Tutorial Sheet 4

Matrix Norms and Iterative Methods

1. Show that the norm defined on the set of all $n \times n$ matrices by
$$\|A\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$
is not subordinate to any vector norm on $\mathbb{R}^n$.

2. For a given vector $(x_1, \ldots, x_n)^T$, the corresponding Vandermonde matrix is defined as
$$V_n = (v_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}}$$
where $v_{ij} = x_i^{j-1}$. For a fixed integer $n > 1$, let $x_k = 1 + k/n$, for $k = 1, 2, \ldots, n$, then show that $\|V_n\|_\infty \to \infty$ as $n \to \infty$.

3. Let A and B be invertible matrices with condition numbers $\kappa(A)$ and $\kappa(B)$ respectively. Show that $\kappa(AB) \le \kappa(A)\kappa(B)$.

4. In solving the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ with matrix
$$A = \begin{pmatrix} 1 & 4 \\ 1 & 4.01 \end{pmatrix},$$
estimate the relative error in the solution vector $\boldsymbol{x}$ in terms of the relative error in $\boldsymbol{b}$. Test your estimate in the case when $\boldsymbol{b} = (4, 4)^T$ and $\tilde{\mathbf{b}} = (3.95, 4.01)^T$. Use the maximum norm for vectors in $\mathbb{R}^2$.

5. For $n \ge 2$, find $\kappa_2(T_n)$ where T_n is the tri-diagonal matrix given by
$$T_n = \begin{pmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & & & \Large 0 & \\ & \ddots & \ddots & \ddots & & & \\ & \Large 0 & & & -1 & 2 & -1 \\ & & & & 0 & -1 & 2 \end{pmatrix}$$

6. Let $\boldsymbol{x}^{(12)}$ be the 12^{th} term of the Gauss-Seidel iterative sequence for the system

$$\begin{aligned}3x_1 + 2x_2 &= 1\\ 4x_1 + 12x_2 + 3x_3 &= -2\\ x_1 + 3x_2 - 5x_3 &= 3\end{aligned}$$

with $\boldsymbol{x}^{(0)} = (0,0,0)^T$. If $\boldsymbol{x}$ denotes the exact solution of the given system, then show that

$$\|\boldsymbol{e}^{(12)}\|_\infty \leq 0.0077073467\|\boldsymbol{x}\|_\infty.$$

7. Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$, where $A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{R})$ be such that $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$, and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$. For a given real number $\omega \neq 0$, define an iterative sequence $\{\boldsymbol{x}^{(k)}\}$ as

$$\begin{aligned}\text{Given } \boldsymbol{x}^{(0)};\quad z_i^{(k+1)} &= \frac{1}{a_{ii}}\left\{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right\},\\ x_i^{(k+1)} &= (1-\omega)x_i^k + \omega z_i^{(k+1)}, \quad \text{for } \begin{array}{l} i = 1, 2, \cdots, n\\ k = 0, 1, 2, \ldots. \end{array}\end{aligned}$$

This iterative method is called the *successive over relaxation method* (SOR method). Find the matrix S_ω and the vector $\boldsymbol{c}$ such that the SOR iterative method is written in the matrix form

$$\boldsymbol{x}^{(k+1)} = S_\omega \boldsymbol{x}^{(k)} + \boldsymbol{c}, \quad k = 0, 1, 2, \ldots.$$

8. Let A be an $n \times n$ matrix with real entries. Let $\kappa_2(A)$ and $\kappa_\infty(A)$ denote the condition numbers of a matrix A that are computed using the matrix norms $\|A\|_2$ and $\|A\|_\infty$, respectively. Answer the following questions.
   i) Determine all the diagonal matrices such that $\kappa_\infty(A) = 1$.
   ii) Let Q be a matrix such that $Q^TQ = I$ (such matrices are called orthogonal matrices). Show that $\kappa_2(Q) = 1$.
   iii) If $\kappa_2(A) = 1$, show that all the eigenvalues of A^TA are equal. Further, deduce that A is a scalar multiple of an orthogonal matrix.