MA111 Tutorial Solutions
Spring 2022

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Contents

Tutorial Sheet 1 (Multiple Integrals)

1.

(a) Let $R := [0,1] \times [0,1]$ and $f(x,y) := [x] + [y] + 1$ for all $(x,y) \in R$, where $[u]$ is the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Using the definition of integration over rectangles, show that f is integrable over R. Also, find its value.

(b) Let $R := [0,1] \times [0,1]$ and $f(x,y) := (x+y)^2$ for all $(x,y) \in R$. Show that f is integrable over R and find its value using Riemann sum.

(c) Let $R := [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be integrable. Show that $|f|$ is also integrable over R.

(d) Check the integrability of the function f over $[0,1] \times [0,1]$;

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{and } y \text{ are rational numbers} \\ -1 & \text{otherwise} \end{cases}$$

What do you conclude about the integrability of $|f|$?

Sol.

(a) We have,

$$f(x,y) = \begin{cases} 1 & x \in [0,1), y \in [0,1) \\ 2 & x = 1, y \in [0,1) \\ 2 & x \in [0,1), y = 1 \\ 3 & x = 1, y = 1 \end{cases}$$

Thus f is bounded on R. Let $P_n := \{(x_i, y_j) \mid i \in \{0,1\ldots n\}, j \in \{0,1\ldots n\}\}$ be a regular partition of R. Then we have

$$R_{ij} := \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right] \forall\, i,j \in \{0,1\ldots n-1\}$$

Clearly,

$$L(f, P_n) = 1$$

Let us compute $U(f, P_n)$.

$$\begin{aligned} U(f,P_n) &= \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \frac{1}{n^2} + \sum_{i=n-1}^{n-1}\sum_{j=0}^{n-1} \frac{2}{n^2} + \sum_{j=n-1}^{n-1}\sum_{i=0}^{n-1} \frac{2}{n^2} - \frac{1}{n^2} \\ &= \frac{(n-1)^2}{n^2} + 2 \times \frac{2n}{n^2} - \frac{1}{n^2} \\ &= 1 + \frac{2}{n} \end{aligned}$$

Thus,

$$U(f,P_n) - L(f,P_n) = \frac{2}{n}$$

Given any $\epsilon > 0$, choose $n = \lfloor \frac{2}{\epsilon} \rfloor + 1$. Thus, we see that

$$U(f,P_n) - L(f,P_n) < \epsilon$$

By the Riemann Condition, we are done. $\square$

(b) A function $f : R \to \mathbb{R}$ is said to be integrable over R if it is Darboux or Riemann integrable over R [lecture 1, slide 17]. Observe that $f(x, y) = (x + y)^2 \leq 2^2 = 4\ \forall\ (x, y) \in R$, hence f is bounded on R. This allows us to use the following theorem [lec. 1, sl. 18]:

> $f : R \to \mathbb{R}$ is Riemann integrable if and only if the Riemann sum
>
> $$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij})\Delta_{ij},$$
>
> where P_n is the sequence of regular partitions of R, tends to the same limit $S \in \mathbb{R}$ as $n \to \infty$ for any choice of tag t, and $\Delta_{ij} = (x_{i+1} - x_i)(y_{i+1} - y_i)$.

That f is integrable on R can be concluded directly from the fact that it is continuous; the value of the integral is then simply given by the limit S. The computation is essentially the same as the examples done in class [lec. 1, sl. 19, 20].

The partitions P_n are given by the subrectangles $\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$ and tags $t = \left\{t_{ij} = \left(\frac{i}{n}, \frac{j}{n}\right)\right\}$ for $i, j = 0, 1, \ldots, n-1$. The Riemann sum is then given by

$$\begin{aligned}
S(f, P_n, t) &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij})\Delta_{ij} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(\frac{i}{n} + \frac{j}{n}\right)^2 \frac{1}{n^2} \\
&= \frac{1}{n^4} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (i + j)^2 = \frac{1}{n^4} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[i^2 + j^2 + 2ij\right] \\
&= \frac{1}{n^4} \left[\sum_{i=0}^{n-1} i^2 \cdot \sum_{j=0}^{n-1} 1 + \sum_{i=0}^{n-1} 1 \cdot \sum_{j=0}^{n-1} j^2 + 2\sum_{i=0}^{n-1} i \cdot \sum_{j=0}^{n-1} j\right] \\
&= \frac{1}{n^4} \left[\frac{(n-1)(n)(2n-1)}{6} n + n\frac{(n-1)(n)(2n-1)}{6} + 2\frac{(n-1)(n-2)}{2} \frac{(n-1)(n-2)}{2}\right] \\
&= \frac{7}{6} - \frac{3}{n} - \frac{3}{2n^2} - \frac{10}{3n^3} + \frac{12}{n^4}
\end{aligned}$$

One can check that $S(f, P_n, t) \to \frac{7}{6} = S$. Hence the value of the integral is $\boxed{\frac{7}{6}}$. □

(c) Given an integrable function $f : R \to \mathbb{R}$, we wish to show that $g := |f|$ must also be integrable over R. First note that for any $(x_1, y_1), (x_2, y_2) \in R$,

$$|g(x_1, y_1) - g(x_2, y_2)| = \big||f(x_1, y_1)| - |f(x_2, y_2)|\big| \leq |f(x_1, y_1) - f(x_2, y_2)|.$$

Further, f must be bounded on R (by definition of integrability [lecture 1, slide 17]). Thus we may use the following theorem, also called the *Riemann condition* [lec. 1, sl. 13]:

> A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that
>
> $$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon,$$
>
> where $U(f, P_\epsilon), L(f, P_\epsilon)$ are the upper and lower double sums with respect to the parition P_ϵ (respectively).

Fix an $\epsilon > 0$. Then from the above theorem, we have a partition P_ϵ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$, i.e.

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{ij}(f)\Delta_{ij} - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij}(f)\Delta_{ij} < \epsilon \implies \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [M_{ij}(f) - m_{ij}(f)]\Delta_{ij} < \epsilon,$$

where $M_{ij}(f), m_{ij}(f)$ denote the supremum and infimum of f over the rectangle R_{ij} of the partition. Since $|g(x) - g(y)| \leq |f(x) - f(y)|$, we have $M_{ij}(g) - m_{ij}(g) \leq M_{ij}(f) - m_{ij}(f)$. This gives us

$$U(g, P_\epsilon) - L(g, P_\epsilon) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [M_{ij}(g) - m_{ij}(g)]\Delta_{ij} \leq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} [M_{ij}(f) - m_{ij}(f)]\Delta_{ij}.$$

Since the quantities on the right and left are both positive, we have $|U(g, P_\epsilon) - L(g, P_\epsilon)| < \epsilon$. Noting that the choice of ϵ was arbitrary, we have shown that for every $\epsilon > 0$, there is a partition P_ϵ such that $|U(g, P_\epsilon) - L(g, P_\epsilon)| < \epsilon$. Once again invoking the Riemann condition, we conclude that $g = |f|$ is integrable. $\square$

(d) Take any partition of $[0, 1] \times [0, 1]$. As we know, $\mathbb{Q} \times \mathbb{Q}$ is a dense subset of $\mathbb{R}^2$, so in any subrectangle there will be a point (x, y) with both coordinates rational, and also there will be point inside the subrectangle with both coordinates irrational.

Observe that the value of f at those points is 1 and -1 respectively which is also the inf and sup on that rectangle. Thus the uppersum $U(f, P)$ for any partition is

$$\sum_{i,j} (1)\, \Delta_{ij} = 1.$$

And the lower sum $L(f, P)$ is

$$\sum_{i,j} (-1)\, \Delta_{ij} = -1.$$

Which shows that $\inf U(f, P) \neq \sup L(f, P)$, or the function isn't Riemann integrable, but $|f|$ is the constant function 1 which is trivially integrable. $\blacksquare$

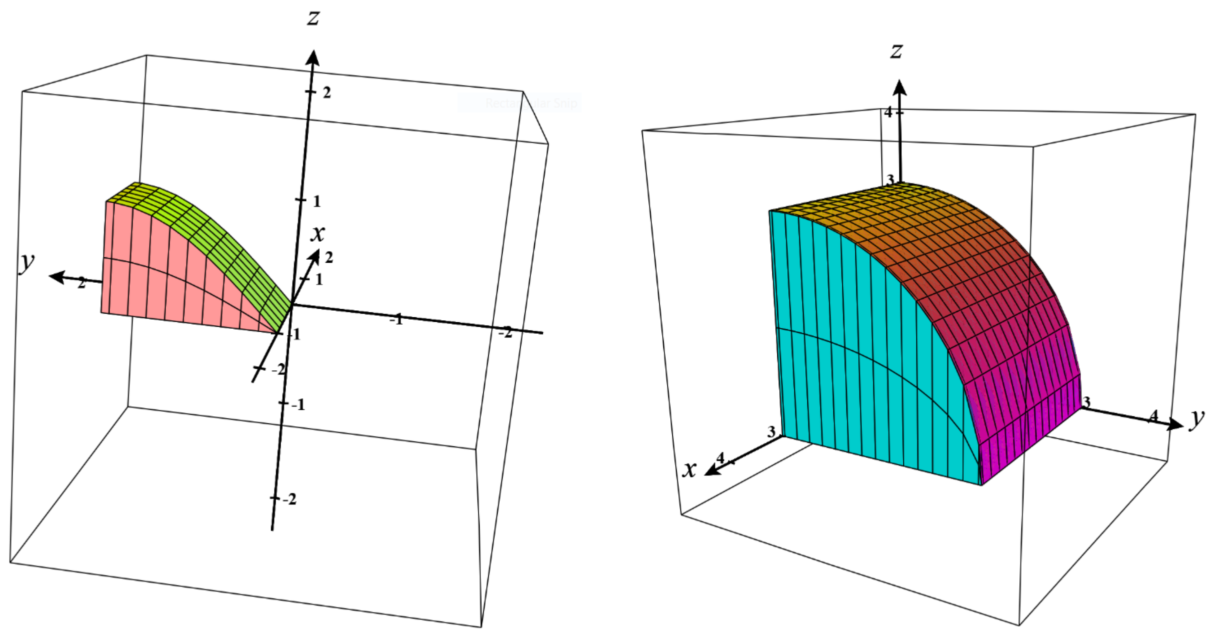
2.

(a) Sketch the solid bounded by the surface $z = \sin y$, the planes $x = -1, x = 0, y = 0$ and $y = \dfrac{\pi}{2}$ and the xy plane and compute its volume.

(b) The integral $\iint_R \sqrt{9 - y^2}\,dx\,dy$, where $R = [0,3] \times [0,3]$, represents the volume of a solid. Sketch the solid and find its volume.

Sol.

The solids are sketched in Fig. 1.



(a) $z = \sin y$

(b) $z = \sqrt{9 - y^2}$

Figure 1: Figure showing the solid surfaces for question 2

(a) $\text{Volume} = \displaystyle\int_{[-1,0]} \int_{[0,\pi/2]} \sin y\,dy\,dx = \int_{[-1,0]} 1\,dx = \boxed{1}$

(b) $\text{Volume} = \displaystyle\int_{[0,3]} \int_{[0,3]} \sqrt{9 - y^2}\,dy\,dx = \int_{[0,3]} \frac{9\pi}{4}\,dx = \boxed{\frac{27\pi}{4}}$

■

3. Consider the function $f : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} 1 - 1/q & \text{if } x = p/q \text{ where } p, q \in \mathbb{N} \text{ are relatively prime and } y \text{ is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

Show that f is integrable but the iterated integrals do not always exist.

Sol.

Clearly $f(x, y) \in [0, 1]\ \forall\ (x, y) \in R = [0, 1] \times [0, 1]$. Take $\mathbb{R} \ni \epsilon > 0$. For some y, define a set S,

$$S = \{x \mid 1 - f(x, y) \geq \epsilon,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ y \in \mathbb{Q}\}$$

Thus, S has all rational numbers $0 \leq \frac{p}{q} \leq 1$ satisfying $q \leq \frac{1}{\epsilon}$, $q \in \mathbb{N}$. (*Note: S is a finite set, with a finite number of elements, say* L.)

Consider a partition $P_\epsilon = \{x_0, x_1, x_2, \dots, x_m\} \times \{y_0, y_1, y_2, \dots, y_n\}$ with $x_0 = y_0 = 0$ and $x_m = y_n = 1$ such that,

$$x_j - x_{j-1} < \frac{\epsilon}{L}\ \forall j \in \{1, 2, 3, \dots, m\}$$

$$y_k - y_{k-1} < \frac{\epsilon}{L}\ \forall k \in \{1, 2, 3, \dots, n\}$$

$$\implies \|P_\epsilon\| < \frac{\epsilon}{L}$$

We define the following for brevity:

$$P_{\epsilon jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$$

$$\Delta_{jk} = (x_j - x_{j-1})(y_k - y_{k-1})$$

$$m_{jk} = \inf_{(x,y) \in P_{\epsilon jk}} f(x, y)$$

$$M_{jk} = \sup_{(x,y) \in P_{\epsilon jk}} f(x, y)$$

$M_{jk} = 1\ \forall\ (j, k)$ since between any 2 numbers, there are infinitely many irrational numbers. $m_{jk} = 1 - \frac{1}{q}$ for some $q \in \mathbb{N}$. Thus,

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{j=1}^{m} \sum_{k=1}^{n} M_{jk}\Delta_{jk} - \sum_{j=1}^{m} \sum_{k=1}^{n} m_{jk}\Delta_{jk}$$

$$\implies U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{j=1}^{m} \sum_{k=1}^{n} (M_{jk} - m_{jk})\,\Delta_{jk}$$

The sum is now split into two parts: one containing at least one point with $x \in S$ and the other with $x \notin S$:

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{j=1}^{m} \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \phi}}^{n} (M_{jk} - m_{jk})\,\Delta_{jk} + \sum_{j=1}^{m} \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] = \phi}}^{n} (M_{jk} - m_{jk})\,\Delta_{jk}$$

In the first part, $m_{jk} \leq 1 - \frac{1}{q}$ and thus $M_{jk} - m_{jk} \geq \frac{1}{q} = \epsilon$. Also, note that $M_{jk} - m_{jk} \leq 1$.

$$\implies \sum_{j=1}^{m} \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \phi}}^{n} (M_{jk} - m_{jk})\,\Delta_{jk} \leq \sum_{j=1}^{m} \sum_{\substack{k=1 \\ S \cap [x_{j-1}, x_j] \neq \phi}}^{n} \Delta_{jk} \ \leq 2L\left(\frac{\epsilon}{L} \times 1\right) \leq 2\epsilon$$

since there are at most $2L$ number of $\left(\frac{\epsilon}{L} \times 1\right)$ rectangles which are proper subset of $P_{\epsilon jk}$ for some (j, k) such that $S \cap [x_{j-1}, x_j] \neq \phi$.

For the second part, we have $m_{jk} > 1 - \frac{1}{q}$ and thus, $M_{jk} - m_{jk} < \frac{1}{q} = \epsilon$. The upper limit of the summation of Δ_{jk} over all the values of j and k is the area of $R = [0,1] \times [0,1]$ which is 1.

$$\sum_{j=1}^{m} \sum_{\substack{k=1 \\ S\cap[x_{j-1},x_j]\neq\phi}}^{n} (M_{jk} - m_{jk})\Delta_{jk} \leq \sum_{j=1}^{m} \sum_{\substack{k=1 \\ S\cap[x_{j-1},x_j]\neq\phi}}^{n} \epsilon\Delta_{jk} \leq \epsilon$$

Consequently,

$$U(f, P_\epsilon) - L(f, P_\epsilon) \leq 2\epsilon + \epsilon = 3\epsilon$$

3ϵ can be made arbitrarily small. Thus $f(x, y)$ is integrable on $R = [0,1] \times [0,1]$.
Before we move to the next part, we will introduce two functions here:

Thomae's function is a real-valued function of a real variable that can be defined as:

$$\mathbf{T}(x) := \begin{cases} \frac{1}{q} & x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N} p \text{ and } q \text{ are coprime} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Dirichlet function is a *cousin* of Thomae's function and can be defined as:

$$\mathbf{1}_{\mathbb{Q}}(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Now, we calculate the iterated integrals.

- **Case I:**
  Define $\Phi^y(x) := f(x, y)$ for some fixed y.

$$\Phi^y(x) = \begin{cases} \begin{cases} 1 - \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases} & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$$

  Thus, for a given y, $\Phi^y(x)$ is either the constant function 1 or a function that can be derived from Thomae's function in x, both of which we know are integrable for $x \in [0,1]$. This yields,

$$\int_0^1 \Phi^y(x)dx = \begin{cases} \int_0^1 1 - \mathbf{T}(x)dx & y \in \mathbb{Q} \\ \int_0^1 1dx & y \notin \mathbb{Q} \end{cases}$$

  Utilizing the fact the integral of $\mathbf{T}(x)$ over any sub-interval of $[0,1]$ is 0 (Justify), we can write,

$$\int_0^1 \Phi^y(x)dx = 1$$

  Consequently,

$$\int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 \left(\int_0^1 \Phi^y(x)dx \right) dy = \int_0^1 1dy = 1$$

- **Case II:**
  Define $\Phi^x(y) := f(x, y)$ for some fixed x.

$$\Phi^x(y) = \begin{cases} \begin{cases} 1 - k_x & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases} & x = \frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

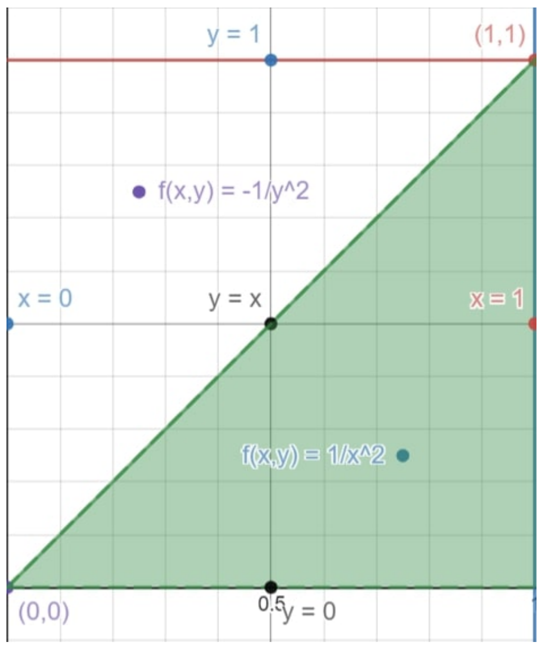
  where $k_x = \frac{1}{q}$.
  Observe that for a given x, $\Phi^x(y)$ is either the constant function 1 or $1 - \mathbf{1}_{\mathbb{Q}}(y)$. However, we know that $\mathbf{1}_{\mathbb{Q}}(y), y \in [0,1]$ is not integrable (Justify). Therefore the iterated integral does not exist. ■

4. Consider the function $f : [0, 1] \times [0, 1] \mapsto \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Is f integrable over the rectangle? Do both iterated integrals exist? If they exist, do they have the same value?

Sol.



Observe that f is not bounded in $R = [0, 1] \times [0, 1]$ near $x = 0$ and $y = 0$. Thus, we can immediately claim that f is not integrable in R. However, this *does not* imply that the iterated integrals do not exist.

Evaluating iterated integrals:

For $y = 0$, $f(x, y) = 0$ and thus $\int_{x=0}^{1} f(x, y)\, dx = 0$.

For $y \in (0, 1]$,

$$\int_{x=0}^{1} f(x, y)\, dx = \int_{x=0}^{y} f(x, y)\, dx + \int_{x=y}^{1} f(x, y)\, dx$$

.

Then,

$$\int_{x=0}^{1} f(x, y)\, dx = \int_{x=0}^{y} \frac{-1}{y^2}\, dx + \int_{x=y}^{1} \frac{1}{x^2}\, dx$$

$$\implies \int_{x=0}^{1} f(x, y)\, dx = \frac{-1}{y} + \frac{1}{y} - 1 = -1$$

Thus, define,

$$A(y) = \int_{x=0}^{1} f(x, y)\, dx = \begin{cases} 0 & \text{for } y = 0 \\ -1 & \text{for } y \in (0, 1) \end{cases}$$

This function is bounded and has a single point of discontinuity, and hence Riemann integrable.

$$\int_{y=0}^{1} \left(\int_{x=0}^{1} f(x,y)\, dx \right) dy = \int_{y=0}^{1} A(y)\, dy = -1$$

Similarly, let's evaluate the iterated integral: $\displaystyle\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\, dy \right) dx$ by repeating the above procedure.

For $x = 0$, $f(x,y) = 0$ and thus $\displaystyle\int_{y=0}^{1} f(x,y)\, dy = 0$.

For $x \in (0, 1]$,

$$\begin{aligned}
\int_{y=0}^{1} f(x,y)\, dy &= \int_{y=0}^{x} f(x,y)\, dy + \int_{y=x}^{1} f(x,y)\, dy \\
&= \int_{y=0}^{x} \frac{1}{x^2}\, dy + \int_{y=x}^{1} \frac{-1}{y^2}\, dy \\
&= \frac{1}{x} + 1 - \frac{1}{x} \\
&= 1
\end{aligned}$$

Thus, define

$$B(x) = \int_{y=0}^{1} f(x,y)\, dy = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \in (0,1) \end{cases}$$

As discussed above, this function is Riemann integrable (Why?). Hence, the iterated integral:

$$\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\, dy \right) dx = \int_{y=0}^{1} B(x)\, dx = 1$$

Clearly both are iterated integral are unequal. ■

5. For the following, write an equivalent iterated integral with the order of integration reversed and verify if their values are equal.

(a) $\int_0^1 \left(\int_0^1 \ln[(x+1)(y+1)]dx\right) dy$

(b) $\int_0^1 \left(\int_0^1 (xy)^2 \cos(x^3) dx\right) dy$.

Sol.

(a)

$$\begin{aligned}\int_0^1 \left(\int_0^1 \ln[(x+1)] + \ln[(y+1)]dx\right) dy &= \int_0^1 [(2\ln 2 - 1) + \ln[(y+1)]]dy \\ &= 4\ln 2 - 2\end{aligned}$$

Since the area considered is a square, interchanging the variables does not affect the limits,

$$\begin{aligned}\int_0^1 \left(\int_0^1 \ln[(x+1)] + \ln[(y+1)]dy\right) dx &= \int_0^1 [(2\ln 2 - 1) + \ln[(x+1)]]dx \\ &= 4\ln 2 - 2\end{aligned}$$

(b) We compute the integral in the given order as follows:

$$\begin{aligned}\int_0^1 \left(\int_0^1 (xy)^2 \cos(x^3) dx\right) dy &= \int_0^1 \left(\int_0^1 y^2x^2 \cos(x^3) dx\right) dy \\ &= \int_0^1 \frac{1}{3} \left(\int_0^1 y^2 \cos(x^3) d(x^3)\right) dy \\ &= \int_0^1 \frac{1}{3} \left(y^2 \sin(1)\right) dy \\ &= \frac{1}{3}\frac{1}{3} \sin(1) \\ &= \frac{\sin(1)}{9}\end{aligned}$$

Now to calculate the iterated integral with reversed order, first note that since it is on $[0,1]x[0,1]$ the limits of the integral will remain the same.

$$\begin{aligned}\int_0^1 \left(\int_0^1 (xy)^2 \cos(x^3) dx\right) dy &= \int_0^1 \left(\int_0^1 y^2x^2 \cos(x^3) dy\right) dx \\ &= \int_0^1 \frac{1}{3} x^2 \cos(x^3) dx \\ &= \int_0^1 \frac{1}{3}\frac{1}{3} \cos(x^3) d(x^3) \\ &= \frac{1}{3}\frac{1}{3} \sin(1) \\ &= \frac{\sin(1)}{9}\end{aligned}$$

We see that the iterated integrals have the same values. ■

6.

(a) Let $R = [a, b] \times [c, d]$ and $f(x, y) = \phi(x)\psi(y)$ for all $(x, y) \in R$ where $\phi(x)$ is continuous on $[a, b]$ and $\psi(y)$ is continuous on $[c, d]$. Show that

$$\int\int_R f(x,y)dxdy = \left(\int_a^b \phi(x)dx\right)\left(\int_c^d \psi(y)dy\right)$$

(b) Compute $\int\int_{[1,2]\times[1,2]} x^r y^s dxdy$ for $r \geq 0$ and $s \geq 0$.

(c) Compute $\int\int_{[0,1]\times[0,1]} xye^{x+y}dxdy$.

Sol.

(a) We will proceed in this problem by using the definition of a Riemann Sum.
Let P be a tagged partition with the tags as (α_i, β_j) where $\alpha_i \in [x_{i-1}, x_i]$ and $\beta_j \in [y_{j-1}, y_j]$.
We have,

$$\begin{aligned}
S(P, f, t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} f(x,y)\Delta_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \phi(\alpha_i)\psi(\beta_j)\Delta_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} [\phi(\alpha_i)(x_i - x_{i-1})][\psi(\beta_j)(y_j - y_{j-1}] \\
&= \left(\sum_{i=1}^{n} \phi(\alpha_i)(x_i - x_{i-1})\right)\left(\sum_{j=1}^{n} \psi(\beta_j)(y_j - y_{j-1})\right)
\end{aligned}$$

We know that f, ϕ and ψ are integrable in their respective domains (Why?). We know that as $||P|| \to 0$, the respective norms of the partitions along the x and y directions go to zero as well.
Thus, the two sums in single variables approach the respective integrals as the double sum approaches the double integral. Hence, we have proved,

$$\int\int_R f(x,y)dxdy = \left(\int_a^b \phi(x)dx\right)\left(\int_c^d \psi(y)dy\right)$$

□

(b) We will use the result we just proved. We have $\phi(x) = x^r$ and $\psi(y) = y^s$. Thus, we can see that

$$\int\int_{[1,2]\times[1,2]} x^r y^s dxdy = \frac{(2^{r+1}-1)(2^{s+1}-1)}{(r+1)(s+1)}$$

□

(c) Again, we will use the result from part (a).
We have $\phi(x) = xe^x$ and $\psi(y) = ye^y$. Thus, we can show that

$$\int\int_{[0,1]\times[0,1]} xye^{x+y}dxdy = 1$$

■

7. Evaluate the following integrals

1. $\int\int_R (x+2y)^2$ where $R = [-1,2] \times [0,2]$
2. $\int\int_R \left(xy + \frac{x}{y+1}\right) dxdy$ where $R = [1,4] \times [1,2]$

Sol.

(a) $R = [-1,2] \times [0,2]$, we need to evaluate

$$\int\int_R (x+2y)^2 dxdy$$

Let it the integral be I

$$\begin{aligned}
I &= \int\int_R (x^2 + 4y^2 + 4xy) dxdy && \text{(Expanding } (x+2y)^2\text{)} \\
&= \int\int_R x^2 dxdy + \int\int_R 4y^2 dxdy + \int\int_R 4xy dxdy && \text{(Seperating the terms)} \\
&= \int_0^2 \left(\int_{-1}^2 x^2 dx\right) dy + \int_0^2 \left(\int_{-1}^2 dx\right) 4y^2 dy + \int_0^2 \left(\int_{-1}^2 4x dx\right) y dy \\
&= \int_0^2 3dy + \int_0^2 12y^2 dy + \int_0^2 6dy \\
&= 6 + 32 + 12 = \boxed{50}
\end{aligned}$$

□

(b) $R = [1,4] \times [1,2]$, we need to evaluate

$$\int\int_R \left(xy + \frac{x}{y+1}\right) dxdy.$$

Now it is easy to observe that

$$\begin{aligned}
\int\int_R \left(xy + \frac{x}{y+1}\right) dxdy &= \int\int_R x \left(y + \frac{1}{y+1}\right) dxdy \\
&= \left(\int_1^4 x\, dx\right) \times \left(\int_1^2 y + \frac{1}{y+1}\right) dy \\
&= \frac{15}{2} \times \left(\frac{3}{2} + \ln\frac{3}{2}\right) \approx \boxed{14.29}
\end{aligned}$$

■

8. Consider the function $f : [-1,1] \times [-1,1]$

$$f(x,y) = \begin{cases} x+y & \text{if } x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the sets of points at which f is discontinuous. Is f integrable over $[-1,1] \times [-1,1]$?

Sol.
We will split the domain into three sets E_1, E_2 and E_3.
On set $E_1 = \{(x,y) : x^2+y^2 < 1\}$, we have the function $f(x,y) = x+y$ which is a continuous function. But on the set $E_2 = \{(x,y) : x^2+y^2 = 1\}$, if we approach any point in E_2 from E_1, we get $f(x,y) = x+y$.
Now if we approach from $E_3 = R - (E_1 \cup E_2)$, we get $f(x,y) = 0$ which is clearly discontinuous except when $x+y=0$ on E_2 i.e. $x = \pm\frac{1}{\sqrt{2}}$ and $y = \mp\frac{1}{\sqrt{2}}$,we have the continuity, since the function goes to zero at these two points.
So $f(x,y)$ is discontinuous on $E_2 - \left\{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$
Recall the concept of content zero. We apply it here and state that the function is integrable over R.

$$\begin{aligned}
\int\int_R f(x,y)dxdy &= \int\int_{E_1} f(x,y)dxdy \\
&= \int\int_{E_1} (x+y)dxdy \\
&= \int\int_{E_1} xdxdy + \int\int_{E_1} ydxdy \\
&= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xdxdy + \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} ydxdy \\
&= 0
\end{aligned}$$

■

---

1. For the following, write an equivalent iterated integral with the order of integration reversed:

(a) $\int_0^1 \left[\int_1^{e^x} dy\right] dx$

(b) $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy$

Sol.

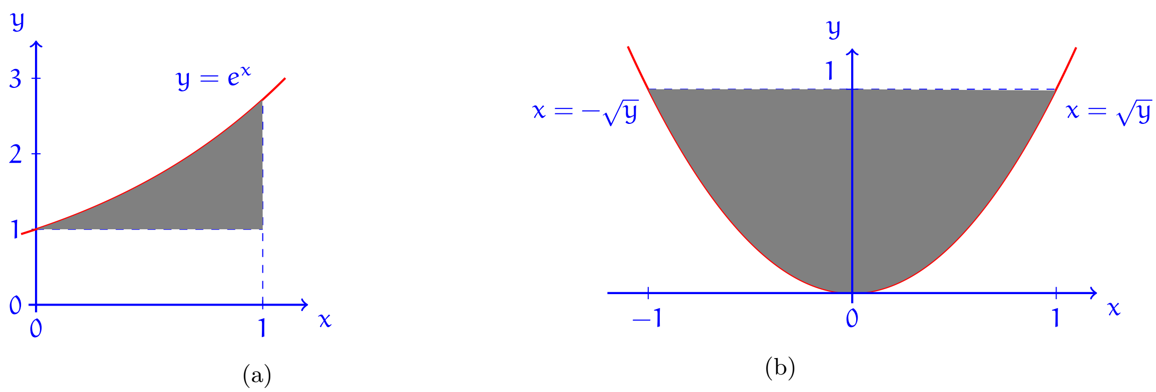


Figure 2: Regions of integration for question 1

(a) By Fubini's theorem, the iterated integrals are equal, Hence we can reverse the order of integration and obtain an equivalent integral,

$$\int_0^1 \left[\int_1^{e^x} dy\right] dx = \boxed{\int_1^e \left[\int_{\ln y}^1 dx\right] dy}.$$

□

(b) The reversed equivalent integral is,

$$\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy = \boxed{\int_{-1}^1 \left[\int_{x^2}^1 f(x,y)dy\right] dx}.$$

■

2. Evaluate the following integrals

(a) $\int_0^{\pi}\left[\int_x^{\pi}\frac{\sin(y)}{y}\,dy\right]dx$

(b) $\int_0^{1}\left[\int_y^{1}x^2e^{xy}\,dx\right]dy$

(c) $\int_0^{2}(\tan^{-1}(\pi x)-\tan^{-1}(x))\,dx$

Sol.

(a) Note that the integral is difficult to evaluate in the order given. So we change the order.

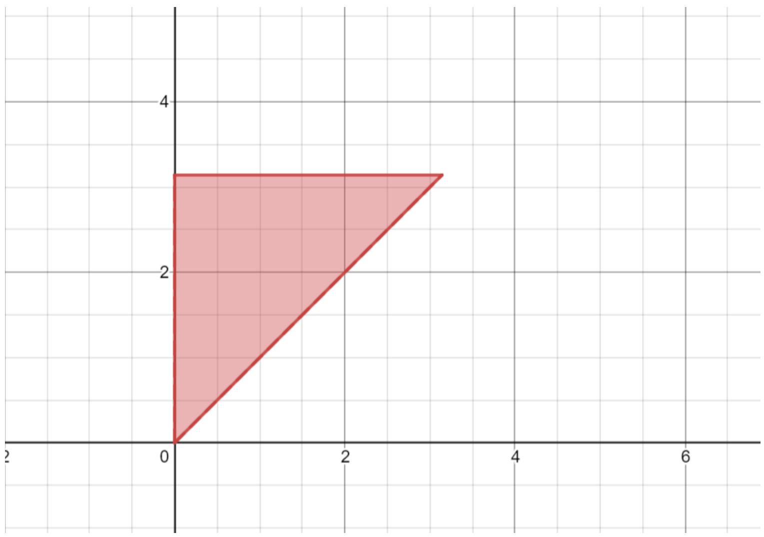


Figure 2(a): Region of integration for question 2(a)

From the figure we get the limits as $0 \le x \le y$ and $0 \le y \le \pi$ .Thus,

$$\begin{aligned}
I &= \int_0^{\pi}\left(\int_0^{y}\frac{\sin(y)}{y}\,dx\right)dy \\
&= \int_0^{\pi}\left[\frac{\sin(y)}{y}(x)\Big|_{x=0}^{x=y}\right]dy \\
&= \int_0^{\pi}\sin(y)\,dy \\
&= \boxed{2}
\end{aligned}$$

□

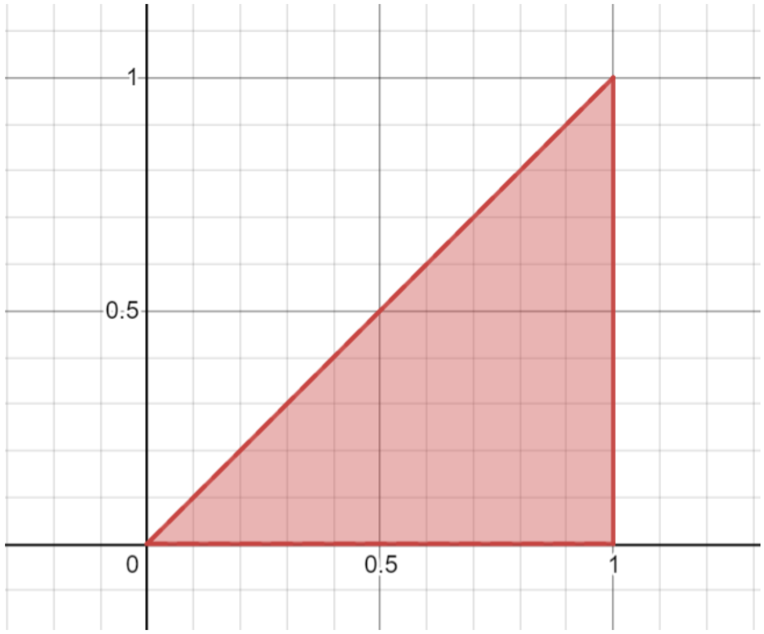


Figure 2(b): Region of integration for question 2(b)

(b) We again reverse the order of integration. From figure 2(b) we can see the limits can also be written as $0 \leq y \leq x$ and $0 \leq x \leq 1$.Thus,

$$
\begin{aligned}
I &= \int_0^1 \left(\int_0^x x^2 e^{xy} \, dy \right) dx \\
&= \int_0^1 \left(x e^{x^2} - x \right) dx \\
&= \left. \frac{e^{x^2} - x^2}{2} \right|_0^1 \\
&= \boxed{\frac{e-2}{2}}
\end{aligned}
$$

$\square$

(c) [1] We are to compute the following integral:

$$\int_0^2 \left(\tan^{-1}\pi x - \tan^{-1} x\right) dx.$$

This is a single variable Riemann integral, which is not easy to compute directly. We convert it into a double integral and compute the double integral using Fubini, hoping that that'll be easier. Now, note that:

$$\int_0^2 \left(\tan^{-1}\pi x - \tan^{-1} x\right) dx = \int_0^2 \left(\int_x^{\pi x} \frac{1}{1+y^2} dy\right) dx$$

This a double integral over the region

$$D := \{(x, y) : 0 \le x \le 2, x \le y \le \pi x\}.$$

Now, note that this region D is the union of two regions D_1 and D_2 where

$$D_1 := \left\{(x, y) : 0 \le y \le 2, \frac{y}{\pi} \le x \le y\right\}, \text{ and}$$

$$D_2 := \left\{(x, y) : 2 \le y \le 2\pi, \frac{y}{\pi} \le x \le 2\right\}.$$

Note that $D_1 \cap D_2$ is of (two-dimensional) content zero.
Therefore, we get that

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f,$$

Where $f : D \to \mathbb{R}$ is defined as $f(x, y) := \left(1 + y^2\right)^{-1}$.
Thus the required integral is given by

$$\begin{aligned}
I &= \int_0^2 \left(\int_{y/\pi}^{y} \frac{1}{1+y^2} dx\right) dy + \int_2^{2\pi} \left(\int_{y/\pi}^{2} \frac{1}{1+y^2} dx\right) dy \\
&= \left(1 - \frac{1}{\pi}\right) \int_0^2 \frac{y}{1+y^2} dy + \int_2^{2\pi} \frac{2}{1+y^2} dy - \frac{1}{\pi} \frac{2}{2\pi} \frac{y}{1+y^2} dy \\
&= \frac{1}{2}\left(1 - \frac{1}{\pi}\right) \left[\ln\left(1+y^2\right)\right]_0^2 + 2\left[\tan^{-1} y\right]_2^{2\pi} - \frac{1}{2\pi}\left[\ln\left(1+y^2\right)\right]_2^{2\pi} \\
&= \boxed{\frac{\ln 5}{2}\left(1 - \frac{1}{\pi}\right) + 2\left[\tan^{-1} 2\pi - \tan^{-1} 2\right] - \frac{1}{2\pi} \ln \frac{1+4\pi^2}{5}}
\end{aligned}$$

■

[1] thanks to integral daddy

3. Find $\int\int_D f(x,y)d(x,y)$ where $f(x,y) = e^{x^2}$ and D is the region bounded by the lines $y = 0, x = 1$ and $y = 2x$

Sol.
We can solve this by looking at one particular iterated integral. We can do this because the function is well behaved (continuous) and both the iterated integrals exist.

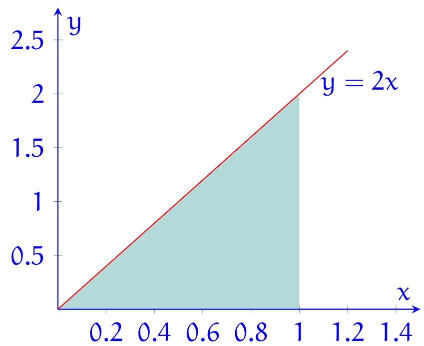


Figure 1: Region of integration for question 3

Note that the integral,

$$\int_\alpha^1 e^{x^2} dx$$

$(0 \le \alpha \le 1)$ does not exist in closed form in terms of α.
We therefore cannot evaluate the integral,

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} dxdy$$

Thus, our only option is to evaluate the other iterated integral. We have:

$$\int\int_D e^{x^2} d(x,y) = \int_0^1 \int_0^{2x} e^{x^2} dydx = \int_0^1 2xe^{x^2} dx = \int_0^1 e^u du = e - 1$$

Hence, we have obtained,

$$\int\int_D e^{x^2} d(x,y) = \boxed{e-1}$$

■

4.

(a) Compute the volume of the solid enclosed by the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

where a, b, c are real numbers.

(b) Find the volume of the region under the graph of $f(x, y) = e^{x+y}$ over the region

$$D = \left\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\right\}.$$

Sol.

(a) We defined D,

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$$

We can compute this volume easily by changing the variables from (x, y, z) to (u, v, w). We will scale the axes to make the region a sphere.

$$x = au$$
$$y = bv$$
$$z = cw$$

The Jacobian $\mathbb{J}$ for the above transformation is,

$$\mathbb{J} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Note that $\|\mathbb{J}\| = abc$.
Thus,

$$\iiint_D dzdydx = abc \iiint_{D'} dwdvdu$$

where D' is the new volume we have to integrate over. Notice that,

$$D' = \left\{(u, v, w) \in \mathbb{R}^3 \,\middle|\, u^2 + v^2 + w^2 \leq 1\right\}$$

This is simply a unit sphere. Therefore,

$$\iiint_{D'} dwdvdu = \frac{4\pi}{3}$$

Thus,

$$\iiint_D dxdydz = \iiint_{D'} abc\ dudvdw = \boxed{\frac{4\pi abc}{3}}$$

□

(b) Observe that the volume will be

$$\iint_D e^{x+y} dxdy$$

where the region D is given by (notice that the region has been written out differently. Justify),

$$\left\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x + y \leq 1, -1 \leq y - x \leq 1\right\}$$

Use the change of variables,

$$x = \frac{u+v}{2}$$
$$y = \frac{u-v}{2}$$

The Jacobian $\mathbb{J}$ corresponding to this transformation will be,

$$\mathbb{J} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$\implies |\mathbb{J}| = \frac{1}{2}$$

Therefore, we can now write,

$$\iint_D e^{x+y}\,dxdy = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}e^u\,dudv$$
$$= \boxed{e - \frac{1}{e}}$$

■

5. Evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y)\,d(x,y)$$

where D is the parallelogram with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Sol.
The region D is given by,

$$\left\{(x,y) \in \mathbb{R}^2 \mid \pi \le x+y \le 3\pi \text{ and } -\pi \le y-x \le \pi\right\}$$

Use the change of variables,

$$x = \frac{u+v}{2}$$
$$y = \frac{u-v}{2}$$
$$\implies \iint_D (x-y)^2 \sin^2(x+y)\,d(x,y) = \int_{-\pi}^{\pi}\int_{\pi}^{3\pi} \frac{1}{2}v^2 \sin^2 u\,dudv$$
$$= \frac{1}{2}\left(\int_{-\pi}^{\pi} v^2\,dv\right)\left(\int_{\pi}^{3\pi} \sin^2(u)\,du\right)$$
$$= \frac{\pi^3}{3} \cdot \pi$$
$$= \boxed{\frac{\pi^4}{3}}$$

■

6. Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1, xy = 9$ and the lines $y = x, y = 4x$. Find $\iint_D dxdy$ by transforming it to $\iint_{D^*} dudv$, where $x = \frac{u}{v}, y = uv, v > 0$.

Sol.
We have,

$$x = \frac{u}{v}$$
$$y = uv$$

Thus, note that,

$$u = \sqrt{xy}$$
$$v = \sqrt{\frac{y}{x}}$$

Define,

$$h_1(u, v) := \frac{u}{v}$$
$$h_2(u, v) := uv$$

Then we have the Jacobian,

$$\mathbb{J}(u, v) := \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix}$$

$$\implies |\mathbb{J}| = 2\frac{u}{v}$$

We define D^* in the following manner:

$$D^* = \{(u, v) \mid u \in [1, 3], v \in [1, 2]\}$$

Notice that D^* corresponds to the region D given in the question, but in the transformed coordinates. We can call D^* the transformed region of integration.
Let,

$$f^* : D^* \to \mathbb{R}$$
$$f^*(u, v) := 2\frac{u}{v}$$

We are now tasked to compute the integral,

$$\iint_{D*} f^*(u, v)dudv$$

Note that f^* is of the form $\phi(u)\psi(v)$, where ϕ and ψ are continuous, and hence integrable functions on their respective domains. Moreover f^* is continuous on D^* and hence integrable on D^*. Recall `Tutorial One, question 6(a)`. Thus,

$$\iint_{D*} f^*(u, v)dudv = \left(\int_1^2 \frac{1}{v}dv\right)\left(\int_1^3 2udu\right)$$
$$= \boxed{8 \log 2}$$

A picture of D is given below. Now contrast this with D^*.

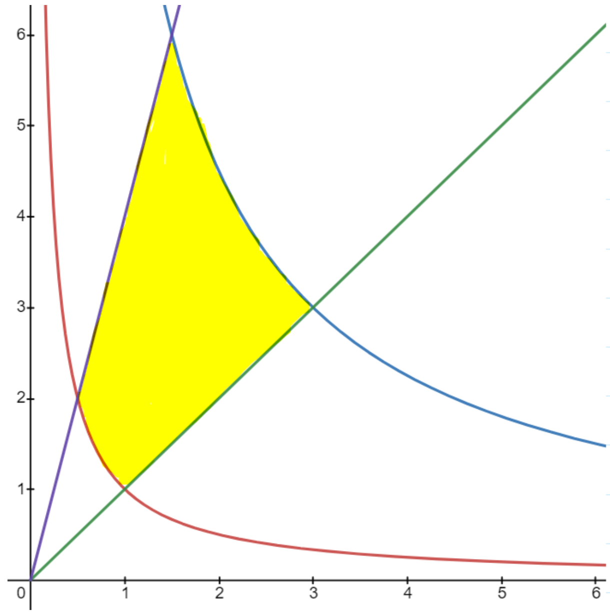


Figure 1: Region of integration D

7. Find

$$\lim_{r\to\infty}\iint_{D(r)} e^{-(x^2+y^2)}\,d(x,y)$$

where $D(r)$ equals:

(a) $\left\{(x,y) : x^2+y^2 \le r^2\right\}$

(b) $\left\{(x,y) : x^2+y^2 \le r^2, x \ge 0, y \ge 0\right\}$

(c) $\{(x,y) : |x| \le r, |y| \le r\}$

(d) $\{(x,y) : 0 \le x \le r, 0 \le y \le r\}$

Sol.

(a) As should be obvious from the form of $D(r)$, we will utilize polar coordinates here. On converting the integral from the Cartesian to the polar form, we get,

$$\begin{aligned}\lim_{r\to\infty}\iint_{D(r)} e^{-(x^2+y^2)}\,d(x,y) &= \lim_{r\to\infty}\int_0^{2\pi}\int_0^r e^{-R^2}R\,dR\,d\theta \\ &= -\pi\lim_{r\to\infty}\int_0^r e^{-R^2}\,d(-R^2) \\ &= \pi\lim_{r\to\infty}\left(1-e^{-r^2}\right) \\ &= \boxed{\pi}\end{aligned}$$

□

(b) Again, we utilize the polar coordinates here. The integral will be the same as before except for the difference in the limits of θ - since $x \geq 0, y \geq 0$, θ will vary from 0 to $\frac{\pi}{2}$,

$$\begin{aligned}\lim_{r\to\infty}\iint_{D(r)} e^{-(x^2+y^2)}d(x,y) &= \lim_{r\to\infty}\int_0^{\frac{\pi}{2}}\int_0^r e^{-R^2}RdRd\theta \\ &= \boxed{\frac{\pi}{4}}\end{aligned}$$

□

(c) Consider the following integrals:

$$\begin{aligned}I_1(r) &= \int_0^{2\pi}\int_0^{r\sqrt{2}} e^{-R^2}RdRd\theta \\ I_2(r) &= \int_0^{2\pi}\int_0^{r} e^{-R^2}RdRd\theta\end{aligned}$$

We can now write (why?),

$$I_1(r) > \iint_{D(r)} e^{-(x^2+y^2)}d(x,y) > I_2(r)$$

We state the sandwich theorem below,

Let I be an interval having the point a as a limit point. Let g, f and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have,

$$g(x) \leq f(x) \leq h(x)$$

and also suppose that,

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

Then $\lim_{x\to a} f(x) = L$.

It is clear that $\lim_{r\to\infty} I_1(r) = \lim_{r\to\infty} I_2(r) = \pi$. If we now make use of the sandwich theorem, we can write,

$$\lim_{r\to\infty}\iint_{D(r)} e^{-(x^2+y^2)}d(x,y) = \boxed{\pi}$$

□

(d) $\lim_{r\to\infty}\iint_{D(r)} e^{-(x^2+y^2)}d(x,y) = \boxed{\frac{\pi}{4}}$ (Justify).

■

8. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

Sol.

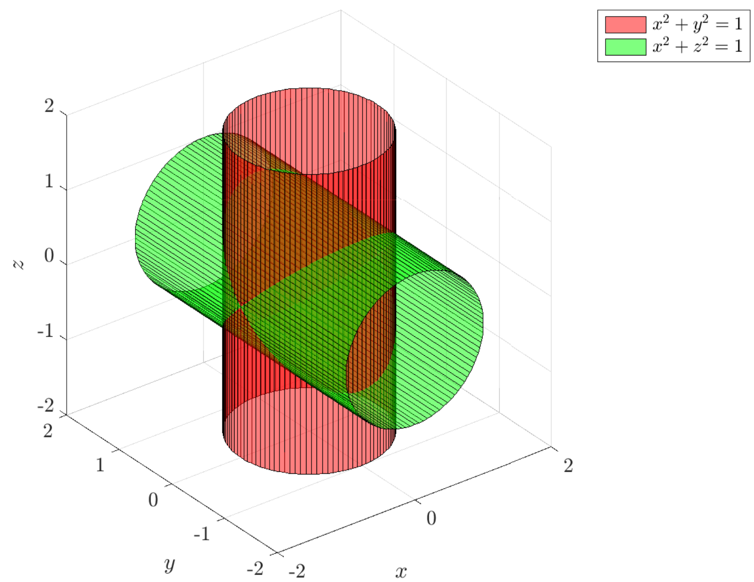


Figure 1: Region of integration for volume in question 8 ($a = 1$ taken for representation)

Notice that the volume of consideration is the same in each of the octants. We will calculate the integral over the first octant where $x, y, z \geq 0$ and thereby get the total volume.
The region of integration in the $x - y$ plane is given by a circular disc centered at the origin and having radius a. The 'height' (along the z-axis) of the volume (in the first octant) at a point (x, y) is $z = \sqrt{a^2 - x^2}$. Thus we may write the integral as,

$$V = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy \right) dx = \int_0^a \left(a^2 - x^2\right) dx = a^2 \cdot a - \frac{a^3}{3} = \frac{2}{3}a^3.$$

Hence we the the total common volume (over all octants) will be, $= 8 \cdot \frac{2}{3}a^3 = \boxed{\frac{16}{3}a^3}$. ■

9. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the region $x^2 + y^2 = 2x$ in the x-y plane.

Sol.
Let the volume be V. We can write,

$$V = \iint_D z\,dy\,dx = \iint_D (x^2 + y^2)\,dy\,dx$$

where $D = \left\{(x, y) : (x-1)^2 + y^2 \leq 1, (x, y) \in \mathbb{R}^2\right\}$
Make the substitutions $x = 1 + r\cos\theta, y = r\sin\theta$. This gives,

$$V = \iint_{D'} (r + r^3 + 2r^2\cos\theta)\,dr\,d\theta$$

where $D' = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$.

$$V = \int_0^{2\pi} \int_0^1 (r + r^3 + 2r^2\cos\theta)\,dr\,d\theta$$
$$= \int_0^{2\pi} \left(\frac{3}{4} + \frac{2\cos\theta}{3}\right) d\theta = \boxed{\frac{3\pi}{2}}$$

■

10. Express the solid $D = \left\{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\right\}$ as

$$\{(x, y, z) \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x), \quad \xi_1(x, y) \leq z \leq \xi_2(x, y)\}.$$

Sol.

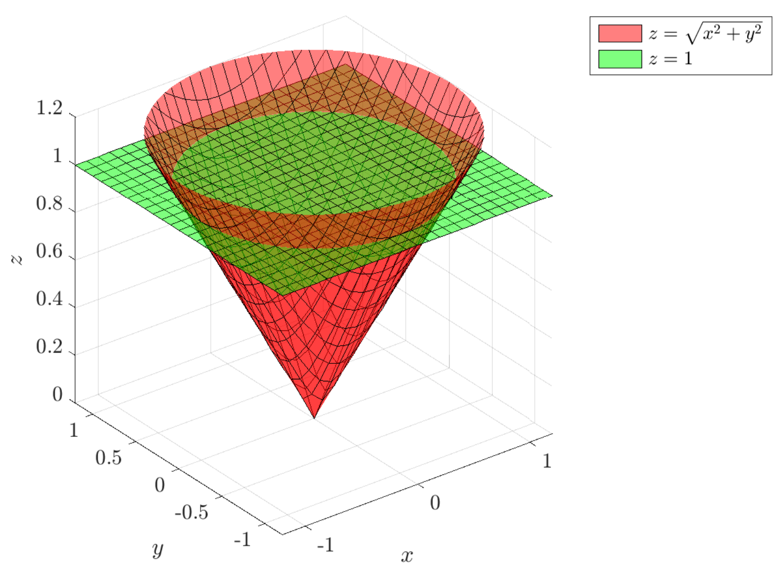


Figure 1: Solid D represented using bounding surfaces

Since $\sqrt{x^2 + y^2} \leq z \leq 1$, we have that $\sqrt{x^2 + y^2} \leq 1 \implies |y| \leq \sqrt{1 - x^2}$, and hence $x^2 \leq 1$, which gives us $|x| \leq 1$. Thus we may write D in the desired form as

$$\boxed{\left\{(x, y, z) \mid -1 \leq x \leq 1, \; -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \; \sqrt{x^2 + y^2} \leq z \leq 1\right\}}.$$

■

11. Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x \, dz \right) dy \right) dx$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

Sol.

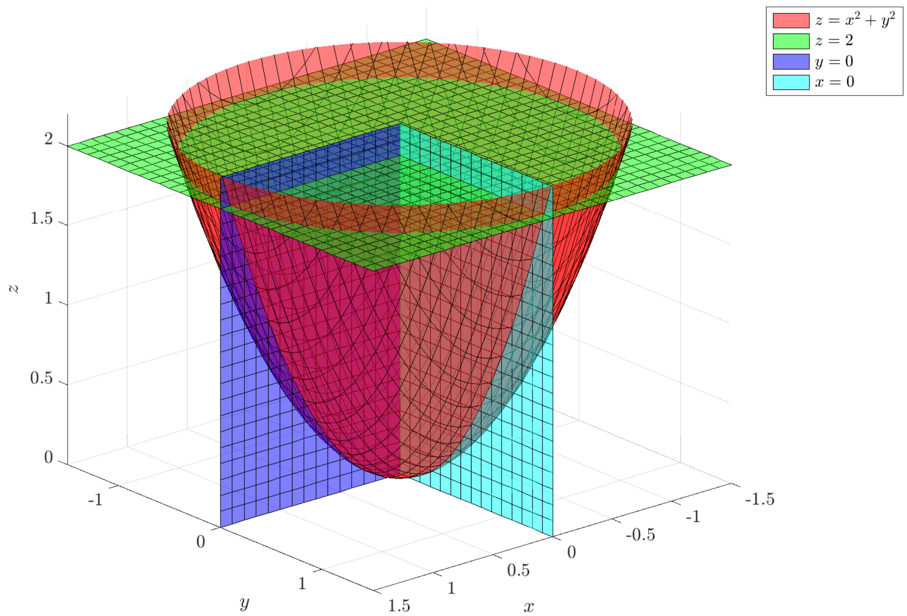


Figure 1: Region of integration for volume in question 11

$$\begin{aligned}
I &= \int_0^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x \, dz \right) dy \right] dx \\
&= \int_0^{2} \left[\int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-y^2}} x \, dx \right) dy \right] dz \\
&= \int_0^{2} \left[\int_0^{\sqrt{z}} \frac{z-y^2}{2} dy \right] dz \\
&= \int_0^{2} \left(\frac{z\sqrt{z}}{2} - \frac{1}{2} \cdot \frac{\sqrt{z}^3}{3} \right) dz \\
&= \int_0^{2} \left(\frac{z\sqrt{z}}{2} - \frac{z\sqrt{z}}{6} \right) dz \\
&= \int_0^{2} \frac{z^{\frac{3}{2}}}{3} dz \\
&= \boxed{\frac{8\sqrt{2}}{15}}.
\end{aligned}$$

■

12. Using suitable change of variables, evaluate the following:

(a) $I = \iiint_D (z^2x^2 + z^2y^2)\,dx\,dy\,dz$

where D is the cylindrical region $x^2 + y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

(b) $I = \iiint_D \exp(x^2 + y^2 + z^2)^{\frac{3}{2}}\,dx\,dy\,dz$

over the region enclosed by the unit sphere in $\mathbb{R}^3$.

Sol.

(a) Perform the substitution,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \mathfrak{z}$$

As seen in class, the determinant of the Jacobian is r. Thus,

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^1\int_{-1}^1 \mathfrak{z}^2 r^2 \times r\,d\mathfrak{z}\,dr\,d\theta \\ &= 2\pi \times \frac{1}{4} \times \frac{2}{3} \\ &= \boxed{\frac{\pi}{3}} \end{aligned}$$

□

(b) Perform the substitution,

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi$$

As seen in class, the determinant of the Jacobian is $r^2|\sin\phi|$. Thus, we have,

$$\begin{aligned} I &= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{r^3} \times r^2|\sin\phi|\,dr\,d\theta\,d\phi \\ &= 2 \times 2\pi \times \frac{1}{3}(e-1) \\ &= \boxed{\frac{4\pi}{3}(e-1)} \end{aligned}$$

■

Tutorial Sheet 3 (I. Multiple integrals and change of variables)

1. Find the volume that lies under the paraboloid $z = x^2 + y^2$, above the xy plane, and inside the cylinder $x^2 + y^2 = 2x$.

Sol.
Though the question is worded differently, the region is exactly the same as that of Question 9 in Tutorial sheet 2, hence the same solution follows.

■

2. Using a suitable change of variables, evaluate the integral $\int\int_D y\,dy\,dx$, where D is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x, y \geq 0$.

Sol.
We can use the following variables:

$$\begin{aligned} x &= 1 + u \\ y &= \sqrt{v} \end{aligned}$$

Since D is the region given by,

$$D = \left\{ (x, y) \;\middle|\; \frac{y^2 - 4}{4} \leq x \leq \frac{4 - y^2}{4}, y \geq 0 \right\}$$

On changing variables to u and v, the D is transformed to a new region D',

$$D' = \left\{ (u, v) \;\middle|\; 0 \leq v \leq 4, \frac{v}{4} - 2 \leq u \leq \frac{-v}{4} \right\}$$

The Jacobian $\mathbb{J}$ is easy enough to find. $|\mathbb{J}| = \frac{1}{2\sqrt{v}}$ (Verify).

Thus the integral $\int\int_D y\,dy\,dx$ can be computed as follows:

$$\begin{aligned} \iint_D y\,dy\,dx &= \iint_{D'} \sqrt{v} \times \frac{1}{2\sqrt{v}} du\,dv \\ &= \int_0^4 \int_{\frac{v}{4}-2}^{\frac{-v}{4}} \frac{1}{2} du\,dv \\ &= \int_0^4 \frac{1}{2} \left(2 - \frac{v}{2} \right) dv \\ &= \boxed{2} \end{aligned}$$

■

3. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Sol.

Since we are supposed to use spherical coordinates, we can write the change of coordinates as,

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned}$$

We define a region D as the following:

$$D := \left\{ (x, y, z) \;\middle|\; x^2 + y^2 \leq \frac{1}{2}, \sqrt{x^2 + y^2} \leq z \leq \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2} \right\}$$

It should be clear that D is the region of which the volume we have to compute. We now represent D in spherical coordinates and call this transformed region D'.

$$D' := \left\{ (r, \phi, \theta) \;\middle|\; 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq \phi \leq 2\pi, 0 \leq r \leq \cos\theta \right\}$$

So the volume to be computed is simply,

$$\begin{aligned} V &= \iiint_{D'} r^2 \sin\theta \, dr d\theta d\phi \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\theta} r^2 \sin\theta \, dr d\theta d\phi \\ &= 2\pi \int_0^{\frac{\pi}{4}} \sin\theta \left(\frac{\cos^3\theta}{3} \right) d\theta \\ &= -\frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \cos^3\theta \, d(\cos\theta) \\ &= \boxed{\frac{\pi}{8}} \end{aligned}$$

■

4. Use cylindrical coordinates to evaluate $\iiint_W (x^2+y^2)\,dz\,dy\,dx$ where

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2\}.$$

Sol.
We perform the following substitution to get the cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Observe that the region W can be written as,

$$W = \left\{ (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 \;\middle|\; 0 \le \theta < 2\pi, 0 \le r \le 2, r \le z \le 2 \right\}$$

The given integral therefore becomes,

$$\begin{aligned} \int_0^{2\pi}\int_0^2\int_r^2 r^3\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^2 r^3(2-r)\,dr\,d\theta \\ &= (2\pi)\times\left(8-\frac{32}{5}\right) \\ &= \boxed{\frac{16\pi}{5}} \end{aligned}$$

■

5. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta,$$

and evaluate the integral.

Sol.
The limits of integral describe the thick spherical shell S shown in Fig. 1 such that,

$$S := \left\{ (\rho,\theta,\phi) \;\middle|\; 1 \le \rho \le 2, 0 \le \phi \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2} \right\}$$

The integration is trivial,

$$\begin{aligned} \iiint_S \rho^2\sin\phi\,d\rho\,d\phi\,d\theta &= \left(\int_0^{\pi/2}\sin\phi\,d\phi\right)\cdot\left(\int_0^{\pi/2}d\theta\right)\cdot\left(\int_1^2\rho^2\,d\rho\right) \\ &= 1\times\frac{\pi}{2}\times\frac{7}{3} \\ &= \boxed{\frac{7\pi}{6}} \end{aligned}$$

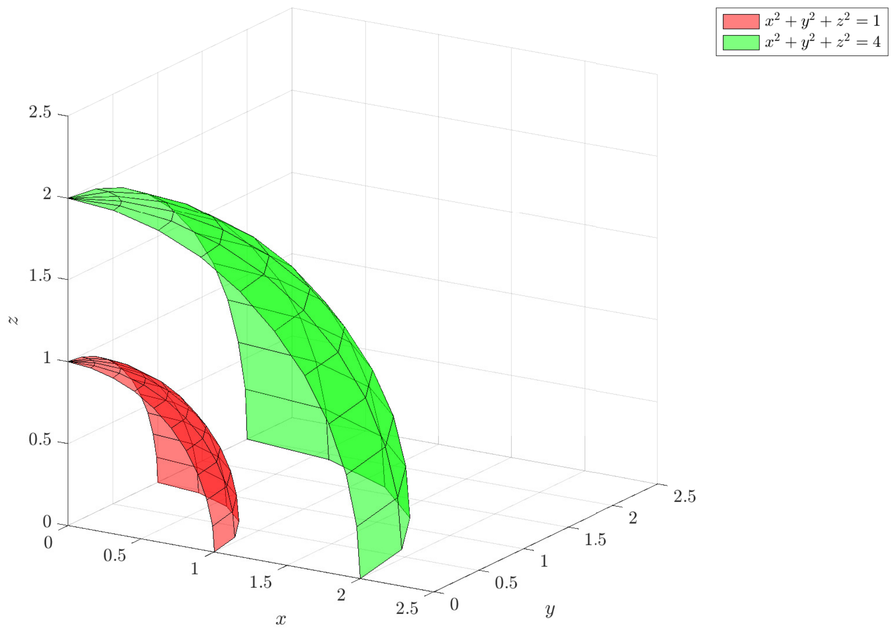


Figure 1: The region between the two spherical boundaries is the volume for question 5

■

6. Find $\iiint_F \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}} dV$ where F is the region bounded by the spheres with the center the origin and radii r and R with $0 < r < R$

Sol.
We will change the coordinates from Cartesian to spherical:

$$\begin{aligned} x &= r_1 \sin\theta \cos\varphi \\ y &= r_1 \sin\theta \sin\varphi \\ z &= r_1 \cos\theta \end{aligned}$$

where $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$ and we are given $r_1 \in (r, R)$.
So the integral after transformation is,

$$\begin{aligned} \iiint_F \frac{1}{(x^2+y^2+z^2)^{\frac{n}{2}}} dV &= \int_0^{\pi}\int_0^{2\pi}\int_r^R \frac{1}{r_1^n} r_1^2 \sin\theta \, dr_1 \, d\theta \, d\varphi \\ &= \left(\int_0^{2\pi} d\varphi\right)\cdot\left(\int_0^{\pi}\sin\theta \, d\theta\right)\cdot\left(\int_r^R r_1^{2-n} dr_1\right) \\ &= 4\pi\wp(n) \end{aligned}$$

where,

$$\wp(n) = \begin{cases} \ln\left(\frac{R}{r}\right) & n = 3 \\ \frac{1}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{otherwise} \end{cases}$$

■

1. Let f, g be differentiable functions on $\mathbb{R}^2$. Show that

i. $\nabla(fg) = f\nabla g + g\nabla f$;

ii. $\nabla f^n = nf^{n-1}\nabla f$;

iii. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

Sol.

i. We use the definition of the gradient to rewrite the expressions in terms of partial derivatives, and then use the previously established product rule of the partial derivative to prove these identities.

$$\begin{aligned}\nabla(fg) &= \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} \\ &= \left(f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}g\right)\mathbf{j} && (\because \text{ product rule for partial derivatives}) \\ &= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j}\right) + g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \\ &= f\nabla g + g\nabla f\end{aligned}$$

□

ii. We use the result obtained in part i. above, and the principle of induction. Let $P(n)$ denote the statement $\nabla f^n = nf^{n-1}\nabla f$. Observe that $P(1)$ is $\nabla f = \nabla f$, which is trivially true.
Now suppose $P(n)$ is true for some $n \geq 1$, i.e. $\nabla f^n = nf^{n-1}\nabla f$ holds. Then we have,

$$\begin{aligned}\nabla f^{n+1} &= \nabla(f^n f) = f\nabla f^n + f^n\nabla f && (\because \text{ i.}) \\ &= fnf^{n-1}\nabla f + f^n\nabla f && (\because P(n) \text{ is true}) \\ &= (n+1)f^n\nabla f,\end{aligned}$$

which is the same as $P(n+1)$. Hence we have that $P(n+1)$ is true whenever $P(n)$ is, for all $n \geq 1$, and that $P(1)$ is trivially true, which completes the proof for all positive integers n.

By the result in iii. (which will be proved in the next part), we may similarly show that $P(n-1)$ is true whenever $P(n)$ is, for any $n \leq 1$; hence by the principle of induction we have that $P(n)$ holds for non-positive integers as well. This completes the proof for all integers n. □

iii. Taking $h = \frac{1}{g}$ (which is possible when $g \neq 0$), we have

$$\nabla h = \frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j} = -\frac{1}{g^2}\frac{\partial g}{\partial x}\mathbf{i} - \frac{1}{g^2}\frac{\partial g}{\partial y}\mathbf{j} = -\frac{1}{g^2}\nabla g.$$

We now apply i. to the product fh, to obtain

$$\nabla\left(\frac{f}{g}\right) = \nabla(fh) = f\nabla h + h\nabla f = -f\frac{1}{g^2}\nabla g + \frac{1}{g}\nabla f = \frac{g\nabla f - f\nabla g}{g^2}$$

■

2. Let **a**, **b** be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:

1. $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n
2. $\mathbf{a} \cdot \nabla(\frac{1}{r}) = -\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}$
3. $\mathbf{b} \cdot \nabla\left(\mathbf{a} \cdot \nabla(\frac{1}{r})\right) = \dfrac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5} - \dfrac{\mathbf{a}\cdot\mathbf{b}}{r^3}$

Sol.

1. Let $n \in \mathbb{Z}$ be given. Define $f : \mathbb{R}^3 \to \mathbb{R}$ as

$$f(x, y, z) := \left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}$$

Thus,

$$\frac{\partial f}{\partial x} = nx\left(x^2 + y^2 + z^2\right)^{\frac{n-2}{2}}$$

Similar will be the face for the other partial derivatives. Therefore,

$$\begin{aligned}\nabla r^n &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= nr^{n-2}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \\ &= nr^{n-2}\mathbf{r}\end{aligned}$$

□

2. Let $n = -1$. By the previously proved part, we have,

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= -\frac{\mathbf{r}}{r^3} \\ \implies \mathbf{a} \cdot \nabla\left(\frac{1}{r}\right) &= -\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\end{aligned}$$

□

3. Let $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. Using the result obtained in part 2.,

$$\begin{aligned}\mathbf{b} \cdot \nabla\left(\mathbf{a} \cdot \nabla\left(\frac{1}{r}\right)\right) &= \mathbf{b} \cdot \nabla\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) \\ &= \mathbf{b} \cdot \nabla\left(-\frac{a_x x + a_y y + a_z z}{r^3}\right) \\ &= \frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5} - \frac{\mathbf{a}\cdot\mathbf{b}}{r^3}\end{aligned}$$

■

3. Calculate the line integral of the vector field

$$\mathbf{F}(x,y) = \left(x^2 - 2xy\right)\mathbf{i} + \left(y^2 - 2xy\right)\mathbf{j}$$

from $(-1,1)$ to $(1,1)$ along $y = x^2$.

Sol.
We can represent the curve $y = x^2$ (let's call it C) in parametric form as,

$$x = \tau$$
$$y = \tau^2$$
$$\mathbf{c}(\tau) = (x,y) = (\tau,\tau^2)$$

where τ is the parameter, varying from -1 to 1.
The tangent $\mathbf{c}'(\tau)$ at every point (τ,τ^2) on C will be,

$$\mathbf{c}'(\tau) = \mathbf{i} + 2\tau\mathbf{j}$$

Therefore, we can write,

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{ds} &= \int \mathbf{F}(\mathbf{c}(\tau)) \cdot \mathbf{c}'(\tau) d\tau \\ &= \int_{-1}^{1} \left(\tau^2 - 2\tau^3 + 2\tau^5 - 4\tau^4\right) d\tau \\ &= \boxed{-\frac{2}{15}}\end{aligned}$$

■

4. Calculate the line integral of

$$\mathbf{F}(x,y) = \left(x^2 + y^2\right)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.
Remark Often line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F.ds}$. A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.
To evaluate $\int_C \mathbf{F.ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F.ds} := \int_{\mathbf{c}} \mathbf{F.ds}$$

'$\oint_C$' means the line integral over a closed curve C.

Sol.
Let's call the ellipse E. Every point on the ellipse can be parametrized in the following manner:

$$x(\theta) = a\cos\theta$$
$$y(\theta) = b\sin\theta$$
$$\mathbf{c}(\theta) = (x,y) = (a\cos\theta, b\sin\theta)$$

where $\theta \in [0,2\pi)$.
We can now calculate the line integral as,

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{ds} &= \int \mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta) d\theta \\ &= \int_0^{2\pi} \left[\left(a^2\cos^2\theta + b^2\sin^2\theta\right)\mathbf{i} + (a\cos\theta - b\sin\theta)\mathbf{j}\right] \cdot (-a\sin\theta\mathbf{i} + b\cos\theta\mathbf{j})\, d\theta \\ &= \int_0^{2\pi} \left(-a^3\cos^2\theta\sin\theta - ab^2\sin^3\theta + ab\cos^2\theta - b^2\sin\theta\cos\theta\right) d\theta \\ &= \boxed{\pi ab}\end{aligned}$$

■

5. Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx-(x-y)dy}{x^2+y^2}.$$

where C is the curve $x^2+y^2=a^2$ traversed once in the counter clockwise direction.

Sol.
The curve C can be parametrised as $\mathbf{c}(t) := (a\cos t, a\sin t)\,, t \in [0, 2\pi]$. Define $\mathbf{F}(x,y) := \left(\frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2}\right)$.
The integral can then be written as,

$$\oint_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

which is equal to,

$$\begin{aligned}\oint_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt &= \int_0^{2\pi} F(a\cos t, a\sin t) \cdot (-a\sin t, a\cos t)dt \\ &= \int_0^{2\pi} -(\cos t+\sin t)\sin t + \cos t(\sin t - \cos t)\,dt \\ &= -\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)dt \\ &= \boxed{-2\pi}\end{aligned}$$

■

6. Calculate

$$\oint_C y dx + z dy + x dz$$

where C is the intersection of two surfaces $z = xy$ and $x^2+y^2=1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

Sol.
Verify that the given curve can be paramaterized as follows:

$$c(t) := (x(t), y(t), z(t)) = (\cos t, \sin t, \cos t \sin t)\,, t \in [-\pi, \pi]$$

It can be seen that this respects the direction given.
Also, $(x'(t), y'(t), z'(t)) = (-\sin t, \cos t, \cos 2t)$. Hence, we can now evaluate our integral as follows:

$$\begin{aligned}\oint_C y dx + z dy + x dz &= \int_{-\pi}^{\pi} [\sin t(-\sin t) + \cos t \sin t(\cos t) + \cos t(\cos 2t]\,dt \\ &= -\int_{-\pi}^{\pi} \sin^2 t dt \\ &= \boxed{-\pi}\end{aligned}$$

7. Let the curve C be given by $x^2+y^2=1, z=0$. Let $\mathbf{c}_1$ be a parametrization defined by $\mathbf{c}_1(t)=(\cos t, \sin t)$ for $t\in[0,2\pi]$. Find the line integral of $\mathbf{F}(x,y,z)=-y\mathbf{i}+x\mathbf{j}$ along this curve. Also find the line integral along the curve parametrized by $\mathbf{c}_2(t)=(\cos t, -\sin t)$, for $t\in[0,\pi]$.

Sol.
For the parametrization $\mathbf{c}_1$, we have,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{s} &= \int \mathbf{F}\left(\mathbf{c}_1(t)\right)\cdot \mathbf{c}_1'(t)dt \\
&= \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)dt \\
&= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt \\
&= \int_0^{2\pi} dt \\
&= \boxed{2\pi}
\end{aligned}$$

For the parametrization $\mathbf{c}_2$, we have,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{s} &= \int \mathbf{F}\left(\mathbf{c}_2(t)\right)\cdot \mathbf{c}_2'(t)dt \\
&= \int_0^{\pi} (\sin t, \cos t)\cdot(-\sin t, -\cos t)dt \\
&= -\int_0^{\pi} \left(\sin^2 t + \cos^2 t\right) dt \\
&= -\int_0^{\pi} dt \\
&= \boxed{-\pi}
\end{aligned}$$

8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle: $x^2+y^2=1$. Is this also true for a force field $\mathbf{F}(x,y,z)=\alpha(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$, for some constant α.

Sol.
Consider the following constant force field,

$$\mathbf{F}' = F_x\mathbf{i} + F_y\mathbf{j}$$

where F_x, F_y are constants.
We can parametrize the path along the circle easily by substituting,

$$\begin{aligned}
x &= \cos\theta \\
y &= \sin\theta \\
\mathbf{c}(\theta) &= (x,y) \\
&= (\cos\theta, \sin\theta)
\end{aligned}$$

where $\theta\in[0,2\pi)$.
We now calculate the line integral for F',

$$\begin{aligned}
\int_C \mathbf{F}'\cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}'(\mathbf{c}(\theta))\cdot \mathbf{c}'(t)dt \\
&= \int_0^{2\pi} \left(-F_x\sin\theta + F_y\cos\theta\right) d\theta \\
&= -F_x\int_0^{2\pi} \sin\theta d\theta + F_y\int_0^{2\pi}\cos\theta d\theta \\
&= \boxed{0}
\end{aligned}$$

We see that, independent of our choice of F_x and F_y, the integral comes out to be zero.
In the case of F, the line integral comes out to be,

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{ds} &= \int_0^{2\pi}\mathbf{F}(\mathbf{c}(\theta))\cdot\mathbf{c}'(t)dt \\ &= \alpha\int_0^{2\pi}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\,d\theta \\ &= \alpha\int_0^{2\pi}(-\cos\theta\sin\theta+\cos\theta\sin\theta)\,d\theta \\ &= \boxed{0}\end{aligned}$$

9. Let $C : x^2+y^2=1$. Find

$$\oint_C \text{grad}\left(x^2-y^2\right)\mathbf{.ds.}$$

Sol.
We can parametrize the given C by substituting,

$$\begin{aligned}x &= \cos\theta \\ y &= \sin\theta \\ \mathbf{c}(\theta) &= (x,y) \\ &= (\cos\theta,\sin\theta)\end{aligned}$$

where $\theta\in(0,2\pi)$.

$$\mathbf{c}'(\theta) = (-\sin\theta,\cos\theta)$$

$$\begin{aligned}\text{grad}\left(x^2-y^2\right) &= \left(\frac{\partial\left(x^2-y^2\right)}{\partial x},\frac{\partial\left(x^2-y^2\right)}{\partial y}\right) \\ &= (2x,-2y) \\ &= (2\cos\theta,-2\sin\theta)\end{aligned}$$

The required line integral is,

$$\begin{aligned}\oint_C \text{grad}\left(x^2-y^2\right)\cdot\mathbf{ds} &= \int_0^{2\pi}(2\cos\theta,-2\sin\theta)\,.\,(-\sin\theta,\cos\theta)\,d\theta \\ &= \int_0^{2\pi}(-2\sin\theta\cos\theta-2\sin\theta\cos\theta)\,d\theta \\ &= \int_0^{2\pi}(-2\sin 2\theta)\,d\theta \\ &= 0\end{aligned}$$

This is to be expected as gradient of a scalar field is a conservative vector field and hence the line integral over a continuous closed loop is 0.

■

10. Evaluate

$$\int_C \text{grad}\left(x^2-y^2\right)\mathbf{.ds}$$

where C is $y=x^3$, joining $(0,2)$ and $(2,8)$.

Sol.
We can parametrize C as follows,

$$\begin{aligned} x &= t \\ y &= t^3 \\ \mathbf{c}(t) &= (x,y) \\ &= \left(t,t^3\right) \end{aligned}$$

where $t \in (0,2)$.

$$\mathbf{c}'(t) = \left(1,3t^2\right)$$

As shown in the previous question,

$$\begin{aligned} \text{grad}\left(x^2-y^2\right) &= (2x,-2y) \\ &= \left(2t,-2t^3\right) \\ \implies \int_C \text{grad}\left(x^2-y^2\right)\mathbf{.ds} &= \int_0^2 \left(2t,-2t^3\right).\left(1,3t^2\right)dt \\ &= \int_0^2 \left(2t-6t^5\right)dt \\ &= \boxed{-60} \end{aligned}$$

■

11. Compute the line integral

$$\oint_C \frac{dx+dy}{|x|+|y|}$$

where C is the square with vertices $(1,0),(0,1),(-1,0)$ and $(0,-1)$ traversed once in the counter clockwise direction.

Sol.

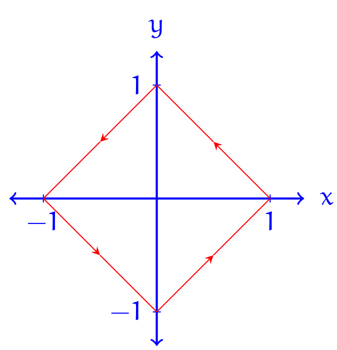


Figure 1: Figure for question 11

The path C is given by the region in Fig. 1. Note that along this path, $|x|+|y|=1$. Hence the line integral reduces to

$$\oint_C (dx+dy), \text{ which can be rewritten as } \oint_C \nabla(x+y)\cdot\mathbf{ds}.$$

The fundamental theorem of calculus (FTC) for gradient fields can then be applied here to conclude that this integral is 0, but not directly, since C is not a smooth path. We break up C into 4 constituent paths $C_1, \ldots, C_4$, which are cyclically from $(1, 0)$ to $(0, 1)$, ..., and from $(0, -1)$ to $(1, 0)$ respectively. Each of these paths are smooth, and thus we may apply FTC to each of them, giving us

$$\begin{aligned}\oint_C \nabla(x+y)\cdot \mathbf{ds} &= \oint_{C_1} \nabla(x+y)\cdot \mathbf{ds} + \oint_{C_2} \nabla(x+y)\cdot \mathbf{ds} + \oint_{C_3} \nabla(x+y)\cdot \mathbf{ds} + \oint_{C_4} \nabla(x+y)\cdot \mathbf{ds} \\ &= (x+y)\Big|_{(x,y)=(1,0)}^{(x,y)=(0,1)} + (x+y)\Big|_{(x,y)=(0,1)}^{(x,y)=(-1,0)} + (x+y)\Big|_{(x,y)=(-1,0)}^{(x,y)=(0,-1)} + (x+y)\Big|_{(x,y)=(0,-1)}^{(x,y)=(1,0)} \\ &= \boxed{0}\end{aligned}$$

■

12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0, 0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.

Sol.

Parametrize the curve as,

$$\begin{aligned} x &= \tau \\ y &= a\tau^b \\ \mathbf{c}(\tau) &= (x, y) \\ &= \left(\tau, a\tau^b\right)\end{aligned}$$

We can therefore write,

$$\begin{aligned}\int_C \mathbf{F.ds} &= \int \mathbf{F}(\mathbf{c}(\tau))\cdot \mathbf{c}'(\tau)d\tau \\ &= \int_0^1 \left(a\tau^{b+1}\mathbf{i} + a^2\tau^{2b+6}\mathbf{j}\right)\cdot\left(\mathbf{i} + ab\tau^{b-1}\mathbf{j}\right)d\tau \\ &= \int_0^1 \left(a\tau^{b+1} + a^3b\tau^{3b+5}\right)d\tau \\ &= \frac{3a + ba^3}{3b+6} = I(a, b)\end{aligned}$$

If we want $I(a, b)$ to be independent of b, then we will have to ensure that $\frac{\partial I}{\partial b} = 0$.

$$\begin{aligned}\frac{\partial I}{\partial b} &= 0 \\ \implies \frac{a^3(3b+6) - 3\left(3a + ba^3\right)}{(3b+6)^2} &= 0 \\ \implies a &= 0, \pm\sqrt{\frac{3}{2}}\end{aligned}$$

We will take the value of $a > 0$. So for $I(a, b)$ to be independent of b, we will need $a = \boxed{\sqrt{\frac{3}{2}}}$.

■
