CS 207

CS

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* Rozens book

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Cardinality

→ A set X is countable if it is either

→ $2^{\mathbb{N}}$ is uncountable

→ Cantor's thm: There is no bijection between $\mathbb{N}$ & $2^{\mathbb{N}}$

X - finite $|2^X| = 2^{|X|}$

Proof: Suppose for contradiction that $2^{\mathbb{N}}$ is countable.
Let A_0, A_1, A_2, A_3 be an ~~enumeration~~ / listing of $2^{\mathbb{N}}$

$$\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
A_0 \rightarrow & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \ldots \\
A_1 \rightarrow & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \ldots \\
A_2 \rightarrow & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_3 \rightarrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots \rightarrow & & & & & & & & \\
\hline
B & 0 & 1 & 1 & 1 & 0 & & &
\end{array}$$

An infinite matrix. row → Subsets in the enumeration
i^{th} row → A_i
columns → element of $\mathbb{N}$
j^{th} column → $j \in \mathbb{N}$
$(i,j)^{th}$ entry $= 1$ $j \in A_i$?
$= 0$ if $j \notin A_i$

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Define $B \subseteq N$ as follows. $B = \{ i \in N \mid (i,i)^{th} \text{ entry} = 0 \}$
$= \{ i \in N \mid i \notin A_i \}$

$B = \{ 1, 2, 3$

~~0~~ $0 \notin B$ $4 \notin B$

claim: There is no $i \in N$ ~~sort~~ such that $B = A_i$

Proof of the claim: _ _ _ $\square$

This provides a contradiction to the fact (Assumption) that $A_0, A_1, A_2, \ldots$ is an enumeration of 2^N.

Cantor's thm: There is no bijection between N & 2^N | 2^N is uncountable.

$|N| < |2^N| \overset{?}{<} (2^{(2^N)}) |N| \le |2^N|$

$|N| < |2^N|$

Definition: $|x| < |y|$ if $|x| \le |y|$ & $|x| \ne |y|$

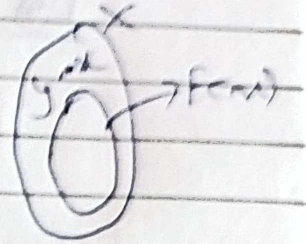
Cantor's theorem: Let X be any set. Then $|X| < |2^X|$.

Proof: We will show that $|X| \ne |2^X|$, & $|X| \le |2^X|$, this will imply that $|X| < |2^X|$. We will prove by contradiction that there is no bijection between X and 2^X.

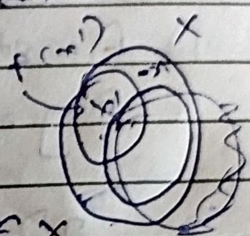
Suppose, for contradiction there is a bijection $f: X \to 2^X$.

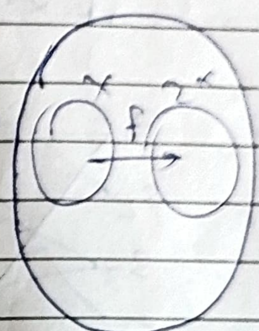
We Define a subset $Y \subseteq X$ as follows:

claim: There is no $x \in X$ such that $f(x) = Y$.



Proof of claim: let $w \in X$ be arbitrary. Then





Consider the subset $f(w) \subseteq X$ and the following two exhaustive cases.

case 1: $\boxed{w \in f(w)}$: By definition of Y, $w \notin Y$. Therefore $f(w) \neq Y$.

case 2: $\boxed{w \notin f(w)}$ By definition of Y, $w \in Y$. Hence $f(w) \neq Y$.

This concludes the proof of the claim. So, ~~the~~ this provides a contradiction to the fact that f is a bijection from X to 2^X.

Therefore, there is no bijection from X to 2^X.

Continuum Hypothesis

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* Relations :

let A, B be sets

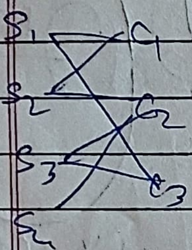
Def: A binary relation, R, from A to B is a subset of $A \times B$

$$R \subseteq A \times B = \{ (a, b) \mid a \in A, b \in B \}$$

We write $a R b$ if $(a, b) \in R$ otherwise we write $a \not R b$.

Ex: 1) A - students B - courses

$$Reg = \{ (s, c) \in A \times B \mid \text{student } s \text{ has registered for course } c \}$$



2) A - courses B - instructors

$$Teach = \{ (c, i) \in A \times B \mid c \text{ is taught by } i \}$$

Obs: 1) A function $f : A \to B$ naturally gives a relation R_f - the graph of f - from $A \times B$

$$R_f = \{ (a, b) \in A \times B \mid b = f(a) \}$$

$\forall\ a \in A$ (there is a unique b) $\exists !\, b$ such that $(a, b) \in R_f$

2) R_1 - from A to B
R_2 - from B to C $\Big\}$ relation

$$R_1 \circ R_2 = \{ (a, c) \in A \times C \mid \exists b \in B \text{ with } (a, b) \in R_1 \text{ and } (b, c) \in R_2 \}$$

Reg → Students to courses
Teach → courses to instructors

Reg ∘ Teach (s, i)
students to instructors

Relations:

Operation on Relations: R_1, R_2 are relations from A to B. $R_1, R_2 \subseteq A \times B$, $R_1 \cup R_2$, $R_1 \cap R_2$, $\overline{R_1}$
$= \{ (a,b) \mid a \not R_1 b \}$
$\in A \times B$

n-ary relations: $A_1, A_2 \ldots A_n$ – n sets
$R \subseteq A_1 \times A_2 \times \ldots \times A_n$

Flight (AI 101, Mumbai, Delhi)
(3-ary relation)

~~Relat~~

Relations on a Set:

Def: A relation R on a set A is a subset of $A \times A$ that is a relation from A to itself.

Examples:
1) $A = \mathbb{Z}$, $R - \leq$ $\boxed{(-3, 5) \in \leq}$ $\boxed{-3 < 5}$

$aRb \equiv (a,b) \in R$

2) $A = 2^X$ where X is a set
R_1: disjoint $= \{ (X_1, X_2) \in A \times A \mid X_1 \cap X_2 = \phi \}$

2.2 subset = $\{(x_1, x_2) \in A \times A \mid x_1 \leq x_2\}$

3)

P - positive integer

R - divides "|" $R = \{(a,b) \in P \times P \mid b$ is an integer multiple of $a\}$

4. Fix a positive integer m. We ~~define~~ define the relation $\equiv_m$ on $\mathbb{Z}$

m: $a \equiv_m b$ if $(a-b)$ is a multiple of m.

If $m=3$ $(1,4) \in \equiv_m$ $1 \equiv_m 4$

Ex) A = student

$R_1 = \{(s_1, s_2) \mid s_1$ and s_2 belong to the same dept$\}$

$R_2 = \{(s_1, s_2) \mid s_1$ and s_2 stay in the same hostel$\}$

$R_3 = \{(s_1, s_2) \mid$ there is a course C such that both s_1 and s_2 are registered for $C\}$

course

Properties of relations

A – a set

R – a relation on A

Def 1: R is reflexive if every element of A is related (by R) to itself.

$\forall a \in A, \; aRa$

ex. $\leq$, Subset, division, $a \equiv_m b$

R_1, R_2, R_3

Def 2: R is symmetric if whenever $(a,b) \in R$ so is (b,a)

$\forall a,b \in A \; (a,b) \in R \Rightarrow (b,a) \in R$

$\forall a,b \in A \quad aRb \rightarrow bRb$

ex. disjoint, $a \equiv_m b$, R_1, R_2, R_3

Def 3: R is transitive if

$\forall a,b,c \in A \; [(aRb) \text{ and } (bRc)] \Rightarrow aRc$

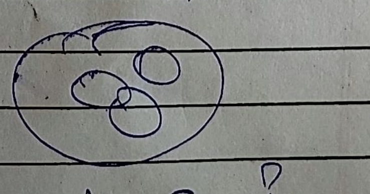
Ex. $\leq$, subset, division, $\equiv_m$, R_1, R_2

(disjoint is not transitive $\because$ (x_1, x_2) (x_2, x_1) then but x_1, x_1 is not in set.

R_3 is not transitive, $\because$ there can be two different course

$s_1 s_2 \rightarrow c_1$

$s_2, s_3 \rightarrow c_2$



$a \equiv_m b \equiv_m c \overset{?}{\Rightarrow} a \equiv_m c$

$a - b = k_1 m$
$\Rightarrow a - c$
$b - c = k_2 m$
$= (k_1 + k_2) m$

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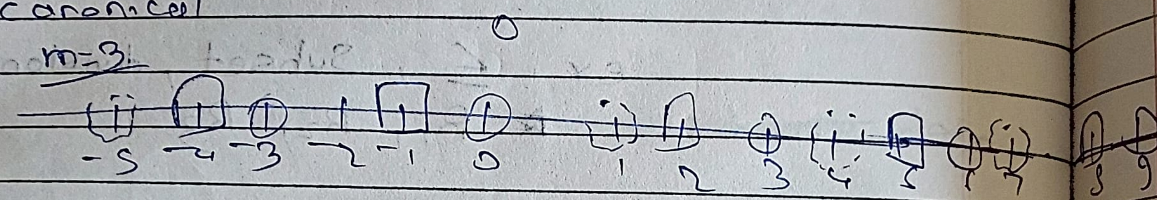
Equivalence relation.

Def. A relation R on A is called an equivalente relation if it is reflexive, symmetric, and transitive.

canonical

m=3



Date 19/01/2024

→ Tutorial sheet will be posted today on moodle.
→ Tut on next wednesday (24th Jan) evening 6 to 7pm.
→ Quiz (-1) on wed (31st Jan) ——

Equivalence relations on a set

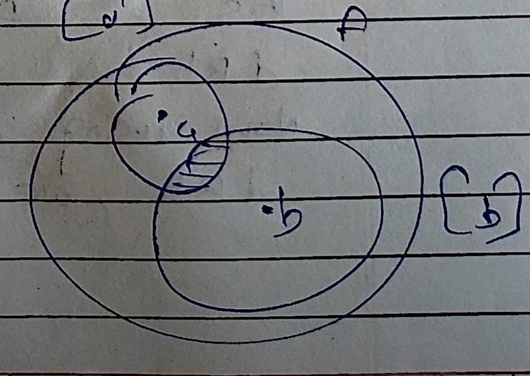
Let A be a set and R be an equivalent relation on A.

$R \subseteq A \times A$ → R is reflexive ($\forall a \in A$, aRa)
→ R is symmetric ($\forall a, b$ $aRb \Rightarrow bRa$)
→ R is transitive ($\forall a, b, c$ aRb and $bRc \Rightarrow aRc$)

Fix $a \in A$. The equivalence class of a is

$[a] = \{ b \in A \mid aRb \}$
Note as R is reflexive, aRa. Therefore $a \in [a]$
Hence $[a] \neq \phi$.

Lemma 4: let $a, b \in A$ Then the following are equivalent
i) aRb
ii) $[a] = [b]$
iii) $[a] \cap [b] \neq \phi$



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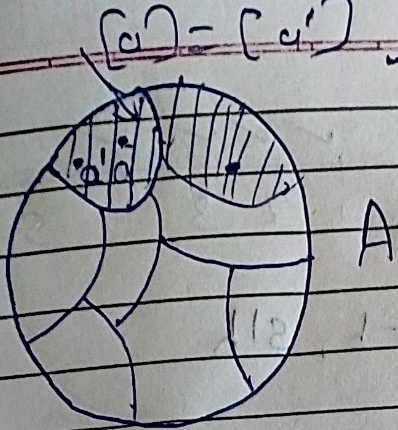
Proof : $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow i)$

($i) \Rightarrow ii)$) Assume aRb. We will show that $[a] = [b]$ by showing that $[a] \subseteq [b]$ & $[b] \subseteq [a]$

We first show that $[a] \subseteq [b]$
Let $c \in [a]$. by definition of $[a]$,
aRc. We need to show that bRc.
As aRb, R is symmetric, bRa. As bRa and aRc, by transitivity of R, bRc
By definition of $[b]$, $c \in [b]$. Thus we have shown that $[a] \subseteq [b]$. Similarly, we have $[b] \subseteq [a]$. Hence $[a] = [b]$.

($ii) \Rightarrow iii)$) Assume $[a] = [b]$ we need to show $[a] \cap [b] \neq \phi$. But as $[a] \neq \phi$
clearly $[a] \cap [b] = [a] \neq \phi$

($iii) \Rightarrow i)$) ~~Ass~~ Assume that $[a] \cap [b] \neq \phi$
We need to show aRb. As $[a] \cap [b] \neq \phi$, there is an element, say d in $[a] \cap [b]$.
As $d \in [a]$ and $d \in [b]$ we have aRd, bRd
As bRd, R is symmetric, we have dRb.
Now, using transitivity & the facts aRd, dRb, we conclude that aRb.

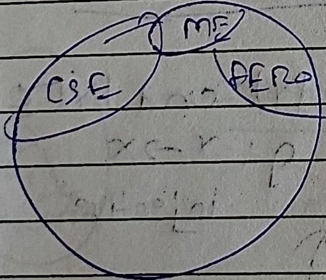


Equivalence relations on a set

An equivalence relation gives rise to a natural partition of A in terms of equivalence classes.

A = students

$R = \{(S_1, S_2) \mid S_1$ and S_2 are from the same ~~dept~~ dept $\}$



A partition of A is a collection of $\mathcal{C} = \{B_1, B_2, B_3 \ldots\}$ of non-empty pairwise disjoint ~~sub~~ subsets of A whose union is A.

A partition of A is a collection of $\mathcal{C} = \{B_1, B_2, B_3 \ldots\}$ of non-empty pairwise disjoint subsets of A whose union is A.

$A = \mathbb{Z}$

$a, b \in \mathbb{Z}$ $\quad a \equiv_5 b$ if $a - b$ is an integer multiple of 5.

R

$\simeq, \equiv$

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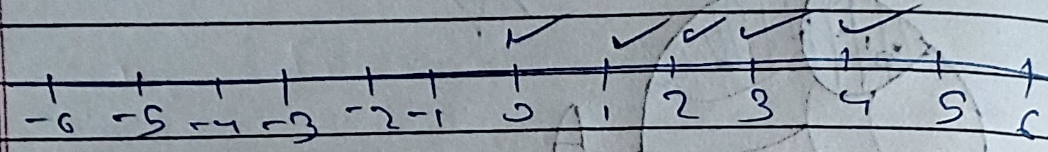
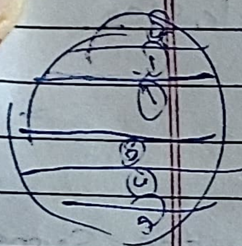


$a \equiv_5 b$ iff $5 \mid -b$, $5 \mid b$

$[0] = \{0, 5, -5, 10, -10, 15, -15, 20, -20, \ldots$

$0 \not\equiv_5 1$ $\quad [1] = \{1, 6, -4 \quad\quad \} = \{x \mid x = k \cdot 5 + 1$



~~Cantor - Schroder~~

Cantor-Schroeder-Bernstein theorem:

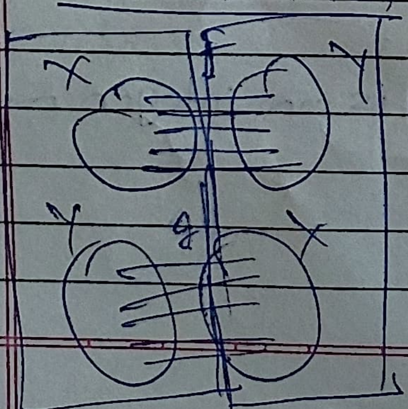
X, Y sets $|X| \le |Y|$ and $|Y| \le |X|$

$f: X \to Y$ $\quad g: Y \to X$

one-to-one/ injective $\quad$ injective

Then $|X| = |Y|$ then there exists bijection between X and Y. $|X| = |Y|$.

Proof sketch



Assume X and Y are disjoint

one to one / injective

$Z = X \cup Y$ $\quad h : Z \to Z$

$h(z) = f(z)$ if $z \in X$

$\quad\quad = g(z)$ if $z \in Y$

claim 1: h is injective

Identity f^n

K.S+1 $\quad h : Z \to Z \quad m \geq 0 \quad h^m = Id_Z$ if $m = 0$

$\quad\quad = h \circ h^{m-1}$ if $m \geq 1$

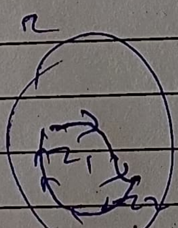
Claim: $\forall m \geq 0$, h^m is injective.

We define the ~~relat~~ relation $\equiv_h$ on Z.

$z_1, z_2 \in Z$

$z_1 \equiv_h z_2$ if $\exists m \geq 0$ such that either

$h^m(z_1) = z_2$ or $h^m(z_2) = z_1$



reflective, symmetric, transitivity

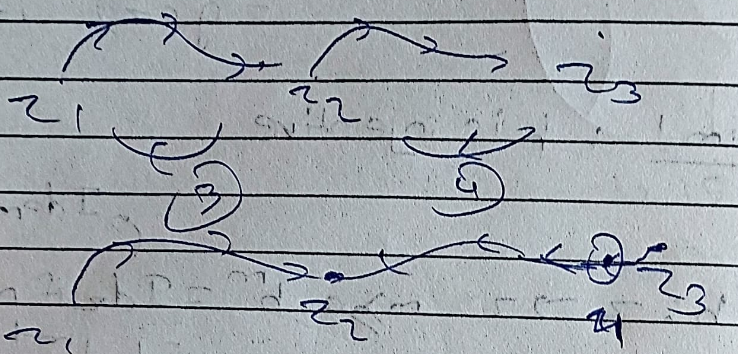
claim: $\equiv_h$ is an equivalence relation.

$z_1 R_h z_2$ $\quad \exists m \geq 0 \quad h^m(z_1) = z_2$

$\quad\quad\quad\quad \exists m \geq 0 \quad h^m(z_2) = z_1$ $\implies$

$z_2 R_h z_1$

$$z_1 R_h z_2 R_h z_3 \overset{?}{\Longrightarrow} z_1 R_h z_3$$



h^m is injective $\therefore$ one to one

Ex: to analyze the eq classes of $\equiv_h$.

Cases :

Case 1 : The seq $\sigma(z)$ is bi-infinite
⇓
(infinite in both direction)

$\sigma(z) = (z_i)_{i \in \mathbb{Z}}$

Case 1.1 : All the entries in $\sigma(z)$ are distinct. ✓

Case 1.2 : There are indices i and j ($i, j \in \mathbb{Z}$) such that $i \neq j$ but $z_i = z_j$

Assume $i < j$

$\ldots z_{-3}\ z_{-2}\ z_{-1}\ z_0\ z_1\ z_2\ z_3\ z_4\ z_5 \ldots$

$z_{-3}\ z_{-2}\ z_{-1}\ z_0\ z_1\ z_2\ z_3\ z_4\ z_5 \ldots$

claim $\forall\, k \in \mathbb{Z},\ z_k = z_{k+(j-i)}$

Case 2 : $\sigma(z)$ is not bi-infinite.

$\exists\ z' \in [z = z_0]$ such that the seq $\sigma(z)$ is of the form

$z' - z_0' \rightarrow z_1' \rightarrow z_2' \rightarrow z_3' \rightarrow \ldots$

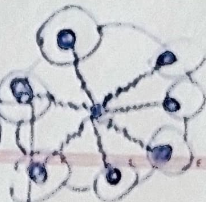
Claim: All the entries in $\sigma(z)$ are distinct.

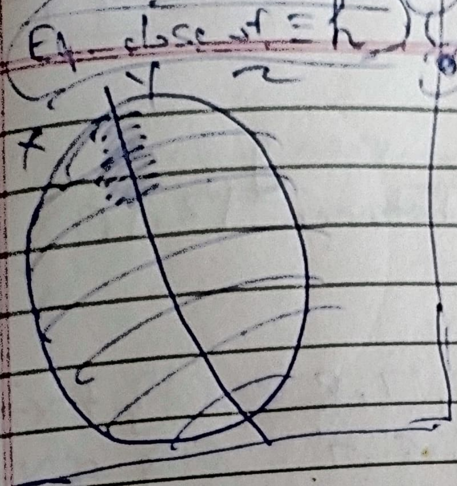
(: If there is repetition then it goes to infinite leftward also.)

$[z_0] = \{z_0', z_1', z_2', z_3' \ldots$

Step 4: Setting up a bijection from X to Y within an ~~[z]~~ eq. ch.

$\sigma(z)$ is a sequence which alternates between X and Y.





case 1.1:

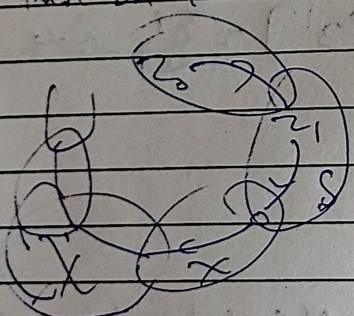
$[z_0] = \{ \ldots z_{-2}, z_{-1}, z_1, z_2, z_3 \ldots \}$

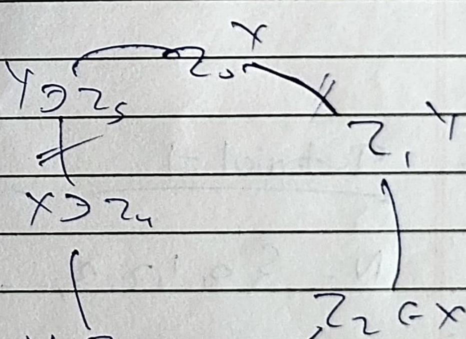
(Even indexed from one set
odd indexed from other set)

case 1.2:

$[z_0] = \{ z_0, z_1, z_2, \ldots \ldots z_k \}$
$\quad X \quad Y \quad X \quad - - - \quad z_k = z_0$

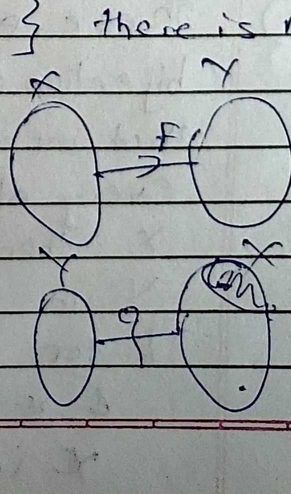
(So there should be even no. of elements
∴ if we want to come back then at last
there should be Y.





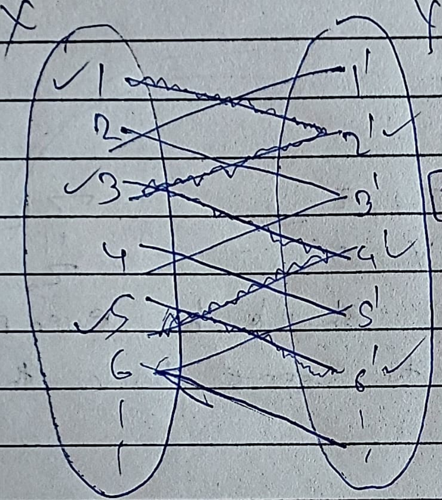
case 2: $[z_0] = \{ z_0' \to z_1' \to z_2' \to z_3' \to z_4' \to$
$z_5' \to \ldots \}$ there is no
$\omega \in Z$ such that $h(\omega) = z_0'$

case 2.1: $z_0' \in X$ use
f on $[z_0]$ to get a
bijection between $[z_0] \cap X$
and $[z_0] \cap Y$



case 2.2 : 2! ∈ X use g on [2₀] to get a bijection between [2₀] ∩ X and [2₀] ∩ Y.

X = {1, 2, 3, 4, 5, 6, 7, 8 ... } f : X → Y
X = {1', 2', 3', 4', 5', 6', 7' ... } g : Y → X



h : X ∪ Y → X ∪ Y

[3] = ~~[3]~~ 1 → 2' → 3 → 4' → 5

~~[3]~~

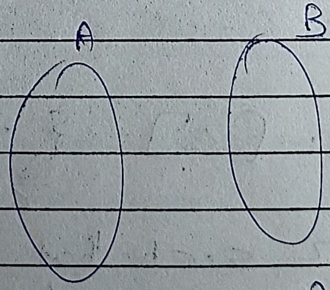
[3'] = {1' → 2 → 3' → 4 → 5' → }

* Tutorial - 1

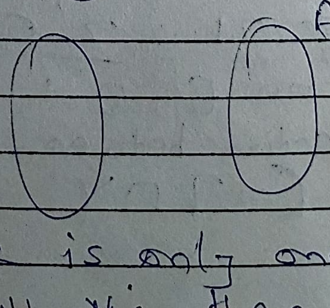
N = {0, 1, 2, 3, ... } N ∈ Natural numbers

1.

A to B



1) There exists a bijection from A to B that means for every element in X there exists a Y in ~~g~~ B & for one x there is only one X. & ~~x to~~ for all y's there exists a x in X.



B) There exists a bijection from ~~A~~ B to A.
~~that means~~
That is equivalent to above statement.

$f(i) = (i+1)$

$g(i') - (i+1)$

∵ for each y there is a x
there exists only one
in X. So ~~its~~ inverse of each y.
& for ~~every~~ y's there exists
a x ∵ X→Y is onto.

→ 6' → 7 → 8' → ... }

∴ Y → X is also one to
one (∵ there is only one x
for each y.) & its onto also (∵ for each
x there was y) ∴ Y→X is bijection.

→ 7' → ... }

Q.2)