Question 4

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a. Blurring of Edges

Let $I(x)$ be a 1D image. Then, the Lagrangian of the image would be the second derivative of the function.

$$\nabla^2 I(x) = \frac{d^2 I(x)}{dx^2} \tag{1}$$

That is, the lagrangian at any point in this image would be rate of change of rate of change of the intensity. In more intuitive sense, it weights locations having abrupt changes in intensity with the start and end of a border. So the lagrangian is a locator for sharp edges. Sharper the edge, more localized and high valued the lagrangian is.

Take an example of a step image. For purposes of differentiation, we will approximate these regions as exponential.

$$I(x) = \frac{1}{1 + e^{-0.25(x-50)}} \tag{2}$$

We extract the following images with this image.

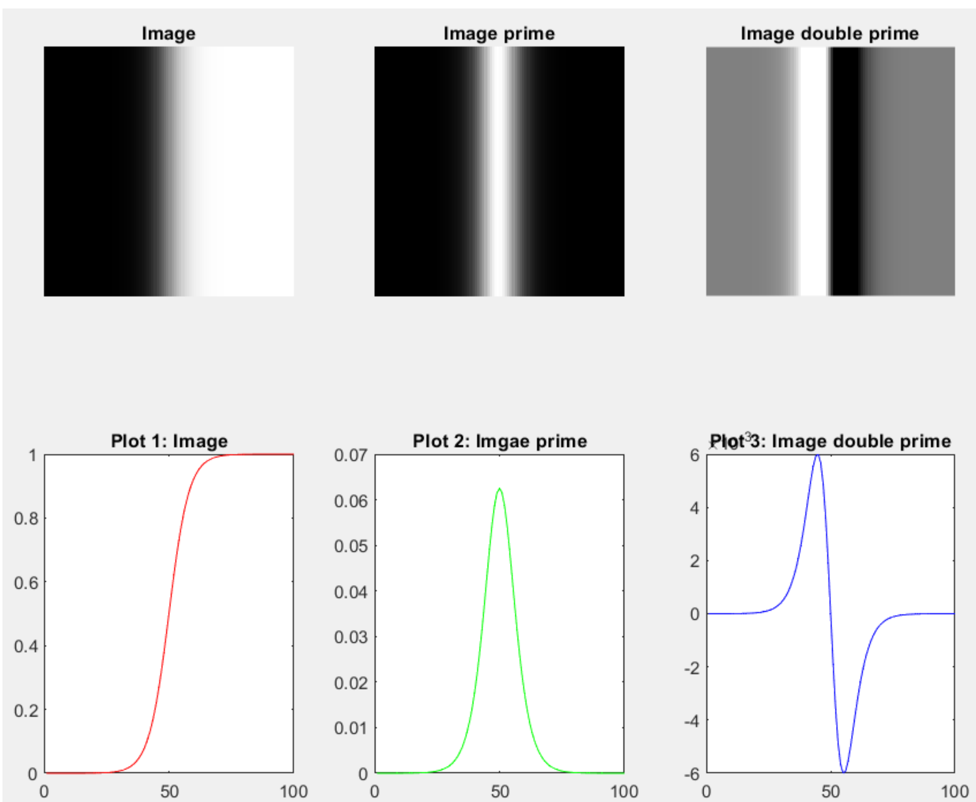

Figure 1: Derivatives of an image normalized

It is clear from the results above that the lagrangian image is whiter near the black side of the edge from the original image. At the same time, the lagrangian is blacker near the whiter side of the edge. Hence upon adding these with a constant, we would expect the black side of the edge to become whiter and the white side of the edge to become darker. This would effectively broaden the border of the original image. There is no effect seen in the parts of the images where the intensity is relatively constant. Overall, this operation has caused the blurring of edges. Additionally, a sharper border has formed midway of the earlier existing border. The effect can be seen below.

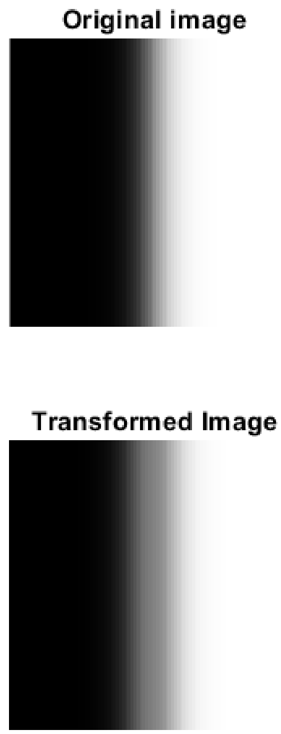

Figure 2: Adding lagrangian of an image to itself with alpha = 25

b. Iterating blurring of edges

Upon iterating this operation, we would expect to see further bluring. Let i denote the iteration number and I_i denote the image after i iterations. Let $I_0 = I$ be the original image. Then

$$I_1 = I_0 + \alpha \nabla^2 I_0 \tag{3a}$$

$$I_2 = I_0 + 2\alpha \nabla^2 I_0 + \alpha^2 \nabla^4 I_0 \tag{3b}$$

$$I_3 = I_0 + 3\alpha \nabla^2 I_0 + 3\alpha^2 \nabla^4 I_0 + \alpha^3 \nabla^6 I_0 \tag{3c}$$

$$I_4 = I_0 + 4\alpha \nabla^2 I_0 + 6\alpha^2 \nabla^4 I_0 + 4\alpha^3 \nabla^6 I_0 + \alpha^4 \nabla^8 I_0 \tag{3d}$$

Generalizing,

$$I_i = \sum_{k=0}^{n} {}^{i}C_k \alpha^k \nabla^{2k} I_0 \tag{4}$$

Let us define an operator κ as,

$$\kappa_i = (1 + \alpha \nabla^2)^i \tag{5}$$

then,

$$I_i = \kappa_i I_0 \tag{6}$$

The following image summarizes the effect of this operator.

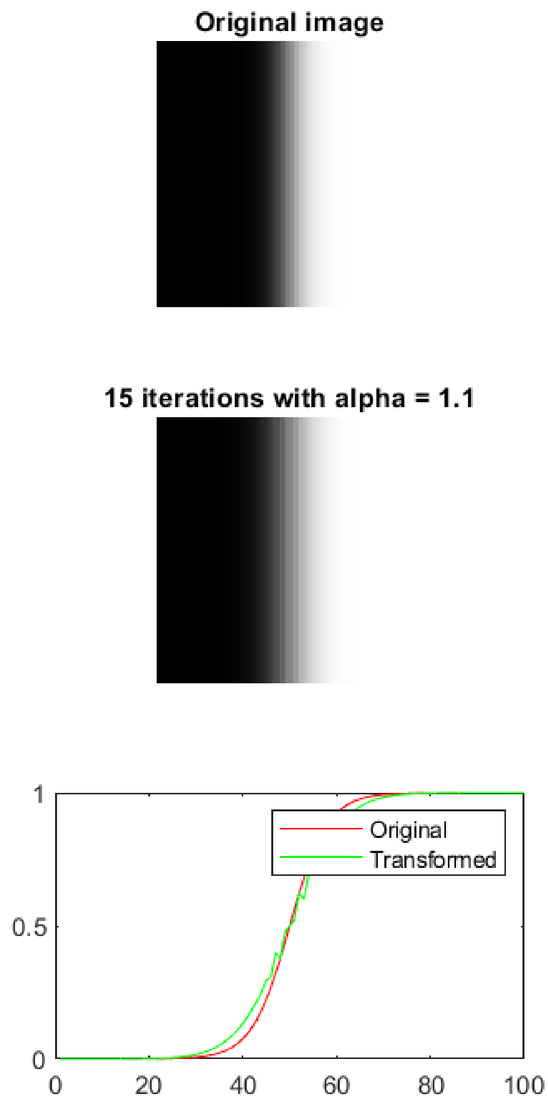

Figure 3: Results of iterating with κ operator

c. Reasoning

It is clearly seen that the operator causes blurring. To understand why this happens, let is look how the image changes in consecutive iterations.

$$I_i - I_{i-1} = (\kappa_i - \kappa_{i-1})I_0 = ((1 + \alpha\nabla^2)^i - (1 + \alpha\nabla^2)^{i-1})I_0 = (1 + \alpha\nabla^2)^{i-1}(1 + \alpha\nabla^2 - 1)I_0 \tag{7}$$

$$I_i - I_{i-1} = \alpha\nabla^2 I_{i-1} \tag{8}$$

Let us assume that these iterations have been developing over time. With infitesimal steps, this becomes

$$\dot{I} = \alpha \nabla^2 I \tag{9}$$

This is similar to the heat equation. The heat equation applied in such a scenario would result is normalizing of the temperature distribution. A similar effect applies on the image and we see the blurring of the edge.

d. Subtracting Lagrangian

Similarly, for the subtraction iteration,

$$\lambda_i = (1 - \alpha \nabla^2)^i \tag{10}$$

and

$$I_i = \lambda_i I_0 \tag{11}$$

Here, we would see the opposite effect. An edge is expected to become sharper.

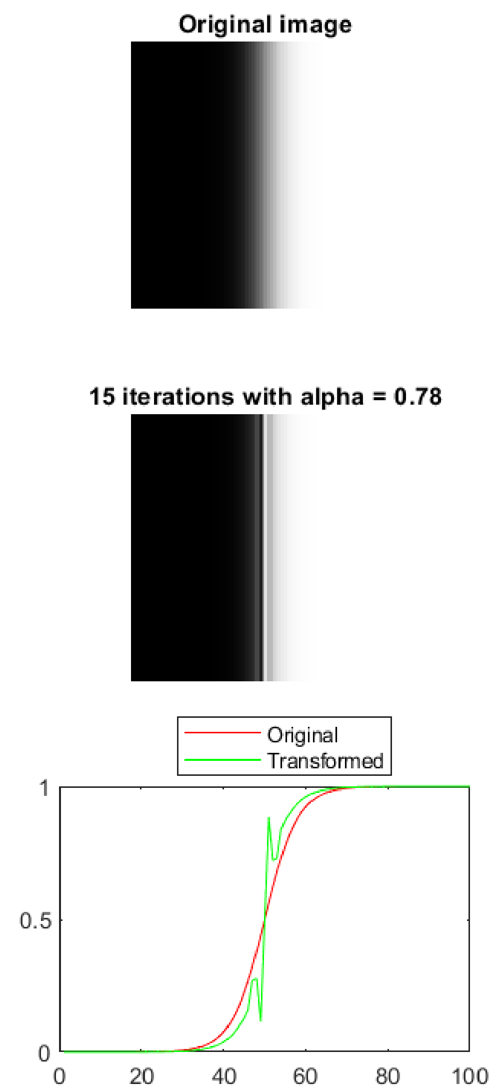

Figure 4: Results of iterating with λ operator