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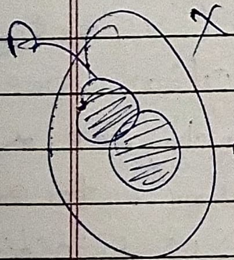
Tut-2 during Friday lecture (Attendance will be recorded) 8th Feb

Basic counting: (finite sets)

→ A and B are subsets of X.

$$|A \cup B| = |A| + |B| - |A \cap B|$$



→ A, B - sets $A \times B = \{(a,b) \mid a \in A, b \in B\}$

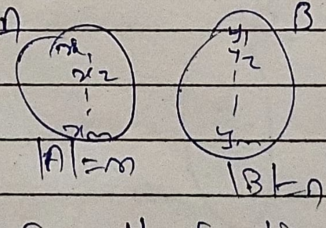
$$|A \times B| = |A| . |B|$$

Ex. 1) There are 2^{11} many binary strings of length 11.

01100010011

2) Let A and B be sets with m & n elements.
There are ______ many functions from A to B?

$$\underbrace{n \times n \times \cdots n}_{m \text{ times}} = n^m$$



B^A = the set of all functions from A to B

$$|B^A| = |B|^{|A|}$$

3) There are 2^n subsets of an n-element set.

How many one-to-one functions from A to B?

If $m > n$, then there is no one-to-one function from A to B.

So henceforth, $m \leq n$

$$\# = n \times (n-1) \times (n-2) \ldots (n-(m-1))$$

$$= \frac{n \times (n-1) \times \ldots (n-m+1)(n-m) \ldots 1}{(n-m) \ldots 1}$$

$$= \frac{n!}{(n-m)!}$$

\# of permutations of n objects = n!

How many passwords of length 4?

60 characters $\Rightarrow 60^4$

k=0

for i_1 = 1 to n_1 {

for i_2 = 1 to n_2 {

for i_k = 1 to n_k {

k

k = k+1

}

}

}

What is final value of k?

$k = n_1 \times n_2 \times \ldots \times n_m$

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k = 0
For $i_1 = 1$ to n_1 {
    k = k+1
}
for $i_2 = 1$ to n_2 {
    k = k+1
}
⋮
for $i_m = 1$ to n_m {
    k = k+1
}

$n_1 + n_2 + n_3 + \ldots + n_m$

→ If m objects are placed into n boxes and $m > n$, then there exists a box which contains two or more objects

→ If $m > n$ then there is no one-to-one function from an m-element set to an n-elt set,

Ex. FACTS: there are 97 students registered for : CS 207 M

There are atleast 49 students of the same gender who have registered for CS 207 M.

There are 92 non-cse students

There are 10 depts in IIT.

⇒ There is a dept which contains 11 or more students in CS 207 M.

If $nk+1$ objects are placed into n boxes, then there is a box which has $k+1$ or more objects in it.

$n = 2$ boys/girls
$k = 48$
$nk+1 = 48 \times 2 + 1 = 97$

~~Theor~~

Theorem: Among any $n+1$ positive integers from $\{1, 2, \ldots 2n\}$ there must be an integer that divides one of the other integers.

Proof: Let $a_1, a_2, \ldots a_{n+1}$ be $n+1$ distinct integers from $\{1, 2, \ldots 2n\}$

Write each of these numbers as a power of 2 times an odd integer.

$a_1 = 2^{k_1} b_1$ where $k_1 \geq 0$ and b_1 is odd
$a_2 = 2^{k_2} b_2$
$\vdots$
$a_{n+1} = 2^{k_{n+1}} b_{n+1}$

for each $1 \leq i \leq n+1$ where $k_i \geq 0$ and b_i is odd.

Note that each b_i is an odd integer from $\{1, \ldots 2n\}$

Pigeonhole principle:

If A has n elements then any sequence of length $n+1$ of elements of A must contain a repetition

$\bar{a} = a_1, a_2 \qquad a_{n+1}$

$\exists\, 1 \le i < j \le l+1$ such that $a_i = a_j$

By pigeonhole principle, $\exists\, 1 \le i < j \le n+1$ such that $b_i = b_j$

Now consider $a_i = 2^{k_i} b_i$ and $a_j = 2^{k_j} b_j$,

If $k_i > k_j$ then a_j divides a_i

If $k_j > k_i$ then a_i divides a_j

Let $\bar{a} = a_1, a_2, \ldots a_m$ be sequence of real numbers

Def: A subsequence of $\bar{a}$ is a sequence of the form $a_{i_1}, a_{i_2}, \ldots a_{i_k}$ where $1 \le i_1 < i_2 < i_3 < \ldots < i_k \le m$

$\bar{a} = 2, 3, 1, 5, 4, 3, 1$

$\bar{b} = 3, 1, 1$

2) $\bar{a}$ is said to be strictly increasing if $a_1 < a_2 < a_3 \ldots < a_m$

3) $\bar{a}$ is said to be strictly decreasing if $a_1 > a_2 > a_3 \ldots > a_m$

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Theorem: Any seq of n^2+1 distinct real number contains a subsequence of length $n+1$ which is either strictly increasing or strictly decreasing.

$n=3$

Proof: Let $\bar{a} = a_1, a_2, \ldots a_{n^2+1}$ be a seq of distinct numbers.

For ~~each~~ k^{th} term, we write (i_k, d_k)

i_k = the length of the longest increasing subsequence starting at a_k.

d_k = has ... decreasing ...

2, 3, 4, 1, 5, 6, 8, 7

$\binom{6}{2}$ $\binom{3}{1}$

Suppose for contradiction that $1 \le i_k \le n$ & $1 \le d_k \le n$

$\forall\ 1 \le k \le n^2+1$

$\exists\ l < m$ such that $(i_l, d_l) = (i_m, d_m)$

$\underset{a_l}{(i_l, d_l)} = \underset{a_m}{(i_m, d_m)}$

4 7

$\binom{3}{6}$ $\binom{3}{6}$

can't possible

2

Classes

missing

before last class.

&

one after last class.

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Thm: Any sequence of n^2+1 distinct reals contains either an increasing or decreasing subsequence of length $n+1$.

Proof: let $a_1, a_2, \ldots, a_{n^2+1}$ be an arb. seq of n^2+1 distinct numbers.

For k: $1 \le k \le n^2+1$

i_k = the length of a longest increasing subsequence starting at a_k.

$a_k \; a_{k+1} \; a_{n^2+1}$

Remark: This is optimal. $n=3$ $n^2=9$ 7

$\boxed{789}$ $\boxed{456}$ $\boxed{123}$

only 3 no. not 4.

Theorem: Among a group of six people, either there are three mutual friends or three mutual strangers.

Proof: Let A be one of those 6 people.

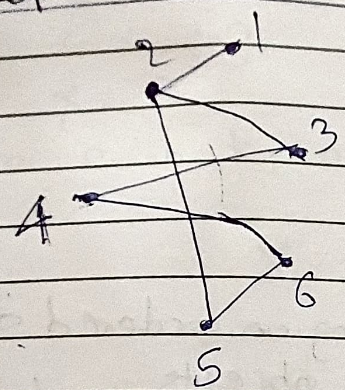
By pigeonhole either A has atleast 3 friends among the remaining 5 or A is stranger to atleast 3 among the ~~ll~~

Case 1: A has at least three friends.
Let B, C, D be three friends of A.
If there is a pair of friends among B, C, D then that pair together with A gives us 3 mutual friends.

Otherwise, B, C, D are three mutual strangers.

Case 2: Similarly argued.

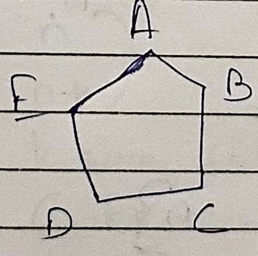
A graph theoretic interpretation



A undirected graph on 6 vertices/nodes contains either a triangle or an independent set of size 3.

$\{2, 3, 6\}$ is an independent set of size 3.

For 5 people the theorem doesn't hold



Ramsey Numbers

*: Permutation & combinations:

In how many IPL team of 11 people players can be selected from a pool of 15 players?

order selection $\underbrace{15 \times 14 \times 13 \times \cdots \times 5}$ $= \frac{15!}{9!}$

$\frac{15!}{9! \, \cancel{11!}} \times \cancel{11!}$

15

# of ways of making an ordered selection of k objects from n objects,

$= n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!} = P(n,k)$

# of ways of selecting k objects from n objects $= \frac{n(n-1) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$

$= C(n,k)$

$= \binom{n}{k}$

n - positive integer

$[n] = \{1, 2, \ldots n\}$

$\binom{n}{k} = C(n,k) = $ # of k-size subsets of $[n]$

X = the set of all subsets of $[n]$ of size k.

$2^{[n]}$ = power set of $[n]$

$|X| = \binom{n}{k}$

$\boxed{0 \le k \le n}$

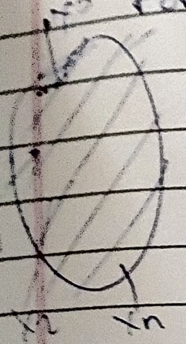
* Combinatorial proofs / bijective proofs

1) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$|X| = 2^n$

$X = 2^{[n]}$

For each k, $0 \le k \le n$,



X_k - the set of k-size subset of $[n]$ $\subseteq X$

$|X_k| = \binom{n}{k}$

$\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k}}$

$|X| = \sum_{k=0}^{n} |X_k| =$

$0 \le k \le n$

2) $\binom{n}{k} = \binom{n}{n-k}$

X - k size subsets

Y - $(n-k)$ size subsets

$f: X \to Y$ complementation is a bijection

3) $1 \le k \le n$ $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

X - the set of k-size subsets of $[n]$

$= \{1, 2, \ldots, n\}$

X_1 - ~~the set of k-size subsets of $[n]$~~ which contain n.

$= \{A \in X \mid n \in A\}$

X_2 = the remaining the set of k-size subsets of $[n]$ which do not contain 'n'

$|X| = |X_1| + |X_2|$

$|X_1| = \binom{n-1}{k-1}$

Y_1 = the set of $(k-1)$ size subsets of $[n-1]$.

$X_1 \leftrightarrow Y_1$

Clear bijection between these two.

$|X_2| = \binom{n-1}{k}$

Pascal's triangle

$\binom{0}{0} = 1$

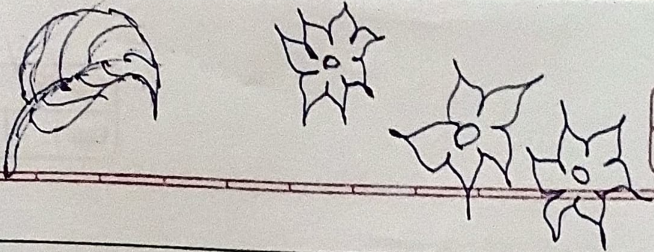
$1 = \binom{1}{0}$ $\binom{1}{1} = 1$

$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$1 =$ $= 2$ $= 1$

Fix n

$$A \longmapsto \begin{cases} A \setminus \{n\} & \text{if } n \in A \\ A \cup \{n\} & \text{if } n \notin A \end{cases}$$



Vandermonde's identity:

$m, n \quad , \quad r \leq \min\{m, n\}$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

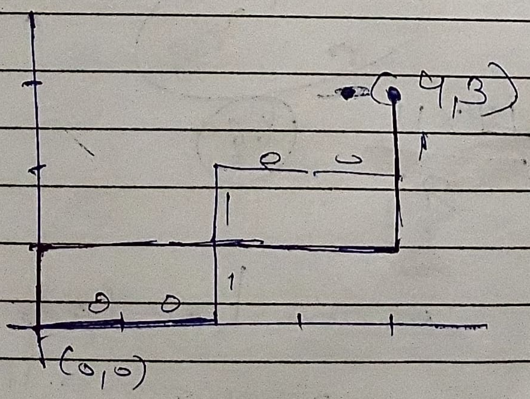
m-girls n-boys total = m+n people
r = students

Decide selecting k girls so have to select r-k boys &
then put them together.

Combinatorial interpretations:

1) Binary strings of length n with exactly k 0's. $\binom{n}{k}$

2) lattice paths:



either move right or up

$\binom{7}{3}$

How many lattice paths from (0,0) to (m,n) - $\binom{m+n}{m}$

0011001
1000011

$= \binom{m+n}{n} \to$ choose vertical steps

choose horizontal steps

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* **Permutation & combinations with repetition allowed.**

$[n] = \{1, 2, \dots, n\}$

r – selection size

ordered selections of r objects from n objects when repetition is allowed $= n^r$.

unordered.

fruit basket with apples & bananas
In how many ways can I select 10 fruits

$a + b = 10$

${}^{10+1}C_1 \quad {}^{11}C_1$

$a + b + c = 10$

${}^{10+2}C_2$

${}^{12}C_2$

$\frac{12 \times 11}{2} \quad 66$

oo ooo | oo | ooo

oooooooooo ||

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$1^{m_1}\ 2^{m_2}\ n^{m_n}$ — $m_1+m_2+m_3+\dots$ times

$\underbrace{0\,0\,0}_{m_1 \text{ times}}\ 1\ \underbrace{0000}_{m_2}\ 1\ \underbrace{00}_{m_3}\ 1\ 1$

$\underbrace{00}_{m_n \text{ times}}$

$\binom{n-1+r}{n-1}$

* Multinomial theorem :

$x_i \geq 0$

$x_1 \geq 0$ $\quad x_1+x_2+\quad+x_n=r$

$x_2 \geq 0$ $\quad x_n \geq 0$

$\binom{n-1+r}{n-1}$

1st factor

$(x_1+x_2+\dots+x_m)^n = (x_1+x_2+\dots+x_m)\times$

$(x_1+x_2+\dots+x_m)\dots(x_1+x_2+\dots+x_m)$

2nd factor $\qquad$ nth factor

$= \sum x_1^{n_1}\ x_2^{n_2}\ x_m^{n_m}$

$$= \sum_{\substack{n_1, n_2, n_m \geq 0 \\ n_1+n_2+n_3+\dots+n_m=n}} \binom{n}{n_1, n_2, n_3 \dots n_m} x_1^{n_1}\, x_2^{n_2}\, x_m^{n_m}$$

→ the multinomial coefficient

Binomial coefficient

$\binom{n}{k} = \binom{n}{k,\, n-k}$

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$$\binom{n}{n_1, n_2 \ldots n_m} = \# \text{ of ways of distributing}$$

n objects into m boxes such that i^{th} box receives n_i objects.

$$= \frac{n!}{n_1! n_2! \ldots n_m!}$$

$$\binom{n}{n_1}\binom{n-n_1}{n_2} \ldots$$

$$\binom{n}{n_1, n_2 \ldots n_m} = \binom{n-1}{n_1 - 1, n_2, \ldots n_m} + \binom{n-1}{n_1, n_2 - 1, n_3 \ldots n_m} + \binom{n-1}{n_1, n_2, \ldots n_m - 1}$$

$$\binom{n}{k, n-k} = \binom{n-1}{k-1, n-k} + \binom{n-1}{k, n-k-1}$$

Tut — 3

Q1. Discussed in class

(a, b) (c, d)

Q.2) $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

Our original coord are integers.

so $\frac{a+c}{2}$ will be integer if a & c have same

parity (both even or odd)

$\frac{b+d}{2}$ is integer

& b & d have same parity

both Even or odd.

So parities of coordinates matter

possibilities (odd, odd) (odd, even) (even, even)

(even, odd)

$2 \times 2 = 4$

(odd, even)

5

odd e.

(odd, odd) (odd, even) (even, odd)

(even, even) (even, odd)

2.) 5 -> 3 △ | 3 complete Strategy

5

1. 4 f, N

1 — F

NF

1 — 1 , 1 + 1

F

N

2 3

Case 1:

1 a — b c d e

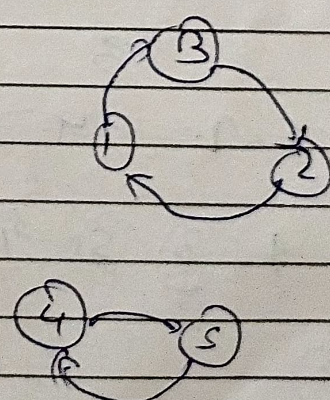
Case 2.

a b

Cyclic permutation

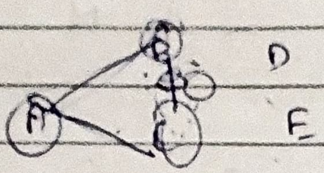
$f: X \to X$

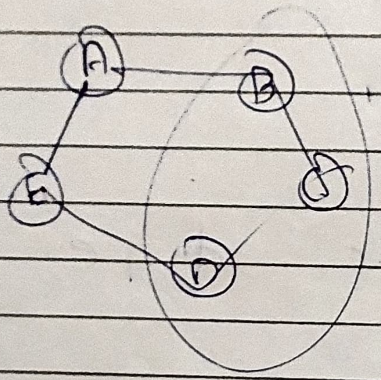
1	2	3	4	5
3	1	2	5	4



$C(n, k)$

Q.1) 1





Q.3) $n+1$ +ve

$2n$ $\leq 2n$

3

$n+1$ ~~2~~ 4 6 8 ——— $2n+2$

1 3 5 7 9 ——— n^{th}

$2k-1$

n^{th}

$2k-1+1$

$2k$

10 H

$a|b$ $\frac{b}{a}$ $\frac{a}{b}$ $\frac{b}{a}$

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2 ... 4 ...

$2n+\cancel{2}$

2.→) also ~~the consecutive gcd~~

∴ If there common factor

2.3) 2k

then diff. will be

min ~~differ~~ (diff.) common factor

min 1 $\leq 2n$ (k)

$+ > 2$

1 3 5 7 , , $2k+1$

d_1 d_2 d_n

$(2k-1)$

nth

2 4 $2n+2$

$d_1 + d_2 \cdots d_n \leq 2n-1$

≥ 2 ≥ 2

then $2n$

so 1.

2.→)

$n_1 \; n_2 \; n_3 \; - \; - \; kn_i$

→ l_1

17 11 13 15 16 17 18

2

$kn_i + d_i$

b/a

$b/a = 0$

n_i $kn_i + d_i$. b $k_{n+1-i} + d_{n+1-i}$

kn

$i-1$ n_i $kn_i + d_i$ $kn_i + d_i$ $2n$

$n+1-i$ $n+1-i$

$n_1 \quad n_2$

$a_1, a_2, a_3, \ldots \ldots, a_{n+1} \leq 2n$

Sorted

$a_1 \quad K$

$a_1, \quad a_1K_1 + d_1, a_1k_2 + d_2, \ldots\ldots, a_1K_n + d_n$

$< 2n$

$a_1 \quad (2K_j)^{q_j}$

K_j

$2 \times q_j$

$\cancel{n_1}$ $\quad n_2$

$a_1, a_2, a_3, \_\_\_\_ \quad a_{n+1} \leq 2n$

Sorted

a_1 $\quad k$

$a_1, \quad a_1 k_1 + d_1, \; a_1 k_2 + d_2, \; \_\_ \; , a_1 k_n + d_n \leq 2n$

$a_1 \qquad (2k_j)^{q_j}$

k_j

$2 \times q_j$

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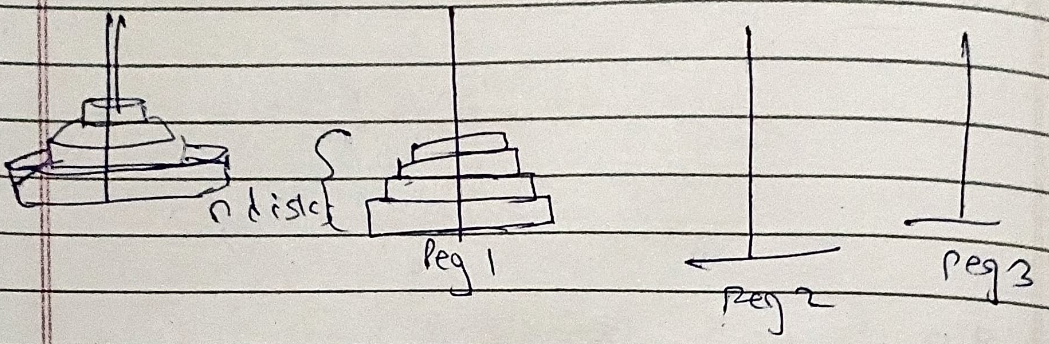
"Recurrence" relations

The tower of Hanoi puzzle



A valid move | step allows to move a disk from one peg to another ensuring that a disk is never placed on top of a smaller disk.
Question: What is the minimum of steps required to move all n disks from peg 1 to peg 2?

- one disk 1
- two disks $p1 \to p3$ $p1 \to p2$ $p3 \to p2$ 3

let a_n be the answer for n disks

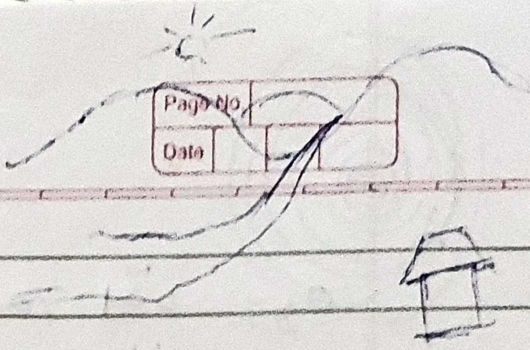
$P_1 \xrightarrow[n]{} P_2$ $P_1 \xrightarrow[n-1]{} P_3$ a_{n-1} $P_1 \to P_2$ largest disks

$P_3 \xrightarrow[n-1]{} P_2$

$n \geq 2$

$a_n = a_{n-1} + 1 + a_{n-1}$

$= 2a_{n-1} + 1$



a_6

a_1	a_2	a_3	a_4	a_5	a_6
1	3	7	15	31	63

$a_1 = 1$

$\boxed{n \geq 2}$ $\boxed{a_1 = 1}$

$$a_n = 2a_{n-1} + 1$$
$$= 2[2a_{n-2} + 1] + 1$$
$$= 2^2 a_{n-2} + 2 + 1$$
$$= 2^3 a_{n-3} + 2^2 + 2 + 1$$
$$= 2^{n-1} a_1 + 2^{n-2} + \cdots + 2 + 1$$
$$= 2^{n-1} + 2^{n-2} + \cdots + 1$$
$$= \frac{2^n - 1}{2 - 1} = 2^n - 1 \qquad n \geq 2$$

$$\boxed{\forall n \geq 1 \quad a_n = 2^n - 1}$$

- Count the # of n-length binary strings which do not have two consecutive 0's.

$n = 1$ 0, 1 $a_1 =$ a_n = # of such strings

$n = 2$ 01, 10, 11, 00 $a_2 = 3$

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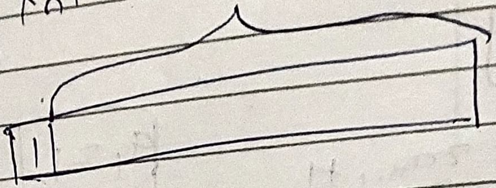
$a_3 = 001, 000, 010, 011, 100, 101,$
$110, 111$ $\quad a_3 = 5$

$a_n = |X_n|$

w'

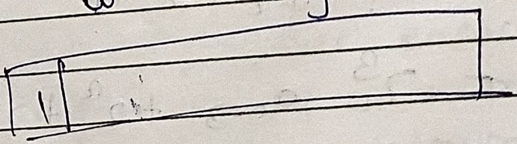


$w =$

X_n $\quad 1\,Y_{n-1}$ $\quad Z_n = 0$ $\quad n \geq 2$

Assume w starts with a '1'.

w' of length $n-1$ and without "00"

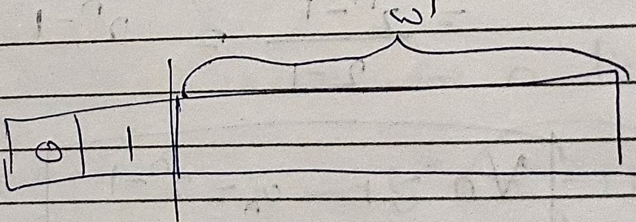


$a_n = |X_n| = |Y_n| + |Z_n|$ $\qquad Y_n \leftrightarrow X_{n-1}$
$1w' \leftarrow w'$

$= a_{n-1} +$

w'

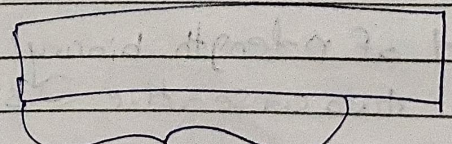
$w =$



$\in Z_n$

w'' $\qquad w'' \in X_{n-2}$

01



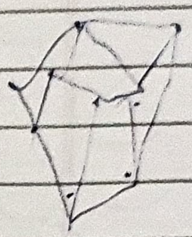
$n-2$ length

$Z_n \leftrightarrow X_{n-2}$
01

$\boxed{n \geq 3}$

$a_1 = 2, \quad a_2 = 3$

$n \geq 3 \quad a_n = a_{n-1} + a_{n-2}$



$a_0 = 1$

$n = 0$ (ε)

fibonacci recurrence relation.

- Count the number of regions in which a plane gets divided by 'n' straight lines in "general position." →

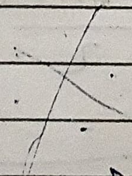
(No two lines are parallel)

(No two lines are ll & No 3 lines are concurrent.)

a_n = # of regions for n lines

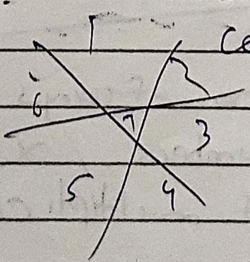
For general situation you should get maximum possible count.

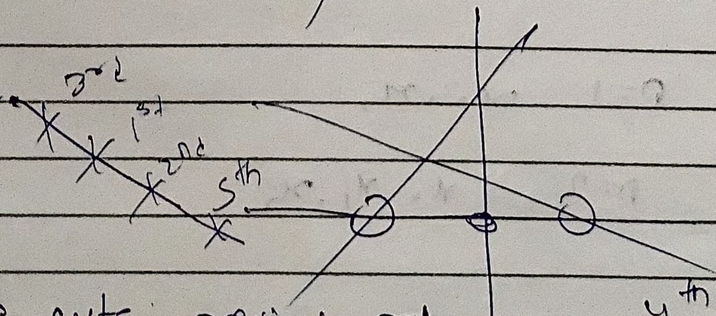
$a_1 = 2$



$a_2 = 4$

$a_3 = 7$.





n^{th} line cuts previous $n-1$ lines.

$n-1$ intersections so

4^{th} line hits last 3 lines.

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So n segments got spliced

$$\therefore n \geq 2 \quad \boxed{a_n = a_{n-1} + n}$$

$$= a_{n-2} + n-1 + n$$

$$\vdots$$

$$= a_1 + 2 + 3 + \cdots + n$$

$$= 2 + 2 + 3 + \cdots + n$$

$$= 1 + \frac{n(n+1)}{2}$$

$$= \boxed{\frac{n^2 + n + 2}{2}}$$

$$\boxed{\forall\ n \geq 1 \quad a_n = \frac{n^2 + n + 2}{2}}$$

Count the no. of ways to parenthesize a product of n+1 numbers $x_0, x_1, \ldots, x_n$ to decide the order of multiplication.

$n = 1 \quad x_0 \cdot x_1$

$n = 2 \quad x_0 \cdot x_1 \cdot x_2$

$(x_0 . x_1) x_2$ $\quad$ $x_0 . (x_1 . x_2)$

$n=0$ $\quad x_0$

$n=3$ $\quad x_0 . x_1 . x_2 . x_3$

$a_0 = 1$

$a_1 = 1$

$a_2 = 2$

$a_3 = 6$

$x_0 \quad x_k$ $\quad$ $x_{k+1} \quad x_n$

$k+1$ $\quad$ $n-k$

$\sum_{k=} a_k \cdot a_{n-k-1}$

Catalan numbers

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Tut - 4

Q.1) $11^2 = 121$

So $n \le 121$ have factors 2, 3, 5, 7 not greater than ~~5~~ 11

$\therefore$ $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6$

$3^1, 3^2, 3^3, 3^4$

2^5

$13 \times 13 = 169$ ≤ 13

2, 3, 5, 7, 11

~~not great~~

not greater than 169

ns $\left(\frac{150}{2}\right) + \left(\frac{150}{3}\right) + \left(\frac{150}{5}\right) + \left(\frac{150}{7}\right) + \left(\frac{150}{11}\right)$

$- \left(\frac{150}{2\times3}\right) - \left(\frac{150}{3\times5}\right) + \left(\frac{150}{2\times3\times5}\right)$

$- \left(150\right.$

Similarly you can do it.

$\boxed{x_1 \geq 3}$

$y_1 \; x_1 \geq 3$

$x_1 \geq 3$

~~$x_1 + x_2 + x_3 + x_4 = 17$~~

x_1

$(x+3)$

$x_1 \geq 3$

$y_1 \geq 0$

$(x_1 - 3) \geq 0$

$(x_1+3) +$

$3 - x_1 + 4 - x_2$

≥ 0

(x_1+3)

3

$3 - x_1 + 4 - x_2 + 5 - x_3 + 8 - x_4 = 17$

$x_1 + x_2 + x_3 + x_4 = 3$

$^6C_3$

$^{3+3}C_3 \quad {}^6C_3$

$\cancel{\frac{6 \times 5 \times 4}{3 \times 2}}$ $\boxed{20}$

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n

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$2^n - (n+1)$

$2^n - a_n$

$2^n - 2a_{n-1}$

$a_1 = 2$

$a_n = a_{n-1} + 1$

2^{n-2} a_2

$2^n \to 2^{n-2}$

a_{n-1}

a_n

a_{n-1}

$a_n = a_{n-1} + 1$

$2^n - a_n$

$T_n = 2^n - (a_n)$

$2^n - (a_{n-1} + 1)$ T_n

$2 \cdot 2^{n-1}$

$2^{n-1} + 2^{n-1} - (a_{n-1} + 1)$

$T_n = 2^{n-1} + T_{n-1} - 1$

T_{n-1}

$2^{n-1} + (2^{n-1} - a_{n-1}) - 1$

$a_n = a_{n-1} + 1$

a_{n-2}

$a_n - a_n \quad 2^n - a n$