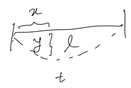
PDE — bd val probs

olve → product meth

Vibrating str'g — 1D wave eqn → superpos thm



$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

assume $z = X(x)Y(y)$

subst $\frac{\partial^2 XY}{\partial x^2} - 2\frac{\partial XY}{\partial x \partial y} + \frac{\partial XY}{\partial y^2} = 0$

$Y\ddot{X} - 2\dot{Y}\dot{X} + X\ddot{Y} = 0$

$\frac{\ddot{X} - 2\dot{X}}{X} = -\frac{\ddot{Y}}{Y} = \lambda$; const independent

$\ddot{X} - 2\dot{X} = \lambda X$ $\quad -\ddot{Y} = \lambda Y$

D operator: $D = \frac{\partial}{\partial x}$, $D^2 = \frac{\partial^2}{\partial x^2}$

$D^2X - 2DX = \lambda X$

$\int \frac{-\ddot{Y}}{\dot{Y}} = \int \lambda$

auxiliary eqn

$m^2 x - 2mx = \lambda x$

$-\ln|Y| = \lambda y + c_1$

$(m^2 - 2m - \lambda)x = 0$

$Y = e^{-\lambda y + c_1}$

$= e^{-\lambda y} C$

$m = \frac{+2 \pm \sqrt{4 + 4\lambda}}{2}$

$= 1 \pm \sqrt{1+\lambda}$

soln $X(x) = A_1 e^{(1+\sqrt{1+\lambda})x} + B_1 e^{(1-\sqrt{1+\lambda})x}$

$m_1 \neq m_2 \Rightarrow A_1 e^{m_1 x} + B_1 e^{m_2 x}$

$m_1 = m_2 \Rightarrow A_1 e^{m_1 x} + x e^{m_2 x}$

complex $\Rightarrow e^{\alpha x}[A_1 \cos\beta x + B_1 \sin\beta x]$

$m = \alpha \pm \beta x$

$Z = XY$

$= e^{-\lambda y} C [A_1 ----]$

$= e^{-\lambda y} [A \_\_ + B \_\_]$ $\quad CA_1 = A$, $CB_1 = B$

$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-2x}$

assume $Z = X(x)T(t)$

subst $\frac{\partial XT}{\partial x} = 2\frac{\partial XT}{\partial t} + XT$

$T\dot{X} = 2X\dot{T} + XT$

$\frac{\dot{X}}{X} = \frac{2\dot{T} + T}{T} = \lambda$; indep const

$\int \frac{\dot{X}}{X} = \int \lambda \quad - 2\int \frac{\dot{T}}{T} + \int 1 = \int \lambda$

$\ln|X| = \lambda x + c_1 \quad 2\ln|T| + T = \lambda T + d_1$

$X = e^{\lambda x + c_1} \quad T = e^{\frac{T(\lambda-1)}{2} + \frac{d_1}{2}}$

$= c e^{\lambda x} \quad = D e^{\frac{T(\lambda-1)}{2}}$

$u = XT = A e^{\lambda x + \frac{T(\lambda-1)}{2}}$

$u(x,0) = 6e^{-2x} = A e^{\lambda x}$

$A = 6$, $\lambda = -2$

$u = 6e^{-2x - \frac{3T}{2}}$ //

$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-x}$

assume $u = X(x)Y(y)$

subst $3\frac{\partial XY}{\partial x} + 2\frac{\partial XY}{\partial y} = 0$

$3Y\dot{X} + 2X\dot{Y} = 0$

$\frac{3\dot{X}}{X} = -\frac{2\dot{Y}}{Y} = \lambda$

$3\int \frac{\dot{X}}{X} = \int \lambda \quad -2\int \frac{\dot{Y}}{Y} = \int \lambda$

$3\ln|X| = \lambda x + c_1 \quad -2\ln|Y| = \lambda y + d_1$

$X = e^{\frac{\lambda x}{3} + c_3} \quad Y = e^{-\frac{\lambda y}{2} - \frac{d_1}{2}}$

$= c e^{\frac{\lambda x}{3}} \quad = d e^{-\frac{\lambda y}{2}}$

$u = A e^{\frac{\lambda x}{3} - \frac{\lambda y}{2}}$

$u(x,0) = 4e^{-x} = A e^{\frac{\lambda x}{3}}$

$A = 4$, $\frac{\lambda}{3} = -1$, $\lambda = -3$

$u = 4e^{-x - \frac{3y}{2}}$ //

$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

can change: initial conditions $t = 0$: $\dot{y}(x,0) = u_0$ ← initial v; $y(x,0) = 0$ ← of rest

bdd conditions: $x = 0$: $y(0,t) = 0$; $x = L$: $y(L,t) = 0$ } same for everything

$y = XT$

$X\ddot{T} = c^2 T \ddot{X}$

$\frac{\ddot{T}}{c^2 T} = \frac{\ddot{X}}{X} = \lambda$

$\lambda = 0$: $\ddot{T} = 0$, $\ddot{X} = 0$

$T = c_1 T + d_1$, $X = c_2 x + d_2$

$y = (c_1 T + d_1)(c_2 x + d_2)$

$\lambda > 0 \Rightarrow \lambda = m^2 > 0$

$\frac{\ddot{T}}{T} = \lambda$, $\frac{\ddot{X}}{X} = \lambda = m^2$

$D^2 T = m^2 T$

$T(D^2 - m^2) = 0$

$D = \pm mc$

$T = A_1 e^{mct} + B_1 e^{-mct}$, $X = A_2 e^{mx} + B_2 e^{-mx}$

$y = (A_1 e^{mct} + B_1 e^{-mct})(A_2 e^{mx} + B_2 e^{-mx})$

$\lambda < 0 \Rightarrow \lambda = -m^2$

$\frac{\ddot{T}}{T} = -m^2 c^2$, $\frac{\ddot{X}}{X} = -m^2$

$\ddot{T} + m^2 c^2 T = 0$, $\ddot{X} + m^2 X = 0$

SHM $\omega = mc$

$X = A_2 \cos mx + B_2 \sin mx$

$\Rightarrow T = A_1 \cos mct + B_1 \sin mct$

$y = (A\cos mx + B\sin mx)(C\cos mct + d\sin mct)$

soln periodic ⇐ string vibration each point repeats with t

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initial conditions: $t = 0$: $y(x,0) = a\sin\frac{n\pi x}{L}$, $\dot{y}(x,0) = 0$

bdd: $y(0,t) = 0$, $y(L,t) = 0$

$y = (A\cos mx + B\sin mx)(C\cos mct + D\sin mct)$

$y(0,t) = 0$

$0 = (A + 0)(C\cos mct + D\sin mct)$ — indep from x $\Rightarrow \neq 0$

$\Rightarrow A = 0$

$\Rightarrow y = B\sin mx (C\cos mct + D\sin mct)$

$y(L,t) = 0 = B\sin mL (C\cos mct + D\sin mct)$ $\neq 0$

$\sin mL = 0$

$\sin mL = \sin 0$

$mL = n\pi$

$m = \frac{n\pi}{L}$

$y = B\sin\left(\frac{n\pi x}{L}\right)\left[C\cos mct + D\sin mct\right]$ — differentiate

$\dot{y}(x,0) = 0 = B\sin\frac{n\pi x}{L}\left[C_1 \sin mc(0) + D_1 \cos mc(0)\right]$

$B \neq 0$, $\sin \neq 0$, $\cos = 1$

$\Rightarrow D = 0$

$y = B\sin\left(\frac{n\pi x}{L}\right) C_1 \cos\frac{mct}{L}$ ($m = \frac{n\pi}{L}$) $\Rightarrow$ y_1 is a soln, y_2 is a soln, ... y_n ✓ $\Rightarrow$ Σ soln by superposition thm

$y(x,t) = \sum C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$; $n = 0, 1, 2, \ldots$

whatever is 0 first

$y(x,0) = \sum c_n \sin\frac{n\pi x}{L} = a\sin\frac{n\pi x}{L}$

$a = c_1 = c_2 = \cdots$

$\Rightarrow y(x,t) = a\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$

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initial velocity ≠ differentiate

$\frac{L u_0}{n\pi c}$ = fourier sin series

$a_n = a_0 = 0$

$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$= \frac{2}{L}\int_0^L \frac{L u_0}{n\pi c}\sin\frac{n\pi x}{L}\,dx$

$= \frac{2u_0}{n\pi c}\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)_0^L$

$= \frac{2u_0 L}{(n\pi)^2 c}\left[-\cos n\pi + 1\right]$ { 0 odd; $\frac{2u_0 L}{(n\pi)^2}(1 - (-1)^n)\sin\frac{n\pi x}{L} = \sum a_n \sin\frac{n\pi x}{L}$ =