Dnyanesh Pawaskar

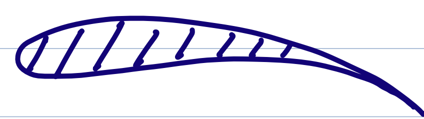
ME 202

10 APR 2023

TORSION OF PRISMATIC SHAFTS (Non-circular cross-sections)

Refs
- Advanced Mechanics of Solids
  Srinath
- Elasticity: Theory, Application, Numerics
  Sadd

Goal:



Airfoil cross-section

Torsional stiffness

$$T = K_t \alpha \qquad \theta = \alpha L$$

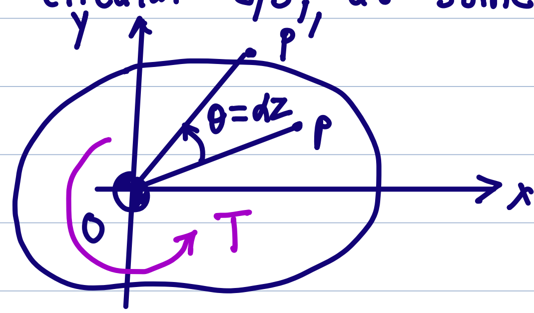
$\uparrow$ goal of theory

Dnyanesh Pawaskar

Torsion of Non-circular cross-sections

- □ Angle of Twist
- □ T_{max} for safe operation   c/s Wing / Turbine blade

Theory based on circular c/s, at some z.



centroid

origin here for now.

$OP = OP'$

$\theta = \alpha z$, α unit angle of twist rad/m

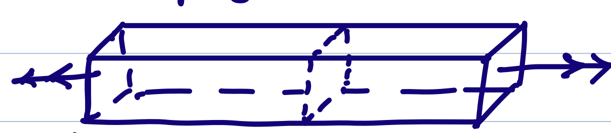
small angles.

Applied torque $\underline{T} = T\,\underline{e}_z$

$u = -\alpha y z$

$v = +\alpha x z$

$w = w(x,y)$ warping.

T  T

from expt observations

or prove formally $w \neq 0$ for non-circ c/s later.

Ang. Disp $\longrightarrow$ Strains $\longrightarrow$ Stresses $\longrightarrow$ Torque
α $\qquad\qquad\qquad\qquad\qquad\qquad$ T

Goal: $T = K_t \alpha$

$\uparrow$ Torsional stiffness (G, geometry)

Stress Formulation

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = 0, \quad \epsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \alpha y\right)$$

$$\epsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \alpha x\right)$$

<u>Stresses</u> Hooke's Law

$$\sigma_{zx} = 2G\epsilon_{zx} = G\left(\frac{\partial w}{\partial x} - \alpha y\right)$$

$$\sigma_{zy} = 2G\epsilon_{zy} = G\left(\frac{\partial w}{\partial y} + \alpha x\right)$$

All other stresses zero.

Equilibrium

$$\cancel{\frac{\partial \sigma_{xx}}{\partial x}} + \cancel{\frac{\partial \sigma_{xy}}{\partial y}} + \cancel{\frac{\partial \sigma_{xz}}{\partial z}} + \cancel{b_x} = 0 \quad ✓$$

$$\cancel{\frac{\partial \sigma_{xy}}{\partial x}} + \cancel{\frac{\partial \sigma_{yy}}{\partial y}} + \cancel{\frac{\partial \sigma_{yz}}{\partial z}} + \cancel{b_y} = 0 \quad ✓$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} + \cancel{b_z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \qquad (1) \quad \text{Eqm}$$

Need another equation.

$$\frac{\partial \sigma_{yz}}{\partial x} = G\left(\frac{\partial^2 w}{\partial x \partial y} + \alpha\right), \quad \frac{\partial \sigma_{xz}}{\partial y} = G\left(\frac{\partial^2 w}{\partial x \partial y} - \alpha\right)$$

$$\frac{\partial \sigma_{zx}}{\partial y} - \frac{\partial \sigma_{zy}}{\partial x} = -2G\alpha \qquad (2)$$

Hooke's Law + Strain Disp.

Need to solve (1), (2)

2 PDEs of 1^{st} order $\longrightarrow$ 1 PDE of 2^{nd} order

Dnyanesh Pawaskar

Introduce a scalar function $\varphi(x,y)$. Standard technique "potential theory"

Prandtl stress function

$$\sigma_{xz} = \frac{\partial \varphi}{\partial y}, \quad \sigma_{yz} = -\frac{\partial \varphi}{\partial x}$$

(1) Eqm is automatically taken care of.

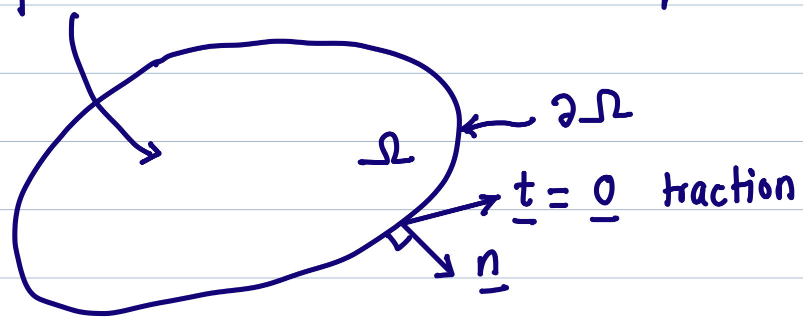
Need to solve (2)

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -2G\alpha \quad \text{in } \Omega$$

$$\nabla^2 \varphi = -2G\alpha \qquad \text{Poisson Eqn in } \Omega$$

given

Ω: c/s of shaft



Need $\varphi(x,y)$

What are BCs on $\partial\Omega$? Lateral surface of shaft

Traction Free $\underline{t} = \underset{\sim}{\sigma}\,\underline{n}$

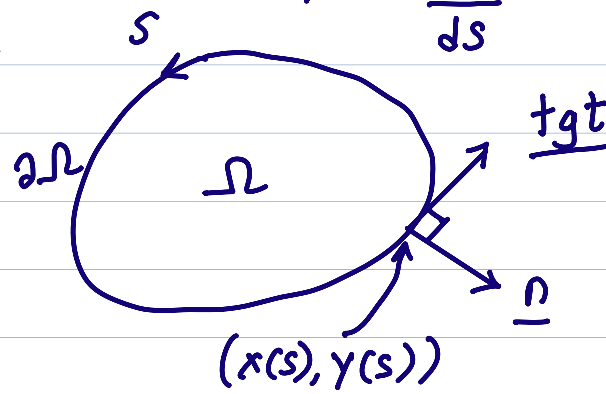
$$\begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{given traction free}} = \begin{pmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix} \begin{matrix} \checkmark \\ \checkmark \\ \checkmark \leftarrow \text{BC} \end{matrix}$$

$$t_z = 0 \Rightarrow \sigma_{xz} n_x + \sigma_{yz} n_y = 0$$

$$\frac{\partial \varphi}{\partial y}\frac{dy}{ds} - \frac{\partial \varphi}{\partial x}\left(-\frac{dx}{ds}\right) = 0 \; , \quad n_x = \frac{dy}{ds}$$

$$n_y = -\frac{dx}{ds}$$

$$\frac{d\varphi}{ds} = 0 \quad \text{along } \partial\Omega$$



$$\underline{tgt} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$$

$$\underline{n} = \underline{tgt} \times \underline{e}_z$$

$$= \left(\frac{dx}{ds}\underline{e}_x + \frac{dy}{ds}\underline{e}_y\right) \times \underline{e}_z$$

$$= \left(\frac{dy}{ds}\underline{e}_x - \frac{dx}{ds}\underline{e}_y\right)$$

$$\underbrace{\frac{\partial \varphi}{\partial x}\frac{dx}{ds} + \frac{\partial \varphi}{\partial y}\frac{dy}{ds}}_{\frac{d\varphi}{ds} = 0} = 0 \quad \text{on } \partial\Omega$$

$$\frac{d\varphi}{ds} = 0 \quad \text{along } \partial\Omega \Rightarrow \underset{\text{constant}}{\varphi = c} \quad \text{on } \partial\Omega$$

Arb set $c = 0$

alt, $\Phi = \varphi - c = 0$ on $\partial\Omega$

Dnyanesh Pawaskar

Torsion Problem Find $\varphi(x,y)$ s.t.

Poisson's Eqn $\nabla^2\varphi = -2G\alpha$ in Ω

with $\varphi = 0$ on $\partial\Omega$

Dirichlet BC

$$T = \int_\Omega (x\,\sigma_{yz} - y\,\sigma_{xz})\,dx\,dy$$

$$T = -\int_\Omega \left(x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y}\right) dx\,dy \longrightarrow (*)$$

An easier way to write $(*)$ is by applying Green / Gauss / Div Thm.

$$\frac{\partial}{\partial x}(\varphi x) = x\frac{\partial\varphi}{\partial x} + \varphi, \quad \frac{\partial}{\partial y}(\varphi y) = y\frac{\partial\varphi}{\partial y} + \varphi$$

$$T = -\int_\Omega \underbrace{\left(\frac{\partial}{\partial x}(\varphi x) + \frac{\partial}{\partial y}(\varphi y)\right)}_{\operatorname{div}\begin{pmatrix}\varphi x\\ \varphi y\end{pmatrix}} dx\,dy + 2\int_\Omega \varphi\,dx\,dy$$

$$= -\cancelto{0}{\int_{\partial\Omega} (\varphi_x n_x + \varphi_y n_y)\,ds} + 2\int_\Omega \varphi\,dx\,dy$$

$\varphi = 0$ on $\partial\Omega$ TBC

$$T = 2\int_\Omega \varphi\,dx\,dy$$

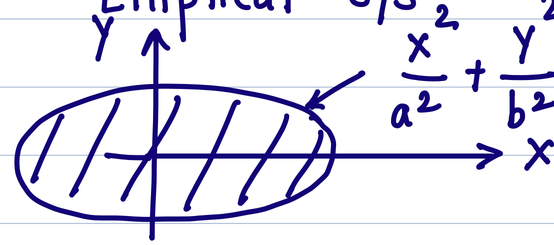
Dnyanesh Pawaskar

Div Thm

$$\int_{\Omega} \underline{\nabla} \cdot \underline{\Psi} \, dx\,dy = \int_{\partial\Omega} \underline{\Psi} \cdot \underline{n} \, ds$$

$$T = 2 \int_{\Omega} \varphi \, dx\,dy$$

<u>Example</u> Elliptical c/s



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Try $\varphi(x,y) = K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$ $\varphi = 0$ on $\partial\Omega$ ✓

$$\nabla^2\varphi = -2G\alpha \Rightarrow \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -2G\alpha$$ ✓

$$K\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = -2G\alpha \Rightarrow K = -\frac{G\alpha a^2 b^2}{a^2 + b^2}$$

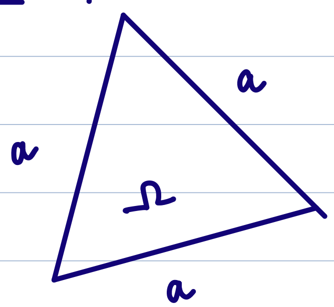
$$T = 2\int_{\Omega} \varphi \, dx\,dy = 2K \int_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dx\,dy$$

$$T = \underbrace{\frac{\pi a^3 b^3}{a^2 + b^2}}_{\text{Torsional stiffness}} G \alpha$$

$$\alpha = \frac{T(a^2 + b^2)}{\pi a^3 b^3 G}$$

can be expt verified in a lab

Problem — Equilateral Triangle cross-section

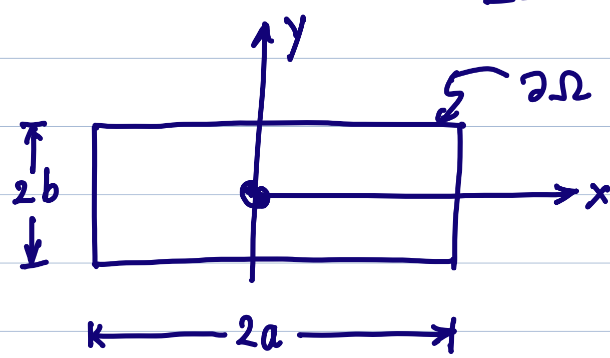


Goal $T = K_t \alpha$

Find $\varphi(x,y)$ s.t. $\nabla^2 \varphi = -2G\alpha$ inside Ω

$\varphi = 0$ on $\partial\Omega$

$\varphi = K \underbrace{F(x,y)}$ $\qquad T = 2 \int_\Omega \varphi \, da$



Try Same trick $\varphi(x,y) = K(x-a)(x+a)(y-b)(y+b)$

$= K(x^2 - a^2)(y^2 - b^2)$

$\downarrow$

$\varphi = 0$ on $\partial\Omega$ — Does not work

Dnyanesh Pawaskar

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = K(y^2-b^2)2 + K(x^2-a^2)2$$

$$\neq -2G\alpha \quad \forall \; x,y \in \Omega$$

for all

Work: Fourier series

Consider, thin rectangle $a \gg b$