CS 207M Tutorial-7

1. Show that if there are two different paths between distinct vertices u and v in a graph G, then there exists a cycle in G. Is it true that if there is a cycle containing u and v, and a cycle containing v and w then there is a cycle containing u and w?

2. Let G be a graph. Show that G is a tree iff there is unique path between any two vertices of G.

3. For a graph $G(V, E)$, the distance $d(v, w)$, between vertices v and w, is the length of the shortest path between v and w. Show that for any vertices v, w, x, we have the triangle law:

$$d(v, w) + d(w, x) \geq d(v, x)$$

   The diameter of a graph is the maximum of the distances between two vertices. Give examples of graphs with very large and very small diameters.

4. Let G be a connected graph. Recall that a spanning tree of G is a tree that has the same vertex set as that of G and its edge-set is a subset of edges of G. Let S and T be two spanning trees of G and $e \in S$ be a tree-edge for S. Show that there is an edge $f \in T$, such that $S - e + f$ and $T - f + e$ are spanning trees.

5. Prove that in any connected graph G with atleast two vertices, there are at least 2 vertices v such that $G - v$ is also connected. Let G be a connected graph such that, the number of connected components in $G - v$ is $\deg(v)$, for all vertices v. Prove that G is a tree.

6. Let G be a graph with atleast three vertices. Show that the following conditions are equivalent.
   (i) G is a cycle.
   (ii) $G - v$ is a tree for all vertices v.
   (iii) $G - e$ is a tree for all edges e.
   (iv) There are exactly two different paths in G between any two vertices.

7. Prove that if k edges are removed from a connected graph, the resulting graph contains at most $k + 1$ connected components.

8. Let G be the graph whose vertex set is the set of k-tuples with elements in $\{0, 1\}$, with vertex x adjacent to vertex y iff x and y differ in exactly two positions. Determine the number of components of G.

9. For $n \geq 3$, determine the minimum number of edges in a connected n-vertex graph in which every edge belongs to a triangle.

10. Let M be a maximal matching in a (not necessarily bipartite) graph G and N be a maximum matching. Show that $|N| \leq 2|M|$.

11. A permutation matrix is a 0-1 matrix such that each row and each column contains exactly one 1. Let A be a 0-1 matrix having exactly k 1's in each row and column. Prove that A can be expressed as a sum of k permutation matrices.

12. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a collection of subsets of $[n]$. A system of distinct representatives for $\mathcal{F}$ is a set of distinct element $a_1, a_2, \ldots, a_m$ in $[n]$ such that $a_i \in A_i$. Prove that $\mathcal{F}$ has a system of distinct representatives iff for every subset S of $[m]$,

$$| \cup_{j \in S} A_j| \geq |S|$$

13. Fix n and k such that $2k < n$. Consider the graph G whose vertex set consists of all subsets of $[n]$ of size k and $k+1$. Two vertices (i.e. subsets) A and B are joined if $A \subset B$. Prove that G has a matching in which all subsets of size k are matched.