PH 107: Quantum Physics and applications

Wave packet, Group velocity and Phase velocity

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Recap

Wave-Particle Duality – Everything, *matter* and *radiation* has both wave and particle property.

Question : Given a matter (or wave) which property do you see ?

Everything (matter and radiation) has both wave and particle properties; which property you see depends on the experiment you perform.

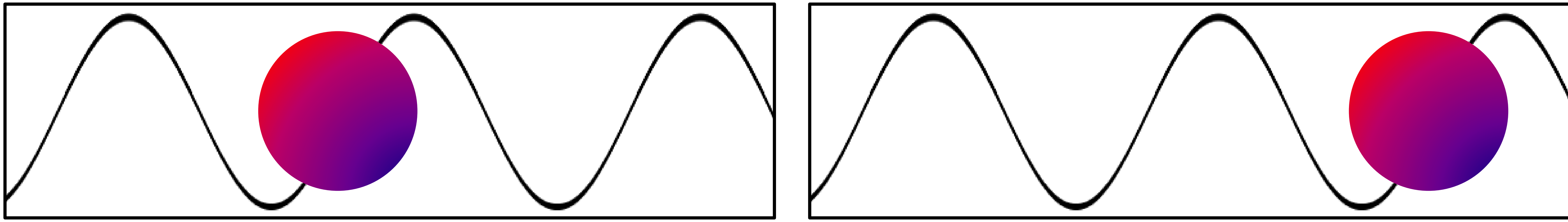
For Photon (radiation)

$$p = \frac{h\upsilon}{c} = \frac{h}{\lambda} = \hbar k$$

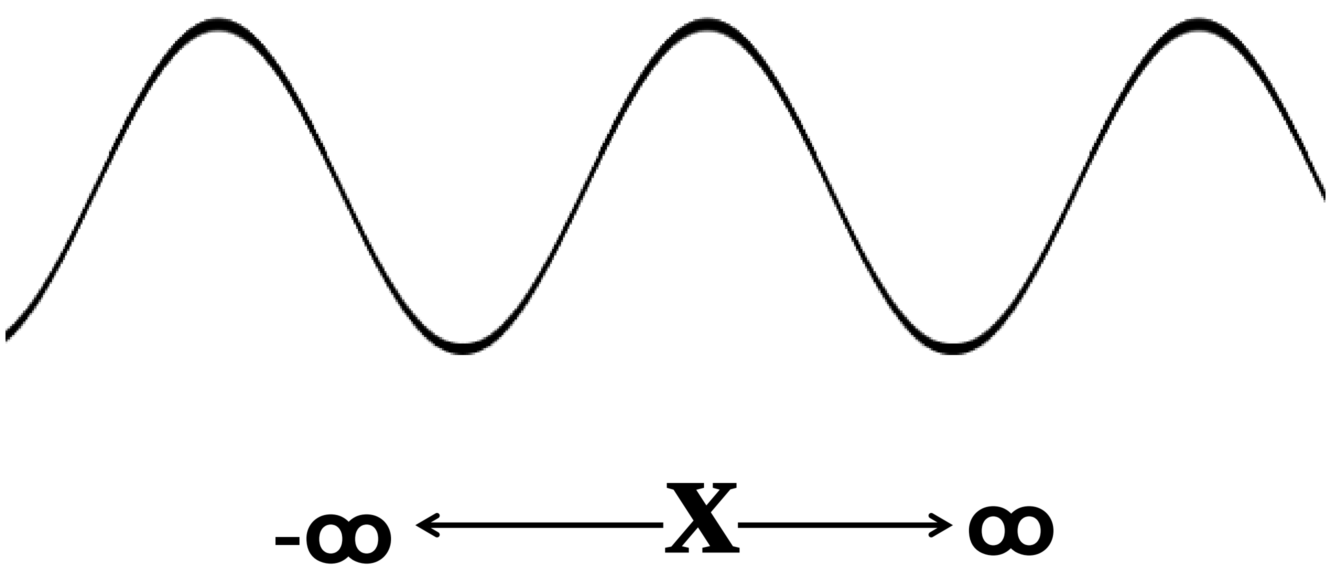
For Particle of momentum , p

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{mv} \; ; \; E = \frac{p^2}{2m}$$

Recap: Matter Wave

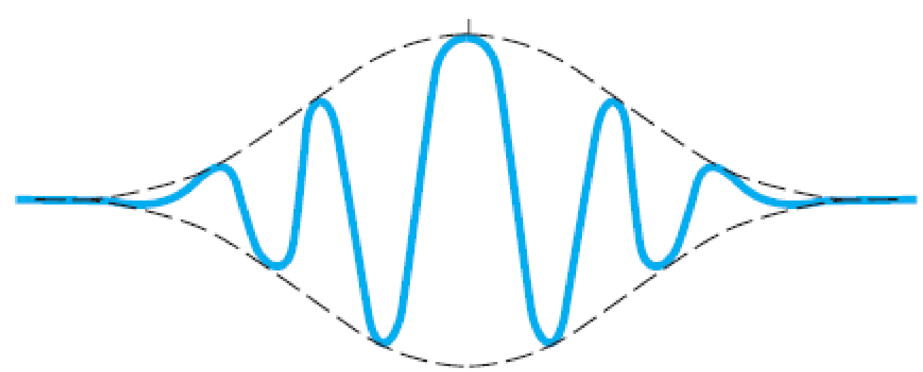


$$y(x,t) = A\sin(kx - \omega t\ )$$



For matter wave

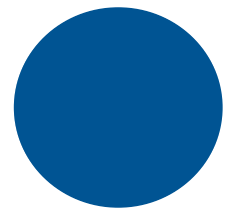
Wave-packet/Wave group



Learning Objectives

- How to define wave associated with particle ?
- Mathematical construction of wave packet
- Definition of Phase velocity and Group velocity.

Particles



Particles are localized. They occupy a well defined region of space, whatever their size.

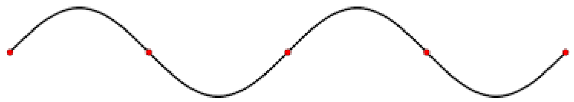
In the idealized picture, we want the particles to have no size at all, so that their position can be described by a single real number.

This makes the mathematical analysis much easier.

Particles

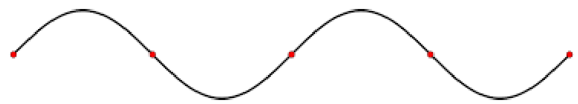
We take this picture so seriously that we picture a macroscopic object as being made of a large number of infinitesimal objects, with tiny but non-zero masses.

Such visualization has been an enormous success. A very large number of phenomena in mechanics and electromagnetism have been explained by treating matter as being made up a large number of "point masses".

Waves 

Waves, on the other hand, are extended objects. In classical wave theory, we do not put spatial restrictions on waves.

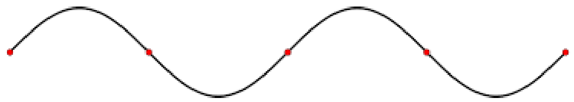
Again, to keep the mathematics simple, we use an idealised picture, where a wave extends in space from minus infinite to plus infinite.

Waves 

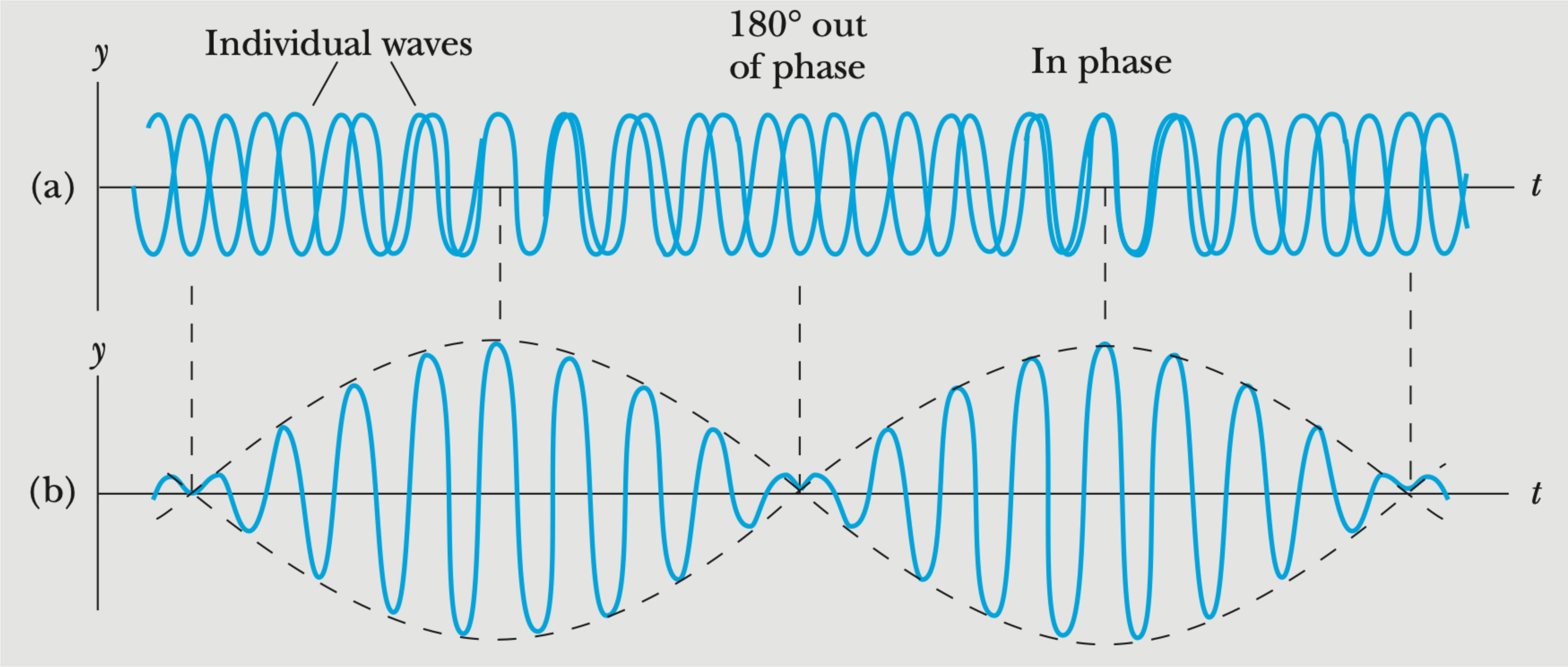
The explanation of interference and diffraction relies on the fact that waves have spatial extent.

We get interference only if the wave goes through (presents itself at) both slits simultaneously.

If physical objects have both particle and wave like properties, how do we build a mathematical description, which brings together these seemingly mutually exclusive properties?

Waves 

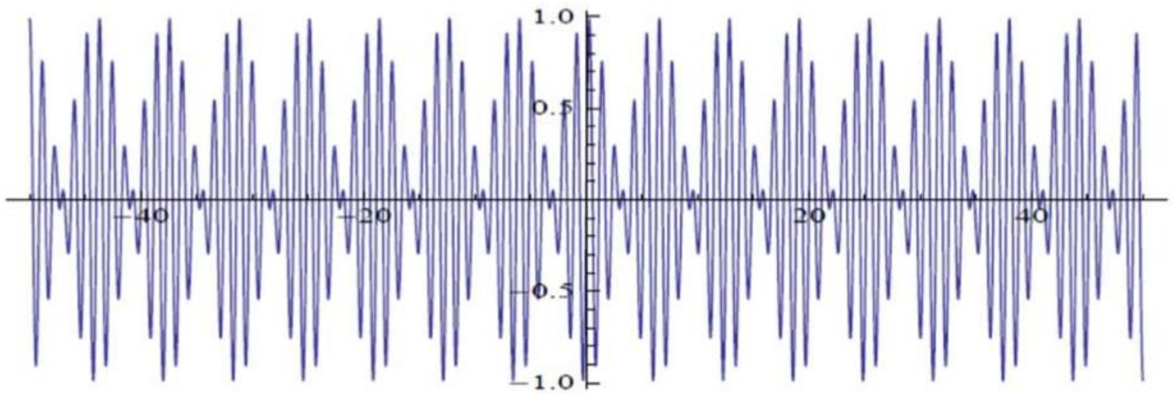
If several waves of different wavelengths and phases are superimposed together, what we get is a localized wave packet.



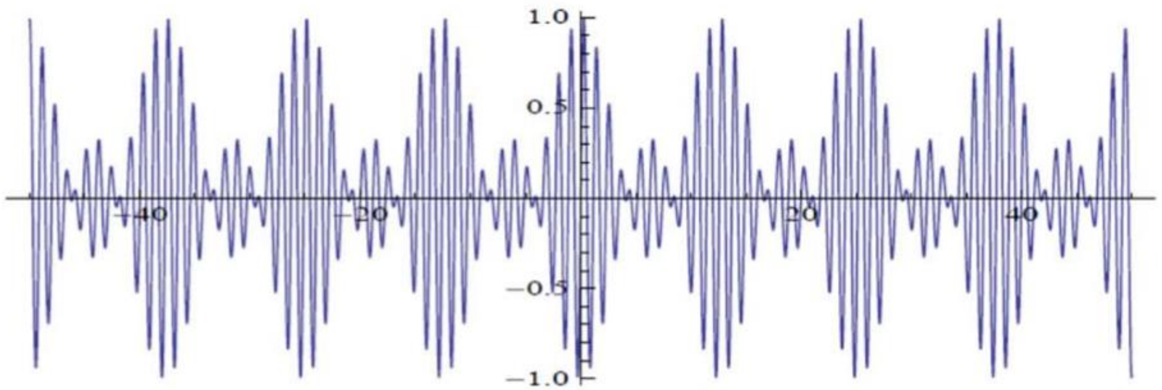
Beat formation in superposition of two sinusoidal waves

Spatial beats by superposition of sinusoidal waves of nearby wavelengths

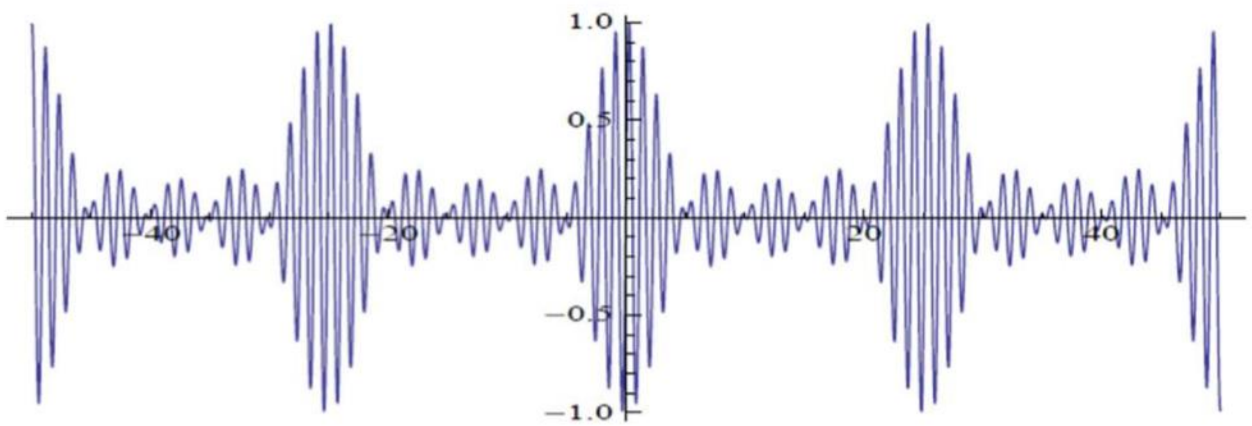
$$\Psi = A \sin\left(\frac{2\pi}{\lambda} x\right) = A \sin(kx)$$



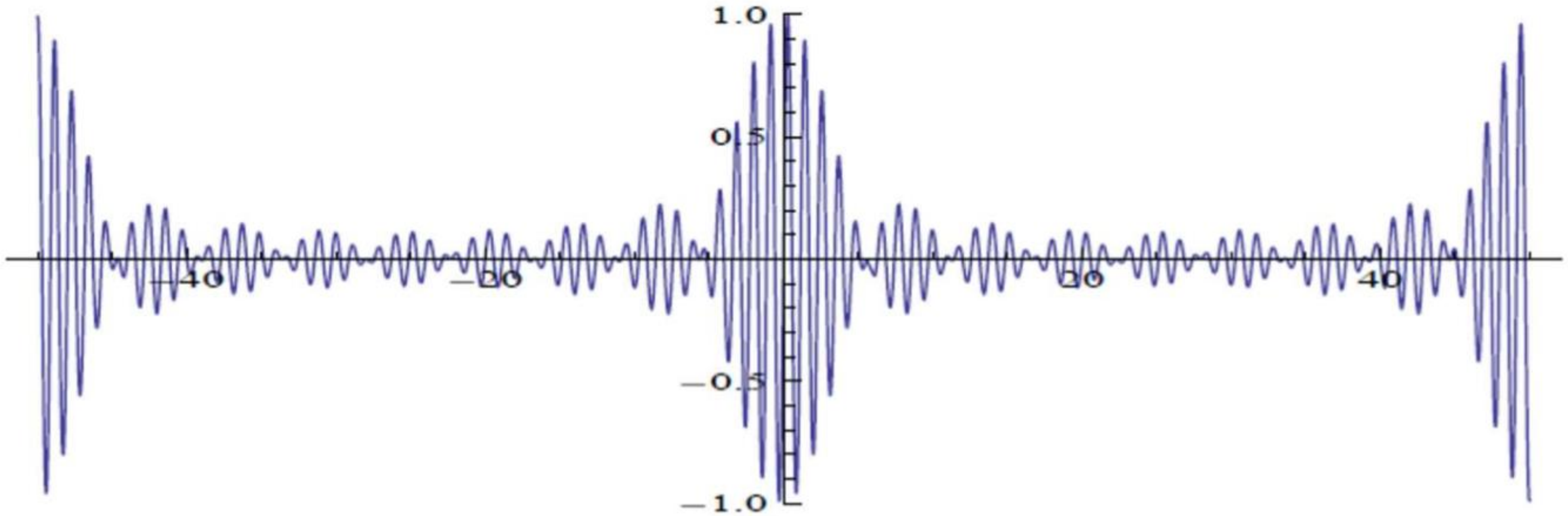
[sin(5x) + sin(6x)]/2



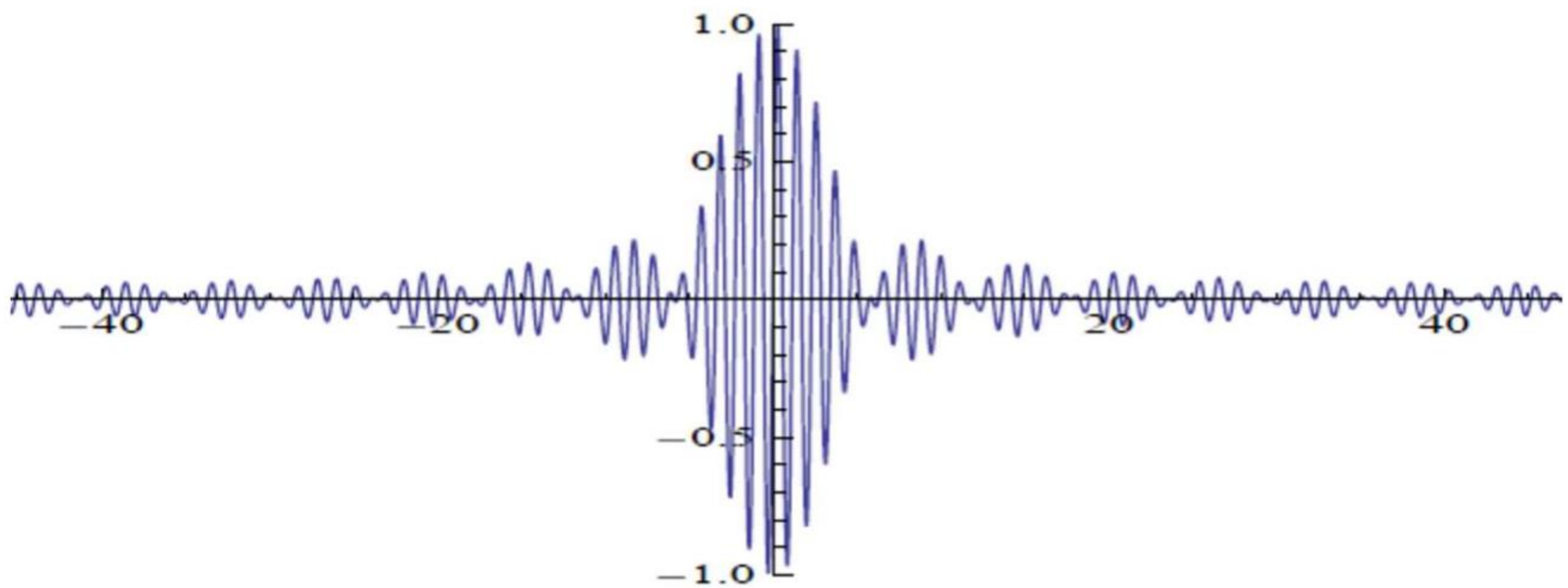
[sin(5x) + sin(5.5x) + sin(6x)]/3



[sin(5x) + sin (5.25x) + sin(5.5x) + sin (5.75x) + sin(6x)]/5

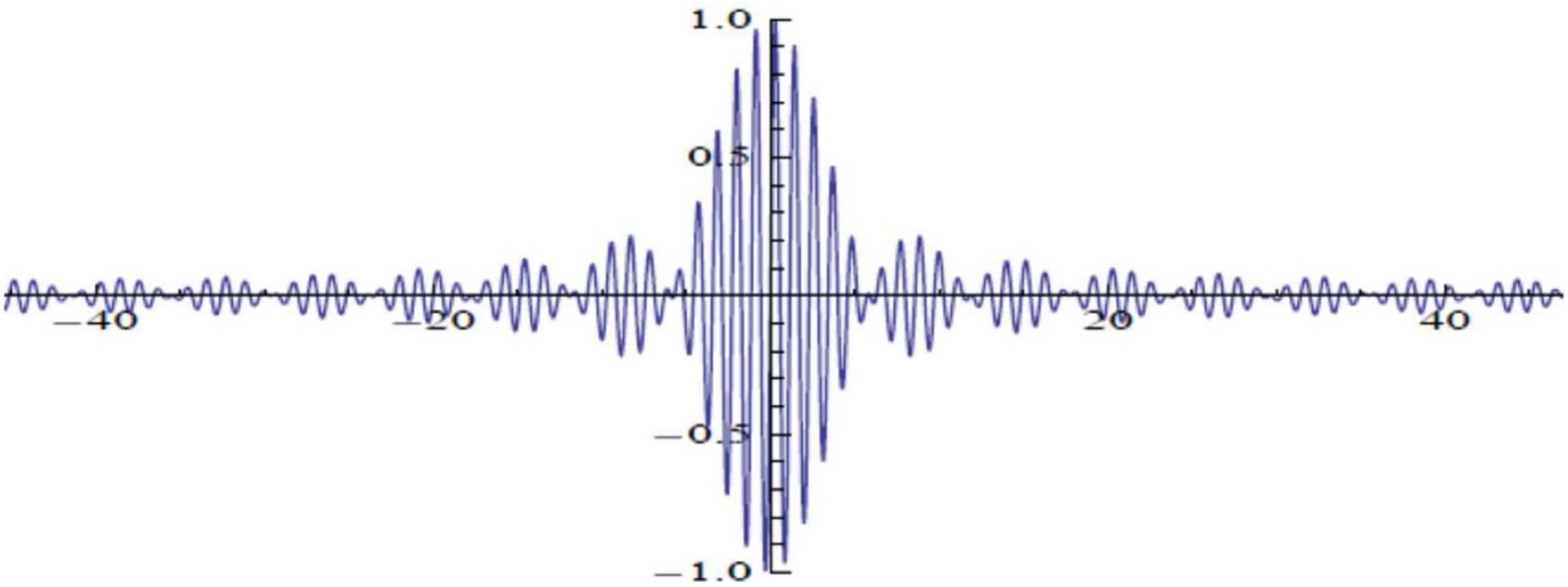


[sin(5x) + sin (5.125x) + sin (5.25x) + sin(5.375x) + sin(5.5x) + sin (5.625x) + sin (5.75x) + sin(5.875x) + sin(6x)]/9



[sin(5x) + sin(5.0625x) + sin (5.125x) + sin(5.1875x) + sin (5.25x) + sin(5.3125x) + sin(5.375x) + sin(5.4375x) + sin(5.5x) + sin(5.5625x) + sin (5.625x) + sin(5.6875x) + sin (5.75x) + sin(5.8125x) + sin(5.875x) + sin(5.9375x) + sin(6x)]/17

Wave Packet

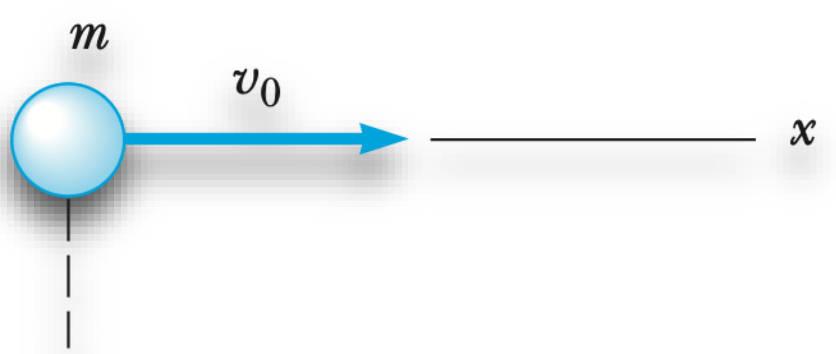


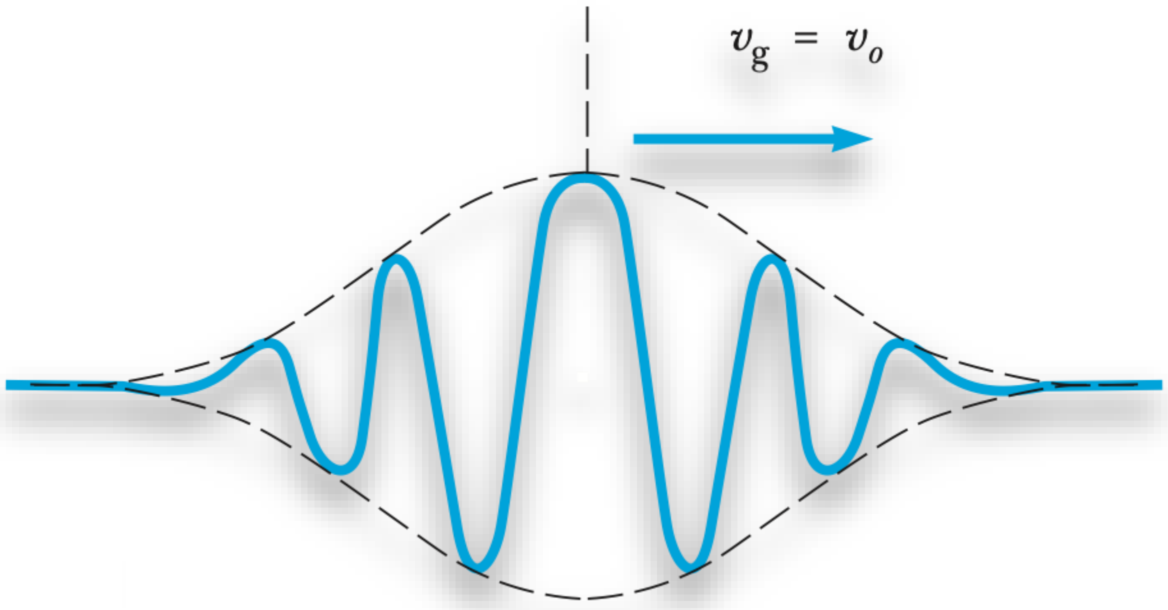
A wave packet is a group of waves with slightly different wavelengths interfering with one another in a way that the amplitude of the group (envelope) is non-zero in the neighbourhood of the particle.

A wave packet is localized; it is a good representation of a particle

Wave Packet

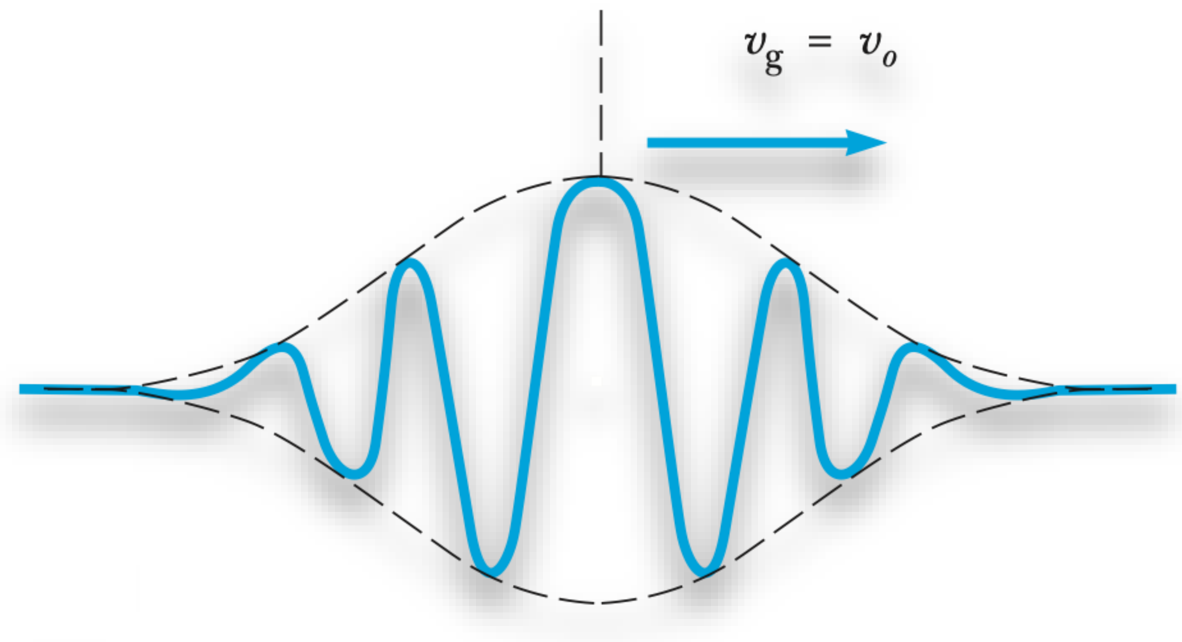
A realistic wave description of a “localized” particle is provided by a **wave packet**, not an ideal sinusoidal wave.





A wave packet has a **finite extent** and within this there is a wave of few cycles.

Wave Packet

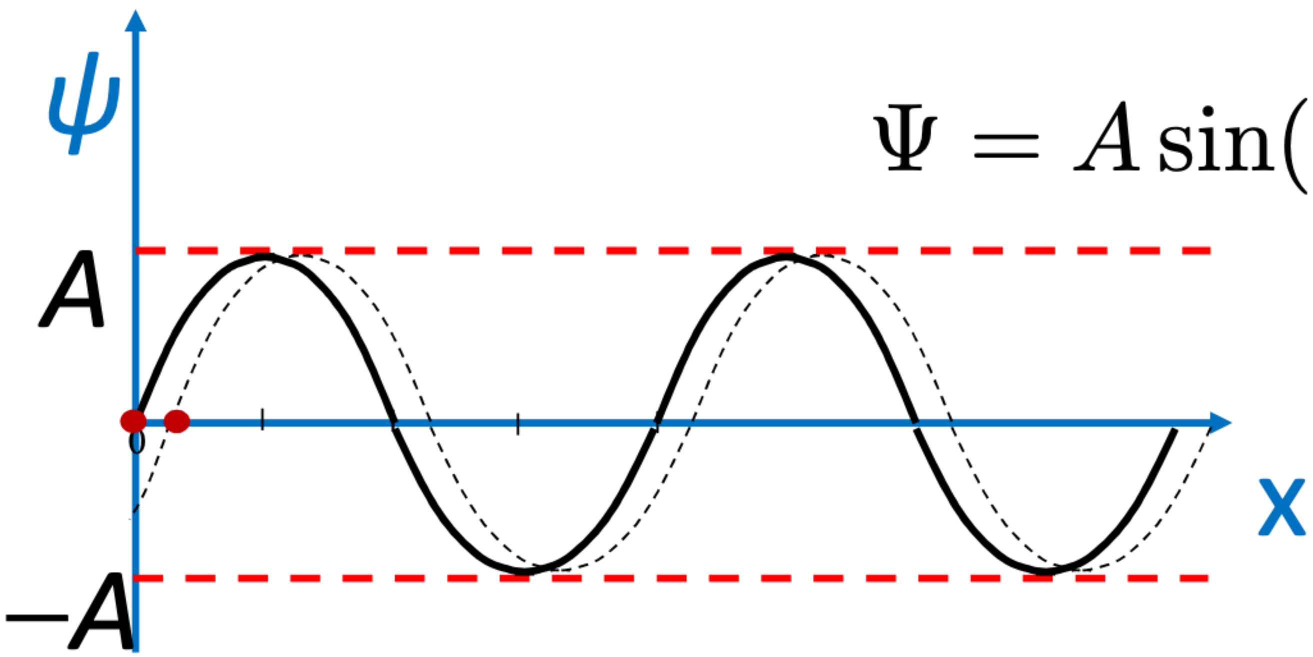


The "wave" in the wave packet does not look like the nice and simple sinusoidal waves we use in doing wave analysis.

The amplitude of the wave is significant only within the extent of the packet.

The “wavelength” changes as a function of the spatial coordinate within the packet.

Phase Velocity



$$\Psi = A\sin(kx - \omega t);\ \ k = \frac{2\pi}{\lambda} \text{ and } \omega = 2\pi\nu$$

Take a point at t = 0 for which ψ = 0. Let time increase to Δt. What would be Δx to maintain ψ = 0.

$$k\Delta x - \omega\Delta t = 0$$

$$v_p = \frac{\Delta x}{\Delta t} = \frac{\omega}{k}$$

Phase velocity is the velocity of a point of constant phase on the wave.

Group Velocity

$$\Psi_1 = A\sin[kx - \omega t]; \text{ and } \Psi_2 = A\sin[(k+\Delta k)x - (\omega + \Delta\omega)t]$$

Let us do superposition of two waves

$$\Psi = \Psi_1 + \Psi_2 = A\sin[kx - \omega t] + A\sin[(k+\Delta k)x - (\omega+\Delta\omega)t]$$

$$\mathit{Using},\ \boldsymbol{\sin a + \sin b = 2\,[\sin(a+b)/2]\,[\cos(a-b)/2]}$$

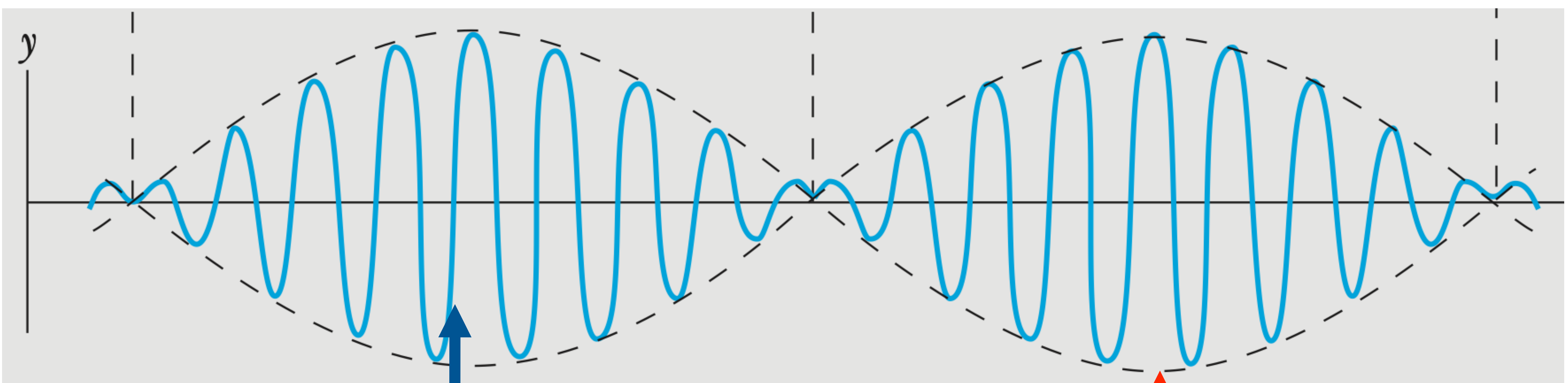
$$\Psi = 2A\sin\left[\frac{(2k+\Delta k)x}{2} - \frac{(2\omega+\Delta\omega)t}{2}\right]\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$

$$\Psi = 2A\sin\left[\left(k + \frac{\Delta k}{2}\right)x - \left(\omega + \frac{\Delta\omega}{2}\right)t\right]\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$

Group Velocity

$$\Psi = 2A\sin\left[\left(k+\frac{\Delta k}{2}\right)x-\left(\omega+\frac{\Delta\omega}{2}\right)t\right]\cos\left(\frac{\Delta kx}{2}-\frac{\Delta\omega t}{2}\right)$$

We see that the first part is like a high frequency wave, modulated by the broad envelope of the second part. The resultant amplitude fluctuates, or “beats”.



High Frequency Wave $\sin\left[\left(k+\frac{\Delta k}{2}\right)x\right]$

Broad Envelope $2A\cos\left(\frac{\Delta kx}{2}\right)$

The envelope and the wave within the envelope move at different speeds

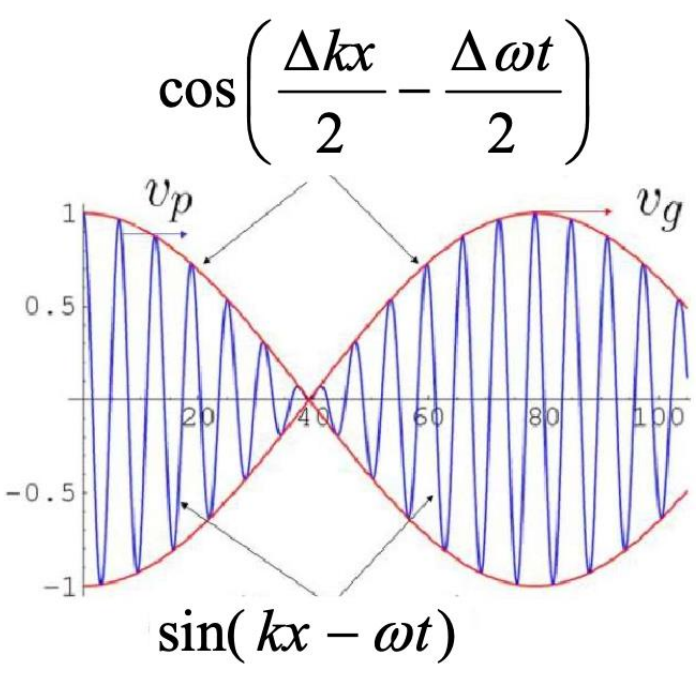
Group Velocity

$$\Psi = 2A\sin\left[\frac{(2k+\Delta k)x}{2} - \frac{(2\omega+\Delta\omega)t}{2}\right]\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$

Δk and $\Delta\omega$ are infinitesimally small quantities

$2k+\Delta k \approx 2k,\ 2\omega+\Delta\omega \approx 2\omega$

$$\Psi = 2A\sin(kx-\omega t)\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$



Slowly varying envelope of frequency $\Delta\omega$

and propagation constant Δk

Group Velocity

Group velocity is the velocity with which the envelope of the wave packet moves.

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

in the limit $\Delta k \to 0; \Delta\omega \to 0$

Velocity of the wave within the envelope (phase velocity)

$$v_p = \frac{\omega + \frac{\Delta\omega}{2}}{k + \frac{\Delta k}{2}} \approx \frac{\omega}{k}$$

$$\omega = k v_p$$

v_g **is the velocity with which the wave packet moves.**

Wave Packet

We can think of adding waves with wavenumber ranging continuously from $k_0 - \Delta k/2$ to $k_0 + \Delta k/2$ and frequencies ranging from $\omega(k_0 - \Delta k/2)$ to $\omega(k_0 + \Delta k/2)$

k_0 is the central wavelength

Δk is the range of the wavelength forming the wave packet

In this case, the group velocity will be given by $v_g = \left(\frac{d\omega}{dk}\right)_{k_0}$

The connection between the group and phase velocities of the composite wave is seen from

$$v_g = \left(\frac{d\omega}{dk}\right)_{k_0} = (v_p)_{k_0} + \left(k\frac{dv_p}{dk}\right)_{k_0}$$

since $\omega = kv_p$

Example

A wave travelling on surface of water has phase velocity proportional to the square root of wavelength. What is the group velocity?

$$v_p = \frac{\omega}{k} = A\sqrt{\lambda} = A\sqrt{\frac{2\pi}{k}} \Longrightarrow \omega = A\sqrt{2\pi k}$$

$$v_g = \frac{d\omega}{dk} = \frac{\pi A}{\sqrt{2\pi k}} = \frac{1}{2} v_p$$

Dispersion Relations

Relation between ω and k is known as dispersion relation.

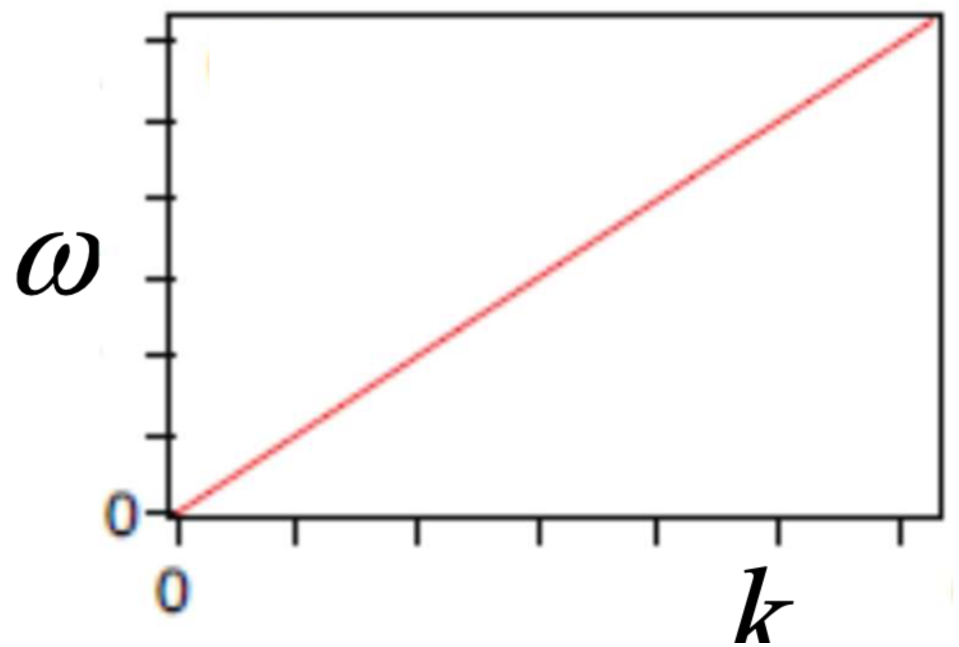
Plot of ω vs k is called the dispersion curve.

Light in vacuum

$$c = \lambda \nu \qquad 2\pi c = 2\pi\lambda\nu$$

$$\therefore (2\pi / \lambda)c = \omega$$

$$\therefore \omega(k) = kc$$



Recommended Readings

Wave Groups and Dispersion, section 5.3 in page 152.