

**MA111 (IIT Bombay) Tutorial Sheet 3 :**  
**Change of variables, Line integrals, February 13, 2022**

**I Multiple integrals and change of variables**

1. Find the volume of the solid that lies under the paraboloid  $z = x^2 + y^2$ , above the  $xy$  plane, and inside the cylinder  $x^2 + y^2 = 2x$ .
2. Using a suitable change of variables, evaluate the integral  $\iint_D y dx dy$ , where  $D$  is the region bounded by the  $x$ -axis and the parabolas  $y^2 = 4 - 4x$  and  $y^2 = 4 + 4x$ ,  $y \geq 0$ .
3. Use spherical coordinates to find the volume of the solid that lies above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$ .
4. Use cylindrical coordinates to evaluate  $\iiint_W (x^2 + y^2) dz dy dx$ , where

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x \leq 2, \quad -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \quad \sqrt{x^2 + y^2} \leq z \leq 2\}.$$

5. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin \phi d\rho d\phi d\theta,$$

and evaluate the integral.

6. Find  $\iiint_F \frac{1}{(x^2 + y^2 + z^2)^{n/2}} dV$ , where  $F$  is the region bounded by the spheres with center the origin and radii  $r$  and  $R$ ,  $0 < r < R$ .

**II Vector analysis and line integrals**

1. Let  $f, g$  be differentiable functions on  $\mathbb{R}^2$ . Show that
  - i.  $\nabla(fg) = f\nabla g + g\nabla f$ ;
  - ii.  $\nabla f^n = n f^{n-1} \nabla f$ ;
  - iii.  $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$  whenever  $g \neq 0$ .
2. Let  $\mathbf{a}, \mathbf{b}$  be two fixed vectors,  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $r^2 = x^2 + y^2 + z^2$ . Prove the following:
  - (i)  $\nabla(r^n) = nr^{n-2}\mathbf{r}$  for any integer  $n$ .
  - (ii)  $\mathbf{a} \cdot \nabla \left( \frac{1}{r} \right) = - \left( \frac{\mathbf{a} \cdot \mathbf{r}}{r^3} \right)$ .
  - (iii)  $\mathbf{b} \cdot \nabla \left( \mathbf{a} \cdot \nabla \left( \frac{1}{r} \right) \right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$ .

3. Calculate the line integral of the vector field

$$\mathbf{F}(x, y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from  $(-1, 1)$  to  $(1, 1)$  along  $y = x^2$ .

4. Calculate the line integral of

$$\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  in the counter clockwise direction.

**Remark:** Often line integral of a vector field  $\mathbf{F}$  along a ‘geometric curve’  $C$  is represented by  $\int_C \mathbf{F} \cdot d\mathbf{s}$ . A geometric curve  $C$  is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.

To evaluate  $\int_C \mathbf{F} \cdot d\mathbf{s}$ , choose a convenient parametrization  $\mathbf{c}$  of  $C$  traversing  $C$  in the given direction and then

$$\int_C \mathbf{F} \cdot d\mathbf{s} := \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

‘ $\oint_C$ ’ means the line integral over a closed curve  $C$ .

5. Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$$

where  $C$  is the curve  $x^2 + y^2 = a^2$  traversed once in the counter clockwise direction.

6. Calculate

$$\oint_C ydx + zdy + xdz$$

where  $C$  is the intersection of two surfaces  $z = xy$  and  $x^2 + y^2 = 1$  traversed once in a direction that appears counter clockwise when viewed from high above the  $xy$ -plane.

7. Let the curve  $C$  be given by  $x^2 + y^2 = 1, z = 0$ . Let  $\mathbf{c}_1$  be a parametrization defined by  $\mathbf{c}_1(t) = (\cos t, \sin t)$  for  $t \in [0, 2\pi]$ . Find the line integral of  $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$  along this curve. Also find the line integral along the curve parametrized by  $\mathbf{c}_2(t) = (\cos t, -\sin t)$ , for  $t \in [0, \pi]$ .
8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle:  $x^2 + y^2 = 1$ . Is this also true for a force field  $\mathbf{F}(x, y, z) = \alpha(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ , for some constant  $\alpha$ .
9. Let  $C : x^2 + y^2 = 1$ . Find

$$\oint_C \text{grad } (x^2 - y^2) \cdot d\mathbf{s}.$$

10. Evaluate

$$\int_C \text{grad } (x^2 - y^2) \cdot d\mathbf{s},$$

where  $C$  is  $y = x^3$ , joining  $(0, 0)$  and  $(2, 8)$ .

11. Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where  $C$  is the square with vertices  $(1, 0), (0, 1), (-1, 0)$  and  $(0, -1)$  traversed once in the counter clockwise direction.

12. A force  $F = xy\mathbf{i} + x^6y^2\mathbf{j}$  moves a particle from  $(0, 0)$  onto the line  $x = 1$  along  $y = ax^b$  where  $a, b > 0$ . If the work done is independent of  $b$  find the value of  $a$ .