6.4 Spectral Radius and Convergence Theorem

In this section, we prove a necessary and sufficient condition for convergence of any stationary iterative method. The condition is based on the spectral radius of the iterative matrix of the method. We start by defining stationary iterative methods.

Definition 6.4.1 [Stationary Iterative Methods].

An iterative method for a system of n linear equation is called a (one-step) ***stationary iterative method*** if the terms of the iterative sequence $\{\boldsymbol{x}^{(k)}\}$ can be written in the form

$$\boldsymbol{x}^{(k+1)} = B\boldsymbol{x}^{(k)} + \boldsymbol{c}, \tag{6.23}$$

for some $B \in M_n(\mathbb{R})$ and $\boldsymbol{c} \in \mathbb{R}^n$.

Remark 6.4.2.

Observe that Jacobi and Gauss-Seidel methods are stationary methods. In fact, we can construct our own stationary iterative method for a linear system of the form $A\boldsymbol{x} = \boldsymbol{b}$ through the following steps:

1. first writing the coefficient matrix A in the form $A = X + Y$, for some $X, Y \in M_n(\mathbb{R})$, with X invertible;
2. then rewriting the given system as $X\boldsymbol{x} = -Y\boldsymbol{x} + \boldsymbol{b}$; and
3. finally get the stationary method by taking $B = -X^{-1}Y$ and $\boldsymbol{c} = X^{-1}\boldsymbol{b}$.

We now define the notion of *spectral radius* in the general context.

Definition 6.4.3 [Spectral Radius].

Let $A \in M_n(\mathbb{C})$ and let $\lambda_i \in \mathbb{C},\ i = 1, 2, \ldots, n$, be the eigenvalues of A. The ***spectral radius*** of A is defined as

$$\rho(A) = \max_{j=1,2,\ldots,n} |\lambda_j|.$$

We now list some of the important results concerning spectral radius.

Lemma 6.4.4.

For any subordinate matrix norm $\|\cdot\| : M_n(\mathbb{C}) \to [0, \infty)$,

$$\rho(A) \leq \|A\|, \quad \text{for all } A \in M_n(\mathbb{C}).$$

Proof of the above lemma is left as an exercise.

The following theorem says that $\rho(A)$ is the greatest lower bound for any subordinate matrix norm of a given matrix A.

Theorem 6.4.5.

For each $A \in M_n(\mathbb{C})$ and each $\epsilon > 0$, there exists a subordinate matrix norm $\|\cdot\|$ such that

$$\rho(A) \leq \|A\| < \rho(A) + \epsilon.$$

Proof of the above theorem is omitted for this course.

Lemma 6.4.6.

Let $B \in M_n(\mathbb{C})$. Then $\rho(B) < 1$ if and only if

$$\lim_{n\to\infty} B^n \boldsymbol{z} = \boldsymbol{0}, \quad \text{for every } \boldsymbol{z} \in \mathbb{C}^n.$$

Proof is omitted for this course.

Lemma 6.4.7.

Let $B \in M_n(\mathbb{C})$ with $\rho(B) < 1$. Then $(I - B)^{-1}$ exists and we have

$$(I - B)^{-1} = I + B + B^2 + \ldots.$$

Proof is omitted for this course.

We now give the main convergence theorem for any stationary iterative method.

Theorem 6.4.8 [Necessary and Sufficient Conditions].

For any $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\boldsymbol{x}^{(k)}\}$ defined by (6.23) converges to the solution of $\boldsymbol{x} = B\boldsymbol{x} + \boldsymbol{c}$ if and only if

$$\rho(B) < 1.$$

Proof.

We first assume that $\rho(B) < 1$ and prove that the sequence $\{\boldsymbol{x}^{(k)}\}$ converges for any

$\boldsymbol{x}^{(0)} \in \mathbb{R}^n$. The proof follows from the above two lemmas.

Conversely, assume that the sequence $\{\boldsymbol{x}^{(k)}\}$ converges for any $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ and prove that $\rho(B) < 1$.

We give the proof only for the particular case when all the eigenvalues of B are real. The proof for the general case is omitted for this course.

On the contrary assume that $\rho(B) \geq 1$. Let I be such that $\rho(B) = |\lambda_I|$. Let $\boldsymbol{v}_I$ be an eigenvector corresponding to the eigenvalue λ_I. Choose the initial guess $\boldsymbol{x}^{(0)}$ as $\boldsymbol{x}^{(0)} := \boldsymbol{x} + \boldsymbol{v}_I$ where $\boldsymbol{x}$ satisfies $\boldsymbol{x} = B\boldsymbol{x} + \boldsymbol{c}$. Then, using (6.23), we get

$$\boldsymbol{x}^{(k+1)} - \boldsymbol{x} \;=\; \lambda_I^{k+1}\boldsymbol{v}_I.$$

Since, we assumed $\rho(B) \geq 1$, the above equation implies $\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}\| \to \infty$ as $k \to \infty$, which is a contradiction.

6.5 Exercises

Iterative Methods

1. Let A be a diagonally dominant matrix such that $a_{ij} = 0$ for every $i, j \in \{1, 2, \cdots, n\}$ such that $i > j + 1$. Does naive Gaussian elimination method preserve the diagonal dominance? Justify your answer.

2. Let A be a diagonally dominant matrix. Show that all the diagonal elements of A are non-zero (*i.e.,* $a_{ii} \neq 0$ for $i = 1, 2, \cdots, n$.). As a consequence, the iterating sequences of Jacobi and Gauss-Seidel methods are well-defined if the coefficient matrix A in the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is a diagonally dominant matrix.

3. Write the formula for the Jacobi iterative sequence of the system

$$\begin{aligned} 7x_1 - 15x_2 - 21x_3 &= 2, \\ 7x_1 - x_2 - 5x_3 &= -3, \\ 7x_1 + 5x_2 + x_3 &= 1. \end{aligned}$$

   Without performing the iterations, show that the sequence does not converge to the exact solution of this system. Can you make a suitable interchange of rows so that the resulting system is diagonally dominants?

4. Find the $n \times n$ matrix B and the n-dimensional vector $\boldsymbol{c}$ such that the Gauss-Seidel method can be written in the form

$$\boldsymbol{x}^{(k+1)} = B\boldsymbol{x}^{(k)} + \boldsymbol{c}, \qquad k = 0, 1, 2, \cdots$$

Here B is called the *iterative matrix* for the Gauss-Seidel method.

5. Check for the convergence of the Jacobi and Gauss-Seidel methods for each of the following systems.

   i)

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0.12, \\ 1.75x_1 + 7x_2 + 0.5x_3 &= 0.1, \\ x_1 + 0.2x_2 + 4.5x_3 &= 0.5. \end{aligned}$$

   ii)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 1, \\ x_1 + x_2 - x_3 &= 1, \\ 2x_1 - 2x_2 + x_3 &= 1. \end{aligned}$$

   iii)

$$\begin{aligned} x_1 + x_2 + 10x_3 &= -1, \\ 2x_1 + 3x_2 + 5x_3 &= -6, \\ 3x_1 + 2x_2 - 3x_3 &= 4. \end{aligned}$$

CHAPTER 7

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with real entries. Eigenvalues of A are defined as the roots of the equation

$$\det(\lambda I - A) = 0. \tag{7.1}$$

Note that $\det(\lambda I - A)$ is a polynomial in λ of degree n, which is known as *characteristic polynomial* of the matrix A. We know that even if A has real entries, the eigenvalues need not be real numbers. It is also a known fact that for every matrix, there are n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (in this list, each eigenvalue is repeated as many times as its algebraic multiplicity, *i.e.,* multiplicity of the eigenvalue as a root of the characteristic polynomial).

When $n = 2$ the characteristic polynomial is a quadratic polynomial for which there is a nice formula for computing roots. When $n = 3$, there is a formula that many of us do not remember. When $n = 4$, none has a formula. But computing eigenvalues is important for applications. Therefore, numerically approximating the eigenvalues is the only way out.

One obvious way of approximating an eigenvalue of a matrix is to first obtain the characteristic polynomial (7.1) explicitly in λ and then use one of the numerical methods discussed in the next chapter (on methods for solving nonlinear equations) to compute a root of this polynomial. But this is not an efficient way of computing eigenvalues because of two reasons. One reason is that obtaining explicit form of (7.1) is itself a difficult task when the dimension of the matrix is very large. Secondly, if we make any small error (like floating-point error) in obtaining the explicit form of the polynomial (7.1), the resulting polynomial may have a root which is entirely different from any of

the eigenvalues that we are looking for. This is illustrated in the following example by Wilkinson where we see that "the roots of polynomials are extremely sensitive to perturbations in the coefficients".

Example 7.0.1 [Wilkinson's example].

Let $f(x)$ and $g(x)$ be two polynomials given by

$$f(x) = (x-1)(x-2)\cdots(x-10), \qquad g(x) = x^{10}.$$

The roots of the polynomial $f(x)$ are $1, 2, \cdots, 10$, and all these roots are simple roots. If we perturb this polynomial as $F(x) = f(x) + 0.01g(x)$, then all the roots lie in the interval $[1, 3.5]$ (verified graphically). In fact, the largest root of the polynomial $f(x)$ is 10 and the largest root of the polynomial $F(x)$ is approximately equal to 3.398067. Thus, if the coefficient of x^{10} is perturbed by a small amount of 0.01, the root 10 of $f(x)$ could move as much a distance as approximately 6.6.

Due to the two reasons discussed above, we look for an alternate method to compute the eigenvalues of a given matrix. One such method is the *power method* that can be used to obtain the eigenvalue which is the largest in magnitude among all the other eigenvalues and the corresponding eigen vector. In Subsection 7.1, we present the power method and discuss the condition under which this method can be applied. In Subsection 7.2 we prove the Gerschgorin theorem which may be used as a tool to find a class of matrices for which power method can be applied successfully.

7.1 Power Method

There are many variations of power method in the literature. We will present the most elementary form of power method. We always deal with matrices with real entries, all of whose eigenvalues are real numbers.

Power method is used to obtain a specific eigenvalue called *dominant eigenvalue* and a corresponding eigenvector for a given $n \times n$ matrix A. The concept of a dominant eigenvalue plays a very important role in many applications. The power method provides an approximation to it under some conditions on the matrix. We now define the concept of a dominant eigenvalue.

Definition 7.1.1 [Dominant Eigenvalue of a Matrix].

An eigenvalue λ of an $n \times n$ matrix A is said to be a ***dominant eigenvalue*** of A if

$$|\lambda| = \max\{\, |z| \,:\, z \text{ is an eigenvalue of } A \,\}.$$

Remark 7.1.2.

1. If a dominant eigenvalue of a matrix is equal to zero, then all its eigenvalues are zero and an eigenvector can be found by solving the linear system $A\boldsymbol{x} = \mathbf{0}$.
2. If λ is a dominant eigenvalue of a matrix A, then there is no other eigenvalue of A whose distance from zero is more than that of λ from zero.
3. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of A (repeated according to their algebraic multiplicities) of an $n \times n$ matrix A. These eigenvalues can be renamed (relabelled, reindexed) as $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that they satisfy the condition:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

Note that λ_1 is a *dominant eigenvalue* of A.

A matrix may have a unique dominant eigenvalue or more than one dominant eigenvalues. Further, even if dominant eigenvalue is unique the corresponding algebraic and geometric multiplicities could be more than one, and also both algebraic and geometric multiplicities may not be the same. All these possibilities are illustrated in the following example.

Example 7.1.3.

1. The matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

has eigenvalues 1, -1, and -2. The matrix A has a unique dominant eigenvalue, which is -2 as this is the largest in absolute value, of all eigenvalues. Note that the dominant eigenvalue of A is a simple eigenvalue.

2. The matrix

$$B = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

has eigenvalues 1, -2, and 2. According to our definition, the matrix B has two dominant eigenvalues. They are -2 and 2. Note that both the dominant eigenvalues of B are simple eigenvalues.

3. Consider the matrices

$$C_1 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}, C_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, C_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The matrix C_1 has a unique dominant eigenvalue 2, which has algebraic multiplicity 2 and geometric multiplicity 1. The matrix C_2 has a unique dominant eigenvalue 2, whose algebraic and geometric multiplicities equal 2. The matrix C_3 has a unique dominant eigenvalue 2, which has algebraic multiplicity 2 and geometric multiplicity 1.

As mentioned above the power method is used to compute the dominant eigenvalue and the corresponding eigen vector of a given $n \times n$ matrix provided this eigenvalue is unique. Thus, in the above examples, power method can be used for the matrices A but not for B even though B has distinct eigenvalues. Let us now detail the ***power method***.

Hypotheses for which the power method can work

Assume that an $n \times n$ matrix A has real eigenvalues λ_1, λ_2, $\cdots$, and λ_n (repeated according to their algebraic multiplicities) with the following properties:

(H1) The eigenvalues are such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \tag{7.2}$$

That is, A has a unique dominant eigenvalue λ_1 which is a simple eigenvalue.

(H2) There exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. That is, there exists $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ satisfying $A\boldsymbol{v}_k = \lambda_k \boldsymbol{v}_k$ for $k = 1, 2, \cdots, n$; and such that for each $\boldsymbol{v} \in \mathbb{R}^n$ there exists unique real numbers $c_1, c_2, \cdots, c_n$ such that

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n.$$

Equivalently, the matrix A is diagonalizable.

(H3) An initial guess $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ be chosen such that

$$\boldsymbol{x}^{(0)} = \sum_{j=1}^{n} c_j \boldsymbol{v}_j, \tag{7.3}$$

for some scalars $c_1, c_2, \cdots, c_n \in \mathbb{R}$ with $c_1 \neq 0$ and $\boldsymbol{x}^{(0)} \notin \bigcup_{k=1}^{\infty} Ker A^k$.

Let us now discuss the key idea behind power method.

- Choose a non-zero vector $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ arbitrarily so that we can find scalars c_1, c_2, $\cdots$, $c_n \in \mathbb{R}$ such that

$$\boldsymbol{x}^{(0)} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n, \quad c_1 \neq 0.$$

- Pre-multiplying by A and substituting $A\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$, $i = 1, \cdots, n$, we get

$$A\boldsymbol{x}^{(0)} = c_1\lambda_1\boldsymbol{v}_1 + \cdots + c_n\lambda_n\boldsymbol{v}_n = \lambda_1\left(c_1\boldsymbol{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)\boldsymbol{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)\boldsymbol{v}_n\right).$$

  Note here that we have assumed $\lambda_1 \neq 0$, which follows from the Assumption (1) above.
- Pre-multiplying by A again and simplying, we get

$$A^2\boldsymbol{x}^{(0)} = \lambda_1^2\left(c_1\boldsymbol{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^2\boldsymbol{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^2\boldsymbol{v}_n\right)$$

- For each $k \in \mathbb{N}$, applying A k-times on $\boldsymbol{x}^{(0)}$ yields

$$A^k\boldsymbol{x}^{(0)} \;=\; \lambda_1^k\left(c_1\boldsymbol{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\boldsymbol{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\boldsymbol{v}_n\right) \tag{7.4}$$

- Using the assumption (7.2), we get $|\lambda_j/\lambda_1| < 1$, for $j = 2, \cdots, n$. Therefore, we have

$$\lim_{k\to\infty}\frac{A^k\boldsymbol{x}^{(0)}}{\lambda_1^k} = c_1\boldsymbol{v}_1. \tag{7.5}$$

  For $c_1 \neq 0$, the right hand side of the above equation is a scalar multiple of the eigenvector.
- From the above expression for $A^k\boldsymbol{x}^{(0)}$, we also see that

$$\lim_{k\to\infty}\frac{(A^{k+1}\boldsymbol{x}^{(0)})_i}{(A^k\boldsymbol{x}^{(0)})_i} = \lambda_1, \tag{7.6}$$

  where i is any index such that the fractions on the left hand side are meaningful (which is the case when $\boldsymbol{x}^{(0)} \notin \cup_{k=1}^{\infty}\mathrm{K}erA^k$).

The power method generates two sequences $\{\mu_k\}$ and $\{\boldsymbol{x}^{(k)}\}$, using the results (7.5) and (7.6), that converge to the dominant eigenvalue λ_1 and the corresponding eigenvectors $\boldsymbol{v}_1$, respectively.

We now describe the steps involved in the power method for generating these two sequences.

Setting up the iterative sequences:

Step 1: Choose a vector $\boldsymbol{x}^{(0)}$ arbitrarily and set $\boldsymbol{y}^{(1)} := A\boldsymbol{x}^{(0)}$.

Step 2: Define $\mu_1 := y_i^{(1)}$, where $i \in \{1, \cdots, n\}$ is the least index such that

$$\|\boldsymbol{y}^{(1)}\|_\infty = |y_i^{(1)}|$$

and set

$$\boldsymbol{x}^{(1)} := \frac{\boldsymbol{y}^{(1)}}{\mu_1}.$$

Step 3: From $\boldsymbol{x}^{(1)}$, we can obtain μ_2 and $\boldsymbol{x}^{(2)}$ as in step 2.

Continue this procedure.

General form of the power method iterative sequences:
After choosing the initial vector $\boldsymbol{x}^{(0)}$ arbitrarily, we generate the sequences $\{\mu^{(k)}\}$ and $\{\boldsymbol{x}^{(k)}\}$ using the formulas

$$\mu_{k+1} = y_i^{(k+1)}, \quad \boldsymbol{x}^{(k+1)} = \frac{\boldsymbol{y}^{(k+1)}}{\mu_{k+1}}, \tag{7.7}$$

where

$$\boldsymbol{y}^{(k+1)} = A\boldsymbol{x}^{(k)} \text{ and } i \text{ is such that } |y_i^{(k+1)}| = \|\boldsymbol{y}^{(k+1)}\|_\infty, \tag{7.8}$$

for $k = 0, 1, \cdots$.

This iterative procedure is called the ***power method***.

Remark 7.1.4.

The scaling factor μ_k introduced in (7.7) makes sure that $\boldsymbol{x}^{(k)}$ has its maximum norm equal to 1, *i.e.*, $\|\boldsymbol{x}^{(k)}\|_\infty = 1$. This rules out the possibilities of $\lim_{k\to\infty} \boldsymbol{x}^{(k)}$ being $\mathbf{0}$ or the vector $\boldsymbol{x}^{(k)}$ escaping to infinity.

We now discuss the sufficient conditions under which the power method converges.

Theorem 7.1.5 [Convergence Theorem for Power method].

Hypothesis: Let A be an $n \times n$ matrix with real eigenvalues having the following properties:

(H1) A has a unique dominant eigenvalue λ_1 which is a simple eigenvalue. That is,

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A (repeated according to their algebraic multiplicities).

(H2) A has n linearly independent real eigenvectors, $\boldsymbol{v}_i$, $i = 1, \cdots, n$.

(H3) An initial guess $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ be chosen such that

$$\boldsymbol{x}^{(0)} = \sum_{j=1}^{n} c_j \boldsymbol{v}_j, \tag{7.9}$$

for some scalars $c_1, c_2, \cdots, c_n \in \mathbb{R}$ with $c_1 \neq 0$ and $\boldsymbol{x}^{(0)} \notin \bigcup_{k=1}^{\infty} \mathrm{K}er A^k$.

Conclusion: Then, in the power method (7.7)-(7.8),

1. the sequence $\{\mu_k\}$ converges to the dominant eigenvalue λ_1 and
2. a subsequence of the sequence $\{\boldsymbol{x}_k\}$ converges to an eigenvector corresponding to the eigenvalue λ_1.

Proof.

From the definition of $\boldsymbol{x}^{(k+1)}$, we have

$$\boldsymbol{x}^{(k+1)} = m_{k+1} A^{k+1} \boldsymbol{x}^{(0)},$$

where $m_{k+1} = 1/(\mu_1 \mu_2 \cdots \mu_{k+1})$.

But, $\boldsymbol{x}^{(0)} = \sum_{j=1}^{n} c_j \boldsymbol{v}_j$, $c_1 \neq 0$. Therefore

$$\boldsymbol{x}^{(k+1)} = m_{k+1} \lambda_1^{k+1} \left(c_1 \boldsymbol{v}_1 + \sum_{j=2}^{n} c_j \left(\frac{\lambda_j}{\lambda_1} \right)^{k+1} \boldsymbol{v}_j \right).$$

Taking maximum norm on both sides and noting that $\|\boldsymbol{x}^{(k)}\|_\infty = 1$, we get

$$1 = \left| m_{k+1} \lambda_1^{k+1} \right| \left\| c_1 \boldsymbol{v}_1 + \sum_{j=2}^{n} c_j \left(\frac{\lambda_j}{\lambda_1} \right)^{k+1} \boldsymbol{v}_j \right\|_\infty.$$

Since $|\lambda_j/\lambda_1|^k \to 0$ as $k \to \infty$, we get

$$\lim_{k \to \infty} \left| m_{k+1} \lambda_1^{k+1} \right| = \frac{1}{|c_1| \|\boldsymbol{v}_1\|_\infty} < \infty.$$

Using this, we get

$$\lim_{k\to\infty} \boldsymbol{x}^{(k+1)} = \lim_{k\to\infty} m_{k+1}\lambda_1^{k+1}c_1\boldsymbol{v}_1 = \begin{cases} \text{either} & +\dfrac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|_\infty} \\ \text{or} & -\dfrac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|_\infty} \\ \text{or} & \text{oscillates between} \\ & \text{the above two vectors} \end{cases} . \quad (7.10)$$

This completes the proof of Conclusion (2).

Let us now prove that $\mu_k \to \lambda_1$. For this, we first note that

$$\lim_{k\to\infty} \boldsymbol{y}^{(k+1)} = K\lambda_1\boldsymbol{v}_1,$$

up to a subsequence, where $K = \pm 1/\|\boldsymbol{v}_1\|_\infty$. Since $\boldsymbol{v}_1$ is an eigen vector, there is at least one non-zero component of $\boldsymbol{v}_1$. We choose one such component of $\boldsymbol{v}_1$ and denote it by $(v_1)_j$. Since $(v_1)_j \neq 0$ and $y_j^{(k+1)} \to K\lambda_1(v_1)_j$, as $k \to \infty$, there exists an integer $N > 0$ such that

$$y_j^{(k+1)} \neq 0, \text{ for all } k \geq N.$$

Similarly, we see that

$$x_j^{(k+1)} \neq 0, \text{ for all } k \geq N.$$

Therefore, we can write

$$\mu_{k+1} = \frac{y_j^{(k+1)}}{x_j^{(k+1)}} = \frac{(A\boldsymbol{x}^{(k)})_j}{(\boldsymbol{x}^{(k+1)})_j},$$

where j denotes the component as given above. Taking limit, we have

$$\lim_{k\to\infty} \mu_{k+1} \quad = \quad \lambda_1.$$

which gives the desired result.

Note that the above theorem does not guarantee the convergence of the sequence $\{\boldsymbol{x}_n\}$ to an eigenvector of the dominant eigenvalue. However, if the dominant eigenvalue has an eigenvector with a unique dominant component, then this sequence converges as discussed in the following theorem.

Theorem 7.1.6 [Second Convergence Theorem for Power Method].

Let A be an $n \times n$ matrix satisfying the hypotheses **(H1)**, **(H2)**, and **(H3)** of Theorem 7.1.5. In addition to these hypotheses,

(H4) let $\boldsymbol{v}_1 = (v_{11}, v_{12}, \cdots, v_{1n})^T$ be such that there exists a unique index $j \in \{1, 2, \cdots, n\}$ with the property

$$|v_{1j}| = \|\boldsymbol{v}_1\|_\infty \tag{7.11}$$

This hypothesis is referred to as $\boldsymbol{v}_1$ **has a single maximal component.**

Conclusion: Then, in the power method (7.7)-(7.8),

1. the sequence $\{\mu_k\}$ converges to the dominant eigenvalue λ_1 and
2. The sequence of vectors $\boldsymbol{x}^{(k)}$ converges to an eigenvector corresponding to the dominant eigenvalue λ_1.

Proof.

Let us first set up some notation. Let the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ be given by

$$\boldsymbol{v}_j = (v_{j1}, v_{j2}, \cdots, v_{jn})^T, \text{ for } j = 1, 2, \cdots, n. \tag{7.12}$$

Since $\boldsymbol{x}^{(0)} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$, we have

$$A\boldsymbol{x}^{(0)} = c_1\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + \cdots + c_n\lambda_n\boldsymbol{v}_n.$$

In coordinate form, we have

$$A\boldsymbol{x}^{(0)} = (\lambda_1 c_1 v_{11} + \lambda_2 c_2 v_{21} + \cdots + \lambda_n c_n v_{n1}, \cdots, \lambda_1 c_1 v_{1n} + \lambda_2 c_2 v_{2n} + \cdots + \lambda_n c_n v_{nn})^T \tag{7.13}$$

In fact, for each $k \in \mathbb{N}$, we have

$$A^k\boldsymbol{x}^{(0)} = (\lambda_1^k c_1 v_{11} + \lambda_2^k c_2 v_{21} + \cdots + \lambda_n^k c_n v_{n1}, \cdots, \lambda_1^k c_1 v_{1n} + \lambda_2^k c_2 v_{2n} + \cdots + \lambda_n^k c_n v_{nn})^T \tag{7.14}$$

Maximum norm of the vector $A^k\boldsymbol{x}^{(0)}$ is going to be the modulus of one of its components. From **(H4)**, we have

$$\frac{|v_{1i}|}{|v_{1j}|} \leq 1 \quad \text{for } i = 1, 2, \cdots, n. \tag{7.15}$$

Observe that

$$\frac{(A^k\boldsymbol{x}^{(0)})_i}{(A^k\boldsymbol{x}^{(0)})_j} = \frac{\lambda_1^k c_1 v_{1i} + \lambda_2^k c_2 v_{2i} + \cdots + \lambda_n^k c_n v_{ni}}{\lambda_1^k c_1 v_{1j} + \lambda_2^k c_2 v_{2j} + \cdots + \lambda_n^k c_n v_{nj}}.$$

The last equation can be written in the form

$$\frac{(A^k\boldsymbol{x}^{(0)})_i}{(A^k\boldsymbol{x}^{(0)})_j} = \frac{v_{1i} + \frac{\lambda_2^k c_2}{\lambda_1^k c_1} v_{2i} + \cdots + \frac{\lambda_n^k c_n}{\lambda_1^k c_1} v_{ni}}{v_{1j} + \frac{\lambda_2^k c_2}{\lambda_1^k c_1} v_{2j} + \cdots + \frac{\lambda_n^k c_n}{\lambda_1^k c_1} v_{nj}} \tag{7.16}$$

Note that the RHS in the equation (7.16) converges to $\dfrac{v_{1i}}{v_{1j}}$ as $k \to \infty$. Since,

$$\left|\frac{v_{1i}}{v_{1j}}\right| < 1, \qquad \text{for} \quad i \neq j, \tag{7.17}$$

we can conclude that there exists a $K \in \mathbb{N}$ such that for $k \geq K$,

$$\left|\frac{(A^k\boldsymbol{x}^{(0)})_i}{(A^k\boldsymbol{x}^{(0)})_j}\right| < 1. \tag{7.18}$$

As a consequence, the maximum norm of $A^k\boldsymbol{x}^{(0)}$ is equal to $|(A^k\boldsymbol{x}^{(0)})_j|$ where j is as given in **(H4)**.

For $k \geq K$, the sequence μ_k is given by

$$\mu_k = \frac{(A^k\boldsymbol{x}^{(0)})_j}{(A^{k-1}\boldsymbol{x}^{(0)})_j} = \lambda_1 \frac{v_{1j} + \frac{\lambda_2^k c_2}{\lambda_1^k c_1} v_{2j} + \cdots + \frac{\lambda_n^k c_n}{\lambda_1^k c_1} v_{nj}}{v_{1j} + \frac{\lambda_2^{k-1} c_2}{\lambda_1^{k-1} c_1} v_{2j} + \cdots + \frac{\lambda_n^{k-1} c_n}{\lambda_1^{k-1} c_1} v_{nj}}. \tag{7.19}$$

Thus,

$$\lim_{k\to\infty} \mu_k = \lambda_1. \tag{7.20}$$

For $k \geq K$, the sequence $\boldsymbol{x}^{(k)}$ is given by

$$\begin{aligned} \boldsymbol{x}^{(k)} &= \frac{A^k\boldsymbol{x}^{(0)}}{(A^k\boldsymbol{x}^{(0)})_j} \\ &= \left(\frac{(A^k\boldsymbol{x}^{(0)})_1}{(A^k\boldsymbol{x}^{(0)})_j}, \cdots, \frac{(A^k\boldsymbol{x}^{(0)})_{j-1}}{(A^k\boldsymbol{x}^{(0)})_j}, 1, \frac{(A^k\boldsymbol{x}^{(0)})_{j+1}}{(A^k\boldsymbol{x}^{(0)})_j}, \cdots, \frac{(A^k\boldsymbol{x}^{(0)})_n}{(A^k\boldsymbol{x}^{(0)})_j}\right)^T. \end{aligned}$$

In view of (7.16), we now conclude that the sequence $\boldsymbol{x}^{(k)}$ converges to $\frac{1}{v_{1j}}\boldsymbol{v}_1$ which is an eigenvector corresponding to the dominant eigenvalue λ_1.

We now give a numerical example illustrating the power method procedure.

Example 7.1.7.

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = 3,\ \lambda_2 = 1 \text{ and } \lambda_3 = 0.$$

The corresponding eigen vectors are

$$\boldsymbol{v}_1 = (1, 0, 2)^T,\ \boldsymbol{v}_2 = (0, 2, -5)^T \text{ and } \boldsymbol{v}_3 = (0, 1, -3)^T.$$

Thus, the hypothesis **(H1)** and **(H2)** are satisfied. Choose the initial guess $\boldsymbol{x}_0 = (1, 0.5, 0.25)^T$, which also satisfies the hypothesis **(H3)**.

The first ten terms of the iterative sequence in power method given by (7.7)-(7.8) for the given matrix A are as follows:

Iteration No: 1

$$\begin{aligned} \boldsymbol{y}_1 &= A\boldsymbol{x}_0 = (3.000000, -0.500000, 7.250000)^T \\ \mu_1 &= 7.250000 \\ \boldsymbol{x}_1 &= \frac{\boldsymbol{y}_1}{\mu_1} = (0.413793, -0.068966, 1.000000)^T \end{aligned}$$

Iteration No: 2

$$\begin{aligned} \boldsymbol{y}_2 &= A\boldsymbol{x}_1 = (1.241379, -0.068966, 2.655172)^T \\ \mu_2 &= 2.655172 \\ \boldsymbol{x}_2 &= \frac{\boldsymbol{y}_2}{\mu_2} = (0.467532, -0.025974, 1.000000)^T \end{aligned}$$

Iteration No: 3

$$\begin{aligned} \boldsymbol{y}_3 &= A\boldsymbol{x}_2 = (1.402597, -0.025974, 2.870130)^T \\ \mu_3 &= 2.870130 \\ \boldsymbol{x}_3 &= \frac{\boldsymbol{y}_3}{\mu_3} = (0.488688, -0.009050, 1.000000)^T \end{aligned}$$

Iteration No: 4

$$\begin{aligned} \boldsymbol{y}_4 &= A\boldsymbol{x}_3 = (1.466063, -0.009050, 2.954751)^T \\ \mu_4 &= 2.954751 \\ \boldsymbol{x}_4 &= \frac{\boldsymbol{y}_4}{\mu_4} = (0.496172, -0.003063, 1.000000)^T \end{aligned}$$

Iteration No: 5

$$\begin{aligned} \boldsymbol{y}_5 &= A\boldsymbol{x}_4 = (1.488515, -0.003063, 2.984686)^T \\ \mu_5 &= 2.984686 \\ \boldsymbol{x}_5 &= \frac{\boldsymbol{y}_5}{\mu_5} = (0.498717, -0.001026, 1.000000)^T \end{aligned}$$

Iteration No: 6

$$\begin{aligned} \boldsymbol{y}_6 &= A\boldsymbol{x}_5 = (1.496152, -0.001026, 2.994869)^T \\ \mu_6 &= 2.994869 \\ \boldsymbol{x}_6 &= \frac{\boldsymbol{y}_6}{\mu_6} = (0.499572, -0.000343, 1.000000)^T \end{aligned}$$

Iteration No: 7

$$\begin{aligned} \boldsymbol{y}_7 &= A\boldsymbol{x}_6 = (1.498715, -0.000343, 2.998287)^T \\ \mu_7 &= 2.998287 \\ \boldsymbol{x}_7 &= \frac{\boldsymbol{y}_7}{\mu_7} = (0.499857, -0.000114, 1.000000)^T \end{aligned}$$

Iteration No: 8

$$\begin{aligned} \boldsymbol{y}_8 &= A\boldsymbol{x}_7 = (1.499571, -0.000114, 2.999429)^T \\ \mu_8 &= 2.999429 \\ \boldsymbol{x}_8 &= \frac{\boldsymbol{y}_8}{\mu_8} = (0.499952, -0.000038, 1.000000)^T \end{aligned}$$

Iteration No: 9

$$\begin{aligned} \boldsymbol{y}_9 &= A\boldsymbol{x}_8 = (1.499857, -0.000038, 2.999809)^T \\ \mu_9 &= 2.999809 \\ \boldsymbol{x}_9 &= \frac{\boldsymbol{y}_9}{\mu_9} = (0.499984, -0.000013, 1.000000)^T \end{aligned}$$

Iteration No: 10

$$\begin{aligned} \boldsymbol{y}_{10} &= A\boldsymbol{x}_9 = (1.499952, -0.000013, 2.999936)^T \\ \mu_{10} &= 2.999936 \\ \boldsymbol{x}_{10} &= \frac{\boldsymbol{y}_{10}}{\mu_{10}} = (0.499995, -0.000004, 1.000000)^T \end{aligned}$$

These ten iterates suggest that the sequence $\{\mu_k\}$ converges to the eigenvalue $\lambda_1 = 3$ and the sequence $\{\boldsymbol{x}^{(k)}\}$ converges to $(0.5, 0, 1) = \frac{1}{2}\boldsymbol{v}_1$.

Remark 7.1.8 [Disadvantages of power method].

1. The Power method requires at the beginning that the matrix has only one dominant eigenvalue, and this information is generally unavailable.

2. Even when there is only one dominant eigenvalue, it is not clear how to choose the initial guess $\boldsymbol{x}^{(0)}$ such that it has a non-zero component (c_1 in the notation of the theorem) along the eigenvector $\boldsymbol{v}_1$.

Note that in the above example, all the hypothesis are satisfied. Now let us ask the question

"What happens when any of the hypotheses of power method is violated?"

We discuss these situations through examples.

Example 7.1.9 [Dominant eigenvalue is not unique (Failure of H1)].

Consider the matrix

$$B = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{pmatrix},$$

which has eigenvalues 1, -2, and 2. Clearly, the matrix B has two dominant eigenvalues, namely, -2 and 2. We start with an initial guess $\boldsymbol{x}^{(0)} = (1, 1, 1)$ and the first five iterations generated using power method are given below:

Iteration No: 1

$$\begin{aligned} \boldsymbol{y}_1 &= A\boldsymbol{x}_0 = (8.000000, 3.000000, -2.000000)^T \\ \mu_1 &= 8.000000 \\ \boldsymbol{x}_1 &= \frac{\boldsymbol{y}_1}{\mu_1} = (1.000000, 0.375000, -0.250000)^T \end{aligned}$$

Iteration No: 2

$$\begin{aligned} \boldsymbol{y}_2 &= A\boldsymbol{x}_1 = (1.125000, 0.500000, 0.500000)^T \\ \mu_2 &= 1.125000 \\ \boldsymbol{x}_2 &= \frac{\boldsymbol{y}_2}{\mu_2} = (1.000000, 0.444444, 0.444444)^T \end{aligned}$$

Iteration No: 3

$$\begin{aligned} \boldsymbol{y}_3 &= A\boldsymbol{x}_2 = (4.111111, 1.333333, -0.888889)^T \\ \mu_3 &= 4.111111 \\ \boldsymbol{x}_3 &= \frac{\boldsymbol{y}_3}{\mu_3} = (1.000000, 0.324324, -0.216216)^T \end{aligned}$$

Iteration No: 4

$$\begin{aligned} \boldsymbol{y}_4 &= A\boldsymbol{x}_3 = (1.108108, 0.432432, 0.432432)^T \\ \mu_4 &= 1.108108 \\ \boldsymbol{x}_4 &= \frac{\boldsymbol{y}_4}{\mu_4} = (1.000000, 0.390244, 0.390244)^T \end{aligned}$$

Iteration No: 5

$$\begin{aligned} \boldsymbol{y}_5 &= A\boldsymbol{x}_4 = (3.731707, 1.170732, -0.780488)^T \\ \mu_5 &= 3.731707 \\ \boldsymbol{x}_5 &= \frac{\boldsymbol{y}_5}{\mu_5} = (1.000000, 0.313725, -0.209150)^T \end{aligned}$$

It is observed that the sequence oscillates even till 1000 iterations as shown below:

Iteration No: 998

$$\begin{aligned} \boldsymbol{y}_{998} &= A\boldsymbol{x}_{997} = (1.103448, 0.413793, 0.413793)^T \\ \mu_{998} &= 1.103448 \\ \boldsymbol{x}_{998} &= \frac{\boldsymbol{y}_{998}}{\mu_{998}} = (1.000000, 0.375000, 0.375000)^T \end{aligned}$$

Iteration No: 999

$$\begin{aligned} \boldsymbol{y}_{999} &= A\boldsymbol{x}_{998} = (3.625000, 1.125000, -0.750000)^T \\ \mu_{999} &= 3.625000 \\ \boldsymbol{x}_{999} &= \frac{\boldsymbol{y}_{999}}{\mu_{999}} = (1.000000, 0.310345, -0.206897)^T \end{aligned}$$

Iteration No: 1000

$$\begin{aligned} \boldsymbol{y}_{1000} &= A\boldsymbol{x}_{999} = (1.103448, 0.413793, 0.413793)^T \\ \mu_{1000} &= 1.103448 \\ \boldsymbol{x}_{1000} &= \frac{\boldsymbol{y}_{1000}}{\mu_{1000}} = (1.000000, 0.375000, 0.375000)^T \end{aligned}$$

and so on. This is a clear indication that the power method is not converging in this case.

Thus we conclude that the power method when applied to a matrix which has more than one dominant eigenvalue may not converge.

Remark 7.1.10 [Dominant eigenvalue is not simple].

It is not necessary to have the dominant eigenvalue of algebraic multiplicity 1 in order that the iterative sequences of power method converge. The important thing is to have a unique dominant eigenvalue and it is allowed to have an algebraic multiplicity r with $r > 1$. However it is necessary that the corresponding geometric multiplicity should also be r to satisfy the hypothesis **(H2)** (also see later on, where we discuss the situation where algebraic and geometric multiplicities do not match). In such a case, power method computes only one eigenvector, as is usual.

Let us now illustrate the situation when the hypothesis **(H3)** is violated.

Example 7.1.11 [Failure of hypothesis (H3): Initial guess $x^{(0)}$ is such that $c_1 = 0$].

Consider the matrix (same as in Example 7.1.7)

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{pmatrix},$$

The eigenvalues of this matrix are

$$\lambda_1 = 3,\ \lambda_2 = 1 \text{ and } \lambda_3 = 0.$$

The corresponding eigenvectors are

$$\boldsymbol{v}_1 = (1, 0, 2)^T,\ \boldsymbol{v}_2 = (0, 2, -5)^T \text{ and } \boldsymbol{v}_3 = (0, 1, -3)^T.$$

Thus, the hypothesis **(H1)** and **(H2)** are satisfied.

Here, we choose a different initial guess $\boldsymbol{x}_0 = (0, 0.5, 0.25)^T$. Note that the hypothesis **(H3)** that $c_1 \neq 0$ is violated here. However, we can see that $c_2 \neq 0$. The first four iterations of the power method are as follows:

Iteration No: 1

$$\begin{aligned} \boldsymbol{y}_1 &= A\boldsymbol{x}_0 = (0.000000, 3.500000, -8.750000)^T \\ \mu_1 &= -8.750000 \\ \boldsymbol{x}_1 &= \frac{\boldsymbol{y}_1}{\mu_1} = (-0.000000, -0.400000, 1.000000)^T \end{aligned}$$

Iteration No: 2

$$\begin{aligned} \boldsymbol{y}_2 &= A\boldsymbol{x}_1 = (0.000000, -0.400000, 1.000000)^T \\ \mu_2 &= 1.000000 \\ \boldsymbol{x}_2 &= \frac{\boldsymbol{y}_2}{\mu_2} = (0.000000, -0.400000, 1.000000)^T \end{aligned}$$

Iteration No: 3

$$\begin{aligned} \boldsymbol{y}_3 &= A\boldsymbol{x}_2 = (0.000000, -0.400000, 1.000000)^T \\ \mu_3 &= 1.000000 \\ \boldsymbol{x}_3 &= \frac{\boldsymbol{y}_3}{\mu_3} = (0.000000, -0.400000, 1.000000)^T \end{aligned}$$

Iteration No: 4

$$\begin{aligned} \boldsymbol{y}_4 &= A\boldsymbol{x}_3 = (0.000000, -0.400000, 1.000000)^T \\ \mu_4 &= 1.000000 \\ \boldsymbol{x}_4 &= \frac{\boldsymbol{y}_4}{\mu_4} = (0.000000, -0.400000, 1.000000)^T \end{aligned}$$

Thus, the power method converges to λ_2, which is the second dominant eigenvalue of the given matrix.

Note that in the chosen initial guess, the first coordinate is zero and therefore, c_1 in (7.9) has to be zero. Thus, (7.4) reduces to

$$A^k\boldsymbol{v} = \lambda_2^k\left(c_2\boldsymbol{v}_2 + c_3\left(\frac{\lambda_3}{\lambda_2}\right)^k\boldsymbol{v}_3 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_2}\right)^k\boldsymbol{v}_n\right).$$

This makes the iteration to converge to λ_2, which is the next dominant eigenvalue.

Remark 7.1.12.

It is important that we understand the hypothesis **(H3)** on the initial guess $\boldsymbol{x}^{(0)}$ correctly. Note that **(H3)** says that the coefficient of $\boldsymbol{v}_1$ (which was denoted by c_1) should be non-zero when $\boldsymbol{x}^{(0)}$ is expressed as

$$\boldsymbol{x}^{(0)} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n.$$

Note that the coefficients $c_1, c_2, \cdots, c_n$ are unique as $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ form a basis for $\mathbb{R}^n$. For such a choice of $\boldsymbol{x}^{(0)}$, it may happen that the first coordinate may be zero. That is, if $\boldsymbol{x}^{(0)}$ is written in coordinate form as $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})^T$, it is possible that $x_1^{(0)} = 0$ and $c_1 \neq 0$. Thus, it is not necessary that the power method will converge to the second dominant eigenvalue if the first coordinate of the initial guess is zero. However, we may expect this to happen if $c_1 = 0$. The following example illustrates this fact.

Example 7.1.13.

Consider the matrix

$$A = \begin{bmatrix} 91.4 & -22.0 & -44.8000 \\ 175.2 & -41.0 & -86.4 \\ 105.2 & -26.0 & -51.4000 \end{bmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = -5$, $\lambda_2 = 3$ and $\lambda_3 = 1$. The corresponding eigenvectors are $\boldsymbol{v}_1 = (3, 5, 4)^T$, $\boldsymbol{v}_2 = (2, 6, 1)^T$ and $\boldsymbol{v}_3 = (1, -2, 3)^T$.

Note that the matrix A satisfies the hypothesis **(H1)** since -5 is the unique dominant eigenvalue and it is also a simple eigenvalue. The matrix A satisfies the hypothesis **(H2)** as all eigenvalues are distinct and hence eigevectors form a basis for $\mathbb{R}^3$. Thus the fate of the power method iterates depends solely on the choice of the initial guess $\boldsymbol{x}^{(0)}$ and whether it satisfies the hypothesis **(H3)**

- Let us take the initial guess $\boldsymbol{x}^{(0)} = (1, 0.5, 0.25)^T$. Note that $c_1 \neq 0$ for this initial guess. Thus the initial guess satisfies the hypothesis **(H3)** and the iterative sequences generated by power method converges to the dominant eigenvalue $\lambda_1 = -5$ and the corresponding eigenvector (with a scalar multiple) $\frac{1}{5}\boldsymbol{v}_1$.

- Let us take the initial guess $\boldsymbol{x}^{(0)} = (0, 0.5, 0.25)^T$. Note that $c_1 \neq 0$ for this initial guess. Thus the initial guess satisfies the hypothesis **(H3)** and the iterative sequences generated by power method converges to the dominant eigenvalue $\lambda_1 = -5$ and the corresponding eigenvector (with a scalar multiple) $\frac{1}{5}\boldsymbol{v}_1$. Compare this with Example 7.1.11. In the present case the first coordinate of the initial guess vector is zero, just as in Example 7.1.11. In Example 7.1.11 the power method iterate converged to the second dominant eigenvalue and the corresponding eigenvector, which does not happen in the present case. The reason is that in the Example 7.1.11, $c_1 = 0$ for the initial guess chosen, but in the current example $c_1 \neq 0$.

7.1.1 Inverse Power Method

From the basic linear algebra, we know that if λ is an eigenvalue of an invertible matrix A, then λ^{-1} is an eigenvalue of A^{-1}. Hence, if the eigenvalues of A (after a re-arrangement) are such that

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{n-1}| > |\lambda_n|,$$

then we can apply the power method to the matrix A^{-1} to approximate the eigenvalue λ_n, provided the other two hypotheses of Theorem 7.1.5 are satisfied.

The main difficulty in this idea is to compute A^{-1}. Instead, of first computing A^{-1} and then applying the power method, we can obtain a LU factorization of A once and then solve the system

$$A\boldsymbol{y}^{(k+1)} = \boldsymbol{x}^{(k)}$$

at every iteration of the power method using one forward substitution and one backward substitution. By keeping all the other steps of the power method unchanged, the resulting method is the *inverse power method* for computing the eigenvalue of A having the smallest absolute value and a corresponding eigen vector.

7.1.2 Shifted Inverse Power Method

We may use the inverse power method to obtain any eigenvalue of a matrix A. The idea is to choose a number ν very close to the required eigenvalue λ and consider applying the inverse power method to the shifted matrix $(A-\nu I)$, provided this matrix is invertible. Note that $\lambda - \nu$ is an eigenvalue of $(A - \nu I)$. Further, if $\lambda - \nu$ happens to be the smallest eigenvalue (in modulus) of $(A-\nu I)$ and if all the hypotheses of Theorem 7.1.5 are satisfied for the matrix $(A - \nu I)^{-1}$, then the power method sequences $\{\mu_k\}$ and $\{\boldsymbol{x}^{(k)}\}$ converge to $1/(\lambda - \nu)$ and a corresponding eigenvector of λ, respectively. The resulting method is the *shifted inverse power method*.

7.2 Gerschgorin's Theorem

An important tool in eigenvalue approximation is the ability to localize the eigenvalues, and the most important tool in eigenvalue localization is the Gerschgorin's theorem. Gerschgorin's Circle theorem helps us in localization of eigenvalues of a matrix. This theorem explicitly constructs n disks in the complex plane with centers at the diagonal elements of the matrix; and all the eigenvalues of the matrix lie in the union of these disks.

Theorem 7.2.1 [Gerschgorin's Circle Theorem].

Let A be an $n \times n$ matrix. For each $k = 1, 2, \cdots, n$, define ρ_k by

$$\rho_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|, \tag{7.21}$$

and D_k denotes the closed disk in the complex plane with centre a_{kk} and radius ρ_k, i.e.,

$$D_k = \{z \in \mathbb{C} : |z - a_{kk}| \leq \rho_k\}. \tag{7.22}$$

1. Each eigenvalue of A lies in one of the disks D_k. That is, no eigenvalue of A lies in $\mathbb{C} \setminus \cup_{k=1}^{n} D_k$.
2. Suppose that among the disks $D_1, D_2, \cdots, D_n$, there is a collection of m disks whose union (denoted by R_1) is disjoint from the union of the rest of the $n - m$ disks (denoted by R_2). Then exactly m eigenvalues lie in R_1 and $n - m$ eigenvalues lie in R_2 (here each eigenvalue is counted as many times as its algebraic multiplicity).

The disks D_k, $k = 1, 2, \ldots, n$ are called the ***Gerschgorin's disks***.

Proof.

We will prove only (i) as it is easy, and the proving (ii) is beyond the scope of this course.

Let λ be an eigenvalue of A. Then there exists a $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ and $\boldsymbol{v} \neq \mathbf{0}$ such that

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \tag{7.23}$$

Let $1 \leq r \leq n$ be such that $|v_r| = \max\{|v_1|, |v_2|, \cdots, |v_n|\}$. The r^{th} equation of the system of equations (7.23) is given by (actually, of $A\boldsymbol{v} - \lambda\boldsymbol{v} = \mathbf{0}$)

$$a_{r1}v_1 + \cdots + a_{r,r-1}v_{r-1} + (a_{rr} - \lambda)v_r + a_{r,r+1}v_{r+1} + \cdots + a_{rn}v_n = 0$$

From the last equation, we get

$$\lambda - a_{rr} = \frac{v_1}{v_r}\, a_{r1} + \cdots + \frac{v_{r-1}}{v_r}\, a_{r,r-1} + \frac{v_{r+1}}{v_r}\, a_{r,r+1} + \cdots + \frac{v_n}{v_r}\, a_{rn} \tag{7.24}$$

Taking modulus on both sides of the equation (7.24), and using the triangle inequality $|a + b| \leq |a| + |b|$ repeatedly we get

$$|\lambda - a_{rr}| \leq \frac{|v_1|}{|v_r|}\, |a_{r1}| + \cdots + \frac{|v_{r-1}|}{|v_r|}\, |a_{r,r-1}| + \frac{|v_{r+1}|}{|v_r|}\, |a_{r,r+1}| + \cdots + \frac{|v_n|}{|v_r|}\, |a_{rn}| \tag{7.25}$$

In view of the choice of r, the components of the vector $\boldsymbol{v}$ satisfy $\frac{|v_s|}{|v_r|} \leq 1$. The last equation (7.25) becomes

$$|\lambda - a_{rr}| \leq |a_{r1}| + \cdots + |a_{r,r-1}| + |a_{r,r+1}| + \cdots + |a_{rn}| \tag{7.26}$$

Observe that the right hand side of the inequality (7.26) is ρ_r. This proves that $\lambda \in D_r$.

Example 7.2.2.

For the matrix

$$\begin{pmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ -2 & 0 & 9 \end{pmatrix},$$

the Gerschgorin's disks are given by

$$\begin{aligned} D_1 &= \{z \in \mathbb{C} : |z-4| \leq 2\}, \\ D_2 &= \{z \in \mathbb{C} : |z-2| \leq 1\}, \\ D_3 &= \{z \in \mathbb{C} : |z-9| \leq 2\}. \end{aligned}$$

Draw a picture of these disks and observe that D_3 neither intersects D_1 nor D_2. By (ii) of Theorem 7.2.1, D_3 has one eigenvalue and $D_1 \cup D_2$ has two eigenvalues counting multiplicities. Note that the eigenvalues are approximately $4.6318,\ 1.8828 \in D_1 \cup D_2$ and $8.4853 \in D_3$.

Remark 7.2.3.

Gerschgorin's circle theorem is helpful in finding bound for eigenvalues. For the matrix in Example 7.2.2, any number z in D_1 satisfies $|z| \leq 6$. Similarly any number z in D_2 satisfies $|z| \leq 3$, and any number z in D_3 satisfies $|z| \leq 11$. Since any eigenvalue λ lies in one of three disks, we can conclude that $|\lambda| \leq 11$.

Remark 7.2.4.

The main disadvantage of the power method discussed in Section 7.1 is that if a given matrix has more than one dominant eigenvalues, then the method may not converge. So, for a given matrix, we do not know whether the power method will converge or not. Also, as the power method is reasonably slow (see Example 7.1.13 for an illustration) we may have to perform reasonably large number of iterations to come to know that the method is not actually converging.

Thus, a tool to find out whether the given matrix has a unique dominant eigenvalue or not is highly desirable. The Gerschgorin theorem (Theorem 7.2.1) can sometimes be used to see if power method can be used for a given matrix. For instance, in Example 7.2.2, we see that the power method can be used to obtain an approximation to the dominant eigenvalue.

Since a matrix A and its transpose (denoted by A^T) have same eigenvalues, we can apply Gerschgorin Circle Theorem to A^T and conclude the following corollary.

Corollary 7.2.5.

Let A be an $n \times n$ matrix. For each $k = 1, 2, \cdots, n$, define τ_k by

$$\tau_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{jk}|, \tag{7.27}$$

and B_k denotes the closed disk in the complex plane with centre a_{kk} and radius τ_k. That is,

$$B_k = \big\{ z \in \mathbb{C} \, : \, |z - a_{kk}| \leq \tau_k \big\}. \tag{7.28}$$

1. Each eigenvalue of A lies in one of the disks B_k. That is, no eigenvalue of A lies in $\mathbb{C} \setminus \cup_{k=1}^{n} B_k$.
2. Suppose that among the disks $B_1, B_2, \cdots, B_n$, there is a collection of m disks whose union (denoted by C_1) is disjoint from the union of the rest of the $n - m$ disks (denoted by C_2). Then exactly m eigenvalues lie in C_1 and $n - m$ eigenvalues lie in C_2 (here each eigenvalue is counted as many times as its algebraic multiplicity).

Optimal Bounds

Using the Gerschgorin theorem, we wish to identify an annular region in the complex plane of the form

$$\big\{ z \in \mathbb{C} \, : \, r \leq |z| \leq R \big\}, \tag{7.29}$$

for some non-negative real numbers r and R, containing all the eigenvalues of a given matrix.

Let $A = (a_{ij})$ be an $n \times n$ matrix with real entries. For $k = 1, 2, \cdots, n$, consider the Gerschgorin disk D_k defined by (7.22). Then, for $z \in D_k$ we have

$$|z - a_{kk}| \leq \rho_k,$$

where ρ_k is given by (7.21). Using the triangle inequality, we get

$$\big|\, |z| - |a_{kk}| \,\big| \leq |z - a_{kk}| \leq \rho_k,$$

from which we get

$$|z| - |a_{kk}| \leq \rho_k \text{ and } |a_{kk}| - |z| \leq \rho_k.$$

Combining the above inequalities, we write

$$|a_{kk}| - \rho_k \leq |z| \leq |a_{kk}| + \rho_k.$$

Therefore, by the Gerschgorin theorem, all the eigenvalues of A are contained in the annular region (7.29) where

$$r = \min_{1\le k\le n} (|a_{kk}| - \rho_k) \text{ and } R = \max_{1\le k\le n} (|a_{kk}| + \rho_k).$$

If r turns out to be negative, then we replace it by 0.

Similarly, we can also obtain an annular region containing all the eigenvalues of A using Corollary 7.2.5. This annular region is given by

$$\{z \in \mathbb{C} \ : \ t \le |z| \le T\},$$

where

$$t = \min_{1\le k\le n} (|a_{kk}| - \tau_k), \quad T = \max_{1\le k\le n} (|a_{kk}| + \tau_k),$$

and τ_k as in (7.27). If t turns out to be negative, then we replace it by 0.

The smallest annular region, obtained using the Gerschgorin theorem, that contains all the eigenvalues of A is given by

$$\{z \in \mathbb{C} \ : \ \Lambda_* \le |z| \le \Lambda^*\},$$

where

$$\Lambda_* := \max\{r, t\} \text{ and } \Lambda^* := \min\{R, T\}.$$

We define Λ_* and Λ^* as the *optimal bounds* for the eigenvalues of A given by the Gerschgorin theorem. Note that this is the best possible information that we can obtain using the Gerschgorin theorem.

Example 7.2.6.

For the matrix

$$\begin{pmatrix} 6 & 0 & 2 \\ -1 & -5 & 0 \\ -2 & 2 & -3 \end{pmatrix}$$

the Gerschgorin disks can by obtained using (7.22) and are given by

$$D_1 = \{z \in \mathbb{C} \ : \ |z-6| \le 2\}, \ D_2 = \{z \in \mathbb{C} \ : \ |z+5| \le 1\}, \ D_3 = \{z \in \mathbb{C} \ : \ |z+3| \le 4\}.$$

The annular region corresponding to these disks is

$$\{z \in \mathbb{C} \ : \ 0 \le |z| \le 8\}.$$

Similarly, the other annular region is given by

$$\{z \in \mathbb{C} \ : \ 1 \leq |z| \leq 9\}.$$

Thus, the smallest annular region, obtained using the Gerschgorin theorem, that contains all the eigenvalues of A is given by

$$\{z \in \mathbb{C} \ : \ 1 \leq |z| \leq 8\}$$

and the optimal bounds are $\Lambda_* = 1$ and $\Lambda^* = 8$.

7.3 Exercises

1. The matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 2$, $\lambda = 1$ and $\lambda_3 = 1$ and the corresponding eigenvectors may be taken as $\mathbf{v}_1 = (1, 2, 3)^T$, $\mathbf{v}_2 = (0, 1, 2)^T$ and $\mathbf{v}_3 = (0, 2, 1)^T$. Perform 3 iterations to find the eigenvalue and the corresponding eigen vector to which the power method converges when we start the iteration with the initial guess $\mathbf{x}^{(0)} = (0, 0.5, 0.75)^T$. Without performing the iteration, find the eigenvalue and the corresponding eigenvector to which the power method converges when we start the iteration with the initial guess $\mathbf{x}^{(0)} = (0.001, 0.5, 0.75)^T$. Justify your answer.

2. The matrix

$$A = \begin{pmatrix} 5.4 & 0 & 0 \\ -113.0233 & -0.5388 & -0.6461 \\ -46.0567 & -6.4358 & -0.9612 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 5.4$, $\lambda_2 = 1.3$ and $\lambda_3 = -2.8$ with corresponding eigenvectors $\mathbf{v}_1 = (0.2, -4.1, 2.7)^T$, $\mathbf{v}_2 = (0, 1.3, -3.7)^T$ and $\mathbf{v}_3 = (0, 2.6, 9.1)^T$. To which eigenvalue and the corresponding eigenvector does the power method converge if we start with the initial guess $\mathbf{x}^{(0)} = (0, 1, 1)$? Justify your answer.

3. Use Gerschgorin's circle theorem to determine the intervals in which the eigenvalues of the matrix

$$A = \begin{pmatrix} 0.5 & 0 & 0.2 \\ 0 & 3.15 & -1 \\ 0.57 & 0 & -7.43 \end{pmatrix}.$$

lie, given that all eigenvalues of A are real. Show that power method can be applied for this matrix to find the dominant eigenvalue without computing eigenvalues explicitly. Compute the first three iterates of Power method sequences.

4. For this question, we take the matrix

$$A = \begin{pmatrix} -2.7083 & -2.6824 & 0.4543 \\ 0.1913 & 0.7629 & 0.1007 \\ -0.3235 & -0.4052 & 5.0453 \end{pmatrix}.$$

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ denote eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respectively of the matrix A which are given by $\lambda_1 \approx 5.0187, \lambda_2 \approx -2.5313, \lambda_3 \approx 0.6125$ and $\boldsymbol{v}_1 \approx (0.25, 0.13, 5.02)^T, \boldsymbol{v}_2 \approx (2.53, -0.15, 0.1)^T, \boldsymbol{v}_3 \approx (-0.49, 0.61, 0.02)^T$. Answer the following questions:

   i) Obtain the Gerschgorin disks associated to the matrix A and pictorially represent them in the complex plane.

   ii) Can Gerschgorin theorem be used to show that the matrix A has a unique dominant eigenvalue? Justify your answer.

   iii) Define the iterative sequences $\{\mu_k\}$ and $\boldsymbol{x}^{(k)}$ using power method that converge to λ_1 and $\alpha\boldsymbol{v}_1$ for some constant α when the initial guess is $\boldsymbol{x}^{(0)} = (1, 1, 1)^T$. Perform one iteration.

   iv) If we take the initial guess $\boldsymbol{x}^{(0)} = (0, 1, 1)^T$, then show that the iterative sequence obtained by power method converges to λ_j and $K\boldsymbol{v}_j$ for some $j \in \{1, 2, 3\}$ and for some constant K. What is the value of j and possible values of K?
   (**Hint:** $\boldsymbol{x}^{(0)} \approx 0.3060\boldsymbol{v}_1 + 0.1864\boldsymbol{v}_2 + 1.6748\boldsymbol{v}_3$.)

   v) Give all possible initial guesses for which the sequence $\{\mu_k\}$ obtained using power method converges to λ_2. Justify your answer.

   vi) Give all possible initial guesses for which the sequence $\{\mu_k\}$ obtained using power method converges to λ_3. Justify your answer.

5. Construct the iterative sequences using inverse power method for the matrix A given in Exercise 4. Staring with $\boldsymbol{x}^{(0)} = (1, 1, 1)^T$ perform five iterations.

6. Using an appropriate shifted inverse power method for the matrix A given in Exercise 4, construct the sequences $\{\mu_k\}$ and $\{\boldsymbol{x}^{(k)}\}$ that converge to λ_2 and an eigenvector of λ_2, respectively. Perform adequate number of iterations to confirm the convergence numerically.

7. Use the Gerschgorin Circle theorem to determine bounds for the eigenvalues for the following matrices. Also find optimum bounds wherever possible. Also draw the pictures of all the regions given by Greschgorin circles.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 4.75 & 2.25 & -0.25 \\ 2.25 & 4.75 & 1.25 \\ -0.25 & 1.25 & 4.75 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 1 & 0 & 1 & 4 \end{pmatrix}.$$

8. Prove that the following two statements concerning $n \times n$ matrices are equivalent.
   i) Every diagonally dominant matrix is invertible.
   ii) Each of the eigenvalues of a matrix A, belongs to at least one Gerschgorin disk corresponding to A.

9. Prove that the eigenvalues of the matrix

$$\begin{pmatrix} 6 & 2 & 1 \\ 1 & -5 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

   satisfy the inequality $1 \leq |\lambda| \leq 9$.

10. Show that the imaginary parts of the eigenvalues of

$$\begin{pmatrix} 3 & 1/3 & 2/3 \\ 1 & -4 & 0 \\ 1/2 & 1/2 & -1 \end{pmatrix}$$

    all lie in the interval $[-1, 1]$.

11. Let A denote the matrix

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 7 & 0 \\ 1 & 0 & \alpha \end{pmatrix}$$

    i) Gerschgorin theorem was used to conclude that the matrix A satisfies **Hypothesis (H1)** of Power method. Find the set of all such values of $\alpha \in \mathbb{R}$.
    ii) Gerschgorin theorem was used to conclude that the matrix A has distinct eigenvalues. Find the set of all such values of $\alpha \in \mathbb{R}$.