Tutorial Sheet 9

Numerical ODEs

1. Write the formula using the Euler's method for approximating the solution to the initial value problem
$$y' = x^2 + y^2, \quad y(x_0) = y_0$$
at the point $x = x_1$ with $h = x_0 - x_1 > 0$. Find the approximate value y_1 of the solution to this initial value problem at the point x_1 when $x_0 = 0$, $y_0 = 1$, and $h = 0.25$. Find an upper bound for the truncation error in obtaining y_1.

2. Using the Simpson's quadrature rule, derive a formula (called *Simpson method*) for obtaining approximate solution to the initial value problem
$$y' = f(x, y), \quad y(x_0) = y_0$$
at the nodes x_j, for $j = 1, 2, \ldots$, when the nodes are equally spaced with $h = x_{j+1} - x_j > 0,\ j \in \mathbb{Z}.$. Is the resulting method an explicit method or an implicit method? Justify your answer.
Using the Simpson method derived above, obtain the nonlinear equation whose solution is the approximate value of the solution $y(0.1)$ of the initial value problem
$$y' = e^y, \quad y(0) = 1,$$
when $h = 0.05$.

3. Consider the initial value problem
$$y' = x^2 + y^2; \quad y(0) = 1.$$
Take $h = 0.05$ and do the following:
   i) Using the appropriate Euler's method (backward or forward), find an approximate value of $y(-0.05)$.
   ii) Using Euler mid-point method, find an approximate value of $y(0.1)$.

4. Consider the IVP
$$\begin{aligned} y' &= \lambda y, \ \text{on} \ (0, \infty) \\ y(0) &= 1. \end{aligned}$$
where $\lambda < 0$.
   i) Let the sequence $\{y_j\}$ of approximate values of the solution to the above IVP at the points $\{x_j\}$, where $x_j = jh$, for some $h > 0$ and for all $j = 1, 2, \ldots$, be generated using the forward Euler's method. Find the condition on $h > 0$ such that $y_j \to 0$ as $j \to \infty$.

ii) If the sequence $\{y_j\}$ is generated using the Euler's trapezoidal rule, then show that $y_j \to 0$ as $j \to \infty$ for any choice of $h > 0$. In other words, show that Euler's trapezoidal method is *absolutely stable*.

5. i) State Runge-Kutta method of order 2.

ii) Let y be the solution of the initial value problem

$$y' = x^2 + y^2, \quad y(0) = 1.$$

Find the value of $y(0.1)$ using Runge-Kutta method of order 2 with $h = 0.05$.