ME 202: Strength of Materials

Amit Singh
(amit.k.singh@iitb.ac.in)

Instructor

- **Name:** Amit Singh
- **Email:** amit.k.singh@iitb.ac.in
- **Phone:** +91-22-2576-5363
- **Office hours:** S26, Wednesday, 3-4 pm

Teaching assistants

- Kunwar Abhikeern 184106014@iitb.ac.in
- Abhishek Kumar 214109001@iitb.ac.in
- Ashwani Kushwaha 193100026@iitb.ac.in
- Hansraj Meena 22M1645@iitb.ac.in
- Shanu Singh 22M1647@iitb.ac.in
- Rahul Anil Chaudhari 22M1724@iitb.ac.in

Lecture hours (LC 302)

- 1A – Monday – 08:30 – 09:25
- 1B – Tuesday – 09:30 – 10:25
- 1C – Thursday – 10:35 – 11:30

Office hours: S26, ME Department, Wednesday, 3-4 pm

Syllabus: Goal of the course

- Why structures fail?
- Develop understanding of mechanics
- Strength of materials
- Behaviour of solid materials subject to loading and constraints
- Stress and strain relations
- Basic understanding of mechanical properties and deformation
- Able to design structures

Textbooks

- **Beer, F., Johnston, E.R., DeWolf, J., and Mazurek, D.F**., “Mechanics of Materials”, 8th or 7th edition McGraw Hill.

- **Hibbeler R. C**., Mechanics of Materials, 9th Edition (SI), Pearson,
- 2018

- **S.P. Timonshenko**, Strength of Materials Vol.I and II, CBS Publishers and Distributors, 1986.

- **J. R. Barber**, Intermediate Mechanics of Materials, 2nd edition, 2011

- **James M. Gere and Barry J. Goodno**, Mechanics of Materials, 8 edition, 2012

Topics

- **Introduction:** A review of forces, moments, free body diagram; Stress, Strain, Constitutive relations; Shear stresses; Bending stresses
- **Torsion of shafts** with circular and non-circular cross sections, thin walled tubes with closed cross sections
- **Shear stresses** in beams and thin-walled members; **Transverse shear**
- **Unsymmetrical bending**; Bending of curved bars; **Shear Center**
- **Combined** loading
- **Deflection** of beams
- **Energy and virtual work** methods
- **Buckling**
- Introduction to **theory of elasticity**

Grades

- Efforts and questions raised in the class will be the deciding factors for the borderline grades.

Homeworks	5%
Quizzes	15%
Group projects (Computational projects or/and paper presentation)	10%
Mid Sem Exam	30%
Final Exam	40%
Attendance (Bonus)	3% (max)

- Relative grading. Approximately uniform distribution across all two sections.

- Students with 80% to 90%, 90% to 95% and more than 95% attendance will get 1, 2 and 3 bonus marks, respectively.

Homework policy

- Homeworks should strictly be on A4 sheets
- Late homework WILL NOT be accepted (unless extreme proper reasons)
- Each problem should be on a separate sheet of paper
- Working in groups is permitted and highly encouraged. Science is social. Science is collaborative.
- But the solution cannot be completely plagiarized.
- Zero grades for copying.

Quizzes

- There will be four quizzes.
- Two before the mid semester exam (19th January and 9th February)
- Two after the mid semester exam (16th March and 6th April).
- Make-up for the quizzes will not be permitted.

Group project/presentation

- Group of four or five students will work on a computational project or/and present a paper.
- A list of possible computational projects will be provided later.
- The papers will be suggested after the mid-sem exam if there will any paper presentation.
- The presentation will be 20 minutes each on a suggested date post mid-sem exam.

Suggestions?

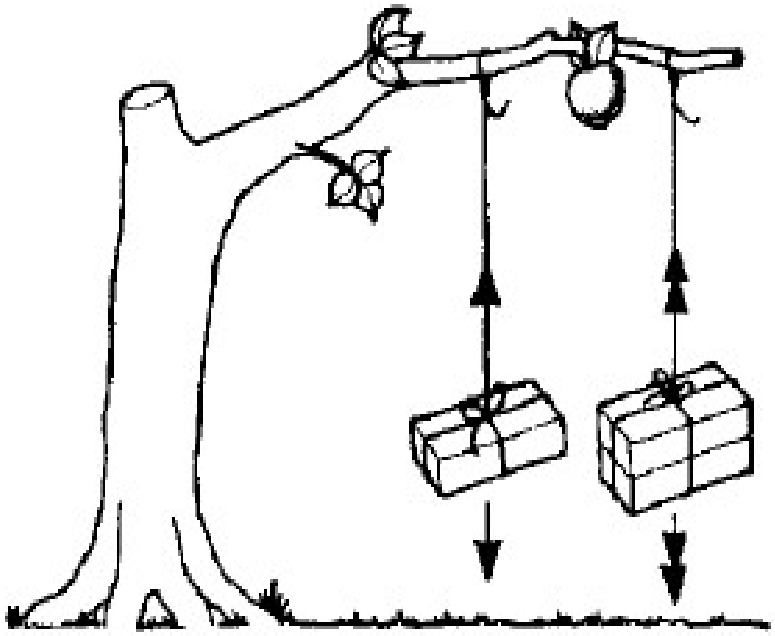
Einstein was 17 when he started thinking about what will happen if he travels with speed of light.

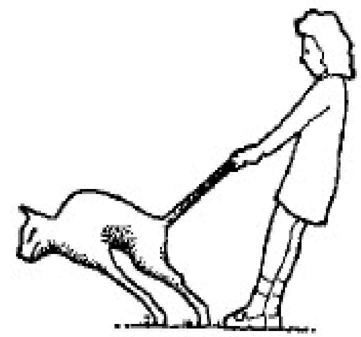
Amit Singh

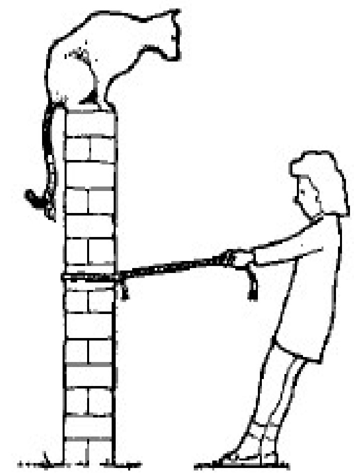
Review

Introduction

- When structures fail? How do they fail?

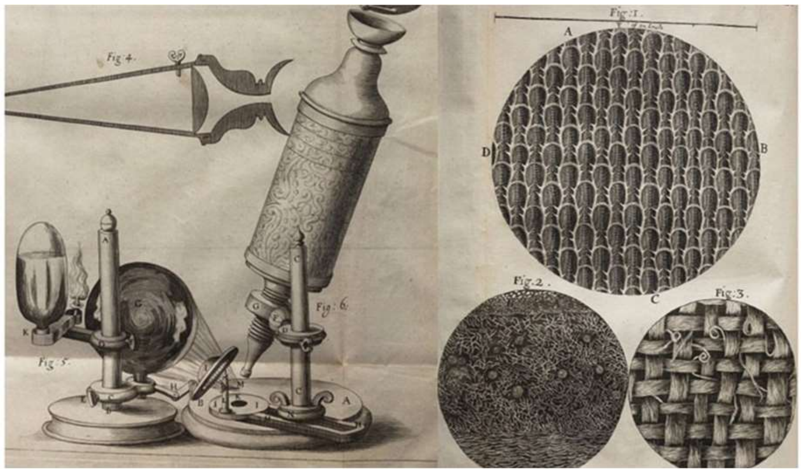


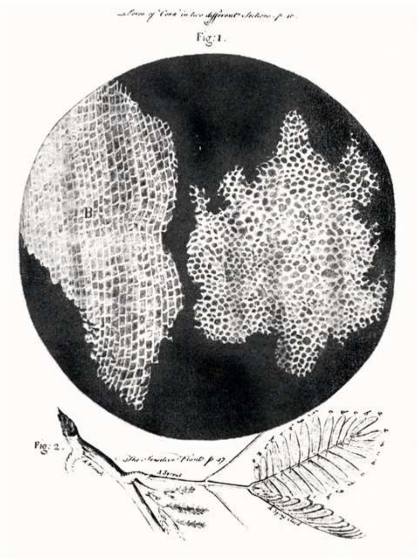




Why materials resist?

- Robert Hooke (1635-1703)



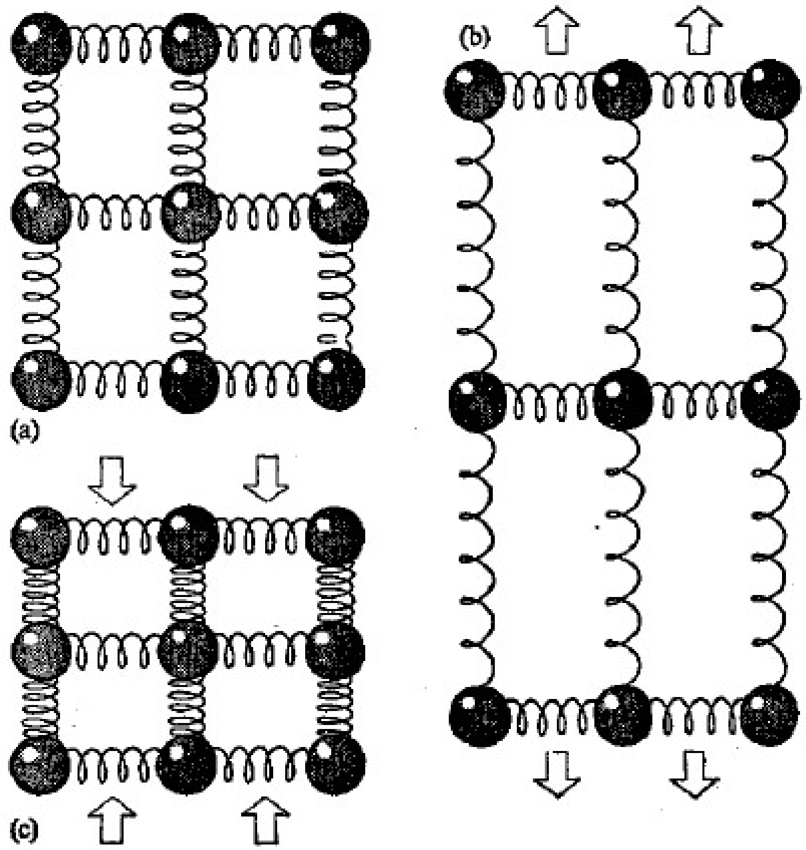


- Hooke in 1676: Why solids resist weight by producing large reactive forces?
- Every kind of solid changes its shape (springiness of solids) under mechanical forces
- This change of shape enables the solid to do pushing back.

Amit Singh

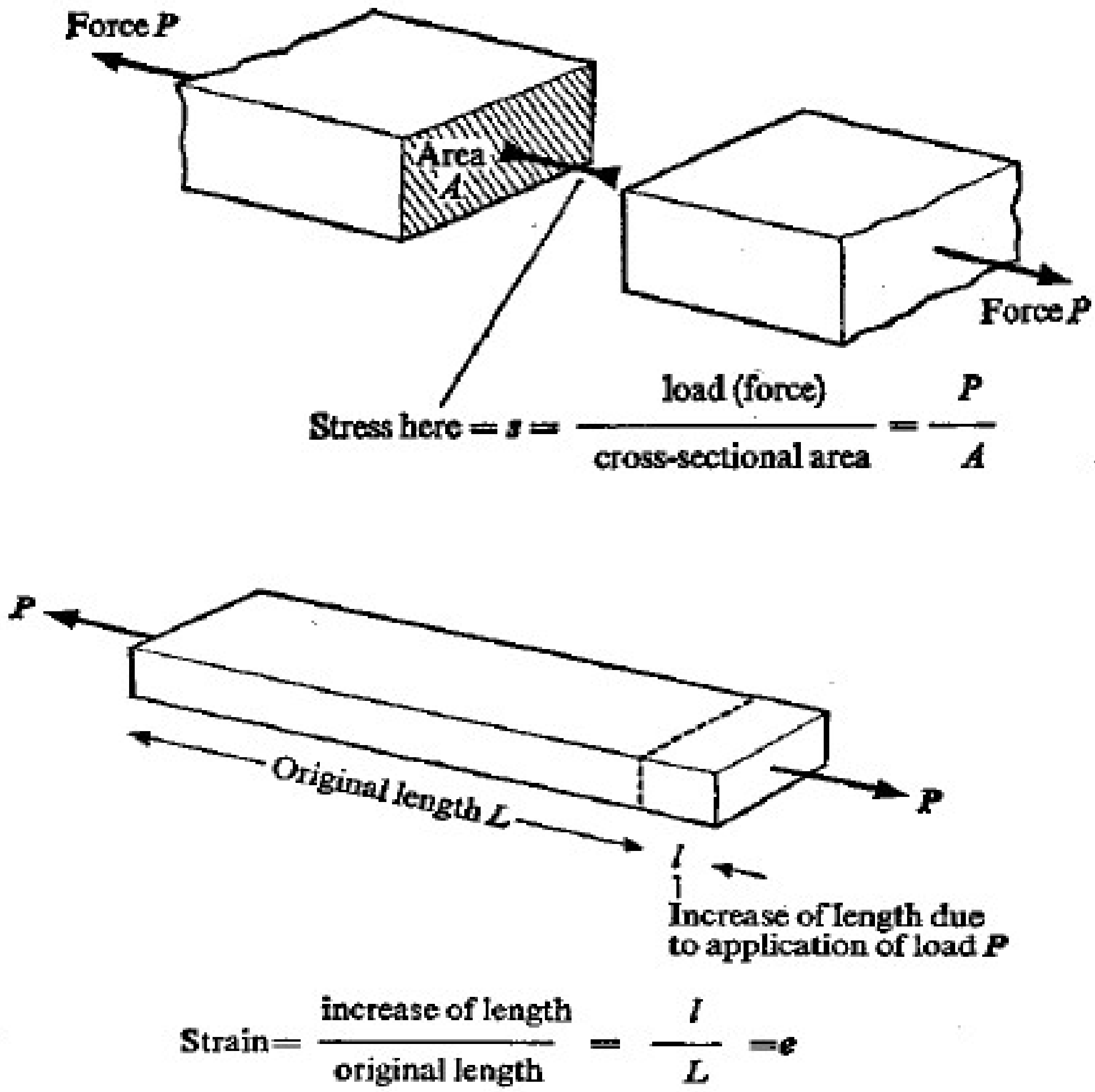
Springiness

•



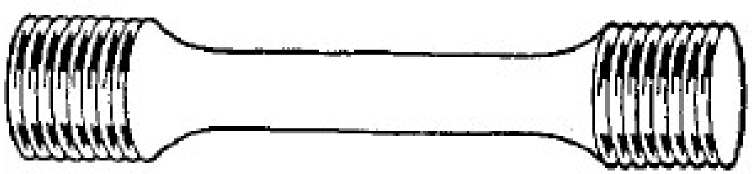
Simplified model of distortion of interatomic bonds under mechanical strain.
(a) Neutral, relaxed or strain-free position.
(b) Material strained in tension, atoms further apart, material gets longer.
(c) Material strained in compression, atoms closer together, material gets shorter.

Stress and strain

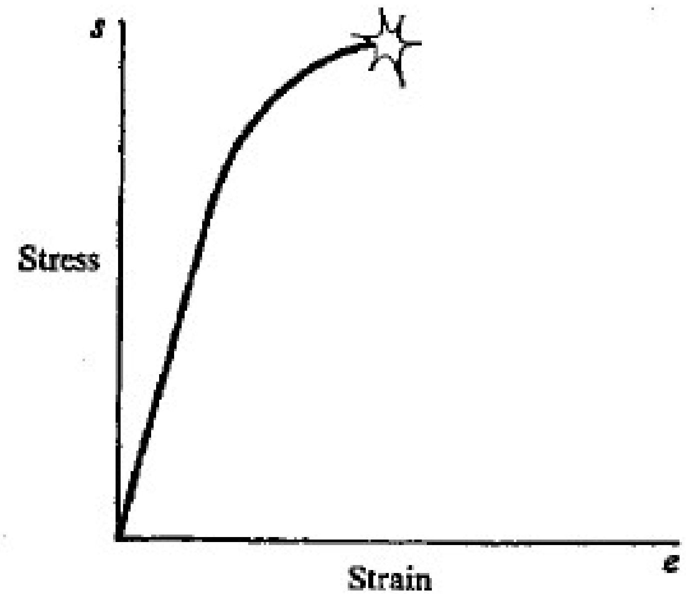
- 

Young’s modulus: How stiff is material

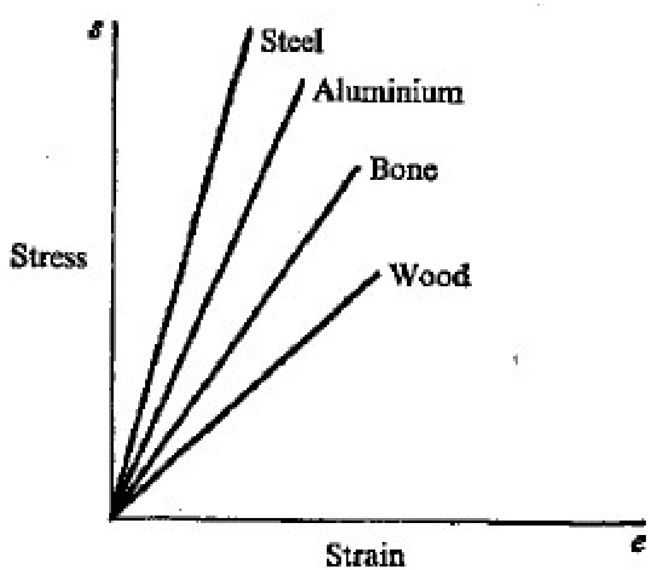
•



A typical tensile test-piece.



A typical 'stress-strain diagram'.



The slope of the straight part of the stress-strain diagram is characteristic of each different material. *E*, the Young's modulus of elasticity, represents this slope.

Hooke’s Law

Amit Singh

Young’s modulus: How stiff is material

•

Approximate Young's moduli of various solids

Material	Young's modulus (*E*) p.s.i.	Young's modulus (*E*) MN/m^2
Soft cuticle of pregnant locust*	30	0.2
Rubber	1,000	7
Shell membrane of egg	1,100	8
Human cartilage	3,500	24
Human tendon	80,000	600
Wallboard	200,000	1,400
Unreinforced plastics, polythene, nylon	200,000	1,400
Plywood	1,000,000	7,000
Wood (along grain)	2,000,000	14,000
Fresh bone	3,000,000	21,000
Magnesium metal	6,000,000	42,000
Ordinary glasses	10,000,000	70,000
Aluminium alloys	10,000,000	70,000
Brasses and bronzes	17,000,000	120,000
Iron and steel	30,000,000	210,000
Aluminium oxide (sapphire)	60,000,000	420,000
Diamond	170,000,000	1,200,000

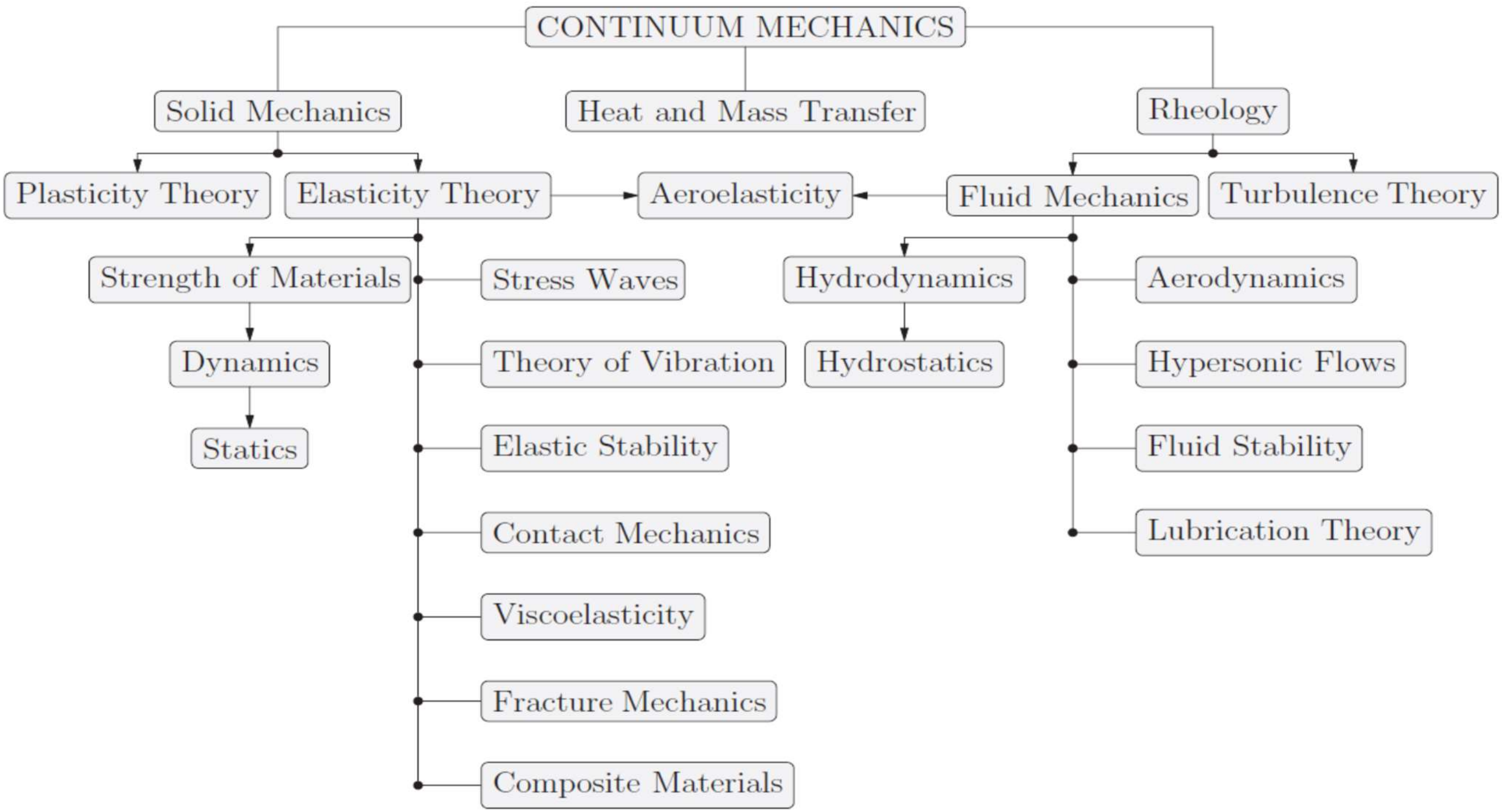
* By courtesy of Dr Julian Vincent, Department of Zoology, University of Reading.

Amit Singh

Continuum Mechanics

- The grand unified theory at the continuum level is Continuum Mechanics

- **Strength of Materials** is a branch of Solid Mechanics with approximations made to specialized geometries such as one-dimensional beam structure and two-dimensional plate and shell structures.



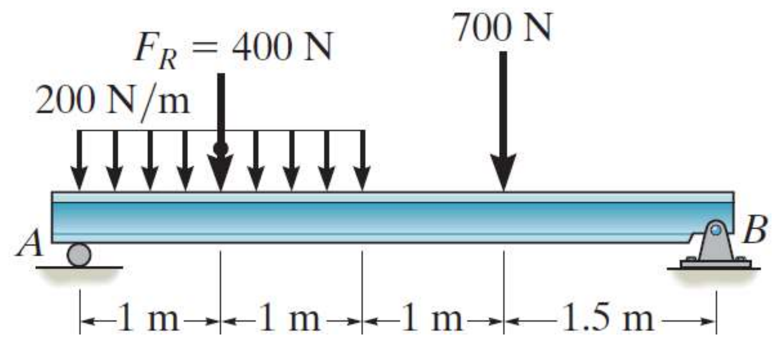
Amit Singh

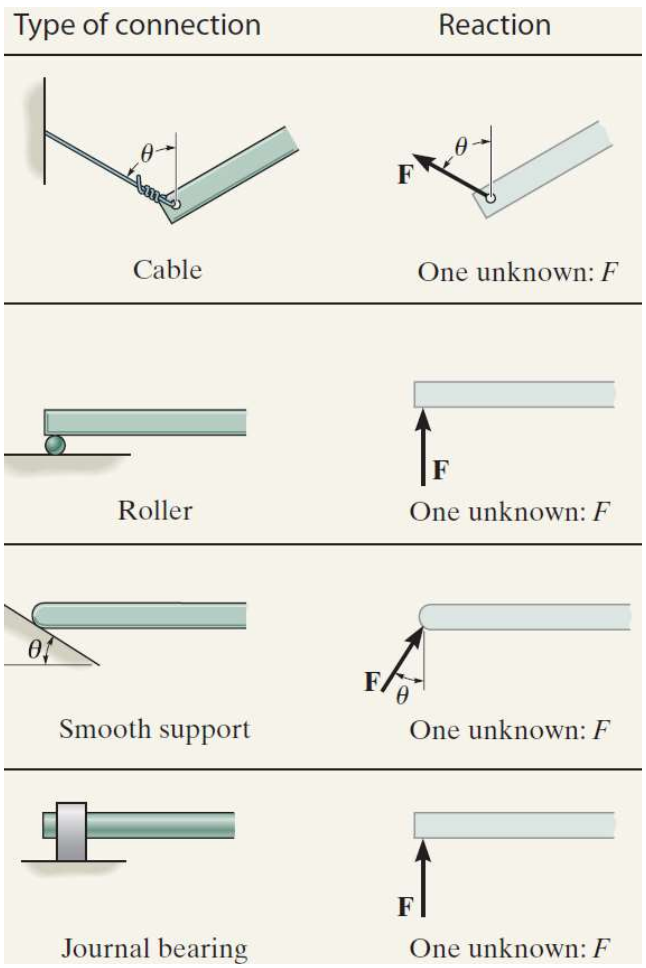
Strength of materials

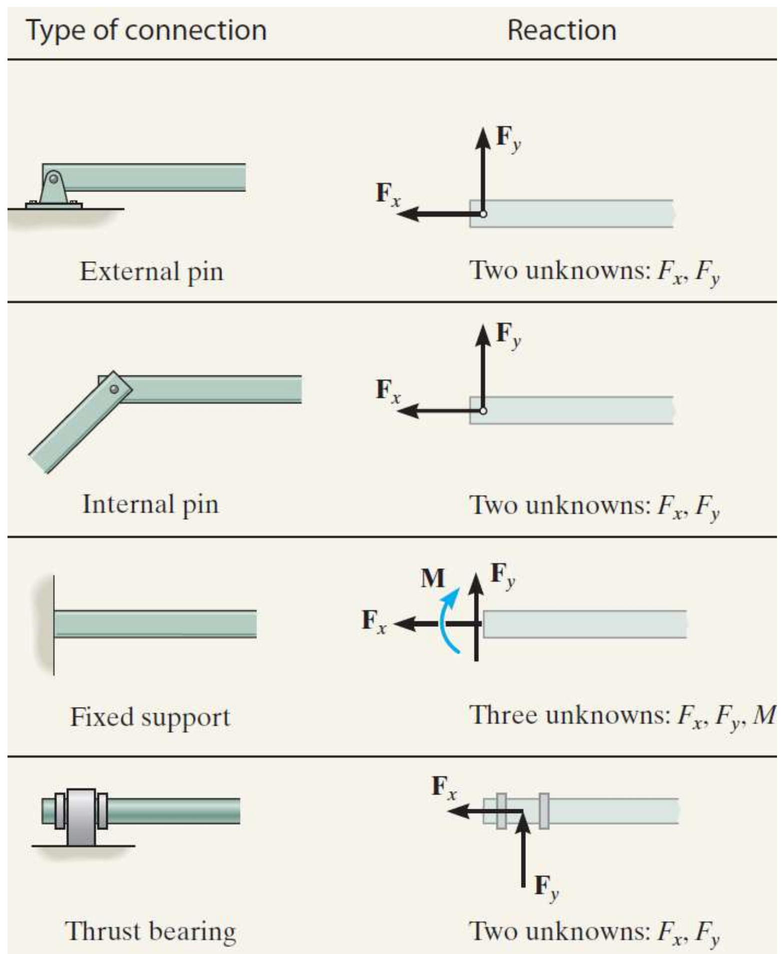
- Studies the behaviour of solid bodies with specialized geometries under different loadings.
- In particular, studies the effect of internal stress and strain within a body.
- Determines the stresses, strains, displacements in the structures and their components due to various loadings.
- The subject is also known as Mechanics of Materials of Mechanics of Deformable Bodies.
- We will start with review of Statics.

Statics

- **Loads**: Surface loads, distributed loadings
- **Loads**: Body force (gravity)
- **Support Reactions**:







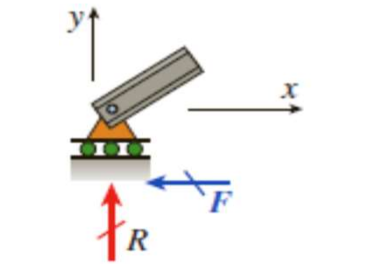
Amit Singh

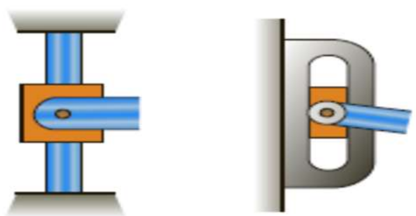
Support reactions

- Develop because the supports **restrict** the corresponding degrees of freedom

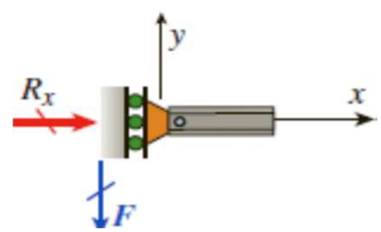
- **Roller support**: (Friction Force F = 0 when smooth surface)

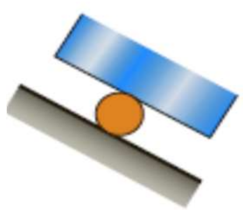
Horizontal



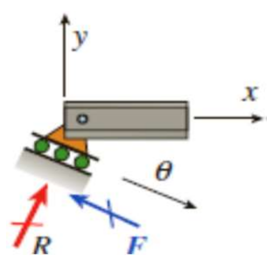


Vertical





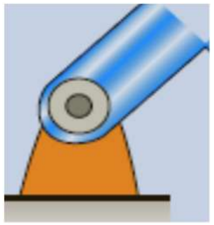
Inclined



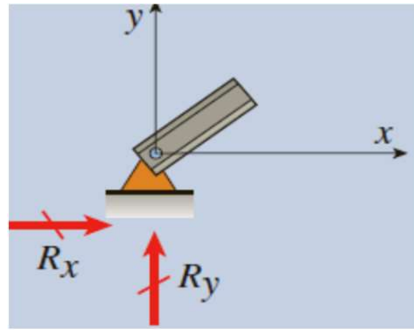
Amit Singh

Support reactions

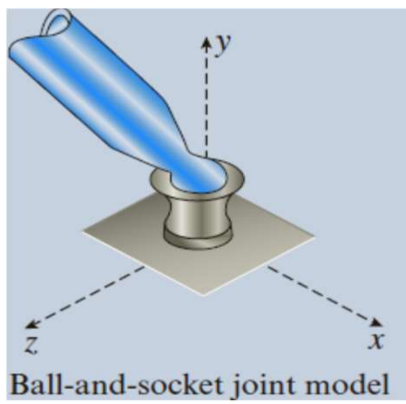
- **Pin support**: Resists motion in any direction normal to the pin. Cannot resist moment, free to rotate.



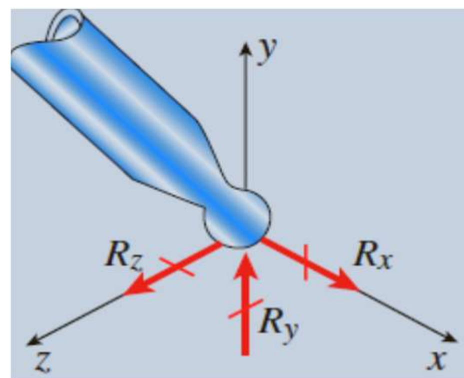
Two-dimensional pin



Camera mount

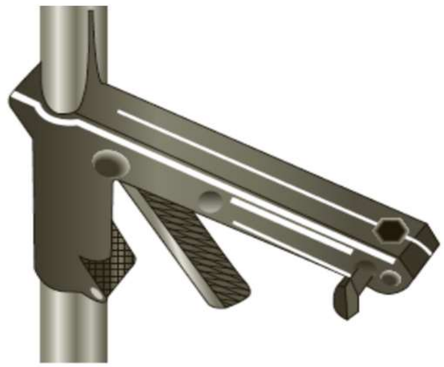


Three-D pin

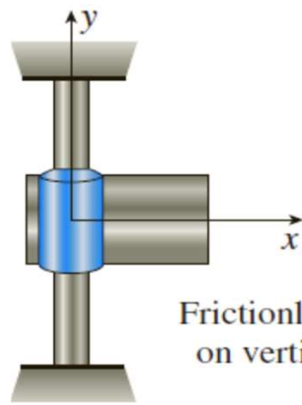


Support reactions

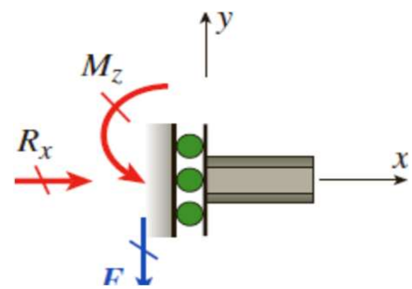
- **Sliding support**: Translates without rotation



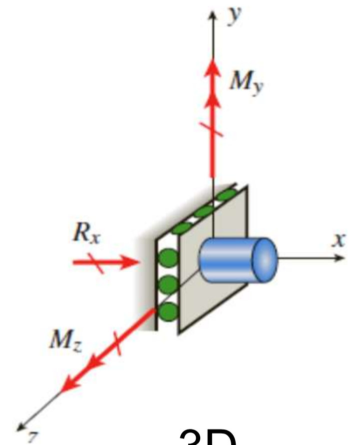
Sliding support for column light stand



Frictionless sleeve on vertical shaft

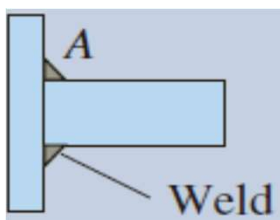


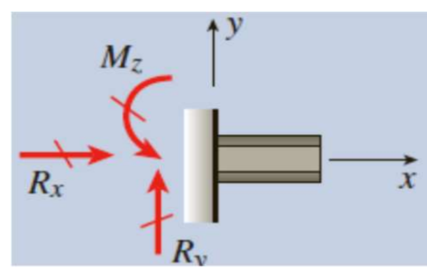
2D

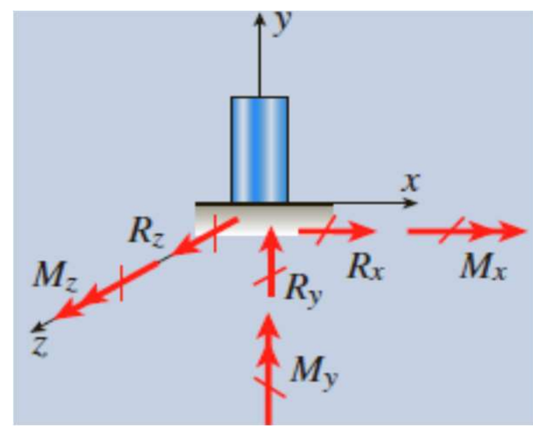


3D

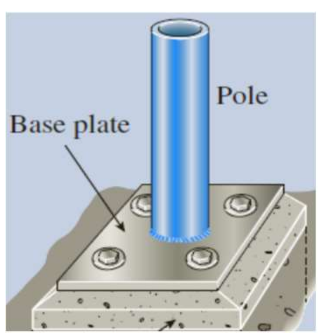
- **Fixed support**: No translation or rotation

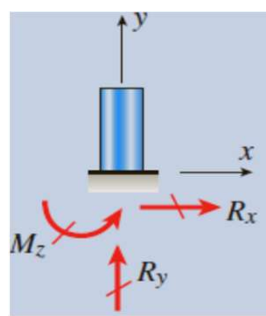






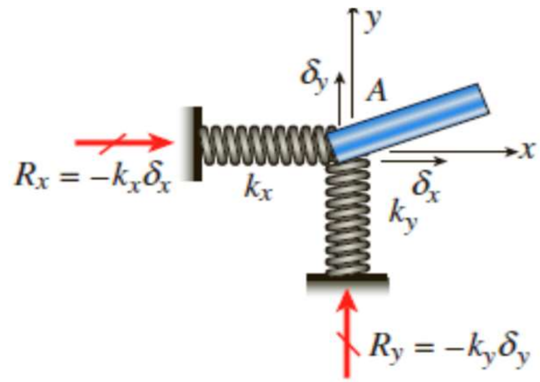
3D





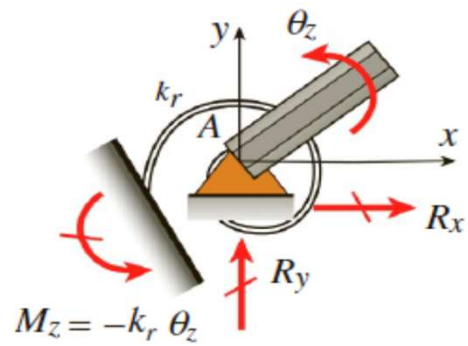
Support reactions

- **Elastic or spring support**:



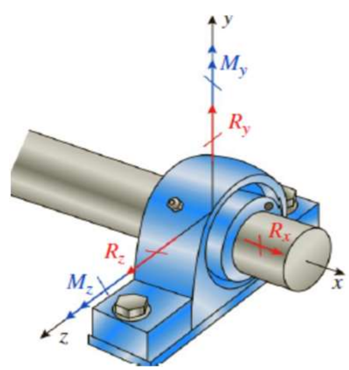
Translational spring 2D

Rotational spring in a clothespin



Rotational spring 2D

- **Thrust-bearing support**: Constrains motion along the shaft axis but allows rotation



For Journal bearing even Rx =0

Amit Singh

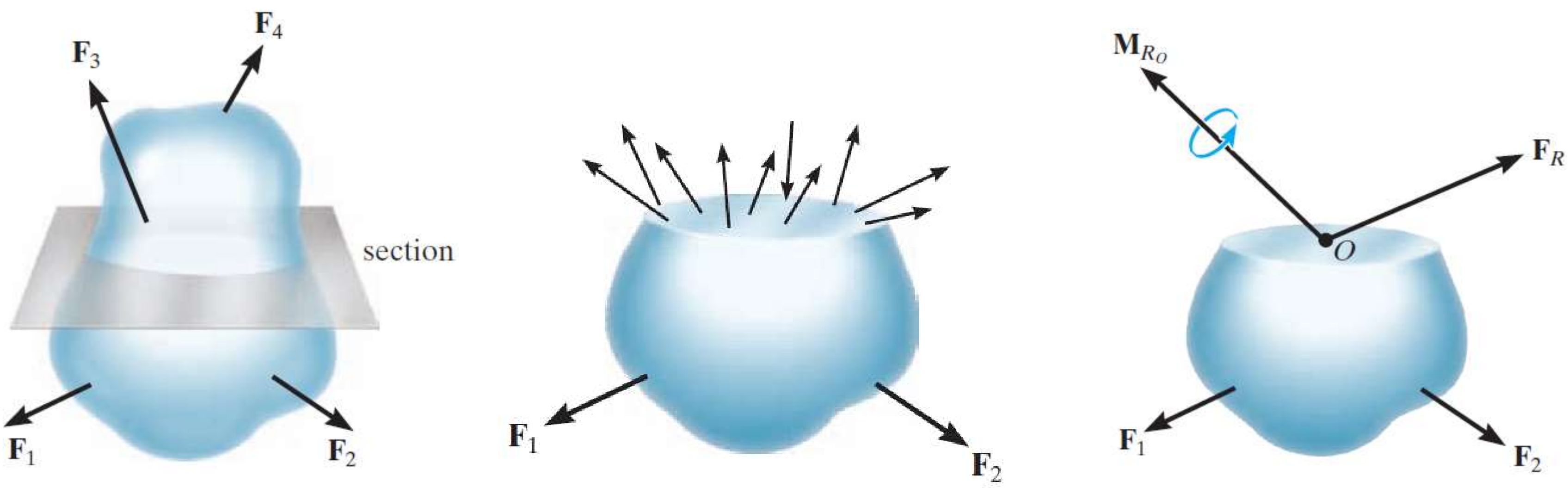
Equilibrium of deformable bodies

- **Equations of equilibrium**:

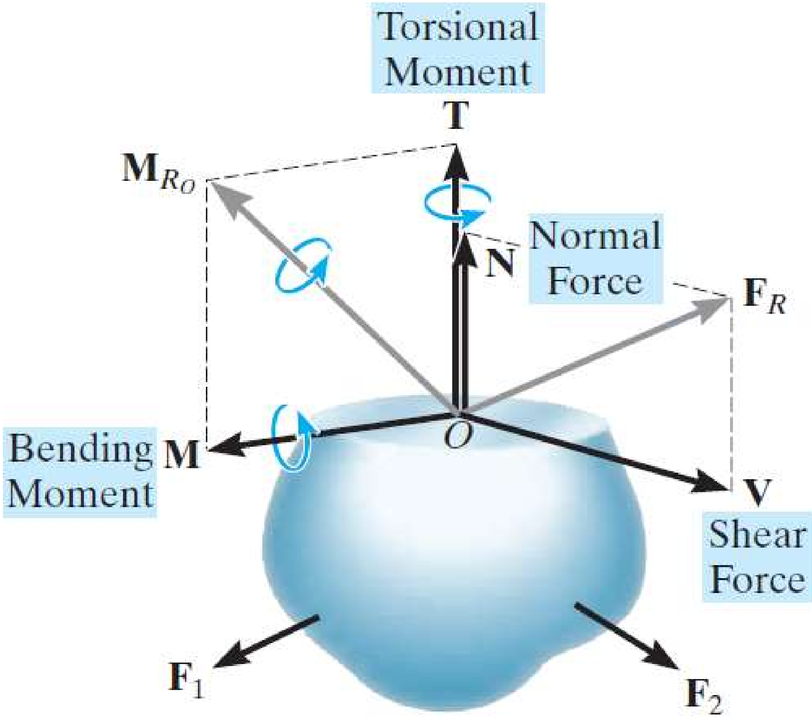
$$\Sigma \mathbf{F} = \mathbf{0}$$
$$\Sigma \mathbf{M}_O = \mathbf{0}$$

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0$$
$$\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0$$

- **Internal resultant loadings (Free-Body Diagram FBD): Pass an imaginary cut**



Internal resultant loadings

- 

Normal force, N.

perpendicular to the area

Shear force, V.

lies in the plane of the area

Torsional moment or torque, T.

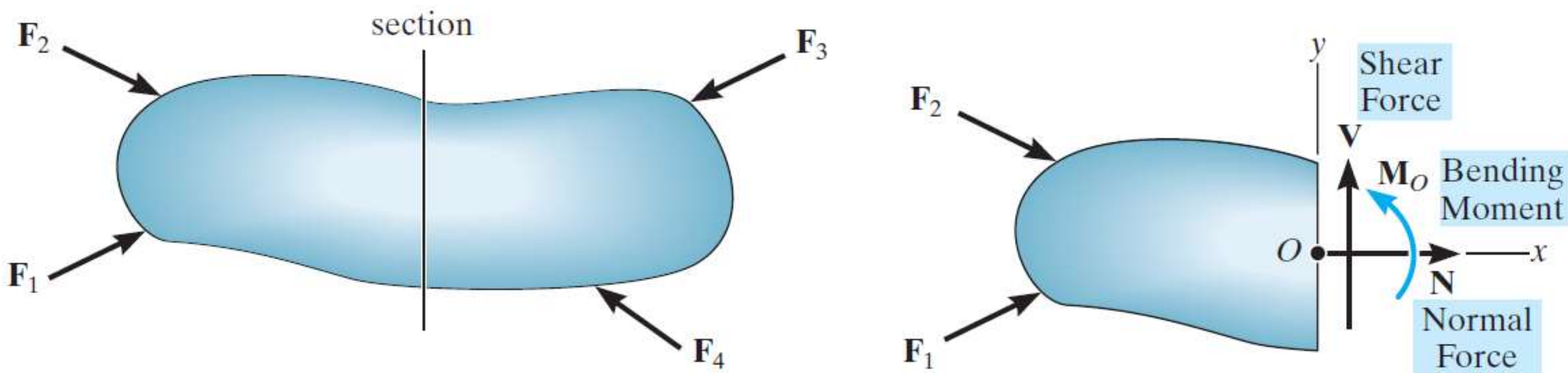
external loads tend to twist one segment of the body with respect to the other about an axis perpendicular to the area.

Bending moment, M.

external loads that tend to bend the body about an axis lying within the plane of the area.

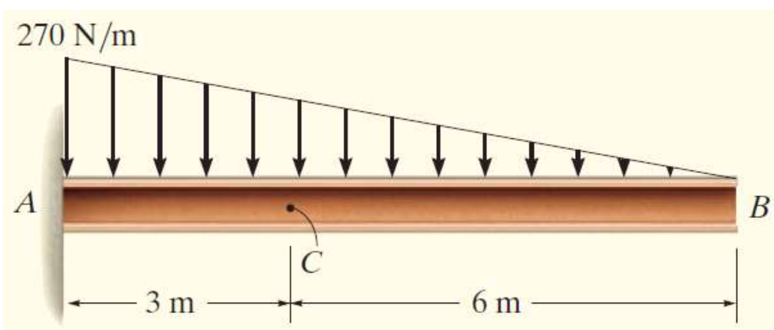
Coplanar loading: examples

- **Coplanar loadings:**



- **Example:**

Determine the resultant internal loadings acting on the cross section at C

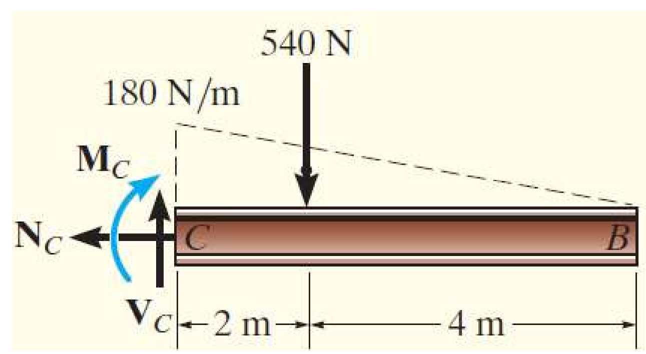


- **Draw the FBD**
- **Eqn of Eqb**

$\xrightarrow{+} \Sigma F_x = 0;$

$+\uparrow \Sigma F_y = 0;$

$\circlearrowleft + \Sigma M_C = 0;$



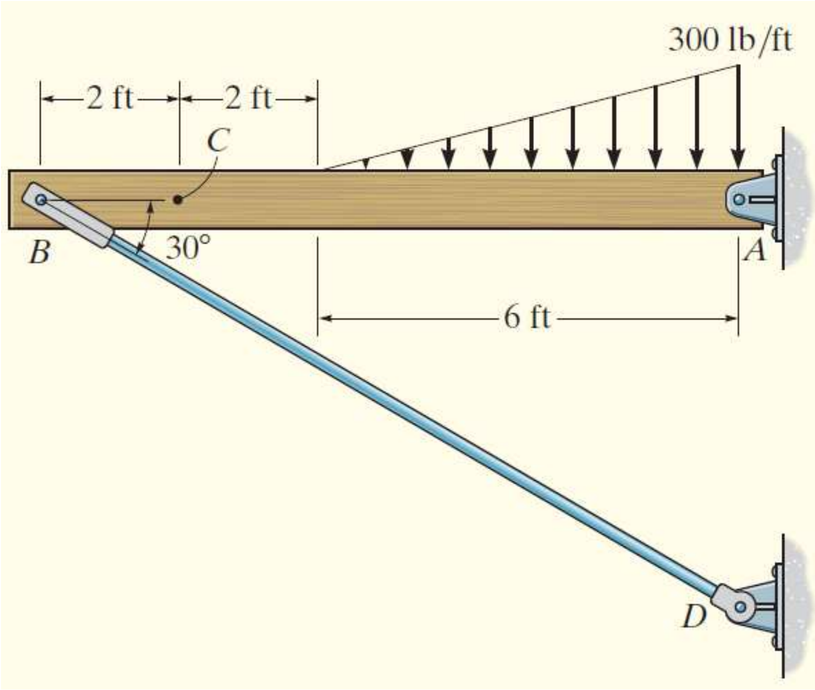
$\xrightarrow{+} \Sigma F_x = 0;$ $\quad -N_C = 0$

$+\uparrow \Sigma F_y = 0;$ $\quad V_C - 540\text{ N} = 0$

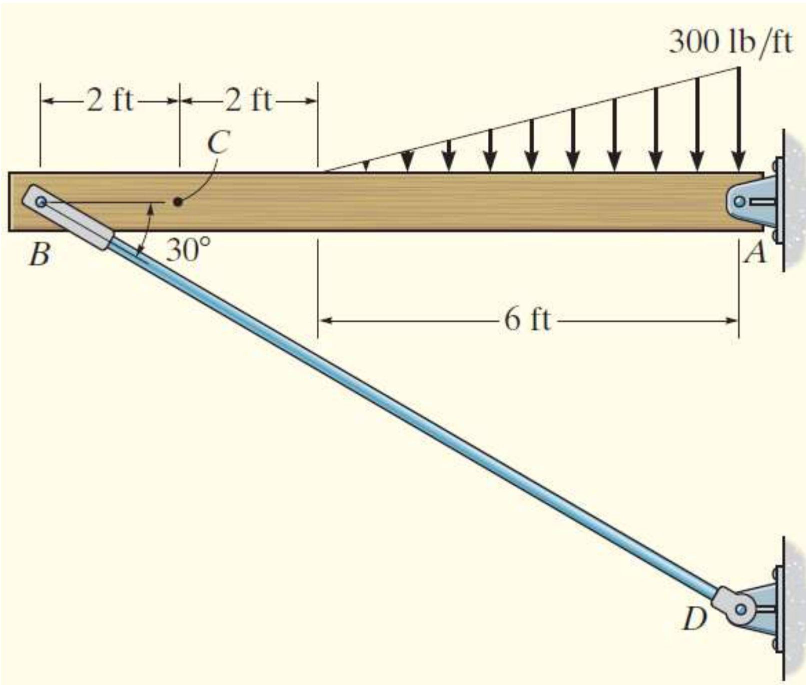
$\circlearrowleft + \Sigma M_C = 0;$ $\quad -M_C - 540\text{ N}(2\text{ m}) = 0$

Amit Singh

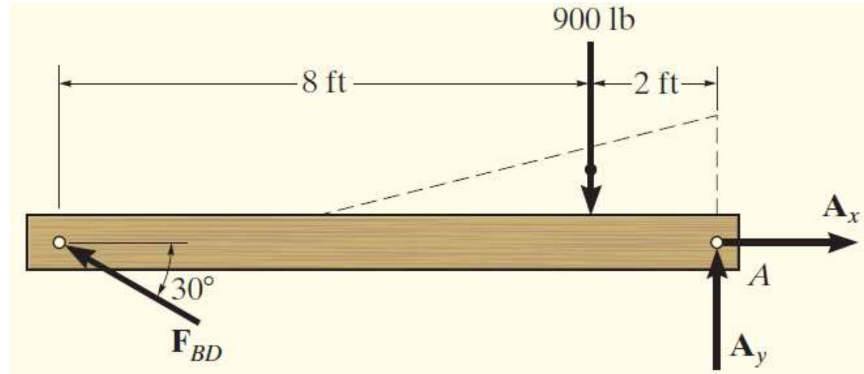
Examples

- 

Examples

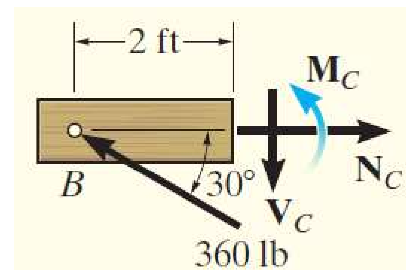
- 

Draw the FBD for full beam



$$\circlearrowleft + \Sigma M_A = 0; \quad (900\text{ lb})(2\text{ ft}) - (F_{BD} \sin 30^\circ)\, 10\text{ ft} = 0 \quad F_{BD} = 360\text{ lb}$$

Draw the FBD for following section:



Eqn of Eqb

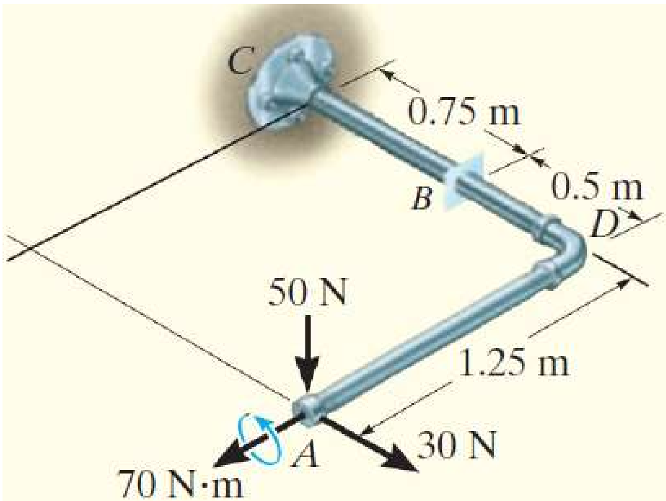
$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad N_C - (360\text{ lb}) \cos 30^\circ = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad (360\text{ lb}) \sin 30^\circ - V_C = 0$$

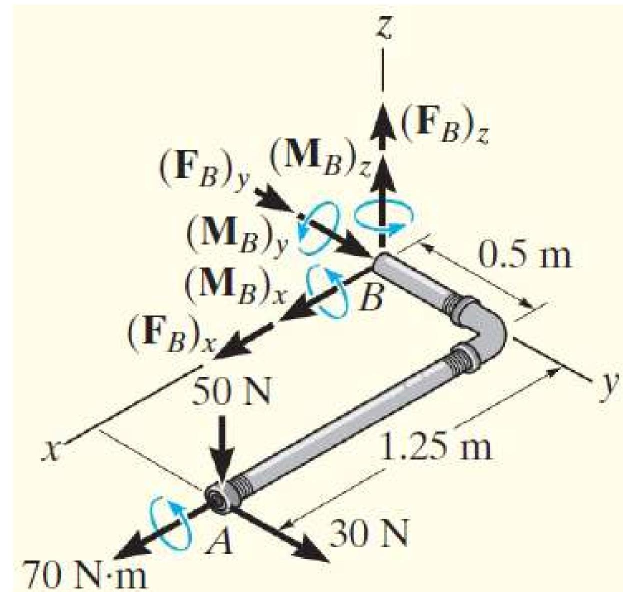
$$\circlearrowleft + \Sigma M_C = 0; \qquad M_C - (360\text{ lb}) \sin 30^\circ (2\text{ ft}) = 0$$

Example

- Determine the resultant internal loadings acting on the cross section at B



Draw the FBD for the following section:



Eqn of

$\Sigma F_x = 0;$ $\quad (F_B)_x = 0$

$\Sigma F_y = 0;$ $\quad (F_B)_y + 30\text{ N} = 0$

$\Sigma F_z = 0;$ $\quad (F_B)_z - 50\text{ N} = 0$

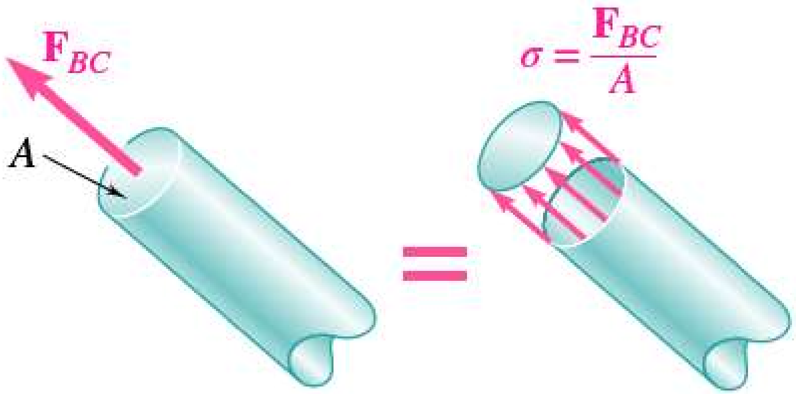
$\Sigma (M_B)_x = 0;$ $\quad (M_B)_x + 70\text{ N}\cdot\text{m} - 50\text{ N}\,(0.5\text{ m}) = 0$

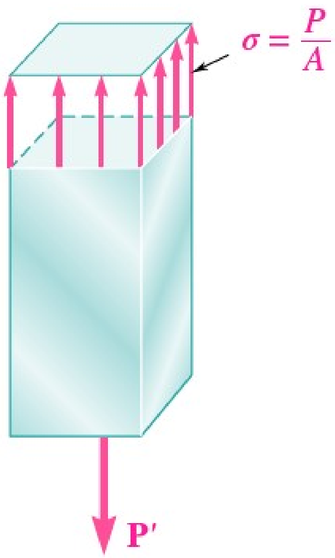
$\Sigma (M_B)_y = 0;$ $\quad (M_B)_y + 50\text{ N}\,(1.25\text{ m}) = 0$

$\Sigma (M_B)_z = 0;$ $\quad (M_B)_z + (30\text{ N})(1.25) = 0$

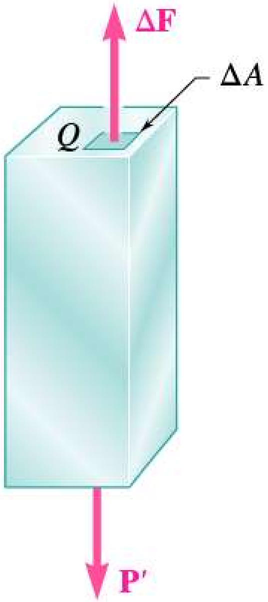
Stresses in members of structures

•



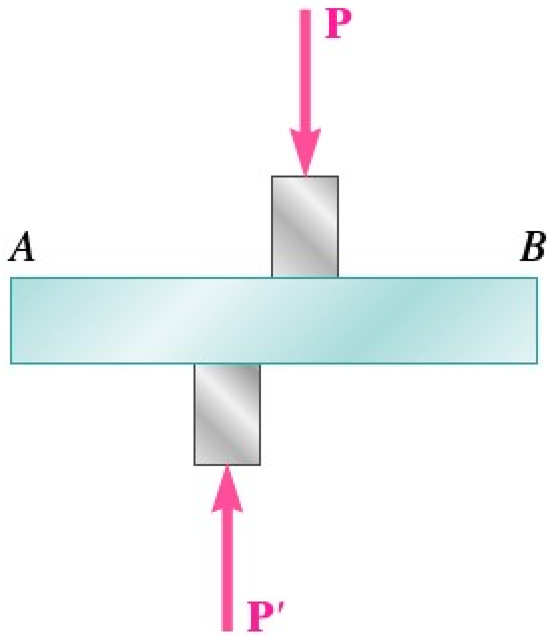


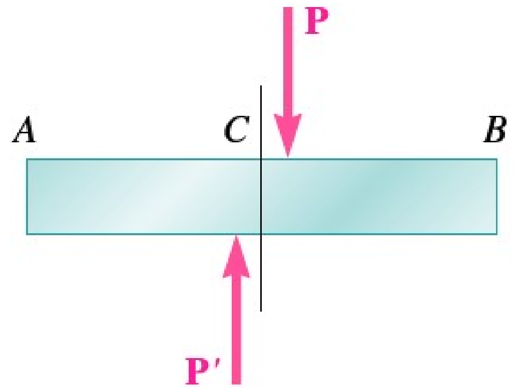
$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$



Amit Singh

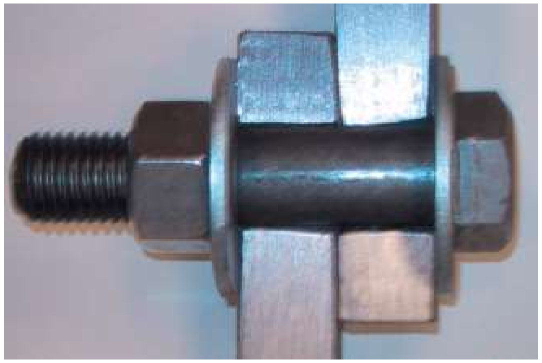
Shearing Stresses: Transverse forces

- 



$$\tau_{\text{ave}} = \frac{P}{A}$$

- Shear stress in bolts:

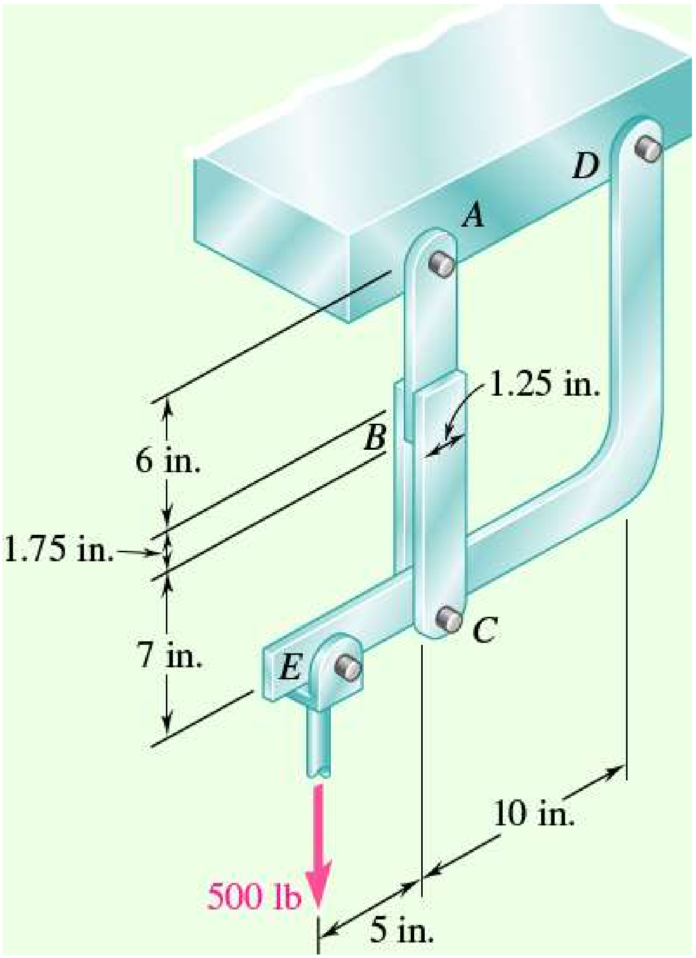


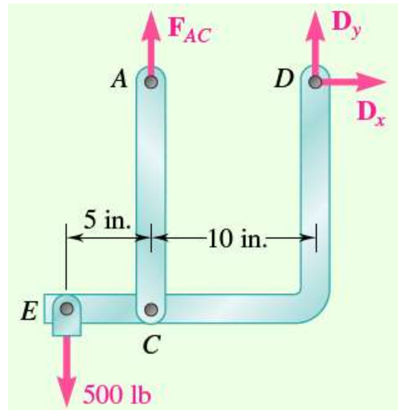
Amit Singh

Example

Determine (*a*) the shearing stress in pin *A*

(*b*) the shearing stress in pin *C*

- 

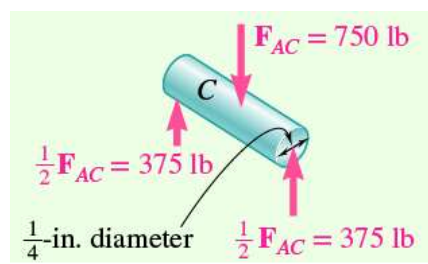


Draw the FBD

$$+\circlearrowleft \Sigma M_D = 0: \qquad (500 \text{ lb})(15 \text{ in.}) - F_{AC}(10 \text{ in.}) = 0$$

$$\tau_A = \frac{F_{AC}}{A} = \frac{750 \text{ lb}}{\frac{1}{4}\pi(0.375 \text{ in.})^2}$$

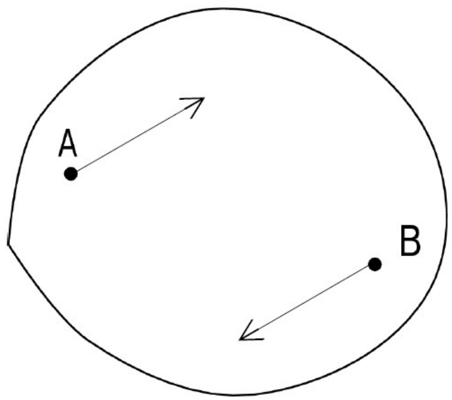
$$\tau_C = \frac{\frac{1}{2}F_{AC}}{A} = \frac{375 \text{ lb}}{\frac{1}{4}\pi(0.25 \text{ in.})^2}$$

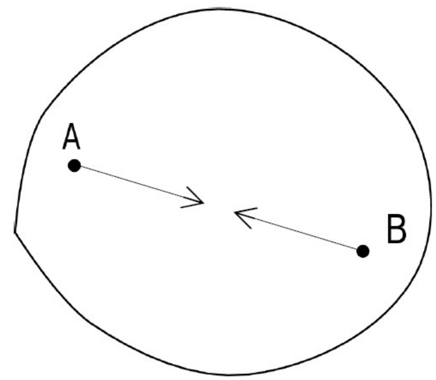


Amit Singh

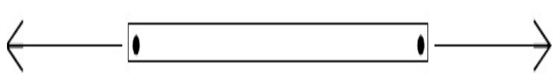
Two-force members

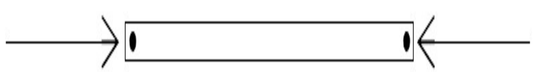
- A structure with exactly two points where external forces are applied
- A truss is a structure consisting of only two-force members



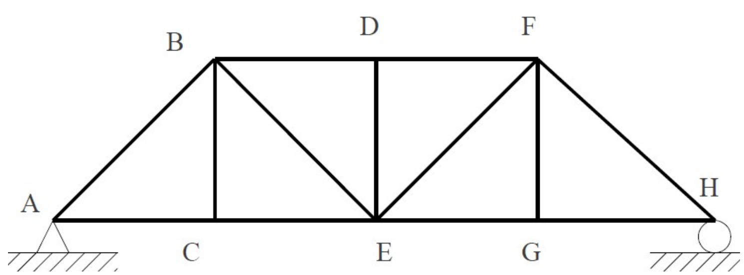


- Can be of any shape, but trusses consists of only straight shape





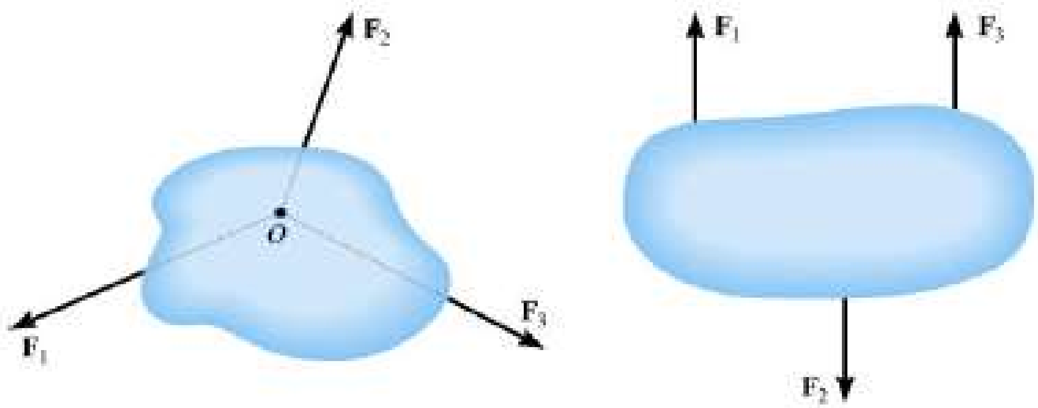
- Pratt truss:



Amit Singh

Three-force members

- A structure with exactly three points where external forces are applied
- Either concurrent or parallel. Why?



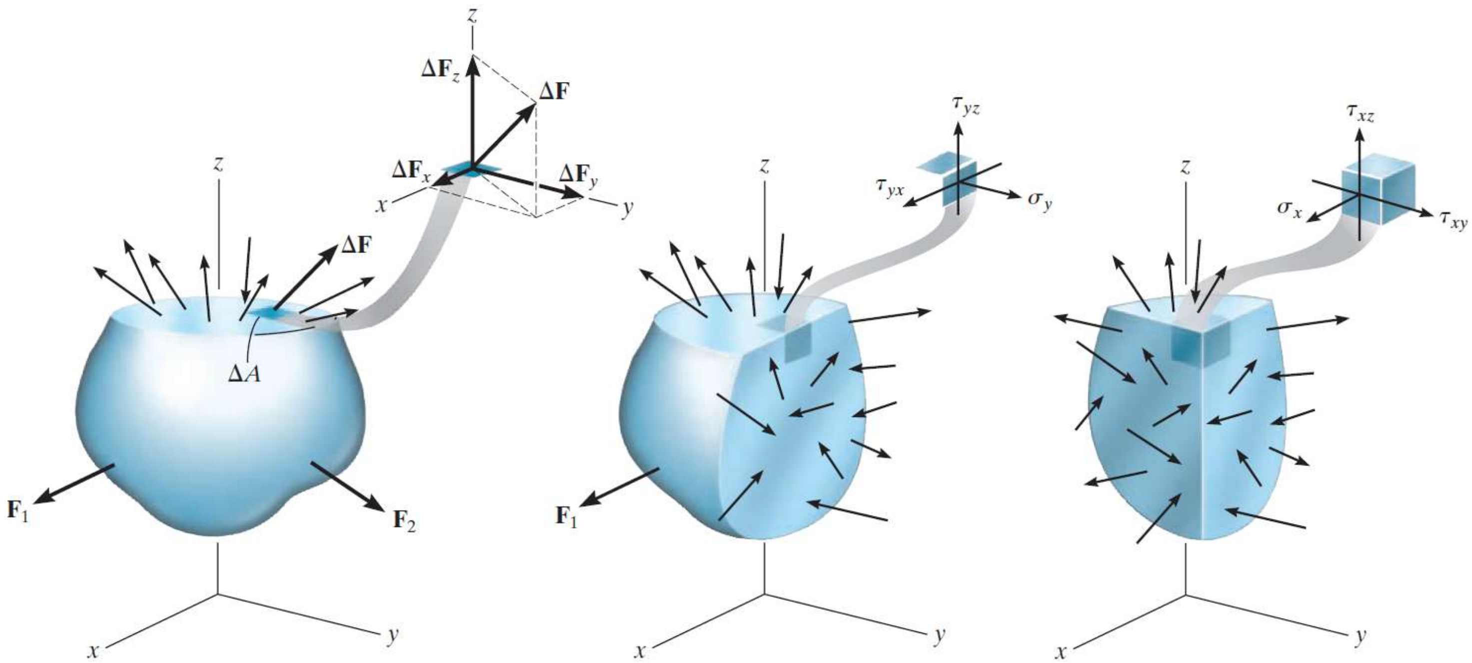
Concurrent forces
(a)

Parallel forces
(b)

- Pick two forces: If their lines of action (LOA) intersect then the third force must also pass through intersection point $\sum \vec{\mathbf{M}}_O = 0$
- If they are parallel then third force must also be parallel. $\sum \vec{\mathbf{F}} = 0$

Stresses: general

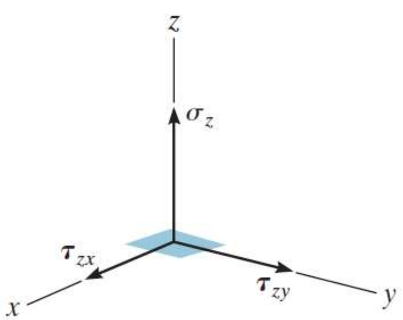
•

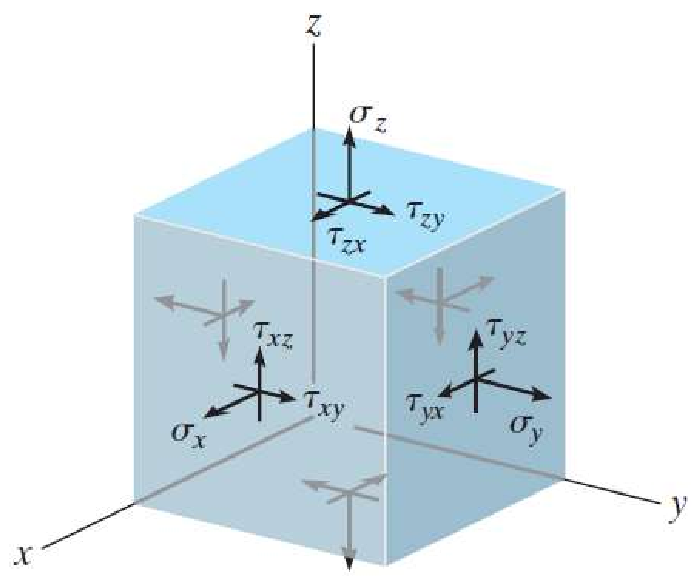


Normal Stress $\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}$

Shear Stress. $\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$

$\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$



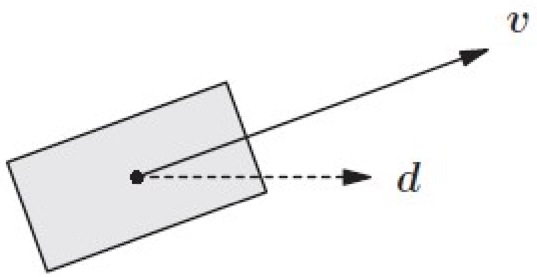


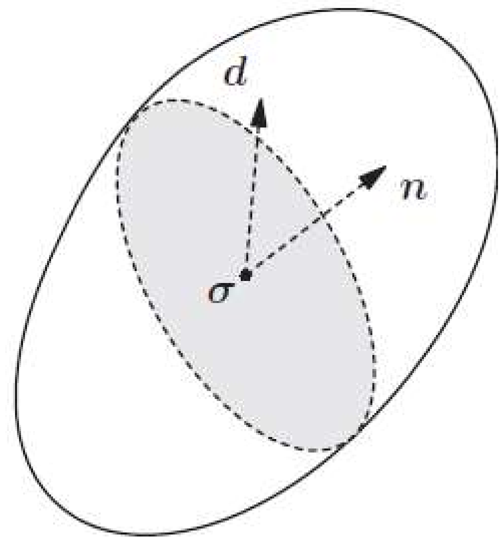
Amit Singh

Tensors

$$[\boldsymbol{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

- **Velocity and stress:**





- **An n-th order tensor is a real-valued n-linear function of vectors:**

$$\boldsymbol{T} : \underbrace{\mathbb{R}^{n_\mathrm{d}} \times \cdots \times \mathbb{R}^{n_\mathrm{d}}}_{n \text{ times}} \to \mathbb{R}.$$

Linear function

- **Linearity**: A function $f(x)$ is **linear** in x if

$$f(x+x') = f(x) + f(x') \; \forall \, x, x' \in \mathbb{R}$$

$$f(\lambda x) = \lambda f(x) \; \forall \, x, \lambda \in \mathbb{R}$$

$$f[\lambda x + \mu x'] = \lambda f[x] + \mu f[x'], \quad \forall \, x, x', \lambda, \mu \in \mathbb{R}$$

$f[x] = Cx$ a linear function

$g(x) = Cx + D$ not linear

Linear function

- **Linearity**: A function $f(x)$ is **linear** in x if

$$f(x+x') = f(x) + f(x')\,\forall\, x, x' \in \mathbb{R}$$

$$f(\lambda x) = \lambda f(x)\,\forall\, x, \lambda \in \mathbb{R}$$

$$f[\lambda x + \mu x'] = \lambda f[x] + \mu f[x'], \quad \forall\, x, x', \lambda, \mu \in \mathbb{R}$$

$$f[x] = Cx \qquad \text{a linear function}$$

$$g(x) = Cx + D \qquad \text{not linear}$$

$$g(x+x') = C(x+x') + D \neq g(x) + g(x') = C(x+x') + 2D$$

Multilinear function

- **Multilinear or n-linear function:** Linear w.r.t. each of its independent variable

$$f[\lambda x+\mu x', y] = \lambda f[x,y] + \mu f[x',y], \quad \forall\, x, x', y, \lambda, \mu \in \mathbb{R}$$
$$f[x, \lambda y+\mu y'] = \lambda f[x,y] + \mu f[x,y'], \quad \forall\, x, y, y', \lambda, \mu \in \mathbb{R}$$

$f[x,y] = Cxy$ is a bilinear function, while $g(x,y) = Cxy + D$ is not

- **Real-valued vector functions** (linear mapping): $\boldsymbol{f} : V \to \mathbb{R}$

$$\boldsymbol{f}[\lambda \boldsymbol{a}+\mu \boldsymbol{a}'] = \lambda \boldsymbol{f}[\boldsymbol{a}] + \mu \boldsymbol{f}[\boldsymbol{a}'], \quad \forall\, \boldsymbol{a}, \boldsymbol{a}' \in V, \forall\, \lambda, \mu \in \mathbb{R}$$

- **Bilinear mapping:** $\boldsymbol{f} : V \times V \to \mathbb{R}$

$$\boldsymbol{f}[\lambda \boldsymbol{a}+\mu \boldsymbol{a}', \boldsymbol{b}] = \lambda \boldsymbol{f}[\boldsymbol{a}, \boldsymbol{b}] + \mu \boldsymbol{f}[\boldsymbol{a}', \boldsymbol{b}], \quad \forall\, \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{b} \in V, \forall\, \lambda, \mu \in \mathbb{R}$$
$$\boldsymbol{f}[\boldsymbol{a}, \lambda \boldsymbol{b}+\mu \boldsymbol{b}'] = \lambda \boldsymbol{f}[\boldsymbol{a}, \boldsymbol{b}] + \mu \boldsymbol{f}[\boldsymbol{a}, \boldsymbol{b}'], \quad \forall\, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{b}' \in V, \forall\, \lambda, \mu \in \mathbb{R}.$$

Multilinear function

- **Multilinear or n-linear function:** $\boldsymbol{f} : \underbrace{V \times \cdots \times V}_{n \text{ times}} \rightarrow \mathbb{R}$,

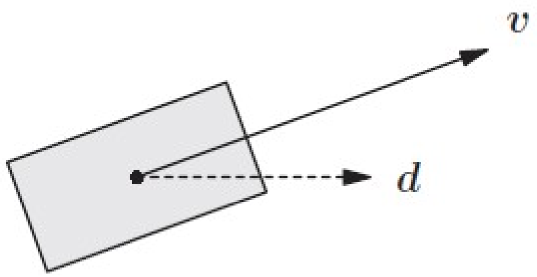
$$\boldsymbol{f}[\boldsymbol{a}_1, \ldots, \lambda \boldsymbol{a}_i + \mu \boldsymbol{a}'_i, \ldots, \boldsymbol{a}_n] = \lambda \boldsymbol{f}[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_n] + \mu \boldsymbol{f}[\boldsymbol{a}_1, \ldots, \boldsymbol{a}'_i, \ldots, \boldsymbol{a}_n]$$

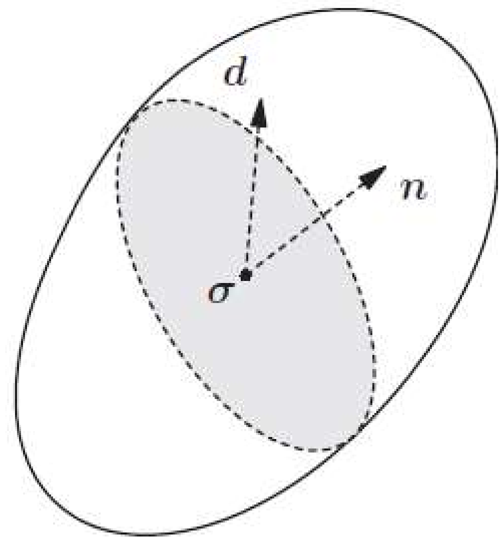
$\forall\, \boldsymbol{a}_i, \boldsymbol{a}'_i \in V$ and $\forall\, \lambda, \mu \in \mathbb{R}$

Tensors

$$[\boldsymbol{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

- **Velocity and stress:**





- **An n-th order tensor is a real-valued n-linear function of vectors:**

$$\boldsymbol{T} : \underbrace{\mathbb{R}^{n_\mathrm{d}} \times \cdots \times \mathbb{R}^{n_\mathrm{d}}}_{n \text{ times}} \to \mathbb{R}.$$

$$\boldsymbol{v}^*[\boldsymbol{d}] = \boldsymbol{v} \cdot \boldsymbol{d}$$

$$\boldsymbol{v}^*[\boldsymbol{e}_i] = v_i$$

We *define* the mapping $\boldsymbol{v}^*[\,]$ to be a *first-order tensor*

Amit Singh

Direct and Indicial notation

- **Tensor** represents physical quantities such as mass, velocity and stress which do not depend upon coordinate system

- **Direct notation:** symbolic notation, tensor operations are defined with symbols

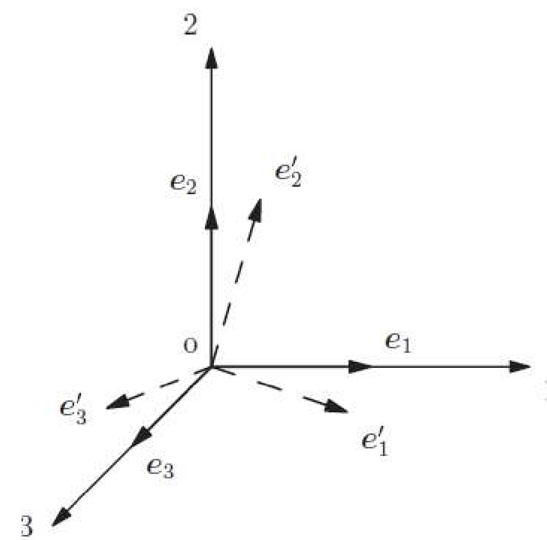
$$m,\ \boldsymbol{v},\ \boldsymbol{\sigma}\ \text{(or } m,\ \underline{v},\ \underline{\sigma} \text{ by hand)}$$

- **Indicial notation**: explicit in terms of components along coordinate axes $m,\ v_i,\ \sigma_{ij}$

- **Rank/order** of a tensor: The number of spatial directions associated

- **Matrix representation**: Vectors and Second-order tensors only

$$[\boldsymbol{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



Summation and dummy indices

- $$S = a_1 x_1 + a_2 x_2 + \cdots + a_{n_\mathrm{d}} x_{n_\mathrm{d}}$$

$$S = \sum_{i=1}^{n_\mathrm{d}} a_i x_i = \sum_{j=1}^{n_\mathrm{d}} a_j x_j = \sum_{m=1}^{n_\mathrm{d}} a_m x_m$$

- No matter what index in the summation is used, meaning is always the same. **Dummy index.**
- **Einstein** during the development of theory of general relativity in 1916 dropped the summation symbol

$$S = a_i x_i = a_j x_j = a_m x_m = a_1 x_1 + a_2 x_2 + \cdots + a_{n_\mathrm{d}} x_{n_\mathrm{d}}$$

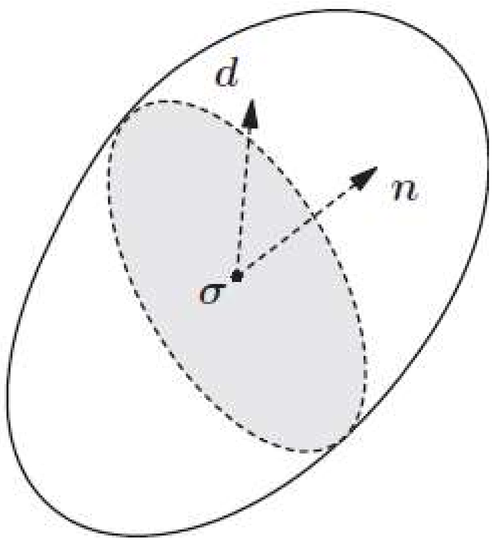
- Einstein's summation convention
- **Example**

$$a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3 .$$
$$a_i a_i = a_1^2 + a_2^2 + a_3^2 .$$
$$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} .$$

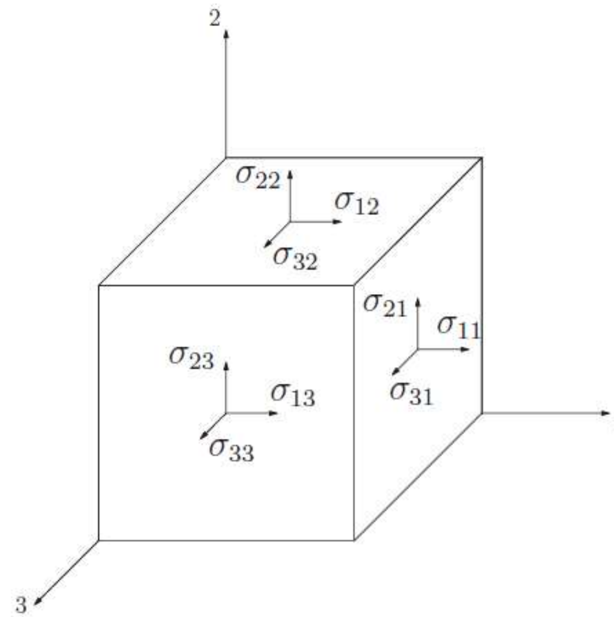
$$\begin{aligned} A_{ij} x_i y_j = & A_{11} x_1 y_1 + A_{12} x_1 y_2 + A_{13} x_1 y_3 \\ & + A_{21} x_2 y_1 + A_{22} x_2 y_2 + A_{23} x_2 y_3 \\ & + A_{31} x_3 y_1 + A_{32} x_3 y_2 + A_{33} x_3 y_3 \end{aligned}$$

Tensors

- **Stress tensor:** A second-order tensor which returns the force per unit area along direction $\boldsymbol{d}$ when cutting a body along a plane with normal $\boldsymbol{n}$



$$\boldsymbol{\sigma}[\boldsymbol{d},\boldsymbol{n}]$$



- **Components of a tensor in a basis** $\{\boldsymbol{e}_i\}$**:** $T_{ij} \equiv \boldsymbol{T}[\boldsymbol{e}_i, \boldsymbol{e}_j]$

$$\sigma_{ij} \equiv \boldsymbol{\sigma}[\boldsymbol{e}_i, \boldsymbol{e}_j]$$

$$\boldsymbol{T}[\boldsymbol{a},\boldsymbol{b}] = \boldsymbol{T}[a_i\boldsymbol{e}_i, b_j\boldsymbol{e}_j] = a_i b_j \boldsymbol{T}[\boldsymbol{e}_i, \boldsymbol{e}_j] = a_i b_j T_{ij}$$

$$\boldsymbol{\sigma}[\boldsymbol{d},\boldsymbol{n}] = \sigma_{ij} d_i n_j$$

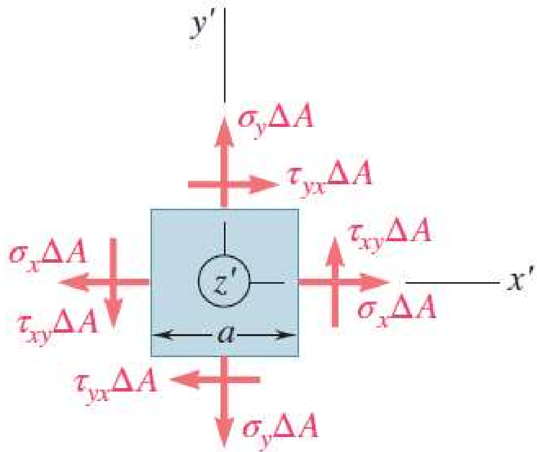
Amit Singh

Stresses: general

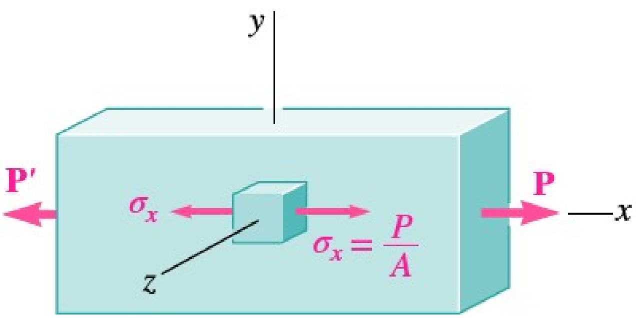
- **Stress tensor is symmetric**

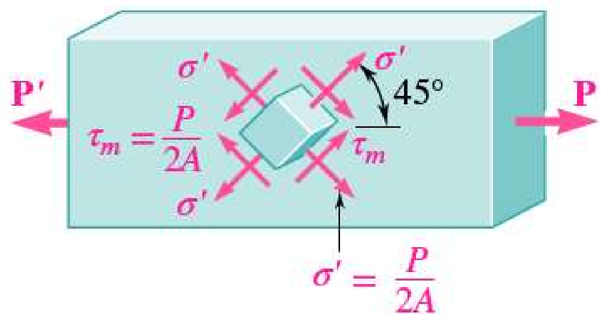
$$+\circlearrowleft \Sigma M_z = 0: \qquad (\tau_{xy}\ \Delta A)a - (\tau_{yx}\ \Delta A)a = 0$$

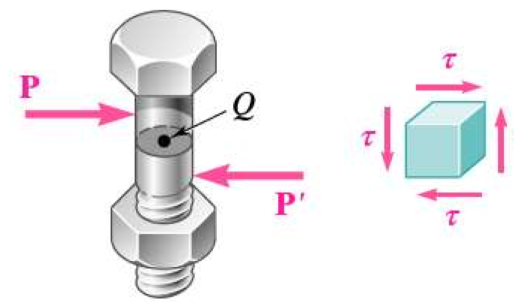
$$\tau_{xy} = \tau_{yx} \qquad \tau_{yz} = \tau_{zy} \qquad \tau_{zx} = \tau_{xz}$$



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$





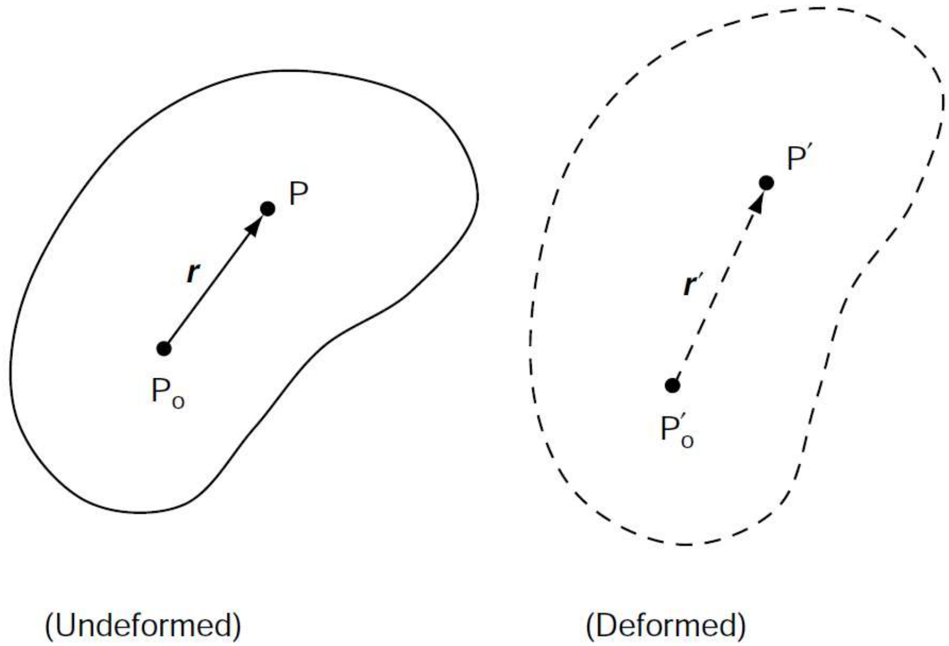


Orientation change

Amit Singh

Deformation: displacement, strain

- Taylor series expansion around point P0



$$u = u^o + \frac{\partial u}{\partial x} r_x + \frac{\partial u}{\partial y} r_y + \frac{\partial u}{\partial z} r_z$$

$$v = v^o + \frac{\partial v}{\partial x} r_x + \frac{\partial v}{\partial y} r_y + \frac{\partial v}{\partial z} r_z$$

$$w = w^o + \frac{\partial w}{\partial x} r_x + \frac{\partial w}{\partial y} r_y + \frac{\partial w}{\partial z} r_z$$

- Define the displacement vectors of points $\mathrm{P_o}$ and P to be $\boldsymbol{u^o}$ and $\boldsymbol{u}$

- The change in relative position vector of P can be written as

$$\Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r} = \boldsymbol{u} - \boldsymbol{u^o}$$

Deformation: displacement, strain

$$\Delta r_x = \frac{\partial u}{\partial x} r_x + \frac{\partial u}{\partial y} r_y + \frac{\partial u}{\partial z} r_z$$

$$\Delta r_y = \frac{\partial v}{\partial x} r_x + \frac{\partial v}{\partial y} r_y + \frac{\partial v}{\partial z} r_z$$

$$\Delta r_z = \frac{\partial w}{\partial x} r_x + \frac{\partial w}{\partial y} r_y + \frac{\partial w}{\partial z} r_z$$

$$\Delta r_i = u_{i,j} r_j$$

The tensor $u_{i,j}$ is called the *displacement gradient tensor*, and may be written out as

$$u_{i,j} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$u_{i,j} = e_{ij} + \omega_{ij}$$

decomposed into symmetric and antisymmetric

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

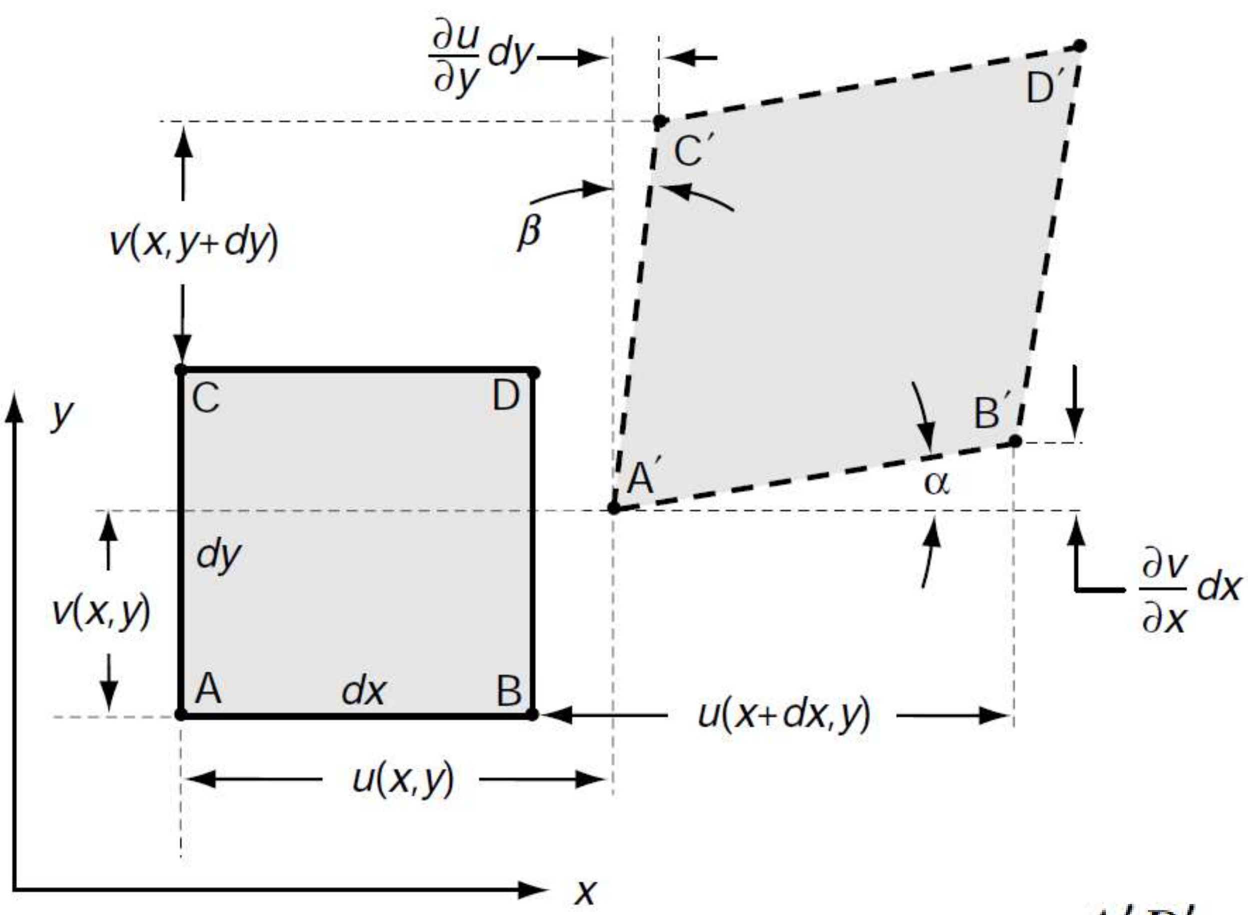
$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

The tensor e_{ij} is called the *strain tensor*, while ω_{ij} is referred to as the *rotation tensor*.

Amit Singh

Deformation: displacement, strain

- *Two-dimensional geometric strain deformation*



The *normal* or *extensional strain component* $\varepsilon_x = \dfrac{A'B' - AB}{AB}$

Engineering normal strain

Amit Singh

Deformation: displacement, strain

- $$A'B' = \sqrt{\left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\,dx \approx \left(1 + \frac{\partial u}{\partial x}\right)dx$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\pi}{2} - \angle C'A'B' = \alpha + \beta$$

Engineering shear strain or shear strain

$$\gamma_{xy} = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} + \frac{\frac{\partial u}{\partial y}dy}{dy + \frac{\partial v}{\partial y}dy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Deformation: displacement, strain

- Define strain tensor e_{ij}

$$e_x = \frac{\partial u}{\partial x}, e_y = \frac{\partial v}{\partial y}, \; e_z = \frac{\partial w}{\partial z}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \; e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \; e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

strain-displacement relations

$$\boldsymbol{e} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right]$$

$$\boldsymbol{e} = [\boldsymbol{e}] = \begin{bmatrix} e_x & e_{xy} & e_{xz} \\ e_{xy} & e_y & e_{yz} \\ e_{xz} & e_{yz} & e_z \end{bmatrix}$$

Strain Compatibility

- Given the six strain components, we cannot obtain three continuous single valued displacement compononents. So there must be extra constraints on strain tensor

$$e_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

$$e_{kl,ij} = \frac{1}{2}(u_{k,lij} + u_{l,kij})$$

$$e_{jl,ik} = \frac{1}{2}(u_{j,lik} + u_{l,jik})$$

$$e_{ik,jl} = \frac{1}{2}(u_{i,kjl} + u_{k,ijl})$$

Saint Venant compatibility equations

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = 2\frac{\partial^2 e_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} = 2\frac{\partial^2 e_{zx}}{\partial z \partial x}$$

$$\frac{\partial^2 e_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right)$$

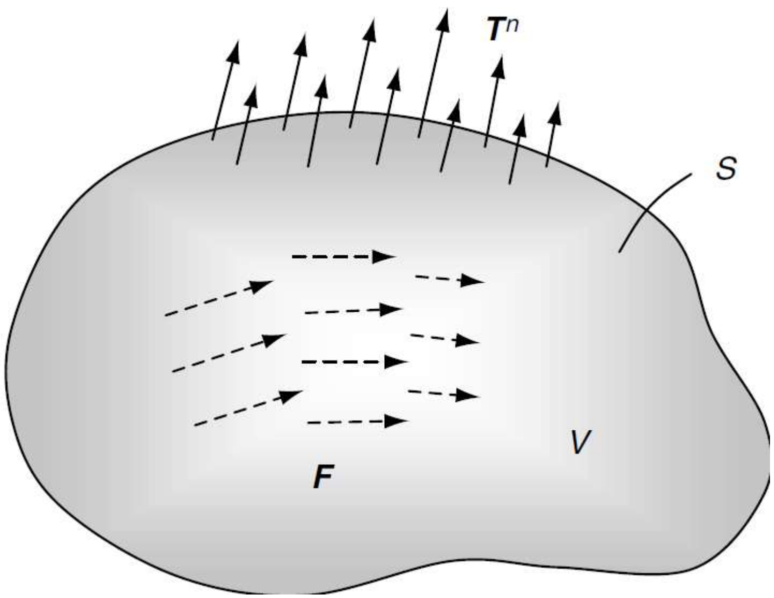
$$\frac{\partial^2 e_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 e_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right)$$

compatibility equations actually represent only three independent relations

Stress and equilibrium

- **Equilibrium equation :** Consider a closed subdomain with volume V and surface S within a body in equilibrium



$$\iint_S T_i^n dS + \iiint_V F_i dV = 0$$

$$\iint_S \sigma_{ji} n_j dS + \iiint_V F_i dV = 0$$

Applying the divergence theorem

$$\iiint_V (\sigma_{ji,j} + F_i) dV = 0$$

$$\sigma_{ji,j} + F_i = 0$$

Stress and equilibrium

- Equation of momentum balance or force equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0$$

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{F} = \boldsymbol{0}$$

- Equation of momentum balance:

$$\sigma_{ij} = \sigma_{ji} \Rightarrow \begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{yz} &= \tau_{zy} \\ \tau_{zx} &= \tau_{xz} \end{aligned}$$

Constitutive equations: Linear elastic solids

- **Unknowns**: 3 displacement components, 6 stress components, 6 strain components, total 15 unknowns
- Number of equations available: 3 strain-displacement + 3 strain compatibility + 3 stress equilibrium; total 9 equations
- We still need 6 more equations: birth of **Constitutive laws or relations**
- **General linear elastic material:**

$$
\begin{aligned}
\sigma_x &= C_{11}e_x + C_{12}e_y + C_{13}e_z + 2C_{14}e_{xy} + 2C_{15}e_{yz} + 2C_{16}e_{zx} \\
\sigma_y &= C_{21}e_x + C_{22}e_y + C_{23}e_z + 2C_{24}e_{xy} + 2C_{25}e_{yz} + 2C_{26}e_{zx} \\
\sigma_z &= C_{31}e_x + C_{32}e_y + C_{33}e_z + 2C_{34}e_{xy} + 2C_{35}e_{yz} + 2C_{36}e_{zx} \\
\tau_{xy} &= C_{41}e_x + C_{42}e_y + C_{43}e_z + 2C_{44}e_{xy} + 2C_{45}e_{yz} + 2C_{46}e_{zx} \\
\tau_{yz} &= C_{51}e_x + C_{52}e_y + C_{53}e_z + 2C_{54}e_{xy} + 2C_{55}e_{yz} + 2C_{56}e_{zx} \\
\tau_{zx} &= C_{61}e_x + C_{62}e_y + C_{63}e_z + 2C_{64}e_{xy} + 2C_{65}e_{yz} + 2C_{66}e_{zx}
\end{aligned}
$$

Constitutive equations: Linear elastic solids

- $$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{61} & \cdot & \cdot & \cdot & \cdot & C_{66} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix}$$

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

C_{ijkl} is a *fourth-order elasticity tensor*

- **For isotropic material:**

$$\begin{aligned} \sigma_x &= \lambda(e_x + e_y + e_z) + 2\mu e_x \\ \sigma_y &= \lambda(e_x + e_y + e_z) + 2\mu e_y \\ \sigma_z &= \lambda(e_x + e_y + e_z) + 2\mu e_z \\ \tau_{xy} &= 2\mu e_{xy} \\ \tau_{yz} &= 2\mu e_{yz} \\ \tau_{zx} &= 2\mu e_{zx} \end{aligned}$$

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}$$

Constitutive equations: Linear isotropic elastic solids

-

$$e_x = \frac{1}{E}\left[\sigma_x - v(\sigma_y + \sigma_z)\right]$$

$$e_y = \frac{1}{E}\left[\sigma_y - v(\sigma_z + \sigma_x)\right]$$

$$e_z = \frac{1}{E}\left[\sigma_z - v(\sigma_x + \sigma_y)\right]$$

$$e_{xy} = \frac{1+v}{E}\tau_{xy} = \frac{1}{2\mu}\tau_{xy}$$

$$e_{yz} = \frac{1+v}{E}\tau_{yz} = \frac{1}{2\mu}\tau_{yz}$$

$$e_{zx} = \frac{1+v}{E}\tau_{zx} = \frac{1}{2\mu}\tau_{zx}$$

$$E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$$

$$v = \lambda/[2(\lambda + \mu)]$$

Field equations

- Strain-displacement relations: $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

  Compatibility relations: $e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$

  Equilibrium equations: $\sigma_{ij,j} + F_i = 0$

  Elastic constitutive law (Hooke's law):

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

Amit Singh

Allowable stress design

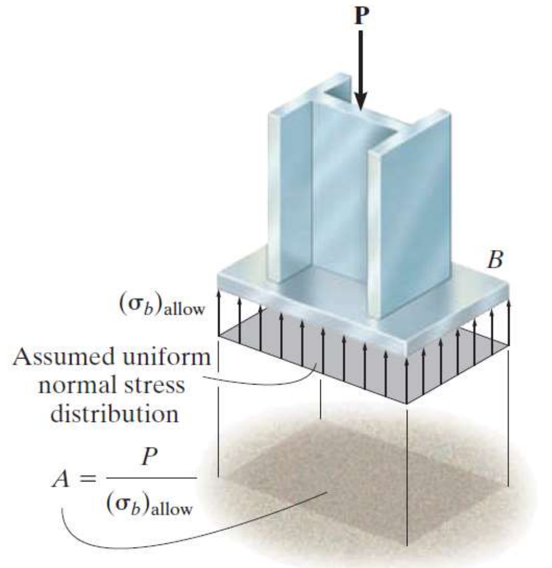
- *factor of safety* (F.S.) $$\text{F.S.} = \frac{F_{\text{fail}}}{F_{\text{allow}}}$$

$$\text{F.S.} = \frac{\sigma_{\text{fail}}}{\sigma_{\text{allow}}} \qquad \text{F.S.} = \frac{\tau_{\text{fail}}}{\tau_{\text{allow}}} \qquad A = \frac{N}{\sigma_{\text{allow}}}$$

$$\text{Margin of safety} = n - 1$$

$$A = \frac{V}{\tau_{\text{allow}}}$$

- **Factor of safety selection:**



The area of the column base plate *B* is determined from the allowable bearing stress for the concrete.

1. *Variations that may occur in the properties of the member.*
2. *The number of loadings expected during the life of the structure*
3. *The type of loadings planned for in the design*
4. *Type of failure.*
5. *Uncertainty due to methods of analysis.*

Load and Resistance Factor Design (LRFD) **Live Load** P_L **Dead Load** P_D

Load and Resistance Factor Design (*LRFD*) allows the engineer to distinguish between the uncertainties associated with the structure and those associated with the load.

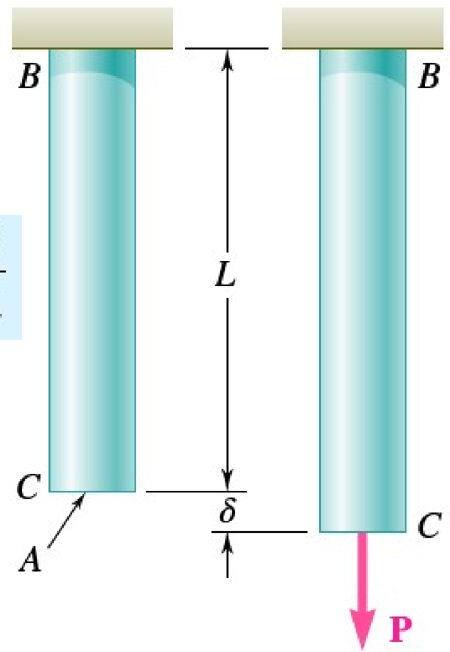
$$\gamma_D P_D + \gamma_L P_L \le \phi P_U$$

The coefficient ϕ is the *resistance factor,*

< 1, accounts for uncertainities with structure

Amit Singh

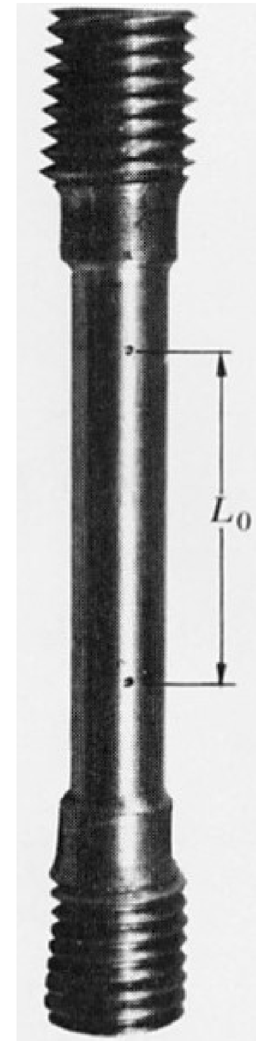
Normal strain under axial loading

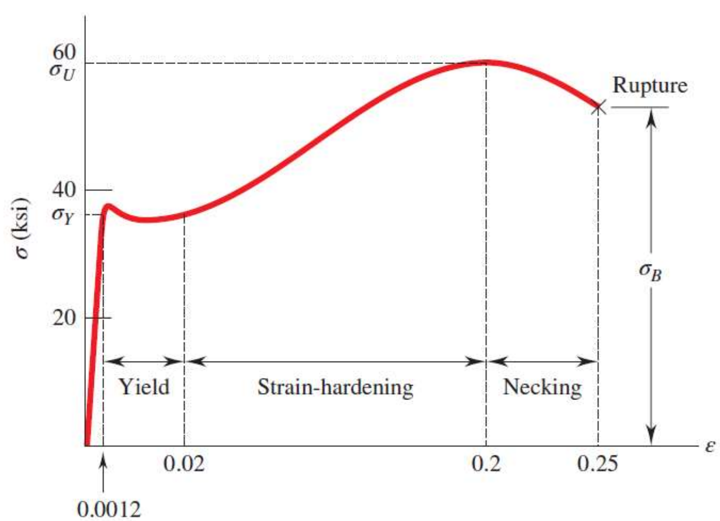


$$\varepsilon = \frac{\delta}{L}$$

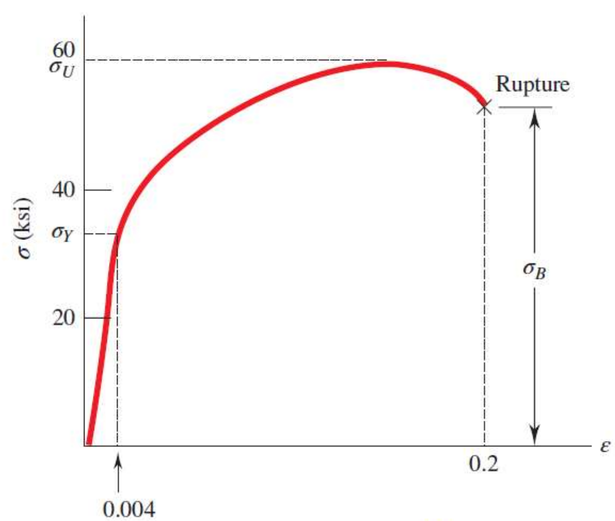
engineering stress $\sigma = \dfrac{P}{A_0}$

engineering strain $\varepsilon = \dfrac{\delta}{L_0}$





Low-carbon steel

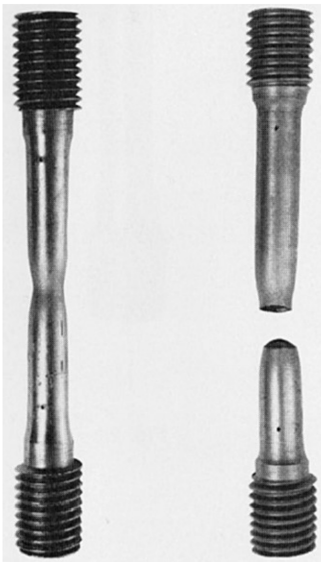


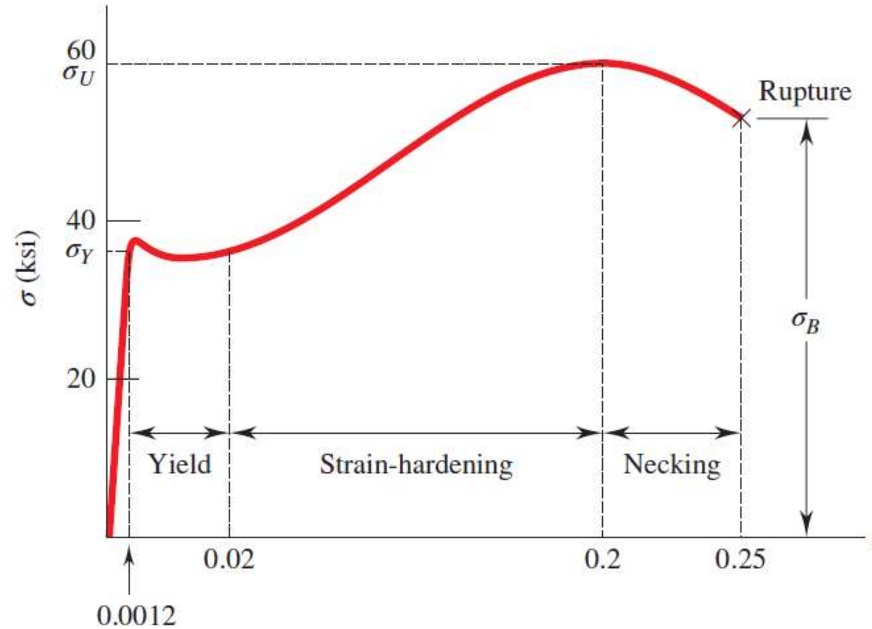
Aluminum alloy

Amit Singh

Ductile material

Ability to yield, ultimate stress, necking, rupture: Steel, alloys of different materials





- **Yield strength** σ_Y**:** Stress at which yield is initiated
- **Ultimate Strength** σ_U: Stress corresponding to maximum load applied
- **Breaking strength** σ_B **:** Stress corresponding to rupture

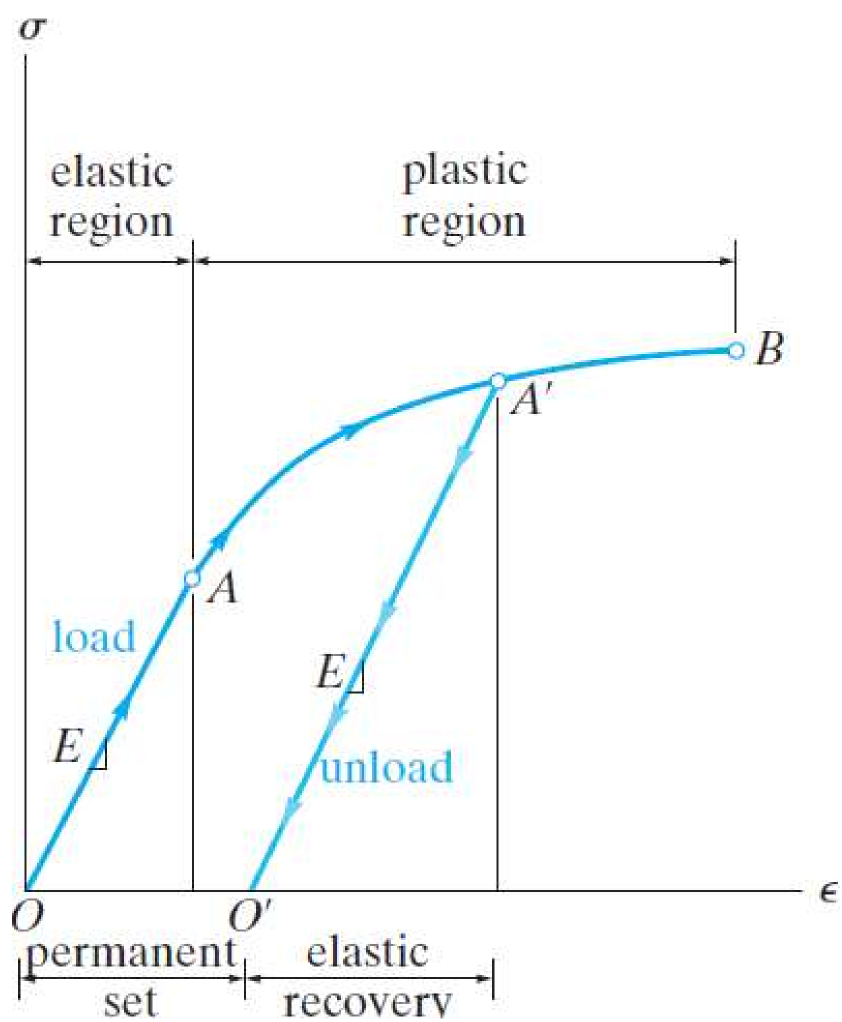
$$\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%)$$

$$\text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%)$$

Amit Singh

Strain hardening

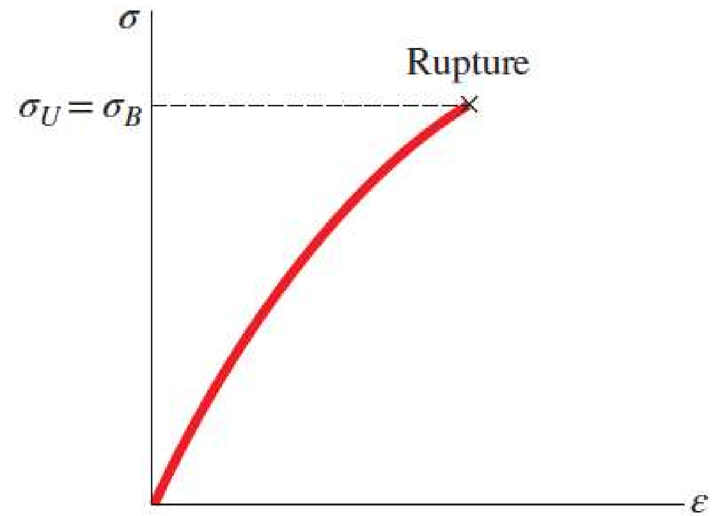
- **Strain hardening:** The yield point of a material can be increased. Go to stress greater than yield stress and then unload.

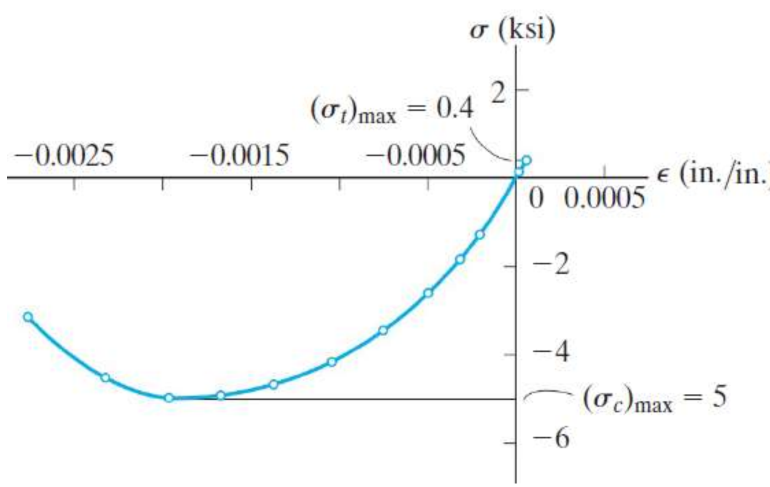


Amit Singh

Brittle material

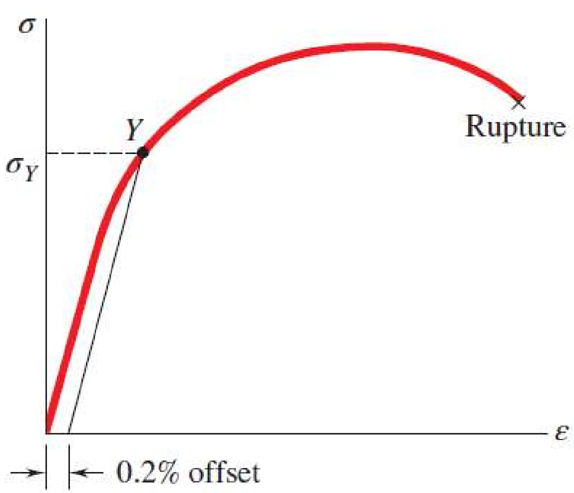
- No diff between ultimate and breaking stress
- Absence of any necking
- Cast iron, glass, stone, wood, concrete
- Compression test: normally different stress-strain behavior





Compression causes material to bulge out

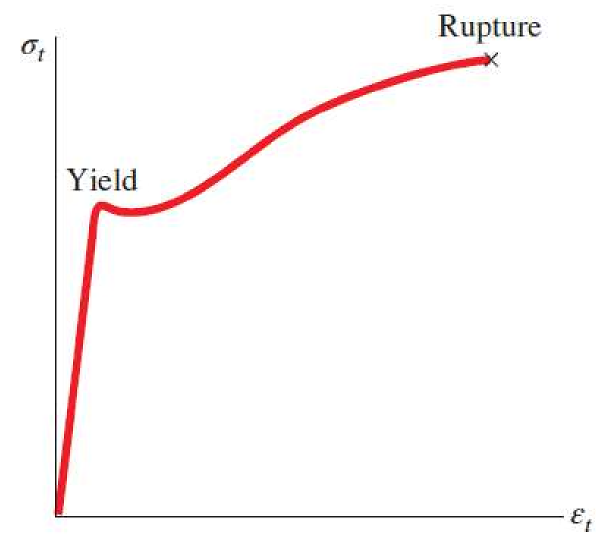
Determination of yield strength



Amit Singh

True stress and true strain

- **True stress** $\sigma_t = P/A$
- **True strain** $\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0}$



typical ductile material

- **Hooke's Law: Modulus of Elasticity**

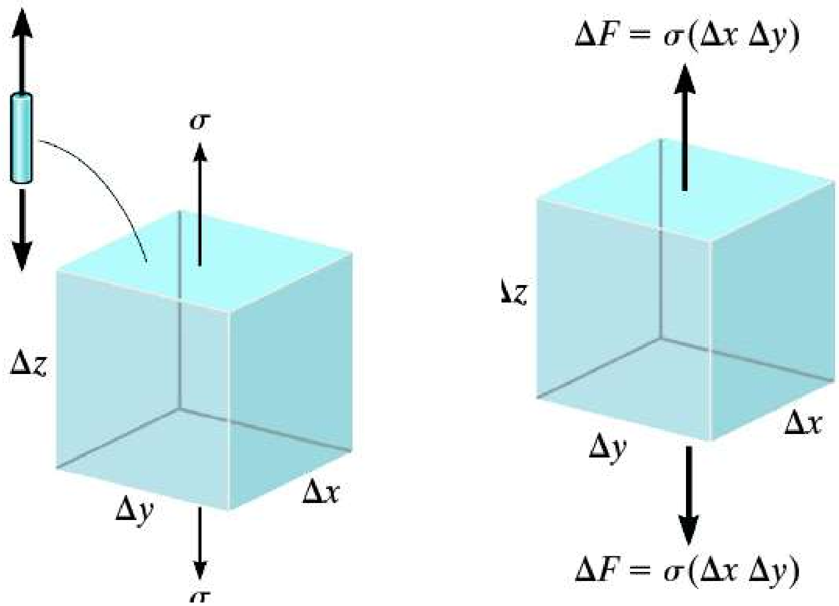
$$\sigma = E\varepsilon$$

E of the material is the *modulus of elasticity* or *Young's modulus*

indicates the ***stiffness*** of a material

Stiff materials have high E, spongy have low E

Strain energy



$$\Delta U = \left(\tfrac{1}{2}\Delta F\right) \epsilon\, \Delta z = \left(\tfrac{1}{2}\, \sigma\, \Delta x\, \Delta y\right) \epsilon\, \Delta z.$$

$$\Delta V = \Delta x\, \Delta y\, \Delta z$$

$$\Delta U = \tfrac{1}{2}\, \sigma \epsilon\, \Delta V$$

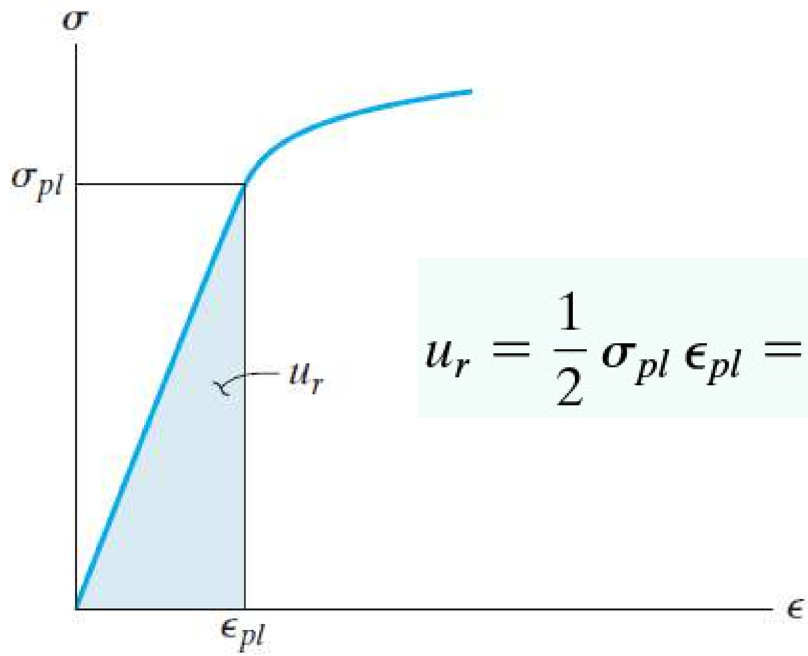
$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon$$

strain energy density

Finally, if the material behavior is *linear elastic*, $\sigma = E\epsilon$, $$u = \frac{1}{2}\frac{\sigma^2}{E}$$

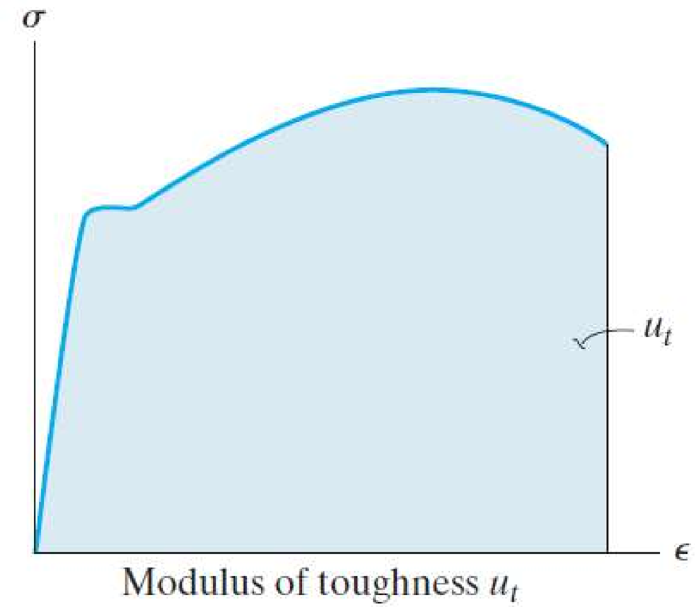
Amit Singh

Modulus of resilience and toughness



$$u_r = \frac{1}{2}\sigma_{pl}\,\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E}$$

Modulus of resilience u_r

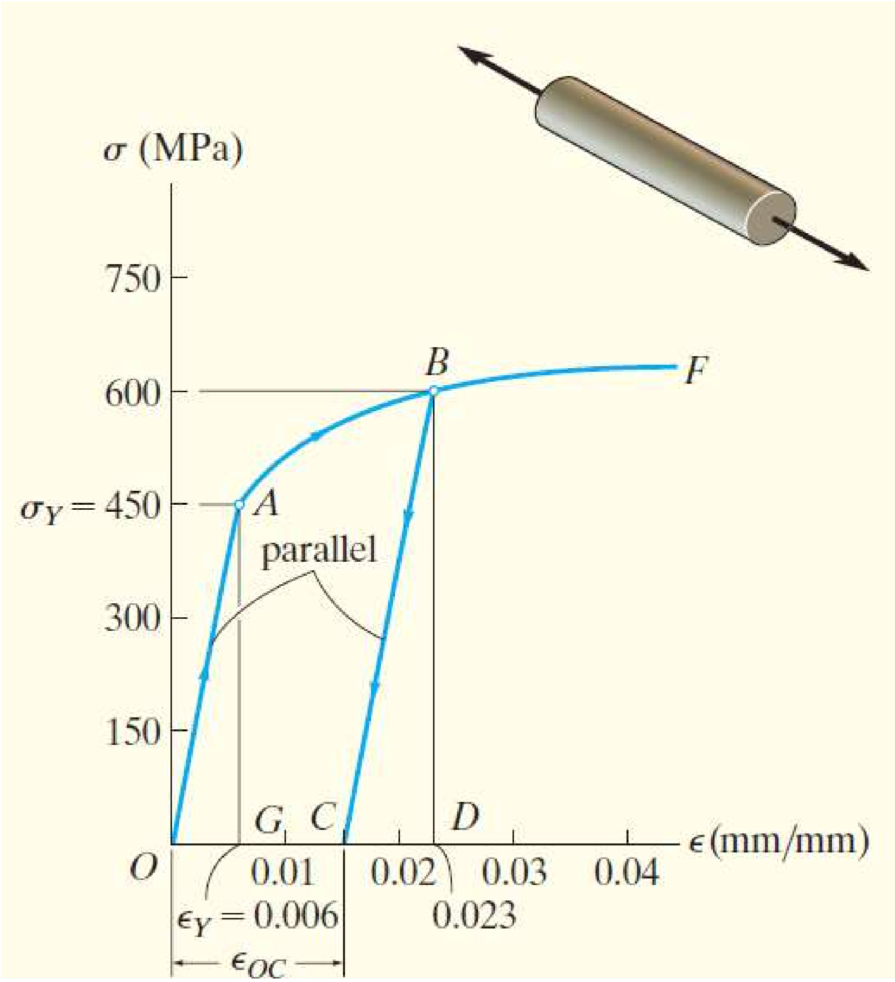


Modulus of toughness u_t

- **Area under stress-strain curve is strain energy density**
- **Area up to yield point is modulus of resilience**
- **Area up to rupture is modulus of toughness**

Example

- **Determine the permanent strain**
- **Find modulus of resilience before and after the load application**



$$E = \frac{450\ \text{MPa}}{0.006\ \text{mm/mm}} = 75.0\ \text{GPa}$$

From triangle CBD,

$$E = \frac{BD}{CD};$$

$$75.0(10^9)\ \text{Pa} = \frac{600(10^6)\ \text{Pa}}{CD}$$

$$CD = 0.008\ \text{mm/mm}$$

$$\epsilon_{OC} = 0.023\ \text{mm/mm} - 0.008\ \text{mm/mm} = 0.0150\ \text{mm/mm}$$

$$(u_r)_{\text{initial}} = \frac{1}{2}\sigma_{pl}\,\epsilon_{pl} = \frac{1}{2}(450\ \text{MPa})(0.006\ \text{mm/mm})$$

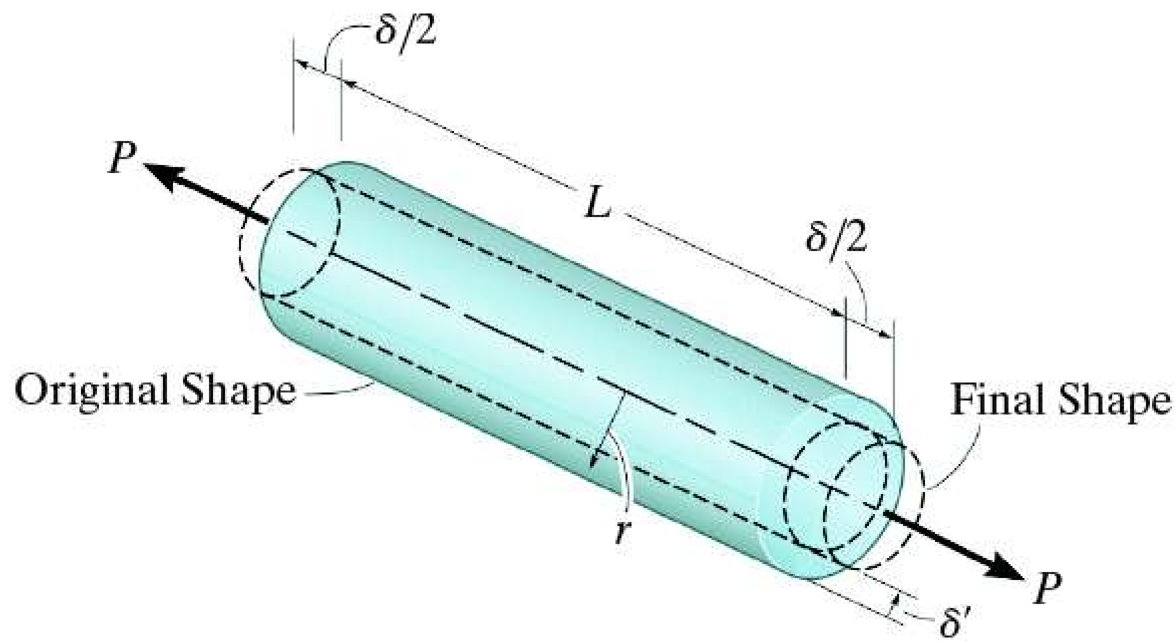
$$(u_r)_{\text{final}} = \frac{1}{2}\sigma_{pl}\,\epsilon_{pl} = \frac{1}{2}(600\ \text{MPa})(0.008\ \text{mm/mm})$$

Poisson's ratio

$$\epsilon_{\text{long}} = \frac{\delta}{L} \quad \text{and} \quad \epsilon_{\text{lat}} = \frac{\delta'}{r}$$

$$\boxed{\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}}$$

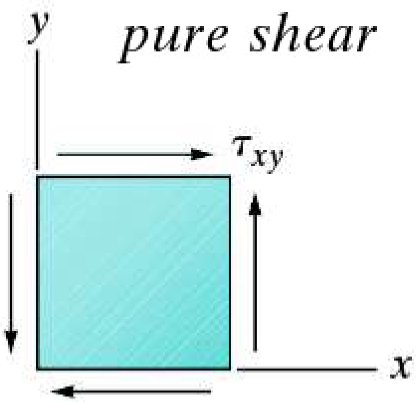
$$0 \leq \nu \leq 0.5$$



Tension

Actually it is $-1 \leq \nu \leq 0.5$ for isotropic elastic material but generally we do not take a negative Poisson ratio.

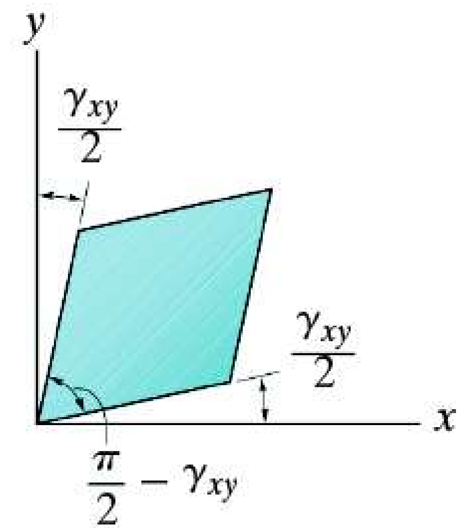
Shear stress-strain diagram

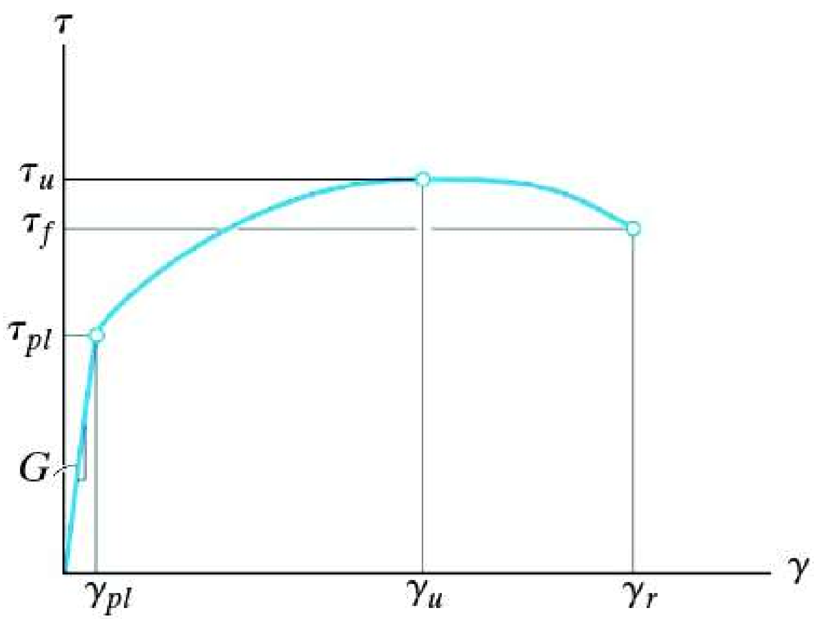


material is *homogeneous* and *isotropic*

$$\tau = G\gamma$$

$$G = \frac{E}{2(1+\nu)}$$



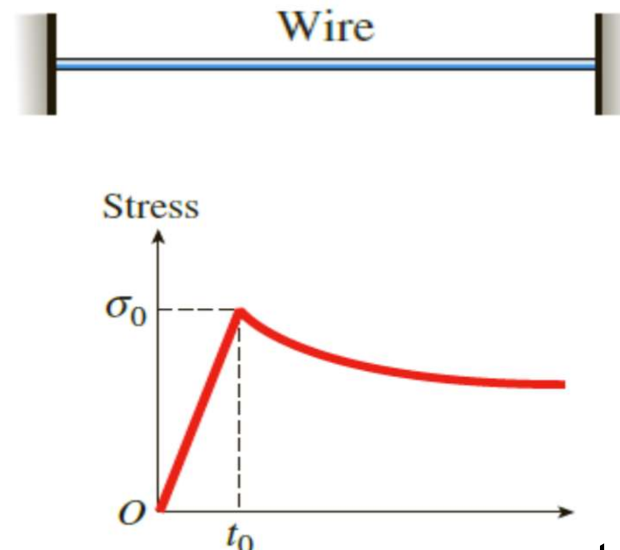


G is called the ***shear modulus of elasticity***

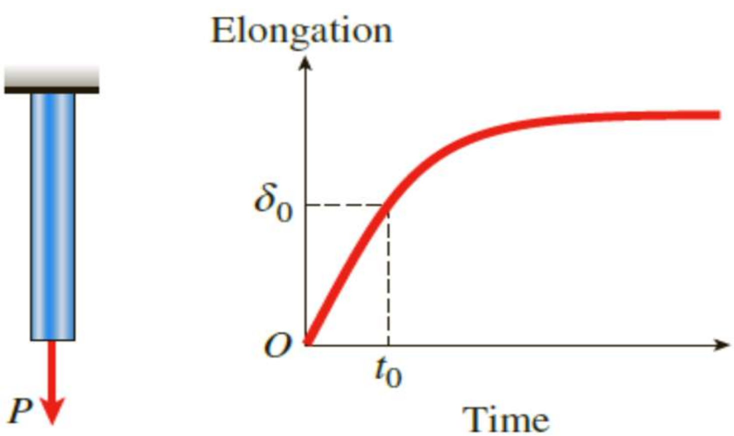
Creep and fatigue

- When material is under load for a long period of time
- Permanent deformation is called **Creep**
- Becomes more important at high temperatures

Relaxation of stress in a wire under constant strain



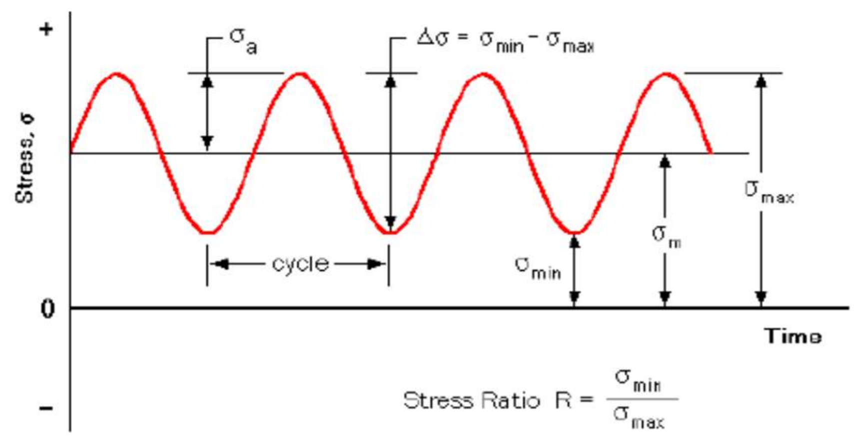
Creep in a bar under constant load

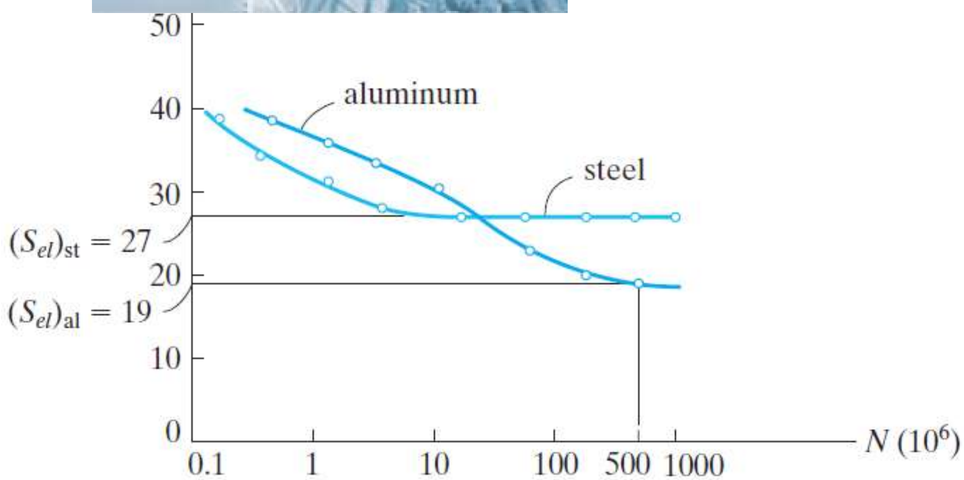


t Singh

Fatigue

- When a metal is subjected to repeated cycle of stress or strain
- Internal structure breaks down, ultimately leads to fracture
- Known as **Fatigue.**
- **Low cycle fatigue and High cycle fatigue**

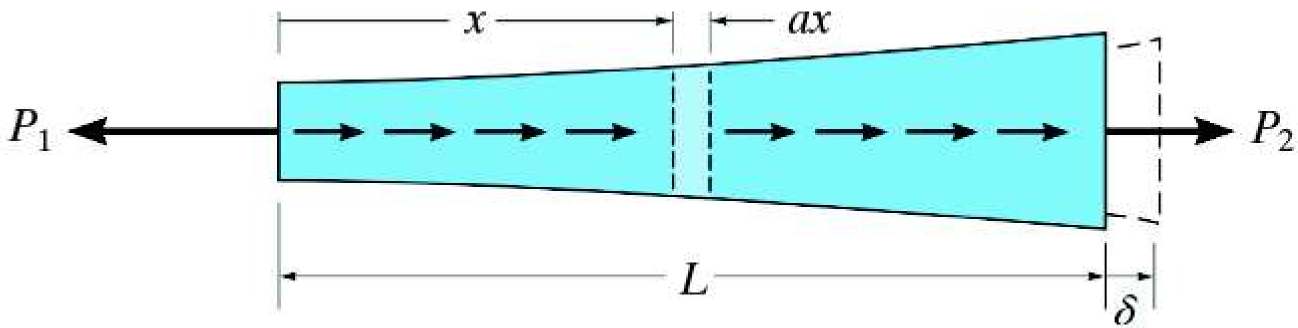


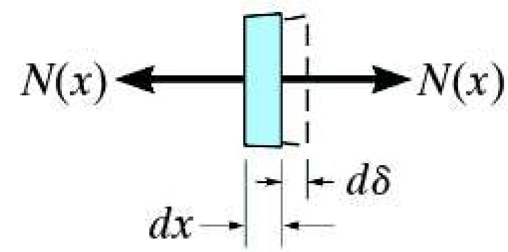


S–N diagram for steel and aluminum alloys
(N axis has a logarithmic scale)

Amit Singh

Axial Load



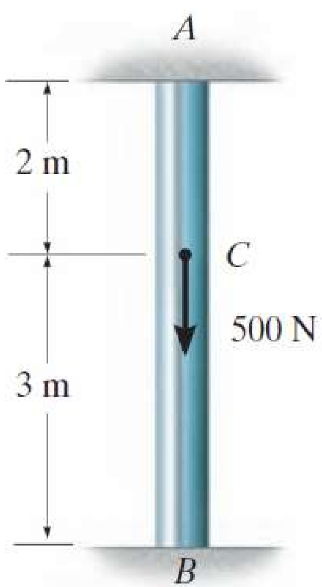


$$\sigma = \frac{N(x)}{A(x)} \text{ and } \epsilon = \frac{d\delta}{dx}$$

$$\frac{N(x)}{A(x)} = E(x)\left(\frac{d\delta}{dx}\right)$$

$$d\delta = \frac{N(x)dx}{A(x)E(x)}$$

$$\delta = \int_0^L \frac{N(x)dx}{A(x)E(x)}$$



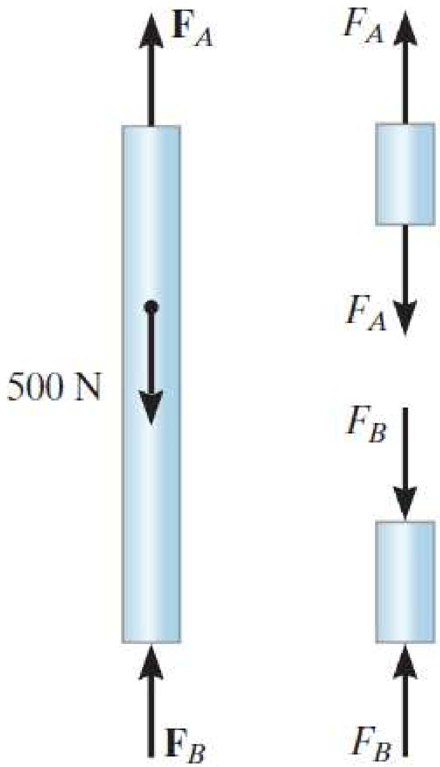
$$+\uparrow \Sigma F = 0; \qquad F_B + F_A - 500\text{ N} = 0$$

statically indeterminate

compatibility or ***kinematic condition*** $\delta_{A/B} = 0$

Amit Singh

Axial Load



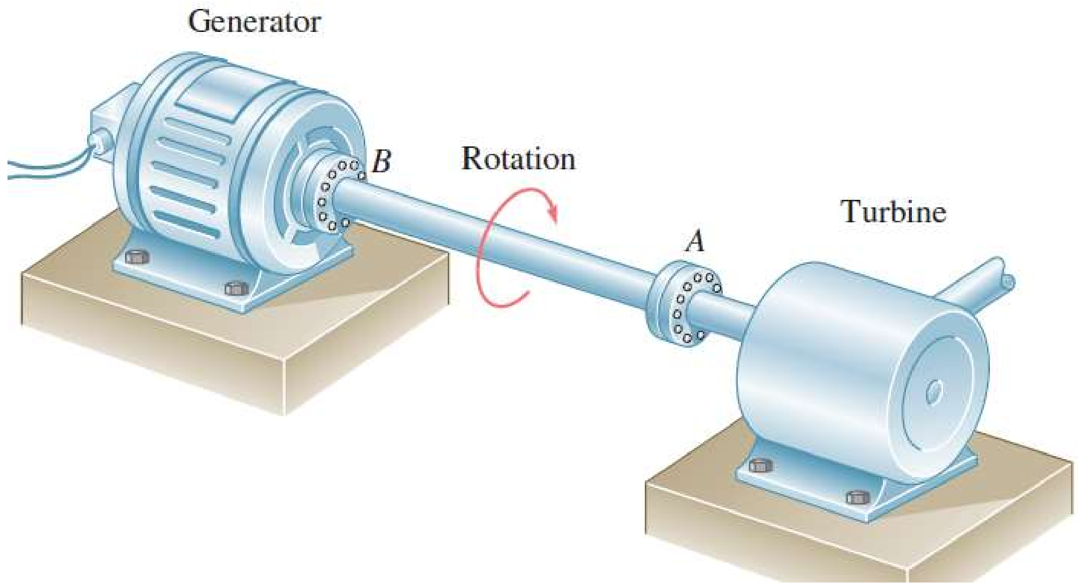
principle of superposition

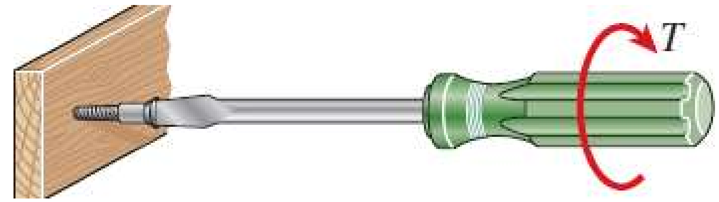
$$\frac{F_A(2\text{ m})}{AE} - \frac{F_B(3\text{ m})}{AE} = 0$$

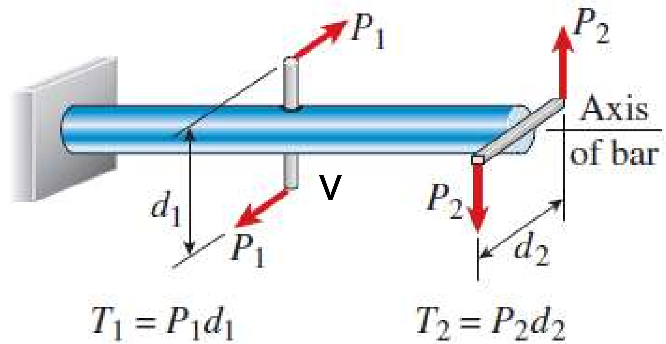
$$F_A = 300\text{ N} \quad \text{and} \quad F_B = 200\text{ N}$$

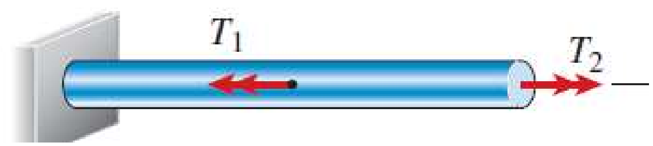
Torsion

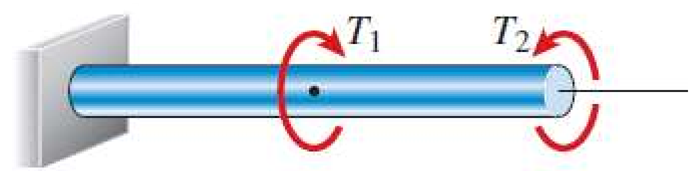
Torsion







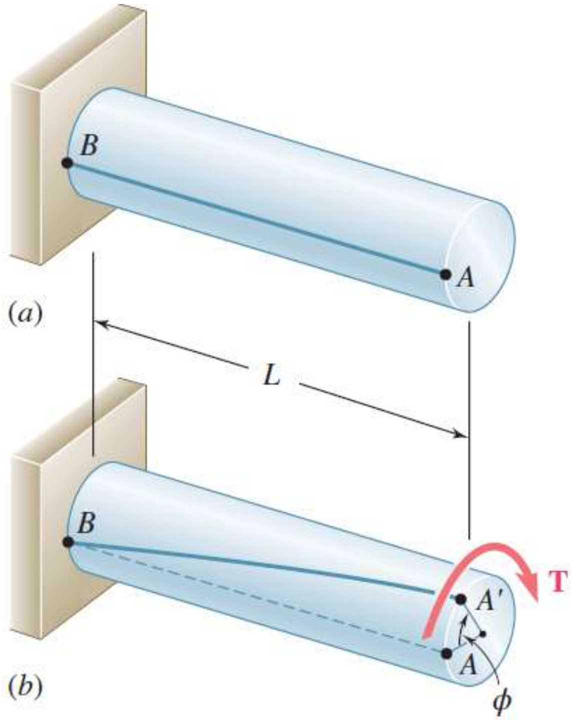


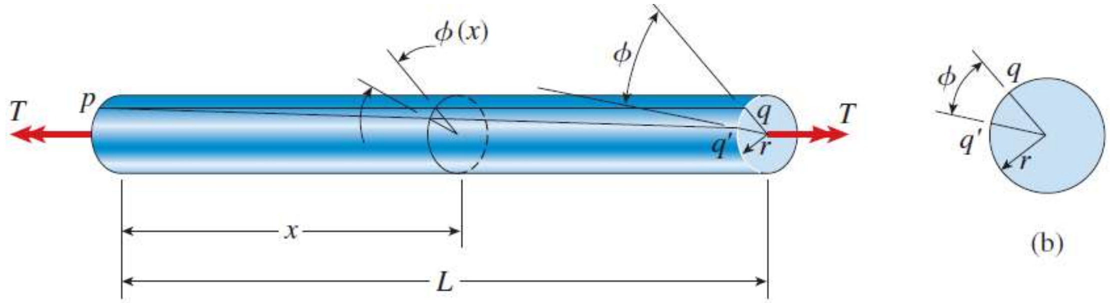


- **Each pair of force forms a couple**
- **The direction (right-hand rule)**
- **Moment that produces "twisting" is called torque or twisting moment**
- **Torsion refers to twisting of a bar when moments produce rotation about the longitudinal axis**

Amit Singh

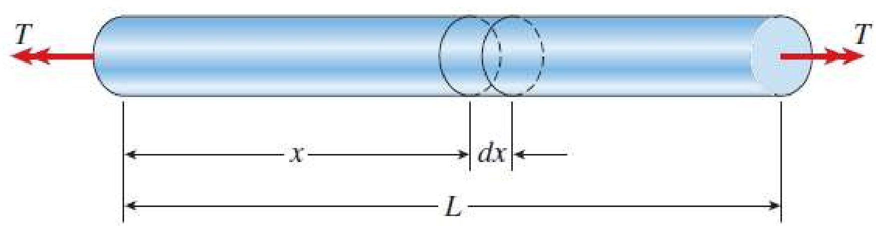
Deformation in circular shaft





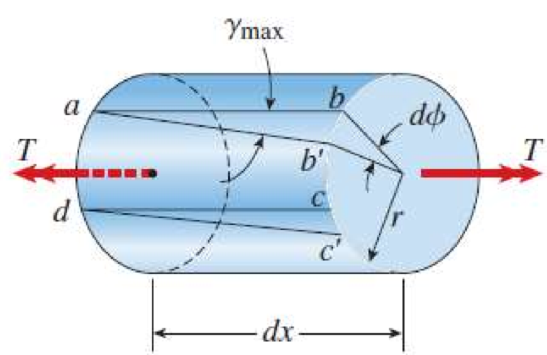
small angle ϕ, known as the **angle of twist**

Shear Strains at the Outer Surface



- Shear strain = Decrease in angle "bad"

$$\gamma_{\max} = \frac{bb'}{ab} \qquad \gamma_{\max} = \frac{r d\phi}{dx}$$



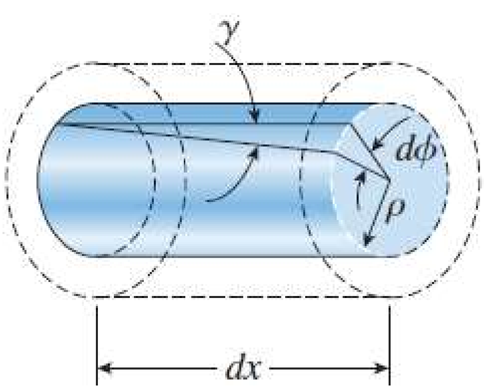
Amit Singh

Shear strains

- **Rate of twist or angle of twist per unit length** $\theta = \frac{d\phi}{dx}$
- **Therefore,** $\gamma_{max} = \frac{r\,d\phi}{dx} = r\theta$
- **Special case: Pure torsion (when the rate of twist is constant, not varying with x)**

$$\gamma_{max} = r\theta = \frac{r\phi}{L}$$

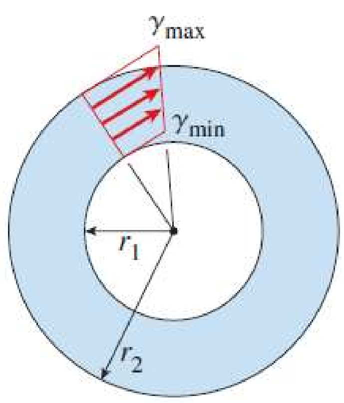
Shear Strains Within the Bar



$$\gamma = \rho\theta = \frac{\rho}{r}\gamma_{max}$$

Circular Tubes

$$\gamma_{max} = \frac{r_2\phi}{L} \qquad \gamma_{min} = \frac{r_1}{r_2}\gamma_{max} = \frac{r_1\phi}{L}$$



Amit Singh

Circular bars of linearly elastic material

- **Hooke's law in shear** $\tau = G\gamma$

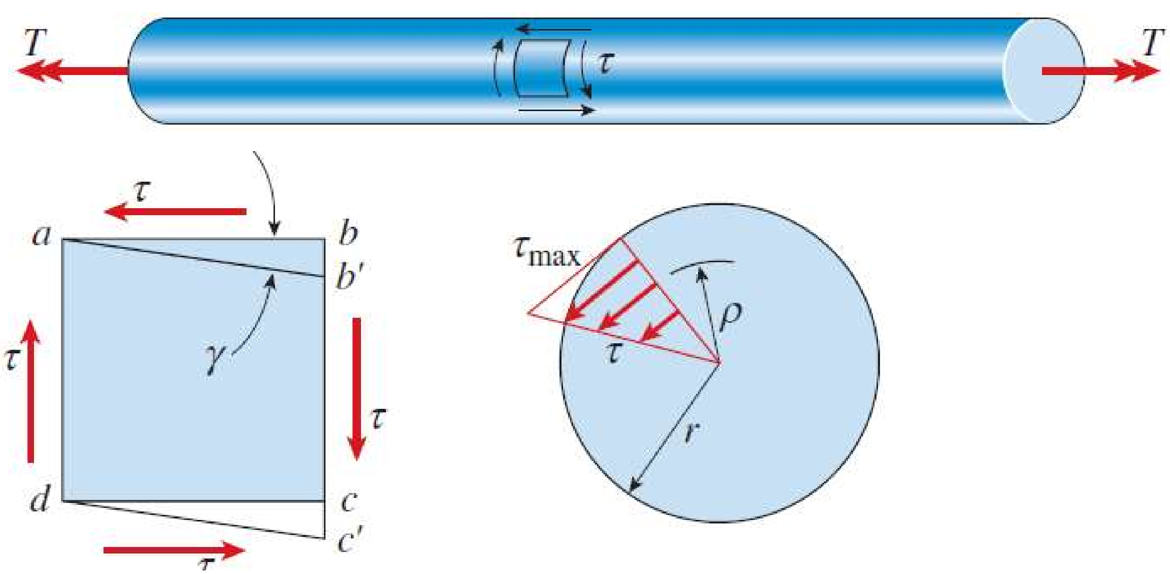
G is the shear modulus of elasticity and γ is the shear strain in radians

$$\tau_{\max} = Gr\theta \qquad \tau = G\rho\theta = \frac{\rho}{r}\tau_{\max}$$

$\tau_{\max}$ is the shear stress at the outer surface of the bar (radius r).

τ is the shear stress at an interior point (radius ρ)

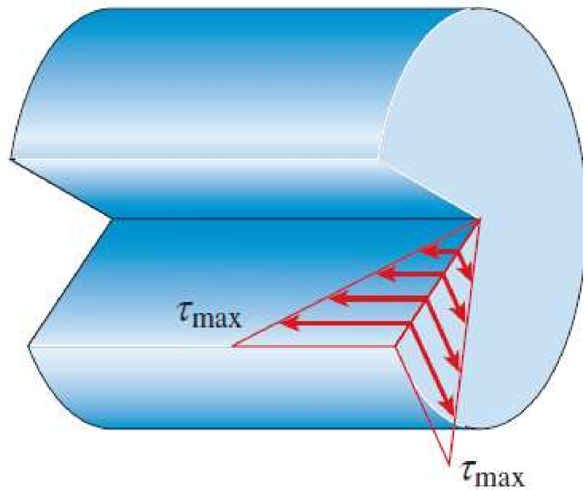
θ is the rate of twist



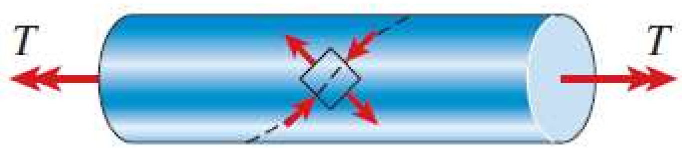
Amit Singh

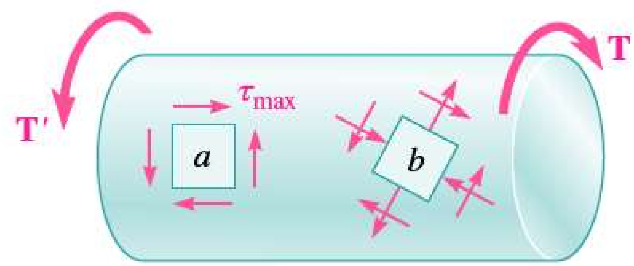
Circular bars of linearly elastic material

- **Longitudinal and trasnverse shear stress in a circular bar subject to torsion**



- **Tensile and compressive stress acting on elements oriented at 45 deg angle to the longitudinal axis**

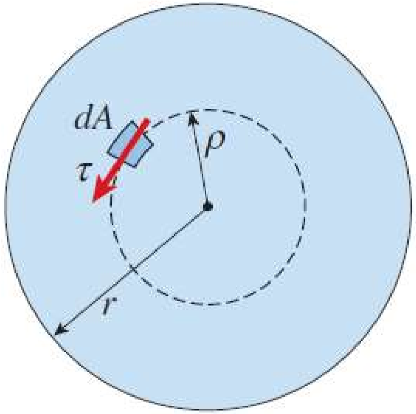




The torsion formula

$$\tau_{max} = \frac{Tr}{I_P}$$

- **What is the relationship between shear stress and torque T?**



The elemental moment about the axis of the bar is

$$dM = \tau\rho dA = \frac{\tau_{max}}{r}\rho^2 dA$$

$$T = \int_A dM = \frac{\tau_{max}}{r}\int_A \rho^2 dA = \frac{\tau_{max}}{r} I_P$$

J or $I_P = \int_A \rho^2 dA$ is the **polar moment of inertia** of the circular cross section

For a **circle** of radius r and diameter d, the polar moment of inertia is $I_P = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$

$\tau_{max} = \frac{Tr}{I_P}$ known as the **torsion formula** $\tau_{max} = \frac{16T}{\pi d^3}$

Make sure you get the units correct. Stress in Pa, torque in N.m

Amit Singh

The torsion formula continued

- The shear stress at distance ρ from the center of the bar is

$$\tau = \frac{\rho}{r}\tau_{\max} = \frac{T\rho}{I_P}$$

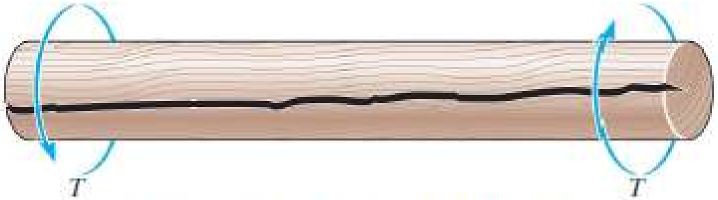
generalized torsion formula

- **Rate of twist** $\theta = \dfrac{T}{GI_P}$ $\quad GI_P$ **torsional rigidity**

- **For a bar in pure torsion, the total angle of twist is** $\phi = \dfrac{TL}{GI_P}$

The quantity GI_P/L, called the **torsional stiffness** of the bar

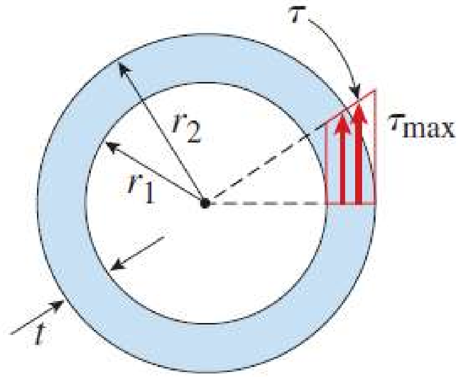
- **Failure: driving shaft in a truck and wooden shaft**



Failure of a wooden shaft due to torsion.

Amit Singh

Circular tubes

- 

$$I_P = \int_A \rho^2 dA = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$I_P = \frac{\pi r t}{2}(4r^2 + t^2) = \frac{\pi d t}{4}(d^2 + t^2)$$

r is the *average radius* of the tube, equal to $(r_1 + r_2)/2$

t is the *wall thickness* equal to $r_2 - r_1$

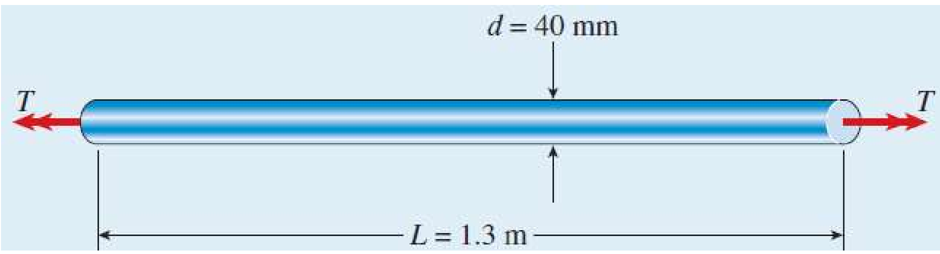
- **For very very thin tube**

$$I_P \approx 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

- **So far only circular, no rectangular, I-shape cross-sections**
- **Torsion for circular bar: Coulomb (1736-1806), Young**
- **Torsion for general cross-sections:** Barré de Saint-Venant (1797–1886)

Example

- A solid steel bar of circular cross section (Fig. 3-11) has diameter $d = 40$ mm, length $L = 1.3$ m, and shear modulus of elasticity $G = 80$ GPa. The bar is subjected to torques T acting at the ends.
  (a) If the torques have magnitude $T = 340\ \mathrm{N \cdot m}$, what is the maximum shear stress in the bar? What is the angle of twist between the ends?
  (b) If the allowable shear stress is 42 MPa and the allowable angle of twist is 2.5°, what is the maximum permissible torque?

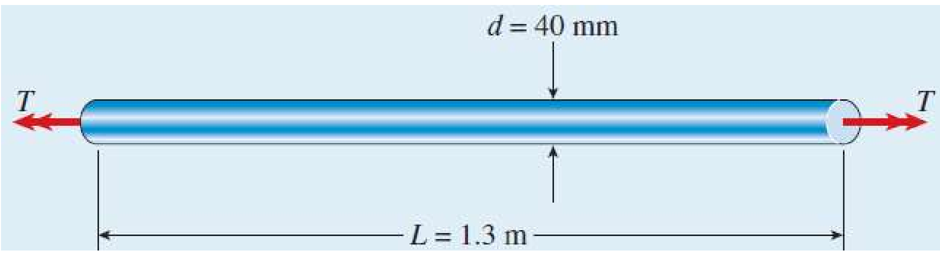


Example

- A solid steel bar of circular cross section (Fig. 3-11) has diameter $d = 40$ mm, length $L = 1.3$ m, and shear modulus of elasticity $G = 80$ GPa. The bar is subjected to torques T acting at the ends.

  (a) If the torques have magnitude $T = 340$ N · m, what is the maximum shear stress in the bar? What is the angle of twist between the ends?

  (b) If the allowable shear stress is 42 MPa and the allowable angle of twist is 2.5°, what is the maximum permissible torque?



$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(340\ \text{N}\cdot\text{m})}{\pi(0.04\ \text{m})^3} = 27.1\ \text{MPa}$$

$$I_P = \frac{\pi d^4}{32} = \frac{\pi(0.04\ \text{m})^4}{32} = 2.51 \times 10^{-7}\ \text{m}^4$$

$$\phi = \frac{TL}{GI_P} = \frac{(340\ \text{N}\cdot\text{m})(1.3\ \text{m})}{(80\ \text{GPa})(2.51 \times 10^{-7}\ \text{m}^4)} = 0.02198\ \text{rad} = 1.26°$$

- **Allowable shear stress:** $T_1 = \frac{\pi d^3 \tau_{allow}}{16} = \frac{\pi}{16}(0.04\ \text{m})^3(42\ \text{MPa}) = 528\ \text{N}\cdot\text{m}$

- **Allowable angle of twist** $T_2 = \frac{GI_P\phi_{allow}}{L} = \frac{(80\ \text{GPa})(2.51 \times 10^{-7}\ \text{m}^4)(2.5°)(\pi\text{rad}/180°)}{1.3\ \text{m}}$ $= 674\ \text{N}\cdot\text{m}$

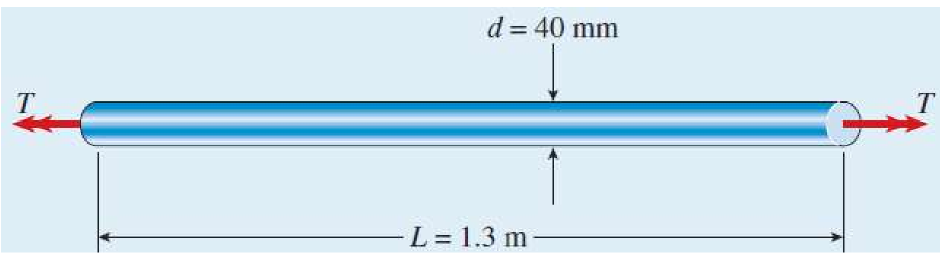
Amit Singh

Example

- A solid steel bar of circular cross section (Fig. 3-11) has diameter $d = 40$ mm, length $L = 1.3$ m, and shear modulus of elasticity $G = 80$ GPa. The bar is subjected to torques T acting at the ends.

  (a) If the torques have magnitude $T = 340\ \text{N} \cdot \text{m}$, what is the maximum shear stress in the bar? What is the angle of twist between the ends?

  (b) If the allowable shear stress is 42 MPa and the allowable angle of twist is 2.5°, what is the maximum permissible torque?



$$T_{max} = 528\ \text{N} \cdot \text{m}$$

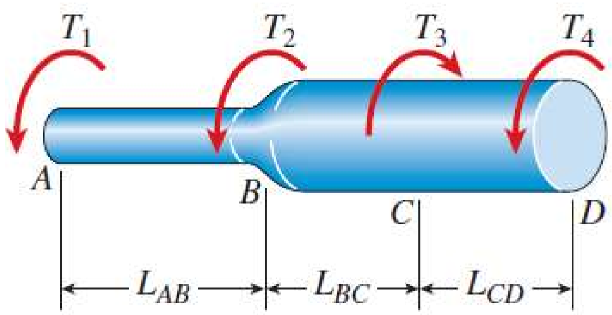
- **Allowable shear stress:** $$T_1 = \frac{\pi d^3 \tau_{allow}}{16} = \frac{\pi}{16}(0.04\ \text{m})^3(42\ \text{MPa}) = 528\ \text{N} \cdot \text{m}$$

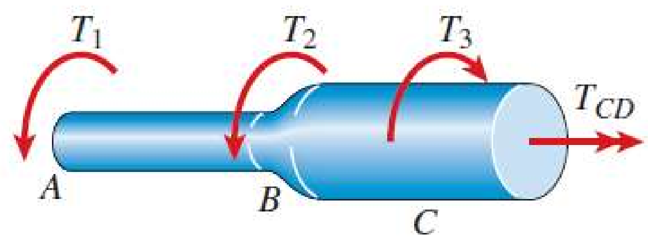
- **Allowable angle of twist** $$T_2 = \frac{G I_p \phi_{allow}}{L} = \frac{(80\ \text{GPa})(2.51 \times 10^{-7}\ \text{m}^4)(2.5°)(\pi \text{rad}/180°)}{1.3\ \text{m}} = 674\ \text{N} \cdot \text{m}$$

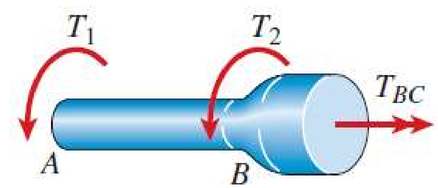
Amit Singh

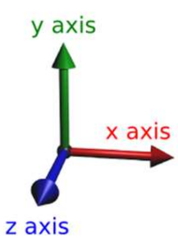
Nonuniform torsion

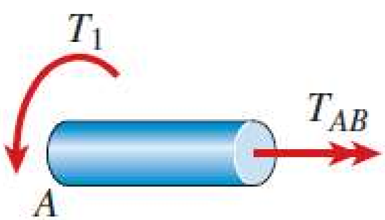
- **Pure torsion of a prismatic bar:** subjected to torques acting only at ends
- **Nonuniform torsion:** Bar need not be prismatic and the applied torque may act anywhere along the axis











Case 1

Bar consisting of prismatic segments

Sign Convention: Positive torque away from the cross-section

$$T_{CD} = -T_1 - T_2 + T_3 \quad T_{BC} = -T_1 - T_2 \quad T_{AB} = -T_1$$

$$\phi = \phi_1 + \phi_2 + \ldots + \phi_n$$

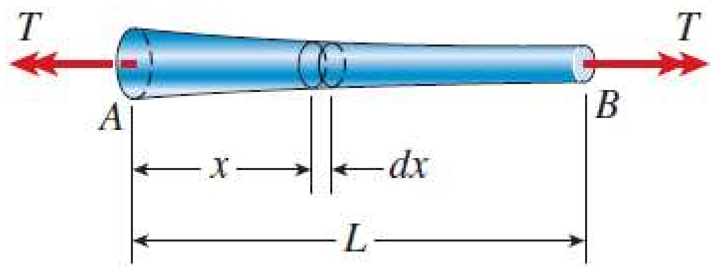
ϕ_1 is the angle of twist for segment 1

$$\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i (I_P)_i}$$

Nonuniform torsion

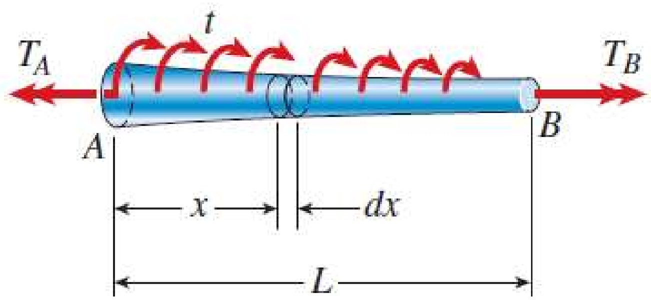
- **Case 2.** *Bar with continuously varying cross sections*

$$d\phi = \frac{T dx}{GI_P(x)} \qquad \phi = \int_0^L d\phi = \int_0^L \frac{T dx}{GI_P(x)}$$

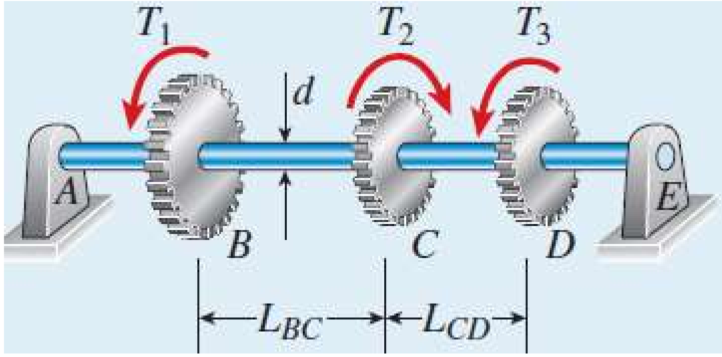


- **Case 3.** *Bar with continuously varying cross sections and continuously varying torque*

$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x) dx}{GI_P(x)}$$



Nonuniform torsion: Example

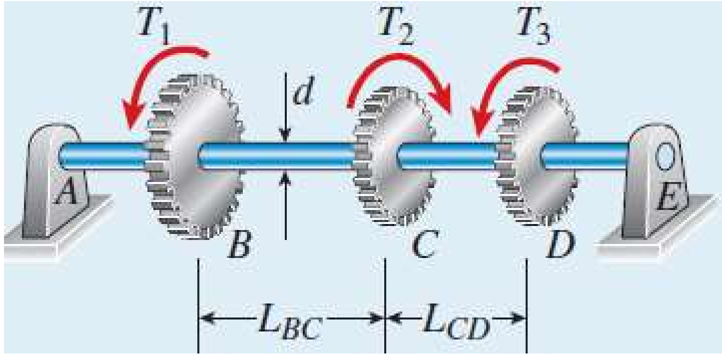
- 

$d = 30$ mm $T_2 = 450$ N·m $T_1 = 275$ N·m $T_3 = 175$ N·m

$L_{BC} = 500$ mm and $L_{CD} = 400$ mm, $G = 80$ GPa

Determine the maximum shear stress in each part of the shaft and the angle of twist between gears B and D.

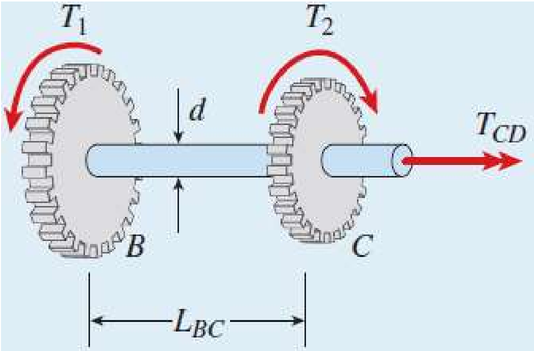
Nonuniform torsion: Example

- 

$d = 30$ mm $T_2 = 450$ N·m $T_1 = 275$ N·m $T_3 = 175$ N·m

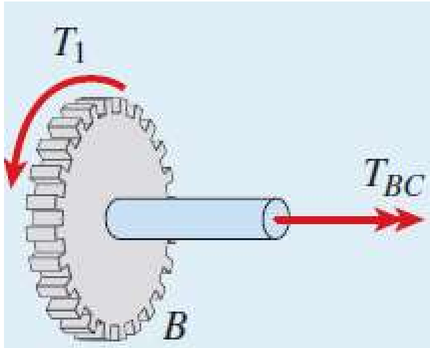
$L_{BC} = 500$ mm and $L_{CD} = 400$ mm, $G = 80$ GPa

Determine the maximum shear stress in each part of the shaft and the angle of twist between gears *B* and *D*.



$$T_{CD} = T_2 - T_1 = 450 \text{ N}\cdot\text{m} - 275 \text{ N}\cdot\text{m} = 175 \text{ N}\cdot\text{m}$$

$$T_{BC} = -T_1 = -275 \text{ N}\cdot\text{m}$$



Amit Singh

Nonuniform torsion: Example

- $$\tau_{BC} = \frac{16T_{BC}}{\pi d^3} = \frac{16(275\ \mathrm{N \cdot m})}{\pi(30\ \mathrm{mm})^3} = 51.9\ \mathrm{MPa}$$

  $$\tau_{CD} = \frac{16T_{CD}}{\pi d^3} = \frac{16(175\ \mathrm{N \cdot m})}{\pi(30\ \mathrm{mm})^3} = 33.0\ \mathrm{MPa}$$

$$\phi_{BD} = \phi_{BC} + \phi_{CD}$$

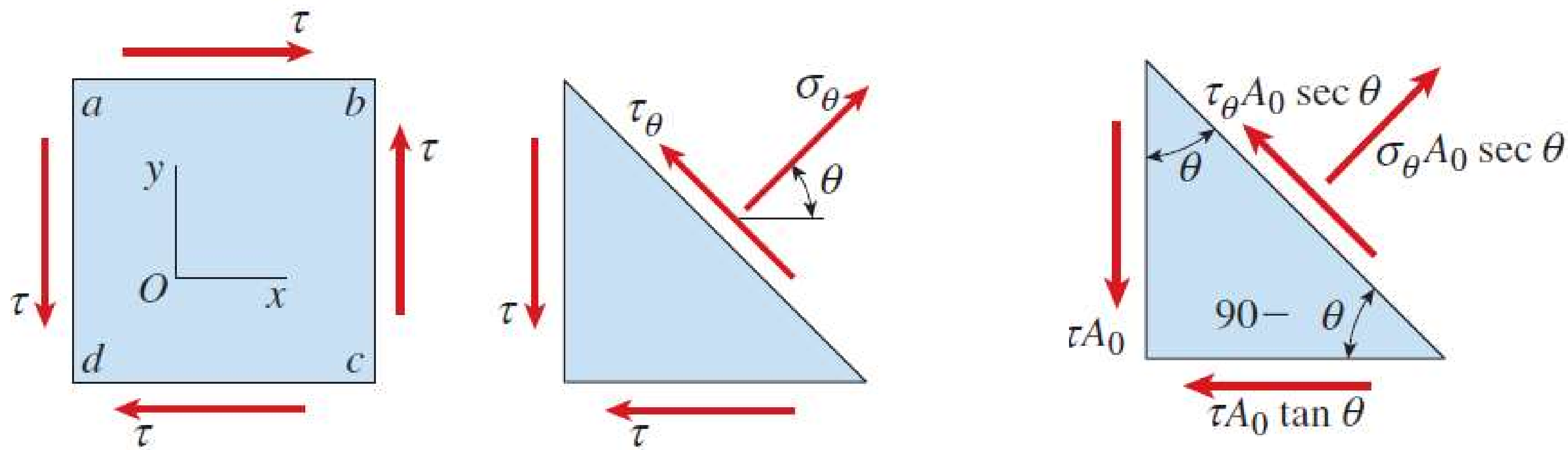
$$I_P = \frac{\pi d^4}{32} = \frac{\pi(30\ \mathrm{mm})^4}{32} = 79{,}520\ \mathrm{mm}^4$$

$$\phi_{BC} = \frac{T_{BC}L_{BC}}{GI_P} = \frac{(-275\ \mathrm{N \cdot m})(500\ \mathrm{mm})}{(80\ \mathrm{GPa})(79{,}520\ \mathrm{mm}^4)} = -0.0216\ \mathrm{rad}$$

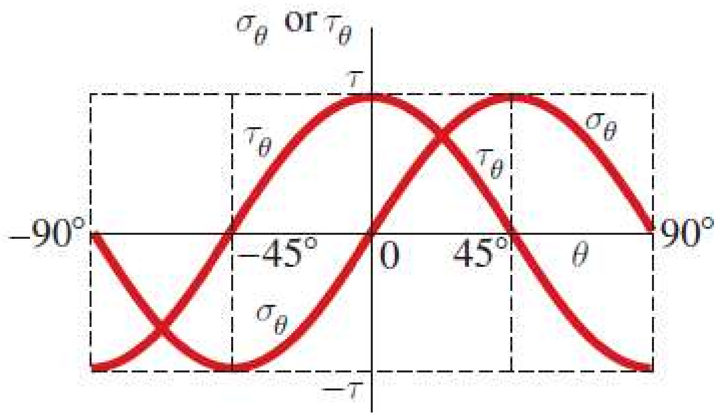
$$\phi_{CD} = \frac{T_{CD}L_{CD}}{GI_P} = \frac{(175\ \mathrm{N \cdot m})(400\ \mathrm{mm})}{(80\ \mathrm{GPa})(79{,}520\ \mathrm{mm}^4)} = 0.0110\ \mathrm{rad}$$

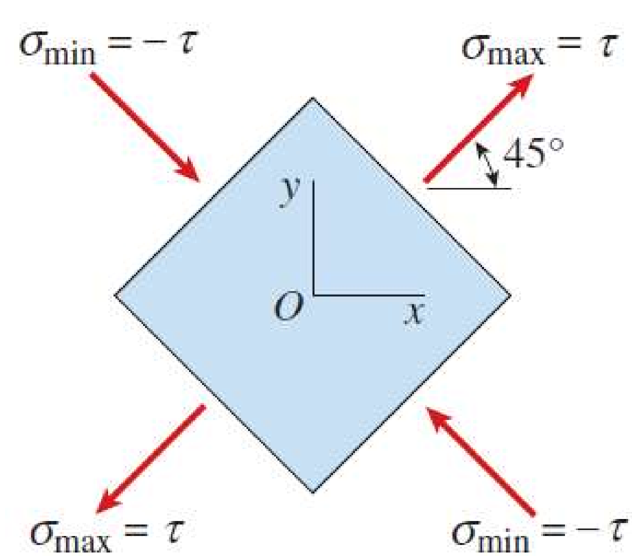
Stresses on inclined plane

•



$$\sigma_\theta = \tau \sin 2\theta \qquad \tau_\theta = \tau \cos 2\theta$$

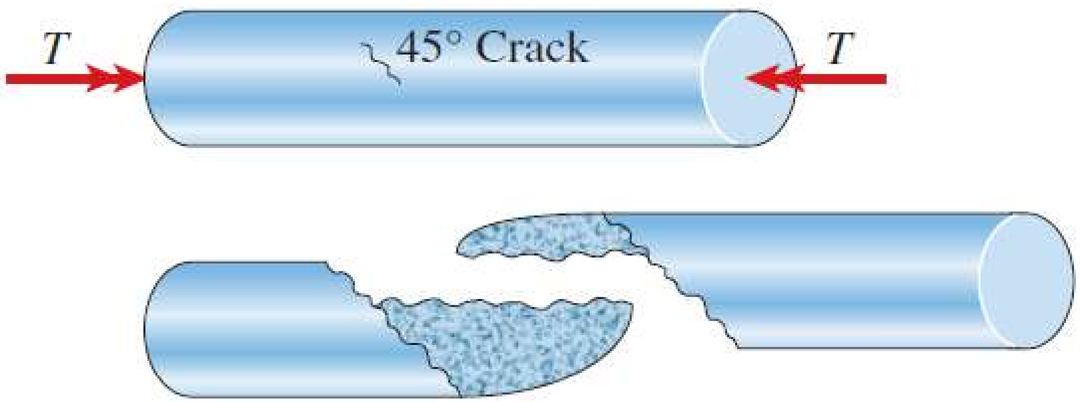




Amit Singh

Stresses on inclined plane

- Do it by yourself:

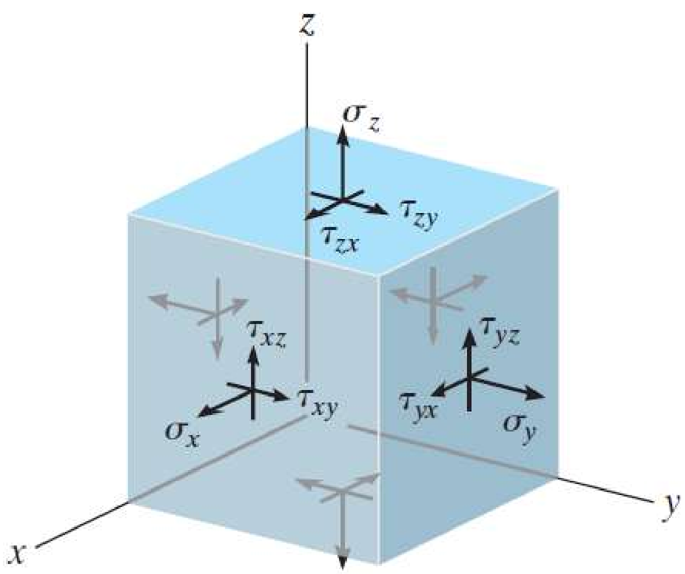


Plane stress, plane strain, anisotropic elasticity

Amit Singh

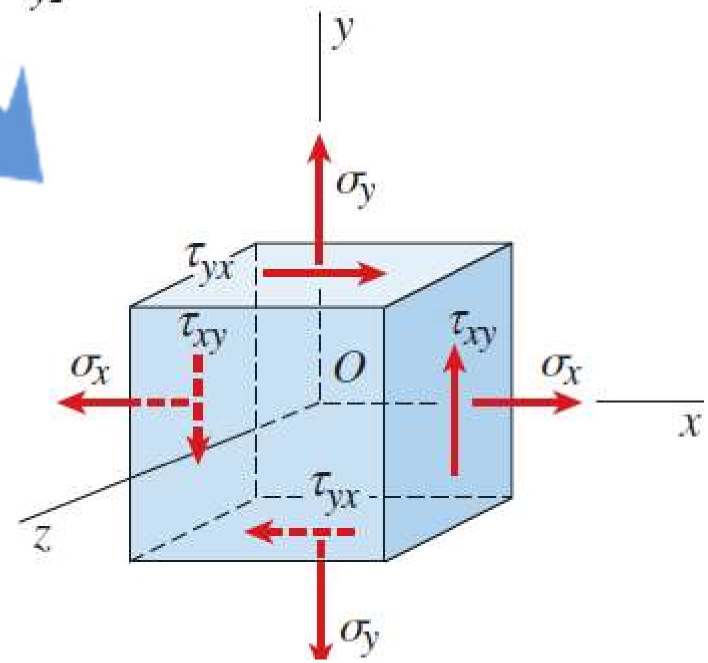
Plane stress

- **General**

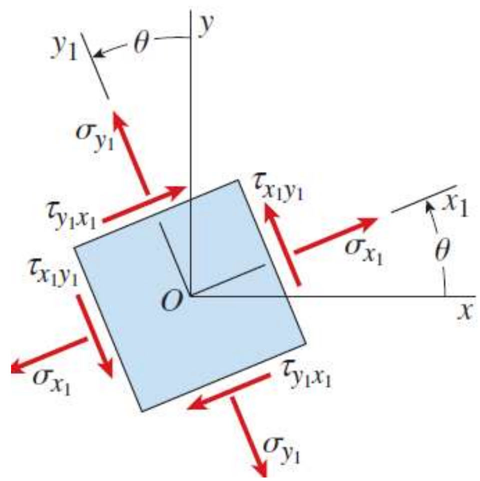


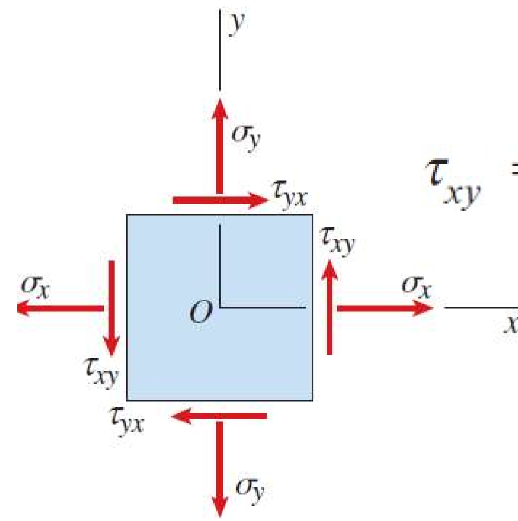
$$\sigma_z = 0 \quad \tau_{xz} = 0 \quad \tau_{yz} = 0$$

Plane Stress



$$\tau_{x_1y_1} = \tau_{y_1x_1}$$





$$\tau_{xy} = \tau_{yx}$$

Plane stress

- **x1 axis:**

$$\sigma_{x_1} A_0 \sec\theta - \sigma_x A_0 \cos\theta - \tau_{xy} A_0 \sin\theta$$
$$-\sigma_y A_0 \tan\theta \sin\theta - \tau_{yx} A_0 \tan\theta \cos\theta = 0$$

- **y1 axis:**

$$\tau_{x_1 y_1} A_0 \sec\theta + \sigma_x A_0 \sin\theta - \tau_{xy} A_0 \cos\theta$$
$$-\sigma_y A_0 \tan\theta \cos\theta + \tau_{yx} A_0 \tan\theta \sin\theta = 0$$

$$\sigma_{x_1} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$\tau_{x_1 y_1} = -(\sigma_x - \sigma_y) \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

- **In other words,**

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

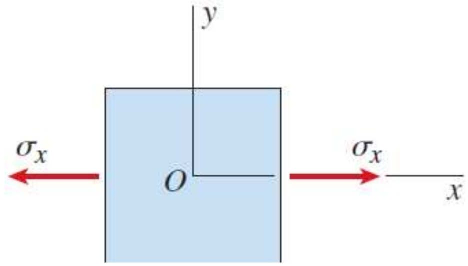
$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y$$

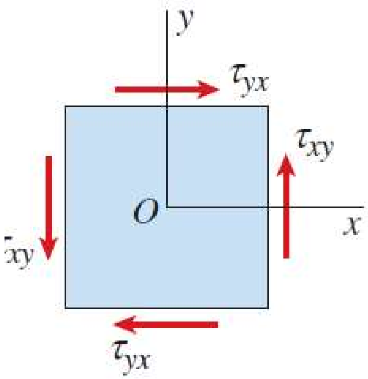
Plane stress

- **uniaxial stress**



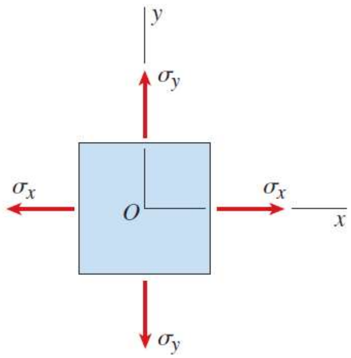
$$\sigma_{x_1} = \frac{\sigma_x}{2}(1 + \cos 2\theta) \qquad \tau_{x_1y_1} = -\frac{\sigma_x}{2}(\sin 2\theta)$$

- **Pure shear:**



$$\sigma_{x_1} = \tau_{xy} \sin 2\theta \quad \tau_{x_1y_1} = \tau_{xy} \cos 2\theta$$

- **Biaxial stress:**



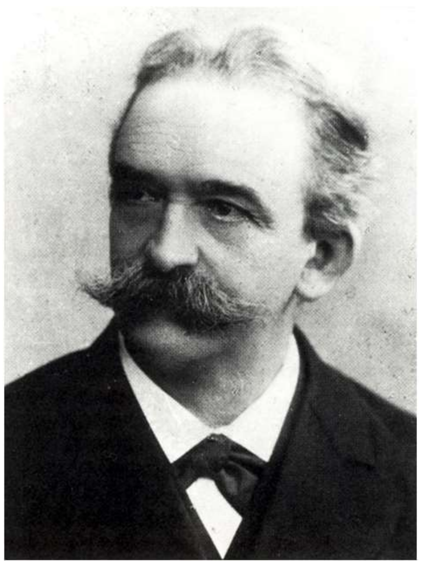
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta$$

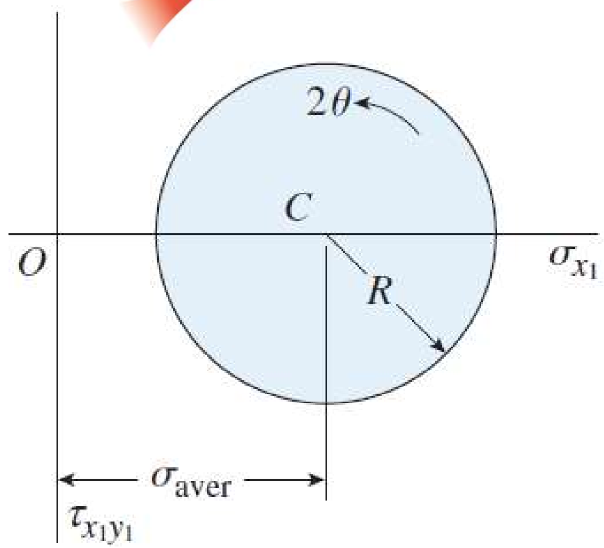
Mohr’s circle

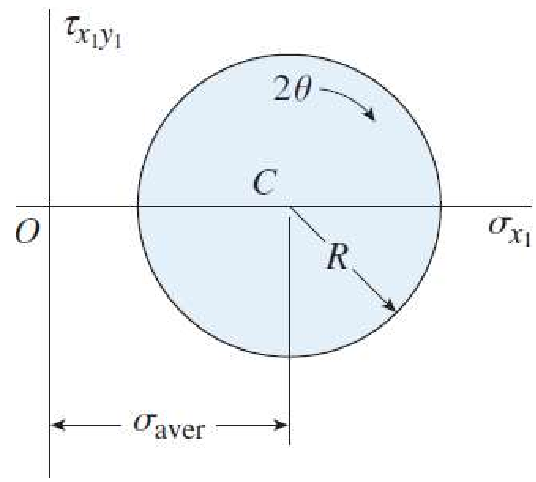
- $$\left(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x_1y_1}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$(\sigma_{x_1} - \sigma_{\text{aver}})^2 + \tau_{x_1y_1}^2 = R^2 \qquad \sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



Christian Otto Mohr (8 October 1835 – 2 October 1918) was a German civil engineer. He is renowned for his contributions to the field of structural engineering, such as Mohr's circle, and for his study of stress.





Mohr's circle

- **Procedure for constructing Mohr's circle**

1. Draw a set of coordinate axes with σ_{x_1} as abscissa (positive to the right) and $\tau_{x_1y_1}$ as ordinate (positive downward).
2. Locate the center C of the circle at the point having coordinates

   $$\sigma_{x_1} = \sigma_{aver} \text{ and } \tau_{x_1y_1} = 0$$

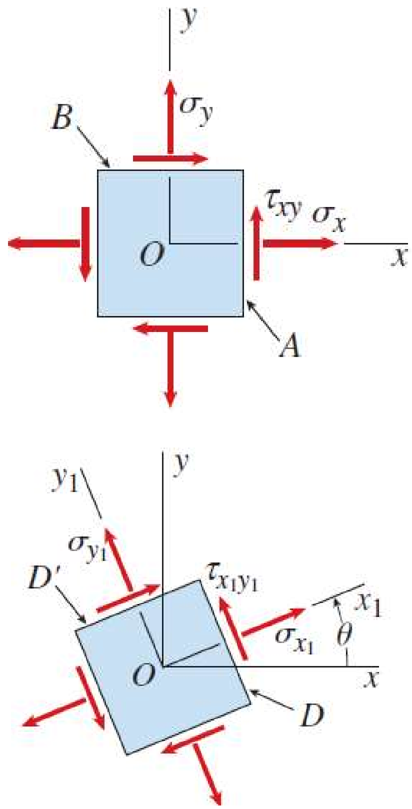
3. Locate point A $\quad \sigma_{x_1} = \sigma_x \quad \tau_{x_1y_1} = \tau_{xy}$

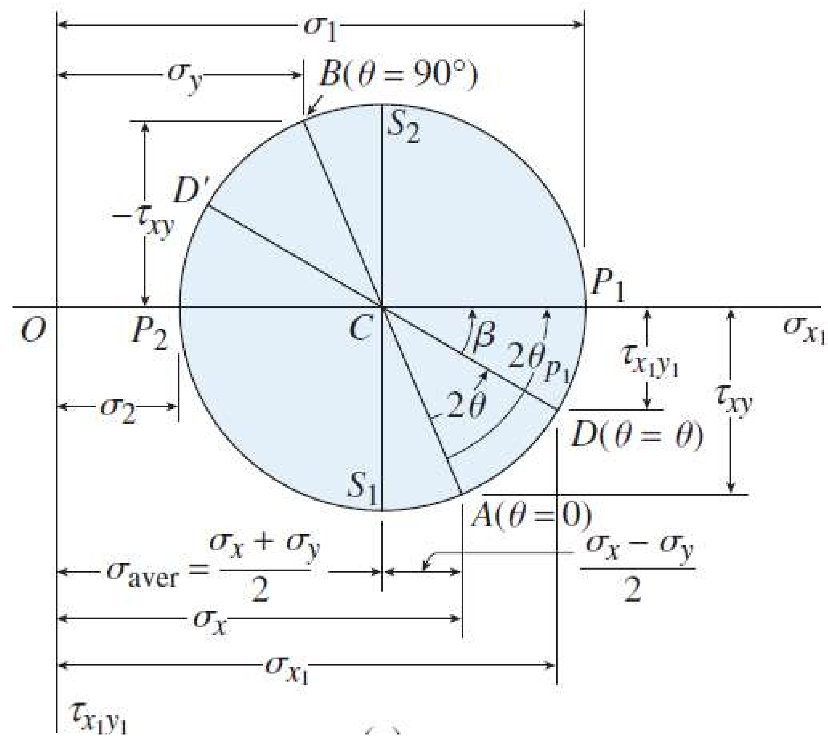
   Note that point A on the circle corresponds to $\theta = 0$
4. Locate point B $\quad \sigma_{x_1} = \sigma_y \quad \tau_{x_1y_1} = -\tau_{xy}$
5. Draw a line from point A to point B. This line is a diameter of the circle and passes through the center C
6. Using point C as the center, draw Mohr's circle through points A and B.

Mohr’s circle

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- **Procedure for constructing Mohr’s circle**



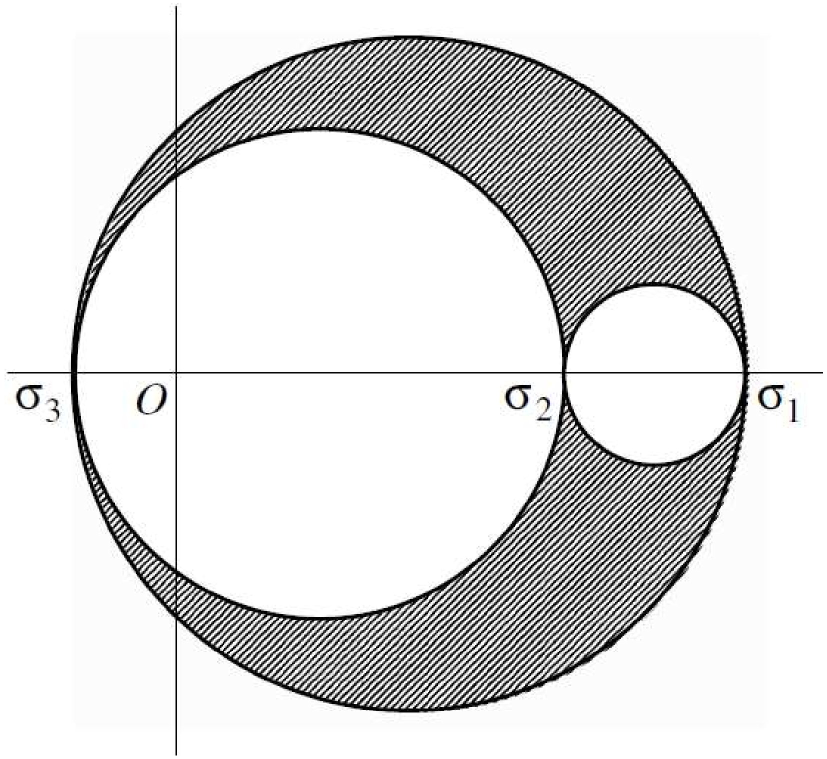


- **Point P1: Normal stress highest, shear stress zero, principal stress, principal plane**
- **Point P2: Normal stress smallest, shear stress zero, principal stress, principal plane**
- **Points S1, S2: Shear stress highest, 45 degree to P1, P2**

Amit Singh

Three dimensional Mohr's circle

•



$$\sigma_{max} = \max(\sigma_1, \sigma_2, \sigma_3)$$

$$\tau_{max} = \max\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}\right)$$

Amit Singh

Principal stresses

Principal Stresses are **eigenvalues** of the second-order stress tensor.

Eigenvalues

Eigenvalues The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0.$$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ → ***Subtract*** λ ***from the diagonal to find*** $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda$$

→ $\det(A - \lambda I) = \lambda^2 - 5\lambda = 0$ yields the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$.

→ Now find the eigenvectors. Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yields an eigenvector $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for $\lambda_1 = 0$

$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yields an eigenvector $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda_2 = 5$.

Second-order tensor: Eigenvalues

- Characteristic equation of **G**:

$$-\lambda^3 + I_1(\boldsymbol{G})\lambda^2 - I_2(\boldsymbol{G})\lambda + I_3(\boldsymbol{G}) = 0$$

where I_1, I_2, I_3 are the *principal invariants* of $\boldsymbol{G}$:

$$\begin{aligned}
I_1(\boldsymbol{G}) &= \quad G_{ii} &&= \operatorname{tr}\boldsymbol{G}, \\
I_2(\boldsymbol{G}) &= \tfrac{1}{2}(G_{ii}G_{jj} - G_{ij}G_{ji}) &&= \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{G})^2 - \operatorname{tr}\boldsymbol{G}^2\right] = \operatorname{tr}\boldsymbol{G}^{-1}\det\boldsymbol{G}, \\
I_3(\boldsymbol{G}) &= \quad \epsilon_{ijk}G_{1i}G_{2j}G_{3k} &&= \det\boldsymbol{G}.
\end{aligned}$$

- Eigenvectors corresponding to distinct eigenvalues of a symmetric tensor are orthogonal: $\boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \cdot \boldsymbol{\Lambda}_\beta^{\boldsymbol{S}} = \delta_{\alpha\beta}$

- For principal stresses, we have $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

- We have already calculated principal values and directions using Mohr's circle for plane stress problems.

Anisotropic elasticity

- Materials such as wood, reinforced composites, biological materials

- **Generalized Hooke's law in Voigt notation** : (Total 36 material constants C_{ij} also called **Stiffness constants**)

$$\begin{bmatrix} \sigma_1 = \sigma_{xx} \\ \sigma_2 = \sigma_{yy} \\ \sigma_3 = \sigma_{zz} \\ \sigma_4 = \sigma_{yz} \\ \sigma_5 = \sigma_{xz} \\ \sigma_6 = \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 = \varepsilon_{xx} \\ \varepsilon_2 = \varepsilon_{yy} \\ \varepsilon_3 = \varepsilon_{zz} \\ \varepsilon_4 = \varepsilon_{yz} \\ \varepsilon_5 = \varepsilon_{xz} \\ \varepsilon_6 = \varepsilon_{xy} \end{bmatrix}$$

- The matrix involving Stiffness constants is called Stiffness matrix.

- This is symmetric. Therefore, only 21 constants.

- This can be inverted.

Anisotropic elasticity

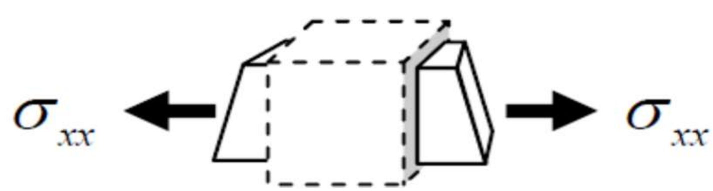
- After inversion, we have

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

- The constants S_{ij}'s in this matrix are called compliances and the matrix is known as **Compliance matrix**.

- So total we have 21 independent material parameters to describe elasticity.

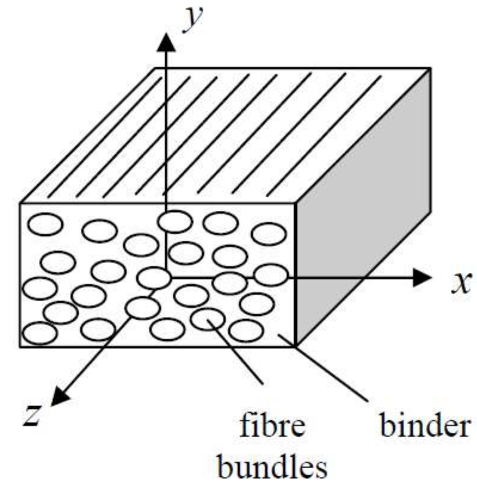
- However, there are further symmetries involved.



Amit Singh

Orthotropic materials

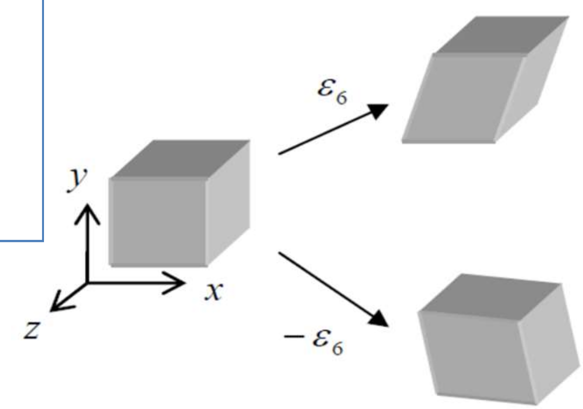
- Three orthogonal planes of microstructural symmetry

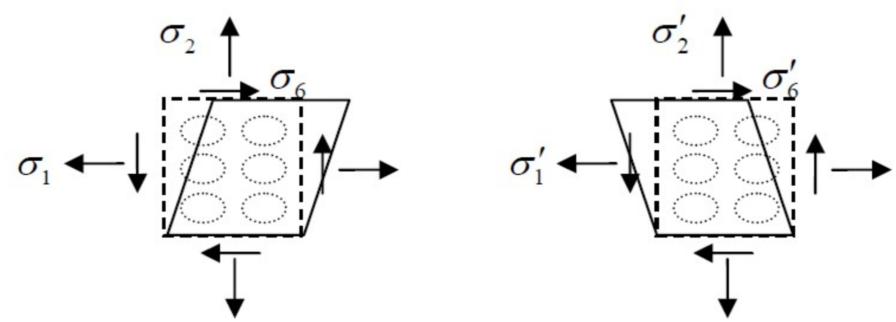


- Consider an element of orthotropic material under shear strains $\varepsilon_6 \left(= \varepsilon_{xy}\right)$ and $-\varepsilon_6 \left(= -\varepsilon_{xy}\right)$
- From the stiffness matrix equation by these two strains:

$$\sigma_1 = C_{16}\varepsilon_6, \quad \sigma_2 = C_{26}\varepsilon_6, \quad \sigma_3 = C_{36}\varepsilon_6$$
$$\sigma_4 = C_{46}\varepsilon_6, \quad \sigma_5 = C_{56}\varepsilon_6, \quad \sigma_6 = C_{66}\varepsilon_6$$

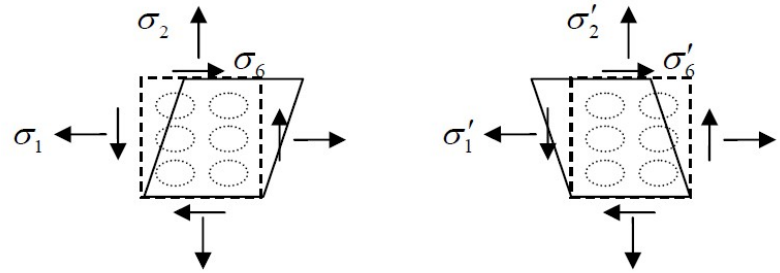
$$\sigma_1' = -C_{16}\varepsilon_6, \quad \sigma_2' = -C_{26}\varepsilon_6, \quad \sigma_3' = -C_{36}\varepsilon_6$$
$$\sigma_4' = -C_{46}\varepsilon_6, \quad \sigma_5' = -C_{56}\varepsilon_6, \quad \sigma_6' = -C_{66}\varepsilon_6$$





Orthotropic materials

- Flip the two figures to see that normal stresses are the same but shear stress will be negative of each other



$$\sigma_1 = \sigma_1' \quad \sigma_2 = \sigma_2' \qquad \sigma_6 = -\sigma_6'$$

- Also, the z axis components should not be affected

$$C_{16} = C_{26} = C_{36} = C_{46} = C_{56} = 0$$

- Use similar arguments for other two planes and show

$$\varepsilon_5 : C_{15} = C_{25} = C_{35} = C_{45} = 0$$
$$\varepsilon_4 : C_{14} = C_{24} = C_{34} = 0$$

- This means we get only 9 independent constants

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

Orthotropic materials

- Invert and then use elastic constants $E,\ \nu$ and G to obtain

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}}
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}
$$

- E_i are Young's modulus in ith direction, $i = 1, 2, 3$
- For uniaxial tensile experiments in direction 1, we will have $\sigma_1 = E_1 \varepsilon_1$
- ν_{ij} is the Poisson's ratio representing ratios of transverse strain to applied strain. For example, $\nu_{12} = -\varepsilon_2 / \varepsilon_1$ for uniaxial tension in the direction 1
- G_{ij} is the shear stiffness in the corresponding plane.

Amit Singh

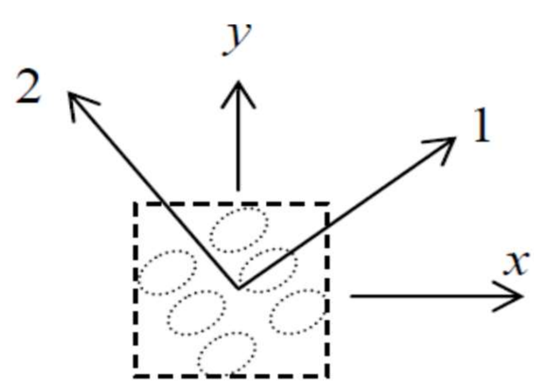
Orthotropic materials

- From the symmetry of the stiffness matrix:

$$\nu_{23}E_3 = \nu_{32}E_2, \quad \nu_{13}E_3 = \nu_{31}E_1, \quad \nu_{12}E_2 = \nu_{21}E_1$$

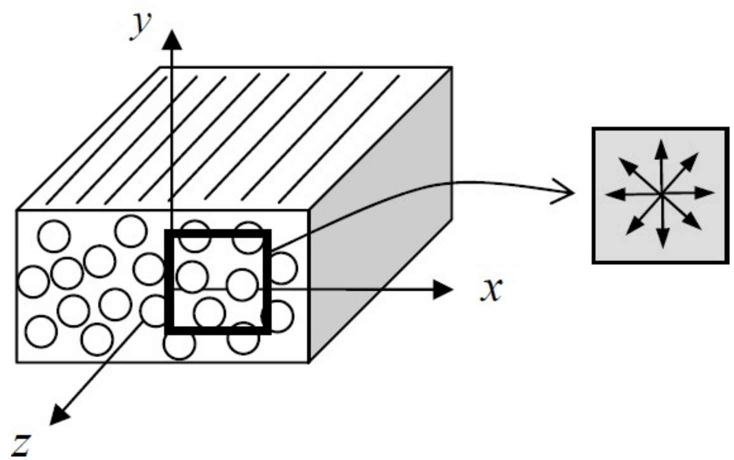
- No shear coupling with respect to material axes. This means normal (shear) stress produces only normal (shear) strain.

- When the coordinate axes are not aligned with the material axes, then the shear coupling will be present:



Transversely isotropic materials

- A single material direction and isotropic in the plane orthogonal to this direction. For example, (same picture as earlier but now the shape of fiber is completely circular):



- Now consider two strains $\epsilon_1 = \epsilon_{xx} = \epsilon$ **and** $\epsilon_2 = \epsilon_{yy} = \epsilon$

- For figure (a), the corresponding induced stresses will be

$$\sigma_1 = C_{11}\varepsilon, \quad \sigma_2 = C_{21}\varepsilon, \quad \sigma_3 = C_{31}\varepsilon$$
$$\sigma_4 = 0, \quad \sigma_5 = 0, \quad \sigma_6 = 0$$

Transversely isotropic materials

- For figure (b), the corresponding induced stresses will be

$$\sigma_1' = C_{12}\varepsilon, \quad \sigma_2' = C_{22}\varepsilon, \quad \sigma_3' = C_{32}\varepsilon$$
$$\sigma_4' = 0, \quad \sigma_5' = 0, \quad \sigma_6' = 0$$

- Now, due to plane isotropy, $\sigma_1 (= \sigma_{xx})$ due to the ε_1 should be the same as $\sigma_2 = (\sigma_{yy})$ due to the ε_2
- Therefore, $C_{11} = C_{22}$ and since $\sigma_3 (= \sigma_{zz})$ should be the same for both, hence $C_{31} = C_{32}$
- Similar arguments can be used for shear deformation and rotation about material axes and one finds $C_{44} = C_{55}$ and $C_{66} = C_{11} - C_{12}$
- So, total 5 independent constants.
- So, if 3 is the material direction, then

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

Transversely isotropic materials

- Invert and obtain

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

- Due to symmetry: $\nu_{13} / E_1 = \nu_{31} / E_3$

- Also, $$G_{12} = \frac{E_1}{2(1+\nu_{12})}$$

Isotropic materials

- Material response independent of orientation.

- Two independent elastic constants:

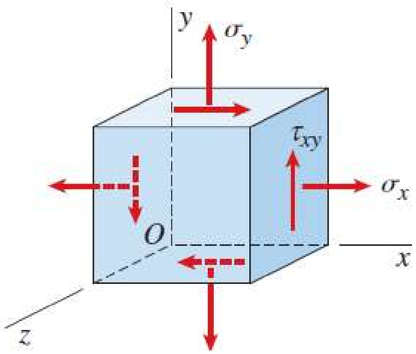
$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{11}-C_{12} & 0 & 0 \\ & & & & C_{11}-C_{12} & 0 \\ & & & & & C_{11}-C_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

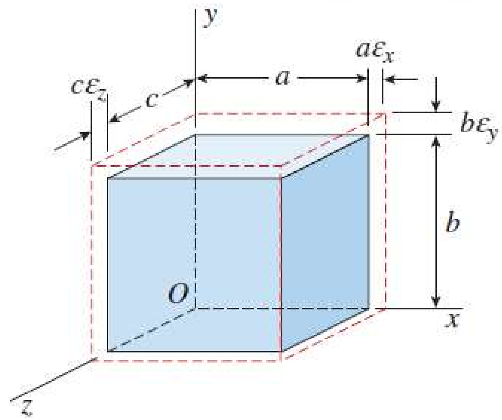
$$\frac{1}{2G} = \frac{1+\nu}{E}$$

Hooke's law for plane stress

- Element of material in plane stress ($\sigma_z = 0$)



Element of material subjected to normal strains ε_x, ε_y, and ε_z



$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\sigma_x = \frac{E}{1 - \nu^2}(\varepsilon_x + \nu\varepsilon_y) \qquad \sigma_y = \frac{E}{1 - \nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$$\tau_{xy} = G\gamma_{xy}$$

$$G = \frac{E}{2(1 + \nu)}$$ two are independent

Volume change and strain energy density

- $V_0 = abc$

$$V_1 = (a + a\varepsilon_x)(b + b\varepsilon_y)(c + c\varepsilon_z)$$

$$= abc(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)$$

$$V_1 = V_0(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)$$

$$V_1 = V_0(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z)$$

$$V_1 = V_0(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)$$

unit volume change e, also known as the **dilatation**. $e = \dfrac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z$

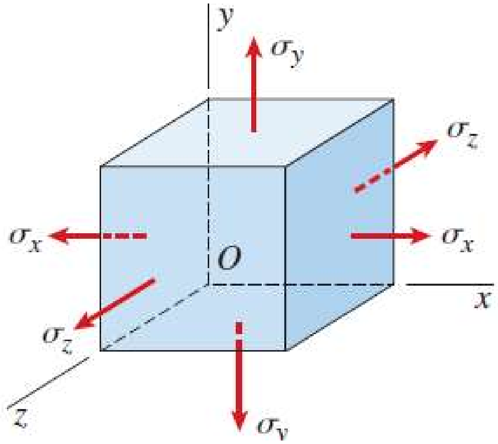
strain-energy density in plane stress: $u = \dfrac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})$

$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{2G}$$

$$u = \frac{E}{2(1 - \nu^2)}(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y) + \frac{G\gamma_{xy}^2}{2}$$

Amit Singh

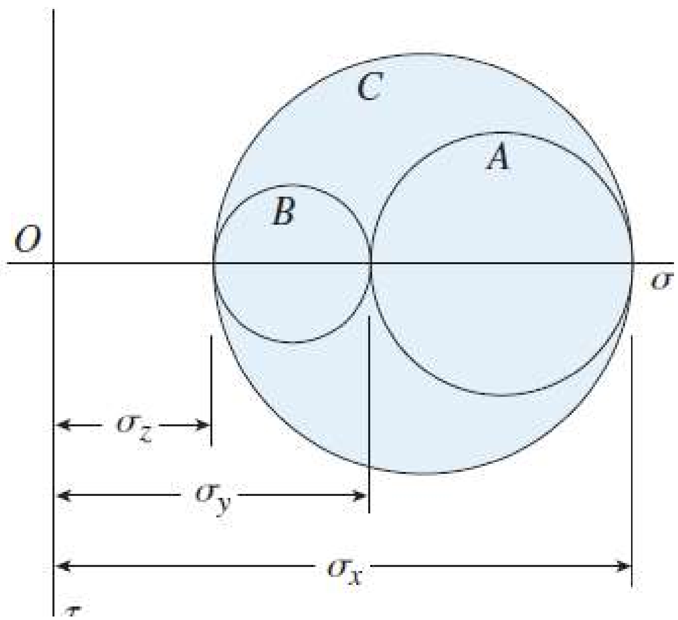
Triaxial stresses

- 

$$(\tau_{max})_z = \pm\frac{\sigma_x - \sigma_y}{2}$$

$$(\tau_{max})_x = \pm\frac{\sigma_y - \sigma_z}{2} \qquad (\tau_{max})_y = \pm\frac{\sigma_x - \sigma_z}{2}$$

Mohr's circles for an element in triaxial stress



Amit Singh

Hooke's law for triaxial stresses

- $$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)}[(1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)}[(1 - \nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)]$$

$$\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)}[(1 - \nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)]$$

Unit Volume Change

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

Strain-Energy Density

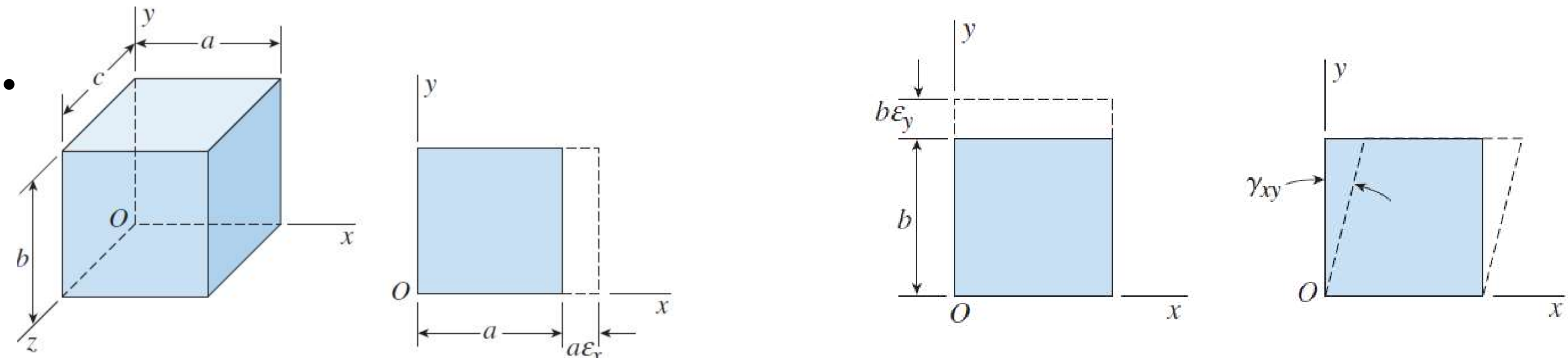
$$u = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z)$$

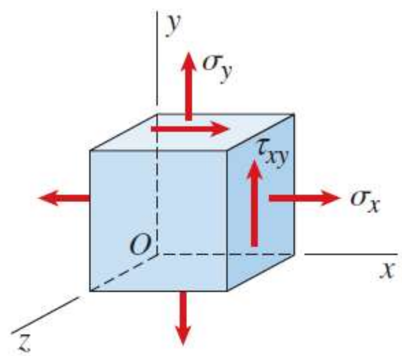
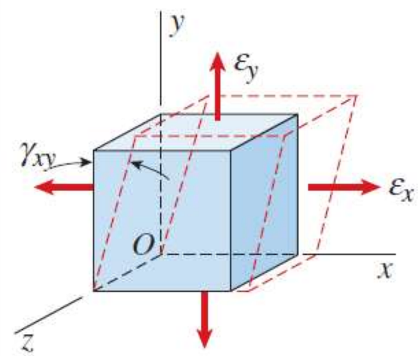
$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z)$$

$$u = \frac{E}{2(1 + \nu)(1 - 2\nu)}[(1 - \nu)(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2\nu(\varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z)]$$

Plane strain

$$\varepsilon_z = 0 \qquad \gamma_{xz} = 0 \qquad \gamma_{yz} = 0$$

- 

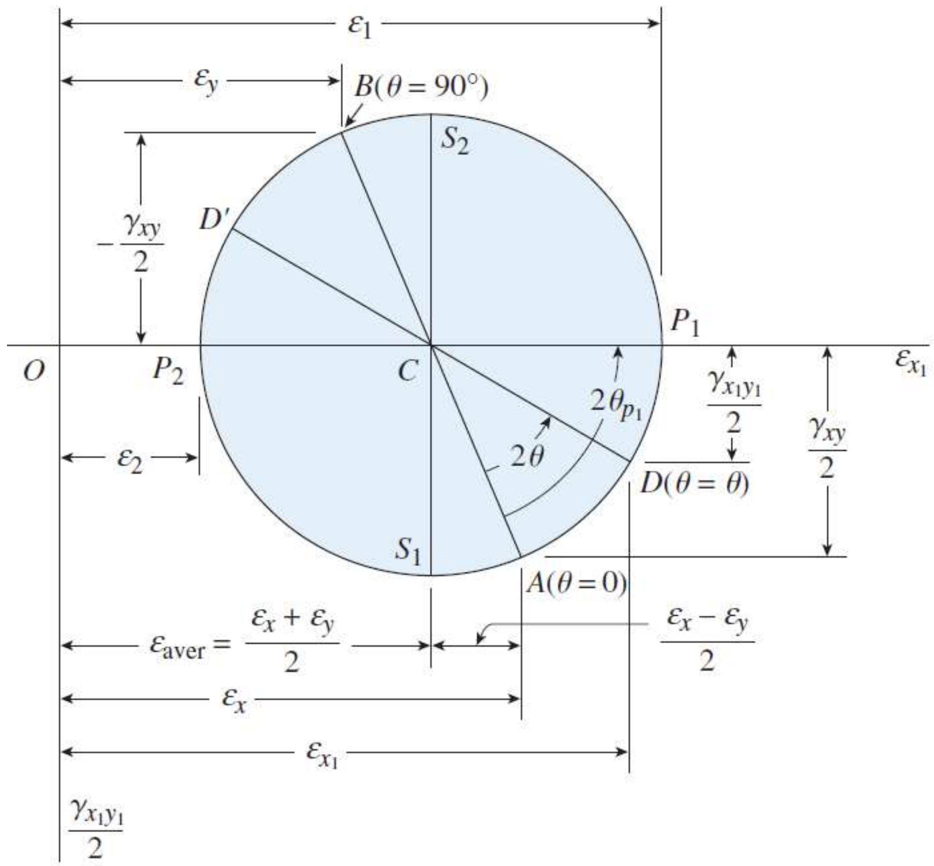
	Plane stress	Plane strain
		
Stresses	$\sigma_z = 0 \quad \tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ε_x, ε_y, ε_z, and γ_{xy} may have nonzero values	$\varepsilon_z = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ε_x, ε_y, and γ_{xy} may have nonzero values

Plane strain transformation: Mohr's circle

- $$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

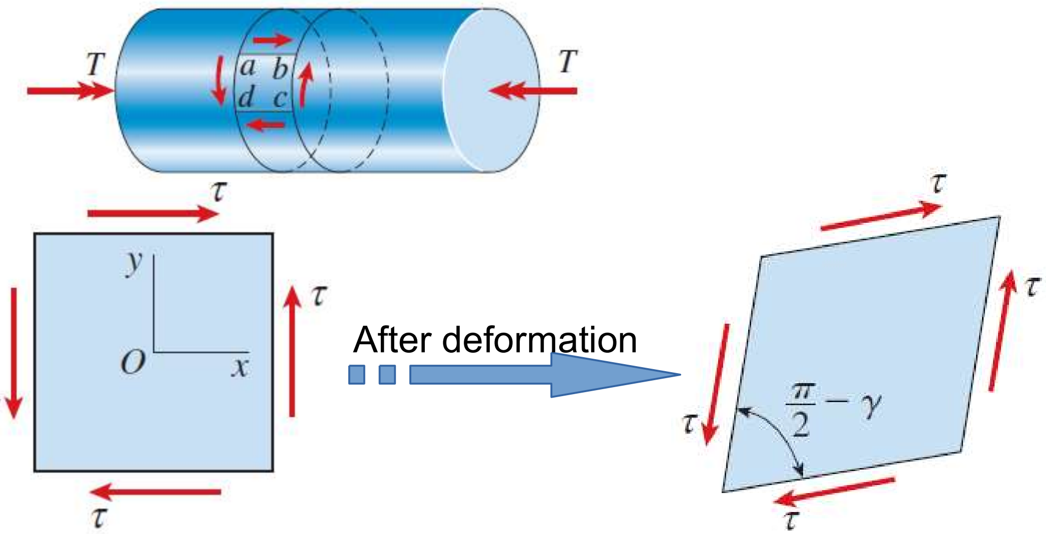
$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

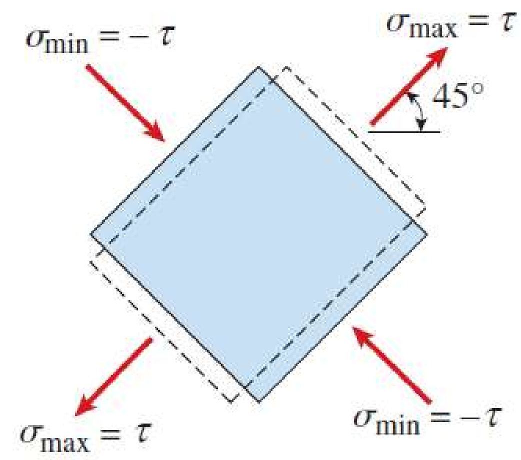
Stresses	Strains
σ_x	ε_x
σ_y	ε_y
τ_{xy}	$\gamma_{xy}/2$
σ_{x_1}	ε_{x_1}
$\tau_{x_1y_1}$	$\gamma_{x_1y_1}/2$



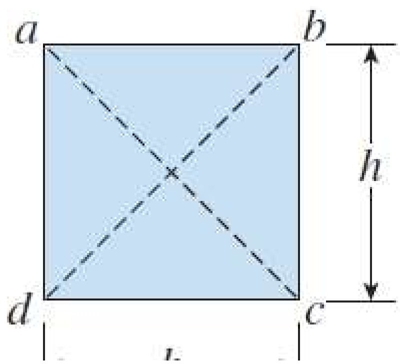
Torsion continued..

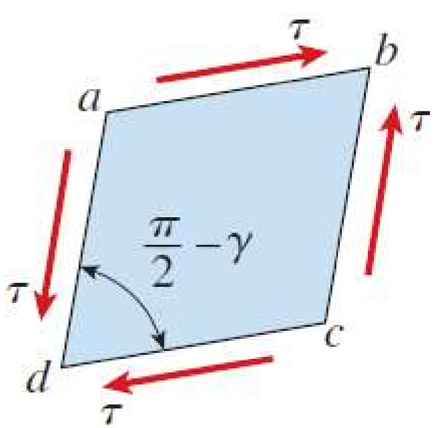
Strain in pure shear

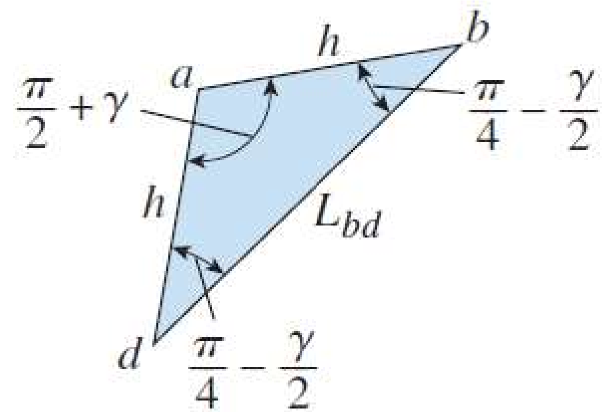
- 



$$\varepsilon_{max} = \frac{\tau}{E} + \frac{\nu\tau}{E} = \frac{\tau}{E}(1 + \nu)$$



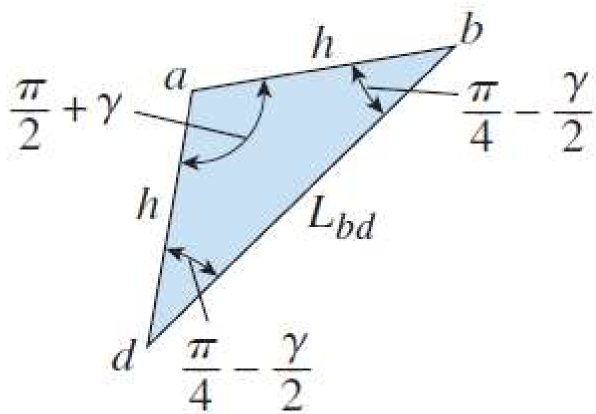




$$L_{bd} = \sqrt{2}h(1 + \varepsilon_{max})$$

$$L_{bd}^2 = h^2 + h^2 - 2h^2\cos\left(\frac{\pi}{2} + \gamma\right)$$

Strain in pure shear

- 

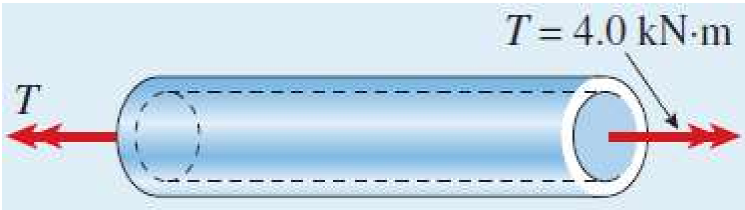
$$(1 + \varepsilon_{\max})^2 = 1 - \cos\left(\frac{\pi}{2} + \gamma\right)$$

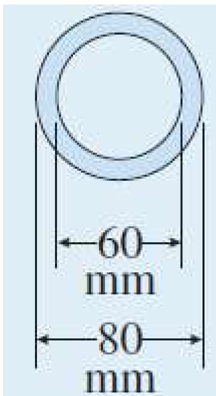
$$1 + 2\varepsilon_{\max} + \varepsilon_{\max}^2 = 1 + \sin\gamma$$

$$\varepsilon_{\max} = \frac{\gamma}{2} = \frac{\tau}{E}(1 + \nu)$$

$$G = \frac{E}{2(1 + \nu)}$$

Problem

- 

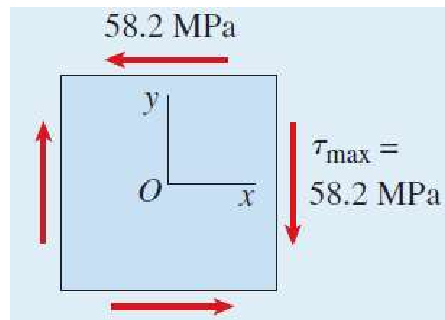


circular tube

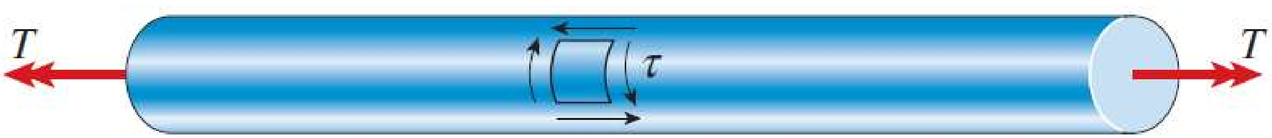
$G = 27$ GPa

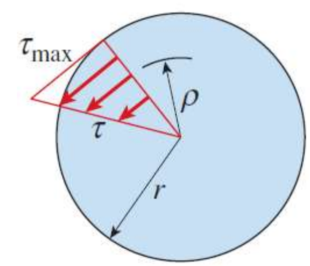
(a) Determine the maximum shear, tensile and compressive stresses. Show these stresses on properly sketched stress elements

$$\tau_{max} = \frac{Tr}{I_P} = \frac{(4000\ \mathrm{N \cdot m})(0.040\ \mathrm{m})}{\frac{\pi}{32}\left[(0.080\ \mathrm{m})^4 - (0.060\ \mathrm{m})^4\right]} = 58.2\ \mathrm{MPa}$$

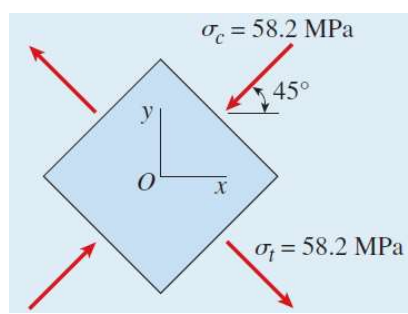


Why the signs as shown here?



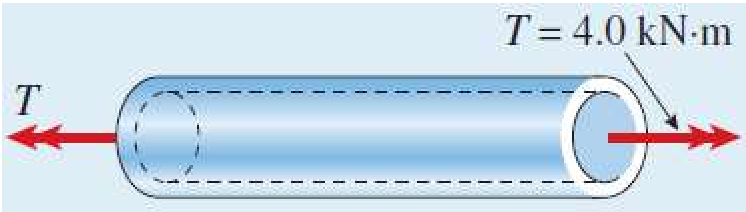


At a plane inclined at 45 degree, we will have normal stress extrema for pure shear.



Amit Singh

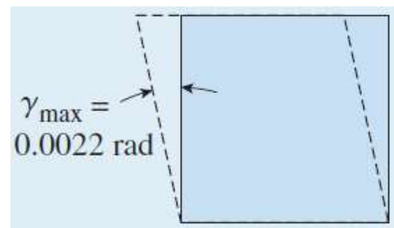
Problem

- 

circular tube

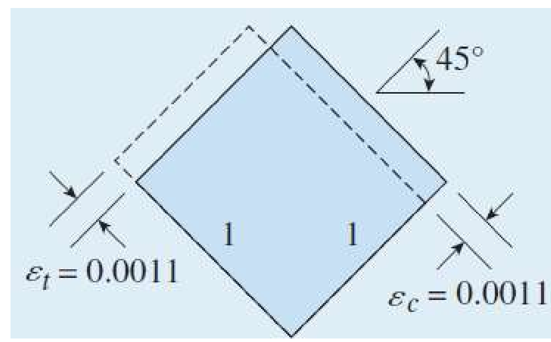
(b) Determine the corresponding strains. Shown these strains on properly sketched deformed elements.

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{58.2\text{ MPa}}{27\text{ GPa}} = 0.0022\text{ rad}$$



$$\varepsilon_{\max} = \frac{\gamma_{\max}}{2} = 0.0011$$

$$\varepsilon_t = 0.0011 \qquad \varepsilon_c = -0.0011$$



(c) Find maximum permissible torque if allowable normal strain is 0.0009.

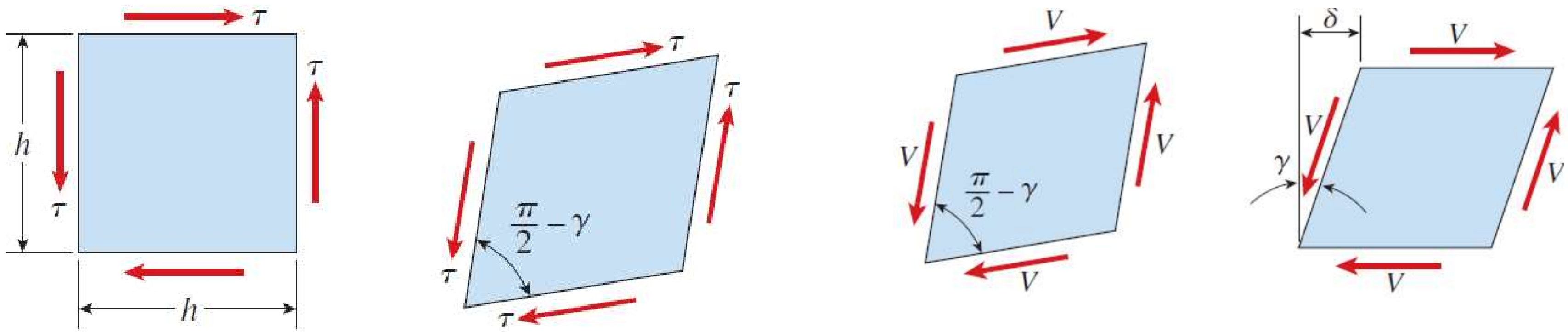
$$\gamma_a = 2\varepsilon_a = 2(0.9 \times 10^{-3}) = 1.8 \times 10^{-3}$$

$$I_p = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$\tau = \frac{T\rho}{I_p} \quad \tau_{\max} = \frac{T d_2/2}{I_p} \quad T_{\max} = \frac{2\tau_{\max} I_p}{d_2} = 3.34 kNm$$

Strain energy density in pure shear

•



$$V = \tau h t$$

$$\delta = \gamma h$$

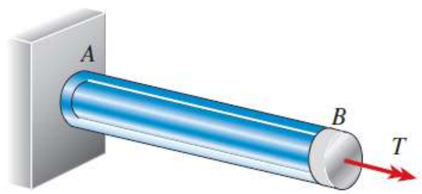
$$U = W = \frac{V\delta}{2}$$

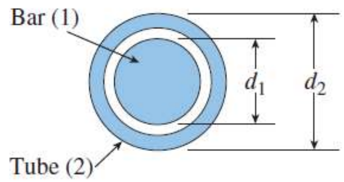
$$U = \frac{\tau\gamma h^2 t}{2}$$

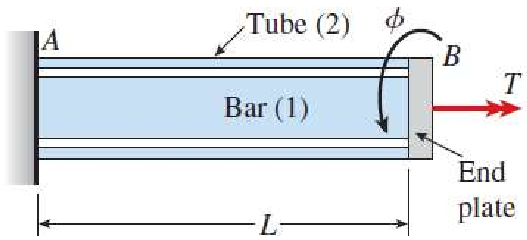
$$u = \frac{\tau\gamma}{2} \qquad u = \frac{\tau^2}{2G} \qquad u = \frac{G\gamma^2}{2}$$

Amit Singh

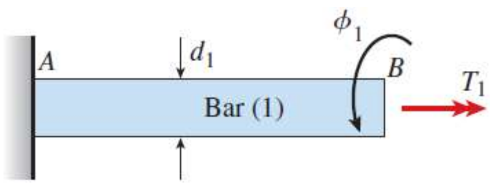
Statically indeterminate torsional members

- 



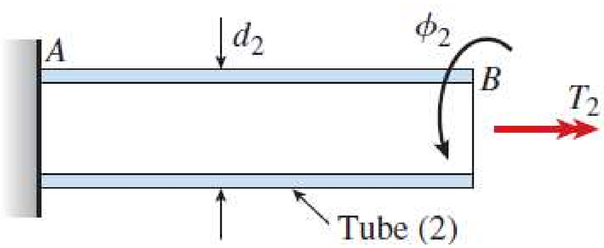


equation of equilibrium $$T_1 + T_2 = T$$



equation of compatibility $$\phi_1 = \phi_2$$

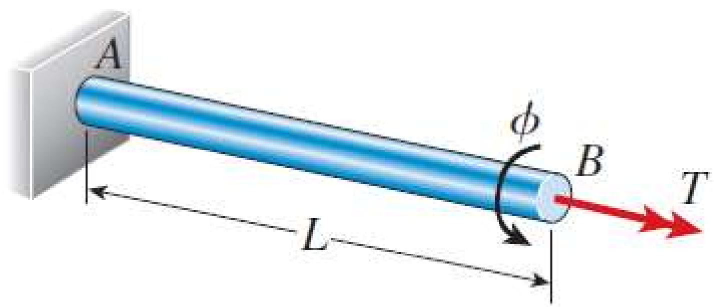
$$\frac{T_1 L}{G_1 I_{P1}} = \frac{T_2 L}{G_2 I_{P2}}$$

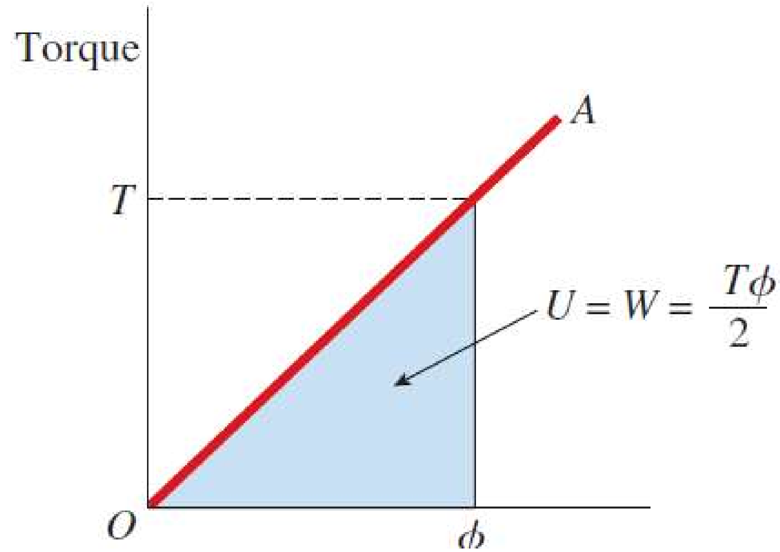


$$T_1 = T\left(\frac{G_1 I_{P1}}{G_1 I_{P1} + G_2 I_{P2}}\right) \qquad T_2 = T\left(\frac{G_2 I_{P2}}{G_1 I_{P1} + G_2 I_{P2}}\right)$$

Amit Singh

Strain energy in torsion

- 



$$\phi = TL/GI_P$$

$$U = \frac{T^2L}{2GI_P} \qquad U = \frac{GI_P\phi^2}{2L}$$

Nonuniform Torsion

$$U = \sum_{i=1}^{n} U_i = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2G_i(I_P)_i}$$

$$U = \int_0^L \frac{[T(x)]^2 dx}{2GI_P(x)}$$

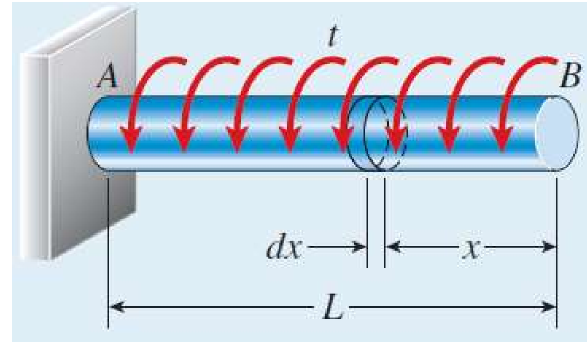
Amit Singh

Strain energy for distributed torque

- **Problem 1**

$$T(x) = tx$$

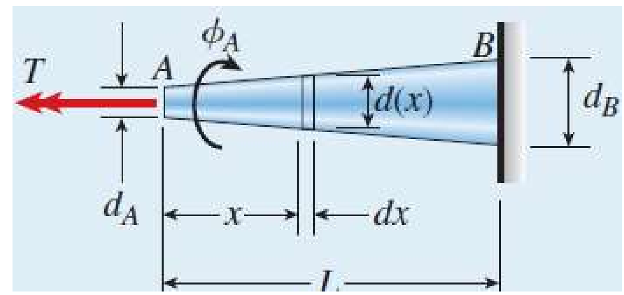
$$U = \int_0^L \frac{[T(x)]^2 dx}{2GI_P} = \frac{1}{2GI_P}\int_0^L (tx)^2 dx = \frac{t^2L^3}{6GI_P}$$



- **Problem 2**

$$I_P(x) = \frac{\pi}{32}\left[d(x)\right]^4$$

$$d(x) = d_A + \frac{d_B - d_A}{L}x$$



$$U = \int_0^L \frac{[T(x)]^2 dx}{2GI_P(x)} = \frac{16T^2}{\pi G}\int_0^L \frac{dx}{\left(d_A + \frac{d_B - d_A}{L}x\right)^4}$$

$$U = \frac{16T^2L}{3\pi G(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

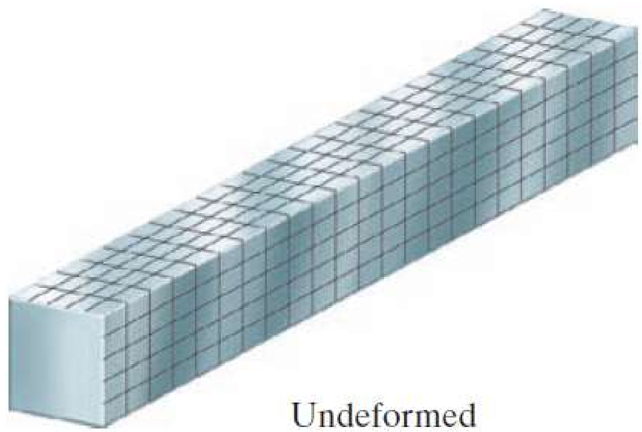
$$U = W = \frac{T\phi}{2}$$

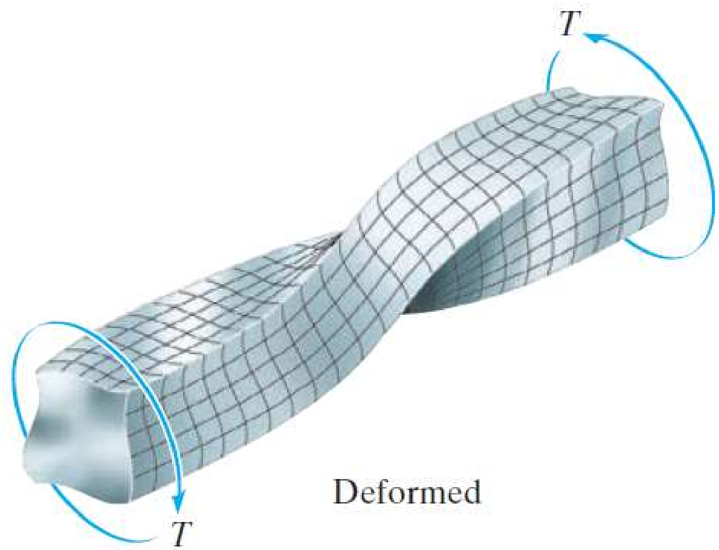
$$\phi_A = \frac{32TL}{3\pi G(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

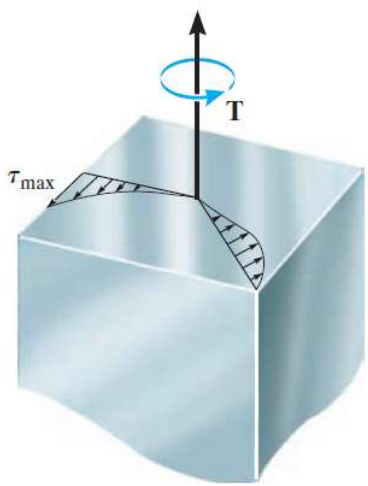
Amit Singh

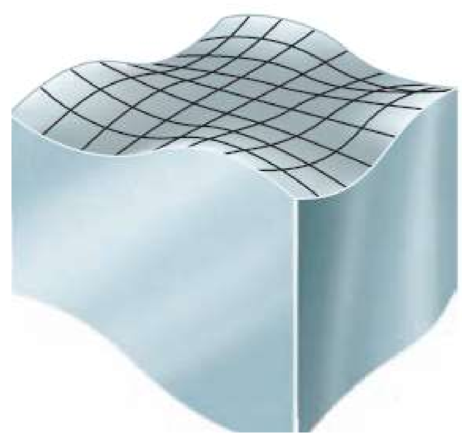
Noncircular prismatic shaft

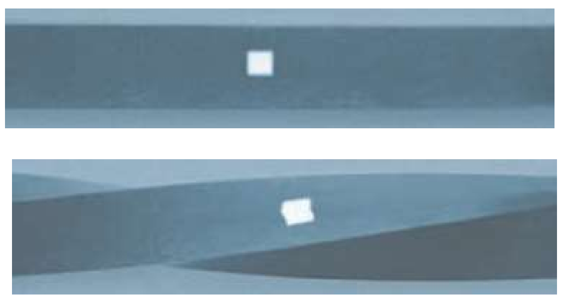
- Cross-sections will warp or bulge when the shaft is twisted







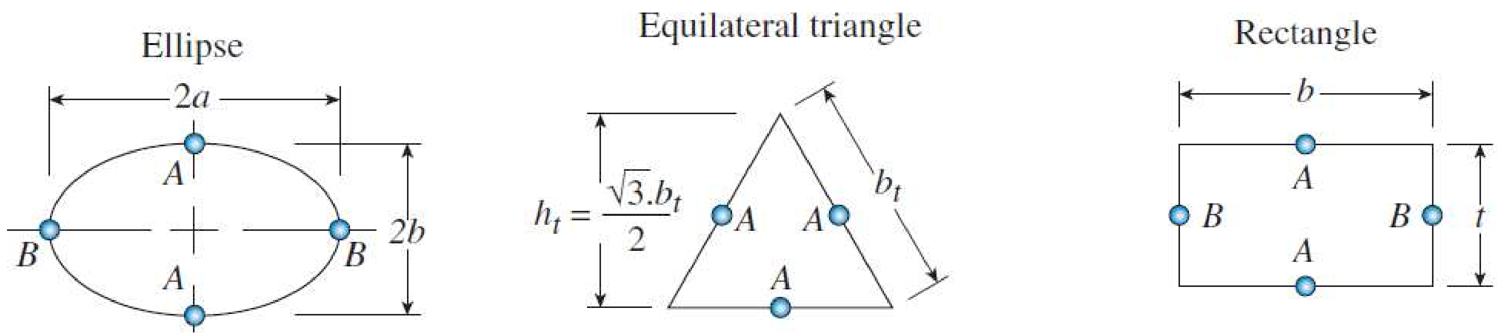




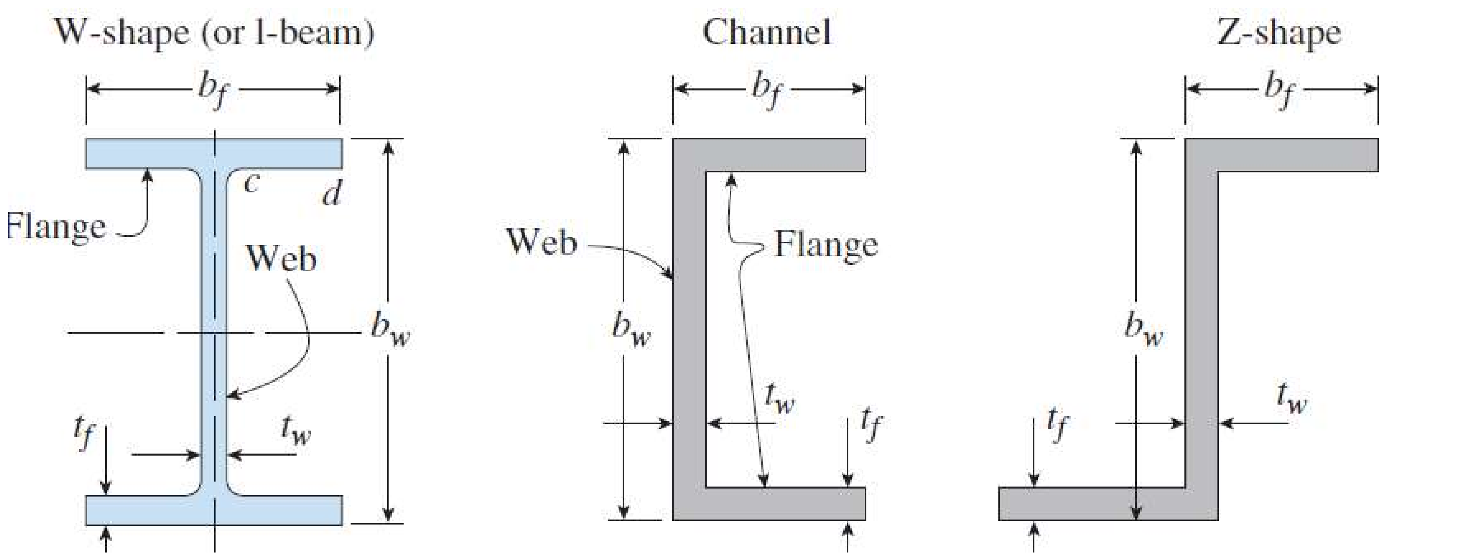
Amit Singh

Noncircular prismatic shaft

- **Solid cross sections**



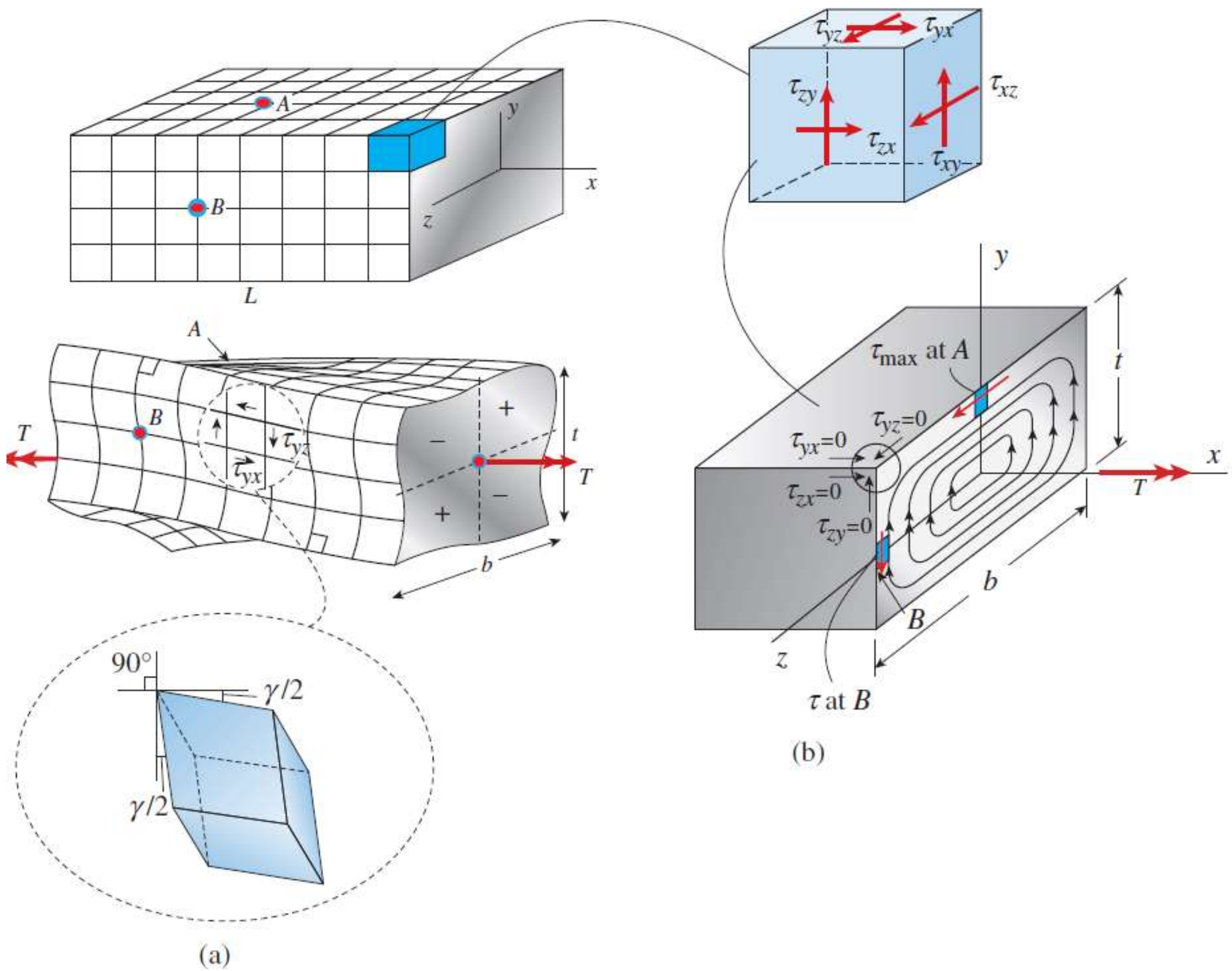
- **Thin-walled open cross sections**

- 

Amit Singh

Noncircular prismatic shaft

- Cannot use earlier torsion formula or torque-angle of twist relations
- For a rectangular bar acted upon by torques T at either end: the shear stress at corners are zero, maximum shear stress is at the midpoint along the longer side (point A)
-



Amit Singh

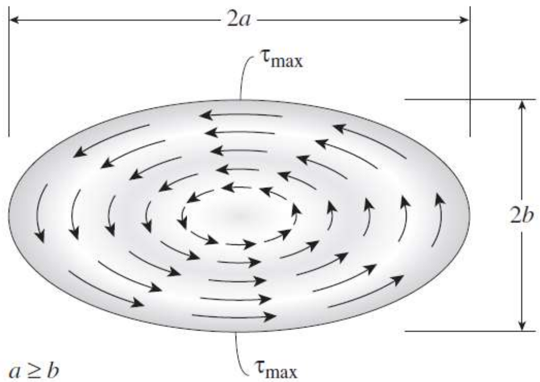
Elliptical, Triangular, and Rectangular Cross Sections

- For prismatic shafts, maximum shear stress for **elliptical** cross sections

$$\tau_{max} = \frac{2T}{\pi a b^2}$$

$$\phi = \frac{TL}{GJ_e}$$

torsion constant J_e $\quad J_e = \frac{\pi a^3 b^3}{a^2 + b^2}$

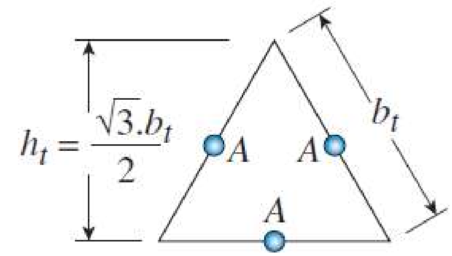


- For **equilateral triangular** cross sections:

$$\tau_{max} = \frac{T\left(\frac{h_t}{2}\right)}{J_t} = \frac{15\sqrt{3}T}{2h_t^3} \qquad J_t = \frac{h_t^4}{15\sqrt{3}}$$

$$\phi = \frac{TL}{GJ_t} = \frac{15\sqrt{3}TL}{Gh_t^4}$$

Equilateral triangle

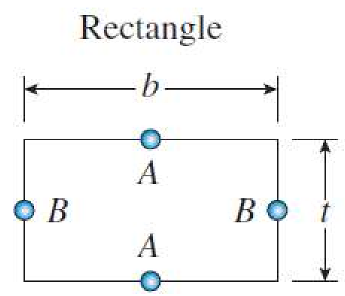


Elliptical, Triangular, and Rectangular Cross Sections

- For **rectangular** cross sections

$$\tau_{\max} = \frac{T}{k_1 b t^2} \qquad \phi = \frac{TL}{(k_2 b t^3)G} = \frac{TL}{GJ_r}$$

$$J_r = k_2 b t^3$$



b/t	1.00	1.50	1.75	2.00	2.50	3.00	4	6	8	10	∞
k_1	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.298	0.307	0.312	0.333
k_2	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.298	0.307	0.312	0.333

Thin-Walled Open Cross Sections: I-beam, Angle, Channel, and Z-shape

- Total torque = sum of torques carried out by webs and flanges
- First compute the flange b_f/t_f ratio and calculate flange torsion constant $J_f = k_1 b_f t_f^3$
- Find constant k1 from this table

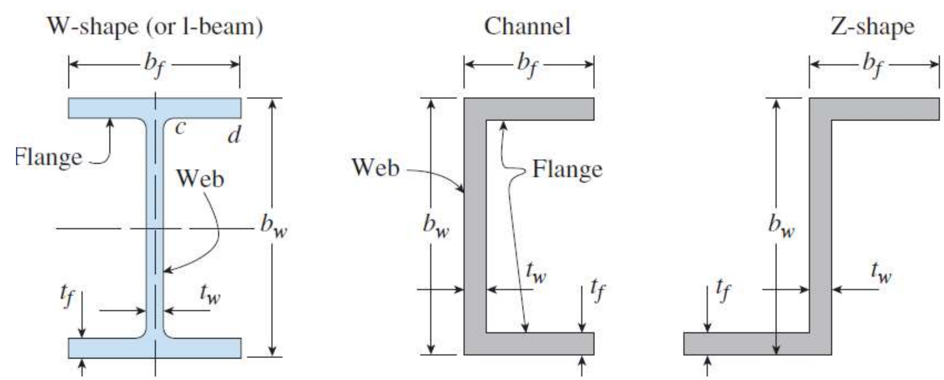
b/t	1.00	1.50	1.75	2.00	2.50	3.00	4	6	8	10	∞
k_1	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.298	0.307	0.312	0.333

- For the web, use ratio $(b_w - 2t_f)/t_w$ to find a new constant k1
- The web torsion constant $J_w = k_1(b_w - 2t_f)t_w^3$
- The combined trosion constant

$$J = J_w + 2J_f$$

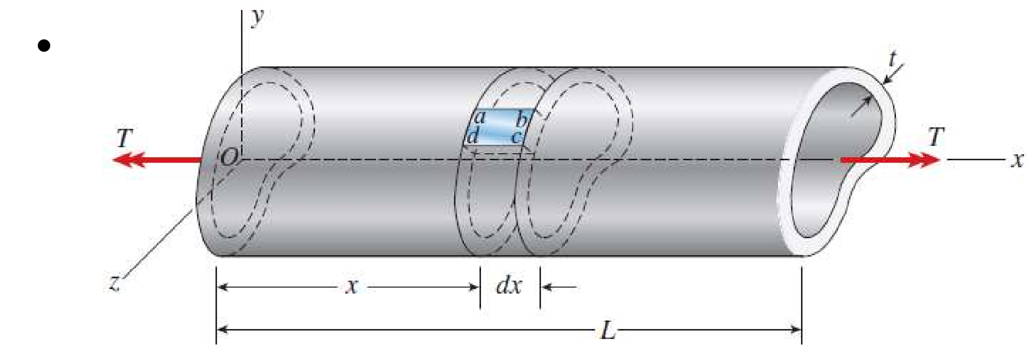
$$\tau_{max} = \frac{2T\left(\frac{t}{2}\right)}{J} \quad \text{and} \quad \phi = \frac{TL}{GJ}$$

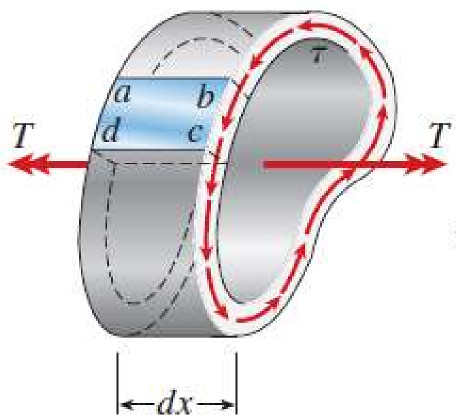
larger of t_f and t_w is used in the formula for τ_{max}

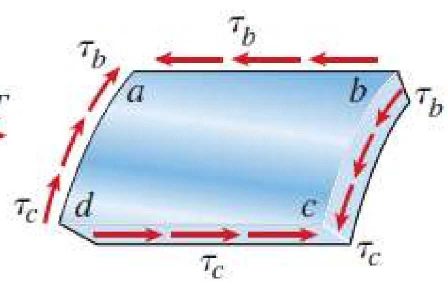


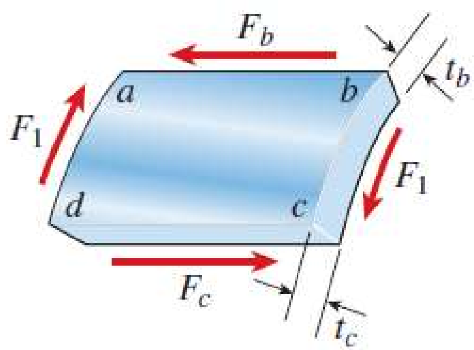
Amit Singh

THIN-WALLED TUBES

- 





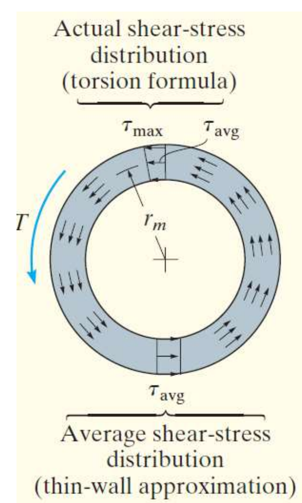


shear flow

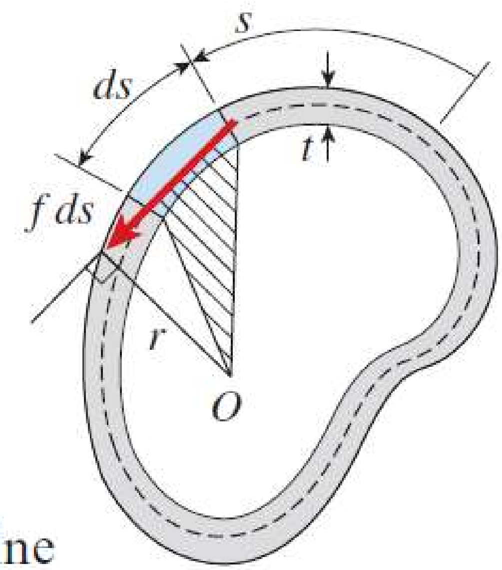
$$f = \tau t = \text{constant}$$

$$F_b = \tau_b t_b dx \qquad F_c = \tau_c t_c dx \qquad \tau_b t_b = \tau_c t_c$$

$$q = \tau_{avg} t$$



THIN-WALLED TUBES



- **Torsion formula:** relates shear flow f to the torque T
- **Median line:** take an element of length ds along this line
- $dT = rfds$ is the moment of the shear force
- acting on this element

$$T = f\int_0^{L_m} r\,ds$$ L_m denotes the length of the median line

- Also , $$\int_0^{L_m} r\,ds = 2A_m$$

area A_m enclosed by the median line of the cross section

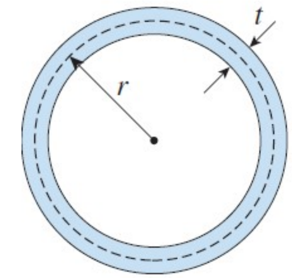
- Finally,

$$\boxed{f = \frac{T}{2A_m}, \quad \tau = \frac{T}{2tA_m}}$$ **Torsion Formula**

$$\boxed{f = \tau t}$$

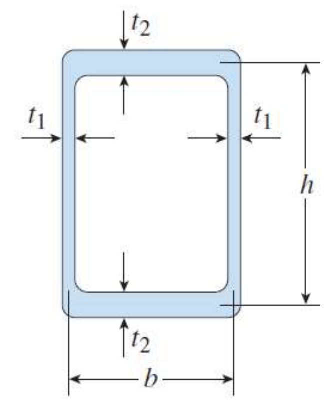
THIN-WALLED TUBES Torsion Formula

- **Thin-walled circular tube**: $A_m = \pi r^2$



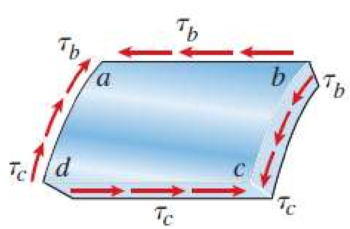
$$\tau = \frac{T}{2\pi r^2 t}$$

- **Thin-walled rectangular tube**: $A_m = bh$



$$\tau_{\text{vert}} = \frac{T}{2t_1bh} \qquad \tau_{\text{horiz}} = \frac{T}{2t_2bh}$$

- **Strain energy:** abcd is in pure shear so strain energy density $\tau^2/2G$



$$dU = \frac{\tau^2}{2G} t\,ds\,dx = \frac{\tau^2 t^2}{2G}\frac{ds}{t}dx = \frac{f^2}{2G}\frac{ds}{t}dx$$

$$U = \int dU = \frac{f^2}{2G}\int_0^{L_m}\frac{ds}{t}\int_0^{L}dx = \frac{f^2 L}{2G}\int_0^{L_m}\frac{ds}{t} = \frac{T^2 L}{8GA_m^2}\int_0^{L_m}\frac{ds}{t}$$

Amit Singh

THIN-WALLED TUBES Torsion Formula

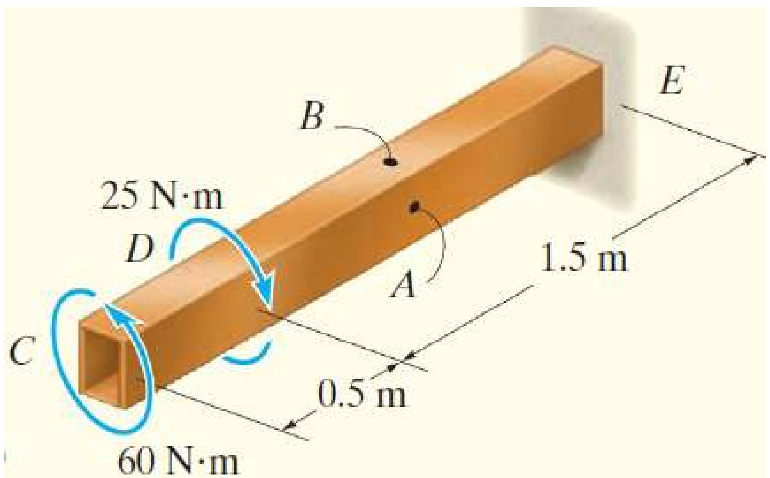
- **Torsion constant:** $$J = \frac{4A_m^2}{\int_0^{L_m} \frac{ds}{t}}$$ $$U = \frac{T^2L}{2GJ}$$
- **For tube with constant thickness t:** $$J = \frac{4tA_m^2}{L_m}$$
- **For thin-walled circular tube:** $L_m = 2\pi r$ and $A_m = \pi r^2$; $J = 2\pi r^3 t$
- **For rectangular tube:** $$\int_0^{L_m} \frac{ds}{t} = 2\int_0^h \frac{ds}{t_1} + 2\int_0^b \frac{ds}{t_2} = 2\left(\frac{h}{t_1} + \frac{b}{t_2}\right)$$ $$J = \frac{2b^2h^2t_1t_2}{bt_1 + ht_2}$$
- **Angle of twist:** $$\phi = \frac{TL}{GJ}$$

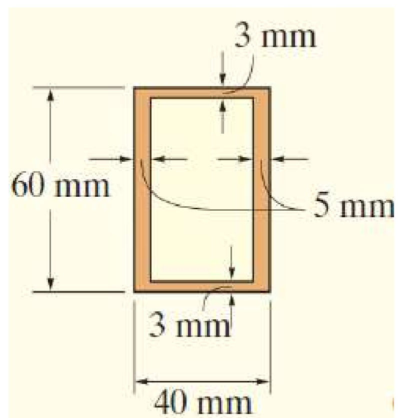
$$\phi = \frac{TL}{4A_m^2 G}\oint \frac{ds}{t}$$

THIN-WALLED TUBES Torsion Formula

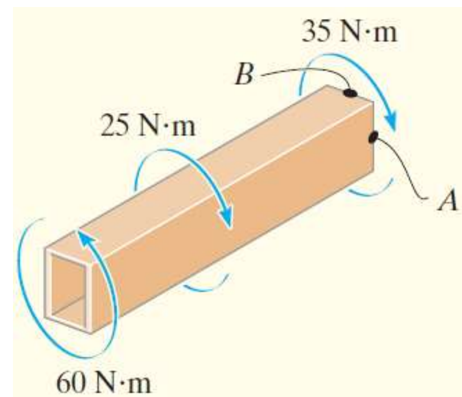
- **Example:**

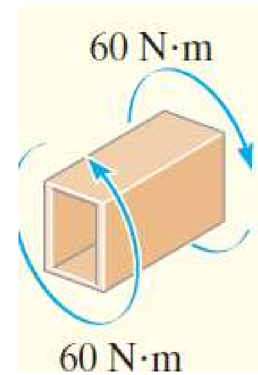
determine the average shear stress in the tube at points A and B. Also, what is the angle of twist of end C? The tube is fixed at E.





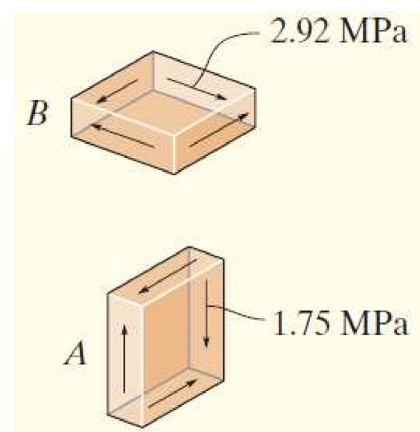
FBD:





$$\tau_B = \frac{T}{2tA_m} = \frac{35\ \text{N}\cdot\text{m}}{2(0.003\ \text{m})(0.00200\ \text{m}^2)} = 2.92\ \text{MPa}$$

$$\tau_A = \frac{T}{2tA_m} = \frac{35\ \text{N}\cdot\text{m}}{2(0.005\ \text{m})(0.00200\ \text{m}^2)} = 1.75\ \text{MPa}$$



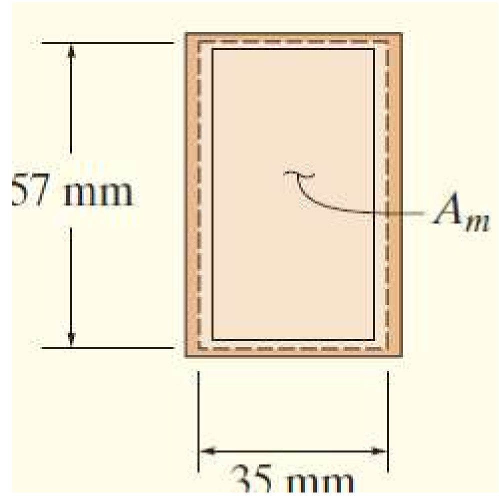
THIN-WALLED TUBES Torsion Formula

- **Example:**

$$\phi = \sum \frac{TL}{4A_m^2\,G}\oint\frac{ds}{t}$$

$$= \frac{60\ \text{N}\cdot\text{m}\ (0.5\ \text{m})}{4(0.00200\ \text{m}^2)^2\ (38(10^9)\ \text{N/m}^2)}\left[2\left(\frac{57\ \text{mm}}{5\ \text{mm}}\right) + 2\left(\frac{35\ \text{mm}}{3\ \text{mm}}\right)\right]$$

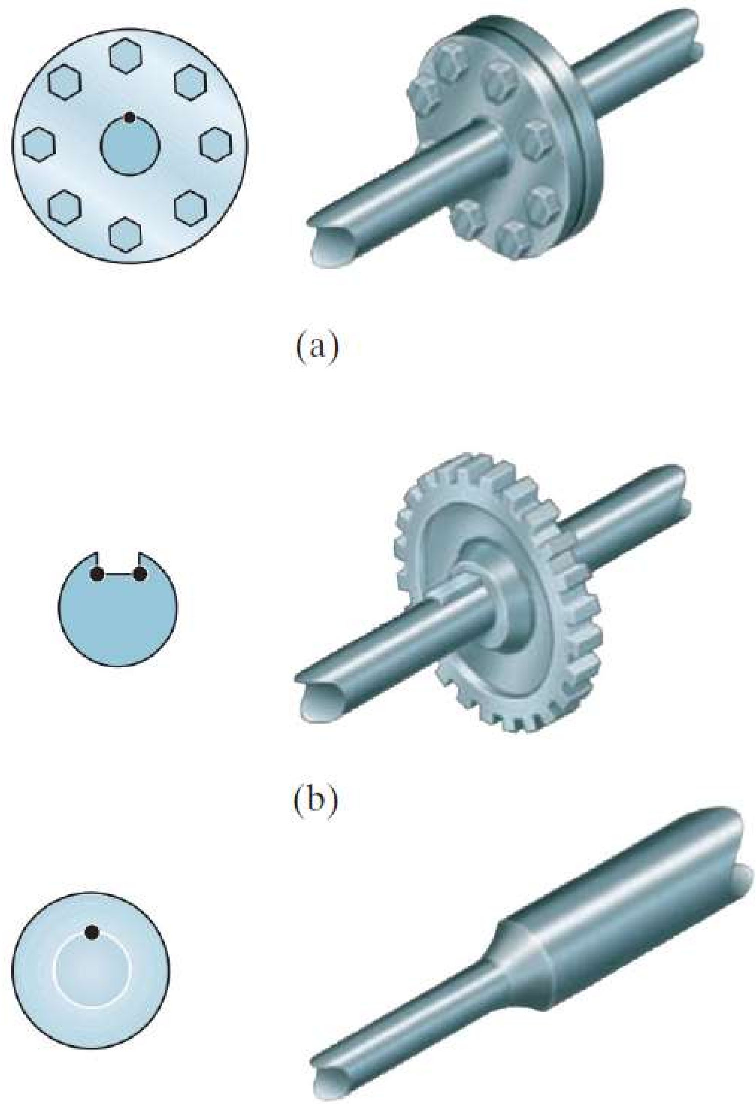
$$+ \frac{35\ \text{N}\cdot\text{m}\ (1.5\ \text{m})}{4(0.00200\ \text{m}^2)^2\ (38(10^9)\ \text{N/m}^2)}\left[2\left(\frac{57\ \text{mm}}{5\ \text{mm}}\right) + 2\left(\frac{35\ \text{mm}}{3\ \text{mm}}\right)\right]$$



$$= 6.29(10^{-3})\ \text{rad} = 0.360°$$

Amit Singh

Torsional stress concentration



(a)

(b)

(c)

Amit Singh

Torsional stress concentration

- The torsion formula cannot be applied if there is a sudden discontinuity in the cross section
- Changes of section, holes or notches cause local stress concentrations
- Full solution using advance theory is difficult to obtain
- Numerical solution possible but takes time

- But results can be tabulated in the form of a multiplier to the nominal stress σ_{nom} or τ_{nom} obtained by elementary mechanics theories such that $\sigma = K_t \sigma_{\text{nom}}$
- This multiplier K or K_t is known as *theoretical stress concentration factor*

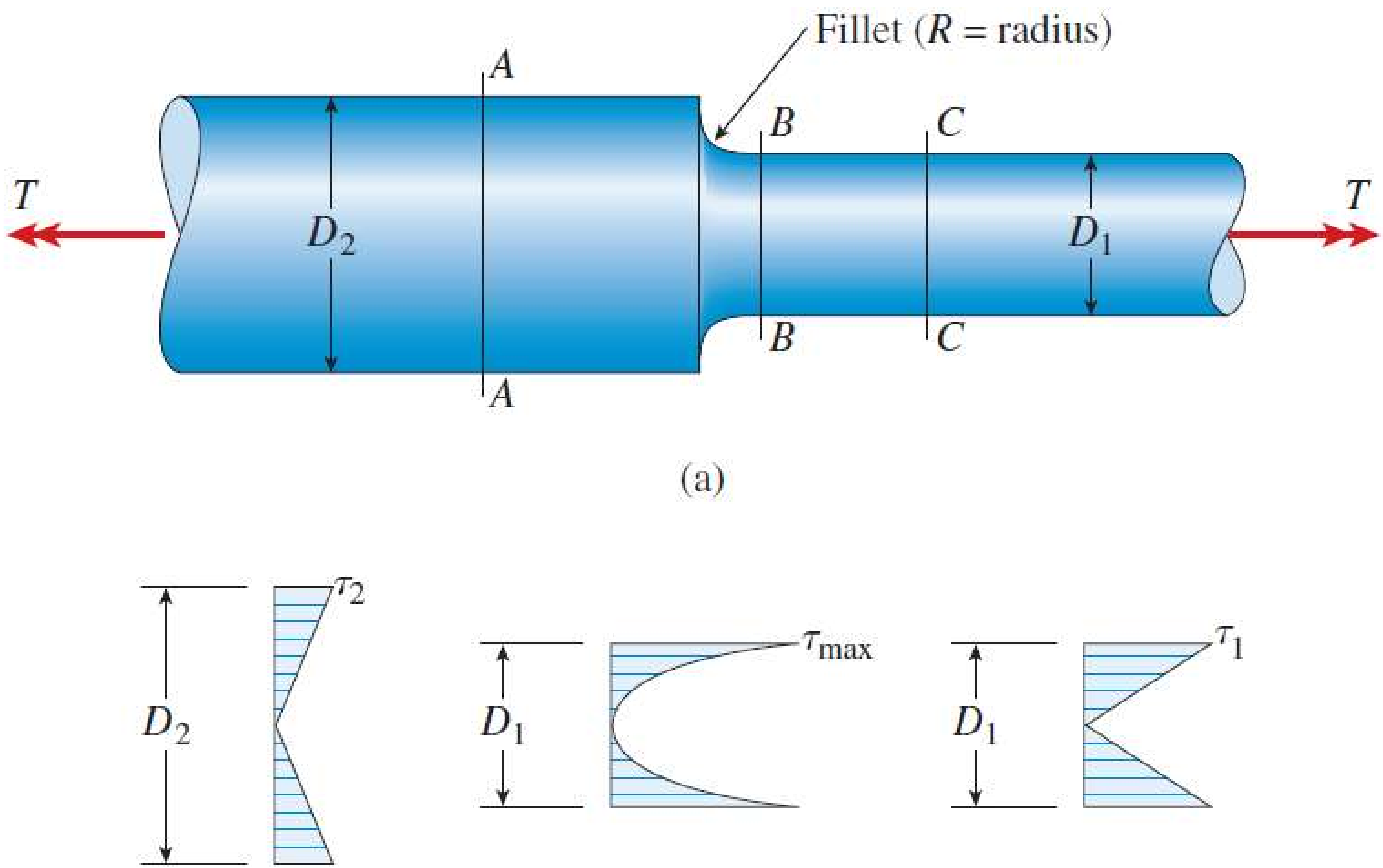
$$\boxed{\tau_{\max} = K\tau_{\text{nom}} = K\frac{T\rho}{J}}$$

***torsional stress concentration factor**, K*

- Applied on the smaller shaft

Amit Singh

Torsional stress concentration



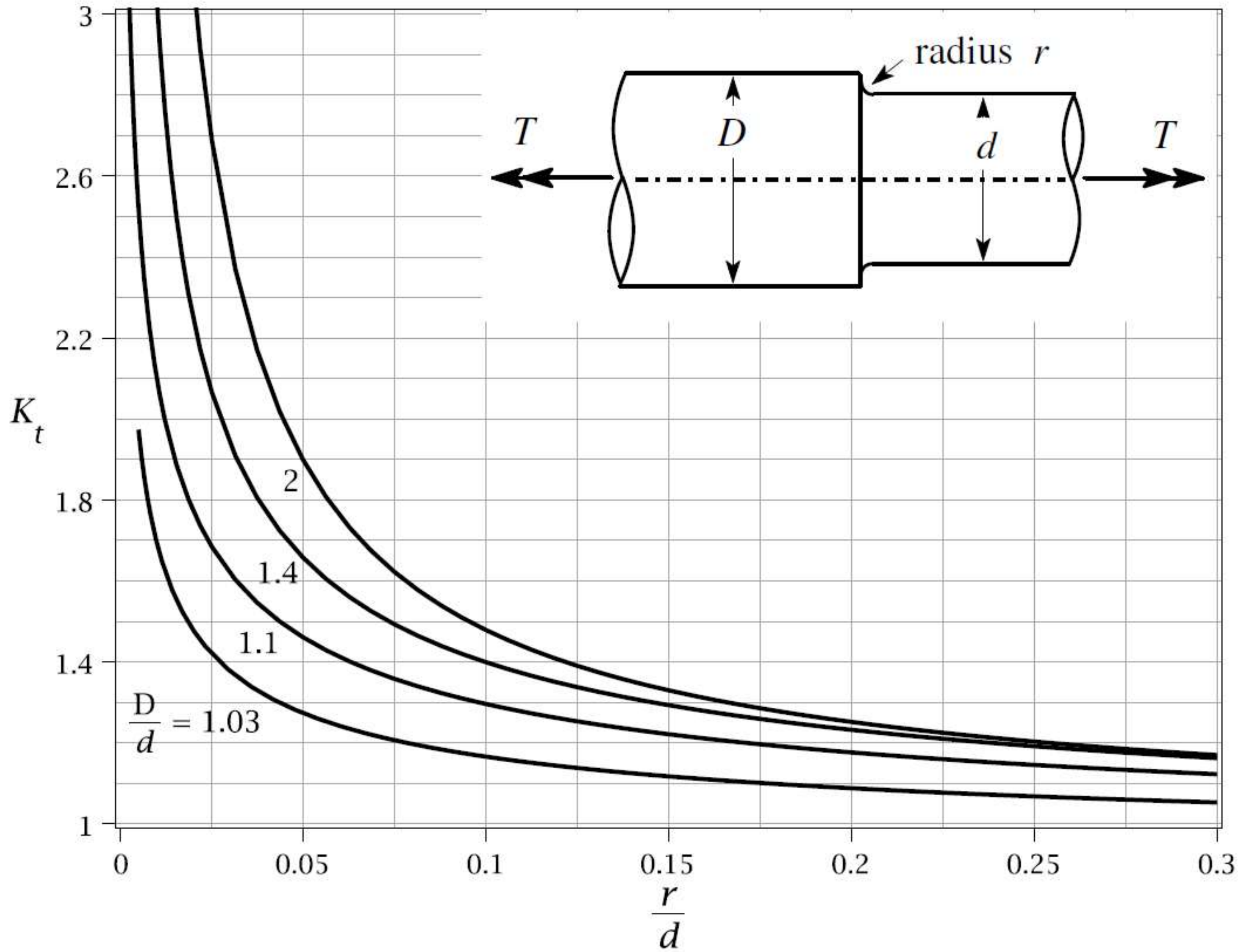
Section A-A
(b)

Section B-B
(c)

Section C-C
(d)

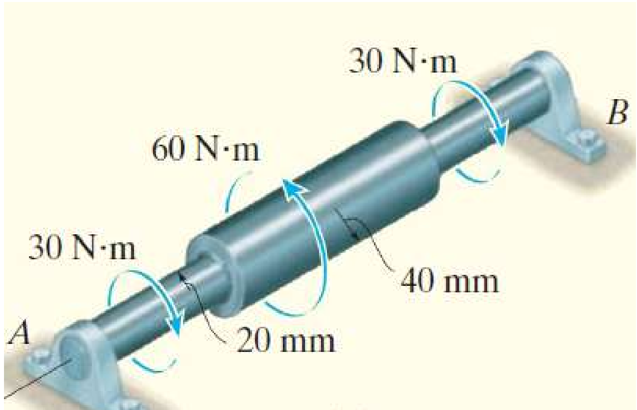
Amit Singh

Torsional stress concentration

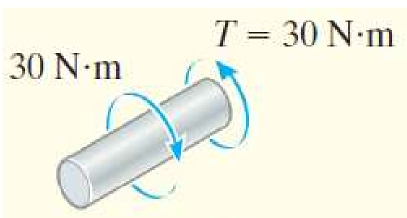


- Filler radius r, Increase in r implies decrease in K
- Must for **brittle material or fatigue (cyclical torsional loading) design**
- **Ductile:** Inelastic strain. Yielding of the material distributes the stress more evenly

Torsional stress concentration

- 

(a)



(b)

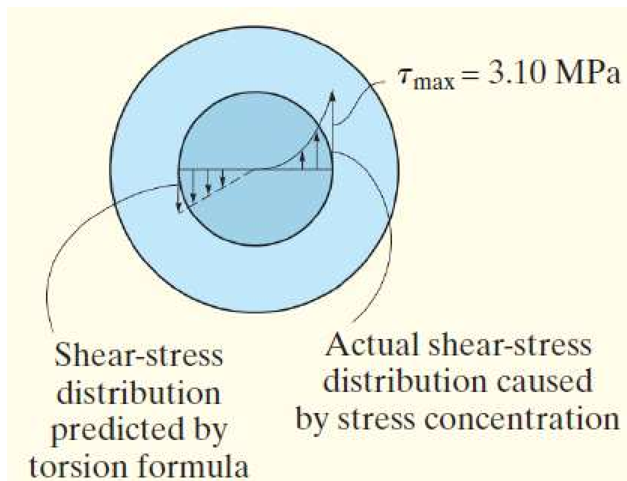
Determine the maximum stress in the shaft due to the applied torques. The shoulder fillet at the junction of each shaft has a radius of $r = 6$ mm.

$$\frac{D}{d} = \frac{2(40 \text{ mm})}{2(20 \text{ mm})} = 2$$

$$\frac{r}{d} = \frac{6 \text{ mm}}{2(20 \text{ mm})} = 0.15$$

$$K = 1.3$$

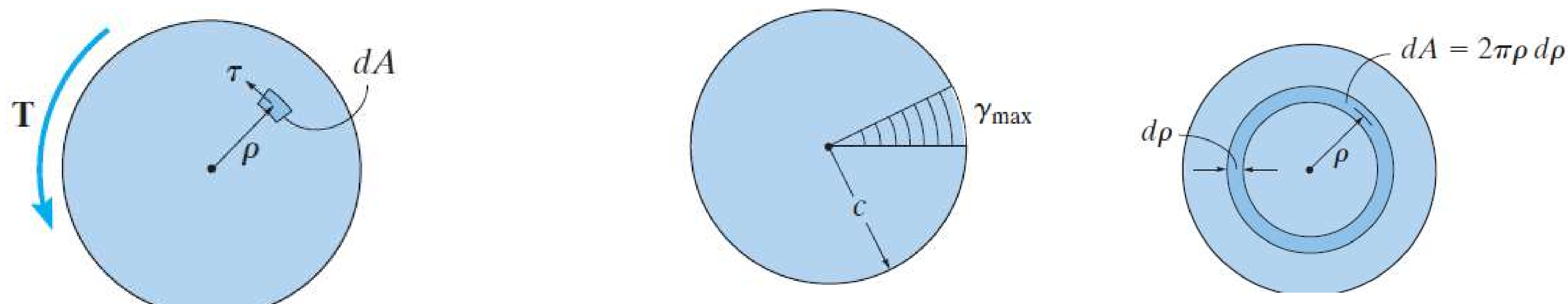
$$\tau_{\max} = K\frac{Tc}{J}; \quad \tau_{\max} = 1.3\left[\frac{30 \text{ N}\cdot\text{m}\,(0.020 \text{ m})}{(\pi/2)(0.020 \text{ m})^4}\right] = 3.10 \text{ MPa}$$



Amit Singh

Inelastic torsion

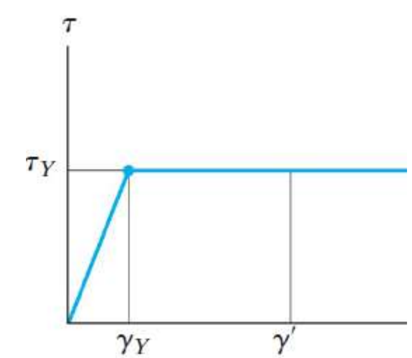
- **If applied torque is excessive, material may yield and Hooke's law is not valid**
- **But shear strain still varies linearly**



Linear shear–strain distribution

$$T = \int_A \rho \tau \, dA = 2\pi \int_0^c \tau \rho^2 \, d\rho$$

Elastic-Plastic Torque

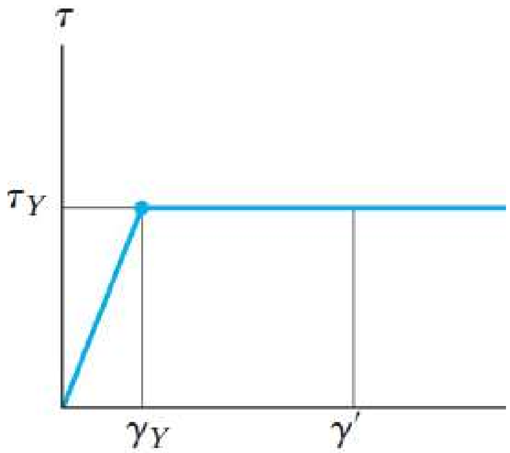


Amit Singh

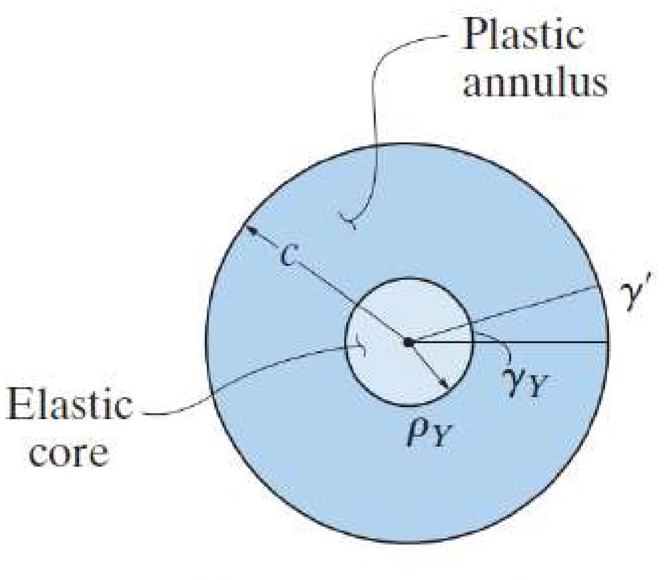
Elastic-Plastic Torque

- If the internal torque produces the maximum shear strain at the outer boundary, then the **maximum elastic torque** is

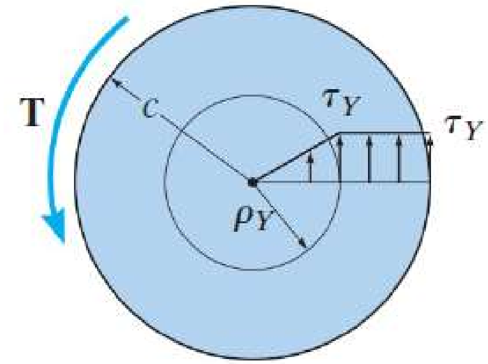
$$T_Y = \frac{\pi}{2} \tau_Y c^3 \qquad \tau_Y = T_Y c / [(\pi/2)c^4]$$



- If applied torque is more, then it starts yielding at the outer boundary.
- For elastic-plastic material, as the toqrue will increase the the yield boundary will shift towards center of the shaft
- This creates Elastic core and Plastic ring



Shear–strain distribution



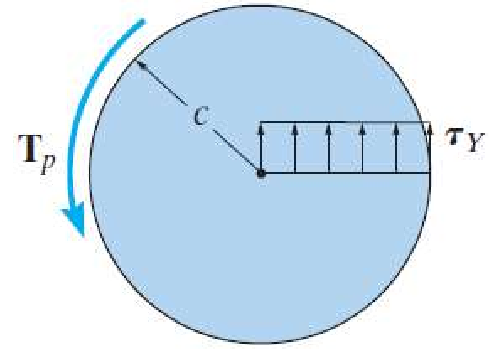
Shear–stress distribution

Elastic-Plastic Torque

- $$T = 2\pi \int_0^c \tau \rho^2 \, d\rho$$
  $$= 2\pi \int_0^{\rho_Y} \left(\tau_Y \frac{\rho}{\rho_Y} \right) \rho^2 \, d\rho + 2\pi \int_{\rho_Y}^{c} \tau_Y \rho^2 \, d\rho$$
  $$= \frac{2\pi}{\rho_Y} \tau_Y \int_0^{\rho_Y} \rho^3 \, d\rho + 2\pi\tau_Y \int_{\rho_Y}^{c} \rho^2 \, d\rho$$
  $$= \frac{\pi}{2\rho_Y} \tau_Y \rho_Y^4 + \frac{2\pi}{3} \tau_Y (c^3 - \rho_Y^3)$$
  $$= \frac{\pi \tau_Y}{6} (4c^3 - \rho_Y^3)$$

Plastic Torque

$$\rho_Y \rightarrow 0$$



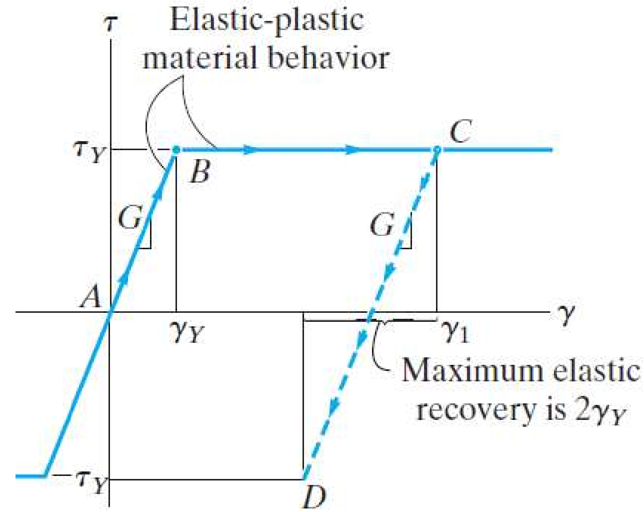
Fully plastic torque

$$T_p = 2\pi \int_0^c \tau_Y \rho^2 d\rho$$
$$= \frac{2\pi}{3} \tau_Y c^3$$
$$T_p = \frac{4}{3} T_Y$$

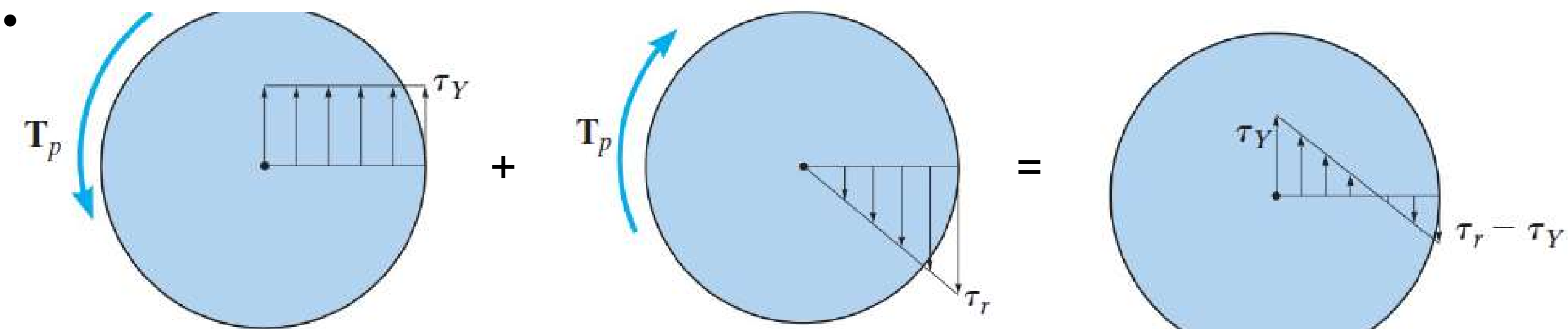
Perfectly plastic behavior: Shear stress distribution is constant over each radial line.

Amit Singh

Residual stress



- A shaft under plastic shear strain caused by torsion
- Removal of torque will cause some residual stress
- Elastic recovery after removal at C
- Draw the plastic torques stress distribution
- Superimpose a linear stress distribution by Tp in opposite direction
-



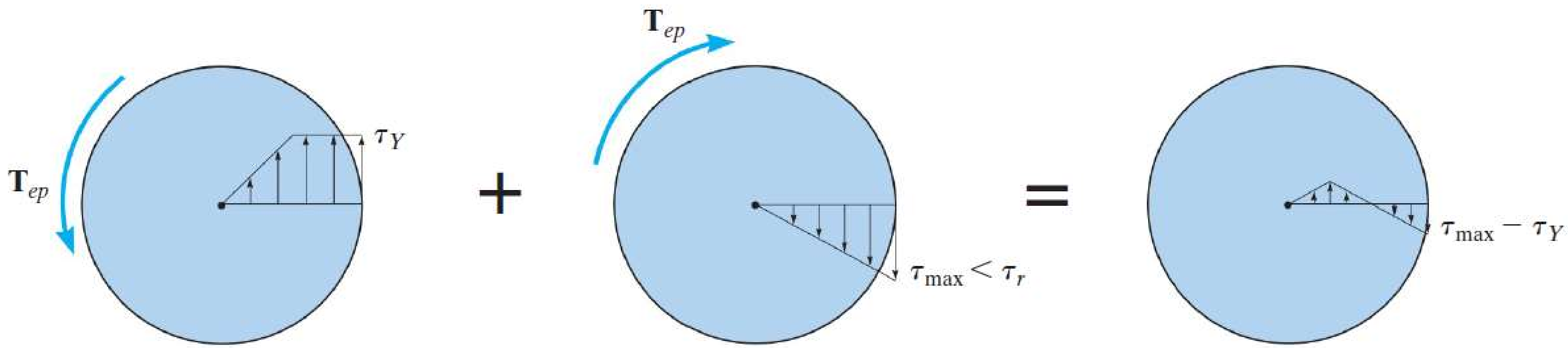
Plastic torque applied causing plastic shear strains throughout the shaft	Plastic torque reversed causing elastic shear strains throughout the shaft	Residual shear–stress distribution in shaft

$$\tau_r = \frac{T_p c}{J} = \frac{T_p c}{(\pi/2)c^4}$$

$$\tau_r = \frac{\left[(2/3)\pi\tau_Y c^3\right]c}{(\pi/2)c^4} = \frac{4}{3}\tau_Y$$

Residual stress

- The shear stress at the center is actually zero. So the shaft will never show perfectly plastic behavior and it goes under elastic-plastic behavior.
-

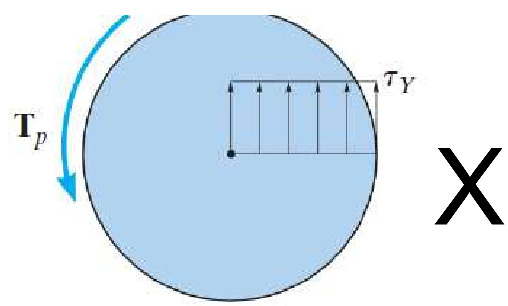


Elastic–plastic torque applied

Elastic–plastic torque reversed

Residual shear–stress distribution in shaft

(d)



Plastic torque applied causing plastic shear strains throughout the shaft

Amit Singh

Power Transmission

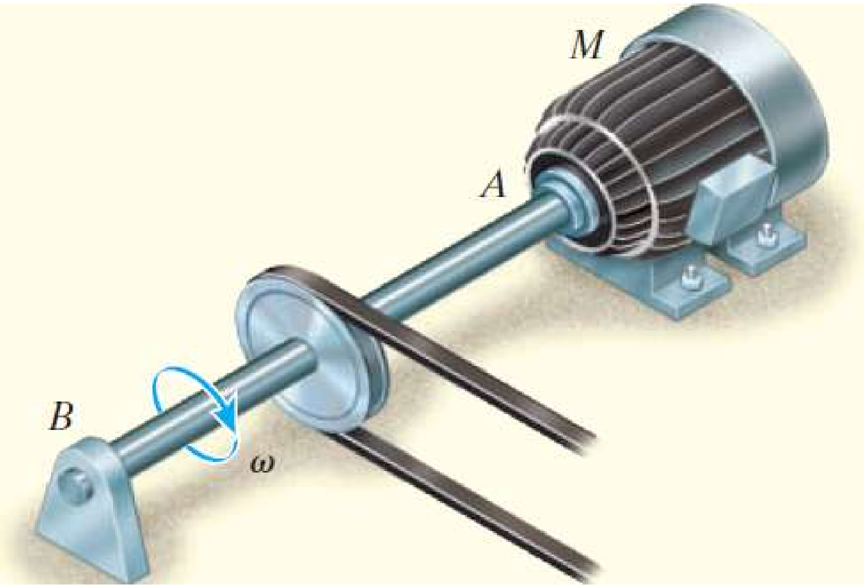
- Normally torque is not known to us in a machine but power is known

- Power: Work performed per unit of time

$$P = \frac{T\,d\theta}{dt} \qquad P = T\omega$$

- Shaft's angular speed is ω

- **Example** :



5 hp

$\omega = 175$ rpm

$\tau_{allow} = 14.5$ ksi

determine required diameter of the shaft

Power Transmission

- $$P = 5 \text{ hp}\left(\frac{550 \text{ ft}\cdot\text{lb/s}}{1 \text{ hp}}\right) = 2750 \text{ ft}\cdot\text{lb/s}$$

$$\omega = \frac{175 \text{ rev}}{\text{min}}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 18.33 \text{ rad/s}$$

$$T = 150.1 \text{ ft}\cdot\text{lb}$$

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

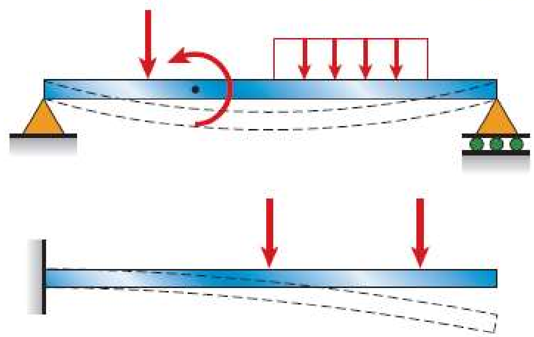
required diameter of the shaft 0.858 in

Bending

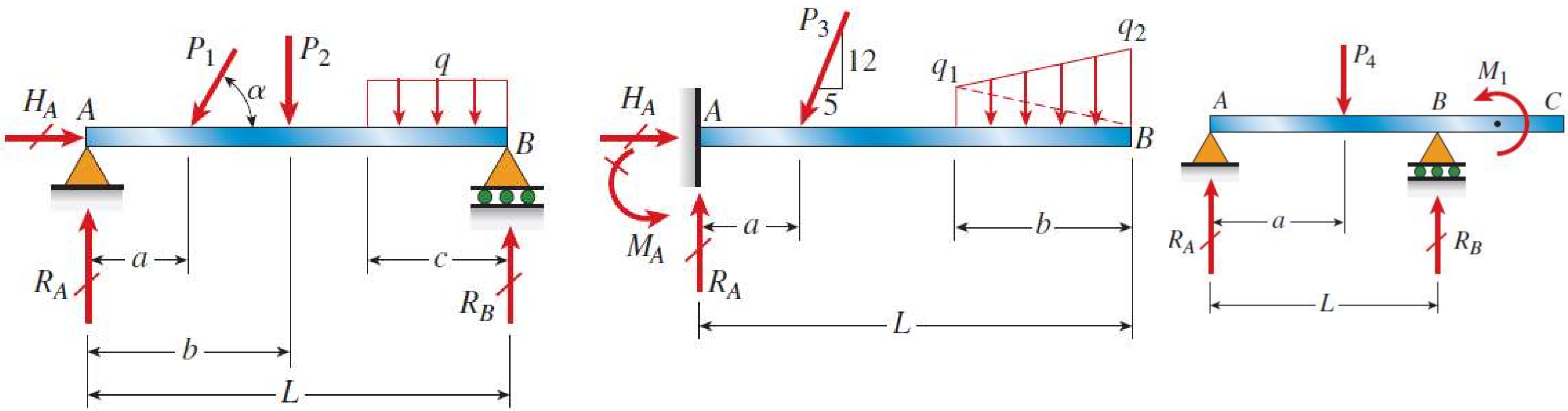
Bending

- **Beams under lateral loading**

Examples of beams subjected
to lateral loads

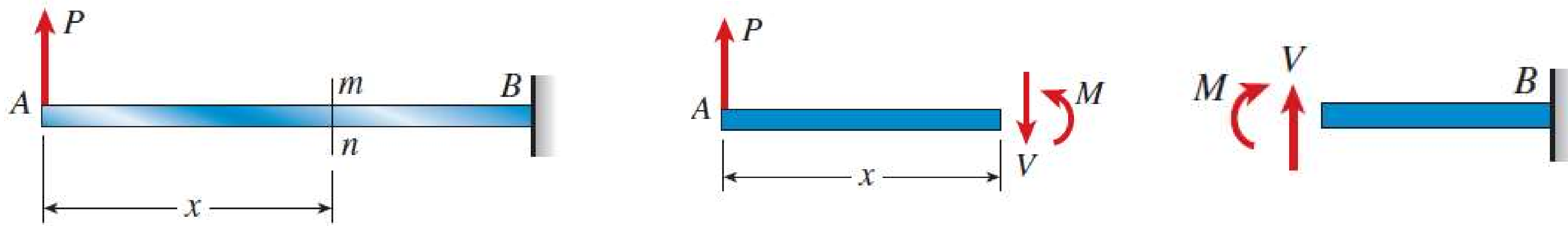


- **Types of beams: simple beam, cantilever beam, beam with a overhang**



pin support   roller support   fixed support

Bending: Shear force and bending moment

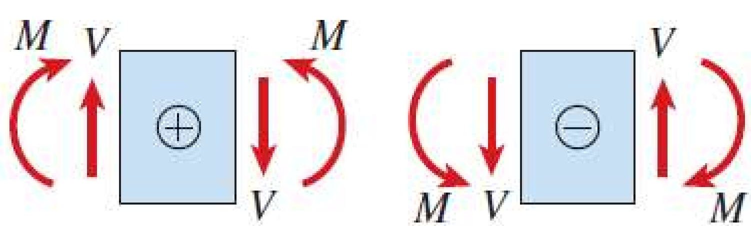
- 

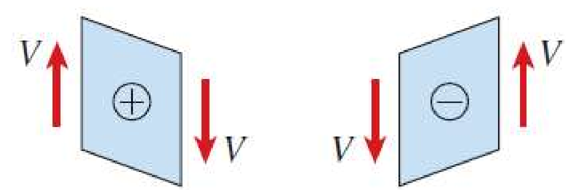
$$\Sigma F_{\text{vert}} = 0 \qquad P - V = 0 \quad \text{or} \quad V = P$$

$$\Sigma M = 0 \qquad M - Px = 0 \quad \text{or} \quad M = Px$$

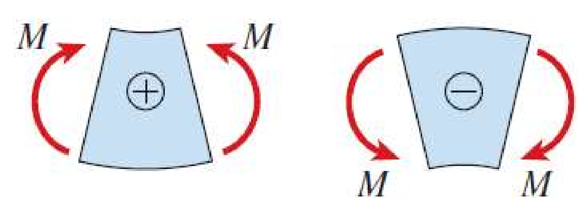
Deformations (highly exaggerated) of a beam element caused by (a) shear forces, and (b) bending moments

Sign conventions for shear force V and bending moment M



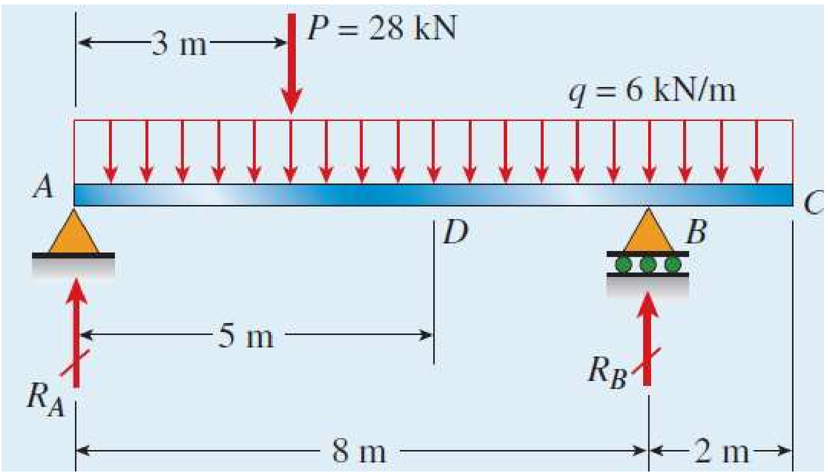


(a)



Shear force and bending moment

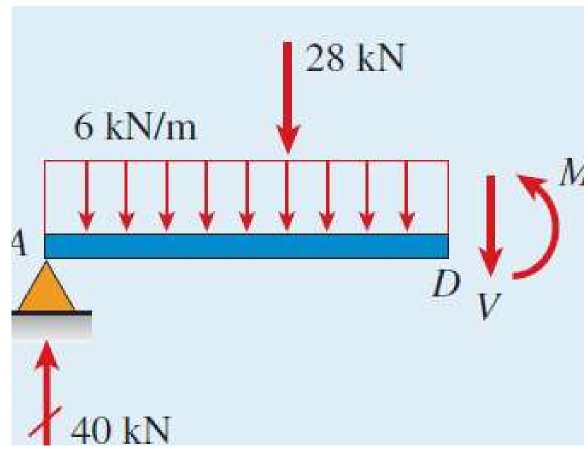
- **Example: Find V and M at D**



First write eqn of eqb for the entire body:

$R_A = 40\ \text{kN} \qquad R_B = 48\ \text{kN}$

Then, cut a section at D:



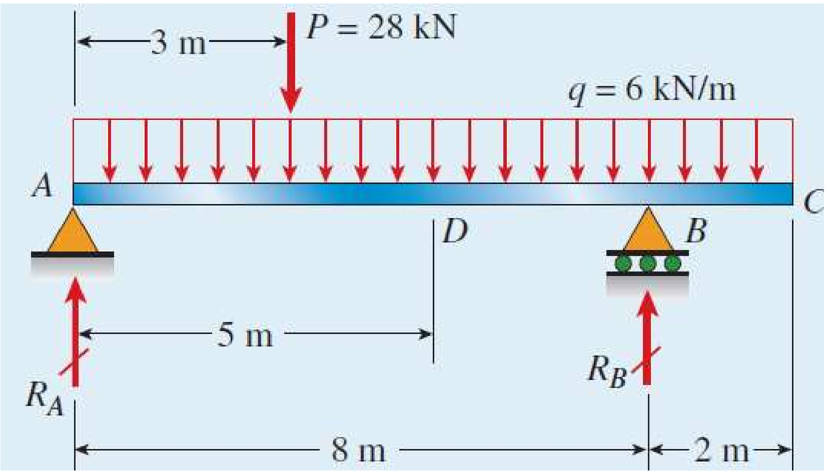
$$\Sigma F_{\text{vert}} = 0 \qquad 40\ \text{kN} - 28\ \text{kN} - (6\ \text{kN/m})(5\ \text{m}) - V = 0$$

$$\Sigma M_D = 0 - (40\ \text{kN})(5\ \text{m}) + (28\ \text{kN})(2\ \text{m}) + (6\ \text{kN/m})(5\ \text{m})(2.5\ \text{m}) + M = 0$$

$V = -18\ \text{kN} \qquad M = 69\ \text{kN}\cdot\text{m}$

Shear force and bending moment

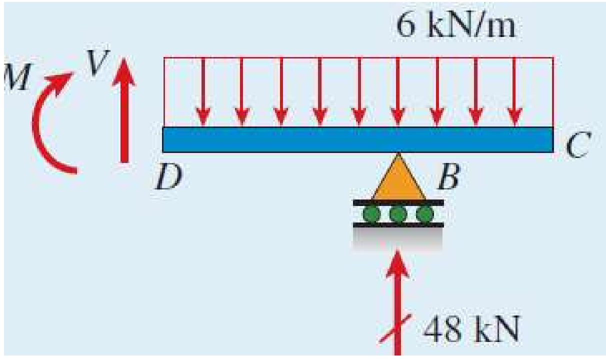
- **Example: Find V and M at D**



First write eqn of eqb for the entire body:

$R_A = 40\ \text{kN} \qquad R_B = 48\ \text{kN}$

Then, cut a section at D (alternately):



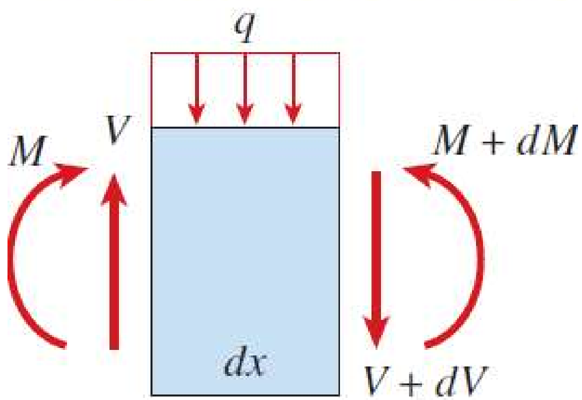
$$\Sigma F_{vert} = 0 \qquad V + 48\ \text{kN} - (6\ \text{kN/m})(5\ \text{m}) = 0$$

$$\Sigma M_D = 0 \qquad -M + (48\ \text{kN})(3\ \text{m}) - (6\ \text{kN/m})(5\ \text{m})(2.5\ \text{m}) = 0$$

$V = -18\ \text{kN} \qquad M = 69\ \text{kN} \cdot \text{m}$

Load, shear force and bending moment

- **Distributed Loads**



$$\Sigma F_{\text{vert}} = 0 \qquad V - q\,dx - (V + dV) = 0$$

$$\frac{dV}{dx} = -q$$

$$\int_A^B dV = -\int_A^B q\ dx$$

$$V_B - V_A = -\int_A^B q\ dx \quad = -(\text{area of the loading diagram between } A \text{ and } B)$$

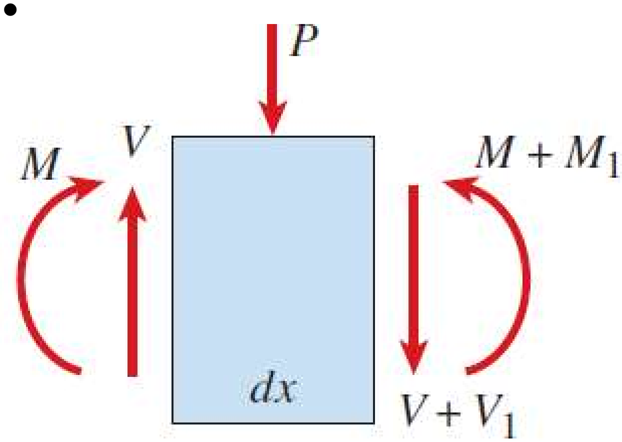
$$\boxed{\frac{dV}{dx} = -q, \qquad \frac{dM}{dx} = V}$$

Bending Moment

$$\Sigma M = 0 \qquad -M - q\ dx\left(\frac{dx}{2}\right) - (V + dV)dx + M + dM = 0$$

$$\frac{dM}{dx} = V \qquad \int_A^B dM = \int_A^B V\ dx \qquad M_B - M_A = \int_A^B V\ dx$$

Concentrated Loads

- 

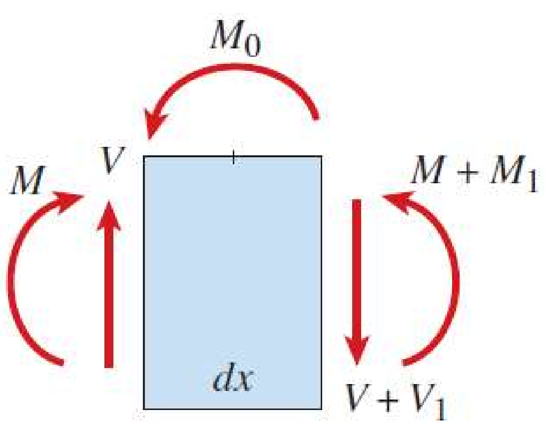
$$V - P - (V + V_1) = 0 \quad \text{or} \quad V_1 = -P$$

$$-M - P\left(\frac{dx}{2}\right) - (V + V_1)dx + M + M_1 = 0$$

$$M_1 = P\left(\frac{dx}{2}\right) + V\,dx + V_1\,dx$$

- **Bending moment does not change but shear force changes abruptly after the point of loading**

Loads in the Form of Couples

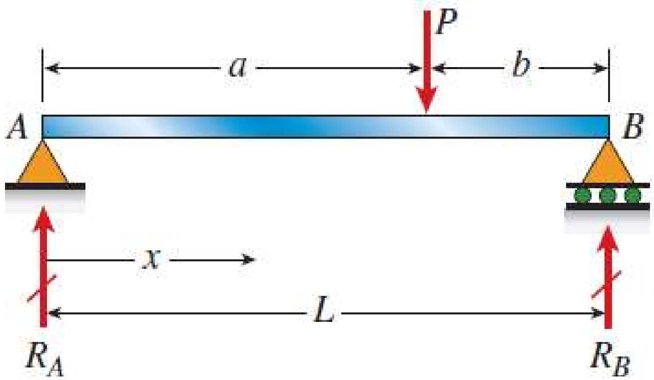


$$-M + M_0 - (V + V_1)dx + M + M_1 = 0$$

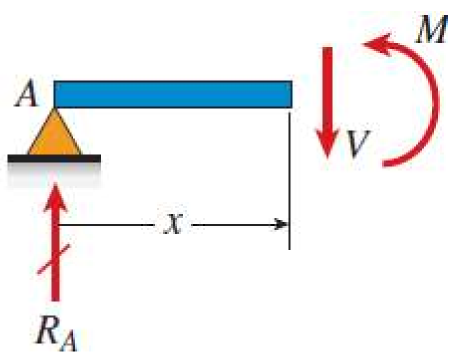
$$M_1 = -M_0$$

Shear and bending moment diagrams

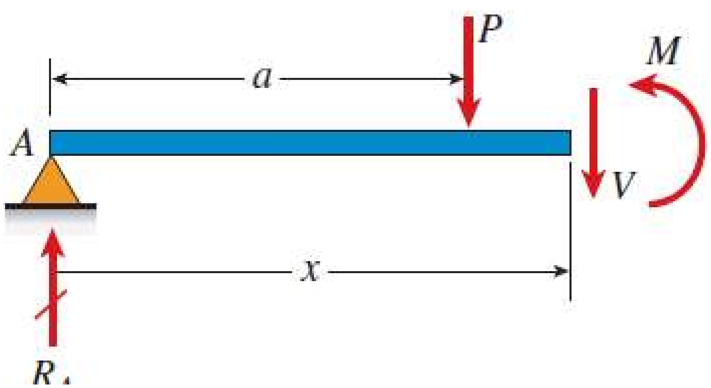
•



$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$



$$V = R_A = \frac{Pb}{L} \qquad M = R_A x = \frac{Pbx}{L} \qquad (0 < x < a)$$

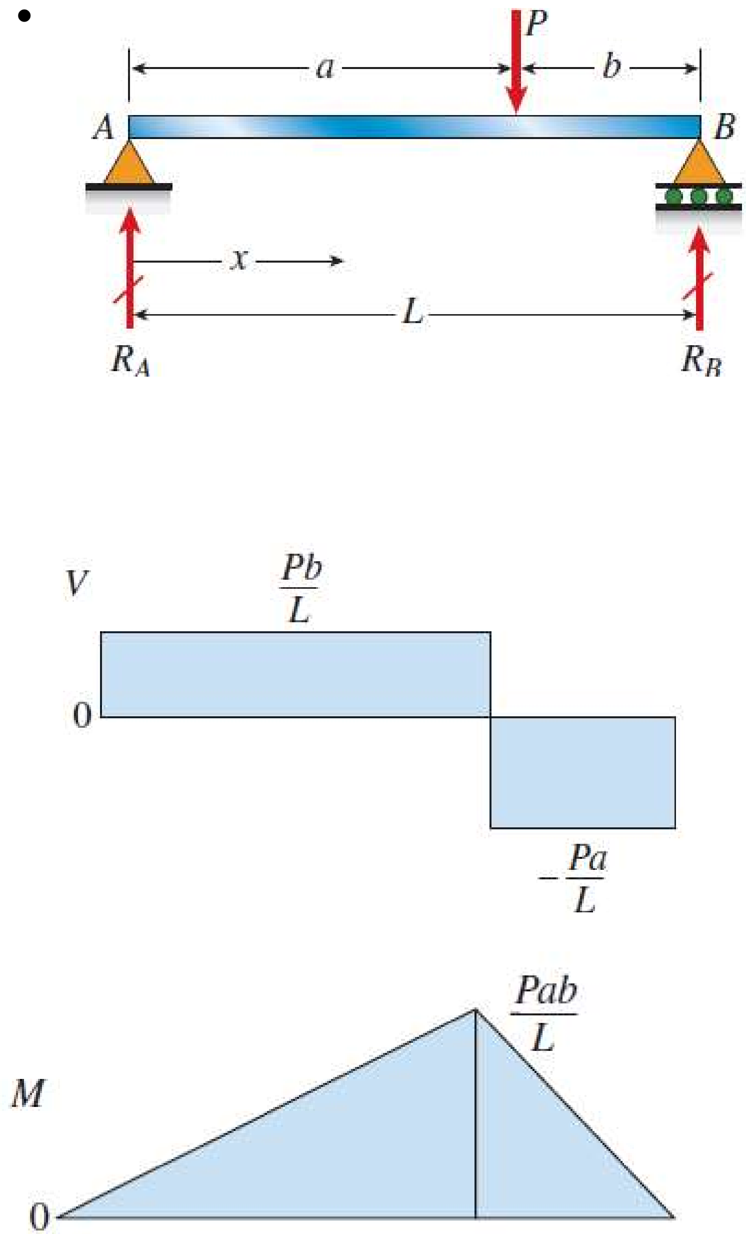


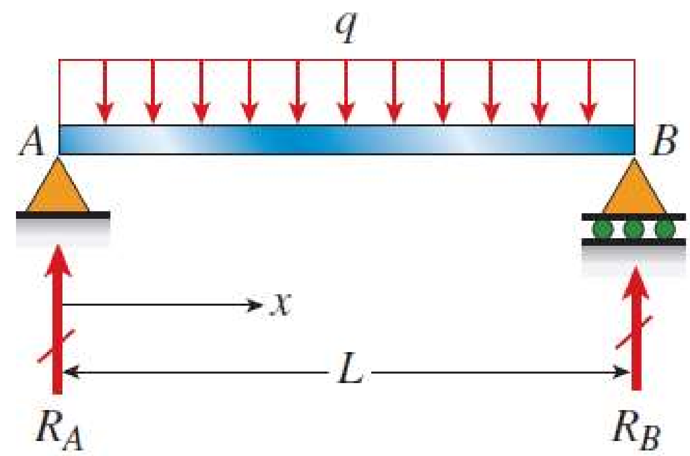
$$V = R_A - P = \frac{Pb}{L} - P = -\frac{Pa}{L} \qquad (a < x < L)$$

$$M = R_A x - P(x - a) = \frac{Pbx}{L} - P(x - a)$$

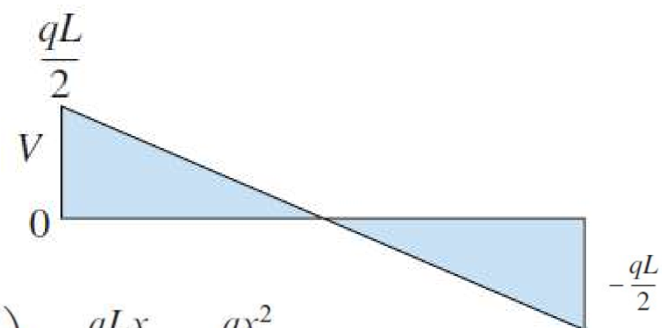
$$= \frac{Pa}{L}(L - x) \qquad (a < x < L)$$

Shear and bending moment diagrams

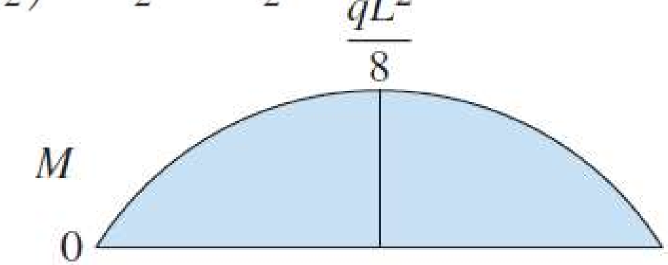
- 



$$V = R_A - qx = \frac{qL}{2} - qx$$

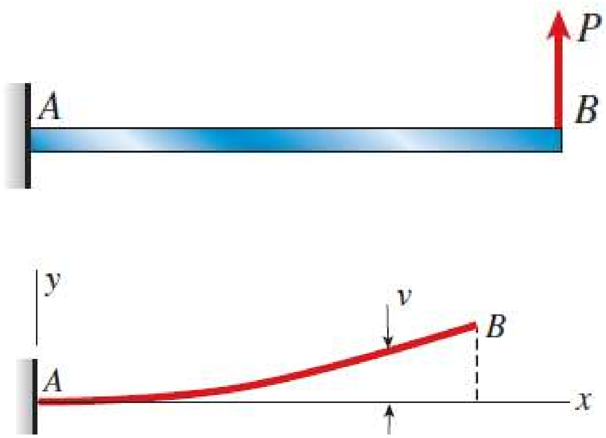


$$M = R_A x - qx\left(\frac{x}{2}\right) = \frac{qLx}{2} - \frac{qx^2}{2}$$



Stress: Bending

- **Bending of a cantilever beam:** Initial straight beam turns into a curved beam

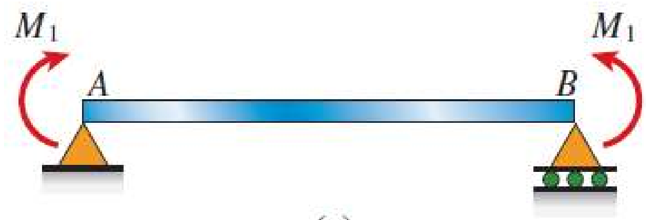


- The cross-section of the beam will be assumed as symmetric. All loads must act in x-y plane.

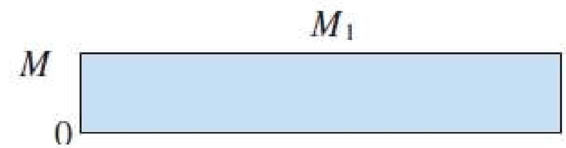
- So the bending deflections will occur in the same plane known as **plane of bending**

Stress: Bending

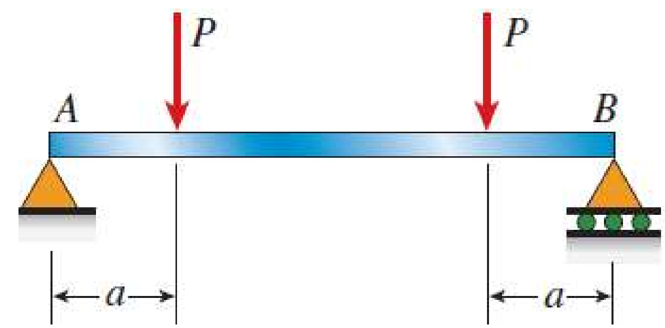
- **Pure Bending: Shear force is zero**

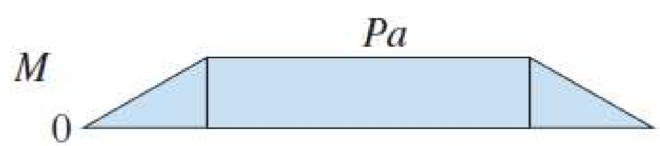


(a)



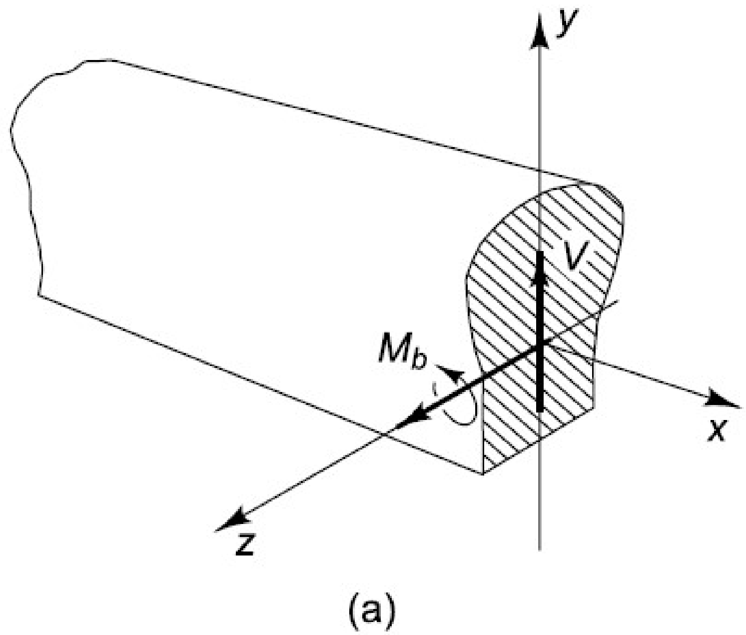
- **Nonuniform bending**: presence of shear forces



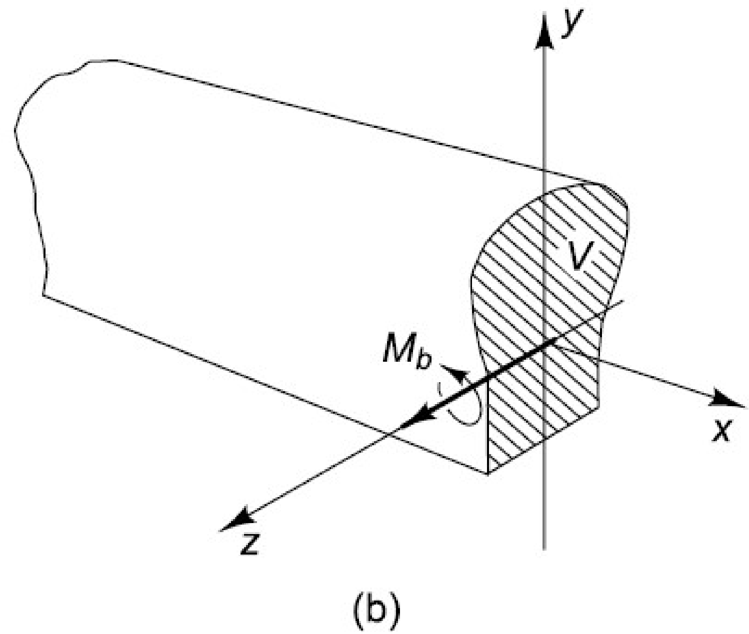


Stress: Bending

•



Non-uniform bending



Uniform bending, V =0

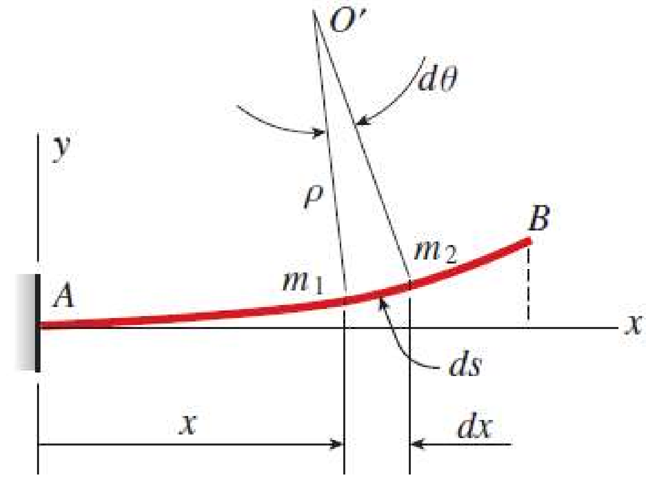
Curvature of a beam

- **Center of curvature O**

radius of curvature ρ

curvature $\kappa = \dfrac{1}{\rho}$

$$\rho\, d\theta = ds$$



deflection curve

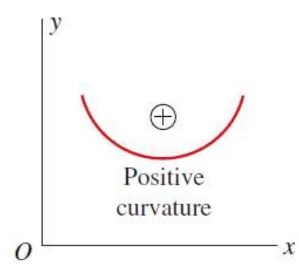
$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$

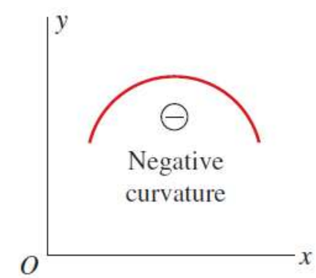
- When the curvature is conatsnt throuhout the length of the curve, we obtain an arc of a circle

- **Small deflection:** Usually the deflection is small, so the deflected curve can be assumed as flat

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}$$

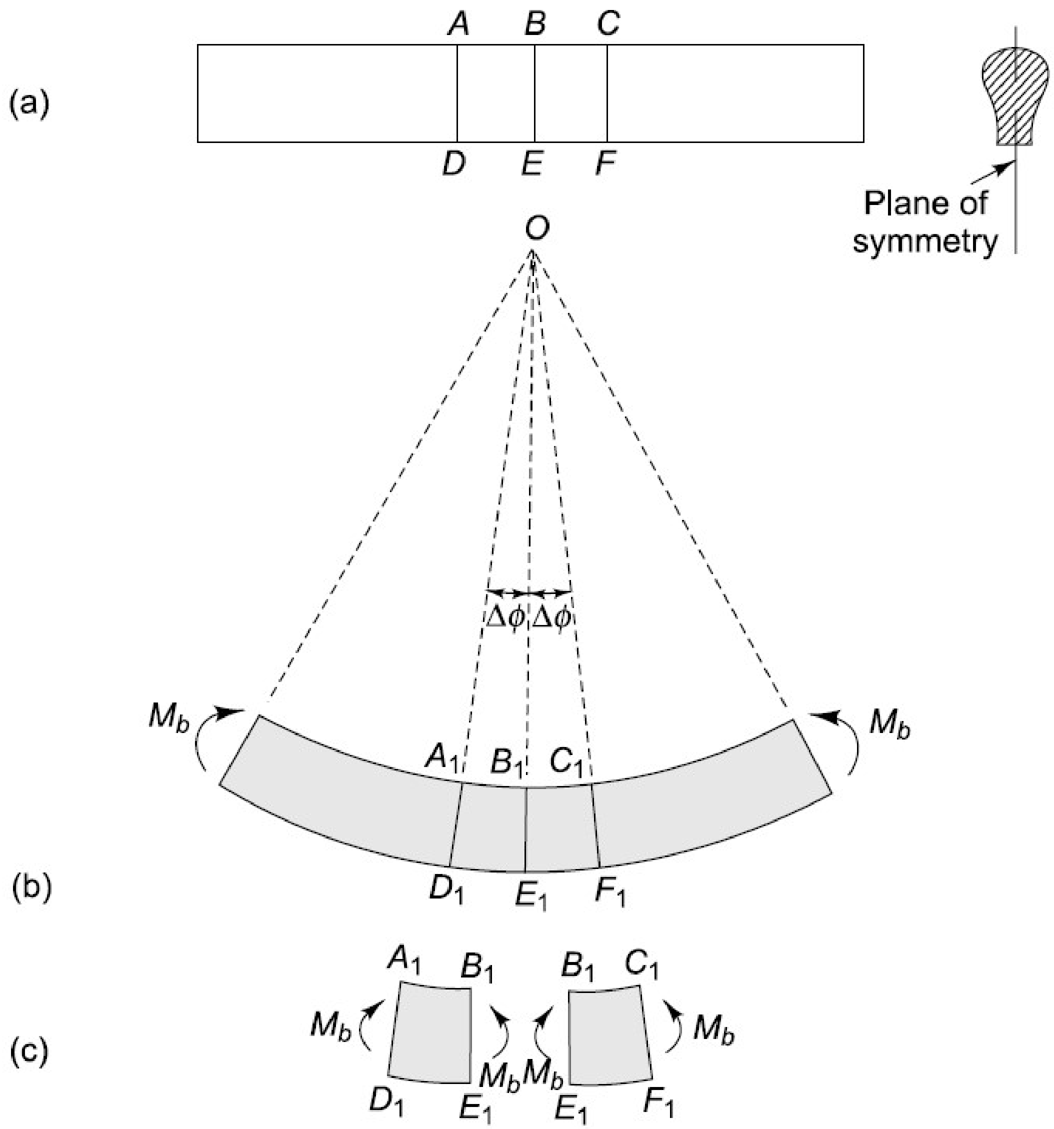
- **Sign convention for curvature**:





Stress: Bending

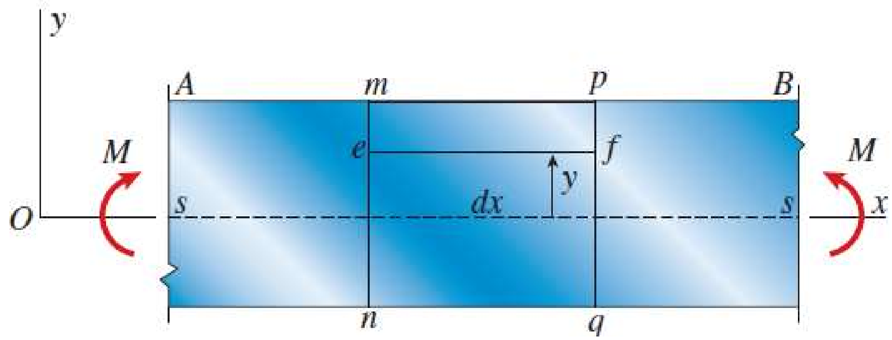
- **Take the case of pure bending: Plane under pure bending remains a plane**



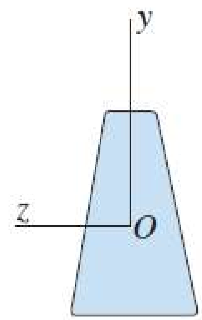
- Surfaces A1D1, B1E1, and C1F1 must be plane surfaces perpendicular to the plane of sym metry.
- In a plane of symmetry plane cross sections remain plane.

Longitudinal strain

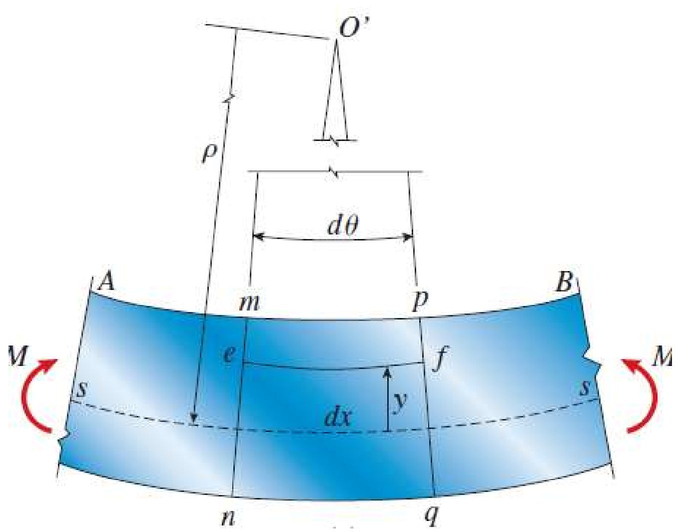
- **Take the case of pure bending: Plane sections under pure bending remains a plane**



side view of beam



cross section of beam

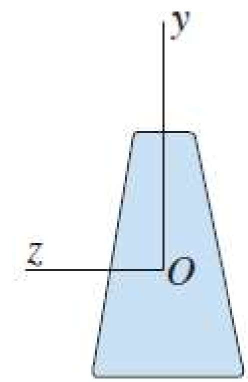


deformed beam

- Cross sections mn and pq rotate w.r.t each other about axis perpendicular to xy plane.
- Longitudinal lines on the lower part are elongated (tension) and on the upper part are shortened (compression).
- In between, there is a neutral surface **ss** on which longitudinal lines do not change length

Longitudinal strain

- **Neutral axis:** The intersection of neutral surface with any cross sectional plane. OZ in the figure

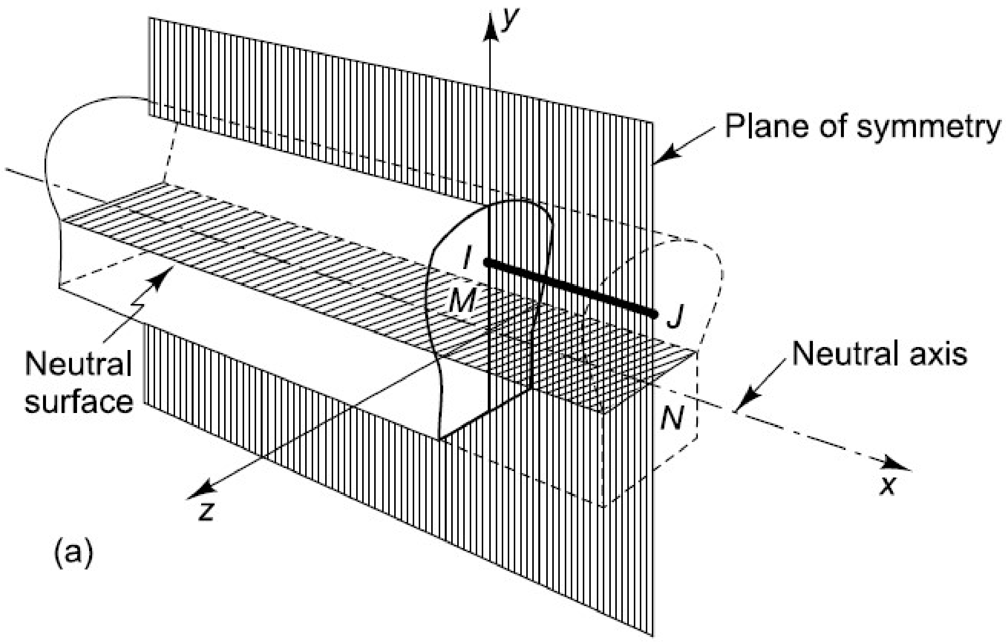


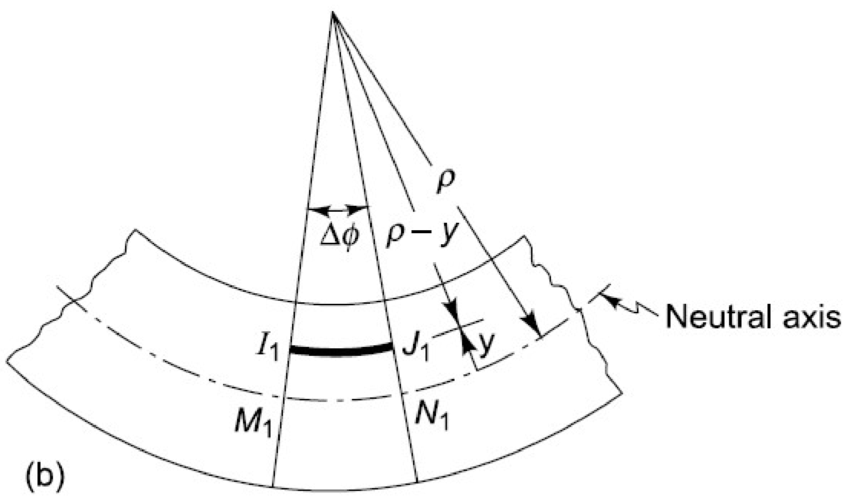
- Assume x-axis along the neutral surface of the undeformed beam
- Take "ef" and calculate strain
- Length L1 of "ef" after deflection is given by

$$L_1 = (\rho - y)d\theta = dx - \frac{y}{\rho}dx \qquad d\theta = dx/\rho$$

- Original length is dx, so elongation = $L_1 - dx$ = $-y\,dx/\rho$
- Therefore, longitduinal strain is (L1-L0)/L0
- $$\varepsilon_x = -\frac{y}{\rho} = -\kappa y$$
- There will also be transverse strain due to Poisson's ratio. But no transverse stresses. Logitudinal elements of a beam in pure bending is in uniaxial stress

Stress and strain: Bending





$$\varepsilon_x = -\frac{y}{\rho} = -\kappa y$$

- This derivation is strictly for the points belonging to the plane of symmetry
- Though we assume that it applied to all points in the cross-section of the beam.
- Since plane sections remain plane, so for all points in the cross-section of the beam:

$$\gamma_{xy} = \gamma_{xz} = 0$$

- For linear isotropic elastic material

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = -\frac{y}{\rho}$$

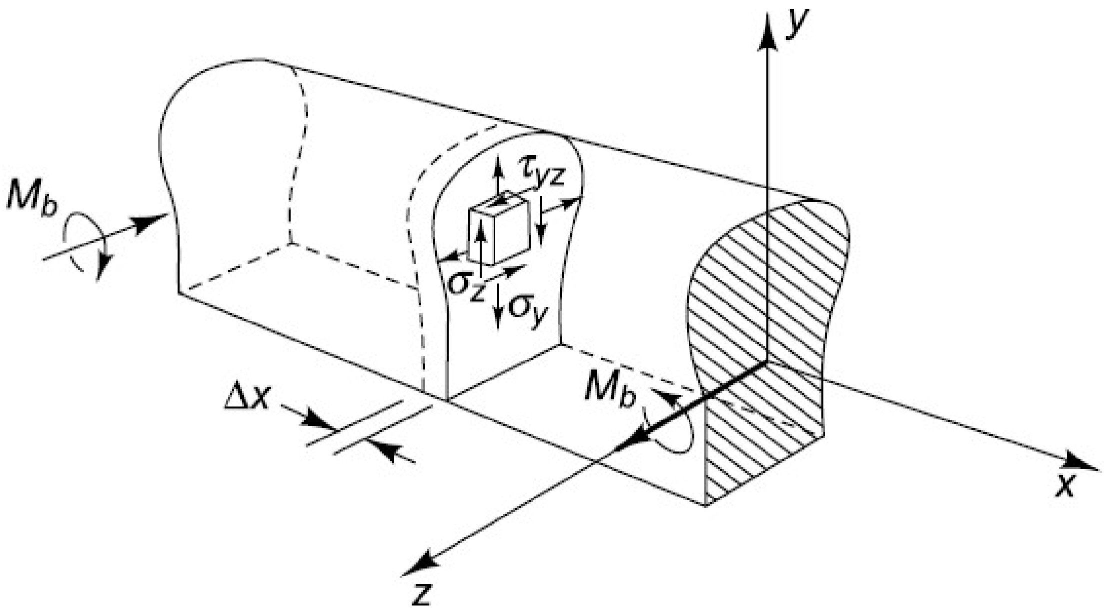
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = 0$$

- We cannot say anything about ϵ_y, ϵ_x, and γ_{yz}
- They must be symmetrical w.r.t xy plane

Transverse Stress: Pure Bending

•



The transverse stresses σ_y, σ_z, and τ_{yz} are assumed to be zero

Transverse Strain

Generalized Hooke's law:

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

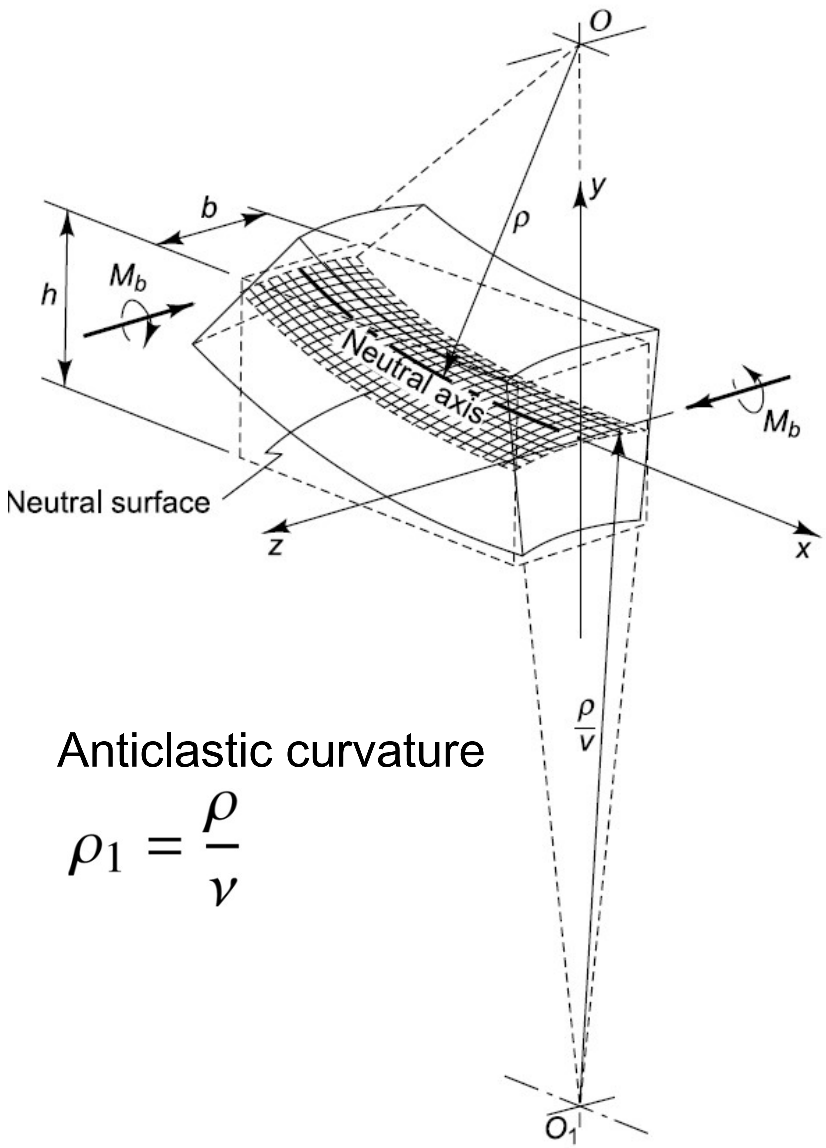
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$



Anticlastic curvature

$$\rho_1 = \frac{\rho}{\nu}$$

$$\epsilon_y = \nu\frac{M_b y}{EI_{zz}} = -\nu\epsilon_x$$

$$\epsilon_x = -\frac{y}{\rho}$$

$$\epsilon_z = \nu\frac{M_b y}{EI_{zz}} = -\nu\epsilon_x$$

$$\epsilon_{zz} = \epsilon_z = \frac{y}{\rho_1}$$

$$\gamma_{yz} = 0$$

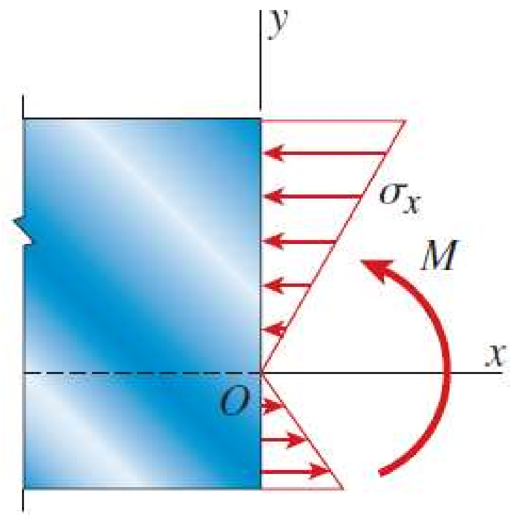
Anticlastic curvature examples

Heydar Aliyev Center, Baku, Azerbaijan

Zaha Hadid

Normal stresses in bending

- **For linearly elastic material, using Hooke's law:** $$\sigma_x = E\varepsilon_x = -\frac{Ey}{\rho} = -E\kappa y$$
- Varies linearly with y.
- We must locate neutral axis to calculate y

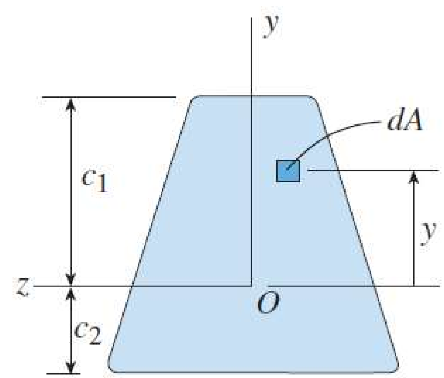


- **Force equilibrium:**

$$\int_A \sigma_x dA = -\int_A E\kappa y dA = 0$$

$$\int_A y\, dA = 0$$

first moment of the area of the cross sectior
evaluated with respect to the z axis, is zero



- When Hooke's law is followed and ***no axial force*** is present then
- Neutral axis passes through the centroid of the plane
- We already assume that y-axis is an axis of symmetry.
- So y-axis also passes through the centroid.
- Origin is centroid.

Centroid

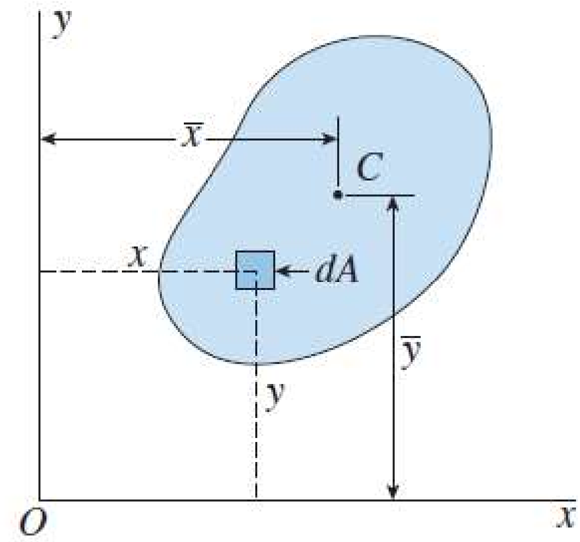
- **Area** $A = \int dA$

- First moments of area w.r.t x and y axes

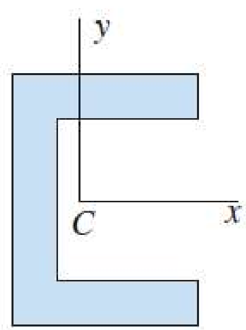
$$Q_x = \int y \, dA \qquad Q_y = \int x \, dA$$

- Coordinates of the centroid are given by

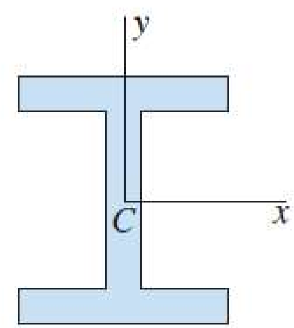
$$\bar{x} = \frac{Q_y}{A} = \frac{\int x \, dA}{\int dA} \qquad \bar{y} = \frac{Q_x}{A} = \frac{\int y \, dA}{\int dA}$$



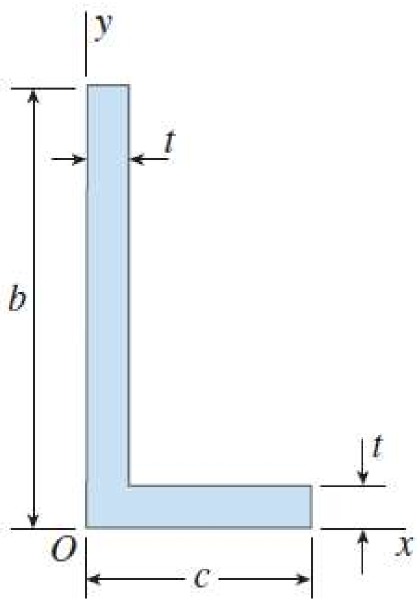
Area with one axis of symmetry



Area with two axes of symmetry



Centroids of composite areas

- 

$$A = \sum_{i=1}^{n} A_i \qquad Q_x = \sum_{i=1}^{n} \bar{y}_i A_i \qquad Q_y = \sum_{i=1}^{n} \bar{x}_i A_i$$

The **coordinates of the centroid** of the composite area

$$\bar{x} = \frac{Q_y}{A} = \frac{\sum_{i=1}^{n} \bar{x}_i A_i}{\sum_{i=1}^{n} A_i} \qquad \bar{y} = \frac{Q_x}{A} = \frac{\sum_{i=1}^{n} \bar{y}_i A_i}{\sum_{i=1}^{n} A_i}$$

-

$$A_1 = +bt \qquad \bar{x}_1 = \frac{t}{2} \qquad \bar{y}_1 = \frac{b}{2}$$

$$A_2 = (c - t)t \qquad \bar{x}_2 = \frac{c + t}{2} \qquad \bar{y}_2 = \frac{t}{2}$$

$$A = A_1 + A_2 = t(b + c - t)$$

$$Q_x = \bar{y}_1 A_1 + \bar{y}_2 A_2 = \frac{t}{2}(b^2 + ct - t^2)$$

$$Q_y = \bar{x}_1 A_1 + \bar{x}_2 A_2 = \frac{t}{2}(bt + c^2 - t^2)$$

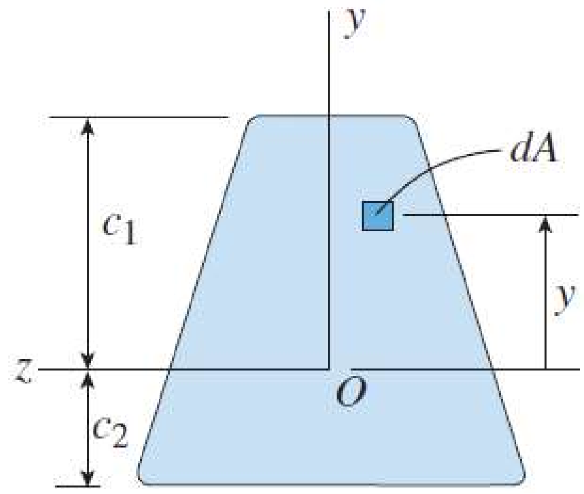
$$\bar{x} = \frac{Q_y}{A} = \frac{bt + c^2 - t^2}{2(b + c - t)}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{b^2 + ct - t^2}{2(b + c - t)}$$

Moment-curvature equation

- $$dM = -\sigma_x y \; dA$$

$$M = -\int_A \sigma_x y \, dA = \int_A \kappa E y^2 \, dA = \kappa E \int_A y^2 \, dA$$



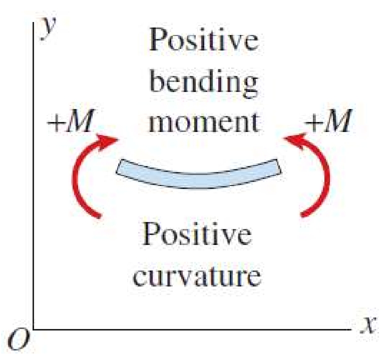
$$M = \kappa E I$$

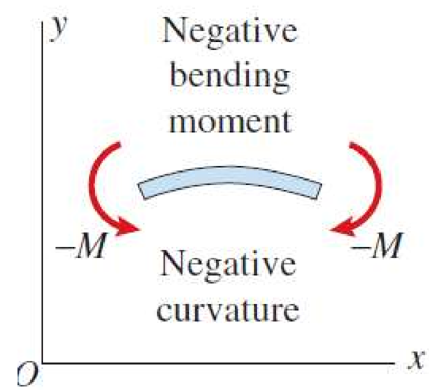
$$I = \int_A y^2 \; dA$$

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

Known as the **moment-curvature equation**

flexural rigidity EI

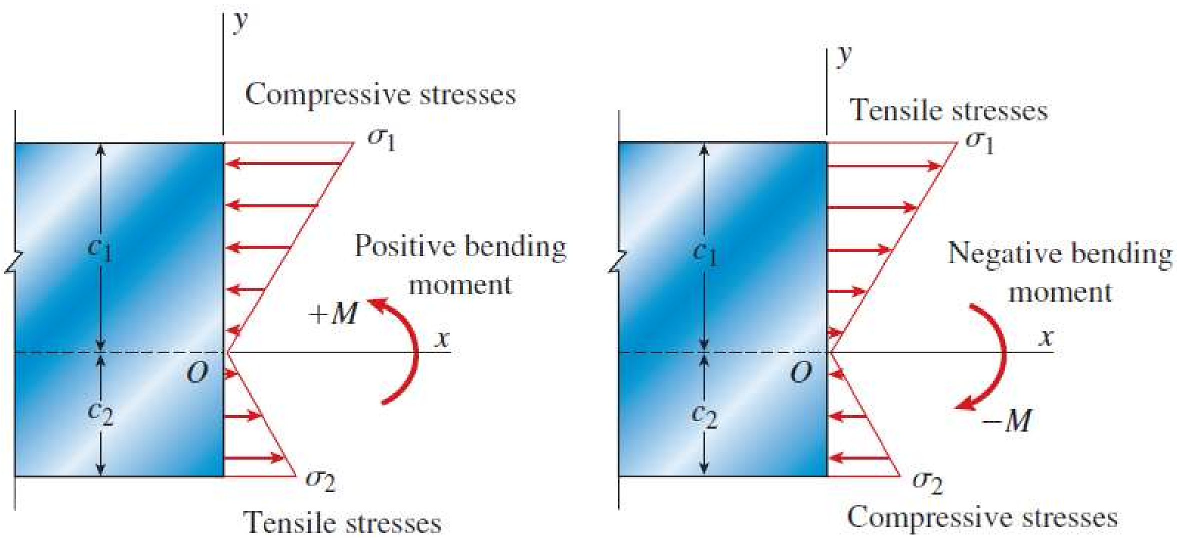




Flexure formula

- **Bending or flexural stresses:** vary linearly with y, directly proportional to bending moment M and inversely proportional to I

$$\sigma_x = -\frac{My}{I}$$



- **Maximum bending stress**: Find y farthest from the neutral axis, c1 and c2

$$\sigma_1 = -\frac{Mc_1}{I} = -\frac{M}{S_1} \qquad \sigma_2 = \frac{Mc_2}{I} = \frac{M}{S_2} \qquad S_1 = \frac{I}{c_1} \qquad S_2 = \frac{I}{c_2}$$

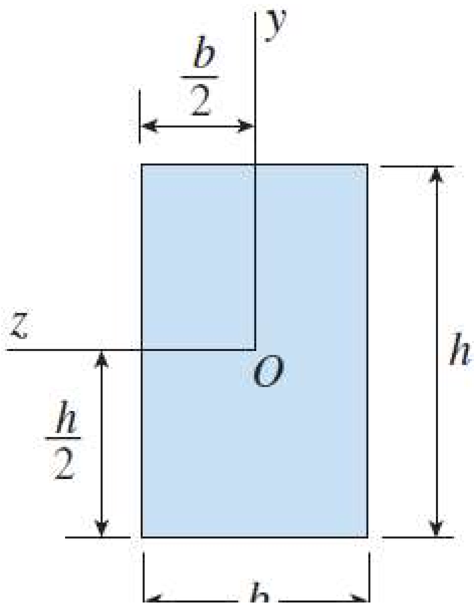
section moduli

Flexure formula

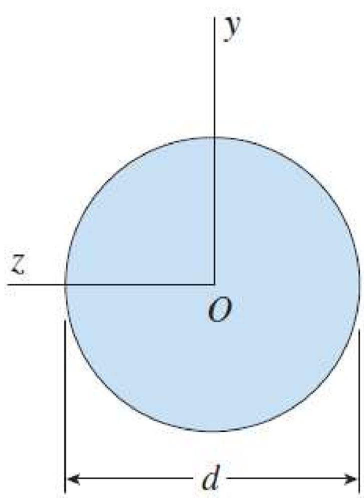
- **Doubly symmetric beams:** $c_1 = c_2 = c$

$$\sigma_1 = -\sigma_2 = -\frac{Mc}{I} = -\frac{M}{S}$$

$$S = \frac{I}{c}$$



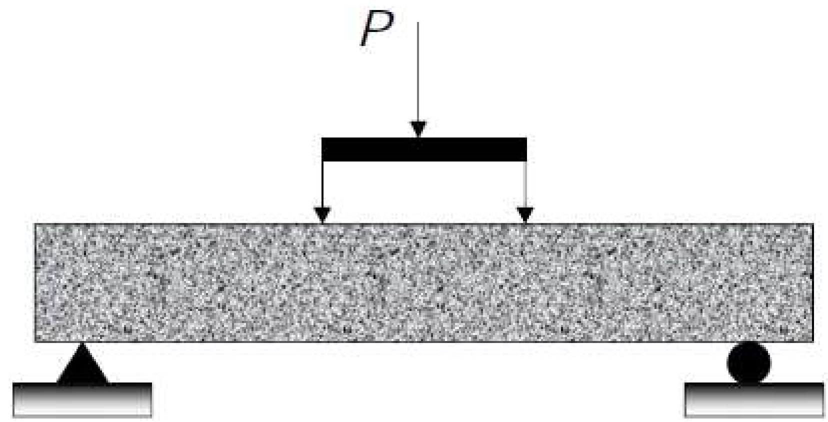
$$I = \frac{bh^3}{12} \qquad S = \frac{bh^2}{6}$$



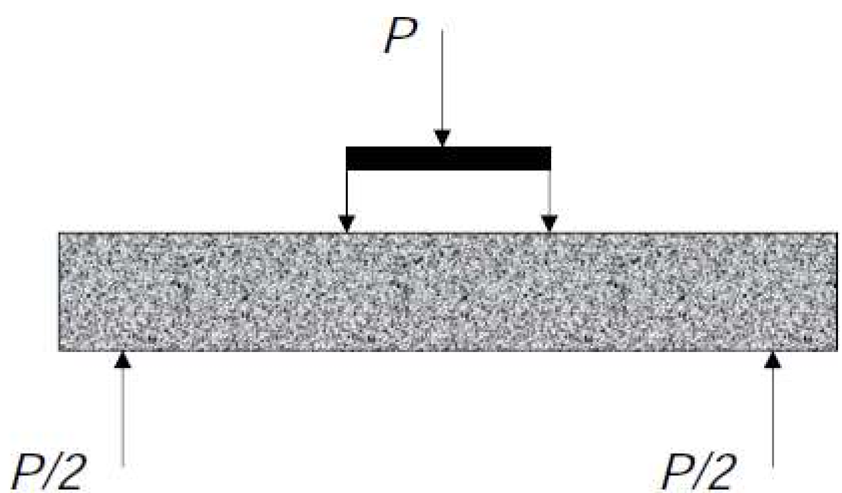
$$I = \frac{\pi d^4}{64} \qquad S = \frac{\pi d^3}{32}$$

Reinforced concrete

Reinforced concrete



The purpose of RC is the reinforcement of areas in concrete that are weak in tension



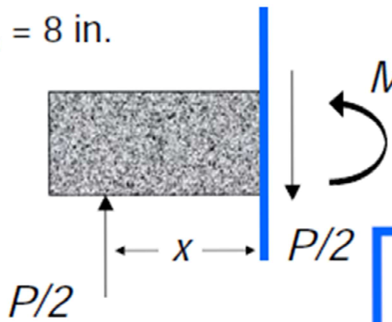
Amit Singh

Reinforced concrete beams

Let's look at the internal moment at section between the supports and applied load

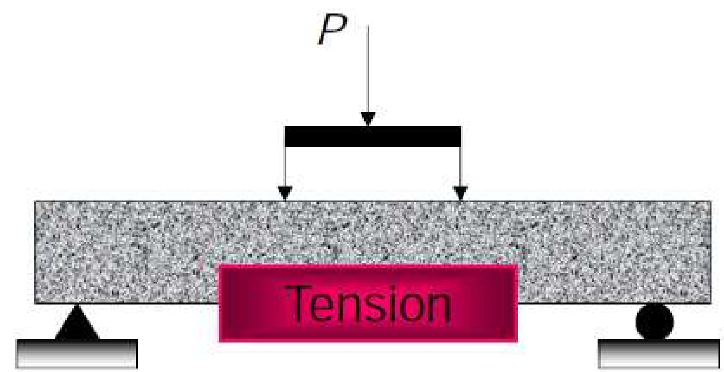
$$\sum M = \frac{P}{2} x$$

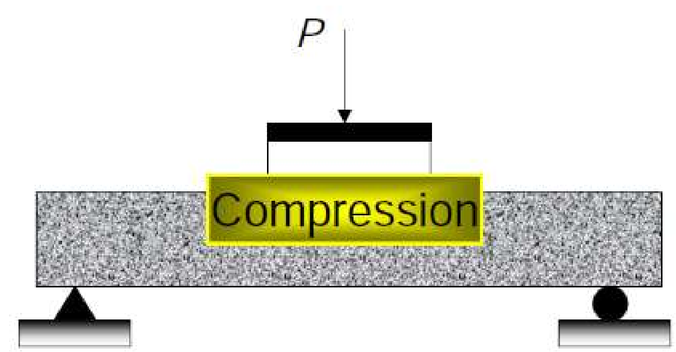
$X_{max} = 8$ in.



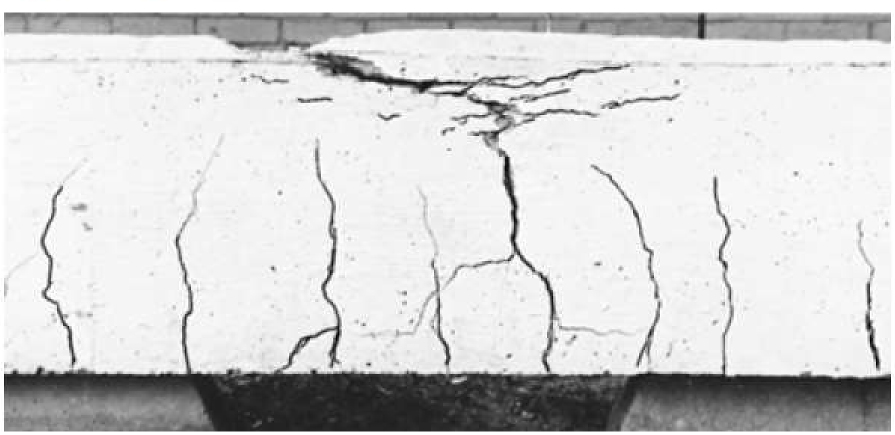
M is the bending moment

$$\boxed{M = 4P \text{ (lb.-in.)}}$$



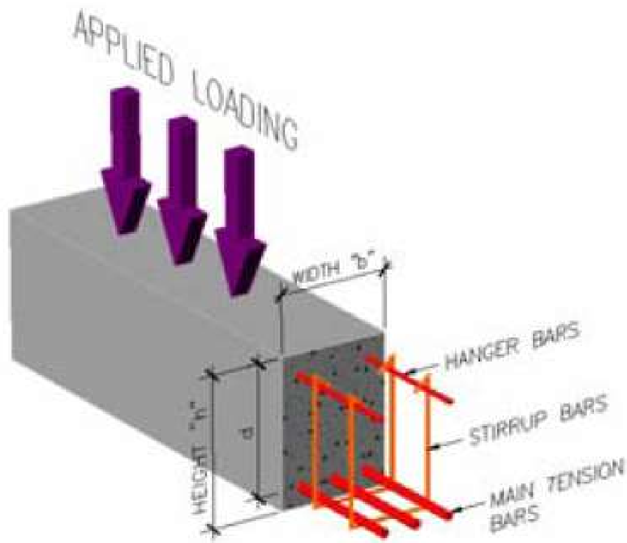


Compression and tension failures in a reinforced concrete beam

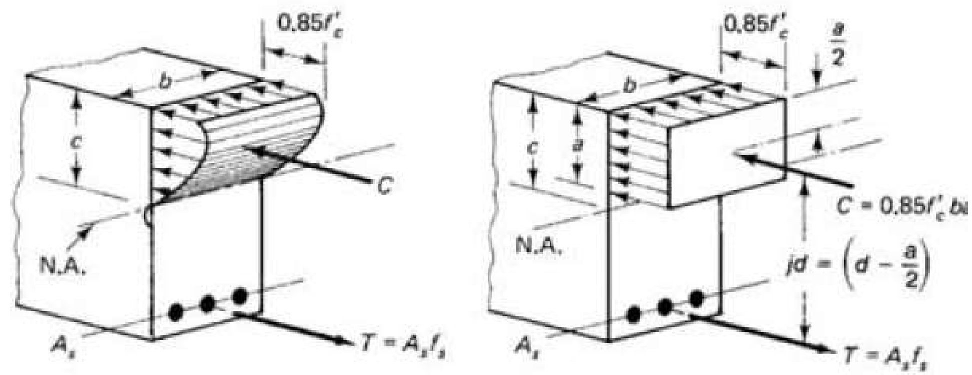


Whitney rectangular compressive stress

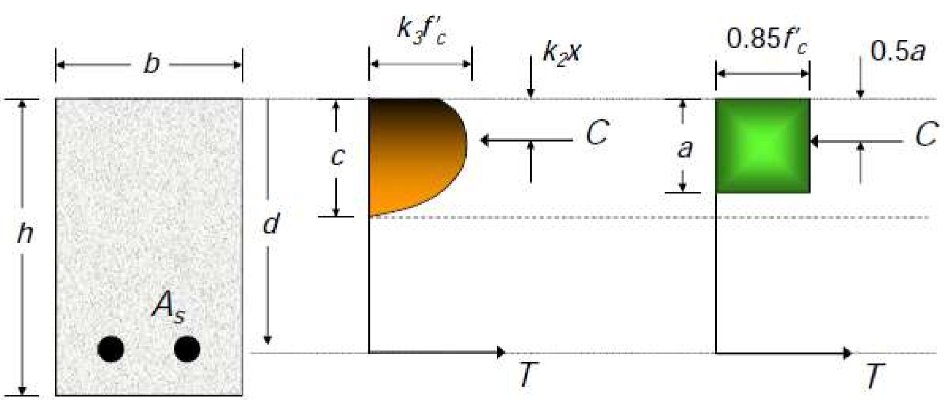
Typical rebar configuration to handle tension and shear loads



In the 1930s, Whitney proposed the use of a rectangular compressive stress distribution



Assume that the concrete contributes nothing to the tensile strength of the beam



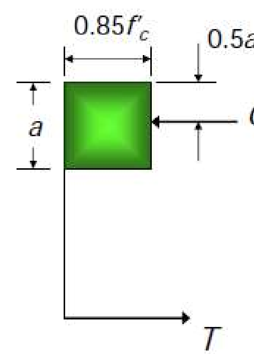
The values of the tension and compression forces are:

$$C = 0.85f'_c\, ba$$

$$T = A_s f_y$$

$$\sum F = 0 = T - C$$

$$a = \frac{A_s f_y}{0.85f'_c\, b}$$

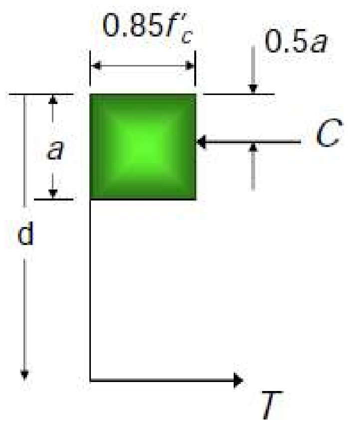


Amit Singh

Whitney rectangular compressive stress

If the tension force capacity of the steel is too high, than the value of ***a*** is large

$$a = \frac{A_s f_y}{0.85 f'_c\, b}$$



If $a > d$, then you have too much steel

$$\sum M = T\left(d - \frac{a}{2}\right)$$

$$M = A_s f_y \left(d - \frac{a}{2}\right)$$

$$M = A_s f_y \left(d - 0.59 \frac{A_s f_y}{f'_c\, b}\right)$$

$$M = 4P$$

$$P_{tension} = \frac{A_s f_y}{4}\left(d - 0.59 \frac{A_s f_y}{f'_c b}\right)$$

Amit Singh

Skateboard

The young skateboarder smashing stereotypes in India

Kamali power

https://www.huckmag.com/art-and-culture/film-2/kamali-the-young-skateboarder-smashing-stereotypes-in-india/

Amit Singh

Skateboard bending

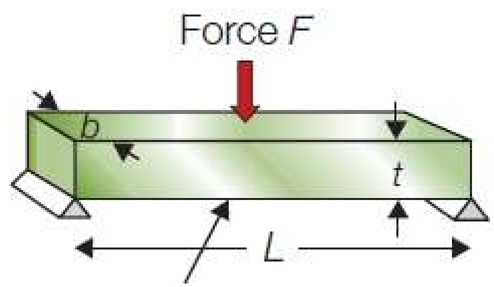
- **Skateboard size:** 32’’ inches long, 8 ‘’ wide, 1/2’’ inches thick

$$I = \frac{bh^3}{12} = 3.47\ \text{cm4}$$

Amit Singh

Skateboard bending

- **Skateboard size:** 32'' inches long, 8 '' wide, 1/2'' inches thick $\quad \mathrm{I} = \frac{bh\text{^}3}{12}$ =3.47 cm4



$$\sigma_{\text{flex}} = \frac{3F_f L}{2b\,t^2}$$

- Let us assume that typical human weight is 70 kg, so the corresponding load is around 690 N.

- So the maximum bending moment is (690/2)N*40.64 cm = 140.2 Nm

- The corresponding maximum bending stress is $\sigma = \frac{MY}{I}$ = 26 MPa

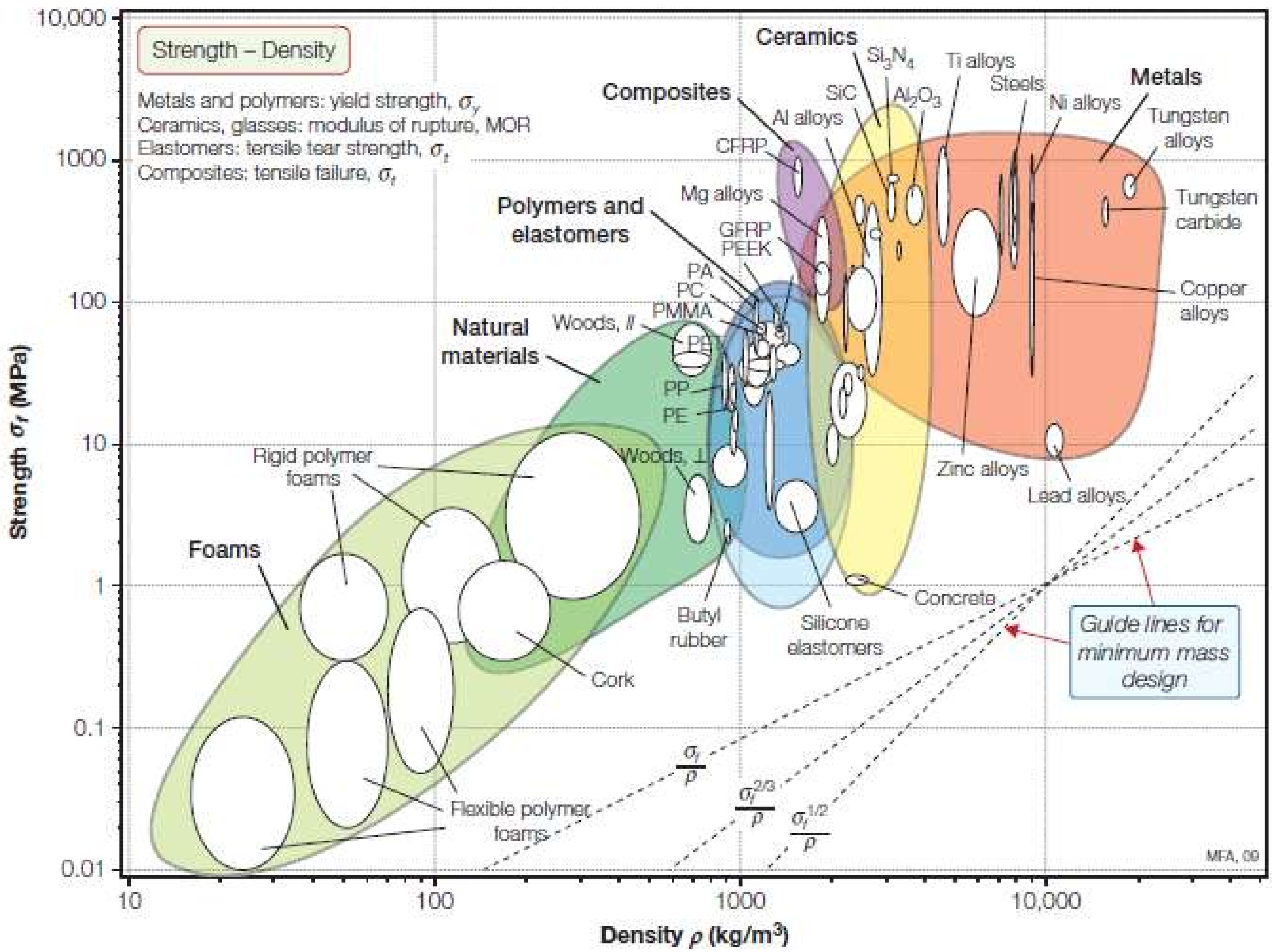
- Let us assume that maximum weight of skateboard is around 2 kg. So the density should be 2 kg/(32*8*0.5*2.54^3 cm^3) = 1000 kg/m^3

- Find the material for which density should not be more than the density chosen above and strength should be larger than calculated bending moment. Optimize $\frac{\sigma}{\rho}$

Amit Singh

Material selection

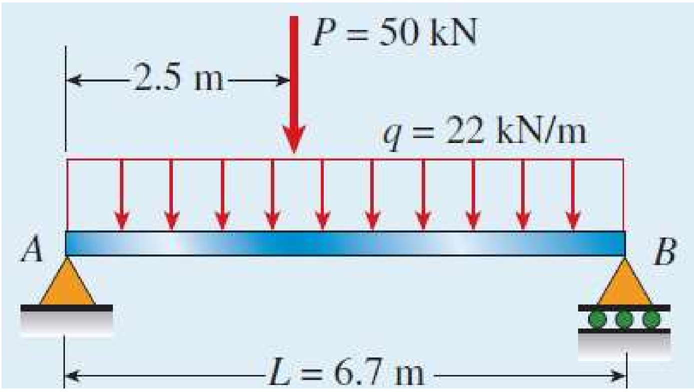
- Ashby, M. F. Materials Selection in Mechanical Design. Butterworth-Heinemann, 2011

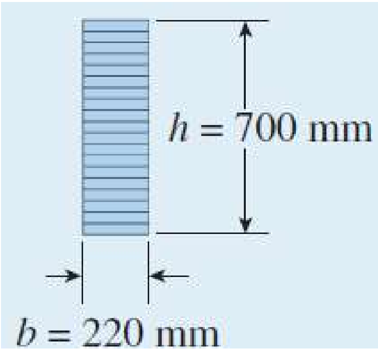


Amit Singh

Example

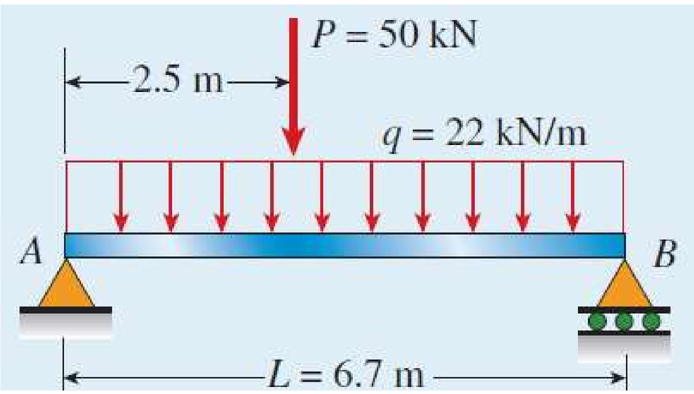
- (a) Determine the maximum tensile and compressive stresses in the beam due to bending.
  (b) If load q is unchanged, find the maximum permissible value of load P if the allowable normal stress in tension and compression is $\sigma_a = 13$ MPa.

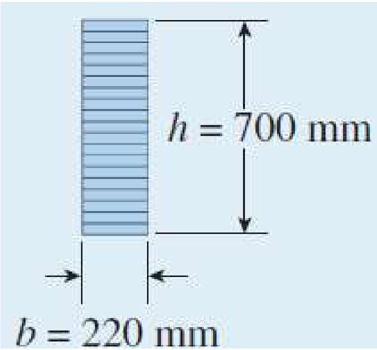


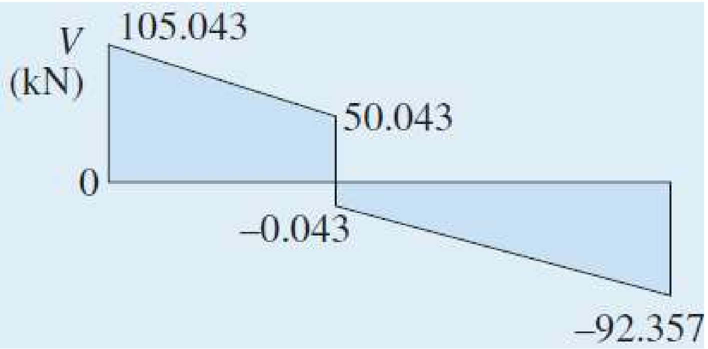


Example

- (a) Determine the maximum tensile and compressive stresses in the beam due to bending.
  (b) If load q is unchanged, find the maximum permissible value of load P if the allowable normal stress in tension and compression is $\sigma_a = 13$ MPa.



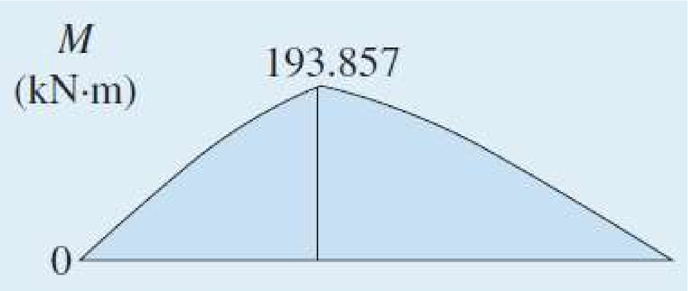




$$R_A = 105 \text{ kN} \qquad R_B = 92.4 \text{ kN}$$

$$M_{max} = 193.9 \text{ kN} \cdot \text{m}$$

$$S = \frac{bh^2}{6} = \frac{1}{6}(0.22 \text{ m})(0.7 \text{ m})^2 = 0.01797 \text{ m}^3$$



Example

- $$\sigma_t = \sigma_2 = \frac{M_{\max}}{S} = \frac{193.9 \text{ kN}\cdot\text{m}}{0.01797 \text{ m}^3} = 10.8 \text{ MPa}$$

  $$\sigma_c = \sigma_1 = -\frac{M_{\max}}{S} = -10.8 \text{ MPa}$$

- **Part(b)**

$$M_{\max} = \frac{a(L - a)(2P + Lq)}{2L} = \sigma_a \times S$$

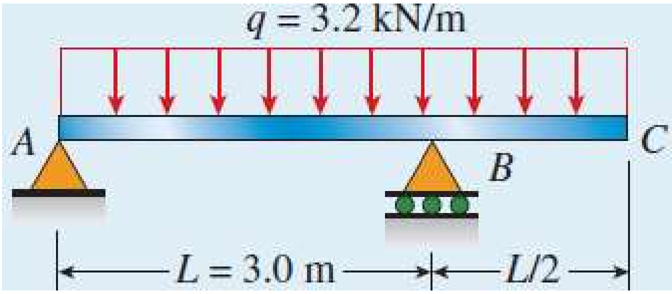
$$L = 6.7 \text{ m and } q = 22 \text{ kN/m}$$

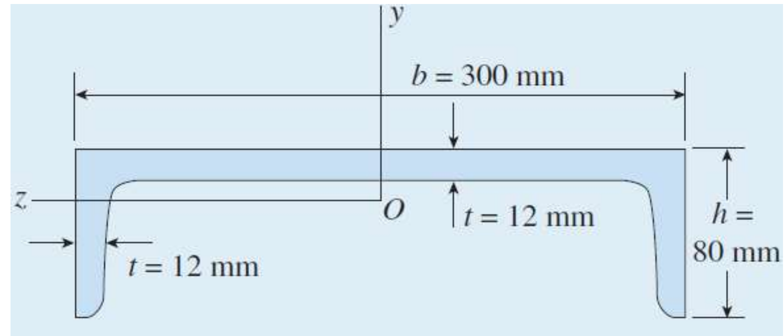
$$P_{\max} = \sigma_a S\left[\frac{L}{a(L - a)}\right] - \frac{qL}{2}$$

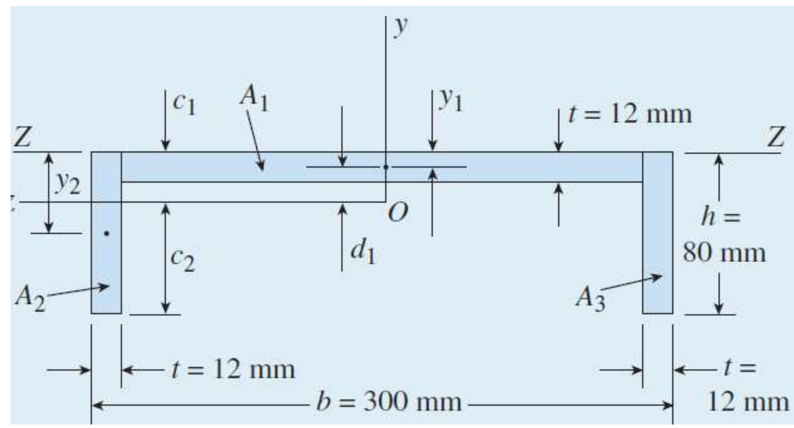
$$= (13 \text{ MPa})(0.01797 \text{ m}^3)\left[\frac{6.7 \text{ m}}{2.5 \text{ m}(6.7 \text{ m} - 2.5 \text{ m})}\right] - 22\,\frac{\text{kN}}{\text{m}}\left(\frac{6.7 \text{ m}}{2}\right)$$

Example

Cross section

- 





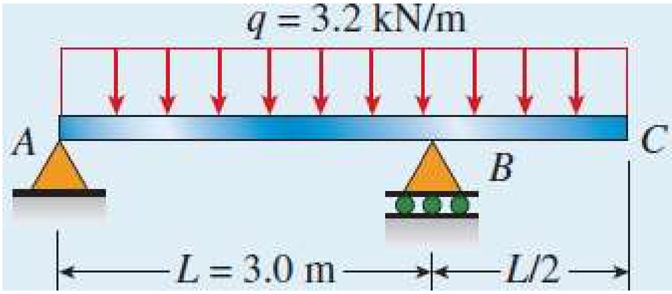
(a) Determine the maximum tensile and compressive stresses in the beam due to the uniform load.

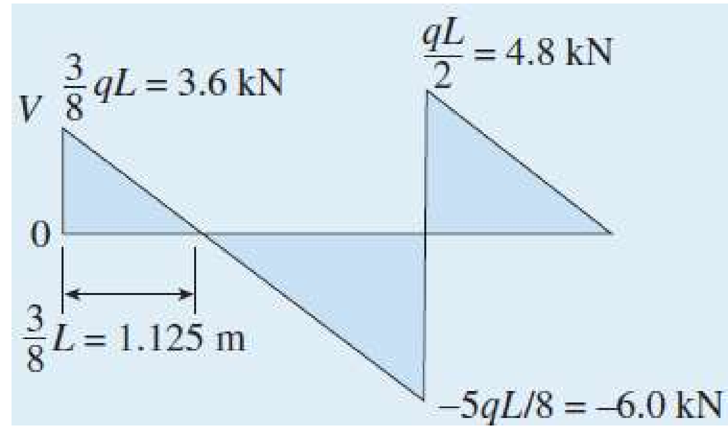
(b) Find the maximum permissible value of uniform load q (in kN/m) if allowable stresses in tension and compression are $\sigma_{aT} = 110$ MPa and $\sigma_{aC} = 92$ MPa, respectively.

Amit Singh

Example

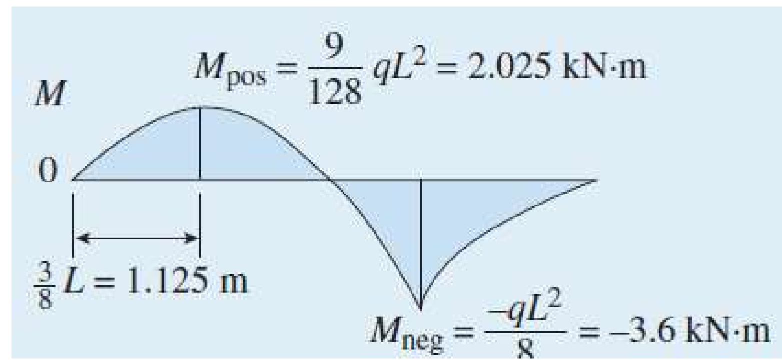
SFD and BMD

- 



$$R_A = \frac{3}{8}qL = 3.6 \text{ kN} \qquad R_B = \frac{9}{8}qL = 10.8 \text{ kN}$$

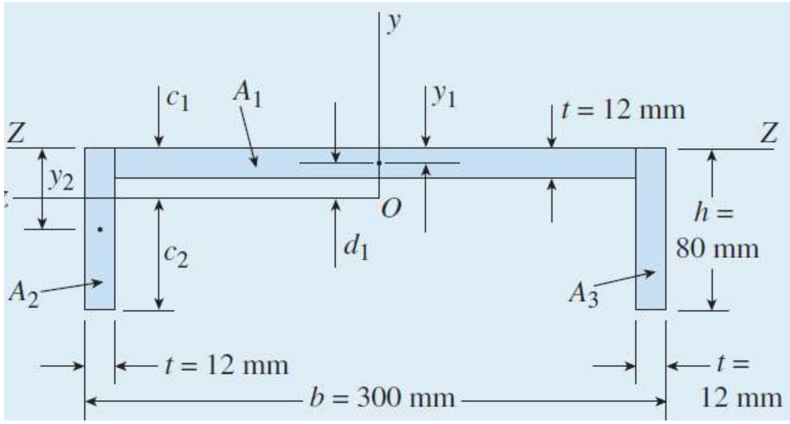
$$M_{pos} = \frac{9}{128}qL^2 = 2.025 \text{ kN}\cdot\text{m} \qquad M_{neg} = \frac{-qL^2}{8} = -3.6 \text{ kN}\cdot\text{m}$$



Neutral axis of the cross section

We need to find where the neutral axis is.

Example

- 

$$y_1 = t/2 = 6 \text{ mm}$$
$$A_1 = (b - 2t)(t) = (276 \text{ mm})(12 \text{ mm}) = 3312 \text{ mm}^2$$

$$y_2 = h/2 = 40 \text{ mm}$$
$$A_2 = ht = (80 \text{ mm})(12 \text{ mm}) = 960 \text{ mm}^2$$

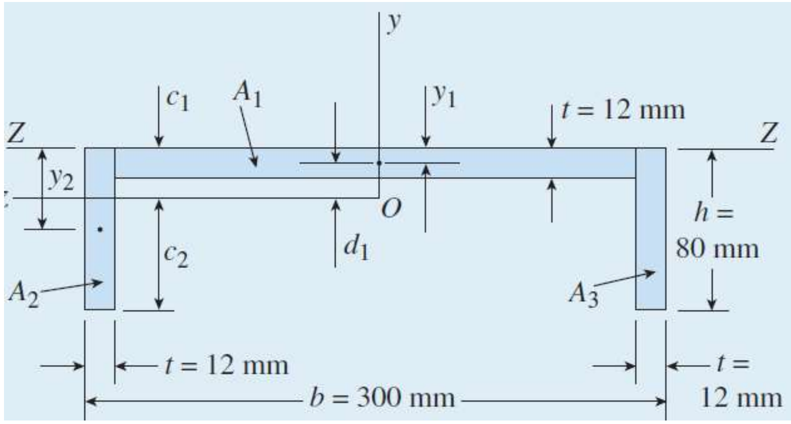
$$y_3 = y_2 \qquad A_3 = A_2$$

$$c_1 = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{y_1 A_1 + 2y_2 A_2}{A_1 + 2A_2}$$
$$= \frac{(6 \text{ mm})(3312 \text{ mm}^2) + 2(40 \text{ mm})(960 \text{ mm}^2)}{3312 \text{ mm}^2 + 2(960 \text{ mm}^2)} = 18.48 \text{ mm}$$
$$c_2 = h - c_1 = 80 \text{ mm} - 18.48 \text{ mm} = 61.52 \text{ mm}$$

Now the next step is to find the moment of inertia for the entire cross section.

Beginning with area A_1

Example

- 

$$y_1 = t/2 = 6 \text{ mm}$$
$$A_1 = (b - 2t)(t) = (276 \text{ mm})(12 \text{ mm}) = 3312 \text{ mm}^2$$

$$y_2 = h/2 = 40 \text{ mm}$$
$$A_2 = ht = (80 \text{ mm})(12 \text{ mm}) = 960 \text{ mm}^2$$

$$y_3 = y_2 \qquad A_3 = A_2$$

$$c_1 = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{y_1 A_1 + 2y_2 A_2}{A_1 + 2A_2}$$
$$= \frac{(6 \text{ mm})(3312 \text{ mm}^2) + 2(40 \text{ mm})(960 \text{ mm}^2)}{3312 \text{ mm}^2 + 2(960 \text{ mm}^2)} = 18.48 \text{ mm}$$
$$c_2 = h - c_1 = 80 \text{ mm} - 18.48 \text{ mm} = 61.52 \text{ mm}$$

Now the next step is to find the moment of inertia for the entire cross section.

Beginning with area A_1 $\qquad (I_z)_1 = (I_c)_1 + A_1 d_1^2$ $\qquad$ Parallel axis theorem

$(I_c)_1$ is the moment of inertia of area A_1 about its own centroidal axis

$$(I_c)_1 = \frac{1}{12}(b - 2t)(t)^3 = \frac{1}{12}(276 \text{ mm})(12 \text{ mm})^3 = 39{,}744 \text{ mm}^4$$

and d_1 is the distance from the centroidal axis of area A_1 to the z axis:

$$d_1 = c_1 - t/2 = 18.48 \text{ mm} - 6 \text{ mm} = 12.48 \text{ mm}$$

Amit Singh

Example

- $(I_z)_1 = 39{,}744 \text{ mm}^4 + (3312 \text{ mm}^2)(12.48 \text{ mm})^2 = 555{,}600 \text{ mm}^4$

$(I_z)_2 = (I_z)_3 = 956{,}600 \text{ mm}^4$

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3 = 2.469 \times 10^6 \text{ mm}^4$$

Section moduli

$$S_1 = \frac{I_z}{c_1} = 133{,}600 \text{ mm}^3 \quad S_2 = \frac{I_z}{c_2} = 40{,}100 \text{ mm}^3$$

Maximum stresses

At the location of maximum positive bending moment

$$\sigma_t = \sigma_2 = \frac{M_{pos}}{S_2} = \frac{2.025 \text{ kN}\cdot\text{m}}{40{,}100 \text{ mm}^3} = 50.5 \text{ MPa}$$

$$\sigma_c = \sigma_1 = -\frac{M_{pos}}{S_1} = -\frac{2.025 \text{ kN}\cdot\text{m}}{133{,}600 \text{ mm}^3} = -15.2 \text{ MPa}$$

At the location of maximum negative bending moment

$$\sigma_t = \sigma_1 = -\frac{M_{neg}}{S_1} = -\frac{-3.6 \text{ kN}\cdot\text{m}}{133{,}600 \text{ mm}^3} = 26.9 \text{ MPa}$$

$$\sigma_c = \sigma_2 = \frac{M_{neg}}{S_2} = \frac{-3.6 \text{ kN}\cdot\text{m}}{40{,}100 \text{ mm}^3} = -89.8 \text{ MPa}$$

$(\sigma_t)_{max} = 50.5 \text{ MPa}$

$(\sigma_c)_{max} = -89.8 \text{ MPa}$

Example

- (b) *Maximum permissible value of uniform load q*

At the location of maximum positive bending moment

$$M_{pos} = \frac{9}{128} q_1 L^2 = \sigma_{aC} S_1 \quad \text{so} \quad q_1 = \frac{128}{9L^2}(\sigma_{aC} S_1) = 19.42 \text{ kN/m}$$ at the *top*

$$M_{pos} = \frac{9}{128} q_2 L^2 = \sigma_{aT} S_2 \quad \text{so} \quad q_2 = \frac{128}{9L^2}(\sigma_{aT} S_2) = 6.97 \text{ kN/m}$$ at the *bottom*

At the location of maximum negative bending moment

$$M_{pos} = \frac{1}{8} q_3 L^2 = \sigma_{aT} S_1 \quad \text{so} \quad q_3 = \frac{8}{L^2}(\sigma_{aT} S_1) = 13.06 \text{ kN/m}$$ at the *top*

$$M_{pos} = \frac{1}{8} q_4 L^2 = \sigma_{aC} S_2 \quad \text{so} \quad q_4 = \frac{8}{L^2}(\sigma_{aC} S_2) = 3.28 \text{ kN/m}$$ at the *bottom*

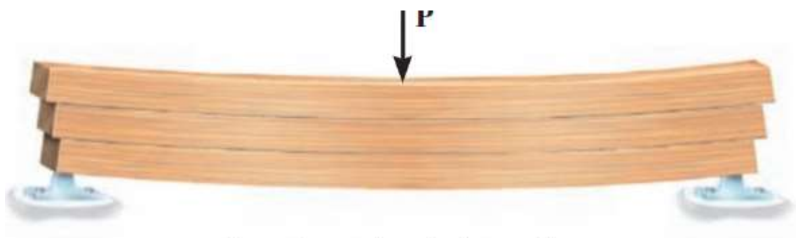
$$q_{max} = 3.28 \text{ kN/m}$$

Amit Singh

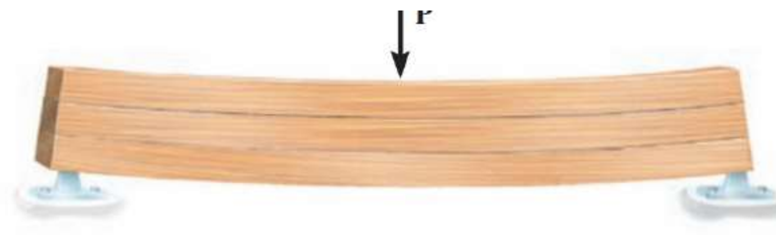
Transverse shear: Non-uniform Bending

Transverse Shear

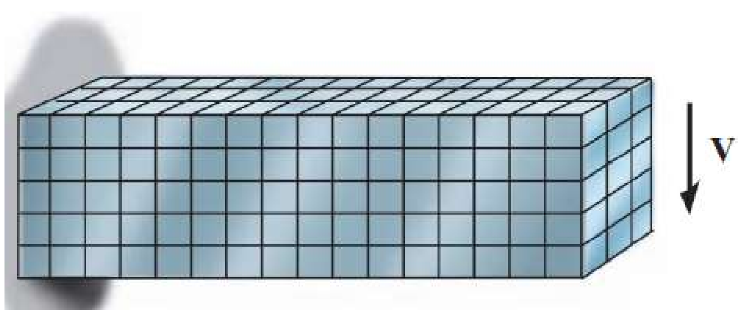
- **Sliding is prevented by shear stresses**



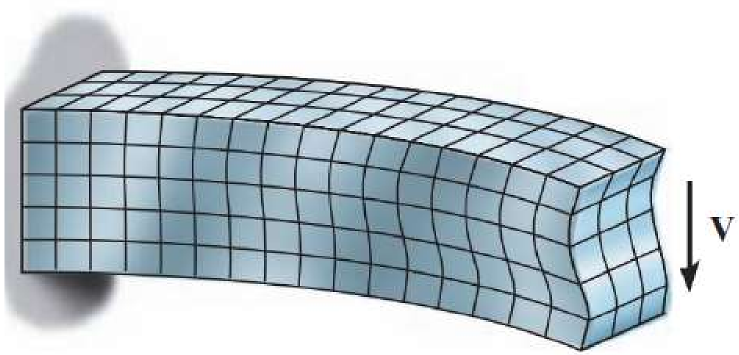
Boards not bonded together



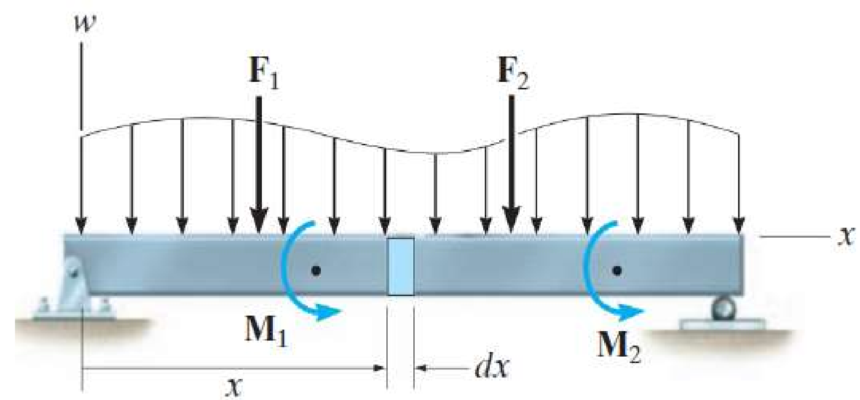
Boards bonded together



(a) Before deformation

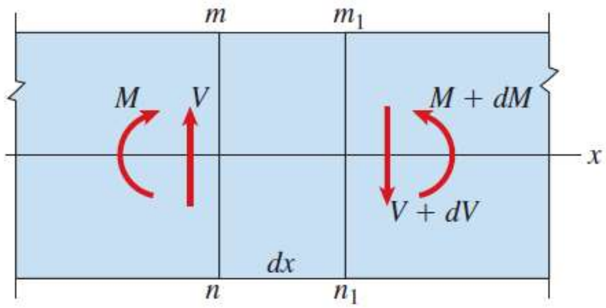


(b) After deformation

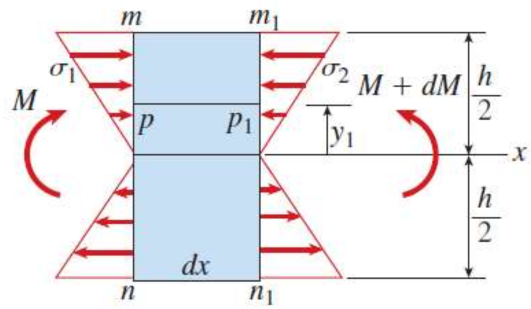


- Many of the symmetry arguments of pure bending theory are no longer applicable.
- But we assume the same bending stress distribution. This is not exact but for most engineering purposes have been found approximately correct.

Transverse Shear in straight members



Side view of beam
(a)

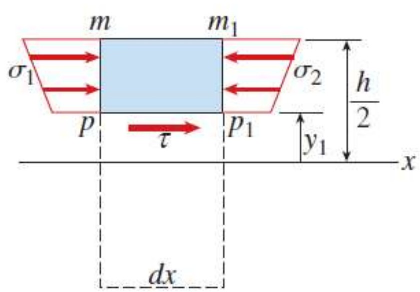


Side view of element
(b)

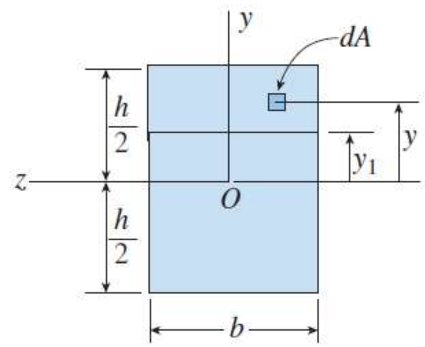
$$F_1 = \int \sigma_1 \, dA = \int \frac{My}{I} dA$$

$$F_2 = \int \sigma_2 dA = \int \frac{(M + dM)y}{I} dA$$

•



Side view of subelement



Cross section of beam at subelement

$$\sigma_1 = -\frac{My}{I} \quad \text{and} \quad \sigma_2 = -\frac{(M + dM)y}{I}$$

- Beam in pure bending (uniform bending) has no shear force and therefore shear stress
- For nonuniform bending: $$\sigma_1 dA = \frac{My}{I} dA$$

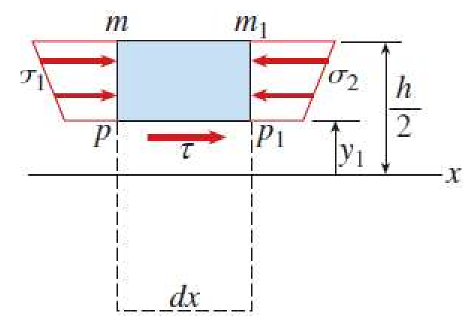
Amit Singh

Transverse Shear in straight members

- $$F_3 = \int \frac{(M + dM)y}{I} dA - \int \frac{My}{I} dA = \int \frac{(dM)y}{I} dA = \frac{dM}{I} \int y\, dA$$

If the shear stresses τ are uniformly distributed across the width b of the beam, the force F_3 is also equal to the following:

$$F_3 = \tau b\, dx$$



Side view of subelement

$$\tau = \frac{dM}{dx}\left(\frac{1}{Ib}\right)\int y\, dA$$

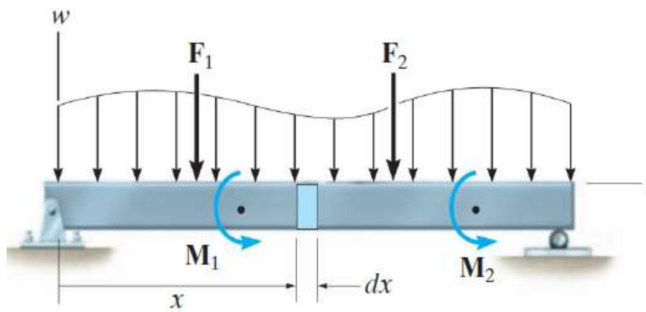
shear formula $\quad \tau = \dfrac{VQ}{Ib} \qquad Q = \displaystyle\int y\, dA$

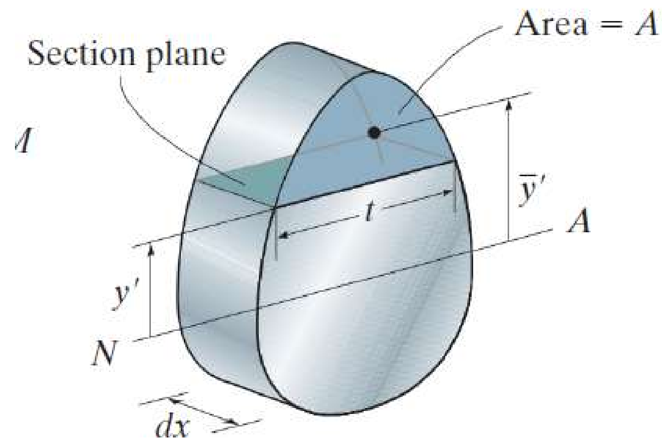
the integral is the first moment of the cross-sectional area above the level at which the shear stress τ is being evaluated.

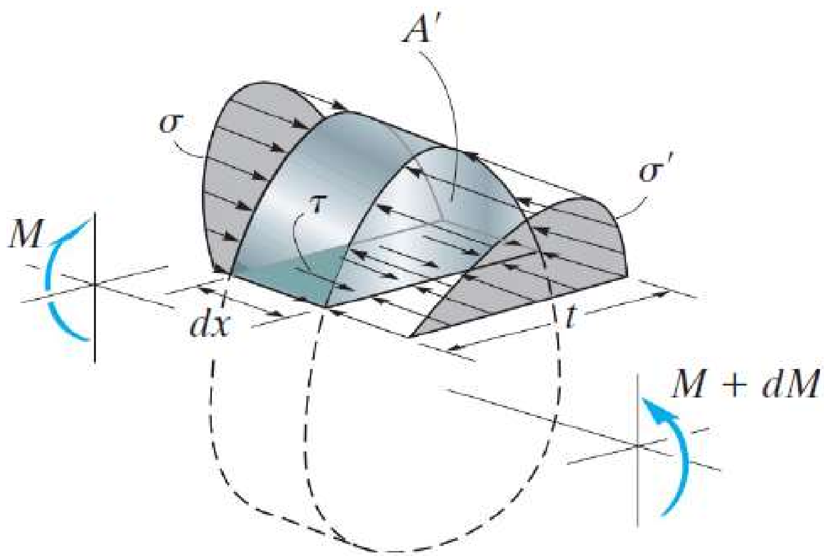
Amit Singh

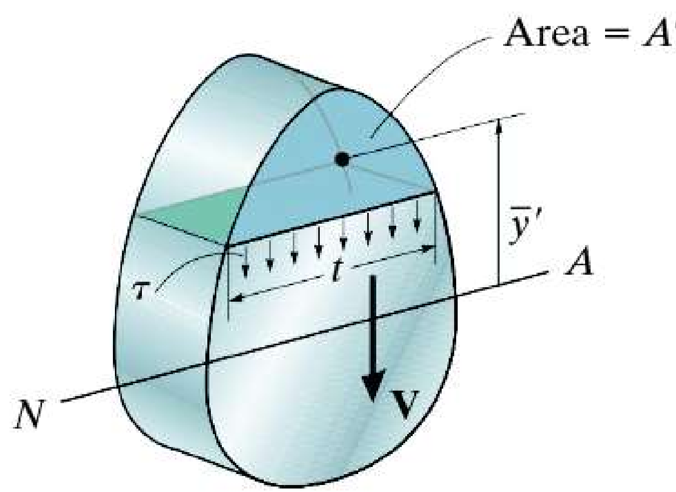
Transverse shear

-





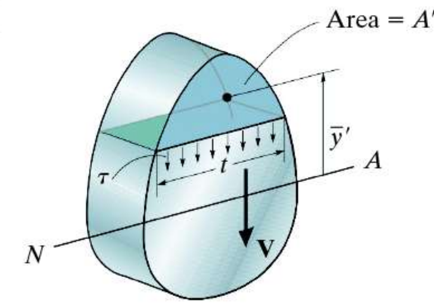




$$\tau = \frac{VQ}{It}$$

Calculation of first moment Q

- τ = the shear stress in the member at the point located a distance y from the neutral axis. This stress is assumed to be constant and therefore *averaged* across the width t of the member



V = the shear force, determined from the method of sections and the equations of equilibrium

I = the moment of inertia of the *entire* cross-sectional area calculated about the neutral axis

t = the width of the member's cross section, measured at the point where τ is to be determined

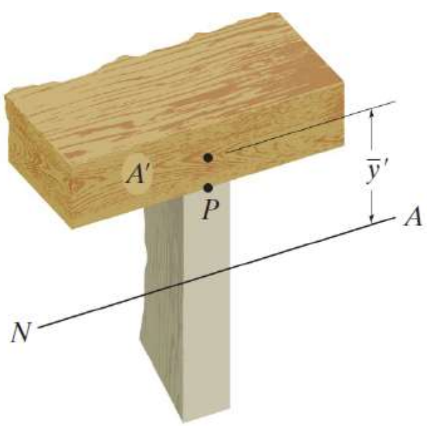
$Q = \bar{y}'A'$, where A' is the area of the top (or bottom) portion of the member's cross section, above (or below) the section plane where t is measured, and $\bar{y}'$ is the distance from the neutral axis to the centroid of A'

$$\tau = \frac{VQ}{It}$$

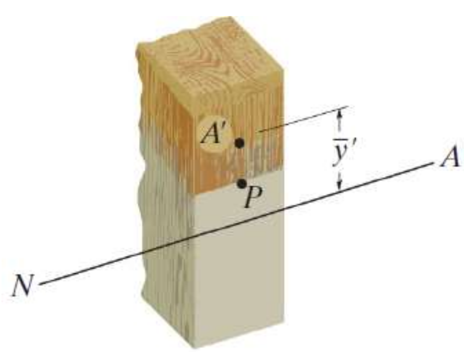
$$Q = \int_{A'} y \, dA' = \bar{y}'A'$$

$$\bar{y}' = \int_{A'} y \, dA' / A'$$

$$I = \int_A y^2 \, dA$$



$Q = \bar{y}'A'$

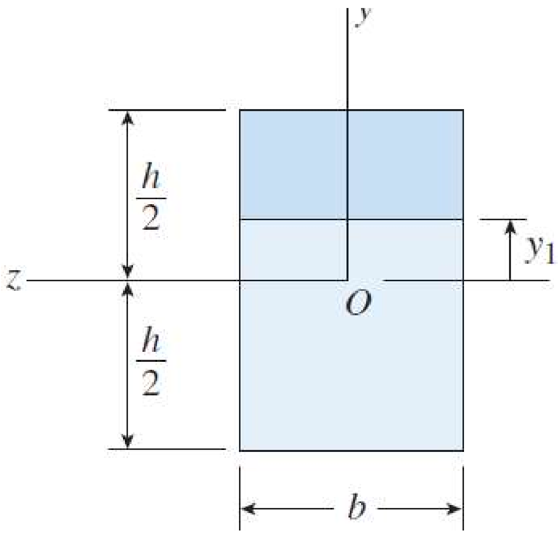


$Q = \bar{y}'A'$

Amit Singh

Calculation of Q for rectangular cross section

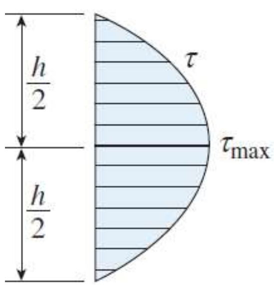
- **For rectangular cross sections**



$$Q = b\left(\frac{h}{2} - y_1\right)\left(y_1 + \frac{h/2 - y_1}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

$$Q = \int y \, dA = \int_{y_1}^{h/2} yb \, dy = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

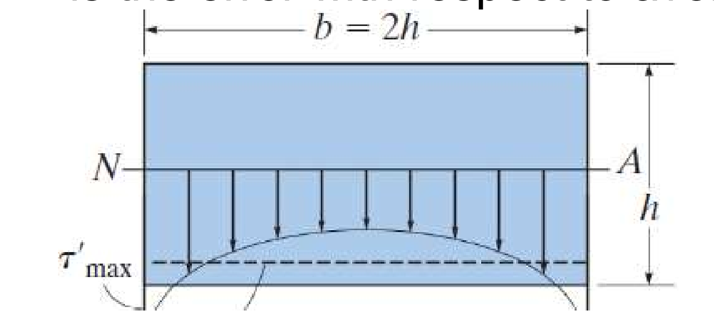


$$\tau_{max} = \frac{Vh^2}{8I} = \frac{3V}{2A}$$

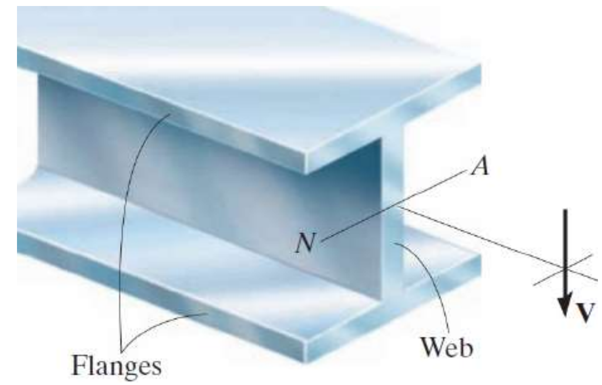
Amit Singh

Limitations

- **Average shear stress is not always a good assumption** (flatter the section greater is the error with respect to average)



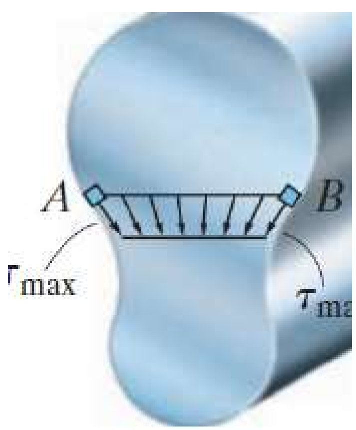
τ'_{max} is about 40% greater than τ_{max}

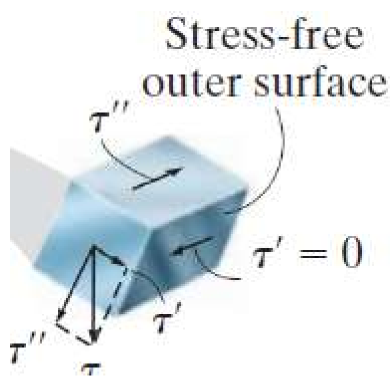


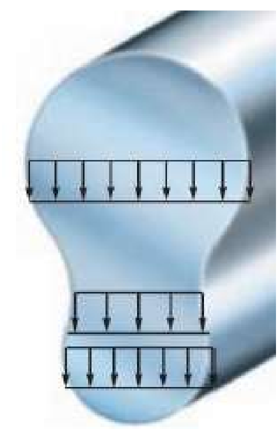
Fortunately, maximum occurs near the neutral axis

- **Irregular boundary**

The shear formula should not be used to determine the shear stress on cross sections that are short or flat, at points of sudden cross-sectional changes, or across a section that intersects the boundary of the member at an angle other than 90º.



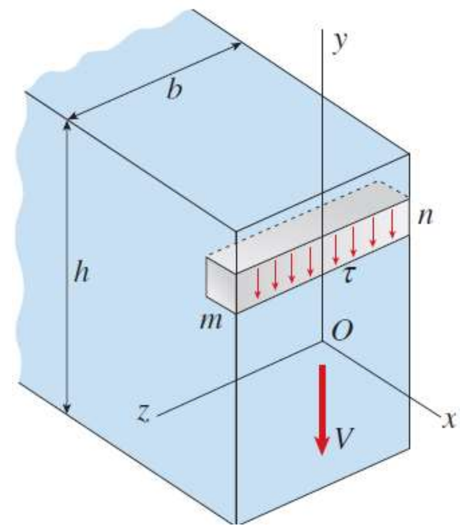




Amit Singh

Limitations

- Valid only when linear elastic material is used as we used the bending stress formula
- The edges of cross section must be parallel to y-axis.
- The shear stress must be uniform across the width of the cross
- section.
- Applicable only to prismatic beams
- Less accurate for beam with rectangular cross section when b
- increases relative to h.



Amit Singh

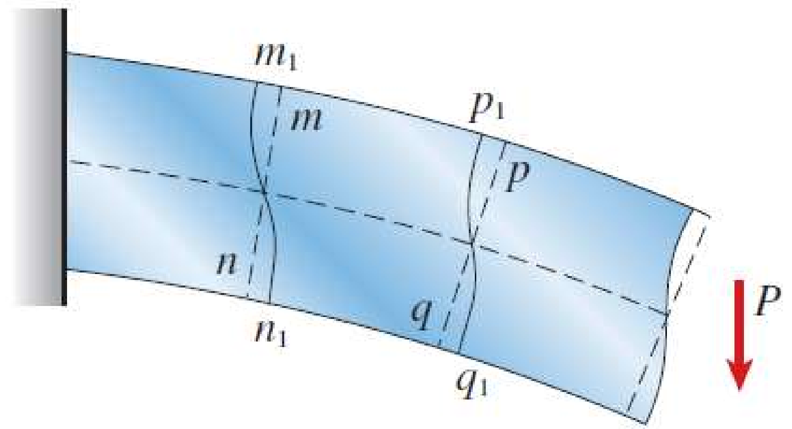
Effect of shear strain

- Shear stress varies parabolically over the height of the beam with rectangular cross section

- So shear strain also varies parabolically

- So plane cross sections warp

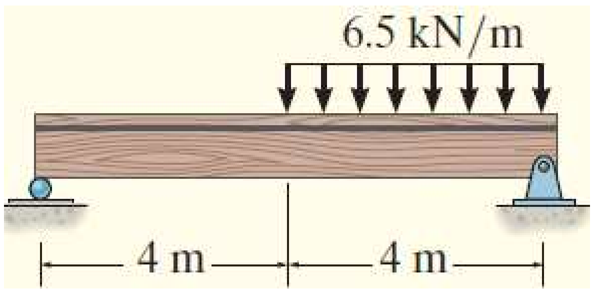
- Warping of cross section does not affect the
- longitudinal strain significantly and therefore
- flexural formula developed for pure bending
- can also be used for nonuniform bending

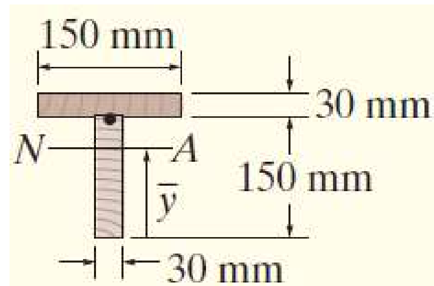


$$\sigma_x = -\frac{My}{I}$$

Example

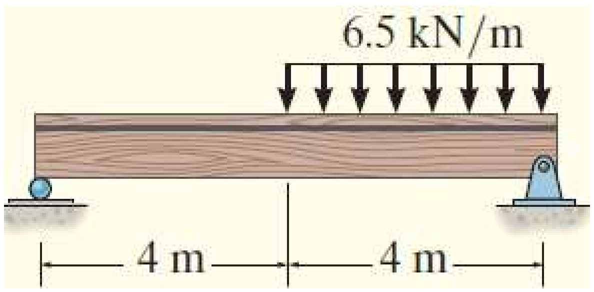
- Determine the ***maximum shear stress*** in the glue

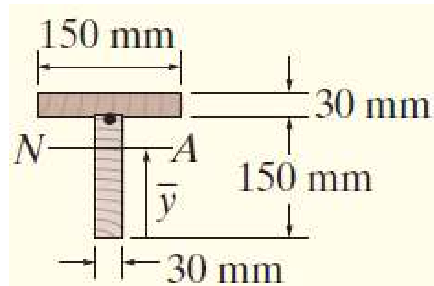




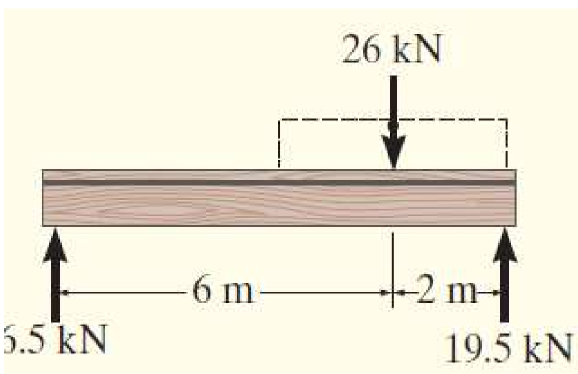
Example

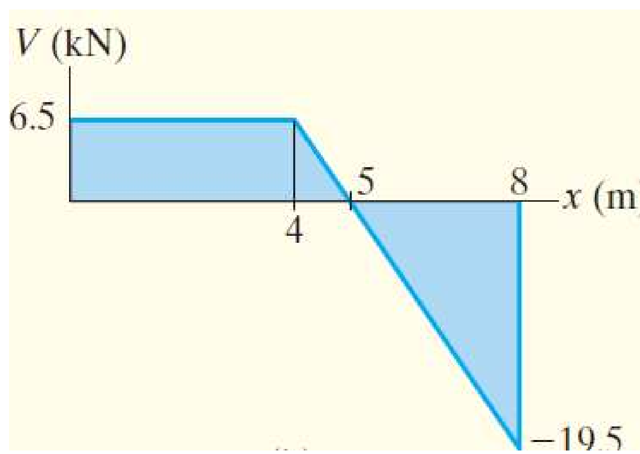
- Determine the ***maximum shear stress*** in the glue





- Bending will cause shear forces to develop and shear stress is $\tau = \dfrac{VQ}{It}$

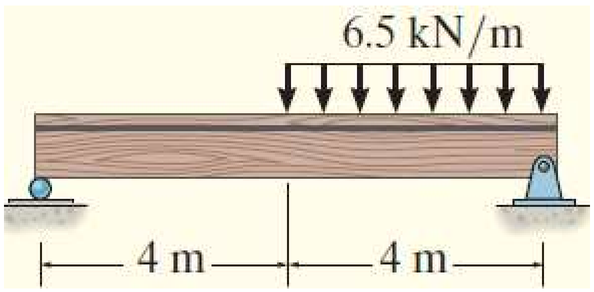




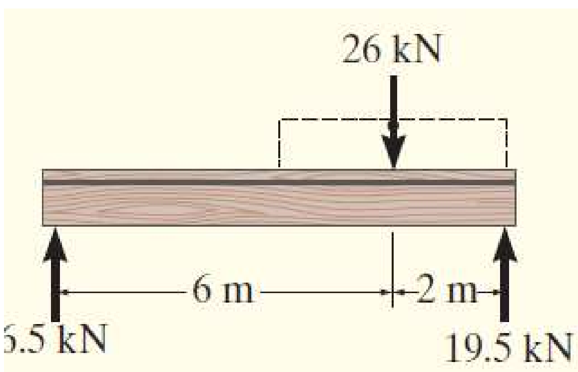
-

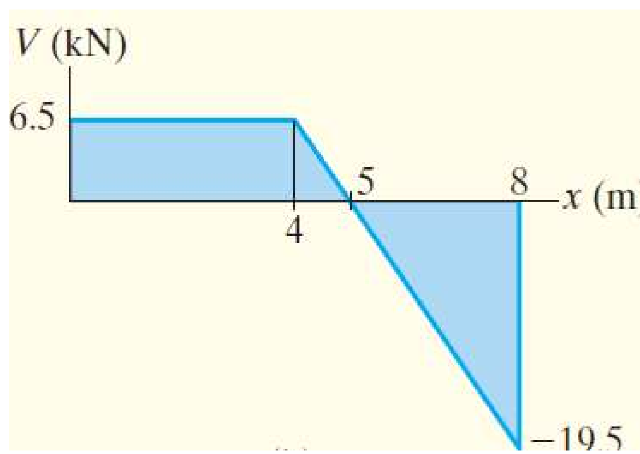
Example

- Determine the ***maximum shear stress*** in the glue



- Bending will cause shear forces to develop and shear stress is $\tau = \dfrac{VQ}{It}$



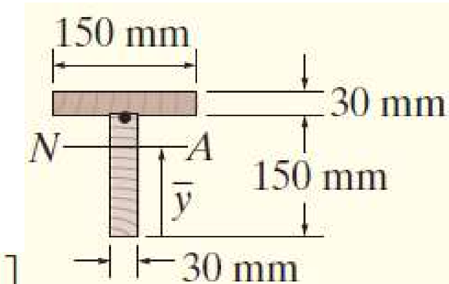


- Find maximum shear force along the length of the beam.
- Q and t are fixed as glue location is known. How to calculate Q and I?

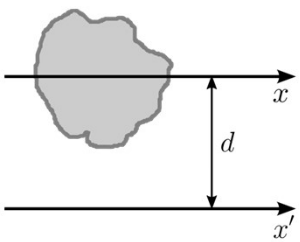
Example

- First locate the neutral axis: coordinates of the centroid

$$\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A} = \frac{[0.075\text{ m}](0.150\text{ m})(0.030\text{ m}) + [0.165\text{ m}](0.030\text{ m})(0.150\text{ m})}{(0.150\text{ m})(0.030\text{ m}) + (0.030\text{ m})(0.150\text{ m})} = 0.120\text{ m}$$

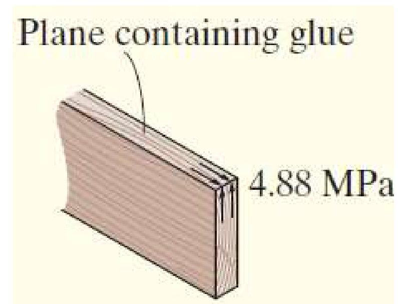


$I_{x'} = I_x + Ad^2$ Parallel axis theorem



$$I = \left[\frac{1}{12}(0.030\text{ m})(0.150\text{ m})^3 + (0.150\text{ m})(0.030\text{ m})(0.120\text{ m} - 0.075\text{ m})^2\right]$$
$$+ \left[\frac{1}{12}(0.150\text{ m})(0.030\text{ m})^3 + (0.030\text{ m})(0.150\text{ m})(0.165\text{ m} - 0.120\text{ m})^2\right]$$
$$= 27.0(10^{-6})\text{ m}^4$$

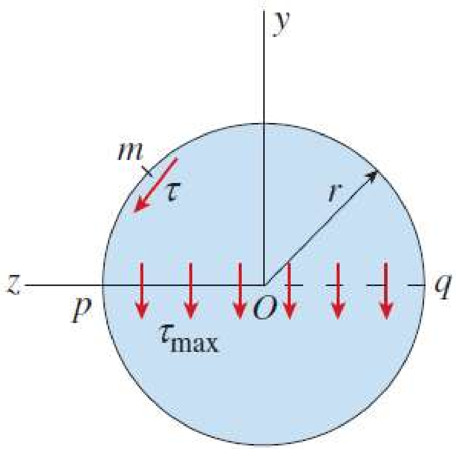
$$Q = \bar{y}'A' = [0.180\text{ m} - 0.015\text{ m} - 0.120\text{ m}](0.03\text{ m})(0.150\text{ m})$$
$$= 0.2025(10^{-3})\text{ m}^3$$



$$\tau_{\max} = \frac{VQ}{It} = \frac{19.5(10^3)\text{ N}(0.2025(10^{-3})\text{ m}^3)}{27.0(10^{-6})\text{ m}^4(0.030\text{ m})} = 4.88\text{ MPa}$$

Shear stress in beam of circular cross section

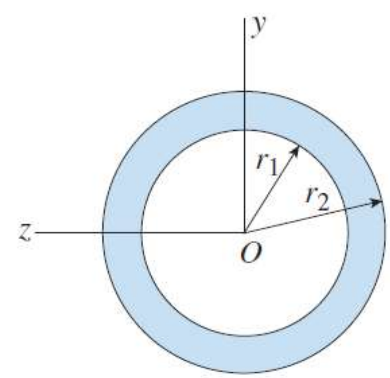
- Shear stress does not act parallel to y-axis
- But shear stress can be determined at the neutral axis
- Use the same formula
- Here, we will further have



$$I = \frac{\pi r^4}{4} \qquad Q = A\bar{y} = \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right) = \frac{2r^3}{3} \qquad b = 2r$$

$$\tau_{max} = \frac{VQ}{Ib} = \frac{V(2r^3/3)}{(\pi r^4/4)(2r)} = \frac{4V}{3\pi r^2} = \frac{4V}{3A}$$

- For hollow circular cross sections

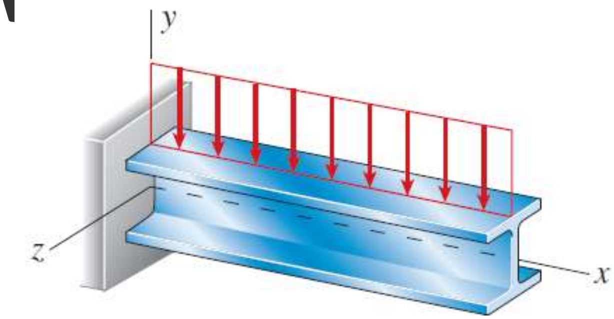


$$I = \frac{\pi}{4}(r_2^4 - r_1^4) \qquad Q = \frac{2}{3}(r_2^3 - r_1^3) \qquad b = 2(r_2 - r_1)$$

$$\tau_{max} = \frac{VQ}{Ib} = \frac{4V}{3A}\left(\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2}\right) \qquad A = \pi(r_2^2 - r_1^2)$$

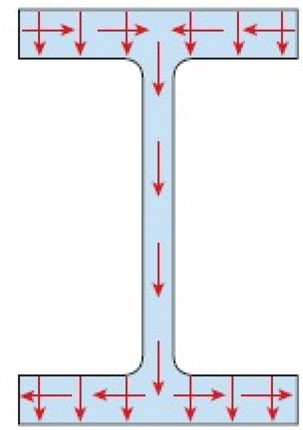
Amit Singh

Shear stress in web of the beam with flanges

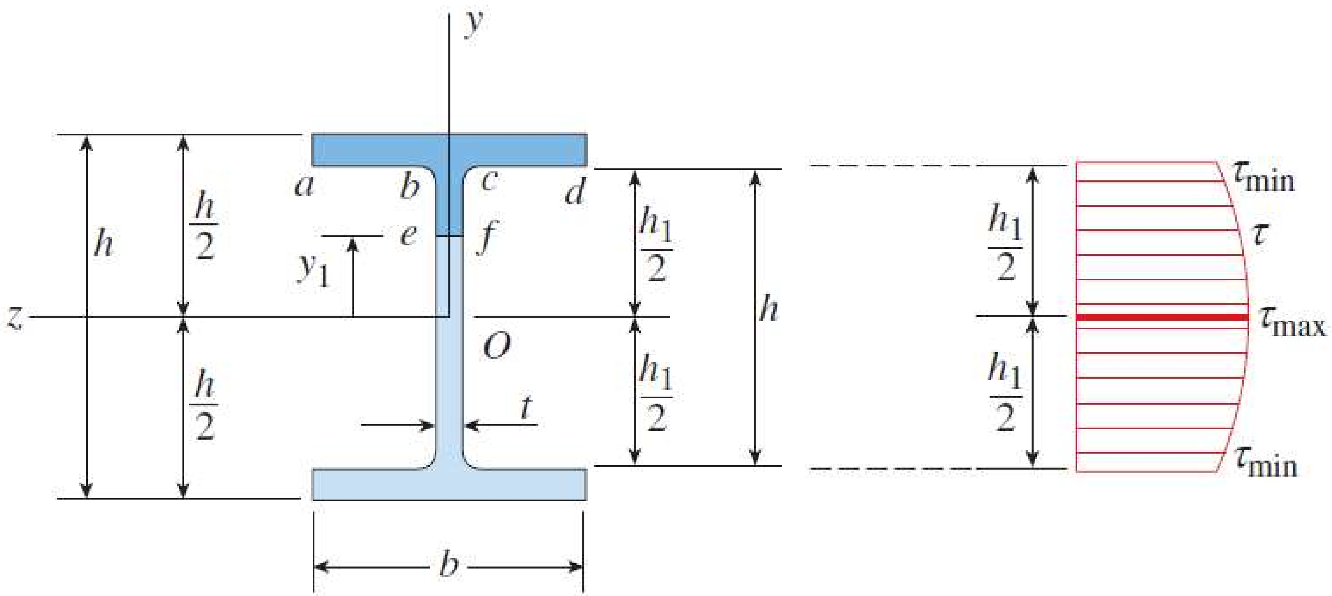


- Shear stress does not act parallel to y-axis

- More complicated, shear stress in flanges acts in not
- only ususal vertical direction but also in horizontal directions.



- **Shear stress in web only**:



$$A_1 = b\left(\frac{h}{2} - \frac{h_1}{2}\right)$$ For upper flange

$$A_2 = t\left(\frac{h_1}{2} - y_1\right)$$ For web, between ef and flange

Amit Singh

Shear stress in web of the beam with flanges

•

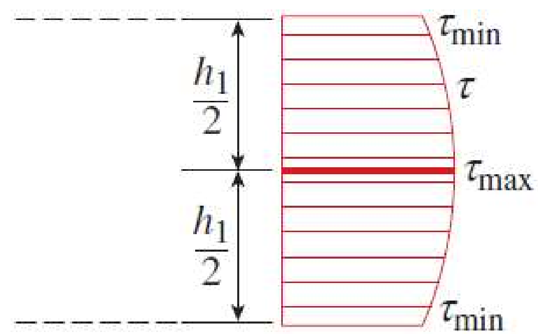
$$Q = A_1\left(\frac{h_1}{2} + \frac{h/2 - h_1/2}{2}\right) + A_2\left(y_1 + \frac{h_1/2 - y_1}{2}\right)$$

$$= \frac{b}{8}(h^2 - h_1^2) + \frac{t}{8}(h_1^2 - 4y_1^2)$$

$$\tau = \frac{VQ}{It} = \frac{V}{8It}\left[b(h^2 - h_1^2) + t(h_1^2 - 4y_1^2)\right]$$

$$I = \frac{bh^3}{12} - \frac{(b - t)h_1^3}{12} = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3)$$

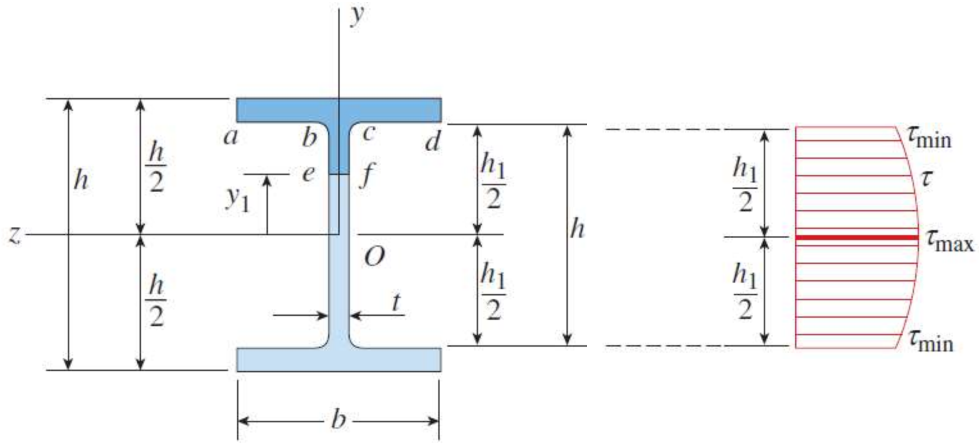
$$\tau_{max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2) \qquad \tau_{min} = \frac{Vb}{8It}(h^2 - h_1^2)$$



Amit Singh

Shear force in the web

- Just integrate the area under the shear stress diagram: $V_{\text{web}} = \int \tau dA$



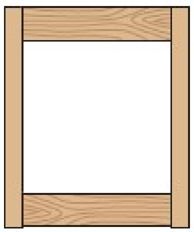
- Two areas per unit thickness involved: rectangular with $h_1\tau_{min}$ and parabolic with $\frac{2}{3}(h_1)(\tau_{max} - \tau_{min})$

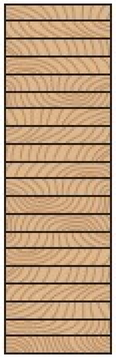
- $V_{web} = \frac{th_1}{3}(2\tau_{max} + \tau_{min})$

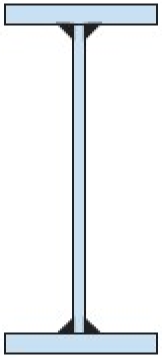
- Most of the shear force is resisted by the web, so approximately $\tau_{aver} = \frac{V}{th_1}$

Built up beams and shear flow

- **Built-up beams**: Fabricated by joining two or more materials together to form a beam
- Wood box beam, glulam beam, plate girder





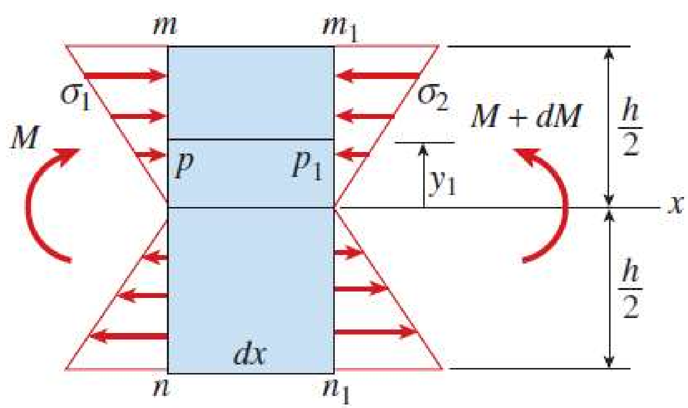


- 

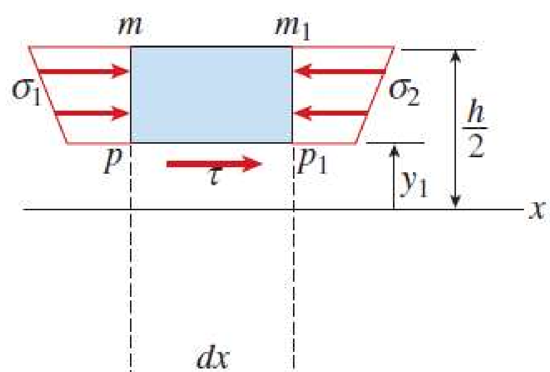
Amit Singh

Built up beams and shear flow

- These beams are designed so that the whole beam behaves as a single member
- First phase of design: as if it were made of a single piece
- Second phase: connections between the parts such as glue, welds, nuts etc are designed
- Connections should be strong so that parts remain bonded together
- This analysis is helped by the concept of shear flow
- **Shear Flow f**:



Side view of element



Side view of subelement

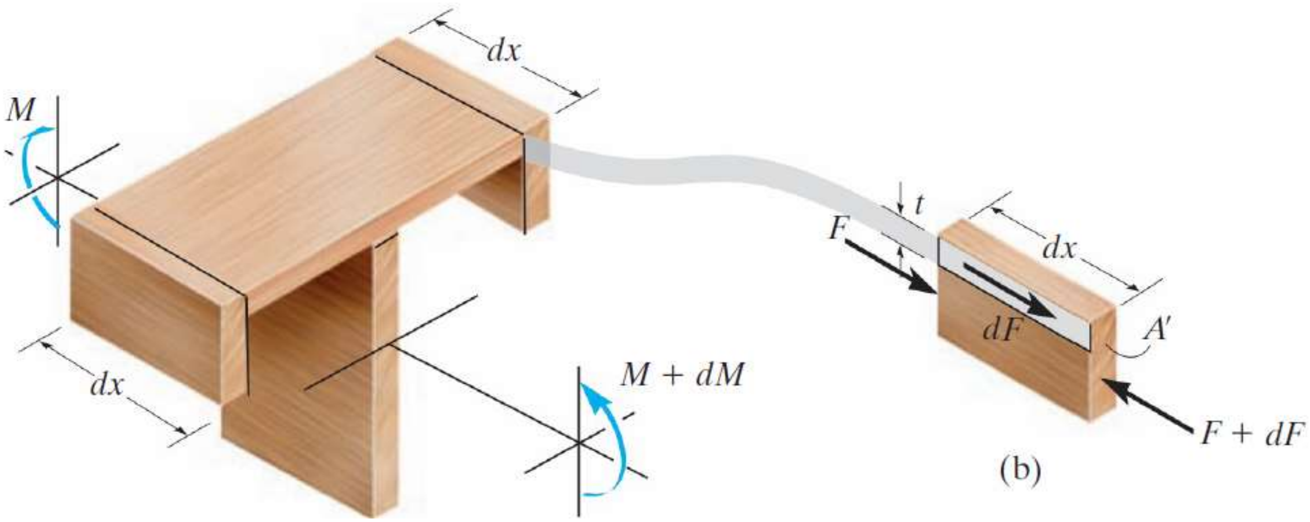
$$F_3 = \frac{dM}{I}\int y\,dA$$

$$f = \frac{F_3}{dx} = \frac{dM}{dx}\left(\frac{1}{I}\right)\int y\,dA$$

$$f = \frac{VQ}{I}$$

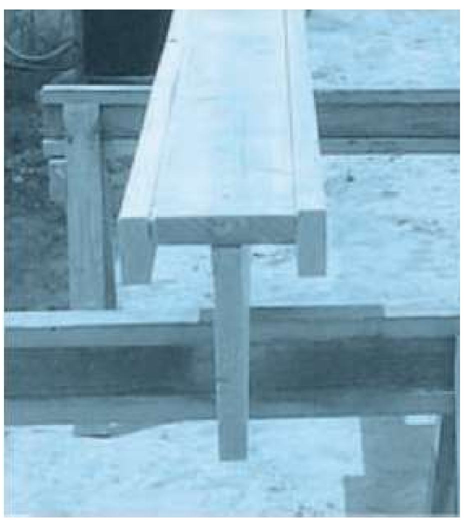
Amit Singh

Built up beams and shear flow



$$dF = \frac{dM}{I}\int_{A'} y\, dA'$$

$$q = dF/dx$$



$$\boxed{q = \frac{VQ}{I}}$$

q = the shear flow, measured as a force per unit length along the beam

Amit Singh

Area used when calculating Q

- **Plate girder:** Here Q is the first moment of the flange area with respect to neutral axis. Once the shear flow is known, the amount of weld can be calculated.
- **Wide-flange beam:** strengthened by rivetting channels to each flange. The horizontal shear force between each channel and beam is transmitted by rivets, which is calculated by shear formula when Q is taken for the shaded area.
- **Wood box beam**: Shear force along both cc and dd, and so Q is calculated for the shaded area.

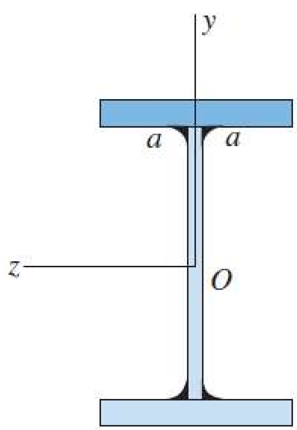
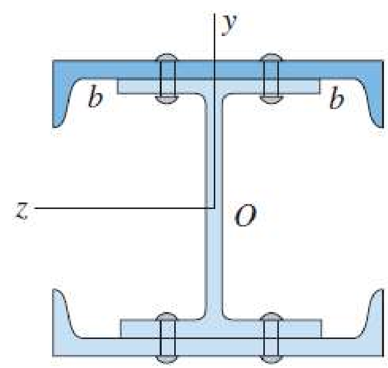
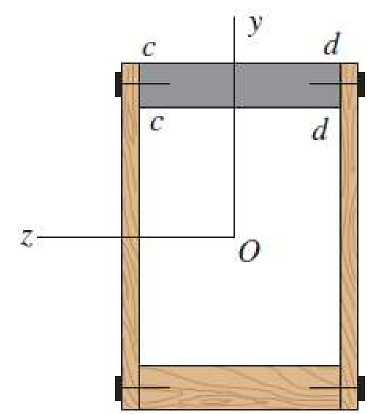


Plate girder



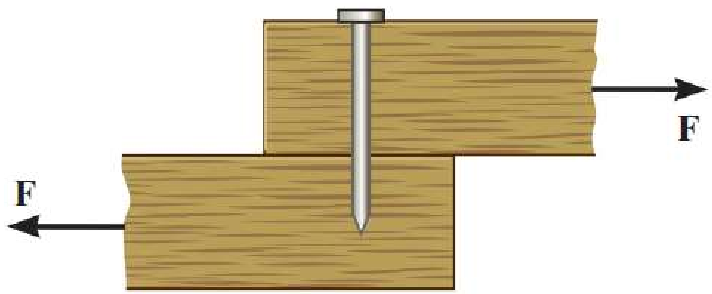
Wide-flange beam

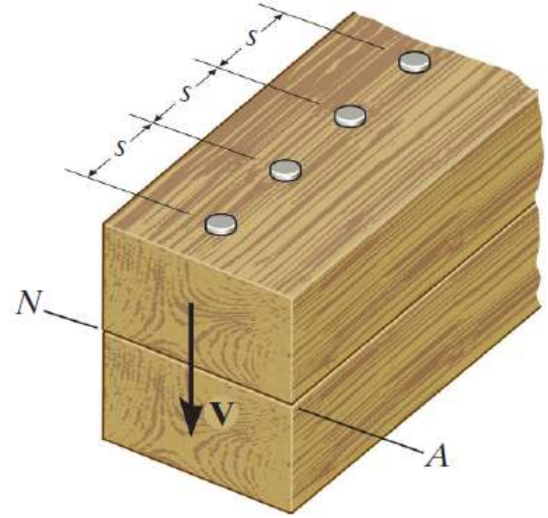


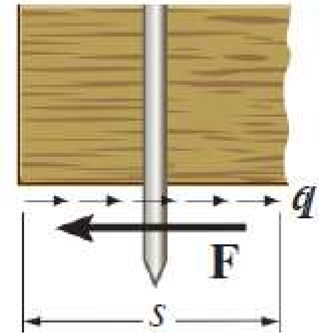
Wood box beam

Amit Singh

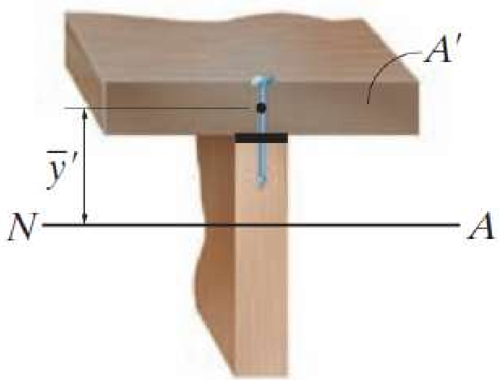
Fastener

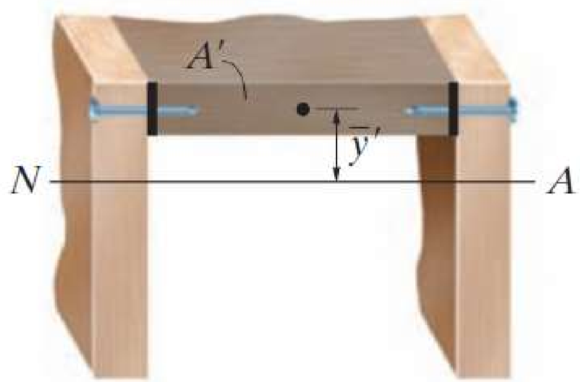
- 

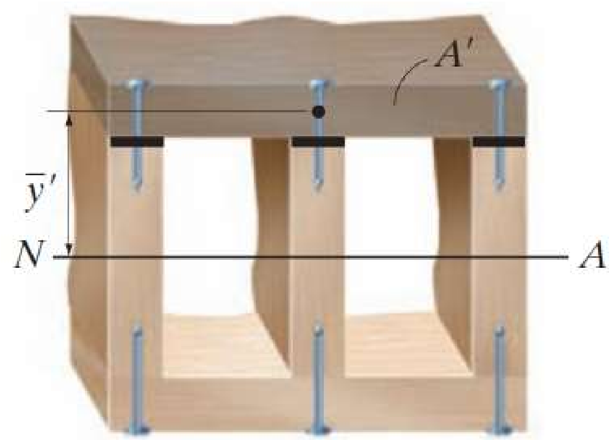




$$F\,(\mathrm{N}) = q\,(\mathrm{N/m})\,s\,(\mathrm{m}) \qquad q = \frac{VQ}{I} \qquad Q = \bar{y}'A'$$



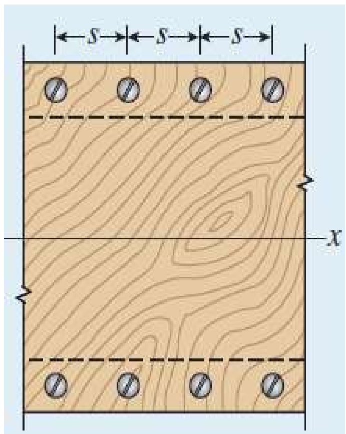




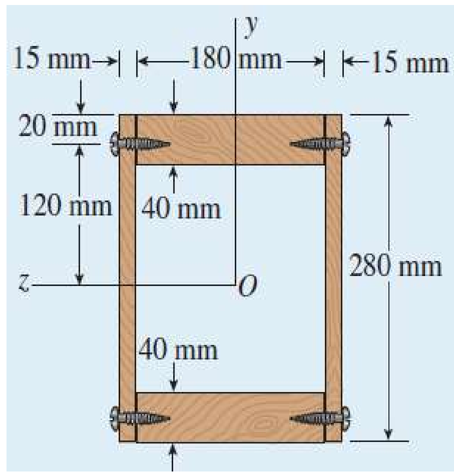
Amit Singh

Built up beams and shear flow

- The plywood is fastened to the flanges by screws with allowable shear load F = 800 N each. If shear force V = 10.5 KN **then determine the longitudinal spacing s beteween the screws**



Side view



Cross section

$$f = \frac{VQ}{I}$$

$$A_f = 40\text{ mm} \times 180\text{ mm} = 7200\text{ mm}^2 \qquad d_f = 120\text{ mm}$$

$$Q = A_f d_f = (7200\text{ mm}^2)(120\text{ mm}) = 864 \times 10^3\text{ mm}^3$$

- $$I = \frac{1}{12}(210\text{ mm})(280\text{ mm})^3 - \frac{1}{12}(180\text{ mm})(200\text{ mm})^3$$

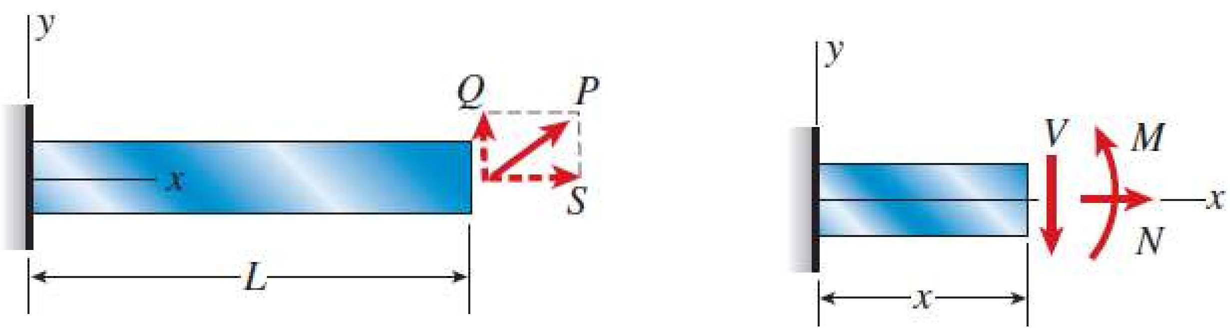
$$f = \frac{VQ}{I} = \frac{(10{,}500\text{ N})(864 \times 10^3\text{ mm}^3)}{264.2 \times 10^6\text{ mm}^4} = 34.3\text{ N/mm}$$

$$s = \frac{2F}{f} = \frac{2(800\text{ N})}{34.3\text{ N/mm}} = 46.6\text{ mm}$$

Amit Singh

Other bending scenarios

Beam with axial loading

- 

- **Stress resultants:** $M = Q(L - x) \quad V = -Q \quad N = S$

- Superposition of bending and axial stresses will be allowed. True if beam is not a slender beam.

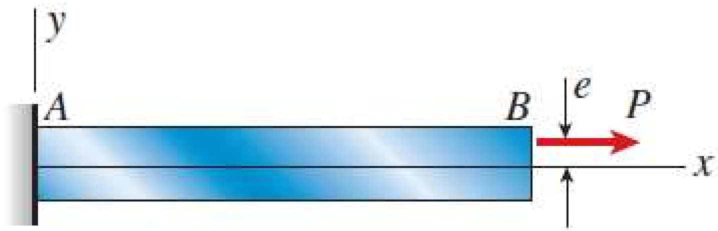
$$\sigma = \frac{N}{A} - \frac{My}{I}$$

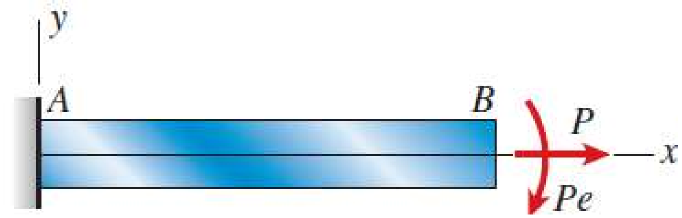
- Here I is centroidal moment of inertia and y is measured from the centroidal axis.

- This is due to superposition of axial and bending stresses. But it shifts the neutral axis.

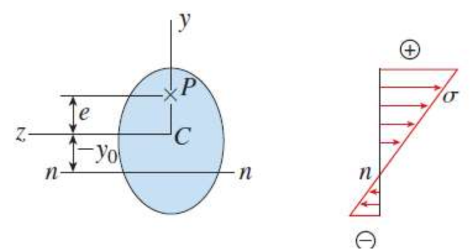
Eccentric axial loading

- **When axial force does not pass through the centroid. The eccentricity is e.**



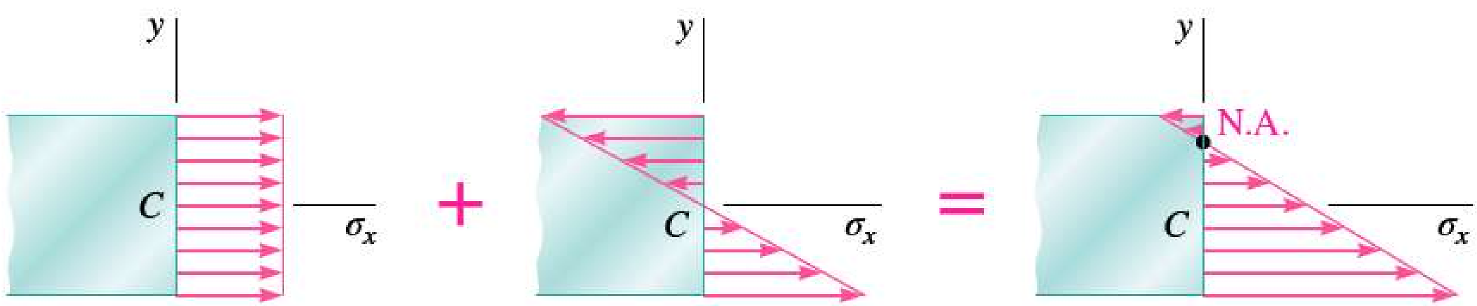


$$\sigma = \frac{P}{A} + \frac{Pey}{I}$$



- **The neutral axis is nn: no resultant stress**

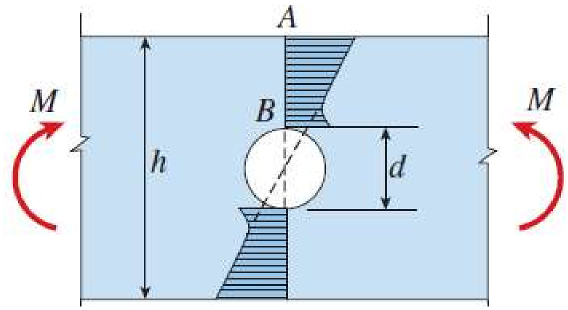
$$y_0 = -\frac{I}{Ae}$$



Amit Singh

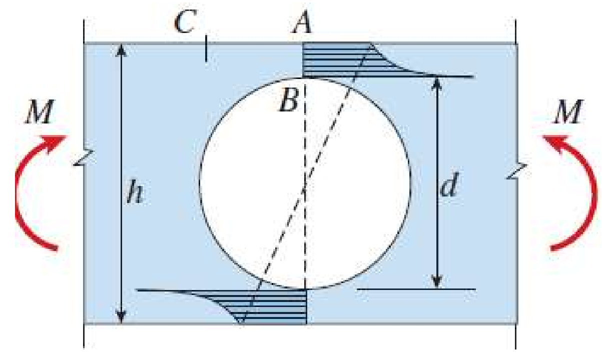
Stress concentration in bending

- **Beam with holes: when diameter d << h**



- **When d and h are comparable:** almost twice of
- nominal stress at edge B

$$\sigma_B \approx 2\frac{My}{I} = \frac{12Md}{b(h^3 - d^3)} \qquad \sigma_C \approx \frac{My}{I} = \frac{6Mh}{b(h^3 - d^3)}$$

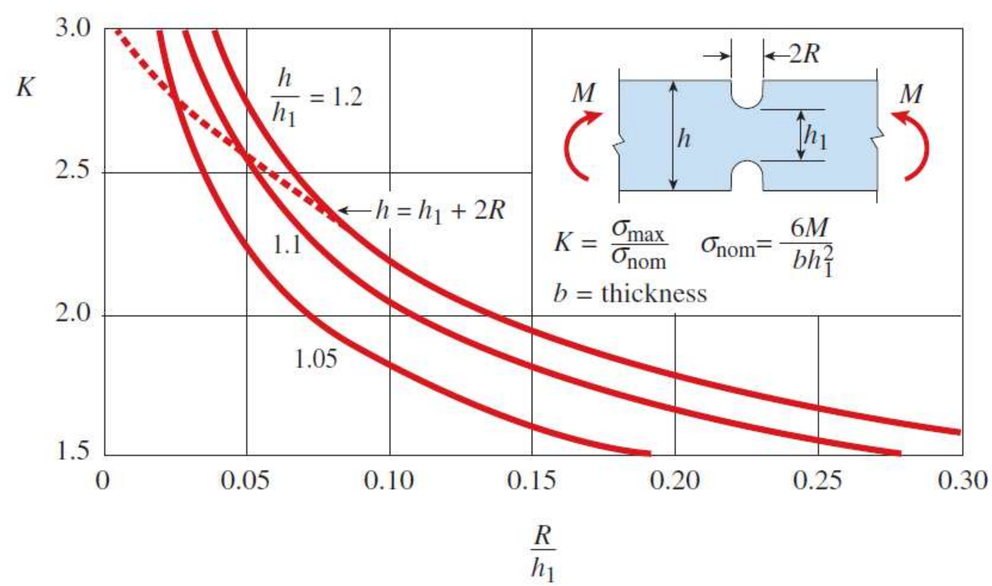


- **Rectangular beams with notches**

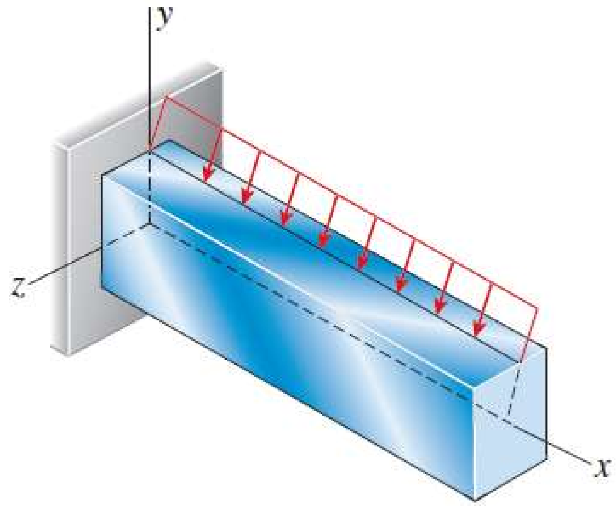
$$\sigma_{\text{nom}} = \frac{My}{I} = \frac{6M}{bh_1^2}$$

$$\sigma_{\max} = K\sigma_{\text{nom}}$$

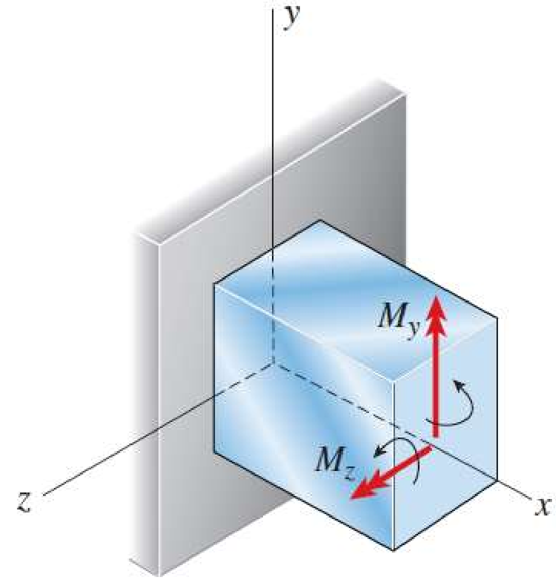
- Max occurs at the notch base



Doubly symmetric beams with inclined loads

- 

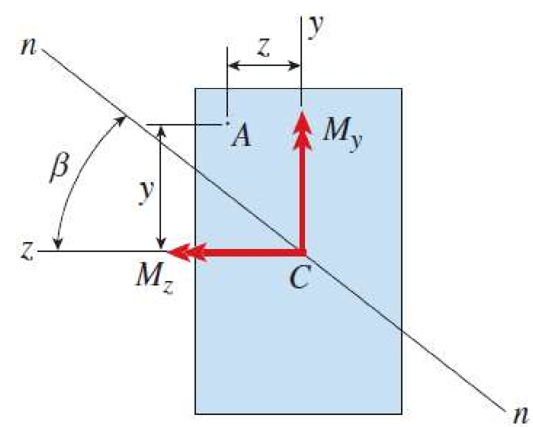
$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$



Neutral Axis

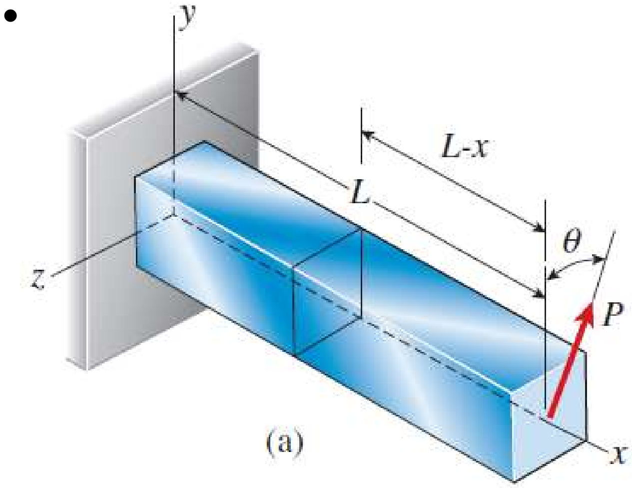
- $$\frac{M_y}{I_y} z - \frac{M_z}{I_z} y = 0$$

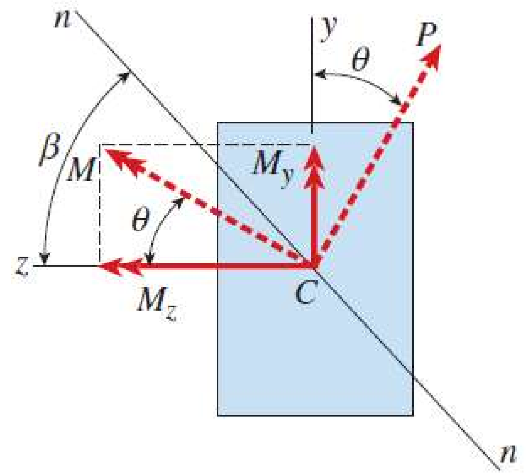
$$\tan \beta = \frac{y}{z} = \frac{M_y I_z}{M_z I_y}$$



Amit Singh

Doubly symmetric beams with inclined loads

- 



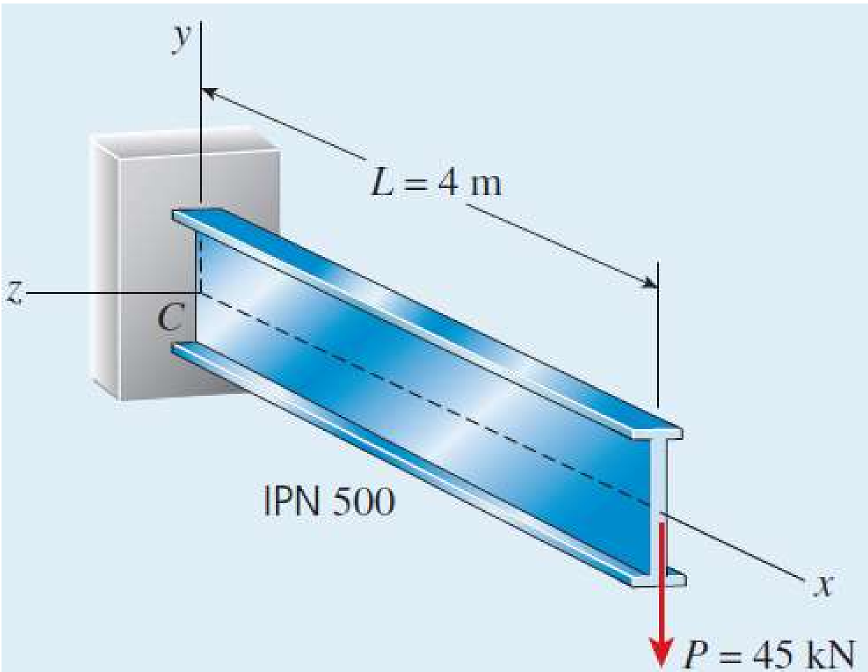
$$M_y = (P \sin \theta)(L - x) \qquad M_z = (P \cos \theta)(L - x) \qquad \frac{M_y}{M_z} = \tan \theta$$

-

Amit Singh

Example

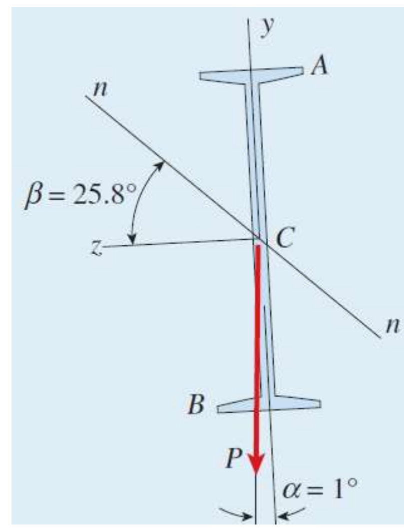
- **Find the max bending stress when P is not inclined to y-axis and when P is inclined.**



when the load is aligned with the y axis

$$\sigma_{max} = \frac{My}{I_z} = \frac{PL(h/2)}{I_z}$$

$$\sigma_{max} = \frac{(45\text{ kN})(4000\text{ mm})(250\text{ mm})}{68740\text{ cm}^4} = 65.5\text{ MPa}$$



when the load is inclined to the y axis

- $M_y = -(P\sin\alpha)L = -(45\text{ kN})(\sin 1°)(4000\text{ mm}) = -3.14\text{ kN}\cdot\text{m}$

  $M_z = -(P\cos\alpha)L = -(45\text{kN})(\cos 1°)(4000\text{ mm}) = -180\text{ kN}\cdot\text{m}$

$$\tan\beta = \frac{y}{z} = \frac{M_y I_z}{M_z I_y} = \frac{(-3.14\text{ kN}\cdot\text{m})(68740\text{ cm}^4)}{(-180\text{ kN}\cdot\text{m})(2480\text{ cm}^4)} = 0.878 \quad \beta = 25.8°$$

$$\sigma_A = \frac{M_y z_A}{I_y} - \frac{M_z y_A}{I_z} = 77.2\text{ MPa}$$

Amit Singh