Linear Regression

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Supervised Machine Learning

- Our goal in supervised machine learning is to extract a relationship from data (ordered pairs of *(y,x)*)

The real relation is

$$y = f(x) + \epsilon$$

ϵ is noise with zero mean.

What we get from learning from data is

$$\hat{y} = h(x)$$

Regression vs Classification

$$y = f(x) + \epsilon$$

- The task of classification differs from regression in that we assign a discrete number of classes (nominal scale or ordinal scale), instead of assigning it a continuous value (interval or ratio scale).

- If y is in interval or ratio scale, then it is regression
- If y is in Nominal or ordinal (?) scale, then it is classification

Regression

- Extract a relationship from data

$(\bar{x}_i, y_i)$

$\bar{x}_i = \begin{Bmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^p \end{Bmatrix}$

$\boxed{y = f(x)}$

Learning arbitrary function is intractable

H is a class of functions

→ parametric function $\quad w \in R^d$

Parameters or the weights

$h_w(x) = \boxed{g(x, w)}$?

Learning is to find w from data (x, y)

Regression

$$h_\omega(x) = g(x, \omega)$$

Case 1 : g_1 : $\underline{\omega_0 + \omega_1 x}$ | ω_0, ω_1

Case 2 : g_2 : $\omega_0 + \omega_1 x + \omega_2 x^2$ | $\omega_0, \omega_1, \omega_2$

Case 3 : g_3 : $\omega_0 + \omega_1 x + \omega_2 x^2 + \omega_3 x^3$ | $\omega_0, \omega_1, \omega_2, \omega_3$

Linear Regression

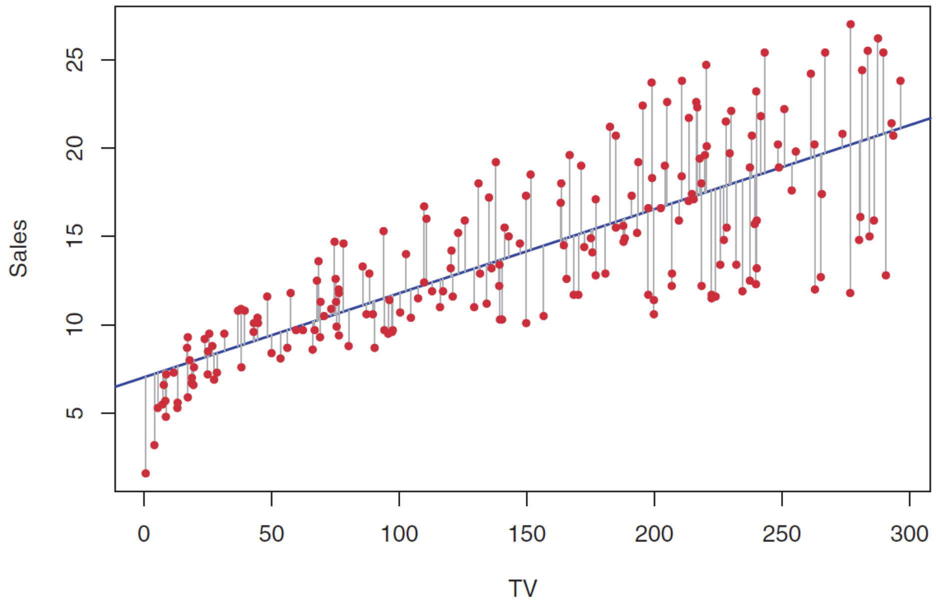
$y = f(x) + \epsilon$ Error/Noise $\quad | \quad y = \beta_0 + \beta_1 x + \epsilon$

- It assumes a linear relation between input x and output y

$$Y \approx \hat{\beta}_0 + \hat{\beta}_1 X$$

Approximately modeled

$\hat{\beta}_0$ and $\hat{\beta}_1$ are unknown coefficients or parameters which are estimated from training data.



Simple Linear Regression

It assumes that there is approximately a linear relationship between *X* and *Y*

$$Y \approx \beta_0 + \beta_1 X \quad \text{or} \quad Y = \beta_0 + \beta_1 X + \epsilon.$$

β0 and β1 are intercept slope known as the model coefficients or parameters

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Hat symbol, ˆ , to denote the estimated value for an unknown parameter or coefficient

Simple Linear Regression

Estimating the Coefficients

- Least squares approach

The least squares approach chooses parameters to minimize the *residual sum of squares* (RSS)

$e_i = y_i - \hat{y}_i$ represents i_{th} residual

$$\mathrm{RSS} = e_1^2 + e_2^2 + \cdots + e_n^2$$

$$\mathrm{RSS} = (y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1)^2 + (y_2 - \hat{\beta}_0 - \hat{\beta}_1 x_2)^2 + \ldots + (y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)^2$$

Simple Linear Regression

Residual sum of squares (RSS)

$$\mathrm{RSS} = e_1^2 + e_2^2 + \cdots + e_n^2$$

$e_i = y_i - \hat{y}_i$

$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

$$\mathrm{RSS} = (y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1)^2 + (y_2 - \underline{\hat{\beta}_0} - \underline{\hat{\beta}_1} x_2)^2 + \ldots + (y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)^2$$

By minimizing RSS we can find

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2},$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x},$$

$\frac{\partial \mathrm{RSS}}{\partial \hat{\beta}_1} = 0, \quad \frac{\partial \mathrm{RSS}}{\partial \hat{\beta}_0} = 0$

where $\bar{y} \equiv \frac{1}{n}\sum_{i=1}^n y_i$ and $\bar{x} \equiv \frac{1}{n}\sum_{i=1}^n x_i$

Simple Linear Regression

Estimating the Coefficients

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x},$$

where $\bar{y} \equiv \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i$

Simple Linear Regression

Assessing the Accuracy of the Model

Residual Standard Error (RSE)

$$\mathrm{RSE} = \sqrt{\frac{1}{n-2}\mathrm{RSS}} = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

$$\mathrm{RSS} = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

R^2 Statistic: The RSE provides an absolute measure, R^2 provides a relative measure

$$R^2 = \frac{\mathrm{TSS} - \mathrm{RSS}}{\mathrm{TSS}} = 1 - \frac{\mathrm{RSS}}{\mathrm{TSS}} \quad \text{where } \mathrm{TSS} = \sum(y_i - \bar{y})^2$$

$$R = \mathrm{Cor}(X, Y) = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \overline{y})^2}}.$$

Simple Linear Regression

$$\boxed{y = f(x, \beta_0, \beta_1, \ldots \beta_j) + \epsilon}$$

$\leftarrow y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

$\uparrow$

- zero mean
- σ^2 var
- Uncorrelated

Population

X: $x_1, x_2 \ldots \ldots \ldots \infty$

Y: $y_1, y_2 \ldots \ldots \ldots \infty$

Sample is of size n

$$\left.\begin{array}{l} x_1, x_2 \ldots \ldots x_n \\ y_1, y_2 \ldots \ldots y_n \end{array}\right\} \Rightarrow \text{Find}$$

β_0, β_1

$\hat{\beta}_0, \hat{\beta}_1$

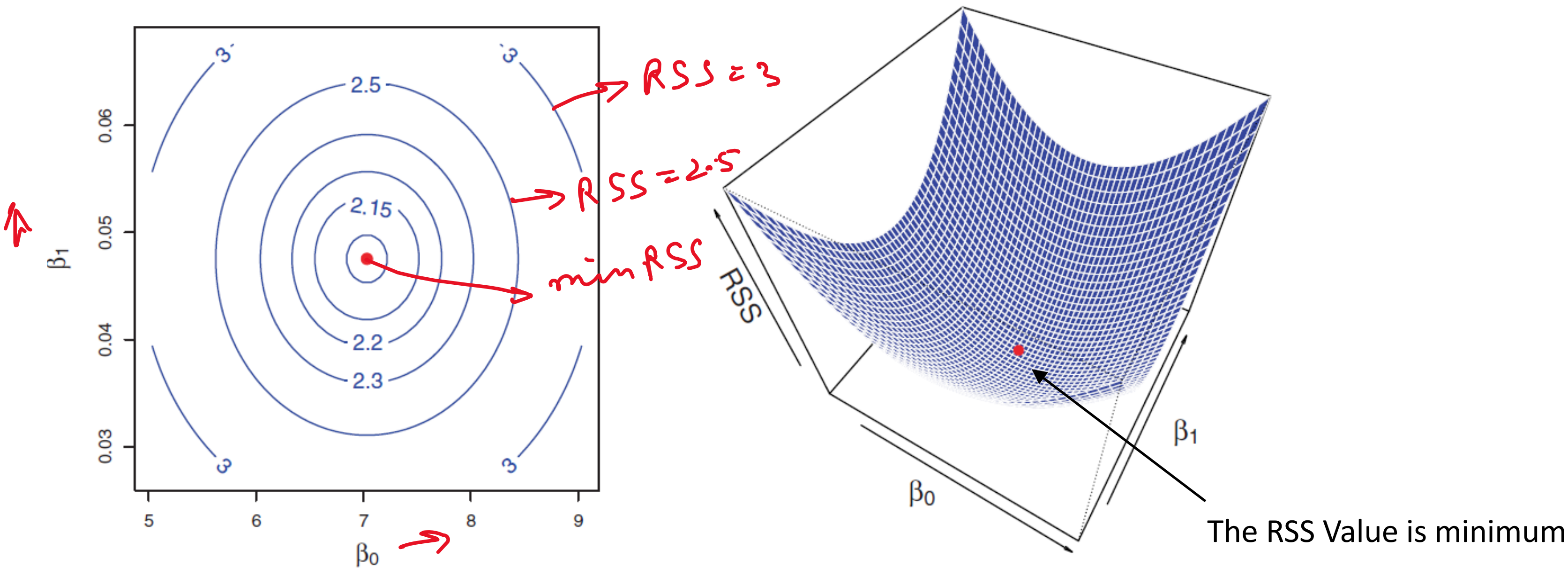
Assessing the accuracy of fit

$\rightarrow \hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

Q1. How close is $\hat{\beta}_0$ to β_0 and $\hat{\beta}_1$ to β_1

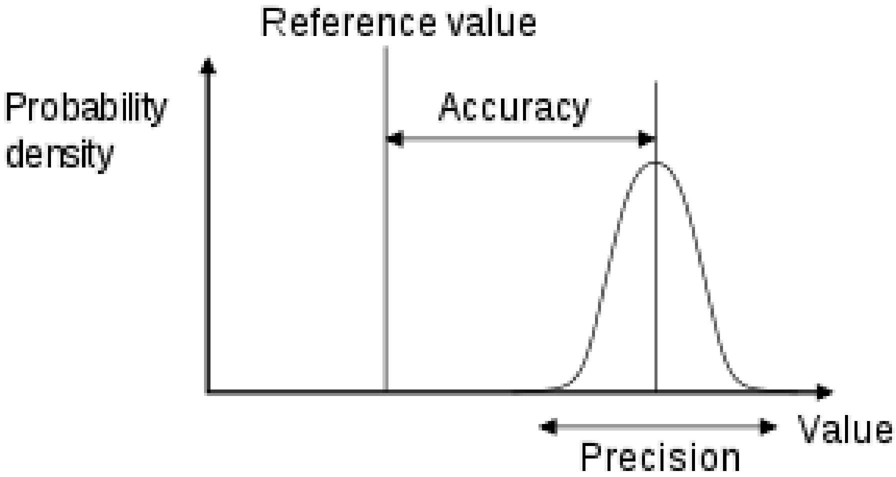
Linear Regression

$RSS = f(\hat{\beta}_0, \hat{\beta}_1)$;

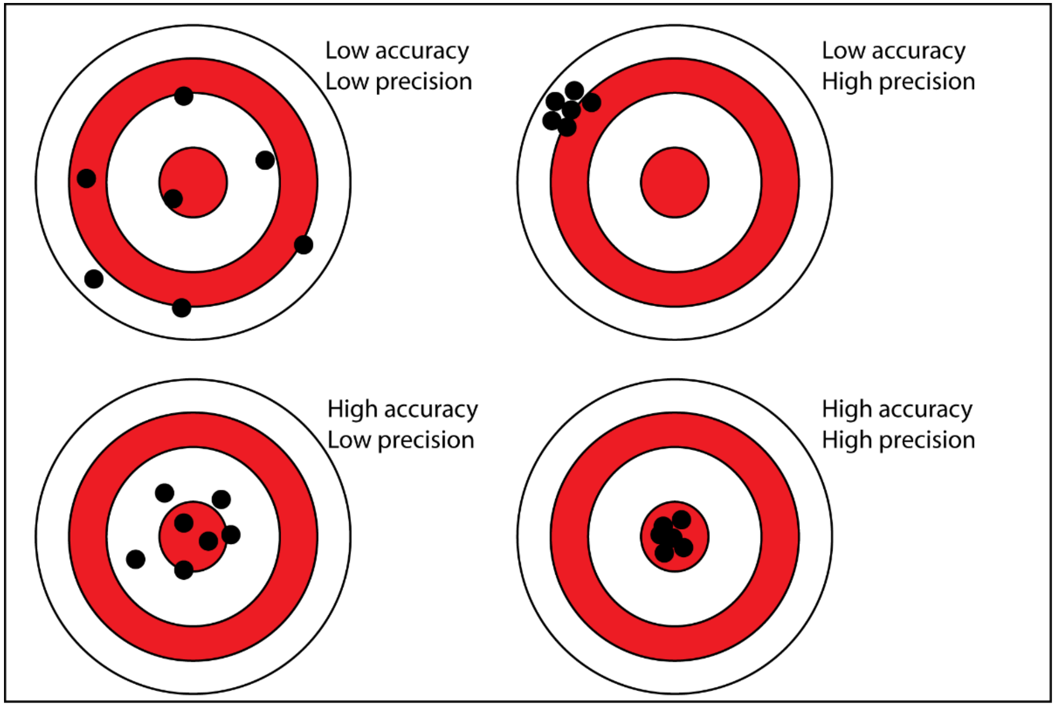


Accuracy and Precision

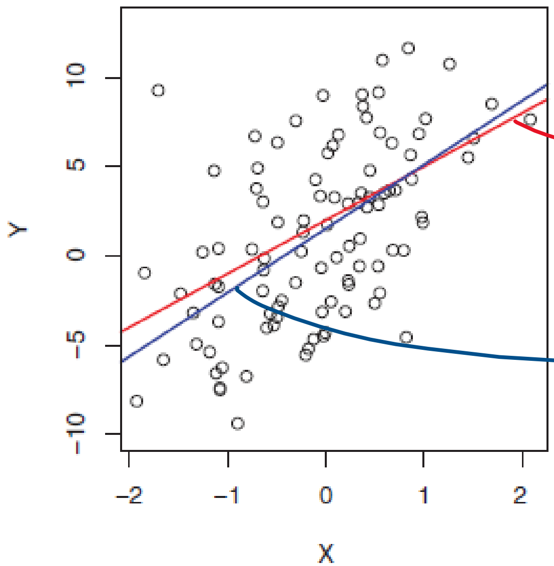
- Accuracy refers to the closeness of a measured value to a standard or known value. Accuracy is a description of systematic errors, a measure of statistical bias.
- Precision refers to the closeness of two or more measurements to each other. Precision is a description of random errors, a measure of statistical variability.



Accuracy Vs. Precision



Linear Regression

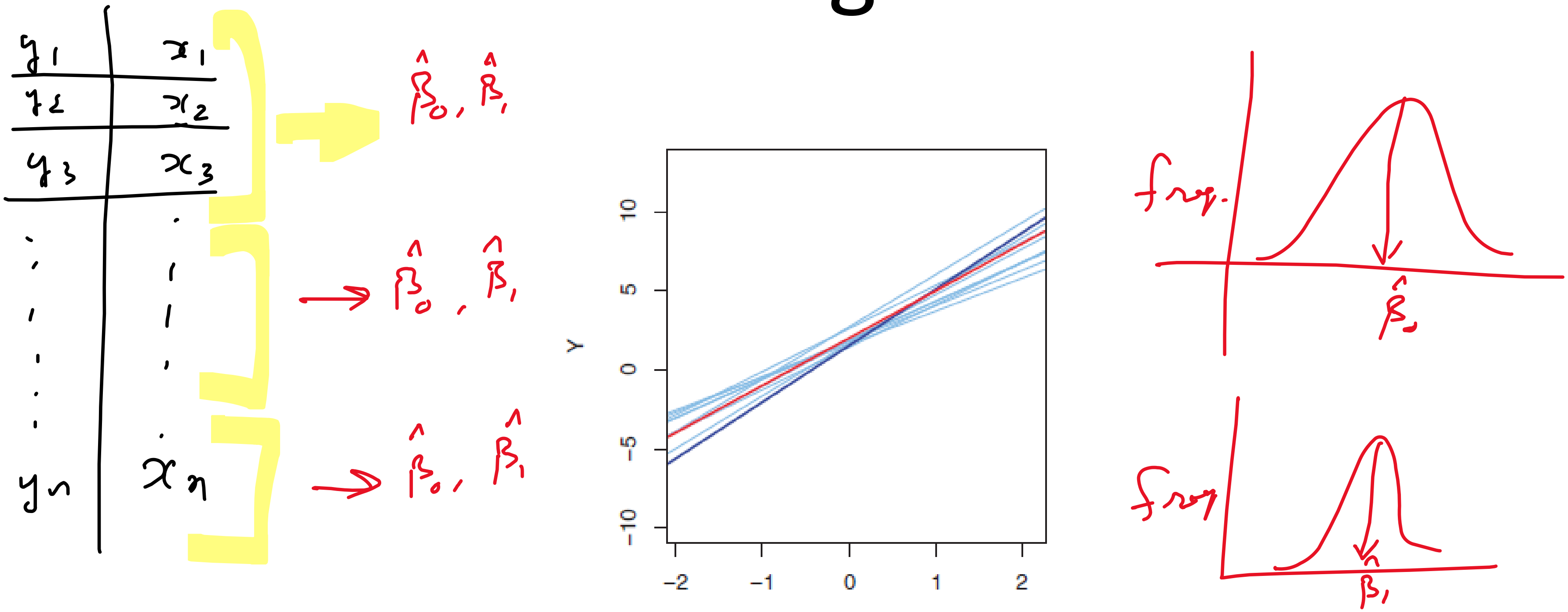


β_0 , β_1

$\hat{\beta}_0 , \hat{\beta}_1$

A simulated data set. Left: *The red line represents the true relationship, f*(*X*) = 2+3*X, which is known as the population regression line. The blue line is the least squares line; it is the least squares estimate for f*(*X*) *based on the observed data, shown in black*

Linear Regression



The population regression line is again shown in red, and the least squares line in dark blue. In light blue, ten least squares lines are shown, each computed on the basis of a <u>separate random set of observations</u>. Each least squares line is different, but on average, the least squares lines are quite close to the population regression line.

Simple Linear Regression

$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

Assessing the Accuracy of the Coefficient Estimates

Standard Errors associated with coefficients

$$\mathrm{SE}(\hat{\beta}_0)^2 = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right], \quad \mathrm{SE}(\hat{\beta}_1)^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

Where $\sigma^2 = var(\epsilon)$ and it is assumed that ϵ_i are uncorrelated and has same variance σ^2

95% confidence interval associated with coefficients

$\hat{\beta}_0 \pm 1.96\, SE(\hat{\beta}_0)$ $\hat{\beta}_1 \pm 1.96\, SE(\hat{\beta}_1)$

Table of the Student's *t*-distribution

The table gives the values of $t_{\alpha;\nu}$ where
$\Pr(T_\nu > t_{\alpha;\nu}) = \alpha$, with ν degrees of freedom

5%

2.5%

ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.076	31.821	63.657	318.310	636.620
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

$\nu \to \infty$

$22 - 2 =$ 20

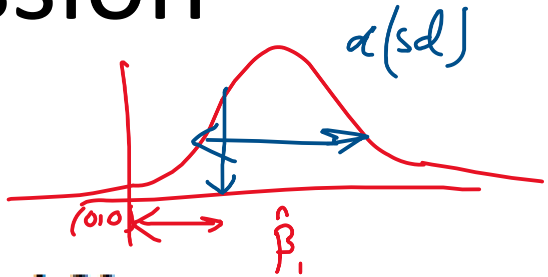
$\pm 2.086\,(SE)$

$\boxed{\pm 1.96\,(SE)}$

1.96

Simple Linear Regression

$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$



Hypothesis tests on the coefficients

$$H_0 : \text{There is no relationship between } X \text{ and } Y$$

versus the *alternative hypothesis*

$$H_a : \text{There is some relationship between } X \text{ and } Y$$

Mathematically, this corresponds to testing

$$H_0 : \beta_1 = 0 \qquad \text{versus} \qquad H_a : \beta_1 \neq 0$$

For this we calculate t statistics which measures the number of standard deviations that $\hat{\beta}_1$ *is away from 0.*

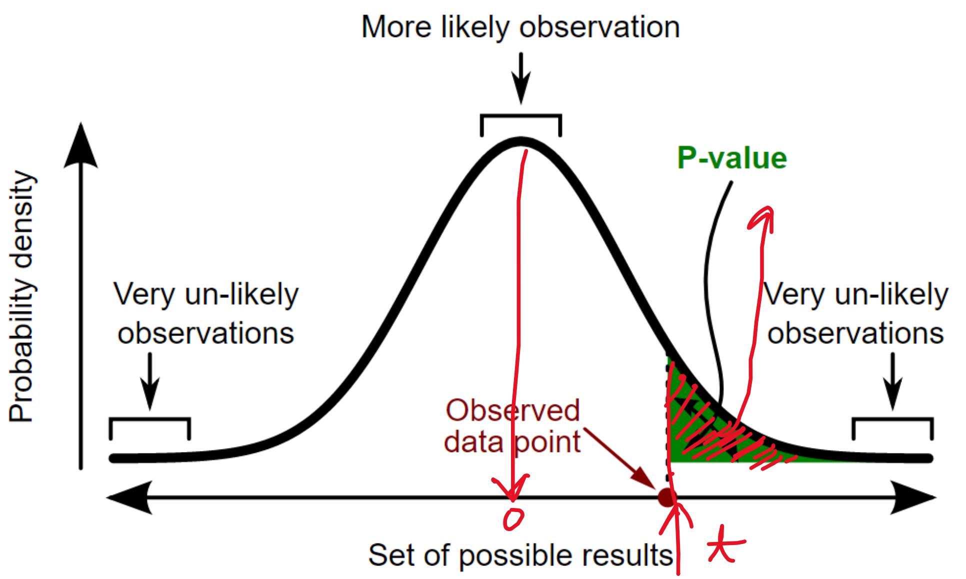
$$t = \frac{\hat{\beta}_1 - 0}{\mathrm{SE}(\hat{\beta}_1)},$$

Simple Linear Regression

P-Value is the probability of observing any value equal to |t| or larger for a t-distribution with n−2 degrees of freedom

Sample of size n $\rightarrow \hat{\beta}_0, \underbrace{\hat{\beta}_1, SE(\hat{\beta}_1)}$

$$t = \frac{\hat{\beta}_1 - 0}{\mathrm{SE}(\hat{\beta}_1)}, \quad t$$



A **p-value** (shaded green area) is the probability of an observed (or more extreme) result assuming that the null hypothesis is true.

t

Sample of size n → $\hat{\beta}_0, \hat{\beta}_1$ → $t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)}$ → compare with std. t-dist of n-2 degrees of freedom

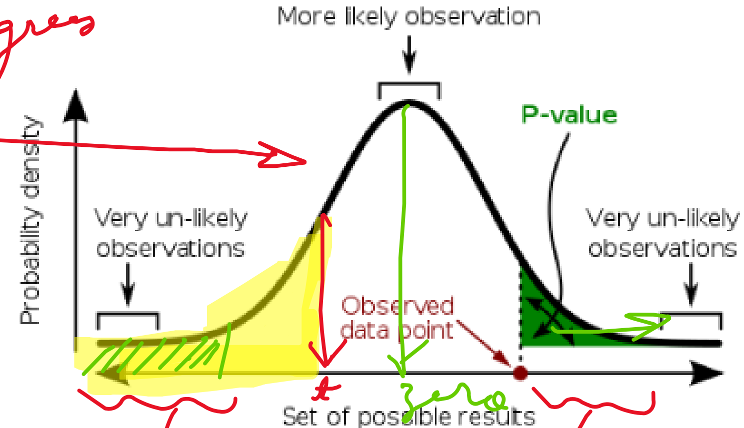
Simple Linear Regression

$$H_0 : \beta_1 = 0 \qquad H_a : \beta_1 \neq 0$$

$$t = \frac{\hat{\beta}_1 - 0}{\mathrm{SE}(\hat{\beta}_1)},$$

$\nu = n - 2$ degrees

t-dist



A **p-value** (shaded green area) is the probability of an observed (or more extreme) result assuming that the null hypothesis is true.

p-value is defined as

- $\Pr(T \geq t|H)$ for a one-sided (right tail) test,
- $\Pr(T \leq t|H)$ for a one-sided (left tail) test,
- $2\min\{\Pr(T \leq t|H), \Pr(T \geq t|H)\}$ for a two-sided test,

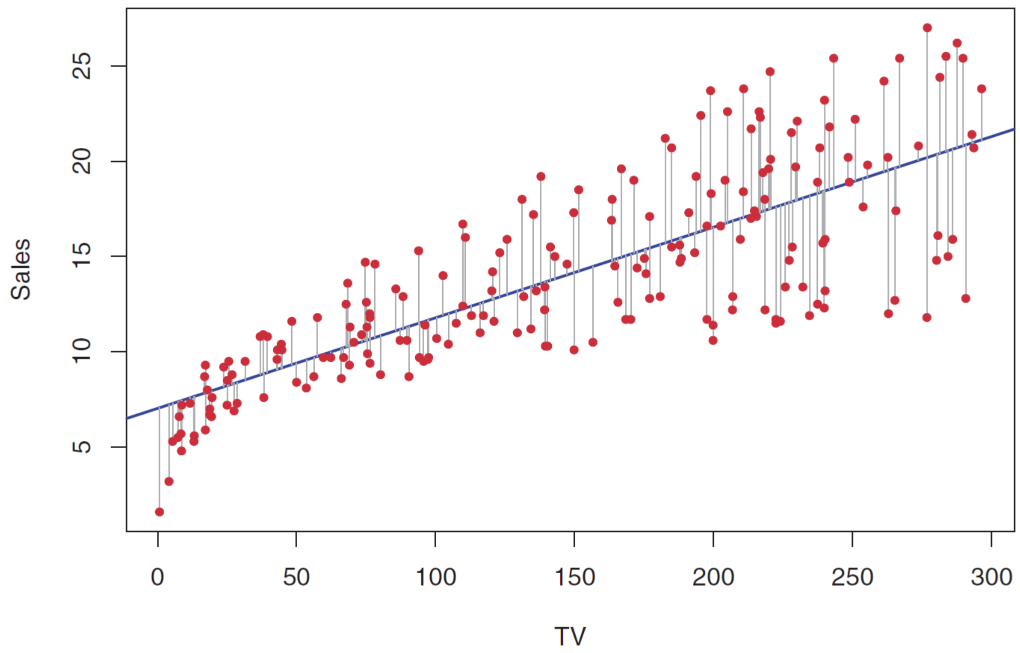
Notice that just by replacing T by $-T$ one converts a test based on extremely large values to a test based on extremely small values; ar by replacing T by $|T|$ one gets a test with *p*-value

- $\Pr(T \leq -|t||H) + \Pr(T \geq +|t||H)\}$.

Simple Linear Regression

- The p-value represents the chance your results could be random (i.e. happened by chance).

- So a small p-value means that there is a small chance that your results are random. Thus, they are not random. So we can infer that there is an association between the predictor and the response (i.e we *reject the null hypothesis)*



For the Advertising *data, the least squares fit for the regression of* sales *onto* TV *is shown. The fit is found by minimizing the sum of squared errors. Each grey line segment represents an error, and the fit makes a compromise by averaging their squares. In this case a linear fit captures the essence of the relationship, although it is somewhat deficient in the left of the plot.*

P-value

	Coefficient	Std. error	t-statistic	p-value
Intercept	7.0325	0.4578	15.36	< 0.0001
TV	0.0475	0.0027	17.67	< 0.0001

For the Advertising *data, coefficients of the least squares model for the regression of number of units sold on TV advertising budget. An increase of* \$1,000 *in the TV advertising budget is associated with an increase in sales by around 50 units (Recall that the* sales *variable is in thousands of units, and the* TV *variable is in thousands of dollars).*