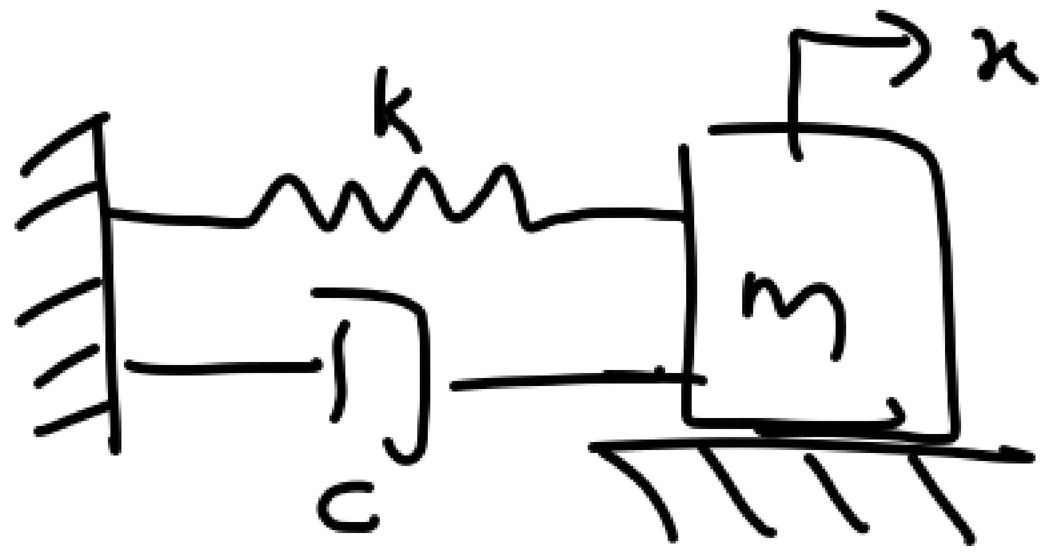
Damped free vibration



$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Damping coefficient

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} \; ; \; \omega_n = \sqrt{\frac{k}{m}}$$

$\zeta < 1 \rightarrow$ Underdamped.

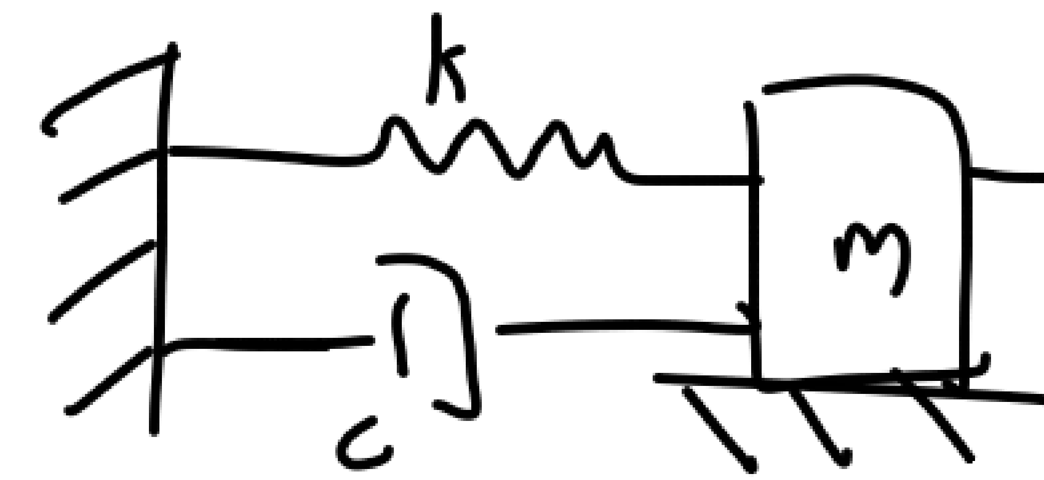
Response :

$$x(t) = e^{-\zeta\omega_n t} X \cos(\underset{\substack{\uparrow \\ \text{Damped natural freq.}}}{\omega_d} t - \phi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Exponentially decaying yet oscillatory response.

(*) Addition of force :



$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

$$x(t) = x_h(t) + x_p(t)$$

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0$$

$$m\ddot{x}_p + c\dot{x}_p + kx_p = F_0 \cos(\omega t)$$

Let $\boxed{x_p(t) = P\cos(\omega t) + Q\sin(\omega t)}$

$$-m\omega^2\left[p\cos(\omega t)+q\sin(\omega t)\right]$$
$$+c\omega\left(-p\sin(\omega t)+q\cos(\omega t)\right]$$
$$+k\left[p\cos(\omega t)+q\sin(\omega t)\right]$$
$$=F_0\cos(\omega t)$$

$$\cos(\omega t)\left[-pm\omega^2+qc\omega+kp\right]$$
$$+\sin(\omega t)\left[-m\omega^2 p-pc\omega+kq\right]=F_0\cos(\omega t)$$

$$\Rightarrow\quad (k-m\omega^2)p+qc\omega=F_0$$
$$-(c\omega+m\omega^2)p+kq=0$$

$$p=\frac{(k-m\omega^2)F_0}{(c\omega)^2+(k-m\omega^2)^2}$$

$$q=\frac{c\omega F_0}{(c\omega)^2+(k-m\omega^2)^2}$$

$$x_p(t)=\frac{F_0}{\left[(c\omega)^2+(k-m\omega^2)^2\right]}\left\{(k-m\omega^2)\cos(\omega t)+(c\omega)\sin(\omega t)\right\}$$

Total response:

$$\boxed{\begin{aligned}x(t)&=x_h(t)+x_p(t)\\&=e^{-\xi\omega_n t}X\cos(\omega_d t-\phi)+x_p(t)\end{aligned}}$$

The unknowns X and ϕ are found using the initial conditions $x(0)$ and $\dot{x}(0)$;

At large times, contribution of $x_h(t)$ becomes negligible and can be ignored. The corresponding response is called "steady state response".

So for large t: $\boxed{x(t) \simeq x_p(t)}$

Eventually the initial conditions will not matter. Response purely governed by the forcing.

Rewriting the particular solution in terms of frequency ratio $r = \frac{\omega}{\omega_n}$ and damping coefficient $\zeta = \frac{c}{c_c}$ as,

$$x_p(t) = X_p \cos(\omega t - \psi)$$

$$\psi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

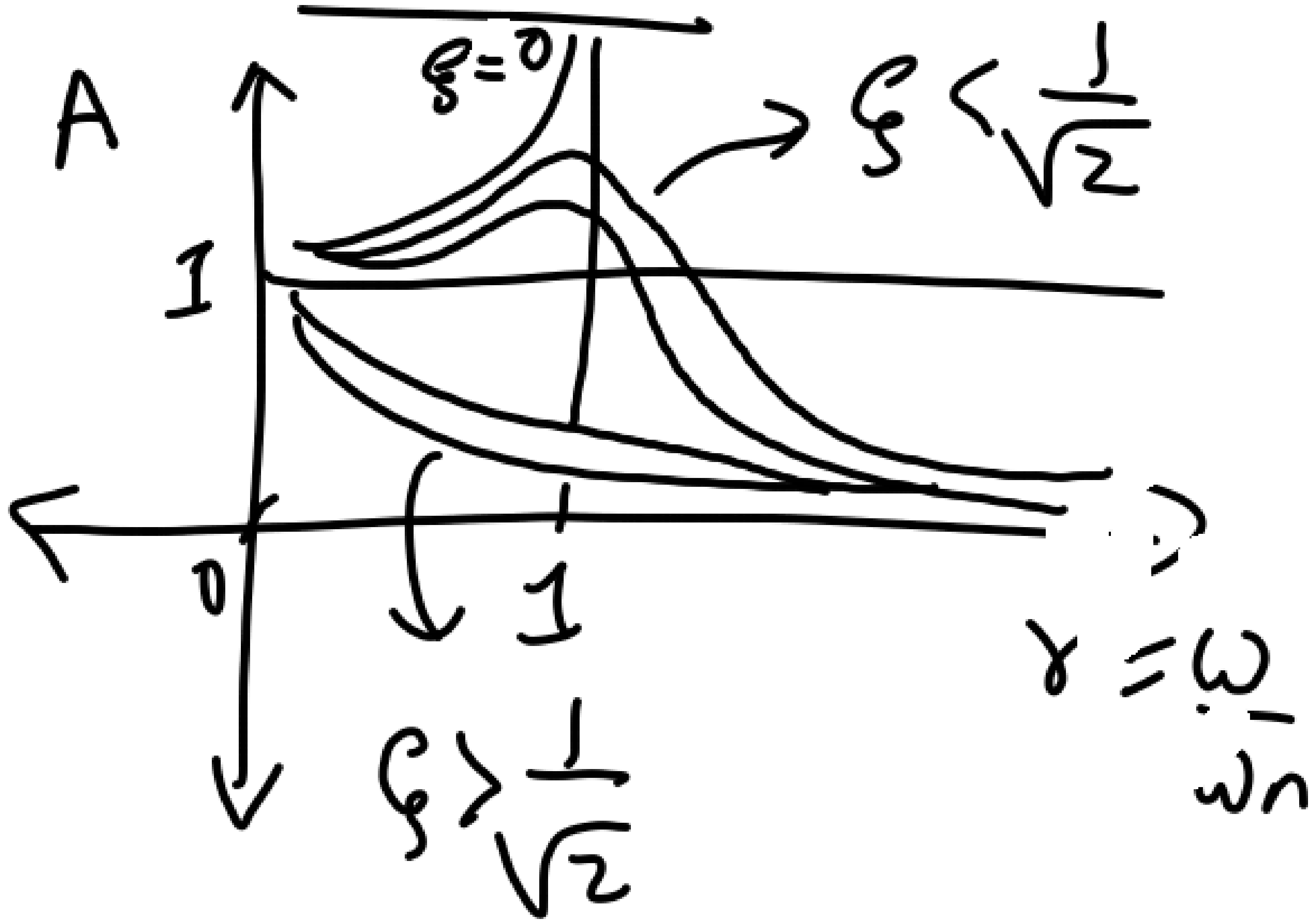
$$X_p = \frac{(F_0/k)}{\left[(1-r^2)^2 + (2\zeta r)^2\right]^{1/2}}$$

$$\frac{X_p}{(F_0/k)} = A = \text{Amplification ratio}$$

$$A = \frac{1}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{1/2}}$$

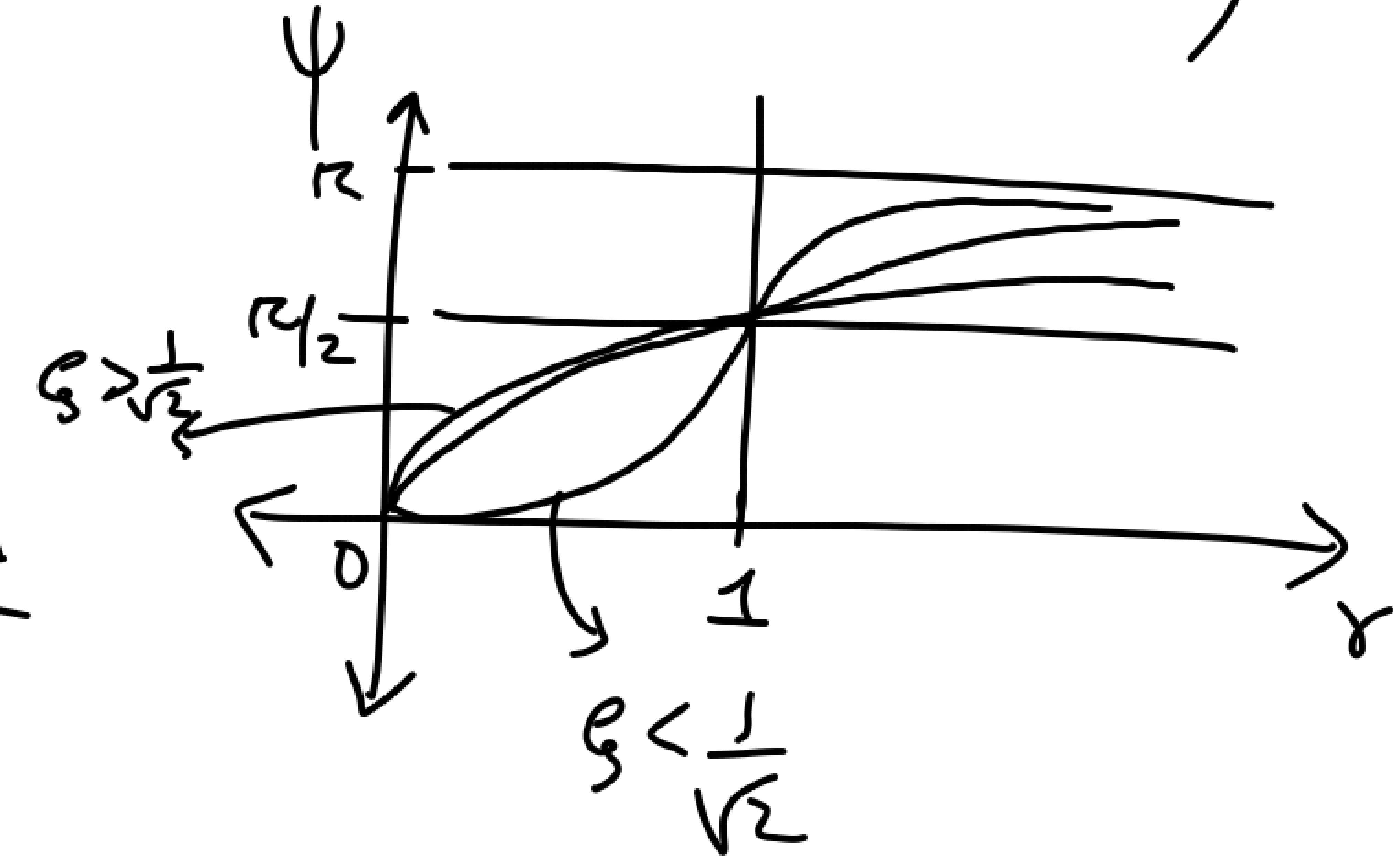
Plot of A and ψ as a function of r and ξ :

$0 < \xi < 1$

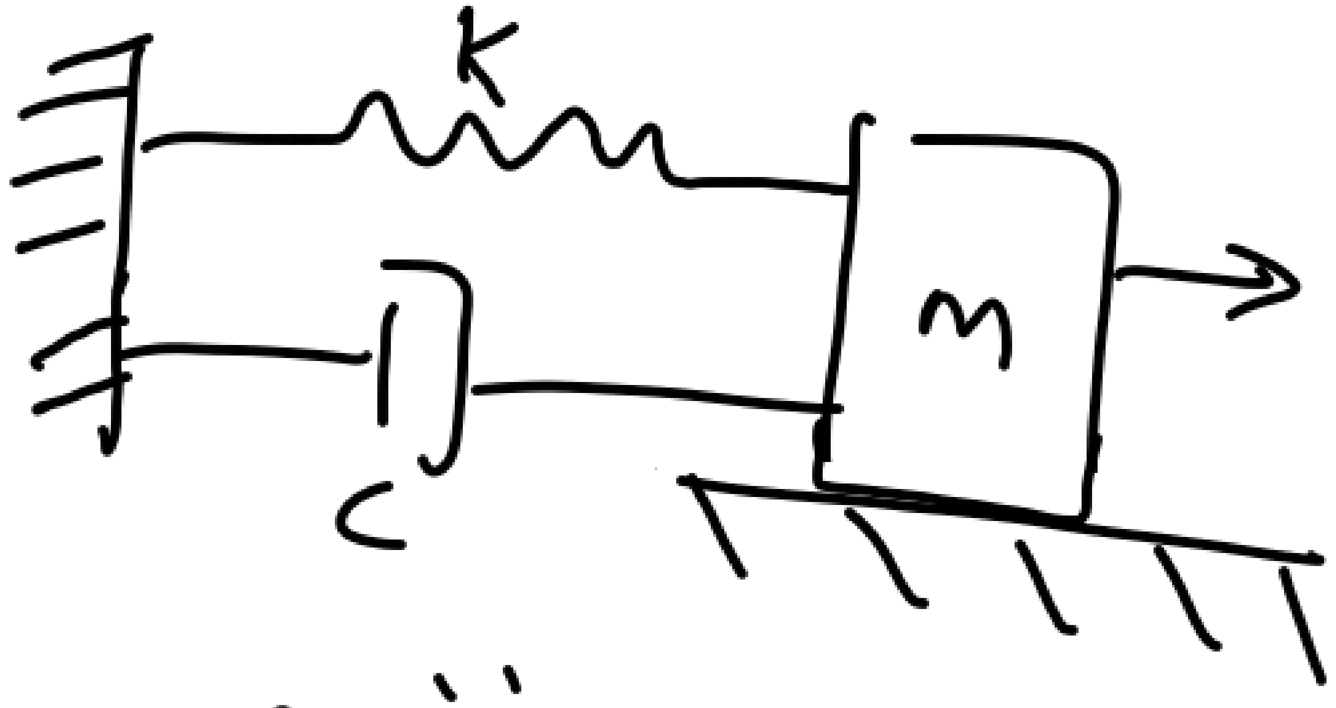


Phase angle plot

$$\psi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right)$$



Phase angle plot



$$F(t) = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t)$$

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t)$$

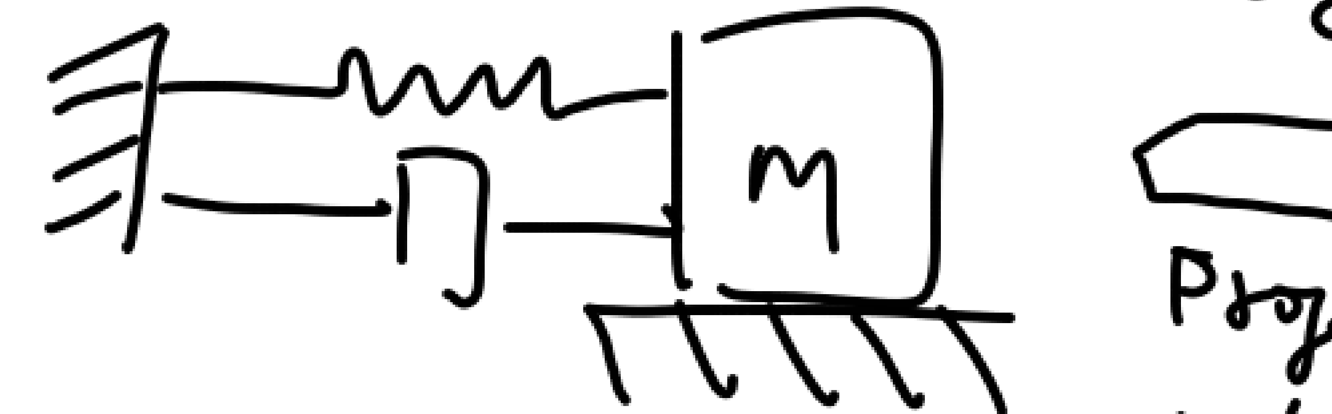
Periodic function as a forcing.

$$f(t) = f_0(t) + \sum_{n=1}^{\infty} f_n^c \cos(\omega_n t) + \sum_{n=1}^{\infty} f_n^s \sin(\omega_n t)$$

Principle of superposition

Forcing as a non-periodic function

e.g.



Projectile impacting the mass

Example of impulse causing vibration :

Newton's law : $\underset{\sim}{G} = m\underset{\sim}{v}$

$$\dot{\underset{\sim}{G}} = \underset{\sim}{F}$$

$$\Delta \underset{\sim}{G} = \underbrace{\int_1^2 \underline{F}\, dt}_{\text{Impulse}}$$

Let $\int_1^2 F\,dt = I_m$ (Impulse)

$$\int_1^2 F\,dt \simeq (F\,\Delta t)$$

$$\Delta G = \int_1^2 F\,dt \quad \text{(along the } x\text{-axis)}$$

$$mv^+ - mv^- = I_m$$

v^+ = velocity just after impulse

v^- = velocity just before impulse

Before impact, system is at rest,

$v^- = 0$

$$\boxed{v^+ = \frac{I_m}{m}}$$

So the initial conditions are;

$$x(0) = 0;$$

$$\dot{x}(0) = v^+ = \frac{I_m}{m}$$

Free vibration response

$$x(t) = e^{-\zeta\omega_n t}\left[E\cos(\omega_d t) + F\sin(\omega_d t)\right]$$

$$x(0) = 0 \Rightarrow E = 0$$

$$\dot{x}(0) = \frac{I_m}{m} \Rightarrow \frac{I_m}{m} = F\omega_d$$

$$\therefore \boxed{F = \left(\frac{I_m}{m\omega_d}\right)}$$

$$\boxed{x(t) = \frac{I_m}{m\omega_d} e^{-\xi\omega_n t} \sin(\omega_d t)}$$

Response post
impulse