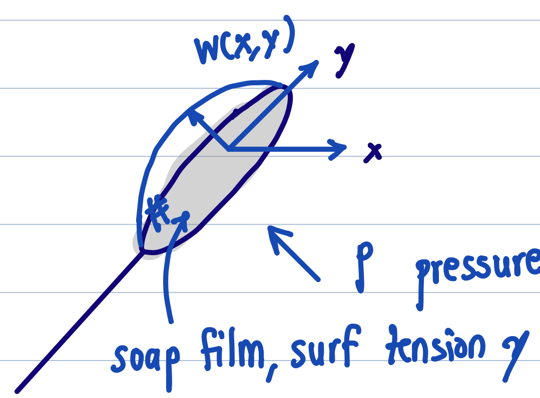
Dnyanesh Pawaskar

ME 202

TORSION OF THIN CROSS-SECTIONS

Q2p1

SOAP BUBBLES ☺



w(x,y) shape of soap bubble

Assume small displacements

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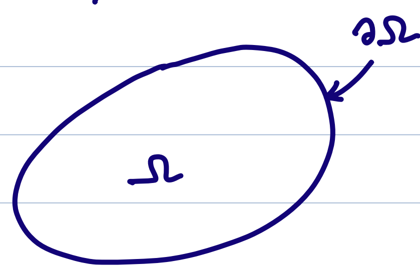
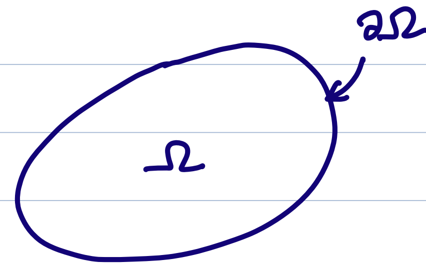
TORSION EQN	MEMBRANE EQN
$\varphi(x,y)$ Prandtl stress func.	$w(x,y)$ z-disp of film/ membrane
$\nabla^2\varphi = -2G\alpha$ in Ω	$\nabla^2 w = -p/\gamma$ in Ω
G shear modulus N/m^2	γ surf tension $N/m \equiv J/m^2$
α intensity of twist rad/m	p pressure N/m^2
$\varphi = 0$ on $\partial\Omega$	$w = 0$ on $\partial\Omega$
$T = 2\int_\Omega \varphi \, dx\,dy$	$V = \int_\Omega w \, dx\,dy$
T torque N-m	V enclosed volume

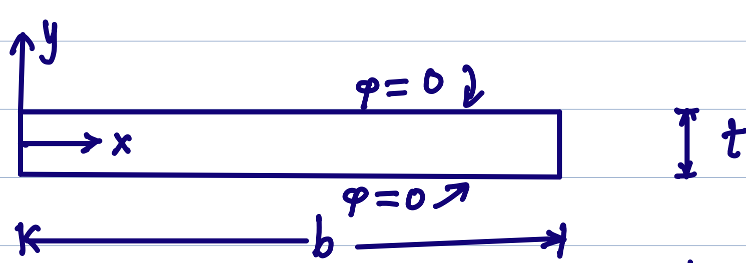




Use this analogy to visualize φ.

<u>Utility</u>

thin rect c/s of shaft

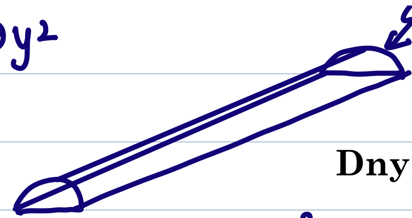


$b \gg t$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -2G\alpha \quad , \varphi$$

$\cancel{\partial x^2}$ $\quad \partial y^2$

ignored



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$$\frac{\partial\varphi}{\partial y} = -2G\alpha y + C_1, \quad \varphi = -2G\alpha\frac{y^2}{2} + C_1 y + C_2$$

$\varphi = 0$ when $y = \pm t/2$

$$\varphi = G\alpha\left(\frac{t^2}{4} - y^2\right)$$

$$\text{Torque} = 2\int_\Omega \varphi \, dx\, dy$$

$$= 2G\alpha \int_0^b dx \int_{-t/2}^{t/2} \left(\frac{t^2}{4} - y^2\right) dy$$

$$T = \frac{G\alpha b t^3}{3} \qquad \frac{N}{m^2}\,\frac{rad}{m}\,mm^3 \equiv Nm \checkmark$$

$$K_t = \frac{G b t^3}{3}$$

Expt

b/t	K_t
1	
5	
10	
....	

$$\tau_{xz} = \frac{\partial\varphi}{\partial y} = -2G\alpha y$$

$$\tau_{yz} = -\frac{\partial\varphi}{\partial x} = 0$$

$$\tau_{max} = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = G\alpha t = \frac{3T}{bt^2}$$

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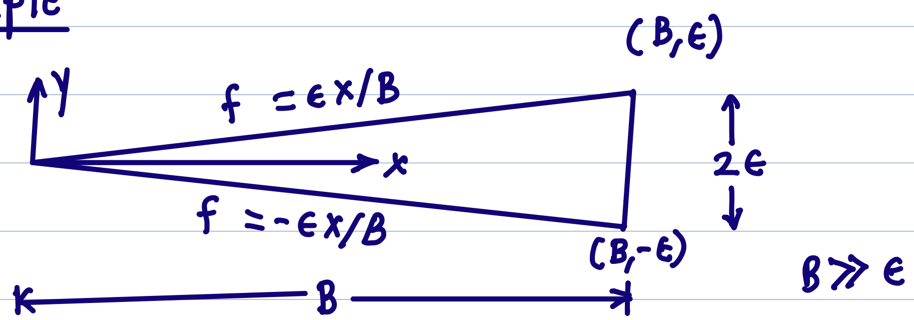
Example



$B \gg \epsilon$

$$\underbrace{\cancel{\frac{\partial^2 \varphi}{\partial x^2}}}_{\approx 0} + \frac{\partial^2 \varphi}{\partial y^2} = -2G\alpha\,, \qquad \varphi = 0 \text{ on } f = \pm \frac{\epsilon x}{B}$$

$$\varphi = G\alpha(f^2 - y^2)$$

$$T = -2G\alpha \int_0^B dx \int_{-f}^{f} dy \, (y^2 - f^2)$$

$$T = \frac{2}{3} G\alpha \epsilon^3 B$$

$$\varphi(x,y) = K\left(y^2 - \frac{\epsilon^2 x^2}{B^2}\right) \cancel{(x - B)}$$

$$\frac{\partial \varphi}{\partial y} = K\,2y\,, \qquad \frac{\partial^2 \varphi}{\partial y^2} = 2K$$

$$\frac{\partial \varphi}{\partial x} = \frac{-K\epsilon^2 \, 2x}{B^2}\,, \qquad \frac{\partial^2 \varphi}{\partial x^2} = -\frac{2K\epsilon^2}{B^2}$$

$$\left|\frac{\partial^2 \varphi}{\partial x^2}\right| \ll \left|\frac{\partial^2 \varphi}{\partial y^2}\right| \qquad \frac{\epsilon^2}{B^2} \ll 1$$

$$\nabla^2 \varphi \approx \frac{\partial^2 \varphi}{\partial y^2} = 2K = -2G\alpha$$

$$\Rightarrow K = -G\alpha$$

$$\varphi = -G\alpha\left(y^2 - \frac{\epsilon^2 x^2}{B^2}\right)$$

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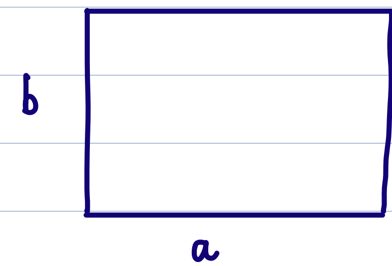
For sections

need approx K_t



$b \not\gg a$

- ☐ Fourier series
- ☑ Principal of Min Potential Energy PMPE

Use PMPE to solve a linear algebraic system of equations rather than the partial differential equation

$$\nabla^2 \varphi = -2G\alpha \quad \text{in } \Omega$$

$$\varphi = 0 \quad \text{on } \partial\Omega$$

$$T = 2\int_{\Omega} \varphi \, dx\, dy$$

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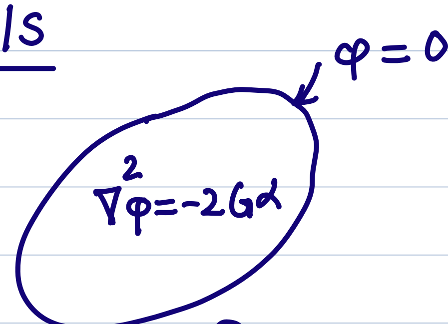
ME 202

APPLICATION OF PMPE TO TORSION OF NON-CIRCULAR C/S



PMPE

- ☐ Linear system

  $$\underset{\sim}{A}\,\underline{x} = \underline{b} \qquad N \times N$$

$$T = 2\int_{\Omega} \varphi \, da$$

- ☐ Approximate method
- ☐ Computational method

$$\Pi_{total} = \Pi_{SE} + \underbrace{\Pi_{Ext\ \cancel{Forces}}}_{\text{Torques}} \qquad L = 1$$

$-T\alpha$ **Dnyanesh Pawaskar**

Recall,

strain/stored elastic energy density (J/m^3) $= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\epsilon_{ij}$

$$\begin{array}{cc} 1 & x \\ 2 & y \\ 3 & z \end{array}$$

$$= \cancel{\frac{1}{2}\sigma_{xx}\epsilon_{xx}} + \cancel{\frac{1}{2}\sigma_{xy}\epsilon_{xy}} + \cancel{\frac{1}{2}\sigma_{yx}\epsilon_{yx}}$$

$$+ \dots + \frac{1}{2}\sigma_{xz}\epsilon_{xz} + \frac{1}{2}\sigma_{zx}\epsilon_{zx} + \frac{1}{2}\sigma_{yz}\epsilon_{yz}$$

$$+ \frac{1}{2}\sigma_{zy}\epsilon_{zy} + \cancel{\frac{1}{2}\sigma_{zz}\epsilon_{zz}}$$

Torsion $\underset{\sim}{\sigma} = \begin{pmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{pmatrix}$

Hooke's Law

$$\epsilon_{xz} = \frac{1}{2G}\sigma_{xz}, \quad \epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

$$\sigma_{xz} = \frac{\partial\varphi}{\partial y}, \quad \sigma_{yz} = -\frac{\partial\varphi}{\partial x}$$

$$SED = \frac{1}{2G}\Bigg[\underbrace{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}_{\underset{\sim}{\nabla}\varphi \cdot \underset{\sim}{\nabla}\varphi}\Bigg] \quad \frac{J}{m^3}$$

This is NOT $\nabla^2\varphi$

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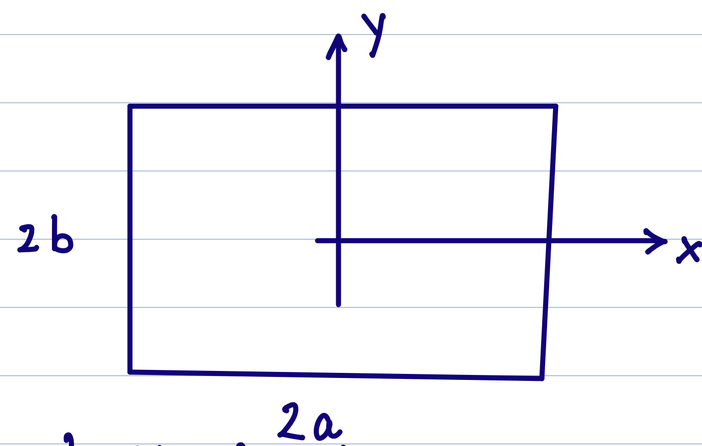
Total PE per unit length

$$\Pi = \int_{\Omega} \frac{1}{2G}\left[\underline{\nabla}\varphi \cdot \underline{\nabla}\varphi\right] da - 2\alpha \int_{\Omega} \varphi \, da$$

Find $\varphi(x,y)$ that approx. minimizes Π
and which obeys BC $\varphi = 0$. ← KBC

Example



$\varphi(x,y) = K\,(x^2 - a^2)(y^2 - b^2)$ zero on $\partial\Omega$

$$\frac{\partial\varphi}{\partial x} = -2Kx\,(b^2 - y^2), \quad \frac{\partial\varphi}{\partial y} = -2Ky\,(a^2 - x^2)$$

$$\Pi = \frac{1}{2G}\int_{x=-a}^{a} dx \int_{y=-b}^{b} dy \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] - 2\alpha \int_{x=-a}^{a} dx \int_{y=-b}^{b} dy \; \varphi$$

SDOF problem
K ← 1 DOF

$$\Pi(K) \underset{\text{approx.}}{=} \frac{64 K^2 a^3 b^3 (a^2+b^2)}{45 G} - \frac{32 K a^3 b^3}{9} \alpha$$

Find K that minimizes Π

$$\frac{\partial \Pi}{\partial K} = 0 \Rightarrow \frac{128 K a^3 b^3 (a^2+b^2)}{45 G} - \frac{32 \alpha a^3 b^3}{9} = 0$$

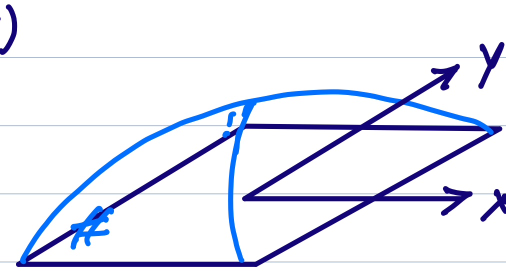
$$K = \frac{5 G \alpha}{4(a^2+b^2)}$$

$$\varphi = \frac{5 G \alpha}{4(a^2+b^2)} (x^2-a^2)(y^2-b^2)$$

$$T = 2 \int_{x=-a}^{a} dx \int_{y=-b}^{b} dx \; \varphi = \frac{40 G \alpha a^3 b^3}{9(a^2+b^2)}$$

$$T = \frac{40 G a^3 b^3}{9(a^2+b^2)} \cdot \alpha$$

Recall,
soap film analogy



□ zero on $\partial \Omega$

□ agrees with analogy □ easy to integrate

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Pick $\varphi = K \cos\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi y}{2b}\right)$

$$\frac{\partial \varphi}{\partial x} = -\frac{K\pi}{2a} \sin\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi y}{2b}\right)$$

$$\frac{\partial \varphi}{\partial y} = -\frac{K\pi}{2b} \sin\left(\frac{\pi y}{2b}\right) \cos\left(\frac{\pi x}{2a}\right)$$

$$\Pi(K) = \frac{K^2 \pi^2 (a^2+b^2)}{8Gab} - \frac{32 \alpha a b K}{\pi^2}$$

PMPE $\Rightarrow$ $\Pi'(K) = 0 \Rightarrow K = \dfrac{128 G a^2 b^2 \alpha}{\pi^4 (a^2+b^2)}$

$$T = 2 \int \varphi \, dx \, dy$$

$$T = \frac{4096 \, G a^3 b^3}{\pi^6 (a^2+b^2)} \alpha$$

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2}$$

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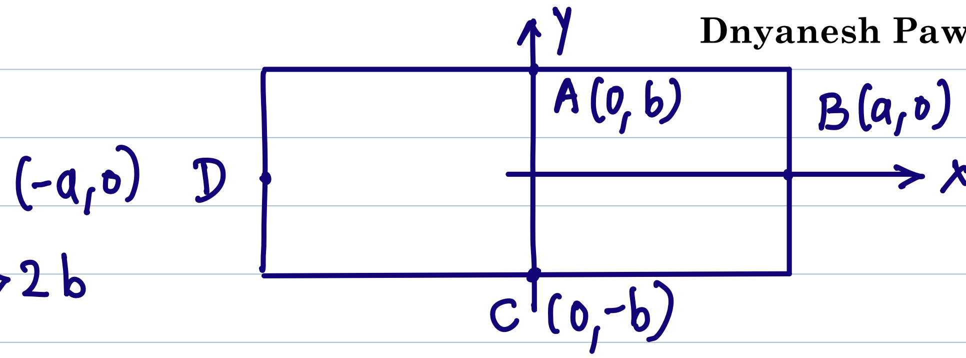

$2a > 2b$

$$\tau = \frac{64\, G\alpha}{\pi^3 (a^2+b^2)} \sqrt{a^2 b^4 \cos^2\left(\frac{\pi y}{2b}\right) \sin^2\left(\frac{\pi x}{2a}\right) + b^2 a^4 \cos^2\left(\frac{\pi x}{2a}\right) \sin^2\left(\frac{\pi y}{2b}\right)}$$

From soap film analogy, expect τ to hit max at A, C.

$$\tau_A = \tau_C = \frac{64\, a^2 b\, G\alpha}{\pi^3 (a^2+b^2)}$$ max shear stress

$$\tau_B = \tau_D = \frac{64\, a b^2\, G\alpha}{\pi^3 (a^2+b^2)}$$

Exercise 1 check τ at corners of rectangle

Exercise 2 $\varphi(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} K_{mn} \cos\left(\frac{m\pi x}{2a}\right) \cos\left(\frac{n\pi y}{2b}\right)$

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Use orthogonality of cosines to simplify integrals

PMPE $\frac{\partial \Pi}{\partial K_{11}} = 0, \; \frac{\partial \Pi}{\partial K_{12}} = 0, \; \ldots\ldots, \text{etc.}$