MA-111 Calculus II
(D3 & D4)

Lecture 4

B.K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

January 31, 2022

Integrable functions

Integrals over any bounded region in $\mathbb{R}^2$

Bonaventura Cavalieri (1598 - 1647)

http://en.wikipedia.org/wiki/File:Bonaventura_Cavalieri.jpeg

Cavalieri's Principle

Suppose two solids are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

http://en.wikipedia.org/wiki/File:Cavalieri

The Slice Method

Cavalieri's basic idea is that we can find the volume of a given solid by slicing it into thin cross sections, calculating the areas of the slices and then adding up these areas.

Let S be a solid and P_x be a family of planes perpendicular to the x-axis with x as x-coordinate such that

1. S lies between P_a and P_b,
2. the area of the slice of S cut by P_x is $A(x)$.

Then the volume of S is given by

$$\int_a^b A(x)dx.$$

Applying this to the solid graph of $z = f(x, y)$ above a rectangle R in the plane, we see that we get exactly the second of our iterated integrals.

Thus Cavalieri's principle is actually a generalization of the method of iterated integrals. Note that in order to apply the principle we do not require the solid to necessarily lie above a rectangular region in the plane.

Cavalieri's principle is particularly useful in computing the volumes of solids of revolution. These are obtained by taking a region B lying between the lines $x = a$ and $x = b$ on the x-axis and the graph of a function $y = f(x)$ and rotating it through an angle 2π around the x-axis.

Solids of revolution

In this case, we can easily compute the cross-sectional area $A(x)$, since each cross section is nothing but a disc. The radius of the circle is nothing but $f(x)$. Hence, the area $A(x)$ is given by

$$A(x) = \pi[f(x)]^2,$$

and the volume V of the solid is given by

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Solids of revolution may also arise by rotating the graph of a function $f(x)$ around the y-axis. In this case, we can follow the procedure above, replacing x by y and the function $f(x)$ by its inverse.

Existence of integrals on $R = [a, b] \times [c, d]$-I

All our statements so far depend on f being integrable on R. Is there any characterization to determine if f is integrable?

Let $f : R \to \mathbb{R}$ be a bounded function. The function 'f is monotonic in each of two variables' means that for each fixed x, $f(x, y)$ is a monotonic function in y variable and similarly, for each fixed y, $f(x, y)$ is a monotonic function in x variable.

Theorem

If f is bounded and monotonic in each of two variables, then f is integrable on R.

Again the proof follows by using Riemann condition.

Example: Let $f(x, y) := [x + y]$, for all $(x, y) \in R$, where $[u]$ means the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Since f is monotonic in each of two variables, f is integrable on R.

However, the previous condition is not that common and seems rather special.

Surely what worked in one variable should work here. In fact, a proof similar to the case of one variable will show the following theorem.

Existence of integrals on $R = [a, b] \times [c, d]$-II

Theorem

If a function $f : R \to \mathbb{R}$ is bounded and continuous on R except possibly finitely many points in R, then f is integrable on R.

Example. Let $R := [-1, 1] \times [-1, 1]$,

$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)}, & (x, y) \in R, \quad (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

What are points of discontinuity for f on R?

In the one variable case, we saw that a bounded function with at most a finite number of discontinuities on a closed bounded interval is Riemann integrable.

The reason that a finite number of discontinuities do not matter is that points have length zero. What might be the analogous result in two variables?

In other words what sets have "zero area"?

A bounded subset E of $\mathbb{R}^2$ has 'zero area' if for every $\epsilon > 0$, there are finitely many rectangles whose union contains E and the sum of whose areas is less than ϵ.

It turns out graph of a continuous function, that is, set of the form $\{(x, \phi(x)) \mid x \in [a, b]\}$ for a continuous function $\phi : [a, b] \to [c, d]$ has 'zero area' or has *content zero.*

Theorem

If a function f is bounded and continuous on a rectangle $R = [a, b] \times [c, d]$ except possibly along a finite number of graphs of continuous functions, then f is integrable on R.

Example: Let $R = [0, 1] \times [0, 1]$ and

$$f(x, y) = \begin{cases} 1, & 0 \leq x < y, \quad y \in [0, 1], \\ 0, & y \leq x \leq 1, \quad y \in [0, 1]. \end{cases}$$

Is f integrable over R?

The slightly more general theorem says that given a rectangle R and a bounded function $f : R \to \mathbb{R}$, the function is integrable over R if the points of discontinuity of f is a set of 'content zero'.

However the converse of the above statement is not true. There are integrable functions whose points of discontinuity is not a set of 'content zero'. (Check Tutorial)

Counter example: Bivariate Thomae function: $f : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1, \quad \text{if} \quad x = 0, \quad y \in \mathbb{Q} \cap [0,1], \\ \frac{1}{q}, \quad x, y \in \mathbb{Q} \cap [0,1] \quad \text{and} \quad x = \frac{p}{q}, \\ \qquad\qquad\qquad\qquad p, q \in \mathbb{N} \quad \text{are relatively prime}, \\ 0, \quad \text{otherwise}. \end{cases}$$

Integrals over any bounded region in $\mathbb{R}^2$

So far we have learnt to integrate bounded functions on any rectangle in $\mathbb{R}^2$.

Let D be any bounded subset (not necessarily rectangle) of $\mathbb{R}^2$.

How to define integral of $f : D \to \mathbb{R}$ on D?

Remedy If D is a bounded subset of $\mathbb{R}^2$, then there exists a rectangle R in $\mathbb{R}^2$ containing D, i.e., $D \subset R$. Why?
Since D is a bounded subset of R^2, there exists $a > 0$ such that any $(x, y) \in D$ satisfies $x^2 + y^2 < a^2$, i.e, $D \subset B_a = \{(x, y) \mid x^2 + y^2 \leq a^2\}$.

Note $B_a \subset [-a, a] \times [-a, a]$

Then the rectangle $R := [-a, a] \times [-a, a]$ contains D.

Extend f from D to R by defining

$$f^*(x,y) := \begin{cases} f(x,y), & (x,y) \in D, \\ 0, & (x,y) \notin D. \end{cases}.$$

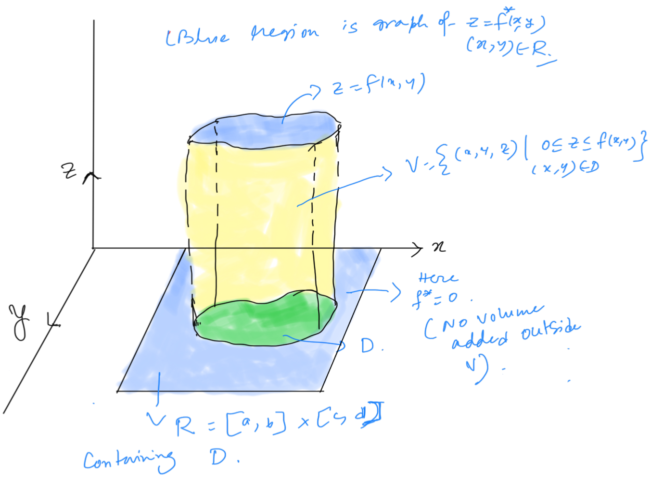
Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be integrable on bounded $D \subset \mathbb{R}^2$, if f^* is integrable on R and the integral of f on D is defined by

$$\int\int_D f(x,y)\,dx\,dy := \int\int_R f^*(x,y)\,dx\,dy.$$

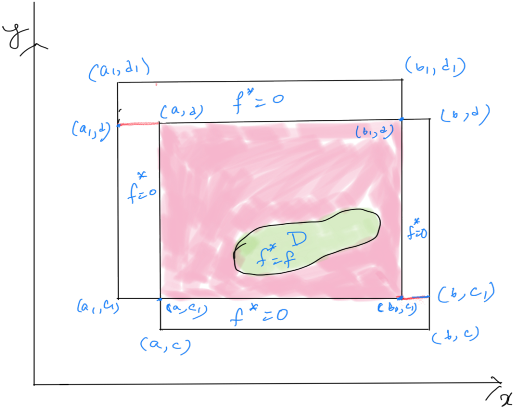
- If $f \geq 0$ on $D \subset \mathbb{R}^2$ and f is integrable on D, then the double integral of f on D is the volume of the solid that lies above D in the x-y plane and below the graph of the surface $z = f(x,y)$ for all $(x,y) \in D$.

$\int\int_D f =$ volume of V



Independent of choice of rectangle

- The choice of rectangle R containing D is not unique.
- But the value of the integral of f on D does not depend on the choice of the rectangle R containing D.
- Use the additivity property of integrals on rectangle and note that only 'zero' is getting added outside D.



Properties of Integrals over bounded sets in $\mathbb{R}^2$

Let D be a bounded subset of $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be an integrable function.

- The algebraic properties for integrals on any bounded set D in $\mathbb{R}^2$ hold similarly to those of the case of integrals on rectangle.

Domain additivity property: Let $D \subseteq \mathbb{R}^2$ be a bounded set. Let $D_1, D_2 \subseteq D$ such that $D = D_1 \cup D_2$. Let $f : D \to \mathbb{R}^2$ be a bounded function. If f is integrable over D_1 and D_2 and $D_1 \cap D_2$ has content zero then f is integrable on D and

$$\int\int_D f = \int\int_{D_1} f + \int\int_{D_2} f.$$

Existence of Integrals over bounded sets in $\mathbb{R}^2$

Theorem

Let $D \subset \mathbb{R}^2$ be a bounded set whose boundary ∂D is given by the finitely continous closed curve then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable over D.

Example. Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $f(x, y) = x^2 + y^2, \quad \forall (x, y) \in D$. Then f is integrable over D.

A slightly more general theorem is as follows:
Let D be a bounded set in $\mathbb{R}^2$ such that ∂D is of content zero. Let $f : D \to \mathbb{R}$ be a bounded function whose points of discontinuity have 'content zero'. Then f is integrable over D.