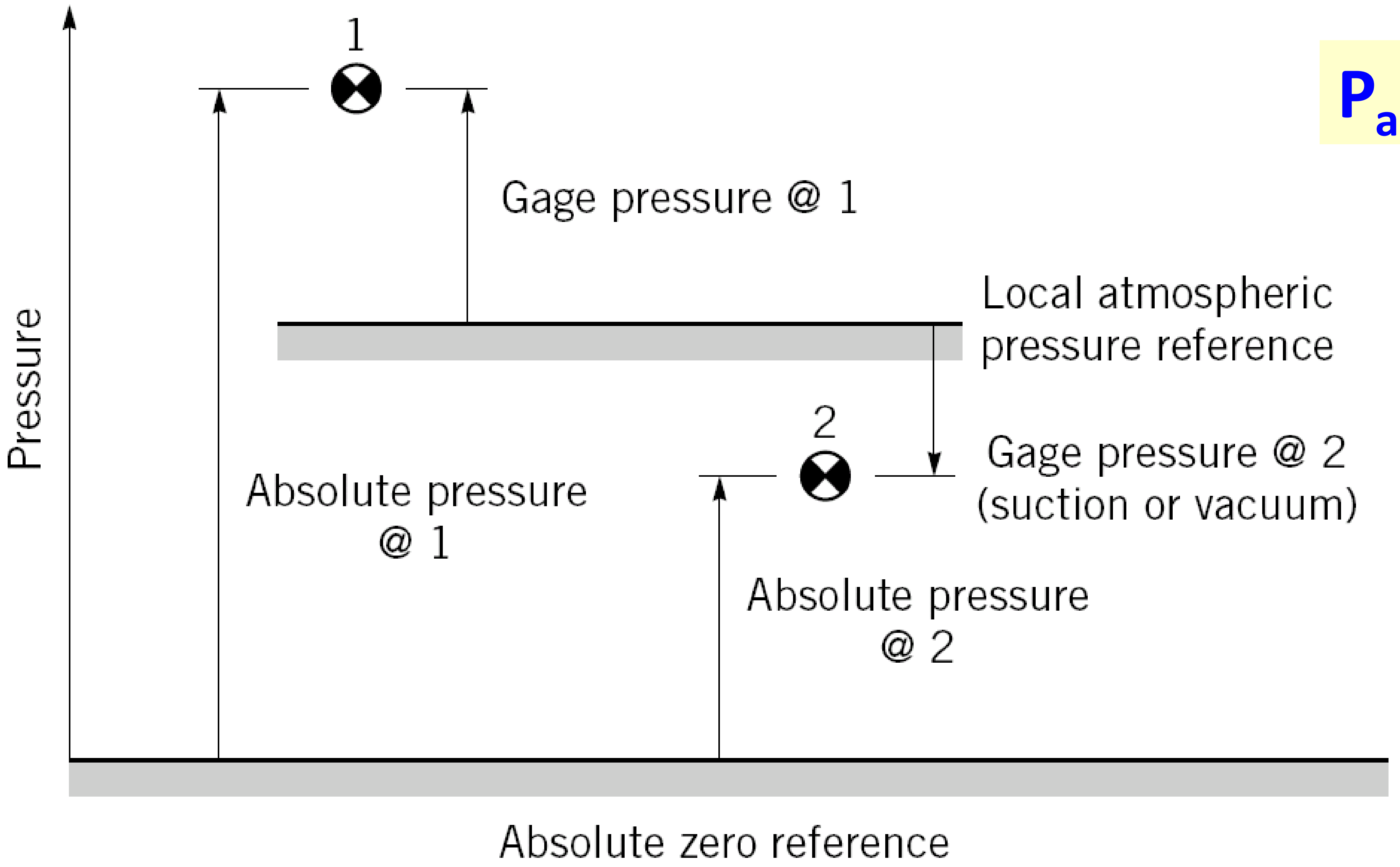
MEASUREMENT OF PRESSURE

GRAPHICAL REPRESENTATION OF PRESSURE



$$P_{abs} = P_{atm} \pm P_{gage}$$

ABSOLUTE PRESSURES – POSITIVE

GAGE PRESSURES – POSITIVE & NEGATIVE

By Newton's Second Law,

$$\sum \delta F = \delta m\,\vec{a}$$

Surface forces (pressure) + Body forces (weight) $= \delta m\,\vec{a}$

$$-\nabla P(\delta x\,\delta y\,\delta z) - \rho g(\delta x\,\delta y\,\delta z)\hat{k} = \rho(\delta x\,\delta y\,\delta z)\vec{a}$$

$$-\nabla P - \rho g\hat{k} = \rho\vec{a}$$

General equation of motion for a fluid – no shearing stress

PRESSURE VARIATION IN A FLUID AT REST

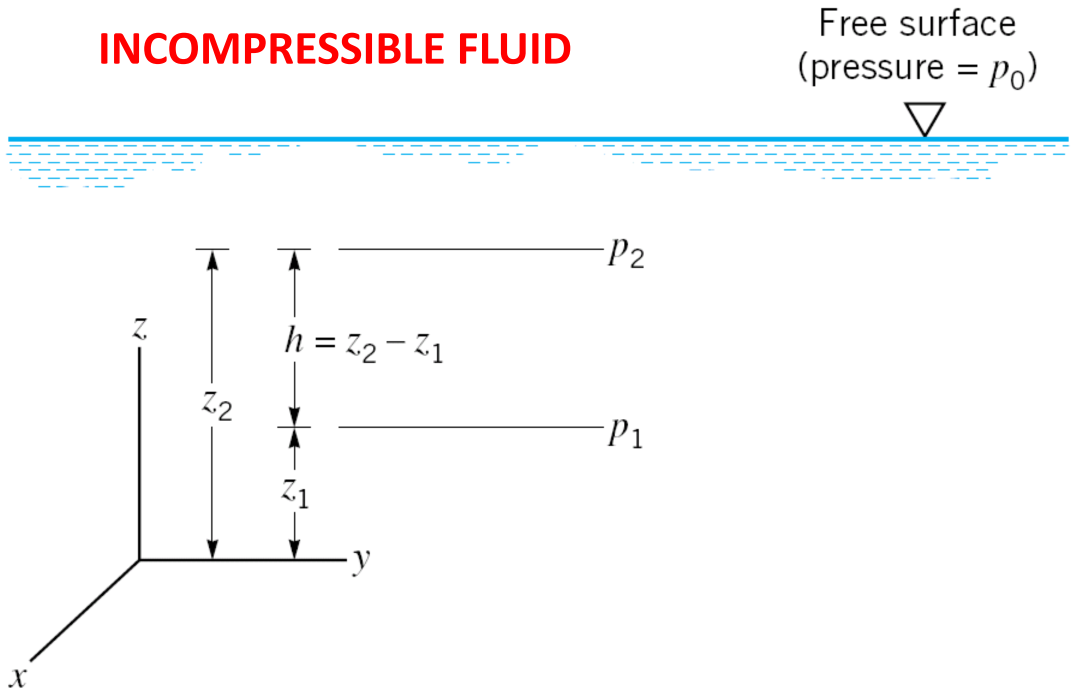
$$-\nabla P - \rho g \hat{k} = \rho \vec{a}$$

$$\frac{\partial P}{\partial x} = 0; \frac{\partial P}{\partial y} = 0; \frac{\partial P}{\partial z} = -\rho g; \vec{a} = 0$$

$$\frac{\partial P}{\partial z} = -\rho g$$

$$z \Uparrow \quad P \Downarrow$$

INCOMPRESSIBLE FLUID



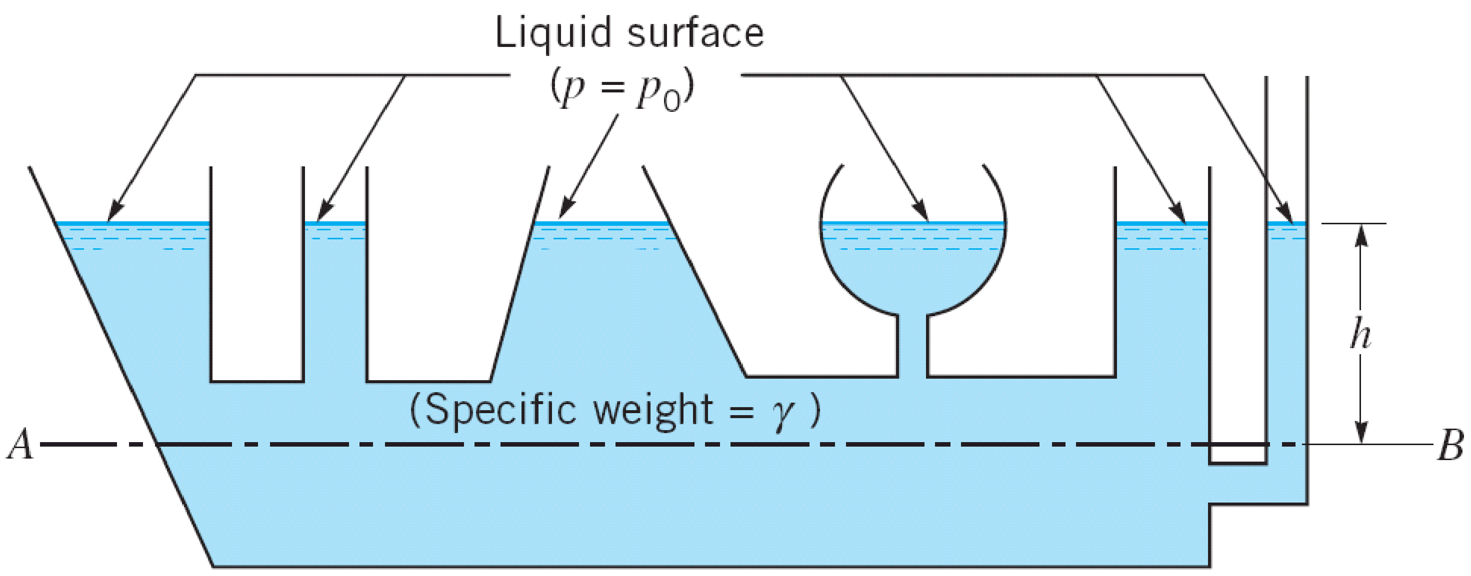
For liquids or gases at rest the pressure gradient in the vertical direction at any point in a fluid depends only on the specific weight of the fluid at that point.

$$\frac{dP}{dz} = -\rho g \Rightarrow \int_{p_1}^{p_2} dp = -\rho g \int_{z_1}^{z_2} dz$$

$$p_1 - p_2 = \rho g (z_2 - z_1) = \rho g h$$

This pressure distribution – HYDROSTATIC DISTRIBUTION

$$h = \frac{p_1 - p_2}{\rho g} = \frac{101325}{13.6 \times 1000 \times 9.81} = 760 \ \ mm\ of\ Hg$$



$$P_{AB} = \rho g h + P_o$$

PRESSURE IN A HOMOGENOUS, INCOMPRESSIBLE FLUID AT REST

- **depends on the depth of the fluid relative to some reference plane**
- **not influenced by the size of shape of the tank in which the fluid is held**

PRESSURE STANDARDS

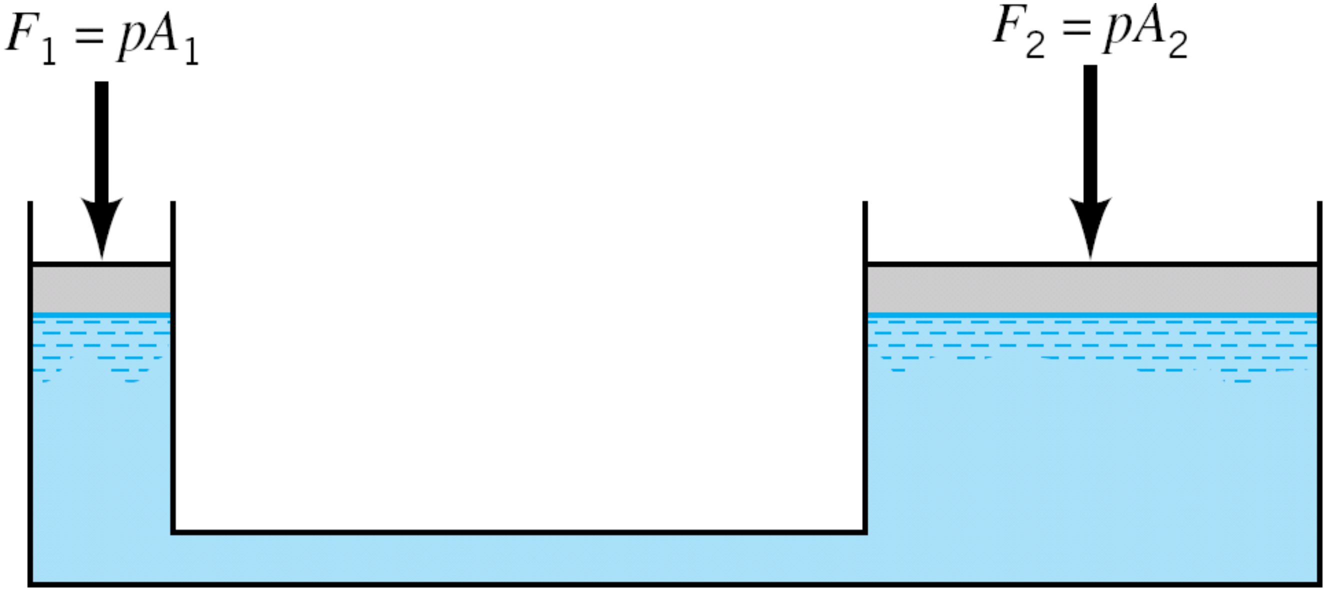
TYPE	RANGE	UNCERTAINTY
Deadweight piston gauge	70 Pa – 70 MPa	0.01 -0.05% of rdg
Barometer	690 – 790 mm Hg	0.001-0.03% of rdg
McLeod Gauge	0.01μm – 1 mm Hg	3-5% of reading
Manometer	700 Pa – 0.7 Mpa	0.02 – 2% of rdg
Micromanometer	0.005 – 500 mm H_2O	1% of reading

DEAD WEIGHT PISTON GAUGE- Amagat - 1893

- Calibration of pressure gauges

PROCEDURE OF CALIBRATION

1. Known mass is loaded on one end of piston
2. Fluid pressure is applied to the other end of the piston until enough force is developed to lift the piston-weight combination
3. Piston is floating freely within the cylinder – piston gauge is in equilibrium with unknown system pressure
4. Piston is rotated to reduce the friction
5. Effective area – average of cylinder and piston areas

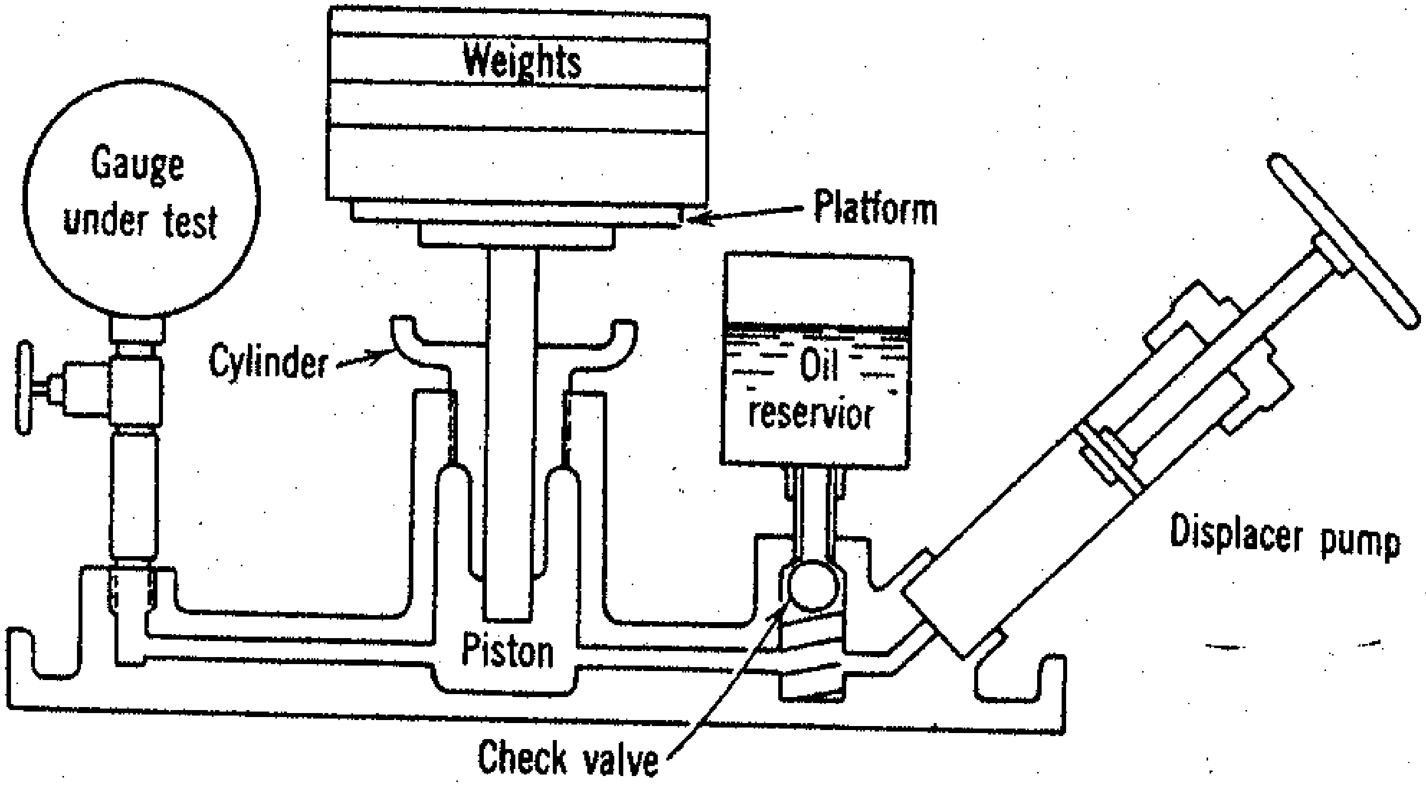


Corrections of Deadweight Pressure Gage

- **Buoyancy correction:**

Archimedes Principle: Air displaced by the weights and the piston exerts a buoyant force that causes the gage to indicate to high a pressure

$$C_{tb} = -\left(\frac{w_{air}}{w_{weights}}\right) \qquad P_{act} = P_1(1 + C_{tb})$$



FORTIN TYPE BAROMETER (FORTIN – 1750-1831)

- **Measures atmospheric pressure**

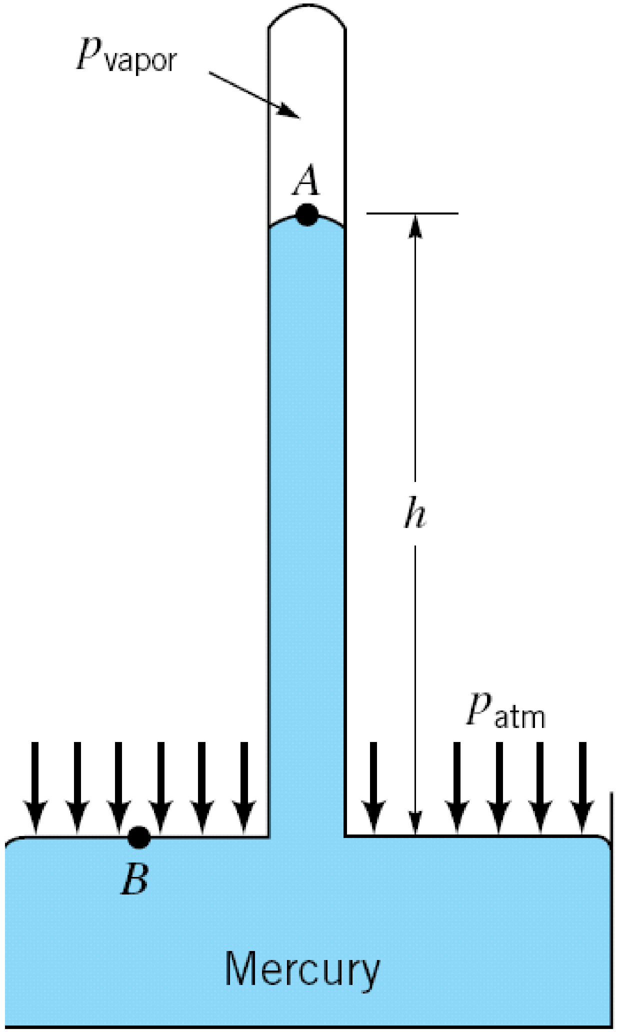
$$P_{atm} = \rho g h + p_{vapour}$$

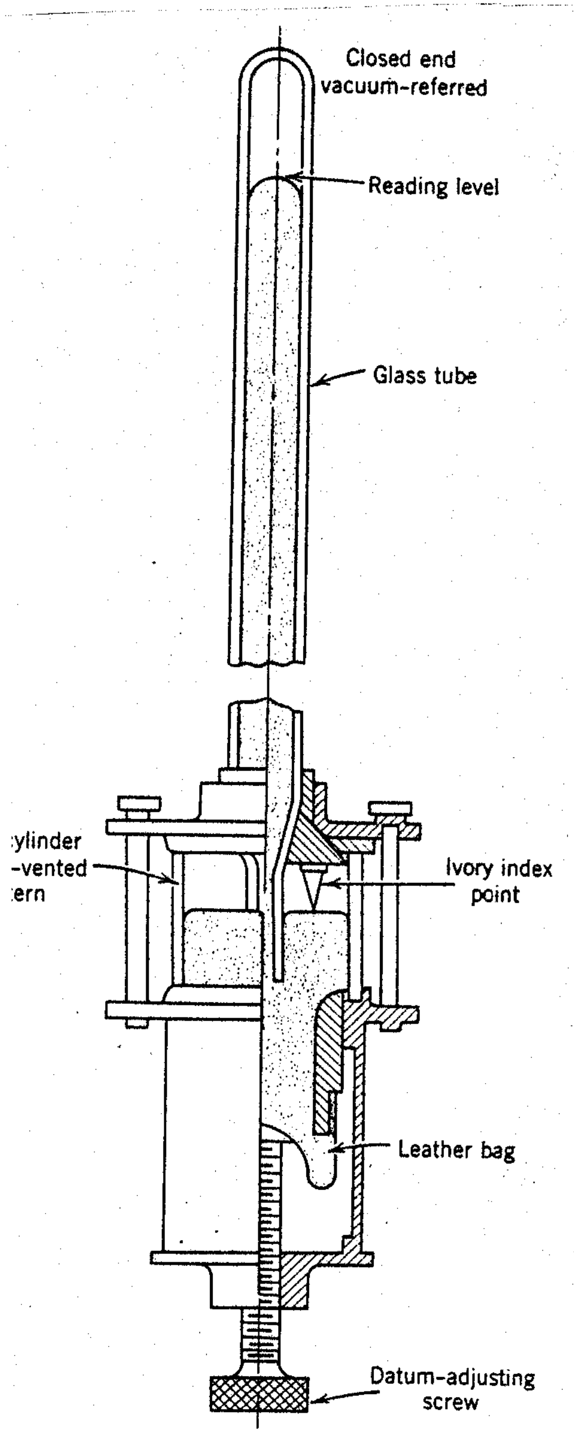
For mercury at 20° C,

$$p_{vapour} = 0.158585\ Pa$$

Evangelista Torricelli - 1644

$$P_{atm} \approx \rho g h$$

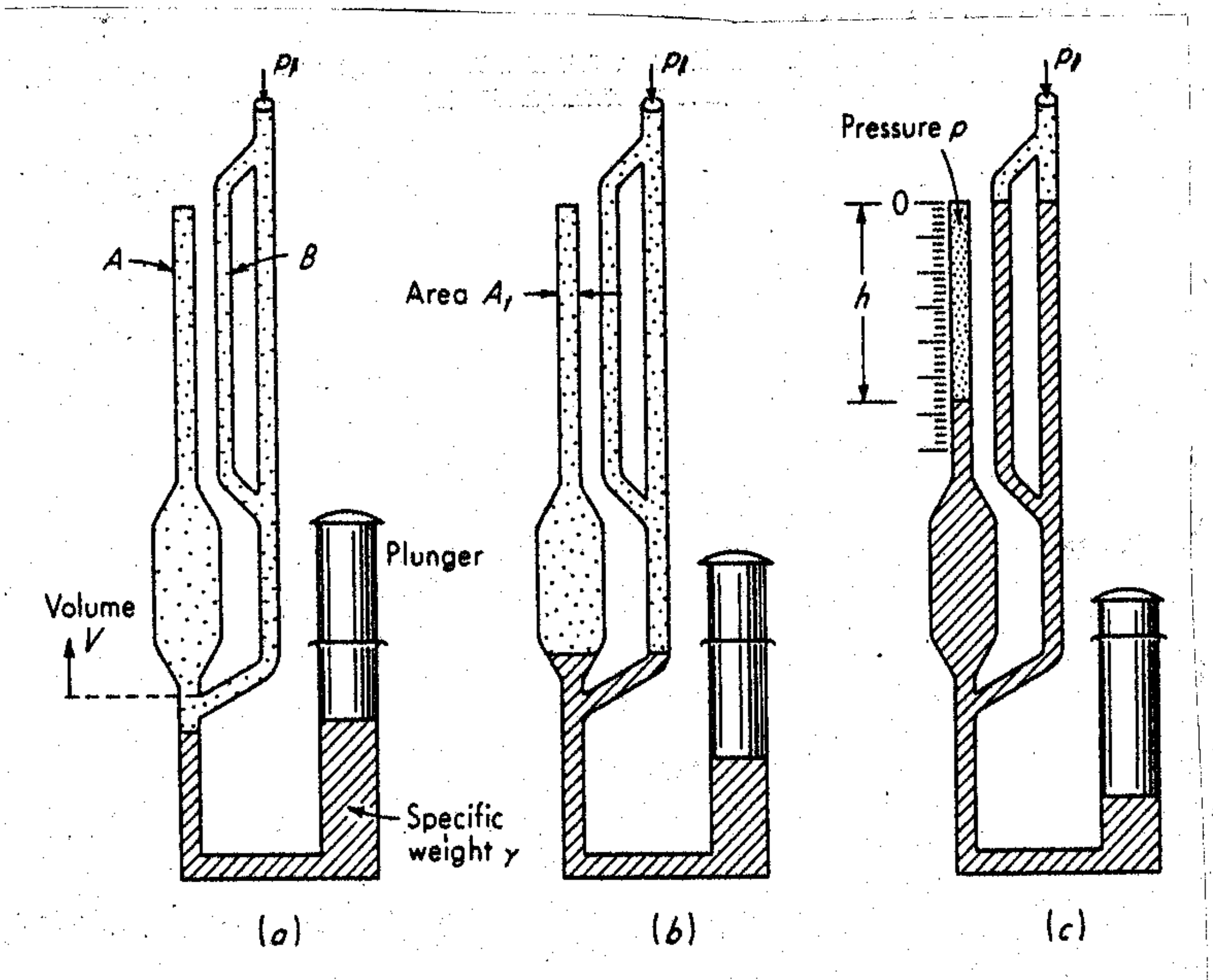




McLEOD GAGE (LOW PRESSURE – VACUUM)

McLeod Gauge **0.01μm – 1 mm Hg** **3-5% of reading**

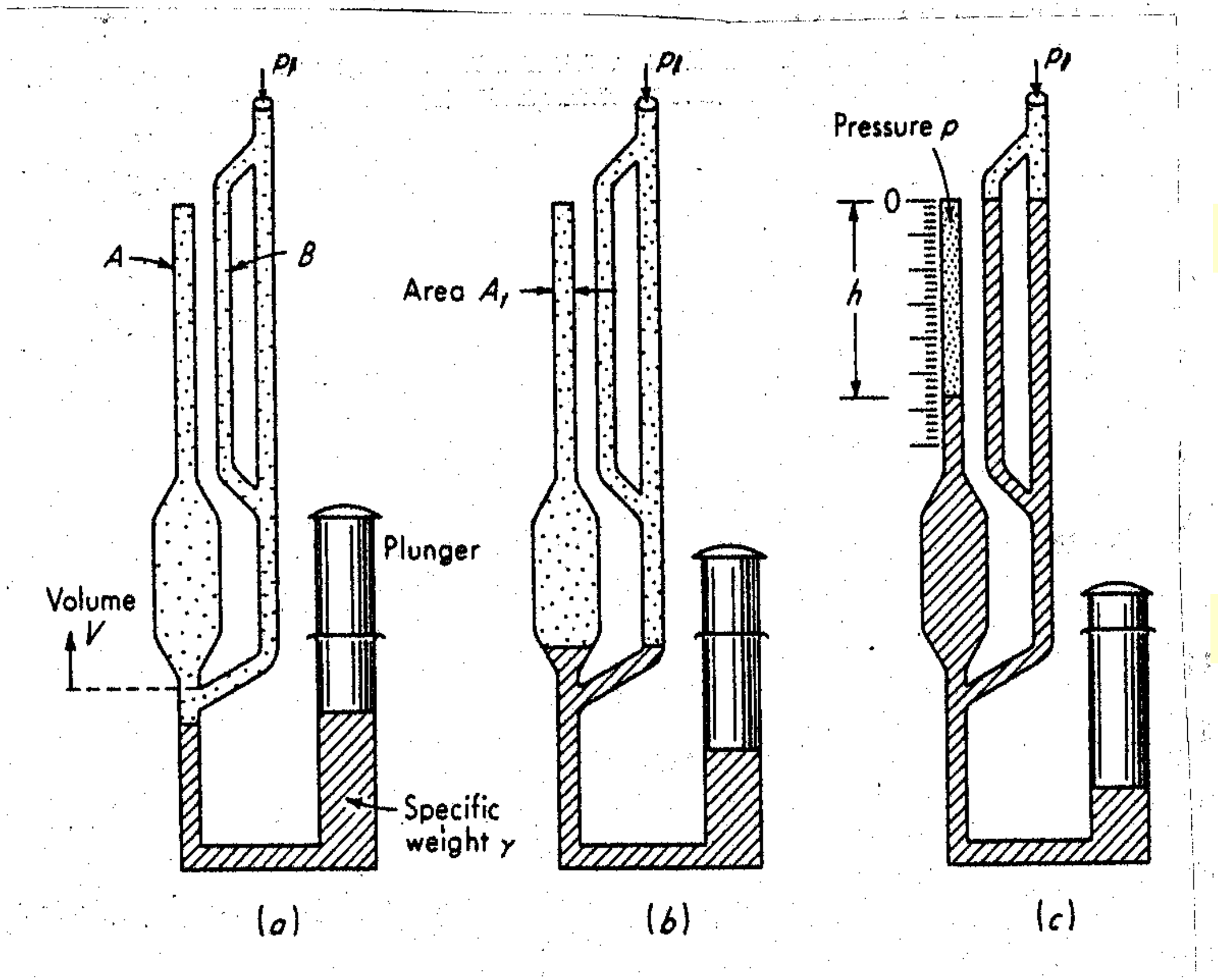
Principle: The compression of a sample of the low pressure gas to a pressure sufficiently high to read with a simple manometer.



By withdrawing the plunger, the mercury level is lowered to the position in Fig. a, admitting the gas at unknown pressure p_i

when the plunger is pushed in, the mercury level goes up, sealing off a gas sample of known volume V in the bulb and capillary tube A.

Further motion of the plunger causes compression of this sample and motion is continued until the mercury level in capillary B is at the zero mark.



Boyles law PV = Constant

$$p_i = Vacuum\ Pressure$$

$$p_iV = pA_th$$

$$p = p_i + \rho gh \quad - Statics$$

$$p_iV = (p_i + \rho gh)A_th$$

$$p_iV = p_iA_th + (\rho gh)A_th$$

$$p_i(V - A_th) = (\rho gh)A_th$$

$$p_i = \frac{\rho g A_t h^2}{V - A_t h}$$

If the measured gas contains any vapours that are condensed by the compression process, then the pressure will be in error

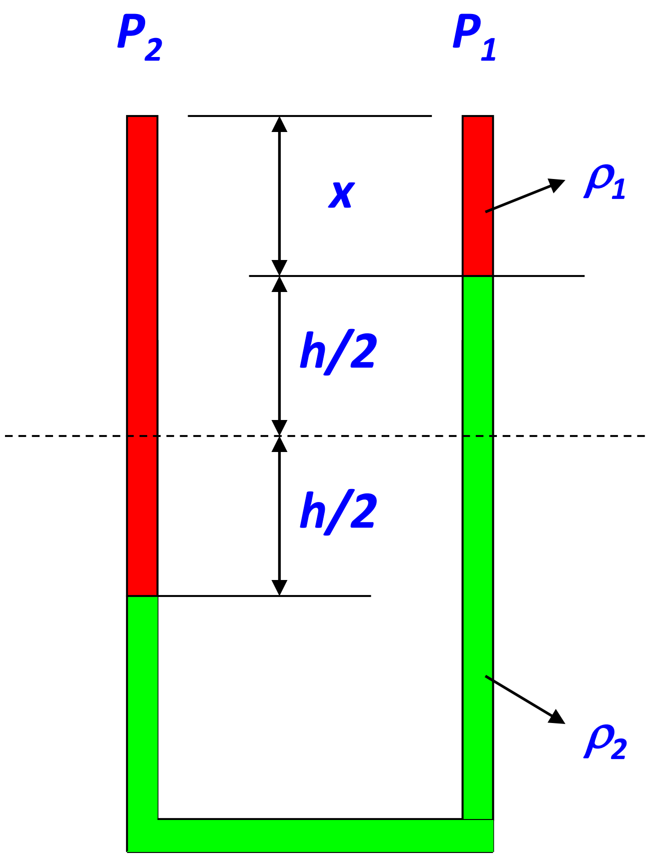
MANOMETER

$$P_1 + x\rho_1 g + h\rho_2 g = P_2 + x\rho_1 g + h\rho_1 g$$

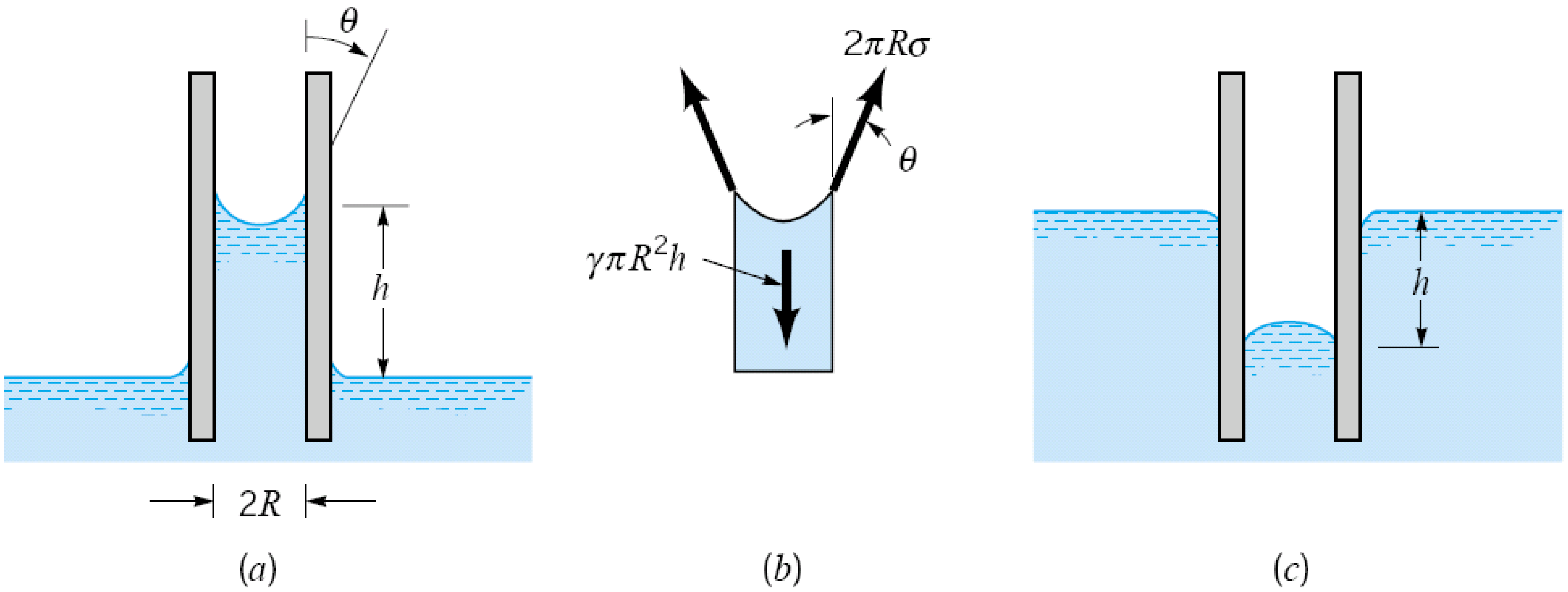
$$P_2 - P_1 = (\rho_2 - \rho_1)hg$$

ρ_1 = Density of working fluid

ρ_2 = Density of manometric fluid



Capillary rise or Depression



Adhesion >> Cohesion

Cohesion >> Adhesion

$$\rho g R^2 h = 2\pi R\sigma cos\theta$$

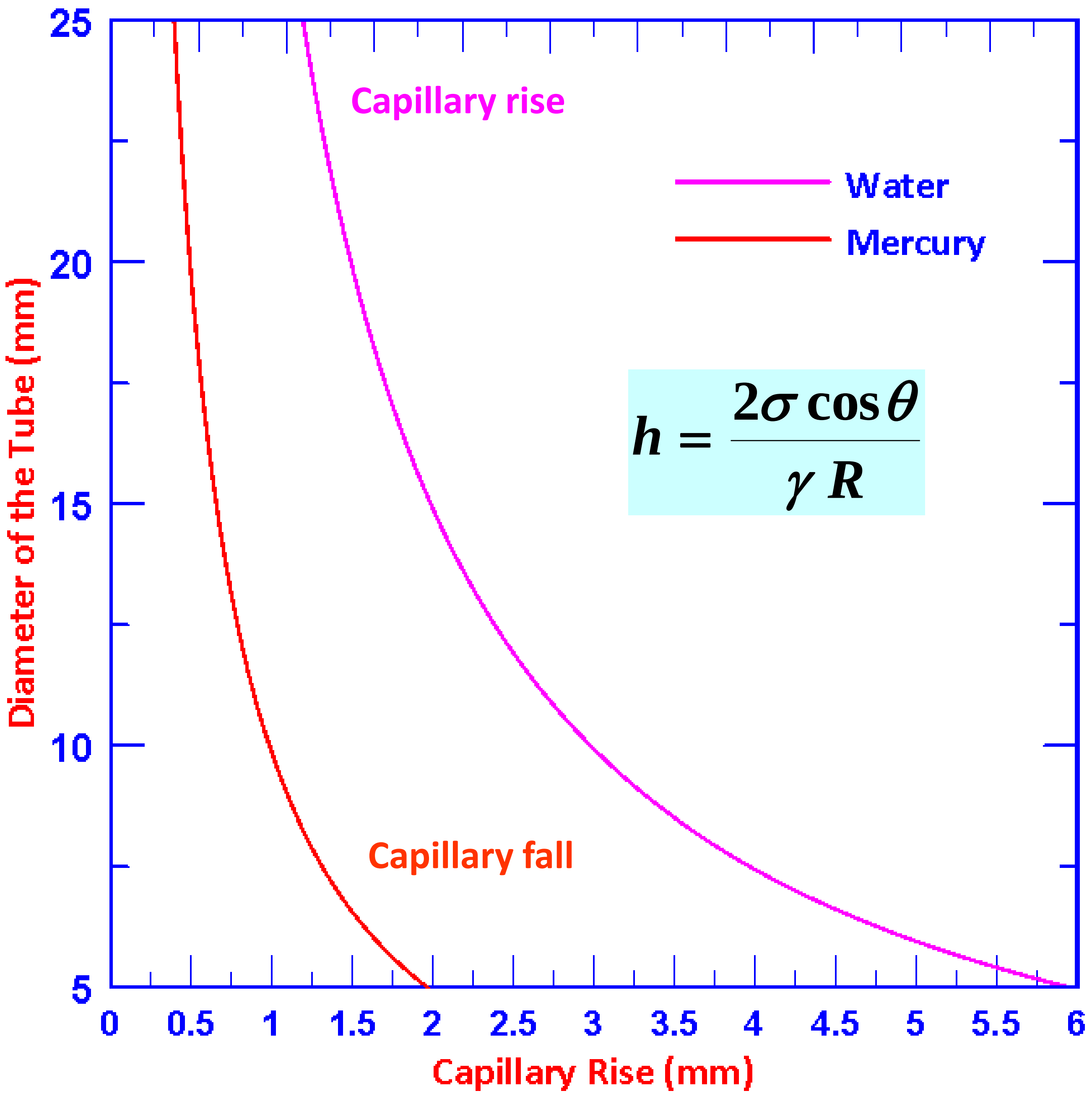
$$h = \frac{2\sigma cos\theta}{\rho g R}$$

For water θ = 0°

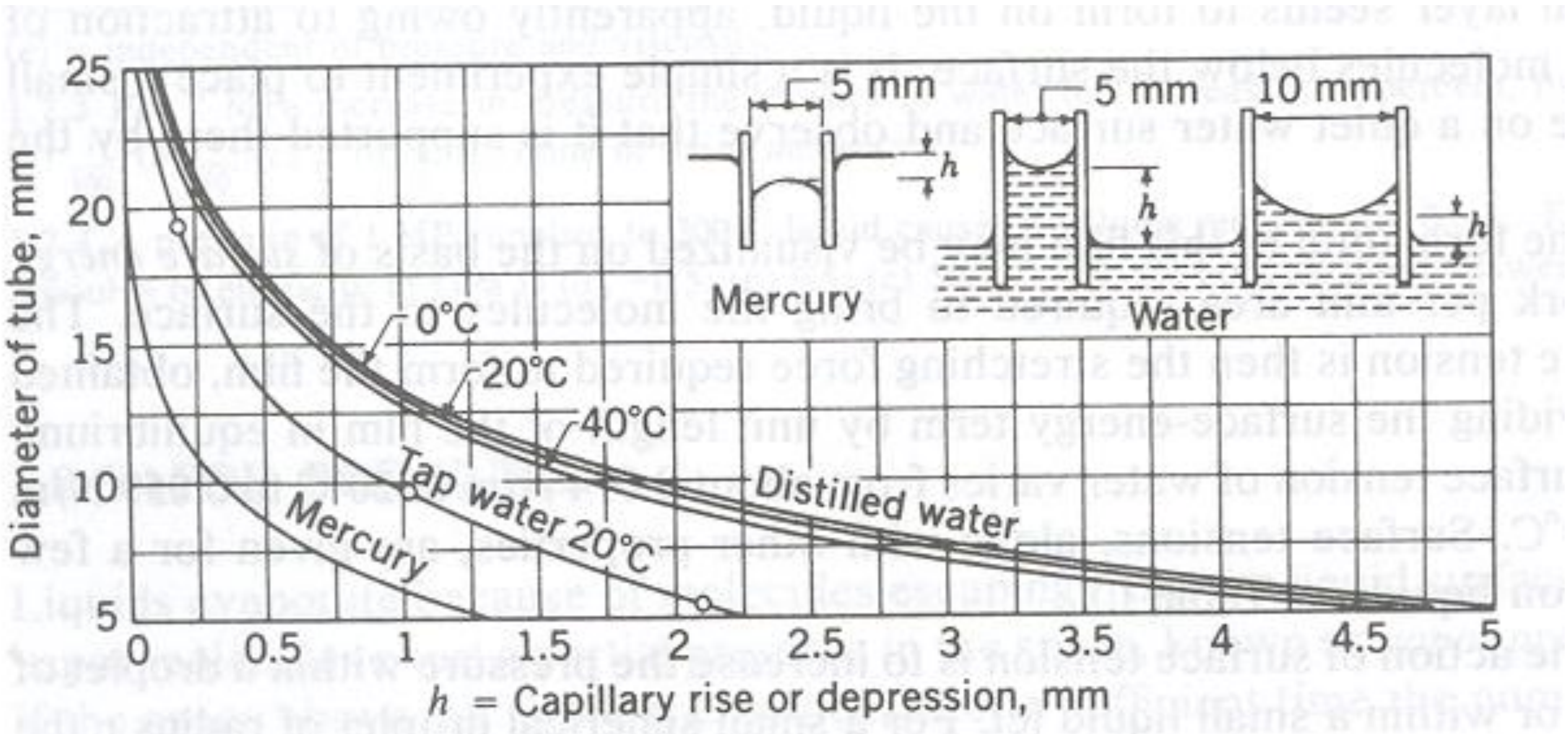
For mercury θ = 130°

Approximate properties of common liquids at 20°C and standard atmospheric pressure

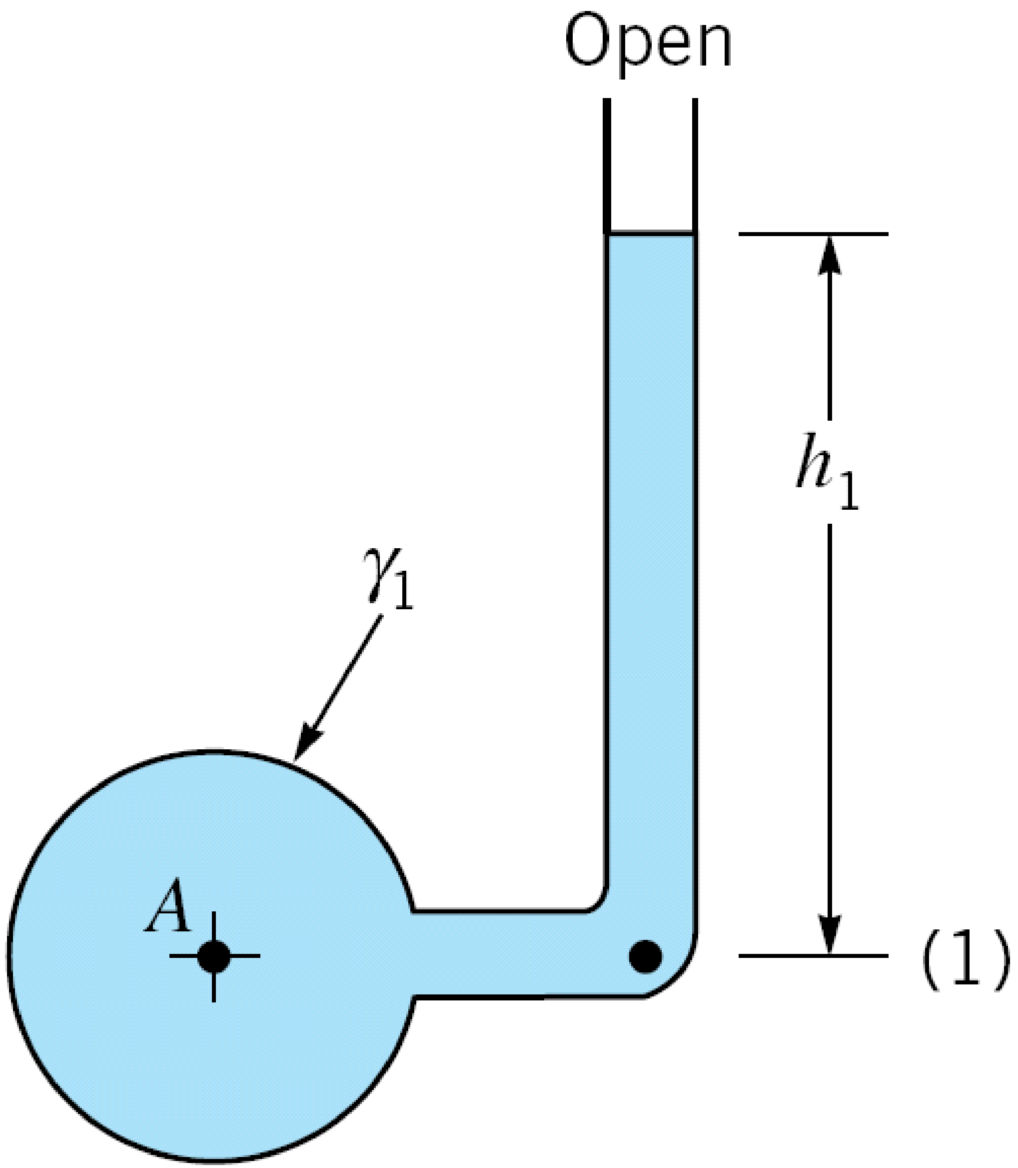
Liquid	Specific gravity SG	Bulk modulus of elasticity (GPa)	Vapour pressure P_v (kPa)	Surface tension σ(N/m) In contact with air
Ethyl alcohol	0.79	1.21	5.86	0.0223
Benzene	0.88	1.03	10.0	0.0289
Carbon tetrachloride	1.59	1.10	13.1	0.0267
Mercury	13.57	26.2	0.00017	0.51
Kerosene	0.81	-	-	0.023-0.038
Crude oil	0.85-0.93	-	-	0.023-0.038
Lubricating oil	0.85-0.88	-	-	0.035-0.038
Water	1.00	2.07	2.45	0.074


Capillary rise
Water
Mercury
$h = \frac{2\sigma\cos\theta}{\gamma R}$
Capillary fall
Diameter of the Tube (mm)
Capillary Rise (mm)
25
20
15
10
5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6

CAPILLARITY IN CIRCULAR GLASS TUBES



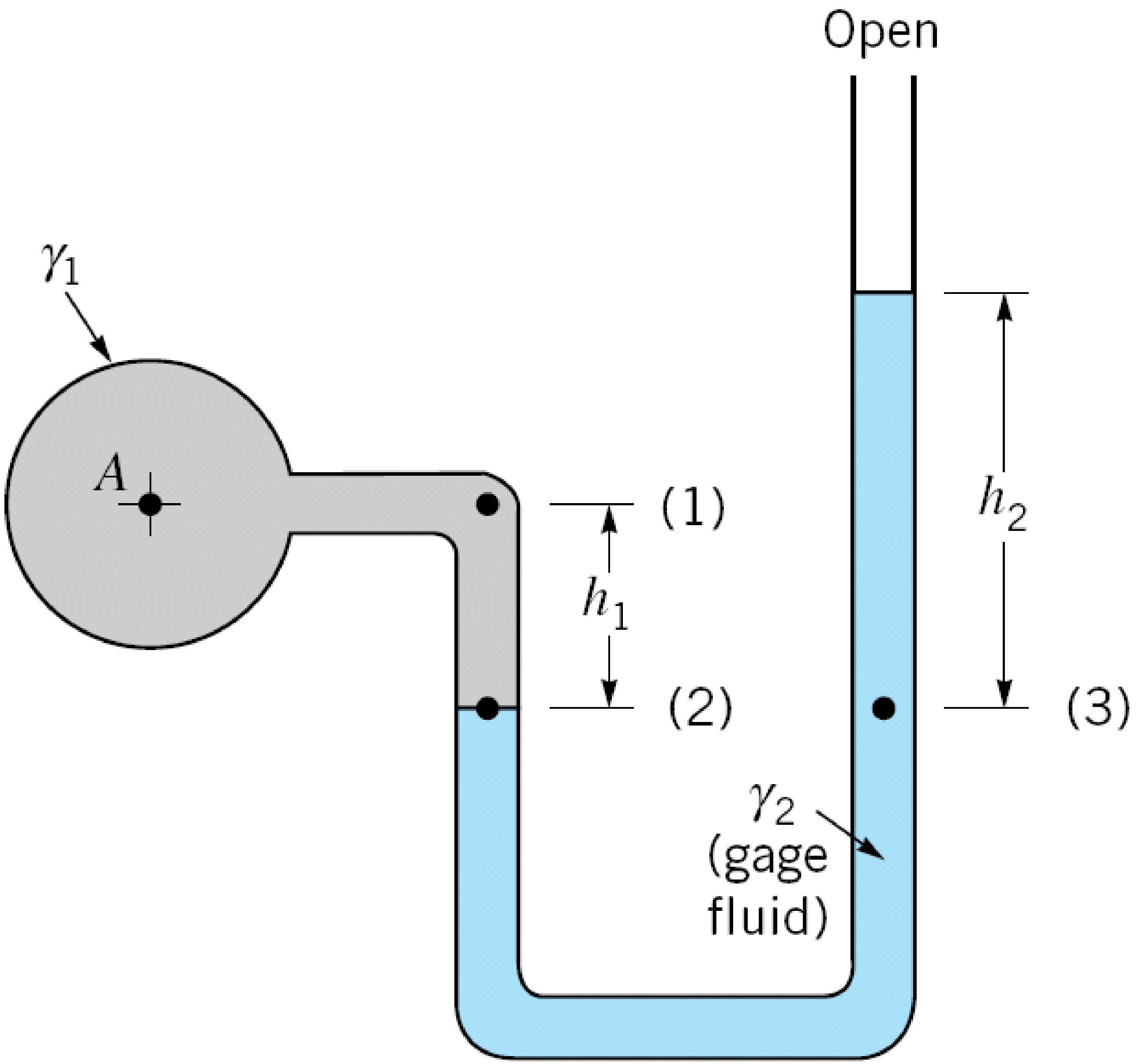
PIEZOMETER TUBE



$$P_A = \rho g h_1 + P_{atm}$$

- **Simple and accurate**
- **Suitable only for pressure greater than atmospheric pressure**
- **Pressure measured to be reasonably small – height will be large**
- **Capillarity effects are negligible for large bore tube ie., diameters greater than 30 mm**

U-TUBE MANOMETER



$$P_A + \rho_1 g h_1 - \rho_2 g h_2 = 0$$

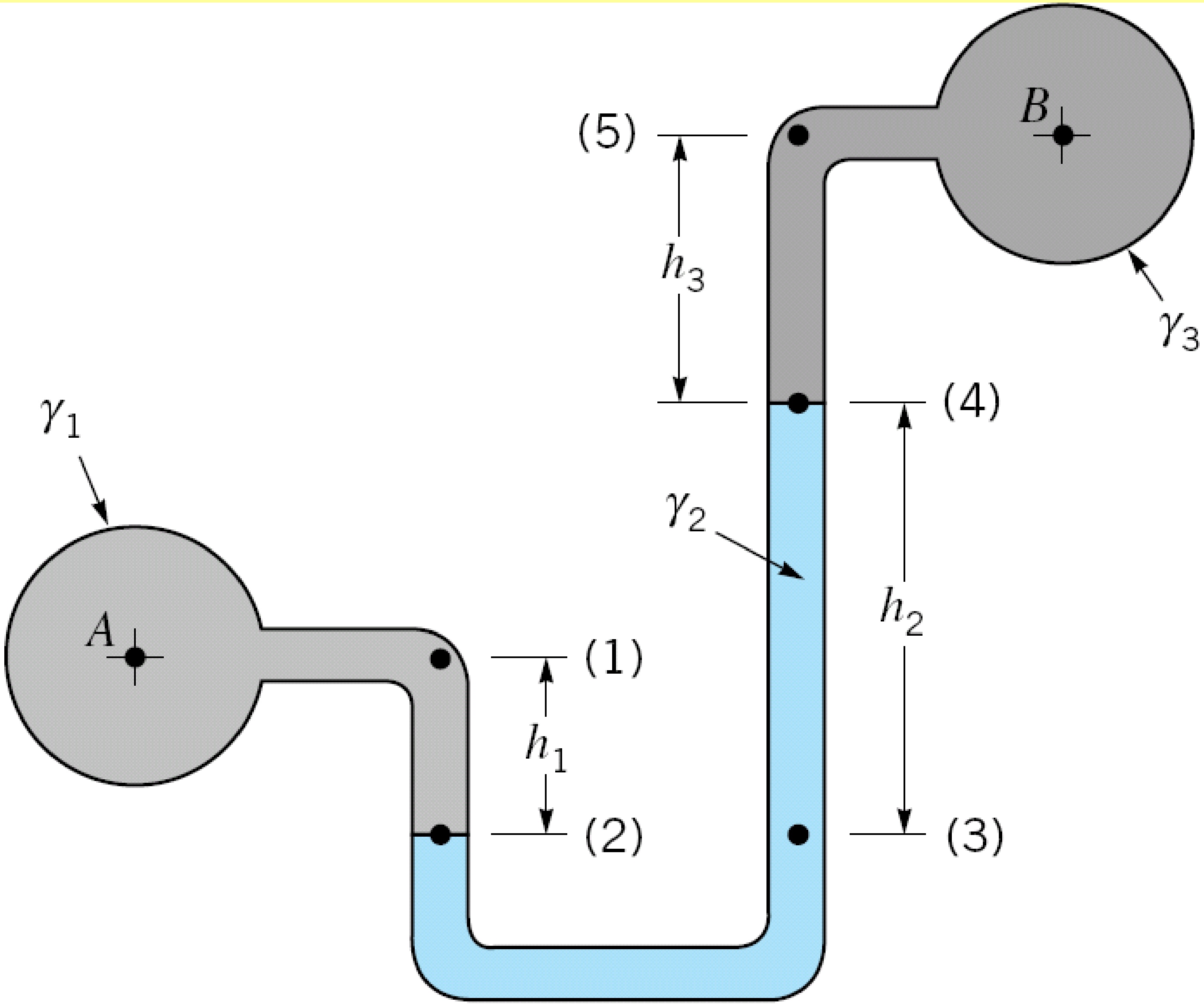
$$P_A = \rho_2 g h_2 - \rho_1 g h_1$$

If A contains gas,

$$P_A = \rho_2 g h_2$$

Capillarity effects are negligible for large bore tube ie., diameters greater than 30 mm

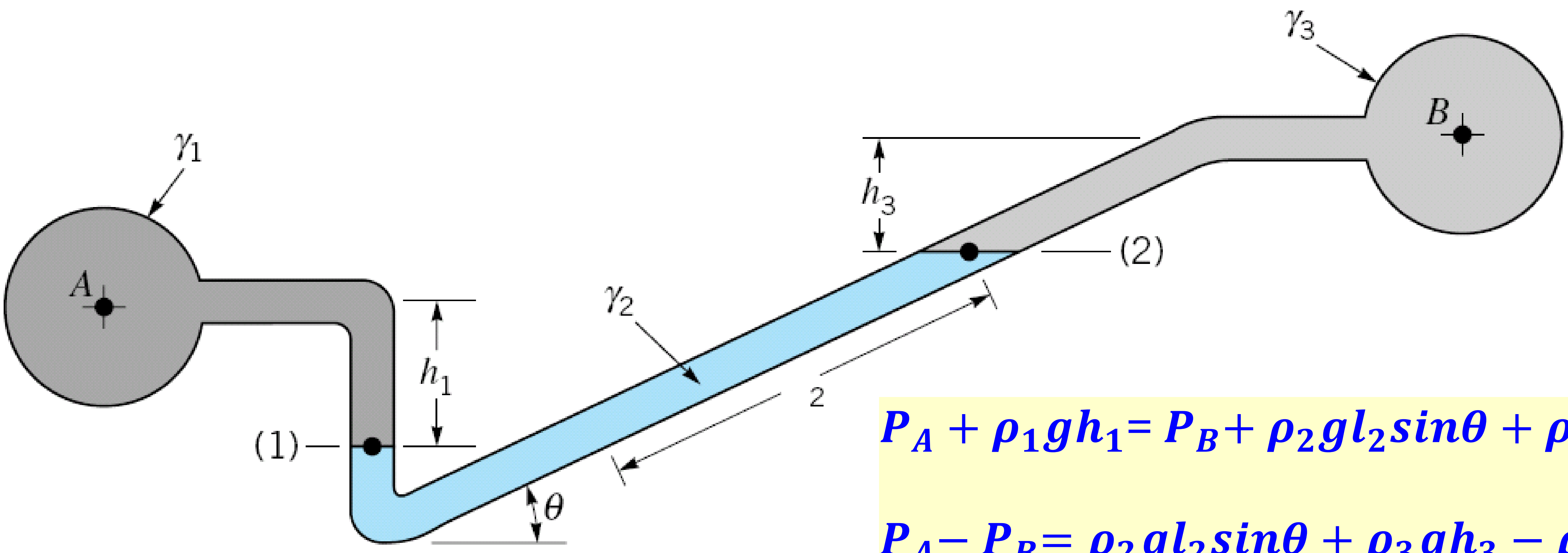
DIFFERENTIAL U-TUBE MANOMETER



$$P_A + \rho_1 g h_1 = P_B + \rho_2 g h_2 + \rho_3 g h_3$$

Capillarity effects cancel

INCLINED TUBE MANOMETER



$$P_A + \rho_1 g h_1 = P_B + \rho_2 g l_2 sin\theta + \rho_3 g h_3$$

$$P_A - P_B = \rho_2 g l_2 sin\theta + \rho_3 g h_3 - \rho_1 g h_1$$

If A and B contain gas

$$l_2 = \frac{P_A - P_B}{\rho_2 g sin\theta}$$

Amplification factor is $\frac{1}{sin\theta}$

MICROMANOMETER

$$2\Delta y A = Ra$$

A – Area of the larger vessel

a – Area of the tube

At BB pressure is same in liquid "ρ_3"

$$P_C + \rho_1 g(k_1 + \Delta y) + \left(k_2 + \frac{R}{2} - \Delta y\right)\rho_2 g =$$

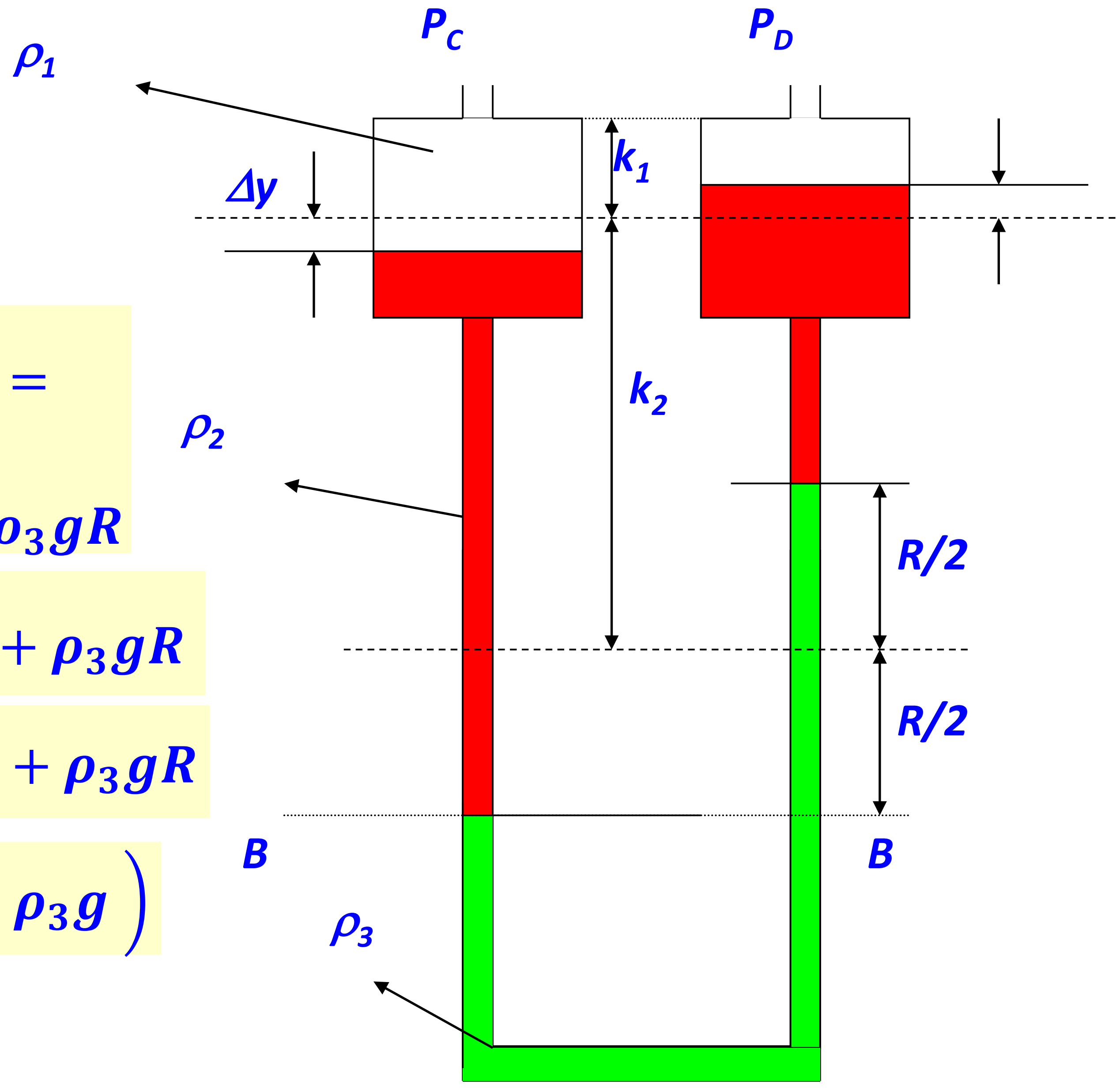
$$P_D + \rho_1 g(k_1 - \Delta y) + \left(k_2 - \frac{R}{2} + \Delta y\right)\rho_2 g + \rho_3 g R$$

$$P_C - P_D = -2\rho_1 g(\Delta y) + 2\left(-\frac{R}{2} + \Delta y\right)\rho_2 g + \rho_3 g R$$

$$P_C - P_D = -\rho_1 g\left(R\frac{a}{A}\right) - \rho_2 g R + \rho_2 g\left(R\frac{a}{A}\right) + \rho_3 g R$$

$$P_C - P_D = R\left(-\rho_1 g\left(\frac{a}{A}\right) - \rho_2 g + \rho_2 g\left(\frac{a}{A}\right) + \rho_3 g\right)$$

$$P_C - P_D = R(\rho_3 - \rho_2)g + \left(\frac{a}{A}\right)(\rho_2 g - \rho_1 g)$$



MICROMANOMETER

$$2\Delta yA = Ra$$

A – Area of the larger vessel

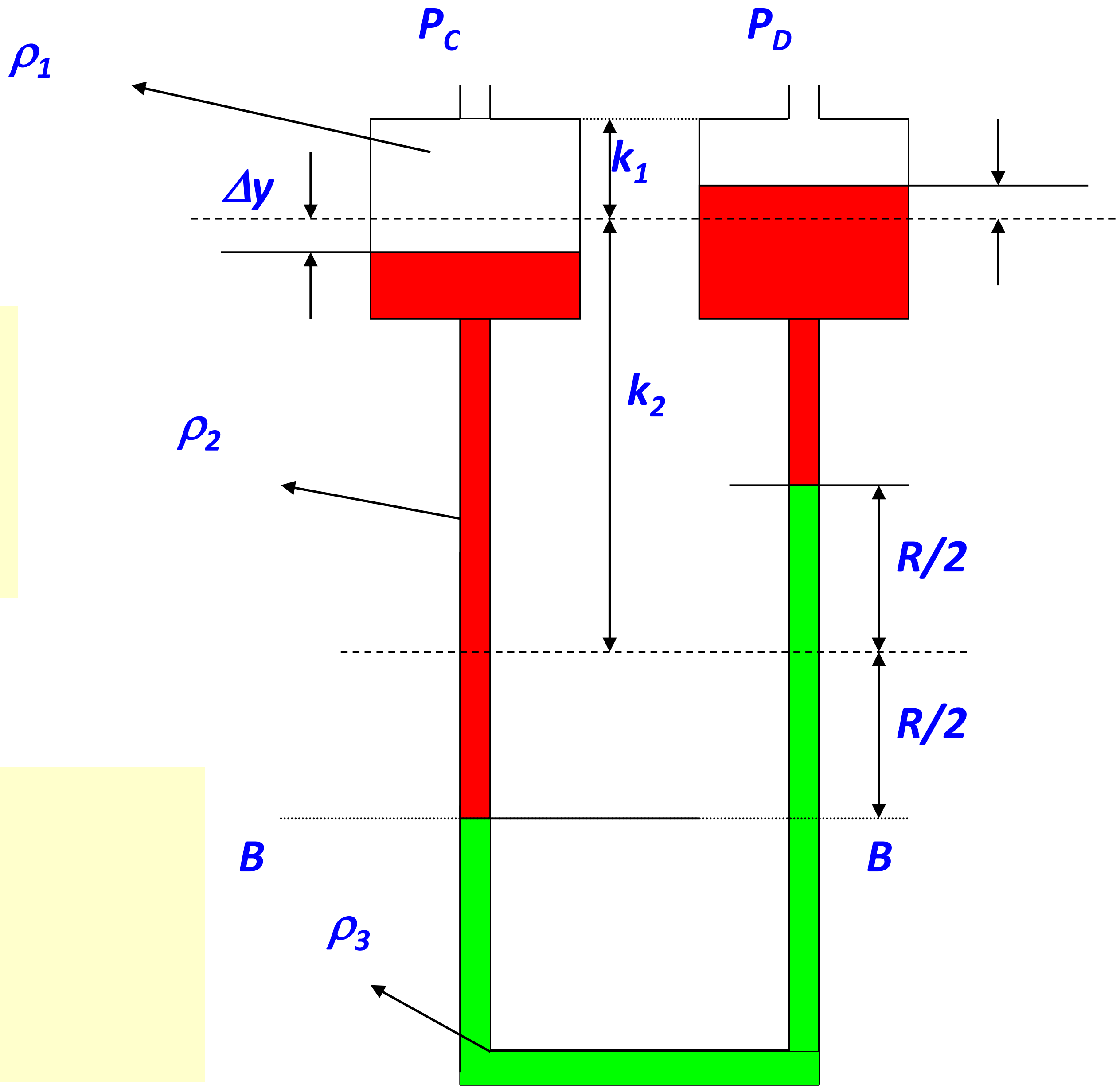
a – Area of the tube

At BB pressure is same in liquid "ρ_3"

$$P_C - P_D = R(\rho_3 - \rho_2)g + \left(\frac{a}{A}\right)(\rho_2 g - \rho_1 g)$$

$$\frac{a}{A} \ll 1 \Rightarrow P_C - P_D = R(\rho_3 - \rho_2)g$$

$$R = \frac{P_C - P_D}{(\rho_3 - \rho_2)g}$$



Output = R, input = $P_C - P_D$

Static sensitivity = $\dfrac{1}{(\rho_3 - \rho_2)}$

Lower is the density difference between the liquids, higher is the sensitivity of the instrument.

MICROMANOMETER

$$R = \frac{P_C - P_D}{(\rho_3 - \rho_2)g} = \frac{P_C - P_D}{\rho_{water}(SG_3 - SG_2)g}$$

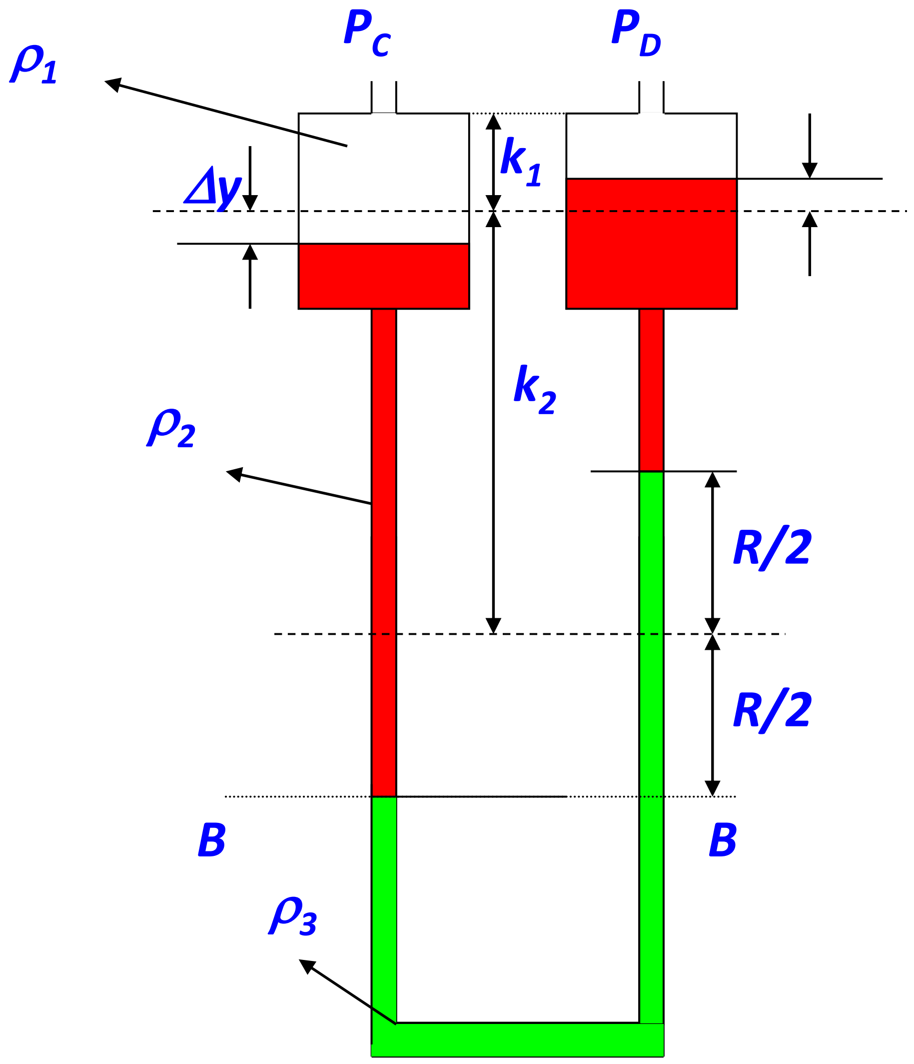
$$Amplificaton = \frac{1}{(SG_3 - SG_2)}$$

Air, water and ethylene glycol

$$Amplificaton = \frac{1}{(SG_3 - SG_2)} = \frac{1}{(1.13 - 1.0)} = 7.7$$

Air, kerosene and water

$$Amplificaton = \frac{1}{(SG_3 - SG_2)} = \frac{1}{(1.00 - 0.82)} = 5.6$$

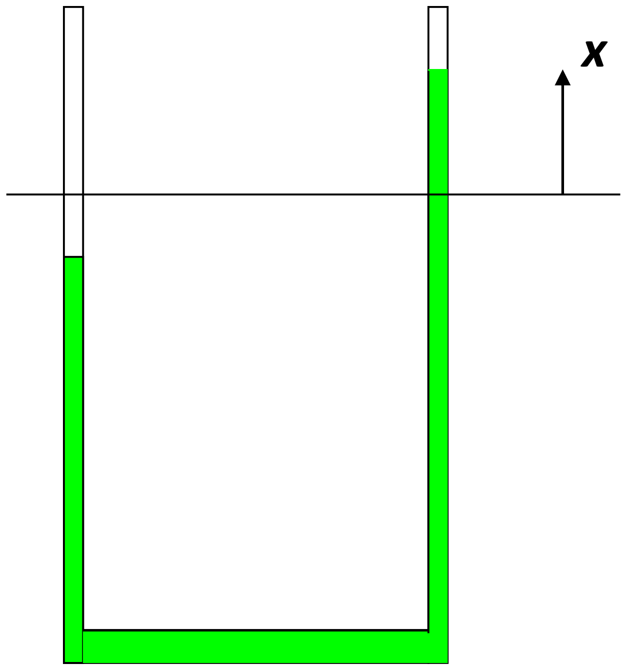


DYNAMIC RESPONSE OF MANOMETER FOR STEP INPUT

Assumptions

- Flow is laminar - Round rube relations are valid
- Effect of curvature is neglected – d/D is small
- Surface tension effects at the two ends of the body of

  the fluid are neglected



Suddenly there is a change in the pressure difference across manometer. Manometer column will oscillate for some time. Friction tries to damp the motion.

Forces Acting On The Manometer Liquid Column

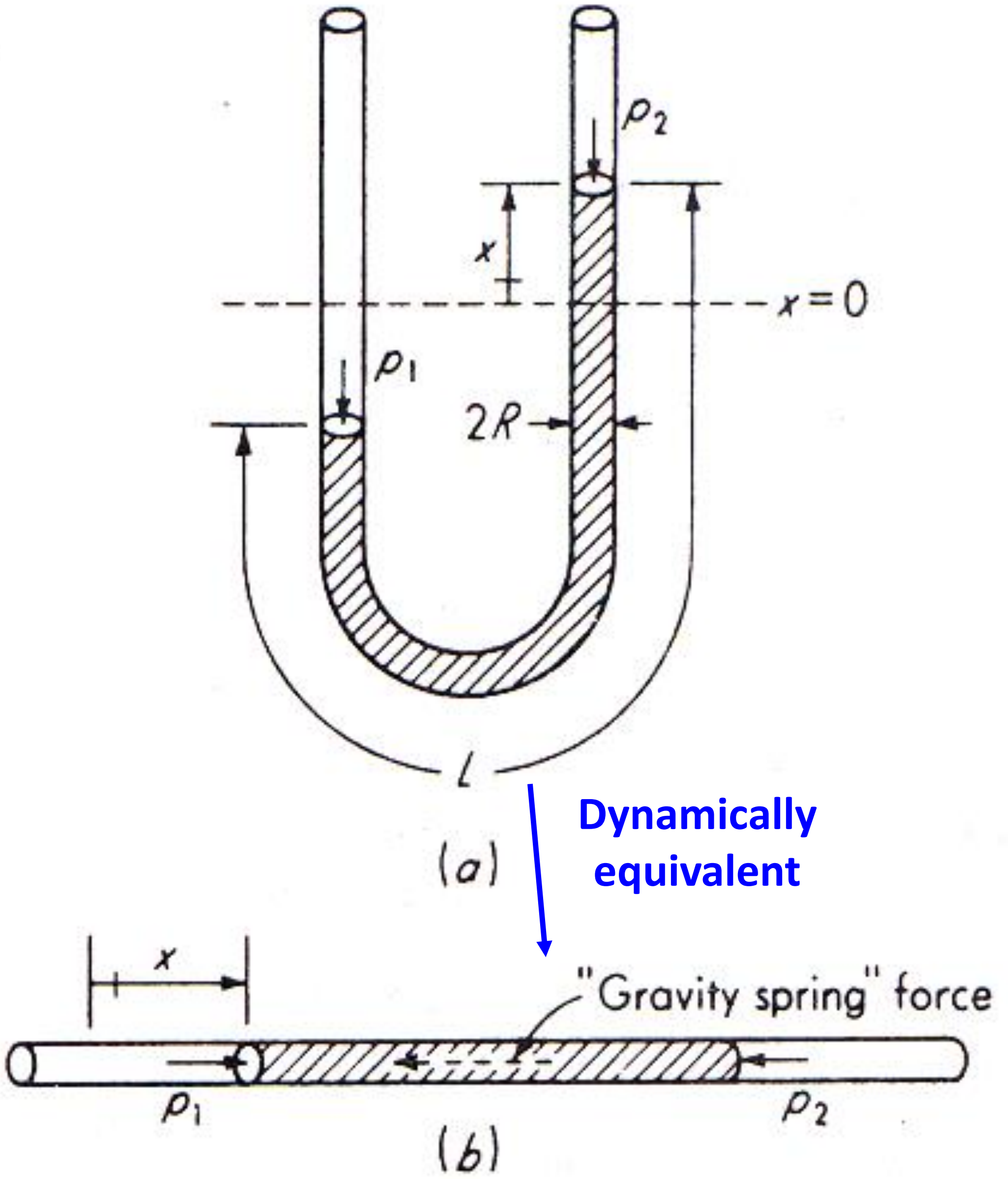
- Gravity force (weight) distributed uniformly over the whole body of fluid

- Forces on the two ends of the free body due to pressure p_1 and p_2

- Drag force due to motion of the fluid within the tube and related to the wall shearing stress

- Gravity forces – Unbalanced gravity force acting on the liquid column tending to restore the level $x = 0$

- $-2\pi R^2 x\rho g$ – force is proportional to displacement and always opposes it – SPRING FORCE



Balance of forces results in

Inertia force = Pressure force + Gravity force + Viscous force

Inertia force

Velocity profile for laminar flow

$$u = u_{cl}\left(1-\frac{r^2}{R^2}\right) \;\&\; u_{cl} = 2u_{av} \Rightarrow u = 2u_{av}\left(1-\frac{r^2}{R^2}\right)$$

$$Momentum = \dot{m}u = (\rho A u)u = \int_0^R \rho\, 4u_{av}^2\left(1-\frac{r^2}{R^2}\right)^2 (2\pi r dr) = 8\pi u_{av}^2 \rho \int_0^R \left(1-\frac{r^2}{R^2}\right)^2 (r dr)$$

$$Momentum = 8\pi u_{av}^2 \rho \int_0^R \left(1-2\frac{r^2}{R^2}+\frac{r^4}{R^4}\right)(r dr) = 8\pi u_{av}^2 \rho \int_0^R \left(r-2\frac{r^3}{R^2}+\frac{r^5}{R^4}\right)(dr)$$

$$Momentum = 8\pi u_{av}^2 \rho \left(\frac{R^2}{2}-2\frac{R^4}{4R^2}+\frac{R^6}{6R^4}\right) = 8\pi u_{av}^2 \rho R^2\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{4}{3}\pi R^2 \rho\, u_{av}^2$$

$$\ddot{x} = u_{av}\frac{du_{av}}{ds} \sim \frac{u_{av}^2}{L}$$

$$Inertia\;Force = \frac{4}{3}\pi R^2 \rho \ddot{x} \mathrm{L}$$

$$Inertia\ Force = \frac{4}{3}\pi R^2 \rho \ddot{x} L$$

$$\text{Pressure Force} = \pi R^2 (p_1 - p_2)$$

$$Gravity\ Force = \pi R^2 \rho g (2x) = 2\pi R^2 \rho g x$$

$$Viscous\ Force = \tau_w A_s = 8\pi\mu u_{av} L = 8\pi\mu \dot{x} L$$

$$f = \frac{64}{Re} \Rightarrow C_f = \frac{16}{Re} = \frac{16\mu}{\rho u_{av} D} \Rightarrow \frac{\tau_w}{\frac{1}{2}\rho u_{av}^2} = \frac{16\mu}{\rho u_{av} D}$$

$$\tau_w = \frac{8\mu u_{av}}{D} \Rightarrow \tau_w A_s = \frac{8\mu u_{av}}{D}\pi D L = 8\pi\mu u_{av} L$$

Inertia force = Pressure force + Gravity force + Viscous force

$$\frac{4}{3}\pi R^2 \rho \ddot{x} L = \pi R^2 (p_1 - p_2) - 2\pi R^2 \rho g x - 8\pi\mu \dot{x} L$$

$$\frac{4}{3}\pi R^2 \rho \ddot{x} L + 8\pi\mu \dot{x} L + 2\pi R^2 \rho g x = \pi R^2 (p_1 - p_2)$$

Divide throughout by $2\pi R^2 \rho g$

$$\frac{\ddot{x}}{\frac{3g}{2L}} + \frac{4\mu L}{R^2 \rho g}\dot{x} + x = \frac{\Delta p}{2\rho g}$$

Gravity Force acts as a spring force
Viscous force acts as a damping force
Pressure force is the input

$$\frac{\ddot{x}}{\frac{3g}{2L}} + \frac{4\mu L}{R^2\rho g}\dot{x} + x = \frac{\Delta p}{2\rho g}$$

Gravity Force acts as a spring force
Viscous force acts as a damping force
Pressure force is the input

At any instant of time, the wall shearing stress can be computed from the instantaneous velocity of the liquid by using the relations used for steady pipe flows

$$q_o(s) \;\; \Delta p \longrightarrow \boxed{\frac{\frac{1}{2\rho g}}{\left(\frac{1}{\omega_n^2}s^2 + \frac{2\xi}{\omega_n}s + 1\right)}} \longrightarrow q_i(s) \;\; x$$

$$\frac{2\xi}{\omega_n} = \frac{4\mu L}{R^2\rho g} = \frac{2\xi}{\sqrt{\frac{3g}{2L}}}$$

$$K = \frac{1}{2\rho g};\quad \omega_n = \sqrt{\frac{3g}{2L}}\;;\xi = \frac{2.45\mu}{R^2\rho}\sqrt{\frac{L}{g}}$$

$$\underset{\Delta p}{q_o(s)} \longrightarrow \boxed{\dfrac{\frac{1}{2\rho g}}{\left(\frac{1}{\omega_n^2}s^2 + \frac{2\xi}{\omega_n}s + 1\right)}} \longrightarrow \underset{x}{q_i(s)}$$

Compare mercury and water manometer:

$\mu_{Hg} = 0.15 \times 10^{-2}\ Pa.s$ at 300 K

$\rho_{Hg} = 13600\ \frac{kg}{m^3}$

$L = 0.67\ m$

$R = 3.3\ mm$

$\mu_{water} = 0.08 \times 10^{-2}\ Pa.s$ at 300 K

$\rho_{water} = 1000\ \frac{kg}{m^3}$

$\omega_n = 4.7 \frac{rad}{s}$; $\xi_{water} = 0.047$; $\xi_{Mercury} = 0.007$

DAMPING FOR MERCURY IS SMALLER!!

If ρ is large, ξ is small. If R is small, ξ is large . If L is large, ξ is large.

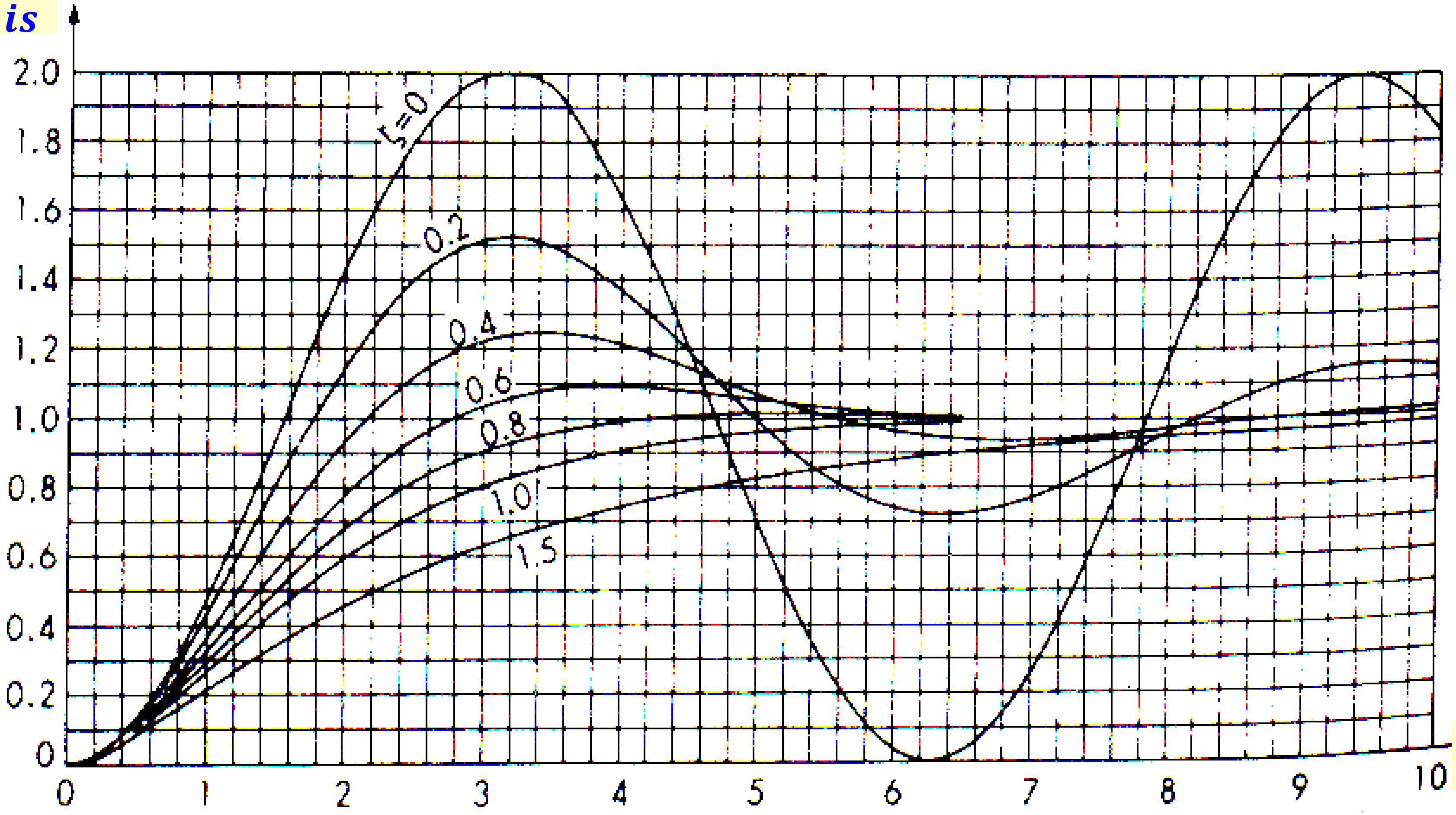
ξ Is large $\Rightarrow$ Time to reach steady state value increases.

Transient measurements pose problems.

Step input response of underdamped second order system

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$\dfrac{q_o(t)}{Kq_{is}}$ ***Damped Natural Frequency*** $\omega_d = \omega_n\sqrt{1-\xi^2}$



$\omega_n t$

Mercury Manometer Dynamics

$\mu_{Hg} = 0.15 \times 10^{-2}\ Pa.s$ at 300 K

$\rho_{Hg} = 13600\ \frac{kg}{m^3}$

$L = 0.67\ m\ R = 3.3\ mm$

$\omega_n = 4.7 \frac{rad}{s}\ ;\ \xi_{Mercury} = 0.007$

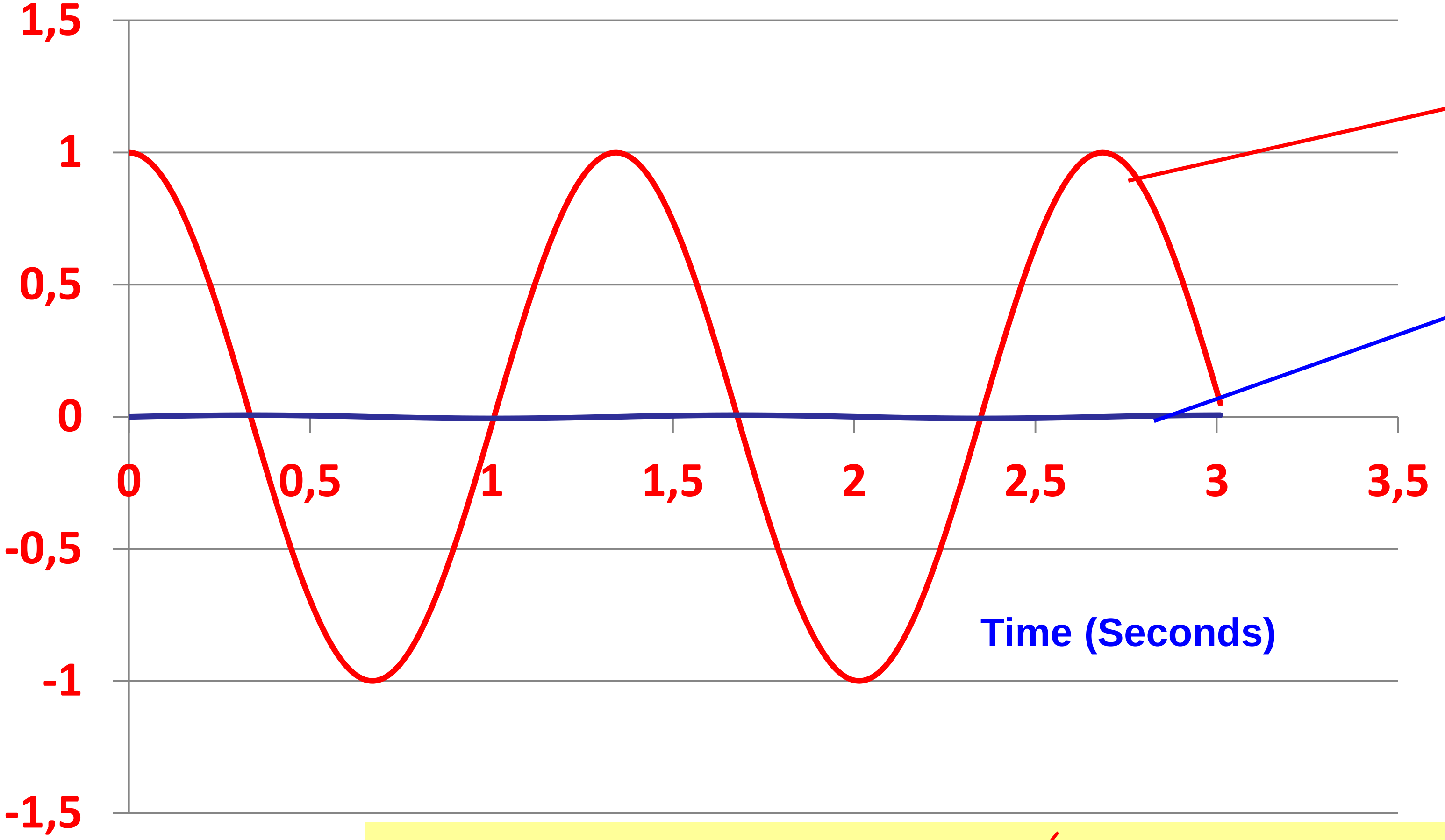
$$\omega_d = \omega_n\sqrt{1-\xi^2} = 4.7\sqrt{1-(0.007)^2} = 4.68\ rad/s$$

$$Period = T = \frac{2\pi}{\omega_d} = \frac{2\pi}{4.68} = 1.34\ seconds$$

$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$\frac{\xi}{\sqrt{1-\xi^2}} = \frac{0.007}{\sqrt{1-(0.007)^2}} = 0.007 \Rightarrow \cos\omega_d t \gg \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t$$

$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}(\cos\omega_d t)$$

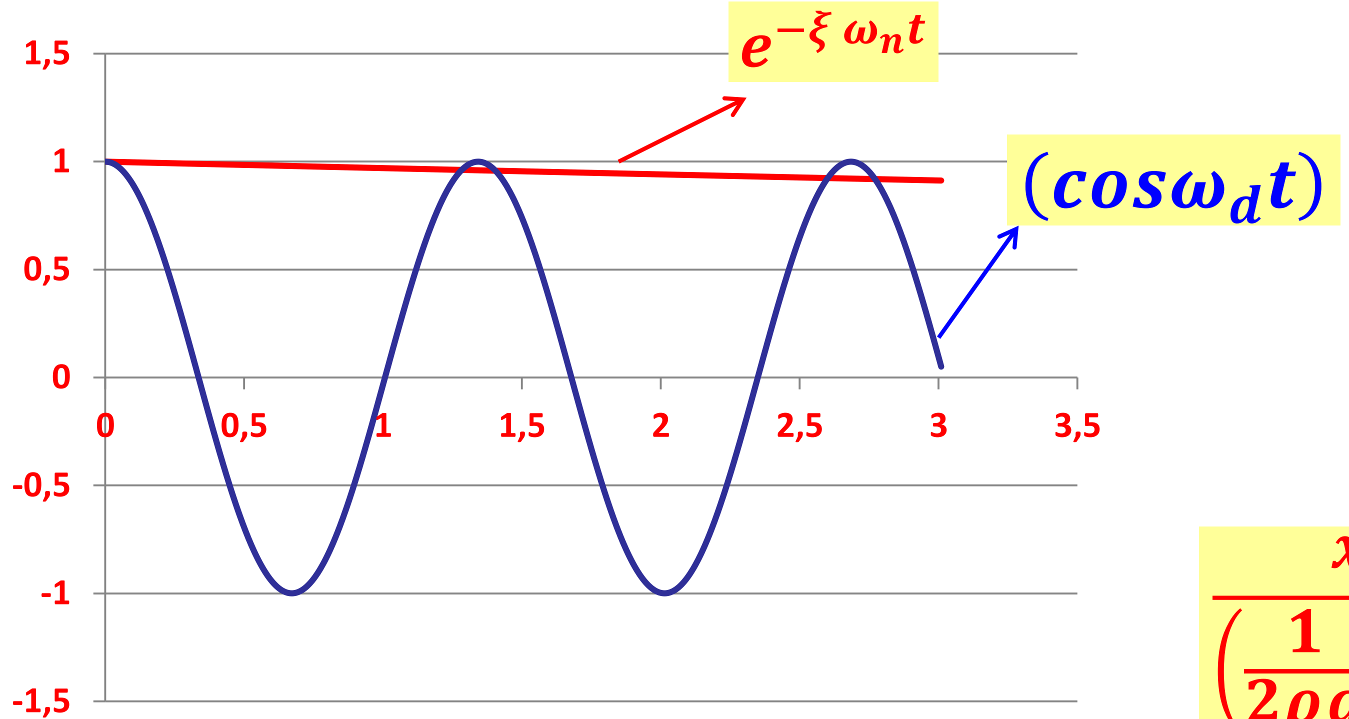


$$(cos\omega_d t)$$

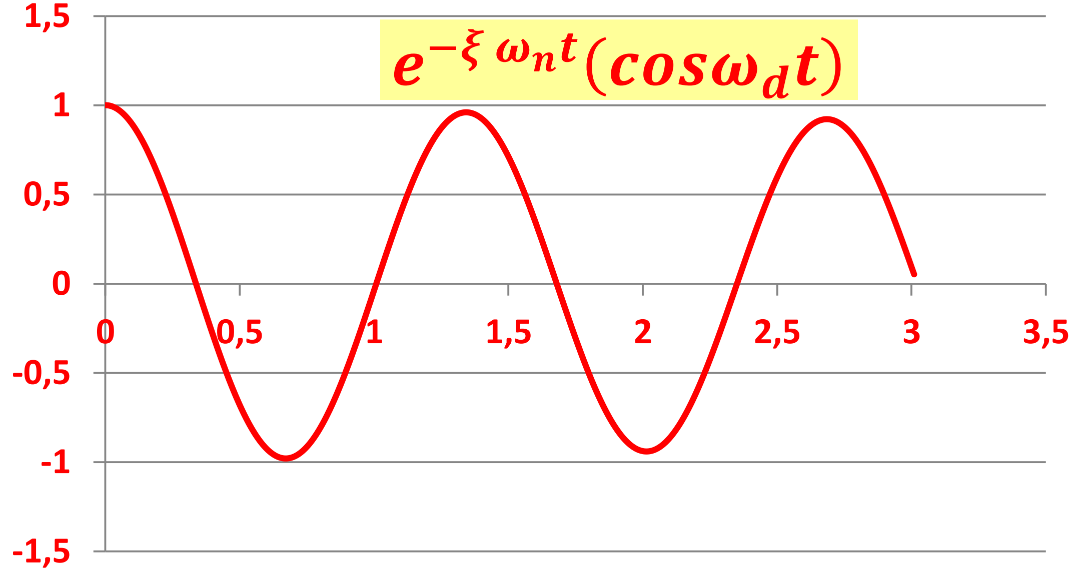
$$\frac{\xi}{\sqrt{1-\xi^2}} sin\omega_d t$$

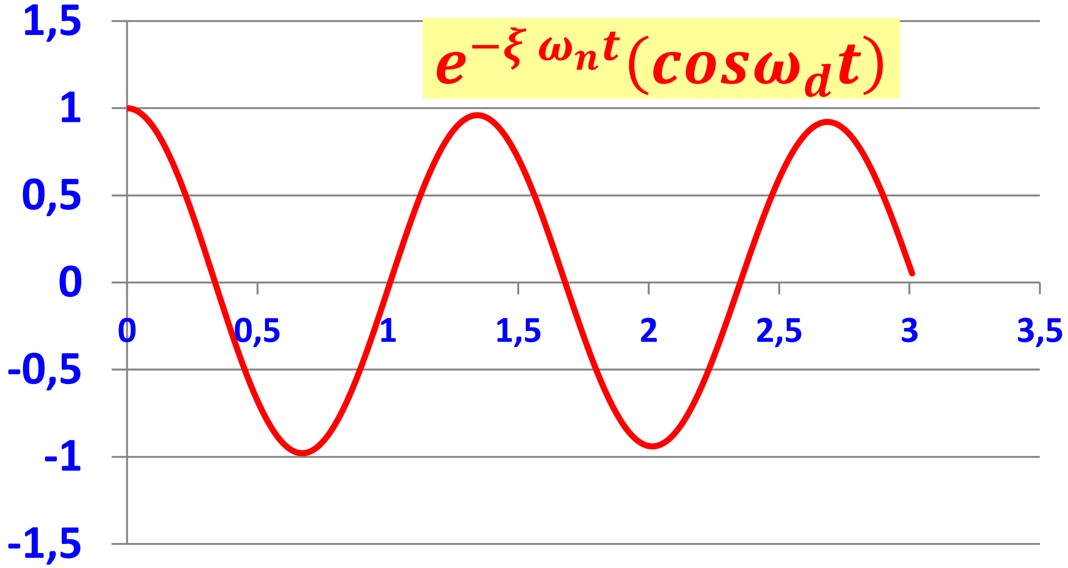
$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}\left(cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} sin\omega_d t\right)$$

$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}(cos\omega_d t)$$

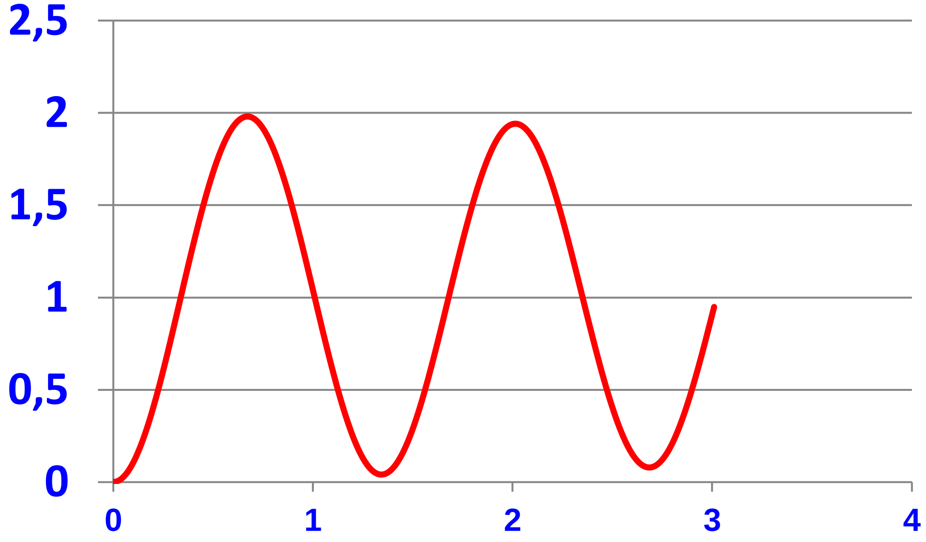


$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}(cos\omega_d t)$$

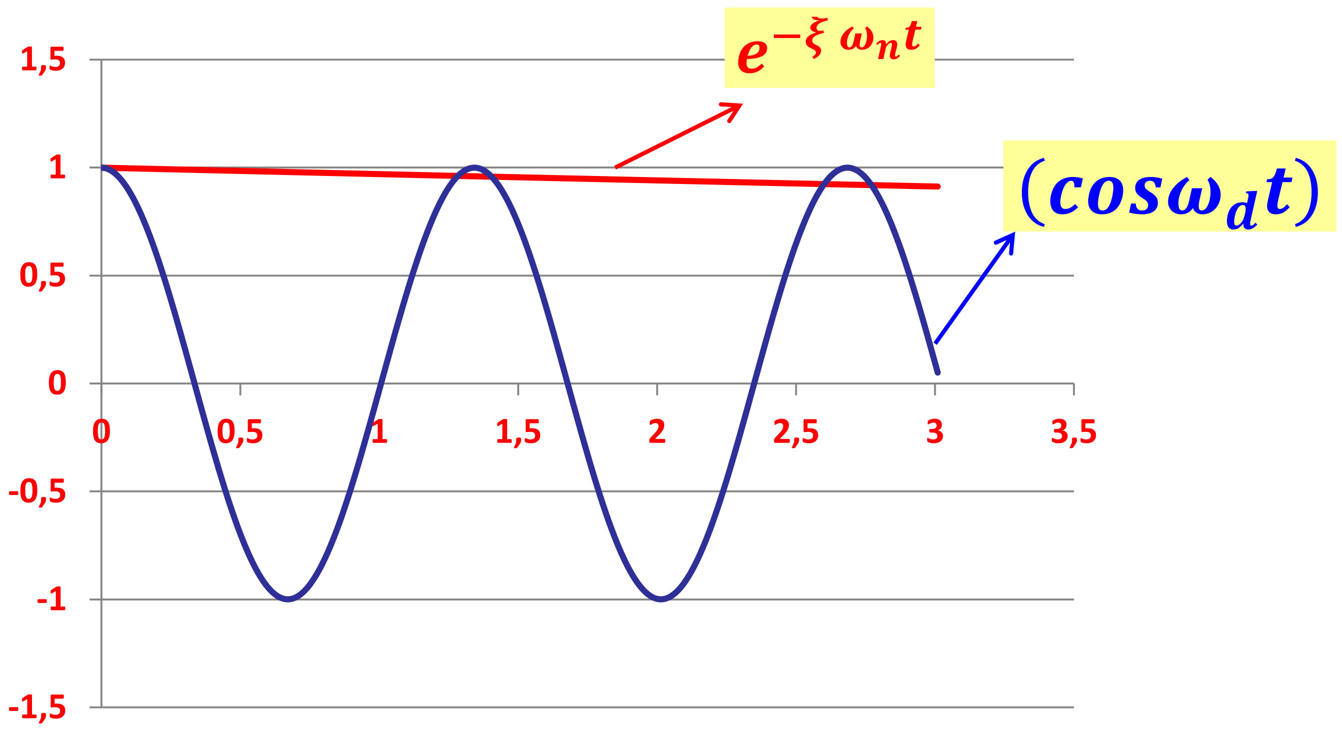


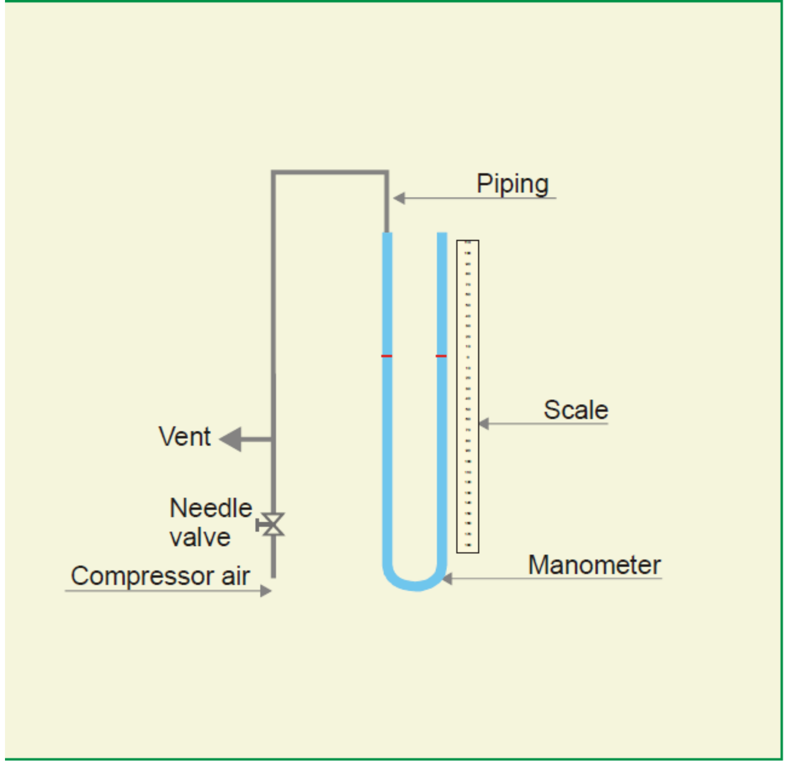


$$e^{-\xi\,\omega_n t}(\cos\omega_d t)$$



$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}(\cos\omega_d t)$$





Measurements

1. Number of oscillations = nearly 8
2. Average time period of oscillations = T
3. peak values at various instants of time ($t = 0, t = t_1, t = t_2$)

Calculations

$$\omega_d \approx \omega_n = \frac{2\pi}{(t_1 - t_o)}$$

$$\frac{Y_1}{Y_o} = \frac{e^{-\xi\,\omega_n t_1}}{e^{-\xi\,\omega_n t_o}} = e^{-\xi\,\omega_n t_1}$$

Comparison with the theory

Compare the value of ξ, ω_n computed with the ξ, ω_n measured

PROBLEM ON THE U-TUBE MANOMETER DYNAMICS

Pressure at the peak (mm of Hg)	Time at the peak (seconds)	Natural frequency	Calculated Natural Frequency	Deviation	Theoretical damping ratio	Calculated Damping ratio	Deviation
36	0.666667						
30	2.133333						
26	3.6						
23	5.1						
21	6.566667						

Diameter of the manometer tube = 4 mm

Length of the mercury column = 1m

Manometric fluid = Mercury

Measured natural frequency – calculated

Find the natural frequency and damping ratio for these measurements and also calculate the theoretical values from the theory and compare these values and comment

Compare mercury and water manometer:

$$\mu_{Hg} = 0.15 \times 10^{-2}\ Pa.s \text{ at 300 K}$$

$$\rho_{Hg} = 13600\ \frac{kg}{m^3}$$

$$L = 1.0\ m$$

$$R = 2.0\ mm$$

THEORETICAL VALUES

$$\omega_n = \sqrt{\frac{3g}{2L}} = \sqrt{\frac{3 \times 9.81}{2 \times 1}} = 3.836 \text{ rad/s;}$$

$$\xi = \frac{2.45\mu}{R^2\rho}\sqrt{\frac{L}{g}} = \frac{2.45 \times 0.15 \times 10^{-2}}{(2.0 \times 10^{-3})^2 \times 13.6 \times 1000}\sqrt{\frac{1}{9.81}} = 0.0216$$

$$\omega_n = 3.836 \text{ rad/s;} \quad \xi = 0.0216$$

Pressure at the peak (mm of Hg)	Time at the peak (seconds)	Natural frequency (rad/s)	Calculated Natural Frequency (rad/s)	Deviation (%)	Theoretical damping ratio	Calculated Damping ratio	Deviation
46	0						
36	0.666667	9.4248	3.836	145	0.0216	0.0390	44.7
30	2.133333	4.284	3.836	10.5	0.0216	0.0290	25.6
26	3.6	4.284	3.836	10.5	0.0216	0.0228	5.30
23	5.1	4.189	3.836	8.4	0.0216	0.0195	-10.6
21	6.566667	4.284	3.836	10.5	0.0216	0.0145	-49.0

$$\omega_d \approx \omega_n = \frac{2\pi}{(t_1 - t_o)} = \frac{2\pi}{(2.1333 - 0.6667)} = 4.284$$

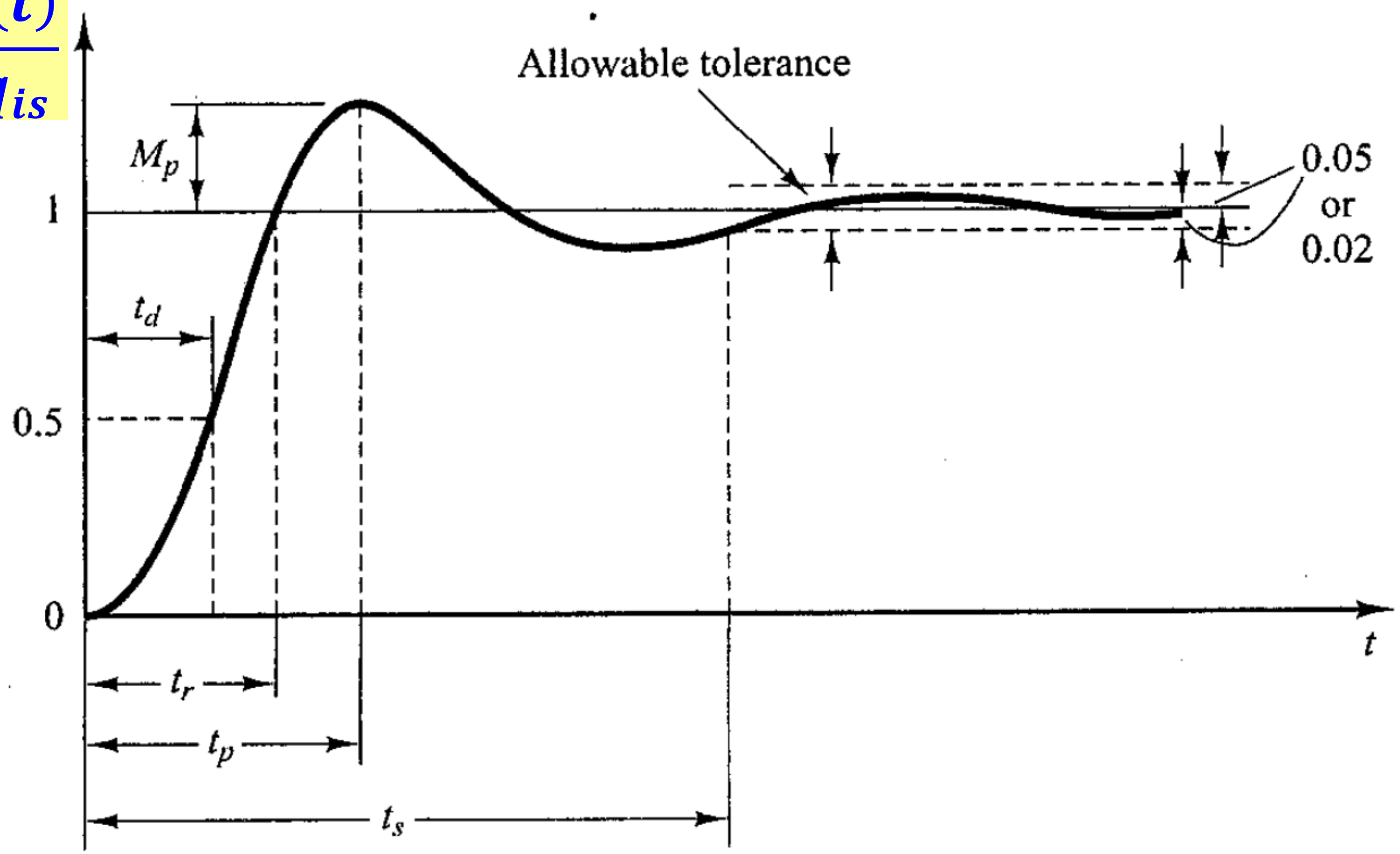
$$\frac{Y_1}{Y_o} = \frac{e^{-\xi\,\omega_n t_1}}{e^{-\xi\,\omega_n t_o}} = e^{-\xi\,\omega_n (t_1 - t_o)}$$

$$\xi = -\frac{1}{\omega_n (t_1 - t_o)} ln \frac{Y_1}{Y_o}$$

$$\xi = -\frac{1}{4.284(2.1333 - 0.6667)} ln \frac{30}{36} = 0.029$$

Different parameters of an underdamped second order system

$$\frac{q_o(t)}{Kq_{is}}$$



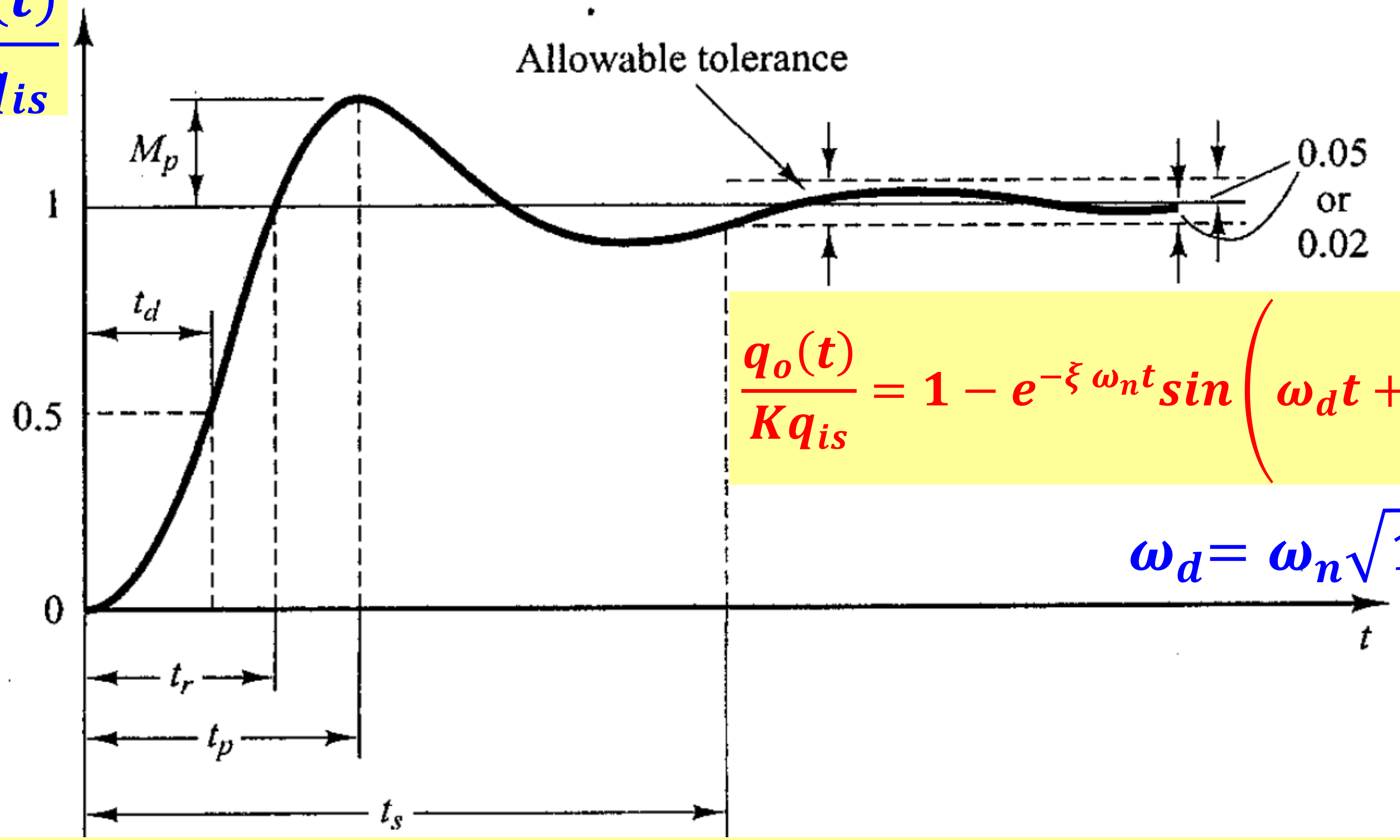
Rise time is the time required to attain the steady state value for the first time (t_r)

Peak time is the time required to reach the first peak (t_p)

Settling time is the time required for the output to reach and stay within a tolerance band $\pm\Delta$ (t_s)

Different parameters of an underdamped second order system

$$\frac{q_o(t)}{Kq_{is}}$$



$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t} sin\left(\omega_d t + Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

Rise time is the time required to attain the steady state value for the first time

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t} sin\left(\omega_d t + Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right) = 1 \Rightarrow sin\left(\omega_d t + Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right) = 0 \Rightarrow sin\pi$$

$$\omega_d t_r + Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \pi$$

$$t_r = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$\omega_d t_r + Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \pi$$

$$t_r = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

Peak time is the time required to reach the first peak

$$\omega_d t_r + Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \pi$$

In this equation, $t_r = t_p$ when $Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = 0$

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\frac{d\left(\frac{q_o(t)}{Kq_{is}}\right)}{dt} = -(-\xi\,\omega_n)e^{-\xi\,\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right) + e^{-\xi\,\omega_n t}\left(-\omega_d\sin\omega_d t + \frac{\xi\omega_d}{\sqrt{1-\xi^2}}\cos\omega_d t\right)$$

The above equation would be zero at $t = t_p$

$$0 = (\xi\,\omega_n)e^{-\xi\,\omega_n t_p}\left(\cos\omega_d t_p + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_p\right) - e^{-\xi\,\omega_n t_p}\left(-\omega_d\sin\omega_d t_p + \frac{\xi\omega_n\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}}\cos\omega_d t_p\right)$$

$$0 = \cancel{(\xi\,\omega_n)\cos\omega_d t_p} + (\xi\,\omega_n)\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_p + \omega_d\sin\omega_d t_p - \cancel{\xi\omega_n\cos\omega_d t_p}$$

$$0 = (\xi\,\omega_n)\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_p + \xi\omega_n\sqrt{1-\xi^2}\sin\omega_d t_p$$

$$0 = \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t_p + \sqrt{1-\xi^2}\sin\omega_d t_p$$

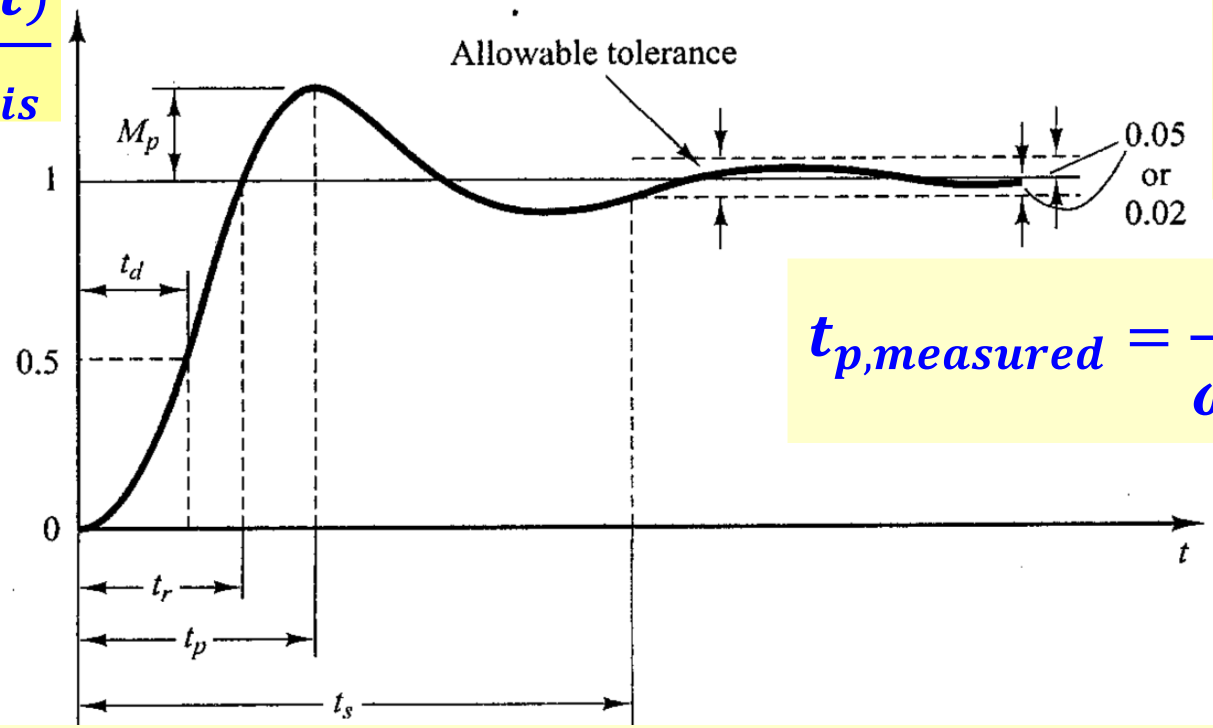
$$\sin\omega_d t_p = 0 \qquad \omega_d t_p = 0, \pi, 2\pi, \ldots$$

Since the peak time corresponds to the first peak overshoot $\omega_d t_p = \pi$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

Determination of ξ and ω_n using measured parameters $t_{r,measured}$ and $t_{p,measured}$

$$\frac{q_o(t)}{Kq_{is}}$$



$$t_r = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$t_{p,measured} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \Rightarrow \omega_n\sqrt{1-\xi^2} = \frac{\pi}{t_{p,measured}}$$

$$t_{r,measured} = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\left(\frac{\pi}{t_{p,measured}}\right)}$$

From the known parameters, parameters $t_{r,measured}$, $t_{p,measured}$ calculate ξ

$$\omega_n == \frac{\pi}{t_{p,measured}\sqrt{1-\xi^2}}$$

Maximum overshoot occurs at $t = t_{p,measured}$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t}\left(cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} sin\omega_d t\right)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

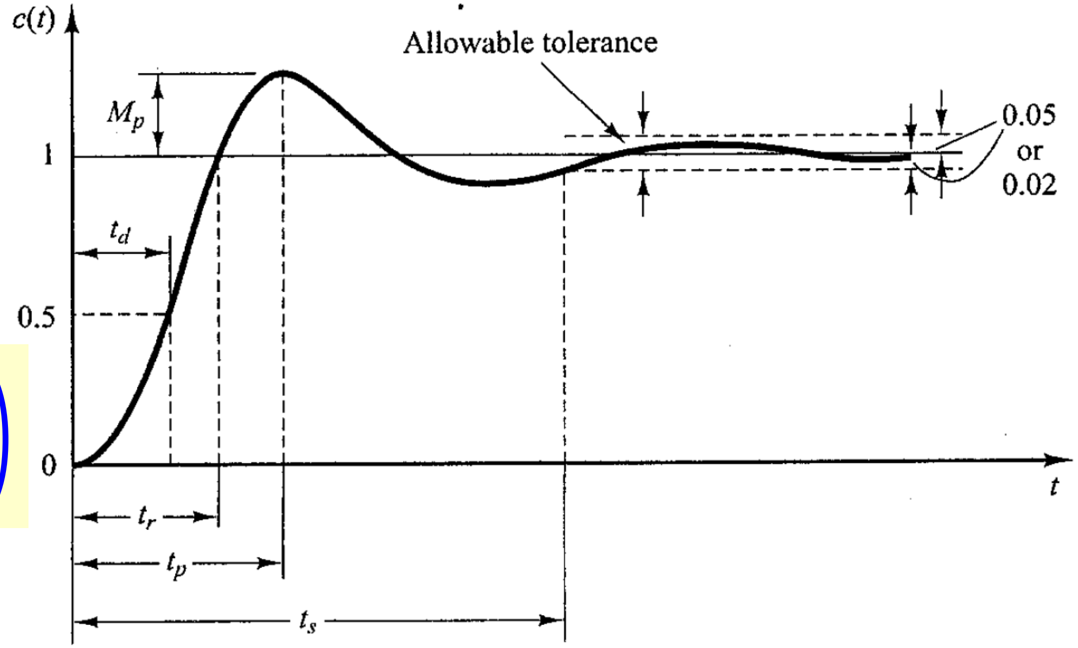
$$\frac{q_o(t_p)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t_p}\left(cos\omega_d \frac{\pi}{\omega_d} + \frac{\xi}{\sqrt{1-\xi^2}} sin\omega_d \frac{\pi}{\omega_d}\right)$$

$$M_p = \frac{q_o(t_p)}{Kq_{is}} - 1 = -e^{-\xi\,\omega_n t_p}\left(cos\pi + \frac{\xi}{\sqrt{1-\xi^2}} sin\pi\right)$$

$$cos\pi = -1 \;\; sin\pi = 0$$

$$M_p = -e^{-\xi\,\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}(-1)$$

$$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$



$$t_r = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d}$$

An underdamped second order system when subjected to unit step input records rise time of 0.55 seconds and a peak time of 0.785 seconds. Find the natural frequency and damping coefficient of this system. Find the maximum overshoot

$$t_r = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$t_{p,measured} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \Rightarrow \omega_n\sqrt{1-\xi^2} = \frac{\pi}{0.785} = 4.002$$

$$\omega_n\sqrt{1-\xi^2} = 4.002$$

$$t_{r,measured} = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$0.55 = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{4.002}$$

$$2.201116 = \pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

$$Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \pi - 2.201116$$

$$Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = 0.9405$$

$$\frac{\sqrt{1-\xi^2}}{\xi} = Tan\left(\frac{180}{\pi} \times 0.9405\right)$$

$$\frac{\sqrt{1-\xi^2}}{\xi} = 1.370606$$

$$\frac{1-\xi^2}{\xi^2} = 1.8786$$

$$1 = 2.8786\xi^2$$

$$\xi = 0.5894$$

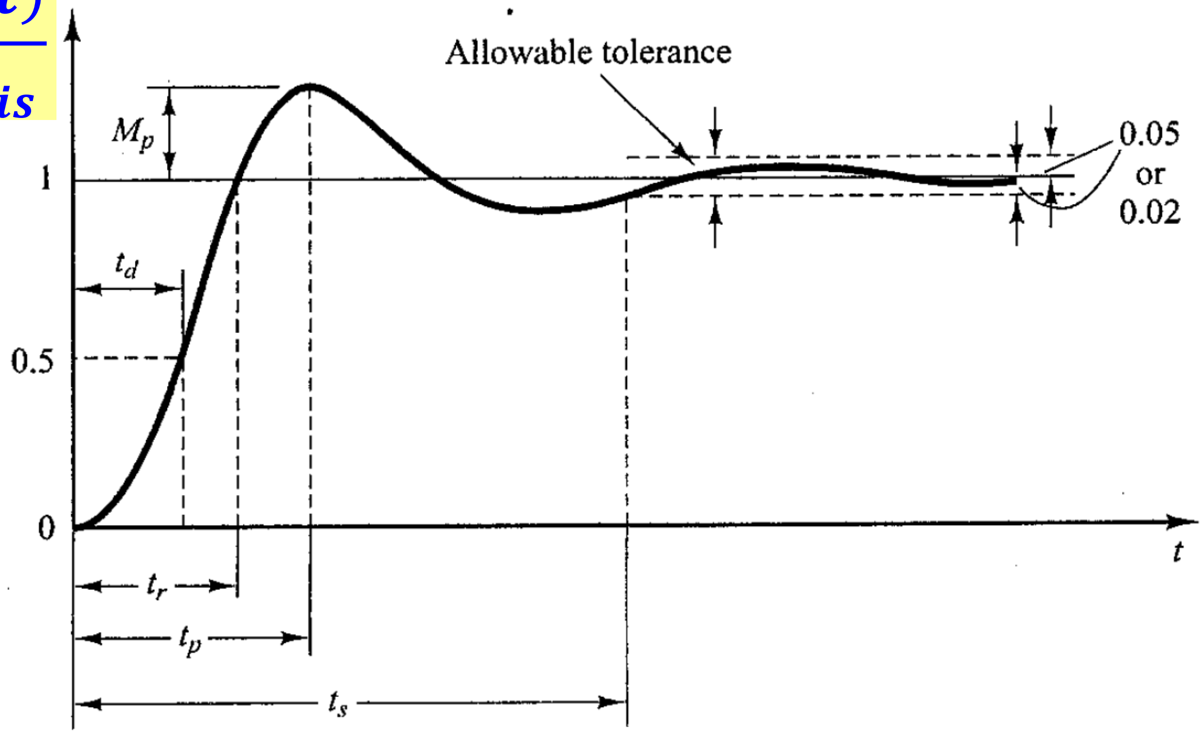
$$\omega_n\sqrt{1-\xi^2} = 4.002$$

$$\xi = 0.5894$$

$$\omega_n\sqrt{1-(0.5894)^2} = 4.002$$

$$\omega_n = 4.954 \text{ rad/s}$$

$$\frac{q_o(t)}{Kq_{is}}$$



Maximum overshoot

$$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

$$t_r = \frac{\pi - Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$M_p = e^{\frac{-0.5894\pi}{\sqrt{1-(0.5894)^2}}}$$

$$M_p = 0.101054$$

The maximum percentage overshoot is 10.1%

Validity of the laminar flow assumption

$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}(\cos\omega_d t)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

Step input response of underdamped second order system

$$\xi = 0 \Rightarrow \frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-0}(\cos\omega_n t\ )$$

$$\omega_d = \omega_n$$

$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - \cos\omega_n t$$

For very low damping (ξ = 0.007), one can estimate the maximum flow velocity by assuming no damping at all. A second order system with no damping executes pure sinusoidal oscillations when subjected to step input function. Thus, its motion would be

X - size of the step function. The velocity (same as average flow velocity)

$$\frac{x}{X} = 1 - \cos\omega_n t$$

$$\dot{x} = X\omega_n \sin\omega_n t$$

$\dot{x} = u_{av} = X\omega_n$ - Maximum average velocity

For very low damping (ξ = 0.007), one can estimate the maximum flow velocity by assuming no damping at all. A second order system with no damping executes pure sinusoidal oscillations when subjected to step input function. Thus, its motion would be

X - size of the step function. The velocity (same as average flow velocity)

$$\dot{x} = X\omega_n sin\omega_n t$$

$$\dot{x} = u_{av} = X\omega_n$$

$$Re = \frac{\rho u_{av} D}{\mu} = \frac{\rho X \omega_n D}{\mu} = \frac{13.6 \times 1000 \times X \times 4.7 \times 6.6 \times 10^{-3}}{0.15 \times 10^{-2}} = 2300$$

$$X = 8.177 \times 10^{-3} m = 8\,mm$$

Thus, to ensure laminar flow at all times during the oscillation, the step input can be no larger than 8 mm (80 Pa).

Validity of the laminar flow assumption

$$\frac{x}{\left(\frac{1}{2\rho g}\right)\Delta p} = 1 - e^{-\xi\,\omega_n t}(cos\omega_d t)$$

For very low damping (ξ = 0.007), one can estimate the maximum flow velocity by assuming no damping at all. A second order system with no damping executes pure sinusoidal oscillations when subjected to step input function. Thus, its motion would be

$$x = X\ Sin\omega_n t$$

X - size of the step function. The velocity (same as average flow velocity)

$$\dot{x} = X\omega_n Cos\omega_n t$$

$$\dot{x} = u_{av} = X\omega_n$$

$$Re = \frac{\rho u_{av} D}{\mu} = \frac{\rho X\omega_n D}{\mu} = \frac{13.6 \times 1000 \times X \times 4.7 \times 6.6 \times 10^{-3}}{0.15 \times 10^{-2}} = 2300$$

$$X = 8.177 \times 10^{-3} m = 8\ mm$$

Thus, to ensure laminar flow at all times during the oscillation, the step input can be no larger than 8 mm (80 Pa).

Flow in manometric limb assumed laminar – could quite as well be turbulent.

Wall shear for turbulent flow

$$f = 0.3164 Re^{-\frac{1}{4}}$$

$$C_f = \frac{f}{4} \Rightarrow \frac{\tau_w}{\frac{1}{2}\rho u_{av}^2} = \frac{0.3164}{4} Re^{-\frac{1}{4}} \Rightarrow \tau_w = \frac{1}{2}\rho u_{av}^2 \left(\frac{0.3164}{4}\right) \frac{\rho^{-\frac{1}{4}} u_{av}^{-\frac{1}{4}} R^{-\frac{1}{4}}}{\mu^{-\frac{1}{4}}}$$

$$\tau_w = 0.03325 \frac{\rho^{\frac{3}{4}} u_{av}^{\frac{7}{4}} \mu^{\frac{1}{4}}}{R^{\frac{1}{4}}}$$

$For\ Hg$ $\mu = 0.15 \times 10^{-2}$; $\rho = 13600$; $L = 0.67$ m; $R = 3.3$ mm

For laminar flow, $\tau_w = \frac{8\mu u_{av}}{D} = \frac{8 \times 0.15 \times 10^{-2} \times u_{av}}{2 \times 3.3 \times 10^{-3}} = 3.64 u_{av}$ $\tau_w = 3.64 u_{av}$

For turbulent flow, $\tau_w = 34.4 u_{av}^{1.75}$

Now get a nonlinear differential equation. Methods for getting linearized versions present.

PRESSURE TRANSDUCER

Transducer – actuated by energy in one from one system, supplies energy in another form to another system.

Passive Transducer

Fluid energy input → Pressure → Elastic element pressure transducer → Displacement → Mechanical energy output

Active Transducer

Fluid energy input → Pressure → Elastic element + Electrical element pressure transducer (Displacement – Voltage) → Voltage → Electrical energy output

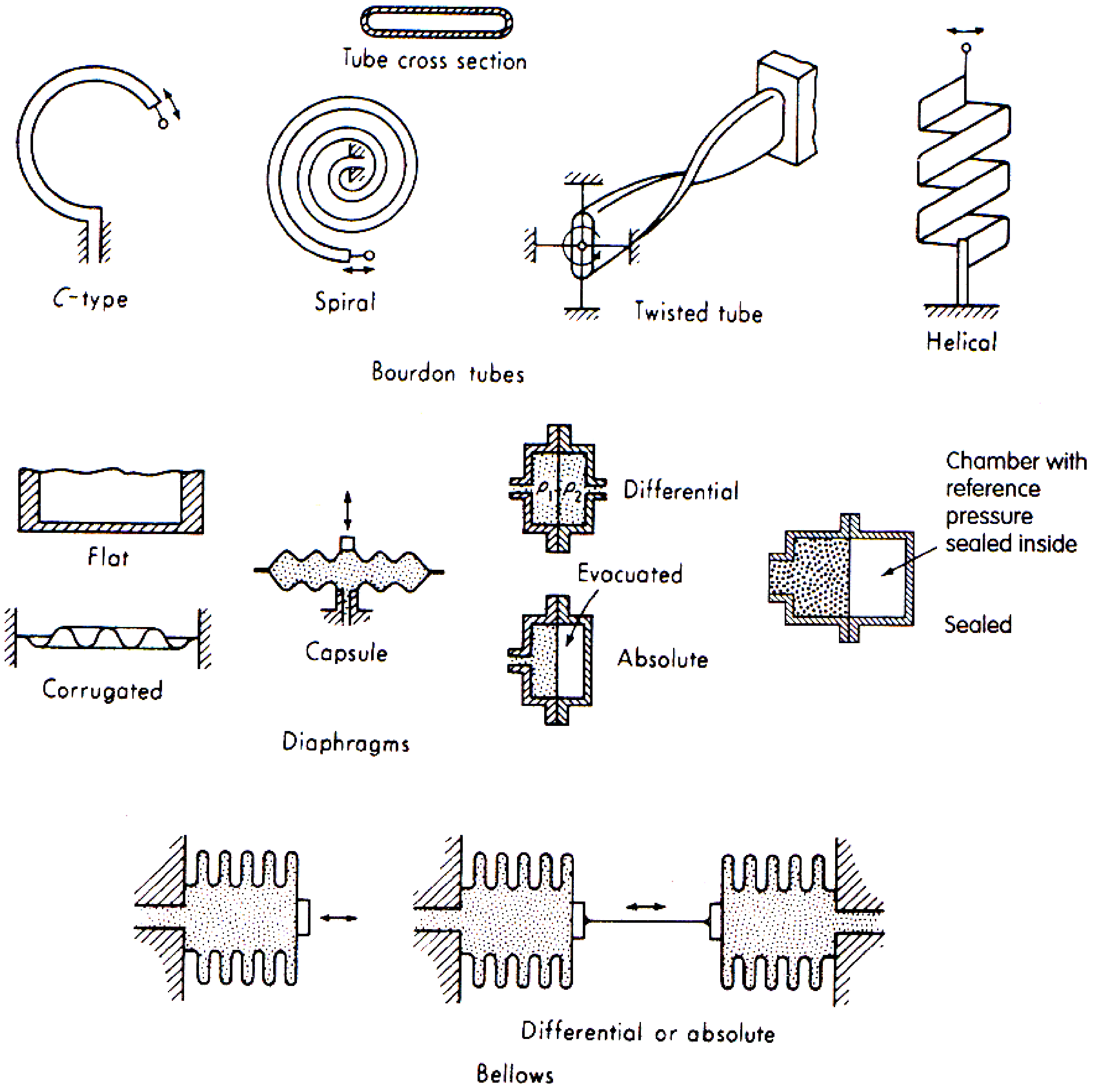
Mechanical Pressure Transducers

- Bourdon Tube
- Bellows and Capsule
- Diaphragms

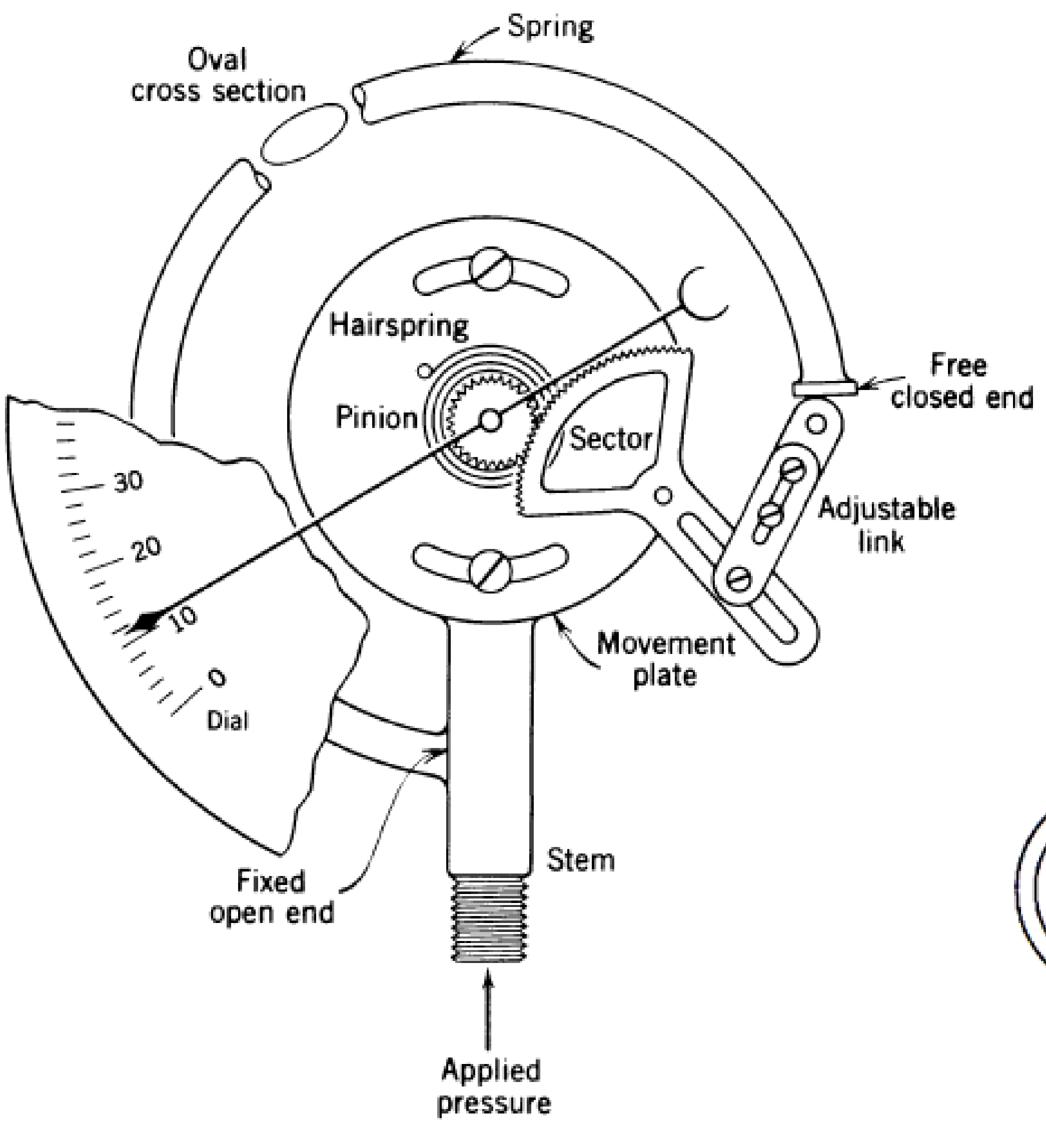
Electrical Pressure Transducers

- Potentiometer Type (Resistive Type)
- Strain Gage Type (Resistive Type)
- LVDT (Inductance Type)
- Capacitance Type

PRIMARY SENSING ELEMENT



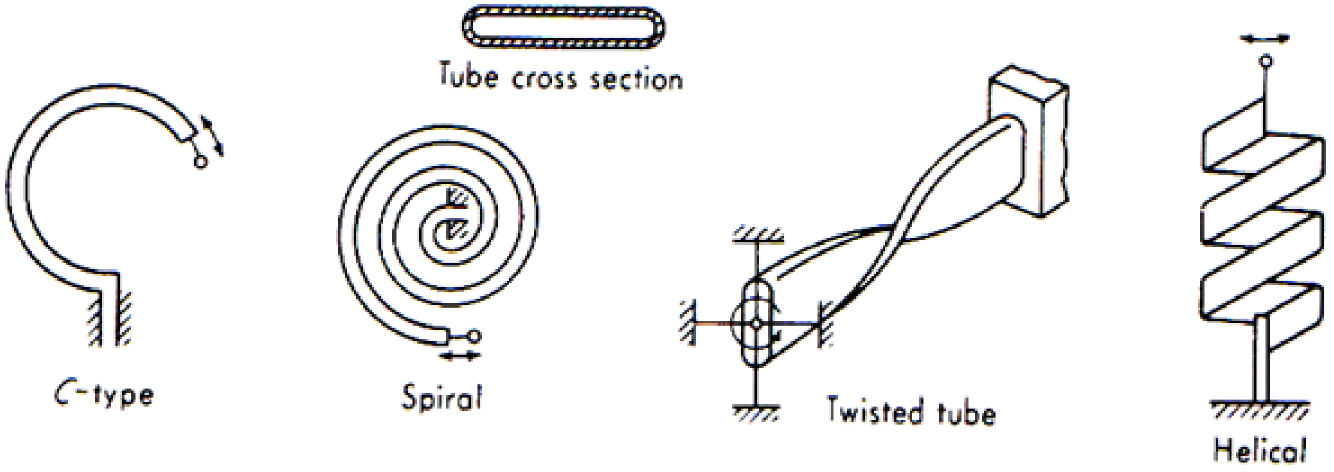
BOURDON TUBE



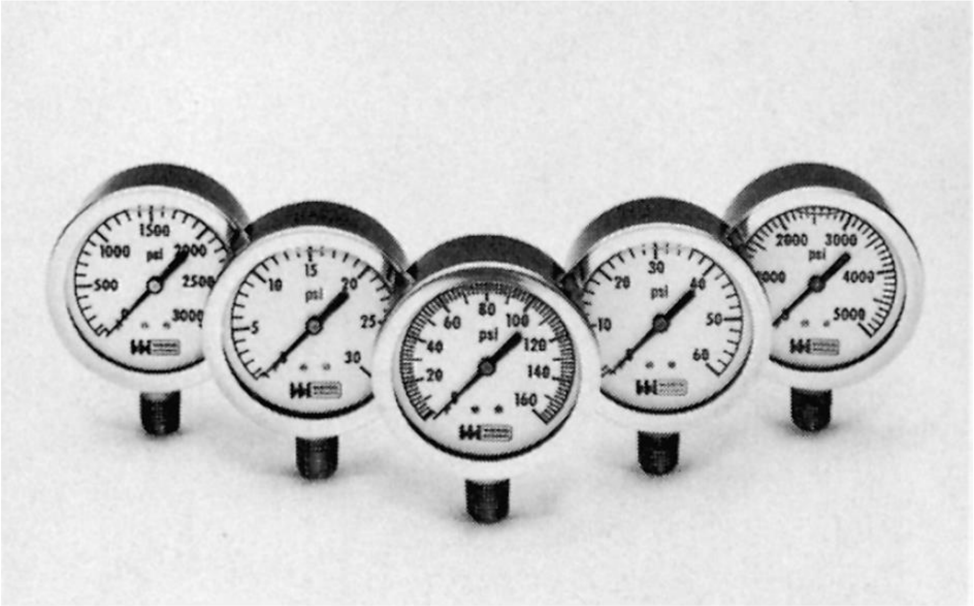
The best Bourdon tube gauges have instrument uncertainties as low as 0.1% of the full-scale deflection of the gauge, with values of 0.5% to 2% more common.

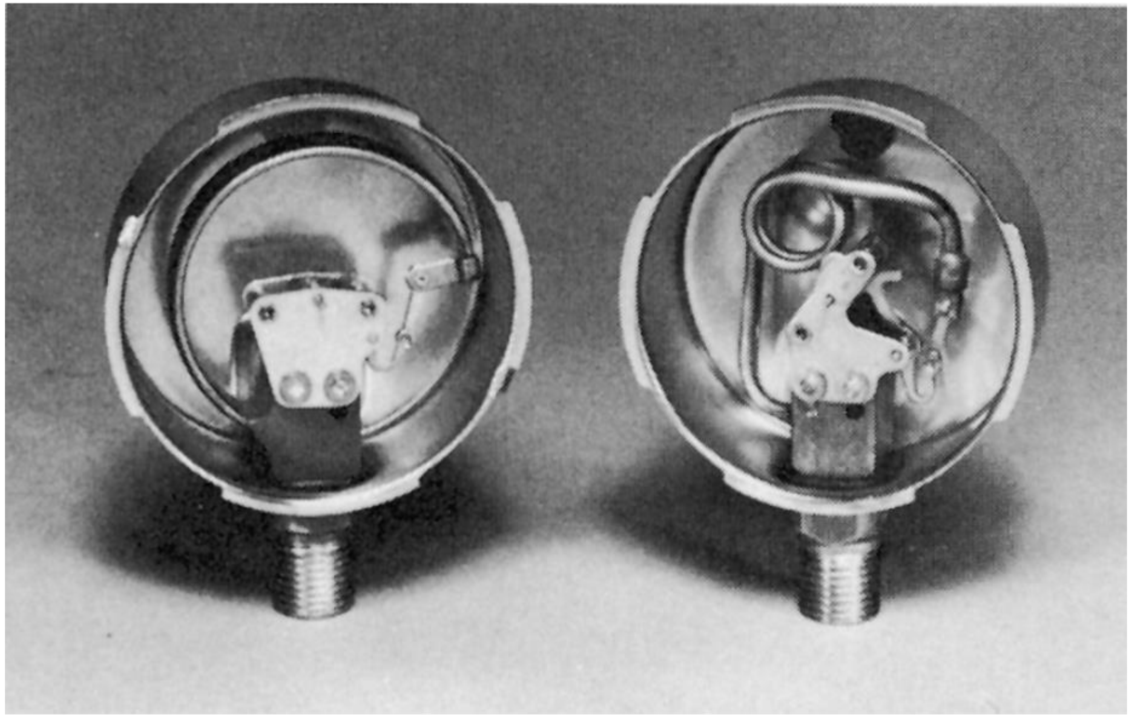
Attractiveness of this device is

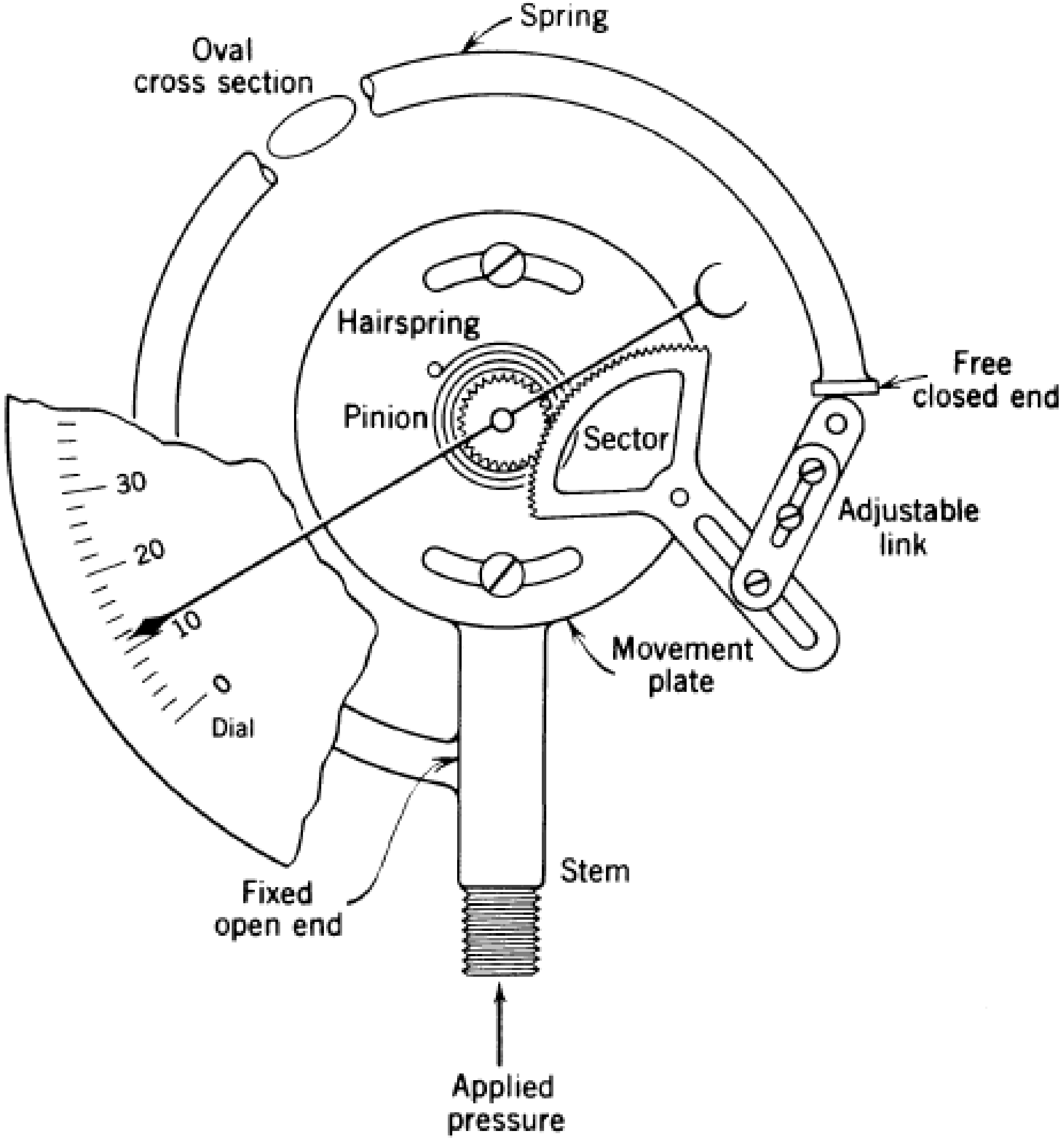
- Simple
- Portable
- Robust, lasting for years of use.



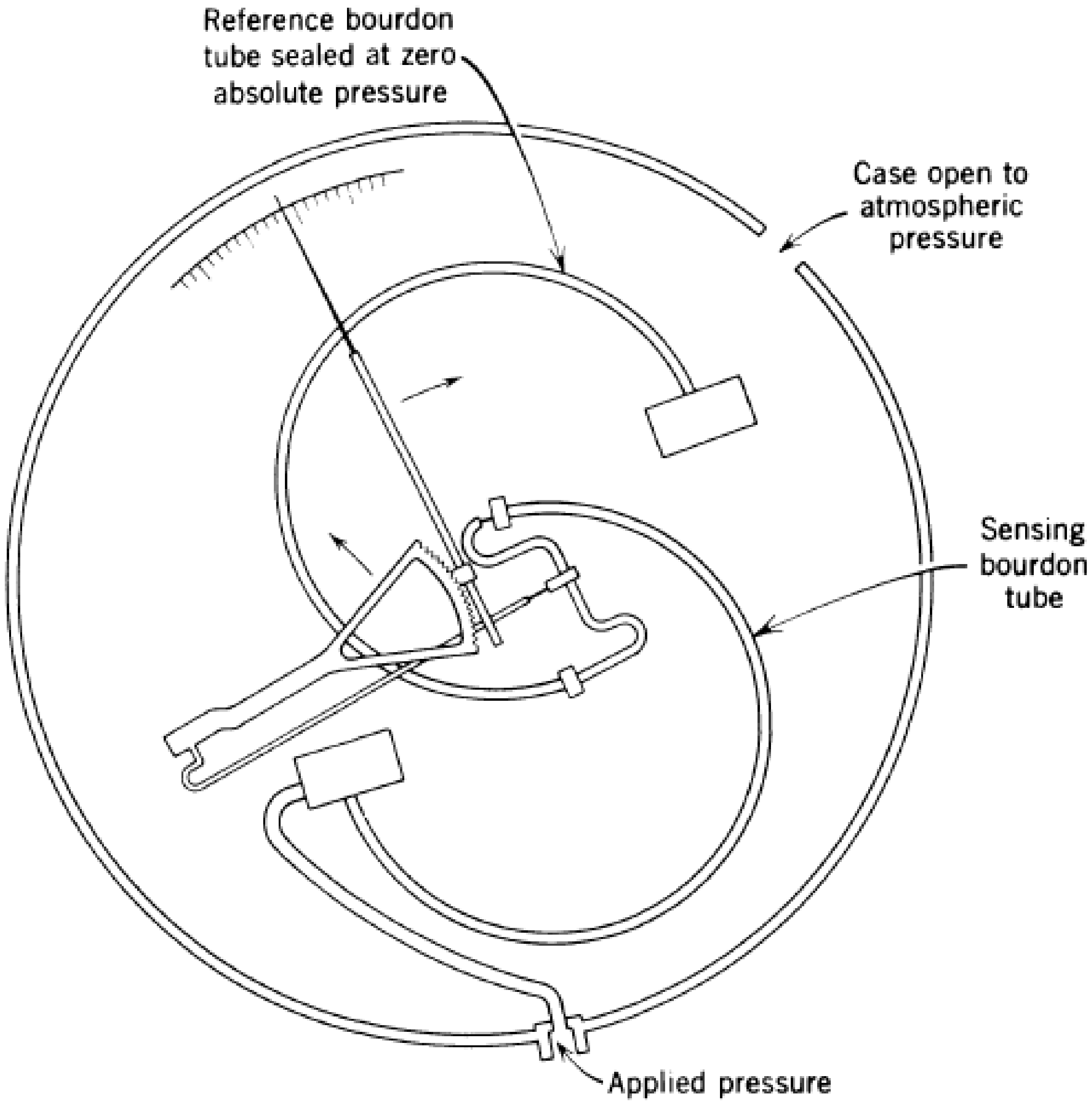
Bourdon tubes



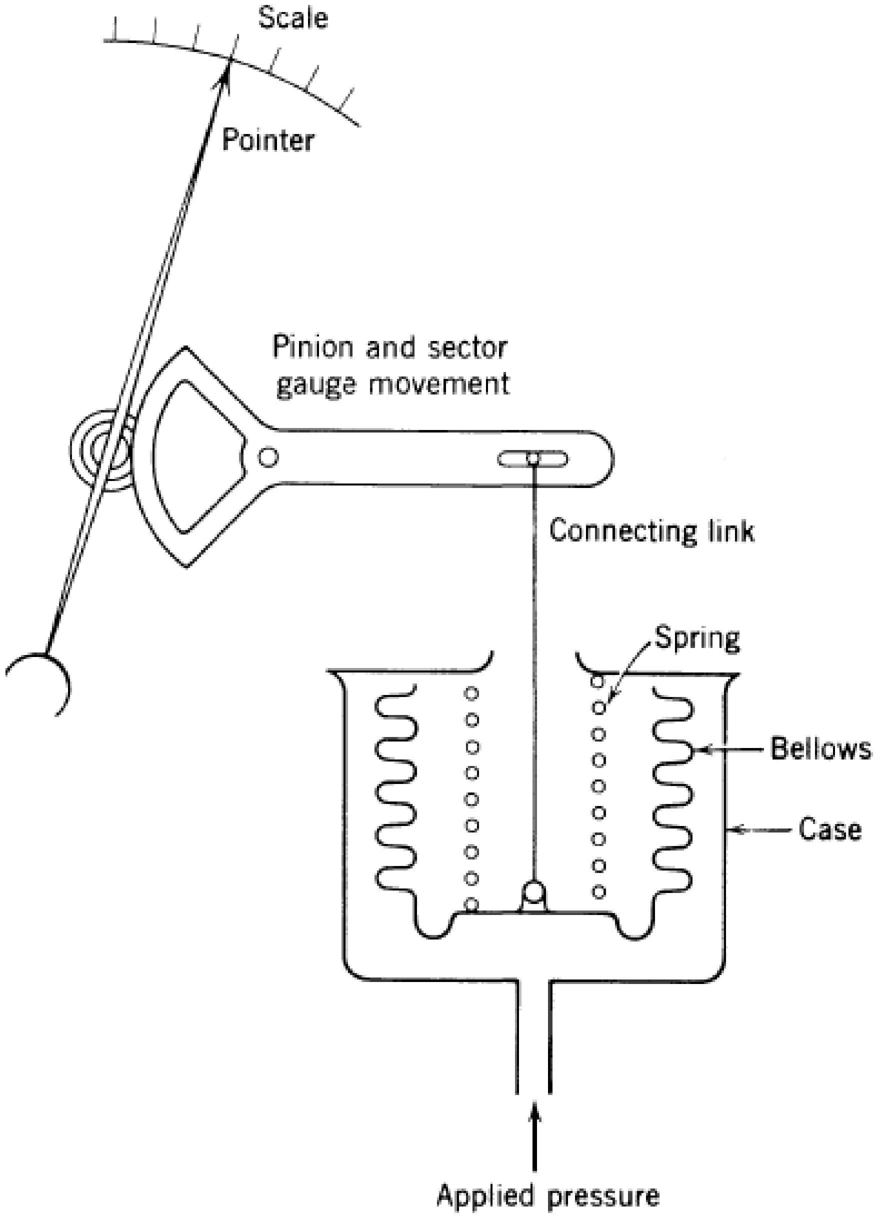



Spring
Oval
cross section
Hairspring
Pinion
Sector
Free
closed end
Adjustable
link
30
20
10
0
Dial
Movement
plate
Stem
Fixed
open end
Applied
pressure

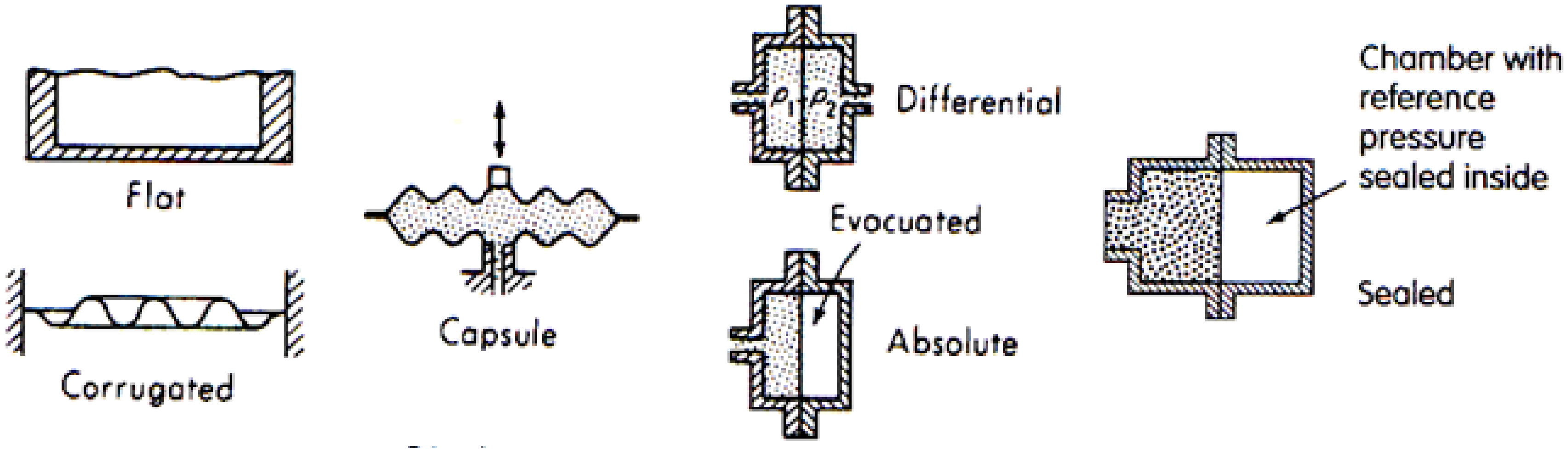
BOURDON TUBES ARRANGED FOR ABSOLUTE PRESSURE MEASUREMENT



BOURDON TUBES WITH COMMON BELLOW GAGE



DIAPHRAGMS



Diaphragm - thin elastic circular plate supported about its circumference.

A pressure differential on the top and bottom diaphragm faces acts to deform it. The magnitude of the deformation is proportional to the pressure difference.

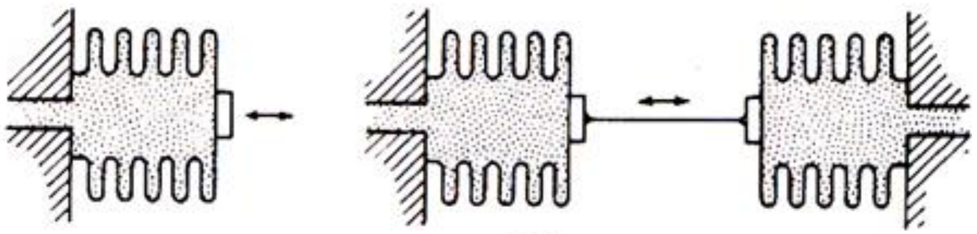
Membranes are made of metal or nonmetallic material, such as plastic or neoprene.

The material chosen depends on the pressure range anticipated and the fluid in contact with it.

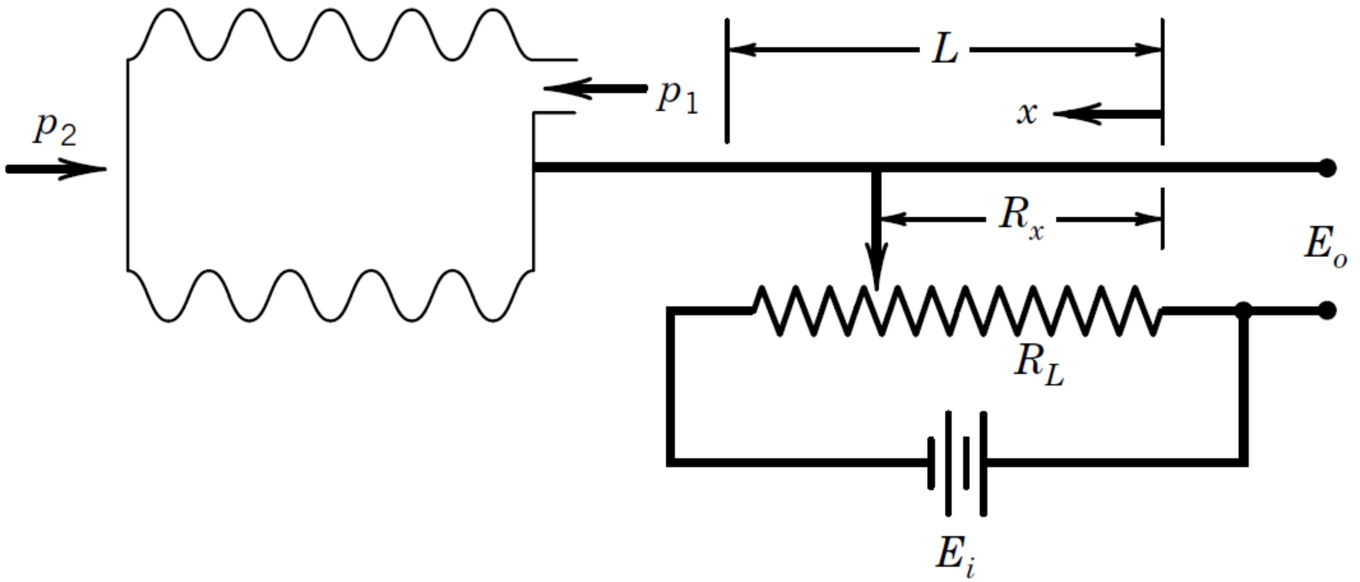
Corrugated diaphragms contain a number of corrugations that serve to increase diaphragm stiffness and to increase the diaphragm effective surface area.

An advantage of the diaphragm sensor is that the very low mass and relative stiffness of the thin diaphragm give the sensor a very high natural frequency with a small damping ratio.

BELLOWS AND CAPSULE ELEMENTS



Differential or absolute

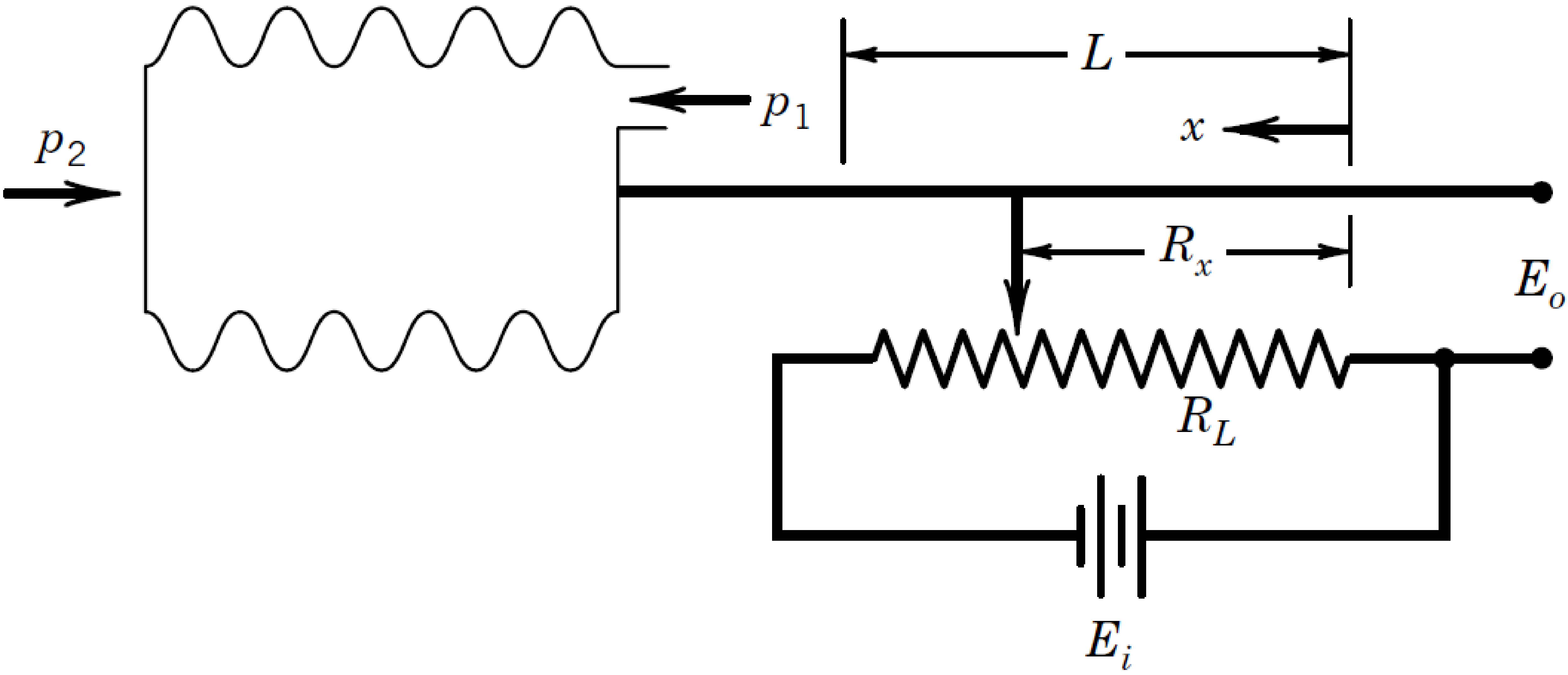


Bellow - thin-walled, flexible metal tube formed into deep convolutions and sealed at one end. One end is held fixed and pressure is applied internally. A difference between the internal and external pressures causes the bellows to change in length.

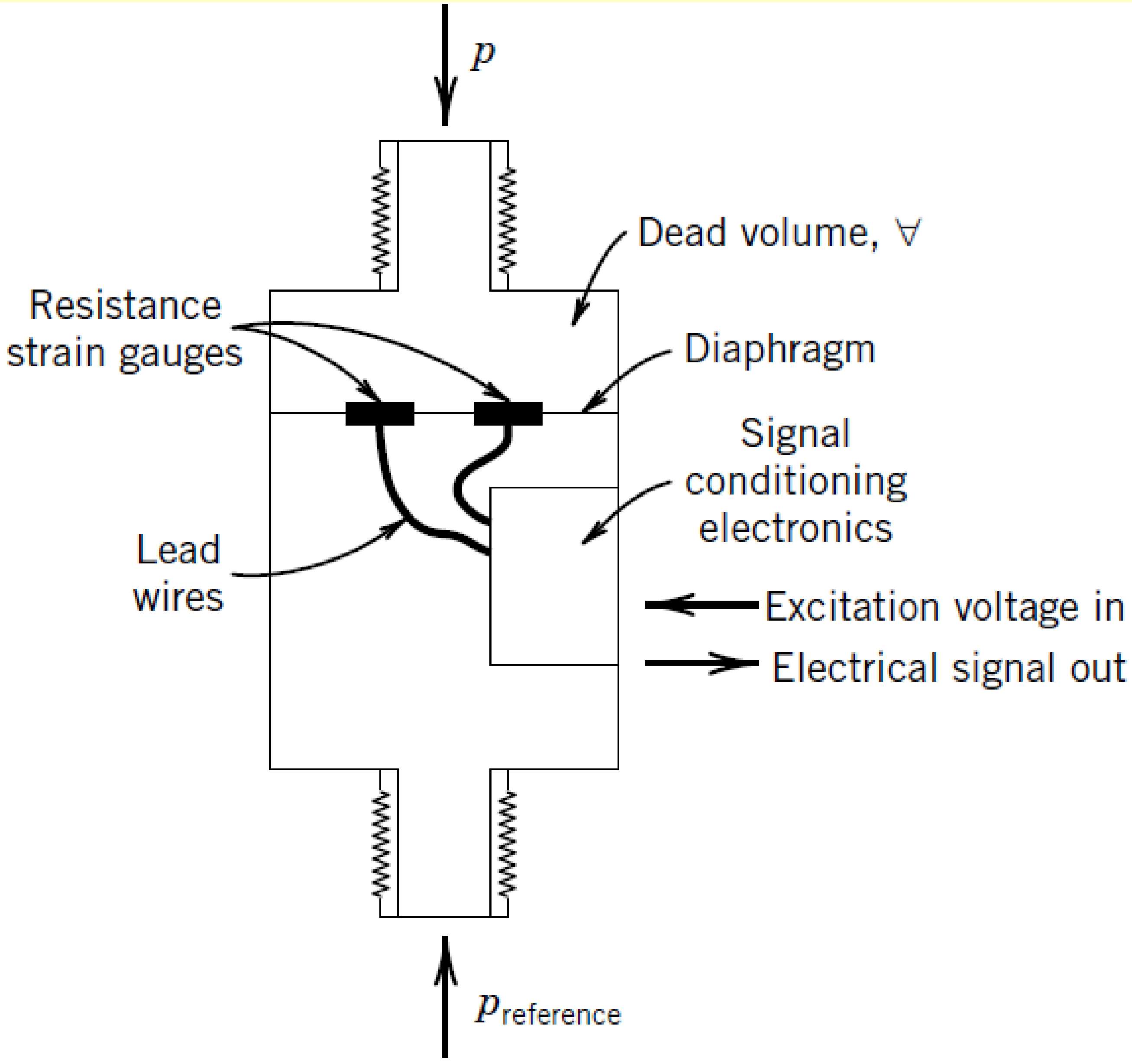
The bellows is housed within a chamber that can be sealed and evacuated for absolute measurements, vented through a reference pressure port for differential measurements, or opened to atmosphere for gauge pressure measurements.

Capsule - thin-walled, flexible metal tube whose length changes with pressure, but its shape tends to be wider in diameter and shorter in length

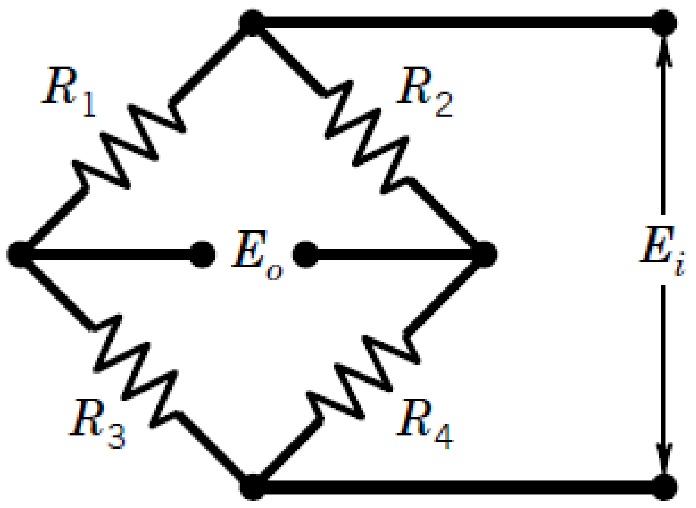
Resistive type - Potentiometer type pressure transducer with bellow as sensing element



Resistive type - Strain gage type with diaphragm as the sensing element

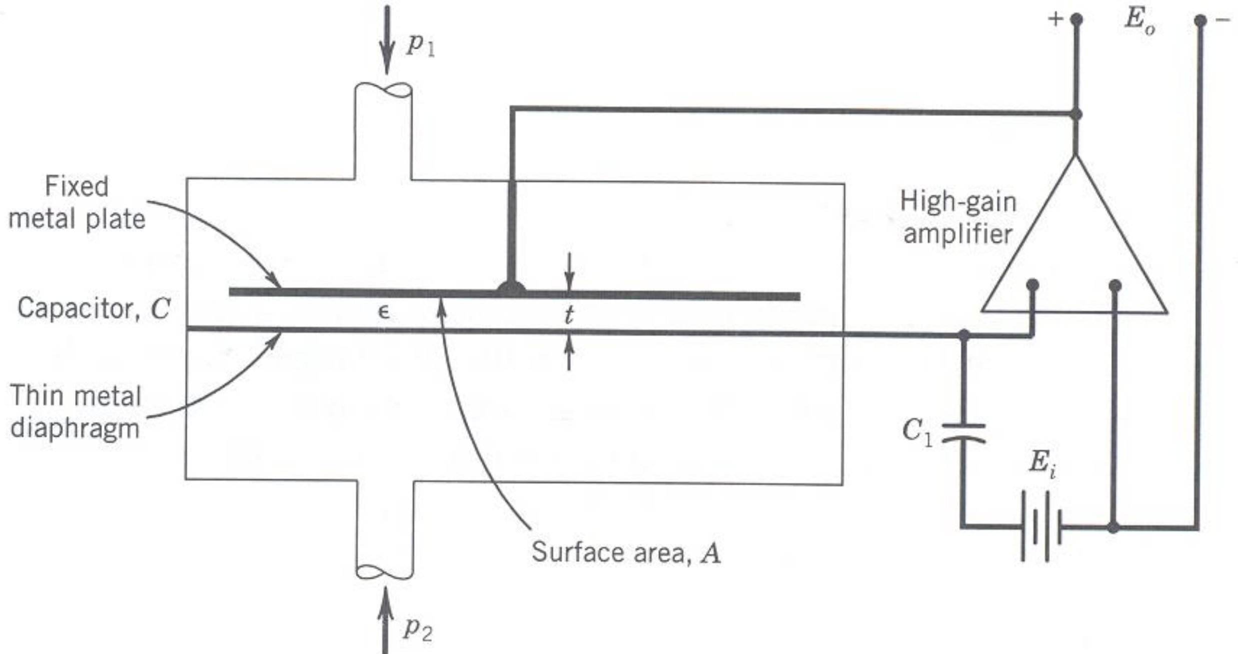


(a) Sensing scheme



(b) Bridge–strain gauge circuit for pressure diaphragms.

Capacitance type pressure transducer



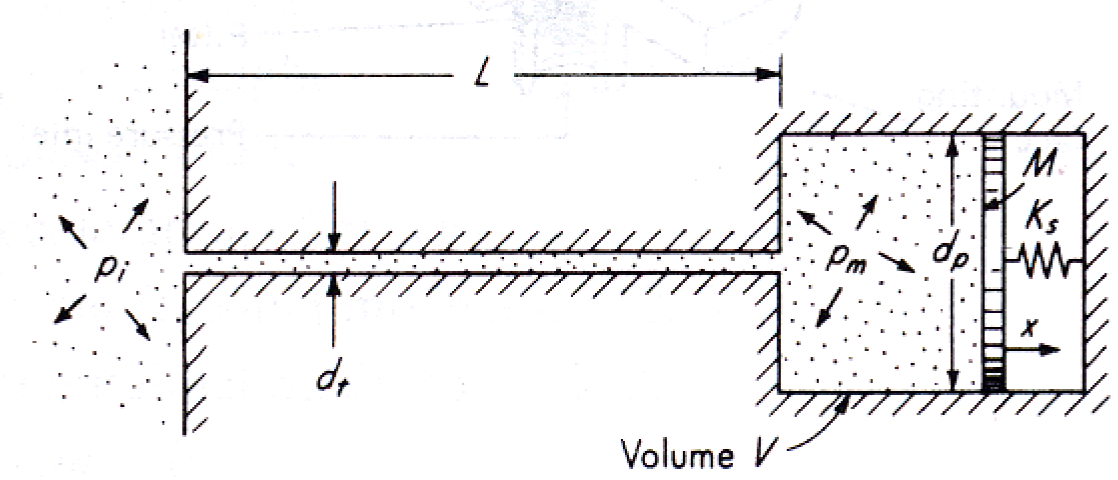
When pressure changes, so as to deflect the diaphragm, the gap between the plates changes, which causes a change in capacitance.

The capacitance C developed between two parallel plates separated by average gap t is determined by

$$C = \frac{c\varepsilon A}{t}$$

where the product $c\varepsilon$ is the permittivity of the material between the plates relative to a vacuum (ε = 8.85 × 10^{-12} F/m), c is dielectric constant and A is the overlapping area of the two plates. The dielectric constant depends on the material in the gap, which for air is c =1 but for water is c = 80.

LIQUID SYSTEMS, HEAVILY DAMPED, SLOW ACTING - TRANSDUCER TUBING MODEL



Inertia force - Neglected
Damping force - Viscous force Stiffness
Forcing function (input)

FIRST ORDER SYSTEM

- spring loaded piston represents the flexible element of the pressure pickup
- P_i - Input pressure from source at connecting tube inlet
- P_m - Pressure measured by transducer
- Assumption – Inertia effects of both the fluid and the moving parts of the pickup are negligible compared with viscous and spring forces

For steady laminar flow within the connecting tube

$$P_i - P_m = \frac{32\mu L u_{av}}{d^2}$$

$$u_{av} = \frac{(P_i - P_m)d^2}{32\mu L}$$

While this equation is exact only for steady flow, it holds quite closely for slowly varying velocities

Volume entering measuring chamber in time dt is

$$dV = \frac{\pi d^2}{4} u_{av} dt = \frac{\pi d^2}{4} \frac{(P_i - P_m) d^2}{32\mu L} dt = \frac{\pi d^4 (P_i - P_m)}{128\mu L} dt$$

Volume added or taken away represents change in pressure of system

$$dP_m = \frac{dV}{C_{vp}}$$

C_{vp} = volume change per unit pressure change for the pickup (m^3/Pa)

Compliance C_{vp}: is a measure of the flexibility in a structure, component, or substance, and so it is the inverse of the system stiffness. It is a direct analog to electrical capacitance

$$dP_m = \frac{dV}{C_{vp}} = \frac{\pi d^4 (P_i - P_m)}{128\mu L} \frac{1}{C_{vp}} dt$$

$$\frac{dP_m}{dt} = \frac{\pi d^4 (P_i - P_m)}{128\mu L} \frac{1}{C_{vp}}$$

$$\frac{128\mu L C_{vp}}{\pi d^4} \frac{dP_m}{dt} = P_i - P_m$$

$$\frac{128\mu L C_{vp}}{\pi d^4} \frac{dP_m}{dt} + P_m = P_i$$

$$\frac{128\mu L C_{vp}}{\pi d^4}\frac{dP_m}{dt} + P_m = P_i$$

$$P_i(s) \longrightarrow \boxed{\frac{K}{\tau s + 1}} \longrightarrow P_m(s)$$

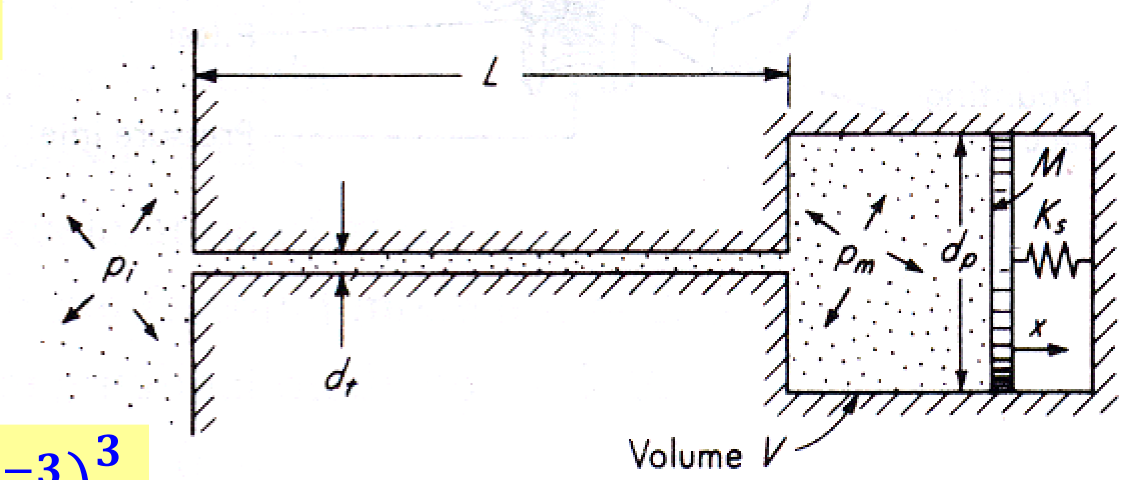
$$\tau_w = \frac{128\mu L C_{vp}}{\pi d^4}$$

τ_w - time constant

To keep time constant small, keep *L* small and *d* large

If the changes are rapid connecting tube dynamics become more important; tube diameter is much smaller than piston diameter

Consider a pressure transducer connected to a pipe through a connecting tube of diameter 3 mm and a length of 50 mm. Water is flowing in the pipe. There are pressure transients which occur every 5 seconds. The compliance of the pressure transducer is 0.001 mm^3/bar. Does this pressure transducer along with the connecting tube modelled as a first order system capture these pressure transients.



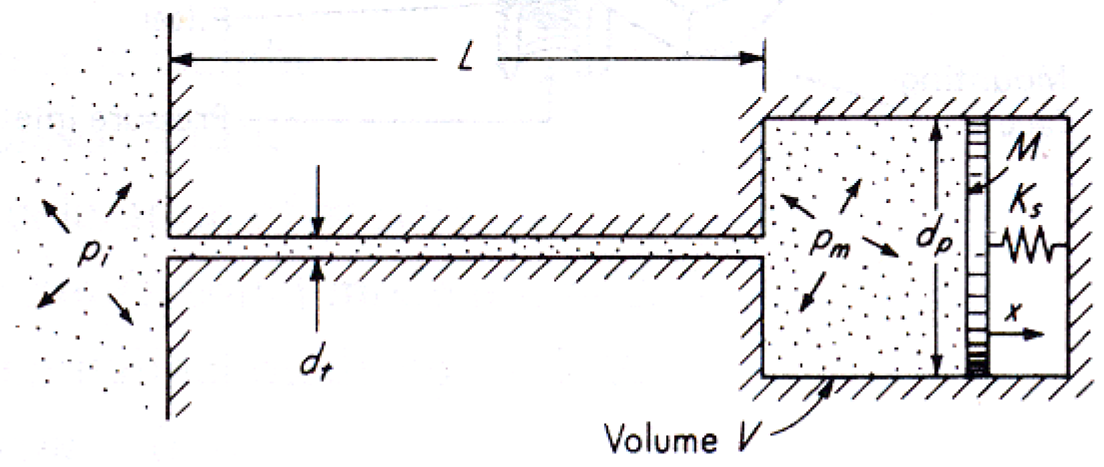
$$\tau_w = \frac{128\mu L C_{vp}}{\pi d^4}$$

$$\tau_w = \frac{128 \times 0.001 \times 50 \times 10^{-3} \times 0.001 \times \frac{\left(10^{-3}\right)^3}{10^{-5}}}{\pi(3 \times 10^{-3})^4}$$

$$\tau_w = 2.52\ seconds$$

LIQUID SYSTEMS, MODERATELY DAMPED, FAST ACTING

- Main assumption in the previous analysis - Inertia effects of both the fluid and moving parts of the pickup are negligible compared with viscous and spring forces
- Motions of the liquid and the pickup element are rapid, their inertia is no longer negligible
- Any change in the pressure p_m results in volume change - causes inflow or outflow of liquid through the tube
- $d_t << d_p$ $\therefore$ tube flow velocity >>> piston velocity $\Rightarrow$ K.E of liquid in tube - major part of total system K.E
- This increase in K.E is equivalent to adding mass to the piston and ignoring fluid inertia
- This added masses lowers the system natural frequency and thereby degrades dynamic response



Transducer volume change = Equivalent piston volume change

Transducer volume change = pC_{vp}

Equivalent piston volume change = Piston Area × Displacement of member

$$= \left(\frac{\pi}{4}d_p^2\right)x = \left(\frac{\pi}{4}d_p^2\right)\frac{pA}{K_s} = \left(\frac{\pi}{4}d_p^2\right)\frac{p\left(\frac{\pi}{4}d_p^2\right)}{K_s} = \frac{p}{K_s}\left(\frac{\pi}{4}d_p^2\right)^2$$

Transducer volume change = Equivalent piston volume change

$$pC_{vp} = \frac{p}{K_s}\left(\frac{\pi}{4}d_p^2\right)^2$$

$$\boxed{K_s = \frac{\pi^2 d_p^4}{16C_{vp}}}$$

- Natural frequency of both the systems ($i.e.$, transducer and equivalent piston volume system with no fluid present) must be equal

$$\omega_{n,t} = \sqrt{\frac{K_s}{M}}$$

- $\omega_{n,t}$ - Transducer natural frequency of the transducer alone without the tubing considered

If C_{vp} is known, K_s can be found out.

By knowing the transducer natural frequency, mass M can be found out

This is the mass of the piston if the connecting tube dynamics are not considered.

Analysis of equivalent system

Volume change dV is related to the piston motion dx by

$$dV = \frac{\pi}{4} d_p^2\, dx;$$

$$\frac{dV}{dt} = \frac{\pi}{4} d_p^2 \frac{dx}{dt} = \frac{\pi}{4} d_t^2\, V_{t,av};\; V_{t,av} = \left(\frac{d_p^2}{d_t^2}\right)\frac{dx}{dt}$$

$V_{t,av}$ – Average flow velocity in the tube

V_{tc} – Centre line velocity in the tube

$$V_{t,av} = \left(\frac{d_p^2}{d_t^2}\right)\frac{dx}{dt}$$

Assuming steady laminar flow – parabolic velocity profile is given by

$$V_t = V_{t,c}\left(1 - \frac{r^2}{R^2}\right) \;\&\; V_{t,c} = 2V_{t,av} \Rightarrow V_t = 2V_{t,av}\left(1 - \frac{r^2}{R^2}\right)$$

$$K.E. = \int \frac{1}{2}\rho V_t^2 (2\pi r dr)L = \int \rho 4V_{t,av}^2 \left(1 - \frac{r^2}{R^2}\right)^2 (\pi r dr)L = 4\pi L\rho V_{t,av}^2 \int \left(1 - 2\frac{r^2}{R^2} + \frac{r^4}{R^4}\right) r dr$$

$$K.E. = 4\pi L\rho V_{t,av}^2 \int \left(r - 2\frac{r^3}{R^2} + \frac{r^5}{R^4}\right) dr = 4\pi L\rho V_{t,av}^2 \left(\frac{R^2}{2} - 2\frac{R^4}{4R^2} + \frac{R^6}{6R^4}\right) = 4\pi L\rho V_{t,av}^2 \frac{R^2}{6}$$

$$K.E. = 4\pi L\rho V_{t,av}^2 \frac{R^2}{6} = 4\pi L\rho V_{t,av}^2 \frac{d_t^2}{6\times 4} = \frac{\pi}{6} L\rho d_t^2 V_{t,av}^2$$

$$K.E. = \frac{\pi}{6} L\rho d_t^2 V_{t,av}^2$$

The rigid mass M_e attached to M, which would have the same kinetic energy as the fluid

$$K.E. = \frac{1}{2} M_e \left(\frac{dx}{dt}\right)^2 = \frac{\pi}{6} L\rho d_t^2 V_{t,av}^2 = \frac{\pi}{6} L\rho d_t^2 \left(\frac{d_p^4}{d_t^4}\right)\left(\frac{dx}{dt}\right)^2$$

$$K.E. = \frac{\pi}{6} L\rho d_t^2 V_{t,av}^2$$

$$V_{t,av} = \left(\frac{d_p^2}{d_t^2}\right)\frac{dx}{dt}$$

$$M_e = \frac{\pi L\rho d_p^4}{3d_t^2}$$

Natural Frequency of the transducer along with tubing system (ω_n)

$$K_s = \frac{\pi^2 d_p^4}{16 C_{vp}}$$

$$\omega_n = \sqrt{\frac{K_s}{M + M_e}}$$

$$\omega_{n,t} = \sqrt{\frac{K_s}{M}}$$

$\omega_{n,t}$ - Transducer natural frequency of the transducer alone without the tubing considered

$$\omega_n = \sqrt{\frac{1}{\frac{M}{K_s} + \frac{M_e}{K_s}}}$$

$$\omega_n = \sqrt{\frac{1}{\frac{1}{\omega_{n,t}^2} + \frac{\frac{\pi L\rho d_p^4}{3d_t^2}}{\frac{\pi^2 d_p^4}{16C_{vp}}}}}$$

$$\omega_n = \sqrt{\frac{1}{\frac{1}{\omega_{n,t}^2} + \frac{16\rho L C_{vp}}{3\pi d_t^2}}}$$

To keep ω_n large, keep L as small as possible and d_t as large as possible

In most of the practical cases, $M_e >>> M$

$$\omega_n = \sqrt{\frac{3\pi d_t^2}{16\rho L C_{vp}}}$$

Damping ratio of transducer and tubing system

- Assume transducer itself has no damping – only damping due to fluid friction in the tube
- Assume the validity of the steady laminar flow relations to calculate the pressure drop due to the fluid viscosity

$$\Delta P = \frac{32\mu L V_{t,av}}{d_t^2}$$

FORCE ON THE PISTON DUE TO THIS PRESSURE DROP = DAMPING FORCE

$$\Delta P A = \frac{32\mu L V_{t,av}}{d_t^2}\frac{\pi}{4}d_p^2 = B\frac{dx}{dt}$$

$$V_{t,av} = \left(\frac{d_p^2}{d_t^2}\right)\frac{dx}{dt}$$

$$\frac{32\mu L}{d_t^2}\left(\frac{d_p^2}{d_t^2}\right)\frac{dx}{dt}\frac{\pi}{4}d_p^2 = B\frac{dx}{dt}$$

$$B = 8\pi\mu L\frac{d_p^4}{d_t^4}$$

$$(M + M_e)\ddot{x} + B\dot{x} + K_s x = A\Delta P$$

$$M_e = \frac{\pi L \rho d_p^4}{3 d_t^2}$$

$$B = 8\pi\mu L \frac{d_p^4}{d_t^4}$$

$$K_s = \frac{\pi^2 d_p^4}{16 C_{vp}}$$

$$\omega_n = \sqrt{\frac{3\pi d_t^2}{16\rho L C_{vp}}}$$

$$\xi = \frac{B}{2\sqrt{(M + M_e)K_s}} = \frac{B}{2K_s\sqrt{\frac{(M + M_e)}{K_s^2}K_s}} = \frac{B}{2K_s\sqrt{\frac{(M + M_e)}{K_s}}}$$

$$\xi = \frac{8\pi\mu L \frac{d_p^4}{d_t^4}}{2\left(\frac{\pi^2 d_p^4}{16 C_{vp}}\right)\sqrt{\frac{1}{\omega_{n,t}^2} + \left(\frac{16\rho L C_{vp}}{3\pi d_t^2}\right)}}$$

$$\xi = \frac{64\mu L C_{vp}}{\pi d_t^4\sqrt{\frac{1}{\omega_{n,t}^2} + \left(\frac{16\rho L C_{vp}}{3\pi d_t^2}\right)}}$$

In most of the practical cases, $M_e >>> M$

$$\xi = \frac{64\mu L C_{vp}}{\pi d_t^4\sqrt{\left(\frac{16\rho L C_{vp}}{3\pi d_t^2}\right)}}$$

$$\xi = \frac{16\mu}{d_t^3}\sqrt{\frac{3 L C_{vp}}{\pi\rho}}$$

For gases, we simplify by assuming that the system is rigid relative to the compressibility of the gas.

Compliance is defined as

$$C_{vp} = \frac{V}{E_m}$$

V – Volume of the transducer
E_m – Bulk Modulus of the gas

$$K_s = \frac{\pi^2 d_p^4}{16 C_{vp}} = \frac{\pi^2 d_p^4 E_m}{16V}$$

$$\omega_n = \sqrt{\frac{3\pi d_t^2 E_m}{16\rho L V}}$$

$$\xi = \frac{16\mu}{d_t^3}\sqrt{\frac{3LC_{vp}}{\pi\rho}} = \frac{16\mu}{d_t^3}\sqrt{\frac{3LV}{\pi\rho E_m}}$$

Velocity of sound in gas

$$c = \sqrt{\frac{E_m}{\rho}} = \sqrt{\gamma RT}$$

$$\xi = \frac{16\mu}{d_t^3}\sqrt{\frac{3LV}{\pi\rho E_m}} = \frac{16\mu}{d_t^3}\sqrt{\frac{3LV}{\pi\rho\rho^2 c^2}}$$

$$\omega_n = \sqrt{\frac{3\pi d_t^2 c^2}{16\rho L V}}$$

$$K_s = \frac{\pi^2 d_p^4 c^2 \rho}{16V}$$

$$\xi = \frac{16\mu}{d_t^3}\sqrt{\frac{3LV}{\pi\rho\rho^2 c^2}}$$