Part - 2

Each question carries 4 mark.

8. The sequence $\{x_n\}$ defined by

$$x_n = \frac{1}{2^{n^2}}$$

converges to 0 as $n \to \infty$. Answer the following questions for the given sequence $\{x_n\}$:

A) Show that the order of convergence of the sequence is atleast superlinear. **(1 mark)**

B) Does the sequence converges quadratically? Justify your answer. **(1.5 marks)**

C) Does there exists an $\alpha > 1$ such that the sequence converges to 0 with order of convergence equals α? Justify your answer. **(1.5 marks)**

Answer.

Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{2^{n^2}}.$$

A) We have

$$\frac{|a_{n+1} - 0|}{|a_n - 0|} = \frac{2^{n^2}}{2^{(n+1)^2}} = \left(\frac{1}{2}\right)\left(\frac{1}{2^{2n}}\right) \to 0 \text{ as } n \to \infty.$$

A $\Rightarrow$ LHS formula; B $\Rightarrow$ RHS conclusion.

(A+B = 0.5+0.5=1 mark)

Therefore the sequence superlinearly converges to zero.

B) We have

$$\frac{|a_{n+1} - 0|}{|a_n - 0|^2} = \frac{(2^{n^2})^2}{2^{(n+1)^2}} = 2^{n^2-2n-1} \to \infty \text{ as } n \to \infty.$$

C $\Rightarrow$ LHS formula; D $\Rightarrow$ RHS conclusion.

(**C+D = 0.5+0.5=1 mark**)

Therefore the sequence **does not** converges quadratically.
If only the conclusion is written without any justification, then Step E marks are not given.

(**E: 0.5 marks**)

C) We now show that the order of convergence cannot be any $\alpha > 1$. For any $\alpha > 1$, we have

$$\frac{|a_{n+1} - 0|}{|a_n - 0|^\alpha} = \frac{(2^{n^2})^\alpha}{2^{(n+1)^2}} = 2^{(\alpha-1)n^2-2n-1} \to \infty \text{ as } n \to \infty.$$

(**F: 0.5 marks**)

Since $\alpha - 1 > 0$, we see that $(\alpha - 1)n^2 - 2n - 1 \to \infty$ as $n \to \infty$. Therefore, the RHS of the above equation tends to ∞ as $n \to \infty$.

(**G: 0.5 marks**)

Hence, the given sequence **cannot** have order of convergence α for any $\alpha > 1$.
If only the conclusion is written without any justification, then Step H marks are not given.

(**H: 0.5 marks**)

9. Consider a computing device that uses n-digit rounding (decimal) arithmetic. Let $\mathrm{fl}(x)$ denote the floating-point approximation of a positive real number x in this device. Let x be a positive real number for which the $(n+1)^{\text{th}}$ digit in the mantissa is greater than or equal to 5, in its floating-point representation. Prove
$$\left|\frac{x - \mathrm{fl}(x)}{x}\right| \le \frac{1}{2}10^{-n+1}.$$

Answer.

Let the floating-point representation of x be given by

$$x = (-1)^s \times (0.d_1 d_2 \cdots)_{10} \times 10^e$$

Let $d_{n+1} \geq 5$. In this case, $\mathrm{fl}(x)$ is given by

$$\mathrm{fl}(x) = (-1)^s \times \Big((0.d_1 d_2 \cdots d_n)_{10} + 10^{-n}\Big) \times 10^e.$$

(A: 0.5 marks)

Therefore we have

$$|x - \mathrm{fl}(x)| \;=\; \Big(10^{-n} - (0.00 \cdots 0 d_{n+1} d_{n+2} \cdots)_{10}\Big) \times 10^e$$

The above can be rewritten as

$$|x - \mathrm{fl}(x)| \;=\; \left(\left\{\frac{9}{10^{n+1}} + \frac{9}{10^{n+2}} + \cdots\right\} - \left\{\frac{d_{n+1}}{10^{n+1}} + \frac{d_{n+2}}{10^{n+2}} + \cdots\right\}\right) \times 10^e$$

(B(RHS Term 1)+C(RHS Term 2): 0.5+0.5 = 1 mark)

Further simplification gives

$$\begin{aligned} |x - \mathrm{fl}(x)| &= \left(\frac{9 - d_{n+1}}{10^{n+1}} + \frac{9 - d_{n+2}}{10^{n+2}} + + \frac{9 - d_{n+3}}{10^{n+3}} + \cdots\right) \times 10^e \\ &\leq \left(\frac{4}{10^{n+1}} + \frac{9}{10^{n+2}} + \frac{9}{10^{n+3}} + \cdots\right) \times 10^e. \end{aligned}$$

(D: 0.5 marks)

Since $9 - d_{n+1} \leq 4$, we can write

$$|x - \mathrm{fl}(x)| \;\leq\; \frac{5}{10^{n+1}} \times 10^e.$$

(E: 1 mark)

As a consequence,

$$\frac{|x - \text{fl}(x)|}{|x|} \leq \frac{\frac{5}{10^{n+1}} \times 10^e}{(0.d_1 d_2 \cdots)_{10} \times 10^e}$$
$$\leq \frac{\frac{5}{10^{n+1}}}{(0.d_1 d_2 \cdots)_{10}}$$

(F: 0.5 marks)

Observe that

$$(0.d_1 d_2 \cdots)_{10} \geq 0.1 \text{ as } d_1 \geq 1$$

(G: 0.5 marks)

As a result, we get

$$\frac{|x - \text{fl}(x)|}{|x|} \leq \frac{1}{2} \times 10^{-n+1}$$