தொண்டைமானாறு வெளிக்கள நிலையம் நடாத்தும்

நான்காம் தவணைப் பரீட்சை - 2022

Field Work Centre, Thondaimanaru

4th Term Examination - 2022

Grade - 13 (2022) — இணைந்த கணிதம் - I — Marking Scheme

PART A

1. Let $f(n) = 3^{2n} + 7$.

For $n=1$, $f(1) = 3^2+7 = 16 = 8\times 2$

$\therefore$ the result is true for $n=1$. 5

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n=p$.

$f(p) = 3^{2p} + 7 = 8k,\ k \in \mathbb{Z}$ 5

For $n = p+1$,

$f(p+1) = 3^{2p+2} + 7$

$= 9 \times 3^{2p} + 7$

$= 9(8k-7) + 7$ 5

$= 8(9k-7),\ 9k-7 \in \mathbb{Z}$ 5

Hence if the result is true for $n=p$ it is also true for $n=p+1$. We have already proved that the result is true for $n=1$. By the Principle of Mathematical Induction the result is true for all $n \in \mathbb{Z}^+$. 5

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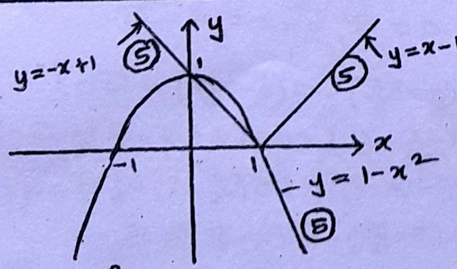
2.



$4 - x^2 > 2|x-2|$

$4 - 4\left(\frac{x}{2}\right)^2 > 4\left|\frac{x}{2} - 1\right|$ 5

$1 - \left(\frac{x}{2}\right)^2 > \left|\frac{x}{2} - 1\right|$

$0 < \frac{x}{2} < 1 \Rightarrow 0 < x < 2$ 5

$\boxed{25}$

3. $f(x) = x^3 + 4x^2 + ax + b$

$f(1) = 3$ 5

$\Rightarrow 1 + 4 + a + b = 3$ 5

$\Rightarrow a + b = -2$ —— ①

$f(2) = 7$ 5

$\Rightarrow 8 + 16 + 2a + b = 7$ 5

$\Rightarrow 2a + b = -17$ —— ②

①, ② $\Rightarrow a = -15,\ b = 13$ 5

$\boxed{25}$

4. $\lim\limits_{x\to 0} \dfrac{\sqrt{\tan x + 1} - \sqrt{\sin x + 1}}{x^3}$

$= \lim\limits_{x\to 0} \dfrac{\tan x + 1 - (\sin x + 1)}{x^3\left(\sqrt{\tan x + 1} + \sqrt{\sin x + 1}\right)}$ 5

$= \lim\limits_{x\to 0} \dfrac{\frac{\sin x}{\cos x} - \sin x}{x^3\left(\sqrt{\tan x + 1} + \sqrt{\sin x + 1}\right)}$ 5

$= \lim\limits_{x\to 0} \dfrac{\sin x(1 - \cos x)}{x^3 \cos x\left(\sqrt{\tan x + 1} + \sqrt{\sin x + 1}\right)}$

$= \lim\limits_{x\to 0} \dfrac{\sin x \cdot 2\sin^2\frac{x}{2}}{x^3 \cos x\left(\sqrt{\tan x + 1} + \sqrt{\sin x + 1}\right)}$ 5

$= \lim\limits_{x\to 0} \dfrac{\sin x}{x}\left(\lim\limits_{\frac{x}{2}\to 0} \dfrac{\sin\frac{x}{2}}{\left(\frac{x}{2}\right)}\right)^2 \lim\limits_{x\to 0} \dfrac{1}{2\cos x\left(\sqrt{\tan x + 1} + \sqrt{\sin x + 1}\right)}$ 5

$= 1 \times 1^2 \times \dfrac{1}{2(2)}$

$= \dfrac{1}{4}$ 5

$\boxed{25}$

5. $\sin^{-1}x + \cos^{-1}y = \frac{3\pi}{4}$

differentiat w.r.t. x, we get

$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$ (5)

$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} + \sqrt{1-y^2} = 0$

$\Rightarrow \sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{1}{2\sqrt{1-x^2}}(-2x) + \frac{1}{2\sqrt{1-y^2}}(-2y)\frac{dy}{dx} = 0$ (10)

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$ (5)

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ (5)

$\boxed{25}$

6. $x = at^2 \qquad y = at^3$

$\frac{dx}{dt} = 2at$ (5) $\qquad \frac{dy}{dt} = 3at^2$

$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{3at^2}{2at} = \frac{3t}{2}$

$\therefore$ the gradient of the normal at $p = -\frac{2}{3t_0}$ (5)

The required eq^n is

$y - at_0^3 = -\frac{2}{3t_0}(x - at_0^2)$

$3t_0y - 3at_0^4 = -2x + 2at_0^2$

$\Rightarrow 2x + 3t_0y - at_0^2(3t_0^2+2) = 0$ (5)

when $y=0$, $x = \frac{at_0^2(3t_0^2+2)}{2}$

when $x=0$, $y = \frac{at_0(3t_0^2+2)}{3}$

$\left|\frac{at_0^2(3t_0^2+2)}{2}\right| = \left|\frac{at_0(3t_0^2+2)}{3}\right|$ (5)

$|t_0| = \frac{2}{3} \Rightarrow t_0 = \pm\frac{2}{3}$

$\left(\frac{4a}{9}, \pm\frac{8a}{27}\right)$ (5)

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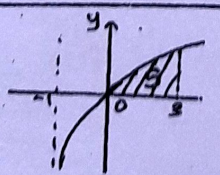
7.



Area $= \int_0^3 \frac{x}{\sqrt{x+1}}\,dx$ (5)

$= \int_0^3 \frac{x+1-1}{\sqrt{x+1}}\,dx$

$= \int_0^3 \left(\sqrt{x+1} - \frac{1}{\sqrt{x+1}}\right)dx = \left[\frac{(x+1)^{3/2}}{3/2} - \frac{(x+1)^{1/2}}{1/2}\right]_0^3$

(5) $= \frac{8}{3}$ square units

Volume $= \int_0^3 \pi\frac{x^2}{x+1}\,dx$ (5)

$= \pi\int_0^3 \frac{x^2-1+1}{x+1}\,dx$

$= \pi\int_0^3\left(x-1+\frac{1}{x+1}\right)dx$

$= \pi\left[\frac{x^2}{2} - x + \ln|x+1|\right]_0^3$ (5)

$= \frac{\pi}{2}(3-4\ln 2).$

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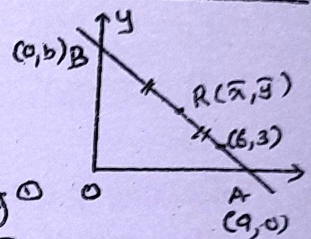
8. Let $R \equiv (\bar{x}, \bar{y})$



Equation of AB is

$\frac{x}{a} + \frac{y}{b} = 1$

$(6,3)$ // $\frac{6}{a} + \frac{3}{b} = 1$ (5) — (1)

(5) $\frac{a}{2} = \bar{x}, \quad \frac{b}{2} = \bar{y}$ (5)

$\frac{6}{2\bar{x}} + \frac{3}{2\bar{y}} = 1$ (5) $\Rightarrow 3\bar{x} + 6\bar{y} - 2\bar{x}\bar{y} = 0$

$\bar{x} \equiv x,\ \bar{y} \equiv y \Rightarrow 3x + 6y - 2xy = 0$ (5)

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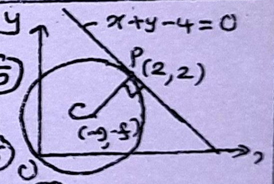
9. $S \equiv x^2+y^2+2gx+2fy+c=0$



$(0,0)$ // $c = 0$ — (1) (5)

$(2,2)$ // $4+4+4g+4f+c=0$

$g+f=-2$ — (2) (5)

$\frac{2+f}{2+g}\times -1 = -1 \Rightarrow g = f$ — (3) (5)

(1),(2),(3) $\Rightarrow g = f = -1$ (5)

$S \equiv x^2+y^2-2x-2y=0$ (5)

$\boxed{25}$

10. $4\sin(x+\frac{\pi}{9})\sin(x-\frac{\pi}{18})$

$= 2\left[\cos\frac{\pi}{6} - \cos(2x+\frac{\pi}{18})\right]$ (5)

$= \sqrt{3} - 2\cos(2x+\frac{\pi}{18})$ (5)

$= a + b\cos(2x+\alpha)$

$a = \sqrt{3},\ b = -2,\ \alpha = \frac{\pi}{18}$

$4\cos(x-\frac{5\pi}{9})\cos(x-\frac{7\pi}{18}) = \sqrt{3}-1$

$4\cos(x+\frac{\pi}{18}-\frac{\pi}{2})\cos(x+\frac{\pi}{9}-\frac{\pi}{2}) = \sqrt{3}-1$ (5)

$4\cos(\frac{\pi}{2}-(x-\frac{\pi}{18}))\cos(\frac{\pi}{2}-(x+\frac{\pi}{9})) = \sqrt{3}-1$

$4\sin(x-\frac{\pi}{18})\sin(x+\frac{\pi}{9}) = \sqrt{3}-1$

$\sqrt{3} - 2\cos(2x+\frac{\pi}{18}) = \sqrt{3}-1$ (5)

$\cos(2x+\frac{\pi}{18}) = \frac{1}{2} = \cos\frac{\pi}{3}$

$2x+\frac{\pi}{18} = 2n\pi \pm \frac{\pi}{3}$; $n\in\mathbb{Z}$

$x = n\pi + \frac{5\pi}{36}$, $n\in\mathbb{Z}$ or $x = n\pi - \frac{7\pi}{36}$, $n\in\mathbb{Z}$ (5)

$\boxed{25}$

11]

a) $x^2 - 2kx + (k-4)^2 = 0$

$\Delta = (-2k)^2 - 4(1)(k-4)^2$ (5)

$= 4\{k^2 - (k-4)^2\}$

$= 4\{8k - 16\}$

$= 32(k-2)$ (5)

> 0 (5) $(\because k > 2)$

$\therefore$ the roots are real and distinct (5) $\boxed{20}$

$\alpha + \beta = 2k$

$\alpha\beta = (k-4)^2$ (5)

$\alpha > 0$ and $\beta > 0$

$\Leftrightarrow \alpha + \beta > 0$ and $\alpha\beta > 0$ (5)

$\Leftrightarrow 2k > 0$ and $(k-4)^2 > 0$ (5)

$\Leftrightarrow k > 2$ and $k \neq 4$ (5) $(\because k > 2)$

$\boxed{20}$

or $2 < k < 4$ or $k > 4$

$2 < k < 4 \Rightarrow \alpha$ and β are both positive

$\sqrt{\alpha} \cdot \sqrt{\beta} = \sqrt{\alpha\beta}$

$= \sqrt{(k-4)^2}$

$= |k-4|$ (5)

$= (4-k)$ (5)

$(\because 2 < k < 4)$

$(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta}$ (5)

$= 2k + 2(4-k)$

$= 8$ (5)

$\sqrt{\alpha} + \sqrt{\beta} = 2\sqrt{2}$ (5)

$\boxed{25}$

The required eq^n is

$x^2 - (\sqrt{\alpha} + \sqrt{\beta})x + \sqrt{\alpha} \cdot \sqrt{\beta} = 0$ (5)

$x^2 - 2\sqrt{2}x + (4-k) = 0$ (5) — $(*)$

take $y = \frac{2}{x}$ (5)

$x = \sqrt{\alpha} \Rightarrow y = \frac{2}{\sqrt{\alpha}}$

$x = \sqrt{\beta} \Rightarrow y = \frac{2}{\sqrt{\beta}}$ (5)

put $x = \frac{2}{y}$ in $(*)$

$\left(\frac{2}{y}\right)^2 - 2\sqrt{2}\left(\frac{2}{y}\right) + (4-k) = 0$

$(4-k)y^2 - 4\sqrt{2}y + 4 = 0$ (5)

$\therefore$ The equation whose roots $\frac{2}{\sqrt{\alpha}}, \frac{2}{\sqrt{\beta}}$ is (5)

$(4-k)x^2 - 4\sqrt{2}x + 4 = 0$ $\boxed{30}$

b) $ax + by + c = 0$ $\quad \alpha, \beta$

$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$

$a'x + b'y + c' = 0 \quad \alpha, \beta$

$\alpha + \beta = -\frac{b'}{a'}, \quad \alpha\beta = \frac{c'}{a'}$ (5)

$\Rightarrow -\frac{b}{a} = -\frac{b'}{a'}, \quad \& \quad \frac{c}{a} = \frac{c'}{a'}$ (5)

$\Rightarrow \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$ (5) $\boxed{15}$

$\alpha(x^2 - \alpha x + \beta) + \beta(x^2 - \beta x + \alpha) = 0$

$(\alpha + \beta)x^2 - (\alpha^2 + \beta^2)x + 2\alpha\beta = 0$ — ①

$3x^2 - 10x + 8 = 0$ — ② (5)

①,② have same roots

$\Rightarrow \frac{\alpha + \beta}{3} = \frac{-(\alpha^2 + \beta^2)}{-10} = \frac{2\alpha\beta}{8} = k$ (say) (5)

$\alpha + \beta = 3k, \quad \alpha^2 + \beta^2 = 10k, \quad \alpha\beta = 4k$ (5)

$\alpha^2 + \beta^2 = 10k$

$(\alpha + \beta)^2 - 2\alpha\beta = 10k$ (5)

$9k^2 - 2(4k) = 10k$

$k(k-2) = 0$

$k - 2 = 0 \quad (\because k \neq 0)$

$k = 2$ (5)

$\alpha + \beta = 6 \quad \alpha\beta = 8$ (5)

The required eq^n is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ (5)

$x^2 - 6x + 8 = 0$ (5) $\boxed{40}$

12. (a)

$$f(x) = (x-a)(x-b)\phi(x) + px + q \quad 10$$

Putting $x = a$ and $x = b$, we get

$f(a) = pa + q$ —— (1) 5

$f(b) = pb + q$ —— (2) 5

$(1) - (2) \Rightarrow f(a) - f(b) = p(a-b)$ 5

$\Rightarrow p = \dfrac{f(a) - f(b)}{a-b}$ 5

$b \times (1) - a \times (2) \Rightarrow bf(a) - af(b) = bq - aq$ 5

$\Rightarrow q = \dfrac{bf(a) - af(b)}{b-a}$

$= \dfrac{af(b) - bf(a)}{a-b}$ 5 $\boxed{40}$

$\dfrac{f(a) - f(b)}{a-b} = \dfrac{f(a) - f(c)}{a-c}$. 5

$\Rightarrow (a-c)f(a) - (a-c)f(b) = (a-b)f(a) - (a-b)f(c)$

$\Rightarrow -cf(a) - (a-c)f(b) = -bf(a) - (a-b)f(c)$ 5

$\Rightarrow (b-c)f(a) + (c-a)f(b) + (a-b)f(c) = 0$ 5

$P(x) = 3x^4 + kx^3 + 2$

$a = 0, b = 1, c = -2$

$P(0) = 2$ 5, $P(1) = 5 + k$ 5, $P(-2) = 58 - 8k$ 5

$(b-c)f(a) + (c-a)f(b) + (a-b)f(c) = 0$

$3(2) + (-2)(5+k) + (-1)(58-8k) = 0$ 5

$6k = 54$ 5

$k = 9$ $\boxed{40}$

$P(x) = 3x^4 + 9x^3 + 2$

$3x^4 + 9x^3 + 2 \equiv (x^2+1)(Ax^2 + Bx + C) + Dx + E$ 10

Comparing the coefficient of power of x;

$$\left.\begin{aligned} x^4 &: 3 = A \\ x^3 &: 9 = B \\ x^2 &: 0 = C + A \\ x &: 0 = B + D \\ x^0 &: 2 = C + E \end{aligned}\right\} \quad 10 \quad \begin{aligned} A &= 3 \\ B &= 9 \\ C &= -3 \\ D &= -9 \\ E &= 5 \end{aligned}$$

Quotient $= 3x^2 + 9x - 3$ 5

Remainder $= -9x + 5$ 5 $\boxed{30}$

(b) $g(x) = 3x^2 - 6kx + 5k^2 - 2$

$= 3(x^2 - 2kx) + 5k^2 - 2$ 5

$= 3(x-k)^2 + 2(k^2-1)$ 5

$(g(x))_{min} = 0 + 2(k^2-1) = 2(k^2-1)$ 5

$[\because (x-k)^2 \geq 0]$ $\boxed{15}$

(i) $(g(x))_{min} > 0$

$\Rightarrow 2(k^2-1) > 0$ 5

$\Rightarrow (k-1)(k+1) > 0$ 5

$\Rightarrow k < -1$ or $k > 1$ 5 $\boxed{15}$

(ii) $(g(x))_{min} = 0$ 5

$\Rightarrow 2(k^2-1) = 0$ 5

$\Rightarrow k = \pm 1$ $\boxed{10}$

13. (a) $f(x) = \dfrac{(x-1)^3}{(x+1)^2}$

$$f'(x) = \frac{(x+1)^2 3(x-1)^2 - (x-1)^3 2(x+1)}{(x+1)^4} \quad 10$$

$$= \frac{(x-1)^2[3(x+1) - 2(x-1)]}{(x+1)^3} \quad 5$$

$$= \frac{(x-1)^2(x+5)}{(x+1)^3} \quad 5$$

$\boxed{20}$

$f'(x) = 0 \Leftrightarrow x = 1$ or $x = -5$ 5

	$-\infty < x < -5$	$-5 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
sign of $f'(x)$	$(+)$	$(-)$	$(+)$	$(+)$
$f(x)$ is	Increasing	Decreasing	Increasing	Increasing
	5	5	5	5

$\therefore f(x)$ is increasing on $(-\infty, -5]$ and $(-1, \infty)$ and dereasing on $[-5, -1)$ 5

Turning point : $(-5, -\frac{27}{2})$ is a local maximum 5

Inflection point : $(1, 0)$ 5

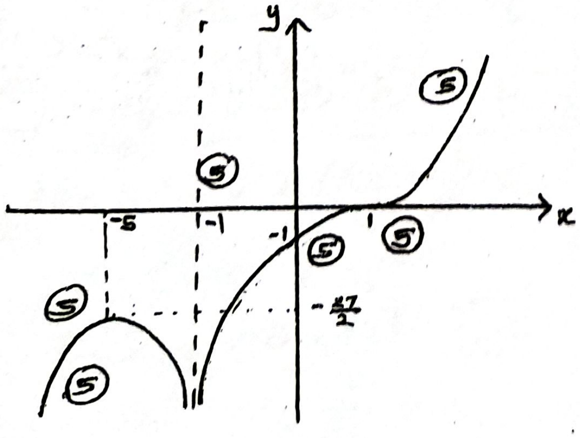
$\boxed{40}$

When $x = 0$, $y = -1$ $\quad (0, -1)$

$x = -1$ is a vertical asymptote 5

$$\lim_{x \to \infty} \frac{(x-1)^3}{(x+1)^2} = \lim_{x \to \infty} \frac{x(1-\frac{1}{x})^3}{(1+\frac{1}{x})^2} = \infty \quad 5$$

$$\lim_{x \to -\infty} \frac{(x-1)^3}{(x+1)^2} = \lim_{x \to -\infty} \frac{x(1-\frac{1}{x})^3}{(1+\frac{1}{x})^2} = -\infty \quad 5$$



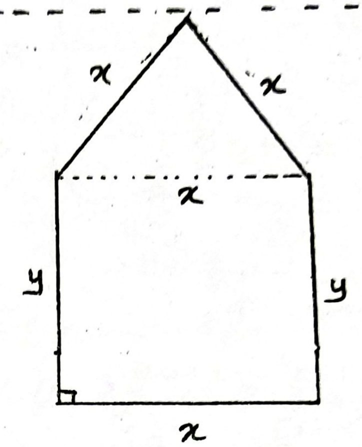
$f(x) - |f(x)| < 0$ 5

$\Leftrightarrow f(x) < 0$

$\Leftrightarrow x < -1$ or $-1 < x < 1$ 5

$\boxed{55}$

(b)



$3x + 2y = a$

$y = \dfrac{a - 3x}{2}$ 5

$y > 0 \Rightarrow \dfrac{a-3x}{2} > 0$ 5

$\Rightarrow x < \dfrac{a}{3}$

$\therefore 0 < x < \dfrac{a}{3}$

$A = xy + \frac{1}{2}x^2 \sin 60^\circ$ 5

$= \dfrac{x(a-3x)}{2} + \dfrac{1}{2}x^2\dfrac{\sqrt{3}}{2}$ 5

$A = \frac{1}{2}ax - \left(\frac{6-\sqrt{3}}{4}\right)x^2$

$\dfrac{dA}{dx} = \frac{1}{2}a - \left(\frac{6-\sqrt{3}}{4}\right)2x$

$= -\left(\frac{6-\sqrt{3}}{2}\right)\left(x - \frac{a}{6-\sqrt{3}}\right)$

$\dfrac{dA}{dx} = 0 \Leftrightarrow x = \dfrac{a}{6-\sqrt{3}}$ 5

For $0 < x < \dfrac{a}{6-\sqrt{3}}$, $\dfrac{dA}{dx} > 0$ and 5

for $\dfrac{a}{6-\sqrt{3}} < x < \dfrac{a}{3}$, $\dfrac{dA}{dx} < 0$

$\therefore A$ is maximum when $x = \dfrac{a}{6-\sqrt{3}}$

5

$\boxed{35}$

14. (a)

$$3x^4+2x^3+23x^2+7x+40 \equiv Ax(x-1)+B(x-1)(x^2+4)+C(x^2+4)^2$$

$x^4: \quad 3 = C$

$x^3: \quad 2 = B$

$x^2: \quad 23 = A - B + 8C$ (10)

$x: \quad 7 = -A + 4B$

$x^0: \quad 40 = -4B + 16C$

$A = 1$ (5)
$B = 2$ (5)
$C = 3$ (5)

$$\frac{3x^4+2x^3+23x^2+7x+40}{(x^2+4)^2(x-1)}$$

$$= \frac{x(x-1)+2(x-1)(x^2+4)+3(x^2+4)^2}{(x^2+4)^2(x-1)}$$

$$= \frac{x}{(x^2+4)^2} + \frac{2}{(x^2+4)} + \frac{3}{x-1} \quad (5)$$

$$\int \frac{3x^4+2x^3+23x^2+7x+40}{(x^2+4)^2(x-1)}\,dx$$

$$= \int \frac{x}{(x^2+4)^2}\,dx + 2\int \frac{1}{x^2+4}\,dx + 3\int \frac{1}{x-1}\,dx \quad (5)$$

$$= \frac{1}{2}\,\frac{(x^2+4)^{-1}}{-1} \ (5) + \frac{2}{2}\tan^{-1}\frac{x}{2} \ (5) + 3\ln|x-1| + c \ (5)$$

$$= -\frac{1}{2(x^2+4)} + \tan^{-1}\frac{x}{2} + 3\ln|x-1| + C$$

$\boxed{50}$

(b) (i) Let $2x = y$. $\frac{dy}{dx} = 2$

$$\int_0^{\frac{\pi}{2}} \ln(\sin 2x)\,dx = \int_0^{\pi} \ln(\sin y)\frac{1}{2}dy \quad (5)$$

$$= \frac{1}{2}\int_0^{\pi} \ln(\sin x)\,dx \quad (5)$$

(ii) $\int_0^{\pi} \ln(\sin x)\,dx$

$$= \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} \ln(\sin x)\,dx \quad (5)$$

Let $\pi - x = u$

$$\frac{du}{dx} = -1$$

$$\int_{\frac{\pi}{2}}^{\pi} \ln(\sin x)\,dx = \int_{\frac{\pi}{2}}^{0} \ln(\sin(\pi-u))(-du) \quad (5)$$

$$= \int_0^{\frac{\pi}{2}} \ln(\sin u)\,du \quad (5)$$

$$= \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx \quad (5)$$

$$\int_0^{\pi} \ln(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx \quad (5)$$

$\boxed{35}$

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx \quad (5)$$

$$= \int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx \quad (5)$$

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx + \int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx \quad (5)$$

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin x\cos x)\,dx \ (5) = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right)dx \ (5)$$

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}} \ln 2\,dx \quad (5)$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin 2x)\,dx - \frac{\pi}{4}\ln 2 \quad (5)$$

$$= \frac{1}{4}\int_0^{\pi} \ln(\sin x)\,dx - \frac{\pi}{4}\ln 2 \quad (5)$$

$$= \frac{1}{4}(2)I - \frac{\pi}{4}\ln 2 \quad (5)$$

$$I = -\frac{\pi}{2}\ln 2 \quad (5)$$

$\boxed{50}$

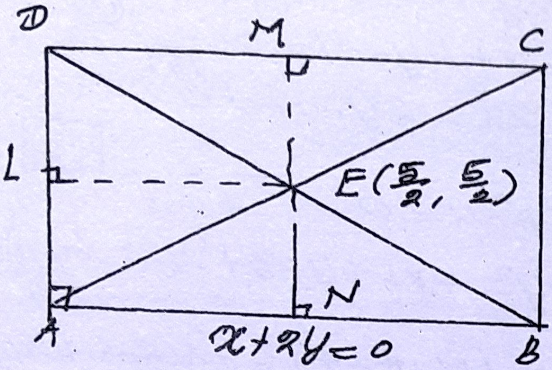
(c) $$\int x\ln x\,dx = \ln x\,\frac{x^2}{2} - \int \frac{x^2}{2}\,\frac{1}{x}\,dx \quad (10)$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C \quad (5)$$

$\boxed{15}$

[15]

a) Theory $\boxed{20}$



$$EN = \left| \frac{\frac{5}{2} + 2\left(\frac{5}{2}\right)}{\sqrt{1^2+2^2}} \right| \quad (5)$$

$$= \frac{\frac{15}{2}}{\sqrt{5}} = \frac{3\sqrt{5}}{2} \quad (5)$$

We can write DC

$$x+2y+k=0 \quad (5)$$

$$EN = EM = \left| \frac{\frac{5}{2}+2\left(\frac{5}{2}\right)+k}{\sqrt{1^2+2^2}} \right| \quad (10)$$

$$\frac{3\sqrt{5}}{2} = \frac{|15+2k|}{2\sqrt{5}}$$

$$15 = |15+2k|$$

$$15+2k = \pm 15 \quad (5)$$

$$\left.\begin{array}{l} (+) \Rightarrow k=0 \\ (-) \Rightarrow 2k=-30 \\ \qquad k=-15 \end{array}\right\} (10)$$

$\underline{DC}$

$$x+2y-15=0 \quad (5) \qquad \boxed{45}$$

$$AB = 2AD = 2(2EN)$$

$$= 4 \times \frac{3\sqrt{5}}{2}$$

$$= 6\sqrt{5}$$

$$EL = \frac{1}{2} AB = 3\sqrt{5} \quad (5)$$

We can write AD

$$2x-y+\lambda=0 \quad (5)$$

$$3\sqrt{5} = EL = \left| \frac{2\left(\frac{5}{2}\right) - \frac{5}{2} + \lambda}{\sqrt{2^2+(-1)^2}} \right| \quad (5)$$

$$3\sqrt{5} = \frac{|5+2\lambda|}{2\sqrt{5}}$$

$$30 = |5+2\lambda|$$

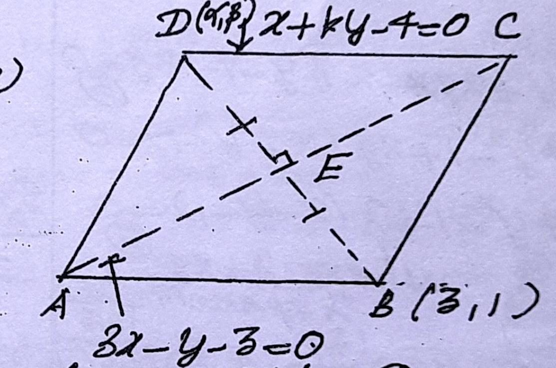
$$5+2\lambda = \pm 30 \quad (5)$$

$$\left.\begin{array}{l} (+) \Rightarrow \lambda = \frac{25}{2} \\ (-) \Rightarrow \lambda = -\frac{35}{2} \end{array}\right\} (5)$$

$$AD \equiv 2x-y+\frac{25}{2} \Rightarrow BC \equiv 2x-y-\frac{35}{2}=0 \quad (5)$$

$$AD \equiv 2x-y-\frac{35}{2}=0 \Rightarrow BC \equiv 2x-y+\frac{25}{2}=0 \quad (5)$$

b) $\boxed{35}$



$$E \equiv \left(\frac{\alpha+3}{2}, \frac{\beta+1}{2}\right) \quad (5)$$

E lies on AC

$$3\left(\frac{\alpha+3}{2}\right) - \left(\frac{\beta+1}{2}\right) - 3 = 0 \quad (5)$$

$$3\alpha-\beta+2=0 \quad (5)$$

$$m_{AC} \times m_{BD} = -1 \quad (5)$$

$$\left(\frac{\beta-1}{\alpha-3}\right)(3) = -1 \quad (5)$$

$$\alpha+3\beta-6=0 \quad (5)$$

$$\Rightarrow \alpha=0 \quad \beta=2 \quad D \equiv (0,2) \quad (5)$$

$$x+ky-4=0$$

$$(0,2) \Rightarrow 0+2k-4=0$$

$$k=2 \quad (5)$$

$\underline{AB}$

$$x+2y+k_1=0 \quad (5)$$

$$(3,1) \Rightarrow 3+2+k_1=0$$

$$k_1=-5$$

$$x+2y-5=0 \quad (5)$$

$\boxed{50}$

Q16] a) (i) Theory $\boxed{20}$

(ii) Theory $\boxed{20}$

b) $S \equiv x^2+y^2+2gx+2fy+c=0$ — (*)

$(1,0) \Rightarrow 1+2g+c=0$ (5)

$c=-1-2g$

Centre $\equiv (-g,-f)$ (5)

which lies on $x+y=0$

$\Rightarrow -g-f=0$

$f=-g$ (5)

(*) $\Rightarrow$

$x^2+y^2+2gx-2gy-1-2g=0$ (5)

$x^2+y^2-1+2g(x-y-1)=0$

$x^2+y^2-1+\lambda(x-y-1)=0$ (5)

where $\lambda=2g$

λ - parameter $\boxed{25}$

$S \equiv x^2+y^2+2\left(\frac{\lambda}{2}\right)x+2\left(-\frac{\lambda}{2}\right)y+(-1-\lambda)=0$ — ①

$S_1 \equiv x^2+y^2+2(1)x+2(-1)y+(-11)=0$ — ②

① & ② intersect orthogonally

$2\left\{\left(\frac{\lambda}{2}\right)(1)+\left(-\frac{\lambda}{2}\right)(-1)\right\}=-1-\lambda-11$ (10)

$2\lambda=-\lambda-12$

$\lambda=-4$ (5)

$\therefore S \equiv x^2+y^2-4x+4y+3=0$ (5)

$\boxed{20}$

The chord of contact of the tangents to S from $P \equiv (0,4)$

$0x+4y-2(0+x)+2(4+y)+3=0$ (10)

$-2x+6y+11=0$ (5)

$u \equiv 2x-6y-11=0$

$\boxed{15}$

$S+\lambda u=0$

$x^2+y^2-4x+4y+3+\lambda(2x-6y-11)=0$ (10)

$(1,1) \Rightarrow 1+1-4+4+3+\lambda(2-6-11)=0$

$5+\lambda(-15)=0$

$\lambda=\frac{1}{3}$ (5)

$x^2+y^2-4x+4y+3+\frac{1}{3}(2x-6y-11)=0$ (5)

$S_2 \equiv 3(x^2+y^2)-10x+6y-2=0$ (5) $\boxed{25}$

$S_3 \equiv x^2+y^2-2x-2y+c=0$

Centre $\equiv (1,1)$ (5)

Common chord of S_2 and S_3 is

$4x-12y+3c+2=0$ (10)

$(1,1) \Rightarrow 4\times1-12\times1+3c+2=0$ (5)

$-8+3c+2=0$

$3c=6$

$c=2$ (5)

$\boxed{25}$

17 (a)

$\cos(A+B) = \cos A\cos B - \sin A\sin B$ 5

$\cos(A-B) = \cos A\cos B + \sin A\sin B$ 5

$\boxed{10}$

$\cos(A+B)\cos(A-B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$ 5

$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 B)\sin^2 B$ 5

$= \cos^2 A - \sin^2 B$

$\boxed{10}$

(i) $\cos(A+A)\cos(A-A) = \cos^2 A - \sin^2 A$ 5

$\cos 2A = \cos^2 A - \sin^2 A \quad [\because \cos 0 = 1]$

$\boxed{05}$

(ii) $\cos^2\frac{3\pi}{24} + \cos^2\frac{5\pi}{24} + \cos^2\frac{7\pi}{24} - 2\sin^2\frac{\pi}{24} - \sin^2\frac{3\pi}{24}$

$= \left(\cos^2\frac{3\pi}{24} - \sin^2\frac{3\pi}{24}\right) + \left(\cos^2\frac{5\pi}{24} - \sin^2\frac{\pi}{24}\right) + \left(\cos^2\frac{7\pi}{24} - \sin^2\frac{\pi}{24}\right)$ 5

$= \cos\frac{\pi}{4}\cos 0 + \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\cos\frac{\pi}{4}$ 5

$= \frac{1}{\sqrt2}(1) + \frac{1}{\sqrt2}\frac{\sqrt3}{2} + \frac{1}{2}\frac{1}{\sqrt2} = \frac{3+\sqrt3}{2\sqrt2}$ 5 $\boxed{15}$

(b) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ 5 $\boxed{05}$

$\sin 2x = 2\sin x\cos x$ 5

$= \frac{2\sin x\cos x}{\cos^2 x + \sin^2 x} = \frac{\frac{2\sin x\cos x}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}} = \frac{2\tan x}{1+\tan^2 x}$ 5 5

$\boxed{15}$

$\cos 2x = \cos^2 x - \sin^2 x$ 5

$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \frac{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}} = \frac{1-\tan^2 x}{1+\tan^2 x}$ 5 5

$\boxed{15}$

$\sin 2B + \cos 2B = \frac{31}{25}$

$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{31}{25}$; where $t = \tan B$ 5

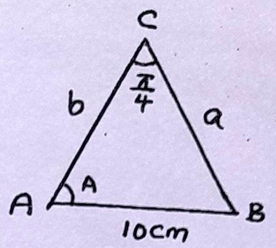
$28t^2 - 25t + 3 = 0$

$\Rightarrow (7t-1)(4t-3) = 0$

$\Rightarrow t = \frac{1}{7}$ or $t = \frac{3}{4}$ 5

$\Rightarrow \tan B = \frac{1}{7}$ or $\tan B = \frac{3}{4}$

As B has two values, two distinct triangles will be there



If $\tan B = \frac{1}{7}$, $\sin B = \frac{1}{5\sqrt2}$, $\cos B = \frac{7}{5\sqrt2}$ $\left[\because 0 < B < \frac{\pi}{2}\right]$ 5

$\sin A = \sin(\pi - (B + \frac{\pi}{4}))$

$= \sin(B + \frac{\pi}{4})$

$= \sin B\cos\frac{\pi}{4} + \cos B\sin\frac{\pi}{4}$

$= \frac{1}{5\sqrt2}\frac{1}{\sqrt2} + \frac{7}{5\sqrt2}\frac{1}{\sqrt2} = \frac{4}{5}$ 5

$a = \frac{c\sin A}{\sin C} = \frac{10\left(\frac{4}{5}\right)}{\frac{1}{\sqrt2}} = 8\sqrt2$ cm 5

$b = \frac{c\sin B}{\sin C} = \frac{10\left(\frac{1}{5\sqrt2}\right)}{\frac{1}{\sqrt2}} = 2$ cm 5

If $\tan B = \frac{3}{4}$, $\sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$ 5

$\sin A = \sin B\cos\frac{\pi}{4} + \cos B\sin\frac{\pi}{4}$

$= \frac{3}{5}\frac{1}{\sqrt2} + \frac{4}{5}\frac{1}{\sqrt2} = \frac{7}{5\sqrt2}$ 5

$a = \frac{c\sin A}{\sin C} = \frac{10\left(\frac{7}{5\sqrt2}\right)}{\frac{1}{\sqrt2}} = 14$ cm 5

$b = \frac{c\sin B}{\sin C} = \frac{10\left(\frac{3}{5}\right)}{\frac{1}{\sqrt2}} = 6\sqrt2$ cm 5

$\boxed{50}$

(c) $\tan^{-1}(e^x) + \tan^{-1}(2e^x) = \frac{3\pi}{4}$

Let $\alpha = \tan^{-1}(e^x)$ and $\beta = \tan^{-1}(2e^x)$

$\tan\alpha = e^x$ and $\tan\beta = 2e^x$

$\alpha + \beta = \frac{3\pi}{4}$

$\tan(\alpha+\beta) = \tan\frac{3\pi}{4}$ 5

$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -1$ 5

$\frac{e^x + 2e^x}{1 - e^x \cdot 2e^x} = -1$ 5

$2(e^x)^2 - 3e^x - 1 = 0$

$e^x = \frac{3 \pm \sqrt{17}}{4}$ 5

$e^x > 0 \Rightarrow e^x = \frac{3+\sqrt{17}}{4}$

$x = \ln\left(\frac{3+\sqrt{17}}{4}\right)$ 5

$\boxed{25}$

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