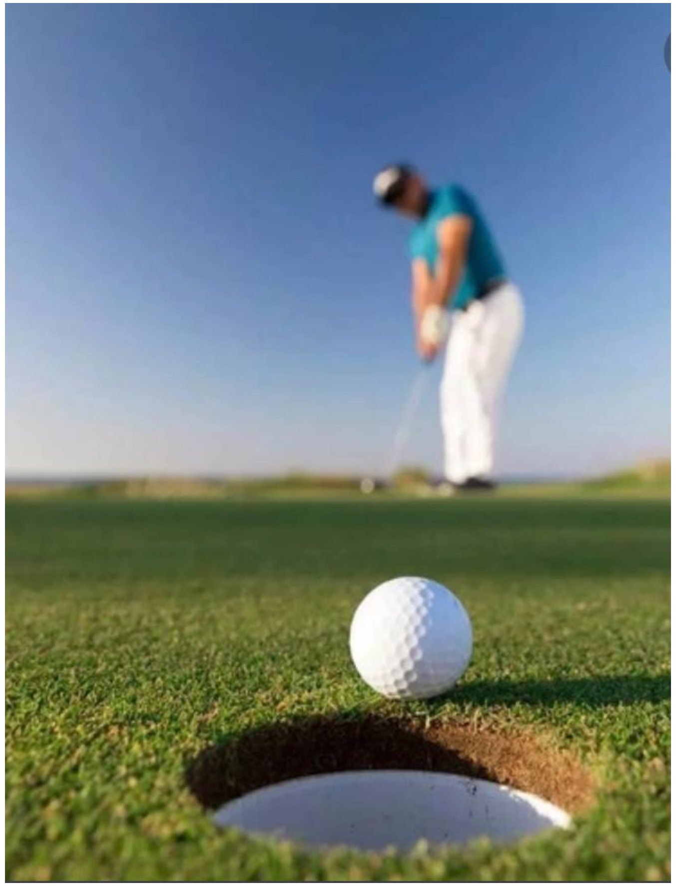
- Evaluation and comparison of basic design configurations
- Evaluation of material alternatives
- Selection of design parameters so that the product will work well under a wide variety of field conditions, that is, so that the product is robust
- Determination of key product design parameters that impact product performance
- Formulation of new products.

Example

Objective: Lowering golf score

Some of the factors that may be important, or that may influence golf score, are as follows:

1. The type of driver used (oversized or regular-sized)
2. The type of ball used (balata or three-piece)
3. Walking and carrying the golf clubs or riding in a golf cart
4. Drinking water or drinking “something else” while playing
5. Playing in the morning or playing in the afternoon
6. Playing when it is cool or playing when it is hot
7. The type of golf shoe spike worn (metal or soft)
8. Playing on a windy day or playing on a calm day.



Statistical Design of Experiments

Statistical design of experiments refers to the process of planning the experiment so that appropriate data will be collected and analyzed by statistical methods, resulting in valid and objective conclusions.

There are two aspects to any experimental problem:

- The **design** of the experiment and
- The statistical **analysis** of the data.

Guidelines for Experiments

Guidelines for Designing an Experiment

1. Recognition of and statement of the problem	Pre-experimental planning
2. Selection of the response variable[a]	
3. Choice of factors, levels, and ranges[a]	
4. Choice of experimental design	
5. Performing the experiment	
6. Statistical analysis of the data	
7. Conclusions and recommendations	

[a]In practice, steps 2 and 3 are often done simultaneously or in reverse order.

Guidelines

1. Recognition of and statement of problem

- A clear statement of the problem often contributes substantially to better understanding of the phenomenon being studied and the final solution of the problem.
- Some (but by no means all) of the reasons for running experiments include:
    a) Factor screening or characterization.
    b) Optimization
    c) Confirmation
    d) Discovery
    e) Robustness
- A **sequential approach** employing a series of smaller experiments, each with a specific objective, such as factor screening, is a better strategy.

Guidelines

2. Selection of the response variable

- In selecting the response variable, the experimenter should be certain that this variable really provides useful information about the process under study.
- Most often, the average or standard deviation (or both) of the measured characteristic will be the response variable.
- Multiple responses are NOT unusual.
- It is usually critically important to identify issues related to defining the responses of interest and how they are to be measured before conducting the experiment.

Guidelines

3. Choice of factors, levels, and range.

- Factors can be classified as either potential design factors or nuisance factors.
- Potential design factors: design factors, held-constant factors, and allowed-to-vary factors.
- Nuisance factors, on the other hand, may have large effects that must be accounted for, yet we may not be interested in them in the context of the present experiment.
- Nuisance factors are often classified as controllable, uncontrollable, or noise factors.
- Next, the ranges over which these factors will be varied and the specific levels at which runs will be made.

Guidelines

3. Choice of factors, levels, and range.

- The cause-and-effect diagram (**fishbone diagram**) can be a useful technique for organizing some of the information generated in pre-experimental planning.

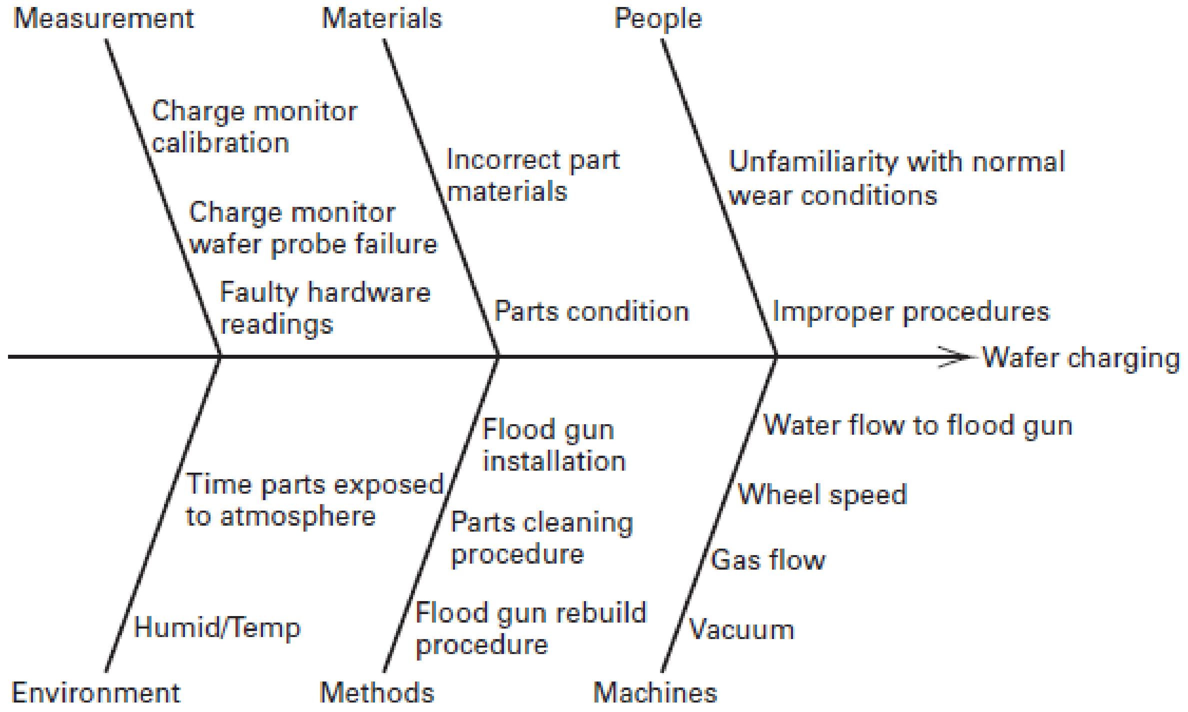


Fig. A cause-and-effect diagram for the etching process experiment

Guidelines

4. Choice of experimental design

- Design selection also involves thinking about and selecting a tentative empirical model to describe the results.
- Some examples of models:
  - First order model: two variables with main effects

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

  - First order model with interaction

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \varepsilon$$

  - Second order model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_{11}^2 + \beta_{22} x_2^2 + \varepsilon$$

Guidelines

5. Performing the experiment

- When running the experiment, it is vital to monitor the process carefully to ensure that everything is being done according to plan.
- Someone should be assigned to check factor settings before each run.
- Coleman and Montgomery (1993) suggest that prior to conducting the experiment a few trial runs or pilot runs are often helpful.
- In initiating an experimental study, we should probably plan in the first experiment to do no more than 25% of the total experiments we have resources for
- One big experiment is not only inefficient but leaves us with no resources left if our initial conjecture was in error.
- We should begin by looking at many factors in a somewhat superficial fashion and more toward the examination of only the few, most relevant, factors in a more comprehensive fashion.

Guidelines

5. Performing the experiment

- The three basic principles of experimental design are:
  - Blocking
  - Randomization
  - Replication

Guidelines: Blocking

- When known sources of extraneous/unwanted variation can be identified, we can design the experiment in such a way as to eliminate their influence and provide more sensitive test of significance test

Example: Imagine two tool materials being tested to determine whether or not there is a real difference in their wear characteristics

Material A

$\stackrel{?}{=}$

Is wear of A same as wear of B?

Material B

Guidelines: Blocking

Material A

$\stackrel{?}{=}$

Is wear of A same as wear of B?

Material B

Experimental Design:

- Twenty tools are made, ten with material A and ten with material B.
- Twenty machine operators are chosen at random, given the tools, and told to use them in machining as they normally do
- At the end of the experiments the mean amount of wear is determined based on the ten measurements for each material type.
- The mean difference is calculated and examined by a statistical test of significance.
- No real difference is found

Do you agree with the results? OR Do you see any problems with this experiment?

Guidelines: Blocking

Problems with First Experiments:

- The twenty operators are very different in terms of their skills and experiences
- Hence, a sizeable nuisance variation is introduced within the measurements comprising the mean wear for each material.
- This nuisance variation markedly increases the level of "chance" variation and hence may be "hiding" the presence of a real difference in materials

Revised Experiment:

- Give each operator a pair of tools with one tool of material A and one tool of material B.
- At the end of experiment measure the wear on each tool for each operator and calculate the difference in wear within each operator.
- Average these differences across all operators and perform a test of significance on this average difference.
- Operator-to-operator nuisance variation is in this way blocked from consideration since only relative differences within each operator block are examined

Guidelines: Randomization

- Unknown sources of nuisance variation are always present but hard to identify – trends from day to day.
- Systematic conduct of tests may violate basic statistical assumptions for significance tests, i.e., Normality/Independence.
- Random sampling imply that the data are independently distributed about their respective means.

- **Block for what you can identify, Randomize for what you cannot.**

Guidelines: Replication

- In the analysis of the results of comparative experiments, it is necessary to assess the magnitude of the test statistic of interest, say a difference in the mean results of two methods, in light of the natural variability inherent in the experiment.
- The question we are really asking is: could this experimental result--difference--have arisen solely due to chance causes or is there a real difference in the methods?
- To answer this question we must measure the level of chance variation, usually through genuine test replication, and then determine the probability that we could have observed the result before us if only these chance causes were at work.

Summary:

- Experiments should be comparative.
- There should be genuine replication: replicates can provide an accurate measure of errors.
- Whenever appropriate, blocking (pairing) should be used to reduce error.
- Randomization is needed for homogeneity or independence.
- Experiments should be designed in such a way as to be able to determine the interaction effects of factors.

Guidelines

6. Statistical analysis of data

- Statistical methods should be used to analyze the data so that results and conclusions are objective rather than judgmental in nature.
- Remember that statistical methods cannot prove that a factor (or factors) has a particular effect. They only provide guidelines as to the reliability and validity of results.
- The primary advantage of statistical methods is that they add objectivity to the decision-making process.

7. Conclusions and recommendations

- Once the data have been analyzed, the experimenter must draw practical conclusions about the results and recommend a course of action.
- Follow-up runs and confirmation testing should also be performed to validate the conclusions from the experiment.