MA-111 Calculus II
(D3 & D4)

Lecture 14

B.K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

February 22, 2022

Conservative field and its curl

Divergence of a vector field

Parametrized surfaces

The tangent plane

Recap

For a C^1 vector field $\mathbf{F} = (F_1, F_2, F_3)$, we define the curl of $\mathbf{F}$:

$$\operatorname{curl}\mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Curl form or Tangential form: Let the region D in $\mathbb{R}^2$ and the vector field $\mathbf{F}$ satisfy the hypothesis in Green's theorem. Then,

$$\int_{\partial D} \mathbf{F}.\mathbf{T}\,ds = \int\int_D (\operatorname{curl}\mathbf{F}).\mathbf{k}\,dxdy,$$

where $\mathbf{T}$ is the unit tangent to the boundary curve ∂D and $\mathbf{k}$ is the unit vector along the positive z-axis.

- If $\mathbf{F}$ is a C^1 conservative field on an open region D, then $\operatorname{curl}\mathbf{F} = 0$.
- Conversely, if $\operatorname{curl}\mathbf{F} = 0$, then can we tell that $\mathbf{F}$ is a conservative vector field? Under some condition on D.

Conservative field and its curl in $\mathbb{R}^2$

Theorem

1. *Let Ω be an open, simply connected region in $\mathbb{R}^2$.*
2. *if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on Ω.*

Then $\mathbf{F}$ is a conservative field in Ω if and only if

$$\mathit{curl}\,\mathbf{F} = 0, \quad \text{in} \quad \Omega.$$

Outline of the proof: Let the assumptions on Ω and $\mathbf{F}$ in the statement hold.

- If $\mathbf{F}$ is C^1 and a conservative field, i.e., $\mathbf{F} = \nabla f$, for some f is C^2. Then a direct calculation gives curl$F = 0$.
- Now conversely, if $\mathbf{F}$ is C^1 and curl $\mathbf{F} = 0$ on Ω. Then by Green's theorem we can show that the line integral of F over any simple closed curve is 0. That is, the line integral of F in Ω is path independent. Hence the result follows.

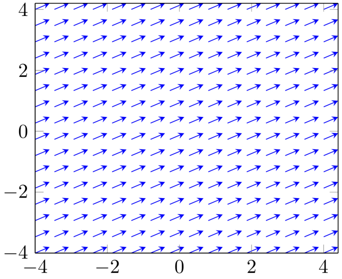
The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

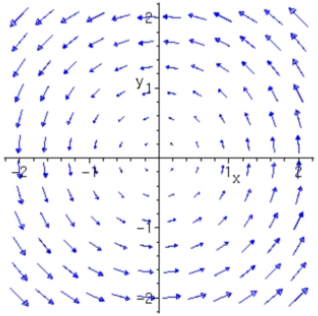
Definition: Let $\mathbf{F} = (F_1, F_2, F_3)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

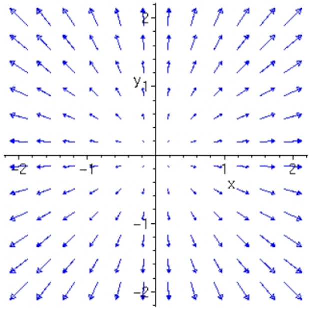
$$\operatorname{div}\mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

One way to interpret divergence of a velocity vector field at a point P as the amount of fluid flowing in versus the amount of fluid flowing out.



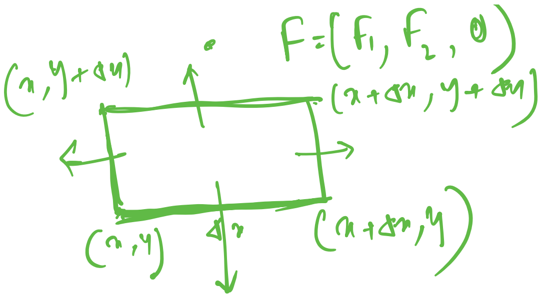
If $\mathbf{F}$ is a constant vector field then at any point what is flowing in is flowing out and the divergence is 0.





Is the divergence for this vector field 0?

This should have non-zero divergence. But what is it measuring?



Bottom: $F(x,y)\cdot(-j)\,\Delta x = -F_2(x,y)\,\Delta x$

Top: $F(x, y+\Delta y)\cdot j\,\Delta x = F_2(x, y+\Delta y)\,\Delta x$

Right: $F(x+\Delta x, y)\cdot i\,\Delta y$

$= F_1(x+\Delta x, y)\cdot\Delta y$

Left: $F(x,y)\cdot(-i)\,\Delta y$

$= -F_1(x,y)\,\Delta y$

Total amount of fluid flowing out of the rectangle is

$$\Big(F_2(x, y+\Delta y) - F_2(x,y)\Big)\Delta x + \Big(F_1(x+\Delta x, y) - F_1(x,y)\Big)\Delta y$$

$$\approx \frac{\partial F_2}{\partial y}\Delta y\,\Delta x + \frac{\partial F_1}{\partial x}\Delta x\,\Delta y$$

Rate of fluid flow per unit area

$$= \left(\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x}\right)$$

- Flux density is $\dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_1}{\partial x}$

Physical interpretation

If $\mathbf{F}$ is the velocity field of a fluid, the divergence of $\mathbf{F}$ gives the rate of expansion of the volume of the fluid per unit volume as the volume moves with the flow. In the case of planar vector fields we get the corresponding rate of expansion of area.

Example : $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. The flow lines of this vector field point radially outward from the origin, so it is clear that the fluid is expanding as it flows. This is reflected in the fact that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

Example : If we look at the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$, we see that $\nabla \cdot \mathbf{F} = -2$. This is consistent with the fact that the flow lines of the vector field all point towards the origin, and the fluid is getting compressed.

Example : $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. In this case the fluid is moving countercolockwise around the origin - so it is neither being compressed, nor is it expanding. One checks easily that $\nabla \cdot \mathbf{F} = 0$.

The divergence of any curl is zero. In other words, if $\mathbf{G}$ is a $\mathcal{C}^2$ vector field,

$$\operatorname{div}(\operatorname{curl}\mathbf{G}) = \nabla \cdot (\nabla \times \mathbf{G}) = 0.$$

Qn : If $\nabla \cdot \mathbf{F} = 0$, does it imply that $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$?

This question is related to the topological properties to of the domain of the vector field as in the case of when a curl free vector field is a gradient vector field. We will be able to show that this is the case when the domain is $\mathbb{R}^n$ for $n = 2, 3$. We postpone it for later.

Next, we mention the Divergence theorem in $\mathbb{R}^2$:
Let ∂D be a non-singular, positively oriented curve in $\mathbb{R}^2$, parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are denoted by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}, \quad \forall\, t \in [a, b].$$

Divergence form of Green's theorem

Divergence form or Normal form:

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int\int_D \operatorname{div}\mathbf{F} dxdy.$$

Gauss's divergence theorem is a 3-dimensional analogue of the above result.

Outline of its proof: Since $\mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}$, for all $t \in [a, b]$, using the definition of $\mathbf{c}(t)$, we get $\mathbf{n}(t) = \left(\frac{y'(t)}{\|\mathbf{c}'(t)\|}, \frac{-x'(t)}{\|\mathbf{c}'(t)\|}, 0\right)$. Thus, for $\mathbf{F} = (F_1, F_2, 0)$, using $ds = \|\mathbf{c}'(t)\| dt$

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int_{\partial D} \left[F_1(\mathbf{c}(t))\frac{y'(t)}{\|\mathbf{c}'(t)\|} - F_2(\mathbf{c}(t))\frac{x'(t)}{\|\mathbf{c}'(t)\|}\right] ds$$

$$= \int_{\partial D} [F_1(\mathbf{c}(t))y'(t) - F_2(\mathbf{c}(t))x'(t)]dt = \int_{\partial D} F_1\, dy - F_2\, dx.$$

$\tilde{F}(c(t)) \cdot c'(t)$

$\tilde{F}(x,y) = (-F_2(x,y), F_1(x,y))$

Now by Green's theorem, we get

$$\int_{\partial D} F_1\, dy - F_2\, dx = \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dxdy = \iint_D \operatorname{div}\mathbf{F} dxdy.$$

Physical interpretation of Divergence theorem

We can interpret the above theorem in the context of fluid flow. If $\mathbf{F}$ represents the flow of a fluid, then the left hand side of the divergence theorem represents the net flux of the fluid across the boundary ∂D. On the other hand, the right hand side represents the integral over D of the rate $\nabla \cdot \mathbf{F}$ at which fluid area is being created. In particular if the fluid is incompressible (or, more generally, if the fluid is being neither compressed nor expanded) the net flow across ∂D is zero.

We can talk about volume analogously in the three dimensional case after proving Stokes theorem.

Surfaces : Definition

A curve is a "one-dimensional" object. Intuitively, this means that if we want to describe a curve, it should be possible to do so using just one variable or parameter.

To do line integration, we further required some extra properties of the curve - that it should be $\mathcal{C}^1$ and non-singular.

We will now discuss the two dimensional analog, namely, surfaces. In order to describe a surface, which is a two-dimensional object, we clearly need two parameters.

Definition

Let D be a path connected subset in $\mathbb{R}^2$. A parametrised surface is a continuous function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

This definition is the analogue of what we called paths in one dimension and what are often called parametrized curves.

Geometric parametrised surfaces

As with curves and paths, we will distinguish between the surface $\mathbf{\Phi}$ and its image. Similarly, the image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.

Note that for a given $(u, v) \in D$, $\mathbf{\Phi}(u, v)$ is a vector in $\mathbb{R}^3$. Each of the coordinates of the vector depends on u and v. Hence we write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on D.

The parametrized surface $\mathbf{\Phi}$ is said to be a smooth parametrized surface if the functions x, y, z have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing D.

Examples

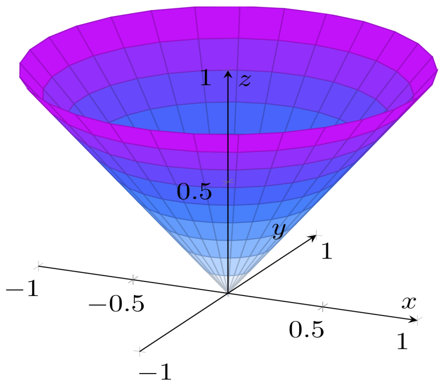
Example 1: Graphs of real valued functions of two independent variables are parametrised surfaces.
Let $f(x, y)$ be a scalar function and let $z = f(x, y)$, for all $(x, y) \in D$, where D is a path connected region in $\mathbb{R}^2$. We can define the parametrised surface $\mathbf{\Phi}$ by

$$\mathbf{\Phi}(u, v) = (u, v, f(u, v)), \quad \forall\, (u, v) \in D.$$

More specifically, we have $x(u, v) = u$, $y(u, v) = v$ and $z(u, v) = f(u, v)$.

Example 2: Consider the cylinder, $x^2 + y^2 = a^2$. Then this is parametrized surface defined by $\mathbf{\Phi} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ defined as $\mathbf{\Phi}(u, v) = (a \cos u, a \sin u, v)$.

Example 3: Consider the sphere of radius $\cancel{a}$, 1
$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Is it a parametrized surface? Recall using spherical coordinates we can represent it using the following parametrization, $\mathbf{\Phi} : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ defined as $\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v)$.



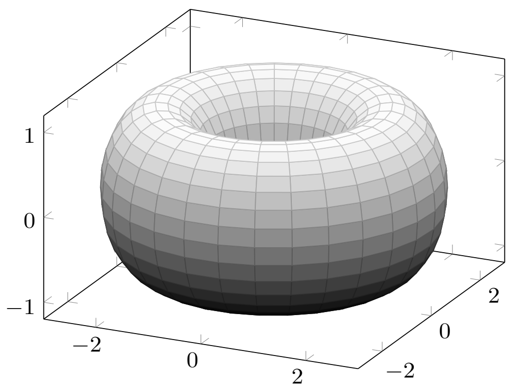
Example 4: The graph of $z = \sqrt{x^2 + y^2}$ can also be parametrized. We use the idea that at each value of z we get a circle of radius z. We can describe the cone as the parametrized surface $\mathbf{\Phi} : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^3$ as $\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u)$.

Example 5: If we have parametrized curve on the z-y-plane $(0, y(u), z(u))$ which we rotate around z-axis, we can parametrise it as follows:

$$x = y(u) \cos v, \quad y = y(u) \sin v, \quad \text{and} \quad z = z(u).$$

Here $a \leq u \leq b$ if $[a, b]$ is the domain of the curve, and $0 \leq v \leq 2\pi$.

Surfaces of revolution around the z-axis



For instance we can parametrize a torus by taking a circle in the y-z plane with center $(0, a, 0)$ of radius b. This is given by the curve $(0, a + b\cos u, b\sin u)$.

Then the parametrization of the torus is then $\mathbf{\Phi}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.

Parametrised surfaces are more general than graphs of functions.

P_0 $\Phi(u_0, v_0)$

Tangent vectors for a parametrised surface

Let $\mathbf{\Phi}(u, v)$ be a smooth parametrised surface. If we fix the variable v, say $v = v_0$, we obtain a curve $\mathbf{c}(u, v_0)$ that lies on the surface. Thus

$$\mathbf{c}(u) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

Since this curve is C^1 we can talk about its tangent vector at the point u_0. This is given by

$$\mathbf{c}'(u_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

We can *define* the partial derivative of a vector valued function as

$$\mathbf{\Phi}_u(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) := \mathbf{c}'(u_0).$$

Similarly, by fixing u $= u_0$ and varying v we obtain a curve $\mathbf{l}(u_0, v)$ and we can set

$$\mathbf{\Phi}_v(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0) := \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

The tangent plane

Let for any given point on the surface, $P_0 = (x_0, y_0, z_0) := \mathbf{\Phi}(u_0, v_0)$ for some $(u_0, v_0) \in D$.
The two tangent vectors $\mathbf{\Phi}_u(u_0, v_0)$ and $\mathbf{\Phi}_v(u_0, v_0)$ at P_0 define a plane.
We call this plane as the tangent plane to the surface at P_0.

The normal to this plane at P_0, $\mathbf{n}(u_0, v_0) = \mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)$.

Thus for a given point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ in $\mathbb{R}^3$ the equation of the tangent plane is given by

$$\mathbf{n}(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

provided $\mathbf{n} \neq 0$.

In particular, if $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0) = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the equation of the tangent plane at P_0 is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Let us find the equation of the tangent plane at points on the various parametrised surfaces we have already looked at.

Example 1: Let D be a path-connected subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a C^1 function. The surface given by the graph of the function $z = f(x, y)$ is parametrized by $\mathbf{\Phi}(x, y) = (x, y, f(x, y))$. In this case, at $P_0 = \mathbf{\Phi}(x_0, y_0)$ for $(x_0, y_0) \in D$,

$$\mathbf{\Phi}_x(x_0, y_0) = \mathbf{i} + \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_y(x_0, y_0) = \mathbf{j} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{k}.$$

Hence,

$$\mathbf{n}(x_0, y_0) = \mathbf{\Phi}_x(x_0, y_0) \times \mathbf{\Phi}_y(x_0, y_0) = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1 \right).$$

Thus the equation of the tangent plane is

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1 \right) = 0;$$

which yields,

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$