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TO DO LIST

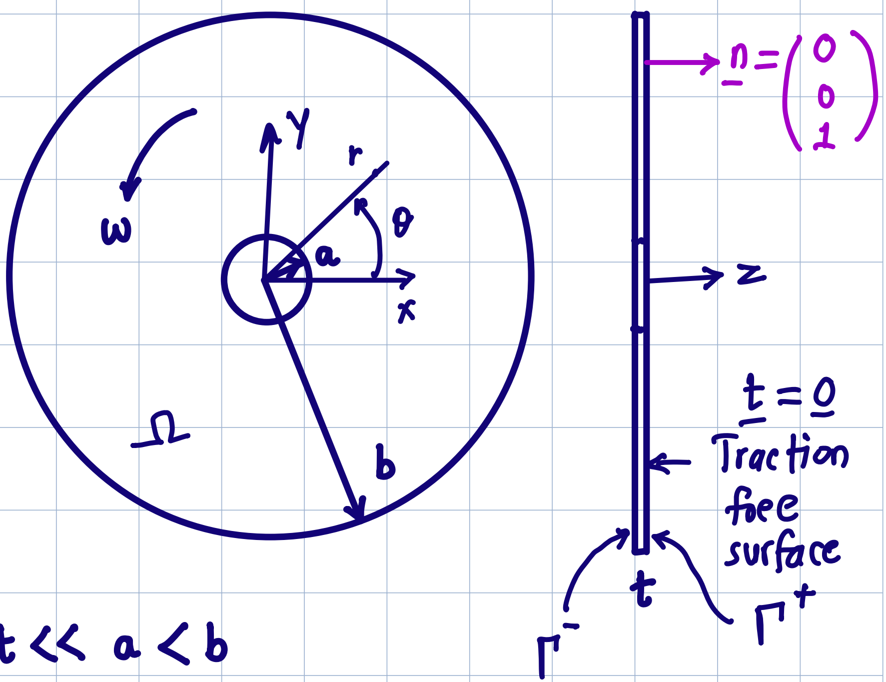
- ☐ 2D axisymmetric problems
  radial coordinates r
- ☐ Energy methods beam deflections
- ☐ Elastic instability buckling
- ☐ Unsymmetric bending
- ☐ Torsion, Back to Square One

ME 202

TUE 28 FEB 2023

Ref: $\frac{\text{Srinath}}{\text{Sadd}}$

In-plane loading



$t \ll a < b$

3D ⟶ 2D ⟶ 1D* Axisymmetric

"Plane stress"

Cylindrical Coordinates r, θ, z

$x = r\cos\theta, \; y = r\sin\theta, \; z = z$

$$\underset{\sim}{\sigma} = \begin{pmatrix} \boxed{\begin{matrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{\theta r} & \sigma_{\theta\theta} \end{matrix}} & \begin{matrix} \cancel{\sigma_{rz}} \\ \cancel{\sigma_{\theta z}} \end{matrix} \\ \begin{matrix} \cancel{\sigma_{zr}} & \cancel{\sigma_{z\theta}} \end{matrix} & \cancel{\sigma_{zz}} \end{pmatrix}$$

$\underline{t} = \underset{\sim}{\sigma}\,\underline{n}$ on Γ

Static Equilibrium on a surface with normal $\underline{n}$ $\sum \underline{F} = \underline{0}$

$$\begin{pmatrix} t_r \\ t_\theta \\ t_z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{given}} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ & \sigma_{\theta\theta} & \sigma_{\theta z} \\ & & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$0 = \sigma_{rz}.1, \quad 0 = \sigma_{\theta z}.1, \quad 0 = \sigma_{zz}.1 \quad \text{on } \Gamma^+$

$0 = \sigma_{rz}(-1), \quad 0 = \sigma_{\theta z}(-1), \quad 0 = \sigma_{zz}(-1) \quad \text{on } \Gamma^-$

$\Rightarrow \sigma_{rz} = 0, \; \sigma_{\theta z} = 0, \; \sigma_{zz} = 0 \quad \text{in } \Omega$

Plane stress assumption

In-plane loading only

Hooke's Law in 3D

$$\epsilon_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu}{E}\left(\sigma_{\theta\theta} + \cancel{\sigma_{zz}}\right) = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}\left(\sigma_{rr} + \cancel{\sigma_{zz}}\right) = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$\epsilon_{zz} = \frac{\cancel{\sigma_{zz}}}{E} - \frac{\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right) = \frac{\partial u_z}{\partial z}$$

$$\downarrow$$

$$\frac{\partial u_z}{\partial z} = \frac{-\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

$$= \frac{-\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\epsilon_{rz} = \frac{1+\nu}{E}\cancel{\sigma_{zr}} = 0$$

$$\epsilon_{\theta z} = \frac{1+\nu}{E}\cancel{\sigma_{\theta z}} = 0$$

$$\epsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta}$$

$$= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\cancel{\partial \theta}} + \frac{\cancel{\partial u_\theta}}{\partial r} - \frac{\cancel{u_\theta}}{r}\right),$$

$$\underset{\sim}{\epsilon} = \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & 0 \\ \epsilon_{r\theta} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}$$

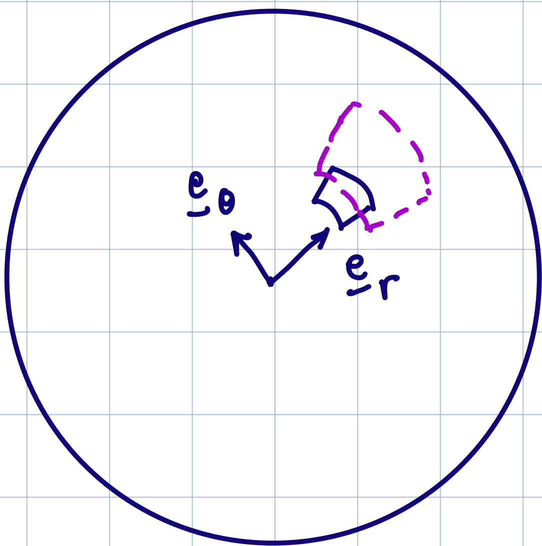
$$u_z \approx \frac{-\nu t}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

$u_\theta = 0 \qquad \frac{\partial (\;)}{\partial \theta} = 0$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\cancel{\frac{\partial \sigma_{r\theta}}{\partial \theta}} + \cancel{\frac{\partial \sigma_{zr}}{\partial z}} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \underset{\rho r \omega^2}{\underset{\shortparallel}{b_r}} = 0$$

$$\cancel{\frac{\partial \sigma_{r\theta}}{\partial r}} + \frac{1}{r}\cancel{\frac{\partial \sigma_{\theta\theta}}{\partial \theta}} + \cancel{\frac{\partial \sigma_{\theta z}}{\partial z}} + \frac{2}{r}\cancel{\sigma_{r\theta}} + \cancel{b_\theta} = 0 \quad \checkmark$$

$$\cancel{\frac{\partial \sigma_{rz}}{\partial z}} + \frac{1}{r}\cancel{\frac{\partial \sigma_{\theta z}}{\partial \theta}} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + \frac{1}{r}\cancel{\sigma_{zr}} + \cancel{b_z} = 0 \quad \checkmark$$



$$\frac{d}{dr}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho r \omega^2 = 0 \qquad r\text{-direction}$$

Stress Formulation | **Displacement Formulation**

ODE in $\sigma_{rr}(r)$ | ODE in $u_r(r)$

Depends on BCs.

① **Spinning freely at constant ω**

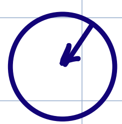
Ensemble

BCs

Inner radius traction free

Outer ——— " ———

@ $r = a$.



$$\underline{n} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Bigg\} \text{ in } r, \theta, z \text{ coords}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \underline{t} = \underline{\underline{\sigma}}\, \underline{n}$$

$$\Rightarrow \quad 0 = \sigma_{rr}(-1) \quad \Rightarrow \quad \sigma_{rr}(a) = 0$$

$$\sigma_{rr}(b) = 0$$

Aside $\quad \omega = 0$, internal pressure $= p_a$

external pressure $= p_b$

$$\underline{t} = \underset{\sim}{\sigma}\,\underline{n} \Rightarrow \begin{pmatrix} +p_a \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{rr} & 0 \\ 0 & \sigma_{\theta\theta} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\sigma_{rr}(a) = -p_a$$

$$\sigma_{rr}(b) = -p_b \qquad\qquad u(r) = u_r(r)$$

$$\epsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right) = \frac{du}{dr}$$

equate

$$\epsilon_{\theta\theta} = \frac{u}{r} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right)$$

$$u = \frac{r}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) \Rightarrow \frac{du}{dr} =$$

$$\Rightarrow \quad \sigma_{\theta\theta} = \sigma_{rr} + \nu\sigma_{rr}' \qquad\qquad ' \equiv \frac{d}{dr}$$

Plug into eqm

$$r \frac{d^2\sigma_{rr}}{dr^2} + 3 \frac{d\sigma_{rr}}{dr} = 0$$

$$\sigma_{rr} = C_1 + \frac{C_2}{r^2}, \quad \sigma_{\theta\theta} = C_1 - \frac{C_2}{r^2}$$

C_1, C_2 from TBCs on σ_{rr}

$$\sigma_{rr} = \frac{P_a a^2 - P_b b^2}{b^2 - a^2} - \frac{a^2 b^2}{r^2}\left(\frac{P_a - P_b}{b^2 - a^2}\right)$$

$$\sigma_{\theta\theta} = \underbrace{}_{} + \underbrace{}_{}$$

$$u = \frac{1-\nu}{E} \frac{P_a a^2 - P_b b^2}{b^2 - a^2} r + \frac{1+\nu}{E} \frac{a^2 b^2}{r^2} \frac{P_a - P_b}{b^2 - a^2}$$

Displacement Approach

Invert Hooke's Law

$$\left\{\begin{aligned} \sigma_{rr} &= \frac{E}{1-\nu^2}\left(\overset{\frac{du}{dr}}{\overset{\parallel}{\epsilon_{rr}}} + \nu\, \overset{\frac{u}{r}}{\overset{\parallel}{\epsilon_{\theta\theta}}}\right) \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left(\epsilon_{\theta\theta} + \nu\, \epsilon_{rr}\right) \end{aligned}\right.$$

$\rightarrow$ Into eqm

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = 0$$

$$u = C_1 r + \frac{C_2}{r}$$

Useful for DBCs.