Midsem Solutions

(2024) CS 207M

March 21, 2024

Problem 2: Let R_1, R_2 be symmetric relations on a set X. Show that $R_1 \circ R_2$ is symmetric if and only if $R_1 \circ R_2 = R_2 \circ R_1$

We have to prove both directions:

(a) If $R_1 \circ R_2$ is symmetric then $R_1 \circ R_2 = R_2 \circ R_1$ (2 marks).

We will show that $R_1 \circ R_2 \subseteq R_2 \circ R_1$, and vice-versa, which will prove that $R_1 \circ R_2 = R_2 \circ R_1$.

Let (a, c) be an arbitrary element in $R_1 \circ R_2$. By symmetry of $R_1 \circ R_2$, $(c, a) \in R_1 \circ R_2$. By definition of composition, there exists $b \in X$ such that $(c, b) \in R_1$ and $(b, a) \in R_2$. By symmetry of R_1 and R_2, $(b, c) \in R_1$ and $(a, b) \in R_2$. Therefore by definition of composition $(a, c) \in R_2 \circ R_1$. This shows that $R_1 \circ R_2 \subseteq R_2 \circ R_1$.

Other side can be done similarly.

(b) If $R_1 \circ R_2 = R_2 \circ R_1$ then $R_1 \circ R_2$ is symmetric (2 marks).

Let (a, c) be an arbitrary element in $R_1 \circ R_2$. Since $R_1 \circ R_2 = R_2 \circ R_1$, $(a, c) \in R_2 \circ R_1$. By definition of composition, there exists $b \in X$ such that $(a, b) \in R_2$ and $(b, c) \in R_1$. By symmetry of R_1 and R_2, $(b, a) \in R_2$ and $(c, b) \in R_1$. By definition of composition, $(c, a) \in R_1 \circ R_2$ and we are done.

Problem 3: Let $f : X \rightarrow X$ be a function. We define a relation $\sim_f$ on X as follows: $x \sim_f y$ if $\exists m \geq 0$ such that either $f^m(x) = y$ or $f^m(y) = x$.

$\sim_f$ is not an equivalence relation as it need not be transitive, although it is trivially reflexive and symmetric. We will show that $\sim_f$ is not transitive by giving a counter-example. Consider $X = \{1, 2, 3\}$ and the constant function $f(x) = 2$. By definition $1 \sim_f 2$ since $f(1) = 2$ and $2 \sim_f 3$ since $f(3) = 2$. But $(1, 3) \notin \sim_f$ since regardless of any number of applications of f, there is no way to reach 3 from 1 or vice-versa.