MA-111 Calculus II
(D3 & D4)

Lecture 2

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Recall from Lecture 1

Let R be any closed, bounded rectangle in $\mathbb{R}^2$: $R = [a, b] \times [c, d]$, where $a, b, c, d \in \mathbb{R}$.
Partition of R: A partition P of a rectangle $R = [a, b] \times [c, d]$ is $P_1 \times P_2$ where P_1 is a partition of $[a, b]$ and $P_2$is a partition of $[c, d]$. Let

$$P_1 = \{x_0, x_1, \cdots x_m\}, \quad \text{with} \quad a = x_0 < x_1 < x_2 < \cdots < x_m = b\},$$

$$P_2 = \{y_0, y_1, \cdots y_n\}, \quad \text{with} \quad c = y_0 < y_1 < y_2 < \cdots < y_n = d\},$$

and $P = P_1 \times P_2$ be defined by

$$P = \{(x_i, y_j) \mid i \in \{0, 1, \cdots m\}, \quad j \in \{0, 1, \cdots, n\}\}.$$

The points of P divide the rectangle R into *nm non-overlapping sub-rectangles* denoted by

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad \forall\, i = 0, \cdots m-1, \quad j = 0, \cdots, n-1.$$

Note $R = \cup_{i,j} R_{ij}$.

Partitions of a Rectangle

Example: Let P_1 denote a partition of $[-3,3]$ into 3 equal intervals and P_2 the partition of $[-3,3]$ into 2 equal intervals. Describe the rectangles in the partition $P_1 \times P_2$. Note $P_1 = \{-3,-1,1,3\}$ and $P_2 = \{-3,0,3\}$ and thus $[-3,3]\times[-3,3]$ is devided into 6 sub-rectangles
$R_{00} = [-3,-1]\times[-3,0]$, $R_{01} = [-3,-1]\times[0,3]$, $R_{10} = [-1,1]\times[-3,0]$, $R_{11} = [-1,1]\times[0,3]$, $R_{20} = [1,3]\times[-3,0]$, $R_{21} = [1,3]\times[0,3]$.

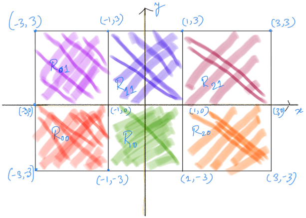

Figure: Partition of $[-3,3]\times[-3,3]$

Partitions of rectangles contd.

The area of each R_{ij}: $\Delta_{ij} := (x_{i+1} - x_i) \times (y_{j+1} - y_j)$, for all $i = 0, \cdots, m-1$, $j = 0, \cdots, n-1$.

Norm of the partition P:
$\|P\| := \max\{(x_{i+1} - x_i), (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$.

Why do we not define the norm by
$\max\{(x_{i+1} - x_i) \times (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$?

Darboux integral

Let $f : R \to \mathbb{R}$ be a bounded function where R is a rectangle . Let $m(f) = \inf\{f(x,y) \mid (x,y) \in R\}$, $M(f) = \sup\{f(x,y) \mid (x,y) \in R\}$. For all $i = 0, 1, \cdots, m-1, \quad j = 0, 1, \cdots, n-1$, let,
$m_{ij}(f) := \inf\{f(x,y) \mid (x,y) \in R_{ij}\}$, and
$M_{ij}(f) := \sup\{f(x,y) \mid (x,y) \in R_{ij}\}$.

Lower double sum: $L(f,P) := \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij}(f)\Delta_{ij}$, and *Upper double sum:* $U(f,P) := \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{ij}(f)\Delta_{ij}$, Note that for any partition P of R

$$m(f)(b-a)(d-c) \leq L(f,P) \leq U(f,P) \leq M(f)(b-a)(d-c).$$

Lower Darboux integral: $L(f) := \sup\{L(f,P) \mid P \text{ is any partition of } R\}$.
Upper Darboux integral:$U(f) := \inf\{U(f,P) \mid P \text{ is any partition of } R\}$.Note $L(f) \leq U(f)$.

Darboux integral contd.

Definition (Darboux integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Darboux integrable* if $L(f) = U(f)$. The Double integral of f is the common value $U(f) = L(f)$ and is denoted by

$$\int\int_R f, \quad \text{or} \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

Theorem (Riemann condition)

Let $f : R \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon.$$

Example

Recall the Dirichlet function for one variable:

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable over $[0,1]$?
Ans. No!

Ex: Check the integrability of Bivariate Dirichlet function over $[0,1] \times [0,1]$

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } \quad y \quad \text{are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Riemann Integral

Riemann integral: Let P be any partition of a rectangle $R = [a, b] \times [c, d]$. We define a tagged partition (P, t) where

$$t = \{t_{ij} \mid t_{ij} \in R_{ij}, \quad i = 0, 1, \dots m-1, \quad j = 0, 1, \dots n-1\}.$$

The *Riemann sum* of f associate to (P, t) is defined by

$$S(f, P, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(t_{ij})\Delta_{ij} \text{ where, } \Delta_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j)$$

Definition (Riemann integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Riemann integrable* if there exists a real number S such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(f, P, t) - S| < \epsilon,$$

for every tagged partition (P, t) satisfying $\|P\| < \delta$ and S is the value of Riemann integral of f.

Riemann Integral contd.

- For any rectangle $R \subseteq \mathbb{R}^2$, let $f : R \to \mathbb{R}^2$ be bounded. The Darboux integrability and Riemann integrability are equivalent.
- A function $f : R \to \mathbb{R}^2$ is called integrable on R if (Darboux or) Riemann integrability condition holds on R.
- In summary, if f is integrable on R, then

$$\int \int_R f(x, y)\; dxdy := S = L(f) = U(f).$$

Examples: Let $R = [a, b] \times [c, d]$.
• The constant function is integrable.

• The projection functions $p_1(x, y) = x$ and $p_2(x, y) = y$ are both integrable on any rectangle $R \subset \mathbb{R}^2$. Why?

• Let $f : R \to \mathbb{R}$ be defined as $f(x, y) = \phi(x)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function. Is f integrable? what is $\int \int_R f\; dxdy$?