CS 207M Tutorial-2

1. Fix a positive integer m and consider the relation $\sim$ on the set of integers, Z:

$$a \sim b \text{ if } m \text{ divides } a - b$$

   Show that $\sim$ is an equivalence relation. How many equivalence classes does $\sim$ admit?

2. Let $f : X \to X$ be a function. We define a relation $\sim$ on X as follows:

$$x \sim y \text{ if } \exists m \geq 0 \text{ such that either } f^m(x) = y \text{ or } f^m(y) = x$$

   Is $\sim$ reflexive/symmetric/anti-symmetric/transitive?

3. Let $f : X \to Y$ and $g : Y \to Z$ be two functions and $Graph(f) \subseteq X \times Y$ and $Graph(g) \subseteq Y \times Z$ be the associated relations. Prove that $Graph(g) \circ Graph(f) = Graph(g \circ f)$.

4. Let R be a relation on the set A. The converse of R, denoted R^{-1} is a relation on A such that $(a, b) \in R$ iff $(b, a) \in R^{-1}$, for all a, b in A. The identity relation on A is denoted by I.

   Let R_1, R_2, R_3 be relations defined on a set A. Prove or disprove the following statements

   - $R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3$.
   - I is a subset of $R_1 \circ R_1^{-1}$.
   - If R_1, R_2 are reflexive/symmetric/transitive/antisymmetric then $R_1 \circ R_2$ also has the same property.
   - If R_1, R_2 are reflexive/symmetric/transitive/antisymmetric then $R_1 \cup R_2$ also has the same property.
   - If R_1, R_2 are reflexive/symmetric/transitive/antisymmetric then $R_1 \cap R_2$ also has the same property.
   - $(R_1 \circ R_2)^{-1} = (R_2)^{-1} \circ (R_1)^{-1}$.

5. Let R be a relation on a set A. Recall that R^n, $n \geq 0$ is a relation on A defined inductively as: $R^0 = I$ and, for $n \geq 1$, $R^n = R^{n-1} \circ R$. We define R^+ to be the relation $\bigcup_{n \geq 1} R^n$. Similarly, $R^* = \bigcup_{n \geq 0} R^n$

   - Show that the relation R^+ is transitive.

- Show that if R is symmetric then so is R^n for all $n \geq 0$.
- Show that R is transitive iff $R^n \subseteq R$ for all $n \geq 1$.

6. Let R be a reflexive , transitive relation on a set A. Define a relation X on A as follows: for all a, b elements of A, aXb iff aRb and bRa. Show that X is an equivalence relation.

7. Show that the reflexive-transitive-closure of a relation R is $R^* = \bigcup_{n \geq 0} R^n$.

8. A relation R on X is said to be a flower if there exists $x \in X$ such that for all $y \in X$, $(x, y) \in R$. Let R be any relation on X. What is the flower-closure of R?

9. Show that the smallest equivalence relation containing R is $(R \cup R^{-1})^*$.