Assignment 3: CS 663, Fall 2024

Due: 7th October before 11:55 pm

Remember the honor code while submitting this (and every other) assignment. You may discuss broad ideas with other students or ask me for any difficulties, but the code you implement and the answers you write must be your own. We will adopt a zero-tolerance policy against any violation.

Submission instructions: Follow the instructions for the submission format and the naming convention of your files from the submission guidelines file in the homework folder. Please see assignment3.zip in the homework folder. For all the questions, write your answers and scan them, or type them out in word/Latex. In eithe case, create a separate PDF file. The last two questions will also have code in addition to the PDF file. Once you have finished the solutions to all questions, prepare a single zip file and upload the file on moodle before 11:55 pm on 7th October. **Only one student per group should submit the assignment.** We will not penalize submission of the files till 10 am on 8th October. **No assignments will be accepted after this time.** Please preserve a copy of all your work until the end of the semester. **Your zip file should have the following naming convention:** RollNumber1_RollNumber2_RollNumber3.zip for three-member groups, RollNumber1_RollNumber2.zip for two-member groups and RollNumber1.zip for single-member groups.

1. Consider the barbara256.png image from the homework folder. Implement the following in MATLAB: (a) an ideal low pass filter with cutoff frequency $D \in \{40, 80\}$, (b) a Gaussian low pass filter with $\sigma \in \{40, 80\}$. Show the effect of these on the image, and display all filtered images in your report. Display the frequency response (in log absolute Fourier format) of all filters in your report as well. Comment on the differences in the outputs. Also display the log absolute Fourier transform of the original and filtered images. Comment on the differences in the outputs. Make sure you perform appropriate zero-padding while doing the filtering! [15 points]
   Solutions and Marking Scheme: 10 points for Gaussian filter and 10 points for ideal LPF. If zero-padding is not properly done and/or if the filter size is inaccurate, deduct 4 points for each part. Note: the size of the filter in the frequency domain must be at least $2H \times 2W$ where the image size is $H \times W$. The image also needs to be zero-padded to attain size at least $2H \times 2W$.

2. Derive the 2D Fourier transform of the correlation of two continuous 2D signals in the continuous domain. Repeat the same for the 2D DFT of two 2D discrete signals. [10 points]
   Solutions: For the first part, let the two real-valued signals be $f(t_1, t_2)$ and $g(t_1, t_2)$. Their correlation is given by $h(t_1, t_2) = (f \star g)(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u_1, u_2) g(t_1 + u_1, t_2 + u_2) du_1 du_2$. Taking Fourier transforms

on both sides, we have the following, denoting the complex conjugate operation by H:

$$
\begin{aligned}
H(\mu_1,\mu_2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u_1,u_2)g(t_1+u_1,t_2+u_2)e^{-j2\pi(\mu_1 t_1+\mu_2 t_2)}dt_1dt_2du_1du_2 && (1)\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u_1,u_2)\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(t_1+u_1,t_2+u_2)e^{-j2\pi(\mu_1 t_1+\mu_2 t_2)}dt_1dt_2\right)du_1du_2 && (2)\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u_1,u_2)G(\mu_1,\mu_2)e^{j2\pi(\mu_1 u_1+\mu_2 u_2)}du_1du_2 \text{ using Fourier shift thm.} && (3)\\
&= G(\mu_1,\mu_2)\int_{-\infty}^{+\infty} f(u_1,u_2)e^{j2\pi(\mu_1 u_1+\mu_2 u_2)}du_1du_2 && (4)\\
&= G(\mu_1,\mu_2)\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^{\mathsf{H}}(u_1,u_2)e^{-j2\pi(\mu_1 u_1+\mu_2 u_2)}du_1du_2\right)^{\mathsf{H}} && (5)\\
&= G(\mu_1,\mu_2)\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u_1,u_2)e^{-j2\pi(\mu_1 u_1+\mu_2 u_2)}du\right)^{\mathsf{H}} && (6)\\
&= G(\mu_1,\mu_2)F^{\mathsf{H}}(\mu_1,\mu_2), && (7)
\end{aligned}
$$

which proves the correlation theorem. Note that since $f(u_1,u_2)$ is real-valued, we have $f^{\mathsf{H}}(u_1,u_2) = f(u_1,u_2)$. Some students used the following definition of correlation which is needed for complex-valued signals, but does not apply to real-valued signals: $h(t_1,t_2) = (f \star g)(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^{\mathsf{H}}(u_1,u_2)g(t_1+u_1,t_2+u_2)du_1du_2$. If you used this definition, no points will be taken off, but this was not required, and as such, we haven't seen this definition in class.

For the discrete part: Let the two discrete signals be $f(x,y)$ and $g(x,y)$ with size $N \times M$ and let their Discrete Fourier Transforms be $F(u,v), G(u,v)$ respectively. Then we have:

$$
\begin{aligned}
DFT(f\star g)(u,v) &= \sum_{x=0}^{N-1}\sum_{y=0}^{M-1}(f\star g)(x,y)\exp(-j2\pi(ux/N+vy/M)) && (8)\\
&= \sum_{x=0}^{N-1}\sum_{y=0}^{M-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)g(x+n,y+m)\exp(-j2\pi(ux/N+vy/M)) && (9)\\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} g(x+n,y+m)\exp(-j2\pi(ux/N+vy/M)) && (10)\\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\exp(j2\pi(nu/N+mv/M))G(u,v) \text{ by Fourier shift theorem} && (11)\\
&= G(u,v)\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}(f(n,m)\exp(-j2\pi(nx/N+my/M)))^{\mathsf{H}} && (12)\\
&= G(u,v)F^{\mathsf{H}}(u,v). \text{ by defn. of the Fourier transform} && (13)
\end{aligned}
$$

Note that the shifts in the above summations are all circular by definition.
Marking scheme: In the continuous case, there are 5 points: 0.5 points each for steps 1 and 2, 2 points for the use of the Fourier shift theorem and 2 points for completing the proof. An analogous marking scheme exists for the discrete case.

3. Consider the two images in the homework folder 'barbara256.png' and 'kodak24.png'. Add zero-mean Gaussian noise with standard deviation $\sigma = 5$ to both of them. Implement a mean shift based filter and show the outputs of the mean shift filter on both images for the following parameter configurations: $(\sigma_s = 2, \sigma_r = 2); (\sigma_s = 15, \sigma_r = 3); (\sigma_s = 3, \sigma_r = 15)$. Comment on your results in your report. Repeat when the image is corrupted with zero-mean Gaussian noise of $\sigma = 10$ (with the same mean shift filter parameters). Comment on your results in your report. Include all image ouputs as well as noisy images in the report. All parameters assume that the images are from 0 to 255 in intensity. [20 points]

Solutions and Marking Scheme: See code in homework folder. Basic mean shift implementation until convergence, 14 points. 5 points to be deducted if the student runs mean shift for a fixed number of iterations. 6 points for results for all parameter settings and noise values to be included in report. If the images are missing in the report, then deduct 3 points. Comments: larger σ_s, σ_r lead to more smoothing and smaller values lead to less smoothing.

4. Consider a 201×201 image whose pixels are all black except for the central column (i.e. column index 101 beginning from 1 to 201) in which all pixels have the value 255. Derive the Fourier transform of this image analytically, and also plot the logarithm of its Fourier magnitude using `fft2` and `fftshift` in `MATLAB`. Use appropriate colorbars. [8+2=10 points]
Marking Scheme: To compute the 2D-DFT, we first compute column-wise 1D-DFTs and then row-wise 1D DFTs of the resultant. Each column is all zeros, except the central column which has all pixels with value 255. Hence the Fourier transform of the central column is a Kronecker delta function with height equal to $255/\sqrt{201}$, whereas the Fourier transform of all other columns is 0. The row-wise 1D-DFTs of this intermediate signal will yield zeros for all rows except the central row, where the 1D DFT will yield a uniform signal consisting of all values equals to 255. The accompanying code is in `fourier_row.m`.
Note that we could also have computed first the row-wise 1D DFT of the image, followed by column-wise DFTs. This will yield the same result as before – try it on your own.
Marking scheme: Full marks to be given if a student plugs in the formula and obtains the same answer. Partial credit for sensible steps. This stepwise answer involving 1D DFTs also deserves full credit. In this case, 4 points for column-wise DFT and 4 points for row-wise DFT. The order of the column-wise and row-wise DFT can be swapped as explained in the solution.
2 points for the code and appropriate plot generated. No points to be given if the plot is not included in the report.

5. If a function $f(x,y)$ is real, prove that its Discrete Fourier transform $F(u,v)$ satisfies $F^*(u,v) = F(-u,-v)$. If $f(x,y)$ is real and even, prove that $F(u,v)$ is also real and even. The function $f(x,y)$ is an even function if $f(x,y) = f(-x,-y)$. [15 points]
Solution: We have $F(u,v) = \sum_{x,y} f(x,y)\exp(-j2\pi(ux+vy)/N)$ and $F(-u,-v) = \sum_{x,y} f(x,y)\exp(j2\pi(ux+vy)/N) = \left(\sum_{x,y} f(x,y)\exp(-j2\pi(ux+vy)/N)\right)^* = F^*(u,v)$. Note that the conjugation step is valid only because $f(x,y)$ is real-valued.

Now consider that f is both real and even, i.e. $f(x,y) = f(-x,-y)$. Now we have:

$$F^*(u,v) = \left(\sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(-j2\pi(ux+vy)/N)\right)^* = \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(ux+vy)/N) \text{ as } f \text{ is real} \tag{14}$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(-u(-x)-v(-y))/N) \tag{15}$$

$$= \sum_{x'=0}^{-N}\sum_{y=0}^{-N} f(x',y')\exp(j2\pi(-ux'-vy')/N) \tag{16}$$

replacing $x' \triangleq -x, y' \triangleq -y$ and also using $f(x',y') = f(-x',-y')$ as f is even

$$= \sum_{x'=0}^{N}\sum_{y=0}^{N} f(x',y')\exp(-j2\pi(ux'+vy')/N) \text{ using the periodic nature of the DFT} \tag{17}$$

$$= F(u,v). \tag{18}$$

This establishes that $F(u,v)$ is real-valued. We had already proved $F^*(u,v) = F(-u,-v)$ which establishes that $F(u,v) = F(-u,-v)$ and proves that F is even. Another way to prove that F is even, is to consider:

$$F(-u,-v) = \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(-j2\pi(-ux-vy)/N) = \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(ux+vy)/N) \tag{19}$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{N} f(x,y)\exp(j2\pi(-u(-x)-v(-y))/N) \tag{20}$$

$$= \sum_{x'=0}^{-N}\sum_{y=0}^{-N} f(-x',-y')\exp(j2\pi(-ux'-vy')/N) \text{ replacing } x' \triangleq -x, y' \triangleq -y \tag{21}$$

$$= \sum_{x'=0}^{N}\sum_{y=0}^{N} f(x',y')\exp(-j2\pi(ux'+vy')/N) \tag{22}$$

using the periodic nature of the DFT and also using $f(x',y') = f(-x',-y')$ as f is even

$$= F(u,v). \tag{23}$$

Marking scheme: 5 points for the proof that if $f(x,y)$ is real, then its Discrete Fourier transform $F(u,v)$ satisfies $F^*(u,v) = F(-u,-v)$. 5 points for the proof that the DFT of a real and even function is real, and 5 points for the proof that the DFT of a real and even function is even.

6. If $\mathcal{F}$ is the continuous Fourier operator, prove that $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(t)$. Hint: Prove that $\mathcal{F}(\mathcal{F}(f(t))) = f(-t)$ and proceed further from there. What could be a practical use of the relationship $\mathcal{F}(\mathcal{F}(f(t))) = f(-t)$ while deriving Fourier transforms of certain functions? [12+3=15 points]
Solution: We have $F(\mu) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi\mu t}dt$ and $f(t) = \int_{-\infty}^{+\infty} F(\mu)e^{j2\pi\mu t}d\mu$.

Hence $\mathcal{F}^2(f(t))(t) = \mathcal{F}(F(\mu))(t) = \int_{-\infty}^{+\infty} F(\mu)e^{-j2\pi\mu t}dt = f(-t)$ (comparing to the inverse Fourier transform expression for $f(t)$). Note that instead of t, we could have used another variable μ' (to distinguish it from μ), but this core argument will not change. This is also called Fourier duality. That is, if $F(\mu)$ is the Fourier transform of $f(t)$, then $f(-\mu)$ is the Fourier transform of $F(t)$.

Now, the Fourier transform of $f(-t)$ is equal to $F(-\mu)$. Hence $\mathcal{F}^3(f(t)) = F(-\mu)$. We now use an argument very similar to the one we used to obtain $\mathcal{F}^2(f(t))$. Hence $\mathcal{F}^4(f(t)) = \int_{-\infty}^{+\infty} F(-\mu)e^{-j2\pi\mu t}dt = x(t)$.

If $F(\mu)$ is the Fourier transform of $f(t)$, then $f(-\mu)$ is the Fourier transform of $F(t)$. This helps in computing the Fourier transform of some function without going through tedious calculations. For example, since we know that the Fourier transform of a rect function is a sinc. Finding the Fourier transform of the sinc function from first principles can be tedious, but the Fourier duality principle helps us to establish that this Fourier transform will be a rect. This is the practical application of the relationship $\mathcal{F}(\mathcal{F}(f(t))) = f(-t)$.
Marking scheme: For $\mathcal{F}^2$, $\mathcal{F}^3$ and $\mathcal{F}^4$, there are 4 points each. 3 marks for stating the practical application of the relationship explicitly.

7. Consider the partial differential equation $\dfrac{\partial I}{\partial t} = c\left(\dfrac{\partial^2 I}{\partial x^2} + \dfrac{\partial^2 I}{\partial y^2}\right)$ where c is some non-negative constant. This is the isotropic heat equation. Using the differentiation theorem in Fourier transforms, prove that running this PDE on an image I is equivalent to convolving it with a Gaussian of zero mean and appropriate standard deviation. What is the value of the standard deviation? You will also need to use the result that the Fourier transform of a Gaussian is also a Gaussian. [15 points]
Answer: We again consider an image $I(x,y,t)$. To this effect, consider the original PDE:

$$\frac{\partial I(x,y,t)}{\partial t} = c\left(\frac{\partial^2 I(x,t)}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2}\right). \tag{24}$$

Taking Fourier transforms on both sides using $\mathcal{F}$ as notation for the Fourier operator, we have

$$\mathcal{F}(\frac{\partial I(x,y,t)}{\partial t})(\mu_1,\mu_2) = c\mathcal{F}[\frac{\partial^2 I(x,y,t)}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2}](\mu_1,\mu_2). \tag{25}$$

The LHS can be written as follows:

$$\begin{aligned}\mathcal{F}(\frac{\partial I(x,y,t)}{\partial t}) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\partial I(x,y,t)}{\partial t}e^{-j2\pi(\mu_1 x+\mu_2 y)}dxdy \\ &= \frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}I(x,y,t)e^{-j2\pi(\mu_1 x+\mu_2 y)}dxdy \\ &= \frac{\partial \hat{I}(\mu_1,\mu_2,t)}{\partial t}\end{aligned} \tag{26}$$

where $\hat{I}(\mu_1, \mu_2, t)$ stands for the Fourier Transform of I at time instant t, and (μ_1, μ_2) stands for the frequency. Note that we could take the partial derivative outside the integral only because the second term $e^{j2\pi(\mu_1 x+\mu_2 y)}$ is independent of t. This result carries 4 points

The RHS of Eqn 25 is given as follows, using integration by parts:

$$\mathcal{F}[c(\frac{\partial^2 I(x,y,t)}{\partial x^2}](\mu_1,\mu_2) = c(I_x(x,y,t)e^{-j2\pi(\mu_1 x+\mu_2 y)})\Big|_{-\infty}^{\infty} - c\int_{-\infty}^{+\infty}I_x(x,y,t)[-j2\pi\mu_1]e^{-j2\pi(\mu_1 x+\mu_2 y)}dx \tag{27}$$

$$= 0 + c[j2\pi\mu_1]\int_{-\infty}^{+\infty}I_x(x,y,t)e^{-j2\pi(\mu_1 x+\mu_2 y)}dx(\text{ using } I_x(-\infty,x,t) = I_x(\infty,y,t) = 0) \tag{28}$$

$$= cj2\pi\mu_1\Big((I(x,y,t)e^{-j2\pi(\mu_1 x+\mu_2 y)})\Big|_{-\infty}^{\infty} - \int_{-\infty}^{+\infty}I(x,y,t)[-j2\pi\mu_1]e^{-j2\pi(\mu_1 x+\mu_2 y)}dx\Big) \tag{29}$$

$$= 0 - 4\pi^2\mu_1^2 c\hat{I}(\mu_1,\mu_2,t) \text{ using } I(\infty,y,t) = I(-\infty,y,t) = 0 \text{ and by defn. of Fourier transform }. \tag{30}$$

Likewise, we have $\mathcal{F}[c(\frac{\partial^2 I(x,y,t)}{\partial x^2}](\mu_1,\mu_2) = -4\pi^2\mu_2^2 c\hat{I}(\mu_1,\mu_2,t)$. These two results carry 4 points

Now, Eqn. 25 becomes:

$$\frac{\partial \hat{I}(\mu_1,\mu_2,t)}{\partial t} = -4\pi^2(\mu_1^2+\mu_2^2)c\hat{I}(\mu_1,\mu_2,t). \tag{31}$$

which is really just an ordinary differential equation (ODE) since we have eliminated the derivatives in x and have only the derivatives in t. This ODE can be solved as follows:

$$\frac{d\hat{I}(\mu_1,\mu_2,t)}{dt} = -4\pi^2(\mu_1^2+\mu_2^2)c\hat{I}(\mu_1,\mu_2,t) \tag{32}$$

$$\therefore \frac{d\hat{I}(\mu,t)}{\hat{I}(\mu,t)} = -4\pi^2(\mu_1^2+\mu_2^2)cdt \tag{33}$$

$$\therefore \log\hat{I}(\mu_1,\mu_2,t) = -4\pi^2(\mu_1^2+\mu_2^2)ct + c'(\mu_1,\mu_2) \text{ taking the integrals on both sides} \tag{34}$$

$$\therefore \hat{I}(\mu_1,\mu_2,t) = e^{-4\pi^2(\mu_1^2+\mu_2^2)\rho t}c'(\mu_1,\mu_2) \tag{35}$$

where $c'(\mu_1, \mu_2)$ is a constant of integration, which could be function of μ_1, μ_2, but will always be completely independent of t. This result carries 3 points
Setting $t = 0$ in the above equation, we now have:

$$\hat{I}(\mu_1,\mu_2,0) = c'(\mu_1,\mu_2), \tag{36}$$

which finally yields:

$$\hat{I}(\mu_1,\mu_2,t) = e^{-4\pi^2(\mu_1^2+\mu_2^2)\rho t}\hat{I}(\mu_1,\mu_2,0). \tag{37}$$

This result carries 2 points

Note that the RHS of the above equation contains the product of two Fourier transforms, one being $\hat{I}(\mu_1, \mu_2, 0)$ and the other being $e^{-4\pi^2(\mu_1^2+\mu_2^2)ct}$. Using the convolution theorem to transfer to the spatial domain, will give rise to the convolution of two functions:

$$I(x,y,t) = \frac{1}{\sqrt{4\pi\rho t}}e^{-\frac{x^2+y^2}{4\rho t}} * I(x,y,0) \tag{38}$$

where the former function is a Gaussian with mean 0 and standard deviation $\sqrt{2ct}$. Note that this was possible because of the following result pertaining to Fourier transforms of a Gaussian:

$$\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2 \mu^2}{a}}. \tag{39}$$

This result carries 2 points. You will find a proof of the above result at here, but the proof is not essential for the rest of the material here, or for the rest of this course.

Hence we can say that:

$$\mathcal{F}^{-1}(e^{-4\pi^2(\mu_1^2+\mu_2^2)ct}) = \mathcal{F}^{-1}(e^{-\frac{\pi^2(\mu_1^2+\mu_2^2)}{1/(4\rho t)}}) \tag{40}$$

$$= \sqrt{\frac{1}{4\pi\rho t}} e^{-\frac{x^2+y^2}{4\rho t}} [\text{ Note that: } a = \frac{1}{4\rho t}] \tag{41}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2\rho t}} e^{-\frac{x^2+y^2}{2(2\rho t)}}. \tag{42}$$

Thus, we have proved that executing the heat equation is equivalent to Gaussian convolution. As mentioned earlier, the standard deviation of the Gaussian is proportional to $\sqrt{t}$. The longer you run the PDE, the more the standard deviation of the Gaussian.

Now, you may then wonder why we should bother about the heat equation at all. After all, we have proved that executing the heat equation is equivalent to Gaussian convolution. In fact, the convolution operation is computationally faster and more numerically stable than executing the heat equation. Well, we still study it for two reasons. Firstly, this is a basic framework for image filtering. There are several different types of PDEs (and amongst them, the heat equation is one of the most basic PDEs), all of which perform varied operations on an image. Most of them <u>cannot</u> be expressed in the form of convolutions or any analytic, closed-form formulae. The only way to solve most of them, is to simulate them numerically. Secondly, the overall idea of simulating diffusion equations on an image is quite interesting!