MA-111 Calculus II (D3 & D4)

Lecture 12

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Green's Theorem

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\Big(D \cup \partial D\Big) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a double integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

Examples.

Example. Compute the line integral $\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy$, where C is the circle in $\mathbb{R}^2$ with origin at $(2,0)$ and radius 1.

Can compute directly using definition of line integral! But is there any better way?

Use Green's theorem: Set $F_1(x,y) = ye^{-x}$ and $F_2(x,y) = (\frac{1}{2}x^2 - e^{-x})$, for all $(x,y) \in D$, where $D = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 \leq 1\}$. Using Green's theorem,

$$\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy.$$

Now see

$$\int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy = \int\int_D xdxdy,$$

and derive the double integral using polar coordinates: Check!

$$\int\int_D xdxdy = 2\pi.$$

Area of a region

Can the area of a region enclosed be expressed as a line integral?
If C is a positively oriented curve that bounds a region D , then the area $A(D)$ is given by (Why?)

$$A(D) = \frac{1}{2}\int_C xdy - ydx.$$

Note if $F_1(x,y) = -\frac{y}{2}$ and $F_2(x,y) = \frac{x}{2}$, for all $(x,y) \in D$, then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and hence $A(D) := \int\int_D 1\, dxdy = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dxdy$.
By Green's theorem,

$$\int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dxdy = \int_C F_1\, dx + F_2\, dy = \frac{1}{2}\int_C xdy - ydx,$$

Thus $A(D) = \frac{1}{2}\int_C xdy - ydx$.
Also note for $F_1 \equiv 0$ and $F_2(x,y) = x$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = \int_C x\, dy$.
Further for $F_1(x,y) = -y$ and $F_2 \equiv 0$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = -\int_C y\, dx$.
In summary, $A(D) = \frac{1}{2}\int_C xdy - ydx = \int_C x\, dy = -\int_C y\, dx$.

Example: Let us use the formula above to find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: We parametrise the curve C by $\mathbf{c}(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$. By the formula above, we get

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_C x dy - y dx \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt \\
&= \frac{1}{2}\int_0^{2\pi} ab dt = \pi ab.
\end{aligned}$$

Polar coordinates

Suppose we are given a simple positively oriented closed curve $C : (r(t), \theta(t))$ in polar coordinates. Thus for $t \in [a, b]$ $x(t) = r(t)\cos(\theta(t))$ and $y(t) = r(t)\sin(\theta(t))$ and using chain rule formula:

$$\frac{dx}{dt}(t) = \cos(\theta(t))\frac{dr}{dt}(t) - r(t)\sin\theta(t)\frac{d\theta}{dt}(t),$$

$$\frac{dy}{dt}(t) = \sin(\theta(t))\frac{dr}{dt}(t) + r(t)\cos\theta(t)\frac{d\theta}{dt}(t).$$

Then, by the area formula above, we know that the area enclosed by C is given by

$$(-y(t), x(t)) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt$$

$$\frac{1}{2}\int_C xdy - ydx := \frac{1}{2}\int_a^b \underbrace{\left(x(t)\frac{dy}{dt}(t) - y(t)\frac{dx}{dt}(t)\right)} dt$$

$$= \frac{1}{2}\int_a^b r(t)\cos\theta(t)\sin\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r^2(t)\cos^2\theta(t)\frac{d\theta}{dt}dt$$

$$-\frac{1}{2}\int_a^b r(t)\sin\theta(t)\cos\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r(t)^2\sin^2\theta(t)\frac{d\theta}{dt}dt$$

$$\boxed{= \frac{1}{2}\int_C r^2 d\theta.}$$

Example:

Exercise: Find the area of the cardioid $r = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$.

Solution: Using the formula we have just derived, the desired area is simply

$$\frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 d\theta = a^2\int_0^{2\pi} -2\cos\theta + \frac{\cos 2\theta}{2} + \frac{3}{2}d\theta$$

$$= \frac{3a^2\pi}{2}.$$

Green's Theorem for Simply connected regions

Theorem (Green's theorem:)

1. *Let D be a connected bounded region in $\mathbb{R}^2$ with positively oriented ∂D a piecewise continuously differentiable simple closed curve.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\Big(D \cup \partial D\Big) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

Here the region above mentioned is simply connected and bounded by a simple closed curve. Rectangles, discs are examples of such domain.

We will give a proof of the above theorem for some special cases. However, note that we can generalize Green's theorem by noting the following. Often regions which are not simply connected can be written as a union of simply connected regions.

A proof of Green's theorem for regions of special type

We give a proof of Green's theorem when the region D is both of type 1 and type 2 .
Examples: Rectangles, Discs are examples of such region.
Assume that D is of Type 1

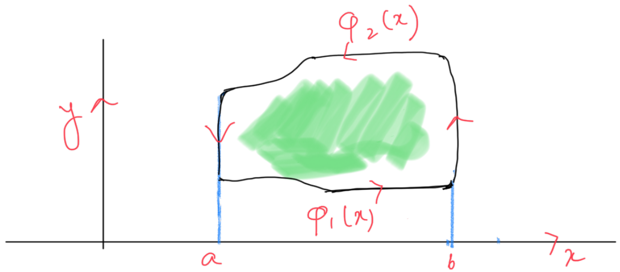
$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

for some continuous functions ϕ_1 and ϕ_2.
Also assume there exist two continuous functions ψ_1 and ψ_2 such that D can be written as Type 2:

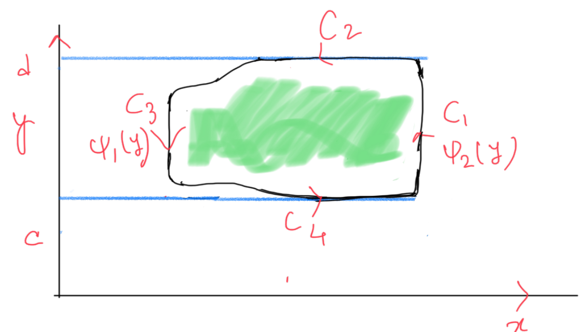
$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y)\}.$$

The proof follows two main steps:

- Double integrals can be reduced to iterated integrals.
- Then the fundamental theorem of calculus can be applied to the resulting one-variable integrals.



Type 1.



Type 2.

The proof of Green's theorem, contd.

To prove

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy,$$

we show

Step 1 Using the fact that D is a region of Type 2,

$$\iint_D \frac{\partial F_2}{\partial x} = \int_{\partial D} F_2 dy.$$

Step 2 Using the fact that D is a region of Type 1,

$$-\iint_D \frac{\partial F_1}{\partial y} = \int_{\partial D} F_1 dx.$$

Then combining the both equalities, we get our result.
Since D is a region of Type 2, it gives

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x, y) dxdy.$$

Using the Fundamental Theorem of Calculus we get

$$\int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x, y) dxdy = \int_c^d F_2(\psi_2(y), y) - F_2(\psi_1(y), y) dy$$

The proof of Green's theorem contd.

Now let us calculate $\int_{\partial D} F_2 dy$. Note that ∂D can be written as union of four curves C_1, C_2, C_3 and C_4 such that

On C_1: $C_1 = \{(\psi_2(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction upwards. So,

$$\int_{C_1} F_2\, dy = \int_c^d F_2(\psi_2(y), y)\, dy.$$

On C_3: $C_3 = \{(\psi_1(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction downwards. So,

$$\int_{C_3} F_2\, dy = -\int_{-C_3} F_2\, dy = -\int_c^d F_2(\psi_1(y), y)\, dy.$$

On C_2 and C_4: $C_2 = \{(x, d) \mid \psi_1(d) \leq x \leq \psi_2(d)\}$ going from right to left and $C_4 = \{(x, c) \mid \psi_1(c) \leq x \leq \psi_2(c)\}$ going from left to right. In particular, they are vertical lines and y is constant along these lines. Thus, for any parametirzation of C_2 and C_4, $\frac{dy}{dt} = 0$, and

$$\int_{C_2} F_2\, dy = 0 = \int_{C_4} F_2\, dy.$$

The proof of Green's theorem contd.

Noting that

$$\int_{\partial D} F_2 dy = \int_{C_1} F_2\, dy + \int_{C_2} F_2\, dy + \int_{C_3} F_2\, dy + \int_{C_4} F_2\, dy,$$

and using previous results, we obtain

$$\int_{\partial D} F_2 dy = \int_c^d F_2(\psi_2(y), y)\, dy - \int_c^d F_2(\psi_1(y), y)\, dy,$$

and thus

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_{\partial D} F_2 dy.$$

Similarly, using the fact that D can be written as a region of Type 1, we get

$$\iint_D \frac{\partial F_1}{\partial y} dxdy = -\int_{\partial D} F_1 dx.$$

Where does the minus sign come from?

From the fact that $y = \phi_2(x)$ is oriented in the direction of decreasing x.

Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_{\partial D} F_1 dx + \int_{\partial D} F_2 dy.$$

A more general case

How does one proceed in general, that is for more general regions which may not be of both type 1 and type 2. We can try proceeding as follows:

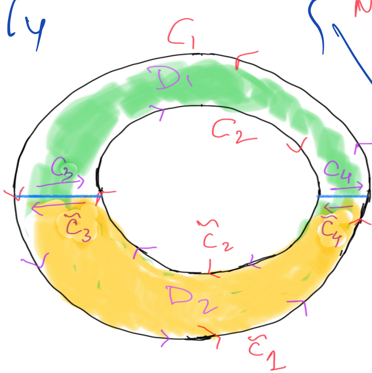
- Break up D into smaller regions each of which is of both type 1 and type 2 but so that any two pieces meet only along the boundary.
- Apply Green's theorem to each piece.
- Observe that the line integrals along the interior boundaries cancel, leaving only the line integral around the boundary of D.

$D = D_1 \cup D_2, \quad \partial D = C_1 \cup C_2.$

$\partial D_1 = C_1 \cup C_3 \cup C_2 \cup C_4, \quad \partial D_2 = \tilde{C}_1 \cup \tilde{C}_4 \cup \tilde{C}_2 \cup \tilde{C}_3$

$$\iint_{D_1} (\quad) = \int_{C_1+C_3+C_2+C_4} (\quad)$$

$$\iint_{D_2} (\quad) = \int_{\tilde{C}_1+\tilde{C}_4+\tilde{C}_2+\tilde{C}_3} (\quad)$$



Note: $C_3 = -\tilde{C}_3$, $C_4 = -\tilde{C}_4$

$$\int_{C_1} + \int_{C_2} + \cancel{\int_{C_3}} + \cancel{\int_{C_4}} + \int_{\tilde{C}_1} + \int_{\tilde{C}_2} + \cancel{\int_{\tilde{C}_3}} + \cancel{\int_{\tilde{C}_4}}$$

$C_1 \cup C_2 \cup \tilde{C}_1 \cup \tilde{C}_2$

Del operator on vector fields

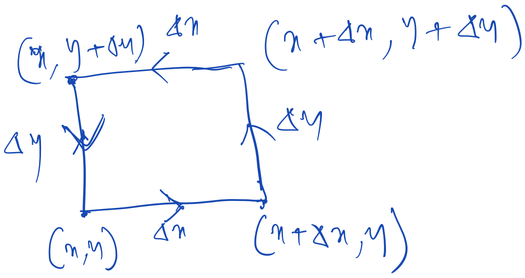
The del operator operates on vector fields as in two different ways. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

$$\operatorname{curl}\mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

It is often written as a determinant;

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

$$F = (F_1, F_2, 0), \quad \nabla \times F = \underline{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)K}$$



$F(x,y) = F_1(x,y)\,i + F_2(x,y)\,j$

Bottom: $(F\cdot i)\,\Delta x = F_1(x,y)\,\Delta x$

Top: $F(x,y+\Delta y)\cdot(-i)\cdot\Delta x = -F_1(x,y+\Delta y)\,\Delta x$

$(\nabla \times F)\cdot k$

$= \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$

Right: $(F(x+\Delta x, y)\cdot(j))\,\Delta y$

$= F_2(x+\Delta x, y)\cdot\Delta y$

Left: $F(x,y)\cdot(-j)\,\Delta y$

$= -F_2(x,y)\,\Delta y$

$-\left(F_1(x,y+\Delta y) - F_1(x,y)\right)\Delta x$

$\simeq -\dfrac{\partial F_1}{\partial y}\,\Delta y\cdot\Delta x$

$\dfrac{\partial F_2}{\partial x}\,\Delta x\,\Delta y$

Circulation density for R $= \dfrac{\left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right)\Delta x\cdot\Delta y}{\Delta x\,\Delta y}$

$= \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$

- Circulation density at a point $= \dfrac{\partial F_2}{\ } - \dfrac{\partial F_1}{\ }$