Ms. WD
Ms. PHI
Ms. AD
Ms. HM

Riemann

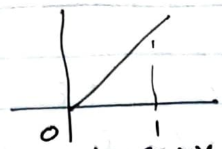
No: Date:

$u(f,\sigma)$ $L(f,\sigma)$

Calculate upper sum & lower sum

① $f(x) = x$

$\sigma = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ on $[0,1]$



increasing

$\boxed{1}$ $x_k = \frac{k}{4}$; $k = 0, 1, 2, 3, 4$

$\boxed{2}$ $I_k = x_k - x_{k-1} = \frac{k}{4} - \frac{k-1}{4} = \frac{1}{4}$

$\boxed{3}$ $u(f,\sigma) = \sum_{k=1}^{n} M_k |I_k|$; where $M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}$

$M_k = \frac{k}{4}$

$= \sum_{k=1}^{4} \frac{k}{4} \times \frac{1}{4}$

$= \frac{1}{16} \frac{n(n+1)}{2}$

$= \frac{4(4+1)}{16 \times 2} = \frac{5}{8}$ //

$\boxed{4}$ $m_k = \inf\{f(x) \mid x \in I_k = [x_{k-1}, x_k]\}$

$= \frac{k-1}{4}$

$L(f,\sigma) = \sum_{k=1}^{4} m_k |I_k| = \sum_{k=1}^{4} \frac{(k-1)}{4} \times \frac{1}{4} = \frac{1}{16} \frac{(n-1)(n-1+1)}{2} = \frac{3}{8}$ //

* take n out before sum

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$	$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$	$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

- mention If fun is bdd.
- take partition $\sigma = \{x_0 = a, x_1, \dots, x_n = b\}$ on $[a,b]$
- $I_k \to M_k \to m_k$
- take $u(f)$, $L(f)$
- $u(f,P) - L(f,P) = \sum_{k=1}^{n} [x_k - x_{k-1}](M_k - m_k)$

Let $P = \dots$ partition on...

Satisfying $|I_k| < \delta = \frac{\varepsilon}{f(a) - f(b)}$

$< \sum_{k=1}^{n} (x_k - x_{k-1}) \frac{\varepsilon}{f(a) - f(b)}$

telescoping $= \varepsilon$ ✓

Thm $\lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\left(\frac{b-a}{n}\right)\right) = \int_a^b f(x)\,dx$ take n out

a, b $f(x) \Rightarrow$ replace $x = a + k\frac{(b-a)}{n}$ is integrable ← mention

Atlas

Th^{m} Cauchy criterion

bdd. fn^{n} on a compact interval $f:[a,b]\to\mathbb{R}$ in R.I $\Leftrightarrow$ $\forall \varepsilon>0$ $\exists$ partition P_ε that may depend on ε s.t. $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$

$\Leftarrow$) suppose $\forall\varepsilon>0$ $\exists$ partition P_ε ~~such~~ that may depend on ε s.t

$U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$

~~$\forall$ partitions in a compact interval~~

$U(f)\le U(f,P_\varepsilon)$ — ① $\quad L(f,P_\varepsilon)\le L(f)$ — ②

~~where~~ since $U(f)=\inf\{U(f,P_\varepsilon)\mid P_\varepsilon\in\mathbb{P}\}$ & $L(f)=\sup\{L(f,P_\varepsilon)\mid P\in\mathbb{P}\}$

$U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$

$U(f)-L(f) < U(f,P_\varepsilon)-L(f,P_\varepsilon)$, from ① & ②

①, ② $\Rightarrow$ $U(f)-L(f)<U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$

$\Rightarrow$ $U(f)-L(f)<\varepsilon$

since $\varepsilon>0$ is arbitrary $U(f)-L(f)=0$ $U(f)=L(f)$

$\Rightarrow$ R.I ✓

$\Rightarrow$) given $\forall\varepsilon>0$ ~~let~~ $\exists$ P_1 & P_2 partitions ~~[illegible]~~ on $[a,b]$ s.t.

~~[illegible]~~ $L(f)-\frac{\varepsilon}{2}<L(f,P_1)\le\int_a^b f=L(f)=U(f)$

$<U(f,P_2)<U(f)+\frac{\varepsilon}{2}$

let $P_\varepsilon=P_1\cup P_2$ common refinement of P_1 & P_2

$U(f,P_\varepsilon)-L(f,P_\varepsilon)<U(f,P_2)-L(f,P_1)$

refinement

$<U(f)+\frac{\varepsilon}{2}-\left(L(f)-\frac{\varepsilon}{2}\right)$

$=U(f)-L(f)+\varepsilon$

Assume f is R.I on $[a,b]$ then $U(f)=L(f)$

$\therefore$ $0+\varepsilon$

$=\varepsilon$

$\therefore$ $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$ //

Lecture Schedule
Semester | 2
J Jayasinghe
T Jayasu
Miura
Sach
IPN
W Ya
J Jo
RV

②

No :
Date :

- bdd on compact interval
- cts $\Rightarrow$ R.I
- upper sum / lower sum calculate, x_k, $|I_k|$, $U(f,P)$, $L(f,P)$
- $L(f,P) \le L(f) = U(f) = \int^b f \le U(f,P)$ proofs
- decreasing / increasing fun} let $f(a) < f(b)$ → partition satisfying $|I_k| < \delta$

$\rightarrow U(f,P) - L(f,P) = \sum_{k=1}^{n} (x_k - x_{k-1})(M_k - m_k)\delta = \frac{\varepsilon}{f(b)-f(a)}$

start with to show cauchy

$< \sum_{k=1}^{n} \underbrace{f(x_k) - f(x_{k-1})}_{} \frac{\varepsilon}{f(b)-f(a)}$

telescoping ✓ cauchy ✓ ✓

- Sequences U_n, L_n $\quad$ $P_\varepsilon = P_n \cup Q_n$ $\quad$ $U(f,P_\varepsilon) - L(f,P_\varepsilon) \le U_n - L_n \rightarrow 0 < \varepsilon$

$U(f,P_n)$ $L(f,Q_n)$ $\quad$ $P_\varepsilon \supseteq P_n \cup Q_n$ $\quad$ when $n \rightarrow \infty$

$\underbrace{L(f,P_n)}_{L_n} < L(f,P_\varepsilon) < L(f) = U(f) = \int^b f$

$< U(f,P_\varepsilon) < \underbrace{U(f,P_n)}_{U_n}$

squeeze thm $n \rightarrow \infty$ ✓

- finite set of discts. points with bdd $\Rightarrow$ R.I $\quad$ # draw the fun if possible
- infinite set of discts points but unique limit point $\Rightarrow$ R.I.

write the set of discts pts: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \frac{1}{2^k}\} \longrightarrow 0$

common term

unique limit pt. $= 0 \Rightarrow$ R.I ✓

<u>infinite sums</u> ⓪ try finding formula / express as a sum

[1] take n out from num / denominator $a + k\frac{(b-a)}{n} = x$

[2] find $f\left(a + k\left(\frac{b-a}{n}\right)\right) = f(x) \rightarrow$ plug back and check

[3] apply thm $\int_a^b f(x)\,dx =$ ✓ $\quad$ write fun is integrable on $[a,b]$

$e^{\ln f(x)}$

$\ln(x) \rightarrow$ expand, rearrange

No
Date: / /

- MVT for integrals $\frac{11}{24} \le \int_0^{1/2} \sqrt{1-x^2}\,dx \le \frac{11\sqrt{3}}{36}$

[1] find $f(x)$, $g(x) = 1-x^2$ } $g(x) \ge 0$ on $[0, \frac{1}{2}]$ & $f, g \in \mathcal{R}$

$f(x) = \frac{1}{\sqrt{1-x^2}}$

by generalized MVT

[2] $m \int_0^{1/2} g(x)\,dx \le \int_0^{1/2} f(x) g(x)\,dx \le M \int_0^{1/2} g(x)\,dx$ —— ①

[3] $f'(x) = \frac{-1}{2} \frac{1}{(1-x^2)^{3/2}} \times -2x = \frac{x}{(1-x^2)^{3/2}} \ge 0$

$f(x)$ is increasing on $[0, \frac{1}{2}]$

then, $m = f(0) = 1$

$M = f(\frac{1}{2}) = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

[4] $\int_0^{1/2} g(x)\,dx = \int_0^{1/2} (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^{1/2} = \frac{1}{2} - \frac{1}{8 \times 3} - 0 = \frac{11}{24}$

[5] ① $\Rightarrow$ $1 \times \frac{11}{24} \le \int_0^{1/2} \sqrt{1-x^2}\,dx \le \frac{2}{3}\sqrt{3} \times \frac{11}{24_{12}} = \frac{11}{36}\sqrt{3}$

- definition of definite integrals

$f(x) = x$ on $[0, 1]$

[1] $\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$

consider equal n # of partitions on $[0,1]$ n #

$\Delta x_k \times k + a$

0 x_k Δx 1

$\frac{1-0}{n}$

[2] $\Delta x_k = |I_k| = \frac{1-0}{n} = \frac{1}{n}$

[3] $x_k = \Delta x_k \times k + a$ } coordinates

$x_k = \frac{1}{n} \times k + 0 = \frac{k}{n}$

$f(x_k) = f(\frac{k}{n}) = \frac{k}{n}$

④ from thm $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n} \frac{1}{n} = \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \lim_{n \to \infty} \frac{1}{n^{\cancel{2}}} \frac{\cancel{n}(n+1)}{2} = \frac{1}{2}$ //

- uniform, dominated, monotone cvg

$$\lim_{n\to\infty}\int_0^1 \frac{n+\cos x}{ne^x+\sin x}\,dx = 1-\frac{1}{e}$$

[1] take n out

$f_n(x) = \dfrac{n+\cos x}{ne^x+\sin x}$ $\quad f_n : [0,1] \longrightarrow \mathbb{R}$

$$\lim_{n\to\infty} \frac{n\left(1+\frac{\cos x}{n}\right)}{n\left(e^x+\frac{\sin x}{n}\right)} = \frac{1+0}{e^x+0} = e^{-x}$$

[2] $f_n \to f$ cvgs. uniformly to $f(x)=e^{-x}$ on $[0,1]$ for $0\le x\le 1$

[3] check $|f_n(x)-f(x)| = \left|\dfrac{n+\cos x}{ne^x+\sin x} - e^{-x}\right|$

$$\left|\frac{\cos x - e^{-x}\sin x}{ne^x+\sin x}\right| \le \frac{1-0}{n+0} = \frac{1}{n} = \varepsilon$$

$\therefore \forall \varepsilon>0\ \exists N=N(\varepsilon)$ s.t. $|f_n(x)-f(x)|<\varepsilon\ \forall n\ge N$

[4] by thm $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{n+\cos x}{ne^x+\sin x} = \int_0^1 \lim_{n\to\infty}\frac{n+\cos x}{ne^x+\sin x} = \int_0^1 e^{-x} = 1-\frac{1}{e^{-x}}$ $\quad \left(\dfrac{e^{-x}}{-1}\right)\Big|_0^1$

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x}\,dx = \int_0^{1/\sqrt{n}} \cdot \int_{1/\sqrt{n}}^1 \cdots$$

$f(x)$ continuos $\longrightarrow$ Generalized MVT

$$\int \underbrace{\frac{n f(x)}{1+n^2x}}_{g(x)f(x)}\,dx = f(\alpha_n)\int_0^{1/\sqrt{n}} \frac{n}{1+n^2x}\ \checkmark \qquad 0\le\alpha\le\frac{1}{\sqrt{n}}$$

$n\to\infty$

$\alpha\to 0$

$f(0)$ ✓

✱ last resort subst. $x=?$ $f(0)$ when $n\to\infty$ ✓

uniform cvgs $\longrightarrow$ exchange limit ✓

- I FTC, II FTC, I by parts
- $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x)) \times b'(x) - f(a(x)) \times a'(x)$ //

- Improper integrals, cvgs tests

$\int_0^1 \frac{\sin(\frac{1}{x})}{\sqrt{x}} dx$ $\quad f(x) = \frac{\sin(\frac{1}{x})}{\sqrt{x}}$

since there is no neighbourhood of point 0 which f(x) keeps the same sign ∴ consider $|f(x)|$ on $\forall x \in (0, 1]$

$\hookrightarrow g \Rightarrow |f| \Rightarrow f$

comparison abs. cvgs

* If integral don't seem possible evaluate by DNE points
$\varepsilon \to 0^+$

$\int_0 \frac{dx}{\sqrt{x(1-x)}}$ check discts.pts & slice them $\varepsilon \to 0^+$ eval if lim exist and finite ⇒ cvg ✓

DNE or $\pm\infty$ ⇒ dvgs.

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g cts, $G(x) = \int_0^{\sin x} f(t)dt$, show G diff.ble & find G'

select $M > 0$ s.t $|g(x)| \leq M$ $\forall x \in \mathbb{R}$ for $x, y \in \mathbb{R}$ $|x-y| < \delta =$

For $x<y$, $|G(x) - G(y)| = \left| \int_0^{\sin x} f(t)dt - \int_0^{\sin y} f(t)dt \right|$

$= \left| \int_0^{\sin x} f(t)dt + \int_{\sin y}^{0} f(t)dt \right| = \left| \int_{\sin y}^{\sin x} f(t)dt \right|$

$\leq \int_{\sin y}^{\sin x} |f(t)|dt \leq M[\sin x - \sin y] = M\, 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$

$\leq 2M = \varepsilon$

* $G \in UC$ on $\mathbb{R}$

$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\int_0^{\sin(x+h)} f(t)dt - \int_0^{\sin x} f(t)dt\right]$

$= \lim_{h\to 0} \int_{\sin x}^{\sin(x+h)} f(t)dt$