Solutions to Tutorial Sheet 4

5. Let $f(x) = 1$ if $x \in [0, 1]$ and $f(x) = 2$ if $x \in (1, 2]$. Show from the first principles that f is Riemann integrable on $[0, 2]$ and find $\int_0^2 f(x)dx$.

   Solution. The given function is integrable as it is monotone. Let P_n be the partition of $[0, 2]$ into 2×2^n equal parts. Then $U(P_n, f) = 3$ and

$$L(P_n, f) = 1 + 1 \times \frac{1}{2^n} + 2 \times \frac{2^n - 1}{2^n} \to 3$$

   as $n \to \infty$. Thus, $\int_0^2 f(x)dx = 3$.

6. (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

   Solution. $f(x) \geq 0 \implies U(P, f) \geq 0$, $L(P, f) \geq 0 \implies \int_a^b f(x)dx \geq 0$. Suppose, moreover, f is continuous and $\int_a^b f(x)dx = 0$. Assume $f(c) > 0$ for some $c \in [a, b]$. Then $f(x) > \frac{f(c)}{2}$ in a δ-nbhd of c for some $\delta > 0$. This implies that

$$U(P, f) > \delta \times \frac{f(c)}{2}$$

   for any partition P, and hence, $\int_a^b f(x)dx \geq \delta \frac{f(c)}{2} > 0$, a contradiction.

   (b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

   Solution. On $[0, 1]$ take

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

7. Evaluate $\lim_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

   (i) $S_n = \frac{1}{n^{5/2}} \sum_{i=1}^{n} i^{3/2}$

   Solution. $S_n = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{3/2} \to \int_0^1 x^{3/2}dx = \frac{2}{5}$

(iii) $S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in+n^2}}$

Solution. $S_n = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{1+\frac{i}{n}}} \to \int_0^1 \frac{dx}{\sqrt{1+x}} = 2(\sqrt{2}-1)$

(iv) $S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{i\pi}{n}$

Solution. $S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{i\pi}{n} \to \int_0^1 \cos\pi x = 0$

(v) $S_n = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}\right\}$

Solution. $S_n \to \int_0^1 x\,dx + \int_1^2 x^{3/2}dx + \int_2^3 x^2 dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2}-1) + \frac{19}{3}$

8. (b) Compute $\frac{dF}{dx}$, if for $x \in \mathbb{R}$:

(i) $F(x) = \int_1^{2x} \cos(t^2)dt$

Solution. $F'(x) = \cos((2x)^2) \times 2 = 2\cos(4x^2)$

(ii) $F(x) = \int_0^{x^2} \cos(t)dt$

Solution. $F'(x) = \cos(x^2) \times 2x = 2x\cos(x^2)$

9. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a.

(*Hint:* Consider $F(a) = \int_a^{a+p} f(t)dt$, $a \in \mathbb{R}$).

Solution. Define $F(x) = \int_x^{x+p} f(t)dt$, $x \in \mathbb{R}$. Then $F'(x) = f(x+p) - f(x) = 0$ for every x.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = g'(0) = 0$.

Solution. Write $\sin\lambda(x-t)$ as $\sin(\lambda x)\cos(\lambda t) - \cos(\lambda x)\sin(\lambda t)$ in the integrand, take terms in x outside the integral, evaluate $g'(x)$ and $g''(x)$, and simplify to show LHS = RHS. From the expressions for $g(x)$ and $g'(x)$ it should be clear that $g(0) = g'(0) = 0$.

Alternate. The problem can also be solved by appealing to the following theorem:

Theorem A. Let $h(t,x)$ and $\dfrac{\partial h}{\partial x}(t,x)$ be continuous functions of t and x on the rectangle $[a,b]\times[c,d]$. Let $u(x)$ and $v(x)$ be differentiable functions of x on $[c,d]$ such that, for each $x\in[c,d]$, the points $(u(x),x)$ and $(v(x),x)$ belong to $[a,b]\times[c,d]$. Then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} h(t,x)dt = \int_{u(x)}^{v(x)} \frac{\partial h}{\partial x}(t,x)dt - u'(x)h(u(x),x) + v'(x)h(v(x),x).$$

Consider now

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

Let $h(t,x) = \dfrac{1}{\lambda}f(t)\sin\lambda(x-t)$, $u(x)=0$ and $v(x)=x$. Then it follows from Theorem A that

$$g'(x) = \int_0^x f(t)\cos\lambda(x-t)dt.$$

Again, applying Theorem A, we have

$$g''(x) = -\lambda\int_0^x f(t)\sin\lambda(x-t) + f(x).$$

Thus $g''(x)+\lambda^2 g(x) = f(x)$. $g(0)=g'(0)=0$ is obvious from the expressions for $g(x)$ and $g'(x)$.

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