Vectors

a : b : c (direction ratios) → const $\langle pa, pb, pc \rangle$ → unit vector (direction cosines ✓)

$$\hat{a} = \frac{a}{|a|} = i\cos\alpha + j\cos\beta + k\cos\gamma$$

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$a \cdot b = |a||b|\cos\theta$$

$$a \times b = |a||b|\sin\theta\ \hat{n}$$

★ $\Rightarrow \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix}$ (+ − +)

// gram Area $= |a \times b|$

coplanar / colinear ★ {equate coefficient, take dot, cross, scalar triple swap, dot product distribute}

$$|\vec{AB} \cdot \hat{n}| = |AC|$$

---

parallelopiped volume $= [a, b, c] = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

tetrahedron vol. $= [a, b, c] \times \frac{1}{6}$

★ same vector in scalar product $= 0$

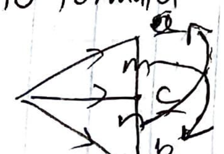
$$a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$$

$$(a \times b) \times c = (a \cdot c)\, b - (c \cdot b)\, a$$

★ coplanar vectors $[a, b, c] = 0$

$[a, b, a] = 0$

- ratio formula



$$c = \frac{mb + na}{m + n}$$

Straight line

- Intersecting lines

eq^n 1, eq^n 2 } solve → eq^n 3 → plug

s, t parameters solve

sol^n exist ✓

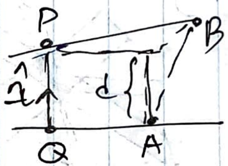
- angle between lines

$$\frac{t_1 \cdot t_2}{|t_1||t_2|} = \cos\theta, \text{ vectors along lines}$$

→ parallel $\langle a, b, c \rangle = \lambda \langle d, e, f \rangle$

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

- shortest distance betwee skew lines
  - → not //
  - → not intersecting



$$r_1 = \langle 1, 1, 0 \rangle + \lambda \langle 2, -1, 1 \rangle$$

$$r_2 = \langle 2, 1, -1 \rangle + \mu \langle 3, -5, 2 \rangle$$

$$d = |\vec{AB} \cdot \hat{n}|$$

★ can calculate P & Q by using λ, μ and dot product with one eq^n's vector along it

---

A (1, 2, 3)

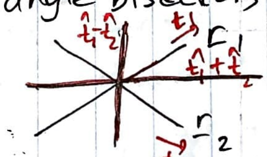
$r = \langle 2, 3, 4 \rangle + \lambda \langle 3, 4, 5 \rangle$ (t_1)

$P(2 + 3\lambda, 3 + 4\lambda, 4 + 5\lambda)$

$\vec{AP} \cdot t_1 = 0 \Rightarrow \lambda$ ✓ $\Rightarrow A$ ✓

- perpendicular distance between point & line

- angle bisectors



vectors along the angle bisectors: $\hat{t}_1 + \hat{t}_2$ ✓, $\hat{t}_1 - \hat{t}_2$ ✓

common point ⇒ solve r_1 & r_2 ✓

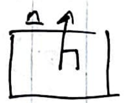
Vector eq^n $= r = \langle 1, 2, 3 \rangle + t\langle 4, 5, 6 \rangle$

parametric eq^n's $=$ $x = 1 + 4t$, $y = 2 + 5t$, $z = 3 + 6t$

Cartesian eq^n $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}$

Symmetric form $\frac{x-1}{4/\sqrt{4^2+5^2+6^2}}$ ...

Plane



$\underline{r} \cdot \underline{n} = \underline{r_0} \cdot \underline{n}$

$\langle x, y, z\rangle \langle a, b, c\rangle = \langle x_0, y_0, z_0\rangle \langle a, b, c\rangle$

$ax + by + cz = d$

- angle between planes

$\pi - \theta$

$\cos(\pi - \theta) = \dfrac{\underline{n}_1 \times \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$

$-\cos\theta =$

- eq^n of line along intersection of planes

take $x = t \in \mathbb{R}$ $\Rightarrow$ π_1, π_2 subst.

arbitrary point on the line of intersection $\longrightarrow$ $y = ?$, $z = ?$

cartesian eq^n ✓

Intersection of plane & line

$t \in \mathbb{R} \xrightarrow{\text{subs.}} l$

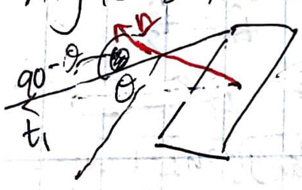
$\downarrow$ subs.

π and t ✓

$\downarrow$

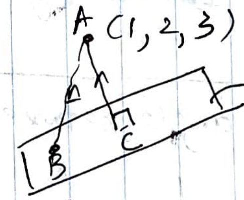
point P ✓

- Angle between plane & line



$\underbrace{\cos(90-\theta)}_{\sin\theta} = \dfrac{\underline{n} \cdot \underline{t_1}}{|\underline{n}||\underline{t_1}|}$

- perpendicular distance from point – plane



$\pi = 2x - y + z = 4$

$\underline{n} = \langle 2, -1, 1\rangle \longrightarrow AC \equiv \langle 1,2,3\rangle + t\langle 2,-1,1\rangle$

x, y, z points $\to \pi \to t$ ✓ $\to C$ ✓

find B using π $(x=1, y=1, z=?)$

$|AC| = |\vec{AB} \cdot \hat{n}|$

- Angle bisectors between 2 planes

$\underline{r} \cdot \underline{n}_1 = d_1$

$\underline{r} \cdot \underline{n}_2 = d_2$

$\dfrac{|\underline{r} \cdot \underline{n}_1 - d_1|}{|n_1|} = \dfrac{|\underline{r} \cdot \underline{n}_2 - d_2|}{|n_2|}$

$|\hat{n}_1 + \hat{n}_2| \to$ new normal vector

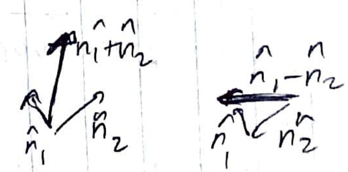
$|\hat{n}_1 - \hat{n}_2|$

method 2

$l \to$ intersecting line $\to t_1$ vector along l_1

$\pi_1 + \lambda\pi_2 \to$ intersecting plane $\to \underline{n}$

$\underline{n} \cdot \underline{t}_1 = 0 \to \lambda$ ✓



point on intersecting line

If asked distance $\Rightarrow |\ |$

remember

parallel vector $\langle -4, -20, -8\rangle \parallel -4\underbrace{\langle 1, 5, 2\rangle}_{\text{take this}}$

Matrices

$A = A^T \rightarrow$ symmetric

$A = -A^T \rightarrow$ skew-symmetric
diagonals $= 0 \Leftarrow i = j$

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{skew symmetric}}$$

show $A = A^T$; show $A = -A^T$

$(AB)^T = B^T A^T$

$(A+B)^T = A^T + B^T$

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij}$$

row expan

co-factor matrix $\{ A_{ij} = (-1)^{i+j} |M_{ij}|$

M_{ij} obtained by deleting ith & jth row

$\text{adj}(A) = (A_{ij})^T$

$$= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$$

Inverse A^{-1} $\rightarrow$ using formulae

$A^{-1} = \frac{\text{adj}(A)}{|A| \neq 0}$ // non singular $\leftarrow |A| \neq 0$

$\rightarrow$ using Aug. matrix

$[A \mid I]$

$[I \mid A^{-1}]$ ✓

$(AB)^{-1} = B^{-1} A^{-1}$

Properties of determinants

① col/row $= 0$ or 2 rows/cols equal
det $= 0$

② col/row operation fine ✓

③ except interchanging row/column
$= -|A|$

④ Scaler multiples from rows/cols can be taken out

⑤ $|AB| = |A||B|$
both square matrices

$|A| = |A^T|$

If some/all elements in row/col of det can be expressed as 2 or more terms $\Rightarrow$ det can be expressed 2 or more terms

$$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$$

---

non-homogenous systems $AX = B \neq 0$

method 1

$|A| \neq 0$, no. eq^n $= n$ (unknowns)

$X = A^{-1} B$ ✓

method 2

crammer's rule

$|A| \neq 0$, no. eq^n $= n$

$$\boxed{x_i = \frac{|A_i|}{|A|}}$$

replace ith col with B

method 3 gaussian elemination

$AX = B$

inverse using row operations

$[A \mid I]$

$[I \mid A^{-1}]$ $\Leftarrow$ no condition

$[A \mid B] \rightarrow$ row echolen form

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

row exchange
don't exchange

$(\updownarrow)$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $(\updownarrow)$

only ✗ only

- each rows 0 befor non-zero elements in increasing order
- each row 1st non-zero element $= 1$ (leading 1)

System of linear eq^{n}s

$A_{m \times n} X_{n \times 1} = B_{m \times 1}$

non-homogeneous
$B \neq 0$
$AX = B$

- consistent
  rank(A) = rank(A|B)
  - unique: rank(A) = rank(A|B) $= n$
    $|A| \neq 0$
  - infinite: rank(A) = rank(A|B) $< n$
    $|A| = 0$
- inconsistent
  rank(A) $\neq$ rank(A|B)
  no sol^n

homogeneous
$B = 0$
$AX = 0$
$\downarrow$
always consistent
$X = 0$
trivial sol^n ✓

- rank(A) = rank(A|B) $= n$
  only trivial
  $\equiv$
  $|A| \neq 0$
  then $X = 0$
- rank(A) = rank(A|B) $< n$
  trivial + infinite sol^n
  $t \in \mathbb{R}$
  $|A| = 0$

Eigen values & vectors

$AX = \lambda X$

$(A - \lambda I) \cdot X = 0 \rightarrow$ homogeneous $\circledast$

for non-trivial sol^{n}s,

$|A - \lambda I| = 0$ } characteristic eq^n $\boxed{2}$

$\boxed{1}$ characteristic polynomial

$\boxed{3}$ find $\lambda_1, \lambda_2, \ldots$ } roots of characteristic eq^n $\rightarrow$ eigen values of A

$\boxed{4}$ for each λ_i; plug back to $\circledast$ by $(A-\lambda_i I)X_i = 0$, $i = 1,2,3$ where X_i is eigen vector $\rightarrow$ gaussian elemination

find the corresponding eigenvector.

check with $(A-\lambda I)X = 0$

- 1 parameter t_i if λ_i is unique — use last variable as $t \in \mathbb{R}$
- 2 parameters (α, β) if λ is repeated r times — take α, β out as a sum — use x, y, z instead of x_1, x_2, x_3

r = alegebraic multiplicity

- Col. vectors satisfying eq^n $(A - \lambda_i I)X_i = 0$ $X_1, X_2, \ldots$ are called eigen vectors of A
- Set of all eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$ of A is called spectrum of A
- Spectral radius of $A = R = max\{|\lambda_1|, |\lambda_2| \ldots\}$

- eigen vectors are always L.I.

$A\underline{V} = \lambda \underline{V}$

to find $A \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

normalization of eigen vectors

$AX = \lambda X$

can be multiplied by non-zero K

$A(KX) = \lambda(KX)$

scale eigenvectors = normalized eigenvectors

For eigen vectors with $\mathbb{R}$ numbers, common normalization is to, take sum of squares of elements = 1

$X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

satisfy

normalizing factor $(K) = \dfrac{1}{\sqrt{a^2+b^2+c^2}}$

normalized vector = $\begin{pmatrix} a/\sqrt{a^2+b^2+c^2} \\ b/\sqrt{a^2+b^2+c^2} \\ c/\sqrt{a^2+b^2+c^2} \end{pmatrix}$

---

when eigen vector is given and corresponding eigen value is given,

use: $A\underline{V} = \lambda_1 \underline{V}$; where $\underline{V}$ eigenvector, λ_1 is corresponding eigen val.

---

eg: $p(x)$ $4x^3 + 8x^2 - 5x + 10$

all possible zeros $\pm \dfrac{1, 2, 5, 10}{1, 2, 4} \Rightarrow \pm 1, \frac{1}{2}, \frac{1}{4}, 2, 5, \frac{5}{2}, \frac{5}{4}, 10$

Norm of an n-element row/column vector:

$$||\omega|| = (\omega_1^2 + \omega_2^2 + \dots + \omega_n^2)^{1/2}$$

- set of row/col n-element vectors $u_1, u_2, \dots, u_n$ are,

eigen vectors are mutually orthogonal $x_i \cdot x_j = 0$ ★

orthogonal $\Rightarrow$ $u_i \cdot u_j = \begin{cases} 0, & i \neq j \\ ||u_i||^2, & i = j \end{cases}$

orthonormal $\Rightarrow$ $u_i \cdot u_j = \begin{cases} 0, & i \neq j \\ ||u_i||^2 = 1, & i = j \end{cases}$

like dot product

checking for orthonormal

If A, nxn symmetric matrix $(A = A^T)$
inverse is also symmetric

$\Rightarrow$ 1) eigenvalues of A all are $\mathbb{R}$

2) eigenvectors of A corresponding to distinct eigenvalues are mutually orthogonal

---

A is orthoganal $\Leftrightarrow$ $A A^T = I$

$\boxed{A^{-1} = A^T}$ } to show it is orthoganal

properties: 1) product of orthoganal matrices = orthoganal matrix

2) rows/cols of orthoganal matrix forms an orthonormal set of vectors.

---

(1) Number of eigenvalues = size of matrix.

(1) eigenvalues of upper triangular matrix /lower $\rightarrow$ diagonal elements

eg $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \rightarrow \lambda_1 = 1, \lambda_2 = 2$

(2) If λ is eigenvalue of A $\Rightarrow \lambda^n$ is an eigen value of A^n

(3) $\sum_{i=1}^{n} \lambda_i = tr(A)$ } sum of eigen values = trace of A
sum of diagonals

(4) product of eigenvalues of $A = |A|$

$\prod_{i=1}^{n} \lambda_i = |A|$

eg: 2 eigenvalues are given & find other one use this

(5) eigenvalues of A & A^T are same because their characteristic eq^{n}s are same.

(6) If λ is eigenvalue of $A \Rightarrow \frac{1}{\lambda}$ is an eigenvalue of A^{-1}
A is invertible

Th^m For 2 square matrices A & B,

1) A is similar to B $\Rightarrow \exists P^{-1}$ s.t. $B = P^{-1} A P$

2) A & B have same eigen values

3) A^n & B^n " , $A \rightarrow \lambda_i$, $A^2 \rightarrow \lambda_i^2$

Diagonalizing a matrix

[1] $A \rightarrow$ characteristic eq^n $|A - \lambda I| = 0$
eigenvalues $\lambda_i \rightarrow$ find
eigenvectors $x_i \rightarrow$ find

[2] diagonalizing matrix $= P = (x_1 \; x_2 \; \dots) \rightarrow P^{-1}$ find
i^{th} column is eigen vector x_i

diagonal matrix $= D = P^{-1} A P$ prove

$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & \\ & & \cdots & \lambda_n \end{pmatrix}$$

$D = P^{-1}AP$
$PD = AP$
check check $\rightarrow$ instead of calculating P^{-1}

Note: diagonalizing matrix P is not unique
$\because$ it depends on the order of eigenvectors corresponding to order of eigenvals

properties of diagonal matrix,

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \lambda_n \end{pmatrix}$$

1) $D^m = \begin{pmatrix} \lambda_1^m & \cdots & 0 \\ \vdots & \lambda_2^m & \vdots \\ 0 & \cdots & \lambda_n^m \end{pmatrix}$; $m > 0$

2) $|D| = |\lambda_1 \cdot \lambda_2 \lambda_3 \cdots \lambda_n|$

3) If D is non-singular $\equiv |D| \neq 0 \equiv$ not any diagonal element $= 0$

$$D^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \vdots \\ \vdots & \ddots & \\ 0 & \cdots & \frac{1}{\lambda_n} \end{pmatrix}$$

A canbe diagonalized means $\Rightarrow \exists P^{-1}$ s.t
$D = P^{-1}AP$
$A = PDP^{-1}$
$A^m = PD^mP^{-1}$ } easily take powers

Find orthogonal diagonalizing matrix for symmetric matrices

[1] eigen values, eigenvectors

[2] normalized eigen vectors $\hat{x}_1 \leftarrow x_1$ divide each element by $\|x_1\|$

form diagonalizing matrix $P \rightarrow$ orthoganal ✓

Cayley-hamilton th^m

If $\underbrace{P_n(\lambda)}_{|A - \lambda I|}$ is characteristic polynomial of A

$\Rightarrow$ A satisfies its own characteristic eq^n

$P_n(\lambda) = 0 \Rightarrow \boxed{P_n(A) = 0}$

can plug in A instead of λ

eg: $\lambda^2 - 4\lambda - 1 = 0$
$A^2 - 4A - I = 0$
$A^{-1}A^2 - 4A^{-1}A - A^{-1}I = 0$
$A - 4I - A^{-1} = 0$
$A^{-1} = A - 4I \rightarrow$ easy way to find A^{-1}

$1 \rightarrow I$
$0 \rightarrow O$ null matrix

Matrix norms (always non-(−)ve $\mathbb{R}$ numbers)

1) 1-norm

$$\boxed{\|A\|_1 = \max_{1\le j\le n}\left(\sum_{i=1}^{m}|a_{ij}|\right)}$$

- max abs. column sum

2) euclidean's norm (2-norm)

$$\boxed{\|A\|_E = \|A\|_2 = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2}}$$

- square root of sum of squares (all elements)

3) Infinity norm

$$\boxed{\|A\|_\infty = \max_{1\le i\le m}\left(\sum_{j=1}^{n}|a_{ij}|\right)}$$

- max abs. row sum

4) Frobenius norm

★

$$\boxed{\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\mathrm{trace}(A^TA)}}$$

$z = x+iy$, $|z| = \sqrt{x^2+y^2}$ } to make norm of complex numbers real

Properties of matrix norms

1) $\|A\| \geq 0$
2) $\|A\| = 0 \iff A = 0$
3) $\|kA\| = |k|\,\|A\|$ for any scalar k
4) $\|A+B\| \leq \|A\| + \|B\|$ triangle inequality
5) $\|AB\| \leq \|A\|\,\|B\|$ ★

- above 5 properties must be fullfilled to define a matrix norm
- norm always $\in \mathbb{R}$, valid for any type of norm

Vector norm (induced matrix norm)

1) $\|X\| > 0$ when $X \neq 0$ ∧ $\|X\| = 0 \iff X = 0$
3) $\|kX\| = |k|\,\|X\|$ for any scalar k
4) $\|X+Y\| \leq \|X\| + \|Y\|$

no 5th property

① 1-norm

$$\|X\|_1 = \sum_{i=1}^{n}|x_i|$$

② 2-norm

$$\|X_2\| = \sqrt{\sum_{i=1}^{n}x_i^2}$$

③ infinity norm (∞-norm)

$$\boxed{\|X_\infty\| = \max_i |x_i|}$$

$\boxed{\text{Th}^m}$ for any vector X in $\mathbb{R}^{n}$ ← n-vector

$$\|X\|_1 \geq \|X\|_2 \geq \|X_\infty\|$$

* only for vector norms

Induced matrix norm

$\|A\| = \max \|AX\|$

[1] eq^n for norm

[2] $\{X \mid X \in \mathbb{R}^{n}, \|X\|_{(\ )} \leq 1\}$ unit ball in $\|X\|_{(\ )}$ given by

[3] for boundary $\|X\|_{(\ )} = 1$

[4] vectors satisfy the eq^n (boundary of unit ball) draw it ◇ □ ○

[5] image of the boundary under the action of A given by $\{Y \mid Y = AX \text{ with } \|X\|_{(\ )} = 1\}$

[6] find $Y = AX$

[7] draw write $\|Y\|_{(\ )} = \|A\|_{(\ )}$

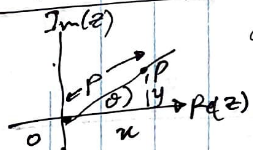
vertices

- unit ball in $\|X\|_{(\ )} = \{X \mid X \in \mathbb{R}^{n}, \|X\|_{(\ )} \leq 1\}$ n-vector (elements)
- boundary points ⇒ vectors satisfying $\|X\|_{(\ )} = 1$
- image of the boundary of unit ball in $\|X\|_{(\ )}$ under the action of A $\{Y \mid Y = AX \text{ with } \|X\|_{(\ )} = 1\}$ calculate with

Euler's formula & euler's identity

$\boxed{e^{ix} = \cos x + i\sin x}$ $\quad x \in \mathbb{R}$, $i = \sqrt{-1}$

proved using power series



cartesian $z = x + iy$

Polar $z = \rho e^{i\theta}$

$\rho = \sqrt{x^2+y^2}$

- when $\rho = 1$, all $\mathbb{C}$ numbers in the form $\boxed{z = e^{i\theta}}$ form a unit circle centered at origin
- when $x = 2\pi n$ used to find different roots of 1 (like $1^{1/3}, 1^{1/5}, 1^{1/6}, \dots$)

euler's identity

$$\boxed{e^{i\pi} + 1 = 0}$$

limits at ∞

$\lim_{z\to z_0} f(z) = \infty \Leftrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0$

$\lim_{z\to\infty} f(z) = L \Leftrightarrow \lim_{z\to 0} f(1/z) = L$

$\lim_{z\to\infty} f(z) = \infty \Leftrightarrow \lim_{z\to 0} \frac{1}{f(1/z)} = 0$

Continuity of complexed valued fu^n

$f(z) = u(x,y) + i\,v(x,y)$

where $z = x + iy$, $x, y \in \mathbb{R}$

$u(x,y)$, $v(x,y) \in \mathbb{R}$ fu^n

limit of a $\mathbb{C}$ fu^n

$\lim_{z\to z_0} f(z_0) = L$ exists

$\Leftrightarrow$ We need to approach along any path to z_0 and get the limit L otherwise lim D.N.E

$\forall \varepsilon > 0\ \exists \delta > 0\ |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$

continuity check first then

Thm

- Any polynomial in z is cts everywhere
- Any rational fu^n is cts everywhere except at 0 of the denominator

To show limit DNE

1. check easy paths if the limit is same
   $z \to z_0$ along x axis then $y = 0$
   $z = x$, $z \to z_0$, $x \to ?$
   $z \to z_0$ along y axis then $z = iy$, $x = 0$

to show lim exists

2. [1] choose $z \to z_0$ along $y = mx + c$
   $\hookrightarrow$ subs. x_0, y_0 find c
   $f(z) \to$ write as x variables
   $z \to z_0$ $\quad x \to ?$
   if limit has m, i.e. the limit changes with m / depend on the path take
   $\Rightarrow$ Limit DNE

   otherwise limit exist

   * can use limit laws like $\mathbb{R}$

3. [2] $f(z) = u(x,y) + i\,v(x,y)$
   if $\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = L_1$ & $\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = L_2$ exists
   $\Rightarrow \lim_{z\to z_0} f(z) = L_1 + i L_2$

differentiability of complex valued fu^n

$f(z)$ is diff·ble at point z_0

$\Leftrightarrow \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists $\Big\}$ $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$

$\Delta z = \Delta x + i\Delta y$

$z = x+iy$

$\downarrow$

lim paths

$\Delta z \to 0$, $\Delta x \to 0$, $\Delta y \to 0$ } else

where $z = x+iy$

---

* rules of differentiation of real functions can be used

* quotient rule

- singular point $\rightarrow$ point where $f'(z)$ DNE } can find if rational fu^n $\rightarrow$ denominators

- analytic $\rightarrow$ analytic at z_0 if it is differentiable througout the neighbourhood of z_0

  $\downarrow$

  reigon contained within some circle $|z-z_0| = r$, $r \in \mathbb{R}$

Note

- $f(z)$ diff'ble at a point only $\not\Rightarrow$ f is analytic at that point
- $f(z)$ not diff'ble anywhere $\Rightarrow$ f is not analytic anywhere.
- $f(z)$ diff'ble at a point $\Rightarrow$ f is cts at that point

  converse might not be True. but contrapositive is True

cauchy riemann eq^ns

original

If $f(z)$ is diff'ble at $z=z_0$ $\wedge$ $f(z) = u(x,y) + i\,v(x,y)$

$\Rightarrow$ all partial derivatives of u & v exist

$\wedge$ at z_0 $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Th^m

If $f(z) = u(x,y) + i\,v(x,y)$ satisfy cauchy riemann eq^ns

$\wedge$ all partial derivatives of u & v cts at z_0 — mention this

$\Leftrightarrow$ $f(z)$ is diff'ble at z_0

* $\wedge$ $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

if cauchy-riemann eq^n satisfied only at a point $\Rightarrow$ $f(z)$ not analytic anywhere

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1. write $f(z) = u(x,y) + i\,v(x,y)$

   check $\frac{\partial u}{\partial x} =$ take derivatives w.r.t x considering all other variables constants $\overset{=}{\longleftrightarrow}$ $\frac{\partial v}{\partial y} =$

   $\frac{\partial u}{\partial y} \overset{=(-)}{\longleftrightarrow} \frac{\partial v}{\partial x}$

2. all partial derivatives of u & v cts. at z_0

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if partial derivatives hard to find take lim def^n

$\frac{\partial u}{\partial x}(x_0,y_0) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x}$

$\frac{\partial u}{\partial y}(x_0,y_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y}$

$y_0)$

$0)$