# MA 105: Calculus
# Lecture 17

Prof. B.V. Limaye
IIT Bombay

Wednesday, 8 March 2017

# Double Integral on a Rectangle

The concept of the Riemann integral of a function was motivated by our attempt to find 'the area under a curve'. We now look for a concept which will let us find 'the volume under a surface'. We shall assume that the **volume** of a **cuboid** $[a, b]\times[c, d]\times[p, q]$ is equal to $(b-a)(d-c)(q-p)$.

Let $R := [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$ with $a < b$, $c < d$. Let $f : R \to \mathbb{R}$ be a bounded function. Define

$$m(f) := \inf\{f(x, y) : (x, y) \in R\}, \; M(f) := \sup\{f(x, y) : (x, y) \in R\}.$$

Let $n, k \in \mathbb{N}$, and consider a **partition** $P$ of $R$ given by $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$, where $a := x_0 < x_1 < \cdots < x_n := b;\; c = y_0 < y_1 < \cdots < y_k = d$.

The points in $P$ divide the rectangle $R$ into $nk$ nonoverlapping subrectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, n$; $j = 1, \ldots, k$.

Define

$$\begin{aligned} m_{i,j}(f) &:= \inf\{f(x,y) : (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}, \\ M_{i,j}(f) &:= \sup\{f(x,y) : (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}. \end{aligned}$$

Clearly, $m(f) \le m_{i,j}(f) \le M_{i,j}(f) \le M(f)$ for $i = 1, \ldots, n$; $j = 1, \ldots, k$.

Define the **lower double sum** and the **upper double sum** of $f$ with respect to $P$ by

$$\begin{aligned} L(P,f) &:= \sum_{i=1}^{n} \sum_{j=1}^{k} m_{i,j}(x_i - x_{i-1})(y_j - y_{j-1}), \\ U(P,f) &:= \sum_{i=1}^{n} \sum_{j=1}^{k} M_{i,j}(x_i - x_{i-1})(y_j - y_{j-1}). \end{aligned}$$

Since $\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$ and $\sum_{j=1}^{k}(y_j - y_{j-1}) = d - c$, we obtain

$$m(f)(b-a)(d-c) \le L(P,f) \le U(P,f) \le M(f)(b-a)(d-c).$$

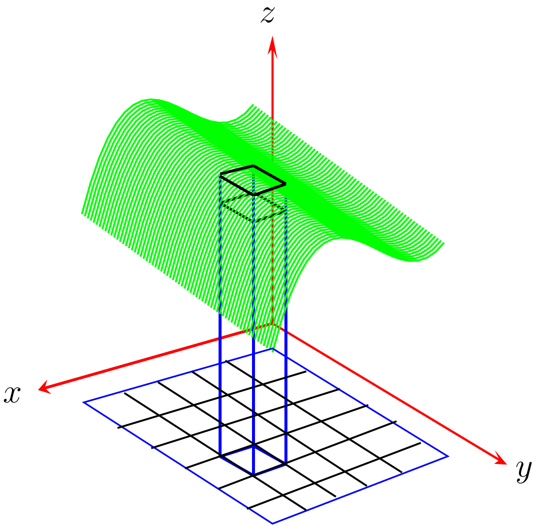



Figure : Summands of lower and upper double sums.

Let $P_1$ and $P_2$ be partitions of the rectangle $R$. A lower double sum increases and a upper double sum decreases when the partition is refined. Considering the common refinement of $P_1$ and $P_2$, we obtain $L(P_1, f) \leq U(P_2, f)$. Define

$$\begin{aligned} L(f) &:= \sup\{L(P, f) : P \text{ is a partition of } R\}, \\ U(f) &:= \inf\{U(P, f) : P \text{ is a partition of } R\}. \end{aligned}$$

$L(f)$ is called the **lower double integral** of $f$ and $U(f)$ is called the **upper double integral** of $f$.

Then $L(f) \leq U(f)$ by using the definitions of sup and inf.

**Definition**

A bounded function $f : R \to \mathbb{R}$ is said to be **(double) integrable** on $R$ if $L(f) = U(f)$.

In this case, the **double integral** of $f$ on $R$ is the common value $U(f) = L(f)$, and it is denoted by

$$\iint_R f \quad \text{or} \quad \iint_R f(x,y)d(x,y).$$

Let $f : R \to \mathbb{R}$ be integrable and nonnegative. The double integral of $f$ on $R$ gives the volume of the solid $E_f$ under the surface $z = f(x,y)$ and above the rectangle $R$.

Examples:

(i) Let $f(x,y) := 1$ for all $(x,y) \in R$. Then $L(P,f) = U(P,f) = (b-a)(d-c)$ for every partition $P$ of $R$. Hence $f$ is integrable on $R$, and its double integral on $R$ is equal to $(b-a)(d-c)$.

(ii) Define the **bivariate Dirichlet function** $f : R \to \mathbb{R}$ by

$$f(x,y) := \begin{cases} 1 & \text{if} \quad x \text{ and } y \text{ are rational numbers,} \\ 0 & \text{if} \quad x \text{ or } y \text{ is an irrational number.} \end{cases}$$

Then $f$ is a bounded function on $R$. For each partition $P$ of $R$,

$$m_{i,j}(f) = 0 \text{ and } M_{i,j}(f) = 1, \quad i = 1, \ldots, n; \; j = 1, \ldots, k,$$

and so $L(P, f) = 0$ and $U(P, f) = (b - a)(d - c)$. Thus $L(f) = 0$ and $U(f) = (b - a)(d - c)$. Since $L(f) \neq U(f)$, $f$ is not integrable.

(iii) Let $\phi : [a, b] \to \mathbb{R}$ be bounded, and define $f : R \to \mathbb{R}$ by $f(x, y) := \phi(x)$ for $(x, y) \in R$. Then $f$ is a bounded function. If $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ is any partition of $R$, then $P_1 := \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, and $m_{i,j}(f) = m_i(\phi)$, $i = 1, \ldots, n$; $j = 1, \ldots, k$, and so

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_{i,j}(f)(x_i - x_{i-1})(y_j - y_{j-1}) = (d - c)\sum_{i=1}^{n} m_i(\phi)(x_i - x_{i-1}).$$

Thus $L(P, f) = (d - c)L(P_1, \phi)$. Also, $U(P, f) = (d - c)U(P_1, \phi)$.

Further, if $Q := \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then $Q = P_1$, where $P := \{(x_i, y_j) : i = 0, 1, \ldots, n, \ j = 0, 1\}$. So

$L(f) = (d - c)L(\phi)$ and $U(f) = (d - c)U(\phi)$. Hence

$f$ is integrable on $R \iff \phi$ is Riemann integrable on $[a, b]$, and in this case, the double integral of $f$ on $R$ is equal to $(d - c)$ times the Riemann integral of $\phi$ on $[a, b]$.

The following result allows us to test the integrability of a bounded function on $R$.

**Theorem (Riemann condition)**

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f$ is integrable if and only if for every $\epsilon > 0$, there is a partition $P_\epsilon$ of $[a, b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

The proof is very similar to the proof of the result about the Riemann condition in the one variable case.

# Double Integration

The Riemann condition can be used to prove many useful results regarding double integration.

**(Domain Additivity)**

Let $R := [a, b]\times[c, d]$, and let $f : R \to \mathbb{R}$ be a bounded function. Let $s \in (a, b)$, $t \in (c, d)$. Then $f$ is integrable on $R$ if and only if $f$ is integrable on the four subrectangles $[a, s]\times[c, t]$, $[a, s]\times[t, d]$, $[s, b]\times[c, t]$ and $[s, b]\times[t, d]$. In this case, the integral of $f$ on $R$ is the sum of the integrals of $f$ on the four subrectangles.

We make the following conventions:

If $a = b$ or $c = d$, then $\iint_{[a,b]\times[c,d]} f := 0$.

$\iint_{[b,a]\times[c,d]} f := -\iint_{[a,b]\times[c,d]} f =: \iint_{[a,b]\times[d,c]} f$, and

$\iint_{[b,a]\times[d,c]} f := \iint_{[a,b]\times[c,d]} f$.

# Integrable functions

Let $f : R \to \mathbb{R}$.

(i) If $f$ is monotonic in each of the two variables, then $f$ is integrable on $R$.

(ii) If $f$ is bounded on $R$, and has at most a finite number of discontinuities in $R$, then $f$ is integrable on $R$.

Examples:

(i) Let $f(x, y) := [x] + [y]$ for $(x, y) \in R$. Since $f$ is increasing in each variable, $f$ is integrable.

(ii) Let $a, c > 0$ and $r, s \geq 0$. Define $f(x, y) = x^r y^s$ for $(x, y) \in R$. Since $f$ is continuous on $R$, it is integrable.

(iii) Let $f(0, 0) := 0$ and $f(x, y) := xy/(x^2 + y^2)$ if $(x, y) \in [-1, 1] \times [-1, 1]$ and $(x, y) \neq (0, 0)$. Since $f$ is bounded on $R$, and it is discontinuous only at $(0, 0)$, $f$ is integrable.

# Albebraic and Order Properties

Let $f, g : R \to \mathbb{R}$ be integrable functions. Then

(i) $f + g$ is integrable, and $\iint_R (f+g) = \iint_R f + \iint_R g$.

(ii) $\alpha f$ is integrable, and $\iint_R \alpha f = \alpha \iint_R f$ for all $\alpha \in \mathbb{R}$.

(iii) $f \cdot g$ is integrable.

(iv) If there is $\delta > 0$ such that $|f(x,y)| \geq \delta$ for all $(x,y) \in R$ (so that $1/f$ is bounded), then $1/f$ is integrable.

(v) If $f \leq g$, then $\iint_R f \leq \iint_R g$.

(vi) $|f|$ is integrable, and $|\iint_R f| \leq \iint_R |f|$.

Proof: (v) For any partition $P$ of $R$, $U(P,f) \leq U(P,g)$, and so $\iint_R f = U(f) \leq U(g) = \iint_R g$.

(vi) $-|f| \leq f \leq |f| \implies -\iint_R |f| \leq \iint_R f \leq \iint_R |f|$.

# Evaluation of a Double Integral

Suppose a function is integrable on a rectangle $R$. How can we find its double integral?

To evaluate the Riemann integral of an integrable function $f$ on an interval $[a, b]$, we used a powerful result known as the fundamental theorem of calculus, Part II: If we find a function $g$ on $[a, b]$ whose derivative is equal to $f$ on $[a, b]$, then

$$\int_a^b f(x)dx = \int_a^b g'(x)dx = g(b) - g(a).$$

Later in this course, we shall see some versions of the fundamental theorem of calculus for functions of two/three variables, known as the Green/Stokes' theorem.

For the present, we consider an easy and widely used method for evaluating double integrals, namely, reduction of the problem to a repeated evaluation of Riemann integrals.

## Theorem (Fubini Theorem on a Rectangle)

Let $R := [a, b]\times[c, d]$, let $f : R \to \mathbb{R}$ be integrable, and let $I$ denote the double integral of $f$ on $R$.

(i) If for each fixed $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)dy$ exists, then the **iterated integral** $\int_a^b \left(\int_c^d f(x, y)dy\right) dx$ exists and is equal to $I$.

(ii) If for each fixed $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)dx$ exists, then the **iterated integral** $\int_c^d \left(\int_a^b f(x, y)dx\right) dy$ exists and is equal to $I$.

(iii) If the hypotheses in both (i) and (ii) above hold, and in particular, if $f$ is continuous on $R$, then

$$\int_a^b\left(\int_c^d f(x, y)dy\right) dx = I = \int_c^d\left(\int_a^b f(x, y)dx\right) dy.$$

The Fubini theorem can be proved by using the Riemann condition for a double integral and for a Riemann integral.

Special case: Let $\phi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$ be Riemann integrable. Define $f : R \to \mathbb{R}$ by $f(x, y) := \phi(x)\psi(y)$, $(x, y) \in R$. Then $f$ is integrable on $R$, and its double integral is equal to

$$\int_a^b \left( \int_c^d \phi(x)\psi(y)dy \right) dx = \left( \int_a^b \phi(x)dx \right) \left( \int_c^d \psi(y)dy \right).$$

In particular, if $r, s \in \mathbb{R}$ with $r \geq 0$ and $s \geq 0$, then

$$\iint_{[a,b]\times[c,d]} x^r y^s d(x, y) = \left( \frac{b^{r+1} - a^{r+1}}{r + 1} \right) \left( \frac{d^{s+1} - c^{s+1}}{s + 1} \right),$$

provided $0 < a < b$ and $0 < c < d$.

Geometrically, the Fubini theorem says that if $f$ is a nonnegative integrable function on $R := [a, b] \times [c, d]$, then the volume of the solid $D$ under the surface $z = f(x, y)$ and above the rectangle $R$ can be found by the slice method: Find the areas $A(x) := \int_c^d f(x, y)dy$, $x \in [a, b]$, of the cross-sections of $D$ perpendicular to the $x$-axis. Alternatively, find the areas $B(y) := \int_a^b f(x, y)dx$, $y \in [c, d]$, of the cross-sections of $D$ perpendicular to the $y$-axis. Then

$$\mathrm{Vol}\,(D) = \int_a^b A(x)dx = \int_c^d B(y)dy.$$

Examples:

(i) Let $R := [0, 1] \times [0, 1]$, and $f(x, y) := (x + y)^2$, $(x, y) \in R$. Then $f$ is continuous on $R$. The double integral of $f$ on $R$ is

$$\begin{aligned} \int_0^1 \left( \int_0^1 (x+y)^2 dx \right) dy &= \frac{1}{3} \int_0^1 (x+y)^3 \Big|_0^1 dy \\ &= \frac{1}{3} \int_0^1 \left( (1+y)^3 - y^3 \right) dy = \frac{7}{6}. \end{aligned}$$

(ii) Let $R := [0,1]\times[0,1]$, $f(0,0) := 0$, and for $(x,y) \neq (0,0)$, let $f(x,y) := xy(x^2-y^2)/(x^2+y^2)^3$. For $x \in [0,1]$, $x \neq 0$,

$$A(x) := \int_0^1 f(x,y)dy = \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}dy = \frac{x}{2(1+x^2)^2},$$

and also, $A(0) = \int_0^1 0dy = 0$. Hence

$$\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = \int_0^1 A(x)dx = \int_0^1 \frac{x}{2(1+x^2)^2}dx = \frac{1}{8}.$$

By interchanging $x$ and $y$, $\displaystyle\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = -\frac{1}{8}$.

Thus the two iterated integrals exist, but they are not equal.

Note that since $f(2/n, 1/n) = 6n^2/125$ for all $n \in \mathbb{N}$, the function $f$ is not bounded on $R$, and so it is not be integrable on $R$. Thus Fubini's theorem is not applicable.

# Double Riemann Sum

If we are not able to evaluate the double integral exactly, we attempt to find its approximations.

Given a bounded function $f : R \to \mathbb{R}$, and a partition $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$, of $R := [a, b] \times [c, d]$, a double sum of the form

$$S(P, f) := \sum_{i=1}^{n} \sum_{j=1}^{k} f(s_i, t_j)(x_i - x_{i-1})(y_j - y_{j-1}),$$

where $s_i \in [x_{i-1}, x_i]$, $t_j \in [y_{j-1}, y_j]$, is called a double Riemann sum for $f$ corresponding to $P$.

Note: $L(P, f) \leq S(P, f) \leq U(P, f)$ for any $s_1, \ldots, s_n \in [a, b]$ and $t_1, \ldots, t_k \in [c, d]$.

Define the **mesh** of a partition
$P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ by
$\mu(P) := \max\{x_1 - x_0, \ldots, x_n - x_{n-1},\ y_1 - y_0, \ldots, y_k - y_{k-1}\}$.

Theorem: Let $f$ be integrable on $R$, and let $\epsilon > 0$. Then there is $\delta > 0$ such that $U(P, f) - L(P, f) < \epsilon$ for every partition $P$ satisfying $\mu(P) < \delta$.

Corollary: Let $(P_n)$ is a sequence of partitions of $R$ such that $\mu(P_n) \to 0$. Then $U(P_n, f) - L(P_n, f) \to 0$. Further, if $S(P_n, f)$ is a double Riemann sum corresponding to $P_n$ and $f$, then $S(P_n, f) \to \iint_R f$.

Proof: Given $\epsilon > 0$, find $\delta > 0$ as in the theorem, and let $n_0 \in \mathbb{N}$ be such that $\mu(P_n) < \delta$ for all $n \geq n_0$, so that $U(P_n, f) - L(P_n, f) < \epsilon$. Thus $U(P_n, f) - L(P_n, f) \to 0$.
Since $L(P_n, f) \leq S(P_n, f) \leq U(P_n, f)$, and
$L(P_n, f) \leq L(f) = \iint_R f = U(f) \leq U(P_n, f)$, we see that
$\left|S(P_n, f) - \iint_R f\right| \leq U(P_n, f) - L(P_n, f) \to 0.$ $\square$

The above result also allows us to find the sums of certain series. For example, let

$$s_n := \frac{1}{n^4}\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)^2 \quad \text{for } n \in \mathbb{N}.$$

Then, as $n \to \infty$,

$$s_n := \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{i}{n}+\frac{j}{n}\right)^2 \to \int_0^1\int_0^1 (x+y)^2 d(x,y) = \frac{7}{6}.$$

Caution:

If we define

$$\mu(P) := \max\{(x_i - x_{i-1})(y_j - y_{j-1}) : 1 \le i \le n;\ 1 \le j \le k\},$$

then we may not have $S(P_n, f) \to \iint_R f$ as $\mu(P_n) \to 0$ for every integrable function $f$ on $R$. An example of this kind is given by the so-called bivariate Thomae function.

# Tutorial 9

1. Let $D \subset \mathbb{R}^2$. Show that $D$ is closed if and only if $\partial D \subset D$.
2. Let $E \subset \mathbb{R}^2$. Then $E$ is called **open** (in $\mathbb{R}^2$) if every point in $E$ is an interior point of $E$. Show that $E$ is open $\iff$ the complement $D := \mathbb{R}^2 \setminus E$ is closed.
3. Are the following subsets of $\mathbb{R}^2$ closed? Are they open in $\mathbb{R}^2$? Find their boundaries.
   (i) $\{(x, y) \in \mathbb{R}^2 : 0 \le x < 1 \text{ and } 0 < y \le 1\}$,
   (ii) $\{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2\}$,
   (iii) $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}\}$.
4. Let $a, b > 0$, $D := \{(x, y) \in \mathbb{R}^2 : b^2x^2 + a^2y^2 \le a^2b^2\}$, and define $f : D \to \mathbb{R}$ by $f(x, y) := x^2 - y^2$. Does $f$ attain its bounds on $D$? If so, find them. (Use the Orthogonal Gradient Theorem.)

# Tutorial 9

5. Maximize $f(x, y, z) := 400\,x\,y\,z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.
6. Let $f(x, y, z) := x^2 + y^2 + z^2$ for $(x, y, z) \in \mathbb{R}^2$. Does $f$ have constrained extrema subject to $x + 2y + 3z = 6$ and $x + 3y + 4z = 9$? If so, find them.
7. Let $(x, y) \in [1, 3] \times [2, 5]$. Define $f(x, y) := 1$ if $x$ and $y$ are both rational and $f(x, y) := -1$ otherwise. Is $f$ integrable? Is $|f|$ integrable?
8. Let $f(x, y) := x^2y^2$ and $g(x, y) := x^2 + y^2$ for $(x, y) \in [a, b] \times [c, d]$. Are $f$ and $g$ integrable on $[a, b] \times [c, d]$? If so, find their double integrals.
9. Find the limit of the sequence $(s_n)$ if for $n \in \mathbb{N}$,

$$s_n := \frac{1}{n^6} \sum_{i=1}^{n} \sum_{j=1}^{n} i^2 j^2 \quad \text{and} \quad s_n := \frac{1}{n^4} \sum_{i=1}^{n} \sum_{j=1}^{n} (i^2 + j^2).$$

# MA 105: Calculus
# Lecture 18

Prof. B.V. Limaye
IIT Bombay

Friday, 10 March 2017

# Double Integral over a Bounded Set

Let $D$ be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a bounded function. Consider a rectangle $R := [a, b] \times [c, d]$ such that $D \subset R$, and define $f^* : R \to \mathbb{R}$ by

$$f^*(x, y) := \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that $f$ is **integrable over** $D$ if $f^*$ is integrable on $R$, and in this case, the **double integral of** $f$ **over** $D$ is defined to be the double integral of $f^*$ on $R$, that is,

$$\iint_D f(x, y) d(x, y) := \iint_R f^*(x, y) d(x, y).$$

By the domain additivity of double integrals on rectangles, the integrability of $f$ over $D$ and the value of its double integral are independent of the choice of a rectangle $R$ containing $D$ and the corresponding extension $f^*$ of $f$ to $R$.

Let $D$ is a bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be integrable. We may also denote the double integral of $f$ over $D$ by $\iint_D f$.

If $f$ is nonnegative, then the **volume** of the solid under the surface given by $z = f(x, y)$ and above the region $D$ is defined to be the double integral of $f$ over $D$. Thus

$$\mathrm{Vol}\,(E_f) := \iint_D f(x,y)d(x,y),$$

where $E_f := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } 0 \le z \le f(x, y)\}$.

Next, if $g : D \to \mathbb{R}$ is integrable, and $f \le g$ on $D$, then

$$\iint_D \big(g(x,y) - f(x,y)\big)d(x,y)$$

is defined to be the **volume between the surfaces** given by $z = f(x, y)$ and $z = g(x, y)$.

The double integral over a bounded subset of $\mathbb{R}^2$ has algebraic and order properties analogous to those of the double integral on a rectangle.

In order to seek conditions under which a bounded function $f$ defined on a bounded subset $D$ of $\mathbb{R}^2$ is integrable over $D$, we introduce a new concept.

A bounded subset $E$ of $\mathbb{R}^2$ is of (two-dimensional) **content zero** if for every $\epsilon > 0$, there are finitely many rectangles whose union contains $E$ and the sum of whose areas is less than $\epsilon$.

A set of (two-dimensional) content zero is a 'thin' subset of $\mathbb{R}^2$.

Examples:

(i) Every finite subset of $\mathbb{R}^2$ is of content zero.

(ii) The infinite subset $\{(1/n, 1/k) : n, k \in \mathbb{N}\}$ of $\mathbb{R}^2$ is of content zero.

(iii) The subset $\{(x, y) \in [0, 1]\times[0, 1] : x, y \in \mathbb{Q}\}$ of $\mathbb{R}^2$ is not of content zero. (Consider any $\epsilon$ less than 1.)

(iv) Let $\varphi : [a, b] \to \mathbb{R}$ be an integrable function. Then its graph $E := \{(x, \varphi(x)) : x \in [a, b]\}$ is of content zero. To see this, let $\epsilon > 0$. By the Riemann condition, there is a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $U(P, \varphi) - L(P, \varphi) < \epsilon$. Then $E \subset \bigcup_{i=1}^{n} R_i$, where $R_i := [x_{i-1}, x_i]\times[m_i(\varphi), M_i(\varphi)]$ and

$$\text{Area}\,(R_1) + \cdots + \text{Area}\,(R_n) = U(P, \varphi) - L(P, \varphi) < \epsilon.$$

**Theorem**

Let $D$ be a bounded subset of $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ be a bounded function. If the set of discontinuities of $f$ in $D$ is of (two-dimensional) content zero and if the boundary $\partial D$ of $D$ is of (two-dimensional) content zero, then $f$ is integrable over $D$.

We omit the proof.

# Elementary Regions

Let $D \subset \mathbb{R}^2$. Suppose there are $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ such that $\phi_1$ and $\phi_2$ are continuous on $[a, b]$, $\phi_1 \le \phi_2$, and

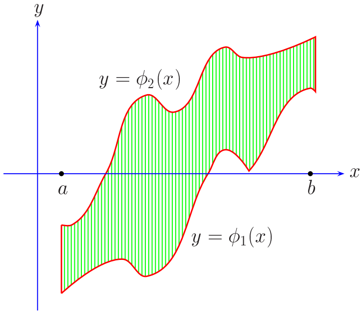
$$D = \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}.$$

Alternatively, suppose there are $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ such that $\psi_1$ and $\psi_2$ are continuous on $[c, d]$, $\psi_1 \le \psi_2$, and

$$D = \{(x, y) \in \mathbb{R}^2 : c \le y \le d \text{ and } \psi_1(y) \le x \le \psi_2(y)\}.$$

In both cases, $D$ is called an **elementary region** in $\mathbb{R}^2$. In the former case, it is said to be of **type** I, and in the latter case, it is said to be of **type** II.

We note that the boundary of an elementary domain in $\mathbb{R}^2$ is of (two-dimensional) content zero. Hence a continuous function defined on such a domain is integrable over it.



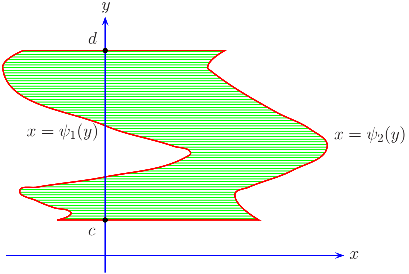




Figure : Elementary regions with boundaries in red colour

Clearly, a rectangle is an elementary region in $\mathbb{R}^2$ of type I as well as of type II.

Also, if $a > 0$, then the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le a^2\}$ is an elementary region in $\mathbb{R}^2$, since

$$D = \left\{(x, y) \in \mathbb{R}^2 : -a \le x \le a \text{ and } -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}\right\}.$$

Essential feature of an elementary region $D$ in $\mathbb{R}^2$ of type I: There are $a, b \in \mathbb{R}$ such that for every $x \in [a, b]$, the vertical cross-section of $D$ at $x$ is a closed and bounded interval.

Essential feature of an elementary region $D$ of type II: There are $c, d \in \mathbb{R}$ such that for every $y \in [c, d]$, the horizontal cross-section of $D$ at $y$ is a closed and bounded interval.

There do exist bounded subsets of $\mathbb{R}^2$ that are not elementary regions. For example, a star-shaped subset of $\mathbb{R}^2$ is not an elementary region, since for some $x \in \mathbb{R}$, the vertical cross-section of $D$ at $x$ is not an interval, and for some $y \in \mathbb{R}$, the horizontal cross-section of $D$ at $y$ is not an interval.

## Proposition (Fubini Theorem over Elementary Regions)

Let $D$ be a subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $D := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous, then the iterated integral $\int_a^b \Big( \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy\Big) dx$ exists and equals $\iint_D f(x, y)d(x, y)$.

(ii) If $D :=\{(x, y) \in \mathbb{R}^2 : c \le y \le d \text{ and } \psi_1(y) \le x \le \psi_2(y)\}$, where $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ are continuous, then the iterated integral $\int_c^d \Big( \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx\Big) dy$ exists and equals $\iint_D f(x, y)d(x, y)$.

Proof: (i) Let $c := \inf\{\phi_1(x) : x \in [a, b]\}$ and $d := \sup\{\phi_2(x) : x \in [a, b]\}$. Then $D \subset R := [a, b]\times[c, d]$, and the extended function $f^* : R \to \mathbb{R}$ is integrable. Use the Fubini theorem for $f^*$. (ii) A similar argument works here.

Examples:

(i) Let $D := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1/2 \text{ and } 0 \le y \le x^2\}$ and $f(x,y) := x + y$ for $(x,y) \in D$. Then $f$ is continuous on the elementary region $D$. By the Fubini theorem,

$$I := \iint_D (x+y)d(x,y) = \int_0^{1/2}\left(\int_0^{x^2}(x+y)dy\right)dx,$$

which is equal to

$$\int_0^{1/2}\left[xy + \frac{y^2}{2}\right]_{y=0}^{y=x^2} dx \ = \ \int_0^{1/2}\left(x^3 + \frac{x^4}{2}\right)dx \ = \ \frac{3}{160}.$$

Alternatively, since
$D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1/4 \text{ and } \sqrt{y} \le x \le 1/2\}$,

$$I = \int_0^{1/4}\left(\int_{\sqrt{y}}^{1/2}(x+y)dx\right)dy = \int_0^{1/4}\left[\frac{x^2}{2} + xy\right]_{x=\sqrt{y}}^{x=1/2} dy = \frac{3}{160}.$$

(ii) Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 2x\}$ and $f(x, y) := e^{x^2}$ for $(x, y) \in D$. Then $f$ is continuous on the elementary region $D$. By the Fubini theorem,

$$\iint_D f = \int_0^1 \left( \int_0^{2x} e^{x^2} dy \right) dx = \int_0^1 2x\, e^{x^2} dx = e - 1.$$

Also, since $D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 2 \text{ and } y/2 \le x \le 1\}$,

$$\iint_D f = \int_0^2 \left( \int_{y/2}^1 e^{x^2} dx \right) dy.$$

However, the integral $\int_{y/2}^1 e^{x^2} dx$ cannot be evaluated in terms of known functions.

This example shows that an iterated integral may not always be useful in evaluating a double integral.

# Constant Function on a Bounded Subset of $\mathbb{R}^2$

Let $D$ be a bounded subset of $\mathbb{R}^2$. Define

$$1_D : D \to \mathbb{R} \quad \text{by} \quad 1_D(x,y) := 1 \text{ for all } (x,y) \in D.$$

If the boundary $\partial D$ of $D$ is of (two-dimensional) content zero, then the continuous function $1_D$ is integrable on $D$. The converse also holds, that is, if the function $1_D$ is integrable on $D$, then $\partial D$ is of (two-dimensional) content zero.

Example: Let $R := [a,b]\times[c,d]$. If $D := R$, then we have seen that $1_D$ is integrable and its double integral is equal to $(b-a)(d-c)$. But if $D := \{(x,y) \in R : x, y \in \mathbb{Q}\}$, then the function $1_D^* : R \to \mathbb{R}$, obtained by extending the function $1_D$ as usual, is the bivariate Dirichlet function on $R$. We have seen that $1_D^*$ is not integrable on $R$, that is, $1_D$ is not integrable over $D$. Note: In this case $\partial D = [a,b]\times[c,d]$ is not of two-dimensional content zero.

# Area of a Bounded Subset of $\mathbb{R}^2$

Let $D \subset \mathbb{R}^2$ be bounded. We say that $D$ has an **area** if the function $1_D$ is integrable over $D$, and then we define

$$\text{Area}\,(D) := \iint_D 1_D(x,y)d(x,y).$$

Thus a bounded subset $D$ of $\mathbb{R}^2$ has an area $\iff$ $\partial D$ is of content zero.

Important special case: Consider an elementary domain $D := \{(x,y) \in \mathbb{R}^2 : a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$, where $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ are continuous. By the Fubini theorem,

$$\text{Area}\,(D) = \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} 1\, dy \right) dx = \int_a^b \big(\phi_2(x) - \phi_1(x)\big)\, dx,$$

which was our definition of the **area between the curves** $y = \phi_1(x)$ and $y = \phi_2(x)$, $x \in [a,b]$.

The following basic inequality is important.

Let $D$ be a bounded subset of $\mathbb{R}^2$ which has an area. If $f : D \to \mathbb{R}$ is an integrable function, and if $|f| \le \alpha$ on $D$, then

$$\left| \iint_D f(x,y)d(x,y) \right| \le \iint_D |f(x,y)|d(x,y) \le \alpha \, \text{Area}\,(D).$$

The following result is useful in evaluating a double integral.

**Theorem (Domain Additivity)**

Let $D$ be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a bounded function. Suppose $D = D_1 \cup D_2$, where $D_1 \cap D_2$ is of (two-dimensional) content zero. If $f$ is integrable over $D_1$ and over $D_2$, then $f$ is integrable over $D$ and

$$\iint_D f(x,y)d(x,y) = \iint_{D_1} f(x,y)d(x,y) + \iint_{D_2} f(x,y)d(x,y).$$

# Change of Variables in a Double Integral

The calculation of the double integral $\iint_D f(x, y)d(x, y)$ can often be simplified by reducing it to another double integral $\iint_E g(u, v)d(u, v)$, where the pair $(u, v)$ of new variables is related to the the pair $(x, y)$ of the given variables by a suitable transformation.

In doing so, we look for a function $g$ which is 'simpler' than the given function $f$, making sure that the domain $E$ of $g$ is also 'simpler' that the given domain $D$ of $f$.

Let us begin with an affine transformation $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Psi := (\psi_1, \psi_2)$, where $\psi_1$ and $\psi_2$ are defined as follows:

$$x = \psi_1(u, v) := x^\circ + a_1 u + b_1 v \quad \text{and} \quad y = \psi_2(u, v) := y^\circ + a_2 u + b_2 v$$

Note that the above equations have a unique solution, that is, the transformation $\Psi$ is one-one $\iff a_1 b_2 - a_2 b_1 \neq 0$.

In this case, the square region $E := [0,1]\times[0,1]$ is transformed to the parallelogram $D := \Psi(E)$ with vertices $\Psi(0,0) = (x^\circ, y^\circ)$, $\Psi(1,0) = (x^\circ + a_1, y^\circ + a_2)$, $\Psi(0,1) = (x^\circ + b_1, y^\circ + b_2)$ and $\Psi(1,1) = (x^\circ + a_1 + b_1, y^\circ + a_2 + b_2)$. Note that the area 1 of $E$ is scaled to the area $|a_1 b_2 - a_2 b_1|$ of $D = \Psi(E)$.

In general, if $E \subset \mathbb{R}^2$ and $\Phi := (\phi_1, \phi_2) : E \to \mathbb{R}^2$, then we may replace $\phi_1$ and $\phi_2$ by their affine approximations around an interior point $Q_0 := (u_0, v_0)$ of $E$.

If $\phi_1(u,v) := x(u,v)$ and $\phi_2(u,v) = y(u,v)$ for $(u,v) \in E$, and the functions $x$ and $y$ have partial derivatives at $(u_0, v_0)$, then the **Jacobian** of $\Phi$ at $(u_0, v_0)$ is given by

$$J(\Phi)(u_0, v_0) := \frac{\partial(x,y)}{\partial(u,v)}(u_0, v_0) := \det \begin{bmatrix} \dfrac{\partial x}{\partial u}(Q_0) & \dfrac{\partial x}{\partial v}(Q_0) \\ \dfrac{\partial y}{\partial u}(Q_0) & \dfrac{\partial y}{\partial v}(Q_0) \end{bmatrix}.$$

In particular, $J(\Psi)(u,v) = a_1 b_2 - a_2 b_1$ for all $(u,v) \in \mathbb{R}^2$.

Let $Q_0 := (u_0, v_0) \in \mathbb{R}^2$, let $E$ be a square region about $Q_0$, and consider a transformation $\Phi := (\phi_1, \phi_2) : E \to \mathbb{R}^2$, where $\phi_1$ and $\phi_2$ are differentiable at $Q_0$.

Let $P_0 := \Phi(u_0, v_0)$. For $(u, v) \in \mathbb{R}^2$, let

$$\begin{aligned}
\psi_1(u, v) &:= \phi_1(u_0, v_0) + \frac{\partial \phi_1}{\partial u}\Big|_{(u_0,v_0)}(u - u_0) + \frac{\partial \phi_1}{\partial v}\Big|_{(u_0,v_0)}(v - v_0), \\
\psi_2(u, v) &:= \phi_2(u_0, v_0) + \frac{\partial \phi_2}{\partial u}\Big|_{(u_0,v_0)}(u - u_0) + \frac{\partial \phi_2}{\partial v}\Big|_{(u_0,v_0)}(v - v_0).
\end{aligned}$$

By the definition of differentiability of $\phi_1$ and $\phi_2$ at $(u_0, v_0)$,

$$\phi_1(u, v) = \psi_1(u, v) + \epsilon_1(u, v) \quad \text{and} \quad \phi_2(u, v) = \psi_2(u, v) + \epsilon_2(u, v),$$

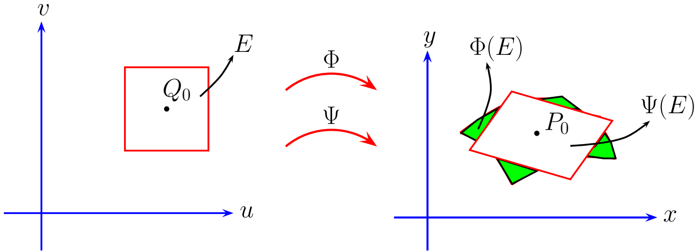
for all $(u, v) \in E$, where $\epsilon_1(u, v)/\|(u - u_0, v - v_0)\| \to 0$ and $\epsilon_2(u, v)/\|(u - u_0, v - v_0)\| \to 0$ as $(u, v) \to (u_0, v_0)$.

Thus the affine transformation $\Psi := (\psi_1, \psi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ maps $Q_0$ to $P_0$, and approximates the transformation $\Phi$ around $Q_0$.

Also,

$$J(\Psi) = \left(\frac{\partial \psi_1}{\partial u}\frac{\partial \psi_2}{\partial v} - \frac{\partial \psi_1}{\partial v}\frac{\partial \psi_2}{\partial u}\right) = \left(\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u}\right)(Q_0) = J(\Phi)(Q_0).$$

Hence Area $(\Psi(E)) = |J(\Psi)|$Area $(E) = |J(\Phi)(Q_0)|$Area $(E)$.




Definition: A subset $\Omega$ of $\mathbb{R}^2$ is called **open** if every point of $\Omega$ is an interior point of $\Omega$.

## Theorem (Change of Variables Formula)

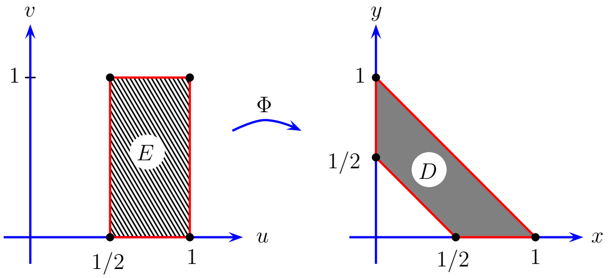
Let $D$ be a closed and bounded subset of $\mathbb{R}^2$ which has an area, and let $f : D \to \mathbb{R}$ be continuous. Suppose $\Omega$ is an open subset of $\mathbb{R}^2$ and $\Phi : \Omega \to \mathbb{R}^2$ is a one-one transformation such that $\Phi := (\phi_1, \phi_2)$, where $\phi_1$ and $\phi_2$ have continuous partial derivatives in $\Omega$ and $J(\Phi)(u, v) \neq 0$ for all $(u, v) \in \Omega$. Let $E \subset \Omega$ be such that $\Phi(E) = D \subset \Phi(\Omega)$. Then $E$ is a closed and bounded subset of $\Omega$ and $E$ has an area. Moreover, $f \circ \Phi : E \to \mathbb{R}$ is continuous, and

$$\iint_D f = \iint_E (f \circ \Phi)(u, v) |J(\Phi)(u, v)| d(u, v).$$

Note: The hypothesis $J(\Phi)(u, v) \neq 0$ for all $(u, v) \in \Omega$ may be weakened by assuming only that $J(\Phi)(u, v) \geq 0$ for all $(u, v) \in \Omega$ or $J(\Phi)(u, v) \leq 0$ for all $(u, v) \in \Omega$, and $J(\Phi)(u, v) = 0$ only for $(u, v)$ in a subset of $\Omega$ of (two-dimensional) content zero.

## Example

Let $D := \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ \ y \geq 0 \text{ and } 1 \leq 2(x + y) \leq 2\}$, and define $f : D \to \mathbb{R}$ by $f(x, y) := y/(x + y)$.




Let $u := x{+}y,\ \ v := y/(x{+}y),$ that is, $x := u(1{-}v),\ \ y := uv$.

We let $\Omega := \{(u, v) \in \mathbb{R}^2 : u > 0\}$, and define $\Phi : \Omega \to \mathbb{R}^2$ by $\Phi(u, v) = \big(u(1 - v), uv\big)$. Then $\Phi$ is one-one from $\Omega$ onto $\{(x, y) \in \mathbb{R}^2 : x + y > 0\}$. If $\Phi = (\phi_1, \phi_2)$, then the partial derivatives of $\phi_1$ and $\phi_2$ exist and are continuous.

Also,

$$J(\Phi)(u,v) = \det \begin{bmatrix} 1-v & -u \\ v & u \end{bmatrix} = u \neq 0 \quad \text{for all } (u,v) \in \Omega.$$

Further, if $E := [1/2, 1] \times [0, 1]$, then $\Phi(E) = D$. Since $f$ is continuous on $D$,

$$\begin{aligned} \iint_D f(x,y)d(x,y) &= \iint_E f(u(1-v), uv)|u|d(u,v) \\ &= \iint_E uv\, d(u,v) \\ &= \left(\int_{1/2}^{1} u\, du\right)\left(\int_0^1 v\, dv\right) \\ &= \frac{3}{16}. \end{aligned}$$

Polar Transformation Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\Phi(r,\theta) := (r\cos\theta,\ r\sin\theta)$. If $\Phi := (\phi_1, \phi_2)$, then $\phi_1$ and $\phi_2$ have continuous partial derivatives in $\mathbb{R}^2$ and

$$J(\Phi)(r,\theta) = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r \quad \text{for all } (r,\theta) \in \mathbb{R}^2.$$

Thus the Jacobian of $\Phi$ is nonzero on $\{(r,\theta) \in \mathbb{R}^2 : r \neq 0\}$. Also, the function $\Phi : \{(r,\theta) \in \mathbb{R}^2 : r > 0 \text{ and } -\pi < \theta \leq \pi\}$ $\to \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$ is one-one and onto. In fact, the following stronger result holds.

## Proposition

Let $D$ be a closed and bounded subset of $\mathbb{R}^2$ which has an area, and let $f : D \to \mathbb{R}$ be continuous. Suppose the set $E := \{(r,\theta) \in \mathbb{R}^2 : r \geq 0, -\pi \leq \theta \leq \pi,\ (r\cos\theta, r\sin\theta) \in D\}$, has an area. Then the function $(r,\theta) \longmapsto f(r\cos\theta, r\sin\theta)$ is continuous on $E$ and
$\iint_D f(x,y)d(x,y) = \iint_E f(r\cos\theta, r\sin\theta)\, r\, d(r,\theta)$.

Examples

Let $D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. Define

$E := \{(r,\theta) \in \mathbb{R}^2 : r \ge 0, -\pi \le \theta \le \pi \text{ and } (r\cos\theta, r\sin\theta) \in D\}$.

Then $E := [0,1] \times [-\pi, \pi]$.

(i) Let $f(x,y) := \sqrt{1 - x^2 - y^2}$ for $(x,y) \in D$. Then

$$\begin{aligned}
\iint_D f(x,y)d(x,y) &= \iint_E f(r\cos\theta, r\sin\theta)\, r\, d(r,\theta) \\
&= \int_{-\pi}^{\pi} \left( \int_0^1 \sqrt{1-r^2}\, r\, dr \right) d\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{2} \left( \int_0^1 \sqrt{s}\, ds \right) d\theta = \frac{2\pi}{3}.
\end{aligned}$$

(ii) Let $f(x,y) := e^{x^2+y^2}$ for $(x,y) \in D$. Then

$$\iint_D f(x,y)d(x,y) = \int_{-\pi}^{\pi} \left( \int_0^1 e^{r^2} r\ dr \right) d\theta = \int_{-\pi}^{\pi} \left( \frac{e-1}{2} \right) d\theta = \pi(e-1).$$

# MA 105: Calculus
# Lecture 19

Prof. B.V. Limaye
IIT Bombay

Wednesday, 15 March 2017

# Triple Integrals

The theory of double integrals readily extends to triple integrals of functions of three variables. No new ideas are needed. We shall, therefore, mention only a few important points.

Consider a **cuboid** $K := [a, b] \times [c, d] \times [p, q]$, where $a < b$, $c < d$, $p < q$. We consider a partition $P$ of $K$ into several subcuboids.

For a bounded function $f : K \to \mathbb{R}$, we define the corresponding lower triple sum $L(P, f)$, the upper triple sum $U(P, f)$, the lower triple integral $L(f)$, and the upper triple integral $U(f)$. As before, $f$ is **(triple) integrable** on $K$, if $L(f) = U(f)$. In this case, its **triple integral** is denoted by

$$\iiint_K f(x, y, z)d(x, y, z) \quad \text{or} \quad \iiint_K f.$$

If $f := 1$ on $K$, then $f$ is integrable on $K$, and its triple integral is equal the volume of $K$, whereas if $f$ is the trivariate Dirichlet function on $K$, then $f$ is not integrable on $K$.

Further, $f$ is integrable on $K$ if and only if it satisfies the Riemann condition. If $f$ is increasing in each of the three variables, or if $f$ is continuous on $K$, then $f$ is integrable.

Let $f$ be integrable on $K$, and let $I$ denote its triple integral. Fubini's theorem on cuboids says that if for each fixed $x \in [a, b]$, the double integral $\iint_{[c,d]\times[p,q]} f(x, y, z)d(y, z)$ exists, and if for each fixed $(x, y) \in [a, b] \times [c, d]$, the Riemann integral $\int_p^q f(x, y, z)dz$ exists, then the **iterated integral** $\int_a^b \left[ \int_c^d \left( \int_p^q f(x, y, z)dz \right) dy \right] dx$ exists and is equal to $I$.

There are other versions of this result with the roles of $x, y, z$ interchanged. It is useful in evaluating triple integrals on $K$.

Triple Riemann sums give approximations of a triple integral.

Let $D$ be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a bounded function. Consider a cuboid $K := [a, b] \times [c, d] \times [p, q]$ such that $D \subset K$, and define $f^* : K \to \mathbb{R}$ by

$$f^*(x, y, z) := \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that $f$ is **integrable** over $D$ if $f^*$ is integrable on $K$, and in this case, the **triple integral** of $f$ (over $D$) is defined to be the triple integral of $f^*$ (on $K$), that is,

$$\iiint_D f(x, y, z) d(x, y, z) := \iiint_K f^*(x, y, z) d(x, y, z).$$

We may also denote the triple integral by $\displaystyle\iiint_D f$.

The triple integral over a bounded subset of $\mathbb{R}^3$ has algebraic and order properties analogous to those of a double integral over a bounded subset of $\mathbb{R}^2$.

In order to seek conditions under which a bounded function $f$ defined on a bounded subset $D$ of $\mathbb{R}^3$ is integrable over $D$, we introduce the following concept.

A bounded subset $E$ of $\mathbb{R}^3$ is of (three-dimensional) **content zero** if for every $\epsilon > 0$, there are finitely many cuboids whose union contains $E$ and the sum of whose volumes is less than $\epsilon$.

Example: Let $D_0$ be a bounded subset of $\mathbb{R}^2$, and let $\varphi : D_0 \to \mathbb{R}$ be integrable over $D_0$. Then its graph $E := \{(x, y, \varphi(x, y)) : (x, y) \in D_0\}$ is of (three-dimensional) content zero. To see this, let $R$ be a rectangle containing $D_0$, and let $\varphi^* : R \to \mathbb{R}$ be the usual extension of $\varphi$. Let $\epsilon > 0$. By the Riemann condition, there is a partition $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ of $R$ such that $U(P, \varphi^*) - L(P, \varphi^*) < \epsilon$. Then the $nk$ cuboids $K_{i,j} := [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [m_{i,j}(\varphi^*), M_{i,j}(\varphi^*)], i = 1, \ldots n;$ $j = 1, \ldots, k$, work.

Remark: The subset $E := [a, b] \times [c, d] \times \{0\}$ of $\mathbb{R}^3$ is of three-dimensional content zero, but the subset $[a, b] \times [c, d]$ of $\mathbb{R}^2$ is not of two-dimensional content zero.

**Theorem**

Let $D$ be a bounded subset of $\mathbb{R}^3$, and $f : D \to \mathbb{R}$ be a bounded function. If the set of discontinuities of $f$ in $D$ is of (three-dimensional) content zero and if the boundary $\partial D$ of $D$ is of (three-dimensional) content zero, then $f$ is integrable over $D$.

We omit the proof.

Suppose $D_0$ is a subset of $\mathbb{R}^2$ having an area, that is, $\partial D_0$ is of two-dimensional content zero. Let $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ be continuous, and let $\psi_1 \leq \psi_2$. Consider $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0 \text{ and } \psi_1(x, y) \leq z \leq \psi_2(x, y)\}$. Then $\partial D$ is of three-dimensional content zero. Hence if a function is continuous on $D$, then it is integrable over $D$.

## Proposition (Cavalieri Principle)

Let $D$ be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be continuous. Suppose $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0$ and $\psi_1(x, y) \le z \le \psi_2(x, y)\}$, where $D_0$ is a subset of $\mathbb{R}^2$ having an area, $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ are continuous and $\psi_1 \le \psi_2$. Then $f$ is integrable on $D$, the **iterated integral** exists, and

$$\iiint_D f = \iint_{D_0} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) dz \right) d(x, y).$$

In particular, if $D_0$ is an elementary region in $\mathbb{R}^2$ and $D_0 := \{(x, y) : \mathbb{R}^2 : a \le x \le b$ and $\phi_1(x) \le y \le \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous and $\phi_1 \le \phi_2$, then

$$\iiint_D f = \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz \right) dy \right) dx.$$

The Cavalieri Principle stated above gives a version of Fubini's theorem for an 'elementary domain' $D$ in $\mathbb{R}^3$. Other versions can be obtained similarly.

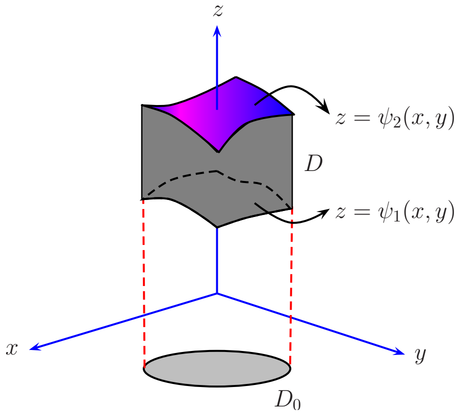



Figure : Illustration of the Cavalieri Principle: a solid between two surfaces defined over $D_0$.

# Constant Function on a Bounded Subset of $\mathbb{R}^3$

Let $D$ be a bounded subset of $\mathbb{R}^3$. Define

$$1_D : D \to \mathbb{R} \quad \text{by} \quad 1_D(x, y, z) := 1 \text{ for all } (x, y, z) \in D.$$

If the boundary $\partial D$ of $D$ is of (three-dimensional) content zero, then the continuous function $1_D$ is integrable on $D$. The converse also holds, that is, if the function $1_D$ is integrable on $D$, then $\partial D$ is of (three-dimensional) content zero.

Examples:

Let $K := [a, b] \times [c, d] \times [p, q]$. If $D := K$, then $1_D$ is integrable and its triple integral is equal to $(b - a)(d - c)(q - p)$. But if $D := \{(x, y, z) \in K : x, y, z \in \mathbb{Q}\}$, then the function $1_D^* : K \to \mathbb{R}$, obtained by extending the function $1_D$ as usual, is the trivariate Dirichlet function on $K$. Now $1_D^*$ is not integrable on $K$, that is, $1_D$ is not integrable over $D$. Note: In this case, $\partial D = K$ is not of three-dimensional content zero.

# Volume of a Bounded Subset of $\mathbb{R}^3$

Let $D \subset \mathbb{R}^3$ be bounded. We say that $D$ has a **volume** if the function $1_D$ is integrable over $D$, and then we define

$$\text{Vol}\,(D) := \iiint_D 1_D(x,y)d(x,y,z).$$

Important special case: Consider an elementary domain $D := \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D_0$ and $\psi_1(x,y) \le z \le \psi_2(x,y)\}$, where $D_0$ is a subset of $\mathbb{R}^2$ having an area, $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ are continuous and $\psi_1 \le \psi_2$.

By the Cavalier Principle, the volume of $D$ is equal to

$$\iint_{D_0} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} dz \right) d(x,y) = \iint_{D_0} \big(\psi_2(x,y) - \psi_1(x,y)\big) d(x,y),$$

which was our definition of the **volume between the surfaces** $z = \psi_1(x,y)$ and $z = \psi_2(x,y)$, $(x,y) \in D_0$.

For instance, let

$$D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 4z^2 \le 6\}.$$

Then $D = \left\{(x, y, z) \in \mathbb{R}^3 : |x| \le \sqrt{6},\ |y| \le \sqrt{6 - x^2} \text{ and } |z| \le \sqrt{6 - x^2 - y^2}/2\right\}$. Hence the volume of the ellipsoidal solid $D$ is equal to

$$8 \int_0^{\sqrt{6}} \left( \int_0^{\sqrt{6-x^2}} \left( \int_0^{\sqrt{6-x^2-y^2}/2} dz \right) dy \right) dx = 4\sqrt{6}\,\pi.$$

Note: It can be shown that the definitions of the volume of a solid used in the slice method, the washer method and the shell method agree with the general definition of the volume of a bounded subset of $\mathbb{R}^3$ given above.

The following basic inequality is important. Let $D$ be a bounded subset of $\mathbb{R}^3$ which has a volume. If $f : D \to \mathbb{R}$ is an integrable function, and if $|f| \leq \alpha$ on $D$, then

$$\left| \iiint_D f(x,y,z)d(x,y,z) \right| \leq \iiint_D |f(x,y,z)| d(x,y,z) \leq \alpha \operatorname{Vol}(D).$$

The following result is useful in evaluating a triple integral.

**Theorem (Domain Additivity)**

Let $D$ be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a bounded function. Suppose $D = D_1 \cup D_2$, where $D_1 \cap D_2$ is of (three-dimensional) content zero. If $f$ is integrable over $D_1$ and over $D_2$, then $f$ is integrable over $D$ and

$$\iiint_D f(x,y)d(x,y) = \iiint_{D_1} f(x,y)d(x,y) + \iiint_{D_2} f(x,y)d(x,y).$$

# Change of Variables in a Triple Integral

To simplify the calculation of $\iiint_D f(x, y, z) d(x, y, z)$, we may reduce it to another triple integral $\iiint_E g(u, v, w) d(u, v, w)$.

Let $E \subset \mathbb{R}^3$ and $\Phi := (\phi_1, \phi_2, \phi_3) : E \to \mathbb{R}^3$. Suppose $Q_0 := (u_0, v_0, w_0)$ is an interior point of $E$.
If $x := \phi_1(u, v, w)$, $y := \phi_2(u, v, w)$ and $z := \phi_3(u, v, w)$ for $(u, v, w) \in E$, and the functions $x$, $y$ and $z$ have partial derivatives at $Q_0$, then the **Jacobian** of $\Phi$ at $Q_0$ is given by

$$J(\Phi)(Q_0) := \frac{\partial(x, y, z)}{\partial(u, v, w)}(Q_0) := \det \begin{bmatrix} \frac{\partial x}{\partial u}(Q_0) & \frac{\partial x}{\partial v}(Q_0) & \frac{\partial x}{\partial w}(Q_0) \\ \frac{\partial y}{\partial u}(Q_0) & \frac{\partial y}{\partial v}(Q_0) & \frac{\partial y}{\partial w}(Q_0) \\ \frac{\partial z}{\partial u}(Q_0) & \frac{\partial z}{\partial v}(Q_0) & \frac{\partial z}{\partial w}(Q_0) \end{bmatrix}.$$

## Theorem (Change of Variables Formula)

Let $D$ be a closed and bounded subset of $\mathbb{R}^3$ which has a volume. Let $f : D \to \mathbb{R}$ be continuous. Suppose $\Omega$ is an open subset of $\mathbb{R}^3$ and $\Phi : \Omega \to \mathbb{R}^3$ is a one-one transformation such that $\Phi := (\phi_1, \phi_2, \phi_3)$, where $\phi_1, \phi_2, \phi_3$ have continuous partial derivatives in $\Omega$ and $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$. Let $E \subset \Omega$ be such that $\Phi(E) = D \subset \Phi(\Omega)$. Then $E$ is a closed and bounded subset of $\Omega$ and $E$ has a volume. Moreover, $f \circ \Phi : E \to \mathbb{R}$ is continuous, and

$$\iiint_D f = \iiint_E (f \circ \Phi)(u, v, w)|J(\Phi)(u, v, w)|d(u, v, w).$$

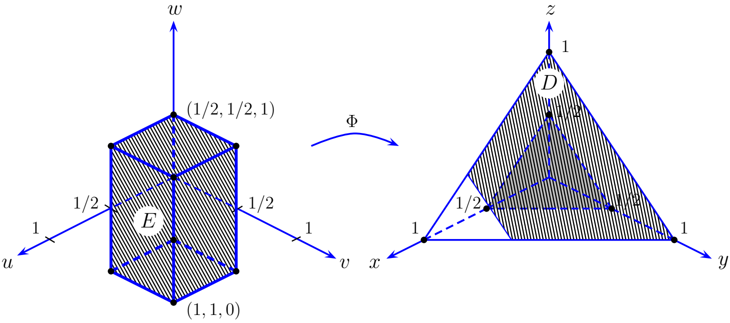
Note: The hypothesis $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$ may be weakened by assuming only that $J(\Phi)(u, v, w) \geq 0$ for all $(u, v, w) \in \Omega$ or $J(\Phi)(u, v, w) \leq 0$ for all $(u, v, w) \in \Omega$, and $J(\Phi)(u, v, w) = 0$ only for $(u, v, w)$ in a subset of $\Omega$ of (three-dimensional) content zero.

Example: Let

$$D := \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x \leq y+z,\ 1 \leq 2(x+y+z) \leq 2\},$$

and define $f : D \to \mathbb{R}$ by $f(x, y, z) := z/(y+z)$.

(Note: $(x, y, z) \in D \implies 0 \leq x \leq x/2 + (y+z)/2 \leq 1/2$.)




$$\text{Let } u := x + y + z, \quad v := \frac{y+z}{x+y+z}, \quad w := \frac{z}{y+z},$$

$$\text{that is, } x := u(1-v), \quad y := u\,v(1-w), \quad z := u\,v\,w.$$

Consider $\Omega := \{(u, v, w) \in \mathbb{R}^3 : u > 0 \text{ and } v > 0\}$, and define $\Phi : \Omega \to \mathbb{R}^3$ by $\Phi(u, v, w) := \big(u(1-v), u\,v(1-w), u\,v\,w\big)$. Now $\Phi : \Omega \to \Phi(\Omega) := \{(x, y, z) \in \mathbb{R}^2 : x+y+z > 0 \text{ and } y+z > 0\}$ is one-one and onto. Also, if $\Phi = (\phi_1, \phi_2, \phi_3)$, then the partials of $\phi_1$, $\phi_2$, and $\phi_3$ are continuous, and for all $(u, v, w) \in \Omega$,

$$J(\Phi)(u, v, w) = \det \begin{bmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -u\,v \\ v\,w & u\,w & u\,v \end{bmatrix} = u^2 v \neq 0.$$

Let $E := [1/2, 1] \times [1/2, 1] \times [0, 1]$. Then $\Phi(E) = D$, and $(f \circ \Phi)(u, v, w) = w$ for $(u, v, w) \in E$. Hence

$$\begin{aligned} \iiint_D f &= \iiint_E f(u(1-v), u\,v(1-w), u\,v\,w)|u^2 v| d(u, v, w) \\ &= \iiint_E u^2 v\,w\,d(u, v, w) \\ &= \left(\int_{1/2}^1 u^2\,du\right)\left(\int_{1/2}^1 v\,dv\right)\left(\int_0^1 w\,dw\right) = \frac{7}{128}. \end{aligned}$$

Two important cases involving a change of variables in triple integrals are given by switching to cylindrical coordinates, and switching to spherical coordinates.

(i) Cylindrical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(r, \theta, z) := (r\cos\theta,\ r\sin\theta,\ z) \quad \text{for } (r, \theta, z) \in \mathbb{R}^3.$$

Then for all $(r, \theta, z) \in \mathbb{R}^3$,

$$J(\Phi)(r, \theta, z) = \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r.$$

The Jacobian of $\Phi$ is nonzero on $\{(r, \theta, z) \in \mathbb{R}^3 : r \neq 0\}$.

The function $\Phi : \{(r, \theta, z) \in \mathbb{R}^3 : r > 0 \text{ and } -\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

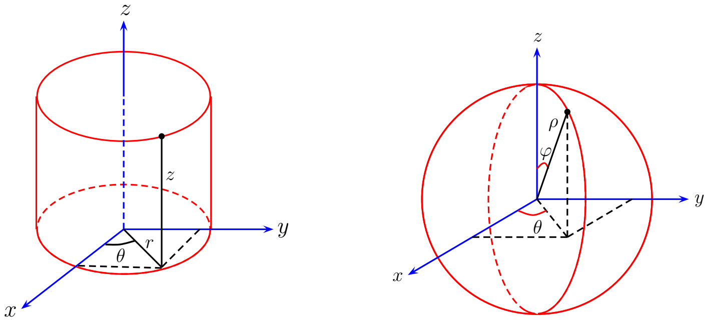



Figure : Illustrations of cylindrical and spherical coordinates.

(ii) Spherical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(\rho, \varphi, \theta) := (\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta,\ \rho \cos\varphi) \quad \text{for } (\rho, \varphi, \theta) \in \mathbb{R}^3.$$

Then for all $(\rho, \varphi, \theta) \in \mathbb{R}^3$,

$$\begin{aligned} J(\Phi)(\rho, \varphi, \theta) &= \det \begin{bmatrix} \sin\varphi \cos\theta & \rho \cos\varphi \cos\theta & -\rho \sin\varphi \sin\theta \\ \sin\varphi \sin\theta & \rho \cos\varphi \sin\theta & \rho \sin\varphi \cos\theta \\ \cos\varphi & -\rho \sin\varphi & 0 \end{bmatrix} \\ &= \rho^2 \sin\varphi. \end{aligned}$$

The Jacobian of $\Phi$ is nonzero on the set
$\{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \neq 0 \text{ and } \varphi \neq m\pi \text{ for any } m \in \mathbb{Z}\}$.

The function $\Phi : \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho > 0,\ 0 < \varphi < \pi$ and $-\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

In fact, the following stronger results hold.

## Proposition

Let $D$ be a closed and bounded subset of $\mathbb{R}^3$ which has a volume, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $E := \{(r, \theta, z) \in \mathbb{R}^3 : r \geq 0, -\pi \leq \theta \leq \pi$ and $(r\cos\theta,\ r\sin\theta,\ z) \in D\}$, and if $E$ has a volume, then the triple integral of $f$ over $D$ is equal to

$$\iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z).$$

(ii) If $E := \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \geq 0,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi$ and $(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \in D\}$, and if $E$ has a volume, then the triple integral of $f$ over $D$ is equal to

$$\iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2\sin\varphi\ d(\rho, \varphi, \theta).$$

Examples

(i) Let $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1 \text{ and } 0 \le z \le 1\}$, and define $f : D \to \mathbb{R}$ by $f(x, y, z) := z\sqrt{1 - x^2 - y^2}$. Then

$$\begin{aligned} E &:= \{(r, \theta, z) \in \mathbb{R}^3 : r \ge 0,\ -\pi \le \theta \le \pi,\ (r\cos\theta, r\sin\theta, z) \in D\} \\ &= [0, 1]\times[-\pi, \pi]\times[0, 1]. \end{aligned}$$

Hence

$$\begin{aligned} \iiint_D f &= \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z) \\ &= \int_0^1 \left[\int_{-\pi}^{\pi} \left(\int_0^1 z\sqrt{1 - r^2}\, r\, dz\right) d\theta\right] dr \\ &= \pi \int_0^1 \sqrt{1 - r^2}\, r\, dr = \frac{\pi}{2}\int_0^1 \sqrt{s}\, ds = \frac{\pi}{3}. \end{aligned}$$

(ii) Let $a > 0$ and $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le a^2\}$. Define $f : D \to \mathbb{R}$ by $f(x, y, z) = z^2$. Then

$$\begin{aligned} E &:= \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \ge 0,\ 0 \le \varphi \le \pi,\ -\pi \le \theta \le \pi \text{ and} \\ &\qquad (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi) \in D\} \\ &= [0, a] \times [0, \pi] \times [-\pi, \pi]. \end{aligned}$$

Hence the triple integral of $f$ over $D$ is equal to

$$\begin{aligned} &\iiint_E f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)\, \rho^2 \sin\varphi \, d(\rho, \varphi, \theta) \\ &= \int_0^a \left[ \int_0^\pi \left( \int_{-\pi}^{\pi} (\rho^2 \cos^2\varphi)\, \rho^2 \sin\varphi \, d\theta \right) d\varphi \right] d\rho \\ &= 2\pi \int_0^a \rho^4 \left( \int_0^\pi \cos^2\varphi \sin\varphi \, d\varphi \right) d\rho = \frac{2\pi a^5}{5} \cdot \frac{2}{3} = \frac{4\pi a^5}{15}. \end{aligned}$$

# Tutorial 10

1. Which of the following subsets of $\mathbb{R}^2$ have areas? Find the areas if they exist.
   (i) $\{(x, y) \in \mathbb{R}^2 : b^2x^2 + a^2y^2 \leq a^2b^2\}$, where $a, b > 0$,
   (ii) $\{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}$ and $x^2 + y^2 \leq 1\}$.

2. Evaluate $\displaystyle\int_0^1 \left( \int_y^1 x^2 e^{xy} dx \right) dy$.

3. Let $D$ be the parallelogram with vertices at $(\pi, 0), (2\pi, \pi)$, $(\pi, 2\pi), (0, \pi)$. Find $\displaystyle\iint_D (x - y)^2 \sin^2(x + y) d(x, y)$.

4. Which of the following subsets of $\mathbb{R}^3$ have volumes? Find the volumes if they exist.
   (i) $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y \leq 1, \ x^2 + y^2 \leq z \leq x + y\}$,
   (ii) $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1$ and $x, y, z \notin \mathbb{Q}\}$.

5. Let $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$. Find the triple integral of the function $f(x, y, z) := e^{(x^2+y^2+z^2)^{3/2}}$ over $D$.

# MA 105: Calculus
# Lecture 20

Prof. B.V. Limaye
IIT Bombay

Friday, 17 March 2017

# Vector Algebra

We begin a part of this course known as '**Vector Analysis**'.
It deals with scalar fields and vector fields.
We begin with some concepts in '**Vector Algebra**'.
In Lecture 12, we have defined the **Euclidean space**

$$\mathbb{R}^m := \{\boldsymbol{x} = (x_1, \ldots, x_m) : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, m\},$$

where $m \in \mathbb{N}$. An element of $\mathbb{R}^1 := \mathbb{R}$ is called a **scalar**, and an element of $\mathbb{R}^m$ is called a **vector** if $m \geq 2$.

Let $\boldsymbol{x} := (x_1, \ldots, x_m)$, $\boldsymbol{y} := (y_1, \ldots, y_m) \in \mathbb{R}^m$, and $a \in \mathbb{R}$. We have already defined the sum $\boldsymbol{x} + \boldsymbol{y}$ and the scalar multiple $a\,\boldsymbol{x}$. Also, we have studied the **scalar product** of $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\boldsymbol{x} \cdot \boldsymbol{y} := x_1 y_1 + \cdots + x_m y_m \in \mathbb{R}.$$

It is often called the **dot product** of $\boldsymbol{x}$ and $\boldsymbol{y}$.

# Vector Product

Let $m := 3$, $\boldsymbol{x} := (x_1, x_2, x_3)$ and $\boldsymbol{y} := (y_1, y_2, y_3)$. We now define the **vector product** of $\boldsymbol{x}$ with $\boldsymbol{y}$:

$$\boldsymbol{x} \times \boldsymbol{y} := (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1) \in \mathbb{R}^3.$$

It is often called the **cross product** of $\boldsymbol{x}$ with $\boldsymbol{y}$. Let

$$\boldsymbol{i} := (1, 0, 0), \quad \boldsymbol{j} := (0, 1, 0), \quad \boldsymbol{k} := (0, 0, 1).$$

Then $\boldsymbol{x} := x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$, $\boldsymbol{y} := y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k}$, and

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Note: $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$.

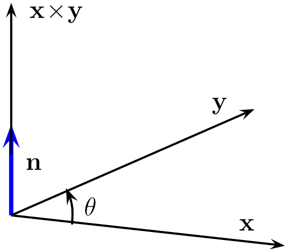
Properties of a determinant show that for every $\boldsymbol{z} \in \mathbb{R}^3$,

$$(\boldsymbol{x} + \boldsymbol{y}) \times \boldsymbol{z} = (\boldsymbol{x} \times \boldsymbol{z}) + (\boldsymbol{y} \times \boldsymbol{z}) \quad \text{and} \quad \boldsymbol{y} \times \boldsymbol{x} = -(\boldsymbol{x} \times \boldsymbol{y}).$$

Let $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{y} \neq \mathbf{0}$. It can be shown that

$$\boldsymbol{x} \times \boldsymbol{y} = \|\boldsymbol{x}\| \, \|\boldsymbol{y}\| (\sin\theta) \, \boldsymbol{n},$$

where $\theta \in [0, \pi]$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{n}$ is the unit vector which is perpendicular to the plane containing $\boldsymbol{x}$ and $\boldsymbol{y}$, and obeys the 'right-hand rule'.




Hence $\|\boldsymbol{x} \times \boldsymbol{y}\|$ = the area of the parallelogram with sides $\boldsymbol{x}$, $\boldsymbol{y}$.

# Scalar Triple Product and Vector triple Product

Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in \mathbb{R}^3$. Then $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) \in \mathbb{R}$ and $\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) \in \mathbb{R}^3$ are called the **scalar triple product** and the **vector triple product** of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ respectively. It is easy to see that if $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3)$, $\boldsymbol{z} = (z_1, z_2, z_3)$, then

$$\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) \;=\; \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Geometrically, $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z})$ can be interpreted as the (signed) volume of the parallelopiped defined by the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

One can prove the **Lagrange formula**

$$\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{z})\,\boldsymbol{y} - (\boldsymbol{x} \cdot \boldsymbol{y})\,\boldsymbol{z}$$

by considering each component of the LHS and the RHS.

# Scalar Fields and Vector Fields

A **scalar field** is an assignment of a scalar to each point in a region in the space. For example, the temperature at a point on the earth is a scalar field (defined on a subset of $\mathbb{R}^3$.)

A **vector field** is an assignment of a vector to each point in a region in the space. For example, the velocity field of a moving fluid is a vector field that associates a velocity vector to each point in the fluid.

**Definition**

Let $m \in \mathbb{N}$, and let $D$ be a subset of $\mathbb{R}^m$.

A **scalar field** is a map from $D$ to $\mathbb{R}$.

A **vector field** is a map from $D$ to $\mathbb{R}^m$.
If $m = 2$, then it is called a **vector field in the plane**, and if $m = 3$, then it is called a **vector field in the space**.

# Smooth Scalar and Vector Fields

Suppose $D$ is an **open** subset of $\mathbb{R}^m$, that is, every point in $D$ is an interior point of $D$.
A scalar field $f : D \to \mathbb{R}$ is called **smooth** if all first order partial derivatives of $f$ exist and are continuous on $D$. The set of all smooth scalar fields on $D$ is denoted by $C^1(D)$. Similarly, the set of all scalar fields on $D$ having continuous partial derivatives of the first and second order is denoted by $C^2(D)$.

Let $\boldsymbol{F} : D \to \mathbb{R}^m$ be a vector field on $D$, and let

$$\boldsymbol{F}(\boldsymbol{x}) = (F_1(\boldsymbol{x}), \ldots, F_m(\boldsymbol{x})) \quad \text{for } \boldsymbol{x} \in D,$$

where $F_i : D \to \mathbb{R}$ is scalar field on $D$, $i = 1, \ldots, m$. Then the vector field $\boldsymbol{F}$ is called **smooth** if each component scalar field $F_i$ is smooth, that is, if each $\dfrac{\partial F_i}{\partial x_j}$, $i, j = 1, \ldots, m$ exists and is continuous on $D$.

# Gradient, Divergence and Curl

Let $f$ be a smooth scalar field on $D \subset \mathbb{R}^3$. Then the vector field

$$\operatorname{grad} f := \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

defined on $D$ is called the **gradient field** of $f$.

Next, let $\boldsymbol{F} := (P, Q, R)$ be a smooth vector field on $D \subset \mathbb{R}^3$. The **divergence field** of $\boldsymbol{F}$ is the scalar field on $D$ defined by

$$\operatorname{div} \boldsymbol{F} := \nabla \cdot \boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

and the **curl field** of $\boldsymbol{F}$ is the vector field on $D$ defined by

$$\operatorname{curl} \boldsymbol{F} := \nabla \times \boldsymbol{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

As in the case of the cross product $\boldsymbol{x} \times \boldsymbol{y}$, we may write

$$\nabla \times \boldsymbol{F} = \nabla \times (P, Q, R) \quad = \quad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

For any $m \in \mathbb{N}$, we can define the gradient field $\operatorname{grad} f$ of a scalar field $f$ on an open subset of $\mathbb{R}^m$ as well as the divergence field $\operatorname{div} \boldsymbol{F}$ of a vector field $\boldsymbol{F}$ on an open subset of $\mathbb{R}^m$ in a similar manner. But the curl field $\operatorname{curl} \boldsymbol{F}$ can only be defined for a vector field $\boldsymbol{F}$ on an open subset of $\mathbb{R}^3$.

Of course, if $D$ is an open subset of $\mathbb{R}$ and $f$ is a smooth scalar field on $D$, then we can let $\boldsymbol{F}(x, y, z) := (f(x), 0, 0)$ for $(x, y, z) \in D \times \mathbb{R}^2$, and define $\operatorname{curl} f := \operatorname{curl} \boldsymbol{F}$. Also, if $D$ is an open subset of $\mathbb{R}^2$ and $\boldsymbol{G}$ is a smooth vector field on $D$, then we can let $\boldsymbol{F}(x, y, z) := (G(x, y), 0)$ for $(x, y, z) \in D \times \mathbb{R}$, and define $\operatorname{curl} \boldsymbol{G} := \operatorname{curl} \boldsymbol{F}$.

# The GCD Sequence

Suppose the first and second order partial derivatives of $f$ and of $P, Q, R$ exist and are continuous on $D$. By the Mixed Partials Theorem, we obtain (i) $\text{curl}\,(\text{grad}\, f) = \nabla \times (\nabla f) = \mathbf{0}$:

$$\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = (0, 0, 0),$$

and also (ii) $\text{div}\,(\text{curl}\, \boldsymbol{F}) = \nabla \cdot (\nabla \times \boldsymbol{F}) = 0$:

$$\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

Thus Gradient, Curl and Divergence give a sequence of maps

$$\left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{grad}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{curl}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{div}} \left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\}$$

which satisfies $\text{curl}\,(\text{grad}\, f) = \mathbf{0}$ and $\text{div}\,(\text{curl}\, \boldsymbol{F}) = 0$, so that the successive composites are zero.

The above phenomenon raises the following basic questions:

(i) If $\boldsymbol{G}$ is a smooth vector field such that $\operatorname{curl} \boldsymbol{G} = \mathbf{0}$, then is $\boldsymbol{G}$ a gradient field, that is, is there a scalar field $f$ such that $\boldsymbol{G} = \operatorname{grad} f$?

(ii) If $\boldsymbol{H}$ is smooth vector field such that $\operatorname{div} \boldsymbol{H} = 0$, then is $\boldsymbol{H}$ a curl field, that is, is there a vector field $\boldsymbol{F}$ such that $\boldsymbol{H} = \operatorname{curl} \boldsymbol{F}$?

These questions are reminiscent of the Fundamental theorem of Calculus, Part I, which answers the following question in the affirmative: If $g$ is a continuous function on $[a, b]$, then is there an antiderivative of $g$, that is, is there a differentiable function $f$ such that $g = f'$?

As such, the two questions raised above call for a suitable theory of integration, to which we now turn. Eventually, we shall come back to these questions.

# Laplacian

Let $f$ be a smooth vector field on $D \subset \mathbb{R}^3$, and suppose the second order partials $f_{xx}, f_{yy}, f_{zz}$ exist on $D$. Let us consider the maps

$$\begin{Bmatrix} \text{scalar} \\ \text{fields} \end{Bmatrix} \xrightarrow{\text{grad}} \begin{Bmatrix} \text{vector} \\ \text{fields} \end{Bmatrix} \xrightarrow{\text{div}} \begin{Bmatrix} \text{scalar} \\ \text{fields} \end{Bmatrix}.$$

The **Laplacian field** of $f$ is the scalar field defined on $D$ by

$$\text{div}\,(\text{grad}\, f) := \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z}.$$

The Laplacian plays a very important role in the theory of partial differential equations, and its various applications.

# Paths

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and let $m \in \mathbb{N}$. A **path** or a **parametrized curve** in $\mathbb{R}^m$ is a continuous map $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$, that is, if $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$, then $\gamma_j : [\alpha, \beta] \to \mathbb{R}$ is continuous for each $j = 1, \ldots, m$.

A path $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ is called **closed** if $\boldsymbol{\gamma}(\alpha) = \boldsymbol{\gamma}(\beta)$. Such a path is also called a **loop**. A path $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ is called **simple** if $\boldsymbol{\gamma}(t_1) \neq \boldsymbol{\gamma}(t_2)$ for all $t_1, t_2 \in (\alpha, \beta)$, $t_1 \neq t_2$.

If for $t \in [\alpha, \beta]$,

$$\frac{d\boldsymbol{\gamma}}{dt} = \boldsymbol{\gamma}'(t) := (\gamma_1'(t), \ldots, \gamma_m'(t))$$

exists, then it is called the **tangent vector** to $\boldsymbol{\gamma}$ at $t$, and if it is nonzero, then $\hat{\boldsymbol{t}} := \boldsymbol{\gamma}'(t)/\|\boldsymbol{\gamma}'(t)\|$ is called the **unit tangent vector** to $\boldsymbol{\gamma}$ at $t$. (We write $\hat{\boldsymbol{t}}$ instead of $\hat{\boldsymbol{t}}_{\gamma}(t)$ for brevity.)

Further, a path $\boldsymbol{\gamma}$ in $\mathbb{R}^m$ is called **smooth** or $C^1$ if each $\gamma_j : [\alpha, \beta] \to \mathbb{R}$ is continuously differentiable for $j = 1, \ldots, m$; in case $\boldsymbol{\gamma}$ is a closed curve, that is, $\boldsymbol{\gamma}(\alpha) = \boldsymbol{\gamma}(\beta)$, we also require $\boldsymbol{\gamma}'(\alpha) = \boldsymbol{\gamma}'(\beta)$.

A smooth path $\boldsymbol{\gamma}$ in $\mathbb{R}^m$ is called **regular** if $\boldsymbol{\gamma}'(t) \neq \mathbf{0}$ for all $t \in [\alpha, \beta]$, that is, if the unit tangent vector to $\boldsymbol{\gamma}$ exists at each $t \in [\alpha, \beta]$.

A path $\boldsymbol{\gamma}$ in $\mathbb{R}^m$ is called **piecewise smooth** if there are $\alpha := t_0 < t_1 < \cdots < t_n =: \beta$ such that $\boldsymbol{\gamma}$ is smooth on each $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, and it is called **piecewise regular**, if $\boldsymbol{\gamma}$ is regular on each $[t_{i-1}, t_i]$, $i = 1, \ldots, n$.

We shall assume hereafter that all paths are piecewise smooth, unless otherwise stated.

# Path-connected and Convex Subsets

A subset $D$ of $\mathbb{R}^m$ is called **path-connected** if for every $\boldsymbol{u}, \boldsymbol{v} \in D$, there is a path $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ such that $\boldsymbol{\gamma}(\alpha) = \boldsymbol{u}$, $\boldsymbol{\gamma}(\beta) = \boldsymbol{v}$ and $\boldsymbol{\gamma}(t) \in D$ for all $t \in (\alpha, \beta)$.

In particular, a **convex** subset $D$ of $\mathbb{R}^m$ is path-connected since for $\boldsymbol{u}, \boldsymbol{v} \in D$, there is the straight-line path $\gamma : [0, 1] \to \mathbb{R}^m$ defined by $\boldsymbol{\gamma}(t) := \boldsymbol{u} + t(\boldsymbol{v} - \boldsymbol{u}) \in D$ for $t \in [0, 1]$.

Examples:

(i) The subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ of $\mathbb{R}^2$ is path-connected; in fact it is convex.

(ii) The subset $\{(x, y) \in \mathbb{R}^2 : 1/2 \leq x^2 + y^2 \leq 1\}$ of $\mathbb{R}^2$ is path-connected, but it is not convex.

(iii) The subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(2, 0)\}$ of $\mathbb{R}^2$ is not path-connected.

## Example

Let $a > 0$. Define $\boldsymbol{\gamma}(t) := (a\cos t, a\sin t)$ for $t \in [-\pi, \pi]$. This path is called the **standard parametrized circle in** $\mathbb{R}^2$ **of radius** $a$. Since $\boldsymbol{\gamma}(-\pi) = \boldsymbol{\gamma}(\pi)$ and $\boldsymbol{\gamma}'(t) := (-a\sin t, a\cos t)$ for $t \in [-\pi, \pi]$, we see that $\boldsymbol{\gamma}$ is closed, simple, smooth, regular, and its unit tangent vector at $t \in [-\pi, \pi]$ is $\hat{\boldsymbol{t}} := (-\sin t, \cos t)$.

If we let $\widetilde{\boldsymbol{\gamma}}(t) := (a\cos 2t, a\sin 2t)$ for $t \in [-\pi, \pi]$, then it is easy to see that $\widetilde{\boldsymbol{\gamma}}$ is also closed, smooth, regular, and its unit tangent vector at $t \in [-\pi, \pi]$ is $(-\sin 2t, \cos 2t)$.

Note that $\widetilde{\boldsymbol{\gamma}}([-\pi, \pi]) = \boldsymbol{\gamma}([-\pi, \pi])$, that is, the functions $\widetilde{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}$ have the same range, namely $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, although they are clearly different paths: one goes around the circle once, but the other does so twice.

# MA 105: Calculus
# Lecture 21

Prof. B.V. Limaye
IIT Bombay

Wednesday, 22 March 2017

# Arc length

Let $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_m) : [\alpha, \beta] \to \mathbb{R}^m$ be a piecewise smooth parametrized curve. We define the **arc length** of $\boldsymbol{\gamma}$ by

$$\ell(\boldsymbol{\gamma}) := \int_\alpha^\beta \|\boldsymbol{\gamma}'(t)\| dt = \int_\alpha^\beta \sqrt{\gamma_1'(t)^2 + \cdots + \gamma_m'(t)^2} dt.$$

This definition agrees with our earlier definition for $m = 2, 3$.

Example

Let $\boldsymbol{\gamma}$ denote the standard parametrized circle in $\mathbb{R}^2$ of radius $a$. Then $\ell(\boldsymbol{\gamma}) = \int_{-\pi}^{\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\, dt = 2\pi a$.

Now let $\widetilde{\boldsymbol{\gamma}}(t) := (a\cos 2t, a \sin 2t)$ for $t \in [-\pi, \pi]$. Then $\ell(\widetilde{\boldsymbol{\gamma}}) = \int_{-\pi}^{\pi} \sqrt{4a^2 \sin^2 2t + 4a^2 \cos^2 2t}\, dt = 4\pi a$.

Note that $\widetilde{\boldsymbol{\gamma}}([-\pi, \pi]) = \boldsymbol{\gamma}([-\pi, \pi])$, but $\ell(\widetilde{\boldsymbol{\gamma}}) = 2\ell(\boldsymbol{\gamma})$.

# Line Integral of a Scalar Field

Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path, and let $C := \gamma([\alpha, \beta])$. Let $f : C \to \mathbb{R}$ be a bounded scalar field. We define the **line integral** of $f$ **along** $\boldsymbol{\gamma}$ by

$$\int_{\boldsymbol{\gamma}} f \, ds := \int_{\alpha}^{\beta} f(\boldsymbol{\gamma}(t)) \|\boldsymbol{\gamma}'(t)\| dt,$$

provided the Riemann integral on the right side exists. In particular, if $f$ is continuous, then $\int_{\boldsymbol{\gamma}} f \, ds$ is well-defined.

Special case: If $f := 1$ on $C$, then clearly $\int_{\boldsymbol{\gamma}} f \, ds = \ell(\boldsymbol{\gamma})$.

The algebraic and the order properties of the Riemann integral imply similar properties of the line integral of a scalar field.

Example: Let $f(x, y) := x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$. If $\boldsymbol{\gamma}$ is the standard parametrized circle of radius $a$, then $\int_{\boldsymbol{\gamma}} f \, ds = 2\pi a^3$.

# Reparametrization of a Curve

Let $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_m) : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth parametrized curve. Let $h : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ be continuously differentiable, $h'(\tilde{t}) \neq 0$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, and let $h([\tilde{\alpha}, \tilde{\beta}]) = [\alpha, \beta]$. Then the smooth path $\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\gamma} \circ h$ is called a reparametrization of $\boldsymbol{\gamma}$.

Define $C := \boldsymbol{\gamma}([\alpha, \beta]$, and let $f : C \to \mathbb{R}$ be a continuous scalar field. By the chain rule and the change of variable formula for Riemann integration,

$$\begin{aligned}
\int_{\widetilde{\boldsymbol{\gamma}}} f\,ds &= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\widetilde{\boldsymbol{\gamma}}(\tilde{t}))\|\widetilde{\boldsymbol{\gamma}}'(\tilde{t})\|d\tilde{t} \\
&= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\boldsymbol{\gamma}(h(\tilde{t}))\|\boldsymbol{\gamma}'(h(\tilde{t}))\||h'(\tilde{t})|d\tilde{t} \\
&= \int_{\alpha}^{\beta} f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|dt = \int_{\boldsymbol{\gamma}} f\,ds.
\end{aligned}$$

Thus the line integral of a continuous scalar field along a smooth path is invariant under reparametrization. In particular, the length of a smooth path is invariant under reparametrization.

Here is a nice and useful way to reparametrize a regular path $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$. For $t \in [\alpha, \beta]$, let

$$s(t) := \int_\alpha^t \|\boldsymbol{\gamma}'(\tau)\| d\tau.$$

Then $s(\alpha) = 0$, $s(\beta) = \ell(\boldsymbol{\gamma})$, and $s'(t) = \|\boldsymbol{\gamma}'(t)\|$ for all $t \in [\alpha, \beta]$ by the FTC, Part I. Since $\boldsymbol{\gamma}$ is a regular path, $s'(t) > 0$ for all $t \in [\alpha, \beta]$, and so $t \longmapsto s(t)$ is a strictlty increasing differentiable function from $[\alpha, \beta]$ onto $[0, \ell(\boldsymbol{\gamma})]$. Let $h : [0, \ell(\boldsymbol{\gamma})] \to [\alpha, \beta]$ denote its inverse. Then $h$ is differentiable and its derivative does not vanish on $[0, \ell(\boldsymbol{\gamma})]$. Define $\widetilde{\boldsymbol{\gamma}}(s) := \boldsymbol{\gamma}(h(s))$ for $s \in [0, \ell(\boldsymbol{\gamma})]$. This reparametrization of a regular curve $\boldsymbol{\gamma}$ is known as the **arc length parametrization**, and $s$ is called the **arc length parameter**.

Example: If $\gamma$ is the standard parametrized circle in $\mathbb{R}^2$ of radius $a$ and centre $(0,0)$, then

$$s(t) = \int_{-\pi}^{t} \|(-a\sin t, a\cos t)\| dt = a(t+\pi) \quad \text{for } t \in [-\pi, \pi],$$

and so the arc length parametrization is given by

$$\widetilde{\gamma}(s) = \left(a\cos\left(\frac{s}{a} - \pi\right), \, a\sin\left(\frac{s}{a} - \pi\right)\right) = \left(-a\cos\frac{s}{a}, \, a\sin\frac{s}{a}\right)$$

for $s \in [0, 2\pi a]$.

In view of the equality $s'(t) = ds/dt = \|\gamma'(t)\|$ for $t \in [\alpha, \beta]$ given above, we introduce the following

Differential Notation:

For a regular parametrized curve $\gamma$, let $ds := \|\gamma'(t)\| dt$. Then

$$\gamma'(t) = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} = \frac{d\gamma}{ds}\|\gamma'(t)\|, \quad \text{and so} \quad \hat{\boldsymbol{t}} = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{d\gamma}{ds}.$$

# Geometric Curve

Let $[\alpha, \beta]$ be an interval in $\mathbb{R}$, and let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a parametrized curve. Then $C := \boldsymbol{\gamma}([\alpha, \beta])$ is called a **geometric curve**, and $\boldsymbol{\gamma}$ gives a **parametrization** of $C$.

Suppose $\boldsymbol{\gamma}$ is simple and smooth. In view of the reparametrization result for continuous scalar fields on parametrized curves proved above, we define

$$\int_C f\, ds := \int_{\boldsymbol{\gamma}} f\, ds = \int_\alpha^\beta f(\boldsymbol{\gamma}(t)) \|\boldsymbol{\gamma}'(t)\| dt$$

for a continuous scalar field $f$ on $C$, and, in particular,

$$\ell(C) := \ell(\boldsymbol{\gamma}) = \int_\alpha^\beta \|\boldsymbol{\gamma}'(t)\| dt.$$

# Line Integral of a Vector Field

Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path, and let $C := \gamma([\alpha, \beta])$. Let $\boldsymbol{F} : C \to \mathbb{R}^m$ be a bounded vector field. We define the **line integral** of $\boldsymbol{F}$ **along** $\boldsymbol{\gamma}$ by

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} := \int_{\alpha}^{\beta} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) dt,$$

provided the Riemann integral on the right side exists. In particular, if $\boldsymbol{F}$ is continuous, then $\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$ is well-defined.

In analogy with the differential notation $ds := \|\boldsymbol{\gamma}'(t)\| dt$, let

$$d\boldsymbol{s} := \boldsymbol{\gamma}'(t) dt.$$

Let $\gamma(t) := (x(t), y(t), z(t))$ for $t \in [\alpha, \beta]$. Then

$$\gamma' := (x', y', z').$$

If $\boldsymbol{F} := (P, Q, R)$, then the line integral is also written as

$$\int_{\alpha}^{\beta} \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt = \int_{\boldsymbol{\gamma}} P\, dx + Q\, dy + R\, dz.$$

If the path $\boldsymbol{\gamma}$ is regular, then

$$d\boldsymbol{s} = \boldsymbol{\gamma}'(t)dt = \hat{\boldsymbol{t}}\, \|\boldsymbol{\gamma}'(t)\| dt = \hat{\boldsymbol{t}}\, ds,$$

and so we write the line integral of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$ as $\displaystyle\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \hat{\boldsymbol{t}}\, ds$.

The algebraic properties of the double integral imply similar properties of the surface integral of a vector field.

Examples:

(i) Let $\gamma(t) := (\cos t, \sin t, \cos t \sin t) \in \mathbb{R}^3$ for $t \in [-\pi, \pi]$, and let $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$. Then since $\cos t \sin t = (\sin 2t)/2$ for $t \in [-\pi, \pi]$, we obtain

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} y\, dx + z\, dy + x\, dz$$

$$= \int_{-\pi}^{\pi} \Big((\sin t)(-\sin t) + \frac{1}{2}(\sin 2t)(\cos t) + (\cos t)(\cos 2t)\Big)\, dt$$

$$= -\pi.$$

(ii) Let $\boldsymbol{F}(x, y) := (x\, y, y^2)$ for $(x, y) \in \mathbb{R}^2$. If $\boldsymbol{\gamma}(t) := (t, 2t^2)$ for $t \in [1, 2]$, then

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_1^2 (2t^3, 4t^4) \cdot (1, 4t) dt = \int_1^2 (2t^3 + 16t^5) dt = \frac{351}{2}.$$

On the other hand, if $\widetilde{\gamma}(t) := (-t, 2t^2)$ for $t \in [-2, -1]$, then

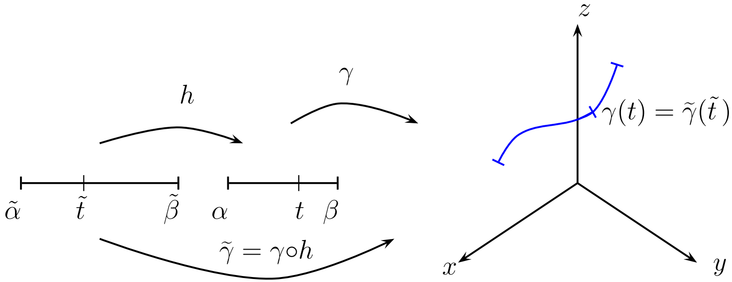
$$\int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{-2}^{-1} (-2t^3, 4t^4) \cdot (-1, 4t) dt = \int_{-2}^{-1} (2t^3 + 16t^5) dt = -\frac{351}{2}.$$

In the above example, $\boldsymbol{\gamma}([1,2]) = \widetilde{\boldsymbol{\gamma}}([-2,-1])$, $\boldsymbol{\gamma}(1) = \widetilde{\boldsymbol{\gamma}}(-1)$, $\boldsymbol{\gamma}(2) = \widetilde{\boldsymbol{\gamma}}(-2)$, and the line integral of the vector field $\boldsymbol{F}$ along $\widetilde{\boldsymbol{\gamma}}$ is the negative of the line integral of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$.

In general, we have the following result.

The line integral of a continuous vector field along a smooth path is invariant under a reparametrization only up to its sign.

Proof. Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth parametrized curve, and let $\boldsymbol{F}$ be a continuous vector field on $C := \boldsymbol{\gamma}([\alpha, \beta])$.




Suppose $\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\gamma}\circ h$ is a reparametrization of $\boldsymbol{\gamma}$. Since $h'(\tilde t) \neq 0$ for all $\tilde t \in [\tilde\alpha, \tilde\beta]$, either $h'(\tilde t) > 0$ for all $\tilde t \in [\tilde\alpha, \tilde\beta]$, or $h'(\tilde t) < 0$ for all $\tilde t \in [\tilde\alpha, \tilde\beta]$.

By the chain rule and the change of variables formula, it follows that in the former case, $\displaystyle\int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{s}$, while in the latter case, $\displaystyle\int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F}\cdot d\boldsymbol{s} = -\int_{\boldsymbol{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{s}$. $\square$

For example, let $\tilde{\alpha} := \alpha/2$, $\tilde{\beta} := \beta/2$, and define $h(\tilde{t}) := 2\tilde{t}$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Then $\widetilde{\boldsymbol{\gamma}}(\tilde{t}) := \boldsymbol{\gamma}(2\tilde{t})$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Since $h'(\tilde{t}) = 2 > 0$ for all $\tilde{t} \in \tilde{[}\alpha, \tilde{\beta}]$, we obtain $\displaystyle\int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$.

Next, let $\tilde{\alpha} := -\beta$, $\tilde{\beta} := -\alpha$, and define $h(\tilde{t}) := -\tilde{t}$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Then $\widetilde{\boldsymbol{\gamma}}(\tilde{t}) := \boldsymbol{\gamma}(-\tilde{t})$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Since $h'(\tilde{t}) = -1 < 0$ for all $\tilde{t} \in \tilde{[}\alpha, \tilde{\beta}]$, we obtain $\displaystyle\int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$.

If $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ is a parametrized curve, and we define $\tilde{\boldsymbol{\gamma}} : [-\beta, -\alpha] \to \mathbb{R}^m$ by $\widetilde{\boldsymbol{\gamma}}(\tilde{t}) := \boldsymbol{\gamma}(-\tilde{t})$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, then the parametrized curve $\tilde{\boldsymbol{\gamma}}$ is called the **negative** of $\boldsymbol{\gamma}$. Note that $\widetilde{\boldsymbol{\gamma}}([-\beta, -\alpha]) = \boldsymbol{\gamma}([\alpha, \beta])$, $\widetilde{\boldsymbol{\gamma}}(-\beta) = \boldsymbol{\gamma}(\alpha)$ and $\widetilde{\boldsymbol{\gamma}}(-\alpha) = \boldsymbol{\gamma}(\beta)$. The negative of the parametrized curve $\boldsymbol{\gamma}$ it is denoted by $-\boldsymbol{\gamma}$. Thus if $\boldsymbol{F}$ is a continuous vector field on $\boldsymbol{\gamma}([\alpha, \beta])$, then

$$\int_{-\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

## Physical interpretation of the line integral of a vector field:

Suppose a vector field $\boldsymbol{F}$ is defined on a domain $D$, $\boldsymbol{\gamma}$ is a regular path, and $C := \boldsymbol{\gamma}([\alpha, \beta]) \subset D$. Then $\boldsymbol{F} \cdot \hat{\boldsymbol{t}}$ is the component of $\boldsymbol{F}$ in the direction of the unit tangent vector $\hat{\boldsymbol{t}}$.

If $\boldsymbol{F}$ is a force field in the domain $D$, then the line integral

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \hat{\boldsymbol{t}}\, ds$$

represents the work done by the force $\boldsymbol{F}$ along the path $\boldsymbol{\gamma}$.

On the other hand, if $\boldsymbol{F}$ is the velocity field of a moving fluid in the domain $D$, then the line integral represents the flow of the fluid along the path $\boldsymbol{\gamma}$; in case the path $\boldsymbol{\gamma}$ is closed, the line integral represents the circulation of the fluid along $\boldsymbol{\gamma}$.

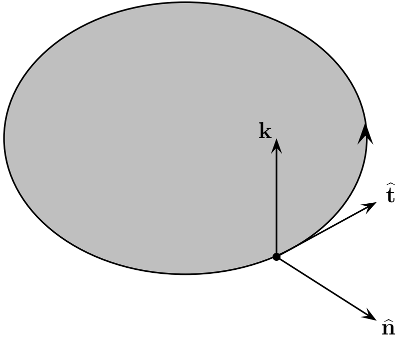
# Flux Integral

There is a variant of the circulation known as the flux of a moving fluid. We shall now consider its version in $\mathbb{R}^2$.

Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^2$ be a simple closed regular path. Let $D$ be an open subset of $\mathbb{R}^2$ containing $C := \boldsymbol{\gamma}([\alpha, \beta])$. Let $\hat{\boldsymbol{n}}$ denote the (continuous) **outward unit normal** to $\boldsymbol{\gamma}$. If $\boldsymbol{F}$ is a continuous vector field on $D$, then

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds$$

is called the **flux integral** of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$. It represents the flux of a velocity field $\boldsymbol{F}$ along a simple closed regular path $\boldsymbol{\gamma}$.




Suppose $\boldsymbol{\gamma}(t)$ moves in the anti-clockwise direction (as seen from high above) as the parameter $t$ goes from $\alpha$ to $\beta$,. Then

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{t}} \times \boldsymbol{k} = \frac{d\boldsymbol{\gamma}}{ds} \times \boldsymbol{k} = \left( \frac{dx}{ds}\, \boldsymbol{i} + \frac{dy}{ds}\, \boldsymbol{j} \right) \times \boldsymbol{k} = \frac{dy}{ds}\, \boldsymbol{i} - \frac{dx}{ds}\, \boldsymbol{j}.$$

Hence if $\boldsymbol{F} := P\, \boldsymbol{i} + Q\, \boldsymbol{j}$, then the flux integral of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$ is

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds = \int_{\boldsymbol{\gamma}} \left( P\, \frac{dy}{ds} - Q\, \frac{dx}{ds} \right) ds = \int_{\boldsymbol{\gamma}} P\, dy - Q\, dx.$$

## Example

Let $\gamma$ denote the standard unit circle in $\mathbb{R}^2$, and let $\boldsymbol{F} := (P, Q)$, where $P(x, y) := x - y$ and $Q(x, y) := x$ for $(x, y) \in \mathbb{R}^2$.

Then the circulation of $\boldsymbol{F}$ along $\gamma$ is

$$\int_{\gamma} P\,dx + Q\,dy = \int_{-\pi}^{\pi} (\cos t - \sin t)(-\sin t) + (\cos t)(\cos t)dt = 2\pi.$$

Also, the flux of $\boldsymbol{F}$ along $\gamma$ is

$$\int_{\gamma} P\,dy - Q\,dx = \int_{-\pi}^{\pi} (\cos t - \sin t)(\cos t) - (\cos t)(-\sin t)dt = \pi.$$

# Tutorial 11

1. Let $\boldsymbol{r}(x, y, z) := (x, y, z)$ and $r(x, y, z) := \|\boldsymbol{r}(x, y, z)\|$ for $(x, y, z) \in \mathbb{R}^3 \setminus (0, 0, 0)$. Show that $\nabla r^n = n r^{n-2} \boldsymbol{r}$ for $n \in \mathbb{N}$.
2. Let $f$, $g$ be a smooth scalar fields, and let $\boldsymbol{F}$ be a smooth vector field defined on a subset of $\mathbb{R}^3$. Show that

   (i) $\operatorname{div}(g\,\boldsymbol{F}) = g \operatorname{div} \boldsymbol{F} + \nabla g \cdot \boldsymbol{F}$,

   (ii) $\operatorname{curl}(g\,\boldsymbol{F}) = g \operatorname{curl} \boldsymbol{F} + \nabla g \times \boldsymbol{F}$,

   (iii) $\operatorname{div}(g \nabla f) - \operatorname{div}(f \nabla g) = g \nabla^2 f - f \nabla^2 g$,

   (iv) $\operatorname{curl}(g \nabla f) + \operatorname{curl}(f \nabla g) = \mathbf{0}$.
3. Let $a > 0$, $b \in \mathbb{R}$ and $\gamma(t) := (a \cos t, a \sin t, bt)$ for $t \in [0, 3\pi]$. Find the arc-length parametrization of $\gamma$.

4. Let $f(x, y, z) := xyz$ for $(x, y, z) \in \mathbb{R}^3$, and $\boldsymbol{\gamma}(t) := (\cos t, \sin t, t)$ for $t \in [0, \pi]$. Find $\int_{\boldsymbol{\gamma}} f\, ds$.
5. Let $\boldsymbol{F}(x, y) := (x^2 - 2xy, y^2 - 2xy)$ for $(x, y) \in \mathbb{R}^2$, and $\boldsymbol{\gamma}(t) := (t, t^2)$ for $t \in [-1, 1]$. Find $\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s}$.
6. Let $\boldsymbol{F}(x, y) := (x, y)$ for $(x, y) \in \mathbb{R}^2$, and $\boldsymbol{\gamma}(t) := (\cos t, 2 \sin t)$ for $t \in [-\pi, \pi]$. Find the circulation of $\boldsymbol{F}$ and the flux of $\boldsymbol{F}$ along $\gamma$.
7. Give examples of (i) a subset of $\mathbb{R}^2$ which is not path-connected, (ii) a subset of $\mathbb{R}^2$ which is path-connected, but not convex, (iii) a subset of $\mathbb{R}^2$ which is convex.

# MA 105: Calculus
# Lecture 22

Prof. B.V. Limaye
IIT Bombay

Friday, 24 March 2017

# Path Independence

Let $m \in \mathbb{N}$, $D \subset \mathbb{R}^m$, and consider a path $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$. We say that $\boldsymbol{\gamma}$ **lies in** $D$ if $\boldsymbol{\gamma}([\alpha, \beta]) \subset D$. Also, we say that $\boldsymbol{\gamma}(\alpha)$ is the **initial point** and $\boldsymbol{\gamma}(\beta)$ is the **final point** of $\boldsymbol{\gamma}$.

Let $\boldsymbol{F} : D \to \mathbb{R}^m$ be a continuous vector field. We say that line integrals of $\boldsymbol{F}$ are **path-independent** in $D$ if

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s}$$

whenever the (piecewise smooth) paths $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$ lie in $D$ and have the same initial point as well as the same final point.

Note: The path independence of line integrals of $\boldsymbol{F}$ in $D$ does not mean that $\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s}$ for any paths $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$. It only says that this equality holds if the paths $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$ both lie in $D$, both have the same initial point, and both have the same final point.

## Proposition

Let $D \subset \mathbb{R}^m$, and $\boldsymbol{F} : D \to \mathbb{R}^m$. Then line integrals of $\boldsymbol{F}$ are path-independent in $D$ $\iff$ $\displaystyle\int_{\boldsymbol{\gamma}_c} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ for every closed path $\boldsymbol{\gamma}_c$ that lies in $D$.

$\implies$ ) Let $\boldsymbol{\gamma}_c : [\alpha, \beta] \to \mathbb{R}^m$ be a closed path lying in $D$. Then $\boldsymbol{\gamma}_c(\alpha) = \boldsymbol{\gamma}_c(\beta) \in D$. Define $\boldsymbol{\gamma}(t) := \boldsymbol{\gamma}_c(\alpha)$ for all $t \in [\alpha, \beta]$. Then both $\boldsymbol{\gamma}_c$ and $\boldsymbol{\gamma}$ lie in $D$. They have the same initial point as well as the same final point $\boldsymbol{\gamma}_c(\alpha)$. By path-independence,

$$\int_{\boldsymbol{\gamma}_c} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\alpha}^{\beta} \boldsymbol{F}(\boldsymbol{\gamma}(\alpha)) \cdot \mathbf{0}\, dt = 0.$$

$\impliedby$ ) Let $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$ be paths lying in $D$ which have the same initial point, and the same final point. Then $\boldsymbol{\gamma}_c := \boldsymbol{\gamma} - \widetilde{\boldsymbol{\gamma}}$ is a closed path in $D$, and so

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{-\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}_c} \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

### Theorem

Let $D$ be an open subset of $\mathbb{R}^m$, and let $\boldsymbol{F}$ be a continuous vector field on $D$. Suppose $\boldsymbol{F}$ is a gradient field, then line integrals of $\boldsymbol{F}$ are path-independent in $D$. In fact, let $f$ be a smooth scalar field $f$ on $D$ such that $\boldsymbol{F} = \nabla f$ on $D$. Then

$$\int_{\boldsymbol{\gamma}} \nabla f \cdot d\boldsymbol{s} = f(B) - f(A) \text{ for every path } \boldsymbol{\gamma} \text{ lying in } D,$$

where $A$ is the initial point and $B$ is the final point of $\boldsymbol{\gamma}$.

Proof: By the chain rule,

$$\int_{\boldsymbol{\gamma}} \nabla f \cdot d\boldsymbol{s} = \int_{\alpha}^{\beta} (\nabla f)(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) dt = \int_{\alpha}^{\beta} (f \circ \boldsymbol{\gamma})'(t) dt.$$

So by Part II of the FTC (for the function $f \circ \boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}$),

$$\int_{\boldsymbol{\gamma}} \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{\gamma}(\beta)) - f(\boldsymbol{\gamma}(\alpha)) = f(B) - f(A). \quad \square$$

The previous result is called Part II of the fundamental theorem of calculus for line integrals.

If $\boldsymbol{F}$ is a gradient field, then the evaluation of a line integral $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$ can be simplified by replacing $\gamma$ by a path having the same initial point and the same final point. If we actually know a scalar field $f$ such that $\nabla f = \boldsymbol{F}$, then it is easy to evaluate $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$ even when the path $\gamma$ is very complicated.

Example: Let $\boldsymbol{F}(x, y, z) := (y\,z, z\,x, x\,y)$ for $(x, y, z) \in \mathbb{R}^3$, and let $\gamma(t) := (\cos^4 t, \sin^4 t, \tan^4 t)$, $t \in [0, \pi/4]$. If we consider $f(x, y, z) := x\,y\,z$ for $(x, y, z) \in \mathbb{R}^3$, then $\nabla f = \boldsymbol{F}$ on $\mathbb{R}^3$, and so

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = f\left(\frac{1}{4}, \frac{1}{4}, 1\right) - f(1, 0, 0) = \frac{1}{16}.$$

We now consider the converse of the previous proposition.

## Theorem

Let $D$ be an open path-connected subset of $\mathbb{R}^m$, and let $\boldsymbol{F}$ be a continuous vector field on $D$. Suppose line integrals of $\boldsymbol{F}$ are path-independent on $D$. Then $\boldsymbol{F}$ is a gradient field. In fact, fix $\boldsymbol{x}_0$ in $D$, and define

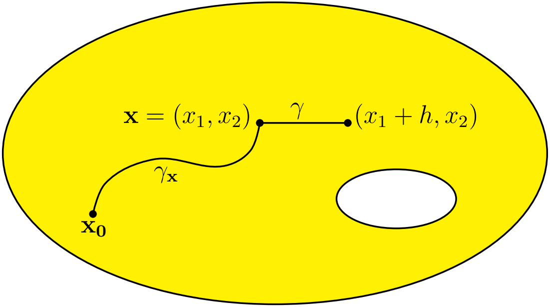
$$f(\boldsymbol{x}) := \int_{\boldsymbol{\gamma}_{\boldsymbol{x}}} \boldsymbol{F} \cdot d\boldsymbol{s}, \quad \text{where } \boldsymbol{\gamma}_{\boldsymbol{x}} \text{ is a path in } D \text{ from } \boldsymbol{x}_0 \text{ to } \boldsymbol{x}.$$

Then $\boldsymbol{F} = \nabla f$ on $D$.

Proof: Since $D$ is path-connected, for every $\boldsymbol{x} \in D$, there is a (piecewise smooth) path $\boldsymbol{\gamma}_{\boldsymbol{x}}$ with initial point $\boldsymbol{x}_0$ and final point $\boldsymbol{x}$. Further, since line integrals of $\boldsymbol{F}$ path-independent on $D$, the function $f : D \to \mathbb{R}$ mentioned in the statement of the theorem is well-defined.

For simplicity, let $m = 2$, and consider $\boldsymbol{x} := (x_1, x_2) \in D$.

Since $D$ is open in $\mathbb{R}^2$, the line segment joining $(x_1, x_2)$ to $(x_1 + h, x_2)$ lies in $D$ for all $h \in \mathbb{R}$ with $|h|$ sufficiently small.




Let $\gamma(t) := (x_1 + th, x_2)$, $t \in [0, 1]$. Now $\gamma_{\boldsymbol{x}}$ followed by $\gamma$ is a (piecewise smooth) path from $\boldsymbol{x}_0$ to $(x_1 + h, x_2)$. Hence

$$f(x_1+h, x_2) - f(x_1, x_2) = \int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(x_1+th, x_2) \cdot (h, 0)\, dt.$$

Let $\boldsymbol{F} := (F_1, F_2)$. It follows that for all $h \neq 0$ and small $|h|$,

$$\frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h} = \int_0^1 F_1(x_1 + th, x_2)dt.$$

Also, since $F_1$ is continuous at $(x_1, x_2)$,

$$\int_0^1 F_1(x_1 + th, x_2)dt \to \int_0^1 F_1(x_1, x_2)dt = F_1(x_1, x_2) \text{ as } h \to 0.$$

Thus the partial of $f$ at $\boldsymbol{x}$ with respect to the first variable equals $F_1(\boldsymbol{x})$. Similarly, for the second variable. Hence $\boldsymbol{F} = (F_1, F_2) = (f_{x_1}, f_{x_2}) = \nabla f$. $\square$

The above result is called Part I of the fundamental theorem of calculus for line integrals for obvious reasons.

The hypothesis of path-connectedness of the open set $D$ can be relaxed by considering separate components of $D$ each of which is open and path-connected.

In view of the results proved above, it is important to know whether a given vector field is indeed a gradient field. First we prove a necessary condition.

**Proposition (Cross-Derivative Test)**

Let $D \subset \mathbb{R}^m$ be open, and let $\boldsymbol{F} := (F_1, \ldots, F_m) : D \to \mathbb{R}^m$ be a smooth vector field. If $\boldsymbol{F}$ is a gradient field, then $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ for all $i, j = 1, \ldots, m$.

Proof: Let $\boldsymbol{F} = \nabla f$ on $D$, where $f : D \to \mathbb{R}$ is a scalar field. Since $\boldsymbol{F}$ is smooth, the partial derivatives of $f$ of orders 1 and 2 are continuous on $D$. Hence by the Mixed Partials Theorem,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial F_j}{\partial x_i} \quad \text{for } i, j = 1, \ldots, m. \qquad \square$$

The above test says that all 'cross partials' are equal.

## Special Cases of the Cross-Derivative Test

$m := 2$, $\boldsymbol{F} := (P, Q)$: $\qquad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}.$

$m := 3$, $\boldsymbol{F} := (P, Q, R)$: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}, \ \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}, \ \dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z},$

that is, $\operatorname{curl} \boldsymbol{F} = \mathbf{0}$.

Caution: Even if $\boldsymbol{F}$ satisfies the cross-derivative test on an open subset $D$ of $\mathbb{R}^m$, $\boldsymbol{F}$ may not be a gradient field!

Examples:

(i) Let $D := \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$, and $\boldsymbol{F} := (P, Q)$, where $P := -y/(x^2 + y^2)$ and $Q := x/(x^2 + y^2)$, $(x, y) \in D$.

Then $P_y = (y^2 - x^2)/(x^2 + y^2)^2 = Q_x$. But $\boldsymbol{F}$ is not a gradient field on $D$ since for the standard unit circle $\gamma$ in $D$,

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{-\pi}^{\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = 2\pi \neq 0.$$

(ii) Let $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$, and $\boldsymbol{F} := (P, Q, R)$, where $P := -y/(x^2 + y^2)$, $Q := x/(x^2 + y^2)$ and $R := 1$ for $(x, y, z) \in D$. As in Example (i), $P_y = Q_x$, and so $\text{curl}\,(\boldsymbol{F}) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \boldsymbol{0}$. But $\boldsymbol{F}$ is not a gradient field on $D$ since for the standard unit circle $\gamma$ in $D$,

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^{2\pi} (-\sin t, \cos t, 1) \cdot (-\sin t, \cos t, 0) dt = 2\pi \neq 0.$$

(iii) Let $D := \mathbb{R}^2$, and $\boldsymbol{F}(x, y) := (P, Q)$, where $P := -y$ and $Q := x$ for $(x, y) \in D$. Then $P_y = -1$ and $Q_x = 1$. Since the cross-derivative test fails, $\boldsymbol{F}$ is not a gradient field on $D$.

Remark

If $D := \mathbb{R}^m$ and $\boldsymbol{F}$ satisfies the cross-derivative test on $\mathbb{R}^m$, then it can be proved that $\boldsymbol{F}$ is indeed a gradient field. In fact, this also holds if $D$ is an open simply connected subset of $\mathbb{R}^m$ and $\boldsymbol{F}$ satisfies the cross-derivative test on $D$. Roughly speaking, $D$ is simply connected if it does not have any holes!

Let $D \subset \mathbb{R}^2$. Then $D$ is **simply connected** if every simple closed path lying in $D$ encloses only points of $D$. If $D$ is bounded, then it is simply connected if and only if the complement $\mathbb{R}^2 \setminus D$ of $D$ in $\mathbb{R}^2$ is path-connected.
For example, a disk or a rectangle in $\mathbb{R}^2$ is simply connected, but an annulus $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ is not.

Let $D \subset \mathbb{R}^3$. Then $D$ is **simply connected** if every simple closed path lying in $D$ is the 'edge' of a bounded surface lying in $D$. For example, a ball or a cuboid in $\mathbb{R}^3$ is simply connected, and so is $D := \{(x, y, z) \in \mathbb{R}^3 : 1 < x^2 + y^2 + z^2 < 4\}$, but the set $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is not.

Let a smooth vector field $\boldsymbol{F}$ satisfy the cross-derivative test on an open simply connected subset $D$ of $\mathbb{R}^2$ or $\mathbb{R}^3$. The theorems of Green and Stokes will show that $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ for every simple closed path $\boldsymbol{\gamma}$ in $D$, and hence for every closed path $\boldsymbol{\gamma}_c$ in $D$. This implies that $\boldsymbol{F}$ is a gradient field on $D$.

Let $D := I \times J \times K$, where $I, J, K$ are open intervals in $\mathbb{R}$. Suppose a smooth vector field $\boldsymbol{F} := (P, Q, R)$ on $D$ satisfies the cross-derivative test, that is, $\operatorname{curl}(\boldsymbol{F}) = \mathbf{0}$ on $D$. How can we find a scalar field $f$ on $D$ such that $\boldsymbol{F} = \nabla f$, that is, $(P, Q, R) = (f_x, f_y, f_z)$? This can be accomplished by repeated integration as follows. Let $x_0 \in I$.

Solve $f_x = P$ by integrating with respect to $x$: For $x \in I$, let

$$f(x, y, z) := \int_{x_0}^{x} P(u, y, z)du + g(y, z),$$

where $g$ is an arbitrary function of $y$ and $z$. Use this expression for $f$ in the equation $f_y = Q$ to obtain an equation for $g_y$. Solve it by integrating with respect to $y$. Its solution will involve an arbitrary function $h$ of $z$. Use this expression for $g$ in the equation $f_z = R$ to obtain an equation for $h'$. Solve it by integrating with respect to $z$. Its solution will involve an arbitrary constant. This yields $f : D \to \mathbb{R}$ such that $\nabla f = \boldsymbol{F}$.

Further, if $\boldsymbol{F} = \nabla g$ for a scalar field $g$ on $D$ as well, then $\nabla(g - f) = 0$ on $D$, and so $g = f + c$, where $c$ is a constant.

Example:

Let $\boldsymbol{F} := (P, Q, R)$ where $P := x^2 + y\,z$, $Q := y^2 + z\,x$, $R := z^2 + x\,y$ for $(x, y, z) \in \mathbb{R}^3$. It is easy to check that $\operatorname{curl}(\boldsymbol{F}) = \boldsymbol{0}$. Let $f$ be a scalar field such that $\nabla f = \boldsymbol{F}$.

$f_x = P = x^2 + y\,z \implies f(x, y, z) = x^3/3 + x\,y\,z + g(y, z)$.
$f_y = Q = y^2 + z\,x \implies x\,z + g_y = y^2 + z\,x$, and so $g_y = y^2$.
Thus $g(y, z) = y^3/3 + h(z)$, and as a consequence,

$$f(x, y, z) = \frac{x^3 + y^3}{3} + x\,y\,z + h(z).$$

$f_z = R = z^2 + x\,y \implies x\,y + h' = z^2 + x\,y$, and so $h' = z^2$.
Thus $h(z) = z^3/3 + c$, and as a consequence,

$$f(x, y, z) = \frac{x^3 + y^3 + z^3}{3} + x\,y\,z + c, \quad \text{where } c \text{ is a constant.}$$

# MA 105: Calculus
# Lecture 23

Prof. B.V. Limaye
IIT Bombay

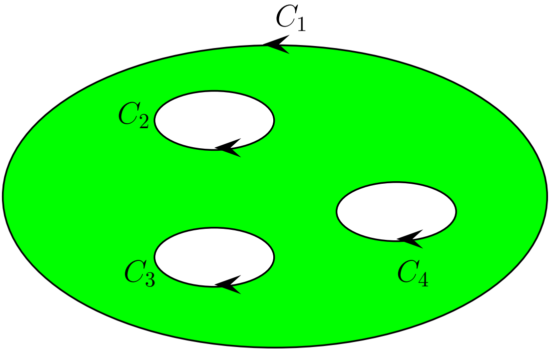
Wednesday, 29 March 2017

Let $\boldsymbol{\gamma}$ be a simple and piecewise smooth path in $\mathbb{R}^2$, and let $C := \boldsymbol{\gamma}([\alpha, \beta]) \subset \mathbb{R}^2$ denote the geometric curve defined by $\gamma$. Suppose $\boldsymbol{F}$ is a continuous vector field defined on $C$.

Since the line integral of a continuous vector field along a smooth path is invariant under a reparametrization only up to its sign, $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ is determined only up to its sign. To determine it completely, we must specify the sense or the direction in which the geometric curve $C$ is traversed, or how it is oriented. For example, we may specify that as we travel along the curve $C$ with $\boldsymbol{k}$ as our upright direction, the region in $\mathbb{R}^2$ enclosed by $C$ lies on our left. In this case, the curve $C$ is traversed **anticlockwise** (as seen from high above the curve).

Suppose $D$ is a bounded region in $\mathbb{R}^2$ whose boundary $\partial D$ in $\mathbb{R}^2$ consists of a finite number of nonintersecting simple closed piecewise smooth geometric curves $C_1, \ldots, C_k$.

Suppose the curve $C_1$ is the outer boundary of $D$.




If the outer boundary curve $C_1$ of $D$ is traversed anticlockwise, but the inner boundary curves $C_2, \ldots, C_k$ are traversed clockwise, then we say that the boundary $\partial D$ of the region $D$ is **positively oriented**. In this case, the region $D$ is on our left as we travel along any part of its boundary (with $\boldsymbol{k}$ as our upright direction).

# The Green Theorem

We have defined line integrals in terms of Riemann integrals. An important result of Green relates line integrals to double integrals. It can be viewed as a two-dimensional analogue of the Fundamental Theorem of Calculus (Part II) for Riemann integrals and for line integrals.

## Theorem

Let $D$ be a region in $\mathbb{R}^2$ such that the boundary of $D$ consists of a finite number of nonintersecting simple closed piecewise smooth geometric curves, and suppose it is positively oriented. Let $\boldsymbol{F} = (P, Q)$ be a smooth vector field on an open subset of $\mathbb{R}^2$ containing $D \cup \partial D$. Then

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} P\, dx + Q\, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y).$$

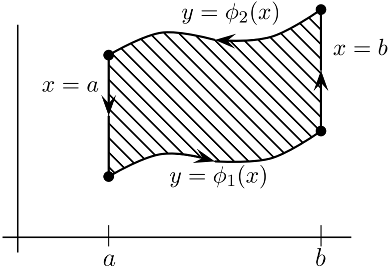
First let $D$ be an elementary region of type I and of type II.

Since $D$ is an elementary region of type I,

$$D := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \ \text{ and } \phi_1(x) \le y \le \phi_2(x)\},$$

where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous and $\phi_1 \le \phi_2$. Clearly,

$$\begin{aligned} \int_{\partial D} P\,dx &= \int_a^b P(x, \phi_1(x))dx + 0 + \int_b^a P(x, \phi_2(x))dx + 0 \\ &= \int_a^b \big(P(x, \phi_1(x)) - P(x, \phi_2(x))\big)\,dx. \end{aligned}$$



On the other hand, the Fubini theorem and the FTC show that

$$\iint_D \frac{\partial P}{\partial y} d(x,y) = \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y} dy \right) dx$$
$$= \int_a^b \big( P(x, \phi_2(x)) - P(x, \phi_1(x)) \big) dx.$$

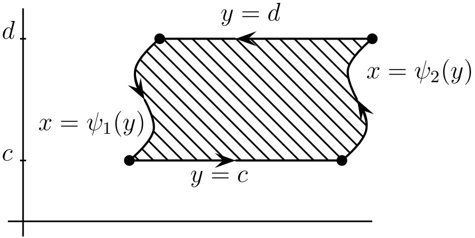
Thus $\displaystyle\int_{\partial D} P\, dx = -\iint_D \frac{\partial P}{\partial y} d(x,y).$

Next, since $D$ is an elementary region of type II,

$$D := \{(x,y) \in \mathbb{R}^2 : c \le y \le d \text{ and } \psi_1(y) \le x \le \psi_2(y)\},$$

where $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ are continuous and $\psi_1 \le \psi_2$. Clearly,

$$\int_{\partial D} Q\, dy = \int_c^d Q(\psi_2(y), y)dy + 0 + \int_d^c Q(\psi_1(y), y)dy + 0$$
$$= \int_c^d \big(Q(\psi_2(y), y) - Q(\psi_1(y), y)\big)dy.$$




On the other hand, the Fubini theorem and the FTC show that

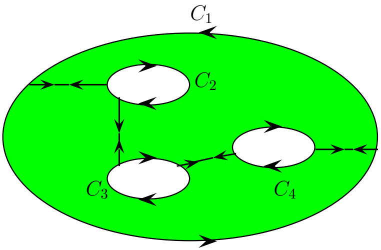
$$\iint_D \frac{\partial Q}{\partial x} d(x, y) = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x} dx \right) dy$$
$$= \int_c^d \big(Q(\psi_2(y), y) - Q(\psi_1(y), y)\big)dy.$$

Thus $\int_{\partial D} Q\, dy = \iint_D \frac{\partial Q}{\partial x} d(x, y)$.

Combining the two equalities proved above, we obtain

$$\int_C P\, dx + Q\, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y).$$

The proof given above applies when $D$ is a rectangle or a disk. It can be modified to treat the case where $D$ is simply connected, so that $\partial D$ has only the outer component. Further, the general case of a **multiply connected** domain $D$ can be treated by introducing cuts as shown below. $\square$



# Uses of the Green Theorem

(i) Calculation of a double integral

Example: Calculation of area of a subset of $\mathbb{R}^2$
Let $D$ be a closed and bounded subset of $\mathbb{R}^2$ whose positively oriented boundary $\partial D$ consists of a finite number of nonintersecting simple piecewise smooth closed curves. Then

$$\text{Area}\,(D) := \iint_D 1_D d(x,y).$$

Let $P$, $Q$ be a smooth functions such that $Q_x - P_y = 1$ on $D$. By the Green theorem,

$$\text{Area}\,(D) = \iint_D (Q_x - P_y) d(x,y) = \int_{\partial D} P\ dx + Q\ dy$$

For instance, for $(x,y) \in D$, we may let

$$P(x,y) := -\frac{y}{2} \quad \text{and} \quad Q(x,y) := \frac{x}{2}.$$

Then clearly $Q_x - P_y = 1$ on $D$, and so

$$\text{Area}\,(D) = \frac{1}{2}\int_{\partial D} x\ dy - y\ dx.$$

We note that Area $(D)$ is also equal to

$$\int_{\partial D} x\,dy \quad \text{and} \quad -\int_{\partial D} y\,dx$$

if we let $Q := x,\ P := 0$, and $P := -y,\ Q := 0$, respectively.

Suppose the positively oriented boundary $\partial D$ of $D$ is parametrized by $(x(t), y(t)),\ t \in E$, where $E$ is a finite union of closed and bounded intervals in $\mathbb{R}$. Then

$$\text{Area}\,(D) = \frac{1}{2}\int_{\alpha}^{\beta}\big(x(t)y'(t) - y(t)x'(t)\big)\,dt = \frac{1}{2}\int_{\alpha}^{\beta} W(x,y)(t)dt,$$

where $W(x,y) := \det\begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$ is called the **Wronskian** of $x, y$.

As a special case, suppose $\partial D$ is given by a polar equation

$$r = p(\theta), \quad \theta \in [\alpha, \beta],$$

so that $x(\theta) = p(\theta)\cos\theta$, $y(\theta) = p(\theta)\sin\theta$. Then

$$\begin{aligned} W(x,y) &= \det \begin{bmatrix} p(\theta)\cos\theta & p(\theta)\sin\theta \\ p'(\theta)\cos\theta - p(\theta)\sin\theta & p'(\theta)\sin\theta + p(\theta)\cos\theta \end{bmatrix} \\ &= p^2(\theta), \end{aligned}$$

and so

$$\text{Area}\,(D) = \frac{1}{2}\int_{\alpha}^{\beta} p^2(\theta)d\theta,$$

as we had defined earlier. For example, the area enclosed by the cardioid given by $r = a(1-\cos\theta)$, $\theta \in [-\pi, \pi]$, is equal to

$$\frac{1}{2}\int_{-\pi}^{\pi} a^2(1 - 2\cos\theta + \cos^2\theta)d\theta = \frac{3\pi a^2}{2}.$$

(ii) Calculation of a line integral along an oriented boundary

Example:

Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } y \geq 0\}$, so that $\partial D$ consists of the line segment from $(-1, 0)$ to $(1, 0)$ followed by the semicircle of radius 1 and centre $(0, 0)$ from $(1, 0)$ to $(-1, 0)$. Let $P(x, y) := y^2$ and $Q(x, y) := 3x\,y$ for $(x, y) \in \mathbb{R}^2$.

To calculate $\int_{\partial D} P\,dx + Q\,dy$, we may use the Green theorem and obtain

$$\begin{aligned}\int_{\partial D} y^2 dx + 3x\,y\,dy &= \iint_D \left[\frac{\partial}{\partial x}(3x\,y) - \frac{\partial}{\partial y}(y^2)\right] d(x, y) \\ &= \int_{-1}^{1}\left(\int_0^{\sqrt{1-x^2}} y\,dy\right)dx = \int_{-1}^{1}\frac{1-x^2}{2}dx = \frac{2}{3}.\end{aligned}$$

# Consequences of the Green Theorem

## Proposition

Let $\boldsymbol{F} := (P, Q)$ be a smooth vector field on an open subset containing a closed and bounded subset $D$ of $\mathbb{R}^2$ such that $Q_x = P_y$ on $D$. If $\partial D$ consists of a finite number of nonintersecting simple closed piecewise smooth curves, and is positively oriented, then

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} P\ dx + Q\ dy = 0.$$

Proof:

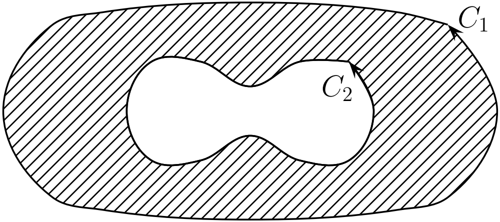
By the Green theorem,

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D (Q_x - P_y) d(x, y) = \iint_D 0\, d(x, y) = 0. \qquad \square$$

# Invariance of Some Line Integrals

Suppose $C_1$ and $C_2$ are simple closed piecewise smooth curves such that $C_2$ lies in the interior of $C_1$, that is, $C_1$ encloses $C_2$, and suppose both are oriented counterclockwise. Let $P, Q$ be smooth scalar fields satisfying $Q_x = P_y$ on an open set containing $C_1$, $C_2$ and the region between them. Then

$$\int_{C_1} P\ dx + Q\ dy = \int_{C_2} P\ dx + Q\ dy.$$



This result follows by applying the Green theorem to the subset $D$ of $\mathbb{R}^2$ consisting of $C_1$, $C_2$ and the region between them, and by noting that $\partial D = C_1 - C_2$.

This is called the deformation principle for line integrals.

Example:

Gauss Law in $\mathbb{R}^2$: Let $C$ be a piecewise smooth simple closed curve in $\mathbb{R}^2$, and let $D$ denote the union of $C$ and the region enclosed by $C$, so that $\partial D = C$, oriented counterclockwise. Suppose $\mathbf{0} \notin C$. Then

$$\int_C \frac{\tilde{\boldsymbol{r}}}{r^2} \cdot d\boldsymbol{s} = \begin{cases} 0 & \text{if } \mathbf{0} \notin D, \\ 2\pi & \text{if } \mathbf{0} \in D, \end{cases}$$

where $\tilde{\boldsymbol{r}}(x, y) := (-y, x)$ and $r(x, y) := \|(x, y)\|$, $(x, y) \in \mathbb{R}^2$.

Proof: Let $\boldsymbol{F} := \tilde{\boldsymbol{r}}/r^2$, that is, $\boldsymbol{F} := (P, Q)$, where $P = -y/r^2 = -y/(x^2 + y^2)$ and $Q := x/r^2 = x/(x^2 + y^2)$ for $\boldsymbol{r} \neq \mathbf{0}$. Then $Q_x - P_y = 0$ on $D \setminus \{(0, 0)\}$ as we saw earlier.

Let $\mathbf{0} \notin D$. Then the vector field $\boldsymbol{F}$ is smooth on $D$. Hence $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ by the Green theorem.
Next, let $\mathbf{0} \in D$. We have seen earlier that $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 2\pi$, where $C_1$ denotes the standard unit circle in $\mathbb{R}^2$, oriented counterclockwise. Since $\mathbf{0}$ is an interior point of $D$, there is $\epsilon > 0$ such that the closed disk of radius $\epsilon$ and center $\mathbf{0}$ lies inside $D$. Let $C_\epsilon$ denote the circle of radius $\epsilon$ and centre $\mathbf{0}$, oriented counterclockwise. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_\epsilon} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 2\pi$$

by the deformation principle for line integrals, applied to the curves $C$ and $C_\epsilon$, and then to the curves $C_\epsilon$ and $C_1$. $\square$

## Proposition

Let $D$ be a simply connected open subset of $\mathbb{R}^2$, and let $\boldsymbol{F} := (P, Q)$ be a smooth vector field such that $Q_x = P_y$ on $D$. Then $\boldsymbol{F}$ is a gradient field.

Proof: Let $C$ denote a simple closed smooth curve in $D$. Since $D$ is simply connected, $C$ encloses only points of $D$. As a consequence of the Green theorem, $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0$. This also holds if a smooth closed curve $C$ in $D$ intersects itself, since we can break up $C$ into several simple closed smooth curves. As a result, line integrals of $\boldsymbol{F}$ are path-independent in $D$, and so $\boldsymbol{F}$ is a gradient field on $D$. $\square$

---

Recall the differential notation for a regular smooth curve $\gamma$:

$ds = \|\gamma'(t)\| dt$, $d\boldsymbol{s} = \gamma'(t) dt$, $\hat{\mathbf{t}} = \gamma'(t)/\|\gamma'(t)\|$, $d\boldsymbol{s} = \hat{\mathbf{t}}\, ds$,

$\hat{\boldsymbol{n}} = \hat{\mathbf{t}} \times \boldsymbol{k}$, If $\boldsymbol{F} := (P, Q)$, then $\boldsymbol{F} \cdot \hat{\boldsymbol{n}}\ ds = P\ dy - Q\ dx$.

# Alternative Formulations of the Green Formula

We can express the conclusion of the Green theorem, namely,

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y).$$

as follows.

(i) Circulation-Curl Form or Tangential Form:

$$\int_{\partial D} \boldsymbol{F} \cdot \hat{\boldsymbol{t}}\, ds = \iint_D (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{k}\, d(x, y). \quad [\text{Let } \boldsymbol{F} := (P, Q, 0).]$$

(The Stokes theorem is a 3-dimensional analogue of this form.)

(ii) Flux-Divergence Form or Normal Form:

$$\int_{\partial D} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds = \iint_D (\operatorname{div} \boldsymbol{F}) d(x, y). \quad [\text{Let } \boldsymbol{F} := (-Q, P, 0).]$$

(The Gauss theorem is a 3-dimensional analogue of this form.)

# Tutorial 12

1. Determine whether the following vector fields are path-independent. Are they gradient fields? If so, find their potential functions.
   (i) For $(x, y) \in \mathbb{R}^2$, $\boldsymbol{F}(x, y) :=$
   $(y \cos x + y^2, \sin x + 2(x-1)y)$,

   (ii) $\boldsymbol{F}(x, y, z) := (y, z \cos yz + x, y \cos yz)$, $(x, y, z) \in \mathbb{R}^3$,

   (iii) For $(x, y) \in \mathbb{R}^2$, $\boldsymbol{F}(x, y) := (e^{xy}, e^{x+y})$.

2. (i) Let $\gamma(t) := (2\cos t, 3\sin t)$ for $t \in [-\pi, \pi]$.
   Find $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F}$ is defined in Problem 1 (i).

   (ii) Let $\gamma(t) := (\cos t, \sin t, t)$ for $t \in [\pi/2, \pi]$.
   Find $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F}$ is defined in Problem 1 (ii).

3. Let $a > 0$, and let $C$ denote the geometric curve (called a **hypocycloid**) given by $\gamma(t) := (a\cos^3 t, a\sin^3 t)$ for $t \in [-\pi, \pi]$. Find the area of the region enclosed by $C$.
4. Let $\boldsymbol{F}(x, y) := (-y, x)$ for $(x, y) \in \mathbb{R}^2$. Suppose a smooth simple closed geometric curve $C$ in $\mathbb{R}^2$ is oriented counterclockwise, and it encloses a region whose area is equal to $A$. Find $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$.
5. Let $C$ be a smooth simple closed geometric curve in $\mathbb{R}^2$ not passing through $(0, 0)$, and oriented counterclockwise. Let $\boldsymbol{r}(x, y) := (x, y)$ and $r(x, y) := \|\boldsymbol{r}(x, y)\|$ for $(x, y) \in \mathbb{R}^2$. Find $\displaystyle\int_C \frac{\boldsymbol{r}}{r^2} \cdot d\boldsymbol{s}$.

# MA 105: Calculus
# Lecture 24

Prof. B.V. Limaye
IIT Bombay

Friday, 31 March 2017

# Surfaces in $\mathbb{R}^3$

We now move up one dimension, and pass from paths and line integrals to surfaces and surface integrals.

Just as a curve in $\mathbb{R}^2$ is given implicitly by an equation $F(x, y) = 0$, or parametrically by $(x(t), y(t))$ for $t \in [\alpha, \beta]$, a surface in $\mathbb{R}^3$ is given implicitly by an equation $F(x, y, z) = 0$, or parametrically by $(x(u, v), y(u, v), z(u, v))$ for $(u, v) \in E$, where $E$ is a subset $\mathbb{R}^2$. Here $u$ and $v$ are independent parameters, that is, they do not depend on each other..

For example, the **unit sphere** in $\mathbb{R}^3$ is given implicitly by the equation $x^2 + y^2 + z^2 - 1 = 0$, and also parametrically by $\big((\sin u)(\cos v), (\sin u)(\sin v), \cos u\big)$ for $(u, v) \in [0, \pi] \times [-\pi, \pi]$.

We have discussed the tangent planes and normal lines for implicitly defined surfaces in Lecture 15.

# Parametrized Surfaces

A **parametrized surface** in $\mathbb{R}^3$ is a continuous map $\Phi : E \to \mathbb{R}^3$, where $E$ be a subset of $\mathbb{R}^2$ that has an area. Let

$$\Phi(u, v) := (x(u, v), y(u, v), z(u, v)) \quad \text{for } (u, v) \in E.$$

The map $\Phi$ is called **smooth** if each $x, y, z$ has continuous partial derivatives on an open subset of $\mathbb{R}^2$ containing $E$.

Given a parametrized surface as above, we may eliminate some parameters to obtain an equation involving only $x, y, z$, which defines the surface implicitly. The converse process is harder.

Examples

(i) Consider a continuous function $f : E \to \mathbb{R}$, and define $\Phi : E \to \mathbb{R}^3$ by $\Phi(x, y) := (x, y, f(x, y))$ for $(x, y) \in E$. This parametrized surface is called the **graph** of $f$, and it is implicitly given by the equation $F(x, y, z) := z - f(x, y) = 0$.

# Examples of Parametrized Surfaces

(ii) Let $a$, $h > 0$ and $E := [-\pi, \pi] \times [0, h]$.

Define $\Phi : E \to \mathbb{R}^3$ by $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$ for $(\theta, z) \in E$. This parametrized surface is called the right circular **cylinder** of radius $a$ and height $h$. It is implicitly given by the equation $F(x, y, z) := x^2 + y^2 - a^2 = 0$, $0 \le z \le h$.

Define $\Phi : E \to \mathbb{R}^3$ by $\Phi(\theta, z) := (z\cos\theta, z\sin\theta, z)$ for $(\theta, z) \in E$. This parametrized surface is called the right circular **cone** of height $h$. It is implicitly given by the equation $F(x, y, z) := x^2 + y^2 - z^2 = 0$, $0 \le z \le h$.

(iii) Let $a > 0$ and $E := [0, \pi] \times [-\pi, \pi]$. Define $\Phi : E \to \mathbb{R}^3$ by $\Phi(\varphi, \theta) := (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. This parametrized surface is called the **standard sphere of radius** $a$. It is implicitly given by the equation $F(x, y, z) := x^2 + y^2 + z^2 - a^2 = 0$.

# Normal Vector to a Parametrized Surface

Let $\Phi : E \to \mathbb{R}^3$ be a parametrized surface. Let $(u_0, v_0) \in E$, and $P_0 := \Phi(u_0, v_0) = (x_0, y_0, z_0)$ be a point in $\Phi(E)$.

Let $\gamma(t) := (u(t), v(t))$, $t \in [\alpha, \beta]$, be any path in $E$ passing through $(u_0, v_0)$, and $t_0 \in (\alpha, \beta)$ be such that $\gamma(t_0) = (u_0, v_0)$.

Consider the path $C$ in $\mathbb{R}^3$ given by $\Phi \circ \gamma$ which 'lies on' $\Phi$. Let $\Phi$ and $\gamma$ be smooth. The tangent vector to $C$ at $P_0$ is given by

$$(\Phi \circ \gamma)'(t_0) = \Phi_u(u_0, v_0)\; u'(t_0) + \Phi_v(u_0, v_0)\; v'(t_0).$$

by the Chain Rule. Since $\Phi_u \times \Phi_v$ is perpendicular to both $\Phi_u$ and $\Phi_v$, it follows that the vector $\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)$ is perpendicular to the tangent vector to $C$ at $P_0$. Hence we call $(\Phi_u \times \Phi_v)(u_0, v_0) := \Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)$ a **normal vector** to $\Phi$ at $P_0$, provided it is nonzero. It is also called the **fundamental product** for $\Phi$ at $(u_0, v_0)$.

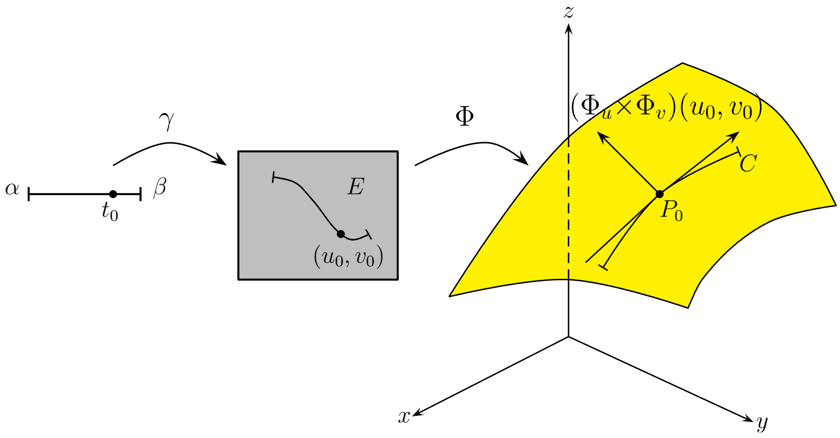



Figure : Normal vector to a parametrized surface at $P_0$

# Tangent Plane to a Parametrized Surface

A smooth parametrized surface $\Phi$ is said to be **regular** if $(\Phi_u \times \Phi_v)(u, v) \neq 0$ for every $(u, v) \in E$. In this case,

$$\hat{\mathbf{n}}\,(u, v) := \frac{(\Phi_u \times \Phi_v)(u, v)}{\|(\Phi_u \times \Phi_v)(u, v)\|}$$

is called a **unit normal vector** to $\Phi$ at $\Phi(u, v)$, $(u, v) \in E$.

Let $P_0 := \Phi(u_0, v_0) = (x_0, y_0, z_0) \in \Phi(E)$. The plane given by

$$(\Phi_u \times \Phi_v)(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

that is, by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$,

where $(a, b, c) := (\Phi_u \times \Phi_v)\,(u_0, v_0)$, is called the **tangent plane** to $\Phi$ at $P_0$.

## Examples

(i) **Graph of a function**

Let $E \subset \mathbb{R}^2$ and $f : E \to \mathbb{R}$ be a smooth function. Since $\Phi(x, y) = (x, y, f(x, y))$ for $(x, y) \in E$, we see that $\Phi_x = (1, 0, f_x)$ and $\Phi_y = (0, 1, f_y)$, and so

$$\Phi_x \times \Phi_y = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1).$$

Since $\|\Phi_\theta \times \Phi_z\| = \sqrt{f_x^2 + f_y^2 + 1} \neq 0$, $\Phi$ is a regular surface.

The tangent plane at $P_0 := (x_0, y_0, f(x_0, y_0))$ is given by

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

(ii) Cylinder

Since $\Phi(\theta, z) = (a\cos\theta, a\sin\theta, z)$ for $(\theta, z) \in [-\pi, \pi] \times [0, h]$,

$$\Phi_\theta \times \Phi_z = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos\theta, a\sin\theta, 0).$$

Since $\|\Phi_\theta \times \Phi_z\| = a \neq 0$, $\Phi$ is a regular surface.

The tangent plane at $P_0 := (a\cos\theta_0, a\sin\theta_0, z_0)$ is given by

$$(\cos\theta_0)x + (\sin\theta_0)y = a.$$

(iii) Sphere

Since $\Phi(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in [0, \pi] \times [-\pi, \pi]$, we see that

$$\Phi_\varphi \times \Phi_\theta = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}.$$

Thus

$$\begin{aligned}\Phi_\varphi \times \Phi_\theta &= (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi) \\ &= (a\sin\varphi)\,\Phi(\varphi,\theta).\end{aligned}$$

Since $\|\Phi(\varphi,\theta)\| = a$ for $(\varphi,\theta) \in [0,\pi] \times [-\pi,\pi]$, we obtain

$$\|\Phi_\varphi \times \Phi_\theta\| = a^2 \sin\varphi \neq 0 \quad \text{if } \varphi \neq 0 \text{ or } \pi,$$

and then the tangent plane at $P_0 := \Phi(\varphi_0, \theta_0)$ is given by

$$(\sin\varphi_0\cos\theta_0)x + (\sin\varphi_0\sin\theta_0)y + (\cos\varphi_0)z = a.$$

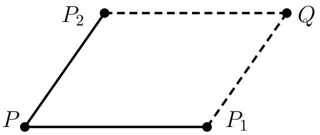
Using the parametrization $\Phi(x,y) := (x, y, \sqrt{a^2 - x^2 - y^2})$ for $x^2 + y^2 \leq a^2$, $y \geq 0$, we may obtain the unit normal vector at the **north pole** $(0,0,a)$ to be $(0,0,1)$ and the equation of the tangent plane to be $z = a$. Similarly we may treat the **south pole** $(0,0,-a)$.

# Surface Area

Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface.
Let $(u, v) \in E$. For $h, k \in \mathbb{R}$ with $|h|, |k|$ small, consider

$$P := \Phi(u, v), \quad P_1 := \Phi(u + h, v) \approx \Phi(u, v) + h\,\Phi_u(u, v),$$

$$P_2 := \Phi(u, v+k) \approx \Phi(u, v)+k\,\Phi_v(u, v), \quad Q := \Phi(u+h, v+k).$$




Area of the parallelogram with sides $PP_1$ and $PP_2$

$$= \|(P_1 - P) \times (P_2 - P)\| \approx \|\Phi_u(u, v) \times \Phi_v(u, v)\|\,|h||k|.$$

In view of this approximation, we define

$$\text{Area}\,(\Phi) = \iint_E \|\Phi_u \times \Phi_v\|(u, v)d(u, v).$$

Since the subset $E$ of $\mathbb{R}^2$ has an area, the two-dimensional content of $\partial D$ is zero. Also, the function $\|\Phi_u \times \Phi_v\|$ is continuous on $E$. Hence the integral in the definition of Area $(\Phi)$ is well-defined.

In analogy with $ds = \|\gamma'(t)\|dt$, we introduce the following

Differential Notation

$$dS = \|\Phi_u \times \Phi_v\| d(u, v), \quad \text{so that} \quad \text{Area}\,(\Phi) = \iint_E dS.$$

Examples

(i) Graph of a function:

Let $E \subset \mathbb{R}^2$, $f : E \to \mathbb{R}$ be a smooth function, and $\Phi(x, y) = (x, y, f(x, y))$ for $(x, y) \in E$. Then

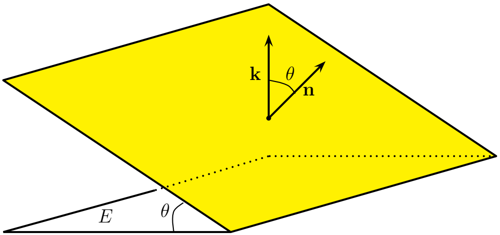
$$\begin{aligned} \text{Area}\,(\Phi) &= \iint_E \|(-f_x, -f_y, 1)\| d(x, y) \\ &= \iint_E \sqrt{1 + f_x^2 + f_y^2}\, d(x, y). \end{aligned}$$

Let $\theta$ be the angle between $\Phi_x \times \Phi_y = (-f_x, -f_y, 1)$ and $\boldsymbol{k} = (0, 0, 1)$. Then $1 = (\Phi_x \times \Phi_y) \cdot \boldsymbol{k} = \|\Phi_x \times \Phi_y\| \cos\theta$. Hence

$$\cos\theta = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}, \quad \text{and} \quad \text{Area}\,(\Phi) = \iint_E \frac{d(x,y)}{|\cos\theta(x,y)|}.$$

In case $\theta(x,y) := \theta$, a constant for all $(x,y) \in E$, then

$$\text{Area}\,(\Phi) = \frac{\text{Area}\,(E)}{\cos\theta} \quad \text{(Area Cosine Principle)}.$$



(ii) Let $E := [-\pi, \pi] \times [0, h]$, $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$, and $\Psi(\theta, z) := (\cos 2\theta, \sin 2\theta, z)$ for $(\theta, z) \in E$. Then

$$\begin{aligned} \text{Area}\,(\Phi) &= \iint_E \|\Phi_\theta \times \Phi_z\| d(\theta, z) = \iint_E a\, d(\theta, z) = 2\pi a\, h, \\ \text{Area}\,(\Psi) &= \iint_E \|\Psi_\theta \times \Psi_z\| d(\theta, z) = \iint_E 2a\, d(\theta, z) = 4\pi a\, h. \end{aligned}$$

We note that $\Psi(E) = \Phi(E)$, but $\text{Area}\,(\Psi) = 2\,\text{Area}\,(\Phi)$.

(iii) Let $[0, \pi] \times [-\pi, \pi]$, and
$\Phi(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$.
Then

$$\begin{aligned} \text{Area}\,(\Phi) &= \iint_E \|\Phi_\varphi \times \Phi_\theta\| d(\varphi, \theta) = \iint_E a^2 \sin\varphi d(\varphi, \theta) \\ &= \int_{-\pi}^{\pi} \left( \int_0^{\pi} a^2 \sin\varphi\, d\varphi \right) d\theta = 4\pi a^2. \end{aligned}$$

(iv) Suppose a smooth curve $C$ in $\mathbb{R}^2$ lies on or above the $x$-axis. If $C$ is given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, and is revolved about the $x$-axis, it generates a surface parametrized by

$$\Phi(t,\theta) := (x(t),\ y(t)\cos\theta,\ y(t)\sin\theta),\ (t,\theta) \in E,$$

where $E := [\alpha,\beta]\times[-\pi,\pi]$. For all $(t,\theta) \in E$,

$$\begin{aligned}\Phi_t \times \Phi_\theta &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\ &= (y(t)y'(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta).\end{aligned}$$

By the Fubini theorem, we obtain (as earlier)

$$\begin{aligned}\text{Area}\,(\Phi) &= \iint_E \sqrt{y(t)^2y'(t)^2 + x'(t)^2y(t)^2}\, d(\theta,t) \\ &= 2\pi \int_\alpha^\beta |y(t)|\sqrt{x'(t)^2 + y'(t)^2}\, dt.\end{aligned}$$

# Surface Integral of a Scalar Field

Let $E \subset \mathbb{R}^2$ and let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface. Let $S := \Phi(E)$, and $f : S \to \mathbb{R}$ be a bounded scalar field. The **surface integral** of $f$ **across** $\Phi$ is defined by

$$\iint_{\Phi} f \, dS := \iint_{E} f(\Phi(u,v)) \|(\Phi_u \times \Phi_v)(u,v)\| \, d(u,v),$$

provided the double integral on the right side exists. In particular, if $f$ is continuous, then $\iint_{\Phi} f \, dS$ is well-defined.

In particular, if $f := 1$ on $S$, then clearly $\iint_{\Phi} f \, dS = \text{Area}\,(\Phi)$.

The algebraic and the order properties of the double integral imply similar properties of the surface integral of a scalar field.

Example:

Let $E := [0, \pi/2] \times [-\pi, \pi]$, and let
$\Phi(\varphi, \theta) := (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a \cos\varphi)$ for $(\varphi, \theta) \in E$.
Then

$$\begin{aligned} \text{Area}\,(\Phi) &= \iint_E \|\Phi_\varphi \times \Phi_\theta\| d(\varphi, \theta) = \iint_E a^2 \sin\phi \, d(\varphi, \theta) \\ &= \int_{-\pi}^{\pi} \left( \int_0^{\pi/2} a^2 \sin\varphi \, d\varphi \right) d\theta = 2\pi a^2. \end{aligned}$$

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) := z$ for $(x, y, z) \in \mathbb{R}^3$. Then

$$\begin{aligned} \iint_\Phi f \, dS &= \int_{-\pi}^{\pi} \left( \int_0^{\pi/2} (a \cos\varphi)(a^2 \sin\varphi) d\varphi \right) d\theta \\ &= \pi a^3 \int_0^{\pi/2} \sin 2\varphi \, d\varphi = \pi a^3. \end{aligned}$$

# Reparametrization of a Surface

Let $E \subset \mathbb{R}^2$ be path-connected and have an area. Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface.

Let $\widetilde{E}$ be a path-connected subset $\mathbb{R}^2$ having an area, and let $h : \widetilde{E} \to \mathbb{R}^2$ be a continuously differentiable and one-one function and such that $h(\widetilde{E}) = E$ and its Jacobian $J(h)$ does not vanish on $\widetilde{E}$. Then the smooth surface $\tilde{\Phi} := \Phi \circ h$ is called a **reparametrization** of $\Phi$.

Using the chain rule, it can be checked that for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$,

$$(\widetilde{\Phi}_{\tilde{u}} \times \widetilde{\Phi}_{\tilde{v}})(\tilde{u}, \tilde{v}) = (\Phi_u \times \Phi_v)(h(\tilde{u}, \tilde{v}))J(h)(\tilde{u}, \tilde{v}).$$

Let $S := \Phi(E)$, and let $f : S \to \mathbb{R}$ be a continuous scalar field. Then by the change of variables formula for double integration,

$$
\begin{aligned}
\iint_{\widetilde{\Phi}} f\, dS &= \iint_{\widetilde{E}} f(\widetilde{\Phi}(\tilde{u},\tilde{v}))\|(\widetilde{\Phi}_{\tilde{u}} \times \widetilde{\Phi}_{\tilde{v}})(\tilde{u},\tilde{v})\| d(\tilde{u},\tilde{v}) \\
&= \iint_{\widetilde{E}} f(\Phi \circ h(\tilde{u},\tilde{v}))\|(\widetilde{\Phi}_{\tilde{u}} \times \widetilde{\Phi}_{\tilde{v}})(\tilde{u},\tilde{v})\| d(\tilde{u},\tilde{v}) \\
&= \iint_{E} f(\Phi(u,v))\|(\Phi_u \times \Phi_v)(u,v)\| d(u,v) \\
&= \iint_{\Phi} f\, dS.
\end{aligned}
$$

Thus the surface integral of a continuous scalar field over a smooth surface is invariant under reparametrization. In particular, the area of a smooth surface is invariant under reparametrization.

# Geometric Surface

Let $E$ be a subset of $\mathbb{R}^2$, and let $\Phi : E \to \mathbb{R}^3$ be a parametrized surface. Then $S = \Phi(E)$ is called a **geometric surface**, and $\Phi$ is called a **parametrization** of $S$.

Suppose $\Phi$ is smooth and one-one except possibly on a subset of $E$ of two-dimensional content zero. In view of the reparametrization result for continuous scalar fields on parametrized surfaces proved above, we define

$$\iint_S f\, dS := \iint_\Phi f\, dS = \iint_E f(\Phi(u,v)) \|(\Phi_u \times \Phi_v)(u,v)\| d(u,v)$$

for a continuous scalar field $f$ on $S$, and in particular,

$$\text{Area}\,(S) := \text{Area}\,(\Phi) = \iint_E \|(\Phi_u \times \Phi_v)(u,v)\| d(u,v).$$

# MA 105: Calculus
# Lecture 25

Prof. B.V. Limaye
IIT Bombay

Wednesday, 5 April 2017

# Surface Integral of a Vector Field

Let $E$ be a subset of $\mathbb{R}^2$ having an area, and $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface. Let $\boldsymbol{F}$ be a bounded vector field on $\Phi(E)$. The **surface integral** of $\boldsymbol{F}$ **across** $\Phi$ is defined by

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} := \iint_{E} \boldsymbol{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v)(u,v)\, d(u,v),$$

provided the double integral on the right exists. In particular, if $\boldsymbol{F}$ is continuous, then $\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}$ is well-defined.

In analogy with the differential notation
$dS := \|\Phi_u \times \Phi_v\| d(u,v)$, we let

$$d\boldsymbol{S} := (\Phi_u \times \Phi_v) d(u,v).$$

Let $\Phi(u,v) := (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in E$. Then

$$\Phi_u \times \Phi_u = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left( \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right).$$

If $\boldsymbol{F} := (P, Q, R)$, then the surface integral is also written as

$$\iint_E \left( P \frac{\partial(y,z)}{\partial(u,v)} + Q \frac{\partial(z,x)}{\partial(u,v)} + R \frac{\partial(x,y)}{\partial(u,v)} \right) d(u,v)$$
$$= \iint_\Phi P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$$

with the differential notation $dy \wedge dz := \dfrac{\partial(y,z)}{\partial(u,v)} d(u,v)$, etc.

If the surface $\Phi$ is regular, then

$$d\boldsymbol{S} = (\Phi_u \times \Phi_v) d(u,v) = \hat{\boldsymbol{n}}\, \|\Phi_u \times \Phi_v\| d(u,v) = \hat{\boldsymbol{n}}\, dS,$$

and we write the surface integral of $\boldsymbol{F}$ across $\Phi$ as $\displaystyle\int_\Phi \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS$.

The algebraic properties of the double integral imply similar properties of the surface integral of a vector field.

Physical interpretation:

Suppose a vector field $\boldsymbol{F}$ is defined on a domain $D$, and $\Phi : E \to \mathbb{R}^3$ is a regular surface in $D$, that is, $S := \Phi(E) \subset D$. If $\boldsymbol{F}$ is an electric field, a magnetic field, or a velocity field, then its surface integral

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\Phi} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS$$

represents the flux, that is, the rate of flow, of the field $\boldsymbol{F}$ across the surface $\Phi$. As a result, this is also called a **flux integral**.

## Examples

(i) Let $E \subset \mathbb{R}^2$, and $f : E \to \mathbb{R}$ be a smooth function. Let $\Phi : E \to \mathbb{R}^3$ represent the graph of $f$, and let $\boldsymbol{F} : \Phi(E) \to \mathbb{R}^3$ be a continuous vector field. If $\boldsymbol{F} := (P, Q, R)$, then

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{E} (-P\, f_x - Q\, f_y + R) d(x, y)$$

since $d\boldsymbol{S} = (\Phi_x \times \Phi_y) d(x, y) = (-f_x, -f_y, 1) d(x, y)$.

For instance, let $E := [0, 1] \times [0, 1]$, $f(x, y) := x + y + 1$ for $(x, y) \in E$, and $F(x, y, z) := (x^2, y^2, z)$, $(x, y, z) \in \mathbb{R}^3$. Then

$$\begin{aligned} \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_{E} \big( -x^2 - y^2 + (x + y + 1) \big) d(x, y) \\ &= \int_0^1 \left( \int_0^1 (x + y + 1 - x^2 - y^2) dy \right) dx \\ &= = \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}. \end{aligned}$$

(ii) Let $E := [-\pi, \pi] \times [0, h]$, and $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$ for $(\theta, z) \in E$. If $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{E} (a^2 \cos\theta \sin\theta + z\, a\sin\theta + 0) d(\theta, z) = 0,$$

since $d\boldsymbol{S} = (\Phi_\theta \times \Phi_z) d(\theta, z) = (a\cos\theta, a\sin\theta, 0) d(\theta, z)$.

(iii) Let $E := [0, \pi] \times [-\pi, \pi]$, and let $\Phi(\varphi, \theta) := (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. If $\boldsymbol{F}(x, y, z) := (x, y, z)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{E} \Phi(\varphi, \theta) \cdot (a\sin\varphi) \Phi(\varphi, \theta) d(\varphi, \theta) = 4a^3\pi,$$

since $d\boldsymbol{S} = (\Phi_\varphi \times \Phi_\theta) d(\varphi, \theta) = a\sin\varphi\, \Phi(\varphi, \theta) d(\varphi, \theta)$.

On the other hand, let $\widetilde{E} := [-\pi, \pi] \times [0, \pi]$, and let $\widetilde{\Phi}(\theta, \varphi) := (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a \cos\varphi)$ for $(\theta, \varphi) \in \widetilde{E}$. If $\boldsymbol{F}(x, y, z) := (x, y, z)$ for $(x, y, z) \in \mathbb{R}^3$ as above, then

$$\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\widetilde{E}} -a \sin\varphi \, \widetilde{\Phi}(\theta, \varphi) \cdot \widetilde{\Phi}(\theta, \varphi) d(\theta, \varphi) = -4a^3\pi,$$

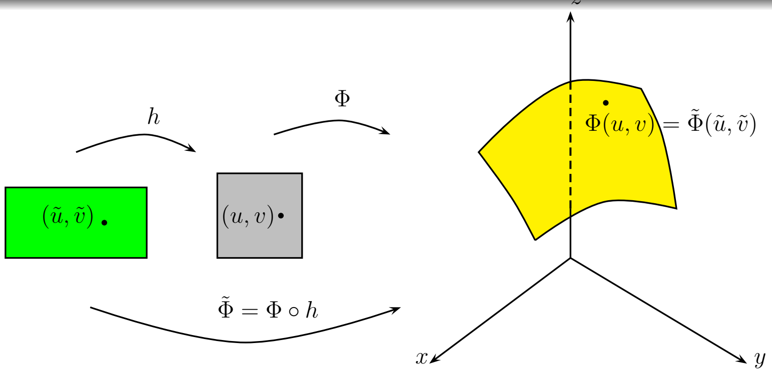
since $d\boldsymbol{S} = (\widetilde{\Phi}_\theta \times \widetilde{\Phi}_\varphi) d(\theta, \varphi) = -a \sin\varphi \, \widetilde{\Phi}(\theta, \varphi) d(\theta, \varphi)$.

The surface integral of a continuous vector field across a smooth parametrized surface is invariant under a reparametrization only up to its sign.

Proof.
Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface. Suppose $\widetilde{\Phi} := \Phi \circ h$ is a reparametrization of $\Phi$. Then for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$,

$$(\widetilde{\Phi}_{\tilde{u}} \times \widetilde{\Phi}_{\tilde{v}})(\tilde{u}, \tilde{v}) = (\Phi_u \times \Phi_v)(h(\tilde{u}, \tilde{v})) J(h)(\tilde{u}, \tilde{v}).$$

Note that the change of variables formula involves $|J(h)(\tilde{u}, \tilde{v})|$.




Since $\widetilde{E}$ is path-connected, and $J(h)(\tilde{u}, \tilde{v}) \neq 0$ for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$, either $J(h)(\tilde{u}, \tilde{v}) > 0$ for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$ or $J(h)(\tilde{u}, \tilde{v}) < 0$ or all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$. Let $\boldsymbol{F}$ be a continuous vector field on $S := \Phi(E)$. By the change of variables formula, we obtain $\displaystyle\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}$ in the former case, while $\displaystyle\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = -\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}$ in the latter. $\square$

For example, let $\widetilde{E} := \{(u/2, v/2) : (u,v) \in E\}$, and define $h(\tilde{u},\tilde{v}) := (2\tilde{u}, 2\tilde{v})$ for $(\tilde{u},\tilde{v}) \in \widetilde{E}$. Then we see that $\widetilde{\Phi}(\tilde{u},\tilde{v}) := \Phi(h(\tilde{u},\tilde{v})) = \Phi(2\tilde{u}, 2\tilde{v})$ for $(\tilde{u},\tilde{v}) \in \tilde{E}$. Since $J(h)(\tilde{u},\tilde{v}) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$ for all $(\tilde{u},\tilde{v}) \in \tilde{E}$, we obtain

$$\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

On the other hand, let $\widetilde{E} := \{(v,u) : (u,v) \in E\}$, and define $h(\tilde{u},\tilde{v}) := (\tilde{v}, \tilde{u})$ for $(\tilde{u},\tilde{v}) \in \widetilde{E}$. Then we see that $\widetilde{\Phi}(\tilde{u},\tilde{v}) := \Phi(h(\tilde{u},\tilde{v})) = \Phi(\tilde{v}, \tilde{u})$, $(\tilde{u},\tilde{v}) \in \widetilde{E}$. Since $J(h)(\tilde{u},\tilde{v}) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$ for all $(\tilde{u},\tilde{v}) \in \widetilde{E}$, we obtain

$$\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = -\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

# Opposite of a Parametrized Surface

Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface, and let $\widetilde{E} := \{(v, u) : (u, v) \in E\}$. Define

$$\widetilde{\Phi}(\tilde{u}, \tilde{v}) := \Phi(\tilde{v}, \tilde{u}) \quad \text{for } (\tilde{u}, \tilde{v}) \in \widetilde{E}.$$

Then the parametrized surface $\widetilde{\Phi}$ is called the **opposite** of $\Phi$. The opposite of $\Phi$ will be denoted by $\Phi^{\rm op}$. Clearly, $\Phi^{\rm op}(\widetilde{E}) = \Phi(E)$. Also, for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$,

$$(\Phi^{\rm op}_{\tilde{u}} \times \Phi^{\rm op}_{\tilde{v}})(\tilde{u}, \tilde{v}) = -(\Phi_u \times \Phi_v)(\tilde{v}, \tilde{u}).$$

If $\boldsymbol{F}$ is a continuous vector field on $\Phi(E)$, then we obtain

$$\iint_{\Phi^{\rm op}} \boldsymbol{F} \cdot d\boldsymbol{S} = -\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

We remark that $(\Phi^{\rm op})^{\rm op} = \Phi$.

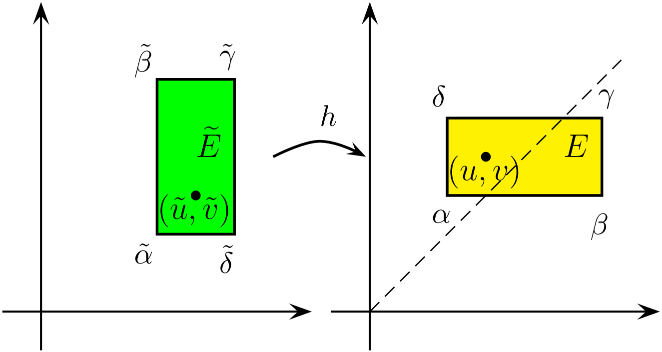



Figure : Parameter domain $\widetilde{E}$ of the opposite $\Phi^{\mathrm{op}}$ of a parametrized surface $\Phi$ is obtained by reflecting the parameter domain $E$ of $\Phi$ in the dashed line $u = v$. Note that the anti-clockwise direction of the movement of the corners of $E$ is changed to the clockwise direction of the movement of the corresponding corners of $\widetilde{E}$ shown above.

# Orientable Surfaces

Consider a regular parametrized surface $\Phi : E \to \mathbb{R}^3$, so that $(\Phi_u \times \Phi_v)(u, v) \neq 0$ for all $(u, v) \in E$. Suppose the function $\Phi$ is one-one. Then

$$\hat{\boldsymbol{n}}\,(u, v) := \frac{(\Phi_u \times \Phi_v)(u, v)}{\|(\Phi_u \times \Phi_v)(u, v)\|}.$$

is a unit normal vector at a point $P := \Phi(u, v)$ on the geometric surface $S := \Phi(E)$, and so is $-\hat{\boldsymbol{n}}\,(u, v)$.

Next, let $D \subset \mathbb{R}^3$, and consider a function $F : D \to \mathbb{R}$. Suppose the equation $F(x, y, z) = 0$ implicitly defines a surface $S$ in $\mathbb{R}^3$. If $F$ has partial derivatives at all points of $D$ and $\nabla F \neq 0$ on $D$, then

$$\frac{\nabla F(P)}{\|\nabla F(P)\|}$$

is a unit normal vector at $P \in S$, and so is $-\nabla F(P)/\|\nabla F(P)\|$.

Let $S$ be a surface in $\mathbb{R}^3$. If there exists a continuous function $P \longmapsto \boldsymbol{n}(P)$ from $S$ to $\mathbb{R}^3$ such that $\boldsymbol{n}(P)$ is a unit normal vector at $P$, then we say that the surface $S$ is **orientable**, and such an assignment is called an **orientation** of $S$. A surface $S$ with a given orientation is called **oriented**.

Let $S$ be an oriented surface. If a parametrization $\Phi$ of $S$ yields the same unit normal vectors as specified by the given orientation, then $\Phi$ is called an **orientation-preserving parametrization** of $S$, and $\Phi^{\mathrm{op}}$ is called an **orientation-reversing parametrization** of $S$.

Let $S$ be an oriented surface in $\mathbb{R}^3$, and let $\boldsymbol{F} : S \to \mathbb{R}^3$ be a continuous vector field. If $\Phi$ is an orientation-preserving parametrization of $S$, then we define

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} := \iint_\Phi \boldsymbol{F} \cdot d\boldsymbol{S}.$$

## Examples

(i) Let $E \subset \mathbb{R}^2$, and $f : E \to \mathbb{R}$ be a smooth scalar field. Consider the graph $S := \{(x, y, f(x, y) : (x, y) \in E\}$ of $f$. For $P := (x, y, z) \in S$, let $\boldsymbol{n}(P) := (-f_x(P), -f_y(P), 1)/\|(-f_x(P), -f_y(P), 1)\|$. This continuous assignment of upward unit normal vectors gives an orientation of $S$. Hence $S$ is orientable. Clearly, the parametrization of $S$ given by $\Phi(x, y) := (x, y, f(x, y))$ for $(x, y) \in E$, is orientation-preserving.

(ii) Let $S := \{(x, y, z) :\in \mathbb{R}^3 : x^2 + y^2 = a^2 \text{ and } 0 \le z \le h\}$. For $P := (x, y, z) \in S$, let $\boldsymbol{n}(P) := (-x/a, -y/a, 0)$. This continuous assignment of inward unit normal vectors gives an orientation of $S$. Hence the cylinder $S$ is orientable.

Let $E := [0, h]\times[-\pi, \pi]$ and

$\Psi(z, \theta) := (a\cos\theta, a\sin\theta, z)$ for $(z, \theta) \in E$.

If $P := \Psi(z, \theta) = (x, y, z) \in S$, then

$$\frac{\Psi_z \times \Psi_\theta}{\|\Psi_z \times \Psi_\theta\|}(z, \theta) = \frac{(-a\cos\theta, -a\sin\theta, 0)}{a} = \left(-\frac{x}{a}, -\frac{y}{a}, 0\right) = \boldsymbol{n}(P).$$

Hence $\Psi$ is an orientation-preserving parametrization.

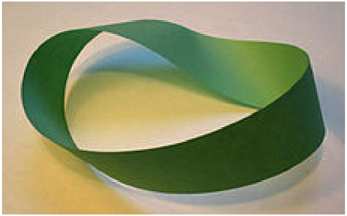
(iii) Let $S := \{(x, y, z) :\in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2\}$. For $P := (x, y, z) \in S$, let $\boldsymbol{n}(P) := (x/a, y/a, z/a)$. This continuous assignment of outward unit normal vectors gives an orientation of the sphere $S$. Hence $S$ is orientable.

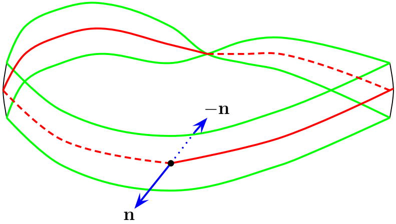
Let $E := (0, \pi) \times [-\pi, \pi]$ and
$\Phi(\varphi, \theta) := (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$.
If $P := \Phi(\varphi, \theta) = (x, y, z) \in S$, then

$$\frac{\Phi_\varphi \times \Phi_\theta}{\|\Phi_\varphi \times \Phi_\theta\|}(\varphi, \theta) = \frac{(a\sin\varphi)\Phi(\varphi, \theta)}{a^2\sin\varphi} = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) = \boldsymbol{n}(P).$$

Hence $\Phi$ is an orientation-preserving parametrization of $S \setminus \{(0, 0, \pm a)\}$.

## (iv) Möbius strip






(Artwork by David Bebbennick)

A parametrization of a Möbius strip is given as follows. Let $E := [-\pi, \pi] \times [-1, 1]$, and for $(u, v) \in E$, let $\Phi(u, v) := (x(u, v), y(u, v), z(u, v))$, where

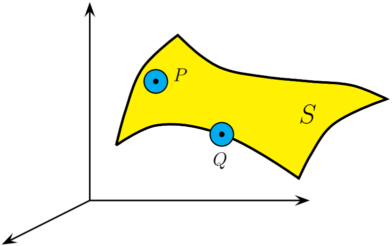
$$\begin{aligned} x(u, v) &:= \big(1 + (v/2)\cos(u/2)\big)\cos u, \\ y(u, v) &:= \big(1 + (v/2)\cos(u/2)\big)\sin u, \\ z(u, v) &:= (v/2)\sin(u/2). \end{aligned}$$

It can be shown that a Möbius strip is not orientable.

# Intrinsic Boundary of a Surface

Let $S$ be a (geometric) surface in $\mathbb{R}^3$, and suppose $S$ is a closed subset of $\mathbb{R}^3$. Typically, no point of $S$ is an interior point of $S$ as a subset of $\mathbb{R}^3$, and so $\partial S = S$.

Let $P$ be a point on $S$. It is called an **intrinsic interior point** of $S$ if there is a disk-like neighbourhood of $P$ that is contained in $S$. Otherwise, it is called an **intrinsic boundary point** of $S$. We shall denote the set of all intrinsic bounadry points of $S$ by $\eth S$. It consists of the 'edges' of $S$, if any.



In the figure above, $P$ is an intrinsic interior point of $S$, whereas $Q$ is an intrinsic boundary point of $S$.

If $\eth S = \emptyset$, then the surface $S$ is called a **surface without edges**. (Sometimes it is also called a **closed surface**.)

For a planar surface $S := D \subset \mathbb{R}^2$, the intrinsic boundary $\eth D$ coincides with the usual boundary $\partial D$.

Examples:

(i) $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2\} \implies \eth S = \emptyset$.

(ii) $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2 \text{ and } z \geq 0\} \implies \eth S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2 \text{ and } z = 0\}$.

(iii) $S := [a, b]\times[c, d]\times[p, q] \implies \eth S = \emptyset$.

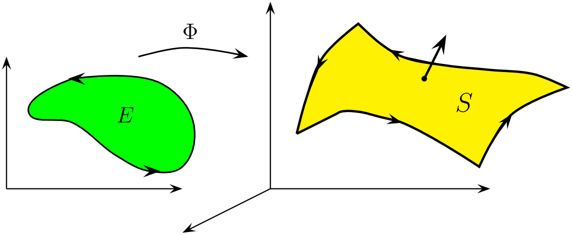
(iv) $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2,\ 0 \leq z \leq h\} \implies \eth S = \{(x, y, z) \in x^2 + y^2 = a^2 \text{ and } z = 0 \text{ or } z = h\}$.

# Orientation of the Intrinsic Boundary

An orientation of a surface induces an orientation on its intrinsic boundary as follows.

Suppose $S$ is an oriented surface, and let $\boldsymbol{n}(P)$ be the specified unit normal vector at a point $P$ on $S$. As we travel along any part of the intrinsic boundary $\eth S$ of the surface $S$ with $\boldsymbol{n}(P)$ as our upright direction, suppose the surface $S$ lies on our left. This gives the **induced orientation** on $\eth S$.

Note: If $\Phi : E \to \mathbb{R}^3$ is an orientation-preserving parametrization of the surface $S$, then the induced orientation on the intrinsic boundary $\eth S$ of $S$ corresponds to the positive orientation of the boundary $\partial E$ of the parameter set $E$.




Example:

Suppose the unit upper hemisphere
$S := \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$ is oriented by the unit normal vector $\boldsymbol{n}(P) := (x, y, z)$ for $P := (x, y, z) \in S$.

Let $E := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $\Phi(x, y) := \left(x, y, \sqrt{1 - x^2 - y^2}\right)$ for $(x, y) \in E$. Then $\Phi$ is an orientation-preserving parametrization of $S$, and the induced orientation on $\partial S = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z = 0\}$ corresponds to the counterclockwise orientation of $\partial E := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

# The Stokes Theorem

An important result of Stokes relates line integrals along the edges of surfaces to integrals across the surfaces. It is called the fundamental theorem of calculus for surface integrals. It can be viewed as a 'curved' version of the Green theorem. In fact, the proof of the Stoke theorem heavily depends on the proof of the Green theorem.

## Theorem

Let $S$ be a piecewise $C^2$, bounded oriented surface in $\mathbb{R}^3$ whose piecewise smooth intrinsic boundary $\eth S$ consists of a finite number of nonintersecting simple closed curves along with the induced orientation. Let $\boldsymbol{F}$ be a smooth vector field defined on an open subset containing $S$. Then

$$\int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$$

Sketch of Proof:

Let $\boldsymbol{F} := (P, Q, R)$, and $\Phi(u, v) := (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in E$. Then

$$\int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\eth S} P\,dx + Q\,dy + R\,dz.$$

By the chain rule,

$$\int_{\eth S} P\,dx = \int_{\partial E} P(\Phi(u, v))(x_u du + x_v dv).$$

Let us define $g(u, v) := P(\Phi(u, v))$ for $(u, v) \in E$. Then

$$\int_{\eth S} P\,dx = \int_{\partial E} (g\,x_u) du + (g\,x_v) dv.$$

The Green theorem shows that

$$\begin{aligned}\int_{\partial E}(g\,x_u)du+(g\,x_v)dv &= \iint_E \big((g\,x_v)_u-(g\,x_u)_v\big)\,d(u,v)\\ &= \iint_E (g_u\,x_v-g_v\,x_u)d(u,v)\end{aligned}$$

since $g\,x_{vu}-g\,x_{uv}=0$ by the Mixed Partials Theorem.

But since $g(u,v)=P(x(u,v),y(u,v),z(u,v))$, we obtain

$$g_u=P_x\,x_u+P_y\,y_u+P_z\,z_u,\quad g_v=P_x\,x_v+P_y\,y_v+P_z\,z_v,$$

by the chain rule. Hence

$$\begin{aligned}g_u\,x_v-g_v\,x_u &= -P_y(x_u\,y_v-x_v\,y_u)+P_z(z_u\,x_v-z_v\,u_u)\\ &= -P_y\,\frac{\partial(x,y)}{\partial(u,v)}+P_z\,\frac{\partial(z,x)}{\partial(u,v)}.\end{aligned}$$

It follows that

$$\int_{\partial S} P\,dx = \iint_E \left(-P_y \frac{\partial(x,y)}{\partial(u,v)} + P_z \frac{\partial(z,x)}{\partial(u,v)}\right) d(u,v).$$

In a similar manner,

$$\int_{\partial S} Q\,dy = \iint_E \left(-Q_z \frac{\partial(y,z)}{\partial(u,v)} + Q_x \frac{\partial(x,y)}{\partial(u,v)}\right) d(u,v).$$

and

$$\int_{\partial S} R\;dz = \iint_E \left(-R_x \frac{\partial(z,x)}{\partial(u,v)} + R_y \frac{\partial(y,z)}{\partial(u,v)}\right) d(u,v).$$

Hence $\displaystyle\int_{\partial S} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\partial S} P\,dx + Q\,dy + R\,dz$ is equal to

$$\iint_E \left((R_y - Q_z)\frac{\partial(y,z)}{\partial(u,v)} + (P_z - R_x)\frac{\partial(z,x)}{\partial(u,v)} + (Q_x - P_y)\frac{\partial(x,y)}{\partial(u,v)}\right) d(u,v).$$

On the other hand,

$$\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_E \operatorname{curl} \boldsymbol{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v)\, d(u,v),$$

where

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y),$$

and

$$\Phi_u \times \Phi_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left( \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right).$$

Thus

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}, \quad \text{as desired.} \quad \square$$

The conclusion of the Stokes theorem is also written as follows:

$$\int_{\partial\Phi} P\,dx + Q\,dy + R\,dz$$
$$= \iint_{\Phi} (R_y - Q_z)\,dy\wedge dz + (P_z - R_x)\,dz\wedge dx + (Q_x - P_y)\,dx\wedge dy.$$

# Tutorial 13

1. Let $E := [-\pi, \pi] \times [0, h]$. Consider the cone given by $\Phi(\theta, z) := (z\cos\theta, z\sin\theta, z)$ for $(\theta, z) \in E$. If $z_0 \neq 0$, then find the equation of the tangent plane at $P_0 := (z_0\cos\theta_0, z_0\sin\theta_0, z_0) \in \Phi(E)$. Also, find Area $(\Phi)$.
2. Find the area of the triangular region in $\mathbb{R}^3$ with vertices at $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. (Use Area Cosine Principle.)
3. Suppose a smooth curve $C$ in $\mathbb{R}^2$ lying on the right of the $y$-axis is revolved about the $y$-axis to generate a surface $S$. If $C$ is given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, find a parametrization $\Phi : E \to \mathbb{R}^3$ of the surface $S$. Find also the fundamental product of $\Phi$, and hence show that Area $(S) = 2\pi \int_\alpha^\beta |x(t)|\sqrt{x'(t)^2 + y'(t)^2}\,dt$.
4. Let $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$ for $(\theta, z) \in [-\pi, \pi] \times [0, h]$. If $f(x, y, z) := x + y + z$ for $(x, y, z) \in \mathbb{R}^3$, find $\iint_\Phi f\,dS$.

# Tutorial 13

5. (i) Let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \le 0\}$. What is the intrinsic boundary $\eth S$ of the unit lower hemisphere $S$? If $S$ is oriented by the unit normal vector $\boldsymbol{n}(P) := (x, y, z)$ for $P := (x, y, z) \in S$, find the induced orientation on $\eth S$?

   (ii) Let $S_1 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2,\ 0 \le z \le h\}$, $S_2 := \{(x, y, 0) : x^2 + y^2 \le a^2\}$, and $S := S_1 \cup S_2$. What is the intrinsic boundary $\eth S$ of the 'can' $S$? If $S_1$ is oriented by the unit normal vector $\boldsymbol{n}(P) := (x/a, y/a, 0)$, $P := (x, y, z) \in S_1$, find the induced orientation on $\eth S$?

6. Let $S$ denote the surface of the cone $z^2 = x^2 + y^2$ intercepted by the cylinder $x^2 + (y - a)^2 = a^2$, $z \ge 0$. Verify Stokes theorem for the vector field $\boldsymbol{F}(x, y, z) := (x - y, z + x, y + z)$ and the surface $S$.

# MA 105: Calculus
# Lecture 26

Prof. B.V. Limaye
IIT Bombay

Friday, 7 April 2017

# The Stokes Theorem

An important result of Stokes relates line integrals along the edges of surfaces to integrals across the surfaces. It is called the fundamental theorem of calculus for surface integrals. It can be viewed as a 'curved' version of the Green theorem. In fact, the proof of the Stoke theorem heavily depends on the proof of the Green theorem.

## Theorem

Let $S$ be a piecewise $C^2$, bounded oriented surface in $\mathbb{R}^3$ whose piecewise smooth intrinsic boundary $\eth S$ consists of a finite number of nonintersecting simple closed curves along with the induced orientation. Let $\boldsymbol{F}$ be a smooth vector field defined on an open subset containing $S$. Then

$$\int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$$

# Surface-Independence

Let $E \subset \mathbb{R}^2$, and consider a parametrized surface $\Phi : E \to \mathbb{R}^3$. Consider $D \subset \mathbb{R}^3$. We say that $\Phi$ lies in $D$ if $\Phi(E) \subset D$.

Let $\boldsymbol{F} : D \to \mathbb{R}^3$ be a continuous vector field. We say that the surface integrals of $\boldsymbol{F}$ are **surface-independent** in $D$ if

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S}$$

whenever the piecewise smooth, bounded parametrized surfaces $\Phi$ and $\widetilde{\Phi}$ lie in $D$, and $S := \Phi(E)$ and $\widetilde{S} := \widetilde{\Phi}(\widetilde{E})$ have the same intrinsic boundary with the same induced orientation on it. Note that $S$ and $\widetilde{S}$ need not be the same.

(Compare the definition of path-independence of a vector field given in Lecture 22.)

## Theorem

Let $D$ be an open subset of $\mathbb{R}^3$, and let $\boldsymbol{F}$ be a continuous vector field on $D$. Suppose $\boldsymbol{F}$ is a curl field. Then surface integrals of $\boldsymbol{F}$ are surface-independent in $D$. In fact, let $\boldsymbol{G}$ be a smooth vector field on $D$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$ on $D$. Then

$$\iint_{\Phi} \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S} = \int_{\eth S} \boldsymbol{G} \cdot d\boldsymbol{s} \text{ for every parametrized surface } \Phi$$

lying in $D$, where the intrinsic boundary $\eth S$ of the geometric surface defined by $\Phi$ has the induced orientation.

Proof: Let $\Phi : E \to \mathbb{R}^3$ be a piecewise $C^2$, bounded parametrized surface in $D$, and let $S := \Phi(E)$.

By the Stokes theorem,

$$\iint_{\Phi} \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S} = \int_{\eth S} \boldsymbol{G} \cdot d\boldsymbol{s} \quad \square$$

If $F$ is a curl field, then the evaluation of a surface integral $\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}$ can be simplified by replacing $\Phi$ by a parametrized surface having the same oriented intrinsic boundary. If we actually know a vector field $\boldsymbol{G}$ such that curl $\boldsymbol{G} = \boldsymbol{F}$, then we can reduce this evaluation to the calculation of a line integral.

Example: Let $F(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, and let $\Phi(u, v) := (u, v, \sqrt{1 - u^2 - v^2})$ for $(u, v) \in E$, where $E := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$.

If we consider $G(x, y, z) := (z^2/2, x^2/2, y^2/2)$ for $(x, y, z) \in \mathbb{R}^3$, then curl $G = \boldsymbol{F}$ on $\mathbb{R}^3$.

Also, if we let $\widetilde{\Phi}(u, v) := (u, v, 0)$ for $(u, v) \in E$, then the oriented intrinsic boundary of $S := \Phi(E)$ and $\widetilde{S} := \widetilde{\Phi}(E)$ is the same, namely the positively oriented standard unit circle $\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\}$.

Since $\widetilde{\Phi}_u \times \widetilde{\Phi}_v = (1,0,0) \times (0,1,0) = (0,0,1)$,

$$\begin{aligned}\iint_{\Phi} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\widetilde{\Phi}} \mathbf{F} \cdot d\mathbf{S} = \iint_{E} (v, 0, u) \cdot (0,0,1)\, d(u,v) \\ &= \iint_{E} u\, d(u,v) = \int_{-\pi}^{\pi} \int_{0}^{1} (r\cos\theta) r\, dr\, d\theta = 0.\end{aligned}$$

Also, using the Stokes theorem directly, we obtain

$$\begin{aligned}\iint_{\Phi} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\Phi} \operatorname{curl} \mathbf{G} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{G} \cdot d\mathbf{s} \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (0, \cos^2 t, \sin^2 t) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos^3 t\, dt = \frac{1}{8} \int_{-\pi}^{\pi} (\cos 3t + 3\cos t) dt = 0,\end{aligned}$$

which agrees with our previous simpler calculation.

Let $D$ be an open subset of $\mathbb{R}^3$. It is important to know when a given vector field $\boldsymbol{F}$ on $D$ is in fact a curl field, that is, when there is a smooth vector field $\boldsymbol{G}$ on $D$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$.

We already know a necessary condition, namely, $\operatorname{div} \boldsymbol{F} = 0$:
If $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$ on $D$, then $\operatorname{div}(\boldsymbol{F}) = \operatorname{div}(\operatorname{curl} \boldsymbol{G}) = 0$ on $D$.

Caution: Even if $\operatorname{div} \boldsymbol{F} = 0$ on $D$, $\boldsymbol{F}$ may not be a curl field!

Example:
Let $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, and let $\boldsymbol{F} := (P, Q, R)$, where

$$P := \frac{x}{(x^2+y^2+z^2)^{3/2}}, \quad Q := \frac{y}{(x^2+y^2+z^2)^{3/2}}, \quad R := \frac{z}{(x^2+y^2+z^2)^{3/2}}.$$

Then $(x^2 + y^2 + z^2)^3 \operatorname{div} \boldsymbol{F} = (x^2 + y^2 + z^2)^3 (P_x + Q_y + R_z) = 3(x^2 + y^2 + z^2)^{3/2} - 3(x^2 + y^2 + z^2)^{1/2}(x^2 + y^2 + z^2) = 0$.
Thus $\operatorname{div} \boldsymbol{F} = 0$ on $D$. But $\boldsymbol{F}$ is not a curl field on $D$ since the standard unit sphere $S := \Phi([0, \pi] \times [-\pi, \pi])$ lies in $D$, and

$$\begin{aligned}\iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} &= \iint_{E} \Phi(u,v)\cdot\big(\sin u\, \Phi(u,v)\big)\, d(u,v)\\ &= \int_{-\pi}^{\pi}\left(\int_{0}^{\pi}\sin u\, du\right)dv = 4\pi \neq 0,\end{aligned}$$

whereas $\iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{\Phi} \operatorname{curl} \boldsymbol{G}\cdot d\boldsymbol{S}$ must equal zero as a consequence of the Stokes theorem. (Note: $\partial S = \emptyset$.)

A geometric surface $S$ such that $\partial S = \emptyset$ is called a **closed** surface. (Compare the definition of a closed geometric curve.)

If $D := I \times J \times K$, where $I, J, K$ are open intervals in $\mathbb{R}$, and $\boldsymbol{F}$ is a smooth vector field on $D$ such that $\operatorname{div} \boldsymbol{F} = 0$ on $D$, then $\boldsymbol{F}$ is indeed a curl field. In fact, this also holds if $D$ is an open subset of $\mathbb{R}^3$ such that every surface in $D$ without any edge is the boundary of a solid lying entirely in $D$.

Let $D := I \times J \times K$, where $I, J, K$ are open intervals in $\mathbb{R}$. Suppose $\boldsymbol{F} := (P, Q, R)$ is a smooth vector field on $D$ such that div $\boldsymbol{F} = 0$ on $D$. We find a vector field $\boldsymbol{G} := (L, M, N)$ on $D$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$, that is,

$$(P, Q, R) = (N_y - M_z, L_z - N_x, M_x - L_y).$$

Let $x_0 \in I$, $z_0 \in K$.

Define $L := 0$ and $N(x, y, z) := -\int_{x_0}^{x} Q(u, y, z)du$.

Then $L_z - N_x = 0 + Q = Q$. Next, define

$M(x, y, z) := \int_{x_0}^{x} R(u, y, z)du - \int_{z_0}^{z} P(x_0, y, v)dv$. Then

$M_x - L_y = R - 0 = R$.

Finally, by differentiating under the integral sign,

$$\begin{aligned} N_y(x, y, z) &= -\int_{x_0}^{x} Q_y(u, y, z)du = \int_{x_0}^{x} (P_x + R_z)(u, y, z)du \\ &= P(x, y, z) - P(x_0, y, z) + \int_{x_0}^{x} R_z(u, y, z)du, \end{aligned}$$

and

$$M_z(x,y,z) = \int_{x_0}^{x} R_z(u,y,z)du - P(x_0,y,z).$$

Hence $N_y - M_z = P$. Thus $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$.

Further, if there is a vector field $H$ on $D$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{H}$ as well, then $\operatorname{curl}(H - G) = 0$ on $D$, and so $H = G + \nabla f$, where $f$ is a smooth scalar field on $D$, as we have seen while considering 'path-independence' of vector fields.

Example:

Let $\boldsymbol{F} := (P, Q, R)$, where $P := x$, $Q = -2\,y$ and $R := z$ for $(x,y,z) \in \mathbb{R}^3$. Clearly, $\operatorname{div} \boldsymbol{F} = 0$ on $\mathbb{R}^3$.
Let $x_0 := 0 =: z_0$. Define $L := 0$, $N := 2\,x\,y$ and $M := z\,x$ as above. If we let $\boldsymbol{G} := (0, z\,x, 2\,x\,y)$, then $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$. Also, if $\operatorname{curl} \boldsymbol{H} = \boldsymbol{F}$, then $\boldsymbol{H} = \boldsymbol{G} + (f_x, f_y, f_z)$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth.

# Uses of Stokes Theorem

(i) Calculation of the surface integral of a curl field

Example:

Let $\boldsymbol{F}(x,y,z) := (y, -x, e^{xz})$ for $(x,y,z) \in \mathbb{R}^3$, and let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z-\sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors.

In order to find $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}$, we note that the induced orientation on $\partial S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}$ is anticlockwise as seen from the point $(0,0,4)$.

By the Stokes theorem,

$$\begin{aligned}\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} &= \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= \int_{-\pi}^{\pi} (\sin t, -\cos t, e^{\cos t \cdot 0}) \cdot (-\sin t, \cos t, 0)\,dt \\ &= \int_{-\pi}^{\pi} -(\sin^2 t + \cos^2 t)\,dt = -2\pi.\end{aligned}$$

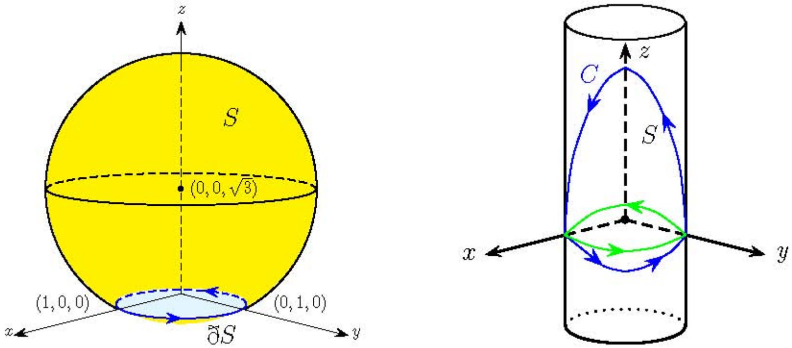



Figure : Examples (i) and (ii) of the use of the Stokes theorem

## (ii) Calculation of the line integral along an oriented boundary

Example:
Let $\boldsymbol{F}(x,y,z) := (-y^3, x^3, -z^3)$ for $(x,y,z) \in \mathbb{R}^3$, and let $C$ denote the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$, oriented by the anticlockwise motion on the projection of $C$ on the $xy$-plane.

In order to find $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$, consider the surface

$$S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \text{ and } x + y + z = 1\}.$$

Let us think of $S$ as the graph of the function
$f(x,y) := 1 - x - y$, $(x,y) \in D_0 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, oriented by the upward unit normal vectors. Then $\partial S = C$, and the induced orientation is the same as the given orientation of $C$. Hence

$$d\boldsymbol{S} = (-f_x, -f_y, 1)d(x,y) = (1,1,1)d(x,y).$$

Also,

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0, 0, 3x^2 + 3y^2),$$

and so $(\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 3(x^2 + y^2) d(x, y)$.

By the Stokes theorem,

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 3 \iint_{D_0} (x^2 + y^2) d(x, y) \\ &= 3 \int_0^1 \int_{-\pi}^{\pi} r^2 r \, dr \, d\theta = \frac{3\pi}{2}. \end{aligned}$$

# Consequences of Stokes Theorem

## Proposition

Let $\boldsymbol{F}$ be a smooth vector field on an open subset $D$ of $\mathbb{R}^3$ such that $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$ on $D$.
(i) Suppose $S$ is a bounded oriented piecewise $C^2$ surface in $D$, and let $\eth S$ denote its intrinsic boundary with the induced orientation, as in the Stokes theorem. Then $\int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

In particular, if $\eth S = C_1 - C_2$, that is, if $\eth S = C_1 \cup C_2$ with induced orientations on $C_1$ and $-C_2$, then $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$.

(ii) If $D$ is simply connected, then $\boldsymbol{F}$ is a gradient field on $D$.

Proof:
(i) By the Stokes theorem,

$$\int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_S \boldsymbol{0} \cdot d\boldsymbol{S} = 0.$$

Next, suppose $\eth S = C_1 - C_2$ as above. Then

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_1 - C_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

(ii) Suppose $D$ is simply connected. Let $C$ denote a simple closed smooth curve in $D$. Then there is a smooth surface $S$ in $D$ such that $\eth S = C$, and so $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

This also holds if a smooth closed curve $C$ in $D$ intersects itself, since we can break up $C$ into several simple closed smooth curves. As a result, line integrals of $\boldsymbol{F}$ are path independent in $D$, and so $\boldsymbol{F}$ is a gradient field on $D$. $\square$

Remark:
The last statement in part (i) above is known as the invariance of line integrals, or as the deformation principle for line integrals along paths in $\mathbb{R}^3$.

Examples: (i) Let $\boldsymbol{F} := (P, Q, R)$, where $P := -y/(x^2+y^2)$, $Q := x/(x^2+y^2)$ and $R := 1$ for $(x, y, z) \in \mathbb{R}^3$ with $x^2+y^2 \neq 0$. We have seen earlier in Lecture 22 that $\text{curl}\,(\boldsymbol{F}) = (0, 0, Q_x - P_y) = \mathbf{0}$. Consider the upper hemisphere $H := \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \text{ and } z \geq 0\}$. Let $C_1$ denote the base circle given by $x^2+y^2 = 1$ and $z = 0$. Also, consider a plane in $\mathbb{R}^3$ which intersects $H$ in a circle $C_2$ such that $C_1 \cap C_2 = \emptyset$ and $(0, 0, 1) \notin C_2$.

Let $S$ denote the part of the hemisphere $H$ bounded by $C_1$ and $C_2$, and oriented by the outward unit normal vectors. Suppose $C_1$ and $-C_2$ have the induced orientation, that is, both $C_1$ and $C_2$ are anticlockwise as seen from the point $(0, 0, 2)$. Then

$$\int_{C_2} -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}\,dy + dz = \int_{C_1} -y\,dx + x\,dy = 2\pi,$$

as we had calculated earlier.

(ii) Let $D := \mathbb{R}^3$, and let $\boldsymbol{F}(x, y, z) := (y\, e^z, x\, e^z, x\, y\, e^z)$ for $(x, y, z) \in D$. It is easy to check that $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$ on $D$.

Since $D$ is simply connected, $\boldsymbol{F}$ is a gradient field. In fact, if we let $f(x, y, z) := x\, y\, e^z$ for $(x, y, z) \in D$, then $\nabla f = \boldsymbol{F}$ on $D$.

We turn to another consequence of the Stokes theorem.

**Proposition**

Let $S$ be an oriented smooth surface in $\mathbb{R}^3$ such that $\partial S = \emptyset$. Suppose $\boldsymbol{F}$ is a smooth vector field on an open subset containing $S$. Then $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 0$.

Proof: Introduce a hole in $S$ by cutting out a small piece along a smooth simple closed curve $C$ on $S$. Let $S_1$ denote the part of $S$ cut out, and let $S_2$ denote the remaining part of $S$. Then

$$\iint_S (\text{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} = \iint_{S_1} (\text{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} + \iint_{S_2} (\text{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S}.$$

By the Stokes theorem,

$$\iint_{S_1} (\text{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} = \int_{\eth S_1} \boldsymbol{F}\cdot d\boldsymbol{s} \text{ and } \iint_{S_2} (\text{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} = \int_{\eth S_2} \boldsymbol{F}\cdot d\boldsymbol{s}.$$

Although $\eth S_1 = C = \eth S_2$, the orientations induced on $C$ by $S_1$ and by $S_2$ are opposite. Hence

$$\iint_S (\text{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} = \int_{\eth S_1} \boldsymbol{F}\cdot d\boldsymbol{s} + \int_{\eth S_2} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_C \boldsymbol{F}\cdot d\boldsymbol{s} - \int_C \boldsymbol{F}\cdot d\boldsymbol{s} = 0.$$

# MA 105: Calculus
# Lecture 27

Prof. B.V. Limaye
IIT Bombay

Wednesday, 12 April 2017

Recall the conclusion of the Green theorem for a region $D$ in $\mathbb{R}^2$. Let $\partial D$ denote the positively oriented boundary of $D$:

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y) d(x,y).$$

(i) (Circulation-Curl form) Let $\boldsymbol{F} := (P, Q, 0)$. Then

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{k}\, d(x,y).$$

(ii) (Flux-Divergence form) Let $\boldsymbol{F} := (-Q, P, 0)$. Then

$$\int_{\partial D} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds = \iint_D (\operatorname{div} \boldsymbol{F})\, d(x,y).$$

We have seen that the Stokes theorem generalizes (i) from a region $D$ in $\mathbb{R}^2$ to a surface in $\mathbb{R}^3$. We shall now show that the Gauss divergence theorem generalizes (ii) from a region $D$ in $\mathbb{R}^2$ to a solid in $\mathbb{R}^3$.

Let $D$ be a closed and bounded subset of $\mathbb{R}^3$. Suppose that the boundary $\partial D$ of $D$ consists of a finite number of piecewise smooth nonintersecting geometric surfaces without any edges. (Note that in this case $D$ has a volume.) If each of these surfaces is so oriented that the unit normal vectors point out of the solid $D$, then we say that the boundary $\partial D$ of $D$ is **positively oriented**. (Compare the definition of the positively oriented boundary of a multiply connected domain in $\mathbb{R}^2$.)

Example: Let $D := \{(x, y, z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 4\}$.
Then $\partial D = S_1 \cup S_2$ is positively oriented if
$S_1 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is oriented by the inward unit normal vectors, and
$S_2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4\}$ is oriented by the outward unit normal vectors. Note that $\eth S_1 = \emptyset = \eth S_2$.

### Theorem (Gauss Divergence Theorem)

Let $D$ be a closed and bounded subset of $\mathbb{R}^3$ whose boundary $\partial D$ consists of a finite number of nonintersecting piecewise smooth surfaces without any edges, and is positively oriented. Let $\boldsymbol{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing $D$. Then

$$\iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S} \;=\; \iiint_D (\operatorname{div} \boldsymbol{F})\, d(x, y, z).$$

Proof: Let $D$ be an elementary region of the following type. Suppose $D_0 \subset \mathbb{R}^2$, and $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ are continuous. Let $D := \{(x, y, z) \in \mathbb{R}^3 : \psi_1(x, y) \le z \le \psi_2(x, y)\}$, and suppose $\partial D$ consists of a single surface $S$ without any edges, and $S := S_1 \cup S_2$ , where $S_1 := \{(x, y, z) \in \mathbb{R}^3 : z = \psi_1(x, y)\}$ is its lower part and $S_2 := \{(x, y, z) \in \mathbb{R}^3 : z = \psi_2(x, y)\}$ is its upper part.

Let $\boldsymbol{F} = (P, Q, R)$. Then the surface integral on the left side is

$$\iint_{\partial D} (P, Q, R) \cdot d\boldsymbol{S}.$$

On the other hand, the triple integral on the right side is

$$\iiint_D \operatorname{div}(\boldsymbol{F}) d(x, y, z) = \iiint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) d(x, y, z).$$

We show that

$$\iint_{\partial D} (0, 0, R) \cdot d\boldsymbol{S} = \iiint_D \frac{\partial R}{\partial z}\, d(x, y, z).$$

The other two equalities would follow similarly.

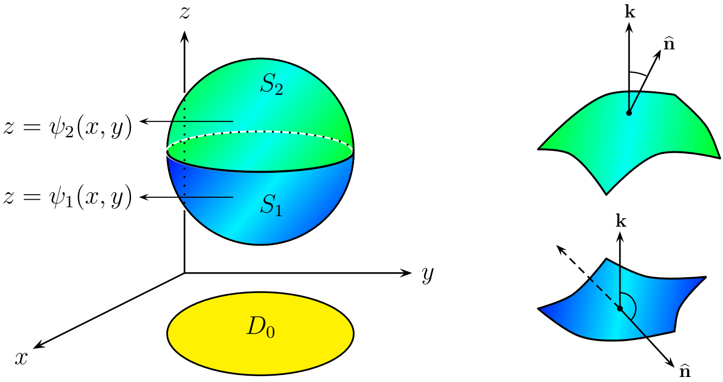



Figure : An outward normal $\hat{\boldsymbol{n}}$ to the upper part $S_2$ is upward, and an outward normal $\hat{\boldsymbol{n}}$ to the lower part $S_1$ is downward.

The upper part $S_2$ is given by $\Phi(x, y) := (x, y, \psi_2(x, y))$ for $(x, y) \in D_0$, and so the normal vector

$$\Phi_x \times \Phi_y = \left(1, 0, \frac{\partial \psi_2}{\partial x}\right) \times \left(0, 1, \frac{\partial \psi_2}{\partial y}\right) = \left(-\frac{\partial \psi_2}{\partial x}, -\frac{\partial \psi_2}{\partial y}, 1\right),$$

points 'upward', that is, 'out of ' the solid $D$. Hence

$$\begin{aligned} \iint_{S_2} (0, 0, R) \cdot d\boldsymbol{S} &= \iint_{D_0} \big(0, 0, R(\Phi(x, y))\big) \cdot (\Phi_x \times \Phi_y) d(x, y) \\ &= \iint_{D_0} R(x, y, \psi_2(x, y)) d(x, y). \end{aligned}$$

The lower part $S_1$ is given by $\Phi(x, y) := (x, y, \psi_1(x, y))$ for $(x, y) \in D_0$. Now the 'upward' normal vector $\Phi_x \times \Phi_y = \left(-\frac{\partial \psi_1}{\partial x}, -\frac{\partial \psi_1}{\partial y}, 1\right)$ points inward to the solid $D$. Hence

$$\iint_{S_1} (0, 0, R) \cdot d\boldsymbol{S} = -\iint_{D_0} R(x, y, \psi_1(x, y)) d(x, y).$$

Combining and using $S_1 \cup S_2 = S = \partial D$, we obtain

$$\begin{aligned}
\iint_{\partial D} (0,0,R)\cdot d\boldsymbol{S} &= \iint_{D_0} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} \frac{\partial R}{\partial z} dz \right) d(x,y) \\
&= \iiint_D \frac{\partial R}{\partial z} d(x,y,z)
\end{aligned}$$

by Part II of the FTC, and by the Fubini theorem for the triple integrals, which is also known as the Cavalieri Principle.

The above proof can be modified to treat the general case. $\square$

Physical Interpretation:

The Gauss divergence theorem says that the flux of a vector field $\boldsymbol{F}$ across an oriented surface $S$ without any edges in the direction of the outward normals is equal to the integral of the divergence of the vector field $\boldsymbol{F}$ over the solid $D$ enclosed by the surface $S$.

The Gauss Divergence theorem is often stated as follows:

$$\iint_{\partial D} P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$$
$$= \iiint_D (P_x + Q_y + R_z) d(x, y, z),$$

where $P, Q, R$ are smooth real-valued functions defined on an open subset of $\mathbb{R}^3$ containing a closed bounded subset $D$ of $\mathbb{R}^3$ whose boundary $\partial D$ consists of a finite number of piecewise smooth surfaces without any edges, and is oriented by normals going out of $D$.

# Uses of the Gauss Divergence Theorem

(i) Calculation of a triple integral

Example: Calculation of volume of a subset of $\mathbb{R}^3$

Let $D$ be a closed and bounded subset of $\mathbb{R}^3$ whose positively oriented boundary $\partial D$ consists of a finite number of piecewise smooth surfaces. Note that

$$\mathrm{Vol}\,(D) := \iiint_D 1_D d(x, y, z).$$

Let $\boldsymbol{F}$ be a smooth vector field such that $\operatorname{div} \boldsymbol{F} = 1$ on $D$. By the Gauss divergence theorem,

$$\mathrm{Vol}\,(D) = \iiint_D (\operatorname{div} \boldsymbol{F}) d(x, y, z) = \iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

Thus if $\boldsymbol{F} := (P, Q, R)$, and $P_x + Q_y + R_z = 1$ on $D$, then

$$\mathrm{Vol}\,(D) = \iint_{\partial D} P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy.$$

For instance, for $(x, y, z) \in D$, we may let

$$P(x,y,z) := \frac{x}{3}, \quad Q(x,y,z) := \frac{y}{3}, \quad R(x,y,z) := \frac{z}{3}.$$

Then clearly $P_x + Q_y + R_z = 1$ on $D$, and so

$$\mathrm{Vol}\,(D) = \frac{1}{3} \iint_{\partial D} x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

Suppose the positively oriented boundary $\partial D$ of $D$ is parametrized by $\Phi(u,v) := (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in E$. Then

$$x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy = \det \begin{bmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix} d(u,v).$$

The determinant of the matrix on the right is called the **Wronskian** of $x, y, z$, and it is denoted by $W(x, y, z)$. Thus

$$\text{Vol}\,(D) = \frac{1}{3} \iint_E W(x, y, z)(u, v) d(u, v).$$

We note that $\text{Vol}\,(D)$ is also equal to

$$\iint_{\partial D} x\, dy \wedge dz; \quad \iint_{\partial D} y\, dz \wedge dx \quad \text{and} \quad \iint_{\partial D} z\, dx \wedge dy$$

if we let $P := x$, $Q := R := 0$; $Q := y$, $P := R := 0$ and $R := z$, $P := Q := 0$, respectively.

Example:

Let $a, b, c, d > 0$ be such that $d < \min\{a, b, c\}$, and let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1 \text{ and } x^2+y^2+z^2 \ge d^2\right\}.$$

Then $\partial D = S_1 \cup S_2$, where
$S_1 := \{(x, y, z) \in \mathbb{R}^3 : x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ with the outward normals, and $S_2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = d^2\}$ with the inward normals.

The surface $S_1$ is given by
$x(u, v) := a \sin u \cos v,\ y(u, v) := b \sin u \sin v,\ z(u, v) := c \cos u$ for $(u, v) \in [0, \pi]\times[-\pi, \pi]$. We can heck that the Wronskian $W(x, y, z)$ at $(u, v) \in [0, \pi]\times[-\pi, \pi]$ is equal to $abc \sin u$.

Also, $S_2$ is obtained by letting $a := b := c := d$. Hence

$$\text{Vol}\,(D) = \frac{1}{3}(abc - d^3) \int_{-\pi}^{\pi} \left( \int_0^{\pi} \sin u \, du \right) dv = \frac{4\pi}{3}(abc - d^3).$$

(ii) Calculation of surface integral across an oriented boundary

Example:

Let $D := \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$, and let $\boldsymbol{F}(x, y, z) := (x^2 z^3,\ 2x\,y\,z^3,\ x\,z^4)$ for $(x, y, z) \in \mathbb{R}^3$.

Direct calculation of $\iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S}$ is long since $\partial D$ has 6 faces.

On the other hand, we note that $\operatorname{div} \boldsymbol{F}(x, y, z) = 2x\,z^3 + 2x\,z^3 + 4x\,z^3 = 8x\,z^3$, and by the Gauss divergence theorem, we obtain

$$\iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_D (\operatorname{div} \boldsymbol{F})\, d(x, y, z) = \iiint_D 8x\,z^3\, d(x, y, z) = 162.$$

# Consequences of the Gauss Divergence Theorem

## Proposition

Let $\boldsymbol{F}$ be a smooth vector field on an open subset containing a closed and bounded subset $D$ of $\mathbb{R}^3$ such that $\operatorname{div} \boldsymbol{F} = 0$ on $D$. If $\partial D$ consists of a finite number of nonintersecting closed piecewise smooth surfaces, and is oriented by normals going out of $D$, then

$$\iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S} = 0.$$

Proof:

By the Gauss divergence theorem,

$$\iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_D (\operatorname{div} \boldsymbol{F}) d(x, y, z) = \iiint_D 0 \, d(x, y, z) = 0. \quad \square$$

# Invariance of Some Surface Integrals

Suppose $S_1$ and $S_2$ are smooth surfaces without edges such that $S_2$ lies in the interior of $S_1$, that is, $S_1$ encloses $S_2$. Let $D$ denote the subset of $\mathbb{R}^3$ consisting of $S_1, S_2$ and the region between them. Let $P, Q, R$ be smooth scalar fields satisfying $P_x + Q_y + R_z = 0$ on an open set containing $D$. If $\boldsymbol{F} := (P, Q, R)$, then $\operatorname{div} F = 0$ on $D$, and so

$$\iint_{\partial D} P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy = \iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = 0,$$

provided $\partial D = S_1 \cup S_2$ is oriented by normals going out of $D$, so that $S_1$ and $S_2$ are oriented by normals going in opposite directions. Hence if $S_1$ and $S_2$ are oriented by normals going in the same direction, then the surface integrals of $P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$ across $S_1$ and $S_2$ are equal.

This is called the deformation principle for surface integrals.

Example: Gauss Law in $\mathbb{R}^3$: Let $S$ be a piecewise smooth surface in $\mathbb{R}^3$ without any edges, and let $D$ denote the union of $S$ and the region enclosed by $S$, so that $\partial D = S$, oriented by outward normals. Suppose $\mathbf{0} \notin S$. Then

$$\iint_S \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} = \begin{cases} 0 & \text{if } \mathbf{0} \notin D, \\ 4\pi & \text{if } \mathbf{0} \in D, \end{cases}$$

where $\boldsymbol{r}(x,y,z) := (x,y,z)$ and $r(x,y,z) := \|\boldsymbol{r}(x,y,z)\|$ for $(x,y,z) \in \mathbb{R}^3$.

Proof: Let $\boldsymbol{F} := \boldsymbol{r}/r^3$, that is, $\boldsymbol{F} := (P,Q,R)$, where $P = x/r^3$, $Q := y/r^3$ and $R := z/r^3$ for $\boldsymbol{r} \neq \mathbf{0}$. We have seen earlier that $P_x + Q_y + R_z = \operatorname{div} \boldsymbol{F} = 0$ and $\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} = 4\pi$, where $S_1$ denotes the standard unit sphere in $\mathbb{R}^3$ (oriented by the outward unit normal vectors).

Let $\mathbf{0} \notin D$. Then $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_D \operatorname{div}(\boldsymbol{F}) d(x,y,z) = 0$ by the Gauss divergence theorem.

Next, let $\mathbf{0} \in D$. Since $\mathbf{0}$ is an interior point of $D$, there is $\epsilon > 0$ such that the closed ball of radius $\epsilon$ and center $\mathbf{0}$ lies inside $D$. Let $S_\epsilon$ denote the sphere of radius $\epsilon$ and centre $\mathbf{0}$, oriented by the outward unit normal vectors. Then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{S_\epsilon} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} = 4\pi$$

by the deformation principle for surface integrals, applied to the surfaces $S$ and $S_\epsilon$, and then to the surfaces $S_\epsilon$ and $S_1$. $\square$

The treatment of calculus presented in these lectures is based on the books

(i) A Course in Calculus and Real Analysis

(ii) A Course in Multivariable Calculus and Analysis

written by Sudhir R. Ghorpade and Balmohan V. Limaye, and published by Springer, New York, in 2006 and 2010 respectively,

and on the book

(iii) Basic Multivariable Calculus

written by Jerrold E. Marsden, Anthony J. Tromba and Alan Weinstein, and published by Springer, New York, in 1993.

Acknowledgement

I have also made use of the lecture notes of similar courses given by Sudhir Ghorpade (Autumn Semester of 2011-2012) and Prachi Mahajan (Autumn Semester of 2015-2016) at the Indian Institute of Technology Bombay.