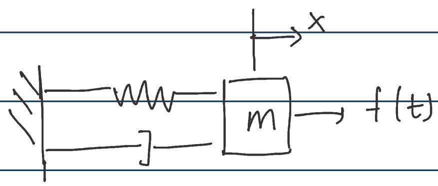
Vibrations

Monday, 8 April 2024 10:37 AM

⇒ <u>Single degree of freedom vibrations</u>



Equation of motion:

$m\ddot{x} + c\dot{x} + kx = f(t)$ → 2^{nd} order linear ODE

(I) Undamped free vibration:

$m\ddot{x} + kx = 0$

Ansatz: $x = ce^{st}$

$ms^2 + k = 0 \Rightarrow s = \pm i\sqrt{\frac{k}{m}}$

$\therefore x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ $\omega = \sqrt{\frac{k}{m}}$: Natural frequency

↳ Simple harmonic motion.

c_1, c_2 can be obtained from the initial conditions $x(0), \dot{x}(0)$.

$x(t) = x(0)\cos(\omega t) + \frac{\dot{x}(0)}{\omega}\sin(\omega t)$

$$= X\cos(\omega t + \phi)$$

(II) Undamped vibrations under constant forcing

$$m\ddot{x} + kx = F$$

substitute $x - \frac{F}{k} = y$

$$m\ddot{y} + ky = 0$$

$$y = A\cos(\omega t) + B\sin(\omega t) \qquad \omega_n = \sqrt{k/m}$$

$$x = \frac{F}{k} + A\cos(\omega_n t) + B\sin(\omega_n t)$$

i.e. Under a constant force, only the equilibrium position changes.

(III) Periodic forcing:

$$m\ddot{x} + kx = F_0\cos(\omega t)$$

$x_p = A\cos(\omega t) + B\sin(\omega t)$ => particular solution.

$$\dot{x}_p = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$\ddot{x}_p = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$

$$\cos(\omega t)\cdot\left[-mA\omega^2 + kA\right] + \sin(\omega t)\left[-mB\omega^2 + kB\right] = F_0\cos(\omega t)$$

$\therefore B = 0 \qquad A = \dfrac{F_0}{k - m\omega^2}$

$$x_p = \frac{F_0}{k - m\omega^2} \cos(\omega t) = \frac{F_0/m}{\omega_n^2 - \omega^2} \cos(\omega t)$$

$$x(t) = \frac{F_0/k}{1 - (\omega/\omega_n)^2} \cos(\omega t) + A\cos(\omega_n t) + B\sin(\omega_n t)$$

If $\omega \to \omega_n$, amplitude blows up and we have resonance.

(IV) Damped free vibrations:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Ansatz: $x = c\, e^{st}$

$$ms^2 + cs + k = 0$$

$$s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

When $\left(\dfrac{c}{2m}\right)^2 - \dfrac{k}{m} = 0$ (critically damped)

$$c_c = \sqrt{4mk}$$

Define $\zeta = \dfrac{c}{\sqrt{\ldots}}$.

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Rewriting equation of motion:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$s_{\pm} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

Case 1: $\zeta < 1$. (Underdamped)

$$s = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$x(t) = e^{-\zeta\omega_n t} \cdot \left(A\cos(\omega_d t) + B\sin(\omega_d t) \right)$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ $\Rightarrow$ Damped Natural frequency

Case 2: $\zeta > 1$ (Overdamped)

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$x(t) = e^{-\zeta\omega_n t} \cdot \left(A\cosh(\omega_n\sqrt{\zeta^2-1}\,t) + B\sinh(\omega_n\sqrt{\zeta^2-1}\,t) \right)$$

↳ Exponential decay.

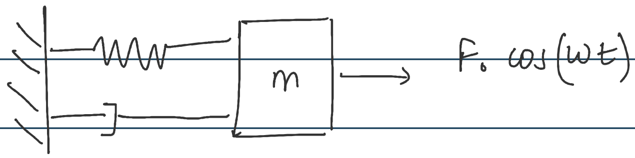
Case 3: $\zeta = 1$ (Critically damped)

$$s = -\omega_n, -\omega_n$$

$\therefore \quad x(t) = (A+Bt)\, e^{-\zeta \omega_n t}$

$\hookrightarrow$ Exponential decay

(v) Forced damped vibrations



$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

Particular solution: $x_p = A\cos(\omega t) + B\sin(\omega t)$

$$-m\omega^2\big(A\cos(\omega t) + B\sin(\omega t)\big) + c\omega\big(-A\sin(\omega t) + B\cos(\omega t)\big)$$

$$+k\big(A\cos(\omega t) + B\sin(\omega t)\big) = F_0\cos(\omega t)$$

$$-m\omega^2 B - A\omega c + kB = 0$$

$$B(k - m\omega^2) = A\omega c \qquad B = \frac{\omega c}{k - m\omega^2} A$$

$$-m\omega^2 A + Bc\omega + kA = F_0$$

$$A\left(k - m\omega^2 + c\omega \cdot \frac{\omega c}{k - m\omega^2}\right) = F_0$$

$$A = \frac{F_0 \left(k - m\omega^2\right)}{\left(k - m\omega^2\right)^2 + \left(\omega c\right)^2}$$

$$x(t) = x_h(t) + x_p(t) \qquad (\zeta < 1)$$

$$= e^{-\zeta \omega_n t} \cdot \left(A\cos(\omega_d t) + B\sin(\omega_d t)\right) +$$

$$\frac{F_0 \left(k - m\omega^2\right)}{\left(k - m\omega^2\right)^2 + \left(\omega c\right)^2} \cdot \left[\cos(\omega t) + \frac{(\omega c)}{\left(k - m\omega^2\right)} \sin(\omega t)\right]$$

$$x_p(t) = X_p \cos(\omega t - \psi)$$

$$\psi = \tan^{-1}\left(\frac{2\zeta\omega_n}{1 - r^2}\right) \qquad r = \omega/\omega_n$$

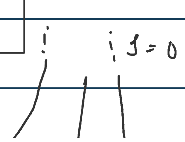
$$X_p = \frac{F_0/k}{\left[\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2\right]^{1/2}} \qquad \zeta = c/c_c$$

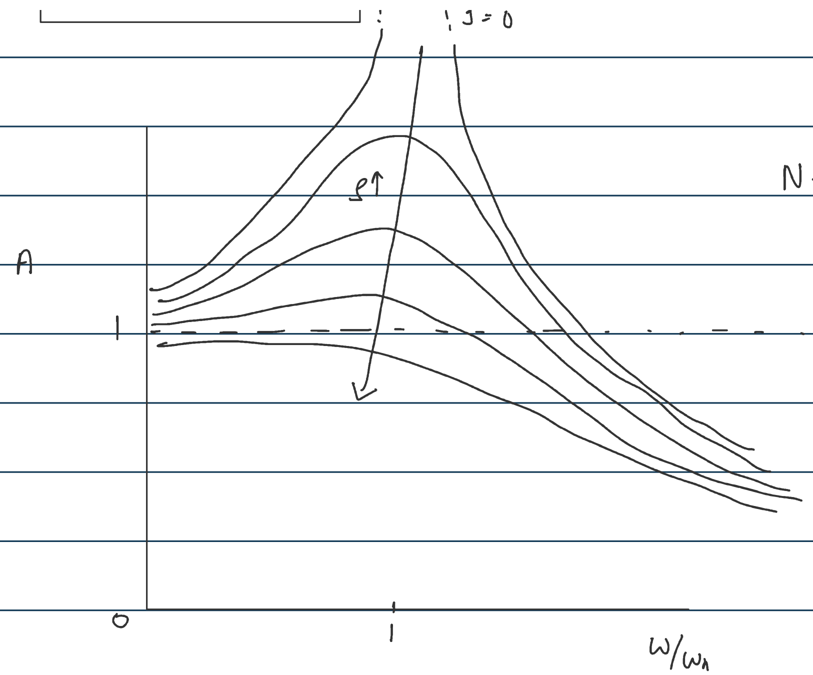
$$\frac{X_p}{\left(F_0/k\right)} = \frac{1}{\left[\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2\right]^{1/2}} = A$$

Homogeneous part of solution decays with time.
At $t = \infty$, system oscillates with forcing frequency.

Resonance: $$\frac{d}{d\omega}\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2\right) = 0$$

$$\boxed{\omega_r = \omega_n \sqrt{1 - 2\zeta^2}}$$





No resonance for $\zeta > 1/\sqrt{2}$.

(vi) Non-periodic forcing.

eg. Impulse. $F(t) = I_m \delta(t)$

$$m\ddot{x} + c\dot{x} + kx = I_m \delta(t)$$

Impulse gives initial condition. $v(0^+) = \dot{x}(0^+) = \frac{I_m}{m}$

$$x(t) = \frac{I_m}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

For a general non-periodic forcing, response can be found by the convolution integral of the function with the impulse response.

$$x(t) = \int^{t} G(t,\tau)\, F(\tau)\, d\tau$$

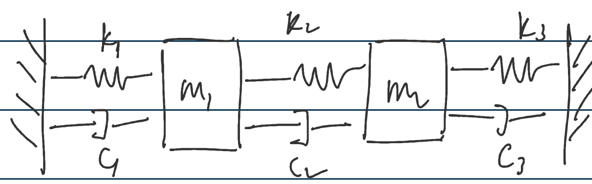
0

where $G(t,\tau)$ is the greens function solution to the ODE.

For the second order system,

$$G(t,\tau) = \frac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_d} \sin\left(\omega_d(t-\tau)\right)$$

⇒ <u>Two Degree of Freedom Systems:</u>



EOM:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

or

$$M\ddot{X} + C\dot{X} + KX = 0$$

Example: Undamped response.

Homogeneous Solution: $X = Ve^{\lambda t}$

$$(M\lambda^2 + K) V e^{\lambda t} = 0$$

$$\therefore \det(M\lambda^2 + K) = 0$$

$$\begin{bmatrix} \lambda^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & \lambda^2 m_2 + k_2 + k_3 \end{bmatrix} = 0$$

$$\lambda^4 m_1 m_2 + \lambda^2 \left[m_1 (k_2 + k_3) + m_2 (k_1 + k_2)\right] + (k_1 + k_2)(k_2 + k_3) - k_2^2 = 0$$

$$\lambda^4 + \lambda^2 \left[\frac{k_2 + k_3}{m_2} + \frac{k_1 + k_2}{m_1}\right] + \frac{(k_1 k_2 + k_1 k_3 + k_2 k_3)}{m_1 m_2} = 0$$

$$\lambda^2 = \frac{-\left(\frac{k_2 + k_3}{m_2} + \frac{k_1 + k_2}{m_1}\right) \pm \sqrt{\left(\frac{k_2 + k_3}{m_2} + \frac{k_1 + k_2}{m_1}\right)^2 - \frac{4(k_1 k_2 + k_2 k_3 + k_3 k_1)}{m_1 m_2}}}{2}$$

For the trivial case $(k_1 = k_2 = k_3)$, $(m_1 = m_2)$

$$\lambda^2 = \frac{-\frac{4k}{m} \pm \sqrt{\frac{16k^2}{m^2} - \frac{12k^2}{m^2}}}{2}$$

$$= -\frac{2k}{m} \pm \frac{k}{m}$$

$$\lambda_1 = \pm i\sqrt{\frac{3k}{m}} \qquad \lambda_2 = \pm i\sqrt{\frac{k}{m}}$$

$$x(t) = V_1 \left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) + V_2 \left(C\cos(\omega_2 t) + D\sin(\omega_2 t)\right)$$

$$X(t) = V_1\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) + V_2\left(C\cos(\omega_2 t) + D\sin(\omega_2 t)\right)$$

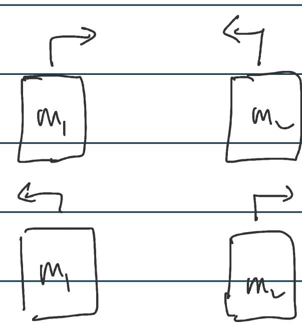
V_1, V_2 : found by solving eigenvalue equation with λ_1, λ_2.

A, B, C, D found from initial conditions.

Modes (shapes) :

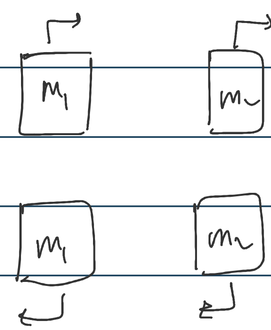
$$\omega = \omega_1 \Rightarrow \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$V_1 = 1 \quad V_2 = -1 \Rightarrow$



$$\omega = \omega_2 \Rightarrow \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$V_1 = 1 \quad V_2 = 1$



Forced undamped vibrations:

Let $F = \begin{pmatrix} F_0 \cos(\alpha t) \\ 0 \end{pmatrix}$

$$M\ddot{x} + Kx = F$$

$$M\ddot{X} + KX = F$$

Let $X_p = \begin{pmatrix} A \\ B \end{pmatrix} \cos(\alpha t)$ (No $\sin(\alpha t)$ because no first derivative on LHS)

$$\therefore \left[-M\alpha^2 + K\right]\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -m\alpha^2+2k & -k \\ -k & -m\alpha^2+2k \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\left(-m\alpha^2+2k\right) A - kB = F_0$$

$$-kA + \left(-m\alpha^2+2k\right) B = 0$$

$$B = \frac{Ak}{2k-m\alpha^2}$$

$$\left(2k-m\alpha^2\right) A - \frac{Ak^2}{2k-m\alpha^2} = F_0$$

$$A\left(\frac{\left(2k-m\alpha^2\right)^2 - k^2}{2k-m\alpha^2}\right) = F_0$$

$$A = \frac{F_0\left(2k-m\alpha^2\right)}{3k^2-4mk\alpha^2+\left(m\alpha^2\right)^2} = \frac{F_0/3k\left(2-\frac{m}{k}\cdot\alpha^2\right)}{\left(1-4\frac{m}{3k}\alpha^2+\frac{m}{3k}\alpha^2\,\frac{m\alpha^2}{k}\right)}$$

$$= F_0/3k\left(2-\left(\alpha/\omega_1\right)^2\right.$$

$$= \frac{F_0/_{3k}\left(2-\left(\alpha/_{\omega_1}\right)^2\right)}{1-\left(\frac{\alpha^2}{\omega_1^2}+\frac{\alpha^2}{\omega_2^2}\right)+\frac{\alpha^2}{\omega_1^2}\frac{\alpha^2}{\omega_2^2}}$$

$$A = \frac{\frac{F_0}{3k}\left(2-\left(\frac{\alpha}{\omega_1}\right)^2\right)}{\left(1-\left(\frac{\alpha}{\omega_1}\right)^2\right)\left(1-\left(\frac{\alpha}{\omega_2}\right)^2\right)}$$

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3k}{m}}$$

$$B = \frac{F_0/_{3k}}{\left(1-\left(\frac{\alpha}{\omega_1}\right)^2\right)\left(1-\left(\frac{\alpha}{\omega_2}\right)^2\right)}$$

Vibration isolation - Make $A = 0$. (Choose m, k such that $\alpha = \sqrt{2}\omega$)