MA-111 Calculus II
(D3 & D4)

Lecture 13

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Curl of a vector field

Divergence of a vector field

Del operator on vector fields

The del operator operates on vector fields as in two different ways. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

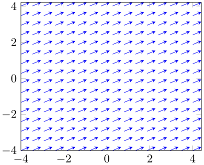
$$\operatorname{curl}\mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

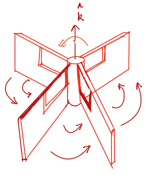
It is often written as a determinant;

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

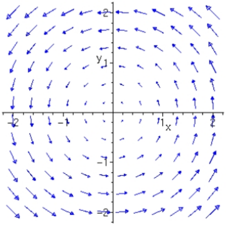
Curl as a measure of rotation

Curl of a vector field is measuring the extent to which the field rotate a particle. For instance ,





Imagine putting a small paddle wheel as shown in the above figure at any point in the plane with the vector field acting on it and visualize how it will rotate. Clearly in this example it will not rotate.



How about in this example?

Angular velocity

Consider a solid body B rotating around the z-axis on the x-y-plane.

Let $\mathbf{v}$ denote the velocity vector, $\mathbf{w}$ the angular velocity vector at a point $\mathbf{r}$ in B. Note $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed. Further, $\|v\| = \|w\|\|r\| \sin\theta$ where θ is the angle made by $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ with the axis of rotation.

Then $\mathbf{v} = \mathbf{w} \times \mathbf{r} = -\omega y\mathbf{i} + \omega x\mathbf{j}$. Check?

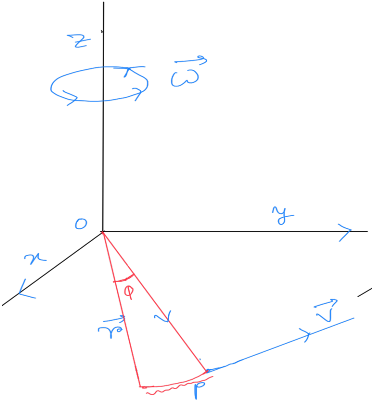
Now,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\mathbf{w}.$$

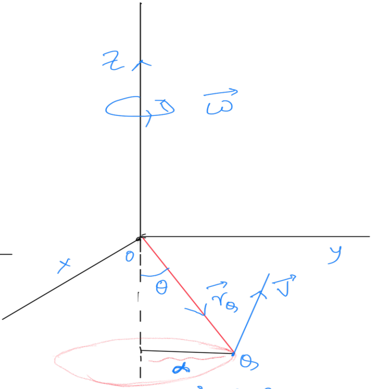
Thus, the curl of velocity is twice the angular velocity.

If a vector field $\mathbf{F}$ represents the flow of a fluid, then the value of $\nabla \times \mathbf{F}$ at a point is twice the rotation vector of a rigid body that rotates as the fluid does near that point. In particular, $\nabla \times \mathbf{F} = 0$ at a point P means that the fluid is free from the rigid rotations at P.
The curl free vector field is called irrotational field.

Angular velocity



$$\vec{V} = \vec{\omega} \times \vec{r}.$$



$$\alpha = \|\vec{r}_\theta\| \sin\theta.$$

The curl of a gradient

Suppose that $\mathbf{F} = \nabla f$ for some scalar function f and f is $\mathcal{C}^2$. Then

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}.$$

Clearly,

$$\nabla \times \mathbf{F} = 0.$$

In particular, this gives a criterion for deciding whether a vector field arises as the gradient of a function. This gives that curl$\mathbf{F} = 0$ is a necessary condition for any smooth vector field $\mathbf{F}$ to be the gradient field.

Is the condition $\nabla \times \mathbf{F} = 0$ sufficient for $\mathbf{F}$ to be a gradient field?

Recall that we have previously looked at the vector field

$$\mathbf{F} = \frac{y}{x^2 + y^2} \cdot \mathbf{i} + \frac{-x}{x^2 + y^2} \cdot \mathbf{j},$$

Exercise 1: Check that $\nabla \times \mathbf{F} = 0$.
Can you express $\mathbf{F}$ as the gradient of a suitable scalar function? Ans. No!

We can conclude that the Image of ∇ operator on scalar functions defined on $D \subset \mathbb{R}^3$ is a proper subset of

$$\ker(\mathrm{curl}) = \{\mathbf{F} \text{ is a vector field on } D \mid \mathrm{curl}\mathbf{F} = 0\}.$$

Definition (Scalar curl:) If $\mathbf{F} := (F_1, F_2)$ (a vector field in $\mathbb{R}^2$), then we define the curl of F by thinking of it as a vector field in $\mathbb{R}^3$ on the x-y plane with $F_3 = 0$.

$$\operatorname{curl}\mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

The function $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ is called the scalar curl of $\mathbf{F}$.

We can now state a vector valued version of Green's theorem using curl.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be a C^1 vector field on an open connected region D with ∂D be positively oriented. Then

$$\int_{\partial D} \mathbf{F}\cdot\mathbf{ds} = \iint_D (\operatorname{curl} F \cdot \mathbf{k})\ dxdy.$$

Other forms of Green's theorem in $\mathbb{R}^2$

Under the hypothesis on the region D and the functions F_1 and F_2 as stated in Green's theorem, we have

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

We assume that ∂D can be parametrised by a single curve - otherwise break up the curve into parametrisable pieces.

Let ∂D be a non-singular, positively oriented curve in $\mathbb{R}^2$, parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are denoted by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}, \quad \forall\, t \in [a, b].$$

Other form Green's theorem

As consequences of Green's theorem in $\mathbb{R}^2$, we have following results:

Curl form or Tangential form

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int\int_D (\text{curl}\mathbf{F}).\mathbf{k} dxdy.$$

The Stokes theorem is a 3-dimensional version of the above result.

Outline of its proof: Considering $\mathbf{F} = (F_1, F_2, 0)$ and noting that $ds = \|c'(t)\| dt$, we get

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int_{\partial D} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t) dt = \int_{\partial D} F_1 dx + F_2 dy.$$

Now using Green's theorem and noting $\text{curl}\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{k}$, the result follows.

Conservative field and its curl

Theorem

1. *Let Ω be an open, simply connected region in $\mathbb{R}^2$.*
2. *if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on Ω.*

Then $\mathbf{F}$ is a conservative field in Ω if and only if

$$curl\,\mathbf{F} = 0, \quad \text{in} \quad \Omega.$$

Outline of the proof: Let the assumptions on Ω and $\mathbf{F}$ in the statement hold.

- If $\mathbf{F}$ is C^1 and a conservative field, i.e., $\mathbf{F} = \nabla f$, for some f is C^2. Then a direct calculation gives $\text{curl} F = 0$.
- Now conversely, if $\mathbf{F}$ is C^1 and $\text{curl}\,\mathbf{F} = 0$ on Ω. Then by Green's theorem we can show that the line integral of F over any simple closed curve is 0. That is, the line integral of F in Ω is path independent. Hence the result follows.

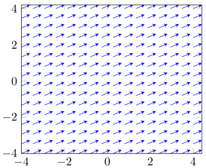
The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

Definition: Let $\mathbf{F} = (F_1, F_2, F_2)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

$$\operatorname{div} \mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

One way to interpret divergence of a velocity vector field at a point P as the amount of fluid flowing in versus the amount of fluid flowing out.



If $\mathbf{F}$ is a constant vector field then at any point what is flowing in is flowing out and the divergence is 0.