Factorial Design

- When we wish to compare two different techniques, we may either use the independent t-test, or we may use a paired t-test if a nuisance variable is to be blocked away.
- Suppose now that we want to compare several techniques, the independent t-tests would no longer be adequate and **a k-variable** analysis should be used.
- There are many experimental designs available which may be chosen to suit particular experimental situations. But these designs, e.g., k-variable analysis, involve certain assumptions and restrictions.
- We will now introduce a general and effective class of experimental design called the **Factorial Design which includes k-variable analysis**, etc. as a special case.
- **To develop a general factorial design, one would select a fixed number of "levels" for each of a number of variables (factors) and then design tests with all possible combinations.**

For example, if there are 'L_1' levels of variable 1, 'L_2' levels of variable 2, and 'L_3' levels of variable 3, …

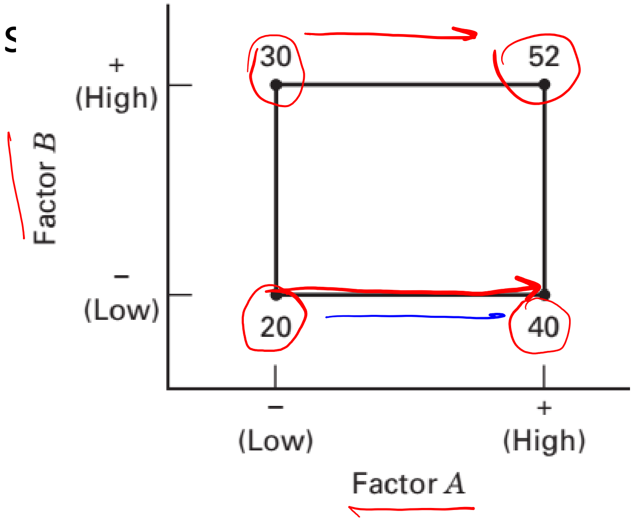
and are 'L_k' levels of variable k; then the total number of observations required will be $L_1 \times L_2 \times \cdots \times L_k$

Factorial Designs

- Factorial designs are preferred over other experimental designs because :
  - They require relatively few runs per variable (factor) even though they do not explore the entire region of interest
  - They deal easily with variable interactions
  - hey can indicate major trends and help determine a possible direction for further experimentation
  - They can be augmented to form composite designs
  - They form the basis for fractional factorial designs
  - The interpretation of the data produced by the designs can be easily analyzed.
- **Variables and Levels**
  - A design with 3 variables at 2 levels would require 8 tests for a $2 \times 2 \times 2 = 2^3$ factorial design
  - A design with 3 variables at 3 levels would need 27 tests for a $3 \times 3 \times 3 = 3^3$ factorial design.
  - A design with 5 variables at 2 levels would need 32 tests for a 2^5 factorial design.

Factorial Design: Example

- Consider a factorial experimental design with 2 variables (A and B) each at 2 levels
- Since there are only two levels of each variable, we can call the 'low' and 'high'



What is the 'main effect' of variable A? i.e., how does the result change when you change variable A from low setting to high?

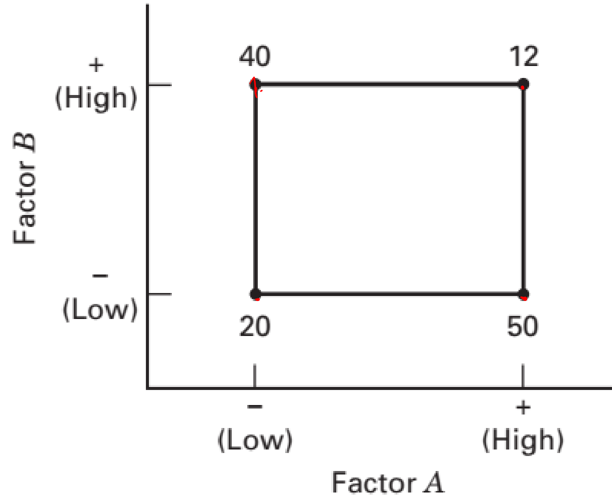
factor B is low +20
factor B is high +22
+21

What is the 'main effect' of variable B? i.e., how does the result change when you change variable A from low setting to high?

factor A is low +10
factor A is high +12
+11

The effect of a factor is defined to be the change in response produced by a change in the level of the factor. This is frequently called the **main effect** because it refers to the primary factors of interest in the experiment.

Interaction Effect

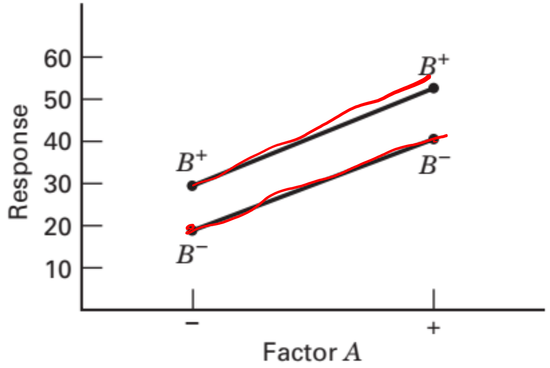


- In some experiments, we may find that the difference in **response between the levels of one factor is not the same at all levels of the other factors**. When this occurs, there is an **interaction between the factors**.
- **Interaction Effect**

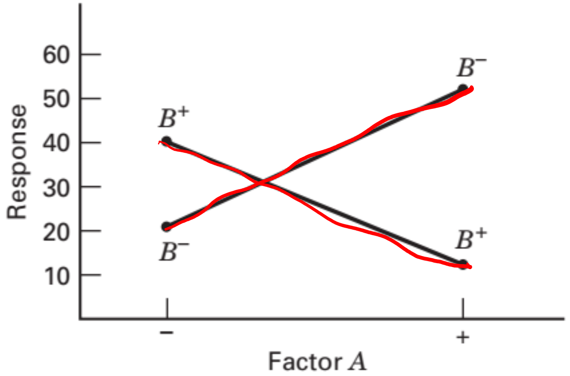
At Low level of B, effect of A is = +30

At High level of B, effect of A is = −28

+1



■ **FIGURE 5.3** A factorial experiment without interaction



■ **FIGURE 5.4** A factorial experiment with interaction

Example

Welding of High Strength Steel Bars

- High carbon steel, because of its high strength and low cost, has been known to have a potential for a "good market". However, because of its high carbon content, it is not easy to weld.
- According to the code of the American Welding Society (AWS), additional steps of pre-heating and post-heating are required in order to have good quality welds and high strength steel.
- A user of this steel was interested to study whether or not these additional steps of pre-heating and post-heating were really needed.
- After a preliminary investigation by manual arc welding tests, it appeared that there were **three variables significantly affecting the ultimate tensile stress of a weld**.
    1. Ambient temperature,
    2. Wind velocity
    3. Bar size
- The evidence, however, was not decisive, and further experiments were therefore planned. **How many tests should be conducted?**

Example

Example 2: Welding of High Strength Steel Bars

- Based upon available funds and time limit, it was decided that 244 tests can be run (maximum).
- In the meantime, an engineering statistician, who was called upon for consultation, suggested that 16 tests be run according to his specified sets of test conditions.
- The statistical experimental design that was formulated by the engineering statistician for this study was **a two-level three-variable factorial design**, simply designated as a 2^3 factorial design.
- The three selected variables were:
  - Ambient temperature, denoted by T
  - Wind velocity, denoted by V
  - Bar size, denoted by B
- Two levels ('low' and 'high') were chosen for each variable based upon desired field conditions to be simulated.

$2 \times 2 \times 2 = 8 = 2^3$

1

repeated 2

Example

Welding of High Strength Steel Bars

- The experimental variables and their two levels (low and high) are shown in the Table.

Variable	Unit	Low	High
Ambient Temperature (T)	°F	0	70
Wind Velocity (V)	mph	0	20
Bar Size (B)	1/8 inch	4	11

-1 +1
-1 +1
-1 +1

Transformation of Levels

- In order to adopt a notation which will be the same for all two-level factorial designs, we use transforming equations to code the variables such that
  - the high level will be denoted by +1,
  - the low level will be denoted by -1.

$$x_T = \frac{T - T_{avg}}{T_{avg}}$$

$$x_T = \frac{T - T_a}{T_a}$$

- By so doing, regardless of the physical conditions represented by the two levels, the basic design of any two-level factorial design becomes a simple arrangement of +1 and -1.

Example

- Construction of 2^k Factorial Design

	Coded Test Conditions			Actual Test Conditions		
Test #	X1	X2	X3	°F	(mph)	(1/8 in)
1	-1	-1	-1	0	0	4
2	1	-1	-1	70	0	4
3	-1	1	-1	0	20	4
4	1	1	-1	70	20	4
5	-1	-1	1	0	0	11
6	1	-1	1	70	0	11
7	-1	1	1	0	20	11
8	1	1	1	70	20	11

Example

Results of the factorial experiment with 2 repetitions (total 8*2 = 16 trials)

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

Example

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

$\Delta y_{1\to 2} = 87.3 - 87.5 = -0.2$ ✓

$\Delta y_{3\to 4}$ 9.2 ✓

$\Delta y_{5\to 6}$

$\Delta y_{7\to 8}$

avg

Average effect of Temperature (E1)

- Note that between Test 1 and Test 2, Wind (X2) and Bar Size (X3) is same. The only change is that Temperature (X1) goes from LOW (-1) to HIGH (+1). Therefore, the difference in these two test results (apart from the intrinsic variation that is present) can be attributed solely to the effect of ambient temperature alone.
- Similarly, for the pairs of Test 3 and 4, 5 and 6, and 7 and 8 in Table; each pair involved similar test conditions with respect to wind velocity and bar size, but different test conditions with respect to ambient temperature. Thus, the differences in the results within each of these four pairs reflect the effect of ambient temperature alone.

Example

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

$$E_1 = \frac{-0.2 + 9.2 + 18.5 + 9.1}{4}$$

$$\left.\begin{aligned}
&(Test\ Nos.\ 1\ and\ 2)\ \ \bar{y}_2 - \bar{y}_1 = 87.3 - 87.5 = -0.2\\
&(Test\ Nos.\ 3\ and\ 4)\ \ \bar{y}_4 - \bar{y}_3 = 87.0 - 77.8 = 9.2\\
&(Test\ Nos.\ 5\ and\ 6)\ \ \bar{y}_6 - \bar{y}_5 = 97.6 - 79.1 = 18.5\\
&(Test\ Nos.\ 7\ and\ 8)\ \ \bar{y}_8 - \bar{y}_7 = 87.7 - 78.6 = 9.1
\end{aligned}\right\}$$

Average Effect of Temperature would be an average of these FOUR numbers

Main Effect

y_1 y_2 $\bar{y}$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

y

Average effect of Temperature (E1)

- OR we can also write he average effect of ambient temperature (called ‘E1’) as:

$$E_1 = 1/4[(\bar{y}_2 - \bar{y}_1) + (\bar{y}_4 - \bar{y}_3) + (\bar{y}_6 - \bar{y}_5) + (\bar{y}_8 - \bar{y}_7)]$$
$$= 1/4[-0.2 + 9.2 + 18.5 + 9.1]$$
$$= 9.15 \text{ units of } 1000 \text{ psi}$$
$$= 9150 \text{ psi.}$$

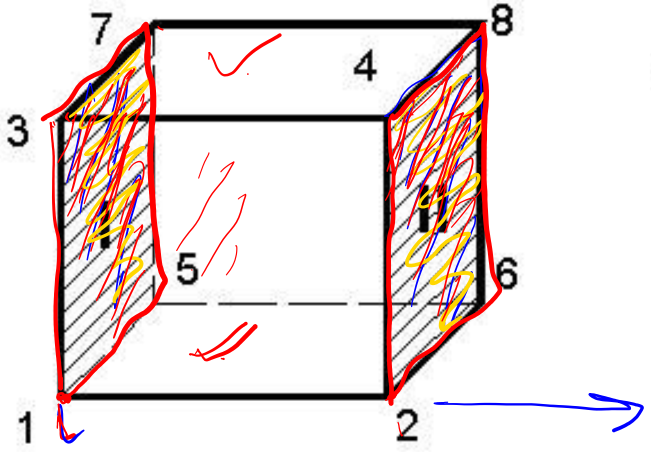
$E_1 = 1/4 \left(y_2 + y_4 + y_6 + y_8 - y_1 + y_3 + y_5 + y_7 \right)$

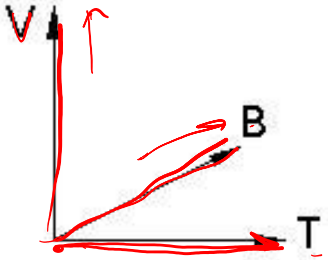
- **Note that the average effect is commonly referred to as *main effect*.**

Main Effect

Average effect of Temperature (E1)

- Geometrically, the average effect of Temperature is the differer LOW) and plane 2 (Temp setting HIGH)





Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

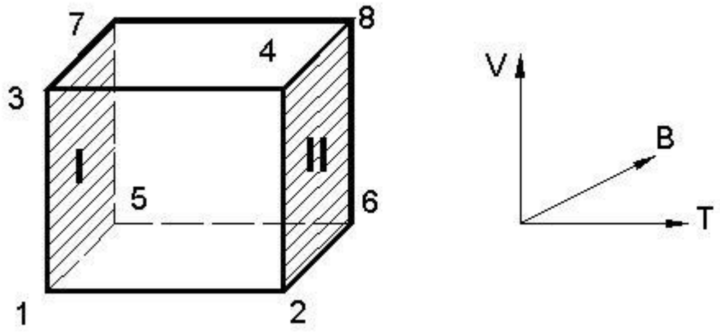
avg when T_H — avg when T_{lo}

$$E_1 = \frac{1}{4}\left[(\bar{y}_2 + \bar{y}_4 + \bar{y}_6 + \bar{y}_8) - \underline{(\bar{y}_1 + \bar{y}_3 + \bar{y}_5 + \bar{y}_7)}\right]$$

high level of ambient temperature — Low level of ambient temperature

Main Effect

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7



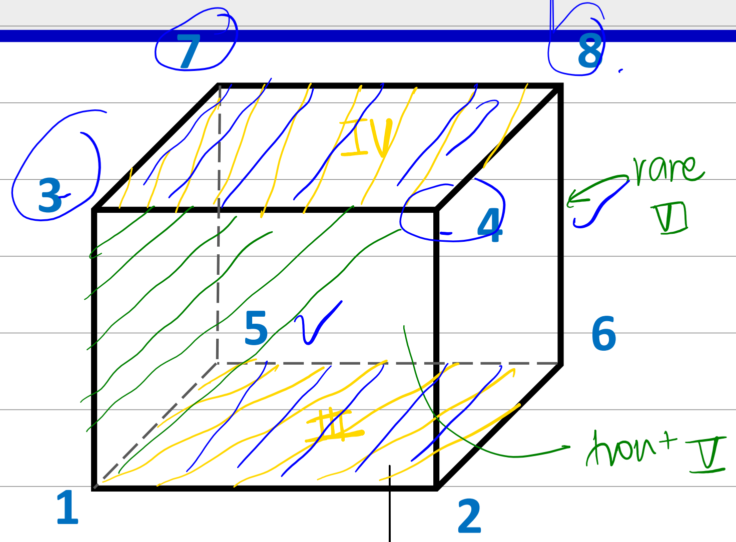
Average effect of Temperature (E1)

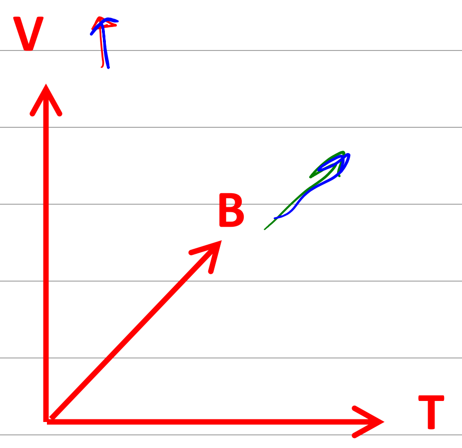
- The average effect of ambient temperature tells us that on the average, over the ranges of other variables in this investigation, the effect of changing the ambient temperature from LOW to HIGH increases the ultimate tensile stress by 9150 psi.
- But, notice that the individual differences (y2 – y1 = -200 psi, y4 – y3 = 9200 psi, y6 – y5 = 18000 psi, y8 - y7 = 9100 psi) are quite erratic.

The average effect, therefore, must be interpreted in conjunction with the intrinsic variabilities that are present in the experimental results.

Main Effect

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





$$E_2 = \frac{1}{4}\left((\bar{Y}_3 - \bar{Y}_1) + (\bar{Y}_4 - \bar{Y}_2) + (\bar{Y}_8 - \bar{Y}_6) + (\bar{Y}_7 - \bar{Y}_5)\right)$$

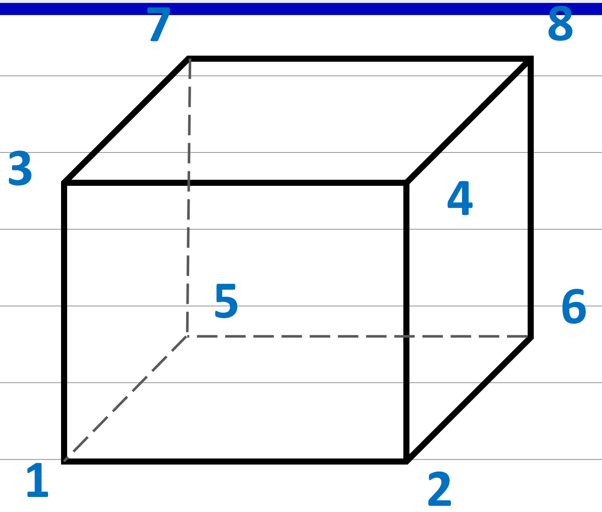
$E_2 =$ avg of plane IV − avg of plane III

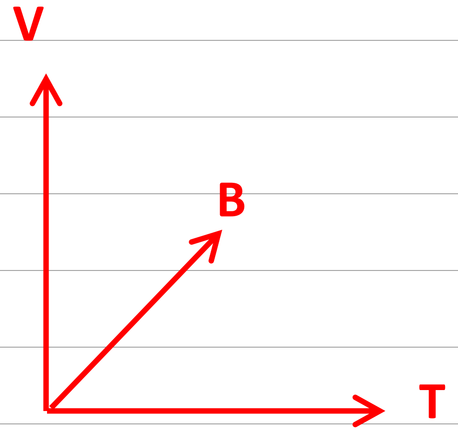
$$= \frac{1}{4}(\bar{Y}_3 + \bar{Y}_4 + \bar{Y}_7 + \bar{Y}_8) - \frac{1}{4}(\bar{Y}_1 + \bar{Y}_2 + \bar{Y}_5 + \bar{Y}_6)$$

$E_3 =$ avg of plane VI (rare) − avg of plane V (front)

Interaction Effect

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7



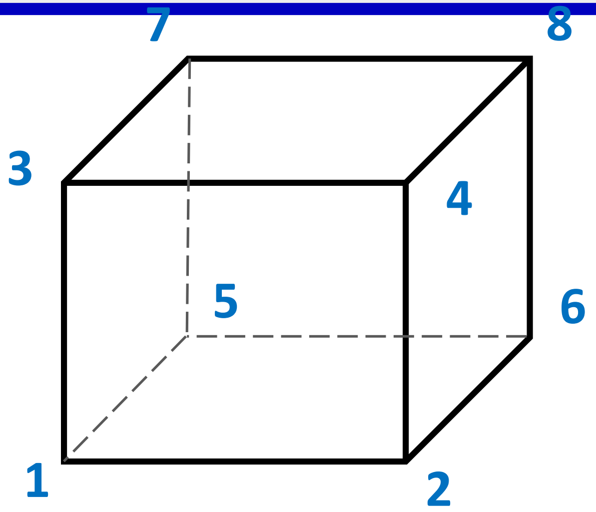


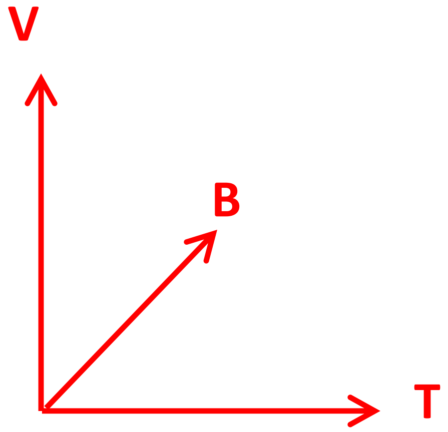
Two-variable Interactions

- The average effects E1, E2, and E3 represent the individual effects of ambient temperature, wind velocity and bar size on the ultimate tensile stress (UTS).
- **But, what about the joint effect of two variables?**
  - Effect of temperature AND wind velocity on the UTS?
  - Effect of wind velocity AND bar size on UTS?
- ***These joint effects are indicated by the two variable interactions.***

Two Variable Interactions

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

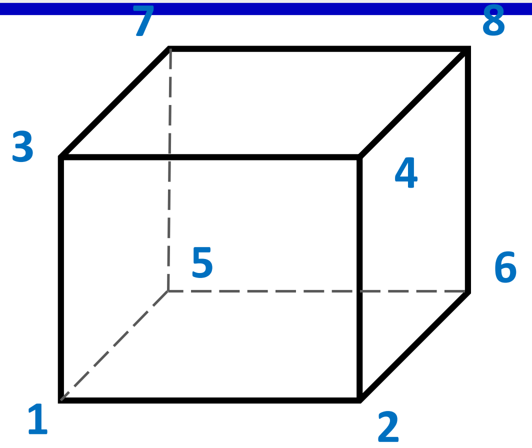


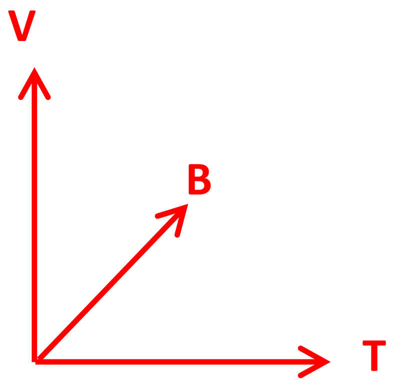


Two-variable Interactions

- There are three two-variable interactions to be calculated,
  - Between temperature and wind velocity, denoted by E_{12}
  - Between wind velocity and bar size, denoted by E_{23}
  - Between bar size and temperature, denoted by E_{31} or E_{13}

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





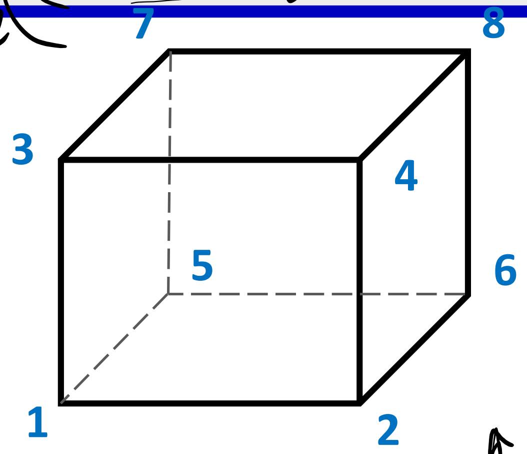
Two-variable Interaction between Temperature and Wind Velocity, E_{12}

- At HIGH level of wind velocity (X2), the two differences in the result are given by $(\overline{y_4} - \overline{y_3})$ and $(\overline{y_8} - \overline{y_7})$
- Both these differences reflect individual effect of change in UTS due to change in temperature from LOW to HIGH.
- The average of these effects (at HIGH wind velocity) is = $\frac{1}{2}[(\overline{y_4} - \overline{y_3}) + (\overline{y_8} - \overline{y_7})]$
- Similarly, the average change due to temperature at LOW wind velocity is = $\frac{1}{2}[(\overline{y_2} - \overline{y_1}) + (\overline{y_6} - \overline{y_5})]$

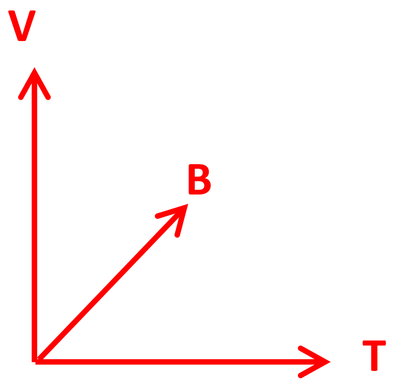
E_{12} = avg diff betw effect x_1 across x_2

$= \frac{1}{2}\left((x_2 \text{ is at } -1)\, \Delta \text{Output}_{x_1} - (x_2 \text{ is at } 1)\, \Delta \text{Output}_{x_1}\right)$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7



x_1 x_2



Two-variable Interaction between Temperature and Wind Velocity, E_{12}

- The interaction between ambient temperature and wind velocity is the average difference between these two averages, i.e.,

$$E_{12} = \frac{1}{2}\left\{\frac{1}{2}\left[(\bar{Y}_2 - \bar{Y}_1) + (\bar{Y}_6 - \bar{Y}_5)\right] - \frac{1}{2}\left[(\bar{Y}_4 - \bar{Y}_3) + (\bar{Y}_8 - \bar{Y}_7)\right]\right\}.$$

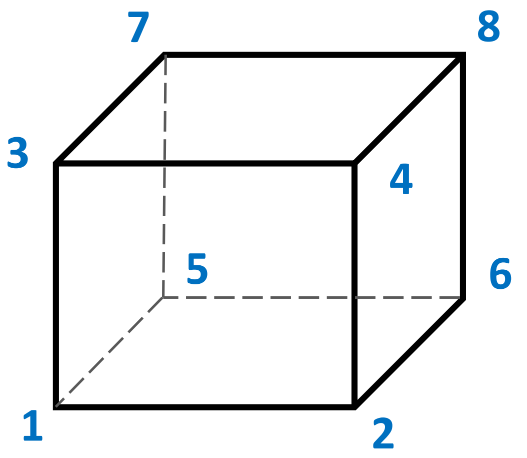
$$= \frac{1}{4}\left\{(\bar{Y}_1 + \bar{Y}_5 + \bar{Y}_4 + \bar{Y}_8) - (\bar{Y}_2 + \bar{Y}_6 + \bar{Y}_3 + \bar{Y}_7)\right\}$$

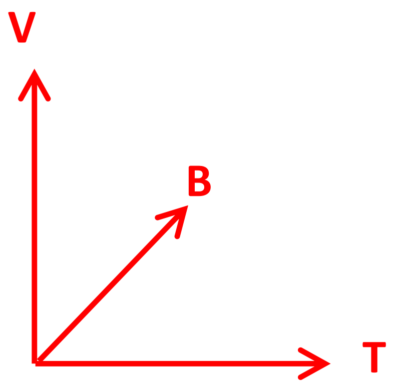
$= \frac{1}{4}$ ($x_1 x_2$ both high or low − $x_1 x_2$ one of them is higher)

- Note, therefore, that the interaction between ambient temperature and wind velocity tells us the average change in ultimate tensile stress that would occur due to a change from the low level to the high level in both the ambient temperature and wind velocity.

$E_{12} = \dfrac{+87.5 - 87.3 - 77.8 + 87 + 79.1 - 97.6 - 78.6 + 87.7}{4}$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7

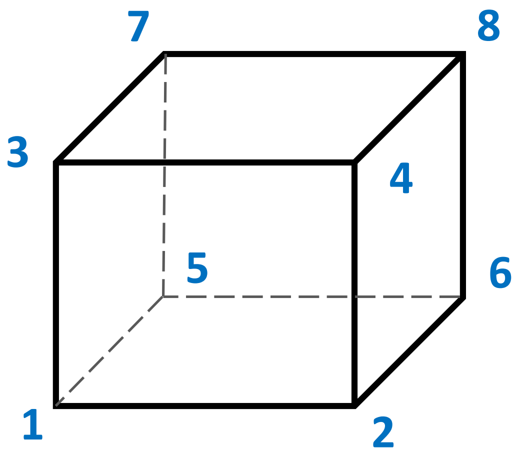


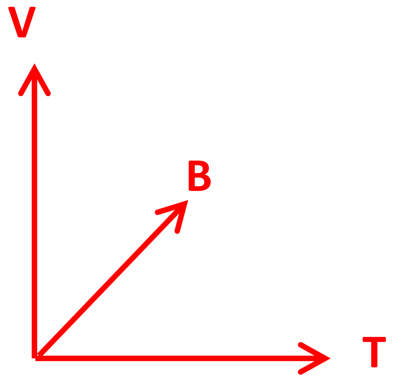


$$E_{13} = \frac{1}{4}(\bar{Y}_1 + \bar{Y}_3 + \bar{Y}_6 + \bar{Y}_8) - \frac{1}{4}(\bar{Y}_2 + \bar{Y}_4 + \bar{Y}_5 + \bar{Y}_7)$$

$$= -4650\, psi$$

Test #	***X1***	***X2***	***X3***	***Test Order***	***Y_{ai} (kpsi)***	***Test Order***	***Y_{bi} (kpsi)***	***Average (kpsi)***
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





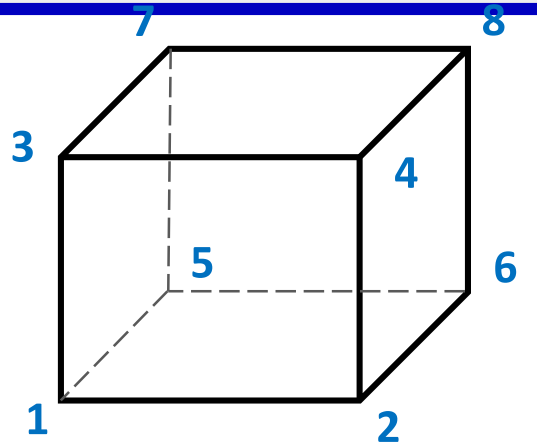
$$E_{23} = \frac{1}{4}(\bar{Y}_1 + \bar{Y}_2 + \bar{Y}_7 + \bar{Y}_8) - \frac{1}{4}(\bar{Y}_3 + \bar{Y}_4 + \bar{Y}_5 + \bar{Y}_6)$$

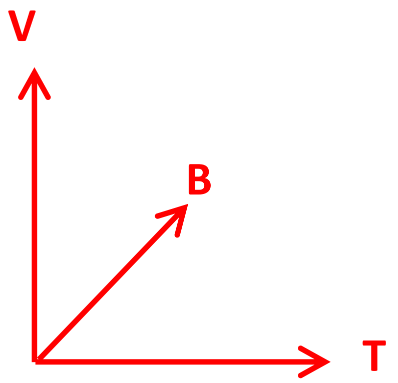
$$= \frac{1}{4}(87.5 + 87.3 + 78.6 + 87.7) - \frac{1}{4}(77.8 + 87.0 + 79.1 + 97.6)$$

$$= \text{-}0.10 \ \text{Kpsi}$$

$$= \text{-}100 \ \text{psi.}$$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





Three-variable Interaction E_{123}

- At HIGH level of bar size (X3), once can calculate the interaction between temperature (X1) and wind velocity (X2)

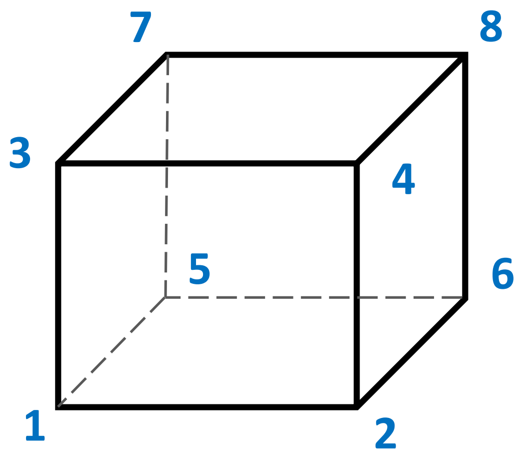
$$\frac{1}{2}[(\bar{Y}_8 - \bar{Y}_6) - (\bar{Y}_7 - \bar{Y}_5)]$$

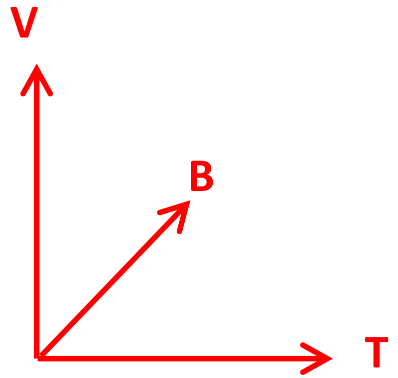
- Similarly, at LOW level of bar size, interaction between the temperature (X1) and wind velocity (X2) is

$$\frac{1}{2}[(\bar{Y}_4 - \bar{Y}_2) - (\bar{Y}_3 - \bar{Y}_1)].$$

- The three-variable interaction among temperature, wind velocity and bar size is the average difference between these two-variable interactions. $E_{123} = \frac{1}{2}\{\frac{1}{2}[(\bar{Y}_8 - \bar{Y}_6) - (\bar{Y}_7 - \bar{Y}_5)] - \frac{1}{2}[(\bar{Y}_4 - \bar{Y}_2) - (\bar{Y}_3 - \bar{Y}_1)]\}$

Test #	X1	X2	X3	Test Order	Y_{ai} (kpsi)	Test Order	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	6	84	3	91	87.5
2	1	-1	-1	8	90.6	7	84	87.3
3	-1	1	-1	1	69.6	5	86	77.8
4	1	1	-1	2	76	4	98	87
5	-1	-1	1	5	77.7	8	80.5	79.1
6	1	-1	1	3	99.7	1	95.5	97.6
7	-1	1	1	4	82.7	2	74.5	78.6
8	1	1	1	7	93.7	6	81.7	87.7





Three-variable Interaction E_{123}

- In common practice *we may* assume that the system is not so complicated such that the tree-variable interaction will be negligible.
- More over, we may contribute it as a part of the intrinsic variation of the test method, or in other terms, as a part of experimental error.
- **But, this should be exercised with caution and should be checked if ever possible. (Model Adequacy Check)**

Calculation Matrix

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Calculation Matrix

	Main Effects			Interactions				
Test	X_1	X_2	X_3	X_1X_2	X_1X_3	X_2X_3	$X_1X_2X_3$	$\bar{y}$
1	-1	-1	-1	1	1	1	-1	87.5
2	1	-1	-1	-1	-1	1	1	87.3
3	-1	1	-1	-1	1	-1	1	77.8
4	1	1	-1	1	-1	-1	-1	87
5	-1	-1	1	1	-1	-1	1	79.1
6	1	-1	1	-1	1	-1	-1	97.6
7	-1	1	1	-1	-1	1	-1	78.6
8	1	1	1	1	1	1	1	87.7

main effect x_1

E_1 = avg when x_1 is high − avg x_1 is low

$= 1/4\,(y_2+y_4+y_6+y_8) - 1/4\,(y_1+y_3+y_5+y_7)$

interaction of x_1 & x_2

E_{12} = avg when x_1x_2 high − avg x_1x_2 low

$= 1/4\,(y_1+y_4+y_5+y_8) - 1/4\,(y_2+y_3+y_6+y_7)$

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Main Effects

Ambient temperature (E_1) 9150 psi

Wind Velocity (E_2) - 5100 psi

Bar Size (E_3) 850 psi

Two-Variable Interactions

Ambient temperature-Wind Velocity (E_{12}) 0 psi

Ambient temperature-Bar Size (E_{13}) - 4650 psi

Wind Velocity-Bar Size (E_{23}) -100 psi

Three-Variable Interaction

Ambient temperature-Wind Velocity-Bar Size (E_{123}) 4700 psi

```
[ 8.53250000e+01 4.57500000e+00 -2.55000000e+00 4.25000000e-01 -3.55271368e-15 2.32500000e+00 -5.00000000e-02 -2.35000000e+00]
```

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Main Effects

Ambient temperature (E_1) 9150 psi

Wind Velocity (E_2) - 5100 psi

Bar Size (E_3) 850 psi

Two-Variable Interactions

Ambient temperature-Wind Velocity (E_{12}) 0 psi

Ambient temperature-Bar Size (E_{13}) 4650 psi

Wind Velocity-Bar Size (E_{23}) -100 psi

Three-Variable Interaction

Ambient temperature-Wind Velocity-Bar Size (E_{123}) -4700 psi

$$y_1 = \beta_0 + \beta_1(-1) + \beta_2(-1) + \beta_3(-1) + \beta_{12}(+1) + \beta_{13}(+1) + \beta_{23}(+1) + \beta_{123}(-1)$$

What is the regression model?

$$y = f(x_1, x_2, x_3)$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3$$

we want to find & β_i ✓

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

8×4

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

For each experimental data, we can find the error (ϵ_i) between the model predicted value ($\hat{y}_\mathrm{i}$) and observed experimental value (y_i)

$$\epsilon_i = y_i - \hat{y}_\mathrm{i}$$

With 'least square fit', the aim is to find such coefficients, that minimizes the total sum of squares of error.

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_\mathrm{i})^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each $\boldsymbol{\beta_i}$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

$\hat{y}_1 = \beta_0 + \beta_1 x_{11}$

$\hat{y}_2$ –

$\hat{y}_3$ –

$\hat{y}_4$

$\hat{y}_8$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each β_i

We can write the model in a matrix format

$$[Y_{exp}] = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_8 \end{bmatrix}, \quad [\hat{Y}] = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_8 \end{bmatrix}_{8\times 1} = [X]_{8\times 4}[\beta]_{4\times 1}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

where,

$$[X] = \begin{bmatrix} 1 & x_{11} & x_{2i} & x_{31} \\ 1 & x_{12} & x_{22} & x_{33} \\ 1 & \vdots & & \\ \vdots & & & \\ 1 & x_{18} & x_{28} & x_{38} \end{bmatrix}$$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each β_i

Thus, $L = \sum \varepsilon_i^2 = [\varepsilon]^T[\varepsilon]$

$= [Y_{exp} - \hat{Y}]^T [Y_{exp} - \hat{Y}]$

$= [Y_{exp} - X\beta]^T [Y_{exp} - X\beta]$

$\Rightarrow \boxed{L = [Y_{exp}]^T [Y_{exp}] - 2[\beta]^T [X]^T [Y_{exp}] + [\beta]^T [X]^T [X][\beta]}$

CEP2022_Notebook (2.3.2)

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_8 \end{bmatrix} = [Y_{exp}] - [\hat{Y}]$$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

Let's say that we want to fit the following model,

$$\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

The least square function,

$$L = \sum \epsilon_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum [y_i - (\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i})]^2$$

Goal is to minimize L with respect to each $\boldsymbol{\beta_i}$

$$\Rightarrow \boxed{L = [Y_{exp}]^T [Y_{exp}] - 2 [\beta]^T [X]^T [Y_{exp}] + [\beta]^T [X]^T [X] [\beta]}$$

minimize L wrt β $\Rightarrow \frac{\partial L}{\partial [\beta]} = 0$

$$-2 [X]^T [Y_{exp}] + 2 [X]^T [X] [\beta] = 0$$

$$\Rightarrow$$

$$\Rightarrow \boxed{[\beta] = \left(\underset{4\times 8}{[X]^T} \underset{8\times 4}{[X]} \right)^{-1} \underset{4\times 8}{[X]^T} \underset{8\times 1}{[Y_{exp}]}}$$

How to Find Regression Coefficients?

Test #	X1	X2	X3	Average (kpsi)
1	-1	-1	-1	87.5
2	1	-1	-1	87.3
3	-1	1	-1	77.8
4	1	1	-1	87
5	-1	-1	1	79.1
6	1	-1	1	97.6
7	-1	1	1	78.6
8	1	1	1	87.7

CEP2022_Notebook (2.3.2)

$$\Rightarrow \boxed{[\beta] = \left([X]^T[X]\right)^{-1}[X]^T[Y_{exp}]}$$

What if we want to fit a model like

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{123} x_1 x_2 x_3 + \beta_{11} x_1^2$$

x_6

then, rename $x_1 x_2 = x_4$, $x_1 x_2 x_3 = x_5$, $x_1^2 = x_6$

$\beta_{12} = \beta_4$, $\beta_{123} = \beta_5$, $\beta_{11} = \beta_6$

then do the same procedure as before

Example

The yield from a certain chemical depends on either the chemical formulation of the input materials or the mixer speed, or both.

A 2-level factorial design was run with three replicates and the yield data are shown below.

Find main effects and interaction effects.

x_1	x_2	y_a	y_b	y_c	$\bar{y}$
-1	-1	10	20	30	20
1	-1	40	30	50	40
-1	1	60	30	60	50
1	1	40	45	50	45

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

Example

Consider following factorial design with 2 variables (k = 2), and 3 levels each

Each combination replicated 4 times (n = 4)

Life (in hours) Data for the Battery Design Example

Material	Temperature (°F)					
Type	15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

What is the effects model and hypothesis test?

In general, for 2-factor design, we could have ‘a’ levels of factor A, and ‘b’ levels of factor B.

Each combination is replicated ‘n’ times

General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

What is the effects model and hypothesis test?

		Factor B			
		1	2	. . .	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	$\vdots$				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

In the two-factor factorial, both row and column factors (or treatments), A and B, are of al interest. Specifically, we are interested in **testing hypotheses** about the equality of row treatment effects, say

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1: \text{at least one } \tau_i \neq 0 \qquad \textbf{(5.2a)}$$

and the equality of column treatment effects, say

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_b = 0$$
$$H_1: \text{at least one } \beta_j \neq 0 \qquad \textbf{(5.2b)}$$

We are also interested in determining whether row and column treatments *interact*. Thus, we also wish to test

$$H_0: (\tau\beta)_{ij} = 0 \qquad \text{for all } i, j$$
$$H_1: \text{at least one } (\tau\beta)_{ij} \neq 0 \qquad \textbf{(5.2c)}$$

We now discuss how these hypotheses are tested using a **two-factor analysis of variance**.

Effects Model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

ANOVA for Two-Factor Factorial Design

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

α

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	
A treatments	SS_A	$a-1$	$MS_A = \dfrac{SS_A}{a-1}$	$F_0 = \dfrac{MS_A}{MS_E}$	$F_{1-\alpha, a-1, ab(n-1)}$
B treatments	SS_B	$b-1$	$MS_B = \dfrac{SS_B}{b-1}$	$F_0 = \dfrac{MS_B}{MS_E}$	$F_{1-\alpha, b-1, ab(n-1)}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \dfrac{MS_{AB}}{MS_E}$	$F_{1-\alpha, (a-1)(b-1), ab(n-1)}$
Error	SS_E	$ab(n-1)$	$MS_E = \dfrac{SS_E}{ab(n-1)}$		
Total	SS_T	$abn-1$			

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
	130	155	539	34	40	229	20	70	230	
1	74	180		80	75		82	58		998
	150	188	623	136	122	479	25	70	198	
2	159	126		106	115		58	45		1300
	138	110	576	174	120	583	96	104	342	
3	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130, 74	155, 180	134.75	34, 80	40, 75	57.25	20, 82	70, 58	57.5	998
2	150, 159	188, 126	155.75	136, 106	122, 115	119.75	25, 58	70, 45	49.5	1300
3	138, 168	110, 160	144	174, 150	120, 139	145.75	96, 82	104, 60	85.5	1501
$y_{.j.}$	1738			1291			770			3799 = $y_{...}$

$N = abn$
$= 3 \times 3 \times 4$
$= 36$

36 terms

$$SS_{Total} = \sum_i \sum_j \sum_k y_{ijk}^2 = (130^2 + 155^2 + 74^2 + 180^2 + 34^2 + 40^2 + \dots + 96^2 + 104^2 + 82^2 + 60^2)$$
$$= 478647$$

$$\text{Grand Mean} = \frac{\sum\sum\sum y_{ijk}}{36} = \frac{3799}{36} = \bar{\bar{y}}$$
$$= 105.53$$

$$SS_{mean} = N\bar{\bar{y}}^2 = 36\left(\frac{3799}{36}\right)^2 = 400900$$

3 terms

$$SS_{material} = \underset{b\ \ n}{3 \times 4} \times \left[\left(\frac{998}{12} - 105.53\right)^2 + \left(\frac{1300}{12} - 105.53\right)^2 + \left(\frac{1501}{12} - 105.53\right)^2\right]$$
$$= 10683.12$$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	134.75	34	40	57.25	20	70	57.5	
	74	180		80	75		82	58		998
2	150	188	155.75	136	122	119.75	25	70	49.5	
	159	126		106	115		58	45		1300
3	138	110	144	174	120	145.75	96	104	85.5	
	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

3 terms

$$SS_{temp} = \underbrace{3}_{a} \times \underbrace{4}_{w} \times \left[\left(\frac{1738}{12} - 105.53\right)^2 + \left(\frac{1291}{12} - 105.53\right)^2 + \left(\frac{770}{12} - 105.53\right)^2 \right]$$

$$= 39118.72$$

36 terms

For replicates,

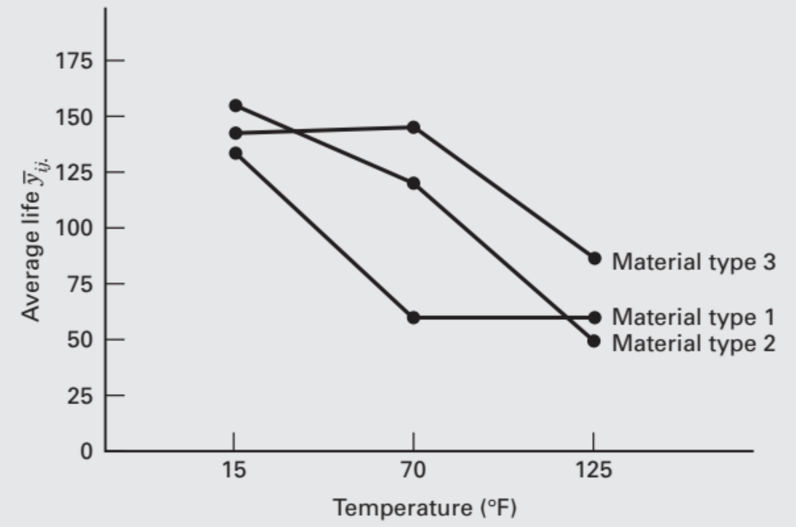
$$SS_E = \sum\sum (y_{ijk} - \bar{y})^2 = \left((130 - 134.75)^2 + (155 - 134.75)^2 + \dots + (34 - 57.25)^2 + (40 - 57.25)^2 \right.$$

$$SS_{interaction} = [SS_T - SS_{mean}] - SS_{material} - SS_{temp} - SS_E$$

Example

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
	130	155	134.75	34	40	57.25	20	70	57.5	
1	74	180		80	75		82	58		998
	150	188	155.75	136	122	119.75	25	70	49.5	
2	159	126		106	115		58	45		1300
	138	110	144	174	120	145.75	96	104	85.5	
3	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$



Analysis of Variance for Battery Life Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Material types	10,683.72	2	5,341.86	7.91	0.0020
Temperature	39,118.72	2	19,559.36	28.97	< 0.0001
Interaction	9,613.78	4	2,403.44	3.56	0.0186
Error	18,230.75	27	675.21		
Total	77,646.97	35			

How would you check Model Adequacy?

Life Data (in hours) for the Battery Design Experiment

Material Type	Temperature (°F) 15			70			125			$y_{i..}$
1	130	155	134.75	34	40	57.25	20	70	57.5	
	74	180		80	75		82	58		998
2	150	188	155.75	136	122	119.75	25	70	49.5	
	159	126		106	115		58	45		1300
3	138	110	144	174	120	145.75	96	104	85.5	
	168	160		150	139		82	60		1501
$y_{.j.}$	1738			1291			770			$3799 = y_{...}$

Significance of Main and Interaction Effects

- With 2-factor ANOVA, we can check if the factors or their interaction have a significant effect on the response
- **But what about the statistical significance of main and interaction effect values?**
  - For example, if we calculate the main effect of a variable to be 500.
  - Our attitude towards this average effect of say, 500, would not be the same if the 95% confidence interval were 500 ± 2 as it would be if the interval were 500 ± 2000.
  - **If it is 500 ± 2**, we would feel that the existence of an average effect has been rather convincingly demonstrated and we could assert with some confidence that its true magnitude is probably fairly close to 500.
  - **If it is 500 ± 2000**, this is not the case at all, because considerable uncertainty is associated with the effect and its magnitude.
- **How to find out the uncertainty (via confidence intervals) in the calculated values of main and interaction effects?**

Significance of Main and Interaction Effects

- To obtain a quantitative measure of the uncertainty in our calculated average effects and interactions, we proceed as follows:
    1. Estimate the variance S^2 of an individual observation
    2. Estimate the variances associated with the average effects and interactions
    3. Calculate the appropriate 95% confidence intervals for the “true” average effects and interactions
- From the 95% confidence intervals, we may be able to interpret the significance of each average effect and interaction, and draw conclusions regarding the experimental study.

Significance of Main and Interaction Effects

- Recall that in the previous welding example, we had 16 observations (8 combinations with 2 replications)
- It is the variance of each of these 16 observations that we will now estimate.
- We shall assume that the true variance σ^2 is the same for all sixteen observations and that the observations are independent.

Test #	X1	X2	X3	Y_{ai} (kpsi)	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	84	91	87.5
2	1	-1	-1	90.6	84	87.3
3	-1	1	-1	69.6	86	77.8
4	1	1	-1	76	98	87
5	-1	-1	1	77.7	80.5	79.1
6	1	-1	1	99.7	95.5	97.6
7	-1	1	1	82.7	74.5	78.6
8	1	1	1	93.7	81.7	87.7

For Test #1, the sample variance can be calculated as

$$S_1^{\,2} = \frac{(y_{a1} - \bar{y}_1)^2 + (y_{b1} - \bar{y}_1)^2}{(2-1)}$$

$$= (84.0 - 87.5)^2 + (91.0 - 87.5)^2$$

$$= 24.50$$

Similarly, we can find 8 sample variances for 8 combinations, S_1^2, S_2^2, …….,S_8^2, one for each test.

S_1^2 = 24.5, S_2^2=21.78, S_3^2=134.48, S_4^2=242.0, S_5^2=3.92, S_6^2=8.82, S_7^2=33.62, S_8^2=72.00

Significance of Main and Interaction Effects

- Since we are assuming same true variance σ^2 for all sixteen observations, an estimate for σ^2 is the pooled sample variance S_p^2 of the eight estimated variances $S_1^2, S_2^2, \ldots\ldots, S_8^2$.

  $S_1^2 = 24.5$, $S_2^2=21.78$, $S_3^2=134.48$, $S_4^2=242.0$, $S_5^2=3.92$, $S_6^2=8.82$, $S_7^2=33.62$, $S_8^2=72.00$

- Here

$$S_p^2 = \frac{[(y_{a1}-\bar{y}_1)^2+(y_{b1}-\bar{y}_1)^2+\ldots\ldots\ldots\ldots\ldots+(y_{a8}-\bar{y}_8)^2+(y_{b8}-\bar{y}_8)^2]}{(2-1)+\ldots\ldots\ldots\ldots+(2-1)}$$

$$= \frac{24.50+21.78+\ldots\ldots+72}{8}$$

$$= 67.64.$$

- *It should be pointed out that when the number of replications are not the same for all eight tests, the pooled sample variance S_p^2 has to be modified properly.*

$$S_p^2 = \frac{SS}{\nu} = \frac{SS_1+SS_2+SS_3 \ldots SS_8}{\nu_1+\nu_2 \ldots \nu_8} = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2 \ldots + (n_8-1)S_8^2}{(n_1-1)+(n_2-1)+(\ldots + (n_8-1)}$$

Significance of Main and Interaction Effects

Estimation of the Variances Associated with the Average Effects and Interactions

- The average effect of ambient temperature, E1 is

$$E_1 = \frac{1}{4}(\bar{y}_2 - \bar{y}_1 + \bar{y}_4 - \bar{y}_3 + \bar{y}_6 - \bar{y}_5 + \bar{y}_8 - \bar{y}_7)$$

- But since each term $\bar{y}_i$ is an average of two observations,

$$E_1 = \frac{1}{4}\left[\frac{(y_{a2} + y_{b2})}{2} - \frac{(y_{a1} + y_{b1})}{2} + \ldots + \frac{(y_{a8} + y_{b8})}{2} - \frac{(y_{a7} + y_{b7})}{2}\right]$$

or

$$E_1 = \frac{1}{8}[y_{a2} + y_{b2} - y_{a1} - y_{b1} + \ldots\ldots\ldots - y_{a7} - y_{b7}]$$

$V(E_1) = \frac{1}{64}(V(\bar{y}_{a2}) + V(y_{b2}) \ldots$
$\sigma^2 \ldots$
$= 16\sigma^2/64 = \sigma^2/4$

- Thus we can show that $V(E_1) = \sigma^2/4$
- In fact, we can show, $V(E_1) = V(E_2) = V(E_3) = V(E_{12}) = V(E_{13}) = V(E_{23}) = V(E_{123}) = \sigma^2/4$

Significance of Main and Interaction Effects

Estimation of the Variances Associated with the Average Effects and Interactions

- $V(E_1) = V(E_2) = V(E_3) = V(E_{12}) = V(E_{13}) = V(E_{23}) = V(E_{123}) = \sigma^2/4$
- Substituting pooled variance in place of true variance, we can obtain confidence intervals
- The confidence interval for the average and interaction effects can be obtained as:

$$E_i \pm t\sqrt{\frac{S_p^{\,2}}{4}} \quad i = 1,2,\ldots$$

$$E_i \pm 2.306\sqrt{\frac{S_p^2}{4}}$$

- We already have the values of E1, E2, ...,E12,....and S_p^2; what is left to be determined is the value of t.
- We have a total of sixteen tests, and we used up eight degrees of freedom in calculating the eight averages,
- Therefore, the appropriate t-value is the value associated with eight degrees of freedom and corresponding to a 95% confidence level, which is $t_{8,\,0.025} = 2.306$.

Significance of Main and Interaction Effects

Test #	X1	X2	X3	Y_{ai} (kpsi)	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	84	91	87.5
2	1	-1	-1	90.6	84	87.3
3	-1	1	-1	69.6	86	77.8
4	1	1	-1	76	98	87
5	-1	-1	1	77.7	80.5	79.1
6	1	-1	1	99.7	95.5	97.6
7	-1	1	1	82.7	74.5	78.6
8	1	1	1	93.7	81.7	87.7

Example

$$E_i \pm t\sqrt{\frac{S_p^{\;2}}{4}} \quad i = 1,2,\ldots$$

Here,

$$S_p^{\;2} = \frac{[(y_{a1}-\bar{y}_1)^2+(y_{b1}-\bar{y}_1)^2+\ldots\ldots\ldots+(y_{a8}-\bar{y}_8)^2+(y_{b8}-\bar{y}_8)^2]}{(2-1)+\ldots\ldots+(2-1)} = 67.64$$

t-value is the value associated with eight degrees of freedom and corresponding to a 95% confidence level, which is $t_{8,\,0.025} = 2.306$

For example,

95% confidence interval for the "true" average effect of ambient temperature (in kpsi) is: E1 ± 9.48= 9.15 ± 9.48

Significance of Main and Interaction Effects

Example

$$E_i \pm t\sqrt{\frac{S_p^{\,2}}{4}} \quad i = 1,2,\ldots$$

Here,

$$S_p^{\,2} = \frac{[(y_{a1}-\bar{y}_1)^2+(y_{b1}-\bar{y}_1)^2+\ldots\ldots+(y_{a8}-\bar{y}_8)^2+(y_{b8}-\bar{y}_8)^2]}{(2-1)+\ldots\ldots+(2-1)} = 67.64$$

Test #	X1	X2	X3	Y_{ai} (kpsi)	Y_{bi} (kpsi)	Average (kpsi)
1	-1	-1	-1	84	91	87.5
2	1	-1	-1	90.6	84	87.3
3	-1	1	-1	69.6	86	77.8
4	1	1	-1	76	98	87
5	-1	-1	1	77.7	80.5	79.1
6	1	-1	1	99.7	95.5	97.6
7	-1	1	1	82.7	74.5	78.6
8	1	1	1	93.7	81.7	87.7

Average Effects	95% Confidence Interval
Ambient temperature (E_1)	9150±9480 psi
Wind Velocity (E_2)	-5100±9480 psi
Bar Size (E_3)	850±9480 psi

Two-Variable Interactions

Ambient temperature-Wind Velocity (E_{12})	0±9480 psi
Ambient temperature-Bar Size (E_{13})	-4650±9480 psi
Wind Velocity-Bar Size (E_{23})	-100±9480 psi

Three-Variable Interaction

Ambient temperature-Wind Velocity-Bar Size (E_{123}) 4700±9480 psi