CS663 - Assignment 5 - Question 1

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Answer:

(1) The paper presents a method for estimating the translation between two images based on the properties of the Fourier Transform. The approach consists of the following three steps:

Conside f_1 as the original image and f_2 as the translated version of f_1.

$$f_2(x, y) = f_1(x - x_0, y - y_0)$$

- **Fourier Transform of Images**: Compute the Fourier transforms of the two images, resulting in F_1 and F_2. According to the Fourier shift theorem, they are related by:

$$F_2(\mu, \nu) = e^{-j2\pi(\mu x_0 + \nu y_0)} \cdot F_1(\mu, \nu)$$

- **Cross-Power Spectrum**: Calculate the cross-power spectrum of the two images:

$$\frac{F_1(\mu, \nu) F_2^*(\mu, \nu)}{|F_1(\mu, \nu) F_2(\mu, \nu)|} = e^{j2\pi(\mu x_0 + \nu y_0)}$$

- **Inverse Fourier Transform**: Compute the inverse Fourier transform of the cross-power spectrum:

$$\mathcal{F}^{-1}\left(e^{j2\pi(\mu x_0 + \nu y_0)}\right) = \delta(x + x_0, y + y_0)$$

The resulting delta function is zero everywhere except at the point $(-x_0, -y_0)$. By reversing the signs of this observed nonzero or maximum location, we can determine the translation between the two images.

Time Complexity for Images of Size $N \times N$:

- **Step 1**: The Fourier Transform can be computed using the Fast Fourier Transform (FFT). For an image of size $N \times N$, the FFT has a time complexity of $\mathcal{O}(MN \log(MN))$. In this case:

$$\mathcal{O}(N^2 \log(N^2)) + \mathcal{O}(N^2 \log(N^2)) = \mathcal{O}(N^2 \log N)$$

- **Step 2**: Calculating the cross-power spectrum is a point-wise operation. If we consider the per-point computation to have an upper-bound constant complexity, the overall time complexity is approximately $\mathcal{O}(N^2)$.

- **Step 3**: The inverse Fourier Transform can also be performed using the FFT, resulting in:

$$\mathcal{O}(N^2 \log(N^2)) \approx \mathcal{O}(N^2 \log N)$$

Thus, the total time complexity of the process is:

$$\mathcal{O}(N^2 \log N) + \mathcal{O}(N^2) + \mathcal{O}(N^2 \log N) \approx \mathcal{O}(N^2 \log N)$$

Pixel-wise image comparison involves iterating over various possible translation values. For each iteration, the following steps are performed:

1. **Image Warping**: One of the images is transformed according to the current translation value. This step has a time complexity of approximately $\mathcal{O}(N^2)$, as it involves accessing the intensity value of each pixel in the transformed image.

2. **MSSD (Mean Squared Successive Difference)**: The mean squared difference between the transformed image and the reference image is computed. This step also requires $\mathcal{O}(N^2)$ time, as it involves calculating the intensity difference for every pixel.

Given that the number of possible translation values is N^2, the overall time complexity for this process is:

$$\mathcal{O}(N^2(N^2 + N^2)) \approx \mathcal{O}(N^4)$$

Finally, finding the translation value that minimizes the MSSD has a time complexity of approximately $\mathcal{O}(N^2)$.

Thus, the total time complexity for the pixel-wise comparison approach is:

$$\mathcal{O}(N^4)$$

In comparison, the FFT-based method significantly reduces the complexity, offering an improvement by a factor of $\frac{N^2}{\log N}$.

(2) Predicting the rotation between two images uses the properties of Fourier translation and rotation. This method is similar to the previous translation prediction technique, with the translation method being a part of this rotation prediction method.

Let the images be f_1 and f_2, where f_2 is a translated and rotated version of f_1:

$$f_2(x, y) = f_1(x \cos\theta_0 + y \sin\theta_0 - x_0, -x \sin\theta_0 + y \cos\theta_0 - y_0)$$

1. **Relation between their Fourier Transforms**:

$$F_2(\mu, \nu) = e^{-j2\pi(\mu x_0 + \nu y_0)} \cdot F_1(\mu \cos\theta_0 + \nu \sin\theta_0, -\mu \sin\theta_0 + \nu \cos\theta_0)$$

2. **Relation between the Magnitudes of $F_1(M_1)$ and $F_2(M_2)$**:

$$M_2(\mu, \nu) = M_1(\mu \cos\theta_0 + \nu \sin\theta_0, -\mu \sin\theta_0 + \nu \cos\theta_0)$$

In polar coordinates:

$$M_1(\rho, \theta) = M_2(\rho, \theta - \theta_0)$$

This equation is analogous to Equation **1** in the translation prediction method, so phase correlation between M_1 and M_2 can be used to determine the rotation angle θ_0.

This method can successfully predict the rotation, even in the presence of translation.

2D Fourier Rotation Theorem:

Aim:

$$f_2(x, y) = f_1(x\cos\theta_0 + y\sin\theta_0, -x\sin\theta_0 + y\cos\theta_0)$$

implies

$$F_2(\mu, \nu) = F_1(\mu\cos\theta_0 + \nu\sin\theta_0, -\mu\sin\theta_0 + \nu\cos\theta_0)$$

where F_1 and F_2 represent the Fourier transforms of f_1 and f_2, respectively.

The Fourier transform of $f_2(x, y)$ is given by:

$$F_2(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(x, y)e^{j2\pi(\mu x+\nu y)}\,dx\,dy$$

Substituting the expression for $f_2(x, y)$, we get:

$$F_2(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x\cos\theta_0 + y\sin\theta_0, -x\sin\theta_0 + y\cos\theta_0)e^{j2\pi(\mu x+\nu y)}\,dx\,dy$$

Next, we perform a change of variables by introducing new coordinates p and q such that:

$$p = x\cos\theta_0 + y\sin\theta_0 \quad \text{and} \quad q = -x\sin\theta_0 + y\cos\theta_0$$

The inverse transformations are:

$$x = -q\sin\theta_0 + p\cos\theta_0$$

$$y = p\sin\theta_0 + q\cos\theta_0$$

Now, we compute the Jacobian of this transformation. The Jacobian determinant is:

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{vmatrix} dp\,dq = \begin{vmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{vmatrix} dp\,dq = dp\,dq$$

Substituting the variables and the Jacobian determinant into the integral, we obtain:

$$F_2(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(p, q)e^{j2\pi\mu(-q\sin\theta_0+p\cos\theta_0)}e^{j2\pi\nu(p\sin\theta_0+q\cos\theta_0)}\,dp\,dq$$

This can be simplified as:

$$F_2(\mu, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(p, q)e^{j2\pi p(\mu\cos\theta_0+\nu\sin\theta_0)}e^{j2\pi q(-\mu\sin\theta_0+\nu\cos\theta_0)}\,dp\,dq$$

Finally, recognizing that the expression inside the integrals is simply the Fourier transform F_1, we conclude:

$$F_2(\mu, \nu) = F_1(\mu\cos\theta_0 + \nu\sin\theta_0, -\mu\sin\theta_0 + \nu\cos\theta_0)$$

Hence proved.