Chapter 2

ELASTIC STRESS ANALYSIS USING LINEAR ELEMENTS

The basis of finite element formulation for analysis of stresses in an object was established in Chapter 1. It was shown that the force equilibrium approach and minimization of potential energy yield identical final equations. Versatility of the potential energy approach makes it an ideal choice and we shall use it in our subsequent formulations. The finite element procedure requires subdivision of an object into elements. The simplest two-dimensional element is a triangle. The corresponding element in three dimensions is a tetrahedron. These two elements are unique in terms of simplicity of formulation. Displacement is assumed to vary linearly throughout the element and hence these elements are also called linear elements. In spite of their simplicity these elements are versatile enough and are capable of modelling any complex object. These features have made such elements very popular for general engineering analysis. This chapter is devoted to linear elements. Although concentrated loads were used in Chapter 1, sufficient hint was given about several other type of loads which may be present in real structures or objects under investigation. We shall first consider examples of loading of some common objects to illustrate the nature of loads which should be considered in finite element analysis.

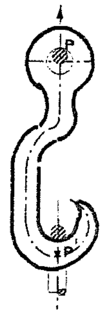
2.1 NATURE OF LOADING

Loading in the case of stress analysis may assume several forms. It may be in the form of concentrated loads acting at several points on the body or it may assume the form of uniformly or non-uniformly distributed load on some area. Alternatively, the load may appear as gravity force, inertial force or electromagnetic force, generally termed body forces. Internal forces may also appear due to non-uniform expansion and contraction resulting from heating and cooling or due to metallurgical changes. Yet another feature which may affect stresses is the presence of residual stresses which may result from certain forming or other operations previously carried out, such as rolling of plates, cooling of casting and forging or quenching during heat treatment. An explanation of these will help to clarify the concepts developed later in this chapter.

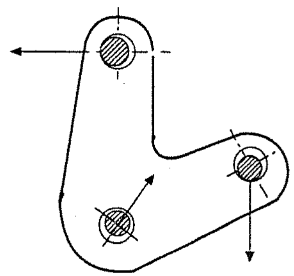
2.1.1 Concentrated or Distributed Loads

The concept of concentrated or uniformly distributed load is well known to a student of mechanics. Examples of concentrated load are shown in Fig. 2.1. The contacts between gear teeth or the supporting and loading pins of a crane hook or a lever, as shown in the Fig 2.1, can be considered point contact. The contact, however, extends uniformly over the whole thickness of the part. Strictly speaking, the contact extends over some width of the pin and supporting member due to elastic deflection, as shown in Fig. 2.1(iv). At points slightly away from the position of load application the effect of concentrated load or slightly spread out load

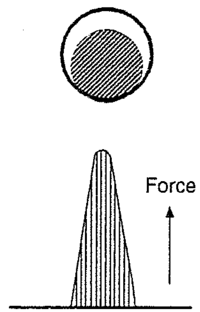
(i) gear tooth



(ii) crane hook



(iii) lever



(iv) General pressure distribution in roller contact

Fig. 2.1 Examples of point contact (concentrated load) (as in iv) will be the same. Moreover, the finite element formulation has a natural tendency to smoothen the load peaks due to the process of subdivision into elements; this tendency brings the result of FEM closer to that due to load distribution shown in Fig. 2.1(iv), although the load may have been treated as a concentrated one. This will become clearer later, during formulation. Examples of distributed load are given in Fig. 2.2. Pressure on the wall of a cylinder filled with gas is an example of uniformly distributed load while that on the wall of a water tank is a linearly varying distributed load. Pressure on the two sides of a turbine blade is an example of non-linearly varying distributed load. In the example of a cutting tool (Fig. 2.2(iv)), the pressure is distributed non-uniformly over a small area of tool chip contact.

2.1.2 Body Force (Gravity etc.)

The concept of body force includes those forces which are distributed over the entire volume of a component in a known manner. This is in contrast to the distributed loads discussed earlier, which are spread on some surface of the component. Examples of body force are force due to

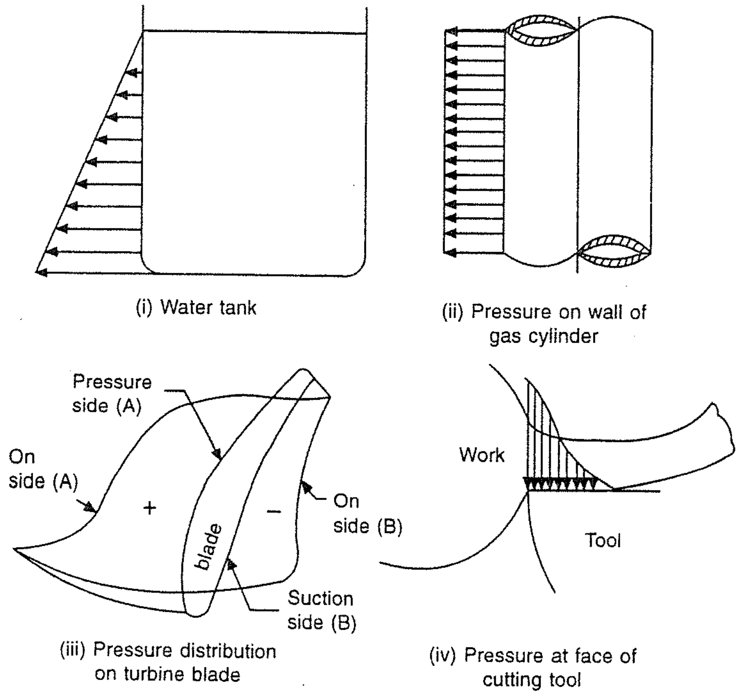


Fig. 2.2 Examples of distributed loads

gravity, inertial force due to motion or electromagnetic force. These forces act on each and every element of the component and are related to elemental volumes. For illustration, if we consider the force due to gravity, it will be represented by '$\rho \cdot g \cdot dx \cdot dy \cdot dz$' where ρ and g are density and acceleration due to gravity.

2.1.3 Loading Due to Thermal Strains etc.

When the component under investigation is subjected to non-uniform heating or cooling, stresses develop in it even if external forces are absent. Thermal expansion per unit length or thermal strain at any point is given by $T\alpha$ in all the three directions x, y and z. T indicates the temperature of the point and α, the coefficient of thermal expansion. Since, thermal strain is in addition to elastic strain, it should be added to the elastic strain to get the total strain at a point, i.e.

$$\varepsilon = \varepsilon_e + T\alpha$$

where ε is total strain and ε_e, the elastic component of strain. Another source of such strains may be metallurgical transformations resulting in microstructure change, leading to volume expansion similar to that observed during heating. All such strains may be grouped together and these are sometimes called *initial strains*, represented as ε_o. Thus

$$\varepsilon = \varepsilon_e + \varepsilon_o \tag{2.1}$$

2.1.4 Residual Stresses

The primary manufacturing operations, such as rolling, casting, welding and forging result in the development of residual stresses and if the part is not stress relieved, these stresses affect the subsequent load-carrying capacity of the component. Hence, the stresses in the part at any instant of loading will be the sum of stresses due to loading and those present initially as residual stresses or

$$\sigma = \sigma_L + \sigma_o \qquad \text{... (2.1a)}$$

where σ_L and σ_o are the stresses due to load and the initial residual stresses respectively; σ_o is also referred to as *initial stress*.

2.2 TWO-DIMENSIONAL ANALYSIS

In 2D analysis the region under study is divided into small triangular elements, such as 1-2-3 shown in Figs. 2.3(a) and 2.4.

During straining the points in the region are displaced from the original positions and the displacement can be represented by two components,

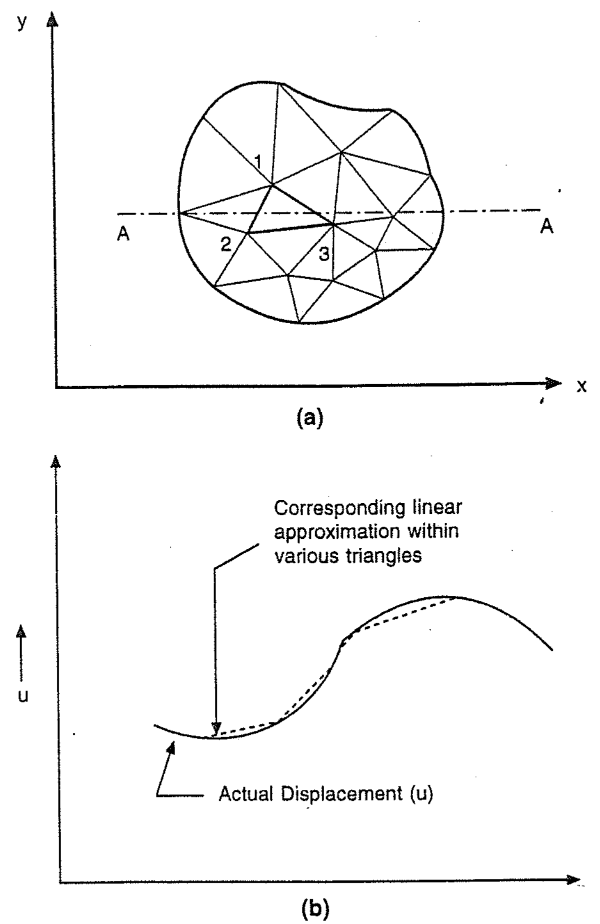


Fig. 2.3 Domain subdivision in triangular element

u in x-direction and v in y-direction. Thus, the displacements at the three corners of the triangle (called nodes) will be given by u_1, v_1, u_2, v_2 and u_3, v_3. Similarly, forces acting at the nodes may be represented as F_{x1}, F_{y1} etc. *It is common to express forces, such as F_{x1}, F_{y1} etc., for an unit thickness of the 2D object. The same convention applies to other quantities, such as potential energy.* The simplicity of approach lies in the fact that the variation of displacement within the element is assumed linear, although overall variation of displacement throughout the region may follow any pattern. It is easy to see, as shown in Figs. 2.3(a) and (b), that as the element size decreases the linear variations of displacements within various triangular elements ultimately approaches its actual distribution in the whole region.

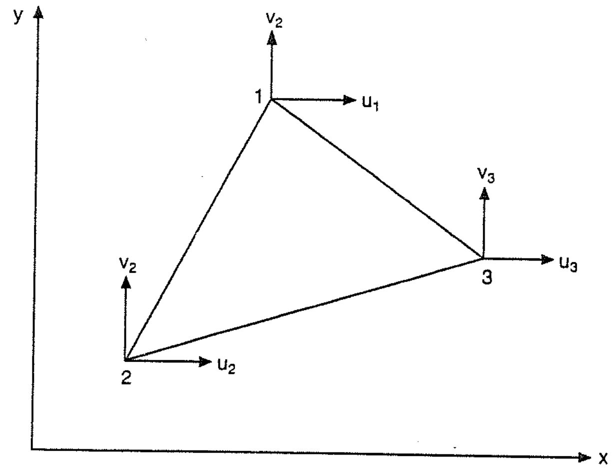


Fig. 2.4 Nomenclature for triangular element

Assumption of linear variation of displacement within the element greatly simplifies subsequent expressions for strains, stresses etc. The final equilibrium equation for element or the expression for potential energy for it will be quite simple. As explained in the first chapter, either we can consider the force equilibrium for all the elements put together or we may minimize the total potential energy of the whole region and thus solve the system of equations for displacements at nodes. The following discussion develops this treatment further.

Writing the general expression for displacements within the elements (1-2-3), we obtain

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \qquad \text{... (2.2)}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6 are constants. Since the nodal displacement should satisfy these expressions we get the following six equations,

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$v_1 = \alpha_4 + \alpha_5 x_1 + \alpha_6 y_1 \qquad \text{... (2.2a)}$$
$$-\;-\;-\;-\;-\;-\;-$$
$$-\;-\;-\;-\;-\;-\;-$$
$$-\;-\;-\;-\;-\;-\;-$$
$$v_3 = \alpha_4 + \alpha_5 x_3 + \alpha_6 y_3$$

The six equations, (2.2a) are solved to determine the six unknowns α_1, α_2, α_3, α_4, α_5 and α_6 in terms of u_1, v_1 .... and x_1, y_1 .... etc.

Rewriting expressions for u_1, u_2 and u_3 from eqs. (2.2a) and using matrix notations, we get

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \text{... (2.3)}$$

Equation (2.3), on inversion, gives α_1, α_2, α_3 as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \quad \text{... (2.4)}$$

Here the matrix formed by a_1...., b_1 ... c_1 is the transpose of the cofactors matrix derived on inversion of the matrix in eq. (2.3) and

$$2\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad \text{... (2.4a)}$$

is the corresponding determinant. In these expressions

$$\begin{aligned} a_1 &= x_2y_3 - x_3y_2, & a_2 &= x_3y_1 - x_1y_3, & a_3 &= x_1y_2 - x_2y_1 \\ b_1 &= y_2 - y_3, & b_2 &= y_3 - y_1, & b_3 &= y_1 - y_2 \\ c_1 &= x_3 - x_2, & c_2 &= x_1 - x_3, & c_3 &= x_2 - x_1 \end{aligned} \quad \text{... (2.4b)}$$

The determinant in (2.4a) has special significance in that it represents twice the area of triangle 1-2-3, as shown in Appendix 1. Here Δ is the area of the triangle. It should be noted that the direction of numbering the nodes has special significance and the area shown by the determinant in eq. (2.4) will become negative if the order is changed to 1-3-2. Hence, if Δ is to be used as area of the triangle, as will be done in subsequent analysis, the ordering of nodes should be in anticlockwise direction (i.e., 1-2-3, as shown). It will not help if we take the area as half the absolute value of the determinant because sometimes when we cancel Δ appearing as area with Δ appearing as half the determinant in the denominator, the error of '*sign*' takes place, especially during calculation of stresses. Hence, it is good practice to renumber the nodes and write these in anticlockwise sequence.

Using the above derivation, the expression for displacement u is written as (see eq. 2.4),

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ &= \frac{1}{2\Delta} [(a_1 + b_1x + c_1y)\, u_1 + (a_2 + b_2x + c_2y)\, u_2 + (a_3 + b_3x + c_3y)\, u_3] \end{aligned} \quad \text{... (2.5)}$$

A similar expression is derived for 'v' using expression for v_1, v_2 and v_3 from eq. (2.2). Thus

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \quad \text{... (2.6)}$$

Similarity between expressions (2.6) and (2.3) can be exploited to write the expression for displacement v as

$$v = \frac{1}{2\Delta} [(a_1 + b_1x + c_1y)\, v_1 + (a_2 + b_2x + c_2y)\, v_2 + (a_3 + b_3x + c_3y)\, v_3] \quad \text{... (2.7)}$$

Writing down $(a_1 + b_1x + c_1y)/2\Delta$ as N_1 and using similar expressions for N_2 and N_3, by changing the suffixes, we obtain

$$\begin{aligned} u &= N_1u_1 + N_2u_2 + N_3u_3 \\ v &= N_1v_1 + N_2v_2 + N_3v_3 \end{aligned} \quad \text{... (2.8)}$$

At this stage the matrix notations are introduced to represent the displacements within the element and at the nodes. The vectors $\{d\}$, $\{d_1\}$, $\{d_2\}$ define general and nodal displacements in the element while $\{d^e\}$ represents the nodal displacement vector for the whole element having u_1, u_2... v_3 as its 6 components. Thus

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix}; \{d_1\} = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}; \{d_2\} = \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}; \{d_3\} = \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}; \{d^e\} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \\ \{d_3\} \end{Bmatrix} \quad \text{... (2.9)}$$

The vector representation within the braces $\{d^e\}$ may be omitted for the sake of simplicity, thus expressing

$$\{d^e\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \quad \text{... (2.10)}$$

It should be recognized here that d_1, d_2 and d_3 are individually vectors having two components given by expression (2.9). With these notations we represent the eqs. (2.8) as

$$\{d\} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{... (2.11)}$$

or,

$$\{d\} = [N]\,\{d^e\} \quad \text{... (2.12)}$$

This equation has special significance. It defines the displacement within the element in terms of the displacements at the nodes. The matrix $[N]$ which defines this relation is called the shape function; it is also called the interpolation function [2]. The components of the shape function N_1, N_2 and N_3 vary within the element and it is interesting to see that these have special values at the three nodes. It can be shown that $N_1 = 1$, $N_2 = 0$ and $N_3 = 0$ at node 1 while $N_2 = 1$, $N_1 = 0$ and $N_3 = 0$ at node 2 and similar values exist at node 3. This is not mere coincidence but illustrates an important characteristic of the shape function. In fact, some shape functions have been developed directly, using the aforesaid characteristics and without defining displacement explicitly, as done in eq. (2.2).

Exercise 2.1: Determine the magnitudes of N_1, N_2 and N_3 at the three nodes of a triangular element and show that these follow the general rule stated in Sec. 2.2, i.e., $N_1 = 1$ at node 1 and zero at other nodes and N_2, N_3 follow a similar pattern.

2.2.1 Strain Displacement Relation

The strain in a two-dimensional situation has only three components ε_x, ε_y and γ_{xy}. The expressions for these components in terms of displacement can be derived [3] by considering the elemental rectangle shown in Fig. 2.5. The rectangle ABCD is distorted to A′B′C′D′ after loading and if the displacement components at corner A are u, v, the corresponding displacements at B, C and D will be

$$u + \frac{\partial u}{\partial y}dy;\quad v + \frac{\partial v}{\partial y}dy \qquad \text{(at } B\text{)}$$

$$u + \frac{\partial u}{\partial x}dx;\quad v + \frac{\partial v}{\partial x}dx \qquad \text{(at } D\text{)}$$

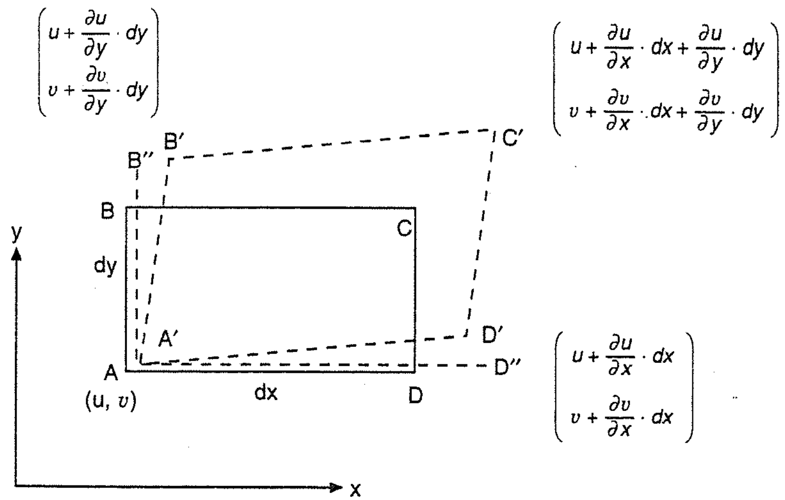


Fig. 2.5 Displacement in a rectangular element due to straining

and,

$$u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy;\quad v + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \qquad \text{(at C)}$$

It can be seen that the change in displacement u between points A and D or B and C is $(\partial u/\partial x) \cdot dx$. Hence, increase in the length of edges AD or BC is $(\partial u/\partial x) \cdot dx$. The strain in x-direction is defined as (*increase in length*)/(*original length*). Thus

$$\varepsilon_x = \frac{\frac{\partial u}{\partial x}dx}{dx} = \frac{\partial u}{\partial x} \quad \text{... (2.13)}$$

It is easy to see that corresponding increase in lengths of edges AB and CD are $(\partial v/\partial y) \cdot dy$. This gives the strain in y-direction as

$$\varepsilon_y = \frac{\frac{\partial v}{\partial y}dy}{dy} = \frac{\partial v}{\partial y} \quad \text{... (2.14)}$$

The shearing strain γ_{xy} is defined as the change in angle BAD which is given by

$$\gamma_{xy} = \text{magnitude of } \angle B''A'B' + \text{magnitude of } \angle D''A'D'$$

$$= \frac{\frac{\partial u}{\partial y}dy}{dy} + \frac{\frac{\partial v}{\partial x}dx}{dx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \text{... (2.15)}$$

The strain vector can thus be written as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{Bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{... (2.16)}$$

Expression (2.16) can be written as $\{\varepsilon\} = [L]\{d\}$, where $[L]$ is the operator matrix given above. The displacement $\{d\}$ is to be operated upon by matrix $[L]$. On substituting for $\{d\}$ from expression (2.12), we get

$$\{\varepsilon\} = [L]\,[N]\,\{d^e\} \quad \text{... (2.17)}$$

The matrix $[L]$ should operate upon the product $[N]\{d^e\}$. However, operator matrix $[L]$ involves differentials with respect to x and y only (eq. 2.16) and product $[N]\{d^e\}$ is such that x and y appear only in $[N]$ (see eq. 2.7). Hence, vector $\{d^e\}$ will appear as a constant term and only matrix $[N]$ should be operated upon (or differentiated) with respect to matrix $[L]$. We may thus obtain the matrix $[L]\,[N]$ as $[B]$ and relation (2.17) can be written as

$$\{\varepsilon\} = [B]\,\{d^e\} \quad \text{... (2.18)}$$

To derive matrix $[B]$ we write the expanded forms of $[L]$ and $[N]$, i.e.

$$[B] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

On substituting for N_1, N_2 and N_3 from eq. (2.7) and differentiating these with respect to x and y, it is easy to obtain the $[B]$ matrix as follows.

$$[B] = \frac{1}{2\Delta} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \quad \text{... (2.19)}$$

Matrix $[B]$ is also known as the strain-displacement matrix.

Exercise 2.2: Obtain the expression for strain ε_x in partial differential form when a rectangular element shown in Fig. 2.6(a) undergoes uniform displacement in x-direction.

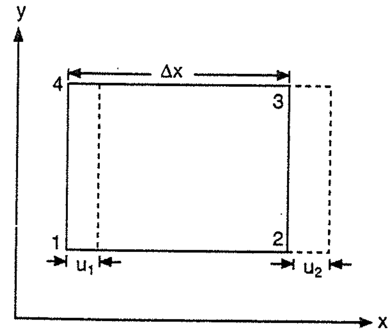


Fig. 2.6(a) Exercise 2.2

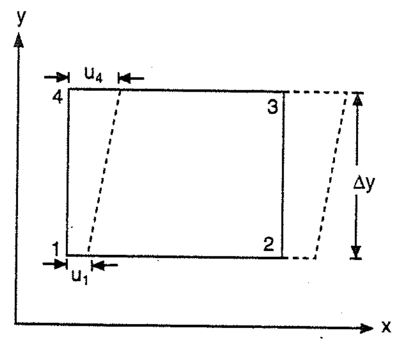


Fig. 2.6(b) Exercise 2.2

Determine the expression for shear strain γ_{xy} in partial differential form when this element is subjected to displacement u, which varies linearly in y-direction also (Fig. 2.6(b)).

Exercise 2.3: Use figures similar to Figs. 2.6(a) and (b) to obtain expressions of ε_y and γ_{xy} by considering displacement in y-direction only (displacement component v).

Show that these expressions, when combined together, give the strains ε_x, ε_y and γ_{xy} derived in Sec. 2.2.1.

Exercise 2.4: If an operator matrix $[L]$ (such as the one given in eq. 2.16) operates on a vector $\{A\}$, which itself is a product of matrix $[B]$ and vector $\{C\}$, then the relation given in the following equation is valid only if matrix $[B]$ is a function of x, y while vector $\{C\}$ contains constants only, i.e.

$$[L]\,\{A\} = [L]\,[B]\,\{C\} = [M]\,\{C\}$$

where, $$[M] = [L]\,[B]$$

[For illustration, take the order of matrix $[B]$ as 2×2 and that of vector $\{C\}$ as 2×1].

2.2.2 Stress-Strain Relation

The stress vector for a two-dimensional case is similar to the strain vector and has three components, σ_x, σ_y and τ_{xy}. We generally encounter two situations in 2-D loading. If the plate is thin and the loading is in the plane of the plate, there exists no stress perpendicular to the plane of the plate; we call this a case of plane stress. However, the strain in z-direction is given by

$$\varepsilon_z = -\,\nu\,\frac{\sigma_x}{E} - \nu\,\frac{\sigma_y}{E} \quad \text{... (2.20)}$$

This normal strain ε_z does not influence the magnitudes of in-plane stresses and strains and is disregarded in the analysis. Another situation is the plane strain case in which the element is constrained from expanding or contracting along the thickness, resulting in strain ε_z being zero. In practice such a situation may arise when the element is a slice of a very thick member, for example, a transverse slice of a large thick cylinder subjected to internal pressure and rigidly clamped at the end faces.

The two cases are dealt with separately.

a) Plane stress: The strains in a plane stress situation, when only elastic strains are present, are given by

$$\begin{aligned}\varepsilon_x &= \frac{(\sigma_x - \nu\sigma_y)}{E}\\ \varepsilon_y &= \frac{(-\nu\sigma_x + \sigma_y)}{E} \qquad \dots (2.21)\\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \tau_{xy}\,\frac{2(1+\nu)}{E}\end{aligned}$$

It may be noted that ε_z is given by expression (2.20). However, in the present situation ε_z is not required for deriving stress-strain relation. Hence, this has not been incorporated in (2.21). On rewriting the first two equations of (2.21) in matrix form, we get

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu\\ -\nu & 1\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix} \qquad \dots (2.22)$$

On inversion it gives

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\end{Bmatrix} \qquad \dots (2.23)$$

Combining this with the third equation of (2.21), the matrix relation is obtained as

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \qquad \dots (2.24)$$

or,

$$\{\sigma\} = [D]\,\{\varepsilon\}$$

Matrix $[D]$ denoted above is known as the stress-strain or elasticity matrix.

b) Plane strain: The stress-strain relation for the plane strain case can be similarly derived by writing the appropriate equations as follows

$$\begin{aligned}\varepsilon_x &= \frac{(\sigma_x - \nu\sigma_y - \nu\sigma_z)}{E}\\ \varepsilon_y &= \frac{(-\nu\sigma_x + \sigma_y - \nu\sigma_z)}{E} \qquad \dots (2.25)\\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \tau_{xy}\,\frac{2(1+\nu)}{E}\\ \varepsilon_z &= \frac{(-\nu\sigma_x - \nu\sigma_y + \sigma_z)}{E} = 0\end{aligned}$$

The fourth equation of (2.25) can be used to eliminate σ_z from first two and thus the expressions for ε_x and ε_y are obtained in terms of σ_x and σ_y only. Hence

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\begin{aligned}\varepsilon_x &= \frac{\{\sigma_x(1-\nu^2) - \sigma_y(1+\nu)\,\nu\}}{E} \qquad \dots (2.26)\\ \varepsilon_y &= \frac{\{-\sigma_x(1+\nu)\,\nu + \sigma_y(1-\nu^2)\}}{E}\end{aligned}$$

On writing ε_x and ε_y in matrix form

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\end{Bmatrix} = \frac{(1+\nu)}{E}\begin{bmatrix}(1-\nu) & -\nu\\ -\nu & (1-\nu)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix} \qquad \dots (2.27)$$

Inversion gives

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu\\ \nu & (1-\nu)\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\end{Bmatrix} \qquad \dots (2.28)$$

Combining (2.28) with the expression for γ_{xy} from (2.25), we obtain

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & 0\\ \nu & (1-\nu) & 0\\ 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \qquad \dots (2.29)$$

or,

$$\{\sigma\} = [D]\,\{\varepsilon\}$$

Equations (2.24) and (2.29) assume the presence of elastic strains only and the initial strains (thermal strains etc.) and initial stresses (residual stresses) are considered absent. If these are also present, then eqs. (2.1) and (2.1a) are used and we obtain the final relation as

$$\{\sigma\} = [D]\,(\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\} \qquad \text{... (2.30)}$$

where $[D]$ is the matrix given by expressions (2.24) or (2.29), as the case may be.

2.2.3 Potential Energy

The principle explained in Chapter 1 that the minimization of potential energy of the system is equivalent to solving for equilibrium displacement and stresses, can now be used here for the two-dimensional problem discussed above. The expression for the potential energy of the system is written as

$$\Pi = \sum_{e=1}^{n} \Pi_{se} - \Pi_w \qquad \text{... (2.31)}$$

where Π_{se} is the strain energy of the element and Π_w represents the energy equivalent to the work done by the external forces which form part of the system, such as concentrated loads, distributed loads and the body forces.

We first consider the case when initial stress is absent and eq. (2.30) becomes

$$\{\sigma\} = [D]\,(\{\varepsilon\} - \{\varepsilon_o\}) \qquad \text{... (2.32)}$$

As explained in Sec. 2.1.3, the elastic component of strain is obtained as the difference between the total strain and initial strain. It is given by

$$\{\varepsilon_e\} = \{\varepsilon\} - \{\varepsilon_0\} \qquad \text{... (2.33)}$$

The strain energy per unit volume is given as $1/2 \times$ *elastic strain* $\times$ *stress*—summed over all the stress and strain components. Thus, for a triangular element it is given as

$$\Pi_{se} = \int_{V^e} \frac{1}{2} \{(\varepsilon_x)_e \cdot \sigma_x + (\varepsilon_y)_e \cdot \sigma_y + (\gamma_{xy})_e \cdot \tau_{xy}\}\, dV$$

$$= \int_{V^e} \frac{1}{2} \{\varepsilon_e\}^T \{\sigma\}\, dV$$

$$= \frac{1}{2} \int_{V^e} \{\{\varepsilon\} - \{\varepsilon_0\}\}^T [D]\,(\{\varepsilon\} - \{\varepsilon_0\})\, dV \qquad \text{... (2.34)}$$

where the integral is taken over the volume of the element, V^e and use is made of eqs. (2.32) and (2.33). *For a 2-D object of uniform thickness dV can be replaced by tdA and integral taken over the area of element A^e. Here t refers to thickness. This has been done later in eq.* (2.46a). Rewriting eq. (2.34),

$$\Pi_{se} = \int_{V^e} \left[\frac{1}{2}\{\varepsilon\}^T [D]\{\varepsilon\} - \frac{1}{2}\{\varepsilon\}^T [D]\{\varepsilon_0\} - \frac{1}{2}\{\varepsilon\}^T [D]\{\varepsilon\} + \frac{1}{2}\{\varepsilon_0\}^T [D]\{\varepsilon_0\}\right] dV$$

Recognizing that $\{\varepsilon\}^T [D] \{\varepsilon_0\}$ and $\{\varepsilon_0\}^T [D] \{\varepsilon\}$ are equal if $[D]$ is a symmetrical matrix (see Exer. 2.5), we obtain

$$\Pi_{se} = \int_{V^e} \frac{1}{2}\{\varepsilon\}^T [D]\{\varepsilon\}\, dV - \int_{V^e} \{\varepsilon\}^T [D]\{\varepsilon_0\}\, dV + \int_{V^e} \frac{1}{2}\{\varepsilon_0\}^T [D]\{\varepsilon_0\}\, dV \qquad \text{... (2.35)}$$

On substituting for $\{\varepsilon\}$ from eq. 2.18 and bringing $\{d^e\}$ outside the integral sign because it represents a constant in so far as integration over the volume of the element is concerned, we obtain

$$\Pi_{se} = \frac{1}{2}\{d^e\}^T \left[\int_{V^e} \{B\}^T [D]\{B\}\, dV\right] \{d^e\} - \{d^e\}^T \left[\int_{V^e} \{B\}^T [D]\{\varepsilon_0\} dV\right] + \int_{V^e} \frac{1}{2}\{\varepsilon_0\}^T [D]\{\varepsilon_0\} dV \qquad \text{... (2.36)}$$

Here care is taken to preserve the order of multiplication of vectors and matrices (see Exer. 2.6). The integral written within first bracket in eq. (2.36) can be identified as a square matrix and that within second bracket as a vector. Representing these by $[K^e]$ and $\{f^e_{\varepsilon_0}\}$, the elemental strain energy is expressed as

$$\Pi_{se} = \frac{1}{2}\{d^e\}^T [K^e]\{d^e\} - \{d^e\}^T \{f^e_{\varepsilon_0}\}^T + \int_{V^e} \frac{1}{2}\{\varepsilon_0\}^T [D]\{\varepsilon_0\} dV \qquad \text{... (2.37)}$$

$[K^e]$ is called the elemental stiffness matrix and $\{f^e_{\varepsilon_0}\}$ symbolizes load vector due to initial strain. Superscript *e* signifies that these are the quantities associated with an element '*e*'.

Exercise 2.5: Show that the product $\{A\}^T[B]\{C\}$ is identical to $\{C\}^T[B]\{A\}$ if $[B]$ is a symmetrical matrix. For illustration take the order of matrix $[B]$ as 2×2 and that of vectors $\{A\}$ and $\{C\}$ as 2×1.

Exercise 2.6: In the above example if we assume matrix $[B]$ to be a function of coordinates x, y while vector $\{C\}$ and $\{A\}$ do not depend on x, y, then show that the integral over an area A^e of the product $\{C\}^T[B]\{A\}$ is given by

$$\int_{A^e} \{C\}^T\ [B]\ \{A\}\ dx\, dy = \{C\}^T \left[\int_{A^e} [B]\, dx\, dy \right] \cdot \{A\}$$

Exercise 2.6(a) If Π represents the product $\{\phi\}^T[A]\{\phi\}$ when $[A]$ is a matrix of constants and $\{\phi\}$ is the vector representing variables ϕ_1, ϕ_2 etc., then show that the following relation is valid if matrix $[A]$ is *symmetrical*.

$$\left\{ \begin{matrix} \dfrac{\partial \Pi}{\partial \phi_1} \\ \dfrac{\partial \Pi}{\partial \phi_2} \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\} = \frac{\partial \Pi}{\partial \{\phi\}} = 2[A]\{\phi\}$$

[Take $[A]$ as a matrix of order 2×2 and $\{\phi\}$ as vector of order 2×1].

Energy of external loads: The second part of expression (2.31) represents the work done by external loads. Fig. 2.7 shows the distributed load P acting along the edge and the body force W per unit volume acting along the volume of the triangular element. Both P and W are vectors with components in x and y directions and P represents load per unit area of

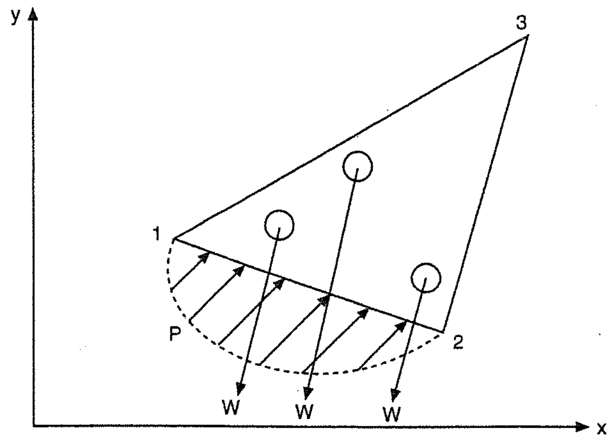


Fig. 2.7 Distributed edge load and body force on an element.

the edge 1-2 which has some thickness t while W represents load (body force) per unit volume. P and W may be uniform or may depend upon the coordinates x and y. The work done by the load P acting along the boundary 1-2 of the triangle is given by

$$\text{work done} = \int_A \{d\}^T \{P\}\, dA$$

where dA is the small element of area A of edge 1-2. Similarly, work done by body force W is given by

$$\text{work done} = \int_{V^e} \{d\}^T \{W\}\, dV$$

in which dV is the small element of the volume V^e of the triangle 1-2-3 (having thickness t). Hence, the total work done by these two types of forces acting on the whole region can be obtained by substituting for $\{d\}^T$ as $\{d^e\}^T[N]^T$ (from eq. 2.12) and summing over all the elements—the surface integral being taken over the boundary face of those surface elements only where the distributed load is acting (say, m). Thus

$$\text{work done} = \sum_{e=1}^{m} \left[\int_A \{d^e\}^T [N]^T\ \{P\}\, dA \right] + \sum_{e=1}^{n} \left[\int_{V^e} \{d^e\}^T [N]^T\ \{W\}\, dV \right] \quad \text{... (2.38)}$$

When $\{d^e\}^T$ is taken out of the integral sign, being independent of the coordinates x and y within the element, we obtain

$$\text{work done} = \sum_{e=1}^{m} \left[\{d^e\}^T \int_A [N]^T\ \{P\}\, dA \right] + \sum_{e=1}^{n} \left[\{d^e\}^T \int_{V^e} [N]^T\ \{W\}\, dV \right] \quad \text{... (2.39)}$$

Here m is the number of those elements which are located at the boundary and on which the distributed load is acting. Representing the quantities within the integral sign by symbols $\{f_p^e\}$ and $\{f_w^e\}$ and calling these the load vectors due to distributed load and body force, we obtain eq. (2.39) as

$$\text{work done} = \sum_{e=1}^{m} [\{d^e\}^T\ \{f_p^e\}] + \sum_{e=1}^{n} [\{d^e\}^T \{f_w^e\}] \quad \text{.. (2.40)}$$

The additional component of the work done by external forces will be that due to concentrated loads. These concentrated loads are assumed to act at various nodes over the whole thickness of object and thus can be easily represented in terms of nodal load vectors $\{T_1\}$, $\{T_2\}$ etc; combining these, we get the nodal load vector for the whole region as $\{T\}$. Also if the nodal displacement vector for the whole region is represented as $\{\delta\}$ which has nodal displacements $\{d_1\}$, $\{d_2\}$ etc. (eq. 2.10) as its components, it is easy to see that this work will simply be equal to the product of force at the node and corresponding displacement at the node. Thus

$$\text{work done by concentrated loads} = \{d_1\}^T\{T_1\} + \{d_2\}^T\{T_2\}\ldots = \{\delta\}^T\{T\}$$

Combining this with the work of distributed forces and body forces, given by expression (2.40), we obtain the expression for external work as

$$\Pi_{\bar{w}} = \sum_{e=1}^{n} \{d^e\}^T\left[\{f_P^e\} + \{f_{\bar{w}}^e\}\right] + \{\delta\}^T\{T\} \qquad \ldots (2.41)$$

(The contribution from elements not having a distributed load will be zero when load $\{f_P^e\}$ is considered).

Energy minimization: The expression for total potential energy in the region can now be written by substituting from eqs.(2.37) and (2.41) into (2.31). Thus

$$\Pi = \sum_{e=1}^{n}\left[\frac{1}{2}\{d^e\}^T[K^e]\{d^e\} - \{d^e\}^T\left(\{f_{\varepsilon_0}^e\} + \{f_P^e\} + \{f_{\bar{w}}^e\}\right)\right]$$
$$- \{\delta\}^T\{T\} + \sum_{e=1}^{n}\int_V \{\varepsilon_0\}^T[D]\{\varepsilon_0\}\,dV \qquad \ldots (2.42)$$

In order to minimize total potential energy of the system, the expression for Π should be differentiated with respect to each of the nodal displacements (because these are the unknown variables) and these differentials should be equated to zero to yield n equations which can be solved to get individual displacements. Thus

$$\begin{array}{ccc} \dfrac{\partial\Pi}{\partial\{d_1\}} = 0 & & \dfrac{\partial\Pi}{\partial u_1} = 0 \\ \dfrac{\partial\Pi}{\partial\{d_2\}} = 0 & \text{or} & \dfrac{\partial\Pi}{\partial v_1} = 0 \\ \ldots & & \dfrac{\partial\Pi}{\partial u_2} = 0 \\ \ldots & & \ldots \end{array} \qquad \ldots (2.43)$$

where u_1, v_1, u_2 etc. are components of $\{d_1\}$, $\{d_2\}$....

An important observation is now made concerning the expression for potential energy given by eq. (2.42). The first term involving $[K^e]$ will have second-order terms in u_1, v_1, u_2 etc. (i.e., u_1u_2, u_1v_1, v_1u_2...) and the second term containing $\{f_{\varepsilon_0}^e\}$, $\{T\}$ will have only first-order terms in u_1, v_1, u_2.... Also the third term containing $\{\varepsilon_0^e\}$ etc. will not have u_1, v_1...; thus, it will appear as constant in so far as differential with respect to u_1, v_1 ... is concerned. On differentiation, second-order terms will change to first-order terms and first-order terms to the terms involving coefficients only as shown in Exer. 2.6(a). The complete set of differentials can be written in the following form:

$$\left\{\begin{array}{c} \dfrac{\partial\Pi}{\partial u_1} \\ \dfrac{\partial\Pi}{\partial v_1} \\ \dfrac{\partial\Pi}{\partial u_2} \\ \ldots \end{array}\right\} = 0 = \sum_{e=1}^{n}[K^e]\{d^e\} - \sum_{e=1}^{n}\left[\{f_{\varepsilon_0}^e\} + \{f_P^e\} + \{f_{\bar{w}}^e\}\right] - \{T\} \qquad \ldots (2.44)$$

If the elemental stiffness matrices $[K^e]$ and f-vectors are arranged in the form of corresponding overall stiffness matrix and f-vectors using the assembly procedure outlined in Chapter 1, the result will be the stiffness relation in the following form. (An alternative explanation of assembly procedure is given in Appendix 2.)

$$[K]\{\delta\} - \{f_{\varepsilon_0}\} - \{f_P\} - \{f_w\} - \{T\} = 0 \qquad \ldots (2.45)$$

where,

$$[K] = \sum_{e=1}^{n}[K^e] = \sum_{e=1}^{n}\int_{V^e}[B]^T[D][B]\,dV$$

$$\{f_{\varepsilon_0}\} = \sum_{e=1}^{n}\{f_{\varepsilon_0}^e\} = \sum_{e=1}^{n}\int_{V^e}[B]^T[D]\{\varepsilon_0\}\,dV$$

$$\{f_P\} = \sum_{e=1}^{n}\{f_P^e\} = \sum_{e=1}^{n}\int_{A^e}[N]^T\{P\}\,dA$$

$$\{f_w\} = \sum_{e=1}^{n}\{f_{\bar{w}}^e\} = \sum_{e=1}^{n}\int_{V^e}[N]^T\{W\}\,dV$$

$\{T\}$ is external concentrated load vector.

The summation here is vector sum as explained in Chapter 1. Equation (2.45) can be rewritten as

$$[K]\,\{\delta\} - \{f\} = 0 \qquad \text{... (2.46)}$$

This equation is solved for displacement $\{\delta\}$. In some cases when the object is restrained at some points so that the displacement is zero there, or some other regions are subjected to known displacement, these boundary conditions should be incorporated in eq. (2.46) before solving it. The method commonly used is to modify the stiffness matrix $[K]$ so that results satisfy automatically the constraints on displacement. The procedure for such matrix modification is explained in Chapter 3 where Gauss elimination procedure of solution is discussed.

For the case of a two-dimensional triangular element the volume integrals given in eq. (2.45) will be evaluated by integrating in x and y direction over the area of the triangle (thickness being constant). Since all the terms of $[B]$ and $[D]$ are constants with respect to x and y within the element (given by eqs. 2.19, 2.24 and 2.29), and also $\{\varepsilon_0\}$ is a vector of constant value within the element, these quantities can be taken out of the integral sign in expressions (2.45). The resultant expressions are

$$[K^e] = [B]^T\,[D]\,[B]\,t\int dx\,dy = [B]^T\,[D]\,[B]\,t\Delta$$

$$\{f^e_{\varepsilon_0}\} = [B]^T\,[D]\,\{\varepsilon_0\}\,t\int dx\,dy = [B]^T\,[D]\,\{\varepsilon_0\}\,t\Delta$$

$$\{f^e_p\} = t\int_{l^e} [N]^T\,\{P\}\,dl\ (l^e\text{, edge where load is acting}) \qquad \text{... (2.46a)}$$

$$\{f^e_w\} = t\int_{A^e} [N]^T\,\{W\}\,dx\,dy\ (A^e\text{, area of element})$$

Inversion of expression (2.46) yields the overall displacement vector, $\{\delta\}$. The components of elemental displacement vector (i.e., $\{d^e\}$) can be used to determine strains and subsequently elemental stresses from eqs.(2.18) and (2.30).

Exercise 2.7: Evaluate the terms of vector $\{f^e_w\}$ by integrating the expressions over the area of a triangle. Assume the intensity of body force W to be constant over the whole element with its values being W_x, W_y in the two directions. Write the final expression in terms of nodal coordinates x_1, y_1, x_2 ....

Hint :
$$\int_A x\,dA = \bar{x}A$$

where $\bar{x} = (x_1 + x_2 + x_3)/3$ is the coordinate of the centroid.

Exercise 2.8: Determine the expression for quantities stated in Exer. 2.7 by shifting the origin of coordinates to the centroid of the element. Check whether these remain the same. Hence, conclude that these calculations are more efficiently performed if the origin is located at the centroid of the element.

Exercise 2.9: If the vectors $\{d\phi\}$, $\{\phi\}$ and $\{A\}$ are defined as

$$\{d\phi\} = \begin{Bmatrix} d\phi_1 \\ d\phi_2 \\ d\phi_3 \end{Bmatrix},\ \{\phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix},\ \{A\} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}$$

where components of $\{A\}$ are constants while ϕ_1, ϕ_2 and ϕ_3 are variables, then show that the following integral relations are valid.

$$\int_0^{\phi_1} d\phi_1 A_1 + \int_0^{\phi_2} d\phi_2 A_2 + \int_0^{\phi_3} d\phi_3 A_3 = \int_0^{\{\phi\}} \{d\phi\}^T\,\{A\} = \{\phi\}^T\,\{A\}$$

and

$$\int_0^{\{\phi\}} \{d\phi\}^T\,[B]\,\{\phi\} = \frac{1}{2}\{\phi\}^T\,[B]\,\{\phi\} \qquad \text{... (a)}$$

where $[B]$ is a 3×3 matrix comprising constants. Note that the limits of integration in (a) should conform to corresponding components of $\{\phi\}$ vector.

Exercise 2.10: Use the results of Exer. 2.9 to determine the expression for elemental strain energy Π_{se} when initial stress is present (Sec. 2.1.4) and initial strain is absent.

Hint: The strain energy for an infinitesimal volume dV is given as

$$\left[\int_0^{\varepsilon_x} d\varepsilon_x\sigma_x + \int_0^{\varepsilon_y} d\varepsilon_y\sigma_y + \int_0^{\gamma_{xy}} d\gamma_{xy}\tau_{xy}\right] dV = \left[\int_0^{\{\varepsilon\}} \{d\varepsilon\}^T\,\{\sigma\}\right] dV$$

Hence, the elemental strain energy is

$$\Pi_{se} = \int_{V^e} \left[\int_0^{\{\varepsilon\}} \{d\varepsilon\}^T\,\{\sigma\}\right] dV \qquad \text{where } \{\sigma\} = \{\sigma_0\} + [D]\,\{\varepsilon\}$$

Thus show that the result (using relation (a)) is

$$\Pi_{se} = \{d^e\}^T \int_{V^e} [B]^T\,\{\sigma_0\}\,dV + \frac{1}{2}\{d^e\}^T \left[\int_{V^e} [B]^T\,[D]\,[B]\,dV\right]\{d^e\}$$

$$= \{d^e\}^T\,\{f^e_{\sigma_0}\} + \frac{1}{2}\{d^e\}^T\,[K^e]\,\{d^e\}$$

where $\{f^e_{\sigma_0}\}$ is identified as load vector due to initial stress, $\{\sigma_0\}$.

Overall stiffness relation: The matrix relation obtained in eqs. (2.44) and (2.45) gives the relation between displacement $\{\delta\}$ and various load vectors due to initial strain, distributed load, body force and external concentrated load. The effect of initial stress was not considered. Based on the results of Exer. 2.10 we can incorporate the effect of initial stress in the form of load vector $\{f_{\sigma_0}\}$ given as

$$\{f_{\sigma_0}\} = \sum_{e=1}^{n}\{f^e_{\sigma_0}\} = \sum_{e=1}^{n}\int_{V^e} [B]^T \{\sigma_0\}\, dV$$

where, σ_0 is the initial stress. The final relation now becomes

$$[K]\{\delta\} + \{f_{\sigma_0}\} - \{f_{\varepsilon_0}\} - \{f_P\} - \{f_w\} - \{T\} = 0 \qquad \ldots (2.47)$$

Although we have not considered initial stress and initial strain together in exer. 2.10, the result will be the same as given in eq. (2.47) even when both are considered together.

2.3 THREE-DIMENSIONAL ANALYSIS

The simplest of the three-dimensional elements is a tetrahedron, shown in Fig. 2.8. It has only four nodes and like the triangular element, formulation of the tetrahedron is based on linear variation of displacements within the element. However, unlike triangular elements, the division of a volume into tetrahedrons is not simple and naming the nodes in proper order is even more difficult. In comparison, division of a volume in brick-shaped elements, shown in Fig. 2.9 is very easy. The characteristics of a brick element can be obtained by subdividing it into several tetrahedra (Figs.

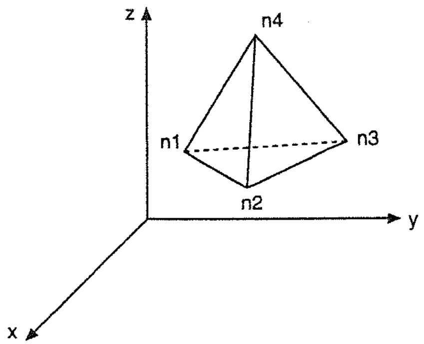


Fig. 2.8 Tetrahedron element

2.10 and 2.11) and then writing stiffness matrix etc. for this element by adding the stiffness matrices of these tetrahedra in a proper manner.

Considering a tetrahedron n_1-n_2-n_3-n_4 (Fig. 2.8), the nodal displacements can be represented as $u_1, v_1, w_1, u_2, v_2, w_2$ etc. in x, y and z directions at nodes n_1, n_2... respectively. Since a linear variation of displacement is assumed within the element, the general expressions for displacement within the tetrahedron element is

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \\ v &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 z \\ w &= \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z \end{aligned} \qquad \ldots (2.48)$$

On substituting the nodal values in these expressions we get the following for u-displacement, with similar sets for v and w.

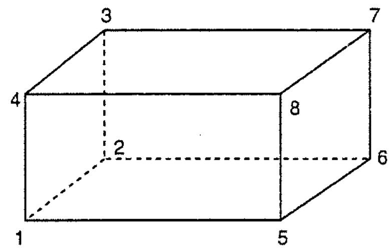


Fig. 2.9 Brick element

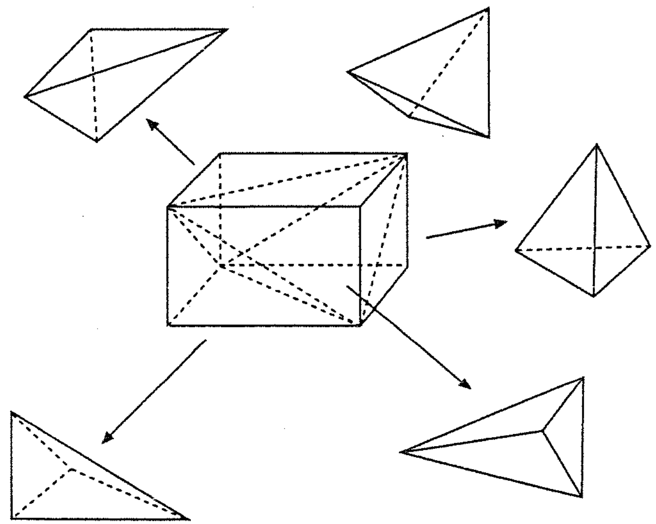


Fig. 2.10 Division of a brick element into 5 tetrahedron elements.

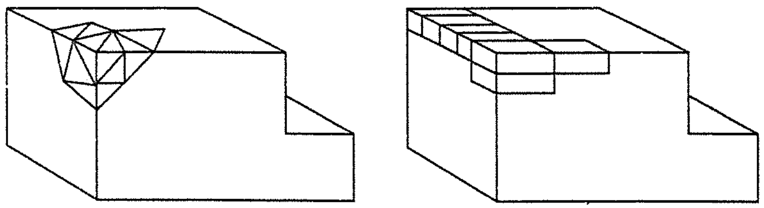


Fig. 2.11 Division of a volume into tetrahedron and brick elements.

$$
\begin{aligned}
u_1 &= \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 z_1\\
u_2 &= \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 z_2\\
u_3 &= \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 z_3\\
u_4 &= \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 z_4
\end{aligned}
\quad \text{... (2.49)}
$$

where x_1, y_1 and z_1 are coordinates of node $n1$ with similar values for nodes $n2$, $n3$ and $n4$. The matrix form of this set of equations is

$$
\begin{Bmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1\\ 1 & x_2 & y_2 & z_2\\ 1 & x_3 & y_3 & z_3\\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \begin{Bmatrix} \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4 \end{Bmatrix} \quad \text{... (2.50)}
$$

The inversion of this gives the following values of coefficients α_1, α_2, α_3 and α_4.

$$
\begin{Bmatrix} \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\\ c_1 & c_2 & c_3 & c_4\\ e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{Bmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{Bmatrix} \quad \text{... (2.51)}
$$

The matrix shown above is the transpose of cofactors-matrix of the matrix given in expression (2.50) and '$6V$' gives the determinant of this. (This follows the usual procedure of matrix inversion.) 'V' also represents the volume of the tetrahedron as shown in Appendix 1. Thus

$$
V = \frac{1}{6} \begin{bmatrix} 1 & x_1 & y_1 & z_1\\ 1 & x_2 & y_2 & z_2\\ 1 & x_3 & y_3 & z_3\\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \quad \text{... (2.52)}
$$

$$
a_1 = \begin{vmatrix} x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\\ x_4 & y_4 & z_4 \end{vmatrix}; a_2 = -\begin{vmatrix} x_1 & y_1 & z_1\\ x_3 & y_3 & z_3\\ x_4 & y_4 & z_4 \end{vmatrix} = \begin{vmatrix} x_3 & y_3 & z_3\\ x_1 & y_1 & z_1\\ x_4 & y_4 & z_4 \end{vmatrix}
$$

[by exchanging rows, which results in change of sign]

$$
a_3 = \begin{vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_4 & y_4 & z_4 \end{vmatrix}; a_4 = -\begin{vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 & z_2\\ x_1 & y_1 & z_1\\ x_3 & y_3 & z_3 \end{vmatrix} \quad \text{... (2.53)}
$$

[by exchanging rows, which results in change of sign]

Similarly, expressions for b_1, c_1, e_1 etc. will be

$$
b_1 = -\begin{vmatrix} 1 & y_2 & z_2\\ 1 & y_3 & z_3\\ 1 & y_4 & z_4 \end{vmatrix}; c_1 = \begin{vmatrix} 1 & x_2 & z_2\\ 1 & x_3 & z_3\\ 1 & x_4 & z_4 \end{vmatrix}; e_1 = -\begin{vmatrix} 1 & x_2 & y_2\\ 1 & x_3 & y_3\\ 1 & x_4 & y_4 \end{vmatrix} \quad \text{... (2.54)}
$$

b_2, c_2, e_2 etc. can be obtained by changing the suffixes in the same manner as in (2.53). A clear and systematic procedure of ordering the nodes and of writing these expression is given in the next section.

2.3.1 Numbering Sequence for Nodes of Elements

Since expression (2.52) also represents the volume of the element and will be used in place of the volume, it is essential that the nodal sequence in the element be written properly so that the volume is positive. All the more so because any exchange in rows (i.e., change in sequence of suffixes) will result in the value of the determinant so represented becoming negative (a property of determinants). The tetrahedron shown in Fig. 2.8 has nodes numbered as $n1$, $n2$, $n3$ and $n4$. The coordinate system used is the *Right-hand Cartesian coordinate system* in which the sequence x-y-z is such that if a right-hand screw is rotated in the direction from x to y it moves axially in the direction of z-axis or alternatively y to z rotation results in screw movement in x-direction and z to x rotation results in right-hand screw movement in y direction. It should be understood that the clockwise, anticlockwise sequence, discussed here, will apply to this coordinate system only (the other system is the Left-hand coordinate system).

A look at Fig. 2.8 and expression (2.52) for the volume of tetrahedron reveals that if we look at the plane $n1$-$n2$-$n3$ from node number $n4$, the sequence 1-2-3 is *anticlockwise* in direction. Thus, if we keep number 4 (corresponding to the node from which we are viewing the plane $n1$-$n2$-$n3$) at the end of above sequence, it becomes 1-2-3-4 which is the sequence of suffixes used in the rows of expression (2.52) representing the volume. It is shown in Appendix 1 that the above ordering sequence gives a positive value of volume represented by (2.52). Again, if we rewrite expression (2.52) by exchanging the rows and use the property of determinant whereby the sign of its value gets changed by every exchange of rows or columns, we get

$$V = -\frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_4 & y_4 & z_4 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_4 & y_4 & z_4 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix} \quad \ldots (2.55)$$

A similar double exchange of rows will result in

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \\ 1 & x_2 & y_2 & z_2 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & x_2 & y_2 & z_2 \\ 1 & x_4 & y_4 & z_4 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_1 & y_1 & z_1 \end{vmatrix} \quad \ldots (2.56)$$

The expressions for V given in (2.55) and (2.56) are consistent with the above-mentioned ordering sequence. If we look from node $n3$, the anticlockwise direction of the opposite plane is 1-4-2. Hence, the overall sequence should be 1-4-2-3, which is the one given in eq. (2.55). Similarly, looking from nodes $n2$ and $n1$ respectively gives the overall sequences 1-3-4-2 and 2-4-3-1 consistent with expression (2.56). Here it should be understood that several other numbering sequences are also valid, such as looking from node $n3$ gives sequence 1-4-2-3 as above. But 2-1-4-3 or 4-2-1-3 are equally valid for this configuration.

2.3.2 Shape Function

Coming back to the derivation of expressions for nodal displacements (2.49, 2.50 and 2.51), we can resubstitute the values of $\alpha_1, \alpha_2, \alpha_3 \ldots$ in (2.48) to get the expression for u as

$$\begin{aligned} u &= \frac{1}{6V}[a_1u_1+a_2u_2+a_3u_3+a_4u_4+b_1u_1x+b_2u_2x \\ &\quad + b_3u_3x + b_4u_4x + c_1u_1y + \ldots + e_1u_1z + \ldots] \\ &= N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 \end{aligned} \quad \ldots (2.57)$$

where N_1 represents $(a_1+b_1x+c_1y+e_1z)/6V$ and N_2, N_3, N_4 are represented by similar expressions obtained by changing the suffixes. It is not difficult to see that an operation similar to the one represented by expressions (2.49) to (2.51) can be carried out to determine $a_5, \ldots a_8$ in the expression for v and $\alpha_9, \ldots \alpha_{12}$ in the expression for w. This yields the following expression.

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

$$w = N_1w_1 + N_2w_2 + N_3w_3 + N_4w_4 \quad \ldots (2.58)$$

Combining we get

$$\{d\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \cdot \\ \cdot \\ \cdot \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

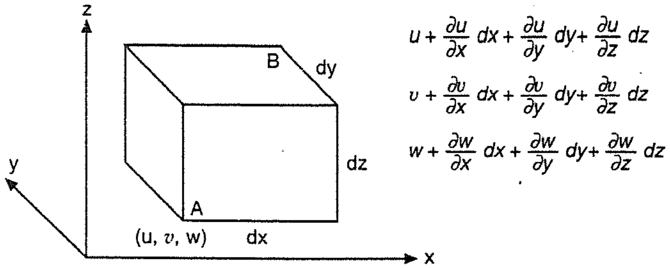
or,
$$\{d\} = [N]\,\{d^e\} \quad \ldots (2.59)$$

This is the equation defining shape function $[N]$; $\{d\}$ and $\{d^e\}$ are displacement vectors representing the general displacement and nodal displacements respectively within the element.

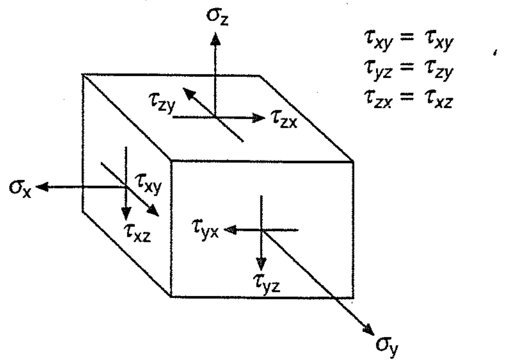
2.3.3 Strain Displacement Relation

There are, in general, six components of stress and strain in three dimensions. The stress components are shown in Fig. 2.12(b) and the corresponding six strain components can be interpreted from this. The displacements at the two extreme corners of the cube are also shown in this Figure. Analyzing it in a manner similar to Sec. 2.2.1 where two-dimensional strains were discussed, the expressions for various strain components in three dimensions will be obtained as

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned} \quad \text{or,} \quad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad \ldots (2.60)$$



(a) Displacement at two extreme corners



(b) Stress components

Fig. 2.12 Displacement and stresses in a cubic element

or,

$$\{\varepsilon\} = [L]\,\{d\} \qquad \text{... (2.61)}$$

Here $[L]$ is the operator matrix given in (2.60) and $\{d\}$ is the general displacement vector.

Reconsidering expression (2.61) and substituting for $\{d\}$ from (2.59)

$$\{\varepsilon\} = [L]\,[N]\,\{d^e\}$$

$$\{\varepsilon\} = [B]\,\{d^e\} \qquad \text{... (2.62)}$$

Here again, since nodal vector $\{d^e\}$ does not depend on x, y or z, the matrix $[B]$ is obtained simply by differentiating the terms of shape function $[N]$ as per the operator matrix $[L]$. Carrying out differentiation term by term the strain displacement matrix $[B]$ is obtained.

$$[B] = \frac{1}{6V}\begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & 0 \\ 0 & 0 & e_1 & 0 & 0 & e_2 & 0 & 0 & e_3 & 0 & 0 & e_4 \\ c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 & 0 \\ 0 & e_1 & c_1 & 0 & e_2 & c_2 & 0 & e_3 & c_3 & 0 & e_4 & c_4 \\ e_1 & 0 & b_1 & e_2 & 0 & b_2 & e_3 & 0 & b_3 & e_4 & 0 & b_4 \end{bmatrix} \qquad \text{... (2.63)}$$

2.3.4 Stress-Strain Relation

The six stress components are related to corresponding strains as follows.

$$\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \varepsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \tau_{xy}\cdot\frac{2(1+\nu)}{E} \\ \gamma_{yz} &= \tau_{yz}\cdot\frac{2(1+\nu)}{E} \\ \gamma_{zx} &= \tau_{zx}\cdot\frac{2(1+\nu)}{E} \end{aligned} \qquad \text{... (2.64)}$$

In these E, G and ν are Young's modulus, shear modulus and Poisson ratio respectively. Considering the first three of these equations and writing in matrix form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} \qquad \text{... (2.65)}$$

On inversion it is easy to see that

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} \qquad \text{... (2.66)}$$

Combining this with the other three equations of (2.64) gives the complete elasticity matrix, i.e.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

or,

$$(\sigma) = [D]\,\{\varepsilon\} \qquad \text{... (2.67)}$$

The relation assumes, however, that the strain vector $\{\varepsilon\}$ contains only elastic strains and that initial strains (such as thermal strains) etc. are absent. If initial strains and initial stresses are also present, as explained in Sec. 2.1, this assumes the form

$$\{\sigma\} = [D]\,(\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\} \qquad \text{... (2.67a)}$$

2.3.5 Solution

The solution procedure now involves writing the expression for potential energy of the system consisting of external concentrated or distributed loads and body forces. The potential energy is then minimized over the whole domain with respect to nodal displacements exactly in the same manner as explained for 2D analysis given in Sec. 2.2.3. Finally, the stiffness relation will be obtained in the form

$$[K]\,\{\delta\} + \{f_{\sigma_0}\} - \{f_{\varepsilon_0}\} - \{f_P\} - \{f_w\} - \{T\} = 0 \qquad \text{... (2.68)}$$

The expressions for stiffness matrix $[K]$ and various 'f' vectors, $\{f_{\sigma_0}\}$, $\{f_{\varepsilon_0}\}$, $\{f_P\}$, $\{f_w\}$ and $\{T\}$, which correspond to initial stress, initial strain, distributed load, body force and external concentrated loads respectively, are given in eqs. (2.45) and (2.47). The final expression is written in a compact form as

$$[K]\,\{\delta\} - \{f\} = 0$$

Matrix inversion is carried out to determine displacements $\{\delta\}$, which can be used in various ways to determine strains and stresses using expressions (2.62, 2.63, 2.67 and 2.67a).

Exercise 2.11: Determine the terms of load vector $\{f_P^e\}$ due to distributed load acting at an exterior boundary (S^e) of a tetrahedron element. Assume that the load is distributed uniformly. Express its first term in a closed form after performing the required integral.

Obtain the expression (i) when the origin of coordinates is located at the centroid of the boundary face and (ii) when the origin lies outside this face.

The load vector due to distributed load is given as

$$\{f_P^e\} = \int_{S^e} [N]^T \{P\}\, dS$$

Solution

The load vector can be written in component form as

$$\{f_P^e\} = \begin{Bmatrix} \int_{S^e} N_1 P_x\, dS \\ \int_{S^e} N_1 P_y\, dS \\ \int_{S^e} N_1 P_z\, dS \\ \vdots \\ \int_{S^e} N_4 P_z\, dS \end{Bmatrix}$$

For illustration, the first term is given as

$$\int_{S^e} N_1 P_x\, dS = \int_{S^e} \frac{(a_1 + b_1 x + c_1 y + e_1 z)}{6V} P_x\, dS$$

where S^e is the area of boundary face, assumed to be face 1-2-3 in Fig. 2.13.

Two approaches are possible. In the first approach we can take the origin of the coordinate system to be located at the centroid of the triangle 1-2-3 and in the second it could be a general origin. In both approaches we shall assume the distribution of force $\{P\}$ to be uniform over the element. Thus P_x will be a constant.

First approach: The integral given above can be expressed as

$$I = \left(\frac{a_1}{6V} P_x\right)\int_{S^e} dS + \left(\frac{b_1}{6V} P_x\right)\int_{S^e} x\,dS + \left(\frac{c_1}{6V} P_x\right)\int_{S^e} y\,dS + \left(\frac{e_1}{6V} P_x\right)\int_{S^e} z\,dS$$

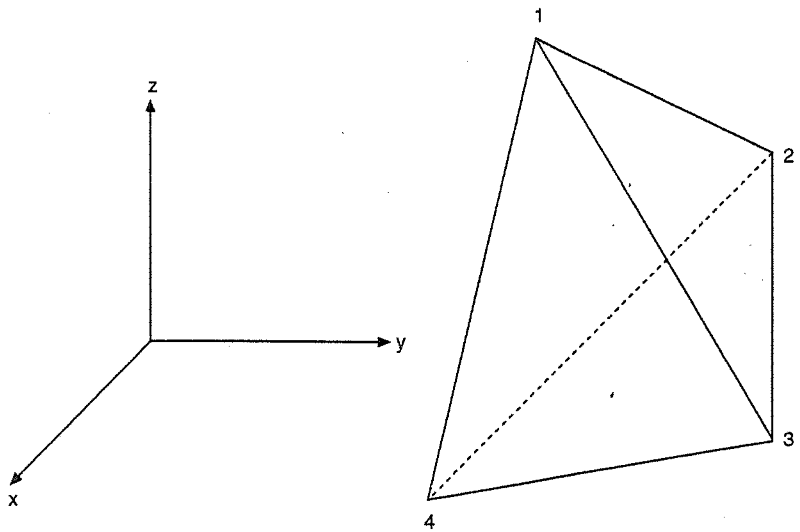


Fig. 2.13 Exercise 2.11

The last three integrals when taken around the centroid of the triangle are zero. [Project area of triangle on y-z plane and take moment about z-axis etc.]

So,
$$I = \frac{a_1}{6V} P_x S^e \qquad \text{... (a)}$$

Now,
$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}$$

Also,
$$6V = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

Adding the 2nd and 3rd rows to the 1st row

$$6V = \begin{vmatrix} 3 & x_1+x_2+x_3 & y_1+y_2+y_3 & z_1+z_2+z_3 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

Since, $\bar{x} = (x_1 + x_2 + x_3)/3 = 0; \bar{y} = 0; \bar{z} = 0$ [origin at centroid of Δ 1-2-3], so

$$6V = \begin{vmatrix} 3 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = 3 \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 3a_1$$

Thus,
$$I = \frac{P_x S^e}{3} \qquad \text{... (b)}$$

Second approach: The integral will now be expressed in terms of centroidal coordinates $\bar{x}, \bar{y}, \bar{z}$ or,

$$I = \frac{a_1 P_x S^e}{6V} + \frac{b_1 P_x \bar{x} S^e}{6V} + \frac{c_1 P_x \bar{y} S^e}{6V} + \frac{e_1 P_x \bar{z} S^e}{6V}$$

$$= \frac{P_x S^e}{6V} [a_1 + b_1\bar{x} + c_1\bar{y} + e_1\bar{z}]$$

$$= \frac{P_x S^e}{6V}\frac{1}{3}[3a_1 + b_1(x_1+x_2+x_3) + c_1(y_1+y_2+y_3) + e_1(z_1+z_2+z_3)]$$

$$= \frac{P_x S^e}{6V}\frac{1}{3}[a_1 + b_1x_1 + c_1y_1 + e_1z_1 + a_1 + b_1x_2 + c_1y_2 + e_1z_2$$
$$+ a_1 + b_1x_3 + c_1y_3 + e_1z_3]$$

$$= \frac{P_x S^e}{3}[N_{1(\text{at node 1})} + N_{1(\text{at node 2})} + N_{1(\text{at node 3})}]$$

$$= \frac{P_x S^e}{3}$$

2.4 AXI-SYMMETRIC ANALYSIS

A large number of engineering problems fall in a special category of 3D analysis where the shape of the object and the nature of loading are both symmetrical about an axis. Typical examples are a cylindrical storage tank for liquid, pressurized gas cylinder, extrusion of rods and wires, cooling of continuous-cast billet in steel plant etc. This symmetry about an axis can be exploited to convert such a 3D analysis into a specialized form of 2D analysis. This results in large saving in computational time.

An axi-symmetric element is a ring of uniform cross-section with its axis coinciding with the z-axis of the r, θ, z cylindrical coordinate system. It is shown as a ring of triangular cross-section in Fig. 2.14. The object can have any configuration; the only limitation is that it should be a solid of revolution. This means that the object is obtained by revolving a 2D plate of any shape about an axis (z-axis here). It goes without saying that in order to exploit the axial symmetry the loading should also be symmetrical about the same axis. The load should be uniformly distributed over a ring around the axis as shown in Fig. 2.14.

Under these conditions the displacement, strains and stresses etc. over any cross-section of a ring-shaped element will be identical and analysis of the stress state over a cross-section of the element will be sufficient to determine the state of stress anywhere within the element. Hence, determination of stresses over a plane which forms the template for a solid of revolution will suffice for complete analysis. In this sense the analysis becomes similar to 2D analysis. A significant difference from the 2D plate-type element is that a thin slice of the ring-shaped element discussed here will not have uniform thickness as in the case of a 2D plane

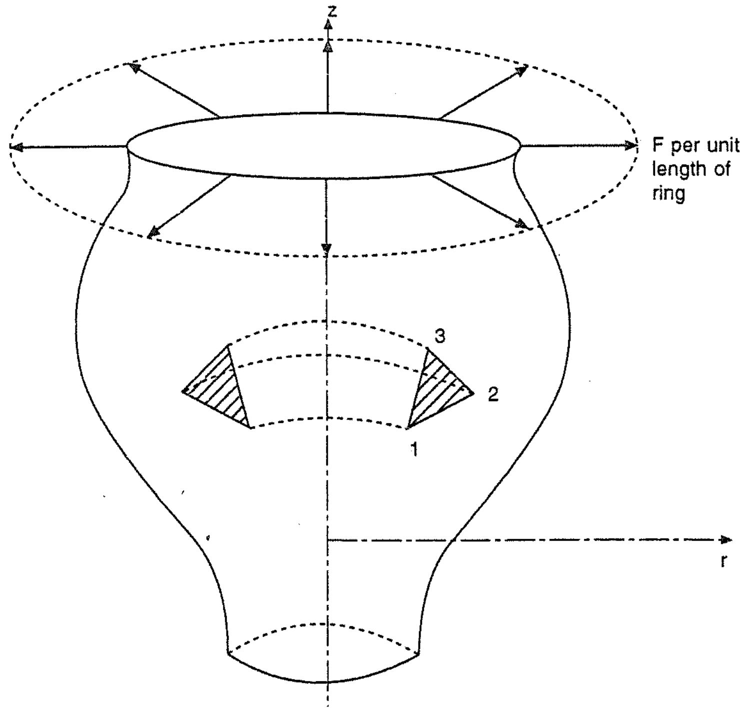


Fig. 2.14 An axi-symmetric element

stress or plane strain element. A slice of an axi-symmetric element subtending an angle $d\theta$ about z-axis will have decreasing thickness tending towards zero as z-axis is approached (Fig. 2.15). This feature needs to be properly considered in the analysis.

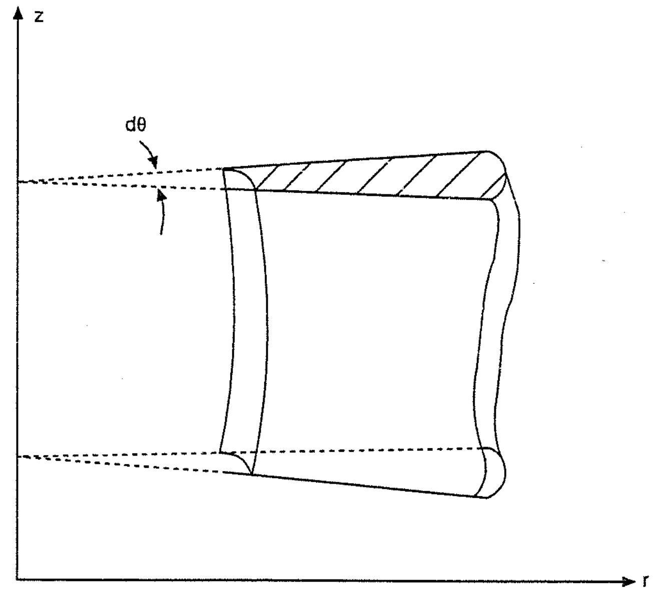


Fig. 2.15 A typical slice of ring-shaped axi-symmetric element.

2.4.1 Shape Function

Ring-shaped axi-symmetric elements with triangular cross-section will appear as triangles in the r-z plane. The displacements and forces in r and z directions are represented as u, v, F_r and F_z respectively and suffixes 1, 2, 3 can be added to these quantities to represent the nodal values. These are shown in Fig. 2.16. F_r and F_z signify the intensities of forces, meaning the force per unit length of the element edge passing through the node under consideration. The nomenclature being similar to the usual 2D triangular element, the procedure of Sec. 2.2 can be used to express displacements as

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

$$v = N_1v_1 + N_2v_2 + N_3v_3$$

or,

$$\{d\} = [N]\{d^e\} \qquad \dots (2.69)$$

Here $\{d\}$ is the displacement vector while $[N]$ and $\{d^e\}$ are shape function

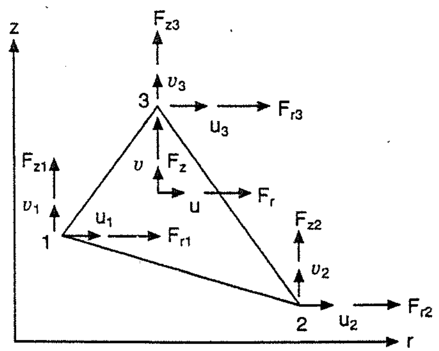


Fig. 2.16 Nomenclature of an axi-symmetric element

and nodal displacement vectors respectively. The components of shape function, N_1, N_2 and N_3 are given by

$$N_1 = \frac{1}{2\Delta}(a_1 + b_1 r + c_1 z)$$

$$N_2 = \frac{1}{2\Delta}(a_2 + b_2 r + c_2 z) \qquad \text{... (2.70)}$$

$$N_3 = \frac{1}{2\Delta}(a_3 + b_3 r + c_3 z)$$

where Δ represents the area of the triangle and a_1, a_2, b_1 etc. can be obtained by replacing x and y by r and z respectively in expressions (2.4b).

2.4.2 Strain Displacement Relation

Although there are six distinct strain components in a 3D situation and three components in a 2D case, the axi-symmetric analysis will not involve all six strain components. Some of these will be absent in order to satisfy the special configuration of axi-symmetric analysis. It can be shown that only four non-zero strain components and corresponding stresses will be present.

Referring to Fig. 2.17 all the components of strain acting on an elemental solid in an axi-symmetric case are shown at (i). At (ii) and (iii) the consequences of the presence of $\gamma_{r\theta}$ and $\gamma_{z\theta}$ are shown pictorially. It is obvious that distortion of an elemental solid under the action of these strains will lead to opening out of the ring in the form of a spiral in the case of $\gamma_{r\theta}$ (Fig. 2.17(ii)) and twisting of it in the form of a helical spring in the case of $\gamma_{z\theta}$ (Fig. 2.17(iii)). Thus, the axi-symmetric shape of a ring is not retained in the presence of these two strains; an axi-symmetric configuration retains its axial symmetry only if these two strain components are absent. A look at the other four strain components reveals that distortion due to ε_z, ε_r and γ_{rz} will be symmetric about the axis, leading to no loss of axial symmetry (Fig. 2.17(iv)). It should be noted that this puts no

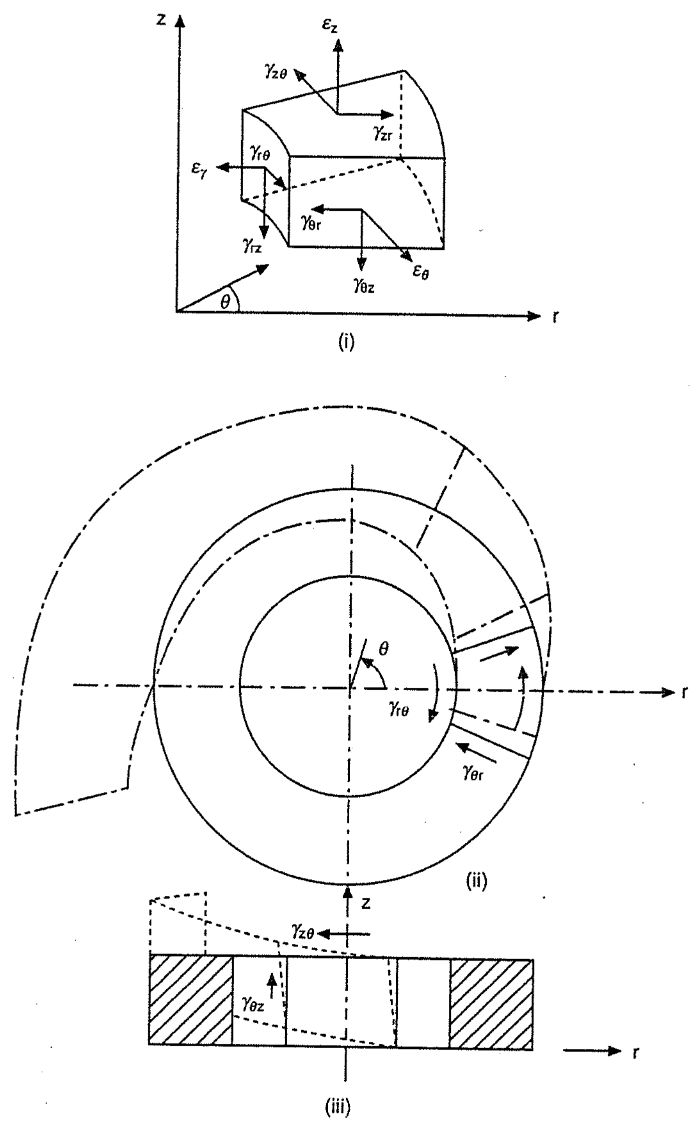


Fig. 2.17 Various components of strain in axi-symmetric case (presence of $\gamma_{r\theta}$ and $\gamma_{z\theta}$ in Figs. ii and iii result in loss of symmetry).

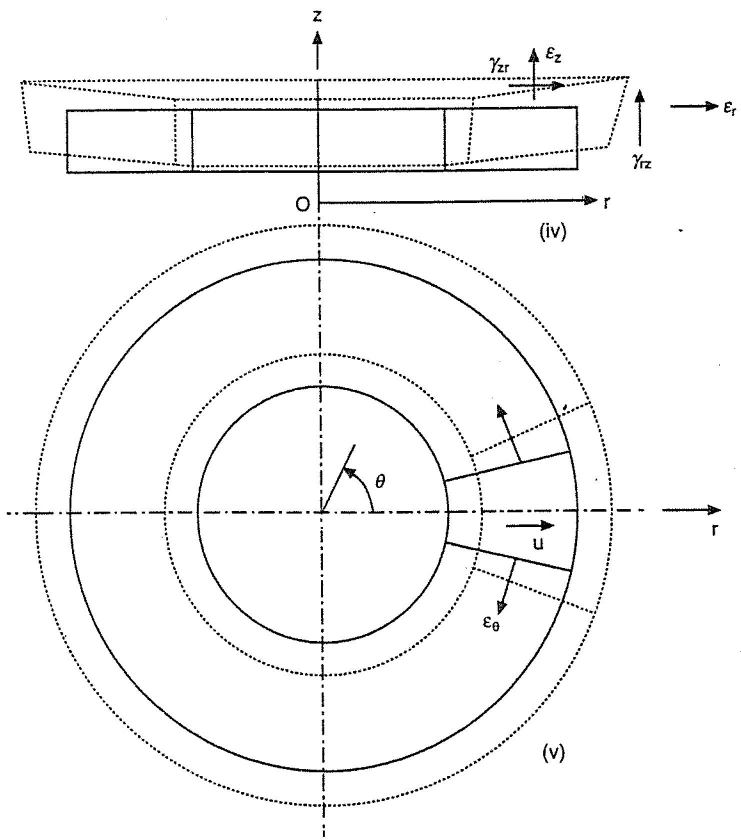


Fig. 2.17 Various components of strain in axi-symmetric case.

restriction on the magnitude and distribution of ε_z, ε_r and γ_{rz} over the cross-section of the object in r or z direction. However, axial symmetry demands that there be no change in stress and strain values in θ direction. Fig. 2.17(v) shows the consequence of the presence of ε_θ which is uniform along the entire circumference of the ring. The presence of ε_θ will result in general increase in diameter of the ring, which means that this strain should be related to outward movement of the element in r direction (displacement u). It can be seen that 'u' displacement results in an increase in circumference from $2\pi r$ to $2\pi(r + u)$. Hence, circumferential strain should be given by $\varepsilon_\theta = \{2\pi(r + u) - 2\pi r\}/2\pi r = u/r$. The three other non-zero components of strain ε_z, ε_r and γ_{rz} can be obtained by following the procedure of Sec. 2.2. Thus, we write the four components of the strain vector as

$$\begin{Bmatrix} \varepsilon_z \\ \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial r} \end{Bmatrix} \qquad \text{... (2.71)}$$

or,

$$\{\varepsilon\} = \begin{Bmatrix} 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial r} & 0 \\ \dfrac{1}{r} & 0 \\ \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial r} \end{Bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \text{... (2.72)}$$

Here the operator matrix is again denoted by $[L]$. On substituting for displacement in terms of nodal displacement vector we obtain

$$\{\varepsilon\} = [L]\,[N]\,\{d^e\} = [B]\,\{d^e\} \qquad \text{... (2.73)}$$

Components of matrix $[B]$ can be obtained by substituting for $[N]$ from (2.11) and (2.70). Thus

$$[B] = \frac{1}{2\Delta}\begin{bmatrix} 0 & c_1 & \\ b_1 & 0 & \\ \dfrac{a_1}{r} + b_1 + \dfrac{c_1 z}{r} & 0 & \text{Plus other terms obtained on replacing suffixes} \\ c_1 & b_1 & \end{bmatrix} \qquad \text{... (2.74)}$$

A significant difference between $[B]$ matrix for an axi-symmetric case and that for 2D analysis is the presence of r and z in the terms of this matrix. This means that $[B]$ matrix does not behave as a matrix of constants over the whole area of the element. Furthermore, the order of the matrix also differs.

2.4.3 Stress-Strain Relation

As the state of stress in an axi-symmetric case is a special case of 3D stress distribution, the stress-strain relation can readily be written by disregarding those components of strain and stress which are absent. Thus, the elasticity matrix for a 3D case (eq. 2.67) can be specialized to the axi-symmetric situation as

$$\begin{Bmatrix} \sigma_z \\ \sigma_r \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_z \\ \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \quad \dots (2.75)$$

or,

$$\{\sigma\} = [D]\{\varepsilon\}$$

The other strain components have been shown to be absent and consequently the corresponding stresses will also be zero. If initial stress and strains are present these can be incorporated in a manner similar to that explained in Sec. 2.21 and we get

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$$

where $\{\varepsilon_0\}$ and $\{\sigma_0\}$ are four component initial strain and initial stress vectors.

2.4.4 Solution

The procedure of Sec. 2.2.3 now applies fully and minimization of potential energy will give the final set of matrix equations as

$$[K]\{\delta\} + \{f_{\sigma_0}\} - \{f_{\varepsilon_0}\} - \{f_P\} - \{f_w\} - [T] = 0 \quad \dots (2.76)$$

The various terms of this equation were given earlier during explanation of eq. (2.45). Individual terms are represented as vector sum of respective elemental quantities. Thus

$$[K] = \sum_{e=1}^{n} [K^e] = \sum_{e=1}^{n} \int_V [B]^T [D][B]\, dV = \sum_{e=1}^{n} \int_V [B]^T [D][B]\, r dr dz$$

where the volume integral is taken over a slice of the element having an angular span $d\theta$ equal to 1 radian (Fig. 2.15); Δ represents the cross-sectional area of the element. Similarly, on integrating other terms over the same slice of span 1 radian, we obtain

$$\{f_{\sigma 0}\} = \sum_{e=1}^{n} \{f_{\sigma_0}^e\} = \sum_{e=1}^{n} \int_{A^e} [B]^T \{\sigma_0\}\, r\, dr\, dz$$

$$\{f_{\varepsilon 0}\} = \sum_{e=1}^{n} \{f_{\varepsilon_0}^e\} = \sum_{e=1}^{n} \int_{A^e} [B]^T [D]\{\varepsilon_0\}\, r\, dr\, dz$$

$$\{f_P\} = \sum_{e=1}^{m} \{f_P^e\} = \sum_{e=1}^{m} \int_{l^e} [N]^T \{P\}\, r\, dl \quad \dots (2.77)$$

$$\{f_w\} = \sum_{e=1}^{n} \{f_w^e\} = \sum_{e=1}^{n} \int_{A^e} [N]^T \{W\}\, r\, dr\, dz$$

where m is the number of elements located at the boundary and $\{T\}$ represents the vector of external concentrated loads but contrary to a two- or three-dimensional case the components of this vector represent load distributed along a circular ring-shaped edge of the element passing through the node 'A' under consideration, shown in Fig. 2.14 as a force of magnitude F per unit length of ring. This means that when we consider a slice of angular span $(d\theta)$ of one radian the forces acting at node A are given by

$$(T_r)_A = F_r r_A\, d\theta = r_A F_r \quad (\text{for } d\theta = 1)$$

and,

$$(T_z)_A = F_z r_A\, d\theta = r_A F_z \quad (2.77a)$$

or,

$$\{T\} = \{rF\}$$

where components of $\{T\}$ at node $A = r_A \{F\}_A = r_A \begin{Bmatrix} F_r \\ F_z \end{Bmatrix}_A$

The distributed load $\{P\}$ forming the component of vector $\{f_P\}$, acts on the external surface of the axi-symmetric solid and represents the load acting on the unit area formed by unit length of the edge ds of the generating cross-section of the axi-symmetric solid in one direction (Fig. 2.14) and the unit length considered along the circumference of the solid passing through the middle of this edge. W, as usual, represents the body force per unit volume of the solid. All the vectors of eq. (2.76) can be combined to give the matrix equation

$$[K]\{\delta\} - \{f\} = 0$$

The assembly procedure explained in Sec. 1.4.2 will be used to form the overall stiffness matrix $[K]$ and vector $\{f\}$ in the above equation. The solution will follow the usual matrix inversion procedure given in a later chapter and nodal displacement vector $\{\delta\}$ obtained in this manner will be used to calculate stresses etc.

2.4.5 Nature of Expressions

The nature of expressions (2.77) differs from similar expressions for two- or three-dimensional case (eq. 2.45) in that the integrand in those expressions contained quantities which did not vary over area of cross-section of the triangular element or over volume of the tetrahedron element. Hence, integration was very easy as given in eq. (2.46a). In an axi-symmetric case the integrands (eq. 2.77) are not constants but 'r' appears as a direct component of it while terms of matrix $[B]$ also depend on 'r' and 'z'. These elemental vectors $\{f_{\sigma_0}^e\}$ etc. are to be obtained by integrating each term separately. In general, such integration may not be easy and instead of attempting a closed form integration, numerical integration is used.

2.4.6 Numerical Integration

Figure 2.18 illustrates the principle of numerical integration which can be carried out for a line, an area or a volume. The illustration shows its application to a triangular area. Integration of a function $F(x, y)$, defined over the whole area, can be expressed as

$$I = \int_{\Delta} F\,dx\,dy \qquad \text{... (2.78)}$$

$$= \Delta \sum_{i=1}^{n} W_i F_i$$

where F_i represents the value of the integrand F at several integration points spread over the domain. In the present case P, Q and R are the three integration points located mid-sides of the triangle and F_i has three different values ($n = 3$) at these points, with W_i representing the weightage corresponding to these integration points. The integral for the triangular element given by eq. (2.78) is thus the weighted sum of the function at the integration points, with the total having been multiplied by the area of the triangle. The weightage for this special set of integration points are 1/3 each. For functions which vary widely over the area of the element we may use more integration points for better accuracy.

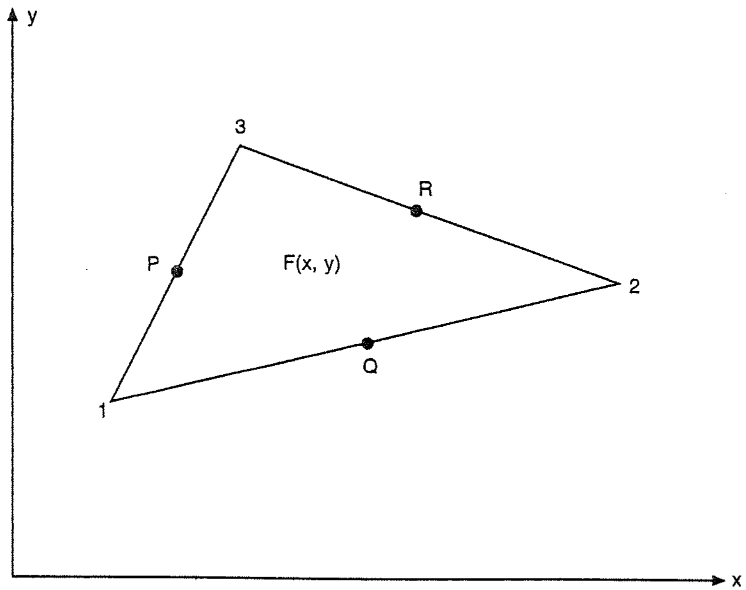


Fig. 2.18 Numerical integration for triangle with 3 integration points

More details of numerical integration are discussed in Chapter 7. At this stage it is useful to recognize that numerical integration provides an easy, unambiguous and accurate means of calculating the terms of various vectors and matrices used in finite element analysis.

Exercise 2.12: A thick cylindrical pressure vessel, shown in Fig. 2.19(a), is subjected to internal pressure p. The vessel is discretized into axi-symmetric triangular elements, one of which is shown as 1-2-3 in the hemispherical end-cap portion of the vessel. The edge 1-2 represents the surface of the element which is subjected to internal pressure.

(i) Assuming that the two nodes 1 and 2 subtend angles θ_1 and θ_2 at the centre of the sphere, derive the expression for the first term of load vector $\{f_p^e\}$ for this element. The expression should be sufficiently general so as to apply to any such element located in the hemispherical cap (say, written in terms of angle θ and integrated through incremental angle $d\theta$). Assume $\theta_1 = 45°$ and $\theta_2 = 30°$ and obtain a numerical expression for this integral.

(ii) Next, develop a similar expression for the end-cap shown in Fig. 2.19(b). Compare these two expressions and explore the possibility of writing a general expression applicable to both types of end-caps.

(iii) Use the procedure of numerical integration (Chapter 7) to evaluate

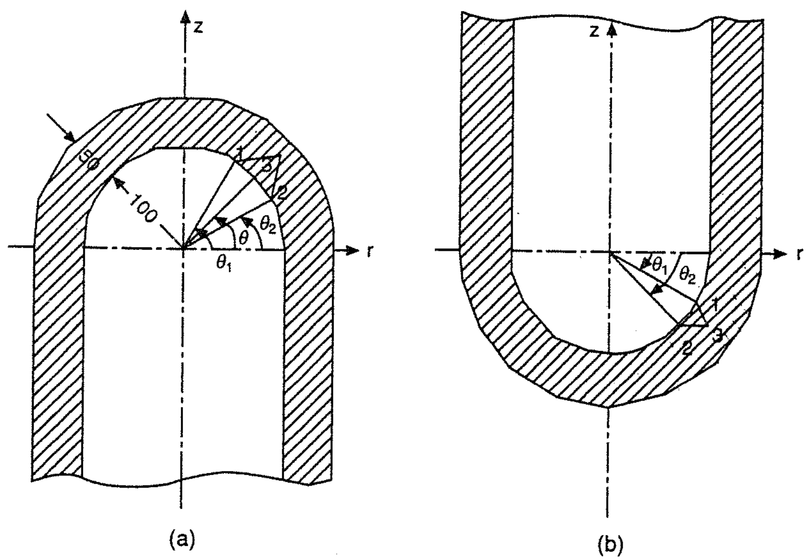


Fig. 2.19 Exercise 2.12

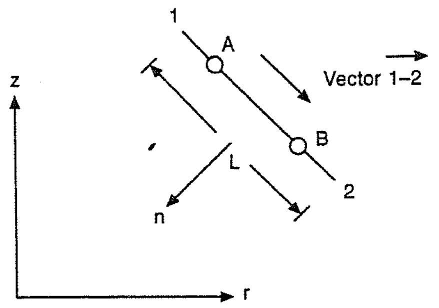


Fig. 2.19(c) Exercise 2.12

this term and check whether this has general applicability to both types of end-caps.

Solutions

(i) The expression for distributed load vector is

$$\{f_P^e\} = \int_{l^e} [N]^T \{P\}\, r\, dl$$

where the integral is taken over the length of the exterior edge of the element (say, 1-2). On expanding it in components, we obtain

$$\{f_P^e\} = \int_{l^e} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}^T \begin{Bmatrix} P_r \\ P_z \end{Bmatrix} r\, dl = \begin{Bmatrix} \int_{l^e} N_1 P_r\, r\, dl \\ \int_{l^e} N_1 P_z\, r\, dl \\ \cdot \\ \cdot \\ \int_{l^e} N_3 P_z\, r\, dl \end{Bmatrix}$$

The first term is expressed as

$$I_1 = \int_{l^e} \frac{1}{2\Delta} (a_1 + b_1 r + c_1 z)\, P_r r\, dl \qquad \text{... (a)}$$

Here r and z represent the coordinates of a point along edge 1-2. If R is taken as the radius of the hemispherical cap, the integral I_1, can be expressed in terms of angle θ as

$$I_1 = \int_{\theta_2}^{\theta_1} \frac{1}{2\Delta} (a_1 + b_1 r + c_1 z)\, P_r r R\, d\theta$$

Also substituting $r = R\cos\theta$, $z = R\sin\theta$ and $P_r = p\cos\theta$ and taking $R = 100$, we obtain

$$I_1 = \int_{\theta_2}^{\theta_1} \frac{1}{2\Delta} (a_1 + 100\, b_1 \cos\theta + 100\, c_1 \sin\theta)\, p \cos^2\theta \cdot 10^4\, d\theta \qquad \text{... (b)}$$

Integrating within limits $\theta_1 = 45°$ and $\theta_2 = 30°$, we obtain

$$I_1 = \frac{10^4 p}{2\Delta} [0.1644\, a_1 + 13.09\, b_1 + 9.865\, c_1] \qquad \text{... (c)}$$

(ii) It is easy to see that a similar integral expression for the end-cap shown in Fig. 2.19(b) is

$$I_1 = \int_{\theta_1}^{\theta_2} \frac{1}{2\Delta} (a_1 + 100\, b_1 \cos\theta - 100\, c_1 \sin\theta)\, p \cos^2\theta \cdot 10^4\, d\theta \qquad \text{... (d)}$$

On integrating within the limits $\theta_1 = 30°$ and $\theta_2 = 45°$ (Fig. 2.19(b)), we obtain

$$I_1 = \frac{10^4 p}{2\Delta} [0.1644\, a_1 + 13.09\, b_1 - 9.865\, c_1] \qquad \text{... (e)}$$

Observe the difference between the integral expressions (b) and (d) and (c) and (e). The sign of the third term has changed. Thus, the two expressions are not identical. It may be possible here to define the positive direction of angle θ always in an anticlockwise sense with respect to r-axis and thus obtain the same expression for integral I_1 in both types of end-caps.

However, the more complex the contour becomes, the more difficult it will be to define the angle appropriately and to use the same expression for all the surfaces. Instead, if this integral is defined in terms of r and z coordinates of two end points 1 and 2, the anomaly can be removed irrespective of the orientation of the surface. This approach is followed in numerical integration, as will be shown below.

(iii) As explained in Chapter 7, the procedure of numerical integration expresses the integral as the weighted sum of the function at a few integration points. For the case of integral given by eq. (a) we take two integration points A and B along the edge 1-2 of element 1-2-3 (Fig. 2.19(c)) and the integral is expressed as

$$I_1 = \int_{l^e} F\, dl = \frac{L}{2} [F_A W_A + F_B W_B] \left(\text{or} = \frac{L}{2} \sum_{i=1}^{2} F_i W_i \right) \qquad \text{... (f)}$$

From the table of Gauss quadrature (Ch. 7) for 2 point integration, both points A and B are located at the distance of $(0.57735)L/2$ from mid-point of element 1-2 and the weightages W_A and W_B are equal to 1. F_A and F_B

represent the values of function F (given by eq. (a)) at points A and B. F_A and F_B can now be expressed completely in terms of coordinates of point A or B, i.e.

$$F_A = \frac{1}{2\Delta}(a_1 + b_1 r_A + c_1 z_A)\, p \cos\left(\tan^{-1}\frac{z_A}{r_A}\right) r_A \qquad \text{... (g)}$$

The tangent of the angle of orientation of surface 1-2 at the point A with respect to r-axis refers to the special case when the normal to surface passes though the origin of coordinates, as is the case here (see Fig. 2.19(a)).

We can evaluate this function as well as integral I_1 for both types of end-cap.

Case of Fig. 2.19(a)

$$r_A = 74.07 \qquad r_B = 83.24$$
$$z_A = 66.33 \qquad z_B = 54.37 \qquad \text{... (h)}$$

and $$I_1 = \frac{10^4\, p}{2\Delta}(0.164\, a_1 + 12.977\, b_1 + 9.777\, c_1)$$

Case of Fig. 2.19(b)

$$r_A = 74.07 \qquad r_B = 83.24$$
$$z_A = -66.33 \qquad z_B = -54.37 \qquad \text{... (i)}$$

and $$I_1 = \frac{10^4\, p}{2\Delta}[0.164\, a_1 + 12.977\, b_1 - 9.777\, c_1]$$

The results of eqs. (h) and (i) are very close to the results obtained in eqs. (c) and (e) respectively. It is to be noted that single expression (g) now applies to both types of end-cap.

In more elaborate formulation vector notations are used to indicate the outward drawn normal to the surface as always lagging with respect to vector $\overrightarrow{1-2}$ (shown in Fig. 2.19(c)) by $\pi/2$. Pressure p acts in the direction of normal but opposite in sense. The function F can then be defined unambiguously for any general surface.

2.5 ILLUSTRATIVE EXAMPLES

Elastic stress analysis finds extensive application in design of mechanical components and structures. Examples given in this section highlight the advantages of this approach. Several considerations are required while modelling a component for stress analysis. The directions and points of application of loads as well as the restraints should be specified with care. The examples given here provide guidelines for this.

2.5.1 Specifying Loads and Restraints

The loading points and directions of applied external loads are generally known in the case of machine parts. Reactions are, however, not so obvious. Consider, for example, a cam mounted on a shaft by means of a key, shown in Fig. 2.20. The point of application of load where the follower touches the cam profile is obtained easily as 'M'. To determine the reactions one has to consider the mechanics of motion of the cam. Even if the inertial force is neglected, the external force applies a torque about the axis of the shaft. This should be balanced by a reaction torque applied by the shaft through the key.

A schematic diagram of the cam, key and shaft system is given in Fig. 2.20. The points through which the load is transmitted from shaft to key and then to cam or from shaft to cam directly are not discernible. A clearer picture emerges when we consider the clearance between various parts. This is demonstrated in Fig. 2.21. At (a) is shown the situation when the key is loosely fitted (sliding key fitted in shaped keyway in the shaft). A clearance is also assumed between the shaft and hole in the cam, although the magnitude shown in the Figure is highly exaggerated. Under this situation the shaft, key and cam will adjust themselves mutually so as to

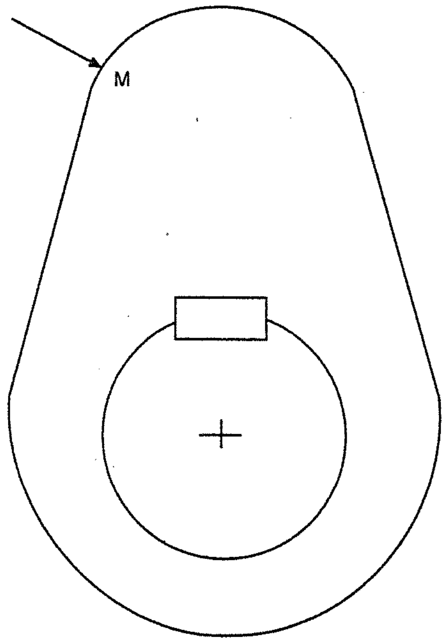


Fig. 2.20

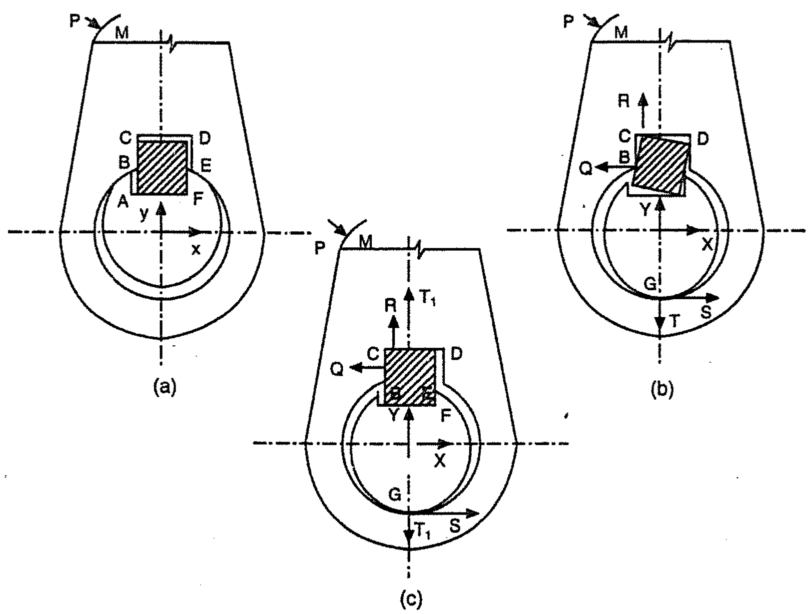


Fig. 2.21 Loading of a cam and shaft system

oppose the external load in the best possible manner. The cam will press against the key and the load will be transmitted through the surface BC. Since the key is fitted in the shaft and the forces act on the key through surfaces BC and FE, these will form a couple and the key will rotate in the keyway, thus pressing the shaft downwards until the shaft touches the bottom of the hole in the cam. The true position is shown in Fig. 2.21(b). Under this situation the reactions acting on the cam are the forces Q, R, S and T shown in this Figure. Forces S and T are related through the coefficient of friction.

If we draw the free body diagram of the cam, the forces acting on it are P, Q, R, S and T. The unknown reactions can be determined using the condition of equilibrium. Assume the line of action of force P to be located at distance d from axis of the shaft and take the following dimensions of the key and camshaft, i.e.,

width of key = w
height of key = h (assuming half of it to be located in the shaft)
radius of shaft = r
angle which direction of force P makes with x-axis = θ

Assuming the clearances to be small compared to the dimensions of the keyway, obtain the equilibrium equation as follows.

Force equilibrium:

$$P \cos \theta + S - Q = 0$$

$$P \sin \theta + T - R = 0$$

Moment equilibrium:

$$Q \cdot r + S \cdot r - R \cdot (w/2) - P \cdot d = 0$$

and friction condition

$$T\mu - S = 0$$

where μ represent the coefficient of friction. These four equations suffice for determining the unknown reactions Q, R, S and T.

(a) Alternative loading patterns: In the explanation given above we have assumed that the dimensions of key and keyway are such that the corner of the key close to point D does not touch the keyway. The tolerances on the key, keyway dimensions and shaft diameter may result in some combinations of key and shaft such that corner D of the key transmits force and under these conditions the equilibrium equation may be different. Similarly, the location and orientation of point of application of force P will change as the follower moves on the cam, thus altering the loading pattern. It is essential that all these alternative loading situations be analyzed for location of maximum stresses and the design decided on the basis of all these considerations. The finite element method comes in quite handy here because it provides quick evaluation of stresses in all these situations. The next paragraph discusses yet another situation which may exist when the key is fitted in the shaft.

The non-sliding key may be force-fitted in the keyway as shown in Fig. 2.21(c). This results in appreciable amounts of initial reaction forces shown as T_1. When load P acts on the cam the reactions through the key faces BC and CD are Q and R respectively. The frictional force S also acts at point G. Assuming that the normal force T_1 is sufficient to generate desired friction force S, we can consider the equilibrium of forces P, Q, R, S to obtain the magnitudes of the reactions. Here again the points of application of reactions Q and R depend on distribution of forces on face BC and CD which is known only when the stresses in the key are also analyzed simultaneously and an iterative analysis considering all three components of the system (cam, key and shaft) is carried out. For more accurate determination of local distribution of stresses such an analysis is necessary. This is sometimes called coupled analysis of the system. In the initial study the average locations of these reaction forces are assumed.

(b) Restraints: The complete specification of forces, determined after analyzing the reactions, should be sufficient for analysis of stresses using

FEM. The procedure of numerical computations demands an additional consideration. During calculation of various reactions or while specifying the coordinates of the points of application of loads, data are rounded off by dropping the higher decimal points. This often results in a small unbalanced residual couple or force. Even, the computer rounds off the results of calculations to a specified number of digits. Unbalanced residual forces cause free translation or rotation of the part, thus giving very large values of displacements. In fact, the stiffness matrix itself becomes unsolvable. To avoid this situation and to make the solution procedure more robust, such that it is not prone to error due to rounding off, a few restraints are specified. The restraints are specified in a manner that free translation and rotation of the body is prevented. For this purpose u/v-displacement or both u and v displacements are specified as zero for at least two points on the object. The choice of the points and the nature of displacement restraint should not influence the stress pattern. One of the two points is fully constrained ($u = 0, v = 0$) and the second is constrained in only one direction, thus preventing free rotation. Generally the points of application of load or reaction are chosen for this purpose. Restraints for the cam in the example given in the preceding section are shown in Fig. 2.22. This applies to the case shown in Fig. 2.21(b). Point B is restrained in both x and y directions and it is symbolically represented by a hinge. Point C, where reaction R is acting, is chosen for specifying the 2nd restraint. Since reaction R acts in y direction, this point should be permitted to move freely in y-direction and thus point C is restrained in x-direction only. It can now be seen that a small unbalanced force in either x- or y-direction or a small couple will not lead to free translation or rotation of the components shown. The small additional reactions generated at points B and C due to the type of restraint specified there will take care of such unbalanced loading. However, the net effect of this small additional load or reaction on the overall stress pattern will be very insignificant and results will not differ from the stress pattern which would have been generated due to application of perfectly balanced loads P, Q, R, S, T without having specified the aforesaid restraints. With these two restraints free translation and rotation of the cam are completely prevented.

Specifying restraints for objects with symmetrical configuration and loading is fairly simple, as shown in the example of in-plane loading of a plate with a central hole (Fig. 2.23). Due to symmetry, only one-half of the plate is required to be analyzed. Roller supports specified along the lines of symmetry (y-axis) with one of the supports being a hinge, are sufficient for this case.

2.5.2 Stress Analysis in Crane Hook

A two-dimensional stress analysis program was used to determine the

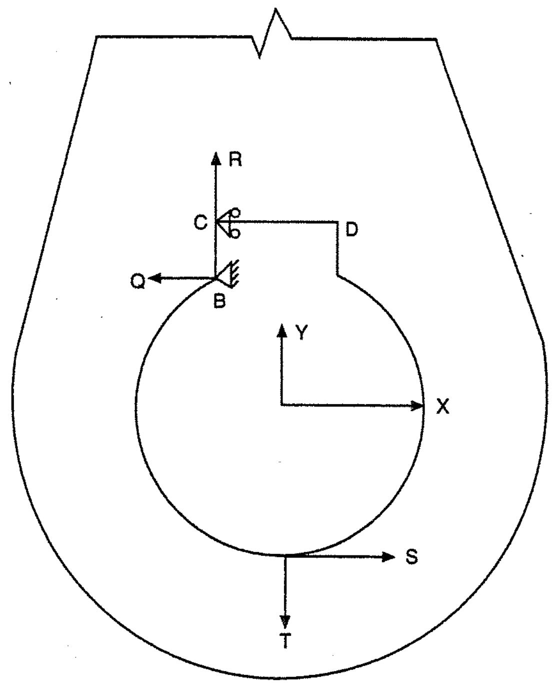


Fig. 2.22 Representation of restraint for a cam loaded under external force

stress distribution in a crane hook lifting a load of 15,000 N. First, a coarse triangular mesh was used. The mesh and resultant distribution of σ_y are shown in Fig. 2.24(i). The central portion of the hook is zoomed to clarify the mesh and stress pattern. The nature of stress distribution in the middle portion of the hook appears fairly reasonable. However, use of a finer mesh revealed the shortcomings of using coarse mesh. The results are shown in Fig. 2.24(ii). Both maximum tensile and compressive stresses increased in magnitude. The distribution of compressive stresses in the outer fibres of the hook showed a somewhat erratic pattern. A further refinement of elements especially in the outer fibres located in the region of highest compressive stress resulted in improvement in the magnitudes of maximum compressive stresses (σ_y) while maximum tensile stress changed marginally. The stress pattern became smoother as shown in Fig. 2.24(iii). Another trial was made using 6-noded quadratic triangular elements (see Ch. 7). Only 105 elements were used but it produced a dramatic improvement in results and the maximum tensile and compressive stresses increased to 508 MPa and – 328 MPa respectively. These results are shown in Fig. 2.24(iv). Another test with 414, 6-noded quadratic

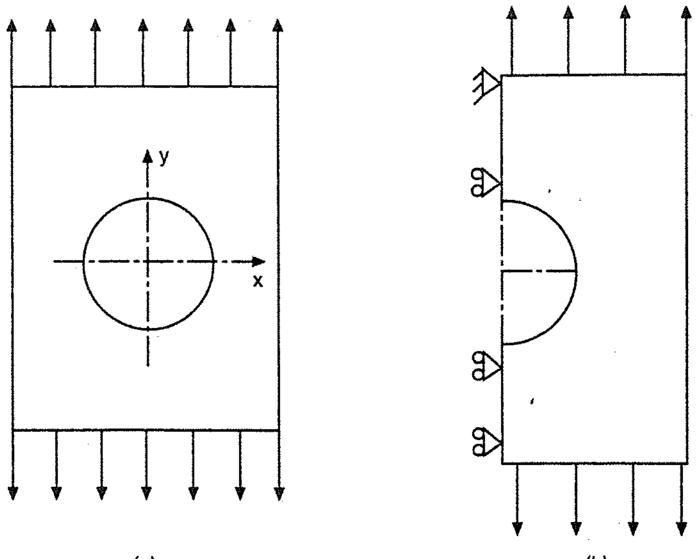
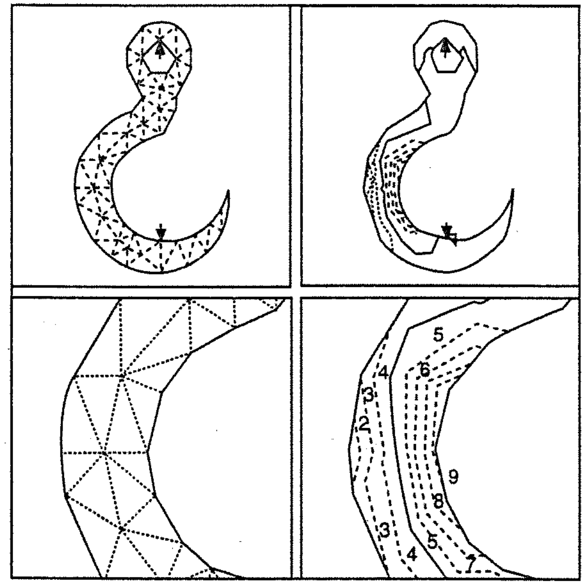


Fig. 2.23 Representation of loading and restraint for a plate with a hole

triangular elements improved the values to only 533 MPa and – 325 MPa. Data given in Table 2.1 showing the maximum and minimum values of other stresses (σ_x and τ_{xy}) present a similar trend.

As shown in these plots a coarse grid represents the general trend of stresses but it may give a highly erroneous magnitude of peak stresses. The region of high stress concentration should have a fine mesh. This feature is part of some of the finite element programs in which the mesh is refined automatically in the regions of high-stress gradient and the process is continued untill the subsequent improvement in results due to further refinement is small. This is called adaptive grid refinement, which we shall discuss in a subsequent chapter. In this example possibly 400–500, 3-noded elements with finer mesh at critical locations will be required for good accuracy. Alternative approach of using 6-noded quadratic triangular elements also seems promising due to small number of elements used.

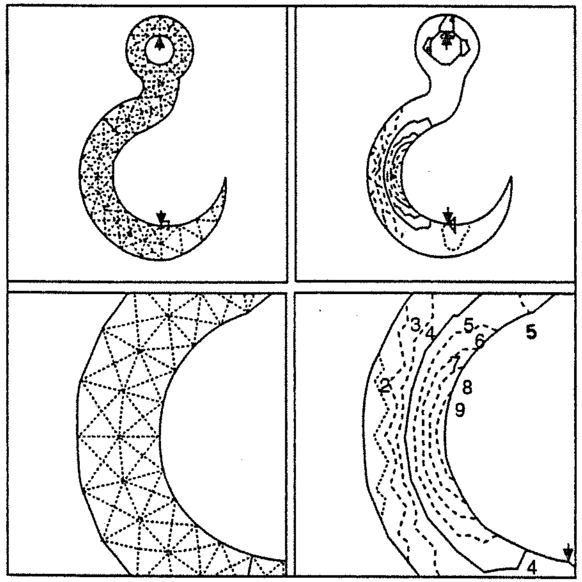
Exercise 2.13 Figure 2.25(a) to (g) shows some common mechanical components with probable points of application of loads and reactions. The directions of load and reactions are also shown. These are not the only possible patterns of reactions for such components and the reactions may change in magnitude, direction and point of application depending on the nature and type of fit between mating parts.



FEM/BEM-2D-1.09
Developed by
Prof. O.P. Gupta
Mech. Engg. Dept.
I.I.T. Kharagpur
India
Write if Help Req.
Programmed
Scale x 0.1E+00
Sigma-y MPa

.... 1 : – 1106
.... 2 : – 780
--- 3 : – 455
--- 4 : –130
— 5 : 194
--- 6 : 520
--- 7 : 845
--- 8 : 1170
... 9 : 1495
.. 10 : 1821

(i) Coarse mesh



FEM/BEM-2D-1.09
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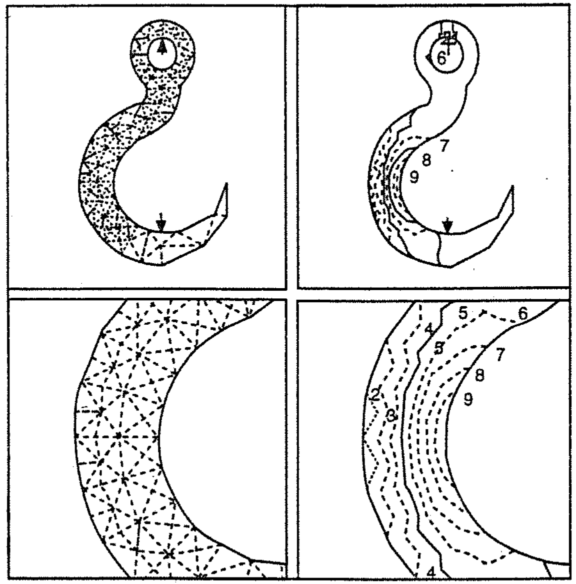
I.I.T. Kharagpur
India
Write if Help Req.
Programmed
Scale x 0.1E+00
Sigma-y MPa

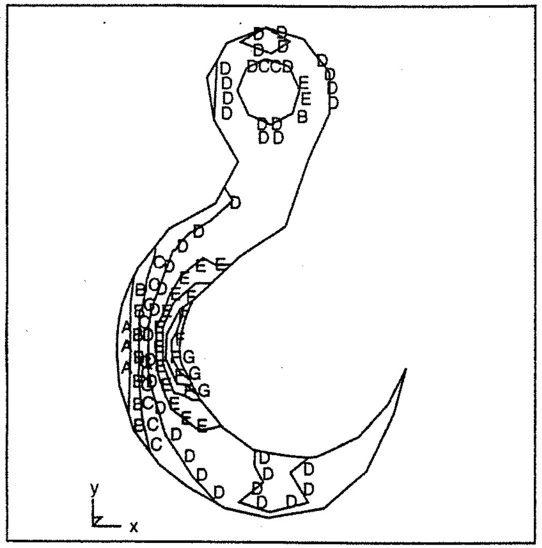
.... 1 : –1574
.... 2 : –1078
--- 3 : –582
--- 4 : –87
— 5 : 408
--- 6 : 903
--- 7 : 1399
--- 8 : 1894
... 9 : 2390
.. 10 : 2885

(ii) Refined mesh

Fig. 2.24 (Contd.)



(iii) Further refined mesh



(iv) Use of 6-noded quadratic triangular element.

Fig. 2.24 Analysis of stresses in Crane Hook

Table 2.1

Sl. No.	Type of mesh	No. of elements	No. of nodes	σ_x (MPa)		σ_y (MPa)		τ_{xy} (MPa)	
				Max	Min	Max	Min	Max	Min
1.	Coarse	65	55	59.5	– 47.0	182.1	– 110.6	71.4	– 41.4
2.	Fine	189	131	76.9	– 74.7	288.5	– 157.4	92.4	– 92.6
3.	Refinement along outer fibres	193	132	78.1	– 94.0	293.4	– 228.2	93.8	– 94.8
4.	Quadratic element	105		149.1	– 161	508.2	– 328.8		

Assuming that these are the reactions in typical situations, determine the magnitudes of reactions and fix up the restraints such that these parts can be analyzed using FEM without permitting free translation and rotation.

Exercise 2.14: The three nodes of a triangular element are given as (1, 1), (3, 5), (5, 2). The first and second digits in the bracket's represent the x and y coordinates of the three nodes in mm. Consider nodes in proper sequence and calculate the magnitudes of the components of [B] matrix.

$$\text{Ans. } [B] = 1/14 \begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 4 \\ -2 & -3 & -2 & 4 & 4 & -1 \end{bmatrix}$$

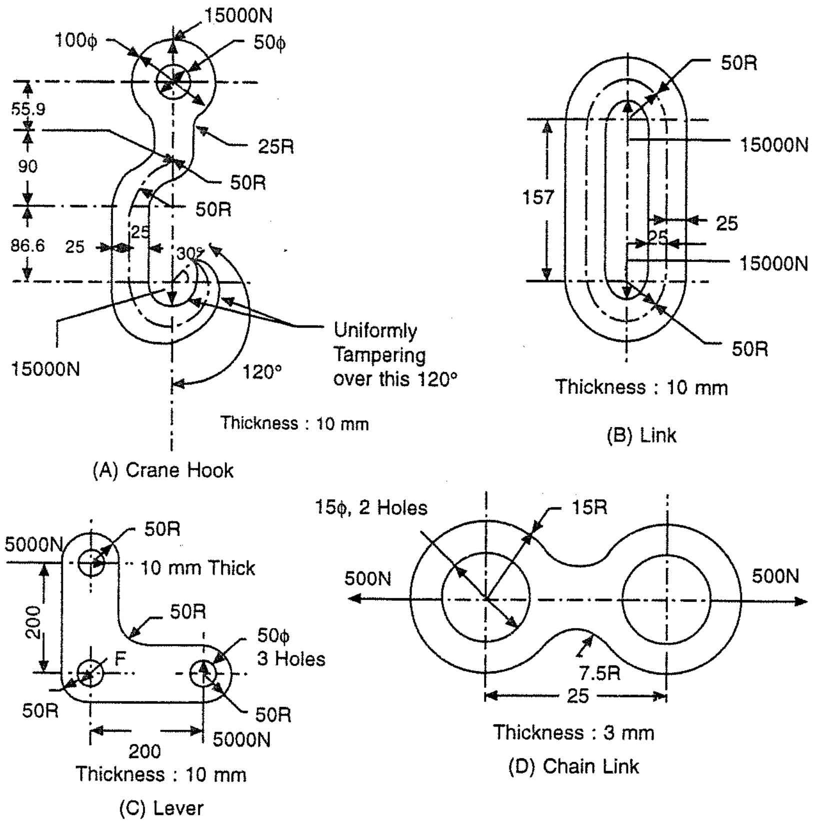


Fig. 2.25 Exercise 2.13—Stress evaluation

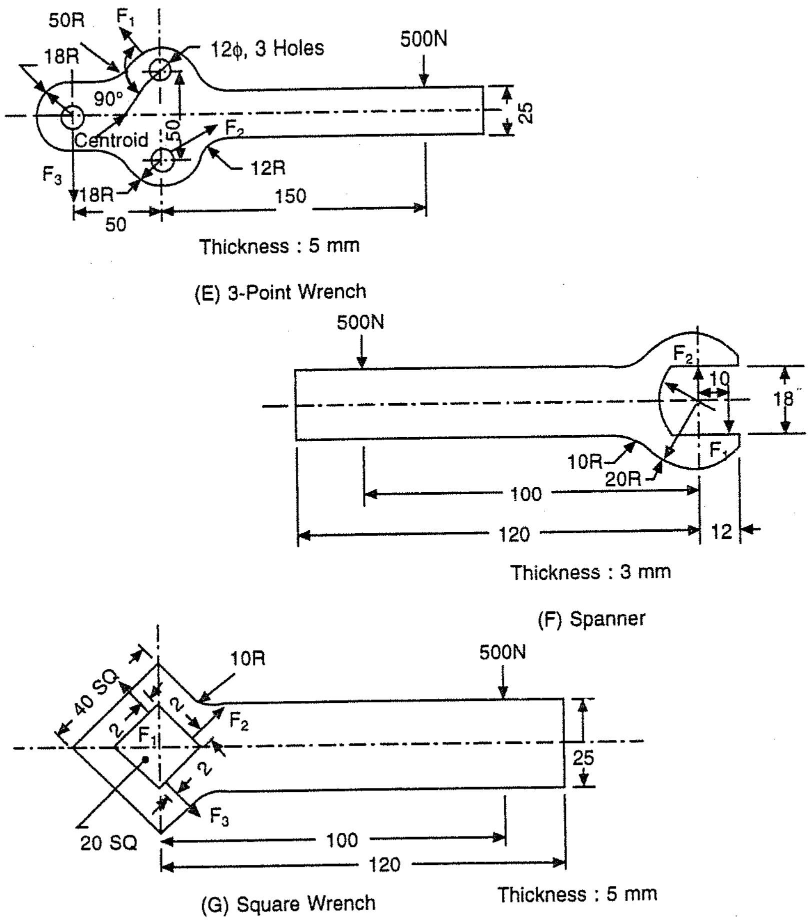


Fig. 2.25

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