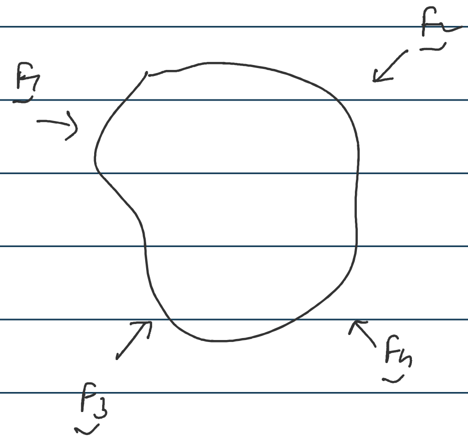
Planar Kinetics

Sunday, 14 April 2024 12:15 AM



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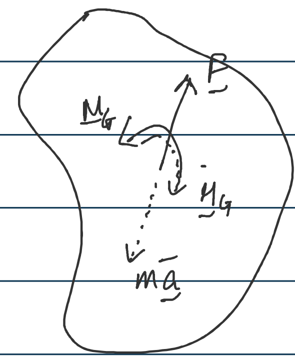


Free body diagram   Kinetic diagram

Angular Momentum:

$$\underset{\sim}{H}_G = \int_m \underset{\sim}{\rho} \times \left(dm\, \dot{\underset{\sim}{\rho}}\right)$$

$$\underset{\sim}{\rho} = \underset{\sim}{r}_{A/G} \qquad \dot{\underset{\sim}{\rho}} = v_{A/G} = \underset{\sim}{\omega} \times \underset{\sim}{r}_{A/G}$$

For planar motion, $\underset{\sim}{\rho} = \rho\, \underset{\sim}{e}_r \qquad \underset{\sim}{\omega} = \omega e_3$

$$\underset{\sim}{\omega} \times \underset{\sim}{\rho} = \rho\omega \left(\underset{\sim}{e}_3 \times \underset{\sim}{e}_r\right) = e_t$$

$$H_G = \int_m \omega \rho^2\, dm \left(\underset{\sim}{e}_r \times \underset{\sim}{e}_t\right)$$

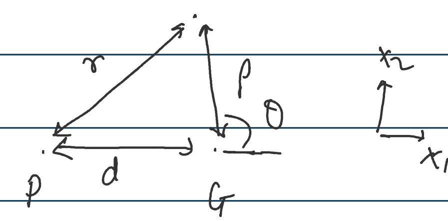
$$= \omega\left(\int_m \rho^2\, dm\right) e_3$$

$$I_G := \int_m \rho^2 \, dm \quad \Rightarrow \quad \text{Mass Moment of Inertia}$$

$$\boxed{\underset{\sim}{H}_G = (I_G \omega) \underset{\sim}{e}_3}$$

$$\boxed{\dot{\underset{\sim}{H}}_G = (I_G \alpha) e_3}$$

<u>Parallel axis theorem:</u>



$$I_P = \int dm \, r^2 =$$

$$= \int dm \left(d^2 + \rho^2 + 2 d \rho \cos\theta \right)$$

$$= \int dm \, \rho^2 + \int dm \, d^2 + 2d \int dm \, \rho \cos\theta$$

$$= I_G + m d^2 + 2d \underbrace{\int \cancel{x_1} \, dm}_{0}$$

$$\boxed{I_P = I_G + m d^2}$$

$\Rightarrow$ <u>Forms of Angular Momentum:</u>

$$\dot{\underset{\sim}{H}}_G = \underset{\sim}{M}_G = I_G \underset{\sim}{\alpha}$$

$$\dot{\underset{\sim}{H}}_0 = \underset{\sim}{M}_0 = I_0 \underset{\sim}{\alpha}$$

$$\underset{\sim}{M}_P = \dot{\underset{\sim}{H}}_P^{rel} + \bar{\underset{\sim}{\rho}} \times m \bar{\underset{\sim}{a}}_P \longrightarrow \underset{\sim}{M}_P = I_P \underset{\sim}{\alpha} + \bar{\underset{\sim}{\rho}} \times m \bar{\underset{\sim}{a}}_P$$

(Think of it as applying a pseudo force in P's frame leads to a moment about P $\Rightarrow$ $\underset{\sim}{M}_P - \bar{\underset{\sim}{\rho}} \times m\bar{a}_P = I_P \alpha$)

$\Rightarrow$ For planar kinematics,

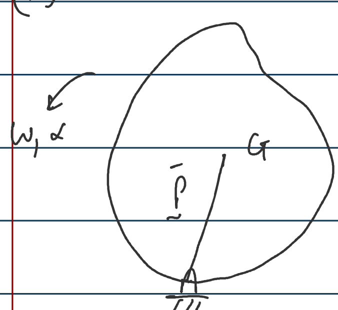
(i) Translation (ii) Rotation (iii) Combined translation & rotation

(i) Pure translation:

$$\omega = \alpha = 0 \implies \Sigma M_P = \left(\bar{\underset{\sim}{\rho}} \times m \bar{\underset{\sim}{a}}_P\right)$$

$$\Sigma \underset{\sim}{F} = m \underset{\sim}{a} \qquad \Sigma \underset{\sim}{M}_G = 0 \qquad \Sigma \underset{\sim}{M}_O = 0$$

(ii) Pure rotation:



$$\Sigma \underset{\sim}{F} = m \bar{\underset{\sim}{a}}$$

$$= m\left(-\omega^2 \bar{\rho}\, e_r + \alpha \bar{\rho}\, e_\theta\right)$$

$\because$ Reactions at O are unknown, balance of angular momentum is generally applied at O.

$\Sigma M_0 = I_0 \alpha = \left(I_G + m\bar{\rho}^2\right)\alpha$

(iii) Combined Rotation & Translation

Velocity of G: $\bar{\underset{\sim}{v}}$ acceleration of G: $\bar{\underset{\sim}{a}}$

$\Sigma \underset{\sim}{F} = m\bar{\underset{\sim}{a}}$

$\Sigma M_G = I_G \alpha$; $\Sigma M_0 = I_0 \alpha$

$\Sigma M_p = I_G \alpha + \left(\bar{\underset{\sim}{\rho}} \times m\bar{\underset{\sim}{a}}\right)$ where $\underset{\sim}{v_p} = \bar{\underset{\sim}{v}} - \underset{\sim}{\omega} \times \bar{\underset{\sim}{\rho}}$

$= I_p \alpha + \left(\bar{\underset{\sim}{\rho}} \times m\underset{\sim}{a_p}\right)$ $\underset{\sim}{a_p} = \bar{\underset{\sim}{a}} - \underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \bar{\underset{\sim}{\rho}}\right) - \underset{\sim}{\alpha} \times \bar{\underset{\sim}{\rho}}$

⇒ <u>Work Energy Theorem</u>:

For a particle: $\underbrace{W}_{\text{work done by external forces}} = \underbrace{\Delta K.E.}_{\text{change in kinetic energy}}$

$W = \Delta V_g + \Delta V_{sp} + \Delta KE$

Work done by forces other than gravity and spring.

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① Translation: $\underset{\sim}{v} = \underset{\sim}{\bar{v}}$

$$KE = \int_m \frac{1}{2} dm\, \underset{\sim}{v} \cdot \underset{\sim}{v} = \frac{m\bar{v}^2}{2}$$

② Rotation:

$$KE = \int_m \frac{1}{2} dm \left(\underset{\sim}{v} \cdot \underset{\sim}{v}\right) = \int_m \frac{1}{2} dm \left(\underset{\sim}{\omega} \times \underset{\sim}{r}\right) \cdot \left(\underset{\sim}{\omega} \times \underset{\sim}{r}\right)$$

$$= \int \frac{1}{2} dm \left|\omega \times r\right|^2 = \int \frac{1}{2} dm\, \omega^2 r^2$$

$$= \frac{I_O \omega^2}{2}$$ O: center of rotation.

③ General motion: $\underset{\sim}{v} = \underset{\sim}{\bar{v}} + \underset{\sim}{\omega} \times \underset{\sim}{\rho}$

$$\underset{\sim}{v} \cdot \underset{\sim}{v} = |\bar{v}|^2 + 2\, \underset{\sim}{\bar{v}} \cdot \left(\underset{\sim}{\omega} \times \underset{\sim}{\rho}\right) + \omega^2 \rho^2$$

$$KE = \frac{1}{2} m\bar{v}^2 + \frac{1}{2} I_G \omega^2 + \int_m dm\, \underset{\sim}{\bar{v}} \cdot \left(\underset{\sim}{\omega} \times \underset{\sim}{\rho}\right)$$

$$= \underset{\sim}{\vec{v}} \cdot \left(\underset{\sim}{\omega} \times \int_m dm\, \underset{\sim}{\rho}\right) = 0$$

$$KE = \frac{1}{2} m\bar{v}^2 + \frac{1}{2} I_G \omega^2$$

= KE of translation + KE of rotation