Singular Value Decomposition (SVD)

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Singular Value Decomposition (SVD)

$$M_{m \times n} = U_{m \times m} \, \Sigma_{m \times n} \, V^*_{n \times n}$$

The diagonal value are called the singular values.

Singular Value Decomposition (SVD)

$$\begin{bmatrix} \Sigma_{ii} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underset{m \times n}{M} = \underset{m \times m}{U} \; \underset{m \times n}{\Sigma} \; \underset{n \times n}{V^*}$$

Both U and V^* are unitary matrix (Real or complex)

$\Rightarrow U^* U = I_{m \times m}$ and $V^* V = I_{n \times n}$

Σ is a diagonal rectangular matrix of non-negative real numbers.

Singular Value Decomposition (SVD)

$$M_{m \times n} = U_{m \times m} \Sigma_{m \times n} V^*_{n \times n}$$

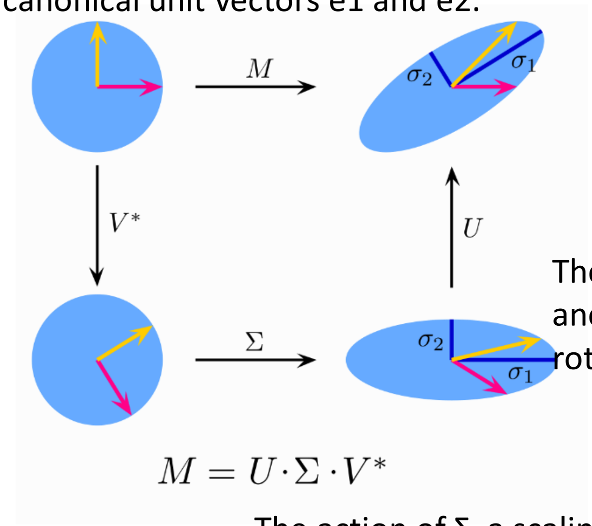
If M is real then U and V are real orthogonal matrices
The diagonal values of Σ are know as the singular values. By convention they are written in descending order. In this case Σ (but not always U and V^*) is uniquely determined by M.

Singular Value Decomposition (SVD)

$$M = U\Sigma V^*$$

Illustration of the singular value decomposition UΣV* of a real 2×2 matrix M.

The action of M, indicated by its effect on the unit disc D and the two canonical unit vectors e1 and e2.



The action of V*, a rotation, on D, e1, and e2.

The action of U, another rotation.

The action of Σ, a scaling by the singular values σ1 horizontally and σ2 vertically.

Compact Singular Value Decomposition

Semi-unitary Matrix $\Rightarrow U^*U = I$

$V^*V = I$

$$\underset{m \times n}{M} = \underset{m \times r}{U} \; \underset{r \times r}{\Sigma} \; \underset{r \times n}{V^*}$$

$V_{n \times r}$

$r \leq \min(m, n)$ is the rank of M

$\Sigma_{r \times r}$ is a non-zero diagonal matrix, it has non-zero singular value

$\Sigma \begin{bmatrix} \text{larger} \\ \to \text{smaller} \end{bmatrix}$

Multiple Linear Regression

$$\beta^* = \underbrace{\left(X^T X\right)^{-1}}\ \underbrace{X^T Y}$$

$$X = \begin{bmatrix} 1 & x_1^1 & x_1^2 & \cdots & x_1^p \\ \vdots & & & & \\ 1 & x_n^1 & x_n^2 & \cdots & x_n^p \end{bmatrix}$$

$n \times (p+1)$

$\underbrace{(p+1) \times n \quad n \times (p+1)}$

$(p+1)\ (p+1)$

$\underbrace{(p+1) \times n \quad n \times 1}$

$(p+1)\ 1$

$(p+1) \times 1$

Collinearity

If singular values are not zero then

$(X^T X)^{-1}$ will exist.

$$X = U \Sigma V^T$$

$$\begin{bmatrix} \Sigma_d \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \Sigma_d & 0 \\ 0 & 0 \end{bmatrix}$$

$X = U \Sigma V^T = U \begin{bmatrix} \Sigma_d \\ 0 \end{bmatrix} V^T$ → diagonal matrix with singular values

Collinearity

$$(X^T X)^{-1} = \left(V \begin{bmatrix} \Sigma_d & 0 \end{bmatrix} \underbrace{U^T U}_{I} \begin{bmatrix} \Sigma_d \\ 0 \end{bmatrix} V^T \right)^{-1}$$

If singular values are very small

$$= \left(V \Sigma_d^2 V^T \right)^{-1}$$

$\Rightarrow$ large β

$$= (V^T)^{-1} (\Sigma_d^2)^{-1} (V)^{-1}$$

$$= V \Sigma_d^{-2} V^T$$

→ diagonal with $\frac{1}{\text{singular}^2}$