Question 7

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For the spatial domain, the 2D Fourier transform is

$$\mathcal{F}_{2D}(I) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I e^{-j(\omega_x x + \omega_y y)} dxdy \tag{1}$$

$$\mathcal{F}_{2D}(\frac{\partial I(x,y,t)}{\partial x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial I(x,y,t)}{\partial x} e^{-j(\omega_x x + \omega_y y)} dxdy \tag{2}$$

$$\mathcal{F}_{2D}(\frac{\partial I(x,y,t)}{\partial x}) = j\omega_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(x,y,t) e^{-j(\omega_x x + \omega_y y)} dxdy \tag{3}$$

$$\mathcal{F}_{2D}(\frac{\partial I}{\partial x}) = j\omega_x \mathcal{F}_{2D}(I) \tag{4}$$

Repeating the same exercise again,

$$\mathcal{F}_{2D}(\frac{\partial^2 I}{\partial x^2}) = -\omega_x^2 \mathcal{F}_{2D}(I) \tag{5}$$

Combining all results,

$$\mathcal{F}_{2D}(\nabla^2 I) = -(\omega_x^2 + \omega_y^2)\mathcal{F}_{2D}(I) \tag{6}$$

Applying 2D fourier transform on the heat equation we get,

$$\frac{\partial}{\partial t}\mathcal{F}_{2D}(I) = -c(\omega_x^2 + \omega_y^2)\mathcal{F}_{2D}(I) \tag{7}$$

$$\mathcal{F}_{2D}(I(x,y,t)) = \mathcal{F}_{2D}(I(x,y,0)) e^{-c(\omega_x^2+\omega_y^2)t} \tag{8}$$

Taking fourier inverse, the multiplication of the 2 signals (functions of t) becomes a convolution.

$$I(x,y,t) = I(x,y,0) * \mathcal{F}_{2D}^{-1}(e^{-c(\omega_x^2+\omega_y^2)t}) \tag{9}$$

Let $G(x,y,t) := \mathcal{F}_{2D}^{-1}(e^{-c(\omega_x^2+\omega_y^2)t})$

$$I(x,y,t) = I(x,y,0) * G(x,y,t) \tag{10}$$

Let us solve for G,

$$G(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-c(\omega_x^2+\omega_y^2)t} e^{j(\omega_x x + \omega_y y)} d\omega_x d\omega_y \tag{11}$$

$$G(x,y,t) = \int_{-\infty}^{\infty} e^{-ct\omega_x^2} e^{j\omega_x x} d\omega_x \int_{-\infty}^{\infty} e^{-ct\omega_y^2} e^{j\omega_y y} d\omega_y \tag{12}$$

$$G(x,y,t) = \mathcal{F}_x^{-1}(e^{-ct\omega_x^2})\mathcal{F}_y^{-1}(e^{-ct\omega_y^2}) \tag{13}$$

Let's define $H_a(x) := \mathcal{F}_x^{-1}(e^{-a\omega_x^2})$. By generality,

$$G(x,y,t) = H_{ct}(x)H_{ct}(y) \tag{14}$$

Clearly, H is fourier inverse of a gaussian. We will use the result,

$$H_a(x) = \frac{1}{2\pi a} e^{-x^2/2a} \tag{15}$$

$$G(x, y, t) = \frac{1}{2\pi ct} e^{-(x^2+y^2)/2ct} \tag{16}$$

Then, from equation 16,

$$I(x, y, t) = I(x, y, 0) * \frac{1}{2\pi ct} e^{-(x^2+y^2)/2ct} \tag{17}$$

Let $\sigma := \sqrt{ct}$

$$I(x, y, t) = I(x, y, 0) * \frac{1}{2\pi \sigma^2} e^{-(x^2+y^2)/2\sigma^2} \tag{18}$$

Clearly, the result of the PDE is the convolution of the original image with a 2D gaussian with a standard deviation t $\sigma = \sqrt{ct}$