PH 107 :Quantum Physics and Applications

Step potential cont... and Finite step potential

Lecture 17: 10-02-2022

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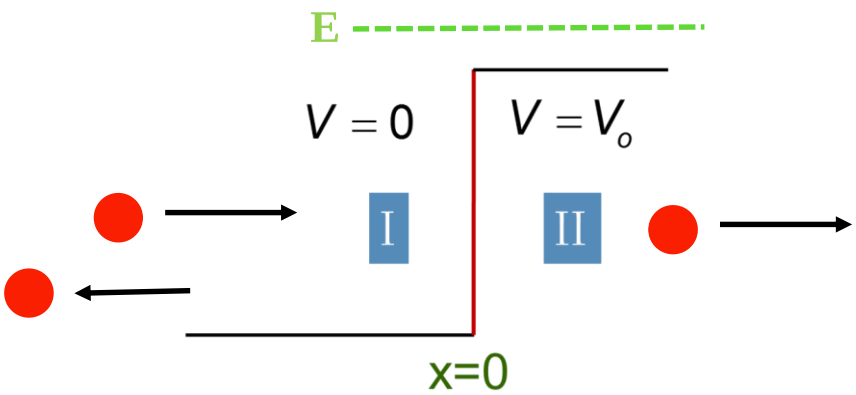
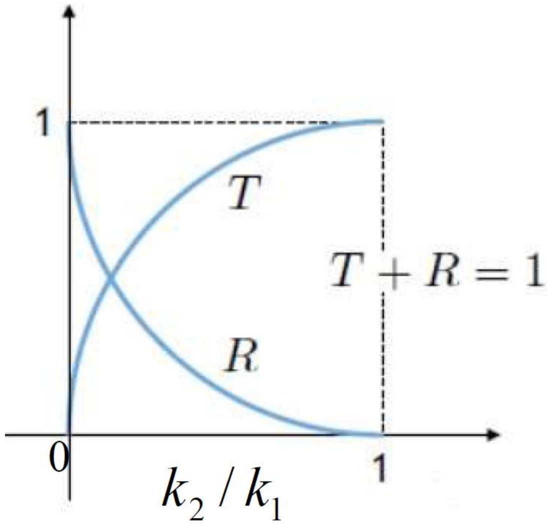
Recap (Step Potential Well, E>V_o)





$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\frac{1 - \kappa}{1 + \kappa}\right)^2 \; ; \; T = \frac{4k_1 k_2}{(k_1 + k_2)^2} = \frac{4\kappa}{(1 + \kappa)^2} \; ; \text{where } \kappa = \frac{k_2}{k_1} = \sqrt{\mathbf{1} - \frac{\boldsymbol{V_o}}{\boldsymbol{E}}}$$

1. Scattering state solutions are non-normalizable. No bound states.
2. Finite probability for reflection and transmission at the boundary (E>Vo). Classically, reflection is forbidden.

3. For E→ $\boldsymbol{V_o}$, T → 0.

4. For large E , (E >> $\boldsymbol{V_o}$); R = 0 and T =1.

Recap (Step Potential Well, E>V_o)

$$J_{refl} = \frac{\hbar k_1}{m}|B|^2 \qquad \frac{B}{A} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)$$

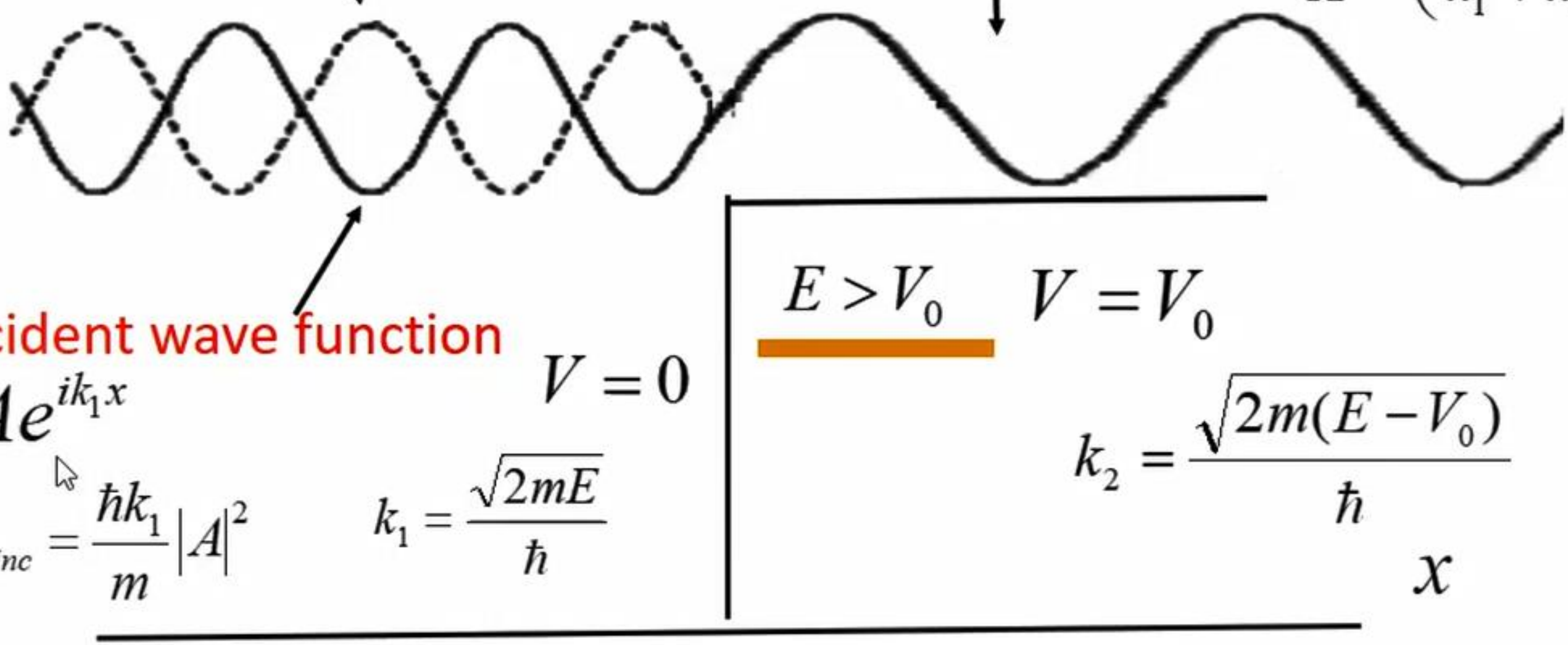
$$J_{trans} = \frac{\hbar k_2}{m}|C|^2$$

Reflected wave function

Be^{-ik_1x}

Transmitted wave function

Ce^{ik_2x} $\quad \frac{C}{A} = \left(\frac{2k_1}{k_1 + k_2}\right)$



$E > V_0$ $\quad V = V_0$

Incident wave function

Ae^{ik_1x}

$V = 0$

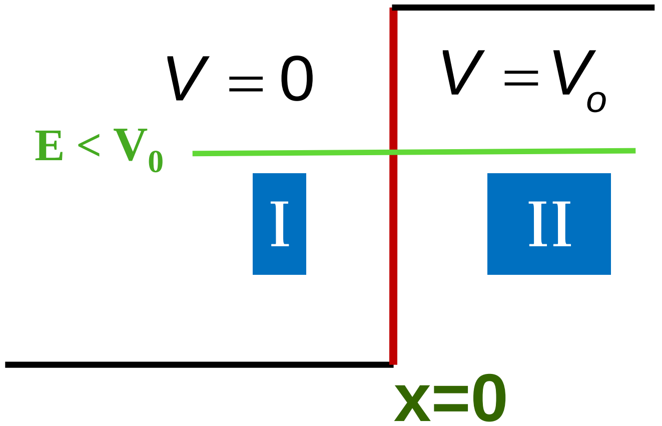
$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

$$J_{inc} = \frac{\hbar k_1}{m}|A|^2 \qquad k_1 = \frac{\sqrt{2mE}}{\hbar}$$

x

$x = 0$

Step Potential Case II: E<V_0 (Ex:7.4; Serway)



I $\varphi_I(x) = Ae^{ikx} + Be^{-ikx};\ k^2 = \frac{2mE}{\hbar^2}$

II $\varphi_{II}(x) = Ce^{-\alpha x} + De^{\alpha x};\ \alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$

1. Since $\varphi_{II}(x) \to 0$ as $x \to \infty$ $\Rightarrow$ $D = 0$

2. Boundary conditions ;

(a) $\varphi_I(0) = \varphi_{II}(0)$ $\Rightarrow$ $A + B = C$

(b) $\varphi_I'(0) = \varphi_{II}'(0)$ $\Rightarrow$ $ik(A - B) = -\alpha C$

Finding the coefficients

Trick: Put $k_1 = k$ **and** $k_2 = i\alpha$

$$\frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

and

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

E > V_0

$$\frac{C}{A} = \frac{2k}{k + i\alpha}$$

and

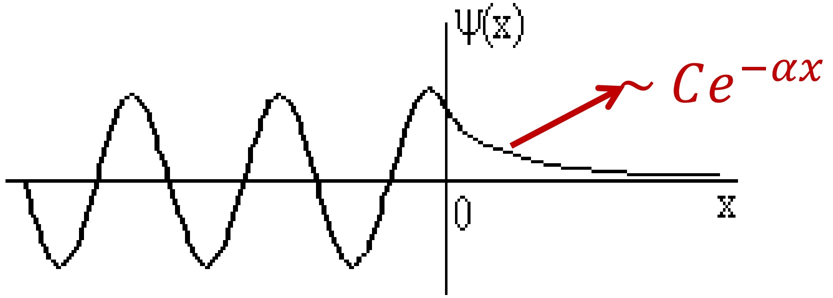
$$\frac{B}{A} = \frac{k - i\alpha}{k + i\alpha}$$

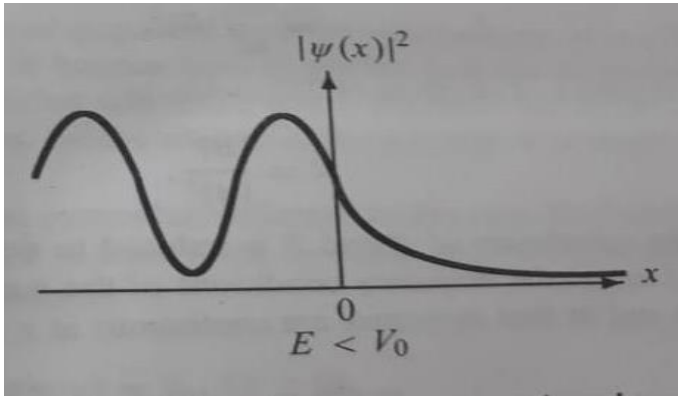
E < V_0

Wave Functions ($E < V_0$)

$$\varphi_I(x) = Ae^{ikx} + Be^{-ikx};\ k^2 = \frac{2mE}{\hbar^2} \qquad \varphi_{II}(x) = Ce^{-\alpha x};\ \alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

$$\boldsymbol{P}_{II}(x) = |\boldsymbol{C}|^2 e^{-2\alpha x}$$





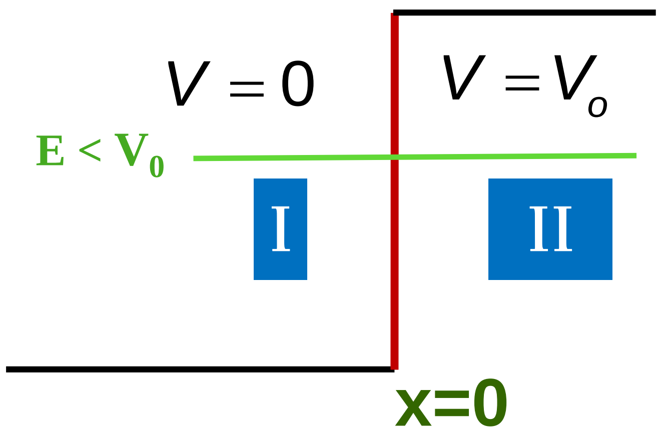
- However, $C \neq 0$ means that the particle penetrates into region II, which again is classically forbidden (E< Vo).
- Wave function rapidly approaches zero beyond $x = (1/\alpha)$.
- The probability density is appreciable only near $x = 0$, in the range.

$$\text{Penetration depth} = (1/\alpha) = \frac{\hbar}{\sqrt{2m(V_0 - E)}}$$

Reflection coefficients (E <Vo)

$$\varphi_I(x) = Ae^{ikx} + Be^{-ikx};\ k^2 = \frac{2mE}{\hbar^2} \qquad \varphi_{II}(x) = Ce^{-\alpha x};\ \alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

$$\frac{C}{A} = \frac{2k}{k+i\alpha} \quad \text{and} \quad \frac{B}{A} = \frac{k-i\alpha}{k+i\alpha}$$



$$j_{incident} = \frac{\hbar k_1}{m}|A|^2$$

$$j_{reflected} = \frac{\hbar k_1}{m}|B|^2$$

- Reflection coefficient ; $R = \left|\frac{B}{A}\right|^2 = \left(\frac{k-i\alpha}{k+i\alpha}\right)\left(\frac{k+i\alpha}{k-i\alpha}\right) = 1$

The de Broglie wave is "totally reflected"

Transmission coefficients (E <Vo)

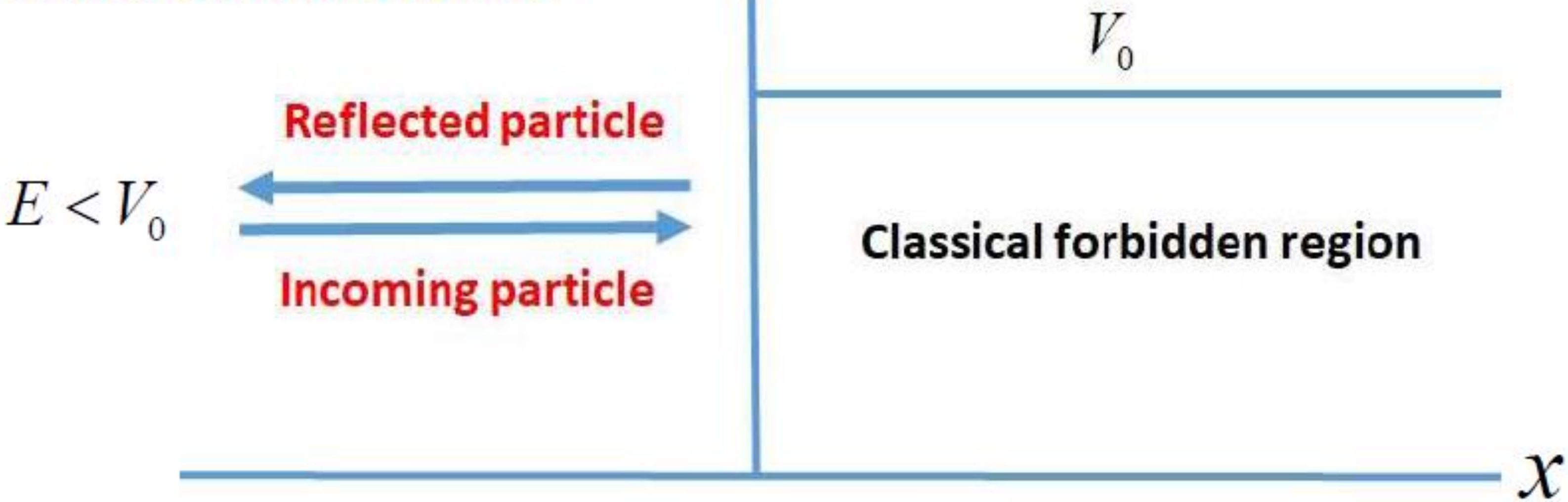
- Since $\varphi_{II}(x) = Ce^{-\alpha x}$

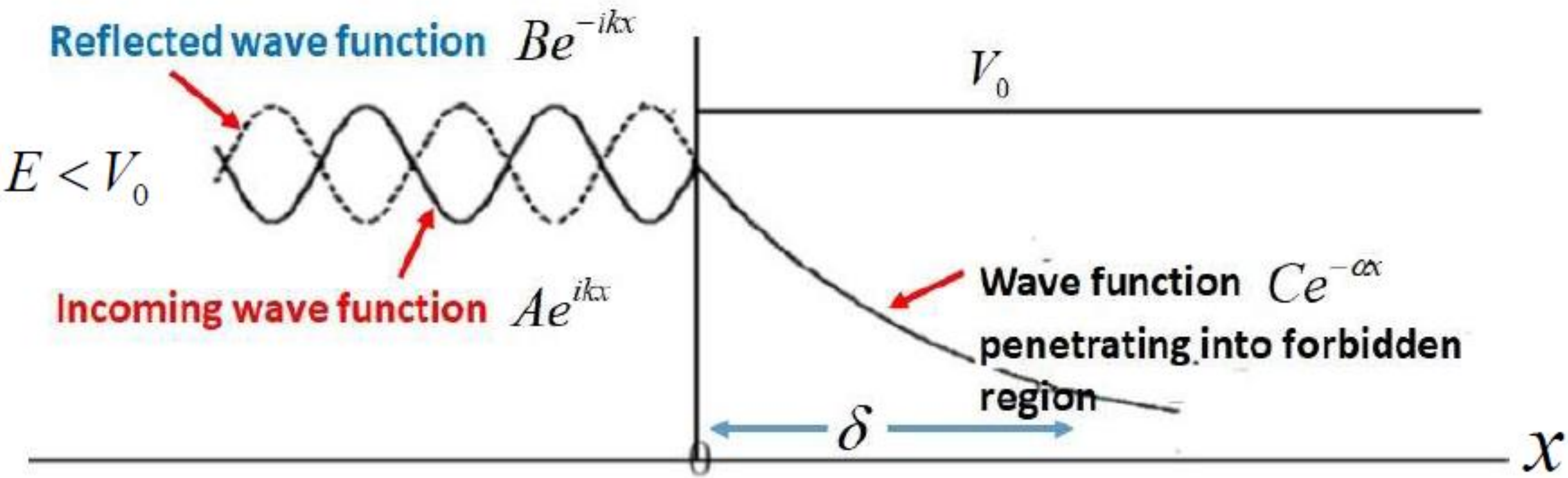
$$J_{transmitted} = \frac{i\hbar}{2m}\left(\Psi(x)\frac{\partial\Psi^*(x)}{\partial x} - \Psi^*(x)\frac{\partial\Psi(x)}{\partial x}\right) = 0$$

Transmission coefficient, $T = \left|\frac{transmitted\ current\ density}{incident\ current\ density}\right| = \left|\frac{J_{transmitted}}{J_{incident}}\right| = 0$

- No transmission current or probability flow due to existing wave function across the potential step.

Classical behaviour



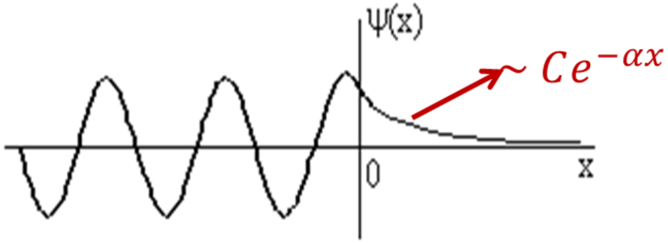


Trying to measure the energy in region II

So one may say that the particle is predominantly localized within the length Δx.

Uncertainty principle then requires that,

$$\Delta p \sim (\hbar / 2\Delta x) \sim \sqrt{2m(V_0 - E)}$$



Uncertainty in the energy of the particle,

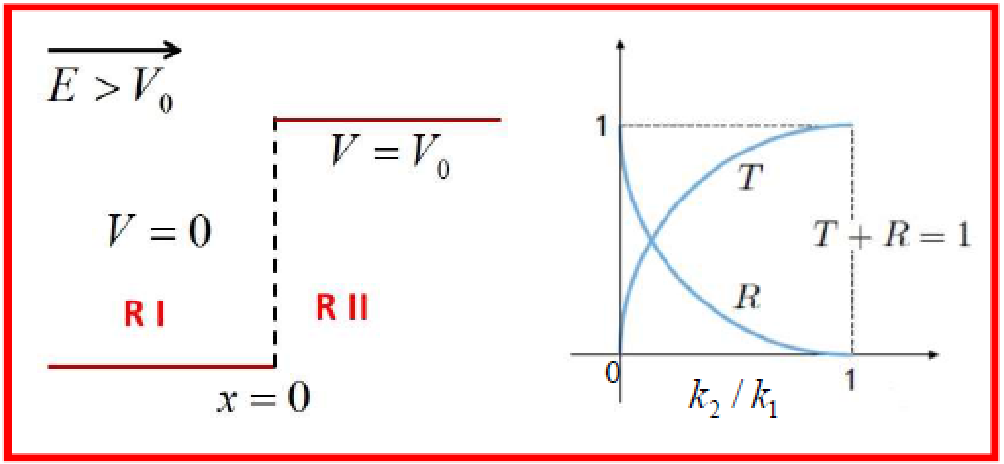
$$\Delta E = \frac{(\Delta p)^2}{2m} \sim (V_0 - E)$$

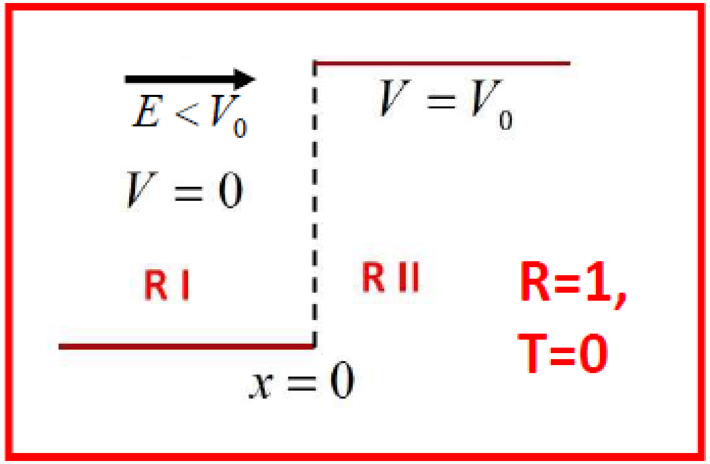
This implies near boundary x > 0, E ~ E + $V_0 - E \geq V_0$.

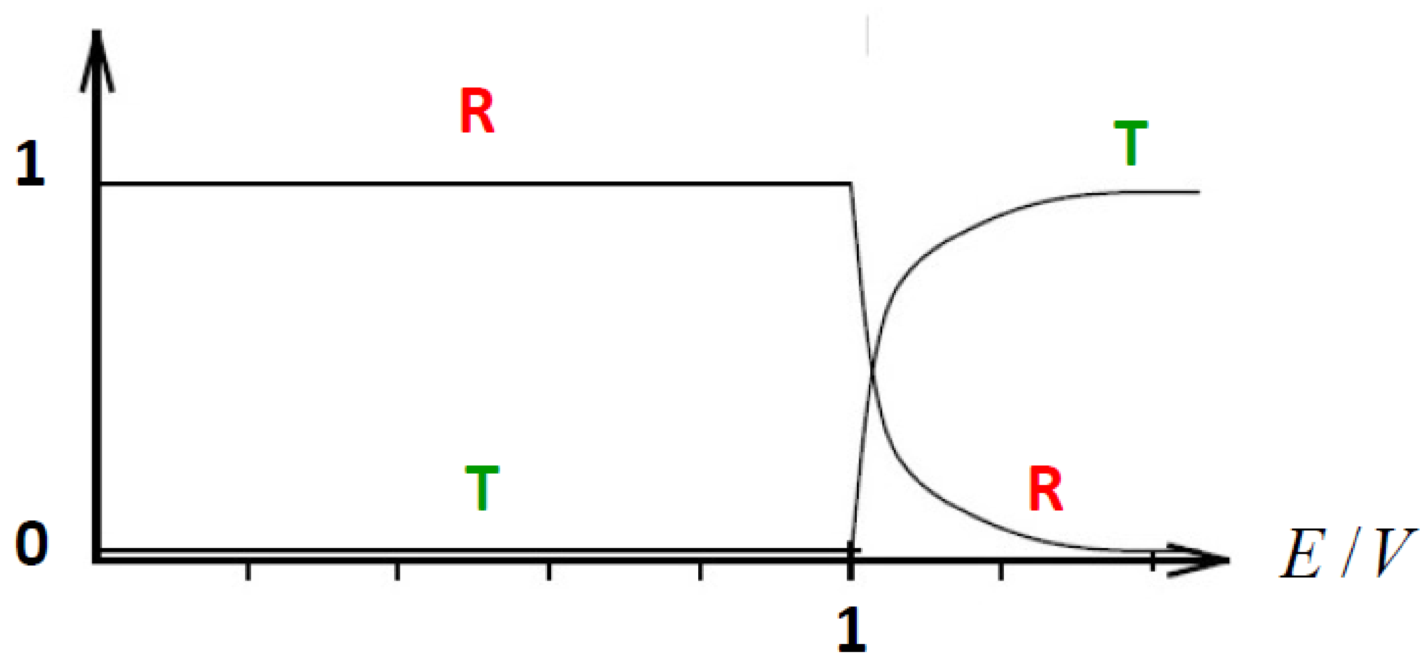
HUP helps to understand the situation with negative kinetic energy.

So, it is impossible to determine whether the energy of the particle is less than or greater than the barrier.

Summary (Step Potential)

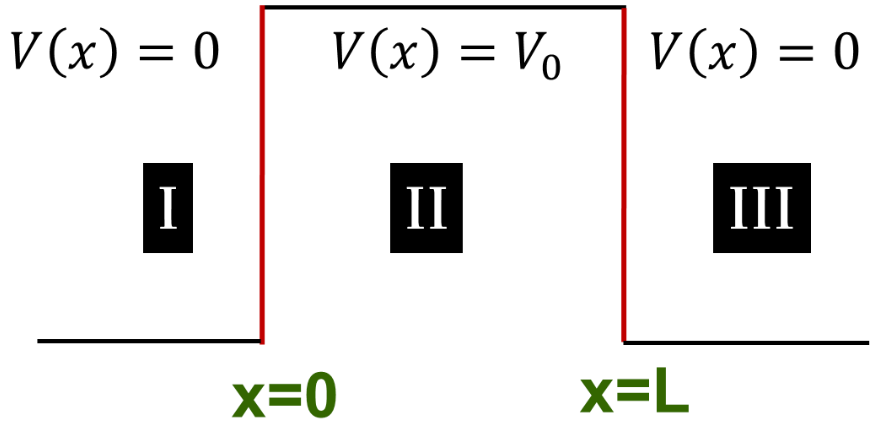




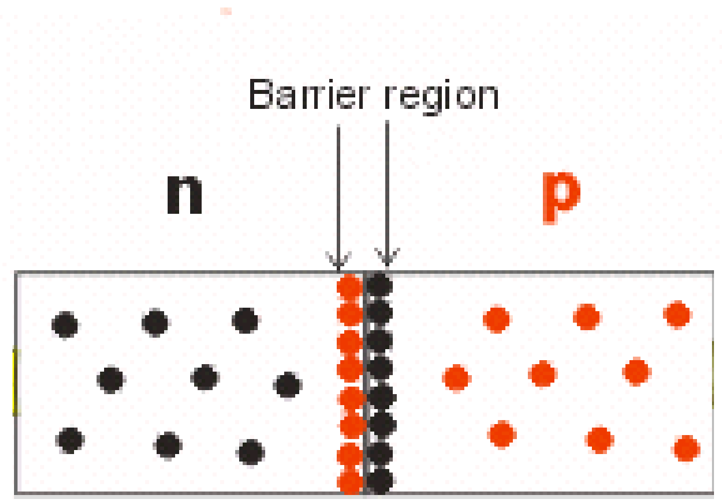


Potential Barrier: Step of finite width

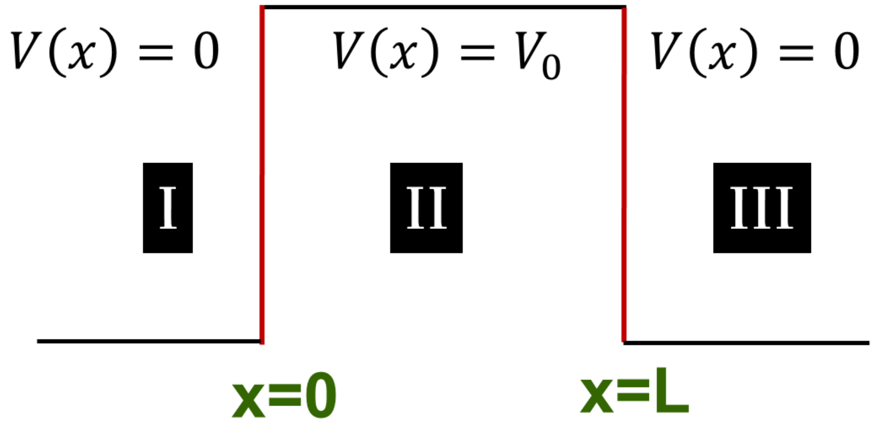
$$V(x) = 0 \quad \forall x \le 0$$
$$= V_0 \quad \forall\, 0 < x < L$$
$$= 0 \quad \forall\, x \ge L$$



- Potential Barrier is the opposite of potential well.
- Consider particle coming from left and moving towards the potential barrier in right direction.
- Region I and Region III : total energy is E = kinetic energy of the particle.
- Region II, the kinetic energy is E- V_0.



Potential Barrier: Classical Particle



$E > V_0$

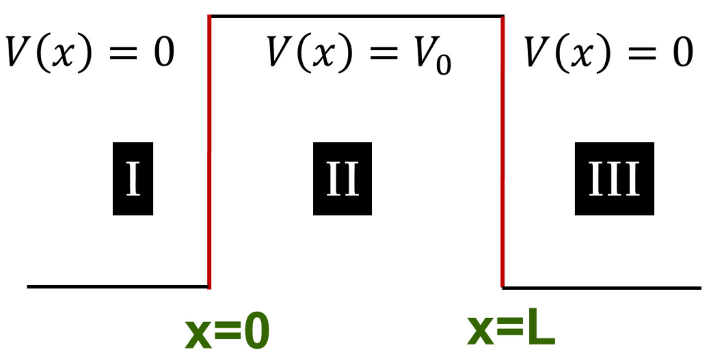
- All particles will pass through the barrier.
- Kinetic energy in region II, will be less due to the potential barrier.

$E < V_0$

- All particles will be reflected back.

Potential Barrier: Quantum particle (E<V$_0$ & E>V$_0$)

$$V(x) = 0 \quad \forall x \le 0$$
$$= V_0 \quad \forall\, 0 < x < L$$
$$= 0 \quad \forall\, x \ge L$$



I

$$\varphi_I(x) = Ae^{ikx} + Be^{-ikx};\ \ k^2 = \frac{2mE}{\hbar^2}$$
$$\varphi_I(x) = Ae^{ikx} + Be^{-ikx};\ \ k^2 = \frac{2mE}{\hbar^2}$$

II

$$\varphi_{II}(x) = Ce^{-\alpha x} + De^{\alpha x};\ \ \alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$$
$$\varphi_{II}(x) = Ce^{-ik'x} + De^{ik'x};\ (k')^2 = \frac{2m}{\hbar^2}(E - V_0)$$

III

$$\varphi_{III}(x) = Fe^{ikx} + Ge^{-ikx};\ \ k^2 = \frac{2mE}{\hbar^2}$$
$$\varphi_{III}(x) = Fe^{ikx} + Ge^{-ikx};\ \ k^2 = \frac{2mE}{\hbar^2}$$

Potential step of finite width $E<V_0$ & $E>V_0$

1. $\varphi_{III}(x) = Fe^{ikx} + Ge^{-ikx}, G = 0$

   $\varphi_{III}(x) = Fe^{ikx} + Ge^{-ikx}, G = 0$

2. $\varphi_I(0) = \varphi_{II}(0)$ **yields** $A + B = \mathrm{C} + \mathrm{D}$

   $$A + B = \mathrm{C} + \mathrm{D}$$

3. $\varphi_I'(0) = \varphi_{II}'(0)$ **yields** $ik(A - B) = \alpha(D - C)$

   $$k(A - B) = k'(D - C)$$

4. $\varphi_{II}(L) = \varphi_{III}(L)$ **yields** $Ce^{-\alpha L} + De^{\alpha L} = Fe^{ikL}$

   $$Ce^{-ik'L} + De^{ik'L} = Fe^{ikL}$$

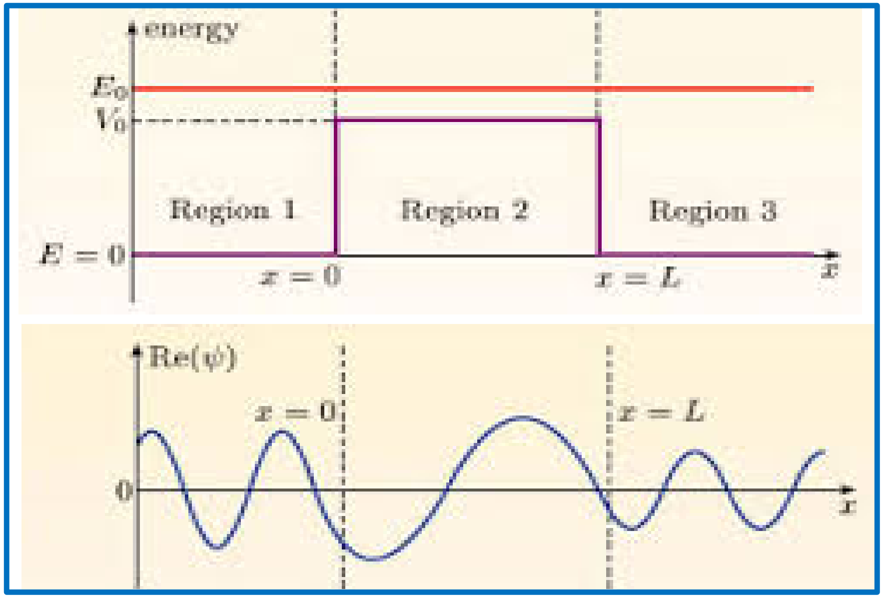
5. $\varphi_{II}'(L) = \varphi_{III}'(L)$ **yields** $\alpha(De^{\alpha L} - Ce^{-\alpha L}) = ikFe^{ikL}$

   $$k'(De^{ik'L} - Ce^{-ik'L}) = kFe^{ikL}$$

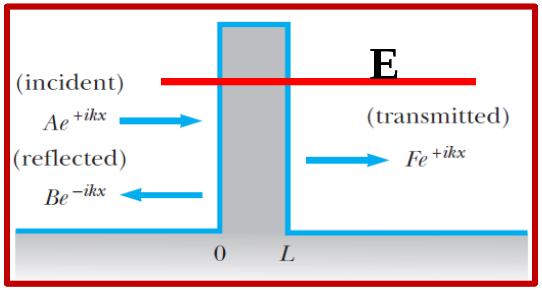
If we solve the 4 equations, in either case, we can get the ratios $B/A, C/A, D/A, F/A$

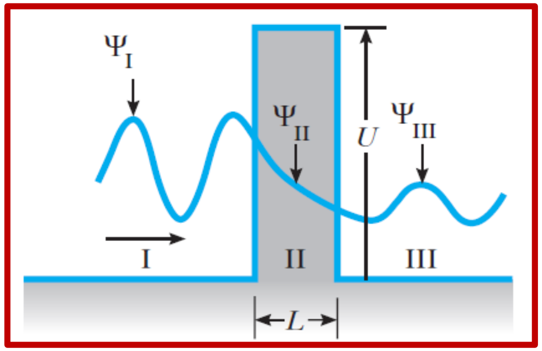
The wavefunctions: $E>V_0$ & $E<V_0$

$E>V_0$



$E<V_0$





$$\varphi_I(x) = Ae^{ikx} + Be^{-ikx};\;\; k^2 = \frac{2mE}{\hbar^2}$$

$$\varphi_{II}(x) = Ce^{-ik'x} + De^{ik'x};\; (k')^2 = \frac{2m}{\hbar^2}(E - V_0)$$

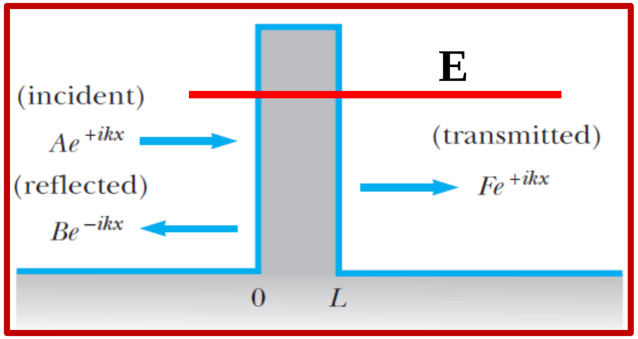
$$\varphi_{III}(x) = Fe^{ikx} + Ge^{-ikx},\, G = 0$$

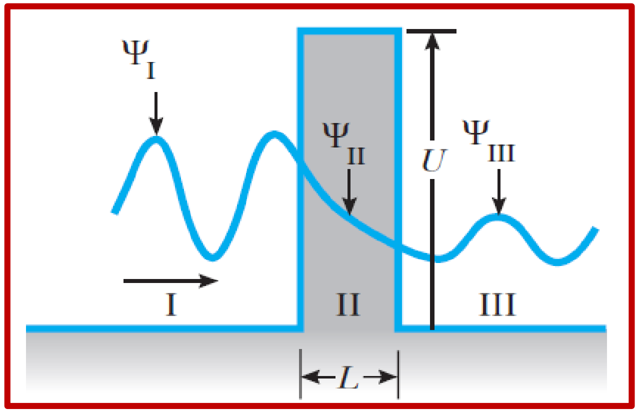
$$\varphi_I(x) = Ae^{ikx} + Be^{-ikx};\;\; k^2 = \frac{2mE}{\hbar^2}$$

$$\varphi_{II}(x) = Ce^{-\alpha x} + De^{\alpha x};\;\; \alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

$$\varphi_{III}(x) = Fe^{ikx} + Ge^{-ikx},\, G = 0$$

Tunneling $E < V_0$





- Classically we would expect total reflection.

- Quantum Mechanically $T(E) \neq 0$; (probability for transmission of particle across barrier is not zero).

- Tunneling effect: consists of propagation of a particle through a region where particle's energy is smaller than the potential energy.

- **Tunneling is also known for light waves.**

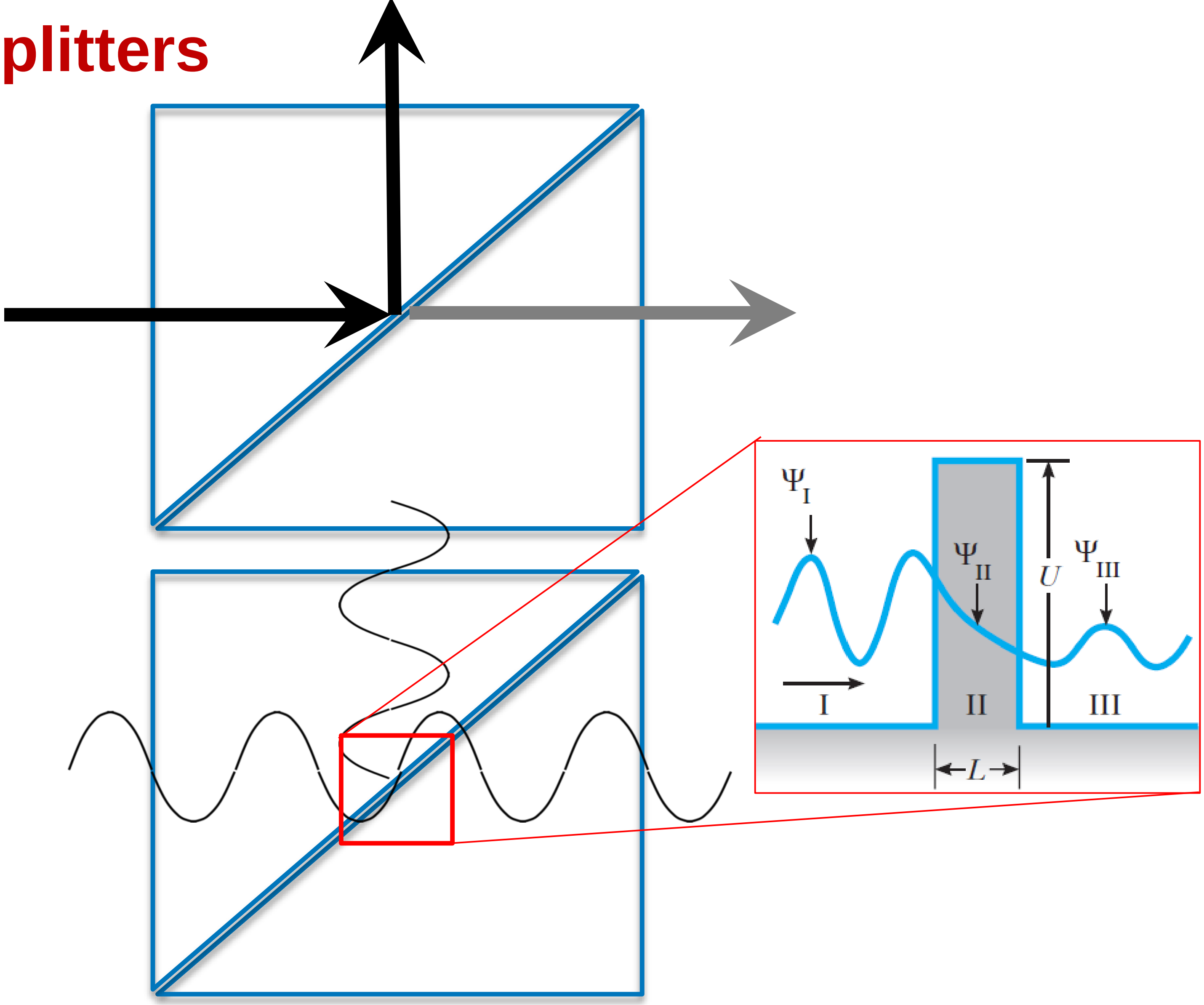
Tunneling Application 1: Frustrated total internal reflection

Light entering a right-angle prism is completely reflected at the hypotenuse face, the evanescent wave, penetrates into the space beyond.

Total internal reflection (a)   (b) **Frustated Total internal reflection**

- Total internal reflection of light waves at a glass – air boundary.
- Optical analog of tunneling, photons have tunneled across the gap separating the two prisms.

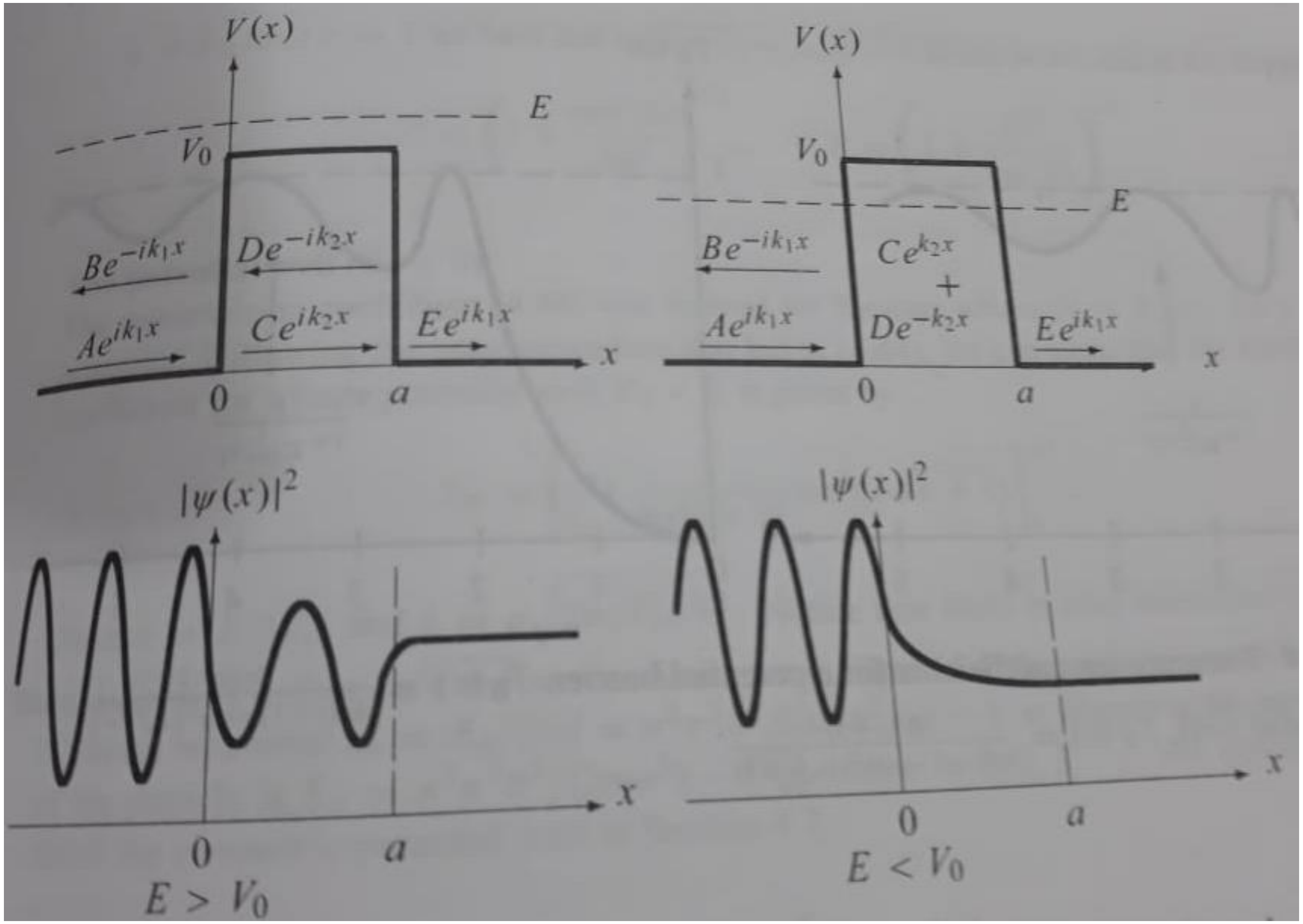
Beam-splitters



- When light is total-internally reflected, Maxwell's equations require that the tangential component of the electric field remains continuous across the boundary of the two media (**Evanescent Wave**). Intensity decreases exponentially

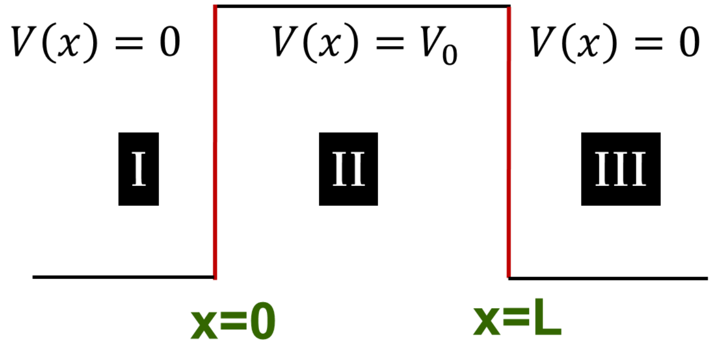
The probability density

$E>V_0$ $E<V_0$



https://demonstrations.wolfram.com/TheCausalInterpretationOfQuantumTunnelingThroughASquareBarri/

Reflection and Transmission coefficients



- The **reflection** coefficient and the **transmission** coefficient are given by

$$R = \left|\frac{B}{A}\right|^2 \text{ and } T = \left|\frac{F}{A}\right|^2,$$

- Both of which are functions of k, and hence the energy E of the particle.
- Note that no additional factor $\frac{k_3}{k_1}$ need to be multiplied to $\left|\frac{F}{A}\right|^2$, since $\frac{k_3}{k_1} = 1$.
- It can be shown that; $R(E) + T(E) = 1$

(because particle is either reflected or tranmistted).

Transmission Coefficients, $E > V_0$

The expression for T(E) can be shown to be :

$$T(E) = \left[1 + \frac{1}{4}\left(\frac{V_0^2}{E(E - V_0)}\right)\sin^2 k'L\right]^{-1}$$

$$where, (k')^2 = \frac{2m}{\hbar^2}(E - V_0)$$

Transmission Coefficients, $E < V_0$

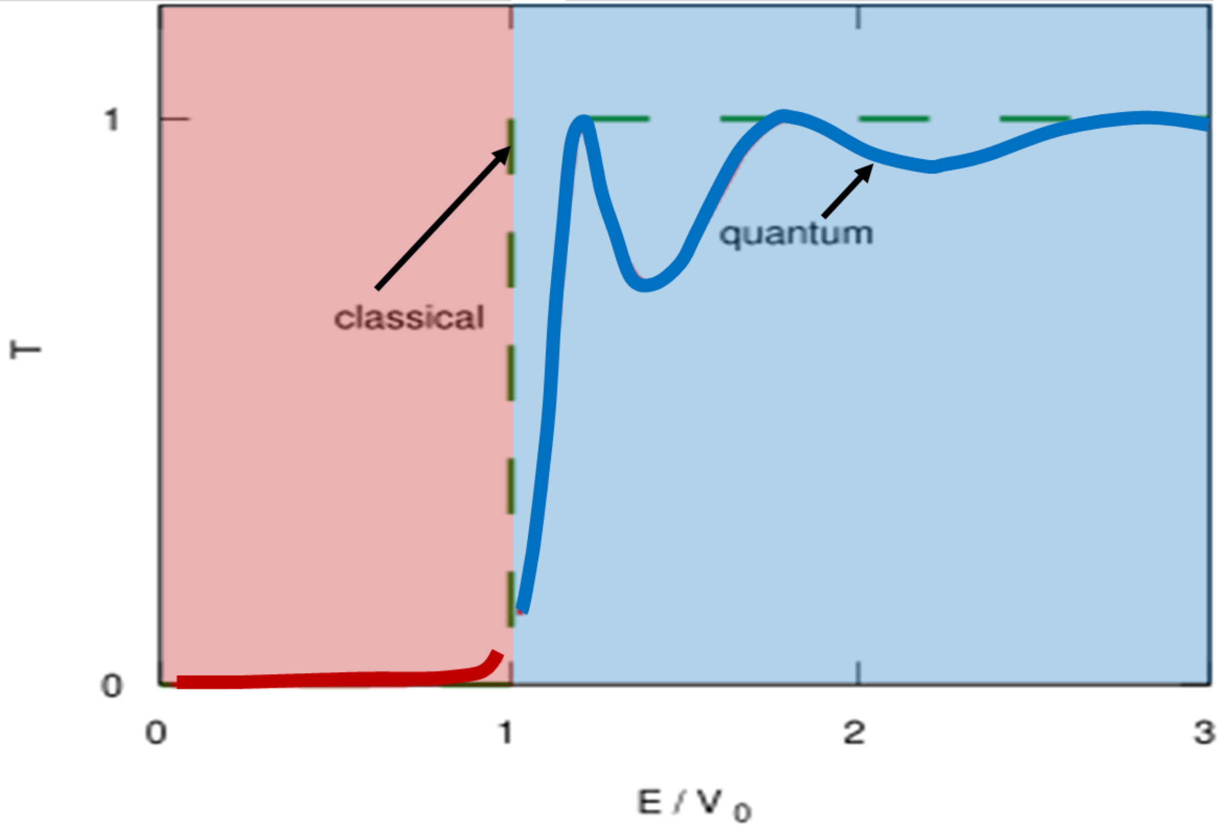
$$T(E) = \left[1 + \frac{1}{4}\left(\frac{V_0^2}{E(V_0 - E)}\right)\sinh^2 \alpha L\right]^{-1}$$

$$\sinh \alpha L = (e^{\alpha L} - e^{-\alpha L})/2 \ ; \ \alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

Plot of Transmission Probability, T(E): E< V_0 & E >V_0

$$\left[1+\frac{1}{4}\left(\frac{V_0^2}{E(V_0-E)}\right)\sinh^2\alpha L\right]^{-1}$$

$$\left[1+\frac{1}{4}\left(\frac{V_0^2}{E(E-V_0)}\right)\sin^2 k'L\right]^{-1}$$



$$\alpha^2=\frac{2m}{\hbar^2}(V_0-E)$$

$$(k')^2=\frac{2m}{\hbar^2}(E-V_0)$$

Tunneling : E<V_0

$$T(E) = \left[1 + \frac{1}{4}\left(\frac{V_0^2}{E(V_0 - E)}\right)\sinh^2\alpha L\right]^{-1} ; \alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

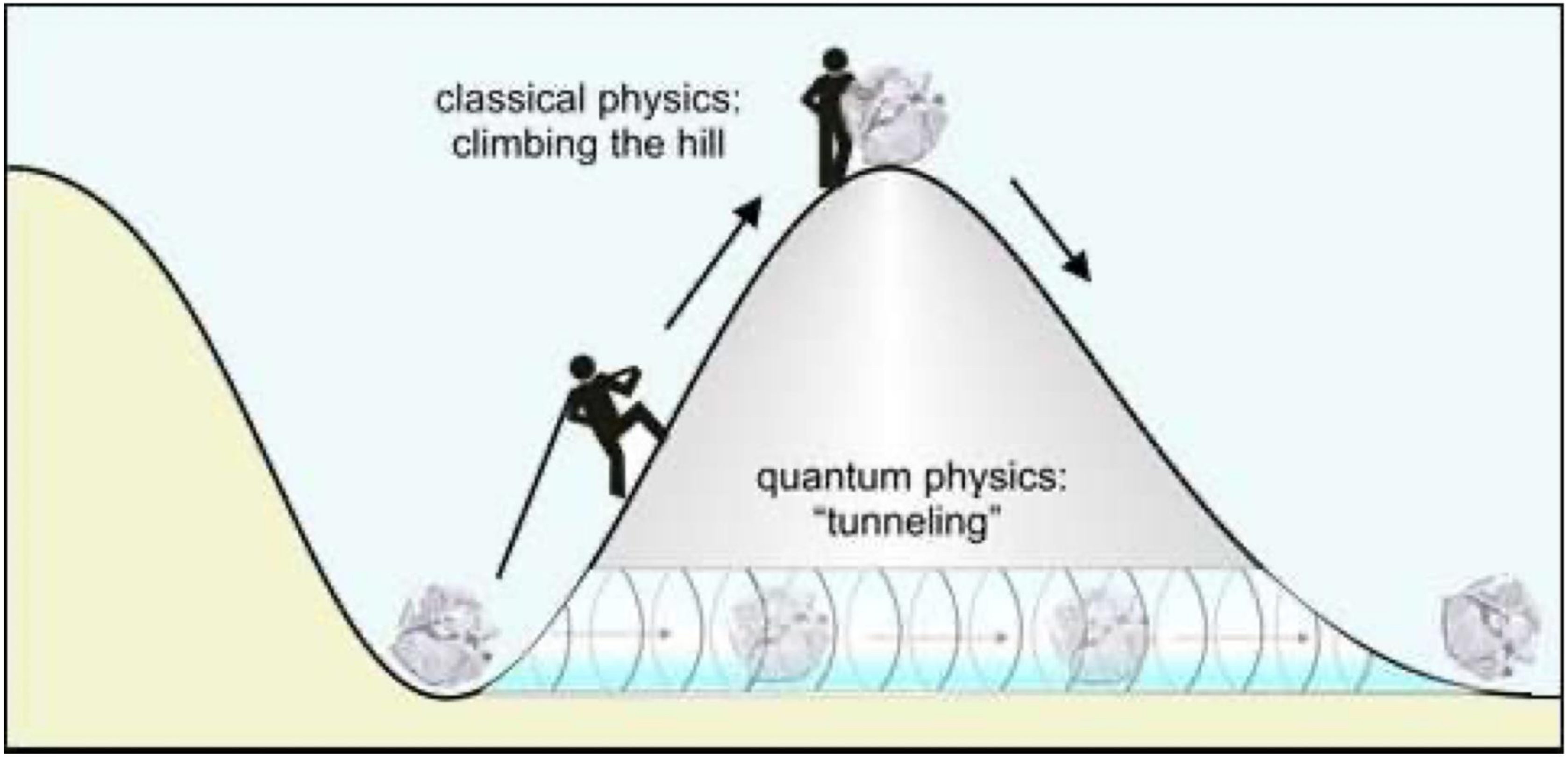
- For E << V_0, one can show, $T(E) \simeq \frac{16E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-(2L/\hbar)\sqrt{2m\,(V_0 - E)}}$

In general for smooth and slowly varying V(x),

$$T(E) \simeq e^{-\frac{2}{\hbar}\sqrt{2m}\int\sqrt{(V(x)-E)}\,dx}$$

- T is finite , probability for transmission is non zero for x > L.
- This is purely quantum mechanical effect due to the wave aspect of microscopic objects, known as tunneling effect.
- Tunneling effect, quantum objects can tunnel through classically impenetrable barriers.
- In the classical limit of $\hbar \longrightarrow 0$, the T $\longrightarrow 0$, the classical result.

Tunnelling



- Non-intuitive, intellectually fascinating and technologically important process.
- Natural phenomena such as radioactive alpha-decay.
- Scanning tunnelling microscope (STM)

Example 7.1 from Serway:

Two conducting copper wires are separated by an insulating layer of copper-oxide. We model the oxide layer as a rectangular barrier of height 10 eV. Calculate the transmission coefficient for penetration by 7 eV electrons, if the layer thickness is (a) 5 nm and (b) 1 nm.

$$\alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar} == \sqrt{2 \times 511000 \times 31973} = 0.9(\mathrm{Angstrom})^{-1}.$$

For $L \gg 1$, $\alpha L \approx 45 \implies \mathrm{Sinh}(\alpha L) \approx \exp(\alpha L)/2$.

Thus we $T \approx 4\exp(-2\alpha L)$ leading to

$$\frac{T(L=50)}{T(L=10)} = 4\exp(-2 \times 0.9 \times 40) \approx 10^{-31}.$$

Because of the exponential factor, small changes of the barrier height or width lead to large changes in the tunnelling probability.