Question 6

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a. Rotational invariance

Let L be the lagrangian operator acting on image $I(x, y)$.

$$L(I(x,y)) = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \tag{1}$$

Consider the following notation.

$$I_{xx} = \frac{\partial^2 I}{\partial x^2} \tag{2a}$$

$$I_{yy} = \frac{\partial^2 I}{\partial y^2} \tag{2b}$$

According to the transformation,

$$u = x\cos\theta - y\sin\theta \tag{3a}$$

$$v = x\sin\theta + y\cos\theta \tag{3b}$$

Inverting,

$$x = u\cos\theta + v\sin\theta \tag{4a}$$

$$y = v\cos\theta - u\sin\theta \tag{4b}$$

Then we find the relevant derivatives,

$$\frac{\partial x}{\partial \theta} = -u\sin\theta + v\cos\theta \tag{5a}$$

$$\frac{\partial y}{\partial \theta} = -v\sin\theta - u\cos\theta \tag{5b}$$

$$x_u = \frac{\partial x}{\partial u} = \cos\theta \tag{5c}$$

$$y_u = \frac{\partial y}{\partial u} = -\sin\theta \tag{5d}$$

$$x_{xu} = \frac{\sin\theta}{u\sin\theta - v\cos\theta} \tag{5e}$$

$$y_{xu} = \frac{\cos\theta}{u\sin\theta - v\cos\theta} \tag{5f}$$

$$x_{yu} = \frac{\sin\theta}{v\sin\theta + u\cos\theta} \tag{5g}$$

$$y_{yu} = \frac{\cos\theta}{v\sin\theta + u\cos\theta} \tag{5h}$$

Solving for I_{uu},

$$I_{uu} = \frac{\partial^2 I}{\partial u^2} = \frac{\partial}{\partial u}\frac{\partial I}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x}(\frac{\partial x}{\partial u}\frac{\partial I}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial I}{\partial y}) + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}(\frac{\partial x}{\partial u}\frac{\partial I}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial I}{\partial y})$$

$$= x_u^2 I_{xx} + I_x x_u x_{xu} + x_u y_u I_{xy} + I_y x_u y_{xu} + y_u x_u I_{yx} + I_x y_u x_{yu} + y_u^2 I_{yy} + I_y y_u y_{yu}$$

Finding more derivatives

$$x_v = \frac{\partial x}{\partial v} = \sin\theta \tag{6a}$$

$$y_v = \frac{\partial y}{\partial v} = \cos\theta \tag{6b}$$

$$x_{xv} = \frac{\partial^2 x}{\partial x \partial v} = \frac{\partial}{\partial x}\frac{\partial x}{\partial v} = \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\sin\theta = \frac{-\cos\theta}{u\sin\theta - v\cos\theta} \tag{6c}$$

$$y_{xv} = \frac{\partial^2 y}{\partial x \partial v} = \frac{\partial}{\partial x}\frac{\partial y}{\partial v} = \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\cos\theta = \frac{\sin\theta}{u\sin\theta - v\cos\theta} \tag{6d}$$

$$x_{yv} = \frac{\partial^2 x}{\partial y \partial v} = \frac{\partial}{\partial y}\frac{\partial x}{\partial v} = \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\sin\theta = \frac{-\cos\theta}{v\sin\theta + u\cos\theta} \tag{6e}$$

$$y_{yv} = \frac{\partial^2 y}{\partial y \partial v} = \frac{\partial}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial \theta}{\partial}\frac{\partial}{\partial \theta}\cos\theta = \frac{\sin\theta}{v\sin\theta + u\cos\theta} \tag{6f}$$

Solving for I_{vv},

$$I_{vv} = \frac{\partial^2 I}{\partial v^2} = \frac{\partial}{\partial v}\frac{\partial I}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x}(\frac{\partial x}{\partial v}\frac{\partial I}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial I}{\partial y}) + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}(\frac{\partial x}{\partial v}\frac{\partial I}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial I}{\partial y})$$

$$= x_v^2 I_{xx} + I_x x_v x_{xv} + x_v y_v I_{xy} + I_y x_v y_{xv} + y_v x_v I_{yx} + I_x y_v x_{yv} + y_v^2 I_{yy} + I_y y_v y_{yv}$$

Adding the above equations with all substitutions gives,

$$I_{uu} + I_{vv} = (x_u^2 + x_v^2)I_{xx} + I_x(x_u x_{xu} + x_v x_{xv}) + (x_u y_u + x_v y_v)I_{xy} + I_y(x_u y_{xu} + x_v y_{xv}) + (y_u x_u + y_v x_v)I_{yx}$$

$$+I_x(y_u x_{yu} + y_v x_{yv}) + (y_u^2 + y_v^2)I_{yy} + I_y(y_u y_{yu} + y_v y_{yv})$$

$$I_{uu} + I_{vv} = (1)I_{xx} + I_x(0) + (0)I_{xy} + I_y(0) + (0)I_{yx} + I_x(0) + (1)I_{yy} + I_y(0)$$

$$I_{uu} + I_{vv} = I_{xx} + I_{yy} \tag{7}$$

b. Directional second derivative

The second directional derivative along u is given by,

$$D_u^2 I = D_u(\nabla I.u) = \nabla(\nabla I.u).u = \nabla\left(I_x u_x + I_y u_y\right).u = ((I_x u_x + I_y u_y)_x, (I_x u_x + I_y u_y)_y)$$

$$= (I_{xx}u_x + I_{yx}u_y, I_{xy}u_x + I_{yy}u_y)u = u^T H u$$

$$D_u^2 I = u^T H u \tag{8}$$

$$H = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix}$$

Note here that $I_x = \frac{\partial I}{\partial x}$ whereas u_x is the x component of u. Evaluating the derivation along the gradient,

$$D_{grad}^2 I = \frac{I_x^2 I_{xx} + 2I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2} \tag{9}$$

Directional second derivative perpendicular to gradient

A vector perpendicular to the gradient vector can be given by,

$$p = (\frac{-I_y}{\sqrt{I_x^2 + I_y^2}}, \frac{I_x}{\sqrt{I_x^2 + I_y^2}})$$

Evaluating the derivative along this we get,

$$D_p^2 I = \frac{I_y^2 I_{xx} - 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2} \tag{10}$$