$\overline{u}_i$, $\overline{T}_i$	Specified displacement and traction at boundary
$(u_i)_k$, $(T_i)_k$	Component of displacement or traction in i-th direction at node k
$\Vert u \Vert$	Displacement norm
w, w (P, Q)	Fundamental solution of a potential problem (for P as source point and Q, observation point)
w_1, w_2, w_3	Weighting functions
x_i (P), x_k (P)	Cartesian coordinates of source point $P(i, k$ may take values 1, 2, 3)
Γ_1, Γ_2, Γ	Boundary of domain
δ	Radius of an infinitesimal circle (or sphere) around source point
$\theta = \partial\phi/\partial n$	Potential gradient normal to boundary
$\{\theta^e\}$, $\{\Theta\}$	Elemental and global gradient vector
$\overline{\theta}$, $\overline{\phi}$	Values of θ and ϕ specified at boundary
θ^n (P), θ^n (Q)	Value of θ at the end of n-th time step
λ, μ	Elastic constants (Sec. 12.2.2)
ρ_i	Body force per unit volume along axial direction i
σ_{ij}	Components of stress tensor
σ	Smoothened stress—supposed to be close to the exact stress value
$\overline{\sigma}$	Approximate stress value—obtained from FE analysis
τ, $\Delta\tau$	Time and time step
ϕ	Potential (for example, temperature)
ϕ_p, ϕ (p), ϕ_P , ϕ (P)	Value of potential ϕ at interior source point, p, or at source point P located at boundary
$\{\phi^e\}$, $\{\Phi\}$	Elemental and global ϕ vectors (global vector includes ϕ for all nodes)
ϕ^n, $\phi^n(P)$, $\phi^n(Q)$	Values of ϕ at the end of n-th time step
Ω	Domain of analysis (object under analysis)

Chapter 1

INTRODUCTION AND BASIC CONCEPTS

1.1 INTRODUCTION

The finite element method (FEM) has now become a very important tool of engineering analysis. Its versatility is reflected in its popularity among engineers and designers belonging to nearly all the engineering disciplines. Whether a civil engineer designing bridges, dams, harbours or a mechanical engineer designing auto engines, rolling mills, machine tools or an aerospace engineer interested in analysis of dynamics of an aeroplane or temperature rise in the heat shield of a space shuttle or a metallurgist concerned about the influence of a rolling operation on the microstructure of a rolled product or an electrical engineer interested in analysis of the electromagnetic field in electrical machinery—all find the finite element method quite handy and useful. It is not that these problems remained unproved before the finite element method came into vogue; rather this method has become popular due to its relative simplicity of approach and accuracy of results. Before going into detail we shall look at the limitations of the traditional approach to design and analysis and then see how neatly FEM overcomes them. The boundary element method is a sister technique inheriting many advantages of FEM.

Traditional methods of engineering analysis, while attempting to solve an engineering problem mathematically, always try for simplified formulation in order to overcome the various complexities involved in exact mathematical formulation. Here we consider some examples which illustrate the approach generally followed for solution of some engineering problems.

The behaviour of an engineering system or its components is governed by various laws of nature, such as Newton's laws of motion for study of dynamic behaviour of moving bodies, Fourier law of heat conduction for analysis of temperature distribution in solids, Stokes' law or the Navier-Stokes' equation for study of motion of viscous fluids, Young's law or various forms of force-stress-strain relationship for study of load-bearing capability of mechanical parts, laws governing the electrical and magnetic field for analyzing parts in the electromagnetic environment and so on. These laws were later rewritten in the form of mathematical governing equations, such as equations of dynamics, Poisson equation of heat conduction, Navier-Stokes' equation of fluid flow, force equilibrium equations, stress-strain relation and continuity equations in the theory of elasticity, Von Mises yield criterion and Prandtl-Reuss flow rule of plasticity, Laplace and Poisson equations governing electromagnetic behaviour etc. These encompass many engineering fields. The conventional approach for analysis of these situations is to solve such governing equations within a set of boundary conditions prevailing in a particular situation by writing mathematical expressions defining boundaries and associated constraints. Simple boundary conditions, such as constant temperature along a plane (i.e., $T = T_0$ at $x = c$) or uniformly distributed load (p) along a circular contour ($F_r = ptrd\theta$ at $r = r_o$ for circular disc of thickness t) are easy to incorporate into the solution to obtain the resultant distribution of the parameter (i.e., temperature and stress pattern in the aforesaid examples).

The moment we encounter more complex realistic boundary conditions (say, for cooling of castings for which we have to specify the rate of heat loss from the surface for very irregular contours, or analysis of stresses in a plate subjected to non-uniform force at the boundary with, say, an elliptical hole in the centre), it becomes nearly impossible to write the boundary conditions in mathematical form, let alone solving the governing equation under these boundary conditions. Furthermore, the governing equations invariably assume constant properties, such as thermal conductivity, specific heat, elastic constants. These values generally vary widely in the practical situations under investigation. For example, thermal conductivity, specific heat and surface heat transfer coefficients all show appreciable variation of an order of magnitude when a wide temperature range, such as that encountered during cooling of castings or forgings is considered. Similarly, in stress analysis problems, if the loads are high, plastic yielding will occur and the coefficient of elasticity cannot be treated as constant. Additionally, the yield point itself will change drastically if temperature changes occur, as seen in the forging operation.

All these factors complicate the analysis to such an extent that exact solutions are possible only in a few cases, generally under simplifying assumptions in spite of considerable ingenuity and effort in formulation of boundary conditions and solving the resultant equations. Undoubtedly, conventional engineering analysis in the past remained choked with empirical rules and rules of thumb. Such analysis was sometimes highly dependent on personal judgement. This practice was prevalent in fields, such as manufacturing process analyses in which the complexities of boundaries and material properties variation went to the extreme, to such an extent that simplifications often led to absurd results.

The finite element method attempts to alleviate this situation and has come a boon to analysts who solve problems prevailing not only in manufacturing process analyses, but in many engineering fields wherein such complexities are expected.

1.2 FINITE ELEMENT METHOD

The basis of FEM is discretization of the object under analysis into a number of elements of finite size. The parameter under consideration (say, temperature in thermal analysis or displacement in stress analysis) is assumed to vary in a known manner within the element. Generally, linear variation is assumed but sometimes quadratic or cubic variation is also used. Fig. 1.1 provides an explanation and justification for this concept. A rod AB subjected to heating at the rate Q_1 and Q_2 at two different locations, as shown in the Figure, will display highly non-linear temperature distribution in x-direction as given in the Figure. However, a subdivision of the rod into elements, such as A-1, 1-2, 2-3 etc. shows that temperature distribution in each of these elements is very close to linear variation. Hence, any solution based on linear variation of temperature within each of the elements and satisfying the fundamental governing equation of heat flow in each element individually, will provide a solution very close to the exact solution. Let it be noted that all the elements need not be equal in size. Accuracy of solution will improve, however, if smaller elements are considered.

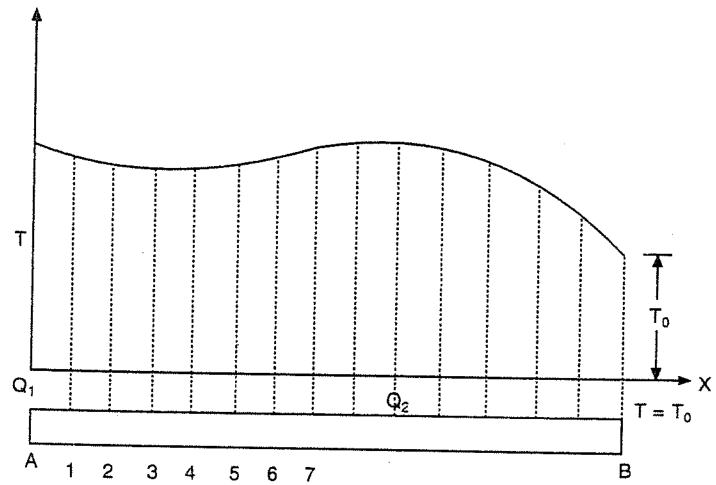


Fig. 1.1 Heat flow in a rod discretized into elements

The assumption of linear temperature distribution within the element means that the temperature at any point inside the element can be written in terms of temperature at the two ends of the element, known as nodes.

Thus, for element 1-2 the temperature anywhere in it can be written in terms of nodal temperatures T_1 and T_2 (i.e., $T = f(T_1, T_2)$). On using this temperature expression in the governing equation, we obtain a simple expression in terms of T_1, T_2 and known material properties. It is a peculiar feature of FEM that such expressions are linear in T_1 and T_2. Repeating this for all the elements gives a number of linear equations in terms of nodal temperatures T_1, T_2, T_3 etc. The solution of these simultaneous equations give results in the form of nodal temperatures. The procedure is straightforward and applies to any type of governing equation, whether in stress analysis, temperature or electromagnetic field analysis or fluid flow analysis etc. Secondly, the basic governing equation being satisfied in all the elements separately, the material properties used in it may vary from element to element. This makes it easy to incorporate any nature of variation in material properties (say, temperature-dependent properties in the above example). Thirdly, any object with irregular boundaries can easily be subdivided into elements. A two-dimensional case is shown in Fig. 1.2, where a simple 3-noded triangular element has been used for subdividing the object. It is easy to see that a complex boundary is very conveniently and accurately represented by the side of these elements.

The three advantages mentioned above are the main driving force for the widespread use of the finite element method. Writing the governing equations for all the elements separately (sometimes with different material properties) and subsequent solution of all these equations demand the use of computers; further, use of a large number of elements for reasonably high accuracy (common practice) implies that the computer should have high computational speed and large memory. The method is thus highly

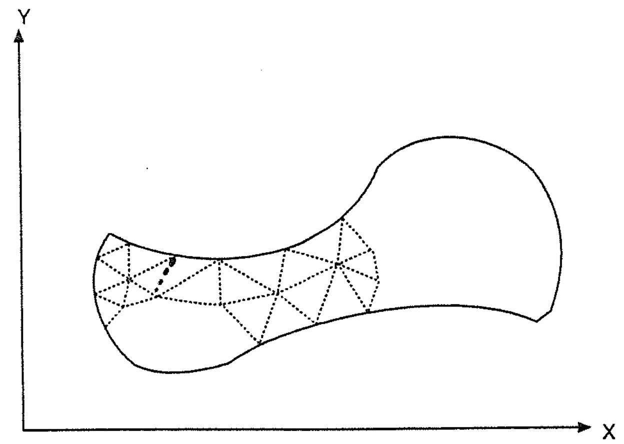


Fig. 1.2 Discretization of two-dimensional object into triangular elements

computer-dependent and development of inexpensive, high-speed, large memory computers has encouraged the extensive use of FEM.

1.3 BOUNDARY ELEMENT METHOD

The finite element analysis in which an attempt is made to satisfy the governing equation in all the elements of the region, can be shown to be equivalent to minimization of certain integrals on the boundary of the region. Hence, the boundary can be subdivided into elements and minimization of a specified boundary integral over these elements will yield the solution. Referring to object in Fig. 1.2, determination of the boundary integral will require forming elements only along the boundary, as shown in Fig. 1.3, and integrating the desired functional over these elements will yield the boundary integral, which, on minimization, will give the desired parameter (say, temperature) at all the boundary nodes (1, 2, 3, 4 .... in Fig. 1.3). Subsequently, this parameter can be calculated at any interior point. This approach is known as the boundary element method (BEM) or sometimes boundary integral technique.

Like FEM any complex two-dimensional (2D) or three-dimensional (3D) boundary can be readily subdivided into line or surface elements. Any governing equation can be written in boundary integral form. Hence, this method offers the same versatility characteristic of FEM. However, the absence of interior elements and nodes reduces the total number of nodes drastically, especially in 3D analysis. This yields great saving in computation time and memory space compared to FEM. However, variation in material properties within the region does create problems and

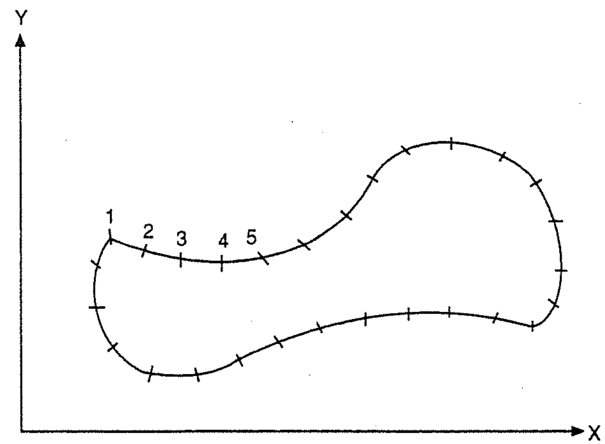


Fig. 1.3 Discretization of two-dimensional object into boundary element

formulation is sometimes more difficult. Nonetheless, use of BEM is increasing. It should be remembered that not all the applications demand consideration of varying material properties. For example, most of the design applications require loading of parts within the elastic limit for which elastic properties remain constant. However, parts in general will have complex boundaries. Such cases can be tackled with advantage using the boundary element method.

1.4 FINITE ELEMENT IMPLEMENTATION

1.4.1 Force Equilibrium Approach

The finite element method was initially applied to the solution of problems of structural design. The basic features of the method can be explained best with an example of structural design. Here we consider the analysis of a structure shown in Fig. 1.4 for the purpose of illustration. It comprises three links hinged to a plate 2-3-6-5, which itself is hinged at corner 3 and has roller support at corner 6. The structure is statically indeterminate and forces in it cannot be determined using the principles of statics.

To apply FEM, we divide the structure into simple elements, such as rods 1-2, 1-5 and 1-6. The plate 2-3-6-5 is subdivided into triangular element 4-5-6 and rectangular element 5-4-3-2. Consideration of rods is simple because these are hinged at corner nodes. The triangle and rectangle forming the plate can also be considered hinged at corner nodes 2, 3, 4, 5 and 6. It is assumed in the analysis that the forces between the elements can be transmitted only through the nodes. This may lead to some ambiguity in the case of plate elements such that the deformation along the edges of elements (such as 4-5) under the action of nodal forces may not be same

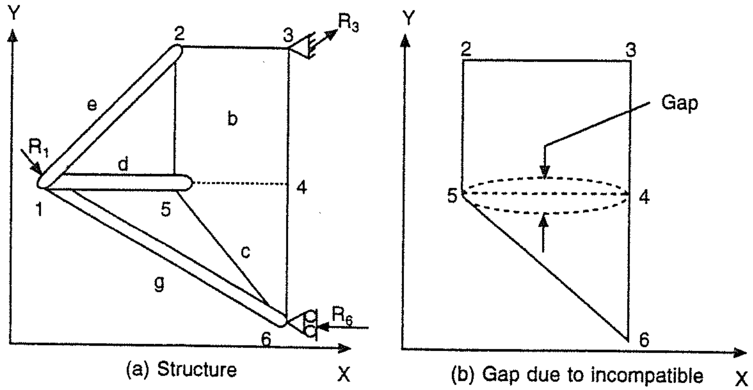


Fig. 1.4 A two-dimensional structure.

for the two adjoining elements and may also lead to gaps in the plate, which of course is unacceptable. Such a situation is shown in Fig. 1.4(b). However, as we shall see later, care is taken to develop displacement functions in the plate elements such that interelement continuity is maintained.

a) Rod element: Now, considering the free-body diagram of an element, we can write the equilibrium equation for it. The free-body diagram of a rod element (e) is shown in Fig. 1.5. The sign convention representing the positive direction of forces and displacements at the nodes is also shown. The rod has uniform cross-sectional area A and length L. The ends being hinged, external force acting at the ends will always produce stretch or compression of the rod. The expression for stretch can be written as

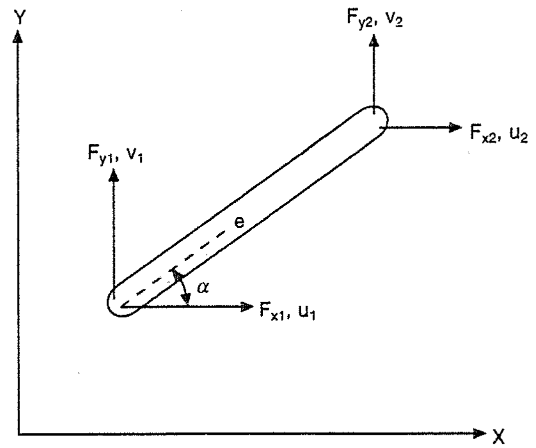


Fig. 1.5 Free-body diagram of rod element

$$\text{elongation } (dL) = (u_2 \cos\alpha + v_2 \sin\alpha) - (u_1 \cos\alpha + v_1 \sin\alpha) \quad \text{... (1.1)}$$

The external tensile force, acting along the axis of the rod, which produces this elongation is,

$$F_R = EA\frac{dL}{L} = \frac{EA}{L}\{(u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha\} \quad \text{... (1.2)}$$

Thus, the x and y components of the external forces acting at the two ends of the rod element can be written as follows after giving due consideration to sign convention, i.e.

$$\left.\begin{aligned} F_{x1} &= -F_R\cos\alpha \\ F_{y1} &= -F_R\sin\alpha \\ F_{x2} &= F_R\cos\alpha \\ F_{y2} &= F_R\sin\alpha \end{aligned}\right\} \quad \text{... (1.3)}$$

After substituting for F_R from eq. (1.2) and rearranging the terms in the forms of a matrix[1] expression, we get

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha & -\cos^2\alpha & -\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & -\sin^2\alpha & \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad \dots (1.4)$$

It is preferable to use matrix and vector notations here in order to write expressions in a more elegant manner. Nodal displacements can be represented by column vector $\{d\}$ such that the displacements at nodes 1 and 2 are respectively,

$$\{d_1\} = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}; \{d_2\} = \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} \quad \dots (1.5)$$

The displacement vector for the whole element (e) comprising nodes 1 and 2 will be written as

$$\{d^e\} = \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad \dots (1.6)$$

Similarly, nodal forces acting at the nodes of element (e) are $\{F_1^e\}$, $\{F_2^e\}$. The overall nodal force vector is $\{F^e\}$. These can be written as

$$\{F_1^e\} = \begin{Bmatrix} F_{x1} \\ F_{y1} \end{Bmatrix}; \{F_2^e\} = \begin{Bmatrix} F_{x2} \\ F_{y2} \end{Bmatrix}; \{F^e\} = \begin{Bmatrix} \{F_1^e\} \\ \{F_2^e\} \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} \quad \dots (1.7)$$

[1]Matrices are explained in Chapter 3.

Substituting these in expression (1.4) we obtain,

$$\{F^e\} = [K^e]\,\{d^e\} \quad \dots (1.8)$$

where, $[K^e]$ is the matrix shown there and is generally called the stiffness matrix for element (e). Eq. (1.8) also signifies the relationship between force and displacement analogous to spring stiffness which defines the relationship between force and displacement in a linear spring.

This expression can also be written in terms of nodal forces by partitioning the elemental stiffness matrix $[K^e]$ into four submatrices such as,

$$[K^e] = \begin{bmatrix} [K_{11}^e] & [K_{12}^e] \\ [K_{21}^e] & [K_{22}^e] \end{bmatrix} \quad \dots (1.9)$$

where submatrices are

$$[K_{11}^e] = \frac{EA}{L}\begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix};$$

$$[K_{12}^e] = \frac{EA}{L}\begin{bmatrix} -\cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & -\sin^2\alpha \end{bmatrix}$$

and so on. Expression (1.8) will now be written as

$$\begin{Bmatrix} \{F_1^e\} \\ \{F_2^e\} \end{Bmatrix} = \begin{bmatrix} [K_{11}^e] & [K_{12}^e] \\ [K_{21}^e] & [K_{22}^e] \end{bmatrix} \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix} \quad \dots (1.10)$$

b) Nodal equilibrium: Referring back to Fig. 1.4(a), it is easy to see that expressions similar to (1.8) and (1.10) can be written for rod elements (d) and (g) by substituting appropriate angle of inclination w.r.t x-axis (α) and using the respective nodal displacement values. In matrix notations these could be written as,

$$\{F^d\} = [K^d]\,\{d^d\}$$

or,

$$\begin{Bmatrix} \{F_1^d\} \\ \{F_5^d\} \end{Bmatrix} = \begin{bmatrix} [K_{11}^d] & [K_{12}^d] \\ [K_{21}^d] & [K_{22}^d] \end{bmatrix} \begin{Bmatrix} \{d_1\} \\ \{d_5\} \end{Bmatrix} \quad \dots (1.11)$$

and,

$$\{F^g\} = [K^g]\,\{d^g\}$$

or,

$$\begin{Bmatrix} \{F_1^g\} \\ \{F_6^g\} \end{Bmatrix} = \begin{bmatrix} [K_{11}^g] & [K_{12}^g] \\ [K_{21}^g] & [K_{22}^g] \end{bmatrix} \begin{Bmatrix} \{d_1\} \\ \{d_6\} \end{Bmatrix} \quad \dots (1.12)$$

Noting that element (b) comprises 4 nodes, a set of equations for nodal forces in this element can also be developed which will be written in similar matrix form in terms of nodal displacement. Such expressions for triangular and rectangular elements will be developed in subsequent chapters. At present, writing these expressions in a form similar to those developed here for rod elements, we obtain

$$\{F^b\} = [K^b]\,\{d^b\}$$

or,

$$\begin{Bmatrix} \{F_2^b\} \\ \{F_3^b\} \\ \{F_4^b\} \\ \{F_5^b\} \end{Bmatrix} = \begin{bmatrix} [K_{11}^b] & [K_{12}^b] & [K_{13}^b] & [K_{14}^b] \\ [K_{21}^b] & - & - & - \\ [K_{31}^b] & - & - & - \\ [K_{41}^b] & - & - & [K_{44}^b] \end{bmatrix} \begin{Bmatrix} \{d_2\} \\ \{d_3\} \\ \{d_4\} \\ \{d_5\} \end{Bmatrix} \qquad \ldots (1.13)$$

and for element (c)

$$\{F^c\} = [K^c]\,\{d^c\}$$

or,

$$\begin{Bmatrix} \{F_4^c\} \\ \{F_5^c\} \\ \{F_6^c\} \end{Bmatrix} = \begin{bmatrix} [K_{11}^c] & [K_{12}^c] & [K_{13}^c] \\ [K_{21}^c] & - & - \\ [K_{31}^c] & - & [K_{33}^c] \end{bmatrix} \begin{Bmatrix} \{d_4\} \\ \{d_5\} \\ \{d_6\} \end{Bmatrix} \qquad \ldots (1.14)$$

The external forces acting at nodes 1, 3 and 6 can also be resolved in their components and on writing these in vector notations, we get

$$\{R_1\} = \begin{Bmatrix} R_{1x} \\ R_{1y} \end{Bmatrix};\ \{R_3\} = \begin{Bmatrix} R_{3x} \\ R_{3y} \end{Bmatrix} \text{ and } \{R_6\} = \begin{Bmatrix} R_{6x} \\ R_{6y} \end{Bmatrix} \qquad \ldots (1.15)$$

The equilibrium of forces acting at various nodes is now considered. For the 2D structure shown in Fig. 1.4(a) there will be two equilibrium equations for every node, one each in x and y direction. Writing in matrix form, the two equilibrium equations for a node are written as a single matrix equation bearing in mind that such a matrix equation is, in effect, made of two separate equations.

Observing that the sum of external forces acting on all elements meeting at any node will be equal to net external force acting there, we write the equilibrium equation for node 1 as,

$$\{F_1^e\} + \{F_1^d\} + \{F_1^g\} = \{R_1\}$$

or,

$$\{F_1^e\} + \{F_1^d\} + \{F_1^g\} - \{R_1\} = 0 \qquad \ldots (1.16)$$

Similar matrix equations for the other five nodes are,

$$\{F_2^e\} + \{F_2^b\} = 0 \text{ (for node 2)} \qquad \ldots (1.17)$$

$$\{F_3^b\} - \{F_3\} = 0 \text{ (for node 3)} \qquad \ldots (1.18)$$

$$\{F_4^b\} + \{F_4^c\} = 0 \text{ (for node 4)} \qquad \ldots (1.19)$$

$$\{F_5^b\} + \{F_5^d\} + \{F_5^c\} = 0 \text{ (for node 5)} \qquad \ldots (1.20)$$

$$\{F_6^c\} + \{F_6^g\} - \{R_6\} = 0 \text{ (for node 6)} \qquad \ldots (1.21)$$

On substituting for the terms of eq. (1.16) from expressions (1.10) to (1.14), we obtain

$$[K_{11}^e]\{d_1\} + [K_{12}^e]\{d_2\} + [K_{11}^d]\{d_1\} + [K_{12}^d]\{d_5\}$$
$$+ [K_{11}^g]\{d_1\} + [K_{12}^g]\{d_6\} - \{R_1\} = 0$$

A rearrangement of terms results in,

$$\Big[[K_{11}^e] + [K_{11}^d] + [K_{11}^g]\Big]\{d_1\} + [K_{12}^e]\{d_2\} + [0]\{d_3\}$$
$$+ [0]\{d_4\} + [K_{12}^d]\{d_5\} + [K_{12}^g]\{d_6\} - \{R_1\} = 0 \qquad \ldots (1.22)$$

The terms with vectors $\{d_3\}$ and $\{d_4\}$ have been added for completeness so that the expression has terms corresponding to all the nodal displacement vectors. Similar equations are written for nodes 2, 3 etc. using expressions (1.17) to (1.21). Some of these are (for nodes 2 and 3)

$$[K_{21}^e]\{d_1\} + \Big\{[K_{22}^e] + [K_{11}^b]\Big\}\{d_2\} + [K_{12}^b]\{d_3\}$$
$$+ [K_{13}^b]\{d_4\} + [K_{14}^d]\{d_5\} + [0]\{d_6\} = 0 \qquad \ldots (1.23)$$

$$[0]\{d_1\} + [K_{21}^b]\{d_2\} + [K_{22}^b]\{d_3\} + [K_{23}^b]\{d_4\}$$
$$+ [K_{24}^b]\{d_5\} + [0]\{d_6\} - \{R_3\} = 0 \qquad \ldots (1.24)$$

On combining (1.22), (1.23), (1.24) and the other three similar equations, the overall matrix expression can be written as

$$\begin{bmatrix} [K_{11}] & [K_{12}] & [0] & [0] & [K_{15}] & [K_{16}] \\ [K_{21}] & [K_{22}] & [K_{23}] & [K_{24}] & [K_{25}] & [0] \\ [0] & [K_{32}] & [K_{33}] & [K_{34}] & [K_{35}] & [0] \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \end{bmatrix} \begin{Bmatrix} \{d_1\} \\ \{d_2\} \\ \{d_3\} \\ \{d_4\} \\ \{d_5\} \\ \{d_6\} \end{Bmatrix} - \begin{Bmatrix} \{R_1\} \\ \{0\} \\ \{R_3\} \\ \{0\} \\ \{0\} \\ \{R_6\} \end{Bmatrix} = 0 \qquad \ldots (1.25)$$

This can be expressed in a more compact form as

$$[K]\{\delta\} - \{R\} = 0 \qquad \ldots (1.26)$$

Here $[K_{11}]$, $[K_{12}]$ etc. are the elements (submatrices) of the overall matrix $[K]$. These submatrices have been derived from equations, such as (1.22), (1.23), (1.24) etc.

For example,

$$[K_{11}] = [K^e_{11}] + [K^d_{11}] + [K^g_{11}] \text{ [from expression (1.22)]}$$

$$[K_{12}] = [K^e_{12}]$$

$$[K_{21}] = [K^e_{21}] \text{ [from expression (1.23)]}$$

and so on. This overall matrix eq (1.26) is solved to determine the displacement vector $\{\delta\}$. After knowing the displacements at all the nodes, equations such as (1.2) can be used to determine forces and stresses in all the elements.

1.4.2 Assembly Procedure

The above-mentioned procedure of assembling the overall stiffness matrix (eq. 1.25) from elemental stiffness matrices (eqs. 1.10 to 1.14) can be described in a simple and more elegant form as explained below. The stiffness matrix $[K^e]$ for element 'e' (eq. 1.8) is made of four terms: $[K^e_{11}]$ etc. (eq. 1.10). Also, the displacement vector associated with matrix eq. (1.10) comprises only two terms, $\{d_1\}$ and $\{d_2\}$, which correspond to nodes 1 and 2. If we associate each term of the elemental stiffness matrix $[K^e]$ with a set of two nodes, expressed both as row and column, as shown in Fig. 1.6, the terms $[K^e_{11}]$, $[K^e_{12}]$, $[K^e_{21}]$ and $[K^e_{22}]$ may be associated with nodal combinations of (1, 1), (1, 2), (2, 1) and (2, 2) respectively. Similarly, if we consider the overall stiffness matrix $[K]$ and associate its terms with a combination of two nodes, the nodal combination for the terms of the first row of this matrix (eq. 1.25) can be written as (1, 1), (1, 2), (1, 3), (1, 4), (1, 5) and (1, 6). Similar interpretation is done for the other terms.

Considering eqs. (1.22) to (1.25) we find that the terms of elemental stiffness matrix $[K^e]$, i.e., $[K^e_{11}]$, $[K^e_{12}]$, $[K^e_{21}]$ and $[K^e_{22}]$, contribute to terms $[K_{11}]$, $[K_{12}]$, $[K_{21}]$ and $[K_{22}]$ of the overall stiffness matrix. These terms of the overall stiffness matrix are located at nodal row-column combinations of (1, 1), (1, 2), (2, 1) and (2, 2). This conforms to the nodal pair combination for various terms of elemental stiffness matrix $[K^e]$ as observed above. These are represented in Fig. 1.6(a). Thus, we can infer from here that the contributions from terms of the elemental stiffness matrix are transferred to those terms of the overall stiffness matrix which are located at the same nodal (or row-column) combination. A consideration of elemental stiffness matrix for rod element d is shown in Fig. 1.6(b). This can also be verified from eqs. (1.22–1.24 etc.). If the contribution from each term of the elemental stiffness matrix is represented by one circle at an appropriate location in the overall stiffness matrix, the situation for the structure shown in Fig. 1.4(a) is represented by the overall stiffness matrix given in Fig. 1.6(c). The terms with no circle are zero or null submatrices. The terms of vector $\{R\}$ (eqs. 1.25, 1.26) can be derived in a similar manner. As an illustration another stiffness matrix for the structure shown in Fig. 1.7(a) is represented symbolically in (b) of this Figure.

1.4.3 Formulation Using Potential Energy Minimization

A more versatile approach to displacement formulation for stress analysis is use of the principle of potential energy minimization [1]. The principle states that the equilibrium stress distribution in a body or structure subjected to external forces will be the one for which the total potential energy of the system, comprising the internal strain energy and work done by external forces, is minimum. It should be noted that straining a body results in increase in its internal energy while the work done by external forces leads to decrease in the energy of the body applying the force. Hence, the potential energy of the system will be given by:

Potential energy of the system (Π) = strain energy in the body (Π_s) – work done by external forces (Π_w) ... (1.27)

The application of this principle to finite element formulation involves writing the strain energy expression for all the elements and the expressions for work done by all the external forces. The potential energy expression for the whole body (or domain) so obtained, is differentiated with respect to variables of the problem and then equated to zero, thus obtaining the conditions for minimization of potential energy. The variables of the problem are the unknown displacements at all the nodes of elements. All the equations obtained in this manner are solved simultaneously. This gives all the nodal displacements which are used to calculate stresses. To illustrate the application of this principle we consider again the structure analyzed in Sec. 1.4.1 which was used to illustrate force-equilibrium approach. This structure, shown in Fig. 1.4(a), has five elements b, c, d, e and g and is subjected to three external forces R_1, R_3 and R_6. Designating elemental strain energies by (Π_{sb}), (Π_{sc}) etc. for elements b, c ... and writing Π_{w1}, Π_{w3} etc. for work done by external forces acting at nodes 1, 3..., the overall potential energy can be expressed as

$$\Pi = \Pi_{sb} + \Pi_{sc} + \Pi_{sd} + \Pi_{se} + \Pi_{sg} - \Pi_{w1} - \Pi_{w3} - \Pi_{w6} \qquad \ldots (1.28)$$

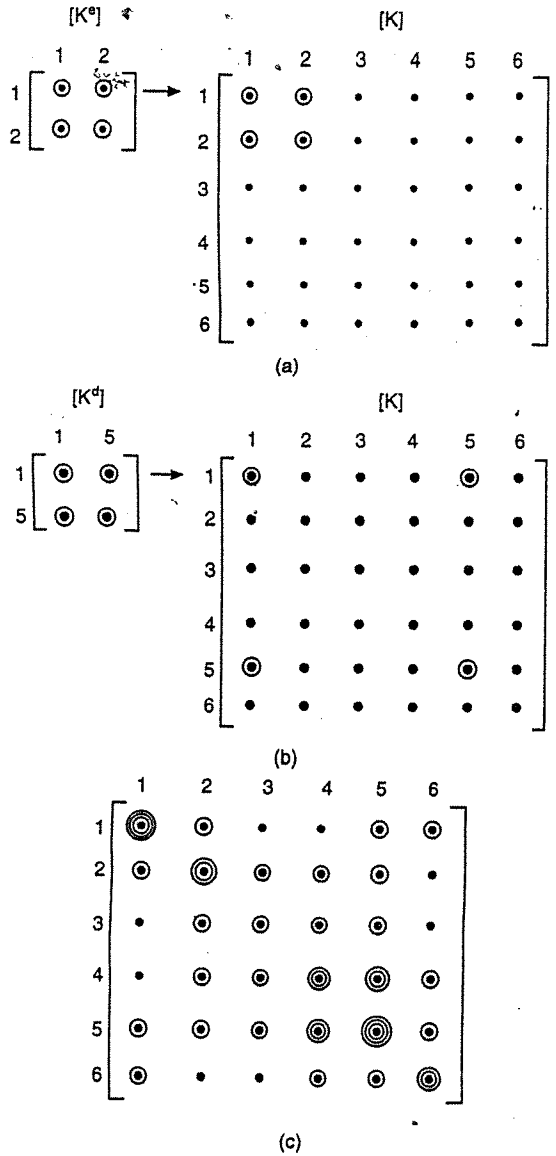


Fig. 1.6 Structure of assembled stiffness matrix

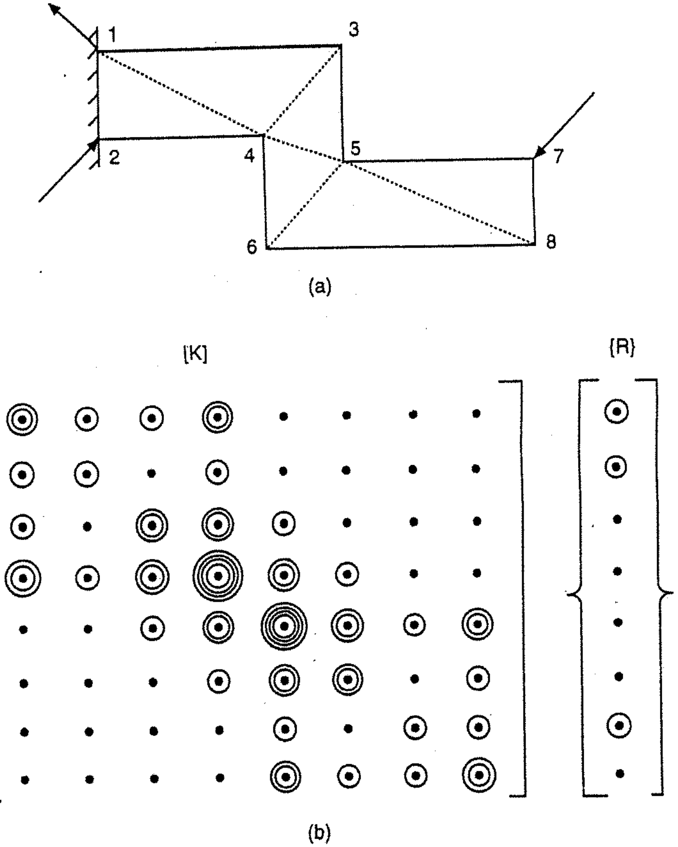


Fig. 1.7 Symbolic representation of assembled stiffness matrix [K] and force vector {R}

or, for a general structure with n elements and m nodes at which external forces are acting, the expression is

$$\Pi = \sum_{k=1}^{n} \Pi_{sk} - \sum_{r=1}^{m} \Pi_{wr} \qquad \dots (1.29)$$

a) Nature of energy expressions: To explain the form of strain energy expression we consider again the element 'e' of the structure and use expressions (1.1) and (1.2) to represent elongation (dL) and axial force (F_R) in the link. Since the strain energy for a one-dimensional link acted upon by a constant axial force and loaded within elastic limit is given by 1/2 × force × elongation, we write,

$$\Pi_{se} = \frac{1}{2}\frac{EA}{L}\{(u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha\}^2 \qquad \dots (1.30)$$

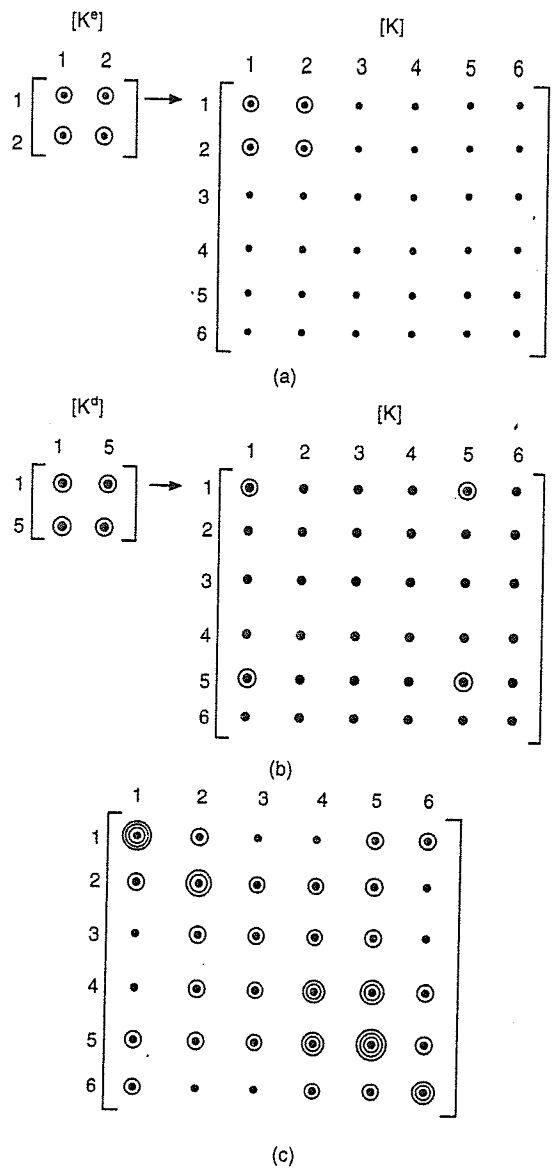
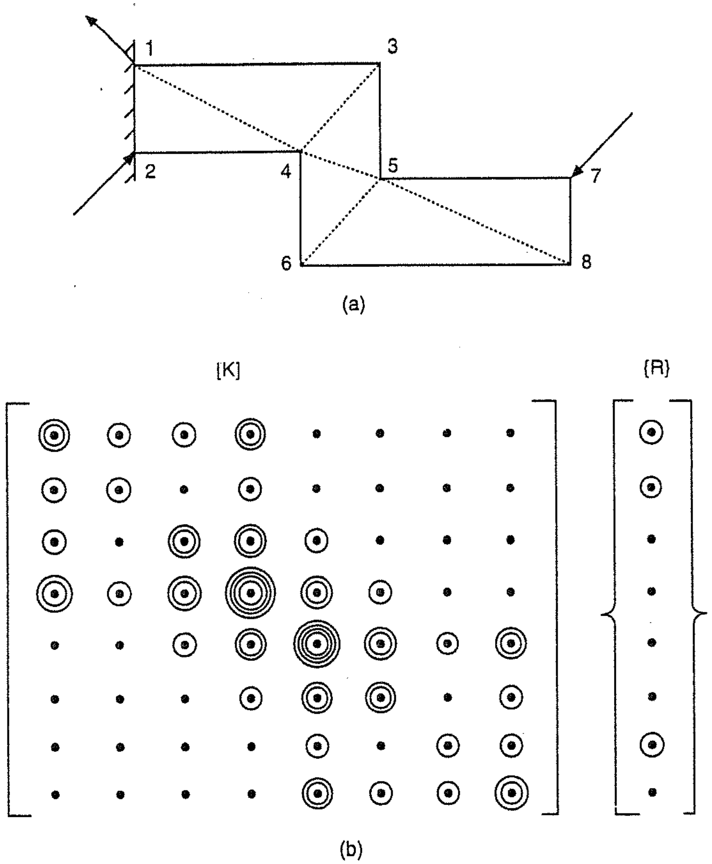


Fig. 1.6 Structure of assembled stiffness matrix



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$$\Pi_{se} = \frac{1}{2}\frac{EA}{L}\{(u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha\}^2 \qquad \dots (1.30)$$

Note that the unknowns of this expression are the nodal displacements u_1, v_1 etc. at the nodes associated with element 'e'. Similarly, the strain energy expressions for other elements will contain only those displacements as unknowns which correspond to the nodes associated with the element. The external work done at node 1 is given by

$$\Pi_{w1} = R_{1x} \cdot u_1 + R_{1y} \cdot v_1 \tag{1.31}$$

where R_{1x} and R_{1y} are the components of force R_1 in x and y direction (eq. 1.15). This again contains displacements associated with node 1 as the only unknowns.

b) Energy minimization: From the discussion of energy expressions it is clear that the expression for total potential energy will have all the nodal displacements as the only unknowns. These unknowns are to be determined by minimization. Since the structure has 6 nodes there will be 12 unknowns u_1, v_1 etc. Differentiating the potential energy expression (eq. 1.28) with respect to each of these displacements and then equating it to zero will give 12 such conditions of minimization of potential energy, i.e.

$$\frac{\partial \Pi}{\partial u_1} = \frac{\partial \Pi_{sb}}{\partial u_1} + \frac{\partial \Pi_{sc}}{\partial u_1} + \frac{\partial \Pi_{sd}}{\partial u_1} + \frac{\partial \Pi_{se}}{\partial u_1} + \frac{\partial \Pi_{sg}}{\partial u_1} - \frac{\partial \Pi_{w1}}{\partial u_1} - \frac{\partial \Pi_{w3}}{\partial u_1} - \frac{\partial \Pi_{w6}}{\partial u_1} = 0$$

$$\frac{\partial \Pi}{\partial v_1} = \frac{\partial \Pi_{sb}}{\partial v_1} + \frac{\partial \Pi_{sc}}{\partial v_1} + \frac{\partial \Pi_{sd}}{\partial v_1} + \frac{\partial \Pi_{se}}{\partial v_1} + \frac{\partial \Pi_{sg}}{\partial v_1} - \frac{\partial \Pi_{w1}}{\partial \varpi_1} - \frac{\partial \Pi_{w3}}{\partial \varpi_1} - \frac{\partial \Pi_{w6}}{\partial \varpi_1} = 0 \text{ etc.} \qquad \ldots (1.32)$$

As explained in the preceding section, the expressions for strain energy Π_{sb} and Π_{sc} will not contain displacements associated with node 1 (i.e., u_1, v_1). Hence, the partial differentiation $\{\partial\Pi_{sb}/\partial u_1\}$ and $\{\partial\Pi_{sc}/\partial u_1\}$ will be zero. Alternatively speaking, the partial differentiation of an elemental energy term will contribute only in those equations from the set of eqs. (1.32) which are obtained on differentiating the total potential energy with respect to displacements associated with nodes belonging to this element. For example, the term representing partial differentiation of strain energy of element 'e', Π_{se}, will be present in the expressions for $\{\partial\Pi/\partial u_1\}$, $\{\partial\Pi/\partial v_1\}$, $\{\partial\Pi/\partial u_2\}$ and $\{\partial\Pi/\partial v_2\}$ only. More clearly, if we rewrite expression (1.32), it gives

$$\frac{\partial \Pi}{\partial u_1} = 0 + 0 + \frac{\partial \Pi_{sd}}{\partial u_1} + \frac{\partial \Pi_{se}}{\partial u_1} + \frac{\partial \Pi_{sg}}{\partial u_1} - \frac{\partial \Pi_{w1}}{\partial u_1} - 0 - 0 = 0$$

$$\frac{\partial \Pi}{\partial v_1} = 0 + 0 + \frac{\partial \Pi_{sd}}{\partial v_1} + \frac{\partial \Pi_{se}}{\partial v_1} + \frac{\partial \Pi_{sg}}{\partial v_1} - \frac{\partial \Pi_{w1}}{\partial v_1} - 0 - 0 = 0 \ldots (1.33)$$

$$\frac{\partial \Pi}{\partial u_3} = \frac{\partial \Pi_{sb}}{\partial u_3} + 0 + 0 + 0 + 0 - 0 - \frac{\partial \Pi_{w3}}{\partial u_3} - 0 = 0 \text{ etc.}$$

Thus, as explained above, the contribution from the strain energy term of element 'e' will be present only in four of the set of 12 equations given as (1.33). These contributions can be written as

$$\frac{\partial \Pi_{se}}{\partial u_1};\ \frac{\partial \Pi_{se}}{\partial v_1};\ \frac{\partial \Pi_{se}}{\partial u_2} \text{ and } \frac{\partial \Pi_{se}}{\partial v_2}$$

We can also write the above expression (eq. 1.33) in terms of the partial differentials of the energy terms with respect to nodal displacement vectors $\{d_1\}$, $\{d_2\}$, ... by combining the set of two equations representing partial differentiation with respect to the nodal displacements belonging to same node (say, u_1, v_1 for node 1 and so on). Thus

$$\left\{\frac{\partial \Pi}{\partial\{d_1\}}\right\} = \{0\} + \{0\} + \left\{\frac{\partial \Pi_{sd}}{\partial\{d_1\}}\right\} + \left\{\frac{\partial \Pi_{se}}{\partial\{d_1\}}\right\} + \left\{\frac{\partial \Pi_{sg}}{\partial\{d_1\}}\right\} - \left\{\frac{\partial \Pi_{w1}}{\partial\{d_1\}}\right\} - \{0\} - \{0\} = 0 \qquad \ldots (1.33a)$$

$$\left\{\frac{\partial \Pi}{\partial\{d_3\}}\right\} = \left\{\frac{\partial \Pi_{sb}}{\partial\{d_3\}}\right\} + \{0\} + \{0\} + \{0\} + \{0\} + \{0\} - \left\{\frac{\partial \Pi_{w3}}{\partial\{d_3\}}\right\} - \{0\} = 0$$

Differentiation of Π_{se} (expression 1.30) with respect to u_1, v_1, u_2 and v_2 will yield four expressions and writing these four in matrix form, we get

$$\begin{Bmatrix} \dfrac{\partial \Pi_{se}}{\partial u_1} \\ \dfrac{\partial \Pi_{se}}{\partial v_1} \\ \dfrac{\partial \Pi_{se}}{\partial u_2} \\ \dfrac{\partial \Pi_{se}}{\partial v_2} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha & -\cos^2\alpha & -\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & -\sin^2\alpha & \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad \ldots (1.34)$$

Equation (1.34) can also be written in the form of submatrices as

$$\begin{Bmatrix} \left\{\dfrac{\partial \Pi_{se}}{\partial \{d_1\}}\right\} \\ \left\{\dfrac{\partial \Pi_{se}}{\partial \{d_2\}}\right\} \end{Bmatrix} = \begin{bmatrix} [K^e_{11}] & [K^e_{12}] \\ [K^e_{21}] & [K^e_{22}] \end{bmatrix} \begin{Bmatrix} \{d_1\} \\ \{d_2\} \end{Bmatrix} \qquad \ldots (1.34a)$$

The right-hand side of this expression is precisely the same as obtained by using the force equilibrium approach (eq. 1.10). The above equation can also be written as

$$\left\{\frac{\partial \Pi_{se}}{\partial \{d_e\}}\right\} = [K^e]\{d^e\} \qquad \ldots (1.35)$$

Removing null (zero) vectors from the expressions (1.33a) we can write these six equations as

$$\left\{\frac{\partial \Pi_{se}}{\partial \{d_1\}}\right\} + \left\{\frac{\partial \Pi_{sd}}{\partial \{d_1\}}\right\} + \left\{\frac{\partial \Pi_{sg}}{\partial \{d_1\}}\right\} - \left\{\frac{\partial \Pi_{w1}}{\partial \{d_1\}}\right\} = 0 \qquad \ldots (1.36a)$$

$$\left\{\frac{\partial \Pi_{se}}{\partial \{d_2\}}\right\} + \left\{\frac{\partial \Pi_{sb}}{\partial \{d_2\}}\right\} = 0 \qquad \ldots (1.36b)$$

$$\left\{\frac{\partial \Pi_{sb}}{\partial \{d_3\}}\right\} - \left\{\frac{\partial \Pi_{w3}}{\partial \{d_3\}}\right\} = 0 \qquad \ldots (1.36c)$$

$$\left\{\frac{\partial \Pi_{sb}}{\partial \{d_4\}}\right\} + \left\{\frac{\partial \Pi_{sc}}{\partial \{d_4\}}\right\} = 0 \qquad \ldots (1.36d)$$

$$\left\{\frac{\partial \Pi_{sb}}{\partial \{d_5\}}\right\} + \left\{\frac{\partial \Pi_{sd}}{\partial \{d_5\}}\right\} + \left\{\frac{\partial \Pi_{sc}}{\partial \{d_5\}}\right\} = 0 \qquad \ldots (1.36e)$$

$$\left\{\frac{\partial \Pi_{sc}}{\partial \{d_6\}}\right\} + \left\{\frac{\partial \Pi_{sg}}{\partial \{d_6\}}\right\} - \left\{\frac{\partial \Pi_{w6}}{\partial \{d_6\}}\right\} = 0 \qquad \ldots (1.36f)$$

On comparing eq. (1.10) with (1.34a) we identify $\{\partial \Pi_{se}/\partial\{d_1\}\}$ and $\{\partial \Pi_{se}/\partial\{d_2\}\}$ to be the same as $\{F^e_1\}$ and $\{F^e_2\}$ respectively. Other terms of eqs. (1.36a) to (1.36'f') can also be identified as being identical with the terms of eqs. (1.16) to (1.21). This means the overall matrix equation obtained by assembling and rearranging the terms of eqs.(1.36a) to (1.36'f') will be same as the matrix equation obtained by assembling eqs. (1.16) to (1.21), i.e.

$$[K]\{\delta\} - \{R\} = 0 \qquad \ldots (1.37)$$

where $\{\delta\}$ is the vector containing all nodal displacements $u_1, v_1, u_2 \ldots$ etc. as its components and $\{R\}$ contains components of the external forces derived on differentiating the external work components $\Pi_{w1}\ldots$ etc. The overall stiffness matrix $[K]$ will be obtained by placing the terms of elemental stiffness matrices (such as the one in eq. 1.34) in proper rows and columns as guided by the assembly procedure given in Sec. 1.4.2.

Expressions (1.34) and (1.37) are same as eqs. (1.4) and (1.26) derived on the consideration of force equilibrium. It can be concluded that the two approaches give the same elemental stiffness matrices and overall equations. For the case of a general element, such as a triangular element etc., the formulation based on potential energy minimization is more convenient and this will be followed in subsequent derivations.

c) Equilibrium equation: The procedure of potential energy minimization can now be identified completely with the force equilibrium approach. The partial differentiation of the expression for elemental strain energy (Π_{se}) with respect to the displacements at nodes of the element give the elemental stiffness matrix (eq. 1.35). Rewriting it,

$$\left\{\frac{\partial \Pi_{se}}{\partial \{d^e\}}\right\} = [K^e]\{d^e\} \qquad \ldots (1.38)$$

Similarly, load vector $\{R\}$ is obtained by partially differentiating the expression for work done by external forces with respect to the nodal displacements or,

$$\frac{\partial \Pi_w}{\partial \{\delta\}} = \{R\} \text{ (where } \Pi_w = \Pi_{w1} + \Pi_{w3} \ldots) \qquad \ldots (1.39)$$

If work done by the external force can be identified with a particular element (such as the work done by gravity force due to mass of element), it becomes possible to express this in terms of nodal displacement at the nodes of the element. Hence, the $\{R\}$ vector given in eq. (1.37) will get contributions from elements also, in addition to obtaining contributions from work done due to forces acting at the node. Such elemental contributions to force vector $\{R\}$ will be

$$\left\{\frac{\partial \Pi_{we}}{\partial \{d^e\}}\right\} = \{f^e\} \qquad \ldots (1.40)$$

where Π_{we} is the work performed by the force distribution on an element 'e'. We shall consider this point again when we take up the different types of loading on the structure. However, the structure of the matrix equilibrium equation (eq. 1.37) is now clear. The first part of this expression is obtained as an assembly of elemental stiffness matrices obtained by partially differentiating the elemental strain energy Π_{se} with respect to the nodal displacement (i.e., eq. 1.38). The second part of eq. (1.37) is obtained as nodal force vector $\{R\}$ due to direct nodal loads expressed by eq. (1.39). If the loading involves some other type of loads, such as gravity force, which acts on all the elements, then vector $\{R\}$ will also get elemental contributions, $\{f^e\}$, from various elements. This is obtained by partially differentiating the expression for external work associated with an element (i.e., Π_{we}) with respect to nodal displacements (eq. 1.40).

1.4.4 Other Approaches

We have discussed finite element formulation using two approaches. One was the force equilibrium approach in which the equilibrium of forces acting at various nodes was considered. It resulted in one set of equations for each node. These equations were solved simultaneously and finally nodal displacements were determined. These displacement values were later used to obtain stresses in various elements. In the other approach the principle of minimization of potential energy was used. The strain energy of all the elements and work done by all external forces were determined. Minimization of total potential energy, the difference between strain energy and external work, resulted in a set of equations. The nodal displacements were again obtained by solving these equations. It was also shown that the two approaches gave identical results.

Equilibrium of forces is the natural basis of stress analysis. Minimization of potential energy is also a well-known principle of stress analysis. However, these are not the only bases for finite element formulation. No obvious basis, such as these, exists in other problems, such as heat conduction or fluid flow. We only know that the governing partial differential equation of the process should be satisfied at every point in the region. As we shall see in subsequent chapters, one powerful technique of satisfying the governing equation everywhere in the region is to use the Euler theorem of variational calculus [2, 3]. This theorem provides a means of determining a function, minimization of which is equivalent to satisfying the governing equation everywhere in the region. Thus, this function can be used in the same manner as the potential energy is used in the case of stress analysis. The Galerkin procedure or weighted residue technique [3] is another method which helps in converting the problem of solving a governing equation into one of minimization. These techniques will be discussed in subsequent chapters where we shall apply finite element methods to the solution of various other types of problems, such as heat flow, fluid flow, torsion, seepage etc.

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