① if $\{|k\rangle\}$ is a basis,

$|a\rangle = \sum_i a_i |i\rangle$, $|b\rangle = \sum_j b_j |j\rangle$

$$\sum_k \langle a|k\rangle\langle k|b\rangle = \sum_K \left\langle \sum_i a_i |i\rangle \,|\, k\right\rangle \left\langle k \,|\, \sum b_j |j\rangle\right\rangle$$

$$= \sum_k a_k^* b_k$$

$$\langle a|b\rangle = \left\langle \sum_i a_i |i\rangle \,\middle|\, \sum_j b_j |j\rangle\right\rangle$$

$$= \sum_{i,j} a_i^\dagger \langle i| b_j |j\rangle$$

$$= \sum_{i,j} a_i^* b_j \delta_{ij} = \sum_i a_i^* b_i$$

Thus, $\sum_k \langle a|k\rangle\langle k|b\rangle = \langle a|b\rangle$

Thus proved one way.

if

$$\langle a|b\rangle = \sum_K \langle a|k\rangle\langle k|b\rangle$$

$$\langle a|b\rangle |a\rangle = \sum_k \langle a|k\rangle |a\rangle \langle k|b\rangle$$

$$\cancel{\langle a|a\rangle} |b\rangle = \sum_k \cancel{\langle a|a\rangle} |k\rangle \langle k|b\rangle$$

$$|b\rangle = \sum |k\rangle \langle k|b\rangle$$

Since any vector $|b\rangle = \sum_k |k\rangle\langle k|b\rangle$ and $\{|k\rangle\}$ are orthonormal, $\{|k\rangle\}$ form a basis.

(2) $e^A = B$

$A + 2\pi i I \neq A$, but

$$e^{A + 2\pi i I} = e^A = B$$

Thus, we have found two logarithms for B already. Hence its not unique.

(3) i) $$\begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 4 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}}_{a} \otimes \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{b}$$

$(2-\lambda)(4-\lambda) - 3 = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$\lambda = 1, 5$

$(-\lambda)(-\lambda) - 1 = 0$

$\lambda = 1, -1$

eigen values are :- $1, -1, 5, -5$

ii) $$\begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$\lambda = 1, -1$

$\lambda = 1, 5$

eigen values are - $1, -1, 5, -5$

iii) $$\begin{pmatrix} 4 & 6 & 6 & 9 \\ 2 & 8 & 3 & 12 \\ 2 & 3 & 8 & 12 \\ 1 & 4 & 4 & 16 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$\lambda = 1, 5$

$\lambda = 1, 5$

eigen values are $1, 5, \cancel{5}, 25$

② 1) $|\psi'\rangle = H|\psi\rangle$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= |1\rangle$$

2) $|\psi\rangle$

before measurement ~~after measurement~~

$$P(|0\rangle) = \|\langle 0|\psi\rangle\|^2 = \frac{1}{2}\|\cancel{a}\langle 0|0\rangle - \langle 0|1\rangle\|^2$$

$$= \frac{1}{2}\|1\|^2 = \frac{1}{2}$$

$$P(|1\rangle) = \|\langle 1|\psi\rangle\|^2 = \frac{1}{2}\|\langle 1|0\rangle - \langle 1|1\rangle\|^2$$

$$= \frac{1}{2}\|-1\|^2 = \frac{1}{2}$$

after measurement:-

if $|0\rangle$ is measured, $P(|0\rangle) = 1$, $P(|1\rangle) = 0$

is $|1\rangle$ is measured, $P(|0\rangle) = 0$, $P(|1\rangle) = 1$

for $|\psi'\rangle$

$$P(|0\rangle) = 0$$
$$P(|1\rangle) = 1$$

here, only $|1\rangle$ could be measured, as $P(|0\rangle) = 0$.

thus, after measurement, $P(|0\rangle) = 0$

$P(|1\rangle) = 1$