CS207 Midsem Solutions

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Question 8

- The most common folly seen was the answer 2^n but this is untrue. The newest great circle added would not be able to cut all the previous regions into two.
- This is similar to the case in 2D where the new line does not intersect all the previous regions.
- So, we use the fact that all great circles intersect twice on the surface of the sphere, draw it out as a sanity check.
- Now, we look at the n^{th} great circle which was inserted. The previous $n-1$ great circles would have intersected this circle twice resulting in $2(n-1)$ "cuts" on this great circle.
- The corresponding $2(n-1)$ regions subtended between these cuts precisely correspond to the regions that have been split on addition of this new great circle.
- Hence, we get the recursion $a_n = a_{n-1} + 2(n-1)$
- The question asked to count this number, and this recursion relation is simple enough to unroll with base case $a_1 = 2$ to obtain the final answer as $n^2 - n + 2$

Question 9

- This recursion turns out to be the famous catalan recursion and let us call the number of making the n^{th} matrices as c_n

- Let us introduce some notation for the matrix

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{bmatrix}$$

It is clear that $a_1 = 1$ and $b_n = 2n$ (check this!!)
Now we look at the smallest i such $b_i = 2i$. This might as well produce the value of $i = n$ as we just showed that $b_n = 2n$, but it could be some smaller value so let us work with that. Since $b_i = 2i$, we can guarantee that all the numbers in the submatrix

$$\begin{bmatrix} a_1 & a_2 & \dots & a_i \\ b_1 & b_2 & \dots & b_i \end{bmatrix}$$

must be exactly the numbers 1 to $2i$ (prove this using the constraints we have on the matrix). On the other hand in the other submatrix

$$\begin{bmatrix} a_{i+1} & a_{i+2} & \dots & a_n \\ b_{i+1} & b_{i+2} & \dots & b_n \end{bmatrix}$$

we have the numbers $2i+1$ to $2n$. There two submatrices are independent of each other and the arrangement in one is not affecting the other as the value of each element of the second is greater than the first submatrix. Hence, in the second submatrix we look at the number of possible of total arrangements as simply c_{n-i}

However, in the first submatrix we see that we cannot simply count this as c_i as we have constraints on the b_j in the fact $b_j > 2j$, so we do the following construction.
Notice that in this submatrix $a_{j+1} < b_j$ for all indices j in the submatrix. This is so because otherwise we will run out of elements to fill the submatrix incase this condition was not satisfied (do convince yourselves of this). Hence we can transform the submatrix as such by cutting out 1 column as follows:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_i \\ b_1 & b_2 & \dots & b_i \end{bmatrix} \rightarrow \begin{bmatrix} a_2 - 1 & a_3 - 1 & \dots & a_i - 1 \\ b_1 - 1 & b_2 - 1 & \dots & b_{i-1} - 1 \end{bmatrix}$$

What this was basically doing was shifting the top layer of the matrix to the left by 1, this will still respect the conditions since $a_{j+1} < b_j$ for all indices j in the submatrix. Now we have no constraint on bottom values of the submatrix and this can simply be treated as c_{i-1} as now the subbmatrix only has $i-1$ columns.

Now, we can finally come up with the whole recursion. If indeed i was the first index where $b_i = 2i$ for the first time, then we know that the number of matrices for this case is $c_{n-i}c_{i-1}$. But this could have happened for any index i in 1 to n. Hence finally we come up with

$$c_n = \sum_{i=1}^{n} c_{n-i}c_{i-1}$$

Looking at the small values, we realize that c_0 should be initialized as 1