D.S. Senanayake College - Colombo 07
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Second Term Test, August 2023
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Combined Maths I
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Grade 13
13 වන ශ්‍රේණිය

Three hours and ten minutes
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Name- ……………………………….

class - ………………….

Instruction:

★ This Question paper consists of two parts.
Part A (Questions 1 -10) and **Part B** (Questions 11 -17).

★ **Part A**
Answer all questions. Write your answer to question in the space provided.

★ **Part B**
Answer any 5 Questions.

★ At the end of the time allotted, tie the answers of the two parts together so that part A is on top of Part B before handing them over to the supervisor.

★ You are permitted to remove only Part B of the question paper from the examination hall.

For Examiner Use Only

Combined Maths I		
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In Numbers	
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Part A

01. Prove that $14^n - 1$ is divisible by 13 for all $n \in \mathbb{Z}^+$ by the principle of mathematical Induction.

02. Sketch the graph of $y = 3 - |x-3|$ and $y = ||x| - 3|$ in the same diagram. Hence find the range of values of $\boldsymbol{x}$ which satisfy the inequality $||x| - 3| \leq 3 - |x-3|$.

03. Express the quadratic function $f(x) = 4(ax-1) - x^2$ in the form $p-(x-q)^2$ for $a \in \mathbb{R}$. And write the maximum values of $f(x)$ in terms of '$\boldsymbol{a}$'. Hence find the value of '$\boldsymbol{a}$' if $y = f(x)$ touches the $\boldsymbol{x}$ axis.

04. Show that,

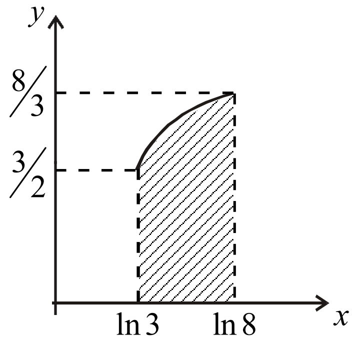
$$\lim_{x \to 0} \frac{\sin\{\pi(1-x)\}}{\sqrt{1+x} - \sqrt{1-x}} = \pi$$

05. If $y = x\tan^{-1}x - \ln\sqrt{1+x^2}$, Show that $(1+x^2)\dfrac{d^2y}{dy^2} - 1 = 0$

06. Given that the Parametric equations of a carve are $x = ct$ and $y = \dfrac{c}{t}$ where ***c*** is a constant and ***t*** is real parameter. $(c, t \neq 0)$). Find $\dfrac{dy}{dx}$ in terms of '***t***'. If the tangent to the curve at $t = 1$ meets the curve again at Q find the coordinates of Q in terms of '***c***'.

07. Let S be the region enclosed by the curves $y = \dfrac{e^x}{\sqrt{1+e^x}}$, $x = \ln 3$, $x = \ln 8$ and $y = 0$.

Show that the area of S is 2 square units. S is rotated through 2π radians about the x axis. Show that the volume of the solid thus generated is $\pi\left[5 - \ln\left(\dfrac{9}{4}\right)\right]$ cubic units.



08. Equation of a diagonal of a rectangle is $x - 5 = 0$. Two vertices are at $(2,\ 3)$, $\left(\lambda,\ 11/2\right)$ and $\lambda > 5$. Find the values of λ. Calculate the length of a diagonal. Hence find the coordinate of the vertices.

09. Find the equation of the circle whose centre is at $(1, 4)$ and bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 21 = 0$.

10. If $\tan^{-1}(x+\alpha) - \tan^{-1}(x+\beta) = \dfrac{\pi}{4}$, Show that $x^2 + (\alpha+\beta)x + \alpha\beta - \alpha + \beta + 1 = 0$. If $\alpha = 1$ and $\beta = (-1)$, find the values of 'x'.

Part - B

Answer 5 questions only.

11. (a) For $p \neq 0$, if the roots of the quadratic equation $x^2 + px + q^2 = 0$ are real and distinct, show that $|p| > 2|q|$. Let those roots λ and μ.

Given that $\alpha\lambda + \beta\mu$ and $\alpha\mu + \beta\lambda$ are the roots of $x^2 + apx + a^2q^2 + bp^2 - 4b^2q^2 = 0$. Find the quadratic equation in terms of a and b whose roots are α and β. And show that the quadratic equation whose roots are $\sqrt{\alpha}$ and $\sqrt{\beta}$ is given by $x^2 - \sqrt{a + 2|b|}\,x + |b| = 0$.

(b) If $f(x) = x^3 - 2ax^2 + (a^2 - b^2)x - ab(a - b - x)$ Show that $f(a-b) = 0$ and write down a factors of $f(x)$ in terms a and b. Find the other factors of $f(x)$ in terms of a and b and write down the roots of $f(x) = 0$.

12. (a) How many different ways the letters of the word "SENANAYA KE" be arranged in a row?
   (i) In how many arrangements all three A's be next to each other?
   (ii) In how many arrangements all three A's be next to each other while E's be not next to each other.

(b) If a Child receives at least Rs. 3, in how many different ways can Rs. 18 be shared among 5 Children? So that each receives and integer multiple of rupees.

(c) Let $U_r = \dfrac{r}{3}(r+1)(r+2)$ for $r \in \mathbb{Z}^+$ and $f(r) = \dfrac{1}{r(r+1)}$; Find the value of k such that $\dfrac{1}{U_r} = k[f(r) - f(r+1)]$. Hence show that, $\displaystyle\sum_{r=1}^{n} \frac{1}{U_r} = \frac{2}{3}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{U_r}$ is convergent and find its sum.

Deduce that $2 \leq 3\left(1 - -\dfrac{2}{(n+1)(n+2)}\right) < 3$

13. (a) Let $f(x) = \dfrac{ax}{x^2 - 2x + b}$; $a \neq 0$. Given that $(-1,\ 1)$ is the turning point of $y = f(x)$, show that $a = (-4)$ and $b = 1$ and show that $f'(x) = \dfrac{4(x+1)}{(x-1)^3}$, where $x \neq 1$. Find the range of values of x for which $f(x)$ is increasing and decreasing.

Let $f''(x) = \dfrac{-8(x+2)}{(x-1)^4}$, find the coordinates of point of inflection of $y = f(x)$. And sketch $y = f(x)$ by indicating clearly assymptoes, turning points and the point of inflection.

(b) A rectangle is inscribed in a quarter circle of radius $3\sqrt{2}$cm. So that two of it's sides are lying along the straight edges of the quarter circle. Calculate the maximum area of the rectangle.

14. (a) It is given that there exist constants A and B such that

$$7x^2-4x+5 \equiv 2(x-2)(x^2+1)-2x(x-2)^2+A(x^2+1)+B(x-2)^2 \text{ for all } x\in\mathbb{R}.$$

Find the values of A and B. Hence write down $\dfrac{7x^2-4x+5}{(x^2+1)(x-2)^2}$ in partial fractions and find

$$\int \frac{7x^2-4x+5}{(x^2+1)(x-2)^2}dx.$$

(b) By using substitution $\theta=\sin^{-1}(\sqrt{x})$ for $0<x<1$ find $\int\sqrt{\dfrac{x}{1-x}}dx$.

Show that $\dfrac{d}{dx}(x\sin^{-1}\sqrt{1-x})=\dfrac{1}{2}\sqrt{\dfrac{x}{1-x}}+\sin^{-1}\sqrt{1-x}$

And deduce that $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{4}}\sin^{-1}\sqrt{1-x}\,dx=\frac{\pi}{24}+\frac{1}{4}-\frac{\sqrt{3}}{8}$

(c) By using integration by parts, Show that $\displaystyle\int_0^{\frac{\pi}{2}} x(\pi-2x)\sin 2x dx=1$.

Let $I=\displaystyle\int_0^{\frac{\pi}{2}}\frac{x(\pi-2x)\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ by using $\displaystyle\int_0^a f(x)dx=\int_0^a f(a-x)dx$

Show that $I=\dfrac{1}{4}\displaystyle\int_0^{\frac{\pi}{2}} x(\pi-2x)\sin x\cos x dx$. Hence deduce the value of *I*.

15. l is a straight line perpendicular to the straight line $ax+by+c=0$ through $p(x_0,\ y_0)$. Show that any point on the line l can be given by $(x_0+at,\ y_0+bt)$.

Hence show that the perpendicular distance from $p(x_0,\ y_0)$ to $ax+by+c=0$ be $\dfrac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

In a ΔABC A, B and C lie on $x+y=0,\ 2x+y=0$ and $3x+y=0$ respectively let abscissa of A be ***t***. The equations of the perpendicular bisectors of AB and AC are $5x-7y-1=0$ and $x+3y-11=0$ respectively.

(i) Show that the equation of AB is given $7x+5y-2t=0$

(ii) Find the equation of AC in terms of t.

(iii) Find the coordinates of B and C in terms of t. show also that BC is parallel to $5x+2y=0$.

(iv) If BC passes through $(0,\ 1)$, find the value of t and deduce that coordinates of A, B and C and the equations of the sides of ΔABC.

(v) Calculate the perpendicular distance from origin to BC and Calculate the area of the ΔBOC.

16. Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

be the equations of two circles. Find the coordinates of the centre and the radius of $S_1 = 0$ from the first principles. Hence write down the coordinates of the centre and the radius of $S_2 = 0$.

Show that they intersect orthogonally if $2(g_1g_2 + f_1f_2) = c_1 + c_2$.

Let $S_1 \equiv x^2 + y^2 - 2x - 6y - 6 = 0$

$S_2 \equiv x^2 + y^2 - 10x - 12y + 60 = 0$

(a) Show that $S_1 = 0$ and $S_2 = 0$ touch each other at A and find the coordinates of **A.**

(b) Find the equation of the common tangent l_1 through A to $S_1 = 0$ and $S_2 = 0$. Show that $B(9, -1)$ does not lie on $S_1 = 0$ and $S_2 = 0$.

(c) Find the equation of the tangent l_2 through B to $S_1 = 0$.

(d) Show that $y - 7 = 0$ is another common tangent to the above circles and find the equation of the third common tangent to the circles.

(e) Find the equation of all the circles which orthogonal to $S_1 = 0$ and $S_2 = 0$. Deduce that they all pass through A.

(f) Show that $S_3 \equiv x^2 + y^2 - 6x - 14y + 54 = 0$ is one of circle orthogonal to $S_1 = 0$ and $S_2 = 0$. with a diameter lies along $y - 7 = 0$.

17. (a) Write down the expressions for $\sin(A+B)$ and $\cos(A+B)$. Hence express $\sin\theta$ and $\cos\theta$ in terms of $\tan\dfrac{\theta}{2}$.

Express $x = \operatorname{cosec}\theta\,(\mu + 1 + \cos\theta)$ in terms of $\tan\dfrac{\theta}{2}$. Where $\theta \in \mathbb{R} - \{n\pi ; n \in \mathbb{Z}\}$ and μ is a positive parameter. Show that $|x| \geq \sqrt{\mu(\mu+2)}$.

(b) Write down the sine rule with usual notations for a triangle ABC. In a triangle ABC, D is a point on BC such that BD : DC = 1 : 2. Let AB : AC = 2 : 3 ; $A\hat{D}C = \theta$; $B\hat{A}D = \alpha$ and $C\hat{A}D = 2\alpha$.

Show that $\cos\alpha = \dfrac{2}{3}$. And find the value of $\tan\theta$.

(c) Find the general solutions of the equation $5\cos^2 x - 2\sqrt{3}\sin x.\cos x + 7\sin^2 x = 5$.