Question 7

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March 2024

What is the size of the largest family of subsets of $[n]$ such that any two sets in the family have a non-empty intersection? Why?

For $A \subseteq [n]$ let $\overline{A} = [n] \setminus A$.

Proof 1: A family of subsets $\mathcal{F} \subseteq 2^{[n]}$ is called *intersecting* if $F \bigcap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$.

If $F \in \mathcal{F}$, then $F \bigcap \overline{F} = \emptyset$ implies $\overline{F} \notin \mathcal{F}$. Thus $\mathcal{F} \bigcap \overline{\mathcal{F}} = \emptyset$, where we can say $\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$. Since, $|\mathcal{F}| = |\overline{\mathcal{F}}|$ and $\mathcal{F} \sqcup \overline{\mathcal{F}} \subseteq 2^{[n]}$, we have $2|\mathcal{F}| = |\mathcal{F}| + |\overline{\mathcal{F}}| = |\mathcal{F} \sqcup \overline{\mathcal{F}}| \leq |2^{[n]}| = 2^n$ implies $|\mathcal{F}| \leq 2^{n-1}$.

Proof 2: *(Using Pigeonhole Principle)* In this problem each hole consists of a pair of sets where every set A paired with its complement $\overline{A}$. We have at most 2^n numbers of subsets of $[n]$ that are possible, and each hole contains exactly 2 sets. This implies we have at most $2^n/2 = 2^{n-1}$ holes possible.

Suppose we have k sets which intersect with each other, these k set can be considered as pigeons. If $k > 2^{n-1}$, there are more pigeons than the holes, implies there will be a hole that contains 2 pigeons. This is not possible, because then there will be a pigeon A and pigeon $\overline{A}$. Since, $A \bigcap \overline{A} = \emptyset$, which means our family of subsets do not all intersect with each other. So $k \leq 2^{n-1}$.

Most simplest example of intersecting families are those for which one fixed element is contained in all members (subsets) of the family. $|\mathcal{F}| = 2^{n-1}$ for such cases.

Marking Scheme: 1 marks for the example, 2 marks for the justification.