Tutorial 3

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Question 1

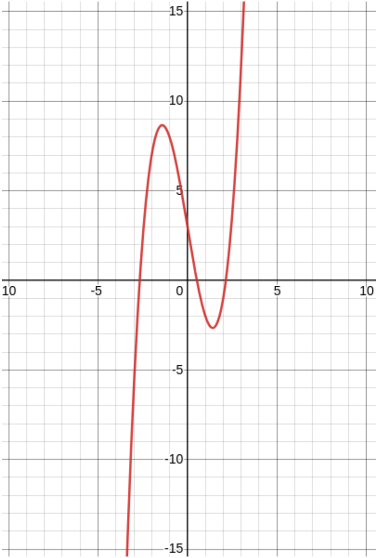
To show : If $f(x) = x^3 - 6x + 3$, then f has all roots real.

- Since, $f(x)$ is an odd-degree polynomial in x, with positive coefficient of the highest power of x, we can always find sufficiently positive/negative numbers s.t. $f(x)$ takes positive/negative values respectively.
- Secondly, the stationary points of f i.e. points where $f'(x) = 0$ are $x = \pm\sqrt{2}$
- Observe, that $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0, f(\sqrt{2}) = -4\sqrt{2} + 3 < 0$ Convince yourself that f satisfies the hypotheses of Intermediate Value Theorem.
  Therefore by IVP, f has roots in each of $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, \infty)$.

Question 4

- Consider the cubic, $f(x) = x^3 + px + q$ where p and q are real numbers.
- To show : If f has three distinct real roots, then $4p^3 + 27q^2 < 0$
- Since f has 3 distinct roots, say $r_1 < r_2 < r_3$, by Rolle's theorem $f'(x)$ has at least **two** real roots, say, x_1 and x_2 such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$.

Question 4



Question 4

- Since $f'(x) = 3x^2 + p$, this implies that $p < 0$ and $x_1 = -\sqrt{-p/3}$, $x_2 = \sqrt{-p/3}$.
- Now, $f''(x_1) = 6x_1 < 0 \implies f$ has a local maximum at $x = x_1$. Similarly, f has a local minimum at $x = x_2$.
- Since the quadratic $f'(x) < 0$ in (x_1, x_2) i.e. f is decreasing over $[x_1, x_2]$ and f has a root r_2 in (x_1, x_2) we must have $f(x_1) > 0$ and $f(x_2) < 0$.

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}, f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

- Finally,

$$\frac{4p^3 + 27q^2}{27} = f(x_1)f(x_2) < 0$$

Question 5

To show : $|\sin a - \sin b| < |a - b|$

Note that for $a = b$, the proof is trivial, the proof for $a \neq b$ is given below.

- Let $f : [a, b] \to \mathbb{R}, f(x) = \sin x$
- The function f satisfies hypotheses of MVT i.e. continuity over $[a, b]$ and differentiability over (a, b)
- So, $\exists x_0 \in (a, b)$ s.t. $\dfrac{\sin a - \sin b}{a - b} = f'(x_0) = \cos x_0$
- Since, $\forall x \in \mathbb{R}, |\cos x| \leq 1$, we have $\left|\dfrac{\sin a - \sin b}{a - b}\right| \leq 1$
- Thus,

$$\forall a, b \in \mathbb{R} |\sin a - \sin b| \leq |a - b|$$

Question 7

By Lagrange's MVT, there exists $c_1 \in (-a, 0)$ and there exists $c_2 \in (0, a)$ such that

$$f(0) - f(-a) = af'(c_1) \text{ and } f(a) - f(0) = af'(c_2)$$

Using the given conditions, we obtain

$$f(0) + a \leq a \text{ and } a - f(0) \leq a$$

which implies $f(0) = 0$.
Optional : Consider $g(x) = f(x) - x.x \in [-a, a]$. Since $g'(x) = f'(x) - 1 \leq 0, g$ is decreasing over $[-a, a]$. As $g(-a) = g(a) = 0$, we have $g \equiv 0$

Question 8

(i) $\forall x \in \mathbb{R}\ f''(x) > 0,\ f'(0) = 1,\ f'(1) = 1$
By Rolle's theorem, $\exists c \in (0,1)$ s.t. $f''(c) = 0$, so no such function exists which can satisfy the given conditions.

(ii) $\forall x \in \mathbb{R}\ f''(x) > 0,\ f'(0) = 1,\ f'(1) = 2$
Take $f(x) = \dfrac{x^2}{2} + x$

(iii) $\forall x \in \mathbb{R}\ f''(x) \geq 0,\ f'(0) = 1,\ \forall x \in \mathbb{R}\ f(x) \leq 100$
$f'' \geq 0 \Rightarrow f'$ is increasing. As $f'(0) = 1$, by Lagrange's MVT we have $f(x) - f(0) \geq x$ for $x > 0$. Hence f with the required properties cannot exist.

(iv) $\forall x \in \mathbb{R}\ f''(x) > 0,\ f'(0) = 1,\ \forall x < 0\ f(x) \leq 1$
Take f to be as follows

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if} \quad x \leq 0 \\ 1 + x + x^2 & \text{if} \quad x > 0 \end{cases}$$

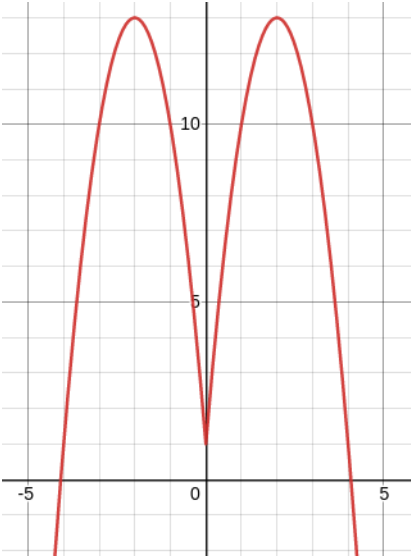
Question 9

Let $f(x) = 1 + 12|x| - 3x^2$
Since f is a continuous function, it will achieve the maximum and minimum on the given closed interval. The points which can possibly achieve these are endpoints of the interval $x = -2, 5$, the stationary points $x = -2, 2$ and the points where f is not differentiable $x = 0$.
Observe that :

- $f(-2) = f(2) = 13$
- $f(5) = -14$
- $f(0) = 1$

So, the absolute maximum $=$ 13 at $x = \pm 2$ and absolute minimum $=$ -14 at $x = 5$

The sketch :



Question 10

Let $2a$ be the width of the window and h be its height. Then $2a + 2h + \pi a = p$, and $0 < a < \dfrac{p}{2+\pi}$. As the area of the colored glass is $\frac{\pi a^2}{2}$ and the area of the plane glass is 2ah, the total light admitted is

$$L(a) = 2ah + \frac{\pi a^2}{4} = 2a\left[\frac{p-(\pi+2)a}{2}\right] + \frac{\pi a^2}{4} \ (0 < a < \frac{p}{2+\pi})$$

Since

$$L'(a) = 0 \Rightarrow a = \frac{2p}{8+3\pi}$$

and

$$L'(a) > 0 \text{ in } (0, \frac{2p}{8+3\pi}) \text{ and } L'(a) < 0 \text{ in } (\frac{2p}{8+3\pi}, \frac{p}{2+\pi}),$$

$a = \dfrac{2p}{8+3\pi}$ must give the global maximum. That yields $h = \dfrac{p(4+\pi)}{2(8+3\pi)}$

Question 10

Note : Here, h or $a = 0$ does not make sense, so we will work with open intervals only, but depending on the problem statement, do not forget to check the values of the function under optimization i.e. L here, on the boundary points as well.

$$L(a = \frac{p}{2+\pi}) < L(a \in (\frac{2p}{8+3\pi}, \frac{p}{2+\pi}))$$

(verify from the sign of the derivative of L when $a \in (\frac{2p}{8+3\pi}, \frac{p}{2+\pi})$)