MA-111 Calculus II
(D3 & D4)

Lecture 7

B.K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

February 7, 2022

Change of variables

Spherical change of variables

Cylindrical change of variables

Change of variables in $\mathbb{R}^2$

Let Ω be an open subset of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}^2$ be an one-one transformation denoted by

$$h(u, v) := (h_1(u, v), h_2(u, v)), \quad \forall\, (u, v) \in \Omega.$$

We now want to make a general change of coordinates given by

$$x = h_1(u, v), \quad y = h_2(u, v).$$

What conditions do we need on h to be able to do a change of coordinates?

Can we compute the area of the image of a rectangle in the u-v plane?

Suppose we have a change in coordinates given by linear functions composed with translations (such functions are called affine linear functions):

$$x = au + bv + t_1 \quad \text{and} \quad y = cu + dv + t_2.$$

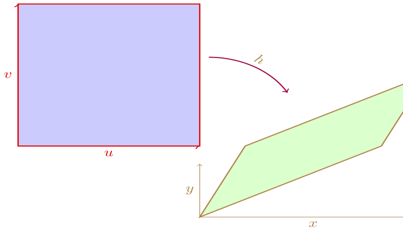
A linear change of coordinates

How does the area of the image of a rectangle under this map compare with the area of the original rectangle?

First, let us write down the affine map in a more compact notation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Clearly, a rectangle $[1, 0] \times [0, 1]$ in the u-v plane is mapped to a parallelogram in the x-y plane. The sides of the parallelogram are given by $(a + t_1, c + t_2)$ and $(b + t_1, d + t_2)$.

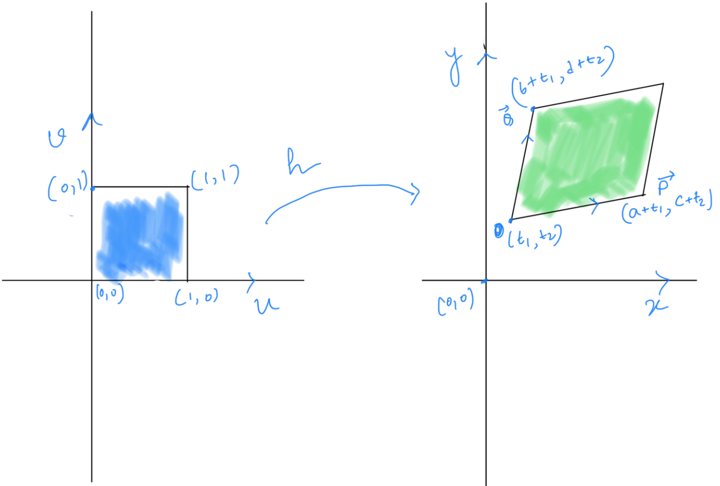


How does one compute the area of this parallelogram?

This is given by the absolute value of the cross product of the vectors,

$$(a, c, 0) \times (b, d, 0) = (ad - bc) \cdot \mathbf{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \mathbf{k}.$$

The area element for a change of coordinates



The area element for a change of coordinates

Let us now suppose that we have a general (not linear any more) change of coordinates given by $x = h_1(u, v)$ and $y = h_2(u, v)$.

How does the area of a rectangle in the u-v plane change? In order to compute the change we need to know the partial derivatives exist.

Let us assume h is a one-one continuously differentiable function .

Noting

$$\Delta x = h_1(u+\Delta u, v+\Delta v) - h_1(u, v), \quad \Delta y = h_2(u+\Delta u, v+\Delta v) - h_2(u, v),$$

and using Taylor's theorem for functions of two variables we see that

$$\Delta x \sim \frac{\partial h_1}{\partial u}\Delta u + \frac{\partial h_1}{\partial v}\Delta v$$

and

$$\Delta y \sim \frac{\partial h_2}{\partial u}\Delta u + \frac{\partial h_2}{\partial v}\Delta v.$$

Using our previous notation, we can write

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \begin{pmatrix}\frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v}\\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v}\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}$$

The Jacobian

You may recognize the matrix

$$J(h) = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

that appears in the preceding formula. The derivative matrix for the function $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is called the Jacobian.

In a neighborhood of the point (u_0, v_0), the function h and the function $J(h)$, behave very similarly (that is, they are the same upto the first order terms - use Taylor's theorem!). In fact, the derivative matrix is the *linear approximation* to the function h, at least in a neighborhood of a point, say (u_0, v_0).

In particular, it is easy to see how the area of a small rectangle changes under h, since we have already done so in the case of a linear map. It simply changes by the (absolute value of) determinant of J!

Theorem (Change of Variables Formula)

- *Let D be a closed and bounded subset of $\mathbb{R}^2$ such that ∂D has content zero. Let $f : D \to \mathbb{R}$ be continuous.*
- *Suppose Ω is an open subset of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}^2$ is a one-one differentiable function such that $h := (h_1, h_2)$, where h_1 and h_2 have continuous partial derivatives in Ω and $det(J(h)(u, v)) \neq 0$ for all $(u, v) \in \Omega$.*
- *Let $D^* \subset \Omega$ be such that $h(D^*) = D$.*

Then D^* is a closed and bounded subset of Ω, and ∂D^* is of content zero. Moreover, $f \circ h : D^* \to \mathbb{R}$ is continuous, and

$$\int\int_D f(x, y)\, dxdy = \int\int_{D^*} (f \circ h)(u, v)|det(J(h)(u, v))|\, dudv.$$

Notation

Often we write $x = x(u, v)$ and $y = y(u, v)$. In this case we use the notation $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$, for the Jacobian determinant.

Let D be a region in the xy plane and D^* a region in the uv plane such that $\phi(D^*) = D$. Then

$$\int\int_D f(x, y)dxdy = \int\int_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv.$$

Remark: Note what we get in the familiar case of polar coordinates: We have $x = r\cos\theta$, $y = r\sin\theta$ and

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r,$$

which is what we obtained previously.

How to choose the change of variables

- Aim: Find h such that a rectangle D^* in $u-v$ plane is getting mapped to the given area D in the xy plane. If D^* is not a rectangle, at least try to have it in the form of the elementary region Type 1 or Type 2.
- Presumably, the boundary D^* in $u-v$ plane should go to the boundary of D in $x-y$ plane.
- The non-vanishing Jacobian determinant of h assures that the properties of D^* is preserved under the transformation and D has similar properties as of D^*.
- In some cases, h can be chosen in a way such that the expression of the integrand becomes simpler after the change of variables.

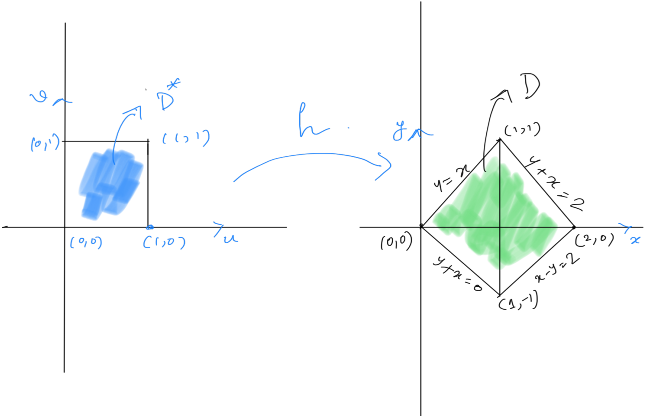
Example Evaluate the integral

$$\iint_D (x^2 - y^2)dxdy$$

where D is the square with vertices at $(0,0)$, $(1,-1)$, $(1,1)$ and $(2,0)$.

Solution: Note D is the region in $x-y$ plane bounded by lines $y=x$, $y+x=0$, $x-y=2$ and $y+x=2$.
Put

$$x = u + v, \quad y = u - v,$$



Then the rectangle

$$D^* = \left\{ (u, v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1, \;\; 0 \leq v \leq 1 \right\}$$

in the uv-plane gets mapped to D, in the xy-plane.
Further,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2.$$

$$\int \int_D (x^2 - y^2)\, dxdy = \int \int_{D^*} (4uv) \times 2\, dudv$$

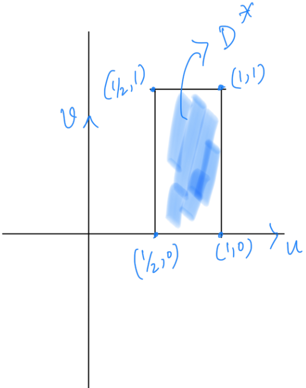
$$= 8 \left(\int_0^1 udu \right) \left(\int_0^1 vdv \right) = 2.$$

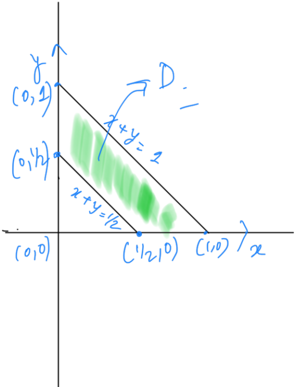
Example

Let D be the region in the first quadrant of the xy-plane bounded by the lines $x + y = \frac{1}{2}$ and $x + y = 1$. Find $\iint_D dA$ by transforming it to $\iint_{D^*} du dv$, where $u = x + y, v = \frac{y}{x+y}$.

Solution: Put

$$x = u(1 - v), \;\; y = uv.$$





Further,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix} = u \neq 0.$$

Hence,

$$\mathrm{Area}(D) = \int\int_D dA = \int\int_{D^*} |u| du dv$$

$$= \left(\int_{\frac{1}{2}}^{1} \frac{u^2}{2} du\right)\left(\int_0^1 dv\right) = \frac{3}{4}.$$

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

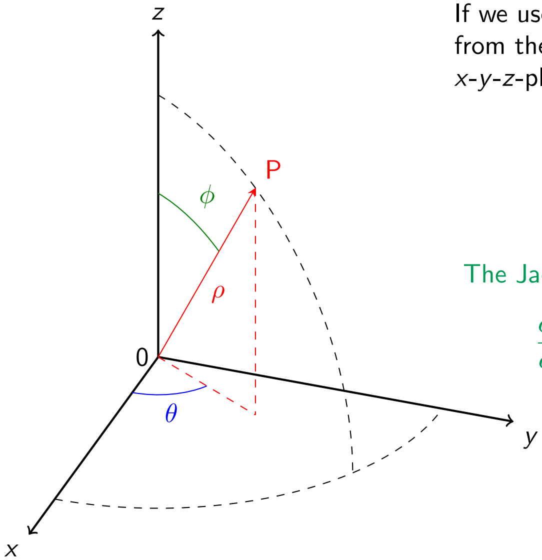
$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw,$$

where $h(P^*) = P$. If the change in coordinates is given by $h = (h_1, h_2, h_3)$, the function g is defined as $g = f(h_1, h_2, h_3)$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

is just the Jacobian determinant for a function of three variables.

Spherical Coordinates



If we use (ρ, θ, ϕ) what is the map from these coordinates to the x-y-z-planes?

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

The Jacobian determinant is

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} x = \rho^2 \sin\phi.$$

Example

Example: It should be much easier computing the volume of the unit sphere now. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\}$.

Then $W^* = \{(\rho, \theta, \phi) \in \mathbb{R}^3 \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$.
Then,

$$\begin{aligned}
\iiint_W dxdydz &= \iiint_{W^*} \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \frac{2\pi}{3} \int_0^{\pi} \sin\phi \; d\phi = \frac{4\pi}{3}
\end{aligned}$$

Cylindrical coordinates in formulae

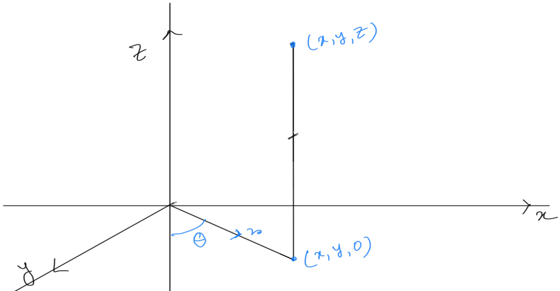
We can also consider a generalization of the polar coordinates. In this case, we use the change of transformation from (r, θ, z) coordinates to $P = (x, y, z) \in \mathbb{R}^3$ given by

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

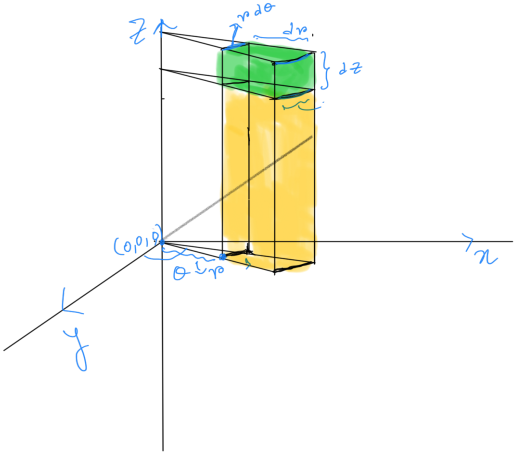
Here $r \geq 0$ and $0 \leq \theta \leq 2\pi$ and the (r, θ, z) are

It is very easy to see that

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = r.$$



The good thing about our convention is that θ means the same thing in both the cylindrical and spherical coordinate systems as well as in the (two-dimensional) polar coordinate system, and r means the same thing in both the cylindrical and (two-dimensional) polar coordinate systems.



Example

Evaluate $\int\int\int_W z^2(x^2+y^2)\,dxdydz$, where W is the cylindrical region determined by $x^2+y^2 \leq 1$ and $-1 \leq z \leq 1$.

Solution. The region W is described in cylindrical coordinates as W^*

$$W^* = \{(r,\theta,z) \mid 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad -1 \leq z \leq 1\}.$$

$$\int\int\int_W z^2(x^2+y^2)\,dxdydz = \int_{z=-1}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} z^2 r^2\, r\, drd\theta dz$$

$$= \int_{-1}^{1} \frac{2\pi}{4} z^2\,dz = \frac{\pi}{3}.$$