Chapter 4

Tangent Cones

In this chapter certain approximations of sets are considered which are very useful for the formulation of optimality conditions. We investigate so-called tangent cones which approximate a given set in a local sense. First, we discuss several basic properties of tangent cones, and then we present optimality conditions with the aid of these cones. Finally, we formulate a Lyusternik theorem.

4.1 Definition and Properties

In this section we turn our attention to the sequential Bouligand tangent cone which is also called the contingent cone. For this tangent cone we prove several basic properties.

First, we introduce the concept of a cone.

Definition 4.1. Let C be a nonempty subset of a real linear space X.
(a) The set C is called a *cone* if

$$x \in C,\ \lambda \geq 0 \implies \lambda x \in C.$$

(b) A cone C is called *pointed* if

$$x \in C,\ -x \in C \implies x = 0_X.$$

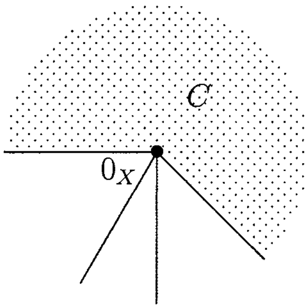

Figure 4.1: Cone.

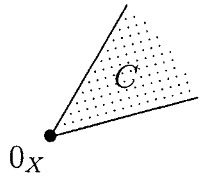

Figure 4.2: Pointed cone.

Example 4.2. (a) The set

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i \in \{1, \ldots, n\}\}$$

is a pointed cone.
(b) The set

$$C := \{x \in C[0,1] \mid x(t) \geq 0 \text{ for all } t \in [0,1]\}$$

is a pointed cone.

In order theory and optimization theory convex cones are of special interest. Such cones may be characterized as follows:

Theorem 4.3. *A cone C in a real linear space is convex if and only if for all $x, y \in C$*

$$x + y \in C. \tag{4.1}$$

Proof. (a) Let C be a convex cone. Then it follows for all $x, y \in C$

$$\frac{1}{2}(x+y) = \frac{1}{2}x + \frac{1}{2}y \in C$$

which implies $x + y \in C$.
(b) For arbitrary $x, y \in C$ and $\lambda \in [0,1]$ we have $\lambda x \in C$ and $(1-\lambda)y \in C$. Then we get with the condition (4.1)

$$\lambda x + (1-\lambda)y \in C.$$

Consequently, the cone C is convex. $\square$

In the sequel we also define cones generated by sets.

Definition 4.4. Let S be a nonempty subset of a real linear space. The set

$$\text{cone}(S) := \{\lambda s \mid \lambda \geq 0 \text{ and } s \in S\}$$

is called the *cone generated* by S.

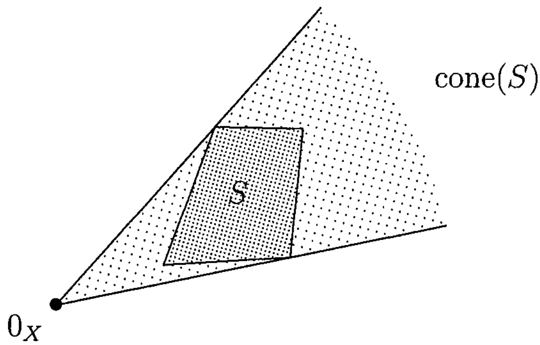

Figure 4.3: Cone generated by S.

Example 4.5. (a) Let $B(0_X, 1)$ denote the closed unit ball in a real normed space $(X, \|\cdot\|)$. Then the cone generated by $B(0_X, 1)$ equals the linear space X.
(b) Let S denote the graph of the function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then the cone generated by S is given as

$$\text{cone}(S) = \{(x, y) \in \mathbb{R}^2 \mid |y| \leq |x|\}.$$

Now we turn our attention to tangent cones.

Definition 4.6. Let S be a nonempty subset of a real normed space $(X, \|\cdot\|)$.
(a) Let $\bar{x} \in \mathrm{cl}(S)$ be a given element. A vector $h \in X$ is called a *tangent vector* to S at $\bar{x}$, if there are a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in S and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers with

$$\bar{x} = \lim_{n\to\infty} x_n$$

and

$$h = \lim_{n\to\infty} \lambda_n(x_n - \bar{x}).$$

(b) The set $T(S, \bar{x})$ of all tangent vectors to S at $\bar{x}$ is called *sequential Bouligand*[4] *tangent cone* to S at $\bar{x}$ or *contingent cone* to S at $\bar{x}$.

Notice that $\bar{x}$ needs only to belong to the closure of the set S in the definition of $T(S, \bar{x})$. But later we will assume that $\bar{x} \in S$.

By the definition of tangent vectors it follows immediately that the contingent cone is in fact a cone.

Before investigating the contingent cone we briefly present the definition of the Clarke tangent cone which is not used any further in this chapter.

Remark 4.7. Let $\bar{x}$ be an element of the closure of a nonempty subset S of a real normed space $(X, \|\cdot\|)$.
(a) The set

$$\begin{aligned} T_{Cl}(S, \bar{x}) \;:=\; \{h \in X \mid\ & \text{for every sequence } (x_n)_{n\in\mathbb{N}} \\ & \text{of elements of } S \text{ with } \bar{x} = \lim_{n\to\infty} x_n \text{ and} \\ & \text{for every sequence } (\lambda_n)_{n\in\mathbb{N}} \text{ of positive} \\ & \text{real numbers converging to 0 there is} \end{aligned}$$

[4]M.G. Bouligand, "Sur les surfaces dépourvues de points hyperlimites (ou: un thèorème d'existence du plan tangent)", *Ann. Soc. Polon. Math.* 9 (1930) 32–41. F. Severi remarked that he has independently introduced this notion (F. Severi, "Su alcune questioni di topologia infinitesimale", *Ann. Soc. Polon. Math.* 9 (1930) 97–108).

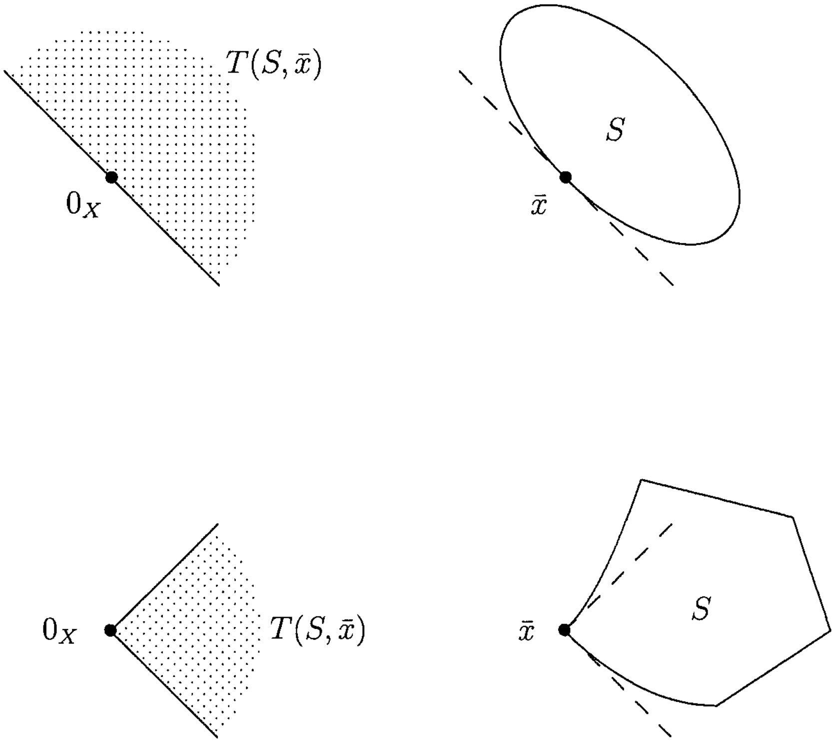

Figure 4.4: Two examples of contingent cones.

$$\text{a sequence } (h_n)_{n\in\mathbb{N}} \text{ with } h = \lim_{n\to\infty} h_n$$
$$\text{and } x_n + \lambda_n h_n \in S \text{ for all } n \in \mathbb{N}\}$$

is called *(sequential) Clarke tangent cone* to S at $\bar{x}$.
(b) It is evident that the Clarke tangent cone $T_{Cl}(S,\bar{x})$ is always a cone.
(c) If $\bar{x} \in S$, then the Clarke tangent cone $T_{Cl}(S,\bar{x})$ is contained in the contingent cone $T(S,\bar{x})$.
For the proof of this assertion let some $h \in T_{Cl}(S,\bar{x})$ be given arbitrarily. Then we choose the special sequence $(\bar{x})_{n\in\mathbb{N}}$ and an arbitrary sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0. Consequently, there is a sequence $(h_n)_{n\in\mathbb{N}}$ with $h = \lim\limits_{n\to\infty} h_n$ and $\bar{x}+\lambda_n h_n \in S$

for all $n \in \mathbb{N}$. Now we set

$$y_n := \bar{x} + \lambda_n h_n \text{ for all } n \in \mathbb{N}$$

and

$$t_n := \frac{1}{\lambda_n} \text{ for all } n \in \mathbb{N}.$$

Then it follows

$$y_n \in S \text{ for all } n \in \mathbb{N},$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (\bar{x} + \lambda_n h_n) = \bar{x}$$

and

$$\lim_{n\to\infty} t_n(y_n - \bar{x}) = \lim_{n\to\infty} \frac{1}{\lambda_n}(\bar{x} + \lambda_n h_n - \bar{x}) = \lim_{n\to\infty} h_n = h.$$

Consequently, h is a tangent vector. □

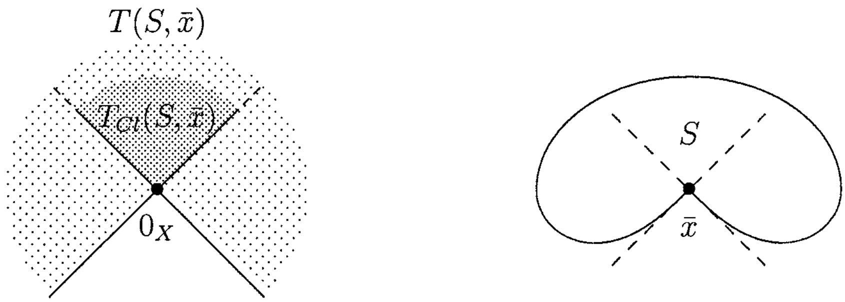

Figure 4.5: Illustration of the result in Remark 4.7,(c).

(d) The Clarke tangent cone $T_{Cl}(S, \bar{x})$ is always a closed convex cone. We mention this result without proof. Notice that this assertion is true without any assumption on the set S.

Next, we come back to the contingent cone and we investigate the relationship between the contingent cone $T(S, \bar{x})$ and the cone generated by $S - \{\bar{x}\}$.

Theorem 4.8. *Let S be a nonempty subset of a real normed space. If S is starshaped with respect to some $\bar{x} \in S$, then it follows*

$$cone(S - \{\bar{x}\}) \subset T(S, \bar{x}).$$

Proof. Let the set S be starshaped with respect to some $\bar{x} \in S$, and let an arbitrary element $x \in S$ be given. Then we define a sequence $(x_n)_{n\in\mathbb{N}}$ with

$$x_n := \bar{x} + \frac{1}{n}(x - \bar{x}) = \frac{1}{n}x + \Big(1 - \frac{1}{n}\Big)\bar{x} \in S \text{ for all } n \in \mathbb{N}.$$

For this sequence we have

$$\lim_{n\to\infty} x_n = \bar{x}$$

and

$$\lim_{n\to\infty} n(x_n - \bar{x}) = x - \bar{x}.$$

Consequently, $x - \bar{x}$ is a tangent vector, and we obtain

$$S - \{\bar{x}\} \subset T(S, \bar{x}).$$

Since $T(S, \bar{x})$ is a cone, we conclude

$$\text{cone}(S - \{\bar{x}\}) \subset \text{cone}(T(S, \bar{x})) = T(S, \bar{x}).$$

$\square$

Theorem 4.9. *Let S be a nonempty subset of a real normed space. For every $\bar{x} \in S$ it follows*

$$T(S, \bar{x}) \subset cl(cone(S - \{\bar{x}\})).$$

Proof. We fix an arbitrary $\bar{x} \in S$ and we choose any $h \in T(S, \bar{x})$. Then there are a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in S and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers with $\bar{x} = \lim_{n\to\infty} x_n$ and $h = \lim_{n\to\infty} \lambda_n(x_n - \bar{x})$. The last equation implies

$$h \in \mathrm{cl}(\mathrm{cone}(S - \{\bar{x}\}))$$

which has to be shown. □

By the two preceding theorems we obtain the following inclusion chain for a set S which is starshaped with respect to some $\bar{x} \in S$:

$$\mathrm{cone}(S - \{\bar{x}\}) \subset T(S, \bar{x}) \subset \mathrm{cl}(\mathrm{cone}(S - \{\bar{x}\})). \tag{4.2}$$

The next theorem says that the contingent cone is always closed.

Theorem 4.10. *Let S be a nonempty subset of a real normed space $(X, \|\cdot\|)$. For every $\bar{x} \in S$ the contingent cone $T(S, \bar{x})$ is closed.*

Proof. Let $\bar{x} \in S$ be arbitrarily chosen, and let $(h_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of tangent vectors to S at $\bar{x}$ with $\lim_{n\to\infty} h_n = h \in X$. For every tangent vector h_n there are a sequence $(x_{n_i})_{i\in\mathbb{N}}$ of elements in S and a sequence $(\lambda_{n_i})_{i\in\mathbb{N}}$ of positive real numbers with $\bar{x} = \lim_{i\to\infty} x_{n_i}$ and $h_n = \lim_{i\to\infty} \lambda_{n_i}(x_{n_i} - \bar{x})$. Consequently, for every $n \in \mathbb{N}$ there is a number $i(n) \in \mathbb{N}$ with

$$\|x_{n_i} - \bar{x}\| \le \frac{1}{n} \text{ for all } i \in \mathbb{N} \text{ with } i \ge i(n)$$

and

$$\|\lambda_{n_i}(x_{n_i} - \bar{x}) - h_n\| \le \frac{1}{n} \text{ for all } i \in \mathbb{N} \text{ with } i \ge i(n).$$

If we define the sequences $(y_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ by

$$y_n := x_{n_{i(n)}} \in S \text{ for all } n \in \mathbb{N}$$

and

$$t_n := \lambda_{n_{i(n)}} > 0 \text{ for all } n \in \mathbb{N},$$

then we obtain $\lim_{n\to\infty} y_n = \bar{x}$ and

$$\begin{aligned} \|t_n(y_n - \bar{x}) - h\| &= \|\lambda_{n_{i(n)}}(x_{n_{i(n)}} - \bar{x}) - h_n + h_n - h\| \\ &\le \frac{1}{n} + \|h_n - h\| \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence we have

$$h = \lim_{n\to\infty} t_n(y_n - \bar{x})$$

and h is a tangent vector to S at $\bar{x}$. □

Since the inclusion chain (4.2) is also valid for the corresponding closed sets, it follows immediately with the aid of Theorem 4.10:

Corollary 4.11. *Let S be a nonempty subset of a real normed space. If the set S is starshaped with respect to some $\bar{x} \in S$, then it is*

$$T(S, \bar{x}) = cl(cone(S - \{\bar{x}\})).$$

If the set S is starshaped with respect to some $\bar{x} \in S$, then Corollary 4.11 says essentially that for the determination of the contingent cone to S at $\bar{x}$ we have to consider only rays emanating from $\bar{x}$ and passing through S.

Finally, we show that the contingent cone to a nonempty convex set is also convex.

Theorem 4.12. *If S is a nonempty convex subset of a real normed space $(X, \|\cdot\|)$, then the contingent cone $T(S, \bar{x})$ is convex for all $\bar{x} \in S$.*

Proof. We choose an arbitrary $\bar{x} \in S$ and we fix two arbitrary tangent vectors $h_1, h_2 \in T(S, \bar{x})$ with $h_1, h_2 \neq 0_X$. Then there are sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ of elements in S and sequences $(\lambda_n)_{n\in\mathbb{N}}$, $(\mu_n)_{n\in\mathbb{N}}$ of positive real numbers with

$$\bar{x} = \lim_{n\to\infty} x_n,\ h_1 = \lim_{n\to\infty} \lambda_n(x_n - \bar{x})$$

and

$$\bar{x} = \lim_{n\to\infty} y_n,\ h_2 = \lim_{n\to\infty} \mu_n(y_n - \bar{x}).$$

Next, we define additional sequences $(\nu_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ with

$$\nu_n := \lambda_n + \mu_n \text{ for all } n \in \mathbb{N}$$

and

$$z_n := \frac{1}{\nu_n}(\lambda_n x_n + \mu_n y_n) \text{ for all } n \in \mathbb{N}.$$

Because of the convexity of S we have

$$z_n = \frac{\lambda_n}{\lambda_n + \mu_n} x_n + \frac{\mu_n}{\lambda_n + \mu_n} y_n \in S \text{ for all } n \in \mathbb{N},$$

and we conclude

$$\begin{aligned}
\lim_{n\to\infty} z_n &= \lim_{n\to\infty} \frac{1}{\nu_n}(\lambda_n x_n + \mu_n y_n) \\
&= \lim_{n\to\infty} \frac{1}{\nu_n}(\lambda_n x_n - \lambda_n \bar{x} + \mu_n y_n - \mu_n \bar{x} + \lambda_n \bar{x} + \mu_n \bar{x}) \\
&= \lim_{n\to\infty} \left(\frac{\lambda_n}{\nu_n}(x_n - \bar{x}) + \frac{\mu_n}{\nu_n}(y_n - \bar{x}) + \bar{x}\right) \\
&= \bar{x}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \nu_n (z_n - \bar{x}) &= \lim_{n\to\infty} (\lambda_n x_n + \mu_n y_n - \nu_n \bar{x}) \\
&= \lim_{n\to\infty} (\lambda_n (x_n - \bar{x}) + \mu_n (y_n - \bar{x})) \\
&= h_1 + h_2.
\end{aligned}$$

Hence it follows $h_1 + h_2 \in T(S, \bar{x})$. Since $T(S, \bar{x})$ is a cone, Theorem 4.3 leads to the assertion. □

Notice that the Clarke tangent cone to an arbitrary nonempty set S is already a convex cone, while we have shown the convexity of the contingent cone only under the assumption of the convexity of S.

4.2 Optimality Conditions

In this section we present several optimality conditions which result from the theory on contingent cones.

First, we show, for example, for convex optimization problems with a continuous objective functional that every minimal point $\bar{x}$ of f on S can be characterized as a minimal point of f on $\{\bar{x}\} + T(S, \bar{x})$.

Theorem 4.13. *Let S be a nonempty subset of a real normed space $(X, \|\cdot\|)$, and let $f : X \to \mathbb{R}$ be a given functional.*
(a) If the functional f is continuous and convex, then for every minimal point $\bar{x} \in S$ of f on S it follows:

$$f(\bar{x}) \leq f(\bar{x} + h) \text{ for all } h \in T(S, \bar{x}). \tag{4.3}$$

(b) If the set S is starshaped with respect to some $\bar{x} \in S$ and if the inequality (4.3) is satisfied, then $\bar{x}$ is a minimal point of f on S.

Proof. (a) We fix an arbitrary $\bar{x} \in S$ and assume that the inequality (4.3) does not hold. Then there are a vector $h \in T(S, \bar{x}) \setminus \{0_X\}$ and a number $\alpha > 0$ with

$$f(\bar{x}) - f(\bar{x} + h) > \alpha > 0. \tag{4.4}$$

By the definition of h there are a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in S and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers with

$$\bar{x} = \lim_{n\to\infty} x_n$$

and

$$h = \lim_{n\to\infty} h_n$$

where

$$h_n := \lambda_n(x_n - \bar{x}) \text{ for all } n \in \mathbb{N}.$$

Because of $h \neq 0_X$ we have $\lim\limits_{n\to\infty} \dfrac{1}{\lambda_n} = 0$. Since f is convex and continuous, we obtain with the inequality (4.4) for sufficiently large $n \in \mathbb{N}$:

$$\begin{aligned}
f(x_n) &= f\Big(\frac{1}{\lambda_n}\bar{x} + x_n - \bar{x} + \bar{x} - \frac{1}{\lambda_n}\bar{x}\Big) \\
&= f\Big(\frac{1}{\lambda_n}(\bar{x} + h_n) + \Big(1 - \frac{1}{\lambda_n}\Big)\bar{x}\Big) \\
&\leq \frac{1}{\lambda_n} f(\bar{x} + h_n) + \Big(1 - \frac{1}{\lambda_n}\Big) f(\bar{x}) \\
&\leq \frac{1}{\lambda_n}(f(\bar{x} + h) + \alpha) + \Big(1 - \frac{1}{\lambda_n}\Big) f(\bar{x}) \\
&< \frac{1}{\lambda_n} f(\bar{x}) + \Big(1 - \frac{1}{\lambda_n}\Big) f(\bar{x}) \\
&= f(\bar{x}).
\end{aligned}$$

Consequently, $\bar{x}$ is not a minimal point of f on S.
(b) If the set S is starshaped with respect to some $\bar{x} \in S$, then it follows by Theorem 4.8

$$S - \{\bar{x}\} \subset T(S, \bar{x}).$$

Hence we get with the inequality (4.3)

$$f(\bar{x}) \leq f(\bar{x} + h) \text{ for all } h \in S - \{\bar{x}\},$$

i.e., $\bar{x}$ is a minimal point of f on S. □

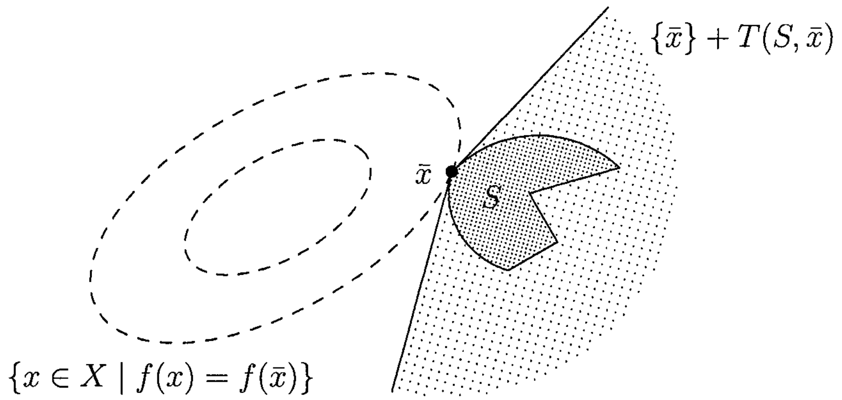

Figure 4.6: Geometric illustration of the result of Theorem 4.13.

Using Fréchet derivatives the following necessary optimality condition can be formulated.

Theorem 4.14. *Let S be a nonempty subset of a real normed space $(X, \|\cdot\|)$, and let f be a functional defined on an open superset of S. If $\bar{x} \in S$ is a minimal point of f on S and if f is Fréchet differentiable at $\bar{x}$, then it follows*

$$f'(\bar{x})(h) \geq 0 \textit{ for all } h \in T(S, \bar{x}).$$

Proof. Let $\bar{x} \in S$ be a minimal point of f on S, and let some $h \in T(S, \bar{x}) \setminus \{0_X\}$ be arbitrarily given (for $h = 0_X$ the assertion is trivial). Then there are a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in S and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers with

$$\bar{x} = \lim_{n\to\infty} x_n$$

and

$$h = \lim_{n\to\infty} h_n$$

where

$$h_n := \lambda_n(x_n - \bar{x}) \text{ for all } n \in \mathbb{N}.$$

By the definition of the Fréchet derivative and because of the minimality of f at $\bar{x}$ it follows:

$$\begin{aligned} f'(\bar{x})(h) &= f'(\bar{x})\Big(\lim_{n\to\infty} \lambda_n(x_n - \bar{x})\Big) \\ &= \lim_{n\to\infty} \lambda_n f'(\bar{x})(x_n - \bar{x}) \\ &= \lim_{n\to\infty} \lambda_n[f(x_n) - f(\bar{x}) - (f(x_n) - f(\bar{x}) - f'(\bar{x})(x_n - \bar{x}))] \\ &\geq -\lim_{n\to\infty} \lambda_n(f(x_n) - f(\bar{x}) - f'(\bar{x})(x_n - \bar{x})) \\ &= -\lim_{n\to\infty} \|h_n\| \frac{f(x_n) - f(\bar{x}) - f'(\bar{x})(x_n - \bar{x})}{\|x_n - \bar{x}\|} \\ &= 0. \end{aligned}$$

Hence, the assertion is proved. □

Next, we investigate under which assumptions the condition in Theorem 4.14 is a sufficient optimality condition. For this purpose we define pseudoconvex functionals.

Definition 4.15. Let S be a nonempty subset of a real linear space, and let $f : S \to \mathbb{R}$ be a given functional which has a directional derivative at some $\bar{x} \in S$ in every direction $x - \bar{x}$ with arbitrary $x \in S$. The functional f is called *pseudoconvex* at $\bar{x}$ if for all $x \in S$

$$f'(\bar{x})(x - \bar{x}) \geq 0 \implies f(x) - f(\bar{x}) \geq 0.$$

Example 4.16. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = xe^x \text{ for all } x \in \mathbb{R}$$

and

$$g(x) = -\frac{1}{1+x^2} \text{ for all } x \in \mathbb{R}$$

are pseudoconvex at every $\bar{x} \in \mathbb{R}$. But the two functions are not convex.

A relationship between convex and pseudoconvex functionals is given by the next theorem.

Theorem 4.17. *Let S be a nonempty convex subset of a real linear space, and let $f : S \to \mathbb{R}$ be a convex functional which has a directional derivative at some $\bar{x} \in S$ in every direction $x - \bar{x}$ with arbitrary $x \in S$. Then f is pseudoconvex at $\bar{x}$.*

Proof. We fix an arbitrary $x \in S$. Because of the convexity of f we get for all $\lambda \in (0, 1]$

$$f(\lambda x + (1-\lambda)\bar{x}) \leq \lambda f(x) + (1-\lambda) f(\bar{x})$$

and

$$\begin{aligned} f(x) &\geq f(\bar{x}) + \frac{1}{\lambda}(f(\lambda x + (1-\lambda)\bar{x}) - f(\bar{x})) \\ &= f(\bar{x}) + \frac{1}{\lambda}(f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x})). \end{aligned}$$

Since f has a directional derivative at $\bar{x}$ in the direction $x - \bar{x}$, we conclude

$$f(x) - f(\bar{x}) \geq f'(\bar{x})(x - \bar{x}).$$

Consequently, if $f'(\bar{x})(x - \bar{x}) \geq 0$, then

$$f(x) - f(\bar{x}) \geq 0.$$

Hence f is pseudoconvex at $\bar{x}$. □

It is also possible to formulate a relationship between quasiconvex and pseudoconvex functionals.

Theorem 4.18. *Let S be a nonempty convex subset of a real normed space, and let f be a functional which is defined on an open superset of S. If f is Fréchet differentiable at every $\bar{x} \in S$ and pseudoconvex at every $\bar{x} \in S$, then f is also quasiconvex on S.*

Proof. Under the given assumptions we prove that for every $\alpha \in \mathbb{R}$ the level set

$$S_\alpha := \{x \in S \mid f(x) \leq \alpha\}$$

is a convex set. For this purpose we fix an arbitrary $\alpha \in \mathbb{R}$ so that S_α is a nonempty set. Furthermore we choose two arbitrary elements $x, y \in S_\alpha$. In the following we assume that there is a $\hat{\lambda} \in [0, 1]$ with

$$f(\hat{\lambda}x + (1 - \hat{\lambda})y) > \alpha \geq \max\{f(x), f(y)\}.$$

Then it follows $\hat{\lambda} \in (0, 1)$. Since f is Fréchet differentiable on S, by Theorem 3.15 f is also continuous on S. Consequently, there is a $\bar{\lambda} \in (0, 1)$ with

$$f(\bar{\lambda}x + (1 - \bar{\lambda})y) \geq f(\lambda x + (1 - \lambda)y) \text{ for all } \lambda \in (0, 1).$$

Using Theorem 3.13 and Theorem 3.8,(a) (which is now applied to a maximum problem) it follows for $\bar{x} := \bar{\lambda}x + (1 - \bar{\lambda})y$

$$f'(\bar{x})(x - \bar{x}) \leq 0$$

and

$$f'(\bar{x})(y - \bar{x}) \leq 0.$$

With

$$\begin{aligned} x - \bar{x} &= x - \bar{\lambda}x - (1 - \bar{\lambda})y = (1 - \bar{\lambda})(x - y), \\ y - \bar{x} &= y - \bar{\lambda}x - (1 - \bar{\lambda})y = -\bar{\lambda}(x - y) \end{aligned} \tag{4.5}$$

and the linearity of $f'(\bar{x})$ we obtain

$$0 \geq f'(\bar{x})(x - \bar{x}) = (1 - \bar{\lambda})f'(\bar{x})(x - y)$$

and

$$0 \geq f'(\bar{x})(y - \bar{x}) = -\bar{\lambda}f'(\bar{x})(x - y).$$

Hence we have

$$0 = f'(\bar{x})(x - y),$$

and with the equality (4.5) it also follows

$$f'(\bar{x})(y - \bar{x}) = 0.$$

By assumption f is pseudoconvex at $\bar{x}$ and therefore we conclude

$$f(y) - f(\bar{x}) \geq 0.$$

But this inequality contradicts the following inequality:

$$\begin{aligned} f(y) - f(\bar{x}) &= f(y) - f(\bar{\lambda}x + (1 - \bar{\lambda})y) \\ &\leq f(y) - f(\hat{\lambda}x + (1 - \hat{\lambda})y) \\ &< f(y) - \alpha \\ &\leq 0. \end{aligned}$$

□

Using Theorem 3.13 the result of the Theorems 4.17 and 4.18 can be specialized in the following way: If $(X, \|\cdot\|)$ is a real normed space and if $f : X \to \mathbb{R}$ is a functional which is Fréchet differentiable at every $\bar{x} \in X$, then the following implications are satisfied:

$$\begin{aligned} f \text{ convex} &\Longrightarrow f \text{ pseudoconvex at every } \bar{x} \in X \\ &\Longrightarrow f \text{ quasiconvex.} \end{aligned}$$

After these investigations we come back to the question leading to the introduction of pseudoconvex functionals. With the next theorem we present now assumptions under which the condition in Theorem 4.14 is a sufficient optimality condition.

Theorem 4.19. *Let S be a nonempty subset of a real normed space, and let f be a functional defined on an open superset of S. If S is starshaped with respect to some $\bar{x} \in S$, if f is directionally differentiable at $\bar{x}$ and pseudoconvex at $\bar{x}$, and if*

$$f'(\bar{x})(h) \geq 0 \text{ for all } h \in T(S, \bar{x}),$$

then $\bar{x}$ is a minimal point of f on S.

Proof. Because of the starshapedness of S with respect to $\bar{x} \in S$ it follows by Theorem 4.8 $S - \{\bar{x}\} \subset T(S, \bar{x})$, and therefore we have

$$f'(\bar{x})(x - \bar{x}) \geq 0 \text{ for all } x \in S.$$

Since f is pseudoconvex at $\bar{x}$, we conclude

$$f(x) - f(\bar{x}) \geq 0 \text{ for all } x \in S,$$

i.e., $\bar{x}$ is a minimal point of f on S. □

Notice that the assumption in Theorem 3.8,(b) under which the inequality (3.1) is a sufficient condition, can be weakened with the aid of the pseudoconvexity assumption. This result is summarized with Theorem 3.8 in the next corollary.

Corollary 4.20. *Let S be a nonempty subset of a real linear space, and let $f : S \to \mathbb{R}$ be a given functional. Moreover, let the functional f have a directional derivative at some $\bar{x} \in S$ in every direction $x - \bar{x}$ with arbitrary $x \in S$ and let f be pseudoconvex at $\bar{x}$. Then $\bar{x}$ is a minimal point of f on S if and only if*

$$f'(\bar{x})(x - \bar{x}) \geq 0 \textit{ for all } x \in S.$$

4.3 A Lyusternik Theorem

For the application of the necessary optimality condition given in Theorem 4.14 to optimization problems with equality constraints we need a profound theorem which generalizes a result given by Lyusternik[5] published in 1934. This theorem says under appropriate assumptions that the contingent cone to a set described by equality constraints is a superset of the set of the linearized constraints. Moreover, it can be shown under these assumptions that both sets are equal.

[5]L.A. Lyusternik, "Conditional extrema of functionals", *Mat. Sb.* 41 (1934) 390–401.

Theorem 4.21. *Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be real Banach spaces, and let $h : X \to Z$ be a given mapping. Furthermore, let some $\bar{x} \in S$ with*

$$S := \{x \in X \mid h(x) = 0_Z\}$$

be given. Let h be Fréchet differentiable on a neighborhood of $\bar{x}$, let $h'(\cdot)$ be continuous at $\bar{x}$, and let $h'(\bar{x})$ be surjective. Then it follows

$$\{x \in X \mid h'(\bar{x})(x) = 0_Z\} \subset T(S, \bar{x}). \tag{4.6}$$

Proof. We present a proof of Lyusternik's theorem which is put forward by Werner [347]. This proof can be carried out in several steps. First we apply an open mapping theorem and then we prove the technical inequality (4.14). In the third part we show the equations (4.26) and (4.27) with the aid of a construction of special sequences, and based on these equations we get the inclusion (4.6) in the last part.

(1) Since $h'(\bar{x})$ is continuous, linear and surjective by the open mapping theorem the mapping $h'(\bar{x})$ is open, i.e. the image of every open set is open. Therefore, if $B(0_X, 1)$ denotes the open unit ball in X, there is some $\varrho > 0$ such that

$$B(0_Z, \varrho) \subset h'(\bar{x})\, B(0_X, 1) \tag{4.7}$$

where $B(0_Z, \varrho)$ denotes the open ball around 0_Z with radius ϱ. Because of the continuity of $h'(\bar{x})$ there is a

$$\varrho_0 := \sup\{\varrho > 0 \mid B(0_Z, \varrho) \subset h'(\bar{x})\, B(0_X, 1)\}.$$

(2) Next we choose an arbitrary $\varepsilon \in (0, \frac{\varrho_0}{2})$. $h'(\cdot)$ is assumed to be continuous at $\bar{x}$, and therefore there is a $\delta > 0$ with

$$\|h'(\tilde{x}) - h'(\bar{x})\|_{L(X,Z)} \leq \varepsilon \text{ for all } \tilde{x} \in B(\bar{x}, 2\delta). \tag{4.8}$$

Now we fix arbitrary elements $\tilde{x}, \tilde{\tilde{x}} \in B(\bar{x}, 2\delta)$. By a Hahn-Banach theorem (Thm. C.4) there is a continuous linear functional $l \in Z^*$ with

$$\|l\|_{Z^*} = 1 \tag{4.9}$$

and

$$l(h(\tilde{\tilde{x}})-h(\tilde{x})-h'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})) = \|h(\tilde{\tilde{x}})-h(\tilde{x})-h'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})\|_Z. \quad (4.10)$$

Next we define a functional $\varphi : [0,1] \to \mathbb{R}$ by

$$\varphi(t) = l(h(\tilde{x}+t(\tilde{\tilde{x}}-\tilde{x})) - th'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})) \text{ for all } t \in [0,1]. \quad (4.11)$$

φ is differentiable on $[0,1]$ and we get

$$\varphi'(t) = l(h'(\tilde{x}+t(\tilde{\tilde{x}}-\tilde{x}))(\tilde{\tilde{x}}-\tilde{x}) - h'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})). \quad (4.12)$$

By the mean value theorem there is a $\bar{t} \in (0,1)$ with

$$\varphi(1) - \varphi(0) = \varphi'(\bar{t}). \quad (4.13)$$

Then we obtain with (4.10), (4.11), (4.13), (4.12), (4.9) and (4.8)

$$\begin{aligned} &\|h(\tilde{\tilde{x}}) - h(\tilde{x}) - h'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})\|_Z \\ = \;& l(h(\tilde{\tilde{x}}) - h(\tilde{x}) - h'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})) \\ = \;& \varphi(1) - \varphi(0) \\ = \;& \varphi'(\bar{t}) \\ = \;& l(h'(\tilde{x}+\bar{t}(\tilde{\tilde{x}}-\tilde{x}))(\tilde{\tilde{x}}-\tilde{x}) - h'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})) \\ \leq \;& \|h'(\tilde{x}+\bar{t}(\tilde{\tilde{x}}-\tilde{x})) - h'(\bar{x})\|_{L(X,Z)} \, \|\tilde{\tilde{x}}-\tilde{x}\|_X \\ \leq \;& \varepsilon\|\tilde{\tilde{x}}-\tilde{x}\|_X. \end{aligned}$$

Hence we conclude

$$\|h(\tilde{\tilde{x}}) - h(\tilde{x}) - h'(\bar{x})(\tilde{\tilde{x}}-\tilde{x})\|_Z \leq \varepsilon\|\tilde{\tilde{x}}-\tilde{x}\|_X \text{ for all } \tilde{x}, \tilde{\tilde{x}} \in B(\bar{x}, 2\delta). \quad (4.14)$$

(3) Now we choose an arbitrary $\alpha > 1$ so that $\alpha(\frac{1}{2} + \frac{\varepsilon}{\varrho_0}) \leq 1$ (notice that $\frac{\varepsilon}{\varrho_0} < \frac{1}{2}$). For the proof of the inclusion (4.6) we take an arbitrary $x \in X$ with $h'(\bar{x})(x) = 0_Z$. For $x = 0_X$ the assertion is trivial, therefore we assume that $x \neq 0_X$. We set $\hat{\lambda} := \frac{\delta}{\|x\|_X}$ and fix an arbitrary $\lambda \in (0, \hat{\lambda}]$. Now we define sequences $(r_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ as follows:

$$r_1 = 0_X,$$

$$h'(\bar{x})(u_n) = h(\bar{x} + \lambda x + r_n) \text{ for all } n \in \mathbb{N}, \tag{4.15}$$

$$r_{n+1} = r_n - u_n \text{ for all } n \in \mathbb{N}. \tag{4.16}$$

Since $h'(\bar{x})$ is assumed to be surjective, for a given $r_n \in X$ there is always a vector $u_n \in X$ with the property (4.15). Consequently, sequences $(r_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are well-defined (although they do not need to be unique). From the inclusion (4.7) which holds for $\varrho = \frac{\varrho_0}{\alpha}$ and the equation (4.15) we conclude for every $n \in \mathbb{N}$

$$\|u_n\|_X \leq \frac{\alpha}{\varrho_0} \|h(\bar{x} + \lambda x + r_n)\|_Z. \tag{4.17}$$

For simplicity we set

$$d(\lambda) := \|h(\bar{x} + \lambda x)\|_Z$$

and

$$q := \frac{\varepsilon \alpha}{\varrho_0}.$$

Since $\|\lambda x\|_X \leq \delta$ we get from the inequality (4.14)

$$\begin{aligned} d(\lambda) &= \|h(\bar{x} + \lambda x) - h(\bar{x}) - h'(\bar{x})(\lambda x)\|_Z \\ &\leq \varepsilon \|\lambda x\|_X \\ &\leq \varepsilon \delta, \end{aligned} \tag{4.18}$$

and moreover, because of $\alpha > 1$ we have

$$q \leq 1 - \frac{\alpha}{2} < \frac{1}{2}. \tag{4.19}$$

Then we assert for all $n \in \mathbb{N}$:

$$\|r_n\|_X \leq \frac{\alpha}{\varrho_0} d(\lambda) \frac{1 - q^{n-1}}{1 - q}, \tag{4.20}$$

$$\|h(\bar{x} + \lambda x + r_n)\|_Z \leq d(\lambda) q^{n-1} \tag{4.21}$$

and

$$\|u_n\|_X \leq \frac{\alpha}{\varrho_0} d(\lambda) q^{n-1}. \tag{4.22}$$

We prove the preceding three inequalities by induction. For $n = 1$ we get

$$\|r_1\|_X = 0,$$
$$\|h(\bar{x} + \lambda x + r_1)\|_Z = d(\lambda)$$

and by the inequality (4.17)

$$\begin{aligned}\|u_1\|_X &\leq \frac{\alpha}{\varrho_0}\|h(\bar{x} + \lambda x + r_1)\|_Z \\ &= \frac{\alpha}{\varrho_0} d(\lambda).\end{aligned}$$

Hence the inequalities (4.20), (4.21) and (4.22) are fulfilled for $n = 1$. Next assume that they are also fulfilled for any $n \in \mathbb{N}$. Then we get with (4.16), (4.20) and (4.22)

$$\begin{aligned}\|r_{n+1}\|_X &= \|r_n - u_n\|_X \\ &\leq \|r_n\|_X + \|u_n\|_X \\ &\leq \frac{\alpha}{\varrho_0} d(\lambda)\left(\frac{1-q^{n-1}}{1-q} + q^{n-1}\right) \\ &= \frac{\alpha}{\varrho_0} d(\lambda)\frac{1-q^n}{1-q}.\end{aligned}$$

Hence the inequality (4.20) is proved. For the proof of the following inequalities notice that from (4.20), (4.18) and (4.19)

$$\begin{aligned}\|\lambda x + r_n\|_X &\leq \|\lambda x\|_X + \|r_n\|_X \\ &\leq \delta + \frac{\alpha}{\varrho_0} d(\lambda)\frac{1-q^{n-1}}{1-q} \\ &\leq \delta + \frac{\alpha\varepsilon\delta}{\varrho_0}\,\frac{1-q^{n-1}}{1-q} \\ &= \delta\Big(1 + \underbrace{\frac{q}{1-q}}_{<1}\underbrace{(1-q^{n-1})}_{<1}\Big) \\ &< 2\delta \end{aligned} \tag{4.23}$$

and from (4.16), (4.20), (4.18) and (4.19)

$$\|\lambda x + r_n - u_n\|_X \leq \|\lambda x\|_X + \|r_{n+1}\|_X$$

$$
\begin{aligned}
&\leq \delta + \frac{\alpha}{\varrho_0} d(\lambda) \frac{1-q^n}{1-q} \\
&\leq \delta\Big(1 + \underbrace{\frac{q}{1-q}}_{<1} \underbrace{(1-q^n)}_{<1}\Big) \\
&< 2\delta. \qquad (4.24)
\end{aligned}
$$

Next with (4.16), (4.15), (4.23), (4.24), (4.14) and (4.22) we conclude

$$
\begin{aligned}
&\|h(\bar{x} + \lambda x + r_{n+1})\|_Z \\
&= \|h(\bar{x} + \lambda x + r_n - u_n)\|_Z \\
&= \| - h'(\bar{x})(-u_n) - h(\bar{x} + \lambda x + r_n) + h(\bar{x} + \lambda x + r_n - u_n)\|_Z \\
&\leq \varepsilon \, \| - u_n\|_X \\
&\leq \varepsilon \, \frac{\alpha}{\varrho_0} d(\lambda) q^{n-1} \\
&= d(\lambda) q^n, \qquad (4.25)
\end{aligned}
$$

and with (4.17) and (4.25) we obtain

$$
\begin{aligned}
\|u_{n+1}\|_X &\leq \frac{\alpha}{\varrho_0} \|h(\bar{x} + \lambda x + r_{n+1})\|_Z \\
&\leq \frac{\alpha}{\varrho_0} d(\lambda) q^n.
\end{aligned}
$$

Consequently, the inequalities (4.21) and (4.22) are fulfilled. From the inequalities (4.22) and (4.18) we get

$$
\begin{aligned}
\|u_n\|_X &\leq \frac{\alpha}{\varrho_0} d(\lambda) q^{n-1} \\
&\leq \frac{\alpha \varepsilon \delta}{\varrho_0} q^{n-1} \\
&= \delta q^n \text{ for all } n \in \mathbb{N},
\end{aligned}
$$

and because of the inequality (4.19) it follows $\lim\limits_{n \to \infty} u_n = 0_X$. With the equation (4.16) and the inequalities (4.22) and (4.19) we see for all $n, k \in \mathbb{N}$

$$
\begin{aligned}
\|r_{n+k} - r_n\|_X &= \|r_n - u_{n+k-1} - u_{n+k-2} - \cdots - u_n - r_n\|_X \\
&\leq \|u_n\|_X + \|u_{n+1}\|_X + \cdots + \|u_{n+k-1}\|_X
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\alpha}{\varrho_0} d(\lambda)(q^{n-1} + q^n + \cdots + q^{n+k-2}) \\
&= \frac{\alpha}{\varrho_0} d(\lambda) q^{n-1}(1 + q + \cdots + q^{k-1}) \\
&= \frac{\alpha}{\varrho_0} d(\lambda) q^{n-1} \frac{1-q^k}{1-q} \\
&< \frac{\alpha d(\lambda)}{\varrho_0(1-q)} q^{n-1},
\end{aligned}
$$

and therefore $(r_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. So, there is a vector $r(\lambda) \in X$ with $\lim\limits_{n\to\infty} r_n = r(\lambda)$. Furthermore, we obtain from the equation (4.15) in the limit

$$
h(\bar{x} + \lambda x + r(\lambda)) = 0_Z. \tag{4.26}
$$

From (4.20) we conclude

$$
\begin{aligned}
\frac{\|r(\lambda)\|_X}{\lambda} &\leq \frac{\alpha}{\lambda\varrho_0} d(\lambda) \frac{1}{1-q} \\
&= \frac{\alpha}{\varrho_0(1-q)} \frac{\|h(\bar{x} + \lambda x) - h(\bar{x}) - \lambda h'(\bar{x})(x)\|_Z}{\lambda},
\end{aligned}
$$

and therefore we have

$$
\lim_{\lambda\to 0_+} \frac{r(\lambda)}{\lambda} = 0_X. \tag{4.27}
$$

(4) Finally we show that x belongs to the contingent cone $T(S, \bar{x})$. Take any sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in (0, \hat{\lambda}]$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \lambda_n = 0$, and define the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ with

$$
\mu_n := \frac{1}{\lambda_n} > 0 \text{ for all } n \in \mathbb{N}
$$

and

$$
x_n := \bar{x} + \lambda_n x + r(\lambda_n) \text{ for all } n \in \mathbb{N}.
$$

From the equation (4.26) we get

$$
x_n \in S \text{ for all } n \in \mathbb{N}.
$$

Moreover, we have with (4.27)

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \bar{x} + \lambda_n x + r(\lambda_n)\\ &= \lim_{n\to\infty} \bar{x} + \lambda_n \left(x + \frac{r(\lambda_n)}{\lambda_n}\right)\\ &= \bar{x},\end{aligned}$$

and we conclude

$$\begin{aligned}\lim_{n\to\infty} \mu_n(x_n - \bar{x}) &= \lim_{n\to\infty} \frac{1}{\lambda_n}(\lambda_n x + r(\lambda_n))\\ &= \lim_{n\to\infty} x + \frac{r(\lambda_n)}{\lambda_n}\\ &= x.\end{aligned}$$

Consequently, we obtain $x \in T(S, \bar{x})$ which completes the proof.

□

With the following theorem we show that the inclusion (4.6) also holds in the opposite direction.

Theorem 4.22. *Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be real normed spaces, and let $h : X \longrightarrow Z$ be a given mapping. Furthermore, let some $\bar{x} \in S$ with*

$$S := \{x \in X \mid h(x) = 0_Z\}$$

be given. If h is Fréchet differentiable at $\bar{x}$, then it follows

$$T(S, \bar{x}) \subset \{x \in X \mid h'(\bar{x})(x) = 0_Z\}.$$

Proof. Let $y \in T(S, \bar{x})\backslash\{0_X\}$ be an arbitrary tangent vector (the assertion is evident for $y = 0_X$). Then there are a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in S and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers with

$$\bar{x} = \lim_{n\to\infty} x_n$$

and

$$y = \lim_{n\to\infty} y_n$$

where

$$y_n := \lambda_n(x_n - \bar{x}) \text{ for all } n \in \mathbb{N}.$$

Consequently, by the definition of the Fréchet derivative we obtain:

$$\begin{aligned}
h'(\bar{x})(y) &= h'(\bar{x})\Big(\lim_{n\to\infty} \lambda_n(x_n - \bar{x})\Big) \\
&= \lim_{n\to\infty} \lambda_n h'(\bar{x})(x_n - \bar{x}) \\
&= \lim_{n\to\infty} \lambda_n[h(x_n) - h(\bar{x}) - (h(x_n) - h(\bar{x}) - h'(\bar{x})(x_n - \bar{x}))] \\
&= -\lim_{n\to\infty} \|y_n\|_X \frac{h(x_n) - h(\bar{x}) - h'(\bar{x})(x_n - \bar{x})}{\|x_n - \bar{x}\|_X} \\
&= 0_Z.
\end{aligned}$$

□

The proof of the preceding theorem is similar to the proof of Theorem 4.14. Since the assumptions of Theorem 4.22 are weaker than those of Theorem 4.21, we summarize the results of the two preceding theorems as follows: Under the assumptions of Theorem 4.21 we conclude

$$T(S, \bar{x}) = \{x \in X \mid h'(\bar{x})(x) = 0_Z\}.$$

Exercises

4.1) Let C be a convex cone in a real normed space with nonempty interior int(C). Show: int$(C)=$ int$(C)+C$.

4.2) Let X be a real linear space. Prove that a functional $f : X \to \mathbb{R}$ is sublinear if and only if its epigraph is a convex cone.

4.3) Let S be a nonempty convex subset of a real linear space. Show that the cone generated by S is convex.

4.4) In $\mathbb{R}^2$ let the set $S := \{(x,y) \in \mathbb{R}^2 \mid -x + y \leq 1,\ 2x + y \leq 4,\ 0 \leq x \leq \frac{3}{2},\ y \geq 0\}$ be given. Determine the cone generated by S.

4.5) Let the set S be given as in Exercise 4.4). Determine the contingent cone to S at $(1,2)$.

4.6) Let S be a nonempty subset of a real normed space $(X, \|\cdot\|)$ with nonempty interior $\mathrm{int}(S)$. For every $\bar{x} \in \mathrm{int}(S)$ show $T(S, \bar{x}) = X$.

4.7) Let S_1 and S_2 be two nonempty subsets of a real normed space. Prove the following implications:
(a) $\bar{x} \in S_1 \subset S_2 \;\Rightarrow\; T(S_1, \bar{x}) \subset T(S_2, \bar{x})$,
(b) $\bar{x} \in S_1 \cap S_2 \;\Rightarrow\; T(S_1 \cap S_2, \bar{x}) \subset T(S_1, \bar{x}) \cap T(S_2, \bar{x})$.

4.8) Let S be a nonempty subset of a real normed space $(X, \|\cdot\|)$, and let some $\bar{x} \in S$ be arbitrarily given. Show:
$T(S, \bar{x}) = \{h \in X \mid$ there are a number $\sigma > 0$ and a mapping $r : (0, \sigma] \to X$ with $\lim\limits_{t \to 0_+} \frac{1}{t} r(t) = 0_X$, and there is a sequence $(t_n)_{n \in \mathbb{N}}$ of positive real numbers converging to 0 so that $\bar{x} + t_n h + r(t_n) \in S$ for all $n \in \mathbb{N}\}$.

4.9) Let $\bar{x}$ be an element of a subset S of a real normed space. Prove that the Clarke tangent cone $T_{Cl}(S, \bar{x})$ is closed and convex.

4.10) Is the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$ for all $x \in \mathbb{R}$ pseudoconvex at an arbitrary $\bar{x} \in \mathbb{R}$?