Tut 3

1 Let $f(x) = x^3 - 6x + 3$

$f(-3) = -6$
$f(0) = 3$
$f(1) = -2$
$f(3) = 12$

f is continuous on $\mathbb{R}$.
By IVT, f has a root between -3 and 0, another between 0 and 1, and another between 1 and 3.
f has atmost 3 real roots.
$\therefore$ f has all roots real.

4 (i) If $p > 0$, $f'(x) = 3x^2 + p > 0 \ \forall x \in \mathbb{R}$
$\therefore$ f is strictly monotonically increasing.
$\Rightarrow$ f can have atmost one real root

If $p = 0$, $f(x) = x^3 + q$, which has a unique real root.
(roots are $(-q)^{1/3}$, $(-q)^{1/3}\omega$, $(-q)^{1/3}\omega^2$)

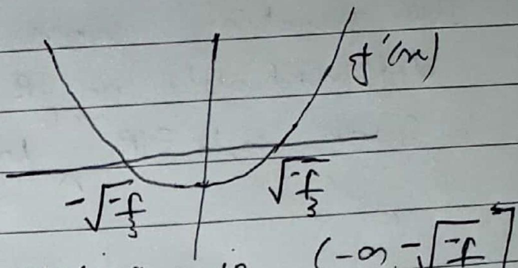
$\therefore$ $p < 0$.

(ii) $f'(x) = 3x^2 + p$
Roots of $f'(x)$ are $x = \pm\sqrt{\frac{-p}{3}}$
$f''(x) \neq 0$ at $x = \pm\sqrt{\frac{-p}{3}}$

$\therefore$ f has max./min. at $\pm\sqrt{-p/3}$

Scanned with CamScanner

(iii) $f'(x) = 3x^2 + p$



f is monotonically increasing in $\left(-\infty, -\sqrt{\frac{-p}{3}}\right]$

$\therefore$ f has atmost one real root in $\left(-\infty, \sqrt{\frac{-p}{3}}\right]$

Similarly, f has atmost one real root in $\left[-\sqrt{\frac{-p}{3}}, \sqrt{\frac{-p}{3}}\right)$, and in $\left[\sqrt{\frac{-p}{3}}, \infty\right)$.

Since f has 3 distinct real roots, it must have exactly one root in each of these 3 intervals.

$\therefore$ f has one real root in $\left(-\sqrt{\frac{-p}{3}}, \sqrt{\frac{-p}{3}}\right)$ and f is strictly decreasing in this interval.

$$\therefore f\left(-\sqrt{\frac{-p}{3}}\right) > 0, \quad f\left(\sqrt{\frac{-p}{3}}\right) < 0$$

In particular, $f\left(-\sqrt{\frac{-p}{3}}\right) f\left(\sqrt{\frac{-p}{3}}\right) < 0$

$$\left(-\sqrt{\frac{-p^3}{27}} - p\sqrt{\frac{-p}{3}} + q\right)\left(\sqrt{\frac{-p^3}{27}} + p\sqrt{\frac{-p}{3}} + q\right) < 0$$

$$\left(\frac{2}{3}\sqrt{\frac{-p^3}{3}} + q\right)\left(-\frac{2}{3}\sqrt{\frac{-p^3}{3}} + q\right) < 0$$

$$q^2 - \frac{4}{9}\left(\frac{-p^3}{3}\right) < 0$$

$$27q^2 + 4p^3 < 0$$

5 The function $\sin x$ is continuous and differentiable on $\mathbb{R}$.

$\therefore$ Given $a, b \in \mathbb{R}$, $a \neq b$, by LMVT, $\exists c \in (a,b)$ s.t.

WLOG, assume $a < b$

$$\frac{\sin a - \sin b}{a-b} = \sin'(c) = \cos(c)$$

$\therefore$ $|\sin a - \sin b| \leq |a-b|$ for $a \neq b$

For $a = b$, obvious.

7 f is continuous on $[-a,a]$ and diff. on $(-a,a)$.

Given $x \in (-a,a)$,

$\exists\, c_1 \in (-a,x)$ s.t. $\dfrac{f(x) - (-a)}{x-(-a)} = f'(c_1) \leq 1$

$\therefore f(x) + a \leq x + a$

$f(x) \leq x$

$\exists\, c_2 \in (x, a)$ s.t. $\dfrac{f(a) - f(x)}{a-x} = f'(c_2) \leq 1$

$\therefore f(x) \geq x$

$\therefore f(x) = x \quad \forall x \in (-a,a)$

8

(i) $f'(0) = 1,\ f'(1) = 1$

If f'' exists, then f' is continuous and diff.

$\therefore$ By LMVT, $\exists\, c \in (0,1)$ s.t. $f''(c) = 0$.

$\therefore$ No such function exists.

(ii) $f'(x) = 1 + x$ works

$\therefore f(x) = x + \dfrac{x^2}{2}$

Suppose such an f exists.

(iii) $f''(x) \geq 0\ \forall x \in \mathbb{R} \Rightarrow f'(x)$ is monotonically increasing

$\Rightarrow f'(x) \geq 1\ \ \forall x > 0.$

Let $x > 0$.

f is cont. and diff. on $\mathbb{R}$,

$\therefore$ By LMVT, $\exists\, c \in (0,x)$ s.t.

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \geq 1$$

$$\therefore f(x) \geq x + f(0)$$

~~Take x~~

$$\therefore f(101 - f(0)) \geq 101 > 100$$

Contradiction

$\therefore$ No such f exists.

9 $f(x) = \begin{cases} 1 - 12x - 3x^2 & x \in [-2, 0) \\ 1 + 12x - 3x^2 & x \in [0, 5] \end{cases}$

f is diff. on $[-2,0) \cup (0,5]$

$f'(x) = -12 - 6x$ for $x \in [-2, 0)$
$f'(x) = 12 - 6x$ for $x \in (0, 5]$

$f'(x) = 0$ at $x = -2$, $x = 2$

Check the values of f at $-2, 0, 2, 5$ and determine the absolute maximum and min.