MA-111 Calculus II
(D3 & D4)

Lecture 18

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Closed surface

Can we have a generalized version of the divergence for of Green's theorem?
Yes!, Gauss's divergence theorem under suitable hypothesis on W, a region in $\mathbb{R}^3$.

We define a closed surface S in $\mathbb{R}^3$ to be a surface which is bounded, whose complement is open and boundary of S is empty. This is analogous to the closed curve.
If S is a closed surface, for example like sphere, then it encloses a 3-dimensional region. Call it W, and then S will be its boundary, ∂W.

This is analogous to a simple closed curve being boundary of a region D in $\mathbb{R}^2$.

Gauss's divergence theorem

Theorem (Gauss's Divergence Theorem)

1. *Let $S = \partial W$ be a closed oriented surface enclosing the region W with the outward normal giving the positive orientation.*
2. *Let F be a smooth vector field defined on on open set containing $W \cup \partial W$.*

Then

$$\iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_{W} (div \boldsymbol{F}) dxdydz.$$

Clearly, the importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.

Consequences of Gauss' theorem

Theorem

Let W be as in Gauss' theorem. Let $\mathbf{F}$ be a smooth vector field on an open set in $\mathbb{R}^3$ containing $W \cup \partial W$ satisfying $div\, \mathbf{F} = 0$ on W. Then $\int\int_{\partial W} \mathbf{F}.\mathbf{dS} = 0.$

Using the above theorem, we have following result:

Corollary

Let S and W be as in Gauss' theorem. Suppose $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \partial S_1 = \partial S_2$ and S_1 and S_2 have the induced orientation from S. If $\mathbf{F}$ be a vector field defined on an open set containing $W \cup \partial W$ with $div\, \mathbf{F} = 0$, then

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{dS} = \iint_{-S_2} \mathbf{F} \cdot \mathbf{dS}$$

where $-S_2$ denotes the surface S_2 with opposite orientation.

Gauss' Law of Electromagnetic theory

Let a point charge Q is located at the origin. Then the electric field created by the charge is

$$\mathbf{E}(x,y,z) = \frac{Q}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3} \quad (\mathbf{r} = (x,y,z) \neq (0,0,0)).$$

- Check that $\nabla \cdot \mathbf{E} = 0$.
- Let S_i and S_o be two spheres of radius a and b respectively with $a < b$.
- Let W be the region enclosed by S_i and S_o.
- Then $\iint_{S_o} \mathbf{E} \cdot d\mathbf{S} = \iint_{-S_i} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon}$ (check it).
- In fact if S is any closed surface that encloses origin and S_o is a large enough sphere containing S then

$$\iint_{S_o} \mathbf{E} \cdot d\mathbf{S} = \iint_{-S} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon}.$$

- Thus the outward flux of E over a closed surface is equal to the total charge enclosed by the surface, which is the Gauss' law.

Physical Interpretation of the Gauss Divergence Theorem:

Suppose a solid body W in $\mathbb{R}^3$ is enclosed by a closed geometric surface S, oriented in the direction of the outward normals. Let $\boldsymbol{F}$ be a vector field on D. The Gauss divergence theorem says that the flux of $\boldsymbol{F}$ across S is equal to the triple integral of the divergence of the vector field $\boldsymbol{F}$ over W.

Example 1 Calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

Solution: Using Gauss's theorem, we see that we need only evaluate

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W (\nabla \cdot \mathbf{F}) dV = \iiint_W 3(x^2 + y^2 + z^2) dxdydz,$$

where W is the unit ball.
This problem is clearly ideally suited to the use of spherical coordinates.
Making a change of variables, we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin\phi \, d\rho d\phi d\theta = \frac{12\pi}{5}$$

Examples

Example 2: Let $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, and let S be the unit sphere. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution: Using Gauss' theorem we see that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W (\nabla \cdot \mathbf{F}) dV,$$

where W is the unit ball bounded by the sphere. Since $\nabla \cdot \mathbf{F} = 2(1 + y + z)$ we get

$$2 \iiint_W (1 + y + z) dV = 2 \iiint_W dV + 2 \iiint_W y dV + 2 \iiint_W z dV.$$

Notice that the last two integrals above are 0, by symmetry. Hence, the flux is simply

$$2 \iiint_W dV = \frac{8\pi}{3}.$$

Curl and divergence

Theorem

1. *If* $\mathbf{F} = \nabla \times \mathbf{G}$, *where* $\mathbf{G}$ *is a* C^2 *vector field defined on an open set* W *in* $\mathbb{R}^3$, *then*
$$\mathit{div}\,\mathbf{F} = 0 \quad \text{on} \quad W.$$
2. *If* $\mathbf{F}$ *is a* C^1 *vector field defined on* $\mathbb{R}^3$ *satisfying* $\mathit{div}\,\mathbf{F} = 0$ *on* $\mathbb{R}^3$, *then there exists a* C^2 *vector filed* $\mathbf{G}$ *defined on* $\mathbb{R}^3$ *such that*
$$\mathbf{F} = \mathit{curl}\,\mathbf{G}, \quad \text{on} \quad \mathbb{R}^3.$$

If $\operatorname{div}\mathbf{F} = 0$ in $\mathbb{R}^3$, how to find $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}\mathbf{G}$?

Example Is $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector filed?
Check $\mathbf{F}$ is smooth vector field satisfying $\operatorname{div}\mathbf{F} = 0$ in $\mathbb{R}^3$. So there exists a smooth vector filed $\mathbf{G}$ such that $\mathbf{F} = \mathbf{G}$ in $\mathbb{R}^3$.

Example contd.

To find $\mathbf{G}$: Let us assume $\mathbf{G}(x, y, z) = G_1(x, y, z)\mathbf{i} + G_2(x, y, z)\mathbf{j}$ for all $(x, y, z) \in \mathbb{R}^3$. Then solve G_1 and G_2 in such a way that $\operatorname{curl} \mathbf{G} = \mathbf{F}$, i.e.,

$$\frac{\partial G_2}{\partial z}(x, y, z) = -F_1(x, y, z) = -x, \quad \frac{\partial G_1}{\partial z}(x, y, z) = F_2(x, y, z) = -2y,$$

$$\Big(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\Big)(x, y, z) = F_3(x, y, z) = z.$$

Now solving the equations, $G_2(x, y, z) = -xz + g(x, y)$ and $G_1(x, y) = -2yz + h(x, y)$. Using the 3rd equation,

$$-z + \partial_x g(x, y) + 2z - \partial_y h(x, y) = z.$$

It yields $\partial_x g(x, y) - \partial_y h(x, y) = 0$. Choosing, $g \equiv 0 \equiv h$, we get

$$\mathbf{G}(x, y, z) = -2yz\mathbf{i} - xz\mathbf{j}, \quad \text{in } \mathbb{R}^3.$$

Fundamental theorems

- Normal form of Green's theorem:

$$\int_{\partial R} \mathbf{F} \cdot \mathbf{n} \; ds = \iint_{R} \nabla \cdot \mathbf{F} \; dA$$

- Divergence theorem:

$$\iint_{\partial W} \mathbf{F} \cdot \mathbf{n} \; dS = \iiint_{W} \nabla \cdot \mathbf{F} \; dxdydz$$

- Tangential form of Green's theorem:

$$\int_{\partial R} \mathbf{F} \cdot d\mathbf{s} = \iint_{R} \nabla \times \mathbf{F} \cdot \; \mathbf{k} \; dA.$$

- Stokes' theorem:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_{S} \nabla \times \mathbf{F} \cdot \; \mathbf{n} \; dS.$$

Underlying principle: The integral of a differential operator acting on a vector field over a region equals to the sum of the appropriate vector field components over the boundary of the region.

Fundamental theorem of calculus revisited

Let f be a differentiable function on $[a, b]$, then the fundamental theorem of calculus says that

$$\int_a^d f'(t)\, dt = f(b) - f(a).$$

We can view Fundamental theorem of calculus as a special case of Green's theorem: Let $\mathbf{F} = f(x)\mathbf{i}$.

- Then $\nabla \cdot F = \frac{df}{dx}$.
- Take the normal vectors at b and a to be $\mathbf{i}$ and $-\mathbf{i}$, respectively.
- Then

$$\begin{aligned} f(b) - f(a) &= f(b)\mathbf{i} \cdot \mathbf{i} + f(a)\mathbf{i} \cdot (-\mathbf{i}) \\ &= \mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} \\ &= \text{Total flux of } \mathbf{F} \text{ across the boundary of } [a, b]. \end{aligned}$$

- Then Fundamental theorem of calculus says that

$$\int_{[a,b]} \nabla \cdot \mathbf{F}\, dx = \text{Total flux of } \mathbf{F} \text{ across the boundary of } [a, b].$$

Tutorial problems: Surface integrals

1. Find a suitable parametrization $\Phi(u, v)$ and the normal vector $\Phi_u \times \Phi_v$ for the following surface:
   (i) The plane $x - y + 2z + 4 = 0$.
   (ii) The right circular cylinder $y^2 + z^2 = a^2$.
2. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$ and $z = u + 2v$ at the point $(1, 1, 3)$.
3. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.
4. Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.
5. Let S denote the plane surface whose boundary is the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot \mathbf{n} \, dS$.

Examples

Example 2.Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$ and $z = u + 2v$ at the point $(1, 1, 3)$.

Ans $E = \mathbb{R}^2$ and $\mathbf{\Phi}(u, v) = (u^2, v^2, u + 2v)$, for all $(u, v) \in E$. Computing $\mathbf{\Phi}_u(u, v) = (2u, 0, 1)$ and $\mathbf{\Phi}_v(u, v) = (0, 2v, 2)$ and $(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u, v) = (-2u, -4v, 4uv)$. For $(x, y, z) = (1, 1, 3)$, i.e., solving $u^2 = 1$, $v^2 = 1$ and $u + 2v = 3$, we get the unique solution, $u = 1$, and $v = 1$. Thus $\mathbf{n}(u, v) = (-2, -4, 4)$ (can be normalized but not necessary to normalize it for obtaining the tangent plane).
Then tangent plane at $(x_0, y_0, z_0) = (1, 1, 3)$,

$$-2(x - x_0) - 4(y - y_0) + 4(z - z_0) = 0.$$

Examples

Example 3. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

Ans. The surface S is given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + (y - \frac{a}{2})^2 \leq \frac{a^2}{4}, \quad z^2 = a^2 - x^2 - y^2\}.$$

Note $\mathrm{Area}(S) = 2\mathrm{Area}(S_+)$, where S_+ is denoted by

$$S_+ = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + (y - \frac{a}{2})^2 \leq \frac{a^2}{4}, \quad z = \sqrt{a^2 - x^2 - y^2}\}.$$

Denoting $f(x, y) = \sqrt{a^2 - x^2 - y^2}$, for all $(x, y) \in E = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - \frac{a}{2})^2 \leq \frac{a^2}{4}\}$, S_+ can be written in the graph form by $\mathbf{\Phi}(x, y) = (x, y, f(x, y))$ for all $(x, y) \in E$. Calculating $f_x(x, y) = \frac{-x}{a^2 - x^2 - y^2}$ and $f_y(x, y) = \frac{-y}{a^2 - x^2 - y^2}$ for all $(x, y) \in E$,

$$\mathrm{Area}(S_+) = \int\int_{x^2 + (y - \frac{a}{2})^2 \leq \frac{a^2}{4}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dxdy.$$

Example 3 contd.

Using polar coordinate, Check

$$\mathrm{Area}(S_+) = \int_{\theta=0}^{\pi} \int_{r=0}^{a \sin\theta} \frac{a\, r}{\sqrt{a^2 - r^2}}\, dr d\theta = a^2 \pi.$$

Thus $\mathrm{Area}(S) = 2a^2\pi$.

Examples

Example 4. Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.

Ans. The surface given in the problem is

$$S = \{(x, f(x,z), z) \in \mathbb{R}^3 \mid 0 \leq x^2 + z^2 \leq 2a^2 \mid y = f(x,z) = \frac{x^2 + z^2}{2a}\}.$$

It can be written in the graph form by $\mathbf{\Phi}(x, y, z) = (x, f(x,z), z)$ for all $(x, y) \in E = \{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 \leq 2a^2\}$.
Calculating $f_x(x,z) = \frac{x}{a}$ and $f_z(x,z) = \frac{z}{a}$ for all $(x,z) \in E$, we have

$$\begin{aligned}
&\mathrm{Area}(S) = \int\int_E \sqrt{1 + f_x^2(x,z) + f_z^2(x,z)}\, dxdz \\
&= \int\int_{x^2+z^2\leq 2a^2} \sqrt{1 + \tfrac{x^2}{a^2} + \tfrac{z^2}{a^2}}\, dxdz \\
&= \tfrac{1}{a}\int\int_{x^2+z^2\leq 2a^2} \sqrt{a^2 + z^2 + x^2}\, dxdz.
\end{aligned}$$

Now using polar coordinate,

$$\mathrm{Area}(S) = \frac{1}{a}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}a} \sqrt{a^2 + r^2}\, r\, drd\theta = \frac{3^{\frac{3}{2}} - 1}{3} 2\pi a^2.$$

(check)

Examples

Example 5. Let S denote the plane surface whose boundary is the triangle with vertices at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and let $\mathbf{F}(x,y,z) \ = \ x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\displaystyle\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$.

Ans The plane enclosed the triangle is given by

$$S = \{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=1\}.$$

Denoting $f(x,y)=1-x-y$ for all $(x,y)\in E=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1, \quad 0\le y\le 1-x\}$, S can be written in graph form by $\mathbf{\Phi}(x,y)=(x,y,f(x,y))$ for all $(x,y)\in E$. Calculation $f_x(x,y)=-1$ and $f_y(x,y)=-1$ for all $(x,y)\in E$, $\mathbf{n}=\frac{1}{\sqrt{3}}(1,1,1)$ and $dS=\sqrt{1+f_x^2+f_y^2}dxdy=\sqrt{3}\,dxdy$. Thus

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS = \int\int_E x+y+(1-x-y)\,dxdy = \int_{x=0}^{1}\int_{y=0}^{1-x} 1dxdy = \frac{1}{2}.$$

(Check)

Tutorial Problems: Stokes theorem

1. Using Stokes Theorem, evaluate the line integral

$$\oint_C yz\,dx + xz\,dy + xy\,dz$$

where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

2. Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.

3. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate

$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$

4. Let $\boldsymbol{F}(x, y, z) := (y, -x, e^{xz})$ for $(x, y, z) \in \mathbb{R}^3$, and let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - \sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find $\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S}$.

Examples

Example 1: Using Stokes Theorem, evaluate the line integral

$$\oint_C yz\,dx + xz\,dy + xy\,dz$$

where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

Ans: To use Stokes theorem, consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 9y^2 \leq 9, \quad z = y^2 + 1\}.$$

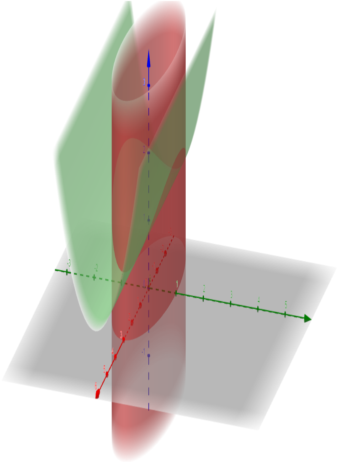
C is the boundary of S. For $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$,

$$\mathrm{curl}\ (\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = 0.$$

Thus the required line integral is

$$\iint_S \mathrm{curl}\ (\mathbf{F}) \cdot \mathbf{n} dS = 0.$$

ExerciseCheck the unit outward normal to S and if S is oriented positively.



Examples

Example 2: Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.

Ans To use Stokes' theorem for the given triangle C, consider S is the surface enclosed by the triangle C. Stokes' theorem says

$$\int_T \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the unit normal vector in the direction of positive orientation. The given triangle lies in the yz-plane. If the surface is to lie to the left of an observer walking around the triangle in the order described, the surface must be oriented so that the unit normal points in the direction of the positive x-axis. So $\mathbf{n} = \mathbf{i}$.
Calculating the curl of $\mathbf{F}$, we get

$$\nabla \times \mathbf{F} = -\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Hence,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -1,$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS = -1 \times A(S) = -2.$$

Examples

Example 3: Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate

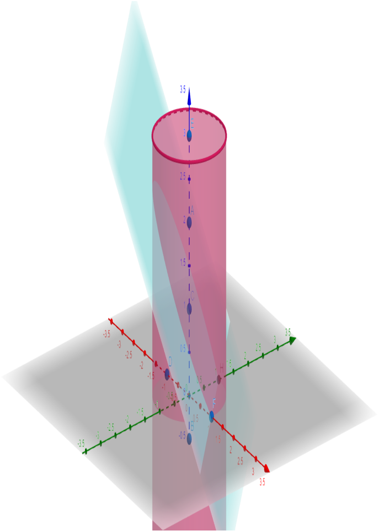
$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$

Ans: Use Stokes theorem.
Consider the surface give by the graph of $z = 1 - x - y$ over $x^2 + y^2 \leq 1$,

$$S = \{(x, y,) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \quad z = 1 - x - y\}.$$

S is enclosed by the curve C.
Check The unit normals to S is given by $\pm\frac{1}{\sqrt{3}}(1, 1, 1)$. To orient S positively so that we traverse C in the counterclockwise direction, we must choose

$$\mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1).$$



Examples

Example 4: Let $\boldsymbol{F}(x,y,z) := (y, -x, e^{xz})$ for $(x,y,z) \in \mathbb{R}^3$, and let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z-\sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find $\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S}$.

Ans. Use Stokes theorem to find $\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S}$. Note

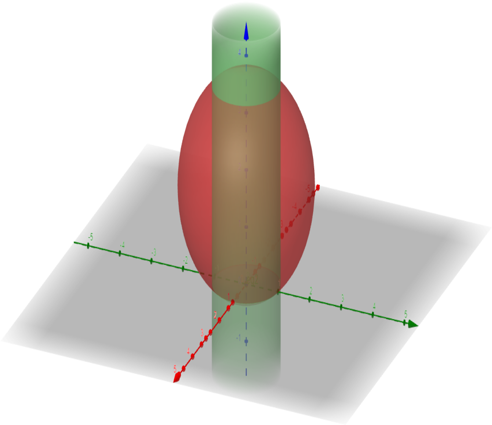
$$\partial S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}$$

is anticlockwise as seen from the point $(0,0,4)$.

∂S is parametrized by $\mathbf{c}(t) = (\cos t, \sin t, 0)$ for all $t \in [0, 2\pi]$ and hence the outward normal to the curve ∂S is $\mathbf{n}(t) = (-\sin t, \cos t, 0)$.

By the Stokes theorem,

$$\begin{aligned}
\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} &= \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} \\
&= \int_{-\pi}^{\pi} (\sin t, -\cos t, e^{\cos t \cdot 0}) \cdot (-\sin t, \cos t, 0) dt \\
&= \int_{-\pi}^{\pi} -(\sin^2 t + \cos^2 t) dt = -2\pi.
\end{aligned}$$



Tutorial problems: Divergence theorem

1. Calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.
2. Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the 'can' W given by $x^2 + y^2 \leq 1$, $-1 \leq z \leq 1$.
3. Evaluate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

   and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$.
4. Find out the flux of $F = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, $z = 1$.
5. Is $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector filed? If yes, find a vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}\mathbf{G}$ in $\mathbb{R}^3$.

Example 1 Calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

Solution: Again if we use Gauss's theorem we see that we need only evaluate

$$\iiint_W (\nabla \cdot \mathbf{F}) dV = \iiint_W 3(x^2 + y^2 + z^2) dxdydz,$$

where W is the unit ball.

This problem is clearly ideally suited to the use of spherical coordinates. Making a change of variables, we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin\phi \, d\rho d\phi d\theta = \frac{12\pi}{5}$$

Examples

Example 2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the 'can' W given by $x^2 + y^2 \leq 1$, $-1 \leq z \leq 1$.
Ans. Here div $\mathbf{F} = x^2 + y^2$. So, by Gauss's divergence theorem

$$\iiint_W \text{div } \mathbf{F} dV = \iiint_W (x^2 + y^2) dV$$

$$= \int_{-1}^{1} \left(\iint_{x^2+y^2\leq 1} (x^2 + y^2) dxdy \right) dz = 2 \iint_{x^2+y^2\leq 1} (x^2 + y^2) dxdy$$

$$= 2 \int_0^{2\pi} \left(\int_0^1 r^3 dr \right) d\theta = \pi.$$

Examples contd.

Example 3. Evaluate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$.

Ans. First compute the divergence

$$\operatorname{div} \mathbf{F} = 3y.$$

Note that the region E is given by

$$E = \{(x, y, z) : \ -1 \leq x \leq 1, \ 0 \leq z \leq 1 - x^2, \ 0 \leq y \leq 2 - z\}.$$

Since the surface is a closed surface, we use the Divergence Theorem and obtain

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_E \operatorname{div} \mathbf{F} \, dV = \int\int\int_E 3y \, dV$$
$$= 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y \, dy \, dz \, dx = 184/35.$$

Examples

Example 4. Find out the flux of $F = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, $z = 1$.

Ans. Let W be the interior of the cube and let S be the surface of the cube, that is, the boundary. Here $\nabla \times F = y + z + x$. Therefore,

$$\text{Flux} = \int\int_S F.\,d\mathbf{S} = \int\int_S F.\mathbf{n}dS = \int\int\int_W \nabla.F\,dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (x + y + z)\,dx\,dy\,dz = \frac{3}{2}.$$

Examples

Example 5. Is $\mathbf{F}(x,y,z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector filed?

Check $\mathbf{F}$ is smooth vector field satisfying $\operatorname{div}\mathbf{F} = 0$ in $\mathbb{R}^3$. So there exists a smooth vector filed $\mathbf{G}$ such that $\mathbf{F} = \mathbf{G}$ in $\mathbb{R}^3$.

To find $\mathbf{G}$: Let us assume $\mathbf{G}(x,y,z) = G_1(x,y,z)\mathbf{i} + G_2(x,y,z)\mathbf{j}$ for all $(x,y,z) \in \mathbb{R}^3$. Then solve G_1 and G_2 in such a way that $\operatorname{curl}\mathbf{G} = \mathbf{F}$, i.e.,

$$\frac{\partial G_2}{\partial z}(x,y,z) = -F_1(x,y,z) = -x, \quad \frac{\partial G_1}{\partial z}(x,y,z) = F_2(x,y,z) = -2y,$$

$$\Big(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\Big)(x,y,z) = F_3(x,y,z) = z.$$

Now solving the equations, $G_2(x,y,z) = -xz + g(x,y)$ and $G_1(x,y) = -2yz + h(x,y)$. Using the 3rd equation,

$$-z + \partial_x g(x,y) + 2z - \partial_y h(x,y) = z.$$

It yields $\partial_x g(x,y) - \partial_y h(x,y) = 0$. Choosing, $g \equiv 0 \equiv h$, we get

$$\mathbf{G}(x,y,z) = -2yz\mathbf{i} - xz\mathbf{j}, \quad \text{in}\, \mathbb{R}^3.$$