PH107 D1T5: Tutorial 5

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OPERATORS AND WAVE FUNCTION

Question 1

Which of the operators A_i defined in the following are linear operators? Which of these are hermitian? All the functions $\psi(x)$ are well behaved functions vanishing at $\pm\infty$.

(a) $\hat{A}_1\psi(x) = \psi(x)^2$

$$\begin{aligned}\hat{A}_1 c\psi(x) &= (c\psi(x))^2 \\ &= c^2\psi(x)^2 \\ &= c^2\hat{A}_1\psi(x)\end{aligned} \tag{1}$$

Not a linear Operator

The operator $\hat{A}_1$ is called hermitian if $\int(\hat{A}_1\psi)^*\psi dx = \int \psi^*\hat{A}_1\psi dx$

$$\begin{aligned}\implies LHS &= \int(\psi^2)^*\psi dx \\ \implies &\neq RHS\end{aligned} \tag{2}$$

Not a Hermitian

(b) $\hat{A}_2\psi(x) = \frac{\partial\psi(x)}{\partial x}$

$$\begin{aligned}\hat{A}_2 c\psi(x) &= \frac{\partial c\psi(x)}{\partial x} \\ &= c\frac{\partial\psi(x)}{\partial x} \\ &= c\hat{A}_2\psi(x) \\ \hat{A}_2(\psi(x)+\phi(x)) &= \frac{\partial(\psi(x)+\phi(x))}{\partial x} \\ &= \frac{\partial\psi(x)}{\partial x} + \frac{\partial\phi(x)}{\partial x} \\ &= \hat{A}_2\psi(x) + \hat{A}_2\phi(x)\end{aligned} \tag{3}$$

Linear Operator

The operator $\hat{A}_2$ is called hermitian if $\int(\hat{A}_2\psi)^*\psi dx = \int \psi^*\hat{A}_2\psi dx$

$$\implies LHS = \int\left(\frac{\partial\psi(x)}{\partial x}\right)^*\psi(x)dx \tag{4}$$

Using Integration by parts and ψ tends to zero as $x \to \infty$

$$
\begin{aligned}
&= -\int \left(\frac{\partial \psi(x)}{\partial x}\right) \psi(x)^* dx \\
&= -RHS
\end{aligned} \tag{5}
$$

Not a Hermitian

(c) $\hat{A}_3\psi(x) = \int_a^x \psi(x')dx'$

$$
\begin{aligned}
\hat{A}_3 c\psi(x) &= \int_a^x c\psi(x')dx' \\
&= c\int_a^x \psi(x')dx' \\
&= c\hat{A}_3\psi(x) \\
\hat{A}_3(\psi(x) + \phi(x)) &= \int_a^x (\psi(x') + \phi(x'))dx' \\
&= \int_a^x \psi(x')dx' + \int_a^x \phi(x')dx' \\
&= \hat{A}_3\psi(x) + \hat{A}_3\phi(x)
\end{aligned} \tag{6}
$$

Linear Operator

The operator $\hat{A}_3$ is called hermitian if $\int(\hat{A}_3\psi)^*\psi dx = \int \psi^*\hat{A}_3\psi dx$

$$
\begin{aligned}
LHS &= \int \left(\int_a^x \psi\right)^* \psi dx \\
&= \int \left(\int_a^x \psi^*\right) \psi dx \\
&\neq \int \psi^* \left(\int_a^x \psi\right) dx
\end{aligned} \tag{7}
$$

Not Hermitian

(d) $\hat{A}_4\psi(x) = 1/\psi(x)$

$$
\begin{aligned}
\hat{A}_4 c\psi(x) &= \frac{1}{c\psi(x)} \\
&= \frac{1}{c}\hat{A}_4\psi(x)
\end{aligned} \tag{8}
$$

Not a linear Operator

The operator $\hat{A}_4$ is called hermitian if $\int(\hat{A}_4\psi)^*\psi dx = \int \psi^*\hat{A}_4\psi dx$

$$\int \frac{1}{\psi^*}\psi dx \neq \int \psi^* \frac{1}{\psi} dx \tag{9}$$

Not Hermitian

(e) $\hat{A}_5\psi(x) = -\psi(x+a)$

$$\begin{aligned} \hat{A}_5 c\psi(x) &= -c\psi(x+a) \\ &= c\hat{A}_5\psi(x) \\ \hat{A}_5(\psi(x)+\phi(x)) &= -(\psi(x+a)+\phi(x+a)) \\ &= \hat{A}_5\psi(x) + \hat{A}_5\phi(x) \end{aligned} \tag{10}$$

Linear Operator

The operator $\hat{A}_5$ is called hermitian if $\int(\hat{A}_5\psi)^*\psi dx = \int \psi^*\hat{A}_5\psi dx$

$$\int -\psi(x+a)^*\psi dx \neq \int \psi^* - \psi(x+a) dx \tag{11}$$

Not Hermitian

(f) $\hat{A}_6\psi(x) = \sin(\psi(x))$

$$\begin{aligned} \hat{A}_6 c\psi(x) &= \sin(c\psi(x)) \\ &\neq c\hat{A}_6\psi(x) \end{aligned} \tag{12}$$

Not Linear Operator

The operator $\hat{A}_6$ is called hermitian if $\int(\hat{A}_6\psi)^*\psi dx = \int \psi^*\hat{A}_6\psi dx$

$$\int \sin(\psi(x))\psi dx \neq \int \psi^* \sin(\psi(x)) dx \tag{13}$$

Not Hermitian

(g) $\hat{A}_7\psi(x) = \frac{\partial^2\psi(x)}{\partial x^2}$

$$\begin{aligned}
\hat{A}_7 c\psi(x) &= \frac{\partial^2 c\psi(x)}{\partial x^2} \\
&= c\frac{\partial^2 \psi(x)}{\partial x^2} \\
&= c\hat{A}_7\psi(x) \\
\hat{A}_7(\psi(x)+\phi(x)) &= \frac{\partial^2(\psi(x)+\phi(x))}{\partial x^2} \\
&= \frac{\partial^2\psi(x)}{\partial x^2} + \frac{\partial^2\phi(x)}{\partial x^2} \\
&= \hat{A}_7\psi(x) + \hat{A}_7\phi(x)
\end{aligned} \tag{14}$$

Linear Operator

The operator $\hat{A}_7$ is called hermitian if $\int(\hat{A}_7\psi)^*\psi dx = \int \psi^*\hat{A}_7\psi dx$

$$\begin{aligned}
LHS &= \int \left(\frac{\partial^2\psi(x)}{\partial x^2}\right)^* \psi(x) dx \\
&= -\int \left(\frac{\partial\psi(x)}{\partial x}\right)^* \left(\frac{\partial\psi(x)}{\partial x}\right) dx \\
&= \int \psi(x)^* \left(\frac{\partial^2\psi(x)}{\partial x^2}\right) dx \\
&= RHS
\end{aligned} \tag{15}$$

Hermitian

Question 2

(a) If $\hat{A}$ and $\hat{B}$ are Hermitian and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C}$, prove that $\hat{C}$ is Hermitian

$$\begin{aligned}
\hat{A}\hat{B} - \hat{B}\hat{A} &= i\hat{C} \\
\implies (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger &= -i\hat{C}^\dagger \\
\implies (\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger &= -i\hat{C}^\dagger \\
\implies \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger &= -i\hat{C}^\dagger \\
\hat{B}\hat{A} - \hat{A}\hat{B} &= -i\hat{C}^\dagger \\
-i\hat{C} &= -i\hat{C}^\dagger \\
\hat{C} &= \hat{C}^\dagger
\end{aligned} \tag{16}$$

(b) An operator is said to be anti-Hermitian if $\hat{O}^\dagger = -\hat{O}$. Prove that $[\hat{A}, \hat{B}]$ is antiHermitian.

$$\begin{aligned} [\hat{A},\hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\ &= (\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger \\ &= \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger \\ &= \hat{B}\hat{A} - \hat{A}\hat{B} \\ &= -[\hat{A},\hat{B}] \end{aligned} \tag{17}$$

Question 3

Prove that if $\hat{K}$ is a Hermitian operator, $\exp(i\hat{K})$ is an unitary operator, and if $\hat{U}$ is an Unitary operator, then there is an operator K such that $\hat{U} = \exp(i\hat{K})$ and this $\hat{K}$ is Hermitian.

Given, $\hat{K}$ is Hermitian. Let, $\hat{U} = \exp(i\hat{K})$

$$\begin{aligned} \hat{U}\cdot\hat{U}^\dagger &= \exp(i\hat{K})\exp(-i\hat{K}^\dagger) \\ &= \exp(i(\hat{K} - \hat{K}^\dagger)) \\ &= 1 \end{aligned} \tag{18}$$

Given, $\hat{U}$ is unitary.Then, by the Spectral Theorem for unitary matrices, there is another **unitary matrix** B such that $U = B\Lambda B^{-1}$, and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. As all $|\lambda_k| = 1$, we write them as $\lambda_k = e^{i\theta_k}$ where θ_k are real numbers. Then set

$$A = Bdiag(\theta_1, \theta_2, \cdots, \theta_n)B^{-1} = B\Lambda_1 B^{-1}$$

Then A is Hermitian:

$$A^\dagger = (B^{-1})^\dagger \Lambda_1 B^\dagger = B\Lambda_1 B^{-1} = A,$$

and evidently $\exp(iA) = U$

Question 4

If $\hat{A}$ and $\hat{B}$ are operators, prove

(a) that $(\hat{A}^\dagger)^\dagger = \hat{A}$

$A^\dagger$ is transpose conjugate of A. So, taking transpose-conjugate twice will not change matrix.

$$(\hat{A}^\dagger)^\dagger = \hat{A}$$

(b) that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$

$$\int \phi^* \hat{A}\hat{B}\psi = \int (\hat{A}^\dagger\phi)^*\hat{B}\psi = \int (\hat{B}^\dagger\hat{A}^\dagger\phi)^*\psi \tag{19}$$
$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$$

(c) that $\hat{A}+\hat{A}^\dagger, i(\hat{A}-\hat{A}^\dagger)$*, and that* $\hat{A}\hat{A}^\dagger$ *are Hermitian operators.*

An operator $\hat{A}$ is Hermitian if $\hat{A} = \hat{A}^\dagger$

$$\begin{aligned}(\hat{A}+\hat{A}^\dagger)^\dagger &= (\hat{A}^\dagger + \hat{A})\\ i(\hat{A}-\hat{A}^\dagger)^\dagger &= -i(\hat{A}^\dagger - \hat{A})\\ &= i(\hat{A}-\hat{A}^\dagger)^\dagger\\ (\hat{A}\hat{A}^\dagger)^\dagger &= (\hat{A}\dagger)^\dagger\hat{A}^\dagger\\ &= \hat{A}\hat{A}^\dagger\end{aligned} \tag{20}$$

Question 5

An operator is given by
$$\hat{G} = i\hbar\frac{\partial}{\partial x} + Bx$$
where B is a constant. Find the eigen function $\phi(x)$. If this eigen function is subjected to a boundary condition $\phi(a) = \phi(-a)$ find out the eigen values.

We need to solve the differential equation

$$\begin{aligned}\hat{G}\phi(x) &= \lambda\phi(x)\\ \implies i\hbar\frac{d\phi(x)}{dx} + Bx\phi(x) &= \lambda\phi(x)\\ \implies \frac{d\phi(x)}{dx} &= \frac{i}{\hbar}(Bx-\lambda)\phi(x)\\ \implies \int\frac{1}{\phi(x)}d\phi(x) &= \frac{i}{\hbar}\int(Bx-\lambda)dx\\ \implies \ln\phi(x) &= \frac{i}{\hbar}\left(\frac{B}{2}x^2-\lambda x\right)+\ln(C)\\ \implies \phi(x) &= Ce^{-i\frac{\lambda}{\hbar}x}e^{i\frac{B}{2\hbar}x^2}\end{aligned} \tag{21}$$

Applying the boundary condition $\phi(a) = \phi(-a)$

$$\begin{aligned}\implies e^{i\frac{2a\lambda}{\hbar}} &= 1\\ \implies \frac{2a\lambda}{\hbar} &= 2n\pi, n\in\mathcal{N}\\ \implies \lambda_n &= \frac{n\pi\hbar}{a}\end{aligned} \tag{22}$$

Question 6

$\psi_1(x)$ and $\psi_2(x)$ are the normalized eigen functions of an operator $\hat{P}$, with eigen values P_1 and P_2 respectively. If the wave function of a particle is $0.25\psi_1(x)+0.75\psi_2(x)$ at $t=0$, find the probability of observing P_1

Since $\hat{P}$ corresponds to an observable, therefore it is a Hermitian Operator. Thus it's eigen-functions with distinct eigenvalues must be orthogonal

$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle = 0$$

$$\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1$$

According to the postulates of QM, the probability of observing P_1 is $\frac{|\langle\psi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle}$ where $\psi(x) = 0.25\psi_1(x) + 0.75\psi_2(x)$

$$\begin{aligned} \langle\psi_1|\psi\rangle &= 0.25\,\langle\psi_1|\psi_1\rangle + 0.75\,\langle\psi_1|\psi_2\rangle = 0.25 \\ \langle\psi|\psi\rangle &= 0.25^2\,\langle\psi_1|\psi_1\rangle + 0.75^2\,\langle\psi_2|\psi_2\rangle + 0.25*0.75\,\langle\psi_1|\psi_2\rangle + 0.25*0.75\,\langle\psi_2|\psi_1\rangle \\ &= 0.25^2 + 0.75^2 \end{aligned} \tag{23}$$

Probability of $P_1 = \frac{|\langle\psi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{0.25^2}{0.25^2+0.75^2} = \frac{1}{10} = 0.1$

Question 7

Consider a large number (N) of identical experimental set-ups. In each of these, a single particle is described by a wave function $\Phi(x) = A\exp(-x^2/b^2)$ at $t=0$, where A is the normalization constant and b is another constant with the dimension of length. If a measurement of the position of the particle is carried out at time $t=0$ in all these set-ups, it is found that in 100 of these, the particle is found within an infinitesimal interval of $x=2b$ to $2b+dx$. Find out, in how many of the measurements, the particle would have been found in the infinitesimal interval of $x=b$ to $b+dx$.

$$\begin{aligned} \frac{N_{[2b,2b+dx]}}{N_{[b,b+dx]}} &= \frac{P(2b<x<2b+dx)}{P(b<x<b+dx)} \\ &= \frac{|\phi(2b)|^2dx}{|\phi(b)|^2dx} = e^{-\frac{4b^2-b^2}{b^2}} = e^{-6} \\ \implies N_{[b,b+dx]} &\approx 100e^6 \approx 40343 \end{aligned} \tag{24}$$

Question 8

An observable A is represented by the operator $\hat{A}$. Two of its normalized eigen functions are given as $\Phi_1(x)$ and $\Phi_2(x)$, corresponding to distinct eigenvalues a_1 and a_2, respectively. Another observable B is represented by an operator $\hat{B}$. Two normalized eigen functions of this operator are given as $u_1(x)$ and $u_2(x)$ with distinct eigenvalues b_1 and b_2, respectively. The eigen functions

$\Phi_1(x)$ and $\Phi_2(x)$ are related to $u_1(x)$ and $u_2(x)$ as, $\Phi_1 = D(3u_1 + 4u_2); \Phi_2 = F(4u_1 - Pu_2)$ At time $t = 0$, a particle is in a state given by $\frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2$.

(a) Find the values of D, F and P

We know that Φ_1, Φ_2, u_1, u_2 are normalized, and Φ_1, Φ_2 are orthogonal and so are u_1, u_2.

$$\begin{aligned}
\langle \Phi_1 | \Phi_1 \rangle &= 1 \\
\implies 25|D|^2 &= 1 \\
\implies D &= 0.2e^{i\theta} \\
\langle \Phi_1 | \Phi_2 \rangle &= 0 \\
\implies \bar{D}F(12\langle u_1|u_1\rangle - 4P\langle u_2|u_2\rangle - 3P\langle u_1|u_2\rangle + 16\langle u_2|u_1\rangle) &= 0 \\
\implies P &= 3 \\
\langle \Phi_2 | \Phi_2 \rangle &= 1 \\
\implies |F|^2(16\langle u_1|u_1\rangle + 9\langle u_2|u_2\rangle - 12\langle u_2|u_1\rangle - 12\langle u_1|u_2\rangle) &= 1 \\
\implies 25|F|^2 &= 1 \\
\implies F &= 0.2e^{i\theta}
\end{aligned} \tag{25}$$

(b) If a measurement of A is carried out at $t = 0$, what are the possible results and what are their probabilities ?

According to the postulates of QM, the possible results of an A measurement are the eigenvalues a_1 and a_2, and the probability is $\frac{|\langle \Phi|\psi\rangle|^2}{\langle\psi|\psi\rangle}$ where Φ is the corresponding eigenfunction and $\psi = \frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2$

$$\begin{aligned}
\langle \Phi_1 | \psi \rangle &= \frac{2}{3} \\
\langle \Phi_2 | \psi \rangle &= \frac{1}{3} \\
\langle \psi | \psi \rangle &= \frac{4}{9} + \frac{1}{9} = \frac{5}{9}
\end{aligned} \tag{26}$$

$\implies$ Probability(a_1) = $\frac{\frac{4}{9}}{\frac{5}{9}} = 0.8$

$\implies$ Probability(a_2) = $\frac{\frac{1}{9}}{\frac{5}{9}} = 0.2$

(c) Assume that the measurement of A mentioned above yielded a value a_1. If a measurement of B is carried out immediately after this, what would be the possible outcomes and what would be their probabilities ?

According to the postulates of QM, once a1 is measured, the wavefunction will collapse to Φ_1. Thus the wavefunction at the moment after measurement is $\Phi_1(x)$.Again we repeat the same exercise as above for u_1 and u_2. The possible outcomes are of course b_1 and b_2.

$$\begin{aligned} \langle u_1|\Phi_1\rangle &= 0.6e^{i\theta} \\ \langle u_2|\Phi_1\rangle &= 0.8e^{i\theta} \\ \langle \Phi_1|\Phi_1\rangle &= 1 \end{aligned} \tag{27}$$

$\implies$ Probability(b_1) = 0.36
$\implies$ Probability(b_2) = 0.64

(d) If instead of following the above path, a measurement of B was carried out initially at $t = 0$, what would be the possible outcomes and what would be their probabilities ?

Following same steps as above, the possible outcomes are same, b_1 and b_2

$$\begin{aligned} \psi &= \frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2 \\ &= \left(\frac{2}{5}e^{i\theta} + \frac{4}{15}e^{i\Phi}\right)u_1 + \left(\frac{8}{15}e^{i\theta} - \frac{1}{5}e^{i\Phi}\right)u_2 \\ \langle u_1|\psi\rangle &= \frac{2}{5}e^{i\theta} + \frac{4}{15}e^{i\Phi} \\ \langle u_2|\psi\rangle &= \frac{8}{15}e^{i\theta} - \frac{1}{5}e^{i\Phi} \\ \langle \psi|\psi\rangle &= \frac{5}{9} \end{aligned} \tag{28}$$

Probability(b_1) = $\frac{9}{5}\left(\frac{4}{25} + \frac{16}{225} + 2\frac{8}{75}\cos(\theta - \Phi)\right)$
Probability(b_2) = $\frac{9}{5}\left(\frac{64}{225} + \frac{1}{25} - 2\frac{8}{75}\cos(\theta - \Phi)\right)$

(e) Assume that after performing the measurements described in (c), the outcome was b_2. What would be the possible outcomes, if A were measured immediately after this and what would be the probabilities ?

So immediately after measurement the wavefunction collapses to u_2. Now once again the possibilities are a_1 and a_2

$$\begin{aligned} \langle \Phi_1|u_2\rangle &= \langle u_2|\Phi_1\rangle^* = 0.8e^{-i\theta} \\ \langle \Phi_2|u_2\rangle &= \langle u_2|\Phi_2\rangle^* = -0.6e^{-i\Phi} \\ \langle u_2|u_2\rangle &= 1 \end{aligned} \tag{29}$$

Probability(a_1) = 0.64
Probability(a_2) = 0.36