Linear models for classification: Perceptron

Problem setting

- Given data points $\mathbf{x}_i, i = 1, 2, \ldots, m$
- Possible class choices: $c_1, c_2, \ldots, c_k$
- Wish to estimate a mapping/classifier,

$$f : \mathbf{x} \to \{c_1, c_2, \ldots, c_k\}$$

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- In general, series of mappings

$$\mathbf{x} \xrightarrow{f(\cdot)} \mathbf{y} \xrightarrow{g(\cdot)} \mathbf{z} \xrightarrow{h(\cdot)} \{c_1, c_2, \ldots, c_k\}$$

  where y, z are in some latent space.

Perceptron Classifier

- Consider a binary classification problem: $f(\mathbf{x}) \in \{-1, +1\}$

- Objective: Learn a linear classifier

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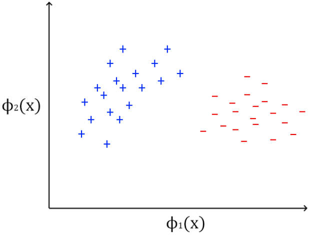
- Desirable: Any new input pattern similar to a seen pattern is **classified** correctly



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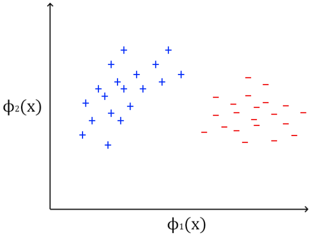

Linear Classification?

$\mathbf{w}^\top \phi(\mathbf{x}) + b \geq 0$ *for +ve points* $(y = +1)$

$\mathbf{w}^\top \phi(\mathbf{x}) + b < 0$ *for -ve points (y= -1)*

$\mathbf{w}, \phi \in \mathbb{R}^m$

Perceptron Classifier

- Often, b is indirectly captured by including it in $\mathbf{w}$, and using a ϕ as: $\phi_{aug} = [\phi, 1]$
- Thus, $\mathbf{w}^\top \phi(\mathbf{x})$

$$= \begin{bmatrix} w_1 & w_2 & w_3 & \dots & w_m & b \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_m \\ 1 \end{bmatrix}$$

- $\mathbf{w}^\top \phi(\mathbf{x}) = 0$ is the separating hyperplane.

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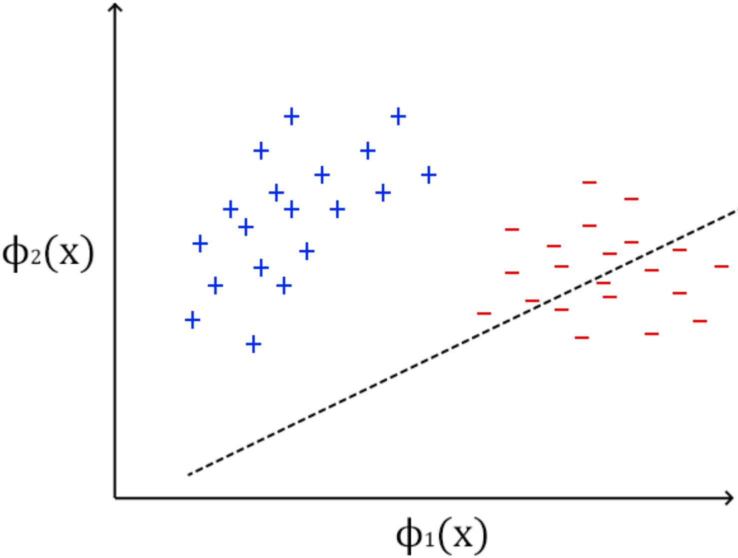
3. If not, marginally correct the weights ....
   - ....by adding to the weights a quantity that is proportional to the product of the input pattern with the desired output $y = \pm 1$

Perceptron Update Rule

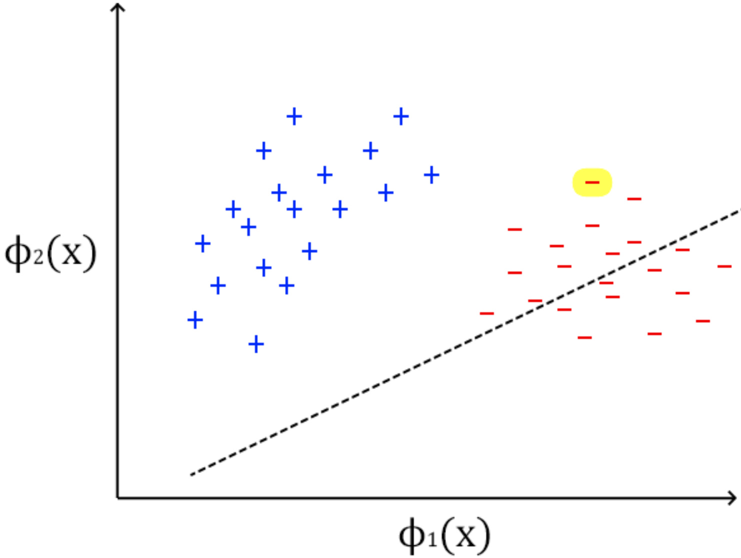
- Start with some weight vector $\mathbf{w}^{(0)}$, and
  for $k = 0, 1, 2, 3, \ldots, n$ (for every example denoted by ($\mathbf{x}$',y')), do:

$$\mathbf{w}^{k+1} = \mathbf{w}^{k} + y'\phi(\mathbf{x}')$$

- only when $\mathbf{x}'$ is misclassified by $(\mathbf{w}^k)^\top \phi(\mathbf{x})$ i.e. $y'(\mathbf{w}^k)^\top \phi(\mathbf{x}') < 0$



$\phi_2(x)$
$\phi_1(x)$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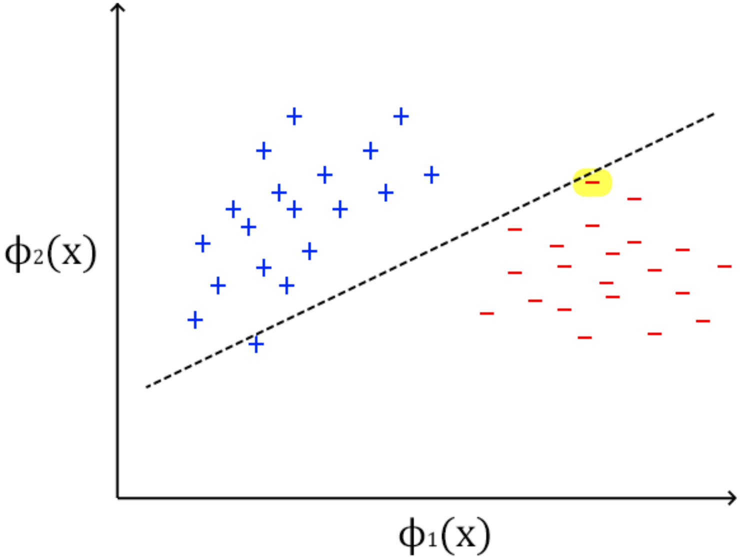

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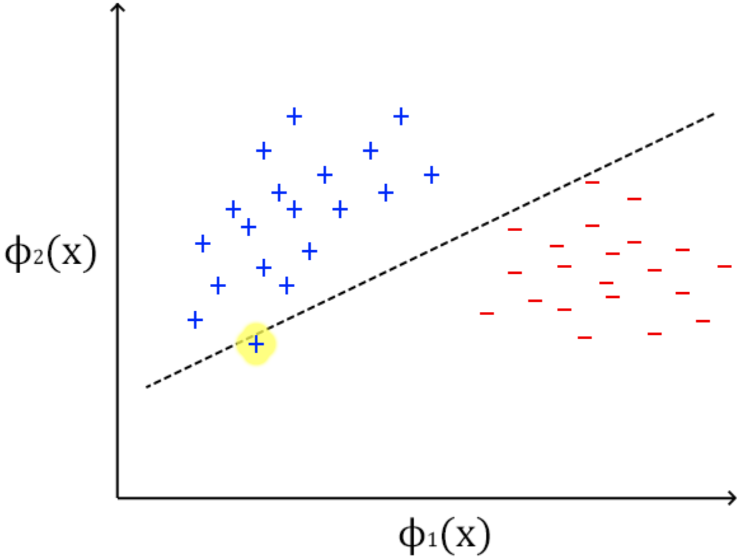

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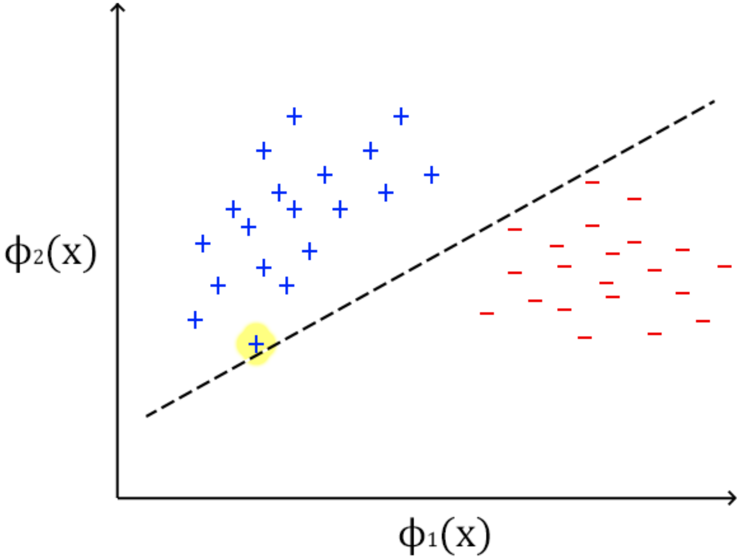

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- Perceptron does not find the *best* separating hyperplane, it finds *any* seperating hyperplane.
- The separating hyperplane does not provide enough breathing space – this is what SVMs address!

Perceptron Update Rule

- Perceptron works for two classes ($y = \pm 1$).
- A point is misclassified if $y\mathbf{w}^T(\phi(\mathbf{x})) < 0$.
- Perceptron Algorithm:
  - INITIALIZE: w=zeros()
  - REPEAT: for each $< \mathbf{x}, y >$
    - If $y\mathbf{w}^T\Phi(\mathbf{x}) < 0$

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    - endif

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$$\begin{aligned} y(\mathbf{w}^{(k+1)})^T\phi(\mathbf{x}) &= y\left(\mathbf{w}^k + \eta y \phi^T(\mathbf{x})\right)\phi(\mathbf{x}) \\ &= y(\mathbf{w}^k)^T\phi(\mathbf{x}) + \eta y^2\|\phi(\mathbf{x})\|^2 \\ &> y(\mathbf{w}^k)^T\phi(\mathbf{x}) \end{aligned}$$

Since $y(\mathbf{w}^k)^T\phi(\mathbf{x}) \leq 0$, we have $y(\mathbf{w}^{(k+1)})^T\phi(\mathbf{x}) \; > \; y(\mathbf{w}^k)^T\phi(\mathbf{x}) \Rightarrow$

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Summarily: Perceptron Update Rule

- Start with zero-weights vector, $\mathbf{w} \leftarrow \vec{0}$
- For each training instance, $\{\mathbf{x}_i, y_i\} \in \mathcal{D}$

  Predict its label using the current weight vector

$$\hat{y}_i = \begin{cases} +1 \text{ if } \mathbf{w}^T\phi(\mathbf{x_i}) \geq 0 \\ -1 \text{ otherwise} \end{cases}$$

  If $y_i \neq \hat{y}_i$ (or equivalently, $y_i\mathbf{w}^T\phi(\mathbf{x}_i) < 0$)

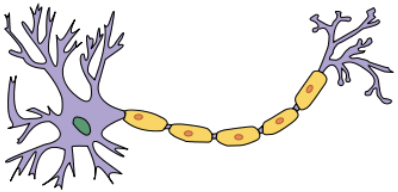
  Adjust the weight vector using the following update rule:

$$\mathbf{w} \leftarrow \mathbf{w} + y_i\phi(\mathbf{x}_i)$$

- Repeat until $\text{sign}(\mathbf{w}^T\mathbf{x}) = y$ for all $(\mathbf{x}, y) \in \mathcal{D}$

Why is it called a perceptron?

Loose inspiration drawn from neurons



Features are inputs; each feature associated with a weight. Activation quantified by the sum of weighted features.

Image from Wikimedia commons

Perceptron can model linearly separable functions

Convergence of Perceptron Update Rule: Formally

Convergence of the perceptron algorithm

Consider the case when data is linearly separable i.e. there exists a weight vector $\mathbf{u}$ s.t. $y = \text{sign}(\mathbf{u}^T\mathbf{x}) \quad \forall \{\mathbf{x}, y\} \in \mathcal{D}$. Without loss of generality, let us assume that $\mathbf{u}$ is a unit-length vector. Let us also assume that the data is scaled to lie in the Euclidean ball of radius 1, i.e., $||\mathbf{x}|| \leq 1 \quad \forall \mathbf{x} \in \mathcal{D}$.

Convergence of the perceptron algorithm

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Define a new quantity, margin of separation $\gamma = \min_{\mathbf{x} \in \mathcal{D}} |\mathbf{u}^T\mathbf{x}|$

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Theorem: If there exists a unit vector $\mathbf{u}$ such that $y\mathbf{u}^T\mathbf{x} \geq \gamma$ for all $\mathbf{x}$, then the number of weight updates made by the perceptron algorithm is at most $\frac{1}{\gamma^2}$.

Proof of the mistake bound for the perceptron algorithm

Let us track the following two quantities: $\mathbf{w}^T\mathbf{u}$ and $||\mathbf{w}||^2$.
Quantity 1: $\mathbf{w}^T\mathbf{u}$ on every update increases by at least γ.
For a positive example:

$$\begin{aligned}(\mathbf{w}+\mathbf{x})^T\mathbf{u} &= \mathbf{w}^T\mathbf{u} + \mathbf{x}^T\mathbf{u} \\ &\geq \mathbf{w}^T\mathbf{u} + \gamma\end{aligned}$$

Proof continued

Quantity 2: $||\mathbf{w}||^2$ after every update increases by at most 1.
For a positive example:

$$\begin{aligned} ||\mathbf{w}+\mathbf{x}||^2 = (\mathbf{w}+\mathbf{x})^T(\mathbf{w}+\mathbf{x}) &= ||\mathbf{w}||^2 + 2\mathbf{w}^T\mathbf{x} + ||\mathbf{x}||^2 \\ &\leq ||\mathbf{w}||^2 + ||\mathbf{x}||^2 \\ &\leq ||\mathbf{w}||^2 + 1 \end{aligned}$$

Proof continued

Putting these two inequalities together i.e. $\mathbf{w}_{i+1}^T\mathbf{u} \geq \mathbf{w}_i^T\mathbf{u} + \gamma$ and $||\mathbf{w}_{i+1}||^2 \leq ||\mathbf{w}_i||^2 + 1$, after k updates, we have

$$\mathbf{w}_k^T\mathbf{u} \geq k\gamma \text{ and } ||\mathbf{w}_k||^2 \leq k$$

Proof continued

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$$\Rightarrow \sqrt{k} \geq ||\mathbf{w}_k|| \geq \mathbf{w}_k^T\mathbf{u} \geq k\gamma$$

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$$\Rightarrow k \leq \frac{1}{\gamma^2}$$

What is the perceptron optimizing?

Hinge loss

Consider the following loss function for the perceptron algorithm:
$L_P(f_{\mathbf{w}}(\mathbf{x}), y) = \max(0, -y f_{\mathbf{w}}(\mathbf{x}))$

Hinge loss

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$L_P(f_{\mathbf{w}}(\mathbf{x}), y) = \max(0, -yf_{\mathbf{w}}(\mathbf{x}))$

Apply the stochastic gradient descent (SGD) algorithm:

Consider a set of examples $\{\mathbf{x}_i, y_i\} \in \mathcal{D}$. Choose an example $\{\mathbf{x}_t, y_t\}$.

1. Compute the gradient of the loss of $f_{\mathbf{w}}(\mathbf{x}_t)$ with respect to $\mathbf{w}$, $\nabla_{\mathbf{w}} L(f_{\mathbf{w}}(\mathbf{x}_t), y)$
2. Update the weight vector: $\mathbf{w} \leftarrow \mathbf{w} - \nabla_{\mathbf{w}} L(f_{\mathbf{w}}(\mathbf{x}_t), y)$

SGD and the perceptron algorithm

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Step 2 of SGD gives $\mathbf{w} \leftarrow \mathbf{w} + y\mathbf{x}$ which is exactly the perceptron update rule!

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- **Averaged Perceptron**: Use the averaged weight vector at the end of the algorithm rather than voting.

Beyond linearly separable data?