CS 207M Tutorial-3

1. Prove or disprove: among a group of five people, either there are three people who are mutual friends or three people who are complete strangers to each other.

2. Let (x_i, y_i), $i = 1, ..5$ be five distinct points with integer co-ordinates in the xy-plane. Show that the mid-point of the line joining atleast one pair of these points has integer co-ordinates.

3. Show that among any $n+1$ positive integers not exceeding $2n$ there must be integers a and b such that $gcd(a, b) = 1$.

4. Show that among any $n+1$ positive integers not exceeding $2n$ there must be distinct integers a and b such that $a|b$.

5. For every m and n, show that there exists a sequence of mn distinct numbers which has neither an increasing subsequence of length $m+1$ nor a decreasing sequence of length $n+1$.

6. Show that among a group of ten people either there are three people who are mutual friends or four people who are complete strangers to each other.

7. How many sequences $a_1, a_2, \ldots, a_n$ are there such that $a_1 = 1, a_n = k$ and every number is either equal to or 1 greater than the previous number?

8. A box contains n pairs of shoes, $2n$ shoes in total. In how many ways can you select m shoes so that exactly k pairs of shoes are selected?

9. How many different anagrams of the word ABRACADABRA can be formed?

10. Prove the following identities by combinatorial arguments:
    - $\sum_{i=0}^{m} \binom{n+i}{n} = \binom{n+m+1}{n+1}$.
    - $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.
    - $M(n_1, n_2, \ldots, n_k) = M(n_1 - 1, n_2, \ldots, n_k) + M(n_1, n_2 - 1, \ldots, n_k) + \cdots + M(n_1, n_2, \ldots, n_k - 1)$.
    - $(p+q)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} p^{[k]} q^{[n-k]}$ where $r^{[l]} = r(r-1)(r-2)\ldots(r-l+1)$.

11. In how many ways can a set with mn elements be partitioned into m parts each of size n?

12. How many subsets of $[n]$ are such that the sum of numbers in the subset is even?

13. For positive integers n and k, let $c(n, k)$ denote the number of permutations of $[n]$ with k cycles and $s(n, k)$ denote the number of partitions of $[n]$ into k sets. Derive analogues of Pascal's identity of binomial numbers for $c(n, k)$ and $s(n, k)$.

14. Consider n points on the circle such that no three diagonals (chords obtained by joining any two of the n points) intersect at the same point inside the circle. Now draw all possible diagonals. How many points of intersections of these diagonals are there inside the circle?

15. A composition of n is an expression of n as an ordered sum of positive integers. For example, there are 8 compositions of 4: 1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2, 1+3, 3+1, 4. Count the number of compositions of n.