Supplementary Document
Asymptotically Stabilizing Virtual Constraint Controllers for Multi-Domain Periodic Gaits of 3D Quadruped Locomotion*

Kaveh Akbari Hamed[1], Wen-Loong Ma[2], and Aaron D. Ames[2]

PROOF OF THEOREM 1

Proof: For the continuous-time domain $v \in \mathcal{V}$, let us define the flow map according to the following policy

$$\mathcal{F}_v\left(x, \xi_v\right) := \varphi_v\left(T_v\left(x, \xi_v\right); x, \xi_v\right) \tag{22}$$

that represents the state solution $\varphi_v(t; x, \xi_v)$ evaluated on the switching manifold $\hat{\mathcal{S}}_{v\to\mu(v)}(\xi_v)$. For single-contact domains $v \in \mathcal{V}$, the decoupling matrix $A_v(x, \xi_v)$ in (13) is square and assumed to be full-rank on an open neighborhood of $\overline{\mathcal{O}}_v$ for every $\xi_v \in \Xi_v$. Hence, the feedback driving $y_v(x, \xi_v)$ to zero is unique on each zero dynamics manifold $\mathcal{Z}_v(\xi_v)$. According to the construction procedure, the orbit $\overline{\mathcal{O}}_v$ is also common to all zero dynamics manifolds, i.e., $\overline{\mathcal{O}}_v \subset \mathcal{Z}_v(\xi_v)$ for all $\xi_v \in \Xi_v$. Therefore, the control restricted to the orbit $\overline{\mathcal{O}}_v$ is independent of ξ_v, i.e.,

$$\frac{\partial}{\partial \xi_v}\Gamma_v\left(x, \xi_v\right)\Big|_{\overline{\mathcal{O}}_v} = 0. \tag{23}$$

This together with [1, Lemma 1] shows that $\overline{\mathcal{O}}_v$ is an invariant integral curve of $\dot{x} = f_v^{\mathrm{cl}}(x, \xi_v)$ for all $\xi_v \in \Xi_v$. If we define the initial and final states on the orbit $\overline{\mathcal{O}}_v$ as $x_{0v}^\star$ and $x_{fv}^\star$, respectively, one can conclude that

$$\mathcal{F}_v\left(x_{0v}^\star, \xi_v\right) = x_{fv}^\star, \quad \forall \xi_v \in \Xi_v. \tag{24}$$

We also remark that

$$\left\{x_{fv}^\star\right\} := \overline{\mathcal{O}}_v \cap \hat{\mathcal{S}}_{v\to\mu(v)}\left(\xi_v\right), \quad \forall \xi_v \in \Xi_v. \tag{25}$$

For multi-contact domains $v \in \mathcal{V}$, the decoupling matrix $A_v(x, \xi_v)$ in (9) is not square and the feedback deriving $y_v(x, \xi_v)$ to zero is not unique. In fact, the I-O linearizing control (11) presents the minimum 2-norm solution that solves the output zeroing problem, i.e.,

$$\begin{aligned}\Gamma_v\left(x, \xi_v\right) = \operatorname{argmin}_u \ & \frac{1}{2}\|u\|_2^2 \\ \text{s.t.} \qquad & A_v\left(x, \xi_v\right) u + b_v\left(x, \xi_v\right) = -w_v.\end{aligned}$$

From Assumption 3, there is a nominal controller parameter $\xi_v^\star$ for which the minimum 2-norm solution $\Gamma_v(x, \xi_v)$ coincides with the nominal control trajectory, i.e.,

$$\Gamma_v\left(x^\star(t, v), \xi_v\right) = u^\star\left(t, v\right). \tag{26}$$

In other words, $\overline{\mathcal{O}}_v$ is an integral curve of the closed-loop ODE $\dot{x} = f_v^{\mathrm{cl}}(x, \xi_v^\star)$. Consequently,

$$\mathcal{F}_v\left(x_{0v}^\star, \xi_v^\star\right) = x_{fv}^\star. \tag{27}$$

Next, let us define the mapping $\mathcal{G}_v : \Xi_v \to \mathbb{R}^{2n_q}$ by

$$\mathcal{G}_v\left(\xi_v\right) := \mathcal{F}_v\left(x_{0v}^\star, \xi_v\right) - x_{fv}^\star. \tag{28}$$

From the transversality assumption and [1, Theorem 2], one can show that $\mathcal{G}_v$ is differentiable with respect to ξ_v at $\xi_v = \xi_v^\star$. This latter fact together with $\mathcal{G}_v(\xi_v^\star) = 0$ and $p_v = \dim(\xi_v) > \dim(\mathcal{G}_v) = 2n_q$ implies that there is a nonempty set $\hat{\Xi}_v \subset \Xi_v$ such that (i) $\xi_v^\star \in \hat{\Xi}_v$ and (ii) $\mathcal{G}_v(\xi_v) = 0$ for all $\xi_v \in \hat{\Xi}_v$, or equivalently,

$$\mathcal{F}_v\left(x_{0v}^\star, \xi_v\right) = x_{fv}^\star, \quad \forall \xi_v \in \hat{\Xi}_v. \tag{29}$$

Equation (29) also implies that

$$P_v\left(x_{f\mu^{-1}(v)}^\star, \xi_v\right) = x_{fv}^\star, \quad \forall \xi_v \in \hat{\Xi}_v \tag{30}$$

which in turn results in the existence of an invariant fixed point $x_1^\star$ for the full-order Poincaré map $P(x, \xi)$. In particular, $P(x_1^\star, \xi) = 0$ for all $\xi \in \hat{\Xi}$, where $x_1^\star := x_{fv_1}^\star$ and $\hat{\Xi} := \hat{\Xi}_{v_1} \times \cdots \times \hat{\Xi}_{v_N}$. The continuous differentiability of the Jacobian matrix $\Psi(\xi)$ with respect to ξ follows from [1, Theorem 2]. ■

REFERENCES

[1] K. Akbari Hamed, B. Buss, and J. Grizzle, "Exponentially stabilizing continuous-time controllers for periodic orbits of hybrid systems: Application to bipedal locomotion with ground height variations," *The International Journal of Robotics Research*, vol. 35, no. 8, pp. 977–999, 2016.

*The work of K. Akbari Hamed is supported by the National Science Foundation (NSF) under Grant Number 1637704. The work of A. D. Ames is supported by the NSF under Grant Numbers 1544332, 1724457, and 1724464 as well as Disney Research LA. The content is solely the responsibility of the authors and does not necessarily represent the official views of the NSF.

[1]K. Akbari Hamed is with the Department of Mechanical Engineering, Virginia Tech, Blacksburg, VA 24061 USA kavehakbarihamed@vt.edu

[2]W. Ma and A. D. Ames are with the Department of Mechanical and Civil Engineering, California Institute of Technology, Pasadena, CA 91125 USA wma@caltech.edu and ames@cds.caltech.edu