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Western Province Educational Department

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, **2023 (2024)**
General Certificate of Education (Adv. Level) Examination, 2023 (2024)

සංයුක්ත ගණිතය I
Combined Mathematics **I**

10 E I

2023.12.14 / 08.30 - 11.40

පැය තුනයි
Three hours

අමතර කියවීම් කාලය - මිනිත්තු 10 යි
Additional Reading Time - 10 minutes

additional reading time to go through the question paper, select the questions and decide on the questions that you give priority in answering.

Index Number							

Instructions:

* *This question paper consists of two parts.*
  Part A (Questions 1 - 10) *and* **Part B** (Questions 11 - 17)
* **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:**
  *Answer **five** questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove only* **Part B** *of the question paper from the Examination Hall.*

For Examiners' Use only

(10) Combined Mathematics I		
Part	**Question No.**	**Marks**
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
	Total	
	Percentage	

Final Marks

In Numbers	
In Words	

Code Numbers

Marking Examiner	
Checked by: 1	
2	
Supervised by:	

Part A

01. Using the **principle of mathematical induction,** prove that $\sum_{r=1}^{n} 2\cdot 3^{r-1} = 3^n - 1$ for all $n \in \mathbb{Z}^+$

02. Sketch the graph of $y = 2-|x-2|$ and $y = \begin{cases} -x; x<0 \\ \frac{1}{3}|x|; x \geq 0 \end{cases}$ in the same diagram. **Hence or otherwise** find all real vales of x sahisfying the inequality $|x+1| + 3|x-1| \leq 6$

03. Sketch in the Argand diagram the locus of the points that represent the complex number z satisfying $\text{Arg}\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4}$. **Hence or otherwise** find the maximum value of $|iz-1|$

04. Let $A=\{a,e,i,o,u\}$ and B ={Prime numbers less than 10}
A secret code with four digits should be make such that first letter should get from set A and rest should get from set B . Find the number of different codes when repetition is allowed and not allowed.

05. Show that $\lim_{x\to 0}\left(\dfrac{\sin 3x}{\sqrt[3]{2x+1}-(x-1)^2}\right)=\dfrac{9}{8}$

06. Let S be the region enclosed by the curves $y=\dfrac{\sqrt{1-\sin x}}{x+\cos x}$, $y=0$, $x=0$ and $2x=\pi$. S is rotated about x axis through $\dfrac{\pi}{3}$ radians. Show that the volume of the soild the generated is $\dfrac{\pi-2}{6}$.

07. Let C be a curve which is parametrically given by $x = 1+\sqrt{2}\cos\theta$ and $y = 1-\sqrt{2}\sin\theta$.
The **normal** to the curve at the point correspond to $\theta = \frac{\pi}{4}$, meets the curve again at P then find the corresponding parameter for P. Further find the constants a and b such that the equation is of the form $\frac{x}{a}+\frac{y}{b}=1$; $(a,b)\in\mathbb{Z}^{+}$

08. Let $a\in\mathbb{R}$ and $a\neq\pm2$.
It is given the straight line $(2+a)x+(2-a)y=2-5a$ passes through a fixed point A, Find the coordinates of A. Show that the coordinates on the straight line which passes through A and inclined an angle $\tan^{-1}\left(\frac{1}{2}\right)$ with $x-y+5=0$ is given by $(3t-2,\ t+3)$. Here t is a real **parameter.**

09. If the tangent drawn to the circle with center as orgina and radius 1 units cuts the x and y coordinate axes at points A and B, then show that the locus of the **circumcenter** is given by $x^2 + y^2 = 4x^2y^2$.

10. Let $T(x) \equiv 16\sin^2 x + 9\operatorname{cosec}^2 x + 1999$ for $x \in \left(0, \frac{\pi}{2}\right)$. Find the integers a, b and c such that $T(x) \equiv (a\sin x + b\operatorname{cosec} x)^2 + c$; $ab < 0$.

Hence or otherwise, write the smallest image of the function of T. **Dedue** the corresponding unique value for that image in the domain.

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Western Province Educational Department

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2023 (2024)
General Certificate of Education (Adv. Level) Examination, 2023 (2024)

සංයුක්ත ගණිතය I
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Part B

- Answer only **five** questions.

11. (*a*) Let $F(x) \equiv px^2 + qx + r$; $p \neq 0, (p,q,r) \in \mathbb{R}$ where $x \in \mathbb{R}$. Solve the equation or otherwise, show that $\Delta \geq 0$, if $F(x) = 0$ has real roots. Here $\Delta = q^2 - 4pr$.

If $(p,q,r) \in \mathbb{Q}$ then write the necessity of having real **rational roots** for $F(x) = 0$ Futhere more, if 1 is a roots of $F(x) = 0$ then show that $p + q + r = 0$

Let α and β are the roots of $2ax^2 - (2a+b+c)x + b + c = 0$ $a \neq 0,$ $(a,b,c) \in \mathbb{Q}$

If the quadratic equation whose roots are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ can be obtained. $G(x) = 0$ then find $G(x)$.

Write the discriminent of $G(x) = 0$ in terms of a, b and c. **Deduce** that $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ are rational. Furthermore, it *b*, *a* and *c* are consecutive terms of arithemetic sequence respectively then deduce that root of $G(x) = 0$ are real and coincident.

(*b*) $H(x)$ is a 3rd order polinormial. When the polynomial $H(x)$ is divided by the liner factor of $(x-1)$, $(x-2)$ and $(x-3)$ the remember is 7. Futhermore $H(x)$ is divided by $(x-4)$ the remainder is 1. If $H(x) \equiv (ax+b)(x-2)(x-3) + c$ then find *a*, *b*, *c* integers

12. (*a*) Let $a, b \in \mathbb{R}$ write down the expansion of $(ax+by)^n : n \in \mathbb{Z}^+$ in **ascending** powers of x.

Considering the exponsion of $\left(ax + \dfrac{1}{x}\right)^n$, write down the coefficient which is independent of x, in terms n and a.

Find values of n and a, if the fourth term of the above expansion is $\dfrac{5}{54}$. Further, write down the **mid term** of the expansion of $\left(ax + \dfrac{1}{x}\right)^n$ Furthermore, **deduce** the integers α and m, Such that the sum of the coefficients of $\left(ax + \dfrac{1}{x}\right)^n$ is $\left(\dfrac{\alpha+1}{\alpha}\right)^m$.

(*b*) Find the real constants. A and B,

such that $A(r+1)(3r+5)-B(r+2)(3r-1)\equiv 6r^2+19r+17$, for $r\in\mathbb{Z}^+$

Now Let $U_r=\dfrac{6r^2+19r+17}{(3r-1)(3r+2)(3r+5)}$ for $r\in\mathbb{Z}^+$ Write V_r such that $U_r=k\cdot V_r-V_{r+1}$.

Here k is a positive integer to be determined. **Hence or otherwise,** By writing $f(r)$ such that $\dfrac{U_r}{3^r}=f(r)-f(r+1)$ Show that $\displaystyle\sum_{r=1}^{n}\frac{U_r}{3^r}=\frac{1}{5}-\frac{(n+2)}{(3n+2)(3n+5)}\cdot\frac{1}{3^n}$ Further, show that the infinite series $\displaystyle\sum_{r=1}^{\infty}\frac{U_r}{3^r}$ is convergent and find the sum of it. **Hence,** deduce the value of $\displaystyle\sum_{r=2}^{\infty}\frac{U_r}{3^{r-1}}$

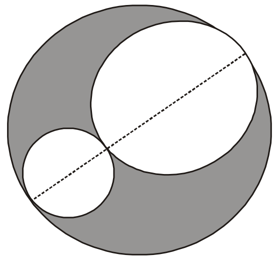
13. (*a*) Let $\mathbf{A}=\begin{pmatrix}2a+1 & 3\\ 2a-1 & a\end{pmatrix}$ such that $a\in\mathbb{R}$ Determine values a can take such that the inverse matrix $\mathbf{A}^{-1}$ **exists.** Given the determinant of $\mathbf{A}$ is 1, find the value of a such that $a\in\mathbb{Z}$.
Write down $\mathbf{A}$ and $\mathbf{A}^{-1}$ for the above value of a. Find **row** matrices $\mathbf{B}$ and $\mathbf{C}$, such that $\mathbf{AB}=\begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{A}^{-1}\mathbf{C}=\begin{pmatrix}1\\1\end{pmatrix}$ Find the matrix $\mathbf{BC}^{\mathbf{T}}$. Here $\mathbf{C}^{\mathbf{T}}$ is the transpose matrix of $\mathbf{C}$.

(*b*) Let $x,y\in\mathbb{R}$ and $z\in\mathbb{C}$. Write down $\mathrm{Re}(z),\mathrm{Im}(z)$ and $|z|$ such that $z=x+iy$.

Show that $|z|^2=(\mathrm{Re}(z))^2+(\mathrm{Im}(z))^2$. Let $z=\cos\theta+i\sin\theta$ for $-\pi<\theta\le\pi$ Represent z in an argand diagram.
Given that $\omega=\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}\mathrm{i}$. Show that complex numbers ω and ω^2 lie on the locus of z. Hence, **deduce** $|\omega-1|=|\omega^2-1|=|\omega-\omega^2|$

(*c*) Let $z_1=1+\sin\left(\dfrac{\pi}{8}\right)+i\cos\left(\dfrac{\pi}{8}\right)$ and $z_2=1+\sin\left(\dfrac{\pi}{8}\right)-i\cos\left(\dfrac{\pi}{8}\right)$. Evaluate the value of θ such that $z_1=(1+\cos\theta)+i\sin\theta$ for $\dfrac{\pi}{4}<\theta<\dfrac{\pi}{2}$. Furthermore express z_1 and z_2 in the form $z_n=r_n(\cos\theta_n+i\sin\theta_n)$; $n=1,2$ Here r_n and $\theta_n\left(\dfrac{-\pi}{4}<\theta_n<\dfrac{\pi}{4}\right)$ are constants to be determined.
Let $\omega=\dfrac{z_1}{z_2}$ using **De Movier's theorem or otherwise** deduce $\mathrm{Re}\left(\omega^{\frac{8}{3}}\right)=-1$ and $\mathrm{Im}\left(\omega^{\frac{8}{3}}\right)=0$

14. (*b*) The figure shows two circular areas touchmg each other and having their centers on the same diameter of a circular compound.
If the shaded area is a lawn, show that the area A of the lawn, is given by $A=2\pi r(1-r)$ where r is the radius of the smaller circle.



Evaluate r such that A is maximum and **deduce** the maximum area of the lawn.

14. (a) Let $f(x)=\dfrac{ax+b}{(x-1)^2};(a,b)\in\mathbb{Z}$ for $x\neq 1$ Determine a and b such that the derivative of $f(x)$ is $f'(x)=\dfrac{-(x+a)}{(x-1)^3}$ for $x\neq 1$ **Hence,** find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing. Also, find the coordinates of the turning point of $f(x)$. **It is given that** $f''(x)=\dfrac{2(x+2a)}{(x-1)^4}$ for $x\neq 1$.

Hence find the intervals on which $f(x)$ concave up and the interval on which $f(x)$ is concave down.

Find the coordinates of the points of inflection of the graph of $y=f(x)$. Sketch the graph of $y=f(x)$ indicating the asymptotes, the turning point, the point of inflection and intercept of the y axis.

Furthermore, **deduce** the graph of $y=\left|f(x)\right|$ in the domain $x\in(-\infty,1)$.

15. (a) Let $8x^3-2x+4\equiv(Ax+B)(2x-1)^2+B\left(4x^2-4x+3\right)(2x-1)+C\left(4x^2-4x+3\right)$ for $x\in\mathbb{R}$.

Find the integers A,B and C. **Hence** write the partial fractions of $\dfrac{8x^3-2x+4}{\left(4x^2-4x+3\right)(2x-1)^2}$.

Find $\displaystyle\int\frac{8x^3-2x+4}{\left(4x^2-4x+3\right)(2x-1)^2}dx$

(b) Show that $\dfrac{d\left(\frac{1}{2}\left(2x+(\ln x)^2\right)\right)}{dx}=\dfrac{x+\ln x}{x}$ for $x\in\mathbb{R}^+$.

Using **integration by parts or otherwise** prove that $\displaystyle\int_1^2\frac{x+1}{x}\cdot\frac{2x+(\ln x)^2}{(x+\ln x)^2}\cdot dx=\frac{L(L+6)}{L+2}$ where L be a real number to be determined

(c) Prove that $\dfrac{1}{1+\sin x}\equiv\sec^2 x-\sec x\cdot\tan x$ for $n\in\mathbb{Z}$ and $x\neq(4n\pm1)\dfrac{\pi}{2}$.

Hence show that $\displaystyle\int_0^{\pi}\frac{1}{1+\sin x}\cdot dx=2$

let $a,b\in\mathbb{R}$ and $a<b$. State the formulae $\displaystyle\int_a^b f(x)dx=\int_a^b f(a+b-x)\cdot dx$. It is given $I=\displaystyle\int_0^{\pi}\frac{x\cdot\sin x}{1+\sin x}\cdot dx$. **Deduce** $I=\dfrac{\pi}{2}(\pi-J)$

Here $J=\displaystyle\int_0^{\pi}\frac{1}{1+\sin x}\cdot dx$, **Hence** evaluate I.

16. Let $y = m_1x + n_1$ and $y = m_2x + n_2$; $m_1 \neq m_2,\ m_1 > m_2$. Prove that the **acute angle** between the two straight lines is $\tan^{-1}\left(\dfrac{m_1 - m_2}{1 + m_1m_2}\right)$

Obtain the equation of the tangent chord drawn from the point (α, β) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ as $\alpha x + \beta y + (\alpha + x)g + (\beta + y)f + c = 0$.

Show that there are two straight lines (l_1, l_2) exist that make an angle of $\tan^{-1}\left(\dfrac{1}{3}\right)$ with the straight line $x + y = 3$ and determine their gradients.

Given that above two straight lines **coincide** at $A \equiv (2,1)$ find l_1 and l_2

Obtain the equstions of two circles (S_1, S_2) in which the center lies on $x + y = 3$, touch the line $l_1 = 0$ and the radius is $\sqrt{5}$ units.

Deduce S,

If $S = 0$ is the circle in which the **abscissa** of the center is positive.

Write down the equation of the tangent chord drawn from point A to the circle $S = 0$

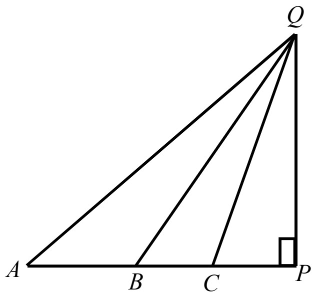
Let B and D are two points of intersection of the tangent chord and the center C of $S = 0$. Show that the equation of the cyclic quadrilateral $ABCD$ is $x^2 + y^2 - 9x + 3y + 10 = 0$.

17. (*a*) Write down $\sin(A + B)$ in terms $\sin A$, $\cos A$, $\sin B$ and $\cos B$. **Hence** obtain $\cos(A + B) \equiv \cos A \cdot \cos B - \sin A \cdot \sin B$. Further more show that $\sin 2A \equiv 2\sin A \cos A$ and $\cos 2A \equiv \cos^2 A - \sin^2 A$ **Deduce** $\sin 2A \equiv \dfrac{2\tan A}{1 + \tan^2 A}$ and $\cos 2A \equiv \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$.

Let $T(x) \equiv \dfrac{2\tan x(1 + \tan x)}{1 + \tan^2 x}$ for all $x \in \mathbb{R}$. Determine real constants a, b and $\alpha \left(0 < \alpha < \dfrac{\pi}{2}\right)$ such that $T(x) \equiv a + b\sin(2x - \alpha)$. **Hence** sketch $y = T(x)$ in the domain $x \in \left[\dfrac{-\pi}{2}, \dfrac{3\pi}{4}\right)$

(*b*) **State** the sine rule for a triangle ABC in standard notation.

The top most position Q of a straight right vertical tower PQ is observed from points A, B and C. Which lies in the same horizontal level of P at angles of elevation α, 2α and 3α respectively.



Using sine rule for $BCQ\Delta$ or otherwise show that $\dfrac{AB}{BC} = \dfrac{\sin 3\alpha}{\sin \alpha}$.

Furthermore showing that $\dfrac{AB}{BC} = 1 + 2\cos 2\alpha$. **Deduce** $AB \leq 3BC$.

(*c*) Prove that $\tan^{-1}(-x) = -\tan^{-1} x$ for $x \in \mathbb{R}$. Show that $\sin^{-1}\left(\dfrac{4x}{x^2 + 4}\right) + 2\tan^{-1}\left(\dfrac{-x}{2}\right)$ is independant from x, and Also find the real values of x.