MA-111 Calculus II (D3 & D4)

Revision

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Conservative vector field

- If $f : D \to \mathbb{R}$ is a differentiable function the for any smooth path $\mathbf{c} : [a, b] \to D \subset \mathbb{R}^n$

$$\int_{\mathbf{c}} \nabla f . \mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

- Suppose the vector field $\mathbf{F}$ is a continuous conservative field in D, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function f. Then

$$\int_{\mathbf{c}} \mathbf{F} . \mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

- Equivalently, the line integral of $\mathbf{F}$ is independent of path in D , i.e. for any closed curve $\mathbf{c}$ in D

$$\int_{\mathbf{c}} \mathbf{F} . \mathbf{ds} = 0.$$

What about the converse?

Conservative vector field contd.

Theorem: Let $\mathbf{F} : D \to \mathbb{R}^3$ be a continuous vector field on a connected open region D in $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field in D if and only if the line integral of $\mathbf{F}$ is independent of path in D.

Theorem (Necessary condition)

- *For $n = 2$, if $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ is a conservative vector field, where F_1 and F_2 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^2$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D.$$

- *For $n = 3$, if $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ is a conservative vector field, where F_1, F_2, F_3 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^3$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D.$$

This can be proved using the fact that $\nabla \times F = 0$.

N-S condition

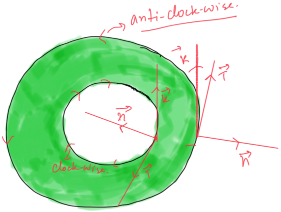
Theorem

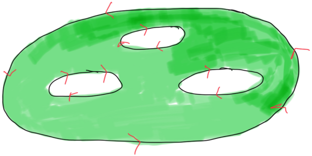
Let $n = 2, 3$ and let D be an open, simply connected region in $\mathbb{R}^n$.

1. *For $n = 2$, let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be such that F_1 and F_2 have continuous first order partial derivatives on D. $\mathbf{F}$ is a conservative field if and only if*
$$\nabla \times \mathbf{F} = 0 \Leftrightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D.$$
2. *For $n = 3$, let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be such that F_1, F_2 and F_3 have continuous first order partial derivatives on D. Then $\mathbf{F}$ is a conservative field if and only if*
$$\nabla \times \mathbf{F} = 0 \Leftrightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D.$$

Orientation of a boundary of region

- A simple closed curve in $\mathbb{R}^2$ encloses a region.
- A simple closed curve in $\mathbb{R}^2$ is positively oriented if the region bounded by the curve always lies to the left of an observer walking along the curve in the chosen direction. otherwise, we say that the curve is negatively oriented.





Fundamental theorem for double integral

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\Big(D \cup \partial D\Big) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

- Curl form or Tangential form

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int \int_D (\text{curl}\mathbf{F}).\mathbf{k} dxdy.$$

- Divergence form or Normal form:

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int \int_D \text{div}\mathbf{F} dxdy.$$

Area of a region

If C is a positively oriented curve that bounds a region D and $A(D)$ is the area of the region D, then

- $A(D) = \frac{1}{2}\int_C xdy - ydx$.
- $A(D) = \int_C x\,dy$.
- $A(D) = \int_C y\,dx$

If the boundary curve is given in the polar co-ordinate, i.e. $C(t) = (r(t), \theta(t))$ for all $t \in [a, b]$, then

$$A(D) = 1/2 \int_C r^2 \, d\theta.$$

Surface

Definition

Let D be a path connected subset in $\mathbb{R}^2$. A parametrised surface is a continuous function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

- The image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.
- $\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v))$ is a smooth parametrized surface if the functions x, y, z have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing D.
- $\mathbf{\Phi}$ is a non-singular parametrized surface if $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0) \neq 0$ for all $(u_0, v_0) \in D$.
- A non-singular and one-to-one parametrization $\mathbf{\Phi} : D \to S$ helps us to define double integration of a surface.
- $\text{Area}(\mathbf{\Phi}) := \int_S \, dS = \iint_D \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u, v)\| \, du \, dv$
- $\iint_S dS = \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv$

Surface integral of a scalar and vector field

- If $f : S \to \mathbb{R}$ is a bounded and continuous function then

$$\iint_S f \; dS = \iint_D f(\mathbf{\Phi}(u,v)) \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u,v)\| \, du \, dv$$
$$= \iint_D f(x,y,z) \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv$$

- If $\mathbf{F}$ is a bounded and continuous vector field on an open set containing S then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_D \mathbf{F}(\mathbf{\Phi}(u,v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) \; dudv.$$

- In compact form

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \; dS.$$

Independence of parametrisation

- An orientable surface together with a specific choice of continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.
- A smooth non-singular parametrisation $\mathbf{\Phi}$ of S gives a natural vector field of unit normal vectors:

$$\hat{\mathbf{n}} = \frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}.$$

  If the unit normal vector $\hat{\mathbf{n}}$ agrees with the given orientation of S we say that the parametrisation $\mathbf{\Phi}$ is orientation preserving.
- Let S be an oriented surface. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two $\mathcal{C}^1$ non-singular parametrisations of S and let $\mathbf{F}$ be a continuous vector field on S.
- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

Orientation of the boundary of a surface

Suppose S is an oriented surface and let $\mathbf{n}(P)$ be the prescribed unit normal vector at a point $P \in S$. We choose the induced orientation of ∂S such that the surface lies to the left of an observer walking along the boundary ∂S with his head in the direction $\mathbf{n}(P)$.

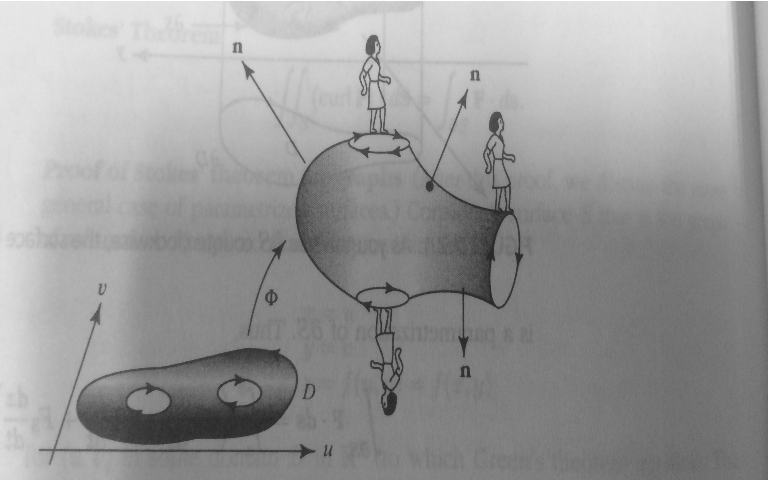


FIGURE 7.2.2. The boundary ∂S of D is oriented so that if you walk along ∂S with $\mathbf{n}$

Stokes theorem

Theorem

1. *Let S be a bounded piecewise smooth oriented surface with non-empty boundary ∂S.*
2. *Let ∂S, the boundary of S, be the disjoint union of simple closed curves each of which is a piecewise non-singular parametrized curve with the induced orientation.*
3. *Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a C^1 vector field defined on an open set containing S.*

Then

$$\int_{\partial S} \mathbf{F}.d\mathbf{s} = \int\int_S (\nabla \times \mathbf{F}).d\mathbf{S}.$$

Consequence of Stokes theorem

- If two different oriented surfaces S_1 and S_2 have the same boundary C, then it follows from Stokes theorem that

$$\int\int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

  where the surfaces S_1 and S_2 are oriented in such a way that the common boundary gets the same orientation.

Gauss's divergence theorem

- A closed surface S in $\mathbb{R}^3$ to be a surface which is bounded, whose complement is open and boundary of S is empty.
- A closed surface encloses a region.

Theorem (Gauss's Divergence Theorem)

1. *Let $S = \partial W$ be a closed oriented surface enclosing the region W with the outward normal giving the positive orientation.*
2. *Let F be a smooth vector field defined on on open set containing $W \cup \partial W$.*

Then

$$\iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_{W} (div\boldsymbol{F}) dxdydz.$$

Consequences of Gauss' theorem

Theorem

Let W be as in Gauss' theorem. Let $\mathbf{F}$ be a smooth vector field on an open set in $\mathbb{R}^3$ containing $W \cup \partial W$ satisfying $div\,\mathbf{F} = 0$ on W. Then
$\int\int_{\partial W} \mathbf{F}.\mathbf{dS} = 0.$

Using the above theorem, we have following result:

Corollary

Let S and W be as in Gauss' theorem. Suppose $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \partial S_1 = \partial S_2$ and S_1 and S_2 have the induced orientation from S. If $\mathbf{F}$ be a vector field defined on an open set containing $W \cup \partial W$ with $div\,\mathbf{F} = 0$, then

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{dS} = \iint_{-S_2} \mathbf{F} \cdot \mathbf{dS}$$

where $-S_2$ denotes the surface S_2 with opposite orientation.