Tutorial 1 Solution

1.

Let L be a real number and let $\{a_n\}$ be a sequence of real numbers. If there exists a positive integer N and a $\mu \in (0,1)$ such that

$$|a_n - L| \leq \mu |a_{n-1} - L|$$

holds for all $n \geq N$, then show that $a_n \to L$ as $n \to \infty$.

This is a statement about the limit of a sequence, and one way to approach it is to use the definition of a limit:

Definition: Let {an} be a sequence of real numbers and let L be a real number. We say that L is the limit of the sequence {an} and write

lim(an) = L as n → ∞

if for every ε > 0, there exists a positive integer N such that for all n ≥ N, |an - L| < ε.

Now, given the condition in the problem statement:

|an − L| ≤ μ|an−1 − L|

We will show this statement satisfies the definition of the limit.

Given any ε > 0, we can find a positive integer N such that 1/μ^(n-N) < ε/(L-L).

By the given condition, we have that
|an − L| ≤ μ|an−1 − L| ≤ ... ≤ μ^(n-N)|aN − L|

so
|an - L| ≤ μ^(n-N) * |aN - L| < ε

for all n ≥ N. This shows that the definition of a limit is satisfied, so we can conclude that lim(an) = L as n → ∞.

2.

Show that the equation $\sin x + x^2 = 1$ has at least one solution in the interval $[0,1]$.

To show that the equation sin x + x^2 = 1 has at least one solution in the interval [0, 1], we can use the Intermediate Value Theorem (IVT). The IVT states that if a continuous function f(x) has different values at two distinct points of an interval [a, b], then it must assume every value between these values at some point in the interval.

In this case, we can see that the function f(x) = sin x + x^2 is continuous in the interval [0, 1], since both sin x and x^2 are continuous functions.

Also, f(0) = sin 0 + 0^2 = 0 + 0 = 0 and f(1) = sin 1 + 1^2 = 0.84147 + 1 = 1.84147

The value of f(0) = 0 is clearly less than 1 and the value of f(1) = 1.84147 is also greater than 1. Therefore, by the IVT, there exists a value x0 in the interval [0,1] such that f(x0) = 1, which is the value of the function at the point (x0,1) is equal to 1. Hence, the equation sin x + x^2 = 1 has at least one solution in the interval [0,1].

3.

Let $f(x)$ be a continuous function on $[a,b]$, let $x_1, \ldots, x_n$ be points in $[a,b]$, and let $g_1, \ldots, g_n$ be non-positive real numbers. Then show that

$$\sum_{i=1}^{n} f(x_i) g_i = f(\xi) \sum_{i=1}^{n} g_i, \quad \text{for some } \xi \in [a,b].$$

Proof is in Video lecture 3.3

4.

Let $f : [a,b] \to [a,b]$ be a continuous function. Prove that the equation $f(x) = x$ has at least one solution lying in the interval $[a,b]$ (Note: A solution of this equation is called a *fixed point* of the function f). Further if $\max\limits_{x \in [a,b]} |f'(x)| < 1$, then show that the equation $f(x) = x$ has a unique solution in $[a,b]$.

The statement that f(x) = x has at least one solution in [a, b] is known as the existence of a fixed point.

Proof of Existence:
Let g(x) = f(x) - x. Then g(x) is a continuous function on [a, b] and g(a) = f(a) - a and g(b) = f(b) - b.

By the Intermediate Value Theorem, for any value y between g(a) and g(b), there exists a point c in [a, b] such that g(c) = y. In particular, if g(a) and g(b) have opposite signs, there exists a point c in [a, b] such that g(c) = 0, which means f(c) = c.

Proof of Uniqueness:
Suppose x1 and x2 are two distinct solutions to the equation f(x) = x in [a, b]. Then, f(x1) = x1 and f(x2) = x2. Subtracting these two equations, we get

f(x1) - f(x2) = x1 - x2, or

f(x1) - x1 = x2 - f(x2)

Let's call the left hand side h(x1) and the right hand side h(x2), so that h(x1)=h(x2),

|h(x1) - h(x2)| = |f(x1) - x1 - (x2 - f(x2))| = |f(x1) - x1 - x2 + f(x2)| = |f(x1) - x1 - x2 + f(x2)| = |f(x1) - x1 - (f(x2) - x2)| = |f(x1) - f(x2)|

Now by the assumption max
x∈[a,b]
|f'(x)| < 1, we can say that |f(x1) - f(x2)| < |x1 - x2|, which implies that h(x1) = h(x2) is not possible.

Therefore, f(x) = x has a unique solution in [a, b].

5.

Let g be a continuously differentiable function (C^1 function) such that the equation $g(x) = 0$ has at least n distinct roots. Show that the equation $g'(x) = 0$ has at least $n-1$ distinct roots.

To prove that the equation g'(x) = 0 has at least n-1 distinct roots, we will use Rolle's theorem.

Rolle's theorem states that if a function g(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), and if g(a) = g(b), then there exists at least one c in

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Rolle's theorem states that if a function g(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), and if g(a) = g(b), then there exists at least one c in the open interval (a, b) such that g'(c) = 0.

We know that g(x) = 0 has at least n distinct roots, let's call them r1, r2, ..., rn. Since g(x) is continuously differentiable, g'(x) is also continuous on [a, b]. And since g(x) = 0 has n distinct roots, we can form n-1 intervals (r1,r2), (r2,r3),...,(rn-1,rn)

Now, by Rolle's theorem, we know that for each of these intervals, there exist at least one c such that g'(c) = 0. Hence, we have n-1 roots of g'(x) = 0.

So, we have shown that the equation g'(x) = 0 has at least n-1 distinct roots.

6.

Evaluate an approximate value of the function $f(x) = e^{x^2}$ at $x = 1$ using $T_2(x)$ about the point $a = 0$. Obtain the remainder $R_2(1)$ in terms of some unknown real number ξ. Compute (approximately) a possible value of ξ.

The Taylor series of f(x) = ex2 about the point a = 0 is given by:

f(x) = T2(x) + R2(x) = 1 + x2 + (x2^2)/2! + R2(x)

So, to evaluate an approximate value of f(x) at x = 1 using T2(x), we substitute x = 1 into the above equation:

f(1) ≈ T2(1) = 1 + 1 + (1^2)/2! = 1 + 1 + 0.5 = 2.5

To find the remainder R2(x) in terms of some unknown real number ξ, we can use the Taylor's theorem which states that for any given function f(x) and point a, there exist a real number ξ in the interval (a,x) such that

R2(x) = (f''(ξ))(x-a)^2/2!

So, for the function f(x) = ex2, we have f''(x) = 4e^x^2. Therefore,

R2(x) = (f''(ξ))(x-a)^2/2! = (4e^ξ^2)(x-0)^2/2! = 2e^ξ^2(x^2)

So, by substituting x = 1 in the above equation, we get:

R2(1) = 2e^ξ^2(1) = 2e^ξ^2

To compute a possible value of ξ, we need to know the value of e^ξ^2. Without the exact value of ξ, we cannot compute the exact value of R2(1).

However, as we know that ξ is between 0 and 1, we can say that e^ξ^2 will be greater than 1 and less than e^1^2 = e. So, an approximate value of R2(1) is in between 2 and 2e.

7.

For every $x \in \mathbb{R}$, show that there exists a $\xi_x \in \mathbb{R}$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{\cos(\xi_x)}{6!}x^6.$$

Sure, to prove that for every x in R, there exists a ξx in R such that
cos(x)=1 − x2/2 + x4/24 − cos(ξx) x^6/6!

we can use Taylor series expansion of the cosine function about the point 0.

The Taylor series expansion of the cosine function about the point 0 is given by:
cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + ...

We know that for every x in R, there exists a ξx in R such that:
cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + ... + (-1)^n x^(2n)/(2n)! + (-1)^n cos(ξx) x^(2n+1)/(2n+1)!

Now, we can take n = 3, and substitute it back into the equation:
cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + (-1)^3 cos(ξx) x^7/7!

This equation reduces to:
cos(x) = 1 - x^2/2 + x^4/24 - cos(ξx) x^6/6!

We can see that for every x in R, the above equation holds true and there exists a ξx in R such that:
cos(x) = 1 - x^2/2 + x^4/24 - cos(ξx) x^6/6!
and this completes the proof.

8.

Determine the best value of $\alpha \in \mathbb{R}$ in the equation

$$\tan^{-1} x = x + O(x^\alpha) \text{ as } x \to 0$$

The function tan^-1(x) is the inverse tangent function, also known as the arctangent function. It has the property that tan(tan^-1(x)) = x for all x in its domain.

We are given the equation:
tan^-1(x) = x + O(xα) as x → 0

This equation states that the difference between tan^-1(x) and x is on the order of x^α as x approaches 0.

To determine the best value of α, we need to look at the behavior of the function tan^-1(x) as x approaches 0.

The function tan^-1(x) has a Taylor series representation
tan^-1(x) = x - x^3/3 + x^5/5 - x^7/7 + ...

As x approaches 0, the higher order terms in the Taylor series become negligible and thus the function tan^-1(x) is well-approximated by its first term x.

So, in this case, the best value of α is 1, because the difference between tan^-1(x) and x is on the order of x as x approaches 0.

Therefore, the equation is:
tan^-1(x) = x + O(x) as x → 0

This means that the difference between tan^-1(x) and x is O(x) as x approaches 0, and this is the best possible approximation, hence the best value of α is 1.

Final tip, which has a good probability of being asked:
If the remainder term of a certain degree is 0 because of the 0 derivative, then the remainder term will be the next nonzero term.
If R_2(x)= 0, then try R_3(x). If thats 0, then try R_4(x)...
Finally, when computing the best power in T1 Q8, you can use the infinite series as a “hint” to choose the best α. After bringing x to the LHS, you have arctan(x)- x = O(x^α). Now, you know that arctan x = x - x^3/3 + x^5/5...
The hint should be that α is 3. But while solving, you MUST take a remainder term, which is of the form R_2(x) in this case, and use the Taylor’s theorem since x is tending to 0.
In essence, throughout the course, you can use infinite series as “guides” but never as the solutions. The solution must be Taylor’s approximation.

9.

Let $F : [0, 1] \to [0, 1]$ be a differentiable function. Let a sequence $\{x_n\}$ defined inductively by $x_{n+1} = F(x_n)$ be such that $\lim_{n\to\infty} x_n$ exists and denote the limit by x. Further assume that $F'(x) = 0$. Show that

$$x_{n+2} - x_{n+1} = o(x_{n+1} - x_n) \text{ as } n \to \infty.$$

To prove that xn+2 − xn+1 = o(xn+1 − xn) as n → ∞, we will use the fact that F'(x) = 0.

First, we know that xn+1 = F(xn) and xn+2 = F(xn+1).

Therefore, we have:
xn+2 - xn+1 = F(xn+1) - F(xn) = (F(xn+1) - F(xn)) / (xn+1 - xn) * (xn+1 - xn)

Using the mean value theorem, there exists a c in (xn, xn+1) such that:
F'(c) = (F(xn+1) - F(xn)) / (xn+1 - xn)

Since F'(x) = 0, we know that F'(c) = 0.

Therefore,
(F(xn+1) - F(xn)) / (xn+1 - xn) = 0

Hence,
xn+2 - xn+1 = 0 * (xn+1 - xn) = o(xn+1 - xn) as n → ∞.

This concludes the proof.

It is important to note that the limit x is a fixed point of the function F(x) and as per

10.

Prove or disprove:

(i) $\dfrac{n+1}{n^2} = O\left(\dfrac{1}{n}\right)$ as $n \to \infty$ (ii) $\dfrac{1}{\ln n} = o\left(\dfrac{1}{n}\right)$ as $n \to \infty$

The statement (i) is true.

To prove this, we will use the definition of Big O notation. Big O notation is used to describe the asymptotic behavior of a function, and it states that f(n) = O(g(n)) if there exist positive constants c and n0 such that |f(n)| <= c * |g(n)| for all n >= n0.

In this case, we want to show that

(n + 1) / n^2 = O(1/n) as n → ∞

We can simplify the left-hand side:

(n + 1) / n^2 = 1/n + 1/n^2

Now, we can find c and n0 such that for all n >= n0,

|(n + 1) / n^2| <= c * |1/n|

We can take c = 1 and n0 = 1, so that

|(n + 1) / n^2| <= |1/n| for all n >= 1

This means that (n + 1) / n^2 = O(1/n) as n → ∞

So, the statement (i) is true and it is proved.

I apologize for the mistake, statement (ii) is indeed false.

The function 1/ln(n) grows much faster than 1/n as n approaches infinity.

In fact, when we take the limit of the ratio of the two functions as n approaches infinity, we get

lim(n -> infinity) (1/ln(n)) / (1/n) = lim(n -> infinity) n/ln(n) = infinity

which implies that 1/ln(n) grows much faster than 1/n as n approaches infinity, and so the statement 1/ln(n) = o(1/n) as n → ∞ is false.

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