DE

ODE

1 independent var

1 or more dependent

non-linear

linear

PDE

2 or more independent var

1 or more dependent var

$P_0(x)y + P_1(x)y' + \dots P_n(x)y^{(n)} + Q(x) = 0$

only depend on x

only y'

✗ y^n, $n>1$

no $y \times \left(\frac{dy}{dx}\right)^n$ ✗

✗ $y \times \frac{d^n y}{dx^n}$

✗ $\sin y$, $\ln y$, e^y

order – highest diff. coefficient $\frac{d^n y}{dx^n} \to n_{max}$

degree – all diff. coefficients ⇒ polynomial
fractional powers removed → must be (+) integer
no transcendental fun^n → log, e^x, trig, ln, inverse trig, hyperbolic
exponent of highest diff coefficient

else degree DNE

★ First convert to polynomial if you can't

Picard's existence & unique th^m

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

In IVP prob.

check $\frac{dy}{dx} = f(x_0, y_0)$ cts near (x_0, y_0)

check $\frac{\partial f}{\partial x}(x_0, y_0)$ cts near (x_0, y_0)

⇒ sol^n of DE is unique

converse might not be true

★ with x anything is fine
$\sin(x)$, $\ln(x)$, e^x ✓

★ $\frac{d^2y}{dx^2} + \sin(y) = x$ → degree 1 defined

no fractions
– powers

$\sin x \frac{dy}{dx}$ ✓ accepted

all diff. coefficients

$\ln\left(\frac{dy}{dx}\right)$ ✗

$\sin\left(\frac{dy}{dx}\right)$ ✗

$e^{\frac{dy}{dx}}$ ✗

- explicit sol^n → don't leave the way it is
- implicit sol^n → find y explicitly
- $\sqrt{\ } \rightleftarrows \pm$
- $+ c_1, A, B$ (arbitrary constant)
- state what type of ODE
- integration → partial fraction → substitution
- $Q \cap (x_{k-1}, x_k] \to Q \supset P$
  $P = P_1 \cup P_2 \to P \supset P_1 \cup P_2$

non elementary integrals

1) $Si(x) = \int \frac{\sin x}{x}$

2) $Ei(x) = \int_{-\infty}^{\infty} \frac{e^{-t}}{t} dt \Rightarrow \int \frac{e^x}{x}$

3) $Li(x) = \int \frac{1}{\ln x}$

4) $\frac{\sqrt{\pi}}{2} erfi(x) = \int e^{x^2} dx$

[1] check the ODE is 1^{st} order / 2^{nd} order

[2] try converting into standard form $f(x,y) = y' + \ldots$

1^{st} order

① variable seperable

$$\frac{dy}{dx} = f(x,y) \checkmark$$

first start with $\frac{dy}{dx}$ check

else replace y'

② homogeneous

$$y' = \frac{x^2+y^2}{x^2}$$

[1] check $f(\lambda x, \lambda y) \rightarrow f(x,y)$

[2] subst. $y = vx$

$$y' = v + x\frac{dv}{dx}$$

$$\frac{x^2+(vx)^2}{x^2} = v + x\frac{dv}{dx}$$

↳ separable ✓

④ linear

$$y' + P(x)\,y = Q(x)$$

no y terms

$$I.F = e^{\int P(x)}$$

$$\int \frac{d\left(e^{\int P(x)}\cdot y\right)}{dx} = \int Q(x)\, e^{\int P(x)} \checkmark$$

I LATE → integration by parts

⑤ bernoulli eqn

$$y' + P(x)\,y = y^{n} f(x) \longrightarrow ①$$

If $n = 0, 1 \Rightarrow$ linear ✓

else:

[1] subs. $v = y^{1-n}$

$$\frac{dv}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$$

[2] $①/y^n \Rightarrow y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = f(x)$

$$\frac{1}{(1-n)}\frac{dv}{dx} + P(x)\,v = f(x)$$

↳ linear ✓ homogeneous

③ reduction to homogeneous

$$y' = \frac{ax+by+c}{Ax+By+C}$$

[1] subs. $x = X + h$

$y = Y + k$

[2] solve h, k

$ah + bk + c = 0$

$Ah + Bk + C = 0$

[3] subs. $Y = vX$

$$Y' = v + X\frac{dv}{dx}$$

$$\frac{aX+bY}{AX+bY}$$

↳ homogeneous

If $\frac{a}{A} = \frac{b}{B} \rightarrow$ no. soln

subs: $ax + by = t$

2nd order

$y = y_c + y_p$

(2)

① homogeneous + linear + constant coefficients

$$y'' + ay' + by = 0 \; ; \; a, b \text{ constants}$$

[1] subs.
$$\left.\begin{cases} y = e^{mx} \\ y' = me^{mx} \\ y'' = m^2 e^{mx} \end{cases}\right\} \text{back to original eq}^n$$

[2] $\underset{\neq 0}{e^{mx}}(m^2 + am + b) = 0$ * check independent is x or t

$m^2 + am + b = 0$

find m *mention c_1/c_2 are constants

If m_1, m_2 real distinct roots

$$\boxed{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

elIf $m_1 = m_2 = m$ real equal roots

$$\boxed{y(x) = c_1 e^{mx} + c_2 x e^{mx}}$$

elif m_1, m_2 complex conjugate

$m = \alpha \pm \beta x$

$$\boxed{y(x) = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]}$$

② non-homogeneous + linear

$$\underbrace{y'' + p(x) y' + q(x) y}_{\text{Complementary sol}^n} = \underbrace{f(x)}_{\text{particular sol}^n}$$

method 1 (undetermined co-efficients)

[1] $y_c \to$ ✓ use $A, B, C, \ldots$ constants

[2] $f(x) = k \to y_p = C$

$f(x) = kx \to y_p = ax + b$

$f(x) = kx^2 \to y_p = ax^2 + bx + C$

$\left.\begin{array}{l} f(x) = k\sin x \\ \text{or } k\cos x \end{array}\right\} y_p = a\cos x + b\sin x$

$f(x) = e^{kx} \} \; y_p = Ce^{kx}$

* If y_c has a term equal to the initial guess of y_p Multiply that same term in y_p with x until there are no tems that solves y_c

[3] diff. $y'p$, $y''p$

[4] plug to original eq^n

[5] compare coefficients find y_p ✓

method 2 - wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = Ce^{\int -p(x)}$$

[1] diff. eq^n solv y_2 ?(remove constants)

[4] $y_c = c_1 y_1 + c_2 y_2$

$y_p = u y_1 + v y_2$

[2] find c_1 using $\omega(x)$

[3]
$$u = -\int \frac{y_2 f}{W}$$
$$v = \int \frac{y_1 f}{W}$$

If $\exists x_0 \in X$ s.t $\omega(x_0) = 0 \Rightarrow c = 0, \omega = 0$

Th^m sol^{n}s y_1, y_2 of DE are L.D $\Leftrightarrow W(y_1, y_2) = 0$

Th^m y_1, y_2 L.D $\Rightarrow W(y_1, y_2) = 0$ If not sol^{n}s

Th^m sol^{n}s y_1, y_2 of DE $W = Ce^{\int -p(x)} = 0 \Rightarrow C = 0$
$W(y_1, y_2)$ is identically 0 or never 0
i.e: If $W(y_1, y_2) = 0$ for even 1 $x \to W = 0 \; \forall x$

def^n $f[a,b] \to \mathbb{R}$ needs to be bounded on a compact (closed $\wedge$ bounded) interval to be integrable

- $\boxed{|I| = b-a = \sum_{k=1}^{n} |I_k| = \sum_{k=1}^{n} (x_k - x_{k-1})}$

- for each partition

def^n

$$M_k = \sup \{ f(x) : x \in [x_{k-1}, x_k] \} = \sup_{I_k} f$$

$$m_k = \inf \{ f(x) : x \in [x_{k-1}, x_k] \} = \inf_{I_k} f$$

Upper riemann sum

$$U(f,P) = \sum_{k=1}^{n} M_k |I_k|$$

darboux

Lower riemann sum

$$L(f,P) = \sum_{k=1}^{n} m_k |I_k|$$

- since $m_k \le M_k$ always

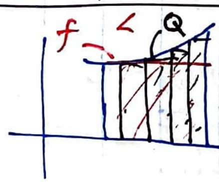
Lemma $\boxed{L(f,P) \le U(f,P)}$

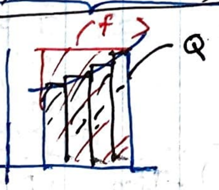
- $\boxed{Q \supseteq P}$, Q is a refinement of P

finer / better / contains more points

If $Q \supseteq P \Rightarrow$

$$\boxed{L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)}$$





If P_1 & P_2 any 2 partitions of $[a,b]$

Lemma $\boxed{L(f,P_1) \le U(f,P_2)}$

proof: If $Q = P_1 \cup P_2$ common ~~partition~~ refinement

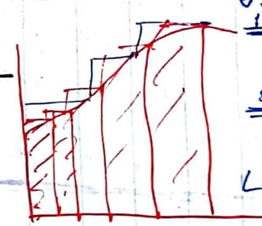
by prev. result

$$L(f,P_1) \le L(f,P_2) \le U(f,P_2) \le U(f,P_1)$$

def^n

$$\int_a^b f \quad U(f) = \inf \{ U(f,P) : P \in \mathbb{P} \}$$

$$\int_a^b f \quad L(f) = \sup \{ L(f,P) : P \in \mathbb{P} \}$$



$U(f,P)$ $\downarrow$ inf

sup $\uparrow$ $L(f,P)$

Lemma for any bounded f on $[a,b]$ compact

$$\boxed{L(f) \le U(f)}$$

we need to have $U(f) = L(f)$

we know any 2 partitions

$$L(f,Q) \le U(f,P)$$

def^n $U(f) = \inf \{ U(f,P) : P \in \mathbb{P} \}$

Similarly $L(f,Q) \le U(f,P)$

def^n $L(f) = \sup \{ L(f,P) : P \in \mathbb{P} \}$

- unbounded function $\Rightarrow$ not R.I
- bounded $f : [a,b] \to \mathbb{R}$ if $L(f) = U(f)$ (compact)

$\Rightarrow$ R.I on $[a,b] = \int_a^b f$

For R.I,

1) check bounded $\to$ If unbounded $f(x)$ $\times$ R.

If bounded

2) Domain of closed interval (unbounded interval) $\to [a,b)$ $\times$ R.I

$[a,\infty)$ $\times$

✓

3) If $L(f) = U(f) \Rightarrow$ R.I (all checked)

If $\underline{\int_a^b} f \ne \overline{\int_a^b} f \Rightarrow$ not R.I even if bounded

Cauchy criterion

Thm

$\left.\begin{array}{l}\text{bounded } f:[a,b]\to\mathbb{R} \\ \text{is R.I.}\end{array}\right\} \Leftrightarrow$ $\forall \epsilon>0\ \exists$ a partition $P\in[a,b]$ which may depend on ϵ s.t. $U(f,P)-L(f,P)<\epsilon$

$L(f)-\frac{\epsilon}{2} < L(f,P_1)$ $\}$ $U(f,P_2) < U(f)+\frac{\epsilon}{2}$

Thm 0.0.1

If f is cts on bounded interval $[a,b]$ $\Rightarrow$ f is uniformly cts. on $[a,b]$

If f is cts. on $[a,b]$ $\Rightarrow$ f is integrable on $[a,b]$

compact set

$f\in D[a,b]$ $\nRightarrow$

• converse might not be true.

$U(f,P)-L(f,P)=\sum_{k=1}^{n}(M_k-m_k)|I_k|$

• Integrals which are RI. but not cts.

↳

Thm If set of discts. points of bounded $f:[a,b]\to\mathbb{R}$ is finite $\}\Rightarrow f$ is R.I

Thm If set of discts. points of bounded $f:[a,b]\to\mathbb{R}$ has finite number of limit points $\}\Rightarrow f$ is R.I.

i.e; f may have infinetly many discts. points but set of all discts. points have finite limit points

$\{x_0, x_1, x_2, \dots\}$ ⟶ x

$\Rightarrow f$ is R.I.

Thm 0.0.2

$\left.\begin{array}{l}\text{bounded } f:[a,b]\to\mathbb{R} \text{ is R.I on } [c,b] \\ \text{where } \forall c\in(a,b)\end{array}\right\}\Rightarrow f$ is R.I on $[a,b]$

a c b

ms. ✓

Thm 0.0.3

$\left.\begin{array}{l}\text{bounded } f:[a,b]\to\mathbb{R} \\ \text{s.t. cts. on every subinterval } [c,b] \\ \text{where } c\in(a,b)\end{array}\right\}\Rightarrow f$ is R.I on $[a,b]$

a c b

f cts. ✓

ms f RI

Thm 0.0.4

$\left.\begin{array}{l}\text{bounded } f:[a,b]\to\mathbb{R} \\ \text{R.I on } [a,c] \wedge [c,b] \\ \text{where } a<c<b\end{array}\right\}\Rightarrow f$ is R.I on $[a,b]$

a c b by 0.0.2, 0.0.3

ms f R.I

Thm 0.1.1

$\left.\begin{array}{l}\text{bounded } f:[a,b]\to\mathbb{R} \\ c\in(a,b) \\ f \text{ is R.I on } [a,b]\end{array}\right\}\Leftrightarrow\left\{\begin{array}{l}f \text{ is R.I on } [a,c] \\ \wedge \\ f \text{ is R.I on } [c,b]\end{array}\right.$

$$\int_a^b f=\int_a^c f+\int_c^b f$$

Thm 0.1.2

$\left.\begin{array}{l}\text{bounded } f:[a,b]\to\mathbb{R} \\ f \text{ is R.I on } [a,b]\end{array}\right.\Leftrightarrow$ $\exists$ sequence (P_n) of partitions s.t. $\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$

A $\int_a^b f=\lim_{n\to\infty}U(f,P_n)=\lim_{n\to\infty}L(f,P_n)=U(f)=L(f)$

inf sup

Properties of integral

Assume f, g R.I on $[a,b]$

① $f+g$ is R.I $\wedge$

$$\boxed{\int_a^b (f+g) = \int_a^b f + \int_a^b g}$$

$L(f,P) + L(g,P) \leq L(f+g,P)$

$U(f+g,P) \leq U(f,P) + U(g,P)$

② $k \in \mathbb{R}$ kf is R.I $\wedge$

$$\boxed{\int_a^b kf = k\int_a^b f}$$

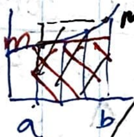
$U(-f,P) = -L(f,P)$

$L(-f,P) = -U(f,P)$

for any $A \subseteq \mathbb{R}$, $\sup(A) = -\inf(-A)$, $\inf(A) = -\sup(-A)$ where $-A = \{-a : a \in A\}$

③ if $m \leq f(x) \leq M$ on $[a,b]$ (i.e. if function is bounded by $m \leq f(x) \leq M$)

$$\Rightarrow \boxed{m(b-a) \leq \int_a^b f \leq M(b-a)}$$



④ if $f(x) \leq g(x)$

$$\Rightarrow \boxed{\int_a^b f \leq \int_a^b g}$$

⑤ $|f|$ R.I.

$$\Rightarrow \boxed{\left|\int_a^b f\right| \leq \int_a^b |f|}$$

If $f \in \mathcal{R}[a,b] \Rightarrow |f| \in \mathcal{R}[a,b]$ ★

converse might not be true

$M(|f|, S) - m(|f|, S) \leq M(f,S) - m(f,S)$

$f - g = f + (-g)$

$\boxed{\text{Th}^m}$ 0.1.2

If f & g are R.I $\Rightarrow$ fg is R.I

$f, g \in \mathcal{R}$ riemann integrable fun

$f \in \mathcal{R} \Rightarrow f^2 \in \mathcal{R}$ ★

$fg = \dfrac{(f+g)^2 - f^2 - g^2}{2}$

$\boxed{\text{Th}^m}$ 0.1.3

If f & g are R.I $\Rightarrow$ $\text{Max}(f,g)$, $\text{Min}(f,g)$ R.I

$\text{Max}(f,g) = \dfrac{|f-g| + g + f}{2}$

$\text{Min}(f,g) = \dfrac{-|f-g| + g + f}{2}$

$\boxed{\text{Th}^m}$ 0.1.4

converse might not be true.

f is integrable on $[a,b]$ $\Rightarrow$ $\wedge$ $\epsilon > 0$ $\exists\, \delta > 0$ s.t. $\forall$ partitions $P = \{x_0, x_1, \dots\}$ with $|P| < \delta$

$$\left|\int_a^b f - \sum_{j=1}^n f(\xi_j) I_j\right| < \epsilon$$

$|P| = \max\limits_{1 \leq i \leq n} \{|\Delta x_i|\}$, where $\xi_j \in [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$

$$\int_a^b f = \sum_{j=1}^n f(\xi_j) I_j$$

//

④

3 ways to R.I

1) cts $\Rightarrow$ RI

2) partitions $\frac{\epsilon}{6}$ - rule, cauchy

3) Limit points, finite discts. points $\Rightarrow$ R.I ✓

infinite discts. but discts. pt $\rightarrow$ finite limit points $\Rightarrow$ R.I ✓

In proofs

define partitions $I_k = [x_{k-1}, x_k]$

$P \subseteq Q$

$L(f,P) = \sum m_k |I_k|$

$U(f,P) = \sum M_k |I_k|$

$\boxed{Th^m}$ IVT for integrals   MVT for integrals

If f is cts on $[a,b]$ $\Rightarrow$ $\exists x \in (a,b)$ s.t.

$$\boxed{f(x) = \frac{1}{(b-a)} \int_a^b f} \equiv f(x)(b-a) = \int_a^b f$$

Average value

$$m(b-a) \le \int_a^b f \le M(b-a)$$

$\boxed{Th^m}$ generalized IVT for integral (aka generalized MVT for integrals)

If $f, g : [a,b] \to \mathbb{R}$ cts $\wedge$ $g(t) \ge 0 \ \forall t \in [a,b]$ $\Rightarrow$ $\exists x \in [a,b]$ s.t

* When selecting $g(t)$

$$\boxed{f(x) = \frac{\int_a^b f(t)g(t)\,dt}{\int_a^b g(t)\,dt}} \equiv f(x)\int_a^b g(t)\,dt = \int_a^b f(t)g(t)\,dt$$

$$m \underbrace{\int_a^b g(t)\,dt}_{\boxed{2}} \le \int_a^b f(t)g(t)\,dt \le M \int_a^b g(t)\,dt$$

$\boxed{1}$ $\boxed{3}$

If f is monotonic   sup & inf ✓
M   m

else differentiate.

Taking limit inside of integral

① $\boxed{Th^m}$ Let f_n be a sequence of R.I fu^n on $[a,b]$

If $f_n \xrightarrow{\text{converges}} f$ uniformly on $[a,b]$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$ ✓

check first

sequence of $f_n(x)$ uniformly converges def^n,

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ s.t $|f_n(x) - f(x)| < \varepsilon$, $\forall n \ge N$

only depend on ε not x

② Dominated convergence th^m

f_n is a sequence of R.I fu^n on $[a,b]$ and $f_n \to f$ pointwise

$\wedge$

f is integrable on $[a,b]$

* must be given for pointwise convergence

$\wedge$

If $\exists M > 0$ s.t. $|f_n(x)| \le M$, $\forall n$, $\forall x \in [a,b]$

must be dominated (bounded)

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$ ✓

③ ~~Monotone convergence th^m~~

same as dominated but $f_n(x)$ is increasing

$f_1(x) \le f_2(x) \le \dots \ \forall x \in [a,b]$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

pointwise convergence def^n

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon, x)$ s.t. $|f_n(x) - f(x)| < \varepsilon$, $\forall n \ge N$

Wk

I - FTC

If $g \in \mathcal{C}[a,b] \wedge g \in \mathcal{D}(a,b)$
$\wedge\ g' \in \mathcal{R}[a,b]$
integrable on $[a,b]$

$$\Rightarrow \boxed{\int_a^b g' = g(b) - g(a)}$$

integral of derivative = fun

II - FTC

Let f be an integrable fun on $[a,b]$
for $x \in [a,b]$
Let $F(x) = \int_a^x f(t)\,dt$

$\Rightarrow F(x)$ is cts on $[a,b]$

If f is cts at $x_0 \in (a,b)$
$\Rightarrow F$ is differentiable at x_0
$\wedge$

$$\boxed{F'(x_0) = f(x_0)}$$

derivative of integral = fun

Integration by parts

$u, v \in \mathcal{C}[a,b] \wedge u', v' \in \mathcal{D}(a,b)$
$\wedge\ u', v'$ integrable on $[a,b]$

$$\boxed{\int_a^b u v' = [uv]_a^b - \int_a^b u'v}$$

$u =$ $v' =$
$u' =$ $v =$

$u \rightarrow$ I LATE
1 included

If $f \in \mathcal{C}(I)$
open interval

$y = b(x)$, $y = a(x) \in I$
must be contain values of $b(x)$, $a(x)$

$$\Rightarrow \frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x)) \times b'(x) - f(a(x)) \times a'(x)$$ //

---

change of variable
accepted ✓

Improper integrals

① $f:(a,b]\to\mathbb{R}$ is integrable on $[c,b]$ for every $a<c<b$

$\Rightarrow \int_a^b f = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f$ → converges if limit exist in $\mathbb{R}$; → diverges if limit $\pm\infty$ or DNE

② $f:[a,b)\to\mathbb{R}$ is integrable on $[c,b]$ for every $a<c<b$

$\Rightarrow \int_a^b f = \lim_{\varepsilon\to 0^+} \int_a^{b-\varepsilon} f$

③ $f:[a,\infty)\to\mathbb{R}$ integrable on $[a,r]$ for every $r>a$

$\Rightarrow \int_a^\infty f = \lim_{r\to\infty} \int_a^r f$

④ $f:(-\infty,b]\to\mathbb{R}$ integrable on $[r,b]$ for every $r<b$

$\Rightarrow \int_{-\infty}^b f = \lim_{r\to\infty} \int_{-r}^b f$

⑤ $f:[a,b]/\{c\}\to\mathbb{R}$ integrable on closed intervals not including $a<c<b$

$\Rightarrow \int_a^b f = \lim_{\delta\to 0^+} \int_a^{c-\delta} f + \lim_{\varepsilon\to 0^+} \int_{c+\varepsilon}^b f$

⑥ $f:\mathbb{R}\to\mathbb{R}$ integrable on every compact interval

$\int_{-\infty}^\infty f = \lim_{s\to\infty} \int_{-s}^{c} f + \lim_{r\to\infty} \int_c^r f$ — can take $c=0$

series (absolutely convergence)

- $\sum |a_n| \to$ converges $\Rightarrow \sum a_n \to$ converges (absolutely)
- conditionally converges if $\sum a_n \to$ converges $\wedge$ $\sum |a_n| \to$ diverges

absolutely convergent improper integrals.

$\int_a^\infty |f|\,dx<\infty \Rightarrow \int_a^\infty f\,dx<\infty$

- improper integral $\int_a^b f$ absolutely converges if $\int_a^b |f| \to$ converges
- condiationally converges if $\int_a^b f \to$ converges but $\int_a^b |f| \to$ diverges

(← also an integral of 2nd kind)

only for improper

$$\int_0^1 \frac{1}{x^p}dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{1}{x^p}dx = \lim_{\varepsilon\to 0^+}\left(\frac{x^{-p+1}}{1-p}\right)\Big|_\varepsilon^1$$

for $0<p<1$ ($p<1$)	$p=1$ ★	$p>1$
$=\lim_{\varepsilon\to 0^+}\frac{1-\varepsilon^{1-p}}{1-p}$ ($\varepsilon^{1-p}>0$)	$\lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{1}{x}dx$	$=\lim_{\varepsilon\to 0^+}\frac{1-\varepsilon^{1-p}}{1-p}$ ($1-p<0$)
$=\frac{1}{1-p}\in\mathbb{R}$	$\lim_{\varepsilon\to 0^+}\ln\lvert x\rvert\Big]_\varepsilon^1$	$=\lim_{\varepsilon\to 0^+}\frac{1-\frac{1}{\varepsilon^{1-p}}}{1-p}$ ($\to\infty$, >0)
$\boxed{\text{converges}}$	$=\lim_{\varepsilon\to 0^+}\ln\left\lvert\frac{1}{\varepsilon}\right\rvert$	$=\infty$
	$=\infty$	$\boxed{\text{diverges}}$
	$\boxed{\text{diverges}}$	

$\boxed{1\le p}$

$$\int_{1(>0)}^\infty \frac{1}{x^p}dx = \lim_{r\to\infty}\int_1^r \frac{1}{x^p}dx = \lim_{r\to\infty}\left[\frac{x^{-p+1}}{1-p}\right]_1^r$$

$$= \lim_{r\to\infty}\left(\frac{r^{1-p}-1}{1-p}\right)$$

for $p\le 1$	$p>1$
$=\infty$	$\boxed{\text{converges}}$
$\boxed{\text{diverges}}$	

absolutely convergent improper integral

★ Improper integral $\int_a^b f$ abs. conv. if $\int_a^b |f|$ conv.

★ " $\int_a^b f$ conditionally conv. if $\int f$ conv.

but $\int |f|$ div.

comparison theorem

f, g are cts functions

s.t $0 \le g(x) \le f(x)$

★ $\int_a^\infty f\,dx$ converges $\Rightarrow \int_a^\infty g\,dx$ converges

★ $\int_a^\infty g\,dx$ diverges $\Rightarrow \int_a^\infty f\,dx$ diverges

1

Limit comparison test

f, g cts. functions $\forall x$ $\wedge$ and $f, g > 0$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = k$$

$0 < k < \infty \Rightarrow \int_a^\infty g\,dx$ converges $\Leftrightarrow \int_a^\infty f\,dx$ converges

$k = 0 \Rightarrow \int_a^\infty g\,dx$ cvgs $\Rightarrow \int_a^\infty f\,dx$ cvgs.

$k = \infty \Rightarrow \int_a^\infty g\,dx$ divgs $\Rightarrow \int_a^\infty f\,dx$ divgs.

Note: choose $g(x)$ well known $\frac{1}{x}$, $\frac{1}{\ln x}$

Gamma fun (eucleriran integral of 2^{nd} kind)

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx}\ ; \quad n>0$$

since it is cvg. for only $n>0$

properties of gamma fun

① $\Gamma(1) = 1$

② $\Gamma(n+1) = n\,\Gamma(n)$; for any real $n>0$

③ $\Gamma(n+1) = n!$; for non negative integers

④ extension of defn of gamma fun

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \ ; \quad \text{valid for } \underset{\text{real}}{n} < 0$$

except $n = 0, -1, -2, \ldots$

transformation of gamma fun

Form I

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}}\,dx \ ; \quad n>0$$

subst. $x^n = t$ (for backwards) ; ✓ $x^{\frac{1}{n}} = t$ ↓ transfer to gamma

when $n=\frac{1}{2}$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$$

Form II

$$\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx} x^{n-1}\,dx \ ; \quad n>0,\ k>0$$

$x = kt$ ↑ ; $kx = t$ ↓ ✓

Form III

$$\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx \ ; \quad n>0$$

$\frac{1}{x} = e^t$

Thm $\forall n>0$ improper integral $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$ is cvgent //

* $\frac{9}{2}\,\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\,\Gamma\left(\frac{9}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$

don't pull this out ✗

$\frac{11}{2} - 1$ out ✓

* $\frac{1}{c^x} = \frac{1}{e^{x\ln c}} = e^{-x\ln c} \to t$ ✓

* Always if $2^{x^{5n}}$ take $e^{x^{5n}\ln 2}$ ✓ substitute

• (−)ve #s (not integers nor 0)

(−)ve

• $\underset{(-)ve\ \#}{\Gamma(n)} = \frac{\Gamma(n+1)}{n}$

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = -\frac{2}{3}\,\Gamma\left(-\tfrac{1}{2}\right) = -\frac{2}{3}\,\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{4}{3}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{4}{3}\sqrt{\pi}$$ //

Beta functions (eulerian integral of the 1st kind)

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx} \quad ; \quad m>0,\ n>0$$

since it is conv. only for above conditions.

Symmetric property,

$$\boxed{B(m,n) = B(n,m)}$$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

relationship between beta & gamma fun

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad ; \quad m>0,\ n>0$$

$\sqrt{a-bx} \rightarrow \frac{at}{b} = x$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin(p\pi)} = \Gamma(m)\,\Gamma(1-m)$$

Tra

For

$n>0$

Fo

B

For

E

For

① $$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}$$

② $$\frac{B(m,n)}{a^m b^n} = \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}$$

③ $$\frac{B(m,n)}{2a^m b^n} = \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}$$

④ $$(a-b)^{m-1+n} B(m,n) = \int_a^b (x-b)^{m-1}(a-x)^{n-1}$$

⑤ $$B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}$$

⑥ $$\frac{B(m,n)}{(a+1)^m a^n} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}$$

⑦ $$\frac{B(m,n)}{(a+b)^m a^n} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}$$

⑧ $$\frac{B(m,n)}{b^m a^n} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{[a+(b-a)x]^{m+n}}$$

$\sin^4\theta = t$) subst.

For

$$\frac{B(m,n)}{a^n(1+a)^m} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}$$

Form VI

$$(a-b)^{(m-1)+n} B(m,n) = \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx$$

Form VII

$$\frac{B(m,n)}{a^n b^m} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}\,dx}{(a+(b-a)x)^{m+n}} \qquad \frac{B(m,n)}{(b+c)^m b^n} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}\,dx}{(b+cx)^{m+n}}$$

No. :

Date