1. Consider a uniform or rectangular probability density function (PDF) **[6 marks]**
   a. Derive its mean and variance. (You may use any symbols to represent the functional parameters)
   b. Suppose the rectangular PDF is bounded between 50 and 100; estimate its mean, median, and standard deviation.

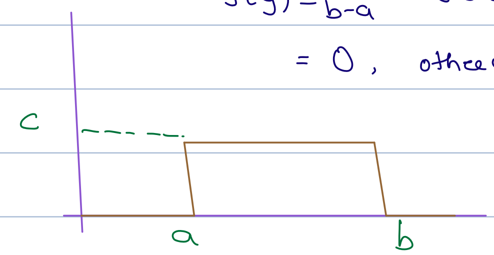
for a PDF

a) $\mu = \int_{-\infty}^{\infty} y \cdot f(y)\, dy$

Rectangular PDF

$f(y) = \frac{1}{b-a}, \quad y \in [a,b]$

$= 0, \quad \text{otherwise}$



$c = \frac{1}{b-a}$

$\therefore$ for a rectangular PDF

$\mu = \int_a^b y \left(\frac{1}{b-a}\right) dy$

$= \frac{1}{b-a} \left[\frac{y^2}{2}\right]_a^b$

$= \frac{1}{2(b-a)} \cdot \left[b^2 - a^2\right]$

$= \frac{(b+a)\cancel{(b-a)}}{2 \cdot \cancel{(b-a)}}$

$= \frac{b+a}{2}$ (Ans) → 2 marks for correct derivation

$V(y) = E\left[(y-\mu)^2\right] = \sigma^2 = \int_{-\infty}^{\infty} (y-\mu)^2 \cdot f(y) \cdot dy$

$= \int_a^b \left[y - \frac{a+b}{2}\right]^2 \cdot \frac{1}{(b-a)}\, dy$ → (1)

(A)

✓ solved below

substitute $y - \frac{a+b}{2} = u$

$\therefore \quad dy = du$

$\therefore \int u^2 \, du = \frac{u^3}{3}$

re-substituting for u we get

$$\left[\frac{\left(y - \frac{a+b}{2}\right)^3}{3} \right]_a^b \Rightarrow \frac{(b-a)^3}{8 \times 3} - \frac{(a-b)^3}{8 \times 3}$$

$$\Rightarrow \frac{(b-a)^3}{24} - \frac{-(b-a)^3}{24}$$

$$\Rightarrow \frac{(b-a)^3}{24} + \frac{(b-a)^3}{24}$$

$$\Rightarrow \frac{(b-a)^3}{12}$$

Substituting A from eqn ① we get

$$V(y) = \frac{1}{(b-a)} \cdot \frac{(b-a)^3}{12}$$

$$= \frac{(b-a)^2}{12}$$ (Ans)

2 marks for correct derivation

b) rectangular PDF bounded between 50 & 100

Here $a = 50$; $b = 100$

mean = median $= \frac{a+b}{2}$

$= \frac{100+50}{2}$

$= 75$ (Ans) 1 mark

standard deviation $= \sqrt{\text{variance}}$

$= \sqrt{\frac{1}{12} \cdot (b-a)^2}$

$= \sqrt{\frac{1}{12} \cdot [100-50]^2}$

$= 14.434$ (Ans) 1 mark

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2. A student wrote two quizzes. In the first quiz, he scored 80 marks, and in the other, he scored 75 marks. The mean and standard deviation of the first quiz are 70 and 15, respectively, while the mean and standard deviation of the second quiz are 54 and 12, respectively. It can be assumed that the quiz scores of the entire class are normally distributed. What can you conclude about the student's performance (i. e. in percentile) in the two quizzes? **[4 marks]**

Scores in two quizzes : $x_1 = 80$

$x_2 = 75$

Quiz 1 data : $\mu_1 = 70$ $\sigma_1 = 15$

Quiz 2 data : $\mu_2 = 54$ $\sigma_2 = 12$

$$Z = \frac{x-\mu}{\sigma}$$

for quiz 1:

$$z_1 = \frac{x_1-\mu_1}{\sigma_1} \Rightarrow \frac{80-70}{15}$$

$$= 0.67$$ (Ans) 1 mark

$$z_2 = \frac{x_2-\mu_2}{\sigma_2} \Rightarrow \frac{75-54}{12}$$

$$= 1.75$$ (Ans) 1 mark

From the table on page 3 of question paper

for $z_1\Big|_{0.67} = 0.7486$

$= 74.86\ \%$ (0.5 mark)

$z_2\Big|_{1.75} = 0.9599$

$= 95.99\ \%$ (0.5 mark)

Comment: Since he has performed better than 95% students in the class in quiz 2. He has performed well in quiz 2. (1 mark)

3. Prove that $V(y_1+y_2) = V(y_1)+V(y_2) + 2E[(y_1-\mu_1)(y_2-\mu_2)]$ where μ_1 and μ_1 are mean values of y_1 and y_1, respectively, and V(y) denotes variance of variable y. **[6 marks]**

$V(y) = E[(y-\mu)^2]$ (1 mark)

$\therefore \quad V(y_1+y_2) = E[(y_1+y_2 - E[y_1+y_2])^2] \Rightarrow \mu = E(y)$ 1 mark

$= E[(\{y_1 - E(y_1)\} + \{y_2 - E(y_2)\})^2]$

(2 marks) $\Rightarrow \quad = E[[y_1 - E(y_1)]^2 + [y_2 - E(y_2)]^2 + 2[y_1 - \underbrace{E(y_1)}_{\mu_1}][y_2 - \underbrace{E(y_2)}_{\mu_2}]]$

$= E[[y_1 - E(y_1)]^2] + E[[y_2 - E(y_2)]^2] +$

$E[2[y_1 - E(y_1)][y_2 - E[y_2]]$

(2 marks) $\Rightarrow \quad = Var(y_1) + Var(y_2) + 2E[(y_1-\mu_1)(y_2-\mu_2)]$

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4. With the help of an example, state and describe the three sources of variability. **[3 marks]**

0.5 mark → stating the sources of variability } (x3)
0.5 mark → describing

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5. An exponential PDF is defined by **[5 marks]**

$f(y) = \lambda e^{-\lambda y}, y \geq 0$

$f(y) = 0, \quad y < 0$

Show that the standard deviation, σ =1/λ

Standard deviation $= \sqrt{\text{Variance}}$ 0.5 mark

Variance $[y] = E[x^2] - E[x]^2$ 0.5 mark

Mean $[y] = \int_0^\infty y \cdot f(y)\,dy$

$= \int_0^\infty y \cdot \lambda e^{-\lambda y} \cdot dy$

$= \lambda \left[\left| \frac{-y \cdot e^{-\lambda y}}{\lambda} \right|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda y}\,dy \right]$

$= \lambda \left[0 + \frac{1}{\lambda} \cdot \frac{(-e^{-\lambda y})}{\lambda} \right]_0^\infty$

$= \lambda \cdot \frac{1}{\lambda^2}$

$= \frac{1}{\lambda}$ (1.5 marks)

||^{ly} $E(x^2) = \frac{2}{\lambda^2}$ (1.5 marks)

$\therefore$ $\text{Var}(x) = E(x^2) - E(x)^2$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

Standard deviation $= \sqrt{\text{variance}}$

$$= \sqrt{\frac{1}{\lambda^2}}$$

$$= \frac{1}{\lambda} \quad \text{(Ans)}$$

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6. From a population of N = 10, shown in the table below, a random sample of size three is drawn. What would you expect the values of the sample mean and sample standard deviation to be? Why? **[4 marks]**

10	7	4	2	9
3	6	5	1	8

1 mark → calculation of sample mean

1 mark → calculation of sample std deviation

$$E(\bar{y}) = E\left(\frac{\sum_{i=1}^{n} y_i}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(y_i)$$

$$= 1/n \sum_{i=1}^{n} \mu$$

$$= \mu$$

$$E(s^2) = E\left[\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{n-1}\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n}(y_i-\bar{y})^2\right]$$

$$= \frac{1}{n-1} E(SS$$

$$E(SS) = E\left[\sum_{i=1}^{n}(y_i-\bar{y})^2\right]$$

$$= E\left[\sum_{i=1}^{n} y_i^2 - n\bar{y}^2\right]$$

$$= \sum_{i=1}^{n}(\mu^2+\sigma^2) - n(\mu^2 + \sigma^2/n)$$

$$= (n-1)\sigma^2$$

7. Suppose two different brands, A and B, of cutting tools are used to machine 20 workpieces, 10 with each brand. The surface finish readings taken on the 20 workpieces are shown in the table below. Based on the sample data, which cutting tool should be preferred and why?

[2 marks]

X_A	X_B
70	70
72	70
71	70
70	69
69	69
70	70
71	70
70	71
68	71
69	70

A B

$\bar{X}_A = \bar{X}_B = 70$

0.5 marks for calculating the average

$$S_A^2 = \frac{\sum_{i=1}^{10} (X_{A_i} - \bar{X}_A)^2}{10-1} = 1.33$$

$$S_B^2 = \frac{\sum_{i=1}^{10} (X_{B_i} - \bar{X}_B)^2}{10-1} = 0.44$$

0.5 marks for S_A^2 calculation

0.5 marks for S_B^2 calculation

Conclusion: tool B gives more consistent (less variable) surface finish

0.5 marks for comment