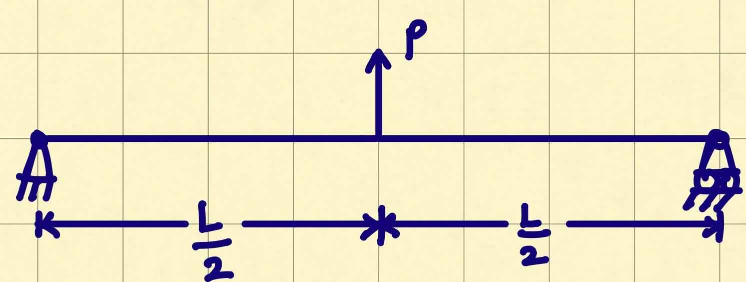
Dnyanesh Pawaskar

ME 202 Spring 2023
Tutorial 8 Solutions
Tue 14 Mar 2023

1.



$$\Pi(u) = \int_0^L \frac{EI}{2} u''^2 \, dz - P\, u\left(\frac{L}{2}\right)$$

$$u(z) = a_1 \sin\frac{\pi z}{L}$$ obeys KBCs u(0) = 0, u(L) = 0

$$u''(z) = -a_1 \left(\frac{\pi}{L}\right)^2 \sin\frac{\pi z}{L}$$

$$\Pi(a_1) = \frac{EIL}{4} a_1^2 \left(\frac{\pi}{L}\right)^4 - P a_1$$

Dnyanesh Pawaskar

$$\Pi'(a_1) = 0 \Rightarrow \quad a_1 = \frac{2PL^3}{\pi^4 EI}$$

$$u(z) = \frac{2PL^3}{\pi^4 EI} \sin\frac{\pi z}{L} \qquad \text{approx.}$$

$$u(z) = \frac{Pz}{48EI}\left(3L^2 - 4z^2\right) \qquad \text{exact}$$

$$0 \leq z \leq \frac{L}{2}$$

Approx $\quad u\left(\frac{L}{2}\right) = 0.02053\,\frac{PL^3}{EI}$

Exact $\quad 0.02083\,\frac{PL^3}{EI}$

Approx $\quad u''\left(\frac{L}{2}\right) = 0.2026\; PL$

Exact $\quad 0.25\; PL$

$$u(z) = a_1 \sin\frac{\pi z}{L} + a_3 \sin\frac{3\pi z}{L} + a_5 \sin\frac{5\pi z}{L}$$

Dnyanesh Pawaskar

$N = 3$

Orthogonality of sines $\int_0^L \sin\frac{m\pi z}{L}\sin\frac{n\pi z}{L}\,dz$

Kronecker delta $= \frac{L}{2}\delta_{mn}$

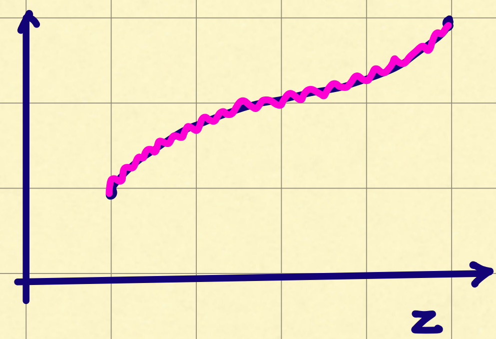
$$\Pi = \frac{EIL}{4}\left[a_1^2\left(\frac{\pi}{L}\right)^4 + a_3^2\left(\frac{3\pi}{L}\right)^4 + a_5^2\left(\frac{5\pi}{L}\right)^4\right] - P(a_1 - a_3 + a_5)$$

$$\frac{\partial\Pi}{\partial a_1} = 0, \quad \frac{\partial\Pi}{\partial a_3} = 0, \quad \frac{\partial\Pi}{\partial a_5} = 0$$

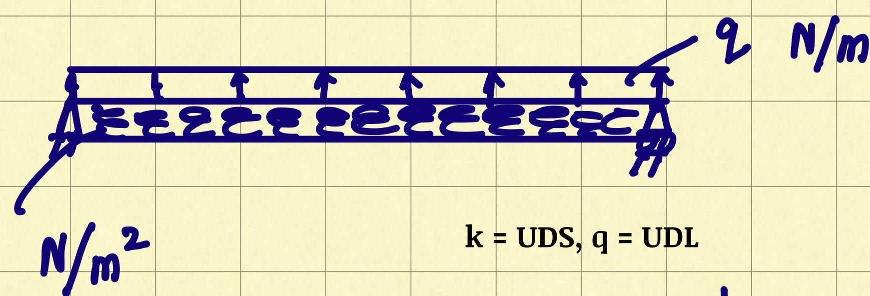
$$a_1 = \frac{2PL^3}{\pi^4 EI}, \quad a_3 = \frac{-2PL^3}{81\pi^4 EI}, \quad a_5 = \frac{2PL^3}{625\pi^4 EI}$$

approx →	u_{max}	M_{max}
1 term	$0.02055\ PL^3/EI$	$0.2026\ PL$
2 terms	$0.02081\ PL^3/EI$	$0.2252\ PL$
3 terms	$0.02083\ PL^3/EI$	$0.2333\ PL$

Dnyanesh Pawaskar



2.



k = UDS, q = UDL

$$\Pi[u] = \int_0^L \frac{EI}{2} u''^2 \, dz - \int_0^L q u \, dz + \int_0^L \frac{1}{2} k u^2 \, dz$$

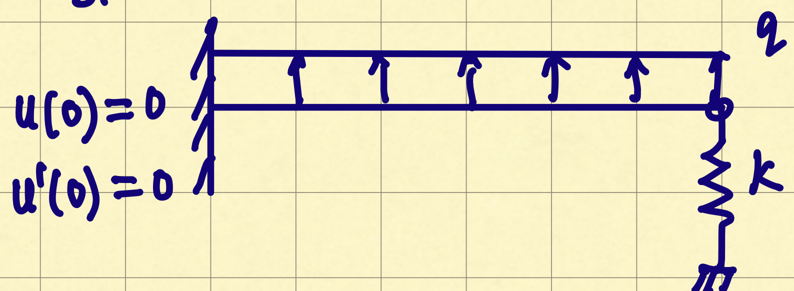
$$u(z) = \sum_{n=1,3,5,\ldots}^{\infty} a_n \sin \frac{n\pi z}{L}$$

Dnyanesh Pawaskar

$$\Pi(a_1, a_3, \ldots) = \sum_{n=1,3,5}^{\infty} \left\{ a_n^2 \left[\left(\frac{EI}{2}\right)\left(\frac{L}{2}\right)\left(\frac{n\pi}{L}\right)^4 + \frac{kL}{4} \right] - a_n \frac{2qL}{n\pi} \right\}$$

$$\frac{\partial \Pi}{\partial a_n} = 0 \Rightarrow a_n = \frac{4qL^4/(n^5\pi^5 EI)}{1 + kL^4/(n^4\pi^4 EI)}$$

3.



$$\Pi = \int_0^L \left(\frac{EI}{2} u'' - qu\right) dz + \frac{1}{2} k u^2(L)$$

$$u = \cancel{a_0} + \cancel{a_1 z} + a_2 z^2 + a_3 z^3$$

physically ok.

Dnyanesh Pawaskar

2 unknowns, 2 DOFs

$$\Pi(a_2, a_3) = 2a_2^2 EIL + 6EI a_3^2 L^3 + 6EI a_3 a_2 L^2$$

$$+ \frac{1}{2} k \left(a_2 L^2 + a_3 L^3\right)^2 - \frac{1}{3} a_2 L^3 q - \frac{1}{4} a_3 L^4 q$$

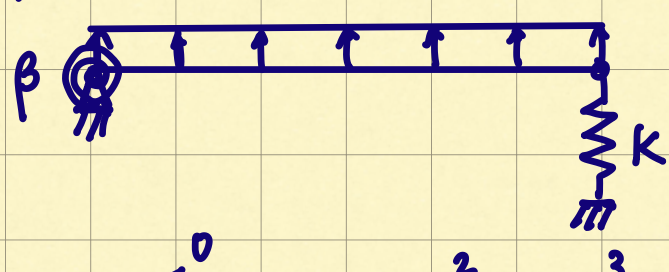
PMPE $\quad \dfrac{\partial \Pi}{\partial a_2} = 0, \quad \dfrac{\partial \Pi}{\partial a_3} = 0$

$\Rightarrow$

$$a_2 = \frac{30EIL^2 q + kL^5 q}{48EI(3EI + kL^3)}$$

$$a_3 = \frac{-L(12EIq + kL^3 q)}{48EI(3EI + kL^3)}$$

If,



KBC u(0) = 0

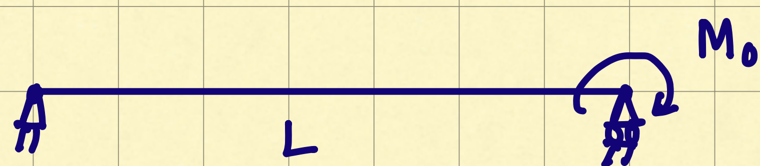
$$u = \cancel{a_0}^{\,0} + a_1 z + a_2 z^2 + a_3 z^3$$

Dnyanesh Pawaskar

$$u'(0) = a_1 \neq 0$$

$$\Pi = \frac{1}{2}\beta\left(u'(0)\right)^2 + \frac{1}{2}k\left(u(L)\right)^2 + \int_0^L \left(\frac{EI}{2}u'' - qu\right)dz$$

4.



KBCS $\quad u(0)=0, \quad u(L)=0$

+

NBC $\quad u''(0) = 0$

$$u = c_0 + c_1 z + c_2 z^2 + c_3 z^3$$

Apply BCs $\qquad c_0 = 0, \quad c_2 = 0$

$$c_1 L + c_3 L^3 = 0$$

shape $\qquad u(z) = c_1 z - \frac{c_1}{L^2} z^3 \qquad N = 1 \qquad$ 1 DOF

Dnyanesh Pawaskar

$$\Pi = \int_0^L \frac{EI}{2} u''^2 dz + (+M_0 u'(L))$$

$$= \frac{6EIc_1^2}{L} + M_0 c_1 (-2)$$

$$\frac{\partial \Pi}{\partial c_1} = 0 \Rightarrow c_1 = \frac{M_0 L}{6EI}$$

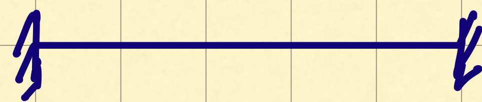
size

$$u(z) = \frac{M_0 L}{6EI}\left(z - \frac{z^3}{L^2}\right)$$

Max $u' = 0 \Rightarrow 3z^2 = L^2 \Rightarrow z = \frac{L}{\sqrt{3}}$

$$u_{max} = \frac{M_0 L^2}{9\sqrt{3}\,EI}$$

5.



$u=0$ $\qquad\qquad\qquad\qquad$ $\left.\begin{array}{l} u=0 \\ u'=0 \end{array}\right\}$ same for $v(z)$

$u'=0$

$u(z)$ exact

$v(z)$ approx $\qquad v(z) = u(z) + \varphi(z)$

BCs $\quad \varphi(0)=0, \ \varphi'(0)=0, \ \varphi(L)=0, \ \varphi'(L)=0$

$$\Pi[u] < \Pi[v]$$

$$\Pi[v] = \int_0^L \frac{EI}{2}\left(u'' + \varphi''\right)^2 dz - \int_0^L q(u+\varphi)\,dz$$

$$= \int_0^L \frac{EI}{2}\left(u''^2 + 2u''\varphi'' + \varphi''^2\right)dz - \int_0^L qu\,dz - \int_0^L q\varphi\,dz$$

$$\Pi[v] = \Pi[u] + \underbrace{\int_0^L \left(EIu''\varphi'' - q\varphi\right) dz}_{I_1} + \underbrace{\int_0^L \frac{EI}{2}\varphi''^2\,dz}_{+ve}$$

u exact $\qquad EIu'''' = q(u)$

v approx $\qquad EIv'''' \neq q(v)$

$$I_1 = -\int_0^L EIu'''\varphi'\,dz + \underbrace{\left[EIu''\varphi'\right]_0^L}_{\substack{0 \\ \text{BCs on } \varphi'}} - \int_0^L q\varphi\,dz$$

$$\int u\,dv = [uv] - \int v\,du$$

$$= \int_0^L EIu''''\varphi\,dz + \underbrace{\left[-EIu'''\varphi\right]_0^L}_{\substack{0 \\ \text{BCs on } \varphi}} - \int_0^L q\varphi\,dz$$

$$= \int_0^L \Big(\underbrace{EIu'''' - q}_{\substack{0 \\ u \text{ exact}}}\Big)\varphi\,dz$$

$$\Pi[u] < \Pi[v]$$