Question 4

Anshika Raman
Roll No: 210050014

Kushal Agarwal
Roll No: 210100087

Kavan Vavadiya
Roll No: 210100166

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Problem Statement

Consider an $n \times n$ image $f(x, y)$ where only $k \ll n^2$ elements are non-zero. The locations of these non-zero elements are known, and we wish to reconstruct the image from a set of only m discrete Fourier transform (DFT) coefficients of known frequencies, where $m < n^2$. We aim to answer the following questions:

(a) How can we reconstruct such an image from the given DFT coefficients?

(b) What is the minimum value of m required for this reconstruction method?

(c) Will this method work if k is known but the locations of the non-zero elements are unknown?

Solution

Part (a): Reconstruction of the Sparse Image Using $m < n^2$ DFT Coefficients

Since the image $f(x, y)$ is sparse, with only $k \ll n^2$ non-zero elements at known positions, we can take advantage of this sparsity to reconstruct the image using a subset of DFT coefficients. Here is a step-by-step approach:

1. **Sparse Signal Representation in the Fourier Domain**: Because we know the positions of the k non-zero elements, we can represent $f(x, y)$ as a vector $\mathbf{f}$ of length k, where $\mathbf{f} = \{f(x_i, y_i)\}_{i=1}^{k}$ are the non-zero values at the known positions $\{(x_i, y_i)\}_{i=1}^{k}$.

2. **Discrete Fourier Transform (DFT)**: The 2D DFT of an image $f(x, y)$ of size $n \times n$ is given by:

$$F(u, v) = \sum_{x=0}^{n-1} \sum_{y=0}^{n-1} f(x, y)\, e^{-2\pi i \left(\frac{ux}{n} + \frac{vy}{n}\right)}$$

where $u, v \in \{0, 1, \ldots, n-1\}$ represent the frequency indices in the Fourier domain.

3. **Simplifying with Known Sparsity**: Since $f(x, y)$ is zero everywhere except at the k known locations (x_i, y_i), we can express the DFT $F(u, v)$ as:

$$F(u, v) = \sum_{i=1}^{k} f(x_i, y_i)\, e^{-2\pi i \left(\frac{ux_i}{n} + \frac{vy_i}{n}\right)}$$

where $f(x_i, y_i)$ are the non-zero values at these positions.

4. **Formulating the Problem as a Linear System**: Given m known DFT coefficients $\{F(u_j, v_j)\}_{j=1}^{m}$, we can set up m equations:

$$F(u_j, v_j) = \sum_{i=1}^{k} f(x_i, y_i)\, e^{-2\pi i \left(\frac{u_j x_i}{n} + \frac{v_j y_i}{n}\right)}, \quad j = 1, \ldots, m.$$

Let $\mathbf{F} \in \mathbb{C}^m$ be the vector of observed DFT coefficients $\{F(u_j, v_j)\}_{j=1}^m$, and let $\mathbf{f} \in \mathbb{C}^k$ be the vector of unknown values at the known non-zero positions $\{f(x_i, y_i)\}_{i=1}^k$. Define the matrix $\mathbf{A} \in \mathbb{C}^{m \times k}$ where

$$\mathbf{A}_{j,i} = e^{-2\pi i\left(\frac{u_j x_i}{n} + \frac{v_j y_i}{n}\right)}.$$

This yields the system:

$$\mathbf{F} = \mathbf{A}\mathbf{f}.$$

5. **Solving the System**: We can solve for $\mathbf{f}$ by inverting or solving this system of equations:

- **Exact Reconstruction**: If $m = k$ and the matrix $\mathbf{A}$ has full column rank, we can exactly solve for $\mathbf{f}$ using standard linear algebra techniques.
- **Overdetermined System**: If $m > k$, we solve the system in a least-squares sense to find the best approximation of $\mathbf{f}$.
- **Underdetermined or Noisy System**: If m is only slightly larger than k or noise is present, we can use regularized methods, such as LASSO, to improve stability.

Part (b): Minimum Value of m

The minimum number of DFT coefficients m required for successful reconstruction depends on the number of unknowns, k:

- In an ideal, noise-free case, the minimum value of m is $m = k$, as we have k unknowns.
- In practical scenarios, a slightly larger m, such as $m = ck$ for a constant $c > 1$ (e.g., $c \approx 1.5$), can help ensure robustness against noise and improve stability.

Thus, the minimum value of m can be expressed as:

$$m \geq k.$$

Part (c): Reconstruction with Unknown Locations of Non-Zero Elements

If the locations of the non-zero elements are unknown, the reconstruction becomes significantly more challenging:

- **Unknown Support**: Without knowing the locations, we lack the "support" (positions) of the non-zero values, which means we cannot directly set up the linear system as in Part (a).
- **Compressed Sensing Approach**: We can potentially reconstruct the image by using compressed sensing techniques, which allow us to recover sparse signals from fewer measurements than conventional methods. In this case, we would need a sufficient number of random DFT coefficients and would have to solve for both the positions and values of the non-zero elements.
- **Increased Measurement Requirement**: Without known locations, more measurements are generally required. To recover both the positions and values of k non-zero elements in an $n \times n$ image, compressed sensing theory suggests that m should be on the order of $O(k \log(n^2/k))$.

In summary:

- With known positions, we can reconstruct the image if $m \geq k$.
- The minimum m for exact reconstruction is $m = k$ in an ideal case.
- If locations are unknown, the required m increases, typically $m = O(k \log(n^2/k))$, to reliably recover both positions and values of the non-zero elements.