Tutorial 1

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Introduction

Hello.

I am Neeraj Jadhav.

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Nice to meet you all.

I will be your TA for the course MA109.

The course is quite interesting where you will be introduced to mathematical rigour and formal methods in mathematics.

You will start to notice how cool it is as the syllabus progresses and in further courses.

Have Fun!

Question 1

1. (iii) $\lim_{n\to\infty} \frac{n^{2/3} sin(n!)}{n+1} = 0$

Let $\epsilon > 0$ be given. We must show that there exists $n_0 \in \mathbb{N}$ such that

$\left| \frac{n^{2/3} sin(n!)}{n+1} - 0 \right| < \epsilon$ for all $n \geq n_0$.

$$\left| \frac{n^{2/3} sin(n!)}{n+1} - 0 \right| < \epsilon \iff \left| \frac{n^{2/3} sin(n!)}{n+1} \right| < \epsilon \Leftarrow \left| \frac{n^{2/3}}{n+1} \right| < \epsilon$$

Note the direction of implication of the red arrow. We have used the fact that $|sinx| < 1$ for all real x.

Question 1

$$\left|\frac{n^{2/3}}{n+1}\right| < \epsilon \Leftarrow \left|\frac{n^{2/3}}{n}\right| < \epsilon \iff \frac{1}{n^{1/3}} < \epsilon \iff \frac{1}{\epsilon^3} < n$$

Thus, we can choose $n_0 = \left\lfloor \frac{1}{\epsilon^3} \right\rfloor + 1$

By our arrows of implication, it can be seen that for $n \geq n_0$, the desired inequality holds.

Question 1

1. (iv) $\lim_{n\to\infty}\left(\frac{n}{n+1}-\frac{n+1}{n}\right)=0$

Let $\epsilon>0$ be given. We must show that there exists $n_0\in\mathbb{N}$ such that $\left|\frac{n}{n+1}-\frac{n+1}{n}-0\right|<\epsilon$ for all $n\geq n_0$.

Observe the following:

$$\left|\frac{n}{n+1}-\frac{n+1}{n}-0\right|=\left|1-\frac{1}{n+1}-1-\frac{1}{n}\right|=\left|-\frac{1}{n+1}-\frac{1}{n}\right|$$

$$=\frac{1}{n+1}+\frac{1}{n}<\frac{2}{n}$$

Thus, if we choose $n_0=\left\lfloor\frac{2}{\epsilon}\right\rfloor+1$, we have it that the desired inequality holds.

Question 2

2. (i) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\cdots+\frac{n}{n^2+n}\right)$

Define $b_n = \left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\cdots+\frac{n}{n^2+n}\right)$.

$a_n = \frac{n^2}{n^2+n} = \frac{n}{n+1}$ and $c_n = \frac{n^2}{n^2+1}$.

We have $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 1$.

Using the Sandwich Theorem (2nd definition), we know b_n converges and

$$\lim_{n\to\infty} b_n = 1$$

Question 2

2. (iv) $\lim_{n\to\infty} (n)^{1/n}$.
Define $h_n := (n)^{1/n} - 1$.
Then, $h_n \geq 0$ for all $n \in \mathbb{N}$.
Now, for $n > 2$, we have

$$
\begin{aligned}
n &= (1 + h_n)^n && (1)\\
&= 1 + nh_n + \binom{n}{2} h_n^2 + \cdots + \binom{n}{n} h_n^n && (2)\\
&\geq 1 + nh_n + \binom{n}{2} h_n^2 && (3)\\
&> \binom{n}{2} h_n^2 && (4)\\
&= \frac{n(n-1)}{2} h_n^2. && (5)
\end{aligned}
$$

Question 2

Thus, $h_n < \sqrt{\dfrac{2}{n-1}}$ for all $n > 2$.
Using Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$ which gives us that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

(Where did we use that $h_n \geq 0$?)

Question 2

2. (v) $\lim_{n\to\infty}\left(\frac{\cos\pi\sqrt{n}}{n^2}\right)$

Define $b_n = \left(\frac{\cos\pi\sqrt{n}}{n^2}\right)$, $a_n = -\frac{1}{n^2}$ and $c_n = \frac{1}{n^2}$.

We have $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$.

Using the Sandwich Theorem (2nd definition), we know b_n converges and

$$\lim_{n\to\infty} b_n = 0$$

Question 2

2. (vi) $\lim_{n\to\infty}(\sqrt{n}(\sqrt{n+1}-\sqrt{n}))$

Define $b_n = \sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, $a_n = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+1}}$ and $c_n = \dfrac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \dfrac{1}{2}$.

We have $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \dfrac{1}{2}$.

Using the Sandwich Theorem (2nd definition), we know b_n converges and

$$\lim_{n\to\infty} b_n = \frac{1}{2}$$

Question 3

3. (i) To show: $\left\{ \dfrac{n^2}{n+1} \right\}_{n \geq 1}$ is not convergent.

We will use the fact that convergent sequences are bounded. We will try to show that the sequence given is not bounded. That would imply that the sequence does not converge. (Why?)

$$\frac{n^2}{n+1} > \frac{n^2 - 1}{n+1} = \frac{(n-1)(n+1)}{n+1} = n - 1$$

Thus, the sequence given is bounded below by $n-1$, but by Archimedean property, we know that n-1 is not bounded above. Thus, our sequence is not bounded (above). As a result, it is not convergent.

Archimedean Property : If $x, y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$.

Question 4

4. (i) $\left\{\dfrac{n}{n^2+1}\right\}_{n\geq 1}$

Take the general term of the above sequence to be a_n and compare a_n with a_{n+1}.

$$\begin{aligned} a_{n+1} - a_n &= \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} \\ &= \frac{(n+1)(n^2+1) - n((n+1)^2+1)}{(n^2+1)((n+1)^2+1)} \\ &= \frac{1 + n(n+1)(n-(n+1))}{(n^2+1)((n+1)^2+1)} \\ &= \frac{1 - n(n+1)}{(n^2+1)((n+1)^2+1)} \leq 0 \end{aligned}$$

Since $a_{n+1} - a_n \leq 0$, the sequence is monotonically decreasing.

Question 4

4. (iii) $\left\{\dfrac{1-n}{n^2}\right\}_{n\geq 2}$

Take the general term of the above sequence to be a_n and compare a_n with a_{n+1}.

$$\begin{aligned}
a_{n+1}-a_n &= \frac{1-(n+1)}{(n+1)^2} - \frac{1-n}{n^2} \\
&= \frac{1}{(n+1)^2} - \frac{1}{n^2} + \frac{1}{n} - \frac{1}{n+1} \\
&= \frac{-2n-1+n(n+1)}{n^2(n+1)^2} \\
&= \frac{n^2-n-1}{n^2(n+1)^2} \\
&= \frac{(n-1)^2-(n-2)}{n^2(n+1)^2} \geq 0 \ \forall n \geq 2.
\end{aligned}$$

Since $a_{n+1}-a_n \geq 0$, the sequence is monotonically increasing.

Question 5

5. (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n} \;\; \forall n \geq 1 \qquad (a_n \geq 0)$

Claim 1 : The sequence is bounded above by 2, i.e. $\forall k \in \mathbb{N}, a_k \leq 2$
Proof :

- Base Case : $a_1 = \sqrt{2} \leq 2$
- Induction : $\forall k \in \mathbb{N}$, if $a_k \leq 2, a_{k+1} \leq 2$
- According to the recurrence relation given above,
  $a_{k+1} = \sqrt{2 + a_k} \leq \sqrt{2 + 2} \leq 2$

Claim 2 : The sequence is monotonically increasing.
Proof : Comparing a_n with a_{n+1}

$$\begin{aligned} a_{n+1}^2 - a_n^2 &= 2 + a_n - a_n^2 \\ &= (2 - a_n)(1 + a_n) \geq 0 \end{aligned}$$

$$(a_{n+1} + a_n)(a_{n+1} - a_n) \geq 0$$

$$a_{n+1} \geq a_n$$

Question 5

Since the sequence is monotonically increasing and bounded above, it converges.
Also, it will converge to its **least upper bound (lub)** i.e. 2 in this case.
Calculations :

$$\begin{aligned}
a_{n+1}^2 - a_n^2 &= (a_{n+1} - a_n)(a_{n+1} + a_n) \\
2 + a_n - a_n^2 &= (a_{n+1} - a_n)(a_{n+1} + a_n) \\
\lim_{n\to\infty} 2 + a_n - a_n^2 &= \lim_{n\to\infty} (a_{n+1} - a_n)(a_{n+1} + a_n) \\
2 + L - L^2 &= (\lim_{n\to\infty} (a_{n+1} - a_n))(\lim_{n\to\infty} (a_{n+1} + a_n)) \\
2 + L - L^2 &= 0 \\
\therefore\ L &= 2 \qquad \text{(Neglecting negative root)}
\end{aligned}$$

Question 7

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \qquad \text{for all } n \geq n_0.$$

Let us choose $\epsilon = \dfrac{|L|}{2}$. (Why is this a valid choice of ϵ ?)
By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq n_0$.

$$\begin{aligned}
&= |a_n - L| < \epsilon && \forall n \geq n_0 \\
&\Rightarrow ||a_n| - |L|| < \epsilon && \forall n \geq n_0 \\
&\Rightarrow -\epsilon < |a_n| - |L| < \epsilon && \forall n \geq n_0 \\
&\Rightarrow |L| - \epsilon < |a_n| && \forall n \geq n_0 \\
&\Rightarrow \frac{|L|}{2} < |a_n| && \forall n \geq n_0
\end{aligned}$$

Question 8

8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.
Let $\epsilon > 0$ be given. This means that $\epsilon^2 > 0$.
By hypothesis, there exists $n_0 \in \mathbb{N}$ such that
$|a_n - 0| = a_n < \epsilon^2 \qquad \forall\ n \geq n_0$. By definition of limit, we have shown that $\lim_{n\to\infty} a_n^{1/2} = 0$.

At what place(s) did we use that $a_n \geq 0$?

Hint for **optional**: Use the inequality $|\sqrt[n]{a} - \sqrt[n]{b}| \leq \sqrt[n]{|a - b|}$ for all $n \in \mathbb{N}$

Question 10

10. To show :
$\{a_n\}_{n\geq 1}$ is convergent $\iff$ $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to same limit.
If $\{a_n\}_{n\geq 1}$ is convergent, then by definition :
i) $\forall \epsilon > 0, \exists N \in \mathbb{N}\ s.t.\ \forall n > N, |a_n - L| < \epsilon$
ii) $\forall \epsilon_1 > 0, \exists N_1 \in \mathbb{N}\ s.t.\ \forall n > N_1, |a_{2n} - L| < \epsilon_1$
iii) $\forall \epsilon_2 > 0, \exists N_2 \in \mathbb{N}\ s.t.\ \forall n > N_2, |a_{2n+1} - L| < \epsilon_2$

Proving the other direction :
let $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to the same limit L.
We have (ii) and (iii),
choose $N_3 = \max\{N_1, N_2\}$, then the following inequalities hold $\forall n \geq N_3$:

- $L - \epsilon_1 < a_{2n} < L + \epsilon_1$
- $L - \epsilon_2 < a_{2n+1} < L + \epsilon_2$

So, $|a_{2n+1} - a_{2n}| < \epsilon_1 + \epsilon_2 = \epsilon_3$

Question 10

Recall the definition of a Cauchy sequence :
A sequence b_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|b_n - b_m| < \epsilon$$

for all $m, n > N$.

a_n in our example is a Cauchy sequence since $\forall \epsilon_3 > 0$ all combinations of m, n from odd/even numbers, satisfy the required condition.
Furthermore, every Cauchy sequence in $\mathbb{R}$ converges, so $\{a_n\}_{n \geq 1}$ is convergent.