Chapter 6

COORDINATE SYSTEMS

All finite element solutions require the evaluation of integrals. Some of these are easily evaluated while others are very difficult. Many are impossible to evaluate analytically so that numerical techniques are employed.

The difficulties associated with evaluating an integral can often be decreased by changing the variables of integration. This involves writing the integral in a new coordinate system. The objective of this chapter is to discuss several coordinate systems that can be used to eliminate some of the difficulties associated with finite element integrals.

Local and natural coordinate systems are discussed in this chapter. These systems are discussed relative to the one-dimensional linear element, and then the two-dimensional triangular and rectangular elements.

6.1 LOCAL COORDINATE SYSTEMS

The linear shape functions developed in Chapter 2,

$$N_i(x)=\frac{X_j-x}{L} \quad \text{and} \quad N_j(x)=\frac{x-X_i}{L} \tag{6.1}$$

are for an element in which the origin of the coordinate system is to the left of node i. These are general equations valid for all linear elements regardless of their location. The disadvantage of these shape functions shows up when evaluating integrals involving products of the shape functions such as

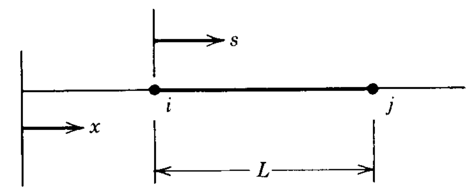
$$\int_{X_i}^{X_j} N_i(x)N_j(x)\,dx \quad \text{or} \quad \int_{X_i}^{X_j} N_i^2(x)\,dx \tag{6.2}$$

Integrals similar to these occur in the consideration of both field problems and solid mechanics problems. The integrations in (6.2) are simplified by developing new shape functions defined relative to a coordinate system whose origin is located on the element. This type of system is called a local coordinate system.

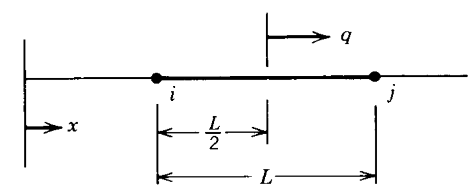
The two most common local coordinate systems for the one-dimensional element have the origin located at node i or at the center of the element (Figure 6.1).

The shape functions for a coordinate system located at node i are obtained from (6.1) by replacing x by $x=X_i+s$. This substitution produces

$$N_i(s)=\frac{X_j-x}{L}=\frac{X_j-(X_i+s)}{L}=1-\frac{s}{L} \tag{6.3}$$



(a)



(b)

Figure 6.1 Local coordinate systems for the one-dimensional element.

and

$$N_j(s)=\frac{x-X_i}{L}=\frac{X_i+s-X_i}{L}=\frac{s}{L} \tag{6.4}$$

Note that each shape function equals one at its own node and is zero at the other node. The two sum to one as did the pair in (6.1).

The shape functions for a coordinate system located at the center of the element are obtained from (6.1) by replacing x by $x=X_i+(L/2)+q$. The shape functions relative to this origin are

$$N_i(q)=\left(\frac{1}{2}-\frac{q}{L}\right) \quad \text{and} \quad N_j(q)=\left(\frac{1}{2}+\frac{q}{L}\right) \tag{6.5}$$

where the coordinate variable q ranges from $-L/2$ to $L/2$.

The shape functions, (6.3) and (6.4), as well as the pair in (6.5), are useful only if a change in the integration variables is performed. The change of variable formula from integral calculus (Olmstead, 1961) is

$$\int_a^b f(x)\,dx=\int_{p_1}^{p_2} f(g(p))\left[\frac{d(g(p))}{dp}\right]dp \tag{6.6}$$

where p is the new coordinate variable and $g(p)$ is the equation relating x and p, that is, $x=g(p)$.

Interpretation of (6.6) relative to the coordinate systems in Figure 6.1 goes as follows. For the coordinate s, where $x=X_i+s$

$$\int_{X_i}^{X_j} f(x)\,dx = \int_{s_1}^{s_2} \frac{d(X_i+s)}{ds}\,ds = \int_0^L h(s)\,ds \tag{6.7}$$

where $h(s)$ is $f(x)$ written in terms of s. The limits of integration were obtained by substituting X_i and X_j for x in $x = X_i + s$ and solving for s.

For the coordinate q, where $x = X_i + L/2 + q$

$$\int_{X_i}^{X_j} f(x)\,dx = \int_{q_1}^{q_2} r(q)\frac{d(X_i+L/2+q)}{dq}\,dq = \int_{-L/2}^{L/2} r(q)\,dq \tag{6.8}$$

where $r(q)$ is $f(x)$ written in terms of q.

The usefulness of (6.7) and (6.8) comes when integrals such as

$$\int_{X_i}^{X_j} N_i^2\,dx$$

are evaluated. Using the coordinate variable s, we obtain

$$\int_{X_i}^{X_j} N_i^2(x)\,dx = \int_0^L N_i^2(s)\,ds = \int_0^L \left(1 - \frac{s}{L}\right)^2 ds = \frac{L}{3}$$

Using the q coordinate, we obtain

$$\int_{X_i}^{X_j} N_i^2(x)\,dx = \int_{-L/2}^{L/2} N_i^2(q)\,dq = \int_{-L/2}^{L/2} \left(\frac{1}{2} - \frac{q}{L}\right)^2 dq = \frac{L}{3}$$

The result, $L/3$, is obtained from

$$\int_{X_i}^{X_j} N_i^2(x)\,dx$$

only after a rather complicated expression is recognized as being L^3.

6.2 NATURAL COORDINATE SYSTEMS

The local coordinate systems s and q can be converted to natural coordinate systems. A natural coordinate system is a local system that permits the specification of a point within the element by a dimensionless number whose absolute magnitude never exceeds unity.

Start with the q coordinate in Figure 6.1 and form the ratio $q/(L/2) = 2q/L = \xi$. The coordinate ξ varies from -1 to $+1$ (Figure 6.2*a*). The shape functions in (6.5) can be written in terms of ξ by replacing q by $q = \xi L/2$. The new shape functions are

$$N_i(\xi) = \tfrac{1}{2}(1-\xi) \qquad \text{and} \qquad N_j(\xi) = \tfrac{1}{2}(1+\xi) \tag{6.9}$$

The change of variables in the integration yields

$$\int_{-L/2}^{L/2} r(q)\,dq = \int_{\xi_1}^{\xi_2} g(\xi)\frac{d(\xi L/2)}{d\xi}\,d\xi = \frac{L}{2}\int_{-1}^{1} g(\xi)\,d\xi \tag{6.10}$$

where $g(\xi)$ is $r(q)$ written in terms of ξ.

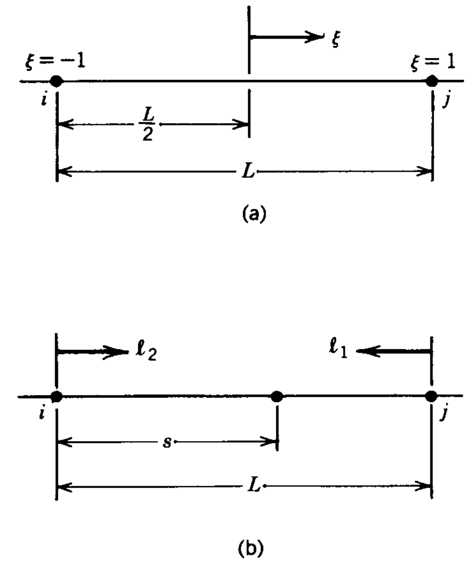


Figure 6.2 Natural coordinate systems for the one-dimensional element.

The advantage of the coordinate variable ξ is the -1 to $+1$ limits of integration. Most computer programs use numerical integration techniques to evaluate the element matrices. A numerical integration scheme used in finite element programs is the Gauss–Legendre method (Conte and deBoor, 1980), which has the sampling points and weighting coefficients defined on a -1, $+1$ interval.

Another interesting natural coordinate system consists of a pair of length ratios, Figure 6.2*b*. If s is the distance from node i, then ℓ_1 and ℓ_2 are defined as the ratios

$$\ell_1 = \frac{L-s}{L} \qquad \text{and} \qquad \ell_2 = \frac{s}{L} \tag{6.11}$$

This pair of coordinates is not independent because

$$\ell_1 + \ell_2 = 1 \tag{6.12}$$

The most important characteristic of (6.11) and (6.12) is that ℓ_1 and ℓ_2 are identical to the shape functions defined by (6.3) and (6.4). The usefulness of these coordinates is associated with the evaluation of integrals of the type

$$\int_0^L N_i^a(s)N_j^b(s)\,ds \tag{6.13}$$

which involve the product of shape functions. The length ratio coordinates result in a simple formula for evaluating an integral similar to (6.13).

The change of variable rule and the relationships $N_i(s) = \ell_1$, $N_j(s) = \ell_2$, $s = L\ell_2$, and $ds/d\ell_2 = L$ give

$$\int_0^L N_i^a(s)N_j^b(s)\,ds = \int_0^1 \ell_1^a \ell_2^b L\,d\ell_2 \tag{6.14}$$

The integral on the right-hand side of (6.14) can be changed to

$$L \int_0^1 (1-\ell_2)^a \ell_2^b \, d\ell_2 \tag{6.15}$$

using (6.12). The integral in (6.15) is of the same form as

$$\int_0^1 t^{z-1}(1-t)^{w-1}\, dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{6.16}$$

where $\Gamma(n+1)=n!$ (Abramowitz and Stegun, 1964). Thus

$$\begin{aligned} L \int_0^1 \ell_1^a \ell_2^b \, d\ell_2 &= L \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1+1)} \\ &= L \frac{a!b!}{(a+b+1)!} \end{aligned} \tag{6.17}$$

Equation (6.17) is useful because it states that a rather complicated integral can be evaluated using an equation which involves only the length of the element and the powers involved in the product.

Evaluation of a pair of integrals illustrates the usefulness of (6.17). Starting with

$$\int_{X_i}^{X_j} N_i^2(x)\, dx = \int_0^L N_i^2(s)\, ds$$

(6.12) gives

$$\int_0^L N_i^2(s)\, ds = L \int_0^1 \ell_1^2 \ell_2^0 \, d\ell_2 = L \frac{2!0!}{(2+0+1)!} = \frac{L}{3}$$

Table 6.1 Coordinate Systems and Limits of Integration for the One-Dimensional Element

Type of System	Coordinate Variable	Shape Functions		Limits of Integration
Global	x	$N_i = \frac{X_j - x}{L}$,	$N_j = \frac{x - X_i}{L}$	X_i, X_j
Local	s	$N_i = 1 - \frac{s}{L}$,	$N_j = \frac{s}{L}$	0, L
Local	q	$N_i = \left(\frac{1}{2} - \frac{q}{L}\right)$,	$N_j = \left(\frac{1}{2} + \frac{q}{L}\right)$	$-\frac{L}{2}$, $\frac{L}{2}$
Natural	ξ	$N_i = \frac{1}{2}(1-\xi)$,	$N_j = \frac{1}{2}(1+\xi)$	-1, 1
Natural	ℓ_2	$N_i = \ell_1$,	$N_j = \ell_2$	0, 1

Another example is

$$\int_0^L N_i^3(s) N_j^2(s)\, ds = L \int_0^1 \ell_1^3 \ell_2^2 \, d\ell_2 = L \frac{3!2!}{(3+2+1)!} = \frac{L}{60}$$

The coordinate systems, shape functions, and limits of integration for the one-dimensional linear element are summarized in Table 6.1.

6.3 RECTANGULAR ELEMENT

Natural coordinate systems can be defined for two-dimensional elements; they have the same advantage as observed for the one-dimensional formulation. They are more convenient for both analytical and numerical integration.

The natural coordinate system for the rectangular element is shown in Figure 6.3. It is located at the center of the element and the coordinates are the length ratios

$$\xi = \frac{q}{b} \quad \text{and} \quad \eta = \frac{r}{a} \tag{6.18}$$

where q and r are the local coordinates. The shape functions in (5.19) are easily converted to the natural coordinate system. The results are

$$\begin{aligned} N_i &= \tfrac{1}{4}(1-\xi)(1-\eta), \qquad N_j = \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_k &= \tfrac{1}{4}(1+\xi)(1+\eta), \qquad N_m = \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{6.19}$$

It should be clear that ξ and η range between plus and minus one, that is,

$$-1 \leqslant \xi \leqslant 1 \quad \text{and} \quad -1 \leqslant \eta \leqslant 1$$

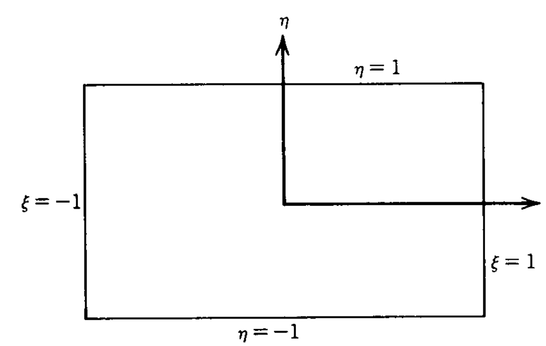


Figure 6.3. A natural coordinate system for the rectangular element.

6.4 TRIANGULAR ELEMENT: AREA COORDINATES

A natural coordinate system for the triangular element is obtained by defining the three length ratios L_1, L_2, and L_3 shown in Figure 6.4*a*. Each coordinate is the ratio of a perpendicular distance from one side, s, to the altitude, h, of that

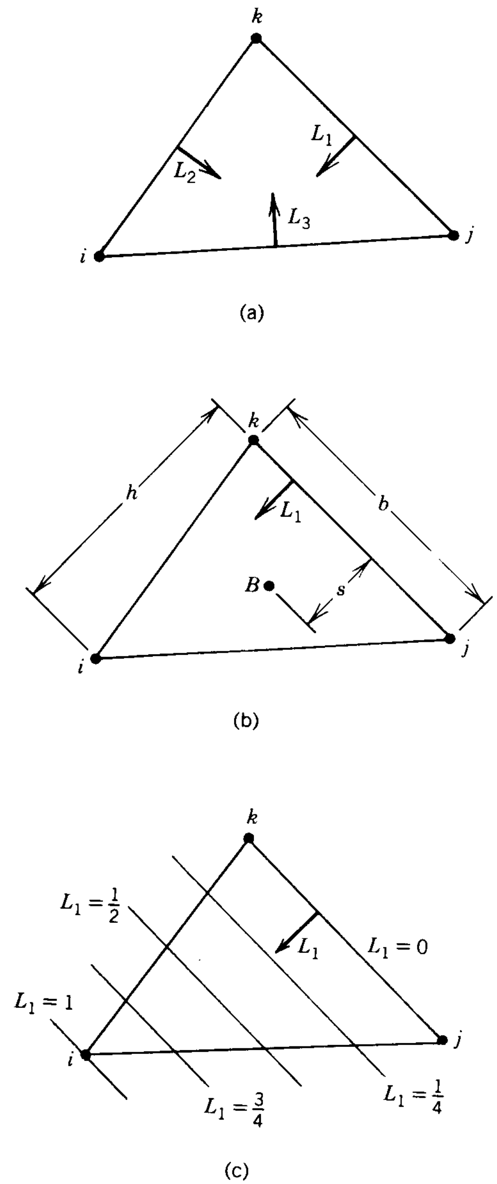


Figure 6.4. The three area coordinates for a triangular element.

same side. This is illustrated in Figure 6.4*b*. Each coordinate is a length ratio that varies between zero and one. The lines of constant L_1 are shown in Figure 6.4*c*. Each of these lines is parallel to the side from which L_1 is measured.

The coordinates L_1, L_2, and L_3 are called area coordinates because their values give the ratio of the area of a subtriangular region to the area of the complete triangle. Consider point B as shown in Figure 6.5. The area of the complete triangle

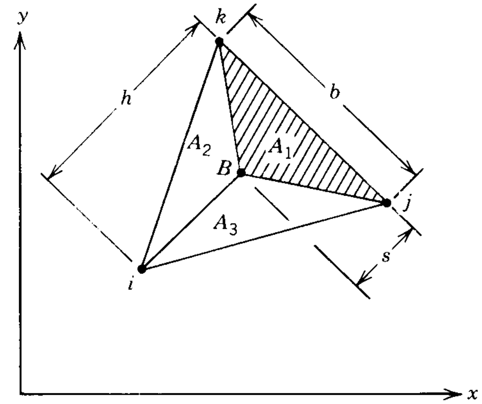


Figure 6.5. A triangle divided into the areas corresponding to the area coordinates.

is A and is given by

$$A = \frac{bh}{2}$$

whereas the area of the shaded triangle (B, j, k) is

$$A_1 = \frac{bs}{2} \tag{6.20}$$

Forming the ratio A_1/A yields

$$\frac{A_1}{A} = \frac{s}{h} = L_1 \tag{6.21}$$

The area coordinate L_1 is the ratio of the shaded area in Figure 6.5 to the total area. Similar equations can be written for L_2 and L_3 giving

$$L_2 = \frac{A_2}{A} \quad \text{and} \quad L_3 = \frac{A_3}{A} \tag{6.22}$$

Since $A_1 + A_2 + A_3 = A$,

$$L_1 + L_2 + L_3 = 1 \tag{6.23}$$

An equation relating the three coordinates was expected because the coordinates are not independent. The location of a point can be specified using two of the coordinates.

Equation (6.21) can be reworked into another form. Multiplying the top and bottom by two gives

$$L_1 = \frac{2A_1}{2A} \tag{6.24}$$

Using the determinant expansion for $2A_1$ produces

$$2A_1 = \begin{vmatrix} 1 & x & y \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix}$$

or

$$2A_1 = (X_j Y_k - X_k Y_j) + (Y_j - Y_k)x + (X_k - X_j)y \tag{6.25}$$

where x and y are the coordinates of B in Figure 6.5. Substituting (6.25) into (6.24) yields

$$L_1 = \frac{1}{2A}\left[(X_j Y_k - X_k Y_j) + (Y_j - Y_k)x + (X_k - X_j)y\right] \tag{6.26}$$

Equation (6.26) is identical to (5.8); thus

$$L_1 = N_i \tag{6.27}$$

A similar analysis for L_2 and L_3 shows that

$$L_2 = N_j \quad \text{and} \quad L_3 = N_k \tag{6.28}$$

The area coordinates for the linear triangular element are identical to the shape functions, and the two sets of quantities can be interchanged.

The advantage of using the area coordinate system is the existence of an integration equation that simplifies the evaluation of area integrals (Eisenberg and Malvern, 1973). This integral equation is related to (6.17) and is

$$\int_A L_1^a L_2^b L_3^c \, dA = \frac{a!b!c!}{(a+b+c+2)!} 2A \tag{6.29}$$

The use of (6.29) can be illustrated by evaluating the shape function product

$$\int_A N_i(x, y) N_j(x, y) \, dA \tag{6.30}$$

over the area of a triangle. The area integral is

$$\int_A N_i N_j \, dA = \int_A L_1^1 L_2^1 L_3^0 \, dA$$

$$= \frac{1!1!0!}{(1+1+0+2)!} 2A = \frac{2A}{4!} = \frac{A}{12}$$

The area coordinates L_1 and L_2 can be substituted for N_i and N_j, respectively. Since N_k was not in the product, L_3 is included to the zero power. Zero factorial is defined as one.

The incorporation of derivative boundary conditions or surface loads into a finite element analysis requires the evaluation of an integral along the edge of an element. These integrals are easy to evaluate once it is known how the area coordinates behave on an edge. Consider point B on the side ij (Figure 6.6). The

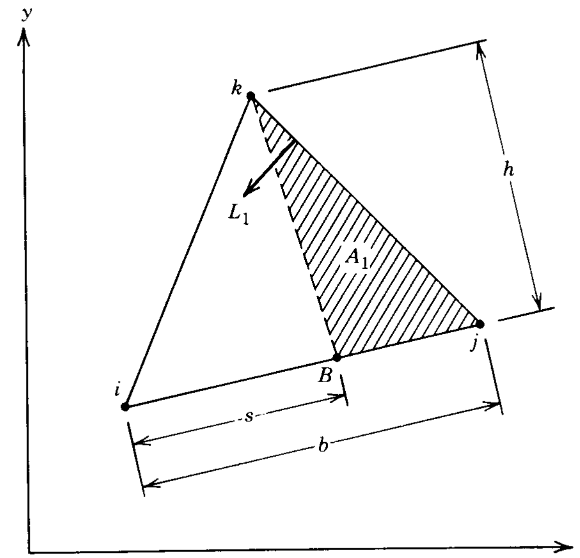


Figure 6.6. The area coordinates for a point on the edge of a triangle.

coordinate L_3 is zero and L_1 is the ratio of the shaded area to the total area. Define the coordinate variable s, which is parallel to side ij and measured from node i. If the coordinate of point B is s, and the length of the side is b, then

$$L_1 = \frac{2A_1}{2A} = \frac{\frac{2h(b-s)}{2}}{\frac{2bh}{2}} = \frac{b-s}{b} = 1 - \frac{s}{b} \tag{6.31}$$

The area coordinate L_2 is

$$L_2 = \frac{s}{b} \tag{6.32}$$

The area coordinates L_1 and L_2 reduce to the one-dimensional shape functions $N_i(s)$ and $N_j(s)$ defined by (6.3) and (6.4). Using the one-dimensional natural coordinates, ℓ_1 and ℓ_2, defined by (6.11), the area coordinates become

$$L_1 = \ell_1 \quad \text{and} \quad L_2 = \ell_2 \quad \text{side } i \to j \tag{6.33}$$

The relationships for the other two sides are

$$L_2 = \ell_1 \quad \text{and} \quad L_3 = \ell_2 \quad \text{side } j \to k \tag{6.34}$$

$$L_3 = \ell_1 \quad \text{and} \quad L_1 = \ell_2 \quad \text{side } k \to i \tag{6.35}$$

The importance of the relationships in (6.33), (6.34), and (6.35) is that any integral over the edge of a triangular element can be replaced by a line integral written in terms of s or ℓ_2, that is,

$$\int_\Gamma f(L_1, L_2, L_3)\, d\Gamma = \int_0^L g(s)\, ds = L \int_0^1 h(\ell_2)\, d\ell_2 \qquad (6.36)$$

and evaluated using the factorial formula (6.17). The boundary of a two-dimensional element is denoted by Γ.

ILLUSTRATIVE EXAMPLE

Evaluate $\int_\Gamma [N]^T d\Gamma$ over side ik of a linear triangular element.

The integral is

$$\int_\Gamma [N]^T d\Gamma = L_{ik} \int_0^1 \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} d\ell_2 = L_{ik} \int_0^1 \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} d\ell_2$$

since the linear triangular shape functions and the area coordinates are equivalent. Along side ik, $L_1 = \ell_1$, $L_2 = 0$, and $L_3 = \ell_2$; thus

$$\int_\Gamma [N]^T d\Gamma = L_{ik} \int_0^1 \begin{Bmatrix} \ell_1 \\ 0 \\ \ell_2 \end{Bmatrix} d\ell_2 = \frac{L_{ik}}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

using (6.35) and then (6.17).

6.5 CONTINUITY

The function for approximating $\phi(x, y)$ consists of a set of continuous piecewise smooth equations, each defined over a single element. The need to integrate this piecewise smooth function places a requirement on the order of continuity between elements.

The integral

$$\int_0^H \frac{d^n\phi}{dx^n}\, dx$$

is defined only if ϕ has continuity of order $(n-1)$ (Olmstead, 1961). This ensures that only finite jump discontinuities exist in the nth derivative. This requirement means that the first derivative of the approximating function must be continuous between elements if the integral contains second-derivative terms, $n = 2$. All of the integrals in this book, except the beam element, contain first-derivative terms. Therefore, ϕ must be continuous between elements, but its derivatives do not have to be continuous. Continuity in the derivative is required for the beam element.

Continuity of ϕ in the one-dimensional element is assured, since two adjacent elements have a common node. Continuity in ϕ along a common boundary between two rectangular elements is relatively easy to prove and is left as an exercise. Continuity of ϕ along a common boundary of two arbitrarily oriented triangular elements is more complicated and is considered here.

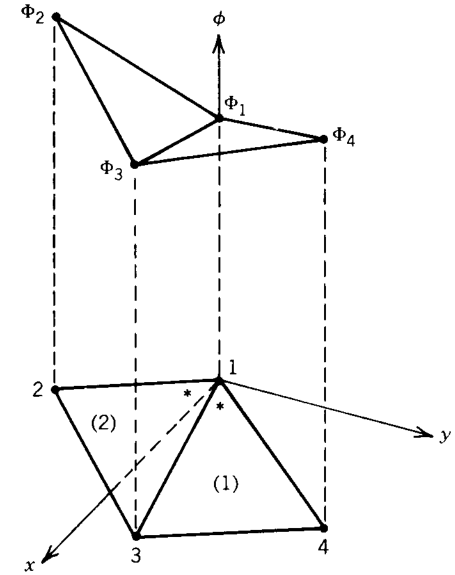


Figure 6.7. A two-element grid.

Consider two adjacent elements (Figure 6.7) with the coordinate system originating at node one. The nodal values are Φ_1, Φ_2, Φ_3, and Φ_4. The equations for ϕ are

$$\begin{aligned} \phi^{(1)} &= N_1^{(1)}\Phi_1 + N_3^{(1)}\Phi_3 + N_4^{(1)}\Phi_4 \\ \phi^{(2)} &= N_1^{(2)}\Phi_1 + N_2^{(2)}\Phi_2 + N_3^{(2)}\Phi_3 \end{aligned} \qquad (6.37)$$

The properties of the shape functions indicate that $N_2^{(2)} = N_4^{(1)} = 0$ along the common boundary. Recalling the equality between the shape functions and the area coordinates, (6.27) and (6.28), allows (6.37) to be written as

$$\begin{aligned} \phi^{(1)} &= L_1^{(1)}\Phi_1 + L_2^{(1)}\Phi_3 \\ \phi^{(2)} &= L_1^{(2)}\Phi_1 + L_3^{(2)}\Phi_3 \end{aligned} \qquad (6.38)$$

Remember that the subscripts on the area coordinates are not related to the node numbers.

Since $L_3^{(1)} = L_2^{(2)} = 0$, (6.38) can be reworked into

$$\begin{aligned} \phi^{(1)} &= L_1^{(1)}\Phi_1 + (1 - L_1^{(1)})\Phi_3 \\ \phi^{(2)} &= L_1^{(2)}\Phi_1 + (1 - L_1^{(2)})\Phi_3 \end{aligned} \qquad (6.33)$$

using (6.23). The proof is completed when it is shown that $L_1^{(1)} = L_1^{(2)}$.

A point on the common boundary is shown in Figure 6.8 with the areas associated with $L_1^{(1)}$ and $L_1^{(2)}$ shaded. Defining the distance from point B to node three as c

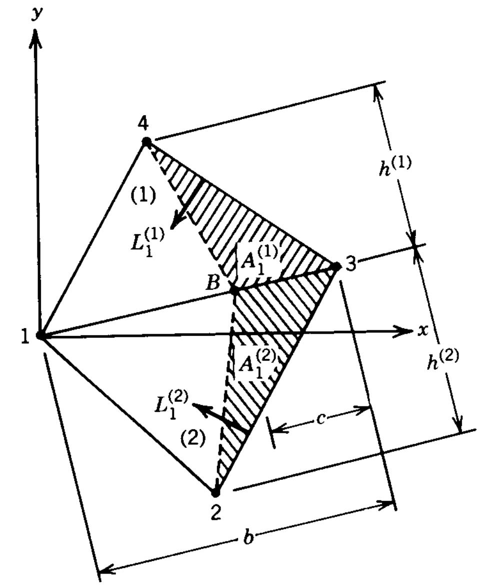


Figure 6.8. The area coordinates $L_1^{(1)}$ and $L_1^{(2)}$ along a common boundary.

and the length of side 1–3 as b,

$$L_1^{(1)}=\frac{2A_1^{(1)}}{2A^{(1)}}=\frac{\dfrac{2ch^{(1)}}{2}}{\dfrac{2bh^{(1)}}{2}}=\frac{c}{b}$$

and

$$L_1^{(2)}=\frac{2A_1^{(2)}}{2A^{(2)}}=\frac{\dfrac{2ch^{(2)}}{2}}{\dfrac{2bh^{(2)}}{2}}=\frac{c}{b}=L_1^{(1)}$$

The proof is complete.

PROBLEMS

6.1 The shape functions for the three node one-dimensional quadratic element (Figure P6.1) relative to the local coordinate s follow. Write these shape functions in terms of the local coordinate q.

$$N_i=\left(1-\frac{2s}{L}\right)\left(1-\frac{s}{L}\right),\qquad N_j=\frac{4s}{L}\left(1-\frac{s}{L}\right)$$

$$N_k=\frac{2s}{L^2}\left(s-\frac{L}{2}\right)$$

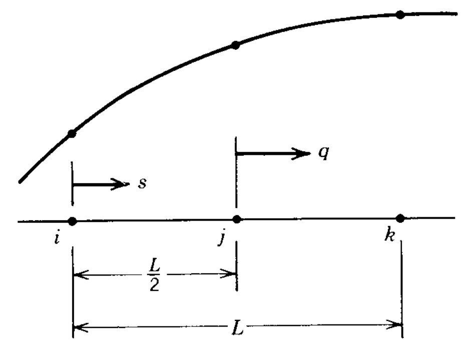


Figure P6.1

6.2 Do Problem 6.1 for the natural coordinate (Figure P6.2).

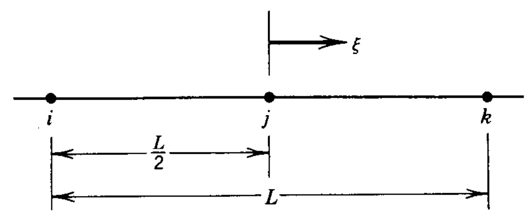


Figure P6.2

6.3 Do Problem 6.1 for the natural coordinates ℓ_1 and ℓ_2 (Figure P6.3).

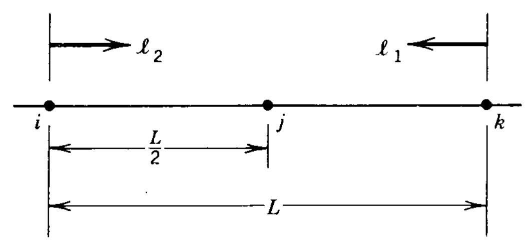


Figure P6.3

6.4 A local coordinate system r has its origin at the one-third point of a linear element (Figure P6.4). Develop the shape functions in terms of r:

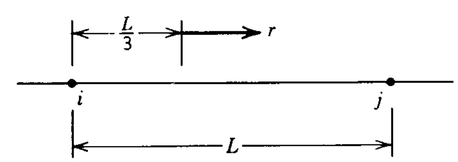


Figure P6.4

(a) Starting with the x-coordinate system.
(b) Starting with the s-coordinate system.
(c) Starting with the q-coordinate system.

6.5 The integration of $d\phi/dx$ occurs in many finite element applications. Whe N_i and N_j are written in terms of a new coordinate variable, $d\phi/dx$ mus be evaluated using the chain rule, that is, for the local coordinate s,

$$\frac{d\phi}{ds}=\frac{d\phi}{dx}\frac{dx}{ds} \quad \text{and} \quad \frac{d\phi}{dx}=\frac{d\phi}{ds}\frac{1}{dx/ds}$$

Show that the following relationships hold for the one-dimensional co ordinate systems studied in this chapter.

$$\frac{d\phi}{ds}=\frac{d\phi}{dx}, \quad \frac{d\phi}{dq}=\frac{d\phi}{dx} \quad \text{and} \quad \frac{d\phi}{d\xi}=\frac{L}{2}\frac{d\phi}{dx}$$

6.6 The quadratic shape functions given in Problem 6.1 are

$$N_i=\ell_1-2\ell_1\ell_2, \qquad N_j=4\ell_1\ell_2 \qquad N_k=2\ell_2^2-\ell_2$$

when written in terms of the natural coordinates ℓ_1 and ℓ_2. Using thes equations for the shape functions and (6.17), evaluate

$$\text{(a)} \int_0^L N_iN_j\,ds \qquad \text{(b)} \int_0^L N_jN_k\,ds \qquad \text{(c)} \int_0^L N_k^2\,ds$$

6.7 Verify that $L_2=N_j$ for the linear triangular element.

6.8 Verify that any line of constant ϕ in a triangular element is a straight line *Hint:* Investigate the value of ϕ along the side of a triangle when two node have the same value.

6.9 Evaluate the following integrals using (6.29).

$$\text{(a)} \int_A N_i^2N_k\,dA \qquad \text{(b)} \int_A N_iN_jN_k\,dA$$

$$\text{(c)} \int_A (N_i^2+N_j)\,dA \qquad \text{(d)} \int_A (N_j^2N_k+N_i)\,dA$$

6.10 Verify that $\phi^{(1)}=\phi^{(2)}$ along the common boundary between the two rec tangular elements shown in Figure P6.10.

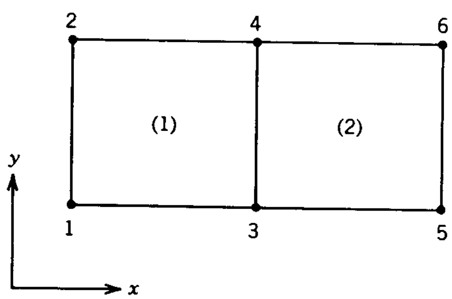


Figure P6.10

6.11 Verify that $\phi^{(1)}=\phi^{(2)}$ along the common boundary between the triangular and rectangular elements in Figure P6.11. *Hint:* Use the st-coordinate system for the rectangular element.

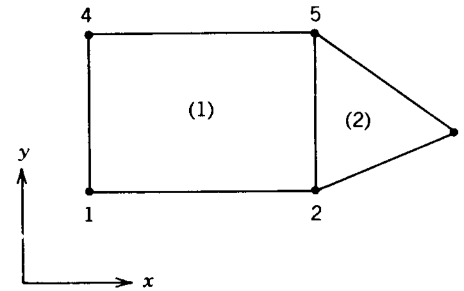


Figure P6.11

6.12 Show that the shape functions for the rectangular element, (6.19), reduce to the linear shape functions, (6.9), along side ij.

6.13 The three shape functions along one edge of a quadratic triangular element are given in Figure P6.13. Show that these equations reduce to the shape functions for the one-dimensional quadratic element that are given in Problem 6.6.

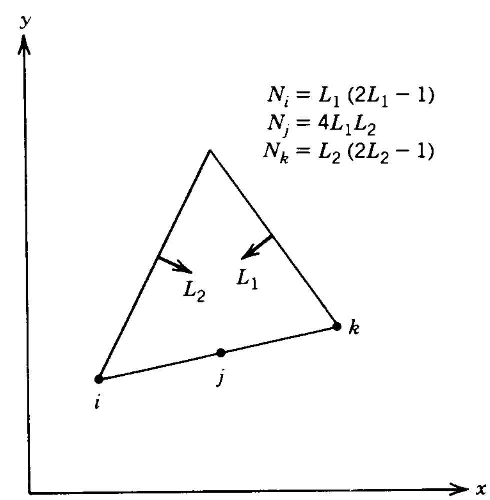


Figure P6.13