CHAPTER 5

Matrix Norms

Solutions to certain linear systems are sensitive to slight variations in their right hand side vectors. The matrices that define such linear systems may be identified using their condition numbers whose definition involves matrix norms. In Section 5.1, we introduce the notion of matrix norms, define subordinate matrix norms, and study some important properties of subordinate matrix norms. In Section 5.2, we define condition number of a matrix.

5.1 Subordinate Matrix Norms

Given any two real numbers α and β, the distance between them is given by $|\alpha - \beta|$. We can also compare their magnitudes $|\alpha|$ and $|\beta|$. Vector norm may be thought of as a generalization of the modulus $|\cdot|$ concept to deal with vectors and in general matrices.

Definition 5.1.1 [Vector Norm].

A ***vector norm*** on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ having the following properties:

1. $\|\boldsymbol{x}\| \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
2. $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.
3. $\|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{R}$.
4. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

The condition 4 in the above definition is called the ***triangle inequality***.

Note
We use the following notation for a vector $\boldsymbol{x} \in \mathbb{R}^n$ in terms of its components:

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T.$$

Example 5.1.2.

There can be many vector norms on $\mathbb{R}^n$. We define three important vector norms on $\mathbb{R}^n$, which are frequently used in matrix analysis.

1. The ***Euclidean norm*** (also called the l_2***-norm***) on $\mathbb{R}^n$ is denoted by $\|\cdot\|_2$, and is defined by

$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}. \tag{5.1}$$

2. The l_∞***-norm***, also called ***maximum norm***, on $\mathbb{R}^n$ is denoted by $\|\cdot\|_\infty$, and is defined by

$$\|\boldsymbol{x}\|_\infty = \max\{\, |x_1|, |x_2|, \cdots, |x_n| \,\}. \tag{5.2}$$

3. l_1***-norm*** on $\mathbb{R}^n$ is denoted by $\|\cdot\|_1$, and is defined by

$$\|\boldsymbol{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|. \tag{5.3}$$

All the three norms defined above are indeed norms; it is easy to verify that they satisfy the defining conditions of a norm given in Definition 5.1.1.

Let us illustrate each of the above defined norms with some specific vectors.

Example 5.1.3.

Let us compute norms of some vectors now. Let $\boldsymbol{x} = (4, 4, -4, 4)^T$, $\boldsymbol{y} = (0, 5, 5, 5)^T$, $\boldsymbol{z} = (6, 0, 0, 0)^T$. Verify that $\|\boldsymbol{x}\|_1 = 16$, $\|\boldsymbol{y}\|_1 = 15$, $\|\boldsymbol{z}\|_1 = 6$; $\|\boldsymbol{x}\|_2 = 8$, $\|\boldsymbol{y}\|_2 = 8.66$, $\|\boldsymbol{z}\|_2 = 6$; $\|\boldsymbol{x}\|_\infty = 4$, $\|\boldsymbol{y}\|_\infty = 5$, $\|\boldsymbol{z}\|_\infty = 6$.

From this example we see that asking which vector is big does not make sense. But once the norm is fixed, this question makes sense as the answer depends on the norm used. In this example each vector is big compared to other two but in different norms.

Remark 5.1.4.

In our computations, we employ any one of the norms depending on convenience. It is a fact that "all vector norms on $\mathbb{R}^n$ are equivalent"; we will not elaborate further on this.

5.1.1 Subordinate Norms

We can also define matrix norms on the vector space $M_n(\mathbb{R})$ of all $n \times n$ real matrices, which helps us in finding distance between two matrices.

Definition 5.1.5 [Matrix Norm].

A ***matrix norm*** on the vector space of all $n \times n$ real matrices $M_n(\mathbb{R})$ is a function $\|\cdot\| : M_n(\mathbb{R}) \to [0, \infty)$ having the following properties:

1. $\|A\| \geq 0$ for all $A \in M_n(\mathbb{R})$.
2. $\|A\| = 0$ if and only if $A = 0$.
3. $\|\alpha A\| = |\alpha|\|A\|$ for all $A \in M_n(\mathbb{R})$ and for all $\alpha \in \mathbb{R}$.
4. $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in M_n(\mathbb{R})$.

Note

As in the case of vector norms, the condition 4 in the above definition is called the *triangle inequality*.

For a matrix $A \in M_n(\mathbb{R})$, we use the notation

$$A = (a_{ij})_{1\leq i\leq n,\ 1\leq j\leq n}$$

where a_{ij} denotes the element in the i^{th} row and j^{th} column of A.

There can be many matrix norms on $M_n(\mathbb{R})$. We will describe some of them now.

Example 5.1.6.

The following define norms on $M_n(\mathbb{R})$.

1. $\|A\| = \sqrt{\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2}$.
2. $\|A\| = \max\{\, |a_{ij}| \;:\; 1 \leq i \leq n,\ 1 \leq j \leq n \,\}$.
3. $\|A\| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|$.

All the three norms defined above are indeed norms; it is easy to verify that they satisfy the defining conditions of a matrix norm of Definition 5.1.5.

Among matrix norms, there are special ones that satisfy very useful and important properties. They are called *matrix norms subordinate to a vector norm*. As the name suggests, to define them we need to fix a vector norm. We will give a precise definition now.

Definition 5.1.7 [Matrix Norm subordinate to a vector norm].

Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$ and let $A \in M_n(\mathbb{R})$. The matrix norm of A ***subordinate*** to the vector norm $\|\cdot\|$ is defined by

$$\|A\| := \sup \left\{ \|A\boldsymbol{x}\| \,:\, \boldsymbol{x} \in \mathbb{R}^n,\ \|\boldsymbol{x}\| = 1 \right\}. \tag{5.4}$$

The formula (5.4) indeed defines a matrix norm on $M_n(\mathbb{R})$. The proof of this fact is beyond the scope of our course. In this course, by matrix norm, we always mean a norm subordinate to some vector norm. An equivalent and more useful formula for the matrix norm subordinate to a vector norm is given in the following lemma.

Lemma 5.1.8.

For any $A \in M_n(\mathbb{R})$ and a given vector norm $\|\cdot\|$, we have

$$\|A\| = \max_{\boldsymbol{z} \neq 0} \frac{\|A\boldsymbol{z}\|}{\|\boldsymbol{z}\|}. \tag{5.5}$$

Proof.

For any $\boldsymbol{z} \neq 0$, we have $\boldsymbol{x} = \boldsymbol{z}/||\boldsymbol{z}||$ as a unit vector. Hence

$$\max_{||\boldsymbol{x}||=1} ||A\boldsymbol{x}|| = \max_{||\boldsymbol{z}||\neq 0} \left\| A\left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)\right\| = \max_{||\boldsymbol{z}||\neq 0} \frac{||A\boldsymbol{z}||}{||\boldsymbol{z}||}.$$

The matrix norm subordinate to a vector norm has additional properties as stated in the following theorem whose proof is left as an exercise.

Theorem 5.1.9.

Let $\|\cdot\|$ be a matrix norm subordinate to a vector norm. Then

1. $\|A\boldsymbol{x}\| \leq \|A\|\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
2. $\|I\| = 1$ where I is the identity matrix.
3. $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in M_n(\mathbb{R})$.

Note

We do not use different notations for a matrix norm and the subordinate vector norm. These are to be understood depending on the argument.

We will now state a few results concerning matrix norms subordinate to some of the vector norms described in Example 5.1.2. We omit their proofs.

Theorem 5.1.10 [Matrix norm subordinate to the maximum norm].

The matrix norm subordinate to the l_∞-norm given in (5.2) on $\mathbb{R}^n$ is denoted by $\|A\|_\infty$ and is given by

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|, \tag{5.6}$$

which is called the ***maximum-of-row-sums norm***.

Theorem 5.1.11 [Matrix Norm Subordinate to the l_1-norm].

The matrix norm subordinate to the l_1-norm given in (5.3) on $\mathbb{R}^n$ is denoted by $\|A\|_1$ and is given by

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|. \tag{5.7}$$

The norm $\|A\|_1$ is called the ***maximum-of-column-sums norm*** of A.

Description and computation of the matrix norm subordinate to the Euclidean vector norm on $\mathbb{R}^n$ is more subtle.

Theorem 5.1.12 [Matrix norm subordinate to the Euclidean norm].

The matrix norm subordinate to the l_2-norm (Euclidean norm) given by (5.1) on $\mathbb{R}^n$ is denoted by $\|A\|_2$ and is given by

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} |\lambda_i|}, \tag{5.8}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of the matrix A^TA. The norm $\|A\|_2$ given by the formula (5.8) is called the ***spectral norm*** of A.

Example 5.1.13.

Let us now compute $\|A\|_\infty$ and $\|A\|_2$ for the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{pmatrix}.$$

1. $\|A\|_\infty = 5$ since

$$\begin{aligned}\sum_{j=1}^{3}|a_{1j}| = |1| + |1| + |-1| &= 3,\\ \sum_{j=1}^{3}|a_{2j}| = |1| + |2| + |-2| &= 5,\\ \sum_{j=1}^{3}|a_{3j}| = |-2| + |1| + |1| &= 4.\end{aligned}$$

2. $\|A\|_2 \approx 3.5934$ as the eigenvalues of A^TA are $\lambda_1 \approx 0.0616$, $\lambda_2 \approx 5.0256$ and $\lambda_3 \approx 12.9128$. Hence $\|A\|_2 \approx \sqrt{12.9128} \approx 3.5934$.

5.2 Condition Number

The following theorem motivates the condition number for an invertible matrix which is similar to the condition number defined for a function in Section 3.5.1.

Theorem 5.2.1.

Let A be an invertible $n \times n$ matrix. Let $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ be the solutions of the systems

$$A\boldsymbol{x} = \boldsymbol{b} \text{ and } A\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}},$$

respectively, where $\boldsymbol{b}$ and $\tilde{\boldsymbol{b}}$ are given vectors. Then

$$\frac{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|} \leq \|A\|\,\|A^{-1}\|\,\frac{\|\boldsymbol{b} - \tilde{\boldsymbol{b}}\|}{\|\boldsymbol{b}\|} \tag{5.9}$$

for any fixed vector norm and the matrix norm subordinate to this vector norm.

Proof.

Since A is invertible, we have

$$\boldsymbol{x} - \tilde{\boldsymbol{x}} = A^{-1}\left(\boldsymbol{b} - \tilde{\boldsymbol{b}}\right).$$

Taking norms on both sides and using the fact that $\|A\boldsymbol{x}\| \leq \|A\|\|\boldsymbol{x}\|$ (see Theorem 5.1.9) holds for every $\boldsymbol{x} \in \mathbb{R}^n$, we get

$$\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\| \leq \|A^{-1}\|\|\boldsymbol{b} - \tilde{\boldsymbol{b}}\| \tag{5.10}$$

The inequality (5.10) estimates the error in the solution caused by error on the right hand side vector of the linear system $A\boldsymbol{x} = \boldsymbol{b}$. The inequality (5.9) is concerned with estimating the relative error in the solution in terms of the relative error in the right hand side vector $\boldsymbol{b}$.

Since $A\boldsymbol{x} = \boldsymbol{b}$, we get $\|\boldsymbol{b}\| = \|A\boldsymbol{x}\| \leq \|A\|\|\boldsymbol{x}\|$. Therefore $\|\boldsymbol{x}\| \geq \frac{\|\boldsymbol{b}\|}{\|A\|}$. Using this inequality in (5.10), we get (5.9).

Remark 5.2.2.

1. In the above theorem, it is important to note that a vector norm is fixed and the matrix norm used is subordinate to this fixed vector norm.
2. The theorem holds no matter which vector norm is fixed as long as the matrix norm subordinate to it is used.
3. In fact, whenever we do analysis on linear systems, we always fix a vector norm and then use matrix norm subordinate to it.

The inequality (5.9) shows the importance of the quantity $\|A\|\,\|A^{-1}\|$.

Definition 5.2.3 [Condition Number of a Matrix].

Let A be an $n \times n$ invertible matrix. Let a matrix norm be given that is subordinate to a vector norm. Then the ***condition number*** of the matrix A (denoted by $\kappa(A)$) is defined as

$$\kappa(A) := \|A\|\,\|A^{-1}\|. \tag{5.11}$$

Note

Notice that the condition number of a matrix A depends very much on the vector norm being used on $\mathbb{R}^n$ and the matrix norm that is subordinate to the vector norm.

From Theorem 5.2.1, it is clear that if the condition number is small, then the relative error in the solution will also be small whenever the relative error in the right hand side vector is small. On the other hand, if the condition number is very large, then the relative error could be large even though the relative error in the right hand side vector is small. We illustrate this in the following example.

Example 5.2.4.

The linear system

$$\begin{aligned} 5x_1 + 7x_2 &= 0.7 \\ 7x_1 + 10x_2 &= 1 \end{aligned}$$

has the solution $x_1 = 0, x_2 = 0.1$. Let us denote this by $\boldsymbol{x} = (0, 0.1)^T$, and the right hand side vector by $\boldsymbol{b} = (0.7, 1)^T$. The perturbed system

$$\begin{aligned} 5x_1 + 7x_2 &= 0.69 \\ 7x_1 + 10x_2 &= 1.01 \end{aligned}$$

has the solution $x_1 = -0.17, x_2 = 0.22$, which we denote by $\tilde{\boldsymbol{x}} = (-0.17, 0.22)^T$, and the right hand side vector by $\tilde{\boldsymbol{b}} = (0.69, 1.01)^T$. The relative error between the solutions of the above systems in the l_∞ vector norm is given by

$$\frac{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_\infty}{\|\boldsymbol{x}\|_\infty} = 1.7,$$

which is too high compared to the relative error in the right hand side vector which is given by

$$\frac{\|\boldsymbol{b} - \tilde{\boldsymbol{b}}\|_\infty}{\|\boldsymbol{b}\|_\infty} = 0.01.$$

The condition number of the coefficient matrix of the system is 289. Therefore the magnification of the relative error is expected (see the inequality (5.9)).

Remark 5.2.5.

A matrix with a large condition number is often referred to as an ***ill-conditioned*** matrix. Whereas a matrix with a small condition number is said to be ***well-conditioned***.

Discussion of condition numbers of matrices is incomplete without the mention of the famous *Hilbert matrix*.

Example 5.2.6.

The ***Hilbert matrix*** of order n is given by

$$H_n = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdot & & & \cdots & \\ \cdot & & & \cdots & \\ \cdot & & & \cdots & \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix} \tag{5.12}$$

For $n = 4$, we have

$$\kappa(H_4) = \|H_4\|_\infty \|H_4^{-1}\|_\infty = \frac{25}{12} 13620 \approx 28000$$

which may be taken as an ill-conditioned matrix. In fact, as the value of n increases, the corresponding condition number of the Hilbert matrix also increases.

An interesting and important question is that what kind of matrices could have large condition numbers. A partial answer is stated in the following theorem.

Theorem 5.2.7.

Let $A \in M_n(\mathbb{R})$ be non-singular. Then, for any singular $n \times n$ matrix B, we have

$$\frac{1}{\kappa(A)} \leq \frac{\|A - B\|}{\|A\|}. \tag{5.13}$$

Proof.

We have

$$\frac{1}{\kappa(A)} = \frac{1}{\|A\|\|A^{-1}\|} = \frac{1}{\|A\|} \left(\frac{1}{\displaystyle\max_{\boldsymbol{x} \neq 0} \frac{\|A^{-1}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}} \right) \leq \frac{1}{\|A\|} \left(\frac{1}{\frac{\|A^{-1}\boldsymbol{y}\|}{\|\boldsymbol{y}\|}} \right)$$

where $\boldsymbol{y} \neq \boldsymbol{0}$ is arbitrary. Take $\boldsymbol{y} = A\boldsymbol{z}$, for some arbitrary vector $\boldsymbol{z}$. Then we get

$$\frac{1}{\kappa(A)} \leq \frac{1}{\|A\|} \left(\frac{\|A\boldsymbol{z}\|}{\|\boldsymbol{z}\|} \right).$$

Let $\boldsymbol{z} \neq \boldsymbol{0}$ be such that $B\boldsymbol{z} = 0$ (this is possible since B is singular), we get

$$\begin{aligned} \frac{1}{\kappa(A)} &\leq \frac{\|(A - B)\boldsymbol{z}\|}{\|A\|\|\boldsymbol{z}\|} \\ &\leq \frac{\|(A - B)\|\|\boldsymbol{z}\|}{\|A\|\|\boldsymbol{z}\|} \\ &= \frac{\|A - B\|}{\|A\|}. \end{aligned}$$

From the above theorem it is apparent that if A is close to a singular matrix, then the reciprocal of the condition number is close to zero, ie., $\kappa(A)$ is large. Let us illustrate this in the following example.

Example 5.2.8.

Clearly the matrix

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is not invertible. For any $\epsilon > 0$, let

$$A = \begin{pmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{pmatrix}.$$

Since $\epsilon > 0$, A is invertible and we have

$$A^{-1} = \epsilon^{-2} \begin{pmatrix} 1 & -1-\epsilon \\ -1+\epsilon & 1 \end{pmatrix}.$$

Let us use the l_∞ norm on $\mathbb{R}^n$. Then $\|A\|_\infty = 2+\epsilon$ and $\|A^{-1}\|_\infty = \epsilon^{-2}(2+\epsilon)$. Hence

$$\begin{aligned} \kappa(A) &= \|A\|_\infty \|A^{-1}\|_\infty \\ &= \left(\frac{2+\epsilon}{\epsilon}\right)^2 \\ &> \frac{4}{\epsilon^2}. \end{aligned}$$

Thus, if $\epsilon \leq 0.01$, then $\kappa(A) \geq 40,000$. As $\epsilon \to 0$, the matrix A tends to approach the matrix B and consequently, the above inequality says that the condition number $\kappa(A)$ tends to ∞.

Also, we can see that when we attempt to solve the system $A\boldsymbol{x} = \boldsymbol{b}$, then the above inequality implies that a small relative perturbation in the right hand side vector $\boldsymbol{b}$ could be magnified by a factor of at least $40,000$ for the relative error in the solution.

5.3 Exercises

1. The following inequalities show that the notion of convergent sequences of vectors in $\mathbb{R}^n$ is independent of the vector norm. Show that the following inequalities hold for each $\boldsymbol{x} \in \mathbb{R}^n$
   i) $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \sqrt{n}\|\boldsymbol{x}\|_\infty$,
   ii) $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_1 \leq n\|\boldsymbol{x}\|_\infty$,
   iii) $\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2$.

2. Show that the norm defined on the set of all $n \times n$ matrices by
$$\|A\| := \max_{\substack{1\le i\le n\\ 1\le j\le n}} |a_{ij}|$$
is not subordinate to any vector norm on $\mathbb{R}^n$.

3. Let A be an invertible matrix. Show that its condition number $\kappa(A)$ satisfies $\kappa(A) \geq 1$.

4. Let A and B be invertible matrices with condition numbers $\kappa(A)$ and $\kappa(B)$ respectively. Show that $\kappa(AB) \leq \kappa(A)\kappa(B)$.

5. Let A be an $n \times n$ matrix with real entries. Let $\kappa_2(A)$ and $\kappa_\infty(A)$ denote the condition numbers of a matrix A that are computed using the matrix norms $\|A\|_2$ and $\|A\|_\infty$, respectively. Answer the following questions.
   i) Determine all the diagonal matrices such that $\kappa_\infty(A) = 1$.
   ii) Let Q be a matrix such that $Q^T Q = I$ (such matrices are called orthogonal matrices). Show that $\kappa_2(Q) = 1$.
   iii) If $\kappa_2(A) = 1$, show that all the eigenvalues of $A^T A$ are equal. Further, deduce that A is a scalar multiple of an orthogonal matrix.

6. Let $A(\alpha)$ be a matrix depending on a parameter $\alpha \in \mathbb{R}$ given by
$$A(\alpha) = \begin{pmatrix} 0.1\alpha & 0.1\alpha \\ 1.0 & 2.5 \end{pmatrix}$$
For each $\alpha \in \mathbb{R}$, compute the condition number of $A(\alpha)$. Determine an α_0 such that the condition number of $A(\alpha_0)$ is the minimum of the set $\{\, \kappa\,(A(\alpha)) : \alpha \in \mathbb{R}\}$. In the computation of condition numbers, use the matrix norm that is subordinate to the maximum vector norm on $\mathbb{R}^2$.

7. In solving the system of equations $A\boldsymbol{x} = \boldsymbol{b}$ with matrix
$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2.01 \end{pmatrix},$$
predict how slight changes in $\boldsymbol{b}$ will affect the solution $\boldsymbol{x}$. Test your prediction in the concrete case when $\boldsymbol{b} = (4,4)^T$ and $\tilde{\mathbf{b}} = (3,5)^T$. Use the maximum norm for vectors in $\mathbb{R}^2$.

8. Consider the following two systems of linear equations
$$x_1 + x_2 = 1,\ x_1 + 2x_2 = 2.$$

and

$$10^{-4}\, x_1 + 10^{-4}\, x_2 = 10^{-4}, \;\; x_1 + 2x_2 = 2.$$

Let us denote the first and second systems by $A_1\mathbf{x} = \mathbf{b}_1$ and $A_2\mathbf{x} = \mathbf{b}_2$ respectively. Use maximum-norm for vectors and the matrix norm subordinate to maximum-norm for matrices in your computations.

i) Solve each of the above systems using Naive Gaussian elimination method.
ii) Compute the condition numbers of A_1 and A_2.
iii) For each of the systems, find an upper bound for the relative error in the solution if the right hand sides are approximated by $\widetilde{\mathbf{b}_1}$ and $\widetilde{\mathbf{b}_2}$ respectively.
iv) Solve the systems

$$A_1\mathbf{x} = \widetilde{\mathbf{b}_1} \text{ and } A_2\mathbf{x} = \widetilde{\mathbf{b}_2}$$

where $\widetilde{\mathbf{b}_1} = (1.02, 1.98)^T$ and $\widetilde{\mathbf{b}_2} = (1.02 \times 10^{-4}, 1.98)^T$ using naive Gaussian elimination method. Compute the relative error in each case. Compare the computed relative errors with the bounds obtained above.

9. Let A and $\tilde{A}$ be non-singular square matrices, and $\boldsymbol{b} \neq 0$. If $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ are the solutions of the systems $A\boldsymbol{x} = \boldsymbol{b}$ and $\tilde{A}\tilde{\boldsymbol{x}} = \boldsymbol{b}$, respectively, then show that

$$\frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\tilde{\boldsymbol{x}}\|} \leq \kappa(A)\frac{\|A - \tilde{A}\|}{\|A\|}.$$

10. In the following problems, the matrix norm $\|\cdot\|$ denotes a matrix norm subordinate to a fixed vector norm.

i) Let A be an invertible matrix and B be any singular matrix. Prove the following inequality.

$$\frac{1}{\|A - B\|} \leq \|A^{-1}\|.$$

ii) Let A be an invertible matrix, and B be a matrix such that

$$\frac{1}{\|A - B\|} > \|A^{-1}\|.$$

Show that B is invertible.

iii) Let C be a matrix such that $\|I - C\| < 1$. Show that C is invertible.
iv) Let D be a matrix such that $\|D\| < 1$. Show that the matrix $I - D$ is invertible.