Quiz - 16

Que. Find the Normalisation constant for $\psi(x)$:-

$$\psi(x) = c_0 e^{-\frac{m\omega x^2}{2\hbar}}$$

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx$$

$$1 = \int_{-\infty}^{\infty} c_0^2 e^{-2\alpha x^2}\,dx \qquad \left(\alpha = \frac{m\omega}{2\hbar}\right)$$

$$1 = c_0^2 \left(\frac{\pi}{2\alpha}\right)^{1/2} \qquad \left(\because \int_{-\infty}^{\infty} e^{-kx^2} = \sqrt{\frac{\pi}{k}}\right)$$

$$\boxed{c_0 = \left(\frac{2\alpha}{\pi}\right)^{1/4}}$$

Que. 2 The ground state wave function for harmonic Oscillator is

$$\psi_0(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2} \quad \text{where } \alpha = \frac{m\omega}{2\hbar}$$

show that $\Delta x \Delta p = \frac{\hbar}{2}$

$\Rightarrow$ For $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^*(x)\, x\, \psi(x)\,dx$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} x\, e^{-2\alpha x^2}\,dx$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \frac{1}{4\alpha} e^{-t}\,dt = \left(\frac{2\alpha}{\pi}\right)^{1/2} \frac{1}{4\alpha} \left(\frac{e^{-2\alpha x^2}}{-1}\right)_{-\infty}^{\infty}$$

$$\boxed{= 0}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \left(\frac{2\alpha}{\pi}\right)^{1/2} x^2 e^{-2\alpha x^2}\,dx$$

$$\langle x^2 \rangle = \left(\frac{2\alpha}{\pi}\right)^{1/2} \frac{\left(\frac{\pi}{2\alpha}\right)^{1/2}}{(2\alpha)(2)}$$

$$\boxed{\langle x^2 \rangle = \frac{1}{4\alpha}}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \left(\frac{2\alpha}{\pi}\right)^{1/2} -i\hbar\left(\frac{d}{dx} e^{-\alpha x^2}\right) e^{-\alpha x^2} dx$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2} (+\alpha) i\hbar \int_{-\infty}^{\infty} 2x\, e^{-2\alpha x^2} dx$$

$$= \boxed{0}$$

$$\langle p^2 \rangle = \left(\frac{2\alpha}{\pi}\right)^{1/2} \cancel{(-i\hbar)} \int_{-\infty}^{\infty} -\hbar^2 \frac{d}{dx}\left(-2\alpha x e^{-\alpha x^2}\right) e^{-\alpha x^2} dx$$

$$= \left(\frac{+2\alpha}{\pi}\right)^{1/2} + \hbar^2 2\alpha \int_{-\infty}^{\infty} \left(e^{-\alpha x^2} \cancel{-2\alpha} x^2 e^{-\alpha x^2}\right) e^{-\alpha x^2} dx$$

$$= \sqrt{\frac{2\alpha}{\pi}}\, \hbar^2 (2\alpha) \left[\int_{-\infty}^{\infty} \left(e^{-2\alpha x^2}\right) - 2\alpha \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} dx\right]$$

$$= \sqrt{\frac{2\alpha}{\pi}}\, \hbar^2 (2\alpha) \left[\sqrt{\frac{\pi}{2\alpha}} - \frac{\cancel{2\alpha}\ \pi^{1/2}}{2(2\alpha)^{1/2}\cancel{(2\alpha)}}\right]$$

$$= \sqrt{\frac{2\alpha}{\pi}}\, (2\alpha\hbar^2) \left[\frac{1}{2}\sqrt{\frac{\pi}{2\alpha}}\right]$$

$$\langle p^2 \rangle = \alpha\hbar^2$$

$$\Delta x = \sqrt{\frac{1}{4\alpha} - 0} \qquad \Delta p = \sqrt{\alpha\hbar^2 - 0}$$

$$\boxed{\Delta x = \frac{1}{2\sqrt{\alpha}}} \qquad \Delta p = \sqrt{\alpha}\,\hbar$$

$$\boxed{\Delta x \cdot \Delta p = \frac{\hbar}{2}}$$