By the above theorem, $\exists \mathcal{C}$ such that $|\mathcal{C}| = k$
and $\mathcal{A} = \{A_1, \ldots, A_k\}$,
$|\mathcal{A}| = k$

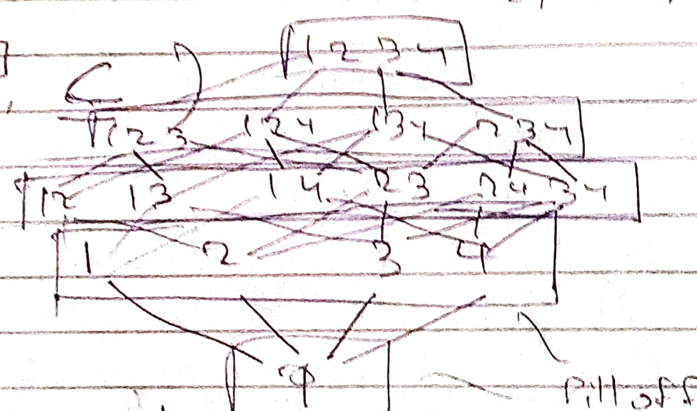
Let D be a chain of P such that $|D| \geq k$

Let D be a chain of P such that $|D| \geq k$

* Dilworth's theorem: let $P = (X, \leq)$. Then there exists K such that P has an antichain of size K and a chain decomposition of size K.

$|A| = K$ $\qquad \mathcal{C} = \{C_1, C_2, \ldots, C_K\}$

$(2^{[3]}, \subseteq)$



Hassy diagram

$|A| \leq |\mathcal{C}|$

Size of Antichain

size of chain decomposition

$\max_A |A| = \min_{\mathcal{C}} |\mathcal{C}|$

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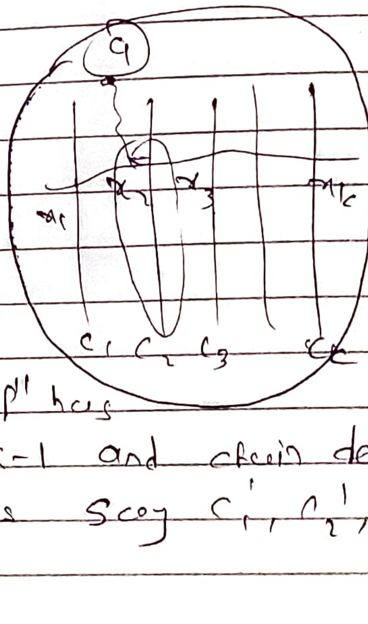
Consider two cases

case 1 : There is an i such that $a \geq x_i$
Fix such an index i. consider the chain $C = \{a\} \cup \{y \in C_i \mid y \leq x_i\}$



~~Note~~ Now consider the poset $P'' = P \setminus C$

Note, by definition of x_i P'' does not have an antichain of size k. So, by induction applied to P'', P'' has an antichain B of size $k-1$ and chain decomposition using $k-1$ chains say $C'_1, C'_2,$ C'_{k-1}

So, $A = \{x_1 \ldots x_n\}$ is an antichain of P of size k and $\{C'_1, C'_2, \ldots C'_{k-1}, C\}$ is a chain decomposition of P using k chains.

case 2 : For each i $a \not\geq x_i$

So, $\{a, x_1, \ldots x_k\}$ is an antichain of P of size $k+1$ and

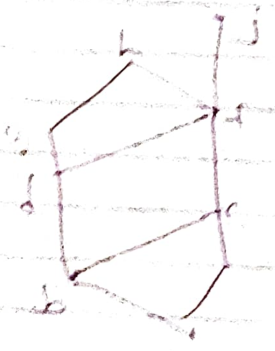
{ ... (X, ≤) ... } ... ... using the ...

Let $P = (X, \leq)$ be a poset

Def. Let $A \subseteq X$ be a subset.
An element u of P is called an upper bound of A if $\forall a \in A$, $a \leq u$.
An element l of P is called a lower bound of A if $\forall a \in A$, $l \leq a$.

$(2^{\{a,b\}}, \subseteq)$

$\{a, b\}$ $\{a, b\}$



$A = \{a, b, c\}$ upper bound $A = \{e, f, h, j\}$

Lower bound of $A = \{a\}$

$A' = \{h, j\}$ upper bound of $A' = \phi$

lower bound = everything except g.

Definition: Let A be a set. An element x is called a "least upper bound of A" if

1) x is an upper bound of A
2) if u is an upper bound of A then $x \leq u$.

Similarly, glb is defined.

$x \leq b$ lub$\{x, y\}$

Page No.
Date glb$(x, y) = x$

$x \vee y =$ lub

$x \wedge y =$ glb

Definition: let $P = (X, \leq)$ be a poset

P is said to be a lattice if $\forall x, y \in X$,
lub$(\{x, y\})$ and glb$(\{x, y\})$ exist.

lub: lowest upper bound
glb: greatest lower bound

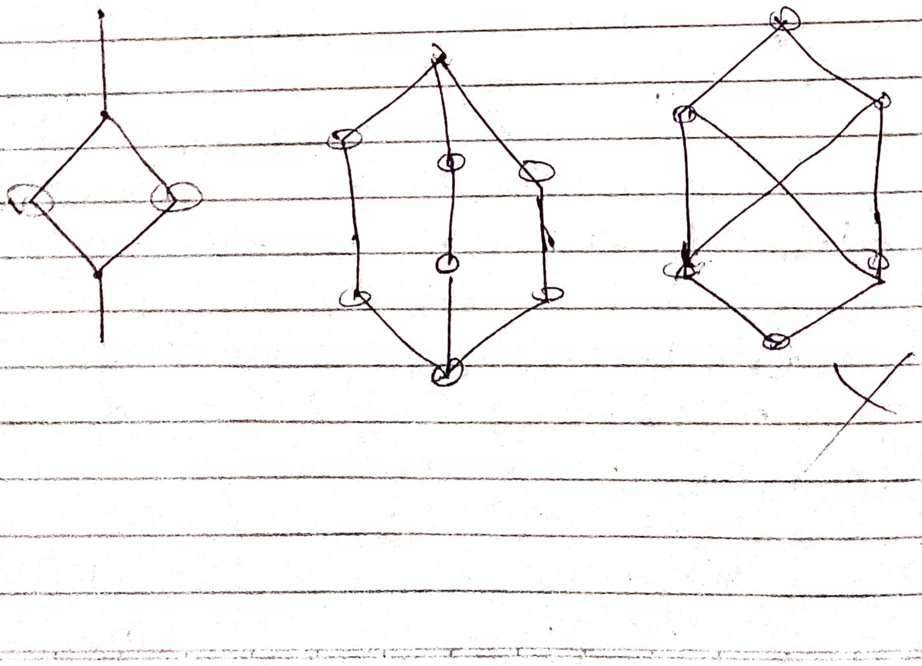
A finite lattice has a unique minimal element & a unique maximal element.

X, Y $\{Z \mid Z \supseteq X \cup Y\}$

$P = \{1, 2, 3, 4, \ldots\}$

$\{(P, |)$
divides

9, 12



Tut - 5

Q 1)

$a_n = 4a_{n-1} - 4a_{n-2} + 2^n$

$\cancel{-4a_{n-2}}$

$\cancel{4(4a_{n-2} - 4(a_{n-2})}$

$4(4a_{n-2} - 4a_{n-3} + 2^{n-1}) - 4a_{n-2} + 2^n$

$a_{n-1} = 16a_{n-2} - 16a_{n-3} + 2\times2^n - 4a_{n-2} + 2^n$

$a_{n+1} = \cancel{16(a_{n-2} - a_n}$

$12a_{n-2} - 16a_{n-3} + 3\times2^n$

$a_n = 4$

$a_2 = 4a_1 - 4a_0 + 2^2$

$= 4 - 0 + 4$

$a_2 = 8$

$a_3 = 4a_2 - 4a_1 + 2^3$

$= 4\times8 - 4\times1 + 8$

$= 32 - 4 + 8$

$= 36$

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Graphs:

Classes of ~~well known graphs~~:

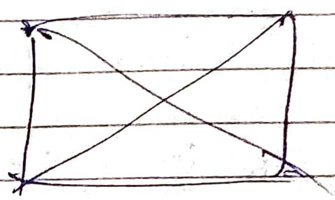
$(n \geq 1)$

- Complete graphs $K_n = (V, E)$ $|V| = n$,

$E = \binom{V}{2}$ = the set of all 2-size subset.

$|E| = \binom{n}{2}$

Clique K_4 :



$\deg_{K_n}(v) = n-1$

~~cliques~~

K

- Paths : P_n $(n \geq 1)$ $P_n = (V, E)$

P_5

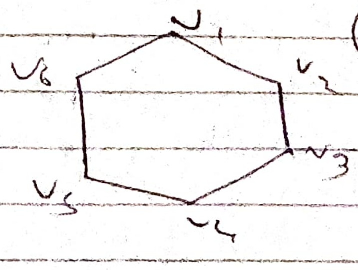
$|V| = n$, V can be ordered say $V = \{v_1, v_2 \ldots v_n\}$

Such that $E = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$

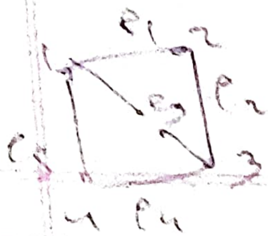
P_n has n vertices and $(n-1)$ edges

$|V| = n$ $|E| =$

Cycles : C_n $(n \geq 3)$



$C_2 = ?$



Connectivity :

Let $G = (V, E)$ be a graph

Definition : 1) A trail in G is a sequence $\pi = (v_1, v_2, v_3 \dots v_k)$ of vertices such that

1) $\forall\ 1 \le i \le k-1$, $(v_i, v_{i+1}) \in E$

Such a trail is said to be of length $k-1$ and is a trail from v_1 to v_k.

2) A path in G is a trail $\pi = (v_1, v_2 \dots v_k)$ such that $v_1, v_2, \dots v_k$ are all distinct.

Note that $\pi = (u)$ is a path of length 0 from u to u.

We define a relation $\sim$ on V:

Given, $u, v \in V$ we say $u \sim v$ is there is a path from u to v in G.

Lemma : $\sim$ is an equivalence relation on V.

Proof : $\sim$ is reflexive on $\pi = (u)$ is a path from u to u

$\sim$ is symmetric $u \sim v \overset{?}{\Rightarrow} v \sim u$

As $u \sim v$ there is a path say $\pi = (u = u_1, u_2, \dots u_k = v)$ from u to v.

Then the seq $\pi^R = (v = u_k, u_{k-1}, \dots u_1 = u)$ is a path from v to u.

Let $u, v, w \in V$ be such that $u \sim v$ & $v \sim w$

We have paths $\alpha = (u = u_1, u_2, \ldots, u_k = v)$ is a path from u to v and $\beta = (v = v_1, v_2, \ldots, v_l = w)$ is a path from v to w.

(Assum $u \neq v$, $v \neq w$)

~~Consider~~ $\Pi = (u =$

Consider the sequence $\Pi = (u = u_1, u_2, \ldots, u_k = v_1, v_2, \ldots, v_l = w)$ Clearly Π is a trail from u to w.

If Π is already a path then we are done. Otherwise there is a $s < k$ and $t > 1$ such that $u_s = v_t$.

Choose smallest s such that $s < k$ and there is a $t > 1$ with $u_s = v_t$

Then $\Pi' = (u = u_1, u_2, \ldots, u_s = v_t, v_{t+1}, \ldots, v_l = w)$ is indeed a path from u to w.

$V = V_1 \cup V_2 \cup V_3 \ldots \cup V_k$ be the equivalence classes of $\sim$.

By definition of $\sim$: no edges in G connects vertices different equivalence class.

$\therefore$ (1) $E = E_1 \cup E_2 \cup \ldots \cup E_k$

E_i is the restriction of E to V_i.

(2) $G_1 = (V_1, E_1)$

33240 (4) 233 (75)

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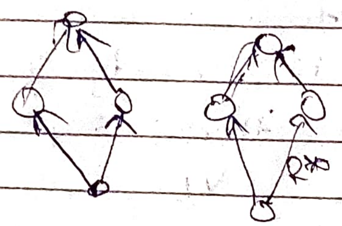
A graph is connected if $\forall u, v \in V$, there is a path from u to v

Tut 1

1. Covering ~~set~~ relation.

Q.1) R^*

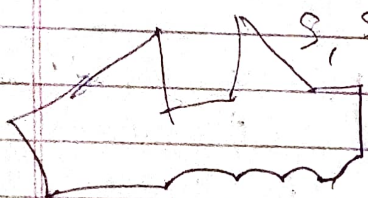


$R^* \cup \{$

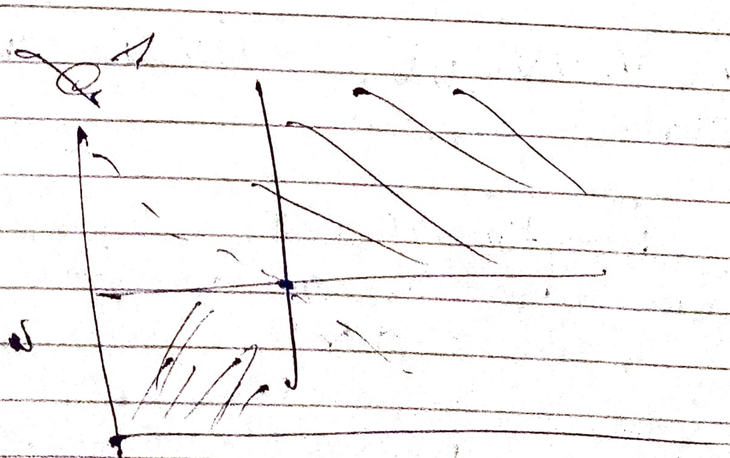
a, b

R

S, S^* $R^* = S^* \cup \{a, b\}$



2.9)



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$B \subseteq A$

$$|N_{G_1}(B)| \overset{?}{\geq} |B|$$

$$N_G(B) = N_{G_1}(B)$$
$$\subseteq N(A)$$

$$|N_G(B)| \geq |B|$$

$$\Rightarrow |N_{G_1}(B)| \geq |B|$$

$B \subseteq V_1 \setminus A$

$$|N_{G_2}(B)| \overset{?}{\geq} |B|$$

$$N_G(A \cup B)$$
$$= N_G(A) \cup N_{G_2}(B)$$

$|N_G(A)| + |N_{G_2}(B)|$
$$|N_G(A \cup B)| \geq |A \cup B|$$
$$|A| + |B|$$

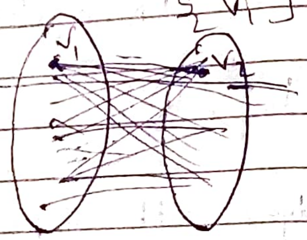
Thm: G_1 & G_2 satisfies Hall's condition.
$\rightarrow$ G_1 satisfies hall's condition
$\rightarrow$ G_2 satisfies hall's condition

By induction:
G_1 has a complete matching from A to $N(A)$
G_2 has — " — from $V_1 \setminus A$ to $V_2 \setminus N(A)$)

Putting the two together gives a complete matching from V_1 to V_2.

$\boxed{\text{case 2}}$: $\forall A \subseteq V_1$, $\emptyset \subsetneq A \subsetneq V_1 \Rightarrow |N(A)| > |A|$

Fix $v_1 \in V_1$. By Hall's condition applied to $\{v_1\}$ $\exists\, v_2 \in V_2$ such that $(v_1, v_2) \in E$



$$G' = (V_1 \setminus \{v_1\} \cup V_2 \setminus \{v_2\}, E')$$

Claim: G' satisfies Hall's cond^n.

Let $\emptyset \subsetneq B \subseteq V_1 \setminus \{v_1\}$ $\quad N_{G'}(B)$

$\subseteq N_G(B)$

$\subseteq N_{G'}(B) \cup \{v_2\}$

$N_G(B) > |B|$

$$|N_{G'}(B)| \geq |B|$$

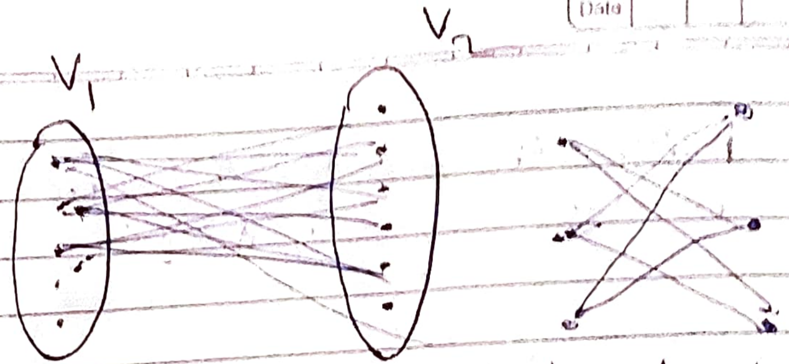
Hall's thm: Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then G has a complete matching from V_1 to V_2. $\Leftrightarrow$ $\forall A' \subseteq V_1$, $|N(A')| \geq |A'|$

$(\Rightarrow)$ ✓

A graph is said to be k-regular if $\forall v \in V$, $\deg(v) = k$.

Let $G = (V_1 \cup V_2, E)$ be a regular bipartite graph. Then G has a perfect matching.

Everybody has found the partner.



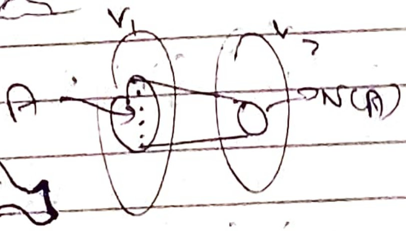
$k|V_1|$ (Every vertex has k edges incident on it.)

$|E| = k|V_1| = k|V_2|$

$|V_1| = |V_2|$.

$|E_A| = k|A|$

$E_{N(A)} = k|N(A)|$



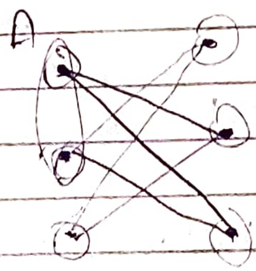
$A \subseteq V_1$

$|N(A)| \overset{?}{\geq} |A|$

E_A = edges with one endpoint in A

$E_{N(A)}$ = edges with —||— in $N(A)$

$E_A \subseteq \cancel{\neq} E_{N(A)}$



$E_A =$

$E_{N(A)}$

$|E_A| \leq |E_{N(A)}|$

$k|A| \leq k|N(A)|$

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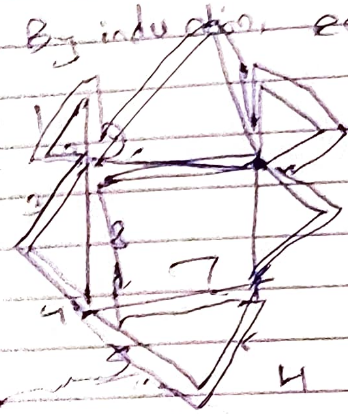
The proof is by the induction on $|E|$.

$$H = (G, E \setminus T) \quad |E \setminus T| < |E|$$

<u>Claim:</u> Every vertex of H has even degree

Let $H_1, H_2, \ldots, H_k$ be the connected components of H.

By induction, each H_i has an eulerian trail.

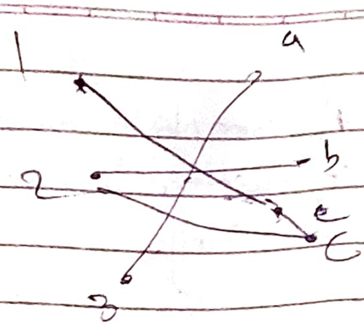


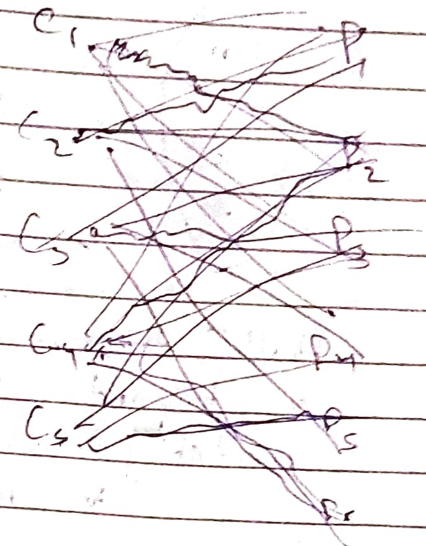
<u>Reading Ex.</u> Hamiltonian graphs

<u>Matchings:</u>

Let $G = (V, E)$ be a graph

<u>Def:</u> A matching M is a collection of edges of G ($M \subseteq E$) such that no two edges in M shares a common endpoint.





$M = \{(2,b),(3,a)\}$

- A matching M is maximal if there is no matching M' such that $M \subseteq M'$ and $M \neq M'$.

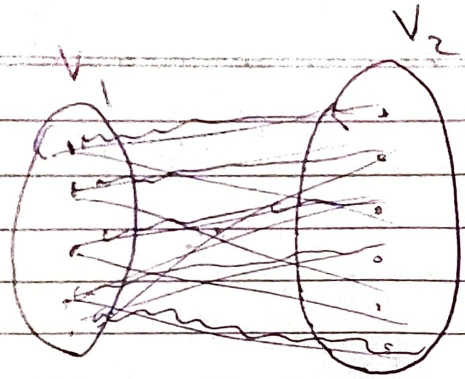
- A matching M is maximum if M has the maximum possible size.

$c_1 - p_1,\ c_2 - p_2,\ c_3 -$

Let $G = (V = V_1 \cup V_2, E)$ be a bipartite graph.

Def: A complete matching M from V_1 to V_2 is a matching in which every vertex of V_1 is matched

- Let M be a matching
- A vertex v is said to matched in M (M-matched) if there is an edge e in M which is incident on v. Otherwise it is said to be unmatched.



A complete matching is in fact a maximum matching.

Hall's theorem: Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. G has a complete matching from V_1 to V_2 $\iff$ $\forall A \subseteq V_1$, $|N(A)| \geq |A|$

$(\Rightarrow)$ $A \subseteq V_1$ $N(A)$ = neighbourhood of A
$= \{ v \in V \mid v$ has a neighbour in $A \}$
v is adjacent to some vertex in A.