Question 3

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1. Separable Filters

(a) The Laplacian Mask with a -8 in the Center Consider the Laplacian mask with a center value of -8. The given Laplacian mask is:

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

This mask represents a 2D convolution kernel. To determine whether this mask is separable, we need to check if it can be expressed as the outer product of two 1D vectors:

$$L = \mathbf{v} \otimes \mathbf{w}$$

One approach to prove the non-separability of a filter is by examining its rank. A 2D matrix is separable if and only if its rank is 1. If the rank of the matrix is greater than 1, then it cannot be expressed as the outer product of two vectors, and thus is non-separable.

Consider the Laplacian filter:

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

To check its rank, we can calculate the determinant of any 2×2 submatrices. Consider the top-left 2×2 submatrix:

$$\text{Submatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -8 \end{bmatrix}$$

The determinant is:

$$\det \begin{bmatrix} 1 & 1 \\ 1 & -8 \end{bmatrix} = 1(-8) - 1(1) = -9$$

Since the determinant is non-zero, the rank of the matrix is greater than 1. Therefore, the Laplacian filter is **non-separable**.

Hence, the Laplacian mask with a -8 in the center **is not** a separable filter.

(b) The Laplacian Mask with a -4 in the Center The mask is:

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A 1D convolution applied sequentially in different directions is equivalent to the convolution with a separable 2D kernel. For a 2D convolution to be separable, the kernel L must be decomposable into the outer product of two 1D vectors. In other words, we should be able to express the mask L as:
To implement this Laplacian mask using 1D convolutions, we need to express it as a combination of separable 1D filters.

Assuming separability, we would have:

$$L = \mathbf{u} \otimes \mathbf{v} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

Matching the entries of the mask L with $\mathbf{u} \otimes \mathbf{v}$, we need to satisfy:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

Testing for possible vectors $\mathbf{u}$ and $\mathbf{v}$: For $u_1v_1 = 0$ to hold, either u_1 or v_1 needs to be 0. If $u_1 = 0$, the entire first row becomes 0, which contradicts $u_1v_2 = 1$. If $v_1 = 0$, the first column becomes 0, which contradicts $u_2v_1 = 1$. Hence, no such vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy this equality. This indicates that the given Laplacian mask is not separable.

Also, the rank of the matrix is not 1.

Hence, given laplacian mask cannot be implemented entirely using 1D convolutions.