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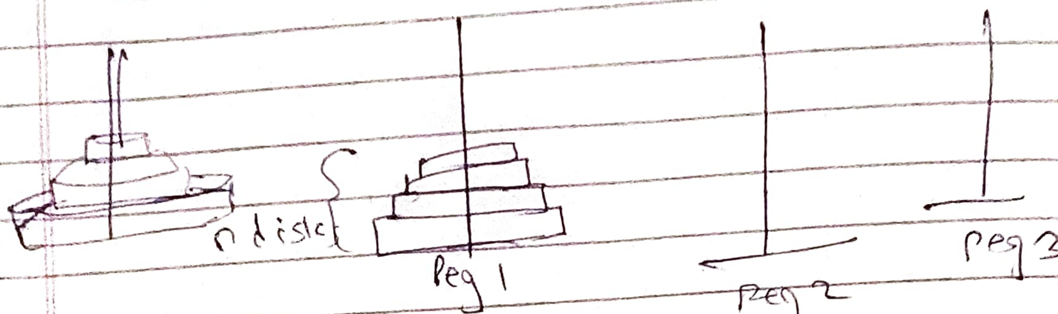
"Recurrence" relations

The tower of Hanoi puzzle



A valid move/step allows to move a disk from one peg to another ensuring that a disk is never placed on top of a smaller disk.
question: What is the minimum of steps required to move all n disks from peg 1 to peg 2?

- one disk ✓ 1
- two disks $p1 \to p3$ $p1 \to p2$ $p3 \to p2$ 3

let a_n be the answer for n disks

$P_1 \xrightarrow[n]{} P_2$ $P_1 \xrightarrow[n-1]{} P_3$ $P_1 \to P_2$ largest disks

a_{n-1}

$P_3 \xrightarrow[n-1]{} P_2$

$n \geq 2$ $a_n = a_{n-1} + 1 + a_{n-1}$

$= 2a_{n-1} + 1$

a_6

a_1	a_2	a_3	a_4	a_5	a_6
1	3	7	15	31	63

$a_1 = 1$

$\boxed{n \geq 2}$ $\boxed{a_1 = 1}$

$$a_n = 2a_{n-1} + 1$$
$$= 2\left[2a_{n-2} + 1\right] + 1$$
$$= 2^2 a_{n-2} + 2 + 1$$
$$= 2^3 a_{n-3} + 2^2 + 2 + 1$$
$$= 2^{n-1} a_1 + 2^{n-2} + \dots + 2 + 1$$
$$= 2^{n-1} + 2^{n-2} + \dots + 1$$
$$= \frac{2^n - 1}{2 - 1} = 2^n - 1 \quad n \geq 2$$

$$\boxed{\forall n \geq 1 \quad a_n = 2^n - 1}$$

- Count the # of n-length binary strings which do not have two consecutive 0's.

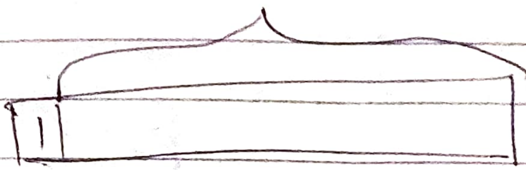
$n = 1$ $\quad$ 0, 1 $\quad$ $a_1 =$ $\qquad$ a_n = # of such strings

$n = 2$ $\quad$ 01, 10, 11, ~~00~~ $\quad$ $a_2 = 3$

$a_3 = 001, 000, 010, 011, 100, 101,$
$110, 111$ $\quad a_3 = 5$

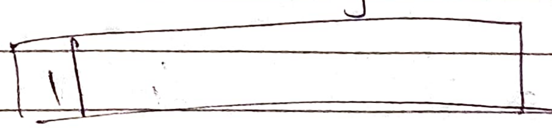
$a_n = |X_n|$

X_n
Y_{n-1}
Z_n
$n \geq 2$



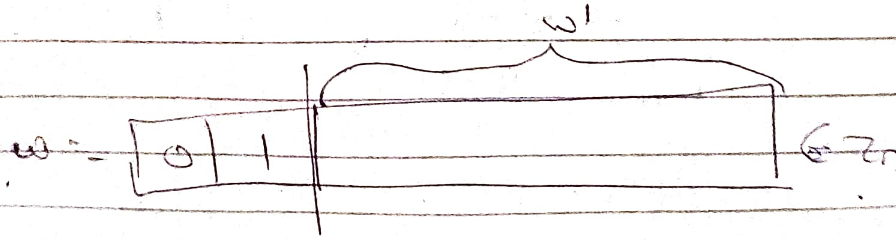
Assume w starts with '1'.

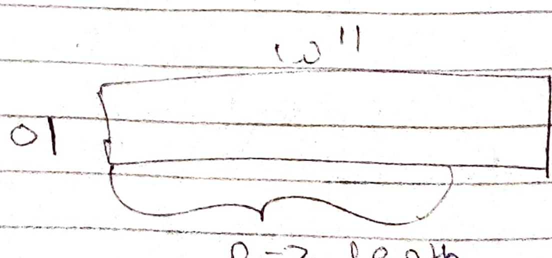
w'' of length n-1 and without '00'



$a_n = |X_n| = |Y_n| + |Z_n|$ $\quad Y_n \longleftrightarrow X_{n-1}$
$\qquad 1w'' \longleftarrow w''$

$= a_{n-1} +$





$w'' \in X_{n-2}$

$Z_n \longleftrightarrow X_{n-2}$
01

$\boxed{n \geq 3}$

$a_1 = 2, \quad a_2 = 3$

$n \geq 3 \qquad a_n = a_{n-1} + a_{n-2}$

$a_0 = 1$

$n = 0$ (1 ε)

fibonacci recurrence relation.

- Count the number of regions in which a plane gets divided by 'n' straight lines in "general position." (No two lines are parallel)

(No two lines are parallel & No 3 lines are concurrent.)

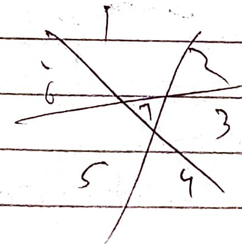
a_n = # of regions for n lines

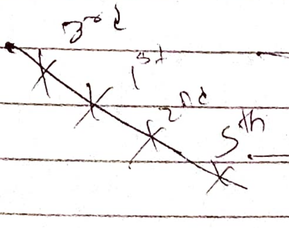
For general situation you should get maximum possible count.

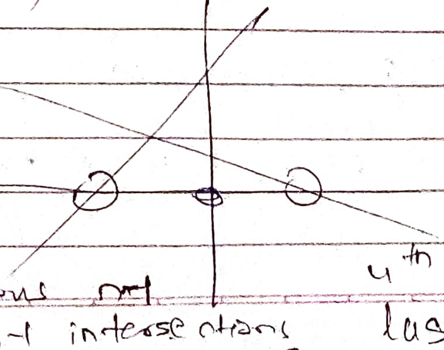
$a_1 = 2$

$a_2 = 4$

$a_3 = 7$







n^{th} line cuts previous $n-1$ lines.

$n-1$ intersections So n segments.

4^{th} line hits last 3 lines.

So n segments got spliced

$\therefore n \geq 2 \quad \boxed{a_n = a_{n-1} + n}$

$= a_{n-2} + n-1 + n$

$\vdots$

$= a_1 + 2 + 3 + \cdots + n$

$= 2 + 2 + 3 + \cdots + n$

$= 1 + \frac{n(n+1)}{2}$

$= \boxed{\frac{n^2+n+2}{2}}$

$$\boxed{\forall\, n \geq 1 \quad a_n = \frac{n^2+n+2}{2}}$$

Count the no. of ways to parenthesize a product of $n+1$ numbers $x_0, x_1, \ldots, x_n$ to decide the order of multiplication.

$n=1 \quad x_0 . x_1$

$n=2 \quad x_0 . x_1 . x_2$

$(x_0 \cdot x_1) \cdot x_2$ $\quad$ $x_0 \cdot (x_1 \cdot x_2)$

$n=0$ $\quad x_0$

$n=3 \quad x_0 \cdot x_1 \cdot x_2 \cdot x_3$

$a_0 = 1$

$a_1 = 1$

$a_2 = 2$

$a_3 = 6$

$x_0 \quad x_k \qquad x_{k+1} \quad x_n$

$k+1 \qquad n-k$

$\sum\limits_{k=} a_k \cdot a_{n-k-1}$

Catalan numbers

Also missing notes of last wednesday

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Linear homogeneous recurrence relation of degree k with "constant" coefs.

Assume $k=2$:

$n \geq 2$ $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ $c_2 \neq 0$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of $(a_n)_{n=0}^{\infty}$

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} c_1 a_{n-1} x^n + \sum_{n=2}^{\infty} c_2 a_{n-2} x^n$$

$A(x) - a_0 - a_1 x = c_1 x [A(x) - a_0] + c_2 x^2 A(x)$

$A(x)[1 - c_1 x - c_2 x^2] = a_0 + a_1 x - c_1 a_0 x$

$$A(x) = \frac{P(x)}{1 - c_1 x - c_2 x^2} \quad \deg P(x) \leq 1$$

$1 - c_1 x - c_2 x^2 = (\ )(\ )$

$r^2 - c_1 r - c_2 = (r - r_1)(r - r_2)$

$\therefore 1 - c_1 x - c_2 x^2 = (1 - r_1 x)\cdot(1 - r_2 x)$

$$= \frac{P(x)}{(1 - r_1 x)(1 - r_2 x)}$$

$$= \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x}$$

$a_n = \underset{\text{coeff.}}{\text{coeff}}$ of x^n of $A(x) = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$\frac{1}{1 - r_1 x} = 1 + r_1 x_1 + r_1^2 x_1^2 + r_1^3 x_1^3 + \dots$$

Ex: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$1 - c_1 x - c_2 x^2 = (1 - r_0 x)(1 - r_0 x)$

$r^2 - c_1 r - c_2 = (r - r_0)^2$

$$A(x) = \frac{P(x)}{(1 - r_0 x)^2} = \frac{\alpha_1}{(1 - r_0 x)^2} + \frac{\alpha_2}{(1 - r_0 x)}$$

$(1+y)^u = \sum_{k=0} \binom{u}{k} y^k$

α_1, α_2 are constants

$a_n = (\gamma_1 n + \delta)x_0^n + \gamma_2 x_0^n$

$a_n = \beta_1 n x_0^n + \beta_2 x_0^n$

$= (\beta_1 n + \beta_2)x_0^n$

$\frac{1}{(1-x_0x)^2} = (1-x_0x)^{-2}$

$= \sum_{k=0}^{\infty} \binom{-2}{k}(-x_0x)^k$

$= \sum_{k=0}^{\infty} (-1)^k (k+1) x_0^k x^k (-1)^k$

$= \sum_{k=0}^{\infty} (k+1) x_0^k x^k$

$\frac{1}{1-y} = 1 + y + y^2 + \ldots$

$\frac{1}{(1-y)^2} = \boxed{k+1}\, y^k$

$\frac{1}{(1-y)^3} = \boxed{k_1^2 + \ldots}\, y^k$

* Linear inhomogeneous recurrence relations of degree K with "constant" coeffs

A valid codeword is a n-digit number in decimal notation with even number of 0's. Let a_n be the number of valid codewords.

$a_1 = 9$ decimal digits 0, 1, 2, ...

$n \geq 2$ $a_n =$

$\neq 0$

$a_n = \underbrace{9a_{n-1}}_{\text{when last digit} \neq 0} + \underbrace{10^{n-1} - a_{n-1}}_{\text{when last digit} = 0}$

$= 8a_{n-1} + 10^{n-1}$

$\frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}$ by definition (a)

$\binom{-2}{k} = \frac{(-2)(-3)\cdots(-2-k+1)}{k!}$

$= (-1)^k \frac{(k+1)!}{k!}$

$= (-1)^k (k+1)$

$a_1 = 8a_0 + 1 \quad 1 \Rightarrow a_0 = 1$

$$\boxed{\begin{array}{l} n \geq 1 \quad a_n = 8a_{n-1} + 10^{n-1} \\ a_0 = 1 \end{array}}$$

$A(x) = \sum_{n=0}^{\infty} a_n x^n$

$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 8 a_{n-1} x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n$

$(A(x) - a_0) = 8x A(x) + x \frac{1}{1-10x}$

$A(x)(1-8x) = 1 + \frac{x}{1-10x} = \frac{1-9x}{1-10x}$

$A(x)(1-8x) = 1 + \frac{x}{1-10x} = \frac{1-9x}{1-10x}$

$1-9x = \alpha_1(1-10x) + \alpha_2(1-8x)$

$x = 1/10$

$a_n = \frac{1}{2}(8^n$

$$A(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

$$= \frac{\alpha_1}{1-8x} + \frac{\alpha_2}{1-10x} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$a_0 = 1$

$$\boxed{a_n = a_0 a_{n-1} + a_1 a_{n-2} + a_2 a_{n-3} + \dots + a_{n-1} a_0}$$

$$= \sum_{k=1}^{n} a_{k-1} a_{n-k}$$

$a_0, a_1, a_2, \dots, a_{n-1}, a_n$

$1 \quad 1 \quad 2 \quad 5$

$a_2 = a_0 a_1 + a_1 a_0 = 1+1 =$

$a_3 = a_0 a_2 + a_1 a_1 + a_2 a_0$

$a_4 =$

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$A(x)A(x) = A^2(x) = (a_0 a_0) + (a_0 a_1 + a_1 a_0)x + (a_0 a_2 + a_1 a_1 + a_2 a_0)x^2 + \dots$$

$$= a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \dots$$

$$x A^2(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots$$
$$= A(x) - a_0$$

$$\boxed{x A^2(x) - A(x) + 1 = 0}$$

$A(x)$

$$\boxed{xy^2 - y + 1 = 0}$$

$$A(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$\sqrt{(1-4x)}$$

$$= (1-4x)^{1/2}$$

$$\frac{1+\sqrt{1-4x}}{2x} \quad \times$$

$$= \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k$$

$$\checkmark \frac{1-\sqrt{1-4x}}{2x} \quad \cancel{2x}$$

$$= \sum_{k=0}^{\infty} b_k x^k$$

$$b_k = (-4)^k \times \binom{1/2}{k}$$

$$= 1 +$$

$$b_1 = (-4)^1 \times \binom{1/2}{1}$$

$+a_1a_0 \quad = 1+1 = 2$

$+a_4, \quad +a_2a_3 = 2+1+2 = 5$

$$= -4 \times 1/2$$

$$= 1 - 2x - 2x^2$$

$$= -2$$

$$b_2 = (-4)^2 \binom{1/2}{2}$$

$$= \frac{16 \times 1/2 \times -1/2}{2 \times 1}$$

$$\frac{1-\sqrt{1-4x}}{2x} = \frac{2x + 2x^2 + \ldots x^3 + \ldots}{2x}$$

$$= \underline{\qquad\qquad}$$

$$\boxed{a_n = \frac{1}{n+1}\binom{2n}{n}}$$

Catalan nos.

final ans something like this

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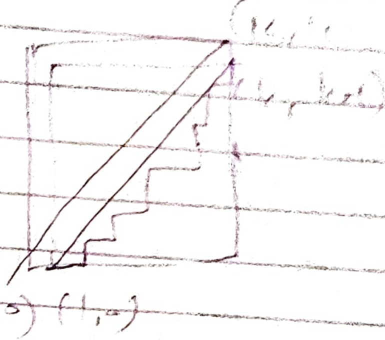
Let b_k be the number of lattice paths from to (k, k) which stay below the diagonal without touching the diagonal.

$$a_n = \sum_{k=1}^{n} b_k a_{n-k} = \sum_{k=1}^{n} a_{k-1} a_{n-k}$$



claim: $b_k = a_{k-1}$

$\boxed{a_0 = 1}$ $(n \geq 1)$

$a_1 = a_0 a_0$ $\quad a_2 = a_0 a_1 + a_1 a_0$

$a_3 = a_0 a_2 + a_1 a_1 + a_2 a_0 = 2 + 1 + 2 = 5$

Let $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

be the g.f. for $(a_n)_{n=0}^{\infty}$

$A^2(x) = a_0 a_0 + (a_0 a_1 + a_1 a_0)x + (a_0 a_2 + a_1 a_1 + a_2 a_0)x^2 + \dots$

$= a_1 + a_2 x + a_3 x^2 + a_4 x^3$

$x(A^2(x)) = A(x) - a_0$

$$\boxed{x A^2(x) - A(x) + 1 = 0}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$n \geq 1$

$= 2n$

$= \frac{(2n-2)!}{n(n-1)!(n-1)!}$ [illegible]

$A(x) = \frac{1-\sqrt{1-4x}}{2x}$ [illegible]

$= \sum_{n=0}^{\infty} c_n x^n$ [illegible]

$c_n = -\frac{C_{n+1}}{2} = \frac{1}{n+1}\binom{2n}{n}$

$c_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$

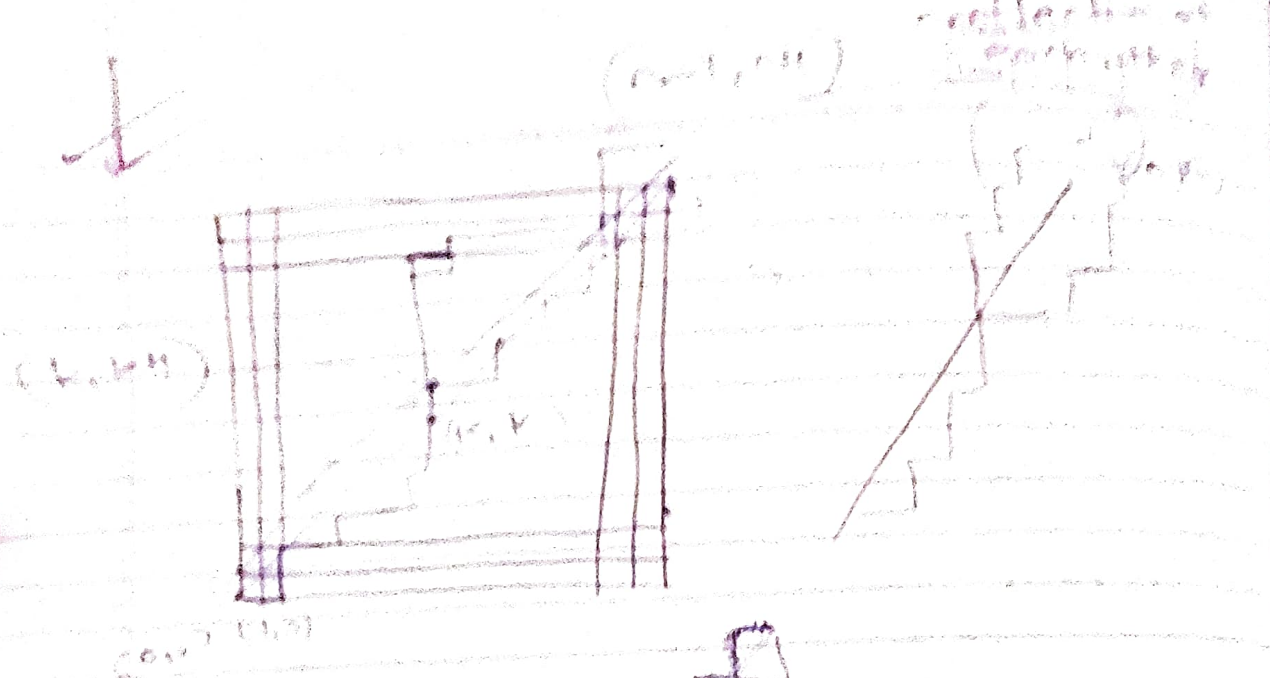
we will show

$c_n = \binom{2n}{n} - \binom{2n}{n+1}$

A path from $(0,0)$ to (n,n) is ~~good~~ good if it stays below the diagonal. otherwise it is bad

To show

The number of ~~bad path~~ bad paths from $(0,0)$ to (n,n) = $\binom{2n}{n-1}$



A bad path when "reflected" at the first point when it ~~comes~~ crosses the diagonal gives # one on the path from (0,2) to (n-1, n+1)

(Catalan numbers proof)

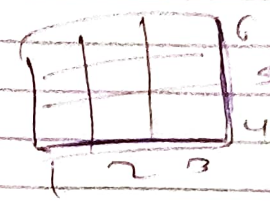
- Number of well-parenthesized strings using n open "(" and n closed ")" brackets

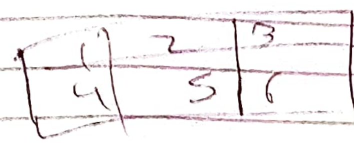
()(()

()() (()) ()()

There is bijection between the previous problem

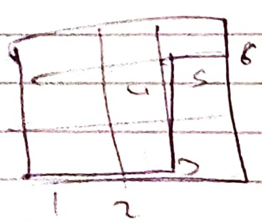
Number of $2 \times n$ arrays with distinct entries elements $1, 2n$





a_i : the step-number of ith horizontal step

b_i - the step number of ith vertical step



1 2 5

3 4 6

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Lemma: Let $P = (X, \leq)$ be a finite poset then minimal & maximal elts exist.

$x_1 < x \in X$

$< x_2 < x_1 < x$

$< x_4 < x_3$

Let X_{min} be the set of all minimal elements of P.

Claim: Any two distinct elements of X_{min} are incomparable.

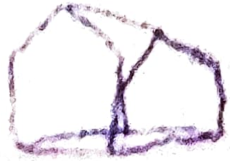
If x & y are in X_{min} and $x \neq y$

If $x \leq y$, then $x < y$ $\therefore$ y is not minimal. That contradicts the minimality of y.

Similly,

Def: An antichain of P is a subset A of X such that no two elements of A are comparable.

Ex: X_{min} & X_{max} are antichains



$P : (2^{\{n\}}, \subseteq)$

$\{\phi\}$

$A : \{ \{1\}, \{2\}, \{3\} \}$

$\{ \{2\}, \{1,3\} \}$

Posets:

Let $P = (X, \leq)$ be a poset

C - the courses

$\leq = \{ (C_1, C_2) \mid$ either $C_1 = C_2$ or C_1 must have complete in order to register for $C_2 \}$

A registration is a sequence $P_1, P_2, \dots P_k$ of subsets of C such that
1) $P_1, \dots P_2, \dots P_k$ is a partition of C.
2) it respects the "$\leq$"

Posets:

Given a poset $P = (M, \leq)$,

Q What is the minimum number of antichain of poset to partition X?

Definition: A chain C of P is a subset of X such that any two elements of C are comparable.
Suppose $\exists$ a chain in P of size K.

Lemma : If A is an Antichain & C is a chain then $|A \cap C| \leq 1$

c = the maximum size of a chain of P

$\boxed{a \geq c}$

a = minimum number of anti-chains required to partition X

c = the maximum size of a chain of P

$\boxed{a \geq c}$

let A = $\{A_1, A_2, \ldots A_l\}$ be a partition of X into l anti-chains and C be a chain of X of size l

Then $a = l = c$

Best antichains

Smallest number of antichains

& largest size chain

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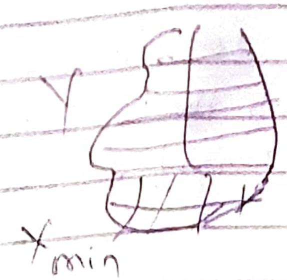
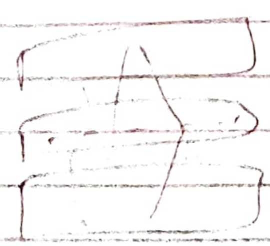
By induction Y has chain say C of size k and an antichain decomposition say $\mathcal{A} = \{A_1, A_2 \ldots A_k\}$ using k antichains.



Let y_0 be the minimum element of chain C. There must exist an element $x_0 \in X_{min}$ such that $x_0 < y_0$.
Now with $C' = C \cup \{x_0\}$ and $\mathcal{A}' = \{A_1, A_2 \ldots A_k, X_{min}\}$ we get a chain C' of P and an antichain decomposition of P with $|C'| = k+1 = |\mathcal{A}'|$



Theorem: Let $P = (X, \leq)$ be a finite poset. Then $\exists$ a k such that the size of the largest chain of P is k and the size of the smallest anti-chain decomposition is k.

By the above theorem, $\exists C$ such that $|C| = k$
and $\mathcal{A} = \{A_1, \ldots, A_k\}$

$|\mathcal{A}| = k$

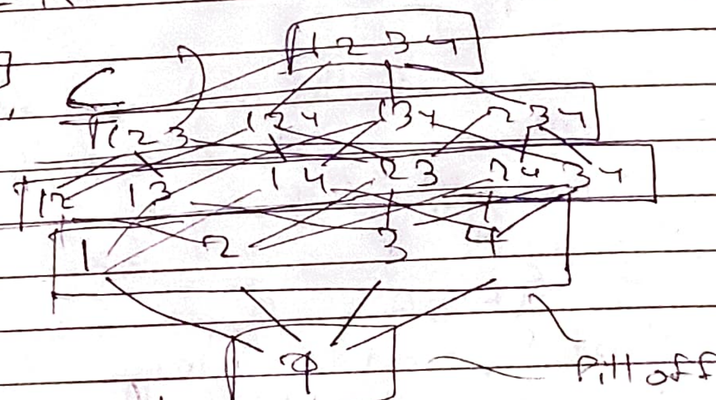
Let D be a chain of P such that $|D| > k$

Let D be a chain of P such that $|D| > k$.

* **Dilworth's theorem**: let $P = (X, \subseteq)$. Then there exists K such that P has an antichain of size K and a chain decomposition of size K.

$|A| = K$ $\qquad \mathcal{C} = \{C_1, C_2, \ldots, C_k\}$

$(2^{[3]}, \subseteq)$



Hassy diagram

$|A| \leq |\mathcal{C}|$

Size of Antichain

size of chain decomposition

$$\max_A |A| = \min_{\mathcal{C}} |\mathcal{C}|$$