set → nonempty $(\mathbb{R},+,\cdot)$

that satisfy

1 Field axioms

① $\forall a,b \in \mathbb{R}$; $a+b \in \mathbb{R}$ → closed under addition

② $\forall a,b,c \in \mathbb{R}$; $(a+b)+c = a+(b+c)$ → addition is associative

③ $\exists 0 \in \mathbb{R}\ \forall a \in \mathbb{R}$; $a+0 = 0+a = a$ → additive identity exist

④ $\forall a \in \mathbb{R}\ \exists -a \in \mathbb{R}$; $a+(-a) = (-a)+a = 0$ → additive inverse exists

group with (any bin. operation) $(G,*)$

⑤ $\forall a,b \in \mathbb{R}$; $a+b = b+a$ → addition is commutative } rare rule to satisfy

abelian group

⑥ $\forall a,b \in \mathbb{R}$; $a\cdot b \in \mathbb{R}$ → closed under multiplication

⑦ $\forall a,b,c \in \mathbb{R}$; $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ → multiplication is associative

⑧ $\exists 1 \in \mathbb{R}\ \forall a \in \mathbb{R}$; $a\cdot 1 = 1\cdot a = a$ → multiplicative identity exist

⑨ $\forall a \in \mathbb{R}-\{0\}$, $\exists a^{-1} \in \mathbb{R}$; $a\cdot a^{-1} = a^{-1}\cdot a = 1$ → multiplicative inverse exists.
where $\mathbb{R}-\{0\} \neq \phi$

⑩ $\forall a,b \in \mathbb{R}$; $a\cdot b = b\cdot a$ → multiplication is commutative

Field $(\mathbb{F},\#,*)$

⑪ $\forall a,b,c \in \mathbb{R}$; $a\cdot(b+c) = a\cdot b + a\cdot c$ → mutiplication is distributive over addition (not converse)

$(\mathbb{R},+,\cdot)$ ✓

but $(\mathbb{R},\cdot,+)$ × not a field

2 order axioms

⑫ $\forall a,b \in \mathbb{R}$; $a=b$, $a<b$, $b>a$
one of this only true
Trichotomy

⑬ $\forall a,b,c \in \mathbb{R}$; $a<b \wedge b<c \Rightarrow a<c$
transitivity

⑭ $\forall a,b,c \in \mathbb{R}$; $a<b \Rightarrow a+c < b+c$
operations with addition

⑮ $\forall a,b \in \mathbb{R}$; $a<b \wedge 0<c \Rightarrow ac < bc$
operations with multiplication

def[n] $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a<0 \end{cases}$
absolute value function

Th[m] triangle inequility
$|a+b| \leq |a|+|b|$
$|a|-|b| \leq |a+b|$

$\boxed{\big||a|-|b|\big| \leq |a-b|}$

$a<b \wedge c<d \Rightarrow a+c < b+d$
$\not\Rightarrow a-c < b-d$

$x \in \mathbb{R}$ $\quad -|x| \leq x \leq |x|$

$|x| \leq \delta \Leftrightarrow -\delta \leq x \leq \delta$
where $\delta > 0$

If $M>0 \Rightarrow \frac{1}{M}>0$

If $a<b \wedge c<d$ ... $\Rightarrow ac < bd$

Solve for x

Domain range

$\log((f(x)))$

not def. for $(-)$ $f(x) \neq 0$ } $f(x) > 0$

$\sqrt{f(x)} > 0$ $\sqrt{f(x)}^2 = \|f(x)\|$

$-1 < \sin, \cos \leq 1$

squaring, reciprocal → 0^+ 0^-

be careful $\frac{1}{0^+} \to \infty$

$\frac{1}{0^-} \to -\infty$

$\frac{1}{f(x)}$ $f(x) \neq 0$

dom $\mathbb{R}^+$ ← $\ln(f(x))$ → ran $(-\infty, \infty)$ $\mathbb{R}$ cause cts.

$\mathbb{R}$ ← $e^{f(x)}$ → ran $(0, +\infty)$

	dom	ran
$\tan(f(x))$	$\neq \frac{n\pi}{2}$, n odd	$(-\infty, \infty)$
$\sin(x)$, $\cos(x)$	$\mathbb{R}$	$[-1, 1]$
$\cot(x)$	$x \neq n\pi$, $n \in \mathbb{Z}$	$\mathbb{R}$
$\sec(x)$	$x \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$ (odd)	$\mathbb{R}$
$\operatorname{cosec}(x)$	$x \neq n\pi$, $n \in \mathbb{Z}$	$(-\infty, -1] \cup [1, \infty)$

$|\sin x| \leq |x|$

$\lim_{x \to a} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 0$ | $\lim_{x \to a} \frac{1 - \cos x}{x} = 0$

* $|\sin x| \leq |x|$ $x \in \mathbb{R}$

proof ↑ MVT $\cos x$ $(0, 1)$

bijection { (surjection) $\wedge$ (injection) }

onto $\operatorname{ran} P = B$ $\equiv \forall y \in B \ \exists x \in A$ s.t. $(x, y) \in P$

one-one $\forall x_1, x_2 \in A$, If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ } ~ many-one def^n not used mostly

$\sim$(one-many) $\wedge$ $\sim$(many-one)

- function needs to be one-one to have an inverse. If f is bijective $\Rightarrow$ f^{-1} exist & bijective

if f is many-one inverse is one-many ✗ not a function

∴ we can restrict domain and find an inverse

$\sin^{-1} x$, $y = x$, $\sin x$

$\sin x : \mathbb{R} \to [-1, 1]$

restricted

$\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

-1.57 $+1.57$

$\sin^{-1} x : [-1, 1] \to [-1.57, 1.57]$

$f\ A \to B$ 1) everywhere defined $\operatorname{dom} f = A$

2) not one-many

codom = B

one-many $\exists x \in A, \exists y_1, y_2 \in B \Leftrightarrow (x, y_1), (x, y_2) \in P \wedge y_1 \neq y_2$

many-one $\exists x_1, x_2 \in A \ \exists y \in B \Leftrightarrow (x_1, y), (x_2, y) \in P \wedge x_1 \neq x_2$

★

f is (one-one), $\sim$(one-many) $\wedge$ $\sim$(many-one)

always True for function check $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

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mpleteness axioms — completeness property

① If set $A \subseteq \mathbb{R} \wedge A \neq \phi \wedge$ A is bdd. above $\Rightarrow$ sup A exist in $\mathbb{R}$

② If set $A \subseteq \mathbb{R} \wedge A \neq \phi \wedge$ A is bdd. below $\Rightarrow$ inf A exists in $\mathbb{R}$

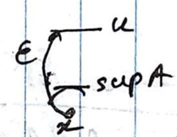
Supremum of A = sup A $\Rightarrow$ least upper bound $\equiv u(A)$

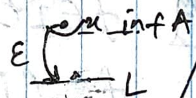
Infimum of A = inf A $\Rightarrow$ largest lower bound $\equiv l(A)$

max A $= u \in A$

min A $= u \in A$

Thm If ~~least upper bound~~ sup A exists $\Leftrightarrow$
$\forall \varepsilon > 0 \ \exists x \in A$ s.t. $x + \varepsilon > \sup A$





Thm If largest lower bound inf A exists $\Leftrightarrow$
$\forall \varepsilon > 0 \ \exists x \in A$ s.t $x - \varepsilon < \inf A$

- $\mathbb{Z}$ is not bdd. above or below

Thm Archemedean property — ✱ $\exists n \in \mathbb{N}; n > \mathbb{R}$

If $x, y \in \mathbb{R}$ $(x > 0)$ $\exists n \in \mathbb{N}$ s.t.
$nx > y \equiv n > \frac{y}{x} \Leftrightarrow \mathbb{N}$ is unbounded

Thm $\forall x \in \mathbb{R} \ \exists [x] \in \mathbb{Z}$ s.t. $\underbrace{[x]}_{\text{floor}} \leq x < \underbrace{[x]+1}_{\text{ceil}}$	$\lfloor x \rfloor$ floor/greatest int. fun greatest int. lesser than or $= x$ greatest int. $\leq x$	$\lceil x \rceil$ ceil/least integer fun least integer $\geq x$

binomial expansion

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \sum_{k=0}^{n} {}^nC_k \, x^{n-k} y^k$$

Squeeze th^m

If $f(x) \le g(x) \le h(x)$ for $0<|x-a|<\delta$ $\wedge$ $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ $\Rightarrow \lim_{x\to a} g(x) = L$

valid for $a^+, a^-, \infty, -\infty$

not fo

one sided squeeze th^m

If f & g are functions s.t. $\exists \delta_1>0$ $\forall x$ s.t. $0<|x-a|<\delta \to f(x) \ge g(x)$,

If $\lim_{x\to a} g(x) = \infty \Rightarrow \lim_{x\to a} f(x) = \infty$

$\underset{x\to a}{g \to -\infty} \Rightarrow f \to -\infty$

$x\to a^+$ $x\to a^-$ $x\to \pm\infty$ ✓ but $g \to \pm\infty$

$\boxed{\text{def}^n}$ $\lim_{x\to a} f(x) = f(a) \Leftrightarrow f$ is cts. at a Continuity

$\underbrace{f(a^-) = f(a)}_{f \text{ is left cts. at } a} = f(a^+) \Leftrightarrow f$ is cts. at a

$f(a) = f(a^+)$: f is right cts. at a

f is cts. on $A \in \mathbb{R} \Leftrightarrow f$ is cts. at each $a \in A$

$f \in \mathcal{C}(A)$

- simply write $f \in \mathcal{C}$ if the set is understood from context

$\boxed{\text{Th}^m}$ f is cts. at a

$\Leftrightarrow$

$\forall \varepsilon>0\ \exists \delta>0\ \forall x,\ 0<|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$

$\Leftrightarrow$

$\forall \varepsilon>0\ \exists \delta>0\ \forall x,\ |x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$

$\boxed{\text{def}^n}$ $f \in \mathcal{C}(A)$ f is cts. at every point $x \in A$

$\Leftrightarrow f$ is cts. $\forall y \in A$

$\Leftrightarrow \forall y \in A\ (\forall \varepsilon>0\ \exists \delta>0\ \forall x,\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

can depend on y & ε

$f \in \mathcal{UC}(A)$ uniform cts.

only depend on ε

$\Leftrightarrow \forall \varepsilon>0\ \exists \delta>0\ \forall x,y \in A,\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

rectangle is same 2ε 2δ for all interval but continuity rectangle changes

$f \in \mathcal{LC}(A)$ lipschitz cts.

$\Leftrightarrow \exists L>0\ \forall x,y \in A,\ |f(x)-f(y)| \le L|x-y|$

$\boxed{\text{Th}^m}$ $f \in \mathcal{LC}(A) \Rightarrow f \in \mathcal{UC}(A) \Rightarrow f \in \mathcal{C}(A)$

$f \in \mathcal{C}[a,b] \Rightarrow f \in \mathcal{UC}[a,b]$

we can prove for a large interval containing a, b

$(\ [\ \ ]\ )$ c a b d

★ If derivative f' exists & bounded on $[a,b]$ $\Rightarrow$ $\mathcal{LC}$ ✓ $\mathcal{UC}$ ✓ $\mathcal{C}$ ✓ on $[a,b]$

Limits

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x$

$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\sim \Rightarrow \exists \varepsilon > 0 \ \forall \delta > 0$ s.t $\forall x$

$0 < |x - a| < \delta \wedge |f(x) - L| \geq \varepsilon$

monotone $\longrightarrow$ increasing / decreasing in their entire dom

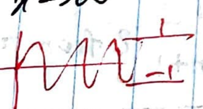
$\log(e^x)$ e^{-x}

increasing $\longrightarrow \forall x_1, x_2 \in A, \ x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$

Strictly increasing $\rightarrow \forall x_1, x_2 \in A, \ x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

can consider derivative $\rightarrow$ + increasing, − decreasing

$\lim_{x \to \infty} \sin x, \cos x$ DNE $\Rightarrow$ $\left.\begin{matrix} \lim_{x \to 0^+} \sin \frac{1}{x} \\ \lim_{x \to 0^+} \cos \frac{1}{x} \end{matrix}\right\}$ DNE



★ not in $\mathbb{R}$

not even as $+\infty$ or $-\infty$

limits

$\boxed{\text{def}^n}$ $\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x$,

$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\underbrace{x \to a^+}$

$x \in (a, a+\delta)$

$\Vert$ $a < x < a + \delta$

$0 < x - a < \delta$

$\underbrace{x \to a^-}$

$x \in (a - \delta, a)$

$\Vert$

$a - \delta < x < a$

$-\delta < x - a < 0$

$x \to \infty$

$\underbrace{\forall \varepsilon > 0 \ \exists N > 0}$

$x > N \Rightarrow$

$\underbrace{x \to -\infty}$

$x < -N \Rightarrow$

$f(x) \to L^+$

$\forall \varepsilon > 0 \ \exists$ - -

$\Rightarrow L < f(x) < L + \varepsilon$

$f(x) \to L^-$

$\forall \varepsilon > 0 \ \exists \delta$ - -

$\Rightarrow L - \varepsilon < f(x) < L$

$f(x) \to \infty$

$\forall M > 0$

$\Rightarrow f(x) > M$

$f(x) \to -\infty$

$\forall M > 0$

$\Rightarrow f(x) < -M$

$|x^2+4| < 24x$

Start always with inside the abs.

find if it is 0

then for that intervals split the inequality

Always input function affects composit f^n

$p \Rightarrow q$ ↑ sufficient

$\sim p \Rightarrow \sim q$ ↑ necessary

real roots → of function

① bisection method (IVT → if f is cts.)

[1] $f(x)$ ⇒ plug in some values and find interval when $f(a) > 0$ and $f(x) < 0$ $[a, b]$

[2]

a	b	$c = \frac{a+b}{2}$	$\boxed{f(c)}$
			If $f(c) > 0$ $b = f(c)$ If $f(c) < 0$ $a = f(c)$

If $|f(c)| < \varepsilon$ ↳ close to 0 } error less than 10^{-2} 0.80 enough

return c

muchfaster but have problems $f'(x) \neq 0$

② Newton's method

[1] intial guess → $x_n = 0$

[2] $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f(x)$ → find $f'(x)$

n	x_n	$\boxed{f(x_n)}$	$f'(x_n)$ →	x_{n+1} } from formula
0	0	✓	✓	✓
1				

$|f(x_n)| < \varepsilon$

↳ close to 0

return x_{n+1}

Differentiability

f is diff·ble at $a \Leftrightarrow \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \in \mathbb{R} = f'(a)$

$x \to a^+ \quad x \to a^-$

Sharp turns
vertical lines
$\times$ diff·ble

$\Leftrightarrow f'_+(a) = f'_-(a) = f'(a)$

right diff·ble　　left diff·ble

Def $f \in \mathcal{D}(A)$, $A \subset \mathbb{R} \Leftrightarrow f$ is diff·ble at each $a \in A$

Thm

$f \in \mathcal{D}(a) \Rightarrow f \in \mathcal{C}(a)$

f is diff·ble at $a \Rightarrow f$ is cts. at a } converse might not be true

f is right diff. at $a \Rightarrow f$ is right cts. at a

f is left 〃 $a \Rightarrow f$ is left 〃

~~$f'(x) \in \mathbb{R}$ for $\forall x \in [a,b] \Rightarrow f \in \mathcal{D}[a,b]$~~

Thm f is continuously diff·ble on $[a,b]$ $\Rightarrow f \in \mathcal{IC}[a,b]$

If f is real valued fuⁿ defined & diff·ble on $[a,b]$, i.e. $|f(x)| \le M$ $\Rightarrow f \in \mathcal{IC}[a,b]$

Thm IVT (intermediate value theorem)

If $f \in \mathcal{C}[a,b] \wedge$ L is strictly between $f(a)$ & $f(b)$

i.e. $f(a) < L < f(b) \vee f(b) < L < f(a)$

$\Rightarrow \exists \xi \in (a,b)$ s.t. $f(\xi) = L$ //

Thm Extreme value thm

If $f \in \mathcal{C}[a,b] \Rightarrow f$ attains a global max/min on $[a,b]$

- c is a global max $\Leftrightarrow \forall x \in A \ f(c) \ge f(x)$
  min $\Leftrightarrow \forall x \in A \ f(c) \le f(x)$
- c is a local max $\Leftrightarrow \exists \delta > 0 \ \forall x, |x-c| < \delta \Rightarrow f(x) \le f(c)$
  min $\Leftrightarrow$ 〃 $\Rightarrow f(c) \le f(x)$

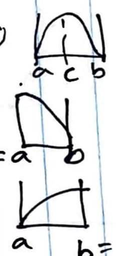
Thm $f \in \mathcal{C}[a,b]$,

c is a local max

① $c \in (a,b) \Rightarrow f'(c) = 0$

② $c = a \Rightarrow f'_+(c) \le 0$

③ $c = b \Rightarrow f'_-(c) \ge 0$



c is a local min

① $c \in (a,b) \Rightarrow f'(c) = 0$

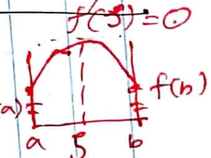
② $c = a \Rightarrow f'_+(a) \ge 0$

③ $c = b \Rightarrow f'_-(a) \le 0$

Thm Rolle's thm

If $f \in \mathcal{C}[a,b] \wedge f \in \mathcal{D}(a,b) \wedge f(a) = f(b)$

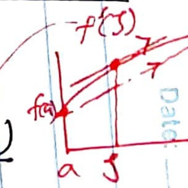
$\Rightarrow \exists \xi \in (a,b)$ s.t. $f'(\xi) = 0$



Thm Mean value thm (MVT)

If $f \in \mathcal{C}[a,b] \wedge f \in \mathcal{D}(a,b) \Rightarrow \exists \xi \in (a,b)$ s.t.

$f'(\xi) = \frac{f(b)-f(a)}{b-a}$ //



Thm Cauchy MVT (extended MVT)

$f, g \in \mathcal{C}[a,b] \wedge f, g \in \mathcal{D}(a,b) \wedge g'(x) \ne 0$ on (a,b)

$\Rightarrow \exists c \in (a,b)$ s.t. $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Taylor series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + f'''(a)(x-a)\ldots$

$f \in C^n \longrightarrow f^{(n)}$ exist and continuous

$f'(x_0)=0 \begin{cases} \text{local max} \xrightarrow{\text{at } x_0} f''(x_0) < 0 \\ \text{local min} \xrightarrow{\text{at } x_0} f''(x_0) > 0 \end{cases}$

$e^{ix} = \cos x + i\sin x$

---

① $\lim_{x\to 0} \frac{1}{x}$ DNE (LHS ≠ RHS)

② $\lim_{x\to 0} \frac{\sin}{\cos}\left(\frac{1}{x}\right)$ (DNE (oscillates)) | $\lim_{x\to\infty} \frac{\sin x}{\cos}$ DNE (oscillates)

③ but $\lim_{x\to 0} x\sin\frac{1}{x} = 0$ (by squeeze Th^m)

L'Hopital

rules/conditions — only for $\boxed{\frac{0}{0}, \frac{\infty}{\infty} \text{ forms}}$

$x \to 0$

① $f, g \in C(-\delta, 0) \cup (0, \delta)$

② $f, g \in D(-\delta, 0) \cup (0, \delta)$

③ $g'(x) \neq 0$ on $(-\delta, 0) \cup (0, \delta)$

④ $f(0) = g(0) = 0$ (or $f(a) = g(a) = 0$)

or $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$

⑤ $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = L$ or $\pm\infty$

$\Rightarrow \lim_{x \to 0} \frac{f(x)}{g(x)} = L$ or $\pm\infty$ ✓

$x \to a^+$	$x \to a^-$	$x \to +\infty$	$x \to -\infty$
enough $(a, a+\delta)$	$(a-\delta, a)$	(N, ∞); $N>0$	$(-\infty, -N)$; $N>0$

$x \to a$

$(a-\delta, a) \cup (a, a+\delta)$

always cts version — $f(n) = n^2$, not $f_n = n$ ✗

$\lim_{x \to 0} \frac{\sin x}{x} = 1$	$\lim_{x \to 0} \frac{\tan x}{x} = 1$	$\lim_{x \to 0} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 0$ by squeeze

① for $\boxed{0^0, \infty^0, 1^\infty \text{ forms}}$

use $y = x^{1/x}$ $\quad y = e^{\ln \boxed{x^{1/x}}} = e^{\frac{1}{x}\ln x}$ ✓

hence $\lim_{x \to a} e^{\frac{1}{x}\ln x} = \exp\left[\lim_{x \to a} \frac{1}{x} \ln x\right]$

since e^x is cts everywhere

② $0 \cdot (\pm\infty)$ and $\infty - \infty$ forms

turn them into quotients

eg: $x \ln x = \frac{\ln x}{1/x} = \frac{-\infty}{\infty}$ form

$x e^x = \left.\frac{e^x}{1/x}\right\}$ won't work $\Rightarrow \frac{x}{e^{-x}} = \frac{0}{0}$ form ✓

try this 1st

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 0$

$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

$\lim_{x \to 0} \frac{\tan x}{x} = 1$

$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

Taylor series

$f^{(n+1)}$ exists $\equiv f \in \mathcal{D}^{n+1}$ on (a,b) $c \in (a, x)$

$\exists\, \xi$ between x and a s.t.

$$\boxed{f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}}_{T_n(x,a)} + \underbrace{\frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}}_{R_n(x,a)}}$$

taylorpolynomial

differential remainder

no fc

- higher the n better the approximation

$|f(x) - T_n(x,a)| = \underbrace{|R_n(x,a)|}_{\text{error}} < 10^{-3}$ find n

$$\boxed{R_n(x,a) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt}$$

$f \in C^{n+1}$

taylorseries

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$= \frac{f(a)(x-a)^1}{1!} + \frac{f'(a)(x-a)^2}{2!} + \dots$$

$$e^x = \frac{1}{1} + \frac{x}{1!} + \frac{x^2}{2!} + \dots = \sum_{n=0} \frac{x^n}{n!}$$

$$\cos x = \sum_{\boxed{n=0}}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{1}{2!}x^2 + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\ln x_{a=0} = \underbrace{\sum_{k=1}^{n} \frac{(-1)^{k-1} x^k}{k}}_{T_n(x,0)} + \frac{(-1)^n}{n+1}\left(\frac{x}{1+\xi}\right)^{n+1}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Sequences

defn $\lim_{n\to\infty} u_k = L \Leftrightarrow$ if $\forall \varepsilon>0\ \exists n \in \mathbb{Z}^+$
$\forall k \in \mathbb{Z}^+,\ k>N \Rightarrow |u_k - L| < \varepsilon$

u_k is converging $\Leftrightarrow \lim_{k\to\infty} u_k = L \in \mathbb{R}$

u_k is diverging $\Leftrightarrow$ u_k is not converging
$\Leftrightarrow \lim_{k\to\infty} u_k = L \in \mathbb{R}$ is not true
$\pm\infty$, DNE

Thm sequence u_k, if we have fun $f(x)$ s.t $f(k) = u_k$
cts. version limit
$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{x\to\infty} u_k = L$ ✓
L'hopital

If $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

$\{r^n\}_{n=0}^{\infty}$ converges if $-1<r\leq 1$
diverges otherwise

Thm monotone convergence theorem

If u_k is ② bounded $\wedge$ ① monotonic (either increasing or decreasing) $\Rightarrow u_k$ is converging (convergent)

* converse might not be true in the case of monotonic
but if u_k is converging $\Rightarrow u_k$ bounded

Thm
every sequence has a monotone subsequence

Thm bolzano weierstrass thm
every bdd sequence has a converging subsequence

Thm Cauchy sequence   analogous to uniform cts.
sequence is cauchy $\Leftrightarrow \forall \varepsilon>0\ \exists N \in \mathbb{Z}^+\ \forall k, n > N \Rightarrow |u_k - u_n| < \varepsilon$

- converging sequences are cauchy   converse true in $\mathbb{R}$
- thm every cauchy sequence in $\mathbb{R}$ converge to a point in $\mathbb{R}$
- any cauchy sequence is bounded

* complete sets : metric space where every cauchy seq. cvgs.
eg: $\mathbb{R}^n$, closed intervals in $\mathbb{R}$
counter eg: $\mathbb{Q}$, $\mathbb{Q}^c$

* to prove sequence convergence
1) prove it is cauchy
2) prove it is monotone & bounded

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Series

$\sum_{n=1}^{\infty} u_n = \sum u_n$

series conv. $\Leftrightarrow$ $\sum u_n = L \in \mathbb{R}$

actual sum (answer)
rarely get that chance

no fc

can slice up
(partial fractions, $\sum a+b = \sum a + \sum b$)

Geometric series

algebric manipulatory reqd
get to form

$\sum_{n=1}^{m} ar^n = ar\left(\frac{1-r^m}{1-r}\right)$ conv. $\Leftrightarrow |r|<1$

① divergence test

If $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges
$\underbrace{}_{\pm\infty,\ DNE}$

· Note If $\lim_{n\to\infty} a_n = 0$ test fails

$\sum a_n$ abs. converges, if $\sum |a_n|$ cvgs.

$\sum a_n$ conditionally cvgs, if $\sum |a_n|$ diverges.

$\hookrightarrow$ thm if $\sum a_n$ abs. cvgs then any rearrangements of $\sum a_n$ will also cvgs have the same value as $\sum a_n$.

• if $\sum a_n$ conditionally cvgs. then rearrangement will result cvg to diff value or div.

telescoping series

$\sum \frac{1}{n^2+3n+2} \to$ partial fractions $\sum \frac{1}{n+1} - \frac{1}{n+2} = (s_1 - s_n)_{n\to\infty}$ ✓

telescoping

harmonic series

$\sum \frac{1}{n} \to$ divgs. p test

② Integral test

$f:[1,\infty)\to(0,\infty)$ ① $\in \mathcal{R}[1,b]\ \forall b>1$ (or cts)

② $\wedge f$ is decreasing to 0

③ $\wedge f > 0$

$\Bigg\}$ $\int_1^{\infty} f(x)dx$ cvgs. $\Leftrightarrow \sum f(n)$ cvgs

③ comparison test (direct)

① $0 < u_n < v_n \to$ take known fun (series) p series

$\sum v_n$ cvgs $\Rightarrow \sum u_n$ cvgs

$\sum u_n$ diverges $\Rightarrow \sum v_n$ dvgs.

④ limit comparison test

$0 < u_n < v_n$ — take well known one $\wedge \lim_{n\to\infty} \frac{u_n}{v_n} = L$

1) If $L \in (0,\infty) \Rightarrow \sum v_n$ cvgs $\Leftrightarrow \sum u_n$ cvgs

2) If $L = 0 \Rightarrow \sum v_n$ cvgs $\Rightarrow \sum u_n$ cvgs

3) If $L = \infty \Rightarrow \sum v_n$ dvgs $\Rightarrow \sum u_n$ dvgs

ratio test

$0 < u_n$

$\sum u_n$, check $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = L$

1) If $L<1 \Rightarrow \sum u_n$ converges

2) If $L>1$ ($\cancel{\infty}$) $\Rightarrow \sum u_n$ diverges

Note $L=1$ Test fails

(6) Root test

$0 < u_n$

check $\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = L$

1) If $L<1 \Rightarrow \sum u_n$ cvgs.

2) If $L>1 \Rightarrow \sum u_n$ diverges

Note $L=1$ test fails.

remark for $L>1$, checking abs. cvg. $\sum|u_n|$ by root/ratio test

it follows $\lim_{n\to\infty} |u_n| = \infty \Rightarrow \lim_{n\to\infty} u_n \neq 0$

by divergence test $\sum u_n$ dvgs.

alternating series

$\sum (-1)^{n+1} a_n$ where $a_n > 0$ (without changing signs)

(7) Alternating series test

$$\left.\begin{array}{l} \sum (-1)^{n+1} a_n \text{ alternating series} \\ \wedge\ a_n \text{ decreasing} \\ \wedge\ \lim_{n\to\infty} a_n = 0 \end{array}\right\} \Rightarrow \sum (-1)^{n+1} a_n \text{ cvgs.}$$

(a_n decreasing $\wedge \lim_{n\to\infty} a_n = 0$: decreasing to 0)

$\cos(n\pi) = (-1)^n$

harmonic series $\sum \frac{1}{n}$ divgs (p series $p=1$)

alternating harmonic series $\sum \frac{(-1)^n}{n}$ (alternating series test)

$\boxed{1}$ divergence check $\lim_{n\to\infty} u_n \neq 0$

$\boxed{2}$ similar to p series, geometric $\rightarrow$ comparison

$\boxed{3}$ rational expression, polynomial $\frac{n^2+3n+2}{6n+3}$, $\sqrt{n^2+\ldots}$ $\rightarrow$ limit comparison test

$\boxed{4}$ c^{α}, $n!$ $\xrightarrow{\text{only}}$ ratio test

$\boxed{5}$ $a_n = (b_n)^n \rightarrow$ root test

$\boxed{6}$ $a_n = (-1)^{n \text{ or } n+1} b_n \rightarrow$ alternating series test

$\boxed{7}$ $f(n) = a_n$ decreasing and (+)ve integral test

Powerseries

|abs.|

1. ratio / root test find inequality $L < 1$
2. test for end points

Note * if $L = \infty > 1$

then $(x-a)^n \to x = a$ only point of cvgs.

hence $R = 0$ for only one point
(radius of cvgence)

abs. cvgs interval $x = a$

$\Rightarrow$ cvgs interval $x = a$

---

* if $L = 0 < 1$

power series cvgs $\forall x \in \mathbb{R}$

$R = \infty$

interval of cvgence $(-\infty, \infty)$

$$\left.\begin{aligned} f(x) &= \sum c_n (x-a)^n \\ f'(x) &= \sum n c_n (x-a)^{n-1} \\ \int f(x)\,dx &= C + \sum c_n \frac{(x-a)^{n+1}}{n+1} \end{aligned}\right\}$$

$R > 0$ all have same R

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1

<u>logic</u>

- $p \to q \equiv \sim p \vee q$
- commutative ✓ $\wedge$
- associative ✓
- $\exists x \forall y \Rightarrow \forall y \exists x$
  converse doesn't hold.
- $-\forall x, P(x) = \exists x, -P(x)$

identity law $\begin{cases} q \vee F = q & q \vee T = T \\ q \wedge F = F & q \wedge T = q \end{cases}$

complement law $\begin{cases} \sim q \vee q = T \\ \sim q \wedge q = F \end{cases}$

$(A \cap B)^c = A^c \cup B^c$

suppose $x \in (A \cap B)^c$

$\Leftrightarrow x \notin (A \cap B)$; def^n of complement

$\Leftrightarrow x \notin A \vee x \notin B$ at least not in one of them

$\Leftrightarrow x \in A^c \vee x \in B^c$

$\Leftrightarrow x \in (A^c \cup B^c)$ i.e $(A \cap B)^c = A^c \cup B^c$ ∎

$\overline{a \cdot b} = \bar{a} + \bar{b}$

$\sim(p \wedge q) = \sim p \vee \sim q$

$\exists x \forall y \, P(x,y) \longrightarrow \forall y \exists x \, P(x,y)$ prove

let p, q, r, s statements $p : P(x_1, y_1), \ldots$

Assume x, y belongs to finite set

$\exists x \forall y P(x,y) = \underbrace{\underbrace{\bigvee_{i=1}^{2} \bigwedge_{j=1}^{2} P(x_i, y_j)}_{①}}_{②}$ } cvt to $p \vee q \longrightarrow$ truth table

True (tautology)

False (contradiction)

---

mod arithmetic

define set $F = \{[0], [1], [2]\}$ for $a \in R$

$[0] = 3k$; $k \in R$

$[1] = 3k+1$ ″

$[2] = 3k+2$ ″

def. binary operations $+_3, \cdot_3$

For $[a], [b], [c] \in F$

Closed $\to$ Associative $\to$ identity $\to$ inverse } distributive $(F, +_3, \cdot_3)$

ms

$A \Rightarrow B \equiv \sim A \vee B$

<u>contrapositive</u>

$\underset{sps}{\sim B} \Rightarrow \underset{ms}{\sim A}$ //

<u>contradiction</u>

sps $\sim(A \Rightarrow B)$

$\sim(\sim A \vee B)$

$\boxed{A \wedge \sim B}$

↓

prove #

for results including $n, m \in \mathbb{Z}^+$ integers try induction

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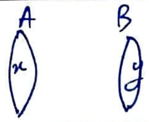
functions



- one – many : $\exists x \in A, \exists y_1, y_2 \in B \Leftrightarrow (x, y_1), (x, y_2) \in P \wedge y_1 \neq y_2$

  many - one : $\exists x_1, x_2 \in A, \exists y \in B \Leftrightarrow (x_1, y), (x_2, y) \in P \wedge x_1 \neq x_2$

  not (many-one) $\Leftrightarrow \boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$

  $\Leftrightarrow x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

(injection) one - one $\equiv \sim$(one-many) $\wedge \sim$(many-one)

✓ fuⁿ

(surjection) onto = ran$P$ = B $\Leftrightarrow \boxed{\forall y \in B \ \exists x \in A \text{ s.t } f(x) = y}$

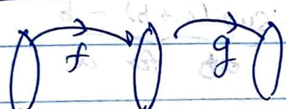
- To find inverse → • fuⁿ needs to be one – one $\equiv$ not(one–many)

  original fuⁿ cannot be many – one → inverse one-many

  • If fuⁿ is bijective ⇒ fuⁿ one–one ⇒ f^{-1} exist

  • restrict dom_f → draw graph replace ~~x~~ with y, y with x

- composite fuⁿ



$A \longmapsto B \longmapsto C$

$a \longmapsto b \longmapsto c$

- $g \circ f = g(f(x))$ } $g \circ f : A \to C$

if g, f one – one,

ms $g \circ f$ is one – one i.e. $g \circ f(x_1) = g \circ f(x_2) \Rightarrow x_1 = x_2$

suppose: $g \circ f(x_1) = g \circ f(x_2)$

$g[f(x_1)] = g(f(x_2))$

Let $f(x_1) = a$ $\quad f(x_2) = b$

then $g(a) = g(b)$

$\Rightarrow a = b$; g is one - one

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$ ∵ f is one-one

$\Rightarrow g \circ f$ is one-one / injective

ms $g \circ f$ is onto / surjective

i.e: $\forall c \in C \ \exists a \in A$ s.t $g \circ f(a) = c$

$g \circ f(a)$

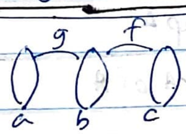
$= g[f(a)]$

$= g(b)$; f is onto $f(a) = b$

$= c$; g " $g(b) = c$

$\Rightarrow g \circ f$ is onto

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$



$(f \circ g^{-1}) \Rightarrow$

$\{(c, a) \mid (a, c) \in f \circ g\}$

$\Leftrightarrow (a, c) \in f \circ g$

$\Leftrightarrow (a, b) \in g \wedge (b, c) \in f$

$\Leftrightarrow (b, a) \in g^{-1} \wedge (c, b) \in f^{-1}$

$\Leftrightarrow (c, b) \in f^{-1} \wedge (b, a) \in g^{-1} \Leftrightarrow (c, a) \in g^{-1} \circ f^{-1}$ ∴ $f \circ g^{-1} = g^{-1} \circ f^{-1}$

$f \circ f \to f \quad f \quad f$

Dom $-1, -\frac{3}{2}$ ✗ $\quad$ Dom -1 ✗ $\quad -\frac{3}{2}$ ✗

ran $1, \frac{3}{2}$ ✗ $\quad$ 1 ✗ ran $\quad \frac{3}{2}$ ✗ ran

$(f \circ f)^{-1}(y) = f^{-1}[f^{-1}(y)]$

② sup / inf

(*)
- completeness property (have to mention)
  to prove $\sup / \inf A = \ldots$
  ① $A \neq \phi \Leftarrow \{ (a,b) \neq \; a<b \Rightarrow a < \frac{a+b}{2} < b$; $\mathbb{R}$ are dense
  ② $A \subseteq \mathbb{R} \Leftarrow \{ a,b \in \mathbb{R} \Rightarrow \frac{a+b}{2} \in \mathbb{R}$
  ③ A is bdd → above $a \leq u \; \forall a \in A \Rightarrow \sup A$ exist
  → below $i \leq a \; \forall a \in A \Rightarrow \inf A$ exist

- $A \subseteq B \Rightarrow a \leq \sup A, \; a \leq \underbrace{\sup B}_{ub} \Rightarrow \sup A \leq \sup B$ ∎
- Archemedian property for finding ub/lb → start with result some $\underbrace{n}_{ub} > R$ for $n \in \mathbb{Z}$, $n \in \mathbb{N}$
- for any $A \in \mathbb{R}$ $\sup(-A) = -\inf A$, $\inf(-A) = -\sup A$ } where $-A = \{a \mid a \in A\}$

use this to replace B with $-B$ and $\Rightarrow \sup(A-B) = \sup A - \inf B$

Type of Problems

① to prove equality eg: $\sup(A+B) = \sup A + \sup B$

(*) for (A+B)
$a \leq \sup A$
$b \leq \sup B$
$a+b \leq \underbrace{\sup A + \sup B}_{ub \text{ for } a+b}$
$\Rightarrow \sup(A+B) \leq \sup A + \sup B$

$\forall a+b \in A+B$
$a+b \leq \sup(A+B)$
$\underbrace{a+b-b}_{a} \leq \underbrace{\sup(A+B)-b}_{ub \text{ for } a}$
$\Rightarrow \sup A \leq \sup(A+B) - b$
$b \leq \underbrace{\sup(A+B) - \sup A}_{ub \text{ for } b}$
$\Rightarrow \sup A + \sup B \leq \sup(A+B)$ ∎

② If $\forall \varepsilon > 0 \; \exists x \in A \;\; x - \varepsilon < L \Leftrightarrow L = \inf A$

$\Leftarrow$) Assume $\sim(\forall \varepsilon>0 \; \exists x \in A \;\; x-\varepsilon<L)$ is True
$\Rightarrow \exists \varepsilon>0 \; \forall x \in A \;\; x - \varepsilon \geq L$
$x \geq \underbrace{L+\varepsilon}_{lb \text{ of } x}$
$\Rightarrow \inf A \geq L+\varepsilon$
$\Rightarrow L \geq L+\varepsilon$
$\Rightarrow 0 \geq \varepsilon$ #

$\Rightarrow$ (*)
$\forall x \in A \;\; \inf A \leq x < L+\varepsilon$
$\inf A < L+\varepsilon$
$\inf A - L < \varepsilon$
$\varepsilon > 0$ arbitrary $\Rightarrow \inf A = L$ ∎
∧ ∵ inf A and L doesn't depend on ε

③ $u = \sup A \Leftrightarrow$ i) u is ub of A
ii) $\forall \varepsilon>0 \; \exists x \in A \;\; x+\varepsilon > u$

$\Rightarrow$) $\forall x \in A \;\; x \leq \sup A = u \Rightarrow u$ is ub of A
Assume not i.e $\exists \varepsilon>0 \; \forall x \in A \;\; x+\varepsilon \leq u$
$\Rightarrow x \leq \underbrace{u-\varepsilon}_{ub \text{ for } x}$
$\Rightarrow u \leq u-\varepsilon$
$\Rightarrow \varepsilon \leq 0$ #

$\Leftarrow$ (*)
$\forall x \in A \;\; x \leq \sup A$
$u < x+\varepsilon \leq \sup A + \varepsilon$
$u \leq \sup A + \varepsilon$
$u - \sup A \leq \varepsilon$
since $\varepsilon > 0$ is arbitrary, doesn't depend on ε
$u = \sup A$ ∎

④ sup / inf $A = \sqrt{2}$ any value find inequality $x < \sqrt{2}$
Assume $u < \sqrt{2}$ any ub
$\Rightarrow u < q < \sqrt{2}$; $\mathbb{R}$ is dense
$u^2 < q^2 < 2$ or element
$\Rightarrow \nexists q \in A \; q < u$ #
$\Rightarrow u \geq \sqrt{2}$
$\Rightarrow \sup A = \sqrt{2}$ by defn

⑤ show $\inf(a,b) = a$, $a<b$
sup → ∞, inf → −∞ } DNE

(*) ✓
Let $i = \inf(a,b)$
by trichotomy,

$i < a$
$i < \frac{i+a}{2} < a$ lb for a
$\frac{i+a}{2} < i$
$a < i$ #

$x > i > a \in A$
$x > i > \frac{i+a}{2} > a$
$\frac{i+a}{2} \ngtr i$
$a \ngtr i$ #

then $i = a$ is True

(3)

take cases

No :

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abs. value proofs

- $\forall a, b \in \mathbb{R} \quad |ab| = |a||b|$

case I : $a \geq 0 \wedge b \geq 0$; $|a| = a$, $|b| = b$, $a.b \geq 0 \Rightarrow |ab| = a.b \Rightarrow |ab| = |a||b|$

case II : $a < 0 \wedge b < 0$; $|a| = -a$, $|b| = -b$, $a.b > 0 \Rightarrow |ab| = ab = (-a)(-b) = |a||b|$

case III : $a \geq 0 \wedge b < 0$; $|a| = a$, $|b| = -b$, $ab < 0 \Rightarrow |ab| = a(-b) = |a||b|$

case IV ; $a < 0 \wedge b \geq 0$ $\rightarrow$ replace a with b for case III

- $\forall x \in \mathbb{R};\ -|x| \leq x \leq |x|$

- $|a| \leq r \iff -r \leq a \leq r$ where $r > 0$

- $|a+b| \leq |a| + |b|$
- $||a| - |b|| \leq |a - b|$; $a = a - b$
  $|a-b| \leq |a| + |b|$; $b = -b$

$|a| - |b| \leq |a+b| \leftarrow$ $a = a - b$, $b = -b$

$|a| - |b| \leq |a+b| \leq |a| + |b|$

$|a| - |b| \leq |a-b| \leq |a| + |b|$

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sequences above / below

- non empty, bdd, $\in \mathbb{R}$ sequences $\rightarrow$ sup $= \lim_{n\to\infty} a_n$
  $\lim_{n\to\infty} a_n = ?$ find

monotonic

① $f: A \to \mathbb{R}$, $C, D \subseteq A$, f is one-one ms. $f(C \cap D) = f(C) \cap f(D)$

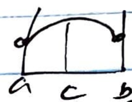
only if one-one

let $y \in f(C \cap D)$	let $y \in f(C) \cap f(D)$
then $\exists x \in C \cap D$ s.t. $f(x) = y$	then $y \in f(C) \wedge y \in f(D)$
$\Rightarrow x \in C \wedge x \in D$	$\Rightarrow \exists x_1 \in C \wedge \exists x_2 \in D$ ∵ f is one-one
$\Rightarrow f(x) \in f(C) \wedge f(x) \in f(D)$	$f(x_1) = y$ $\quad f(x_2) = y$ $\quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
$\Rightarrow f(x) \in f(C) \cap f(D)$	$\Rightarrow x_1 \in (C \cap D)$
$\Rightarrow y \in f(C) \cap f(D)$	$\Rightarrow f(x_1) \in f(C \cap D)$
∴ $f(C \cap D) \subseteq f(C) \cap f(D)$ — ①	$\Rightarrow y \in f(C \cap D)$
	∴ $f(C) \cap f(D) \subseteq f(C \cap D)$ — ②

①, ② ✓

Rolle → $f \in \mathcal{C}[a,b]$ $f \in D(a,b)$ $\wedge$ $f(a) = f(b)$

$\Rightarrow \exists c \in (a,b)$ s.t $f'(c) = 0$



proof $f \in \mathcal{C}[a,b] \Rightarrow f$ attains global max/min on $[a,b]$ (extreme value thm)

case I → global max/min on boundary (a or b)

$gl.\ min\{f(x)\} \le f(x) \le gl.\ max\{f(x)\}$

$f(x) = f(b) = f(a) = K$ const $\Rightarrow f'(c) = 0 \ \forall c \in (a,b)$ $\blacksquare$

case II → at least one ~~gl~~ max/min on interior point let $c \in (a,b)$

by thm $\Rightarrow f'(c) = 0$ $\blacksquare$

MVT → $f \in \mathcal{C}[a,b] \wedge f \in D(a,b) \Rightarrow \exists c \in (a,b)$ s.t $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

proof define $g(x) = (f(b) - f(a))(x-a) - f(x)(b-a)$

$g(a)$, $g(b) \to g(a) = g(b)$ $\quad$ state $g(x)$ diff (a,b) cts $(a,b$

by rolle → $g'(c) = 0$ for $c \in (a,b)$

Cauchy MVT extended → proof $f, g \in \mathcal{C}[a,b]$ $\wedge$ $f, g \in D(a,b)$ $\wedge$ $g'(x) \neq 0$ on $[a,b]$

$\Rightarrow \exists c \in (a,b)$ s.t $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$

proof: $F(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x)$

6 $\forall \varepsilon>0\ \exists \delta>0\ \forall x,\ 0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$

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1 let $\varepsilon>0$

limits

try finding

- ms $\lim_{x\to a} f(x)g(x) = LM \rightarrow$ start with $|f(x)g(x)-LM|<\varepsilon$
- write down know lim. def^n change ε
- finally $\delta = \min\{\delta_1, \delta_2, \delta_3\} \rightarrow$ proof done

Note
- when dividing use $|M|+1$ instead of $|M|$ (we don't know if $M=0$)
- $||a|-|b|| \le |a-b| \Rightarrow |a|-|b| \le |a-b|$ ✓ can use this
- cts at a point reqd $0<|x-a|<\delta$
- $f(x) \to \pm\infty$ start with results $f(x)>M$ find back to $x>N$, $x<-N$

$|f(x)-b|<\varepsilon$ use negative side

$-\varepsilon < f(x)-b$

$b-\varepsilon < f(x)$

$M_1 < g(x)$

ms $f(x)g(x) > M > 0$

$M = (b-\varepsilon)M_1 > 0$

$b > \frac{b}{2}$

---

composite

$\lim_{x\to b} f(g(x)) = L$

$0<|x-b|<\delta \Rightarrow |f(g(x))-L|<\varepsilon$

$\delta_1 \rightarrow |f(x)-L|<\varepsilon$

$\delta_2 \rightarrow |g(x)-a|<\delta_1$

$g(x)\neq a \Rightarrow \delta_3 \Rightarrow 0<|g(x)-a|<\delta_1$ ✓

$\Rightarrow \delta = \min\{\delta_2, \delta_3\}$

$0<|x-a|<\delta \Rightarrow |g(x)-a|<\delta_1$

$\Rightarrow |f(g(x))-L|<\varepsilon$ ✓

---

continuous at a point

check left lim = right lim

$f(a^-) = f(a^+)$

$\forall\varepsilon>0\ \exists\delta>0\ |f(x)-f(a)|<\varepsilon$

$\forall x\ |x-a|<\delta \Rightarrow$

- $\delta \le 1$
- $|x|-|a| \le |x-a|$; a unknown
- $|x-a|<1 \Rightarrow 1<x<3$

$\frac{x^2+5x+3}{x^2+1} < \frac{3^2+5\times3+3}{1+1}$

* careful choosing δ when $x\to a$ $f(x)$ when around a $f(x)$ has discts choose smaller δ

reciprocal proofs

* $|x-a|<\frac{a}{2} \Rightarrow \frac{a}{2}<x<\frac{3a}{2}$

$\Rightarrow \frac{2}{a} > \frac{1}{x} > \frac{2}{3a} > -\frac{2}{a}$

$\Rightarrow \left|\frac{1}{x}\right| < \frac{2}{a}$

or $|f(x)-L|<\frac{|L|}{2} \Rightarrow \left|\frac{1}{f(x)}\right| < \frac{2}{|L|}$

$|f(x)-2|<\frac{1}{2}$ ε_1

- $|f(x)-L|<|L| \Rightarrow |f(x)|-|L|<|L|$

$\Rightarrow |f(x)|<2|L|$

$\Rightarrow -2|L|<f(x)<2|L|$

$\rightarrow \infty$ limits

- $|f(x)-b|<\varepsilon \Rightarrow$

$b-\varepsilon<f(x)<\varepsilon+b$

final results choose ε ✓

$\lim_{x \to c} x^2 = c^2 \equiv \forall \varepsilon > 0 \ \exists \delta > 0$ such that $0 < |x-c| < \delta \Rightarrow |x^2 - c^2| < \varepsilon$

$\boxed{1}$ let $\varepsilon > 0$

$\boxed{2}$ known results (eg: $\lim_{x \to a} f(x) = L \to |f(x) - L| < \varepsilon \Rightarrow$ choose

Suppose $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x$ s.t $0 < |x - c| < \delta$

let $\delta \leq 1$ then $0 < |x - c| < 1$;

$\Rightarrow |x| - |c| < 1$

$\Rightarrow |x| < 1 + |c|$

$\Rightarrow |x| + |c| < 1 + 2|c|$

$\Rightarrow |x + c| \leq |x| + |c| < 1 + 2|c|$

$\boxed{3}$ $|x^2 - c^2| = |x - c||x + c|$

$< \delta \times (1 + 2|c|) \leq \varepsilon$

$\Leftarrow \delta \leq \frac{\varepsilon}{1 + 2|c|} > 0$

$\boxed{4}$ let $\delta = \min\left\{1, \frac{\varepsilon}{1 + 2|c|}\right\} > 0$

$\therefore \forall \varepsilon > 0 \ \exists \delta = \min\left\{1, \frac{\varepsilon}{1 + 2|c|}\right\} > 0 \ \forall x$ s.t $0 < |x - c| < \delta \Rightarrow |x^2 - c^2| < \varepsilon$

$\boxed{5}$ $\Rightarrow \lim_{x \to c} x^2 = c^2$

$x \to 0^- = x - 0^- = h$ ($x = 0 - h$)

- $\boxed{2}$ limit DNE $\to$ Assume limit exists $\lim_{x \to a} f(x) = L$

case I : $L \in \mathbb{R}$ according to ε-δ def^n exists

case II : $L = +\infty$ find #

case III : $L = -\infty$

$\lim_{x \to 0^+} \sin \frac{1}{x}$, $\lim_{x \to \infty} \sin x$ or $\cos x$

* check L.H limit = R.H limit $\boxed{1}$

- one sided limit proofs $x \to 2^-$ $\forall \varepsilon > 0 \ \exists \delta > 0 \ 0 < \delta$ the $2 - \delta < x < 2$

let $x < 2$ } since left sided limit

$|x - 2||x + 2| = \underbrace{-(x - 2)}\ |x + 2|$

(8)

$f'_+(a) = f$ ... Differen...

Continuity

sandwich or to prove cts. L·H lim = R·H lim, $f(a^+) = f(a^-)$, $f'(a^+) = f'(a$...

- define the ~~an~~ step fuⁿ if fuⁿ is discts at point only because its not defined at that point but lim does exists ✓
- cts at a

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x$ s.t $|x-a| < \delta \Rightarrow \underbrace{|f(x) - f(a)| < \varepsilon}_{\text{start with this}}$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$ //

- discts at a

$\sim [\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x$ s.t $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]$

my $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x$ s.t $|x-a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon]$

$\boxed{1}$ choose $\varepsilon = 1, \frac{1}{2}$ usually

use archemedian property

$|f(x) - f(a)| = \dots > 1$ ✓

- first check diff bility $\rightarrow$ f real val.d fuⁿ defined $\in \mathbb{R}$ and bdd ($f(x)$ not f(x) $\leq |M|$) on $[a,b]$

$\Rightarrow IC \Rightarrow UC \Rightarrow C$ ✓ on $[a,b]$

otherwize check IC

↳ UC

↳ C

- If cts (C) on a closed interval $[a,b]$ $\Rightarrow UC \ [a,b]$

or

$(_c \ [_a \ ]_b \ )_d$

larger open interval (c,d) contaning $[a,b]$

Date: / /

<u>to show $\mathcal{LC}$</u>

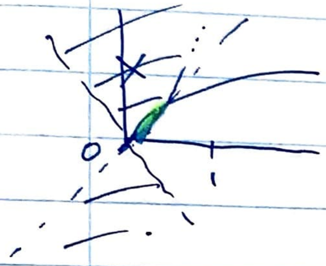
$f \in \mathcal{LC}[a,b] \iff \exists L>0 \ \forall x,y \in [a,b] \ s.t. \ |f(x)-f(y)| \le L|x-y|$

start & prove
find L

<u>to show not $\mathcal{LC}$</u>

$\sim(\exists L>0 \ \forall x,y \in A \ s.t. \ |f(x)-f(y)| \le L|x-y|)$

$\forall L>0 \ \underbrace{\exists x,y}_{} \in A \ s.t. \ |f(x)-f(y)| > L|x-y|$



$\sqrt{x}$ on $[0,1]$

$\boxed{1}$ choose x,y $\quad y=0, \quad 0<x<1$

$\boxed{2}$ $\left|\dfrac{f(x)-f(y)}{x-y}\right| = \left|\dfrac{\sqrt{x}-0}{x-0}\right| = \dfrac{1}{\sqrt{x}} \to \infty$ as $x \to 0^+$

$\boxed{3}$ $\lim\limits_{x\to 0^+} \dfrac{1}{\sqrt{x}} = \infty \Rightarrow \exists \delta>0 \ \forall x \in (0,1), \ 0<x<\delta<1$

$\Rightarrow \dfrac{1}{\sqrt{x}} > L$ ✓

$\mathcal{C}(A) \iff \forall \varepsilon>0 \ \forall y \in A \ \exists \delta>0 \ \forall x \in A \ s.t. \ |x-y|<\delta \Rightarrow |f(x)-f(y)| < \varepsilon$

<u>to show $\mathcal{UC}$</u>

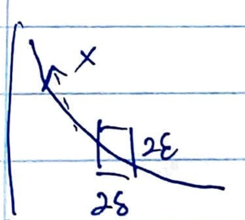
$\forall \varepsilon>0 \ \exists \delta>0 \ \forall x,y \in A \ s.t. \ |x-y|<\delta \Rightarrow |f(x)-f(y)| < \varepsilon$

<u>to show not $\mathcal{UC}$</u>

$\sim(\forall \varepsilon>0 \ \exists \delta>0 \ \forall x,y \in A \ s.t. \ |x-y|<\delta \Rightarrow |f(x)-f(y)| < \varepsilon)$

$\exists \varepsilon>0 \ \forall \delta>0 \ \exists x,y \in A \ s.t. \ |x-y|<\delta \wedge |f(x)-f(y)| \ge \varepsilon$

$\boxed{1}$ choose x,y $\quad x=\delta, \ y=\delta/2$



$\boxed{2}$ $|x-y| = |\delta-\delta/2| = \delta/2 < \delta < 1 \to \varepsilon$

$\boxed{3}$ $|f(x)-f(y)| = \left|\dfrac{1}{x}-\dfrac{1}{y}\right| = \dfrac{|x-y|}{xy} = \dfrac{\delta/2}{\delta \times \delta/2} = \dfrac{1}{\delta} > 1 = \varepsilon$ ✓