Tutorial 2

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Question 2

Suppose $\lim_{x \to \alpha} f(x) = L$. Then, $\lim_{h \to 0} f(\alpha + h) = L = \lim_{h \to 0} f(\alpha - h)$ and since

$$|f(\alpha + h) - f(\alpha - h)| \leq |f(\alpha + h) - L| + |f(\alpha - h) - L|$$

it follows that

$$\lim_{h \to 0} |f(\alpha + h) - f(\alpha - h)| = 0$$

The converse is **false** e.g. consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ \frac{1}{|x|} & \text{if} \quad x \neq 0 \end{cases}$$

Question 3

3. (i) f(x) $= \sin(\frac{1}{x})$, if x $\neq 0$ and f(0) $= 0$

Continuous everywhere except at x $= 0$. To see that f is not continuous at 0, consider the sequences $\{x_n\}_{n \geq 1}$, $\{y_n\}_{n \geq 1}$ where

$$x_n := \frac{1}{n\pi} \text{ and } y_n := \frac{1}{2n\pi + \frac{\pi}{2}}$$

Note that both $x_n, y_n \to 0$, but $f(x_n) \to 0$ and $f(y_n) \to 1$ as $n \to \infty$

3. (ii) f(x) $=$ x $\sin(\frac{1}{x})$, if x $\neq 0$ and f(0) $= 0$

f is continuous everywhere.
For ascertaining the continuity of f at x $= 0$, note that
$|f(x)| \leq |x|$ and $f(0) = 0$

Question 4

$f : \mathbb{R} \rightarrow \mathbb{R}\ s.t.\ f(x+y) = f(x) + f(y)\ \forall x, y \in \mathbb{R}$

Taking x = y = 0, we get $f(0+0) = 2f(0)$ so that $f(0) = 0$. By the assumption of the continuity of f at 0, $\lim_{x\rightarrow 0} f(x) = 0$. Thus,

$$\lim_{h\rightarrow 0} f(c+h) = \lim_{h\rightarrow 0}[f(c) + f(h)] = f(c)$$

showing that f is continuous at $x = c$.

Optional: First verify the equality for all $k \in \mathbb{Q}$ and then use the continuity of f to establish it for all $k \in \mathbb{R}$

To show the equality for rational numbers, let $k = \frac{p}{q}\ q \neq 0$

To show : $qf(kx) \iff qf(px/q) = qkf(x) \iff pf(x)$

Since f is distributive over summation, we know that

$$f(px) = pf(x)\ p \in \mathbb{N}$$

Question 4

Similarly,

$$f(px) = f(q(px/q)) = qf(px/q)$$

i.e. $f(px/q) = f(px)/q = (p/q)f(x)$.
Thus, proved.

Question 5

$f(x) = x^2 \sin(1/x)$ for x $\neq 0$ and $f(0) = 0$

It can be clearly observed that f is differentiable for all $x \neq 0$ and the derivative is

$$f'(x) = 2x \sin(1/x) - \cos(1/x), \ x \neq 0$$

Also,

$$f'(0) = \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = 0$$

Clearly, f' is continuous at any $x \neq 0$. However, $\lim_{x \to 0} f'(x)$ does not exist.
Indeed, for any $\delta > 0$, we can choose $n \in \mathbb{N}$ such that $x := 1/n\pi, y := 1/(n+1)\pi$ are in $(-\delta, \delta)$, but $|f'(x) - f'(y)| = 2$

Question 7

$f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f'(c)$.

$$\begin{aligned}
\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h} &= \lim_{h \to 0^+} \frac{1}{2}\left[\frac{f(c+h) - f(c)}{h} + \frac{f(c-h) - f(c)}{-h}\right] \\
&= \frac{1}{2}[f'(c) + f'(c)] = f'(c)
\end{aligned}$$

The converse is **false**; consider, for example, $f(x) = |x|$ and $c = 0$. The given expression exists and equals 0, but f is not differentiable at $\mathrm{c} = 0$.

Question 9

(i) Let $f(x) = \cos(x)$. Then, $f'(x) = -\sin(x) \neq 0$ for $x \in (0, \pi)$.
Thus, $g(y) = f^{-1}(y) = \cos^{-1}(y),\ -1 < y < 1$ is differentiable and

$$g'(y) = \frac{1}{f'(x)}, \qquad \text{where x is such that } f(x) = y$$

Therefore,

$$g'(y) = \frac{-1}{\sin(x)} = \frac{-1}{\sqrt{1 - \cos^2(x)}} = \frac{-1}{1 - y^2}$$

Question 9

(ii) Note that

$$\operatorname{cosec}^{-1}(x) = \sin^{-1}\frac{1}{x} \;\text{ for } |x| > 1$$

Since

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \text{ for } |x| < 1$$

one has, by the Chain rule,

$$\frac{d}{dx}\operatorname{cosec}^{-1}(x) = \frac{1}{(1-\frac{1}{x^2})}\left(\frac{-1}{x^2}\right), |x| > 1$$

Question 10

By the Chain rule,

$$\frac{dy}{dx} = f'\left(\frac{2x-1}{x+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x+1}\right)$$
$$= \sin\left(\frac{2x-1}{x+1}\right)^2\left[\frac{3}{(x+1)^2}\right] = \frac{3}{(x+1)^2}\sin\left(\frac{2x-1}{x+1}\right)^2$$

Question 11

One way to proceed is to know that

- Since the function is to be continuous everywhere, we can add two continuous functions to obtain another continuous function.
- Now, carefully choosing two functions each of which are not differentiable at a single point can give us the required function.

E.g. $f(x) = |x| + |1 - x|$
Another example can be $f(x) = x^{2/3} + (1 - x)^{2/3}$

Question 12

We will use the concept of sequential continuity here. Recall that sequential continuity and continuity are the same things.
So, if some function is discontinuous even for one sequence, then it has to be discontinuous.
Now, for $c \in \mathbb{R}$, select a sequence $\{a_n\}_{n\geq 1}$ of rational numbers and a sequence $\{b_n\}_{n\geq 1}$ of irrational numbers, both converging to c. Then, $f(a_n)$ converges to 1 whereas $f(b_n)$ converges to 0, showing that limit of f at c does not exist.
Therefore, f is not continuous.

Question 15

Below, we would show that the statements imply each other in a cyclic manner, thus, all three are equivalent.

$(i) \Rightarrow (ii)$: Choose $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq (a, b)$. Take $\alpha = f'(c)$ and

$$\epsilon_1(h) := \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

$(ii) \Rightarrow (iii)$: $\displaystyle\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h\to 0} |\epsilon_1(h)| = 0$

$(iii) \Rightarrow (i)$: $\displaystyle\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \Rightarrow \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists and is equal to α.