DYNAMIC CHARACTERISTICS OF MEASUREMENT SYSTEMS

Learning outcomes from Dynamic Characteristics of Measurement Systems Module

- Mathematical Modelling of Zero order, first order and second order systems
- Laplace Transforms for solving Initial value problems
- Response characteristics of First order system for inputs like
  - Step input
  - Ramp input
  - Sinusoidal input
  - Impulse input
- Steady state response to a combination of sinusoidal inputs using method of superposition applicable for linear systems
- Fourier Series
  - Representation of square wave, triangular wave, saw tooth wave as combination of sine and cos functions
- Utility of analogy of First order Hydraulic, Pneumatic, Electrical and thermal system

DYNAMIC CHARACTERISTICS OF MEASUREMENT SYSTEM

Relation between an input and output in general can be represented as a differential equation (linear)

$$a_n \frac{d^n q_o}{dt^n} + a_{n-1} \frac{d^{n-1} q_o}{dt^{n-1}} + \cdots + a_1 \frac{dq_o}{dt} + a_o q_o = b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + \cdots + b_1 \frac{dq_i}{dt} + b_o q_i$$

a's and b's are assumed constant

$$q_i(t) \longrightarrow \boxed{\text{Measurement System}} \longrightarrow q_o(t)$$

ZERO ORDER MEASUREMENT SYSTEM

$$a_o q_o = b_o q_i \;\Rightarrow\; q_o = \frac{b_o}{a_o} q_i$$

$\frac{b_o}{a_o}$ **- static Sensitivity**

- **The moment input is changed, output will follow**
- **Ideal or perfect dynamic system**

FIRST ORDER MEASUREMENT SYSTEM

$$a_1 \frac{dq_o}{dt} + a_o q_o = b_o q_i$$

$$\frac{a_1}{a_o} \frac{dq_o}{dt} + q_o = \frac{b_o}{a_o} q_i$$

$$\frac{b_o}{a_o} = K$$

Static sensitivity - has the dimensions of output divided by input

K - amount of output per unit input when input is static – all derivatives are zero

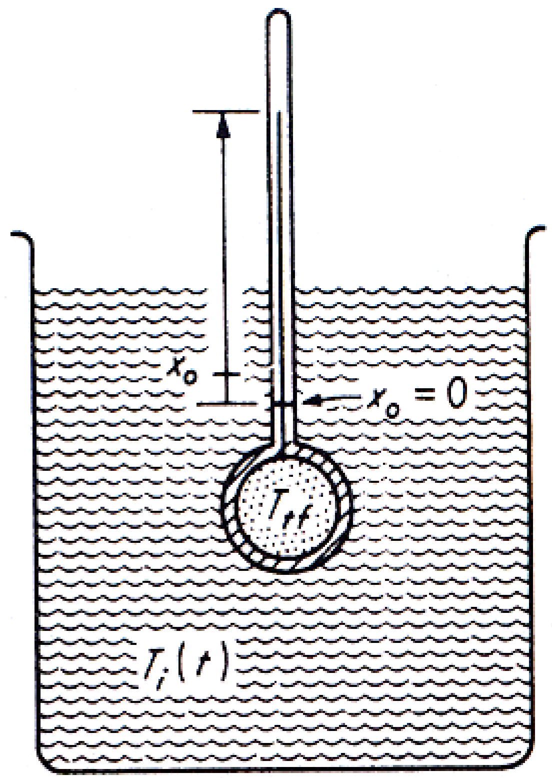
$$\frac{a_1}{a_o} = \tau$$

Time constant – has the dimensions of time

$$\tau \frac{dq_o}{dt} + q_o = K q_i$$

When $\tau = 0$; system reduces to zero order system

MERCURY IN GLASS THERMOMETER



$$\frac{\Delta V}{V} = \beta \Delta T$$

β - coefficient of volume expansion
V – Volume of the bulb
A_c - Cross sectional area of the bulb

$$A_c \Delta x = \beta V \Delta T$$

$$\Delta x = \frac{\beta V}{A_c} \Delta T \Rightarrow x - x_o = \frac{\beta V}{A_c}(T - T_{in})$$

$$Assume\ that\ at\ x_o = 0, T_{in} = 0 \ \Rightarrow x = \frac{\beta V}{A_c} T$$

$$T = \frac{A_c}{\beta V} x$$

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

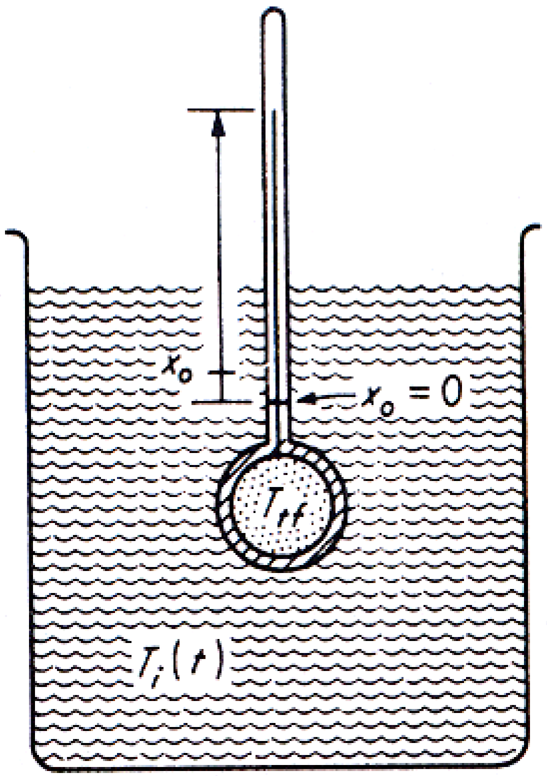
$$hA_s T_f = hA_s \frac{A_c}{\beta V} x + \rho V C_p \frac{d}{dt}\left(\frac{A_c}{\beta V} x\right)$$

$$hA_s(T_f - T) = \rho V C_p \frac{dT}{dt}$$

$$hA_s T_f = hA_s T + \rho V C_p \frac{dT}{dt}$$

h - Heat transfer coefficient
A_s - Bathing area of the bulb

MERCURY IN GLASS THERMOMETER



$$hA_sT_f = hA_s\frac{A_c}{\beta V}x + \rho VC_p\frac{d}{dt}\left(\frac{A_c}{\beta V}x\right)$$

$$T_f = \frac{A_c}{\beta V}x + \frac{\rho VC_p}{hA_s}\frac{A_c}{\beta V}\frac{dx}{dt}$$

$$\frac{\beta V}{A_c}T_f = x + \frac{\rho VC_p}{hA_s}\frac{dx}{dt}$$

$$\frac{\rho VC_p}{hA_s}\frac{dx}{dt} + x = \frac{\beta V}{A_c}T_f$$

$$\tau\frac{dx}{dt} + x = KT_f$$

$$\tau\frac{dq_o}{dt} + q_o = Kq_i$$

$$\tau = \frac{\rho VC_p}{hA_s}seconds$$

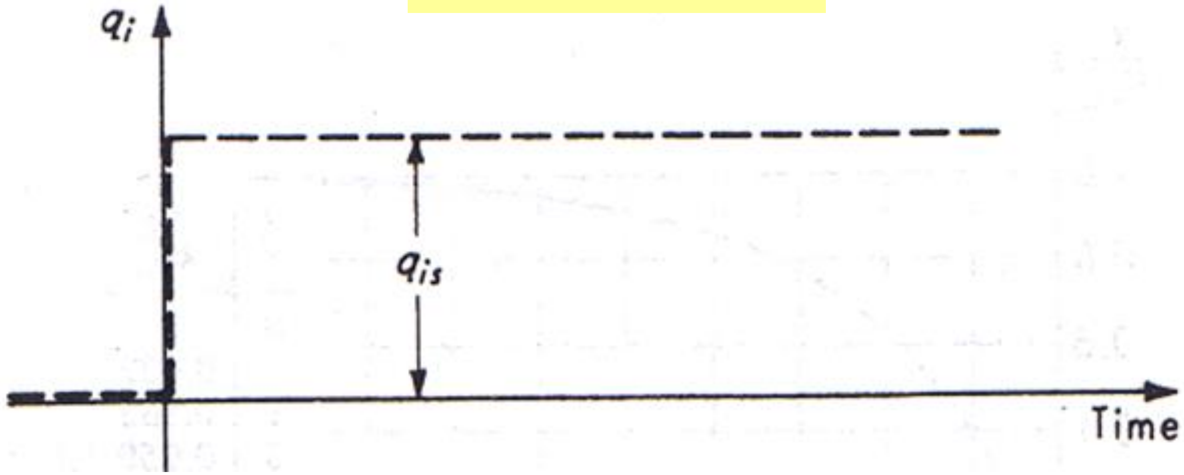
$$K = \frac{\beta V}{A_c}\frac{m}{degC}$$

K – Static Sensitivity – has the dimensions of output divided by input

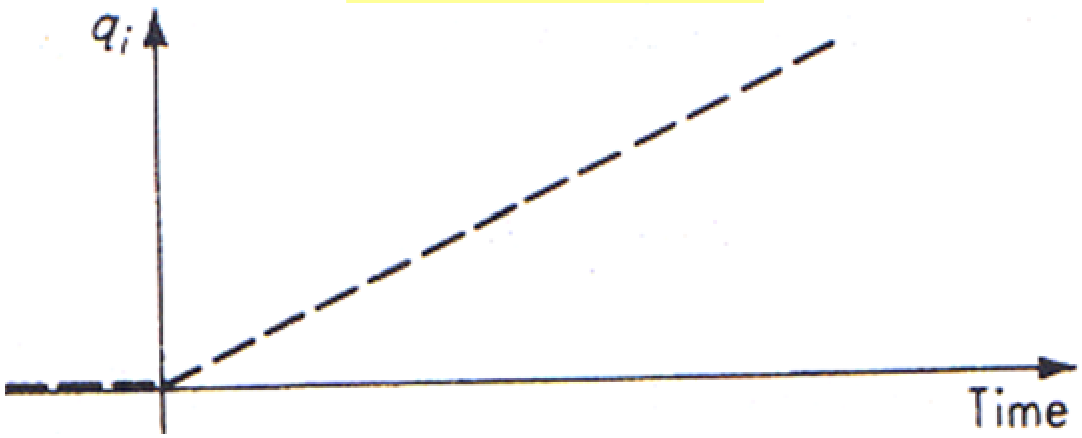
τ – Time constant – has the dimensions of time

TYPES OF INPUTS

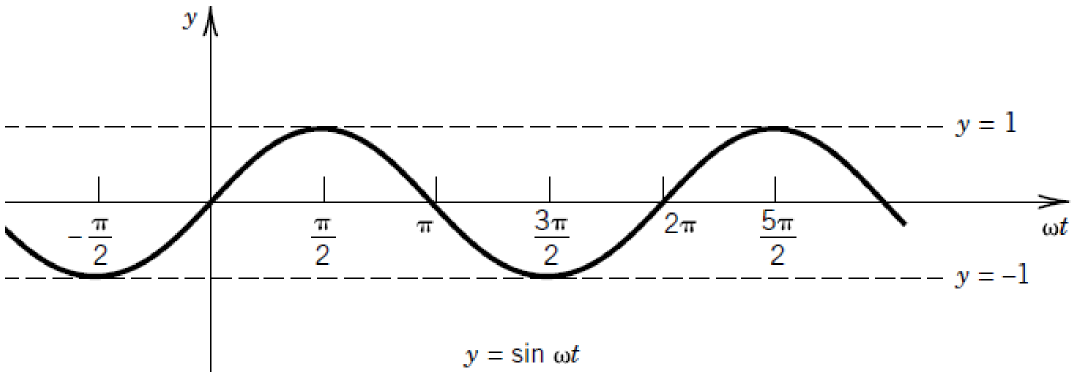
Step Input



Ramp Input

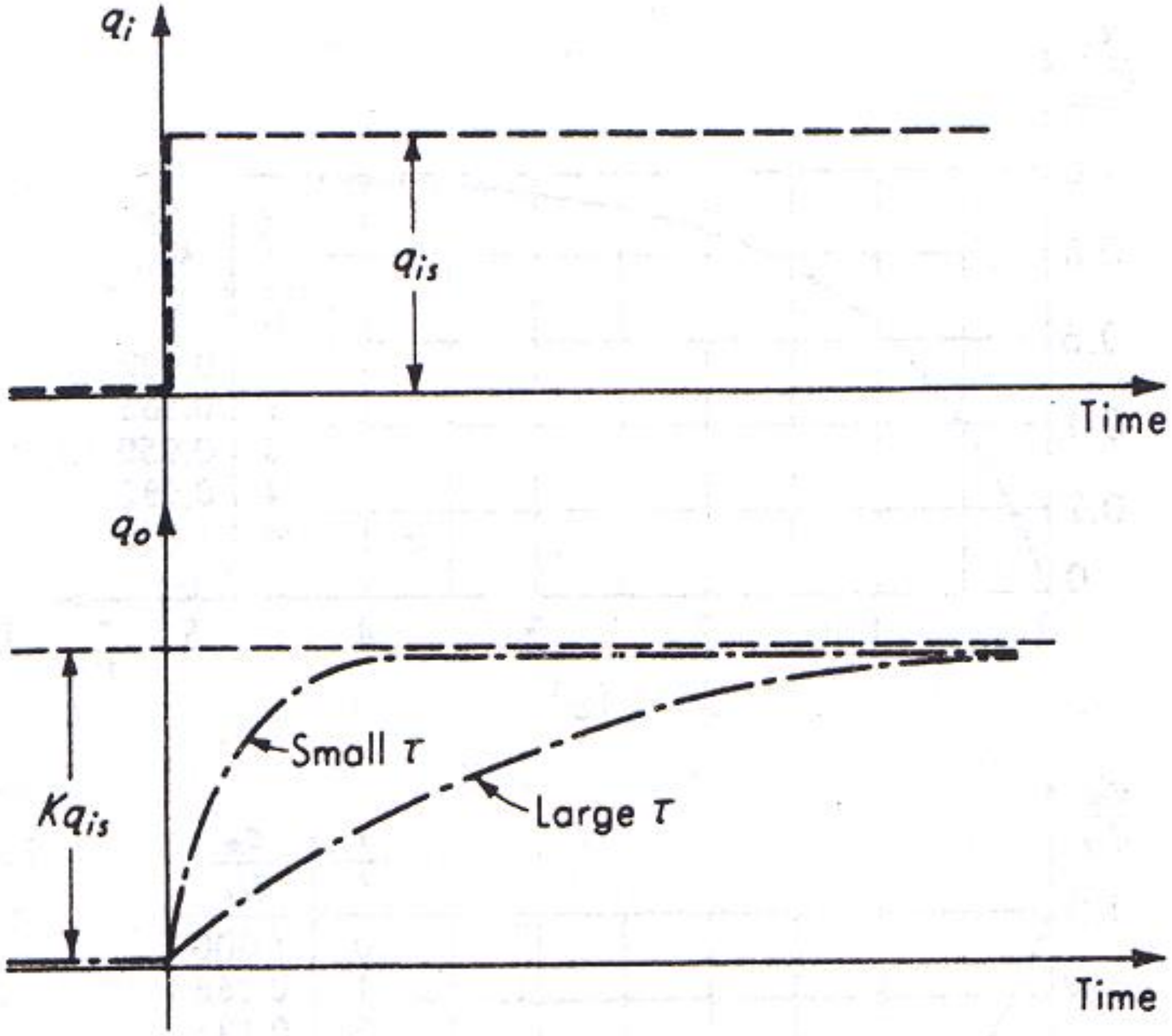


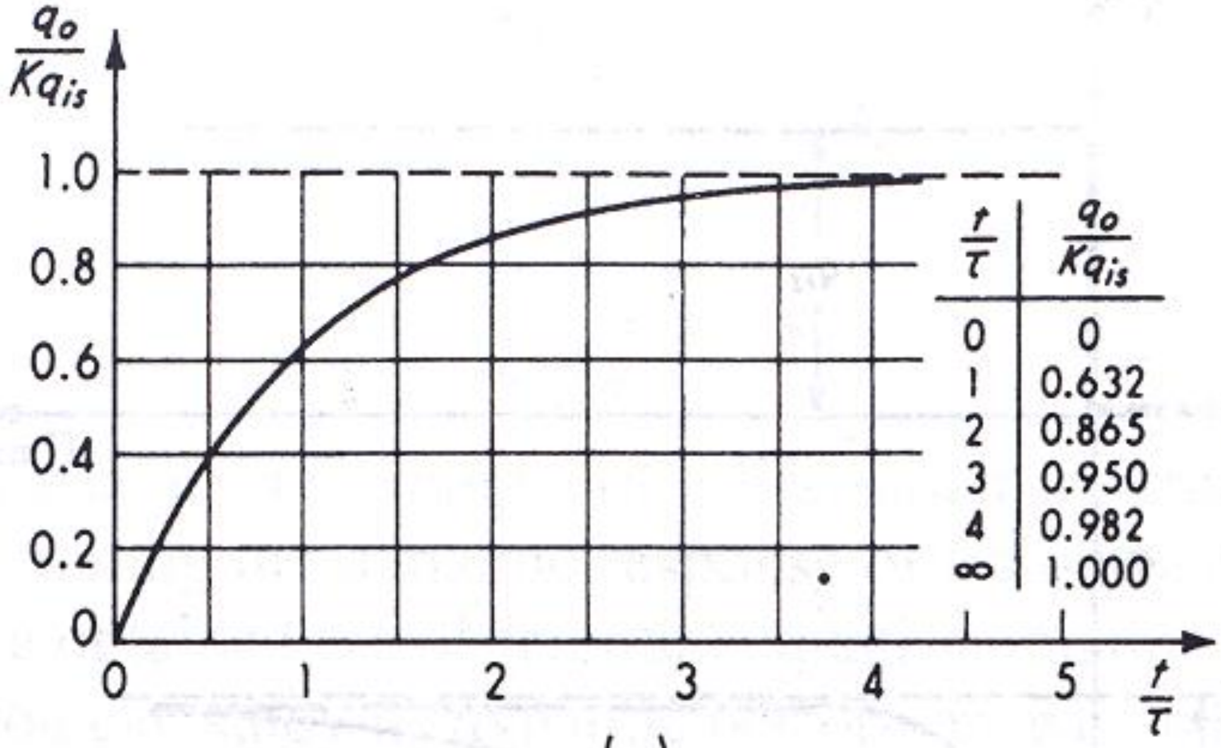
Sinusoidal Input



Impulse Input

STEP RESPONSE OF FIRST ORDER SYSTEM





(a)

$$\tau \frac{dq_o}{dt} + q_o = Kq_{is}$$

$$\tau\frac{dq_o}{dt}+q_o=Kq_{is}$$

$$\mathcal{L}\left(\frac{dq_o(t)}{dt}\right)=sq_o(s)-q_o(0)$$

$$\tau[sq_o(s)-\cancel{q_o(0)}]+q_o(s)=\frac{Kq_{is}}{s}$$

$$\mathcal{L}(q_o(t))=q_o(s)$$

$$q_o(s)[\tau s+1]=\frac{Kq_{is}}{s}$$

$$\mathcal{L}(Kq_{is})=\frac{Kq_{is}}{s}$$

$$\frac{q_o(s)}{q_{is}}=\frac{K}{s(\tau s+1)}=K\left(\frac{A}{s}+\frac{B}{(\tau s+1)}\right)$$

$$\frac{q_o(s)}{q_{is}}=\frac{K}{s(\tau s+1)}=K\left(\frac{A(\tau s+1)+Bs}{s(\tau s+1)}\right)$$

$$K=K(A(\tau s+1)+Bs)$$

$$\frac{q_o(s)}{q_{is}}=K\left(\frac{1}{s}+\frac{-\tau}{(\tau s+1)}\right)$$

$$1=A(\tau s+1)+Bs$$

$$s=0\ \Rightarrow 1=A(\tau(0)+1)+B(0)\Rightarrow A=1$$

$$s=1\ \Rightarrow 1=1(\tau(1)+1)+B(1)\Rightarrow 1=\tau+1+B\Rightarrow B=-\tau$$

$$\frac{q_o(s)}{q_{is}} = K\left(\frac{1}{s} + \frac{-\tau}{(\tau s + 1)}\right)$$

$$\frac{q_o(s)}{q_{is}} = K\left(\frac{1}{s} - \frac{1}{\left(s + \frac{1}{\tau}\right)}\right)$$

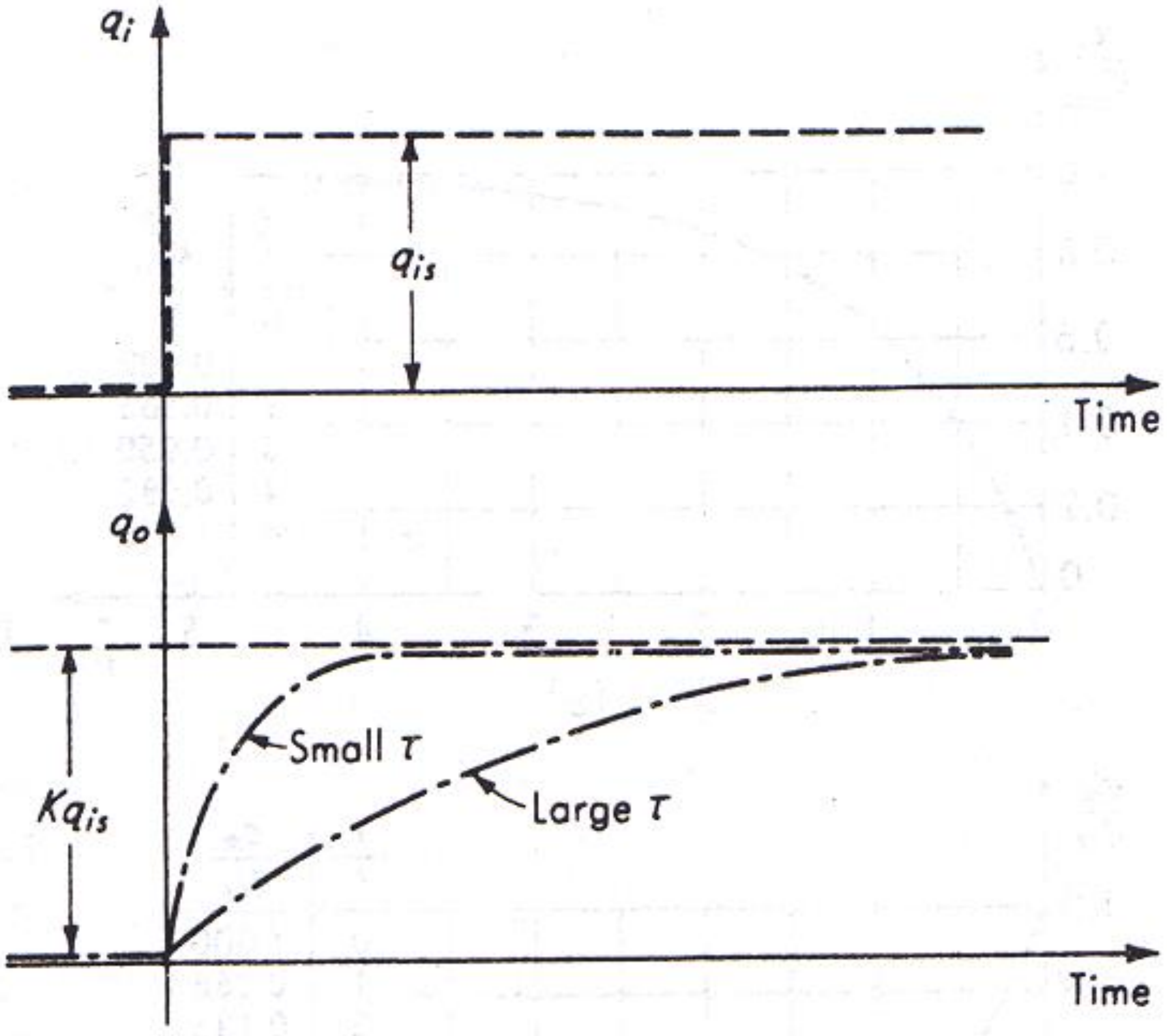
$$\frac{1}{K}\mathcal{L}^{-1}\left(\frac{q_o(s)}{q_{is}}\right) = \frac{1}{K}\mathcal{L}^{-1}\left(K\left(\frac{1}{s} - \frac{1}{\left(s + \frac{1}{\tau}\right)}\right)\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{\tau}}\right)$$

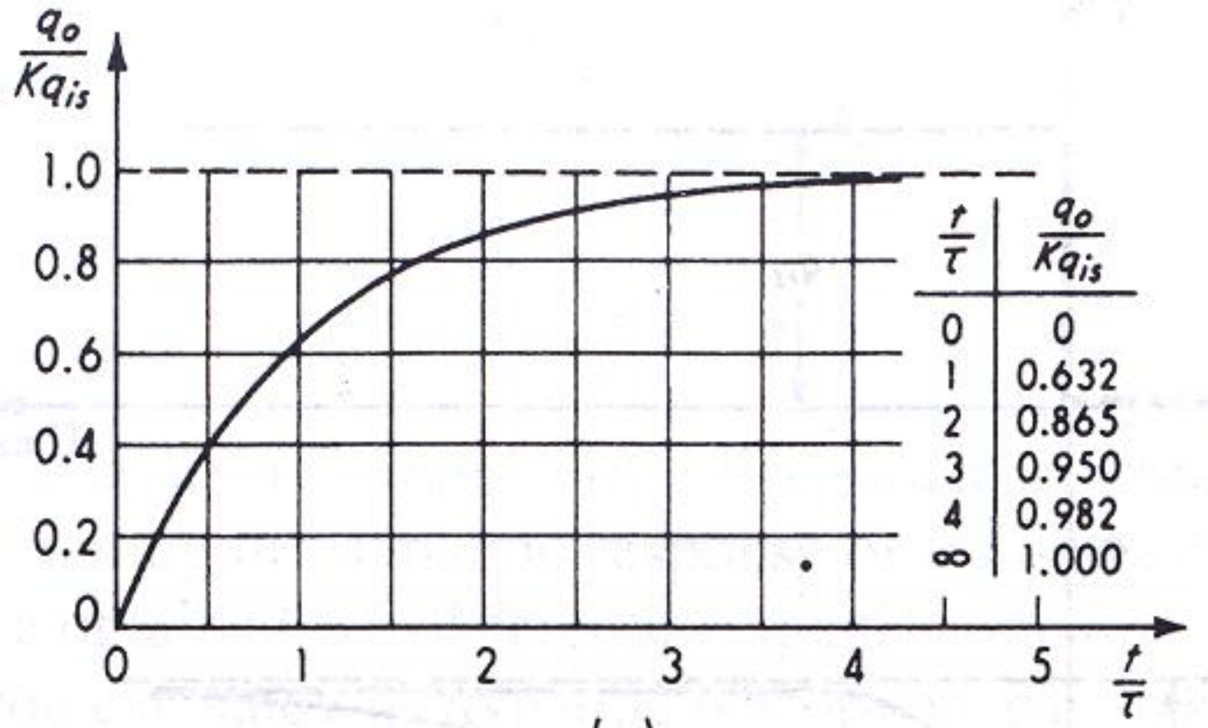
$$\frac{1}{K}\frac{q_o(t)}{q_{is}} = 1 - e^{-\frac{t}{\tau}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s - a}$$

$$\frac{q_o(t)}{q_{is}} = K\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\mathcal{L}(1) = \frac{1}{s}$$





(a)

$$\frac{q_o(t)}{q_{is}} = K\left(1 - e^{-\frac{t}{\tau}}\right)$$

- **Response speed depends only on the value of τ**
- **Good measurements using first order instruments must minimize "τ"**
- **Usually, three time constants are acceptable**

$$\frac{\rho V C_p}{h A_s}\frac{dx}{dt} + x = \frac{\beta V}{A_c} T_f$$

$$\tau = \frac{\rho V C_p}{h A_s}\, seconds$$

$$K = \frac{\beta V}{A_c}\; \frac{m}{degC}$$

$$\tau \frac{dx}{dt} + x = K T_f$$

Properties	C_p (J/kg.K)	ρ (kg/m^3)	h W/m.K	$\frac{V}{A_s} = \frac{\frac{4}{3}\pi R^3}{4\pi R^2} = \frac{R}{3}$	τ s
Mercury	137	13579	30	$\frac{1 \times 10^{-3}}{3}$	20
Alcohol	2.47	790	30	$\frac{1 \times 10^{-3}}{3}$	0.02

Time constant may be reduced by

1. Reduce ρ, V and C_p
2. Increase h and A_s

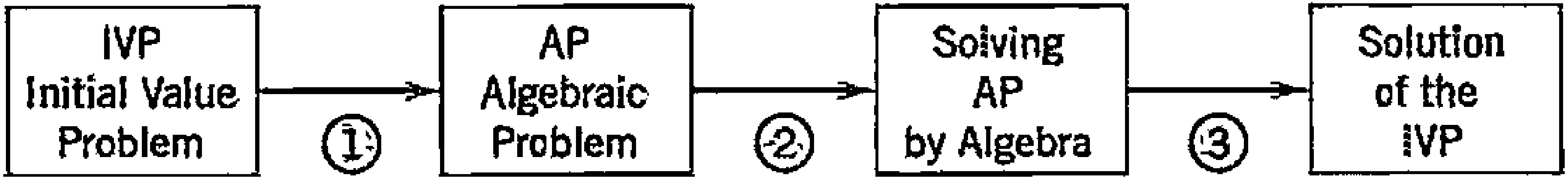
Thermometer in stirred oil - τ = 5 seconds

Thermometer in stagnant air - τ = 100 seconds

LAPLACE TRANSFORMS

The Laplace transform method is a powerful method for solving linear ODEs and corresponding initial value problems, as well as system of ODEs arising in engineering.

The process of solution consists of three steps



Step 1: The given ODE is transformed into an algebraic equation ("subsidiary equation").

Step 2: The subsidiary equation is solved by purely algebraic manipulations.

Step 3: The solution in Step 2 is transformed back, resulting in the solution of the given problem.

LAPLACE TRANSFORMS

If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral of $f(t)$ times e^{-st} from t = 0 to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$; thus

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt$$

The given function $f(t)$ in the above equation is called the inverse transform of $F(s)$and is denoted by $\mathcal{L}^{-1}(F)$. That is, we shall write it as

$$f(t) = \mathcal{L}^{-1}(F)$$

Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}\left(e^{-\infty} - e^0\right) = -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$\mathcal{L}(1) = \frac{1}{s}$$

Let $f(t) = e^{at}$ when $t \geq 0$. Find $F(s)$

$$\mathcal{L}(f) = \mathcal{L}(e^{at}) = \int_0^\infty e^{-st}\, e^{at} dt = \int_0^\infty e^{-(s-a)t}\, dt = -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^\infty$$

$$\mathcal{L}(f) = -\frac{1}{s-a}\left(e^{-\infty} - e^{0}\right) = -\frac{1}{s-a}(0-1) = \frac{1}{s-a}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

Let $f(t) = sinat$ when $t \geq 0$. Find $F(s)$

$$\mathcal{L}(f) = \mathcal{L}(sinat) = \int_0^{\infty} e^{-st}\, sinat\, dt = \frac{e^{-st}}{s^2+a^2}(-ssinat - acosat)\Big|_0^{\infty}$$

$$\mathcal{L}(f)=\left(\frac{e^{-\infty}}{s^2+a^2}(-ssin\infty - acos\infty)\right) - \left(\frac{e^{-0}}{s^2+a^2}(-ssin0 - acos0)\right)$$

$$\mathcal{L}(f)=\left(\frac{0}{s^2+a^2}(-ssin\infty - acos\infty)\right) - \left(\frac{1}{s^2+a^2}(-s(0) - a(1))\right)$$

$$\mathcal{L}(f)=0 - \left(\frac{1}{s^2+a^2}(0 - a)\right)$$

$$\mathcal{L}(sinat) = \frac{a}{s^2+a^2}$$

$$\int e^{ax}\, sinbxdx = \frac{e^{ax}}{a^2+b^2}(asinbx + bcosbx) + C$$

Let $f(t) = sinat$ when $t \geq 0$. Find $F(s)$

$$\mathcal{L}(f) = \mathcal{L}(sinat) = \int_0^\infty e^{-st}\, sinat\, dt = \frac{e^{-st}}{s^2+a^2}(-ssinat - acosat)\Big|_0^\infty$$

$$\mathcal{L}(f)=\left(\frac{e^{-\infty}}{s^2+a^2}(-ssin\infty - acos\infty)\right) - \left(\frac{e^{-0}}{s^2+a^2}(-ssin0 - acos0)\right)$$

$$\mathcal{L}(f)=\left(\frac{0}{s^2+a^2}(-ssin\infty - acos\infty)\right) - \left(\frac{1}{s^2+a^2}(-s(0) - a(1))\right)$$

$$\mathcal{L}(f)=0 - \left(\frac{1}{s^2+a^2}(0 - a)\right)$$

$$\mathcal{L}(sinat) = \frac{a}{s^2+a^2}$$

Let $F(s) = \frac{1}{s-a}$ Find $f(t)$

$$f(t) = \mathcal{L}^{-1}(F)$$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)$$

$$f(t) = e^{at}$$

Let $F(s) = \frac{a}{s^2+a^2}$ Find $f(t)$

$$f(t) = \mathcal{L}^{-1}(F)$$

$$f(t) = \mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$f(t) = sinat$$

INTEGRATION BY PARTS

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx \;-\; \int\left[f'(x)\int g(x)\,dx\right]dx$$

The integral of the product of two functions $=$ **(first function) $\times$ (integral of the second function)** $+$ **Integral of [(differential coefficient of the first function) $\times$ (integral of the second function)]**

LINEARITY OF A LAPLACE TRANSFORM

$$\mathcal{L}\{af(t)+bf(t)\}=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$$

$$coshat=\frac{e^{at}+e^{-at}}{2}$$

$$\mathcal{L}(coshat)=\frac{1}{2}\mathcal{L}(e^{at})+\frac{1}{2}\mathcal{L}(e^{-at})$$

$$\mathcal{L}(coshat)=\frac{1}{2}\left(\frac{1}{s-a}\right)+\frac{1}{2}\left(\frac{1}{s+a}\right)$$

$$\mathcal{L}(coshat)=\frac{1}{2}\left(\frac{(s+a)+(s-a)}{(s+a)(s-a)}\right)$$

$$\mathcal{L}(coshat)=\frac{s}{s^2-a^2}$$

$$f(t)=\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right)=coshat$$

SOME FUNCTIONS AND THEIR TRANSFORMS

SL No.	$f(t)$	$\mathcal{L}(f(t))$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	t^n (n = 0,1,2,...)	$\frac{n!}{s^{n+1}}$
5	t^a (a is positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$

SL No.	$f(t)$	$\mathcal{L}(f(t))$
7	$cos\omega t$	$\frac{s}{s^2+\omega^2}$
8	$sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$coshat$	$\frac{s}{s^2-a^2}$
10	$sinhat$	$\frac{a}{s^2-a^2}$
11	$e^{at}cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at}sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

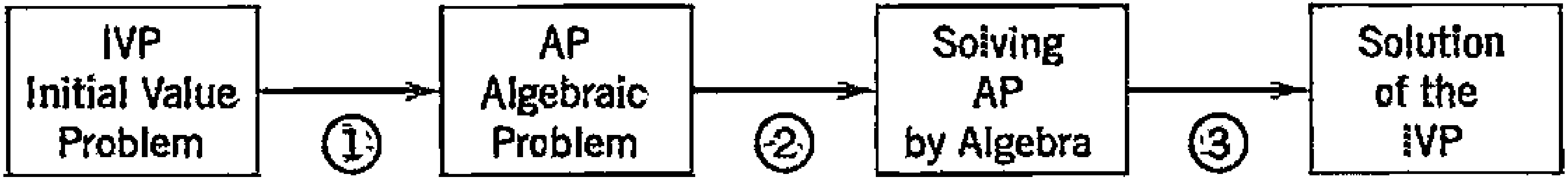
LAPLACE TRANSFORM OF DERIVATIVES

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

INITIAL VALUE PROBLEM : THE BASIC LAPLACE STEPS



Step 1: The given ODE is transformed into an algebraic equation ("subsidiary equation").

$$\tau \frac{dq_o}{dt} + q_o = Kq_{is}$$

$$\mathcal{L}\left(\frac{dq_o(t)}{dt}\right) = sq_o(s) - q_o(0)$$

$$\mathcal{L}\big(q_o(t)\big) = q_o(s)$$

$$\mathcal{L}(Kq_{is}) = \frac{Kq_{is}}{s}$$

$$\tau[sq_o(s) - \cancel{q_o}(0)] + q_o(s) = \frac{Kq_{is}}{s}$$

$$q_o(s)[\tau s + 1] = \frac{Kq_{is}}{s}$$

S.No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\dfrac{A}{x-a} + \dfrac{B}{x-b}$
2.	$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$
4.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b}$
5.	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+bx+c}$,
	where $x^2 + bx + c$ cannot be factorised further	

Step 2: **The subsidiary equation is solved by purely algebraic manipulations.**

$$\frac{q_o(s)}{q_{is}} = \frac{K}{s(\tau s+1)} = K\left(\frac{A}{s} + \frac{B}{(\tau s+1)}\right)$$

$$\frac{q_o(s)}{q_{is}} = \frac{K}{s(\tau s+1)} = K\left(\frac{A(\tau s+1) + Bs}{s(\tau s+1)}\right)$$

$$K = K(A(\tau s+1) + Bs)$$

$$1 = A(\tau s+1) + Bs$$

$$s = 0 \;\Rightarrow 1 = A(\tau(0)+1) + B(0) \Rightarrow A = 1$$

$$s = 1 \;\Rightarrow 1 = 1(\tau(1)+1) + B(1) \Rightarrow 1 = \tau + 1 + B \Rightarrow B = -\tau$$

$$\frac{q_o(s)}{q_{is}} = K\left(\frac{1}{s} + \frac{-\tau}{(\tau s+1)}\right)$$

Step 3: **The solution in Step 2 is transformed back, resulting in the solution of the given problem.**

$$\frac{q_o(s)}{q_{is}} = K\left(\frac{1}{s} + \frac{-\tau}{(\tau s + 1)}\right) \Rightarrow \frac{q_o(s)}{q_{is}} = K\left(\frac{1}{s} - \frac{1}{\left(s + \frac{1}{\tau}\right)}\right)$$

$$\frac{1}{K}\mathcal{L}^{-1}\left(\frac{q_o(s)}{q_{is}}\right) = \frac{1}{K}\mathcal{L}^{-1}\left(K\left(\frac{1}{s} - \frac{1}{\left(s + \frac{1}{\tau}\right)}\right)\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{\tau}}\right)$$

$$\frac{1}{K}\frac{q_o(t)}{q_{is}} = 1 - e^{-\frac{t}{\tau}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s - a}$$

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\frac{q_o(t)}{q_{is}} = K\left(1 - e^{-\frac{t}{\tau}}\right)$$

LAPLACE TRANSFORM METHOD

t-space

s-space

Given Problem

$$\tau \frac{dq_o}{dt} + q_o = Kq_{is}$$

Subsidiary equation

$$\frac{q_o(s)}{q_{is}} = \frac{K}{s(\tau s + 1)}$$

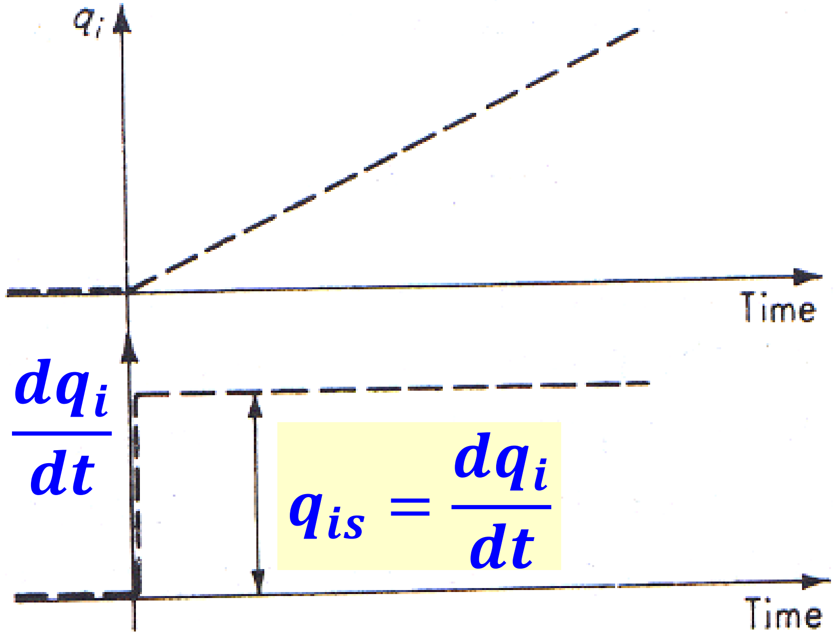
Solution of the Subsidiary equation

$$\frac{q_o(s)}{q_{is}} = K\left(\frac{1}{s} - \frac{\tau}{(\tau s + 1)}\right)$$

Solution of the given problem

$$\frac{q_o(t)}{q_{is}} = K\left(1 - e^{-\frac{t}{\tau}}\right)$$

RAMP RESPONSE OF FIRST ORDER SYSTEM



$$\frac{dq_i}{dt}$$

$$q_{is} = \frac{dq_i}{dt}$$

$$q_i = 0 \quad t \le 0$$

$$q_i = q_{is}t \quad t \ge 0$$

$$\tau\frac{dq_o}{dt} + q_o = Kq_{is}t$$

$$\tau[sq_o(s) - \cancel{q_o}(0)] + q_o(s) = \frac{Kq_{is}}{s^2}$$

$$\frac{q_o(s)}{q_{is}} = \frac{K}{s^2(\tau s + 1)} = K\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(\tau s + 1)}\right)$$

$$\mathcal{L}\left(\frac{dq_o(t)}{dt}\right) = sq_o(s) - q_o(0)$$

$$\mathcal{L}(q_o(t)) = q_o(s)$$

$$\mathcal{L}(Kq_{is}t) = \frac{Kq_{is}}{s^2}$$

$$\frac{q_o(s)}{q_{is}} = \frac{K}{s^2(\tau s + 1)} = K\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(\tau s + 1)}\right)$$

$$\frac{1}{s^2(\tau s + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(\tau s + 1)}$$

$$1 = As(\tau s + 1) + B(\tau s + 1) + \ Cs^2$$

$$Coefficient\, of s^2; \quad 0 = A\tau + C \Rightarrow \ C = -A\tau \Rightarrow \ C = -(-\tau)(\tau\,) \Rightarrow \ C = \tau^2$$

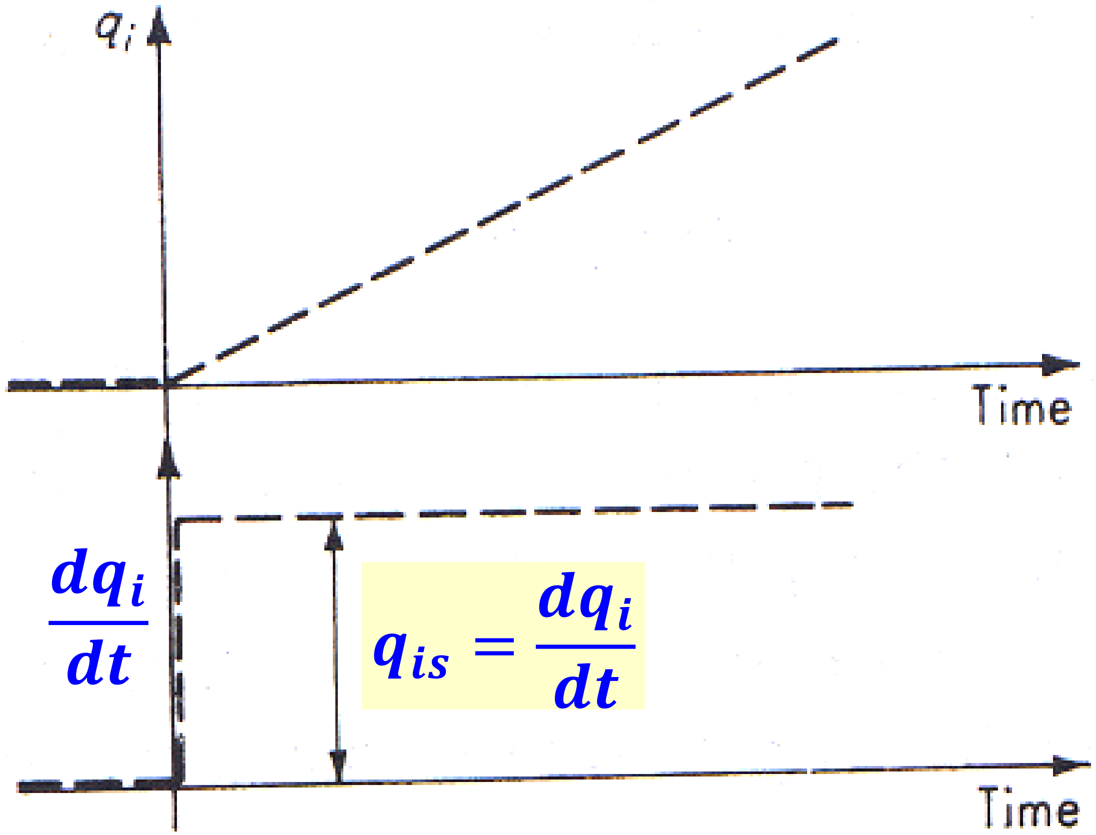
$$Coefficient\, of s; \quad 0 = A + B\tau \Rightarrow \ A = \ -B\tau \Rightarrow \ A = \ (1)(-\tau\,) \Rightarrow \ A = -\tau$$

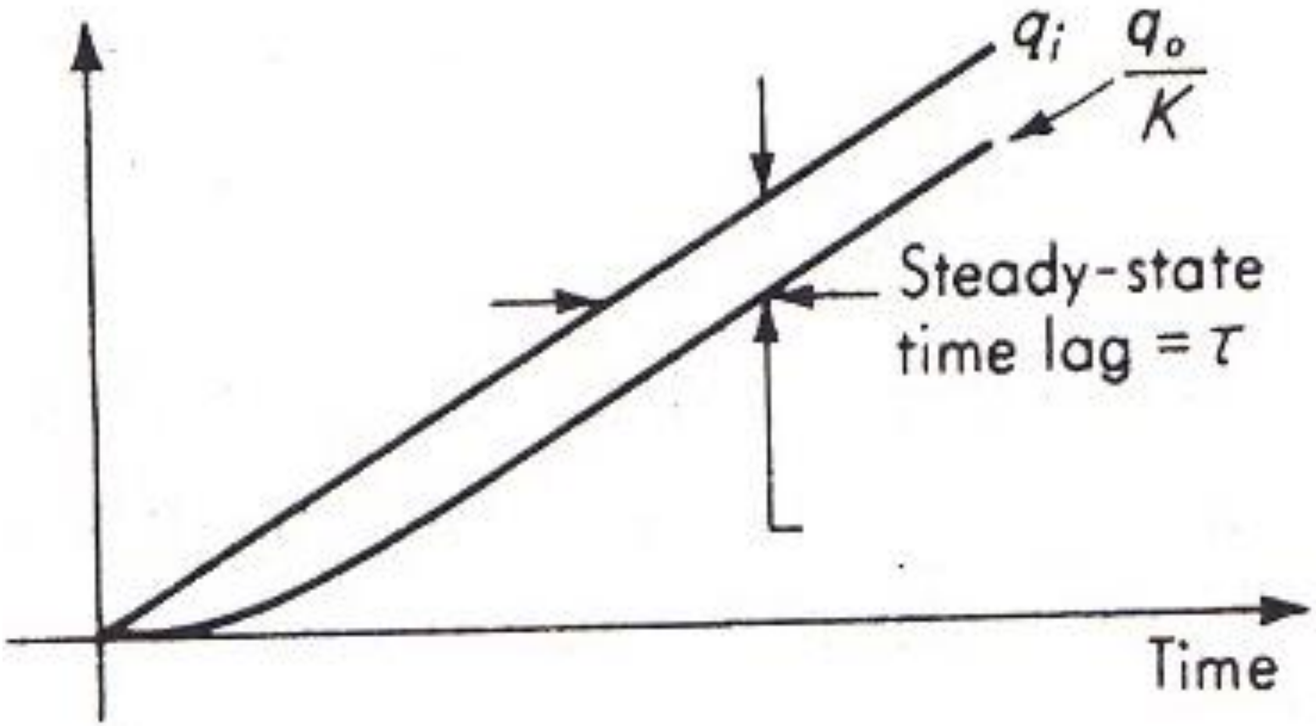
$$Coefficient\, of s^0; \quad 1 = B$$

$$\frac{q_o(s)}{q_{is}} = \frac{K}{s^2(\tau s + 1)} = K\left(\frac{-\tau}{s} + \frac{1}{s^2} + \frac{\tau^2}{(\tau s + 1)}\right) = K\left(-\frac{\tau}{s} + \frac{1}{s^2} + \frac{\tau^2}{\tau\left(s + \frac{1}{\tau}\right)}\right)$$

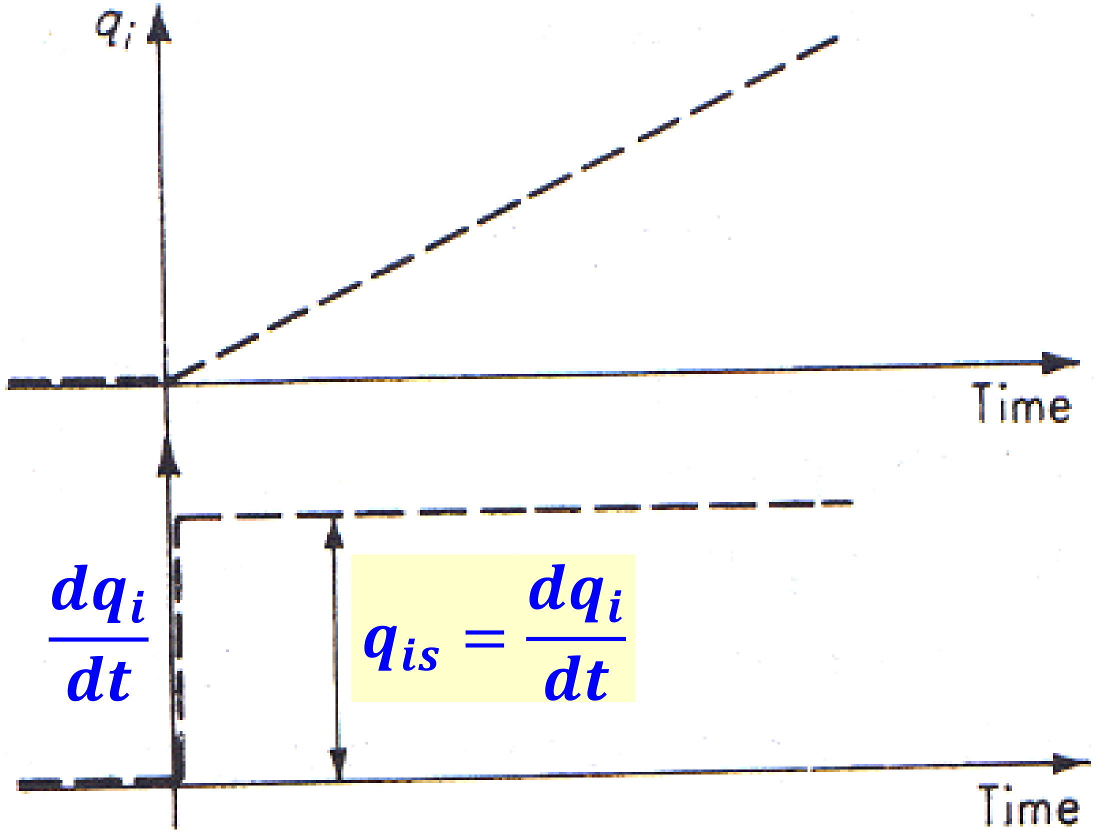
$$\frac{q_o(s)}{q_{is}} = \frac{K}{s^2(\tau s+1)} = K\left(\frac{-\tau}{s} + \frac{1}{s^2} + \frac{\tau^2}{(\tau s+1)}\right) = K\left(-\frac{\tau}{s} + \frac{1}{s^2} + \frac{\tau^2}{\tau\left(s+\frac{1}{\tau}\right)}\right)$$

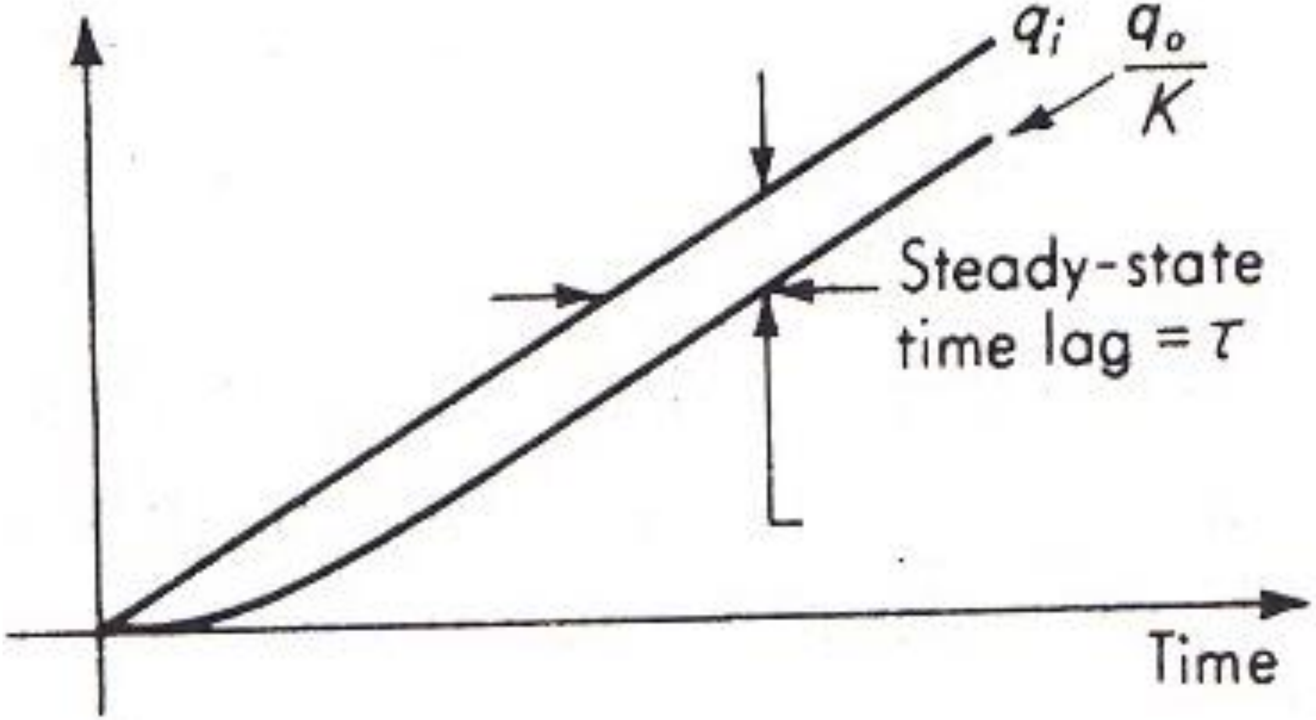
$$q_o(t) = Kq_{is}\left(-\tau + t + \tau e^{-\frac{t}{\tau}}\right)$$





Even after steady state is reached, the output is always lesser compared to input except a case where time constant is very less.





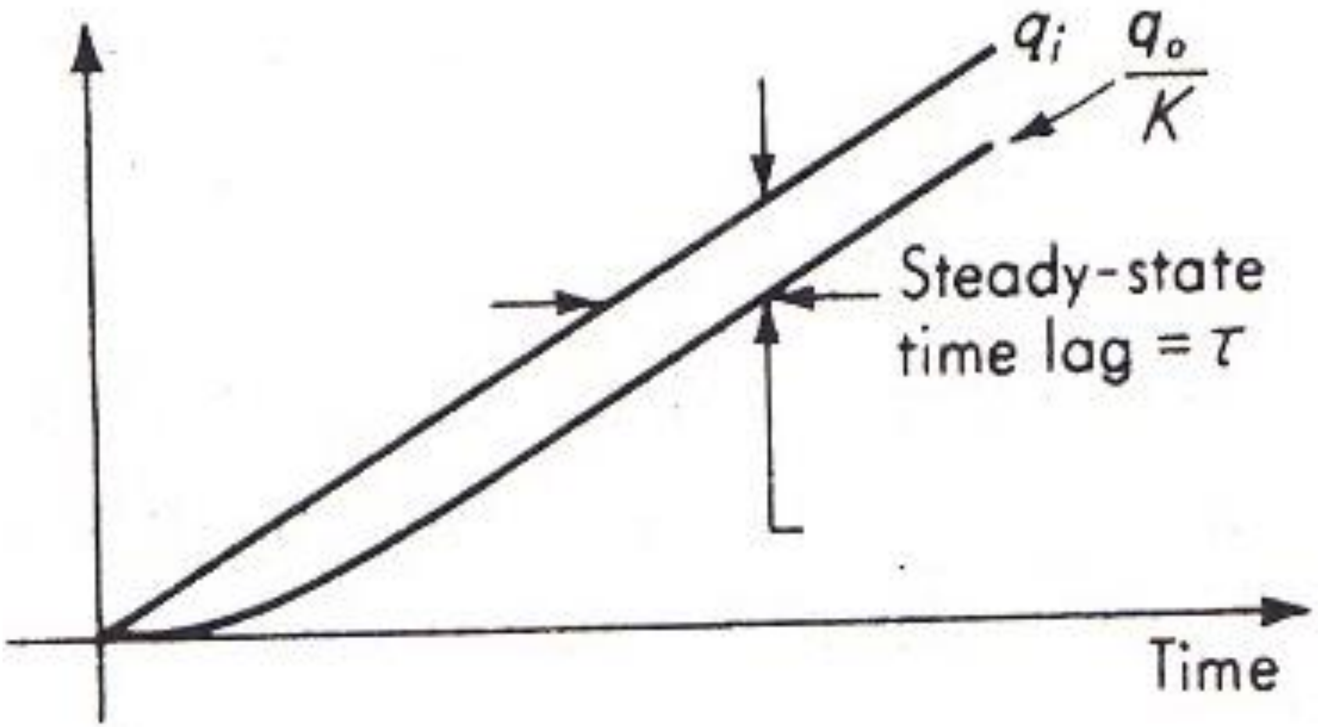
$$q_o(t) = \underbrace{K q_{is} \tau e^{-\frac{t}{\tau}}}_{Unsteady\ Portion} + \underbrace{K q_{is}(t-\tau)}_{Steady\ State\ Portion}$$

$$e_{mo}(t) = q_i - \frac{q_o}{K} = q_{is}t - \frac{K q_{is}\left(-\tau + t + \tau e^{-\frac{t}{\tau}}\right)}{K} = q_{is}\,\tau - q_{is}\tau e^{-\frac{t}{\tau}}$$

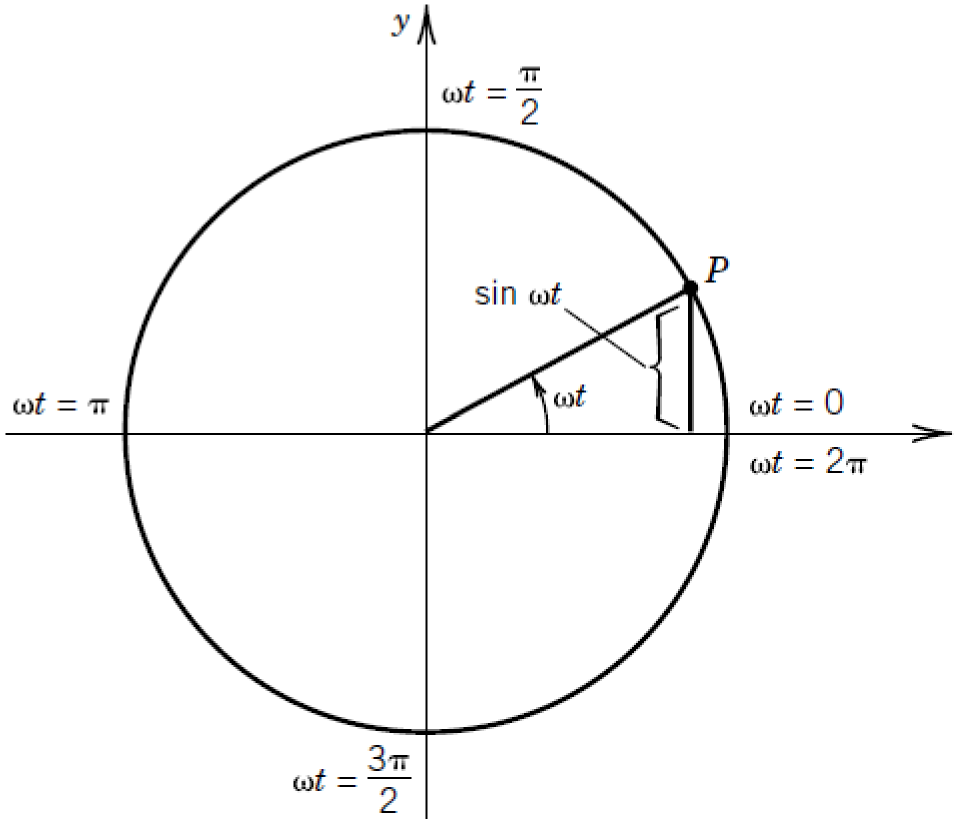
$$e_{mo}(t) = q_{is}\,\tau - q_{is}\tau e^{-\frac{t}{\tau}}$$

$$e_{mo}(t) = q_{is}\,\tau - q_{is}\tau e^{-\frac{t}{\tau}}$$

$$e_{mo}(t) = \underbrace{-q_{is}\tau e^{-\frac{t}{\tau}}}_{\text{Transient error}} + \underbrace{q_{is}\,\tau}_{\text{Steady State Error}}$$



FREQUENCY, PERIOD AND CIRCULAR FREQUENCY



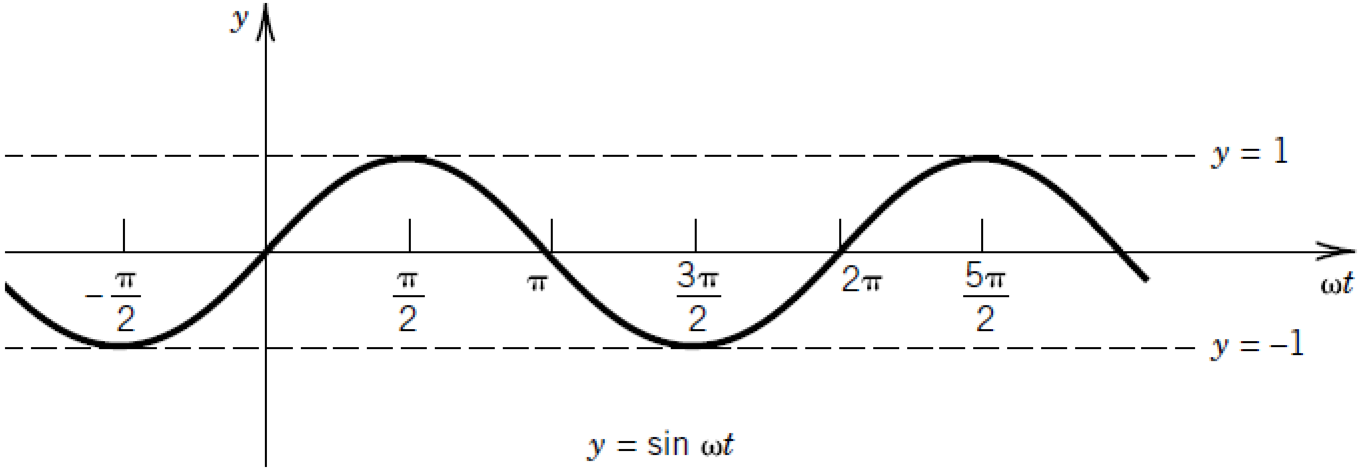
Frequency is related to the period and is defined as the number of complete cycles of the motion per unit time.

This frequency, f, is measured in cycles per second (Hz; 1 cycle/s = 1 Hz).

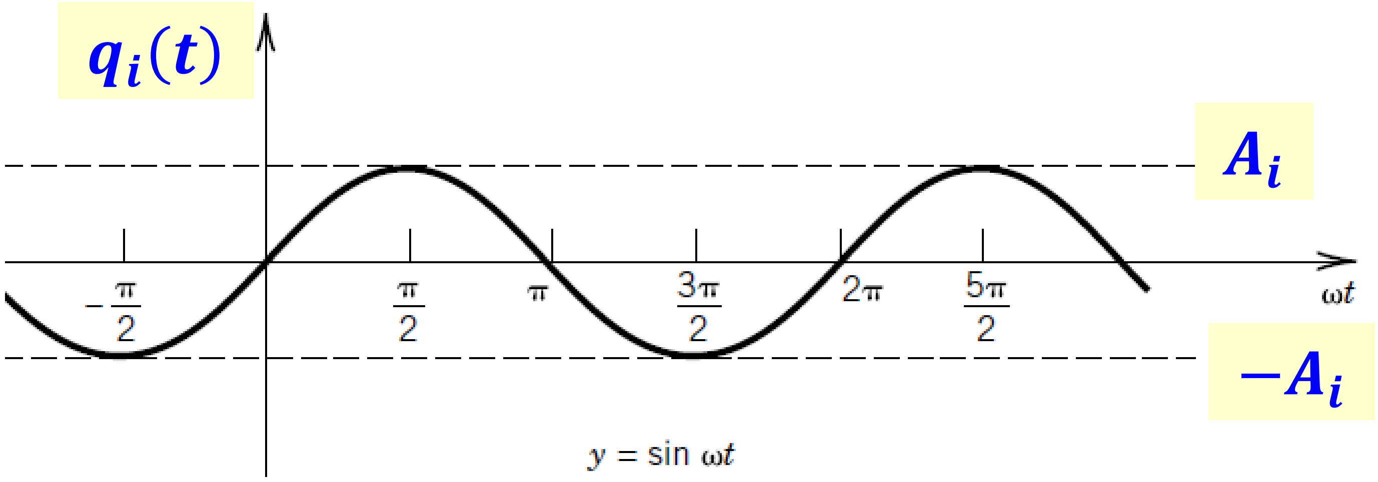
The term ω is also a frequency, but instead of having units of cycles per second it has units of radians per second.

This frequency, ω, is called the circular frequency since it relates directly to cycles on the unit circle

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$



FREQUENCY RESPONSE OF FIRST ORDER SYSTEM : RESPONSE TO SINUSOIDAL INPUT



$$q_i(t) = A_i Sin\omega t$$

$$\tau \frac{dq_o}{dt} + q_o = KA_i Sin\omega t$$

$$\mathcal{L}\left(\frac{dq_o(t)}{dt}\right) = sq_o(s) - q_o(0)$$

$$\mathcal{L}(q_o(t)) = q_o(s)$$

$$\mathcal{L}(KA_i Sin\omega t) = KA_i \frac{\omega}{s^2 + \omega^2}$$

$$\tau[sq_o(s) - q_o(0)] + q_o(s) = KA_i \frac{\omega}{s^2 + \omega^2}$$

$$\frac{q_o(s)}{KA_i} = \frac{\omega}{s^2 + \omega^2} \frac{1}{(\tau s + 1)} = \frac{As + B}{s^2 + \omega^2} + \frac{C}{(\tau s + 1)}$$

$$(As + B)(\tau s + 1) + C(s^2 + \omega^2) = \omega$$

$$A\tau s^2 + B\tau s + As + B + Cs^2 + C\omega^2 = \omega$$

$$A\tau s^2 + B\tau s + As + B + Cs^2 + C\omega^2 = \omega$$

$Coefficient\ of\ s^2;$ $\quad 0 = A\tau + C \Rightarrow \ C = -A\tau \Rightarrow \ C = -(-B\tau)(\tau) \Rightarrow \ C = B\tau^2$

$Coefficient\ of\ s;$ $\quad 0 = A + B\tau \Rightarrow \ A = \ -B\tau$

$Coefficient\ of\ s^0;$ $\quad \omega = B + C\,\omega^2 \Rightarrow \omega = B + B\tau^2\,\omega^2 \Rightarrow \ B = \frac{\omega}{1+\tau^2\omega^2}$

$$C = B\tau^2 = \frac{\omega\tau^2}{1+\tau^2\omega^2}; \qquad A = -B\tau = -\frac{\omega\tau}{1+\tau^2\omega^2}$$

$$A = -\frac{\omega\tau}{1+\tau^2\omega^2}; \ B = \frac{\omega}{1+\tau^2\omega^2}; C = \frac{\omega\tau^2}{1+\tau^2\omega^2};$$

$$\frac{q_o(s)}{KA_i} = \frac{As + B}{s^2+\omega^2} + \frac{C}{(\tau s + 1)}$$

$$\frac{q_o(s)}{KA_i} = -\frac{\omega\tau s}{1+\tau^2\omega^2}\frac{1}{(s^2+\omega^2)} + \frac{\omega}{1+\tau^2\omega^2}\frac{1}{(s^2+\omega^2)} + \frac{\omega\tau^2}{1+\tau^2\omega^2}\frac{1}{(\tau s + 1)}$$

$$\frac{q_o(s)}{KA_i} = -\frac{\omega\tau s}{1+\tau^2\omega^2}\frac{s}{(s^2+\omega^2)} + \frac{\omega}{1+\tau^2\omega^2}\frac{1}{(s^2+\omega^2)} + \frac{\omega\tau^2}{1+\tau^2\omega^2}\frac{1}{(\tau s+1)}$$

$$\frac{q_o(t)}{KA_i} = -\frac{\omega\tau}{1+\tau^2\omega^2}cos\omega t + \frac{\omega}{1+\tau^2\omega^2}\frac{sin\omega t}{\omega} + \frac{\omega\tau^2}{1+\tau^2\omega^2}\frac{1}{\tau}e^{-\frac{t}{\tau}}$$

$$\frac{q_o(t)}{KA_i} = -\frac{\omega\tau}{1+\tau^2\omega^2}cos\omega t + \frac{1}{1+\tau^2\omega^2}sin\omega t + \frac{\omega\tau}{1+\tau^2\omega^2}e^{-\frac{t}{\tau}}$$

$$\frac{q_o(t)}{KA_i} = \underbrace{\frac{\omega\tau}{1+\tau^2\omega^2}e^{-\frac{t}{\tau}}}_{Transient\ part}\underbrace{-\frac{\omega\tau}{1+\tau^2\omega^2}cos\omega t + \frac{1}{1+\tau^2\omega^2}sin\omega t}_{Steady\ State\ part}$$

Consider only steady state portion

$$\frac{q_o(t)}{KA_i} = -\frac{\omega\tau}{1+\tau^2\omega^2}cos\omega t + \frac{1}{1+\tau^2\omega^2}sin\omega t$$

$$\frac{q_o(t)}{KA_i} = BSin(\omega t + \phi) = B(sin\omega t\ Cos\phi + cos\omega t sin\phi)$$

Comparing above equations

$$BCos\phi = \frac{1}{1+\tau^2\omega^2}; BSin\phi = -\frac{\omega\tau}{1+\tau^2\omega^2}$$

$$B^2\left(cos^2\phi + sin^2\phi\right) = \frac{1+\tau^2\omega^2}{(1+\tau^2\omega^2)^2} \Rightarrow B = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

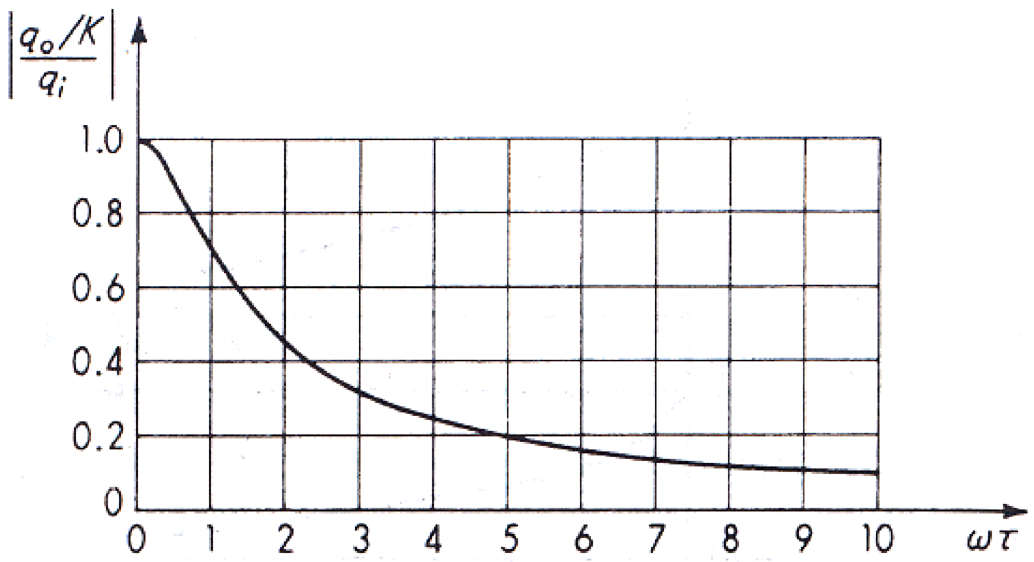
$$Tan\phi = -\omega\tau \Rightarrow \phi = Tan^{-1}(-\omega\tau)$$

$$\left|\frac{q_o(t)}{A_i}\right| = \frac{K}{\sqrt{1+\tau^2\omega^2}}; \quad \phi = Tan^{-1}(-\omega\tau)$$

Positive ϕ - Angle by which the output leads the input
Negative ϕ - Angle by which the output lags the input

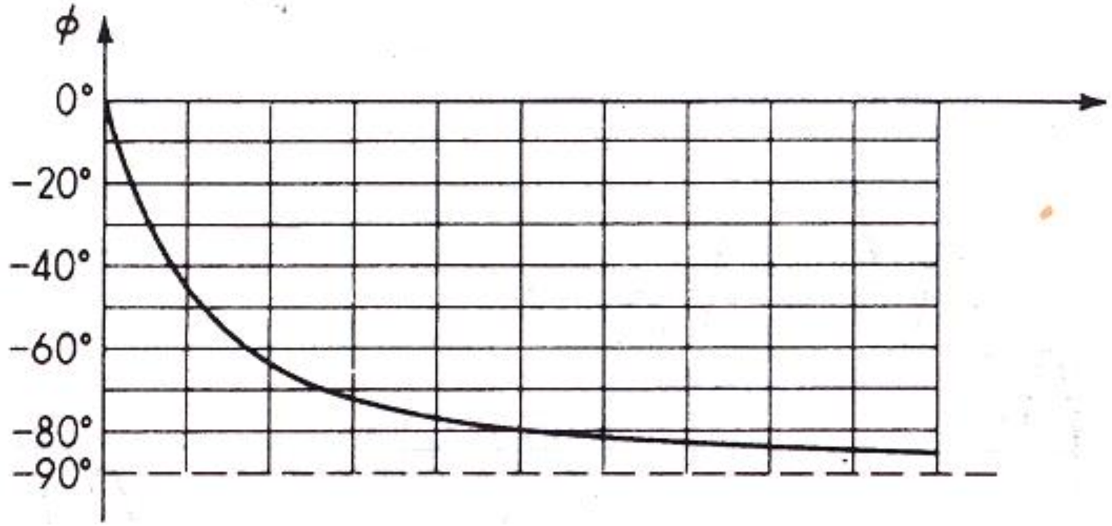
Amplitude attenuation and Phase Shift

$$\left|\frac{q_o(t)}{Kq_i}\right| = \frac{1}{\sqrt{1+\tau^2\omega^2}}; \qquad \phi = Tan^{-1}(-\omega\tau)$$



Ideal Frequency response:

$$\frac{q_o(t)}{Kq_i} = 1 \qquad \phi = 0$$



$\tau\omega$ - small; For high ω, τ - small

GENERAL TRANSFER FUNCTION OF A FIRST ORDER SYSTEM

$$\tau \frac{dq_o}{dt} + q_o = Kq_i(t)$$

$$\mathcal{L}\left(\frac{dq_o(t)}{dt}\right) = sq_o(s) - q_o(0)$$

$$\mathcal{L}(q_o(t)) = q_o(s)$$

$$\mathcal{L}(Kq_i(t)) = Kq_i(s)$$

$$\tau[sq_o(s) - \cancel{q_o(0)}] + q_o(s) = Kq_i(s)$$

$$q_o(s)[\tau s + 1] = Kq_i(s)$$

$$\frac{q_o(s)}{q_i(s)} = \frac{K}{\tau s + 1}$$

$$q_i(s) \longrightarrow \boxed{\frac{q_o(s)}{q_i(s)} = \frac{K}{\tau s + 1}} \longrightarrow q_o(s)$$

SIMPLE METHOD TO GET STEADY STATE FREQUENCY RESPONSE

$$q_i(s) \longrightarrow \boxed{\frac{q_o(s)}{q_i(s)} = \frac{K}{\tau s + 1}} \longrightarrow q_o(s)$$

$$\frac{q_o(s)}{q_i(s)} = \frac{K}{\tau s + 1}$$

Replace s by iω

$$\frac{q_o(i\omega)}{Kq_i(i\omega)} = \frac{1}{(1+i\tau\omega)} \times \frac{(1-i\tau\omega)}{(1-i\tau\omega)} = \frac{1}{1+\tau^2\omega^2} - i\frac{\tau\omega}{1+\tau^2\omega^2}$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\left(\frac{1}{1+\tau^2\omega^2}\right)^2 + \left(\frac{\tau\omega}{1+\tau^2\omega^2}\right)^2} = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$\frac{q_o(i\omega)}{Kq_i(i\omega)} = \frac{1}{(1+i\tau\omega)} \times \frac{(1-i\tau\omega)}{(1-i\tau\omega)} = \frac{1}{1+\tau^2\omega^2} - i\frac{\tau\omega}{1+\tau^2\omega^2}$$

$$Tan\phi = \frac{-\frac{\tau\omega}{1+\tau^2\omega^2}}{\frac{1}{1+\tau^2\omega^2}} \Rightarrow \phi = Tan^{-1}(-\tau\omega)$$

$$\left|\frac{q_o}{Kq_i}\right| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$\phi = Tan^{-1}(-\tau\omega)$$

STEADY STATE FREQUENCY RESPONSE

$$q_i(s) \longrightarrow \boxed{\frac{q_o(s)}{q_i(s)} = \frac{K}{\tau s + 1}} \longrightarrow q_o(s)$$

$$\left|\frac{q_o}{Kq_i}\right| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$\phi = Tan^{-1}(-\tau\omega)$$

- Amplitude attenuation and phase shift - corrected for pure sine wave
- Actual practice, combination of several sine waves of different frequencies

LINEAR SYSTEMS

A system is called linear if the principle of superposition applies.

The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses.

For linear systems, the response to the several inputs can be calculated by treating one input at a time and adding the results.

This principle allows one to build up complicated solutions to the linear differential equation from simple solutions.

HOW DOES ONE KNOW THAT THE SYSTEM IS LINEAR ?

In an experimental investigation of a dynamic system, if the cause and effect are proportional, thus implying that the principle of superposition holds, then the system can be considered linear.

LINEAR DIFFERENTIAL AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation is linear, if the coefficients are constants or only functions of the independent variable.

$$\frac{dy}{dt} + p(t)y = \mathrm{r}(t)$$

This equation is linear in both the unknown function *y* and its derivative *dy/dt*.

A differential equation is linear if the unknown function y(t) and its derivatives dy/dt appear to the power 1 (products of the unknown function and its derivatives are not allowed) and nonlinear otherwise.

$$\frac{dy}{dt} - y^2 = 4$$

$$y\frac{dy}{dt} + \left(\frac{dy}{dt}\right)^2 = 0$$

First order instrument with a time constant of 0.2 seconds

$$q_i = 1sin2t + 0.3sin20t$$

t - seconds

$$\left.\frac{q_o(t)}{Kq_i}\right|_{\omega=2} = \frac{1}{\sqrt{1+\tau^2\omega^2}} = \frac{1}{\sqrt{1+(0.2)^2(2)^2}} = 0.9285;$$

$$\phi|_{\omega=2} = Tan^{-1}(-\omega\tau) = Tan^{-1}(-0.2\times 2) = -21.8^{\circ}$$

$$\left.\frac{q_o(t)}{Kq_i}\right|_{\omega=20} = \frac{1}{\sqrt{1+\tau^2\omega^2}} = \frac{1}{\sqrt{1+(0.2)^2(20)^2}} = 0.2425;$$

$$\phi|_{\omega=20} = Tan^{-1}(-\omega\tau) = Tan^{-1}(-0.2\times 20) = -75.96^{\circ}$$

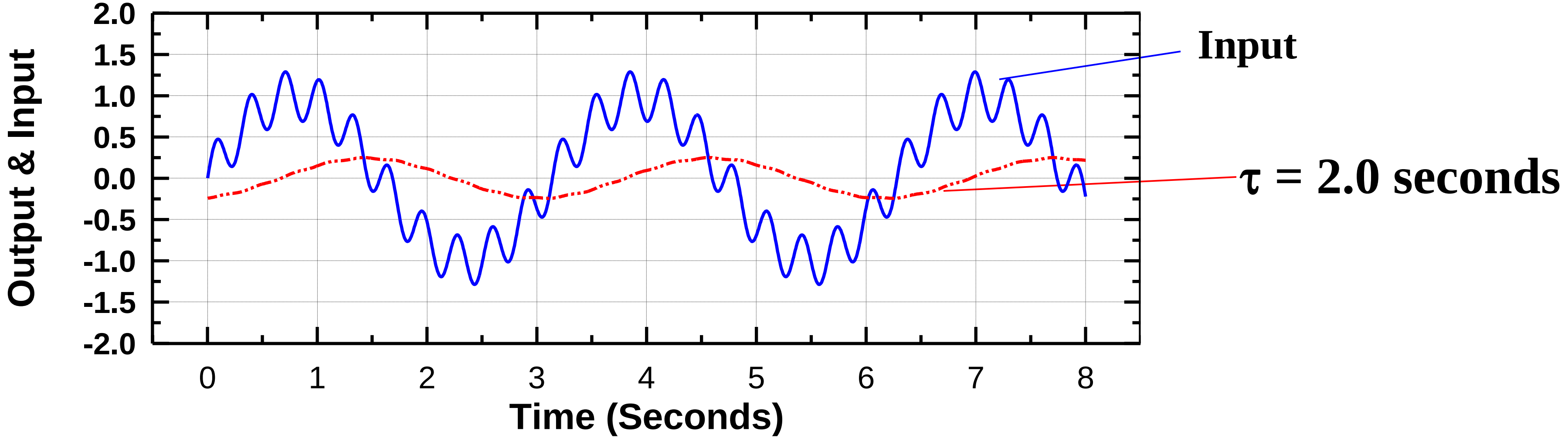
By superposition principle

$$\frac{q_o}{K} = 1(0.9285)sin(2t-21.8) + 0.3(0.2425)sin(20t\ -75.96)$$

$$\frac{q_o}{K} = 0.9285\ sin(2t-21.8) + 0.07275\ sin(20t\ -75.96)$$

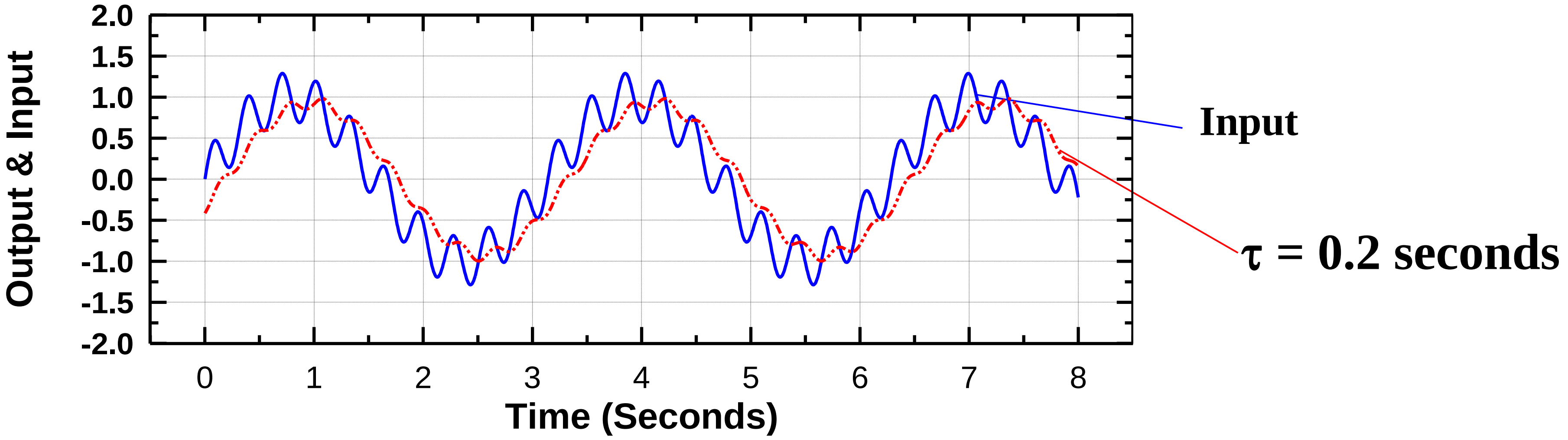
$$q_i = 1\sin 2t + 0.3\sin 20t$$

$$\frac{q_o}{K} = 0.2425\,\sin(2t - 75.96) + 0.0075\,\sin(20t\ - 88.57)$$



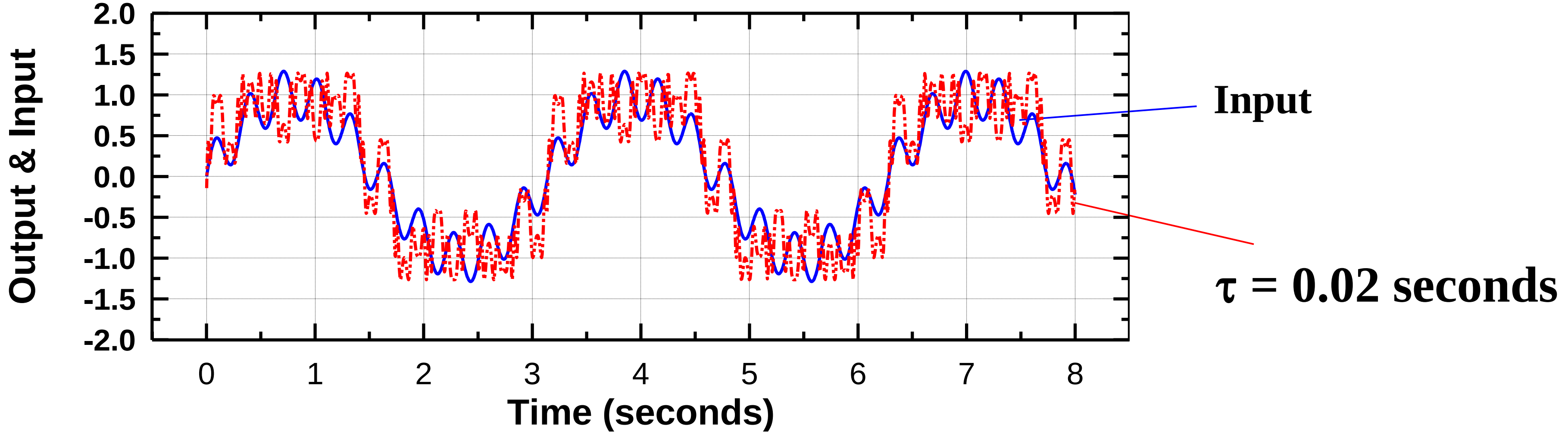
$$q_i = 1\sin 2t + 0.3\sin 20t$$

$$\frac{q_o}{K} = 0.9285\,\sin(2t - 21.8) + 0.07275\,\sin(20t\ - 75.96)$$



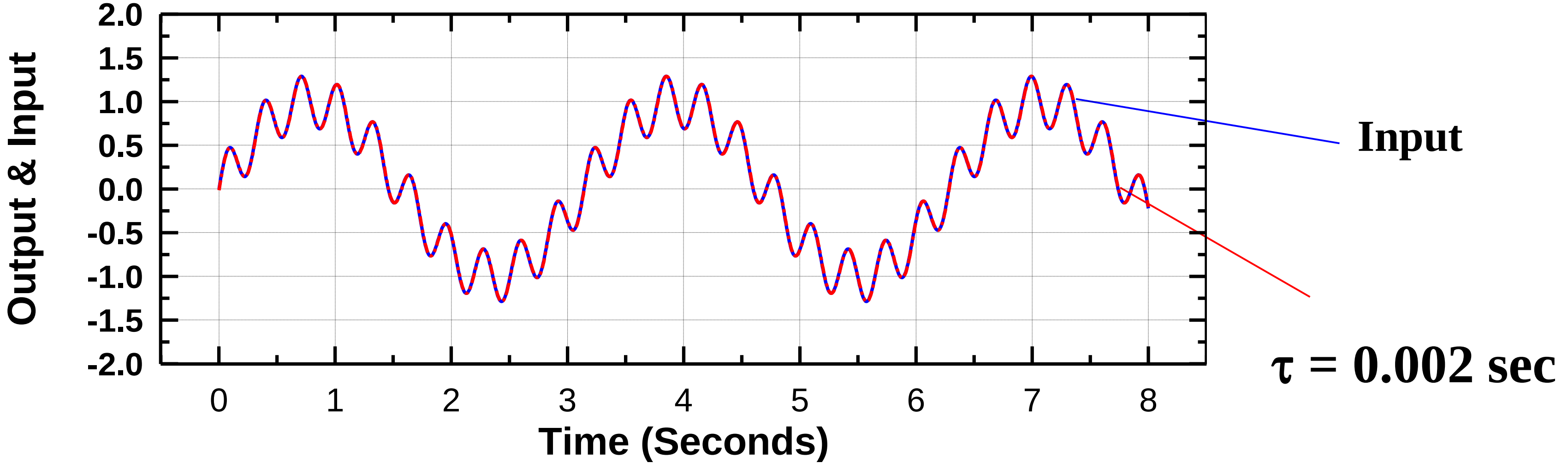
$$q_i = 1\sin 2t + 0.3\sin 20t$$

$$\frac{q_o}{K} = 0.99921\,\sin(2t - 2.29) + 0.27855\,\sin(20t\ - 21.8)$$



$$q_i = 1\sin 2t + 0.3\sin 20t$$

$$\frac{q_o}{K} = 0.99992\,\sin(2t - 0.22918) + 0.29976\,\sin(20t\ - 2.29)$$



Output of the first order instrument with a time constant of 0.002 seconds faithfully follows the input

FREQUENCY ANALYSIS

Many signals that result from the measurement of dynamic variables are nondeterministic in nature and have a continuously varying rate of change.

Any complex signal can be thought of as made up of sines and cosines of differing periods and amplitudes, which are added together in an infinite trigonometric series. This representation of a signal as a series of sines and cosines is called a Fourier series.

In theory, Fourier analysis allows essentially all mathematical functions of practical interest to be represented by an infinite series of sines and cosines.

Classification of waveforms

I. Static $y(t) = A_o$

II. Dynamic

Periodic Waveform

Simple periodic waveform $y(t) = A_o + C sin(\omega t + \phi)$

Complex periodic waveform $y(t) = A_o + \sum_{n=1}^{\infty} C_n sin(n\omega t + \phi_n)$

Aperiodic waveforms

Step[a] $y(t) = A_o U(t)$

$y(t) = A_o \ for\ t > 0$

Ramp $y(t) = A_o t \text{ for } 0 < t < t_f$

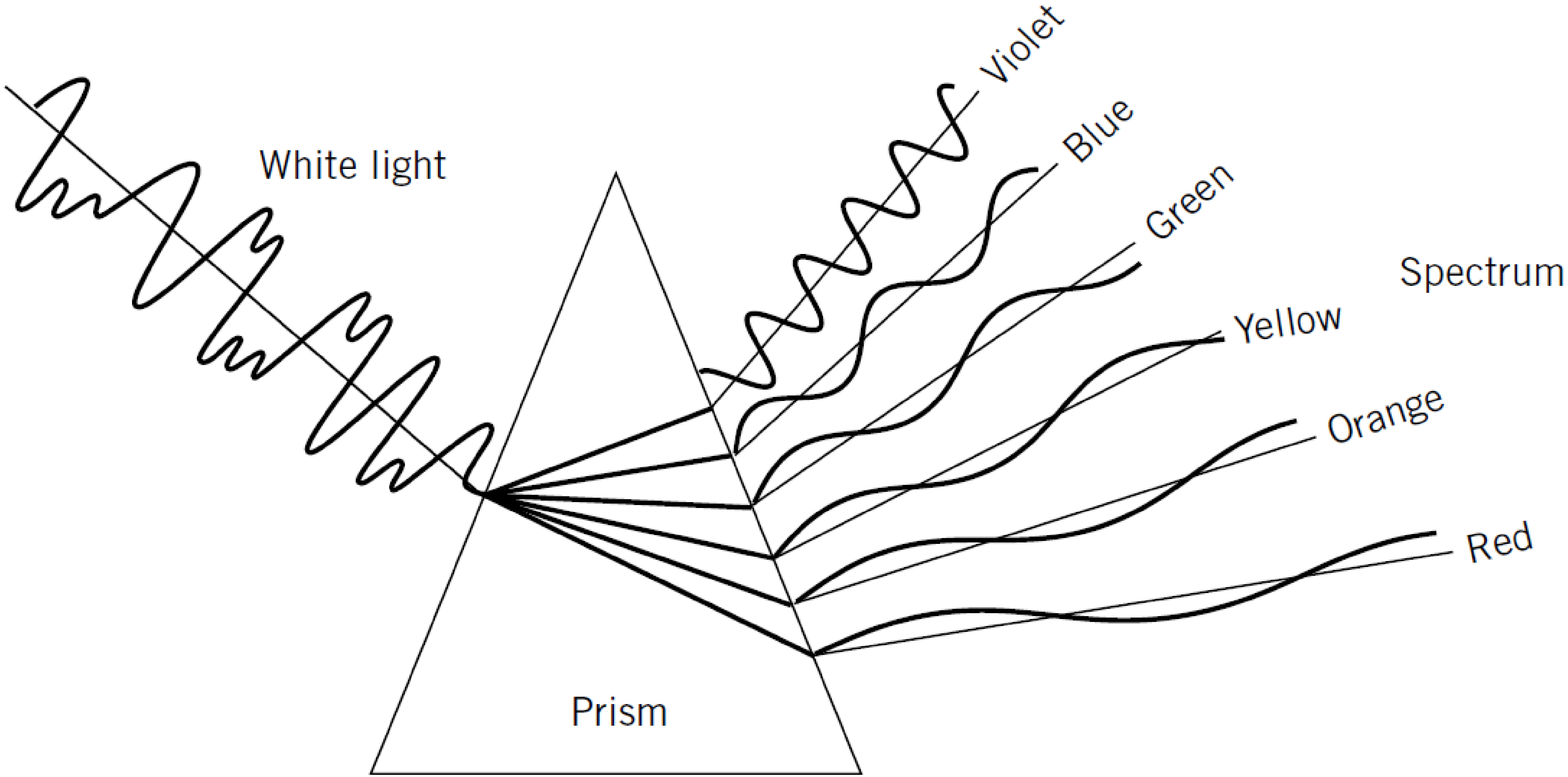
Pulse[b] $y(t) = A_o U(t) - A_o U(t - t_1)$

III. Nondeterministic waveform $y(t) \approx A_o + \sum_{n=1}^{\infty} C_n sin(n\omega t + \phi_n)$

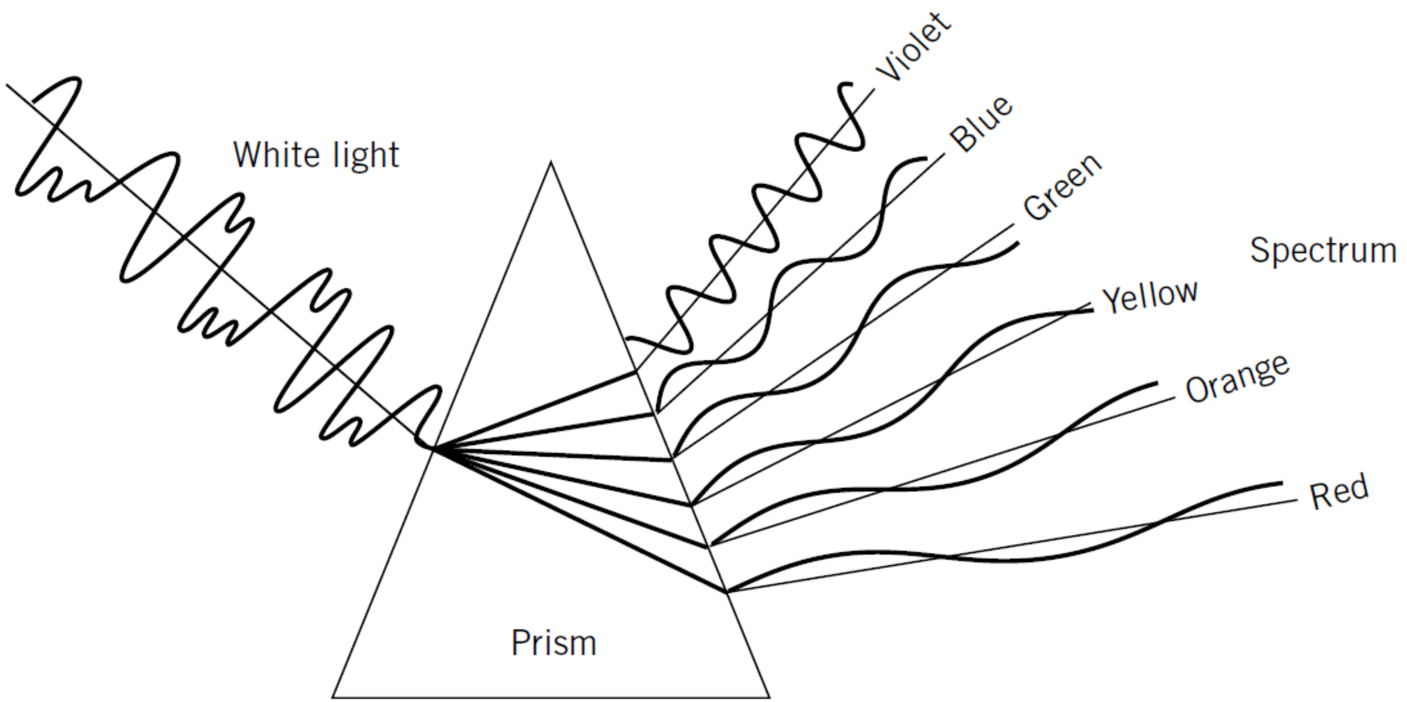
[a] $U(t)$ - unit step function, which is 0 for $t < 0$ and 1 for $t > 0$

[b] t_1 − Pulse width

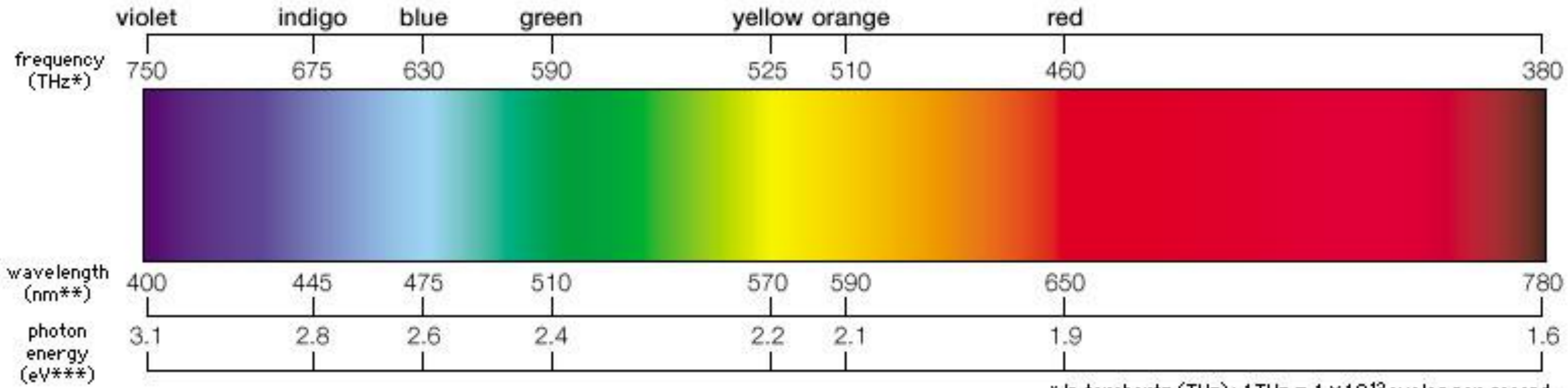
SIGNAL AMPLITUDE AND FREQUENCY



Separation of white light into its colour spectrum.
Colour corresponds to a particular frequency or wavelength
Light intensity corresponds to varying amplitudes.

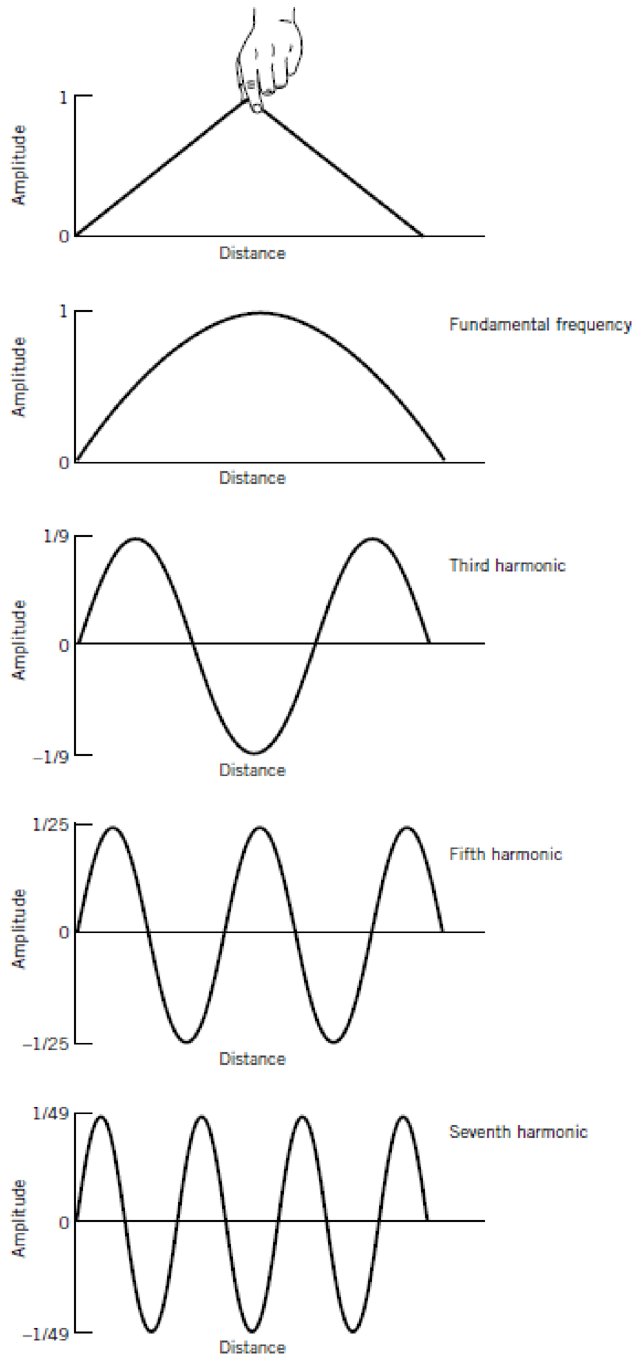


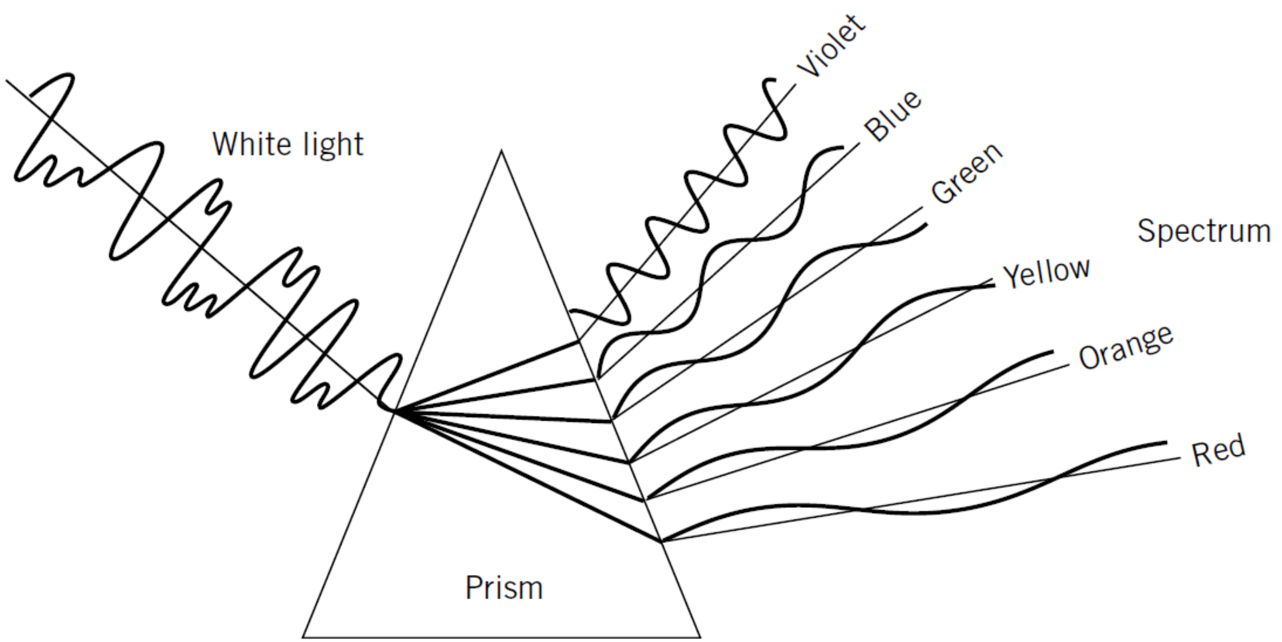
Light, the visible spectrum



* In terahertz (THz); 1 THz = 1×10^{12} cycles per second.
** In nanometres (nm); 1 nm = 1×10^{-9} metre.
*** In electron volts (eV).

MODES OF VIBRATION FOR A STRING PLUCKED AT ITS CENTER





COMPLEX SIGNAL

FOURIER SERIES

$$y(t) = A_o + \sum_{n=1}^{\infty} (A_n cosnt + B_n sinnt)$$

sines and cosines of differing periods and amplitudes

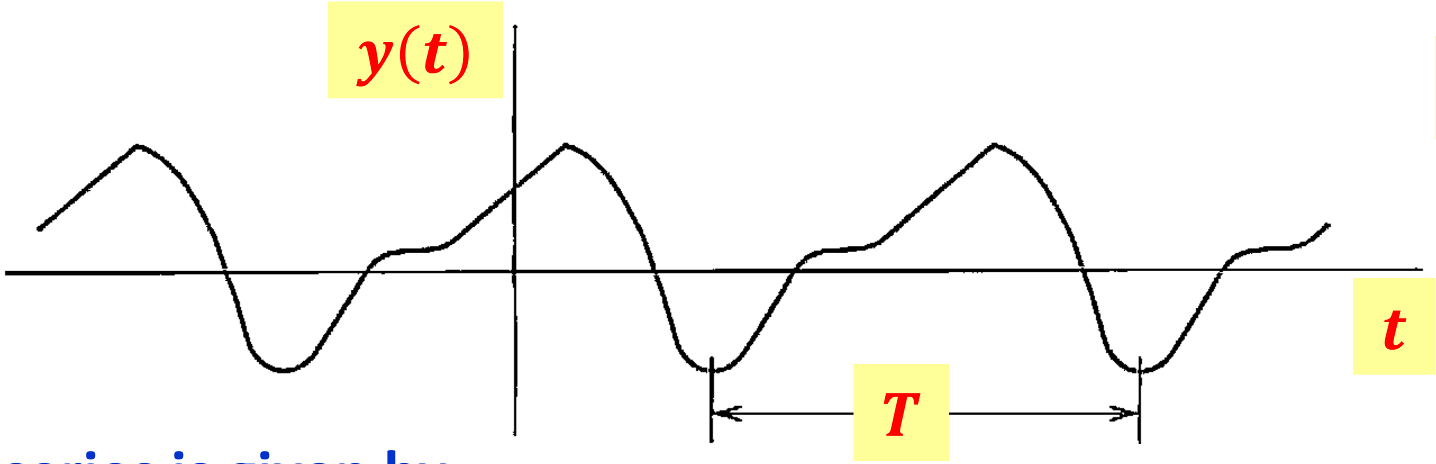
A function $y(t)$ is a periodic function if there is some positive number T such that

$$y(t+T) = y(t)$$

The period of $y(t)$ is T. If both $y_1(t)$ and $y_2(t)$ have period T, then

$$ay_1(t) + by_2(t)$$

Also has a period of T (a and b are constants)



$$Ex: \ Sint, Cost$$

A trigonometric series is given by

$$A_o + A_1 cost + B_1 sint + A_2 cos2t + B_2 sin2t + \cdots.$$

Where A_n and B_n are the coefficients of the series

FOURIER SERIES AND COEFFICIENTS

A periodic function $y(t)$ with a period $T = 2\pi$ is to be represented by a trigonometric series, such that for any t,

$$y(t) = A_o + \sum_{n=1}^{\infty}(A_n cosnt + B_n sinnt)$$

with $y(t)$ known, the coefficients A_n and B_n are to be determined.

Integrating the above equation between the limits -π to π,

$$\int_{-\pi}^{\pi} y(t)dt = A_o \int_{-\pi}^{\pi} dt + \sum_{n=1}^{\infty}\left(A_n \int_{-\pi}^{\pi} cosntdt + B_n \int_{-\pi}^{\pi} sinntdt\right)$$

$$\int_{-\pi}^{\pi} cosnt\, dt = 0$$

$$\int_{-\pi}^{\pi} sinnt\, dt = 0$$

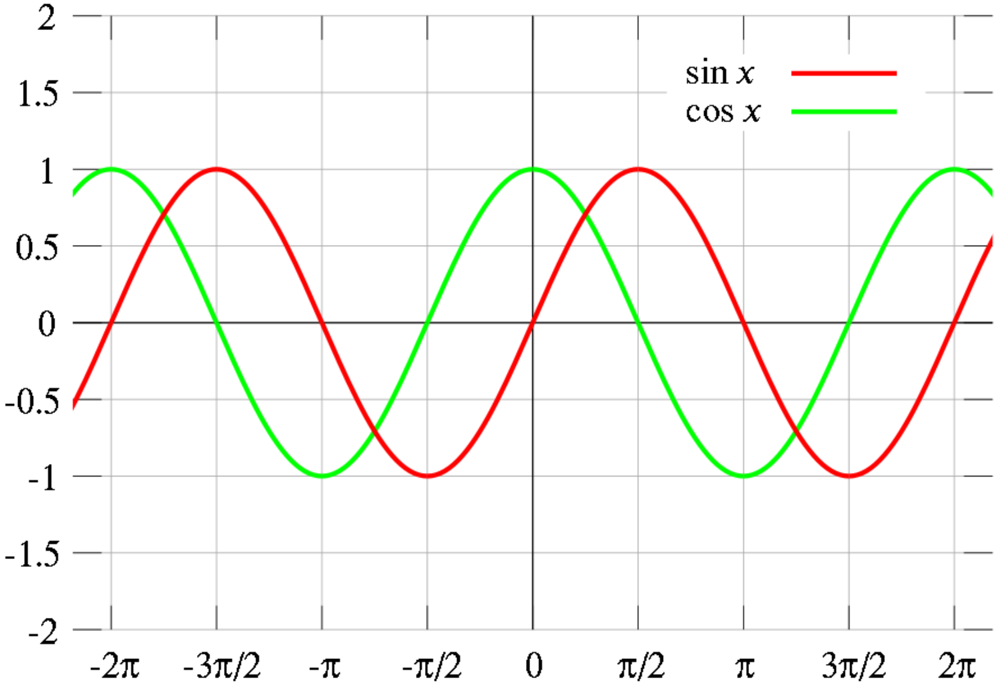
$$\int_{-\pi}^{\pi} y(t)dt = A_o\big(\pi - (-\pi)\big) = 2\pi A_o$$

$$A_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t)dt$$

$$\text{To show that } \int_{-\pi}^{\pi} cosnt\, dt = 0;\ \int_{-\pi}^{\pi} sinnt\, dt = 0$$

$$\int_{-\pi}^{\pi} cosnt\, dt = \frac{sinnt}{n}\Big|_{-\pi}^{\pi} = \frac{1}{n}\big(sinn\pi - sin(-n\pi)\big) = \frac{1}{n}(sinn\pi + sinn\pi) = \frac{2}{n} sinn\pi = 0$$

$$\int_{-\pi}^{\pi} sinnt\, dt = -\frac{cosnt}{n}\Big|_{-\pi}^{\pi} = -\frac{1}{n}\big(cosn\pi - cos(-n\pi)\big) = -\frac{1}{n}(cosn\pi + cosn\pi) = 0$$



$$sinn\pi = sin(-n\pi) = 0$$
$$cosn\pi = cos(-n\pi)$$
$$cosn\pi = cos(-n\pi) = -1 \text{ for n odd}$$
$$cosn\pi = cos(-n\pi) = +1 \text{ for n even}$$

TO DETERMINE A_n

$$y(t) = A_o + \sum_{n=1}^{\infty} (A_n cosnt + B_n sinnt)$$

$$\int_{-\pi}^{\pi} y(t)dt = A_o \int_{-\pi}^{\pi} dt + \sum_{n=1}^{\infty} \left(A_n \int_{-\pi}^{\pi} cosntdt + B_n \int_{-\pi}^{\pi} sinntdt \right)$$

Multiply by $cosmt$

$$\int_{-\pi}^{\pi} y(t)cosmtdt = A_o \cancel{\int_{-\pi}^{\pi} cosmtdt} + \sum_{n=1}^{\infty} \left(A_n \int_{-\pi}^{\pi} cosntcosmtdt + B_n \int_{-\pi}^{\pi} sinntcosmtdt \right)$$

$$\int_{-\pi}^{\pi} cosntcosmt\, dt \;\; and \;\; \int_{-\pi}^{\pi} sinntcosmt\, dt - \text{To find}$$

$$\int_{-\pi}^{\pi} cosnt\, dt = 0$$

$$\int_{-\pi}^{\pi} cosntcosmt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} cos(n+m)t\,dt + \frac{1}{2}\int_{-\pi}^{\pi} cos(n-m)t\,dt$$

$$\int_{-\pi}^{\pi} cosntcosmt\,dt = \frac{1}{2}\frac{sin(n+m)t}{(n+m)}\Bigg|_{-\pi}^{\pi} + \frac{1}{2}\frac{sin(n-m)t}{(n-m)}\Bigg|_{-\pi}^{\pi}$$

$$sinn\pi = sin(-n\pi) = 0$$

$$\int_{-\pi}^{\pi} cosntcosmt\,dt = \frac{1}{2(n+m)}\big(sin(n+m)\pi - sin(n+m)(-\pi)\big) + \frac{1}{2(n-m)}\big(sin(n-m)\pi - sin(n-m)(-\pi)\big)$$

For $m = n$

$$\int_{-\pi}^{\pi} cosntcosmt\,dt = \int_{-\pi}^{\pi} cos^2mt\,dt = \int_{-\pi}^{\pi} \frac{1+cos2mt}{2}dt = \frac{1}{2}\left(t + \overset{\text{zero}}{\frac{sin2mt}{2m}}\right)\Bigg|_{-\pi}^{\pi} = \frac{1}{2}\big(\pi-(-\pi)\big) = \pi$$

$$\int_{-\pi}^{\pi} sinnt cosmt\, dt = \frac{1}{2}\int_{-\pi}^{\pi} sin(n+m)t\, dt + \frac{1}{2}\int_{-\pi}^{\pi} sin(n-m)t\, dt$$

$$\int_{-\pi}^{\pi} sinnt cosmt\, dt = -\frac{1}{2}\frac{cos(n+m)t}{(n+m)}\Bigg|_{-\pi}^{\pi} - \frac{1}{2}\frac{cos(n-m)t}{(n-m)}\Bigg|_{-\pi}^{\pi}$$

$$\int_{-\pi}^{\pi} sinnt cosmt\, dt = -\frac{1}{2(n+m)}\big(cos(n+m)\pi - cos(n+m)(-\pi)\big) - \frac{1}{2(n-m)}\big(cos(n-m)\pi - cos(n-m)(-\pi)\big)$$

$$\int_{-\pi}^{\pi} sinnt cosmt\, dt = -\frac{1}{2(n+m)}\big(-1-(-1)\big) - \frac{1}{2(n-m)}\big(-1-(-1)\big) = 0 \text{ if } n \text{ is odd}$$

$$\int_{-\pi}^{\pi} sinnt cosmt\, dt = -\frac{1}{2(n+m)}\big(1-(1)\big) + \frac{1}{2(n-m)}\big(1-(1)\big) = 0 \text{ if } n \text{ is even}$$

$cosn\pi = cos(-n\pi)$

$cosn\pi = cos(-n\pi) = -1$ for n odd

$cosn\pi = cos(-n\pi) = +1$ for n even

For $m = n$

$$\int_{-\pi}^{\pi} sinmt cosmt\, dt = \frac{1}{2}\int_{-\pi}^{\pi} sin2mt\, dt = \frac{1}{2}\left(\frac{cos2mt}{2m}\right)\Bigg|_{-\pi}^{\pi} = \frac{1}{4m}\big(-1-(-1)\big) = 0$$

$$\int_{-\pi}^{\pi} y(t)cosmtdt = A_o \int_{-\pi}^{\pi} cosmtdt + \sum_{n=1}^{\infty}\left(A_n \int_{-\pi}^{\pi} cosntcosmtdt + B_n \int_{-\pi}^{\pi} sinntcosmtdt\right)$$

$$\int_{-\pi}^{\pi} cosmt\, dt = 0$$

For $m = n$

$$\int_{-\pi}^{\pi} cosntcosmt\, dt = \pi$$

For $m \neq n$

$$\int_{-\pi}^{\pi} cosntcosmt\, dt = 0$$

For $m = n$

$$\int_{-\pi}^{\pi} sinntcosmt\, dt = 0$$

For $m \neq n$

$$\int_{-\pi}^{\pi} sinntcosmt\, dt = 0$$

$$\int_{-\pi}^{\pi} y(t)cosmtdt = A_n\pi$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)cosmtdt$$

TO DETERMINE B_n

$$y(t) = A_o + \sum_{n=1}^{\infty}(A_n cosnt + B_n sinnt)$$

$$\int_{-\pi}^{\pi} y(t)dt = A_o \int_{-\pi}^{\pi} dt + \sum_{n=1}^{\infty}\left(A_n \int_{-\pi}^{\pi} cosntdt + B_n \int_{-\pi}^{\pi} sinntdt\right)$$

Multiply by $sinmt$

$$\int_{-\pi}^{\pi} y(t)sinmtdt = A_o \int_{-\pi}^{\pi} sinmtdt + \sum_{n=1}^{\infty}\left(A_n \int_{-\pi}^{\pi} cosntsinmtdt + B_n \int_{-\pi}^{\pi} sinntsinmtdt\right)$$

$$\int_{-\pi}^{\pi} cosntsinmt\,dt \;\; and \;\; \int_{-\pi}^{\pi} sinntsinmt\,dt - \text{To find}$$

$$\int_{-\pi}^{\pi} sinmt\,dt = 0$$

$$\int_{-\pi}^{\pi} sinmtcosnt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} sin(m+n)t\,dt + \frac{1}{2}\int_{-\pi}^{\pi} sin(m-n)t\,dt$$

$$\int_{-\pi}^{\pi} sinmtcosnt\,dt = -\frac{1}{2}\frac{cos(m+n)t}{(m+n)}\bigg|_{-\pi}^{\pi} - \frac{1}{2}\frac{cos(m-n)t}{(m-n)}\bigg|_{-\pi}^{\pi}$$

$$\int_{-\pi}^{\pi} sinmtcosnt\,dt = -\frac{1}{2(m+n)}\big(cos(m+n)\pi - cos(m+n)(-\pi)\big) - \frac{1}{2(m-n)}\big(cos(m-n)\pi - cos(m-n)(-\pi)\big)$$

$$= 0$$

$$\int_{-\pi}^{\pi} sinmtcosnt\,dt = -\frac{1}{2(m+n)}\big(-1-(-1)\big) + \frac{1}{2(m-n)}\big(-1-(-1)\big) = 0\ if\ n\ is\ odd$$

$$\int_{-\pi}^{\pi} sinmtcosnt\,dt = -\frac{1}{2(m+n)}\big(1-(1)\big) + \frac{1}{2(m-n)}\big(1-(1)\big) = 0\ if\ n\ is\ even$$

$cosn\pi = \ cos(-n\pi)$

$cosn\pi = \ cos(-n\pi) = -1$ for n odd

$cosn\pi = \ cos(-n\pi) = +1$ for n even

For $m = n$

$$\int_{-\pi}^{\pi} sinmtcosmt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} sin2mt\,dt = \frac{1}{2}\left(\frac{cos2mt}{2m}\right)\bigg|_{-\pi}^{\pi} = \frac{1}{4m}\big(-1-(-1)\big) = 0$$

$$\int_{-\pi}^{\pi} sinntsinmt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} cos(n\ -m)t\,dt\ - \frac{1}{2}\int_{-\pi}^{\pi} cos(n+m)t\,dt$$

$$\int_{-\pi}^{\pi} sinntsinmt\,dt = \frac{1}{2}\frac{sin(n\ -m)t}{(n\ -m)}\Bigg|_{-\pi}^{\pi} - \frac{1}{2}\frac{sin(n+m)t}{(n+m)}\Bigg|_{-\pi}^{\pi}$$

$$sinn\pi = \ sin(-n\pi) = 0$$

$$\int_{-\pi}^{\pi} sinntsinmt\,dt = \frac{1}{2(n+m)}\big(sin(n\ -m)\pi - sin(n\ -m)(-\pi)\big) - \frac{1}{2(n-m)}\big(sin(n+m)\pi - sin(n+m)(-\pi)\big) = 0$$

For $m = n$

$$\int_{-\pi}^{\pi} sinntsinmt\,dt = \int_{-\pi}^{\pi} sin^2mt\,dt = \int_{-\pi}^{\pi} \frac{1-cos2mt}{2}dt = \frac{1}{2}\left(t - \overset{\text{zero}}{\frac{sin2mt}{2m}}\right)\Bigg|_{-\pi}^{\pi} = \frac{1}{2}\big(\pi - (-\pi)\big) = \pi$$

$$\int_{-\pi}^{\pi} y(t) sinmtdt = A_o \int_{-\pi}^{\pi} sinmtdt + \sum_{n=1}^{\infty} \left(A_n \int_{-\pi}^{\pi} cosntsinmtdt + B_n \int_{-\pi}^{\pi} sinntsinmtdt \right)$$

$$\int_{-\pi}^{\pi} sinmt\, dt = 0$$

For $m = n$

$$\int_{-\pi}^{\pi} cosntsinmt\, dt = 0$$

For $m \neq n$

$$\int_{-\pi}^{\pi} cosntsinmt\, dt = 0$$

For $m = n$

$$\int_{-\pi}^{\pi} sinntsinmtdt = \pi$$

For $m \neq n$

$$\int_{-\pi}^{\pi} sinntsinmtdt = 0$$

$$\int_{-\pi}^{\pi} y(t) sinmtdt = B_n \pi$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t) sinmtdt$$

Summary of these calculations: Fourier Series and Fourier coefficients

$$y(t) = A_o + \sum_{n=1}^{\infty} (A_n cosnt + B_n sinnt)$$

$$A_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t)dt \qquad A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t)cosmtdt \qquad B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t)sinmtdt$$

When n = 1, the corresponding terms in the Fourier series are called fundamental and have the lowest frequency in the series. Frequencies corresponding to n = 2, 3, 4, . . . are known as harmonics, with, for example, n = 2 representing the second harmonic.

Fourier Coefficients for Functions Having Arbitrary Period $T = \frac{2\pi}{\omega}$

$$y(t) = A_o + \sum_{n=1}^{\infty}(A_n cos n\omega t + B_n sin n\omega t)$$

$$A_o = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)dt$$

$$A_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) cos n\omega t dt$$

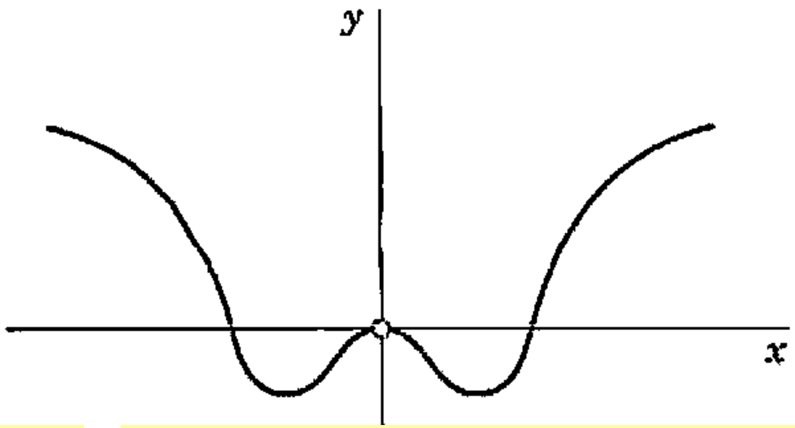
$$B_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) sin n\omega t dt$$

EVEN FUNCTIONS

A function $g(t)$ is even if it is symmetric about the origin, which may be stated, for all t,

$$g(-t) = g(t)$$

Ex: $cosnt$



$$A_o = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)dt \qquad A_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)cosn\omega tdt$$

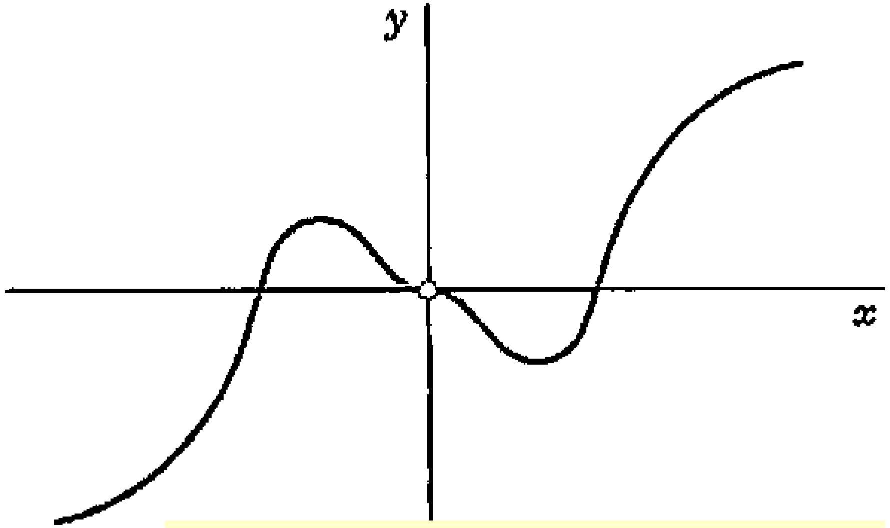
$$y(t) = A_o + \sum_{n=1}^{\infty} A_n cosn\omega t = A_o + \sum_{n=1}^{\infty} A_n cos\frac{2\pi nt}{T}$$

ODD FUNCTIONS

A function $h(t)$ is odd if, for all t,

$$h(-t) = -h(t)$$

Ex: $sinnt$



$$B_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)sinn\omega tdt$$

$$y(t) = \sum_{n=1}^{\infty} B_n sinn\omega t = \sum_{n=1}^{\infty} B_n sin\frac{2\pi nt}{T}$$

A particular function or waveform may be even, odd, or neither even nor odd.

Fourier Cosine Series

If $y(t)$ is even, its Fourier series will contain only cosine terms:

$$y(t) = A_o + \sum_{n=1}^{\infty} A_n cosn\omega t = \sum_{n=1}^{\infty} A_n cos\frac{2\pi nt}{T}$$

If $y(t)$ is even, its Fourier series will contain only cosine terms:

Fourier Sine Series

If $y(t)$ is odd, its Fourier series will contain only sine terms:

$$y(t) = \sum_{n=1}^{\infty} B_n sinn\omega t = \sum_{n=1}^{\infty} B_n sin\frac{2\pi nt}{T}$$

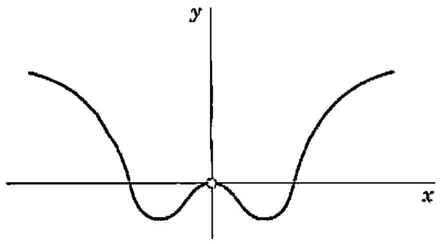
If $y(t)$ is odd, its Fourier series will contain only sine terms:

Functions that are neither even nor odd result in Fourier series that contain both sine and cosine terms.

A_o is zero only for odd function

A function $g(t)$ is even if it is symmetric about the origin, which may be stated, for all t,

$$g(-t) = g(t)$$

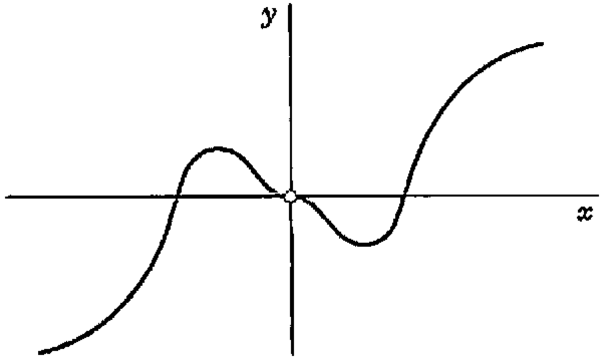


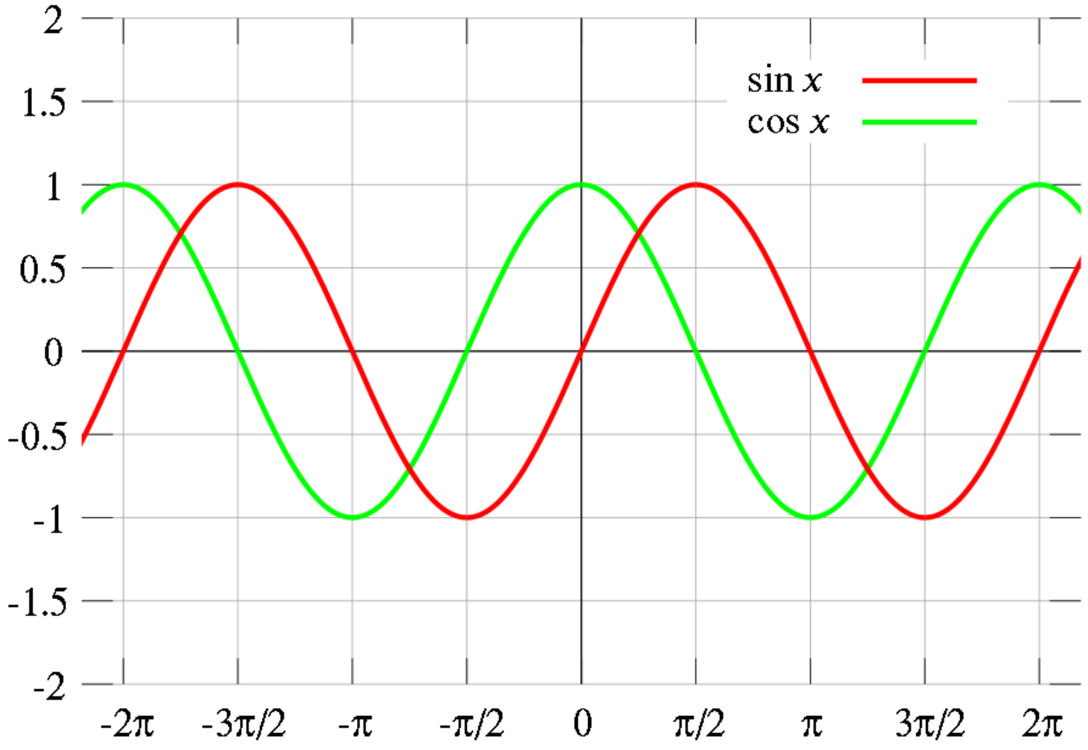
Ex: $cosnt$

A function $h(t)$ is odd if, for all t,

$$h(-t) = -h(t)$$

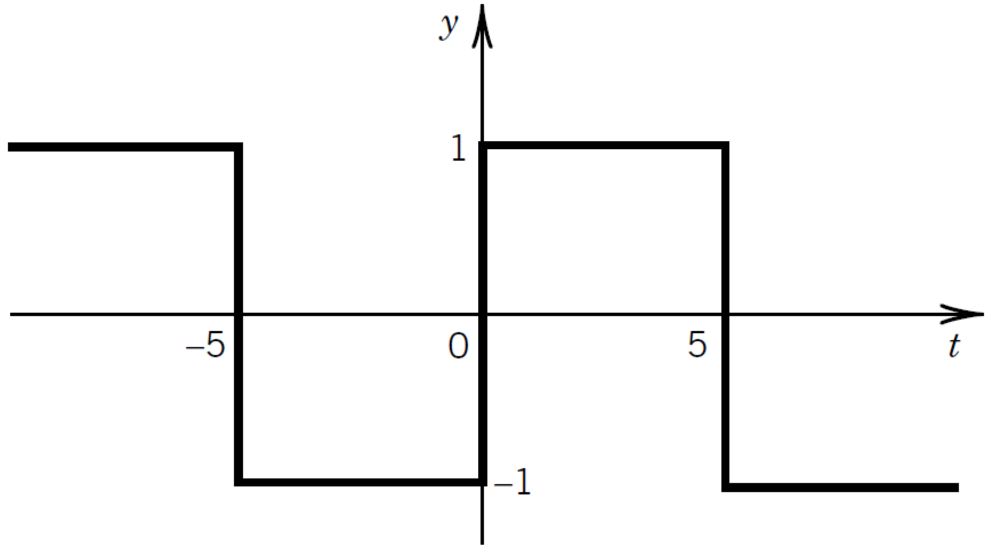
Ex: $sinnt$





$$A_o = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)dt$$

Determine the Fourier series that represents the function shown in Figure



$$A_o = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)dt = 0$$

$$T = 10\ seconds$$

A function $h(t)$ is odd if, for all t, $h(-t) = -h(t)$

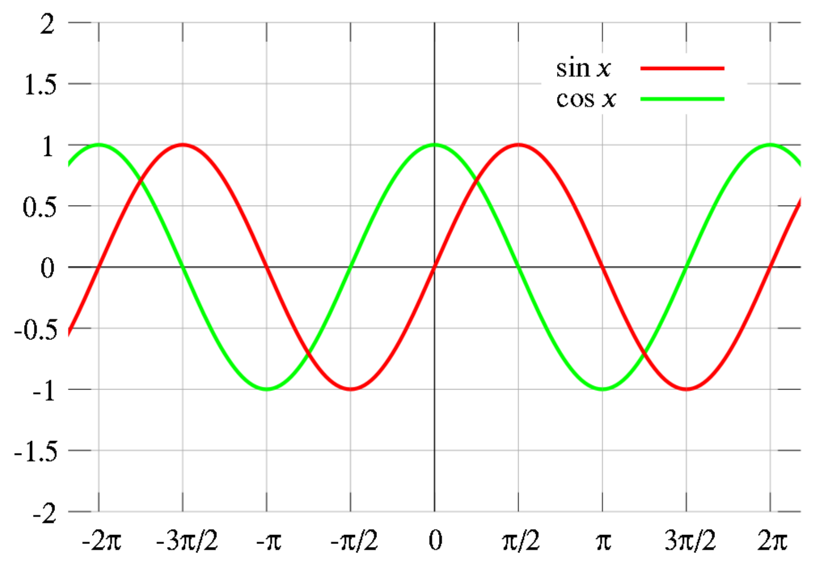
Since the function shown in Figure is odd, the Fourier series will contain only sine terms

$$y(t) = \sum_{n=1}^{\infty} B_n sinn\omega t = \sum_{n=1}^{\infty} B_n sin\frac{2\pi nt}{T}$$

$$B_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) sin\frac{2\pi nt}{T} dt$$

$$B_n = \frac{2}{10}\int_{-5}^{0} (-1) sin\frac{2\pi nt}{10} dt + \frac{2}{10}\int_{0}^{5} (1) sin\frac{2\pi nt}{10} dt$$

$$B_n = \frac{2}{10}\int_{-5}^{0}(-1)\sin\frac{2\pi nt}{10}dt + \frac{2}{10}\int_{0}^{5}(1)\sin\frac{2\pi nt}{10}dt$$



$$B_n = \frac{2}{10}\left(\frac{10}{2n\pi}\right)\cos\left(\frac{2\pi nt}{10}\right)\Bigg|_{-5}^{0} + \frac{2}{10}\left(\frac{-10}{2n\pi}\right)\cos\left(\frac{2\pi nt}{10}\right)\Bigg|_{0}^{5}$$

$$B_n = \left(\frac{1}{n\pi}\right)(\cos 0 - \cos(-n\pi)) + \left(-\frac{1}{n\pi}\right)(\cos(n\pi) - \cos 0)$$

$$B_n = \left(\frac{1}{n\pi}\right)(1 - \cos(-n\pi)) + \left(\frac{1}{n\pi}\right)(1 - \cos(n\pi))$$

$$B_n = \left(\frac{1}{n\pi}\right)(2 - \cos(-n\pi) - \cos(n\pi))$$

$$\cos n\pi = \cos(-n\pi)$$

$$\cos n\pi = \cos(-n\pi) = -1 \text{ for n odd}$$

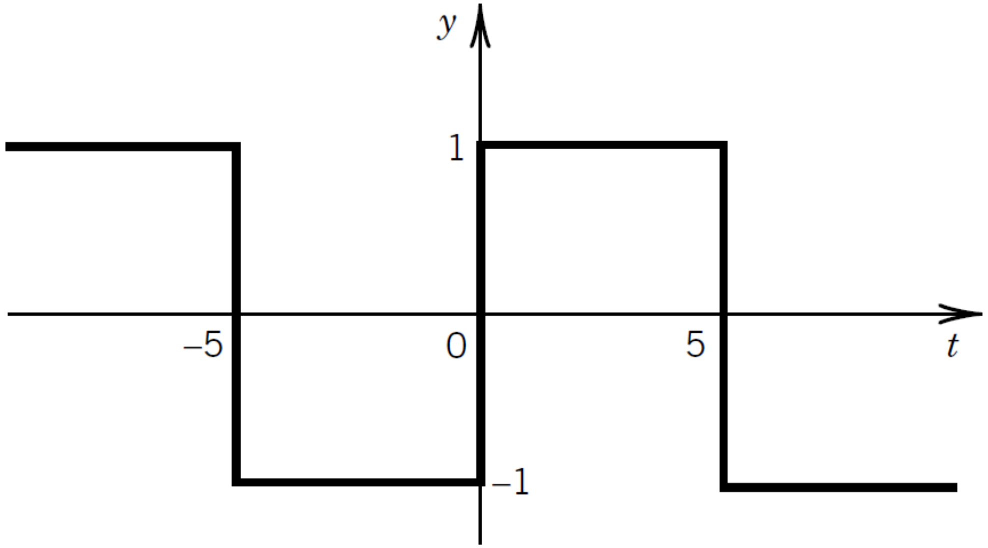
$$\cos n\pi = \cos(-n\pi) = +1 \text{ for n even}$$

For even values of n, B_n is identically zero and for odd values of n, B_n exists

$$B_n = \underbrace{\left(\frac{1}{n\pi}\right)(2 - (-1) - (-1))}_{\text{Odd values of n}} + \underbrace{\left(\frac{1}{n\pi}\right)(2 - (+1) - (+1))}_{\text{Even values of n}} = \frac{4}{n\pi}$$

$$y(t) = \sum_{n=1}^{\infty} B_n \sin\frac{2\pi nt}{10}$$

$$y(t) = \frac{4}{\pi}\sin\frac{2\pi}{10}t + \frac{4}{3\pi}\sin\frac{6\pi}{10}t + \frac{4}{5\pi}\sin\frac{10\pi}{10}t + \cdots$$



$$T = 10\ seconds$$

$$y(t) = \frac{4}{\pi} sin \frac{2\pi}{10} t + \frac{4}{3\pi} sin \frac{6\pi}{10} t + \frac{4}{5\pi} sin \frac{10\pi}{10} t + \cdots$$

$$\omega_1 = \frac{2\pi}{10} \frac{rad}{s}$$

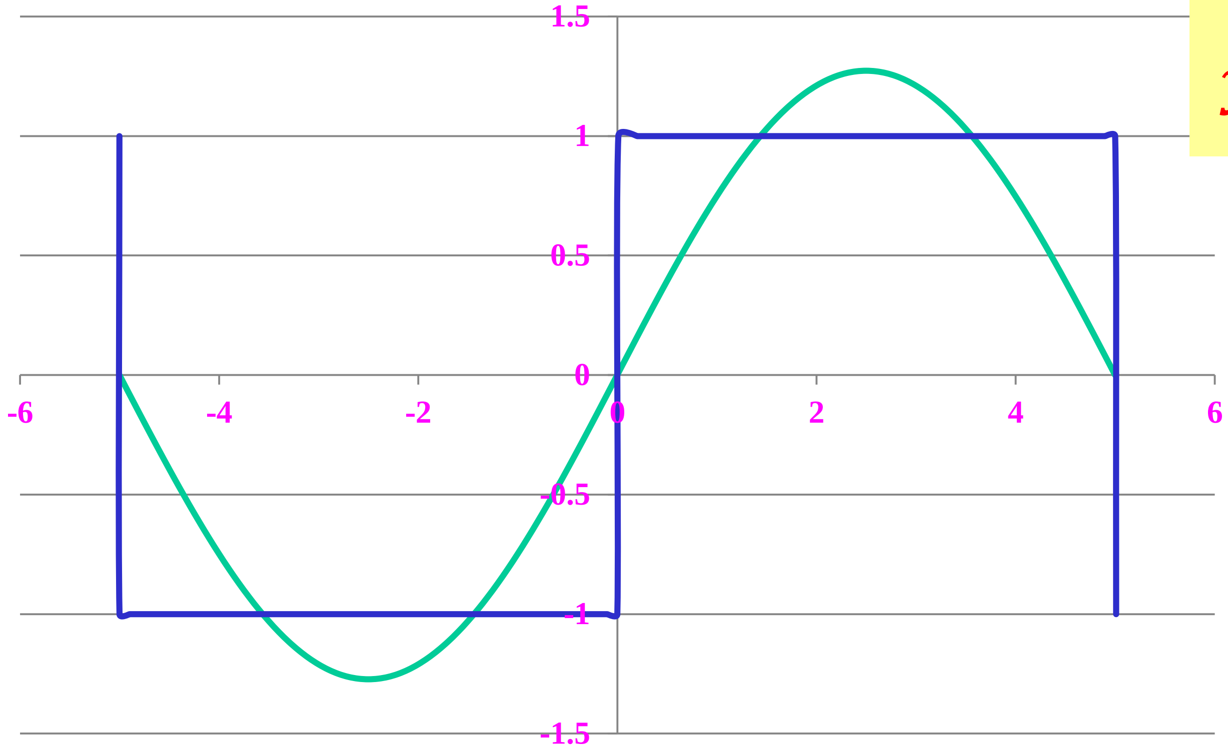
Fundamental frequency

$$\omega_3 = 3\left(\frac{2\pi}{10}\right) \frac{rad}{s}$$

Third Harmonic

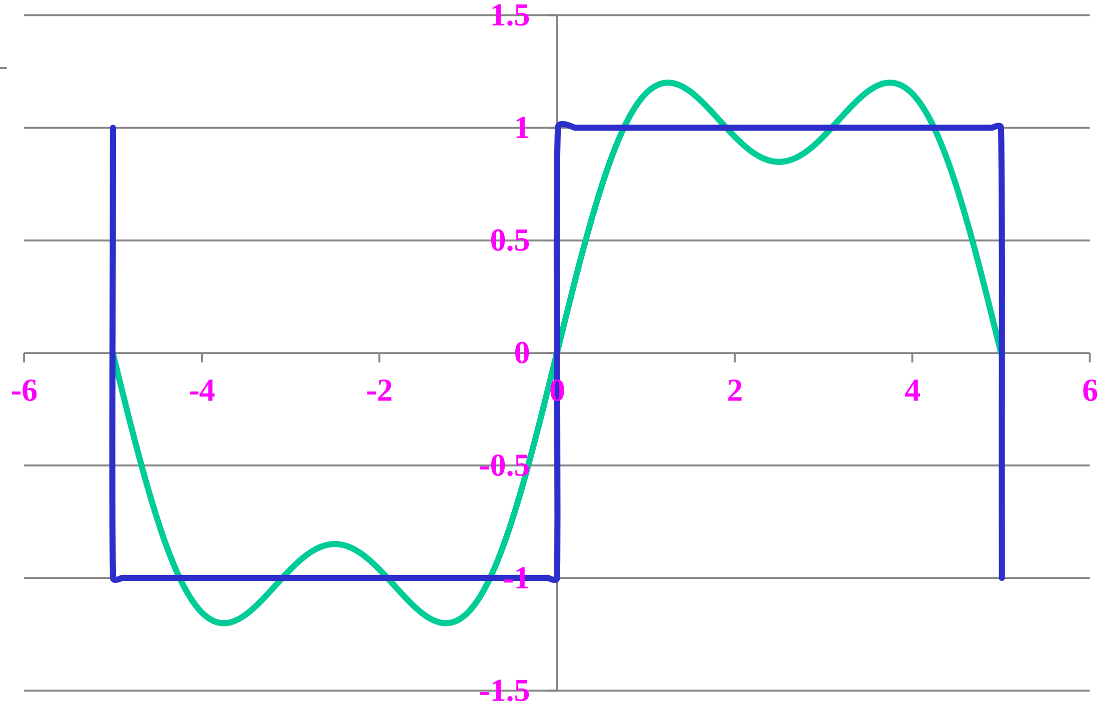
$$\omega_5 = 5\left(\frac{2\pi}{10}\right) \frac{rad}{s}$$

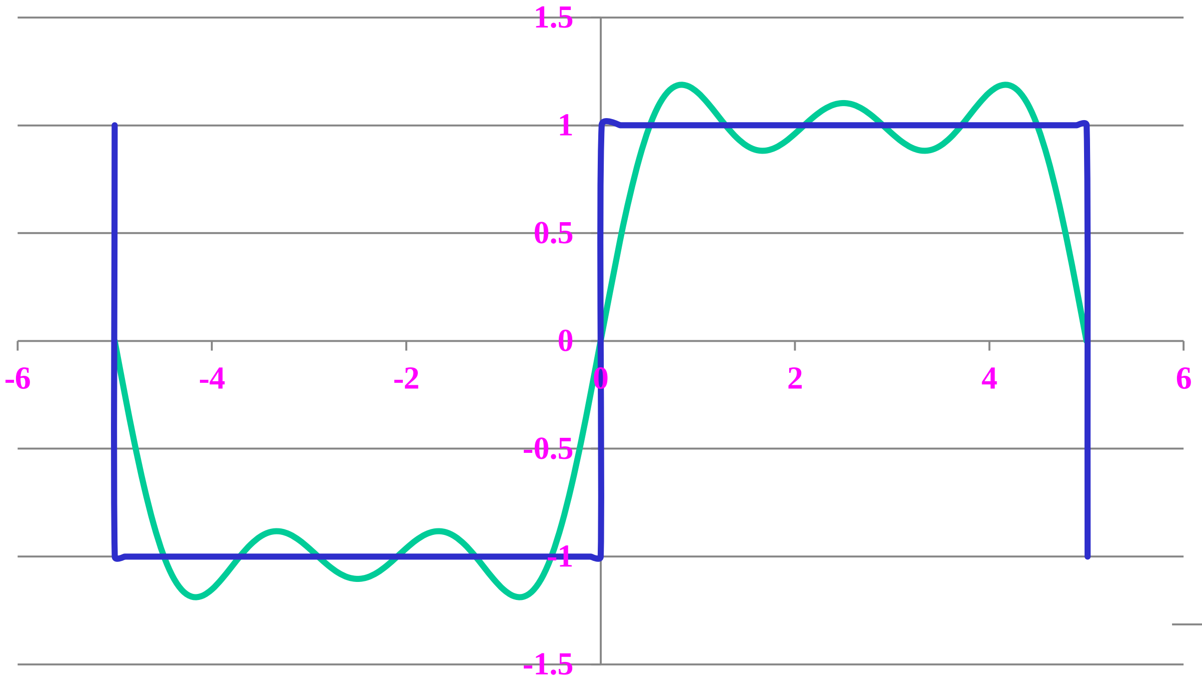
Fifth Harmonic



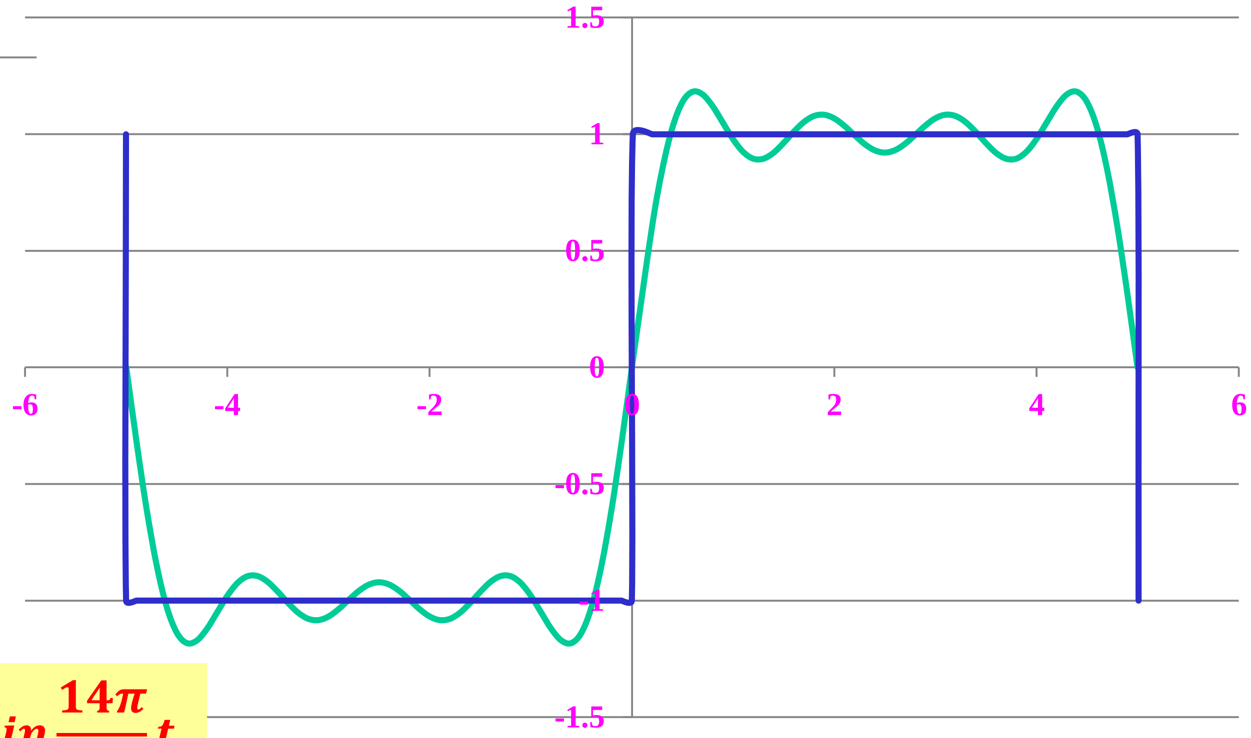
$$y(t) = \frac{4}{\pi} sin \frac{2\pi}{10} t$$

$$y(t) = \frac{4}{\pi} sin \frac{2\pi}{10} t + \frac{4}{3\pi} sin \frac{6\pi}{10} t$$



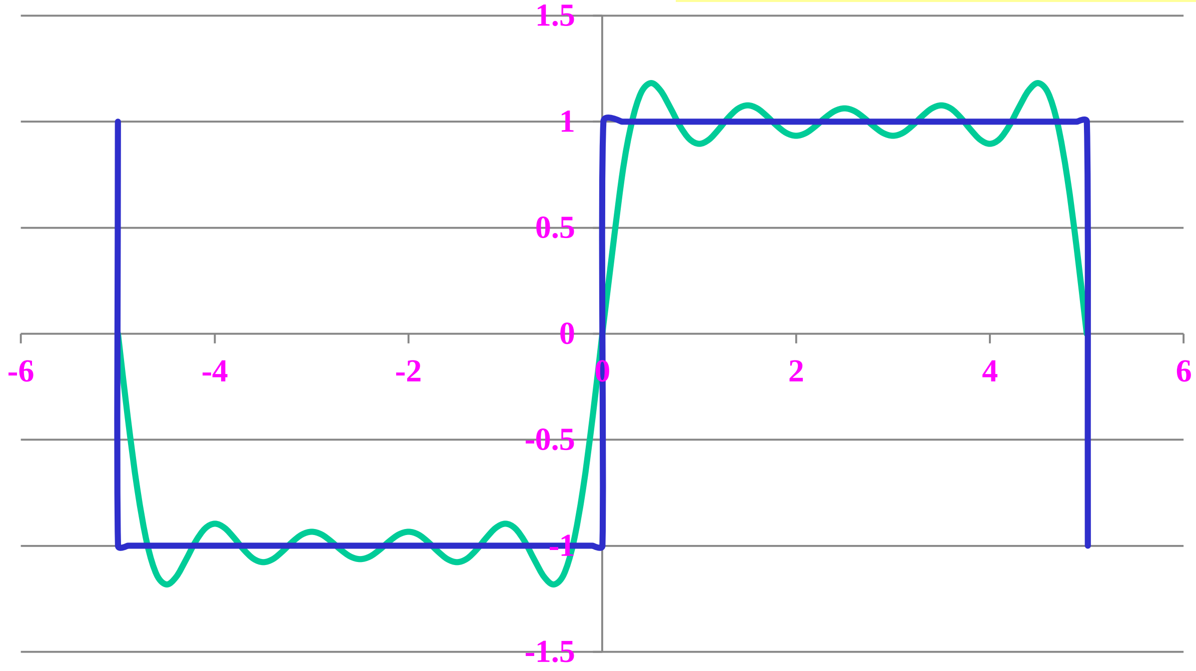


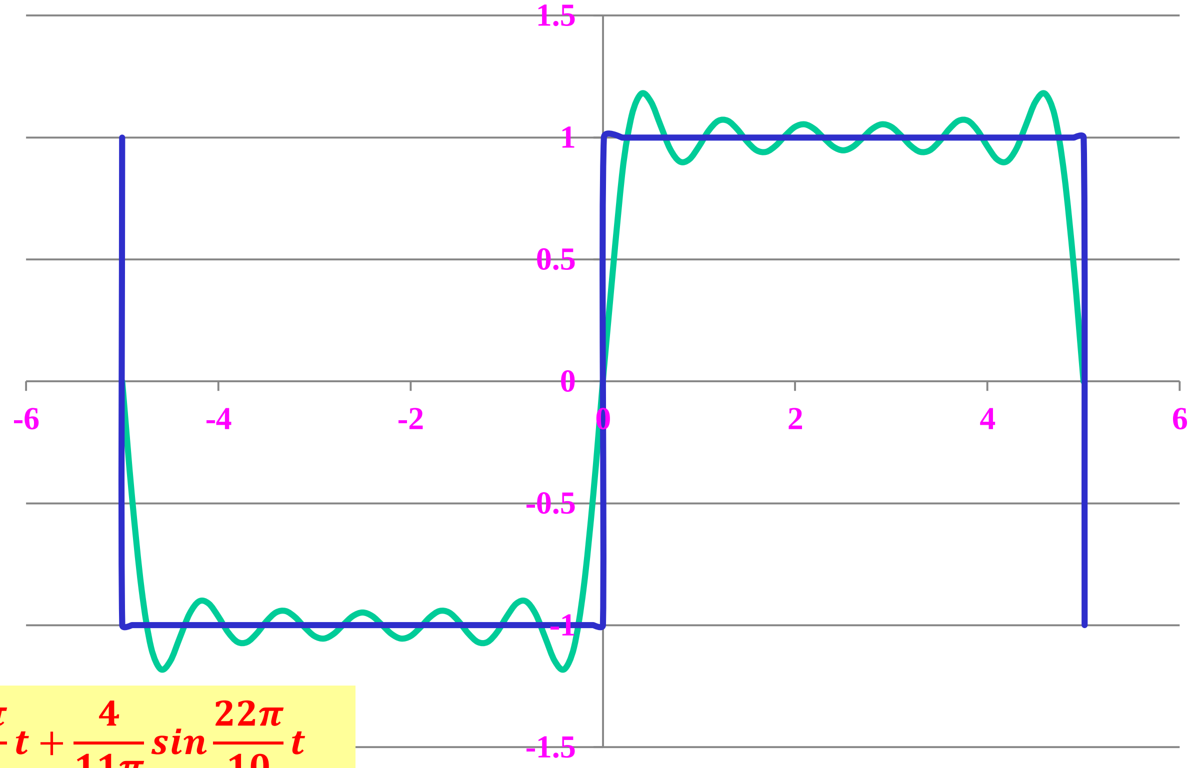
$$y(t) = \frac{4}{\pi} sin \frac{2\pi}{10} t + \frac{4}{3\pi} sin \frac{6\pi}{10} t + \frac{4}{5\pi} sin \frac{10\pi}{10} t$$



$$y(t) = \frac{4}{\pi} sin \frac{2\pi}{10} t + \frac{4}{3\pi} sin \frac{6\pi}{10} t + \frac{4}{5\pi} sin \frac{10\pi}{10} t + + \frac{4}{7\pi} sin \frac{14\pi}{10} t$$

$$y(t) = \frac{4}{\pi} sin \frac{2\pi}{10} t + \frac{4}{3\pi} sin \frac{6\pi}{10} t + \frac{4}{5\pi} sin \frac{10\pi}{10} t + \frac{4}{7\pi} sin \frac{14\pi}{10} t + + \frac{4}{9\pi} sin \frac{18\pi}{10} t$$

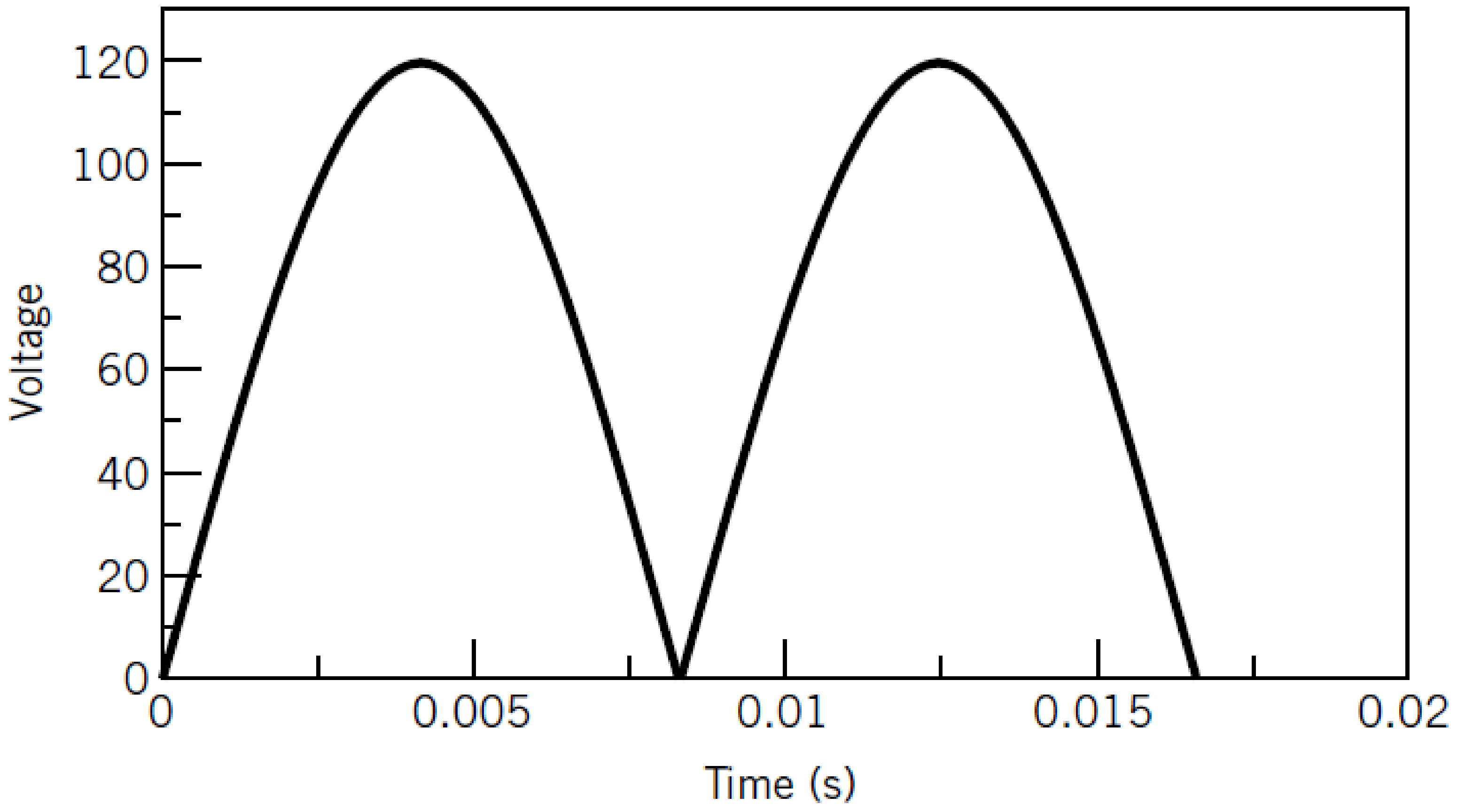




$$y(t) = \frac{4}{\pi} sin \frac{2\pi}{10} t + \frac{4}{3\pi} sin \frac{6\pi}{10} t + \frac{4}{5\pi} sin \frac{10\pi}{10} t + \frac{4}{7\pi} sin \frac{14\pi}{10} t + \frac{4}{9\pi} sin \frac{18\pi}{10} t + \frac{4}{11\pi} sin \frac{22\pi}{10} t$$

As an example of interpreting the frequency content of a given signal, consider the output voltage from a rectifier. A rectifier functions to “flip” the negative half of an alternating current (AC) into the positive half plane, resulting in a signal that appears as shown in Figure. For the AC signal the voltage is given by $E(t) = 120sin120\pi t$

The period of the signal is 1/60 s, and the frequency is 60 Hz



The rectified signal can be expressed as

$$E(t) = |120sin120\pi t\ |$$

$$A_o = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)dt$$

$$E(t) = 120sin120\pi t$$

$$A_o = 2\frac{1}{T}\int_{0}^{\frac{T}{2}} E(t)dt = 2\frac{1}{\left(\frac{1}{60}\right)}\int_{0}^{\frac{T}{2}} 120sin120\pi t\, dt = (-2\times 60\times 120)\left(\frac{cos120\pi t}{120\pi}\Bigg|_{0}^{\frac{1}{120}}\right)$$

$$A_o = -\frac{120}{\pi}(cos\pi - cos0) = -\frac{120}{\pi}(-1-1) = \frac{240}{\pi}$$

$$A_o = \frac{240}{\pi}$$

$$A_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)cosn\omega tdt$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\left(\frac{1}{60}\right)} = 120\pi\frac{rad}{s}$$

$$E(t) = 120sin120\pi t$$

$$A_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 120\ sin120\pi t\ cosn\omega tdt = \frac{4}{T}\int_{0}^{\frac{T}{2}} 120\ sin120\pi t\ cosn\omega tdt$$

$$A_n = \frac{4\times 120}{\left(\frac{1}{60}\right)}\int_{0}^{\frac{1}{2\times 60}} sin120\pi t\ cos120\pi ntdt = \frac{4\times 120}{\left(\frac{1}{60}\right)}\int_{0}^{\frac{1}{120}} sin120\pi t\ cos120\pi ntdt$$

$$A_n = \frac{4\times 120}{\left(\frac{1}{60}\right)}\int_{0}^{\frac{1}{120}} sin(1+n)120\pi t + sin(1-n)120\pi tdt$$

$$A_n = \frac{4 \times 120}{\left(\frac{1}{60}\right)} \int_0^{\frac{1}{120}} sin(1+n)120\pi t + sin(1-n)120\pi t dt$$

$$A_n = \frac{240 \times 120}{2} \left(-\frac{cos(1+n)120\pi t}{(1+n)120\pi} - \frac{cos(1-n)120\pi t}{(1-n)120\pi} \right) \Bigg|_0^{\frac{1}{120}}$$

$$A_n = 120 \left(-\frac{cos(1+n)120\pi t}{(1+n)\pi} - \frac{cos(1-n)120\pi t}{(1-n)\pi} \right) \Bigg|_0^{\frac{1}{120}}$$

$$A_n = \frac{120}{\pi} \left(-\frac{cos(1+n)120\pi t}{(1+n)} - \frac{cos(1-n)120\pi t}{(1-n)} \right) \Bigg|_0^{\frac{1}{120}}$$

$$A_n = \frac{120}{\pi} \left(\frac{-cos(1+n)120\pi \frac{1}{120} + 1}{(1+n)} + \frac{-cos(1-n)120\pi \frac{1}{120} + 1}{(1-n)} \right)$$

$$A_n = \frac{120}{\pi}\left(\frac{-cos(1+n)120\pi\frac{1}{120}+1}{(1+n)}+\frac{-cos(1-n)120\pi\frac{1}{120}+1}{(1-n)}\right)$$

$$A_n = \frac{120}{\pi}\left(\frac{-cos(1+n)120\pi\frac{1}{120}+1}{(1+n)}+\frac{-cos(1-n)120\pi\frac{1}{120}+1}{(1-n)}\right)$$

$$A_n = \frac{120}{\pi}\left(\frac{-cos(1+n)\pi+1}{(1+n)}+\frac{-cos(1-n)\pi+1}{(1-n)}\right)$$

$$A_n = \frac{120}{\pi}\left(\frac{-cos(1+n)\pi+1}{(1+n)}+\frac{-cos(1-n)\pi+1}{(1-n)}\right)$$

$$cosn\pi = \; cos(-n\pi)$$
$$cosn\pi = \; cos(-n\pi) = -1 \text{ for n odd}$$
$$cosn\pi = \; cos(-n\pi) = +1 \text{ for n even}$$

$$n \; is \; odd \;\; \Rightarrow (n+1) \; is \; even \Rightarrow cos(n+1)\pi = +1$$

$$A_n = \frac{120}{\pi}\left(\frac{-(+1)+1}{(1+n)}+\frac{-(+1)+1}{(1-n)}\right) = 0$$

$$n \; is \; even \;\; \Rightarrow (n+1) \; is \; odd \Rightarrow cos(n+1)\pi = -1$$

$$A_n = \frac{120}{\pi}\left(\frac{-(-1)+1}{(1+n)}+\frac{-(-1)+1}{(1-n)}\right)$$

$$A_n = \frac{120}{\pi}\left(\frac{-cos(1+n)\pi + 1}{(1+n)} + \frac{-cos(1-n)\pi + 1}{(1-n)}\right)$$

$$cosn\pi = cos(-n\pi)$$
$$cosn\pi = cos(-n\pi) = -1 \text{ for n odd}$$
$$cosn\pi = cos(-n\pi) = +1 \text{ for n even}$$

$$n \; is \; odd \;\Rightarrow (n+1) \; is \; even \Rightarrow cos(n+1)\pi = +1$$

$$A_n = \frac{120}{\pi}\left(\frac{-(+1)+1}{(1+n)} + \frac{-(+1)+1}{(1-n)}\right) = 0$$

$$n \; is \; even \;\Rightarrow (n+1) \; is \; odd \Rightarrow cos(n+1)\pi = -1$$

$$A_n = \frac{120}{\pi}\left(\frac{-(-1)+1}{(1+n)} + \frac{-(-1)+1}{(1-n)}\right) = \frac{120}{\pi}\left(\frac{2}{(1+n)} + \frac{2}{(1-n)}\right)$$

$$A_n = \frac{120}{\pi}\left(\frac{2}{(1+n)} + \frac{2}{(1-n)}\right) \text{only for n is even from } 2, 4, 6..$$

$$B_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) sin\frac{2\pi nt}{T}dt$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\left(\frac{1}{60}\right)} = 120\pi\frac{rad}{s}$$

$$E(t) = 120sin120\pi t$$

$$sinn\pi = sin(-n\pi) = 0$$

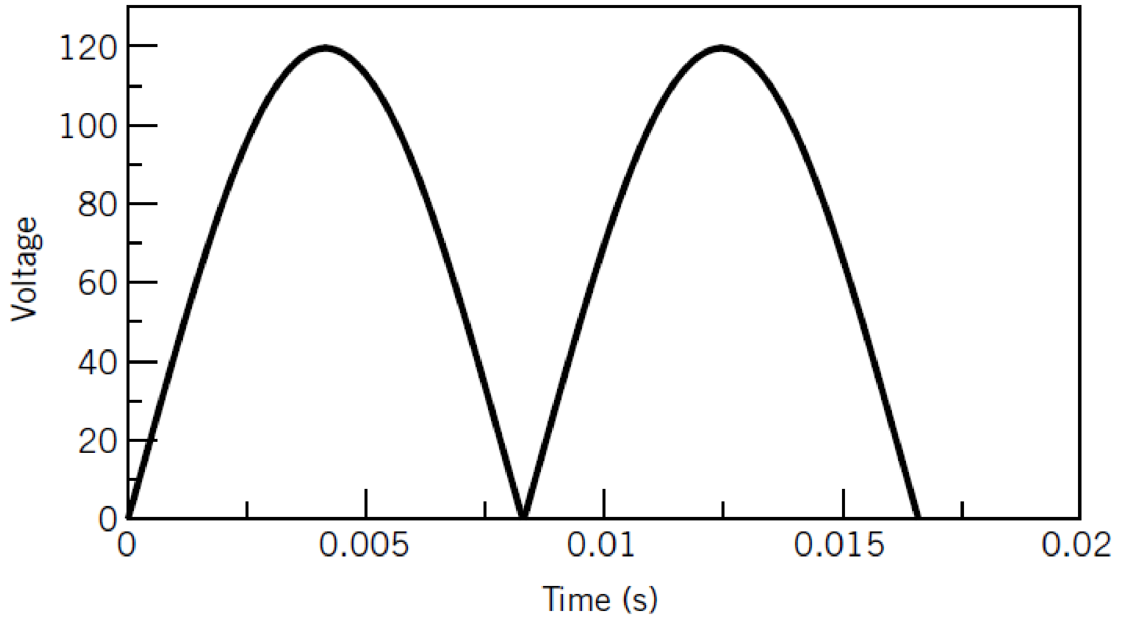
$$B_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 120\ sin120\pi t\ sinn\omega tdt = \frac{4}{T}\int_{0}^{\frac{T}{2}} 120\ sin120\pi t\ sinn\omega tdt$$

$$sinAsinB = \frac{1}{2}\left(cos(A+B) - cos(A-B)\right)$$

$$B_n = \frac{4\times 120}{\left(\frac{1}{60}\right)}\int_{0}^{\frac{1}{2\times 60}} sin120\pi t\ sin120\pi ntdt = \frac{4\times 120}{\left(\frac{1}{60}\right)}\int_{0}^{\frac{1}{120}} sin120\pi t\ sin120\pi ntdt$$

Integration of cos terms results in sin terms and knowing that $sinn\pi = sin(-n\pi) = 0$ implies that B_n is zero

$$E(t) = 120\sin 120\pi t$$



$$y(t) = A_o + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t)$$

$$A_o = \frac{240}{\pi}$$

$$A_n = \frac{120}{\pi}\left(\frac{2}{(1+n)} + \frac{2}{(1-n)}\right)$$

only for n is even 2, 4, 6 …

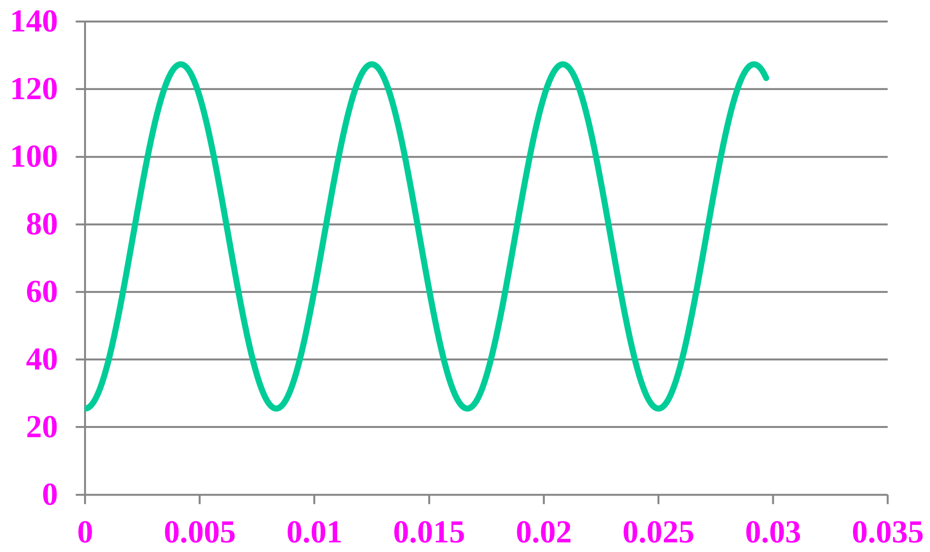
$$B_n = 0$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\left(\frac{1}{60}\right)} = 120\pi \frac{rad}{s}$$

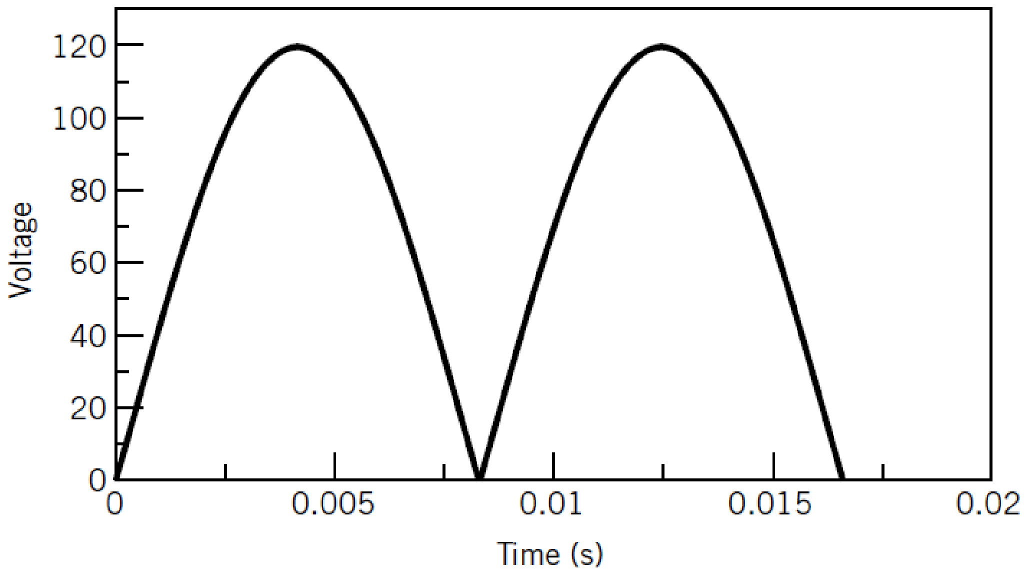
$$y(t) = \frac{240}{\pi} + \sum_{\substack{n=2 \\ for\ even}}^{\infty} \frac{120}{\pi}\left(\frac{2}{(1+n)} + \frac{2}{(1-n)}\right) \cos 120\pi n t$$

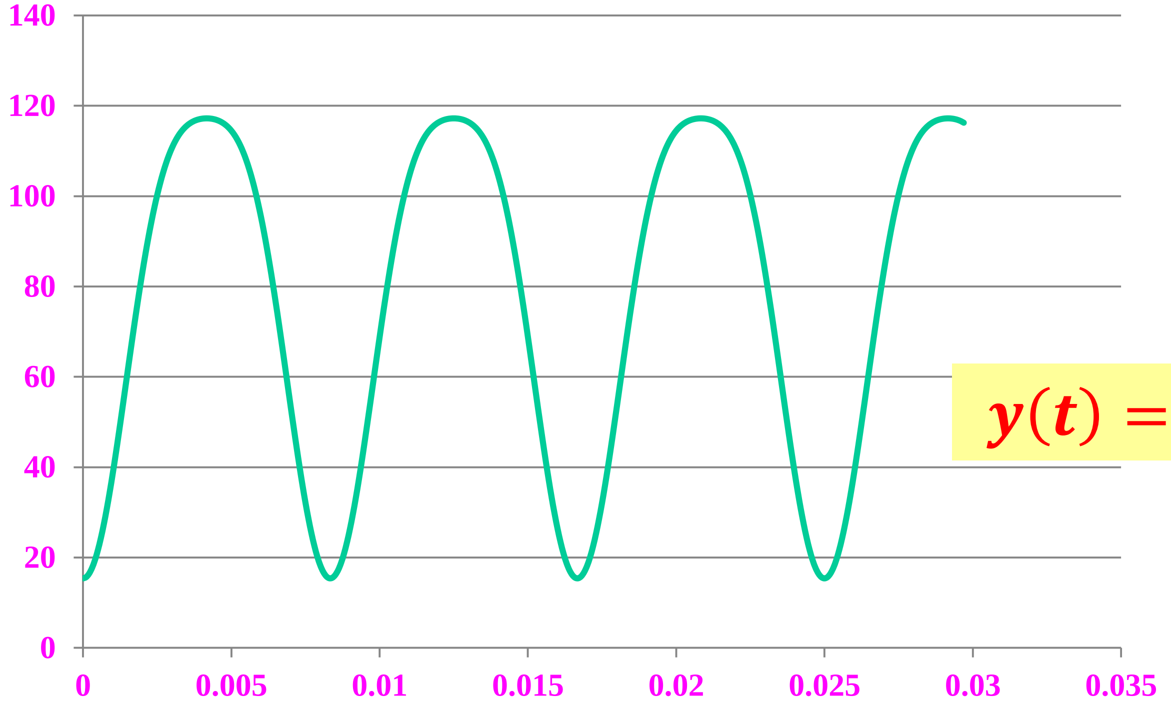
$$y(t) = \frac{240}{\pi} + \sum_{\substack{n=2 \\ for\ even}}^{\infty} \frac{120}{\pi}\left(\frac{2}{(1+n)} + \frac{2}{(1-n)}\right) \cos 120\pi n t$$

$$y(t) = 76.4 - 50.93\cos 240\pi t - 10.19\cos 480\pi t - 4.37\cos 720\pi t$$



$$y(t) = 76.4 - 50.93cos240\pi t$$





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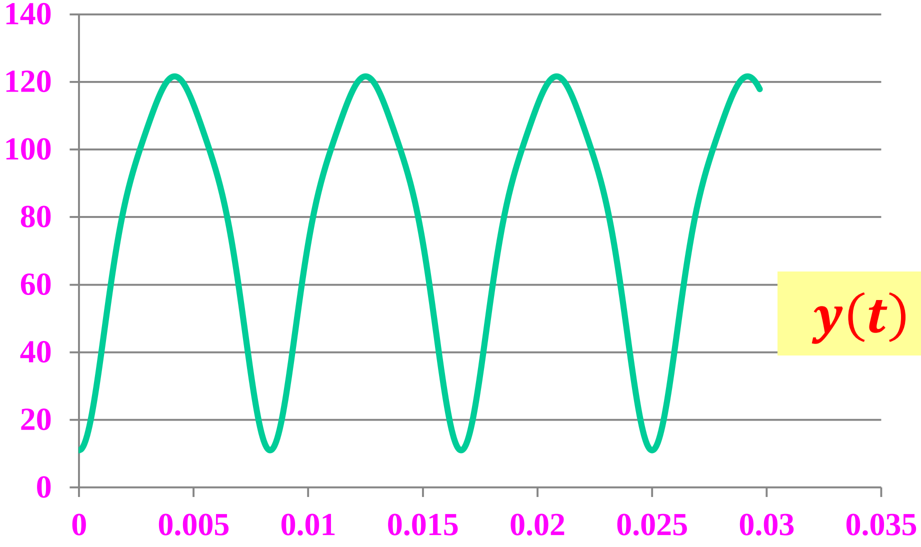

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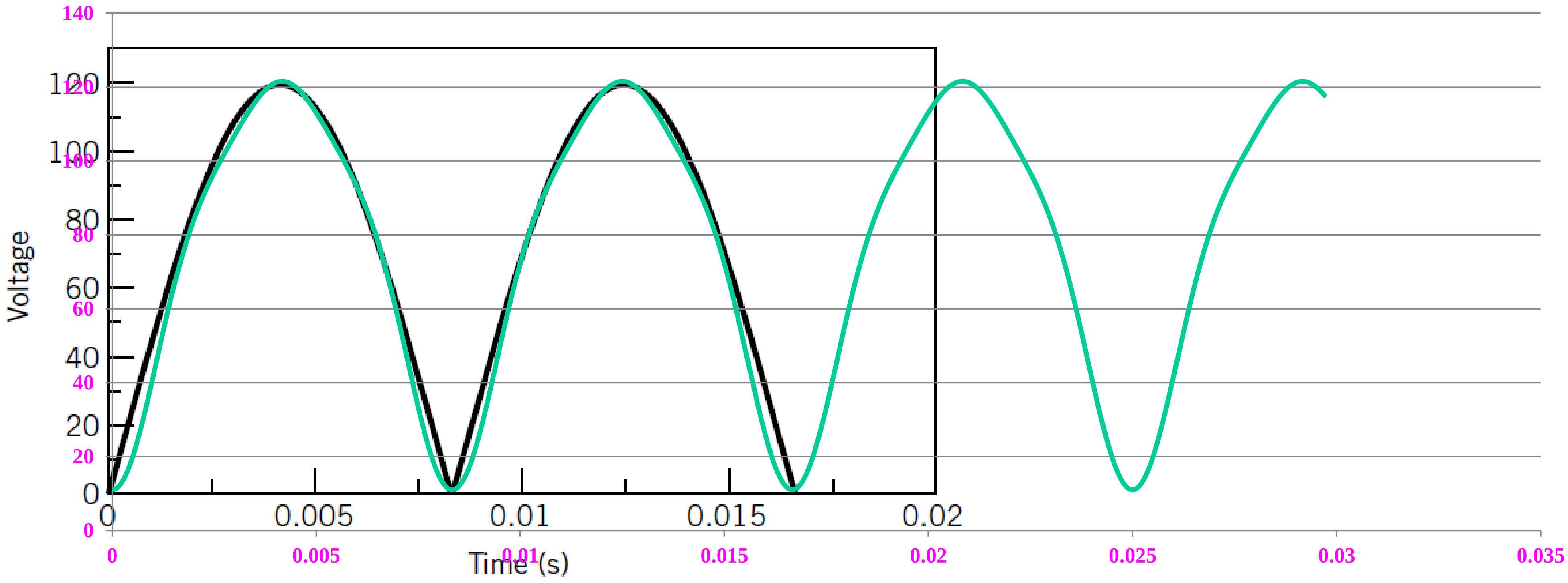
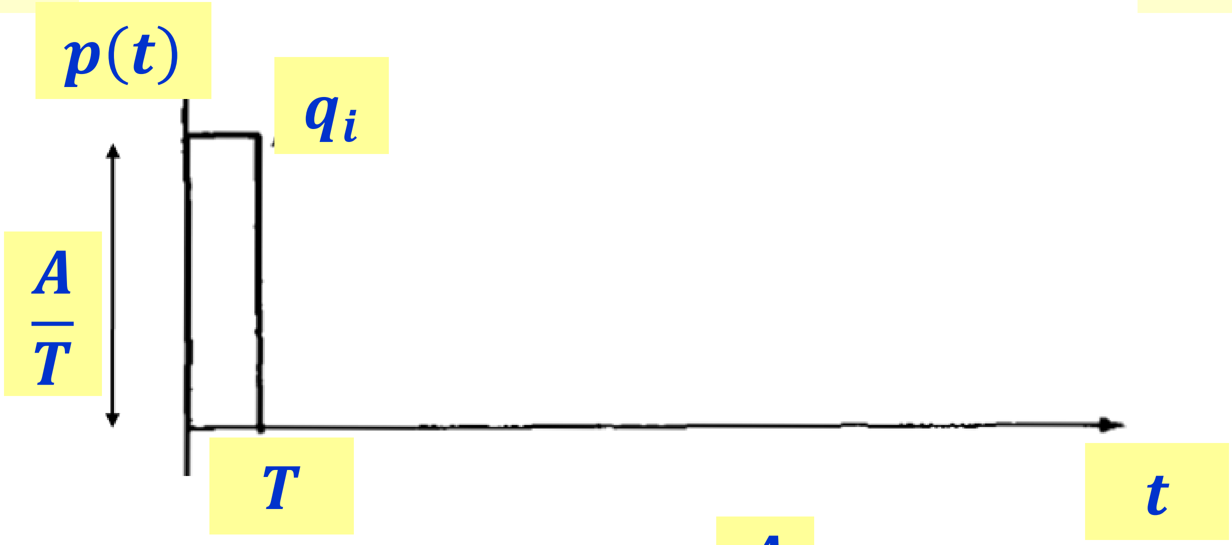


IMPULSE FUNCTION



$$g(t) = \lim_{T\to 0} \frac{A}{T} \quad for \;\; 0 < t < T$$

$$= 0 \quad for \;\; t < 0, T < t$$

Height of the impulse function = $\frac{A}{T}$

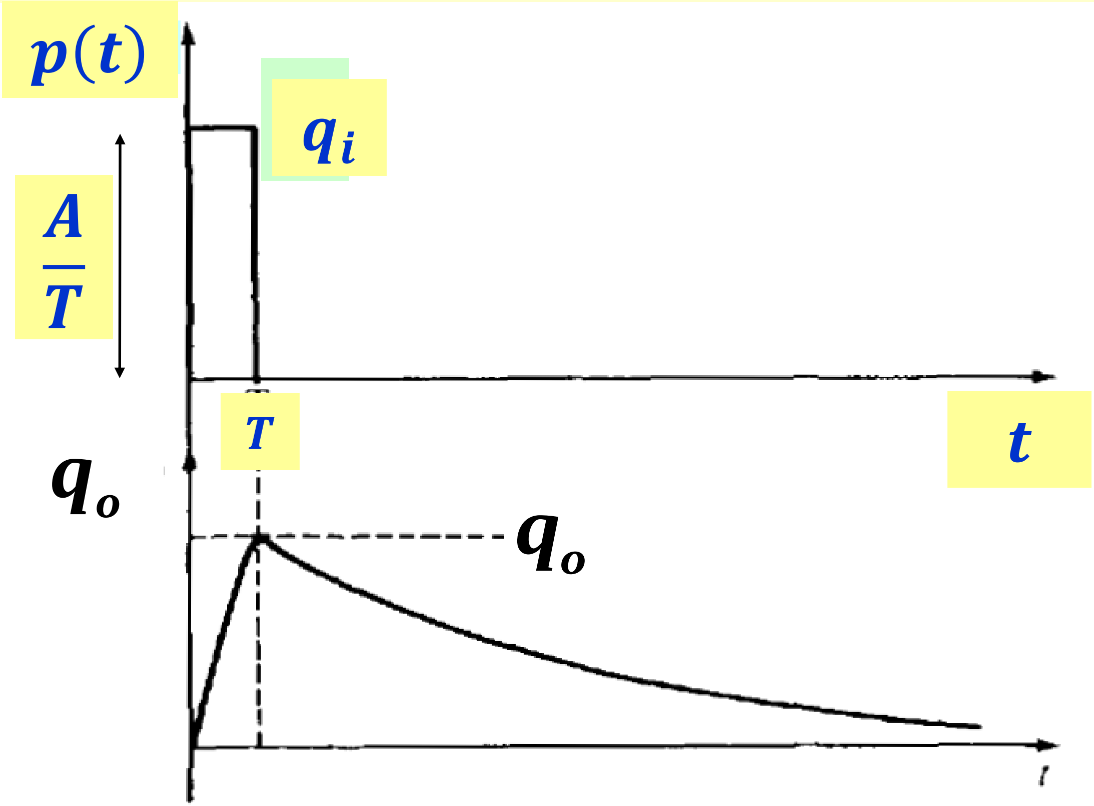
Duration of the impulse function = T

Area under the impulse = A

As the duration to approaches zero, the height $\frac{A}{T}$ approaches infinity, but the area under the impulse remains equal to A.

Note that the magnitude of the impulse is measured by its area.

IMPULSE RESPONSE OF FIRST ORDER SYSTEM



$$\tau \frac{dq_o}{dt} + q_o = Kq_i(t)$$

$$q_i = \frac{A}{T}$$

A – a constant

$$\tau \frac{dq_o}{dt} + q_o = \frac{KA}{T}$$

$$\tau[sq_o(s) - \cancel{q_o}(0)] + q_o(s) = \frac{KA}{T}\frac{1}{s}$$

$$\frac{q_o(s)}{\frac{KA}{T}} = \frac{1}{s(\tau s + 1)} = \frac{A}{s} + \frac{B}{(\tau s + 1)}$$

$$\mathcal{L}\left(\frac{dq_o(t)}{dt}\right) = sq_o(s) - q_o(0)$$

$$\frac{1}{s(\tau s + 1)} = \frac{A}{s} + \frac{B}{(\tau s + 1)} \Rightarrow A(\tau s + 1) + Bs = 1$$

$$\mathcal{L}(q_o(t)) = q_o(s)$$

$$\mathcal{L}\left(K\frac{A}{T}\right) = \frac{KA}{T}\frac{1}{s}$$

$$\frac{1}{s(\tau s+1)} = \frac{A}{s} + \frac{B}{(\tau s+1)} \Rightarrow A(\tau s+1) + Bs = 1$$

$$Coefficient\ of\, s; \quad 0 = A\tau + B \Rightarrow \ 0 = \ (1)\tau + B \Rightarrow B = -\tau$$

$$Coefficient\ of\, s^0; \quad A = 1$$

$$\frac{q_o(s)}{\frac{KA}{T}} = \frac{1}{s(\tau s+1)} = \frac{1}{s} + \frac{-\tau}{(\tau s+1)} = \frac{1}{s} - \frac{1}{\left(s+\frac{1}{\tau}\right)}$$

$$\frac{q_o(t)}{\frac{KA}{T}} = 1 - e^{-\frac{t}{\tau}}$$

$$q_o(t) = \frac{KA}{T}\left(1 - e^{-\frac{t}{\tau}}\right)$$

However, this solution is valid only upto time T.

At t $> T$, .

$$\tau \frac{dq_o}{dt} + q_o = 0$$

$$\tau \int \frac{dq_o}{q_o} = -\int dt \Rightarrow \tau \ln q_o = -t + C_1 \Rightarrow \ln q_o = -\frac{t}{\tau} + \frac{C_1}{\tau} \Rightarrow q_o = Ce^{-\frac{t}{\tau}}$$

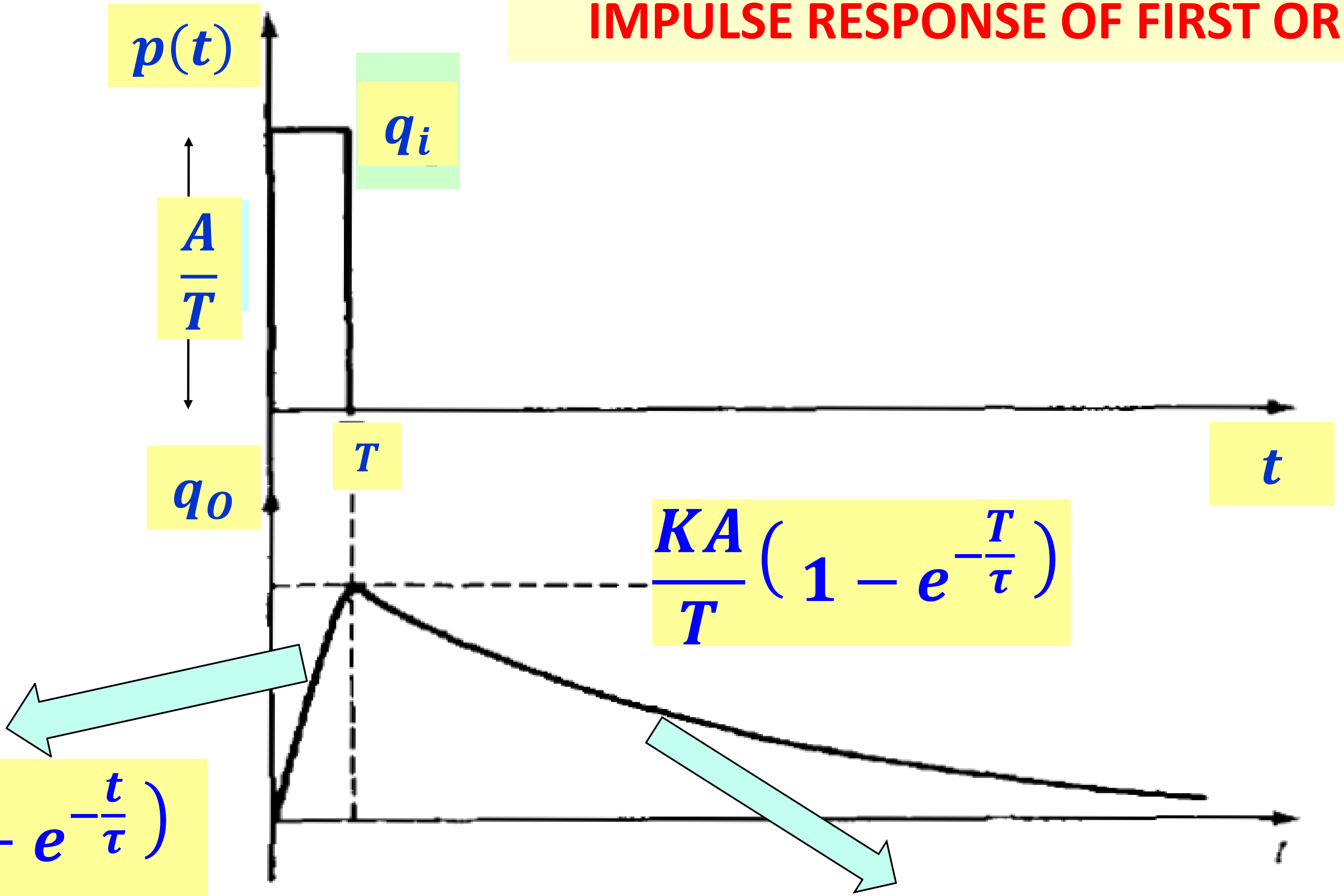
Initial Condition $t = T$

$$q_o\Big|_T = \frac{KA}{T}\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\frac{KA}{T}\left(1 - e^{-\frac{T}{\tau}}\right) = Ce^{-\frac{T}{\tau}} \Rightarrow C = \frac{\frac{KA}{T}\left(1 - e^{-\frac{T}{\tau}}\right)}{e^{-\frac{T}{\tau}}}$$

$$q_o = \frac{\frac{KA}{T}\left(1 - e^{-\frac{T}{\tau}}\right)}{e^{-\frac{T}{\tau}}} e^{-\frac{t}{\tau}}$$

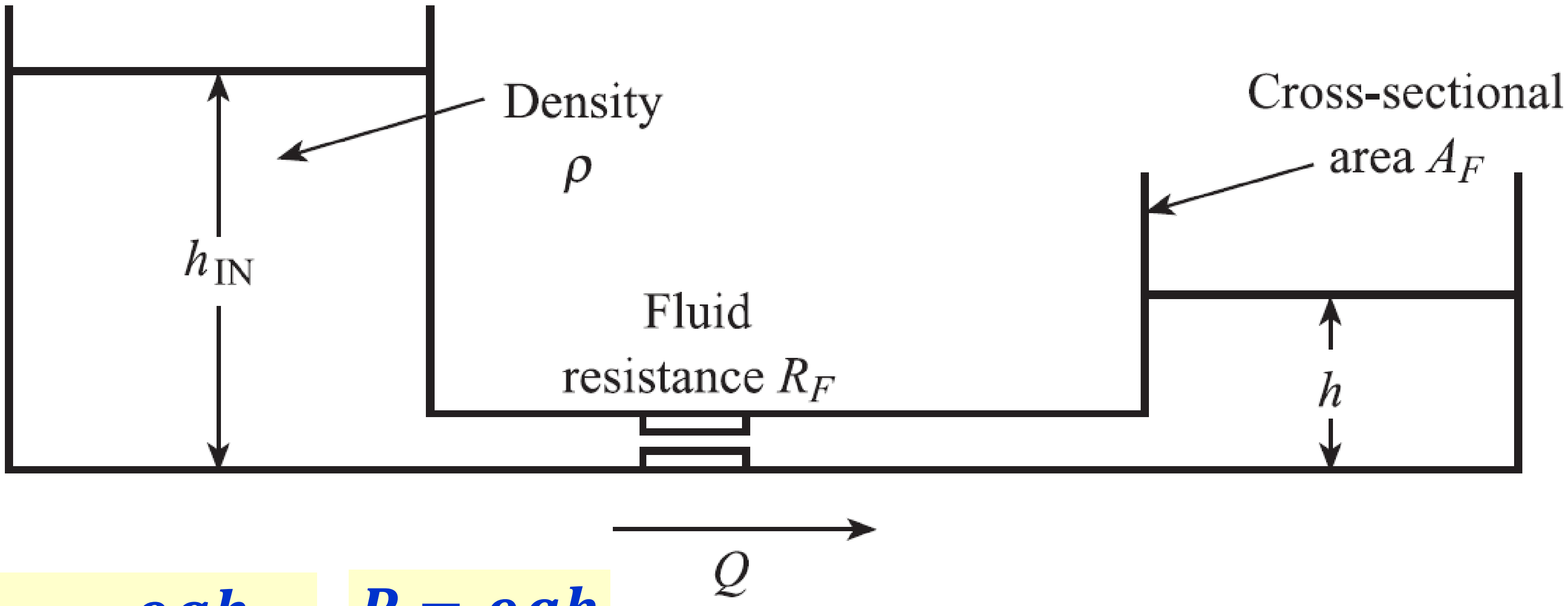
IMPULSE RESPONSE OF FIRST ORDER SYSTEM



$$q_o(t)=\frac{KA}{T}\left(1-e^{-\frac{t}{\tau}}\right)$$

$$q_o=\frac{\frac{KA}{T}\left(1-e^{-\frac{T}{\tau}}\right)}{e^{-\frac{T}{\tau}}}e^{-\frac{t}{\tau}}$$

FIRST ORDER HYDRAULIC SYSTEM



$$\dot{Q} = \frac{P_{in} - P}{R_F}$$ $$P_{in} = \rho g h_{IN}$$ $$P = \rho g h$$

$$\dot{Q} = A_F \frac{dh}{dt} = \frac{\rho g}{R_F}(h_{in} - h)$$

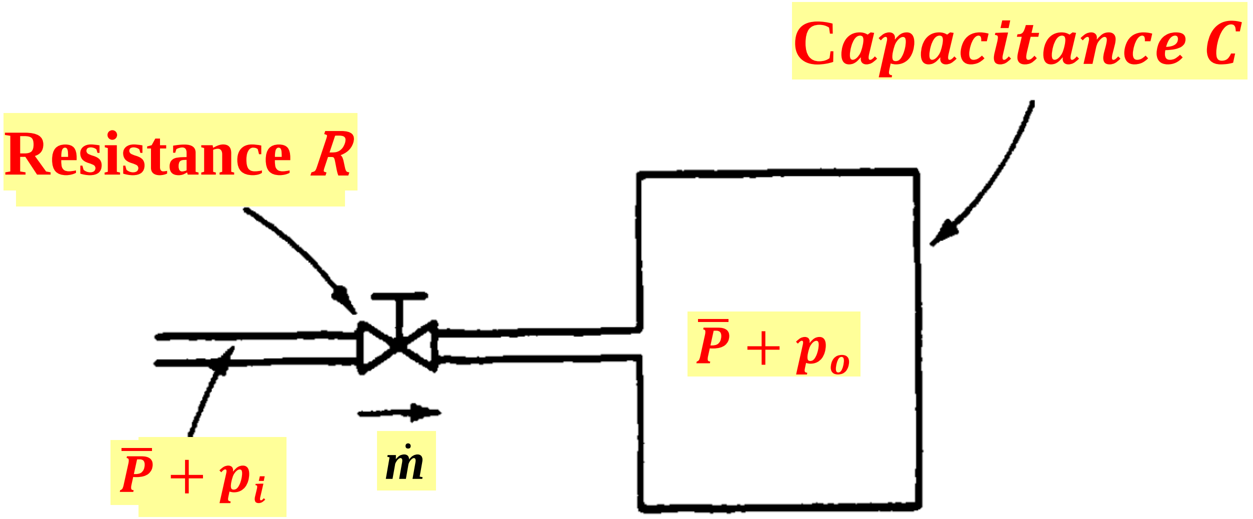
$$\frac{A_F R_F}{\rho g}\frac{dh}{dt} = (h_{in} - h)$$

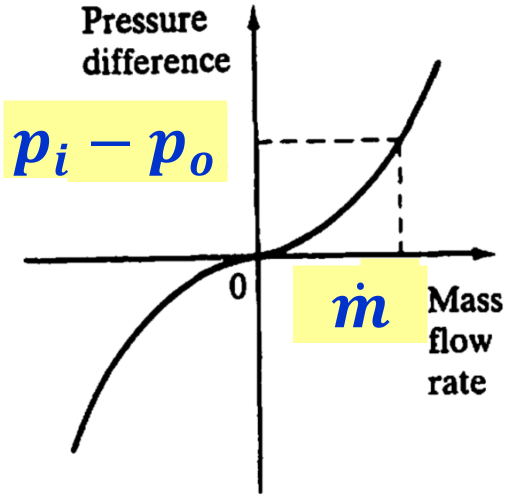
$$\frac{A_F R_F}{\rho g}\frac{dh}{dt} + h = h_{in}$$

$$\frac{A_F R_F}{\rho g}\frac{dh}{dt} + h = h_{in}$$ $$\tau \frac{dq_o}{dt} + q_o = K q_i(t)$$

$$\tau \frac{dh}{dt} + h = h_{in}$$ $$\tau = \frac{A_F R_F}{\rho g};\quad K = 1$$

FIRST ORDER PNEUMATIC SYSTEM





$$R = \frac{p_i - p_o}{\dot{m}}$$

$$C = \frac{dm}{dp_o} = \frac{Change\ in\ mass\ of\ air}{Change\ in\ Pressure}$$

$$RC\frac{dp_o}{dt} + p_o = p_i$$

$$Cdp_o = dm$$

$$Cdp_o = \dot{m}dt$$

$$C\frac{dp_o}{dt} = \dot{m}$$

$$Cdp_o = dm$$

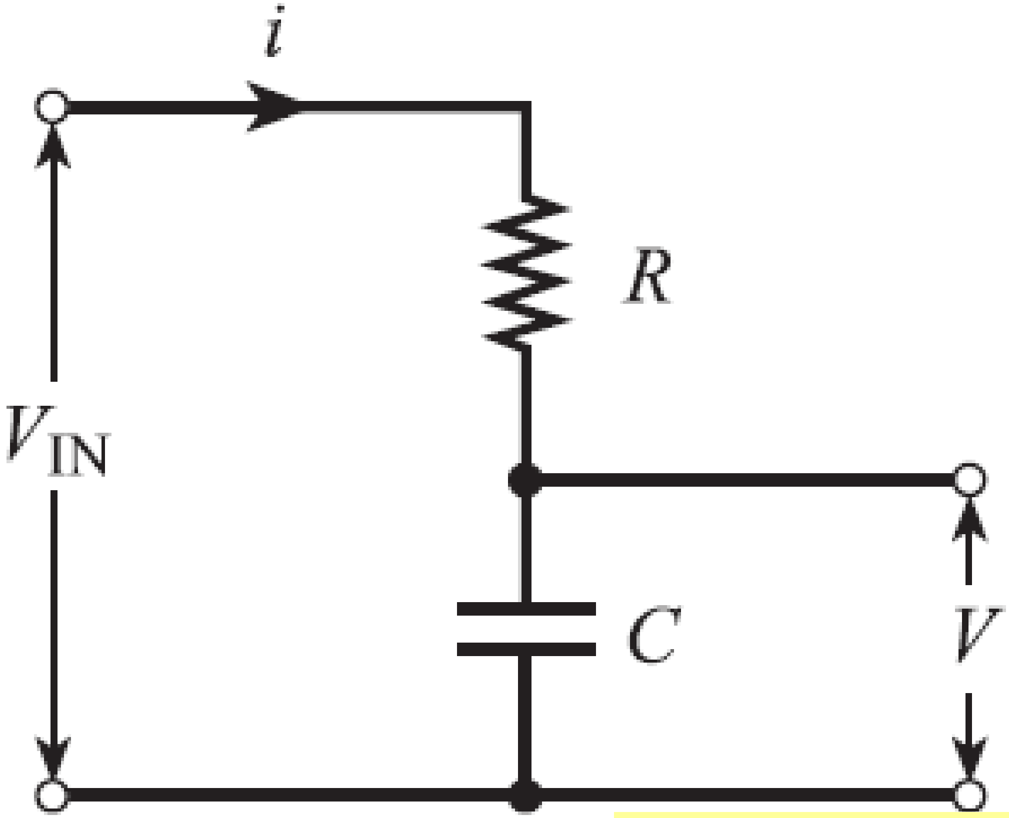
$$Cdp_o = \dot{m}dt$$

$$C\frac{dp_o}{dt} = \frac{p_i - p_o}{R}$$

$$RC\frac{dp_o}{dt} + p_o = p_i$$

$$\tau = RC; \quad K = 1$$

$$\tau\frac{dq_o}{dt} + q_o = Kq_i(t)$$



$$V_{IN} - V = iR$$

$$q = CV$$

$$V_{IN} - V = C\frac{dV}{dt}R$$

$$i = \frac{dq}{dt} = C\frac{dV}{dt}$$

$$RC\frac{dV}{dt} + V = V_{IN}$$

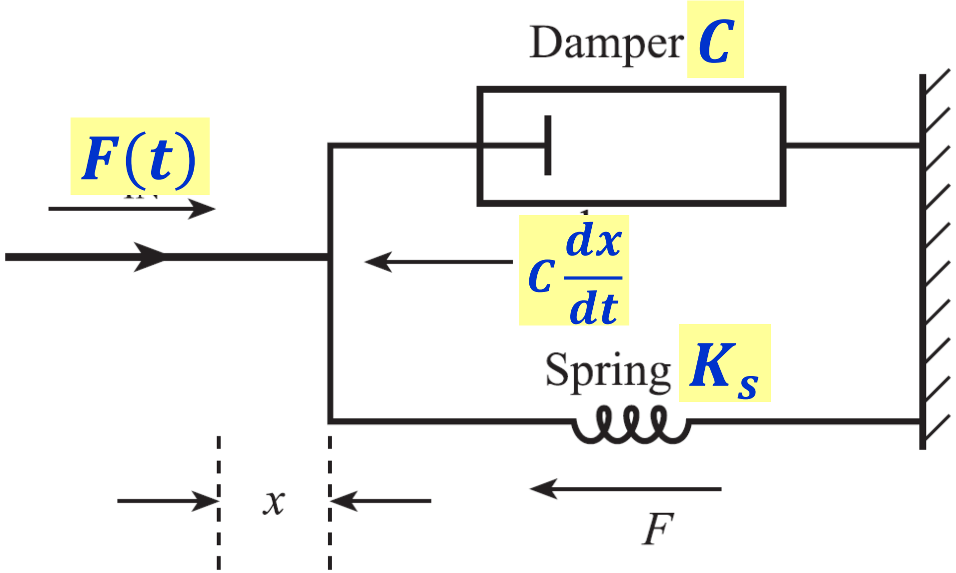
KIRCHOFF'S LAWS

- The algebraic sum of all currents entering a junction point is zero
- The algebraic sum of all voltage drops taken in a given direction around a closed circuit is zero

$$RC\frac{dV}{dt} + V = V_{IN} \qquad \tau\frac{dq_o}{dt} + q_o = Kq_i(t)$$

$$\tau = RC; \quad K = 1$$

FIRST ORDER MECHANICAL SYSTEM



C – Damping Constant $\left(\frac{N.s}{m}\right)$

K_s – Spring Stiffness $\left(\frac{N}{m}\right)$

$$C\frac{dx}{dt} + K_s x = F(t)$$

$$\frac{C}{K_s}\frac{dx}{dt} + x = \frac{F(t)}{K_s}$$

$$\tau\frac{dq_o}{dt} + q_o = Kq_i(t)$$

$$\tau\frac{dx}{dt} + x = KF(t)$$

$$\tau = \frac{C}{K_s} \qquad K = \frac{F(t)}{K_s}$$

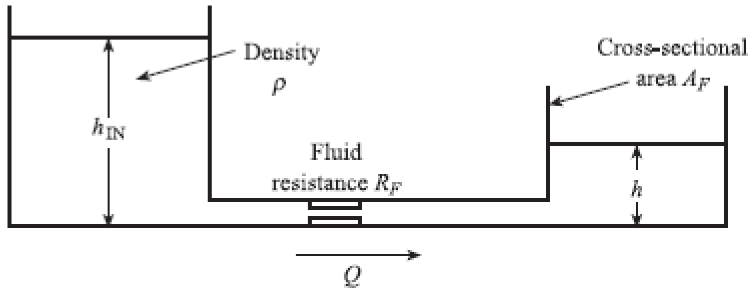
$$Damping\ Force = C\frac{dx}{dt}$$

$$Restoring\ Force = K_s x$$

$$Externally\ Applied\ Force = F(t)$$

UTILITY OF ANALOGY OF VARIOUS TYPES OF FIRST ORDER SYSTEMS

Fluidic



Volume flow rate $Q = \dfrac{1}{R_F}(P_{IN} - P)$

Pressures $P_{IN} = h_{IN}\rho g$, $P = h\rho g$

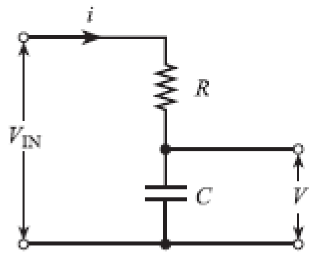
$$A_F \frac{dh}{dt} = Q = \frac{\rho g}{R_F}(h_{IN} - h)$$

$$\frac{A_F R_F}{\rho g}\frac{dh}{dt} + h = h_{IN}$$

i.e.

$$\tau_F \frac{dh}{dt} + h = h_{IN},\ \tau_F = \frac{A_F R_F}{\rho g}$$

Electrical



$V_{IN} - V = iR$

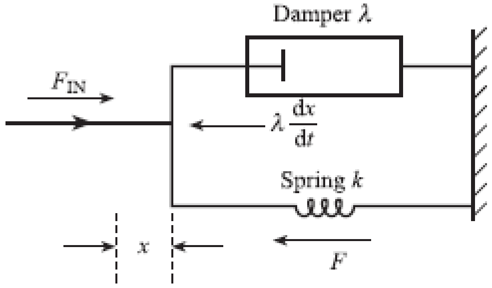
Charge $q = CV$, current $i = \dfrac{dq}{dt} = \dfrac{CdV}{dt}$

$$RC\frac{dV}{dt} + V = V_{IN}$$

i.e.

$$\tau_E \frac{dV}{dt} + V = V_{IN},\ \tau_E = RC$$

Mechanical



$F_{IN} - F = \lambda \dfrac{dx}{dt}$, λ N s m^{-1} = damping constant

Displacement $x = \dfrac{F}{k}$, k N m^{-1} = spring stiffness

$$\frac{\lambda}{k}\frac{dF}{dt} + F = F_{IN}$$

$$\tau_M \frac{dF}{dt} + F = F_{IN},\ \tau_M = \frac{\lambda}{k}$$

Thermal	$\tau_{Th} = \dfrac{MC}{UA} = R_{Th}C_{Th};\ R_{Th} = \dfrac{1}{UA},\ C_{Th} = MC$
Fluidic	$\tau_F = \dfrac{A_F R_F}{\rho g} = R_F C_F;\ R_F = R_F,\ C_F = \dfrac{A_F}{\rho g}$
Electrical	$\tau_E = RC = R_E C_E;\ R_E = R,\ C_E = C$
Mechanical	$\tau_M = \dfrac{\lambda}{k} = R_M C_M;\ R_M = \lambda,\ C_M = \dfrac{1}{k}$

THERMAL SYSTEM

Thermocouple

$$\rho = 8500\frac{kg}{m^3}$$

$$h = 400\frac{W}{m^2}.K$$

$$C_p = 400\frac{J}{kg.K}$$

$$D = 1\,mm$$

$$\tau\frac{dq_o}{dt} + q_o = Kq_i(t)$$

$$\frac{\rho V C_p}{hA_s}\frac{d\theta}{dt} + \theta = \theta_\infty$$

$$\frac{T - T_{init}}{T_\infty - T_{init}} = \frac{\theta}{\theta_\infty} = 1 - e^{-\frac{t}{\tau}}$$

$$R = \frac{1}{hA_s} = \frac{1}{h(4\pi R^2)}$$

$$\mathrm{R} = \frac{1}{400\left(4\pi\left(0.5\times10^{-3}\right)^2\right)} = 795.77\ \frac{K}{W}$$

$$C = \rho V C_p = \rho\frac{4}{3}\pi R^3 C_p = 8500\times\frac{4}{3}\pi\left(0.5\times10^{-3}\right)^3\times400 = 1.78\times10^{-3}J$$

$$\tau = RC = 795.77\ \frac{K}{W}\times1.78\times10^{-3}J = 1.417\text{ seconds}$$

Analogy between thermal and electrical system

Thermocouple

$\rho = 8500 \dfrac{kg}{m^3}$

$h = 400 \dfrac{W}{m^2}.K$

$C_p = 400 \dfrac{J}{kg.K}$

$D = 1\ mm$

$\tau\ = 1.417\ seconds$

$$\tau \frac{dq_o}{dt} + q_o = Kq_i(t)$$

$$\frac{\rho V C_p}{hA_s}\frac{d\theta}{dt} + \theta = \theta_\infty$$

$$\frac{T - T_{init}}{T_\infty - T_{init}} = \frac{\theta}{\theta_\infty} = 1 - e^{-\frac{t}{\tau}}$$

Sl No	Thermal Parameter	Electrical parameter
1	$R = \dfrac{1}{hA_s}$	R
2	$C = \rho V C_p$	C
3	θ	V

RC circuit

$R = 795.77\Omega$

$C = 1.78\ \times\ 10^{-3}\ F$

$\tau\ = 1.417\ seconds$

$$RC\frac{dV}{dt} + V = V_{IN}$$

$$\tau = RC;\ \ K = 1$$

Differential equations are similar
Initial conditions are similar
Solution also needs to be similar

Questions to be answered after completing this module

1. Model the clinical thermometer as a first order system with appropriate assumptions
2. Draw the typical inputs (time varying) provided to first order measurement systems
3. Determine output characteristics of the first order instrument with the following input
    a. Step Input
    b. Ramp Input
    c. Sinusoidal Input
    d. Impulse input
4. What is a linear system ?
5. Show with an example, how method of superposition can be used to find the output of a first order system provided with a combination of sinusoidal inputs
6. What is Fourier Series ?
7. What is a periodic function ?
8. What is the difference between complex periodic wave form and Non deterministic wave form
9. What is the difference between odd function and an even function
10. Determine the Fourier coefficients between the limits $-\pi\ to + \pi$ (period = 2π)
11. Determine the Fourier coefficients between the limits $-T/2\ to + T/2$ (period = T)

Questions to be answered after completing this module

13. **From first principles, model first order**
    a. **Hydraulic system**
    b. **Pnuematic system**
    c. **Electrical system**
    d. **Thermal system**

14. State the conditions under which the following systems can be considered as analogous systems
    a. **Hydraulic system and Electrical system**
    b. **Pnuematic system and Electrical system**
    c. **Electrical system and Thermal system**