MEASUREMENT OF DISPLACEMENT VELOCITY AND ACCELERATION

Motion Parameter	Linear motion	Angular motion
Displacement	$s = f(t)$	$\theta = g(t)$
Velocity	$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Displacement → $v = \frac{ds}{dt}$ → Velocity → $a = \frac{dv}{dt}$ → Acceleration

Measurement of velocity may be obtained by differentiating displacement

Measurement of acceleration may be obtained by successive differentiation of displacement.

If the displacement variation with time is a smooth function, it is not a difficult task.
In practice, the rate of change of displacement varies arbitrarily and the functional form is rarely smooth. So, finding out the slope of the signal at every instant is indeed difficult.

Inverse procedure

Acceleration → $\int \frac{dv}{dt}$ → Velocity → $\int \frac{ds}{dt}$ → Displacement

Step integration – finding the area under a signal versus time curve. This task is rather easy, whatever be the shape of the curve.
Usually, displacement is not found by successively integrating the acceleration twice, but the velocity at any instant can be found with ease this way.

VIBRATION: Displacement-time variation is of a generally continuous form with some degree of repetitive nature.

SHOCK: Action is a single event form, a transient, with the motion generally decaying or damping out before further dynamic action takes place

Both shock and vibration measurements involve the basic measurement of displacement, velocity and acceleration as functions of time

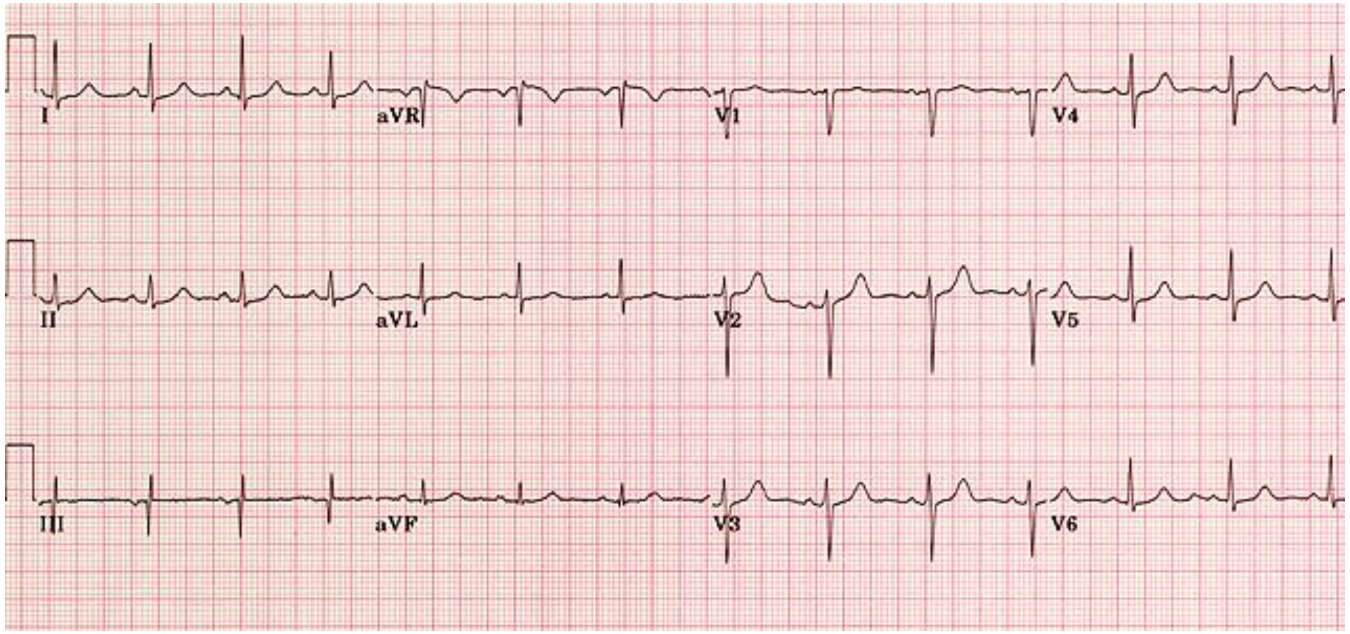
SEISMIC TRANSDUCER

MEASUREMENT OF DISPLACEMENT, VELOCITY AND ACCELERATION

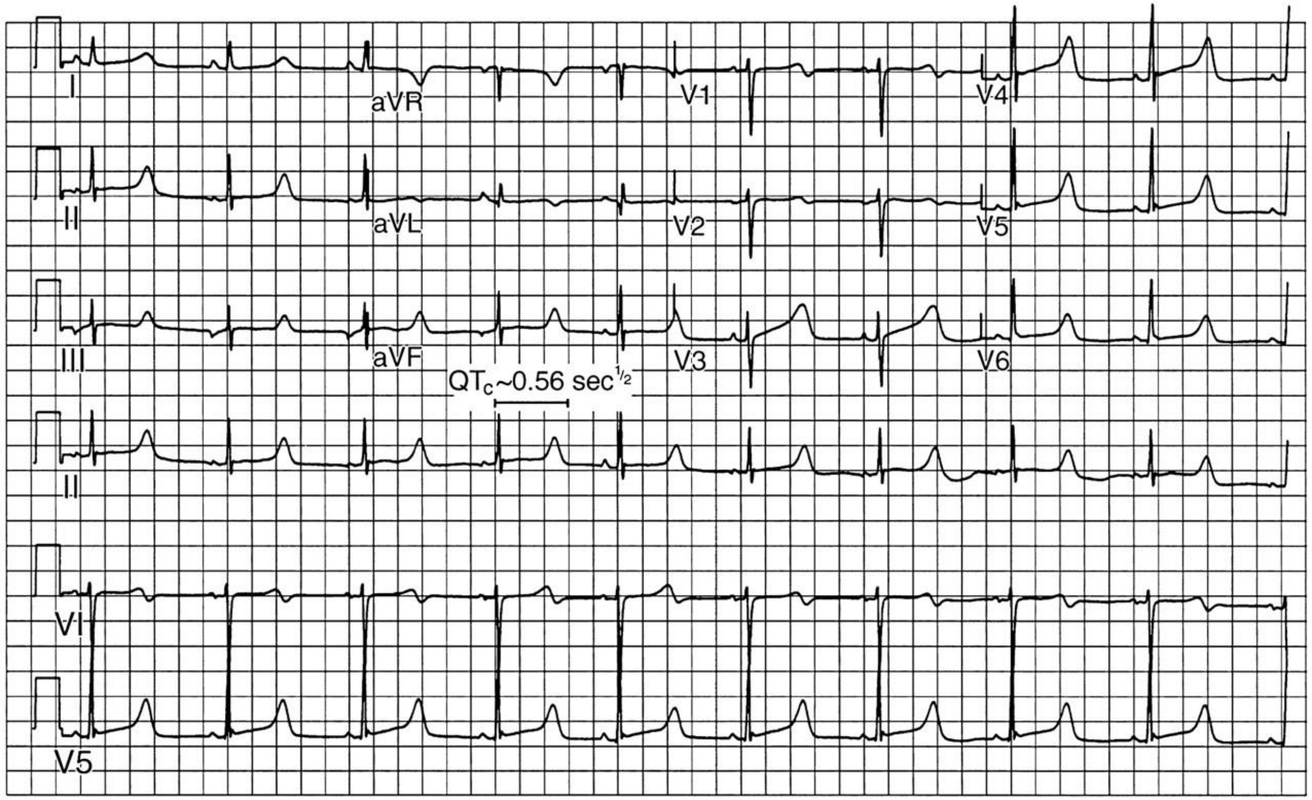
IMPORTANCE OF THE MEASUREMENT OF VIBRATION

- Vibratory motion can occur in all types of machines and structures
- Vibrations may result from
  - slight unbalance of forces in rotating machine components
  - action of wind load on transmission lines
  - suspension bridges
- Loss of efficiency, increasing bearing loads and failure – effects of unwanted vibration
- CONDITION MONITORING (something like ECG for machines)
  - Part of preventive maintenance
  - standard international standards (signature to problem correlation)

Normal ECG



Abnormal ECG



VIBRATION – cyclic or periodically repeated motion about a position of equilibrium

FREE VIBRATION – periodic motion that occurs when an elastic system is displaced from its equilibrium position and released. Frequency of free vibration is called NATURAL FREQUENCY of the system (ω_n)

FORCED VIBRATION – vibration resulting from application of an external periodic force. Frequency of excitation - (ω)

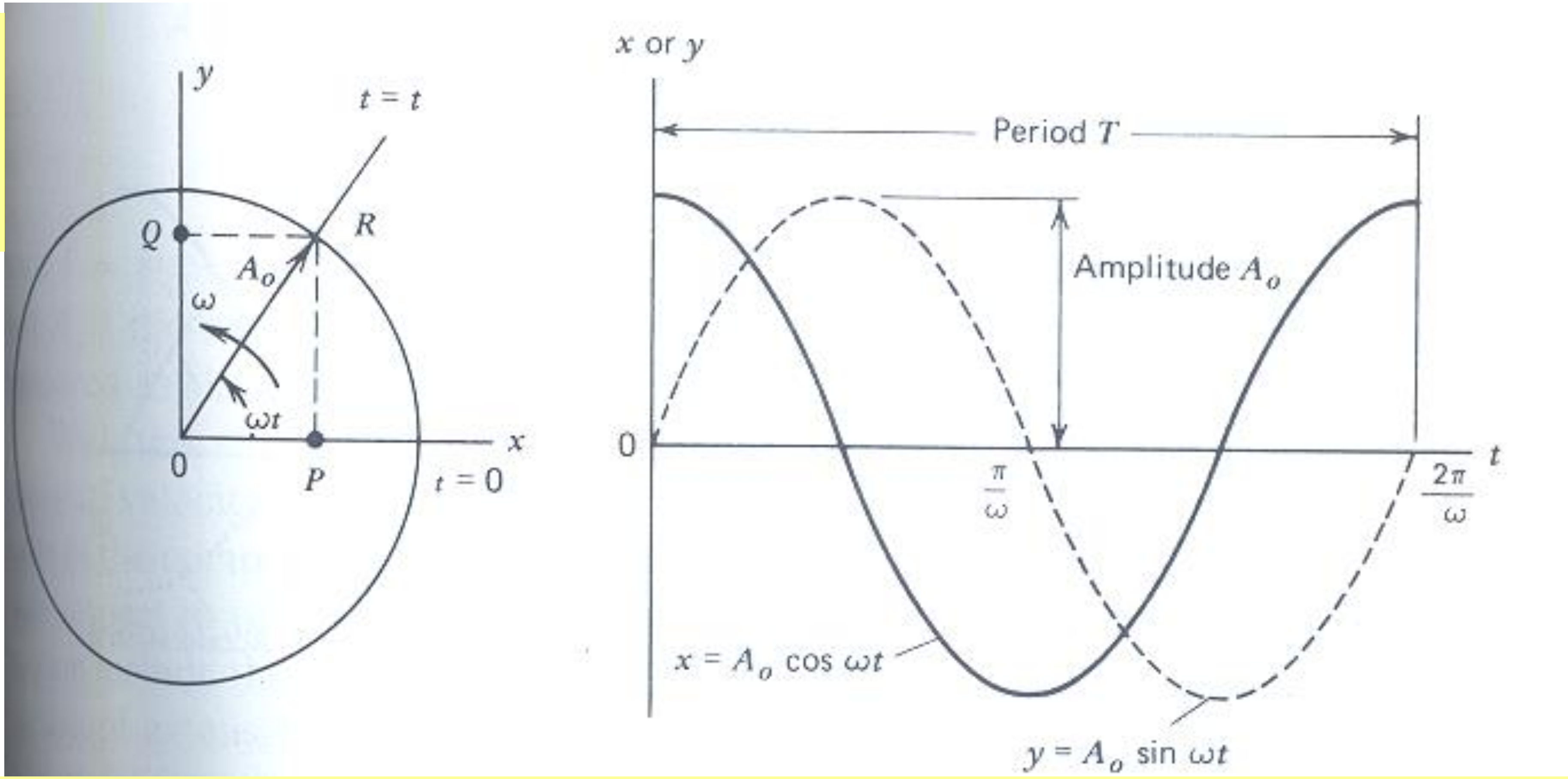
$\omega = \omega_n$ - Maximum amplitude – state of resonance

Single DOF – motion of the body is constrained so that its position can be completely specified by one co-ordinate

Two DOF - if the two coordinates are required to specify the position of the system at any instant of time or the if the system can vibrate in two direction

Common line representation of a simple type of vibratory motion

Line OR of magnitude A_o rotates in a counterclockwise direction about point O with a constant angular velocity ω



Projection of line OR onto the horizontal axis ox – represents motion of particle P having a single DOF in the x-direction

Position of particle P at any time 't' is $x = A_o cos\omega t$

Projection of line OR onto the vertical oy – represents motion of particle Q having a single DOF in the y-direction;

Position of particle P at any time 't' is $y = A_o sin\omega t$

Velocity of P

$$v_p = \dot{x} = -A_o\omega sin\omega t = A_o\omega cos\left(\omega t + \frac{\pi}{2}\right)$$

$$x = A_o cos\omega t$$

Velocity of Q

$$v_q = \dot{y} = A_o\omega cos\omega t = A_o\omega sin\left(\omega t + \frac{\pi}{2}\right)$$

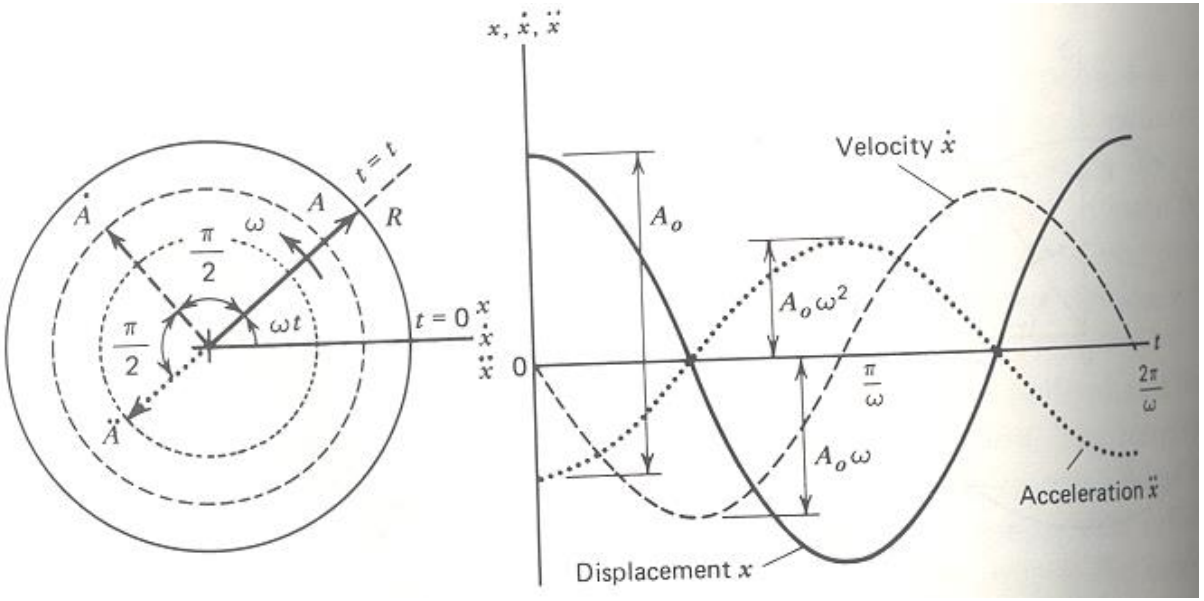
$$y = A_o sin\omega t$$

Acceleration of P

$$a_p = \ddot{x} = -A_o\omega^2 cos\omega t = A_o\omega^2 cos(\omega t + \pi) = -\omega^2 x$$

Acceleration of Q

$$a_q = \ddot{y} = -A_o\omega^2 sin\omega t = A_o\omega^2 sin(\omega t + \pi) = -\omega^2 y$$



Velocity is out of phase (leading) with displacement by 90°

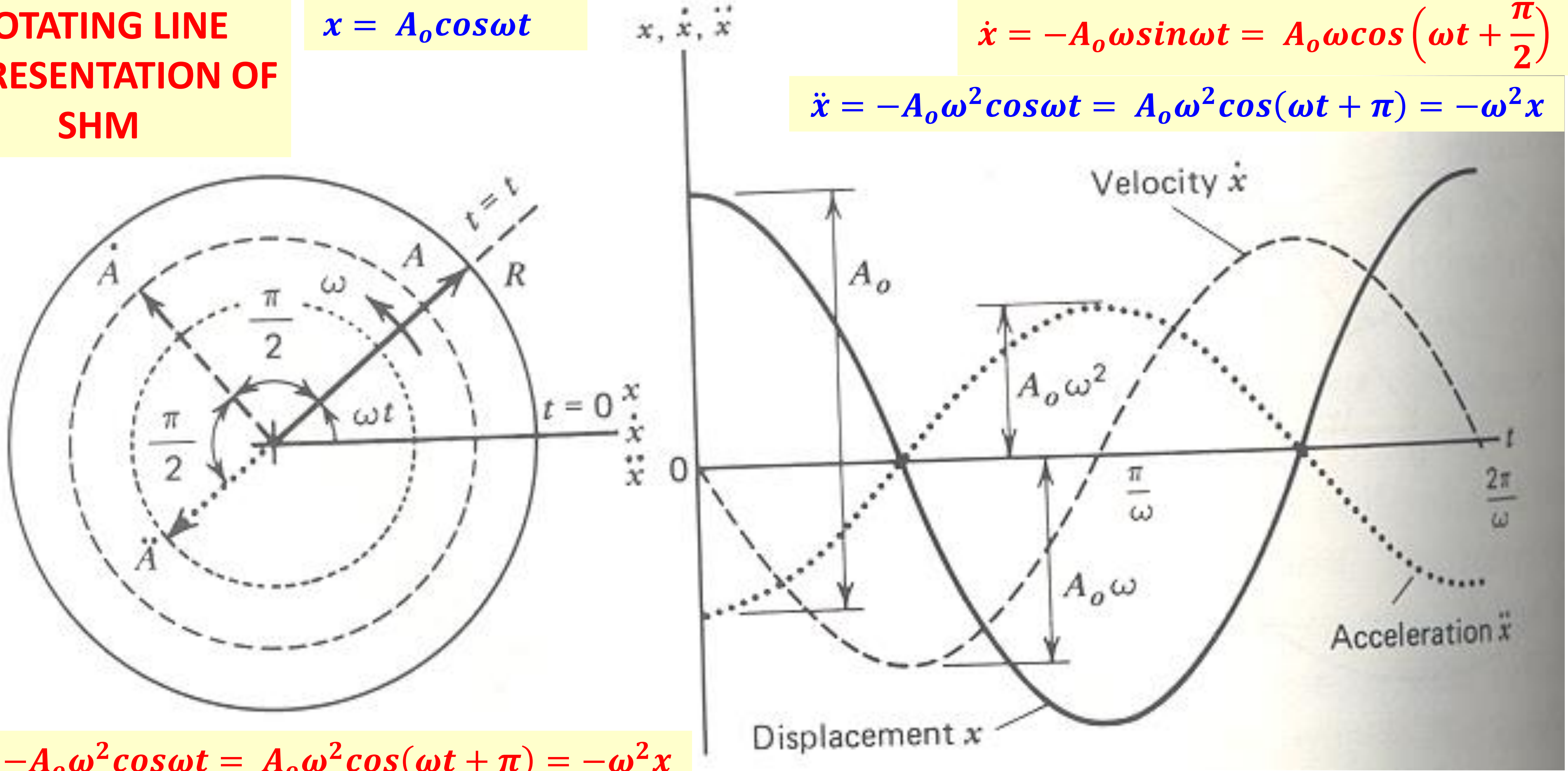
Acceleration is out of phase (leading) with displacement by 180°

ROTATING LINE REPRESENTATION OF SHM

$x = A_o cos\omega t$

$\dot{x} = -A_o \omega sin\omega t = A_o \omega cos\left(\omega t + \frac{\pi}{2}\right)$

$\ddot{x} = -A_o \omega^2 cos\omega t = A_o \omega^2 cos(\omega t + \pi) = -\omega^2 x$



$\ddot{x} = -A_o \omega^2 cos\omega t = A_o \omega^2 cos(\omega t + \pi) = -\omega^2 x$

Velocity is out of phase (leading) with displacement by 90°

Acceleration is out of phase (leading) with displacement by 180°

PHASE ANGLES associated with velocity and acceleration (with displacement as the reference vector) are POSITIVE (counterclockwise) – LEADING ANGLES

LAGGING PHASE ANGLES – would be NEGATIVE (CLOCKWISE)

SHM is –

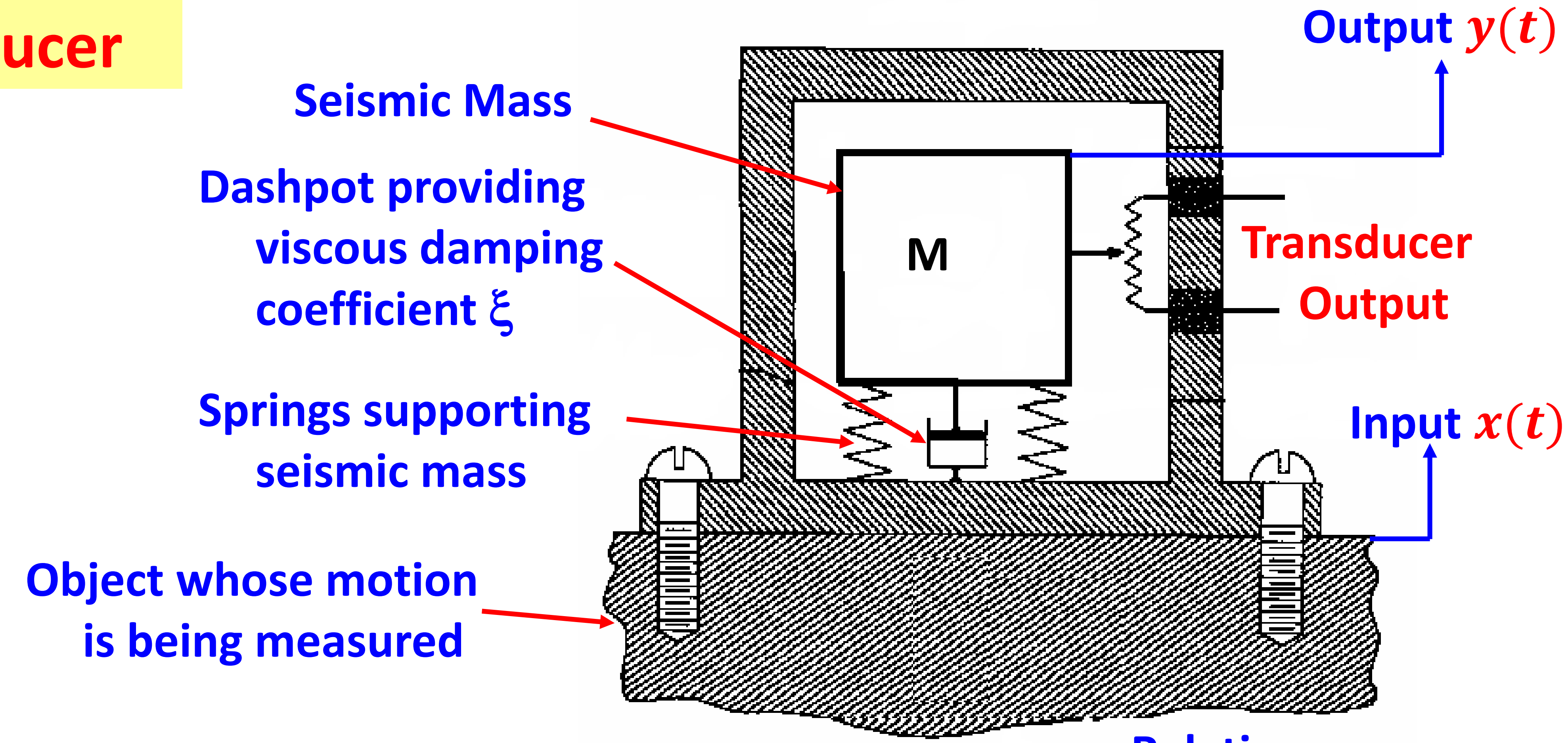
any motion for which acceleration is proportional to the displacement from a fixed point on the path of motion

Always directed toward the fixed point

Any periodic motion which is not SHM can be considered as the SUM OF SHMS of frequencies that are multiples of frequency of the fundamental motion

$$x = A_o + A_1 sin\omega t + A_2 sin2\omega t + \cdots \ldots\ldots + B_1 cos\omega t + B_2 cos2\omega t$$

Seismic Transducer



Motion of support → Primary transducer → Secondary transducer → Relative displacement between mass and support

Proper selection of damping and natural frequency of the system

-Relative displacement is proportional to acceleration of the base

-Relative displacement is proportional to actual displacement of base

SEISMIC TRANSDUCER THEORY

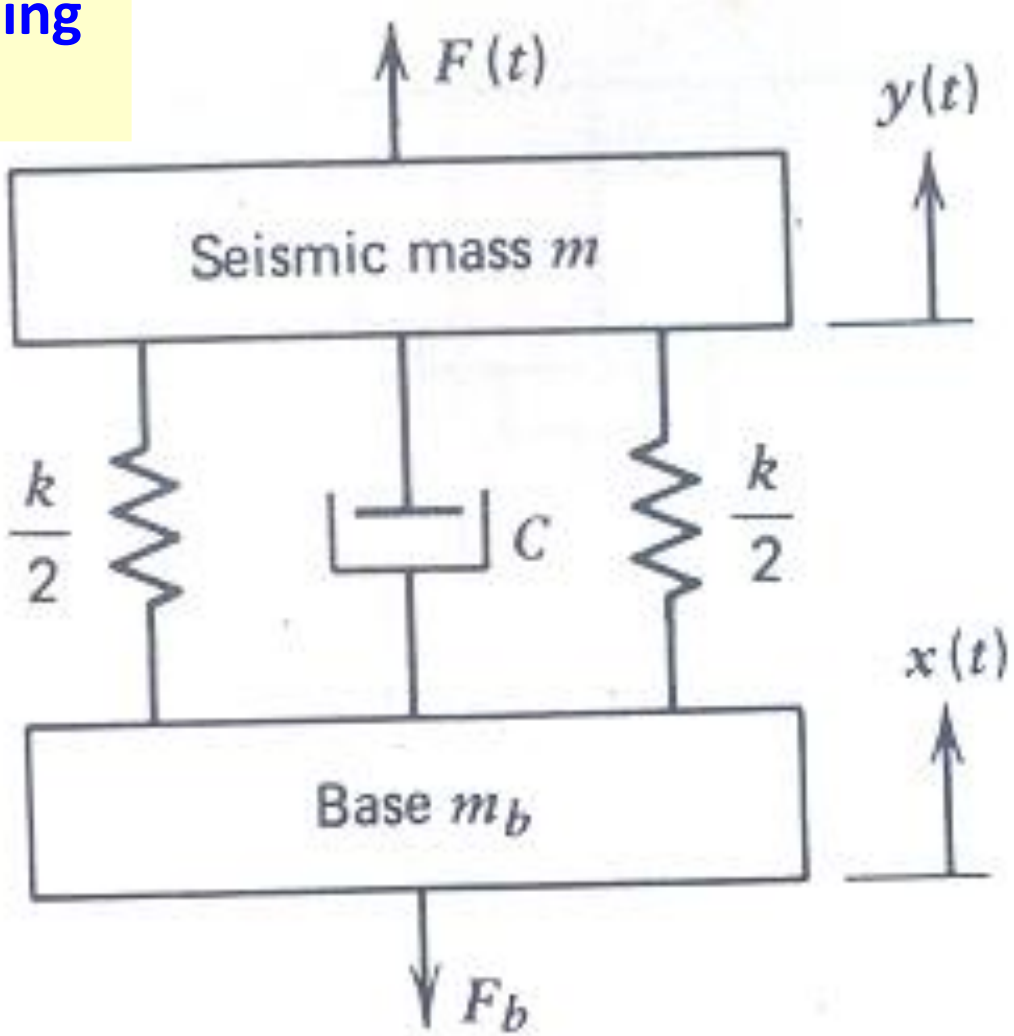
m - Seismic mass

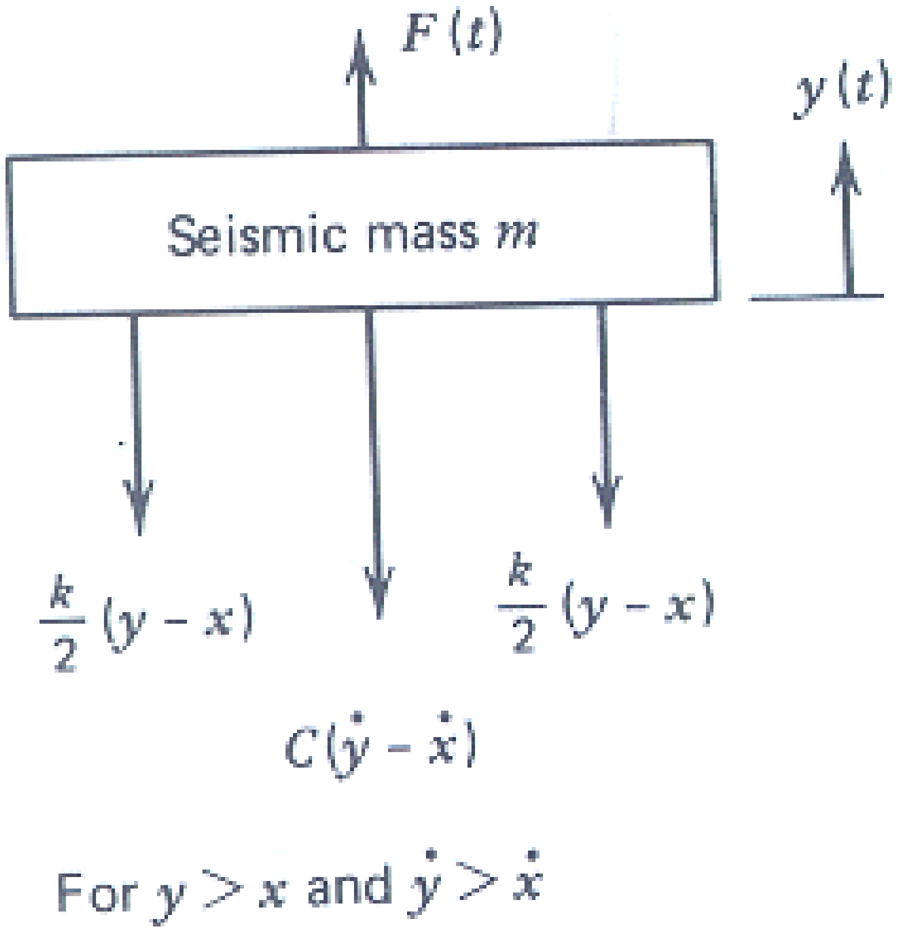
k - Spring stiffness

C - viscous damping coefficient

$x, \dot{x}, \ddot{x}$ – Displacement, velocity and acceleration of the base

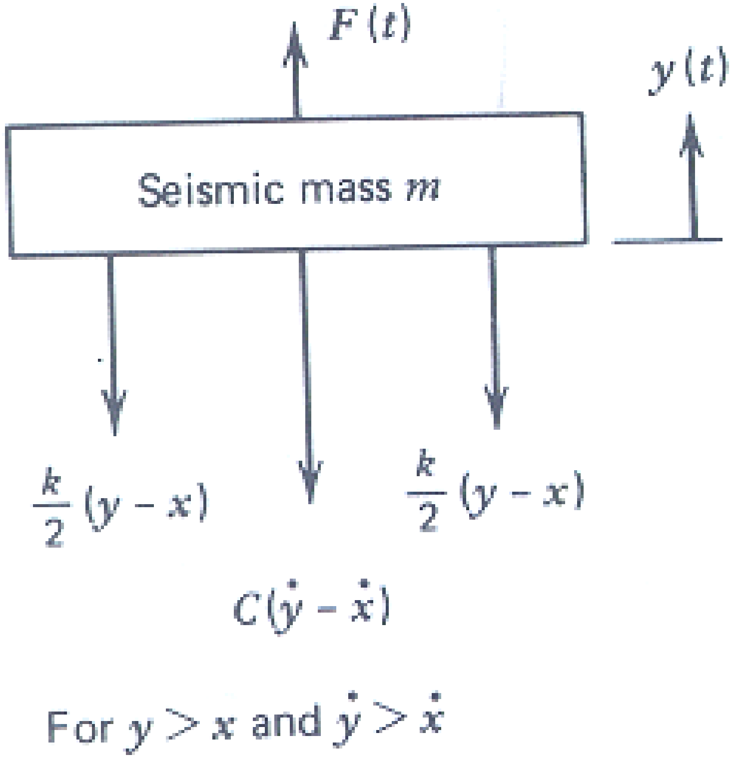
$y, \dot{y}, \ddot{y}$ - Displacement, velocity and acceleration of seismic mass





$m\ddot{y} = -k(y - x) - C(\dot{y} - \dot{x})$

$m\ddot{z} + c\dot{z} + kz = -m\ddot{x}$



$$z = y - x$$
$$\dot{z} = \dot{y} - \dot{x}$$
$$\ddot{z} = \ddot{y} - \ddot{x}$$

z – Relative displacement

$-m\ddot{x}$ - Transducer excitation owing to the inertial force produced by the base acceleration

Seismic elements – secondary transducer responds to relative motion between the seismic mass and the base ie., $z = y - x$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x}$$

$$\ddot{z} + \frac{c}{m}\dot{z} + \frac{k}{m}z = -\ddot{x}$$

$$\ddot{z} + \frac{c}{2\sqrt{mk}}\frac{2\sqrt{k}}{\sqrt{m}}\dot{z} + \frac{k}{m}z = -\ddot{x}$$

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -\ddot{x}$$

$$\xi = \frac{c}{2\sqrt{mk}}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Seismic elements – secondary transducer responds to relative motion between the seismic mass and the base ie., $z = y - x$

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$$\ddot{z} + \frac{c}{2\sqrt{mk}}\frac{2\sqrt{k}}{\sqrt{m}}\dot{z} + \frac{k}{m}z = -\ddot{x}$$

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -\ddot{x}$$

$$\frac{\ddot{z}}{\omega_n^2} + \frac{2\xi}{\omega_n}\dot{z} + z = -\frac{\ddot{x}}{\omega_n^2}$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\frac{1}{\omega_n^2}\left[s^2 z(s) - sz(0) - z'(0)\right] + \frac{2\xi}{\omega_n}\left(sz(s) - z(0)\right) + z(s) = -\frac{1}{\omega_n^2}\left[s^2 x(s) - sx(0) - x'(0)\right]$$

$$\frac{1}{\omega_n^2}\left[s^2 z(s)\right] + \frac{2\xi}{\omega_n}\left(sz(s)\right) + z(s) = -\frac{1}{\omega_n^2}\left[s^2 x(s)\right]$$

$$\left[\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n}s + 1\right]z(s) = -\frac{1}{\omega_n^2}\left[s^2 x(s)\right]$$

$$\left[\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n} s + 1\right] z(s) = -\frac{1}{\omega_n^2}\left[s^2 x(s)\right]$$

$$\frac{z}{x}(s) = \frac{-\dfrac{s^2}{\omega_n^2}}{\dfrac{s^2}{\omega_n^2} + \dfrac{2\xi}{\omega_n} s + 1}$$

FREQUENCY RESPONSE

$$\frac{z}{x}(i\omega) = \frac{\dfrac{\omega^2}{\omega_n^2}}{-\dfrac{\omega^2}{\omega_n^2} + i\dfrac{2\xi}{\omega_n}\omega + 1} = \frac{\dfrac{\omega^2}{\omega_n^2}}{\underbrace{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)}_{a} + i\underbrace{\left(2\xi\dfrac{\omega}{\omega_n}\right)}_{b}}\frac{a - ib}{a - ib} = \frac{\omega^2}{\omega_n^2}\left(\frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}\right)$$

$$\left|\frac{z}{x}\right| = \frac{\omega^2}{\omega_n^2}\sqrt{\frac{a^2}{(a^2 + b^2)^2} + \frac{b^2}{(a^2 + b^2)^2}} = \frac{\dfrac{\omega^2}{\omega_n^2}}{\sqrt{a^2 + b^2}}$$

$$\left|\frac{z}{x}\right| = \frac{\dfrac{\omega^2}{\omega_n^2}}{\sqrt{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\dfrac{\omega}{\omega_n}\right)^2}}$$

$$\left|\frac{z}{x}\right| = \frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$|z| = |x|\frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} - Constant\ \Psi$$

$$Tan\phi = \frac{-2\xi\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)}$$

Output lags behind the input

SEISMIC VIBROMETER – measures displacement

$$|z| = |x| \frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$\frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} - Constant\ \psi$$

$$Tan\phi = \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

$$|z| = |x|\ \psi$$

$$\xi = 0.707 \qquad \frac{\omega}{\omega_n} > 2.0$$

LOW NATURAL FREQUENCY

SOFT SPRUNG MASS

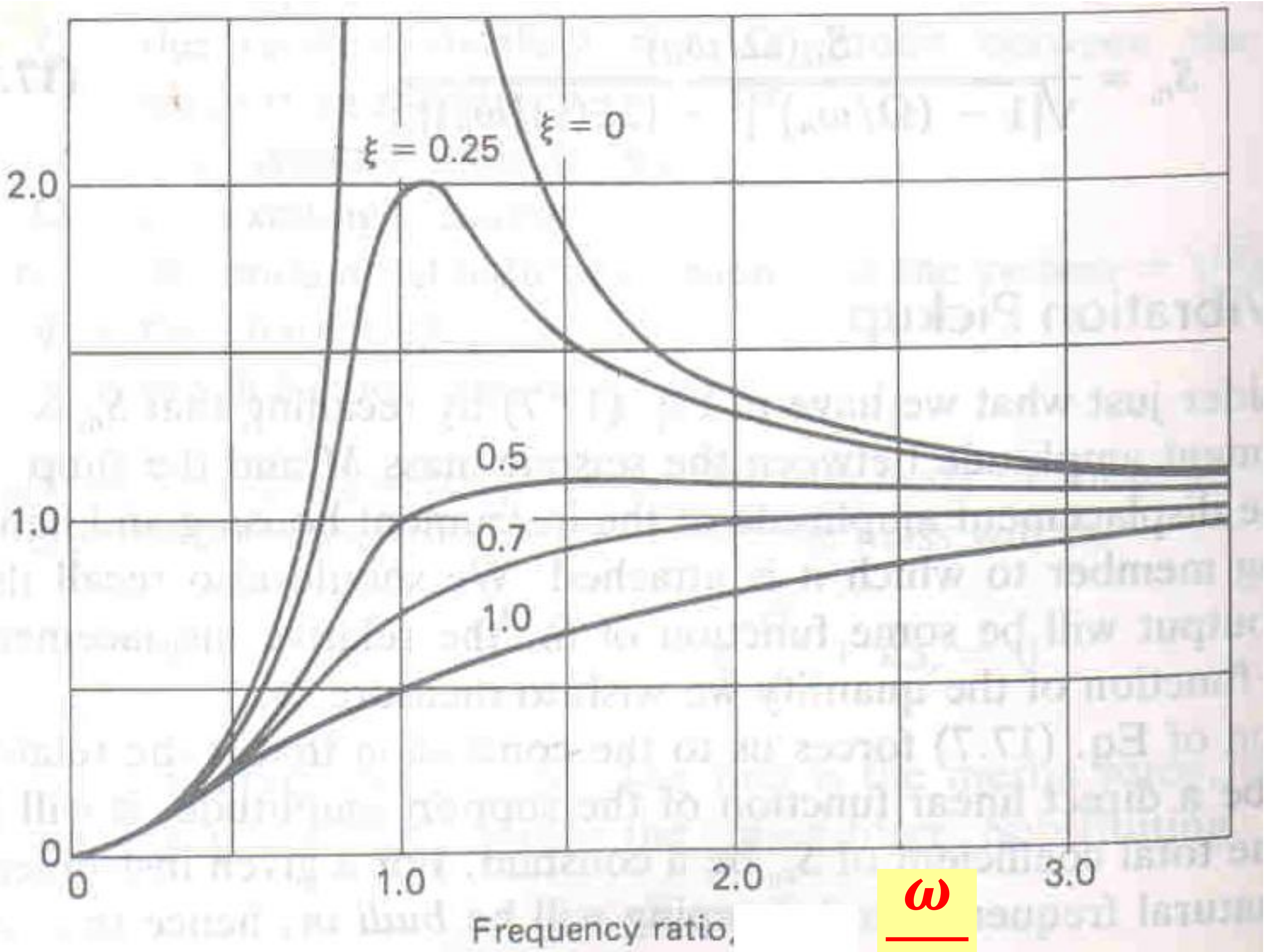
HIGHLY SENSITIVE

PHASE SHIFT – CONSTANT LEADS BY 180 DEG

RESPONSE OF A SEISMIC INSTRUMENT TO HARMONIC DISPLACEMENT

$$\frac{|z|}{|x|} = \frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$\frac{z}{x}$

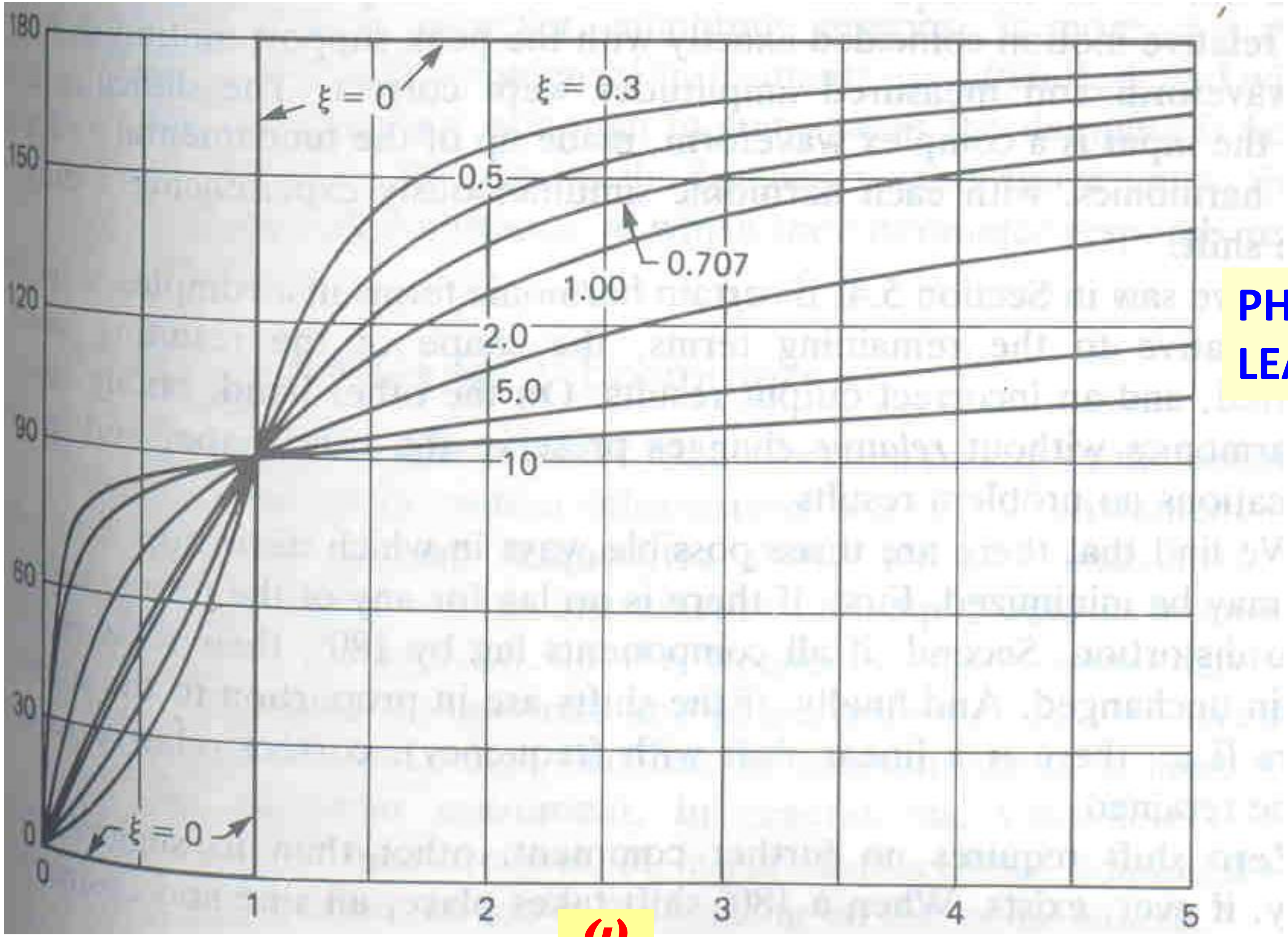


$\frac{\omega}{\omega_n}$

$$Tan\phi = \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

PHASE SHIFT – CONSTANT LEADS BY 180 DEG

ϕ



$\frac{\omega}{\omega_n}$

SEISMIC ACCELEROMETER

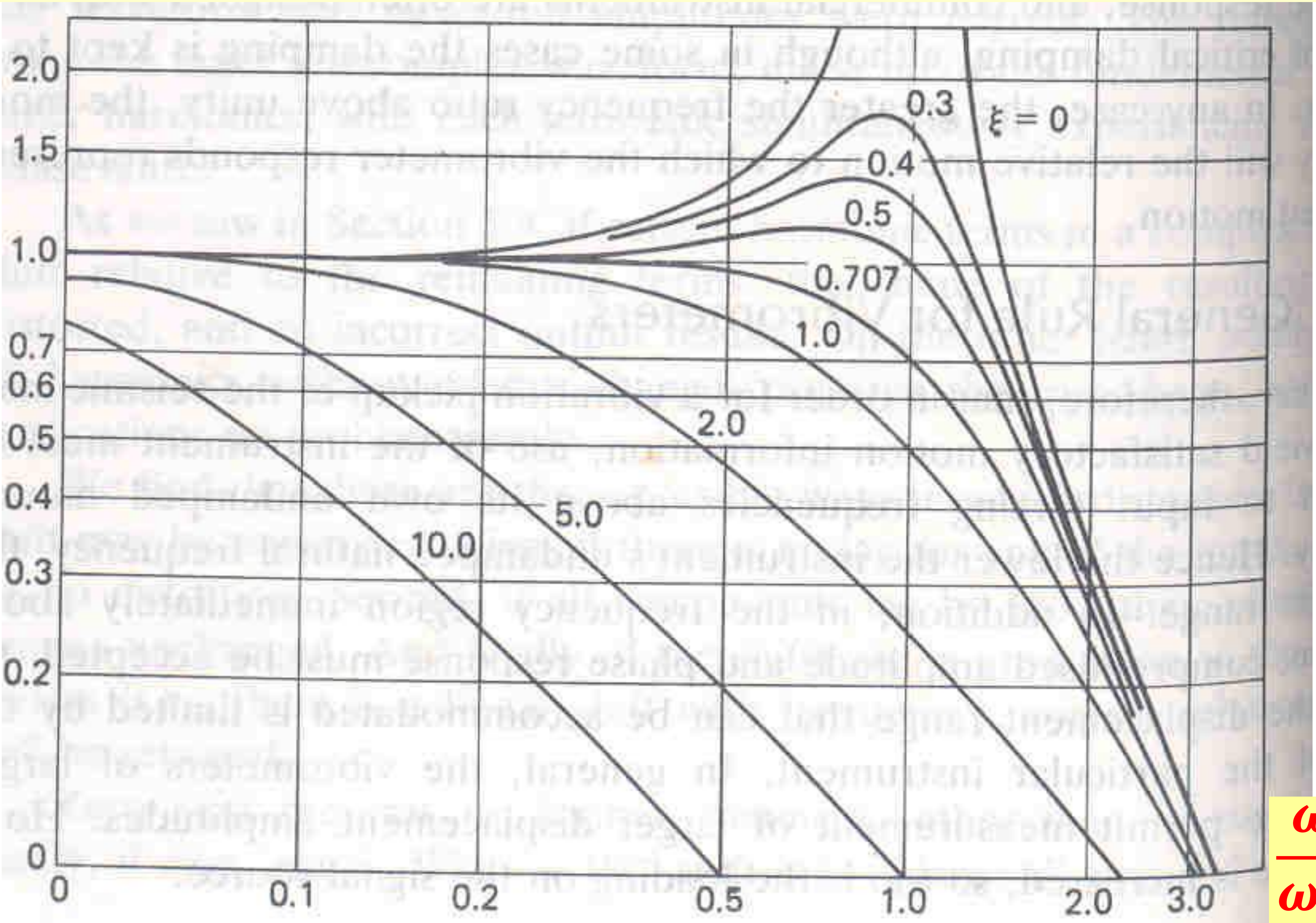
$$|z| = |x|\omega^2 \frac{\frac{1}{\omega_n^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\frac{\frac{1}{\omega_n^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} = \psi\ constant$$

$$|z| = |x|\omega^2\psi$$

$$|z| = \ddot{x}\psi$$

RESPONSE OF A SEISMIC INSTRUMENT TO SINUSOIDAL ACCELERATION



$\frac{|z|}{\ddot{x}}$

$\frac{\omega}{\omega_n}$

SEISMIC ACCELEROMETER	SEISMIC VIBROMETER
MEASURES ACCELERATION	MEASURES DISPLACEMENT
$\xi = 0.707$ and $\omega/\omega_n < 0.4$	$\xi = 0.707$ and $\omega/\omega_n > 2.0$
HIGH NATURAL FREQUENCY	LOW NATURAL FREQUENCY
STIFF SPRUNG MASS	SOFT SPRUNG MASS
RUGGED	HIGHLY SENSITIVE
PHASE SHIFT VARIES LINEARLY	PHASE SHIFT – CONSTANT
WITH EXCITATION FREQUENCY ω	LEADS BY 180 DEG

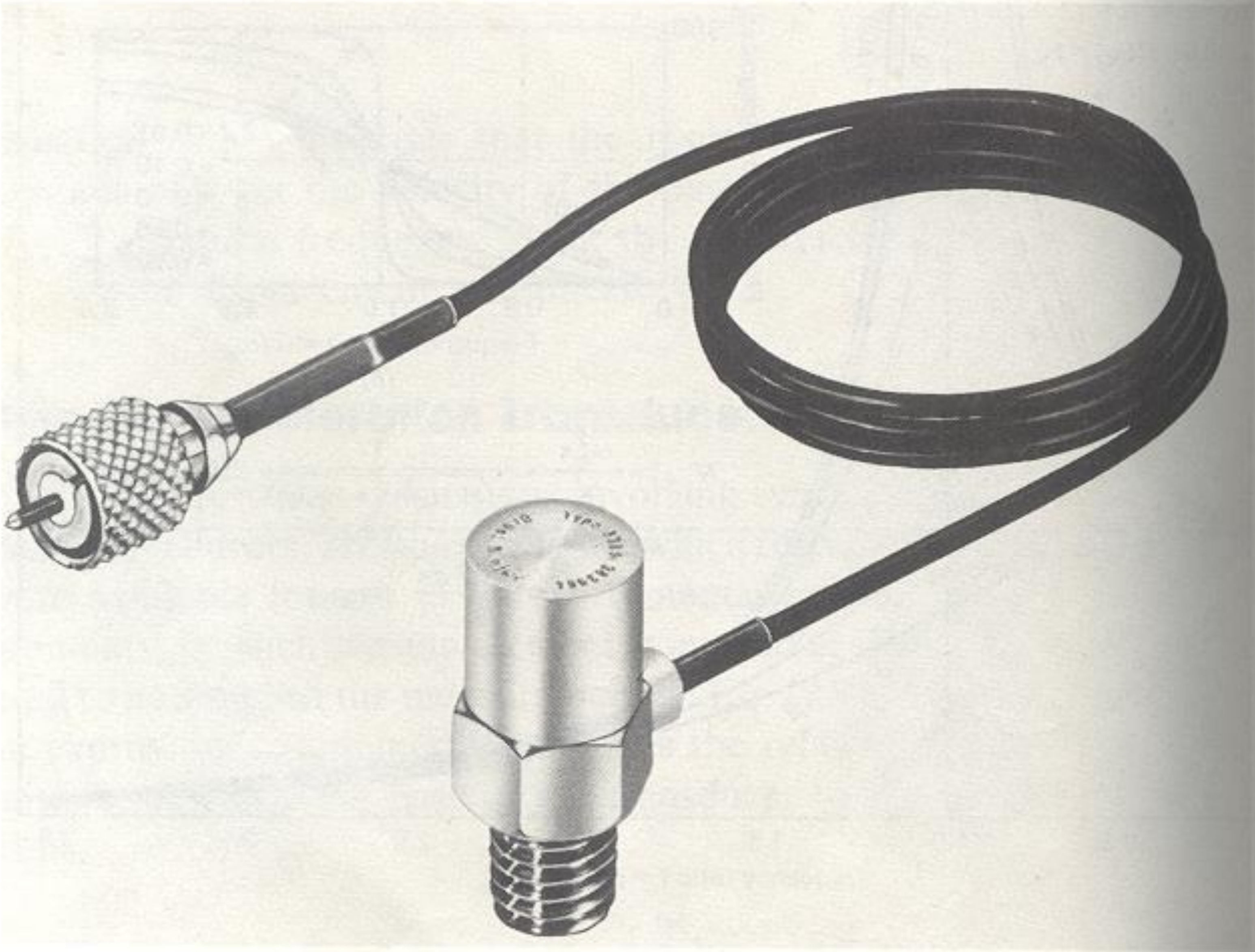


Figure 7.41 An accelerometer with a piezoelectric sensing element. (Courtesy of Brue
and Kjaer.)

A seismic instrument is to be used to measure a periodic vibration having an amplitude of 1.27 mm and a frequency of 15 Hz.

1. Specify an appropriate combination of natural frequency and damping ratio such that the dynamic error in the output is less than 5%. (choose a mass of 22.7 gms)
2. What spring constant and damping constant would yield these values of natural frequency and damping ratio?
3. Determine the phase lag for the output signal. Would the phase lag change if the input frequency were changed?

KNOWN

Input function

$$y_{hs} = 1.27 sin30\pi t$$

$$\xi = 0.7$$

$$|z| = |x| \frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$Tan\phi = \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

$$\xi = \frac{c}{2\sqrt{mk}}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$\frac{\omega}{\omega_n}$	$\frac{z}{x}$	ϕ
10	1	172
8	1	169.9
6	1	166.5
4	0.999	159.5
3	0.996	152.3
2	0.975	137.0
1.7	0.951	128.5

$$\xi = 0.7$$

$$|z| = |x| \frac{\frac{\omega^2}{\omega_n^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$Tan\phi = \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

$$\frac{\omega}{\omega_n} \geq 1.7 \qquad \frac{\omega}{1.7} \geq \omega_n \qquad \frac{2\pi(15)}{1.7} = \omega_n$$

$$\omega_n = 55.44\, rad/s$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$55.44 = \sqrt{\frac{k}{22.7 \times 10^{-3}}}$$

$$k = 69.77\, \frac{N}{m}$$

$$\xi = \frac{c}{2\sqrt{mk}}$$

$$0.7 = \frac{c}{2\sqrt{22.7 \times 10^{-3}} \times 69.77}$$

$$c = 1.7619\, \frac{N.s}{m}$$