Tutorial Sheet 5

Eigenvalues and Eigenvectors

1. Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} 6 & -2 & 1 \\ -4 & 12 & -8 \\ 4 & -16 & 24 \end{pmatrix}$$

and $\boldsymbol{b} \in \mathbb{R}^3$ is any given vector. Does the Jacobi iterative sequence of the given system converge for every initial guess $\boldsymbol{x}^{(0)} \in \mathbb{R}^3$? Justify your answer.

2. Let A be a matrix whose eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 2$, and $\lambda_3 = 9$ with corresponding eigenvectors $\boldsymbol{v}_1 = (1,0,1)^T$, $\boldsymbol{v}_2 = (0,1,-1)^T$, and $\boldsymbol{v}_3 = (0,0,1)^T$. To which eigenvalue and the corresponding eigenvector does the power method converge (up to a subsequence) if we start with the initial guess $\boldsymbol{x}^{(0)} = (2,-3,5)$? Justify your answer.

3. Consider the matrix

$$A = \begin{pmatrix} 12.25 & 0.125 & 0.42 \\ -1.05 & -14 & 0.5 \\ 0.006 & 0.045 & 2.25 \end{pmatrix}$$

whose eigenvalues are real.

   i) Without calculating the eigenvalues explicitly, show that the eigenvalues of A can be labelled as λ_1, λ_2, and λ_3 satisfying

$$|\lambda_1| > |\lambda_2| > |\lambda_3|.$$

   ii) Construct the iterative sequences $\{\mu_k\}$ and $\{\boldsymbol{x}^{(k)}\}$ (based on power method) converging to λ_3 (as in (3i)) and a corresponding eigenvector, respectively.

   iii) Starting with the initial guess $\boldsymbol{x}^{(0)} = (1,2,3)$, perform one iteration of the sequences constructed in (3ii) using Doolittle factorization.

4. Find the optimal bounds for the eigenvalues of the matrix

$$A = \begin{pmatrix} -7 & 0.5 & -0.75 \\ 0.65 & 5 & 0.4 \\ 0 & 0.1 & 1 \end{pmatrix}$$

given by the Gerschgorin theorem.