MA-111 Calculus II
(D3 & D4)

Revision

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Revision of Quiz syllabus

Double Integrals

Assume $f : R \to \mathbb{R}^2$ is a bounded function on a closed bounded rectangle R.

- Recall a partition P of R and its norm.
- Let $U(f, p)$ and $L(f, P)$ be the upper double sum and Lower double sum of f with respect to partition P.
- Then f is integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon.$$

- The Darboux integrability and Riemann integrability of f on R are equivalent.
- The function $f : R \to \mathbb{R}$ is called *integrable* on R if (Darboux or) Riemann integrability condition holds on R.
- If f is integrable on R, then

$$\int \int_R f(x, y)\, dxdy := S = L(f) = U(f),$$

where $U(f)$ and $L(f)$ are upper Darboux integral and lower Darboux integral. And S is the limit of Riemann sum $R(f, P, t)$ for any tagged partition (P, t) satisfying $\|P\| \to 0$.

Properties

- The constant function, the projection functions, are integrable on any rectangle $R \subset \mathbb{R}^2$.

- Geometric interpretation: If $f \geq 0$ on f is integrable on R, then the double integral of f on R is the volume of the solid that lies above R in the x-y plane and below the graph of the surface $z = f(x, y)$ for all $(x, y) \in R$.
- In particular, if $f \equiv 1$, constant function on R, then $\mathrm{Area}(R) = \int\int_R 1 dxdy$.

- Domain additivity:Let R be a rectangle and $f : R \to \mathbb{R}$ be a bounded function. Partition R into finitely many (non-overlapping) subrectangles. Then f is integrable on R if and only if it is integrable on each subrectangle. When it exists, the integral of f on R is the sum of the integrals of f on the subrectangles.

Algebraic Properties

Let f and g be both integrable function on R.

- Sum of integrable functions, scalar multiples of integral functions are integrable.
- Note, $|f|$ is integrable and $|\int\int_R f| \leq \int\int_R |f|$.
- The function $f.g$ is integrable.
- If $\frac{1}{f}$ is well defined and bounded on R, then $\frac{1}{f}$ is integrable on R.
- An immediate consequence is that all polynomial functions are integrable.
- If $f(x,y) \leq g(x,y)$ for all $(x,y) \in R$, then $\int\int_R f \leq \int\int_R g$.

Conditions for integrability

- Let R be a rectangle. If $f : R \to \mathbb{R}^2$ is a bounded function, monotonic in each of two variables, then f is integrable on R.
- If a function $f : R \to \mathbb{R}$ is bounded and continuous on R except possibly finitely many points in R, then f is integrable on R.
- If a function f is bounded and continuous on a rectangle $R = [a, b] \times [c, d]$ except possibly along a finite number of graphs of continuous functions, then f is integrable on R.
- A slightly more general theorem says that : Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the function is integrable if the points of discontinuity of f is a set of content zero.
- A bounded subset E of $\mathbb{R}^2$ is said to be of content zero if for every $\epsilon > 0$, there are finitely many rectangles whose union contains E and the sum of whose areas is less than ϵ.

Evaluation of integrals

Theorem (Fubini's Theorem on Rectangles)

Let $R := [a, b] \times [c, d]$ and $f : R \to \mathbb{R}$ be integrable. Let I denote the integral of f on R.

- *If for each $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists, then the iterated integral $\int_a^b (\int_c^d f(x, y)\, dy)\, dx$ exists and is equal to I.*
- *If for each $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)\, dx$ exists, then the iterated integral $\int_c^d (\int_a^b f(x, y)\, dx)\, dy$ exists and is equal to I.*

As a consequence, if f is integrable on R and if both iterated integrals exist in the above theorem, then

$$\int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx = I = \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

In particular, if f is continuous on R, then f is integrable on R and both iterated integrals exist and

$$\int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx = I = \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

Integrating on general bounded regions

Let D be any bounded region in $\mathbb{R}^2$. Extend f from D to R by defining

$$f^*(x,y) := \begin{cases} f(x,y), & (x,y) \in D, \\ 0, & (x,y) \notin D. \end{cases}.$$

- The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be integrable on bounded $D \subset \mathbb{R}^2$, if f^* is integrable on R and the integral of f on D is defined by

$$\int\int_D f(x,y)\,dx\,dy := \int\int_R f^*(x,y)\,dx\,dy.$$

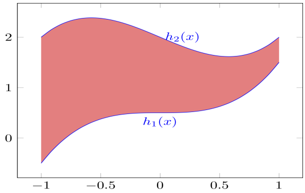
- The value of the integral of f on D does not depend on the choice of the rectangle R containing D.
- The algebraic properties for integrals on any bounded set in $\mathbb{R}^2$ hold similarly to those of the case of integrals on rectangle.
- If $f \geq 0$ on $D \subset \mathbb{R}^2$ and f is integrable on D, then the double integral of f on D is the volume of the solid that lies above D in the x-y plane and below the graph of the surface $z = f(x,y)$ for all $(x,y) \in D$. Also $\mathrm{Area}(D) = \int\int_D 1dxdy$.

- A *path* γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) for will mean a continuous function from $\gamma : [a, b] \to \mathbb{R}^2$ (or $\gamma : [a, b] \to \mathbb{R}^3$) for $a, b \in \mathbb{R}$. It is said to be *closed* if $\gamma(a) = \gamma(b)$.
- By a *curve* γ we mean the image of a path γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$).
- If $D \subset \mathbb{R}^2$ is a bounded set whose boundary ∂D is given by the *continuous closed curve* then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable on D.
- Let $D \subseteq \mathbb{R}^2$ be a bounded set. Let $D_1, D_2 \subseteq D$ such that $D = D_1 \cup D_2$. Let $f : D \to \mathbb{R}^2$ be a bounded function. If f is integrable over D_1 and D_2 and $D_1 \cap D_2$ has content zero then f is integrable on D and

$$\int \int_D f = \int \int_{D_1} f + \int \int_{D_2} f.$$

Evaluating integrals over bounded regions

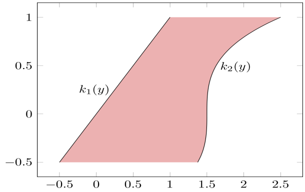
- There are simple types of regions known as *elementary regions* for which ∂D has content zero and the integral can be evaluated easily.
- Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions such that $h_1 \leq h_2$. Consider the set of points $D_1 = \{(x, y) \,|\, a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$. Such a region is said to be of *Type 1* and for every $x \in \mathbb{R}$ vertical cross-section of D_1 is an interval.



- Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions such that $k_1 \leq k_2$. The set of points

$$D_2 = \{(x, y) \,|\, c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$

is called a region of *Type 2* and for every $y \in \mathbb{R}$ horizontal cross-section of D_2 is an interval.



• Any continuous function defined on D_1 or D_2 is integrable over the elementary region.

• If $f : D \to \mathbb{R}$ is bounded, continuous and D is a Type 1 region then,

$$\int_{\alpha}^{\beta} \left[\int_{h_1(x)}^{h_2(x)} f^*(x, y) dy \right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x, y) dy \right] dx,$$

• If $f : D \to \mathbb{R}$ is bounded ,continuous and D is a Type 2 region then,

$$\int \int_D f(x, y) dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x, y) dx \right] dy.$$

Change of variables

• If f is continuous then often we can change into polar coordinates to solve the problem

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)rdrd\theta,$$

where D is the image of the region D^*.

•Let D be a closed and bounded subset of $\mathbb{R}^2$ such that ∂D has content zero. Let $f : D \to \mathbb{R}$ be continuous.

• Suppose Ω is an open subset of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}^2$ is a one-one differentiable function such that $h := (h_1, h_2)$, where h_1 and h_2 have continuous partial derivatives in Ω and $\det(J(h)(u,v)) \neq 0$ for all $(u,v) \in \Omega$.

• Let $D^* \subset \Omega$ be such that $h(D^*) = D$.

Conclusion: Then D^* is a closed and bounded subset of Ω, and ∂D^* is content zero. Moreover, $f \circ h : D^* \to \mathbb{R}$ is continuous, and

$$\int\int_D f(x,y)\ dxdy = \int\int_{D^*} (f \circ h)(u,v)\left|\det(J(h)(u,v))\right|\ dudv.$$

How to choose the change of variables

- Aim: Find h such that a rectangle D^* in $u - v$ plane is getting mapped to the given area D in the xy plane. If D^* cannot be chosen as a rectangle, choose D^* as an elementary region Type 1 or Type 2.
- The boundary D^* in u-v plane should map to the boundary of D in x-y plane.
- The non-vanishing Jacobian determinant of h assures that the properties of D^* is preserved under the transformation and D has similar properties as of D^*.
- In some cases, h can be chosen in a way such that the expression of the integrand becomes simpler after the change of variables.

Triple integrals

• Let f be a bounded function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$ We say f is integrable if $\lim_{n\to\infty} S(f, P_n, t)$ converges to some fixed $S \in \mathbb{R}$ for any choice of tag t. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

• All the theorems for integrals over rectangles go through for integrals over rectangular cuboids.

• If $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous in B, except possibly on (a finite union of) graphs of $\mathcal{C}^1$ functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable on B.

Evaluating triple integrals: Fubini's Theorem

Fubini's Theorem can be generalized - that is, triple integrals can usually be expressed as iterated integrals, this time by integrating functions of a single variable three times.

Thus, if f integrable on the cuboid B we have

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx.$$

There are, in fact, five other possibilities for the iterated integrals.

We again have a theorem saying if f is integrable whenever any of these iterated integral exists, it is equal to the value of the integral of f over B.

Elementary regions in $\mathbb{R}^3$

The triple integrals that are easiest to evaluate are those for which the region P in space can be described by bounding one variable between between the graphs of two functions in the other two variables with the domain of these functions being an elementary region in two variables.

For example,

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid \alpha(x, y) \leq z \leq \beta(x, y), (x, y) \in D\},$$

where α and β are continuous on $D \subset \mathbb{R}^2$ and D is an elementary region in $\mathbb{R}^2$.

Volume of a bounded region W in $\mathbb{R}^3$: $\mathrm{Volume}(W) = \int\int\int_W 1 dxdydz$.

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where $h(P^*) = P$.

If the change in coordinates is given by $h = (h_1, h_2, h_3) =$ also written as $(x(u,v,w), y(u,v,w), z(u,v,w))$,the function g is defined as $g = f \circ h$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

is just the Jacobian determinant for a function of three variables.

Spherical and Cylindrical coordinates

- If we use (ρ, θ, ϕ) what is the map from these coordinates to the x-y-z-planes?

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

The Jacobian determinant is

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} x = \rho^2 \sin\phi.$$

- In this case, we use the change of transformation from (r, θ, z) coordinates to $P = (x, y, z) \in \mathbb{R}^3$ given by

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

Here $r \geq 0$ and $0 \leq \theta \leq 2\pi$ and the (r, θ, z) are as earlier.

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = r.$$

• Let D be a subset of $\mathbb{R}^n$.

Definition: A scalar field on D is a map $f : D \to \mathbb{R}$.

Definition A vector field on D is a map $\mathbf{F} : D \to \mathbb{R}^n$. We choose $n \geq 2$.

• Examples : $\mathbf{F}_1(x, y) = (2x, 2y)$, $\mathbf{F}_2(x, y) = (\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2})$ • We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give a gradient vector field :

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.
• Examples, $\mathbf{F}_1(x, y) = (2x, 2y) = \nabla(x^2 + y^2)$,
$\mathbf{F}_2(x, y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = \nabla(-\ln\left(\sqrt{x^2 + y^2}\right))$. The field $\mathbf{F}_5(x, y) = (\sin y, \cos x)$, this vector field is not ∇f for any f.

• If $\mathbf{F}$ is a vector field defined from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e., a map $\mathbf{c} : [a, b] \to D$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \quad \forall t \in [a, b].$$

In particular, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}$. • Finding the flow line for a given vector field involves solving a system of differential equations,

Recall a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n$. A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n$. Both the curve and path are denoted by the same symbol $\mathbf{c}$.

• Let $n = 3$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b]$. The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous. Similarly, $\mathbf{c}$ is a C^1 path, i.e., continuously differentiable if and only if each component is C^1.

• A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

• A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.

• If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrised curve.

Line integrals of vector fields

• Assume that the vector field $\mathbf{F}$ is continuous and the curve $\mathbf{c}$ is C^1.

Then we define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

• If $\mathbf{c}_1$ is a path joining two points P_0 and P_1, $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

Here $\mathbf{c}$, the union of two C^1 paths $\mathbf{c}_1$ and $\mathbf{c}_2$ is need not be C^1 but picewise C^1. The line integral of a continuous vector field is defined along piecewise C^1 curves.

• Let the curve $\mathbf{c}$ be a union of curves $\mathbf{c}_1, \ldots, \mathbf{c}_n$. We often write this as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \ldots \mathbf{c}_n$, where end point of $\mathbf{c}_i$ is the starting point of $\mathbf{c}_{i+1}$ for all $i = 1, \ldots, n-1$.

Then we can define

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot ds.$$

• Let $\mathbf{c}$ be a curve on $[a, b]$ and $-\mathbf{c}(t) = \mathbf{c}(b + a - t)$, that is the curve $\mathbf{c}$ traversed in the reverse direction. Then $\int_{\mathbf{c}} \mathbf{F} \cdot ds + \int_{-\mathbf{c}} \mathbf{F} \cdot ds = 0$.
• Let $\mathbf{c}(t) : [t_1, t_2] \to \mathbb{R}^n$ be a path which is non-singular, that is, $\mathbf{c}'(t) \neq 0$ for all $t \in [t_1, t_2]$.

- Suppose we now make change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism (this means that h is bijective, $\mathcal{C}^1$ and so is its inverse) from $[u_1, u_2]$ to $[t_1, t_2]$.
- We let $\gamma(u) = \mathbf{c}(h(u))$. Then γ is called a reparametrization of $\mathbf{c}$. We will assume that $h(u_i) = t_i$ for $i = 1, 2$
- Since a path between P and Q is a mapping $\mathbf{c} : [a, b] \to \mathbb{R}^n$ with $\mathbf{c}(a) = P$ and $\mathbf{c}(b) = Q$, (or vice-versa), it allows us to determine the direction in which the path is traversed. This direction of the path is called its Orientation.
- If the reparametrization $\gamma = \mathbf{c}(h)$ preserves the orientation of $\mathbf{c}$, then $\int_\gamma \mathbf{F}.d\mathbf{s} = \int_\mathbf{c} \mathbf{F}.d\mathbf{s}$.
- If the reparamtrization reverses the orientation, then $\int_\gamma \mathbf{F}.d\mathbf{s} = -\int_\mathbf{c} \mathbf{F}.d\mathbf{s}$.
- Let $f : D \to \mathbb{R}$ be a continuous scalar function and $\mathbf{c} : [a, b] \longrightarrow D$ be a non-singular path. Then the path integral of f along $\mathbf{c}$ is defined by $\int_\mathbf{c} f\, ds := \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt$.