Tut 1

1

(iii) $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$

Given $\epsilon > 0$, choose $N_0 \in \mathbb{N}$ s.t. $N_0 > 1/\epsilon^3$.

$\forall n \geq N_0$, $\left|\frac{n^{2/3}\sin(n!)}{n+1}\right| \leq \left|\frac{n^{2/3}}{n+1}\right| \leq \left|\frac{n^{2/3}}{n}\right|$

$\leq \frac{1}{N_0^{1/3}} < \epsilon$

$\therefore \lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$

(iv) $\lim_{n\to\infty}\left(\frac{n}{n+1} - \frac{n+1}{n}\right)$

$\left|\frac{n}{n+1} - \frac{n+1}{n}\right| = \left|\frac{2n+1}{n(n+1)}\right| < \left|\frac{2n+2}{n(n+1)}\right| = \frac{2}{n}$

Given $\epsilon > 0$, choose $N_0 \in \mathbb{N}$ s.t. $N_0 > 2/\epsilon$

Then, $\forall n \geq N_0$, $\left|\frac{n}{n+1} - \frac{n+1}{n}\right| < \frac{2}{n} < \frac{2}{N_0} < \epsilon$

$\therefore \lim_{n\to\infty}\left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$

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2

(i) $$\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \le n \cdot \left(\frac{n}{n^2+1}\right)$$

$$\frac{n}{n^2+1} + \cdots + \frac{n}{n^2+n} \geq n \cdot \left(\frac{n}{n^2+n}\right)$$

$$\lim_{n \to \infty} \frac{n^2}{n^2+1} = 1 \quad (\text{Prove it})$$

$$\lim_{n \to \infty} \frac{n^2}{n^2+n} = 1 \quad (\text{Prove it})$$

By Sandwich theorem, the required limit exists, and is equal to 1.

(iv) $n^{1/n} \geq 1 \quad \forall n \in \mathbb{N}$

Apply AM - GM on $\underbrace{1, 1, \ldots, 1}_{n-2}, \sqrt{n}, \sqrt{n}$

$$\left(1 \cdot 1 \cdots 1 \cdot \sqrt{n} \cdot \sqrt{n}\right)^{1/n} \le \frac{n-2+2\sqrt{n}}{n}$$

$$\therefore n^{1/n} \le 1 - \frac{2}{n} + \frac{2}{\sqrt{n}}$$

$$\lim_{n \to \infty} \left(1 - \frac{2}{n} + \frac{2}{\sqrt{n}}\right) = 1$$

$\therefore$ By Sandwich theorem, $\lim_{n \to \infty} n^{1/n} = 1$

(v) $\lim_{n \to \infty} \left(\frac{\cos \pi \sqrt{n}}{n^2} \right)$

$$\left| \frac{\cos \pi \sqrt{n}}{n^2} \right| \le 1/n^2$$

Is this enough to complete the proof?

(vi) $\lim_{n \to \infty} \sqrt{n} \left(\sqrt{n+1} - \sqrt{n} \right)$

$$\sqrt{n} \left(\sqrt{n+1} - \sqrt{n} \right) = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$\cancel{\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \le \sqrt{n}}$$

Prove that $\lim_{n \to \infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n}} = 2.$

Then, we will have $\lim_{n \to \infty} \sqrt{n} \left(\sqrt{n+1} - \sqrt{n} \right) = 1/2$

3

(i) $$\frac{n^2}{n+1} \geq \frac{n^2}{2n} = \frac{n}{2}$$

Suppose $\frac{n^2}{n+1} \leq M \quad \forall n \in \mathbb{N}$, for some $M \in \mathbb{R}$

But, there exists $N_0 \in \mathbb{N}$ s.t. $N_0 > 2M$

$$\therefore \frac{N_0^2}{N_0^2+1} \geq \frac{N_0}{2} > M$$

4

(i) Let $a_n = \frac{n}{n^2+1} \quad \forall n \in \mathbb{N}$

$$a_{n+1} - a_n = \frac{n+1}{n^2+2n+2} - \frac{n}{n^2+1}$$

$$= \frac{(n^2+1)(n+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{n^3+n^2+n+1-n^3-2n^2-2n}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \quad \forall n$$

$\therefore \{a_n\}_{n \geq 1}$ is a decreasing sequence.

(iii) Let $a_n = \frac{1-n}{n^2} \quad \forall n \geq 2$

$$a_{n+1} - a_n = \frac{-n}{(n+1)^2} - \frac{(1-n)}{n^2}$$

$$= \frac{-n^3 + (n-1)(n+1)^2}{(n+1)^2 n^2} > 0 \quad \forall n \geq 2$$

check

5

(ii) $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n} \quad \forall n \geq 1$

First, prove by induction that $0 \leq a_n \leq 2 \quad \forall n \geq 1$

Base case : $a_1 = \sqrt{2}$

Assume that $0 \leq a_n \leq 2$

Then, clearly $a_{n+1} \geq 0$ and $a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2+2} = 2$

$\therefore$ Proved by induction.

$$a_{n+1} \geq a_n \iff \sqrt{2 + a_n} \geq a_n$$

$$\iff 2 + a_n \geq a_n^2$$

$$\iff a_n^2 - a_n - 2 \leq 0$$

$$\iff (a_n - 2)(a_n + 1) \leq 0$$

$\therefore a_{n+1} \geq a_n$

$\therefore \{a_n\}$ is an increasing seq. bounded above.

Let $\lim_{n\to\infty} a_n = L$,

then $L = \sqrt{2+L} \Rightarrow L = 2$

7 $\lim_{n\to\infty} a_n = L \neq 0$

Let $\epsilon = \frac{|L|}{2}$

$\exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$, $|a_n - L| < \epsilon$

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \quad \forall n \geq N_0$$

$$\therefore |L| - |a_n| < \epsilon \quad \forall n \geq N_0$$

$$\Rightarrow |a_n| > |L| - \epsilon = \frac{|L|}{2} \quad \forall n \geq N_0$$

8 $\lim_{n\to\infty} a_n = 0$

Given $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$, $|a_n| < \epsilon^2$

$$\Rightarrow |a_n^{1/2}| \leq \epsilon \quad \forall n \geq N_0$$

$$\therefore \lim_{n\to\infty} a_n^{1/2} = 0$$

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10 ($\Rightarrow$) Suppose $\{a_n\}$ is convergent.

Let $L = \lim_{n\to\infty} a_n$.

Given $\epsilon > 0$, $\exists\, N_0 \in \mathbb{N}$ s.t. $|a_n - L| < \epsilon \;\; \forall n \geq N_0$.

Then $|a_{2n} - L| < \epsilon \;\; \forall n \geq N_0$

$|a_{2n+1} - L| < \epsilon \;\; \forall n \geq N_0$

$\therefore \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$

($\Leftarrow$) Suppose $\{a_{2n}\}_{n\geq 1}$, $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

Let $L = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1}$

Given $\epsilon > 0$, $\exists\, N_1, N_2 \in \mathbb{N}$ s.t.

$|a_{2n} - L| < \epsilon \;\; \forall\, n \geq N_1$

$|a_{2n+1} - L| < \epsilon \;\; \forall\, n \geq N_2$

$\therefore \forall n \geq \max\{2N_1, 2N_2+1\}$, $|a_n - L| < \epsilon$

$\therefore \lim_{n\to\infty} a_n = L$