

# MA111 TUTORIAL SOLUTIONS SPRING 2022

by MA111 TAs

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# TUTORIAL SHEET 1 (MULTIPLE INTEGRALS)

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1.

- (a) Let  $R := [0, 1] \times [0, 1]$  and  $f(x, y) := [x] + [y] + 1$  for all  $(x, y) \in R$ , where  $[u]$  is the greatest integer less than equal to  $u$ , for any  $u \in \mathbb{R}$ . Using the definition of integration over rectangles, show that  $f$  is integrable over  $R$ . Also, find its value.
- (b) Let  $R := [0, 1] \times [0, 1]$  and  $f(x, y) := (x + y)^2$  for all  $(x, y) \in R$ . Show that  $f$  is integrable over  $R$  and find its value using Riemann sum.
- (c) Let  $R := [a, b] \times [c, d]$  be a rectangle in  $\mathbb{R}^2$  and let  $f : R \rightarrow \mathbb{R}$  be integrable. Show that  $|f|$  is also integrable over  $R$ .
- (d) Check the integrability of the function  $f$  over  $[0, 1] \times [0, 1]$ ;

$$f(x, y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational numbers} \\ -1 & \text{otherwise} \end{cases}$$

What do you conclude about the integrability of  $|f|$ ?

**Sol.**

- (a) We have,

$$f(x, y) = \begin{cases} 1 & x \in [0, 1), y \in [0, 1) \\ 2 & x = 1, y \in [0, 1) \\ 2 & x \in [0, 1), y = 1 \\ 3 & x = 1, y = 1 \end{cases}$$

Thus  $f$  is bounded on  $R$ . Let  $P_n := \{(x_i, y_j) \mid i \in \{0, 1 \dots n\}, j \in \{0, 1 \dots n\}\}$  be a regular partition of  $R$ . Then we have

$$R_{ij} := \left[ \frac{i}{n}, \frac{i+1}{n} \right] \times \left[ \frac{j}{n}, \frac{j+1}{n} \right] \forall i, j \in \{0, 1 \dots n-1\}$$

Clearly,

$$L(f, P_n) = 1$$

Let us compute  $U(f, P_n)$ .

$$\begin{aligned} U(f, P_n) &= \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} \frac{1}{n^2} + \sum_{i=n-1}^{n-1} \sum_{j=0}^{n-1} \frac{2}{n^2} + \sum_{j=n-1}^{n-1} \sum_{i=0}^{n-1} \frac{2}{n^2} - \frac{1}{n^2} \\ &= \frac{(n-1)^2}{n^2} + 2 \times \frac{2n}{n^2} - \frac{1}{n^2} \\ &= 1 + \frac{2}{n} \end{aligned}$$

Thus,

$$U(f, P_n) - L(f, P_n) = \frac{2}{n}$$

Given any  $\epsilon > 0$ , choose  $n = \lfloor \frac{2}{\epsilon} \rfloor + 1$ . Thus, we see that

$$U(f, P_n) - L(f, P_n) < \epsilon$$

By the Riemann Condition, we are done. □

- (b) A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to be integrable over  $\mathbb{R}$  if it is Darboux or Riemann integrable over  $\mathbb{R}$  [lecture 1, slide 17]. Observe that  $f(x, y) = (x + y)^2 \leq 2^2 = 4 \forall (x, y) \in \mathbb{R}$ , hence  $f$  is bounded on  $\mathbb{R}$ . This allows us to use the following theorem [lec. 1, sl. 18]:

$f : \mathbb{R} \rightarrow \mathbb{R}$  is Riemann integrable if and only if the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij},$$

where  $P_n$  is the sequence of regular partitions of  $\mathbb{R}$ , tends to the same limit  $S \in \mathbb{R}$  as  $n \rightarrow \infty$  for any choice of tag  $t$ , and  $\Delta_{ij} = (x_{i+1} - x_i)(y_{i+1} - y_i)$ .

That  $f$  is integrable on  $\mathbb{R}$  can be concluded directly from the fact that it is continuous; the value of the integral is then simply given by the limit  $S$ . The computation is essentially the same as the examples done in class [lec. 1, sl. 19, 20].

The partitions  $P_n$  are given by the subrectangles  $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$  and tags  $t = \{t_{ij} = (\frac{i}{n}, \frac{j}{n})\}$  for  $i, j = 0, 1, \dots, n-1$ . The Riemann sum is then given by

$$\begin{aligned} S(f, P_n, t) &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left( \frac{i}{n} + \frac{j}{n} \right)^2 \frac{1}{n^2} \\ &= \frac{1}{n^4} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (i+j)^2 = \frac{1}{n^4} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} [i^2 + j^2 + 2ij] \\ &= \frac{1}{n^4} \left[ \sum_{i=0}^{n-1} i^2 \cdot \sum_{j=0}^{n-1} 1 + \sum_{i=0}^{n-1} 1 \cdot \sum_{j=0}^{n-1} j^2 + 2 \sum_{i=0}^{n-1} i \cdot \sum_{j=0}^{n-1} j \right] \\ &= \frac{1}{n^4} \left[ \frac{(n-1)(n)(2n-1)}{6} n + n \frac{(n-1)(n)(2n-1)}{6} + 2 \frac{(n-1)(n-2)}{2} \frac{(n-1)(n-2)}{2} \right] \\ &= \frac{7}{6} - \frac{3}{n} - \frac{3}{2n^2} - \frac{10}{3n^3} + \frac{12}{n^4} \end{aligned}$$

One can check that  $S(f, P_n, t) \rightarrow \frac{7}{6} = S$ . Hence the value of the integral is  $\boxed{\frac{7}{6}}$ . □

- (c) Given an integrable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we wish to show that  $g := |f|$  must also be integrable over  $\mathbb{R}$ . First note that for any  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}$ ,

$$|g(x_1, y_1) - g(x_2, y_2)| = ||f(x_1, y_1)| - |f(x_2, y_2)|| \leq |f(x_1, y_1) - f(x_2, y_2)|.$$

Further,  $f$  must be bounded on  $\mathbb{R}$  (by definition of integrability [lecture 1, slide 17]). Thus we may use the following theorem, also called the *Riemann condition* [lec. 1, sl. 13]:

A bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Riemann integrable if and only if for every  $\epsilon > 0$  there is a partition  $P_\epsilon$  of  $\mathbb{R}$  such that

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon,$$

where  $U(f, P_\epsilon), L(f, P_\epsilon)$  are the upper and lower double sums with respect to the partition  $P_\epsilon$  (respectively).

Fix an  $\epsilon > 0$ . Then from the above theorem, we have a partition  $P_\epsilon$  such that  $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$ , i.e.

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{ij}(f) \Delta_{ij} - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij}(f) \Delta_{ij} < \epsilon \implies \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [M_{ij}(f) - m_{ij}(f)] \Delta_{ij} < \epsilon,$$

where  $M_{ij}(f)$ ,  $m_{ij}(f)$  denote the supremum and infimum of  $f$  over the rectangle  $R_{ij}$  of the partition. Since  $|g(x) - g(y)| \leq |f(x) - f(y)|$ , we have  $M_{ij}(g) - m_{ij}(g) \leq M_{ij}(f) - m_{ij}(f)$ . This gives us

$$U(g, P_\epsilon) - L(g, P_\epsilon) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [M_{ij}(g) - m_{ij}(g)] \Delta_{ij} \leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [M_{ij}(f) - m_{ij}(f)] \Delta_{ij}.$$

Since the quantities on the right and left are both positive, we have  $|U(g, P_\epsilon) - L(g, P_\epsilon)| < \epsilon$ . Noting that the choice of  $\epsilon$  was arbitrary, we have shown that for every  $\epsilon > 0$ , there is a partition  $P_\epsilon$  such that  $|U(g, P_\epsilon) - L(g, P_\epsilon)| < \epsilon$ . Once again invoking the Riemann condition, we conclude that  $g = |f|$  is integrable.  $\square$

- (d) Take any partition of  $[0, 1] \times [0, 1]$ . As we know,  $\mathbb{Q} \times \mathbb{Q}$  is a dense subset of  $\mathbb{R}^2$ , so in any subrectangle there will be a point  $(x, y)$  with both coordinates rational, and also there will be point inside the subrectangle with both coordinates irrational.

Observe that the value of  $f$  at those points is 1 and  $-1$  respectively which is also the inf and sup on that rectangle. Thus the uppersum  $U(f, P)$  for any partition is

$$\sum_{i,j} (1) \Delta_{ij} = 1.$$

And the lower sum  $L(f, P)$  is

$$\sum_{i,j} (-1) \Delta_{ij} = -1.$$

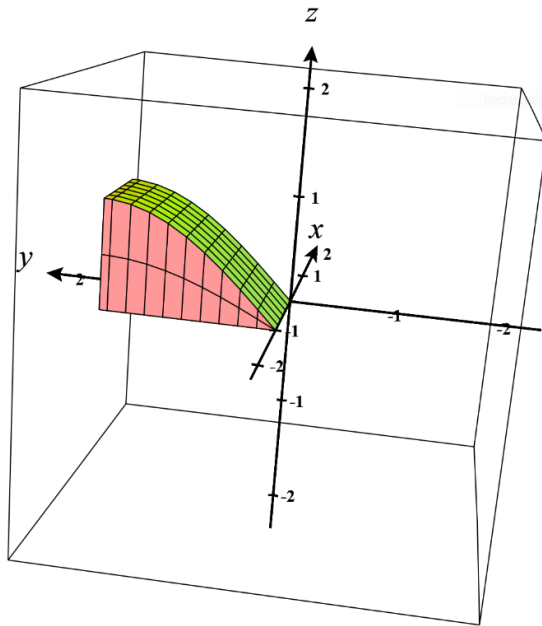
Which shows that  $\inf U(f, P) \neq \sup L(f, P)$ , or the function isn't Riemann integrable, but  $|f|$  is the constant function 1 which is trivially integrable.  $\blacksquare$

2.

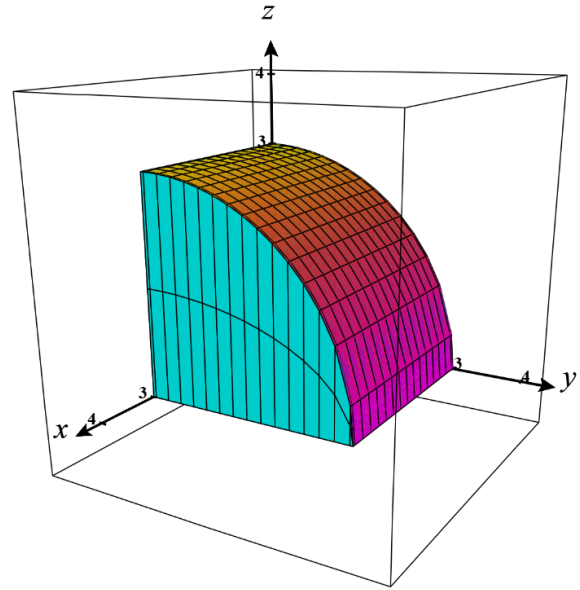
- (a) Sketch the solid bounded by the surface  $z = \sin y$ , the planes  $x = -1, x = 0, y = 0$  and  $y = \frac{\pi}{2}$  and the  $xy$  plane and compute its volume.
- (b) The integral  $\iint_R \sqrt{9 - y^2} dx dy$ , where  $R = [0, 3] \times [0, 3]$ , represents the volume of a solid. Sketch the solid and find its volume.

**Sol.**

The solids are sketched in Fig. 1.



(a)  $z = \sin y$



(b)  $z = \sqrt{9 - y^2}$

Figure 1: Figure showing the solid surfaces for question 2

$$(a) \text{ Volume} = \int_{[-1,0]} \int_{[0,\pi/2]} \sin y dy dx = \int_{[-1,0]} 1 dx = \boxed{1}$$

$$(b) \text{ Volume} = \int_{[0,3]} \int_{[0,3]} \sqrt{9 - y^2} dy dx = \int_{[0,3]} \frac{9\pi}{4} dx = \boxed{\frac{27\pi}{4}}$$

■

3. Consider the function  $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  defined as

$$f(x, y) = \begin{cases} 1 - 1/q & \text{if } x = p/q \text{ where } p, q \in \mathbb{N} \text{ are relatively prime and } y \text{ is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

Show that  $f$  is integrable but the iterated integrals do not always exist.

**Sol.**

Clearly  $f(x, y) \in [0, 1] \forall (x, y) \in R = [0, 1] \times [0, 1]$ . Take  $\mathbb{R} \ni \epsilon > 0$ . For some  $y$ , define a set  $S$ ,

$$S = \{x \mid 1 - f(x, y) \geq \epsilon, 0 \leq x \leq 1, 0 \leq y \leq 1, y \in \mathbb{Q}\}$$

Thus,  $S$  has all rational numbers  $0 \leq \frac{p}{q} \leq 1$  satisfying  $q \leq \frac{1}{\epsilon}$ ,  $q \in \mathbb{N}$ . (Note:  $S$  is a finite set, with a finite number of elements, say  $L$ .)

Consider a partition  $P_\epsilon = \{x_0, x_1, x_2, \dots, x_m\} \times \{y_0, y_1, y_2, \dots, y_n\}$  with  $x_0 = y_0 = 0$  and  $x_m = y_n = 1$  such that,

$$\begin{aligned} x_j - x_{j-1} &< \frac{\epsilon}{L} \quad \forall j \in \{1, 2, 3, \dots, m\} \\ y_k - y_{k-1} &< \frac{\epsilon}{L} \quad \forall k \in \{1, 2, 3, \dots, n\} \\ \implies \|P_\epsilon\| &< \frac{\epsilon}{L} \end{aligned}$$

We define the following for brevity:

$$\begin{aligned} P_{\epsilon jk} &= [x_{j-1}, x_j] \times [y_{k-1}, y_k] \\ \Delta_{jk} &= (x_j - x_{j-1})(y_k - y_{k-1}) \\ m_{jk} &= \inf_{(x, y) \in P_{\epsilon jk}} f(x, y) \\ M_{jk} &= \sup_{(x, y) \in P_{\epsilon jk}} f(x, y) \end{aligned}$$

$M_{jk} = 1 \forall (j, k)$  since between any 2 numbers, there are infinitely many irrational numbers.  $m_{jk} = 1 - \frac{1}{q}$  for some  $q \in \mathbb{N}$ . Thus,

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &= \sum_{j=1}^m \sum_{k=1}^n M_{jk} \Delta_{jk} - \sum_{j=1}^m \sum_{k=1}^n m_{jk} \Delta_{jk} \\ \implies U(f, P_\epsilon) - L(f, P_\epsilon) &= \sum_{j=1}^m \sum_{k=1}^n (M_{jk} - m_{jk}) \Delta_{jk} \end{aligned}$$

The sum is now split into two parts: one containing at least one point with  $x \in S$  and the other with  $x \notin S$ :

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{j=1}^m \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \emptyset}}^n (M_{jk} - m_{jk}) \Delta_{jk} + \sum_{j=1}^m \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] = \emptyset}}^n (M_{jk} - m_{jk}) \Delta_{jk}$$

In the first part,  $m_{jk} \leq 1 - \frac{1}{q}$  and thus  $M_{jk} - m_{jk} \geq \frac{1}{q} = \epsilon$ . Also, note that  $M_{jk} - m_{jk} \leq 1$ .

$$\implies \sum_{j=1}^m \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \emptyset}}^n (M_{jk} - m_{jk}) \Delta_{jk} \leq \sum_{j=1}^m \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \emptyset}}^n \Delta_{jk} \leq 2L \left( \frac{\epsilon}{L} \times 1 \right) \leq 2\epsilon$$

since there are at most  $2L$  number of  $\left(\frac{\epsilon}{L} \times 1\right)$  rectangles which are proper subset of  $P_{\epsilon jk}$  for some  $(j, k)$  such that  $S \cap [x_{j-1}, x_j] \neq \emptyset$ .

For the second part, we have  $m_{jk} > 1 - \frac{1}{q}$  and thus,  $M_{jk} - m_{jk} < \frac{1}{q} = \epsilon$ . The upper limit of the summation of  $\Delta_{jk}$  over all the values of  $j$  and  $k$  is the area of  $R = [0, 1] \times [0, 1]$  which is 1.

$$\sum_{j=1}^m \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \emptyset}}^n (M_{jk} - m_{jk}) \Delta_{jk} \leq \sum_{j=1}^m \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \emptyset}}^n \epsilon \Delta_{jk} \leq \epsilon$$

Consequently,

$$U(f, P_\epsilon) - L(f, P_\epsilon) \leq 2\epsilon + \epsilon = 3\epsilon$$

$3\epsilon$  can be made arbitrarily small. Thus  $f(x, y)$  is integrable on  $R = [0, 1] \times [0, 1]$ .

Before we move to the next part, we will introduce two functions here:

Thomae's function is a real-valued function of a real variable that can be defined as:

$$\mathbf{T}(x) := \begin{cases} \frac{1}{q} & x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } q \text{ are coprime} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Dirichlet function is a *cousin* of Thomae's function and can be defined as:

$$\mathbf{1}_{\mathbb{Q}}(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Now, we calculate the iterated integrals.

• **Case I:**

Define  $\Phi^y(x) := f(x, y)$  for some fixed  $y$ .

$$\Phi^y(x) = \begin{cases} \begin{cases} 1 - \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases} & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$$

Thus, for a given  $y$ ,  $\Phi^y(x)$  is either the constant function 1 or a function that can be derived from Thomae's function in  $x$ , both of which we know are integrable for  $x \in [0, 1]$ . This yields,

$$\int_0^1 \Phi^y(x) dx = \begin{cases} \int_0^1 1 - \mathbf{T}(x) dx & y \in \mathbb{Q} \\ \int_0^1 1 dx & y \notin \mathbb{Q} \end{cases}$$

Utilizing the fact the integral of  $\mathbf{T}(x)$  over any sub-interval of  $[0, 1]$  is 0 (Justify), we can write,

$$\int_0^1 \Phi^y(x) dx = 1$$

Consequently,

$$\int_0^1 \int_0^1 f(x, y) dx dy = \int_0^1 \left( \int_0^1 \Phi^y(x) dx \right) dy = \int_0^1 1 dy = 1$$

• **Case II:**

Define  $\Phi^x(y) := f(x, y)$  for some fixed  $x$ .

$$\Phi^x(y) = \begin{cases} \begin{cases} 1 - k_x & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases} & x = \frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

where  $k_x = \frac{1}{q}$ .

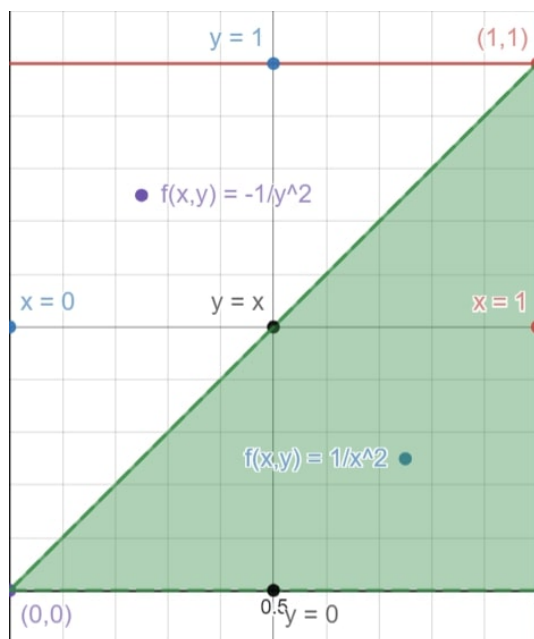
Observe that for a given  $x$ ,  $\Phi^x(y)$  is either the constant function 1 or  $1 - \mathbf{1}_{\mathbb{Q}}(y)$ . However, we know that  $\mathbf{1}_{\mathbb{Q}}(y), y \in [0, 1]$  is not integrable (Justify). Therefore the iterated integral does not exist. ■

4. Consider the function  $f : [0, 1] \times [0, 1] \mapsto \mathbb{R}$  defined as

$$f(x, y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Is  $f$  integrable over the rectangle? Do both iterated integrals exist? If they exist, do they have the same value?

**Sol.**



Observe that  $f$  is not bounded in  $R = [0, 1] \times [0, 1]$  near  $x = 0$  and  $y = 0$ . Thus, we can immediately claim that  $f$  is not integrable in  $R$ . However, this *does not* imply that the iterated integrals do not exist.

**Evaluating iterated integrals:**

For  $y = 0$ ,  $f(x, y) = 0$  and thus  $\int_{x=0}^1 f(x, y) dx = 0$ .

For  $y \in (0, 1]$ ,

$$\int_{x=0}^1 f(x, y) dx = \int_{x=0}^y f(x, y) dx + \int_{x=y}^1 f(x, y) dx$$

Then,

$$\int_{x=0}^1 f(x, y) dx = \int_{x=0}^y \frac{-1}{y^2} dx + \int_{x=y}^1 \frac{1}{x^2} dx$$

$$\Rightarrow \int_{x=0}^1 f(x, y) dx = \frac{-1}{y} + \frac{1}{y} - 1 = -1$$

Thus, define,

$$A(y) = \int_{x=0}^1 f(x, y) dx = \begin{cases} 0 & \text{for } y = 0 \\ -1 & \text{for } y \in (0, 1) \end{cases}$$

This function is bounded and has a single point of discontinuity, and hence Riemann integrable.

$$\int_{y=0}^1 \left( \int_{x=0}^1 f(x, y) dx \right) dy = \int_{y=0}^1 A(y) dy = -1$$

Similarly, let's evaluate the iterated integral:  $\int_{x=0}^1 \left( \int_{y=0}^1 f(x, y) dy \right) dx$  by repeating the above procedure.



For  $x = 0$ ,  $f(x, y) = 0$  and thus  $\int_{y=0}^1 f(x, y) \, dy = 0$ .

For  $x \in (0, 1]$ ,

$$\begin{aligned} \int_{y=0}^1 f(x, y) \, dy &= \int_{y=0}^x f(x, y) \, dy + \int_{y=x}^1 f(x, y) \, dy \\ &= \int_{y=0}^x \frac{1}{x^2} \, dy + \int_{y=x}^1 \frac{-1}{y^2} \, dy \\ &= \frac{1}{x} + 1 - \frac{1}{x} \\ &= 1 \end{aligned}$$

Thus, define

$$B(x) = \int_{y=0}^1 f(x, y) \, dy = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \in (0, 1) \end{cases}$$

As discussed above, this function is Riemann integrable (Why?). Hence, the iterated integral:

$$\int_{x=0}^1 \left( \int_{y=0}^1 f(x, y) \, dy \right) \, dx = \int_{y=0}^1 B(x) \, dx = 1$$

Clearly both iterated integrals are unequal. ■

5. For the following, write an equivalent iterated integral with the order of integration reversed and verify if their values are equal.

(a)  $\int_0^1 \left( \int_0^1 \ln[(x+1)(y+1)] dx \right) dy$

(b)  $\int_0^1 \left( \int_0^1 (xy)^2 \cos(x^3) dx \right) dy.$

**Sol.**

(a)

$$\begin{aligned} \int_0^1 \left( \int_0^1 \ln[(x+1)] + \ln[(y+1)] dx \right) dy &= \int_0^1 [(2 \ln 2 - 1) + \ln[(y+1)]] dy \\ &= 4 \ln 2 - 2 \end{aligned}$$

Since the area considered is a square, interchanging the variables does not affect the limits,

$$\begin{aligned} \int_0^1 \left( \int_0^1 \ln[(x+1)] + \ln[(y+1)] dy \right) dx &= \int_0^1 [(2 \ln 2 - 1) + \ln[(x+1)]] dx \\ &= 4 \ln 2 - 2 \end{aligned}$$

(b) We compute the integral in the given order as follows:

$$\begin{aligned} \int_0^1 \left( \int_0^1 (xy)^2 \cos(x^3) dx \right) dy &= \int_0^1 \left( \int_0^1 y^2 x^2 \cos(x^3) dx \right) dy \\ &= \int_0^1 \frac{1}{3} \left( \int_0^1 y^2 \cos(x^3) d(x^3) \right) dy \\ &= \int_0^1 \frac{1}{3} (y^2 \sin(1)) dy \\ &= \frac{1}{3} \frac{1}{3} \sin(1) \\ &= \frac{\sin(1)}{9} \end{aligned}$$

Now to calculate the iterated integral with reversed order, first note that since it is on  $[0, 1] \times [0, 1]$  the limits of the integral will remain the same.

$$\begin{aligned} \int_0^1 \left( \int_0^1 (xy)^2 \cos(x^3) dx \right) dy &= \int_0^1 \left( \int_0^1 y^2 x^2 \cos(x^3) dy \right) dx \\ &= \int_0^1 \frac{1}{3} x^2 \cos(x^3) dx \\ &= \int_0^1 \frac{1}{3} \frac{1}{3} \cos(x^3) d(x^3) \\ &= \frac{1}{3} \frac{1}{3} \sin(1) \\ &= \frac{\sin(1)}{9} \end{aligned}$$

We see that the iterated integrals have the same values. ■

6.

- (a) Let  $R = [a, b] \times [c, d]$  and  $f(x, y) = \phi(x)\psi(y)$  for all  $(x, y) \in R$  where  $\phi(x)$  is continuous on  $[a, b]$  and  $\psi(y)$  is continuous on  $[c, d]$ . Show that

$$\iint_R f(x, y) dx dy = \left( \int_a^b \phi(x) dx \right) \left( \int_c^d \psi(y) dy \right)$$

- (b) Compute  $\iint_{[1,2] \times [1,2]} x^r y^s dx dy$  for  $r \geq 0$  and  $s \geq 0$ .

- (c) Compute  $\iint_{[0,1] \times [0,1]} xy e^{x+y} dx dy$ .

**Sol.**

- (a) We will proceed in this problem by using the definition of a Riemann Sum.  
Let  $P$  be a tagged partition with the tags as  $(\alpha_i, \beta_j)$  where  $\alpha_i \in [x_{i-1}, x_i]$  and  $\beta_j \in [y_{j-1}, y_j]$ .  
We have,

$$\begin{aligned} S(P, f, t) &= \sum_{i=1}^n \sum_{j=1}^n f(x_i, y_j) \Delta_{ij} \\ &= \sum_{i=1}^n \sum_{j=1}^n \phi(\alpha_i) \psi(\beta_j) \Delta_{ij} \\ &= \sum_{i=1}^n \sum_{j=1}^n [\phi(\alpha_i)(x_i - x_{i-1})][\psi(\beta_j)(y_j - y_{j-1})] \\ &= \left( \sum_{i=1}^n \phi(\alpha_i)(x_i - x_{i-1}) \right) \left( \sum_{j=1}^n \psi(\beta_j)(y_j - y_{j-1}) \right) \end{aligned}$$

We know that  $f, \phi$  and  $\psi$  are integrable in their respective domains (Why?). We know that as  $\|P\| \rightarrow 0$ , the respective norms of the partitions along the  $x$  and  $y$  directions go to zero as well.

Thus, the two sums in single variables approach the respective integrals as the double sum approaches the double integral. Hence, we have proved,

$$\iint_R f(x, y) dx dy = \left( \int_a^b \phi(x) dx \right) \left( \int_c^d \psi(y) dy \right)$$

□

- (b) We will use the result we just proved. We have  $\phi(x) = x^r$  and  $\psi(y) = y^s$ . Thus, we can see that

$$\iint_{[1,2] \times [1,2]} x^r y^s dx dy = \frac{(2^{r+1} - 1)(2^{s+1} - 1)}{(r+1)(s+1)}$$

□

- (c) Again, we will use the result from part (a).

We have  $\phi(x) = xe^x$  and  $\psi(y) = ye^y$ . Thus, we can show that

$$\iint_{[0,1] \times [0,1]} xy e^{x+y} dx dy = 1$$

■

7. Evaluate the following integrals

1.  $\iint_{\mathbf{R}} (x + 2y)^2$  where  $\mathbf{R} = [-1, 2] \times [0, 2]$

2.  $\iint_{\mathbf{R}} \left( xy + \frac{x}{y+1} \right) dx dy$  where  $\mathbf{R} = [1, 4] \times [1, 2]$

**Sol.**

(a)  $\mathbf{R} = [-1, 2] \times [0, 2]$ , we need to evaluate

$$\iint_{\mathbf{R}} (x + 2y)^2 dx dy$$

Let it the integral be I

$$\begin{aligned} I &= \iint_{\mathbf{R}} (x^2 + 4y^2 + 4xy) dx dy && \text{(Expanding } (x + 2y)^2) \\ &= \iint_{\mathbf{R}} x^2 dx dy + \iint_{\mathbf{R}} 4y^2 dx dy + \iint_{\mathbf{R}} 4xy dx dy && \text{(Seperating the terms)} \\ &= \int_0^2 \left( \int_{-1}^2 x^2 dx \right) dy + \int_0^2 \left( \int_{-1}^2 dx \right) 4y^2 dy + \int_0^2 \left( \int_{-1}^2 4x dx \right) y dy \\ &= \int_0^2 3 dy + \int_0^2 12y^2 dy + \int_0^2 6 dy \\ &= 6 + 32 + 12 = \boxed{50} \end{aligned}$$

□

(b)  $\mathbf{R} = [1, 4] \times [1, 2]$ , we need to evaluate

$$\iint_{\mathbf{R}} \left( xy + \frac{x}{y+1} \right) dx dy.$$

Now it is easy to observe that

$$\begin{aligned} \iint_{\mathbf{R}} \left( xy + \frac{x}{y+1} \right) dx dy &= \iint_{\mathbf{R}} x \left( y + \frac{1}{y+1} \right) dx dy \\ &= \left( \int_1^4 x dx \right) \times \left( \int_1^2 y + \frac{1}{y+1} dy \right) \\ &= \frac{15}{2} \times \left( \frac{3}{2} + \ln \frac{3}{2} \right) \approx \boxed{14.29} \end{aligned}$$

■

8. Consider the function  $f : [-1, 1] \times [-1, 1]$

$$f(x, y) = \begin{cases} x + y & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the sets of points at which  $f$  is discontinuous. Is  $f$  integrable over  $[-1, 1] \times [-1, 1]$ ?

**Sol.**

We will split the domain into three sets  $E_1, E_2$  and  $E_3$ .

On set  $E_1 = \{(x, y) : x^2 + y^2 < 1\}$ , we have the function  $f(x, y) = x + y$  which is a continuous function. But on the set  $E_2 = \{(x, y) : x^2 + y^2 = 1\}$ , if we approach any point in  $E_2$  from  $E_1$ , we get  $f(x, y) = x + y$ .

Now if we approach from  $E_3 = \mathbb{R} - (E_1 \cup E_2)$ , we get  $f(x, y) = 0$  which is clearly discontinuous except when  $x + y = 0$  on  $E_2$  i.e.  $x = \pm \frac{1}{\sqrt{2}}$  and  $y = \mp \frac{1}{\sqrt{2}}$ , we have the continuity, since the function goes to zero at these two points.

So  $f(x, y)$  is discontinuous on  $E_2 - \left\{ \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}$

Recall the concept of content zero. We apply it here and state that the function is integrable over  $\mathbb{R}$ .

$$\begin{aligned} \iint_{\mathbb{R}} f(x, y) dx dy &= \iint_{E_1} f(x, y) dx dy \\ &= \iint_{E_1} (x + y) dx dy \\ &= \iint_{E_1} x dx dy + \iint_{E_1} y dx dy \\ &= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x dx dy + \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y dx dy \\ &= 0 \end{aligned}$$

■

## TUTORIAL SHEET 2 (MULTIPLE INTEGRALS)

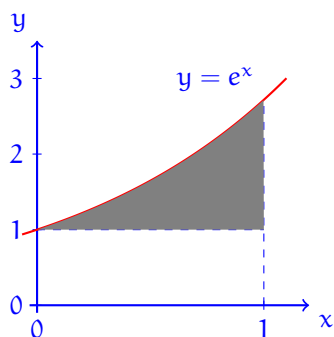
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1. For the following, write an equivalent iterated integral with the order of integration reversed:

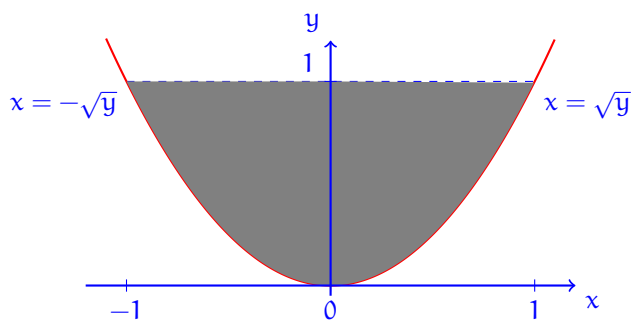
(a)  $\int_0^1 \left[ \int_1^{e^x} dy \right] dx$

(b)  $\int_0^1 \left[ \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx \right] dy$

**Sol.**



(a)



(b)

Figure 2: Regions of integration for question 1

(a) By Fubini's theorem, the iterated integrals are equal, Hence we can reverse the order of integration and obtain an equivalent integral,

$$\int_0^1 \left[ \int_1^{e^x} dy \right] dx = \boxed{\int_1^e \left[ \int_{\ln y}^1 dx \right] dy}.$$

□

(b) The reversed equivalent integral is,

$$\int_0^1 \left[ \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx \right] dy = \boxed{\int_{-1}^1 \left[ \int_{x^2}^1 f(x, y) dy \right] dx}.$$

■

2. Evaluate the following integrals

$$(a) \int_0^\pi \left[ \int_x^\pi \frac{\sin(y)}{y} dy \right] dx$$

$$(b) \int_0^1 \left[ \int_y^1 x^2 e^{xy} dx \right] dy$$

$$(c) \int_0^2 (\tan^{-1}(\pi x) - \tan^{-1}(x)) dx$$

**Sol.**

(a) Note that the integral is difficult to evaluate in the order given. So we change the order.

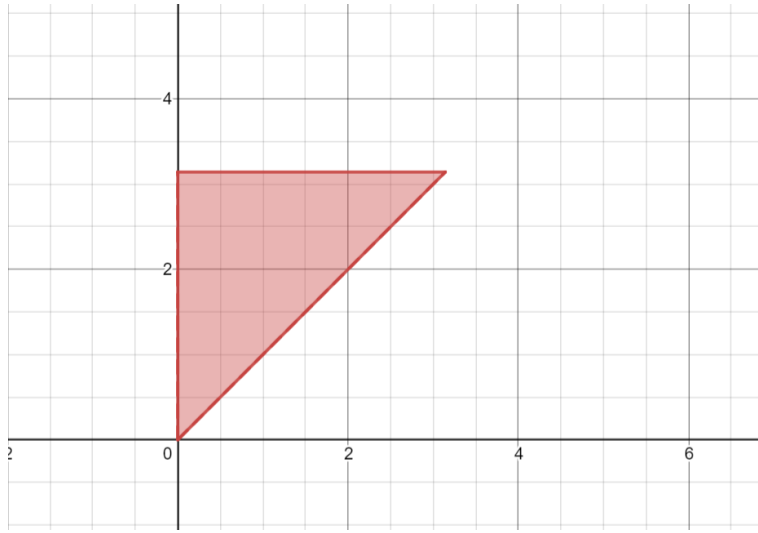


Figure 2(a): Region of integration for question 2(a)

From the figure we get the limits as  $0 \leq x \leq y$  and  $0 \leq y \leq \pi$ . Thus,

$$\begin{aligned} I &= \int_0^\pi \left( \int_0^y \frac{\sin(y)}{y} dx \right) dy \\ &= \int_0^\pi \left[ \frac{\sin(y)}{y} (x) \right]_{x=0}^{x=y} dy \\ &= \int_0^\pi \sin(y) dy \\ &= \boxed{2} \end{aligned}$$

□

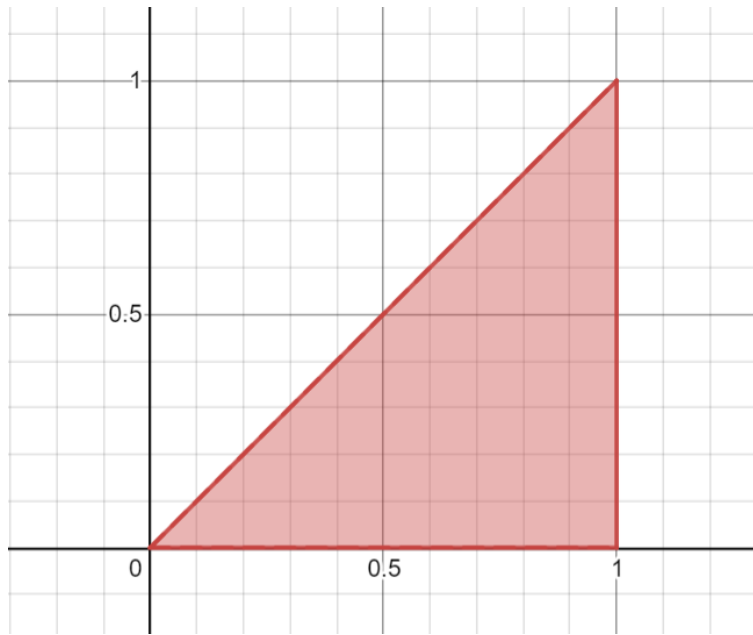


Figure 2(b): Region of integration for question 2(b)

- (b) We again reverse the order of integration. From figure 2(b) we can see the limits can also be written as  $0 \leq y \leq x$  and  $0 \leq x \leq 1$ . Thus,

$$\begin{aligned}
 I &= \int_0^1 \left( \int_0^x x^2 e^{xy} dy \right) dx \\
 &= \int_0^1 (xe^{x^2} - x) dx \\
 &= \frac{e^{x^2} - x^2}{2} \Big|_0^1 \\
 &= \boxed{\frac{e-2}{2}}
 \end{aligned}$$

□



(c) <sup>1</sup> We are to compute the following integral:

$$\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) dx.$$

This is a single variable Riemann integral, which is not easy to compute directly. We convert it into a double integral and compute the double integral using Fubini, hoping that that'll be easier. Now, note that:

$$\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) dx = \int_0^2 \left( \int_x^{\pi x} \frac{1}{1+y^2} dy \right) dx$$

This a double integral over the region

$$D := \{(x, y) : 0 \leq x \leq 2, x \leq y \leq \pi x\}.$$

Now, note that this region  $D$  is the union of two regions  $D_1$  and  $D_2$  where

$$D_1 := \left\{ (x, y) : 0 \leq y \leq 2, \frac{y}{\pi} \leq x \leq y \right\}, \text{ and}$$

$$D_2 := \left\{ (x, y) : 2 \leq y \leq 2\pi, \frac{y}{\pi} \leq x \leq 2 \right\}.$$

Note that  $D_1 \cap D_2$  is of (two-dimensional) content zero.

Therefore, we get that

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f,$$

Where  $f : D \rightarrow \mathbb{R}$  is defined as  $f(x, y) := (1 + y^2)^{-1}$ .

Thus the required integral is given by

$$\begin{aligned} I &= \int_0^2 \left( \int_{y/\pi}^y \frac{1}{1+y^2} dx \right) dy + \int_2^{2\pi} \left( \int_{y/\pi}^2 \frac{1}{1+y^2} dx \right) dy \\ &= \left( 1 - \frac{1}{\pi} \right) \int_0^2 \frac{y}{1+y^2} dy + \int_2^{2\pi} \frac{2}{1+y^2} dy - \frac{1}{\pi} \frac{2}{2\pi} \frac{y}{1+y^2} dy \\ &= \frac{1}{2} \left( 1 - \frac{1}{\pi} \right) [\ln(1+y^2)]_0^2 + 2 [\tan^{-1} y]_2^{2\pi} - \frac{1}{2\pi} [\ln(1+y^2)]_2^{2\pi} \\ &= \boxed{\frac{\ln 5}{2} \left( 1 - \frac{1}{\pi} \right) + 2 [\tan^{-1} 2\pi - \tan^{-1} 2] - \frac{1}{2\pi} \ln \frac{1+4\pi^2}{5}} \end{aligned}$$

■

---

<sup>1</sup>thanks to integral daddy

3. Find  $\iint_D f(x, y) d(x, y)$  where  $f(x, y) = e^{x^2}$  and  $D$  is the region bounded by the lines  $y = 0$ ,  $x = 1$  and  $y = 2x$

**Sol.**

We can solve this by looking at one particular iterated integral. We can do this because the function is well behaved (continuous) and both the iterated integrals exist.

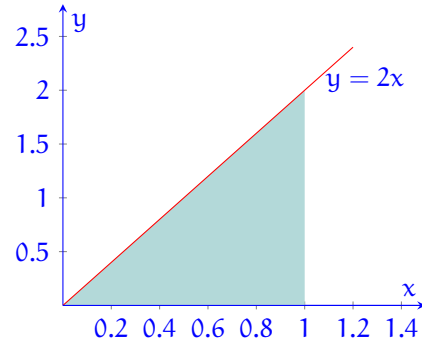


Figure 1: Region of integration for question 3

Note that the integral,

$$\int_{\alpha}^1 e^{x^2} dx$$

( $0 \leq \alpha \leq 1$ ) does not exist in closed form in terms of  $\alpha$ .

We therefore cannot evaluate the integral,

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} dx dy$$

Thus, our only option is to evaluate the other iterated integral. We have:

$$\iint_D e^{x^2} d(x, y) = \int_0^1 \int_0^{2x} e^{x^2} dy dx = \int_0^1 2x e^{x^2} dx = \int_0^1 e^u du = e - 1$$

Hence, we have obtained,

$$\iint_D e^{x^2} d(x, y) = \boxed{e - 1}$$

■

4.

- (a) Compute the volume of the solid enclosed by the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

where  $a, b, c$  are real numbers.

- (b) Find the volume of the region under the graph of  $f(x, y) = e^{x+y}$  over the region

$$D = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}.$$

**Sol.**

- (a) We defined  $D$ ,

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

We can compute this volume easily by changing the variables from  $(x, y, z)$  to  $(u, v, w)$ . We will scale the axes to make the region a sphere.

$$x = au$$

$$y = bv$$

$$z = cw$$

The Jacobian  $\mathbb{J}$  for the above transformation is,

$$\mathbb{J} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Note that  $\|\mathbb{J}\| = abc$ .

Thus,

$$\iiint_D dz dy dx = abc \iiint_{D'} dw dv du$$

where  $D'$  is the new volume we have to integrate over. Notice that,

$$D' = \left\{ (u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 \leq 1 \right\}$$

This is simply a unit sphere. Therefore,

$$\iiint_{D'} dw dv du = \frac{4\pi}{3}$$

Thus,

$$\iiint_D dx dy dz = \iiint_{D'} abc \, du dv dw = \boxed{\frac{4\pi abc}{3}}$$

□

- (b) Observe that the volume will be

$$\iint_D e^{x+y} dx dy$$

where the region  $D$  is given by (notice that the region has been written out differently. Justify),

$$\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x + y \leq 1, -1 \leq y - x \leq 1\}$$

Use the change of variables,

$$\begin{aligned}x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2}\end{aligned}$$

The Jacobian  $\mathbb{J}$  corresponding to this transformation will be,

$$\begin{aligned}\mathbb{J} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ \Rightarrow |\mathbb{J}| &= \frac{1}{2}\end{aligned}$$

Therefore, we can now write,

$$\begin{aligned}\iint_D e^{x+y} dx dy &= \int_{-1}^1 \int_{-1}^1 \frac{1}{2} e^u du dv \\ &= \boxed{e - \frac{1}{e}}\end{aligned}$$

■

5. Evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y) d(x,y)$$

where  $D$  is the parallelogram with vertices at  $(\pi, 0)$ ,  $(2\pi, \pi)$ ,  $(\pi, 2\pi)$  and  $(0, \pi)$ .

**Sol.**

The region  $D$  is given by,

$$\{(x, y) \in \mathbb{R}^2 \mid \pi \leq x+y \leq 3\pi \text{ and } -\pi \leq y-x \leq \pi\}$$

Use the change of variables,

$$\begin{aligned}x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2} \\ \Rightarrow \iint_D (x-y)^2 \sin^2(x+y) d(x,y) &= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} \frac{1}{2} v^2 \sin^2 u du dv \\ &= \frac{1}{2} \left( \int_{-\pi}^{\pi} v^2 dv \right) \left( \int_{\pi}^{3\pi} \sin^2(u) du \right) \\ &= \frac{\pi^3}{3} \cdot \pi \\ &= \boxed{\frac{\pi^4}{3}}\end{aligned}$$

■

6. Let  $D$  be the region in the first quadrant of the  $xy$ -plane bounded by the hyperbolas  $xy = 1$ ,  $xy = 9$  and the lines  $y = x$ ,  $y = 4x$ . Find  $\iint_D dx dy$  by transforming it to  $\iint_{D^*} du dv$ , where  $x = \frac{u}{v}$ ,  $y = uv$ ,  $v > 0$ .

**Sol.**

We have,

$$\begin{aligned}x &= \frac{u}{v} \\ y &= uv\end{aligned}$$

Thus, note that,

$$\begin{aligned}u &= \sqrt{xy} \\ v &= \sqrt{\frac{y}{x}}\end{aligned}$$

Define,

$$\begin{aligned}h_1(u, v) &:= \frac{u}{v} \\ h_2(u, v) &:= uv\end{aligned}$$

Then we have the Jacobian,

$$\begin{aligned}\mathbb{J}(u, v) &:= \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix} \\ \implies |\mathbb{J}| &= 2\frac{u}{v}\end{aligned}$$

We define  $D^*$  in the following manner:

$$D^* = \{(u, v) \mid u \in [1, 3], v \in [1, 2]\}$$

Notice that  $D^*$  corresponds to the region  $D$  given in the question, but in the transformed coordinates. We can call  $D^*$  the transformed region of integration.

Let,

$$\begin{aligned}f^* : D^* &\rightarrow \mathbb{R} \\ f^*(u, v) &:= 2\frac{u}{v}\end{aligned}$$

We are now tasked to compute the integral,

$$\iint_{D^*} f^*(u, v) du dv$$

Note that  $f^*$  is of the form  $\phi(u)\psi(v)$ , where  $\phi$  and  $\psi$  are continuous, and hence integrable functions on their respective domains. Moreover  $f^*$  is continuous on  $D^*$  and hence integrable on  $D^*$ . Recall **Tutorial One, question 6(a)**. Thus,

$$\begin{aligned}\iint_{D^*} f^*(u, v) du dv &= \left( \int_1^2 \frac{1}{v} dv \right) \left( \int_1^3 2u du \right) \\ &= \boxed{8 \log 2}\end{aligned}$$

A picture of  $D$  is given below. Now contrast this with  $D^*$ .

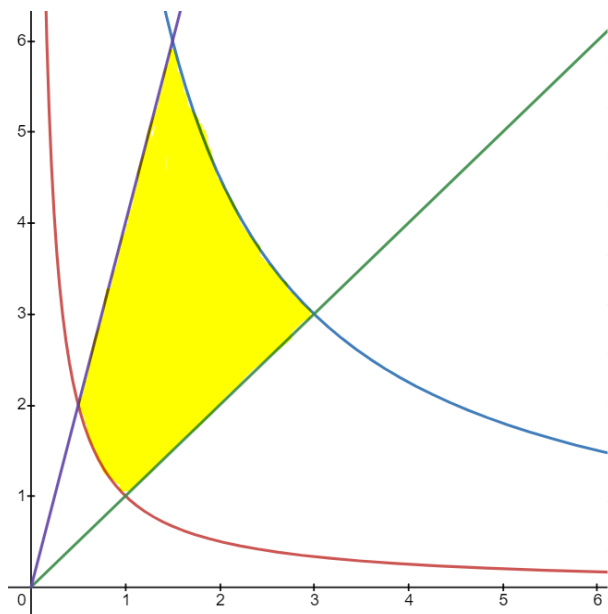


Figure 1: Region of integration  $D$

7. Find

$$\lim_{r \rightarrow \infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x, y)$$

where  $D(r)$  equals:

- (a)  $\{(x, y) : x^2 + y^2 \leq r^2\}$
- (b)  $\{(x, y) : x^2 + y^2 \leq r^2, x \geq 0, y \geq 0\}$
- (c)  $\{(x, y) : |x| \leq r, |y| \leq r\}$
- (d)  $\{(x, y) : 0 \leq x \leq r, 0 \leq y \leq r\}$

**Sol.**

- (a) As should be obvious from the form of  $D(r)$ , we will utilize polar coordinates here. On converting the integral from the Cartesian to the polar form, we get,

$$\begin{aligned} \lim_{r \rightarrow \infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x, y) &= \lim_{r \rightarrow \infty} \int_0^{2\pi} \int_0^r e^{-R^2} R dR d\theta \\ &= -\pi \lim_{r \rightarrow \infty} \int_0^r e^{-R^2} d(-R^2) \\ &= \pi \lim_{r \rightarrow \infty} (1 - e^{-r^2}) \\ &= \boxed{\pi} \end{aligned}$$

- (b) Again, we utilize the polar coordinates here. The integral will be the same as before except for the difference in the limits of  $\theta$  - since  $x \geq 0, y \geq 0$ ,  $\theta$  will vary from 0 to  $\frac{\pi}{2}$ ,

$$\begin{aligned}\lim_{r \rightarrow \infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x, y) &= \lim_{r \rightarrow \infty} \int_0^{\frac{\pi}{2}} \int_0^r e^{-R^2} R dR d\theta \\ &= \boxed{\frac{\pi}{4}}\end{aligned}$$

□

- (c) Consider the following integrals:

$$\begin{aligned}I_1(r) &= \int_0^{2\pi} \int_0^{r\sqrt{2}} e^{-R^2} R dR d\theta \\ I_2(r) &= \int_0^{2\pi} \int_0^r e^{-R^2} R dR d\theta\end{aligned}$$

We can now write (why?),

$$I_1(r) > \iint_{D(r)} e^{-(x^2+y^2)} d(x, y) > I_2(r)$$

We state the sandwich theorem below,

Let  $I$  be an interval having the point  $a$  as a limit point. Let  $g, f$  and  $h$  be functions defined on  $I$ , except possibly at  $a$  itself. Suppose that for every  $x$  in  $I$  not equal to  $a$ , we have,

$$g(x) \leq f(x) \leq h(x)$$

and also suppose that,

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$$

Then  $\lim_{x \rightarrow a} f(x) = L$ .

It is clear that  $\lim_{r \rightarrow \infty} I_1(r) = \lim_{r \rightarrow \infty} I_2(r) = \pi$ . If we now make use of the sandwich theorem, we can write,

$$\lim_{r \rightarrow \infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x, y) = \boxed{\pi}$$

□

- (d)  $\lim_{r \rightarrow \infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x, y) = \boxed{\frac{\pi}{4}}$  (Justify).

■

8. Find the volume common to the cylinders  $x^2 + y^2 = a^2$  and  $x^2 + z^2 = a^2$  using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

**Sol.**

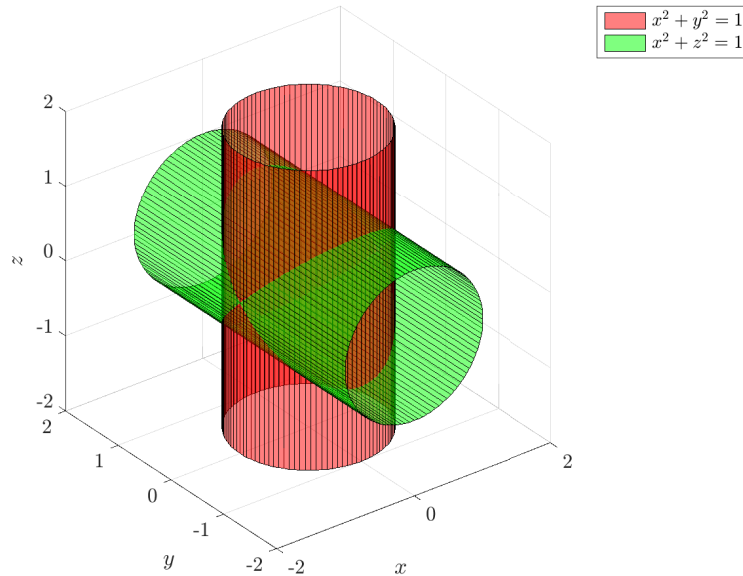


Figure 1: Region of integration for volume in question 8 ( $a = 1$  taken for representation)

Notice that the volume of consideration is the same in each of the octants. We will calculate the integral over the first octant where  $x, y, z \geq 0$  and thereby get the total volume.

The region of integration in the  $x-y$  plane is given by a circular disc centered at the origin and having radius  $a$ . The 'height' (along the  $z$ -axis) of the volume (in the first octant) at a point  $(x, y)$  is  $z = \sqrt{a^2 - x^2}$ . Thus we may write the integral as,

$$V = \int_0^a \left( \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2} \, dy \right) dx = \int_0^a (a^2 - x^2) \, dx = a^2 \cdot a - \frac{a^3}{3} = \frac{2}{3}a^3.$$

Hence we the the total common volume (over all octants) will be,  $= 8 \cdot \frac{2}{3}a^3 = \boxed{\frac{16}{3}a^3}$ . ■



9. Find the volume of the solid that lies under the paraboloid  $z = x^2 + y^2$  above the region  $x^2 + y^2 = 2x$  in the  $x$ - $y$  plane.

**Sol.**

Let the volume be  $V$ . We can write,

$$V = \iint_D z \, dy \, dx = \iint_D (x^2 + y^2) \, dy \, dx$$

where  $D = \{(x, y) : (x - 1)^2 + y^2 \leq 1, (x, y) \in \mathbb{R}^2\}$

Make the substitutions  $x = 1 + r \cos \theta$ ,  $y = r \sin \theta$ . This gives,

$$V = \iint_{D'} (r + r^3 + 2r^2 \cos \theta) \, dr \, d\theta$$

where  $D' = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ .

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (r + r^3 + 2r^2 \cos \theta) \, dr \, d\theta \\ &= \int_0^{2\pi} \left( \frac{3}{4} + \frac{2 \cos \theta}{3} \right) \, d\theta = \boxed{\frac{3\pi}{2}} \end{aligned}$$

10. Express the solid  $D = \{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$  as

$$\{(x, y, z) \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x), \quad \xi_1(x, y) \leq z \leq \xi_2(x, y)\}.$$

**Sol.**

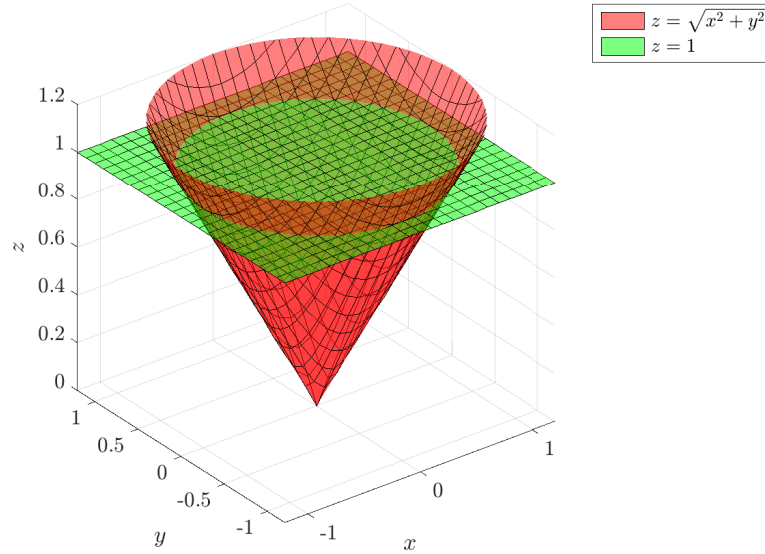


Figure 1: Solid D represented using bounding surfaces

Since  $\sqrt{x^2 + y^2} \leq z \leq 1$ , we have that  $\sqrt{x^2 + y^2} \leq 1 \implies |y| \leq \sqrt{1 - x^2}$ , and hence  $x^2 \leq 1$ , which gives us  $|x| \leq 1$ . Thus we may write  $D$  in the desired form as

$$\boxed{\{(x, y, z) \mid -1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \sqrt{x^2 + y^2} \leq z \leq 1\}}.$$

11. Evaluate

$$I = \int_0^{\sqrt{2}} \left( \int_0^{\sqrt{2-x^2}} \left( \int_{x^2+y^2}^2 x \, dz \right) dy \right) dx$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as  $dx dy dz$ .

**Sol.**

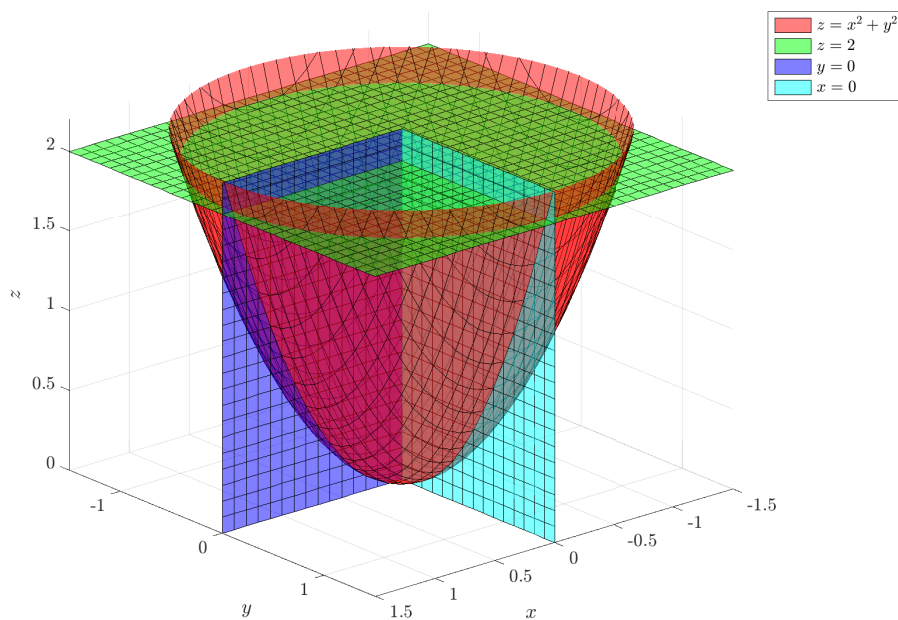


Figure 1: Region of integration for volume in question 11

$$\begin{aligned} I &= \int_0^{\sqrt{2}} \left[ \int_0^{\sqrt{2-x^2}} \left( \int_{x^2+y^2}^2 x \, dz \right) dy \right] dx \\ &= \int_0^2 \left[ \int_0^{\sqrt{z}} \left( \int_0^{\sqrt{z-y^2}} x \, dx \right) dy \right] dz \\ &= \int_0^2 \left[ \int_0^{\sqrt{z}} \frac{z-y^2}{2} dy \right] dz \\ &= \int_0^2 \left( \frac{z\sqrt{z}}{2} - \frac{1}{2} \cdot \frac{\sqrt{z}^3}{3} \right) dz \\ &= \int_0^2 \left( \frac{z\sqrt{z}}{2} - \frac{z\sqrt{z}}{6} \right) dz \\ &= \int_0^2 \frac{z^{\frac{3}{2}}}{3} dz \\ &= \boxed{\frac{8\sqrt{2}}{15}}. \end{aligned}$$

■

12. Using suitable change of variables, evaluate the following:

$$(a) \ I = \iiint_D (z^2 x^2 + z^2 y^2) dx dy dz$$

where  $D$  is the cylindrical region  $x^2 + y^2 \leq 1$  bounded by  $-1 \leq z \leq 1$ .

$$(b) \ I = \iiint_D \exp(x^2 + y^2 + z^2)^{\frac{3}{2}} dx dy dz$$

over the region enclosed by the unit sphere in  $\mathbb{R}^3$ .

**Sol.**

(a) Perform the substitution,

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

As seen in class, the determinant of the Jacobian is  $r$ . Thus,

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 \int_{-1}^1 z^2 r^2 \times r dz dr d\theta \\ &= 2\pi \times \frac{1}{4} \times \frac{2}{3} \\ &= \boxed{\frac{\pi}{3}} \end{aligned}$$

□

(b) Perform the substitution,

$$x = r \cos \theta \sin \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \phi$$

As seen in class, the determinant of the Jacobian is  $r^2 |\sin \phi|$ . Thus, we have,

$$\begin{aligned} I &= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{r^3} \times r^2 |\sin \phi| dr d\theta d\phi \\ &= 2 \times 2\pi \times \frac{1}{3} (e - 1) \\ &= \boxed{\frac{4\pi}{3} (e - 1)} \end{aligned}$$

■

## TUTORIAL SHEET 3 (I. MULTIPLE INTEGRALS AND CHANGE OF VARIABLES)

---

1. Find the volume that lies under the paraboloid  $z = x^2 + y^2$ , above the  $xy$  plane, and inside the cylinder  $x^2 + y^2 = 2x$ .

**Sol.**

Though the question is worded differently, the region is exactly the same as that of Question 9 in Tutorial sheet 2, hence the same solution follows. ■

2. Using a suitable change of variables, evaluate the integral  $\iint_D y \, dy \, dx$ , where  $D$  is the region bounded by the  $x$ -axis and the parabolas  $y^2 = 4 - 4x$  and  $y^2 = 4 + 4x, y \geq 0$ .

**Sol.**

We can use the following variables:

$$\begin{aligned} x &= 1 + u \\ y &= \sqrt{v} \end{aligned}$$

Since  $D$  is the region given by,

$$D = \left\{ (x, y) \mid \frac{y^2 - 4}{4} \leq x \leq \frac{4 - y^2}{4}, y \geq 0 \right\}$$

On changing variables to  $u$  and  $v$ , the  $D$  is transformed to a new region  $D'$ ,

$$D' = \left\{ (u, v) \mid 0 \leq v \leq 4, \frac{v}{4} - 2 \leq u \leq \frac{-v}{4} \right\}$$

The Jacobian  $\mathbb{J}$  is easy enough to find.  $|\mathbb{J}| = \frac{1}{2\sqrt{v}}$  (Verify).

Thus the integral  $\iint_D y \, dy \, dx$  can be computed as follows:

$$\begin{aligned} \iint_D y \, dy \, dx &= \iint_{D'} \sqrt{v} \times \frac{1}{2\sqrt{v}} \, du \, dv \\ &= \int_0^4 \int_{\frac{v}{4}-2}^{\frac{-v}{4}} \frac{1}{2} \, du \, dv \\ &= \int_0^4 \frac{1}{2} \left( 2 - \frac{v}{2} \right) \, dv \\ &= \boxed{2} \end{aligned}$$

■

**3.** Use spherical coordinates to find the volume of the solid that lies above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$ .

**Sol.**

Since we are supposed to use spherical coordinates, we can write the change of coordinates as,

$$x = r \cos \phi \sin \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \theta$$

We define a region  $D$  as the following:

$$D := \left\{ (x, y, z) \mid x^2 + y^2 \leq \frac{1}{2}, \sqrt{x^2 + y^2} \leq z \leq \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2} \right\}$$

It should be clear that  $D$  is the region of which the volume we have to compute. We now represent  $D$  in spherical coordinates and call this transformed region  $D'$ .

$$D' := \left\{ (r, \phi, \theta) \mid 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq \phi \leq 2\pi, 0 \leq r \leq \cos \theta \right\}$$

So the volume to be computed is simply,

$$\begin{aligned} V &= \iiint_{D'} r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos \theta} r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= 2\pi \int_0^{\frac{\pi}{4}} \sin \theta \left( \frac{\cos^3 \theta}{3} \right) d\theta \\ &= -\frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \cos^3 \theta \, d(\cos \theta) \\ &= \boxed{\frac{\pi}{8}} \end{aligned}$$

■

4. Use cylindrical coordinates to evaluate  $\iiint_W (x^2 + y^2) dz dy dx$  where

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}, \sqrt{x^2+y^2} \leq z \leq 2\}.$$

**Sol.**

We perform the following substitution to get the cylindrical coordinates,

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta \\z &= z\end{aligned}$$

Observe that the region  $W$  can be written as,

$$W = \left\{ (r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 \mid 0 \leq \theta < 2\pi, 0 \leq r \leq 2, r \leq z \leq 2 \right\}$$

The given integral therefore becomes,

$$\begin{aligned}\int_0^{2\pi} \int_0^2 \int_r^2 r^3 dz dr d\theta &= \int_0^{2\pi} \int_0^2 r^3 (2-r) dr d\theta \\&= (2\pi) \times \left(8 - \frac{32}{5}\right) \\&= \boxed{\frac{16\pi}{5}}\end{aligned}$$

■

5. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin \phi d\rho d\phi d\theta,$$

and evaluate the integral.

**Sol.**

The limits of integral describe the thick spherical shell  $S$  shown in Fig. 1 such that,

$$S := \left\{ (\rho, \theta, \phi) \mid 1 \leq \rho \leq 2, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2} \right\}$$

The integration is trivial,

$$\begin{aligned}\iiint_S \rho^2 \sin \phi d\rho d\phi d\theta &= \left( \int_0^{\pi/2} \sin \phi d\phi \right) \cdot \left( \int_0^{\pi/2} d\theta \right) \cdot \left( \int_1^2 \rho^2 d\rho \right) \\&= 1 \times \frac{\pi}{2} \times \frac{7}{3} \\&= \boxed{\frac{7\pi}{6}}\end{aligned}$$

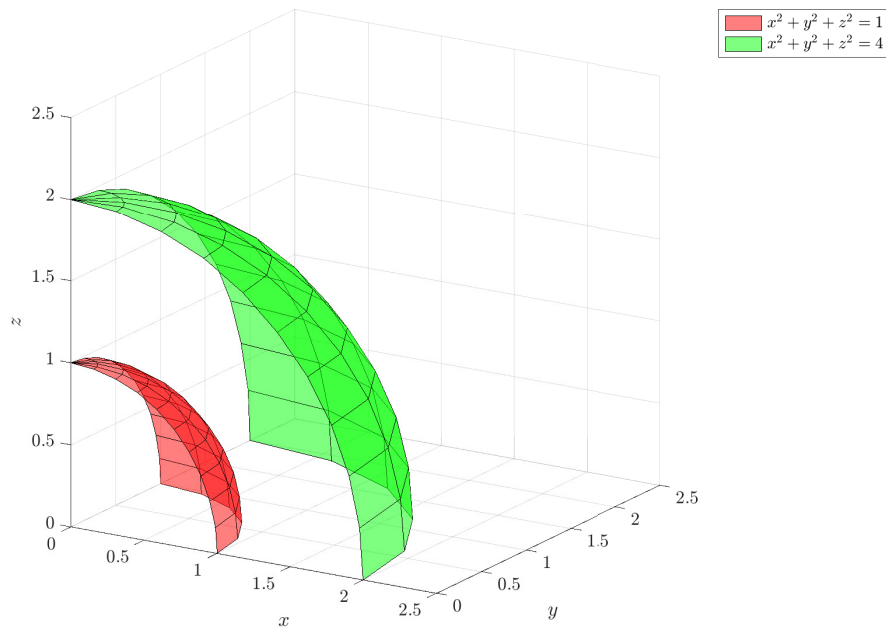


Figure 1: The region between the two spherical boundaries is the volume for question 5

6. Find  $\iiint_F \frac{1}{(x^2 + y^2 + z^2)^{\frac{n}{2}}} dV$  where F is the region bounded by the spheres with the center the origin and radii  $r$  and  $R$  with  $0 < r < R$

**Sol.**

We will change the coordinates from Cartesian to spherical:

$$\begin{aligned}x &= r_1 \sin \theta \cos \varphi \\y &= r_1 \sin \theta \sin \varphi \\z &= r_1 \cos \theta\end{aligned}$$

where  $\theta \in [0, \pi)$ ,  $\varphi \in [0, 2\pi)$  and we are given  $r_1 \in (r, R)$ .

So the integral after transformation is,

$$\begin{aligned}\iiint_F \frac{1}{(x^2 + y^2 + z^2)^{\frac{n}{2}}} dV &= \int_0^\pi \int_0^{2\pi} \int_r^R \frac{1}{r_1^n} r_1^2 \sin \theta dr_1 d\theta d\varphi \\&= \left( \int_0^{2\pi} d\varphi \right) \cdot \left( \int_0^\pi \sin \theta d\theta \right) \cdot \left( \int_r^R r_1^{2-n} dr_1 \right) \\&= 4\pi \varphi(n)\end{aligned}$$

where,

$$\varphi(n) = \begin{cases} \ln\left(\frac{R}{r}\right) & n = 3 \\ \frac{1}{3-n} (R^{3-n} - r^{3-n}) & \text{otherwise} \end{cases}$$

## TUTORIAL SHEET 3 (II. VECTOR ANALYSIS AND LINE INTEGRALS)

---

1. Let  $f, g$  be differentiable functions on  $\mathbb{R}^2$ . Show that

- i.  $\nabla(fg) = f\nabla g + g\nabla f$ ;
- ii.  $\nabla f^n = n f^{n-1} \nabla f$ ;
- iii.  $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$  whenever  $g \neq 0$ .

**Sol.**

- i. We use the definition of the gradient to rewrite the expressions in terms of partial derivatives, and then use the previously established product rule of the partial derivative to prove these identities.

$$\begin{aligned}
 \nabla(fg) &= \frac{\partial(fg)}{\partial x} \mathbf{i} + \frac{\partial(fg)}{\partial y} \mathbf{j} \\
 &= \left( f \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x} g \right) \mathbf{i} + \left( f \frac{\partial g}{\partial y} + \frac{\partial f}{\partial y} g \right) \mathbf{j} && (\because \text{product rule for partial derivatives}) \\
 &= f \left( \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} \right) + g \left( \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \right) \\
 &= f\nabla g + g\nabla f
 \end{aligned}$$

□

- ii. We use the result obtained in part i. above, and the principle of induction. Let  $P(n)$  denote the statement  $\nabla f^n = n f^{n-1} \nabla f$ . Observe that  $P(1)$  is  $\nabla f = \nabla f$ , which is trivially true. Now suppose  $P(n)$  is true for some  $n \geq 1$ , i.e.  $\nabla f^n = n f^{n-1} \nabla f$  holds. Then we have,

$$\begin{aligned}
 \nabla f^{n+1} &= \nabla(f^n f) = f\nabla f^n + f^n \nabla f && (\because \text{i.}) \\
 &= n f^{n-1} \nabla f + f^n \nabla f && (\because P(n) \text{ is true}) \\
 &= (n+1) f^n \nabla f,
 \end{aligned}$$

which is the same as  $P(n+1)$ . Hence we have that  $P(n+1)$  is true whenever  $P(n)$  is, for all  $n \geq 1$ , and that  $P(1)$  is trivially true, which completes the proof for all positive integers  $n$ .

By the result in iii. (which will be proved in the next part), we may similarly show that  $P(n-1)$  is true whenever  $P(n)$  is, for any  $n \leq 1$ ; hence by the principle of induction we have that  $P(n)$  holds for non-positive integers as well. This completes the proof for all integers  $n$ . □

- iii. Taking  $h = \frac{1}{g}$  (which is possible when  $g \neq 0$ ), we have

$$\nabla h = \frac{\partial h}{\partial x} \mathbf{i} + \frac{\partial h}{\partial y} \mathbf{j} = -\frac{1}{g^2} \frac{\partial g}{\partial x} \mathbf{i} - \frac{1}{g^2} \frac{\partial g}{\partial y} \mathbf{j} = -\frac{1}{g^2} \nabla g.$$

We now apply i. to the product  $fh$ , to obtain

$$\nabla \left( \frac{f}{g} \right) = \nabla(fh) = f\nabla h + h\nabla f = -f \frac{1}{g^2} \nabla g + \frac{1}{g} \nabla f = \frac{g\nabla f - f\nabla g}{g^2}$$

■



2. Let  $\mathbf{a}, \mathbf{b}$  be two fixed vectors,  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $r^2 = x^2 + y^2 + z^2$ . Prove the following:

1.  $\nabla(r^n) = nr^{n-2}\mathbf{r}$  for any integer  $n$
2.  $\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}$
3.  $\mathbf{b} \cdot \nabla\left(\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right)\right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$

**Sol.**

1. Let  $n \in \mathbb{Z}$  be given. Define  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  as

$$f(x, y, z) := (x^2 + y^2 + z^2)^{\frac{n}{2}}$$

Thus,

$$\frac{\partial f}{\partial x} = nx(x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

Similar will be the case for the other partial derivatives. Therefore,

$$\begin{aligned}\nabla r^n &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= nr^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= nr^{n-2}\mathbf{r}\end{aligned}$$

□

2. Let  $n = -1$ . By the previously proved part, we have,

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= -\frac{\mathbf{r}}{r^3} \\ \Rightarrow \mathbf{a} \cdot \nabla\left(\frac{1}{r}\right) &= -\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\end{aligned}$$

□

3. Let  $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ . Using the result obtained in part 2.,

$$\begin{aligned}\mathbf{b} \cdot \nabla\left(\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right)\right) &= \mathbf{b} \cdot \nabla\left(-\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right) \\ &= \mathbf{b} \cdot \nabla\left(-\frac{a_x x + a_y y + a_z z}{r^3}\right) \\ &= \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}\end{aligned}$$

■

3. Calculate the line integral of the vector field

$$\mathbf{F}(x, y) = (x^2 - 2xy) \mathbf{i} + (y^2 - 2xy) \mathbf{j}$$

from  $(-1, 1)$  to  $(1, 1)$  along  $y = x^2$ .

**Sol.**

We can represent the curve  $y = x^2$  (let's call it  $C$ ) in parametric form as,

$$\begin{aligned} x &= \tau \\ y &= \tau^2 \\ \mathbf{c}(\tau) &= (x, y) = (\tau, \tau^2) \end{aligned}$$

where  $\tau$  is the parameter, varying from  $-1$  to  $1$ .

The tangent  $\mathbf{c}'(\tau)$  at every point  $(\tau, \tau^2)$  on  $C$  will be,

$$\mathbf{c}'(\tau) = \mathbf{i} + 2\tau\mathbf{j}$$

Therefore, we can write,

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s} &= \int \mathbf{F}(\mathbf{c}(\tau)) \cdot \mathbf{c}'(\tau) d\tau \\ &= \int_{-1}^1 (\tau^2 - 2\tau^3 + 2\tau^5 - 4\tau^4) d\tau \\ &= \boxed{-\frac{14}{15}} \end{aligned}$$

■

4. Calculate the line integral of

$$\mathbf{F}(x, y) = (x^2 + y^2) \mathbf{i} + (x - y) \mathbf{j}$$

once around the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  in the counter clockwise direction.

**Remark** Often line integral of a vector field  $\mathbf{F}$  along a 'geometric curve'  $C$  is represented by  $\int_C \mathbf{F} \cdot d\mathbf{s}$ . A geometric curve  $C$  is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.

To evaluate  $\int_C \mathbf{F} \cdot d\mathbf{s}$ , choose a convenient parametrization  $\mathbf{c}$  of  $C$  traversing  $C$  in the given direction and then

$$\int_C \mathbf{F} \cdot d\mathbf{s} := \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

' $\oint_C$ ' means the line integral over a closed curve  $C$ .

**Sol.**

Let's call the ellipse  $E$ . Every point on the ellipse can be parametrized in the following manner:

$$\begin{aligned} x(\theta) &= a \cos \theta \\ y(\theta) &= b \sin \theta \\ \mathbf{c}(\theta) &= (x, y) = (a \cos \theta, b \sin \theta) \end{aligned}$$

where  $\theta \in [0, 2\pi)$ .

We can now calculate the line integral as,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \int \mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) d\theta \\ &= \int_0^{2\pi} [(a^2 \cos^2 \theta + b^2 \sin^2 \theta) \mathbf{i} + (a \cos \theta - b \sin \theta) \mathbf{j}] \cdot (-a \sin \theta \mathbf{i} + b \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} (-a^3 \cos^2 \theta \sin \theta - ab^2 \sin^3 \theta + ab \cos^2 \theta - b^2 \sin \theta \cos \theta) d\theta \\ &= \boxed{\pi ab} \end{aligned}$$

■

5. Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}.$$

where  $C$  is the curve  $x^2 + y^2 = a^2$  traversed once in the counter clockwise direction.

**Sol.**

The curve  $C$  can be parametrised as  $\mathbf{c}(t) := (a \cos t, a \sin t), t \in [0, 2\pi]$ . Define  $\mathbf{F}(x, y) := \left( \frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2} \right)$ .

The integral can then be written as,

$$\oint_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

which is equal to,

$$\begin{aligned} \oint_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt &= \int_0^{2\pi} F(a \cos t, a \sin t) \cdot (-a \sin t, a \cos t) dt \\ &= \int_0^{2\pi} -(\cos t + \sin t) \sin t + \cos t (\sin t - \cos t) dt \\ &= - \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt \\ &= \boxed{-2\pi} \end{aligned}$$

■

6. Calculate

$$\oint_C y dx + z dy + x dz$$

where  $C$  is the intersection of two surfaces  $z = xy$  and  $x^2 + y^2 = 1$  traversed once in a direction that appears counter clockwise when viewed from high above the  $xy$ -plane.

**Sol.**

Verify that the given curve can be parameterized as follows:

$$\mathbf{c}(t) := (x(t), y(t), z(t)) = (\cos t, \sin t, \cos t \sin t), t \in [-\pi, \pi]$$

It can be seen that this respects the direction given.

Also,  $(x'(t), y'(t), z'(t)) = (-\sin t, \cos t, \cos 2t)$ . Hence, we can now evaluate our integral as follows:

$$\begin{aligned} \oint_C y dx + z dy + x dz &= \int_{-\pi}^{\pi} [\sin t(-\sin t) + \cos t \sin t(\cos t) + \cos t(\cos 2t)] dt \\ &= - \int_{-\pi}^{\pi} \sin^2 t dt \\ &= \boxed{-\pi} \end{aligned}$$

7. Let the curve  $C$  be given by  $x^2 + y^2 = 1, z = 0$ . Let  $\mathbf{c}_1$  be a parametrization defined by  $\mathbf{c}_1(t) = (\cos t, \sin t)$  for  $t \in [0, 2\pi]$ . Find the line integral of  $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$  along this curve. Also find the line integral along the curve parametrized by  $\mathbf{c}_2(t) = (\cos t, -\sin t)$ , for  $t \in [0, \pi]$ .

**Sol.**

For the parametrization  $\mathbf{c}_1$ , we have,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{s} &= \int \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}'_1(t) dt \\ &= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt \\ &= \int_0^{2\pi} dt \\ &= \boxed{2\pi}\end{aligned}$$

For the parametrization  $\mathbf{c}_2$ , we have,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{s} &= \int \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}'_2(t) dt \\ &= \int_0^{\pi} (\sin t, \cos t) \cdot (-\sin t, -\cos t) dt \\ &= -\int_0^{\pi} (\sin^2 t + \cos^2 t) dt \\ &= -\int_0^{\pi} dt \\ &= \boxed{-\pi}\end{aligned}$$

8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle:  $x^2 + y^2 = 1$ . Is this also true for a force field  $\mathbf{F}(x, y, z) = \alpha(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ , for some constant  $\alpha$ .

**Sol.**

Consider the following constant force field,

$$\mathbf{F}' = F_x\mathbf{i} + F_y\mathbf{j}$$

where  $F_x, F_y$  are constants.

We can parametrize the path along the circle easily by substituting,

$$\begin{aligned}x &= \cos \theta \\ y &= \sin \theta \\ \mathbf{c}(\theta) &= (x, y) \\ &= (\cos \theta, \sin \theta)\end{aligned}$$

where  $\theta \in [0, 2\pi)$ .

We now calculate the line integral for  $\mathbf{F}'$ ,

$$\begin{aligned}\int_C \mathbf{F}' \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}'(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) d\theta \\ &= \int_0^{2\pi} (-F_x \sin \theta + F_y \cos \theta) d\theta \\ &= -F_x \int_0^{2\pi} \sin \theta d\theta + F_y \int_0^{2\pi} \cos \theta d\theta \\ &= \boxed{0}\end{aligned}$$

We see that, independent of our choice of  $F_x$  and  $F_y$ , the integral comes out to be zero. In the case of  $F$ , the line integral comes out to be,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) d\theta \\ &= \alpha \int_0^{2\pi} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta \\ &= \alpha \int_0^{2\pi} (-\cos \theta \sin \theta + \cos \theta \sin \theta) d\theta \\ &= \boxed{0}\end{aligned}$$

9. Let  $C : x^2 + y^2 = 1$ . Find

$$\oint_C \text{grad}(x^2 - y^2) \cdot d\mathbf{s}.$$

**Sol.**

We can parametrize the given  $C$  by substituting,

$$\begin{aligned}x &= \cos \theta \\ y &= \sin \theta \\ \mathbf{c}(\theta) &= (x, y) \\ &= (\cos \theta, \sin \theta)\end{aligned}$$

where  $\theta \in (0, 2\pi)$ .

$$\mathbf{c}'(\theta) = (-\sin \theta, \cos \theta)$$

$$\begin{aligned}\text{grad}(x^2 - y^2) &= \left( \frac{\partial (x^2 - y^2)}{\partial x}, \frac{\partial (x^2 - y^2)}{\partial y} \right) \\ &= (2x, -2y) \\ &= (2 \cos \theta, -2 \sin \theta)\end{aligned}$$

The required line integral is,

$$\begin{aligned}\oint_C \text{grad}(x^2 - y^2) \cdot d\mathbf{s} &= \int_0^{2\pi} (2 \cos \theta, -2 \sin \theta) \cdot (-\sin \theta, \cos \theta) d\theta \\ &= \int_0^{2\pi} (-2 \sin \theta \cos \theta - 2 \sin \theta \cos \theta) d\theta \\ &= \int_0^{2\pi} (-2 \sin 2\theta) d\theta \\ &= 0\end{aligned}$$

This is to be expected as gradient of a scalar field is a conservative vector field and hence the line integral over a continuous closed loop is 0. ■

10. Evaluate

$$\int_C \text{grad}(x^2 - y^2) \cdot d\mathbf{s}$$

where  $C$  is  $y = x^3$ , joining  $(0, 2)$  and  $(2, 8)$ .

**Sol.**

We can parametrize  $C$  as follows,

$$\begin{aligned} x &= t \\ y &= t^3 \\ \mathbf{c}(t) &= (x, y) \\ &= (t, t^3) \end{aligned}$$

where  $t \in (0, 2)$ .

$$\mathbf{c}'(t) = (1, 3t^2)$$

As shown in the previous question,

$$\begin{aligned} \text{grad}(x^2 - y^2) &= (2x, -2y) \\ &= (2t, -2t^3) \\ \Rightarrow \int_C \text{grad}(x^2 - y^2) \cdot d\mathbf{s} &= \int_0^2 (2t, -2t^3) \cdot (1, 3t^2) dt \\ &= \int_0^2 (2t - 6t^5) dt \\ &= \boxed{-60} \end{aligned}$$

■

11. Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where  $C$  is the square with vertices  $(1, 0), (0, 1), (-1, 0)$  and  $(0, -1)$  traversed once in the counter clockwise direction.

**Sol.**

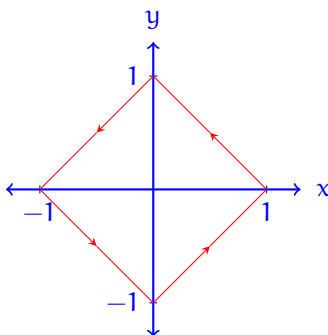


Figure 1: Figure for question 11

The path  $C$  is given by the region in Fig. 1. Note that along this path,  $|x| + |y| = 1$ . Hence the line integral reduces to

$$\oint_C (dx + dy), \text{ which can be rewritten as } \oint_C \nabla(x + y) \cdot d\mathbf{s}.$$

The fundamental theorem of calculus (FTC) for gradient fields can then be applied here to conclude that this integral is 0, but not directly, since  $C$  is not a smooth path. We break up  $C$  into 4 constituent paths  $C_1, \dots, C_4$ , which are cyclically from  $(1, 0)$  to  $(0, 1)$ ,  $\dots$ , and from  $(0, -1)$  to  $(1, 0)$  respectively. Each of these paths are smooth, and thus we may apply FTC to each of them, giving us

$$\begin{aligned} \oint_C \nabla(x+y) \cdot ds &= \oint_{C_1} \nabla(x+y) \cdot ds + \oint_{C_2} \nabla(x+y) \cdot ds + \oint_{C_3} \nabla(x+y) \cdot ds + \oint_{C_4} \nabla(x+y) \cdot ds \\ &= (x+y) \Big|_{(x,y)=(1,0)}^{(x,y)=(0,1)} + (x+y) \Big|_{(x,y)=(0,1)}^{(x,y)=(-1,0)} + (x+y) \Big|_{(x,y)=(-1,0)}^{(x,y)=(0,-1)} + (x+y) \Big|_{(x,y)=(0,-1)}^{(x,y)=(1,0)} \\ &= \boxed{0} \end{aligned}$$

**12.** A force  $F = xy\mathbf{i} + x^6y^2\mathbf{j}$  moves a particle from  $(0, 0)$  onto the line  $x = 1$  along  $y = ax^b$  where  $a, b > 0$ . If the work done is independent of  $b$  find the value of  $a$ . ■

**Sol.**

Parametrize the curve as,

$$\begin{aligned} x &= \tau \\ y &= a\tau^b \\ \mathbf{c}(\tau) &= (x, y) \\ &= (\tau, a\tau^b) \end{aligned}$$

We can therefore write,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \int \mathbf{F}(\mathbf{c}(\tau)) \cdot \mathbf{c}'(\tau) d\tau \\ &= \int_0^1 (a\tau^{b+1}\mathbf{i} + a^2\tau^{2b+6}\mathbf{j}) \cdot (\mathbf{i} + ab\tau^{b-1}\mathbf{j}) d\tau \\ &= \int_0^1 (a\tau^{b+1} + a^3b\tau^{3b+5}) d\tau \\ &= \frac{3a + ba^3}{3b+6} = I(a, b) \end{aligned}$$

If we want  $I(a, b)$  to be independent of  $b$ , then we will have to ensure that  $\frac{\partial I}{\partial b} = 0$ .

$$\begin{aligned} \frac{\partial I}{\partial b} &= 0 \\ \Rightarrow \frac{a^3(3b+6) - 3(3a+ba^3)}{(3b+6)^2} &= 0 \\ \Rightarrow a &= 0, \pm\sqrt{\frac{3}{2}} \end{aligned}$$

We will take the value of  $a > 0$ . So for  $I(a, b)$  to be independent of  $b$ , we will need  $a = \boxed{\sqrt{\frac{3}{2}}}$ . ■

## TUTORIAL SHEET 4 (LINE INTEGRALS AND CONSERVATIVE FIELDS)

---

1. Determine whether or not the given set is a) open, b) path connected and c) simply connected.

(a)  $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 3\}$

(b)  $D = \{(x, y) \in \mathbb{R}^2 \mid 1 < |x| < 2\}$

(c)  $D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$

(d)  $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (1, 4)\}$

**Sol.**

(a) The given set is open, path connected and simply connected.

(b) The given set is open. It is not path connected since there is no path between a point in  $1 < x < 2$  and a point in  $-2 < x < -1$ . This implies the set is also not simply connected.

(c) The given set is not open, but is path connected and simply connected.

(d) The set is open, and path connected. It is not simply connected - if we considered a closed curve such that  $(1, 4)$  belonged to the interior of the curve, we will fail in continuously shrinking the curve to a point. ■

2. Determine whether or not the vector field  $\mathbf{F}(x, y) = 3xy\mathbf{i} + x^3y\mathbf{j}$  is a gradient on any open subset of  $\mathbb{R}^2$ .

**Sol.**

$$\nabla \times \mathbf{F} = \left[ \frac{\partial}{\partial x} (x^3y) - \frac{\partial}{\partial y} (3xy) \right] \mathbf{k} = 3x(xy - 1) \mathbf{k} \neq \mathbf{0}$$

Hence, the given vector field is not a gradient on any open subset of  $\mathbb{R}^2$ . ■

3. Show the line integral is path-independent and evaluate the integral:

$$\int_C 2xe^{-y} dx + (2y - x^2e^{-y}) dy.$$

where C is any path from  $(1, 0)$  to  $(2, 1)$ .

**Sol.**

$$\mathbf{F} = (2xe^{-y})\mathbf{i} + (2y - x^2e^{-y})\mathbf{j}$$

Using partial integration of the  $\mathbf{i}$  and  $\mathbf{j}$  components with respect to  $x$  and  $y$  respectively, and comparing we see,

$$f = x^2e^{-y} + y^2 + c$$

where  $f$  is such that,

$$\nabla f = \mathbf{F}$$

and  $c$  is a constant of integration.

So,

$$\int_C 2xe^{-y} dx + (2y - x^2e^{-y}) dy = f((2, 1)) - f((1, 0)) = \boxed{\frac{4}{e}}$$

4. Is the line integral  $\int_C y dx + x dy + xyz dz$  path independent in  $\mathbb{R}^3$ ? ■

**Sol.**



We have,

$$\mathbf{F} = y\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$$

Thus,

$$F_1(x, y, z) = y$$

$$F_2(x, y, z) = x$$

$$F_3(x, y, z) = xyz$$

Note that,  $\mathbf{F}$  is defined on  $\mathbb{R}^3$ , which is a *simply connected* region. For conservativeness, we only need to check the *sufficient* condition (refer **Lecture 7 Slide 16**).

We see that,

$$\frac{\partial F_1}{\partial z} = 0$$

$$\frac{\partial F_3}{\partial x} = yz$$

This tells us that  $\mathbf{F}$  is not conservative. Further, by the theorem at **Lecture 7 Slide 7** we know that on path connected domains,

$$\text{path independence} \Leftrightarrow \text{conservative}$$

Hence, the given line integral is *not* path independent in  $\mathbb{R}^3$ . (*Note:* If we did not use the theorem, we could not have ruled out path independence, as, in general path independence does not imply conservative. Thus, in general, the contrapositive of that logical sentence (that not conservative implies not path connected) is not true either.)

■

5. Let  $\mathbf{F} = \nabla f$ , where  $f(x, y) = \sin(x - 2y)$ . Find curves  $C_1$  and  $C_2$  that are not closed and satisfy

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = 0,$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 1.$$

**Sol.**

Consider  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ . Also consider a curve  $\mathbf{c}(t), 0 \leq t \leq 1$  such that  $\mathbf{c}(0) = \mathbf{a}$  and  $\mathbf{c}(1) = \mathbf{b}$ . Then,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{b}) - f(\mathbf{a})$$

The following curves fulfill our requirements (Check).

$$C_1(t) = (2\pi t, 0), \quad 0 \leq t \leq 1$$

$$C_2(t) = \left(2t + \frac{\pi}{2}, t\right), \quad 0 \leq t \leq 1$$

■

6. Determine whether or not  $\mathbf{F}$  is a conservative vector field. If it is, find a function  $f$  such that  $\mathbf{F} = \nabla f$ .

- (a)  $\mathbf{F}(x, y) = y^2 e^{xy} \mathbf{i} + (1 + xy) e^{xy} \mathbf{j}$ , for all  $(x, y) \in \mathbb{R}^2$ .  
(b)  $\mathbf{F}(x, y) = (ye^x + \sin y) \mathbf{i} + (e^x + x \cos y) \mathbf{j}$ , for all  $(x, y) \in \mathbb{R}^2$ .  
(c)  $\mathbf{F}(x, y) = (2xy + y^{-2}) \mathbf{i} + (x^2 - 2xy^{-3}) \mathbf{j}$ , for all  $(x, y) \in \mathbb{R}^2$  and  $y > 0$ .

**Sol.**

A sufficient condition to check that  $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} : D \rightarrow \mathbb{R}^2$  for open, simply connected  $D \subset \mathbb{R}^2$  is conservative, is that the first order partial derivatives are continuous and satisfy

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \text{ on } D.$$

Since  $D = \mathbb{R}^2$  and  $D = \mathbb{R} \times \mathbb{R}_{\geq 0}$  here are open and simply connected, we may use this sufficient condition to test whether the  $\mathbf{F}$ s given are conservative.

- (a)  $\mathbf{F}(x, y) = y^2 e^{xy} \mathbf{i} + (1 + xy) e^{xy} \mathbf{j}$ , for all  $(x, y) \in \mathbb{R}^2$ :

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= 2ye^{xy} + xy^2 e^{xy}, \\ \frac{\partial F_2}{\partial x} &= ye^{xy} + (1 + xy)ye^{xy} = 2ye^{xy} + xy^2 e^{xy}. \end{aligned}$$

Hence  $\mathbf{F}$  is conservative. Let  $f$  be the scalar function whose gradient is  $\mathbf{F}$ , i.e.

$$\begin{aligned} \mathbf{F} &= F_1 \mathbf{i} + F_2 \mathbf{j} = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ \implies \frac{\partial f}{\partial x} &= y^2 e^{xy}, \quad \frac{\partial f}{\partial y} = (1 + xy) e^{xy} \\ \frac{\partial f}{\partial x} &= y^2 e^{xy} \implies f(x, y) = \int y^2 e^{xy} dx = ye^{xy} + g(y), \end{aligned}$$

where  $g(y)$  is some function of  $y$  (constant with respect to  $x$ ). Continuing,

$$\frac{\partial f}{\partial y} = (1 + xy) e^{xy} \implies \frac{\partial}{\partial y} (ye^{xy} + g(y)) = (1 + xy) e^{xy} \implies g'(y) = 0 \implies g(y) = c,$$

where  $c$  may be any constant.

We take  $c = 0$  and get,

$$f(x, y) = ye^{xy}$$

□

- (b)  $\mathbf{F}(x, y) = (ye^x + \sin y) \mathbf{i} + (e^x + x \cos y) \mathbf{j}$ , for all  $(x, y) \in \mathbb{R}^2$ :

Following the same steps as in (a), we see that  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$  on  $\mathbb{R}^2$ , and hence  $\mathbf{F}$  is conservative. Further,

$$f = ye^x + x \sin y.$$

□

- (c)  $\mathbf{F}(x, y) = (2xy + y^{-2}) \mathbf{i} + (x^2 - 2xy^{-3}) \mathbf{j}$ , for all  $(x, y) \in \mathbb{R}^2$  and  $y > 0$ :

Once again, with same steps as in (a), we have that  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$  on  $\mathbb{R}^2$ , and hence  $\mathbf{F}$  is conservative. We

also get  $f = x^2 y + xy^{-2}$ . ■

7. Let  $\mathbf{F}$  be a vector field on  $\mathbb{R}^2$ . Find a function  $f$  such that  $\mathbf{F} = \text{grad } f$  and using it evaluate  $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ , where  $\mathbf{F}$  and  $\mathbf{c}$  are given below:

(a)  $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$  and  $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4), 0 \leq t \leq \pi$ .

(b)  $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$  and  $\mathbf{c}(t) = (\cos t, 2\sin t), 0 \leq t \leq \frac{\pi}{2}$ .

(c)  $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$  and  $\mathbf{c}$  is the line segment from  $(1, 0, -2)$  to  $(4, 6, 3)$ .

**Sol.**

The same steps as detailed in the solution to 6(a) can be followed to obtain the scalar function  $f$  such that  $\nabla f = \mathbf{F}$ , with the only difference for  $\mathbb{R}^3$  being that the first constant (after integrating with respect to  $x$ ) will be  $g(y, z)$  (as opposed to  $g(y)$ ). The line integral in each case then reduces to  $\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s}$ .

For smooth  $\mathbf{c} : [a, b] \rightarrow D$ , and continuous  $\nabla f$  (as we have here), we may use the FTC for line integrals, which gives us,

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

(a)  $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$  and  $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4), 0 \leq t \leq \pi$ :

It can be checked that  $f(x, y, z) = x^2yz - \cos x$  is valid, and hence

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(\cos^5 \pi, \sin^3 \pi, \pi^4)) - f(\mathbf{c}(\cos^5 0, \sin^3 0, 0^4)) = (x^2yz - \cos x) \Big|_{(1,0,0)}^{(-1,0,\pi^4)} = \boxed{0}.$$

□

(b)  $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$  and  $\mathbf{c}(t) = (\cos t, 2\sin t), 0 \leq t \leq \frac{\pi}{2}$ :

$f(x, y, z) = xe^{xy}$  works, and hence

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(\cos \frac{\pi}{2}, 2\sin \frac{\pi}{2})) - f(\mathbf{c}(\cos 0, 2\sin 0)) = (xe^{xy}) \Big|_{(1,0)}^{(0,2)} = \boxed{-1}.$$

□

(c)  $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$  and  $\mathbf{c}$  is the line segment from  $(1, 0, -2)$  to  $(4, 6, 3)$ :

Once again, it can be checked that  $f(x, y, z) = xyz + z^2$  is valid, and hence

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f((4, 6, 3)) - f((1, 0, -2)) = (xyz + z^2) \Big|_{(1,0,-2)}^{(4,6,3)} = \boxed{77}.$$

■

8. For  $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ , show that  $\nabla \phi = \mathbf{v}$  for some  $\phi$  and hence calculate  $\oint_C \mathbf{v} \cdot d\mathbf{s}$  where  $C$  is any arbitrary smooth closed curve.

**Sol.**

Should follow the exact same process as in the previous two questions and check,

$$\phi = x^2 + xz^3 + c$$

where  $c \in \mathbb{R}$  is a constant.

$$\oint_C \mathbf{v} \cdot d\mathbf{s} = \boxed{0}$$

■

9. Let  $S = \mathbb{R}^2 \setminus (0,0)$ . Let

$$\mathbf{F} = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j} := F_1(x, y) \mathbf{i} + F_2(x, y) \mathbf{j}$$

- (a) Show that  $\frac{\partial F_1(x, y)}{\partial y} = \frac{\partial F_2(x, y)}{\partial x}$  on  $S$ .
- (b) Compute  $\int_C \mathbf{F} \cdot d\mathbf{s}$  where  $C$  is the circle:  $x^2 + y^2 = 1$ .
- (c) Is  $\mathbf{F}$  a conservative field on  $S$ ?

**Sol.**

- (a) Is quite trivial.

$$\begin{aligned} \frac{\partial F_1(x, y)}{\partial y} &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ \frac{\partial F_2(x, y)}{\partial x} &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned}$$

□

- (b) Take the parametrization  $\mathbf{c}(t) = (\cos t, \sin t)$  for  $0 \leq t \leq 2\pi$ .  $\mathbf{c}'(t) = (-\sin t, \cos t)$ ,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} [-\sin t(-\sin t) + \cos t(\cos t)] dt \\ &= \int_0^{2\pi} dt = \boxed{2\pi} \end{aligned}$$

□

- (c) Even though the curl is  $\mathbf{0}$  on  $S$ , the field  $\mathbf{F}$  is not conservative, as we have in part (b) a closed loop such that the line integral of  $\mathbf{F}$  along it is non-zero.

■

10. A radial force field is one which can be expressed as  $\mathbf{F}(x, y, z) = f(r)\mathbf{r}$  where  $\mathbf{r} = (x, y, z)$  is the position vector and  $r = \|\mathbf{r}\|$ . Show that, if  $f$  is continuous,  $\mathbf{F}$  is conservative in  $\mathbb{R}^3$ . (Hint. Try to guess what the potential function could be, provided  $f$  is continuous.)

**Sol.**

We need to find scalar potential  $\phi(r)$  such that:

$$\begin{aligned} \nabla \phi(r) &= \frac{\phi'(r)}{r} \mathbf{r} = \mathbf{F} = f(r)\mathbf{r} \\ \implies \phi'(r) &= rf(r) \end{aligned}$$

Since  $f(r)$  is continuous, so will be  $rf(r)$  and hence from 2nd FTC we know that there exists an antiderivative of  $rf(r)$  i.e  $\exists \phi(r)$  such that,

$$\phi(r) = \phi(a) + \int_a^r r' f(r') dr'$$

where  $a \in \mathbb{R}^3$ .

The conservative nature of  $\mathbf{F}$  follows.

■

## TUTORIAL SHEET 5 (GREEN'S THEOREM)

---

1. Verify Green's theorem in each of the following cases:

(i)  $F_1(x, y) = -xy$ ;  $F_2(x, y) = xy$ ;  $R: x \geq 0, 0 \leq y \leq 1 - x^2$ ;

(ii)  $F_1(x, y) = 2xy$ ;  $F_2(x, y) = e^x + x^2$ ; where  $R$  is the triangle with vertices  $(0,0)$ ,  $(1,0)$ , and  $(1,1)$ .

**Sol.**

For this question, we just have to verify that,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

(i)  $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \frac{31}{60} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$  (Check). □

(ii)  $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = e = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$  (Check again!). ■

2. Use Green's Theorem to evaluate the integral

$$\oint_{\partial R} y^2 dx + x dy$$

where

(i)  $R$  is the square with vertices  $(0,0)$ ,  $(2,0)$ ,  $(2,2)$ ,  $(0,2)$ .

(ii)  $R$  is the square with vertices with  $(\pm 1, \pm 1)$ .

(iii)  $R$  is the disc of radius 2 and center  $(0,0)$  oriented positively.

**Sol.**

Here, we can use Green's theorem to obtain,

$$\oint_{\partial R} y^2 dx + x dy = \iint_R (1 - 2y) d(x, y)$$

We now use this in the above given scenarios:

(i)

$$\oint_{\partial R} y^2 dx + x dy = \int_0^2 \int_0^2 (1 - 2y) dy dx = \boxed{-4}$$

□

(ii)

$$\oint_{\partial R} y^2 dx + x dy = \int_{-1}^1 \int_{-1}^1 (1 - 2y) dy dx = \boxed{4}$$

□

(iii) Here, switch to plane-polar coordinates to obtain

$$\oint_{\partial R} y^2 dx + x dy = - \int_0^2 \int_0^{2\pi} r(1 - r \sin \theta) dr d\theta = \boxed{-4\pi}$$

■

3. For a simple closed curve given in polar coordinates show using Green's Theorem that the area enclosed is given by

$$A = \frac{1}{2} \oint_C r^2 d\theta$$

Use this to compute the area enclosed by the following curves:

(i) The cardioid:  $r = a(1 - \cos\theta)$ ;  $0 \leq \theta \leq 2\pi$

(ii) The lemniscate:  $r^2 = a^2 \cos 2\theta$ ;  $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

**Sol.**

Consider the curve  $C$  given by the parametrisation  $\mathbf{c} : [a, b] \rightarrow \mathbb{R}^2$  by  $\mathbf{c}(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t))$ . Then we may write the area, given by

$$\begin{aligned} \frac{1}{2} \oint_C x dy - y dx &= \frac{1}{2} \oint_C (-y, x) \cdot d\mathbf{s} = \frac{1}{2} \int_a^b (-y(t), x(t)) \cdot \mathbf{c}'(t) dt \\ &= \frac{1}{2} \int_a^b (-r(t) \sin \theta(t), r(t) \cos \theta(t)) \cdot \left( \frac{dr}{dt} \cos \theta(t) - r(t) \sin \theta(t) \frac{d\theta}{dt}, \frac{dr}{dt} \sin \theta(t) + r(t) \cos \theta(t) \frac{d\theta}{dt} \right) dt \\ &= \frac{1}{2} \int_a^b r(t) (\cos^2 \theta(t) + \sin^2 \theta(t)) \frac{d\theta}{dt} dt = \boxed{\frac{1}{2} \oint_C r^2 d\theta}. \end{aligned}$$

Computing the areas is now straightforward, as

(i)

$$A = \frac{1}{2} \oint_C r^2 d\theta = \frac{a^2}{2} \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta = \boxed{\frac{3\pi a^2}{2}}; \text{ and}$$

□

(ii)

$$A = \frac{1}{2} \oint_C r^2 d\theta = \frac{a^2}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta = \boxed{\frac{a^2}{2}}.$$

■

4. Find the area of the following regions:

(i) The area lying in the first quadrant of the cardioid  $r = a(1 - \cos\theta)$ .

(ii) The region under one arch of the cycloid,

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}, 0 \leq t \leq 2\pi.$$

(iii) The region bounded by the limaçon on

$$r = 1 - 2\cos\theta, 0 \leq \theta \leq \frac{\pi}{2}.$$

and the two axes.

**Sol.**

(i) The area will be given by,

$$\begin{aligned} \frac{1}{2} \oint_C r^2 d\theta &= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 - \cos\theta)^2 d\theta \\ &= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \cos^2\theta d\theta \\ &= \boxed{a^2 \left( \frac{3\pi}{8} - 1 \right)} \end{aligned}$$

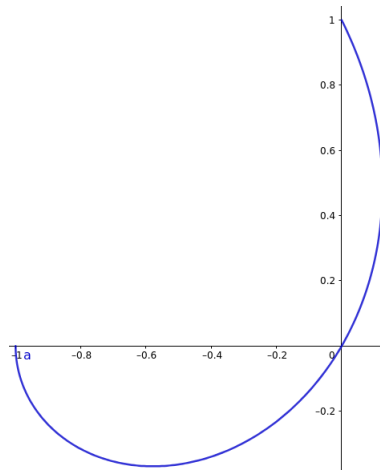
□

(ii) The area  $A$  is equal to  $\oint_C -y dx$  according to Green's theorem. This gives,

$$\begin{aligned}\oint_C -y dx &= \int_0^{2\pi} a(1 - \cos t) \frac{dx}{dt} dt \\ &= a^2 \int_0^{2\pi} (1 - \cos t)^2 dt \\ &= \boxed{3\pi a^2}\end{aligned}$$

□

(iii) Below is a diagram for the locus.



Note: If  $r$  is negative then we are reflecting through the origin.

Now the area is  $\frac{1}{2} \oint_C r^2 d\theta$ , which is the same as (here justification is needed and question 3 is not applicable directly)

$$\begin{aligned}A &= \frac{1}{2} \int_0^{\pi} (1 - 2\cos(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi} 1 - 4\cos(\theta) + 4\cos^2(\theta) d\theta \\ &= \frac{1}{2} \left( \frac{\pi}{2} - 4 + \pi \right) \\ &= \boxed{\frac{3\pi}{4} - 2}\end{aligned}$$

■

5. Let  $D = \{(x, y) \in \mathbb{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2\}$ , where  $0 < a < b$ . Evaluate

$$\int_{\partial D} x e^{-y^2} dx + \left[ -x^2 y e^{-y^2} + 1/(x^2 + y^2) \right] dy$$

where  $\partial D$  is positively oriented.

**Sol.**

Grouping the first two terms together, and making a neat observation, the above integral reduces to:

$$\begin{aligned} & \int_{\partial D} \left[ x e^{-y^2} dx - x^2 y e^{-y^2} dy \right] + \int_{\partial D} \frac{1}{x^2 + y^2} dy \\ &= \int_{\partial D} \frac{1}{2} \nabla(x^2 e^{-y^2}) \cdot ds + \int_{\partial D} \frac{1}{x^2 + y^2} dy. \end{aligned}$$

We deal with each term separately.

$$\int_{\partial D} \frac{1}{2} \nabla(x^2 e^{-y^2}) \cdot ds = 0 \quad (\because \text{Fundamental Theorem of Calculus, since } x^2 e^{-y^2} \text{ and } \partial D \text{ are (piecewise) smooth}).$$

To evaluate the second term, consider the positively oriented (check!) parametrisation of  $\partial D$  given by  $\mathbf{c} : [0, 4\pi] \rightarrow \partial D$  as

$$\mathbf{c}(t) = \begin{cases} (b \cos t, b \sin t) & t \in [0, 2\pi) \\ (a \cos t, -a \sin t) & t \in [2\pi, 4\pi] \end{cases}.$$

We now evaluate the integral, as

$$\begin{aligned} \int_{\partial D} \frac{1}{x^2 + y^2} dy &= \int_{\partial D} \left( 0, \frac{1}{x^2 + y^2} \right) \cdot ds = \int_{\partial D} \left( 0, \frac{1}{x(t)^2 + y(t)^2} \right) \cdot \mathbf{c}'(t) dt \\ &= \int_0^{2\pi} \frac{1}{b^2} b \cos t dt - \int_{2\pi}^{4\pi} \frac{1}{a^2} a \cos t dt = 0 - 0 = 0. \end{aligned}$$

Putting the above together, we have

$$\int_{\partial D} x e^{-y^2} dx + \left[ -x^2 y e^{-y^2} + 1/(x^2 + y^2) \right] dy = \boxed{0}.$$

■

6. Let  $C$  be a simple closed curve in the  $xy$ -plane. Show that

$$3I_0 = \oint_C x^3 dy - y^3 dx$$

where  $I_0 = \frac{1}{3} \iint_D r^2 dx dy$ ,  $D$  is the region enclosed by  $C$ . This  $I_0$  is often called ‘polar moment of inertia’ of the region  $D$ .

**Sol.**

$$I_0 = \iint_D r^2 dx dy = \frac{1}{3} \iint_D (3x^2 + 3y^2) dx dy = \frac{1}{3} \iint_D \left( \frac{\partial x^3}{\partial x} - \frac{\partial(-y^3)}{\partial y} \right) dx dy$$

Using Green’s theorem (which may do since  $x^3$  and  $-y^3$  are  $C^1$  over all  $\mathbb{R}^2$ , in particular over some open set  $\Omega \supset D \cup C$ ), we have,

$$3I_0 = \iint_D \left( \frac{\partial x^3}{\partial x} - \frac{\partial(-y^3)}{\partial y} \right) dx dy = \oint_C -y^3 dx + x^3 dy = \boxed{\oint_C x^3 dy - y^3 dx}.$$

■



7. If  $C$  is the line segment connecting  $(x_1, y_1)$  to the point  $(x_2, y_2)$ , show that

$$\int_C (x dy - y dx) = x_1 y_2 - x_2 y_1.$$

Sol.

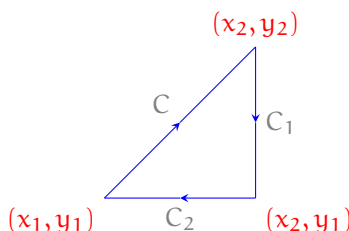


Figure 1: Closed curves  $C, C_1, C_2$  as needed for Question 7

Completing the triangle by adding the point  $(x_2, y_1)$  and taking closed curve going from  $(x_1, y_1)$  to  $(x_2, y_2)$  to  $(x_2, y_1)$  and then back to  $(x_1, y_1)$ . Check that this has a negative orientation with respect to the triangular region inside.

$$\int_C x dy - y dx = \int_{C \cup C_1 \cup C_2} (x dy - y dx) - \int_{C_1 \cup C_2} (x dy - y dx)$$

The first integral is  $-2 \cdot (\text{area of triangle})$ . For  $C_1$ ,  $x$  is constant and for  $C_2$ ,  $y$  is constant.

$$\begin{aligned} \int_C x dy - y dx &= -2 \cdot \frac{1}{2} (x_2 - x_1) (y_2 - y_1) - [x_2 (y_1 - y_2) - y_1 (x_1 - x_2)] \\ &= \boxed{x_1 y_2 - x_2 y_1} \end{aligned}$$

8. Let  $C$  be any counterclockwise closed curve and let  $\mathbf{n}$  be the outward unit normal to the curve  $C$ . Compute

$$\oint_C \nabla(x^2 - y^2) \cdot \mathbf{n} ds.$$

Sol.

Recall the divergence form of the Green's Theorem:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \nabla \cdot \mathbf{F} \, dx dy$$

In the case we have here,  $\mathbf{F} = 2x\mathbf{i} - 2y\mathbf{j}$ . It has divergence zero (check). Therefore,

$$\oint_C \nabla(x^2 - y^2) \cdot \mathbf{n} ds = \boxed{0}$$

9. Let  $D$  be a region in  $\mathbb{R}^2$  with boundary  $\partial D$  satisfying hypothesis stated in the ‘Green’s theorem’. Let  $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$  be a  $C^2$  function.

1. Show that  $\nabla^2 \phi = \text{div}(\text{grad } \phi)$ , where

$$\nabla^2 \phi(x, y) = \frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2}$$

. The operator  $\nabla^2$  is called the Laplace operator.

2. Show that the Green’s Identity holds:

$$\iint_D \nabla^2 \phi(x, y) dx dy = \oint_{\partial D} \frac{\partial \phi}{\partial \mathbf{n}} ds.$$

3. Using the above identity, compute

$$\oint_C \frac{\partial \phi}{\partial \mathbf{n}} ds.$$

For  $\phi = e^x \sin(y)$ , and  $D$  the triangle with vertices  $(0, 0), (4, 2), (0, 2)$ .

**Sol.**

1. We have,

$$\begin{aligned} \nabla \phi &= \phi_x \mathbf{i} + \phi_y \mathbf{j} \\ \nabla \cdot \nabla \phi &= \phi_{xx} + \phi_{yy}, \end{aligned}$$

which is exactly the RHS given. □

2. Recall the divergence form of the Green’s Theorem, i.e.

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \nabla \cdot \mathbf{F} dx dy.$$

Let  $\mathbf{F} = \nabla \phi$ . By the first part, and the fact that  $\frac{\partial \phi}{\partial \mathbf{n}} = \nabla \phi \cdot \mathbf{n}$ , the result follows. □

3. Note that,

$$\phi_{xx} = e^x \sin y, \quad \phi_{yy} = -e^x \sin y$$

Thus  $\nabla^2 \phi = 0$  and hence the answer is 0. ■

10. Let us consider the region  $\Omega = \{(x, y) \mid x^2 + y^2 > 1\}$  and the vector field defined on  $\Omega$ . Evaluate the following line integrals where the loops are traced in the counter clockwise sense

(i)

$$\oint_C \frac{y dx - x dy}{x^2 + y^2},$$

where  $C$  is any closed curve in  $\Omega$  enclosing the origin.

(ii)

$$\oint_C \frac{y dx - x dy}{x^2 + y^2},$$

where  $C$  is any closed curve in  $\Omega$  not enclosing the origin.

(iii) Let  $C$  be a smooth closed curve lying in  $\Omega$ . Find

$$\oint_C \frac{\partial \ln(r)}{\partial y} dx - \frac{\partial \ln(r)}{\partial x} dy.$$

**Sol.**

- (i) Let  $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  in the clockwise sense. Then note that  $C \cup S$  forms a positively oriented boundary for the region enclosed between  $C$  and  $S$ , say  $D$ . Define  $\mathbf{F}(x, y) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j}$ . Observing that  $D \cup C \cup S \subset \Omega' = \{(x, y) \mid x^2 + y^2 > 0.5\}$ , an open set, and that  $\mathbf{F}$  is  $\mathcal{C}^1$  on  $\Omega'$ , we may apply Green's theorem to get

$$\oint_{C \cup S} \frac{y dx - x dy}{x^2 + y^2} = \oint_{C \cup S} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = 0.$$

Further noting that  $C$  and  $S$  are disjoint, and that  $\oint_S \mathbf{F} \cdot d\mathbf{s} = 2\pi$  (from Question 9 of Tutorial sheet 4), we have that

$$\oint_C \frac{y dx - x dy}{x^2 + y^2} = \oint_C \mathbf{F} \cdot d\mathbf{s} = \boxed{-2\pi}.$$

□

- (ii) With  $\mathbf{F}$  and  $\Omega'$  as in (i) above, we have that if  $C$  does not enclose the origin, then  $\mathbf{F}$  is  $\mathcal{C}^1$  on  $\Omega' \supset D \cup C$ , where  $D$  is now the region enclosed by the closed curve  $C$ . Applying Green's theorem, we have

$$\oint_C \frac{y dx - x dy}{x^2 + y^2} = \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \boxed{0}.$$

□

- (iii) This reduces to a generalization of parts (i) and (ii) with the observation that  $\frac{\partial \ln r}{\partial x} = \frac{\partial \ln \sqrt{x^2 + y^2}}{\partial x} = \frac{x}{x^2 + y^2}$  and  $\frac{\partial \ln r}{\partial y} = \frac{y}{x^2 + y^2}$ . We then conclude that the integral required is

$$\oint_C \frac{\partial \ln(r)}{\partial y} dx - \frac{\partial \ln(r)}{\partial x} dy = \boxed{\begin{cases} -2\pi, & \text{if } C \text{ encloses } (0, 0) \\ 0, & \text{if } C \text{ does not enclose } (0, 0) \end{cases}}.$$

■

11. Is there a vector field  $\mathbf{G}$  in  $\mathbb{R}^3$  such that

(i)  $\text{curl } \mathbf{G} = (x \sin y)\mathbf{i} + (\cos y)\mathbf{j} + (z - xy)\mathbf{k}$

(ii)  $\text{curl } \mathbf{G} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

**Sol.**

In each case, let  $\mathbf{V}$  denote the vector given as  $\text{curl } \mathbf{G}$ . We make use of the result that  $\text{div}(\text{curl } \mathbf{G}) = 0$ . Hence if the divergence of  $\mathbf{V}$  is non-zero, we may safely conclude that there does not exist such a  $\mathbf{G}$ . Additionally, if the divergence-free vector field (i.e. the vector field with zero divergence) is  $\mathcal{C}^1$  on a simply connected domain, we have that it must be the curl of some vector field. Since our vector fields are both smooth on  $\mathbb{R}^3$ , we have that  $\mathbf{V}$  being divergence-free is both necessary and sufficient for the required  $\mathbf{G}$  to exist.

(i) Calculating the divergence, we have

$$\text{div } \mathbf{V} = \sin y - \sin y + 1 = 1 \neq 0,$$

hence there does NOT exist such a  $\mathbf{G}$ . □

(ii) We have

$$\text{div } \mathbf{V} = 1 + 1 + 1 = 3 \neq 0,$$

which lets us conclude once again that there does NOT exist such a  $\mathbf{G}$ . ■

12. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$$

in  $\mathbb{R}^3$

where  $f, g, h$  are differentiable functions, is irrotational, i.e.  $\text{curl } \mathbf{F} = 0$ .

**Sol.**

For the functions  $f, g$  &  $h$ , the following hold:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial z} = \frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$$

For a vector field with  $x, y$  and  $z$  components as  $F_1, F_2$  and  $F_3$  respectively, we have the  $x$  component of  $\text{curl } \mathbf{F}$  is given by

$$(\text{curl } \mathbf{F})_1 = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right),$$

and other components are given by cyclically permuting  $x, y, z$ . Since the derivatives all involve only ‘cross-terms’, i.e. involving the derivative of a component of  $\mathbf{F}$  along the other components, we have that each term in the curl evaluates to zero, giving us  $\text{curl } \mathbf{F} = 0$ . ■

13. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$$

in  $\mathbb{R}^3$

where  $f, g, h$  are differentiable functions, is incompressible, i.e.  $\text{div } \mathbf{F} = 0$ .

**Sol.**

Serving as a neat converse to Question 12, here we use the fact that the divergence only involves terms where the derivative of each component of  $\mathbf{F}$  is taken along the same component, i.e.

$$\text{div } \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \boxed{0}.$$
■