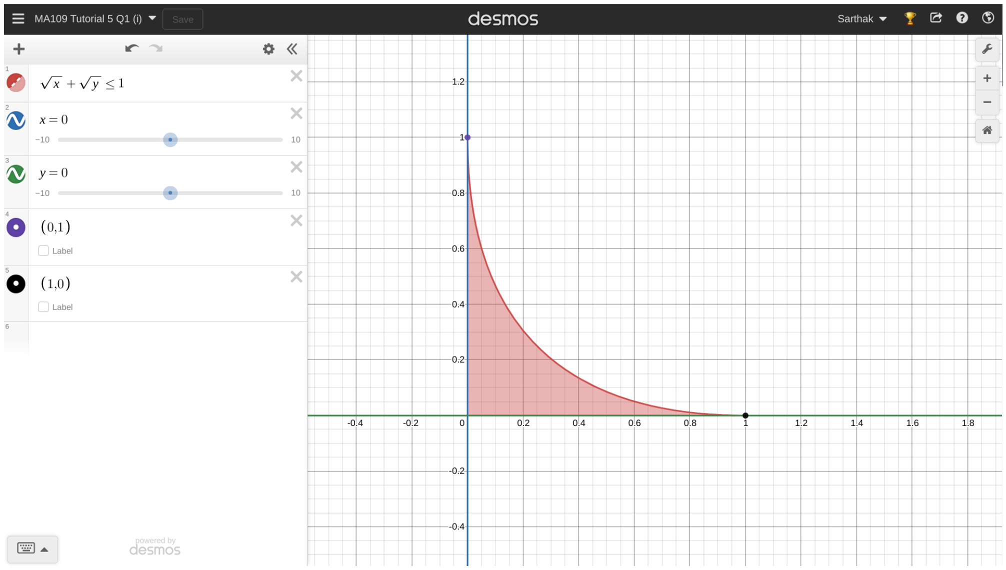
Solutions to Tutorial Sheet 5

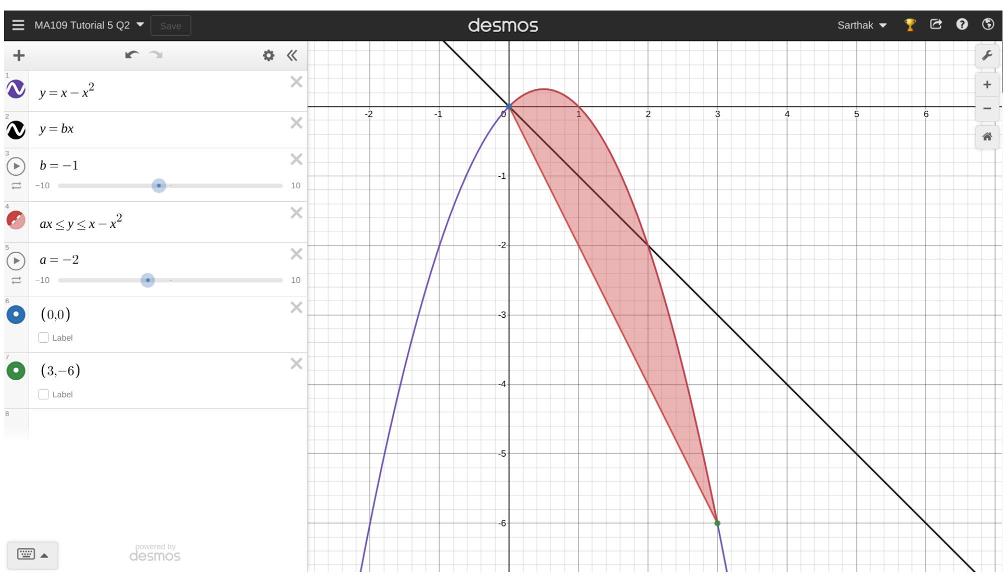
1. Find the area of the region bounded by the given curves in each of the following cases:

   (i) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$ and $y = 0$.



Solution. $\displaystyle\int_0^1 y\,dx = \int_0^1 \left(1 + x - 2\sqrt{x}\right) dx = 1 + \frac{1}{2} - 2 \times \frac{2}{3} = \frac{1}{6}.$

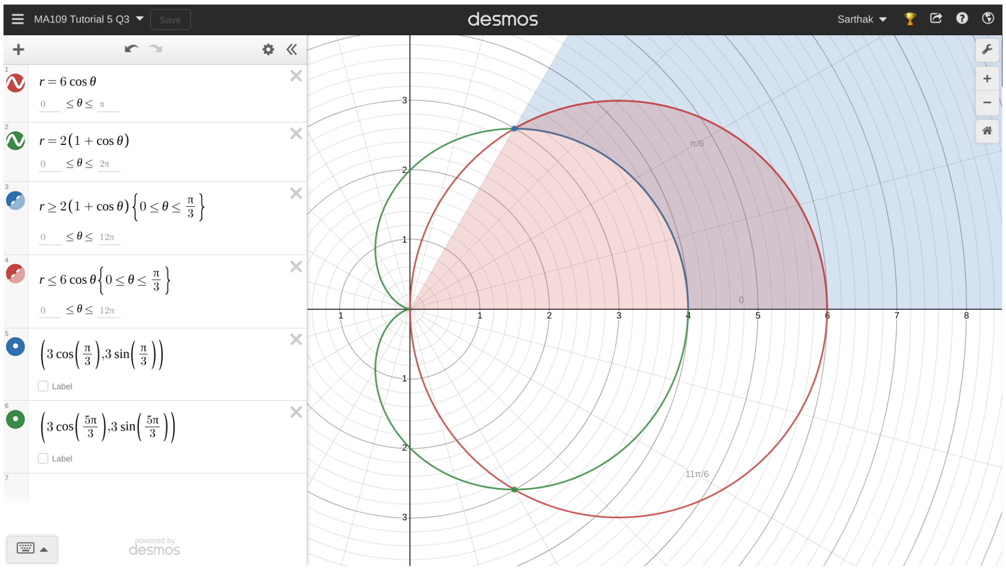
2. Let $f(x) = x - x^2$ and $g(x) = ax$. Determine a so that the region above the graph of g and below the graph of f has area $\dfrac{9}{2}$.



Solution. $\left|\int_0^{1-a} (x - x^2 - ax)\,dx\right| = \left|\int_0^{1-a} ((1-a)x - x^2)\,dx\right| = \frac{9}{2} \implies \left|\frac{(1-a)^3}{6}\right| = \frac{9}{2}$

$$a = -2, 4.$$

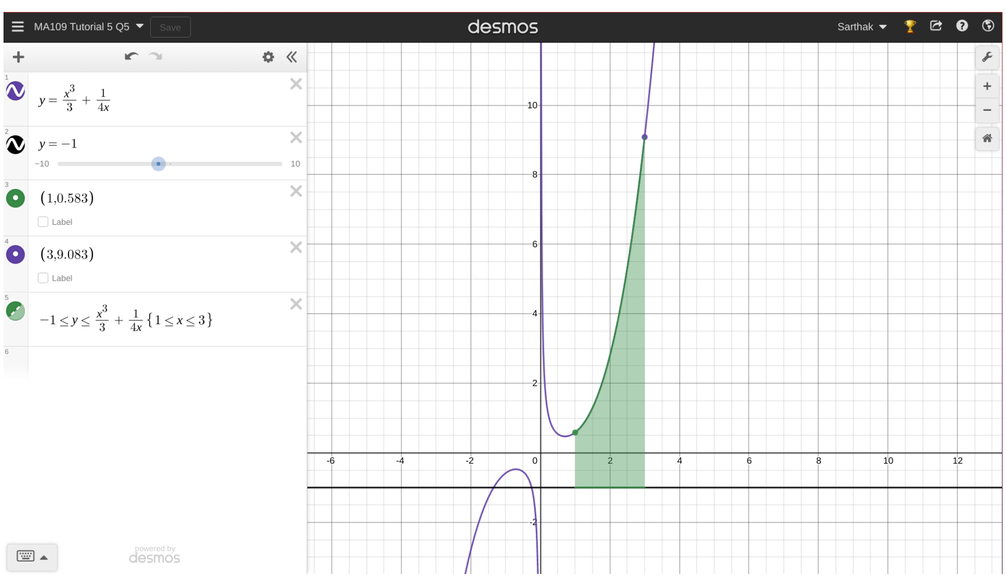
3. Find the area of the region inside the circle $r = 6a\cos\theta$ and outside the cardioid $r = 2a(1+\cos\theta)$.



Solution. Required area $= 2 \times \int_0^{\pi/3} \frac{1}{2}(r_2^2 - r_1^2)d\theta = 4a^2 \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)d\theta = 4\pi a^2.$

5. For the following curve, find the arc length as well as the area of the surface generated by revolving it about the line $y = -1$:

$$y = \frac{x^3}{3} + \frac{1}{4x},\ 1 \le x \le 3$$



Solution. $\dfrac{dy}{dx} = x^2 + \left(-\dfrac{1}{4x^2}\right) \implies \dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \dfrac{1}{16x^4} - \dfrac{1}{2}} = x^2 + \dfrac{1}{4x^2}.$

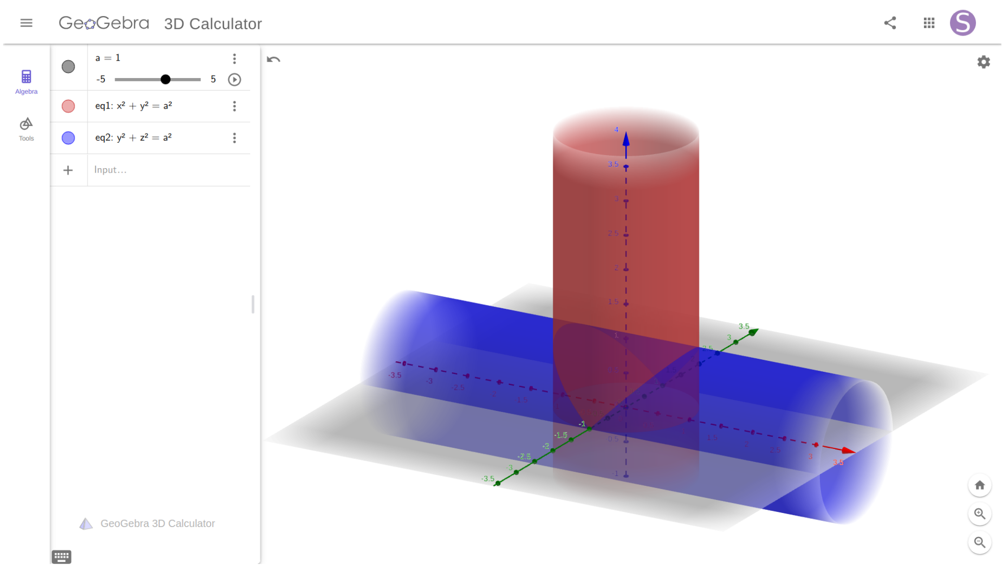
Therefore, the arc length is given by,

$$\int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \frac{53}{6}.$$

The surface area is,

$$\begin{aligned} S &= \int_1^3 2\pi(y+1)\frac{ds}{dx}dx = \int_1^3 2\pi\left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\left(x^2 + \frac{1}{4x^2}\right) dx \\ &= 2\pi\left[\frac{x^6}{18} + \frac{x^3}{3} + \frac{x^2}{6} - \frac{1}{32x^2} - \frac{1}{4x}\right]_1^3 = \frac{1823}{18}\pi \end{aligned}$$

7. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.



Solution. In the first octant, the sections perpendicular to the y-axis are squares with

$$0 \le x \le \sqrt{a^2 - y^2},\ 0 \le z \le \sqrt{a^2 - y^2},\ 0 \le y \le a.$$

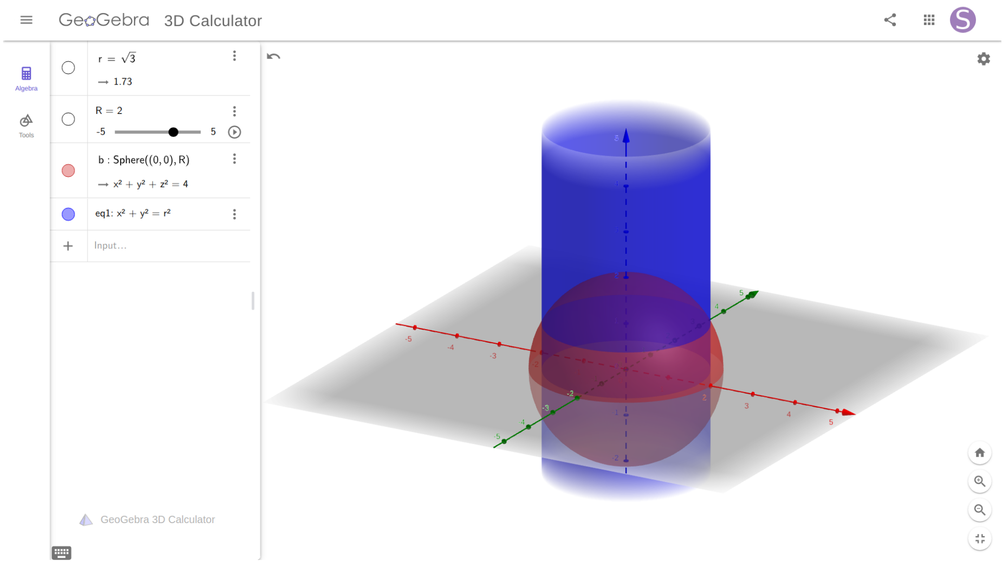
Since the squares have sides of length $\sqrt{a^2 - y^2}$, the area of the cross-section at y is $A(y) = 4(a^2 - y^2)$. Thus the required volume is

$$\int_{-a}^{a} A(y)dy = 8\int_0^a (a^2 - y^2)dy = \frac{16a^3}{3}.$$

8. A fixed line L in 3-space and a square of side r in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion.

Solution. Let the line be along z-axis, $0 \le z \le h$. For any fixed z, the section is a square of area r^2. Hence the required volume is $\int_0^h r^2 dz = r^2 h$.

10. A round hole of radius $r = \sqrt{3}$ cm is bored through the center of a solid ball of radius $R = 2$ cm. Find the volume cut out.



Solution. Required volume = Volume of the sphere − Volume generated by revolving the shaded region around the y-axis.

Washer Method: Integrating x as a function of y (using horizontal solid circular washers)

$$\frac{32}{3}\pi - \left[\int_{-1}^{1} \pi x^2 dy - 2 \times \pi \left(\sqrt{3}\right)^2\right] = \frac{32}{3}\pi - 2\pi\left[\int_0^1 (4-y^2)dy - 3\right] = \frac{32}{3}\pi - 2\pi\left[\frac{11}{3} - 3\right] = \frac{28}{3}\pi.$$

Shell Method: Integrating y as a function of x (using vertical hollow cylindrical shells)

$$\frac{32}{3}\pi - \int_{\sqrt{3}}^{2} 2\pi x \times 2y dx = \frac{32}{3}\pi - 4\pi \int_{\sqrt{3}}^{2} x\sqrt{4-x^2}dx = \frac{32}{3}\pi - 4\pi\frac{1}{3} = \frac{28}{3}\pi.$$

Credits: MA105 Tutorial Solutions (2014)