MA111 (IIT Bombay) End-sem exam, 26th February, 2021

1. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1, \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

[2+1]

   (a) Evaluate iterated integral $\int_{y=0}^{1} \left(\int_{x=0}^{1} f(x,y)dx \right) dy$.

   (b) Mark True or False: The function f is Darboux integrable on $[0,1] \times [0,1]$.

      i. FALSE.

      ii. TRUE.

   Ans. a) -1, b) i).

2. Mark all the statements which are correct. Let $R = [0,10] \times [0,10]$ and $f(x,y) = [x] + [y]$. Then [2]

   (a) the function f is integrable since it is monotonic in each variable on R.

   (b) the function f is not integrable since it is not continuous on R.

   (c) the function f is integrable since it has only finitely many points of discontinuity on R.

   (d) the function f is integrable since it is discontinuous on a finite union of graphs of functions.

   Ans. a), d)

3. Evaluate $\int\int_R 2x \cos y \, dxdy$, where $R = [0,n] \times [0, \frac{n\pi}{2}]$. [2]

   Vary n, $n = 1, 2, 3$

   Ans. $n = 1$, Ans.1; $n = 3$, Ans. -9; $n = 2$, Ans.=0.

4. Let the region D in $\mathbb{R}^2$ be bounded by the curves $y = x^2$ and $y = |x|$. [2+2]

   (a) Which of the following integral computes the Area of the region bounded by $y = x^2$ and $y = |x|$.

      i. $2 \int_0^1 \int_{x^2}^{x} dydx$

      ii. $\int_0^1 \int_y^{\sqrt{y}} dxdy$

      iii. $\int_0^1 \int_{x^2}^{x} dydx$

   (b) Find the value of α, where $\alpha = n\text{Area}(D)$.

   vary n as $n = 3, 6$

   Ans. a) i). b) when n=3, ans= 1; when n=6, ans=2.

5. Let I denote the integral $I = \iint_D f(x,y)dxdy$ where $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x+y \leq 1, 0 \leq x-y \leq 1\}$ and $f(x,y) = \frac{2e^{x+y}}{(e-1)}$ defined on $\mathbb{R}^2$. [2+2]

   (a) Mark the correct statement.

i. $2I = \iint_{[0,1]\times[0,1]} f(u,v)dudv$

ii. $2I = \iint_{[0,1]\times[0,1]} f((u+v)/2,(u-v)/2)dudv$

iii. $2I = \iint_{[0,1]\times[0,1]} f(u+v,u-v)dudv$

(b) Find the value of I.

Ans. a) ii). b) 1.

6. Let $I = \frac{1}{\pi}\int_0^2 \left(\int_0^{\sqrt{4-x^2}} \left(\int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\right) dy\right) dx.$ [2+2]

(a) Then which of the following statements are true.

i. $I = \frac{1}{\pi}\int_0^{\pi/2} \left(\int_0^{\pi/2} \left(\int_0^2 \rho^3 \sin(\phi)\, d\rho\right) d\theta\right) d\phi.$

ii. $I = \frac{1}{\pi}\int_0^{\pi} \left(\int_0^{2\pi} \left(\int_0^4 \rho^3 \sin(\phi)\, d\rho\right) d\theta\right) d\phi.$

iii. $I = \frac{1}{\pi}\int_0^{\pi} \left(\int_0^{2\pi} \left(\int_0^2 \rho^3 \sin(\phi)\, d\rho\ \right) d\theta\right) d\phi.$

(b) Find I.

Ans. $a)i)$, $b)$2.

7. Evaluate the line integral $\int_C \mathbf{F}.\mathbf{ds}$, where C is given by $(2t, 3t, -t^2)$ for all $-a \le t \le a$, and $\mathbf{F}(x,y,z) = x\mathbf{i} + z\mathbf{j} - y\mathbf{k}$ defined in $\mathbb{R}^3$. [2]

Vary a positive integer. $a = 1, 2, 3$.

Ans. $2a^3$. For $a = 1$, ans= 2; For $a = 2$, ans=16, For $a = 3$, ans=54.

8. Let $\mathbf{F}(x,y) = y\cos(xy)\mathbf{i} + x\cos(xy)\mathbf{j}$ be defined in $\mathbb{R}^2$. [2+2]

(a) Mark all correct statements below.

i. There exists a C^1 scalar field ϕ defined from $\mathbb{R}^2$ to $\mathbb{R}$ such that $\mathbf{F} = \nabla\phi$ in $\mathbb{R}^2$.

ii. Let C be a circle with center at origin and radius $a > 0$, then $\int_C \mathbf{F}.\mathbf{ds} = \pi a^2$.

iii. There exists a C^1 scalar field ϕ defined from $\mathbb{R}^2$ to $\mathbb{R}$ such that $\int_C \mathbf{F}.\mathbf{ds} = \phi(Q) - \phi(P)$, where C is a smooth path in $\mathbb{R}^2$ traversed from P to Q.

iv. There exist two smooth paths C_1 and C_2 joining two points P and Q in $\mathbb{R}^2$, such that

$$\int_{C_1} \mathbf{F}.\mathbf{ds} \neq \int_{C_2} \mathbf{F}.\mathbf{ds}.$$

(b) Let the path C be defined by $x = \cos^3\theta, y = \sin^3\theta$, for $0 \le \theta \le \frac{\pi}{2}$. Find $\int_C \mathbf{F}.\mathbf{ds}$.

Ans. a) i), iii) where $\phi(x,y) = \sin(xy)$.

(iv. is False if 'path joining two points P and Q' is considered as 'path from P to Q')

However, 'path joining two points P and Q ' may be interpreted as a path joining the two points but without any prescribed orientation. Then iv) is TRUE because C_1 (joining P to Q) and C_2 (joining Q to P) can be with reversed orientation and the line integrals differ by '-'sign.

Keeping this in mind, in the end sem, for a) i), iii), iv) are also taken as correct answers.

In summary, for a) if anyone chooses i) and iii), they get 2 Marks. If anyone chooses i), iii) and iv) all 3 options, they also get 2 marks.

Ans. b) 0.

9. Consider the vector field $\mathbf{F}(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$ defined in $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 4\}$, where $F_1(x,y) = (x^3 - 2xy^3)$ and $F_2(x,y) = -3x^2y^2$, for all (x,y) in Ω. Then the vector field $\mathbf{F}$ is conservative in Ω. Mark the correct statement below. [2]

   (a) The above statement is TRUE because $\mathbf{F}$ is a C^1 vector field defined in Ω and $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ in Ω and Ω is an open, simply connected subset in $\mathbb{R}^2$.

   (b) The above statement is TRUE but Ω is not path-connected in $\mathbb{R}^2$.

   (c) The above statement is FALSE because there exists a circle C such that $\int_C \mathbf{F}.\mathbf{ds} \neq 0$.

   (d) The above statement is FALSE because $\mathbf{div}\,\mathbf{F} \neq 0$ in Ω.

   Ans. a).

10. Let $\mathbf{F}$ be a C^1 vector field defined in $\mathbb{R}^2$ and let $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \pi, \quad 0 \le y \le \sin x\}$, defined with the boundary ∂D traced anti-clockwise. [2+2]

    (a) Mark all the correct statements:

       i. D is a bounded region in $\mathbb{R}^2$.

       ii. The boundary of D is not a simple closed curve.

       iii. Green's theorem can be used to find $\int_{\partial D} \mathbf{F}.\mathbf{ds}$ because D and $\mathbf{F}$ satisfy all hypothesis in Green's theorem.

       iv. Green's theorem cannot be used to find $\int_{\partial D} \mathbf{F}.\mathbf{ds}$ because D does not satisfy the hypothesis in Green's theorem.

    (b) Find the line integral: $\int_{\partial D} y\,dx + (\alpha+1)x\,dy$.

    Vary α, positive integer. Ans. 2α

    Ans. a) i), iii). b) $\alpha = 1$, Ans. 2; $\alpha = 2$, Ans.4.

11. Let $D = \{(x,y) \in \mathbb{R}^2 \mid a^2 \le x^2+y^2 \le b^2\}$, where $0 < a < b$, with the boundary ∂D. Note $\partial D = C_1 \cup C_2$, where $C_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = a^2\}$ and $C_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = b^2\}$. Mark the correct statement: For any C^1 vector field $\mathbf{F}$, using Green's theorem, $\int_{\partial D} \mathbf{F}.\mathbf{ds} = \int\int_D (\mathrm{curl}\mathbf{F}).\mathbf{k}\,dxdy$, if [2]

    (a) C_1 is traced clockwise and C_2 is traced anti-clockwise.

    (b) Both C_1 and C_2 are traced anti-clockwise.

    (c) C_1 is traced anti-clockwise but C_2 is traced clockwise.

    (d) Both C_1 and C_2 are traced clockwise.

    Ans. a).

12. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$ defined in $\mathbb{R}^3$. There is a vector field $\mathbf{G}$ defined on $\mathbb{R}^3$ such that $\mathbf{F} = \mathrm{curl}\,\mathbf{G}$ in $\mathbb{R}^3$. Mark the correct answer. [2]

    (a) The above statement is TRUE because $\mathbf{F}$ is defined in whole $\mathbb{R}^3$ and $\mathbf{div}\,\mathbf{F} = 0$ in $\mathbb{R}^3$.

    (b) The above statement is FALSE because $\mathbf{div}\,(\mathbf{curl}\,\mathbf{F}) \neq 0$ in $\mathbb{R}^3$.

    (c) The above statement is FALSE because $\mathbf{curl}\,\mathbf{F} \neq 0$ in $\mathbb{R}^3$.

    Ans. a).

13. Let S be a surface given by the portion of the surface $z = xy$ that lies within the cylinder $x^2+y^2 = 1$. If $\mathrm{Area}(S) = \alpha\frac{\pi(2\sqrt{2}-1)}{3}$, find α. [2]

    Ans. 2.

14. Let S be a bounded smooth oriented surface with the boundary ∂S which is a smooth simple closed curve. Let $\mathbf{F}$ be perpendicular to the tangent to ∂S. Find the value α, where $\alpha = \int\int_S \operatorname{curl} \mathbf{F}.\mathbf{dS}$. Mark the correct answer. [2]

    (a) $\alpha = 0$.

    (b) $\alpha = 1$.

    (c) $\alpha = -1$.

    (d) $\alpha =$ Arc length of ∂S.

    Ans. $\alpha = 0$.

15. Let $f(x, y, z) = x^2 - y^2 + z$ be defined on $\mathbb{R}^3$ and S be the unit sphere oriented with the outward unit normal. Find the value of $\int\int_S \nabla f.\mathbf{dS}$. [2]

    Ans. 0

16. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$ oriented with the outward unit normal. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ be the region enclosed by S in $\mathbb{R}^3$. [2+2]

    (a) Mark the correct statement.

        i. The region W is not bounded in $\mathbb{R}^3$.

        ii. S is the surface with boundary.

        iii. The boundary of W is S.

    (b) Evaluate $\int\int_S \mathbf{F}.\mathbf{dS}$, where $\mathbf{F}(x, y, z) = \frac{5x^3}{\pi}\mathbf{i} + \frac{5y^3}{\pi}\mathbf{j} + \frac{5z^3}{\pi}\mathbf{k}$.

    Ans. a) iii). b) 12.

17. The upper hemisphere $S = \{x^2 + y^2 + z^2 = 1, \quad z \geq 0\}$, is oriented by the outward normal (along the radial vector). Find $\int\int_S \operatorname{curl} \mathbf{F}.\mathbf{dS}$, where $\mathbf{F}(x, y, z) = -\frac{y}{\pi}\mathbf{i} + \frac{x}{\pi}\mathbf{j} + zx^3y^2\mathbf{k}$ is defined in $\mathbb{R}^3$. [2]

    Ans. 2

18. Let $W = [0, 1]\times[0, 1]\times[0, 1]$ and $f(x, y, z) = 8x(y+k)z$ defined in $\mathbb{R}^3$. Let $I = \int\int\int_W f(x, y, z)dxdydz$. [1+2]

    (a) Mark True or False: Fubini's theorem can be used to evaluate I.

        i. TRUE.

        ii. FALSE.

    (b) Find the value of I.

    vary $k = 1, 2$.

    Ans. a)i). b)k=1, ans=3. k=2, ans =5.