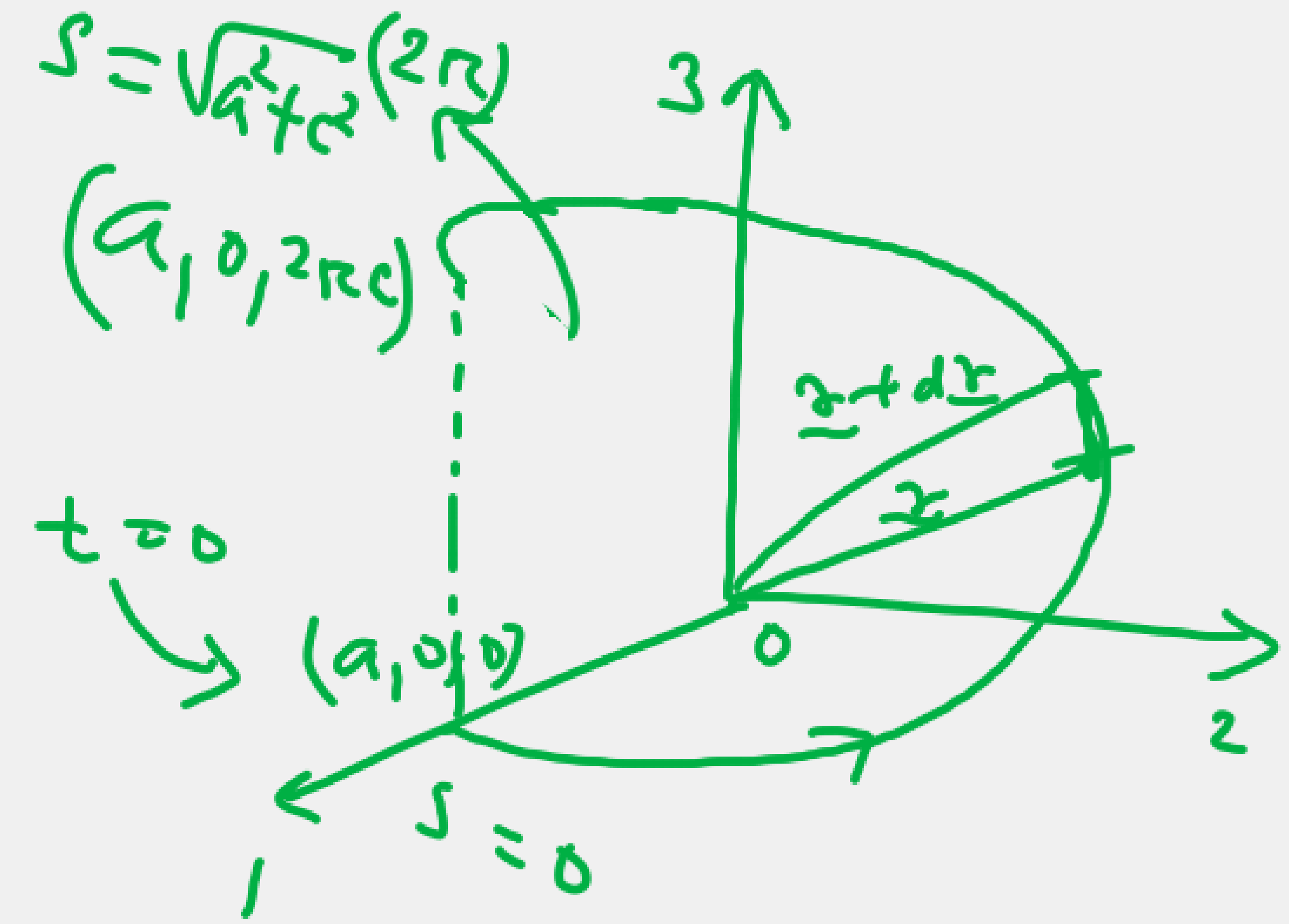
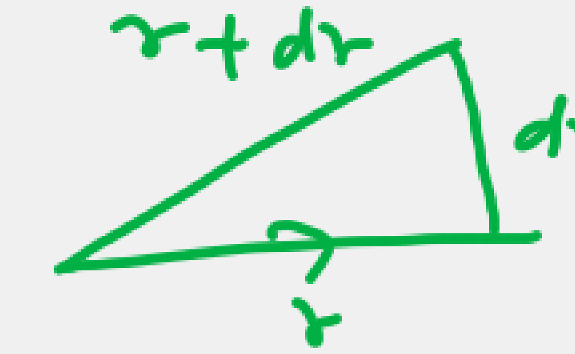
Tangential - Normal frame

Position vector expressed in terms of arc length "s"

$$\underset{\sim}{r} = \underset{\sim}{r}(s(t))$$

Example: Helix, $x_1 = a\cos(t)$, Time
$x_2 = a\sin(t)$,
$x_3 = ct$





Arc length expression

$$(ds)^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r}$$
$$= (dx_1)^2 + (dx_2)^2 + (dx_3)^2$$
$$= (-a\sin t\, dt)^2 + (a\cos t\, dt)^2 + (c\, dt)^2$$
$$ds = (a^2+c^2)^{1/2}\, dt$$

Integrating, $\boxed{s = (a^2+c^2)^{1/2}\, t}$

$$x_1 = a\cos t = a\cos\left(\frac{s}{\sqrt{a^2+c^2}}\right)$$

$$x_2 = a\sin t = a\sin\left(\frac{s}{\sqrt{a^2+c^2}}\right)$$

$$x_3 = ct = \frac{cs}{\sqrt{a^2+c^2}}$$

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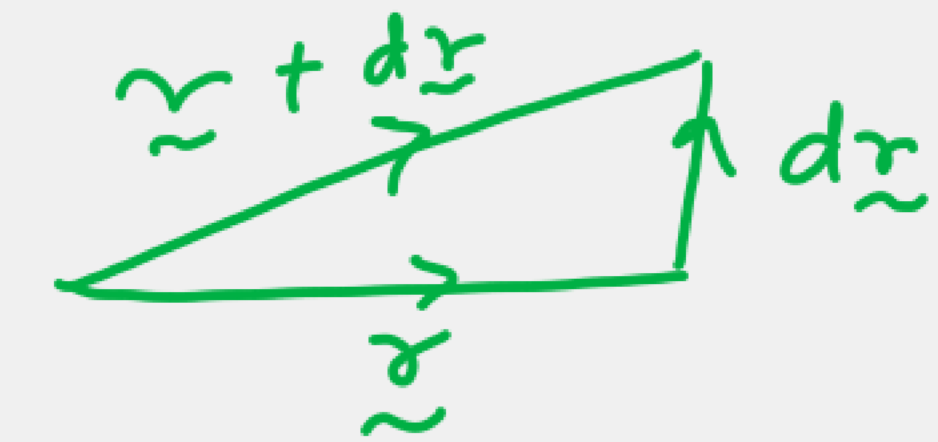
x

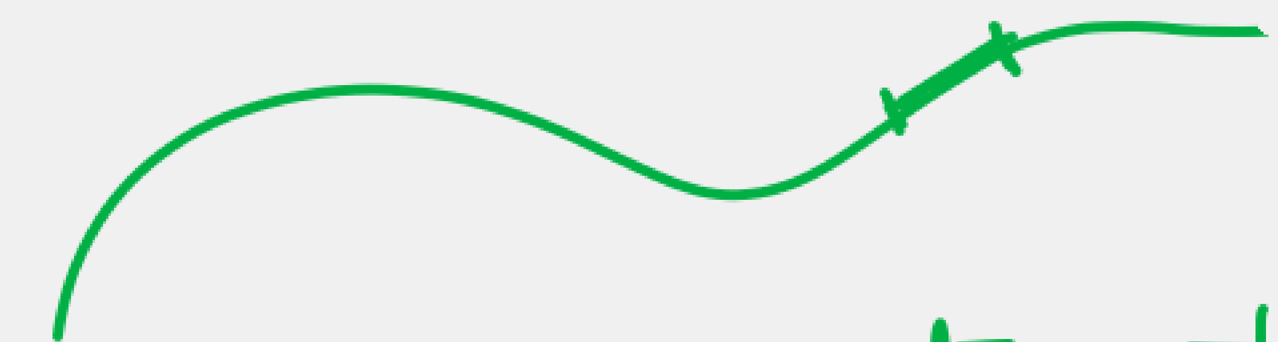
Velocity $$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \underbrace{\left(\frac{d\underset{\sim}{r}}{ds}\right)}_{\text{vector}}\left(\frac{ds}{dt}\right) \rightarrow \text{SCALAR}$$

$$\underset{\sim}{a} = \lambda \underset{\sim}{b} \qquad |\underset{\sim}{a}| = |\lambda|\,|\underset{\sim}{b}|$$

Magnitude of $\frac{d\underset{\sim}{r}}{ds}$

$$\left|\frac{d\underset{\sim}{r}}{ds}\right| = \lim_{\Delta s\to 0} \frac{|\Delta \underset{\sim}{r}|}{\Delta s} = 1$$

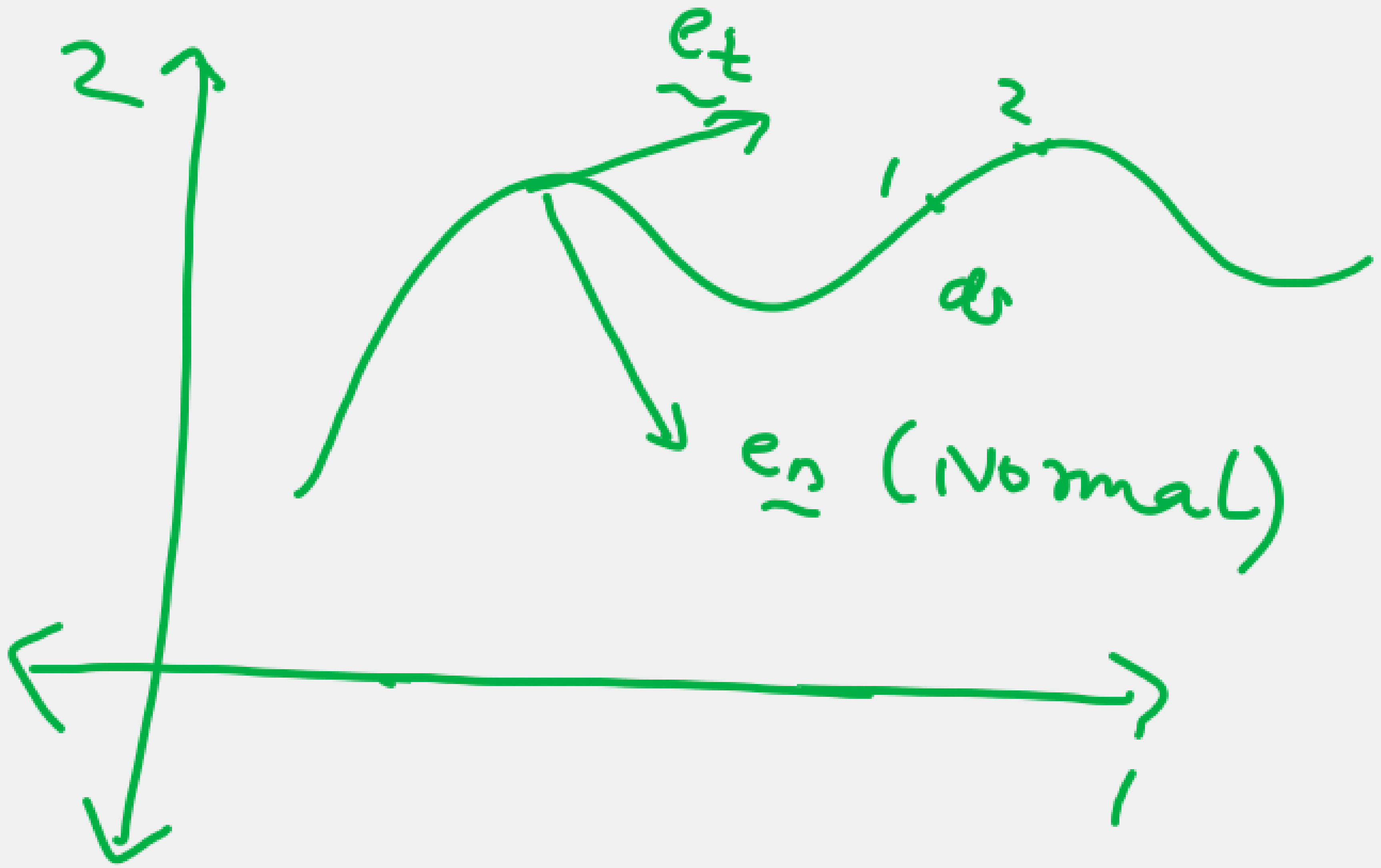




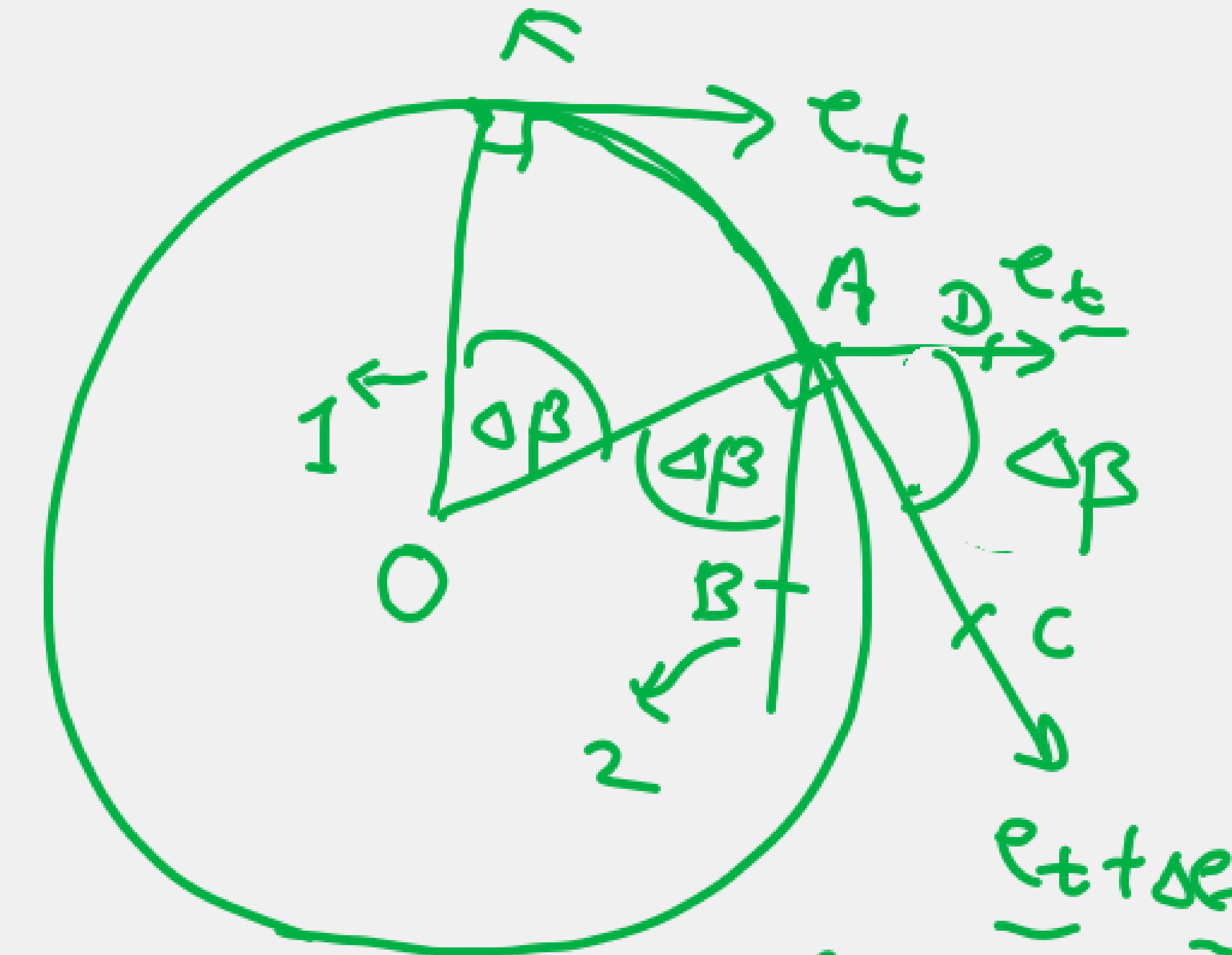
Vector $\frac{d\underset{\sim}{r}}{ds}$ = unit tangent vector

We will denote it as

$$\underset{\sim}{e_t} = \frac{d\underset{\sim}{r}}{ds}$$



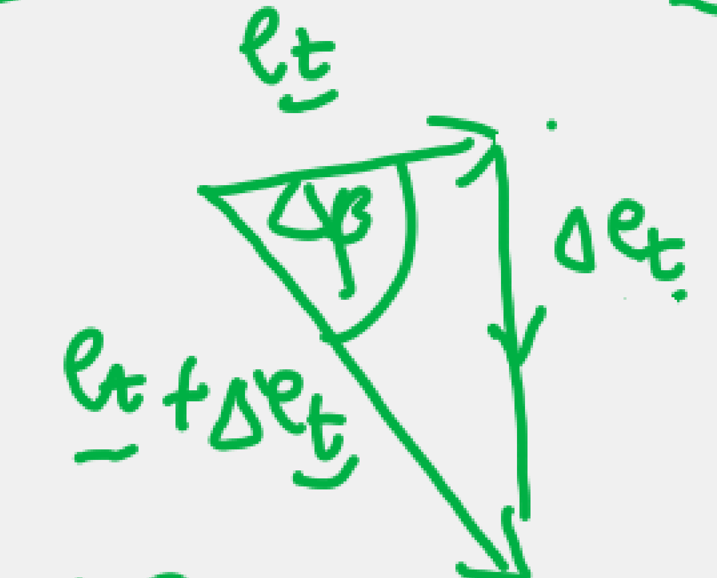
$\underset{\sim}{e_t} \perp \dfrac{d\underset{\sim}{e_t}}{ds}$



$$\boxed{\frac{d\underset{\sim}{e_t}}{ds} := \kappa\, \underset{\sim}{e_n}}$$

Curvature

Definition

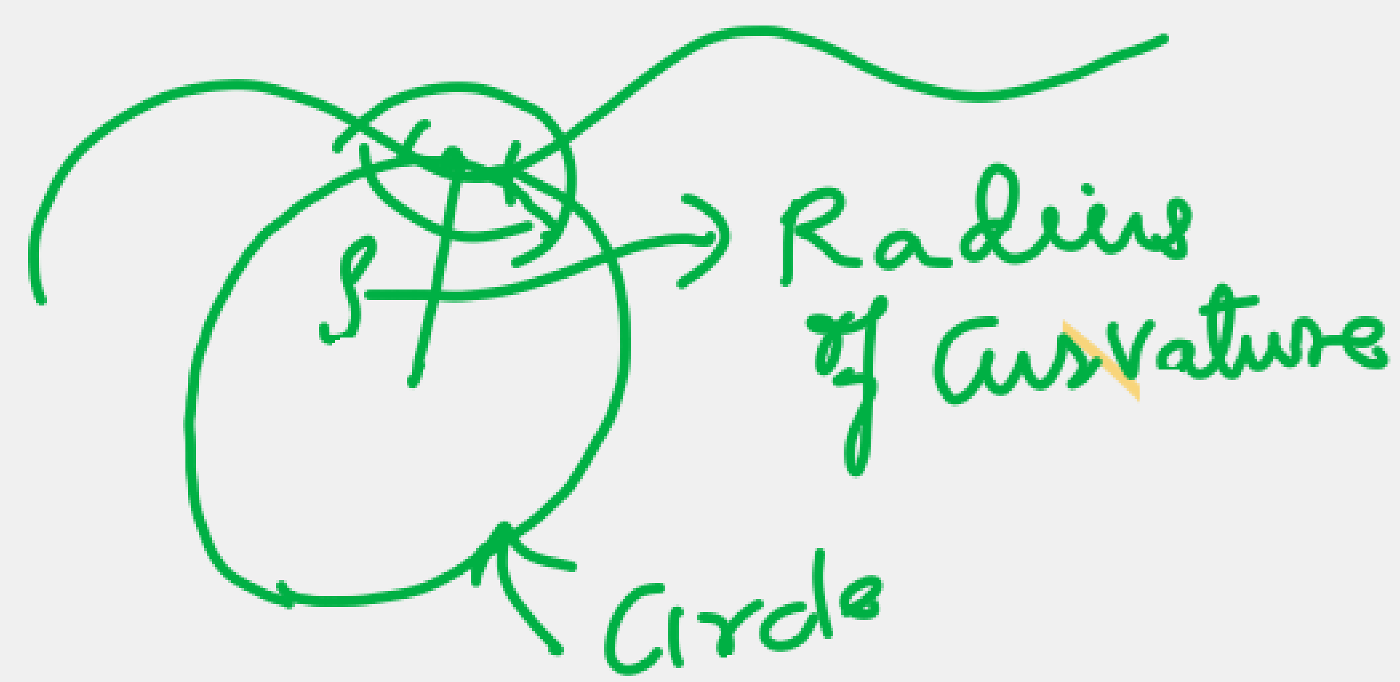


$\underset{\sim}{e_t} \cdot \underset{\sim}{e_t} = 1$

Differentiate w.r.t s

$$\frac{d\underset{\sim}{e_t}}{ds} \cdot \underset{\sim}{e_t} + \underset{\sim}{e_t} \cdot \frac{d\underset{\sim}{e_t}}{ds} = 0$$

$$\underset{\sim}{e_t} \cdot \frac{d\underset{\sim}{e_t}}{ds} = 0$$



$$\frac{d\underset{\sim}{e_t}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \underset{\sim}{e_t}}{\Delta s}$$

Line 1 || Line 2

$\measuredangle OAB + \measuredangle \underline{BAC} = \pi/2$

$\measuredangle \underline{BAC} + \measuredangle CAD = \pi/2$

$\measuredangle OAB = \measuredangle CAD$

$|\Delta \underset{\sim}{e_t}| = |\underset{\sim}{e_t}| \tan(\Delta\beta)$

$\simeq 1 (\Delta\beta) \to \underline{\text{Radians}}$

$|\Delta \underset{\sim}{e_t}| = \Delta\beta$

$$\frac{|\Delta \underset{\sim}{e_t}|}{\Delta s} = \frac{\Delta\beta}{\Delta s} = \frac{\cancel{\Delta\beta}}{\rho \cancel{\Delta\beta}}$$

$$\frac{d\underset{\sim}{e_t}}{ds} = \kappa \underset{\sim}{e_n}$$

$$\frac{|\Delta \underset{\sim}{e_t}|}{\Delta s} = \frac{1}{\rho}$$

$$\frac{d\underset{\sim}{e_t}}{ds} = \kappa \underset{\sim}{e_n} = \frac{1}{\rho} \underset{\sim}{e_n}$$

Inverse of radius of curvature

$\underset{\sim}{r} = \underset{\sim}{r}(s(t))$

$$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{d\underset{\sim}{r}}{ds}\frac{ds}{dt}$$

$$\boxed{\underset{\sim}{v} = \dot{s}\, \underset{\sim}{e_t}}$$

$|\underset{\sim}{v}| = |\dot{s}| \to$ Magnitude

$$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt}$$

$$\underset{\sim}{a} = \frac{d}{dt}\left(\dot{s}\,\underset{\sim}{e_t}\right) = \ddot{s}\,\underset{\sim}{e_t} + \dot{s}\,\frac{d\underset{\sim}{e_t}}{dt}$$

$$\underset{\sim}{a} = \ddot{s}\,\underset{\sim}{e_t} + \dot{s}\,\frac{d\underset{\sim}{e_t}}{ds}\,\frac{ds}{dt}$$

$$= \ddot{s}\,\underset{\sim}{e_t} + (\dot{s})^2\,\frac{1}{\rho}\,\underset{\sim}{e_n}$$

$$\underset{\sim}{a} = \underbrace{\ddot{s}\,\underset{\sim}{e_t}}_{\text{Tangential}} + \underbrace{\frac{(\dot{s})^2}{\rho}\,\underset{\sim}{e_n}}_{\text{Norm. compone.}}$$

$\underset{\sim}{e_t} - \underset{\sim}{e_n}$ frame is known as Serret-Fernet frame.

Bi-normal

Normal to both $\underset{\sim}{e_n}$ and $\underset{\sim}{e_t}$

The direction of $\underset{\sim}{e_n}$ is towards the centre of curvature.

$\underset{\sim}{b} = \underset{\sim}{e_t} \times \underset{\sim}{e_n}$

Crossproduct

$\dot{s}\,\underset{\sim}{e_t} = \cancel{\frac{d\underset{\sim}{r}}{ds}}$ dr/dt

Rigid body:

Collection of ∞ no. of particles.

Distance between any two points remain same.

Rigid body motion = Translation + Rotation