MA 109 Final exam review

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General Advice

1. Concentrate on understanding the statements of the theorems. You will not be asked to reproduce long proofs.
2. When trying to understand a definition, make sure you know plenty of examples.
3. When trying to understand a theorem, make sure you know counter-examples to the conclusion of the theorem when you drop some of the hypotheses.
4. In general, the statement of the theorem is more important than its proof. And examples might be more important than theorems!

Limits of sequences

1. Learn the definition.
2. When proving a fact/theorem/etc. about some limit being l start with an $\epsilon > 0$ and find an N so that the sequence x_n you are dealing with satisfies
$$|x_n - l| < \epsilon,$$
for every $n > N$.
3. To prove that a sequence does not converge you have to show that no real number can be a limit. Thus you must take an arbitrary l and find some fixed $\epsilon > 0$ - this ϵ can be chosen to your convenience so that $|a_n - l| > \epsilon$ for infinitely many n.
4. Theorems to remember for showing that limits exist: the sum, difference, product and quotient and the Sandwich Theorem. In this case you will already know that some sequence has a limit and deduce that another sequence has a limit by comparing it to the known one.

Theorems that abstractly guarantee that the limit of a sequence exists:

A monotonically increasing/decreasing sequence bounded above/below converges to its supremum/infimum.

Every Cauchy sequence converges. It is a good idea to know the definition of a Cauchy sequence. However, you will not be asked questions on Cauchy sequences.

Unless we explicitly mention that you must use the ϵ-N definition to prove that a limit exists, you do not have to. You may use the rules for limits and other theorems instead. You can use simple facts without proving them: e.g. $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$ if $\alpha > 0$.

Exercise 1

If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} \sqrt{a_n} = 0$ using epsilon-N definition.

Solution: Fix $\epsilon > 0$ arbitrarily. We want to find an N such that $n \geq N \implies |\sqrt{a_n}| < \epsilon$.

Since $\lim_{n\to\infty} a_n = 0$, for ϵ^2, the square of ϵ that we fixed earlier, there exists N_1 such that

$$n \geq N_1 \implies |a_n| < \epsilon^2.$$

Then for $N = N_1$ we have

$$n \geq N \implies |\sqrt{a_n}| < \epsilon.$$

Limits of functions

The ideas behind proving or disproving the existence of limits are the same as for sequences (of course, there is no analogue of monotonic bounded sequences or Cauchy sequences).

You can use the basic limits you learnt in 11th/12th standard like $\lim_{x\to 0} \sin x/x = 1$.

Remember that there is a nice algebra for limits, and sandwich/squeeze theorems still apply.

At "endpoints" of intervals, one can make sense of right-handed or left-handed limits.

Continuity

Of course, you have to know the definition. You may use basic facts about limits of functions to prove what you want.

Two basic theorems are:

1. A continuous function on a closed bounded interval is bounded and attains its infimum and supremum
2. Continuous functions have the IVP (remember again, that the converse is not true)

   The sum, difference, product etc. of continuous functions is continuous. The composition of continuous functions is continuous.

Differentiation

Know the definition. Again, here you can use the basic facts about limits.

The basic theorems are:

1. Fermat's Theorem,
2. Rolle's theorem and the MVT,
3. Darboux's theorem.

Know the basic examples and counter-examples: a function that is continuous but not differentiable, a function that is differentiable but not continuously differentiable.

Exercise 2

Show that $x^3 - 10x + 4$ has three real roots.

Solution: Let $f(x) = x^3 - 10x + 4$. Then $f_x = 3x^2 - 10$ which has two roots, namely, $\pm\sqrt{10/3}$.

By the second derivative test we find that $-\sqrt{10/3}$ is a local maximum for f and $\sqrt{10/3}$ is a local minimum.
Since we have only two critical points, it follows that

$$f(-\sqrt{10/3}) > 0 > f(\sqrt{10/3}).$$

By the IVP of f, there exists a zero of f in the interval $(-\sqrt{10/3}, \sqrt{10/3})$.
Since the given function is a cubic, $f(x) \to \pm\infty$ as $x \to \pm\infty$, hence again by IVP we get two more zeros of f in the intervals $(-\infty, -\sqrt{10/3})$ and $(\sqrt{10/3}, \infty)$.

Alternate solution: Show that $f(x)$ changes sign three times. Note that $f(-10) < 0$, $f(-1) > 0$, $f(1) < 0$ and $f(10) > 0$.

Exercise 3

Show that the function $x^4 + 3x + 1$ has exactly one zero in the interval $[-2, -1]$.

Solution: By observing that $f(-2) > 0$ and $f(-1) < 0$, we conclude by IVP that f has a zero in the interval $[-2, -1]$.

Further, the derivative, $4x^3 + 3$, is non-zero on $[-2, -1]$, so by Rolle's theorem, f has no more zeros in the given interval.

Maxima, minima, convex, concave

Remember that the definitions of maxima, minima, concavity, convexity, inflection points etc. have nothing to do with differentiation.

IF the function is (twice) differentiable then one can apply the various derivative tests. Otherwise, one can't.

Note that the existence of maxima and minima usually follows from the fact that we are dealing with continuous functions on a closed bounded interval.

Remember the difference between supremum and maximum (and of course, between infimum and minimum - know the relevant examples).

Taylor's theorem: Know how to compute the Taylor polynomials. Know the form of the Remainder term. Recall that there are smooth functions for which the Taylor series about a point converges but does not converge to the function ($e^{-1/x}$).

Exercise 4

Find the first three terms of the Taylor series of the function $1/x^2$ at 1.

Solution: If the Taylor series of the function f at $x = a$ is $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$,

then $a_n = \dfrac{f^{(n)}(a)}{n!}$.

Using these notations, for $f(x) = 1/x^2$ and $a = 1$, we get $a_0 = 1$, $a_1 = -2$ and $a_2 = 3$.

Integration

Remember what partitions and tagged partitions are.

Recall the definitions of the (Darboux) lower sums, upper sums, lower integrals, upper integrals and Riemann sums.

Learn all three definitions of the Riemann integral.

Basic fact: Bounded functions on closed intervals with at most a finite number of discontinuities are Riemann/Darboux integrable.

The Fundamental Theorem of calculus.

Exercise 5

3. For the function $f(x) = 3x^2$ and the partition

$$P_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1\right\}$$

of $[0,1]$ find the lower sum, $L(f, P_n)$, upper sum, $U(f, P_n)$. Compute $\sup_n L(f, P_n)$ and $\inf_n U(f, P_n)$.

Solution:

$$L(f, P_n) = \sum_{i=0}^{n-1} 3\frac{i^2}{n^2}\frac{1}{n} = 3\frac{1}{n^3}\frac{n(n-1)(2n-1}{6}$$

So

$$L(f, P_n) = \frac{2n^2 - 3n + 1}{2n^2} \quad \text{and} \quad U(f, P_n) = \frac{2n^2 - 3n + 1}{2n^2}$$

and

$$\sup_n L(f, P_n) = 1 \quad \text{and} \quad \inf_n U(f, P_n) = 1.$$

Exercise 6

Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{i^2+n^2}$ by identifying it as a Riemann sum for a certain continuous function on a certain interval and with respect to a certain partition.

Solution: We observe that

$$\sum_{i=1}^{n} \frac{n}{i^2+n^2} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{(i/n)^2+1}.$$

Thus, the given sum is the Riemann sum for the function $\dfrac{1}{x^2+1}$ over the interval $[0,1]$ with respect to the partition $0 < \dfrac{1}{n} < \dfrac{2}{n} < \cdots < \dfrac{n-1}{n} < 1$.

Since the function $1/(1+x^2)$ is continuous on $[0,1]$, it is Riemann integrable.

Hence the limit of the given sum is $\displaystyle\int_0^1 \frac{1}{x^2+1}dx = \pi/4$.

Multivariable Calculus

Functions from $\mathbb{R}^2/\mathbb{R}^3 \to \mathbb{R}$: recall what limits and continuity mean for these functions.

Remember what partial and directional derivatives are. A function may have both partial derivatives or even all directional derivatives and still not be continuous (look at the various examples in Tutorial sheet 6- especially 6.1-6.5).

Learn the definition of differentiability for functions of 2 and 3 variables. Learn the definition of the derivative matrix.

The main point is that if a function has continuous partial derivatives in the neighbourhood of a point, then it is differentiable at that point.

Exercise 7

4. Using the epsilon-delta definition of the limit show that $f(x) + g(y)$ is a continuous function of two variables if $f(x)$ and $g(x)$ are continuous functions of one variable.

Answer: Let $(a, b) \in \mathbb{R}^2$ and ϵ be positive.
Since f is continuous at a and g is continuous at b, for $\epsilon/2$ we get $\delta_f > 0$ and $\delta_g > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \epsilon/2$$

and

$$|y - b| < \delta_g \implies |g(y) - g(b)| < \epsilon/2.$$

Now define $\delta = \min\{\delta_f, \delta_g\}$, then it is clear that

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |x - a| < \delta_f \text{ and } |y - b| < \delta_g.$$

Hence,

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies \left|\big(f(x) + g(y)\big) - \big(f(a) + g(b)\big)\right| < \epsilon$$

Exercise 8

Show from first principles that the functions $f(x,y) = xy$ is differentiable at every point in $\mathbb{R}^2$ and find its Derivative matrix $Df(x,y)$.

Solution: We note that $f_x(x_0,y_0) = y_0$ and $f_y(x_0,y_0) = x_0$. Note that

$$f(x_0+h, y_0+k) = (x_0+h)(y_0+k) = x_0y_0 + hy_0 + kx_0 + hk.$$

$$f(x_0+h, y_0+k) - f(x_0,y_0) - hf_x(x_0,y_0) - kf_y(x_0,y_0) = hk.$$

Since $|hk| \leq (h^2+k^2) = \|(h,k)\|^2$, we see that

$$0 \leq \lim_{(h,k)\to 0} \frac{|f(x_0+h, y_0+k) - f(x_0,y_0) - hf_x(x_0,y_0) - kf_y(x_0,y_0)|}{\|(h,k)\|}$$

$$= \lim_{(h,k)\to 0} \frac{\|(h,k)\|^2}{\|(h,k)\|} = \lim_{(h,k)\to 0} \|(h,k)\| = 0.$$

we see that $f(x,y)$ is differentiable at all point in $\mathbb{R}^2$ and that its derivative is is the 1×2 matrix $(y_0 \ x_0)$.

The gradient

The gradient controls a lot of information about the function and its graph.

If we take $\nabla f \cdot u$ for a unit vector u, we get $\nabla_u f$, the directional derivative of f which measures the rate of change of f in the direction u.

The function $f(x, y, z)$ increases fastest in the direction of the gradient.

If $f(x, y, z) = c$ is a surface, $\nabla f(x_0, y_0, z_0)$ represents a vector normal to the surface at a point (x_0, y_0, z_0) on the surface. We can thus easily derive the equation of the tangent plane to the surface.

Make sure you have understood the Chain rule.

Random problems

State whether the following statements are True or False with a few lines of reasoning or with a counter-example, as the case may be. If you give a counter-example, no further justification is necessary.

1.

$$\left(\frac{\pi}{22}\right)\cos\left(\frac{\pi}{22}\right)+\left(\frac{2\pi}{11}\right)\cos\left(\frac{5\pi}{22}\right)+\left(\frac{2\pi}{11}\right)\cos\left(\frac{9\pi}{22}\right)+$$
$$\left(\frac{\pi}{22}\right)\cos\left(\frac{5\pi}{11}\right)<\left(\frac{\pi}{26}\right)+\left(\frac{3\pi}{13}\right)\cos\left(\frac{\pi}{26}\right)+\left(\frac{3\pi}{13}\right)\cos\left(\frac{7\pi}{26}\right)$$

2. If the sequence a_n is convergent and b_n is monotonically increasing and bounded, then there exists some $N \in \mathbb{N}$ such that $a_n b_n < a_{n+1} b_{n+1}$ for all $n > N$.

3. The largest value of δ such that the inequality $|x^2 - 1| < 10^{-100}$ holds for all $x \in (1-\delta, 1+\delta)$ is $\delta = 10^{-101}$.