Test 2

2 Let $L = \lim_{x \to \alpha} f(x)$

Given $\epsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \epsilon/2$
$\forall x \in (\alpha - \delta, \alpha + \delta)$

Let $h \in \mathbb{R}$,

$$
\begin{aligned}
&|f(\alpha + h) - f(\alpha - h)| \\
&= |f(\alpha + h) - L + L - f(\alpha - h)| \\
&\leq |f(\alpha + h) - L| + |f(\alpha - h) - L|
\end{aligned}
$$

Suppose $h \in (-\delta, \delta)$.

Then $\alpha + h \in (\alpha - \delta, \alpha + \delta)$
$\alpha - h \in (\alpha - \delta, \alpha + \delta)$

$\therefore \forall h \in (-\delta, \delta)$, $|f(\alpha + h) - f(\alpha - h)|$
$\leq |f(\alpha + h) - L| + |f(\alpha - h) - L|$
$< \epsilon/2 + \epsilon/2 = \epsilon$

$\therefore \forall h \in (-\delta, \delta)$, $|f(\alpha + h) - f(\alpha - h)| < \epsilon$

$\therefore \lim_{h \to 0} (f(\alpha + h) - f(\alpha - h)) = 0$

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(i) $f(x) = \sin(1/x)$ if $x \neq 0$, and $f(0) = 0$

$f(x)$ is continuous for all $x \neq 0$.

Consider the sequence $x_n = \dfrac{2}{(4n+1)\pi}$ $\forall n \in \mathbb{N}$

$f(x_n) = \sin(1/x_n) = 1$

$\therefore \lim_{n \to \infty} f(x_n) = 1$ and $\lim_{n \to \infty} x_n = 0$

But $1 \neq f(0)$.

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

(ii) $f(x) = x \sin \dfrac{1}{x}$ if $x \neq 0$, and $f(0) = 0$

$f(x)$ is continuous for all $x \neq 0$

Using $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$, can you prove that $\lim_{x \to 0} f(x) = 0$?

Exercise.

4 f is cont. at 0 $\Rightarrow \lim_{x \to 0} f(x) = f(0)$

Given $\epsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(0)| < \epsilon$

$\forall x$ s.t. $|x| < \delta$.

For $x, c \in \mathbb{R}$, $|f(x) - f(c)|$

$= |f(x - c)|$

Prove that $f(0) = 0$.

$\therefore \forall x \in (c - \delta, c + \delta)$,

$|f(x) - f(c)| = |f(x-c)| < \epsilon$

$\therefore \lim_{x \to c} f(x) = f(c)$

$\therefore f$ is cont. at every $c \in \mathbb{R}$.

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5 $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$

f is clearly diff. at $x \neq 0$ by product, composition rules.

At $x = 0$,

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$ (seen earlier)

$\therefore f$ is diff. at $x = 0$ as well.

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is f' continuous?

$x_n = \frac{1}{2n\pi}$, $n \in \mathbb{N}$

7 $$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$$

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h}$$

$$= \lim_{h\to 0^+}\left(\left(\frac{f(c+h)-f(c)}{2h}\right) + \left(\frac{f(c-h)-f(c)}{-h}\right)\right)$$

-2h

$$= \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{2h} + \lim_{h\to 0^+} \frac{f(c-h)-f(c)}{-2h}$$

This can be done because both limits exist.

$$= \frac{f'(c)}{2} + \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{2h}$$

$$= \frac{f'(c)}{2} + \frac{f'(c)}{2} = f'(c)$$

Converse : $f(x) = |x|$, $c=0$

10 $y = f\left(\frac{2x-1}{x+1}\right)$ let $g(x) = \frac{2x-1}{x+1}$

For $x \neq -1$, g is diff. at x, and f is diff. at $g(x)$

Now apply chain rule.

$(f \circ g)'(x) = f'(g(x))\, g'(x) \quad \forall x \neq -1$

11 $f(x) = |x-1| + |x-2|$

12 $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Let $\epsilon = 1/2$.

Suppose f is cont. at some $c \in \mathbb{R}$.

Then $\exists \delta > 0$ s.t. $\forall x \in (c-\delta, c+\delta)$, $|f(x) - f(c)| < \epsilon$.

① Suppose $c \in \mathbb{Q}$. Choose $x \in (c-\delta, c+\delta)$ s.t. $x \notin \mathbb{Q}$

Then $|f(x) - f(c)| = 1 > \epsilon$

② Suppose $c \notin \mathbb{Q}$. Choose $x \in (c-\delta, c+\delta)$ s.t. $x \in \mathbb{Q}$

Then $|f(x) - f(c)| = 1 > \epsilon$

$\therefore$ Contradiction.

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We will show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii):

Choose $\delta = \min\{c-a,\ b-c\}$.

Let $\alpha = f'(c)$.

Define $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ as

$$\epsilon_1(x) = \begin{cases} \dfrac{f(c+x) - f(c)}{x} - \alpha, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

($f(c+h)$ makes sense because $(c-\delta, c+\delta) \subset (a,b)$)

$$\lim_{h \to 0} \epsilon_1(h) = 0 \quad (\text{why?})$$

Clearly, $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$ $\forall h \in (-\delta, \delta)$

($h=0$ must be checked separately)

(ii) $\Rightarrow$ (iii)

$$\lim_{h \to 0} \left| \frac{f(c+h) - f(c) - \alpha h}{h} \right| = \lim_{h \to 0} |\epsilon_1(h)| = 0$$

(iii) $\Rightarrow$ (i) :

$$\lim_{h \to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0$$

Hence, $$\lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{h} - \alpha \right) = 0$$

(Why? P.T. if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

$\therefore f'(c)$ exists and $f'(c) = \alpha$.

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