Latin Square Designs

Example:

Four cars and four drivers are employed to study the possible differences among four gasoline additives.

Even though we may use four identical cars and drivers that may have similar skills, slight systematic differences can occur from driver to driver.

The **Latin square design** arrangement could be used to study this problem. The design is defined as follows:

		Cars			
		1	2	3	4
Drivers	I	A	B	D	C
	II	D	C	A	B
	III	B	D	C	A
	IV	C	A	B	D

Additives (primary variable):

A, B, C, D

It is assumed that there is NO interaction among cars, drives and additives

Latin Square Designs

Additives (primary variable):

A, B, C, D

		Cars			
		1	2	3	4
Drivers	I	A	B	D	C
	II	D	C	A	B
	III	B	D	C	A
	IV	C	A	B	D

- Note that each additive appears EXACTLY ONCE in each row and column
- The Latin Square Designs are NOT unique. For example,

A	B	D	C
D	C	A	B
B	D	C	A
C	A	B	D

A	B	C	D
D	A	B	C
C	D	A	B
B	C	D	A

A	B	C	D
D	A	B	C
B	C	D	A
C	D	A	B

A	B	C	D
C	D	A	B
D	C	B	A
B	A	D	C

Latin Square Designs

Additives (primary variable): A, B, C, D

		Cars				avg ↓
		1	2	3	4	
Drivers	I	A 21	B 26	D 20	C 25	
	II	D 23	C 26	A 20	B 27	
	III	B 15	D 13	C 16	A 16	
	IV	C 17	A 15	B 20	D 20	
avg →						

Grand Mean

Averages for Additives

A =

B =

C =

D =

ANOVA: Latin Square Design

p = number of levels of each blocking variable

Source	SSQ	DoF	MS	F ratio
Cars	24	4-1=3 =p-1	8	3.00
Drivers	216	4-1=3=p-1	72	27
Additives	40	4-1=3=p-1	13.35	5.00 ✓
Average	6400	1		
Residual	16 (by sub.)	6=(p-2)(p-1)	2.67	
Total	6696	$16 = p^2$		

$$SS_{mean} = 16(20)^2 = 6400$$

$$SS_{cars} =$$

Mathematical Model of Latin Square Design

$$x_{ijk} = \mu + \tau_i + \beta_j + \gamma_k + \varepsilon_{ijk} \qquad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

Observed Value | Grand mean | Effect of treatment | Effect of block 1 | Effect of block 2 | Error

$\varepsilon_{ijk} \sim NID(0, \sigma^2)$

$$SS_T = SS_{Mean} + SS_{treatment} + SS_{block\,1} + SS_{block\,2} + SS_E$$

DOF: p^2 1 $p-1$ $p-1$ $p-1$ $(p-2)(p-1)$

ANOVA of Latin Square Design

Analysis of Variance for the Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}} = \frac{1}{p}\sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{N}$	p^2-1		

Example:

treatment

Consider the rocket propellant problem, where the goal is to study the effect of five different formulations (A, B, C, D, E). Each formulation is mixed from a batch of raw material that is only large enough for five tests. Furthermore, the formulations are prepared by several operators and there maybe substantial difference in their skills and experience.

p=5

Coded Data for the Rocket Propellant Problem

Batches of Raw Material	Operators (block 1) 1	2	3	4	5	$y_{i..}$
1	$A = -1$	$B = -5$	$C = -6$	$D = -1$	$E = -1$	-14
2	$B = -8$	$C = -1$	$D = 5$	$E = 2$	$A = 11$	9
Block 2 3	$C = -7$	$D = 13$	$E = 1$	$A = 2$	$B = -4$	5
4	$D = 1$	$E = 6$	$A = 1$	$B = -2$	$C = -3$	3
5	$E = -3$	$A = 5$	$B = -5$	$C = 4$	$D = 6$	7
$y_{..k}$	-18	18	-4	5	9	$10 = y_{...}$

Latin Square Design

Example:

Coded Data for the Rocket Propellant Problem

Batches of Raw Material	Operators 1	2	3	4	5	$y_{i..}$	avg
1	$A = -1$	$B = -5$	$C = -6$	$D = -1$	$E = -1$	-14	-2.8
2	$B = -8$	$C = -1$	$D = 5$	$E = 2$	$A = 11$	9	1.8
3	$C = -7$	$D = 13$	$E = 1$	$A = 2$	$B = -4$	5	1
4	$D = 1$	$E = 6$	$A = 1$	$B = -2$	$C = -3$	3	0.6
5	$E = -3$	$A = 5$	$B = -5$	$C = 4$	$D = 6$	7	1.4
$y_{..k}$	-18	18	-4	5	9	$10 = y_{...}$	
	-3.6	3.6	-0.8	1	1.8	0.4	

averages for treatments

$A = \frac{(-1+5+1+2+11)}{5}$

$B =$

$C =$

$D =$

$E =$

$$SS_T = \sum\sum x_{ij}^2$$

$$SS_{mean} = 5^2 \left(10/25\right)^2$$

$$SS_{raw\,mat} = 5\left((-3.6-0.4)^2 + (3.6-0.4)^2 + (-0.8-0.4)^2 + (1-0.4)^2 + (1.8-0.4)^2\right)$$

$SS_{op} =$

$SS_{formulation} =$

Latin Square Design

Example:

Coded Data for the Rocket Propellant Problem

Batches of Raw Material	Operators 1	2	3	4	5	$y_{i..}$
1	$A = -1$	$B = -5$	$C = -6$	$D = -1$	$E = -1$	-14
2	$B = -8$	$C = -1$	$D = 5$	$E = 2$	$A = 11$	9
3	$C = -7$	$D = 13$	$E = 1$	$A = 2$	$B = -4$	5
4	$D = 1$	$E = 6$	$A = 1$	$B = -2$	$C = -3$	3
5	$E = -3$	$A = 5$	$B = -5$	$C = 4$	$D = 6$	7
$y_{..k}$	-18	18	-4	5	9	$10 = y_{...}$

■ **TABLE 4.12**

Analysis of Variance for the Rocket Propellant Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Formulations	330.00	4	82.50	7.73	0.0025
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Error	128.00	12	10.67		
Total	676.00	24			

Replicates in Latin Square Design

- The residual number of degrees of freedom in Latin square design is given by (k-1)(k-2), where k is the size of Latin Square. Hence, if one uses a small Latin square design, the error degrees of freedom would be small.

Size of Latin square (k)	2	3	4	5	6
Residual DoF (r) (k-1)(k-2)	0	2	6	12	20

- **Therefore, if a small Latin Square Design is used, it is desirable to replicate the trials.**
- There are three ways to replicate the Latin Square Designs

1. Use the same blocks (e.g., cars and operators) in each replicate
2. One of the blocks is same (e.g. cars), but the other block is different
3. Both the blocks are different in replicates

Replicates in Latin Square Design

Case 1: Use the same blocks (e.g., cars and operators) in each replicate (p replicates)

Let y_{ijkl} be the observation in row i, treatment j, column k, and replicate l. There are $N = np^2$ total observations

Analysis of Variance for a Replicated Latin Square, Case 1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{np}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p+1)-3]$	$\frac{SS_E}{(p-1)[n(p+1)-3]}$	
Total	$\sum\sum\sum\sum y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Replicates in Latin Square Design

DIY

Case 2: Same batches but different operators in each replicate (or same operator different batches)

Analysis of Variance for a Replicated Latin Square, Case 2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)(np-1)$	$\frac{SS_E}{(p-1)(np-1)}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Replicates in Latin Square Design

DIY

Case 3: Different batches and different operators in each replicate

Analysis of Variance for a Replicated Latin Square, Case 3

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{p}\sum_{l=1}^{n}\sum_{k=1}^{p} y_{..kl}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Columns}}}{n(p-1)}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p-1)-1]$	$\frac{SS_E}{(p-1)[n(p-1)-1]}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Missing Data in Latin Square Design

DIY

Estimate the missing data such that its contribution to SS_E is minimum

$$X_{ijk} = \frac{k\left(\overset{\text{Row}}{X'_{i\cdot\cdot}} + \overset{\text{Column}}{X'_{\cdot j\cdot}} + \overset{\text{Third variable}}{X'_{\cdot\cdot k}}\right) - 2\overset{\text{Total sum of data}}{X'_{\cdot\cdot\cdot}}}{(k-2)(k-1)}$$