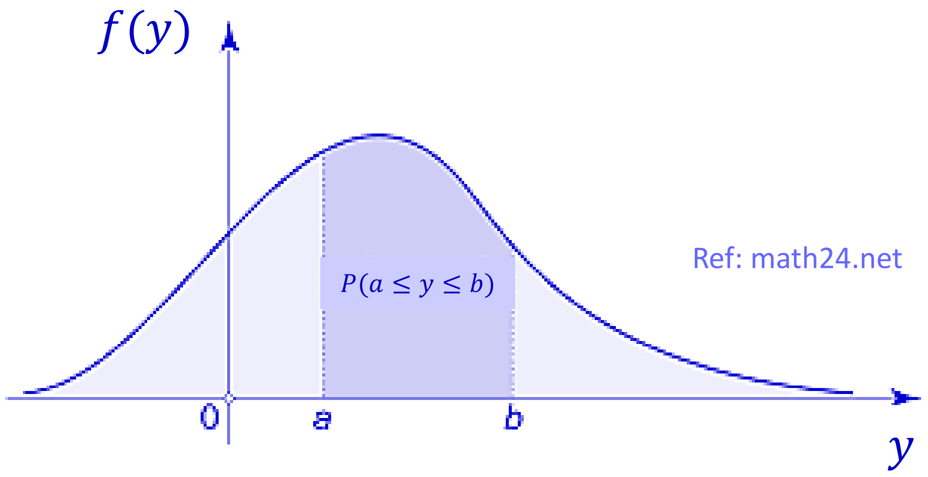
Probability Density/Distribution Function

- For a continuous random variable 'y', the probability behavior is described by a function called 'probability density function' (PDF) = f(y)
- What are the properties of such PDF?

$$f(y) \geq 0$$

$$\int_{-\infty}^{\infty} f(y)dy = 1$$

$$Probability\,(a \leq y \leq b) = \int_a^b f(y)dy$$



- Cumulative distribution function (CDF) for a continuous random variable x with pdf $f(X)$

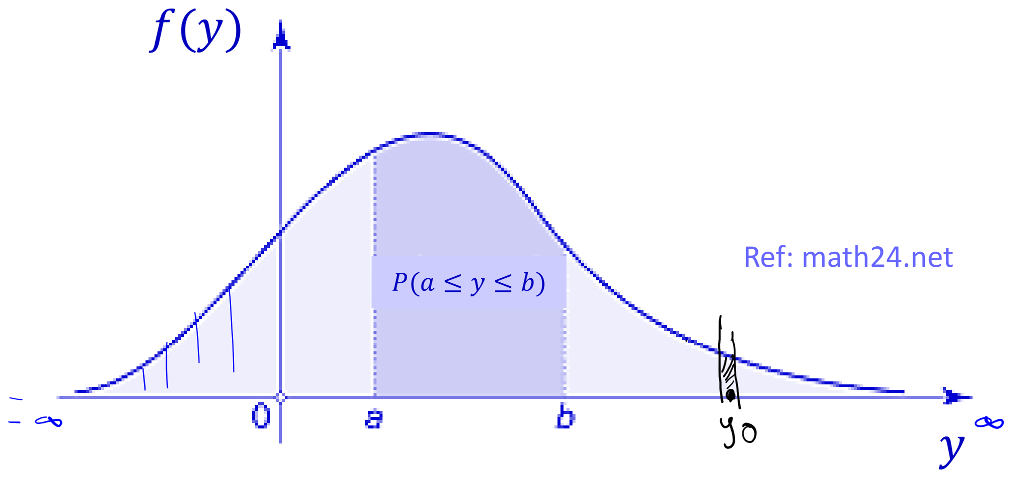
$$F(y) = Probability(Y \leq y) = \int_{-\infty}^{y} f(Y)dY$$

Note: $f(y) = \dfrac{dF(y)}{dy}$

Probability Density Function

- Given $f(y)$, how would you find the *true arithmetic mean (μ)* value of 'y'?

$$\mu = \int_{-\infty}^{\infty} y\, f(y)\, dy$$



- What about *true variance (σ^2)*?

$$\sigma^2 = \int_{-\infty}^{\infty} (y-\mu)^2 f(y)\, dy$$

- The expectation of a function g(y) of a random variable 'y' with pdf 'f(y)' is defined as,

$$\boldsymbol{E}(g(y)) = \int_{-\infty}^{\infty} g(y) f(y) dy$$

$E(y) = \mu$

$E((y-\mu)^2) = \sigma^2$

Rules for Expectation

DIY

Mean (Population)

$$\mu = E(y) = \begin{cases} \int_{-\infty}^{\infty} yf(y)\,dy & y \text{ continuous} \\ \sum_{\text{all } y} yp(y) & y \text{ discrete} \end{cases}$$

Variance (Population)

$$V(y) = E[(y-\mu)^2] = \sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (y-\mu)^2 f(y)\,dy & y \text{ continuous} \\ \sum_{\text{all } y} (y-\mu)^2 p(y) & y \text{ discrete} \end{cases}$$

Identities

1. $E(c) = c$
2. $E(y) = \mu$
3. $E(cy) = cE(y) = c\mu$
4. $V(c) = 0$
5. $V(y) = \sigma^2$
6. $V(cy) = c^2V(y) = c^2\sigma^2$
7. $E(y_1 + y_2) = E(y_1) + E(y_2) = \mu_1 + \mu_2$
8. $V(y_1 + y_2) = V(y_1) + V(y_2) + 2\,\text{Cov}(y_1, y_2)$

$$\text{Cov}(y_1, y_2) = E[(y_1 - \mu_1)(y_2 - \mu_2)]$$

11. $E(y_1 \cdot y_2) = E(y_1) \cdot E(y_2) = \mu_1 \cdot \mu_2$

However, note that, in general

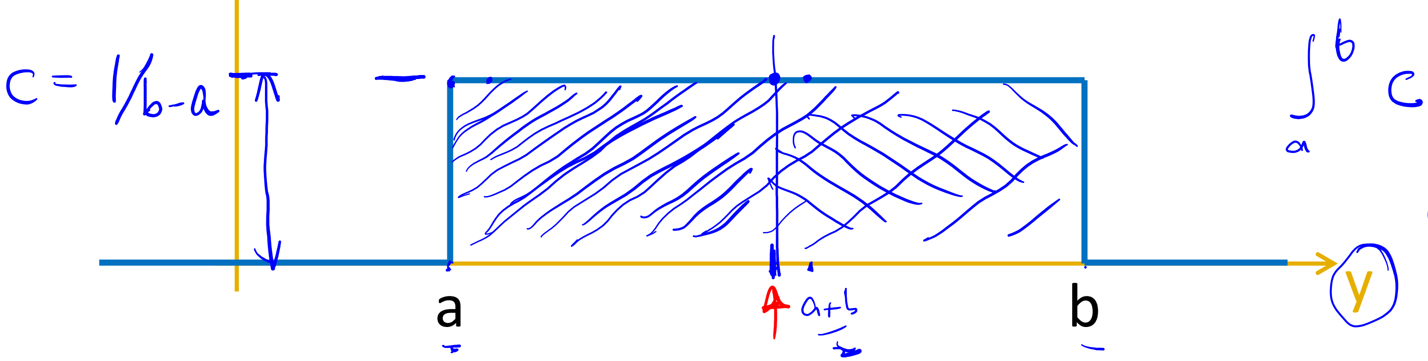
12. $E\left(\frac{y_1}{y_2}\right) \neq \frac{E(y_1)}{E(y_2)}$

regardless of whether or not y_1 and y_2 are independent.

Uniform or Rectangular PDF

$$f(y) = \begin{cases} 1/(b-a) & \text{when } y \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(y)\,dy = 1$$



$$\int_a^b c\,dy = 1$$

$$c\, y\Big|_a^b = 1$$

$$c\,(b-a) = 1$$

- What is mean and variance?

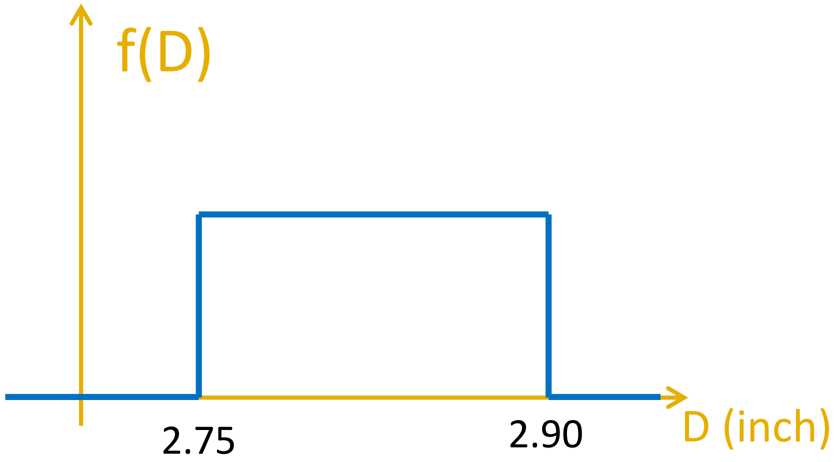
$$\mu = E(y) = \int_{-\infty}^{\infty} y\, f(y)\,dy = \int_a^b y\left(\frac{1}{b-a}\right)dy = \frac{1}{(b-a)}\,\frac{y^2}{2}\Big|_a^b = \frac{1}{2(b-a)}(b^2-a^2) = \left(\frac{a+b}{2}\right)$$

$$V(y) = E((y-\mu)^2) = \int_a^b \left(y - \frac{a+b}{2}\right)^2 \frac{1}{(b-a)}\,dy =$$

- What is median and mode?

$$\text{median} = \left(\frac{a+b}{2}\right) \qquad \text{mode} = \text{any } y \in [a,b]$$

Uniform PDF Example



Suppose a cricket ball manufacturer is making cricket balls of a specified diameter of 2.83 inches.

BUT due to inaccuracies/variations in the making process, the actual diameter of the balls made is uniformly distributed over the range of 2.75 inches to 2.90 inches.

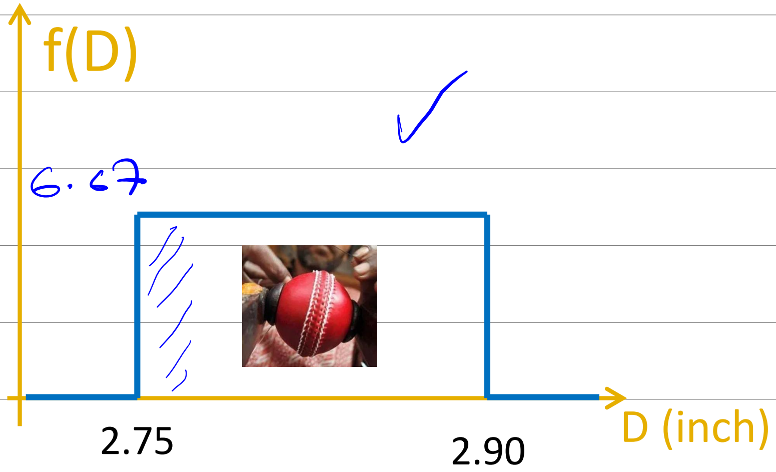
Now, the balls with diameters between 2.80-2.86 inches are still acceptable to BCCI and can be sold for a profit of 100 Rs/ball.

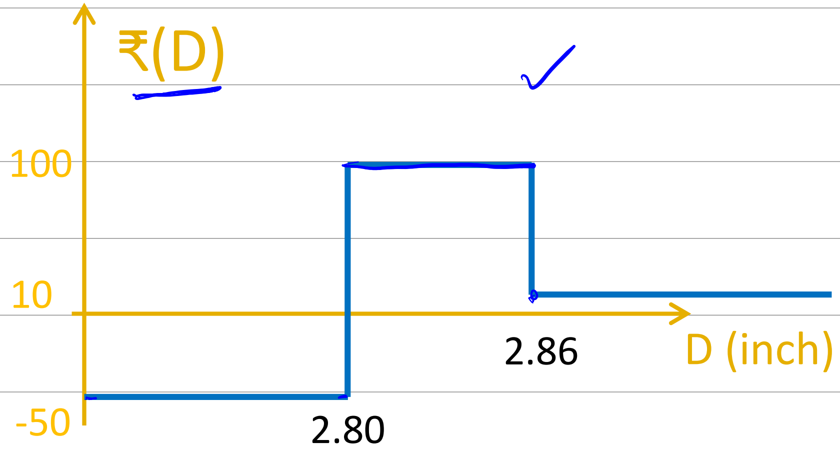
If the ball is oversized (D > 2.86), it can be sold, but at a smaller profit of 10 Rs/ball.

If the ball is undersized (D < 2.80), it needs to be discarded, and there is a loss of 50 Rs/ball.

Question: What is the expected profit (Rs/ball)?

Uniform PDF Example





$$E(\text{₹}(D)) = \int_{-\infty}^{\infty} \text{₹}(D)\, f(y)\, dy$$

$$= \int_{-\infty}^{2.75} \cancel{-50 \times 0}\, dy + \int_{2.75}^{2.80} (-50)\, 6.67\, dy + \int_{2.80}^{2.86} 100 \times 6.67\, dy$$

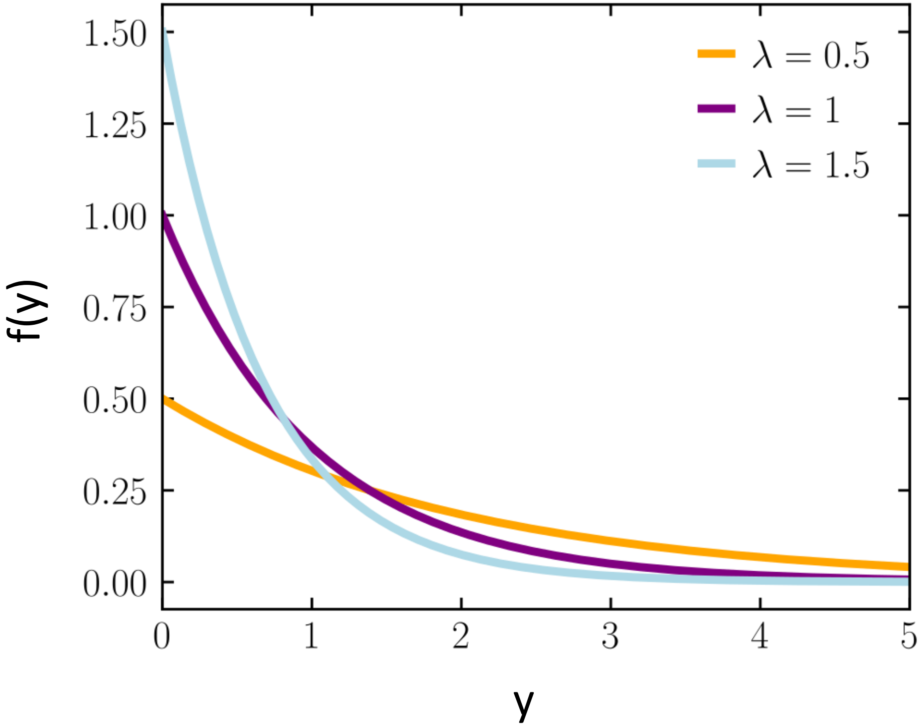
$$+ \int_{2.86}^{2.90} 10 \times 6.67\, dy + \int_{2.90}^{\infty} \cancel{10 \times 0}\, dy$$

$$= 0.05 \times 6.67 \times (-50) + 0.06 \times 6.67 \times 100 + 0.04 \times 10 \times 6.67$$

$$= 26.01 \quad \text{Rs/ball}$$

Exponential PDF

Ref: wikipedia



$$f(y) = \lambda\, e^{-\lambda y}, \qquad y \geq 0$$

$$f(y) = 0, \qquad y < 0$$

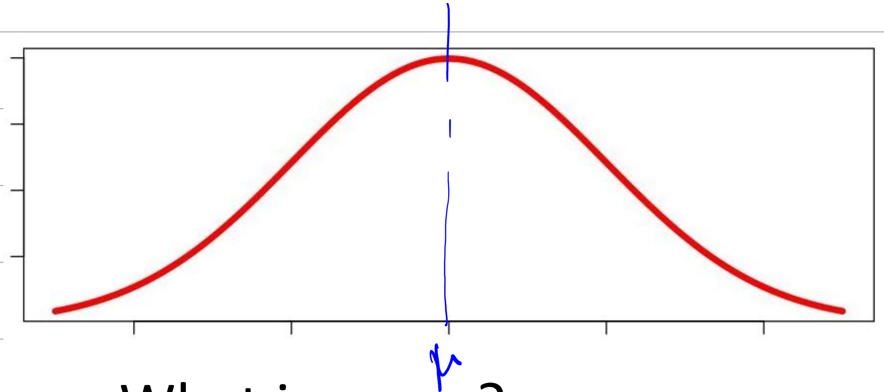
Find mean, std. deviation, median and mode **DIY**

$\text{Mean} = \mu = \frac{1}{\lambda}$ $\qquad$ $\text{Std. Dev} = \sigma = \frac{1}{\lambda}$

$\text{Median} = \frac{\ln(2)}{\lambda}$ $\qquad$ $\text{Mode} = 0$

Normal or Gaussian PDF

DIY



DIY

$$f(y) = \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-b}{a}\right)^2\right) \quad y \in [-\infty, \infty]$$

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)$$

- What is mean?

$\mu = b$

- What is variance and std. deviation?

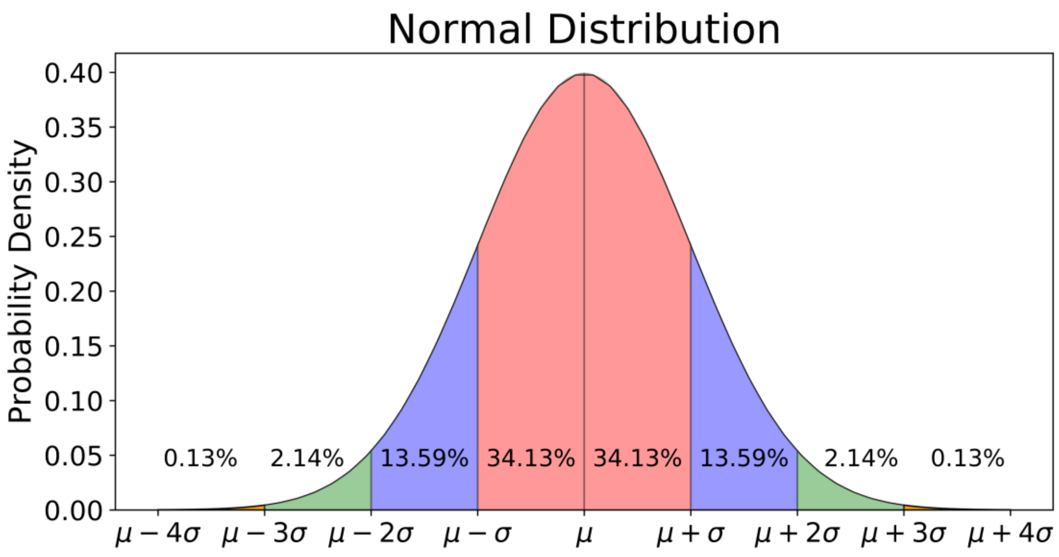
$\sigma^2 = a^2 \quad , \quad \sigma = a$

- What are median and mode?

mean = median = mode = μ = b

Normal PDF

- The 'Normal' distribution, does an excellent job of approximating the relative frequencies of many natural and "man-made" phenomena, e.g. dimension of machined parts, the strength of steel samples, etc.
- It is a bell-shaped curve, symmetric about the objects, which fell off quite rapidly beyond a distance of about one standard deviation from the mean.



NOTE: Although the PDF is defined from –∞ to +∞, most of the density is distributed over a narrow range near the mean (μ)

- **68.26% of the observations fall between $\mu-\sigma$ and $\mu+\sigma$**
- **95.46% of the observations fall between $\mu-2\sigma$ and $\mu+2\sigma$**
- **99.73% of the observations fall between $\mu-3\sigma$ and $\mu+3\sigma$**

Normal PDF Example

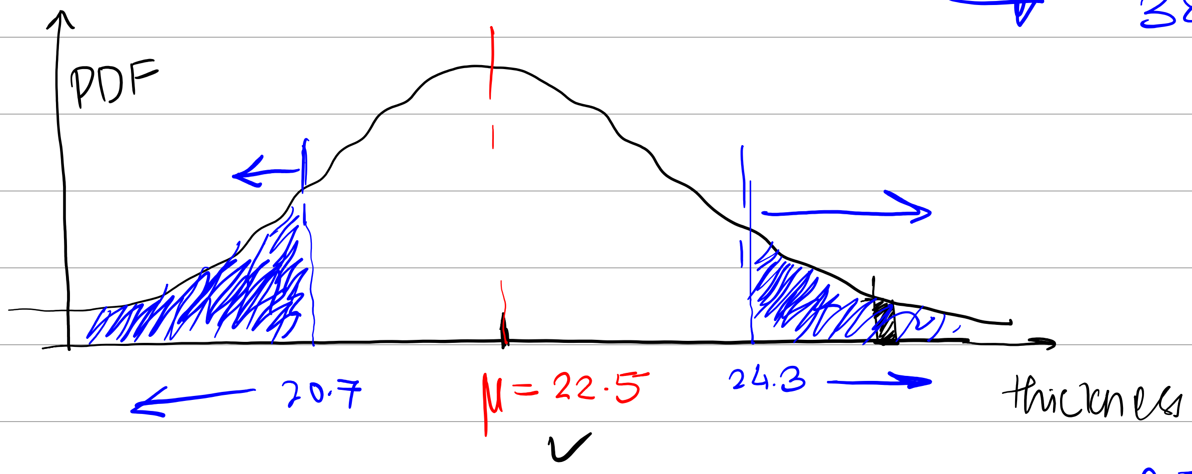
The heat shield plates for the space shuttle must have a closely measured thickness in order to withstand the rigors of heat from re-entry.

After testing 400 of them, the engineer found the thickness was normally distributed with a mean of 22.5 mm. It was also found that 382 plates were within 22.5 ± 2.50 mm.

If the defective plates deviate more than 1.80 mm from the mean, find the number of plates to be rejected during the testing

$\rightarrow$ $382/400$ are within 22.5 ± 2.50 mm

How many out of 400 beyond $(20.7, 24.3)$



$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)$$

~95% plates are within 22.5 ± 2.50 mm

I also know that in ND, 95% population lies bet $\mu \pm 2\sigma$

$\Rightarrow 2\sigma = 2.50$ mm $\Rightarrow \sigma = 1.25$

Normal PDF Example

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If the defective plates deviate more than 1.80 mm from the mean, find the number of plates to be rejected during the testing

Once we know μ & σ we have $f(y)$

fraction of plates rejected = $\int_{-\infty}^{20.7} f(y)\,dy + \int_{24.3}^{\infty} f(y)\,dy$