SVAE LDS KL Divergence

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1 Starting Loss Function

Start with the KL Divergence between two sequences of length T (predicted and updated):

$$
\begin{aligned}
KL(q||p) &= \mathbb{E}_{q(z|x)}\left[\log \frac{q(z|x)}{p(z)}\right] && (1)\\
&= \mathbb{E}_{q(z|x)}\left[\log q(z|x) - log p(z)\right] && (2)\\
&= \mathbb{E}_{q(z|x)}\left[\log q(z_1|x) + \sum_{i=2}^{T} \log q(z_i|z_{i-1}, x) - \log p(z_1) - \sum_{i=2}^{T} \log p(z_i|z_{i-1})\right] && (3)
\end{aligned}
$$

The terms outside of the summation can be separated and form their own KL divergence:

$$
= \underbrace{KL(q(z_1|x)||p(z_1))}_{\text{Term A}} + \sum_{i=2}^{T} \mathbb{E}_{q(z_i, z_{i-1}|x)}\left[\underbrace{\log q(z_i|z_{i-1}, x)}_{\text{Term B}} - \underbrace{\log p(z_i|z_{i-1})}_{\text{Term C}}\right] \tag{4}
$$

The KL divergence in term A can be calculated in closed form, terms B and C are calculated below. Term C is simpler than term B so we will calculate it first.

2 Kalman Filter Equations

Prediction Step Equations:

$$
\begin{aligned}
z_{i|i-1} &= \mathbf{A} z_{i-1|i-1} + \mathbf{b} && (5)\\
P_{i|i-1} &= \mathbf{A} P_{i-1|i-1} + \mathbf{A}^T + Q && (6)\\
& && (7)
\end{aligned}
$$

Update Step Equations

$$
\begin{aligned}
K_i &= P_{i|i-1} H^T (H P_{i|i-1} H^T + R)^{-1} \qquad \text{(Kalman Gain)} && (8)\\
z_{i|i} &= z_{i|i-1} + K_i(x_i - H z_{i|i-1}) && (9)\\
P_{i|i} &= P_{i|i-1} - K_i H P_{i|i-1} = (I - K_i H) P_{i|i-1} && (10)
\end{aligned}
$$

3 Calculating Term C

Lets start with term C from section 1. From the kalman filter update step $\mu_i = \mathbf{A} z_{i-1|i-1} + \mathbf{b}$ and $P_i = \mathbf{A} P_{i-1} \mathbf{A}^T + Q$. Additionally we use the shorthand $\mathbb{E}_q = \mathbb{E}_{q(z_i, z_{i-1}|x)}$.

$$\begin{aligned}
\mathbb{E}_q\left[\log p(z_i|z_{i-1})\right] &= -\frac{1}{2}\mathbb{E}_q\left[k\log(2\pi) + \log\det P_i\right] - \frac{1}{2}\mathbb{E}_q\left[(z_{i|i-1} - \mu_i)^T P_i (z_{i|i-1} - \mu_i)\right] && (11)\\
&= -\frac{1}{2}\left[k\log(2\pi) + \log\det P_i\right] - \frac{1}{2}\mathbb{E}_q\left[(z_{i|i-1} - \mu_i)^T P_i (z_{i|i-1} - \mu_i)\right] && (12)\\
&= -\frac{1}{2}\left[k\log(2\pi) + \log\det P_i\right]\\
&\quad - \frac{1}{2}\mathbb{E}_q\Bigg[\underbrace{z_{i|i-1}^T P_i^{-1} z_{i|i-1}}_{①} - \underbrace{z_{i|i-1}^T P_i^{-1}\mathbf{A} z_{i-1|i-1}}_{②} - \underbrace{z_{i|i-1}^T P_i^{-1}\mathbf{b}}_{③} - \underbrace{\mathbf{A}^T z_{i-1|i-1}^T P_i^{-1} z_{i|i-1}}_{④}\\
&\quad + \underbrace{\mathbf{A}^T z_{i-1|i-1}^T P_i^{-1}\mathbf{A} z_{i-1|i-1}}_{⑤} + \underbrace{\mathbf{A}^T z_{i-1|i-1}^T P_i^{-1}\mathbf{b}}_{⑥} - \underbrace{\mathbf{b}^T P_i^{-1} z_{i|i-1}}_{⑦} + \underbrace{\mathbf{b}^T P_i^{-1}\mathbf{A} z_{i-1|i-1}}_{⑧} + \underbrace{\mathbf{b}^T P_i^{-1}\mathbf{b}}_{⑨}\Bigg] && (13)
\end{aligned}$$

Term ⑨ is constant and can be pulled out of the expectation, all other terms can be calculated with expectation identities from section A.

4 Calculating Term B

The process for calculating term B starts the same way as term A. $q(z_i|z_{i-1})$ represents the update step of the kalman filter, so the equations for μ_i and Σ_i are more complicated.

$$\mathbb{E}_q\left[\log p(z_i|z_{i-1})\right] = -\frac{1}{2}\mathbb{E}_q\left[k\log(2\pi) + \log\det P_i\right] - \frac{1}{2}\mathbb{E}_q\left[(z_i - \mu_i)^T P_i (z_i - \mu_i)\right] \qquad (14)$$

Using traditional kalman filter from section 2 we get:

$$\begin{aligned}
\mu_i &= \mathbf{A} z_{i-1|i-1} + \mathbf{b} + K_i(x_i - H(\mathbf{A} z_{i-1|i-1} + \mathbf{b})) && (15)\\
P_{i|i} &= P_{i|i-1} - K_i H P_{i|i-1} = (I - K_i H) P_{i|i-1} && (16)
\end{aligned}$$

$P_{i|i}$ is constant with respect to z, so we don't need to expand the full equation. It will be helpful to simplify the equation for μ_i as follows:

$$\begin{aligned}
\mu_i &= \mathbf{A} z_{i-1|i-1} + \mathbf{b} + K_i(x_i - H(\mathbf{A} z_{i-1|i-1} + \mathbf{b})) && (17)\\
&= \mathbf{A} z_{i-1|i-1} + \mathbf{b} + K_i x_i - K_i H \mathbf{A} z_{i-1|i-1} - K_i H \mathbf{b} && (18)\\
&= \mathbf{A} z_{i-1|i-1} - K_i H \mathbf{A} z_{i-1|i-1} + \underbrace{\mathbf{b} - K_i H \mathbf{b} + K_i x_i}_{\text{constant}} && (19)
\end{aligned}$$

Now we can expand equation 14 into:

$$\mathbb{E}_q\left[\log p(z_i|z_{i-1})\right] = -\frac{1}{2}\left[k\log(2\pi) + \log\det P_i\right] \tag{20}$$

$$-\frac{1}{2}\mathbb{E}_q\left[(z_i-\mu_i)^T P_i (z_i-\mu_i)\right] \tag{21}$$

A Expectation Identities

$$\mathbb{E}_{z\sim\mathcal{N}(\mu,\Sigma)}[z] = \mu \tag{22}$$

$$\mathbb{E}_{z\sim\mathcal{N}(\mu,\Sigma)}[zz^T] = \mu\mu^T + \Sigma \tag{23}$$

$$\mathbb{E}_{y\sim\mathcal{N}(\mu_y,\Sigma_y),z\sim\mathcal{N}(\mu_z,\Sigma_z)}[zy^T] = \mu_z\mu_y^T \tag{24}$$

B Matrix Identities

The following matrix identities are copied from Max Welling's wonderful notes on Kalman Filters (stat.columbia.edu/ liam/teaching/neurostat-spr12/papers/hmm/KF-welling-notes.pdf)

$$\mathbf{a}^T\mathbf{A}\mathbf{b} = \mathrm{tr}[\mathbf{A}\mathbf{b}\mathbf{a}^T] \tag{25}$$

$$\mathrm{tr}[\mathbf{A}\mathbf{B}] = \mathrm{tr}[\mathbf{B}\mathbf{A}] \tag{26}$$

$$\log\det[\mathbf{A}] = -\log\det[\mathbf{A}^{-1}] \tag{27}$$