Fourier

periodic (no bd conditions stated)

odd/even

$f(-x) = -f(x)$

$g(-x) = g(x)$

$\phi(-x) = f(-x)\,g(-x) = -f(x)\,g(x)$ → $\phi(x)$ odd

euler's form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) \quad | \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \quad | \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

half: $-\ell$ 0 ℓ, period

Half: $a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)$ | $a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)$ | $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)$

full: $\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)$ — full range

half

identical: $\frac{2}{\ell}\int_{-\ell}^{\ell}$ ✓full twice } itf odd/even

dirichlet (check for valid apply)

1) $\int |f(x)| < \infty$

2) finite # max/minima

3) finite # discontinuity

half range

Fourier sin/series, cos

$0 - \ell$ } half range

parsvel's identity (higher order series gen)

$$\frac{1}{\ell}\int_{-\ell}^{\ell} [f(x)]^2 = \frac{a_0^2}{2} + \sum (a_n^2 + b_n^2)$$

normal fourier square: a_0, a_n, b_n

get from fourier (half range whatever stated)

$$\frac{2}{\ell}\int_0^{\ell} (f(x))^2$$

half range

mult both sides $\int f(x)$ & cal ^{2}

⇓

$\int fx^2 = ----$

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finite disc ⟹ drichlet ✓

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 1\,dx \;;\; -\pi \text{ to } 0 \;\; f(x) = 0$$

$$= \frac{x}{\pi}\Big|_0^{\pi} = \frac{\pi}{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\sin nx\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx$$

$$= \frac{1}{\pi}\left(-\cos nx\right)_0^{\pi}$$

$$= \frac{1}{\pi}(-\cos n\pi + 1)$$

$$= \frac{1}{\pi}(1 - \cos n\pi) \begin{cases} 0 & n = \text{odd} \\ \frac{1}{\pi}(1-(-1)^n) & n = \text{even} \end{cases}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{\pi}\sin nx \;;\; n = \text{even}$$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

cos → even $\boxed{\text{only}}$

even ⇒ $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{x^3}{3\pi}\Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}(\cos n\pi) + \frac{2}{n^3}(\cos \pi n - 1)\right]$$

$= 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\,\frac{2}{n^2}\left[\pi\cos \pi n - (-\pi \cos n\pi)\right]$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n$$

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$-\pi \le x \le \pi$

$x = 0$

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad (1)$$

$x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \dots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \dots \quad (2)$$

(1)+(2)

$$\frac{3\pi^2}{12_4} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \dots\right]$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x + \int_{\pi/2}^{\pi} \pi - x\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{\pi}{2}$$

$$\frac{a_0}{2} = \frac{\pi}{4} \checkmark$$