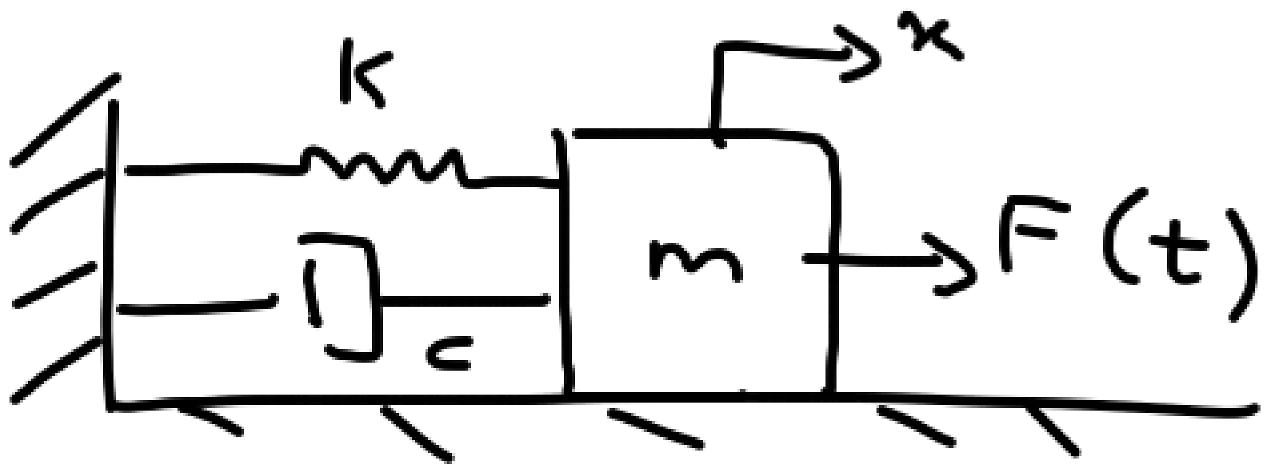
Single degree of freedom Vibration:



Equation of motion:

$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

2nd order linear differential equation.

(I) Undamped, free vibration

$c = 0; \quad F = 0$

$$\boxed{m\ddot{x} + kx = 0} \rightarrow \boxed{\ddot{x} + \frac{k}{m}x = 0}$$

Harmonic equation

$x = e^{st}$

$m s^2 \cancel{e^{st}} + k \cancel{e^{st}} = 0$

$m s^2 + k = 0$

$s^2 = -k/m$

$s = \pm i\sqrt{k/m}$

$i = \sqrt{-1}$ (Imaginary number)

$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$= A_1 e^{i\sqrt{k/m}\,t} + A_2 e^{-i\sqrt{\frac{k}{m}}\,t}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\boxed{x(t) = A\cos\left(\sqrt{\frac{K}{m}}\,t\right) + B\sin\left(\sqrt{\frac{K}{m}}\,t\right)}$$

Initial conditions

$x(0)$, $\dot{x}(0)$

Disp. Velocity

$$x(0) = A$$

$$\dot{x}(0) = B\sqrt{\frac{K}{m}}$$

$$\boxed{x(t) = x(0)\cos\left(\sqrt{\frac{K}{m}}\,t\right) + \frac{\dot{x}(0)}{\sqrt{K/m}}\sin\left(\sqrt{\frac{K}{m}}\,t\right)}$$

$$x(0) = X\cos\phi \; ; \quad \frac{\dot{x}(0)}{\sqrt{K/m}} = X\sin\phi$$

$$\boxed{x(t) = X\cos\left(\sqrt{\frac{K}{m}}\,t - \phi\right)}$$

where $X = \sqrt{[x(0)]^2 + \left(\frac{\dot{x}(0)}{\sqrt{K/m}}\right)^2}$

$$\phi = \tan^{-1}\left(\frac{\dot{x}(0)}{\sqrt{\frac{K}{m}}\,x(0)}\right)$$

mass vibrates at a frequency $\sqrt{\frac{K}{m}}$ governed solely by the system.

No role of external.

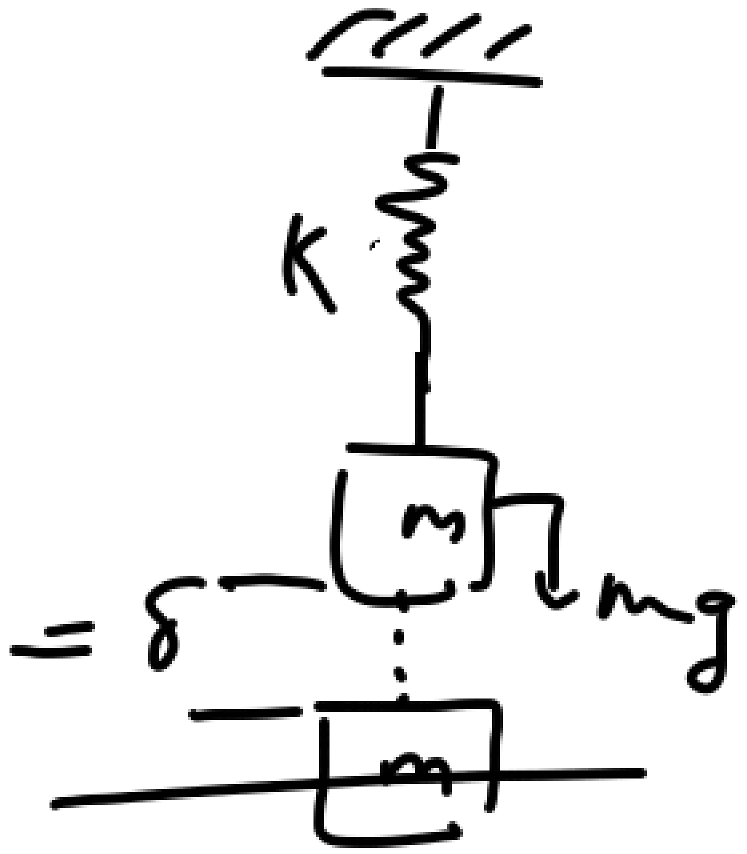
Natural frequency

$$\omega_n = \sqrt{\frac{k}{m}}$$

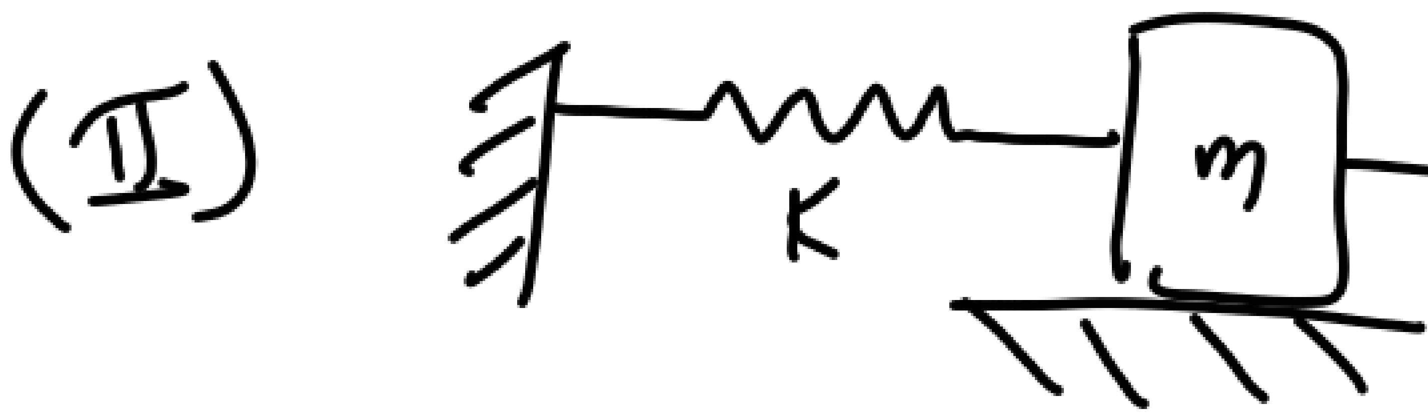
$$f = \frac{\omega_n}{2\pi} \text{ Hz}$$

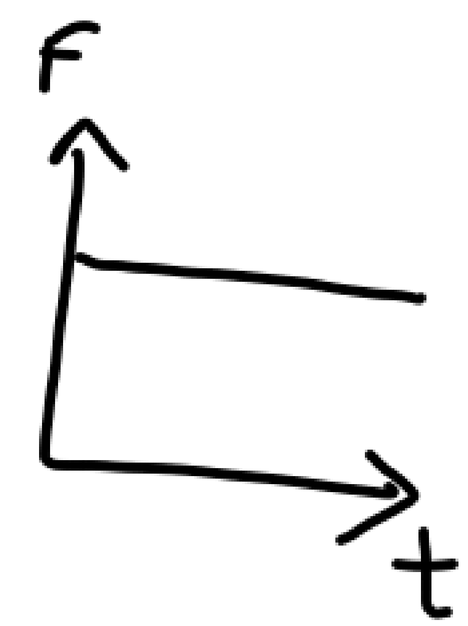
$$T = \frac{2\pi}{\omega_n}$$

$$\frac{mg}{k} = \delta$$



(II)





e.g.: weight in the vertical orientation

$$m\ddot{x} + kx = F$$

$$m\ddot{x} + k\left(x - \frac{F}{k}\right) = 0$$

$$\bar{x} = \left(x - \frac{F}{k}\right)$$

$$\dot{\bar{x}} = \dot{x}$$

$$\ddot{\bar{x}} = \ddot{x}$$

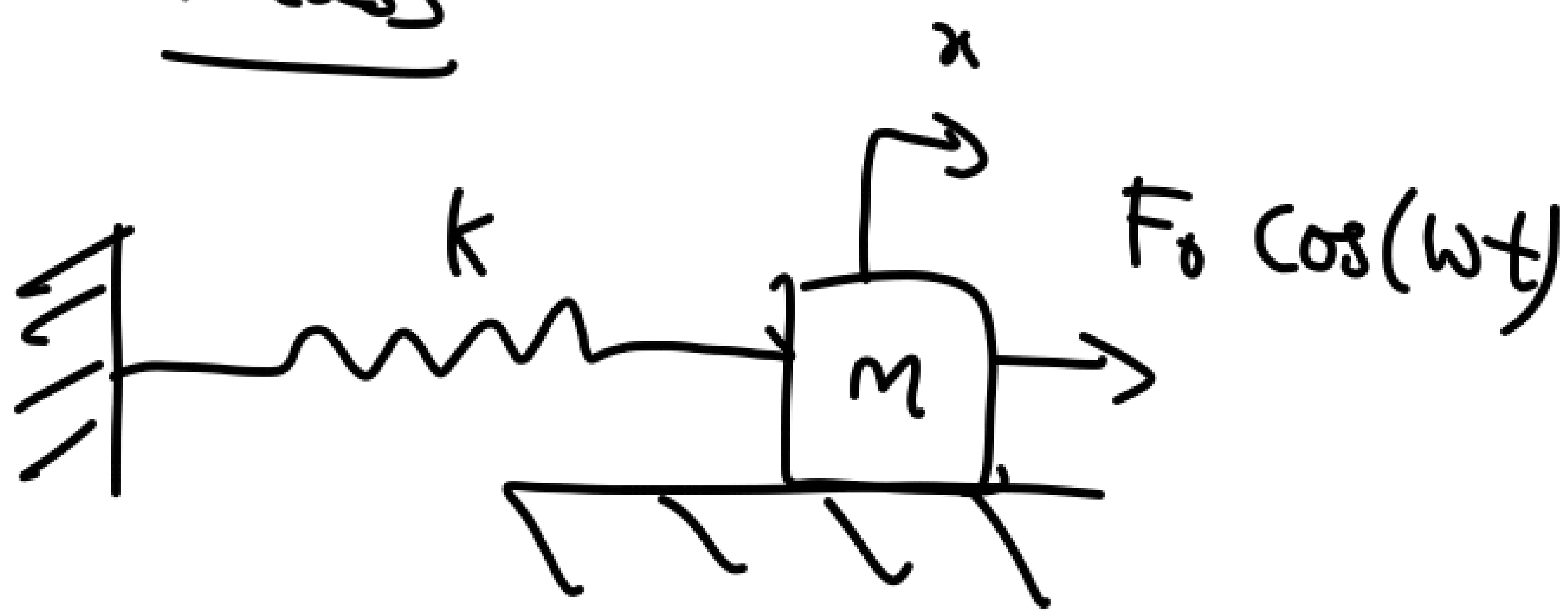
$$m\ddot{\bar{x}} + k\bar{x} = 0$$

$$\bar{x} = X\cos(\omega_n t - \phi)$$

$$\boxed{x = \frac{F}{k} + X\cos(\omega_n t - \phi)}$$

X and ϕ are obtained based on $x(0)$ and $\dot{x}(0)$

(III) Periodic force on the mass



$$\boxed{m\ddot{x} + kx = F_0\cos(\omega t) + 0}$$

$$x(t) = x_h(t) + x_p(t)$$

$x_h(t)$: Homogeneous part

$x_p(t)$: Particular sol'n

$$m\ddot{x}_h + kx_h = 0$$

$$x_h(t) = X\cos(\omega_n t - \phi)$$

$$m\ddot{x}_p + kx_p = F_0\cos(\omega t)$$

$$x_p(t) = P\cos(\omega t)$$

$$\dot{x}_p = -P\omega\sin(\omega t)$$

$$\ddot{x}_p = -P\omega^2\cos(\omega t)$$

$$-P\omega^2 m\cancel{\cos(\omega t)} + kP\cancel{\cos(\omega t)} = F_0\cancel{\cos(\omega t)}$$

$$P = \frac{F_0}{(k - m\omega^2)}$$

$$P = \frac{F_0/k}{1 - \frac{\omega^2}{(k/m)}} = \frac{(F_0/k)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$x(t) = x_h(t) + x_p(t)$$

$$\boxed{x(t) = X\cos(\omega_n t - \phi) + \frac{(F_0/k)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\cos(\omega t)}$$

X, ϕ can be obtained based on $x(0)$ and $\dot{x}(0)$.

If $\omega \rightarrow \omega_n$, then the amplitude blows and we have "RESONANCE".

Similar process if forcing function is $F_0 \sin(\omega t)$

$$m\ddot{x} + kx = F_1\cos(\omega_1 t) + F_2\sin(\omega_2 t)$$

$$x(t) = x_h(t) + x_1(t) + x_2(t)$$

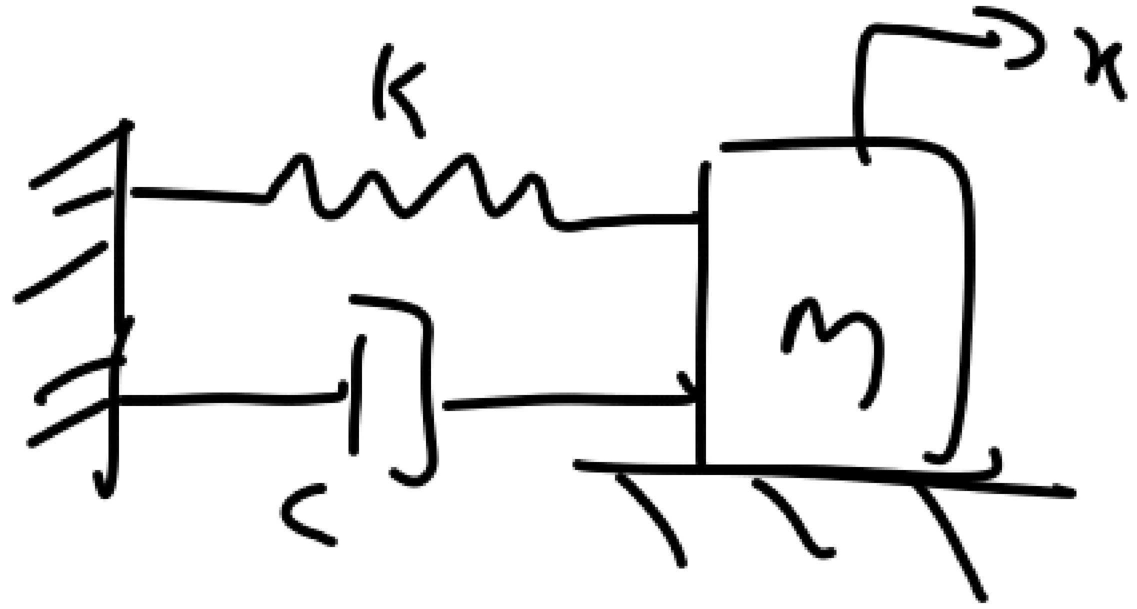
$$m\ddot{x}_h + kx_h = 0$$

$$m\ddot{x}_1 + kx_1 = F_1\cos(\omega_1 t)$$

$$m\ddot{x}_2 + kx_2 = F_2\cos(\omega_2 t)$$

Principle of superposition works because governing eq'n is linear.

(IV) Damped Vibration (Natural Case)



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x = e^{st}$$

$$ms^2\cancel{e^{st}} + cs\cancel{e^{st}} + k\cancel{e^{st}} = 0$$

$$ms^2 + cs + k = 0$$

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$s_{1/2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

When $\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$

$c_c \to$ critical damping coefficient

$$c_c^2 = 4m^2 \frac{k}{m}$$

$$\boxed{c_c = \sqrt{4mk} = 2\sqrt{mk}}$$

We will have three cases:

a) $c > c_c$ (b) $c = c_c$

(c) $c < c_c$

Damping ratio $\xi = \left(\frac{c}{c_c}\right)$

(Xi)

$$s_{1/2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

$$= \frac{-c_c \xi}{2m} \pm \sqrt{\left(\frac{c_c \xi}{2m}\right)^2 - \frac{k}{m}}$$

$$\frac{c_c}{2m} = \frac{2\sqrt{mk}}{2m} = \sqrt{\frac{k}{m}} = \omega_n$$

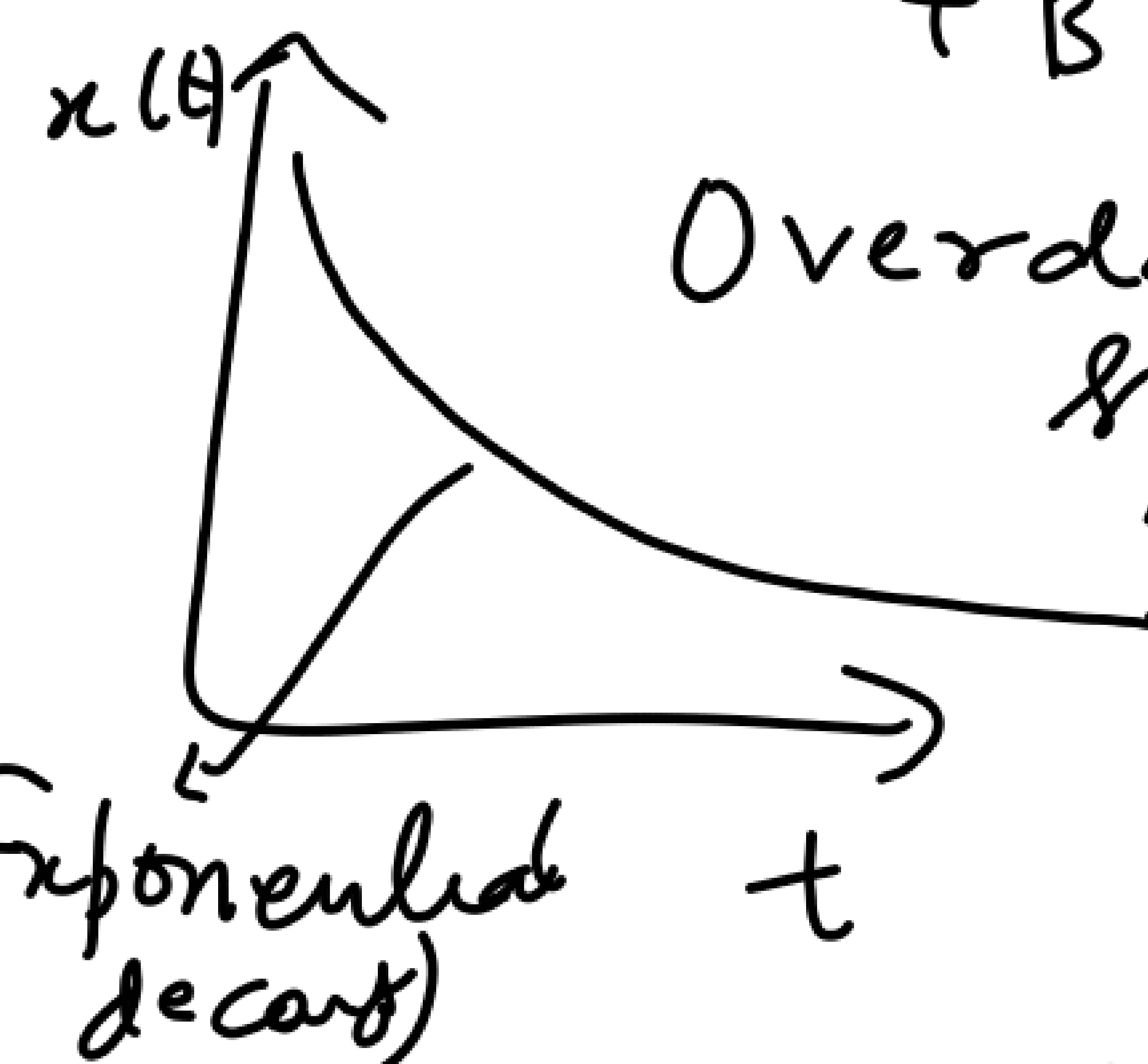
$$s_{1/2} = -\omega_n \xi \pm \sqrt{\omega_n^2 \xi^2 - \omega_n^2}$$

$$\boxed{s_{1/2} = \omega_n\left[-\xi \pm \sqrt{\xi^2 - 1}\right]}$$

(a) $\xi > 1$

$$x(t) = A e^{s_1 t} + B e^{s_2 t}$$

$$= e^{-\omega_n \xi t}\left[A e^{\omega_n \sqrt{\xi^2-1}\, t} + B e^{-\omega_n \sqrt{\xi^2-1}\, t}\right]$$



Overdamped system $\boxed{\xi > 1}$

A and B are obtained using initial condition

(b) $\xi = 1$

$s_{1/2} = -\omega_n$

Repeated root.

$$x(t) = (P + tQ)\, \underline{\underline{e^{-\omega_n t}}}$$

P, Q based on Initial conditions (ICs)

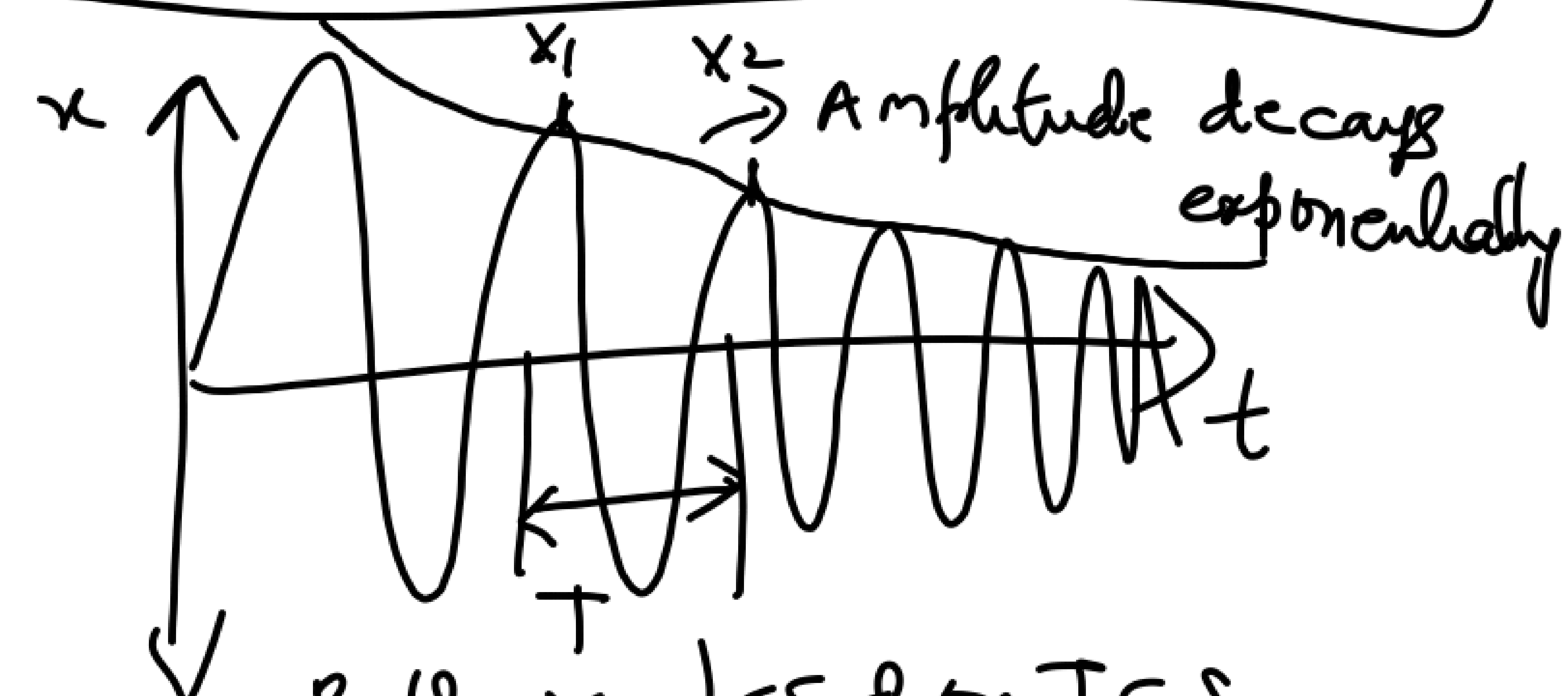
$\to$ Exponentially decaying sol'n.

(c) $\xi < 1$

$$s_{1/2} = \omega_n\left[-\xi \pm i\sqrt{1-\xi^2}\right]$$

$$x(t) = P_1 e^{+\omega_n\left[-\xi + i\sqrt{1-\xi^2}\right]t} + Q_1 e^{-\omega_n\left[-\xi - i\sqrt{1-\xi^2}\right]t}$$

$$\boxed{x(t) = e^{-\omega_n \xi t}\left[P\cos\left(\omega_n\sqrt{1-\xi^2}\, t\right) + Q\sin\left(\omega_n\sqrt{1-\xi^2}\, t\right)\right]}$$



P, Q are based on ICs.

$$P = X\cos\phi$$

$$Q = X\sin\phi$$

$$x(t) = X e^{-\omega_n \xi t}\cos(\omega_d t - \phi)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

→ Damped natural frequency.

$$\frac{x_1}{x_2} = \frac{\cancel{X} e^{-\omega_n \xi t_1}}{\cancel{X} e^{-\omega_n \xi (t_1 + T)}}$$

$$T = \left(\frac{2\pi}{\omega_d}\right)$$

→ Ratio of successive peaks

$$\frac{x_1}{x_2} = e^{\omega_n \xi T}$$

$$= e^{\omega_n \xi \frac{2\pi}{\omega_d}}$$

$$= e^{\cancel{\omega_n} \xi \frac{2\pi}{\cancel{\omega_n}\sqrt{1-\xi^2}}}$$

$$\boxed{\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$

→ Expression to recover damping coefficient