MA-111 Calculus II
(D3 & D4)

Lecture 8

B.K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

February 8, 2022

Vector analysis

Curve and path

Line integrals of vector fields

Orientation of Curves

$\mathbb{R}^n$

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean space defined by

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n) \mid x_j \in \mathbb{R}; \quad \forall\, j = 1, 2, \cdots, n\},$$

equipped with the norm

$$\|x\| := \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{\frac{1}{2}}.$$

- Any real number is called a scalar.
- For $n \in \mathbb{N}$, any element from $\mathbb{R}^n$ is called vector. Note this means elements of $\mathbb{R}$ can be thought of both as a scalar and vector. To avoid confusion we will talk about vectors in $\mathbb{R}^n$ for $n > 1$.

Basic structure:

For any $x := (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y := (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and any $a \in \mathbb{R}$:

$$x+y := (x_1+y_1, x_2+y_2, \cdots, x_n+y_n) \in \mathbb{R}^n, \quad \text{sum of two elements in} \quad \mathbb{R}^n$$

$$ax := (ax_1, ax_2, \cdots, ax_n) \in \mathbb{R}^n, \quad \text{Scalar multiplication.}$$

Scalar fields and Vector fields

Let D be a subset of $\mathbb{R}^n$.

Definition: A scalar field on D is a map $f : D \to \mathbb{R}$.

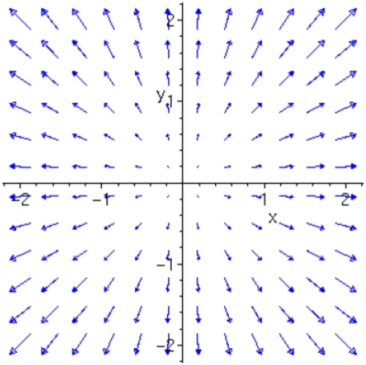
Definition A vector field on D is a map $\mathbf{F} : D \to \mathbb{R}^n$. We choose $n \geq 2$.

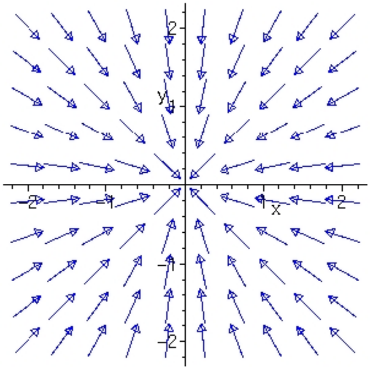
- A scalar field associates a number to each point of D, whereas a vector field associates a vector (of the same space) to each point of D.
- The temperature at a point on the earth is a scalar field.
- The velocity field of a moving fluid, a field describing heat flow, the gravitational field, the magnetic field etc are examples of various vector fields.

Vector fields: Examples

$\mathbf{F}_1(x, y) = (2x, 2y)$

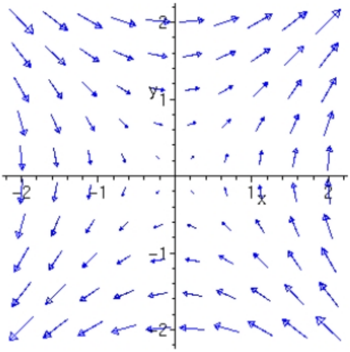
$\mathbf{F}_2(x, y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$



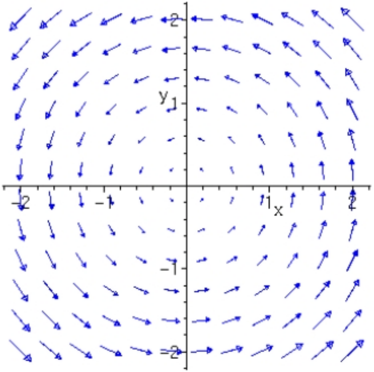


Vector fields: Examples

$\mathbf{F}_3(x, y) = (y, x)$

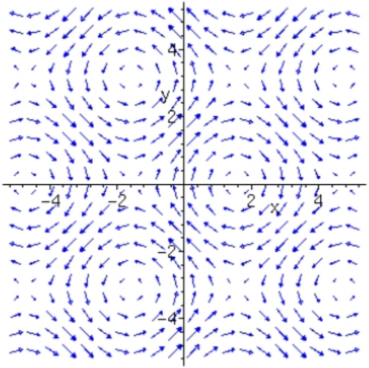


$\mathbf{F}_4(x, y) = (-y, x)$



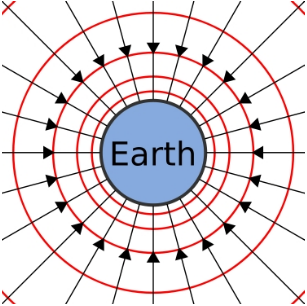
Vector fields: Examples

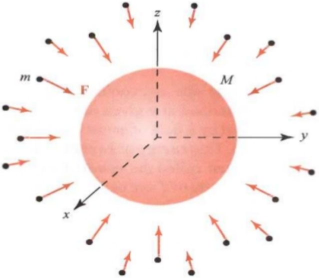
$$\mathbf{F}_5(x, y) = (\sin y, \cos x)$$



The vector fields also occur in nature. Some of this you may have seen in MA 109 as well.

Gravitation fields





The first figure describes the gravitational field of the earth whereas the second one describes that of a body with mass M. The red lines denote the direction of the force exerted on the small particles around the body.

Del operator on Functions

We will assume from now on that our vector fields are smooth wherever they are defined.

One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

The del operator on functions: Gradient field We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give a gradient vector field :

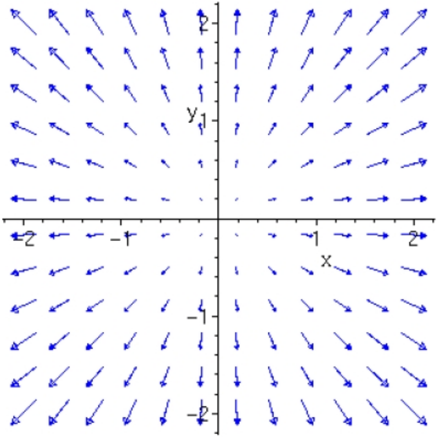
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.

Definition

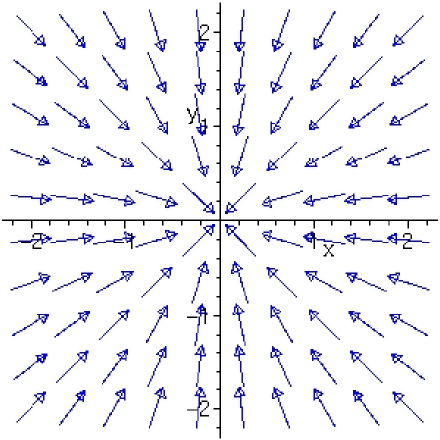
Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$, $n = 2, 3$ be a differentiable function. Then the vector field associated to ∇f is called a gradient vector field.

Gradient fields



$$\mathbf{F}_1(x, y) = (2x, 2y) = \nabla(x^2 + y^2)$$

Gradient fields



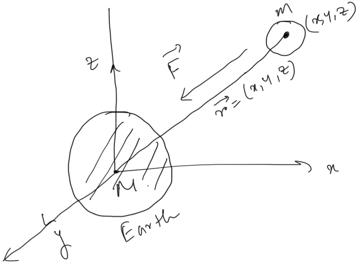
$$\mathbf{F}_2(x, y) = \left(\frac{-x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) = \nabla \left(-\ln(\sqrt{x^2 + y^2}) \right)$$

Gradient Vector fields

Gravitational force field is a gradient field: Placing the origin of a coordinate system at the center of the earth (assumed spherical) with mass M, the force of attraction of the earth on a mass m whose position vector is $\mathbf{r}(x, y, z) = (x, y, z)$ can be given by Newton's Law

$$\mathbf{F}(x, y, z) = -\frac{mMG}{|\mathbf{r}(x, y, z)|^3}\mathbf{r}(x, y, z) = -\nabla V(x, y, z),$$

where $V(x, y, z) = -\frac{mMG}{|\mathbf{r}(x,y,z)|}$. Note that the gravitational force exerted on the mass m acts towards the origin.

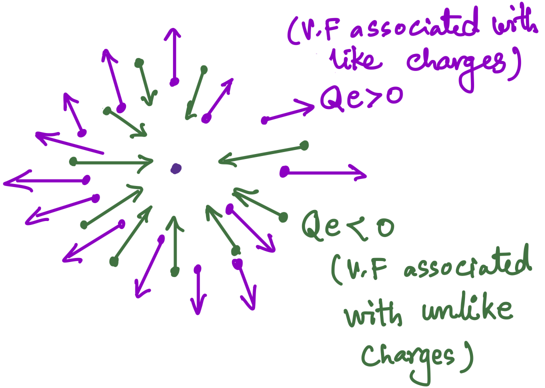


Gradient Vector fields contd.

Coulomb's law says that the force acting on a charge e at a point $\mathbf{r}(x,y,z) = (x,y,z)$ due to a charge Q at the origin is

$$\mathbf{F}(x,y,z) = -\nabla V(x,y,z)$$

where $V(x,yz) = \epsilon Qe/|\mathbf{r}(x,y,z)|$ is the potential. For like charges $Qe > 0$ force is repulsive and for unlike charges $Qe < 0$ the force is attractive.



Definition (Conservative vector field)

A vector field $\mathbf{F}$ is called a conservative vector field if it is a gradient of some scalar function, i..e, there exists a differentiable scalar function f such that $\mathbf{F} = \nabla f$. In this case, f is called a potential function for $\mathbf{F}$.

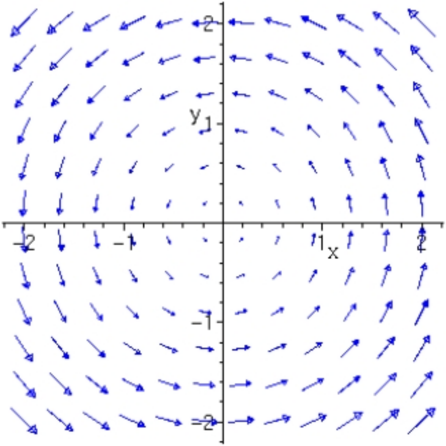
Conservative forces are important as work done along a path will be only dependent on the end points.

Several of the examples we have seen turn out to be gradient vector fields/ conservative vector field. The natural question to ask is which vector field is a gradient field.

There is a neat answer to the above question, which we will see later. Application of the 'Fundamental theorem for line integrals'.

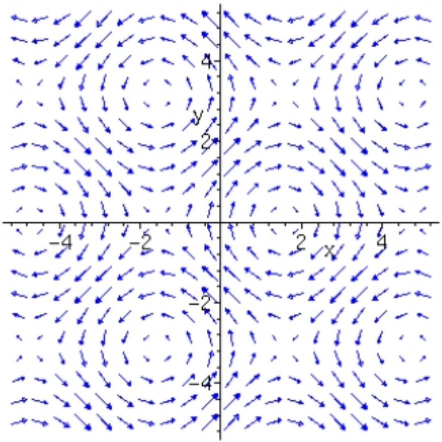
Not all vector fields will turn out to be gradient vector field.

Not gradient fields



$\mathbf{F}_4(x, y) = (-y, x)$, this vector field is not ∇f for any f.

Not gradient fields



$\mathbf{F}_5(x, y) = (\sin y, \cos x)$, this vector field is not ∇f for any f.

How do you check this?

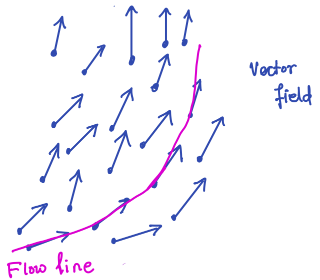
Flow lines for vector field

Vector fields also arise as the tangent vectors to the fluid flow.
Or conversely, given a vector field we can talk about its flow lines.

Definition If $\mathbf{F}$ is a vector field defined from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e., a map $\mathbf{c} : [a, b] \to D$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \quad \forall t \in [a, b].$$

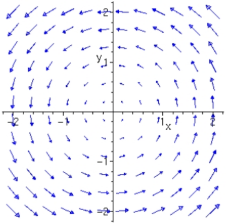
In particular, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}$.



Example: Show that $\mathbf{c}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$, is a flow line for the vector field $\mathbf{F}(x, y) = (-y, x)$.

Ans $\mathbf{c}'(t) = (-\sin t, \cos t)$ and $\mathbf{F}(\mathbf{c}(t)) = (-\sin t, \cos t)$.

Does it have other flow lines? Can you guess by looking at the vector field?



Ans Yes! $\mathbf{c}(t) = (a \cos t, \sin t)$, $t \in [0, 2\pi]$ and any $a > 0$.

Flow line: System of ODEs

Finding the flow line for a given vector field involves solving a system of differential equations, if $\mathbf{c}(t) = (x(t), y(t), z(t))$ then

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)) \\ y'(t) &= Q(x(t), y(t), z(t)) \\ z'(t) &= R(x(t), y(t), z(t)), \end{aligned}$$

where the vector field is given by $\mathbf{F} = (P, Q, R)$.

Such questions are dealt with in MA108.

Curve and path

Recall a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n$.

A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n$.

Both the curve and path are denoted by the same symbol $\mathbf{c}$.

• Let $n = 3$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b]$. The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous. Similarly, $\mathbf{c}$ is a C^1 path, i.e., continuously differentiable if and only if each component is C^1.

• A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

• A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.

• If we write $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in vector notation, the tangent vector to $\mathbf{c}(t)$ is $\mathbf{c}'(t)$, i.e.,

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

• If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrised curve.

Examples of curves

• Let $\mathbf{c}(t) = (\cos 2\pi t, \sin 2\pi t)$ where $0 \leq t \leq 1$. This is a simple closed C^1 (actually smooth) curve.

• Let $\mathbf{c}(t) = (t, t^2)$ where $-1 \leq t \leq 5$ is a simple curve but not closed.

• Let $\mathbf{c}(t) = (\sin(2t), \sin t)$ where $-\pi \leq t \leq \pi$. It traces out a figure 8. It is not a simple but a closed C^1 curve.

• Let $\mathbf{c}(t) = (t^3, t)$ where $-1 \leq t \leq 1$ for some real numbers a, b is a part of the graph of the function $y = x^{1/3}$. This is simple but not a closed curve. Though the function $y = x^{\frac{1}{3}}$ is not a smooth function at origin, but this parametrization is regular.

Work done along a curve

- Recall from Physics, that work done by a particle on which force $\mathbf{F}$ is applied is given by the $\mathbf{F}.ds$ where ds is the displacement.

- If this is in one variable it is just the product and given by dot-product when it is in 2D or 3D space. This idea works when the displacement is straight line.

- If the particle is moving along a curve $\mathbf{c}$ then locally the curve can be approximated by a straight line.

- For a path $\mathbf{c} : [0,1] \to \mathbb{R}^n$ for $n = 2$ or 3 if $\Delta t = t_2 - t_1$ is *very very* small then

$$\Delta s = \mathbf{c}(t_2) - \mathbf{c}(t_1) = \mathbf{c}'(\widehat{t})(t_2 - t_1)$$

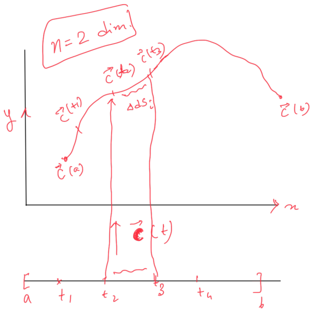
for some $\widehat{t} \in [t_1, t_2]$ by mean value theorem.

- Then work done will have to be computed over these small intervals $[t_i, t_{i+1}]$ for $i = 1, \ldots, n$.

- Total work done $= \sum_{i=1}^{n} \mathbf{F}(\widehat{t_i}) \cdot \mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)$
$= \sum_{i=1}^{n} (\mathbf{F}(\widehat{t_i}) \cdot \mathbf{c}'(\widehat{t_i}))(t_{i+1} - t_i)$.
Does this remind you of something?

Riemann sum: The limit of these Riemann sums as the length of the subintervals tends to zero, if it exists, is defined to be the line integral of the vector field **F** over the curve **c** and is denoted by

$$\int_{\mathbf{c}} \mathbf{F}.d\mathbf{s}.$$



Line integrals of vector fields

Assume that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2$, is continuous and the curve $\mathbf{c} : [a, b] \to D$ is C^1.

Then we define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

If $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, we see that

$$\begin{aligned} &\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\ &= \int_a^b \left(F_1(\mathbf{c}(t)) \frac{dx(t)}{dt} + F_2(\mathbf{c}(t)) \frac{dy(t)}{dt} + F_3(\mathbf{c}(t)) \frac{dz(t)}{dt} \right) dt. \end{aligned}$$

Because of the form of the right hand side the line integral is sometimes written as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b F_1 dx + F_2 dy + F_3 dz.$$

The expression on the right hand side is just alternate notation for the line integral. It does not have any independent meaning.

An example

Example 1: Evaluate
$$\int_{\mathbf{c}} x^2 dx + xydy + dz,$$
where $\mathbf{c} : [0,1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1)$.

Solution: Let $\mathbf{c}(t) = (t, t^2, 1)$.
Let $\mathbf{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) = (x^2, xy, 1)$.

Thus $F_1(t, t^2, 1) = t^2$, $F_2(t, t^2, 1) = t^3$ and $F_3(t, t^2, 1) = 1$.

We have $\mathbf{c}'(t) = (1, 2t, 0)$, hence
$$(F_1(t, t^2, 1), F_2(t, t^2, 1), F_3(t, t^2, 1)) \cdot \mathbf{c}'(t) = t^2 + 2t^4 + 0.$$

$$\int_{\mathbf{c}} x^2 dx + xydy + dz = \int_0^1 (t^2 + 2t^4) dt = 11/15.$$

Example 2 (Marsden, Tromba, Weinstein): Find the work done by the force field $\mathbf{F} = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Solution: The work done is given by

$$\begin{aligned} W &= \int_{\mathbf{c}} \mathbf{F} \cdot ds \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t)\ dt \\ &= \int_0^{2\pi} (\cos t + \sin t) dt \\ &= (\sin t - \cos t)|_0^{2\pi} = 0 \end{aligned}$$

Integrating along successive paths

It is easy to see that if $\mathbf{c}_1$ is a path joining two points P_0 and P_1 and $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), which is C^1 then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

This property follows directly from the corresponding property for Riemann integrals:

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt,$$

where c is a point between a and b.

This allows us to define integration over peicewise differentiable curves for example the perimeter of a square.

Let the curve $\mathbf{c}$ be a union of curves $\mathbf{c}_1, \ldots, \mathbf{c}_n$. We often write this as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \ldots \mathbf{c}_n$, where end point of $\mathbf{c}_i$ is the starting point of $\mathbf{c}_{i+1}$ for all $i = 1, \ldots, n-1$.

Then we can define

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot ds.$$

Divide the curve $\mathbf{c}$ at a point p into two curves $\mathbf{c}_1$ and c_2. Then there it is easy to verify that $\int_{\mathbf{c}} \mathbf{F} \cdot = \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \int_{\mathbf{c}_2} \mathbf{F} \cdot ds$.

Let $\mathbf{c}$ be a curve on $[a, b]$ and $\widetilde{\mathbf{c}}(t) = \mathbf{c}(a + b - t)$, that is the curve $\widetilde{\mathbf{c}}$ traversed in the reverse direction and is denoted by $-\mathbf{c}$. What is $\int_{\mathbf{c}} \mathbf{F} \cdot ds + \int_{-\mathbf{c}} \mathbf{F} \cdot ds$?

$\int_{c} \mathbf{F} \cdot ds = -\int_{-c} \mathbf{F} \cdot ds$ (use change of variables formula).

Different parametrizations of the same path

Example 1: Let $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$. Then $c_2(t) = (\cos 2t, \sin 2t)$ for $0 \le t \le \pi$, the paths are different as a function but the curves traversed are the same.

Example: 2: Here is an example of a simple path with three different parametrisations with the same domain.
Take the straight line segment between $(0,0,0)$ and $(1,0,0)$.

Here are three different ways of parametrising it:

$$\{t,0,0)\}, \quad \{(t^2,0,0)\} \quad \text{and} \quad \{(t^3,0,0)\},$$

where $0 \le t \le 1$.

Reparametrisation

Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a path which is non-singular, that is, $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$.

- Suppose we now make change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism (this means that h is bijective, $\mathcal{C}^1$ and so is its inverse) from $[\alpha, \beta]$ to $[a, b]$. We let $\gamma(u) = \mathbf{c}(h(u))$.
- We will assume that $h(\alpha) = a$ and $h(\beta) = b$.
- Then γ is called a reparametrisation of $\mathbf{c}$.
- Because h is a C^1 diffeomorphism, γ is also a C^1curve.

The line integral of a vector field $\mathbf{F}$ along γ is given by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\alpha^\beta \mathbf{F}(\gamma(u)) \cdot \gamma'(u) du = \int_\alpha^\beta \mathbf{F}(c(h(u)) \cdot \mathbf{c}'(h(u)) h'(u) du,$$

where the last equality follows from the chain rule. Using the fact that $h'(u)du = dt$, we can change variables from u to t to get

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_\mathbf{c} \mathbf{F} \cdot d\mathbf{s}.$$

Orientation of Curves

For given two points P and Q on $\mathbb{R}^n$ for $n = 2, 3$, and a path connecting them, we can ask whether the path is traversed from P to Q or from Q to P?

Since a path from P to Q is a mapping $\mathbf{c} : [a, b] \to \mathbb{R}^n$ with $\mathbf{c}(a) = P$ and $\mathbf{c}(b) = Q$, (or vice-versa), it allows us to determine the direction in which the path is traversed. This direction of the path is called its Orientation.

If the reparametrization $\gamma(\cdot) = \mathbf{c}(h(\cdot))$ preserves the orientation of $\mathbf{c}$, then

$$\int_\gamma \mathbf{F}.\mathbf{ds} = \int_\mathbf{c} \mathbf{F}.\mathbf{ds}.$$

If the reparamtrization reverses the orientation, then

$$\int_\gamma \mathbf{F}.\mathbf{ds} = -\int_\mathbf{c} \mathbf{F}.\mathbf{ds}.$$

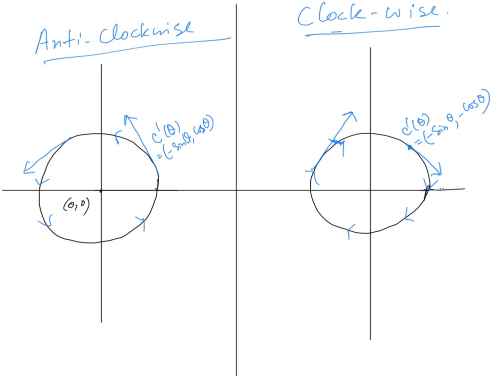
Curves on plane

Let us consider the paths lying in $\mathbb{R}^2$, namely, Planar curves.

For a simple closed planar curve, we get a choice of direction- clockwise or anti-clockwise.

Ex. $\gamma(\theta) = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi]$. This is a circle with direction anti-clockwise.

Set $\gamma_1(\theta) = (\cos(\theta), -\sin(\theta))$, $\theta \in [0, 2\pi]$. It is circle with clockwise direction.



The argument just made shows that the line integral is independent of the choice of parametrisation - it depends only on the image of the non-singular parametrised path.

- A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction. Often the line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$ or by $\int_C F_1 dx + F_2 dy + F_3 dz$.
- To evaluate $\int_C \mathbf{F}.\mathbf{ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F}.\mathbf{ds} := \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$

- '$\oint_C$' means the line integral over a closed curve C.