PH 107: Quantum Physics and applications

Heisenberg Uncertainity Principles

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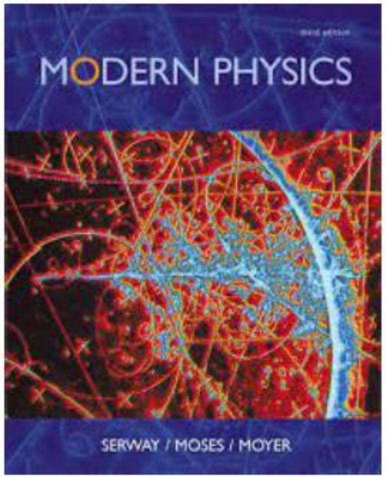
Lecture06: 28-12-2021

Learning Objectives

- HUP and its applications.

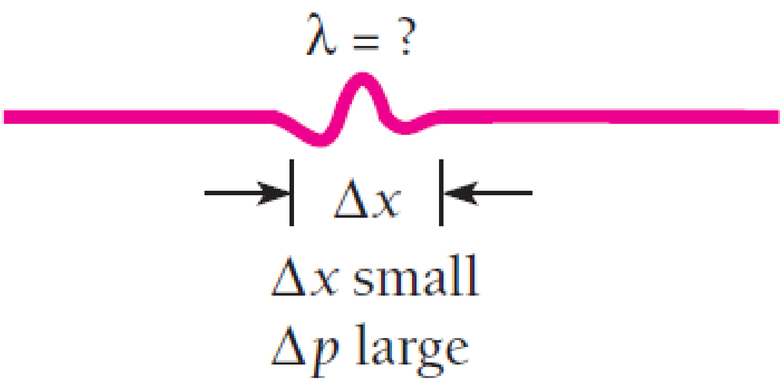
Heisenberg Uncertainty Principle, section 5.5.

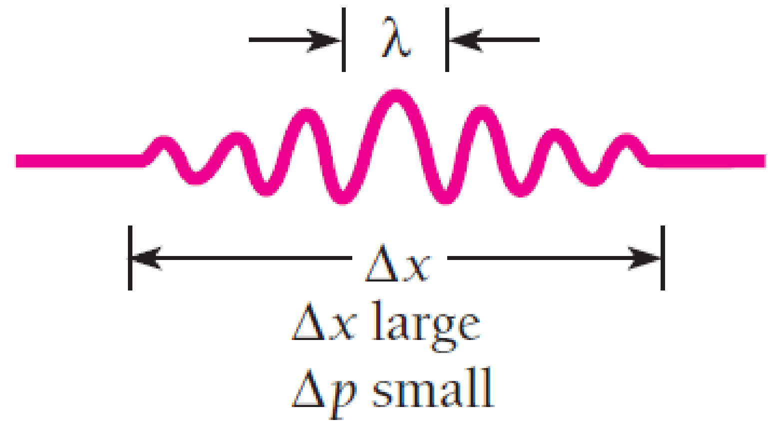


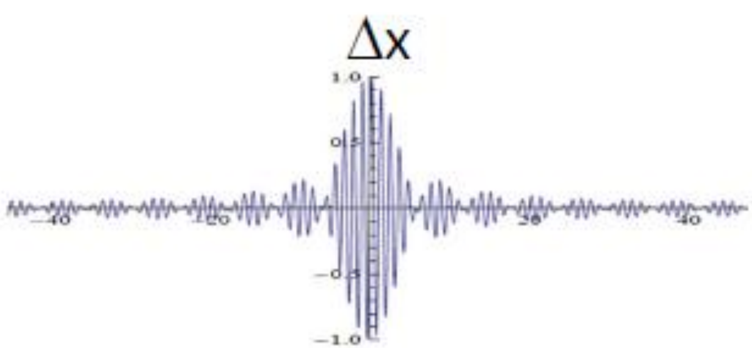
Doubt clearing session every Saturday @ 11am.

The Heisenberg Uncertainty Principle (HUP)

In 1927 Heisenberg introduced the notion that it is impossible to determine simultaneously with unlimited precision the position and momentum of a particle.







$$\Delta p \Delta x \ are \ reciprocally \ realted$$

Δx : *uncertainity in position*

Δp *uncertainity in momentum*

The statement of HUP and the Heisenberg Microscope

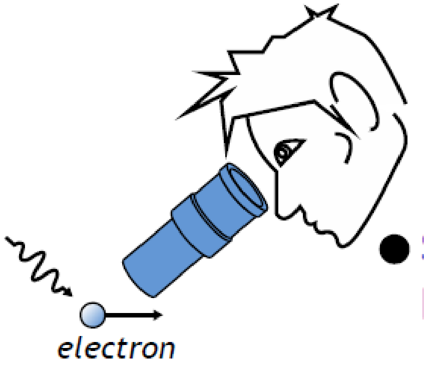
In Quantum regime, it is impossible to carry out an experiment that allows the position and momentum of a particle to be simultaneously measured, with an accuracy that violates the uncertainty principle.

The uncertainties do not arise from limitations of practical measuring instruments; rather from the need to use a **large range of wavenumbers** to represent a **matter wave packet** localized in a small region.

This is illustrated by the **Heisenberg Microscope**

The Heisenberg Microscope

A (thought) apparatus to "view" electrons.



- Shine light on electron and detect reflected light using microscope.

- Minimum uncertainity in position is given by wavelength of light.

- By the relation $E = hc/\lambda$, a photon of short wavelength has large energy. Thus it would impart a large 'kick' to the electron to make momentum uncertain.

- But to determine its momentum accurately, electron must only be given a small kick

- This means using light of long wavelength, which will make position uncertain!

The Heisenberg Microscope

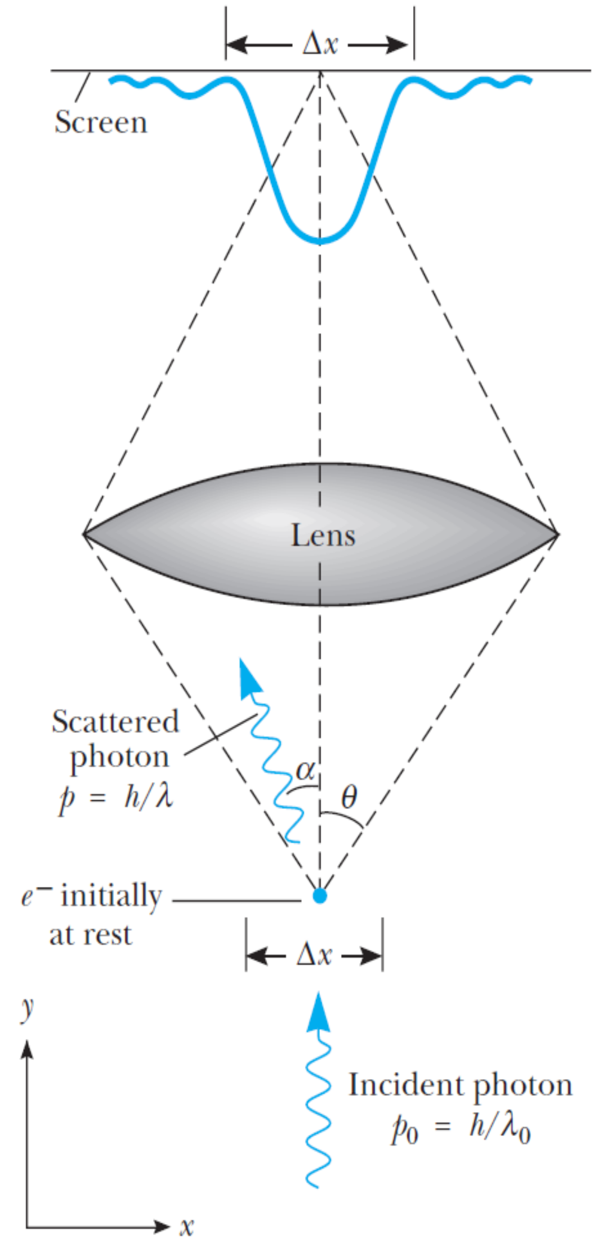
Measurement process disturbs the object being observed. The state of the object changes as a result of the measurement.

Here, the electron is initially assumed to be at rest. Due to the photon scattering process, it recoils.

If the scattered photon is to be collected by the lens, it has to be scattered within at an angular range (-θ, +θ).

The scattering imparts a x-momentum to the electron which is in the range

$$\left(\frac{h}{\lambda}Sin\theta, -\frac{h}{\lambda}Sin\theta\right)$$



The Heisenberg Microscope

Momentum imparted to electron is in the range

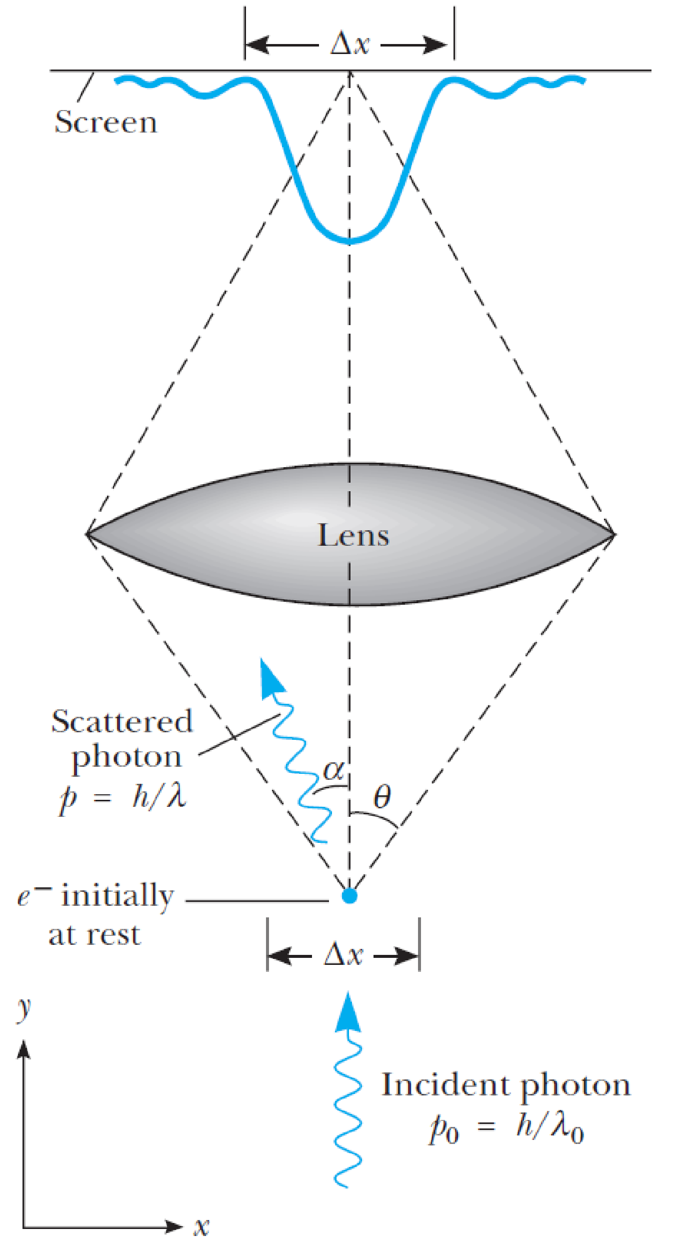
$$\Delta p_x = \frac{2h}{\lambda} Sin\theta$$

The uncertainty in position of the electron

$$\Delta x = \frac{Wavelength}{2\ Numerical\ Apperture} = \frac{\lambda}{2\ Sin\theta}$$

Thus, $$\Delta x \Delta p = \frac{\lambda}{2\ Sin\theta} \text{ X } \frac{2h}{\lambda} Sin\theta$$
$$= h$$

$$\boldsymbol{\Delta x \Delta p \geq \frac{\hbar}{2}}$$



Summary : The Heisenberg Uncertainty Principle

Any measurement to determine p_x doesnot introduce an uncertainity in "y" or "z".

$$\Delta \boldsymbol{x} \Delta p_x \geq \frac{\hbar}{\mathbf{2}}$$

$$\Delta \boldsymbol{y} \Delta p_y \geq \frac{\hbar}{\mathbf{2}}$$

$$\Delta \boldsymbol{z} \Delta p_z \geq \frac{\hbar}{\mathbf{2}}$$

$$\Delta \boldsymbol{E} \Delta t \geq \frac{\hbar}{\mathbf{2}}$$

HUP :

Position-Momentum Uncertainty relation

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

HUP : What does the inequality mean ?

$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

It sets the intrinsic lowest possible limits on the uncertainties in knowing the values of p_x and x, no matter how good an experiments is made.

These uncertainties are inherent in the physical world and have nothing to do with the skill of the observer.

It is impossible to specify simultaneously and with infinite precision the linear momentum and the corresponding position of a particle.

HUP : What does the inequality mean ?

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

If a system is known to exist in a state of energy ***E*** over a limited period Δt, then this energy is uncertain by at least an amount $\frac{\hbar}{2\Delta t}$

Therefore, the energy of an object or system can be measured with infinite precision (ΔE = 0) only if the object of system exists for an infinite time (Δt→∞).

Example : HUP (quantum particle vs macroscopic object)

An electron

m = 9.11 x 10^{-31} kg

v = 40 m/s

p = mv = 3.6 x 10^{-29} kg m/s

Suppose the momentum is measured to an accuracy of 1 percent,

Δp = 0.01 p = 3.6 x 10^{-31} kg m/s

The uncertainty in position is

$$\Delta x \geq \frac{h}{4\pi\Delta p} = 1.4 \times 10^{-4}\mathrm{m}$$

Electron cannot exist within the nucleus

A typical atomic nucleus is ~ $5\text{x}10^{-15}$ m in radius.

Therefore, if an electron resided in the nucleus, its uncertainty in position would be,

$$\Delta x \sim 5\text{x}10{-}15\ \text{m}$$

Therefore , $\Delta p \ \geq \hbar/(2\,\Delta x) \geq 1.1\times 10^{-20}\text{kg. m/s}$

If this is the uncertainty in the momentum, the momentum itself must be at least comparable to this value.

$$E^2 = p^2c^2 + (m_e c^2)^2 = (20\text{MeV})^2 + (0.511\,\text{MeV})^2 \approx (20\,\text{MeV})^2$$

Therefore, in order to reside in the nucleus, the electron must have, $KE \sim \mathrm{E} - \mathrm{mc}^2 = 19.5\text{MeV}$

Experiments show that electrons emitted by certain unstable nuclei never have more than a small fraction of this energy.

Electrons emitted in beta decay have energies ~ 1 MeV or less!

Estimating size of an atom using HUP

Let an electron be confined in a spherical shell of radius a

Uncertainty in position: $\Delta x \approx a$

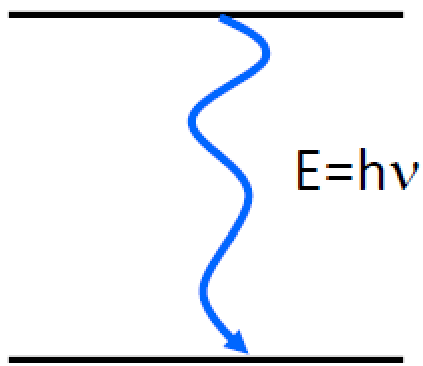
Uncertainty in momentum: $\Delta p \approx \hbar / a$

$$p \sim \Delta p = \hbar / a$$

A charged π meson has rest energy of 140 MeV and a lifetime of 26 ns. What is the energy uncertainty?

Radiative decay of atomic levels

Hydrogen Atom



n=2 Lifetime 0.625 ns

Photon energy = $13.6\left(\frac{1}{1^2}-\frac{1}{2^2}\right)\text{eV} = 10.2\ \text{eV}$

n=1 Lifetime ∞

Planck's Constant, Quantization and Matter waves

- Wave nature of a particle is also a quantum phenomenon.
- The smallness of h in the relation ($\lambda = h/p$) makes wave characteristic of macroscopic particles hard to be observed
- The statement that when $h \longrightarrow 0$, λ becomes vanishingly small means that the wave nature will becomes effectively ``shutoff'' and there would be loss of its wave nature whenever the relevant scale (e.g. the p of the particle) is too large in comparison with $h \sim 10^{-34}$ J.s
- The wave nature of a particle will only show up when the linear momentum scale ***p*** of the particle times the length dimension characterizing the experiment ***(p x d)*** is comparable (or smaller) to the quantum scale of ***h***.
- Whenever ***h*** is not negligible compared to the characteristic scales of the experimental setup, particle behaves like wave; whenever ***h*** is negligible compared to ***pxd***, particle behave like just a conventional particle.
- ***h*** characterises the scale at which quantum nature of particles starts to take over from macroscopic physics.

Back to wave packet

Now it is easy to see how the wave packet is constructed

The high frequency constituent waves

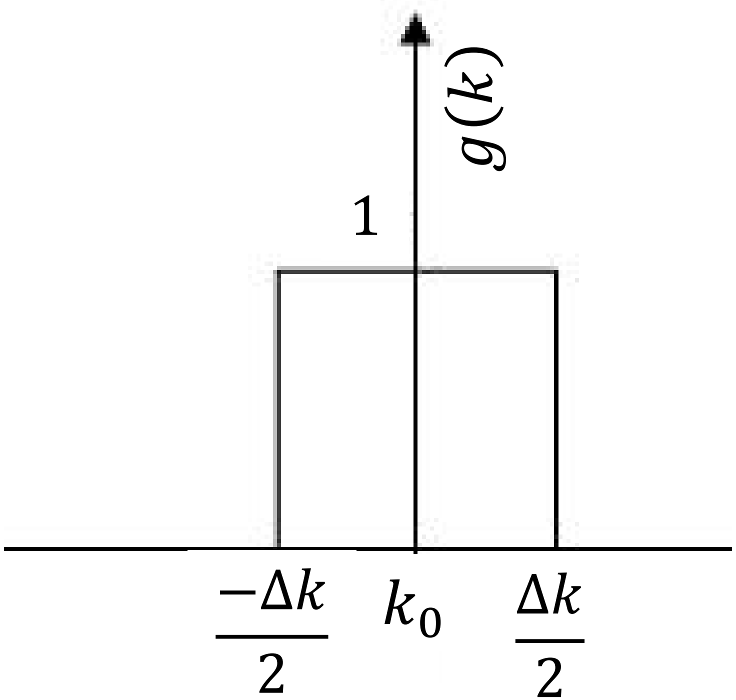
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk$$

The wave packet

Amplitude of the constituent waves

Fourier Integral: An example (Serway Ex 5.7)

Let us define $$g(k) = \begin{cases} 1\ for\ \ k_o - \dfrac{\Delta k}{2} \le k\ \le\ \ k_o + \dfrac{\Delta k}{2} \\ \ 0\ otherwise \end{cases}$$

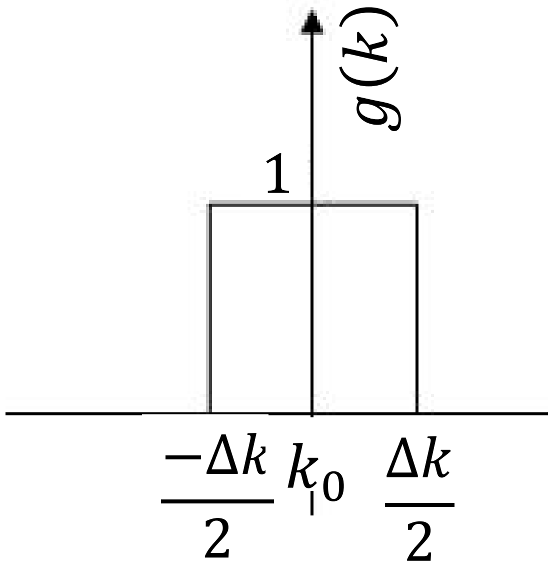


So, how do we calculate $f(x)$?

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} e^{ikx}\, dk$$

$$= \frac{\Delta k}{\sqrt{2\pi}} \left(\frac{\sin\left(\frac{\Delta k}{2} x\right)}{\frac{\Delta k}{2} x} \right) e^{ik_0 x}$$



Reverse FT

$$\mathbf{f(x)} = \frac{\Delta k}{\sqrt{2\pi}}\left(\frac{Sin\,(\frac{\Delta k x}{2})}{\Delta k x/2}\right)e^{ik_o x}$$

This is a complex function. Lets focus on the real part.

$$\mathrm{Re}\big(f(x)\big) = \frac{\Delta k}{\sqrt{2\pi}}\left(\frac{\sin\left(\frac{\Delta k}{2}x\right)}{\frac{\Delta k}{2}x}\right)\cos k_0 x$$

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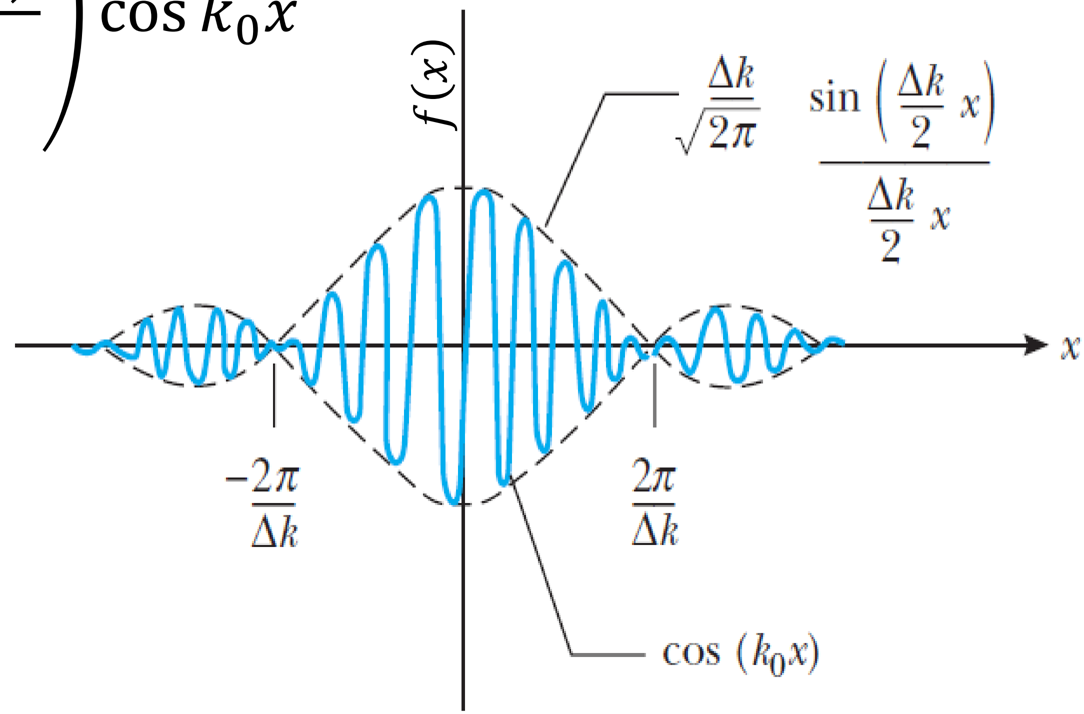


Spatial extent of wave packet: $\Delta x = \dfrac{4\pi}{\Delta k}$

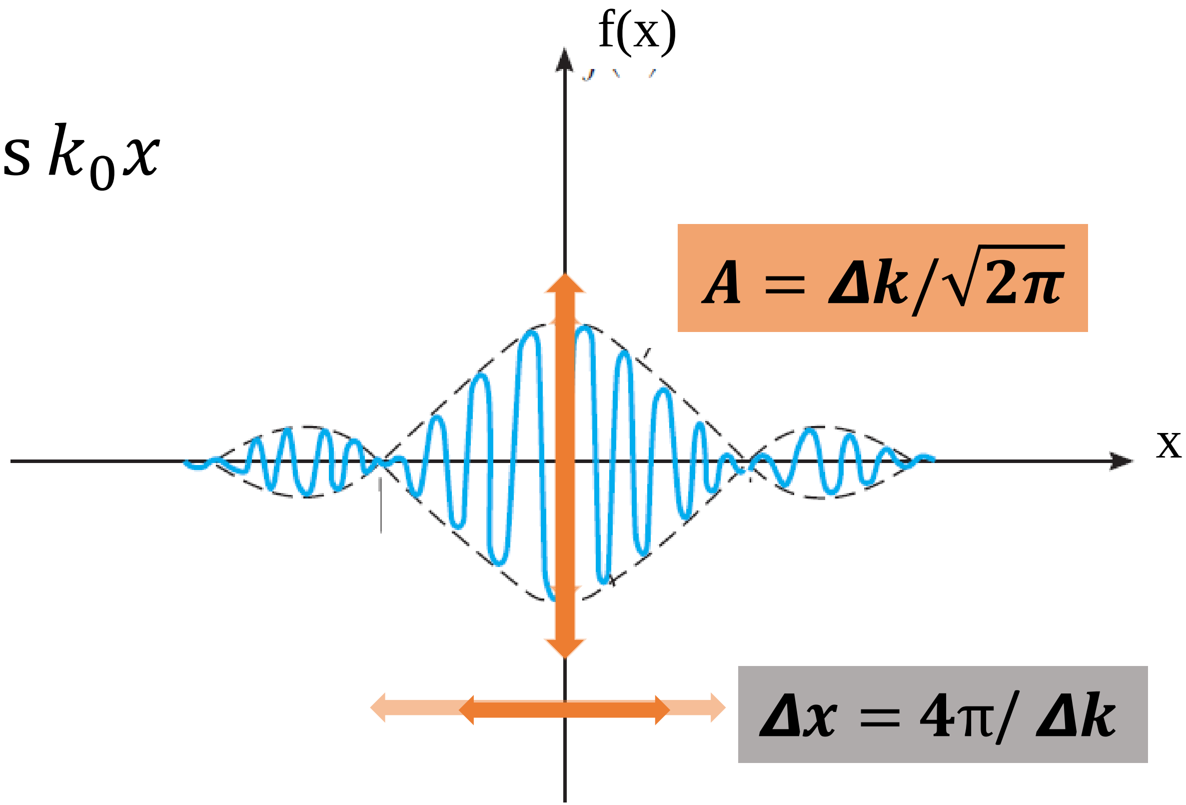
Amplitude of wave packet: $A = \dfrac{\Delta k}{\sqrt{2\pi}}$

1. Properties of the wave packet

What happens if we increase Δk?

Width decreases; *Δk increases* ➡ Amplitude increases

$$\frac{\Delta k}{\sqrt{2\pi}}\left(\frac{\sin\left(\frac{\Delta k}{2}x\right)}{\frac{\Delta k}{2}x}\right)\cos k_0 x$$



Reciprocity relations

The spatial extent of the function $\mathrm{Re}\big(f(x)\big)$, *i.e.* Δx decreases as Δk increases: **Reciprocity relation**

Both Δx and Δk cannot become arbitrarily small: As one decreases, the other increases.

In case of the function that we just constructed

$$\Delta x \Delta k = (4\pi/\Delta k)\Delta k = 4\pi$$

$$\Delta x \Delta p = 4\pi\,\hbar$$

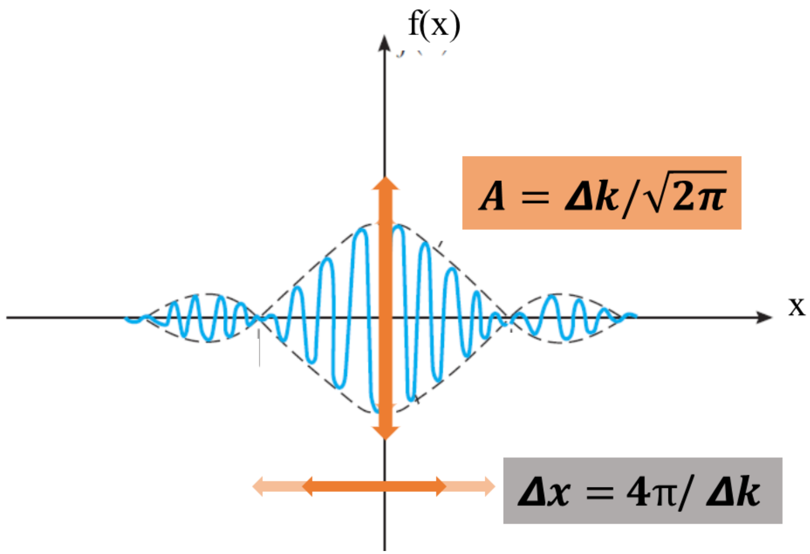
Note that the constant on the right-hand side of the uncertainty relation depends on the shape chosen for g(k) and the precise definition of *x* and *k*.

2. Properties of the wave packet

A localized particle is represented by a wave packet.

The amplitude of the wave packet is large where the particle is likely to be found and small where the particle is less likely to be found. *The probability of finding the particle at any point depends on the amplitude of its de Broglie wave at that point.*

In analogy with classical physics, in which the intensity of any wave is proportional to the square of its amplitude, we have



probability to observe particles $\propto |\, de\ Broglie\ wave\ amplitude\ |^2$