# Linear Algebra

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## Vector, matrix and tensor

Volta

Notice

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ a_{31} & \dots & & \\ a_{mi} & \dots & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{ij} \\ a_{mi} \\ & & & \\ & & & \\ & & & \\ & & \\ & &$$

## Vector, matrix and tensor

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Xi > vector

# Transpose, Addition, Subtraction and Scalar Multiplication

A = 
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
,  $A^{T} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$   $A^{$ 

Matrix Multiplication

$$C_{mxp} = A B$$

$$R_{mxn} = \sum_{k} a_{ik} \times k_{kj} = a_{ik} k_{kj}$$

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$$R_{mxn}$$

vertor Dot produt

$$x = \begin{bmatrix} x_1 \\ y_2 \\ \vdots \\ x_n \end{bmatrix}; y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

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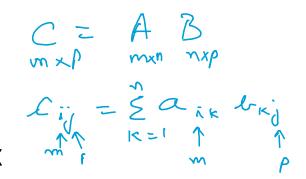
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#### **Einstein Summation Notation**



- Summation is performed over repeated index
- No indices appear more than two times in the equation
- Indices which is summed over is called dummy indices appear only in one side of equation
- Indices which appear on both sides of the equation is free indices.

indices. 
$$b_{jk} = \sum_{i=1}^{2} a_{ij} b_{jk}$$

$$a_{ii} = \sum_{i=1}^{2} a_{ii} = a_{i1} + a_{22} + \cdots + a_{2d}$$

# Matrix Multiplication

$$x \cdot y = \begin{bmatrix} x_1, x_2 & \vdots & x_n \\ y_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots & y_n \end{bmatrix} = x_i y_i$$

$$(x^{T}y) = (y^{T}x)$$

$$x_{i}y_{i}$$

$$y_{j}x_{i}$$

# Square matrix, main diagonal, trace

## **Identity and Inverse Matrices**

In 
$$E$$
  $R^{n\times n}$   $A \cdot R$ .

In  $X = X$ 

And  $X = X$ 

And  $X = X$ 

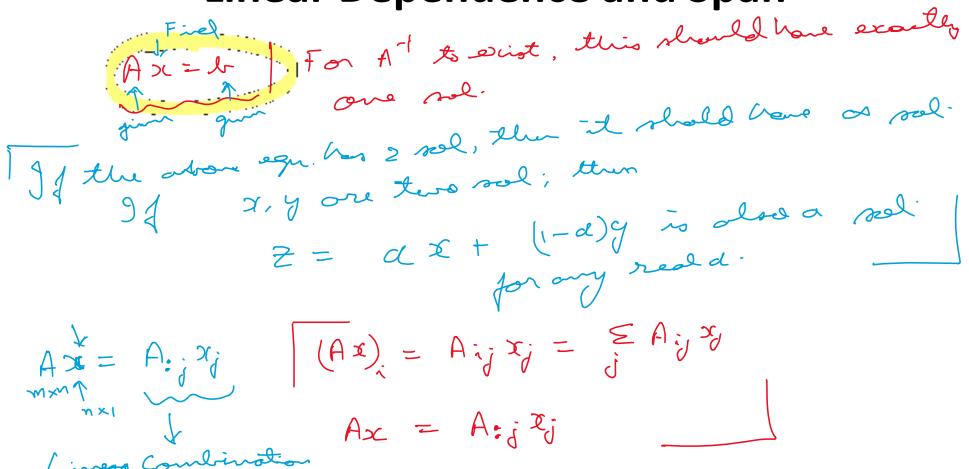
In  $X = X$ 

Ax=lr for A to exist, this should have exactly If the above egn has 2 seel, then I shold have 94 x, y ore two seel; then  $Z = \alpha x + (1-\alpha)y$  is also a self.

for any real  $\alpha$ . (Ax) = Aijxj = EAijx

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ME 781: Statistical Machine Learning and Data Mining



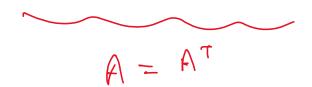
it should be a part of man of M. => Column men og A tole Rm nym { recensory but not official. A has de la square matirie de have A-1 (Ax) = A-1& => (x = A-1&)

#### **Norms**

$$\|x\|_{p} = \left[\sum_{i} |x_{i}|^{p}\right]^{p}$$

Eg. 
$$\|x\|_2 = \sqrt{\sum x_i^2}$$
  
Norm is a furthan  $f$  that ordinary  
 $-f(x) = 0 \Rightarrow x = 0$   
 $-f(x+y) \leq f(x) + f(y)$   
 $+d \in R$ ;  $f(ax) = |a|f(x)$ 

# Symmetric matrix, unit vector and orthogonal



Unit Jector: ||X||, =1

Two restorsore orthogonal if XT y = 0

O rethonord

O retrogord motrix: Columno are mutually orthonormal

ATA = I = A AT } AT = A

#### **General Inner Products**

Let V be a vector space and  $\Omega: V \times V \to R$  be a bilinear mapping that takes two vectors and maps them onto a real number. Then

- $\Omega$  is called symmetric if  $\Omega(x, y) = \Omega(y, x)$  for all  $x, y \in V$ , i.e., the order of the arguments does not matter.
- $\Omega$  is called positive definite if  $\forall x \in V \setminus \{0\} : \Omega(x, x) > 0$ ,  $\Omega(0, 0) = 0$
- A positive definite, symmetric bilinear mapping  $\Omega: V \times V \to R$  is called an inner product on V. We typically write  $\langle x, y \rangle$  instead of  $\Omega(x, y)$ .
- The pair  $(V, \langle \cdot, \cdot \rangle)$  is called an inner product space or (real) vector space with inner product. If we use the dot product definition, we call  $(V, \langle \cdot, \cdot \rangle)$  a Euclidean vector space.

#### Inner Product of Functions

$$\langle u,v\rangle := \int_a^b u(x)v(x)dx$$

 $u: \mathbb{R} \to \mathbb{R} \text{ and } v: \mathbb{R} \to \mathbb{R}$ 

Kronecker delta Sii = 3 Sii Sik = Sik

Permutation tensor, also called the Levi-Civita tensor or isotropic tensor

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any two labels are the same} \\ 1, & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \end{cases}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

#### **Determinant**

$$\det(\mathbf{A}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} \qquad = \qquad \underbrace{\mathcal{Z}}_{ij} \underbrace{\mathcal{Z}}_{i$$

## **Vector cross product**

$$\mathbf{a} imes \mathbf{b} = egin{array}{cccc} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \ a^1 & a^2 & a^3 \ b^1 & b^2 & b^3 \ \end{array} = egin{array}{cccc} arepsilon_{ijk} \mathbf{e}_i a^j b^k \ \end{array}$$

#### **Partial Differentiation and Gradients**

$$\frac{\partial f}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \dots, x_n) - f(x)}{h}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} = \lim_{h \to 0} \frac{f(x_1, \dots, x_{n-1}, x_n + h) - f(x)}{h}$$

$$\nabla_{x} f = \operatorname{grad} f = \frac{\mathrm{d}f}{\mathrm{d}x} = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} & \cdots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{1 \times n}$$

#### **Gradients of Vector-Valued Functions**

$$f(x) = egin{bmatrix} f_1(x) \ dots \ f_m(x) \end{bmatrix} \in \mathbb{R}^m$$
 .

$$\frac{\partial f}{\partial w_i} = \begin{bmatrix} \frac{\partial f_i}{\partial w_i} \\ \vdots \\ \frac{\partial f_{in}}{\partial w_i} \end{bmatrix} = \begin{bmatrix} \lim_{k \to 0} \frac{f_i(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_{i-1}, x_{i-1}) - f_i(x_i)}{h} \\ \vdots \\ \lim_{k \to 0} \frac{f_i(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_{i-1}, x_{i-1}) - f_i(x_i)}{h} \end{bmatrix} \in \mathbb{R}^m$$

#### **Gradients of Vector-Valued Functions**

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \in \mathbb{R}^m. \qquad \frac{hf}{\partial x_i} = \begin{bmatrix} \frac{\partial x_i}{\partial x_i} \\ \vdots \\ \frac{\partial x_{i+1}}{\partial x_i} \end{bmatrix} = \begin{bmatrix} \lim_{x \to \infty} \frac{1}{2} \frac{1}{2$$

### **Gradients of Vector-Valued Functions**

$$J = \nabla_x f = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \left[\frac{\partial f(x)}{\partial x_1} \cdots \frac{\partial f(x)}{\partial x_n}\right]$$

$$= \left[\frac{\partial f_1(x)}{\partial x_1} \cdots \frac{\partial f_1(x)}{\partial x_n}\right]$$

$$= \left[\frac{\partial f_2(x)}{\partial x_1} \cdots \frac{\partial f_2(x)}{\partial x_n}\right]$$