MA 109 Week 5

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Consequences of the FTC

Theorem (Integration by parts)

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable. Suppose $g : [a, b] \to \mathbb{R}$ is integrable and has an antiderivative G on $[a, b]$. Then

$$\begin{aligned}\int_a^b f(x)g(x)dx &= f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)dx \\ &= f(x)G(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)G(x)\, dx.\end{aligned}$$

Proof: For $x \in [a, b]$, define $H(x) := f(x)G(x)$, so that $H'(x) = f(x)G'(x) + f'(x)G(x) = f(x)g(x) + f'(x)G(x)$. Then H' is integrable on $[a, b]$. Hence by the FTC (Part II),

$$\int_a^b H'(x)dx = H(b) - H(a) = f(b)G(b) - f(a)G(a). \quad \square$$

Theorem (Integration by substitution)

Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a continuously differentiable function such that $\phi([\alpha, \beta]) = [a, b]$. Then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$ is integrable, and

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\left(\phi(t)\right)\phi'(t)dt.$$

In particular, if $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$, then

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))|\phi'(t)|dt.$$

Proof: Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(u)du$ for $x \in [a, b]$, and $H : [\alpha, \beta] \to \mathbb{R}$ by $H(t) := F(\phi(t))$ for $t \in [\alpha, \beta]$.

By the chain rule and by the FTC (Part I),

$$H'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t) \quad \text{for } t \in [\alpha, \beta].$$

Then H' is integrable on $[\alpha, \beta]$. Hence by the FTC (Part II),

$$\int_\alpha^\beta H'(t)dt = H(\beta) - H(\alpha) = F(\phi(\beta)) - F(\phi(\alpha)) = \int_{\phi(\alpha)}^{\phi(\beta)} f(u)du,$$

that is,

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_\alpha^\beta f\big(\phi(t)\big)\phi'(t)dt.$$

Now suppose $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$. By the Intermediate Value Property of ϕ', either $\phi'(t) > 0$ for all $t \in (\alpha, \beta)$, or $\phi'(t) < 0$ for all $t \in (\alpha, \beta)$. In the former case, ϕ is strictly increasing on $[\alpha, \beta]$, $\phi(\alpha) = a$, $\phi(\beta) = b$ and $|\phi'| = \phi'$. In the latter case, ϕ is strictly decreasing on $[\alpha, \beta]$, $\phi(\alpha) = b$, $\phi(\beta) = a$ and $|\phi'| = -\phi'$. Hence the desired result follows. $\square$

Examples:

(i) To evaluate $\int_0^1 x\sqrt{1-x}\,dx$, let $f(x) := x$ and $g(x) := \sqrt{1-x}$ for $x \in [0,1]$. Then $f'(x) = 1$ for $x \in [0,1]$. Also, if $G(x) := -(2/3)(1-x)^{3/2}$, then $G'(x) = g(x)$ for $x \in [0,1]$, that is, $G' = g$. Integration by Parts yields

$$\int_0^1 x\sqrt{1-x}\,dx = 0 - 0 - \int_0^1 \left(-\frac{2}{3}\right)(1-x)^{3/2}dx = \frac{2}{3}\int_0^1 (1-x)^{3/2}dx.$$

If we let $F(x) := -(2/5)(1-x)^{5/2}$ for $x \in [0,1]$, then $F'(x) = (1-x)^{3/2}$ for $x \in [0,1]$. By the FTC, Part II, the integral equals $(2/3)(F(1) - F(0)) = (2/3)(2/5) = 4/15$.

(ii) To evaluate $\int_0^1 t\sqrt{1-t^2}\,dt$, let $\phi(t) := 1 - t^2$ for $t \in [0,1]$ and $f(x) := \sqrt{x}$ for $x \in [0,1]$. Then $\phi(0) = 1$, $\phi(1) = 0$, and $\phi'(t) = -2t$ for $t \in [0,1]$. Let $F(x) := (2/3)x^{3/2}, x \in [0,1]$. Then $F'(x) = \sqrt{x}$, $x \in [0,1]$. Integration by Substitution yields

$$\int_0^1 t\sqrt{1-t^2}\,dt = \frac{1}{2}\int_0^1 f(\phi(t))|\phi'(t)|dt = \frac{1}{2}\int_0^1 \sqrt{x}\,dx = \frac{1}{3}.$$

Using Riemann integration, we can define 'new' functions. The FTC (Part I) allows us to investigate their properties.

If $x \geq 1$, then the function $t \longmapsto 1/t$ is continuous on $[1, x]$, and if $0 < x < 1$, it is continuous on $[x, 1]$.

Definition

The **(natural) logarithmic function** defined by

$$\ln x := \int_1^x \frac{1}{t} dt \quad \text{for } x \in (0, \infty).$$

Properties of the (natural) logarithmic function $\ln : (0, \infty) \to \mathbb{R}$

(i) $\ln 1 = 0$, $x \in (0, 1) \Longrightarrow \ln x < 0$, $x \in (1, \infty) \Longrightarrow \ln x > 0$.

(ii) $\ln$ is differentiable and $(\ln)'(x) = 1/x$ for $x \in (0, \infty)$.

(iii) $\ln$ is strictly increasing and strictly concave on $(0, \infty)$ as $(\ln)'(x) > 0$ and $(\ln)''(x) = -1/x^2 < 0$ for $x \in (0, \infty)$.

(iv) Let $x_1, x_2 \in (0, \infty)$. By substituting $t = x_1 s$, we obtain

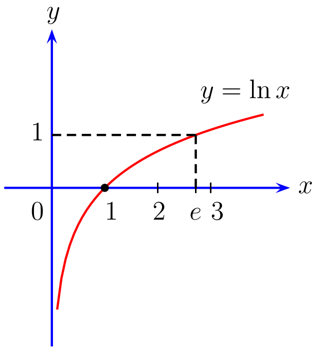
$$\ln x_1 x_2 = \int_1^{x_1 x_2} \frac{dt}{t} = \int_1^{x_1} \frac{dt}{t} + \int_{x_1}^{x_1 x_2} \frac{dt}{t} = \ln x_1 + \ln x_2.$$

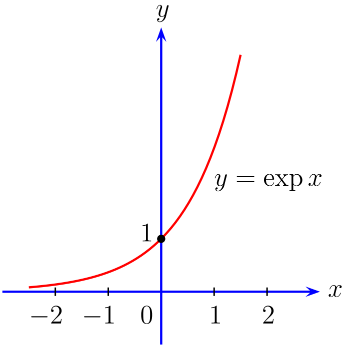
(v) $\ln x \to \infty$ as $x \to \infty$, and $\ln x \to -\infty$ as $x \to 0^+$: $\ln 2^n = n \ln 2$ for $n \in \mathbb{N}$ and $\ln(1/x) = -\ln x$.

(vi) $\ln : (0, \infty) \to \mathbb{R}$ is one-one and onto: $0 < x_1 < x_2 \implies \ln x_1 < \ln x_2$. Also, for $y \in \mathbb{R}$, there are $x_1, x_2 \in (0, \infty)$ such that $\ln x_1 < y < \ln x_2$. Using the IVP of the function $\ln$, there is $x \in (x_1, x_2)$ with $y = \ln x$.

The **exponential function** $\exp : \mathbb{R} \to (0, \infty)$ is the inverse of the logarithmic function $\ln : (0, \infty) \to \mathbb{R}$. Thus

$$\exp x = y \iff \ln y = x \quad \text{for } x \in \mathbb{R}.$$





Properties of exponential function:

(i) $\exp 0 = 1$, $x \in \mathbb{R} \implies \exp x > 0$.

(ii) exp is differentiable and $(\exp)'(x) = \exp x$ for $x \in \mathbb{R}$:
If $x = \ln y$, then $(\exp)'(x) = 1/(\ln)'(y) = y = \exp x$.

(iii) exp is strictly increasing and strictly convex on $\mathbb{R}$ as $\exp' x > 0$ and $\exp'' x > 0$ for $x \in \mathbb{R}$.

(iv) $\exp(x_1 + x_2) = (\exp x_1)(\exp x_2)$ for all $x_1, x_2 \in \mathbb{R}$:
If $\exp x_1 = y_1$ and $\exp x_2 = y_2$, then $\ln(y_1 y_2) = \ln y_1 + \ln y_2 = x_1 + x_2$.

(v) $\exp : \mathbb{R} \to (0, \infty)$ is one-one and onto.

(vi) $\exp x \to \infty$ as $x \to \infty$, and $\exp x \to 0$ as $x \to -\infty$:

Let e denote the unique number in $(0, \infty)$ such that $\ln e = 1$. Then
$2 < e < 4$: $\ln 2 = \int_1^2 (1/t)dt < \int_1^2 1\, dt = 1 = \ln e$, and
$\ln 4 = \int_1^4 (1/t)dt = \int_1^2 (1/t)dt + \int_2^4 (1/t)dt > \frac{1}{2} + \frac{2}{4} = 1 = \ln e$.

Real powers of a positive real number

Definition

Let $a > 0$. For $x \in \mathbb{R}$, define

$$a^x := \exp(x \ln a).$$

Thus $a^x = y \iff \ln y = x \ln a$.

In particular, let $a := e$. Then

$$e^x = \exp(x \ln e) = \exp x \text{ for all } x \in \mathbb{R}.$$

Thus $a^x = e^{x \ln a}$ for $a > 0$ and $x \in \mathbb{R}$.

Let $a > 0$, and $f(x) := a^x$ for $x \in \mathbb{R}$. It follows that f is differentiable on $\mathbb{R}$ and $f'(x) = (\ln a) f(x)$ for $x \in \mathbb{R}$.

Ler $r \in \mathbb{R}$, and $g(x) := x^r = e^{r \ln x}$ for $x \in (0, \infty)$. Then g is differentiable on $(0, \infty)$ and $g'(x) = e^{r \ln x} r/x = r\, x^{r-1}$ for $x \in (0, \infty)$.

In particular, let $k \in \mathbb{N}$ and $r := 1/k$. Then for $x \in (0, \infty)$, $x^{1/k} > 0$ and $(x^{1/k})^k = (e^{(1/k) \ln x})^k = e^{\ln x} = x$, so that $x^{1/k}$ is the unique positive k-th root of x.

Let $k \in \mathbb{N}$ and define $h : [0, \infty) \to \mathbb{R}$ by $h(0) := 0$ and $h(x) := x^{1/k}$ for $x \in (0, \infty)$. Since $\ln x \to -\infty$ as $x \to 0^+$, and $e^y \to 0$ as $y \to -\infty$, we see that $h(x) \to 0$ as $x \to 0^+$. It follows that h is continuous on $[0, \infty)$.

Inverse Trigonometric and Trigonometric Functions

Noting that the function $t \in \mathbb{R} \longmapsto 1/(t^2+1) \in \mathbb{R}$ is continuous on $\mathbb{R}$, we proceed as follows.

Definition

The **arctangent function** is defined by

$$\arctan x := \int_0^x \frac{1}{1+t^2} dt \quad \text{for } x \in \mathbb{R}.$$

We can then find properties of the function arctan, and of its inverse function tan in a manner similar to the way we found properties of the functions ln and its inverse function exp.

The theory of inverse trigonometric and trigonometric functions can be developed on these lines. This also allows us to define the polar coordinates (r, θ) of a point $(x, y) \neq (0, 0)$.

Area between Curves

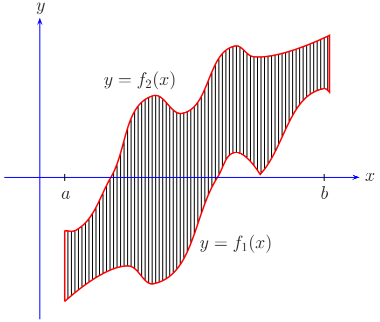
Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions such that $f_1 \le f_2$. Let $R := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } f_1(x) \le y \le f_2(x)\}$ be the **region between the curves** $y = f_1(x)$ and $y = f_2(x)$. Define

$$\text{Area}\,(R) := \text{Area}\,(R_{f_2 - f_1}) = \int_a^b \big(f_2(x) - f_1(x)\big)\,dx.$$

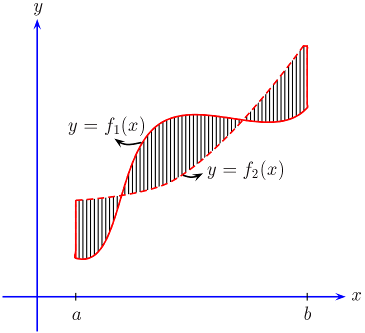
Let $g_1, g_2 : [c, d] \to \mathbb{R}$ be integrable functions such that $g_1 \le g_2$. Let $R := \{(x, y) \in \mathbb{R}^2 : c \le y \le d \text{ and } g_1(y) \le x \le g_2(y)\}$ be the **region between the curves** $x = g_1(y)$ and $x = g_2(y)$. Define

$$\text{Area}\,(R) := \int_c^d \big(g_2(y) - g_1(y)\big)\,dy.$$

If two curves cross each other a finite number of times, then we must find areas of several regions between them separately, and add them up.



y
x
a
b
y = f2(x)
y = f1(x)



y
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y = f1(x)
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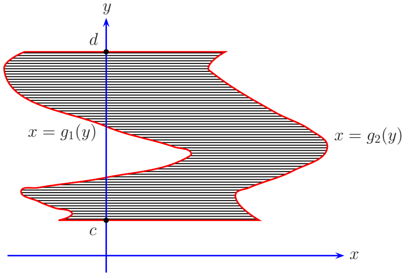

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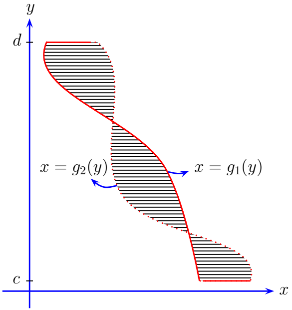
x = g1(y)
x = g2(y)



y
x
d
c
x = g2(y)
x = g1(y)

Examples

(i) Let R denote the region enclosed by the loop of the curve $y^2 = x(1-x)^2$, that is, the region bounded by the curves $y = -\sqrt{x}(1-x)$ and $y = \sqrt{x}(1-x)$.
Now $\sqrt{x}(1-x) = -\sqrt{x}(1-x) \iff x = 0$ or 1, and $\sqrt{x}(1-x) \geq -\sqrt{x}(1-x)$ for $x \in [0,1]$. Hence

$$\text{Area}\,(R) = \int_0^1 \left(\sqrt{x}(1-x) - (-\sqrt{x}(1-x))\right) dx = \frac{8}{15}.$$

(ii) Let R denote the region bounded by the curves $x = -2y^2$ and $x = 1 - 3y^2$.
Now $-2y^2 = 1 - 3y^2 \iff y = \pm 1$, and $-2y^2 \leq 1 - 3y^2$ if $y \in [-1,1]$.
Hence

$$\text{Area}\,(R) = \int_{-1}^1 \left(1 - 3y^2 - (-2y^2)\right) dy = \int_{-1}^1 (1 - y^2)\; dy = \frac{4}{3}.$$

Polar coordinates

Review:

The function $\cos^{-1} : [-1, 1] \to [0, \pi]$ is one-one and onto.

Let $P := (x, y) \neq (0, 0)$. There are unique r, $\theta \in \mathbb{R}$ such that

$$r > 0, \ \theta \in (-\pi, \pi], \ x = r\cos\theta \text{ and } y = r\sin\theta.$$

In fact, $r := \sqrt{x^2 + y^2}$ and

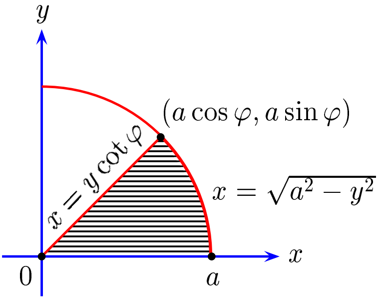
$$\theta := \begin{cases} \cos^{-1}(x/r) & \text{if } y \geq 0, \\ -\cos^{-1}(x/r) & \text{if } y < 0. \end{cases}$$

(If $y < 0$, then $|x/r| < 1$, and $-\cos^{-1}(x/r) \in (-\pi, 0)$.)

The pair (r, θ) is defined as the **polar coordinates** of P.

Area of a sector of a disk

Let $0 \leq \varphi \leq \pi/2$, and let R denote the sector of a disc of radius a, marked by the points $(0,0),(a,0)$ and $(a\cos\varphi, a\sin\varphi)$, that is, the region bounded by the curves $x = (\cot\varphi)y$ and $x = \sqrt{a^2 - y^2}$ for $y \in [0, a\sin\varphi]$, and by the x-axis.



$$\text{Then Area}\,(R) \;=\; \int_0^{a\sin\varphi} \left(\sqrt{a^2 - y^2} - (\cot\varphi)y\right) dy = \frac{a^2\varphi}{2}.$$

By symmetry, this result holds for $\varphi \in (\pi/2, \pi]$ as well.

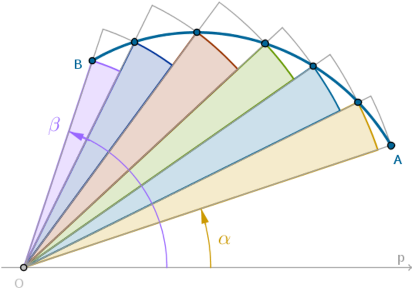
Let R denote the region bounded by the curve $r = p(\theta)$ and the rays $\theta = \alpha$, $\theta = \beta$, where $-\pi \leq \alpha < \beta \leq \pi$. Thus

$$R := \{(r\cos\theta, r\sin\theta) : \alpha \leq \theta \leq \beta \text{ and } 0 \leq r \leq p(\theta)\}.$$

Suppose $p : [\alpha, \beta] \to \mathbb{R}$ is integrable. (If $\alpha = -\pi$ and $\beta = \pi$, then we suppose $p(-\pi) = p(\pi)$.)

- Partition $[\alpha, \beta]$ into $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$.
- Pick sample points $\gamma_i \in [\theta_{i-1}, \theta_i]$ for $i = 1, \ldots, n$.
- Area between the rays $\theta = \theta_{i-1}$ and $\theta = \theta_i$ is approximated by the area of a sector of a disc of radius $r_i := p(\gamma_i)$, that is, by $\dfrac{p(\gamma_i)^2\,(\theta_i - \theta_{i-1})}{2}$.
- The sum of areas of these sectors is a Riemann sum, namely $\displaystyle\sum_{i=1}^{n} \frac{p(\gamma_i)^2\,(\theta_i - \theta_{i-1})}{2}$.

Area inside a polar curve



We define

$$\text{Area}\,(R) := \frac{1}{2}\int_{\alpha}^{\beta} p(\theta)^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$$

Examples: (i) Let $a > 0$. Area of the disc enclosed by the circle $r = a$ is equal to $1/2 \int_{-\pi}^{\pi} a^2 d\theta = \pi a^2$.

(ii) Let $a > 0$, and let R denote the region enclosed by the cardioid $r = a(1 + \cos\theta)$. Then

$$\text{Area}\,(R) = \frac{1}{2}\int_{-\pi}^{\pi} a^2(1+\cos\theta)^2 d\theta = \frac{3a^2\pi}{2}.$$

(iii) Let R denote the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$, where $\theta \in [0, \pi]$. Now $3\sin\theta = 1 + \sin\theta \iff \theta \in \{\pi/6, 5\pi/6\}$, and $1 + \sin\theta \leq 3\sin\theta$ if $\theta \in [\pi/6, 5\pi/6]$. Hence

$$\text{Area}\,(R) = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((3\sin\theta)^2 - (1+\sin\theta)^2\right) d\theta = \pi.$$

Volume of a solid

Let D be a bounded subset of $\mathbb{R}^3$. A cross-section of D obtained by cutting D by a plane in $\mathbb{R}^3$ is called a **slice** of D.

Let $a < b$, and suppose D lies between the planes $x = a$ and $x = b$, which are perpendicular to the x-axis. For $s \in [a, b]$, consider the slice of D by the plane $x = s$, namely $\{(x, y, z) \in D : x = s\}$, and suppose it has an 'area' $A(s)$.

To find the volume of D, we proceed as follows.

- Partition $[a, b]$ into $a = x_0 < x_1 < \cdots < x_n = b$.
- Pick sample points $s_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$.
- Volume between the planes $x = x_{i-1}$ and $x = x_i$ is approximated by the volume of a rectangular slab of width $x_i - x_{i-1}$ and base area $A(s_i)$, that is, by $A(s_i)(x_i - x_{i-1})$.
- The sum of volumes of these slabs is $\displaystyle\sum_{i=1}^{n} A(s_i)(x_i - x_{i-1})$.

Slice Method: We define the **volume** of D by

$$\mathrm{Vol}\,(D) := \int_a^b A(x)\,dx,$$

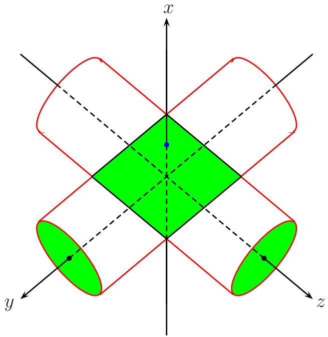
provided the 'area function' $A : [a, b] \to \mathbb{R}$ is integrable.

Examples
(i) If D is a cylinder with cross-sectional area A and height h, then $\mathrm{Vol}\,(D) = Ah$. (The 'area function' is the constant A.)

(ii) Let $a > 0$, and let D denote the solid enclosed by the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. Then D lies between the planes $x = -a$ and $x = a$. For $s \in [-a, a]$, the slice $\{(x, y, z) \in D : x = s\}$ is the square

$$\left\{(s, y, z) \in \mathbb{R}^3 : |y| \le \sqrt{a^2 - s^2} \text{ and } |z| \le \sqrt{a^2 - s^2}\right\},$$

and so $A(s) = \left(2\sqrt{a^2 - s^2}\right)^2 = 4\left(a^2 - s^2\right)$. Hence
$\mathrm{Vol}\,(D) = \int_{-a}^{a} A(s)ds = 4\int_{-a}^{a} \left(a^2 - s^2\right) ds = \dfrac{16a^3}{3}.$



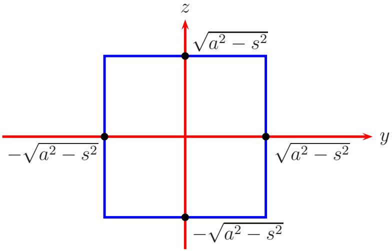


Figure: Solid enclosed by two cylinders and a slice resulting in a square region

Solids of Revolution

If a subset D of $\mathbb{R}^3$ is generated by revolving a planar region about an axis, then D is known as a **solid of revolution**.

Examples

- Let $a > 0$. The spherical ball

  $$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2\}$$

  is obtained by revolving the semidisc

  $$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2 \text{ and } y \geq 0\}$$

  about the x-axis.
- Let $b > 0$. The cylindrical solid

  $$\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq b^2 \text{ and } 0 \leq y \leq h\}$$

  is obtained by revolving the rectangle $[0, b] \times [0, h]$ about the y-axis.

Volume of a Solid of Revolution: Washer Method

- Let D be the solid obtained by revolving the region between the curves $y = f_1(x)$ and $y = f_2(x)$, and the lines $x = a$ and $x = b$, about the x-axis, where $0 \leq f_1 \leq f_2$.
- The slice of D at $x \in [a, b]$ looks like a circular washer, that is, a disc of radius $f_2(x)$ from which a smaller disc of radius $f_1(x)$ has been removed, and so the area of the slice is $A(x) := \pi \left(f_2(x)^2 - f_1(x)^2 \right)$.
- Suppose f_1 and f_2 are integrable on $[a, b]$. Then the area function A is integrable on $[a, b]$, and by the slice method,

$$\mathrm{Vol}\,(D) = \int_a^b A(x)dx = \pi \int_a^b \left(f_2(x)^2 - f_1(x)^2 \right) dx.$$

This special case of the slice method is called the washer method.

If the inner radius of a washer is equal to 0, then the washer is in fact a disk. If this is the case for every $x \in [a, b]$, then the washer method is also known as the disk method.

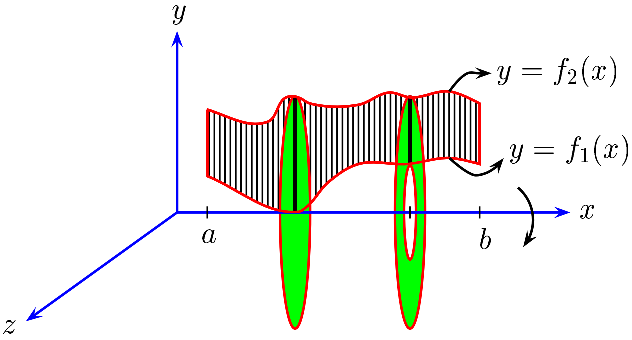


Figure: Illustrations of a disk and of a washer

Examples

(i) Let D denote the solid obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the x-axis.

- Let $f_1(x) := x^2$ and $f_2(x) := x$ for $x \in [0, 1]$. The curves $y = f_1(x)$ and $y = f_2(x)$ intersect at $x = 0$ and $x = 1$, and $0 \leq f_1 \leq f_2$ on $[0, 1]$.
- By the washer method,

$$\mathrm{Vol}\,(D) = \pi \int_0^1 \left((x)^2 - (x^2)^2\right) dx = \pi \int_0^1 (x^2 - x^4) dx = \frac{2\pi}{15}.$$

(ii) Let D denote the spherical ball with centre at $(0, 0, 0)$, and radius $a > 0$. Then D is obtained by revolving the semidisc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2 \text{ and } y \geq 0\}$ about the x-axis.

- Let $f_1(x) := 0$ and $f_2(x) := \sqrt{a^2 - x^2}$ for $x \in [-a, a]$. The curves $y = f_1(x)$ and $y = f_2(x)$ intersect at $x = -a$ and $x = a$, and $0 \leq f_1 \leq f_2$ on $[-a, a]$.
- By the disc method,

$$\mathrm{Vol}\,(D) = \pi \int_{-a}^{a} \left(\left(\sqrt{a^2 - x^2}\right)^2 - 0^2\right) dx = \pi \int_{-a}^{a} (a^2 - x^2) dx = \frac{4\pi a^3}{3}.$$

Let D be the solid obtained by revolving the region between the curves $x = g_1(y)$ and $x = g_2(y)$, $c \leq y \leq d$, about the y-axis, where $0 \leq g_1 \leq g_2$. Then, as before,

$$\text{Vol}\,(D) = \pi \int_c^d \left(g_2(y)^2 - g_1(y)^2\right) dy.$$

Example

Let D denote the solid obtained by revolving the region in the first quadrant between the parabolas $y = x^2$ and $y = 2 - x^2$ about the y-axis. Now $\sqrt{y} = \sqrt{2-y} \iff y = 1$. By the disk method,

$$\text{Vol}\,(D) = \pi \int_0^1 (\sqrt{y})^2 dy \,+\, \pi \int_1^2 \left(\sqrt{2-y}\right)^2 dy = \pi \left(\frac{1}{2} + \frac{1}{2}\right) = \pi.$$

Volume of a Solid of Revolution: Shell Method

Let D be a bounded subset of $\mathbb{R}^3$. A cross-section of D obtained by piercing a cylinder through D is called a **sliver** of D.

- Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be such that $f_1 \leq f_2$, and suppose $0 \leq a < b$. Let D denote the solid generated by revolving the region between the curves $y = f_1(x)$ and $y = f_2(x)$, and the lines $x = a$ and $x = b$ about the y-axis.
- For $s \in [a, b]$, the sliver $\{(x, y, z) \in D : x^2 + z^2 = s^2\}$ of D by the cylinder $x^2 + z^2 = s^2$ is a right circular cylinder having height $f_2(s) - f_1(s)$ and radius s, and so its surface area is $2\pi s\big(f_2(s) - f_1(s)\big)$. (To be justified later)
- Suppose f_1 and f_2 are integrable on $[a, b]$. Since D is 'made up' of these cylindrical slivers (or shells), we define

$$\mathrm{Vol}\,(D) := 2\pi \int_a^b x\big(f_2(x) - f_1(x)\big)\,dx,$$

given by the shell method.

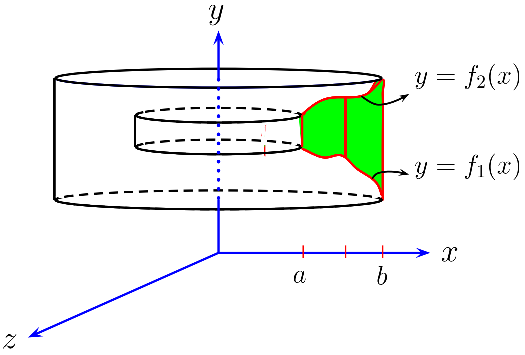


Figure: Illustration of the Shell Method

$$\mathrm{Vol}\,(D) := 2\pi \int_a^b x\big(f_2(x) - f_1(x)\big)\,dx.$$

Shell Method (continued)

Let $g_1, g_2 : [c, d] \to \mathbb{R}$ be integrable functions such that $g_1 \leq g_2$, and suppose $0 \leq c < d$. Let D denote the solid obtained by revolving the region between the curves $x = g_1(y)$ and $x = g_2(y)$ and between the lines $y = c$ and $y = d$ about the x-axis.

For $t \in [c, d]$, the sliver $\{(x, y, z) \in D : y^2 + z^2 = t^2\}$ of D by the cylinder $y^2 + z^2 = t^2$ is a right circular cylinder having height $g_2(t) - g_1(t)$ and radius t, and so its surface area is $2\pi t\big(g_2(t) - g_1(t)\big)$.

Since D is 'made up' of these cylindrical slivers (or shells), we define

$$\mathrm{Vol}\,(D) := 2\pi \int_c^d y\big(g_2(y) - g_1(y)\big)\,dy.$$

Examples

(i) Let D denote the solid obtained by revolving the region bounded by the curves $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. Now $2x^2 - x^3 = 0 \iff x = 0$ or $x = 2$. By the shell method,

$$\text{Vol}\,(D) = 2\pi \int_0^2 x(2x^2 - x^3 - 0)dx = \frac{56\pi}{5}.$$

(ii) Let D denote the solid obtained by revolving the rectangle $[0, b] \times [0, h]$ about the y-axis. By the shell method,

$$\text{Vol}\,(D) = 2\pi \int_0^b x\,(h - 0)\,dx = \pi hb^2.$$

(iii) Let E denote the solid obtained by revolving the rectangle $[0, b] \times [0, h]$ about the x-axis. By the shell method,

$$\text{Vol}\,(E) = 2\pi \int_0^h y\,(b - 0)\,dy = \pi bh^2.$$

(iv) Let D denote the solid in $\mathbb{R}^3$ obtained by revolving the region in the first quadrant bounded by the curve $y = x^3$ and the line $y = 4x$ about the line $x = -1$.

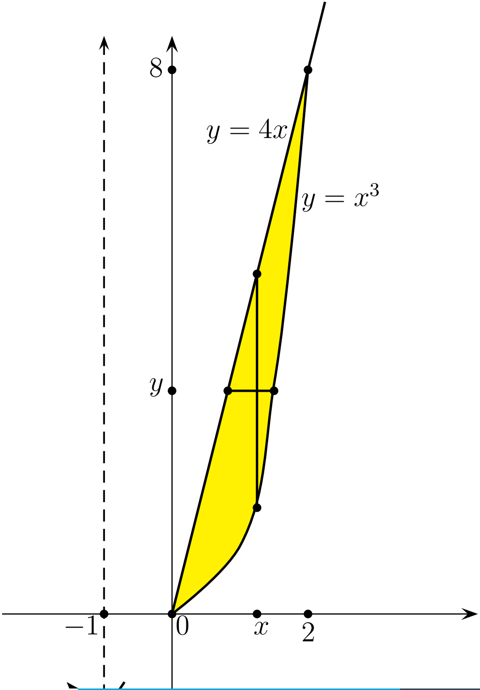
Now in the first quadrant, $x^3 = 4x \iff x = 0$ or $x = 2$.

- By the washer method,

$$\begin{aligned} \mathrm{Vol}\,(D) &= \pi \int_0^8 \left(\left(y^{1/3} + 1\right)^2 - \left((y/4) + 1\right)^2 \right) dy \\ &= \pi \int_0^8 \left(y^{2/3} + 2y^{1/3} - (y^2/16) - (y/2)\right) dy = \frac{248\pi}{15}. \end{aligned}$$

- By the shell method,

$$\begin{aligned} \mathrm{Vol}\,(D) &= 2\pi \int_0^2 (x+1)\left(4x - x^3\right) dx \\ &= 2\pi \int_0^2 (4x + 4x^2 - x^3 - x^4)\, dx = \frac{248\pi}{15}. \end{aligned}$$



8
$y = 4x$
$y = x^3$
y
-1
0
x
2

Remarks on Volume of a Solid of Revolution

Let D denote a bounded solid of revolution.

Remark 1. In the washer method, the slices (which look like washers) are taken perpendicular to the axis of revolution. On the other hand, in the shell method, the slivers (which look like cylindrical shells) are taken parallel to the axis of revolution.

Remark 2. The basic expression in the washer method is $\pi(r_2^2 - r_1^2)$, where r_2 and r_1 are outer and inner radii of the washer. The basic expression in the shell method is $2\pi r(h_2 - h_1)$, where r is the radius of the sliver, while $h_2 - h_1$ is the height of the sliver.

Remark 3. The volume of D found by the washer method and by the shell method must be the same! This result would follow from a general definition of the volume of a solid in $\mathbb{R}^3$. This can serve as a check on your calculations.

Parametrized Curve

A **parametrized curve** or a **path** C in $\mathbb{R}^2$ is given by $(x(t), y(t))$, where $x, y : [\alpha, \beta] \to \mathbb{R}$ are continuous functions.

Here $[\alpha, \beta]$ is called the **parameter interval**.

We wish to define the 'length' of C.

Basic assumption: The (Euclidean) length of a line segment joining points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is equal to

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We shall assume that the curve C is continuously differentiable, that is, the functions x, y are **continuously differentiable** on $[\alpha, \beta]$. This means that x, y are differentiable on $[\alpha, \beta]$, and their derivatives x', y' are continuous on $[\alpha, \beta]$.

Arc Length of a Smooth Curve

- Partition $[\alpha, \beta]$ into $\alpha = t_0 < t_1 < \cdots < t_n = \beta$.
- Let $P_i := \big(x(t_i), y(t_i)\big)$ for $i = 1, \ldots, n$, and draw the line segments joining P_0 to P_1, P_1 to P_2, …, P_{n-1} to P_n.
- The sum of the lengths of these line segments is

$$
\begin{aligned}
& \sum_{i=1}^{n} \sqrt{\left(x(t_i) - x(t_{i-1})\right)^2 + \left(y(t_i) - y(t_{i-1})\right)^2} \\
= \; & \sum_{i=1}^{n} \sqrt{\left(x'(s_i)\right)^2 + \left(y'(u_i)\right)^2}\,(t_i - t_{i-1}),
\end{aligned}
$$

  for some s_i, $u_i \in (t_{i-1}, t_i)$ for $i = 1, \ldots, n$ by the MVT.
- We define the **arc length** of C by

$$\ell(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Special Cases

Special cases:

(i) Let a curve C be given by $y = f(x)$, $x \in [a, b]$.
Here $\alpha := a$, $\beta := b$, $x(t) := t$ and $y(t) := f(t)$ for $t \in [a, b]$. Suppose f is continuously differentiable on $[a, b]$. Then

$$\ell(C) := \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

(ii) Let a curve C be given by $x = g(y)$, $y \in [c, d]$.
Here $\alpha := c$, $\beta := d$, $x(t) := g(t)$ and $y(t) := t$ for $t \in [c, d]$. Suppose g is continuously differentiable on $[c, d]$. Then

$$\ell(C) := \int_c^d \sqrt{g'(y)^2 + 1}\, dy.$$

Arc Length in Polar coordinates

Let C be given by a polar equation $r = p(\theta)$, $\theta \in [\alpha, \beta]$. As a parametrized curve, C is given by $(x(\theta), y(\theta))$, where

$$x(\theta) := p(\theta)\cos\theta \quad \text{and} \quad y(\theta) := p(\theta)\sin\theta, \quad \theta \in [\alpha, \beta].$$

Suppose the function p is continuously differentiable on $[\alpha, \beta]$.

For $\theta \in [\alpha, \beta]$, we note that $\sqrt{x'(\theta)^2 + y'(\theta)^2}$ is equal to

$$\begin{aligned} & \sqrt{\left(p'(\theta)\cos\theta - p(\theta)\sin\theta\right)^2 + \left(p'(\theta)\sin\theta + p(\theta)\cos\theta\right)^2} \\ = \; & \sqrt{p(\theta)^2 + p'(\theta)^2}. \end{aligned}$$

Hence

$$\ell(C) := \int_\alpha^\beta \sqrt{p(\theta)^2 + p'(\theta)^2}\, d\theta.$$

Examples

(i) Let C be given by $y = x^2$, $x \in [0,1]$. Then

$$\begin{aligned} \ell(C) &= \int_0^1 \sqrt{1+(2x)^2}\,dx = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du \\ &= \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln\left(2+\sqrt{5}\right). \end{aligned}$$

(Use Integration by Parts. Also, if $f(u) := \ln(u + \sqrt{1+u^2})$ for $u \in \mathbb{R}$, then note that $f'(u) = 1/\sqrt{1+u^2}$ for $u \in \mathbb{R}$, and so $\int_0^x \sqrt{1+u^2}\,du = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2}\right)\right)$ for $x \in \mathbb{R}$.)

(ii) Let C be given by $x = (2y^6+1)/8y^2$, $y \in [1,2]$. Then

$$\int_1^2 \left(1 + \left(y^3 - \frac{1}{4y^3}\right)^2\right)^{1/2} dy = \int_1^2 \left(y^3 + \frac{1}{4y^3}\right) dy = \frac{123}{32}.$$

(iii) Let $a > 0$ and $\varphi \in [0, \pi]$. Let C denote the arc of a circle of radius a given by $x(\theta) := a\cos\theta$, $y(\theta) := a\sin\theta$ for $\theta \in [0, \varphi]$. Then C is given by the polar equation $r = p(\theta)$, where $p(\theta) = a$ for $\theta \in [0, \phi]$, and so

$$\ell(C) = \int_0^{\varphi} \sqrt{a^2 + 0^2}\, d\theta = a\,\varphi.$$

Hence the length of a circle of radius a is $\int_{-\pi}^{\pi} a\, d\theta = 2\pi a$.

(iv) Let C be given by $r = 1 + \cos\theta$ for $\theta \in [0, \pi]$. Then

$$\begin{aligned} \ell(C) &= \int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta = 2\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 4. \end{aligned}$$

(Note: $\cos(\theta/2) \geq 0$ for $\theta \in [0, \pi]$.)

Suppose C is a smooth parametrized curve in $\mathbb{R}^3$ given by $(x(t), y(t), z(t))$, where $x,\ y,\ z : [\alpha, \beta] \to \mathbb{R}$ are continuously differentiable functions on $[\alpha, \beta]$.

In analogy with the definition of the arc length of a curve in $\mathbb{R}^2$, we define the **arc length** of C by

$$\ell\,(C) := \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Example

Let C denote a **helix** in $\mathbb{R}^3$ given by $x(t) := a\cos t,\ \ y(t) := a\sin t,\ \ z(t) := b\,t,\ \ t \in [\alpha, \beta]$, where $a,\ b, \in \mathbb{R}$, $a > 0$ and $b \neq 0$. Then

$$\ell\,(C) = \int_\alpha^\beta \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\, dt = (\beta - \alpha)\sqrt{a^2 + b^2}.$$

Surface of Revolution

A **surface of revolution** is generated when a curve C in $\mathbb{R}^2$ is revolved about a line L in $\mathbb{R}^2$.

First suppose the curve C is a slanted line segment P_1P_2 of length λ_2, and C does not cross L. Let d_1 and d_2 denote the distances of P_1 and P_2 from L with $d_1 \leq d_2$. Then the surface of revolution is a frustum F of a right circular cone with base radii d_1 and d_2, and slant height λ_2. We find its surface area.

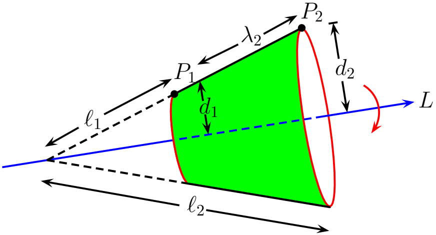


Figure: Frustum of a right circular cone

Consider a cone with base radius d and slant height ℓ. If we slit open this cone, we obtain a sector of a disk of radius ℓ.

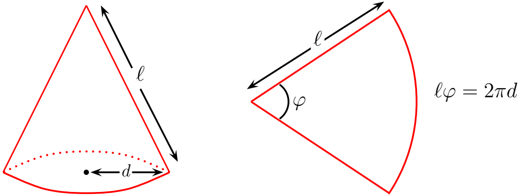


Figure: Right circular cone and sector of a disk

Since $\ell\varphi = 2\pi d$, the surface area of the cone is equal to

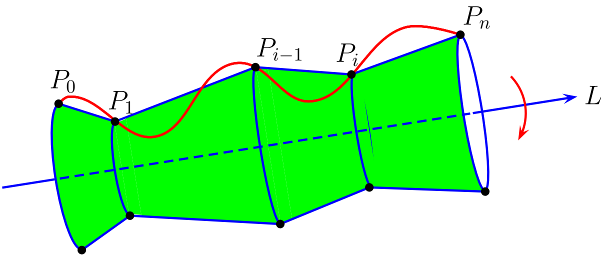
$$\frac{1}{2}\ell^2\varphi = \frac{1}{2}\ell^2\frac{2\pi d}{\ell} = \pi\ell d.$$

Hence the surface area of the frustrum F of the cone is $\pi\ell_2 d_2 - \pi\ell_1 d_1 = \pi(d_1 + d_2)(\ell_2 - \ell_1) = \pi(d_1 + d_2)\lambda_2$.

Now suppose C is parametrized by $(x(t), y(t))$, $t \in [\alpha, \beta]$.

- Partition $[\alpha, \beta]$ into $\alpha = t_0 < t_1 < \cdots < t_n = \beta$.
- Let $P_i := \big(x(t_i), y(t_i)\big)$ for $i = 0, 1, \ldots, n$, and draw the line segments P_0P_1, P_1P_2, …, $P_{n-1}P_n$.

Let $d_0, d_1, d_2, \ldots, d_n$ be the distances of $P_0, P_1, P_2, \ldots, P_n$ from the line L. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the lengths of the line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$. Suppose they don't cross L.



Fix $i \in \{1, \ldots, n\}$. When the line segment $P_{i-1}P_i$ is revolved about the line L, it generates a frustum F_i (of a right circular cone) whose surface area is $\pi(d_{i-1} + d_i)\lambda_i$.

Let $\rho(t)$ denote the distance of the point $(x(t), y(t))$ on the curve C from the line L. Then $d_i = \rho(t_i)$ for $i = 0, 1, \ldots, n$.

Thus the sum of the surface areas of the frustrums $F_1, \ldots, F_n$ is

$$\pi \sum_{i=1}^{n} \left(\rho(t_{i-1}) + \rho(t_i)\right) \lambda_i,$$

If the functions x' and y' are continuously differentiable on $[\alpha, \beta]$, then the length λ_i of the line segment $P_{i-1}P_i$ is given by

$$\begin{aligned}
\lambda_i &= \sqrt{\left(x(t_i) - x(t_{i-1})\right)^2 + \left(y(t_i) - y(t_{i-1})\right)^2} \\
&= \sqrt{x'(s_i)^2 + y'(u_i)^2}\,(t_i - t_{i-1})
\end{aligned}$$

for some s_i, $u_i \in (t_{i-1}, t_i)$ for $i = 1, \ldots, n$ (by the MVT).

Area of Surface of Revolution

Let C be a smooth curve parametrized by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Suppose the curve C does not cross the line L given by $ax + by + c = 0$. We define the **area of the surface** S generated by revolving C about the line L by

$$\text{Area}\,(S) := 2\pi \int_{\alpha}^{\beta} \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt,$$

where $\rho(t)$ is the distance of $(x(t), y(t))$ from the line L,

that is, $\rho(t) := |ax(t) + by(t) + c|/\sqrt{a^2 + b^2}$ for $t \in [a, b]$.

Note: Since the curve C does not cross the line L, the curve C lies entirely on one of the sides of the line L, that is,
either $ax(t) + by(t) + c \geq 0$ for all $t \in [\alpha, \beta]$,
or $ax(t) + by(t) + c \leq 0$ for all $t \in [\alpha, \beta]$.

Special Cases:

(i) Let the line L be the x-axis, and let the curve C be given by $y = f(x)$ for $x \in [a, b]$, where f is continuously differentiable. If $f \geq 0$ on $[a, b]$ or $f \leq 0$ on $[a, b]$, then

$$\text{Area}(S) = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\, dx.$$

(ii) Let the line L be the y-axis, and let the curve C be given by $x = g(y)$ for $y \in [c, d]$, where g is continuously differentiable. If $g \geq 0$ on $[c, d]$ or $g \leq 0$ on $[c, d]$, then

$$\text{Area}(S) = 2\pi \int_c^d |g(y)| \sqrt{1 + g'(y)^2}\, dy.$$

(iii) Let the line L be given by $\theta = \gamma$, where $\gamma \in (-\pi, \pi]$, and let the curve C be given by $r = p(\theta)$ for $\theta \in [\alpha, \beta]$, where p is continuously differentiable on $[\alpha, \beta]$. Suppose C does not cross L. Now the curve C is also given by $(p(\theta)\cos\theta, p(\theta)\sin\theta)$ for $\theta \in [\alpha, \beta]$.

Also, $\rho(\theta) = p(\theta)|\sin(\theta - \gamma)|$ for $\theta \in [\alpha, \beta]$.

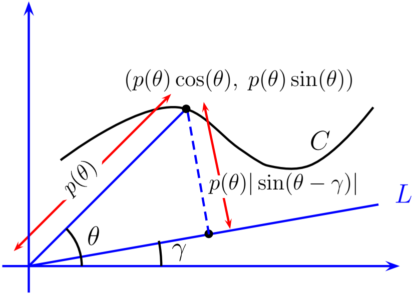


Figure: Distance of a point on a polar curve from a ray.

Thus $\text{Area}(S) = 2\pi \int_\alpha^\beta p(\theta)|\sin(\theta - \gamma)|\sqrt{p(\theta)^2 + p'(\theta)^2}\,d\theta.$

Examples

(i) Let S denote the surface generated by revolving the curve $y = (x^3/3) + (1/4x)$, $x \in [1,3]$, about the line $y = -1$. Then

$$\begin{aligned}
\text{Area}(S) &= 2\pi \int_1^3 (y+1)\sqrt{1+(y')^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\sqrt{1+\left(x^2 - \frac{1}{4x^2}\right)^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\left(x^2 + \frac{1}{4x^2}\right)dx \\
&= 1823\pi/18.
\end{aligned}$$

(iii) Let $0 < b < a$ and let C denote the circle given by $(a + b\cos t, b\sin t)$, $t \in [-\pi, \pi]$. Let S denote the surface generated by revolving the curve C about the y-axis. Then $a + b\cos t > 0$ for all $t \in [-\pi, \pi]$, and so

$$\begin{aligned}\text{Area}\,(S) &= 2\pi \int_{-\pi}^{\pi} (a + b\cos t)\sqrt{(-b\sin t)^2 + (b\cos t)^2}\, dt \\ &= 2\pi b \int_{-\pi}^{\pi} (a + b\cos t) dt \\ &= 4\pi^2 ab.\end{aligned}$$

Note: S is in fact the surface of a torus in $\mathbb{R}^3$.

(iii) Let $a > 0$, and S denote the surface generated by revolving the semicircle $p(\theta) = a$, $\theta \in [0, \pi]$, about the x-axis. Then

$$\text{Area}(S) = 2\pi \int_0^{\pi} a\sin\theta \sqrt{a^2 + 0^2}\, d\theta = 4\pi a^2.$$

Note: S is in fact the sphere of radius a in $\mathbb{R}^3$.

Functions with range contained in $\mathbb{R}$

We will be interested in studying functions $f : \mathbb{R}^m \to \mathbb{R}$, when $m = 2, 3$. We have already mentioned how limits of such functions can be studied in the first few lectures. Before doing this in detail, however, we will study certain other features of functions in two and three variables.

The most basic thing one needs to understand about a function is the domain on which it is defined. Very often a function is given by a formula which makes sense only on some subset of $\mathbb{R}^m$ and not on the whole of $\mathbb{R}^m$. When studying functions of two or more variables given by formulæ it makes sense to first identify this subset, which is sometimes call the natural domain of the function, and to describe it geometrically if possible.

Exercise 5.1: Find the natural domains of the following functions:
(i) $\frac{xy}{x^2-y^2}$

Clearly this function is defined whenever the denominator is not zero, in other words when $x^2 - y^2 \neq 0$.
The natural domain is thus

$$\mathbb{R}^2 \setminus \{(x,y) \,|\, x^2 - y^2 = 0\},$$

that is, $\mathbb{R}^2$ minus the pair of straight lines with slopes ± 1.
(ii) $f(x,y) = \log(x^2 + y^2)$
This function is defined whenever $x^2 + y^2 \neq 0$, in other words, in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Level curves and contour lines

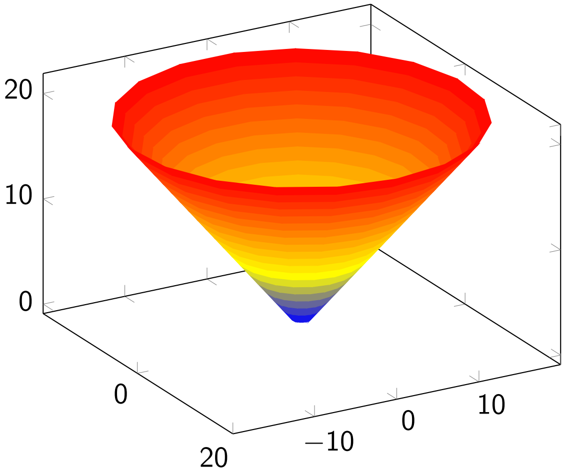
The second thing one should do with a function from $\mathbb{R}^2 \to \mathbb{R}$ is to study its range. This is done in different ways.

One way is to study the level sets of the functions. These are the sets of the form $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$, where c is a constant. The level set "lives" in the xy-plane.

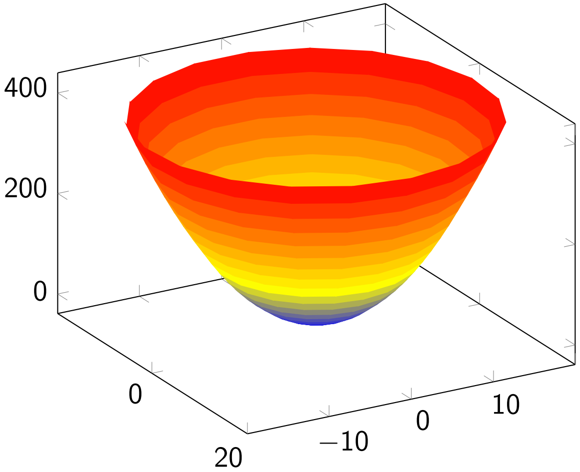
One can also plot (in three dimensions) the surface $z = f(x, y)$. By varying the value of c in the level curves one can get a good idea of what the surface looks like.

When one plots the $f(x, y) = c$ for some constant c one gets a curve. Such a curve is usually called a contour line (the contour "lives" in the $z = c$ plane).

I have a couple of pictures in the next two slides to illustrate the point.



This is the graph of the function $z = \sqrt{x^2 + y^2}$ lying above the xy-plane.
It is a right circular cone.
The contour lines $z = c$ give circles lying on planes parallel to the xy-plane. The curves given by $z = f(x, 0)$ and $z = f(0, y)$ give pairs of straight lines in the planes $y = 0$ and $x = 0$.



This is the graph of the function $z = x^2 + y^2$ lying above the xy-plane. It is a paraboloid of revolution.
The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves $z = f(x, 0)$ or $z = f(y, 0)$ give parabolæ lying in the planes $y = 0$ and $x = 0$. Exercise 6.2.(ii).

Limits

We have already said what it means for a function of two or more variables to approach a limit. We simply have to replace the absolute value function on $\mathbb{R}$ by the distance function on $\mathbb{R}^m$. We will do this in two variables. The three variable definition is entirely analogous. We will denote by U a set in $\mathbb{R}^2$.

Definition: A function $f : U \to \mathbb{R}$ is said to tend to a limit l as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon,$$

whenever $0 < \|x - c\| < \delta$.

We recall that

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

Continuity

Before talking about continuity we remark the following. In the plane $\mathbb{R}^2$ it is possible to approach the point c from infinitely many different directions - not just from the right and from the left. In fact, one may not even be approaching the point c along a straight line! Hence, to say that a function from $\mathbb{R}^2$ to $\mathbb{R}$ possesses a limit is actually imposing a strong condition - for instance, the limits along all possible curves leading to the point must exist and all these (infinitely many) limits must be equal.

Once we have the notion of a limit, the definition of continuity is just the same as for functions of one variable.

Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

The rules for limits and continuity

The rules for addition, subtraction, multiplication and division of limits remain valid for functions of two variables (or three variables for that matter). Nothing really changes in the statements or the proofs.

Using these rules, we can conclude, as before, that the sum, difference, product and quotient of continuous functions are continuous (as usual we must assume that the denominator of the quotient is non zero).

Continuity through examples

Once again, we emphasise that continuity at a point c is a very powerful condition (since the existence of a limit is implicit).

Exercise 6.3.(i) asks whether the function

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution: Let us look at the sequence of points $z_n = (\frac{1}{n}, \frac{1}{n^3})$, which goes to 0 as $n \to \infty$. Clearly $f(z_n) = \frac{1}{2}$ for all n, so

$$\lim_{n\to\infty} f(z_n) = \frac{1}{2} \neq 0.$$

This shows that f is not continuous at 0.
But does the limit exist?

Iterated limits

When evaluating a limit of the form $\lim_{(x_1,x_2)\to(c_1,c_2)} f(x_1,x_2)$ one may naturally be tempted to let x_1 go to c_1 first, and then let x_2 go to c_2. Does this give the limit in the previous sense?

Exercise 6.5: Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}.$$

we have

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0} 0 = 0$$

Similarly, one has $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$.

However, choosing $z_n = (\frac{1}{n}, \frac{1}{n})$, shows that $f(z_n) = 1$ for all $n \in \mathbb{N}$. Now choose $z_n = (\frac{1}{n}, \frac{1}{2n})$ to see that the limit cannot exist.