CS 207M Tutorial-1

Notation: $N = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.

1. Let A and B be sets. Prove that the following statements are equivalent.
   - There exists a bijection from A to B.
   - There exists a bijection from B to A.
2. Prove that if there is a bijection from A to B, then there is a bijection from 2^A to 2^B.
3. Let A be any set. Show that there is a bijection from 2^A to the set of all functions from the set A to the set $[2] = \{1, 2\}$.
4. Prove that the set A^* of finite sequences of elements from a finite set A is countable. Is the set A^ω of infinite sequences of elements from a finite set A countable?
5. Is the set A^* of finite sequences of elements from a countable set A countable?
6. Which of the following sets are countable/uncountable?
   - the set of all infinite subsets of N whose complement is finite.
   - the set of all infinite subsets of N whose complement is infinite.
7. Show that the set of all infinite sequences $(A_1, A_2, A_3, \ldots)$ such that each $A_i \subseteq N$ has the same cardinality as 2^N.
8. Show that if there is an injection from A to B, and also a surjection from A to B, then there is a bijection from A to B.
9. Let $f : X \to X$ be a function. A subset Y of X is called invariant if $f(Y) \subseteq Y$. An invariant subset Y is called irreducible if no proper subset of Y is also invariant. An invariant subset Y is called decomposable if its complement is also invariant.

   Now let $f : X \to X$ be an injection. Show that if X is finite, then X is a disjoint union of irreducible decomposable subsets. Given an example of an infinite set X and an injection $f : X \to X$ where there are infinitely many non-trivial invariant subsets of X. However, there is no non-trivial decomposable or irreducible subset.