MA-111 Calculus II (D3 & D4)

Lecture 6

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The mean value theorem for double integrals

Triple integral

Example Continued

Example: Evaluate $\int\int_D (3x+4y^2)\,dxdy$, where D is the region in the upper half-plane bounded by the circled $x^2+y^2=1$ and $x^2+y^2=4$.
Ans: The region

$$D = \{(x,y) \mid y \geq 0, \quad 1 \leq x^2+y^2 \leq 4\}.$$

In polar coordinate, after using change of variables $x = r\cos\theta$ and $y = r\sin\theta$, in $r-\theta$ plane, D becomes

$$D^* = \{(r,\theta) \mid 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi\}.$$

$$\begin{aligned}\int\int_D (3x+4y^2)\,dxdy &= \int_{\theta=0}^{\pi}\int_{r=1}^{2} (3r\cos\theta + 4r^2\sin^2\theta)r\,drd\theta \\ &= \int_0^{\pi} [r^3\cos\theta + r^4\sin^2\theta]_{r=1}^{2}\,d\theta = \int_0^{\pi} [7\cos\theta + 15\sin^2\theta]\,d\theta = \frac{15\pi}{2}.\end{aligned}$$

The mean value theorem for double integrals

Theorem

If D is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exists (x_0, y_0) in D such that

$$f(x_0, y_0) = \frac{1}{A(D)} \int\int_D f(x, y) dA.$$

The proof follows using the boundedness of $f(x, y)$ and mean value theorem for continuous functions .

Sketch of Proof Since D is closed and bounded and f is continuous, the function attains its maximum and minimum at some points $(x_0, y_0) \in D$ and $(x_1, y_1) \in D$ respectively. Since D is an elementary region, there exists a path $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(0) = (x_0, y_0) \in D$ and $\gamma(1) = (x_1, y_1)$.

Now apply the intermediate value theorem function $f \circ \gamma : [0, 1] \to \mathbb{R}$.

Average value contd.

How does one interpret the above statement geometrically?
If $f(x, y) \geq 0$, $f(x_0, y_0)$, the solid region under the graph of f and over the region D is same as the volume of the region over D whose height is the average value or mean value of f defined above.i.e.,

$$f(x_0, y_0) \times A(D) = \int \int_D f(x, y) dxdy.$$

Application: Center of Mass of a thin plate: (Weighted average): Let a plate occupies a region D of the $x - y$ plane and $\rho(x, y)$ be its density at a point (x, y) in D. Let ρ be a positive continuous function on D. The the coordinate of the center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\int \int_D x\rho(x, y)\, dxdy}{\int \int_D \rho(x, y)\, dxdy}, \quad \bar{y} = \frac{\int \int_D y\rho(x, y)\, dxdy}{\int \int_D \rho(x, y)\, dxdy}.$$

Note that for $\rho \equiv 1$, $\bar{x}$ is the average of $f(x, y) = x$ over the region D and $\bar{y}$ is the average of $g(x, y) = y$ over the region D.

Generalizing integration for $n \geq 3$

Recall our definition of Darboux integrals and Reimann integral. Both these definitions have an analogue in dimensions $n \geq 3$.

In this course, we only extend these ideas to functions on 3 variables. Note we already cannot imagine the graph of a function of 3 variables and much of the geometry is lost.

As an exercise you can think about which of the following definitions are specific to $n = 3$ and which can be generalized further.

If we have a bounded function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$, we can integrate it over this rectangular cuboid (which we often refer to as a cuboid.) We divide the rectangular cuboid into smaller ones B_{ijk}, making sure that the length, breadth and height of the subcuboids are all small.

Integrals over rectangular cuboids

In particular, we can use the regular partition of order n to obtain the Riemann sum

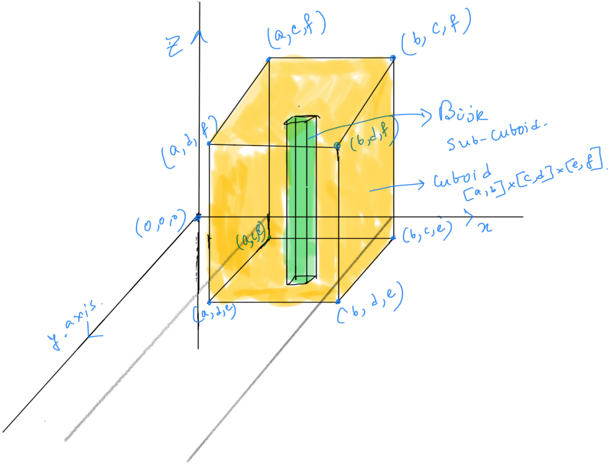
$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \Delta B_{ijk},$$

where ΔB_{ijk} is the volume of B_{ijk}, and $t = \{t_{ijk} \in B_{ijk}\}$ is an arbitrary tag.

As before we say that f is integrable if $\lim_{n\to\infty} S(f, P_n, t)$ converges to some fixed $S \in \mathbb{R}$ for any choice of tag t. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

All the theorems for integrals over rectangles go through for integrals over rectangular cuboids.



Z
(a,c,f)
(b,c,f)
(a,d,f)
(b,d,f)
Brick
sub-cuboid.
cuboid
[a,b]×[c,d]×[e,f].
(0,0,0)
(a,c,e)
(b,c,e)
x
(a,d,e)
(b,d,e)
y. axis.

Integrating over bounded regions B in $\mathbb{R}^3$

First, if $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous in B, except possibly on (a finite union of) graphs of continuous functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable.

This allows us to define the integral of (say) a continuous function on any bounded region B whose boundary is a set of content zero in $\mathbb{R}^3$. Let B^* be a cuboid enclosing the bounded region and $f^* : B^* \to \mathbb{R}$ be defined as f on B and 0 elsewhere.

Then integral of f over B exists if integral of f^* over B^* exists and

$$\iiint_{B^*} f^* = \iiint_B f.$$

Once we have defined the triple integral in this way, it remains to evaluate it.

Evaluating triple integrals: Fubini's Theorem

Fubini's Theorem can be generalized - that is, triple integrals can usually be expressed as iterated integrals, this time by integrating functions of a single variable three times.

Let f be integrable on the cuboid B. Then any iterated integral that exists is equal to the triple integral; i.e.,

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx,$$

provided the right hand side iterated integral exists.
There are, in fact, five other possibilities for the iterated integrals.

We have a theorem saying if f is integrable, whenever any of these iterated integral exists, it is equal to the value of the integral of f over B.
If f is continuous on B, then f is integrable on B and all iterated integrals exist and their values are equal to the integral of f on B.

Elementary regions in $\mathbb{R}^3$

The triple integrals that are easiest to evaluate are those for which the region W in space can be described by bounding z between the graphs of two functions in x and y with the domain of these functions being an elementary region in two variables.

For example,

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid \gamma_1(x, y) \leq z \leq \gamma_2(x, y), (x, y) \in D\},$$

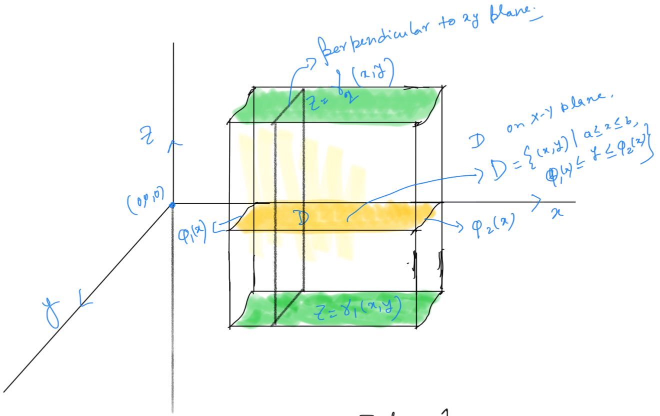
where γ_1 and γ_2 are continuous on $D \subset \mathbb{R}^2$ and D is an elementary region in $\mathbb{R}^2$. For rxample, if D is Type 1, then

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

where $\phi_1 : [a, b] \to \mathbb{R}$ and $\phi_2 : [a, b] \to \mathbb{R}$ are continuous functions. The region D can be Type 2 also.

Example:

- The region W between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$.
- The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$ and $x = z - y - 1$.



When D is Type 1.

Elementary regions (Example)

Suppose that the region W lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of W on the xy plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then for a continuous function f defined over W, we have

$$\iiint_W f(x, y, z) dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z) dzdydx.$$

Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region W, where W is the unit sphere, i.e.,

$$\int\int\int_W 1 dxdydz = ?, \quad \text{where} \quad W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

The volume of the unit sphere

The sphere can be described as the region lying between $z = -\sqrt{1 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

The projection of the sphere onto the xy plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1 - x^2}$ and $\sqrt{1 - x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2 \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2} dy \right] dx.$$

After evaluating the inner integral we obtain

$$2\pi \int_{-1}^{1} \frac{1 - x^2}{2} dx = \frac{4}{3}\pi.$$