Non-parametric copula-based estimator of transition matrices for high-order Markov chains

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September 19, 2020

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ABSTRACT

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Journal of Economic Classification: C12, C13, C14

Keywords: Causality measures; Non-parametric estimation; Vine decomposition; Markov chains; Bernstein copula density; Local bootstrap.

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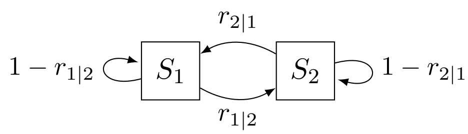
2 Motivation

Consider the simple case of a two-state first-order stationary Markov chain, such as the one depicted below



with an *unknown* transition matrix

$$\mathbf{P} = \begin{bmatrix} 1 - r_{1|2} & r_{1|2} \\ r_{2|1} & 1 - r_{2|1} \end{bmatrix}$$

where

$$r_{i|j} = P[X_t = i \mid X_{t-1} = j], \quad i, j \in \{1, 2\} \tag{1}$$

In such scenarios, estimating the conditional probabilities $r_{i|j}$ is rather straightforward. Consider observing a sequence of states for individual entities at times $t-1$ and t. The probability of transitioning from one state to another can be estimated by calculating the ratio of the entities that transitioned from, say, $S1$ to $S2$ in the period $t-1$ to t to the total number of entities that began at state $S1$ at $t-1$. In other words:

$$r_{i|j} = \frac{n_{ij}}{\sum_j n_{ij}} \tag{2}$$

Now, instead let us consider that the Markov chain under consideration is of order p with a transition probability

$$r_{j_0|j_1,\cdots,j_p} = P[X_t = j_0 \mid X_{t-1} = j_1, \cdots, X_{t-p} = j_p], \quad j_k \in \{1, 2\}, \quad \text{for } k = 0, 1, \cdots, p \tag{3}$$

Evidently, estimating the transition probabilities is no longer trivial. In what follows, we propose methods for estimating the transition probabilities for Markov chains of high-order.

3 Copula-based transition matrices

First, let us consider the case of a third-order stationary Markov chain with a transition matrix

$$r_{j_0|j_1,\cdots,j_3} = P[X_t = j_0 \mid X_{t-1} = j_1, X_{t-2} = j_2, X_{t-3} = j_3]$$

we may express this transition matrix as

$$r_{j_0|j_1,\cdots,j_3} = P[X_t = j_0, X_{t-1} = j_1, X_{t-2} = j_2 \mid X_{t-3} = j_3] \Big/ P[X_{t-1} = j_1, X_{t-2} = j_2 \mid X_{t-3} = j_3] \quad (4)$$

Furthermore, note that

$$P[X_t = j_0, X_{t-1} = j_1, X_{t-2} = j_2 \mid X_{t-3} = j_3] = \sum_{l=0,1} \sum_{m=0,1} \sum_{n=0,1} (-1)^{(l+m+n)} \times P[X_t \leq j_0 - l, X_{t-1} \leq j_1 - m, X_{t-2} \leq j_2 - n \mid X_{t-3} = j_3] \quad (5)$$

where we know from the Theorem of Sklar (1959) that the joint conditional distribution function on the right hand side of relationship (4) can be expressed by its marginal conditional distributions and a copula function capturing their respective dependence, - i.e.

$$\begin{aligned} P[X_t \leq j_0 - l, X_{t-1} \leq j_1 - m, X_{t-2} \leq j_2 - n \mid X_{t-3} = j_3] &= F(j_0 - l, j_1 - m, j_2 - n \mid j_3) \\ &= C(F(j_0 - l \mid j_3), F(j_1 - m \mid j_3), F(j_2 - n \mid j_3)) \end{aligned} \quad (6)$$

Therefore, using relationship (6), we may express (5) as follows

$$P[X_t = j_0, X_{t-1} = j_1, X_{t-2} = j_2 \mid X_{t-3} = j_3] = \sum_{l=0,1} \sum_{m=0,1} \sum_{n=0,1} (-1)^{(l+m+n)} \times C(F(j_0 - l \mid j_3), F(j_1 - m \mid j_3), F(j_2 - n \mid j_3)) \quad (7)$$

Therefore, the transition matrix (4) can alternatively be expressed as

$$r_{j_0|j_1,\cdots,j_3} = \frac{\sum\limits_{l=0,1} \sum\limits_{m=0,1} \sum\limits_{n=0,1} (-1)^{(l+m+n)} C(F(j_0 - l \mid j_3), F(j_1 - m \mid j_3), F(j_2 - n \mid j_3))}{\sum\limits_{m=0,1} \sum\limits_{n=0,1} (-1)^{(m+n)} C(F(j_1 - m \mid j_3), F(j_2 - n \mid j_3))} \quad (8)$$

In the following Proposition these results are extended to a Markov chain of order p.

Proposition 1 *For a Markov chain of order p, the copula-based transition matrix $r_{j_0|j_1,\cdots,j_p}$ is expressed as follows*

$$r_{j_0|j_1,\cdots,j_p} = \frac{\sum\limits_{l_0=0,1} \cdots \sum\limits_{l_p=0,1} (-1)^{(l_0+\cdots+l_p)} C(F(j_0 - l_0 \mid j_p), \cdots, F(j_{p-1} - l_{p-1} \mid j_p))}{\sum\limits_{l_1=0,1} \cdots \sum\limits_{l_p=0,1} (-1)^{(l_1+\cdots+l_p)} C(F(j_1 - l_1 \mid j_p), \cdots, F(j_{p-1} - l_{p-1} \mid j_p))} \quad (9)$$

where $F(. \mid .)$ is a conditional distribution function and $C(u_0, \cdot, u_p)$ is a copula function in $[0,1]^{p+1}$.

4 Estimation and inference

4.1 Estimation

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6.1.1 Bootstrap bias-correction

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References

Sklar, M. (1959). Fonctions de repartition an dimensions et leurs marges. *Publ. inst. statist. univ. Paris*, 8:229–231.

9 Appendix: Proofs