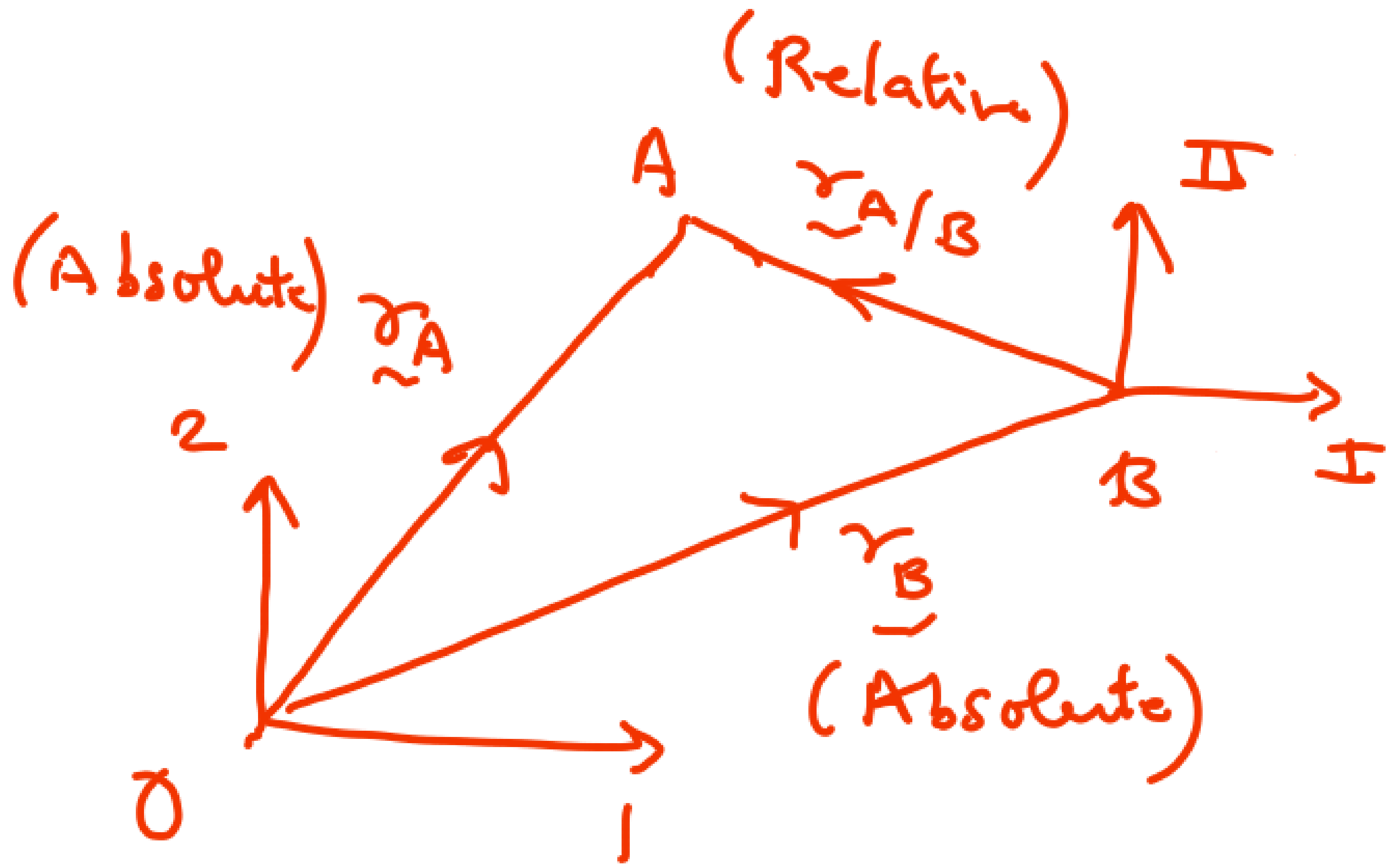
Relative motion



$$r_A = r_B + r_{A/B}$$

Diff. w.r.t time

$$V_A = V_B + V_{A/B}$$

$$a_A = a_B + a_{A/B}$$

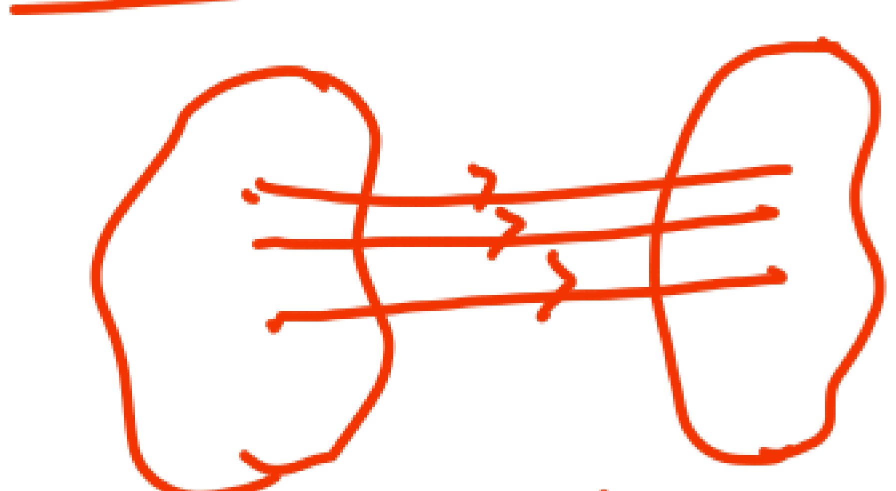
If B is stationary or undergoing translation at uniform speed, $a_A = a_{A/B}$

Inertial frames

Rigid body motion:

Rotation + Translation.

(a) Translation:



Rectilinear

Curvilinear translation

Knowledge of one point is enough.

All points are tracing paths which

are parallel to each other.

(b) Rotation:

(i) About a fixed axis

All the points are tracing circular path

Centre is the fixed point.

Radius = Distance of point from fixed point

$\measuredangle\ POP_1$

$\measuredangle\ gOg_1$

Angular velocity

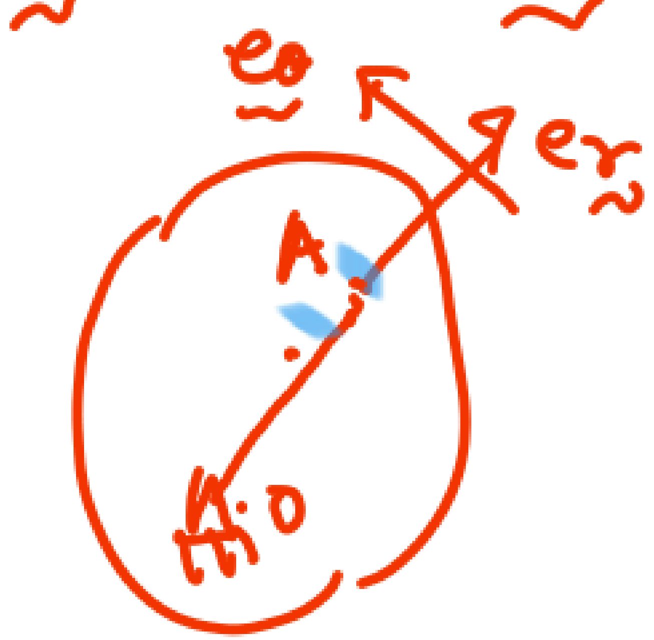
$$\omega = \frac{d\theta}{dt}$$

Angular acceleration

$$\alpha = \frac{d\omega}{dt}$$

Velocity of point P:

$$\underset{\sim}{v} = r\dot{\theta}\,\underset{\sim}{e_\theta}$$



out-of-plane unit vector

$$\underset{\sim}{e_3} = \underset{\sim}{e_1} \times \underset{\sim}{e_2}$$
$$= \underset{\sim}{e_r} \times \underset{\sim}{e_\theta}$$

$$\underset{\sim}{e_\theta} = \underset{\sim}{e_3} \times \underset{\sim}{e_r}$$

$$\underset{\sim}{v} = r\dot{\theta}\,(\underset{\sim}{e_3} \times \underset{\sim}{e_r})$$
$$= (\dot{\theta}\,\underset{\sim}{e_3}) \times (r\,\underset{\sim}{e_r})$$

$$\boxed{\underset{\sim}{v} = \underset{\sim}{\omega} \times \underset{\sim}{r}}$$

Angular velocity vector

In general case,

$$\underset{\sim}{\omega} = \dot{\theta}\,\underset{\sim}{e}^*$$

where $\mathbf{e}^*$ is the unit vector along axis of rotation.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{e}_r)$$

$$\mathbf{v} = r\frac{d\mathbf{e}_r}{dt} = \boldsymbol{\omega}\times\mathbf{r} = \boldsymbol{\omega}\times r\mathbf{e}_r$$

$$\Rightarrow \boxed{\frac{d\mathbf{e}_r}{dt} = \boldsymbol{\omega}\times\mathbf{e}_r}$$

i.e. Rate of change of a vector = cross-product of its angular velocity vector and the vector itself.

<u>Acceleration</u>: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$

$$\mathbf{a} = \frac{d}{dt}(\boldsymbol{\omega}\times\mathbf{r}) = \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + \boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}$$

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt}(\dot{\theta}\mathbf{e}_3) = \frac{d\dot{\theta}}{dt}\mathbf{e}_3 = \alpha\mathbf{e}_3 = \boldsymbol{\alpha}$$

$$\therefore \mathbf{a} = \boldsymbol{\alpha}\times\mathbf{r} + \boldsymbol{\omega}\times\mathbf{v}$$

$$\mathbf{a} = \boldsymbol{\alpha}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

$\underset{\sim}{r} = r\,\underset{\sim}{e_r}$;

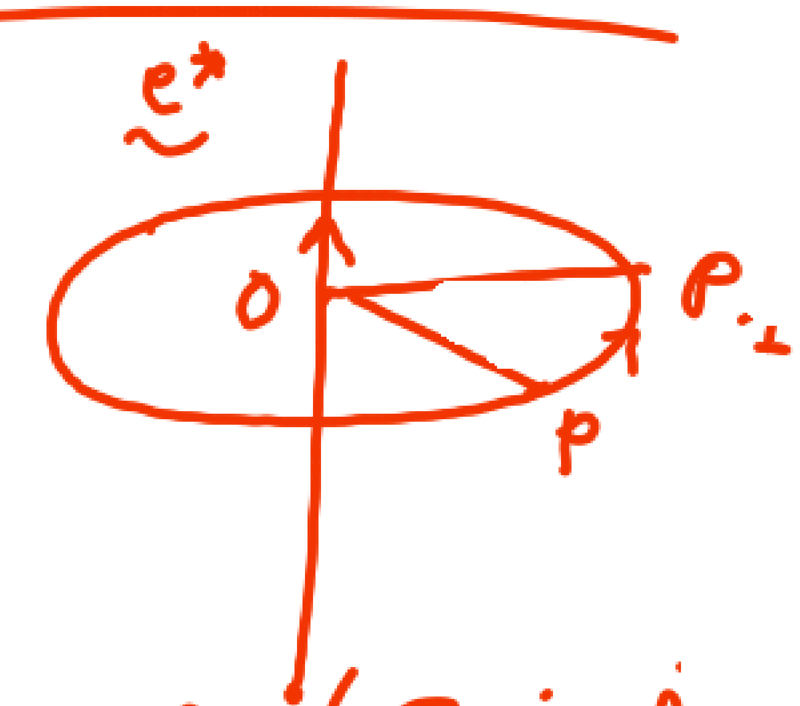
$\underset{\sim}{\omega} = \dot{\theta}\,\underset{\sim}{e_3}$; $\underset{\sim}{\alpha} = \alpha\,\underset{\sim}{e_3}$

$$\underset{\sim}{a} = \alpha r (\underset{\sim}{e_3} \times \underset{\sim}{e_r}) + \dot{\theta}\,\underset{\sim}{e_3} \times (\dot{\theta}\,\underset{\sim}{e_3} \times r\,\underset{\sim}{e_r})$$

$$\underset{\sim}{a} = r\alpha\,\underset{\sim}{e_\theta} + (\dot{\theta})^2 r\,(\underset{\sim}{e_3} \times \underset{\sim}{e_\theta})$$

$$\boxed{\underset{\sim}{a} = \underbrace{r\alpha\,\underset{\sim}{e_\theta}}_{\text{Tangential}} - \underbrace{(\dot{\theta})^2 r\,\underset{\sim}{e_r}}_{\text{Radial}}}$$

Rotation about a fixed point :



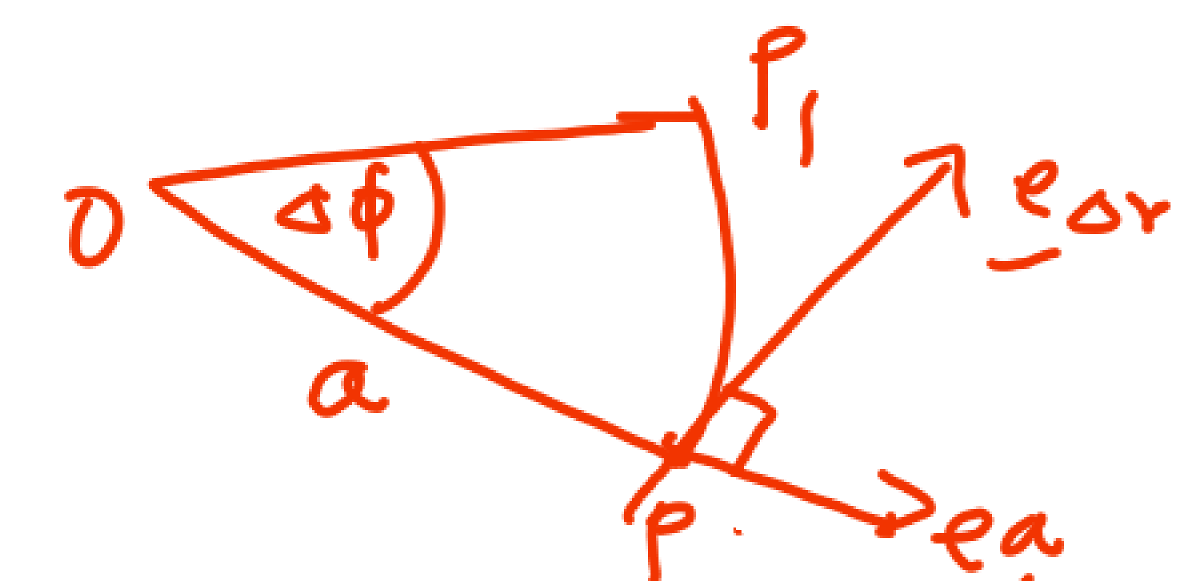
At the given in, let $\underset{\sim}{e^*}$ be the axis of rotation at time t.

$\measuredangle\, POP_1 = \Delta\phi$

P_1 is the location of point P after time Δt

$$\underset{\sim}{v_p} = \lim_{\Delta t \to 0} \frac{\Delta \underset{\sim}{r}}{\Delta t}$$

$\Delta \underset{\sim}{r}$ is the vector joining P_1 to P



$$\Delta \underset{\sim}{r} = a\,\Delta\phi\,(\underset{\sim}{e^*} \times \underset{\sim}{e_a})$$

Direction: Tangent to the circle @ point P.

Direction $e_{\delta r}$

$= e^* \times e_a$

$$V_p = \lim_{\Delta t \to 0} \frac{a\Delta\phi\,(e^* \times e_a)}{\Delta t}$$

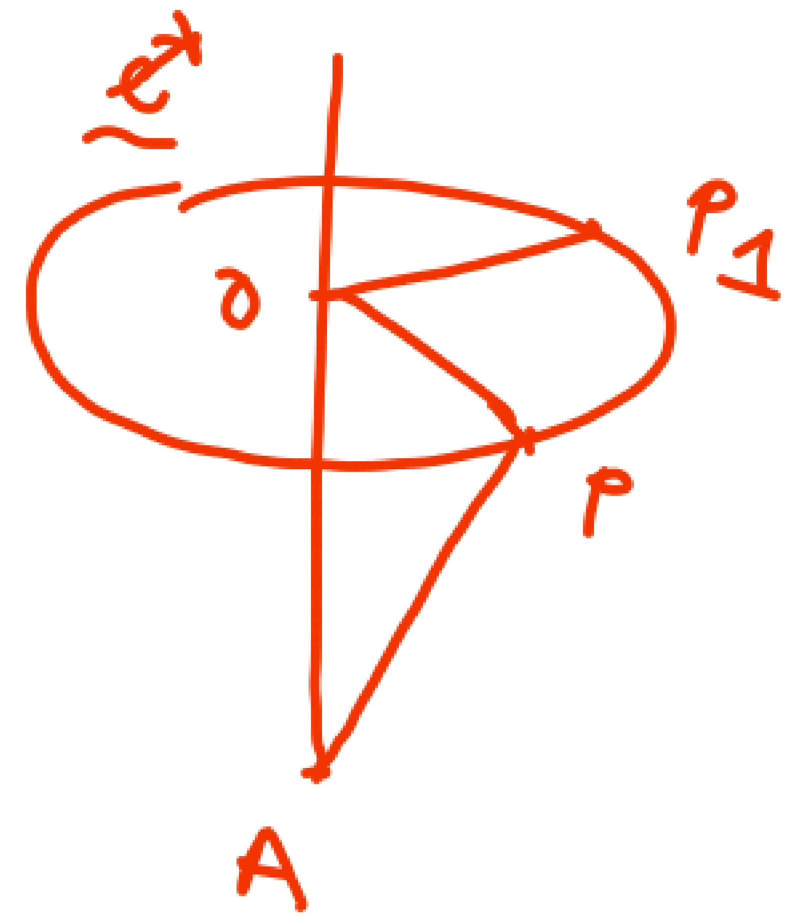
$$= \lim_{\Delta t \to 0} \frac{\Delta\phi\, e^* \times a e_a}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t}\, e^* \times r_{p/o}$$

$$= \omega e^* \times r_{p/o}$$

$$= \omega \times r_{p/o}$$

$\omega = \omega e^*$



Lets focus on Δ OAP

$$r_{p/A} = r_{o/A} + r_{p/o}$$

$$V_p = \omega \times (r_{p/A} - r_{o/A})$$

$$V_p = \omega \times r_{p/A} - \cancel{\omega \times r_{o/A}} \to 0$$

Last term is zero as ω and $r_{o/A}$ are co-axial vectors

$$\therefore \boxed{V_p = \omega \times r_{p/A}}$$

$$\boxed{\frac{d(r_{p/A})}{dt} = \omega \times r_{p/A}}$$

Acceleration:

$$a = \frac{dV}{dt}$$

$$\underset{\sim}{a} = \frac{d}{dt}\left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{P/A}\right)$$

$$= \frac{d\underset{\sim}{\omega}}{dt} \times \underset{\sim}{r}_{P/A}$$

$$+ \underset{\sim}{\omega} \times \frac{d\underset{\sim}{r}_{P/A}}{dt} \rightarrow \underset{\sim}{r}_P - \underset{\sim}{r}_A$$

$$\left(\frac{d\underset{\sim}{r}_P}{dt} - \cancel{\frac{d\underset{\sim}{r}_A}{dt}}^{\,0}\right)$$

$$= \underset{\sim}{\alpha} \times \underset{\sim}{r}_{P/A} + \underset{\sim}{\omega} \times \underset{\sim}{v}$$

$$\boxed{\underset{\sim}{a} = \underset{\sim}{\alpha} \times \underset{\sim}{r}_{P/A} + \underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{P/A}\right)}$$