Department of Physics, Indian Institute of Technology Bombay

5-01-2021 **PH 107: MidSemester** **9:30 - 12:00 hrs**

1. The phase velocity of ripples on a liquid surface is given by $v_p = \sqrt{2\pi \mathrm{S}/\lambda\rho}$, where S is the surface tension, λ is the wavelength and ρ is the density of the liquid. Find the group velocity (v_g) of the ripples. Obtain the relation between v_p and v_g. [3 marks]

   Answer We have:

   $$\mathrm{v_p} = \frac{\omega}{\mathrm{k}} = \sqrt{\frac{2\pi\mathrm{S}}{\lambda\rho}} \Rightarrow \frac{\omega}{\mathrm{k}} = \sqrt{\frac{\mathrm{kS}}{\rho}} \Rightarrow \omega = \mathrm{k}^{\frac{3}{2}}\sqrt{\frac{\mathrm{S}}{\rho}}$$

   The group velocity is given by:

   $$\begin{aligned}\mathrm{v_g} = \frac{\mathrm{d}\omega}{\mathrm{dk}} =& \frac{\mathrm{d}}{\mathrm{dk}}\left(\mathrm{k}^{\frac{3}{2}}\sqrt{\frac{\mathrm{S}}{\rho}}\right) = \frac{3}{2}\mathrm{k}^{\frac{1}{2}}\sqrt{\frac{\mathrm{S}}{\rho}} \\ =& \frac{3}{2}\sqrt{\frac{\mathrm{kS}}{\rho}} = \frac{3}{2}\sqrt{\frac{2\pi\mathrm{S}}{\lambda\rho}}\end{aligned}$$

   We then have:

   $$\mathrm{v_g} = \frac{3}{2}\mathrm{v_p}$$

2. What is the minimum kinetic energy of a non-relativistic electron confined in a region of width $L = 0.1\mathrm{nm}$? [2 marks]

   Answer From the Uncertainty principle, we have:

   $$\Delta \mathrm{x} \Delta \mathrm{p} \sim \tfrac{\hbar}{2} \Rightarrow \Delta \mathrm{p} = \tfrac{\hbar}{2\Delta \mathrm{x}}$$

   $$\Rightarrow \Delta \mathrm{p} = \tfrac{1.055\times 10^{-34}}{2\times 0.1\times 10^{-9}} = 5.275 \times 10^{-25}\mathrm{Ns}$$

   $\therefore$ Minimum kinetic energy $\dfrac{\mathrm{p}^2}{2\ \mathrm{m}} = \dfrac{(5.275 \times 10^{-25})^2}{2 \times 9.1095 \times 10^{-21}} = 1.527 \times 10^{-19}\ \mathrm{J} = 0.953\mathrm{eV}$

3. The one dimensional time-independent wave function of a particle confined in a region is given by

$$\Psi(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

where n is an integer.

(a) Find A.

(b) Consider an operator $\hat{O} = \frac{d^2}{dx^2}$. Show that $\Psi(x)$ is an eigen function of this operator. Find its eigen value.

(c) Find the expectation value of position $< x >$ and momentum $< p >$ of the particle with the above wave function $\Psi(x)$. [1 + 2 + 2 marks]

Answer (a) Using normalisation

$$1 = \int_0^L \Psi^*(x)\Psi(x)dx = A^2 \int_0^L \sin^2 \frac{n\pi x}{L} dx, \qquad A = \sqrt{\frac{2}{L}}$$

(b)

$$\hat{O}\Psi(x) = \frac{d^2}{dx^2}\Psi(x) = \frac{d^2}{dx^2}\sin(kx) = -k^2\Psi(x)$$

Hence $\Psi(x)$ is an eigen function with eigen value $-k^2$.

(c)

$$< x >= \int_0^L \Psi^*(x)x\Psi(x)dx = \frac{2}{L}\int_0^L x\sin^2\frac{n\pi x}{L}dx = \frac{L}{2}$$

$$\begin{aligned} < p > &= \int_0^L \Psi^*(x)\hat{p}\Psi(x)dx \\ &= \frac{h}{2\pi i}\frac{L}{2}\int_0^L \sin\frac{n\pi x}{L}\frac{d}{dx}\sin\frac{n\pi x}{L}dx \\ &= \frac{h}{2\pi i}\frac{2n\pi}{L^2}\int_0^L \sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}dx = 0 \end{aligned}$$

$< x >= \frac{L}{2}; < p >= 0.$

4. The time-independent wave function of a particle moving along the x-direction is given by

$$\Psi(x) = \frac{1+ix}{1+ix^2}\,.$$

Determine the position(s) where the particle is most likely to be found. [5 marks]

Solution The probability density of finding the particle at a point x is given by

$$|\Psi(x)|^2 = \Psi(x)\Psi^*(x) = \frac{1+ix}{1+ix^2}\frac{1-ix}{1-ix^2} = \frac{1+x^2}{1+x^4}$$

The particle is most likely to be found at points for which $d|\Psi|^2/dx = 0$ (which is a definition of maxima). From the above expression, we have

$$\frac{d|\Psi|^2}{dx} = \frac{2x\left(1+x^4\right) - 4x^3\left(1+x^2\right)}{\left(1+x^4\right)^2}$$

we find that $d|\Psi|^2/dx = 0$ when

$$2x\left(1+x^4\right) - 4x^3\left(1+x^2\right) = 0$$

This equation can be simplified to

$$x^4 + 2x^2 - 1 = 0$$

which, after substituting $x^2 = z$, reduces to a quadratic equation

$$z^2 + 2z - 1 = 0 \quad \Longrightarrow \quad z_1 = -1+\sqrt{2} \quad \text{and} \quad z_2 = -1-\sqrt{2}$$

Thus $d|\Psi|^2/dx = 0$ when $x_1^2 = -1+\sqrt{2}$ and $x_2^2 = -1-\sqrt{2}$ since $x^2 > 0$, physically acceptable solutions are

$$x_1 = \pm\sqrt{-1+\sqrt{2}}$$

5. Consider the following triangular wave packet

$$\Phi(x) = \begin{cases} 1 + \frac{x}{L} & -L \leq x \leq 0 \\ \\ 1 - \frac{x}{L} & 0 < x < L \\ \\ 0 & \text{elsewhere} \end{cases}$$

where L is a constant.

(a) Find the inverse Fourier transform

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(x) \mathrm{e}^{-ikx} dx \,.$$

(b) Draw qualitative graphs of $A(k)$ and $\Phi(x)$. How will the graphs get modified if L is increased to $2L$. [3 + 3 marks]

Solution: With the wave packet of the form

$$\Phi(x,0) = \begin{cases} 1 + \frac{x}{L} & -L \leq x \leq 0 \\ \\ 1 - \frac{x}{L} & 0 < x < L \\ \\ 0 & \text{elsewhere} \end{cases}$$

the amplitude $A(k)$ takes the form

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(x,0) \mathrm{e}^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-L}^{0} \left(1 + \frac{x}{L}\right) \mathrm{e}^{-ikx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{L} \left(1 - \frac{x}{L}\right) \mathrm{e}^{-ikx} dx \end{aligned}$$

We can change the variable x to $-x$ in the first integral and obtain

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{L} \left(1 - \frac{x}{L}\right) \left(\mathrm{e}^{ikx} + \mathrm{e}^{-ikx}\right) dx \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{L} \left(1 - \frac{x}{L}\right) \cos(kx) dx \end{aligned}$$

Performing the integration, we get

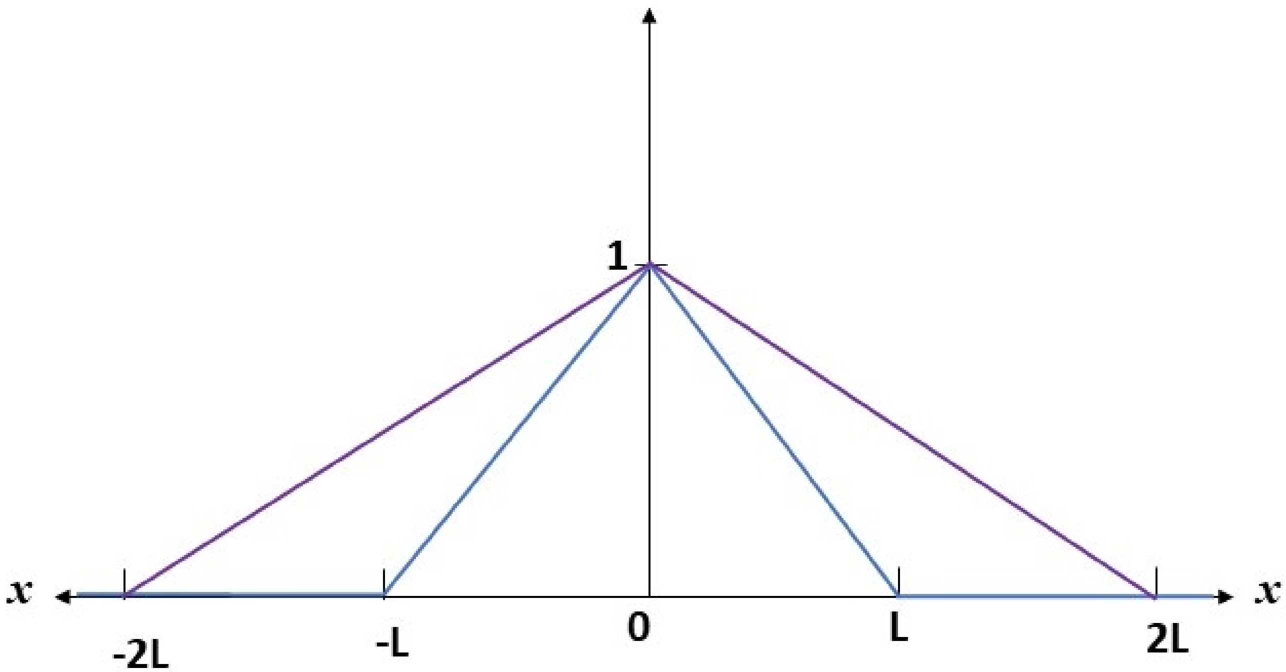
$$A(k) = \frac{2}{\sqrt{2\pi}} \frac{1}{k^2 L} [1 - \cos(kL)]$$

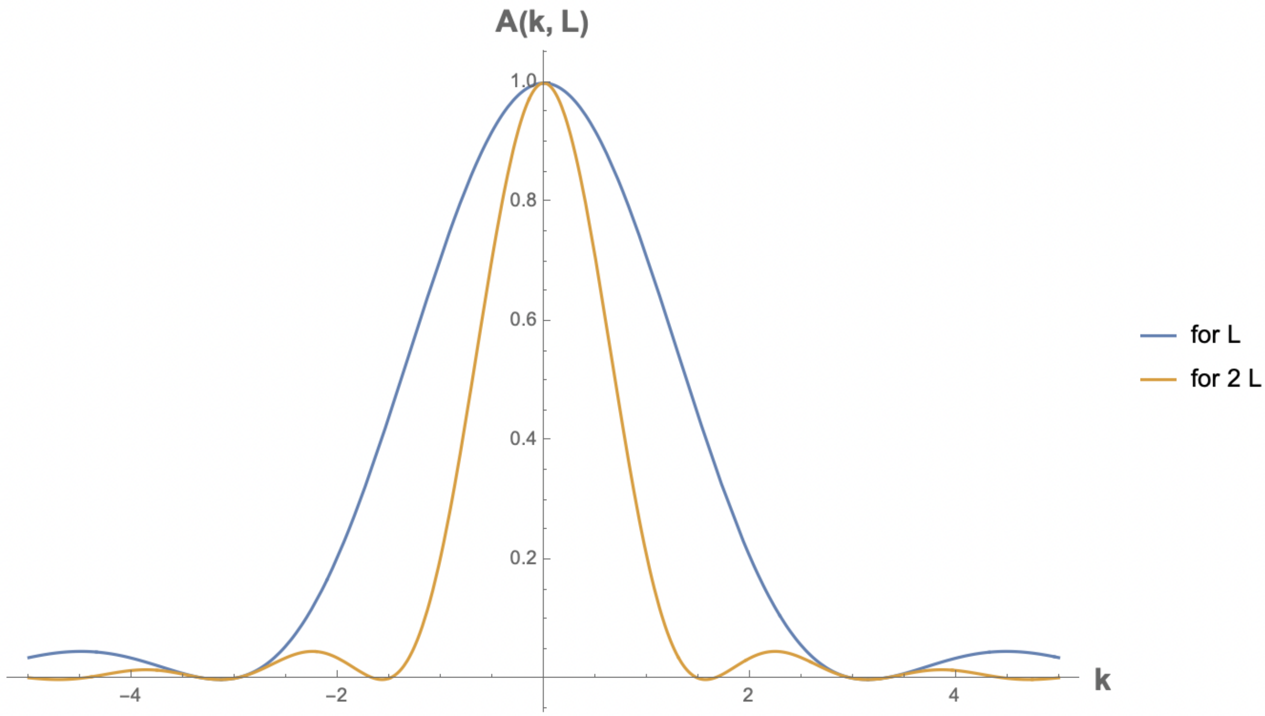
which can be simplified to

$$
\begin{aligned}
A(k) &= \frac{2}{\sqrt{2\pi}}\frac{1}{k^2L}[1-\cos(kL)] = \frac{4}{\sqrt{2\pi}}\frac{1}{k^2L}\sin^2\left(\frac{1}{2}kL\right) \\
&= \frac{1}{\sqrt{2\pi}}\frac{L}{\frac{1}{4}k^2L^2}\sin^2\left(\frac{1}{2}kL\right) = \frac{L}{\sqrt{2\pi}}\frac{\sin^2\left(\frac{1}{2}kL\right)}{\left(\frac{1}{2}kL\right)^2} \\
&= \frac{L}{\sqrt{2\pi}}\left[\frac{\sin\left(\frac{1}{2}kL\right)}{\frac{1}{2}kL}\right]^2
\end{aligned}
$$

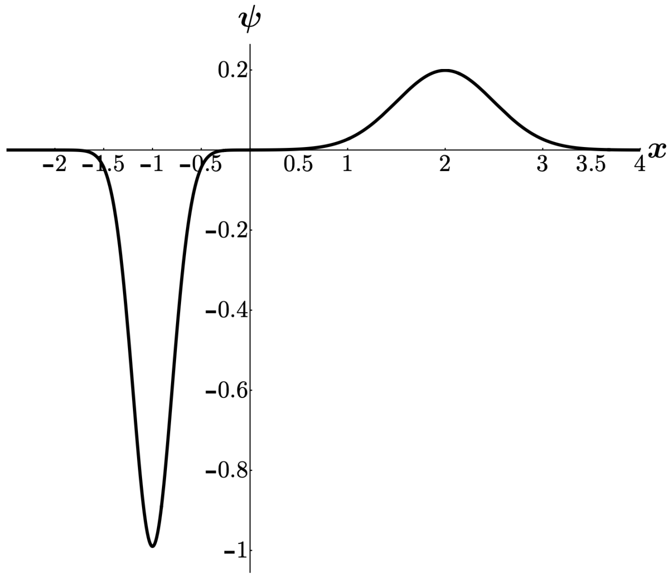
The width of the wave packet is $2b$, whereas the width of the amplitude $A(k)$ is 2π.

b)




A(k, L)
1.0
0.8
0.6
0.4
0.2
−4
−2
2
4
k
for L
for 2 L

6. The wave function of a particle in the range $-2 \leq x \leq 4$ is shown in the Figure below. [2+2 marks]



(a) Find the region(s) where the particle is **most likely** to be found.
Ans: for x values $-1.5 \rightarrow -0.5$

(b) Find the region(s) where the particle is **least likely** to be found
Ans: for x values $-2 \rightarrow -1.5$ and $-0.5 \rightarrow +0.5$ and $+3.5 \rightarrow +4.0$

(even if a student has used more accurate values like -1.75, etc., it is fine as long as the range is correct. Also, if you missed the first range in (b) from -2 to -1.5, that is also fine)

7. Consider Sun and Earth as ideal black bodies in empty space. The Sun's temperature is $T_S = 6000\,\mathrm{K}$ and the heat transfer by oceans and Earth's atmosphere keep the Earth's surface at a uniform temperature. Radius of Earth $R_E = 6.4 \times 10^6\,\mathrm{m}$, radius of Sun $R_S = 7 \times 10^8\,\mathrm{m}$, mass of Sun $M_S = 2 \times 10^{30}\,\mathrm{kg}$ and Earth-Sun distance $d = 1.5 \times 10^{11}\,\mathrm{m}$.

   a) Find the temperature of the Earth.

   b) Find the radiation force on the Earth.

   c) Find the distance from the Sun at which a substance, having a melting point of $T_m = 1550\,\mathrm{K}$, will melt.

   [1 + 2 + 2 marks]

   Answer

   a) The total radiation flux from the Sun is

$$J_S = \sigma T_S^4 4\pi R_S^2$$

   where σ is the Stefan-Boltzmann constant. Only a fraction $\pi R_E^2/4\pi d^2$ of this flux reaches the Earth. In equilibrium this fraction equals the total flux radiated from the Earth at temperature T_E. So

$$\sigma T_S^4 4\pi R_S^2 \frac{\pi R_E^2}{4\pi d^2} = \sigma T_E^4 4\pi R_E^2$$

   From the above expression, we get

$$T_E = \sqrt{\frac{R_S}{2d}} T_S \approx 290\ K$$

   b) The radiation pressure on the Earth is given by

$$P_r = \tfrac{4}{3c}\sigma T_S^4 \tfrac{R_S^2}{d^2}$$

$$= \tfrac{4}{3\cdot 3\cdot 10^8} 5.67 \cdot 10^{-8} \cdot \left(6 \cdot 10^3\right)^4 \left(\tfrac{7\cdot 10^8}{1.5\cdot 10^{11}}\right)^2$$

$$= 7 \cdot 10^{-6}\ N/m^2$$

   where $(R_S/d)^2$ is the ratio of the total flux from the Sun to the flux that reaches the Earth. The radiation force on the Earth

$$f_E = P_r A_E = P_r \pi R_E^2 = 7 \cdot 10^{-6} \cdot \pi \left(6.4 \cdot 10^6\right)^2$$

$$= 9 \cdot 10^8\ N$$

   where A_{E} is the cross section of the Earth.

c) Using the temperature from point (a) and denoting the melting temperature of the metallic particle T_m and the distance from the Sun d_C, we obtain

$$d_C = \frac{1}{2} R_S \left(\frac{T_S}{T_m}\right)^2 = \frac{1}{2} 7 \cdot 10^8 \left(\frac{6000}{1550}\right)^2$$
$$\approx 5 \cdot 10^9 \ m = 5 \cdot 10^6 \ km$$