Solving Discrete Logarithm Problem in Elliptic Curve Cryptography Using Variation of Pollard Rho's Method

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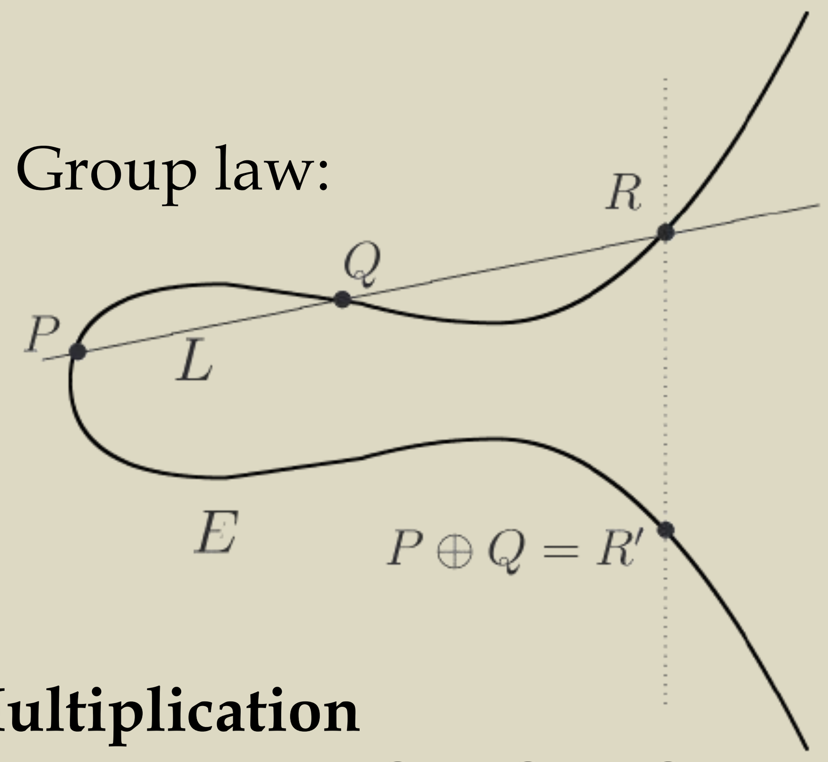
Elliptic Curve Overview

Elliptic curve equation

$$Y^2 = X^3 + AX + B$$

Group law:



Multiplication

$$nP = \underbrace{P \oplus P \oplus \cdots \oplus P}_{n \text{ addition on } P}$$

Elliptic Curve Discrete Logarithm Problem (ECDLP)

Given points P and Q on an elliptic curve $E(\mathbb{F}_p)$, compute n:

$$nP = Q$$

It is difficult to compute n as the standard way requires a lot of storage and is computationally expensive.

Elliptic ElGamal Public Key Cryptosystem

Bob wants to encrypt and send Alice a plaintext message m using a public key from Alice.

Public
1. A prime number p
2. $E(\mathbb{F}_p)$
3. P $\in E(\mathbb{F}_p)$

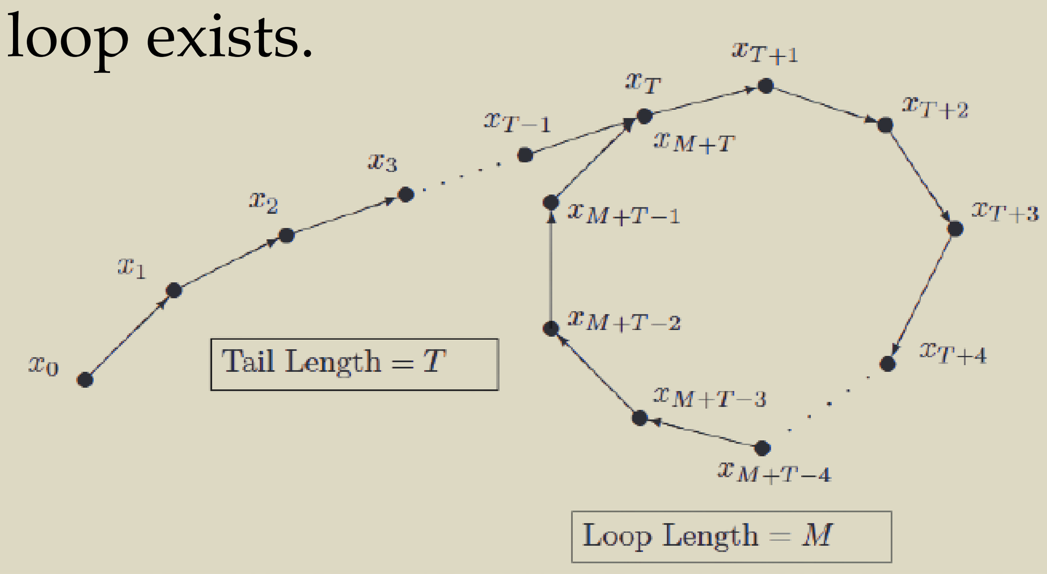
Secret
Private key n_A

Computations
Alice computes public key
$$Q_A = n_A P$$
And publishes Q_A
Bob computes
$$c_1 = kP \in E(\mathbb{F}_p)$$
$$c_2 = m \oplus kQ_A \in E(\mathbb{F}_p)$$
where k is temporary.

Decryption
Alice computes $m = c_2 \ominus n_A c_1 \in E(\mathbb{F}_p)$ to obtain plaintext m.

Pollard ρ Method

$f: \langle P \rangle \to \langle P \rangle$ (A sufficiently random function.)
∵ order of P finite, loop exists.



Series x	Series y
x_0	$y_0 = x_0$
$x_1 = f(x_0)$	$y_1 = f \circ f(y_0)$
$x_2 = f(x_1)$	$y_2 = f \circ f(y_1)$
$x_3 = f(x_2)$	$y_3 = f \circ f(y_2)$

Collision $x_i = y_i$ happens in approximately $O\sqrt{N}$ steps where N is order of P.

$$\Rightarrow a_1 P \oplus b_1 Q = a_2 P \oplus b_2 Q$$
$$(a_1 - a_2)(b_2 - b_1)^{-1} P = Q$$

Pollard ρ method is the most efficient collision detection method to date and requires least storage space as it only stores the latest iteration of x and y values.

Change Number of Iterations for y Series

Classical Pollard ρ method

$$x_i = f(x_{i-1})$$
$$y_i = f \circ f(y_{i-1})$$
$$\therefore y_i = x_{2i}$$

Variation

$$x_i = f(x_{i-1})$$
$$y_i = (\underbrace{f \circ f \circ \cdots \circ f}_{j \text{ iteration of } f})(y_{i-1})$$
$$\therefore y_i = x_{ji}$$

Generate Pseudorandom Points by Modifying $f(R)$

Standard $f(R)$

$$f(R) = \begin{cases} P \oplus R & \text{if } 0 \le R_x < p/3 \\ 2R & \text{if p/3} \le R_x < 2p/3 \\ Q \oplus R & \text{if } 2p/3 \le R_x < p \end{cases}$$

,where $R = (R_x, R_y)$

Store coefficients of P and Q into an array of (a_1, b_1, a_2, b_2)
Update array with each iteration.

Results

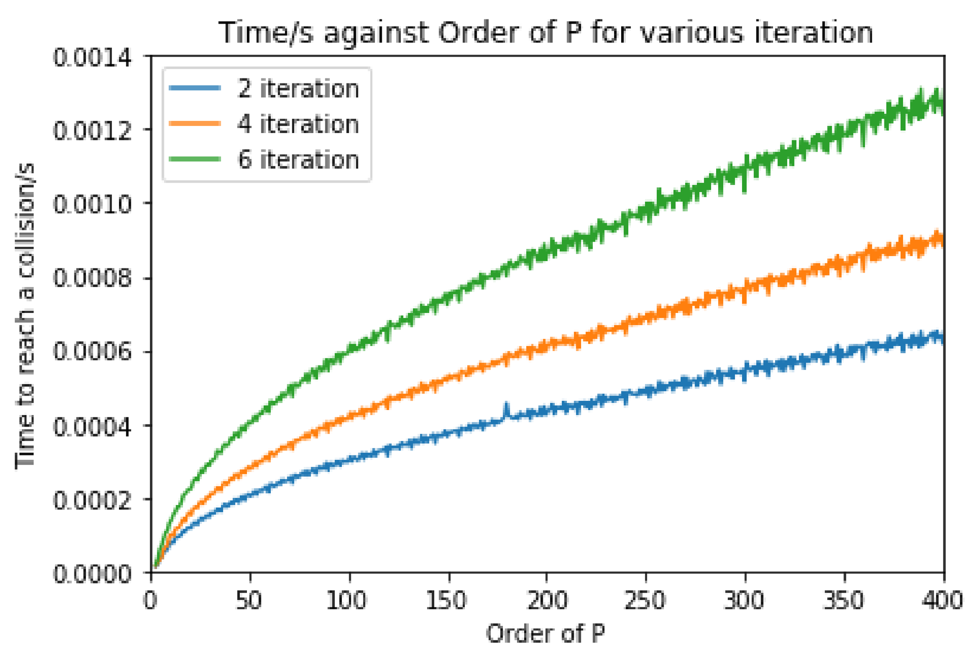


Fig. 1: Graph of time in seconds against $ord(P)$ for different number of iterations

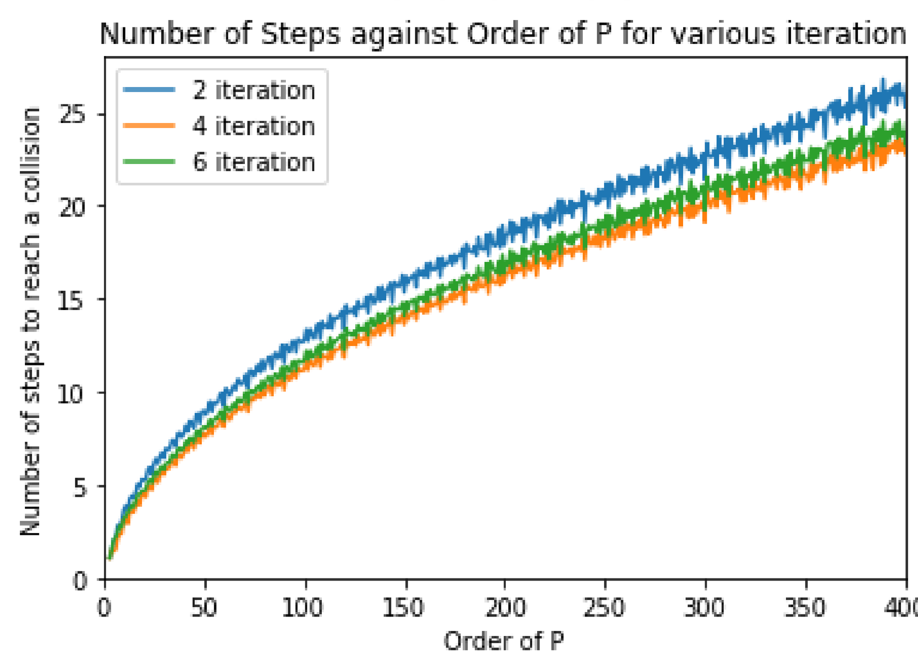


Fig. 2: Graph of N against $ord(P)$ for different number of iterations

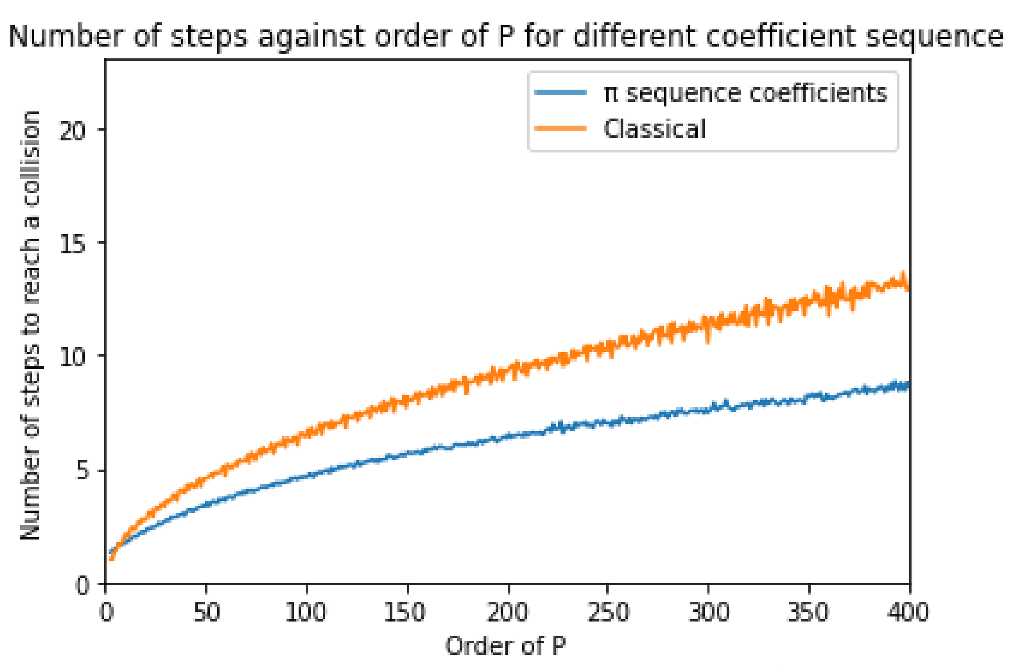


Fig. 3: Graph of N against $ord(P)$ for different iteration function

Conclusion and Future Work

The number of iterations should not be increased since the time taken increases (Fig. 1) with only a marginal decrease in the number of steps (Fig. 2) needed to solve ECDLP.

A different iteration function where the coefficients of P, Q and R are chosen from the digits of π reached a collision in less steps than the standard function (Fig. 3).

Future work in this area can be to:
1. Investigate randomness of f by changing the coefficients and number of partitions
2. Use C rather than python to run code

References

Hoffstein, J., Pipher, J., & Silverman, J. H. (2014). An Introduction to Mathematical Cryptography (Undergraduate Texts in Mathematics) (2nd ed. 2014 ed.). Springer.

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