ME 202
SPRING 2023
09 JAN 2023

Previously,

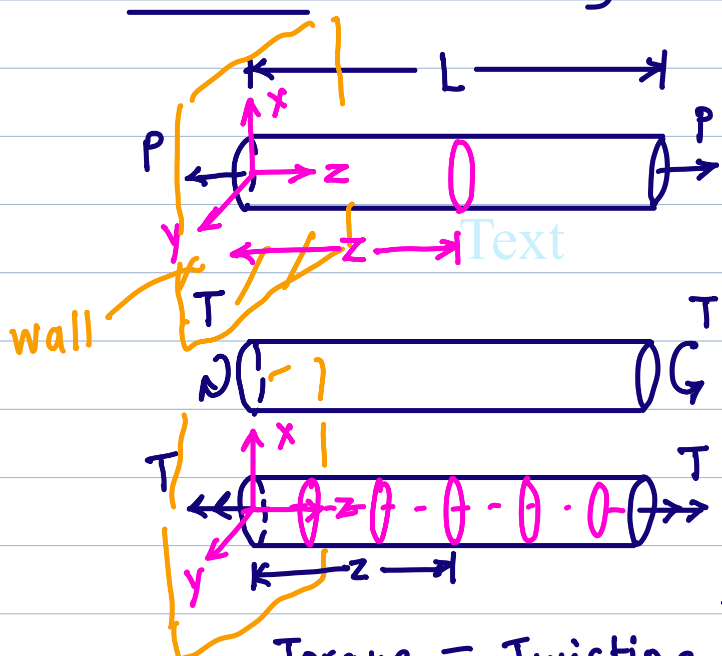
- ☐ Equilibrium of stresses/tractions
- ☐ Kinematics / Strain-Displacement Relationship
- ☐ Stress-strain relationship/Hooke's Law
- ☐ Boundary conditions

Now

- ☐ Axial deformation review
- ☐ Torsion

TORSION

Twisting



$\underline{P} = P\,\underline{e}_z$

Axial Bar

Extension

Torsion

Shaft

Torsion

$\underline{T} = T\,\underline{e}_z$

Torque $\equiv$ Twisting Moment

$$\underline{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\hat{i}$ $\hat{j}$ $\hat{k}$

Review of Axial Extension

$$\left.\begin{array}{l} u \\ v \\ w \end{array}\right\} \text{disp. along} \quad \begin{array}{l} \underline{e}_x, \ x \text{ axis} \\ \underline{e}_y, \ y \text{ axis} \\ \underline{e}_z, \ z \text{ axis} \end{array}$$

Assumption $\quad u = 0$

Kinematics $\quad v = 0$

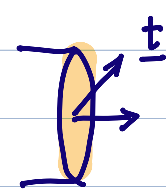
$w = c_1 z$ $\qquad$ Ansatz

Strain $\quad \epsilon_{zz} = \dfrac{dw}{dz} = c_1$

Hooke's Law

Stress $\quad \sigma_{zz} = E c_1$

At $z = L$,



$$\underline{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{t} = \begin{pmatrix} 0 \\ 0 \\ P/A \end{pmatrix} \quad \text{applied traction}$$

From equilibrium at $z = L$,

resultant of tractions $\quad \displaystyle\int_{\Omega} \underline{t}\, da = \underline{P}$

Recall,

$$\underline{t} = \underset{\sim}{\sigma}\,\underline{n} = \begin{pmatrix} 0 \\ 0 \\ \sigma_{zz} \end{pmatrix} = \sigma_{zz}\,\underline{e}_z$$

$$\underset{\sim}{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} \leftarrow \text{uniaxial loading}$$

$$\int_{\Omega} \sigma_{zz}\, da\, \underline{e}_z = P\,\underline{e}_z$$

$$\int_{\Omega} \sigma_{zz}\, da = P$$

$$\int_{\Omega} E c_1\, da = P$$

$$E c_1 A = P \qquad A \text{ area c/s}$$

$$c_1 = \frac{P}{AE}$$

$$w(z) = \frac{Pz}{AE}, \qquad \delta = w(L) = \frac{PL}{AE}$$

$$\sigma_{zz} = \frac{P}{A}$$

Kinematic Assumptions

$\downarrow$

Strains $\xrightarrow{\text{Hooke's Law}}$ Stresses $\xrightarrow{\text{Equilibrium}}$ Ext. Force

Same logic for torsion.

P.S.

check static equilibrium equation in 1D

$$\frac{d}{dz}\sigma_{zz} + b = 0 \quad \text{along } z \text{ direction}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma} + \underset{\sim}{b} = \underset{\sim}{0}$$

$$\frac{d}{dz}(Ec_1) + 0 = 0 \; \checkmark$$

This is the 1D approx solution to the 3D problem.

Theory of Failure

If brittle, $\sigma_{max} = \sigma^*$ max normal/principal stress

If ductile, $\tau_{max} = \tau^*$ max shear stress

Say ductile (metal) $\tau_{max} = \frac{\sigma_{zz}}{2} = \underbrace{\frac{P}{2A}}_{\text{from analysis}} = \underbrace{\tau^*}_{\text{from expts}}$

Max load that can be applied $P^* = 2A\tau^*$

Note: $\tau_{max} = \max\left\{\frac{|\sigma_1-\sigma_2|}{2}, \frac{|\sigma_2-\sigma_3|}{2}, \frac{|\sigma_3-\sigma_1|}{2}\right\}$

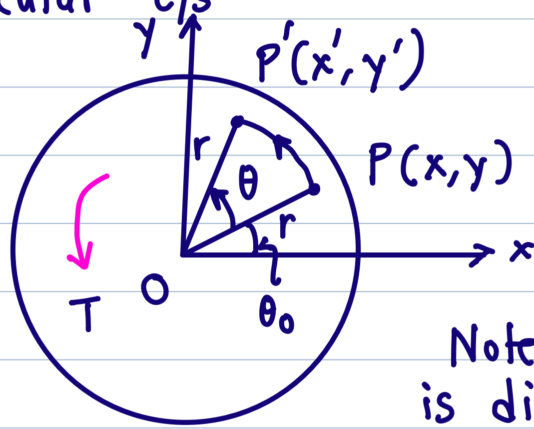
where $\sigma_1, \sigma_2, \sigma_3$ are three principal stresses (eigenvalues) of $\underset{\sim}{\sigma}$

Questions

1. How much does the shaft deflect?
   Angular deflection
2. What is the max torque that can applied?

Kinematic Assumptions

1. Circular c/s



Rigid Body Rotation for each c/s

Note: $\theta = \alpha z$ is diff for every c/s

c/s of shaft at some z coordinate

2. Let α unit angle of twist for the entire shaft $\qquad \alpha \equiv$ rad/m

$\theta(z) =$ angle of twist for c/s at z
$= \alpha z$

Total angle of twist $= \theta(L) = \alpha L$

3. Each c/s twists as if it undergoes a rigid body rotation about z-axis through angle $\theta(z)$.

$$x = r\cos\theta_0, \quad y = r\sin\theta_0$$
$$x' = r\cos(\theta+\theta_0), \quad y' = r\sin(\theta+\theta_0)$$

$$u = x' - x = \cancel{r\cos\theta\cos\theta_0} - r\sin\theta\sin\theta_0 - \cancel{r\cos\theta_0}$$
$$v = y' - y = r\sin\theta\cos\theta_0 + \cancel{r\cos\theta\sin\theta_0} - \cancel{r\sin\theta_0}$$
$w = 0$ assume, no warping

3. small angular deflections

θ small, $\cos\theta \approx 1$, $\sin\theta \simeq \theta$

$$u = -r\sin\theta_0 . \theta = -y\alpha z = -\alpha yz$$
$$v = r\cos\theta_0 . \theta = +x.\alpha z = +\alpha xz$$
$$w = 0$$

Strains from displacements

2D → 3D extend

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = 0, \quad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = 0 \;, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(-\alpha y + 0\right) = -\frac{\alpha y}{2}$$

$$\epsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\alpha x + 0\right) = \frac{\alpha x}{2}$$

strain at any point in shaft

$$\underset{\sim}{\epsilon} = \begin{pmatrix} 0 & 0 & -\alpha y/2 \\ 0 & 0 & \alpha x/2 \\ -\frac{\alpha y}{2} & \frac{\alpha x}{2} & 0 \end{pmatrix}$$

Get stresses from Hooke's Law

Isotropic. $\sigma_{xx} = 0, \; \sigma_{yy} = 0, \; \sigma_{zz} = 0$

$$\sigma_{xy} = 2G\epsilon_{xy} = 0 \qquad \mu = G = \frac{E}{2(1+\nu)}$$

$$\sigma_{xz} = 2G\epsilon_{xz} = -\cancel{2}G\frac{\alpha y}{\cancel{2}} = -G\alpha y$$

$$\sigma_{yz} = 2G\epsilon_{yz} = 2G\frac{\alpha x}{2} = G\alpha x$$

Stress / Stress tensor at any point in the shaft

$$\underset{\sim}{\sigma} = \begin{pmatrix} 0 & 0 & -G\alpha y \\ 0 & 0 & +G\alpha x \\ -G\alpha y & +G\alpha x & 0 \end{pmatrix}$$

Note: α unknown as of now.

Note: Hooke's Law for isotropic body in 3D

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}$$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{zz}$$

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{2G} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\epsilon_{yz} = \frac{\sigma_{yz}}{2G} = \frac{1+\nu}{E}\sigma_{yz}$$

$$\epsilon_{zx} = \frac{\sigma_{zx}}{2G} = \frac{1+\nu}{E}\sigma_{zx}$$

Note: We use tensor shear strains in this course and not engineering shear strains

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\gamma_{xy}}{2}$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{\gamma_{yz}}{2}$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{\gamma_{zx}}{2}$$