Supplementary Document
Dynamically Stable 3D Quadrupedal Walking with Multi-Domain Hybrid System Models and Virtual Constraint Controllers*

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PROOF OF THEOREM 1

Proof: For the continuous-time domain $v \in \mathcal{V}$, let us define the flow map according to the following policy

$$\mathcal{F}_v\left(x, \xi_v\right) := \varphi_v\left(T_v\left(x, \xi_v\right); x, \xi_v\right) \tag{22}$$

that represents the state solution $\varphi_v(t; x, \xi_v)$ evaluated on the switching manifold $\hat{\mathcal{S}}_{v \to \mu(v)}(\xi_v)$. For single-contact domains $v \in \mathcal{V}$, the decoupling matrix $A_v(x, \xi_v)$ in (13) is square and assumed to be full-rank on an open neighborhood of $\overline{\mathcal{O}}_v$ for every $\xi_v \in \Xi_v$. Hence, the feedback driving $y_v(x, \xi_v)$ to zero is unique on each zero dynamics manifold $\mathcal{Z}_v(\xi_v)$. According to the construction procedure, the orbit $\overline{\mathcal{O}}_v$ is also common to all zero dynamics manifolds, i.e., $\overline{\mathcal{O}}_v \subset \mathcal{Z}_v(\xi_v)$ for all $\xi_v \in \Xi_v$. Therefore, the control restricted to the orbit $\overline{\mathcal{O}}_v$ is independent of ξ_v, i.e.,

$$\frac{\partial}{\partial \xi_v} \Gamma_v\left(x, \xi_v\right)\Big|_{\overline{\mathcal{O}}_v} = 0. \tag{23}$$

This together with [1, Lemma 1] shows that $\overline{\mathcal{O}}_v$ is an invariant integral curve of $\dot{x} = f_v^{\mathrm{cl}}(x, \xi_v)$ for all $\xi_v \in \Xi_v$. If we define the initial and final states on the orbit $\overline{\mathcal{O}}_v$ as $x_{0v}^\star$ and $x_{fv}^\star$, respectively, one can conclude that

$$\mathcal{F}_v\left(x_{0v}^\star, \xi_v\right) = x_{fv}^\star, \quad \forall \xi_v \in \Xi_v. \tag{24}$$

We also remark that

$$\left\{x_{fv}^\star\right\} := \overline{\mathcal{O}}_v \cap \hat{\mathcal{S}}_{v \to \mu(v)}\left(\xi_v\right), \quad \forall \xi_v \in \Xi_v. \tag{25}$$

For multi-contact domains $v \in \mathcal{V}$, the decoupling matrix $A_v(x, \xi_v)$ in (9) is not square and the feedback deriving $y_v(x, \xi_v)$ to zero is not unique. In fact, the I-O linearizing control (11) presents the minimum 2-norm solution that solves the output zeroing problem, i.e.,

$$\begin{aligned}\Gamma_v\left(x, \xi_v\right) = \ & \mathrm{argmin}_u \ \frac{1}{2}\|u\|_2^2 \\ & \text{s.t.} \qquad A_v\left(x, \xi_v\right) u + b_v\left(x, \xi_v\right) = -w_v.\end{aligned}$$

From Assumption 3, there is a nominal controller parameter $\xi_v^\star$ for which the minimum 2-norm solution $\Gamma_v(x, \xi_v)$ coincides with the nominal control trajectory, i.e.,

$$\Gamma_v\left(x^\star(t, v), \xi_v\right) = u^\star\left(t, v\right). \tag{26}$$

In other words, $\overline{\mathcal{O}}_v$ is an integral curve of the closed-loop ODE $\dot{x} = f_v^{\mathrm{cl}}(x, \xi_v^\star)$. Consequently,

$$\mathcal{F}_v\left(x_{0v}^\star, \xi_v^\star\right) = x_{fv}^\star. \tag{27}$$

Next, let us define the mapping $\mathcal{G}_v : \Xi_v \to \mathbb{R}^{2n_q}$ by

$$\mathcal{G}_v\left(\xi_v\right) := \mathcal{F}_v\left(x_{0v}^\star, \xi_v\right) - x_{fv}^\star. \tag{28}$$

From the transversality assumption and [1, Theorem 2], one can show that $\mathcal{G}_v$ is differentiable with respect to ξ_v at $\xi_v = \xi_v^\star$. This latter fact together with $\mathcal{G}_v(\xi_v^\star) = 0$ and $p_v = \dim(\xi_v) > \dim(\mathcal{G}_v) = 2n_q$ implies that there is a nonempty set $\hat{\Xi}_v \subset \Xi_v$ such that (i) $\xi_v^\star \in \hat{\Xi}_v$ and (ii) $\mathcal{G}_v(\xi_v) = 0$ for all $\xi_v \in \hat{\Xi}_v$, or equivalently,

$$\mathcal{F}_v\left(x_{0v}^\star, \xi_v\right) = x_{fv}^\star, \quad \forall \xi_v \in \hat{\Xi}_v. \tag{29}$$

Equation (29) also implies that

$$P_v\left(x_{f\mu^{-1}(v)}^\star, \xi_v\right) = x_{fv}^\star, \quad \forall \xi_v \in \hat{\Xi}_v \tag{30}$$

which in turn results in the existence of an invariant fixed point $x_1^\star$ for the full-order Poincaré map $P(x, \xi)$. In particular, $P(x_1^\star, \xi) = 0$ for all $\xi \in \hat{\Xi}$, where $x_1^\star := x_{fv_1}^\star$ and $\hat{\Xi} := \hat{\Xi}_{v_1} \times \cdots \times \hat{\Xi}_{v_N}$. The continuous differentiability of the Jacobian matrix $\Psi(\xi)$ with respect to ξ follows from [1, Theorem 2]. ■

REFERENCES

[1] K. Akbari Hamed, B. Buss, and J. Grizzle, "Exponentially stabilizing continuous-time controllers for periodic orbits of hybrid systems: Application to bipedal locomotion with ground height variations," *The International Journal of Robotics Research*, vol. 35, no. 8, pp. 977–999, 2016.

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