SS

Wednesday, May 25, 2022 7:52 AM

errors → cvgs → rate of cvgs $|\alpha - \alpha_n| \le K\beta_n$ } $\alpha_n = \alpha + O(\beta_n)$

→ stable: small chg to I ⇒ small chg to O } conditionally stable: stable only for some I

→ unstable: " ⇒ large "

truncation → taylor series $f(x) = \underbrace{\sum_{n=0}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}}_{\text{approximation}} + \underbrace{\frac{f^{(n+1)}(\epsilon(x))(x-x_0)^{n+1}}{(n+1)!}}_{\text{truncation error}}$ (ϵ btw x, x_0)

* $\frac{\sin(\epsilon(x))x^4}{4!} \le \frac{\sin(0.4)\,x^4}{4!}$ since $\epsilon(x)$ btw x & $x_0 = 0.4$

or $\le \frac{x^4}{4!}$ ✓

* within 0.00005: err ≤ 0.00005

$\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} + \ldots$ (6th degree; 6th term)

rounding off → chopping: chop the k^{th} digit

rounding: $0.d_1d_2d_3d_4 \cdots d_k d_{k+1} \times 10^n$ (d_1: 1 to 9; others: 0 to 9)

k-digit decimal machine number

if $d_{k+1} \ge 5$: $d_k + 1$ and chop

else: chop

* every step of machine calculations round or chop
* subtraction errors
* quadratic eq^n check b sign
* abs err $= |p - p^*|$
* rel err $= \frac{|p-p^*|}{p}$ → commonly used
* $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
  $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

must be true without rounding or chopping → upto 3 decimal places (0.0005) *

$\sin(35°) = 0.5735764$ $35° = \frac{7\pi}{36} = c$

$= c - \frac{c^3}{3!} + \frac{c^5}{5!}$

$0.572|8738$ ✗

$0.5735|8$ ✓

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① Bisection Thm

① arrange $f(x) = 0$

③ check cts on $[a,b]$ (cond 1), $f(a)f(b) < 0$ (IVT) (cond 2)

be reasonable when choosing a, b { time always > 0

② graph or differentiate: find interval a & b where $f(a)f(b) < 0$ } must contain only one root in interval

④

i	a	f(a)	b	f(b)	$p = \frac{a+b}{2}$	f(p)	rel err $= \frac{p_n - p_{n-1}}{2}$
1	3	+ sign enough	4	−	3.5	+5.001	1st iteration $\left\|\frac{p-a}{p}\right\|$

check $f(a)f(p)$: if both have same sign: choose $a = p$, $b = b$ ✓; else $a = a$, $b = p$

⑤ continue untile rel err $\le 10^{-3} = 0.001$ } 0.0000 ⇒ stop

# of iterations

bisection thm

$|p - p^*| \le 10^{-3}$

$\le \frac{b-a}{2^n}$

find $\boxed{\frac{b-a}{2^n} \le err}$ } $n = ?$

adv	disadv
* always cvgt	* complex roots ✗
* err always bdd	* discts ✗
	* $f(a)f(b) > 0$ ✗

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② fixed pt method

check $f(x) = 0$ for cvgs ⇒ $g(x) = x$ check $|g'(x)| < 1 \ \forall x \in [a,b]$

fixed pt ⇒ $g(x) = x$ ⇒ $x = ?$

existence: at least one

① $g \in C[a,b]$

② $g \in [a,b]$ → extremas (diff critical pt, end pts) $\forall x \in [a,b]$

uniqueness: only one

③ $g(x) \in (a,b)$ $C(a,b)$ } $f(x)$

④ $|g'(x)| \le K < 1$ (K: (+)ve) } $\forall x \in [a,b]$

* exponential fun strictly increasing $3^x, e^x$
* abs. extremas occurs at critical pts (end pts, $f'(x) = 0$)

find $g(x)$

$\boxed{x = g(x)} = x - f(x)$ —①

find $g(x)$

$\boxed{x = g(x)} = x - f(x)$ —①

or $f(x) = 0$ → subject x

$\boxed{|g'(x)| \le k < 1}$ (true) $\forall x \in [a,b]$ fixed pt th^m^ ⇒ cvg + $g(x) \in [a,b]$ maps to itself

lower the k, faster cvg

disadv
- not always cvg
- more iterations req'd for accurate ans

err term $\boxed{\frac{k^n}{1-k}|p_1 - p_0| \le err}$ $p_0 = a$ or b

$g(x_0)$ $g(x_0)$ then $p_1 = g(p_0)$

interval finding

$g(x) = x$ ⇒ $g'(x) < 1$ to cvg

can find x → $[a,b]$ } if root exist ✓ / if not ✗

③ newton's method

① initial guess p_0 (closer to p) cts on $[a,b]$

② $$\boxed{p_n \approx p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}}$$

$f'(p) \ne 0$ } cvgs ✓

- not fixed cvgs criteria
- totally depend on initial guess (closer to root)

$\boxed{f - f}$

differentiation

h = step size

forward, backward → } $O(h)$ err) better

- not too small h → round off err
- need to truncation err

first finite difference

centered / middle, equally spaced — $O(h^2)$ error but less truncation err

x_0-h, x_0, x_0+h → to get a nice formula

forward (x_0, x_0+h)

$$\boxed{f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}}$$

$\boxed{-\frac{h}{2}f''(\xi)}$ $O(h)$

backward (x_0-h, x_0)

$$\boxed{f'(x_0) \approx \frac{f(x_0) - f(x_0-h)}{h}}$$

$+ O(h)$

centered (x_0-h, x_0, x_0+h)

$$\boxed{f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}}$$

$+ O(h^2)$

high accuracy diff formula

fwd

$$\boxed{f'(x_0) \approx \frac{1}{h}\left[-\frac{3}{2}f(x_0) + 2f(x_0+h) - \frac{1}{2}f(x_0+2h)\right]}$$

$+O(h^2)$ improved

(x_0, x_0+h, x_0+2h)

Bwd

$$\boxed{f'(x_0) \approx \frac{1}{h}\left[\frac{3}{2}f(x_0) - 2f(x_0-h) + \frac{1}{2}f(x_0-2h)\right]}$$

$+O(h^2)$

(x_0-2h, x_0-h, x_0)

2nd order

fwd

$x_0 \quad x_0+h \quad x_0+2h$

$$\boxed{f''(x_0) \approx \frac{1}{h^2}\left[f(x_0+2h) - 2f(x_0+h) + f(x_0)\right]} + O(h)$$

bwd

$x_0-2h \quad x_0-h \quad x_0$

$$\boxed{f''(x_0) \approx \frac{1}{h^2}\left[f(x_0) - 2f(x_0-h) + f(x_0-2h)\right]} + O(h)$$

centered

$x_0-h \quad x_0 \quad x_0+h$

$$\boxed{f''(x_0) \approx \frac{1}{h^2}\left[f(x_0+h) - 2f(x_0) + f(x_0-h)\right]} + O(h^2)$$

Integration

trapizoidal rule 1st order

2 pts

$h = b-a$

$x_0 \quad x_1$

$a \quad b$

$$\boxed{\int_{x_0}^{x_1} f(x)\,dx \approx h\left[\frac{f(x_0)+f(x_1)}{2}\right]} + O(h^3)$$

Simpson's 1/3 2nd order

3pts

$x_0 \quad x_1 \quad x_2$

$a \quad b$

$h = \frac{b-a}{2}$

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]} + O(h^5)$$

Simpson's 3/8 3rd order

pts 4

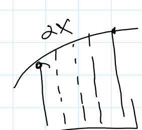
$h = \frac{b-a}{3}$

use when you have odd segments

$x_0 \quad x_1 \quad x_2 \quad x_3$

$a \quad b$

$$\boxed{\int_{x_0}^{x_3} f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]} + O(h^5)$$



composite trapezoidal

still $O(h^3)$ but small h

$\boxed{h = \frac{b-a}{n}}$

n segments/subintervals

step size

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]}$$

$\frac{h}{2}\left[f(x_0)+f(x_1)\right]$

$\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$

still $O(h^5)$ but small h

composite simpson's

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 2\sum_{i=1}^{\left(\frac{n}{2}-1\right)} f(x_{2i}) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + f(x_n)\right]}$$

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① diagonally dominant

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(interchange rows)

↳ diag $\neq 0$

↳ cvg ✓

② Jacobi || Gauss seidal

Stopping

$$\frac{\|X^k - X^{k-1}\|_\infty}{\|X^k\|_\infty} < \varepsilon$$