MA-111 Calculus II
(D3 & D4)

Lecture 17

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Recap

- If $\mathbf{\Phi} : E \to \mathbb{R}^3$ is a non-singular parametrization of S
$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv,$$
- If S is an oriented surface and $\mathbf{F}$ is the velocity field of a fluid moving in three dimensions, then
$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$
is the flux i.e, the net rate (units of volume/units of time) at which the fluid is crossing the surface in the outward direction (if the value of the integral is negative, this means that the net flow is inward).
- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then
$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$
- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then
$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

An example: flow through a hemisphere

Example: Find the flux of the vector field $\mathbf{j}$ across the hemisphere H defined by $x^2 + y^2 + z^2 = 1$, $x \geq 0$, oriented in the direction of increasing x.

Solution: We have already calculated the normal to the sphere for this parametrisation by spherical coordinates. It is

$$-(\sin\phi)\,\mathbf{r} = -\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

Hence,

$$-(\sin\phi)\,\mathbf{r}\cdot\mathbf{j} = -\sin^2\phi\sin\theta.$$

Note, that since we are dealing only with a hemisphere, we have $-\pi/2 \leq \theta \leq \pi/2$. We know that $\mathbf{\Phi}$ is orientation reversing. Hence,

$$\iint_H \mathbf{j}\cdot d\mathbf{S} = \int_0^{\pi}\int_{-\pi/2}^{\pi/2} \sin^2\phi\sin\theta d\theta d\phi = 0.$$

Gauss's Law

The flux of an electric field $\mathbf{E}$ over a "closed" surface is equal to the net charge Q enclosed by the surface. In terms of surface integrals we get

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = Q.$$

As a special case, let us consider an electric field of the form $\mathbf{E} = E\hat{\mathbf{n}}$, where E is a constant scalar. Then Gauss's law takes the form

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_S E dS = Q.$$

It follows that

$$E = \frac{Q}{A(S)}.$$

Fourier's Law

Let $T(x, y, z)$ denote the temperature at a point of a surface S. The famous law of heat flow due to Fourier says that heat flows from regions of higher temperature to regions of lower temperature. More specifically, the heat flow vector field is proportional to the gradient field ∇T.

We write $\mathbf{F} = -k\nabla T$ for this vector field. Hence, if S is a surface through which heat is flowing,

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

is the total rate of heat flow or flux across S.

Example: Suppose the scalar field $T(x, y, z) = x^2 + y^2 + z^2$ represents the temperaturre function at each point, and let S be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward normal vector. Find the heat flux across the surface if $k = 1$.

Solution: The heat flow field is given by

$$\mathbf{F} = -\nabla T(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$

The outward unit normal vector on S is simply given by $\hat{\mathbf{n}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

We have

$$\mathbf{F} \cdot \hat{\mathbf{n}} = -2x^2 - 2y^2 - 2z^2 = -2$$

as the normal component of $\mathbf{F}$. Now the surface integral is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -2 \iint_S dS = -8\pi.$$

In what direction is the heat flux flowing?

Homeomorphism

We now introduce the notion of 'Homeomorphism'.

Let ψ be function from $U_1 \subset \mathbb{R}^n$ to $U_2 \subset \mathbb{R}^m$.
We call the mapping $\psi : U_1 \longrightarrow U_2$ is a homeomorphism if ψ is continuous, bijective map such that ψ^{-1} is also continuous.

Example. $\psi : (-a, a) \to (-a^3, a^3)$, defined by $\psi(x) = x^3$ is a homeomorphism.

Example. $U_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \quad z > 0\}$. Consider $U_1 = \{(x, y) \mid x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ and the mapping

$$\psi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), \quad \forall\, (x, y) \in U_1.$$

Then this is a homeomorphism. Check.

Example. The spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$.

Example: The function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a homeomorphism.

Exercise: (i) Show that the open unit disc is homeomorphic to $\mathbb{R}^2$.
(ii) If D is the upper half of the unit disc, then show that $D \cup [-1, 1]$ is homeomorphic to the closed upper half plane.
Under homeomorphism many properties of a domain, are preserved.

Surface with bounday

Definition: A surface S is a surface without boundary if for every point P in S there exists an open set U in $\mathbb{R}^3$ such that $P \in U \cup S$ is homeomorphic to $\mathbb{R}^2$. A surface S is a surface with boundary if for every point P in S there exists an open set in $\mathbb{R}^3$ such that $P \in U \cap S$ is homeomorphic to either $\mathbb{R}^2$ or the closed upper half plane.

Example: Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2, 0 \leq z \leq h\}$, be a cylinder of height h. Then ∂S is the union of the following two sets
$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2, \quad z = 0\}$ and
$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2, \quad z = h\}$.
Example: Let $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = a^2\}$, a sphere. Then $\partial S = \varnothing = \text{empty}$. Why? A sphere, and a torus have no boundary.
What about an upper hemisphere?

Orientation of the boundary of a surface

Let S be an oriented surface with a boundary that is a simple, closed, non-singular parametrized curve (more generally, a disjoint union of simple, closed, piecewise non-singular parametrized curves). For instance, the cylinder of height h or the upper hemisphere.

Let an orientation of S be prescribed. How is the boundary of S oriented?

Suppose S is an oriented surface and let $\mathbf{n}(P)$ be the prescribed unit normal vector at a point $P \in S$. We choose the induced orientation of ∂S such that the surface lies to the left of an observer walking along the boundary ∂S with his head in the direction $\mathbf{n}(P)$.
The boundary of an oriented surface automatically acquires an orientation.

Note: If D is a path-connected subset on $\mathbb{R}^2$ and $\mathbf{\Phi} : D \to \mathbb{R}^3$ is a smooth non-singular orientation-preserving parametrization of the surface S, then $\mathbf{\Phi}(\partial D) = \partial S$ and the induced orientation of ∂S corresponds to the positive orientation of ∂D.

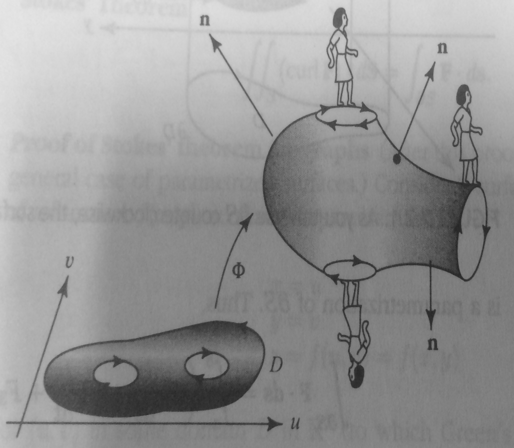


FIGURE 7.2.2. The boundary ∂S of D is oriented so that if you walk along ∂S with $\mathbf{n}$ your upright direction, the surface is on your left.

Example. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \quad z \geq 0\}$, the unit upper hemisphere. Let S be oriented by

$$\mathbf{n}(P) := (x, y, z), \quad \text{for} \quad P := (x, y, z) \in S.$$

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and
$\mathbf{\Phi}(x, y) := (x, y, \sqrt{1 - x^2 - y^2})$ for all $(x, y) \in D$. Note the boundary of the hemisphere S is the circle in x-y plane

$$\partial S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \quad z = 0\}.$$

The induced orientation ∂S by the oriented-parametrization $\mathbf{\Phi}$ corresponds to the counter clock-wise orientation of

$$\partial D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Stokes theorem

A surface S is called piecewise smooth if it is a finite union of smooth surfaces joining along smooth curves. Smooth here means n-times continuously differentiable for al n.

Theorem

1. *Let S be a bounded piecewise smooth oriented surface with non-empty boundary ∂S.*
2. *Let ∂S, the boundary of S, be the disjoint union of simple closed curves each of which is a piecewise non-singular parametrized curve with the induced orientation.*
3. *Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a C^1 vector field defined on an open set containing S.*

Then

$$\int_{\partial S} \mathbf{F}.d\mathbf{s} = \int\int_S (\nabla \times \mathbf{F}).d\mathbf{S}.$$

Remark: It is sufficient to assume the surface is C^2 for this theorem.

Remarks

- If two different oriented surfaces S_1 and S_2 have the same boundary C, then it follows from Stokes theorem that

$$\int\int_{S_1} (\nabla \times \mathbf{F}).\mathbf{dS} = \int\int_{S_2} (\nabla \times \mathbf{F}).\mathbf{dS},$$

  where the surfaces are correctly oriented.

- Green's theorem is the analogous version of Stokes theorem for the planar regions.

Stokes theorem for closed surface

Corollary

Let S be a closed oriented smooth surface in $\mathbb{R}^3$ (and so $\partial S = \emptyset$). Suppose $\mathbf{F}$ *is a smooth vector field on an open subset containing S. Then*

$$\iint_S (curl\boldsymbol{F})\cdot d\boldsymbol{S} = 0.$$

Proof: Introduce a hole in S by cutting out a small piece along a smooth simple closed curve C on S. Let S_1 denote the part of S cut out, and let S_2 denote the remaining part of S. Then

$$\iint_S (\mathrm{curl}\boldsymbol{F})\cdot d\boldsymbol{S} = \iint_{S_1} (\mathrm{curl}\boldsymbol{F})\cdot d\boldsymbol{S} + \iint_{S_2} (\mathrm{curl}\boldsymbol{F})\cdot d\boldsymbol{S}.$$

by the domain additivity.
Now the Stokes theorem shows that

$$\iint_{S_1} (\mathrm{curl}\boldsymbol{F})\cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F}\cdot d\boldsymbol{s} \text{ and } \iint_{S_2} (\mathrm{curl}\boldsymbol{F})\cdot d\boldsymbol{S} = \int_{\partial S_2} \boldsymbol{F}\cdot d\boldsymbol{s}.$$

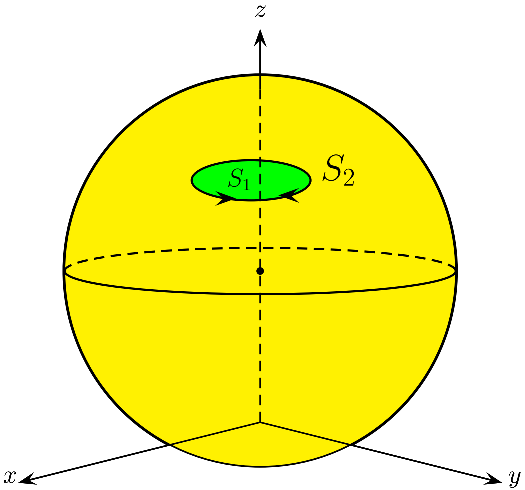


Figure: The Stokes theorem for S with $\partial S = \emptyset$.

We observe that the boundary ∂S_1 of S_1 is the closed curve C with the orientation induced by the orientation on S_1.

Since $\partial S = \emptyset$, the boundary ∂S_2 of S_2 is also C. But the orientations induced on C by the orientations on S_1 and on S_2 are opposite.

Hence

$$\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\partial S_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \boldsymbol{F} \cdot d\boldsymbol{s} - \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

$\square$

Examples.

Example Calculate

$$\oint_C ydx + zdy + xdz,$$

where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, for $b \neq 0, a \neq 0$, oriented counter clockwise as viewed from a point high upon the positive z-axis.

Ans: We have

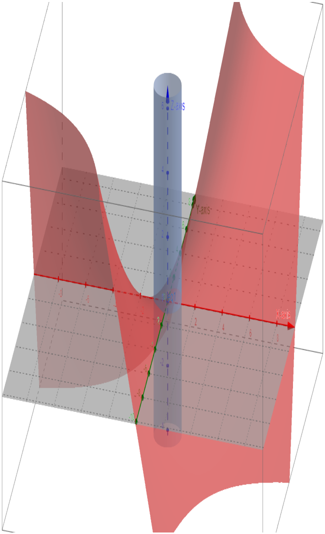
$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \quad \text{and} \quad \mathrm{curl}\ \mathbf{F} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Parametrize the surface lying on the hyperbolic paraboloid $z = xy/b$ and bounded by the curve C as

$$S = \{(x, y, z) \mid \mathbb{R}^3 \mid x^2 + y^2 \leq a^2 \quad z = \frac{xy}{b}\}$$

Then $\mathbf{n}\, dS = (-\frac{y}{b}, -\frac{x}{b}, 1)dxdy$ and

$$\iint_S \mathrm{curl}\ \mathbf{F} \cdot \mathbf{n}\, dS = \frac{1}{b} \iint_{x^2+y^2 \leq a^2} (y + x - b)dxdy$$
$$= \frac{1}{b} \int_0^{2\pi} \int_0^a (r \sin\theta + r\cos\theta - b) r dr d\theta = -\pi a^2.$$


Z-axis
Y-axis
X-axis

Examples

Example Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate

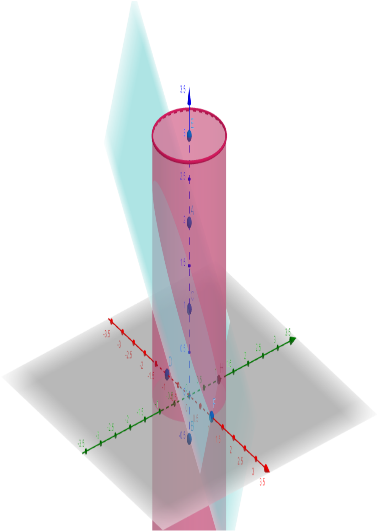
$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$

Ans: Use Stokes theorem.
Consider the surface give by the graph of $z = 1 - x - y$ over $x^2 + y^2 \leq 1$,

$$S = \{(x, y,) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \quad z = 1 - x - y\}.$$

S is enclosed by the curve C.
Check The unit normals to S is given by $\pm\frac{1}{\sqrt{3}}(1, 1, 1)$. To orient S positively so that we traverse C in the counterclockwise direction, we must choose

$$\mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1).$$



A more involved example

Example Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

where S is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The outward normal is used to orient S.

How does one proceed? One can do this directly as a surface integral or use Stokes' theorem but in either case the evaluation is quite tedious.

Idea: Change the surface, keeping the boundary (and its orientation) unchanged!

After all, Stokes' theorem does not care what surface is being bounded by the curve. The surface integral (no matter what the surface is) is equal to the line integral on the boundary.

Example contd.

With this idea in mind, we let C be the curve of intersection of the sphere and the plane $x + y + z = 1$, and we let S_1 be the region of this plane enclosed by C which is just a disc. We have to make sure that we orient S_1 so that C has the same orientation as in the given problem. The normals to S_1 are given by

$$\mathbf{n}_1 = \pm\frac{1}{\sqrt{3}}(1, 1, 1).$$

Which normal should we take for orienting S_1? Clearly $\frac{1}{\sqrt{3}}(1, 1, 1)$. Now $\nabla \times \mathbf{F} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 = -2\sqrt{3}$. Hence

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_1} -2\sqrt{3}dS = -2\sqrt{3}A(S_1)$$

where $A(S_1)$ is the surface area of the surface S_1 which we can easily compute!

Maxwell's equation

Let $\mathbf{E}$ and $\mathbf{H}$ be time-dependent electric and magnetic fields, respectively. One of Maxwell's equations is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}.$$

Let S be a surface with boundary C. Define

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \text{voltage drop around C}$$

and

$$\iint_S \mathbf{H} \cdot d\mathbf{S} = \text{magenetic flux across S.}$$

We will show that Faraday's Law can be derived from this equation of Maxwell.

Faraday's Law

Faraday's Law: The voltage (drop) around C equals the negative rate of change of magnetic flux through S.

Using Stokes' theorem

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S}.$$

Now we use Maxwell's equation to obtain

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \iint_S -\frac{\partial \mathbf{H}}{\partial t} \cdot d\mathbf{S}.$$

The key observation is that we can move the $\frac{\partial}{\partial t}$ across the integral sign. We can do this because the parameter t is independent of the variables dS occurring in the surface integral. This is a very useful trick called "differentiating under the integral sign".

We will not justify this step of differentiating under the integral sign. What we get is

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \iint_S -\frac{\partial \mathbf{H}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_S \mathbf{H} \cdot d\mathbf{S}.$$

And this is nothing but Faraday's Law.

Consequences of Stokes theorem

Proposition

Let $\boldsymbol{F}$ be a smooth vector field on an open subset D of $\mathbb{R}^3$ such that curl$\boldsymbol{F} = \mathbf{0}$ on D.

1. *Suppose S is a bounded oriented piecewise C^2 surface in D, and let ∂S denote its boundary with the induced orientation, as in the Stokes theorem. Then $\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.*

   In particular, if $\partial S = C_1 \cup (-C_2)$, so that C_1 and $-C_2$ have the induced orientation, then $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$.

2. *If F is a vector field defined on $\mathbb{R}^3$, then $\boldsymbol{F}$ is a gradient field on D.*

Closed surface

Can we have a generalized version of the divergence for of Green's theorem?

Yes!, Gauss's divergence theorem under suitable hypothesis on W, a region in $\mathbb{R}^3$.

We define a closed surface S in $\mathbb{R}^3$ to be a surface which is bounded, whose complement is open and boundary of S is empty. This is analogous to the closed curve.

If S is a closed surface, for example like sphere, then it encloses a 3-dimensional region. Call it W, and then S will be its boundary, ∂W.

This is analogous to a simple closed curve being boundary of a region D in $\mathbb{R}^2$.

Let us consider a region W in $\mathbb{R}^3$ which is simultaneously Type 1, Type 2, Type 3 and the boundary of the region as a subset of $\mathbb{R}^3$ is a closed surface. We call such region in $\mathbb{R}^3$ as simple solid region.

For example, regions bounded by ellipsoids, spheres, or rectangular boxes are simple solid regions.

We state the Gauss's divergence theorem for the simple solid regions.

Gauss's divergence theorem

If W is a simple solid region, W is a closed and bounded region in $\mathbb{R}^3$.

Theorem (Gauss's Divergence Theorem)

1. *Let W be a simple solid region of $\mathbb{R}^3$ whose boundary $S = \partial W$ is a closed surface.*
2. *Suppose ∂W is positively oriented.*
3. *Let $\boldsymbol{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing W.*

Then

$$\iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_{W} (div\boldsymbol{F}) dxdydz.$$

Clearly, the importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.