Question 5

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To Prove:

$F^*(u, v) = F(-u, -v)$ when $f(x, y)$ is real

Let $f(x, y)$ be a real-valued function. The Discrete Fourier Transform (DFT) of $f(x, y)$, denoted as $F(u, v)$, is given by:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

where M and N are the dimensions of the signal, and u and v are the frequency indices.

The complex conjugate of $F(u, v)$, denoted by $F^*(u, v)$, is:

$$F^*(u, v) = \left(\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}\right)^*$$

By the properties of complex conjugates, this becomes:

$$F^*(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

Computing $F(-u, -v)$:

$$F(-u, -v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{-ux}{M} + \frac{-vy}{N}\right)}$$

Simplifying the exponents:

$$F(-u, -v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

Comparing $F^*(u, v)$ and $F(-u, -v)$
From the above expressions, we can see that:

$$F^*(u, v) = F(-u, -v)$$

Thus, we have shown that for a real-valued function $f(x, y)$, the Discrete Fourier Transform satisfies:

$$F^*(u, v) = F(-u, -v)$$

To Prove:

The 2D Fourier transform $F(u, v)$ is real and even if $f(x, y)$ is real and even

Fourier Transform Definition (2D):
The 2D Fourier transform of $f(x, y)$ is defined as:

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)}\, dx\, dy$$

where j is the imaginary unit.

Given Properties of $f(x, y)$:
- **Real:** $f(x, y) \in R$, i.e., $f(x, y) = f^*(x, y)$, where $*$ denotes complex conjugation.
- **Even:** $f(x, y) = f(-x, -y)$.

We need to show that
$F(u, v) \in R$, i.e., $F(u, v) = F^*(u, v)$.
- $F(u, v)$ is **even**, i.e., $F(u, v) = F(-u, -v)$.

Proof That $F(u, v)$ Is Real:
Take the complex conjugate of the Fourier transform $F(u, v)$:

$$F^*(u, v) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)}\, dx\, dy \right)^*$$

By properties of complex conjugation, we get:

$$F^*(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^*(x, y)e^{j2\pi(ux+vy)}\, dx\, dy$$

Since $f(x, y)$ is real, $f^*(x, y) = f(x, y)$, so:

$$F^*(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{j2\pi(ux+vy)}\, dx\, dy$$

Next, change variables by letting $x' = -x$ and $y' = -y$. Under this transformation, $dx' = -dx$ and $dy' = -dy$, so the limits of integration remain unchanged. Also, $ux+vy = -(ux' + vy')$, so the exponential term becomes:

$$e^{j2\pi(ux+vy)} = e^{-j2\pi(ux'+vy')}$$

Thus, the expression for $F^*(u, v)$ becomes:

$$F^*(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(-x', -y')e^{-j2\pi(ux'+vy')}\, dx'\, dy'$$

Since $f(x, y)$ is even, $f(-x', -y') = f(x', y')$, so:

$$F^*(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y')e^{-j2\pi(ux'+vy')}\, dx'\, dy' = F(u, v)$$

Thus, $F^*(u, v) = F(u, v)$, meaning that $F(u, v)$ is **real**.

Proof That $F(u, v)$ Is Even:
We now show that $F(u, v) = F(-u, -v)$.
Using the definition of the Fourier transform:

$$F(-u, -v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(-ux - vy)} \, dx \, dy$$

This simplifies to:

$$F(-u, -v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{j2\pi(ux + vy)} \, dx \, dy$$

Next, change variables again by setting $x' = -x$ and $y' = -y$. Under this transformation:

$$F(-u, -v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(-x', -y') e^{-j2\pi(ux' + vy')} \, dx' \, dy'$$

Since $f(x, y)$ is even, $f(-x', -y') = f(x', y')$, so:

$$F(-u, -v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') e^{-j2\pi(ux' + vy')} \, dx' \, dy' = F(u, v)$$

Thus, $F(u, v) = F(-u, -v)$, meaning that $F(u, v)$ is **even**.

Conclusion:
If $f(x, y)$ is **real** and **even**, its 2D Fourier transform $F(u, v)$ is also **real** and **even**.