MA-111 Calculus II (D3 & D4)

Lecture 1

B. K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

January 24, 2022

Welcome to course MA 111!

- The lectures and interaction session will be held on Zoom. The Zoom link has been shared via Moodle.
  Moodle website:
  `https://moodle.iitb.ac.in/login/index.php`
- The common lectures for both $D3$ and $D4$ in each week:

Division	Lecture Slot	Timing
D3 and D4	2A	Monday: 9:30 a.m. - 10:25 a.m.
	2B	Tuesday: 10:35 a.m. - 11:30 a.m.
	2C	Thursday: 11:35 a.m. - 12:30 p.m.

- All lectures will be recorded. The recordings will be uploaded to either Zoom or Google Drive. The link for videos will be shared via Moddle. You can download them according to your convenience.
- Slides of the lectures will be uploaded to Moodle for MA111.
- During these common lectures, interaction with students will be a bit difficult. On Zoom, you may 'raise your hand'. At the end of the class, your questions can be taken by the instructor.
- That is why a separate interaction session is scheduled.

- The following slots will be used for interactions as and when needed. Interaction session:

Division	Interaction Slot	Timing
D3	9B	Thursday: 3:30 p.m. - 4:30 p.m.
D3	11B	Friday: 3:30 p.m. - 4:30 p.m.
D4	4C	Thursday: 9:30 a.m. - 10:25 a.m.
D4	9B	Thursday: 3:30 p.m. - 4:30 p.m.

- Tutorials: D3 and D4: 3:00 p.m -4:00 p.m on Wednesdays. Every week a set of tutorial problems will be assigned and posted on the Moodle class page. *Please attend the tutorial section assigned to you. This is same as your batch from MA 109.* Your TA will discuss some of these problems. You are advised to try problems in advance and use this time to ask questions and doubts.
- If you see any mistakes and inconsistencies in slides, please let me know.
- To contact me, you can send an email on Moodle.

Evaluation policy

There is a total of 100 marks to be earned in this course. The following breakup is tentative*.

Quiz	40 marks
Final	60 marks
Total	100 marks

The platform for the quizzes and final exam along with other details will be announced later during the course.

Academic Honesty: It is obligatory on your part to be honest and not to violate the academic integrity of the Institute. Any form of academic dishonesty, including, but not limited to cheating, plagiarism, submitting as one's own the same or substantially similar work of another, will not be tolerated, and will invite the harshest possible penalties as per institute norms.

Disclaimer: The instructors reserve the right to modify the schedules (e.g. breakup of total marks, scheduling weekly quizzes) and procedures announced in this syllabus. Any such changes will be announced in the class. It is the responsibility of the student to keep informed of such things.

Course objectives

Calculus can be broadly divided into two parts: Differential calculus and integral calculus. This course will be focused on integral calculus of several variables and vector analysis, mainly,

- Double and triple integration, Jacobians and change of variables.
- Parametrization of curves , vector fields, line integrals.
- Parametrization of surfaces and surface integrals.
- Divergence and curl of vector fields, theorems of Green, Gauss, and Stokes.

Goal: To achieve the rigorous understanding of the above topics along with some techniques and tools which are useful in applications.

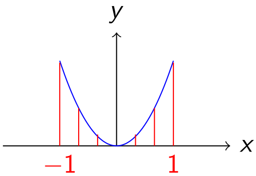
References:

1. [RR] Ravi Raghunathan, *Lecture slides for MA 105*.
2. [MTW] J.E Marsden, A. J. Tromba, A. Weinstein. *Basic Multivariable Calculus*, South Asian Edition, Springer (2017).
3. [GL] S.R. Ghorpade and B. V. Limaye, *A course in Multivariable Calculus and Analysis*, Springer UTM (2017).
4. [Apo] T.M. Apostol, *Calculus, Volumes 1 and 2*, 2nd ed., Wiley (2007).

Recall : One variable Integration from MA 109

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $a, b \in \mathbb{R}$.

- The area enclosed by the graph of a non-negative function over the region of the interval is $\int_a^b f(t)\ dt$.



The area in the figure on the left is $\int_{-1}^{1} x^2\ dx = 2/3$.

- A partition of the interval $[a, b]$ is a set of points $P = \{a = x_0 \leq x_1 \leq \dots x_n = b\}$ for some $n \in \mathbb{N}$.

- The lower Darboux integral and upper Darboux integral of f are $L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\}$, and $U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\}$, respectively.

- When $L(f) = U(f)$ then f is Darboux integrable and

$$\int_a^b f := L(f) = U(f).$$

- A tagged partition is a partition P with a set of points $t = \{t_1, \dots t_n\}$ where $t_j \in [x_{j-1}, x_j]$ for all $j = 1, \dots, n$.
- Define $S(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$ and define the *norm* of a partition P as $\|P\| = \max_j\{|x_j - x_{j-1}|\}, \quad 1 \le j \le n.$
- A function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if for some $S \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, P, t) - S| < \epsilon$, whenever $\|P\| < \delta$. The Riemann integral of f is then S.
- The Riemann integral exists if and only if the Darboux integral exists. Further, the two integrals are equal.
- Unlike the Darboux integral, Reimann integral can be computed as a limit. This is clearly advantageous in computations.
- Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.
- For computing integrals, we use the Fundamental theorem of calculus. If $f : [a, b] \to \mathbb{R}$ is continuous and $f = g'$ for some continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b), then $\displaystyle\int_a^b f = g(b) - g(a)$.

Integrating functions on two variables

Any *closed, bounded rectangle R* in $\mathbb{R}^2$:

$$R = [a, b] \times [c, d],$$

the Cartesian product of two closed intervals $[a, b]$ and $[c, d]$, where $a, b, c, d \in \mathbb{R}$.
Let us consider a real valued function f defined on R i.e.,

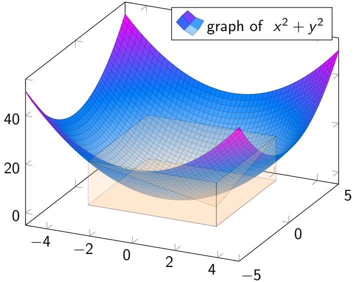
$$f : R \subset \mathbb{R}^2 \to \mathbb{R}.$$

- Graph of f: The subset $\{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in R\}$ in $\mathbb{R}^3$ is called the graph of f.

Double integral of non-negative functions

Consider the function $f(x,y) = x^2 + y^2$, for all $(x,y) \in \mathbb{R}^2$.

We want to compute volume of the region below the graph of f over the rectangle $[-3,3] \times [-3,3]$. The volume of the figure in the shaded region is $V := \{(x,y,z) \mid (x,y) \in [-3,3] \times [-3,3], \quad 0 \le z \le f(x,y)\}$.



The integral of the non-negative function f over $[-3,3] \times [-3,3]$ can be defined as the volume V;
$\int\int_{[-3,3]\times[-3,3]} f(x,y)\,dxdy := \text{Volume} \quad \text{of} \quad V.$

Integration on a Rectangle

Example: Let $g(x, y) = \alpha$, for some non-negative constant $\alpha \in \mathbb{R}$. Then for the rectangle $[0, b] \times [0, d]$ it is easy to see that $\int\int_{[0,b]\times[0,d]} g(x, y)\, dxdy = bd\alpha$.

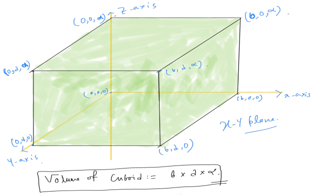


Figure: Cuboid: $[0, b] \times [0, d] \times [0, \alpha]$

Clearly for $f(x, y) = x^2 + y^2$, the computing of the volume is not that simple and we want to be able to define integral for all bounded functions instead of only non-negative ones.

Partitions for rectangles

Partition of R: A partition P of a rectangle $R = [a, b] \times [c, d]$ is the Cartesian product of a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$. Let

$$P_1 = \{x_0, x_1, \cdots x_m\}, \quad \text{with} \quad a = x_0 < x_1 < x_2 < \cdots < x_m = b\},$$

$$P_2 = \{y_0, y_1, \cdots y_n\}, \quad \text{with} \quad c = y_0 < y_1 < y_2 < \cdots < y_n = d\},$$

and $P = P_1 \times P_2$ be defined by

$$P = \{(x_i, y_j) \mid i \in \{0, 1, \cdots m\}, \quad j \in \{0, 1, \cdots, n\}\}.$$

The points of P divide the rectangle R into *nm non-overlapping sub-rectangles* denoted by

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad \forall\, i = 0, \cdots m-1, \quad j = 1, \cdots, n-1.$$

Note $R = \cup_{i,j} R_{ij}$.