Solutions to Tutorial Sheet 2

2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0}[f(\alpha+h) - f(\alpha-h)] = 0.$$

Analyse the converse.

Solution. Suppose $\lim_{x\to\alpha} f(x) = L$. Then $\lim_{h\to 0} f(\alpha+h) = L$, and since

$$|f(\alpha+h) - f(\alpha-h)| \leq |f(\alpha+h) - L| + |f(\alpha-h) - L|$$

it follows that

$$\lim_{h\to 0} |f(\alpha+h) - f(\alpha-h)| = 0.$$

The converse is *false*. For a counter-example, consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{|x|} & x \neq 0 \end{cases}$$

For this example, $\lim_{h\to 0}[f(\alpha+h)-f(\alpha-h)] = 0$ holds, but individually, the left-hand limit and right-hand limit are not finite. Hence limit does not exist, and neither is the function continuous at $x = 0$.

3. Discuss the continuity of the following functions:

(i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

Solution. Continuous everywhere except at $x = 0$. To see that f is not continuous at $x = 0$, consider the sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ where

$$x_n := \frac{1}{n\pi} \text{ and } y_n := \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Note that both x_n, $y_n \to 0$, but $f(x_n) \to 0$ and $|f(y_n)| \to 1$.

Since there exists a finite difference (equal to 1 in absolute value) between two infinitesimally close values of x, the function f is discontinuous at $x = 0$. The value that the function converges to should be exactly the same for *any* choice of sequence converging to the point of concern (here $x = 0$).

(ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

Solution. Continuous everywhere. For proving the continuity of f at $x = 0$, note that $|f(x)| \leq |x|$, and $f(0) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.

Optional. Show that the function f satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.

Solution. Taking $x = y = 0$, we get $f(0+0) = 2f(0)$ so that $f(0) = 0$.

By the assumption of the continuity of f at 0, $\lim_{x\to 0} f(x) = 0$.

Thus,

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0}[f(c)+f(h)] = f(c)$$

showing that f is continuous at $x=c$.

Hint. For the optional part, first verify the equality for all $k \in \mathbb{Q}$ and then use the continuity of f to establish it for all $k \in \mathbb{R}$.

Solution. To show for rational numbers, assume $k = p/q$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Since f is distributive over summation, $f(px) = pf(x)$. Also, $f(px) = f(q(px/q)) = qf(px/q)$.

Hence, $pf(x) = qf(px/q) \implies f(kx) = kf(x)$. This works for rational numbers because p and q are natural numbers and hence can be counted (for distribution of summation). Now since f is continuous for every $c \in \mathbb{R}$, this relation should hold even for $k \notin \mathbb{Q}$ (for now ignore the rigorous proof for why rational + continuity implies real).

5. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

   Solution. Clearly, f is differentiable for all $x \neq 0$ and the derivative is

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), x \neq 0.$$

   Also,

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = 0.$$

   Clearly, f' is continuous at any $x \neq 0$. However $\lim_{x\to 0} f'(x)$ does not exist. Indeed, for any $\delta > 0$, we can choose $n \in \mathbb{N}$ such that $x := 1/n\pi$, $y := 1/(n+1)\pi$ are in $(-\delta, \delta)$, but $|f'(x) - f'(y)| = 2$.

7. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h\to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

   exists and equals $f'(c)$. Is the converse true? [Hint: Consider $f(x) = |x|$.]

   Solution. For the first part, observe that

$$\begin{aligned}\lim_{h\to 0^+} \frac{f(c+h) - f(c-h)}{2h} &= \lim_{h\to 0^+} \frac{1}{2}\left[\frac{f(c+h) - f(c)}{h} + \frac{f(c-h) - f(c)}{-h}\right] \\ &= \frac{1}{2}[f'(c) + f'(c)] = f'(c)\end{aligned}$$

   The converse is *false*. Consider, for example, $f(x) = |x|$ and $c = 0$.

9. Using the theorem on derivatives of inverse function, compute the derivative of

   (i) $cos^{-1}x$, $-1 < x < 1$

   Solution. Let $f(x) = cos(x)$. Then $f'(x) = -\sin(x) \neq 0$ for $x \in (0, \pi)$.

Thus $g(y) = f^{-1}(y) = cos^{-1}(y)$, $-1 < y < 1$ is differentiable and

$$g'(y) = \frac{1}{f'(x)} \text{ where } x \text{ is such that } f(x) = y.$$

Therefore,

$$g'(y) = \frac{-1}{\sin(x)} = \frac{-1}{\sqrt{1 - cos^2(x)}} = \frac{-1}{\sqrt{1 - y^2}}.$$

(ii) $cosec^{-1}x$, $|x| > 1$

Solution. Note that

$$cosec^{-1}(x) = \sin^{-1}\frac{1}{x} \text{ for } |x| > 1$$

Since

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}} \text{ for } |x| < 1,$$

one has, by the chain rule

$$\frac{d}{dx}cosec^{-1}(x) = \frac{1}{\sqrt{1 - \frac{1}{x^2}}}\left(\frac{-1}{x^2}\right), |x| > 1.$$

10. Compute $\frac{dy}{dx}$, given

$$y = f\left(\frac{2x - 1}{x + 1}\right) \text{ and } f'(x) = \sin(x^2)$$

Solution. By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= f'\left(\frac{2x - 1}{x + 1}\right)\frac{d}{dx}\left(\frac{2x - 1}{x + 1}\right) \\ &= \sin\left(\frac{2x - 1}{x + 1}\right)^2\left[\frac{3}{(x + 1)^2}\right] \\ &= \frac{3}{(x + 1)^2}\sin\left(\frac{2x - 1}{x + 1}\right)^2\end{aligned}$$

11. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous every where and is differentiable everywhere except at 2 points.

Solution. Consider $f(x) := |x| + |1 - x|$ for $x \in \mathbb{R}$.

12. Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Show that f is discontinuous at every $c \in \mathbb{R}$.

Solution. For $c \in \mathbb{R}$, select a sequence $\{a_n\}_{n\geq 1}$ of rational numbers and a sequence $\{b_n\}_{n\geq 1}$ of irrational numbers, both converging to c. Then $\{f(a_n)\}_{n\geq 1}$ converges to 1 while $\{f(b_n)\}_{n\geq 1}$ converges to 0, showing that limit of f at c does not exist.

15. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent:

(i) f is differentiable at c

(ii) There exists $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and

$$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta)$$

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \left(\frac{|f(c + h) - f(c) - \alpha h|}{|h|} \right) = 0$$

Solution. To prove equivalence, we need to prove (i) $\iff$ (ii) $\iff$ (iii). We can prove it in a cyclic manner, as (i) $\implies$ (ii), (ii) $\implies$ (iii) and (iii) $\implies$ (i).

(i) $\implies$ (ii): Choose $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$. Take $\alpha = f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \frac{f(c+h) - f(c) - \alpha h}{h} & h \neq 0 \\ 0 & h = 0 \end{cases}$$

(ii) $\implies$ (iii): $\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h \to 0} |\epsilon_1(h)| = 0$

(iii) $\implies$ (i): $\lim_{h \to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \implies \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists and is equal to α.

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