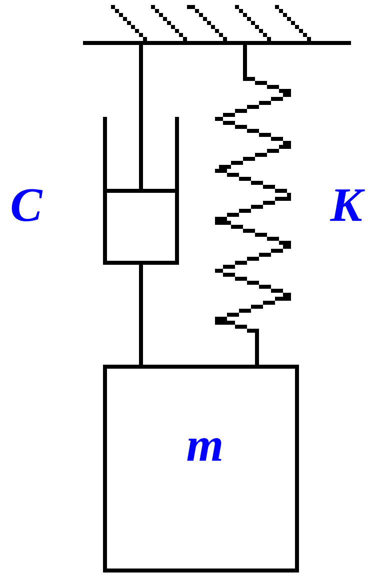
SECOND ORDER SYSTEM



$$a_2 \frac{d^2 q_o}{dt^2} + a_1 \frac{dq_o}{dt} + a_o q_o = b_o q_i$$

$$\frac{a_2}{a_o}\frac{d^2 q_o}{dt^2} + \frac{a_1}{a_o}\frac{dq_o}{dt} + q_o = \frac{b_o}{a_o} q_i$$

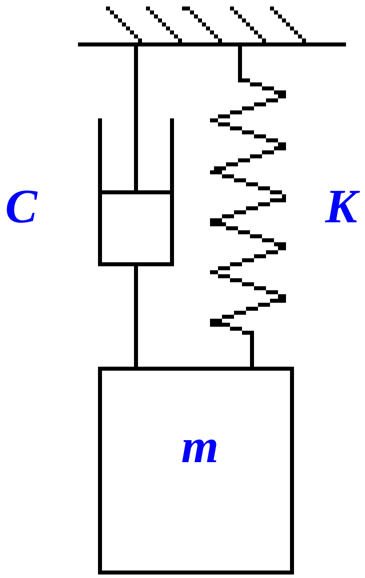
$$\frac{1}{\omega_n^2}\frac{d^2 q_o}{dt^2} + \frac{2\xi}{\omega_n}\frac{dq_o}{dt} + q_o = K q_i$$

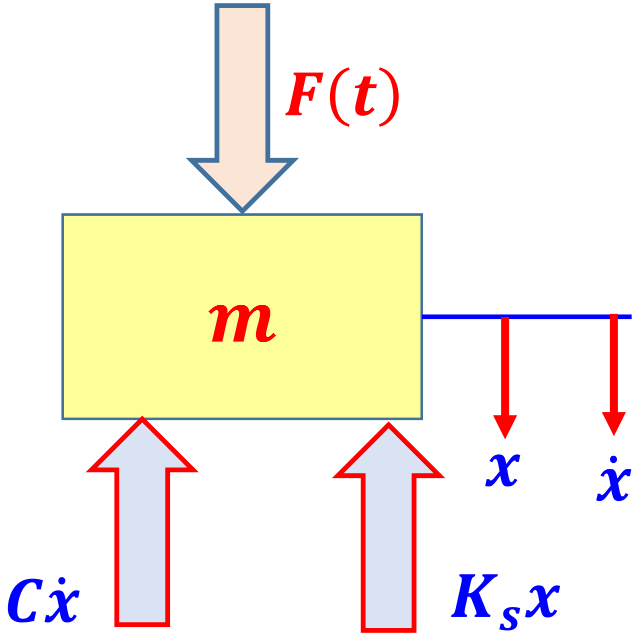
$\omega_n = \sqrt{\frac{a_o}{a_2}}$	*Undamped Natural Frequency*
$K = \frac{b_o}{a_o}$	*Static Sensitivity*
$\xi = \frac{a_1}{2\sqrt{a_o a_2}}$	**Damping Ratio**

$$\frac{a_1}{a_o} = \frac{a_1}{2\sqrt{a_o a_2}} \frac{2\sqrt{a_2}}{\sqrt{a_o}}$$

$$\frac{a_1}{a_o} = \xi \frac{2}{\omega_n} = \frac{2\xi}{\omega_n}$$

FREE BODY DIAGRAM OF THE SPRING MASS DAMPER SYSTEM





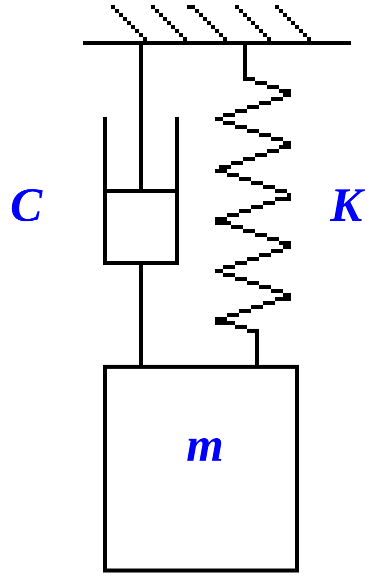
m – Mass
C — damping coefficient
K_s - Spring constant
x - Dispacement
$\dot{x}$ - Velocity $\left(\frac{dx}{dt}\right)$
$\ddot{x}$ - Acceleration $\left(\frac{d^2x}{dt^2}\right)$

Applying Newton's Law of Motion, the force equation can be written as

$$m\ddot{x} = F(t) - C\dot{x} - K_s x$$

$$m\ddot{x} + C\dot{x} + K_s x = F(t)$$

SECOND ORDER MECHANICAL SYSTEM



$$m\ddot{x} + C\dot{x} + K_s x = F(t)$$

$$m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + K_s x = F(t)$$

$$\frac{m}{K_s}\frac{d^2x}{dt^2} + \frac{C}{K_s}\frac{dx}{dt} + x = \frac{1}{K_s}F(t)$$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n}\frac{dx}{dt} + x = KF(t)$$

$$\omega_n = \sqrt{\frac{K_s}{m}};\ K = \frac{1}{K_s};\ \xi = \frac{C}{2\sqrt{mK_s}}$$

$$\frac{2\xi}{\omega_n} = 2\frac{C}{2\sqrt{mK_s}}\sqrt{\frac{m}{K_s}} = \frac{C}{K_s}$$

$\omega_n = \sqrt{\frac{K_s}{m}}$	*Undamped Natural Frequency*
$K = \frac{1}{K_s}$	*Static Sensitivity*
$\xi = \frac{C}{2\sqrt{mK_s}}$	**Damping Ratio**

STEP RESPONSE OF SECOND ORDER INSTRUMENT

$$m\ddot{x} + C\dot{x} + K_s x = F(t)$$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n}\frac{dx}{dt} + x = KF(t)$$

$$\omega_n = \sqrt{\frac{K_s}{m}};\, K = \frac{1}{K_s};\, \xi = \frac{C}{2\sqrt{mK_s}}$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\frac{1}{\omega_n^2}\frac{d^2q_o}{dt^2} + \frac{2\xi}{\omega_n}\frac{dq_o}{dt} + q_o = Kq_{is}$$

$$\frac{d^2q_o}{dt^2} + 2\xi\,\omega_n\frac{dq_o}{dt} + \omega_n^2 q_o = Kq_{is}\omega_n^2$$

$$s^2q_o(s) - \cancel{sq_o(0)} - \cancel{q_o'(0)} + 2\xi\,\omega_n\left(sq_o(s) - \cancel{q_o(0)}\right) + \omega_n^2 q_o(s) = \frac{Kq_{is}\omega_n^2}{s}$$

$$s^2q_o(s) + 2\xi\,\omega_n s q_o(s) + \omega_n^2 q_o(s) = \frac{Kq_{is}\omega_n^2}{s}$$

$$q_o(s)\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right) = \frac{Kq_{is}\omega_n^2}{s}$$

$$q_o(s)\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right) = \frac{K q_{is}\omega_n^2}{s}$$

$$\frac{q_o(s)}{K q_{is}} = \frac{\omega_n^2}{s\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right)}$$

The dynamic behaviour of the second order system can then be described in terms of two parameters namely ξ and ω_n

$0 < \xi < 1 \Rightarrow$ Underdamped system

$\xi = 1 \quad \Rightarrow$ Critically damped system

$\xi > 1 \quad \Rightarrow$ Overdamped system

The transient response of critically damped and overdamped system do not oscillate. If $\xi = 0$, the transient response does not die out.

UNDERDAMPED SYSTEM (0 < ξ < 1)

$$\frac{q_o(s)}{Kq_{is}} = \frac{\omega_n^2}{s\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right)} = \left(\frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\,\omega_n s + \omega_n^2}\right)\omega_n^2$$

$$\frac{q_o(s)}{Kq_{is}} = \frac{\omega_n^2}{s\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right)} = \left(\frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\,\omega_n s + \omega_n^2}\right)$$

$$A\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right) + (Bs + C)s = \omega_n^2$$

$Coefficient\ of\ s^2;\quad A + B = 0$

$Coefficient\ of\ s;\quad A(2\xi\,\omega_n) + C = 0$

$Coefficient\ of\ s^0;\quad A\omega_n^2 = \omega_n^2$

$A = 1;\ \ \mathrm{B} = -1;\ \ \mathrm{C} = -2\xi\,\omega_n$

$$\frac{q_o(s)}{Kq_{is}} = \left(\frac{1}{s} + \frac{(-s - 2\xi\,\omega_n)}{s^2 + 2\xi\,\omega_n s + \omega_n^2}\right) = \left(\frac{1}{s} - \frac{s + 2\xi\,\omega_n}{s^2 + 2\xi\,\omega_n s + \omega_n^2}\right)$$

UNDERDAMPED SYSTEM ($0 < \xi < 1$)

$$\frac{q_o(s)}{Kq_{is}} = \left(\frac{1}{s} - \frac{s + 2\xi\,\omega_n}{s^2 + 2\xi\,\omega_n s + \omega_n^2}\right)$$

$$\frac{q_o(s)}{Kq_{is}} = \frac{1}{s} - \frac{s + 2\xi\,\omega_n}{(s + \xi\,\omega_n)^2 - \xi^2\omega_n^2 + \omega_n^2} = \frac{1}{s} - \frac{s + 2\xi\,\omega_n}{(s + \xi\,\omega_n)^2 + \omega_n^2(1 - \xi^2)}$$

Damped Natural Frequency $\omega_d = \omega_n\sqrt{1 - \xi^2}$

$$\frac{q_o(s)}{Kq_{is}} = \frac{1}{s} - \frac{s + 2\xi\,\omega_n}{(s + \xi\,\omega_n)^2 + \omega_d^2} = \frac{1}{s} - \frac{s + \xi\,\omega_n}{(s + \xi\,\omega_n)^2 + \omega_d^2} - \frac{\xi\,\omega_n}{(s + \xi\,\omega_n)^2 + \omega_d^2}$$

$$\frac{q_o(s)}{Kq_{is}} = \frac{1}{s} - \frac{s + \xi\,\omega_n}{(s + \xi\,\omega_n)^2 + \omega_d^2} - \frac{\xi\,\omega_n}{(s + \xi\,\omega_n)^2 + \omega_d^2}$$

$f(t)$	$\mathcal{L}(f(t))$
$e^{at}\cos\omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
$e^{at}\sin\omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t}\cos\omega_d t - \frac{\xi\,\omega_n}{\omega_d}e^{-\xi\,\omega_n t}\sin\omega_d t$$

UNDERDAMPED SYSTEM ($0 < \xi < 1$)

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t} cos\omega_d t - \frac{\xi\,\omega_n}{\omega_d} e^{-\xi\,\omega_n t} sin\omega_d t$$

Damped Natural Frequency $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t} cos\omega_d t - \frac{\xi\,\omega_n}{\omega_n\sqrt{1-\xi^2}} e^{-\xi\,\omega_n t} sin\omega_d t$$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t}\left(cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} sin\omega_d t\right)$$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\xi\,\omega_n t} sin\left(\omega_d t + Tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)$$

CRITICALLY DAMPED SYSTEM (ξ = 1)

$$\frac{q_o(s)}{Kq_{is}} = \frac{\omega_n^2}{s\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right)} = \frac{\omega_n^2}{s\left(s^2 + 2\,\omega_n s + \omega_n^2\right)} = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$\frac{q_o(s)}{Kq_{is}} = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$$

$$A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs = \omega_n^2$$

$$Coefficient\ of\, s^2; \quad A + B = 0$$

$$Coefficient\ of\, s; \quad 2\,\omega_n A + B\,\omega_n + C = 0$$

$$Coefficient\ of\, s^0; \quad A\omega_n^2 = \omega_n^2$$

$$A = 1;\ \ \mathrm{B} = -1;\ \ \mathrm{C} = -\,\omega_n$$

$$\frac{q_o(s)}{Kq_{is}} = \left(\frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}\right) = \left(\frac{1}{s} + \frac{-1}{(s+\omega_n)} + \frac{-\,\omega_n}{(s+\omega_n)^2}\right)$$

$$\frac{q_o(s)}{Kq_{is}} == \left(\frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{(s+\omega_n)^2}\right)$$

CRITICALLY DAMPED SYSTEM (ξ = 1)

$$\frac{q_o(s)}{Kq_{is}} = \left(\frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{(s+\omega_n)^2}\right)$$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t}\frac{t^{2-1}}{(2-1)!} = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+a)^n}\right) = \frac{1}{(n-1)!} t^{n-1} e^{-at} \quad n = 1, 2, 3$$

OVER DAMPED SYSTEM (ξ > 1)

$$\frac{q_o(s)}{Kq_{is}} = \frac{\omega_n^2}{s\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right)}$$

$$s^2 + 2\xi\,\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-2\xi\,\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = -\xi\,\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$\frac{q_o(s)}{Kq_{is}} = \frac{\omega_n^2}{s\left(s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)\left(s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)}$$

$$\frac{q_o(s)}{Kq_{is}} = \omega_n^2\left(\frac{A}{s} + \frac{B}{s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}} + \frac{C}{s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}}\right)$$

$$\frac{q_o(s)}{Kq_{is}} = \omega_n^2\left(\frac{A}{s} + \frac{B}{s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}} + \frac{C}{s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}}\right)$$

$$\frac{q_o(s)}{Kq_{is}} = \omega_n^2\left(\frac{A}{s} + \frac{B}{s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}} + \frac{C}{s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}}\right)$$

$$\frac{q_o(s)}{Kq_{is}\omega_n^2} = \frac{1}{s\left(s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)\left(s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)}$$

$$A\left(s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)\left(s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}\right) + Bs\left(s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}\right) + Cs\left(s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}\right) = 1$$

$$s = 0 \;\Rightarrow A\left(\xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)\left(\xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}\right) = 1$$

$$s = 0 \;\Rightarrow A\left(\xi^2\omega_n^2 - \omega_n^2(\xi^2 - 1)\right) = 1 \Rightarrow A(\xi^2\omega_n^2 - \xi^2\omega_n^2 + \omega_n^2) = 1 \Rightarrow A(\omega_n^2) = 1$$

$$A = \frac{1}{\omega_n^2}$$

$$A\left(s+\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}\right)+Bs\left(s+\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}\right)$$
$$+Cs\left(s+\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}\right)=1$$

$$s=-\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}$$

$$\Rightarrow B\left(-\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}\right)\left(-\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}+\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}\right)=1$$

$$B\left(-\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}\right)\left(-\,2\omega_n\sqrt{\xi^2-1}\right)=1$$

$$B\left(2\xi\omega_n^2\sqrt{\xi^2-1}+2\omega_n^2(\xi^2-1)\right)=1$$

$$B\left(2\omega_n^2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)\right)=1$$

$$\boxed{B=\frac{1}{2\omega_n^2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)}}$$

$$A\left(s+\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}\right)+Bs\left(s+\xi\,\omega_n-\omega_n\sqrt{\xi^2-1}\right)$$
$$+\,Cs\left(s+\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}\right)=1$$

$$s=-\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}$$

$$\Rightarrow C\left(-\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(-\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}+\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}\right)=1$$

$$C\left(-\xi\,\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(2\omega_n\sqrt{\xi^2-1}\right)=1$$

$$C\left(-2\xi\omega_n^2\sqrt{\xi^2-1}+2\omega_n^2(\xi^2-1)\right)=1$$

$$C\left(2\omega_n^2\sqrt{\xi^2-1}\left(-\xi+\sqrt{\xi^2-1}\right)\right)=1$$

$$C=\frac{1}{2\omega_n^2\sqrt{\xi^2-1}\left(-\xi+\sqrt{\xi^2-1}\right)}$$

$$\frac{q_o(s)}{Kq_{is}} = \omega_n^2\left(\frac{A}{s} + \frac{B}{s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}} + \frac{C}{s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}}\right)$$

$$A = \frac{1}{\omega_n^2} \qquad B = \frac{1}{2\omega_n^2\sqrt{\xi^2 - 1}\left(\xi + \sqrt{\xi^2 - 1}\right)} \qquad C = \frac{1}{2\omega_n^2\sqrt{\xi^2 - 1}\left(-\xi + \sqrt{\xi^2 - 1}\right)}$$

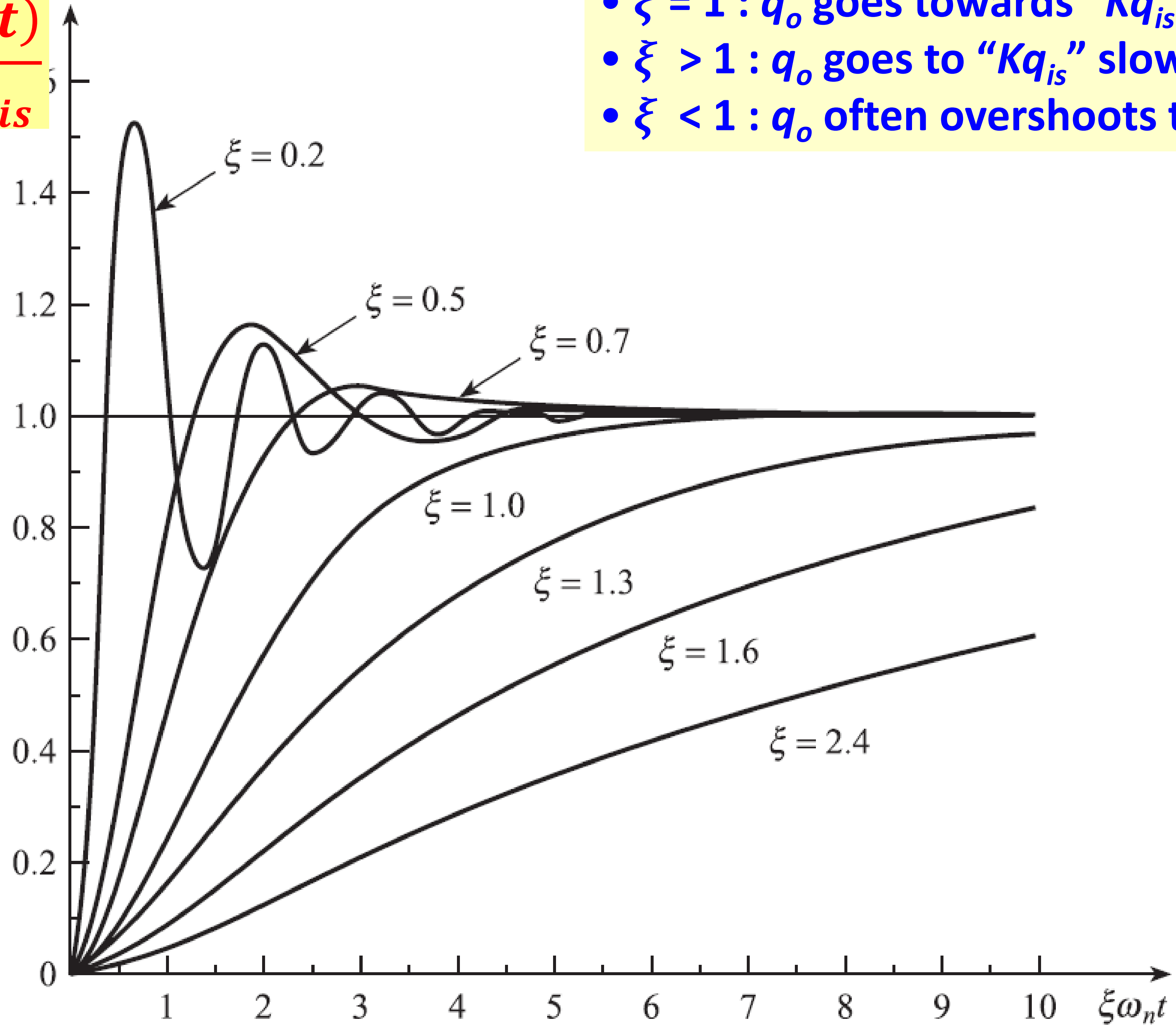
$$\frac{q_o(s)}{Kq_{is}} = \omega_n^2\left(\frac{\frac{1}{\omega_n^2}}{s} + \frac{\frac{1}{2\omega_n^2\sqrt{\xi^2 - 1}\left(\xi + \sqrt{\xi^2 - 1}\right)}}{s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}} + \frac{\frac{1}{2\omega_n^2\sqrt{\xi^2 - 1}\left(-\xi + \sqrt{\xi^2 - 1}\right)}}{s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}}\right)$$

$$\frac{q_o(s)}{Kq_{is}} = \left(\frac{1}{s} + \frac{1}{2\sqrt{\xi^2 - 1}\left(\xi + \sqrt{\xi^2 - 1}\right)}\frac{1}{s + \omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)} + \frac{1}{2\sqrt{\xi^2 - 1}\left(-\xi + \sqrt{\xi^2 - 1}\right)}\frac{1}{s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}}\right)$$

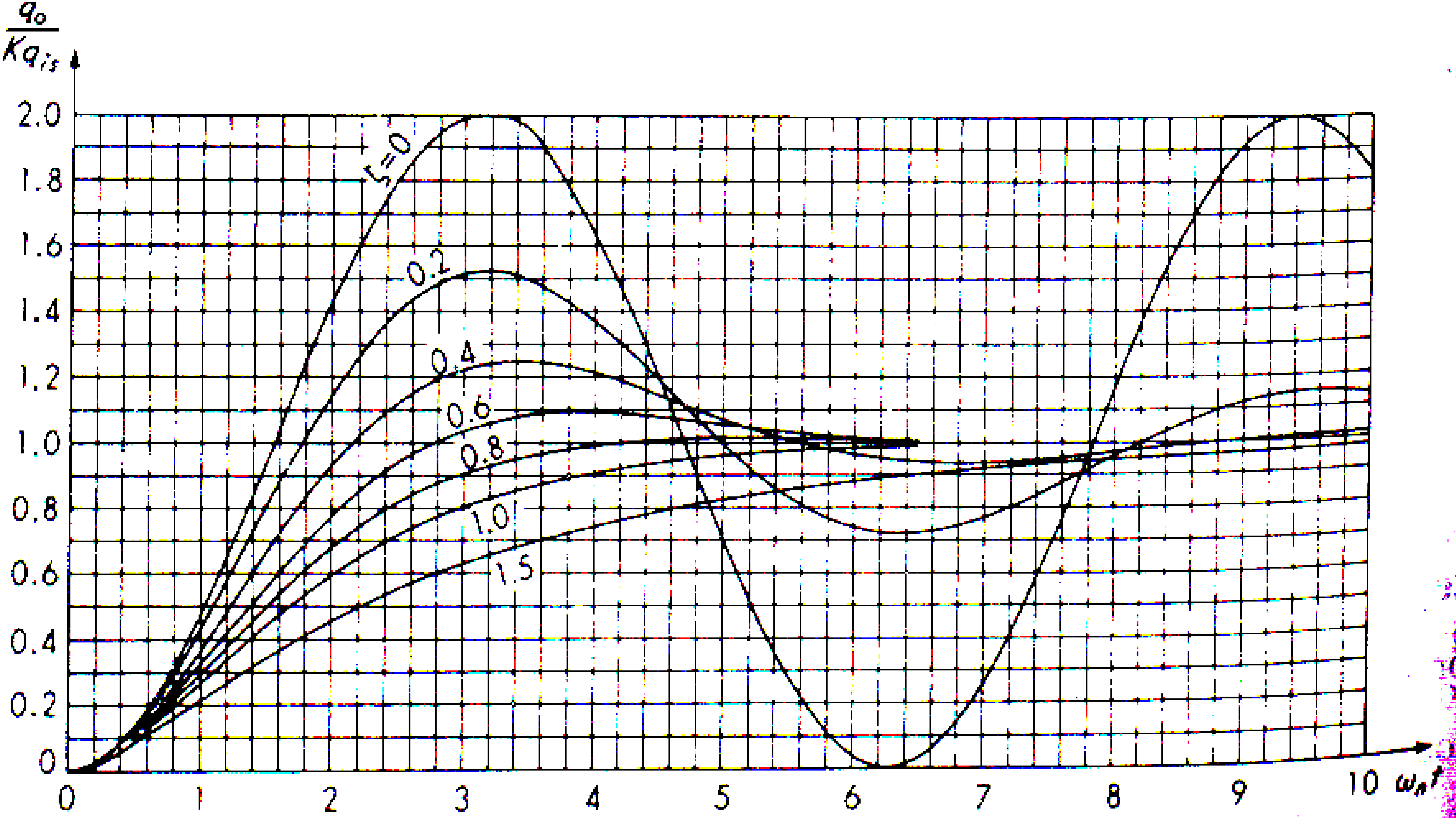
$$\frac{q_o(s)}{Kq_{is}} = \left(\frac{1}{s} + \frac{1}{2\sqrt{\xi^2 - 1}\left(\xi + \sqrt{\xi^2 - 1}\right)}\frac{1}{s + \omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)} + \frac{1}{2\sqrt{\xi^2 - 1}(-1)\left(\xi - \sqrt{\xi^2 - 1}\right)}\frac{1}{s + \omega_n\left(\xi - \sqrt{\xi^2 - 1}\right)}\right)$$

$$\frac{q_o(t)}{Kq_{is}} = 1 + \frac{1}{2\sqrt{\xi^2 - 1}\left(\xi + \sqrt{\xi^2 - 1}\right)}e^{-\left(\xi + \sqrt{\xi^2 - 1}\right)\omega_n t} - \frac{1}{2\sqrt{\xi^2 - 1}\left(\xi - \sqrt{\xi^2 - 1}\right)}e^{-\left(\xi - \sqrt{\xi^2 - 1}\right)\omega_n t}$$

$$\frac{q_o(t)}{Kq_{is}}$$



- $\xi = 1$: q_o goes towards "Kq_{is}" in a particular position
- $\xi > 1$: q_o goes to "Kq_{is}" slower
- $\xi < 1$: q_o often overshoots the final value "Kq_{is}"



$\frac{q_o}{Kq_{is}}$
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
$\zeta=0$
0.2
0.4
0.6
0.8
1.0
1.5
0
1
2
3
4
5
6
7
8
9
10
$\omega_n t$

- ξ = 1 - Good for $\omega_n t$ >7
- ξ > 1 - Sluggish response
- ξ < 1 - Response has oscillatory behaviour. Output can go higher that the input magnitude.

- For ξ = 0.6, around $\omega_n t$ = 7 - Good for the magnitude.

- ξ = 0.6 even for $\omega_n t$ = 2.5 or so, the magnitude ratio is also quite good - one has to fix the acceptable tolerance levels.

- For all underdamped cases, there are certain discrete values of $\omega_n t$ where the solution is acceptable.

- Lowest possible values of $\omega_n t$ are desirable since for a given ω_n, the time required to get the proper output is smallest

- ξ = 0 amplitude does not blow up since forcing function itself provides damping since it is constant.

FREQUENCY RESPONSE OF SECOND ORDER INSTRUMENT

$$m\ddot{x} + C\dot{x} + K_s x = F(t)$$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n}\frac{dx}{dt} + x = KF(t)$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\frac{1}{\omega_n^2}\frac{d^2q_o}{dt^2} + \frac{2\xi}{\omega_n}\frac{dq_o}{dt} + q_o = Kq_i$$

$$\frac{d^2q_o}{dt^2} + 2\xi\,\omega_n\frac{dq_o}{dt} + \omega_n^2 q_o = Kq_i\omega_n^2$$

$$s^2q_o(s) - s\cancel{q_o(0)} - \cancel{q_o'}(0) + 2\xi\,\omega_n\big(sq_o(s) - q_o(0)\big) + \omega_n^2 q_o(s) = Kq_i(s)\omega_n^2$$

$$\frac{q_o(s)}{Kq_i(s)} = \frac{\omega_n^2}{\left(s^2 + 2\xi\,\omega_n s + \omega_n^2\right)}$$

$$\frac{q_o(s)}{Kq_i(s)} = \frac{\omega_n^2}{(s^2 + 2\xi\,\omega_n s + \omega_n^2)}$$

For frequency response, replace s by iω to get the response for sinusoidal input

$$\frac{q_o(i\omega)}{Kq_i(i\omega)} = \frac{\omega_n^2}{(i^2\omega^2 + 2\xi\,\omega_n(i\omega) + \omega_n^2)} = \frac{1}{\left(\left(-\frac{\omega^2}{\omega_n^2}\right) + \frac{2\xi\omega}{\omega_n}(i) + 1\right)}$$

$$\frac{q_o(i\omega)}{Kq_i(i\omega)} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + i\frac{2\xi\omega}{\omega_n}} \times \frac{\left(1 - \frac{\omega^2}{\omega_n^2}\right) - i\frac{2\xi\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) - i\frac{2\xi\omega}{\omega_n}} = \frac{\left(1 - \frac{\omega^2}{\omega_n^2}\right) - i\frac{2\xi\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}{\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2\right)^2}}$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

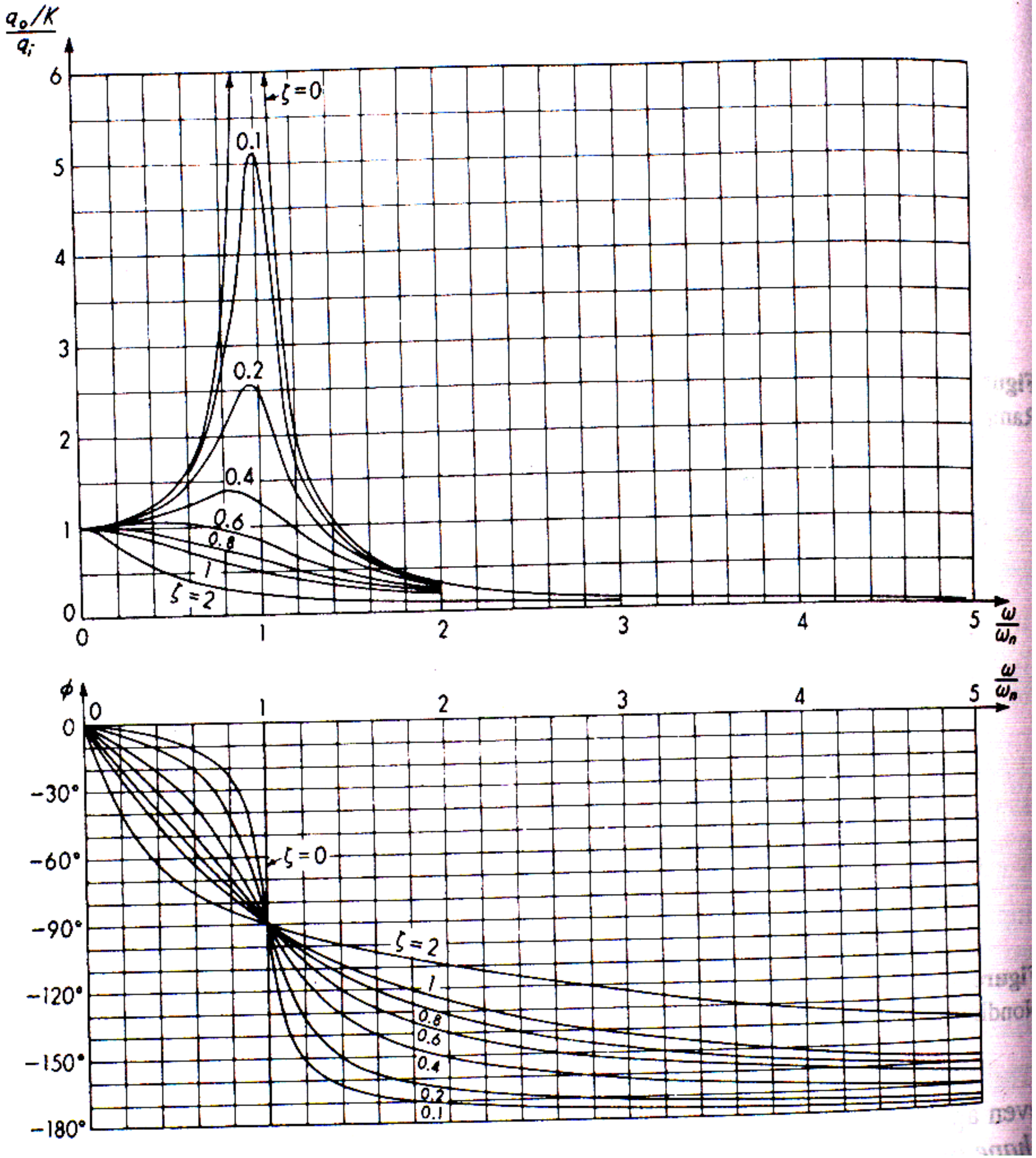
$$\frac{q_o(i\omega)}{Kq_i(i\omega)} = \frac{\left(1-\frac{\omega^2}{\omega_n^2}\right) - i\frac{2\xi\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

$$Tan\phi = \left(\frac{-\frac{2\xi\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

Positive ϕ - Angle by which the output leads the input

Negative ϕ - Angle by which the output lags the input



- Small (ω/ω_n) $\rightarrow$ shape is flat, desirable
- True for ξ = 0.6 – 0.7
- If need to measure higher "ω" need high "ω_n"
- When input is periodic at (ω/ω_n) there is resonance and therefore amplitude ratio is very large for ξ= 0 ;

At resonance, the input and output are in phase and amplitude keeps growing

PHASE LINEARITY

Signal distortion can be illustrated by considering a particular complex waveform represented by a general function, $u(t)$:

$$u(t) = \sin\omega t + sin2\omega t$$

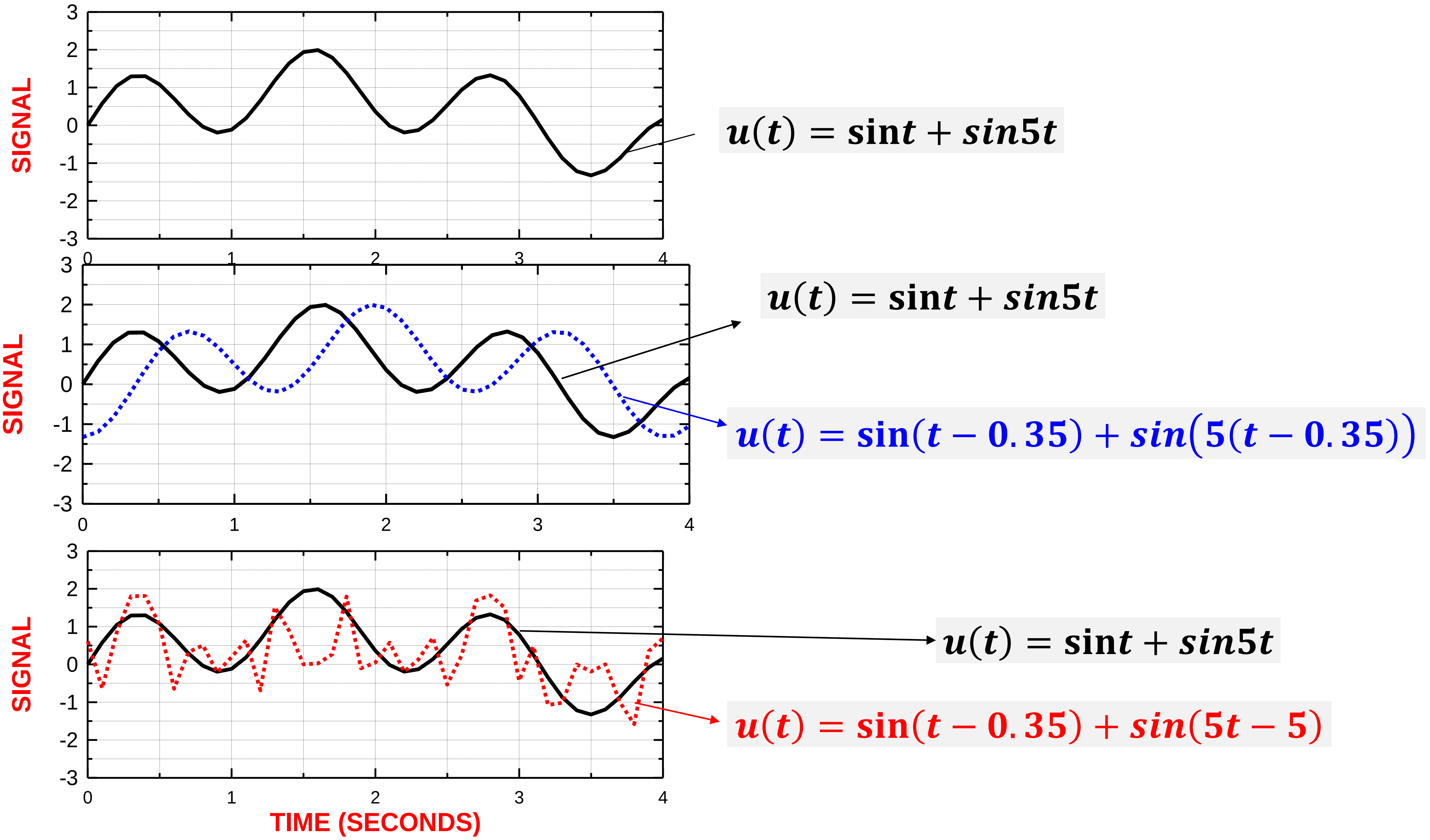
Suppose during a measurement a phase shift of this signal were to occur such that the phase shift remained linearly proportional to the frequency; that is, the measured signal, v(t), could be represented by

$$v(t) = \sin(\omega t - \phi) + sin(2\omega t - 2\phi)$$

$$\theta = (\omega t - \phi)$$

$$v(t) = \sin\theta + sin2\theta$$

$v(t)$ in the above equation is equivalent to the original signal, $u(t)$.
If the phase shift were not linearly related to the frequency, this would not be so.


SIGNAL
$u(t) = \sin t + \sin 5t$
SIGNAL
$u(t) = \sin t + \sin 5t$
$u(t) = \sin(t - 0.35) + \sin(5(t - 0.35))$
SIGNAL
$u(t) = \sin t + \sin 5t$
$u(t) = \sin(t - 0.35) + \sin(5t - 5)$
TIME (SECONDS)

An accelerometer is to be selected to measure a time dependent motion. In particular, input signal frequencies below 100 Hz are of prime interest. Select a set of acceptable parameter specifications for the instrument, assuming a dynamic error of ± 10%. $\xi = 0.707$

$$f \leq 100\ Hz \Rightarrow \omega \leq 2\pi(100) \leq 628\ rad/s$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

$$0.9 = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

$$x = \frac{\omega^2}{\omega_n^2}$$

$$\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2 = \frac{1}{(0.9)^2} \Rightarrow (1-x)^2+4\xi^2 x = 1.235$$

$$1-2x+x^2+4\xi^2 x = 1.235$$

$$x^2-\left(2-4\xi^2\right)x-0.235 = 0$$

$$x^2-\left(2-4(0.707)^2\right)x-0.235 = 0$$

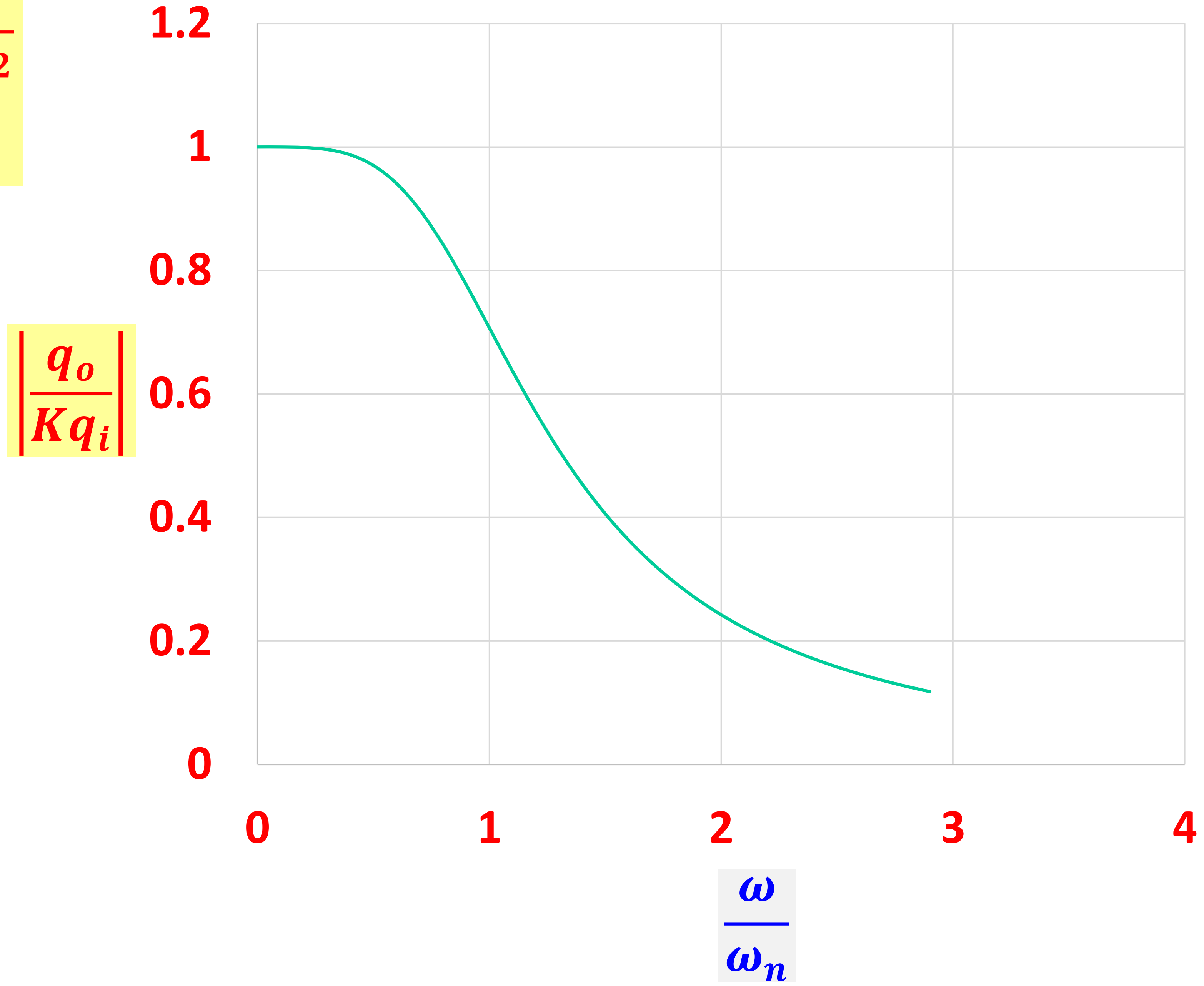
$$x^2-6.04\times10^{-4}x-0.235 = 0$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

$\left|\frac{q_o}{Kq_i}\right|$ *is always less than* 1

Hence, take 0.9 but not 1.1

$$0.9 = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$



$$x^2 - 6.04 \times 10^{-4} x - 0.235 = 0$$

$$x = \frac{6.04 \times 10^{-4} \pm \sqrt{(6.04 \times 10^{-4})^2 - 4(0.235)}}{2}$$

$$x = 0.4851 = \frac{\omega^2}{\omega_n^2}$$

$$\frac{\omega}{\omega_n} = 0.6965 \Rightarrow \omega_n = \frac{628}{0.6965} = 901.65\, rad/s$$

$$\omega_n = 901.65\, rad/s$$

A second order instrument is subjected to a sinusoidal input. Undamped natural frequency is 3 Hz and the damping ratio is 0.5. calculate amplitude ratio and phase angle for an input of 2 Hz.

$$f = 2\ Hz \Rightarrow \omega = 2\pi(2) = 12.57\frac{rad}{s}$$

$$f_n = 3\ Hz \Rightarrow \omega_n = 2\pi(3) = 18.85\ rad/s$$

$$\frac{\omega}{\omega_n} = \frac{2}{3};\ \xi = 0.5$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{1}{\left(1-\frac{2^2}{3^2}\right)^2 + \left(\frac{2\times 0.5\times 2}{3}\right)^2}} = 1.153$$

$$\left|\frac{q_o}{Kq_i}\right| = 1.153$$

$$\theta = -50.6^\circ$$

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

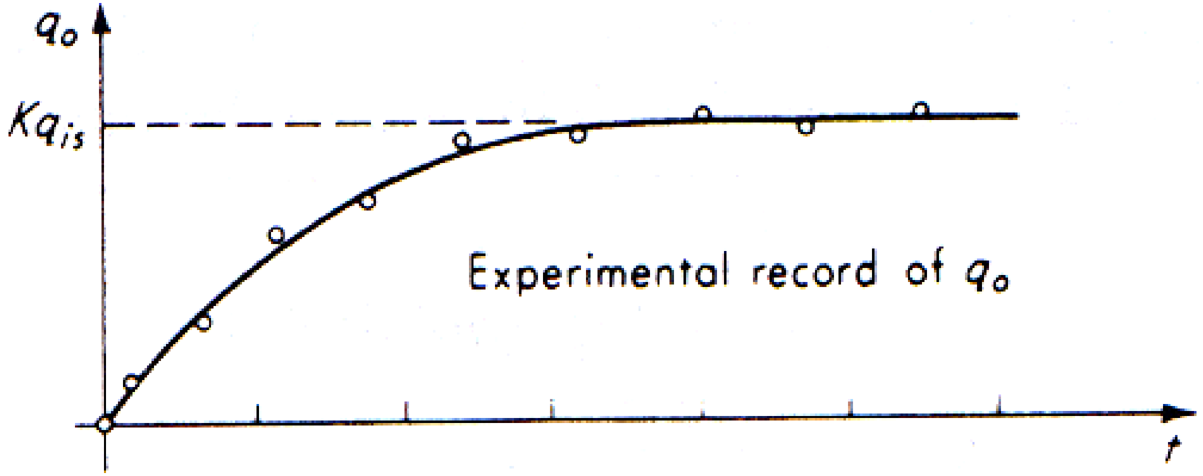
$$Tan\theta = \left(\frac{-\frac{2\xi\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

$$Tan\theta = \left(\frac{-\left(\frac{2\times 0.5\times 2}{3}\right)}{1-\frac{2^2}{3^2}}\right)$$

First order Instruments: Step response of a first order system

$$\tau \frac{dq_o}{dt} + q_o = Kq_{is}$$

$$\frac{q_o(t)}{Kq_{is}} = 1 -$$



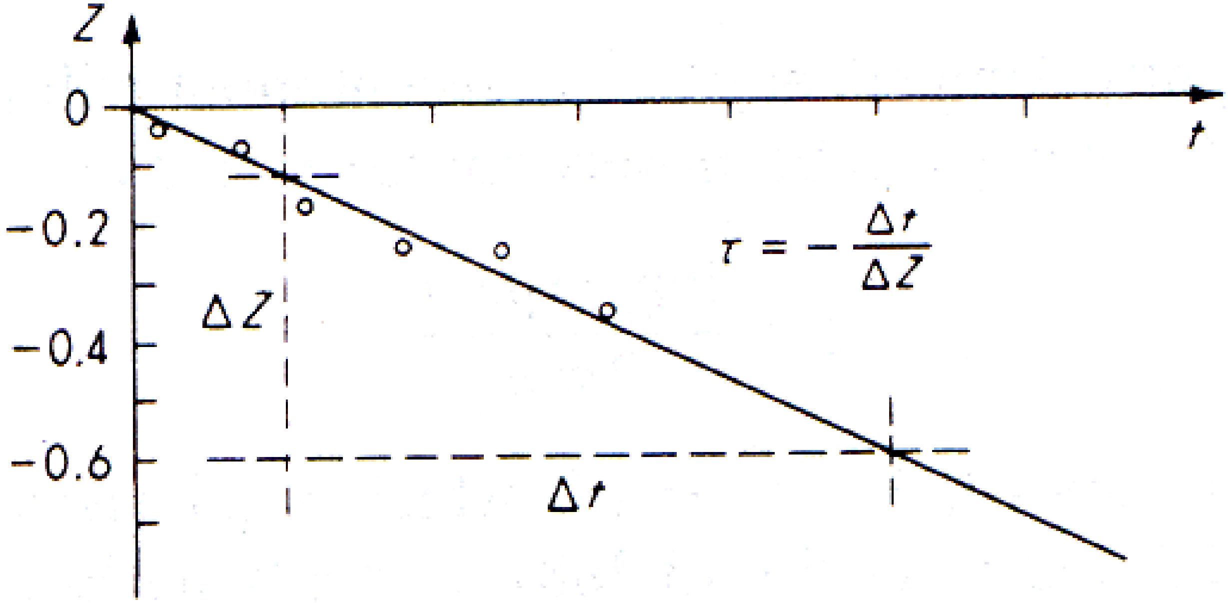
$$\frac{q_o(t)}{Kq_{is}} = 1 - e^{-\frac{t}{\tau}}$$

$$\frac{q_o(t)}{Kq_{is}} - 1 = -e^{-\frac{t}{\tau}}$$

$$1 - \frac{q_o(t)}{Kq_{is}} = e^{-\frac{t}{\tau}}$$

$$\ln\left(1 - \frac{q_o(t)}{Kq_{is}}\right) = -\frac{t}{\tau}$$

$$\ln\left(1 - \frac{q_o(t)}{Kq_{is}}\right) = Z = -\frac{t}{\tau}$$



Bode diagram representation of the frequency response for a system

Bode diagram consists of two graphs

- a curve of the logarithm of the magnitude of the sinusoidal transfer function
- a curve for the phase angle

Both the curves are plotted against the frequency on a logarithmic scale

$$\frac{q_o(s)}{q_i(s)} = \frac{K}{1+\tau s} = \frac{K}{1+\tau(i\omega)} = G(i\omega)$$

$$\left|\frac{q_o(i\omega)}{q_i(i\omega)}\right| = |G(i\omega)|$$

$$dB = 20\log\left|\frac{q_o(i\omega)}{q_i(i\omega)}\right| = 20\, log|G(i\omega)|$$

Bode diagram representation of the frequency response for a system

$$\left|\frac{q_o(i\omega)}{q_i(i\omega)}\right| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$dB = 20\log\left|\frac{q_o(i\omega)}{q_i(i\omega)}\right| = 20\log\left(\frac{1}{\sqrt{1+\tau^2\omega^2}}\right) = -20\log\left(\sqrt{1+\tau^2\omega^2}\right)$$

$$For\ low\ frequencies\ \ or\ \omega\tau\ \lll 1 \Rightarrow dB = -20\log\left(\sqrt{1+\tau^2\omega^2}\right) \approx -20\log(\sqrt{1}) = 0$$

$$For\ high\ frequencies\ \ or\ \omega\tau\ \ggg 1 \Rightarrow dB = -20\log\left(\sqrt{\tau^2\omega^2}\right) \approx -20\log(\tau\omega)$$

$$For\ \omega = \frac{1}{\tau};\quad dB \approx -20\log(\tau\omega) \approx -20\log\left(\tau\frac{1}{\tau}\right) = -20\log(1) = 0$$

$$For\ \omega = \frac{10}{\tau};\quad dB \approx -20\log(\tau\omega) \approx -20\log\left(\tau\frac{10}{\tau}\right) = -20\log(10) = -20$$

Bode diagram representation of the frequency response for a system

$$dB = -20\log\left(\sqrt{1+\tau^2\omega^2}\right)$$

$For\ low\ frequencies\ \ or\ \omega\tau\ \lll 1 \Rightarrow dB = 0$

$For\ high\ frequencies\ \ or\ \omega\tau\ \ggg 1 \Rightarrow dB = \ -20\log(\tau\omega)$

$For\ \omega = \frac{1}{\tau};\quad dB = 0$

$For\ \omega = \frac{10}{\tau};\quad dB = \text{-}20$

Hence, the value of $-20\log(\tau\omega)\ dB$ decreases by 20 dB for every decade

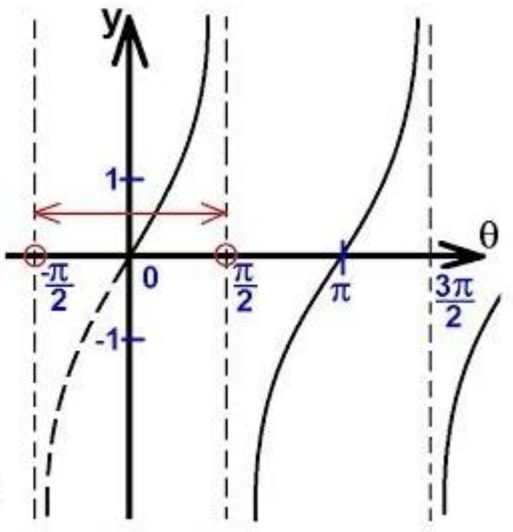
$$\phi = Tan^{-1}(-\tau\omega)$$

$\omega \longrightarrow 0, \phi = Tan^{-1}(-\tau\omega) = Tan^{-1}(0) = 0°\,;$

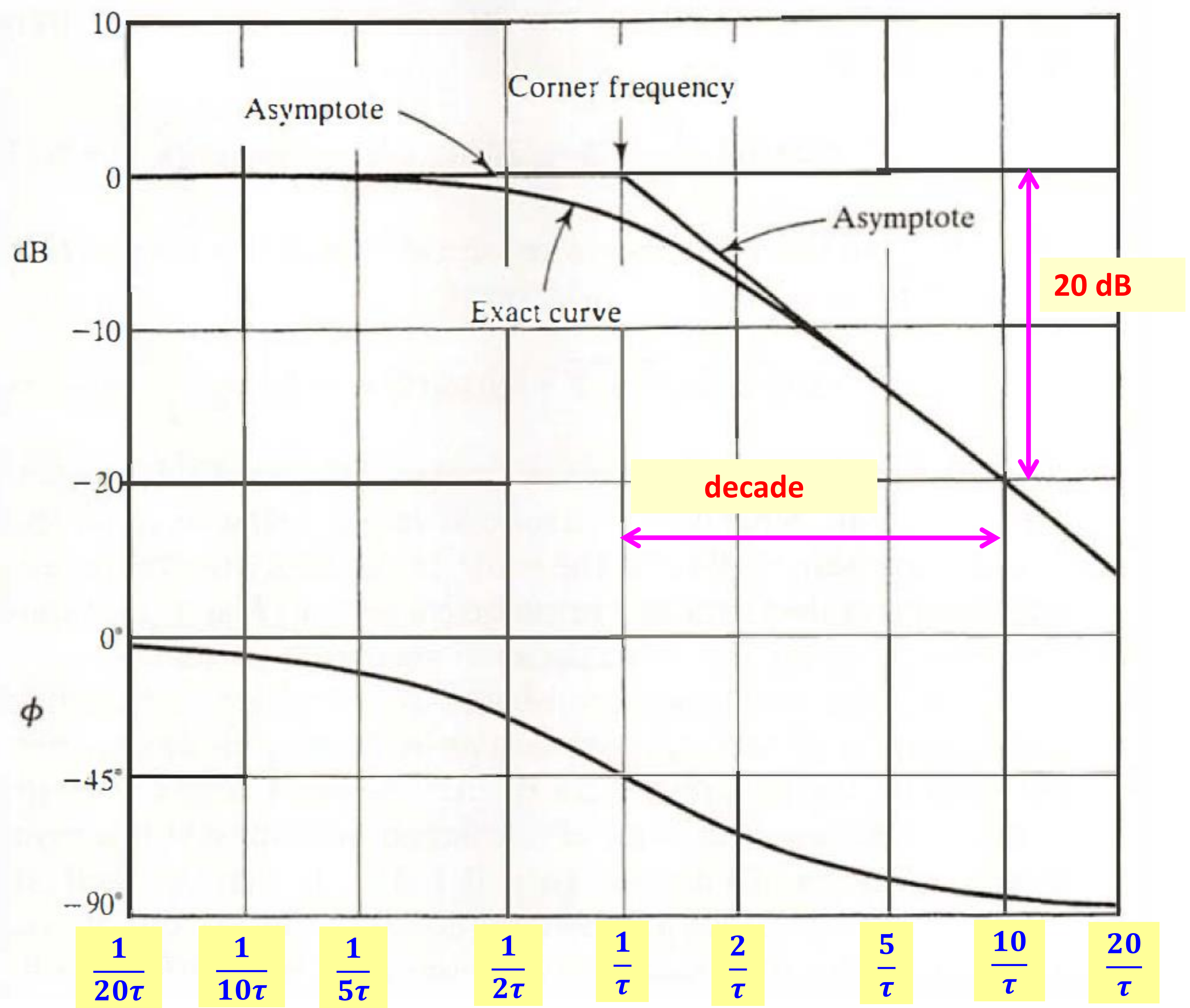
$\omega = \frac{1}{\tau}, \phi = Tan^{-1}(-\tau\omega) = Tan^{-1}\left(-\tau\frac{1}{\tau}\right) = Tan^{-1}(-1) = -45°\,;$

$\omega \longrightarrow \infty, \phi = Tan^{-1}(-\tau\omega) = Tan^{-1}(-\infty) = -90°\,;$

θ (degree)	θ (radian)	tanθ	
0	0	0	0
30	$\pi/6$	$1/\sqrt{3}$	0.577
45	$\pi/4$	1	1
60	$\pi/3$	$\sqrt{3}$	1.732
90	$\pi/2$	∞	∞
120	$2\pi/3$	$-\sqrt{3}$	-1.732
135	$3\pi/4$	-1	-1
150	$5\pi/6$	$-1/\sqrt{3}$	-0.577



Bode diagram representation of the frequency response for a system



Bode diagram representation of the frequency response for a second order system

$$\left|\frac{q_o}{Kq_i}\right| = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

$$dB = 20\log\left|\frac{q_o(i\omega)}{q_i(i\omega)}\right| = 20\log\left(\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}}\right) = -20\log\left(\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\xi\omega}{\omega_n}\right)^2}\right)$$

Bode diagram representation of the frequency response for a second order system

$$\textit{For low frequencies or } \omega \lll \omega_n \Rightarrow dB = -20\log\left(\sqrt{1}\right) \approx -20\log(1) = 0$$

$$\textit{For high frequencies or } \omega \ggg \omega_n \Rightarrow dB = -20\log\left(\frac{\omega^2}{\omega_n^2}\right) \approx -40\log\left(\frac{\omega}{\omega_n}\right)$$

$$dB = -20\log\left(\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}\right) = -20\log\left(\sqrt{\left(1-2\frac{\omega^2}{\omega_n^2}+\frac{\omega^4}{\omega_n^4}\right) + 4\xi^2\frac{\omega^2}{\omega_n^2}}\right)$$

$$1 \ll \frac{\omega^4}{\omega_n^4};\ 2\frac{\omega^2}{\omega_n^2} \ll \frac{\omega^4}{\omega_n^4};\quad 4\xi^2\frac{\omega^2}{\omega_n^2} \ll \frac{\omega^4}{\omega_n^4}$$

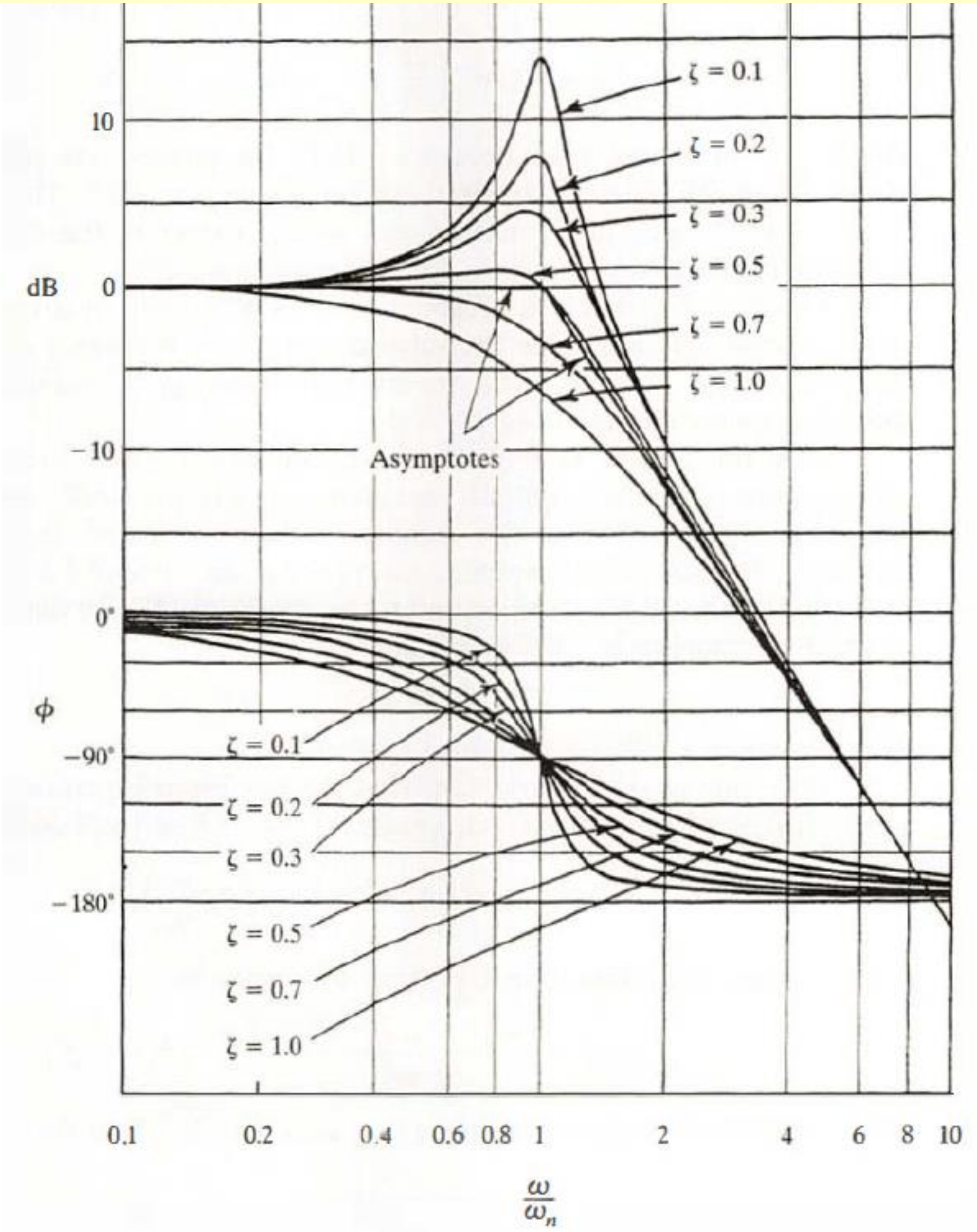
$$dB = -20\log\left(\sqrt{\left(\frac{\omega^4}{\omega_n^4}\right)}\right) = -20\log\left(\frac{\omega^2}{\omega_n^2}\right) = -40\log\left(\frac{\omega}{\omega_n}\right)$$

High frequency asymptote – Straight line having a slope = $-40\ dB/decade$

Low frequency asymptote – Straight line having a slope = $0\ dB/decade$

Asymptotes are independent of ξ

Bode diagram representation of the frequency response for a second order system



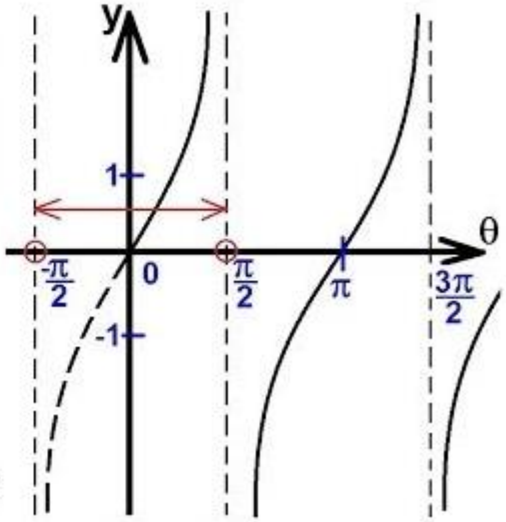
$$Tan\phi = \left(\frac{-\frac{2\xi\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}} \right)$$

$$\omega = 0, \phi = Tan^{-1}(0) = 0° \,;$$

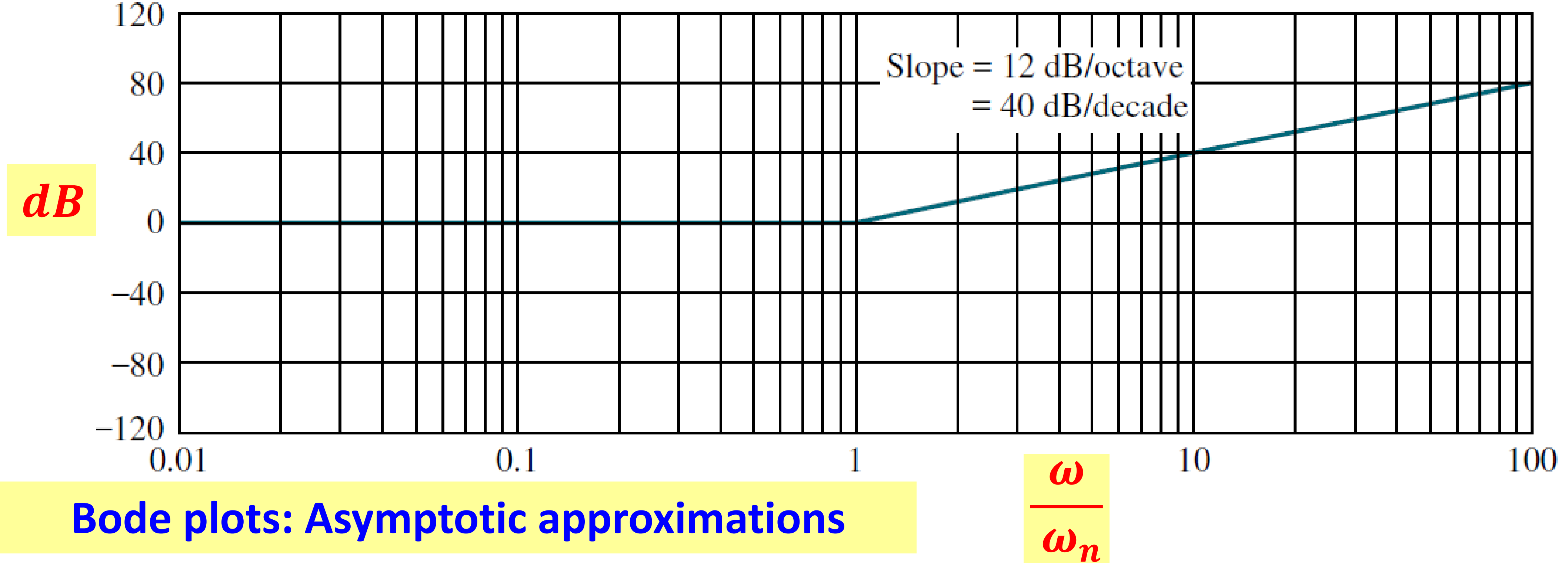
$$\frac{\omega}{\omega_n} = 1, \phi = Tan^{-1}\left(-\frac{2\xi}{0}\right) = Tan^{-1}(-\infty) = -90° \,;$$

$$\frac{\omega}{\omega_n} \longrightarrow \infty, \phi = Tan^{-1}(0) = -180° \,;$$

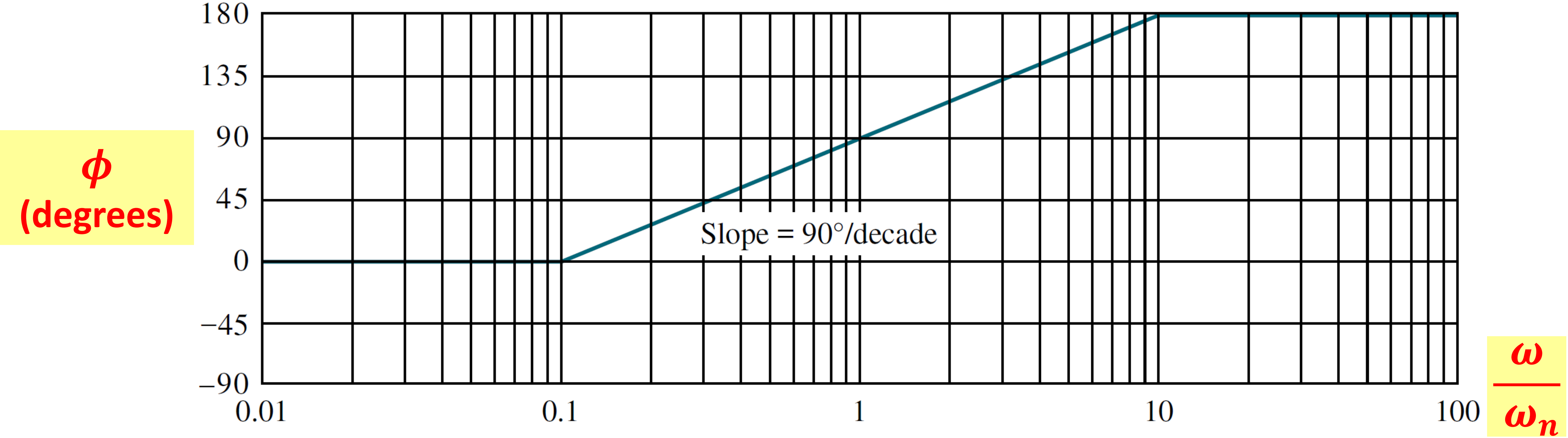
θ (degree)	θ (radian)	tanθ	
0	0	0	0
30	$\pi/6$	$1/\sqrt{3}$	0.577
45	$\pi/4$	1	1
60	$\pi/3$	$\sqrt{3}$	1.732
90	$\pi/2$	∞	∞
120	$2\pi/3$	$-\sqrt{3}$	-1.732
135	$3\pi/4$	-1	-1
150	$5\pi/6$	$-1/\sqrt{3}$	-0.577



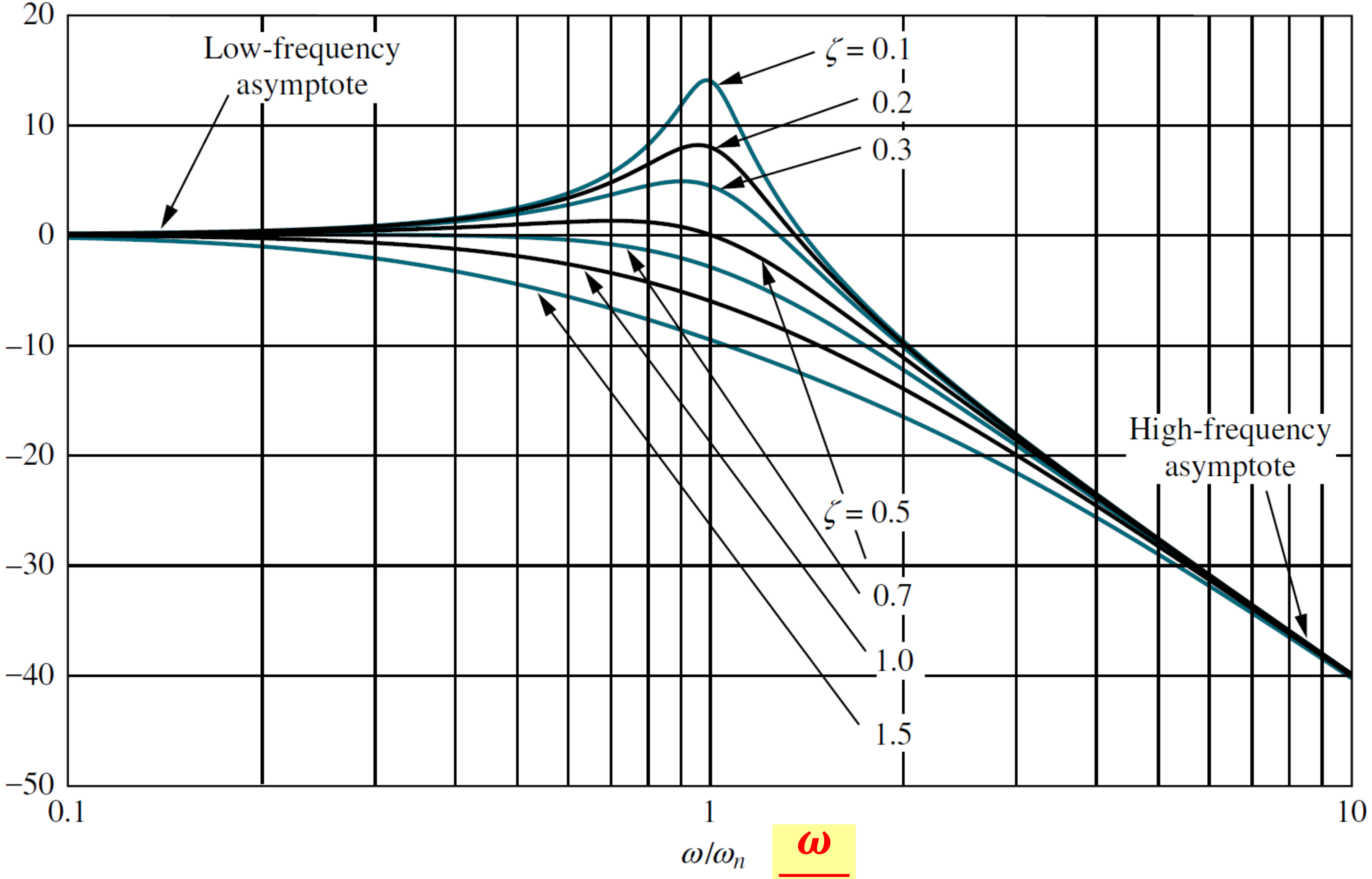
Bode diagram representation of the frequency response for a second order system



Bode plots: Asymptotic approximations

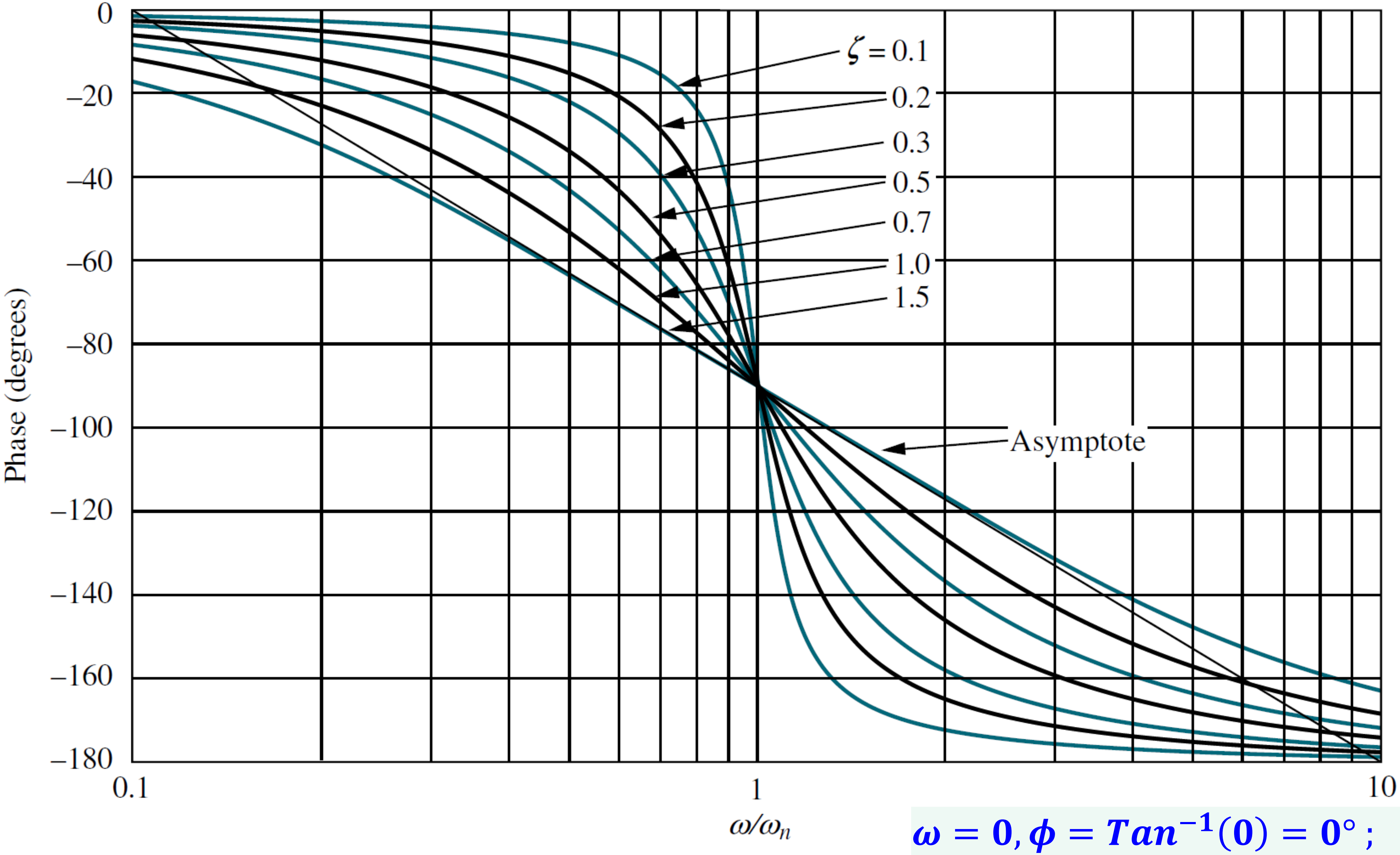


Bode diagram representation of the frequency response for a second order system



dB

$\frac{\omega}{\omega_n}$



Bode diagram representation of the frequency response for a second order system

$$Tan\phi = \left(\frac{-\frac{2\xi\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

$\omega = 0, \phi = Tan^{-1}(0) = 0°$;

$\omega = 1, \phi = Tan^{-1}\left(-\frac{2\xi}{0}\right) = Tan^{-1}\infty = 0$;

$\omega \longrightarrow \infty, \phi = Tan^{-1}(0) = 180°$;

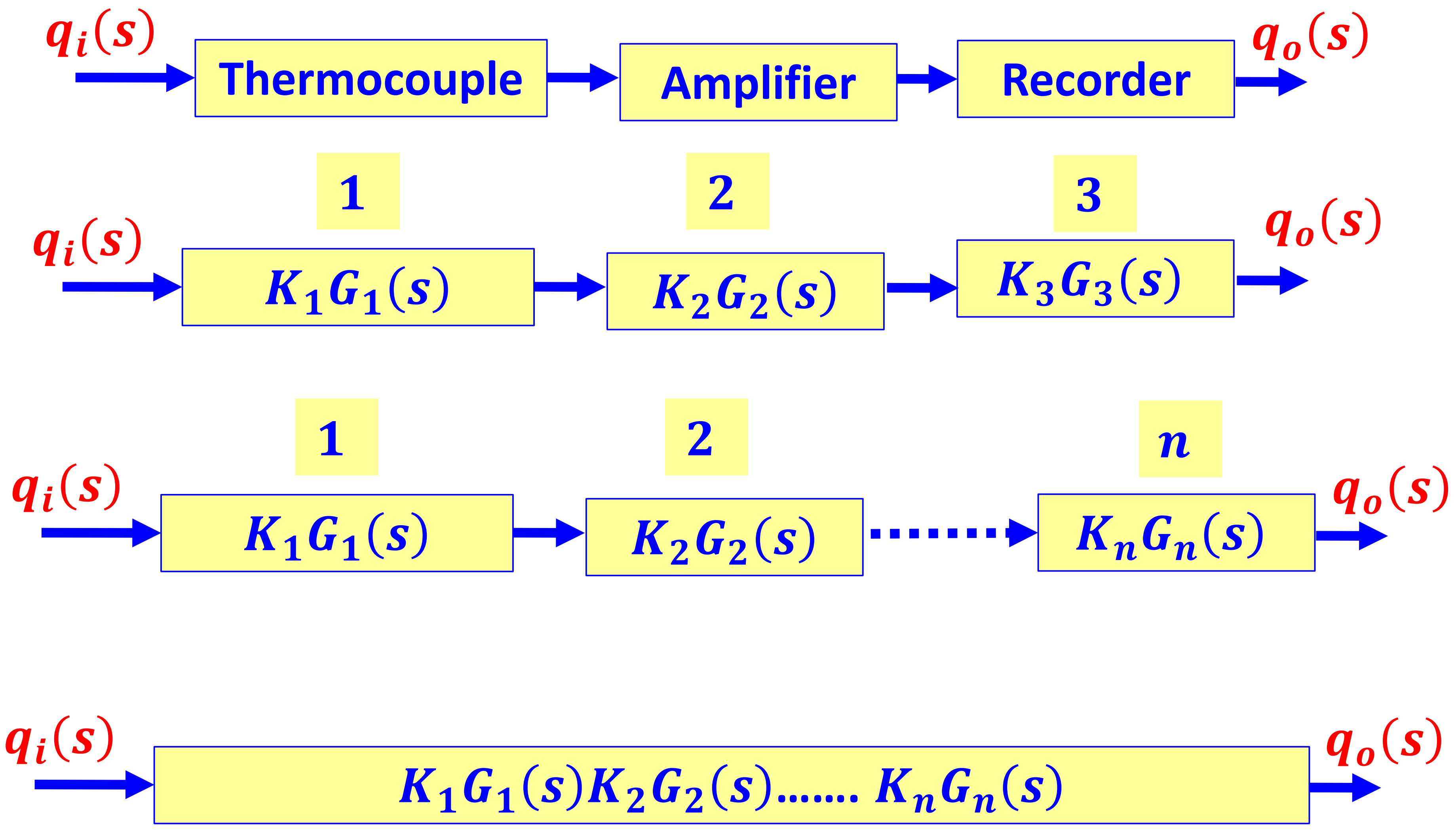
COUPLED SYSTEMS

As instruments in each stage of a measurement system are connected (transducer, signal conditioner, output device, etc.), the output from one stage becomes the input to the next stage to which it is connected, and so forth.

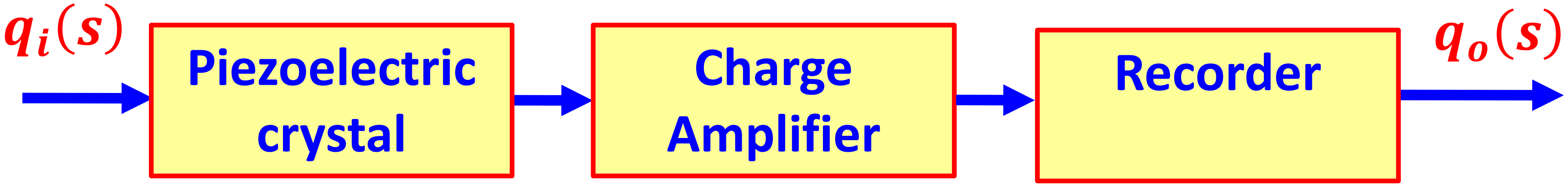
The overall measurement system will have a coupled output response to the original input signal that is a combination of each individual response to the input.

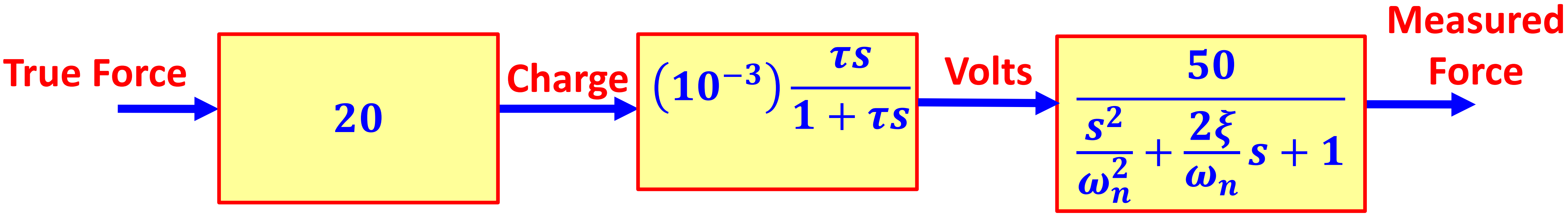
The system concepts of zero-, first-, and second-order systems studied previously can still be used for a case-by-case study of the coupled measurement system.

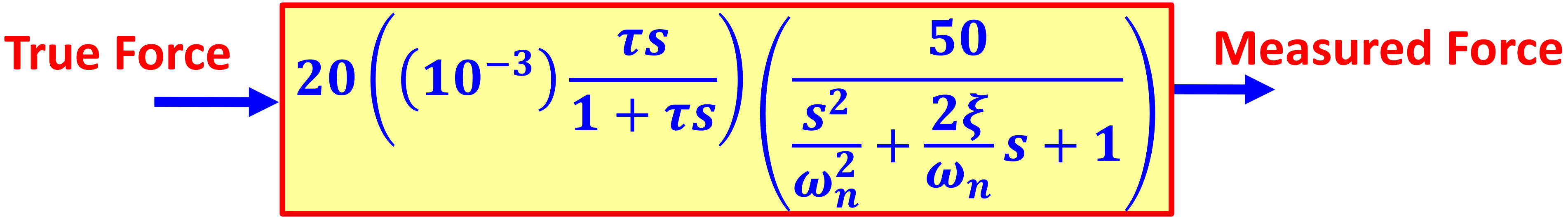
COUPLED SYSTEMS



FORCE MEASUREMENT SYSTEM







This holds true provided that significant loading effects do not exist

LOGARITHMIC PLOTTING OF FREQUENCY RESPONSE CURVES

$$q_i(s) \longrightarrow \boxed{\dfrac{K_1}{1 + i\omega\tau_1}} \longrightarrow \boxed{\dfrac{K_2}{1 + i\omega\tau_2}} \longrightarrow \boxed{\dfrac{K_3}{1 + i\omega\tau_3}} \dashrightarrow q_o(s)$$

$$\frac{q_o(s)}{q_i(s)} = \left(\frac{K_1}{1 + i\omega\tau_1}\right)\left(\frac{K_2}{1 + i\omega\tau_2}\right)\left(\frac{K_3}{1 + i\omega\tau_3}\right) \ldots\ldots\ldots\ldots$$

$$\frac{q_o(s)}{q_i(s)} = \left(\frac{K_1}{1 + i\omega\tau_1}\right)\left(\frac{K_2}{1 + i\omega\tau_2}\right)\left(\frac{K_3}{1 + i\omega\tau_3}\right) \ldots\ldots\ldots\ldots$$

$$\left|\frac{q_o(s)}{q_i(s)}\right| = \left(\frac{K_1}{\sqrt{1 + \tau_1^2\omega^2}}\right)\left(\frac{K_2}{\sqrt{1 + \tau_2^2\omega^2}}\right)\left(\frac{K_3}{\sqrt{1 + \tau_3^2\omega^2}}\right) \ldots\ldots\ldots\ldots$$

$$dB = 20\log\left|\frac{q_o(s)}{q_i(s)}\right| = 20\log\left(\frac{K_1}{\sqrt{1 + \tau_1^2\omega^2}}\right) + 20\log\left(\frac{K_2}{\sqrt{1 + \tau_2^2\omega^2}}\right) + 20\log\left(\frac{K_3}{\sqrt{1 + \tau_3^2\omega^2}}\right) \ldots\ldots\ldots\ldots$$

LOGARITHMIC PLOTTING OF FREQUENCY RESPONSE CURVES

$$q_i(s) \rightarrow \boxed{\frac{K_1}{1+i\omega\tau_1}} \rightarrow \boxed{\frac{K_2}{1+i\omega\tau_2}} \rightarrow \boxed{\frac{K_3}{1+i\omega\tau_3}} \dashrightarrow q_o(s)$$

$$\frac{q_o(s)}{q_i(s)} = \left(\frac{K_1}{1+i\omega\tau_1}\right)\left(\frac{K_2}{1+i\omega\tau_2}\right)\left(\frac{K_3}{1+i\omega\tau_3}\right)\dots\dots\dots\dots$$

$$dB = 20\log\left|\frac{q_o(s)}{q_i(s)}\right| = 20\log\left(\frac{K_1}{\sqrt{1+\tau_1^2\omega^2}}\right) + 20\log\left(\frac{K_2}{\sqrt{1+\tau_2^2\omega^2}}\right) + 20\log\left(\frac{K_3}{\sqrt{1+\tau_3^2\omega^2}}\right)\dots\dots\dots\dots$$

$$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$$

$$Tan\theta = \frac{ad+bc}{ac-bd}; \quad Tan\theta_1 = \frac{b}{a}; \; Tan\theta_2 = \frac{d}{c};$$

$$Tan(\theta_1+\theta_2) = \frac{Tan\theta_1 + Tan\theta_2}{1 - Tan\theta_1 Tan\theta_2} = \frac{\frac{b}{a}+\frac{d}{c}}{1-\frac{bd}{ac}} = \frac{bc+ad}{ac-bd} = Tan\theta$$

Positive ϕ - Angle by which the output leads the input

Negative ϕ - Angle by which the output lags the input

LOGARITHMIC PLOTTING OF FREQUENCY RESPONSE CURVES

$$q_i(s) \rightarrow \boxed{\frac{K_1}{1+i\omega\tau_1}} \rightarrow \boxed{\frac{K_2}{1+i\omega\tau_2}} \rightarrow \boxed{\frac{K_3}{1+i\omega\tau_3}} \dashrightarrow q_o(s)$$

$$\frac{q_o(s)}{q_i(s)} = \left(\frac{K_1}{1+i\omega\tau_1}\right)\left(\frac{K_2}{1+i\omega\tau_2}\right)\left(\frac{K_3}{1+i\omega\tau_3}\right) \dots \dots \dots \dots$$

$$dB = 20\log\left|\frac{q_o(s)}{q_i(s)}\right| = 20\log\left(\frac{K_1}{\sqrt{1+\tau_1^2\omega^2}}\right) + 20\log\left(\frac{K_2}{\sqrt{1+\tau_2^2\omega^2}}\right) + 20\log\left(\frac{K_3}{\sqrt{1+\tau_3^2\omega^2}}\right) \dots \dots \dots \dots$$

$$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$$

$$Tan\theta = \frac{ad+bc}{ac-bd}; \quad Tan\theta_1 = \frac{b}{a}; \; Tan\theta_2 = \frac{d}{c};$$

$$Tan(\theta_1+\theta_2) = Tan\theta$$

Positive ϕ - Angle by which the output leads the input
Negative ϕ - Angle by which the output lags the input

∴ Phase angles just add up

Composite : algebraic addition of the logarithmic amplitudes and the phase angles

Questions to be answered after completing this module

1. Determine output characteristics of the second order instrument with the following input
   a. Step Input
   b. Ramp Input
   c. Sinusoidal Input
   d. Impulse input
2. How does one determine a given system is first order system by applying step input ?
3. How does one determine the order of the system by applying frequency response testing and drawing Bode plots ?
4. How does one determine the transfer function of coupled systems ?
5. How is logarithmic plotting beneficial in handling coupled systems ?