Tut 6

Thm: $f: \mathbb{R}^2 \to \mathbb{R}$

f is cont. at (x_0, y_0) iff for every seq. $((x_n, y_n))$ converging to (x_0, y_0), we have that

$$\lim_{n\to\infty} f(x_n, y_n) = f(x_0, y_0).$$

2

(i) Level set : $\{(x,y) : x-y=c\}$

Contour line: $\{(x,y,c) : x-y=c\}$

(ii) Level set is empty for $c = -3, -2, -1$

For $c=0$, level set is $\{(0,0)\}$

Contour line is $\{(0,0,0)\}$

For $c>0$, level set is $\{(x,y) : x^2+y^2=c\}$

Contour line is $\{(x,y,c) : x^2+y^2=c\}$

3

(i) Consider the seq. $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^3}\right)$

$$\lim_{n\to\infty} (x_n, y_n) = (0,0)$$

$$f(x_n, y_n) = \frac{1/n^6}{2/n^6} = \frac{1}{2}$$

$$\therefore \lim_{n\to\infty} f(x_n, y_n) = \frac{1}{2} \neq f(0,0)$$

$\therefore f$ is not cont. at $(0,0)$.

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(ii) Let (x_n, y_n) be a seq. in $\mathbb{R}^2$ which converges to $(0,0)$. Then $x_n \to 0$ and $y_n \to 0$.

~~$f(x_n, y_n) =$~~

If $(x_n, y_n) \neq 0$, then $\left| \frac{x_n^2 - y_n^2}{x_n^2 + y_n^2} \right| \leq 1$

$\therefore 0 \leq |f(x_n, y_n)| \leq |x_n y_n| \; \forall n \in \mathbb{N}$

By Sandwich theorem, $\lim\limits_{n\to\infty} |f(x_n, y_n)| = 0$

$$\Rightarrow \lim_{n\to\infty} f(x_n, y_n) = 0$$

$\therefore f$ is cont. at $(0,0)$.

(iii) $\lim\limits_{n\to\infty} \left(\left||x_n| - |y_n|\right| - |x_n| - |y_n| \right)$

Use that $|.|$ is a cont. fn.

6

(i) $$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0 - 0}{h} = 0$$

Similarly, $f_y(0,0) = 0$

(ii) $$f_x(0,0) = \lim_{h\to 0} \frac{\frac{\sin^2 h}{|h|} - 0}{h} = \lim_{h\to 0} \frac{\sin^2 h}{h|h|}$$

Let $s_n = \frac{1}{n} \; \forall n$ $\quad \lim\limits_{n\to\infty} \frac{\sin^2 s_n}{s_n |s_n|} = \lim\limits_{n\to\infty} n^2 \sin\left(\frac{1}{n^2}\right) = 1$

Let $t_n = \frac{-1}{n} \; \forall n$ $\quad \lim\limits_{n\to\infty} \frac{\sin^2 t_n}{t_n |t_n|} = \lim\limits_{n\to\infty} -n^2 \sin\left(\frac{1}{n^2}\right) = -1$

$\therefore f_x(0,0)$ does not exist.

Similar for $f_y(0,0)$.

7) Let (x_n, y_n) be a seq. in $\mathbb{R}^2$ s.t. $(x_n, y_n) \to (0,0)$ as $n \to \infty$.

Then $\left|\lim_{n\to\infty} f(x_n, y_n)\right| \leq \left|\lim_{n\to\infty} (x_n^2 + y_n^2)\right| = 0$

$\therefore \lim_{n\to\infty} f(x_n, y_n) = 0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h^2}}{h} = \lim_{h\to 0} h \sin \frac{1}{h^2} = 0$$

Similarly $f_y(0,0) = 0$

For $(x,y) \neq (0,0)$, $f_x(x,y) = (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right)\left(\frac{-1}{(x^2+y^2)^2}\right) \times 2x + 2x \sin \frac{1}{x^2+y^2}$

$$= \frac{-2x}{x^2+y^2} \cos \frac{1}{x^2+y^2} + 2x \sin \frac{1}{x^2+y^2}$$

~~Given $M \in \mathbb{R}$, $M > 0$~~

~~Take $(x,y) = \ldots$~~

8) Cont. is easy to prove (using seq.)

$f_x(0,0) = \lim_{h\to 0} \dfrac{h\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} \sin\frac{1}{h}$, which does not exist

Similar for $f_y(0,0)$

10) Let $(x,y) \neq (0,0)$

If $y \neq 0$, $|f(x,y) - f(0,0)| = \sqrt{x^2+y^2}$

If $y = 0$, $|f(x,y) - f(0,0)| = 0$

$\therefore |f(x,y) - f(0,0)| \leq \sqrt{x^2+y^2}$

Now use sequences to prove that f is cont. at $(0,0)$.

Let $\underline{u} = (u_1, u_2)$, $u_1^2+u_2^2 = 1$

If $u_2 \neq 0$, then $D_u f = \lim_{t\to 0} \dfrac{f(u_1 t, u_2 t) - f(0,0)}{t}$

$= \lim_{t\to 0} \dfrac{1}{t} \dfrac{u_2 t}{|u_2 t|} \sqrt{(u_1^2+u_2^2)t^2}$

$= \lim_{t\to 0} \dfrac{1}{t} \dfrac{u_2 t}{|u_2 t|} \times |t| = \dfrac{u_2}{|u_2|}$

If $u_2 = 0$, then $D_u f = \lim_{t\to 0} \dfrac{0-0}{t} = 0$

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If f is diff. at 0, the total derivative must be

$$\left[\frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(0,0)\right] = (0, 1)$$

We need to check if

$$\lim_{(h,k)\to(0,0)} \frac{\left| f(h,k) - f(0,0) - [0 \ 1]\begin{bmatrix} h \\ k \end{bmatrix} \right|}{\sqrt{h^2+k^2}} = 0$$

For $k \neq 0$,

$$\lim_{(h,k)\to(0,0)} \left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right|$$

Let $h_n = \frac{1}{n}$, $k_n = 1/n$

$$\lim_{n\to\infty} \left| \frac{k_n}{|k_n|} - \frac{k_n}{\sqrt{h_n^2+k_n^2}} \right| = 1 - \frac{1}{\sqrt{2}} \neq 0$$

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