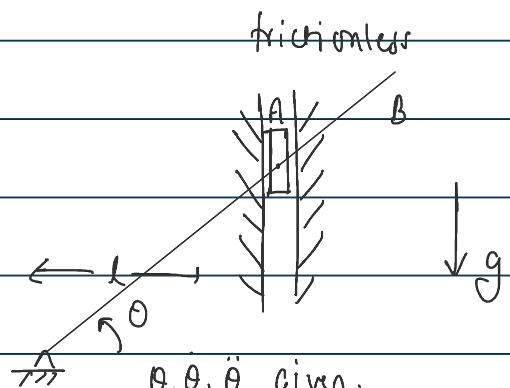
Kinetics

Tuesday, 12 March 2024 12:03 PM

⇒ Newton's 2^{nd} Law:

$$\boxed{\sum \vec{F} = m\vec{a}}$$

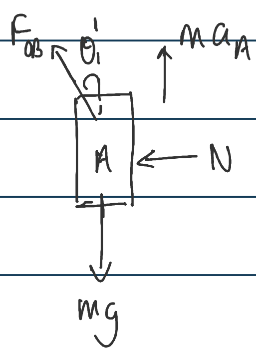
eg.



$\theta, \dot{\theta}, \ddot{\theta}$ given.

Kinematics: $x_A = l$ $\quad y_A = l\tan\theta$

$V_A = l\sec^2\theta \cdot \dot{\theta}$ $\quad a_A = l\sec\theta \cdot \ddot{\theta} + l\, 2\sec^2\theta \cdot \tan\theta\, (\dot{\theta})^2$



F_{OB} are perpendicular to the rod OB since motion along OB is unrestricted (pin joint)

$$ma_A + F_{OB}\cos\theta = mg$$

$$F_{OB} = mg\sec\theta - ml\ddot{\theta}\sec^2\theta - 2ml\sec^3\theta\tan\theta(\dot{\theta})^2$$

⇒ Work Energy relation:

$$dW = \vec{F} \cdot d\vec{r}$$

$$W_{1-2} = \int_1^2 \vec{F} \cdot d\vec{r}$$

In tangential-normal coordinates,

$$\vec{F} \cdot d\vec{r} = (F_n \hat{n} + F_t \hat{t}) \cdot (ds\, \hat{t}) = F_t\, ds$$

$$W_{A-B} = \int_A^B F_t\, ds = \int_A^B m a_t\, ds$$

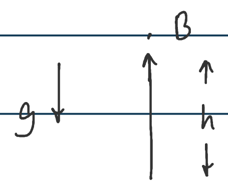
$$= \int_A^B m \frac{dv}{dt}\, ds = m \int_A^B v\, dv$$

$$= \frac{m}{2}\left(v_B^2 - v_A^2\right) = KE_B - KE_A = \Delta KE$$

$$\boxed{W_{net} = \Delta KE}$$

Two important cases:

1) Gravity



$$W = -mgh$$

1) gravity: $g\downarrow$ $h\downarrow$ A $W = mgh$

We define a potential V such that $\Delta V = -W = mgh$

$\Delta V^{(g)}$: change in potential energy due to work done against gravity.

2) In the presence of elastic springs:

(Linear elastic spring)

$$dW_{sp} = F\,dx = -kx\,dx$$

$$W_{sp_{A-B}} = -\int_A^B kx\,dx = -\Delta\left(\frac{1}{2}kx^2\right)$$

We define a potential V^{sp} s.t. $\Delta V^{sp} = -W_{sp} = \Delta\left(\frac{1}{2}kx^2\right)$

$V^{sp} = \frac{1}{2}kx^2$: potential energy due to work done against spring force.

Work energy relation:

$$W = W_{other} - \Delta V^{(g)} - \Delta V^{(sp)} = \Delta KE$$

$$\boxed{W_{other} = \Delta KE + \Delta V^{(g)} + \Delta V^{(sp)}}$$

=> Momentum:

1) Linear Momentum :

$$\underline{G} = m\underline{v}$$

$$\frac{d\underline{G}}{dt} = \frac{d(m\underline{v})}{dt} = m\frac{d\underline{v}}{dt} = \underline{F}_{net}$$

$$\boxed{\dot{\underline{G}} = \underline{F}}$$ (Rate of change of linear momentum = Net external unbalanced force)

$$\Delta G = \int F\,dt \qquad F\,dt: \text{Linear impulse}$$

2) Angular momentum:

$$\underline{H}_0 = \underline{r}_0 \times (m\underline{v})$$

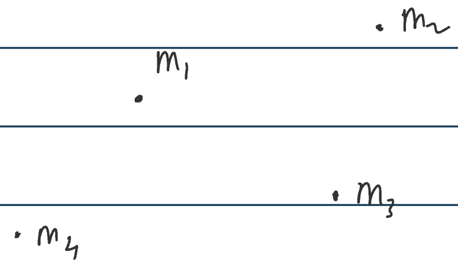
$$\dot{\underline{H}}_0 = \dot{\underline{r}} \times (m\underline{v}) + \underline{r} \times (m\dot{\underline{v}})$$

$$= \underline{v} \times (m\underline{v}) + \underline{r} \times (m\underline{a})$$

$$= \underline{r} \times \underline{F} = \underline{M}_0$$

$$\boxed{\dot{\underline{H}}_0 = \underline{M}_0}$$

⇒ **System of Particles:**



$\underline{F}^{(i)}$: External force on i^{th} particle.

$\underline{f}^{(i-j)}$: Pairwise interaction force b/w particle 1 and 2.

$$\underline{f}^{(2-1)} = -\underline{f}^{(1-2)}$$

For i^{th} particle:

$$\underline{F}^{(i)} + \underline{f}^{(i)} = m^i \underline{a}^{(i)}$$

Interaction force
due to remaining particles.

$$\sum \underline{F}^{(i)} + \underbrace{\cancel{\sum \underline{f}^{(i)}}}_{0} = \sum m_i a^{(i)}$$

$$\boxed{\sum \underline{F}^{(i)} = \sum m_i a^{(i)}}$$

Center of gravity :

$$\underline{r}_c = \frac{\sum m_i \underline{r}_i}{\sum m_i} = \frac{\sum m_i \underline{r}_i}{m}$$

$$\underline{\ddot{r}}_c = \frac{\sum m_i \underline{\ddot{r}}_i}{m} \Rightarrow m \vec{a}_G = \sum m_i \vec{a}_i$$

$$\boxed{\vec{F} = m \vec{a}_G}$$

Linear Momentum :

$$\underline{G}_i = m_i \underline{v}_i$$

$$\sum \underline{G}_i = \sum m_i \underline{v}_i = m \underline{V}_G$$

$$\boxed{\underset{\sim}{G} = m\, \underset{\sim}{v}_G}$$

⇒ <u>Angular Momentum</u> $(\underset{\sim}{H})$:

$r_i = \rho_i + \bar{\underset{\sim}{\rho}}$ $\quad$ r_i: Position vector of i^{th} particle w.r.t. O.

$$\underset{\sim}{H}_0 = \sum (\underset{\sim}{H}_0)_i = \sum \underset{\sim}{r}_i \times (m_i \dot{\underset{\sim}{r}}_i)$$

$$\underset{\sim}{H}_G = \sum (\underset{\sim}{H}_G)_i = \sum \underset{\sim}{\rho}_i \times (m_i \dot{r}_i)$$

$$\dot{\underset{\sim}{H}}_0 = \cancelto{0}{\sum \dot{\underset{\sim}{r}}_i \times (m_i \dot{\underset{\sim}{r}}_i)} + \sum \underset{\sim}{r}_i \times (m_i \ddot{\underset{\sim}{r}}_i)$$

$$= \sum \underset{\sim}{r}_i \times (m_i a) = \sum (M_0)_i$$

$\therefore$ $\boxed{\dot{\underset{\sim}{H}}_0 = \underset{\sim}{M}_0}$ ⇒ Rate of change of angular momentum about a fixed point O = Sum of moments due to external forces at the fixed point O.

For G:

$$\dot{\underset{\sim}{H}}_G = \sum \dot{\underset{\sim}{\rho}}_i \times (m_i \dot{r}_i) + \sum \underset{\sim}{\rho}_i \times (m_i \ddot{r}_i)$$

$$= \cancelto{0}{\sum \dot{\underset{\sim}{\rho}}_i \times (m_i \dot{\underset{\sim}{\rho}}_i)} + \cancelto{0}{\sum m_i \dot{\underset{\sim}{\rho}}_i} \times \dot{\bar{\underset{\sim}{\rho}}} + \underset{\sim}{M}_G$$

$\therefore$ $\boxed{\dot{\underset{\sim}{H}}_G = \underset{\sim}{M}_G}$ $\Rightarrow$ Rate of change of angular momentum about center of mass = Moment about center of mass due to external forces.

$\Rightarrow$ Balance of Angular Momentum:

$$\dot{\underline{H}}_0 = \sum \underline{M}_0$$

$$\dot{\underline{H}}_G = \sum \underline{M}_G$$

Moment due to external forces

How does the balance of angular momentum change about a point P other than O and G?

Suppose P is any point other than the C.G.

$\underline{r}_i'$: position of particle i wrt P.

$\underline{r}_i' := \underline{\bar{\rho}} + \underline{\rho}_i$ $\qquad \bar{\rho}$: Center of gravity

$$\underset{\sim}{H}_P = \sum \underline{r}_i' \times (m_i \underline{v}_i)$$

$$= \sum (\underline{\bar{\rho}} + \underline{\rho_i}) \times (m_i \underline{\dot{r}_i'})$$

$$= \underline{\bar{\rho}} \times \sum m_i \underline{\dot{r}_i} + \sum \rho_i \times m_i \underline{\dot{r}_i'}$$

$$= \underline{\bar{\rho}} \times m \underline{\bar{v}} + \underline{H_G}$$

$$\therefore \boxed{\underline{H_P} = \underline{r_G} \times \underline{G} + \underline{H_G}}$$

$\Rightarrow$ Moment at P in terms of moment at G:

$$\underline{F} = \sum \underline{F_i} \qquad \underline{M_G} = \sum \underline{\rho_i} \times \underline{F_i}$$

$$\underline{M_P} = \sum (\underline{\bar{\rho}} + \underline{\rho_i}) \times \underline{F_i} = \underline{\bar{\rho}} \times \underline{F} + \underline{M_G}$$

$$\boxed{\underline{M_P} = \underline{M_G} + \underline{\bar{\rho}} \times \underline{F}}$$

$\Rightarrow$ Rate of change of H_P (P is moving):

$$H_P^{rel} = \sum_i \underset{\sim}{r_i'} \times (m_i \underset{\sim}{\dot{r}_i'})$$

(relative angular velocity) (Relative Linear Momentum)

$$\dot{\underset{\sim}{H_P}}^{rel} = \sum \underset{\sim}{\dot{r}_i} \times (m_i \underset{\sim}{\dot{r}_i'}) + \sum \underset{\sim}{r_i} \times (m_i \underset{\sim}{\ddot{r}_i})$$

0

$$= \sum (\bar{\rho} + \rho_i) \times m_i (\ddot{\bar{\rho}} + \ddot{\rho}_i)$$

$$= \bar{\rho} \times m \ddot{\bar{\rho}} + \bar{\rho} \times \cancelto{0}{\sum m_i \ddot{\rho}_i} + \sum \rho_i \times m_i \ddot{r}_i$$

$$= \bar{\rho} \times m \ddot{\bar{\rho}} + M_G$$

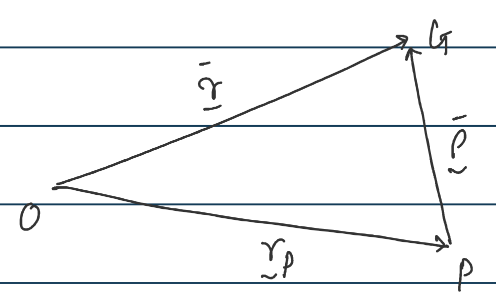
$$M_p = M_G + \bar{\rho} \times m \bar{a}$$

$$\dot{H}_p^{rel} = \bar{\rho} \times m \ddot{\bar{\rho}} + M_p - \bar{\rho} \times m \bar{a}$$

$$\boxed{\dot{H}_p^{rel} = \bar{\rho} \times m (\ddot{\bar{\rho}} - \bar{a}) + M_p}$$

At $P = G$, $\quad r_i' = \bar{\rho} + \rho_i = \rho_i$

$$\therefore \dot{H}_G^{rel} = \dot{H}_G = M_G$$



$$\bar{r} = r_p + \bar{\rho}$$

$$\therefore \vec{a} = \vec{a}_p + \ddot{\bar{\rho}}$$

$$\therefore \quad \boxed{\underset{\sim}{M}_P = \dot{\underset{\sim}{H}}_P^{rel} + \bar{\underset{\sim}{\rho}} \times m\underset{\sim}{a}_P} \Rightarrow \text{Most general form.}$$

$\Rightarrow$ <u>Summary</u> :

① $\underset{\sim}{F} = m\bar{\underset{\sim}{a}}$

② $\Sigma \underset{\sim}{M}_O = \dot{\underset{\sim}{H}}_O \rightarrow$ about fixed origin

$\Sigma \underset{\sim}{M}_G = \dot{\underset{\sim}{H}}_G \rightarrow$ about center of gravity

$\Sigma \underset{\sim}{M}_P = \dot{\underset{\sim}{H}}_P^{rel} + \bar{\underset{\sim}{\rho}} \times m\underset{\sim}{a}_P \rightarrow$ about general point P.