MA-111 Calculus II (D3 & D4)

Lecture 9

B.K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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Parametrisation of curves

Characterization of conservative fields

Recap: Line integral

- Line integral of continuous vector field $\mathbf{F}$ on D over a smooth path $\mathbf{c} : [a, b] \to D$

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

- Line integral is independent of the parametrisation of the curve provided it preserves the orientation of the curve.
- If the reparametrization $\gamma(\cdot) = \mathbf{c}(h(\cdot))$ does not preserves the orientation of $\mathbf{c}$, then

$$\int_{\gamma} \mathbf{F}.\mathbf{ds} = -\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}$$

The arc length parametrisation

There is a natural choice of parametrisation we can make which is useful in many situations. This is the parametrisation by arc length.

Recall the length of a curve $\mathbf{c}$ for a path $\mathbf{c} : [a, b] \to \mathbb{R}^3$, called its arc length ,is given by

$$\ell(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\| \, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$$

We now set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $s'(t) = \|\mathbf{c}'(t)\|$. Why? Fundamental theorem of Calculus.

It is easy to see that s is a strictly increasing differentiable function. Let $h : [0, \ell(\mathbf{c})] \to [a, b]$ be its inverse. Then it is differentiable and its derivative is not vanishing. Define $\tilde{\mathbf{c}}(u) := \mathbf{c}(h(u))$ for $u \in [0, \ell(\mathbf{c})]$. This is called the arc length parametrization.

Let $h(u) = t \in [a, b]$ or $s(t) = u$.
Note that

$$\begin{aligned}
\frac{d\tilde{\mathbf{c}}(u)}{du} &= \mathbf{c}'(h(u))h'(u) \\
&= \mathbf{c}'(h(u))\frac{1}{s'(h(u))} \\
&= \mathbf{c}'(t)\frac{1}{\|\mathbf{c}'(t)\|}
\end{aligned}$$

Using the reparmetrization theorem we get that

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = \int_{\tilde{\mathbf{c}}} \mathbf{F}.\mathbf{ds}.$$

Note,

$$\begin{aligned}
\int_{\tilde{\mathbf{c}}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\tilde{\mathbf{c}}(u)) \cdot \tilde{\mathbf{c}}'(u) du \\
&= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u))) \cdot \frac{\mathbf{c}'(h(u))}{\|\mathbf{c}'(h(u))\|} du \\
&= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u)) \cdot \mathbf{T}(h(u))\ du
\end{aligned}$$

where $\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t\|}$ is the unit tangent vector along the curve.

In other words, the line integral is nothing but the (Riemann) integral of the tangential component of $\mathbf{F}$ with respect to arc length.

Note for this reparametrization we need to assume $\mathbf{c}$ is a non singular curve.

Integrals of scalar functions along path

To this end, as before we set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $ds = \|\mathbf{c}'(t)\| dt$.

Integrals of scalar functions along path: Let $f : D \to \mathbb{R}$ be a continuous scalar function and $\mathbf{c} : [a, b] \longrightarrow D$ be a non-singular path. Then the path integral of f along $\mathbf{c}$ is defined by

$$\int_{\mathbf{c}} f \, ds := \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\| \, dt.$$

Example. Find the circumference of the circle in $\mathbb{R}^2$ whose center is at origin and radius is r, for some $r > 0$.

Ans. Check $\int_{\mathbf{c}} f \, ds$ for $f = 1$ and $\mathbf{c}(t) = (r \cos t, r \sin t)$, for $t \in [0, 2\pi]$.

Quiz

- The quiz on next Friday, 18th February will include everything from Lecture 1 to the previous slide.

- The quiz will be conducted on SAFE. Please make sure when you login with IITB id into SAFE, MA1112022 is visible.

- The question format is *likely* to be objective.

- Further instructions will follow on Moodle.

- Fill the Google form circulated by Tutors before 10 pm Friday, 11th February (essential for you to take the quiz).

Characterization of gradient fields

The main observation about line integrals of a gradient field is the following. This is a form of Fundamental theorem of calculus.

Theorem

Let $n = 2, 3$ and let $D \subset \mathbb{R}^n$.

1. *Let $\mathbf{c} : [a, b] \to D \subset \mathbb{R}^n$ be a smooth path.*
2. *Let $f : D \to \mathbb{R}$ be a differentiable function and let ∇f be continuous on $\mathbf{c}$.*

Then $\int_{\mathbf{c}} \nabla f . \mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$.

Proof. From definition, it follows that

$$\int_{\mathbf{c}} \nabla f . \mathbf{ds} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

Now the integrand on the right hand side is nothing but the directional derivative of f in the direction of $\mathbf{c}(t)$. Hence, we obtain

$$\int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_a^b \frac{d}{dt} f(\mathbf{c}(t)) dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

- Suppose the vector field $\mathbf{F}$ is a continuous conservative field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function f. Then for any smooth path $\mathbf{c}$, we have
$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$
- This shows that the value of the line integral of a conservative field depends only on the value of the function at the end points of the curve, not on the curve itself.

Definition

The line integral of a vector field $\mathbf{F}$ is independent of path in a domain if for any $\mathbf{c}_1$ and $\mathbf{c}_2$ paths in D with the same initial and terminal points,

$$\int_{\mathbf{c}_1} \mathbf{F}.\mathbf{ds} = \int_{\mathbf{c}_2} \mathbf{F}.\mathbf{ds}.$$

Equivalently, the line integral of $\mathbf{F}$ is independent of path in D if for any closed curve $\mathbf{c}$ (why?)
$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = 0.$$

Example Evaluate $\int_C y^2 dx + x dy$, where

1. $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$,
2. $C = C_2$ is the part of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Are the line integrals along C_1 and C_2 same?

Ans 1.) Consider parametrization for C_1,
$\mathbf{c}_1(t) = (5t-5, 5t-3), \quad t \in [0,1]$. Thus $\mathbf{c}_1'(t) = (5,5)$ for all $t \in [0,1]$. So, $\mathbf{F}(\mathbf{c}_1(t)) = ((5t-3)^2, 5t-5)$ and

$$\int_{C_1} y^2 dx + x dy = \int_0^1 [(5t-3)^2 .5 + (5t-5).5]\, dt = -\frac{5}{6}.$$

2. Consider parametrization for C_2, $\mathbf{c}_2(t) = (4-t^2, t), \quad t \in [-3,2]$.
Thus $\mathbf{c}_2'(t) = (-2t, 1)$ for all $t \in [-3,2]$. So, $\mathbf{F}(\mathbf{c}_2(t)) = (t^2, 4-t^2)$ and

$$\int_{C_2} y^2 dx + x dy = \int_{-3}^2 [t^2(-2t) + (4-t^2)]\, dt = 40\frac{5}{6}.$$

Line integrals along C_1 and C_2 are Not same! Though the endpoints of C_1 and C_2 are same!

Conservative vector fields

In general, the line integral of a vector field depends on the path.

Fundemental theorem of calculus for line integrals yields that the line integral of a conservative field is independent of path in D.

What about the converse?

We will now prove the converse to our previous assertion under some assumption on D.

Definition: A subset D of $\mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two non-empty subsets $D_1 \cup D_2$, with $D_1 = D \cap U_1$ and $D_2 = D \cap U_2$, where U_1 and U_2 are open sets.

Definition: A subset of D of $\mathbb{R}^n$ is said to be path connected if any two points in the subset can be joined by a path (that is the image of a continuous curve) inside D.

In $\mathbb{R}^n$ we can show that that an open subset is connected if and only if it is path connected. So it is sufficient to assume domain of the vector field is open and connected.

Examples

Example. $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < a^2\}$ is path-connected.

Ans. If $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ are in D, then which path lying in D can be defined connecting P and Q?

Path connected implies connected.

Example. $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \cup \{(2, 2)\}$ is connected in $\mathbb{R}^2$?
Ans No. (Why?)

Example. $D = \{(x, \sin(\frac{1}{x}) \in \mathbb{R}^2 \mid x \in (0, 1]\} \cup \{(0, 0)\}$ is connected in $\mathbb{R}^2$ but not path-connected.