MA-111 Calculus II (D3 & D4)

Lecture 11

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Characterization of conservative fields Contd.

Green's theorem

Various examples

Necessary condition for conservative fields

Theorem

- *For $n = 2$, if $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ is a conservative vector field, where F_1 and F_2 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^2$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D.$$

- *For $n = 3$, if $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ is a conservative vector field, where F_1, F_2, F_3 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^3$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D.$$

The theorem follows from a direct calculation using the fact that $\mathbf{F} = \nabla V$ and using the properties of the mixed partial derivatives of V.

Example Determine whether or not the vector field

$$\mathbf{F}(x,y) = (x-y)\mathbf{i} + (x-2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.
Ans Here $F_1(x,y) = x - y$ and $F_2(x,y) = x - 2$. Then

$$\frac{\partial F_1}{\partial y} = -1, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1.$$

So by previous theorem, $\mathbf{F}$ cannot be a conservative field.

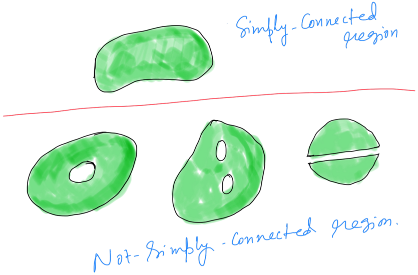
What about the converse of the theorem?
The converse is true under some additional hypothesis on D. However, it is often a convenient method verifying if a vector field is conservative.

Simply connected domain

Definition

A subset D of $\mathbb{R}^n$ for $n = 2, 3$, is simply connected, if D is a connected region such that any simple closed curve lying in D encloses a region that is in D.

Basically, a simply-connected region contains no hole and cannot consist of two separate pieces.



Sufficient condition for conservative field

Theorem

Let $n = 2, 3$ and let D be an open, simply connected region in $\mathbb{R}^n$.

1. *For $n = 2$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

2. *For $n = 3$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is such that F_1, F_2 and F_3 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

We postpone the proof of the theorem for later as it can be derived using Green's theorem.

Examples

Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.
Ans Note that the region $\mathbb{R}^2$ is open and simply-connected and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable.
Let $F_1(x, y) = (3 + 2xy)$ and $F_2(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial F_1}{\partial y}(x, y) = 2x = \frac{\partial F_2}{\partial x}.$$

Thus using the previous theorem, we conclude that $\mathbf{F}$ is a conservative field.
How to find a potential function f such that $\mathbf{F} = \nabla f$, for above example?

Example contd.

Let $\mathbf{F} = \nabla f$, then $\frac{\partial f}{\partial x}(x, y) = F_1(x, y)$ and $\frac{\partial f}{\partial y}(x, y) = F_2(x, y)$.
Step 1 Fixing y, solve the ODE with respect to x-variable:

$$\frac{\partial f}{\partial x}(x, y) = F_1(x, y).$$

Integrating with respect to x in both side, we get

$$f(x, y) = \int_0^x F_1(s, y)\, ds + c(y) = 3x + x^2 y + c(y).$$

Step 2 Determine the $c(y)$ using $\frac{\partial f}{\partial y}(x, y) = F_2(x, y)$. Differentiating $f(x, y)$ with respect to y,

$$\frac{\partial f}{\partial y}(x, y) = x^2 + c'(y),$$

and it has to be equal to $F_2(x, y)$.
so, $x^2 + c'(y) = x^2 - 3y^2$ and thus $c'(y) = -3y^2$. Now solving this ODE with respect to y variable:

$$c(y) = -y^3 + K,$$

for some constant K.
Thus $f(x, y) = 3x + x^2 y - y^3 + K$ such that $\mathbf{F} = \nabla f$.

In summary, for a given vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$:

1. If $\mathbf{F}$ is a continuous, conservative vector field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function, then the line integral of $\mathbf{F}$ along any path C from P to Q in D given by

$$\int_C \mathbf{F}.\mathbf{ds} = f(Q) - f(P),$$

   and it only depends on the value of f, the potential function, at the initial and terminal points of the path.

2. Let $\mathbf{F}$ be a continuous field and let D be an open connected set in $\mathbb{R}^n$. Then $\mathbf{F}$ is a conservative field if and only if the line integral of $\mathbf{F}$ is path-independent in D.
3. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a C^1 conservative vector field on an open region D, then $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.
4. Let D be an open, simply connected region in $\mathbb{R}^2$ and let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be C^1 on D. Then $\mathbf{F}$ is conservative in D if and only if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.

The goal is now to obtain a two-dimensional analog of the Fundamental theorem of calculus to express a double integral over a 'plane region D' as a line integral along the closed curve which is the boundary of D.

To state and prove 'Green's theorem', we need to introduce some concepts and terminology.

Green's theorem is applicable to the region satisfying particular conditions.

For now, we shall consider the region in $\mathbb{R}^2$.

Orientation of planer curves

By convention, the positive orientation of a simple closed curve on a plane corresponds to the anti-clockwise direction and the negative orientation of a simple closed curve on a plane corresponds to the clock-wise direction.

Convention: A simple closed curve in $\mathbb{R}^2$ is positively oriented if the region bounded by the curve always lies to the left of an observer walking along the curve in the chosen direction. otherwise, we say that the curve is negatively oriented.

Why is it clear that a simple closed curve in $\mathbb{R}^2$ bounds a region?

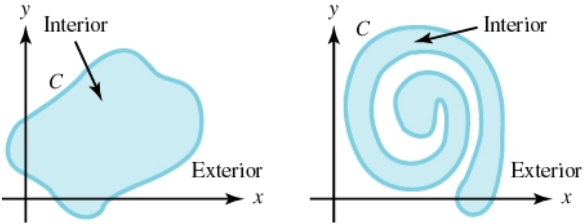
It may be intuitively clear. But it is not easy to prove rigorously! For planar curves, we have clear answer due to 'Jordon curve theorem'.

Jordan curve theorem

This is a celebrated theorem in topology:

Theorem

If $\mathbf{c} : [a, b] \to \mathbb{R}^2$ *is a simple closed path then* $\mathbb{R}^2 - \mathbf{c}\big([a, b]\big)$ *is divided in two connected parts, 'interior' and 'exterior', such that any path from one of them to the other would have to intersect* $\mathbf{c}\big([a, b]\big)$*.*



The bounded part is called the **interior** of the curve and the unbounded part is called the **exterior** of the curve.

Orientation of the boundary of an enclosed region in plane

The Jordan curve theorem perhaps strikes one as intuitively obvious, but is quite difficult to prove.

Once we accept Jordan's theorem, we see that C encloses a bounded region D in the plane. There is a natural notion of positive orientation of the region D - clearly it is given by the vector field $\mathbf{k}$ - the unit normal vector pointing in the direction of the positive z axis.

A curve C can be obtained as the boundary of a region D in the plane, but C may now consists of several components or pieces and D may have "holes".

Then how to define the orientation of the boundary curve C?

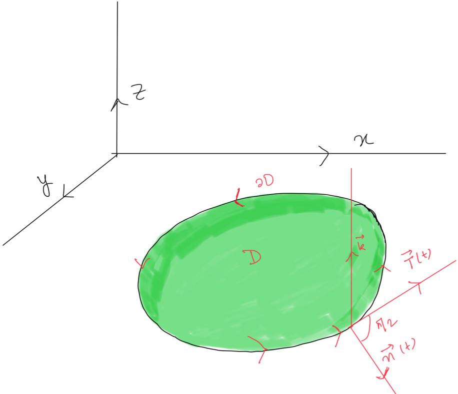
Orienting the boundary curve

Definition: The positive orientation of a curve C in $\mathbb{R}^2$ is given by the vector field

$$\mathbf{k} \times \mathbf{n}_{\text{out}},$$

where $\mathbf{n}_{\text{out}}$ is the normal vector field pointing outward along the curve.

Though the planar curve C lies in $\mathbb{R}^2$, here the path is parametrized as $\mathbf{c}(t) = (x(t), y(t), 0)$ for $t \in [a, b]$ with range in $\mathbb{R}^3$, taking 0 in the 3rd component.



Orienting the boundary curve contd.

Physically, this means that if one walks along C in the direction of the positive orientation, the region D is always on one's left.

As we shall see later, if C is a closed curve in space bounding an oriented surface S, the orientation of S naturally induces an orientation on the boundary C. The above example is a special case of this.

Example: Simple closed curve

Positively oriented curve

Ex. $\gamma(\theta) = (\cos\theta, \sin\theta, 0)$, $\theta \in [0, 2\pi]$. Then $\gamma'(\theta) = (-\sin\theta, \cos\theta, 0)$ and $\mathbf{n}_{\text{out}}(\theta) = (\cos\theta, \sin\theta, 0)$.
Note

$$\mathbf{k} \times \mathbf{n}_{\text{out}}(\theta) = (-\sin\theta, \cos\theta, 0) = \gamma'(\theta),$$

and so the curve is positively oriented.

Negatively Oriented curve

Ex. $\gamma_1(\theta) = (\cos\theta, -\sin\theta, 0)$, $\theta \in [0, 2\pi]$. Then $\gamma_1'(\theta) = (-\sin\theta, -\cos\theta, 0)$ and $\mathbf{n}_{\text{out}}(\theta) = (\cos\theta, -\sin\theta, 0)$.
Note

$$\mathbf{k} \times \mathbf{n}_{\text{out}}(\theta) = (\sin\theta, \cos\theta, 0) = -\gamma_1'(\theta),$$

and so the curve is negatively oriented.

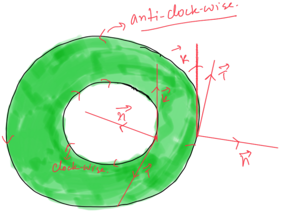
Example: Orientation of boundary of a region with hole

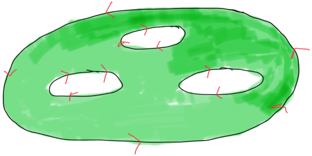
Annulus: For $D = \{(x, y) \in \mathbb{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2\}$, the boundary $\partial D = C_1 \cup C_2$, where C_1 is the outer boundary $C_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = b^2\}$ and C_2 is the inner boundary $C_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = a^2\}$.
what is the positive orientation of ∂D, boundary of D?

C_1 oriented anti-clockwise: $\mathbf{c}_1(\theta) = (b\cos\theta, b\sin\theta, 0)$ for $\theta \in [0, 2\pi]$ and $\mathbf{n}_{\text{out}}(\theta) = (b\cos\theta, b\sin\theta, 0)$ at C_1 and $\mathbf{c}_1'(\theta) = \mathbf{k} \times \mathbf{n}_{\text{out}}(\theta)$.

C_2 oriented clockwise $\mathbf{c}_2(\theta) = (a\cos\theta, -a\sin\theta, 0)$ for $\theta \in [0, 2\pi]$ and $\mathbf{n}_{\text{out}}(\theta) = (-a\cos\theta, a\sin\theta, 0)$ at C_2 and $\mathbf{c}_2'(\theta) = \mathbf{k} \times \mathbf{n}_{\text{out}}(\theta)$.

Then the outer boundary curve is given the anti-clockwise orientation, while the inner boundary curves are oriented in the clockwise direction.





Positive-orientation:

Outer boundary anti-clock-wise.

Inner boundary clock-wise.

Green's Theorem

With the preliminaries out of the way, we are now in a position to state the first major theorem of vector calculus, namely Green's Theorem.

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\Big(D \cup \partial D\Big) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a double integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

Example: Let C be the circle of radius r oriented in the counterclockwise direction, and let $F_1(x, y) = -y$ and $F_2(x, y) = x$. Evaluate

$$\int_C F_1(x, y)dx + F_2(x, y)dy.$$

Solution: Let D denote the disc of radius r. Then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2$. Hence, by Green's theorem

$$\int_C F_1(x, y)dx + F_2(x, y)dy = \iint_D 2dxdy = 2\pi r^2.$$

Also by the direct calculation, denoting $\mathbf{F} = (F_1, F_2)$, check
$\int_C \mathbf{F}.\mathbf{ds} = \int_C F_1(x, y)dx + F_2(x, y)dy =?.$

Examples.

Example. Compute the line integral $\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy$, where C is the circle in $\mathbb{R}^2$ with origin at $(2,0)$ and radius 1.

Can compute directly using definition of line integral! But is there any better way?

Use Green's theorem: Set $F_1(x,y) = ye^{-x}$ and $F_2(x,y) = (\frac{1}{2}x^2 - e^{-x})$, for all $(x,y) \in D$, where $D = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 \leq 1\}$. Using Green's theorem,

$$\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy.$$

Now see

$$\int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy = \int\int_D xdxdy,$$

and derive the double integral using polar coordinates: Check!

$$\int\int_D xdxdy = 2\pi.$$

Area of a region

Can the area of a region enclosed be expressed as a line integral?
If C is a positively oriented curve that bounds a region D , then the area $A(D)$ is given by (Why?)

$$A(D) = \frac{1}{2}\int_C xdy - ydx.$$

Note if $F_1(x, y) = -\frac{y}{2}$ and $F_2(x, y) = \frac{x}{2}$, for all $(x, y) \in D$, then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and hence $A(D) := \int\int_D 1\, dxdy = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dxdy$.
By Green's theorem,

$$\int_D\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \int_C F_1\, dx + F_2\, dy = \frac{1}{2}\int_C xdy - ydx,$$

Thus $A(D) = \frac{1}{2}\int_C xdy - ydx$.
Also note for $F_1 \equiv 0$ and $F_2(x, y) = x$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = \int_C x\, dy$.
Further for $F_1(x, y) = -y$ and $F_2 \equiv 0$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = -\int_C y\, dx$.
In summary, $A(D) = \frac{1}{2}\int_C xdy - ydx = \int_C x\, dy = -\int_C y\, dx$.