Mechanisms

Friday, 26 January 2024 1:54 PM

Mechanisms : Combination of rigid bodies so shaped and connected such that they move upon each other with definite relative motion.

Machine : Collection of mechanisms that transmits force from source of power to the load to be overcome and thus perform useful work.

Degrees of Freedom (DOF) : Number of independent quantities / coordinates required to completely specify the relative motion.

Connection is called joint or Kinematic pair.

Kinematic Pair

- Form closed (joint due to geometric constraint)
- Force-closed (joint due to force constraint)

=) Types of joints :
- Lower pair (Area contact)
- Higher pair (Point / Line contact)

1) Revolute / Turning pair (Hinged joint) :

Relative rotation . 1 DOF - θ

2) Prismatic pair (Slider joint) :

Relative translation 1 DOF – s.

3) Screw / Helical pair :

Relative motion . 1 D.O.F – θ/s.

4) Cylindrical pair :

Relative rotation & translation . 2 DOF : θ, s

5) Spherical pair :

Relative rotation about 3 axes . – θ, φ, ψ

6) Planar pair :

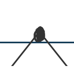
Relative translation along 2 axes, rotation about 1 axis . – x, y, θ

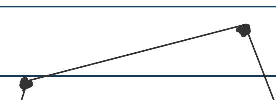
⇒ Elements of a Mechanism :

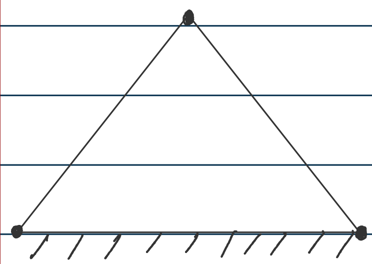
Link : Body common to two or more kinematic pairs .

Kinematic chain : Series of links connected by kinematic pairs.

Closed link : Every link is connected to atleast two other links.


Simplest closed


Simplest

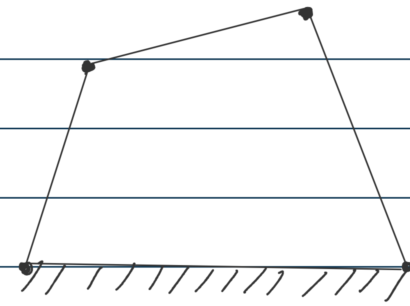


Simplest closed chain.

3 links with 3 joints.

DOF = 0

Structure.



Simplest mechanism.

DOF = 1

Mechanism: Closed kinematic chain with one link fixed.

$\Rightarrow$ Linkage Mechanism:

Kutzbach / Grubler criterion:

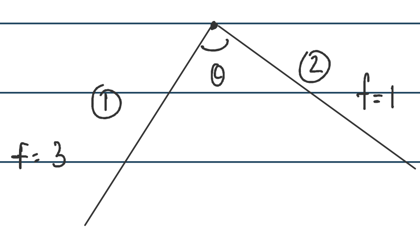
Planar case:

Let n be the number of links.

Each link $\rightarrow$ 3 DOFs. (x, y, θ)

$\therefore$ n links $\rightarrow$ $3n$

1 link is grounded $\rightarrow$ $3(n-1)$



Each lower pair reduces DOF by 2.

If we have j joints, $\boxed{F = 3(n-1) - 2j}$

Each higher pair reduces DOF by 1.

$\therefore$ $\boxed{F = 3(n-1) - 2j - h}$ $\rightarrow$ Kutzbach criterion.

If $F = 1$, we have a constrained mechanism.

$\boxed{1 = 3(n-1) - 2j - h}$ $\rightarrow$ Grubler criterion.

In 3D,

$$\boxed{F = 6(n-1) - 5j_1 - 4j_2 - 3j_3 - 2j_4 - j_5}$$

where j_i denotes joint with i DOFs.

Number Synthesis: Deciding n, j

$j = \dfrac{3(n-1) - F}{2}$ $\rightarrow$ $\therefore$ $3(n-1) - F$ = even

Case 1: $3(n-1)$ is even & F is even

n is odd

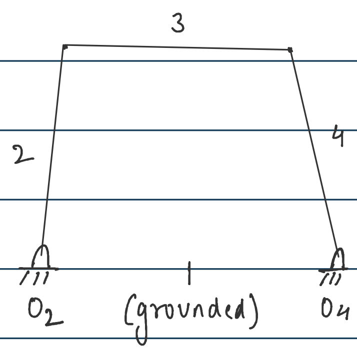
Case 2: $3(n-1)$ is odd & F is odd

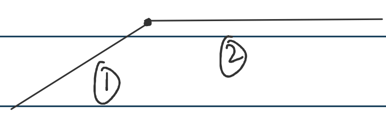
n is even

For a closed linkage,

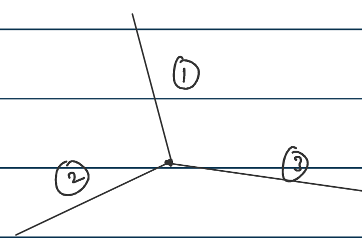
$n_{min} = 3 \rightarrow j = 3 \quad F = 0 \Rightarrow$ structure. (used in trusses)

For non zero F, $n_{min} = 4$, $j = 4$, $F = 1$.





Simple joint



Not a simple joint

Hereon, we assume that the joint is simple.

Assume we have n_2 binary, n_3 ternary, ..., n_k.

Total no. of links $= \sum_{i=2}^{k} n_i$

Let the number of joints be j.

$\therefore$ Number of elements $= 2j$.

For a binary link, no. of elements $= 2n_2$

" n_3 link, " $= 3n_3$

$$\therefore \boxed{2j = 2n_2 + 3n_3 + \dots + k\,n_k}$$

$$\therefore \quad F = 3\sum(n_i - 1) - \sum i\,n_i$$

$$\boxed{F = \sum(3-i)\,n_i - 3}$$

For DOF $=1$, next even $n = 6$

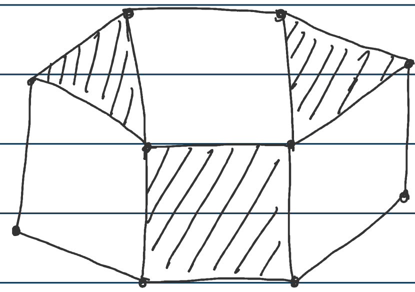
$$4 = \sum(3-i)\,n_i$$

Should all links be binary, or is there a need for ternary, quaternary, etc?

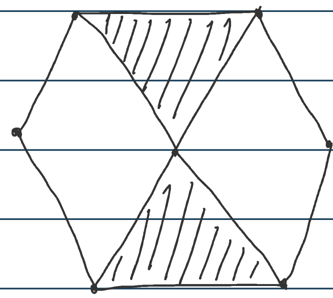
$$4 = n_2 - n_4 \quad \text{(assuming no higher links)}$$

Q. Given 'n' no. of links we want to find out what is the maximum no. of hinges a link can have?

We go backwards.



min no elements = 8



min no. of links = 6

For an n_k link, min no of links = $2k$

i.e. Given "n" no. of links, we can have a link with max

$$\begin{cases} n/2 & \text{(even)} \\ \frac{n+1}{2} & \text{(odd)} \end{cases} \quad \text{hinges.}$$

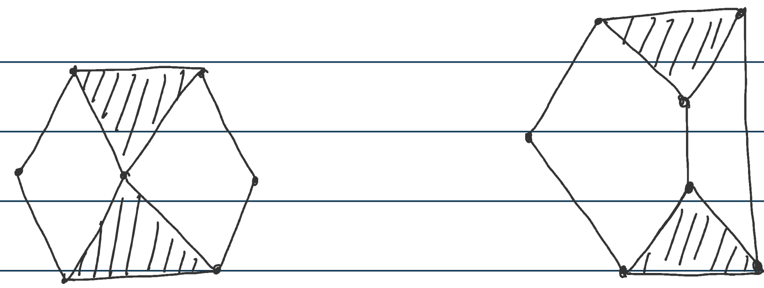
eg. Given $F=1$, $n=6$.

$1 = 3(n-1) - 2j \Rightarrow j = 7$ $\quad n/2 = 3$

So we can have binary & ternary links.

$n_2 + n_3 = 6$ $\quad 2n_2 + 3n_3 = 14$

$n_2 = 4 \qquad n_3 = 2$



Stephenson Mechanism   Watt Mechanism

eg. $n = 7 \qquad \min F = 2 \quad (\because F = \text{even})$

$F = 3(n-1) - 2j \qquad j = 8$

Link with max no of hinges $= \frac{n+1}{2} = 4$

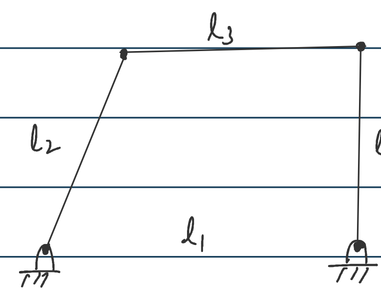
$n_2 + n_3 + n_4 = 7 \qquad 2n_2 + 3n_3 + 4n_4 = 16$

$2n_2 + 2n_3 + 2n_4 = 14$

$n_3 + 2n_4 = 2 \qquad n_3 = 0 \quad n_4 = 1 \quad n_2 = 6$

$n_3 = 2 \quad n_4 = 0 \quad n_2 = 5$

⇒ <u>4-bar Mechanism</u> :



l_1, l_2, l_3, l_4 in ascending order are s, p, q, l.

If $l+s < p+q$, then one of the links can undergo full rotation.

If link adjacent to s is fixed, then s will undergo full rotation and we have crank-rocker mechanism. (s-crank, l_4-rocker)

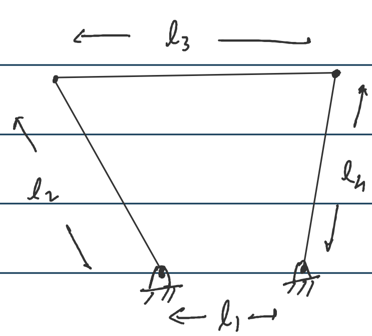
If link opposite to s is fixed, we will have the double rocker mechanism and s will undergo full rotation

If s is fixed, we have a double crank mechanism / drag-link.

Inversion of a mechanism: obtained by changing the fixed link.

If $l+s > p+q$, we have the triple rocker mechanism.

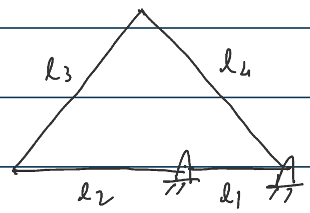
Proof:

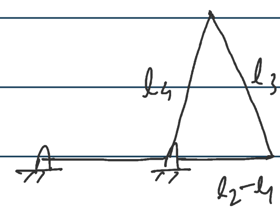


Assume $l_2 > l_1$.

We want to find conditions for l_2 to undergo full revolution.

l_2 has to attain two extreme positions.





$l_1 + l_2 < l_3 + l_4 \quad -①$

$l_2 - l_1 < l_3 + l_4$

$l_1 + l_3 < l_2 + l_4 \quad -②$

$l_1 + l_4 < l_2 + l_3 \quad -③$

Adding ① & ②, $l_1 < l_3$

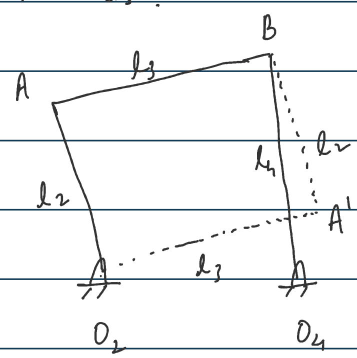
① & ③, $l_1 < l_4$ i.e. l_1 is the shortest link.

② & ③, $l_1 < l_2$

Based on our notation, $l_1 = s$. $\max(l_2, l_3, l_4) = l$

if $s + l < p + q$, automatically other of ①, ②, ③ are satisfied.

What happens to l_2?

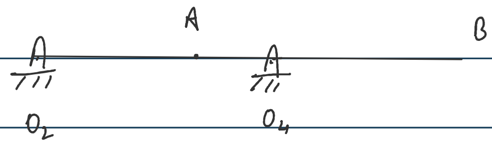


$O_2A'BA$ is a parallelogram.

If we now focus on the 4-bar $O_2A'BA$, if $l_3 < l_1$, then we can show O_2A' will complete full revolution.

$O_2A' \parallel AB$ will also complete full revolution.

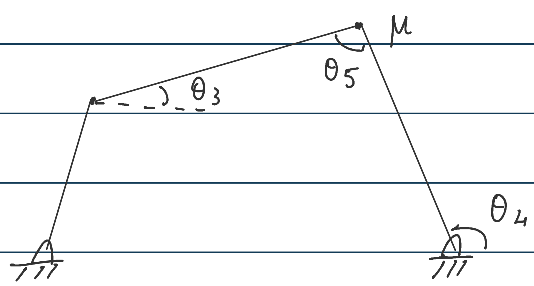
Parallelogram linkage: $s + l = p + q$



All links being inline is the transition point.

Problem: locking in the transition position.

Transmission Angle (μ): $\in [0, \pi/2]$



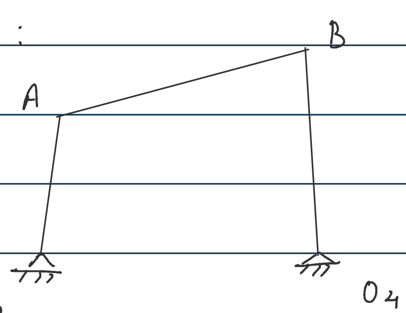
$$\theta_5 = |\theta_3 - \theta_4|$$

$$\mu = \begin{cases} \theta_5, & \theta_5 \in [0, \pi/2] \\ \pi - \theta_5, & \theta_5 \in [\pi/2, \pi] \end{cases}$$

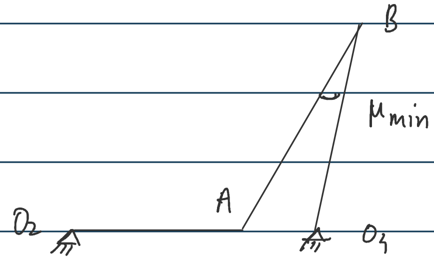
If $\mu \ll 90^\circ$, not ideal since larger force required to provide torque.

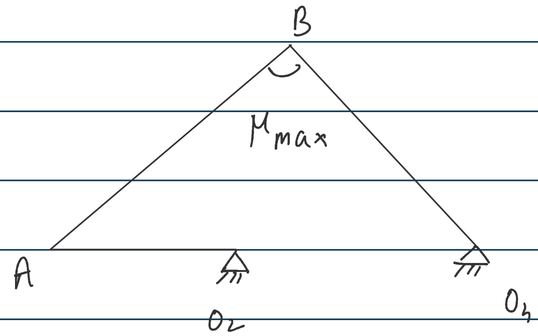
=> ## Inversions of 4-bar mechanism:

1) Crank Rocker:

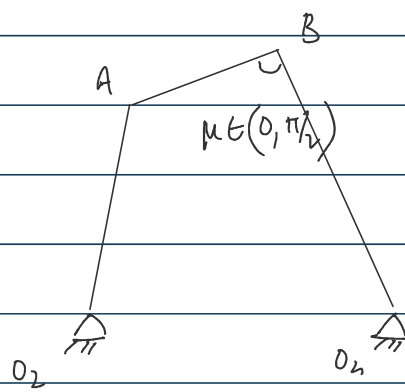


extremes:

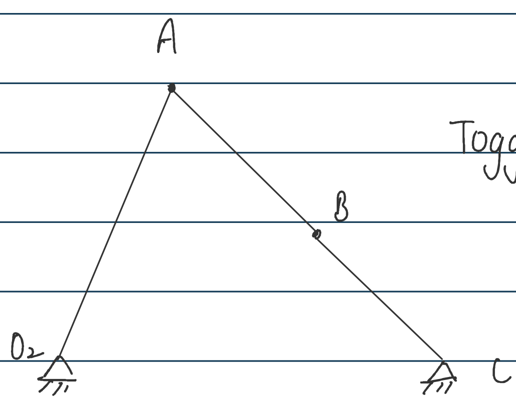




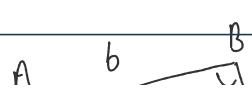
2) Double Crank :



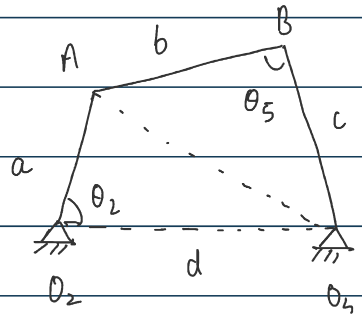
3) Non Grashoff Chain :



Toggle position (links are collinear)



$\cos\theta = a^2 + d^2 - (AO_2)^2$



$$\cos\theta_2 = \frac{a^2 + d^2 - (AO_4)^2}{2ad}$$

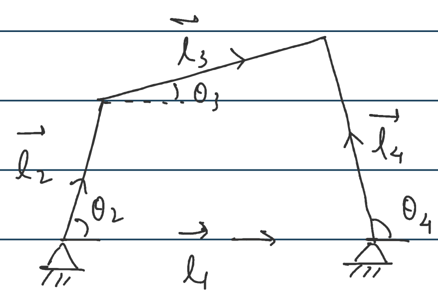
$$\cos\mu = \frac{b^2 + c^2 - (AO_4)^2}{2bc}$$

$$2ad\cos\theta_2 - 2bc\cos\mu = (a^2+d^2) - (b^2+c^2)$$

We can look for $\cos\mu = \pm 1$ for toggle positions.

⇒ **Position, Velocity & Acceleration:**

(1) Position



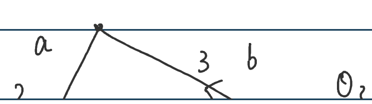
$$\vec{l_1} + \vec{l_2} - \vec{l_3} - \vec{l_4} = 0$$

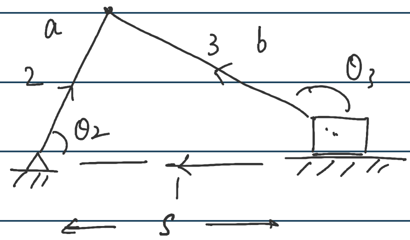
Freudenstein's equation:

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4)$$

where $K_1 = \dfrac{l_1}{l_2}$; $K_2 = \dfrac{l_1}{l_4}$; $K_3 = \dfrac{l_2^2 + l_4^2 + l_1^2 - l_3^2}{2 l_2 l_4}$

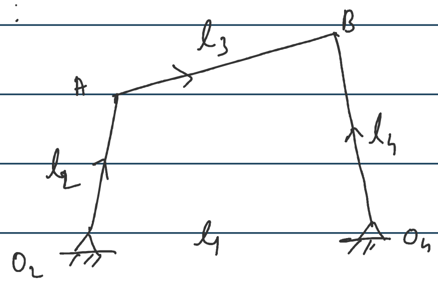
Slider crank:





$$\cos\theta_2 = \frac{s^2 + a^2 - b^2}{2as}$$

(2) Velocity :



$$K_1 \cos\theta_4 - K_2 \cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4)$$

$$\dot{\vec{l}}_2 + \dot{\vec{l}}_3 - \dot{\vec{l}}_4 - \underbrace{\cancel{\dot{\vec{l}}_1}}_{0} = 0$$

$$\vec{V}_A + \vec{V}_{B/A} - \vec{V}_B = 0$$

or $\vec{V}_B = \vec{V}_A + \vec{V}_{B/A}$

$$\vec{\omega}_4 \times \vec{l}_4 = \vec{\omega}_2 \times \vec{l}_2 + \vec{\omega}_3 \times \vec{l}_3$$

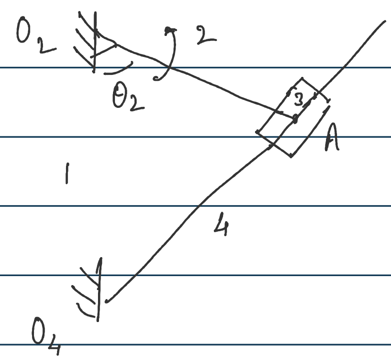
3) Acceleration :

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

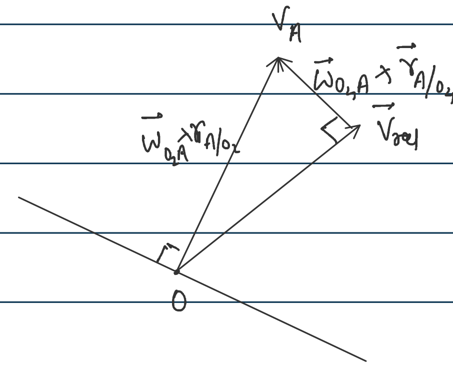
$\Rightarrow$ Inversion of slider crank mechanism:

V_A

⇒ Inversion of slider crank mechanism:



To find: V_A, a_A.



$$V_A = \vec{\omega}_{O_2A} \times \vec{r}_{A/O_2}$$

$$V_A = \vec{\omega}_{O_4A} \times \vec{r}_{A/O_4} + \vec{V}_{rel}$$

$$\vec{V}_{rel} = k \cdot \vec{r}_{A/O_4} \qquad \therefore V_A = \vec{\omega}_{O_4A} \times \vec{r}_{A/O_4} + k \cdot \vec{r}_{A/O_4}$$

$$\therefore \vec{\omega}_{O_2A} \times \vec{r}_{A/O_2} = \vec{\omega}_{O_4A} \times \vec{r}_{A/O_4} + k \cdot \vec{r}_{A/O_4}$$

2 eqns, 2 unknowns (ω_{O_4A}, k)

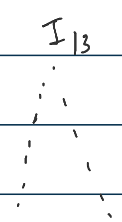
Acceleration:

$$a_A = \vec{\omega}_{O_2A} \times (\vec{\omega}_{O_2A} \times \vec{r}_{A/O_2}) + \vec{\alpha}_{O_2A} \times \vec{r}_{A/O_2}$$

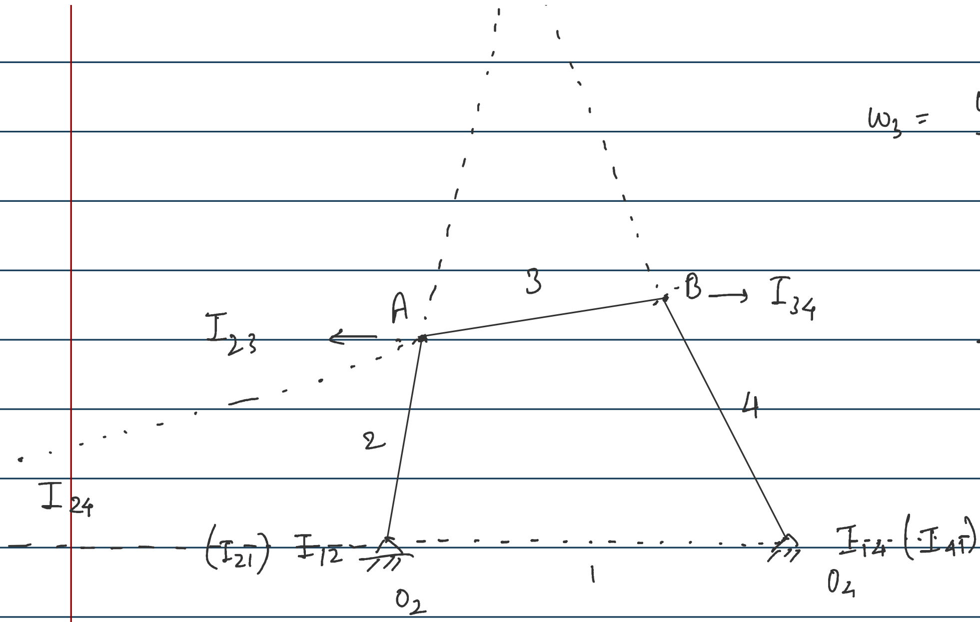
$$a_A = \vec{\omega}_{O_4A} \times (\vec{\omega}_{O_4A} \times \vec{r}_{A/O_4}) + \vec{\alpha}_{O_4A} \times \vec{r}_{A/O_4} + 2\,\vec{\omega}_{O_4A} \times \underbrace{\vec{v}_{rel}}_{k \cdot \vec{r}_{O_4A}} + \underbrace{\vec{a}_{rel}}_{k' \cdot \vec{r}_{O_4A}}$$

Again 2 eqns, 2 unknowns (α_{O_4A}, k')

⇒ Instantaneous centre of velocity:



$$V_A = \omega_2 (O_2A) = \omega_3 (I_{13}A)$$



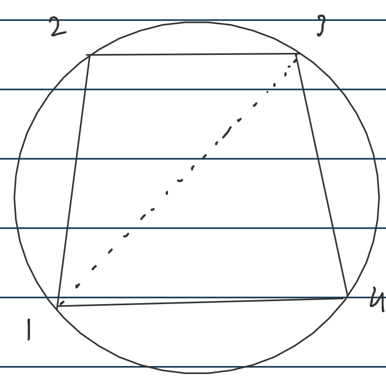
$$\omega_3 = \frac{\omega_2 (O_2A)}{(I_{13}A)} \qquad \omega_3 = \frac{\omega_4 (O_4B)}{(I_{13}B)}$$

$$\frac{\omega_4}{\omega_2} = \left(\frac{I_{13}B}{I_{13}A}\right)\left(\frac{O_2A}{O_4B}\right)$$

Instantaneous centre of velocity for two links is the point w.r.t. which either of the links undergoes pure rotation in the frame of the other.

No. of instantaneous centres $= {}^4C_2 = 6$

Kinematic diagram:



Theorem: For planar motion, three bodies undergoing rigid motion have instantaneous centres of rotation along a straight line. (collinear)

$\therefore I_{24}$ lies on lines (O_2O_4) and (AB).

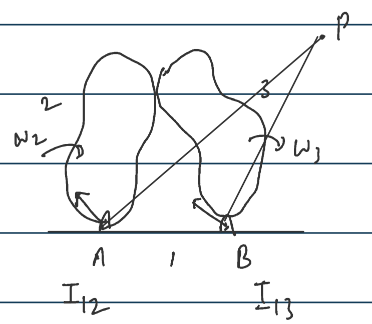
$\because O_2, O_4$ are I_{12}, I_{14} . A, B are I_{23}, I_{34}.

At istantaneous centre of velocity for two links, absolute velocity of both links is the same.

(i.e. absolute velocity of a hypothetical rigid body with link embedded is the same.)

$$\therefore \omega_2 (O_2 I_{24}) = \omega_4 (O_4 I_{24})$$

Proof of Aronhold Kennedy Theorem:



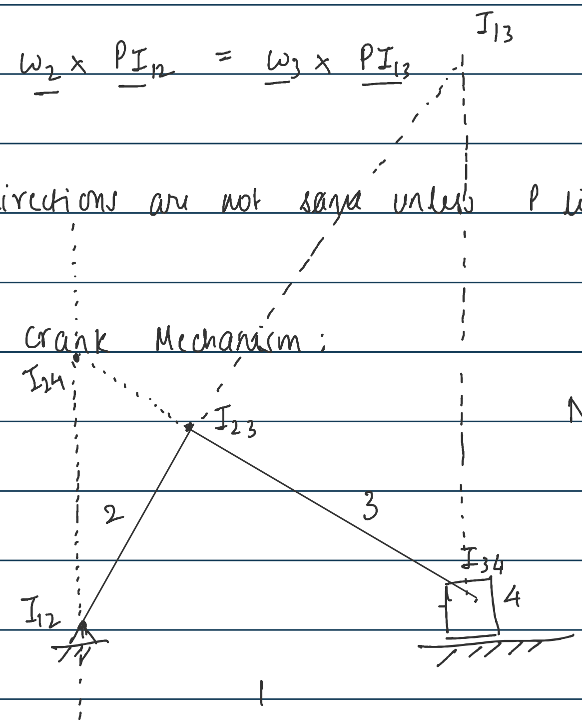
Let P be I_{23}, such that it does not lie on the line $I_{12}I_{13}$.

At P, $\underline{V_2} = \underline{V_3}$

$$\therefore \underline{\omega_2} \times \underline{PI_{12}} = \underline{\omega_3} \times \underline{PI_{13}}$$

directions are not same unless P lies on $I_{12}I_{13}$.

Eg. Slider crank Mechanism:



No. of ICs $= {}^4C_2 = 6$

I_{41} at infinity in -y.

⋮

At point I_{24}, $\underline{v_4} = \underline{v_2}$

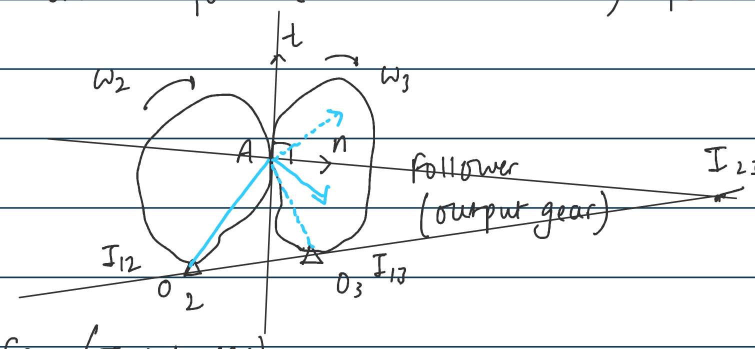
$$\therefore \underline{v_4} = \underline{\omega_2} \times (I_{24} I_{12})$$

$$\underline{v_4} = \omega_2 e_3 \times (I_{24} I_{12} e_2) \quad \therefore v_4 = (\omega_2 I_{24} I_{12})\, e_1$$

⇒ Mechanisms with higher pairs

1) Cam & follower

2) Gears



$$F = 3(n-1) - 2j - h$$

$n = 3 \quad j = 2 \quad h = 1$

$$F = 6 - 4 - 1 = 1$$

At point A, sliding occurs between 2 & 3.

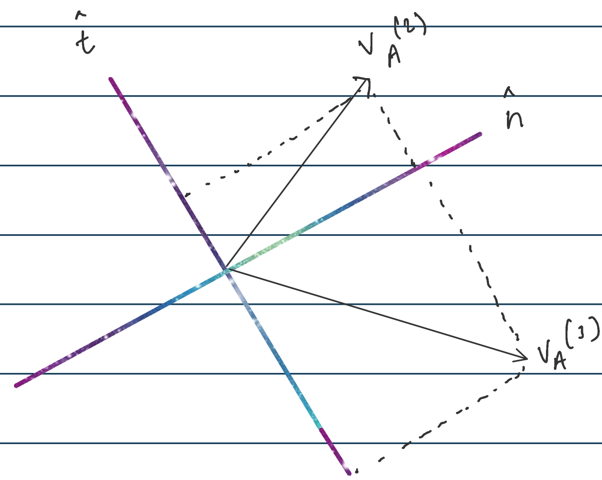
Calculation of instantaneous centre of velocities:

Calculation of instantaneous centre of velocities:

With respect to link 2, velocity of point A is tangential to the contour (along common tangent $\hat{t}$). $\therefore$ I_{23} also lies on common normal $\hat{n}$.

i.e. I_{23} is at the intersection of line $\hat{n}$ and O_2O_3.

$$V_A^{(2)} = \omega_2 \times (AO_2) \qquad V_A^{(3)} = \omega_3 \times (AO_3)$$



For no loss in contact, $\boxed{V_A^{(2)} \cdot \hat{n} = V_A^{(3)} \cdot \hat{n}}$

to avoid a) seperation of bodies b) overlap of bodies

$$V_A^{(2)} - V_A^{(3)} = \left[\left(V_A^{(2)} - V_A^{(3)}\right) \cdot \hat{t}\right] \hat{t}$$

$$\text{Velocity ratio} = \frac{\omega_3}{\omega_2} = \frac{(I_{23}O_2)}{(I_{23}O_3)}$$

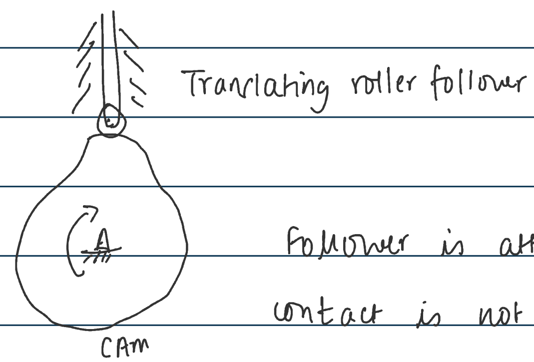
For cams, location of I_{23} keeps changing. Consequently, the velocity

For cams, location of I_{23} keeps changing. Consequently, the velocity ratio also changes.

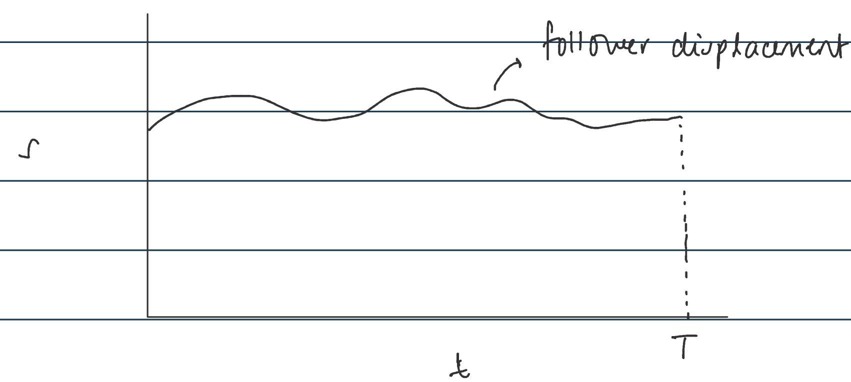
For gears, the surface profile is chosen such that I_{23} is also fixed and the velocity ratio remains constant.

⇒ Cam & Follower :

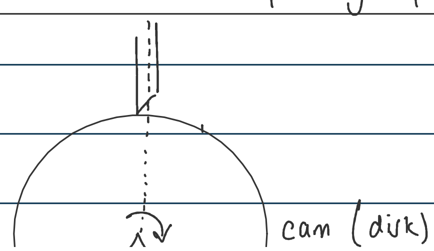
Cam is the input link, driven by a motor at constant speed. Follower usually follows the displacement as:



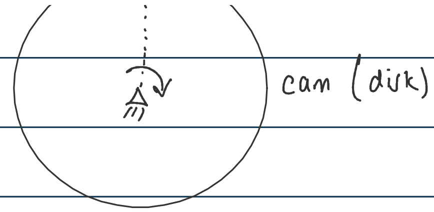
Follower is attached with a spring so that contact is not lost.



Circular disk and knife edge follower: (simplified example)

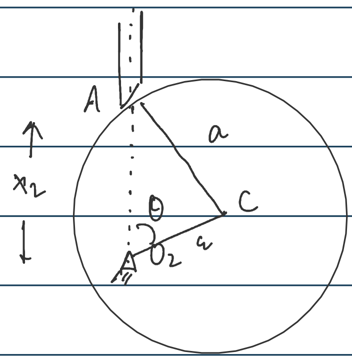


If axis of follower is along



If axis of follower is along the diameter, we have no displacement of the follower.

If some eccentricity is provided to the disk itself,



Follower displacement $\in [a-\epsilon, a+\epsilon]$

$$\cos\theta = \frac{(O_2A)^2 + \epsilon^2 - a^2}{2(O_2A)\,\epsilon}$$

$$O_2A = \varepsilon\cos\theta \pm \sqrt{a^2 - \varepsilon^2\sin^2\theta}$$

For C to lie above the horizontal axis through O_2 at $t=0$,

$$O_2A = \varepsilon\cos\theta + \sqrt{a^2 - \varepsilon^2\sin^2\theta} \qquad \theta = \omega t$$

$$= \varepsilon\left[\cos\theta + \sqrt{(a/\varepsilon)^2 - \sin^2\theta}\right]$$

$$x_2 = \varepsilon\left[\cos\theta + \sqrt{(a/\varepsilon)^2 - \sin^2\theta}\right]$$

$$v = \frac{dx_2}{d\theta}\,\omega = \omega\varepsilon\left[-\sin\theta + \frac{1}{2\sqrt{(a/\varepsilon)^2 - \sin^2\theta}} \cdot -2\sin\theta\cos\theta\right]$$

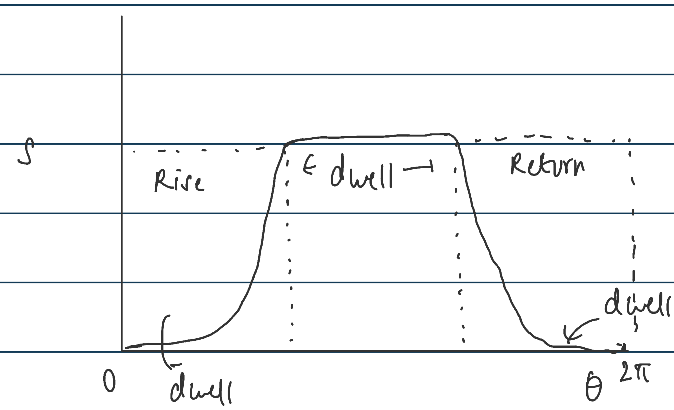
$$= -\omega\varepsilon\sin\theta\left[1 + \frac{\cos\theta}{\sqrt{(a/\varepsilon)^2 - \sin^2\theta}}\right]$$

Similarly, a can be computed.

Similarly, a can be computed.

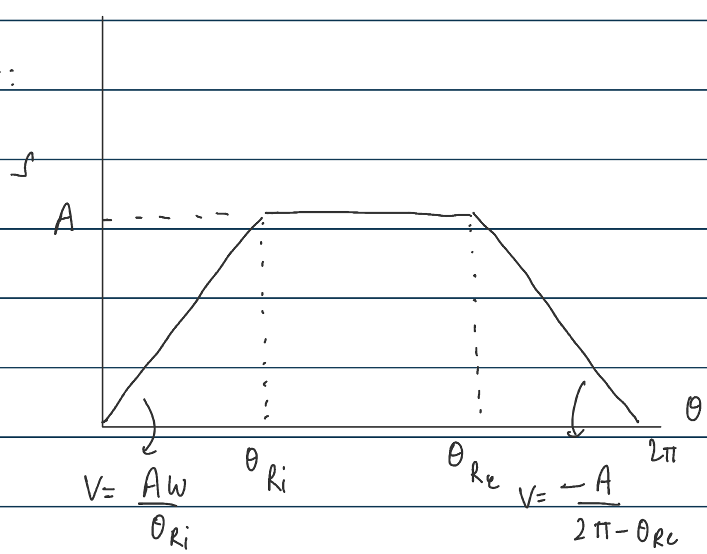
In this example, we have no control over the velocity or acceleration of the follower (except for ω).
High values of v & a may lead to vibrations and noise.

For translating followers, (s) can usually be expressed as



Approximate as:

(1) Linear:



$s = A\left(\theta/\theta_{Ri}\right) \quad 0 \leq \theta \leq \theta_{Ri}$

$s = A \quad \theta_{Ri} \leq \theta \leq \theta_{Re}$

$s = A\left(\frac{2\pi - \theta}{2\pi - \theta_{Re}}\right) \quad \theta_{Re} \leq \theta \leq 2\pi$

(2) Higher order polynomials

(2) Higher order polynomials

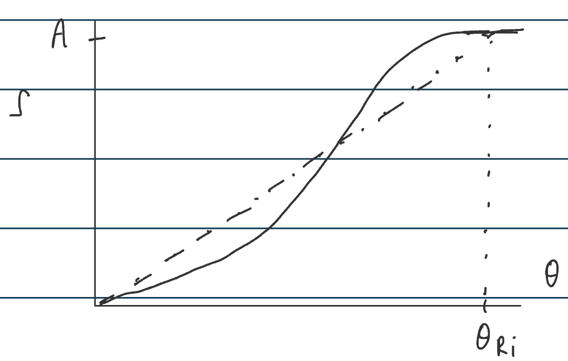
(3) Trignometric functions:

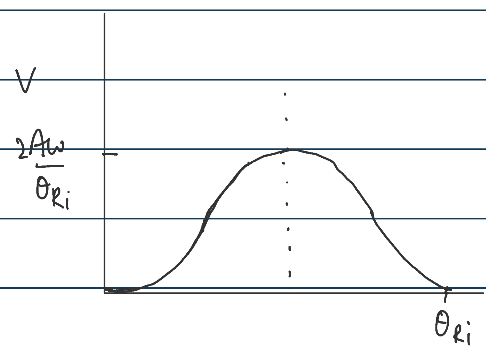
eg. Cycloidal function (Engineering curve)

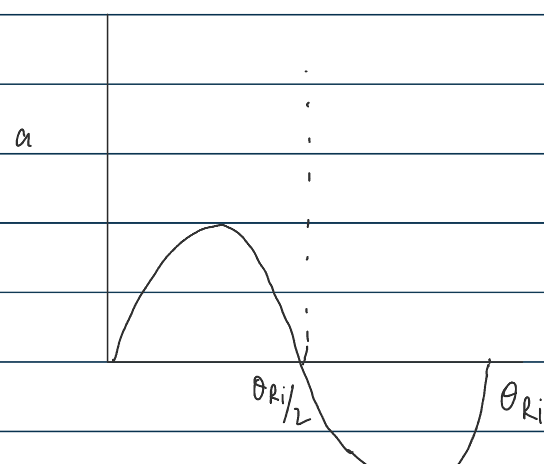
$$S = A\left[\frac{\theta}{\theta_{Ri}} - \frac{1}{2\pi}\sin\left(\frac{2\pi\theta}{\theta_{Ri}}\right)\right] \rightarrow \text{Rise part}$$

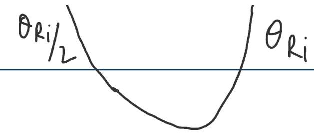
$$V = \frac{ds}{d\theta}\omega = \frac{\omega A}{\theta_{Ri}}\left[1 - \cos\left(\frac{2\pi\theta}{\theta_{Ri}}\right)\right]$$

$$a = \omega\frac{dv}{d\theta} = \frac{2\pi A\omega^2}{\theta_{Ri}^2}\sin\left(\frac{2\pi\theta}{\theta_{Ri}}\right)$$









Cycloid curve ensures zero velocity & accelerations at the start and end of rise.

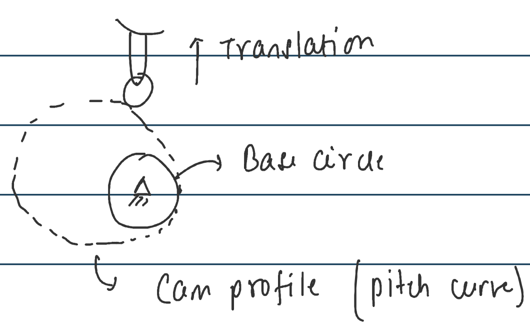
Combination of trigonometric / polynomial functions is recommended for displacement curve.

Given a desired displacement curve, what should be the corresponding CAM profile?

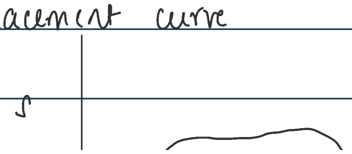
CAM Synthesis:

Principle of inversion:

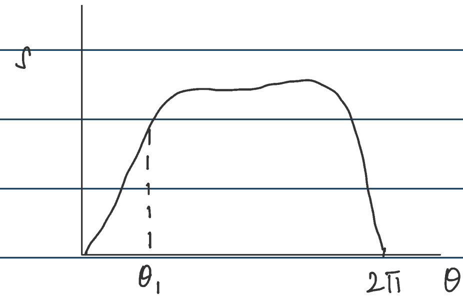
1) Translating knife edge follower (no effect)
2) Roller follower (no offset) – translating
   ↳ Radial follower



Given: displacement curve

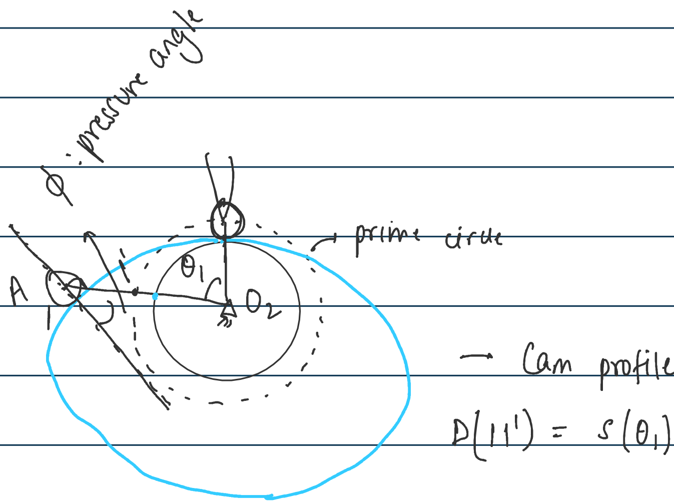


Cam profile:



Cam profile:

$r(\theta) = r_b + s(\theta)$



Follow method of inversion (cam fixed)

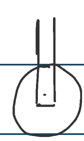
$\overline{O_2A}$ is the axis of the follower ($\because$ follower is radial in the original mechanism).

Angle between the follower axis & common normal is called the pressure angle. Due to the pressure angle, a sideways force acts on the follower (even in absence of friction) tending to bend the follower. Thus, our cam design should also ensure that ϕ is limited ($<30°$ in general).

Choice of base circle:

Greater the value of r_b, smaller will be ϕ (for given s-curve).

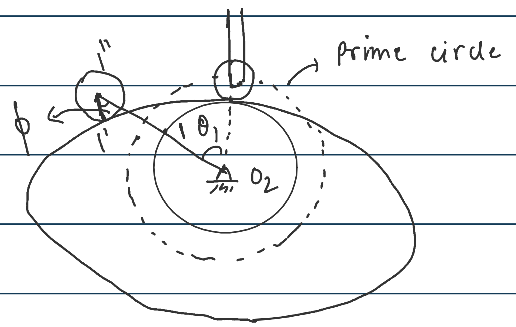
⇒ Roller Follower :



Translating and radial.

Translating and radial.

Start with some appropriate values for roller and base circle.

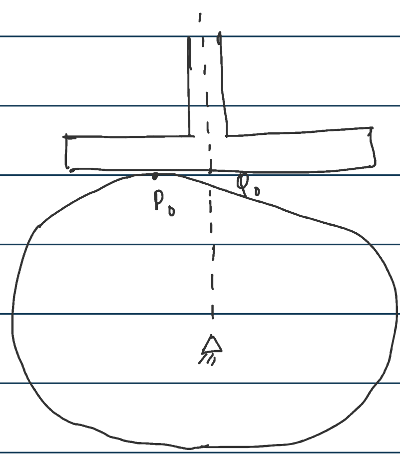


$11'' = s_1$

First we find points 1'' for several values of θ.
Next step is to find a curve that is tangent to all circles centred at these points with radius of roller.

⇒ Analytical Approach:

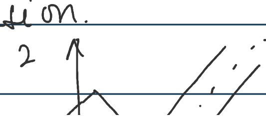
eg. Translating flat face follower.

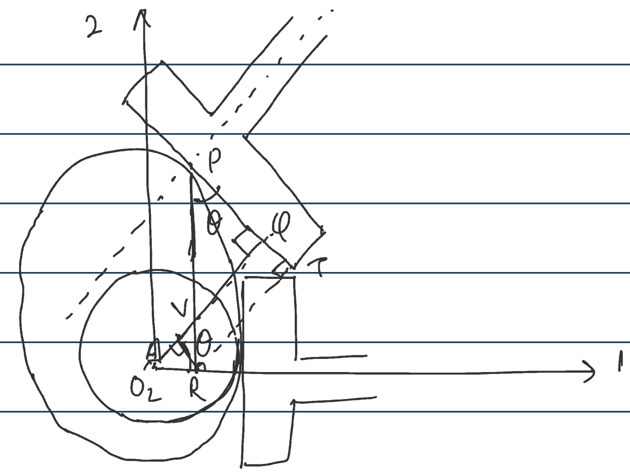


The flat line of the follower is the common tangent at the point of contact.

The contact point P changes as the cam rotates.

Consider the following starting orientation and the subsequent inversion.





$O_2Q = R(\theta) = r_b + f(\theta)$ $\quad$ $f(\theta)$: displacement function of follower

$P = (x, y)$ $\quad$ $PQ = t$

Tangent to the cam at $P(x,y) \perp O_2Q$.

i.e. $\boxed{\left(\frac{dy}{dx}\right)_P \cdot (\tan\theta) = -1}$

$O_2Q = O_2V + VQ = x\cos\theta + y\sin\theta = R(\theta)$

$\therefore \boxed{R(\theta) = x\cos\theta + y\sin\theta}$

$PQ = t = PT - QT = y\cos\theta - x\sin\theta$

$\boxed{t = -x\sin\theta + y\cos\theta}$

$\frac{dy}{d\theta} \sin\theta = -1 \implies \frac{dy}{d\theta}\sin\theta + \frac{dx}{d\theta}\cos\theta = 0$

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad \frac{\sin\theta}{\cos\theta} = -1 \quad \Rightarrow \quad \frac{dy}{d\theta}\sin\theta + \frac{dx}{d\theta}\cos\theta = 0$$

$$\frac{dR(\theta)}{d\theta} = \frac{dx}{d\theta}\cos\theta - x\sin\theta + \frac{dy}{d\theta}\sin\theta + y\cos\theta$$

$$= -x\sin\theta + y\cos\theta = t$$

$$R(\theta) = r_b + f(\theta) \qquad \boxed{t = f'(\theta)}$$

Note: Max and minimum values of $f'(\theta)$ govern the length of the follower required.

$$\begin{pmatrix} R \\ R' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} R \\ R' \end{pmatrix}$$

$$\boxed{x = R\cos\theta - R'\sin\theta \,, \quad y = R\sin\theta + R'\cos\theta}$$

Estimate of base circle dimension:

$$\text{Radius of curvature } (\rho) := \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}$$

For convex cams : $\rho \leqslant 0$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \quad ; \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy/d\theta}{dx/d\theta}\right)}{\frac{dx}{d\theta}}$$

After doing the algebra, we get

$$\rho = r_b + f(\theta) + f''(\theta)$$

For convex cams, $r_b + f(\theta) + f''(\theta) \geqslant 0$

$$\therefore \ r_b \geqslant -f(\theta) - f''(\theta)$$

↳ Impose constraint on base circle.

$\left(r_b > \min\left(-f(\theta) - f''(\theta)\right)\right)$