Tutorial - 1

1. Eigenvalue eqⁿ: $c^2\Psi = \Psi$ —— (1)

If we consider an eigenvalue eqⁿ; $c\Psi = a\Psi$ [a = Eigenvalue]

then from eqⁿ (1) –

$$c^2\Psi = \hat{c}.\hat{c}.\Psi = \hat{c}.a\Psi = a\hat{c}\Psi = a^2\Psi.$$

So, comparing the eqⁿ; $a^2 = 1$; $a = \pm 1$.

Eigenvalues of operator $\hat{c}$ are ± 1.

Another approach:

Eigenvalue eqⁿ: $c^2\Psi = \Psi$

$$\Rightarrow \det(c^2 - I) = 0.$$

$$\Rightarrow \det(c - I).\det(c + I) = 0.$$

Eigenvalues of operator $\hat{c}$ are ± 1.

2. i) Eigenvalue eqⁿ given: $A\Psi = a\Psi$.

$$\hat{A} = -i\hbar.\partial/\partial q$$

Suitable eigenfunction for this op, $\Psi = \exp(aiq)$.

So,

$$\left(-i\hbar\, \partial/\partial q\right)\exp(aiq) = (-i\hbar).ai\,\exp(aiq).$$

$$= (a\hbar)\exp(aiq).$$

It satisfies the eigenvalue eqⁿ with the eigenvalue $a\hbar$.

2. ii) $\hat{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

Suitable wavefn for this operator, $\Psi = \exp[\pm i(ax+by+cz)]$.

So,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \exp[\pm i(ax+by+cz)]$$

$$= \pm(a^2+b^2+c^2)\exp[\pm i(ax+by+cz)]$$

The eigenvalue is $\pm(a^2+b^2+c^2)$.

iii) $\hat{A} = \frac{1}{\sin\theta} \cdot \frac{d}{d\theta}\left(\sin\theta \cdot \frac{d}{d\theta}\right)$

Suitable eigenfn for this operator, $\Psi = a\cos\theta$ (a = Const.).

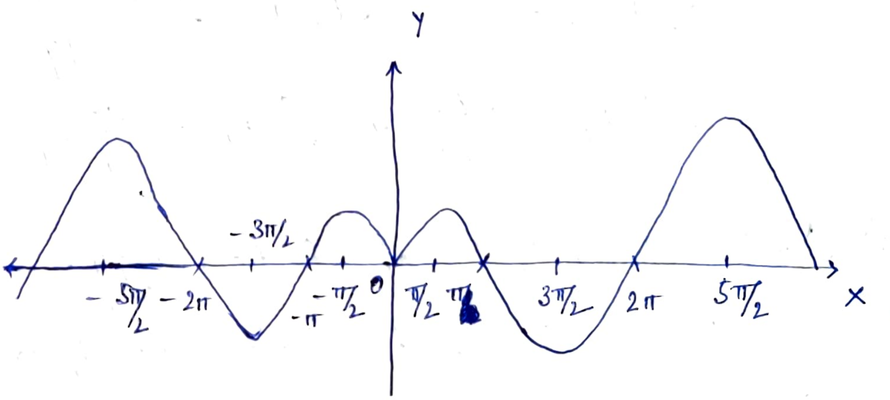
So, $\frac{1}{\sin\theta} \cdot \frac{d}{d\theta}\left(\sin\theta \cdot \frac{d}{d\theta}\right)(a\cos\theta)$

$$= \frac{1}{\sin\theta}\left[\cos\theta \cdot \frac{d}{d\theta} + \sin\theta \cdot \frac{d^2}{d\theta^2}\right](a\cos\theta)$$

$$= \frac{\cos\theta}{\sin\theta} \cdot \frac{d}{d\theta}(a\cos\theta) + \frac{d^2}{d\theta^2}(a\cos\theta) = -2a\cos\theta.$$

3. i) $y = x\sin x$.

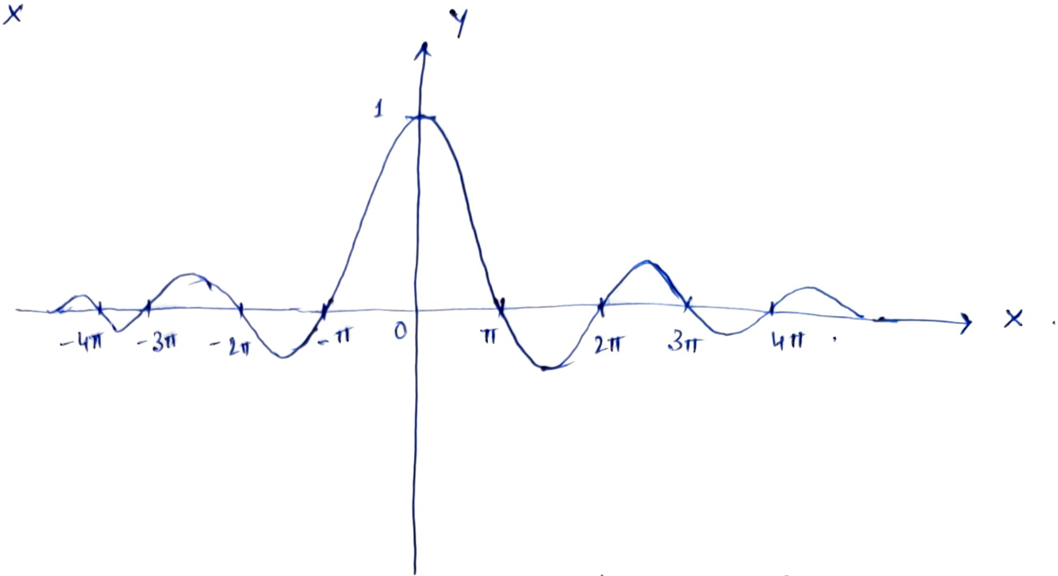
X	Y
0	0
$\pi/2$	$\pi/2$
π	0
$3\pi/2$	$-3\pi/2$
2π	0.



at $x = \infty$: $y = \infty$; So area under the curve becomes infinite; So it is not square integrable.

This is not an acceptable wavefunction.

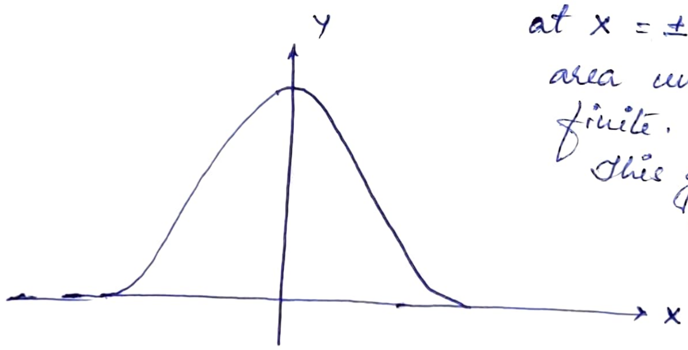
3. ii) $y = \frac{1}{x} \cdot \sin x$.



at $x = 0$; the area under the curve becomes infinite but if we remove that ; the curve becomes continuous. It is known as removable discontinuity.

This is an acceptable wavefunction.

iii) $y = e^{-x^2}$

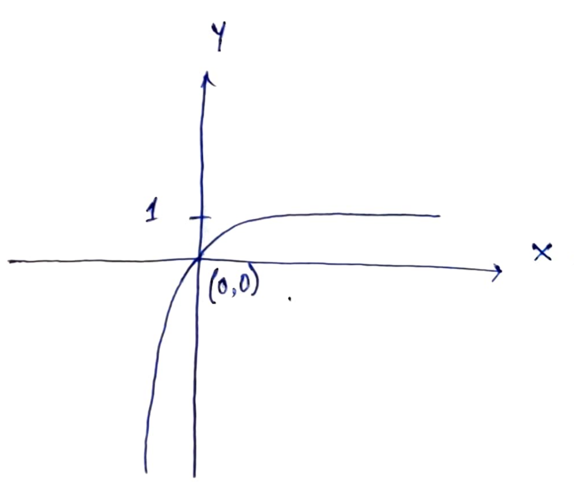


at $x = \pm\infty$; $y = 0$
area under the curve is finite.
This guassian f^n is an acceptable wavefunction.

iv) $y = 1 - e^{-x}$.

at $x = 0$; $y = 0$
$x = \infty$; $y = 1$
but $x = -\infty$; $y = -\infty$.

Area under the curve is infinite :
This wavefunction is not an acceptable wavef^n.



4. Let, the eigenfunctions of the quantum mechanical operator $\hat{A}$ be $\psi_1, \psi_2 \dots \psi_n$. So that -

$\hat{A}\psi_1 = a_1\psi_1$; $\hat{A}\psi_2 = a_2\psi_2$ ... ; $\hat{A}\psi_n = a_n\psi_n$. —— (1)

(Where $a_1, a_2, \dots a_n$ are the corresponding eigenvalues).

Linear Combination of the eigenf^n

$= (c_1\psi_1 + c_2\psi_2 + \dots + c_n\psi_n) = \sum_i c_i\psi_i$

So, $\hat{A}(c_1\psi_1 + c_2\psi_2 + \dots + c_n\psi_n)$

$= c_1\hat{A}\psi_1 + c_2\hat{A}\psi_2 + \dots + c_n\hat{A}\psi_n$

$= a_1c_1\psi_1 + a_2c_2\psi_2 + \dots + a_nc_n\psi_n$.

The eq^n holds only when all the eigenvalues are equal i.e.

$\boxed{a_1 = a_2 = \dots = a_n}$

5. $\psi(x) = \frac{1}{2}\phi_1(x) + \frac{1}{4}\phi_2(x) + \frac{3+\sqrt{2}i}{2}\phi_3(x)$.

(a) For $\psi(x)$ to be normalized, the following condition needs to be satisfied :-

$\langle\psi|\psi\rangle = 1$; Now, ϕ_1, ϕ_2, ϕ_3 are orthogonal to each other ; thus $\sum_i |c_i|^2 = 1$.

$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{3+\sqrt{2}i}{2}\right)\left(\frac{3-\sqrt{2}i}{2}\right)$

$= \frac{1}{4} + \frac{1}{16} + \frac{11}{4} = \frac{49}{16} \neq 1$.

So, $\psi(x)$ is not normalized.

(b) $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are the wave function of the kinetic energy of which are orthogonal to each other. So upon measurement the wavef^n will collapse to the eigenstates. As a result, energies corresponding to those states which are observable quantities i.e. E_1, $3E_1$ and $7E_1$ will be obtained.

5. (c) As the wavef^n is not normalized, normalization const —

$$c^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \frac{(3+\sqrt{2}i)(3-\sqrt{2}i)}{4}$$

$$= \frac{1}{4} + \frac{1}{16} + \frac{11}{4} = \frac{49}{16}$$

$$c = \left(\frac{7}{4}\right).$$

So, the normalized wavefunction,

$$\Psi_N(x) = \frac{4}{7}\left(\frac{1}{2}\phi_1(x) + \frac{1}{4}\phi_2(x) + \frac{3+\sqrt{2}i}{2}\phi_3(x)\right).$$

$$= \frac{2}{7}\phi_1(x) + \frac{1}{7}\phi_2(x) + \left(\frac{6+2\sqrt{2}i}{7}\right)\phi_3(x).$$

i) Average value of kinetic energy,

$$\langle K \rangle = \langle \Psi_N | K | \Psi_N \rangle$$

$$= \left\langle \left(\frac{2}{7}\phi_1(x) + \frac{1}{7}\phi_2(x) + \left(\frac{6+2\sqrt{2}i}{7}\right)\phi_3(x)\right| K \right| \left(\frac{2}{7}\phi_1(x) + \frac{1}{7}\phi_2(x) + \left(\frac{6+2\sqrt{2}i}{7}\right)\phi_3(x)\right).$$

Now, as $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are orthogonal to each other and ~~wavef~~^{n} eigenf^n of K.E. op.

$$= \left(\frac{2}{7}\right)^2 E_1 + \left(\frac{1}{7}\right)^2 3E_1 + \frac{(6+2\sqrt{2}i)(6-2\sqrt{2}i)}{49} \cdot 7E_1$$

$$= \frac{4}{49}E_1 + \frac{3E_1}{49} + \frac{44}{49}E_1 = \frac{45}{7}E_1.$$

ii) Most probable value of kinetic energy.

Probability corresponding to the ϕ_1 state $= \left(\frac{2}{7}\right)^2 = \frac{4}{49}$

" " ϕ_2 " $= \left(\frac{1}{7}\right)^2 = \frac{1}{49}$

" " ϕ_3 " $= \frac{(6+2\sqrt{2}i)(6-2\sqrt{2}i)}{49}$

$$= \frac{44}{49}.$$

So, the most probable value will be associated to the ϕ_3 state and $7E_1$ will be the MPV of kinetic energy.