PH 107: Quantum Physics and applications

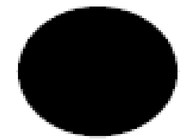
Wave packet, Group velocity and Phase velocity continued…

Sunita.srivsatava@phy.iitb.ac.in

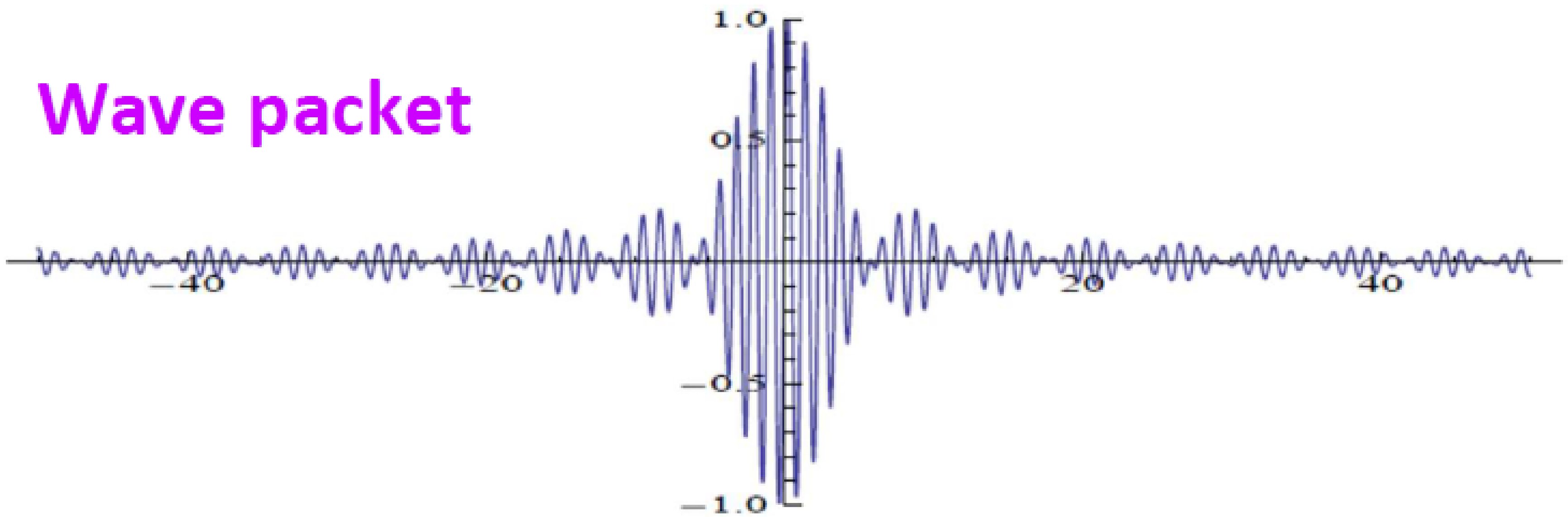
Lecture05: 21-12-2021

Real wave packet

Particle



Wave packet

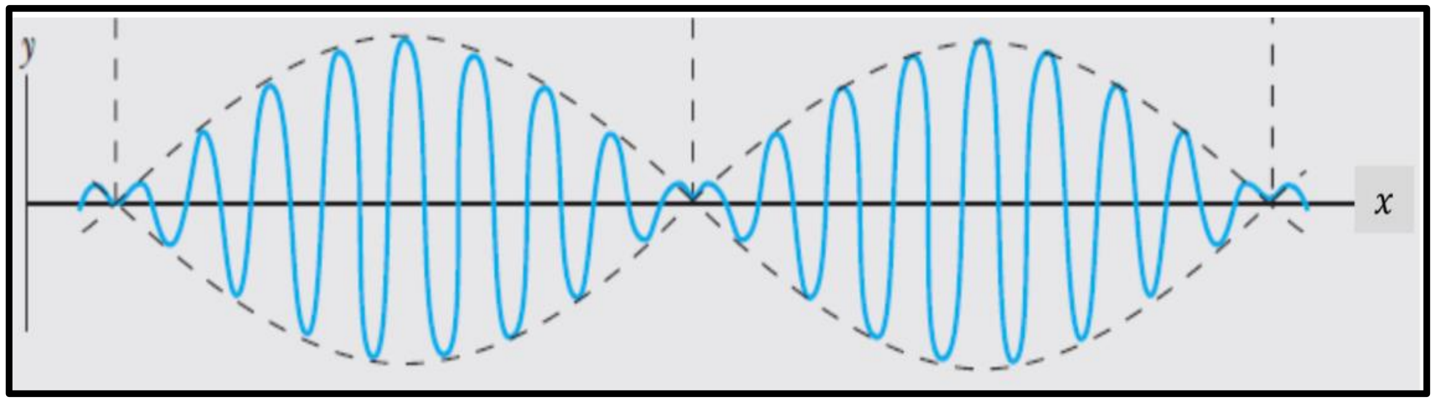


A wave packet is a **group of waves** with **slightly different wavelengths** interfering with one another in a way that the **amplitude of the group (envelope)** is **non-zero** in the neighbourhood of the particle.

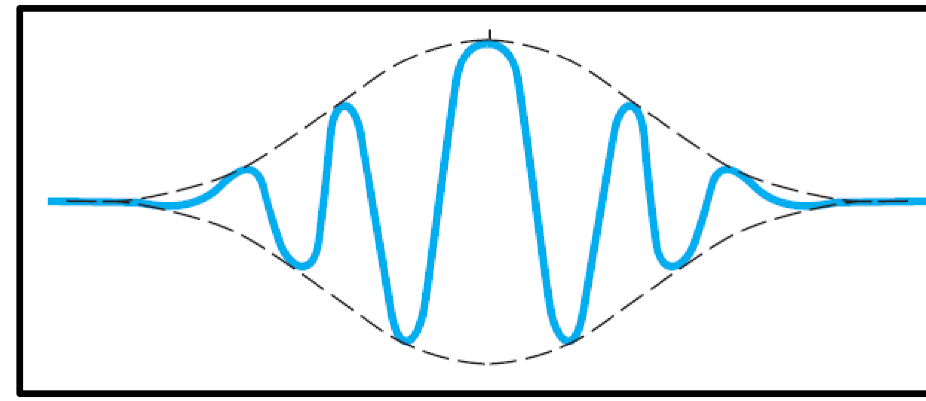
A wave packet is localized; it is a good representation of a particle

Wave Packet so far

So far we discussed the mathematical construction of beats and a “wave packet”, limited to a small region in space.



Pulse or beats



Real wave packet

If we add waves with a *continuous* distribution of wavelengths, we get a true wave packet.

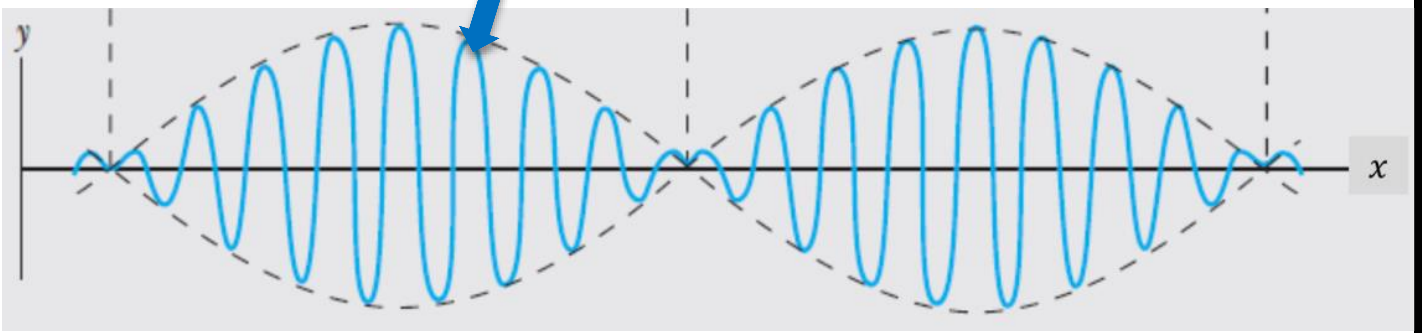
Phase Velocity (High Frequency Wave)

$$\Psi = 2A\cos\left[\frac{\delta k}{2}x - \frac{\delta\omega}{2}t\right]\sin\left[\left(k+\frac{\delta k}{2}\right)x - (\omega+\frac{\delta\omega}{2})t\right]$$

High Frequency wave

Velocity of the wave within the envelope (phase velocity)

$$V_p = \frac{\omega + \frac{\delta\omega}{2}}{k + \frac{\delta k}{2}} \approx \omega/k$$



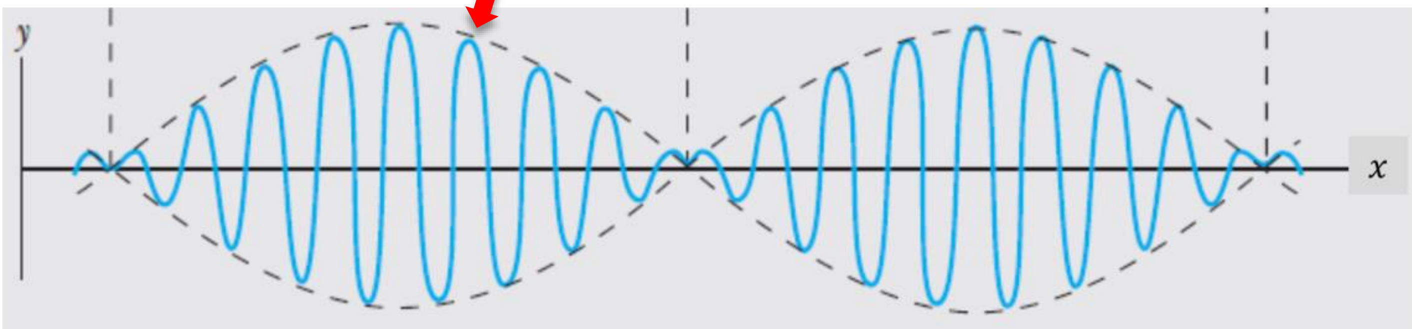
Group Velocity (Broad Envelop)

$$\Psi = 2A\cos\left[\frac{\delta k}{2}x - \frac{\delta\omega}{2}t\right]\sin\left[\left(k + \frac{\delta k}{2}\right)x - (\omega + \frac{\delta\omega}{2})t\right]$$

Broad Envelope

Velocity of the envelope (Group velocity)

$$V_g = \frac{\delta\omega}{\delta k}$$



Velocity at which the wave packet moves

Group velocity, V_g of a relativistic particle

$$\textbf{Since } \boldsymbol{E} = c\sqrt{p^2 + m_o^2 c^2}$$

Since the energy ($E = \hbar\omega$) and momentum of a particle ($p = \hbar k$) are connected to its wave characteristics, the group velocity of matter wave

$$\boldsymbol{V_g = \frac{d\omega}{dk}} \Longrightarrow \boldsymbol{V_g = \frac{dE(p)}{dp}}$$

$$\Longrightarrow \boldsymbol{V_g} = \frac{\boldsymbol{d}\left(c\sqrt{p^2 + m_o^2 c^2}\right)}{\boldsymbol{dp}} = \frac{pc}{\sqrt{p^2 + m_o^2 c^2}} = \boldsymbol{v}$$

de Broglie wave group associated with a moving body travels with the same velocity as that of the body ! ($v_g = v$)

Phase velocity, V_p of a relativistic particle

The phase velocity of the matter waves is

$$V_p = \frac{\omega}{k} = \frac{E}{p}$$

Substituting

$$E = \sqrt{(m_o c^2)^2 + (pc)^2}$$

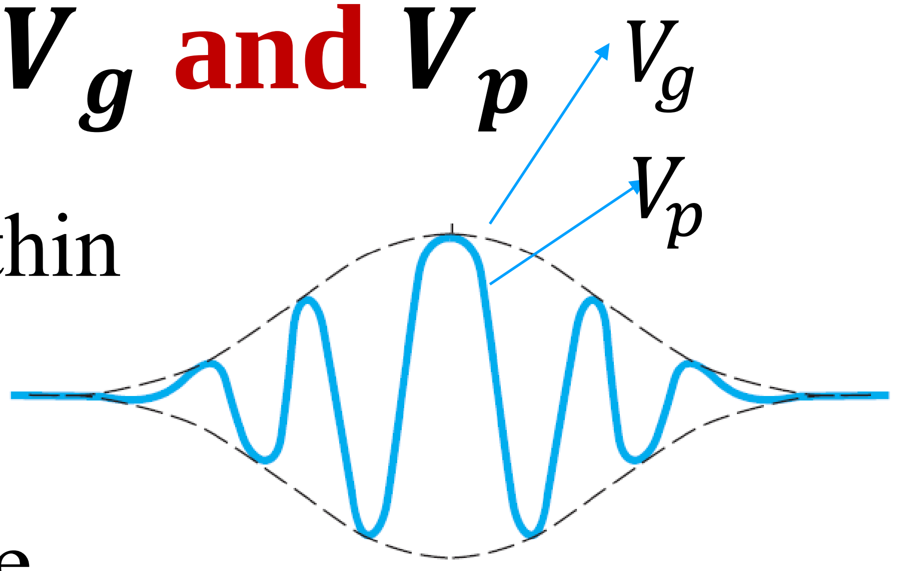
the phase velocity can be written in terms of p only

$$\boldsymbol{V_p} = c\sqrt{1 + \left(\frac{m_o c}{p}\right)^2}$$

$\boldsymbol{V_p}$ **> c; since v < c**

This implies the phase velocity of the *de Broglie* wave is greater than or at least equal to c. This is against the wave concept of the particle hence $\boldsymbol{v} \neq \boldsymbol{v_g}$

General Relationship between $\boldsymbol{V_g}$ and $\boldsymbol{V_p}$



Phase Velocity, V_p: Velocity of the wave within the envelope.

Group Velocity, V_g: Velocity of the envelope.

We can think of adding waves with wavenumbers ranging continuously from $k_0 - \frac{\Delta k}{2}$ to $k_0 + \frac{\Delta k}{2}$, and frequencies ranging from $\omega(k_0 - \frac{\Delta k}{2})$ to $\omega(k_0 + \frac{\Delta k}{2})$, k_0 is the central wavenumber and Δk the range of wavenumbers forming the wave packet. In this case the group velocity,

$$V_g = \left(\frac{d\omega}{dk}\right)_{k_0}$$

Since $\omega = kV_p$ $\boxed{V_g = (V_p)_{k_0} + \left(k\frac{dV_p}{dk}\right)_{k_0}}$

General Relationship between $\boldsymbol{V_g}$ and $\boldsymbol{V_p}$

$$V_g = \left(V_p\right)_{k_0} + \left(k\frac{dV_p}{dk}\right)_{k_0}$$

Since $k = 2\pi/\lambda$

$$V_g = \left[\mathrm{V_p} - \lambda\frac{\mathrm{d}V_p}{\mathrm{d}\lambda}\right]_{\lambda_o}$$

In terms of $\boldsymbol{p}$

$$V_g = \left[\mathrm{V_p} + \mathrm{p}\frac{\mathrm{d}V_p}{\mathrm{dp}}\right]_{\mathrm{p_o}}$$

We see, the group velocity, V_g **can be larger, smaller or equal to** V_p depending on the medium.

V_g and V_p : Non − Dispersive medium

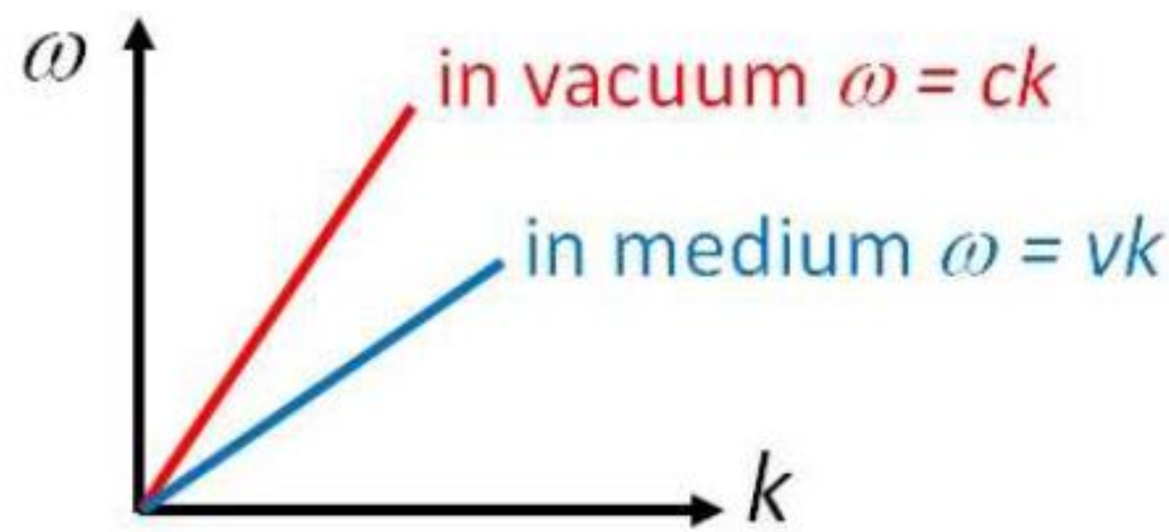
$$\frac{dv_p}{dk} = 0; \quad v_g = v_p$$

All component waves have the same speed!

Example, light in a medium with a constant refractive index (*n*)

$$\omega(k) = \frac{kc}{n}$$

$$\frac{d\omega}{dk} = \frac{c}{n} = v$$



V_g and V_p : Dispersive medium

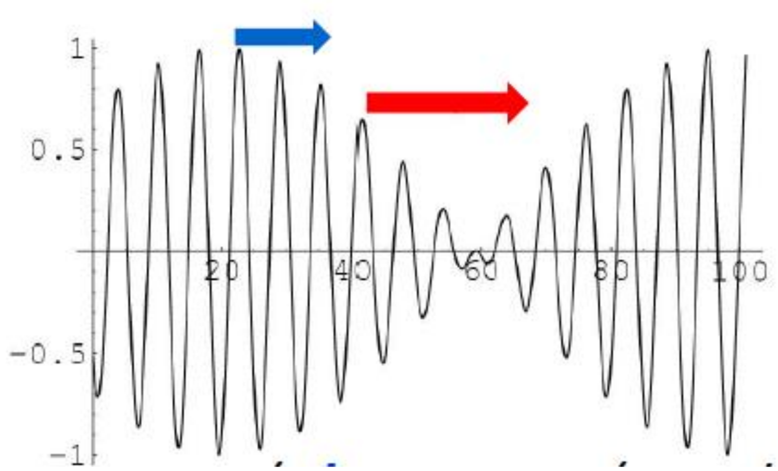
If $\frac{dV_p}{d\lambda}$ **# 0, from** $V_g = \left[V_p - \lambda \frac{dV_p}{d\lambda}\right]_{\lambda_o}$

$$V_g \# V_p$$

Such medium are known as dispersive medium.
Dispersive occurs when phase velocity depends on k or λ.
For such medium, V_g can be smaller or larger than V_p .

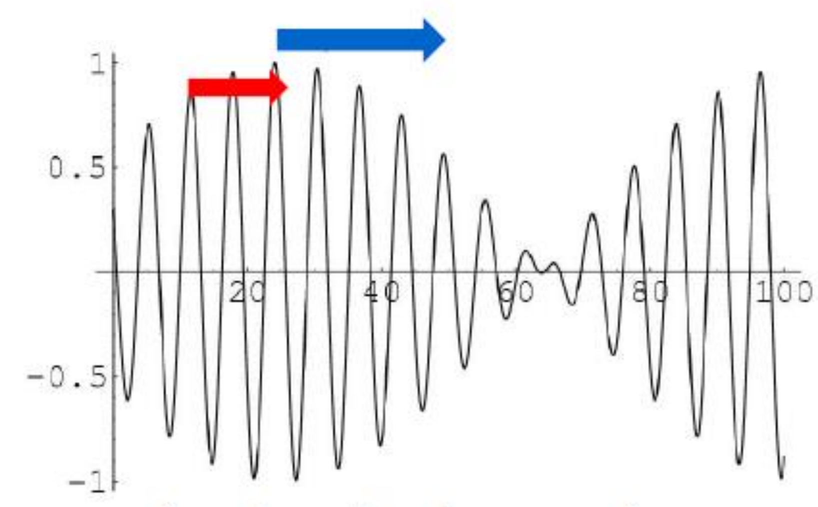
Normal dispersion

$dv_p / d\lambda > 0 \qquad v_g < v_p$



Anomalous dispersion

$dv_p / d\lambda < 0 \qquad v_g > v_p$



(*Blue arrow* (Envelope), *Red arrow* (individual wave)

$\boldsymbol{Example}$: V_g and V_p of a free particle

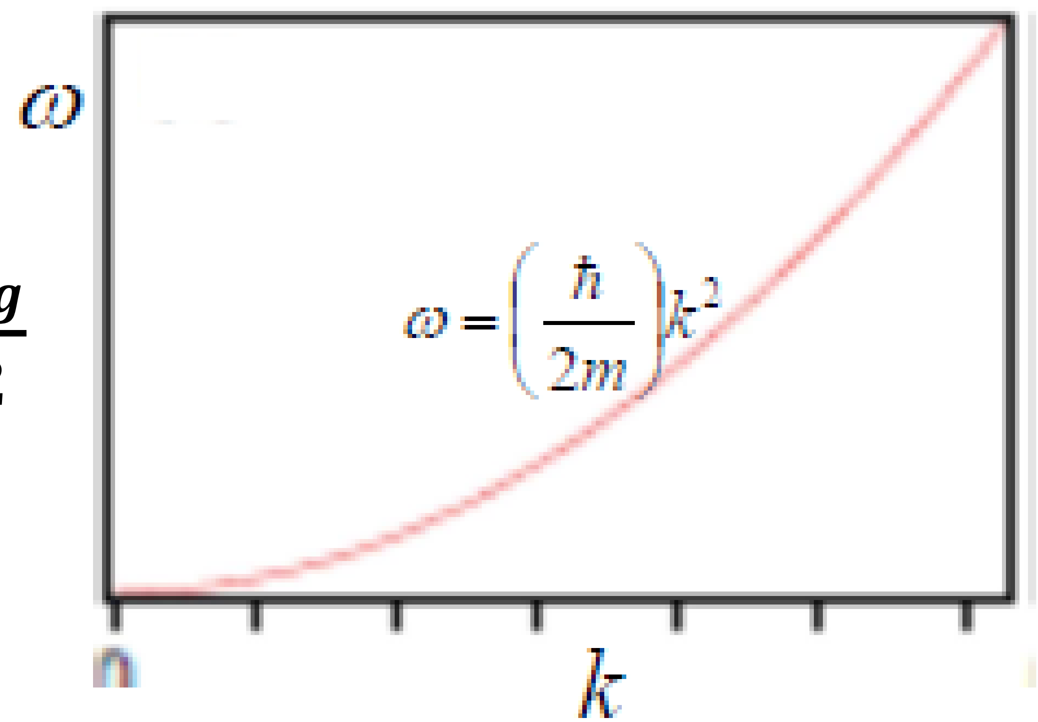
General equation for energy of a particle ; $E = \frac{p2}{2m} + V$

For free particle, (V= 0) ; $E = \frac{p2}{2m} \Longrightarrow \omega(k) = \left(\frac{\hbar}{2m}\right)k^2$

Since, $\mathbf{V}_g = \frac{dE(p)}{dp} \Longrightarrow \mathbf{V}_g = \frac{p}{m}$

Since, $\mathbf{V}_p = \frac{E(p)}{p} \Longrightarrow \mathbf{V}_p = \frac{p}{2m} = \frac{\mathbf{V}_g}{2}$

Dispersion Curve



This implies the inner waveforms travels at half the speed of the speed of the particle. This phase velocity in general has no physical significance.

Example : Water surface wave

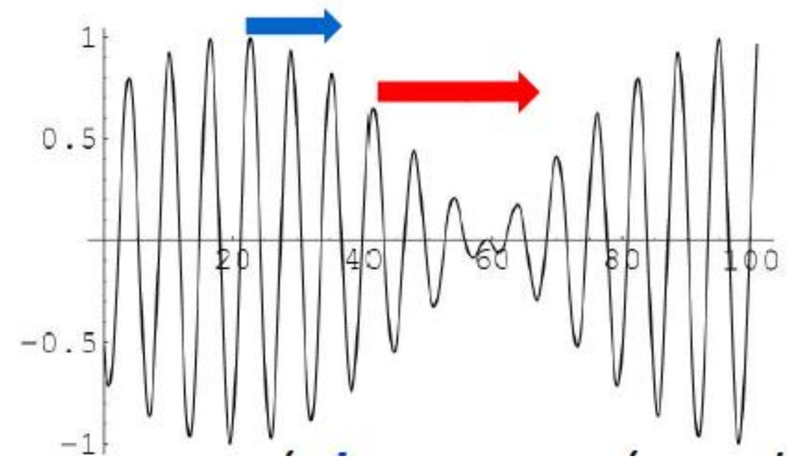
A wave travelling on water surface has phase velocity proportional to the square root of wavelength. What is the group velocity ?

$$V_p = \frac{\omega}{k} = A\lambda^{0.5} = A\sqrt{\frac{2\pi}{k}}$$

$$V_g = \frac{d\omega}{dk} = \frac{A}{2}\sqrt{\frac{2\pi}{k}} = \frac{V_p}{2}$$

Normal dispersion

$dv_p / d\lambda > 0 \qquad v_g < v_p$



Unlike previous example what you see here is that the phase velocity is larger than the group velocity.

Recommended Readings

Wave Groups and Dispersion, section 5.3 in page 152.