1. An aluminum master alloy manufacturer produces grain refiners in an ingot form. The company produces the product in **three furnaces**. Each furnace is known to have its own unique operating characteristics, so any experiment run in the foundry that involves more than one furnace will consider furnaces as a nuisance variable. The process engineers suspect that the stirring rate affects the grain size of the product. Each furnace can be run at **three different stirring rates**. A randomized block design is run for a particular refiner, and the resulting grain size data is as follows.
   a. State the effects model used in randomized complete blocking design.
   b. Is there any evidence that stirring rate affects grain size (Use α=0.05)? (*Hint:* Complete the ANOVA table) **[2+11 marks]**

Stirring Rate (rpm)	Furnace 1	Furnace 2	Furnace 3		
10	14	5	6	25	8.33
15	14	5	9	28	9.33
20	17	9	3	29	9.67
	45	19	18		
	15	6.33	6		82
					9.11

ANOVA table:

Source of Variation	**Sum of Squares**	**Degrees of Freedom**	**Mean Square**	F_0
Treatments	2.91	2	1.46	0.18
Blocks	156.28	2	78.14	9.85
Error	31.7	4	7.93	
Total	190.89	8		
		(2 mark)	(1 mark)	

$a = 3$; $b = 3$; $N = ab = 9$ (1 mark)

Corrected sum of squares:

$$\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{..})^2}_{SS_T} = \underbrace{b\sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})}_{SS_{Treatments}} + \underbrace{a\sum_{j=1}^{b}(\bar{y}_{.j} - \bar{y}_{..})}_{SS_{blocks}} +$$

$$\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{.j} - \bar{y}_{i.} + \bar{y}_{..})^2}_{SS_{error}}$$

$$\therefore SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{..})^2$$

$$= (14 - 9.11)^2 + (5 - 9.11)^2 + (6 - 9.11)^2$$

$$(14 - 9.11)^2 + (5 - 9.11)^2 + (9 - 9.11)^2$$

$$(17 - 9.11)^2 + (9 - 9.11)^2 + (3 - 9.11)^2$$

$= 50.47 + 40.82 + 99.60$

$= 190.89$ (2 marks)

$SS_{\text{Treatments}} = b \cdot \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})$

$= 3 \cdot [(8.33 - 9.11)^2 + (9.33 - 9.11)^2 + (9.67 - 9.11)^2]$

$= 2.91$ (1 mark)

$SS_{\text{blocks}} = a \cdot \sum_{j=1}^{b} (\bar{y}_{.j} - \bar{y}_{..})$

$= 3 [(15 - 9.11)^2 + (6.33 - 9.11)^2 + (6 - 9.11)^2]$

$= 156.28$ (1 mark)

$SS_{\text{error}} = SS_T - SS_{\text{treatments}} - SS_{\text{blocks}}$

$= 190.89 - 2.91 - 156.28$

$= 31.7$ (1 mark)

$F_0 = 0.18$ (1 mark) $F_{0.05, 2, 4} = 6.94$ (0.5 mark)

does not affect (0.5 mark)

2. The effect of **three different ingredients** (A, B, C) on the reaction time of a chemical process is being studied. Each batch of new material is only large enough to permit **three runs** to be made. Furthermore, each run requires approximately 1.5 hours, so only three runs can be made in one day. The experimenter decides to run the experiment as a ***Latin square*** so that day and batch effects may be systematically controlled. She obtains the data that follow. **[2+11 marks]**

a. State the effects model used in the Latin square method.
b. Analyse the data from this experiment (Use α= 0.05) and conclude whether the effect of assembly time affects reaction time. (*Hint:* complete the below table in the process of analyzing the data)

Batch	Day 1	Day 2	Day 3	
1	A=8	B=7	C=7	22
2	C=11	A=7	B=8	26 (1m)
3	B=4	C=10	A=9	23
	23	24 (1m)	24	71

Two factor ANOVA table:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	13.56	2	6.78	
Rows	2.89	2	1.445	
Columns	0.23	2	0.115	0.01
Error	16.21	2	8.11	
Total	32.89	8		
		(2 marks)	(1 mk)	

Here $p = 3$

Latin square terms

A	24
B	19
C	28

$$SS_T = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{..}^2}{N}$$

$$= \left[\begin{array}{lll} 8^2 + 7^2 + 7^2 \\ 11^2 + 7^2 + 8^2 \\ 4^2 + 10^2 + 9^2 \end{array} \right] - \frac{71^2}{9}$$

$$= \{162 + 234 + 197) - \frac{71^2}{9}$$

$$= 32.89$$ (1mk)

$$SS_{Treatmens} = \frac{1}{p} \sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{..}^2}{N}$$

$$= 1/3 \left[24^2 + 19^2 + 28^2 \right) - \frac{71^2}{9}$$

$$= 13.56$$ (1mk)

$$SS_{Row} = \frac{1}{p} \sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N}$$

$$= \frac{22^2 + 26^2 + 23^2}{3} - \frac{71^2}{9} = 2.89$$ (1mk)

$$SS_{columns} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{23^2 + 24^2 + 24^2}{3} - \frac{71^2}{9}$$

$$= 0.23$$ (1 mk)

$$SS_{Error} = SS_T - SS_{Treatment} - SS_{rows} - SS_{columns}$$

$$= 32.89 - 13.56 - 2.89 - 0.23$$

$$= 16.21$$ (1 mk)

$F_{0.05,2,2} = 19$ (0.5) $F_0 < F_{0.05,2,2}$

does not affect (0.5)

3. Answer the following questions with respect to the evaluation of missing data [2+2 **marks**]
   a. Suppose that in question 1 of this exam, observation of Furnace 1 with respect to 15 rpm stirring rate is missing, estimate the missing value.
   b. Suppose that in question 2 of this exam, observation from batch 1 on day 3 is missing, estimate the missing value.

a) $$x_{ij} = \frac{a\, y_{i.}' + b\, y_{.j}' - y_{..}'}{(a-1)(b-1)}$$

$a = b = 3$

$y_{1.}' = 11$ $\qquad$ $y_{.J}' = 31$

$y_{..}' = 68$

$$\therefore \; x_{11} = \frac{3 \times 14 + 3 \times 31 - 68}{(3-1)(3-1)}$$

$$= \frac{42 + 93 - 68}{2 \times 2}$$

$$= 16.75$$

b) $P = 3$

$y_{1..} = 15$ $\qquad$ $y_{.J.} = 21$ $\qquad$ $y_{..K} = 17$

$y_{...} = 64$

$$y_{IJK} = \frac{P(y_{1..} + y_{.J.} + y_{..K}) - 2y_{...}'}{(P-2)(P-1)}$$

$$= \frac{3\{15 + 21 + 17) - 2 \times 64}{(3-2)(3-1)}$$

$$= \frac{3 \times 53 - 128}{2 \times 1} = 15.5$$

Q1 a)

Effects model :

$y_{iJ} = \mu + \tau_i + \beta_J + \varepsilon_{iJ}$ $\longrightarrow$ for $i \to$ treatments

$J \to$ blocks

$\mu =$ overall mean

$\tau =$ effect of i^{th} treatment

$\beta =$ effect of J^{th} block

$\varepsilon_{iJ} =$ usual random error term

Q2 a)

$y_{iJK} = \mu + \alpha_i + \tau_J + \beta_K + \varepsilon_{iJK}$

$i \to$ row

$J \to$ treatment

$K \to$ column