```
[ 8.53250000e+01 4.57500000e+00 -2.55000000e+00 4.25000000e-01 -3.55271368e-15 2.32500000e+00 -5.00000000e-02 -2.35000000e+00]
```

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Main Effects

Ambient temperature (E_1) 9150 psi

Wind Velocity (E_2) - 5100 psi

Bar Size (E_3) 850 psi

Two-Variable Interactions

Ambient temperature-Wind Velocity (E_{12}) 0 psi

Ambient temperature-Bar Size (E_{13}) 4650 psi

Wind Velocity-Bar Size (E_{23}) -100 psi

Three-Variable Interaction

Ambient temperature-Wind Velocity-Bar Size (E_{123}) -4700 psi

$$y_1 = \beta_0 + \beta_1(-1) + \beta_2(-1) + \beta_3(-1) + \beta_{12}(+1) + \beta_{13}(+1) + \beta_{23}(+1) + \beta_{123}(-1)$$

What is the regression model?

$$y = f(x_1, x_2, x_3)$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3$$

we want to find & β_i

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

8×4

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

For each experimental data, we can find the error (ϵ_i) between the model predicted value ($\hat{y}_\mathrm{i}$) and observed experimental value (y_i)

$$\epsilon_i = y_i - \hat{y}_\mathrm{i}$$

With 'least square fit', the aim is to find such coefficients, that minimizes the total sum of squares of error.

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_\mathrm{i})^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each $\boldsymbol{\beta_i}$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each β_i

$\hat{y}_1 = \beta_0 + \beta_1 x_{11}$

$\hat{y}_2$ -

$\hat{y}_3$ -

$\hat{y}_4$

$\hat{y}_8$

We can write the model in a matrix format

$$[Y_{exp}] = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_8 \end{bmatrix}, \quad [\hat{Y}] = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_8 \end{bmatrix}_{8\times 1} = [X]_{8\times 4}[\beta]_{4\times 1}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

where, $$[x] = \begin{bmatrix} 1 & x_{11} & x_{2i} & x_{31} \\ 1 & x_{12} & x_{22} & x_{33} \\ 1 & \vdots & & \\ \vdots & & & \\ 1 & x_{18} & x_{28} & x_{38} \end{bmatrix}$$

How to Find Regression Coefficients?

Test #	***X1***	***X2***	***X3***	***Average (kpsi)***
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each β_i

Thus, $L = \sum \varepsilon_i^2 = [\varepsilon]^T [\varepsilon]$

$= [Y_{exp} - \hat{Y}]^T [Y_{exp} - \hat{Y}]$

$= [Y_{exp} - X\beta]^T [Y_{exp} - X\beta]$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_8 \end{bmatrix} = [Y_{exp}] - [\hat{Y}]$$

$$\Rightarrow \boxed{L = [Y_{exp}]^T [Y_{exp}] - 2[\beta]^T [X]^T [Y_{exp}] + [\beta]^T [X]^T [X][\beta]}$$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each β_i

$$\Rightarrow \boxed{L = [Y_{exp}]^T [Y_{exp}] - 2[\beta]^T [X]^T [Y_{exp}] + [\beta]^T [X]^T [X] [\beta]}$$

minimize L wrt β $\Rightarrow \dfrac{\partial L}{\partial [\beta]} = 0$

$$-2[X]^T [Y_{exp}] + 2[X]^T [X][\beta] = 0$$

$$\Rightarrow$$

$$\Rightarrow \boxed{[\beta] = \left(\underbrace{[X]^T}_{4\times 8} \underbrace{[X]}_{8\times 4}\right)^{-1} \underbrace{[X]^T}_{4\times 8} \underbrace{[Y_{exp}]}_{8\times 1}}$$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

$\Rightarrow$ $$\boxed{[\beta] = \left([X]^T[X]\right)^{-1}[X]^T[Y_{exp}]}$$

What if we want to fit a model like

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{123} x_1 x_2 x_3 + \beta_{11} x_1^2$$

x_4 x_5 x_6

then, rename $x_1 x_2 = x_4$, $x_1 x_2 x_3 = x_5$, $x_1^2 = x_6$

$\beta_{12} = \beta_4$, $\beta_{123} = \beta_5$, $\beta_{11} = \beta_6$

then do the same procedure as before

Example

The yield from a certain chemical depends on either the chemical formulation of the input materials or the mixer speed, or both.

A 2-level factorial design was run with three replicates and the yield data are shown below.

Find main effects and interaction effects.

x_1	x_2	y_a	y_b	y_c	$\bar{y}$
-1	-1	10	20	30	20
1	-1	40	30	50	40
-1	1	60	30	60	50
1	1	40	45	50	45

$$\underline{y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2}$$

Example

Consider following factorial design with 2 variables (k = 2), and 3 levels each

Each combination replicated 4 times (n = 4)

Life (in hours) Data for the Battery Design Example

Material Type	Temperature (°F) 15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

What is the effects model and hypothesis test?

In general, for 2-factor design, we could have 'a' levels of factor A, and 'b' levels of factor B.

Each combination is replicated 'n' times

General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

What is the effects model and hypothesis test?

		Factor B			
		1	2	$\ldots$	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	$\vdots$				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

In the two-factor factorial, both row and column factors (or treatments), A and B, are of al interest. Specifically, we are interested in **testing hypotheses** about the equality of row treatment effects, say

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1: \text{at least one } \tau_i \neq 0 \qquad \textbf{(5.2a)}$$

and the equality of column treatment effects, say

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_b = 0$$
$$H_1: \text{at least one } \beta_j \neq 0 \qquad \textbf{(5.2b)}$$

We are also interested in determining whether row and column treatments *interact*. Thus, we also wish to test

$$H_0: (\tau\beta)_{ij} = 0 \qquad \text{for all } i, j$$
$$H_1: \text{at least one } (\tau\beta)_{ij} \neq 0 \qquad \textbf{(5.2c)}$$

We now discuss how these hypotheses are tested using a **two-factor analysis of variance**.

Effects Model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

ANOVA for Two-Factor Factorial Design

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

α

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	
A treatments	SS_A	$a - 1$	$MS_A = \dfrac{SS_A}{a-1}$	$F_0 = \dfrac{MS_A}{MS_E}$	$F_{1-\alpha,\, a-1,\, ab(n-1)}$
B treatments	SS_B	$b - 1$	$MS_B = \dfrac{SS_B}{b-1}$	$F_0 = \dfrac{MS_B}{MS_E}$	$F_{1-\alpha,\, b-1,\, ab(n-1)}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \dfrac{MS_{AB}}{MS_E}$	$F_{1-\alpha,\, (a-1)(b-1),\, ab(n-1)}$
Error	SS_E	$ab(n-1)$	$MS_E = \dfrac{SS_E}{ab(n-1)}$		
Total	SS_T	$abn - 1$			

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
	130	155	539	34	40	229	20	70	230	
1	74	180		80	75		82	58		998
	150	188	623	136	122	479	25	70	198	
2	159	126		106	115		58	45		1300
	138	110	576	174	120	583	96	104	342	
3	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130 74	155 180	134.75	34 80	40 75	57.25	20 82	70 58	57.5	998
2	150 159	188 126	155.75	136 106	122 115	119.75	25 58	70 45	49.5	1300
3	138 168	110 160	144	174 150	120 139	145.75	96 82	104 60	85.5	1501
$y_{.j.}$	1738			1291			770			3799 $= y_{...}$

$N = abn$
$= 3 \times 3 \times 4$
$= 36$

36 terms

$$SS_{Total} = \sum_i \sum_j \sum_k y_{ijk}^2 = (130^2 + 155^2 + 74^2 + 180^2 + 34^2 + 40^2 + \ldots + 96^2 + 104^2 + 82^2 + 60^2)$$

$$= 478647$$

$$\text{Grand Mean} = \frac{\sum\sum\sum y_{ijk}}{36} = \frac{3799}{36} = \bar{\bar{y}}$$

$$= 105.53$$

$$SS_{mean} = N\bar{\bar{y}}^2 = 36\left(\frac{3799}{36}\right)^2 = 400900$$

3 terms

$$SS_{material} = \underset{b}{3} \times \underset{n}{4} \times \left[\left(\frac{998}{12} - 105.53\right)^2 + \left(\frac{1300}{12} - 105.53\right)^2 + \left(\frac{1501}{12} - 105.53\right)^2\right]$$

$$= 10683.12$$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	134.75	34	40	57.25	20	70	57.5	
	74	180		80	75		82	58		998
	150	188	155.75	136	122	119.75	25	70	49.5	
2	159	126		106	115		58	45		1300
	138	110	144	174	120	145.75	96	104	85.5	
3	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			3799 = $y_{...}$

3 terms

$$SS_{temp} = \underset{a\ w}{3\times 4} \times \left[\left(\frac{1738}{12}-105.53\right)^2+\left(\frac{1291}{12}-105.53\right)^2+\left(\frac{770}{12}-105.53\right)^2\right]$$

$$= 39118.72$$

36 terms

For replicates,

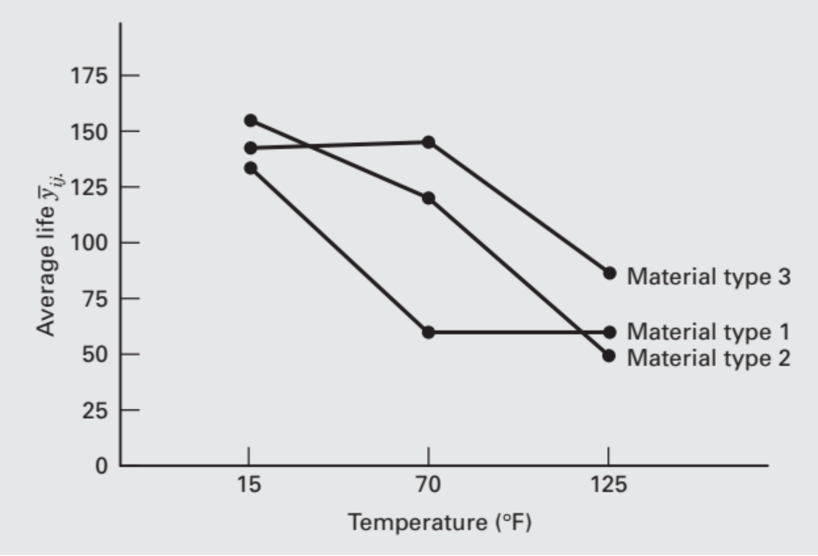
$$SS_E = \sum\sum(y_{ijk}-\bar{y})^2 = \left((130-134.75)^2+(155-134.75)^2+\ldots+(34-57.25)^2+(40-57.25)^2\right.$$

$$SS_{interaction} = [SS_T - SS_{mean}] - SS_{material} - SS_{temp} - SS_E$$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
	130	155	134.75	34	40	57.25	20	70	57.5	
1	74	180		80	75		82	58		998
	150	188	155.75	136	122	119.75	25	70	49.5	
2	159	126		106	115		58	45		1300
	138	110	144	174	120	145.75	96	104	85.5	
3	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$



Analysis of Variance for Battery Life Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Material types	10,683.72	2	5,341.86	7.91	0.0020
Temperature	39,118.72	2	19,559.36	28.97	< 0.0001
Interaction	9,613.78	4	2,403.44	3.56	0.0186
Error	18,230.75	27	675.21		
Total	77,646.97	35			

How would you check Model Adequacy?

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	134.75	34	40	57.25	20	70	57.5	
	74	180		80	75		82	58		998
2	150	188	155.75	136	122	119.75	25	70	49.5	
	159	126		106	115		58	45		1300
3	138	110	144	174	120	145.75	96	104	85.5	
	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

Significance of Main and Interaction Effects

- With 2-factor ANOVA, we can check if the factors or their interaction have a significant effect on the response
- **But what about the statistical significance of main and interaction effect values?**
  - For example, if we calculate the main effect of a variable to be 500.
  - Our attitude towards this average effect of say, 500, would not be the same if the 95% confidence interval were 500 ± 2 as it would be if the interval were 500 ± 2000.
  - **If it is 500 ± 2**, we would feel that the existence of an average effect has been rather convincingly demonstrated and we could assert with some confidence that its true magnitude is probably fairly close to 500.
  - **If it is 500 ± 2000**, this is not the case at all, because considerable uncertainty is associated with the effect and its magnitude.
- **How to find out the uncertainty (via confidence intervals) in the calculated values of main and interaction effects?**

Significance of Main and Interaction Effects

- To obtain a quantitative measure of the uncertainty in our calculated average effects and interactions, we proceed as follows:
  1. Estimate the variance S^2 of an individual observation
  2. Estimate the variances associated with the average effects and interactions
  3. Calculate the appropriate 95% confidence intervals for the "true" average effects and interactions
- From the 95% confidence intervals, we may be able to interpret the significance of each average effect and interaction, and draw conclusions regarding the experimental study.

Significance of Main and Interaction Effects

- Recall that in the previous welding example, we had 16 observations (8 combinations with 2 replications)
- It is the variance of each of these 16 observations that we will now estimate.
- We shall assume that the true variance σ^2 is the same for all sixteen observations and that the observations are independent.

Test #	X1	X2	X3	Y_{ai} (kpsi)	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	84	91	87.5
2	1	-1	-1	90.6	84	87.3
3	-1	1	-1	69.6	86	77.8
4	1	1	-1	76	98	87
5	-1	-1	1	77.7	80.5	79.1
6	1	-1	1	99.7	95.5	97.6
7	-1	1	1	82.7	74.5	78.6
8	1	1	1	93.7	81.7	87.7

For Test #1, the sample variance can be calculated as

$$S_1^{\,2} = \frac{(y_{a1} - \bar{y}_1)^2 + (y_{b1} - \bar{y}_1)^2}{(2-1)}$$

$$= (84.0 - 87.5)^2 + (91.0 - 87.5)^2$$

$$= 24.50$$

Similarly, we can find 8 sample variances for 8 combinations, S_1^2, S_2^2, …….,S_8^2, one for each test.

$S_1^2 = 24.5$, $S_2^2=21.78$, $S_3^2=134.48$, $S_4^2=242.0$, $S_5^2=3.92$, $S_6^2=8.82$, $S_7^2=33.62$, $S_8^2=72.00$

Significance of Main and Interaction Effects

- Since we are assuming same true variance σ^2 for all sixteen observations, an estimate for σ^2 is the pooled sample variance S_p^2 of the eight estimated variances $S_1^2, S_2^2, \ldots\ldots, S_8^2$.

  $S_1^2 = 24.5$, $S_2^2=21.78$, $S_3^2=134.48$, $S_4^2=242.0$, $S_5^2=3.92$, $S_6^2=8.82$, $S_7^2=33.62$, $S_8^2=72.00$

- Here

$$S_p^2 = \frac{[(y_{a1}-\bar{y}_1)^2+(y_{b1}-\bar{y}_1)^2+\ldots\ldots\ldots\ldots\ldots+(y_{a8}-\bar{y}_8)^2+(y_{b8}-\bar{y}_8)^2]}{(2-1)+\ldots\ldots\ldots\ldots+(2-1)}$$

$$= \frac{24.50+21.78+\ldots\ldots+72}{8}$$

$$= 67.64.$$

- *It should be pointed out that when the number of replications are not the same for all eight tests, the pooled sample variance S_p^2 has to be modified properly.*

$$S_p^2 = \frac{SS}{\nu} = \frac{SS_1+SS_2+SS_3 \cdots SS_8}{\nu_1+\nu_2 \cdots \nu_8} = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2 \cdots +(n_8-1)S_8^2}{(n_1-1)+(n_2-1)+(\cdots +(n_8-1)}$$

Significance of Main and Interaction Effects

Estimation of the Variances Associated with the Average Effects and Interactions

- The average effect of ambient temperature, E1 is

$$E_1 = \frac{1}{4}(\bar{y}_2 - \bar{y}_1 + \bar{y}_4 - \bar{y}_3 + \bar{y}_6 - \bar{y}_5 + \bar{y}_8 - \bar{y}_7)$$

- But since each term $\bar{y}_i$ is an average of two observations,

$$E_1 = \frac{1}{4}\left[\frac{(y_{a2} + y_{b2})}{2} - \frac{(y_{a1} + y_{b1})}{2} + \ldots + \frac{(y_{a8} + y_{b8})}{2} - \frac{(y_{a7} + y_{b7})}{2}\right]$$

or

$$E_1 = \frac{1}{8}[y_{a2} + y_{b2} - y_{a1} - y_{b1} + \ldots\ldots\ldots - y_{a7} - y_{b7}]$$

$V(E_1) = \frac{1}{64}\left(V(y_{a2}) + V(y_{b2}) \ldots\right.$
σ^2 - - -
$= 16\sigma^2/64 = \sigma^2/4$

- Thus we can show that $V(E_1) = \sigma^2/4$
- In fact, we can show, $V(E_1) = V(E_2) = V(E_3) = V(E_{12}) = V(E_{13}) = V(E_{23}) = V(E_{123}) = \sigma^2/4$

Significance of Main and Interaction Effects

Estimation of the Variances Associated with the Average Effects and Interactions

- $V(E_1) = V(E_2) = V(E_3) = V(E_{12}) = V(E_{13}) = V(E_{23}) = V(E_{123}) = \sigma^2/4$
- Substituting pooled variance in place of true variance, we can obtain confidence intervals
- The confidence interval for the average and interaction effects can be obtained as:

$$E_i \pm t\sqrt{\frac{S_p^{\;2}}{4}} \quad i = 1,2,\ldots$$

$$E_i \pm 2.306\sqrt{\frac{S_p^2}{4}}$$

- We already have the values of E1, E2, ...,E12,....and $S_p^{\;2}$; what is left to be determined is the value of t.
- We have a total of sixteen tests, and we used up eight degrees of freedom in calculating the eight averages,
- Therefore, the appropriate t-value is the value associated with eight degrees of freedom and corresponding to a 95% confidence level, which is $t_{8,\,0.025} = 2.306$.