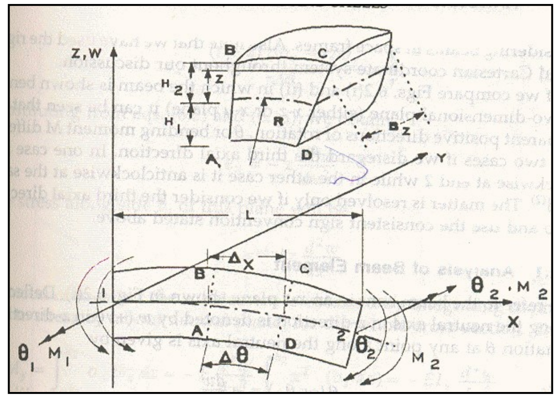
IIT Bombay / Department of Mechanical Engineering / ME 756 – Numerical Modeling of Mfg. Processes

Time: 75 mins QUIZ – 3 (2024) Max. Marks: 30

1. Following figure shows an one-dimensional beam element (along with its cross-section) and applied moments (M1 and M2) on both ends of the element in a schematic manner.



(a) Show that the strain energy of bending for the element can be written as

$$\boxed{\Pi_{se} = \frac{1}{2}\int_L \left(\frac{d^2w}{dx^2}\right)\left(EI_y\frac{d^2w}{dx^2}\right)dx}$$

where E and I_y are Young's Modulus and the area moment of inertia w.r.t. y-axis, respectively. (5)

(b) Develop the elemental shape functions with the displacement and rotation as the nodal degrees of freedoms. (5)

(c) Simplify the strain energy expression, as given above, in terms of x, E and Iy, and the length of the beam element. (5)

2. The final algebraic equation, which is required to solve for a two-dimensional steady-state heat conduction problem using finite element method, can be written as

$$[h^e]\{T^e\} + [\overline{h^e}]\{T^e\} + \{f_Q^e\} + \{f_q^e\} = 0$$

where the elemental conductivity matrix $[h^e]$ is expressed as

$$h_{ij}^e = \int_{A^e} k\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)dA$$

where k is thermal conductivity of the material, A is the area of the element, and N_i and N_j are elemental shape functions for the nodes i and j, and $\{T^e\}$ is the nodal temperature vector.

(a) Find the final expression for h_{ij}^e for a two-dimensional triangular element with three nodes, and for an one dimensional rod element with two nodes. (7.5 + 7.5)

Question-1

(a) In the figure, let the deflection due to bending along the neutral axis in z direction be w. Rotation at any point along the neutral axis is

$$\theta = -\frac{dw}{dx}$$

Change in angle due to rotation is

$$\Delta\theta_y = \frac{d(\theta_y)}{dx}\Delta x = -\frac{d^2w}{dx^2}\Delta x$$

Also,

$$r\Delta\theta_y = \Delta x$$

Strain along the neutral plain is,

$$\varepsilon_x = \frac{(r+z)\Delta\theta_y - r\Delta\theta_y}{r\Delta\theta_y} = \frac{z}{r}$$

$$\varepsilon_x = -z\frac{d^2w}{dx^2}$$

Stress developed is:

$$\sigma_x = -Ez\frac{d^2w}{dx^2}$$

Integrating the moment along the neutral plane from h1 to h2 for the whole cross section, the total bending moment is obtained as:

$$M_y = \int_{-h_1}^{h_2} \sigma_x\, z\, b_z\, dz = -E\frac{d^2w}{dx^2}\int_{-h_1}^{h_2} z^2\, b_z\, dz = -EI_y\frac{d^2w}{dx^2}$$

Strain energy in bending

$$\Delta\Pi_{se} = \frac{1}{2}(M_y\,.\,\Delta\theta_y)$$

Along the length, L

$$\Pi_{se} = \frac{1}{2}\int_L \left(\frac{d^2w}{dx^2}\right)\left(EI_y\frac{d^2w}{dx^2}\right)dx$$

(b) Considering displacement and rotation as nodal degree of freedoms, for an element with two nodes, there will be four nodal variables: w_1, w_2, θ_{y1} and θ_{y2}.

The expressions for these variables can be formed as:

$$w = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

$$\theta = -\frac{dw}{dx} = -\beta_2 - 2x\,\beta_3 - 3x^2\beta_4$$

Considering the element as the reference, with x coordinates at 1 and 2 as $x = 0$ and $x = L$, these variables can be written in matrix form as:

$$\begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & -1 & -2L & -3L^2 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{Bmatrix}$$

The coefficients found out by inverting the matrix gives;

$$w = \left\{1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right\} w_1 - \left\{x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right\} \theta_{y1} + \left\{\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right\} w_2 - \left\{-\frac{x^2}{L} + \frac{x^3}{L^2}\right\} \theta_{y2}$$

This can be written as:

$$w = [[N_1], [N_2]] \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix}$$

Where,

$$[N_1] = \left[\left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right), -\left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)\right]$$

$$[N_2] = \left[\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right), -\left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right)\right]$$

$$\{d_1\} = \begin{Bmatrix} w_1 \\ \theta_{y1} \end{Bmatrix}$$

$$\{d_2\} = \begin{Bmatrix} w_2 \\ \theta_{y2} \end{Bmatrix}$$

Or

$$w = [N]\,\{d^e\}$$

$[N]$ is the shape function and $\{d^e\}$ are the displacement vectors.

(c)

$$\frac{d^2w}{dx^2} = \left[\frac{d^2[N_1]}{dx^2}, \frac{d^2[N_2]}{dx^2}\right] \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix} = [[B_1], [B_2]] \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix}$$

$$[B] = [[B_1], [B_2]]$$

$$\frac{d^2w}{dx^2} = [B]\{d^e\}$$

The strain energy is given by:

$$\Pi_{se} = \frac{1}{2}\int_L EI_y \left(\frac{d^2w}{dx^2}\right)^2 dx$$

Let $[D] = [EI_y]$

$$\Pi_{se} = \frac{1}{2}\int_L \{d^e\}^T [B]^T [D][B]\{d^e\}\, dx^{(4)}$$

Is the expression of strain energy in terms of x, L, E and I_y

Question-2

Expression for elemental conductivity for steady state heat transfer analysis in two-dimensional region is given by:

$$\boxed{h_{ij}^{e} = \int_{A^e} K\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\cdot\frac{\partial N_j}{\partial y}\right) dA}$$

Since, this is for a 3 node triangular element, the elemental conductivity matrix will be $[h] = \begin{bmatrix} h_{11}^e & h_{12}^e & h_{13}^e \\ h_{21}^e & h_{22}^e & h_{23}^e \\ h_{31}^e & h_{32}^e & h_{33}^e \end{bmatrix}$

$$\Rightarrow h_{11}^{e} = \int_{A^e} K\left(\left(\frac{\partial N_1}{\partial x}\right)^2 + \left(\frac{\partial N_1}{\partial y}\right)^2\right) \cdot dA$$

$$h_{12}^{e} = \int_{A^e} K\left(\frac{\partial N_1}{\partial x}\cdot\frac{\partial N_2}{\partial x} + \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial y}\right) \cdot dA$$

$$h_{13}^{e} = \int_{A_e} K\left(\frac{\partial N_1}{\partial x}\cdot\frac{\partial N_3}{\partial x} + \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial y}\right) \cdot dA$$

$$h_{21}^{e} = \int_{A^e} K\left(\frac{\partial N_2}{\partial x}\cdot\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial y}\right) \cdot dA$$

$$h_{22}^{e} = \int_{A^e} K\left(\left(\frac{\partial N_2}{\partial x}\right)^2 + \left(\frac{\partial N_2}{\partial y}\right)^2\right) \cdot dA$$

$$h^e_{23} = \int_{A^e} K \left(\frac{\partial N_2}{\partial x} \cdot \frac{\partial N_3}{\partial x} + \frac{\partial N_2}{\partial x} \frac{\partial N_3}{\partial y} \right) \cdot dA$$

$$h^e_{31} = \int_{A^e} K \left(\frac{\partial N_3}{\partial x} \cdot \frac{\partial N_1}{\partial x} + \frac{\partial N_3}{\partial x} \frac{\partial N_1}{\partial y} \right) \cdot dA$$

$$h^e_{32} = \int_{A^e} K \left(\frac{\partial N_3}{\partial x} \cdot \frac{\partial N_2}{\partial x} + \frac{\partial N_3}{\partial x} \frac{\partial N_2}{\partial y} \right) \cdot dA$$

$$h^e_{33} = \int_{A^e} K \left(\left(\frac{\partial N_3}{\partial x} \right)^2 + \left(\frac{\partial N_3}{\partial y} \right)^2 \right) \cdot dA$$

(<u>Note</u>: Expression of any three terms from above will be sufficient.)

Expression for elemental conductivity for steady state heat transfer analysis in one-dimensional region is given by:

$$\boxed{h^e_{ij} = \int_{L^e} K \left(\frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} \right) dL}$$

Since, this is for a 2 node bar element, the elemental conductivity matrix will be $[h] = \begin{bmatrix} h^e_{11} & h^e_{12} \\ h^e_{21} & h^e_{22} \end{bmatrix}$

$$\Rightarrow h^e_{11} = \int_{L^e} K \left(\frac{\partial N_1}{\partial x}\right)^2 . dL$$

$$h^e_{12} = \int_{L^e} K \left(\frac{\partial N_1}{\partial x} . \frac{\partial N_2}{\partial x}\right) . dL$$

$$h^e_{21} = \int_{L^e} K \left(\frac{\partial N_2}{\partial x} . \frac{\partial N_1}{\partial x}\right) . dL = h^e_{12}$$

$$h^e_{22} = \int_{L^e} K \left(\left(\frac{\partial N_2}{\partial x}\right)^2\right) . dL$$