MA 109 Week 4

Mayukh Mukherjee

Department of Mathematics

December 20, 2021

Partitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}.$$

We can think of the points of the partition as dividing the original interval $[a, b]$ into sub-intervals $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq n$. Indeed $I = \cup_j I_j$ and if two sub-intervals intersect, they have at most one point in common. Hence, the notation "partition".

Definition: A partition $P' = \{a = x'_0 < x'_1 < \ldots < x'_m = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x'_j \in P'$ such that $x_i = x'_j$.

Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals. Any two partitions have a common refinement.

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Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \ldots < x_{b-1} < x_n = b\}$ and a function $f : [a, b] \to \mathbb{R}$, we define two associated quantities. First we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n$$

Defintion: We define the Lower sum as

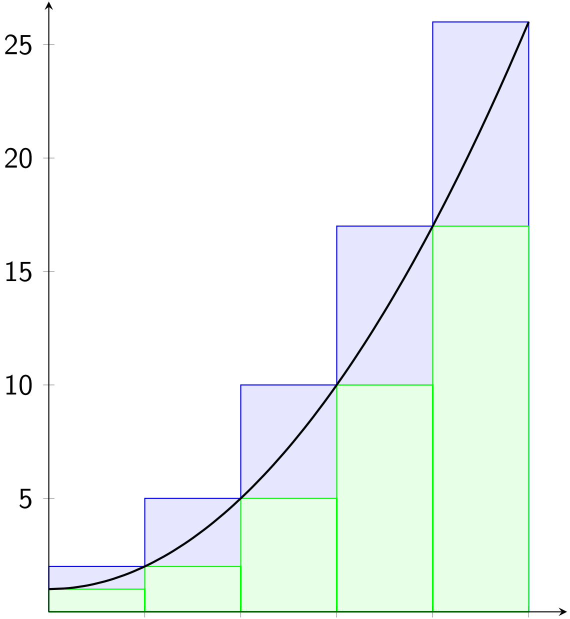
$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

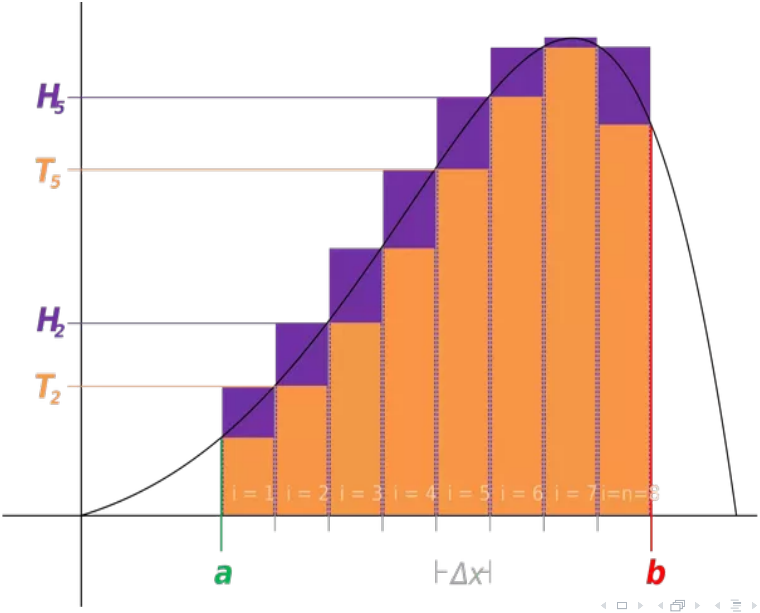
$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

In case the words "infimum" and "supremum" bother you, you can think "minimum" and "maximum" most of time since we will usually be dealing with continuous functions on $[a, b]$.

Lower and Upper Darboux sums



A picture for a non-monotonic function



One basic example

In order to illustrate what we are saying we will take the following basic example. Let $[a, b] = [0, 1]$ and let $f(x) = x$.

One of the most natural partitions on an interval is a partition that divides the interval into sub-intervals of equal length. For $[0, 1]$, this is

$$P_n = \{0 < 1/n < 2/n < \ldots < (n-1)/n < 1\}.$$

On the interval $I_j = [\frac{j-1}{n}, \frac{j}{n}]$, where does the function $f(x) = x$ take its minimum? its maximum?

Clearly, the minimum $m_j = \frac{j-1}{n}$ is attained at $\frac{j-1}{n}$ and the maximum $M_j = \frac{j}{n}$ at $\frac{j}{n}$. And finally, $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$, for all $1 \leq j \leq n$.

An example of a refinement of P_n is P_{2n}, or, more generally, P_{kn} for any natural number k.

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The Darboux integrals

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$.

and similarly the upper Darboux integral of f by

$$U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$. (This time there is no escaping inf and sup!)

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\, dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

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Back to the example

Let us calculate $L(f, P_n)$ and $U(f, P_n)$ in the example we gave.

$$L(f, P_n) = \sum_{j=1}^{n} \frac{(j-1)}{n} \cdot \frac{1}{n} = \sum_{j=0}^{n-1} \frac{j}{n^2}.$$

This can be evaluated explicitly:

$$L(f, P_n) = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} - \frac{1}{2n}.$$

Similary, we can check that

$$U(f, P_n) = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} + \frac{1}{2n}.$$

Can we conclude that the Darboux integral is 1/2 by letting $n \to \infty$? Unfortunately, no.

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Useful properties of the Darboux sums

Since, for any partition P, $L(f,P) \leq U(f,P)$, we have

$$L(f) \leq U(f).$$

In fact, for any two partitions P_1 and P_2, we have

$$L(f,P_1) \leq U(f,P_2).$$

This is easy to see - the lower sum computes the sum of the areas of rectangles that lie entirely below the curve while the upper sum computes the sum of the areas of rectangles whose "tops" lie above the curve.

One of the most useful properties of the Darboux sums is the following. If P' is a refinement of P then obviously

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Riemann Sums

There is another way of getting at the integral due to Riemann which may be a little more intuitive and is better for calculation. This is done via Riemann sums.

To define the notion of a Riemann sum we need one more piece of data. Suppose that for each of the intervals I_j we are given a point $t_j \in I_j$. We will denote the collection of points t_j by t. The pair (P, t) is sometimes called a tagged partition.

Definition: We define the Riemann sum associated to the function f, and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

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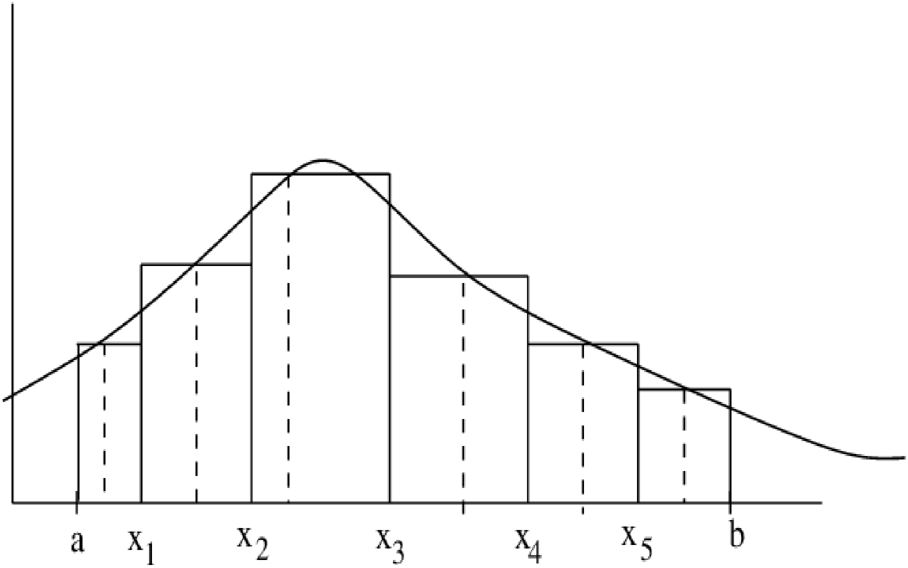

The norm of a partition

As must be clear, the Lower sum, Upper sum and Riemann sum all give approximations to the area between the lines $x = a$ and $x = b$ and between the curve $y = f(x)$ and the x-axis and

$$L(f, P) \leq R(f, P, t) \leq U(f, P).$$

The point is to make this statement quantitatively precise.

We define the norm of a partition P (denoted $\|P\|$) by

$$\|P\| = \max_j \{|x_j - x_{j-1}|\}, \quad 1 \leq j \leq n.$$

The norm gives some measure of the "size" of a partition, in particular, it allows us to say whether a partition is big or small.

When the size of the partition is small, it means that every interval in the partition is small.

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The Riemann integral

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

In other words, for all sufficiently "small" or "fine" partitions, the Riemann sums must be within ϵ of R.

Notice, that as long as $\|P\|$ is small, it doesn't matter exactly where the x_j's or the t_j's are in the interval $[a, b]$.

Also notice that if P' is a refinement of P, then $\|P'\| \leq \|P\|$.

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Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

whenever $\|P\| < \delta$. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

In other words, for all sufficiently "small" or "fine" partitions, the Riemann sums must be within ϵ of R.

Notice, that as long as $\|P\|$ is small, it doesn't matter exactly where the x_j's or the t_j's are in the interval $[a, b]$.

Also notice that if P' is a refinement of P, then $\|P'\| \leq \|P\|$.

The Riemann integral continued

Intuitively, we can see that the smaller or finer the partition, the better the area under the curve is represented by the Riemann sum.

The reason that the Riemann integral is useful is because the definition we have given is actually equivalent to the following apparently weaker definition.

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists a partition P such that for every tagged refinement of (P', t') of P with $\|P'\| \leq \delta$,

$$|R(f, P', t') - R| < \epsilon.$$

The nice thing about the above definition is that one only has to check that $|R(f, P', t') - R|$ is small for for refinements of a fixed partition, and not for all partitions.

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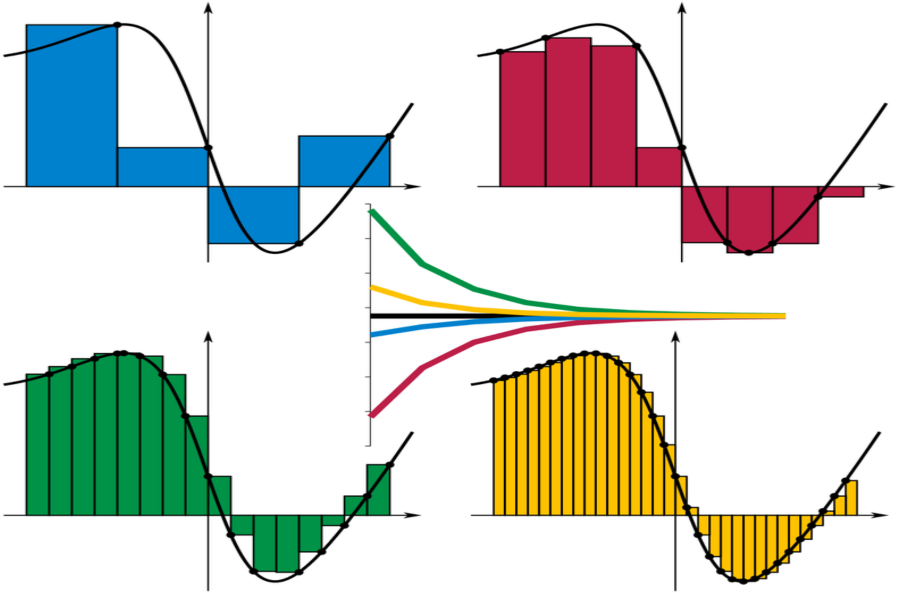
Effect of refinement of partition



Back to our example

Using Definition 2 of the Riemann integral it is easy to see that the function $f(x) = x$ is Riemann integrable.

Let $\epsilon > 0$ be arbitrary. For our fixed partition we take $P = P_n$ where $n > \frac{1}{2\epsilon}$ is some fixed number.

Moreover, if (P', t') is any refinement of P_n we have

$$L(f, P_n) \leq L(f, P') \leq R(f, P', t') \leq U(f, P') \leq U(f, P_n),$$

whence it follows that

$$|R(f, P', t') - \frac{1}{2}| < \epsilon.$$

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The example continued

As the preceding example shows, Definition 2 of the Riemann integral is really easy to work with. Why do we then care about Definition 1 or the Darboux integral?

The reason is that while Definition 2 is good for showing that a given function is Riemann integrable, the other definitions are often better for proving the *abstract properties* of integrals.

In fact, this will be clear in the tutorial exercises. You will see that sometimes the Darboux integral is better than the Riemann integral.

Before going any further we will formally state what we have already been referring to for several slides.

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Comparison with the Darboux integral

Theorem

The Riemann integral (using either definition) exists if and only if the Darboux integral exists and in this case the two integrals are equal.

With this theorem in hand, we see that the function $f(x) = x$ is also Darboux integrable.

How does one prove the above Theorem? It is not too hard but it takes some work and is roundabout.

The easiest way is to proceed as follows. It is clear that if f is Riemann integrable in the sense of Definition 1, it is Riemann integrable in the sense of Definition 2. Next, one shows that if f is Riemann integrable in the sense of Definition 2, then it is Darboux integrable. And finally, one can show that if the Darboux integral exists, then the Riemann integral exists in the sense of Definition 1. An interested student can try this as an exercise.

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The main theorem for Riemann integration

From now on we will use any of the three definitions - the Darboux definition, Definition 1 and Definition 2 for the integral interchangeably and we will use only the words Riemann integral.

The main theorem of Riemann integration is the following:

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

In fact, one can allow even countably many discontinuities and the Theorem will remain true.

Exercise 1: Those of you who have an extra interest in the course should think about trying to prove both Theorem 21 and the extension to countably many discontinuities. You can of course, also look up math stackexchange, where I am certain this has been answered.

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An example of a function that is not Darboux integrable

Here is a function that is not Darboux integrable of $[0, 1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It should be clear that no matter what partition one takes the infimum on any sub-interval in the partition will be 0 and the supremum will be 1. From this one can see immediately that $L(f, P) = 0 \neq 1 = U(f, P)$ for every P, and hence that $L(f) = 0 \neq 1 = U(f)$.

The general result about Riemann integration is the following: a bounded function is Riemann integrable if and only if the set of discontinuities is of "Lebesgue measure 0". Though we are not going to define this term, you can read it as being "negligible" in some sense. Clearly, here the set of discontinuities is $[0, 1]$, which is not negligible (I misspoke about this last point earlier, my apologies!)

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Another property of the Riemann Integral

Theorem

Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P' we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (resp. $L(f)_{[c,b]}$) the Darboux lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).

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Now the supremum on the left hand side is taken only over partitions P' having the special form $P_1 \cup P_2$. Hence it is less than or equal to $\sup_P L(f, P)$ where this supremum is taken over all partitions P. We thus obtain

$$L(f)_{[a,c]} + L(f)_{[c,b]} \leq L(f).$$

On the other hand, for any partition $P = \{a < x_1 < \ldots x_{n-1} < b\}$ we can consider the partition $P' = P \cup \{c\}$. This will be a refinement of the partition P and can be written as a union of two partitions P_1 of $[a, c]$ and P_2 of $[c, b]$.

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By the property for refinements for Darboux sums we know that $L(f, P) \leq L(f, P')$.

Thus, given any partition P of $[a, b]$, there is a refinement P' which can be written as the union of two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, and by the above inequality,

$$\sup_P L(f, P) \leq \sup_{P'} L(f, P'),$$

where the first supremum is taken over all partitions of $[a, b]$ and the second only over those partitions P' which can be written as a union of two partitions as above. This shows that

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so, together with the previous inequality, we get

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The same kind of reasoning applies to the Upper sums which allows us to prove the required property.

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Motivation

The Fundamental Theorem of calculus allows us to relate the process of Riemann integration to the process of differentiation. Essentially, it tells us that integrating and differentiating are inverse processes. This is a tremendously useful theorem for several reasons.

It turns out that (Riemann) integrating even simple functions is much harder than differentiating them (if you don't believe me, try integrating $(\tan x)^3$ via Riemann sums!). In practice, however, integration is what we need to do to solve physical problems. Usually, when we are studying the motion of a particle or a planet what we find is that the position of a particle, which is a function of time, satisfies some differential equation. Solving the differential equation involves performing the inverse operation of taking some combination of derivatives. The simplest such inverse operation is taking the inverse of the first derivative, which the Fundamental Theorem says, is the same as integrating.

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Calculating Integrals

Thus, calculating integrals is one of the basic things one needs to do for solving even the simplest physics and engineering problems. The problem is that this is quite difficult to do.

Once we know the derivatives of some basic functions (polynomials, trigonometric functions, exponentials, logarithms) we can differentiate a wide class of functions using the rules for differentiation , especially the product and chain rules. By contrast, the only rule for Riemann integration that can be proved from the basic definitions is the sum rule.

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The Fundamental Theorem - Part I

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Proof: We know that $f(t)$ is Riemann integrable for any $x \in [a, b]$ because every continuous function is Riemann integrable.

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The proof of Part I continued

We know that

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

for $x + h \in [a, b]$. Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f(t)dt.$$

We know that if $f(t) \leq g(t)$ on $[a, b]$, then $\int f(t)dt \leq \int g(t)dt$. We apply this to the three functions $m(h)$, f and $M(h)$, where $m(h)$ and $M(h)$ are the constant functions given by the minimum and maximum of the function f on $[x, x + h]$ to get:

$$m(h) \cdot h \leq \int_x^{x+h} f(t)dt \leq M(h) \cdot h.$$

Dividing by h and taking the limit gives

$$\lim_{h \to 0} m(h) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq \lim_{h \to 0} M(h).$$

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But f is a continuous function, so $\lim_{h\to 0} m(h) = \lim_{h\to 0} M(h) = f(x)$. By the Sandwich theorem for limits, we see that limit in the middle exists and is equal to $f(x)$, that is $F'(x) = f(x)$. This proves the first part of the Fundamental Theorem of Calculus. $\square$

This first form of the Fundamental Theorem allows us to compute definite integrals. Keeping the notation as in the Theorem we obtain

Corollary:

$$\int_c^d f(t)dt = F(d) - F(c),$$

for any two points $c, d \in [a, b]$.

The Fundamental Theorem of Calculus Part 2

Theorem

Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(x) = f(x)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Note that this statement does not assume that the function $f(t)$ is continuous, and is thus stronger than the corollary just stated.

Proof: We can write:

$$g(b) - g(a) = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})],$$

where $\{a = x_0, x_1, \ldots, x_n = b\}$ is an arbitrary partition of $[a, b]$. Using the mean value theorem for each of the intervals $I_j = [x_j, x_{j-1}]$, we can write

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The proof of the Fundamental Theorem part II continued

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

where $c_i \in (x_{i-1}, x_i)$.

Substituting this in the previous expression and using the fact that $g'(c_i) = f(c_i)$, we get

$$g(b) - g(a) = \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})].$$

The calculation above is valid for any partition The right hand side obviously represents a Riemann sum. By hypothesis f is Riemann integrable. It follows (using Definition 1, for example) that as $\|P\| \to 0$, the right hand side goes to the Riemann integral. □

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