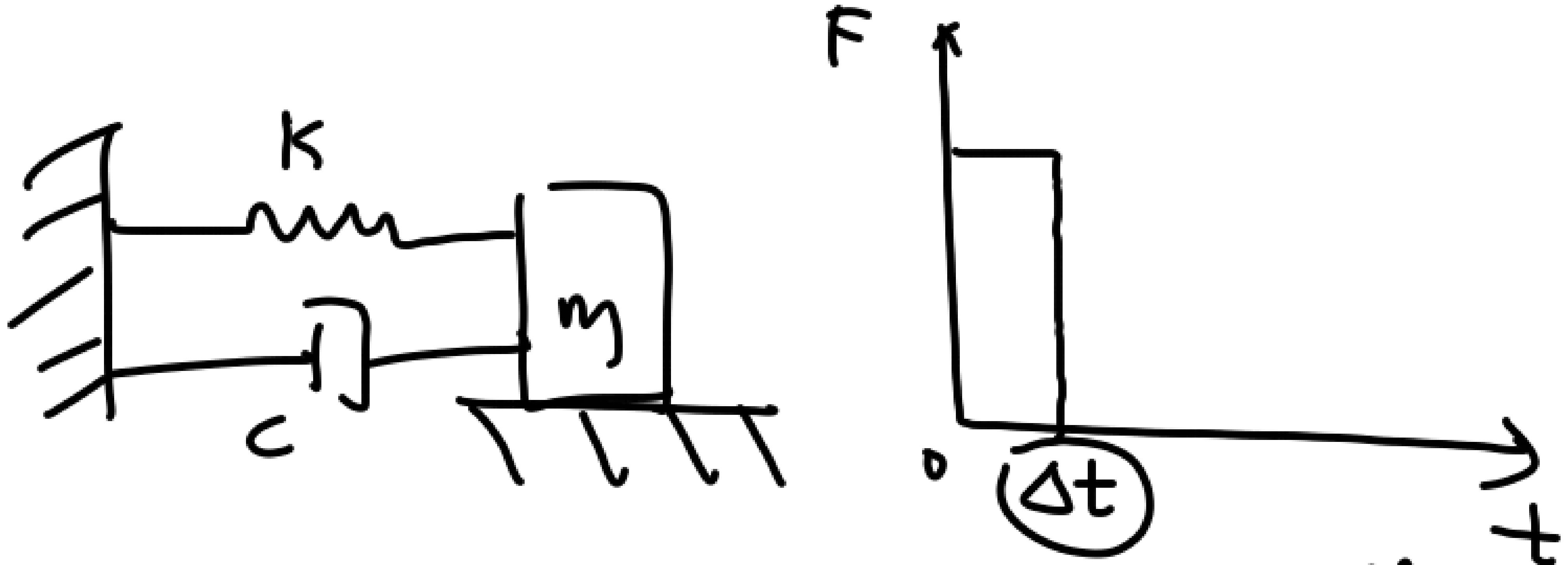
Response of single degree of freedom to Impulse



very small

$$\text{Impulse } I_m = \int_0^{\delta t} F\,dt \simeq F\,\delta t$$

Using conservation of linear momentum

Initial condition
$$\begin{cases} \dot{x}(0) = \dfrac{I_m}{m} = \dfrac{F\,\delta t}{m} \\ x(0) = 0 \end{cases}$$

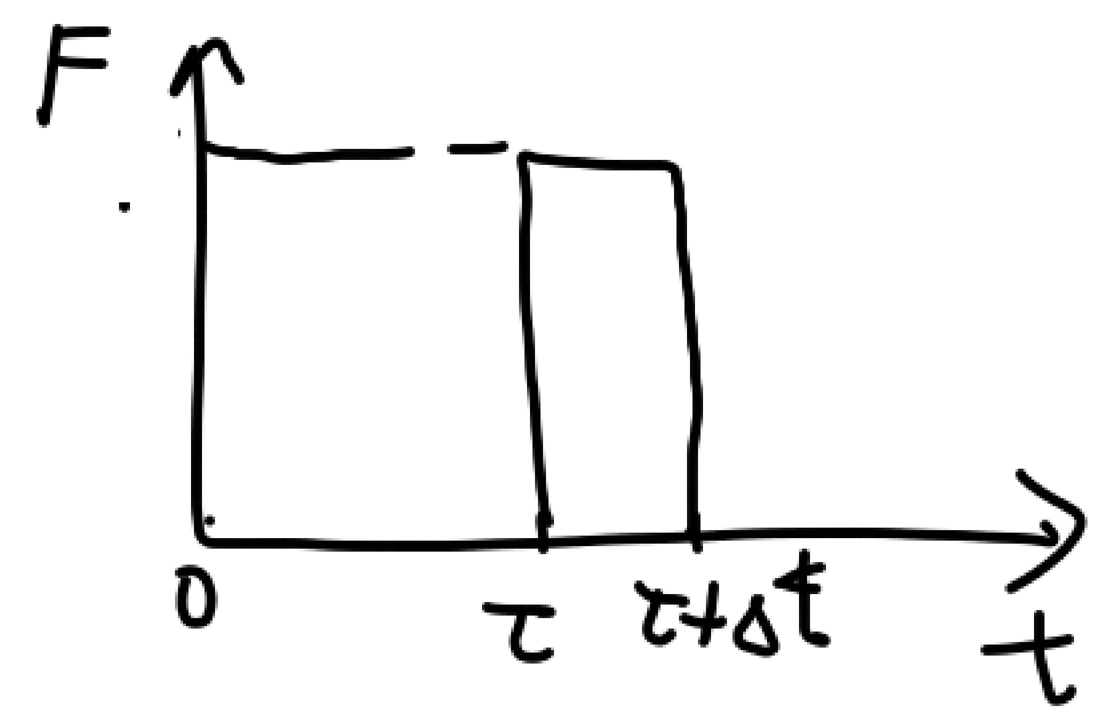
Response:

$$\boxed{x(t) = \frac{e^{-\xi\omega_n t}\,\overbrace{(F\Delta t)}^{\text{Impulse}}\,\sin(\omega_d t)}{m\omega_n\sqrt{1-\xi^2}} \qquad \omega_d = \omega_n\sqrt{1-\xi^2}}$$

Impulse response

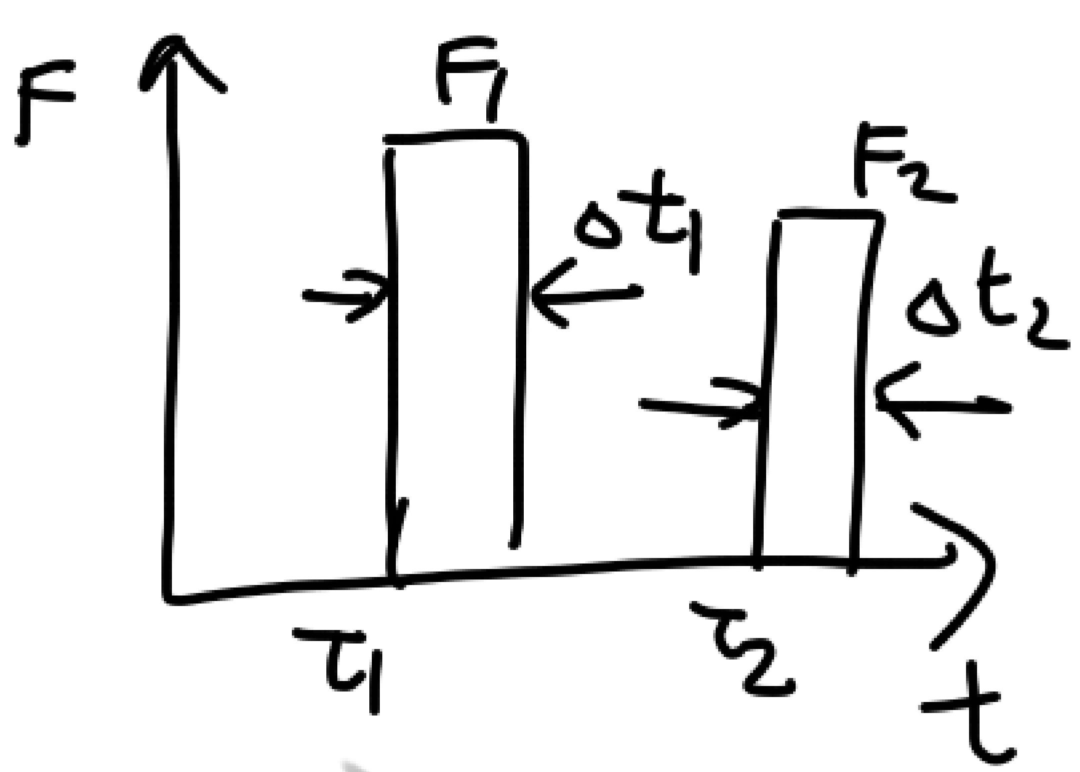
Impulse at time $t=\tau$

$$\bar{t} = (t-\tau)$$



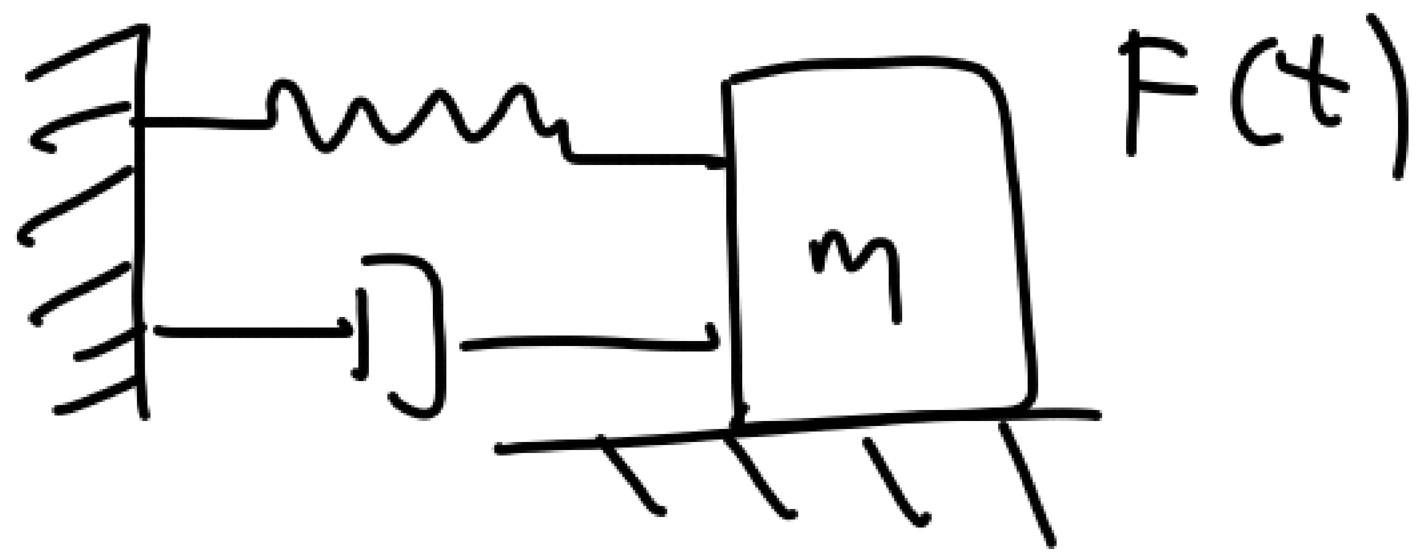
$$x(\bar{t}) = \frac{e^{-\xi\omega_n \bar{t}}(F\Delta t)\sin(\omega_d \bar{t})}{m\omega_n\sqrt{1-\xi^2}}$$

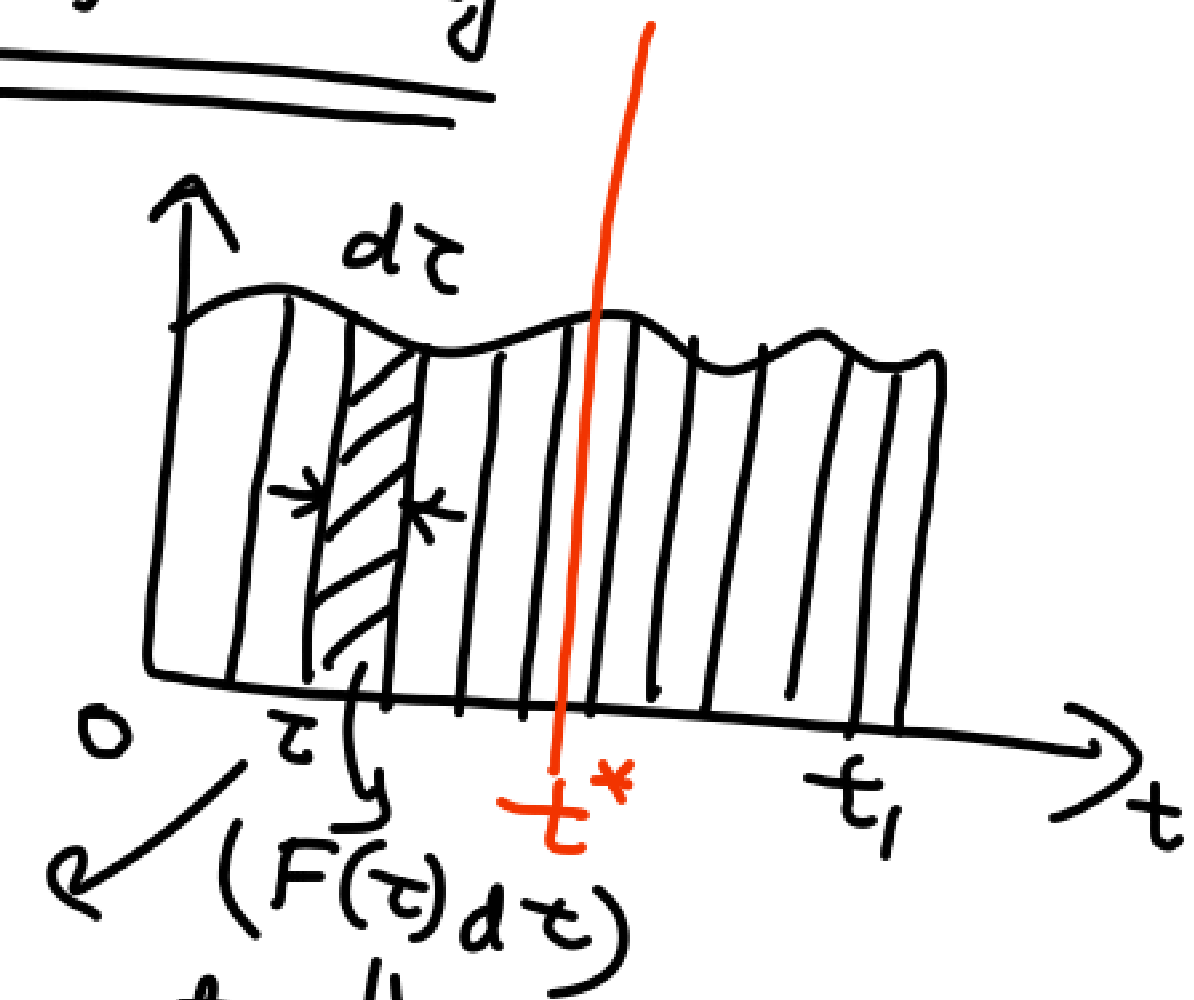
$$\boxed{x(t) = \frac{e^{-\xi\omega_n(t-\tau)}(F\Delta t)\sin(\omega_d(t-\tau))}{m\omega_n\sqrt{1-\xi^2}}}$$



$$x(t) = \begin{cases} 0 & 0 \le t < \tau_1 \\ \dfrac{e^{-\xi\omega_n(t-\tau_1)}(F_1\Delta t_1)\sin(\omega_d(t-\tau_1))}{m\omega_n\sqrt{1-\xi^2}} & \tau_1 < t < \tau_2 \\ \dfrac{e^{-\xi\omega_n(t-\tau_1)}F_1\Delta t_1\sin(\omega_d(t-\tau_1))}{m\omega_n\sqrt{1-\xi^2}} + \dfrac{e^{-\xi\omega_n(t-\tau_2)}F_2\Delta t_2}{m\omega_n\sqrt{1-\xi^2}}\sin(\omega_d(t-\tau_2)) & \tau_2 < t \end{cases}$$

So for a non-periodic forcing





We can represent the non-periodic forcing as series of impulse acting one after the other.

So the response is summation or integration of all such responses:

$$\therefore x(t^*) = \int_0^{t^*} \frac{e^{-\xi\omega_n(t^*-\tau)} F(\tau)\,d\tau \,\sin(\omega_d(t^*-\tau))}{m\omega_n\sqrt{1-\xi^2}}$$

Integral is called as convolution integral or $\underbrace{\text{Du Hamel}}_{\text{mathematician}}$ integral

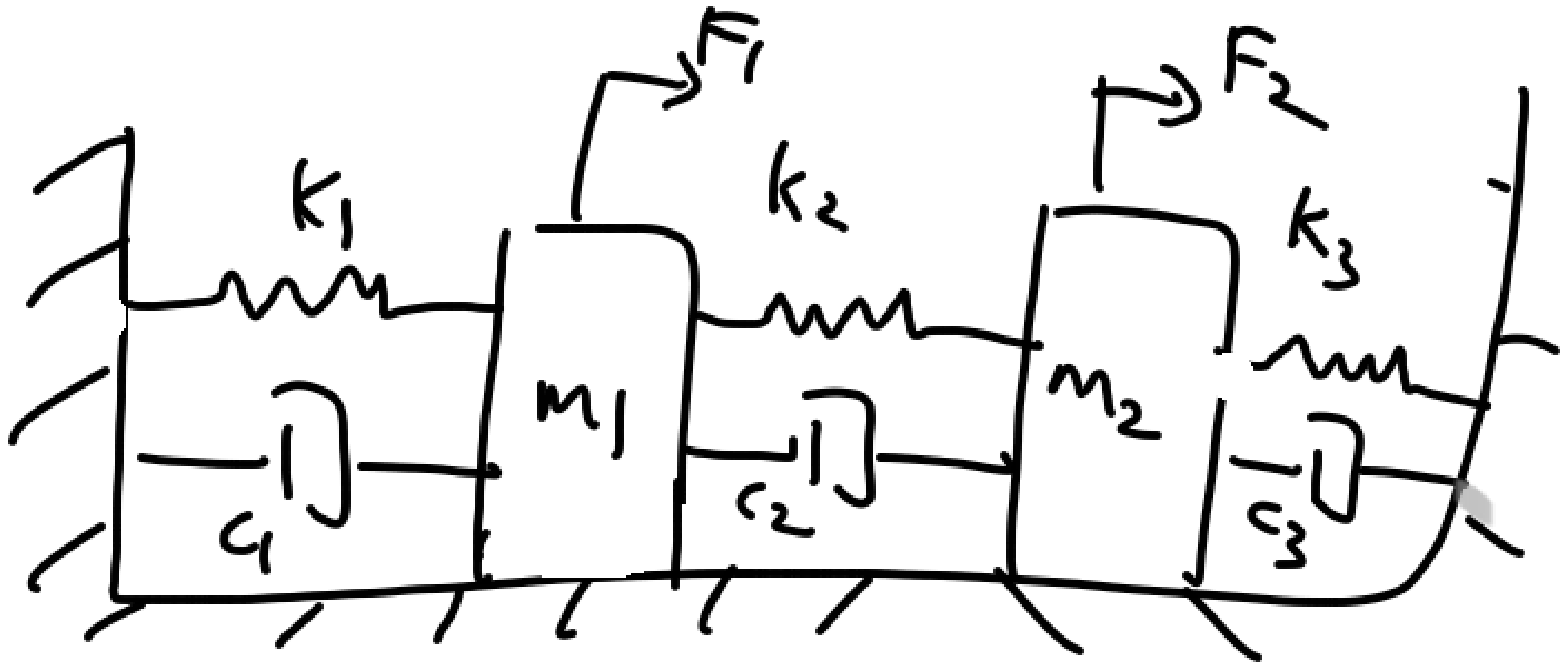
$$x(t^*) = \int_0^{t^*} \underbrace{G(t^*, \tau)} F(\tau)\,d\tau$$

$$G(t^*, \tau) = \frac{e^{-\xi\omega_n(t^*-\tau)} \sin(\omega_d(t^*-\tau))}{m\omega_n\sqrt{1-\xi^2}}$$

↓

Green's function

Two degree of freedom system



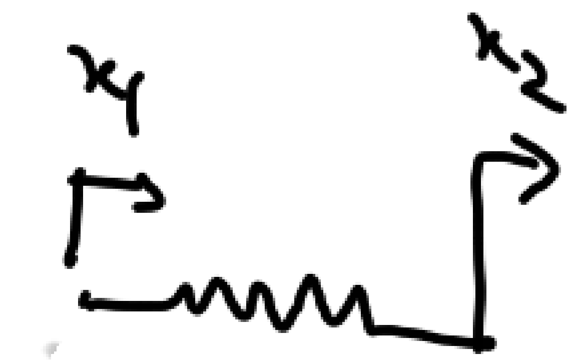
mathematical representation / idealization of an actual system.

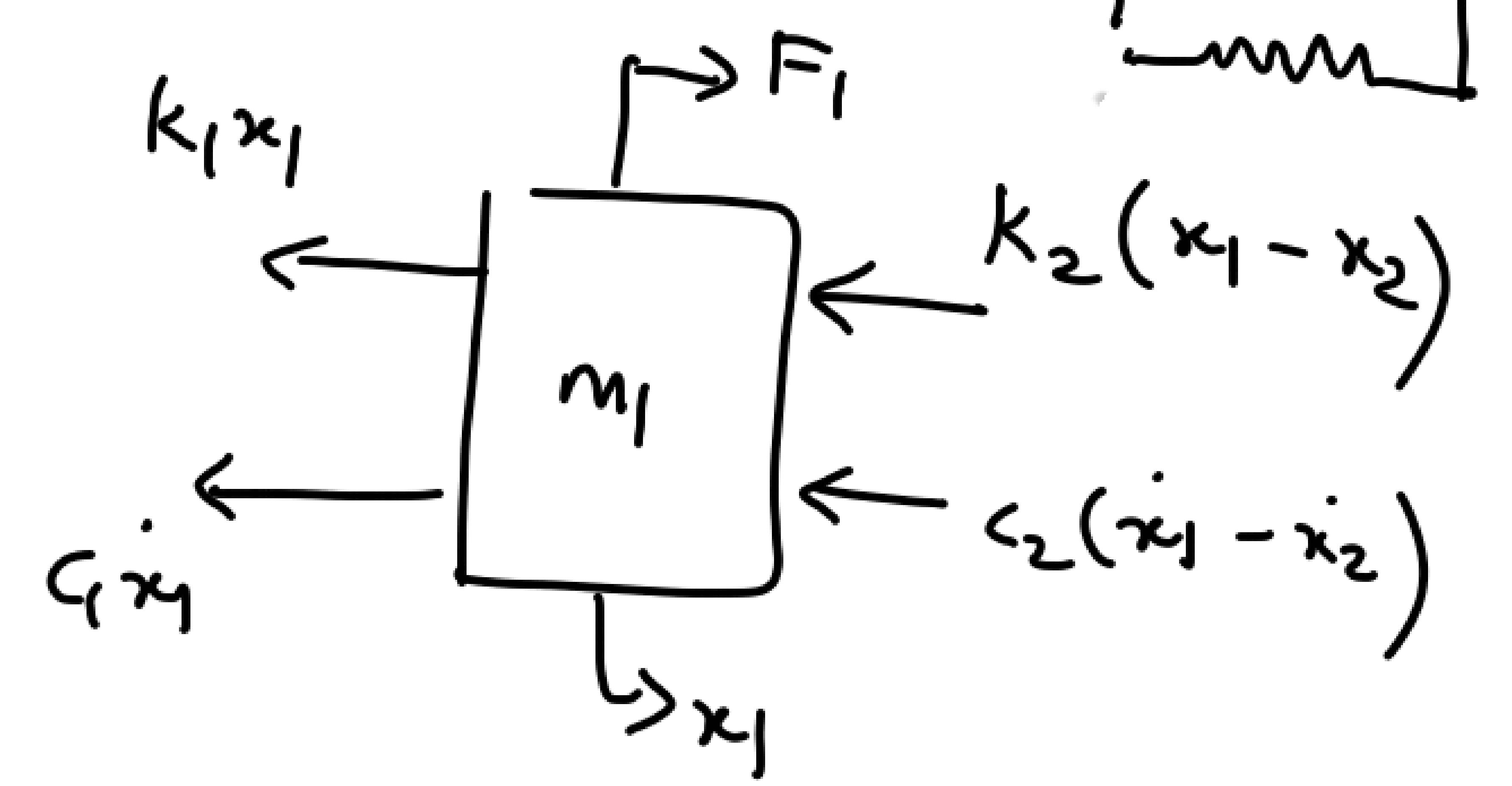
Vibration response due to

a) initial conditions

(b) Forcing on either or both masses

F.B.D of m_1 :

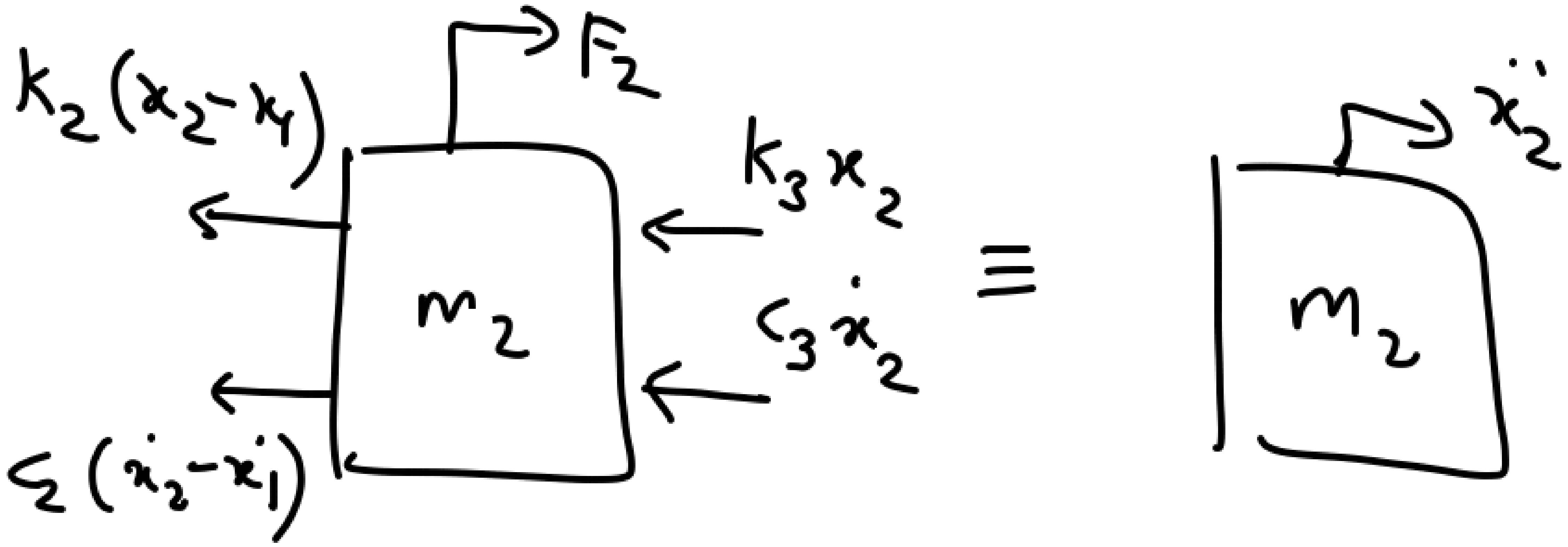




$$m_1\ddot{x}_1 = F_1 - k_1x_1 - c_1\dot{x}_1 - k_2(x_1 - x_2) - c_2(\dot{x}_1 - \dot{x}_2)$$

$$\boxed{m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 = F_1}$$

F.B.D of m_2 :



$$m_2 \ddot{x}_2 = F_2 - k_3 x_2 - c_3 \dot{x}_2 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1)$$

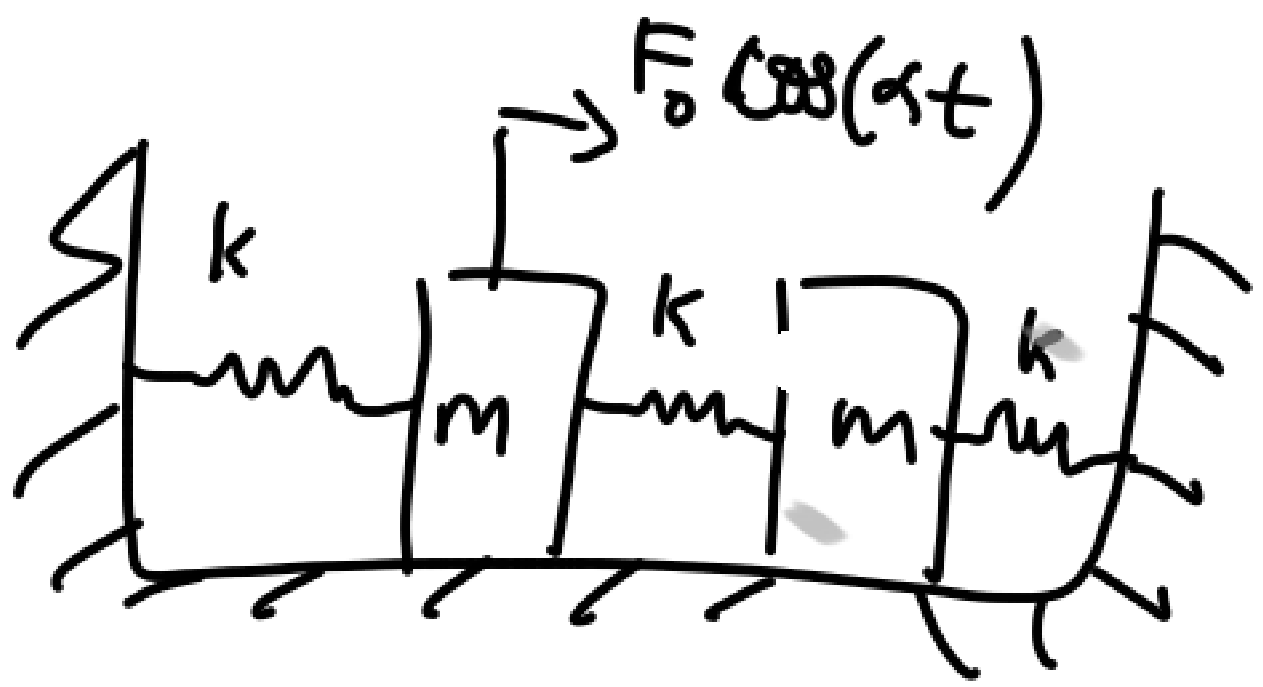
$$\boxed{m_2 \ddot{x}_2 + (k_2+k_3)\dot{x}_2 - k_2 x_1 + (c_2+c_3)\dot{x}_2 - c_2 \dot{x}_1 = F_2}$$

Combining the equations and writing in matrix form :

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$$M\ddot{X} + C\dot{X} + KX = F$$

Example: Undamped response



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_0 \cos(qt) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} F_0 \cos(qt) \\ 0 \end{pmatrix}$$

$$x_1(t) = x_1^h(t) + x_1^p(t)$$

$$x_2(t) = x_2^h(t) + x_2^p(t)$$

For homogeneous sol'n

$$x_1^h = X_1 \cos(\omega t)$$

$$x_2^h = X_2 \cos(\omega t)$$

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \cancel{\cos(\omega t)} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \cancel{\cos(\omega t)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For non-zero or non-trivial sol'n

$$\det\begin{vmatrix} 2k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{vmatrix} = 0$$

$$(2k-m\omega^2)^2 - k^2 = 0$$

or $2k - m\omega^2 = \pm k$

or $\omega_1 = \sqrt{\frac{k}{m}}$; $\omega_2 = \sqrt{\frac{3k}{m}}$

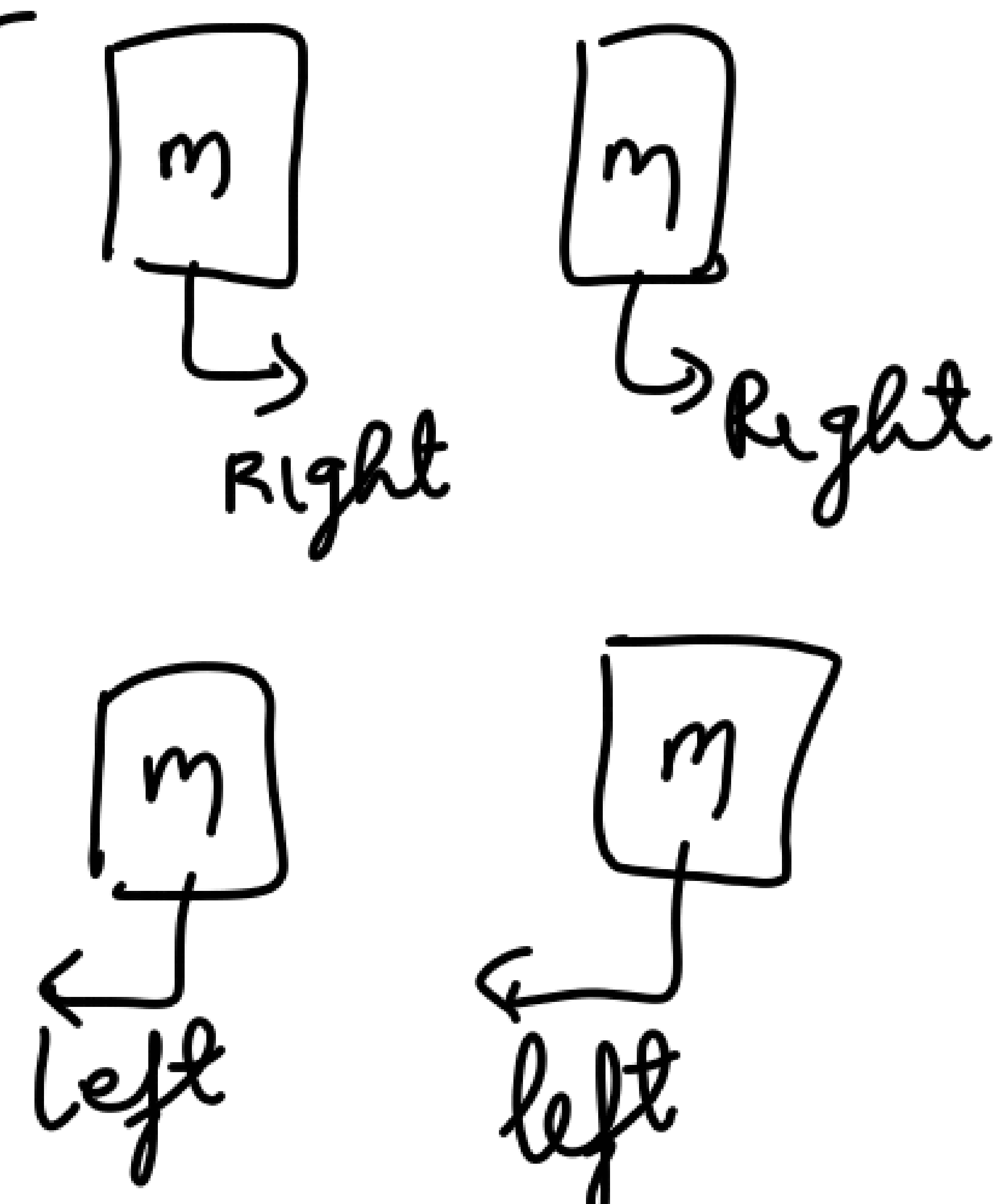
→ Eigen values or characteristic values or natural frequency of system

For $\omega = \omega_1$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = 1$; $x_2 = 1$

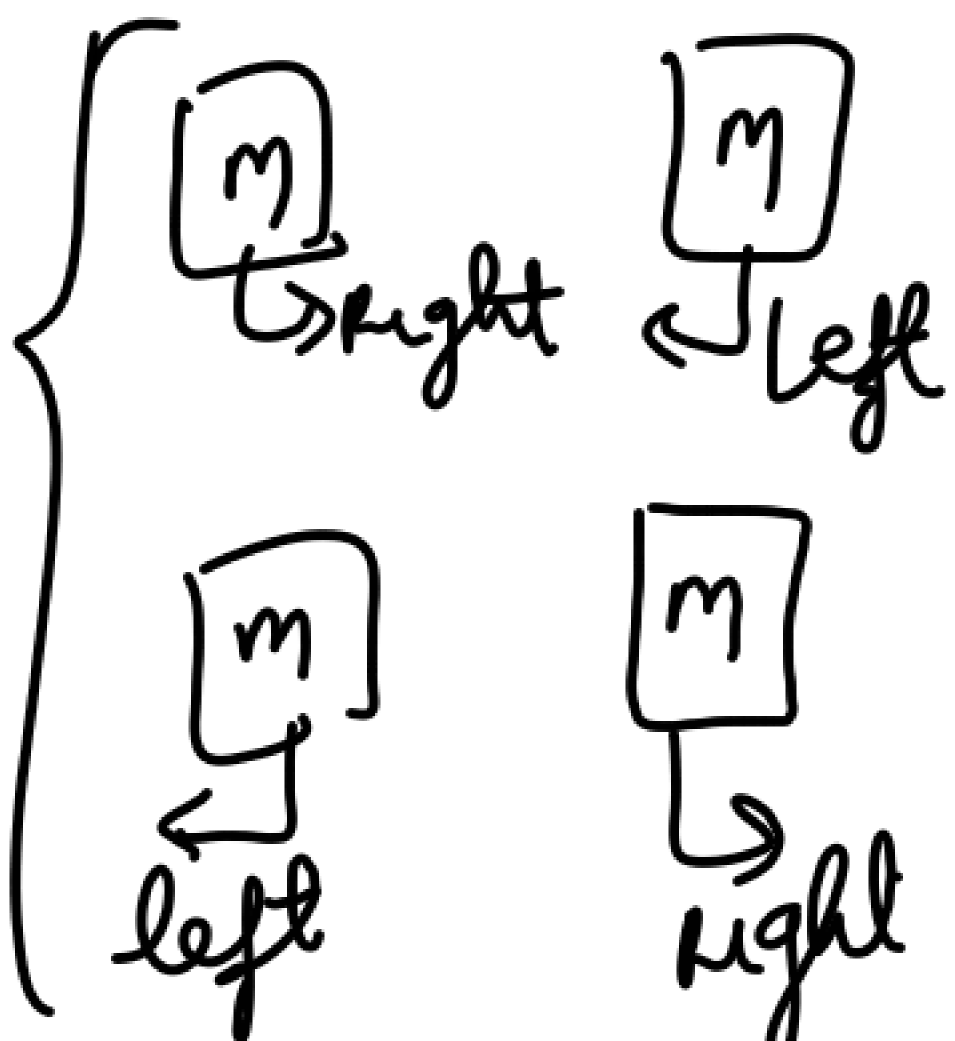
Mode Shape



$\omega = \omega_2$

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = 1$; $x_2 = -1$



Particular sol'n

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_0 \cos(\alpha t) \\ 0 \end{pmatrix}$$

$$x_1 = A\cos(\alpha t); \quad x_2 = B\cos(\alpha t)$$

$$-\alpha^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \cancel{\cos(\alpha t)} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \cancel{\cos(\alpha t)} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix} \cancel{\cos(\alpha t)}$$

$$-m\alpha^2 A + 2KA - KB = F_0$$

$$-\alpha^2 mB + (-KA) + 2KB = 0$$

$$A = \frac{\left[2 - \left(\frac{\alpha}{\omega_1}\right)^2\right]\left(\frac{F_0}{3k}\right)}{\left[1 - \left(\frac{\alpha}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\alpha}{\omega_2}\right)^2\right]}$$

$$B = \frac{(F_0/3K)}{\left[1 - \left(\frac{\alpha}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\alpha}{\omega_2}\right)^2\right]}$$

We can either have $\alpha = \sqrt{2}\omega_1$

or ω_1, ω_2 (m, k) are

chosen suitable to make

the response of first mass

i.e. $A\cos(\alpha t)$ to be zero or

minimal.

This is an example of vibration

isolation.