PH 107: Quantum Physics and applications

Fourier Series and Fourier Transform

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Lecture06: 23-12-2021

Learning Objectives

- Introduction to Fourier Integrals and Fourier Transforms.

Doubt clearing session Saturday (25-12-2021) @ 11am.

Example 1

An electron has de Broglie wavelength of 2 X 10^{-12} m. Find its Kinetic energy, phase velocity and group velocity.

$$E_0 = m_0 c^2 \qquad E_0 = 511\,\text{keV (for electron)}$$

$$E = \sqrt{E_0^2 + p^2 c^2}$$

$$\therefore KE = E - E_0 = \sqrt{E_0^2 + p^2 c^2} - E_0$$

$$pc = \frac{hc}{\lambda} = \frac{(4.136\times10^{-15} eV.s)\times(3\times10^{8} m/s)}{2\times10^{-12} m} = 620 keV$$

$$\therefore E = \sqrt{(511)^2 + (620)^2} = 803\,\text{keV}$$

$$\therefore KE = 803 - 511 = 292\,\text{keV}$$

$$\boxed{\text{KE} = 292\,\text{keV}}$$

$$E = \gamma m_0 c^2 = \gamma E_0$$

$$\gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2 / c^2}}$$

$$\therefore 1 - \frac{\mathrm{v}^2}{\mathrm{c}^2} = \frac{E_0^2}{E^2} \Rightarrow$$

$$\mathrm{v} = c\sqrt{1 - \frac{E_0^2}{E^2}}$$

$$\therefore \mathrm{v} = c\sqrt{1 - \frac{(511)^2}{(803)^2}} \Rightarrow \mathrm{v} = 0.77c$$

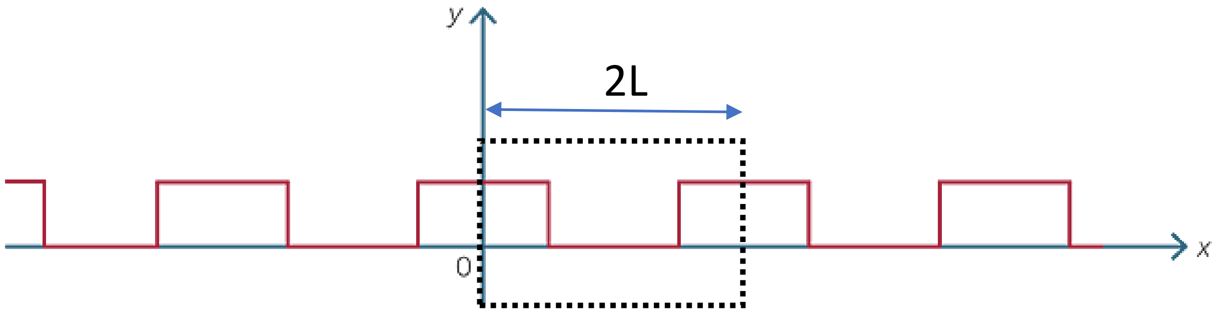
$$\boxed{\mathrm{v}_g = \mathrm{v} = 0.77\mathrm{c}}$$

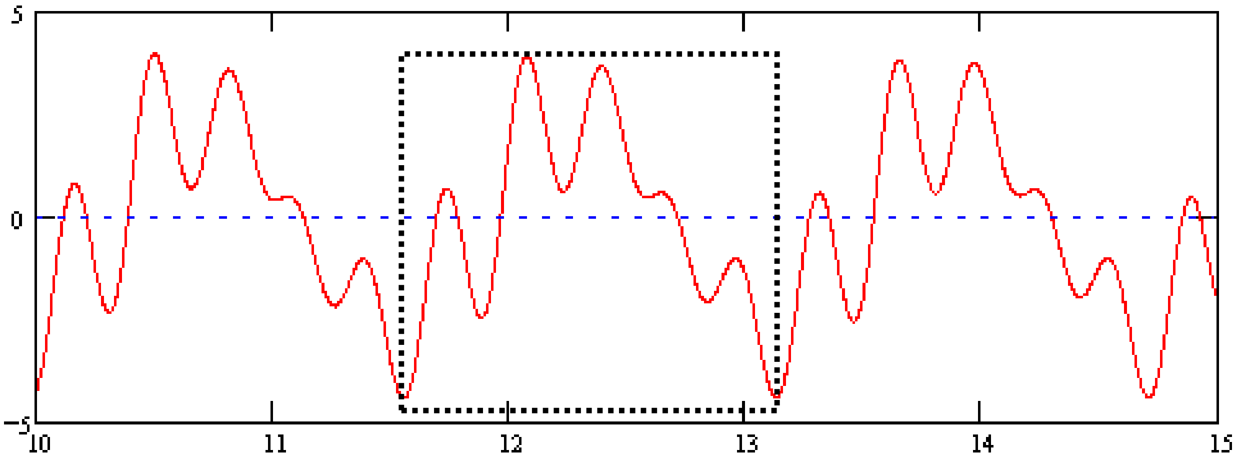
$$\mathrm{v} = \frac{c^2}{\mathrm{v}_p} \qquad \mathrm{v}_p = \frac{c^2}{\mathrm{v}} = \frac{c^2}{0.77c}$$

$$\boxed{\mathrm{v}_p = 1.3\mathrm{c}}$$

Periodic Functions

Consider periodic functions $f(x) = f(x + 2L)$





Periodic but not sinusoidal waves (functions)

Fourier Series

Any given periodic function $y = f(x)$ can be written as a superposition of sinusoidal (sine and cosine) functions

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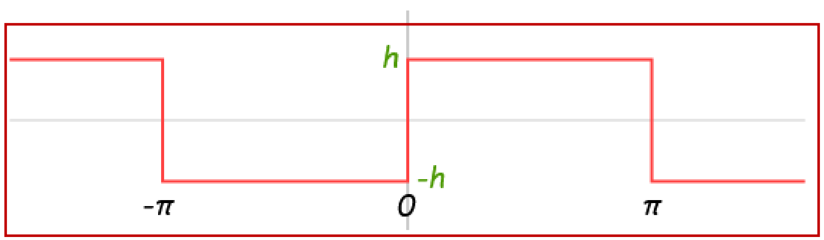
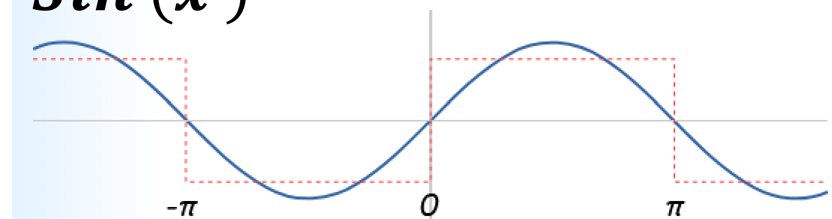
Lets verify pictorially !!

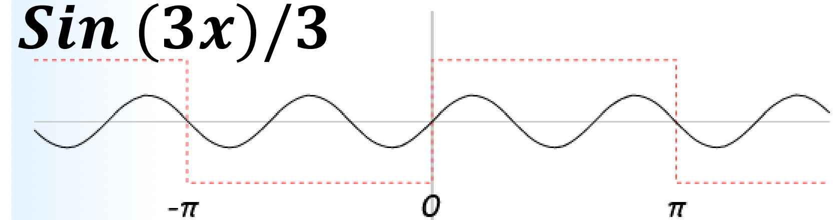


$Sin\ (x)$

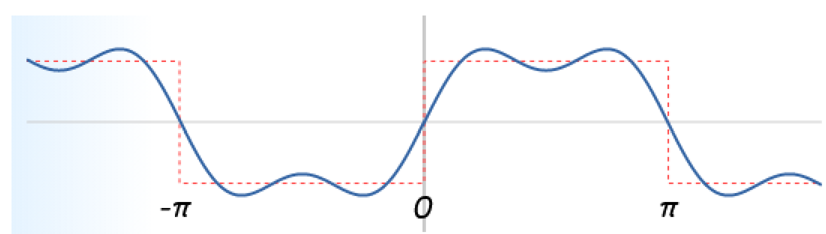


+

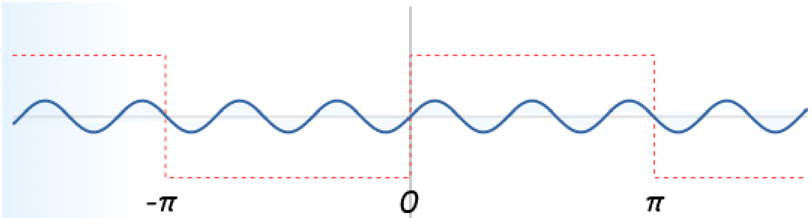
$Sin\ (3x)/3$



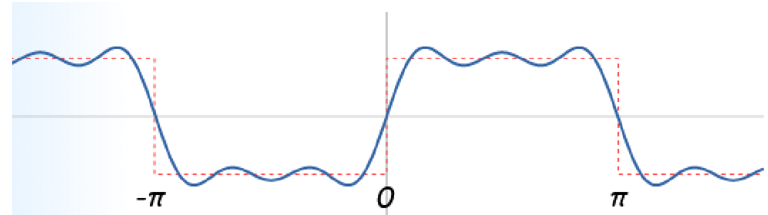
$Sin\ (3x)/3 + Sin\ (x)$ =



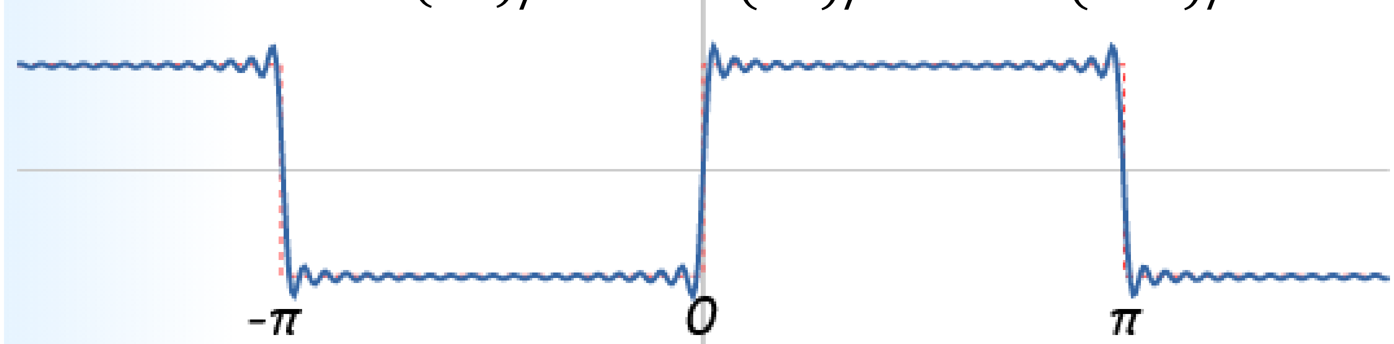
$Sin\ (5x\ )/5$



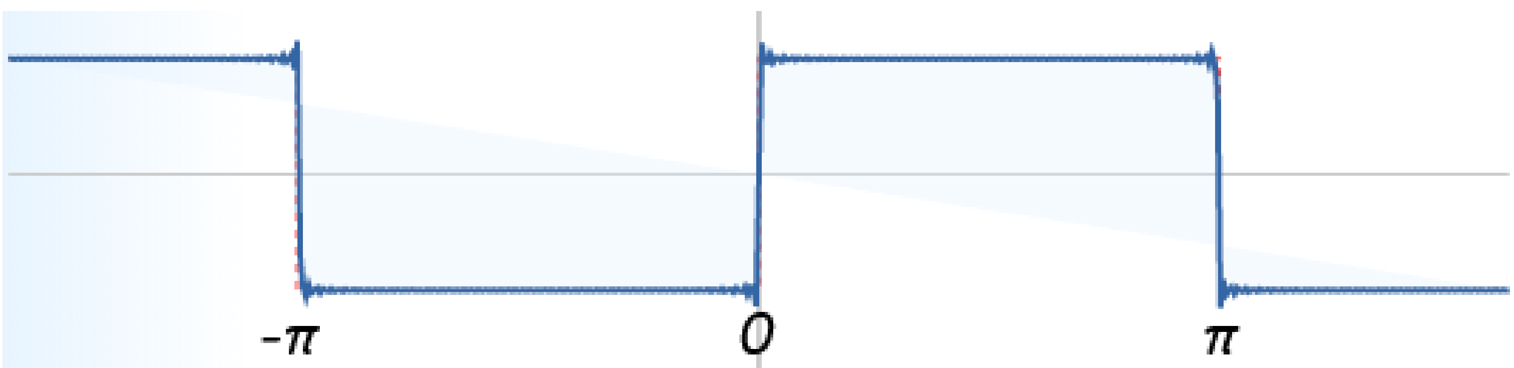
$Sinx + Sin\ (3x)/3 + Sin\ (5x)/5$



20 $Sine\ waves$: $Sinx + Sin\ (3x)/3 + Sin\ (5x)/5\ ....Sin\ (39x)/39$

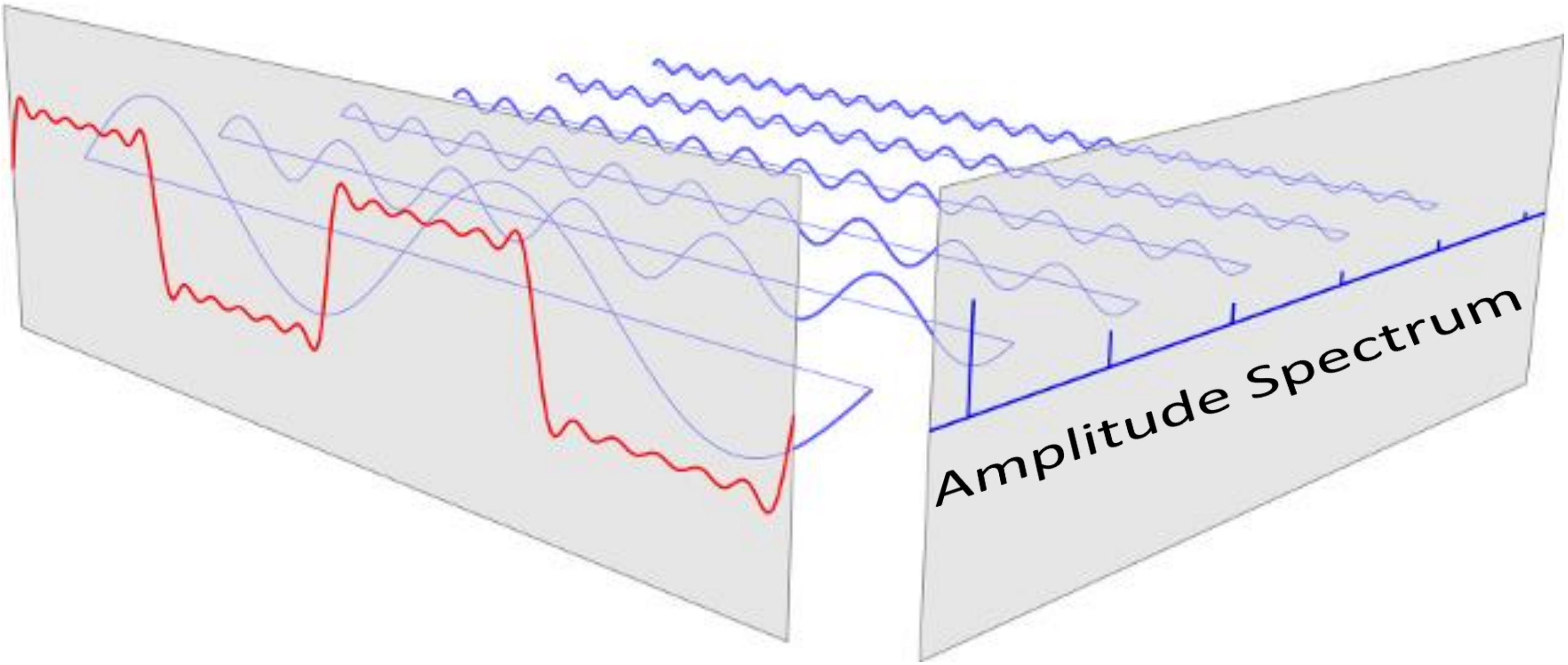


100 $Sine\ waves$: $Sinx + Sin\ (3x)/3 + Sin\ (5x)/5\ ....Sin\ (199x)/199$



A Square Wave = $Sinx + Sin\ (3x)/3 + Sin\ (5x)/5$ **....infinity**

Amplitude Spectrum



Superposing oscillatory waves of different frequency and amplitude

Fourier Series

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Any given periodic function $y = f(x)$ can be written as a superposition of sinusoidal (sine and cosine) functions

Mathematically, if $f(x) = f(x + 2L)$ (i.e. period = $2L$), then $f(x)$ can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

Note, that we are superposing sinusoidal waves of different wavenumbers (wavelengths) and **amplitudes**

Fourier Coefficients

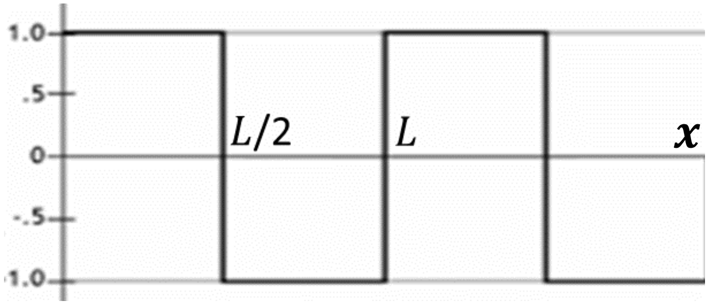
The coefficients a_0, a_n, and b_n are given by (You can verify this)

Period 2L=2π

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx \quad n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx \quad n = 1, 2, 3 \ldots$$

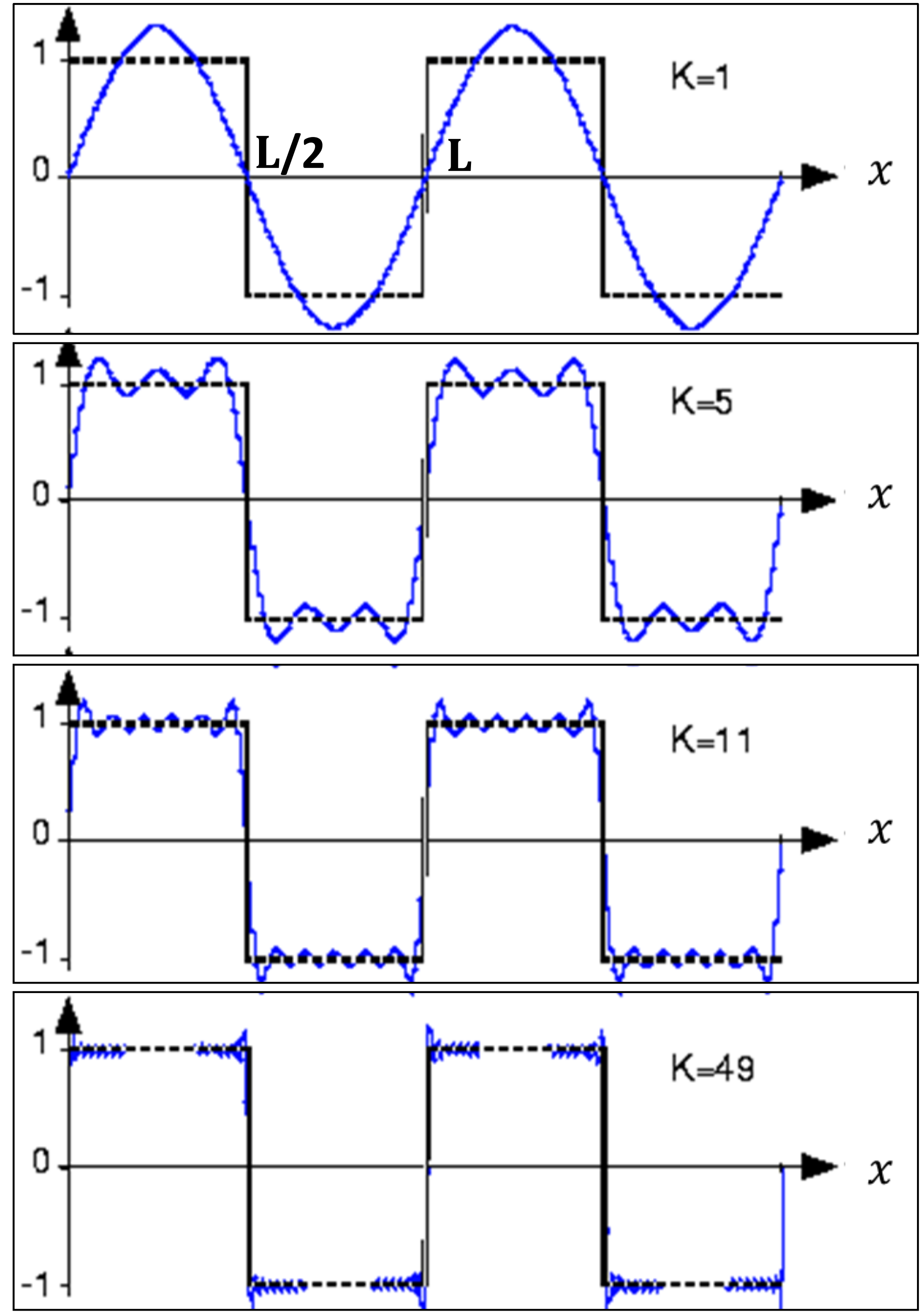


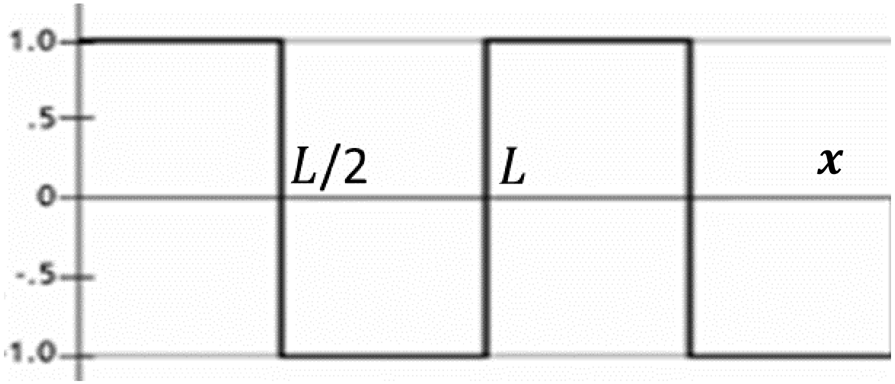
$$\boldsymbol{f(x)} = \begin{cases} \mathbf{1} & \mathbf{0 < x < L/2} \\ \mathbf{-1} & \mathbf{\frac{L}{2} < x < L} \end{cases}$$

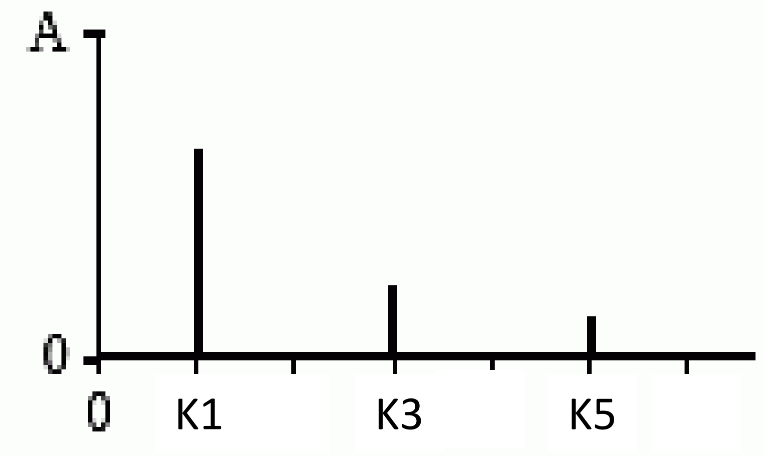
$$\sum_{n=1}^{K} \left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x \right)$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx = 0$$

$$f(x) = \sum_{n=1}^{odd} \left(b_n \sin\frac{n\pi}{L}x \right)$$

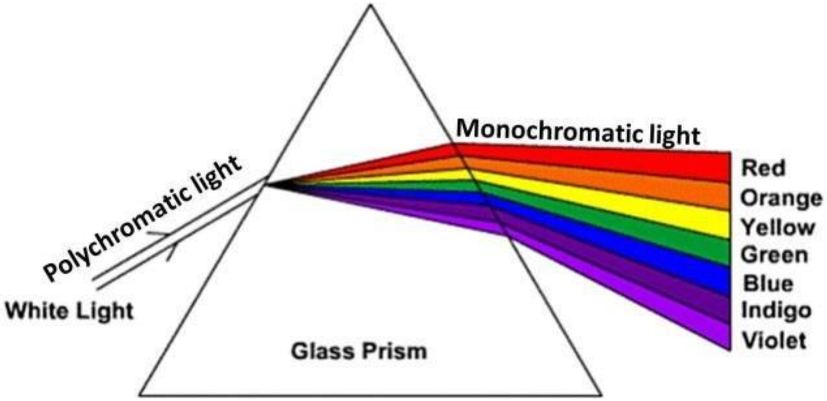




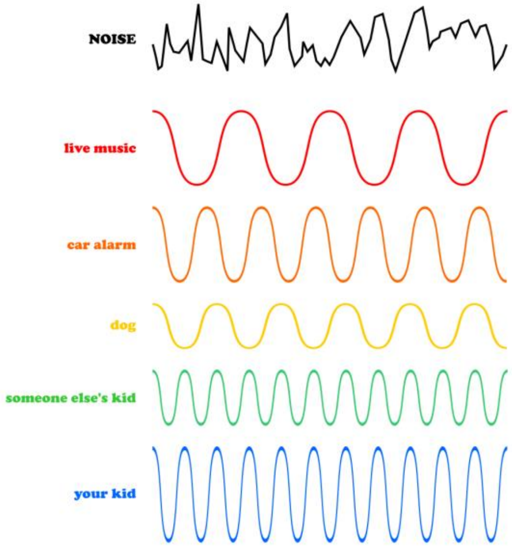


$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

Optical Spectra



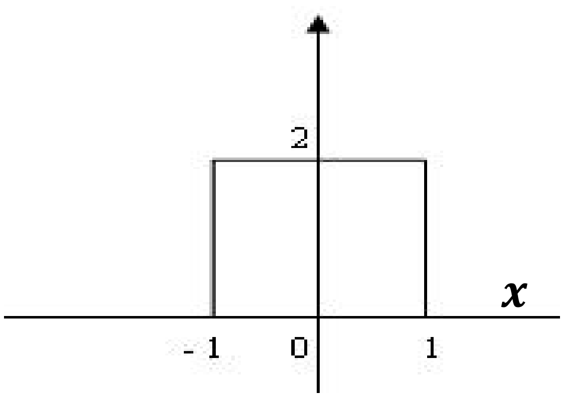
Human Ear

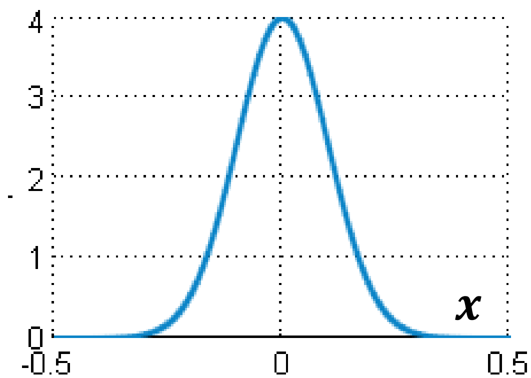


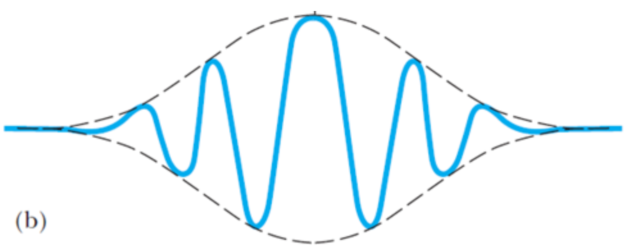
What about non-periodic functions ?

Can we express them as superposition of sinusoidal waves ?

Note that this will solve our problem. A wave packet is localized in space and not a periodic function (like those shown below).

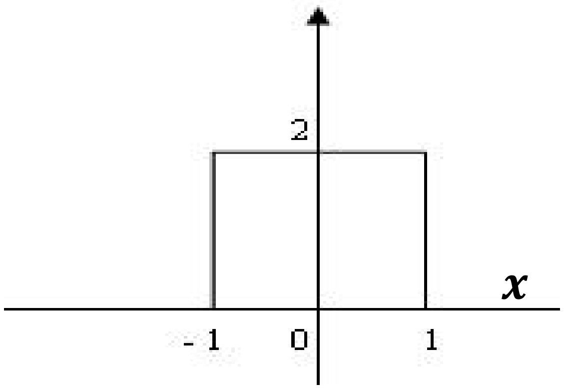






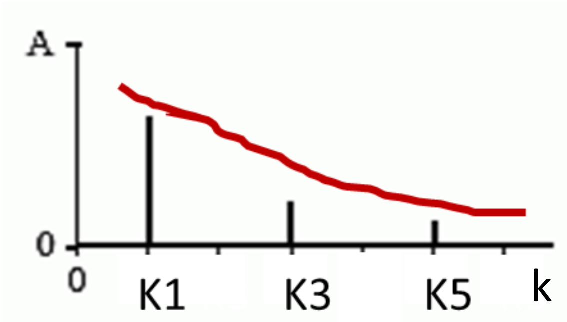
Answer: YES (Fourier Integral)

It is same as imagining a periodic function with $L \rightarrow \infty$. It can be shown that in this limit



$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos k_n x + b_n \sin k_n x)$$



should be replaced by an integral

$$f(x) = \int_0^{\infty} A(k) \cos(kx) + B(k) \sin(kx)\, dk$$

We notice that now we are considering a continuous distribution of wavenumbers for the constituent sinusoidal waves. Coefficients $A(k)$ and B(k), is the weighted amplitude of the oscillatory wave of different frequency (wavelength)

Fourier Integral and Fourier Transform

$$A(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(kx)dx$$

$$B(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(kx)dx$$

Using,

$\left[\cos(kx) = \left(e^{ikx} + e^{-ikx}\right)/2\right]$ and $\left[\sin(kx) = \left(e^{ikx} - e^{-ikx}\right)/2i\right]$

In the complex form, the Fourier integral can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)e^{ikx}\,dk$$ **Inverse Fourier Transform**

With $$g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$ **Fourier Transform**

$g(k)$ is known as the **Fourier transform** of $f(x)$ and vice versa.

Fourier Transform is the change of ‘representation” from x-space to k-space.

Fourier Transform (Time Domain)

In terms of time "t" and frequency "ω"

$$f(t) = \sum_n a_n \, Sin\, nt + \sum_n b_n \, Cos\, nt$$

$$e^{it} = Cos\, t + i\, Sin\, t$$

F(t) as an infinite "sum" (integral) of sine and cosine functions

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega$$

Coefficient, $$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, d\omega$$

FT is a change of "representation" from 't-space' to ' ω – space'.

Fourier Integrals

Wave number, k is reciprocal of wavelength

Frequency, ω is inverse of time, t.

In Real Space; x

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk$$

In Reciprocal Space; k

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dk$$

In Time Domain; t

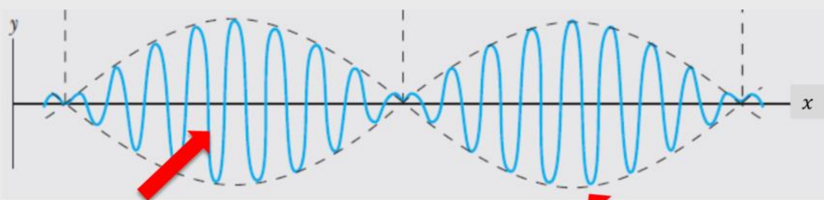
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega$$

In Frequency Domain; ω

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, d\omega$$

Application in Quantum Mechanics

Can FT and Fourier series form an mathematical ground for the construction of the wave packet ?



High Frequency Wave

$$\sin\left[\left(k+\frac{\delta k}{2}\right)x\right]$$

Broad Envelope

$$2A\cos\left[\frac{\delta k}{2}x\right]$$

[sin(5x) + sin(5.0625x) + sin (5.125x) + sin(5.1875x) + sin (5.25x) + sin(5.3125x) + sin(5.375x) + sin(5.4375x) + sin(5.5x) + sin(5.5625x) + sin (5.625x) + sin(5.6875x) + sin (5.75x) + sin(5.8125x) + sin(5.875x) + sin(5.9375x) + sin(6x)]/17

$$\boldsymbol{f}(\boldsymbol{x})=\frac{\mathbf{1}}{\sqrt{\mathbf{2}\boldsymbol{\pi}}}\int_{-\infty}^{\infty}\boldsymbol{g}(\boldsymbol{k})\,\boldsymbol{e}^{\boldsymbol{ikx}}\,dk$$

Wave Packet

Amplitude
(F. Coefficients)

High frequency plane wave

Example: General Periodic Function

$$f(t) = \sum_n c_n e^{i\omega_n t}$$

$$g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \sum_n c_n \int_{-\infty}^{\infty} e^{-i(\omega - \omega_n)t} dt$$

$$= \sum_n c_n \delta(\omega - \omega_n)$$

Note : $\boldsymbol{\delta\ function}$

$$\boldsymbol{\delta(\omega - \omega_n) = \int_{-\infty}^{\infty} e^{-i(\omega-\omega_n)t}\, dt}$$

This $function\ exits\ for$ $\omega = \omega_n$ and is zero for all other values of ω.

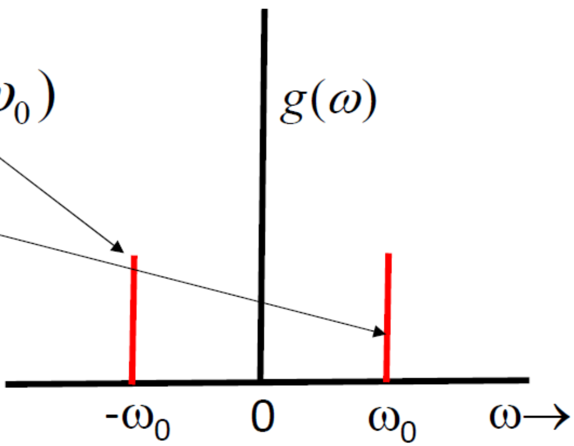
Oscillatory Functions

$$f(t) = \cos \omega_0 t \qquad \cos \omega_0 t = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$$

$$g(\omega) = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)e^{-i\omega t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}\right)dt$$

$$= \frac{1}{2}\delta(\omega-\omega_0) + \frac{1}{2}\delta(\omega+\omega_0)$$



Exercise:

$f(t) = \sin \omega_0 t$ Find g(ω)

Use $\sin \omega_0 t = \frac{1}{2i}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$

Some more examples of FT

Constant $function$

$$f(t) = \alpha$$

$$g(\omega) = \alpha \int_{-\infty}^{\infty} e^{-i\omega t} dt$$

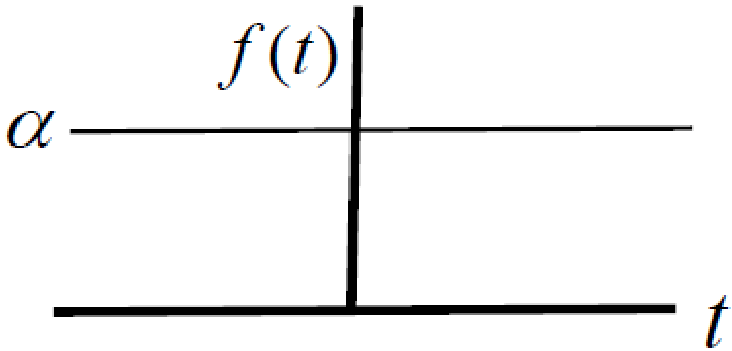
$$= \alpha\delta(\omega)$$

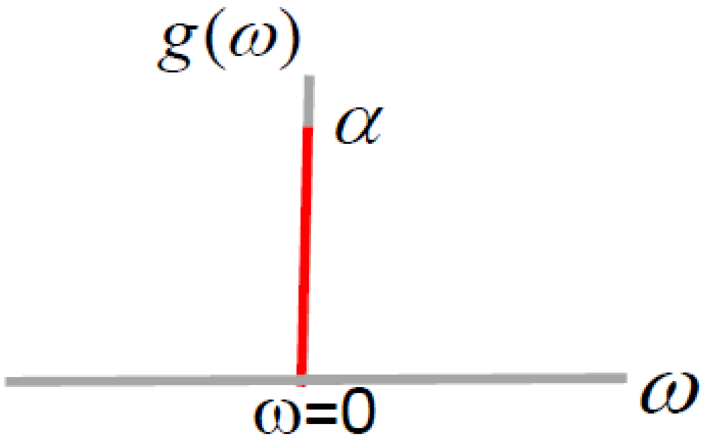
Using,

$$\int_{-\infty}^{\infty} e^{-i(\omega-\omega_n)t} dt = \delta(\omega-\omega_n)$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} dt = \delta(\omega)$$

Zero everywhere except at w =0

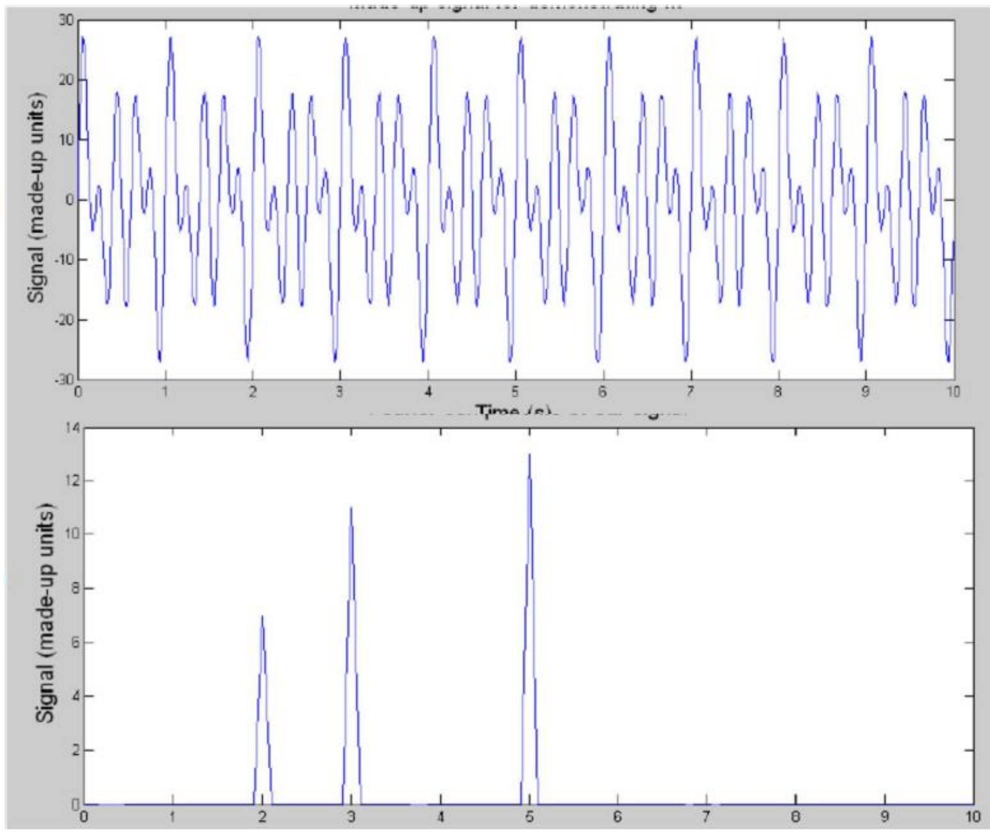




Fourier Transform (time domain)

$$f(t) = 7\sin(2\pi \bullet 2t) + 11\sin(2\pi \bullet 3t) + 13\sin(2\pi \bullet 5t)$$

$f(t)$

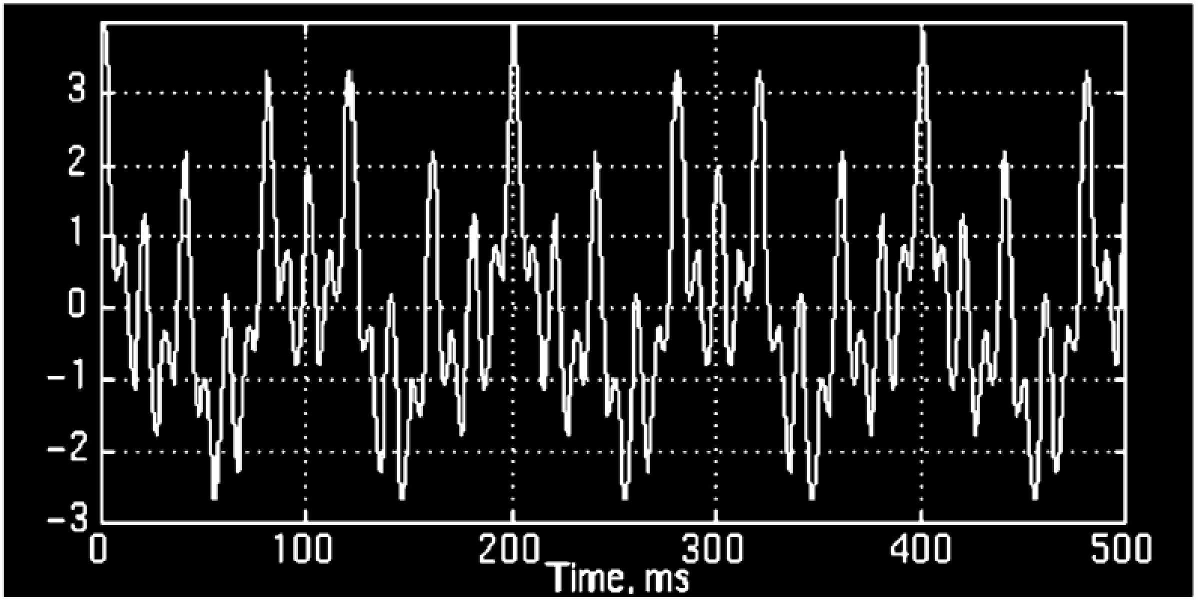


g(ω)

$$f(t) = \sum_n c_n e^{i\omega_n t}$$

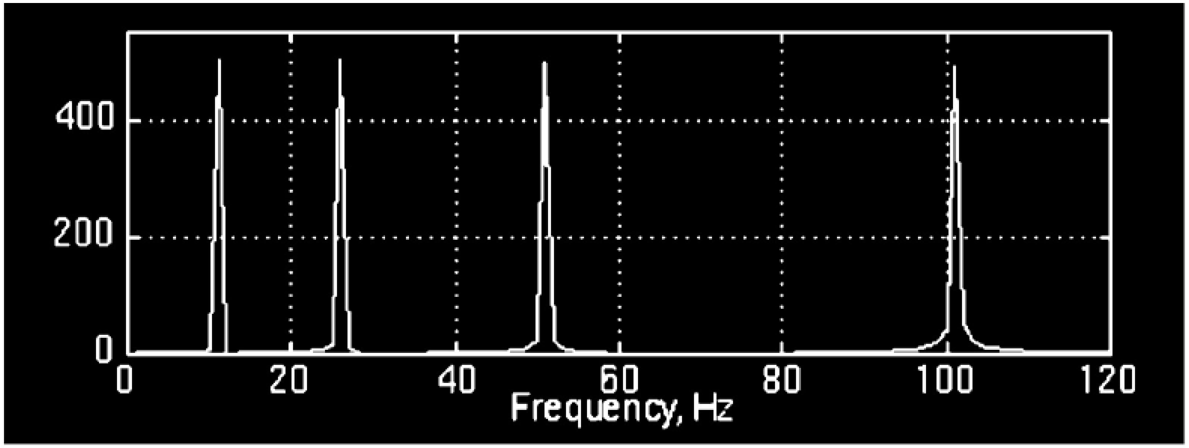
g(ω) = ?

Fourier Transform (time domain)



Periodic signal f(t)

This could be a signal of vibrating floor or an ECG

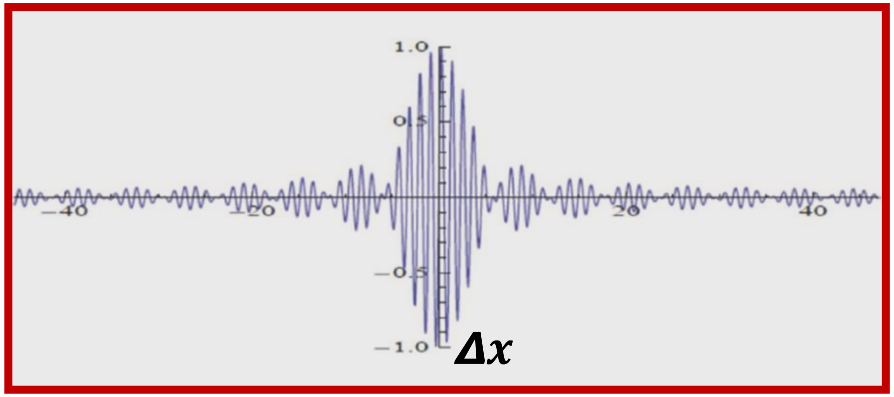


Fourier Transform g(ω) of f(t)

(g(ω) is also called as Spectrum)

By taking Fourier transform, we have identified the frequencies and their 'weights' in the signal.

Application in Quantum Mechanics : Back to wave packet



To form a true wave-packet that is zero everywhere outside a finite range $\boldsymbol{\Delta x}$, requires adding together an infinite number of harmonic waves with continuously varying wavelength and amplitudes.

Fourier Integrals

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk$$

Wave Packet

Amplitude (F. Coefficients)

High frequency plane wave

Supplementary Information

The Exponential Fourier Series
(From Advanced Engineering Mathematics by Kreyszig)

Refer to supplementary notes for derivation of Fourier integrals for non-periodic functions