ss

Wednesday, May 25, 2022 7:53 AM

funⁿ of several var

$$\left[\begin{array}{l} D_f = \{(x,y) \in \mathbb{R}^2 \mid x > y,\ x \neq y\} \\ ran_f = \{\mathbb{R}^2\} \end{array}\right.$$

* $\frac{1}{\sqrt{x-y}} > 0$ sqrt ≥ 0 denominator $\neq 0$ $\ln(x > 0)$

Surfaces

$\boxed{\text{ellipsoid}}$

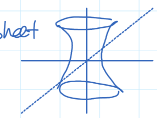
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$x^2 + y^2 + z^2 = 1$ } sphere

$\boxed{\text{hyperboloid}}$

wrap around axis

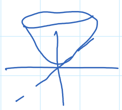
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \ \} \text{ one sheet}$$



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ \} \text{ two sheets}$$

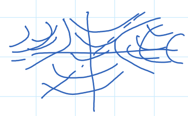
$\boxed{\text{elliptic paraboloid}}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} > 0$$



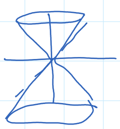
$\boxed{\text{hyperbolic paraboloid}}$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c} > 0$$



$\boxed{\text{cone}}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \ \} +/-$$



level curves

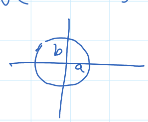
* $\underbrace{f(x,y)}_{} = x^2 + y^2$
  k change step by step } 2D (reduce 1 dimension and draw)
* diff. } critical pts

neighbourhoods

circular

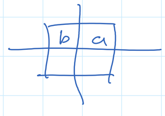
$\forall \varepsilon > 0\ \exists \delta > 0$

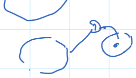
s.t. $\sqrt{(x-a)^2 + (y-b)^2} < \delta$



square

$|x-a| < \delta,\ |y-b| < \delta$



* ε-disk
* interior pt  for some $\varepsilon > 0$
* boundary pt  $\forall\ \varepsilon > 0$
* isolated pt  for some $\varepsilon > 0$
* subset → open (only interior pts)
  → closed (interior + boundary)

  ↓

  bd ⇒ $[a,b]$ ✓
  (a,b) ✗

Limits

$$\left\{\begin{array}{l} \forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } \forall (x,y) \in \mathbb{R}^2 \\ \sqrt{(x-a)^2 + (y-b)^2} < \delta \\ \Rightarrow |f(x,y) - L| \ \varepsilon \end{array}\right.$$

① plug

② try $(x,0)$
$(0,y)$

③ try to match order xy^2 x^4+y^4

② try $(x,0)$
$(0,y)$

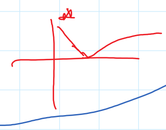
③ try to match order $\dfrac{xy^2}{\underbrace{x^2+y^4}_{x=y^2 m}}$ $\quad \dfrac{x^4+y^4}{x+y}$ } $y=-x+x^4$

④ * $|y| \le \sqrt{x^2+y^2} \Rightarrow \dfrac{|y|}{\sqrt{x^2+y^2}} \le 1$ } $|x| \le x^2+y^2$ × dnt hold

* $|x|<\delta$, $|y|<\delta$

* $\left|\sin\left(\frac{1}{y}\right)\right| \le 1$ or $|\sin(x)| \le |x|$

* $\ln|x| \le x^n$ in the nbhd $(0,0)$



$\ln|x^2+y^2| \le -\ln|x^2+y^2|$

repeated lim

* if rep lim exist but $\ne$ $\Rightarrow$ double lim DNE
* if doub lim exist & 2 rep lim exist $\Rightarrow$ rep lim $=$
* doub lim existence $\not\Rightarrow$ rep lim existence ($\not\Leftarrow$)
* Thm
  doub lim exist ✓ + one rep lim exist $x \to a$ $\forall y \in [a,b]$ $\Rightarrow$ rep lim exist ✓ $=$ double lim

$\not\Leftarrow$ $\dfrac{xy}{x+y}$

---

partial derivatives

$f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$ } $\boxed{f_x(a,b) = \lim_{h\to 0} \dfrac{f(a+h,b)-f(a,b)}{h}}$ } look for problematic pts $(0,0)$ and apply *

MVT multi var $\Rightarrow$ f_x, f_y exist , $\Delta x^2+\Delta y^2 < \delta^2$ & $\theta, \alpha \in (0,1)$

$\Delta f = \Delta x\, f_x(a+\theta\Delta x, b) + \Delta y\, f_y(a+\Delta x, b+\alpha\Delta x)$

Clairaut Thm

$f_{xy} = f_{yx}$ } not always true

cts 1st order, 2nd order partial derivatives

$\boxed{\text{but if } f_x \,\&\, f_y \text{ exist} \wedge \text{cts} \Rightarrow f_{xy} = f_{yx}}$
f_{xy} & f_{yx}

---

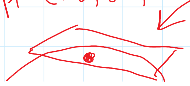
differentiation at (a,b)

$\boxed{\Delta z = \Delta x f_x(a,b) + \Delta y\, f_y(a,b)} + \underbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}_{\text{error}}$

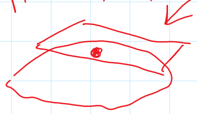
eq of tangent plane at pt (x_0, y_0, z_0)

linear approximation

$\boxed{dz = \Delta x f_x + \Delta y\, f_y}$



at pt (x_0, y_0, z_0)



$$\boxed{\Delta z = \Delta x f_x + \Delta y f_y}$$

Differentiation

Thm

check ① if f_x, f_y exist $\wedge$ cts around (a,b) $\Rightarrow$ f is diff'ble at (a,b)

② if one of f_x or f_y DNE $\Rightarrow$ f is not diff'ble

only pt that can be not defined

use

Example 1.4.6 $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2} & if\ (x,y)\neq(0,0)\\ 0 & if\ (x,y)=(0,0).\end{cases}$

Discuss the differentiability of f at $(0,0)$.

Cond 1

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^3/h^2}{h} = 1$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^3/h^2}{h} = 1$$

cond 2 check ③

① assume diff'ble

def by Δ $\quad \Delta x\, f_x(0,0) + \Delta y\, f_y(0,0) + \epsilon_1 \Delta x + \epsilon_2 \Delta y$; for $\epsilon_1, \epsilon_2 > 0$

$$\frac{\Delta x^3 + \Delta y^3}{\Delta x^2 + \Delta y^2} = \Delta x + \Delta y + \epsilon_1 \Delta x + \epsilon \Delta y$$

② let $\Delta x = r\cos\theta$

$\Delta y = r\sin\theta$

$$\frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = r\cos\theta + r\sin\theta + \epsilon_1 r\cos\theta + \epsilon_2 r\sin\theta$$

③ $\rightarrow$

$\cos^3\theta + \sin^3\theta = \cos\theta + \sin\theta$ } not true for $\forall\theta$

$\Rightarrow$ $f(x,y)$ not diff'ble

---

differential approximation

① $f(x,y) =$ write

② choose x_0, y_0 } easier closer } to calculate f_x, f_y, $f(x_0, y_0)$

③ find $\Delta x = x - x_0$

Δy

④ $\Delta z \approx \Delta x f_x + \Delta y f_y$ $\quad f_x = ?$ $\quad f_y = ?$

$f(x,y) \simeq f(x_0,y_0) + \Delta x f_x(x_0,y_0) + \Delta y f_y(x_0,y_0)$ //

Chain rule

$z = f(x,y)$

$$* \; z_r \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \Bigg\} \equiv \frac{\Delta z}{\partial r} = \frac{\Delta x}{\partial r} f_x + \frac{\Delta y}{\partial r} f_y$$

* $z_{ry} = \dfrac{\partial (z_r)}{\partial y} \to \checkmark$ product rule

* keep the final ans in parameters

$z = f(x, y)$ $\quad x = f(t, s) \quad y = f(t, s)$

$\uparrow \quad \uparrow$

$t, s \quad t, s \checkmark$

---

thm directional derivative

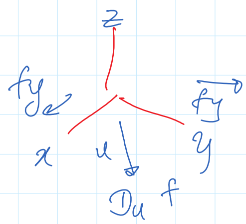
unit vector

$$\boxed{D_{u} f(x,y) = \nabla f(x,y) \cdot \hat{u}}$$

if f is diff'ble

nabla

$$\underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \bar{u}}_{\text{gradient vector along } \hat{u}} = f_x(x_0, y_0)\hat{i} + f_y(x_0, y_0)\hat{j}$$



projection of gradient along unit vector u

* $\boxed{\nabla f = \text{nabla} / \text{gradient of } f}$ ← vector fu^n

* $D_u f(x_0 y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$ } is a value (const) not a vector gradient *

---

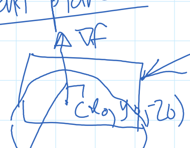
max directional der ⇒ ∇f direction $= \langle f_x, f_y \rangle$ ⇒ unit vector $\hat{\nabla f}$

max rate of change $= |\nabla f| |\hat{u}| \cancel{\cos\theta}$ 1

min directional der ⇒ $-\nabla f$

---

tangent plane



vector eqn

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \nabla f = 0$$

$\langle f_x, f_y, f_z \rangle$

parametric

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t \cdot \nabla f$$

symmetrical

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

Vector eqn normal line

Jacobi

$$J_p f = \begin{bmatrix} f_{1x_1} & f_{1x_2} & \cdots & f_{1x_n} \\ f_{2x_1} & \cdots & & \\ \vdots & & & \\ f_{nx_1} & & & \end{bmatrix}$$

→ one var

Hessian

one var →

$$D \left\{ H(f) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \right.$$

---

local min / max — global min / max

Thm: local min/max exist at (a,b) & f_x, f_y exist $\Rightarrow$ $f_x = f_y = 0$ critical pts

* critical pt → if $f_x, f_y = 0$ or f_x or f_y DNE

⇓

local / global min/max | saddle

2nd der test

① $f_x = f_y = 0$ critical pts

↓ check

② $\boxed{D\big|_{a,b} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(a,b)}}$

③ if $D > 0$: check f_{xx} or $f_{yy} > 0 \Rightarrow$ local min

easiest one

f_{xx} or $f_{yy} < 0 \Rightarrow$ local max

$D < 0 \Rightarrow$ saddle pt

$D = 0 \Rightarrow$ inconclusive (nothing can be said)

langrange multiplier

* remove z, make $f(x, y) \rightarrow$ do above

* extremas ① consider critical pts (check if they're in D) } abs max
② along boundaries eq^n (check max, min) } abs min