Test 4

5 Take any partition P of $[0,2]$

$$P = \{ 0 = x_0 < x_1 < \ldots < x_n = 2 \}$$

Suppose $1 \in (x_{i-1}, x_i)$ for some i.

$$\text{Then, } L(f,P) = \sum_{j=1}^{i} 1 (x_j - x_{j-1}) + \sum_{j=i+1}^{n} 2(x_j - x_{j-1})$$

$$= x_i - x_0 + 2(x_n - x_i)$$

$$= 4 - x_i < 3$$

$$U(f,P) = \sum_{j=1}^{i-1} 1 (x_j - x_{j-1}) + \sum_{j=i}^{n} 2(x_j - x_{j-1})$$

$$= x_{i-1} - x_0 + 2(x_n - x_{i-1})$$

$$= 4 - x_{i-1} > 3$$

Suppose $1 = x_i$ for some i

Then $L(f,P) = 3$, $U(f,P) = 3$

$$\therefore L(f) = \sup_P L(f,P) = 3$$

$$U(f) = \inf_P U(f,P) = 3$$

$\therefore$ f is Darboux integrable on $[0,3]$

and hence Riemann integrable,

and $\int_0^2 f(x)\,dx = 3$.

6

(a) Given any partition $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ of $[a, b]$,

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \text{ where } m_i = \inf_{[x_{i-1}, x_i]} f(x)$$

$$m_i \geq 0$$
$$\therefore L(f,P) \geq 0$$
$$L(f) \geq L(f,P)$$

Since f is Riemann integrable,

$$\int_a^b f(x)\,dx = L(f) \geq 0$$

Suppose $f(c) \neq 0$ for some $c \in [a, b]$

Since f is continuous, $\exists\, \delta$ s.t.
$$|f(x) - f(c)| < \frac{f(c)}{2}$$
$\forall x \in [a, b]$ s.t. $|x - c| < \delta$.

Note that this implies that we can assume $c \in (a, b)$.
Further, we can choose δ small enough so that $(c - \delta, c + \delta) \subset (a, b)$.

Now consider the partition $\{a < c - \delta < c + \delta < b\}$
$L(f,P) > 0$ (check)
$\therefore L(f) > 0$ and $\int_a^b f(x)\,dx > 0$

Contradiction. $\therefore f(x) = 0 \;\; \forall x \in [a, b]$.

(b) ~~f(x)~~ $f : [-1, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Check that f is Riemann integrable and $\int_{-1}^{1} f(x)\, dx = 0$

7

(i) $$S_n = \frac{1}{n^{5/2}} \sum_{i=1}^{n} i^{3/2} = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{3/2} \times \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{3/2} \left(\frac{i}{n} - \frac{i-1}{n}\right)$$

Thm :

f : Riemann integrable on $[a, b]$

(P_n, t_n) : seq. of tagged partitions of $[a, b]$ such that $\|P_n\| \to 0$ $\boxed{\text{as n tends to inf}}$

Then, $$\lim_{n \to \infty} R(f, P_n, t_n) = \int_a^b f(x)\, dx$$

Define $f : [0, 1] \to \mathbb{R}$, $f(x) = x^{3/2}$

f is cont. and bounded on $[0, 1]$, $\therefore$ f is Riemann integrable.

Let $P_n = \{0, 1/n, \ldots, \frac{n-1}{n}, 1\}$

For each interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, pick the tag $\frac{i}{n}$

Then $S_n = R(f, P_n, t_n)$

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$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} R(f, P_n, t_n)$$

$$= \int_0^1 x^{3/2}\, dx$$

$$= \frac{2}{5} \quad (\text{by FTC Part II})$$

(iii) $$S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in+n^2}}$$

$$= \sum_{i=1}^{n} \frac{1}{\sqrt{\frac{i}{n}+1}} \times \frac{1}{n}$$

$$= \sum_{i=1}^{n} \frac{1}{\sqrt{\frac{i}{n}+1}} \left(\frac{i+1}{n} - \frac{i}{n}\right)$$

Take $f(x) = \dfrac{1}{\sqrt{x+1}}$, $x \in [0,1]$

(iv) $f(x) = \cos \pi x$, $x \in [0,1]$

(v) $$f(x) = \begin{cases} x & x \in [0,1] \\ x^{3/2} & x \in (1,2] \\ x^2 & x \in (2,3] \end{cases}$$

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8

(b) (i) $F(x) = \int_1^{2x} \cos(t^2)\, dt$

(ii) $F(x) = \int_0^{x^2} \cos(t)\, dt$

Define $G(x) = \int_a^x f(t)\, dt$ for a Riemann integrable function f on $[a,b]$ and for $x \in [a,b]$

By FTC I, $G'(x) = f(x)$

Now, suppose $F(x) = \int_a^{u(x)} f(t)\, dt$

Then $F(x) = G(u(x))$

$\therefore F'(x) = G'(u(x))\, u'(x)$

$= f(u(x))\, u'(x)$

For (i), $f(x) = \cos(x^2)$, $u(x) = 2x$

For (ii), $f(x) = \cos x$, $u(x) = x^2$

9 Let $F(a) = \int_a^{a+p} f(t)\, dt$, $a \in \mathbb{R}$

Using FTC I $F'(a) = f(a+p) - f(a) = 0$

10 Brute force

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \sin \lambda t \cos \lambda x\right) dt$$

$$= \frac{1}{\lambda} \left(\sin \lambda x \int_0^x f(t) \cos \lambda t \, dt - \cos \lambda x \int_0^x f(t) \sin \lambda t \, dt \right)$$