MA 109 Week 1

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November 29, 2021

Course objectives

Welcome to IIT Bombay. The aim of this course is:

- To help the students achieve a better and more rigourous understanding of the calculus of one variable.
- To introduce the ideas and theorems in the calculus of several variable.
- To help students achieve a working knowledge of the tools and techinques of the calculus of several variables in view of the applications they are likely to encounter in the future.

For details about the syllabus, tutorials, assignments, quizes, exams and procedures for evaluation please refer to the course booklet. The course booklet can also be found on moodle:
http://moodle.iitb.ac.in/login/index.php

The emphasis of this course will be on the underlying ideas and methods rather than very intricate problem solving involving formal manipulations (of course, there will be plenty of problems - just not many with lots of algebra tricks). The aim is to get you to think about calculus, in particular, and mathematics in general.

Ask questions! There is a good chance that if you don't understand something, many other people also do not understand it.

Sequences

Definition: A sequence in a set X is a function $a : \mathbb{N} \to X$, that is, a function from the natural numbers to X.

In this course X will usually be a subset of (or equal to) $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$, though we will also have occassion to consider sequences of functions sometimes. In later mathematics courses X may be the complex numbers $\mathbb{C}$ (MA 205), vector spaces (whatever those maybe) the set of continuous functions on an interval $\mathcal{C}([a, b])$ or other sets of functions (MA 106, MA 108, MA 207, MA 214).

Rather than write the value of the function at n as $a(n)$, we often write a_n for the members of the sequence. A sequence is often specified by listing the first few terms

$$a_1, a_2, a_3, \ldots$$

or, more generally by describing the n^{th} term a_n. When we want to talk about the sequence as a whole we sometimes write $\{a_n\}_{n=1}^{\infty}$, but more often we once again just write a_n.

Examples of sequences

1. $a_n = n$ (here we can take $X = \mathbb{N} \subset \mathbb{R}$ if we want, and the sequence is just the identity function. Of course, we can also take $X = \mathbb{R}$).
2. $a_n = 1/n$ (here we can take $X = \mathbb{Q} \subset \mathbb{R}$ if we want, where $\mathbb{Q}$ denotes the rational numbers, or we can take $X = \mathbb{R}$ itself).
3. $a_n = \frac{n!}{n^n}$ ($X = \mathbb{Q}$ or $X = \mathbb{R}$).
4. $a_n = n^{1/n}$ (here the values taken by a_n are irrational numbers, so it best to take $X = \mathbb{R}$).
5. $a_n = \sin\left(\frac{1}{n}\right)$ (again the values taken by a_n are in general irrational numbers, so it best to take $X = \mathbb{R}$).

These are all examples of sequence of real numbers.

More examples

6. $a_n = \left(n^2, \frac{1}{n}\right)$ (here $X = \mathbb{R}^2$ or $X = \mathbb{Q}^2$).

   This is a sequence in $\mathbb{R}^2$.

7. $f_n(x) = \cos(nx)$ (here X is the set of continuous functions on any interval $[a, b]$ or even on $\mathbb{R}$).

   This is a sequence of functions. More precisely, it is a sequence of continuous functions.

Series

Given a sequence a_n of real numbers, we can manufacture a new sequence, namely its sequence of partial sums:

$$s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \ldots$$

More precisely, we have the sequence

$$s_n = \sum_{k=1}^{n} a_k.$$

8. We can take $a_n = r^n$, for some r, i.e., a geometric progression. Then $s_n = \sum_{k=0}^{n} r^k$.
9. $s_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$, or writing it out $s_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$.

   We get a sequence of polynomial functions.

Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

From the examples in the previous slide, Example 1 is a monotonically increasing sequence, Example 2 is a monotonically decreasing sequence. How about Example 3?

In Example 3 we notice that if $a_n = \frac{n!}{n^n}$,

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}} = a_n \times \frac{(n+1)n^n}{(n+1)^{(n+1)}} \leq a_n,$$

so the sequence is monotonically decreasing.

Eventually monotonic sequences

In Example 4 ($a_n = n^{1/n}$), we note that

$$a_1 = 1 < 2^{1/2} = a_2 < 3^{1/3} = a_3,$$

(raise both a_2 and a_3 to the sixth power to see that $2^3 < 3^2$!).
However, $3^1/3 > 4^{1/4} > 5^{1/5}$. So what do you think happens as n gets larger?

In fact, $a_{n+1} \leq a_n$, for all $n \geq 3$. Prove this fact as an exercise. Such a sequence is called an eventually monotonic sequence, that is, the sequence becomes monotonic(ally decreasing) after some stage. One can similarly define eventually monotonically increasing sequences.

Let us quickly run through the other examples. Example 5 - monotonically decreasing. Example 6 - is not a sequence of real numbers. Example 7 - is a sequence of real numbers if we fix a value of x . Can it be monotonic for some x? Example 8 is monotonic for any fixed value of r and so is Example 9 for any non-negative value of x.

Limits: Preliminaries

While all of you are familiar with limits, most of you have probably not worked with a rigourous definition. We will be more interested in limits of functions of a real variable (which is what arise in the differential calculus), but limits of sequences are closely related to the former, and occur in their own right in the theory of Riemann integration.

So what does it mean for a sequence to tend to a limit? Let us look at the sequence $a_n = 1/n^2$. We wish to study the behaviour of this sequence as n gets large. Clearly as n gets larger and larger, $1/n^2$ gets smaller and smaller and seems to approach the value 0, or more precisely

the distance between $1/n^2$ and 0 becomes smaller and smaller.

In fact (and this is the key point), by choosing n large enough, we can make the distance between $1/n^2$ and 0 smaller than any prescribed quantity.

Let us examine the above statement, and then try and quantify it.

More precisely:

The distance between $1/n^2$ and 0 is given by $|1/n^2 - 0| = 1/n^2$.

Suppose I require that $1/n^2$ be less that 0.1 (that is 0.1 is my prescribed quantity). Clearly, $1/n^2 < 1/10$ for all $n > 3$.

Similarly, if I require that $1/n^2$ be less than $0.0001(= 10^{-4})$, this will be true for all $n > 100$.

We can do this for any number, no matter how small. If $\epsilon > 0$ is any number,

$$1/n^2 < \epsilon \iff 1/\epsilon < n^2 \iff n > 1/\sqrt{\epsilon}.$$

In other words, given any $\epsilon > 0$, we can always find a natural number N (in this case any $N > 1/\sqrt{\epsilon}$) such that for all $n > N$, $|1/n^2 - 0| < \epsilon$.

The rigorous definition of a limit

Motivated by the previous example, we define the limit as follows.

Definition: A sequence a_n tends to a limit l/converges to a limit l, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

This is what we mean when we write

$$\lim_{n \to \infty} a_n = l.$$

If we just want to say that the sequence has a limit without specifying what that limit is, we simply say $\{a_n\}_{n=1}^{\infty}$ converges, or that it is convergent.
A sequence that does not converge is said to diverge, or to be divergent.

Remarks on the definition

Remarks

1. Note that the N will (of course) depend on ϵ, as it did in our example, so it would have been more correct to write $N(\epsilon)$ in the definition of the limit. However, we usually omit this extra bit of notation.
2. We have already shown that $\lim_{n\to\infty} 1/n^2 = 0$. The same argument works for $\lim_{n\to\infty} 1/n^{\alpha}$, for any real $\alpha > 0$. We just take N to be any integer bigger than $1/\epsilon^{1/\alpha}$ for a given ϵ.
3. For a given ϵ, once one N works, any larger N will also work. In order to show that a sequence tends to a limit l we are not obliged to find the best possible N for a given ϵ, just some N that works. Thus, for the sequence $1/n^2$ and $\epsilon = 0.1$, we took $N = 3$, but we can also take $N = 10, 100, 1729$, or any other number bigger than 3.
4. Showing that a sequence converges to a limit l is not easy. One first has to guess the value l and then prove that l satisfies the definition. We will see how to get around this in various ways.

More examples of limits

Let us show that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$.

For this we note that for $x \in [0, \pi/2]$, $0 \leq \sin x \leq x$ (try to remember why this is true).
Hence,

$$|\sin 1/n - 0| = |\sin 1/n| < 1/n.$$

Thus, given any $\epsilon > 0$, if we choose some $N > 1/\epsilon$, $n > N$ implies $1/n < 1/N < \epsilon$. It follows that $|\sin 1/n - 0| < \epsilon$.

Let us consider Exercise 1.1.(ii) of the tutorial sheet. Here we have to show that $\lim_{n\to\infty} 5/(3n+1) = 0$. Once again, we have only to note that

$$\frac{5}{3n+1} < \frac{5}{3n},$$

and if this is to be smaller than ϵ,
we must have $n > N > 5/3\epsilon$.

Formulæ for limits

If a_n and b_n are two convergent sequences then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}(a_n / b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$, provided $\lim_{n\to\infty} b_n \neq 0$

Implicit in the formulæ is the fact that the limits on left hand side exist.

Note that the constant sequence $a_n = c$ has limit c, so as a special case of (2) above we have

$$\lim_{n\to\infty}(c \cdot b_n) = c \cdot \lim_{n\to\infty} b_n.$$

Using the formulæ above we can break down the limits of more complicated sequences into simpler ones and evaluate them.

The Sandwich Theorem(s)

Theorem 1: If a_n, b_n and c_n are convergent sequences such that $a_n \leq b_n \leq c_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n.$$

A second version of the theorem is especially useful:
Theorem 2: Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \leq b_n \leq c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Note that we do not assume that b_n converges in this version of the theorem - we get the convergence of b_n for free . Together with the rules for sums, differences, products and quotients, this theorem allows us to handle a large number of more complicated limits.

An example using the theorems above

Consider Exercise 1.2.(iii) on the tutorial sheet. We have to show that

$$\lim_{n\to\infty} \frac{n^3+3n^2+1}{n^4+8n^2+2}$$

exists and to evaluate it.

It is clear that

$$0 < \frac{n^3+3n^2+1}{n^4+8n^2+2} \leq \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}.$$

(How do we get this?)
Note that $n^3/(n^4+8n^2+2) < n^3/n^4 = 1/n$, and the other two terms can be handled similarly.)

Hence, applying the Sandwich Theorem (Theorem 2) to the sequences

$$a_n = 0, \quad b_n = \frac{n^3+3n^2+1}{n^4+8n^2+2} \quad \text{and} \quad c_n = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}$$

we see that the limit we want exists provided $\lim_{n\to\infty} c_n$ exists, so this is what we must concentrate on proving.

The limit $\lim_{n\to\infty} c_n$ exists provided each of the terms appearing in the sum has a limit and in that case it is equal to the sum of the limits (by the first formula). But each of these limits is quite easy to evaluate.

We already know that

$$\lim_{n\to\infty} 1/n = 0 = \lim_{n\to\infty} 1/n^4,$$

while

$$\lim_{n\to\infty} 3/n^2 = 3 \cdot \lim_{n\to\infty} 1/n^2 = 0$$

where we have used the special case of the second formula (limit of the product is the product of the limits) for the first equality in the equation above. Since all three limits converge to 0, it follows the given limit is $0 + 0 + 0 = 0$.

Bounded Sequences

The formulæ and theorems stated above can be easily proved starting from the definitions. We will prove the second formula and leave the other proofs as exercises.

Definition: A sequence a_n is said to be bounded if there is a real number $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$. A sequence that is not bounded is called unbounded.

In our list of examples, Example 1 ($a_n = n$) is an example of an unbounded sequence, while Examples 2 - 5 ($a_n = 1/n, \sin(1/n), n!/n^n, n^{1/n}$) are examples of bounded sequences.

Bounded sequence don't necessarily converge - for instance $a_n = (-1)^n$. However,

Convergent sequences are bounded

Lemma: Every convergent sequence is bounded.

Proof: Suppose a_n converges to l. Choose $\epsilon = 1$. There exists $N \in \mathbb{N}$ such that $|a_n - l| < 1$ for all $n > N$. In other words, $l - 1 < a_n < l + 1$, for all $n > N$, which gives $|a_n| < |l| + 1$ for all $n > N$. Let

$$M_1 = \max\{|a_1|, |a_2|, \ldots, |a_N|\}$$

and let $M = \max\{M_1, |l| + 1\}$. Then $|a_n| < M$ for all $n \in \mathbb{N}$. $\square$

(Some absolute value signs were missing when this slide was displayed in class. I have now put these in.)

We will use this Lemma to prove the product rule for limits.

The proof of the product rule

We wish to prove that $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

Suppose $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} b_n = l_2$. We need to show that $\lim_{n\to\infty} a_n b_n = l_1 l_2$.

Fix $\epsilon > 0$. We need to show that we can find $N \in \mathbb{N}$ such that $|a_n b_n - l_1 l_2| < \epsilon$, whenever $n > N$. Notice that

$$\begin{aligned} |a_n b_n - l_1 l_2| &= |a_n b_n - a_n l_2 + a_n l_2 - l_1 l_2| \\ &= |a_n(b_n - l_2) + (a_n - l_1) l_2| \\ &\leq |a_n||b_n - l_2| + |a_n - l_1||l_2|, \end{aligned}$$

where the last inequality follows from the triangle inequality. So in order to guarantee that the left hand side is small, we must ensure that the two terms on the right hand side together add up to less than ϵ. In fact, we make sure that each term is less than $\epsilon/2$.

The proof of the product rule, continued

Since a_n is convergent, it is bounded by the lemma we have just proved. Hence, there is an M such that $|a_n| < M$ for all $n \in \mathbb{N}$.

Assume $l_2 \neq 0$ (If $l_2 = 0$, the proof becomes even simpler). Given the quantities $\epsilon/2|l_2|$ and $\epsilon/2M$, there exist N_1 and N_2 such that

$$|a_n - l_1| < \epsilon/2|l_2| \quad \text{and} \quad |b_n - l_2| < \epsilon/2M.$$

Let $N = \max\{N_1, N_2\}$. If $n > N$, then both the inequalities above hold. Hence, we have

$$|a_n||b_n - l_2| \leq M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2} \quad \text{and} \quad |a_n - l_1||l_2| \leq |l_2| \cdot \frac{\epsilon}{2|l_2|} = \frac{\epsilon}{2}.$$

Now it follows that

$$|a_n b_n - l_1 l_2| \leq |a_n||b_n - l_2| + |a_n - l_1||l_2| < \epsilon,$$

for all $n > N$, which is what we needed to prove. $\square$

The proofs of the other rules for limits are similar to the one we proved above. Try them as exercises.

A guarantee for convergence

As we mentioned earlier, proving that a limit exists is hard because we have to guess what its value might be and then prove that it satisfies the definition. The following theorem guarantees the convergence of a sequence without knowing the limit beforehand.

Definition: A sequence a_n is said to be bounded above (resp. bounded below) if $a_n < M$ (resp. $a_n > M$) for some $M \in \mathbb{R}$.

A sequence that is bounded both above and below is obviously bounded.

Theorem 3: A montonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

Remarks on Theorem 3

Theorem 3 clearly makes things very simple in many cases. For instance, if we have a monotonically decreasing sequence of positive numbers, it must have a limit, since 0 is always a lower bound!

Can we guess what the limit of a monotonically increasing sequence a_n bounded above might be?
It will be the supremum or least upper bound (lub) of the sequence. This is the number, say M which has the following properties:

1. $a_n \leq M$ for all n and
2. If M_1 is such that $a_n < M_1$ for all n, then $M \leq M_1$.

The point is that a sequence bounded above may not have a maximum but will always have a supremum. As an example, take the sequence $1 - 1/n$. Clearly there is no maximal element in the sequence, but 1 is its supremum.

Another monotonic sequence

Let us look at Exercise 1.5.(i) which considers the sequence

$$a_1 = 3/2 \quad \text{and} \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

$$\begin{aligned} a_{n+1} \quad < a_n \quad &\iff \quad \tfrac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) < a_n \\ &\iff \quad \sqrt{2} < a_n. \end{aligned}$$

On the other hand,

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2}, \quad \text{(Why is this true?)}$$

so $a_{n+1} \geq \sqrt{2}$ for all $n \geq 1$ and $a_1 > \sqrt{2}$ is given.
Hence, $\{a_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence, bounded below by $\sqrt{2}$. By Theorem 3, it converges.
Exercise 1. What do you think is the limit of the above sequence (Refer to the supplement to Tutorial 1)?

More remarks on limits

Exercise 2. More generally, what is the limit of a monotonically decreasing sequence bounded below? How can you describe it?
This number is called the infimum or greatest lower bound (glb) of the sequence.

The proof of Theorem 3 is not so easy and more or less involves understanding what a real number is. It is related to the notion of Cauchy sequences about which I will try to say something a little later (again, refer to the supplement to Tutorial 1).

An important remark: If we change finitely many terms of a sequence it does not affect the convergence and boundedness properties of a sequence. If it is convergent, the limit will not change. If it is bounded, it will remain bounded though the supremum may change. Thus, an eventually monotonically increasing sequence bounded above will converge (formulate the analogue for decreasing sequences).
Bottomline: From the point of view of the limit, only what happens for large N matters.

Cauchy sequences

As we have seen, it is not easy to tell whether a sequence converges or not because we have to first guess what the limit might be, and then try and prove that the sequence actually converges to this limit. For a monotonic sequence, we have a criterion, but what about more general sequences?

There is another very useful notion which allows us to decide whether the sequence converges by looking only at the elements of the sequence itself. We describe this below.

Definition: A sequence a_n in $\mathbb{R}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon,$$

for all $m, n > N$.

Cauchy sequences: the theorem

Theorem 4: Every Cauchy sequence in $\mathbb{R}$ converges.

Remark 1: One can now check the convergence of a sequence just by looking at the sequence itself!

Remark 2: One can easily check the converse:

Theorem 5: Every convergent sequence is Cauchy.

Remark 3: Remember that when we defined sequences we defined them to be functions from $\mathbb{N}$ to X, for any set X. So far we have only considered $X = \mathbb{R}$, but as we said earlier we can take other sets, for instance, susbets of $\mathbb{R}$. For instance, if we take $X = \mathbb{R} \setminus 0$, Theorem 4 is not valid. The sequence $1/n$ is a Cauchy sequence in this X but obviously does not converge in X. If we take $X = \mathbb{Q}$, the example given in 1.5.(i) $(a_{n+1} = (a_n + 2/a_n)/2)$ is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$. Thus Theorem 4 is really a theorem about real numbers.

The completeness of $\mathbb{R}$

A set in which every Cauchy sequence converges is called a a complete set. Thus Theorem 4 is sometimes rewritten as

Theorem 4': The real numbers are complete.

We will see other examples of complete sets, but we can now address (very briefly) the question of what a real number is. More precisely, we can *construct* the set of real numbers starting with the rational numbers.

We let S be the set of all sequences with values in $\mathbb{Q}$. We will put a relation on this set.

The definition of a real number

Two sequence $\{a_n\}$ and $\{b_n\}$ will be related to each other (and we write $a_n \sim b_n$) if

$$\lim_{n\to\infty} |a_n - b_n| = 0$$

You can check that this is an equivalence relation and it is a fact that it *partitions* the set S into disjoint classes. The set of disjoint classes is denote $S/\sim$.

You can easily see that if two sequences converge to the same limit, they are necessarily in the same class.

Definition: A real number is an equivalence class in $S/\sim$.

So a real number should be thought of as the collection of all rational sequences which converge to it.

Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

Most of our definitions for sequences in $\mathbb{R}$ are actually valid for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function $|\ |$ on $\mathbb{R}$ by the distance functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ all the definitions of convergent sequences and Cauchy sequences remain the same.

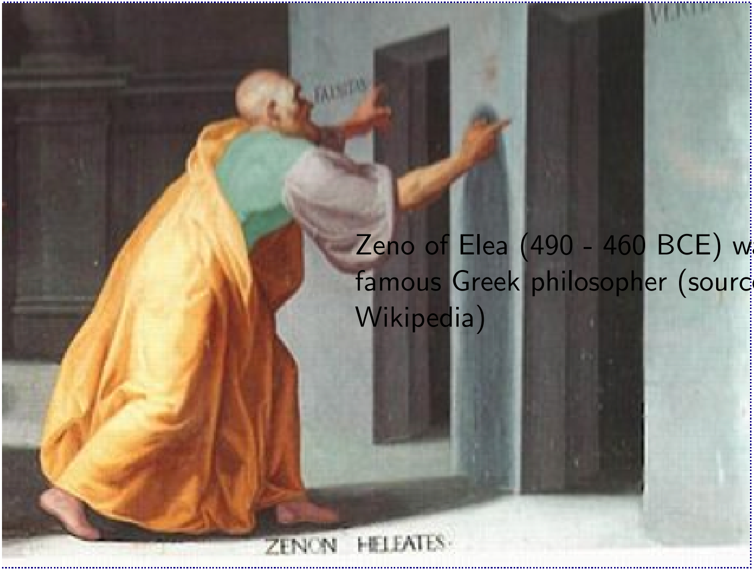
For instance, a sequence $a(n) = (a(n)_1, a(n)_2)$ in $\mathbb{R}^2$ is said to converge to a point $l = (l_1, l_2)$ (in $\mathbb{R}^2$) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{(a(n)_1 - l_1)^2 + a(n)_2 - l_2)^2} < \epsilon$$

whenver $n > N$. A similar defintion can be made in $\mathbb{R}^3$ using the distance function on $\mathbb{R}^3$.

Theorems 2 (the Sandwich Theorem) and 3 (about monotonic sequences) don't really make sense for $\mathbb{R}^2$ or $\mathbb{R}^3$ because there is no ordering on these sets, that is, it doesn't really make sense to ask if one point on the plane or in space is less than the other.

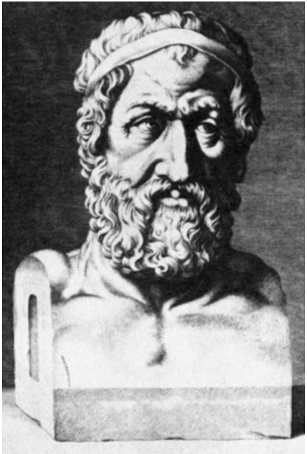
The first man to think about limits?

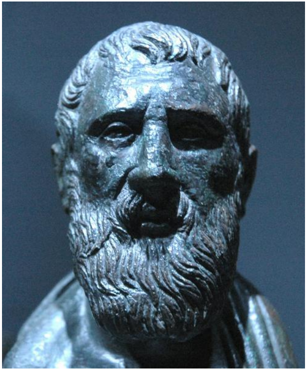


Zeno of Elea (490 - 460 BCE) was a famous Greek philosopher (source: Wikipedia)

Zeno of Elea

First let us record that we have no idea what Zeno looked like. The picture above was painted in the period 1588 - 1594 CE in Spain, about two thousand years after Zeno's time. Here are two more images of Zeno (also from Wikipedia)





Zeno's Paradoxes

I couldn't find out where the first statue came from and when it was made. The second seems to have come from Herculaneum in Italy (incidentally, Elea (modern Vilia) is a town in Italy). Now Herculaneum was destroyed by a volcanic erruption from the nearby volcano Vesuvius in 79 CE, so it looks like the bust was created within 500 years of Zeno's death. Maybe it was even made during his lifetime and was lying around in some wealthy Roman's house for the next few centuries. Unfortunately, it is not clear whether this statue is one of Zeno of Elea or of another Zeno (of Citium) who lived about 150 years later.

The important about Zeno is that it would appear that he was the first human to think about limits and limiting processes, at least in recorded history. Most of what we know about him is through his paradoxes, nine of which survive in the works of another famous Greek philosopher Aristotle (384 - 322 CE), the official guru/tutor of Alexander.

Achilles and the tortoise

One of Zeno's motivations for stating his paradoxes seems to have been to defend his own guru Parmenides' philosophy (whatever that was). Anyway here is his most famous paradox as recorded by Aristotle.

Achilles and the tortoise:
In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.

Aristotle, Physics VI:9, 239b15

A gateway to infinite series

Nowadays, this line of argument does not really bother us, since we understand that an infinite number of terms (in this case consisting of the time travelled in each segment or the distance travelled in each segment) can add up to something finite. Nevertheless there are other philosophical issues that continued to bother mathematicians and physicists for a long time. After all, this kind of discussion does lead us to question whether intervals of time and space can be infinitely subdivided, or if "instantaneous motion" makes sense.

Since we are learning mathematics, we won't speculate on physics or philosophy, but we note that Zeno's argument gives a good way to derive the sum of an infinite geometric series. The geometric series is one of the simplest examples of infinite series, so let us see how this is done.

Geometric series - the formula

Let us suppose that the speed of achilles is v and that the speed of the tortoise is rv for some $0 < r < 1$. We will assume that the tortoise was given a headstart of distance "a".

- The distance covered by Achilles in time t is vt.
- The distance covered by the tortoise in time t is rvt.
- Achilles catches up with the tortoise when $vt = a + rvt$, that is, at time $t = a/(v - rv)$ and when the total distance covered by Achilles is $vt = a/(1 - r)$.

On the other hand,

- Distance covered by the tortoise by the time Achilles has covered distance a is ar.
- Distance covered by the tortoise by the time Achilles has covered distance ar is ar^2 ....
- Total distance covered by Achilles when he has caught up with the tortoise is $a + ar + ar^2 + \ldots$.
- Thus we get $a + ar + ar^2 + \cdots = a/(1 - r)$.

Infinite series - a more rigourous treatment

Let us recall what we mean when we write

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}.$$

Another way of writing the same expression is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

The precise meaning is the following. Form the partial sums

$$s_n = \sum_{k=0}^{n} ar^k.$$

These partial sums $s_1, s_2, \ldots s_n, \ldots$ form a sequence and by $\sum_{k=0}^{\infty} ar^k = a/(1-r)$, we mean $\lim_{n\to\infty} s_n = a/1-r$.
So when we speak of the sum of an infinite series, what we really mean is the limit of its partial sums.

Convergence of the geometric series

So to justify our formula we should show that $\lim_{n\to\infty} s_n = a/1 - r$, that is, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon,$$

for all $n > N$.
In other words we need to show that

$$\left| \frac{a(1-r^{n+1})}{1-r} - \frac{a}{1-r} \right| = \left| \frac{ar^{n+1}}{1-r} \right| < \epsilon$$

if n is chosen large enough.
But $\lim_{n\to\infty} r^n = 0$, so there exists N such that $r^{n+1} < (1-r)\epsilon/a$ for all $n > N$, so for this N, if $n > N$,

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon.$$

This shows that the geometric series converges to the given expression.