MA-111 Calculus II
(D3 & D4)

Lecture 15

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Recap: Surfaces

Definition

Let E be a path connected subset in $\mathbb{R}^2$ with non-zero area. A parametrised surface is a continuous function $\mathbf{\Phi} : E \to \mathbb{R}^3$.

This definition is the analogue of what we called paths in one dimension and what are often called parametrized curves.

Examples:

- Graphs of real valued functions of two independent variables.
- A cylinder, A sphere, A cone.
- Surface of revolution.

Note that for a given $(u, v) \in E$, $\mathbf{\Phi}(u, v)$ can be written as

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on E.

The parametrized surface $\mathbf{\Phi}$ is said to be a smooth parametrized surface if the functions x, y, z have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing E.

Tangent vectors for a parametrised surface

Let $\mathbf{\Phi}(u, v)$ be a smooth parametrised surface. If we fix the variable v, say $v = v_0$, we obtain a curve $\mathbf{c}(u, v_0)$ that lies on the surface. Thus

$$\mathbf{c}(u) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

Since this curve is C^1 we can talk about its tangent vector at the point u_0. This is given by

$$\mathbf{c}'(u_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

We can *define* the partial derivative of a vector valued function as

$$\mathbf{\Phi}_u(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) := \mathbf{c}'(u_0).$$

Similarly, by fixing u and varying v we obtain a curve $\mathbf{l}(u_0, v)$ and we can set

$$\mathbf{\Phi}_v(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0) := \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

The tangent plane

Let for any given point on the surface, $P_0 = (x_0, y_0, z_0) := \mathbf{\Phi}(u_0, v_0)$ for some $(u_0, v_0) \in D$.
The two tangent vectors $\mathbf{\Phi}_u(u_0, v_0)$ and $\mathbf{\Phi}_v(u_0, v_0)$ at P_0 define a plane.
We call this plane as the tangent plane to the surface at P_0.

The normal to this plane at P_0, $\mathbf{n}(u_0, v_0) = \mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)$.

Thus for a given point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ in $\mathbb{R}^3$ the equation of the tangent plane is given by

$$\mathbf{n}(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

provided $\mathbf{n} \neq 0$.

In particular, if $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0) = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the equation of the tangent plane at P_0 is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Tangent Plane: Examples

Let us find the equation of the tangent plane at points on the various parametrised surfaces we have already looked at.

Example 1: Let D be a path-connected subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a C^1 function. The surface given by the graph of the function $z = f(x, y)$ is parametrized by $\mathbf{\Phi}(x, y) = (x, y, f(x, y))$. In this case, at $P_0 = \mathbf{\Phi}(x_0, y_0)$ for $(x_0, y_0) \in D$,

$$\mathbf{\Phi}_x(x_0, y_0) = \mathbf{i} + \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_y(x_0, y_0) = \mathbf{j} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{k}.$$

Hence,

$$\mathbf{n}(x_0, y_0) = \mathbf{\Phi}_x(x_0, y_0) \times \mathbf{\Phi}_y(x_0, y_0) = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right).$$

Thus the equation of the tangent plane is

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right) = 0;$$

which yields,

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Tangent Plane: Examples

Example 2: Let us consider a cylinder parametrized as

$$\mathbf{\Phi}(u, v) = (a\cos u, a\sin u, v), \quad \forall (u, v) \in [0, 2\pi] \times [0, h],$$

where $a > 0$. Then

$$\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos u, a\sin u, 0).$$

Since this is non-zero on $[0, 2\pi] \times [0, h]$ for any $h > 0$, we can define the tangent plane to $\mathbf{\Phi}$ at any point $P_0 = (x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ as

$$(a\cos u_0, a\sin u_0, 0).(x - x_0, y - y_0, z - z_0) = 0.$$

Now using $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0) = (a\cos u_0, a\sin u_0, v_0)$, we get the equation for the tangent plane to $\mathbf{\Phi}$ at P_0 is

$$(\cos u_0)x + (\sin u_0)y = a.$$

Example 3: The sphere: $x^2 + y^2 + z^2 = a^2$, for some $a > 0$. Let us consider the parametrization

$$\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v), \quad \forall\,(u, v) \in [0, 2\pi] \times [0, \pi].$$

Check $\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_u(u, v) = (a \sin v)\mathbf{\Phi}(u, v)$, for all $(u, v) \in [0, 2\pi] \times [0, \pi]$.

Note for $(u_0, v_0) \in [0, 2\pi] \times (0, \pi)$, $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_u(u_0, v_0) \neq (0, 0, 0)$ and the tangent plane at $P_0 = \mathbf{\Phi}(u_0, v_0)$ is

$$(\sin v_0 \cos u_0)x + (\sin v_0 \sin u_0)y + (\cos v_0)z = a.$$

Example 4: This was the example of the right circular cone. The parametric surface was given by

$$\mathbf{\Phi}(u,v) = (u\cos v, u\sin v, u), \quad (u,v) \in [0,\infty) \times [0,2\pi].$$

In this case we get

$$\mathbf{\Phi}_u(u,v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_v(u,v) = -u\sin v\mathbf{i} + u\cos v\mathbf{j},$$

where $\mathbf{n}(u,v) = \mathbf{\Phi}_u(u,v) \times \mathbf{\Phi}_v(u,v) = (-u\cos v, -u\sin v, u)$.

For any $(u_0, v_0) \in (0,\infty) \times [0,2\pi]$, $\mathbf{n}(u_0,v_0) \neq (0,0,0)$ and the tangent plane check

$$(\cos v_0)x + (\sin v_0)y = z.$$

Note that if $(u,v) = (0,0)$, then $\mathbf{n}(0,0) = 0$, so the tangent plane is not defined at the origin. However, it is defined at any other point.

Non-singular surfaces

In analogy with the situation for curves, we will call $\mathbf{\Phi}$ a regular or non-singular parametrised surface if $\mathbf{\Phi}$ is C^1 and $\mathbf{\Phi}_u \times \mathbf{\Phi}_v \neq 0$ at all points.

As we just saw, the right circular cone is not a regular parametrised surface.

For a regular surface parametrized by $\mathbf{\Phi} : D \to \mathbb{R}^3$, the unit normal $\hat{\mathbf{n}}$ to the surface at any point $P_0 = \mathbf{\Phi}(u_0, v_0)$ is defined by

$$\hat{\mathbf{n}}(u_0, v_0) := \frac{\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)}{\|\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)\|}.$$

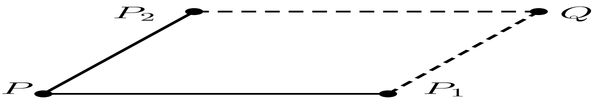
Surface Area

Let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrized surface, where E is a path-connected, bounded subset of $\mathbb{R}^2$ having a non-zero area. Also assume ∂E, the boundary of E, is of content zero.

Let $(u, v) \in E$. For $h, k \in \mathbb{R}$ with $|h|, |k|$ small, assuming $\mathbf{\Phi}$ is C^1 we can get the following approximations;

$$P := \mathbf{\Phi}(u, v), \quad P_1 := \mathbf{\Phi}(u + h, v) \approx \mathbf{\Phi}(u, v) + h\, \mathbf{\Phi}_u(u, v),$$

$$P_2 := \mathbf{\Phi}(u, v + k) \approx \mathbf{\Phi}(u, v) + k\, \mathbf{\Phi}_v(u, v), \quad Q := \mathbf{\Phi}(u + h, v + k).$$



Area of the parallelogram with sides PP_1 and PP_2

$$= \|(P_1 - P) \times (P_2 - P)\| \approx \|\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v)\|\, |h||k|.$$

In view of this approximation, we define

$$\text{Area}(\mathbf{\Phi}) := \iint_E \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u, v)\| \, du \, dv.$$

Since the subset E of $\mathbb{R}^2$ is bounded with boundary ∂E which is of content zero and the function $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$ is continuous on E, the integral in the definition of $\text{Area}(\mathbf{\Phi})$ is well-defined.

In analogy with the differential notation $ds = \|\gamma'(t)\|dt$, we introduce the following differential notation:

$$dS = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\| \, dudv.$$

Thus $\text{Area}(\mathbf{\Phi}) := \iint_E dS$.

Examples

• Graph of a function: Given a subset E of $\mathbb{R}^2$ have an area, $f : E \to \mathbb{R}$ be a smooth function, and $\mathbf{\Phi}(u, v) = (u, v, f(u, v))$ for $(u, v) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|(-f_u, -f_v, 1)\| dudv \\ &= \iint_E \sqrt{1 + f_u^2 + f_v^2} \, dudv \end{aligned}$$

Example: Let $E := [0, 2\pi] \times [0, h]$, $\mathbf{\Phi}(\theta, z) := (a\cos\theta, a\sin\theta, z)$, and $\mathbf{\Psi}(\theta, z) := (a\cos 2\theta, a\sin 2\theta, z)$ for $(\theta, z) \in E$. Then

$$\begin{aligned}
\mathrm{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\theta \times \mathbf{\Phi}_z\| \, d\theta dz = \iint_E a \, d\theta dz = 2\pi a \, h, \\
\mathrm{Area}(\mathbf{\Psi}) &= \iint_E \|\mathbf{\Psi}_\theta \times \mathbf{\Psi}_z\| \, d\theta dz = \iint_E 2a \, d\theta dz = 4\pi a \, h.
\end{aligned}$$

We note that $\mathbf{\Psi}(E) = \mathbf{\Phi}(E)$, but $\mathrm{Area}(\mathbf{\Psi}) = 2\,\mathrm{Area}(\mathbf{\Phi})$.

Example: Let $E := [0, \pi] \times [0, 2\pi]$, and $\mathbf{\Phi}(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. Then

$$\begin{aligned}
\mathrm{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\varphi \times \mathbf{\Phi}_\theta\| \, d\varphi d\theta = \iint_E a^2 \sin\varphi \, d\varphi d\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} a^2 \sin\varphi \, d\varphi \right) d\theta = 4\pi a^2.
\end{aligned}$$

Let C be a smooth curve in $\mathbb{R}^2 \times \{0\}$ given by $\boldsymbol{\gamma}(t) := (x(t), y(t))$, $t \in [\alpha, \beta]$. If C lies on or above the x-axis, and C is revolved about the x-axis, then it generates a surface parametrized by

$$\mathbf{\Phi}(t,\theta) := (x(t),\ y(t)\cos\theta,\ y(t)\sin\theta) \quad \text{for } (t,\theta) \in E,$$

where $E := [\alpha, \beta] \times [0, 2\pi]$. For all $(t, \theta) \in E$,

$$\begin{aligned} (\mathbf{\Phi}_t \times \mathbf{\Phi}_\theta)(t,\theta) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\ &= (y(t)y'(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta). \end{aligned}$$

By the Fubini theorem, we obtain

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \sqrt{y(t)^2 y'(t)^2 + x'(t)^2 y(t)^2}\, d(t,\theta) \\ &= 2\pi \int_\alpha^\beta y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt, \end{aligned}$$

Note: $\mathbf{\Phi}$ is non-singular $\iff$ $\boldsymbol{\gamma}$ is non-singular and $y(t) \neq 0$ for $t \in [\alpha, \beta]$.

The area vector of an infinitesimal surface element

We see that $\mathbf{\Phi}$ takes the small rectangle R to the parallelogram given by the vectors $\mathbf{\Phi}_u \Delta u$ and $\mathbf{\Phi}_v \Delta v$.

It follows that the 'area vector' $\Delta \mathbf{S}$ of this parallelogram is

$$\Delta \mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) \Delta u \Delta v.$$

Thus the surface 'area vector' is to be thought of as a vector pointing in the direction of the normal to the surface and in differential notation:

$$d\mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v)\, du\, dv.$$

The magnitude of the surface 'area vector' is given by

$$dS = \|d\mathbf{S}\| = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, du\, dv.$$

If the parametric surface $\mathbf{\Phi}$ is non-singular, we can write

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

The magnitude of the area vector

It remains to compute the magnitude dS. To do this we must find $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$. Writing this out in terms of x, y and z, we see that

$$d\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv.$$

Hence,

$$dS = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

where $\frac{\partial(y,z)}{\partial(u,v)}$, $\frac{\partial(x,z)}{\partial(u,v)}$, $\frac{\partial(x,y)}{\partial(u,v)}$ are the determinant of corresponding Jacobian matrix. For example

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v},$$

$$\frac{\partial(x,z)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u},$$

The surface area integral

Because of the calculations we have just made, the surface area is given by the double integral

$$\iint_S dS = \iint_E \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The area is nothing but the integral of the constant function 1 on the surface S. We integrate any bounded scalar function $f : S \to \mathbb{R}$:

$$\iint_S fdS = \iint_E f(x,y,z)\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

provided the R.H.S double integral exists. If Σ is a union of parametrised surfaces S_i that intersect only along their boundary curves, then we can define

$$\iint_\Sigma fdS = \sum_i \iint_{S_i} fdS.$$

The surface integral of a vector field

Let $\mathbf{F}$ be a bounded vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : E \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\mathbf{\Phi}(u,v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv,$$

provided the R.H.S double integral exists. This can also be written more compactly as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over S.

Examples

(i) Let a subset E of $\mathbb{R}^2$ have an area, and let $f : E \to \mathbb{R}$ be a smooth function. Let the smooth parametrized surface $\Phi : E \to \mathbb{R}^3$ represent the graph of f, and let $\boldsymbol{F} : \boldsymbol{\Phi}(E) \to \mathbb{R}^3$ be a continuous vector field. If $\boldsymbol{F} := (P, Q, R)$, then

$$\iint_{\boldsymbol{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_E (-P\, f_x - Q\, f_y + R) d(x, y)$$

since $d\boldsymbol{S} = (\boldsymbol{\Phi}_x \times \boldsymbol{\Phi}_y) dxdy = (-f_x, -f_y, 1) dxdy$.

Using above result, let $E := [0,1] \times [0,1]$, $f(x, y) := x + y + 1$ for $(x, y) \in E$. If $\boldsymbol{F}(x, y, z) := (x^2, y^2, z)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned} \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_E \big(- x^2 - y^2 + (x + y + 1)\big) d(x, y) \\ &= \int_0^1 \left(\int_0^1 (x + y + 1 - x^2 - y^2) dy \right) dx \\ &= = \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}. \end{aligned}$$

Examples Contd.

(ii) Let $E := [0, 2\pi] \times [0, h]$, and $\mathbf{\Phi}(u, v) := (a \cos u, a \sin u, v)$ for $(u, v) \in E$. If $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\boldsymbol{S} = \iint_E (a^2 \cos u \sin u + v\, a \sin u + 0) du dv = 0,$$

since $d\boldsymbol{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv = (a \cos u, a \sin u, 0) du dv$.