Plane Kinetics

Equation of motion

① $\sum \underset{\sim}{F} = m\underset{\sim}{\bar{a}}$

$\underset{\sim}{\bar{a}}$ } Acceleration of Centre of mass G

② $\sum M_G = I_G \alpha$

③ $\sum M_O = I_O \alpha$ → O is fixed point

} Rate of change of angular momentum

④ About any other point P ;

(i) $\sum M_P = \underbrace{\sum M_G}_{I_G \alpha} + \underbrace{(\underset{\sim}{\bar{\rho}} \times m\underset{\sim}{\bar{a}})_3}_{3^{rd}\ \text{component}}$

(ii) $\sum M_P = I_P \alpha + (\underset{\sim}{\bar{\rho}} \times m\underset{\sim}{a_P})_3$

$\underset{\sim}{\bar{\rho}}$ } Vector joining P to G

$\underset{\sim}{a_P}$ } Acceleration of P

$I = \int dm\, r^2$ } Mass moment of inertia

For planar case

Kinematics

(i) Translation

(ii) Rotation

(iii) Translation + Rotation

For (i), Translation:

$\omega = \alpha = 0$

$\Sigma \underset{\sim}{F} = m \bar{\underset{\sim}{a}}$

$\Sigma M_G = 0$

$\Sigma M_O = 0$

$\Sigma M_P = \left(\underset{\sim}{\rho} \times m \bar{\underset{\sim}{a}} \right)_3$

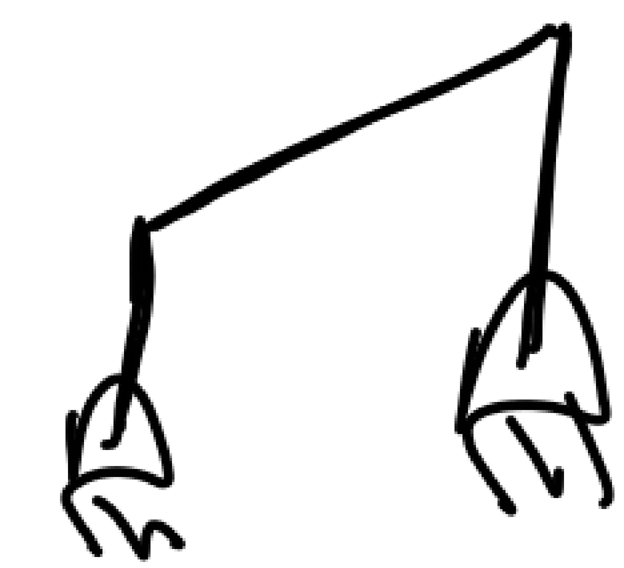
(ii) Rotation case:

O (Fixed point)

$\Sigma \underset{\sim}{F} = m \bar{\underset{\sim}{a}}$

$\bar{r} = |\bar{\underset{\sim}{r}}| = OG$

G } centre of mass



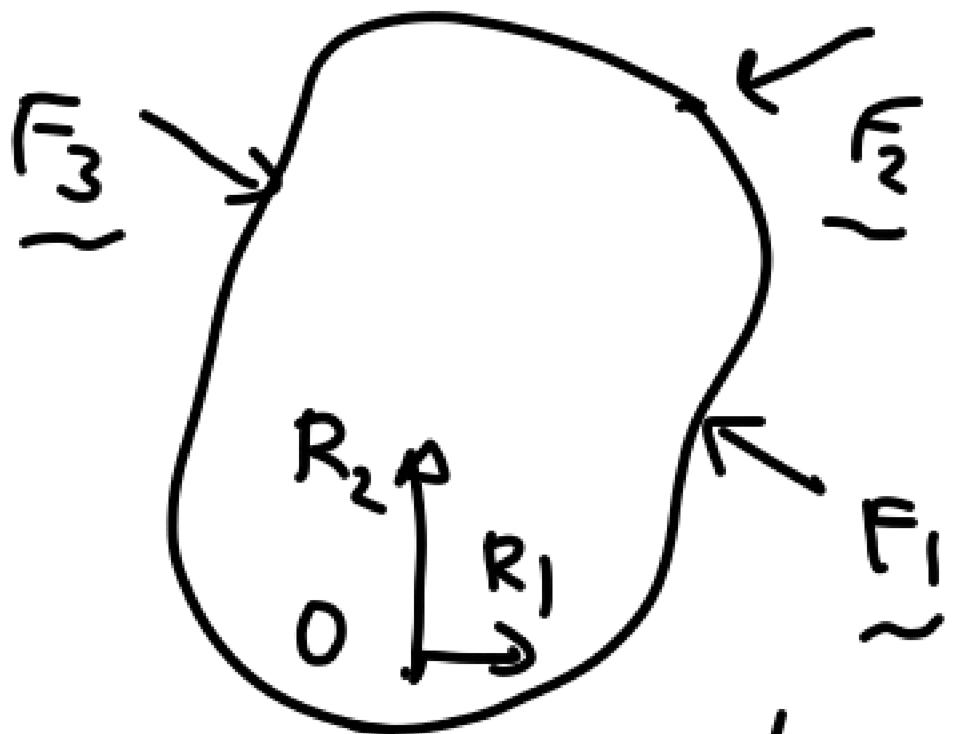
$\bar{\underset{\sim}{v}} = (\underset{\sim}{\omega} \times \bar{\underset{\sim}{r}})$

$\bar{\underset{\sim}{v}} = \omega \bar{r} \underset{\sim}{e_\theta}$

$\bar{\underset{\sim}{a}} = \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) + \underset{\sim}{\alpha} \times \underset{\sim}{r}$

$= -\omega^2 r \underset{\sim}{e_r} + \alpha r e_\theta$

$$\Sigma \mathbf{F} = m\left(-\omega^2 r\, \mathbf{e}_r + \alpha r\, \mathbf{e}_\theta\right)$$



Free body diagram

(Reaction at O) (unknown)

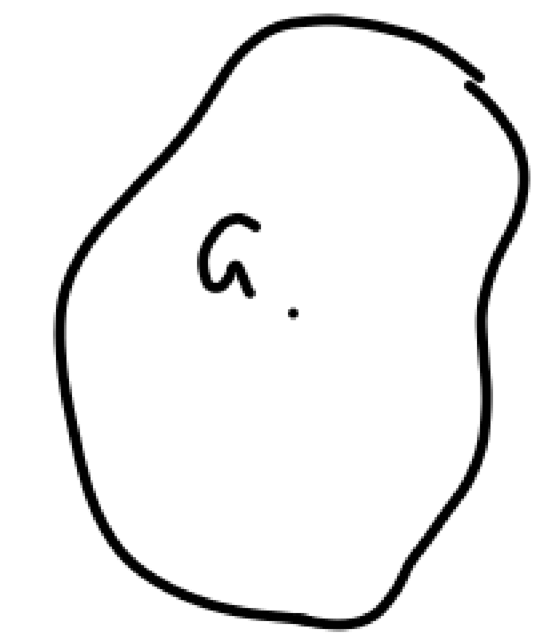
So balance of angular momentum is usually implemented at O.

$$\Sigma M_O = I_O \alpha$$

$$I_O = I_G + m(\bar{r})^2; \quad \bar{r} = |\bar{\mathbf{r}}|$$

(iii) <u>General case</u>

Both translation and rotation



$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

velocity of G: $\bar{\mathbf{v}}$

Acceleration of G: $\bar{\mathbf{a}}$

<u>Balance of angular momentum</u>

$$\Sigma M_G = I_G \alpha; \quad \Sigma M_O = I_O \alpha$$

$$\Sigma M_P = I_G \alpha + \left(\bar{\boldsymbol{\rho}} \times m\bar{\mathbf{a}}\right)_3$$

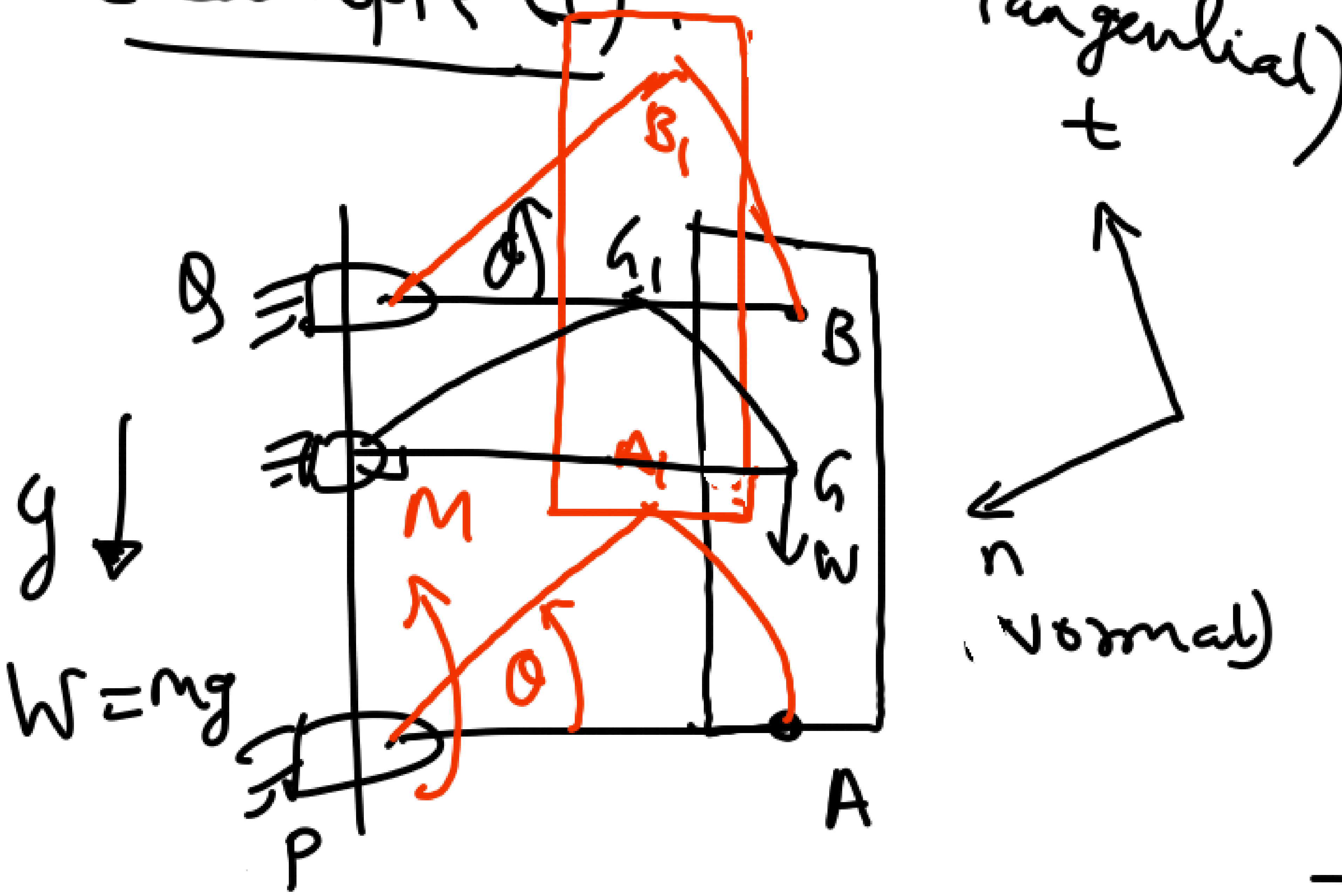
$$= I_P \alpha + \left(\bar{\boldsymbol{\rho}} \times m\mathbf{a}_P\right)_3$$

$$\mathbf{v}_P = \bar{\mathbf{v}} + \underbrace{\boldsymbol{\omega} \times \bar{\mathbf{r}}}_{v_{P/G}}$$

whether in the 2nd term, should we have $\bar{\mathbf{r}}$ or $-\bar{\mathbf{r}}$

$$\mathbf{a}_P = \bar{\mathbf{a}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \bar{\mathbf{r}}) + \boldsymbol{\alpha} \times \mathbf{r}$$

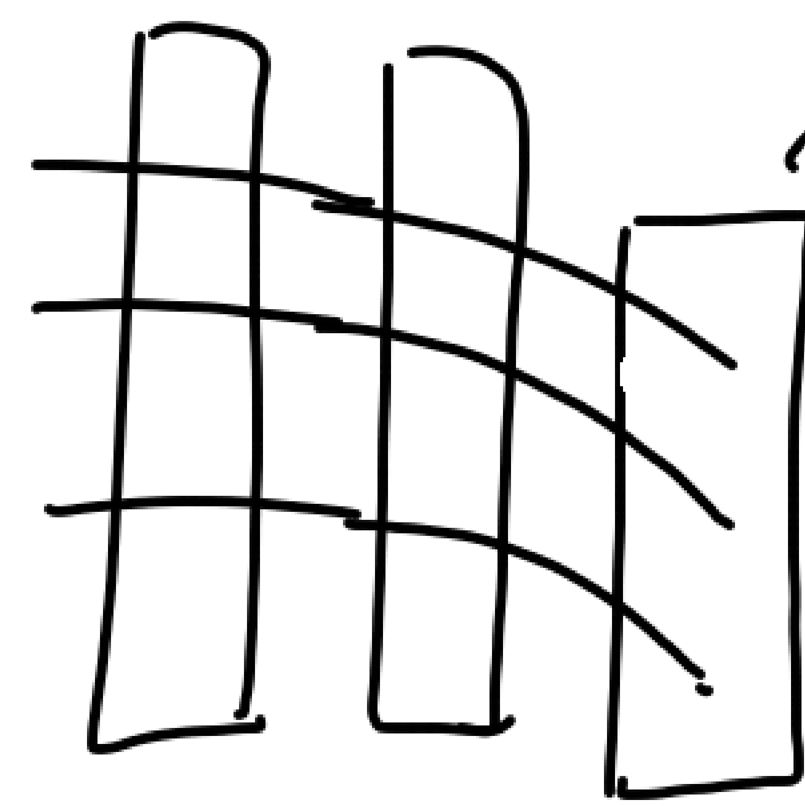
Example (1):



M } Input torque through motor at point P

Motion of bar AB attached firmly to the connecting rod;

Assume AP, QB to be massless.



→ Curvilinear translation

Trajectory of bar AB

PABQ is a parallelogram linkage

$PA = QB = l$

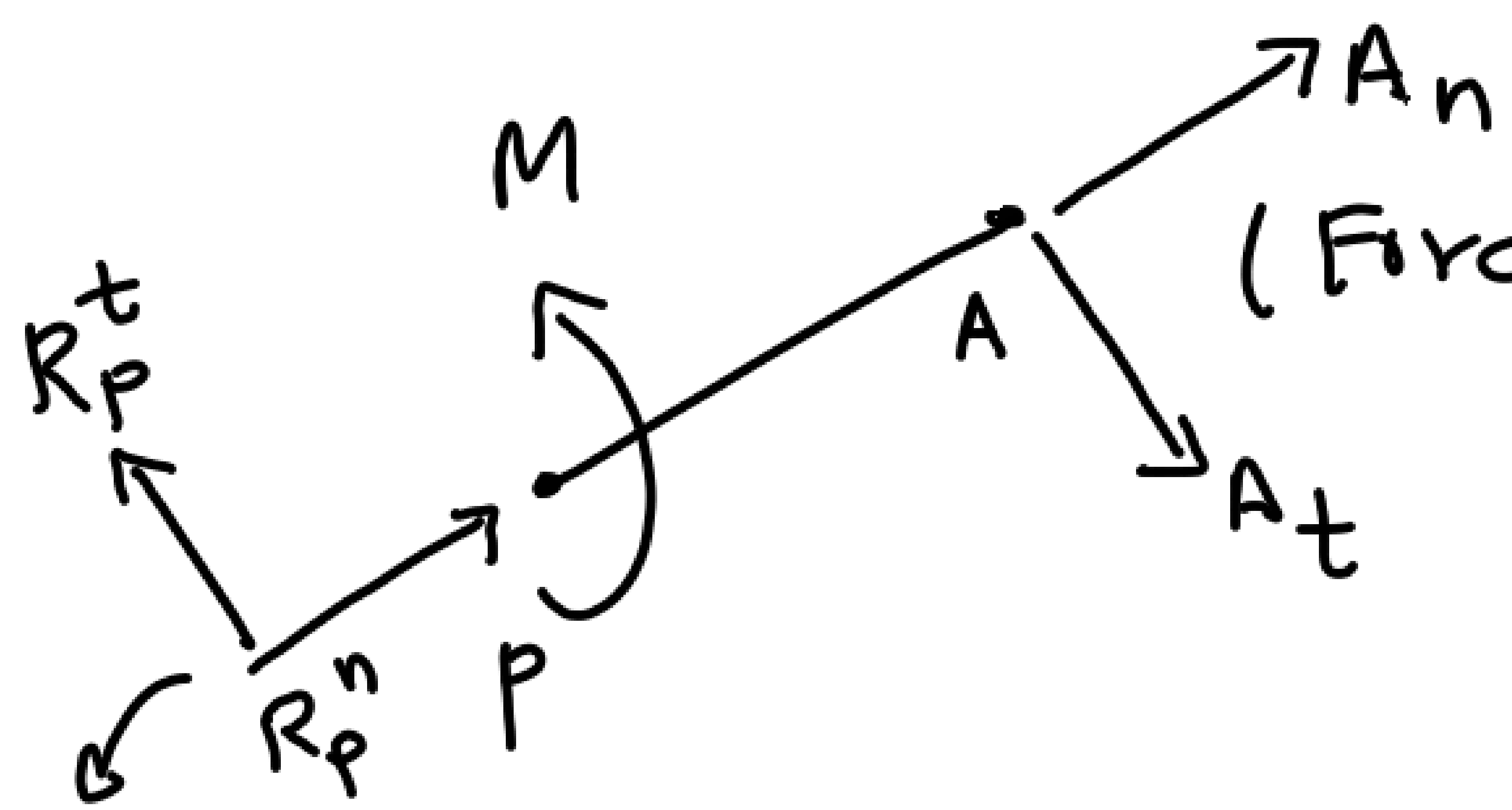
$PQ = AB = L$

To find: ① Forces in link QB as a function of θ

② Angular acceleration of link PA

Link AP :

In the position "θ" :



(Force betⁿ AP and bar AB)

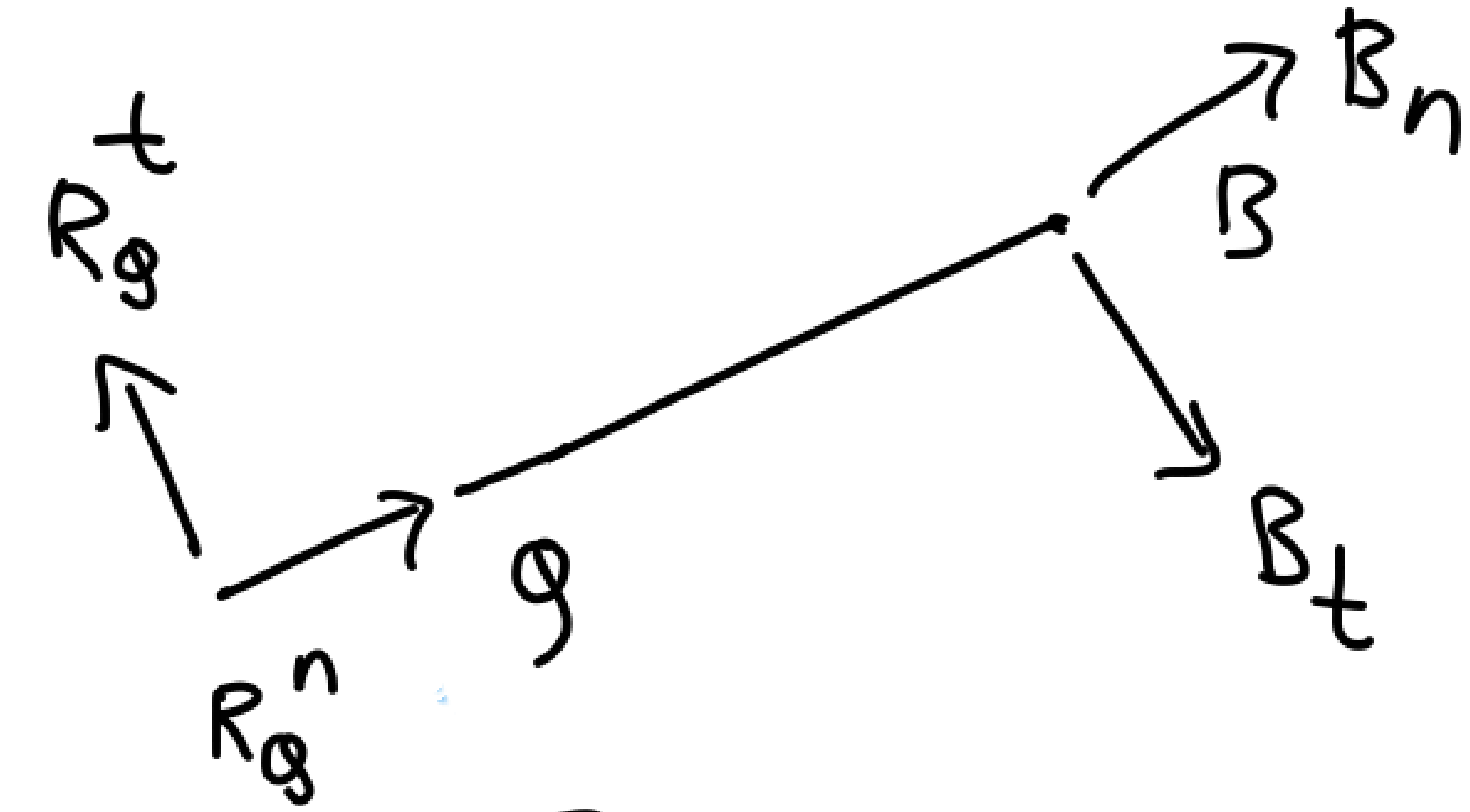
(Reaction forces)

"AP" is under static equilibrium.

$$\Sigma M_p = 0 \Rightarrow M - A_t l = 0$$

$$\boxed{A_t = \frac{M}{l}}$$

Link BQ :



$$\Sigma M_Q = 0 \rightarrow \boxed{B_t = 0}$$

$$\rightarrow R_Q^t = 0$$

$$R_Q^n = -B_n$$

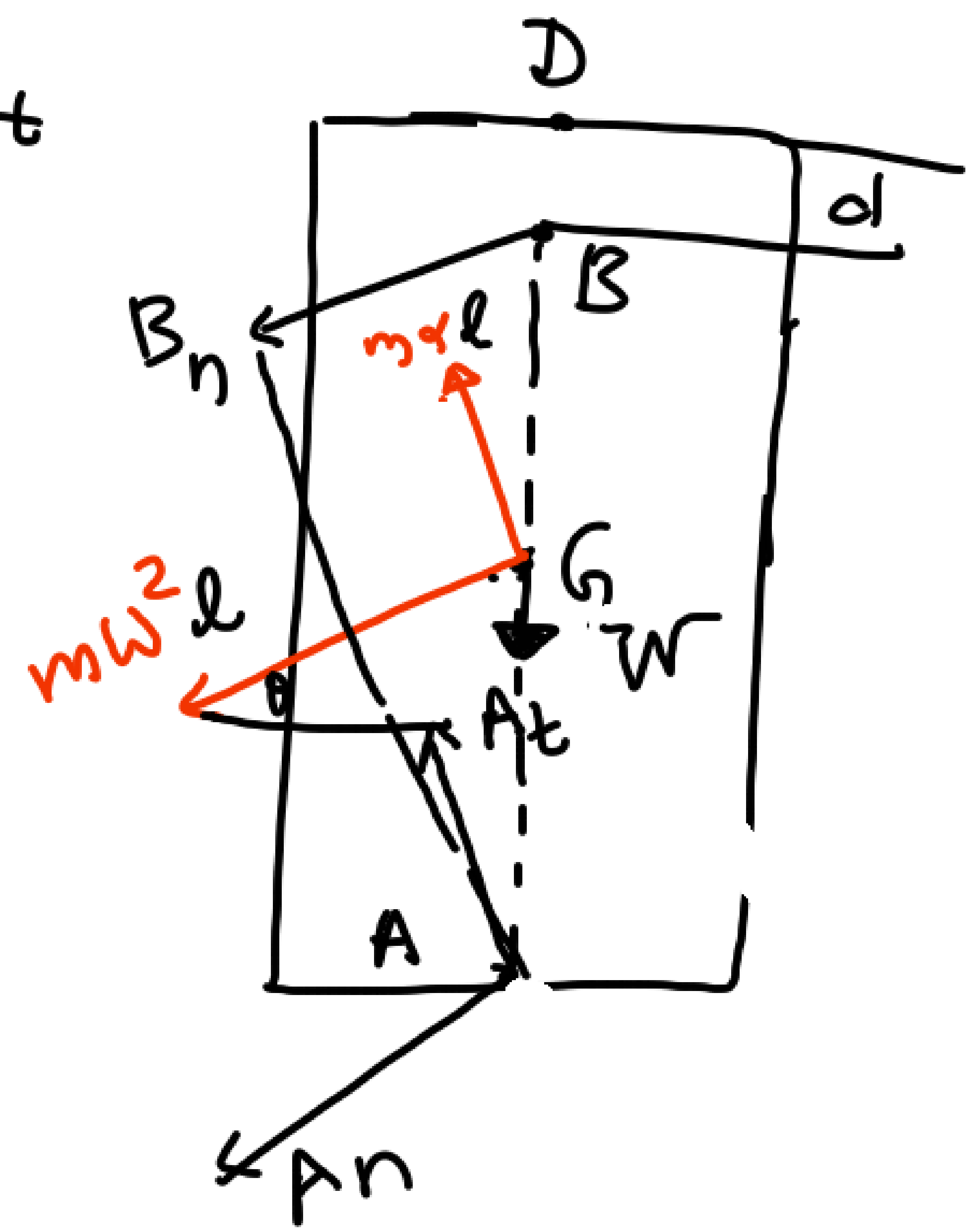
So "BQ" is a two force member.

F.B.D of bar AB :

Known: A_t

Unknown:

A_n, B_n



$AB = L$

$AG = \frac{(L+d)}{2}$

Calculation of α

Acceleration of G

G follows the circular path.

so $\underset{\sim}{\bar{a}} = \omega^2 l \underset{\sim}{\hat{n}} + \alpha l \underset{\sim}{\hat{t}}$

$$\Sigma \underset{\sim}{F} = m \underset{\sim}{\bar{a}}$$

$$(\Sigma \underset{\sim}{F})_n \underset{\sim}{n} + (\Sigma \underset{\sim}{F})_t \underset{\sim}{\hat{t}} = m\omega^2 l \underset{\sim}{\hat{n}} + m\alpha l \underset{\sim}{\hat{t}}$$

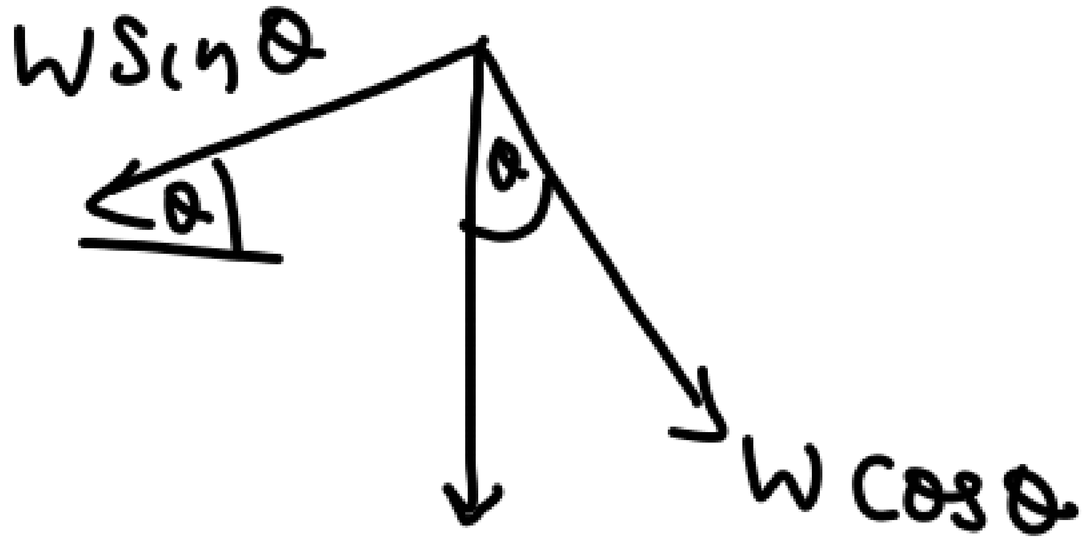
Focussing on tangential part;

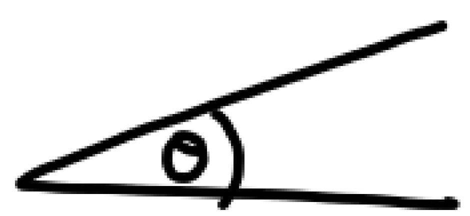
$$(\Sigma \underset{\sim}{F})_t = m\alpha l$$

$$(\Sigma \underset{\sim}{F})_t = A_t - W\cos\theta$$

$$A_t - W\cos\theta = m\alpha l$$

$$\alpha = \frac{A_t - W\cos\theta}{m l}$$





$$\alpha = \frac{d\omega}{dt} = \frac{M}{ml^2} - \frac{W}{ml}\cos\theta$$

$$\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\left(\frac{d\theta}{dt}\right) = \omega\frac{d\omega}{d\theta}$$

$$\omega\frac{d\omega}{d\theta} = \frac{M}{ml^2} - \frac{W}{ml}\cos\theta$$

Integrating both sides

$$\frac{\omega^2}{2} = \frac{M}{ml^2}\theta - \frac{W}{ml}\sin\theta + C_1$$

$$\omega(\theta = 0) = 0 \Rightarrow C_1 = 0$$

$$\omega = \sqrt{\frac{M\theta}{ml^2} - \frac{W}{ml}\sin\theta}$$

<u>Using the normal part:</u>

$$(\Sigma \underset{\sim}{F})_n = m\omega^2 l$$

$$A_n + B_n + W\sin\theta = m\omega^2 l$$

This is not helpful since both A_n and B_n are unknowns.

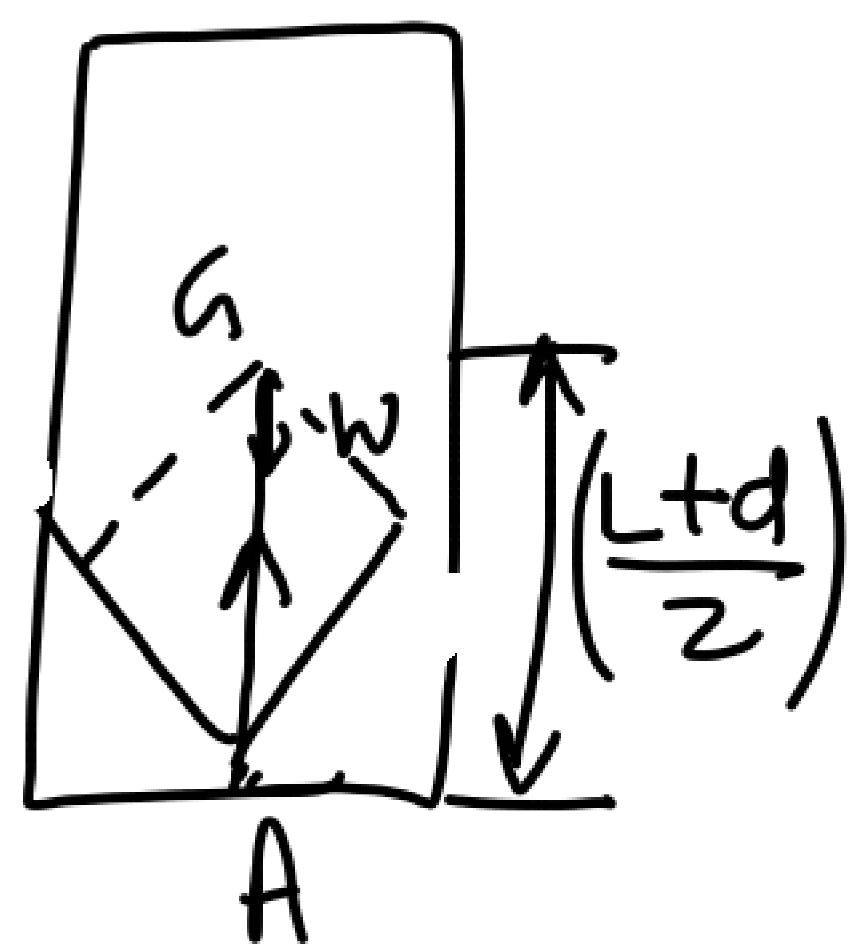
Moment balance about point A;

$$(\Sigma M)_A = (\bar{\rho} \times m\bar{a})_3 \quad (*)$$

$$(\Sigma M_A) = (L\sin(90-\theta)\,\hat{t} \times A_n\hat{n}) = -A_n L\cos\theta\,\hat{e}_3$$

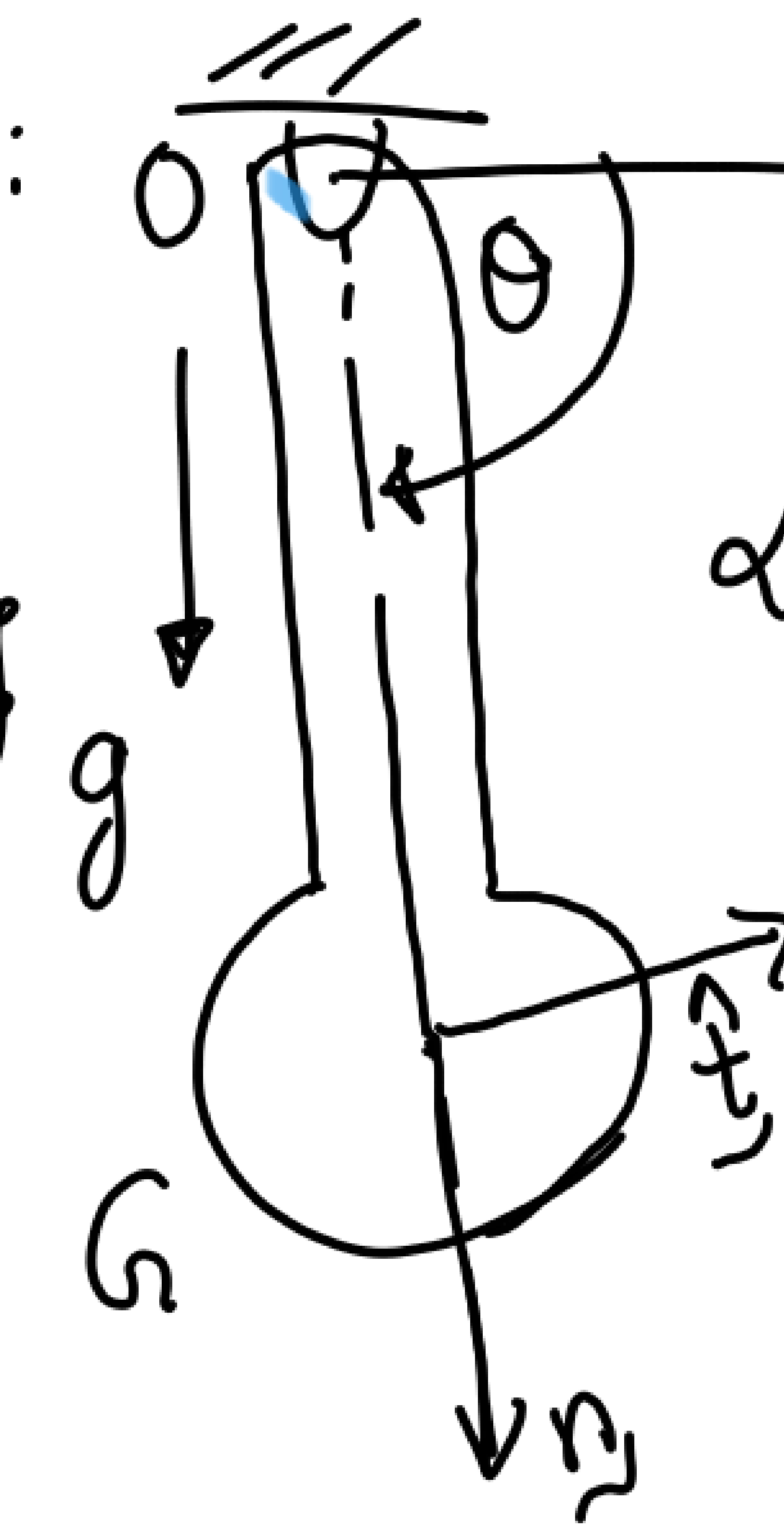
$$|\bar{\rho}| = \frac{(L+d)}{2}$$

$\bar{\rho}$ needs to be resolved along n and t direction



Substituting into (*), will give us A_n.

Example 2:



$OG = \bar{r}$

G } Centre of mass

k = Radius of gyration

$I_G = mk^2$

Pendulum is hanged at O. Initial position is $\theta = 0°$ i.e. horizontal

α and ω as a function of θ;

This is an example of rotation;

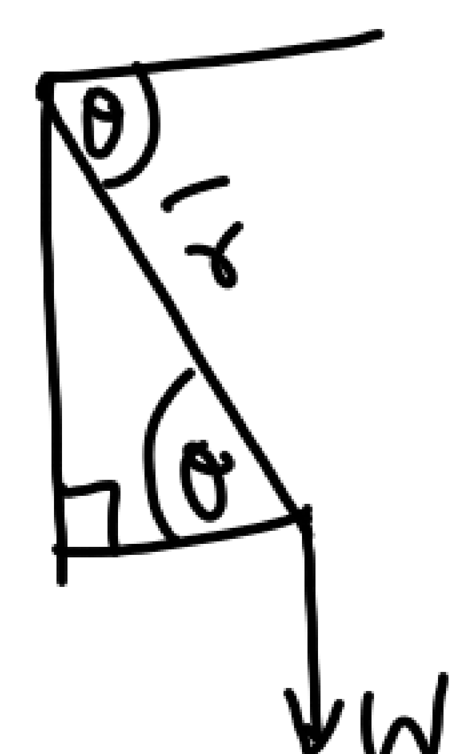
$$\underset{\sim}{\bar{a}} = -\omega^2 \bar{r}\,\underset{\sim}{n} - \alpha \bar{r}\,\underset{\sim}{\hat{t}}$$

$$\Sigma \underset{\sim}{F} = m\,\underset{\sim}{\bar{a}}$$

$$(\Sigma M)_0 = I_0 \alpha$$

$$I_0 = I_G + m(\bar{r})^2 = m\left(k^2 + (\bar{r})^2\right)$$

F.B.D:



$$\Sigma M_0 = (W\bar{r}\cos\theta)$$

↳ Should the sign be positive or negative

$$\alpha = \frac{(\overset{?}{\text{sign}})\, W\bar{r}\cos\theta}{I_0}$$

↓

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$$

$$\omega\frac{d\omega}{d\theta} = \frac{W\bar{r}\cos\theta\,(\text{sign})}{I_0}$$

Integration will give us $\omega(\theta)$