MA-111 Calculus II (D3 & D4)

Lecture 5

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Evaluating integrals over Elementary regions

The integral in polar coordinate

Recap

- If D is a bounded region in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is bounded, then consider any rectangle R containing the region D in $\mathbb{R}^2$ and extend f to the rectangle by 0 outside D and denote it by f^*. The integral of f over D is defined by the integral of f^* on the rectangle R.
- The above definition is consistent because the definition of integral of f on D is independent of the choice of rectangle R.
- To determine the integrability of f over region D, conditions on f and D? The boundary of D should be 'well-behaved'. The set containing points of discontinuity of f is of 'content zero'.
- Algebraic properties of integrals on D are similar to that of the integrals on rectangle.

We will now discuss two types of regions for which D is a bounded set in $\mathbb{R}^2$ with its boundary ∂D of content zero and the integral can be evaluated easily.

We will describe two simple types of regions known as *elementary regions*.

Existence of Integrals over bounded sets in $\mathbb{R}^2$

Theorem

Let $D \subset \mathbb{R}^2$ be a bounded set whose boundary ∂D is given by the finitely continous closed curve then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable over D.

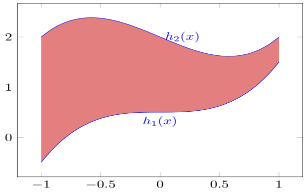
Example. Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $f(x, y) = x^2 + y^2, \quad \forall (x, y) \in D$. Then f is integrable over D.

A slightly more general theorem is as follows:
Let D be a bounded set in $\mathbb{R}^2$ such that ∂D is of content zero. Let $f : D \to \mathbb{R}$ be a bounded function whose points of discontinuity have 'content zero'. Then f is integrable over D.

Elementary region: Type 1

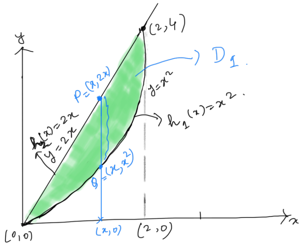
Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions such that $h_1(x) \le h_2(x)$ for all $x \in [a, b]$. Consider the set of points

$$D_1 = \{(x, y) \mid a \le x \le b \text{ and } h_1(x) \le y \le h_2(x)\}.$$



Such a region is said to be of *Type 1* and for every $x \in \mathbb{R}$ vertical cross-section of D_1 is an interval.

Example. $D_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 2, \quad x^2 \le y \le 2x\}$. Here for all $x \in [0, 2]$, $h_1(x) = x^2$ and $h_2(x) = 2x$. Note $h_1(x) \le h_2(x)$ for $x \in [0, 2]$.

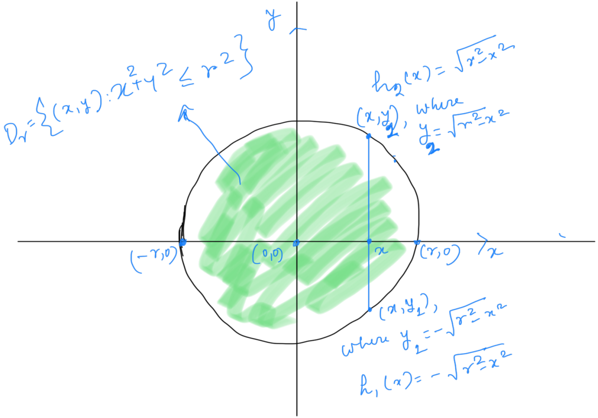


Type 1 contd.

Example. The closed disc D_r of radius r around the origin,

$$D_r := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}.$$

Take $h_1(x) = -\sqrt{r^2 - x^2}$ and $h_2(x) = \sqrt{r^2 - x^2}$. We see that D_r is of Type 1.



For Type 1, let $D_1 = \{(x, y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$, where h_1 and h_2 are continuous.

The region D_1 is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$ for $x \in [a, b]$).

Thus ∂D_1 is of 'content zero' in $\mathbb{R}^2$.

Hence any continuous function defined on D_1 is integrable over the elementary region D_1.

Evaluating integrals on regions of Type 1

Let D be a region of Type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let f^* be the corresponding function on R (obtained by extending f by zero).

The region D is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$). Hence we can conclude that f^* is integrable on R. Applying Fubini's theorem on f^* we get,

$$\int\int_D f(x,y)dxdy := \int\int_R f^*(x,y)dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta f^*(x,y)dy\right] dx.$$

In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} f^*(x,y)dy\right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx,$$

since $f^*(x,y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$. Finally, we get

$$\int\int_D f(x,y)dxdy = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx.$$

Examples

Example Let $D = \{(x,y) \mid 0 \le x \le 2, \quad x^2 \le y \le 2x\}$ and $f(x,y) = x + y$. Find $\int\int_D f(x,y)dxdy$.

Ans Note D is a bounded set in $\mathbb{R}^2$ enclosed by the graphs of the curves $y = x^2$ and $y = 2x$ and hence ∂D is of content zero. Since f is continuous over D and D is bounded with ∂D of content zero, f is integrable over D.

$$\int\int_D f(x,y)\,dxdy = \int_0^2 \Big(\int_{x^2}^{2x} (x+y)\,dy\Big)dx = \int_0^2 [xy + \frac{y^2}{2}]_{y=x^2}^{y=2x}\,dx$$
$$= \int_0^2 [2x^2 + 4\frac{x^2}{2} - x^3 - \frac{x^4}{2}]\,dx$$

Example Let $D = \{(x,y) \mid x^2 + y^2 \le 1, \quad x \ge 0, \quad y \ge 0\}$ and $f(x,y) = \sqrt{1-y^2}$. Find $\int\int_D f(x,y)dxdy$.

Ans Type 1, i.e, $D = \{(x,y) \mid 0 \le x \le 1, \quad 0 \le y \le \sqrt{1-x^2}\}$. Then

$$\int\int_D f(x,y)\,dxdy = \int_0^1 \Big(\int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\Big)\,dx.$$

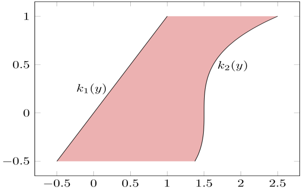
Not easy to compute!

Elementary region: Type 2

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions such that $k_1(y) \leq k_2(y)$, for all $y \in [c, d]$. The set of points

$$D_2 = \{(x, y) \,|\, c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$

is called a region of Type 2 and for every $y \in \mathbb{R}$ horizontal cross-section of D_2 is an interval.



Example $D_2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$. If we take $k_1(y) = -\sqrt{1 - y^2}$ and $k_2(y) = \sqrt{1 - y^2}$, we see that D_2 is of Type 2.

Evaluating integrals on regions of type 2

Note that the boundary of D_2 is of content zero in $\mathbb{R}^2$. Hence any continuous function defined on D_2 is integrable over the elementary region.

Using exactly the same reasoning as in the previous case (basically, interchanging the roles of x and y) we can obtain a formula for regions of Type 2.

Let D be a bounded set of Type 2 in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be a continuous function on D. We get

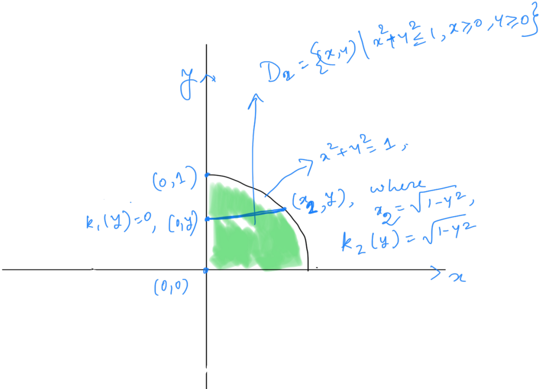
$$\int\int_D f(x,y)dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)dx\right] dy.$$

Example Let $D = \{(x, y) \mid x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0\}$. Evaluate the integral

$$\int\int_D \sqrt{1-y^2}\,dxdy.$$

Ans.

$$\begin{aligned}\int\int_D \sqrt{1-y^2}\,dxdy &= \int_0^1 \Big(\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\Big)\,dy \\ &= \int_0^1 [x\sqrt{1-y^2}]_{x=0}^{\sqrt{1-y^2}}\,dy = \int_0^1 (1-y^2)\,dy = \frac{2}{3}.\end{aligned}$$



Remark

Both of these formulæ can be viewed as special cases of Cavalieri's principle when $f(x, y) \geq 0$. In the first case we are slicing by planes perpendicular to the x-axis, while in the second case, we are slicing by planes perpendicular to the y-axis.

Caution! There exist bounded subsets of $\mathbb{R}^2$ which are not elementary regions; for example, *star-shaped subset* of $\mathbb{R}^2$ or an *annulus*.

Often we can write D as a union of regions of Types 1 and 2 and then we call it a region of type 3.

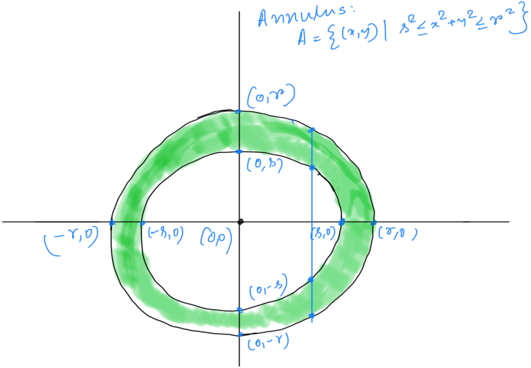
Note the Domain Additivity theorem will then allow us to evaluate integrals which are defined over finite union of such sets.

We could also view the disc as a region of type 3, by dividing it into four quadrants.

Remark contd.

What about the *annulus* $A = \{(x, y) \in \mathbb{R}^2 \mid s^2 \leq x^2 + y^2 \leq r^2\}$?

Is it a type 3 region? yes



Example1: Compute the integral of $f(x, y) = x^2 + y^2$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we compute this integral using iterated integrals ?

Example2: Compute the integral of $g(x, y) = e^{x^2+y^2}$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we use substitution like we did in one variable?

Let us see what happens when we use polar coordinates.

Polar Coordinates

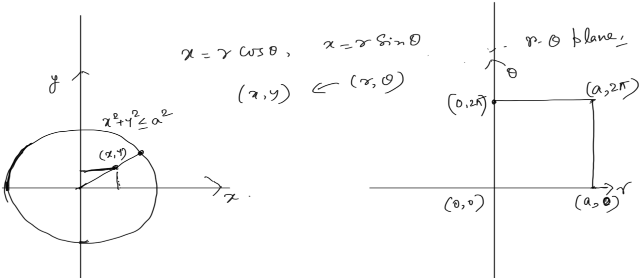
Change of variables from Cartesian coordinate system to polar coordinate system, any $(x, y) \in \mathbb{R}^2$ in Cartesian coordinate can be written as

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r > 0, \theta \in [0, 2\pi].$$

Transformation of region under change of variables:

Ex. $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}$ is transformed in polar coordinate system as a rectangle

$$D^* = \{(r, \theta) \mid 0 \leq r \leq a, \quad \theta \in [0, 2\pi]\}.$$



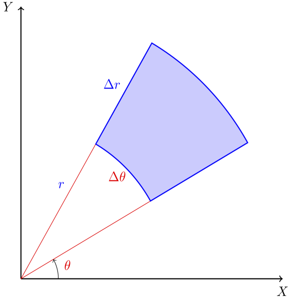
The integral in polar coordinates

Let D^* be a subset of $\mathbb{R}^2$ in polar coordinate system, such that for all $(r, \theta) \in D^*$, $(r\cos(\theta), r\sin(\theta)) \in D$, for $0 \leq r \leq 1$, and

$$g(r, \theta) := f(r\cos(\theta), r\sin(\theta)), \quad (r, \theta) \in D^*.$$

To integrate the function g on a domain D^* we need to cut up D^* into small rectangles, but these will be rectangles in the r-θ coordinate system.

What shape does a rectangle $[r, r + \Delta r] \times [\theta, \theta + \Delta\theta]$ represent in the x-y plane? A part of a sector of a circle.



Then we will be integrating over this sector instead of rectangle.

What is the area of this part of a sector?

Ans: It is $\frac{1}{2} \cdot [(r+\Delta r)^2 \Delta\theta - r^2 \Delta\theta] \sim r^* \Delta r \Delta\theta$, $r \leq r^* \leq r + \Delta r$.

Partitioning the region into subrectangles is equivalent to partitioning the region into parts of sectors as shown earlier.

It follows that the integral we want is approximated by a sum of the form

$$\sum_i \sum_j g(r_i^*, \theta_j^*) r_i^* \Delta r_i \Delta\theta_j,$$

where $\{(r_i^*, \theta_j^*\}$ is a tag for the partition of the "rectangle" in polar coordinates and

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta) r dr d\theta,$$

where D is the image of the region D^*.

This is the change of variable formula for polar coordinates.

Examples

Example1: Integrate $f(x,y) = x^2 + y^2$ on $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$.

Solution: Let us use polar coordinates. Let

$$D^* = \{(r,\theta) \mid 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi\}.$$

Denoting $x = r\cos\theta$ and $y = r\sin\theta$, the polar coordinates will transform D^* to D and

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = r^2.$$

$$\begin{aligned} \int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} r^2.r\ drd\theta \\ &= \int_0^{2\pi}\int_0^1 r^3\ drd\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1\ d\theta = \frac{\pi}{2} \end{aligned}$$

Examples contd.

Example 2: Integrate $f(x,y) = e^{x^2+y^2}$ on $D = \{(x,y) \mid x^2+y^2 \le 1\}$.
Solution: Using the same transformation as above

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get

$$\begin{aligned}
\int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} e^{r^2} r\ drd\theta \\
&= \int_0^{2\pi}\int_0^1 e^{r^2} r\ drd\theta = \int_0^{2\pi} \frac{e^{r^2}}{2}\Big|_0^1\ d\theta = \pi(e-1)
\end{aligned}$$

An Applicaton:The integral of the Gaussian

We would like to evaluate the following integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

What does this integral mean? - so far we have only looked at Riemann integrals inside closed bounded intervals, so the end points were always finite numbers a and b.

An integral like the one above is called an improper integral. We can assign it a meaning as follows. It is defined as

$$\lim_{T \to \infty} \int_{-T}^{T} e^{-x^2} dx,$$

provided, of course, this limit exists. We will see how to evaluate this.

The most amazing trick ever

Consider

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy.$$

We view this product as an iterated integral!

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

Now under polar coordinates, the plane is sent to the plane. Hence, we can write this as

$$\int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2} rdr \right] d\theta.$$

But we can now evaluate the inner integral. Hence, we get

$$\int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right] d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

The answer

Since $I^2 = \pi$, we see that $I = \sqrt{\pi}$.

Using the above result you can easily conclude that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

The integral above arises in a number of places in mathematics - in probability, the study of the heat equation, the study of the Gamma function (next semester) and in many other contexts.

There are many other ways of evaluating the integral I, but the method above is easily the cleverest.