UNCERTAINTY ANALYSIS FOR ENGINEERS

STATIC AND DYNAMIC CHARACTERISTICS OF INSTRUMENTS

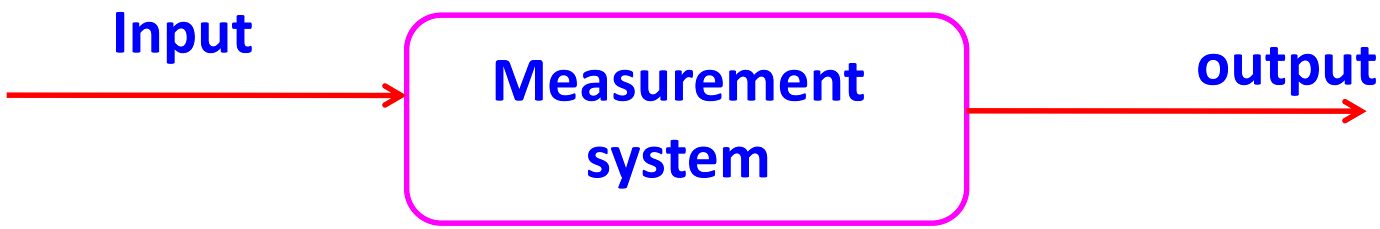
PERFORMANCE CHARACTERISTICS OF INSTRUMENTS

- Static Characteristics
- Dynamic Characteristics

STATIC CALIBRATION

- All inputs except one kept at some constant value
- Input under study varied over some range
- Output under study varies over some other range of constant values
- Calibration standard should be atleast 10 times as accurate as the instrument being calibrated

Measurement system



Thermocouple put in a constant temperature bath

Input – Temperature

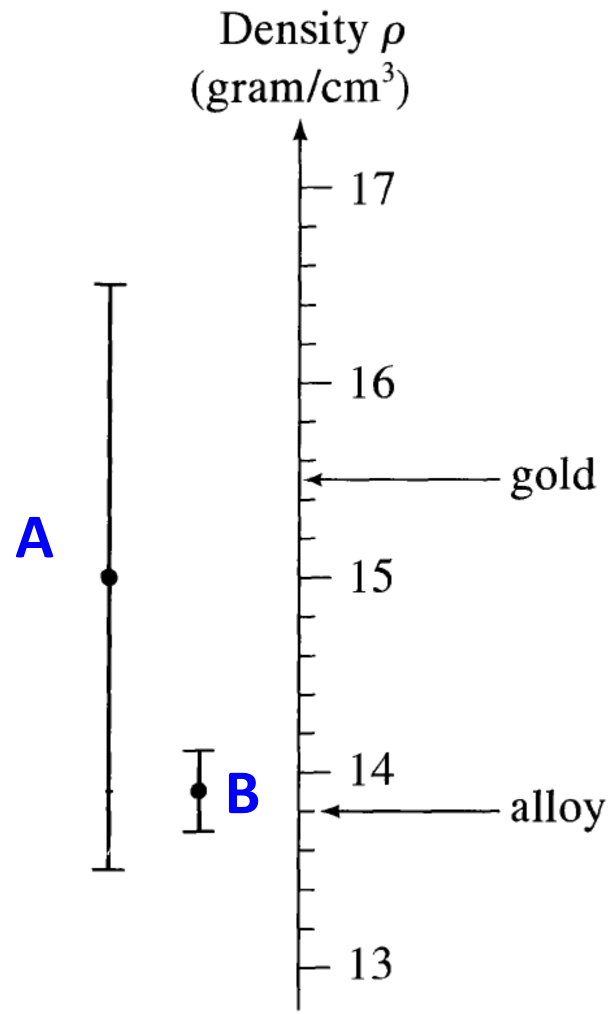
Output – Millivolts

Constant temperature bath – temperature is constant

Static Characteristics – temperature is constant

Dynamic characteristics – temperature is varying

IMPORTANCE OF KNOWING THE UNCERTAINTY



Density of crown

ρ= 15.5 gm/cm^3 (Gold)

ρ = 13.8 gm/cm^3 (alloy)

Error band of B is less than that of A – Believe B

Conclusion – Crown is made of alloy not gold

		Best Estimate	Range
Experimenter A	**15.0**	**13.5-16.5**	**15 ± 1.5**
Experimenter B	**13.9**	**13.7-14.1**	**13.9 ± 0.2**

Uncertainty tag is essential for every measured parameter

Error = True value – Measured value

Very difficult to quantify error because truth is always elusive

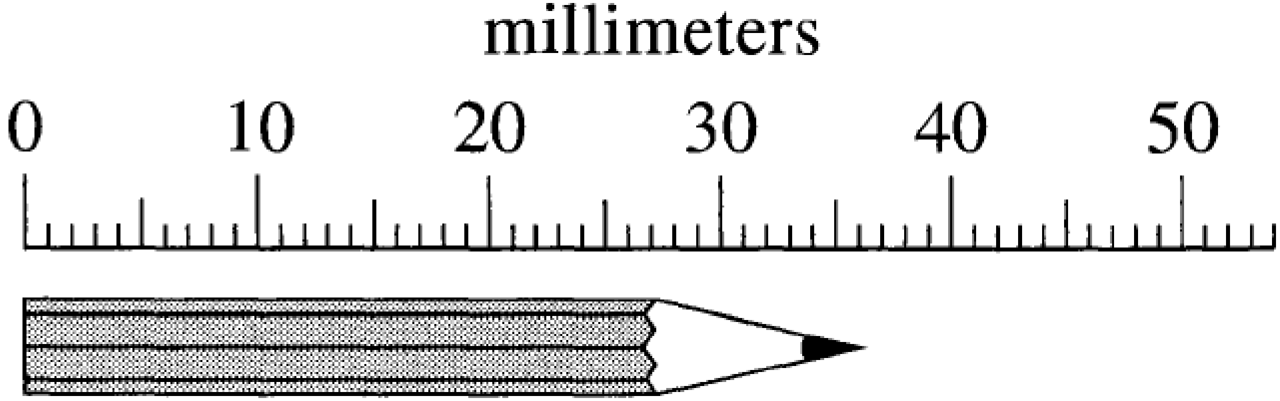
Hence, we try to quantify UNCERTAINTY

ERROR (UNCERTAINTY)

SYSTEMATIC ERROR (BIAS)

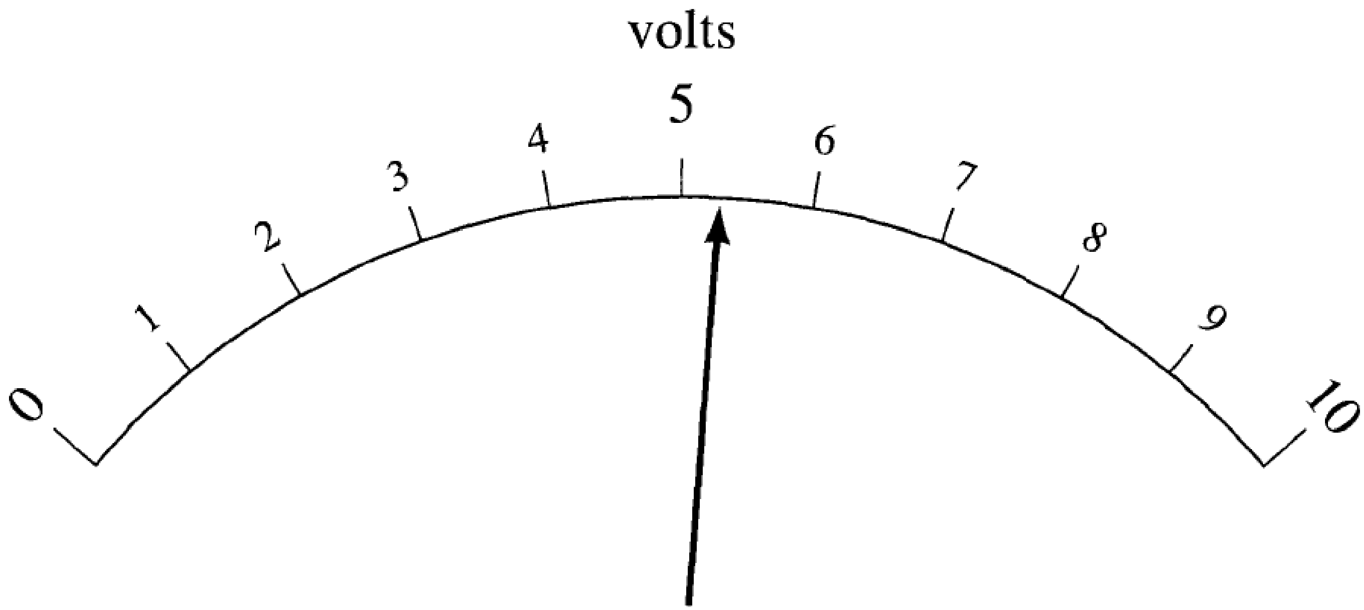
RANDOM ERROR (PRECISION)

MEASURED VALUE = TRUE VALUE + BIAS + PRECISION



Best estimate of length = 36 mm

Probable range = 35.5 to 36.5 mm



Best estimate of voltage = 5.3 Volts

Probable range = 5.2 to 5.4 Volts

Actual – measured = bias this is easily removed by calibration

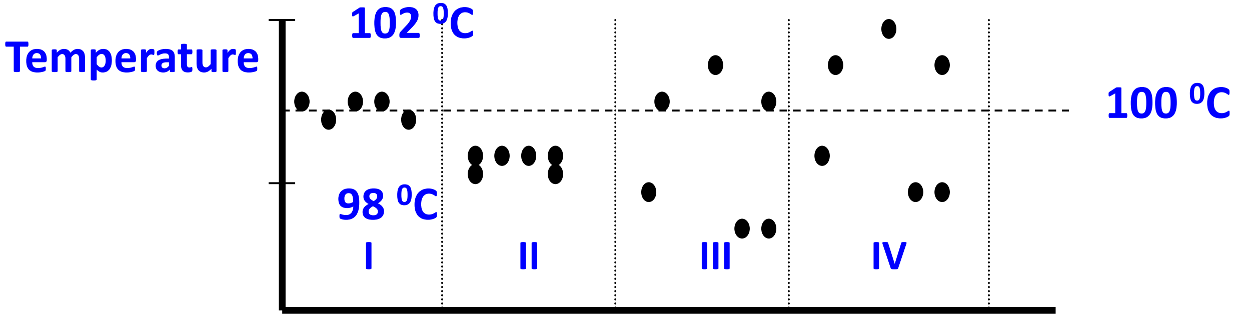
Calibration – act of applying a known value of input to measurement system to observe the output

A calibrated instrument has only Precision

Accuracy → Combination of bias + precision

PRECISION

- **Ability of the system to indicate a particular value upon repeated but independent applications of a specific value of input**
- **Measure of random variation to be expected during repeatability trials**



I → PRECISE + NO BIAS: ACCURATE

II → PRECISE + BIAS : INACCURATE

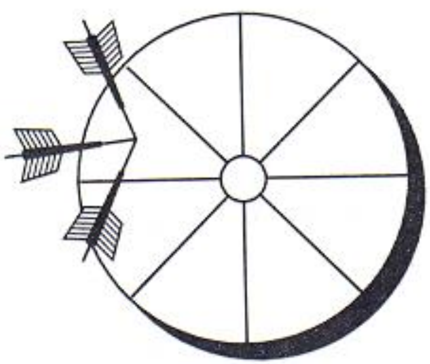
III → IMPRECISE + BIAS : INACCURATE

IV → IMPRECISE + NO BIAS : ACCURATE

Instrument – accuracy ± 1% of full scale

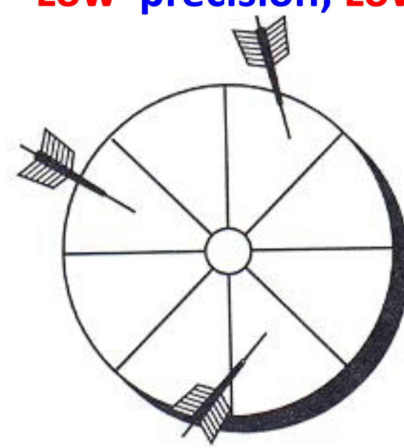
- Usually is the largest error
- No 3σ limits have been established
- A reading at 10% of full scale error is 10%

High precision;Low accuracy

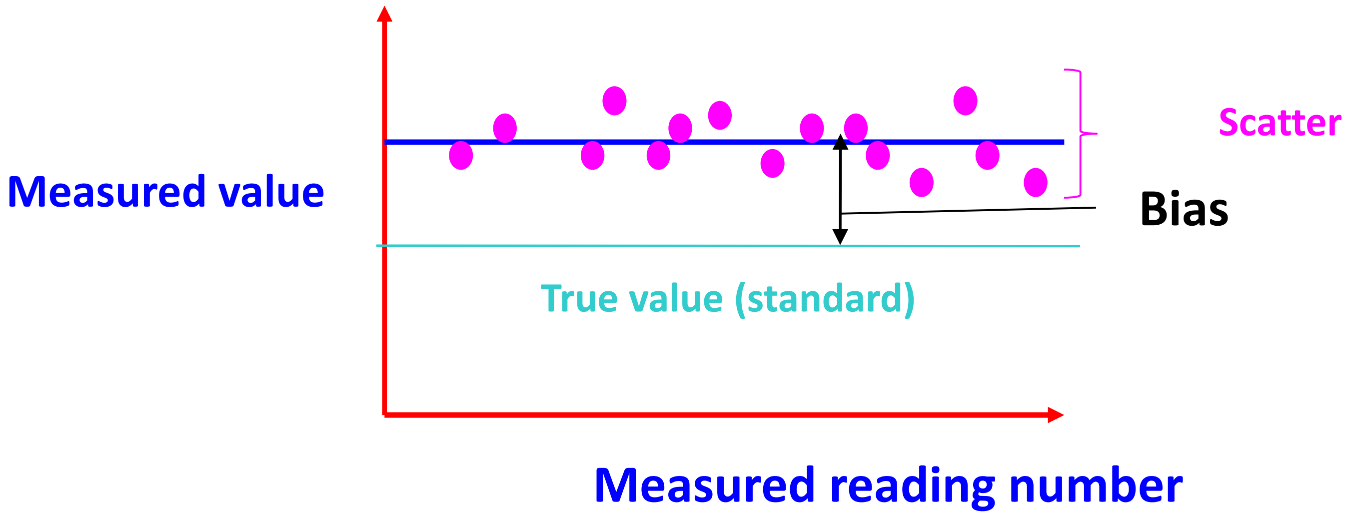


High precision; High accuracy

Low precision; Low accuracy



Pictorial depiction of Bias and precision



A measured value is a random variable with mean μ and standard deviation σ.

- The bias in the measuring process is the difference between the mean measurement and the true value:

$$\text{Bias} = \mu - \text{True value}$$

- The uncertainty in the measuring process is the standard deviation *σ.*

- The smaller the bias, the more accurate the measuring process.

- The smaller the uncertainty, the more precise the measuring process.

IILUSTRATION OF IMPORTANCE OF ACCURATE EXPERIMENTAL RESULTS

Why are experiments necessary?

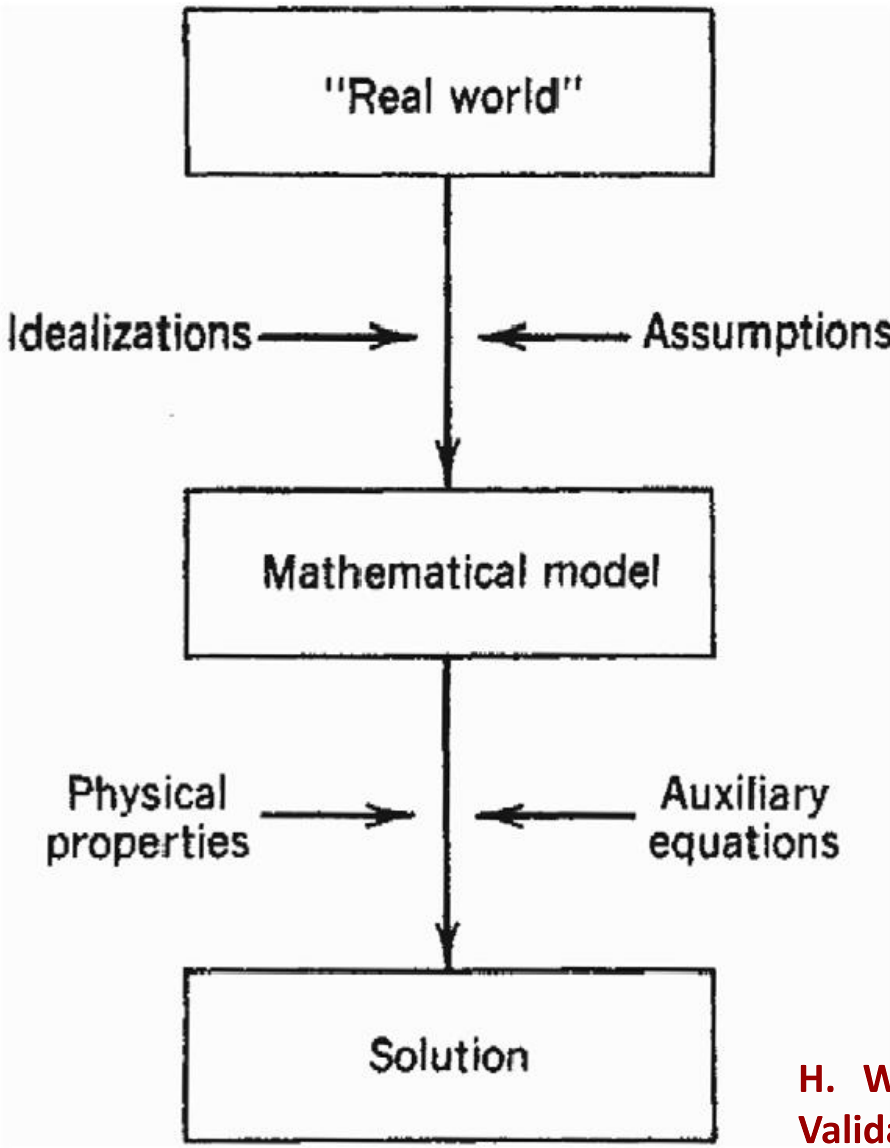
Why do we need to study the subject of experimentation?

The experiments run in science and engineering courses demonstrate physical principles and processes, but once these demonstrations are made and their lessons taken to heart, why bother with experiments?

With the laws of physics we know, with the sophisticated analytical solution methods we study, with the increasing knowledge of numerical solution techniques, and with the awesome computing power available, is there any longer a need for experimentation in the real world?

The scientific method is the systematic attempt to construct theories that correlate wide groups of observed facts and are capable of predicting the results of future observations. Such theories are tested by controlled experimentation and are accepted only so long as they are consistent with all observed facts.

ANALYTICAL APPROACH TO A SOLUTION OF A PROBLEM



MEASUREMENT OF BENDING OF LIGHT AS IT PASSES THROUGH THE SUN

Einsteen – Theory of relativity (1916) Predicted that a light from a start would be bent through an angle of 1.8” as it passes through the sun

Refined Classical theory – predicted a bending angle of 0.9”

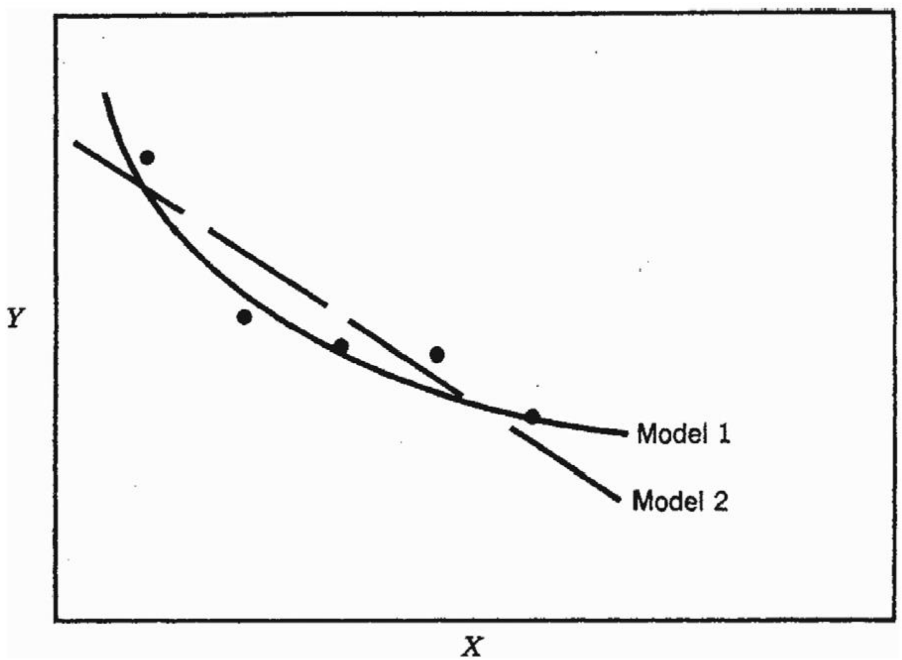
1919, Dyson, Eddington and Davidson measured the bending of light by the sun during solar eclipse.

α= 2” ± 0.3”

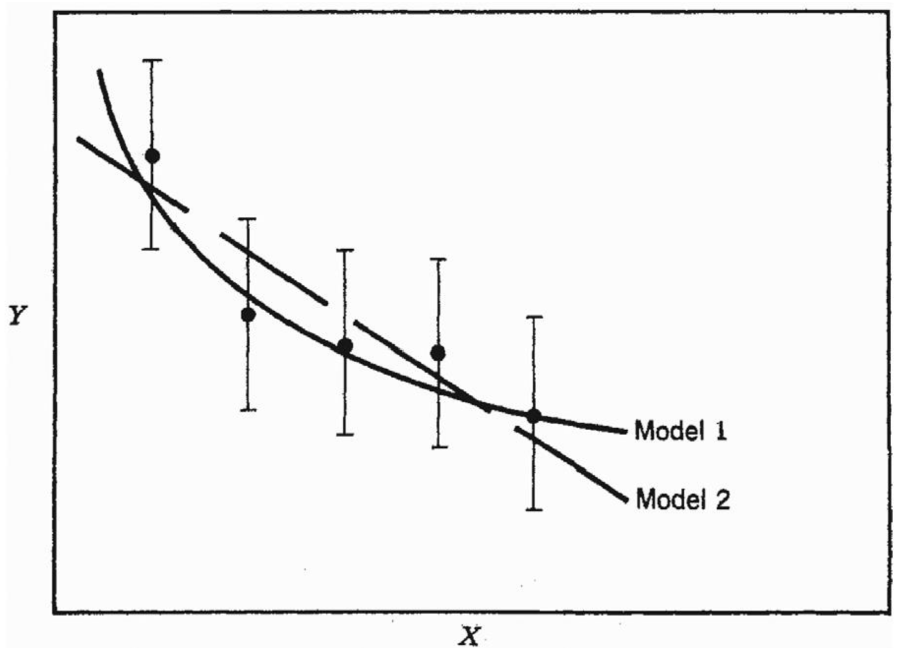
Einsteen’s theory of relativity was accepted.

H. W. Coleman and W. G. Steele, Experimentation, Validation, And Uncertainty Analysis For Engineers, A John Wiley & Sons, Third edition, 2000.

Comparison of model results with experimental data without with consideration of uncertainty in Y



Comparison of model results with experimental data with consideration of uncertainty in Y



Data uncertainty effectively sets the resolution at which such comparisons can be made

H. W. Coleman and W. G. Steele, Experimentation, Validation, And Uncertainty Analysis For Engineers, A John Wiley & Sons, Third edition, 2000.

HOW TO REPORT AND USE UNCERTAINTIES

BEST ESTIMATE ± UNCERTAINTY

Ex: Best estimate of time = 2.4 s

Probable range = 2.3 to 2.5 s

Measured value of time = 2.4 ± 0.1 s

$$\text{Measured Value of } x = x_{best} \pm \delta x$$

SIGNIFICANT FIGURES

The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty

Ex:

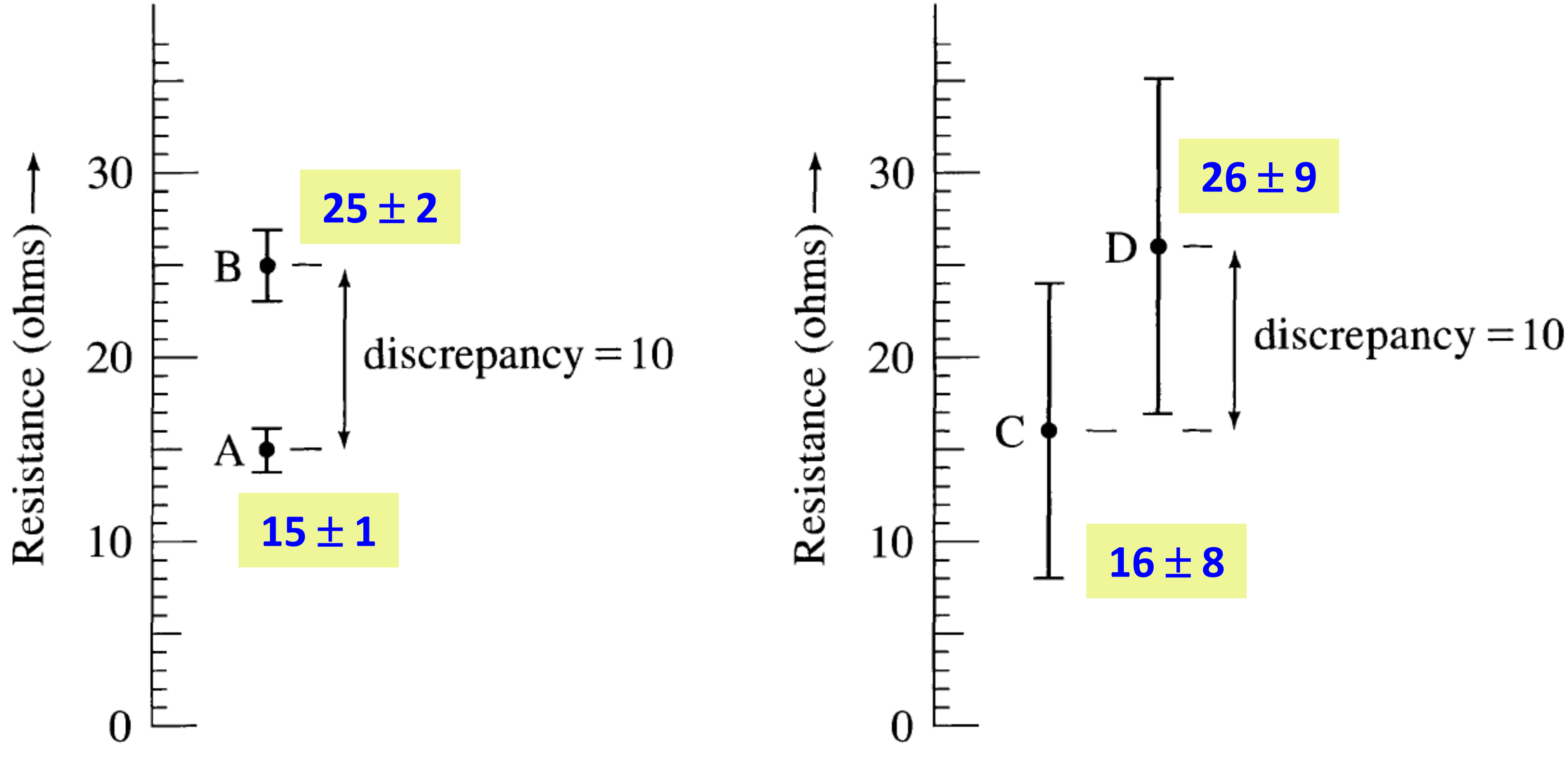
92.81 with an uncertainty of 0.3 is written as 92.8 ± 0.3

92.81 with an uncertainty of 3 is written as 93 ± 3

92.81 with an uncertainty of 30 is written as 90 ± 30

To reduce the inaccuracies caused by rounding, *any numbers to be used in subsequent calculations should normally retain at least one significant figure more than is finally justified.*

DISCREPANCY is the difference between two measured values of the same quantity

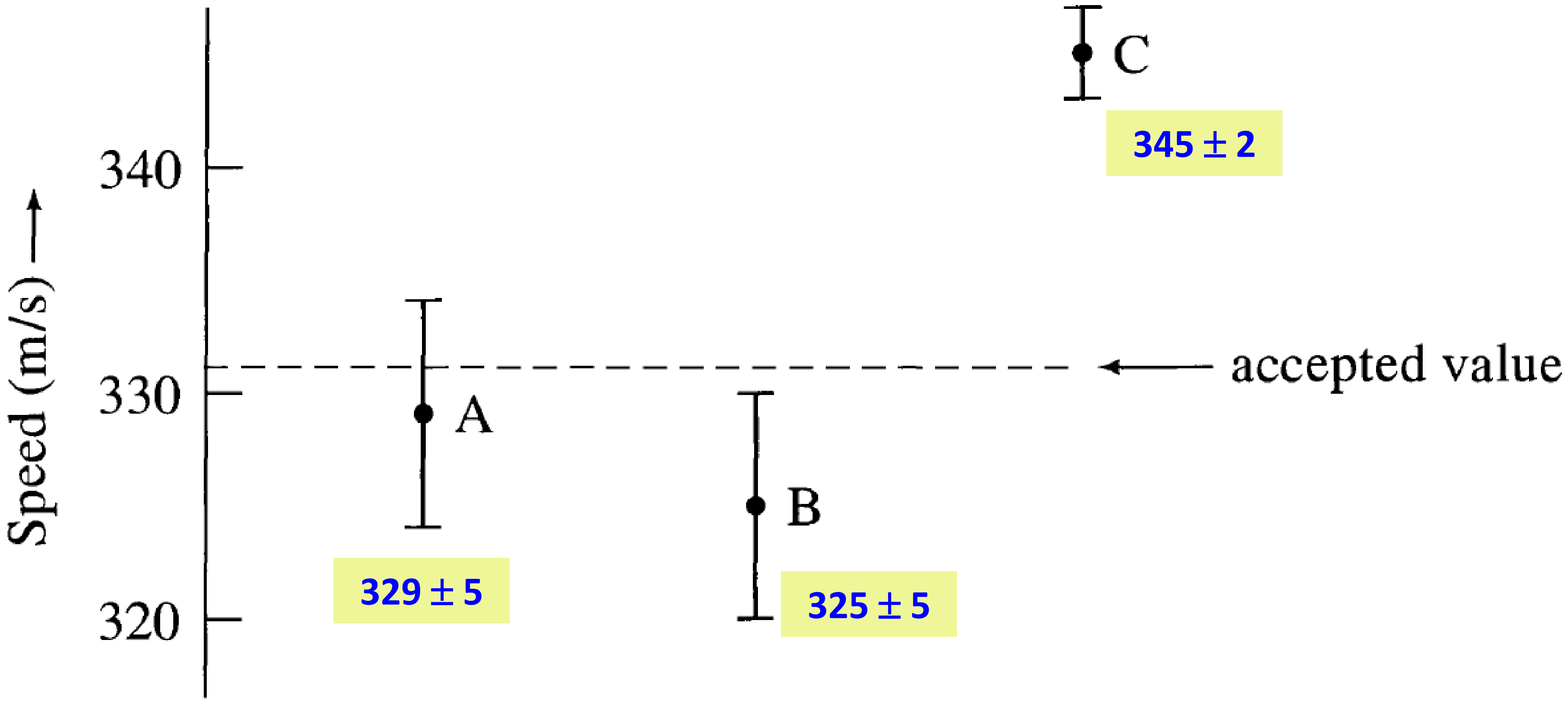


(a) (b)

Two measurements are of same resistance
Discrepancy (10 Ω) > combined uncertainty (3 Ω)
One of the experimenters made a mistake

Two measurements are of same resistance
Stated margins of error overlap
Measurements are accurate, but imprecise.

COMPARISON OF MEASURED AND ACCEPTED VALUE



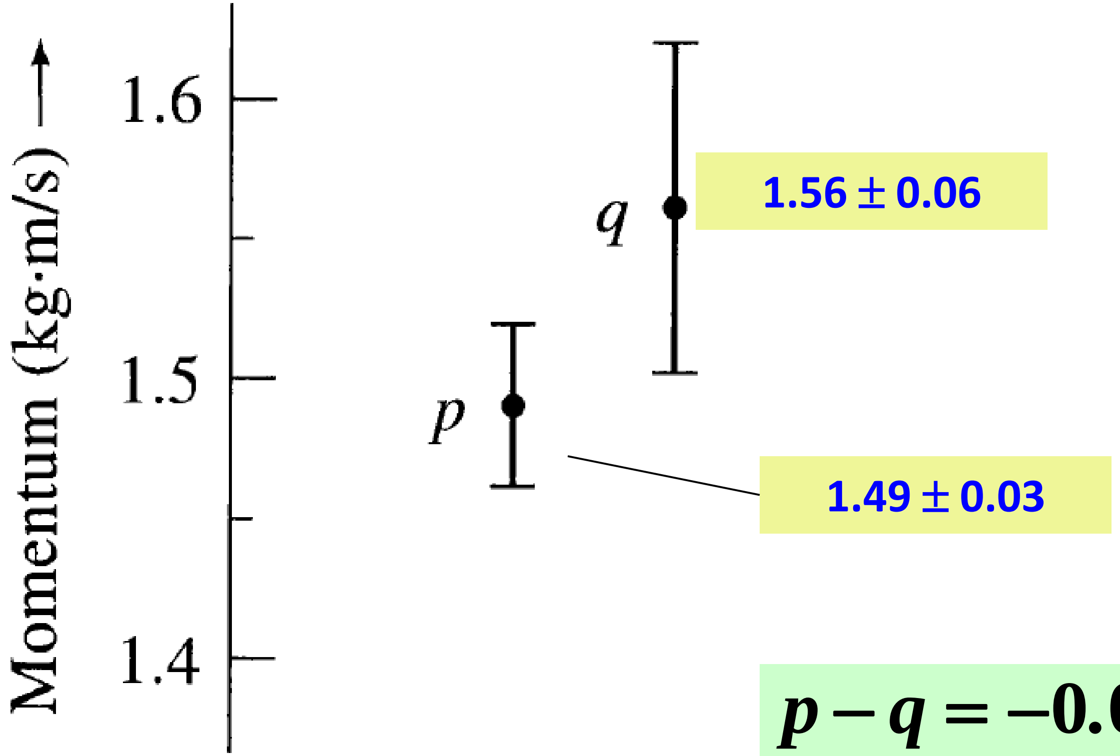
Three measurements of the speed of sound at standard temperature and pressure.

Because the accepted value (331 m/s) is within student A's margins of error, his result is satisfactory

The accepted value is just outside Student B's margin of error, but his measurement is nevertheless acceptable.

The accepted value is far outside Student C's stated margins, and his measurement is definitely unsatisfactory

COMPARISON OF TWO MEASURED NUMBERS



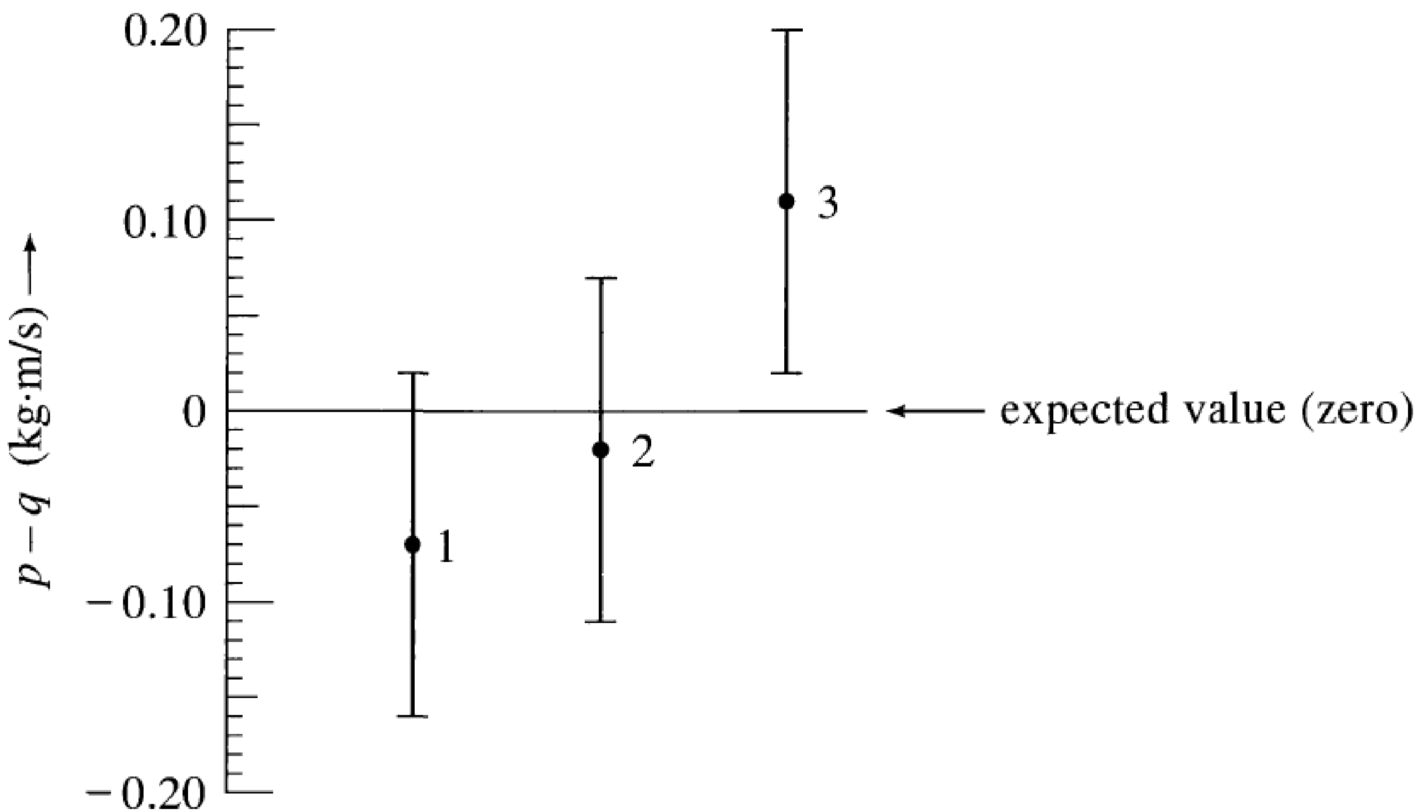
$$p - q = -0.07 \pm 0.09$$

The law of conservation of momentum states that the total momentum of an isolated system is constant.

To test it, a series of experiments with two carts that collide as they move along a frictionless track are conducted.

Because the margins of error for p and q overlap, these measurements are certainly consistent with conservation of momentum (which implies that p and q should be equal)

COMPARISON OF TWO MEASURED NUMBERS



Trial 1 and Trial 2 are within the margins of error

Trial 3 is only slightly outside in trial 3

The results are consistent with the conservation of momentum.

PROPOGATION OF UNCERTAINTIES

SUMS AND DIFFERENCES

$$q = x + y$$

The highest probable value of $x + y$ is $x_{best} + y_{best} + (\delta x + \delta y)$

The lowest probable value of $x + y$ is $x_{best} + y_{best} - (\delta x + \delta y)$

Thus, the best estimate for $q = x + y$ is

$$q_{best} = x_{best} + y_{best}$$

Uncertainty is

$$\boxed{\delta q \approx \delta x + \delta y}$$

$$q = x - y$$

The highest probable value of $x + y$ is $x_{best} - y_{best} + (\delta x + \delta y)$

The lowest probable value of $x + y$ is $x_{best} - y_{best} - (\delta x + \delta y)$

Thus, the best estimate for $q = x - y$ is

$$q_{best} = x_{best} - y_{best}$$

Uncertainty is

$$\delta q \approx \delta x + \delta y$$

$$\delta q \approx \sqrt{(\delta x)^2 + (\delta y)^2}$$

Suppose an experimenter mixes together the liquids in two flasks having first measured their separate masses when full and empty, as follows:

M_1 = mass of first flask and contents = 540 ± 10 gm

m_1 = mass of first empty flask = 72 ± 1 gm

M_2 = mass of second flask and contents = 940 ± 20 gm

m_2 = mass of second empty flask = 97 ± 1 gm

$$M = M_1 - m_1 + M_2 - m_2 = 540 - 72 + 940 - 97 = 1311\ grams$$

$$\delta M \approx \delta M_1 + \delta m_1 + \delta M_2 + \delta m_2 = 10 + 1 + 20 + 1 = 32\ grams$$

$$\boxed{Total\ \ Mass\ (rounded) = 1310 \pm 32\ grams}$$

Smaller uncertainties in the masses of the empty flasks made a negligible contribution to the final uncertainty

PRODUCTS AND QUOTIENTS

$$q = xy$$

Measured value of x is $x_{best} \pm \delta x$

Fractional uncertainty in x is $\frac{\delta x}{|x_{best}|} = \frac{\delta x}{|x|}$

We omit best because it makes it look clumsy

Value of x = $x_{best}\left(1 \pm \frac{\delta x}{|x|}\right)$

$$\text{Value of } q = q_{best}\left(1 \pm \frac{\delta q}{|q|}\right) = \frac{x_{best}}{y_{best}} \frac{1 \pm \frac{\delta x}{|x|}}{1 \pm \frac{\delta y}{|y|}}$$

$$Largest\ Value\ of\ \ q = \frac{x_{best}}{y_{best}} \frac{1+\frac{\delta x}{|x|}}{1-\frac{\delta y}{|y|}} = \frac{x_{best}}{y_{best}} \frac{1+a}{1-b}$$

$$\frac{1}{1-b} \approx 1+b$$

$$\frac{1}{1-b} = 1+b+b^2+b^3+\ldots\ldots.$$

$$\frac{1+a}{1-b} = (1+a)(1+b) = 1+a+b+ab \approx 1+a+b$$

$$Largest\ Value\ of\ \ q = \frac{x_{best}}{y_{best}} \frac{1+\frac{\delta x}{|x|}}{1-\frac{\delta y}{|y|}} \approx \frac{x_{best}}{y_{best}} \left(1+\frac{\delta x}{|x|}+\frac{\delta y}{|y|}\right)$$

$$Value\ of\ \ q = q_{best}\left(1 \pm \frac{\delta q}{|q|}\right)$$

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}$$

In surveying, sometimes a value can be found for an inaccessible length l (such as the height of the tall tree) by measuring three other lengths l_1, l_2, l_3 in terms of which

$$l = \frac{l_1 l_2}{l_3}$$

$$l_1 = 200 \pm 2,\ \ l_2 = 5.5 \pm 0.1,\ l_3 = 10.0 \pm 0.4$$

$$l_{best} = \frac{200 \times 5.5}{10} = 110\ \ ft$$

$$\frac{\delta l}{l} \approx \frac{\delta l_1}{l_1} + \frac{\delta l_2}{l_2} + \frac{\delta l_3}{l_3} = \frac{2}{200} \times 100 + \frac{0.1}{5.5} \times 100 + \frac{0.4}{10} \times 100 = 1\% + 2\% + 4\% = 7\%$$

$$l = 110 \pm \frac{7}{100} \times 110 = 110 \pm 7.7 = 110 \pm 8$$

$$\boxed{l = 110 \pm 8}$$

MEASURED QUANTITY TIMES THE EXACT NUMBER

$$q = Bx$$

If the quantity x is measured with uncertainty δx and is used to compute $q = Bx$, where B has no uncertainty, then the uncertainty in q is just absolute value of B times that in x

$$\delta q = |B|\delta x$$

Measure the thickness of 200 sheets of paper and get the answer

$$T = 33 \pm 1\ mm$$

Thickness of one sheet of paper

$$t = \frac{1}{200} \times T = \frac{1}{200} \times 33 = 0.165\ mm$$

$$t = 0.165 \pm 0.005\ mm$$

Uncertainty in power

If the quantity x is measured with uncertainty δx and the measured values is used to compute the power $q = x^n$, then the fractional uncertainty in q is n times that in x

$$\boxed{\frac{\delta q}{|q|} = n\frac{\delta x}{|x|}}$$

Suppose a student measures acceleration due to gravity, by measuring the time *t* for a stone to fall from a height *h* above the ground. After making several timings, she concludes that t = 1.6 ± 0.1s and h = 12.6 ± 0.3 m

$$h = \frac{1}{2}gt^2 \Rightarrow g = \frac{2h}{t^2} = \frac{2\times 12.6}{1.6^2} = 9.8\ m/s^2$$

$$\frac{\delta h}{h} = \frac{0.3}{12.6}\times 100 = 2.4\% \qquad \frac{\delta t}{t} = \frac{0.1}{1.6}\times 100 = 6.3\%$$

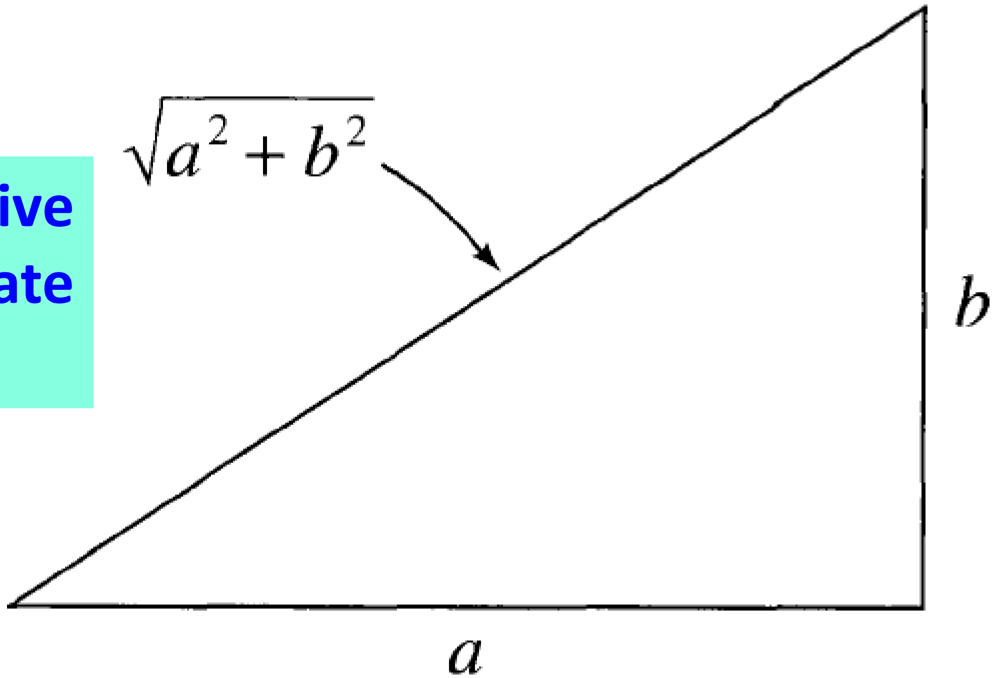
$$\frac{\delta g}{g} = \frac{\delta h}{h} + 2\frac{\delta t}{t} = 2.4 + 2\times 6.3 = 15\% \Rightarrow \delta g = \frac{15}{100}\times 9.8 = 1.5$$

$$\boxed{g = 9.8 \pm 1.5\ m/s^2}$$

$$q = x + y$$

$$\delta q \approx \delta x + \delta y$$

Uncertainties need not be additive always, as a conservative estimate quadrature relation is used.



Because any side of a triangle is less than the sum of the other two sides, the inequality $\sqrt{a^2 + b^2} < a + b$ **is always true**

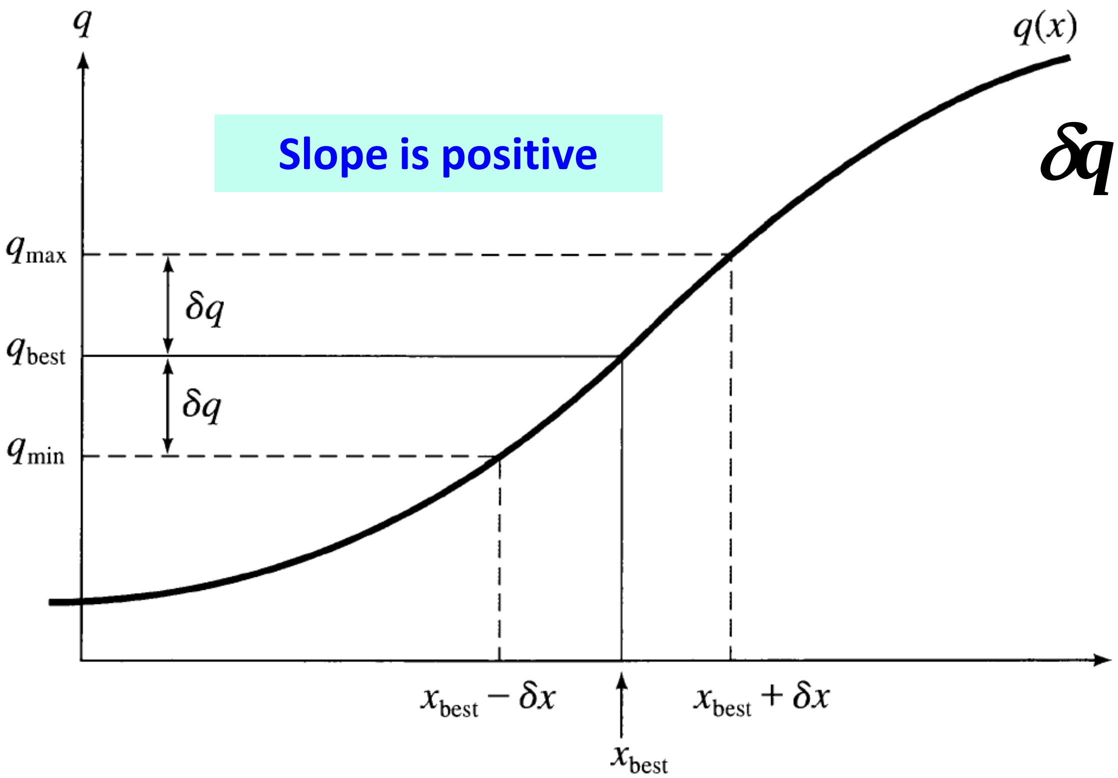
$$\delta q \approx \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$q = xy$$

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}$$

$$\frac{\delta q}{|q|} \approx \sqrt{\left(\frac{\delta x}{|x|}\right)^2 + \left(\frac{\delta y}{|y|}\right)^2}$$

ARBITRARY FUNCTIONS OF ONE VARIABLE



$$\delta q = q\left(x_{best} + \delta x\right) - q\left(x_{best}\right)$$

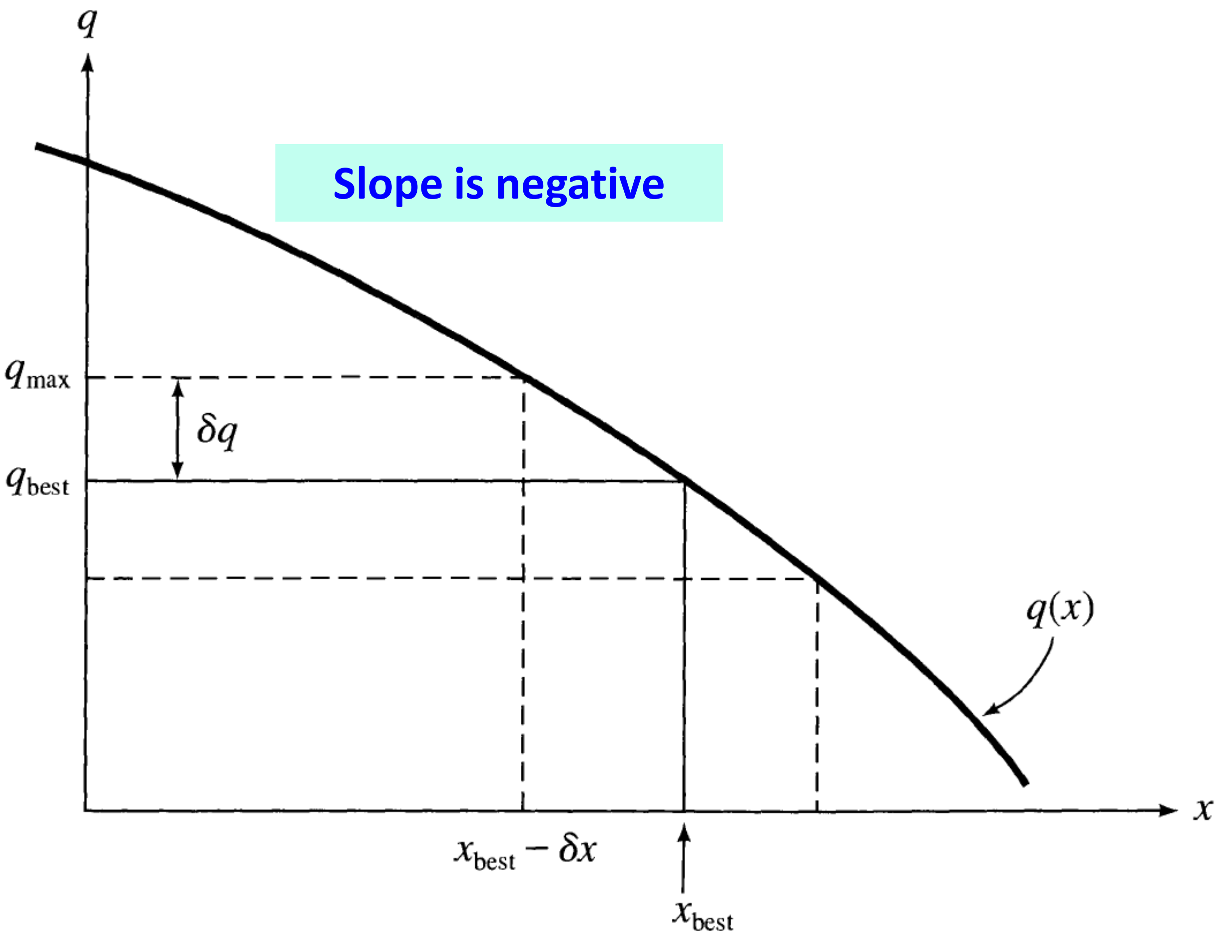
$$\delta q = \frac{dq}{dx}\delta x$$

To find uncertainty, calculate derivate dq/dx and multiply by uncertainty δx

If x is measured as $x_{best} \pm \delta x$, then the best estimate for $q(x)$ is $q_{best} = q(x_{best})$.

The largest and smallest probable values of q(x) correspond to the values $x_{best} \pm \delta x$ of x

J.R. Taylor, An Introduction to Error Analysis, University Science Books, California, Second, Third edition, 1997.



$$\delta q = -\frac{dq}{dx}\delta x$$

Uncertainty in any function of one variable

If x is measured with uncertainty δx and is used to calculate the function $q(x)$, then the uncertainty δq is

$$\boxed{\delta q = \left|\frac{dq}{dx}\right|\delta x}$$

Uncertainty in the measured angle θ = 20 ± 3. Find uncertainty in cosθ

$$\boxed{\delta q = \left|\frac{dq}{dx}\right|\delta x}$$

$$\delta(\cos\theta) = \left|\frac{d(\cos\theta)}{d\theta}\right|\delta\theta$$

$$\delta(\cos\theta) = |\sin\theta|\delta\theta \ (in\ rad)$$

$$\delta(\cos\theta) = |\sin 20|\ 3\times\frac{\pi}{180}\ (in\ rad) = 0.02\ rad$$

$$\cos\theta = 0.94 \pm 0.02$$

Uncertainty in power

If the quantity x is measured with uncertainty δx and the measured values is used to compute the power $q = x^n$, then the fractional uncertainty in q is n times that in x

$$\boxed{\frac{\delta q}{|q|} = n\frac{\delta x}{|x|}}$$

$$\boxed{\delta q = \left|\frac{dq}{dx}\right|\delta x}$$

$$\delta\left(x^n\right) = \left|\frac{d\left(x^n\right)}{dx}\right|\delta x = nx^{n-1}\delta x$$

$$\frac{\delta\left(x^n\right)}{x^n} = \left|\frac{nx^{n-1}}{x^n}\right|\delta x = n\frac{\delta x}{|x|}$$

Uncertainty in a function of several variables

$$q = q(x, y, z)$$

$$q(x+\delta x, y+\delta y, z+\delta z) = q(x, y, z) + \delta q = q(x, y, z) + \frac{\partial q}{\partial x}\delta x + \frac{\partial q}{\partial y}\delta y + \frac{\partial q}{\partial z}\delta z$$

$$\delta q = \frac{\partial q}{\partial x}\delta x + \frac{\partial q}{\partial y}\delta y + \frac{\partial q}{\partial z}\delta z$$

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2 + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

In any case, it is never larger than the ordinary sum

$$\delta q = \frac{\partial q}{\partial x}\delta x + \frac{\partial q}{\partial y}\delta y + \frac{\partial q}{\partial z}\delta z$$

$$U = X_1^{m_1} X_2^{m_2}$$

$$\frac{\sigma_U}{U} = \sqrt{\left(m_1 \frac{\sigma_{X_1}}{X_1}\right)^2 + \left(m_2 \frac{\sigma_{X_2}}{X_2}\right)^2}$$

Measurement of g with simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$l = 92.95\ cm \pm 0.1\ cm$$

$$T = 1.936 \pm 0.004\ sec$$

$$g = \frac{4\pi^2 l}{T^2} \qquad g = \frac{4\pi^2(92.95)}{(0.004)^2} = 979$$

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2 + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

$$\frac{\delta g}{g} = \sqrt{\left(\frac{\partial}{\partial l}\left(\frac{4\pi^2 l}{T^2}\right)\frac{1}{g}\delta l\right)^2 + \left(\frac{\partial}{\partial T}\left(\frac{4\pi^2 l}{T^2}\right)\frac{1}{g}\delta T\right)^2}$$

$$\frac{\delta g}{g} = \sqrt{\left(\frac{4\pi^2}{T^2}\frac{T^2}{4\pi^2 l}\delta l\right)^2 + \left(-\frac{2}{T^3}\frac{T^2}{4\pi^2 l}\delta T\right)^2}$$

$$\frac{\delta g}{g} = \sqrt{\left(\frac{\delta l}{l}\right)^2 + 4\left(\frac{\delta T}{T}\right)^2} = \sqrt{\left(\frac{0.1}{92.95}\right)^2 + 4\left(\frac{0.004}{1.936}\right)^2} = 0.43\%$$

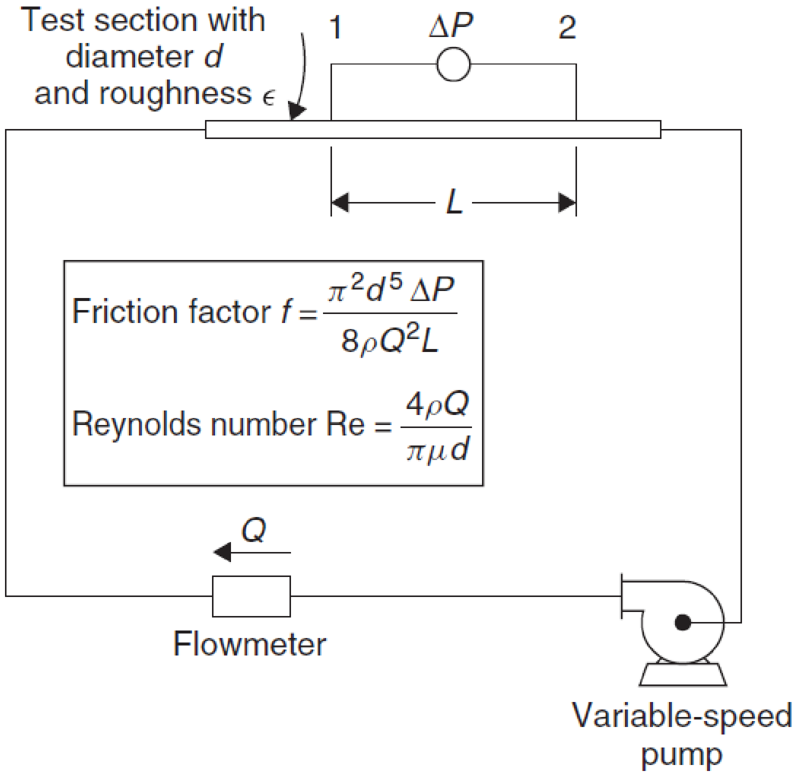
$$\delta g = \frac{0.43}{100} \times 979 = 4.18 \approx 4$$

$$g = 979 \pm 4\ cm/s^2$$

Uncertainty in the measurement of friction factor

$$f = \frac{\pi^2 d^5 (\Delta P)}{8 \rho Q^2 L}$$

$$\mathrm{Re} = \frac{4 \rho Q}{\pi \mu d}$$



- d, the pipe diameter
- ϵ, the roughness "size"
- L, the distance between pressure taps
- ΔP, the directly measured pressure drop over distance L
- Q, the volumetric flow rate of the fluid
- ρ, the density of the fluid
- μ, the dynamic viscosity of the fluid

H. W. Coleman and W. G. Steele, Experimentation, Validation, And Uncertainty Analysis For Engineers, A John Wiley & Sons, Third edition, 2000.

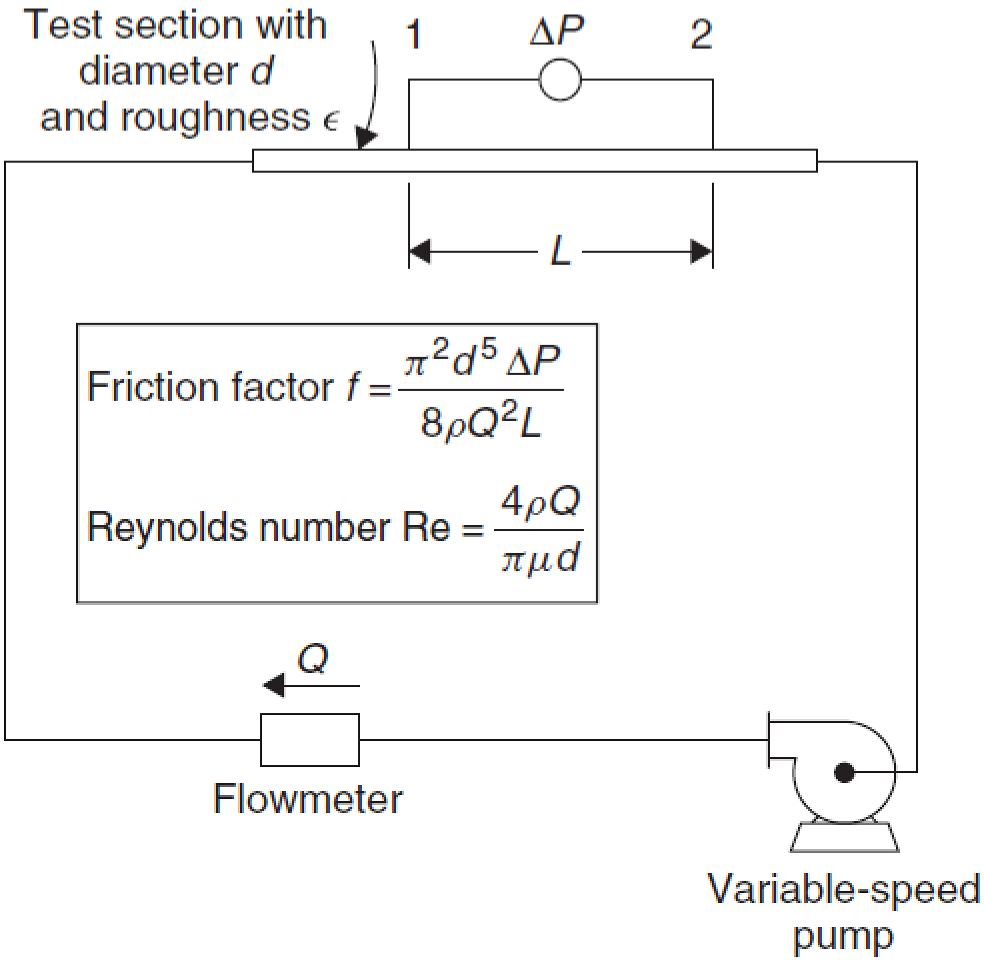
$$\delta d = 1\ mm$$

$$\delta(\Delta P) = 0.1\% \ \ of \ \ FSR = \frac{0.1}{100}\times 6.25\times 1000 = 6.25\ Pa$$

$$\delta\rho \approx zero$$

$$\delta\dot{Q} = 0.0001 m^3/s$$

$$\delta L = 3\ mm$$



$d = 0.05$ m
$\Delta P = 80$ Pa
$\rho = 1000$ kg/m^3
$Q = 0.003$ m^3/s
$L = 0.20$ m
$f = 0.0171$

$$\frac{\delta f}{f} = \sqrt{\left(5\frac{\delta d}{d}\right)^2 + \left(\frac{\delta(\Delta P)}{\Delta P}\right)^2 + \left(\frac{\delta\rho}{\rho}\right)^2 + \left(2\frac{\delta\dot{Q}}{\dot{Q}}\right)^2}$$

H. W. Coleman and W. G. Steele, Experimentation, Validation, And Uncertainty Analysis For Engineers, A John Wiley & Sons, Third edition, 2000.

$$\delta d = 0.1\ mm$$

$$\delta(\Delta P) = 0.1\%\ \ of\ \ FSR = \frac{0.1}{100}\times 6.25\times 1000 = 6.25\ Pa$$

$$\delta\rho \approx zero$$

$$\delta\dot{Q} = 0.0001 m^3/s$$

$$\delta L = 3\ mm$$

$$d = 0.05 m$$

$$\Delta P = 800\ Pa$$

$$\delta\rho \approx zero$$

$$\delta\dot{Q} = 0.00001 m^3/s$$

$$\delta L = 1\ mm$$

$$f = \frac{\pi^2 d^5 (\Delta P)}{8\rho Q^2 L}$$

$$\frac{\delta f}{f} = \sqrt{\left(5\frac{\delta d}{d}\right)^2 + \left(\frac{\delta(\Delta P)}{\Delta P}\right)^2 + \left(\frac{\delta\rho}{\rho}\right)^2 + \left(2\frac{\delta\dot{Q}}{\dot{Q}}\right)^2 + \left(\frac{\delta L}{L}\right)^2}$$

$$\frac{\delta f}{f} = \sqrt{\left(5\frac{0.1\times 10^{-3}}{0.05}\right)^2 + \left(\frac{6.25}{800}\right)^2 + (0)^2 + \left(2\frac{0.00001}{0.003}\right)^2 + \left(\frac{1\times 10^{-3}}{0.2}\right)^2}$$

$$\frac{\delta f}{f} = \sqrt{\left(\frac{1}{100}\right)^2 + (0.078)^2 + (0)^2 + (0.0066)^2 + (0.005)^2}$$

$$\frac{\delta f}{f} = 7.9\times 10^{-2}\ \ or\ \ 7.9\%$$

IMPORTANT CONTINUOUS DISTRIBUTIONS

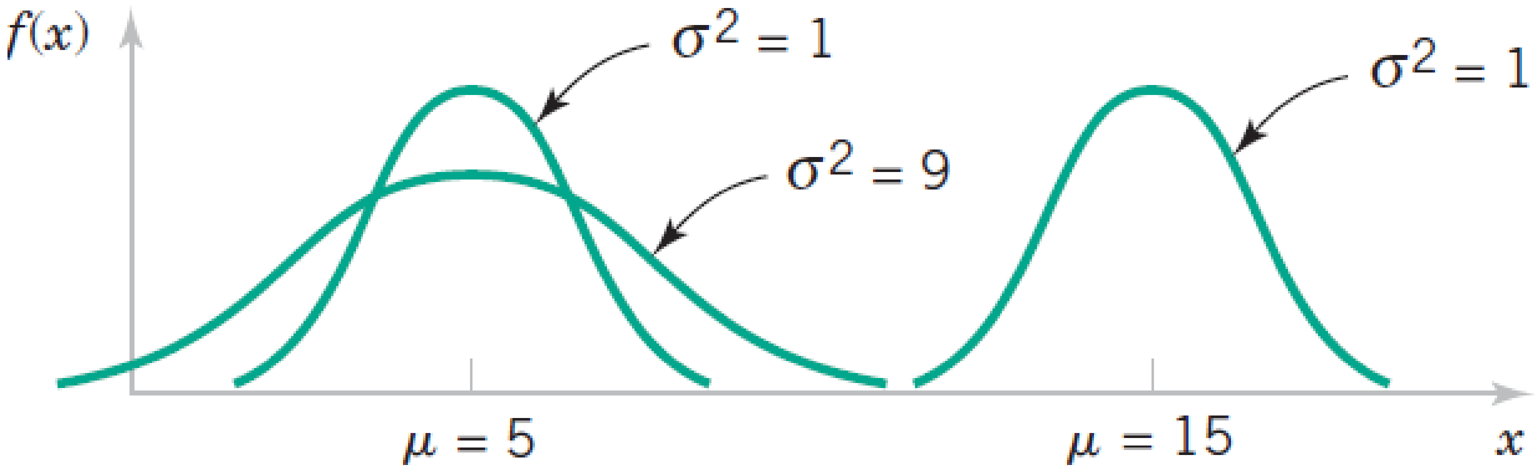
NORMAL DISTRIBUTION (Gaussian distribution)

A fundamental result, known as the central limit theorem, implies that histograms often have bell shape, at least approximately.

Whenever a random experiment is replicated, the random variable that equals the average (or total) result over the replicates tends to have a normal distribution as the number of replicates becomes large.

De Moivre presented initial results in 1733 and Gauss developed a normal distribution nearly 100 years later.

Objective now is to calculate probabilities for a normal random variable. The central limit theorem will be stated later



Normal probability density functions for selected values of the parameters μ and σ^2

HISTOGRAM

Ten measurements of length are given below

26,24,26,28,23,24,25,24,26,25

Measured lengths x and their numbers of occurrences

Different values of x	23	24	25	26	27	28
Number of times found, n_k	1	3	2	3	0	1

$$\bar{x} = \frac{\sum_i x_i}{N} = \frac{23+24+24+24+25+25+26+26+26+28}{10}$$

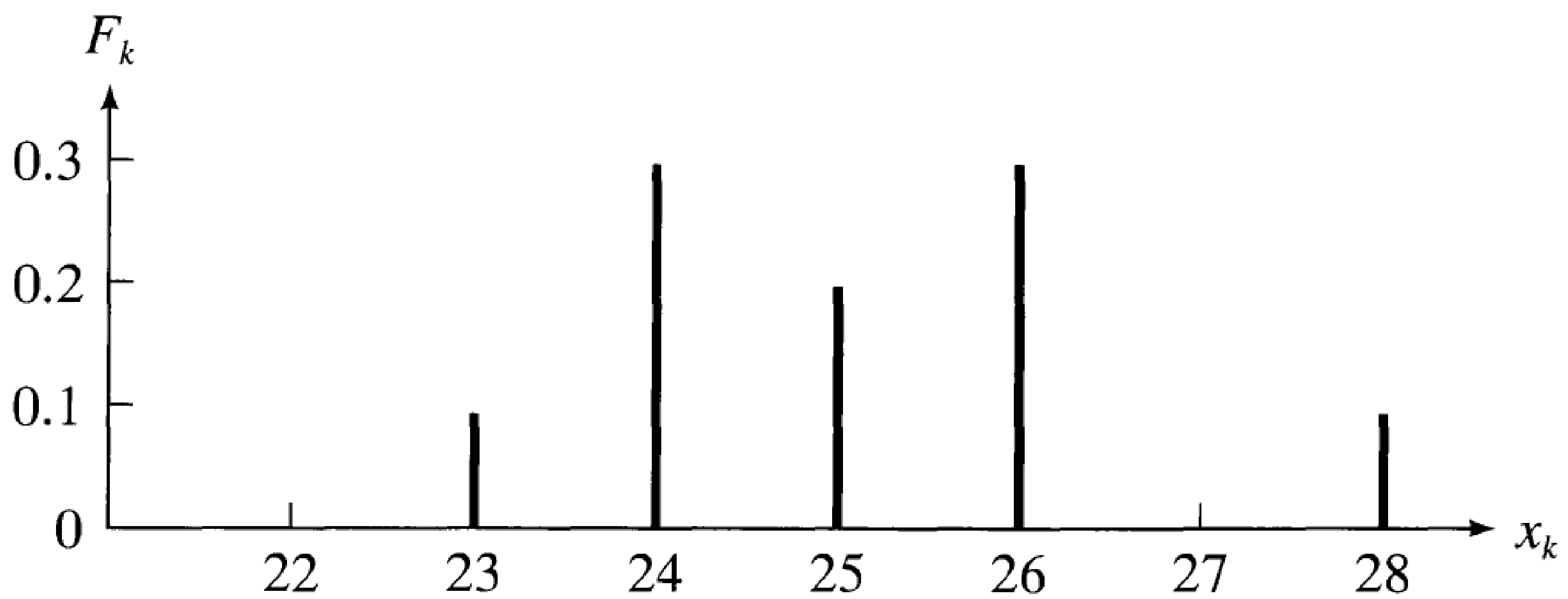
$$\bar{x} = \frac{\sum_i x_i}{N} = \frac{23\times1+24\times3+25\times1+26\times3+27\times0+28\times1}{10}$$

$$\bar{x} = \frac{\sum_k x_k n_k}{N} \qquad N = \sum_k n_k \qquad F_k = \frac{n_k}{N} \qquad \bar{x} = \sum_k x_k F_k$$

$$\bar{x} = \sum_k x_k F_k$$

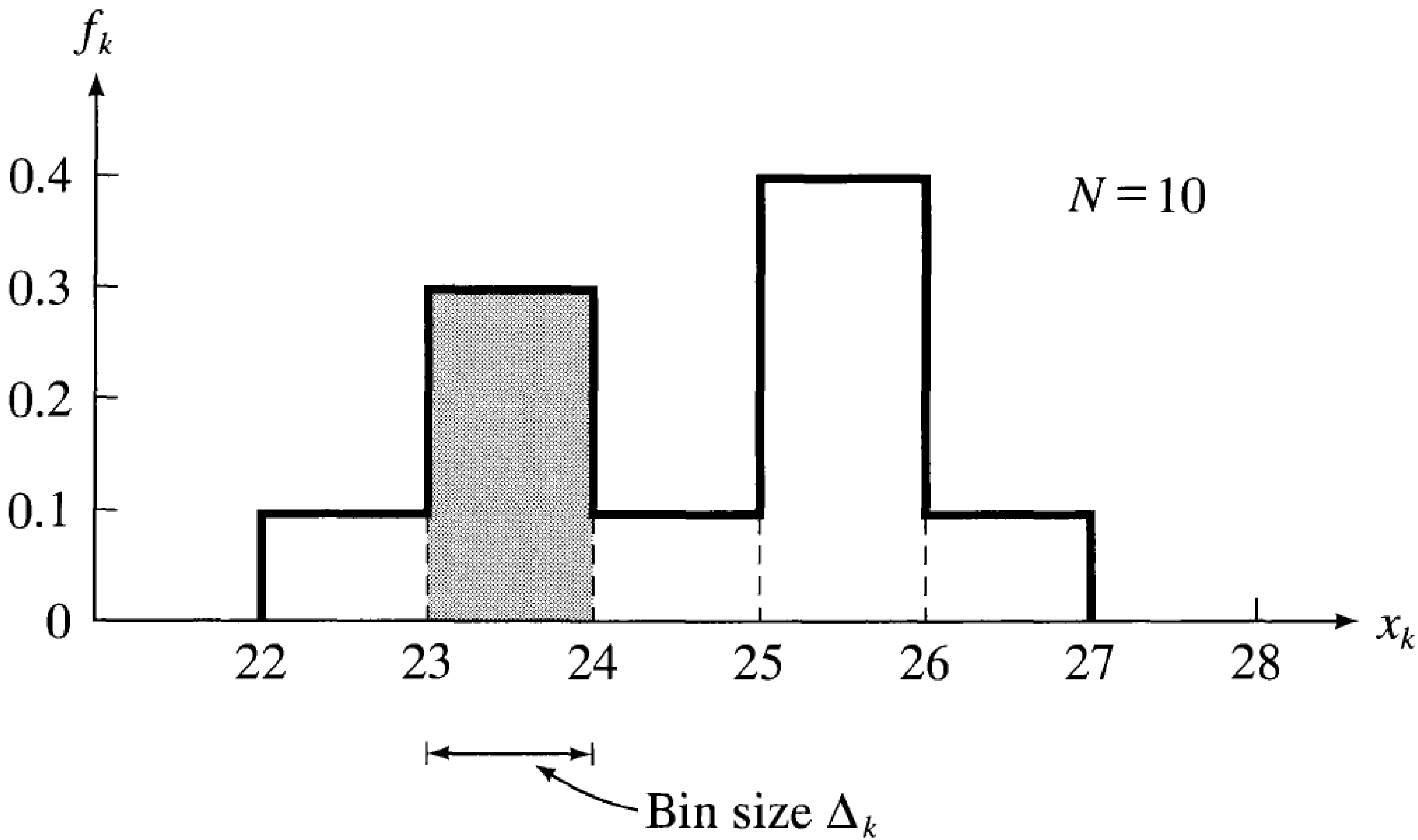
The mean is just the weighted sum of all the different values of x with each x_k weighted by the fraction of times it occurred, F_k

$$\sum_k F_k = 1$$

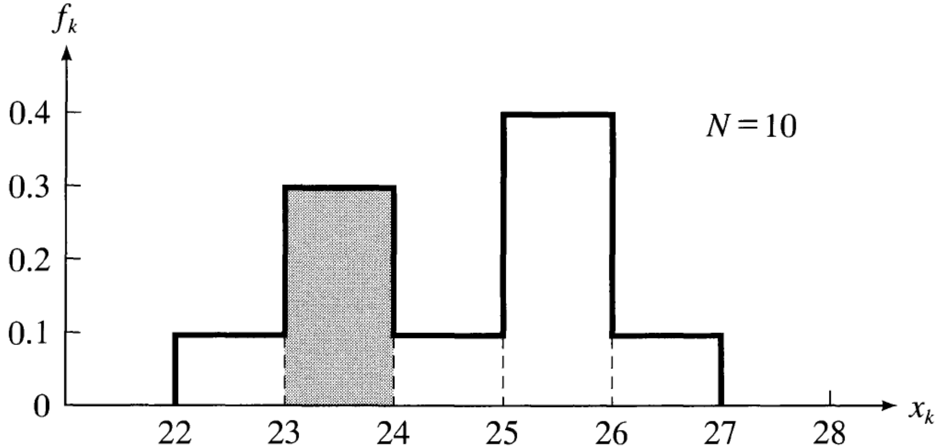


Histogram for 10 measurements of a length x. The vertical axis shows the fraction of times F_k that each value x_k was observed

Bins	22 to 23	23 to 24	24 to 25	25 to 26	26 to 27	27 to 28
Observations in bin	1	3	1	4	1	0

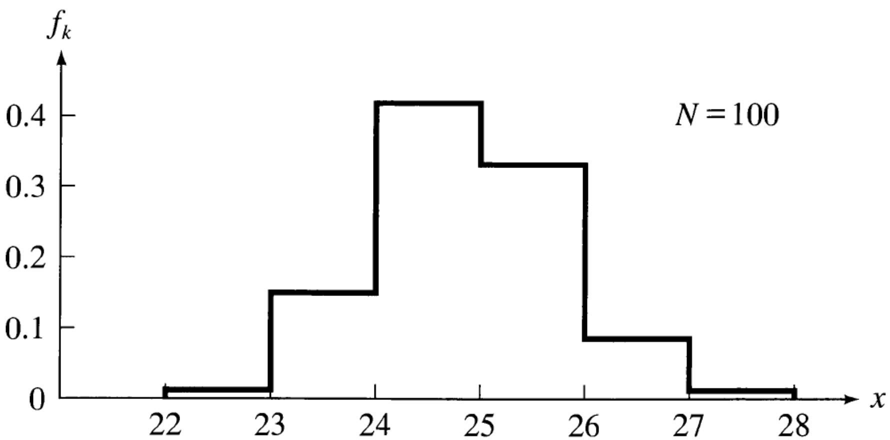


J.R. Taylor, An Introduction to Error Analysis, University Science Books, California, Second, Third edition, 1997.

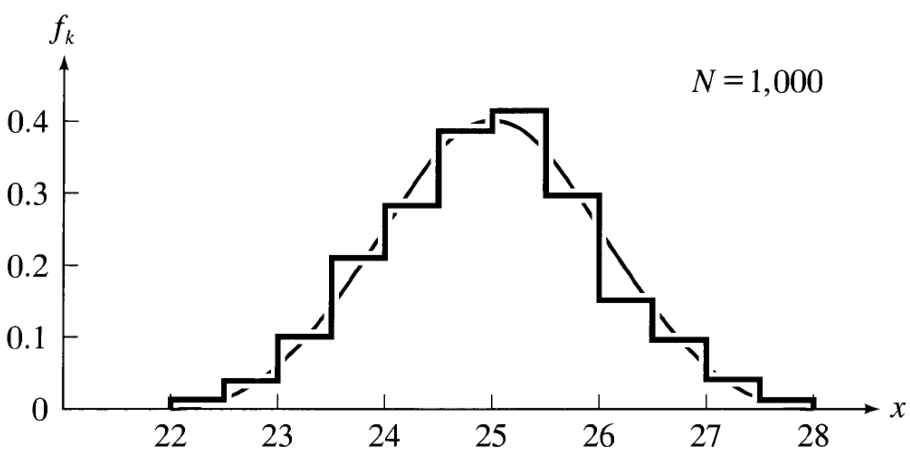


Broken curve is the limiting distribution

The limiting distribution is a theoretical construct that can never itself be measured exactly

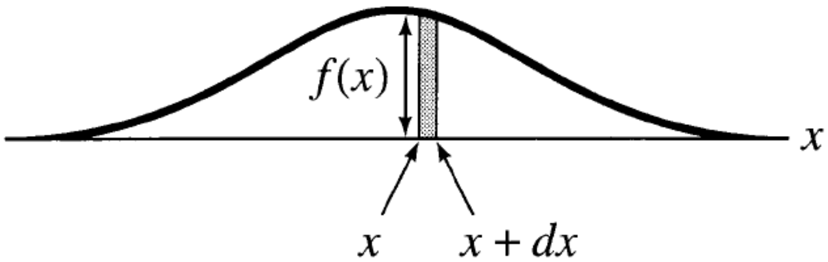


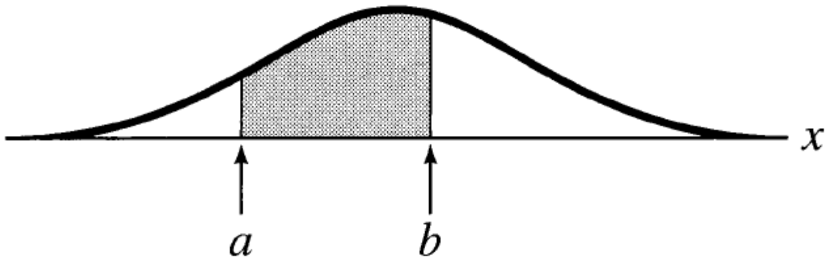
The more measurements we make, the closer our histogram approaches the limiting distribution



If we were to make an infinite number of measurements and use infinitesimally narrow bins would we actually obtain the limiting distribution itself. The limiting distribution such as the smooth curve defines a function which we call $f(x)$.

Normal Distribution





After very many measurements, the fraction falls between x and $x+dx$ is the area $f\,(x)$ dx of the narrow strip

The fraction that falls between x = a and x = b is the shaded area

$$\int_a^b f(x)\,dx = \textit{Fraction of measurements that fall between } x = a \textit{ and } x = b$$

$$f(x) = \textit{Probability that any one measurement will give an answer between } x \textit{ and } x + dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

This identity is the natural analog of the normalisation sum $\sum_k F_k = 1$

Function f(x) satisfying $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ is said to be normalised

$$\overline{x} = \sum_k x_k F_k$$

$$\mu = \int_{-\infty}^{+\infty} x \; f(x)\,dx$$

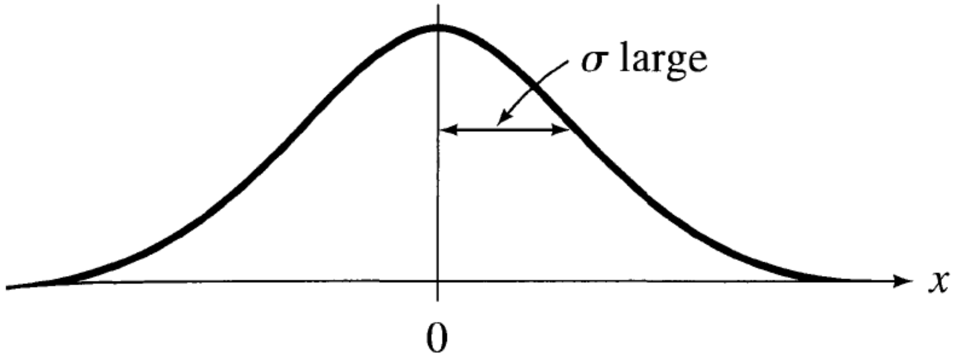
$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

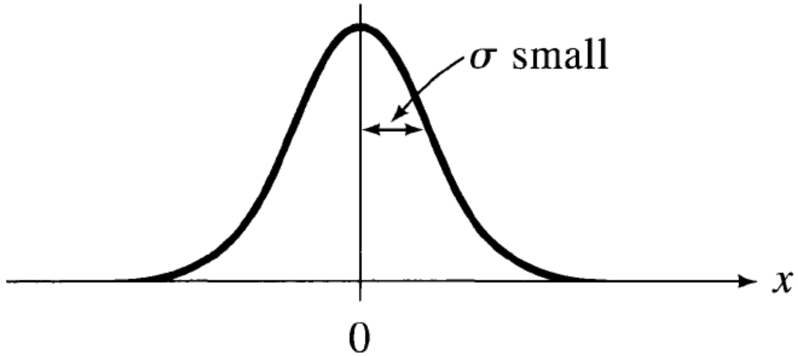
EVOLUTION OF NORMAL DISTRIBUTION FUNCTION

The mathematical function that describes the bell shaped curve is called the NORMAL DISTIBUTION or GAUSS FUNCTION. The protype of this equation is

$$e^{-\frac{x^2}{2\sigma^2}}$$

σ is the fixed parameter that can be visualised as the width parameter.



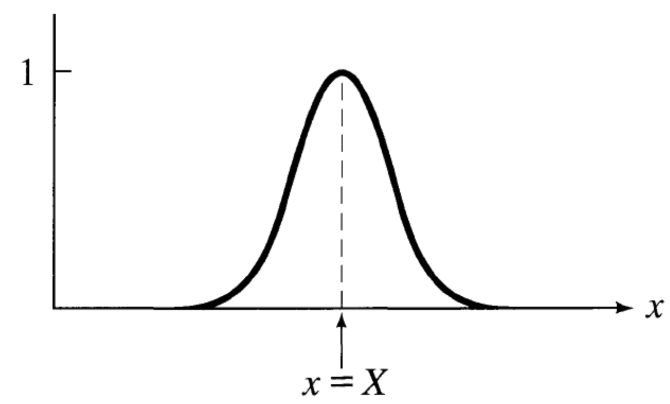


The Gauss function is a bell shaped curved centred on x = 0. To obtain a bell shaped curve centered on some other point $x = X$, x is replaced by x-X.

$$e^{-\frac{(x-X)}{2\sigma^2}}$$

This is not normalised, that is, it must satisfy

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$



We set

$$f(x) = Ne^{-\frac{(x-X)^2}{2\sigma^2}}$$

N does not change the shape or shift the maximum at $x = X$

Normalisation involves

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} Ne^{-\frac{(x-X)^2}{2\sigma^2}}\,dx$$

Let $x - X = y \Rightarrow dx = dy$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} Ne^{-\frac{y^2}{2\sigma^2}}\,dy$$

Let $y/\sigma = z \Rightarrow dy = \sigma\,dz$

$$\int_{-\infty}^{\infty} f(x)\,dx = N\sigma\int_{-\infty}^{\infty} e^{-z^2}\,dz = N\sigma\sqrt{2\pi} \quad \because \int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{2\pi}$$

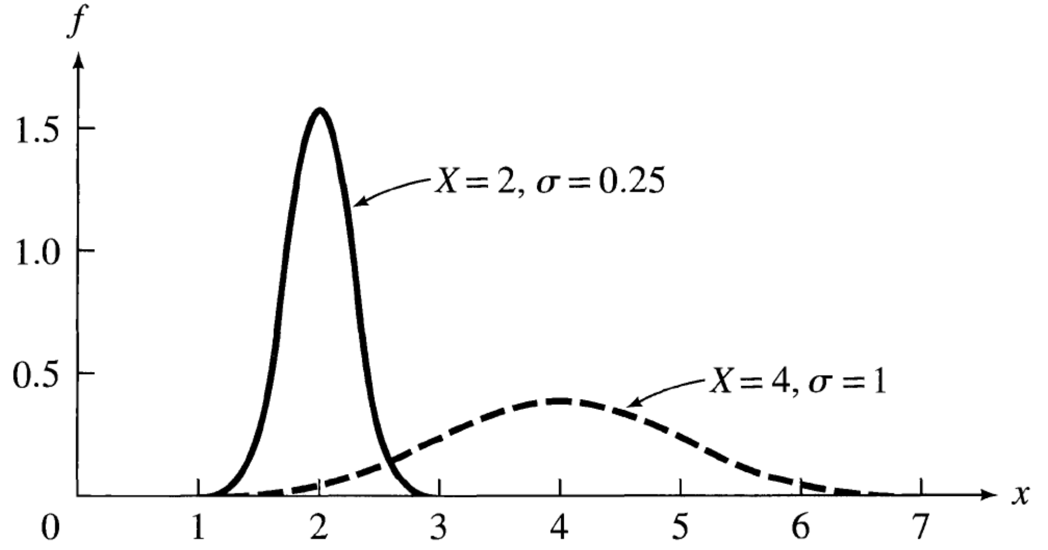
$$\int_{-\infty}^{\infty} f(x)\,dx = N\sigma\sqrt{2\pi}$$

Since this integral has to be one, the normalisation factor N is to be chosen such that

$$N = \frac{1}{\sigma\sqrt{2\pi}}$$

$$f(x) = Ne^{-\frac{(x-X)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-X)^2}{2\sigma^2}}$$

$$\boxed{f_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-X)^2}{2\sigma^2}}}$$



Mean of Normal Distribution: Mean value is the true value *X* on which the Gauss function is centered

$$\mu = \int_{-\infty}^{+\infty} x \; f(x)dx$$

$$f_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

$$\mu = \int_{-\infty}^{+\infty} x \; f_{X,\sigma}(x)dx = \int_{-\infty}^{+\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$z = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma z + \mu; \; dx = \sigma dz$$

$$\mu = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\sigma z + \mu) e^{-\frac{z^2}{2}} \sigma dz = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z e^{-\frac{z^2}{2}} dz + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz$$

$$\mu = \frac{\sigma}{\sqrt{2\pi}} \times 0 + \frac{\mu}{\sqrt{2\pi}} \sqrt{2\pi}$$

$$\int_{-\infty}^{+\infty} z e^{-\frac{z^2}{2}} dz = 0; \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$$

$$\mu_X = \mu$$

This result would be exactly true only if we could make infinitely many measurements. Its practical usefulness is that if we make a large (but finite) number of trials

Variance Normal Distribution:

$$\boxed{\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx}$$

$$f_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$z = \frac{x-\mu}{\sigma};dx = \sigma\,dz$$

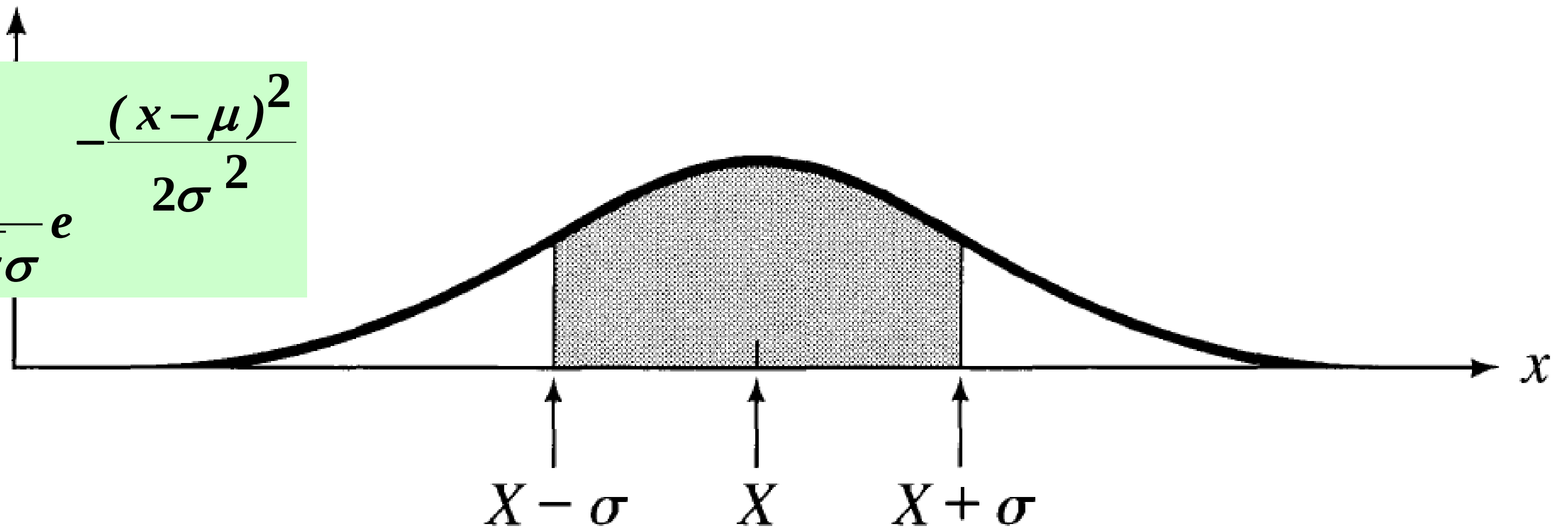
$$V(X) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}\,dz$$

$$V(X) = \frac{\sigma^2}{\sqrt{2\pi}}\left(\sqrt{2\pi}\right)$$

$$V(X) = \sigma^2$$

Standard deviation as 68% confidence limit

$$f(x)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$$f(x)=\int_{-\sigma}^{+\sigma}\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$$Let\ \frac{x-\mu}{\sigma}=z;\frac{dx}{\sigma}=dz$$

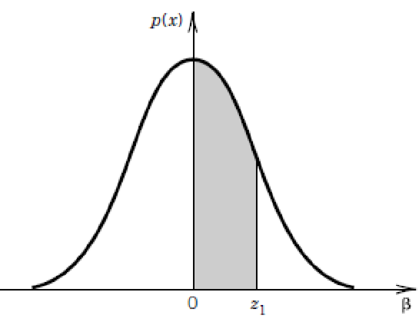
$$f(z)=\int_{-1}^{+1}\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{z^2}{2}}dz\sigma=\int_{-1}^{+1}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=(z)=\int_{-1}^{0}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz+\int_{0}^{+1}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=0.3413+0.3413$$

$$f(z)=0.6826$$

J.R. Taylor, An Introduction to Error Analysis, University Science Books, California, Second, Third edition, 1997.

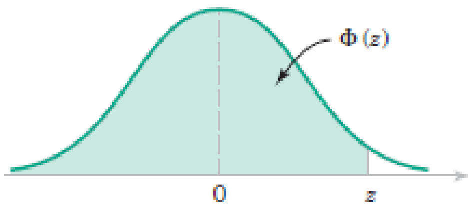
Normal error function

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$



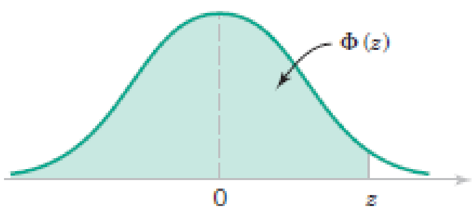
$z_1=\frac{x_1-x'}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1809	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2794	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4292	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4758	0.4761	0.4767
2.0	0.4772	0.4778	0.4803	0.4788	0.4793	0.4799	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.49865	0.4987	0.4987	0.4988	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} \, du$$

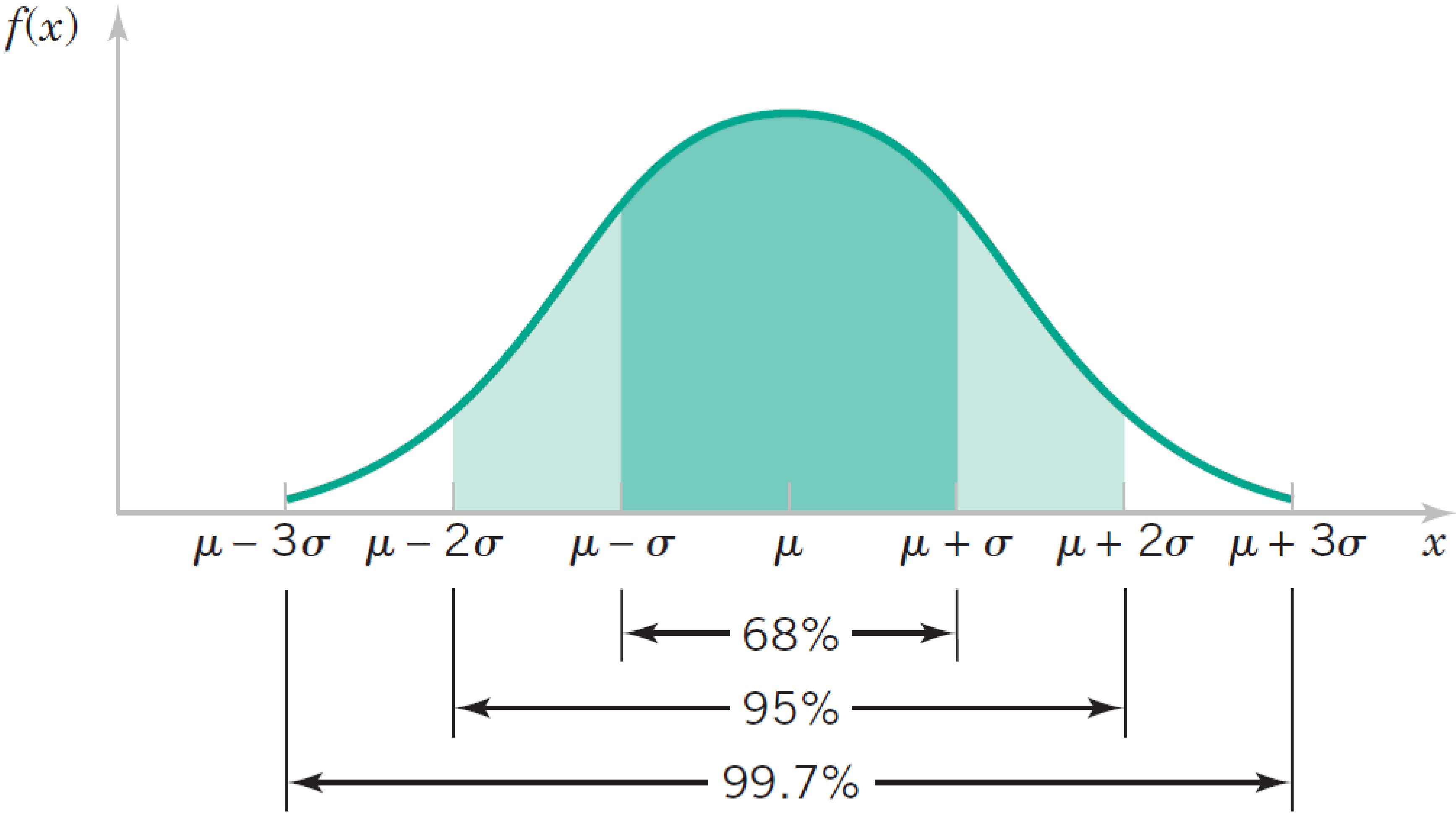


z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00	z
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048	−3.9
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072	−3.8
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108	−3.7
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159	−3.6
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233	−3.5
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337	−3.4
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483	−3.3
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687	−3.2
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968	−3.1
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350	−3.0
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866	−2.9
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555	−2.8
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467	−2.7
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661	−2.6
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210	−2.5
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198	−2.4
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724	−2.3
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903	−2.2
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864	−2.1
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750	−2.0
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717	−1.9
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930	−1.8
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565	−1.7
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799	−1.6
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807	−1.5
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757	−1.4
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801	−1.3
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070	−1.2
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666	−1.1
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655	−1.0
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060	−0.9
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855	−0.8
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964	−0.7
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253	−0.6
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538	−0.5
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578	−0.4
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089	−0.3
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740	−0.2
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172	−0.1
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000	0.0

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}}\, du$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	z
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.523922	0.527903	0.531881	0.535856	0.0
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345	0.1
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092	0.2
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732	0.3
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933	0.4
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405	0.5
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903	0.6
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236	0.7
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267	0.8
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913	0.9
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143	1.0
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977	1.1
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475	1.2
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736	1.3
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888	1.4
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083	1.5
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486	1.6
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273	1.7
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621	1.8
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705	1.9
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691	2.0
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738	2.1
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989	2.2
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576	2.3
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613	2.4
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201	2.5
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427	2.6
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365	2.7
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074	2.8
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605	2.9
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999	3.0
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289	3.1
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499	3.2
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650	3.3
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758	3.4
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835	3.5
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888	3.6
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925	3.7
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950	3.8
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967	3.9



$$P(\mu-\sigma < X < \mu+\sigma) = 0.6827$$
$$P(\mu-2\sigma < X < \mu+2\sigma) = 0.9545$$
$$P(\mu-3\sigma < X < \mu+3\sigma) = 0.9973$$

A random variable *X* with probability density function

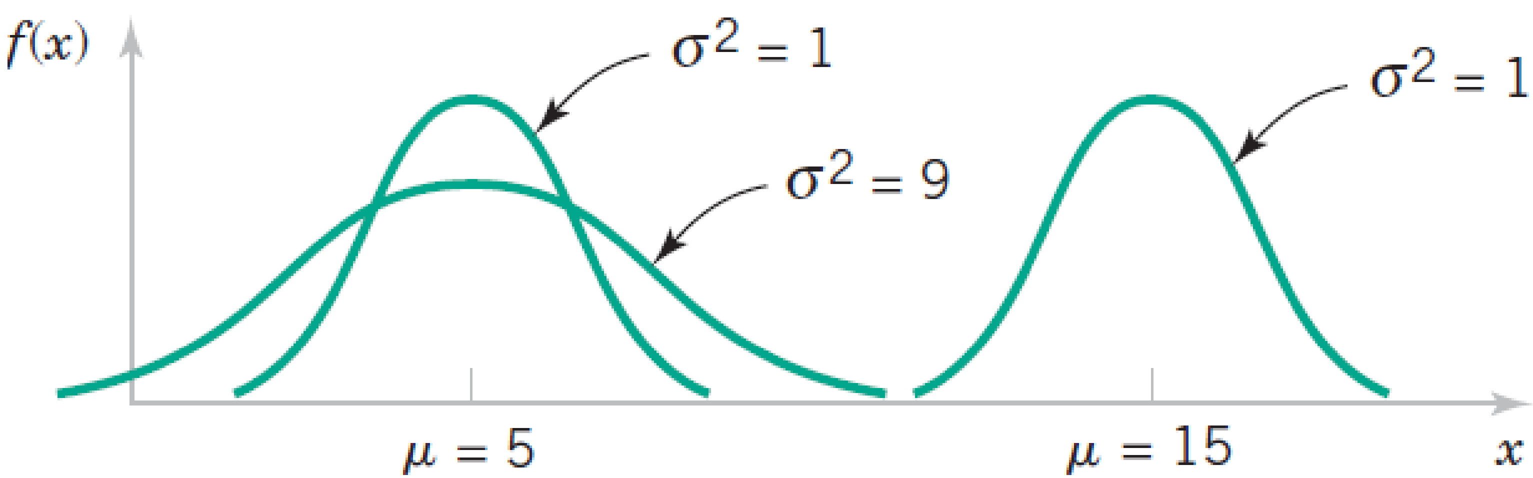
$$f(x)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad for \quad -\infty< x< \infty$$

has a normal distribution (and is called a normal random variable) with parameters μ and σ, where $-\infty < \mu > \infty$ and $\sigma > 0$. Also,

$E(X) = \mu$ and $V(X) = \sigma^2$

$E(X) = \mu$ - center of the probability density function

$V(X) = \sigma^2$ – width

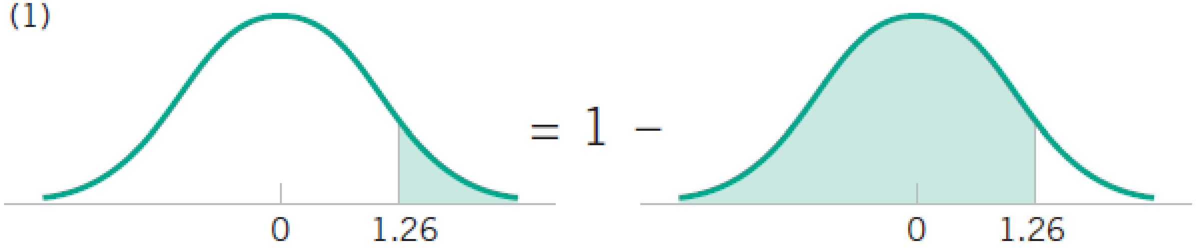


A random variable *X* with a mean μ and variance σ^2. Then

$$\boxed{P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z)}$$

where Z is a standard normal random variable, and $z = (x\text{-}\mu)/\sigma$ obtained by standardising x

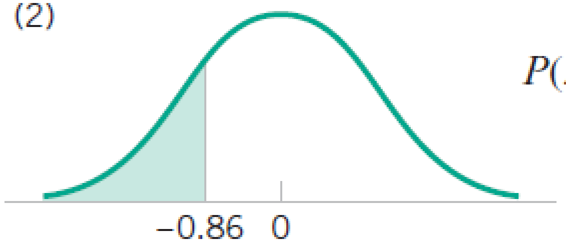
(1)



$$P(Z > 1.26) = 1 - P(Z \le 1.26) = 1 - 0.896165 = 0.10384$$

$$P(Z > 1.26) = 1 - P(Z \le 1.26) = 1 - (0.5 + 0.3962) = 0.10384$$

(2)

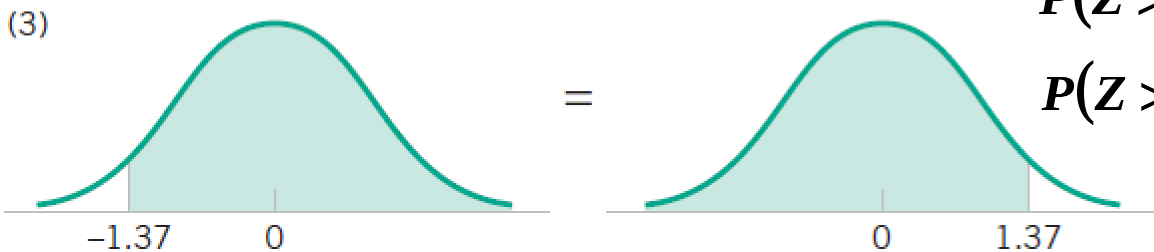


$$P(Z > -1.37) = P(Z < 1.37) = 0.91465$$

$$P(Z < -0.86) = 1 - P(Z \le 1.26) = 1 - 0.805106 = 0.1949$$

$$P(Z < -0.86) = 0.5 - 0.3051 = 0.1949$$

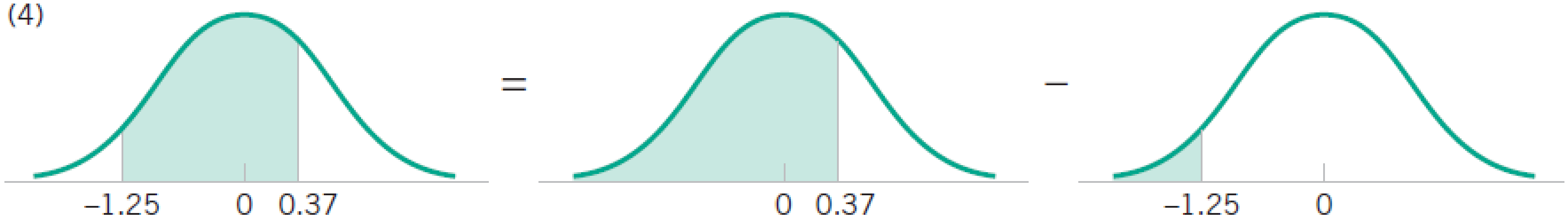
(3)



$$P(Z > -1.37) = (Z < 1.37) = 0.914657$$

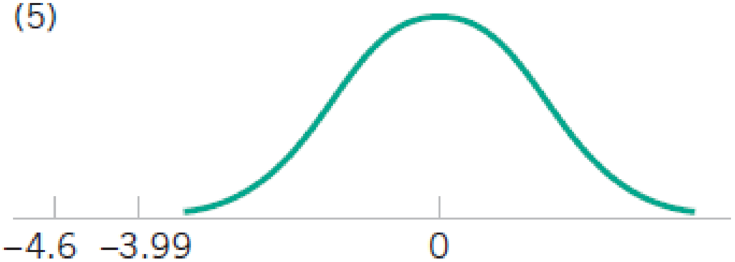
$$P(Z > -1.37) = 0.5 + 0.4147 = 0.9147$$

(4)



$$P(-1.25 < Z < 0.37) = P(-1.25 < Z) - P(Z < 0.37) = 0.64431 - 0.10565 = 0.53866$$

(5)



$$P(Z \leq -4.6) < P(Z \leq -3.99) = 0.00003$$

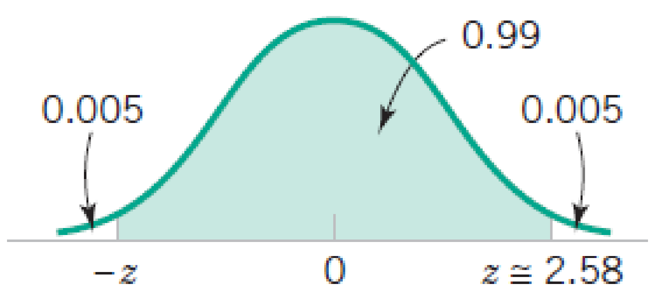
$$P(Z \leq -4.6) \approx 0$$

(6)

$$P(Z > z) = 0.05 \Rightarrow P(Z \leq z) = 0.95$$

Nearest Value is **0.95053**

(7)



$$P(-z < Z < z) = 0.99$$

The area in each tail of distribution $= 0.005$

The value for z *corresponds to a probability* $= 0.995$

$$z = 2.58$$

HISTORICAL PERSPECTIVE

Karl F. Gauss

The normal distribution was introduced by the French mathematician Abraham De Moivre in 1733. De Moivre, who used this distribution to approximate probabilities connected with coin tossing, called it the exponential bell-shaped curve.

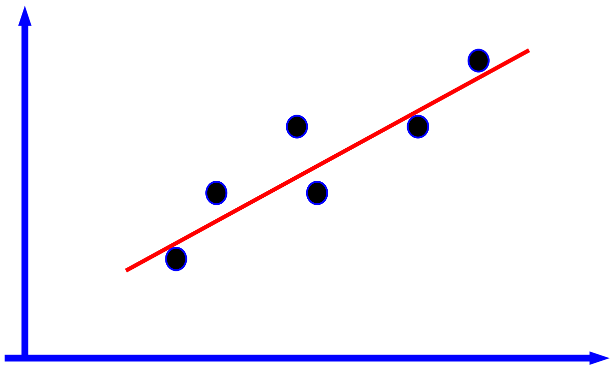
Its usefulness, however, only became truly apparent in 1809 when the famous German mathematician K. F. Gauss used it as an integral part of his approach to predicting the location of astronomical entities. As a result, it became common after this time to call it the Gaussian distribution.

During the middle to late 19th century, however, most statisticians started to believe that the majority of data sets would have histograms conforming to the Gaussian bell-shaped form. Indeed, it came to be accepted that it was “normal” for anywell-behaved data set to follow this curve. As a result, following the lead of Karl Pearson, people began referring to the Gaussian curve as simply the normal curve.

Karl Friedrich Gauss (1777–1855), one of the earliest users of the normal curve, was one of the greatest mathematicians of all time.

ACTUAL CALIBRATION PRACTICE

- A given true value is not repeated several times.
- True value is varied over a range – measured value, therefore, varies over a range.



LEAST SQUARES - DETERMINE THE FIT

LEAST SQUARES EQUATION - STRAIGHT LINE

$y = mx + C$ is the fit (x_k, y_k) are the experimental points

$$Error = E = \sum_{k=1}^{N}\left(\left(m\,x_k + C\right) - y_k\right)^2$$

$$\frac{\partial E}{\partial m} = \frac{\partial E}{\partial C} = 0 \text{ to minimize this error}$$

$$\frac{\partial E}{\partial m} = 0 \Rightarrow \sum 2\left(m\,x_k + C - y_k\right)x_k = 0$$

$$\frac{\partial E}{\partial C} = 0 \Rightarrow \sum 2\left(m\,x_k + C - y_k\right)1 = 0$$

$$m\sum x_k^{\,2} + C\sum x_k - \sum x_k y_k = 0$$

$$m\sum x_k + CN - \sum y_k = 0$$

$$m = \frac{1}{D}\left(N\sum x_k y_k - \sum x_k \sum y_k\right)$$

$$C = \frac{1}{D}\left(\sum x_k^{\,2}\sum y_k - \sum x_k \sum x_k y_k\right)$$

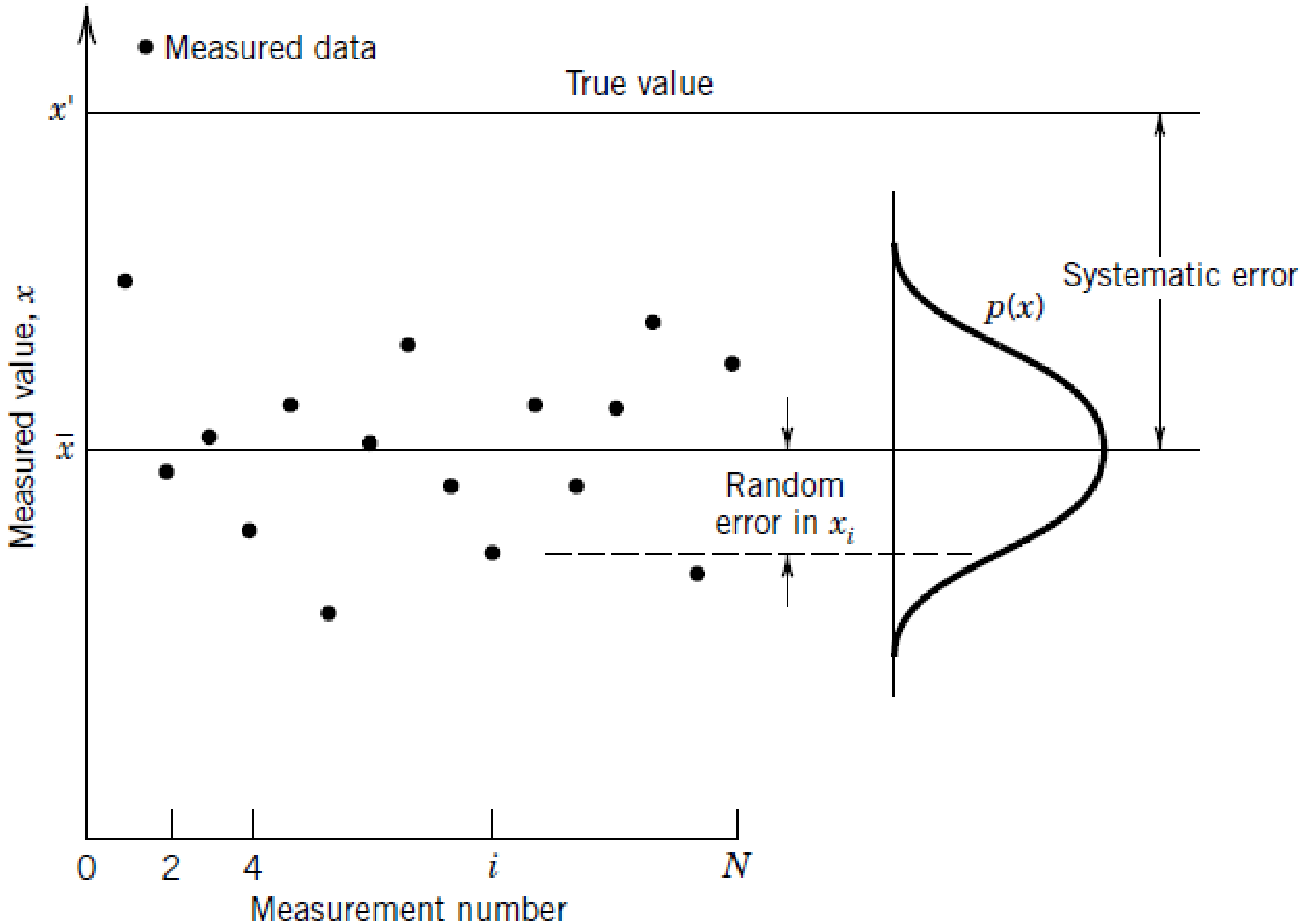
$$D = N\sum x_k^{\,2} - \left(\sum x_k\right)^2$$

$$\text{Standard deviation of data } \sigma^2 = \frac{1}{N-1}\sum_{k=1}^{N}\left[\left(mx_k + C\right) - y_k\right]^2$$

This standard deviation would possibly be the same if one repeated the "actual point – measured point" calibration over and over again.
Assume same standard deviation would occur even if a particular reading were to be repeated over and over again.

Use 99.7% limits (3σ); $\therefore$ Accurate value = $(mx + C) \pm 3\sigma$

Summary of the uncertainty



Uncertainty = ((Bias)^2 + (Precision)^2)$^{1/2}$

R.S.Figliola and D.E. Beasley, Theory and Design for Mechanical Measurements, Fifth Edition, John Wiley and Sons, 2011.