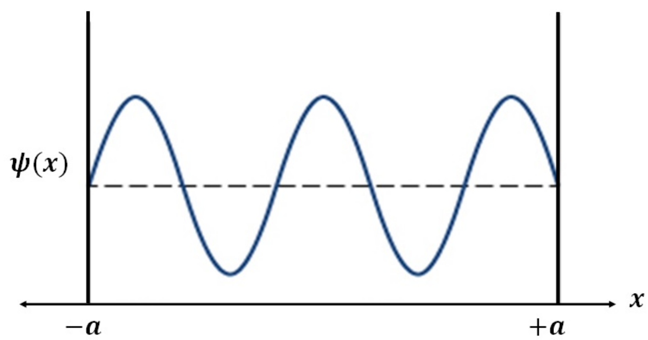
PH 107: End Semester examination

Total Marks: 45

Q1. [3 marks] Figure below shows one of the possible wave functions $\psi(x)$ for an electron located in the region between two impenetrable walls at $x = -a$ and $x = +a$. $V = 0$ for $|x| < a$. If the energy of the electron is 50 eV when it is in this quantum state, the ground state energy (in eV) is ....



Answer:

For a particle in an infinite potential box, $E_n = n^2 E_1$. The given wavefunction is for $n = 5$ (4 nodes).

Ground state energy, $E_1 = 50/25 = 2\,\text{eV}$

Q2. [2 marks] Consider a particle with 1-D wave function given by

$$\psi(x) = A \exp\left(-\frac{x^2}{a^2} + 2ik_0 x\right)$$

where A, a and k_0 are constants with appropriate dimensions. The expectation values of position $\langle x \rangle = \alpha\, a$ and momentum $\langle p \rangle = \beta\, \hbar k_0$. α is ..... and β is .....

(you may use the properties of the following integral: $\int_{-\infty}^{+\infty} e^{-ty^2} dy = \sqrt{\frac{\pi}{t}}$.)

Answer:

$$< x >= \int \psi^* x \psi \, dx = A^2 \int_{-\infty}^{+\infty} x \, \exp\left(-\frac{2x^2}{a^2}\right) dx = 0$$

$$< p >= \int \psi^* (-i\hbar \frac{\partial}{\partial x}) \psi \, dx = A^2 \int_{-\infty}^{+\infty} \psi^* (-i\hbar) \psi (\frac{-2x}{a^2} + 2ik_0) = 2\hbar k_0$$

since $\int \psi^* \psi dx = 1$

Q3. [2 marks] Consider a rigid diatomic molecule in the x-y plane, rotating about the z-axis passing through the centre of the line joining the two atoms. The wave functions of the diatomic molecule are

$$\psi_n(\theta) = \frac{A}{\sqrt{8\pi}} e^{in\theta},$$

where n is a quantum number and A is a constant. The value of A is ....

Answer:

The condition for normalizing a wavefunction is

$$\int \left|\psi^2(x)\right| \, dx = 1$$

Here the integral limits will be from $-\pi$ to $+\pi$

$$\int_{-\pi}^{\pi} \left|\psi^2(\phi)\right| d\phi = \frac{1}{8\pi}\int_{-\pi}^{\pi} A^2 d\phi = 1$$

$$2\pi\frac{A^2}{8\pi} = 1 \Longrightarrow A = 2$$

Q4. [3 marks]* Consider a ~~3-D box~~ of sides L and infinite potential walls. Three particles with total energy of $59E_0$ ($E_0 = \hbar^2/2mL^2$) are to be distributed in the possible energy levels (1,3,7) and (3,5,5). Number of states available for distinguishable particles, Bosons and Fermions are ...., ... and ....., respectively.

Answer:of

For classica lparticles, (1,3,7) gives 6 microstates and (3,5,5) gives 3 microstates (total 9)

For Bosons, (1,3,7) gives 1 microstate and (3,5,5) gives 1 microstate (total 2)

For Femions, (1,3,7) gives 1 microstate and (3,5,5) gives 0 microstate (total 1)

* since there was a confusion about the statement, marks will be awarded to those who attempted and given the staement in the uploaded answer work sheet or those who put 0, 0, 0 for all three. Those who answered as above will also get full marks

Q5. [3 marks] Consider an energy level in a metal lying 0.015 eV below the Fermi energy. Probability of an electron **NOT** occupying this energy level at 300 K is......

[take k_BT as 0.025 eV]

Answer:

Probability of occupation

$$f_{FD} = \frac{1}{1+\exp([E-EF]/kT)} = \frac{1}{1+\exp(-.015/.025)} = \frac{1}{1.5488} = 0.64566$$

Probability of NOT occupying, $p = 1 - f = .354$

Q6. [3 marks] Choose the correct statement(s) about the distribution function $f(E)$:

A. $f(E)$ is the average number of particles in a given single particle state with energy E.

B. $f(E) \propto \exp(-E/k_BT)$ in situations in which the particles can be treated as distinguishable where k_B is the Boltzmann constant.

C. $0 \leq f(E) \leq 1$ for all single particle states with energy E regardless of whether the particles are Bosons or Fermions.

D. When $n_i/g_i \gg 1$,quantum distribution function $f(E)$ changes into classical distribution function .

Answer: A, B and C are correct

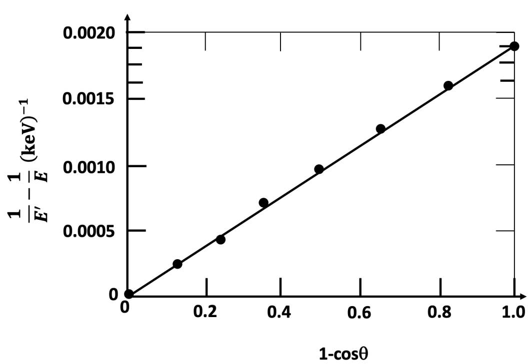
Q7. [3 marks] Let $\hat{A}$ and $\hat{B}$ be two operators. Let us define $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Using this property for any wave function $\psi(x)$, we get $[\hat{x}, \frac{i\hat{p}}{\hbar}]\psi(x) = \alpha\psi(x)$, where α is ......

[$\hat{x}$ is the position operator and $\hat{p} = \hat{p_x}$ is the momentum operator].

Answer:

$$[\hat{x}, i\hat{p}/\hbar]\psi = x\frac{i}{\hbar}(-i\hbar\frac{\partial}{\partial x}\psi) - [\frac{i}{\hbar}(-i\hbar\frac{\partial}{\partial x}x\psi)] = x\frac{\partial}{\partial x}\psi - x\frac{\partial}{\partial x}\psi - \psi = -\psi$$

Q8. [3 marks] Figure below shows the data taken during a Compton scattering experiment. E and E' are the incident and scattered energy of the photon, respectively and θ is the scattering angle. From the graph, the rest mass energy of the electron (in keV) is ——



Answer:

For Compton scattering,

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_0 c^2}(1 - \cos\theta)$$

Slope of the graph gives inverse of rest mass energy. From the graph, slope = 0.001875. Hence $m_e = 1/0.001875 = 533.33$ keV.

Q9. [3 marks] In the original Davisson-Germer experimental setup, the first peak was observed at $\Phi = 50°$ when the accelerating potential was 54 V (see the figure below). The maximum accelerating voltage (in V) for which they could have observed the first peak is ....

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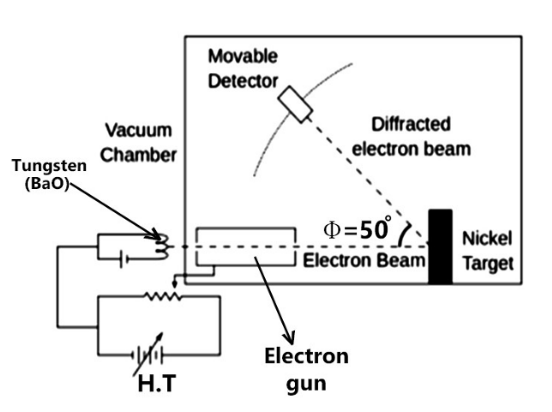
[Take $d = 0.091$ nm as the spacing between adjacent crystal planes in Ni for the given geometry, rest mass energy of electron as 511 keV and $hc = 1240$ eV nm]



Answer:

They can observe the first peak till $\phi = 90°$. This implies Bragg angle $\theta = 45°$.

From Bragg's law,

$$2d\sin\theta = \lambda \implies \lambda = 0.12869\,\text{nm}$$

for the incident electrons. Now

$$\lambda = hc/\sqrt{2m_0c^2V} \implies V = 90.8\,\text{V}$$

OR

$$V = \frac{(12.27)^2}{\lambda^2(\mathring{A}^2)} = 90.91\,\text{V}$$

OR

$$\lambda = hc/\sqrt{2m_0c^2V} = 2d\sin\theta = 2d\cos\phi/2$$

$$V \propto \frac{1}{\cos^2\phi/2} \implies V_2 = V_1\frac{\cos^2(25)}{\cos^2(45)} = 88.7\,\text{V}$$

Q10. [3 marks] Consider a 1-D simple harmonic oscillator. The Hamiltonian H and the wave function Ψ are given by

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$$

where ψ_n denotes the eigen function with energy $E_n = \hbar\omega(n + \frac{1}{2})$. At $t = 0$, if the expectation value of energy $\langle E\rangle = \alpha\hbar\omega$, value of α is ......

Answer:

The expectation value for the energy is

$$\langle E\rangle = \int \Psi^* H\Psi dx = \int \Psi H\left(\frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2\right)dx$$

$$= \frac{1}{2}\int (\psi_1 + \psi_2)(E_1\psi_1 + E_2\psi)\,dx = \frac{1}{2}(E_1 + E_2) = \frac{1}{2}(\frac{3}{2} + \frac{5}{2})\hbar\omega = 2\hbar\omega$$

We have used the orthonormality of the wavefunctions:

$$\int \psi_n \psi_m dx = \delta_{nm}$$

Q 11. [3 marks] Consider a thermodynamic system of N identical particles. Each particle can be in any of the three states with energy eigenvalues E_1, E_2 or E_3. At a given temperature T, the probabilities of finding a particle in a state with E_1, E_2 and E_3 are $P_1 = 0.9, P_2 = 0.09$ and $P_3 = 0.01$, respectively. Assuming Boltzmann distribution, the ratio $\frac{E_2-E_1}{E_3-E_1}$ is ......

Answer:

For particles satisfying Boltzmann distribution, the average number of particles in a quantum state is given by:

$$\frac{n_i}{N} = \frac{e^{-\beta\epsilon_i}}{\sum_j e^{-\beta\epsilon_j}} \qquad i, j = 1, 2 \text{ and } 3$$

This is also the probability for the particles to be in that state. We thus have:

$$\frac{n_1}{N} = 0.9; \ \frac{n_2}{N} = 0.09; \ \frac{n_3}{N} = 0.01$$

$$n_1 = 0.9N \propto e^{-E_1/kT}, \quad n_2 = 0.09N \propto e^{-E_2/kT} \quad n_3 = 0.01N \propto e^{-E_3/kT}$$

$$\frac{n_1}{n_2} = 10 = e^{E_2-E_1/kT}, \quad \frac{n_1}{n_3} = 90 = e^{E_3-E_1/kT}$$

$$\frac{E_2 - E_1}{E_3 - E_1} = \ln 10 / \ln 90 = 0.5117$$

Q12. [2 marks] The wave function for an electron in the Hydrogen atom is

$$\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

where a_0 is the Bohr radius and r is the distance from the origin. If the most probable value for r is given by $r = \alpha a_0$, the value of α is ....

Answer

Probability density of the wave function in a spherical shell is

$$dp = \left[\frac{1}{\sqrt{\pi} a_0^{3/2}} e^{-r/a_0}\right]^2 4\pi r^2 dr$$
$$= \frac{4}{a_0^3} r^2 e^{-2r/a_0} dr$$

differentiating dp with respect to r and setting it equal to 0 allows us to solve for the minimum radius

$$2re^{-2r/a_0} - \frac{2}{a_0} r^2 e^{-2r/a_0} = 0$$

$$r = a_0$$

$$2re^{-2r/a_0}\left[1 - \frac{r}{a_0}\right] = 0$$

Q13. [4 marks] Consider a cavity of volume V filled with electromagnetic radiation. ω_i is the frequency at which the energy density shows a maximum. When the volume of the cavity is increased adiabatically to $8V$, the energy density maximum shifts to ω_f. ω_i/ω_f is .....

[Hint: $U = 3PV$; Assume cavity to be a perfect blackbody]

Answer

$$\lambda_{max}T = \text{constant} \implies \frac{\omega_i}{\omega_f} = \frac{T_i}{T_f}$$

From thermodynamics, $dU = TdS - PdV$ and given $U = 3PV$.

For adiabatic process

$$dS = 0 : dU = 3VdP + 3P\,dV = -PdV \quad \implies \quad 3VdP = -4PdV$$
$$\implies \frac{1}{3}\ln V = -\frac{1}{4}\ln P + \text{const} \quad \implies \quad V^4P^3 = \text{const.}$$

Since $P \propto T^4$,(from the given equeation $U = 3PV$, $\frac{U}{V} = u = 3P \propto T^4$) we have

$$\begin{aligned} VT^3 &= \text{const.} \\ T_f &= \left(\frac{V_i}{V_f}\right)^{1/3} T_i = \frac{T_i}{2} \\ \frac{\omega_i}{\omega_f} &= 2 \end{aligned}$$

Q14. [3 marks] Consider a stream of electrons moving in the $+x$-direction with energy E towards a step potential given by $V = 0$ for $x \leq 0$ and $V = V_0 = 5E/9$ for $x > 0$. If A, B and C are the amplitudes of incident, reflected and transmitted wave functions, respectively, then $A/B =$.........

Answer:

Since $E > V$,

$$\psi_I = Ae^{ikx} + Be^{-ikx}, \; \psi_{II} = Ce^{ikx\frac{2}{3}}$$

Boundary conditions:

$$A + B = C \quad A - B = C\frac{2}{3}$$
$$A = \frac{5C}{6} \quad B = \frac{C}{6} \quad A/B = 5$$

Q15 [2 marks] If the velocity of an electron can be measured only with an accuracy of $\hat{i}(4.00 \pm 0.18) \times 10^5$ ms^{-1}, then the electron should be localized to a space of .... nm.

($m_e = 9.11 \times 10^{-31}$ kg, $h = 6.64 \times 10^{-34}$ Js).

Answer:

$$\Delta p_x = m\Delta v_x = 9.11 \times 10^{-31} \times \; 0.18 \times 10^5 = 1.6398 \times 10^{-26}$$

Now the smallest uncertainty in the position can be estimated as

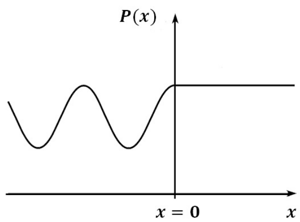
$$\Delta x_{\min} = \frac{\hbar}{2\Delta p_x} = \frac{6.64 \times 10^{-34}}{2 \times 2 \times \pi \times 1.6398 \times 10^{-26}} = 3.225 \times 10^{-9} \text{ m}$$

The smallest space to which the electron can be localised (due to $\pm$in error)

$$x = 2\Delta x_{\text{min}} = 2 \times 3.225 \times 10^{-9} = 6.45\,\text{nm}$$

Q16. [3 marks] For a step function potential given by $V = 0$ for $x \leq 0$ and $V = V_0$ for $x > 0$, choose the correct option(s)

A. When the particle is moving in the $+$ve x-direction from the left of the potential with $E > V_0$, probability has the form



B. Some energy is lost in tunneling when $E < V_0$

C. Reflection happens at $x = 0$ if the particle is moving in the $-$ve x-direction with $E > V_0$

D If the step has a finite width ($V = 0$, for $-L \leq x \leq 0$ and $V = V_0$, for $0 \leq x \leq +L$) and the particle is moving in the $-$ve x-direction with $E < V_0$, then particle can tunnel the barrier.

Answer: A, C and D