Question 4

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Image Representation

The image is a matrix of size 201×201, where all pixels are black except for the central column (at column index 101) where all the pixel values are 255. Let $I(x, y)$ represent the intensity of the pixel at row x and column y. Therefore, the image can be expressed as:

$$I(x, y) = \begin{cases} 255 & \text{if } y = 101 \\ 0 & \text{otherwise} \end{cases}$$

where $x \in [0, 200]$ and $y \in [0, 200]$.

2D Fourier Transform Definition

The 2D Discrete Fourier Transform (DFT) of an image $I(x, y)$ is given by:

$$F(u, v) = \sum_{x=0}^{200} \sum_{y=0}^{200} I(x, y) e^{-2\pi i \left(\frac{ux}{201} + \frac{vy}{201}\right)}$$

where u and v are the spatial frequency indices in the Fourier domain.

Simplifying the Fourier Transform

Since the image is non-zero only when $y = 101$, we can reduce the summation over y as follows:

$$F(u, v) = \sum_{x=0}^{200} I(x, 101) e^{-2\pi i \frac{ux}{201}} e^{-2\pi i \frac{v \cdot 101}{201}}$$

Given that $I(x, 101) = 255$ for all x, we simplify the expression to:

$$F(u, v) = 255 \sum_{x=0}^{200} e^{-2\pi i \frac{ux}{201}} e^{-2\pi i \frac{v \cdot 101}{201}}$$

Summation over x

The summation over x is a geometric series of the form:

$$S(u) = \sum_{x=0}^{200} e^{-2\pi i \frac{ux}{201}}$$

This is a geometric progression with the first term as 1 and the common ratio $e^{-2\pi i \frac{u}{201}}$. The sum of such a geometric series for non-zero u is given by:

$$S(u) = \frac{1 - e^{-2\pi i u}}{1 - e^{-2\pi i \frac{u}{201}}}$$

At $u = 0$, the terms collapse to:

$$S(0) = 201$$

For other values of u, this summation tends to 0 because of destructive interference caused by the oscillating complex exponentials. Therefore, we can approximate the result as:

$$S(u) = 201\delta(u)$$

where $\delta(u)$ is the Kronecker delta function, which is 1 when $u = 0$ and 0 otherwise.

Final Fourier Transform Expression

Substituting this back into the expression for $F(u, v)$, we get:

$$F(u, v) = 255 \cdot 201 \cdot \delta(u) e^{-2\pi i \frac{v \cdot 101}{201}}$$

Thus, the Fourier transform is a vertical line (along the v-axis) located at $u = 0$, modulated by the phase factor $e^{-2\pi i \frac{v \cdot 101}{201}}$, which corresponds to a phase shift due to the position of the non-zero column in the spatial domain.

Magnitude of the Fourier Transform

The magnitude of the Fourier transform is given by:

$$|F(u, v)| = 255 \cdot 201 \cdot |\delta(u)|$$

Since $\delta(u) = 1$ only at $u = 0$, we have:

$$|F(u, v)| = \begin{cases} 255 \cdot 201 & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases}$$

Fourier Magnitude Plot

Here is the plot of the Fourier magnitude obtained from MATLAB:

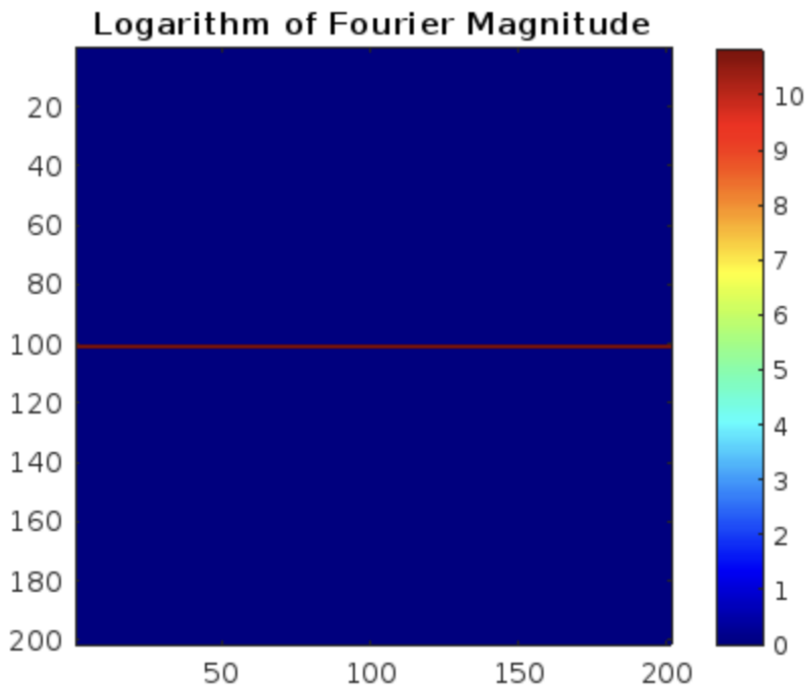

Figure 1: Logarithm of Fourier Magnitude of the 201x201 Image