MA 109 Quiz/Midsemester exam review

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1. Limits
2. Continuity
3. Differentiation

General Advice

1. Concentrate on understanding the statements of the theorems. You will not be asked to reproduce long proofs.
2. When trying to understand a definition, make sure you know plenty of examples.
3. When trying to understand a theorem, make sure you know counter-examples to the conclusion of the theorem when you drop some of the hypotheses.
4. In general, the statement of the theorem is more important than its proof. And examples might be more important than theorems!

Limits of sequences

1. Learn the definition.

2. When proving a fact/theorem/etc. about some limit being l start with an $\epsilon > 0$ and find an N so that the sequence x_n you are dealing with satisfies
$$|x_n - l| < \epsilon,$$
for every $n > N$.

3. To prove that a sequence does not converge you have to show that no real number can be a limit. Thus you must take an arbitrary l and find some fixed $\epsilon > 0$ - this ϵ can be chosen to your convenience so that $|a_n - l| > \epsilon$ for infinitely many n.

4. Theorems to remember for showing that limits exist: the sum, difference, product and quotient and the Sandwich Theorem. In this case you will already know that some sequence has a limit and deduce that another sequence has a limit by comparing it to the known one.

Theorems that abstractly guarantee that the limit of a sequence exists:

A monotonically increasing/decreasing sequence bounded above/below converges to its supremum/infimum.

Every Cauchy sequence converges. It is a good idea to know the definition of a Cauchy sequence. However, you will not be asked questions on Cauchy sequences.

Unless we explicitly mention that you must use the ϵ-N definition to prove that a limit exists, you do not have to. You may use the rules for limits and other theorems instead. You can use simple facts without proving them: e.g. $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$ if $\alpha > 0$.

Exercise 1

If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} \sqrt{a_n} = 0$ using epsilon-N definition.

Solution: Fix $\epsilon > 0$ arbitrarily. We want to find an N such that $n \geq N \implies |\sqrt{a_n}| < \epsilon$.

Since $\lim_{n\to\infty} a_n = 0$, for ϵ^2, the square of ϵ that we fixed earlier, there exists N_1 such that

$$n \geq N_1 \implies |a_n| < \epsilon^2.$$

Then for $N = N_1$ we have

$$n \geq N \implies |\sqrt{a_n}| < \epsilon.$$

Limits of functions

The ideas behind proving or disproving the existence of limits are the same as for sequences (of course, there is no analogue of monotonic bounded sequences or Cauchy sequences).

You can use the basic limits you learnt in 11th/12th standard like $\lim_{x \to 0} \sin x/x = 1$.

Remember that there is a nice algebra for limits, and sandwich/squeeze theorems still apply.

At "endpoints" of intervals, one can make sense of right-handed or left-handed limits.

Continuity

Of course, you have to know the definition. You may use basic facts about limits of functions to prove what you want.

Two basic theorems are:

1. A continuous function on a closed bounded interval is bounded and attains its infimum and supremum

2. Continuous functions have the IVP (remember again, that the converse is not true)

   The sum, difference, product etc. of continuous functions is continuous. The composition of continuous functions is continuous.

Differentiation

Know the definition. Again, here you can use the basic facts about limits.

The basic theorems are:

1. Fermat's Theorem,
2. Rolle's theorem and the MVT,
3. Darboux's theorem.

Know the basic examples and counter-examples: a function that is continuous but not differentiable, a function that is differentiable but not continuously differentiable.

Exercise 3

Show that $x^3 - 10x + 4$ has three real roots.

Solution: Let $f(x) = x^3 - 10x + 4$. Then $f_x = 3x^2 - 10$ which has two roots, namely, $\pm\sqrt{10/3}$.

By the second derivative test we find that $-\sqrt{10/3}$ is a local maximum for f and $\sqrt{10/3}$ is a local minimum.
Since we have only two critical points, it follows that

$$f(-\sqrt{10/3}) > 0 > f(\sqrt{10/3}).$$

By the IVP of f, there exists a zero of f in the interval $(-\sqrt{10/3}, \sqrt{10/3})$. Since the given function is a cubic, $f(x) \to \pm\infty$ as $x \to \pm\infty$, hence again by IVP we get two more zeros of f in the intervals $(-\infty, -\sqrt{10/3})$ and $(\sqrt{10/3}, \infty)$.

Alternate solution: Show that $f(x)$ changes sign three times. Note that $f(-10) < 0$, $f(-1) > 0$, $f(1) < 0$ and $f(10) > 0$.

Exercise 4

Show that the function $x^4 + 3x + 1$ has exactly one zero in the interval $[-2, -1]$.

Solution: By observing that $f(-2) > 0$ and $f(-1) < 0$, we conclude by IVP that f has a zero in the interval $[-2, -1]$.

Further, the derivative, $4x^3 + 3$, is non-zero on $[-2, -1]$, so by Rolle's theorem, f has no more zeros in the given interval.

Maxima, minima, convex, concave

Remember that the definitions of maxima, minima, concavity, convexity, inflection points etc. have nothing to do with differentiation.

IF the function is (twice) differentiable then one can apply the various derivative tests. Otherwise, one can't.

Note that the existence of maxima and minima usually follows from the fact that we are dealing with continuous functions on a closed bounded interval.

Remember the difference between supremum and maximum (and of course, between infimum and minimum - know the relevant examples).

Taylor's theorem: Know how to compute the Taylor polynomials. Know the form of the Remainder term. Recall that there are smooth functions for which the Taylor series about a point converges but does not converge to the function ($e^{-1/x}$).

Exercise 5

Find the first three terms of the Taylor series of the function $1/x^2$ at 1.

Solution: If the Taylor series of the function f at $x = a$ is $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$,

then $a_n = \dfrac{f^{(n)}(a)}{n!}$.

Using these notations, for $f(x) = 1/x^2$ and $a = 1$, we get $a_0 = 1$, $a_1 = -2$ and $a_2 = 3$.