Tut 7

1 $F(x,y,z) = x^2 + 2xy - y^2 + z^2$

$\nabla F(x,y,z) = (F_x(x,y,z), F_y(x,y,z), F_z(x,y,z))$
$= (2x+2y, 2x-2y, 2z)$

$\nabla F(1,-1,3) = (0, 4, 6)$
$= 4\hat{j} + 6\hat{k}$

Tangent plane: $0.(x-1) + 4.(y+1) + 6(z-3) = 0$
$2y + 3z = 7$

Normal line: $x=1, \quad \frac{y+1}{4} = \frac{z-3}{6}$
$x=1, \quad 6y - 4z + 18 = 0$
$x=1, \quad 3y - 2z + 9 = 0$

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2 $F(x,y,z) = 3x - 5y + 2z$

Direction of $\underline{u} = (2x, 2y, 2z)$ at $(2, 2, 1)$

$= (4, 4, 2)$

$\therefore \underline{u} = \dfrac{4\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36}} = \dfrac{2\hat{i}}{3} + \dfrac{2\hat{j}}{3} + \dfrac{\hat{k}}{3}$

$\nabla F(2,2,1) = (3, -5, 2)$

$\therefore D_{\underline{u}} F(2,2,1) = (3,-5,2) \cdot \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$= -2/3$

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3 $\sin(x+y) + \sin(y+z) = 1$ —— (1)

$\cos(x+y) + \cos(y+z)\left(\frac{\partial z}{\partial y} + 1\right) = 0$ on diff. wrt y —— (2)

Now diff. wrt x,

$$-\sin(x+y) + \cos(y+z)\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(-\sin(y+z)\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y} + 1\right) = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\cos(y+z)}\left(\sin(x+y) + \sin(y+z)\frac{\partial z}{\partial x}\left(\frac{\partial z}{\partial y} + 1\right)\right)$$ —— (3)

Diff. (1) wrt x,

$$\cos(x+y) + \cos(y+z)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-\cos(x+y)}{\cos(y+z)}$$

From (2), $1 + \frac{\partial z}{\partial y} = \frac{-\cos(x+y)}{\cos(y+z)}$

Substituting in (3), we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\cos(y+z)}\left(\sin(x+y) + \sin(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}\right)$$

$$= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}$$

4 $f(0,0) = 0$

$$f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0)$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_x(x,y) = \frac{(x^2+y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} \quad \text{for } (x,y) \neq (0,0)$$

$$= \frac{x^4y + 5x^2y^3 - x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \quad \text{for } (x,y) \neq (0,0)$$

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-k - 0}{k} = -1$$

For $(x,y) \neq (0,0)$,

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Consider the seq. $\left(\frac{1}{n}, 0\right)$ for $n \in \mathbb{N}$

$$\lim_{n\to\infty} f_{xy}(1/n, 0) = 1 \neq f_{xy}(0,0)$$

$\therefore f_{xy}$ is discont. at $(0,0)$.

Similarly, compute f_{yx}. It turns out that $f_{yx}(0,0) = 1$ and f_{yx} is not cont. at $(0,0)$.

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(i) $f(x,y) = x^4 + y^4 + 4x - 32y - 7$

$f_x(x,y) = 4x^3 + 4$

$f_{xx}(x,y) = 12x^2$

$f_{xy}(x,y) = 0$

$f_y(x,y) = 4y^3 - 32$

$f_{yy}(x,y) = 12y^2$

~~for~~ $f_x(-1,2) = f_y(-1,2) = 0$

$$\begin{vmatrix} f_{xx}(-1,2) & f_{xy}(-1,2) \\ f_{xy}(-1,2) & f_{yy}(-1,2) \end{vmatrix}$$

$$= \begin{vmatrix} 12 & 0 \\ 0 & 48 \end{vmatrix} = 576 > 0, \; f_{xx}(-1,2) > 0$$

$\therefore$ $(-1,2)$ is a local minima for $f(x,y)$.

(ii) Proceed similarly.

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(i) $f(x,y) = (x^2-y^2)e^{-(x^2+y^2)/2}$

f is defined on all of $\mathbb{R}^2$, and all partials of order 2 exist and are cont.
∴ Second derivative test can be applied.

$$f_x(x,y) = xe^{\frac{1}{2}(-x^2-y^2)}(-x^2+y^2+2)$$

$$f_y(x,y) = ye^{(-x^2-y^2)}(-x^2+y^2-2)$$

Solving $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$ gives

$$(x_0,y_0) \in \{(0,0), (\sqrt{2},0), (-\sqrt{2},0), (0,\sqrt{2}), (0,-\sqrt{2})\}$$

Compute $D(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)$ for each of these points.

For $(0,0)$, $D(0,0) = -4$ ∴ saddle point

For $(0,\sqrt{2})$ and $(0,-\sqrt{2})$, $D(x_0,y_0) = -e^{-y_0^2}(y_0^6 - 3y_0^4 - 8y_0^2 + 4)$

∴ $D(0,\sqrt{2}) = D(0,-\sqrt{2}) > 0$
Also, $f_{xx}(0,\sqrt{2}) > 0$, $f_{xx}(0,-\sqrt{2})$
∴ local minima

For $(\pm\sqrt{2},0)$, $D > 0$, $f_{xx} < 0$ ∴ local maxima

7 $f(x,y) = (x^2-4x)\cos y \qquad 1 \le x \le 3, \ -\frac{\pi}{4} \le y \le \pi/4$

For interior points (x,y),

$f_x(x,y) = (2x-4)\cos y$

$f_y(x,y) = (x^2-4x)\sin y$

Only critical point is $(2,0)$

Now restrict to each boundary

On right boundary, fn. is $-3\cos y$, $-\frac{\pi}{4} \le y \le \pi/4$

Find critical points using single variable calculus.

$(3,0)$, $(3,\pi/4)$, $(3,-\pi/4)$

Left boundary: $(1,0)$, $(1,\pi/4)$, $(1,-\pi/4)$

Bottom: $\left(1,\frac{-\pi}{4}\right)$, $\left(2,\frac{-\pi}{4}\right)$, $\left(3,\frac{-\pi}{4}\right)$

Top: $(1,\pi/4)$, $(2,\pi/4)$, $(3,\pi/4)$

Compute f at each of these points.