CS 207M Tutorial-6

1. Recall that, for a relation T, $T^* = \bigcup_{n=0}^{\infty} T^n$ is the reflexive-transitive closure of T.

   Let R be a partial order on a finite set A. Show that there exists a relation R' on A such that $R = R'^*$ and for any other relation Q such that $Q^* = R$, $R' \subseteq Q$. The relation R' is called the transitive reduction of R. Is this true if A is not finite?

2. Let $P = (A, \leq)$ be a finite poset. Show that $l(P).w(P) \geq |A|$ where the length $l(P)$ of P (resp. width $w(P)$ of P) is the size of the largest chain (resp. antichain) of P.

   Use this to show that any sequence of distinct $mn+1$ numbers contains either an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$. Hint: For a sequence $(a_i)_{i=1}^{mn+1}$, define a relation $\ll$ on $[mn+1]$ as follows: $i \ll j$ if $i \leq j$ and $a_i \leq a_j$. Show that $\ll$ is a partial order and apply the above result to this partial order.

3. Show that, for any sequence of distinct numbers, the minimum number of increasing subsequences into which this sequence can be partitioned is the length of the longest decreasing subsequence.

4. Let $P = (A, \leq_P)$ and $Q = (B, \leq_Q)$ be two posets. Consider the binary relation $\leq$ on $A \times B$ defined as follows:

   $$(a, b) \leq (a', b') \text{ iff } a \leq_P a' \text{ and } b \leq_Q b'$$

   Show that $R = (A \times B, \leq)$ is a poset. This poset is called the direct product of P and Q, and is denoted by $P \times Q$. Assume that P and Q are finite. Show that $P \times Q$ is finite and describe the covering relation of $P \times Q$ in terms of covering relations of P and Q.

5. Let $P = (N, \leq)$ be the poset of natural numbers ordered by the usual 'less than or equal to' relation. Show that every antichain of the poset $P \times P$ is finite. Can we partition $P \times P$ into finitely many chains?

6. Let S be a set and $\mathcal{F}$ be a collection subsets of S such that S and the empty set are in $\mathcal{F}$ and the intersection of any sets in $\mathcal{F}$ is also in $\mathcal{F}$. Show that the subsets in $\mathcal{F}$ ordered by inclusion form a lattice.

7. A lattice is said to be distributive if for all a, b, c

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Show that in a distributive lattice

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

8. Show that the boolean poset $\mathcal{B}_n = (2^{[n]}, \subseteq)$ is a distributive lattice.

9. Fix a positive integer n. Consider the set Π_n of all partitions of $[n]$. We define the refinement relation $\leq$ on Π_n as follows. Let $\pi = \{A_1, A_2, \ldots, A_p\}$ and $\sigma = \{B_1, B_2, \ldots B_q\}$ in Π_n. We say $\sigma \leq \pi$ if each B_i is contained in some A_j. Show that $(\Pi_n, \leq)$ is a lattice. Give an example to show $(\Pi_n, \leq)$ is not necessarily distributive.

10. Consider the division poset P on $[2n]$. Compute $w(P)$ and a minimum cardinality decomposition of P into chains.