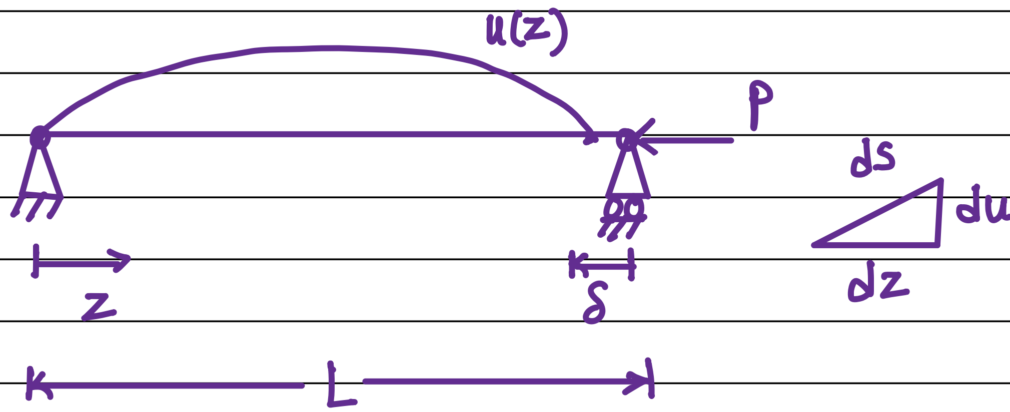
Approximate method to calculate the buckling load

Potential Energy



δ : dist thru which P traveled

Finite Element Method

$$\Pi = \int_0^L \frac{EI}{2} u''^2 \, dz - P\delta$$

Assume beam is inextensible.

$$\int_0^{L-\delta} dz \sqrt{1+u'^2} = L$$

$$ds = dz \sqrt{1+\left(\frac{du}{dz}\right)^2} = dz \sqrt{1+u'^2}$$

Assume $|u'| << 1 \Rightarrow \sqrt{1+u'^2} \approx 1 + \frac{u'^2}{2}$

$$\int_0^{L-\delta} dz \left(1 + \frac{u'^2}{2}\right) = L$$

$$\cancel{L} - \delta + \int_0^{L-\delta} \frac{u'^2}{2} = \cancel{L}$$

$$\delta = \int_0^{L-\cancel{\delta}} \frac{u'^2}{2}\, dz \qquad \delta << L$$

$$\delta = \int_0^{L} \frac{u'^2}{2}\, dz$$

$$\Pi = \int_0^{L} \left(\frac{EI}{2} u''^2 - \frac{P}{2} u'^2\right) dz$$

Assume $u(z)$ which obeys KBCs.

Apply KBCs to orig config.

Find coeffs in $u(z)$ that result in stationary PE.

Example

Assume $u(z) = a z (L-z) = \cancel{a_0} + a_1 z + a_2 z^2$

1 DOF

$u(0) = 0$

$u(L) = 0$

$$u' = aL - 2az, \quad u'' = -2a$$

$$\Pi(a) = \frac{a^2 L^3}{6}\left(\frac{12EI}{L^2} - P\right)$$

$$\frac{d\Pi}{da} = 0 \Rightarrow \frac{2aL^3}{6}\left(\frac{12EI}{L^2} - P\right) = 0$$

$$a = 0 \quad \text{OR} \quad \frac{12EI}{L^2} - P = 0$$

trivial soln ✓ $\Rightarrow P^* = \frac{12EI}{L^2}$

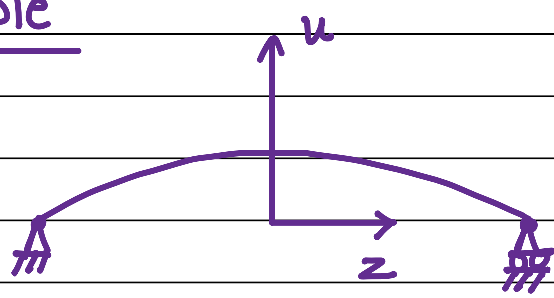
$$\frac{d^2\Pi}{da^2} = \frac{2L^3}{6}\left(\underbrace{\frac{12EI}{L^2}}_{P^*} - P\right)$$

stability (sign of second deriv) switch

Exact soln $P^* = \dfrac{\pi^2 EI}{L^2}$

$\simeq \dfrac{10\, EI}{L^2}$

Example



Assume $u = a_0 \left[1 + a \left(\dfrac{z}{L/2} \right)^2 + b \left(\dfrac{z}{L/2} \right)^4 \right]$

$u(\pm L/2) = 0$ KBC

Either solve 2 DOF problem OR

$EI\, u''(\pm L/2) = 0$ NBC extra

$0 = 1 + a + b$

$0 = \dfrac{2a}{(L/2)^2} + \dfrac{12\, b}{(L/2)^2}$

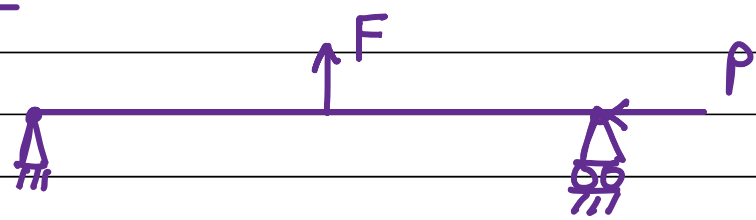
$$u = a_0 \left[1 - \frac{6}{5}\left(\frac{z}{L/2}\right)^2 + \frac{1}{5}\left(\frac{z}{L/2}\right)^4 \right]$$

$$\frac{d\Pi}{da_0} = 0 \Rightarrow P^* = \frac{168}{17}\frac{EI}{L^2}$$

$$= 9.882 \frac{EI}{L^2} > P^*_{exact}$$

$$\pi^2 \simeq 9.87$$

Example



$$\Pi = \int_0^L \left(\frac{EI}{2} u''^2 - \frac{P}{2} u'^2 \right) dz - F u\left(\frac{L}{2}\right)$$

Assume $u = az(L-z)$

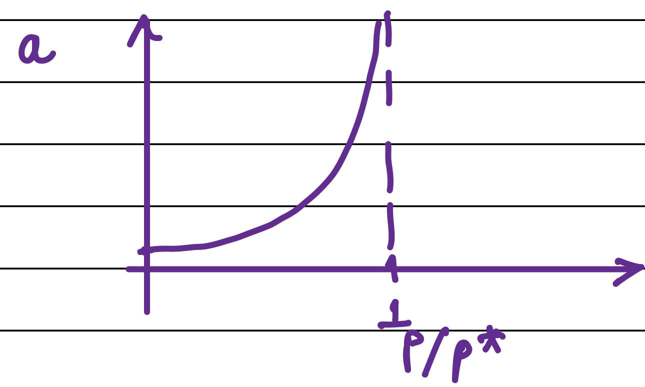
$$u\left(\frac{L}{2}\right) = \frac{aL^2}{4}$$

$$\Pi = 2EIa^2L - \frac{Pa^2L^3}{6} - \frac{FaL^2}{4}$$

Eqm $\frac{d\Pi}{da} = 0$

$$\Rightarrow \quad a = \frac{FL^2}{4\left(4EIL - \frac{PL^3}{3}\right)} = \frac{FL}{16EI\left(1 - \frac{P}{P^*}\right)}$$

$P^* = \frac{12EI}{L^2}$



Example

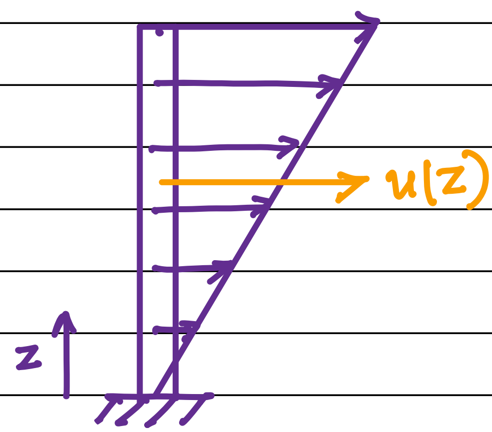
$w(z) = w_0 \frac{z}{H}$ wind load/length

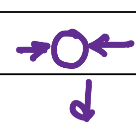
$A = \frac{\pi d^2}{4}$

$I = \frac{\pi d^4}{64}$

E, I, A

$g \downarrow$ H





Instability

As $H \to H^*$, $u \to \infty$

$$\Pi = \int_0^H \left(\frac{EI}{2} u''^2 - wu - \frac{P}{2} u'^2 \right) dz$$

$$P = mg(H-z)$$

$m = \rho A$ mass/ ht

Approx $u = cz^2$ KBCs $u(0)=0$ ✓

$u'(0)=0$ ✓

$$\Pi = 2c^2 EIH - w_0 \frac{cH^3}{4} - \frac{c^2 H^4 mg}{6}$$

Eqm $\frac{d\Pi}{dc} = 0$

$$C = \frac{w_0 H^3/4}{\left(4EIH - \frac{H^4 mg}{3}\right)}$$

As $H \to H^* = \left(\frac{12EI}{\rho A g}\right)^{1/3}$, $C \to \infty$

Max tower ht beyond which it buckles under its own weight