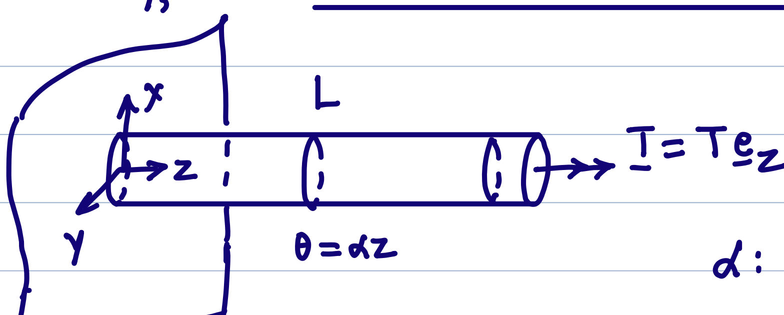
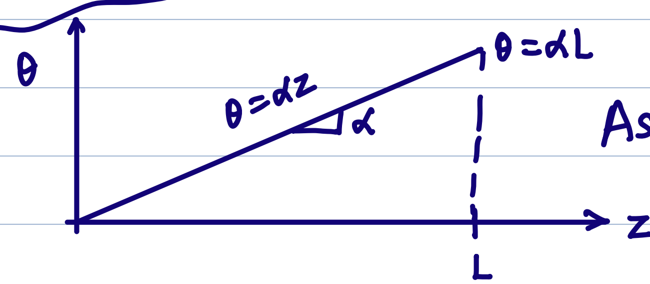
Previously,

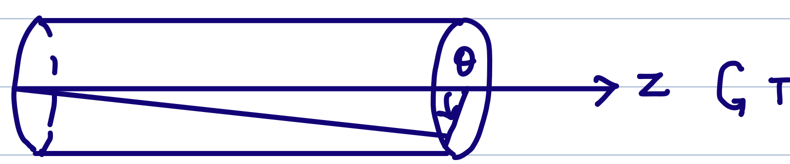
TORSION OF CYL SHAFTS



α : unit $\measuredangle$ twist rad/m



Assumption



Kinematic assumption $\left\{ \begin{aligned} u &= -\alpha y z \\ v &= +\alpha x z \\ w &= 0 \end{aligned} \right.$

Hooke's Law

Strains $\epsilon_{yz} = \alpha x/2$, $\epsilon_{xz} = -\alpha y/2$

Stresses $\sigma_{yz} = G\alpha x$, $\sigma_{xz} = -G\alpha y$

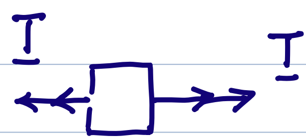
check if stresses are in pointwise equilibrium

3D

$$\Sigma F_x = 0 \Rightarrow \frac{\partial \cancel{\sigma_{xx}}}{\partial x} + \frac{\partial \cancel{\sigma_{xy}}}{\partial y} + \frac{\partial \sigma_{xz}}{\cancel{\partial z}} + \cancel{b_x} = 0 \quad \checkmark$$

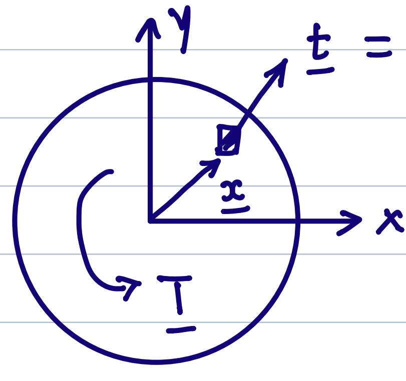
$$\Sigma F_y = 0 \Rightarrow \frac{\partial \cancel{\sigma_{yx}}}{\partial x} + \frac{\partial \cancel{\sigma_{yy}}}{\partial y} + \frac{\partial \sigma_{yz}}{\cancel{\partial z}} + \cancel{b_y} = 0 \quad \checkmark$$

$$\Sigma F_z = 0 \Rightarrow \cancel{\frac{\partial \sigma_{xz}}{\partial x}} + \cancel{\frac{\partial \sigma_{yz}}{\partial y}} + \frac{\partial \cancel{\sigma_{zz}}}{\partial z} + \cancel{b_z} = 0 \quad \checkmark$$



Every c/s ~~carries~~ transmits torque $\underline{T}$

$$\int_{\Omega} \underline{x} \times \underline{t} \, da$$



$\underline{t} = \underset{\sim}{\sigma} \, \underline{n}$ $\quad N/m^2$

$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\underline{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

c/s at some z

$$\underline{T} = \int_{\Omega} \underline{x} \times \underline{t}\, da \qquad (\underline{T} = \underline{r} \times \underline{F})$$

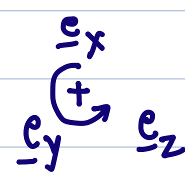
$$\begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \begin{pmatrix} \cancel{\sigma_{xx}} & \cancel{\sigma_{xy}} & \sigma_{xz} \\ \cancel{\sigma_{yx}} & \cancel{\sigma_{yy}} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \cancel{\sigma_{zz}} \end{pmatrix} \begin{pmatrix} n_x = 0 \\ n_y = 0 \\ n_z = 1 \end{pmatrix}$$

$$t_x = \sigma_{xz}, \quad t_y = \sigma_{yz}, \quad t_z = 0$$

$$\underline{x} = x\, \underline{e}_x + y\, \underline{e}_y, \quad \underline{t} = \sigma_{xz}\, \underline{e}_x + \sigma_{yz}\, \underline{e}_y$$

$$\underline{T} = T \underline{e}_z = \int_{\Omega} (x\, \underline{e}_x + y\, \underline{e}_y) \times (\sigma_{xz}\, \underline{e}_x + \sigma_{yz}\, \underline{e}_y)\, dx\, dy$$

$$\boxed{T = \int_{\Omega} (x\, \sigma_{yz} - y\, \sigma_{xz})\, dx\, dy}$$



Equate $\underline{e}_z$ cpts

$$T = \int_{\Omega} \left(x\, G\alpha x - y(-G\alpha y)\right) dx\, dy$$

$$= G\alpha \int_{\Omega} (x^2 + y^2)\, dx\, dy$$

J = polar second moment of area = polar moment of inertia

$T = G\alpha J$

$$\boxed{\alpha = \frac{T}{GJ}}$$

$J \equiv$ geometry

Recall, axial deformation

$$A = \int_{\Omega} dx\,dy$$

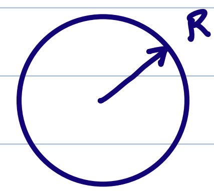
Torsion $J = \int_{\Omega} (x^2+y^2)\,dx\,dy$

Total angle of twist

$$\theta = \alpha L = \frac{TL}{GJ}$$

J for circular c/s. cylindrical

$$J = \int_{\Omega} (x^2+y^2)\,dx\,dy$$



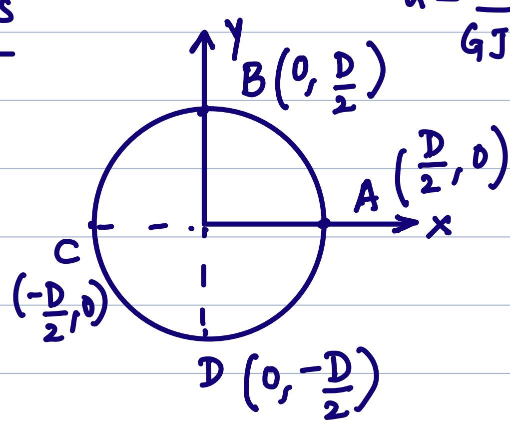
$D = 2R$

HW

$$J = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

Max shear stress

$$\alpha = \frac{T}{GJ} = \frac{32T}{G\pi D^4}$$



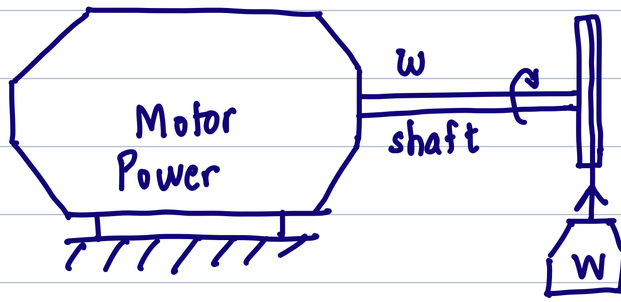
	A	B	C	D
$\tau_{zx} = -G\alpha y$	0	$-\tau$	0	$+\tau$
$\tau_{zy} = +G\alpha x$	$+\tau$	0	$-\tau$	0

$$\tau = 16T/\pi D^3$$

Recall Axial deformation $\underset{\sigma_{zz}}{\overset{\sigma}{\shortparallel}} = \frac{P}{A} = \frac{4P}{\pi D^2}$

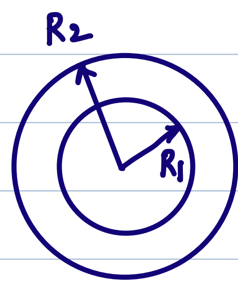
Applications Hollow Cylindrical Shafts

- ☐ minimize cost
- ☐ minimize self-weight



ω rad/sec

Power = Tω



$$T = G\alpha J = G\alpha \int_{R_1}^{R_2} \int_{0}^{2\pi} \underbrace{(x^2+y^2)}_{r^2} \, r\,dr\,d\varphi$$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$T = G\alpha \frac{\pi}{32} \left(D_2^4 - D_1^4\right)$$

$$= G\alpha \frac{\pi}{2} \left(R_2^4 - R_1^4\right)$$

Power Transmission by shafts

$$\text{Power} = T\omega = T\,\frac{2\pi N}{60}$$

Tachometer

T Nm
ω rad/s
Power W
N RPM

$$1\ \text{HP} \approx 746\ \text{W} = 550\ \text{ft lb/s}$$

$$\text{HP} = \frac{2\pi N T}{33{,}0000}$$

T lb ft
N RPM
$1\ \text{ft} = 12\ \text{in}$

$$\tau_{max} = \frac{16\,T}{\pi D^3}$$

psi
T lb-in
D in

Shear Traction/Shear Stress Due to Torque

$$\underline{t} = \underset{\sim}{\sigma}\,\underline{n} = \begin{pmatrix} 0 & 0 & \sigma_{zx} \\ 0 & 0 & \sigma_{zy} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

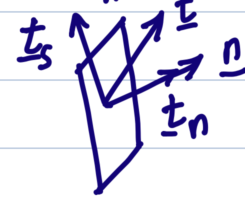
$$\underline{t} = \sigma_{zx}\,\underline{e}_x + \sigma_{zy}\,\underline{e}_y$$

in-plane cpt only
No normal cpt.

normal cpt — shear cpt

$\underline{t}_n = \underline{0}$, $\underline{t}_s = \underline{t}$

$\underline{t} = \underline{t}_n + \underline{t}_s$



$$\sigma_s = \|\underline{t}_s\|_2 = \sqrt{\underline{t}_s \cdot \underline{t}_s}$$

$$\sigma_s = \tau$$

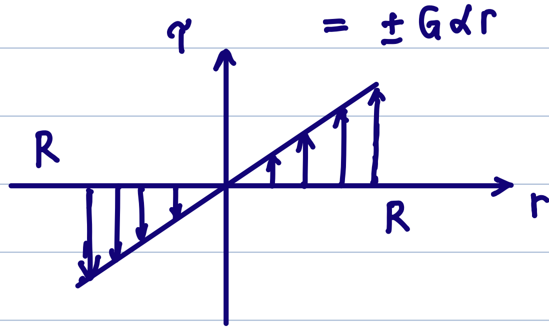
$$= \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2}$$

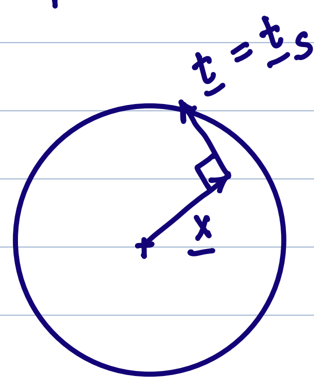
$$= \sqrt{(-G\alpha y)^2 + (G\alpha x)^2}$$

$$= G\alpha \sqrt{x^2 + y^2}$$

$$= \pm G\alpha r$$

$r =$ radial coordinate





$$\underline{t} = -G\alpha y \, \underline{e}_x + G\alpha x \, \underline{e}_y$$

$$\underline{x} = x \, \underline{e}_x + y \, \underline{e}_y$$

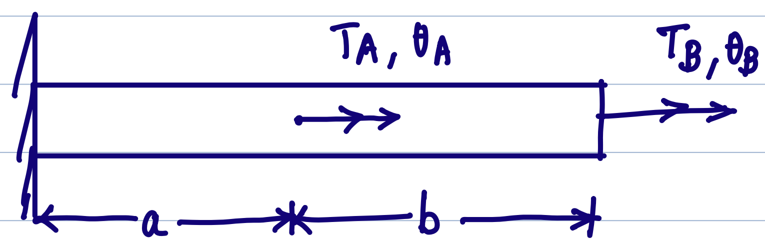
Note that $\underline{t} \cdot \underline{x} = 0$, $\underline{t} \perp \underline{x}$ in x-y plane

Torsional Stiffness

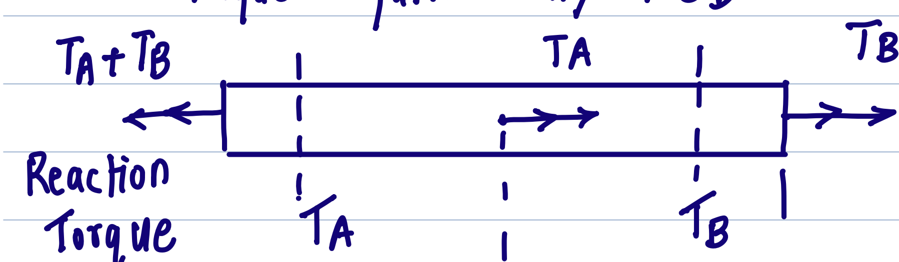
$$\theta = \frac{TL}{GJ}, \qquad k_T = \frac{T}{\theta} = \frac{GJ}{L}$$

Multiple Torques

Linear $\equiv$ whole = sum of its parts



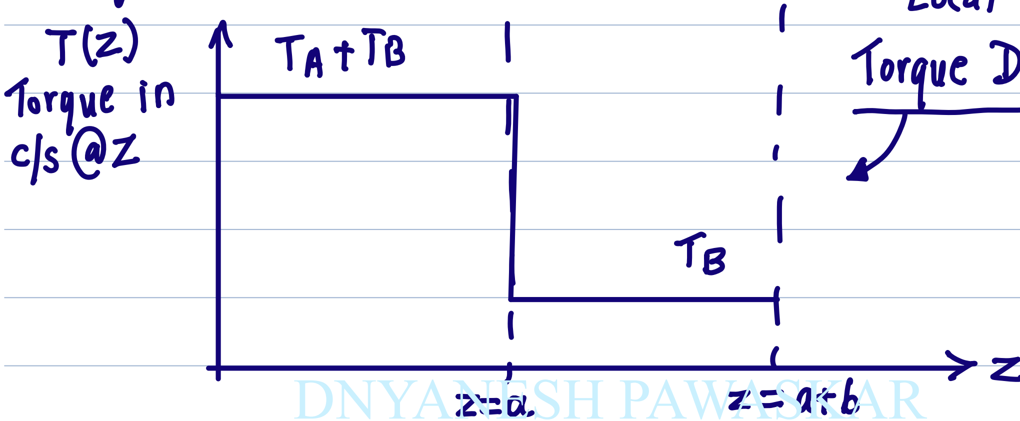
Torque Equilibrium / FBD



Global Equilibrium

Local Equilibrium

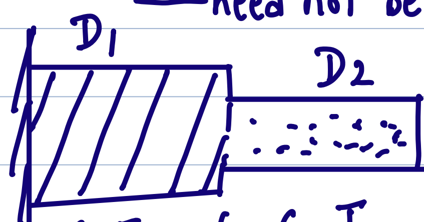
Torque Diagram



Total angle of twist at $z = a+b$ $\qquad \frac{TL}{GJ}$

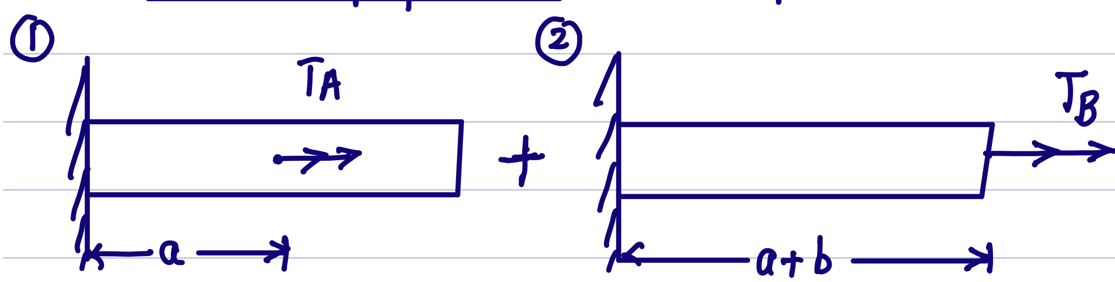
$$= \frac{(T_A+T_B)a}{\underbrace{GJ}} + \frac{T_B\, b}{\underbrace{GJ}}$$

need not be equal



$G_1J_1 \neq G_2J_2$

<u>Linear Superposition</u> $\qquad$ Given problem =



① $\quad \frac{T_A\, a}{GJ} + \frac{T_B(a+b)}{GJ}$ ②

Total angle of twist at free end