(iv) The cylinder liner was cooled through water circulating around it with the surface heat-transfer coefficient taken as 5000 $W/m^2.K$ and water temperature as 380 K.

(v) Finite element analysis was performed in two stages. During the initial first stage a steady-state analysis was performed using an average heat-transfer coefficient and mean gas temperature. This was necessary in order to establish the initial conditions for start of the iteration process for subsequent unsteady-state heat-transfer analysis. Later, the analysis was performed for different piston positions during the engine cycle while using the instantaneous values of combustion gas temperature and its enhanced heat-transfer coefficient for each such piston positions. The program was iterated through a series of complete engine cycles until the nodal temperatures for each piston position came within a permissible range with respect to the same piston position as in the previous cycle.

(vi) The temperatures so calculated at the surface of the ceramic coating on the piston varied within 700–900 K compared to similar variation for an uncoated piston within the range of 540–580 K.

The basic features of the analysis are the extreme care taken during modelling of surface heat-transfer conditions (boundary conditions) along different regions and the fine grid used at critical locations. Needless to say, the accuracy of such calculations depends greatly on the way boundary conditions and other real effects are simulated.

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Chapter 6

BEAMS, PLATES AND SHELLS

6.1 INTRODUCTION

Beams and plates form important elements of structures, pressure vessels and machines. Analysis of indeterminate structures formed of beams of uniform and non-uniform cross-section subjected to various types of loading including bending, twisting or direct load calls for very special efforts and the finite element method comes in very handy in these situations. Automobile chassis, cranes, underframe of railway bogies and machine beds are a few typical examples from the field of mechanical engineering besides building structures, bridges etc. which can be analyzed using beam elements. All pressure vessels containing gas or liquid under pressure, boilers, submarines, ship decks and machine frames are examples of plate structures. The complex configuration and nature of loading makes the finite element technique the only viable approach for solving such problems. Problems of sheet metal forming of automobile body parts (including its die design) have also been tackled using large-displacement plate bending analysis [1, 2]. Beam and plate problems are similar in that both are loaded in bending and other types of stress may often play a secondary role. This makes the form of deflection expression and subsequent analysis similar in the two cases.

6.2 BENDING OF BEAMS

A beam of non-uniform cross-section loaded in the transverse direction is shown in Fig. 6.1. It is divided into elements, such as 1-2, 2-3 etc. One of the elements (say 2-3) is redrawn in Fig. 6.2 and is denoted as 1-2 for convenience. We shall analyze the stresses due to bending of this element. Before that let us define a few related quantities, such as bending moment, slope, deflection, rotation and the sign conventions for these. Generally the beam, subjected to bending, is assumed to deflect in a plane passing through the neutral axis. If the beam is aligned along the x-axis, it may deflect either in x-z or x-y planes, as shown in Fig. 6.2(i) and (ii) respectively. These Figures also show the three coordinate axes.

Considering Fig. 6.2(i), deflection of the beam in z-direction is designa-

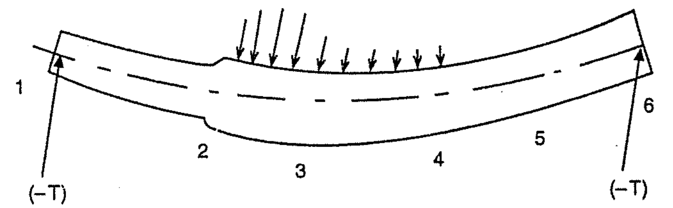


Fig. 6.1 Beam subjected to transverse load

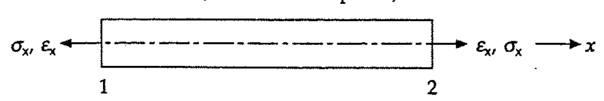
ted by w. The slope along the neutral axis is $(\partial w/\partial x)$. The slope is caused by the application of a bending moment M, the magnitude of which is M_2 at end 2 of the element. Application of bending moment also causes rotation of the transverse plane. The transverse plane at point 2 is rotated by angle θ_2. We now introduce the sign convention for bending moment and rotation. As shown in Fig. 6.2(i), giving a rotation θ_2 to the transverse plane along end 2 of the element is equivalent to rotating it about y-axis in a clockwise direction.

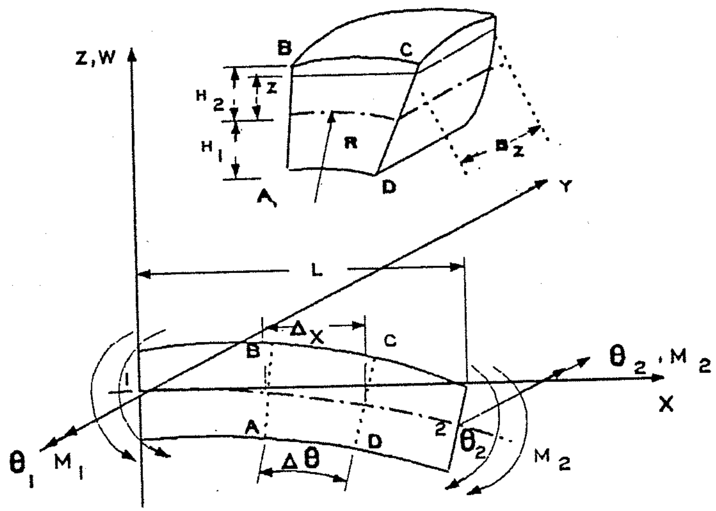
Thus we define the positive direction as the direction of rotation of a right-hand screw which results in its forward motion in the positive direction of y-axis. We also observe that this definition applies to end 2 only where the outward normal to transverse plane is pointed in the positive direction of x-axis. At end 1 where such outward normal to transverse plane is pointed in the 'negative' x-direction, the corresponding positive direction of rotation is also reversed. The rotation can thus be represented as an arrow in the direction of the axis about which rotation takes place. In order to distinguish it from linear motion along the axis, we use a double arrow to indicate rotation, such as shown in Fig. 6.2. The same sign convention applies to the bending moment[1].

Again, considering bending of beam in the x-y plane, as in Fig. 6.2(ii), and using the sign convention stated above, the positive direction of rotation at end 2 of the element is as shown. Note that when the right-hand screw rotates in this direction it moves along $+z$ axis.

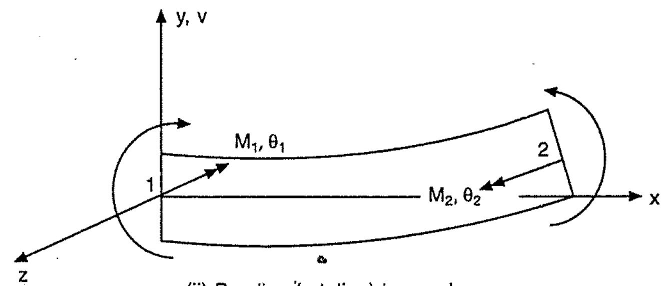
Figure 6.2(iii) shows the positive directions of rotation at ends 1 and 2 of a general beam located along x-axis. This Figure also includes rotation about x-axis. This sign convention will be found very useful while

[1]The negative direction of M_1, θ_1 at end 1 is not difficult to visualize. If we consider a similar situation in which a rod is subjected to tensile force F, the + direction of strain ε_x and stress σ_x are along + x-axis at end 2 while the same acts in negative direction of the axis at end 1. (see the sign convention for stress, strains in Chapter 2).

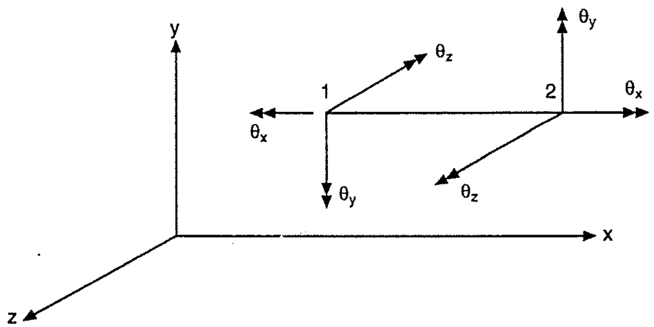




(i) Bending (rotation) in x-z plane



(ii) Bending (rotation) in x-y plane



(iii) Rotation in all three planes

Fig. 6.2 Bending of beam in various planes considering beams in space frames. Also note that we have used the right-hand Cartesian coordinate system throughout our discussion.

If we compare Figs. 6.2(i) and (ii) in which the beam is shown bent in a two-dimensional plane (either x-z or x-y plane) it can be seen that the apparent positive directions of rotation, θ or bending moment M differ in the two cases if we disregard the third axial direction. In one case it is clockwise at end 2 while in the other case it is anticlockwise at the same end[2]. The matter is resolved only if we consider the third axial direction also and use the consistent sign convention stated above.

6.2.1 Analysis of Beam Element

We refer to the beam bent in an x-z plane shown in Fig. 6.2(i). Deflection along the neutral axis in z-direction is denoted by w (+ve in z-direction). Rotation θ at any point along the neutral axis is given by

$$\theta\,(\text{or } \theta_y) = -\frac{dw}{dx} \qquad \text{... (6.1)}$$

Observe the negative sign and note that we follow the sign convention for θ as shown in Fig. 6.2(i). Since θ represents rotation about y-axis we may also represent it by θ_y. Considering a small subelement ABCD of length Δx along its neutral axis, we observe that due to bending its two faces AB and CD rotate by angle $\Delta\theta$ with respect to each other. Thus

$$\Delta\theta_y = \frac{d(\theta_y)}{dx}\cdot\Delta x = -\frac{d^2w}{dx^2}\cdot\Delta x \qquad \text{... (6.1a)}$$

The bending moment M acting at this subelement is derived on the basis of stresses acting on the transverse plane, AB or CD. For this we observe the exploded view of this subelement shown in Fig. 6.2(i). If the neutral plane is bent to a radius r and since the undeformed length of the element (i.e., length along neutral plane) is Δx, we obtain the following relations

$$r\Delta\theta_y = \Delta x \qquad \text{... (6.2)}$$

Strain along a plane distance z from the neutral plane is

[2]This difference has sometimes resulted in somewhat contradictory sign conventions being used by several authors [3, 4, 5] when bending of beam has been analyzed in two-dimensional plane (say x-z plane) without regard to third axis. In these cases different sign conventions have been used when bending in three dimensions (either plate bending or beam bending) is considered and this sign convention usually confirms to the one stated above. In any case the logic and justification of sign convention explained here will become clearer when we consider three-dimensional beams later.

$$\varepsilon_x = \frac{(r+z)\,\Delta\theta_y - r\Delta\theta_y}{r\Delta\theta_y} = \frac{z}{r} \qquad \text{... (6.2b)}$$

Substituting from eqs. (6.2) and (6.1a), we obtain

$$\varepsilon_x = -z\frac{d^2w}{dx^2}$$

The stress along side b_z of this plane will be

$$\sigma_x = -Ez\frac{d^2w}{dx^2} \qquad \text{... (6.3)}$$

The total moment at the subelement is obtained by integrating moment due to stress σ_x (about neutral plane) for the whole cross-section. Thus

$$M_y = \int_{-h_1}^{h_2} \sigma_x z b_z\,dz = -E\frac{d^2w}{dx^2}\int_{-h_1}^{h_2} z^2\cdot(b_z\,dz) = -EI_y\frac{d^2w}{dx^2} \qquad \text{... (6.4)}$$

Here I_y is the area moment of inertia of the cross-section about y-axis. Since the incremental rotation for subelement ABCD ($\Delta\theta_y$) and bending moment (M_y) are linearly related (eqs. 6.1a and 6.4), the strain energy of bending will be $(M_y\cdot\Delta\theta_y)/2$ and for the whole element

$$\Pi_s^e = \frac{1}{2}\int_L \left(\frac{d^2w}{dx^2}\right)\left(EI_y\frac{d^2w}{dx^2}\right)dx \qquad \text{... (6.5)}$$

Summation of strain energy for all the elements and the work done against external force will give the potential energy of the system, which can be minimized over the entire region in order to get the solution. However, before proceeding further it would be useful at this point to discuss the nature of the expression for displacement w.

6.2.2 Interelement Continuity of Displacement and Slope – C_1 Continuity

In the analysis of systems with linear elements, where displacement varies linearly within the element, the interelement continuity of displacement is ensured due to the uniqueness of nodal displacements. This situation is represented in Fig. 6.3(a) where the linear variation of displacement w is assumed in the line elements A and B. The first derivative of displacement (dw/dx) is discontinuous at node 2 but has finite values within the elements. In contrast, the second derivative (d^2w/dx^2) tends to infinity at node 2 due to a sudden change in the value of dw/dx at this node. Its value thus becomes indeterminate within the element. The expression for strain energy, while analyzing stresses in solids, contains

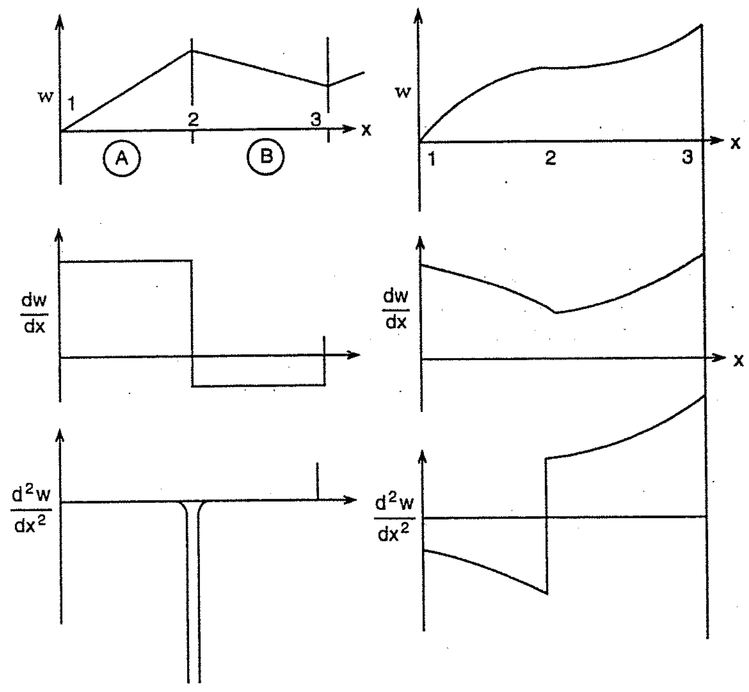


Fig. 6.3 Variation of first and second slopes for (a) linear and (b) non-linear variation in displacement w.

strain terms, $\{\varepsilon_e\}^T$ (or ε_x, ε_y, γ_{xy} in 2D case) (eq. 2.34, Chapter 2). These strains can be written in terms of first derivatives of u, v, i.e. $\partial u/\partial x$, $\partial v/\partial y$ etc. Hence, the expression for strain energy in this case contains only first derivatives of displacements. These derivatives have finite values for linear variation of displacement (Fig. 6.3(a)). The strain energy will thus have finite value within the elements and can be evaluated. The situation is similar for a 3D case also. However, strain energy terms during analysis of bending (eq. 6.5) contain second derivatives which tend to infinity at interelement boundaries and cannot be evaluated.

Considering a somewhat different type of distribution of w in the two elements shown in Fig. 6.3(b), we find that this distribution has interelement continuity even for dw/dx. Thus d^2w/dx^2, although being discontinuous at node 2, has finite values within the elements and the expressions containing d^2w/dx^2 (such as eq. 6.5) can now be evaluated within the elements. The conditions in 2D triangular elements are similar and these are represented in Fig. 6.4 for a section through two adjoining elements A and B.

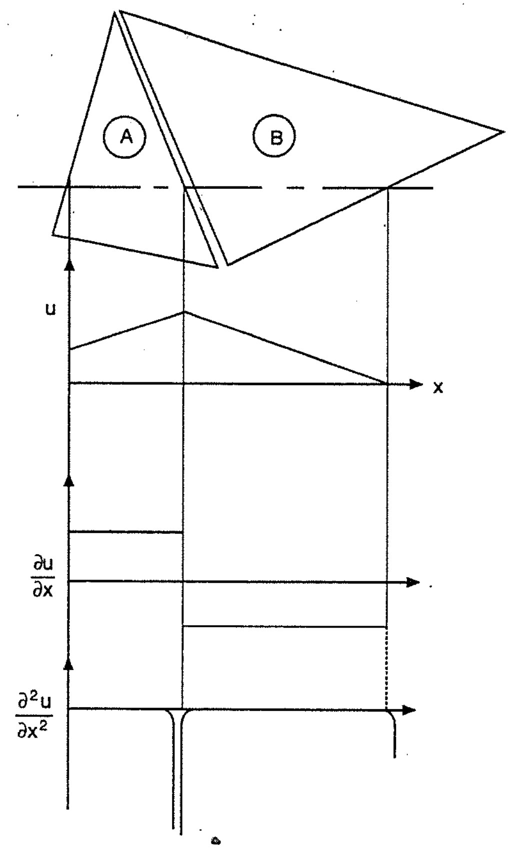


Fig. 6.4 Effect of discontinuity in first slope for triangular elements

A general conclusion can now be drawn that if the terms of potential energy (or functional) contain n-th order derivatives (of displacement), then (n-1)th derivative should be continuous at the interelement boundaries. It is stated that displacement expressions for such elements should have C_{n-1} continuity. Thus for bending problems we should have C_1 continuity while for stress analysis in solids or for conductive heat-flow analysis we may have C_0 continuity.

6.2.3 Displacement Function

The expression for displacement w within the element should now satisfy C_1 continuity. Since the beam element (Fig. 6.2) is a line element, displacement is a function of x only. A linear expression will thus be

$$w = \beta_1 + \beta_2 x$$

It is obvious that the first derivative (dw/dx) will be a constant within an element. Hence its value in two adjoining elements will not be the same, in general. Thus interelement continuity of slope dw/dx cannot be ensured. In order that the slope variation within the element be such that continuity of slope is ensured between elements (Fig. 6.3b), the expression for w should have x^2 or higher order terms. One way of ensuring the continuity of this type can be to specify both displacement w and rotation θ_y ($-dw/dx$) at the nodes. This means that for a two-noded line element (such as 1-2 of Fig. 6.2) there will exist four nodal variables, w_1, θ_{y1}, w_2 and θ_{y2}. In order to determine these four uniquely the expression for w should have four constants. Thus, a possible expression, consisting of lowest order terms, would be

$$w = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3 \qquad \text{... (6.6)}$$

Also

$$\theta_y = -\beta_2 - 2x\beta_3 - 3x^2\beta_4 \left(= -\frac{\partial w}{\partial x} \right) \text{ from eq. (6.1)}$$

In non-linear expressions, such as (6.6), it is convenient to consider the local coordinate system with reference to the element itself. Thus one end of the line element can be taken as origin with x-direction laid along the length of element. The x coordinates of the two nodes 1 and 2 will be $x = 0$ and $x = L$ respectively (Fig. 6.2). Substituting these in eq. (6.6) and on rearranging, we get

$$\begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & -1 & -2L & -3L^2 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{Bmatrix} \qquad \text{... (6.7)}$$

The inversion of this matrix equation gives us

$$\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{3}{L^2} & \frac{2}{L} & \frac{3}{L^2} & \frac{1}{L} \\ \frac{2}{L^3} & -\frac{1}{L^2} & -\frac{2}{L^3} & -\frac{1}{L^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix} \qquad \text{... (6.8)}$$

Substituting β_1, β_2 etc. from (6.8) into (6.6) yields

$$w = \left\{1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right\} w_1 - \left\{x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right\} \theta_{y1} + $$
$$+ \left\{\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right\} w_2 - \left\{-\frac{x^2}{L} + \frac{x^3}{L^2}\right\} \theta_{y2} \qquad \text{... (6.9)}$$

This can be written in the matrix form as

$$w = \Big[[N_1], [N_2] \Big] \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix} \qquad \text{... (6.10)}$$

where $\{d_1\}$ and $\{d_2\}$ are nodal displacement vectors, as usual, denoted by the following expressions, and $[N_1]$, $[N_2]$ are row vectors(3) of order (1×2) derived from eq. (6.9) and stated below.

$$\{d_1\} = \begin{Bmatrix} w_1 \\ \theta_{y1} \end{Bmatrix}; \{d_2\} = \begin{Bmatrix} w_2 \\ \theta_{y2} \end{Bmatrix}; \{d^e\} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix}$$

$$[N_1] = \left[\left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right), -\left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right) \right]$$

$$[N_2] = \left[\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right), \left(\frac{x^2}{L} - \frac{x^3}{L^2}\right) \right],$$

and

$$[N] = \Big[[N_1], [N_2] \Big].$$

Using these notations displacement w can be written in terms of shape function $[N]$ as

$$w = [N] \{d^e\} \qquad \text{... (6.10a)}$$

6.2.4 Strain Energy of Deformation

If we look at the expression for bending moment in the element (eq. 6.4) and the strain energy (eq. 6.5) in it, the similarities between these and corresponding strain and strain energy expression for a 2D solid element (Sec. 2.2.3) will be obvious. If d^2w/dx^2 and $EI_y d^2w/dx^2$ are considered equivalent to ε and $D \cdot \varepsilon$, an expression similar to $\{\varepsilon\} = [B] \{d^e\}$ (eq. 2.18) can be written using d^2w/dx^2. Thus substituting for w from eq. (6.9) or (6.10), we obtain

$$\frac{d^2w}{dx^2} = \left[\frac{d^2[N_1]}{dx^2}, \frac{d^2[N_2]}{dx^2} \right] \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix} = \Big[[B_1], [B_2] \Big] \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix} \qquad \text{... (6.11)}$$

---

[3] A row vector is considered a matrix of order ($1 \times n$) and is denoted by brackets[] [6].

such that

$$[B_1] = \left[-\left(\frac{6}{L^2} - \frac{12x}{L^3}\right), \left(\frac{4}{L} - \frac{6x}{L^2}\right) \right]$$

$$[B_2] = \left[\left(\frac{6}{L^2} - \frac{12x}{L^3}\right), \left(\frac{2}{L} - \frac{6x}{L^2}\right) \right] \quad \dots (6.12)$$

and $[B] = [[B_1], [B_2]]$

Also considering $[D]$ as (1×1) matrix comprising term EI_y only, the expression for strain energy for the element (eq. 6.5) is written as

$$\Pi_s^e = \frac{1}{2} \int_L \{d^e\}^T [B]^T [D] [B] \{d^e\} \, dx^{(4)} \quad \dots (6.13)$$

Proceeding in a manner similar to the analysis conducted in Chapter 2 (eq. 2.36), the expression for elemental strain energy will be

$$\Pi_s^e = \frac{1}{2} \{d^e\}^T [K^e] \{d^e\} \quad \dots (6.14)$$

where

$$[K^e] = \int_L [B]^T [D] [B] \, dx \quad \dots (6.15)$$

Since we are not considering the expansion (displacement) along the length of the element the strains arising out of thermal heating or similar causes cannot be incorporated in this analysis.

6.2.5 Potential Energy Due to External Loads

External loads may appear in the form of distributed loads, P (Fig. 6.1) or may be assumed concentrated at nodes (say T). Loads may also appear in the form of external moments acting at the nodes, Q. These are shown again in Fig. 6.5 with the values at nodes 1 and 2 being T_1 and Q_2 and the positive direction of these are represented therein. The work done by the distributed load is given below (note that load P is taken positive in +z direction)

[4]Although $d^2w/dx^2 = [B]\{d^e\}$ from eq. (6.11) it can also be expressed as $d^2w/dx^2 = [[B]\{d^e\}]^T = [\{d^e\}^T [B]^T]$, since the left-hand side is a single quantity or (1×1) matrix and its transpose will be equal to itself. These two different expressions are used for d^2w/dx^2 in eq. (6.13) with a view to simplifying further analysis in order to utilize the similarity with 2D analysis given in Chapter 2.

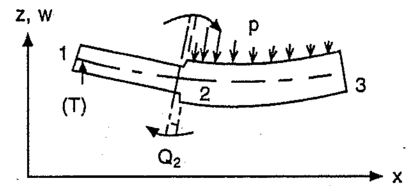


Fig. 6.5

$$\text{work done} = \int_L P \cdot w \cdot dx = \int_L \{d^e\}^T [N]^T P \, dx \quad \dots (6.16)$$

$$= \{d^e\}^T \{f_p^e\} \quad \dots (6.17)$$

The work done by external concentrated loads and moments will simply be $T_1 \cdot w_1$, $Q_1 \cdot \theta_{y1}$ (if present) $T_2 \cdot w_2$, $Q_2 \cdot \theta_{y2}$ etc. and on combining these in vector form the work done can be shown as

$$\text{work done} = \{\delta\}^T \{T'\}, \quad \dots (6.18)$$

where $\{T'\}$ is the overall combined load vector consisting of T and Q as its components. Similarly, $\{\delta\}$ is overall displacement vector with $\{d^1\}$ etc. as its components. The total external work will be

$$\Pi_w^e = \sum_{e=1}^{n} \{d^e\}^T \{f_p^e\} + \{\delta\}^T \{T'\} \quad \dots (6.19)$$

6.2.6 Stiffness Relation

The total potential energy being composed of elemental strain energies and external work done, it can be expressed by using eqs. (6.14) and (6.19) as

$$\Pi = \sum_{e=1}^{n} \left[\frac{1}{2} \{d^e\}^T [K^e] \{d^e\} - \{d^e\}^T \{f_p^e\} \right] - \{\delta\}^T \{T'\} \quad \dots (6.20)$$

Minimization of potential energy over the whole region, carried out in the manner explained in Sec. 2.2.3, results in the following final expression

$$[K]\{\delta\} - \{f_p'\} - \{T'\} = 0$$

or

$$[K]\{\delta\} - \{f\} = 0, \quad \dots (6.21)$$

where the various terms can be rewritten as follows.

$$[K] = \sum_{e=1}^{n} [K^e] = \sum_{e=1}^{n} \int_L [B]^T [D] [B] \, dx$$

$$\{f_p'\} = \sum_{e=1}^{n} \{f_p^e\} = \sum_{e=1}^{n} \int_L [N]^T P\, dx$$

and

$$\{T'\} = \begin{Bmatrix} T_1 \\ Q_1 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

where $\{T'\}$ is the overall load vector comprising direct load T and moment Q as its components.

(a) Elemental stiffness matrix: The expression for the elemental stiffness matrix is easily obtained by using eqs. (6.12) and (6.14). The term by term multiplication and subsequent integration over the length of element L gives the following:

$$K_{11}^e = \int_0^L \left(\frac{6}{L^2} - \frac{12x}{L^3}\right)\left(\frac{6}{L^2} - \frac{12x}{L^3}\right) EI_y\, dx = \frac{12EI_y}{L^3}$$

$$K_{12}^e = \int_0^L -\left(\frac{6}{L^2} - \frac{12x}{L^3}\right)\left(\frac{4}{L} - \frac{6x}{L^2}\right) EI_y\, dx = -\frac{6EI_y}{L^2}$$

etc.

The final stiffness matrix is

$$[K^e] = \frac{EI_y}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ & 4L^2 & 6L & 2L^2 \\ & \text{Symm} & 12 & 6L \\ & & & 4L^2 \end{bmatrix}; \text{also } \{d^e\} = \begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix} \quad \ldots (6.22)$$

This stiffness matrix has been derived for constant cross-sectional area of the element over its length L.

(b) Stress in the element: The stress in the element is given by expression (6.3) and its magnitude will be maximum at the two extremities of the section, i.e., at $z = -h_1$ or $z = h_2$. Thus the two extreme values of σ_x are given as

$$\sigma_{x1} = Eh_1\left(\frac{d^2w}{dx^2}\right) = Eh_1\,[B]\,\{d^e\} \text{ and}$$

$$\sigma_{x2} = -Eh_2\,[B]\,\{d^e\} \quad \ldots (6.22a)$$

Knowing from expression (6.12) that terms of $[B]$ vary linearly with x, the maximum value of σ_{x1} or σ_{x2} will exist at any of the nodes. These nodal values can thus be calculated and the maximum stress in the element or its distribution can be obtained.

Exercise 6.1: Considering the bending of a beam in x-y plane (Fig. 6.2(ii)) and using the sign convention shown in this Figure and also using a third-order expression similar to eq. (6.6) to represent displacement v, derive the expressions for shape function $[N]$, $[B]$ matrix and the stiffness matrix $[K^e]$. Show that these are given by

$$[N] = \Big[[N_1], [N_2]\Big]; [B] = \Big[[B_1], [B_2]\Big]$$

where

$$[N_1] = \left[\left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right), \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)\right]$$

$$[N_2] = \left[\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right), -\left(\frac{x^2}{L} - \frac{x^3}{L^2}\right)\right] \quad \ldots (a1)$$

and

$$[B_1] = \left[-\left(\frac{6}{L^2} - \frac{12x}{L^3}\right), -\left(\frac{4}{L} - \frac{6x}{L^2}\right)\right]$$

$$[B_2] = \left[\left(\frac{6}{L^2} - \frac{12x}{L^3}\right), -\left(\frac{2}{L} - \frac{6x}{L^2}\right)\right] \quad \ldots (b1)$$

$$[K^e] = \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & \text{Symm} & 12 & -6L \\ & & & 4L^2 \end{bmatrix}; \text{also } \{d^e\} = \begin{Bmatrix} y_1 \\ \theta_{z1} \\ y_2 \\ \theta_{z2} \end{Bmatrix} \ldots (c1)$$

Hint:

$$\theta_z = \frac{dv}{dx}; \Delta\theta_z = \frac{d^2v}{dx^2}\Delta x;$$

$$\varepsilon_x = -y\frac{d^2v}{dx^2}; M_z = -\int_{-t_1}^{t_2} \sigma_x \cdot y \cdot b_y\, dy$$

Exercise 6.2: Considering that a rod element, discussed in Chapter 1 (Fig. 1.5) is oriented along x-axis and is also subjected to twisting moment about the axis in addition to direct axial load, write the expression for stiffness matrix in terms of the only possible nodal displacements, in this case u_1, θ_{x1}, u_2, θ_{x2}. You may use potential energy minimization approach in order to write the expression for strain energy in torsion of the rod subjected to twist $(\theta_{x2} - \theta_{x1})$.

Hint: Taking $\alpha = 0$, $[K^e]$ and $\{d^e\}$ in eq. (1.34) become

$$[K^e] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } \{d^e\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \dots \text{(a2)}$$

In torsion,

$$\text{Torque } (T) = \frac{GJ\theta}{L},$$

where G = shear modulus; J = polar moment of inertia; L = length of element and torsional strain energy

$$\Pi_{S^e} = \frac{1}{2} T\theta = \frac{1}{2}\frac{GJ}{L}(\theta_{x2} - \theta_{x1})^2$$

In order to minimize torsional strain energy, we differentiate Π_{S^e} with respect to θ_{x1} and θ_{x2}. This will give the contributions from this element in the partial differential of total potential energy (see eqs. 1.33, 1.34).

i.e.,

$$\begin{Bmatrix} \dfrac{\partial \Pi_{S^e}}{\partial \theta_{x1}} \\ \dfrac{\partial \Pi_{S^e}}{\partial \theta_{x2}} \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} = [K^e]\{d^e\} \qquad \dots \text{(b2)}$$

Here J can also be written as I_x, i.e., area moment of inertia about x-axis. Equations (a2) and (b2) above are assembled to obtain the stiffness matrix for combined axial load and torsion.

Exercise 6.3: If a beam element oriented along x-axis, as shown in Fig. 6.6, is subjected to combined displacements u, v, w and rotations θ_x, θ_y, θ_z along the three axes such that the nodal displacements-cum-rotation vectors now are

$$\{d_1\}^T = [u_1\ v_1\ w_1\ \ \theta_{x1}\ \theta_{y1}\ \theta_{z1}]$$
$$\{d_2\}^T = [u_2\ v_2\ w_2\ \theta_{x2}\ \theta_{y2}\ \ \theta_{z2}] \qquad \dots \text{(a3)}$$

show that the stiffness matrix under combined bending, torsional and axial loading is given by

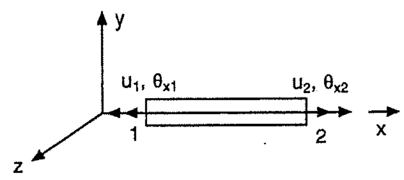


Fig. 6.6 Exercise 6.3

$$[K^e] = \begin{bmatrix} [K^e_{11}] & [K^e_{12}] \\ [K^e_{21}] & [K^e_{22}] \end{bmatrix} \qquad \dots \text{(b3)}$$

for elemental displacement vector

$$\{d^e\} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix}$$

Here these submatrices are 6×6 matrices given by

$$[K^e_{11}] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix}$$

$$[K^e_{12}] = [K^e_{21}]^T = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{-GI_x}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix}$$

$$[K_{22}^e] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix}$$

Hint: If the combined stress in the element does not exceed the yield point then stresses due to all the types of loadings can be superimposed. Under these conditions the combined elemental stiffness matrix can be obtained by assembling the stiffness matrices due to bending in x-z, x-y planes (eqs. 6.22 and c1) and those due to axial loading and torsion (eqs. a2 and b2). This will give the 12 × 12 matrix $[K^e]$ stated in eq. (b3).

6.2.7 Beam Element with General Orientation in 3D Space

A beam in space frame or general structure will normally be oriented at some angle with respect to the three axes of the system (Fig. 6.7). The coordinate system of the overall structure is called the global coordinate system. The stiffness matrix for the beam can be obtained easily in a Cartesian coordinate system with its x-axis located along the axis of the beam (see Exer. 6.3). This coordinate system is referred to as the local coordinate system. As explained later in Sec. 7.11.4 the stiffness matrix written in the local coordinate system can be transformed easily into the global coordinate system by using transformation matrix, $[T]$. Thus representing global elemental stiffness matrix as $\{K_g^e\}$, we use the following relation to obtain it.

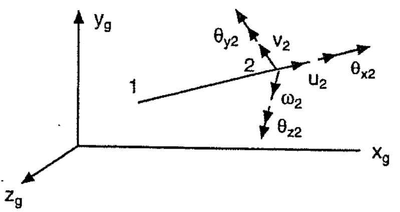


Fig. 6.7 Local displacement and rotation at node 2 of element 1-2 in the global coordinate system

$$[K_g^e] = [T]^T\,[K^e]\,[T] \qquad \ldots (6.23)$$

If the global coordinate system is also a Cartesian coordinate system, the transformation matrix is easily obtained in terms of direction cosines of the local axes with respect to the global coordinate system (eqs. 7.85 and 7.88).

6.3 BENDING OF PLATES

In most applications the plates are not thick and subject to bending only. Analysis proceeds along the same lines as presented for beams. The simple theory of bending of plates considers the plate subjected to bending moments in both x and y directions. The external loads are generally (i) pressure perpendicular to the plane of plate and occasionally, (ii) bending moments and external loads concentrated at a few points. An example of such a bending moment is the load-carrying bracket mounted on the wall of a pressure vessel subjected to internal pressure, shown in Fig. 6.8.

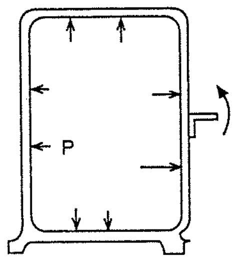


Fig. 6.8 Vessel subjected to internal pressure and bending moment.

6.3.1 Theory of Plate Bending

The simple plate bending theory is based on the assumption that the middle plane of the plate, called the neutral plane, remains unstretched while the fibres along one side of the neutral plane get stretched and those on the opposite side get compressed during the bending of the plate. The magnitude of strain varies linearly along the cross-section of the plate from maximum tensile to maximum compressive value. The effect of stresses, if any, in the direction of plate thickness is neglected. From the point of view of finite elements formulation we are more interested in knowing the expression for potential energy of deformation. In this section we shall derive the expression for potential energy in an element of thin plate. We shall use the same sign conventions which were used to represent the rotation and bending moment in the case of a beam (Sec. 6.2). A further explanation of these sign conventions is provided here to clarify their application in the context of plate bending.

(a) Sign convention: During initial development of the theory of plate bending, only horizontal plates were analyzed and such concepts as 'concave upwards' or 'convex upwards' were introduced to represent the '+' and '–' curvature and corresponding rotations of the transverse section of the plate[7]. These definitions fail to satisfy the requirements when plates are initially curved or various portions of plate structure are located in horizontal, vertical or inclined planes. For such a structure the coordinate system located in the plane of the plate also changes with change in orientation. Since a nodal point is associated with more than one element, it is essential that a change in orientation of local coordinate plane from one element to another not result in change in total magnitude or the sign of parameters, such as bending moment and rotation at the nodes. It is well known that a vector is invariant in any coordinate system although its individual components along the three orthogonal axes may vary with a change in coordinate system. Thus we shall try to seek a vector representation for bending moment (M) and plate rotation (θ) such that these quantities can readily be transferred from one coordinate system to other, using the standard rules for coordinate transformation for vectors.

The sign convention for rotation and bending moment were introduced in Sec. 6.2 (Fig. 6.2) and the general convention for representing these quantities by double arrow was also explained. These quantities are shown for a plate element in Fig. 6.9. A right-hand screw rotated in the positive direction of M_x or θ_x should produce a forward motion of the screw in x direction. The symbolic double arrow representation of these quantities in vector form is also given. In fact, moment or rotation about z-axis can also be represented in the same manner. If M_x, M_y and M_z are considered as three components of a vector, the length and direction of that vector should represent the overall bending moment acting at a point of the plate (say P). A more rigorous proof that moment and rotation are actually

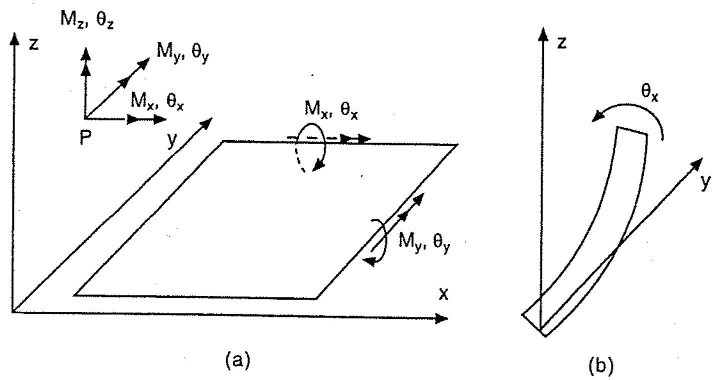


Fig. 6.9 Sign convention for bending moment and rotation.

vectors and follow the parallelogram law of addition is given in Appendix-3. We can designate as M the vector which has M_x, M_y and M_z as its components. It is now easy to see that if M is known, its components can be determined in any Cartesian coordinate system $x' - y' - z'$. The same concept holds good for rotation θ with its components θ_x, θ_y and θ_z.

θ_x and θ_y can also be related to the slopes of a bent plate along appropriate axes. The cross-section of a plate rotated by angle θ_x is shown in Fig. 6.9(b). Figure 6.2(i) similarly shows the cross-section of a plate rotated along y-axis by an angle (say θ_y). It is easy to see that θ_x and θ_y are now represented as rotation of the curved plate given as

$$\theta_x = \frac{\partial w}{\partial y}$$

$$\theta_y = -\frac{\partial w}{\partial x} \qquad \text{... (6.24)}$$

(b) Rotation and twist in plate bending: A plate subjected to bending deforms in several ways. It not only bends about x and y axes, it is also subject to twisting. An element of deformed plate is shown in Fig. 6.10. The twist of the faces BC and CD with respect to the opposite faces of element AD and AB are shown as $\Delta\phi_x$ and $\Delta\phi_y$. These twists are actually the consequence of shearing stresses τ_{xy} and τ_{yx} acting on these faces[5]. These stresses are shown in Fig. 6.11(a). The twisting moments M_{xy} acting

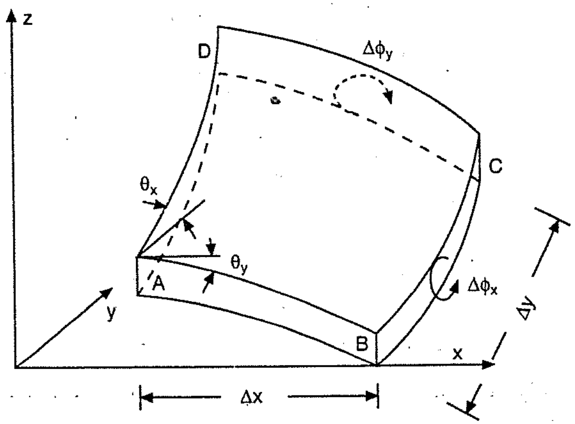


Fig. 6.10 A plate element.

[5]Note that these stresses vary from top to bottom of faces, such as BC, CD, thus resulting in twisting moment along these faces.

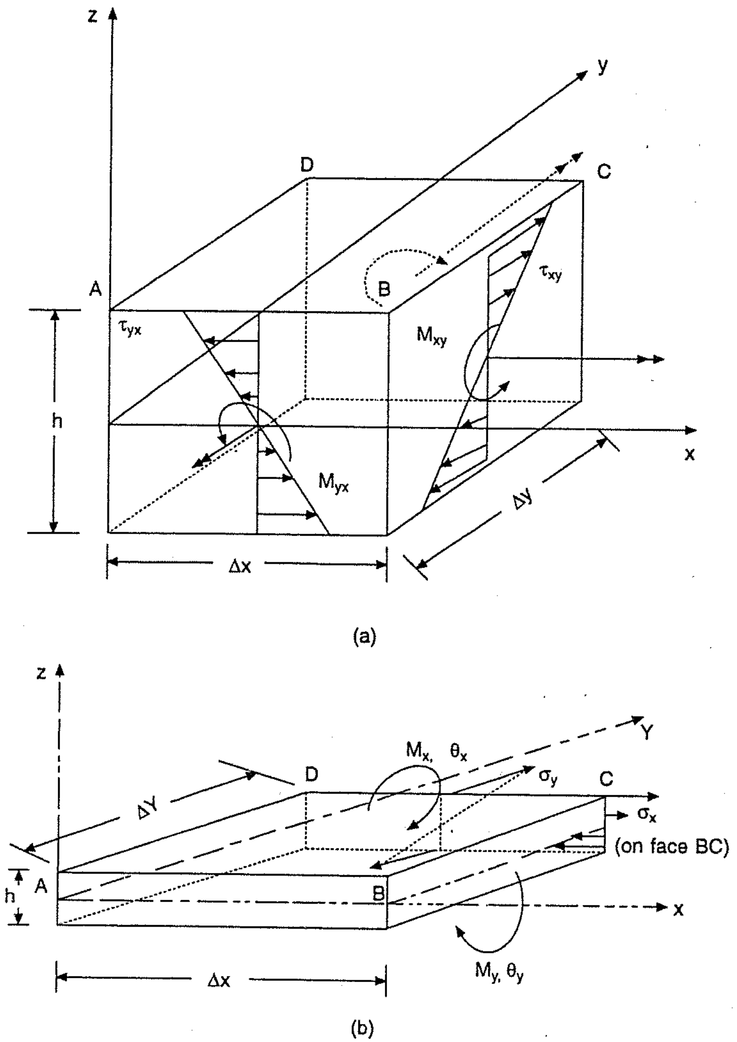


Fig. 6.11 Loading of a plate element

on various faces are also shown. The sign convention again follows the rule stated earlier.

The slopes θ_x and θ_y at point A are also shown in Fig. 6.10. The twist $\Delta\phi_y$ can now be visualized as change of rotation θ_y along edge AD (or over the distance Δy). Similarly, twist $\Delta\phi_x$ can be represented as change of rotation θ_x over distance Δx. Thus

$$\Delta\phi_x = \frac{\partial}{\partial x}(\theta_x)\cdot\Delta x = \frac{\partial^2 w}{\partial x\,\partial y}\cdot\Delta x$$

and
$$\Delta\phi_y = \frac{\partial}{\partial y}(\theta_y)\cdot\Delta y = -\frac{\partial^2 w}{\partial x\,\partial y}\cdot\Delta y \qquad \dots (6.25)$$

At this stage we consider the deformation of the plate element as a whole. Fig. 6.11 shows the moments acting on the plate element ABCD. It is subjected to bending moments M_x, M_y and twisting moments M_{xy}, M_{yx}. Resulting from the action of these moments will be change of twist $\Delta\phi_x$ and $\Delta\phi_y$, as stated in eq. (6.25), and change in rotation $\Delta\theta_y$ and $\Delta\theta_x$ along edges AB and BC respectively. These rotations are also given by (also see eq. 6.1a)

$$\Delta\theta_y = \frac{\partial}{\partial x}(\theta_y)\cdot\Delta x = -\frac{\partial^2 w}{\partial x^2}\cdot\Delta x$$

and
$$\Delta\theta_x = \frac{\partial}{\partial y}(\theta_x)\cdot\Delta y = \frac{\partial^2 w}{\partial y^2}\cdot\Delta y \qquad \dots (6.26)$$

Thus the net strain energy of deformation is given as

$$\text{Strain energy} = \frac{1}{2}M_x\Delta\theta_x + \frac{1}{2}M_y\Delta\theta_y + \frac{1}{2}M_{xy}\Delta\phi_x + \frac{1}{2}M_{yx}\Delta\phi_y \qquad \dots (6.27)$$

(c) Strain, stress and bending moment: Bending and twisting moments as well as the normal and shear stresses along the various faces of the plate element are shown in Figs. 6.11(a) and (b). The two Figures show the same plate element of size $\Delta x\cdot\Delta y$. These are drawn separately only to improve the clarity of presentation. It is assumed in simple plate bending theory that normal and shear stresses vary along the thickness of the plate, being maximum at one face to zero in the middle plane to maximum in the opposite direction at the other face. The various stresses are represented in the Figure in the respective positive direction along the edges of top face, ABCD. From this we can easily deduce the expressions for various bending and twisting moments. Thus

$$M_{yx} = \int_{-h/2}^{h/2} \tau_{yx}\cdot z\cdot\Delta x\cdot dz$$

$$M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy}\cdot z\cdot\Delta y\cdot dz$$

$$M_x = -\int_{-h/2}^{h/2} \sigma_y \cdot z \cdot \Delta x \cdot dz$$

$$M_y = \int_{-h/2}^{h/2} \sigma_x \cdot z \cdot \Delta y \cdot dz \qquad \ldots (6.28)$$

We now proceed to determine the magnitudes of strains and stresses within the plate element. Fig. 6.12(a) shows the cross-section of the plate through the x-z plane. An element AB along the mid-section of the plate (neutral plane) remains free of any deformation or stresses. A rectangular element located there is shown in the x-y plane as ABCD in Fig. 6.12(b). Another plane at distance z from neutral plane deforms to a shape A′B′C′D′. If we can write the expressions for u and v, being displacement in x and y directions at point A, then we could use the strain-displacement relation developed in Chapter 2 to write the expressions for strains ε_x, ε_y, γ_{xy}. Since rotation at point A is θ_y, the displacement u of point A′ with respect to A is given as $z \cdot \theta_y$. Similarly, consideration of plate bending in y direction, shown in Fig. 6.9(b), gives the magnitude of v as $-z \cdot \theta_x$. Thus

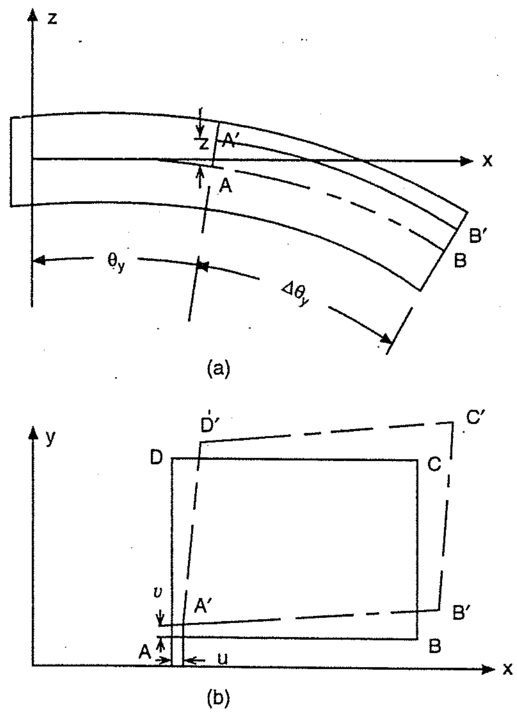


Fig. 6.12 Displacement in a plate element.

$$u = z\theta_y$$

$$v = -z\theta_x \qquad \ldots (6.29)$$

The expressions for various strain components are obtained as

$$\varepsilon_x = \frac{\partial u}{\partial x} = z\frac{\partial(\theta_y)}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial(\theta_x)}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \qquad \ldots (6.30)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = z\frac{\partial(\theta_y)}{\partial y} - z\frac{\partial(\theta_x)}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

The stress-strain relations are given as

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \qquad \ldots (6.31)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy}$$

On substituting (6.30) into (6.31) and after simplification, we obtain

$$\sigma_x = -\frac{E}{1-\nu^2} \cdot z\left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\}$$

$$\sigma_y = -\frac{E}{1-\nu^2} \cdot z\left\{\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right\} \qquad \ldots (6.32)$$

$$\tau_{xy} = \tau_{yx} = -\frac{E}{1+\nu} \cdot z\frac{\partial^2 w}{\partial x \partial y}$$

Again, after substituting the values of stresses from eq. (6.32) into the expression for bending and twisting moments (eq. 6.28) we get

$$M_x = D\left[\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]\Delta x$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]\Delta y$$

$$M_{xy} = D(1-\nu)\left[\frac{\partial^2 w}{\partial x \partial y}\right]\Delta y$$

$$M_{yx} = -D(1-\nu)\left[\frac{\partial^2 w}{\partial x \partial y}\right]\Delta x \quad \ldots (6.33)$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

(d) Strain energy: The expression for total strain energy can be obtained by substituting values of moments from eq. (6.33) and incremental rotations and twists from eqs. (6.26) and (6.25) into eq. (6.27). Thus

$$\text{strain energy} = \frac{D}{2}\frac{\partial^2 w}{\partial y^2}\left[\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]\Delta x \cdot \Delta y + \frac{D}{2}\frac{\partial^2 w}{\partial x^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]\Delta x \cdot \Delta y + D(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \Delta x \cdot \Delta y \quad \ldots (6.34)$$

The strain energy for small volume ($\Delta x \cdot \Delta y \cdot h$) having thickness h is now known. It can be integrated over the area of element and then on summing over the whole domain we shall obtain the total strain energy. Thus

$$\Pi_{S^e} = \int_{A^e} \frac{1}{2}\left[\frac{\partial^2 w}{\partial x^2}\cdot D \cdot \left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\} + \frac{\partial^2 w}{\partial y^2}\cdot D \cdot \times \left\{\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right\} + 2\frac{\partial^2 w}{\partial x \partial y}\cdot D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\right] dA \quad \ldots (6.35)$$

This expression has a peculiar feature which makes it quite similar to corresponding strain energy expression for a 2D plane stress case. If we define two hypothetical vectors $\{\varepsilon\}$ and $\{M\}$ and a matrix $[D_b]$ as[6]

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}; \{M\} = \begin{Bmatrix} D\left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\} \\ D\left\{\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right\} \\ D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$[D_b] = D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \quad \ldots (6.36)$$

we observe that

$$\Pi_{S^e} = \int_{A^e} \frac{1}{2}\{\varepsilon\}^T\{M\}\,dA = \int_{A^e} \frac{1}{2}\{\varepsilon\}^T\{D_b\}\{\varepsilon\}\,dA \quad \ldots (6.37)$$

The similarity between this expression and corresponding plane stress expression can be exploited for the finite element formulation of a plate bending problem simply by replacing (ε) and [D] by the expressions given in (6.36).

6.4 FINITE ELEMENT IMPLEMENTATION

Among the early applications of the finite element method to the plate bending problem, use of the 12 degrees of freedom rectangular element represents a classical example which neatly depicts the general approach[8]. The method was used for analysis of a rectangular flat plate and the plate edges were oriented along the two axes, x and y. In this case the rectangular elements formed after subdivision have edges parallel to the axes. The out-of-plane displacement w should have at least 2nd-order terms both in x and y in order to ensure C_1 continuity along the edges of the element shown in Fig. 6.13. As in the case of beam bending analysis in Sec. 6.2, if we take w, θ_x and θ_y as independent nodal parameters there will exist 12 independent quantities associated with 4 nodes of the element. This means the expression for w can be a 12-term expression with as many coefficients. Thus

$$w = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^3 + + \beta_8 x^2 y + \beta_9 xy^2 + \beta_{10} y^3 + \beta_{11} x^3 y + \beta_{12} xy^3 \quad \ldots (6.38)$$

It can be seen that the expression for w along any of the edges of the

[6] Some investigators have writen vector $\{\varepsilon\}^T$ as $\left[-\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, 2\frac{\partial^2 w}{\partial x \partial y}\right]$ instead of the one given in eq. (6.36). The vector $\{M\}$ is similarly changed to

$$\{M\}^T = \left[-D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\right]$$

It should be observed that the expression for strain energy (eq. 6.35) remains the same in both cases. It can also be observed that such a change in signs in vector $\{\varepsilon\}$ does not affect expression (6.37) due to the special nature of matrix $[D_b]$ which is symmetrical and has zeros in the non-diagonal terms of the last row and column. The matrices resulting from this analysis (eqs. 6.41 and 6.44) may appear slightly different in terms of signs of few terms but it can be shown that the overall result remains unaltered. The expression for $\{\varepsilon\}$ used here is followed in the text because this appears to be more logical to the author.

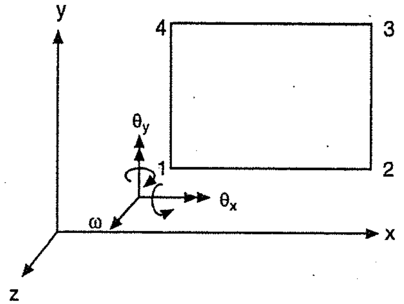


Fig. 6.13 Rectangular plate element

element is a four-term expression. For example, y being constant along edge 1-2 (say y_1) w can be expressed over this edge as

$$w = (\beta_1 + \beta_3 y_1 + \beta_6 y_1^2 + \beta_{10} y_1^3) + (\beta_2 + \beta_5 y_1 + \beta_9 y_1^2 + \beta_{12} y_1^3)\,x +$$
$$(\beta_4 + \beta_8 y_1)x^2 + (\beta_7 + \beta_{11} y_1)x^3$$
$$= \alpha_1 + \alpha_2 x + \alpha_3 x^3 + \alpha_4 x^3 \qquad \ldots (6.39)$$

and
$$\theta_y = -\,\alpha_2 - 2\alpha_3 x - 3\alpha_4 x^2$$

Four independent parameters along the nodes of edge 1-2, w_1, θ_{y1}, w_2 and θ_{y2} completely specify these four coefficients α_1, α_2, α_3, α_4 and thus interelement continuity of w and θ_y is ensured along this edge. Note that continuity of θ_x is not ensured along edge 1-2. Similarly, we can show the continuity of w and θ_x along edge 1-4 and so on. Since the normal slope along the edge (θ_x along 1-2 and θ_y along 1-4) is not continuous, full C_1 continuity is not ensured by the displacement expression given in (6.38). However, the element displays reasonably satisfactory results and can be used for rectangular plates.

The nodal displacement vector will now comprise w, θ_x and θ_y and we may define nodal and elemental vectors in the following manner:

$$\{d_1\} = \begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \end{Bmatrix} = \begin{Bmatrix} w_1 \\ \left(\dfrac{\partial w}{\partial y}\right)_1 \\ \left(-\dfrac{\partial w}{\partial x}\right)_1 \end{Bmatrix} \quad \text{etc.}$$

and
$$\{d^e\} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \\ \{d_3\} \\ \{d_4\} \end{Bmatrix} \qquad \ldots (6.40)$$

Substituting the nodal coordinates in the expression for w (eq. 6.38) and its derivatives we obtain

$$\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \end{Bmatrix} =$$

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & x_1^3 & x_1^2y_1 & x_1y_1^2 & y_1^3 & x_1^3y_1 & x_1y_1^3 \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 & 0 & x_1^2 & 2x_1y_1 & 3y_1^2 & x_1^3 & 3x_1y_1^2 \\ 0 & -1 & 0 & -2x_1 & -y_1 & 0 & -3x_1^2 & -2x_1y_1 & -y_1^2 & 0 & -3x_1^2y_1 & -y_1^3 \end{bmatrix} \{\beta\}$$
$$= [G_1]\,\{\beta\} \qquad \ldots (6.41)$$

Here $\{\beta\}$ is the 12-term vector comprising $\beta_1, \ldots \beta_{12}$. On writing similar matrices for nodes 2, 3 and 4 the final relation will be obtained as

$$\begin{Bmatrix} \{d_1\} \\ \{d_2\} \\ \{d_3\} \\ \{d_4\} \end{Bmatrix} = \begin{Bmatrix} [G_1]\,\{\beta\} \\ [G_2]\,\{\beta\} \\ [G_3]\,\{\beta\} \\ [G_4]\,\{\beta\} \end{Bmatrix} \text{ or } \{d^e\} = [G]\,\{\beta\} \qquad \ldots (6.42)$$

Inversion of this matrix $[G]$ gives the vector $\{\beta\}$. The inverse of $[G]$ cannot be obtained in an explicit form but all its terms being known an inverse can be obtained by using an appropriate inversion subroutine. Thus

$$\{\beta\} = [G]^{-1}\{d^e\} \qquad \ldots (6.43)$$

The vector $\{\varepsilon\}$ used in the expression for strain energy and defined in eq. (6.36) is now obtained by using eq. (6.38) for w.

$$\{\varepsilon\} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 \end{bmatrix} \{\beta\}$$
$$= [H]\,\{\beta\} = [H]\,[G]^{-1}\{d^e\} \qquad \ldots (6.44)$$

Elemental strain energy is obtained on substituting (6.44) into (6.37). Also, recognizing that vector $\{d^e\}$ remains invariant over the area of element, this results in

$$\Pi_{S^e} = \frac{1}{2}\{d^e\}^T \left\{\int_{A^e} [G]^{-1T}[H]^T[D_b][H][G]^{-1}dA\right\}\cdot\{d^e\}$$

$$= \frac{1}{2}\{d^e\}^T[K^e]\{d^e\} \qquad \dots (6.45)$$

The expression for $[K^e]$ contains matrices $[G]^{-1}$ and $[G]^{-1T}$, both of which are matrices of constants for an element. Hence separating these from the integral expression gives

$$[K^e] = [G]^{-1T}\left\{\int_{A^e} [H]^T[D_b][H]\,dA\right\}\cdot[G]^{-1} \qquad \dots (6.46)$$

6.4.1 External Work Done

Loading on a plate structure is generally in the form of distributed and concentrated loads normal to the plane of plate as shown in Fig. 6.14. The work done by the distributed force of intensity P per unit area is given by

$$W^e = \int_{A^e} w \cdot P\,dA \qquad \dots (6.47)$$

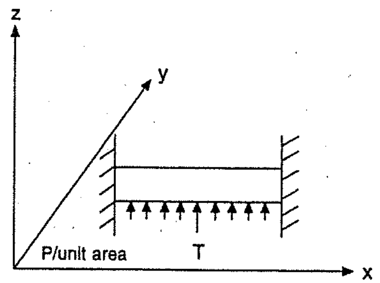


Fig. 6.14 Loads on a plate structure.

Expressing w in compact matrix form of eq. (6.38)

$$w = [X]\{\beta\} \qquad \dots (6.48)$$

where matrix $[X]$ is given as

$$[X] = [1\ x\ y\ x^2\ xy\ y^2\ x^3\ x^2y\ xy^2\ y^3\ x^3y\ xy^3]$$

Rewriting,

$$w = [[X]\{\beta\}]^T = \{\beta\}^T[X]^T = \{d^e\}^T[G]^{-1T}[X]^T$$

Thus work done by distributed force over an element is given by

$$\{d^e\}^T[G]^{-1T}\int_{A^e}[X]^T P\,dA$$

$$= \{d^e\}^T\{f_p^e\} \qquad \dots (6.49)$$

where $\{f_p^e\}$ can be called the elemental distributed load vector. The concentrated loads 'T' may be assumed located at nodes and the net external work due to nodal concentrated loads is given by

$$T_1w_1 + T_2w_2 + \cdot + \cdot + \cdot + T_nw_n$$

$$= [w_1\ \theta_{x1}\ \theta_{y1}\ w_2\ \theta_{x2}\ \theta_{y2}\ \cdots\cdots]\begin{Bmatrix} T_1 \\ 0 \\ 0 \\ T_2 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \\ \cdot \\ \cdot \\ \cdot \\ \{d_n\} \end{Bmatrix}^T \cdot \{T\} = \{\delta\}^T\{T\} \qquad \dots (6.50)$$

Here $\{T\}$ represents overall nodal load vector for the whole domain, as defined above. The potential energy for the domain comprising n elements can be expressed by combining eqs. (6.45, 6.49, 6.50). Thus

$$\Pi = \text{Strain energy} - \text{external work done}$$

$$= \frac{1}{2}\sum_{e=1}^{n}\{d^e\}^T[K^e]\{d^e\} - \sum_{e=1}^{n}\{d^e\}^T\{f_p^e\} - \{\delta\}^T\{T\} \qquad \dots (6.51)$$

Minimization of Π in a manner similar to the procedure followed in Sec. 2.2.3 gives

$$\left[\sum_{e=1}^{n}[K^e]\right]\{\delta\} - \sum_{e=1}^{n}\{f_p^e\} - \{T\} = 0$$

or

$$[K]\{\delta\} - \{f\} = 0 \qquad \dots (6.52)$$

On inversion of eq. (6.52) we obtain the vector $\{\delta\}$ which comprises nodal displacements and rotation values.

6.5 OTHER TYPES OF ELEMENTS

The element used for illustration in the previous section suffers from several limitations. Obvious constraint of edges being parallel to the axes makes it unsuitable for non-rectangular areas or areas wherein edges are oriented in directions different from the directions of axes. It is also incapable of modelling a general shell or plate oriented in space, such as the shell of a spherical pressure vessel. Hence it is required to develop plate (or shell) elements which have general quadrilateral or triangular shape and which can be oriented easily in any direction in space. Such elements are discussed in the next chapter.

6.6 APPLICATION EXAMPLE

An example of a structure which can be analyzed using the elements discussed in this chapter is a ship deck which consists of a deck plate supported by welding on longitudinal girders and transverse girders (Fig. 6.15).

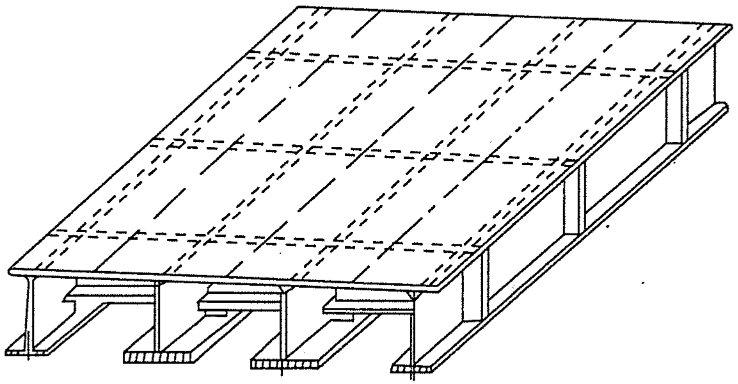


Fig. 6.15 Layout of a ship deck.

For the purpose of analysis the longitudinal and transverse girders can be assumed connected to each other in a manner that the neutral axes lie in a horizontal (x-y) plane. The deck plate can also be assumed to lie in this plane. With this simplification, the deck structure reduces to two-dimensional configuration made of beams and rectangular plate elements. The girders (beam elements) are shown by broken lines in Fig. 6.16 while the plate elements are represented by dotted lines. For better accuracy a greater number of plate elements can be placed between the girders. Each node will have three degrees of freedom (w, θ_x, θ_y) and assembled stiffness matrix will be obtained by combining beam and plate elemental stiffness matrices.

By placing the deck plate along the neutral plane of the girders the

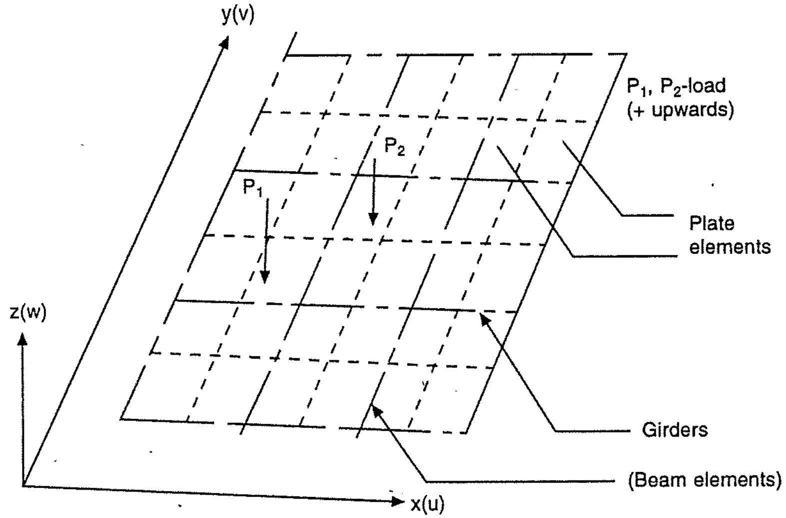


Fig. 6.16 Schematic representation of a ship deck (model for finite element analysis).

apparent stresses in the plate will be lessened. A more accurate analysis should consider the girders also made of plate elements with the deck plate located at the top of girders, as shown in Fig. 6.15. Analysis now becomes quite expensive computationally.

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