Example 1: Mortar Formula

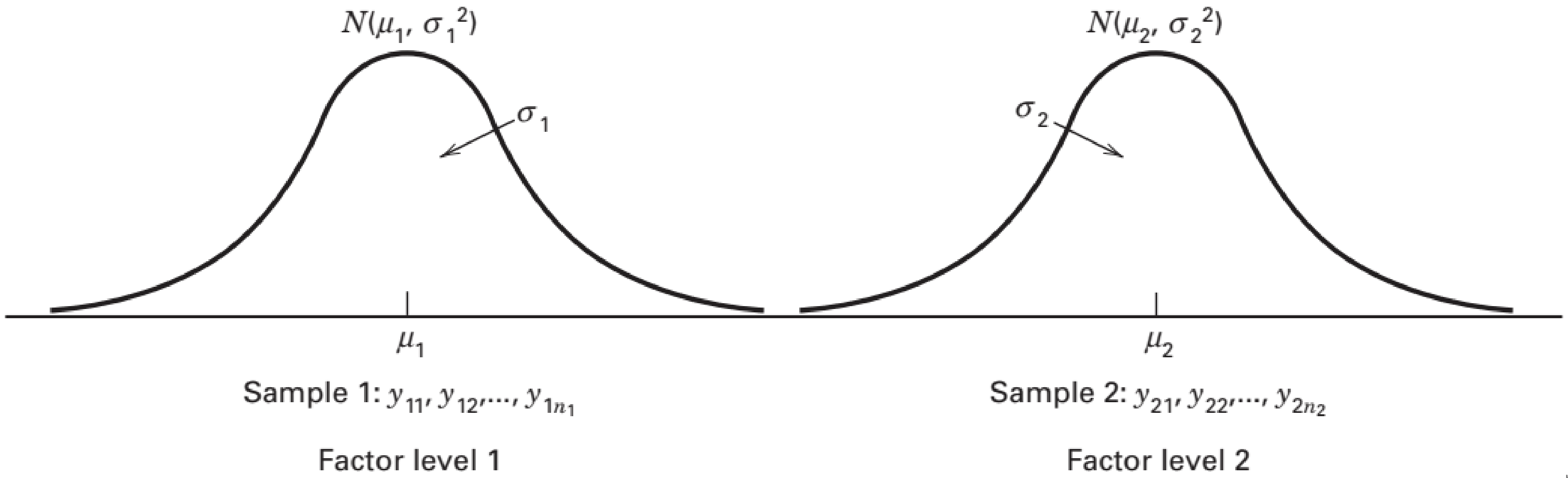
Now the question is whether μ_1 & μ_2 are statistically different

Hypothesis Testing

$H_0 : \mu_1 = \mu_2$ Null Hypothesis

$H_1 : \mu_1 \neq \mu_2$ Alternate Hypothesis (two-sided)

$\mu_1 < \mu_2$ or if $\mu_1 > \mu_2$.



■ **TABLE 2.1**

Tension Bond Strength Data for the Portland Cement Formulation Experiment

j	Modified Mortar y_{1j}	Unmodified Mortar y_{2j}
1	16.85	16.62
2	16.40	16.75
3	17.21	17.37
4	16.35	17.12
5	16.52	16.98
6	17.04	16.87
7	16.96	17.34
8	17.15	17.02
9	16.59	17.08
10	16.57	17.27

Two-Sample t-Test

Suppose that we could assume that the variances of tension bond strengths were identical for both mortar formulations. $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Then the **appropriate test statistic** to use for comparing two treatment means in the completely randomized design is

$\frac{SS}{V} = S^2$

$\sum y_{1j} - \bar{y}_1 = 0$

$\sum y_{2j} - \bar{y}_2 = 0$

$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{1/n_1 + 1/n_2}}$

Pooled Var $= \frac{SS}{V}$

$$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Where

$= \frac{SS}{V}$

$= \frac{SS_1 + SS_2}{n_1 + n_2 - 2}$

$= \frac{S_1^2 (n_1 - 1) + S_2^2 (n_2 - 1)}{n_1 + n_2 - 2}$

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

TABLE 2.1
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10	16.57	17.27

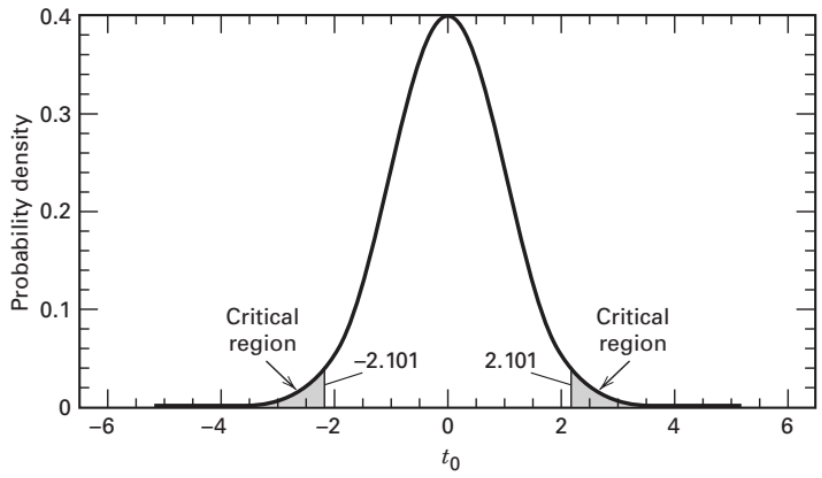
S_1^2, $\bar{y}_1$ S_2^2, $\bar{y}_2$

S_p^2 is an estimate of the common variance $\sigma_1^2 = \sigma_2^2 = \sigma^2$

t-Test

Two-Sample t-Test Procedure

$$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$



1) To determine whether to reject $H_0: \mu_1 = \mu_2$, we would compare t_0 to the t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

2) If $t_0 > t_{\frac{\alpha}{2}, n_1+n_2-2}$ OR $t_0 < -t_{\frac{\alpha}{2}, n_1+n_2-2}$, then we will reject $H_0: \mu_1 = \mu_2$

t-Test

Justification of Two-Sample t-Test

If we were sampling from two independent normal distributions, then the distribution of $\overline{y_1} - \overline{y_2}$ will be a normal distribution with mean $\mu_1 - \mu_2$ and variance $\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$

If σ^2 were known, and if $H_0: \mu_1 = \mu_2$ were true, then the Z_0 distribution would be a normal distribution with mean 0 and variance 1

$$Z_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

But since we do NOT know σ^2, we use S_p^2

and the normal distribution changes to t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

Two-Sample t-Test

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{1/n_1 + 1/n_2}} = \frac{16.76 - 17.04}{\sqrt{0.081}\sqrt{\frac{2}{10}}} = -2.21$$

In this example

Modified Mortar	Unmodified Mortar
$\bar{y}_1 = 16.76\ \text{kgf/cm}^2$	$\bar{y}_2 = 17.04\ \text{kgf/cm}^2$
$S_1^2 = 0.100$	$S_2^2 = 0.061$
$S_1 = 0.316$	$S_2 = 0.248$
$n_1 = 10$	$n_2 = 10$

$$S_p^2 = \frac{S_1^2(n_1-1) + S_2^2(n_2-1)}{n_1+n_2-2} = \frac{0.1\times 9 + 0.061\times 9}{18}$$

$$= 0.161/2 = 0.0805 \rightarrow S_p =$$

■ **TABLE 2.1**
Tension Bond Strength Data for the Portland Cement Formulation Experiment

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1	16.85	16.62
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8	17.15	17.02
9	16.59	17.08
10	16.57	17.27

t-Test

Two-Sample t-Test

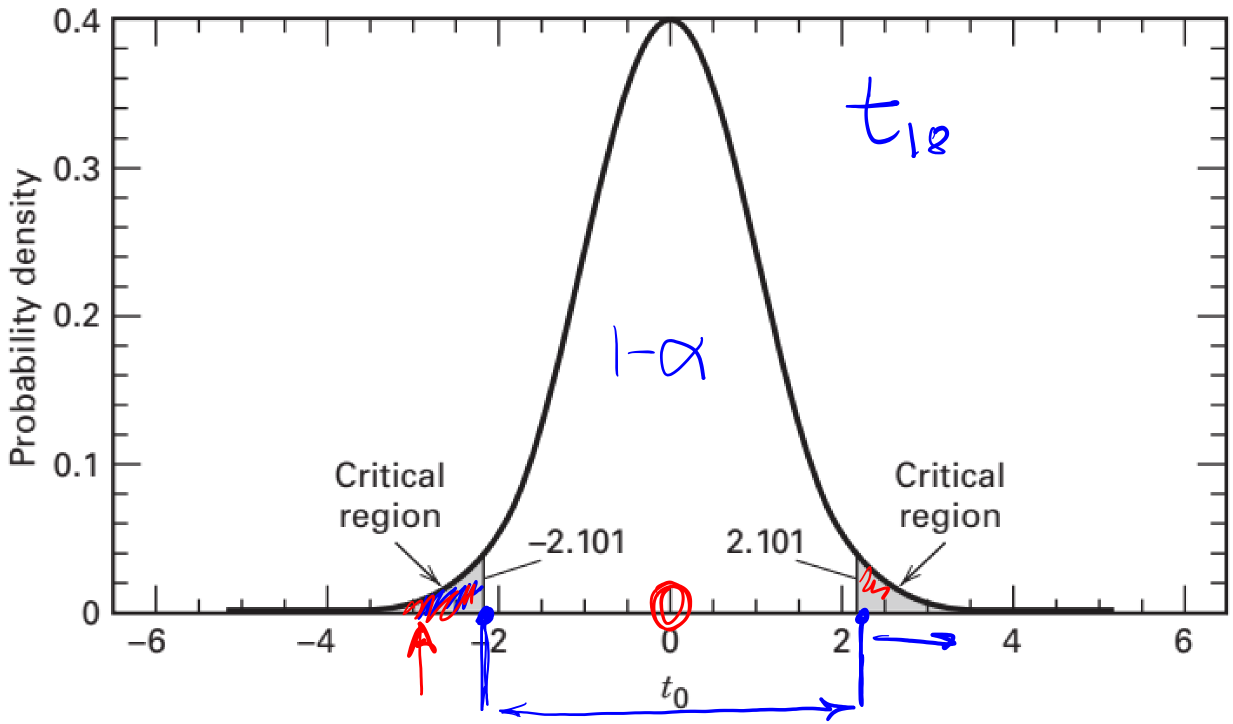
$\alpha = 5\%$

In this example

Modified Mortar	Unmodified Mortar
$\bar{y}_1 = 16.76\ \text{kgf/cm}^2$	$\bar{y}_2 = 17.04\ \text{kgf/cm}^2$
$S_1^2 = 0.100$	$S_2^2 = 0.061$
$S_1 = 0.316$	$S_2 = 0.248$
$n_1 = 10$	$n_2 = 10$

your $t_0 = -2.21$

H_0 is Rejected.



■ **FIGURE 2.10** The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

Furthermore, $n_1 + n_2 - 2 = 10 + 10 - 2 = 18$, and if we choose $\alpha = 0.05$, then we would reject $H_0: \mu_1 = \mu_2$ if the numerical value of the test statistic $t_0 > t_{0.025,18} = 2.101$, or if $t_0 < -t_{0.025,18} = -2.101$. These boundaries of the critical region are shown on the reference distribution (t with 18 degrees of freedom) in Figure 2.10.

t-Test Calculations

Two-Sample t-Test

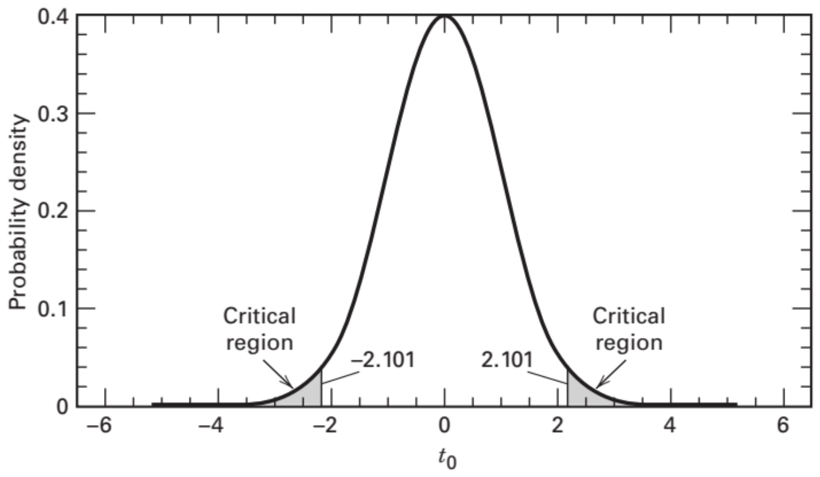
In this example

Modified Mortar	Unmodified Mortar
$\bar{y}_1 = 16.76 \text{ kgf/cm}^2$	$\bar{y}_2 = 17.04 \text{ kgf/cm}^2$
$S_1^2 = 0.100$	$S_2^2 = 0.061$
$S_1 = 0.316$	$S_2 = 0.248$
$n_1 = 10$	$n_2 = 10$

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

$$= \frac{9(0.100) + 9(0.061)}{10 + 10 - 2} = 0.081$$

$$S_p = 0.284$$



■ **FIGURE 2.10** The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{16.76 - 17.04}{0.284\sqrt{\frac{1}{10} + \frac{1}{10}}}$$

$$= \frac{-0.28}{0.127} = -2.20$$

We Reject $H_0: \mu_1 = \mu_2$ at Significance level of 0.05

Two-Sample t-Test

In this example, we concluded that we Reject $H_0: \mu_1 = \mu_2$ at significance level of α = 0.05

Do you see any problem/limitation of this?

For example, what will be the conclusion if the significance level is 0.04 or 0.03 or 0.01?

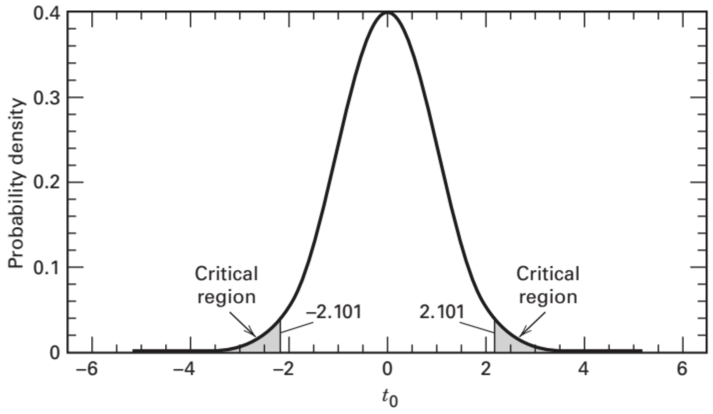
We do not know whether the test-statistic t0 lies just barely in the rejection region OR very far into the rejection region

Thus, we can specify P-value, which is the minimum significance value which will

Result in rejection of the null hypothesis

For example, in the mortar experiments, the null hypothesis will be rejected for

any level of significance > 0.0411



■ **FIGURE 2.10** The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

Concept of Confidence Interval

- Given a random sample of '*n'* observations from some process of interest and an estimate of the process mean, it is of interest to make some statement about the *"goodness" of that sample mean*, as an estimate of μ, i.e., the degree of belief or confidence that can be placed on it.
- One way of approaching this problem is through *the concept of the confidence interval*.

- **Remember:** Distribution of sample means is a normal distribution (CLT)
- *That means,* for random samples of size 'n' drawn from a population, we expect that 95% of all sample means will be within an interval of $\mu \pm 1.96$ standard deviations of the distribution of the sample mean, i.e., $\mu \pm \frac{1.96\sigma_x}{\sqrt{n}}$

 https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Concept of Confidence Interval

In other words, $\bar{y} \pm \frac{1.96\sigma_y}{\sqrt{n}}$ is called a 95% confidence interval for the true mean μ

$\mu \pm 1.96\sigma_{\bar{y}}$

$\bar{y}$

In general,

$$\bar{y} \pm \left(z_{1-\frac{\alpha}{2}}\right)\frac{\sigma_y}{\sqrt{n}}$$ **is a 100*(1-α)% confidence interval for the true mean μ**

μ

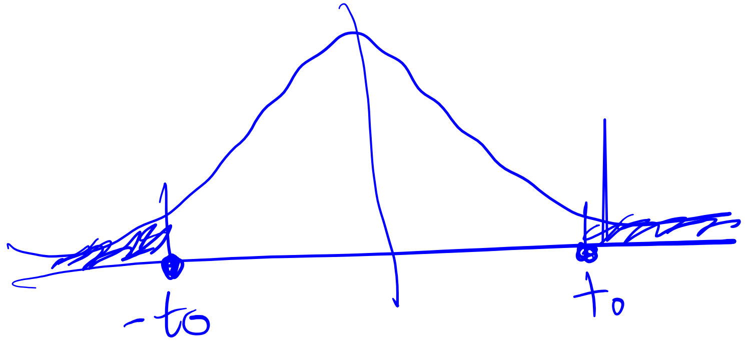
$\bar{y} \pm 1.96\sigma_{\bar{y}}$

95%

When sample size is small and σ_y is UNKNOWN,

the confidence interval is given by $\bar{y} \pm \left(t_{v,1-\frac{\alpha}{2}}\right)\frac{s}{\sqrt{n}}$

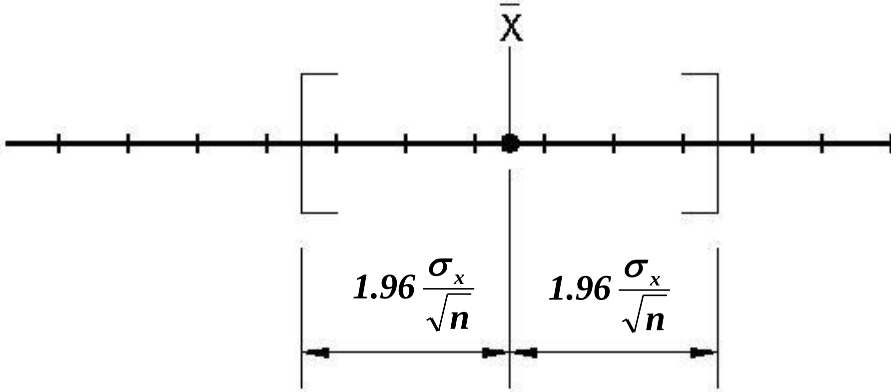
Where v = n-1 is the degree of freedom

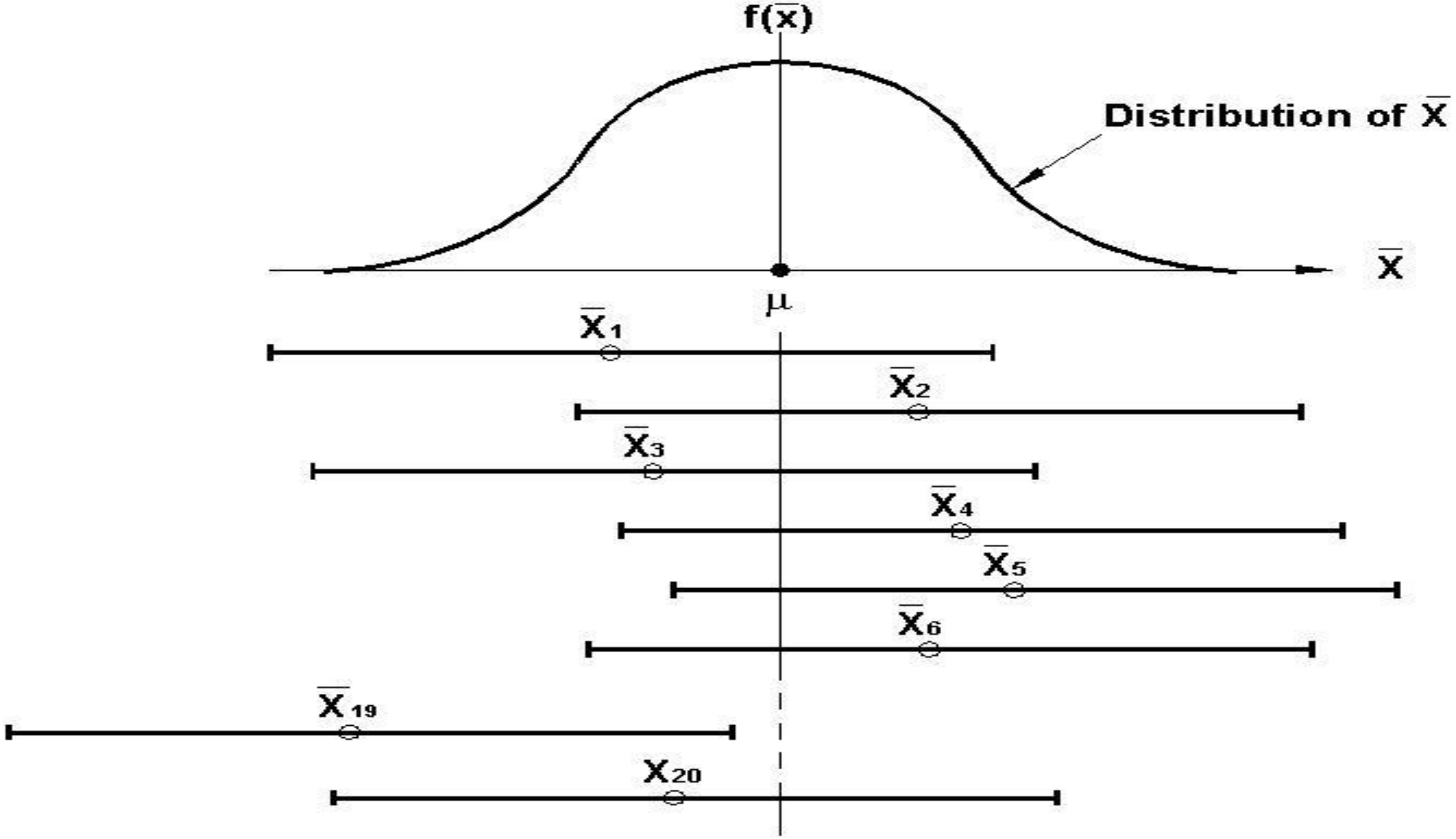


$t_{v,\alpha/2} = -t_{v,1-\alpha/2}$

https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Confidence Interval





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Confidence Interval Approach

To define a confidence interval, suppose that θ is an unknown parameter. To obtain an interval estimate of θ, we need to find two statistics L and U such that the probability statement

$$P(L \leqslant \theta \leqslant U) = 1 - \alpha \quad \textbf{(2.27)}$$

is true. The interval

$$L \leqslant \theta \leqslant U \quad \textbf{(2.28)}$$

is called a **100(1 − α) percent confidence interval** for the parameter θ. The interpretation of this interval is that if, in repeated random samplings, a large number of such intervals are constructed, $100(1 - \alpha)$ percent of them will contain the true value of θ. The statistics L and U are called the **lower** and **upper confidence limits**, respectively, and $1 - \alpha$ is called the **confidence coefficient**. If $\alpha = 0.05$, Equation 2.28 is called a 95 percent confidence interval for θ. Note that confidence intervals have a frequency interpretation; that is, we do not know if the statement is true for this specific sample, but we do know that the *method* used to produce the confidence interval yields correct statements $100(1 - \alpha)$ percent of the time.

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7	16.96	17.34
8	17.15	17.02
9	16.59	17.08
10	16.57	17.27

$-t_{\alpha/2} = t_{1-\alpha/2}$

Suppose that we wish to find a $100(1-\alpha)$ percent confidence interval on the true difference in means $\mu_1 - \mu_2$ for the Portland cement problem. The interval can be derived in the following way. The statistic

$$\frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

is distributed as $t_{n_1+n_2-2}$. Thus,

$$P\left(-t_{\alpha/2,n_1+n_2-2} \le \frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \le t_{\alpha/2,n_1+n_2-2}\right) = 1-\alpha$$

or

$$P\left(\bar{y}_1 - \bar{y}_2 - t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \right.$$
$$\left. \le \bar{y}_1 - \bar{y}_2 + t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) = 1-\alpha$$

$\Delta y = y_1 - y_2$

$\overline{\Delta y} = \bar{y}_1 - \bar{y}_2$

$$\frac{\overline{\Delta y} - \Delta\mu}{S_p\sqrt{1/n_1 + 1/n_2}}$$

Comparing Equations 2.29 and 2.27, we see that

$$\bar{y}_1 - \bar{y}_2 - t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2$$
$$\le \bar{y}_1 - \bar{y}_2 + t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

is a $100(1-\alpha)$ percent confidence interval for $\mu_1 - \mu_2$.

The actual 95 percent confidence interval estimate for the difference in mean tension bond strength for the formulations of Portland cement mortar is found by substituting in Equation 2.30 as follows:

$$
\begin{aligned}
16.76 - 17.04 - (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}} &\leq \mu_1 - \mu_2 \\
&\leq 16.76 - 17.04 + (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}} \\
-0.28 - 0.27 \leq \mu_1 - \mu_2 &\leq -0.28 + 0.27 \\
-0.55 \leq \mu_1 - \mu_2 &\leq -0.01
\end{aligned}
$$

95% $\alpha = 5\%$

$\mu_1 \neq \mu_2$

$\mu_1 - \mu_2 \neq 0$

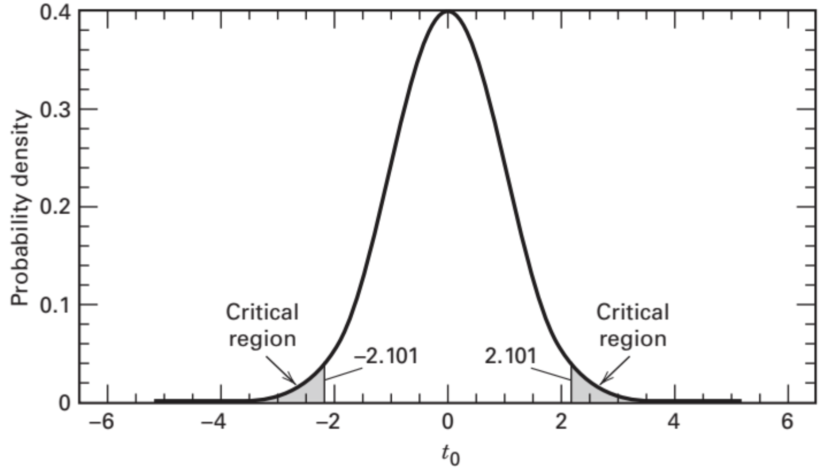
Note that because $\mu_1 - \mu_2 = 0$ is *not* included in this interval, the data do not support the hypothesis that $\mu_1 = \mu_2$ at the 5 percent level of significance (recall that the *P*-value for the two-sample *t*-test was 0.042, just slightly less than 0.05).

$\cancel{H_0}$

Recap: Comparison when we do NOT know σ

Two-Sample t-Test Procedure (Two-Sided)

$$t_0 = \frac{(\bar{y}_1 - \bar{y}_2) - 0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$



1) To determine whether to reject $H_0: \mu_1 = \mu_2$, we would compare t_0 to the t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

2) If $t_0 > t_{\frac{\alpha}{2}, n_1+n_2-2}$ OR $t_0 < -t_{\frac{\alpha}{2}, n_1+n_2-2}$, then we will reject $H_0: \mu_1 = \mu_2$

Recap: Comparison when we do NOT know σ

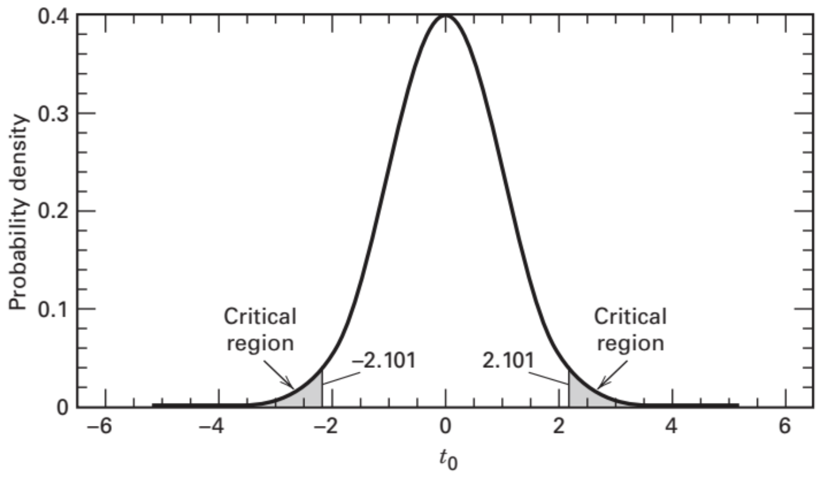
Two-Sample t-Test Procedure (Two-Sided) using Confidence Interval

$$P\left(-t_{\alpha/2,n_1+n_2-2} \le \frac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \le t_{\alpha/2,n_1+n_2-2}\right) = 1 - \alpha$$

or

$$P\left(\bar{y}_1 - \bar{y}_2 - t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \right.$$

$$\left. \le \bar{y}_1 - \bar{y}_2 + t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) = 1 - \alpha$$



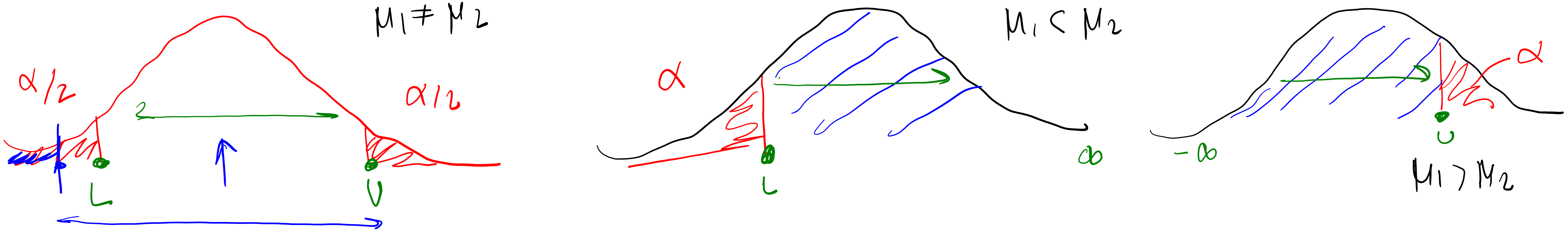
Comparing Equations 2.29 and 2.27, we see that

$$\bar{y}_1 - \bar{y}_2 - t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2$$

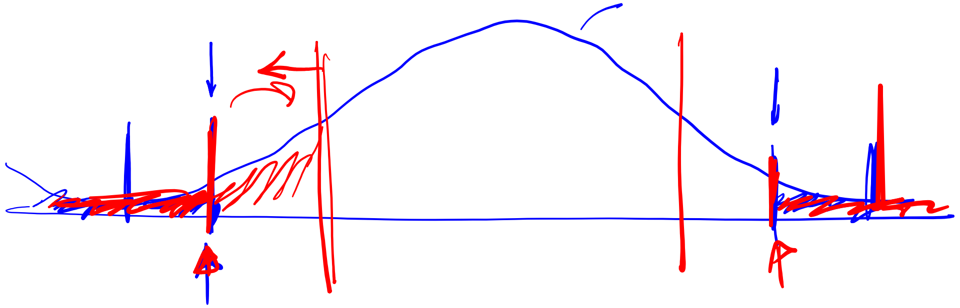
$$\le \bar{y}_1 - \bar{y}_2 + t_{\alpha/2,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

is a $100(1 - \alpha)$ percent confidence interval for $\mu_1 - \mu_2$.

Recap



what is p value? $\Rightarrow$ minimum α at which H_0 is rejected.



any $\alpha < p \rightarrow H_0$ ✓

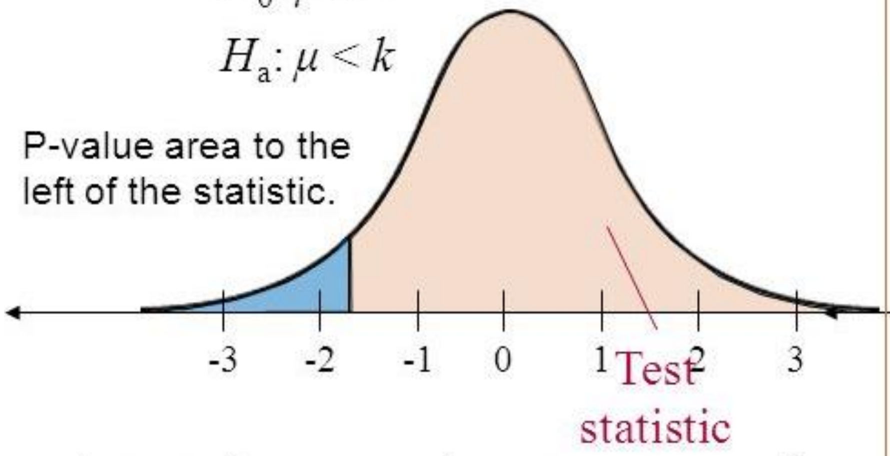
$p = 2CDF(t_0)$

One-sided Tests

Left Tailed Test: The alternative hypothesis H_a contains the less-than inequality symbol (<).

$$H_0: \mu \geq k$$

$$H_a: \mu < k$$



A water faucet manufacturer announces that the mean flow rate of a certain type of faucet is less than 2.5 gallons per minute.

$$H_0: \mu \geq 2.5$$

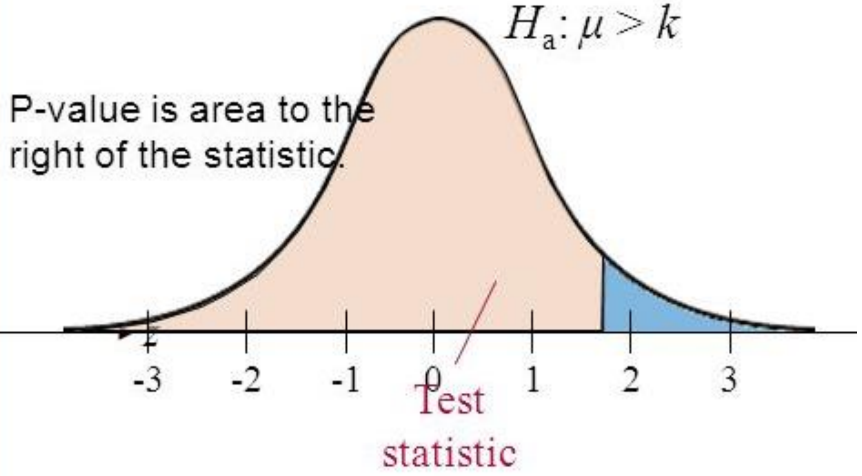
$$H_a: \mu < 2.5$$

Larson/Farber 4th ed.

Right Tailed Test: The alternative hypothesis H_a contains the less-than inequality symbol (>).

$$H_0: \mu \leq k$$

$$H_a: \mu > k$$



A cereal company says: Mean weight of box is more than 20 oz.

$$H_0: \mu \leq 20$$

$$H_a: \mu > 20$$

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