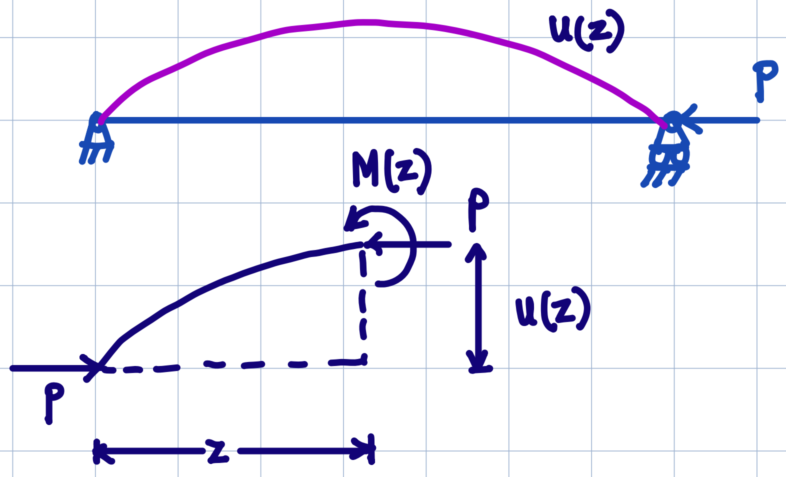
Dnyanesh Pawaskar

ME 202

BUCKLING OF ELASTIC COLUMNS/BEAMS



$u = u(z)$

$$EIu'' = M(z) = -Pu,$$

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$EI\,u'' + Pu = 0$ $\quad$ 2^{nd} order ODE

$$-EI \frac{d^2}{dz^2} u = P u$$ eigenvalue problem

P eigenvalues, $u(z)$ eigenfunctions

$u(0) = 0, \quad u(L) = 0$

$$u'' + \lambda^2 u = 0 \qquad \lambda^2 = \frac{P}{EI}$$

$$u = A \cos \lambda z + B \sin \lambda z$$

$u(0) = 0 \Rightarrow A = 0$

$u(L) = 0 \Rightarrow B \sin \lambda L = 0$

$B = 0$ or $\sin \lambda L = 0$, $B \neq 0$ ✓

trivial soln

$\lambda L = n\pi \qquad n = 1, 2, 3$

$$P_1 = \frac{\pi^2 EI}{L^2}, \quad P_2 = \frac{4\pi^2 EI}{L^2}, \quad P_3 = \frac{9\pi^2 EI}{L^2}$$

$$u_1 = B \sin \frac{\pi z}{L}, \quad u_2 = B \sin \frac{2\pi z}{L}, \quad u_3 = B \sin \frac{3\pi z}{L}$$

B undetermined

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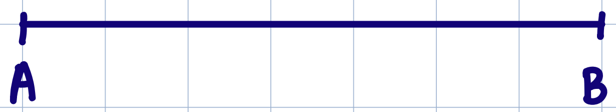
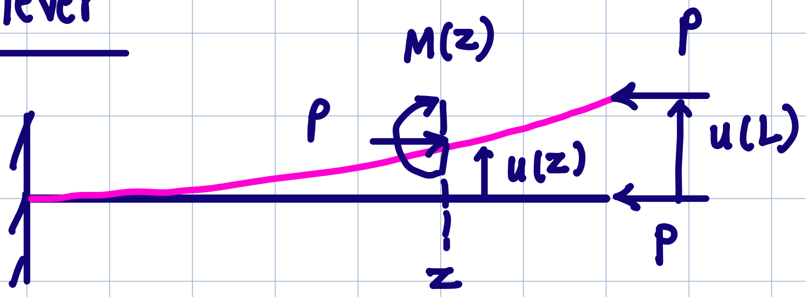
Euler Load $P_E = \dfrac{\pi^2 EI}{L^2}$ $\qquad P_n = n^2 P_E$



Cantilever



$$M(z) = P\left(u(L) - u(z)\right) = EI\,u''$$

$$EI\,u'' + Pu = Pu_L \qquad u_L = u(L)$$

$$u = A\cos\lambda z + B\sin\lambda z + u_L$$

$$\lambda^2 = \frac{P}{EI}, \qquad u' = -A\lambda\sin\lambda z + B\lambda\cos\lambda z$$

BCs $\quad u(0) = 0, \quad u'(0) = 0, \quad u(L) = u_L$

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$$u_L + A = 0$$

$$B\lambda = 0 \quad \Rightarrow \quad B = 0$$

$$\cancel{u(L)} + A\cos\lambda L + \cancel{B}\sin\lambda L = \cancel{u(L)}$$

$$A\cos\lambda L = 0 \quad \Rightarrow \quad \lambda L = \frac{n\pi}{2}$$

n odd

$$u(z) = -A + A\cos\frac{n\pi z}{2L}$$

$$u(z) = A\left(1 - \cos\frac{n\pi z}{2L}\right)$$

$$P_n = \frac{n^2\pi^2 EI}{4L^2}$$

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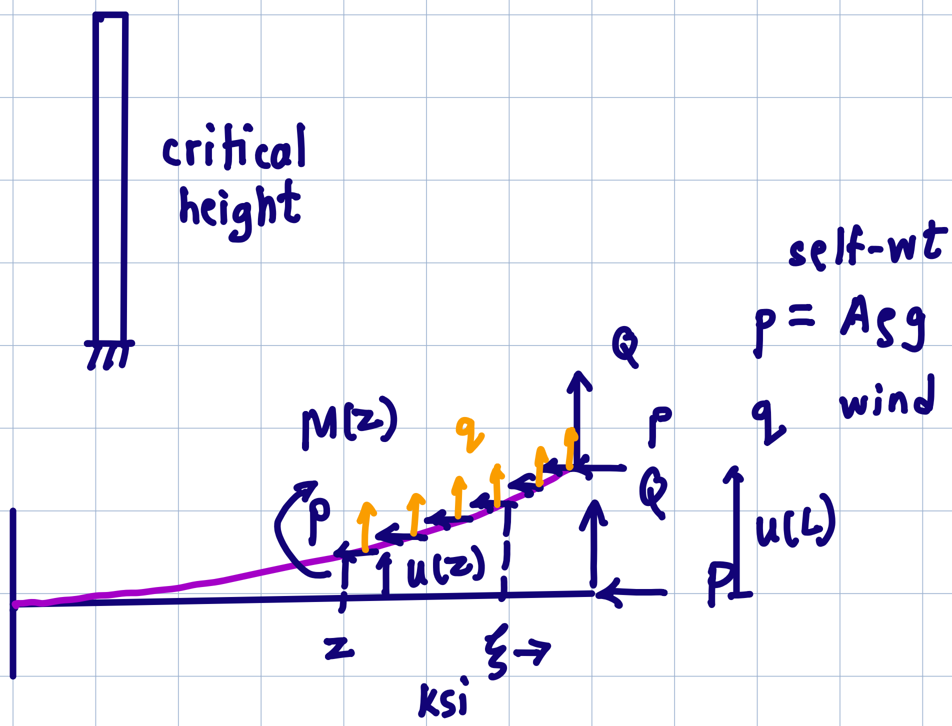
Self-buckling (under own weight)



$$M(z) = Q(L-z) + P\left(u(L) - u(z)\right) + \int_z^L q(\xi)(\xi - z)\,d\xi + \int_z^L p(\xi)\left(u(\xi) - u(z)\right)d\xi$$

$$= EI\,u''$$

Take derivs wrt z

DUIS / Leibniz / Reynolds Transport Thm
Newton

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$$\frac{d}{dz}\int_{\varphi_1(z)}^{\varphi_2(z)} F(\xi, z)\,d\xi = \int_{\varphi_1(z)}^{\varphi_2(z)} \frac{\partial F}{\partial z}\,d\xi + F(\varphi_2, z)\frac{d\varphi_2}{dz} - F(\varphi_1, z)\frac{d\varphi_1}{dz}$$

$$\frac{d}{dz}\int_{z}^{L} F\,d\xi = \int_{z}^{L} -\frac{du}{dz}\,d\xi + F(L,z)\overbrace{\cancel{\frac{dL}{dz}}}^{0} - \overbrace{\cancel{F(z,z)}}^{0}\frac{dz}{dz}$$

$$F = u(\xi) - u(z)$$

$$= (L-z)\left(-\frac{du}{dz}\right)$$

$$EI\frac{d^3u}{dz^3} + Q + P\frac{du}{dz} + q(L-z) + \underbrace{p(L-z)}_{\text{variable coefficient}}\frac{du}{dz} = 0$$

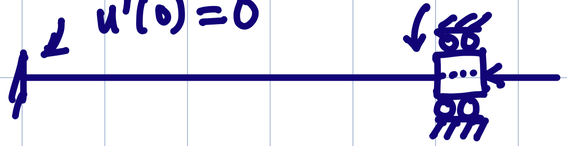
Example

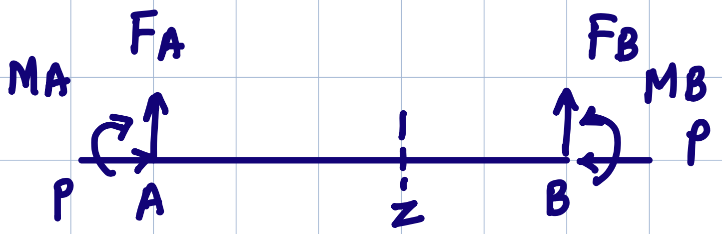
ΔT

$u(0)=0$

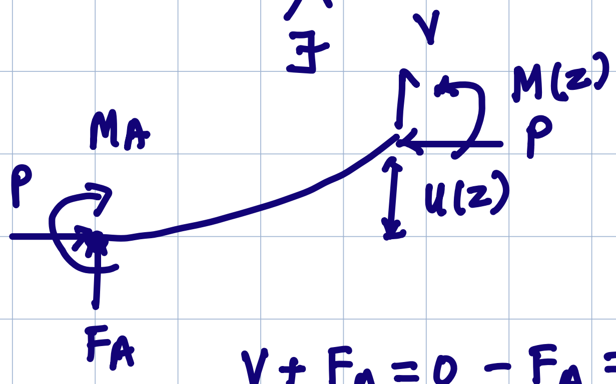
$u'(0)=0$

$u(L)=0, \; u'(L)=0$





Find P for which $\exists$ non-trivial $u(z)$



$$V + F_A = 0, \quad -F_A = V$$

$$-M_A - F_A z + M + P\,u(z) = 0$$

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$$M = -Pu(z) + M_A + F_A z = EIu''$$

$$EIu'' + Pu = M_A + F_A z$$

$$\lambda^2 = \frac{P}{EI}$$

$$u(z) = A\cos\lambda z + B\sin\lambda z + \frac{M_A}{P} + \frac{F_A}{P} z$$

A, B, M_A, F_A

4 BCs $\quad u(0)=0, \; u'(0)=0, \; u(L)=0,$
$u'(L)=0$

$$\frac{M_A}{P} + A = 0, \quad M_A = -AP$$

$$\frac{F_A}{P} + B\lambda = 0, \quad F_A = -B\lambda$$

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$$-A - B\lambda L + A\cos\lambda L + B\sin\lambda L = 0$$

$$-B\lambda - A\lambda\sin\lambda L + B\lambda\cos\lambda L = 0$$

$$\underbrace{\begin{pmatrix} \cos\lambda L - 1 & \sin\lambda L - \lambda L \\ -\lambda\sin\lambda L & \lambda\cos\lambda L - \lambda \end{pmatrix}}_{\det = 0} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$