MA111 Tutorial Solutions
Spring 2022

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Last updated : March 4, 2022

Contents

Tutorial Sheet 6 (I. Surface and surface integrals)

1. Find a suitable parameterization $\Phi(u, v)$ and the normal vector $\Phi_u \times \Phi_v$ for the following surface:

(i) The plane $x - y + 2z + 4 = 0$.

(ii) The right circular cylinder $y^2 + z^2 = a^2$.

Sol.

1. Parameterise the surface $\Phi(u, v)$ as
$$\left(u, v, \frac{v}{2} - \frac{u}{2} - 2\right)$$
then $\Phi_u = (1, 0, -\frac{1}{2}), \Phi_v = (0, 1, \frac{1}{2})$ which in turn gives
$$\Phi_u \times \Phi_v = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.$$

2. Parameterise the surface Φ as
$$\Phi(u, v) = (u, a\cos(v), a\sin(v)).$$
then
$$\Phi_u = (1, 0, 0) \& \Phi_v = (0, -a\sin(v), a\cos(v)).$$
Which gives $\Phi_u \times \Phi_v = -a\cos(v)\mathbf{j} - a\sin(v)\mathbf{k}$ And the normal vector remains same by scaling so multiply by $-1/a$ as it is non zero.

   which gives the normal vector as $(0, y, z)$ which is the final answer to all the cases.

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2. Find the tangent plane to the surface with parametric equations $x = u^2, y = v^2$ and $z = u + 2v$ at the point $(1, 1, 3)$.

Sol. The surface $\Phi(u, v)$ is given as
$$(u^2, v^2, u + 2v)$$
So $\Phi_u = (2u, 0, 1)$ & $\Phi_v = (0, 2v, 2)$ Which gives
$$\Phi_u \times \Phi_v = (-2v, -4u, 4uv) = (-2, -4, 4)$$

■

3. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

Sol.

The sphere intersects the cylinder at two equal surfaces, one for $z > 0$ and one for $z < 0$. We first calculate the area for $z > 0$. The surface may be parametrized by $\Phi(u, v) = (u, v, f(u, v))$ (the surface along the sphere), with $f(u, v) = \sqrt{a^2 - u^2 - v^2}$, on $E = \{(u, v) \in \mathbb{R}^2 \mid -\sqrt{av - v^2} \le u \le \sqrt{av - v^2},\ 0 \le v \le a\}$ (the restriction to the cylinder). Then we have that the area is given by
$$\iint_S dS = \iint_E \|\Phi_u \times \Phi_v\|\, dudv = \iint_E \sqrt{1 + f_u^2 + f_v^2} dudv = \iint_E \frac{a}{\sqrt{a^2 - u^2 - v^2}} dudv,$$
where S is half of our total surface. This can be solved by converting to polar coordinates, with $u = r\cos\theta, v = r\sin\theta$, for $0 \le \theta \le \pi, 0 \le r \le a\sin\theta$. Substituting this into the above expression, we get
$$\frac{1}{2}\text{Area} = \iint_S dS = \int_0^\pi \left(\int_0^{a\sin\theta} \frac{ar}{\sqrt{a^2 - r^2}} dr\right) d\theta = (\pi - 2)a^2.$$

Hence the total required area is given by $\boxed{2(\pi-2)a^2}$. ■

4. Compute the area of that portion of the paraboloid $x^2+z^2=2ay$ which is between the planes $y=0$ and $y=a$.

Sol.

The following parametrisation is valid (check): $\Phi(u,v)=(u,f(u,v),v)$ (the surface along the paraboloid), with $f(u,v)=\frac{1}{2a}(u^2+v^2)$, on $E=\{(u,v)\in\mathbb{R}^2 \mid 0\le u^2+v^2\le 2a^2\}$ (the restriction between the planes). Then we have that the area is given by

$$\iint_S dS=\iint_E \|\Phi_u\times\Phi_v\|\,dudv=\iint_E\sqrt{1+f_u^2+f_v^2}\,dudv=\iint_E\frac{\sqrt{a^2+u^2+v^2}}{a}\,dudv,$$

where S is our total surface. This can be solved by converting to polar coordinates, with $u=r\cos\theta, v=r\sin\theta$, for $0\le\theta\le 2\pi, 0\le r\le\sqrt{2}a$. Substituting this into the above expression, we get

$$\text{Area}=\iint_S dS=\int_0^{2\pi}\left(\int_0^{\sqrt{2}a}\frac{1}{a}\sqrt{1+\frac{r^2}{a^2}}\,r\,dr\right)d\theta=\boxed{\frac{3\sqrt{3}-1}{3}2\pi}.$$

■

5. Let S denote the plane surface whose boundary is the triangle with vertices at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and let $\mathbf{F}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}dS$.

Sol.

Verify that (intuition?),

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS=\iint_S \mathbf{F}\cdot(dxdy\mathbf{k}+dydz\mathbf{i}+dzdx\mathbf{j})$$

Therefore,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}dS&=\iint_S \mathbf{F}\cdot\mathbf{k}dxdy+\iint_S \mathbf{F}\cdot\mathbf{j}dzdx+\iint_S \mathbf{F}\cdot\mathbf{i}dzdy\\
&=\iint_S zdxdy+\iint_S ydzdx+\iint_S xdzdy\\
&=\int_0^1\int_0^{1-x}(1-x-y)\,dydx+\int_0^1\int_0^{1-x}(1-z-y)\,dzdx+\int_0^1\int_0^{1-y}(1-z-y)\,dzdy\\
&=\boxed{\frac{1}{2}}
\end{aligned}$$

■

Tutorial Sheet 6 (II. Application of Stokes theorem)

1. Consider the vector field $\mathbf{F}=(x-y)\mathbf{i}+(x+z)\mathbf{j}+(y+z)\mathbf{k}$. Verify Stokes theorem for $\mathbf{F}$ where S is the surface of the cone: $z^2=x^2+y^2$ intercepted by

(a) $x^2+(y-a)^2+z^2=a^2 : z\ge 0$

(b) $x^2+(y-a)^2=a^2$

Sol.

As per Stokes theorem, given a bounded piecewise smooth, closed oriented surface S with non-empty, piecewise non-singular parametrised boundary ∂S, and a $\mathcal{C}^1$ vector field $\mathbf{F}$, we have

$$\int_{\partial S}\mathbf{F}\cdot d\mathbf{s}=\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}.$$

It is now an exercise in calculation to check that the terms on both sides are equal to $2\pi a^2$ for both (a) and (b). ■

2. Using Stokes Theorem, evaluate the line integral

$$\oint_C yzdx + xzdy + xydz$$

where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

Sol.

Define $\mathbf{F}(x, y, z) = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$. Then our required integral is

$$\oint_C yzdx + xzdy + xydz = \oint_C \mathbf{F} \cdot \mathbf{ds}.$$

Observe that $\mathbf{F}$ is $\mathcal{C}^1$ and $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \mathbf{0}$. Since C is a simple, closed curve which allows a non-singular parametrisation, and there exists some bounded, smooth, oriented surface S for which C is the boundary, we may use Stokes theorem, and write

$$\oint_C yzdx + xzdy + xydz = \oint_C \mathbf{F} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = \boxed{0}.$$

■

3. Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.

Sol.

Check that,

$$\nabla \times \mathbf{F} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$$

Let S denote the surface of the given triangle. The positively oriented boundary ∂S of S will be the one that has been given i.e. $(0, 0, 0) \to (0, 2, 0) \to (0, 0, 2) \to (0, 0, 0)$. Therefore, we can now apply Stoke's theorem and say,

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot \mathbf{dl} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{dS} \\ &= \int_0^2 \int_0^{2-y} (z\mathbf{i} - x\mathbf{j} - y\mathbf{k}) \cdot \mathbf{i} dzdy \\ &= \int_0^2 \int_0^{2-y} zdzdy \\ &= \boxed{\frac{4}{3}}\end{aligned}$$

■

4. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate

$$\int_C -y^3 dx + x^3 dy - z^3 dz$$

Sol.

Take $F = (-y^3, x^3, z^3)$ and verify that

$$\nabla \times \mathbf{F} = 3(x^2 + y^2)\mathbf{k}$$

Now taking the surface ϕ(u,v) = (u,v,1-u-v) we get the normal

$$n = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

Using Stoke's theorem, we get:

$$\oint_C \mathbf{F} \cdot \mathbf{dl} = \iint_\phi \nabla \times \mathbf{F} \cdot \mathbf{dS}$$
$$= \iint 3(x^2 + y^2) dx dy$$

Now using polar co-ordinates, we get:

$$\oint_C \mathbf{F} \cdot \mathbf{dl} = \int_0^{2\pi} \int_0^1 3r^2 r dr d\theta$$
$$= \boxed{\frac{3\pi}{2}}$$

■

5. Let $\mathbf{F}(x, y, z) := (y, -x, e^{xz})$ for $(x, y, z) \in \mathbb{R}^3$ and let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (x - \sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{dS}$$

Sol.
Note that, S is the portion of sphere centered at $(0, 0, \sqrt{3})$ with radius 2 with the constraint of $z \geq 0$. It intersects the $x - y$ plane at the points

$$\partial S = \{(x, y) \mid x^2 + y^2 = 1^2\}$$

Now, by Stokes Theorem,

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{ds} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{dS}$$

Parametrise ∂S as $\mathbf{c}(t) = (\cos t, \sin t, 0) \forall t \in [0, 2\pi]$. Thus, we have

$$\int_0^{2\pi} (\sin t \times -\sin t + -\cos t \times \cos t) = -2\pi$$

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Tutorial Sheet 6 (III. Application of Gauss divergence theorem)

1. Calculate the flux of $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

Sol.
We have,

$$\nabla \cdot \mathbf{F} = 3(x^2 + y^2 + z^2) = 3r^2$$

By the divergence theorem, we need to compute,

$$I = \iiint_D \nabla \cdot \mathbf{F} dV$$

where D is the unit sphere. Applying the spherical transformation, we see,

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^1 3r^2 \times r^2 \sin \phi \, dr d\phi d\theta$$

Thus,

$$I = 2\pi \times 2 \times 3/5 = 12\pi/5$$

■

2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the 'can' W given by $x^2 + y^2 \leq 1$, $-1 \leq z \leq +1$.

Sol.
We have,

$$\nabla \cdot \mathbf{F} = (y^2 + x^2)$$

By the divergence theorem, we need to compute,

$$I = \iiint_W \nabla \cdot \mathbf{F} dV$$

Use the cylindrical transformation to see that,

$$\begin{aligned} I &= \int_0^{2\pi} \int_{-1}^{+1} \int_0^1 \rho^2 \times \rho d\rho dz d\theta \\ &= 2\pi \times 2 \times 1/4 \\ &= \pi \end{aligned}$$

■

3. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0, y = 0$ and $y + z = 2$.

Sol. Verify that

$$\nabla \cdot F = 3y$$

Thus using Gauss divergence theorem we have

$$\iint_S F \cdot dS = \iiint_E \nabla \cdot F dV$$

where E is the region enclosed by S. Thus

$$\begin{aligned} \iint_S F \cdot dS = \iiint 3y \, dx dy dz \\ &= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, dy dz dx \\ &= \boxed{\frac{98}{35}} \end{aligned}$$

■

4. Find out the flux of $F = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1, y = 1, z = 1$.

Sol.
The total flux will be the sum of the flux from the three sides, Let S_1 be the face represented by $x = 1$ and similarly S_2 for $y = 1$ and S_3 for $z = 1$.

Then observe that the normal vector of S_1, S_2, S_3 are $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. So the flux is

$$
\begin{aligned}
\text{Flux} &= \int\limits_{S_1} F \cdot \mathbf{n} dS + \int\limits_{S_2} F \cdot \mathbf{n} dS + \int\limits_{S_3} F \cdot \mathbf{n} dS \\
&= \int_{S_1} F \cdot \mathbf{i} dS + \int_{S_2} F \cdot \mathbf{j} dS + \int_{S_3} F \cdot \mathbf{k} dS \\
&= \int_{S_1} xy dy dz + \int_{S_2} yz dx dz + \int_{S_3} zx dx dy \\
&= \int_0^1 \int_0^1 y dy dz + \int_0^1 \int_0^1 z dz dx + \int_0^1 \int_0^1 x dx dy \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}
$$

■

5. Is $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector filed? If yes, find a vector field $\mathbf{G}$ such that $\mathbf{F} =$ curl $\mathbf{G}$ in $\mathbb{R}^3$.

Sol. Observe that

$$\text{div}(F) = 1 - 2 + 1 = 0.$$

And looking at F you can certainly say it is curl of some vector field, but to find a G such that $\nabla \times G = F$ looks hard, so we attempt at a general solution of the form

$$(a_1 xy + a_2 yz + a_3 zx, b_1 xy + b_2 yz + b_3 zx, c_1 xy + c_2 yz + c_3 zx)$$

using which we get one of the solutions as $(-yz, 0, xy)$ So setting $G(x, y, z) = (-yz, 0, xy)$ gives the answer.

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