CS663: Fundamentals of Digital Image Processing Homework II

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Question 7)

Answer:
Rotate (x, y) by θ to get (u, v):

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We get:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x\cos\theta + -y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

We can define a new function $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ that

$$\tilde{f}(u, v) = f(x(u, v), y(u, v))$$

If Laplace's Operator is rotationally invariant, then

$$\nabla^2 \tilde{f}(u, v) = \nabla^2 f(x, y)$$

This can be proved by the following:

$$\begin{aligned}
\nabla^2 \tilde{f}(u, v) &= \tilde{f}_{uu} + \tilde{f}_{vv} \\
&= \frac{\partial}{\partial u}\left(f_x \frac{\partial x}{\partial u} + f_y \frac{\partial y}{\partial u}\right) + \frac{\partial}{\partial v}\left(f_x \frac{\partial x}{\partial v} + f_y \frac{\partial y}{\partial v}\right) \\
&\quad + f_{xx}\left(\frac{\partial x}{\partial u}\right)^2 + f_{yy}\left(\frac{\partial y}{\partial u}\right)^2 + f_{xy}\frac{\partial y}{\partial u}\frac{\partial x}{\partial u} \\
&\quad + f_x \frac{\partial^2 x}{\partial u^2} + f_y \frac{\partial^2 y}{\partial u^2} \\
&\quad + f_{xx}\left(\frac{\partial x}{\partial v}\right)^2 + f_{yy}\left(\frac{\partial y}{\partial v}\right)^2 + f_{xy}\frac{\partial y}{\partial v}\frac{\partial x}{\partial v} \\
&\quad + f_x \frac{\partial^2 x}{\partial v^2} + f_y \frac{\partial^2 y}{\partial v^2} \\
&= \tilde{f}_{uu} + \tilde{f}_{vv} + f_{xx}\cos^2\theta - 2\sin\theta\cos\theta f_{xy} + \sin^2\theta f_{yy} \\
&\quad + f_{xx}\sin^2\theta + 2\sin\theta\cos\theta f_{xy} + \cos^2\theta f_{yy} \\
&= f_{xx}\sin^2\theta + 2\sin\theta\cos\theta f_{xy} + \cos^2\theta f_{yy} + \tilde{f}_{uu} + \tilde{f}_{vv} \\
&= \nabla^2 f(x, y)
\end{aligned}$$