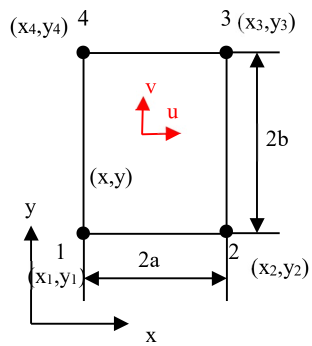
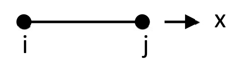
ME 756: Quiz 02 Solution

1. *(a) Define appropriate bilinear shape functions for the following 4 node quadrilateral element to present the elemental displacement {d} in terms of the elemental shape function [N] and the nodal displacement {de}, and also show that all the shape functions follow the completeness norms.* *(5)*
   (b) Considering plane stress and following generalized Hook's law, show that the elemental stress {σ} and the elemental strain {ε} vectors can be presented in a matrix form by the use of an elasticity matrix [D], where [D] is a function of elastic modulus and Poisson's ratio. *(5)*
   (c) What would be the nature of variations of elemental displacement {d}, stress {σ} and strain {ε} within the element and along the elemental boundaries. *(5)*



Solution:

(a) The linear shape functions are given by



$$N_i(x) = \frac{x_j - x}{x_j - x_i},\ N_j(x) = \frac{x - x_i}{x_j - x_i}$$

where, N is the shape function, x_i and x_j are the x-coordinate for the point i and j, respectively.

The bilinear shape functions for the 4-node quadrilateral element to present the elemental displacement {d} in terms of the elemental shape function [N] and the nodal displacement $\{d^e\}$are defined as follows

$$N_1 = \frac{1}{4ab}(x_2 - x)(y_4 - y) \qquad (1)$$

$$N_2 = \frac{1}{4ab}(x - x_1)(y_3 - y) \qquad (2)$$

$$N_3 = \frac{1}{4ab}(x - x_4)(y - y_2) \qquad (3)$$

$$N_4 = \frac{1}{4ab}(x_3 - x)(y - y_1) \qquad (4)$$

Properties of the shape function for the bilinear rectangular element are given by

- Each shape function varies linearly along the edges between its node and the two adjacent nodes.
- Each shape function is zero along the sides its node does not touch.
- Each shape function has a value of one at its own node and zero at other nodes.
- $N_1 + N_2 + N_3 + N_4 = 1$

(b) In plane stress condition, the elastic strains in x- and y-directions are given by

$$\epsilon_x = \frac{(\sigma_x - \nu\sigma_y)}{E} \quad (1)$$

$$\epsilon_y = \frac{(-\nu\sigma_x + \sigma_y)}{E} \quad (2)$$

where, E is the elastic modulus, ν is the Poisson's ratio, σ is the stress.

In the matrix form, the equations (1) and (2) is written as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} \quad (3)$$

The equation (3) can be written after the inversion as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \epsilon_x \\ \epsilon_y \end{Bmatrix} \quad (4)$$

The shear stress-strain relationship is given by

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \tau_{xy}\frac{2(1+\nu)}{E} \quad (5)$$

Combining the equation (4) and (5), the matrix relationship is obtained as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (6)$$

or,

$$\{\sigma\} = [D]\{\epsilon\} \quad (7)$$

where, D is the elasticity matrix.

(c) The elemental displacement {d}is linear in x along any line of constant y and linear in y along any line of constant x. Because of these properties, the element is called as bilinear. The strain {ε} and {σ} also varies linearly within the element and along the elemental boundaries.

2. *(a) Consider an axi-symmetric ring-type element with uniform triangular cross-section and three vertices in the cylindrical coordinate system (r, θ, z). The axis of the ring element coincides with the z-axis of the cylindrical coordinate system. State in two bulleted points how such an element can be useful for any kind of numerical modelling of either elastic stress analysis or heat conduction analysis?* *(5)*

   (b) Derive elemental displacement {d} for such an element in terms of the elemental shape function [N] and nodal displacement vector {d^e}. *(5)*

   (c) What would be the nature of variations of elemental displacement {d}, stress {σ} and strain {ε} within the element and along the elemental boundaries. *(5)*

Solution:

(a) Figure 1 shows an axi-symmetric ring-type element with uniform triangular cross-section and three vertices in the cylindrical coordinate system (r, θ, z).

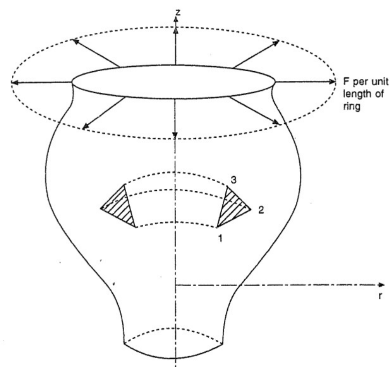


Figure 1 An axi-symmetric element

- Axi-symmetric elements allow for the simplification of complex 3D problems into 2D analyses by exploiting the symmetry about an axis.
- The axi-symmetric elements are useful for any kind of numerical modelling of either elastic stress analysis or heat conduction analysis by saving in computational time.

(b) Figure 2 shows the ring-shaped axi-symmetric elements with triangular cross-section in the r-z plane. The displacements and forces in r and z directions are represented as u, v, F_r, F_z, respectively. The nodal values are given as suffixes as 1, 2, 3 for the three nodes of the triangular element.

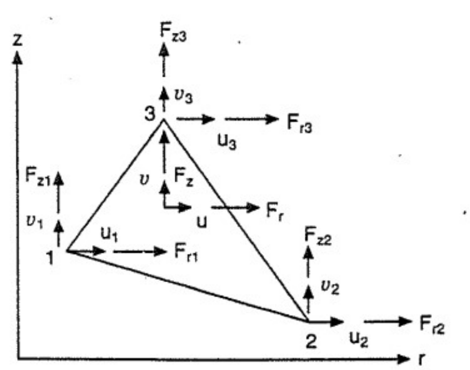


Figure 2 Nomenclature of an axi-symmetric element

The general expansion for the displacements within the elements (1-2-3) are written as

$$u = \alpha_1 + \alpha_2 r + \alpha_3 z \tag{1}$$

$$v = \alpha_4 + \alpha_5 r + \alpha_6 z \tag{2}$$

where, α_1, α_2, α_3, α_4, α_5, and α_6 are constants.

The nodal displacements in x-direction can be written in the matrix notation as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{3}$$

Equation (3), on inversion, gives α_1, α_2, α_3 as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{4}$$

where,

$$2\Delta = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \tag{5}$$

$$\begin{aligned} a_1 &= r_2 z_3 - r_3 z_2, & a_2 &= r_3 z_1 - r_1 z_3, & a_3 &= r_1 z_2 - r_2 z_1 \\ b_1 &= z_2 - z_3, & b_2 &= z_3 - z_1, & b_3 &= z_1 - z_2 \\ c_1 &= r_3 - r_2, & c_2 &= r_1 - r_3, & c_3 &= r_2 - r_1 \end{aligned} \tag{6}$$

The expression for the displacement u is written as

$$u = \frac{1}{2\Delta} [(a_1 + b_1 r + c_1 z)\, u_1 + (a_2 + b_2 r + c_2 z)\, u_2 + (a_3 + b_3 r + c_3 z)\, u_3] \tag{7}$$

Similarly, the displacement v is written as

$$v = \frac{1}{2\Delta} [(a_1 + b_1 r + c_1 z)\, v_1 + (a_2 + b_2 r + c_2 z)\, v_2 + (a_3 + b_3 r + c_3 z)\, v_3] \tag{8}$$

By considering $(a_1 + b_1 r + c_1 z)/\, 2\Delta$ as N_1 and so on, Equations (7) and (8) can be written as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \tag{9}$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 \tag{10}$$

The vectors {d}, $\{d_1\}$, $\{d_2\}$, $\{d_3\}$ define general and nodal displacements in the element while nodal displacement vector is $\{d^e\}$, can be written as

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix};\ \{d_1\} = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix};\ \{d_2\} = \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix};\ \{d_3\} = \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix};\ \{d^e\} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \\ \{d_3\} \end{Bmatrix} \tag{11}$$

Equation (9) and (10) can be written as

$$\{d\} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{12}$$

or, $$\{d\} = [N]\{d^e\} \tag{13}$$

(c) The elemental displacement {d}, strain {ε}, and stress {σ} over any cross-section of a ring-shaped element is identical and varies linearly within the element and along the elemental boundaries.