MA-111 Calculus II
(D3 & D4)

Lecture 3

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Properties of integrals over rectangles

Evaluating Integrals: Iterated integrals

Integrable functions

Riemann Integral contd.

- For any rectangle $R \subseteq \mathbb{R}^2$, let $f : R \to \mathbb{R}^2$ be bounded. The Darboux integrability and Riemann integrability are equivalent.
- A function $f : R \to \mathbb{R}^2$ is called integrable on R if (Darboux or) Riemann integrability condition holds on R.
- In summary, if f is integrable on R, then

$$\int\int_R f(x, y)\, dxdy := S = L(f) = U(f).$$

Examples: Let $R = [a, b] \times [c, d]$.

• The constant function is integrable.

• The projection functions $p_1(x, y) = x$ and $p_2(x, y) = y$ are both integrable on any rectangle $R \subset \mathbb{R}^2$. Why?

• Let $f : R \to \mathbb{R}$ be defined as $f(x, y) = \phi(x)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function. Is f integrable? what is $\int\int_R f\, dxdy$?

Regular partitions

Because the current way of taking partitions isn't truly helpful in making computations, we define *Regular* partitions.

The regular partition of R of order any $n \in \mathbb{N}$ is defined by $x_0 = a$ and $y_0 = c$, and for $i = 0, 1, \cdots, n-1, \quad j = 0, 1, \cdots, n-1$,

$$x_{i+1} = x_i + \frac{b-a}{n}, \quad y_{j+1} = y_j + \frac{d-c}{n}.$$

We take $t = \{t_{ij} \in R_{ij} \mid i, j \in \{0, 1, \cdots, n-1\}\}$ any arbitrary tag.
To check the integrability of a function f, it is enough to consider a sequence of regular partitions P_n of R.

Theorem

A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij},$$

tends to the same limit $S \in \mathbb{R}$ as $n \to \infty$, for any choice of tag t.

An Example

Example: Let $f(x, y) = x^2 + y^2$. Is it a continuous function on $\mathbb{R}^2$?
Ans. Yes! Suppose the function is integrable on $[0, 1] \times [0, 1]$. Compute the integral using the theorem.

Let $R = [0, 1] \times [0, 1]$ and P_n be a regular partition. Then for tag $t = \{(\frac{i}{n}, \frac{j}{n}) \mid i = 0, \ldots n-1, j = 0, \ldots, n-1\}$,

$$S(f, P_n, t) = (\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (\frac{i}{n})^2 + (\frac{j}{n})^2) \frac{1}{n^2}.$$

Compute $\lim_{n\to\infty} S(f, P_n, t)$. How would you go about it? Answer is

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Conventions

Based on our definition, we make the following convention: Let $a, b, c, d \in \mathbb{R}$

- If $a = b$ or $c = d$, then $\int\int_{[a,b]\times[c,d]} f(x,y)dxdy := 0$.
- If $a < b$ and $c < d$:
  $\int\int_{[b,a]\times[c,d]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,

  $\int\int_{[a,b]\times[d,c]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,

  $\int\int_{[b,a]\times[d,c]} f(x,y)dxdy := \int\int_{[a,b]\times[c,d]} f(x,y)dxdy$.

Properties of integrals over rectangles

(Domain Additivity Property:) Let R be a rectangle and $f : R \to \mathbb{R}$ be a bounded function. Partition R into finitely many non-overlapping sub-rectangles. Then f is integrable on R if and only if it is integrable on each sub-rectangle. When it exists, the integral of f on R is the sum of the integrals of f on the sub-rectangles.

Algebraic properties :

Let $R := [a, b] \times [c, d]$. Let f and g be integrable on R.

- If f is defined as $f(x, y) = \alpha \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$ then $\int\int_R f = \alpha A(R)$ where A is the area of R.
- The function $f + g$ is integrable, and $\int\int_R f + g = \int\int_R f + \int\int_R g$.
- For all $\alpha \in \mathbb{R}$, αf is integrable and $\int\int_R \alpha f = \alpha \int\int_R f$.
- If $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, then $\int\int_R f \leq \int\int_R g$.
- $|f|$ is integrable and $|\int\int_R f| \leq \int\int_R |f|$.
- The function $f.g$ is integrable.
- If $\frac{1}{f}$ is well defined and bounded on R, then $\frac{1}{f}$ is integrable on R.

All these follow by applying the definition and properties of limits. An immediate consequence is that all polynomial functions are integrable.

Evaluating Integrals

Suppose $f : R \to \mathbb{R}$ is integrable. How do we compute its double integral?

Geometrically, if f is non-negative then the double integral is the volume of the region D between the rectangle and under the solid $z = f(x, y)$. Cavalier's method was to compute this volume slice by slice.

That is, first compute area of each slice $A(x) = \int_c^d f(x, y)\ dy$ of the cross section of D perpendicular to the x-axis (or alternately the area $B(y) = \int_a^b f(x, y)\ dx$ of the cross section perpendicular to the y-axis)

Then the volume of $D = \int_a^b A(x)\ dx = \int_c^d B(y)\ dy$.

Fubini theorem and Iterated integrals

Theorem

Let $R := [a, b] \times [c, d]$ and $f : R \to \mathbb{R}$ be integrable. Let I denote the integral of f on R.

1. *If for each $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists, then the iterated integral $\int_a^b (\int_c^d f(x, y)\, dy)\, dx$ exists and is equal to I.*
2. *If for each $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)\, dx$ exists, then the iterated integral $\int_c^d (\int_a^b f(x, y)\, dx)\, dy$ exists and is equal to I.*

As a consequence, if f is integrable on R and if both iterated integrals exist in 1. and 2. in above theorem, then

$$\int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx = I = \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

Sketch of the proof

The proof is using Riemann condition.

- Since f is double integrable over R, for any given $\epsilon > 0$, there exists a partition $P_\epsilon = \{(x_i, y_j) \mid i = 0, 1, \cdots k-1, \quad j = 0, \cdots n-1\}$ of R such that
$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$
- Assume for each fixed $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists. Define
$$A(x) := \int_c^d f(x, y)\, dy, \quad \forall\, x \in [a, b].$$
- Claim: The function A is integrable over $[a, b]$. Note that $m(f)(d-c) \leq A(x) \leq M(f)(d-c)$ for all $x \in [a, b]$ and hence A is bounded. Also by domain additivity, $A(x) = \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} f(x, y)\, dy$, for all $x \in [a, b]$.
- Thus for each fixed $i \in \{0, \cdots k-1\}$, for $x \in [x_i, x_{i+1}]$, we obtain
$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \leq A(x) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

Sketch of the proof contd.

- Denoting $m_i(A) := \inf\{A(x) \mid x \in [x_i, x_{i+1}]\}$ and $M_i(A) := \sup\{A(x) \mid x \in [x_i, x_{i+1}]\}$, we have

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \leq m_i(A) \leq M_i(A) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

  Multiplying by $(x_{i+1} - x_i)$ and summing over $i = 0, \ldots, k-1$, we obtain

$$L(f, P_\epsilon) \leq \sum_{i=0}^{k-1} m_i(A)(x_{i+1} - x_i) \leq \sum_{i=0}^{k-1} M_i(A)(x_{i+1} - x_i) \leq U(f, P_\epsilon).$$

  and it yields that there exists a partition $P_1 := \{x_0, \cdots, x_{k-1}\}$ of $[a, b]$ such that

$$U(A, P_1) - L(A, P_1) < \epsilon.$$

- Thus the function of A is integrable and

$$\int\int_R f \, dx \, dy = \int_a^b A(x) \, dx = \int_a^b \Big(\int_c^d f(x, y) \, dy \Big) \, dx.$$

Remarks on Fubini's theorem

- The both iterated integrals may exist but the function f may not be double integrable.
  Example 1: $R := [0,1] \times [0,1]$,
  $$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$
  Compute both the iterated integrals Are they same? Is f integrable?

  The function f may be double integrable. But one of the iterated integrals may not exist. (Check Tutorial problems).

  Let R be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be a continuous function. Then both iterated integrals of f exist and are equal to the double integral of f over R.

Examples:

Example : Find the integral of $f(x,y) = x^2 + y^2$ on the rectangle $[0,1] \times [0,1]$ if it exists.

Solution: Check the integrability of f using the definition. Let us now compute the integral using iterated integrals.

$$
\begin{aligned}
\int\int_{[0,1]\times[0,1]} x^2 + y^2 \ dxdy &= \int_0^1 \int_0^1 x^2 + y^2 \ dxdy \\
&= \int_0^1 [\frac{x^3}{3} + xy^2]_0^1 \\
&= \int_0^1 (\frac{1}{3} + y^2) \\
&= [\frac{y}{3} + \frac{y^3}{3}]_0^1 = \frac{2}{3}
\end{aligned}
$$

Example (Marsden, Tromba and Weinstein page 288): Compute $\int\int_R \sin(x+y)dxdy$, where $R = [0,\pi] \times [0,2\pi]$.

Solution:

$$\begin{aligned}
\int\int_R \sin(x+y)dxdy &= \int_0^{2\pi} \left[\int_0^{\pi} \sin(x+y)dx\right] dy \\
&= \int_0^{2\pi} [-\cos(x+y)|_{x=0}^{\pi}]dy \\
&= \int_0^{2\pi} [\cos y - \cos(y+\pi)]dy \\
&= [\sin y - \sin(y+\pi)]|_{y=0}^{2\pi} = 0
\end{aligned}$$

Example (Marsden, Tromba and Weinstein, page 289): If D is a plate defined by $1 \leq x \leq 2, 0 \leq y \leq 1$ (measured in centimeters), and the mass density $\rho(x, y) = ye^{xy}$ grams per square centimeter. Find the mass of the plate.

Solution: The total mass of the plate is got by integrating over the rectangular region covered by D:

$$\begin{aligned}\int\int_D \rho(x,y)dxdy &= \int_0^1\int_0^2 ye^{xy}dxdy = \int_0^1 (e^{xy}|_{x=1}^2 dy \\ &= \int_0^1 (e^{2y} - e^y)dy = \tfrac{e^2}{2} - e + \tfrac{1}{2}\end{aligned}$$

Special case Let $\phi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$ be Riemann integrable. Define $f(x, y) := \phi(x)\psi(y)$, for all $(x, y) \in R = [a, b] \times [c, d]$. Then f is integrable on R and

$$\int\int_R f(x,y)\,dx\,dy = \Big(\int_a^b \phi(x)\,dx\Big)\Big(\int_c^d \psi(y)\,dy\Big).$$

Example Let $0 < a < b$ and $0 < c < d$ and $r \geq 0$ and $s \geq 0$. Denote $R = [a, b] \times [c, d]$. Compute $\int\int_R x^r y^s \,dx\,dy$.

Existence of integrals on $R = [a, b] \times [c, d]$-I

All our statements so far depend on f being integrable on R. Is there any characterization to determine if f is integrable?

Let $f : R \to \mathbb{R}$ be a bounded function. The function 'f is monotonic in each of two variables' means that for each fixed x, $f(x, y)$ is a monotonic function in y variable and similarly, for each fixed y, $f(x, y)$ is a monotonic function in x variable.

Theorem

If f is bounded and monotonic in each of two variables, then f is integrable on R.

Again the proof follows by using Riemann condition.

Example: Let $f(x, y) := [x + y]$, for all $(x, y) \in R$, where $[u]$ means the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Since f is monotonic in each of two variables, f is integrable on R.

However, the previous condition is not that common and seems rather special.

Surely what worked in one variable should work here. In fact, a proof similar to the case of one variable will show the following theorem.

Existence of integrals on $R = [a, b] \times [c, d]$-II

Theorem

If a function $f : R \to \mathbb{R}$ is bounded and continuous on R except possibly finitely many points in R, then f is integrable on R.

Example. Let $R := [-1, 1] \times [-1, 1]$,

$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)}, & (x, y) \in R, \quad (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

What are points of discontinuity for f on R?

In the one variable case, we saw that a bounded function with at most a finite number of discontinuities on a closed bounded interval is Riemann integrable.

The reason that a finite number of discontinuities do not matter is that points have length zero. What might be the analogous result in two variables?

In other words what sets have "zero area"?

A bounded subset E of $\mathbb{R}^2$ has 'zero area' if for every $\epsilon > 0$, there are finitely many rectangles whose union contains E and the sum of whose areas is less than ϵ.

It turns out graph of a continuous function, that is, set of the form $\{(x, \phi(x)) \mid x \in [a, b]\}$ for a continuous function $\phi : [a, b] \to [c, d]$ has 'zero area' or has *content zero.*

Theorem

If a function f is bounded and continuous on a rectangle $R = [a, b] \times [c, d]$ except possibly along a finite number of graphs of continuous functions, then f is integrable on R.

Example: Let $R = [0, 1] \times [0, 1]$ and

$$f(x, y) = \begin{cases} 1, & 0 \leq x < y, \quad y \in [0, 1], \\ 0, & y \leq x \leq 1, \quad y \in [0, 1]. \end{cases}$$

Is f integrable over R?

The slightly more general theorem says that given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the function is integrable over R if the points of discontinuity of f is a set of 'content zero'.

However the converse of the above statement is not true. There are integrable functions whose points of discontinuity is not a set of 'content zero'. (Check Tutorial)

Counter example: Bivariate Thomae function: $f : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1, & \text{if} \quad x = 0, \quad y \in \mathbb{Q} \cap [0,1], \\ \frac{1}{q}, & x, y \in \mathbb{Q} \cap [0,1] \quad \text{and} \quad x = \frac{p}{q}, \\ & \qquad\qquad p, q \in \mathbb{N} \quad \text{are relatively prime}, \\ 0, & \text{otherwise.} \end{cases}$$