MA-111 Calculus II
(D3 & D4)

Lecture 16

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Parametrized surfaces

Surface integral of a vector field

Orientability

Recall

Let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a non-singular smooth parametrized surface.

- $$\text{Area}(\mathbf{\Phi}) := \int\int_E dS = \iint_E \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u,v)\| \, du \, dv.$$
- If $\mathbf{\Phi}(u,v) = (x(u,v), y(u,v), z(u,v))$,

$$\iint_S dS = \iint_E \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

- If f is a continuous scalar field on S, then

$$\iint_S f dS = \iint_E f(x,y,z) \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The surface integral of a vector field

Let $\mathbf{F}$ be a bounded vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : E \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv,$$

provided the R.H.S double integral exists. This can also be written more compactly as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over S.

Examples

(i) Let a subset E of $\mathbb{R}^2$ have an area, and let $f : E \to \mathbb{R}$ be a smooth function. Let the smooth parametrized surface $\Phi : E \to \mathbb{R}^3$ represent the graph of f, and let $\boldsymbol{F} : \boldsymbol{\Phi}(E) \to \mathbb{R}^3$ be a continuous vector field. If $\boldsymbol{F} := (P, Q, R)$, then

$$\iint_{\boldsymbol{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_E (-P\, f_x - Q\, f_y + R) d(x, y)$$

since $d\boldsymbol{S} = (\boldsymbol{\Phi}_x \times \boldsymbol{\Phi}_y) dxdy = (-f_x, -f_y, 1) dxdy$.

Using above result, let $E := [0,1] \times [0,1]$, $f(x,y) := x + y + 1$ for $(x, y) \in E$. If $\boldsymbol{F}(x,y,z) := (x^2, y^2, z)$ for $(x,y,z) \in \mathbb{R}^3$, then

$$\begin{aligned}
\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_E \big(-x^2 - y^2 + (x + y + 1)\big) d(x, y) \\
&= \int_0^1 \left(\int_0^1 (x + y + 1 - x^2 - y^2) dy \right) dx \\
&= = \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}.
\end{aligned}$$

Examples Contd.

(ii) Let $E := [0, 2\pi] \times [0, h]$, and $\mathbf{\Phi}(u, v) := (a \cos u, a \sin u, v)$ for $(u, v) \in E$. If $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, then

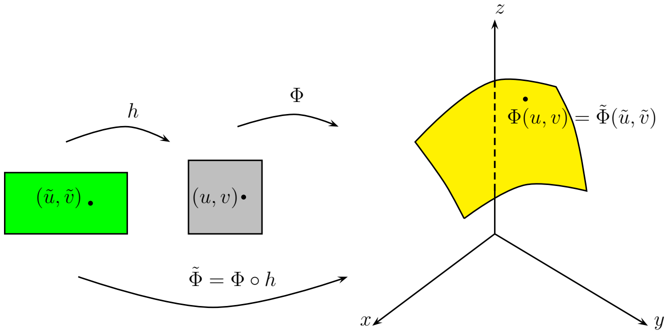
$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\boldsymbol{S} = \iint_E (a^2 \cos u \sin u + v\, a \sin u + 0) du dv = 0,$$

since $d\boldsymbol{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv = (a \cos u, a \sin u, 0) du dv$.

Reparametrization of a Surface

Let E be a path-connected subset of $\mathbb{R}^2$ having an area, and let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrized surface.
Let $\widetilde{E}$ be a path-connected subset of $\mathbb{R}^2$ having an area, and let $h : \widetilde{E} \to \mathbb{R}^2$ be a continuously differentiable and one-one function such that $h(\widetilde{E}) = E$ and its Jacobian $J(h)$ does not vanish on $\widetilde{E}$. Then the smooth surface $\tilde{\mathbf{\Phi}} := \mathbf{\Phi} \circ h$ is called a **reparametrization** of $\mathbf{\Phi}$. Note that

$$\left(\tilde{\mathbf{\Phi}}_{\tilde{u}} \times \tilde{\mathbf{\Phi}}_{\tilde{v}}\right)(\tilde{u}, \tilde{v}) = \left(\mathbf{\Phi}_u \times \mathbf{\Phi}_v\right)(h(\tilde{u}, \tilde{v}))J(h)(h(\tilde{u}, \tilde{v})).$$



Examples: Let $E := (0, \pi) \times [-\pi, \pi]$, and define
$\mathbf{\Phi}(\varphi, \theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\varphi, \theta) \in E$.

If $\widetilde{E} := [-\pi, \pi] \times (0, \pi)$, and we define
$\widetilde{\mathbf{\Phi}}(\theta, \varphi) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\theta, \varphi) \in \widetilde{E}$,
then $\widetilde{\mathbf{\Phi}}$ is a reparametrization of $\mathbf{\Phi}$ since $\widetilde{\mathbf{\Phi}}(\theta, \varphi) = \mathbf{\Phi}(h(\theta, \varphi))$, where
$h : \widetilde{E} \to E$ is given by $h(\theta, \varphi) := (\varphi, \theta)$ with $J(h) = -1$.

Similarly, if $\widetilde{E} := (0, \pi/2) \times [-\pi/2, \pi/2]$, and we define
$\widetilde{\mathbf{\Phi}}(\varphi, \theta) := (\sin 2\varphi \cos 2\theta, \sin 2\varphi \sin 2\theta, \cos 2\varphi)$ for $(\varphi, \theta) \in \widetilde{E}$,
then $\widetilde{\mathbf{\Phi}}$ is a reparametrization of $\mathbf{\Phi}$ since $\widetilde{\Phi}(\varphi, \theta) = \Phi(h(\varphi, \theta))$, where
$h : \widetilde{E} \to E$ is given by $h(\varphi, \theta) := (2\varphi, 2\theta)$ with $J(h) = 4$.

THEOREM: The surface integral of a continuous vector field over a smooth surface is invariant under reparametrization upto a sign. In particular, the area of a smooth surface is invariant under reparametrization.

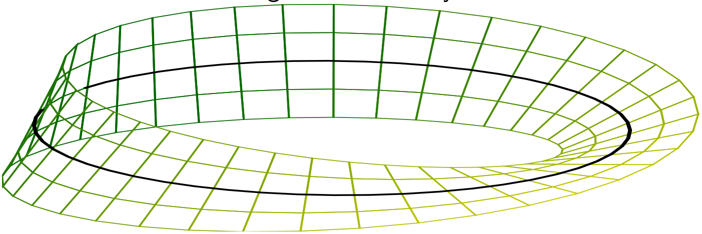
Oriented surfaces

In what follows we will assume that any parametrised surface $\mathbf{\Phi}$ is $\mathcal{C}^1$ and non-singular. We now are setting up to state Stokes' theorem

We need to define orientation on a surface as we did for curves. One plausible way is to imitate the idea of interior and exterior for a surface. An oriented surface S could be defined as two-sided surface with one side specified as the outside (or positive side) and the other side as the inside (or negative side).

What is the problem with this definition?

There are surfaces which have only one side! The simples such surface is the Möbius strip, named after its discoverer, a famous Swedish mathematician of the eighteenth century.



Orientable surfaces -definition

This tells us that we need a better way to make sense of terms like "inside" and "outside".

Recall that for us a vector field on a surface S is a vector field is a function $\mathbf{F} : U \to \mathbb{R}^3$ defined on a open set containing $S \subseteq U$. We say $\mathbf{F}$ is continuous (or C^1) if it is continuous on U.

Definition: A surface S is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P.

At each point of S there are two possible directions for the normal vector to S. The question is whether the normal vector field be can be chosen so that the resulting vector field is continuous.
Note this definition is independent of the choice of parametrization and only dependent on the geometric surface

Examples of orientable surfaces

Example: For the unit sphere in $\mathbb{R}^3$ we can choose an orientation by selecting the unit vector $\hat{\mathbf{n}}(x, y, z) = \hat{\mathbf{r}}$, where $\mathbf{r}$ points outwards from the surface of the sphere.

More explicitly, we define

$$\mathbf{F}(x, y, z) = (x, y, z).$$

This obviously defines a continuous vector field on S. Hence, we see that the unit sphere in $\mathbb{R}^3$ is orientable.

Notice, that we can also define a vector field $\mathbf{G}(x, y, z) = -(x, y, z)$. The vector field $\mathbf{G} = -\mathbf{F}$ is also obviously continuous. There are two possible choices of orientation.

Non-orientable surfaces

Definition: A surface on which there exists no continuous vector field consisting of unit normal vectors is called non-orientable.

Exercise 1: Make a Möbius strip out of a piece of paper. Starting at the top draw a series of stick figures, head to toe, and label their left and right hands. When the stick figure comes back to the top (on the underside) compare the left and right hands of the two stick figures at the top.

Choosing an orientation

As we have just seen in the preceding example, if S is an orientable surface and $\mathbf{F}$ is a continuous vector field of unit normal vectors, so is $-\mathbf{F}$.

An orientable surface together with a specific choice of continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.

Once one has chosen a particular vector field of normal vectors it makes sense to talk about the "outside" or "positive side" of the surface: usually, it is the side given by the direction of the unit normal vector. The other side is then called the "inside" or "negative side". However, which side one calls "positive" or "negative" is a matter of choice.

The orientation of parametrised surfaces

Let us suppose that we are given an oriented geometric surface S that is described as a $\mathcal{C}^1$ non-singular parametrised surface $\mathbf{\Phi}(u, v)$.

Notice that a smooth non-singular parametrisation $\mathbf{\Phi}$ of S gives a natural vector field of unit normal vectors:

$$\hat{\mathbf{n}} = \frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}.$$

Definition: If the unit normal vector $\hat{\mathbf{n}}$ agrees with the given orientation of S we say that the parametrisation $\mathbf{\Phi}$ is orientation preserving. Otherwise we say that $\mathbf{\Phi}$ is orientation reversing.

Examples

(i) Let $E \subset \mathbb{R}^2$ have an area, and $f : E \to \mathbb{R}$ be a smooth scalar field.
Consider the graph $S := \{(x, y, f(x, y)) : (x, y) \in E\}$ of f.
For $P := (x, y, z) \in S$, define

$$\hat{\mathbf{n}}(P) := (-f_x(P), -f_y(P), 1)/\|(-f_x(P), -f_y(P), 1)\|.$$

This continuous assignment of upward unit normal vectors gives an orientation of S. Hence S is orientable.

Clearly, the parametrization of S given by $\mathbf{\Phi}(x, y) := (x, y, f(x, y))$ for $(x, y) \in E$, is orientation-preserving.

(ii) Let $S := \{(x, y, z) :\in \mathbb{R}^3 : x^2 + y^2 = a^2 \text{ and } 0 \leq z \leq h\}$. For $P := (x, y, z) \in S$, define $\hat{\mathbf{n}}\,(P) := (x/a, y/a, 0)$.
This continuous assignment of outward unit normal vectors gives an orientation of S. Hence the cylinder S is orientable.

Let $E := [0, 2\pi] \times [0, h]$ and $\mathbf{\Phi}(u, v) := (a \cos u, a \sin u, v)$ for $(u, v) \in E$.

Examples contd.

If $P := \mathbf{\Phi}(u, v) = (x, y, z) \in S$, then

$$\frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}(u, v) = \frac{(a\cos u, a\sin u, 0)}{a} = \left(\frac{x}{a}, \frac{y}{a}, 0\right) = \hat{\mathbf{n}}(P).$$

Hence $\mathbf{\Phi}$ is an orientation-preserving parametrization.

(iii) Let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2\}$. For
$P := (x, y, z) \in S$, define $\hat{\mathbf{n}}(P) := (x/a, y/a, z/a)$.
This continuous assignment of outward unit normal vectors gives an orientation of the sphere S. Hence S is orientable.
Let $E := [0, 2\pi] \times (0, \pi)$ and
$\mathbf{\Phi}(u, v) := (a\cos u \sin v, , a\sin u \sin v, a\cos v)$ for $(u, v) \in E$.
If $P := \mathbf{\Phi}(u, v) = (x, y, z) \in S$, then

$$\frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}(u, v) = -\frac{(a\sin v)\mathbf{\Phi}(u, v)}{a^2 \sin v} = -\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) = -\hat{\mathbf{n}}(P).$$

Hence $\mathbf{\Phi}$ is an orientation-reversing parametrization of $S \setminus \{(0, 0, \pm a)\}$.

Independence of parametrisation

Let S be an oriented surface. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two $\mathcal{C}^1$ non-singular parametrisations of S and let $\mathbf{F}$ be a continuous vector field on S.

- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

For an oriented surface, the notation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS,$$

is unambiguous.