Interaction Effects

	Thermometer			
Analyst	A	B	C	D
1	2.0 1.5	1.0 1.5	−0.5 0.5	1.5 1.5
2	1.0 1.0	0.0 1.0	−0.5 0.0	−1.0 0.0
3	1.5 1.0	1.0 1.5	1.0 1.0	0.5 1.0

We can find column/row means after finding mean of repeated trials first

	Thermometer				
Analyst	A	B	C	D	Mean
1	2.0 1.75 1.5	1.0 1.25 1.5	−0.5 0.00 0.5	1.5 1.50 1.5	1.125
2	1.0 1.00 1.0	0.0 0.50 1.0	−0.5 -0.50 -0.5	−1.0 -0.50 0.0	0.125
3	1.5 1.25 1.0	1.0 1.25 1.5	1.0 1.00 1.0	0.5 0.75 1.0	1.063
Mean	1.333	1.000	0167	0.583	0.771

Mathematical Model

$$x_{ijk} = \mu + \tau_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$$

Observed Value — Grand mean — Effect of factor 1 — Effect of factor 2 — "Effect of Interaction" — Residual Error

here, $i = 1, 2, \ldots, a$ (number of factor 1) $\quad j = 1, 2, \ldots, b$ (number of factor 2) $\quad k = 1, 2, \ldots, r$ (number of replicates)

$$x_{ijk} = \bar{x}_{...} + (\bar{x}_{i.} - \bar{x}_{...}) + (\bar{x}_{.j} - \bar{x}_{...}) + \underbrace{(\bar{x}_{ij.} - \bar{x}_{i.} - \bar{x}_{.j} - \bar{x}_{...})}_{\text{"Interaction term"}} + (x_{ijk} - \bar{x}_{...})$$

Mean Values

Grand Mean $= \bar{\bar{x}}_{...} = \dfrac{\sum_i \sum_j \sum_k x}{N}$ here, $N = abr$

Means of replicates $= \bar{x}_{ij\cdot} = \sum_k x_{ij} / r$

Means of factor 1 $= \bar{x}_{i\cdot} = \sum_j \bar{x}_{ij\cdot} / b$

Means of factor 2 $= \bar{x}_{\cdot j} = \sum_i \bar{x}_{ij\cdot} / a$

	Thermometer					
Analyst	A	B	C	D	Mean	
1	2.0 1.75 1.5	1.0 1.25 1.5	−0.5 0.00 0.5	1.5 1.50 1.5	1.125	
2	1.0 1.00 1.0	0.0 0.50 1.0	−0.5 -0.50 -0.5	−1.0 -0.50 0.0	0.125	
3	1.5 1.25 1.0	1.0 1.25 1.5	1.0 1.00 1.0	0.5 0.75 1.0	1.063	
Mean	1.333	1.000	0167	0.583	0.771	

Sum of Squares $SS_T = SS_M + SS_1 + SS_2 + SS_{interaction} + SS_{Error}$

found by subtraction

Total $SS_T = \sum_i \sum_j \sum_k x_{ijk}$

Grand Mean $SS_M = N\bar{x}_{...}^2 = abr\,\bar{x}_{...}^2$

Factor 1 $SS_1 = (rb)\sum_{i=1}^{a} (\bar{x}_{i.} - \bar{x}_{...})^2$

Factor 2 $SS_2 = (ra)\sum_{j=1}^{b} (\bar{x}_{.j} - \bar{x}_{...})^2$

Pure error $SS_E = \sum_i \sum_j \sum_{k=1}^{r} (x_{ijk} - \bar{x}_{ij.})$

ANOVA Table

$$SS_{analyst} = 2\times4[(1.125-0.771)^2+(0.125-0.771)^2+(1.063-0.771)^2]$$
$$=5.021$$

$$SS_{therm} = 6[(1.333-0.771)^2+(1.000-0.771)^2+(0.167-0.771)^2+(0.583-0.771)^2]$$
$$=4.615$$

$$SS_E = [(2.0-1.75)^2+(1.5-1.75)^2]+[(1.0-1.25)^2+(1.5-1.25)^2]+\cdots$$
$$+[(0.5-0.75)^2+(1.0-0.75)^2]$$

	Thermometer				
Analyst	A	B	C	D	Mean
1	2.0 1.75 1.5	1.0 1.25 1.5	−0.5 0.00 0.5	1.5 1.50 1.5	1.125
2	1.0 1.00 1.0	0.0 0.50 1.0	−0.5 -0.50 -0.5	−1.0 -0.50 0.0	0.125
3	1.5 1.25 1.0	1.0 1.25 1.5	1.0 1.00 1.0	0.5 0.75 1.0	1.063
Mean	1.333	1.000	0167	0.583	0.771

Source	SSQ	DoF	M.S.	*F*
Analyst	5.021	3-1=2	2.51	15.06
Thermometer	4.615	4-1=3	1.54	9.24
Interaction	3.354 (By subtraction)	6	0.56	3.354
Grand mean	14.26	1		
Pure Error	2.00	12	0.167	
Total	29.25	24		

- $F_{0.95}(2, 12) = 3.89$
- $F_{,0.95}(3, 12) = 3.49$
- $F_{0.95}(6, 12) = 3.00$

Conclusions

- Significant differences in analysts and thermometers
- Significant interaction effect

Chapter 2.4

Classical Design of Experiments

Factorial Design

Factorial Design

- When we wish to compare two different techniques, we may either use the independent t-test, or we may use a paired t-test if a nuisance variable is to be blocked away.
- Suppose now that we want to compare several techniques, the independent t-tests would no longer be adequate and **a k-variable** analysis should be used.
- There are many experimental designs available which may be chosen to suit particular experimental situations. But these designs, e.g., k-variable analysis, involve certain assumptions and restrictions.
- We will now introduce a general and effective class of experimental design called the **Factorial Design which includes k-variable analysis**, etc. as a special case.
- **To develop a general factorial design, one would select a fixed number of "levels" for each of a number of variables (factors) and then design tests with all possible combinations.**

For example, if there are 'L_1' levels of variable 1, 'L_2' levels of variable 2, and 'L_3' levels of variable 3, …

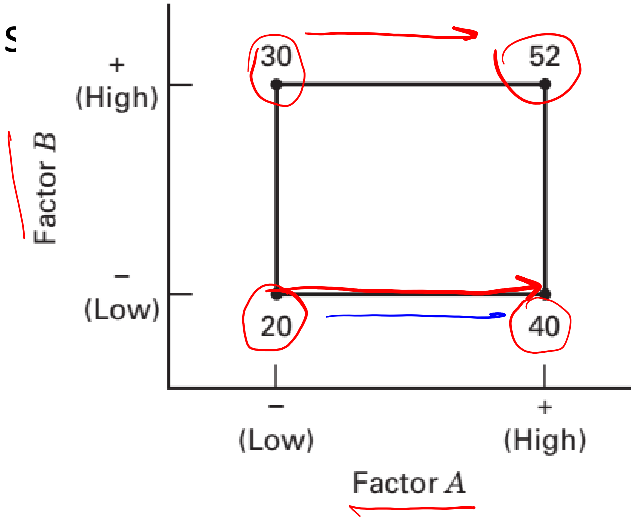
and are 'L_k' levels of variable k; then the total number of observations required will be $L_1 \times L_2 \times \cdots \times L_k$

Factorial Designs

- Factorial designs are preferred over other experimental designs because :
  - They require relatively few runs per variable (factor) even though they do not explore the entire region of interest
  - They deal easily with variable interactions
  - hey can indicate major trends and help determine a possible direction for further experimentation
  - They can be augmented to form composite designs
  - They form the basis for fractional factorial designs
  - The interpretation of the data produced by the designs can be easily analyzed.
- **Variables and Levels**
  - A design with 3 variables at 2 levels would require 8 tests for a 2 x 2 x 2 = 2^3 factorial design
  - A design with 3 variables at 3 levels would need 27 tests for a 3 x 3 x 3 = 3^3 factorial design.
  - A design with 5 variables at 2 levels would need 32 tests for a 2^5 factorial design.

Factorial Design: Example

- Consider a factorial experimental design with 2 variables (A and B) each at 2 levels
- Since there are only two levels of each variable, we can call the 'low' and 'high'



What is the 'main effect' of variable A? i.e., how does the result change when you change variable A from low setting to high?

factor B is low +20
factor B is high +22] +21

What is the 'main effect' of variable B? i.e., how does the result change when you change variable A from low setting to high?

factor A is low +10
factor A is high +12] +11

The effect of a factor is defined to be the change in response produced by a change in the level of the factor. This is frequently called the **main effect** because it refers to the primary factors of interest in the experiment.

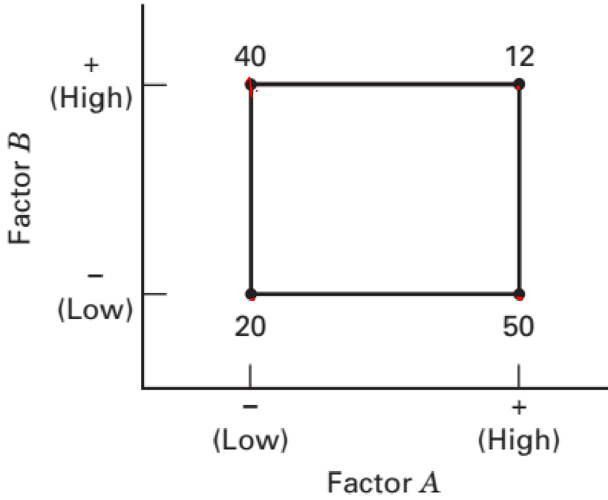
Interaction Effect

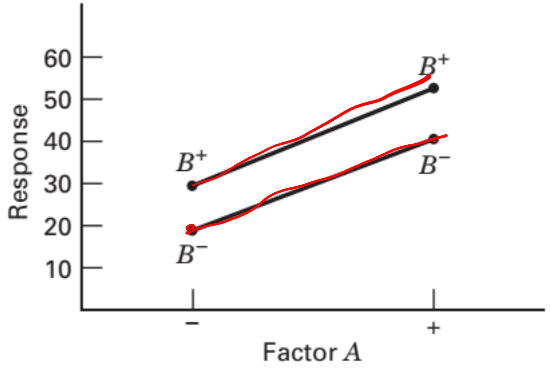
- In some experiments, we may find that the difference in **response between the levels of one factor is not the same at all levels of the other factors**. When this occurs, there is an **interaction between the factors**.
- **Interaction Effect**

At Low level of B, effect of A is = +30

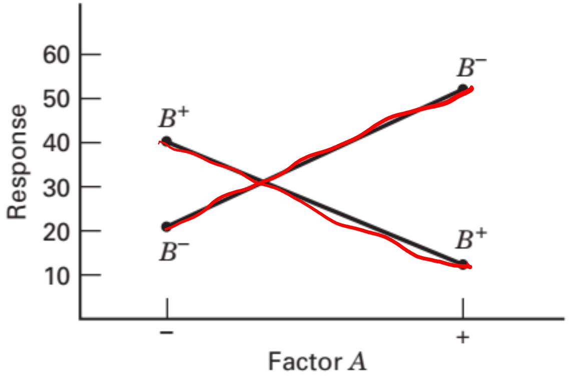
At High level of B, effect of A is = −28

+1





■ **FIGURE 5.3** A factorial experiment without interaction



■ **FIGURE 5.4** A factorial experiment with interaction

Example

Welding of High Strength Steel Bars

- High carbon steel, because of its high strength and low cost, has been known to have a potential for a "good market". However, because of its high carbon content, it is not easy to weld.
- According to the code of the American Welding Society (AWS), additional steps of pre-heating and post-heating are required in order to have good quality welds and high strength steel.
- A user of this steel was interested to study whether or not these additional steps of pre-heating and post-heating were really needed.
- After a preliminary investigation by manual arc welding tests, it appeared that there were **three variables significantly affecting the ultimate tensile stress of a weld**.
    1. Ambient temperature,
    2. Wind velocity
    3. Bar size
- The evidence, however, was not decisive, and further experiments were therefore planned. **How many tests should be conducted?**

Example

Example 2: Welding of High Strength Steel Bars

- Based upon available funds and time limit, it was decided that 244 tests can be run (maximum).
- In the meantime, an engineering statistician, who was called upon for consultation, suggested that 16 tests be run according to his specified sets of test conditions.
- The statistical experimental design that was formulated by the engineering statistician for this study was **a two-level three-variable factorial design**, simply designated as a 2^3 factorial design.
- The three selected variables were:
  - Ambient temperature, denoted by T
  - Wind velocity, denoted by V
  - Bar size, denoted by B
- Two levels ('low' and 'high') were chosen for each variable based upon desired field conditions to be simulated.

$2 \times 2 \times 2 = 8 = 2^3$

1

repeated 2

Example

Welding of High Strength Steel Bars

- The experimental variables and their two levels (low and high) are shown in the Table.

Variable	Unit	Low	High
Ambient Temperature (T)	°F	0	70
Wind Velocity (V)	mph	0	20
Bar Size (B)	1/8 inch	4	11

-1 +1
-1 +1
-1 +1

Transformation of Levels

- In order to adopt a notation which will be the same for all two-level factorial designs, we use transforming equations to code the variables such that
  - the high level will be denoted by +1,
  - the low level will be denoted by -1.

$$x_T = \frac{T - T_{avg}}{T_{avg}}$$

$$x_T = \frac{T - T_a}{T_a}$$

- By so doing, regardless of the physical conditions represented by the two levels, the basic design of any two-level factorial design becomes a simple arrangement of +1 and -1.

Example

- Construction of 2^k Factorial Design

	Coded Test Conditions			Actual Test Conditions		
Test #	X1	X2	X3	°F	(mph)	(1/8 in)
1	-1	-1	-1	0	0	4
2	1	-1	-1	70	0	4
3	-1	1	-1	0	20	4
4	1	1	-1	70	20	4
5	-1	-1	1	0	0	11
6	1	-1	1	70	0	11
7	-1	1	1	0	20	11
8	1	1	1	70	20	11

Example

Results of the factorial experiment with 2 repetitions (total 8*2 = 16 trials)

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

Example

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

$\Delta y_{1\to 2} = 87.3 - 87.5 = -0.2$

$\Delta y_{3\to 4}$ 9.2

$\Delta y_{5\to 6}$

$\Delta y_{7\to 8}$

avg

Average effect of Temperature (E1)

- Note that between Test 1 and Test 2, Wind (X2) and Bar Size (X3) is same. The only change is that Temperature (X1) goes from LOW (-1) to HIGH (+1). Therefore, the difference in these two test results (apart from the intrinsic variation that is present) can be attributed solely to the effect of ambient temperature alone.
- Similarly, for the pairs of Test 3 and 4, 5 and 6, and 7 and 8 in Table; each pair involved similar test conditions with respect to wind velocity and bar size, but different test conditions with respect to ambient temperature. Thus, the differences in the results within each of these four pairs reflect the effect of ambient temperature alone.

Example

Test #	*X1*	*X2*	*X3*	*Test Order*	*Y_{ai} (kpsi)*	*Test Order*	*Y_{bi} (kpsi)*	*Average (kpsi)*
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

$$E_1 = \frac{-0.2 + 9.2 + 18.5 + 9.1}{4}$$

$$(Test\ Nos.\ 1\ and\ 2)\ \ \bar{y}_2 - \bar{y}_1 = 87.3 - 87.5 = -0.2$$

$$(Test\ Nos.\ 3\ and\ 4)\ \ \bar{y}_4 - \bar{y}_3 = 87.0 - 77.8 = 9.2$$

$$(Test\ Nos.\ 5\ and\ 6)\ \ \bar{y}_6 - \bar{y}_5 = 97.6 - 79.1 = 18.5$$

$$(Test\ Nos.\ 7\ and\ 8)\ \ \bar{y}_8 - \bar{y}_7 = 87.7 - 78.6 = 9.1$$

Average Effect of Temperature would be an average of these FOUR numbers

Main Effect

y_1 y_2 $\bar{y}$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

y

Average effect of Temperature (E1)

- OR we can also write he average effect of ambient temperature (called 'E1') as:

$$E_1 = 1/4[(\bar{y}_2 - \bar{y}_1) + (\bar{y}_4 - \bar{y}_3) + (\bar{y}_6 - \bar{y}_5) + (\bar{y}_8 - \bar{y}_7)]$$
$$= 1/4[-0.2 + 9.2 + 18.5 + 9.1]$$
$$= 9.15 \text{ units of } 1000 \text{ psi}$$
$$= 9150 \text{ psi.}$$

$E_1 = 1/4 \left(\underline{y_2 + y_4 + y_6 + y_8} - \underline{y_1 + y_3 + y_5 + y_7} \right)$

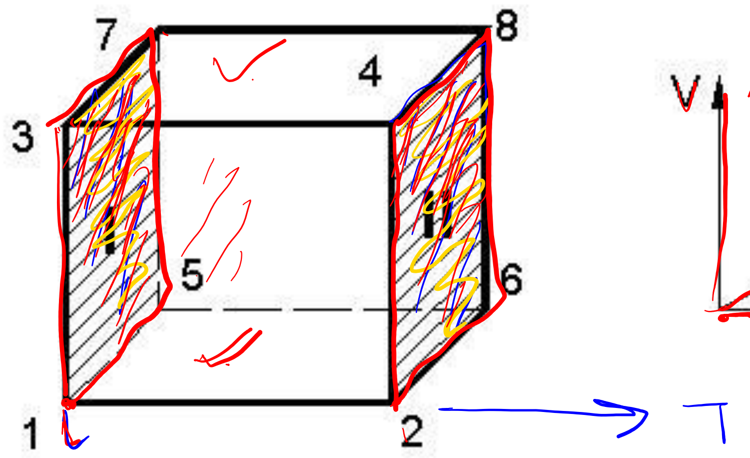
- **Note that the average effect is commonly referred to as *main effect.***

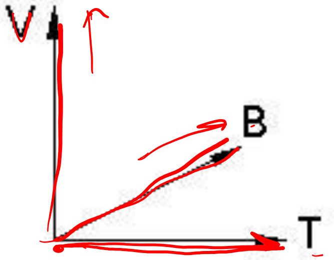
Main Effect

Average effect of Temperature (E1)

- Geometrically, the average effect of Temperature is the differer LOW) and plane 2 (Temp setting HIGH)

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





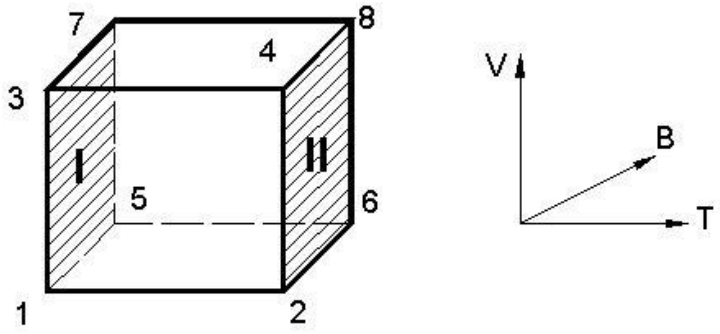
avg when T_H — avg when T_{lo}

$$E_1 = \frac{1}{4}\left[(\bar{y}_2 + \bar{y}_4 + \bar{y}_6 + \bar{y}_8) - \underline{(\bar{y}_1 + \bar{y}_3 + \bar{y}_5 + \bar{y}_7)}\right]$$

high level of ambient temperature — Low level of ambient temperature

Main Effect

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7



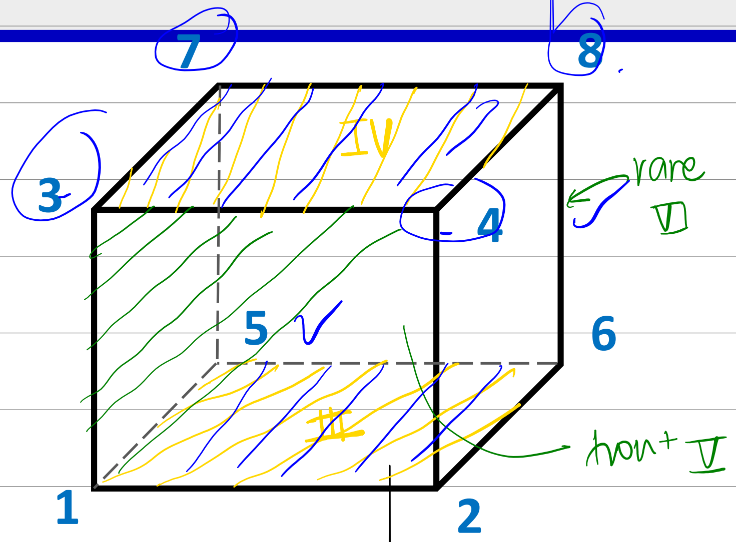
Average effect of Temperature (E1)

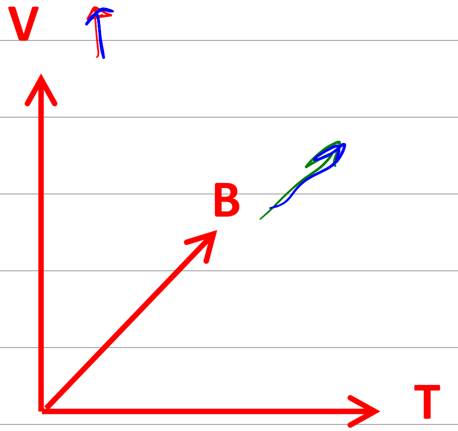
- The average effect of ambient temperature tells us that on the average, over the ranges of other variables in this investigation, the effect of changing the ambient temperature from LOW to HIGH increases the ultimate tensile stress by 9150 psi.
- But, notice that the individual differences (y2 – y1 = -200 psi, y4 – y3 = 9200 psi, y6 – y5 = 18000 psi, y8 - y7 = 9100 psi) are quite erratic.

The average effect, therefore, must be interpreted in conjunction with the intrinsic variabilities that are present in the experimental results.

Main Effect

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





$$E_2 = \frac{1}{4}\left((\bar{Y}_3 - \bar{Y}_1) + (\bar{Y}_4 - \bar{Y}_2) + (\bar{Y}_8 - \bar{Y}_6) + (\bar{Y}_7 - \bar{Y}_5)\right)$$

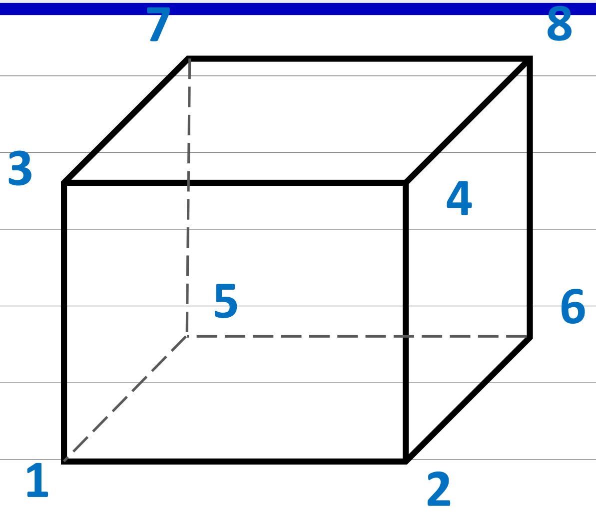
$E_2 =$ avg of plane IV − avg of plane III

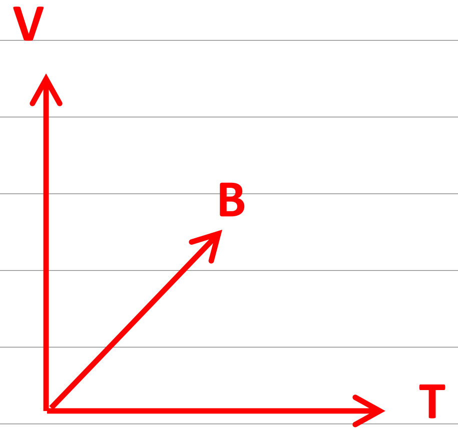
$$= \frac{1}{4}\left(\bar{Y}_3 + \bar{Y}_4 + \bar{Y}_7 + \bar{Y}_8\right) - \frac{1}{4}\left(\bar{Y}_1 + \bar{Y}_2 + \bar{Y}_5 + \bar{Y}_6\right)$$

$E_3 =$ avg of plane VI (rare) − avg of plane V (front)

Interaction Effect

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7



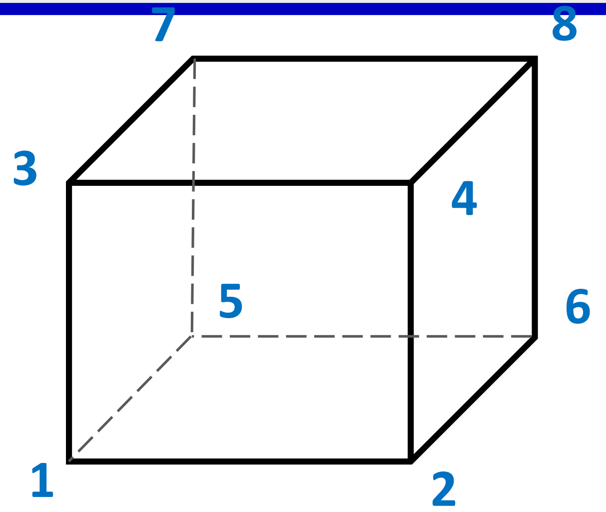


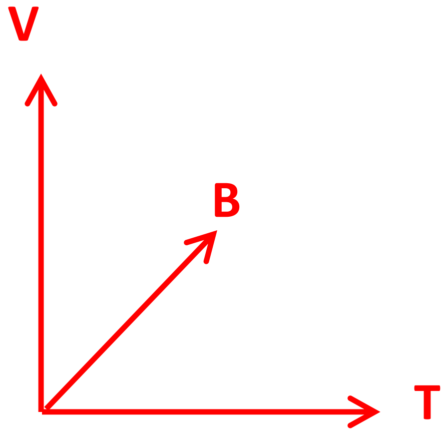
Two-variable Interactions

- The average effects E1, E2, and E3 represent the individual effects of ambient temperature, wind velocity and bar size on the ultimate tensile stress (UTS).
- **But, what about the joint effect of two variables?**
  - Effect of temperature AND wind velocity on the UTS?
  - Effect of wind velocity AND bar size on UTS?
- ***These joint effects are indicated by the two variable interactions.***

Two Variable Interactions

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

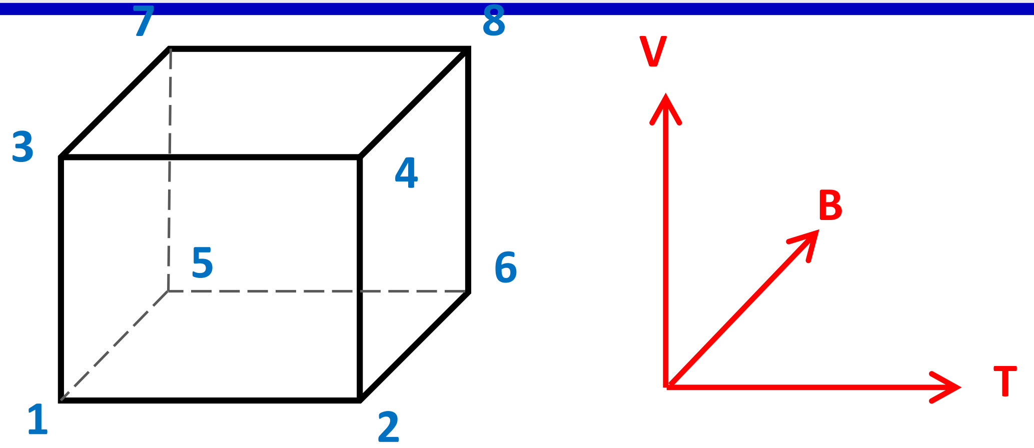




Two-variable Interactions

- There are three two-variable interactions to be calculated,
  - Between temperature and wind velocity, denoted by E_{12}
  - Between wind velocity and bar size, denoted by E_{23}
  - Between bar size and temperature, denoted by E_{31} or E_{13}

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7



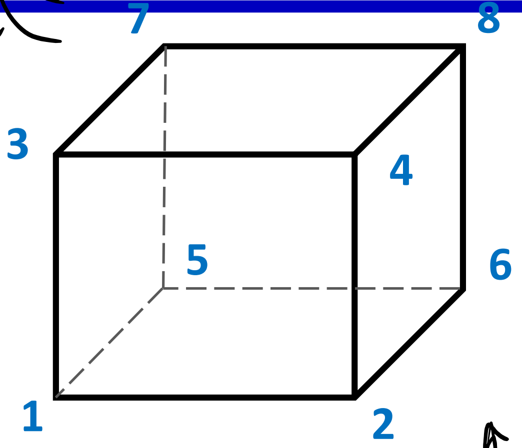
Two-variable Interaction between Temperature and Wind Velocity, E_{12}

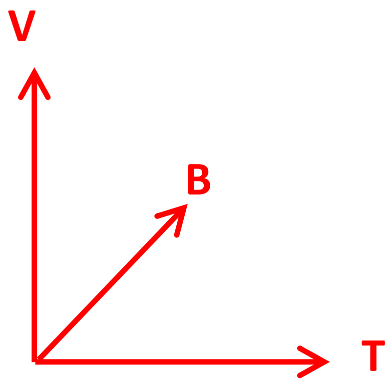
- At HIGH level of wind velocity (X2), the two differences in the result are given by $(\overline{y_4} - \overline{y_3})$ and $(\overline{y_8} - \overline{y_7})$
- Both these differences reflect individual effect of change in UTS due to change in temperature from LOW to HIGH.
- The average of these effects (at HIGH wind velocity) is = $\frac{1}{2}[(\overline{y_4} - \overline{y_3}) + (\overline{y_8} - \overline{y_7})]$
- Similarly, the average change due to temperature at LOW wind velocity is = $\frac{1}{2}[(\overline{y_2} - \overline{y_1}) + (\overline{y_6} - \overline{y_5})]$

E_{12} = avg diff betw effect x_1 across x_2

$= \frac{1}{2}\left((x_2 \text{ is at } -1)\ \Delta \text{Output}_{x_1} - (x_2 \text{ is at } 1)\ \Delta \text{Output}_{x_1}\right)$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





X1 X2

Two-variable Interaction between Temperature and Wind Velocity, E_{12}

- The interaction between ambient temperature and wind velocity is the average difference between these two averages, i.e.,

$$E_{12} = \frac{1}{2}\left\{\frac{1}{2}[(\bar{Y}_2 - \bar{Y}_1) + (\bar{Y}_6 - \bar{Y}_5)] - \frac{1}{2}[(\bar{Y}_4 - \bar{Y}_3) + (\bar{Y}_8 - \bar{Y}_7)]\right\}.$$

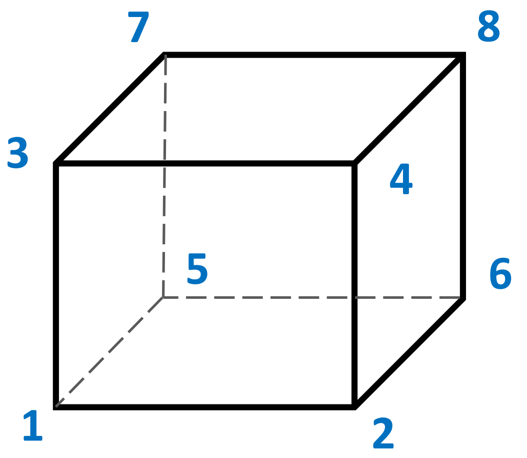
$$= \frac{1}{4}\{(\bar{Y}_1 + \bar{Y}_5 + \bar{Y}_4 + \bar{Y}_8) - (\bar{Y}_2 + \bar{Y}_6 + \bar{Y}_3 + \bar{Y}_7)\}$$

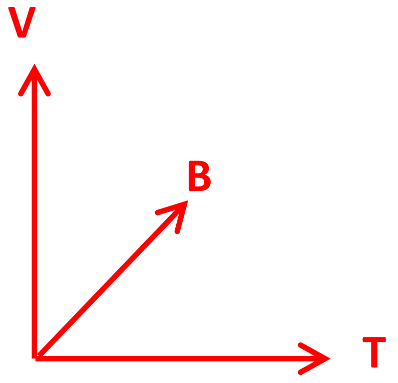
$= 1/4$ ($x_1 x_2$ both high or low − $x_1 x_2$ one of them is larger)

- Note, therefore, that the interaction between ambient temperature and wind velocity tells us the average change in ultimate tensile stress that would occur ~~due to a change from the low level to~~ the high level in both the ambient temperature and wind velocity.

$$E_{12} = \frac{+87.5 - 87.3 - 77.8 + 87 + 79.1 - 97.6 - 78.6 + 87.7}{4}$$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

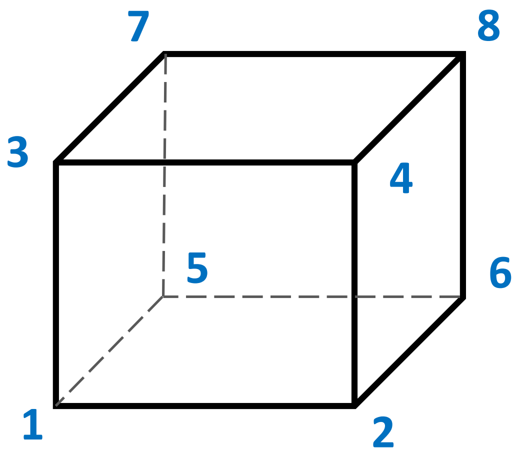


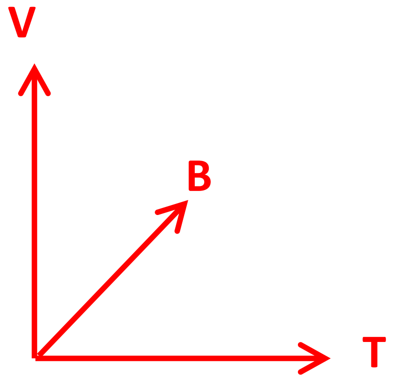


$$E_{13} = \frac{1}{4}(\bar{Y}_1 + \bar{Y}_3 + \bar{Y}_6 + \bar{Y}_8) - \frac{1}{4}(\bar{Y}_2 + \bar{Y}_4 + \bar{Y}_5 + \bar{Y}_7)$$

$$= -4650\,psi$$

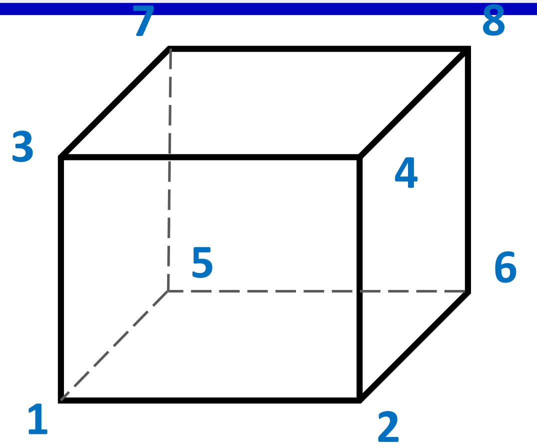
Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

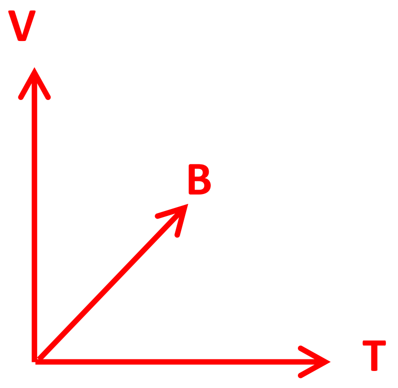




$$
\begin{aligned}
E_{23} &= \frac{1}{4}(\bar{Y}_1 + \bar{Y}_2 + \bar{Y}_7 + \bar{Y}_8) - \frac{1}{4}(\bar{Y}_3 + \bar{Y}_4 + \bar{Y}_5 + \bar{Y}_6) \\
&= \frac{1}{4}(87.5 + 87.3 + 78.6 + 87.7) - \frac{1}{4}(77.8 + 87.0 + 79.1 + 97.6) \\
&= \text{-}0.10 \ \text{ Kpsi} \\
&= \text{-}100 \text{ psi.}
\end{aligned}
$$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





Three-variable Interaction E_{123}

- At HIGH level of bar size (X3), once can calculate the interaction between temperature (X1) and wind velocity (X2)

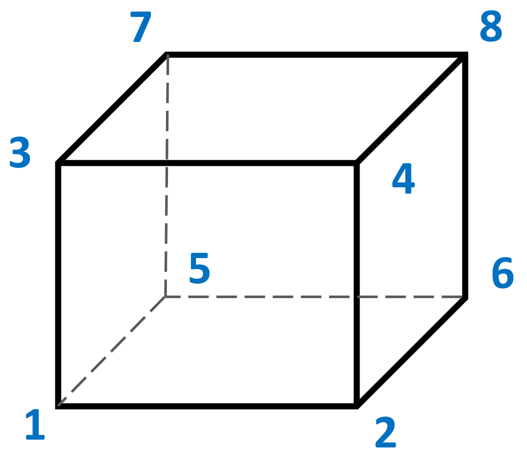
$$\frac{1}{2}[(\bar{Y}_8 - \bar{Y}_6) - (\bar{Y}_7 - \bar{Y}_5)]$$

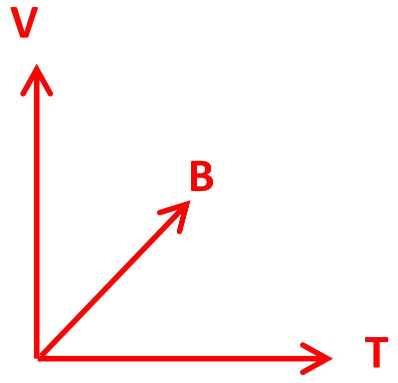
- Similarly, at LOW level of bar size, interaction between the temperature (X1) and wind velocity (X2) is

$$\frac{1}{2}[(\bar{Y}_4 - \bar{Y}_2) - (\bar{Y}_3 - \bar{Y}_1)].$$

- The three-variable interaction among temperature, wind velocity and bar size is the average difference between these two-variable interactions. $E_{123} = \frac{1}{2}\{\frac{1}{2}[(\bar{Y}_8 - \bar{Y}_6) - (\bar{Y}_7 - \bar{Y}_5)] - \frac{1}{2}[(\bar{Y}_4 - \bar{Y}_2) - (\bar{Y}_3 - \bar{Y}_1)]\}$

Test #	***X1***	***X2***	***X3***	***Test Order***	***Y_{ai} (kpsi)***	***Test Order***	***Y_{bi} (kpsi)***	***Average (kpsi)***
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





Three-variable Interaction E_{123}

- In common practice *we may* assume that the system is not so complicated such that the tree-variable interaction will be negligible.
- More over, we may contribute it as a part of the intrinsic variation of the test method, or in other terms, as a part of experimental error.
- **But, this should be exercised with caution and should be checked if ever possible. (Model Adequacy Check)**

Calculation Matrix

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Calculation Matrix

	Main Effects			Interactions				
Test	X_1	X_2	X_3	X_1X_2	X_1X_3	X_2X_3	$X_1X_2X_3$	$\bar{y}$
1	-1	-1	-1	1	1	1	-1	87.5
2	1	-1	-1	-1	-1	1	1	87.3
3	-1	1	-1	-1	1	-1	1	77.8
4	1	1	-1	1	-1	-1	-1	87
5	-1	-1	1	1	-1	-1	1	79.1
6	1	-1	1	-1	1	-1	-1	97.6
7	-1	1	1	-1	-1	1	-1	78.6
8	1	1	1	1	1	1	1	87.7

main effect x_1

E_1 = avg when x_1 is high − avg x_1 is low

$= 1/4\,(y_2+y_4+y_6+y_8) - 1/4\,(y_1+y_3+y_5+y_7)$

interaction of x_1 & x_2

E_{12} = avg when x_1x_2 high − avg x_1x_2 low

$= 1/4\,(y_1+y_4+y_5+y_8) - 1/4\,(y_2+y_3+y_6+y_7)$

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Main Effects

Ambient temperature (E_1) 9150 psi

Wind Velocity (E_2) - 5100 psi

Bar Size (E_3) 850 psi

Two-Variable Interactions

Ambient temperature-Wind Velocity (E_{12}) 0 psi

Ambient temperature-Bar Size (E_{13}) - 4650 psi

Wind Velocity-Bar Size (E_{23}) -100 psi

Three-Variable Interaction

Ambient temperature-Wind Velocity-Bar Size (E_{123}) 4700 psi

[8.53250000e+01 4.57500000e+00 -2.55000000e+00 4.25000000e-01 -3.55271368e-15 2.32500000e+00 -5.00000000e-02 -2.35000000e+00]

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Main Effects

Ambient temperature (E_1) 9150 psi

Wind Velocity (E_2) - 5100 psi

Bar Size (E_3) 850 psi

Two-Variable Interactions

Ambient temperature-Wind Velocity (E_{12}) 0 psi

Ambient temperature-Bar Size (E_{13}) 4650 psi

Wind Velocity-Bar Size (E_{23}) -100 psi

Three-Variable Interaction

Ambient temperature-Wind Velocity-Bar Size (E_{123}) -4700 psi

$$y_1 = \beta_0 + \beta_1(-1) + \beta_2(-1) + \beta_3(-1) + \beta_{12}(+1) + \beta_{13}(+1) + \beta_{23}(+1) + \beta_{123}(-1)$$

What is the regression model?

$$y = f(x_1, x_2, x_3)$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3$$

we want to find & β_i

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

8×4

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

For each experimental data, we can find the error (ϵ_i) between the model predicted value ($\hat{y}_i$) and observed experimental value (y_i)

$$\epsilon_i = y_i - \hat{y}_i$$

With 'least square fit', the aim is to find such coefficients, that minimizes the total sum of squares of error.

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each β_i

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each $\boldsymbol{\beta_i}$

$\hat{y}_1 = \beta_0 + \beta_1 x_{11}$

$\hat{y}_2$ -

$\hat{y}_3$ -

$\hat{y}_4$

$\hat{y}_8$

We can write the model in a matrix format

$$[Y_{exp}] = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_8 \end{bmatrix}, \quad [\hat{Y}] = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_8 \end{bmatrix}_{8\times 1} = [X]_{8\times 4}[\beta]_{4\times 1}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

where, $$[X] = \begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} \\ 1 & x_{12} & x_{22} & x_{33} \\ 1 & \vdots & & \\ \vdots & & & \\ 1 & x_{18} & x_{28} & x_{38} \end{bmatrix}$$