රාජකීය විද්‍යාලය - කොළඹ 07
Royal College Colombo 07

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය
General Certificate of Education (Adv. Level)Examination

13 වන ශ්‍රේණිය - පළමු වාර පරීක්ෂණය 2019 නොවැම්බර්
Grade 13 First Term Test November 2019

Time : 3 hours and 10 minutes

Combined Mathematics - I

Name / Index No :- .... Grade:- ....

- **Answer all questions in the part A and any five questions in the part B.**

Part A

1. α, β be the roots of quadratic equation $ax^2 + bx + c = 0$ and γ, δ be the roots of quadratic equation $px^2 + qx + r = 0$. Also, Δ_1 and Δ_2 are their discriminants respectively. If α, β, γ and δ are in an **arithmetic** progression, show that $\Delta_1 : \Delta_2 = a^2 : p^2$

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2. Find the **range** of the function.

$f(x) = \sqrt{1-\cos x\sqrt{1-\cos x\sqrt{1-\cos x\ldots}}}$

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3. Evaluate $\lim_{x \to 1}\left(\dfrac{\cos \pi x + \sin \dfrac{\pi x}{2}}{(x-1)^2}\right)$

4. Solve.

$\log_{4\sqrt{x}} x \;\; + \log_{4x} \sqrt{x} = 0.\; ; x \in \mathbb{R}^+$

5. $ABCD$ is a rhombus. The equations of AB and AC are $x - y + 1 = 0$, $2x - y - 1 = 0$ respectively. If the side BC passes through the point $(5, -6)$, then find the equations of the sides BC, CD, and DA without finding the vertices of the rhombus explicitly.

6. The region enclosed by the curves $y = -|x^2 - 3x|$, $x = 0$, $x = 3$ and $y = 0$ is rotated about the x axis through 2π radians. Show that the **volume** of the solid thus generated is $\frac{3^4}{10}\pi$

7. Show that the normal drawn to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, at the point $P \equiv \left[\frac{2(1-t^2)}{1+t^2}, \frac{6t}{1+t^2}\right]$ is

$3y(1-t^4)-4t(1+t^2)x-10t(1-t^2)=0$.

Where $t \in \mathbb{R}-\{-1, 1\}$ is a parameter.

8. Evaluate $\int_{-2}^{2} \frac{x^2+1}{x^4-2x^2+1}dx$

9. Solve $\tan\theta + \tan 2\theta + \tan 3\theta = 0 \; ; \; \theta \in \mathbb{R}$

10. Show that $\cot^{-1}\left[\dfrac{xy+1}{x-y}\right] + \cot^{-1}\left[\dfrac{yz+1}{y-z}\right] + \cot^{-1}\left[\dfrac{zx+1}{z-x}\right] = n\pi \quad ; n \in \mathbb{Z}$

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අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය
General Certificate of Education (Adv. Level)Examination
13 වන ශ්‍රේණිය - පළමු වාර පරීක්ෂණය 2019 නොවැම්බර්
Grade 13 First Term Test November 2019

Combined Mathematics - I

- Answer **five** questions only.

Part B

11. (*a*) $a(\neq 0)$, b and c are real coefficients of the equation $ax^2 + bx + c = 0$.

Show that $\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta}{4a^2} = 0$ where $\Delta = b^2 - 4ac$

Hence Prove that $\Delta \geq 0$, **if and only if** $x \in \mathbb{R}$

Furthermore, by solving above equation, obtain $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

If $a, b, c \in \mathbb{Q}$ and $\Delta \geq 0$, state the necessary and sufficient condition to be $x \in \mathbb{Q}$,

Hence deduce the condition to be $x \notin \mathbb{Q}$

Now, consider the cubic function $F(x) \equiv ax^3 + bx^2 + cx + a.$ where $a, b, c \in \mathbb{Q}$

If $(1 - \sqrt{2})$ is a root of $F(x) = 0$, then find b and c in terms of a.

Hence find all the roots of the equation $F(x) = 0$

Given that α, β and γ are the roots of the equation $F(x) = 0$ and $\alpha < \beta < \gamma$, **deduce that** the quadratic equation with roots α and β is $(1 + \sqrt{2})x^2 - \sqrt{2}\,x - 1 = 0$.

(*b*) $f(x) = x^4 + x^3 - (2 + d)x^2 + d(c + 1)x - cd$, where $c,\ d \in \mathbb{R}^+$. Given that $(x - c)$ is a factor of $f(x)$ and when $f(x)$ is divided by $(x - d)$, the remainder is $cd(d - 1)$. Find the values of c and d.

~~**Using the division algorithm and differentiation** find the remainder, when $f(x)$ is divided by~~ $[x + (c + d)]^3$

12. (*a*) Let $f(x) = |3^x - 1| + |3^x - 9|$ and $g(x) = 3^x$

By chooing a suitable coordinate system and domain sketch both graphs in the same plane.

Show that the equation $3^x - |3^x - 1| - |3^x - 9| = 0$ has two real solutions and find them.

Furthermore **deduce** that if $3^x \geq |3^x - 1| + |3^x - 9|$ then $\dfrac{\ln 8}{\ln 3} \leq x \leq \dfrac{\ln 10}{\ln 3}$

(b)(i) The function f is defined as $2f(x) + 3f\left(\frac{1}{x}\right) = \frac{1-x^2}{1+x^2}$ for $x \in \mathbb{R}$. Find the function $f(x)$ by replacing x in a suitable manner.

(ii) State the domain and the range of f. Find the minimum value of $f(x)$ and the corresponding x value.

Find $\lim_{x \to \pm\infty} f(x)$ and sketch the graph of $y = f(x)$.

(iii) Let $g(x) = f(\tan x)$.

Find g and sketch it within $0 \le x \le \frac{\pi}{2}$. What is the range of g ?

(iii) Find the inverse function g^{-1} of g. Write the domain and the range of g^{-1}. Sketch the graph of $y = g^{-1}(x)$.

13. (a) Using the **first principals** find the derivative of $\dfrac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}$

A and B are constants of the equation $y = Ae^{-2x} + Be^{-x} + \sin x$.

When $x = 0$, the value of the function and its first derivative both are zero.

Show that $A = 1$ and $B = -1$

Find C and D if $\dfrac{d^2y}{dx^2} + C\dfrac{dy}{dx} + Dy = 3\cos x + \sin x$

Hence find $\left(\dfrac{dy}{dx}\right)_{x=\pi}$ and $\left(\dfrac{d^2y}{dx^2}\right)_{x=\pi}$

(b) (i) If $\sqrt{y+x} + \sqrt{y-x} = c$, ($c \in \mathbb{R}^+$) then show that $\dfrac{dy}{dx} = \dfrac{y}{x} - \sqrt{\dfrac{y^2}{x^2} - 1}$

(ii) If $y = x e^{\frac{-1}{x}}$ then show that $x^3 \dfrac{d^3y}{dx^3} + x\dfrac{d^2y}{dx^2}(3x - 1) = 0$

(c) A hollow right conical shaped water tank with radius 10 m and height 25 m is kept upside down in a vertical plane with its axis vertical. At the beginning, it is filled with water up to height x m from the bottom vertex. If the rate of reduction of the height of water in the tank is t %. Show that the rate of reduction of volume of water in the tank $12\,t$ %.

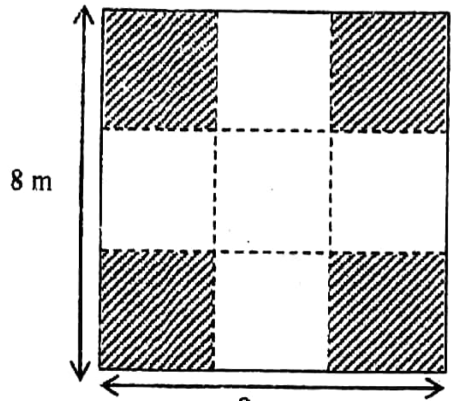
14. (a) Let $f(x) = \dfrac{6(x^2 - 8x + 1)}{(x-2)^3}$ for $x \neq 2$

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{-6(x^2 - 12x - 13)}{(x-2)^4}$ for $x \neq 2$.

Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points.

Find $f''(x)$ and find the x coordinates of the points of inflection of the graph of $y = f(x)$.

The below diagram shows a square shaped lamina [illegible] four squares are shaded in [illegible] corners in the lamina. By removing all four shaded [illegible] the lamina, an open box is made by folding the remaining lamina along the edges. Find dimensions of the box in order to get the maximum possible volume?



15. (*a*) Using a suitable substitution or using any other method.

i) Show that $\int_{-1}^{1} x(1-x)^{2019}dx = \dfrac{2}{2021}$

ii) Using intergration by parts show that $\int_{0}^{\pi/2} x^4\sqrt[3]{\dfrac{x^3}{x^3+1}}\,dx = \dfrac{(\pi^3-12)(\pi^3+8)^{\frac{2}{3}}+96}{320}$.

(*b*) Let $y_n = \cos^n x$. When n is an even number, show that $\dfrac{d^2y_n}{dx^2} = -n^2 y_n + n(n-1)y_{n-2}$

Given that $I_n = \int_0^{\pi} e^{-x}y_n dx$, Then show that $I_n = 1 - e^{-\pi} + \int_0^{\pi} e^{-x}\dfrac{d^2y_n}{dx^2}dx$.

Hence show that, $n(n-1)I_{n-2} = (n^2+1)I_n - (1-e^{-\pi})$

Deduce the value I_4

(*c*) Show that the area enclosed by curves $y = \ln x$, $y = 1$ and $x = e^2$ is ≈ 2.718 square units.

16. (*a*) Find the two straight lines represents $2x^2 - 9xy + 9y^2 = 0$.

Find the equations of two straight lines.

If above two straight lines are representing two adjacent sides of a parallelogram and $x + y + 4 = 0$ is an equation of a diagonal of that parallelogram,

Then find the equations of the other two sides and the remaining diagonal.

Show that the area of parallelogram is $\frac{12}{5}$ square units.

Find the equations of of line which passes through point of to diagonals and while making an angle 45° with the line $2x - y - 5 = 0$

(*b*) $u_1 \equiv l_1x + m_1y + n_1 = 0$ and $u_2 \equiv l_2x + m_2y + n_2 = 0$ are two intersecting straight lines.

If $u_1 = 0$ is the equation of the bisector of the lines $u = 0$ and $u_2 = 0$.

Then show that the equation of the straight line $u = 0$ is $2(l_1l_2 + m_1m_2)u_1 - (l_1^2 + m_1^2)u_2 = 0$

In the square $ABCD$, $A = (3, -1)$ and $B = (6, 4)$.

(i) Find the equations of the diagonals of the square.

(ii) Also, find equations of the sides AD and DC.

17. (a) Prove the following trigonometric identity.

$$\frac{1}{1+2\cos\left(\frac{\pi}{3}+\theta\right)}+\frac{1}{1+2\cos\left(\frac{\pi}{3}-\theta\right)}\equiv\frac{1}{2\cos\theta-1}$$

(b) If $\alpha+\beta-\gamma=\pi$,
Prove that $\sin^2\alpha+\sin^2\beta-\sin^2\gamma=2\sin\alpha\sin\beta\cos\gamma$.

(c) **State** the sine rule for any triangle in usual notation.
Prove that $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$ for any triangle ABC in usual notation.

(d) Rearrange $f(\theta)=4\cos^2\theta+6\sin\theta\cos\theta+12\sin^2\theta$ in the form $a+b\cos(2\theta+\alpha)$ where a, b and α $\left(0<\alpha<\frac{\pi}{2}\right)$ are constants to be determined.
Sketch the graph of $y=f(\theta)$ for $[0,\pi]$.
Hence, find the set of values of k, which $f(\theta)=k$, has
(i) Only one solution.
(ii) Two solutions.
(iii) Three solutions.
(iv) No solutions.

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ROYAL COLLEGE

GRADE 13

1^{st} TERM TEST – NOVEMBER 2019

COMBINED MATHS I

MARKING SCHEME

Part A

(1) $ax^2+bx+c=0$ $\quad$ 05 $\quad \alpha+\beta=-b/a$

$\Delta_1=b^2-4ac \qquad \alpha\beta=c/a$

$px^2+qx+r=0$ $\quad$ 05 $\quad \gamma+\delta=-q/p$

$\Delta_2=q^2-4pr \qquad \gamma\delta=r/p$

When $\alpha, \beta, \gamma, \delta$ are in the arithmetic progression.

$$\beta-\alpha=\delta-\gamma \quad 05$$

$$(\beta-\alpha)^2=(\delta-\gamma)^2$$

$$(\beta+\alpha)^2-4\alpha\beta=(\delta+\gamma)^2-4\delta\gamma \quad 05$$

$$\frac{b^2}{a^2}-\frac{4c}{a}=\frac{q^2}{p^2}-\frac{4r}{p}$$

$$\frac{b^2-4ac}{a^2}=\frac{q^2-4pr}{p^2}$$

$$\frac{\Delta_1}{a^2}=\frac{\Delta_2}{p^2} \quad 05$$

$$\Delta_1/\Delta_2=a^2/p^2$$

$$\therefore \Delta_1:\Delta_2=a^2:p^2$$

$\boxed{25}$

(2) $y=\sqrt{1-\cos x\sqrt{1-\cos x\sqrt{1-\cos x\cdots}}}$

$$y^2=1-y\cos x \quad 05$$

$$\therefore \cos x=\frac{1-y^2}{y}$$

Range of $\cos x$ is

$$-1\le \cos x\le 1$$

$$-1\le \frac{1-y^2}{y}\le 1 \quad 05$$

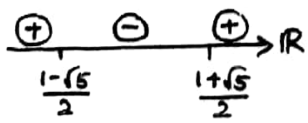
Let consider

$$-y\le 1-y^2$$

$$y^2-y-1\le 0$$

$$\left(y-\tfrac{1}{2}\right)^2-\left(\tfrac{\sqrt5}{2}\right)^2\le 0$$

$$\left(y-\left(\frac{\sqrt5+1}{2}\right)\right)\left(y-\left(\frac{1-\sqrt5}{2}\right)\right)\le 0 \quad 05$$

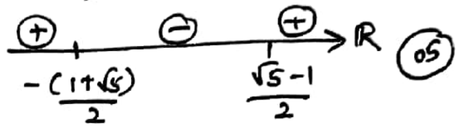


Also $1-y^2\le y$

$$y^2+y-1\ge 0$$

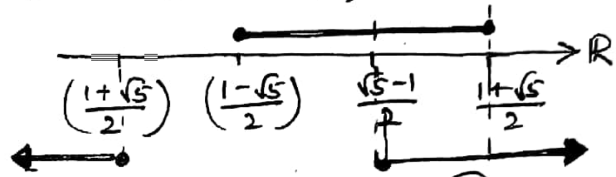
$$\left(y+\tfrac{1}{2}\right)^2-\left(\tfrac{\sqrt5}{2}\right)^2\ge 0$$

$$\left[y+\left(\frac{1+\sqrt5}{2}\right)\right]\left[y+\left(\frac{1-\sqrt5}{2}\right)\right]\ge 0$$



05

$\therefore$ Solution of y;



05

$$y\in\left[\frac{\sqrt5-1}{2},\frac{\sqrt5+1}{2}\right] //$$

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(3) $\lim\limits_{x\to1}\left[\dfrac{\cos\pi x+\sin\frac{\pi x}{2}}{(x-1)^2}\right]$

$$=\lim_{x\to1}\frac{\sin\left(\frac{\pi}{2}-\pi x\right)+\sin\left(\frac{\pi x}{2}\right)}{(x-1)^2} \quad 05$$

$$=\lim_{x\to1}\frac{2\sin\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)\cos\left(\frac{\pi}{4}-\frac{3\pi x}{4}\right)}{(x-1)^2}$$

$$=-2\lim_{\frac{\pi x}{4}\to\frac{\pi}{4}}\frac{\sin\left(\frac{\pi x}{4}-\frac{\pi}{4}\right)}{\left(\frac{\pi x}{4}-\frac{\pi}{4}\right)}\cdot\frac{\pi}{4}\lim_{x\to1}\frac{\cos\left(\frac{\pi}{4}-\frac{3\pi x}{4}\right)}{(x-1)} \quad 05$$

$$=-2\cdot\times\frac{\pi}{4}\lim_{x\to1}\frac{\sin\left(\frac{\pi}{2}+\frac{\pi}{4}-\frac{3\pi x}{4}\right)}{(x-1)} \quad 05$$

$$=\frac{\pi}{2}\lim_{\frac{3\pi x}{4}\to\frac{3\pi}{4}}\frac{\sin\left(\frac{3\pi x}{4}-\frac{3\pi}{4}\right)}{\left(\frac{3\pi x}{4}-\frac{3\pi}{4}\right)}\times\frac{3\pi}{4} \quad 05$$

$$=\frac{\pi}{2}\times\frac{3\pi}{4}=\frac{3\pi^2}{8} // \quad 05$$

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(4) $\log_{4\sqrt{x}} x + \log_{4x} \sqrt{x} = 0$

Let; $\log_{4\sqrt{x}} x = \dfrac{\log_4 x}{\log_4 4\sqrt{x}}$

$= \dfrac{\log_4 x}{\log_4 4 + \log_4 \sqrt{x}}$ (05)

$= \dfrac{\log_4 x}{1 + \frac{1}{2}\log_4 x}$

$\log_{4x} \sqrt{x} = \dfrac{\log_4 \sqrt{x}}{\log_4 4x}$

$= \dfrac{\frac{1}{2}\log_4 x}{\log_4 4 + \log_4 x}$ (05)

$= \dfrac{\frac{1}{2}\log_4 x}{1 + \log_4 x}$

$\log_4 x = t$

$\dfrac{t}{1+t/2} + \dfrac{t/2}{1+t} = 0$

$\dfrac{4t(1+t) + t(2+t)}{2(2+t)(1+t)} = 0$ (05)

$\dfrac{(5t^2+6t)}{2(2+t)(1+t)} = 0$

$\dfrac{t(5t+6)}{2(2+t)(1+t)} = 0$

$t = 0$ or $t = -6/5$ (05)

$\therefore \log_4 x = t = 0$

$x = 4^0 = 1$

$\log_4 x = t = -6/5$ (05)

$x = 4^{-6/5}$ //

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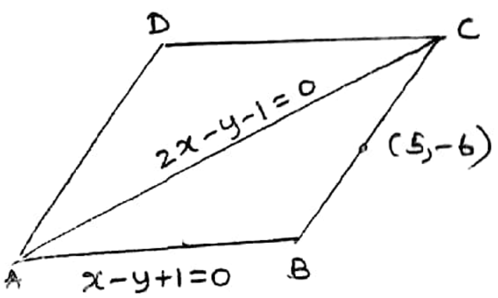
(5)



equation of AD

$2x - y - 1 + \lambda(x - y + 1) = 0$

$m_{AD} = \dfrac{2+\lambda}{1+\lambda}$ (05)

$\left| \dfrac{\frac{2+\lambda}{1+\lambda} - 2}{1 + 2\left(\frac{2+\lambda}{1+\lambda}\right)} \right| = \left| \dfrac{2-1}{1+2} \right|$ (05)

$\left| \dfrac{-\lambda}{3\lambda+5} \right| = \left| \dfrac{1}{3} \right|$

$\dfrac{-\lambda}{3\lambda+5} = \pm \dfrac{1}{3}$ (05)

$(+) \Rightarrow \lambda = -5/6 \quad (-) \Rightarrow \#$

eqⁿ of AD (05)

$2x - y - 1 - \frac{5}{6}[x - y + 1] = 0$

$7x - y - 11 = 0$

Also; $m_{BC} = 7$

$\therefore$ eqⁿ of BC $\Rightarrow$ (05)

$\dfrac{y+6}{x-5} = 7$

$7x - y - 41 = 0$ //

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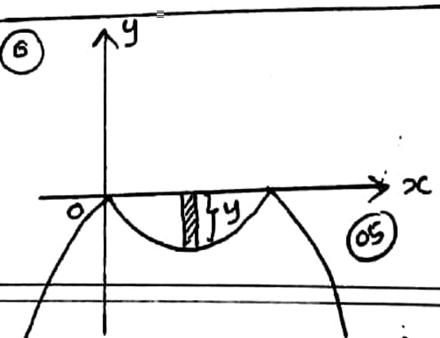
(6)



(05)

$V = \int_0^3 \pi \left(-|x^2 - 3x|\right)^2 dx$ (10)

$= \int_0^3 \pi (x^4 - 6x^3 + 9x^2)\, dx$

$= \pi \int_0^3 (x^4 - 6x^3 + 9x^2)\, dx$

$V = \pi \left[\dfrac{x^5}{5} - \dfrac{6x^4}{4} + \dfrac{9x^3}{3} \right]_0^3$ (05)

$= \pi \left[\dfrac{3^5}{5} - \dfrac{3 \cdot 3^4}{2} + 3 \cdot 3^3 \right]$

$= \pi 3^4 \left[\frac{3}{5} - \frac{3}{2} + 1 \right]$

$V = \dfrac{3^4 \pi}{10}$ // (05)

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(7) $\frac{x^2}{4}+\frac{y^2}{9}=1$

differentiation with respect to x

$\frac{1}{4}\times 2x+\frac{1}{9}\times 2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-9x}{4y}$ (05)

$\left(\frac{dy}{dx}\right)_P=\frac{-9}{4}\times\frac{2(1-t^2)}{(1+t^2)}\times\frac{(1+t^2)}{6t}$

$=\frac{-3(1-t^2)}{4t}$ (05)

$\therefore$ gradient of normal $=\frac{4t}{3(1-t^2)}$ (05)

$\therefore$ equation of the normal (05)

$y-\frac{6t}{1+t^2}=\frac{4t}{3(1-t^2)}\left[x-\frac{2(1-t^2)}{1+t^2}\right]$

$3(1-t^4)y-4t(1+t^2)x-10t(1-t^2)=0$ (05)

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(8) $I=\int_{-2}^{2}\frac{x^2+1}{x^4-2x^2+1}\,dx$

$I=\int_{-2}^{2}\left(\frac{1+1/x^2}{x^2-2+1/x^2}\right)dx$ (05)

$=\int_{-2}^{2}\left(\frac{1+1/x^2}{(x-1/x)^2}\right)dx$ (05)

$x-\frac{1}{x}=t$

$dt=1+\frac{1}{x^2}$

$x=-2 \quad t=-2+\frac{1}{2}=-3/2$ (05)

$x=2 \quad t=2-\frac{1}{2}=3/2$

$I=\int_{-3/2}^{3/2}\frac{dt}{t^2}$

$=-\left[\frac{1}{t}\right]_{-3/2}^{3/2}$ (05)

$=-\left[\frac{2}{3}+\frac{2}{3}\right]=\frac{-4}{3}$ (05)

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(9) $\tan\theta+\tan 2\theta+\tan 3\theta=0$ (05)

$\tan\theta+\tan 2\theta+\frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=0$

$(\tan\theta+\tan 2\theta)\left[1+\frac{1}{1-\tan\theta\tan 2\theta}\right]=0$ (05)

$(\tan\theta+\tan 2\theta)(2-\tan\theta\tan 2\theta)=0$

$\tan\theta=-\tan 2\theta$

$\theta=n\pi+(\pi-2\theta) \quad n\in\mathbb{Z}$

$3\theta=(n+1)\pi$ (05)

$\theta=\frac{\pi}{3}(n+1); \quad n\in\mathbb{Z}$

or

$\tan\theta\tan 2\theta=2$

$\cancel{2}\tan^2\theta=\cancel{2}(1-\tan^2\theta)$

$2\tan^2\theta=1$ (05)

$\tan\theta=\pm 1/\sqrt{2}$

$\theta=m\pi+\alpha, \quad m\in\mathbb{Z}$

$\alpha=\tan^{-1}\left(\pm\frac{1}{\sqrt{2}}\right)$ (05)

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(10) $\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)=n\pi$

$\alpha=\cot^{-1}\left(\frac{xy+1}{x-y}\right) \qquad \tan\alpha=\frac{x-y}{xy+1}$

$\beta=\cot^{-1}\left(\frac{yz+1}{y-z}\right) \qquad \tan\beta=\frac{y-z}{yz+1}$ (05)

$\gamma=\cot^{-1}\left(\frac{zx+1}{z-x}\right) \qquad \tan\gamma=\frac{z-x}{zx+1}$

$\therefore \alpha+\beta+\gamma=n\pi$

$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ (05)

$=\frac{\frac{x-y}{xy+1}+\frac{y-z}{yz+1}}{1-\left(\frac{x-y}{xy+1}\right)\left(\frac{y-z}{yz+1}\right)}$

$=\frac{x-z}{1+xz}=-\frac{(z-x)}{1+xz}$ (05)

$=-\tan\gamma$

$\therefore \tan(\alpha+\beta)=-\tan\gamma$ (05)

$\tan(\alpha+\beta)=\tan(n\pi-\gamma)$ (05)

$\therefore \alpha+\beta=n\pi-\gamma \quad \therefore \alpha+\beta+\gamma=n\pi$

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11. a) Part B

Let $ax^2+bx+c=0$. (05)

then $\left(x+\frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}$; $a \neq 0$

$x \in \mathbb{R}$; then $x+\frac{b}{2a} \in \mathbb{R}$

$\therefore \left(x+\frac{b}{2a}\right)^2 \geq 0$ (05)

$\therefore \frac{\Delta}{4a^2} \geq 0$ (05)

$\Delta \geq 0$ (05)

$\left(x+\frac{b}{2a}\right)^2 = \left(\frac{\sqrt{\Delta}}{2a}\right)^2$ (05)

$\left(x+\frac{b}{2a}\right) = \pm\frac{\sqrt{\Delta}}{2a}$

$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$

$x = \frac{-b \pm\sqrt{\Delta}}{2a}$ // (05)

To be $x \in \mathbb{Q}$ then $\Delta = k^2$ $k \in \mathbb{Q}$

that is Δ is a perfect square $(a, b, c \in \mathbb{Q})$ (05)

To be $x \notin \mathbb{Q}$ it should be satisfy $\Delta > 0$ and $\Delta \neq k^2$ (05)

$\boxed{40}$

Let $F(x) \equiv ax^3+bx^2+cx+a$

$1-\sqrt{2} \in \mathbb{Q}'$ is a root of $f(x)=0$

then the other root should be $1+\sqrt{2}$. (05)

$F(x) \equiv [x-(1-\sqrt{2})][x-(1+\sqrt{2})](Ax+B)$

$F(x) \equiv (x^2-2x-1)(Ax+B)$ (05)

$ax^3+bx^2+cx+a \equiv (x^2-2x-1)(Ax+B)$

$\left.\begin{array}{ll} ((x^3)); & A=a \\ ((x^2)); & -2A+B=b \end{array}\right\}$ (05)

$\left.\begin{array}{ll} ((x)); & -2B-A=c \\ ((x^0)); & -B=a \end{array}\right\}$ (05)

$\underbrace{A=a \quad B=-a}_{(05)}$ $\underbrace{b=-3a \quad c=a}_{(05)}$

$F(x) \equiv ax^3+bx^2+cx+a$

$= ax^3-3ax^2+ax+a$ (05)

$F(x)=0$

$ax^3-3ax^2+ax+a=0$; $a \neq 0$

$x^3-3x^2+x+1=0$ (05)

$(x^2-2x-1)(x-1)=0$

$\alpha < \beta < \gamma$ (05)

$\alpha = 1-\sqrt{2}$ $\beta = 1$ $\gamma = 1+\sqrt{2}$

Equation of roots α and β

$(x-\alpha)(x-\beta)=0$ (05)

$x^2-(\alpha+\beta)x+\alpha\beta=0$

$x^2-(2-\sqrt{2})x+1-\sqrt{2}=0$

$(1+\sqrt{2})x^2-\sqrt{2}x-1=0$ // (05)

$\boxed{55}$

b) $f(x)=x^4+x^3-(2+d)x^2+d(c+1)x-cd$

$f(c)=0$ (05) ($\because (x-c)$ is a factor of $f(x)$)

$c^4+c^3-(2+d)c^2+d(c+1)c-cd=0$

$c^2(c^2+c-2)=0$

$c^2(c+2)(c-1)=0$ (05)

$\therefore c=0$ or $c=-2$ or $c=1$

But $c \in \mathbb{R}^+$ $\therefore c=1$ //

$f(d) = cd(d-1)$ (05)

$d^4+d^3-(2+d)d^2+d^2(c+1)-cd = cd^2-cd$

$d^4-d^2=0$ (05)

$d^2(d^2-1)=0$

$d^2(d-1)(d+1)=0$

$d=0$ or $d=1$ or $d=-1$

But $d \in \mathbb{R}^+$ (05) $\therefore d=1$ //

$f(x)=x^4+x^3-3x^2+2x-1$ (05)

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$f(x) = x^4 + x^3 - 3x^2 + 2x - 1$

$\equiv (x+2)^3 Q(x) + Ax^2 + Bx + C$

$x = -2 \quad 4A - 2B + C = -9$ (05)

differentiate $f(x)$;

$f'(x) = 4x^3 + 3x^2 - 6x + 2$

$\equiv (x+2)^3 Q'(x) + 3Q(x)(x+2)^2$

$+ 2Ax + B$

$x = -2; \quad -6 = -4A + B$ (05)

second derivative;

$f''(x) = 12x^2 + 6x - 6$

$\equiv (x+2)^3 Q''(x) + 3Q'(x)(x+2)^2$ (05)

$+ 3Q'(x)(x+2) + Q(x) \cdot 6(x+2) + 2A$

$x = -2; \quad 12(-2)^2 + 6(-2) - 6 = 2A$

$A = 15$

$B = 54$ (05)

$C = 39$

The remainder (05)

$= 15x^2 + 54x + 39$

$\boxed{55}$

(12) $f(x) = |3^x - 1| + |3^x - 9|$

$g(x) = 3^x$ (05)

Substitute; $3^x = t$

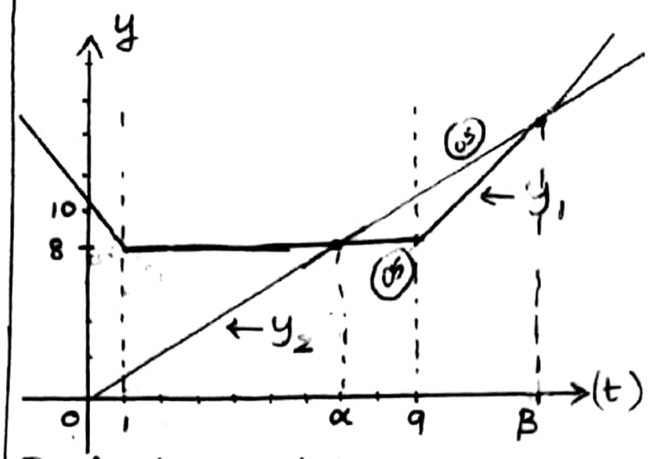
~~$f(x) = |t-1| + |t-9| = y_1$~~ (05)

$g(x) = t = y_2$

$y_1 \begin{cases} -(t-1)-(t-9) & t < 1 \\ (t-1)-(t-9) & 1 \leq t < 9 \\ (t-1)+(t-9) & t \geq 9 \end{cases}$ (05) (05)

$y_1 = \begin{cases} -2t + 10 & t < 1 \\ 8 & 1 \leq t < 9 \\ 2t - 10 & t \geq 9 \end{cases}$ (05)

$y_2 = 8$



To find α and β

$y_1 = y_2$

$\alpha; \quad 8 = t$ $\qquad$ $\beta; \quad 2t - 10 = t$

$t = 10$

$3^x = 8$ (05) $\qquad$ $3^x = 10$

$x \cdot \ln 3 = \ln 8$ (05) $\qquad$ $x \ln 3 = \ln 10$

$x = \frac{\ln 8}{\ln 3} = \alpha$ $\qquad$ $x = \frac{\ln 10}{\ln 3} = \beta$

If $g(x) \geq f(x)$

$\alpha \leq t \leq \beta$ (05)

$\frac{\ln 8}{\ln 3} \leq x \leq \frac{\ln 10}{\ln 3}$

$\boxed{50}$

b) i) $2f(x) + 3f\left(\frac{1}{x}\right) = \frac{1-x^2}{1+x^2}$ — ①

$x \to \frac{1}{x}$; $2f\left(\frac{1}{x}\right) + 3f(x) = \frac{x^2-1}{x^2+1}$ (05) — ②

~~①×2 − ②×3~~

$f(x) = \frac{x^2-1}{x^2+1}$ (05)

ii) $D_f = \mathbb{R}$

$f(x) = \frac{x^2-1}{x^2+1} = y$

$x^2 = \frac{y+1}{1-y}$ (05)

$x = \sqrt{\frac{y+1}{1-y}}$; $y \neq 1$

To $x \in \mathbb{R}$

$\frac{y+1}{1-y} \geq 0$ (05)

$\ominus \quad -1 \quad \oplus \quad 1 \quad \ominus$

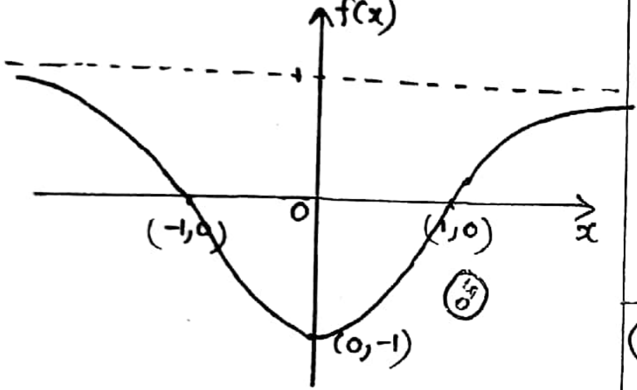
$\therefore R_f = \{y / -1 \leq y < 1,\ y \in \mathbb{R}\}$ (05)

$f(x)_{min} = -1$ (05)

$y = -1;\quad x = \sqrt{\frac{y+1}{1-y}} = 0$ (05)

$\lim_{x \to \pm\infty}[f(x)] = \lim_{x \to \pm\infty}\left(\frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}}\right)$ (05)

$= 1$ // (05)



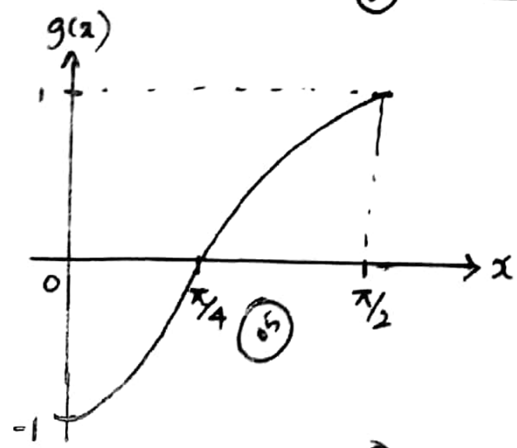
(05)

iii) $g(x) = f(\tan x)$

$= \frac{\tan^2 x - 1}{\tan^2 x + 1}$ (05)

$= \sin^2 x - \cos^2 x$

$g(x) = -\cos 2x$ (05)



(05)

$D_g = (0 \leq x \leq \pi/2)$ (05)

$R_g = -1 \leq x \leq 1$ (05)

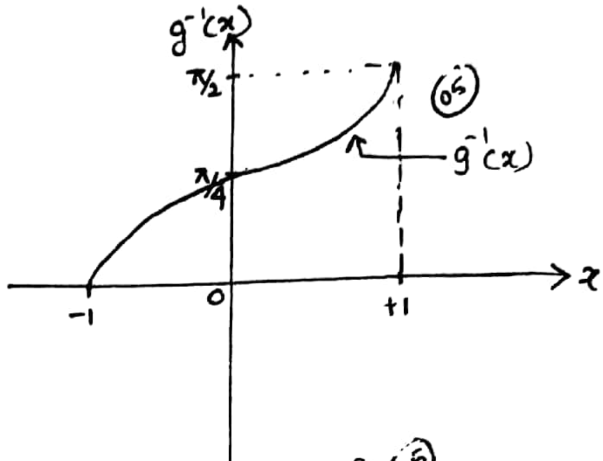
iv) $y = -\cos(2x)$

$y = \cos(\pi - 2x)$ (05)

$\cos^{-1} y = \pi - 2x$

$x = \frac{\pi}{2} - \frac{1}{2}\cos^{-1}(y)$ (05)

$g^{-1}(x) = y = \frac{\pi}{2} - \frac{1}{2}\cos^{-1}(x)$ //



(05)

$D_{g^{-1}} = \{-1 \leq x \leq 1\}$ (05)

$R_{g^{-1}} = \{0 \leq x \leq \pi/2\}$ (05)

— 100

(13) a) Theory —— 20

$y = Ae^{-2x} + Be^{-x} + \sin x$

$y|_{x=0} = 0$

$Ae^0 + Be^0 + \sin 0 = 0$ (05)

$A + B = 0$ — (1)

$\frac{dy}{dx} = A \cdot e^{-2x} \cdot (-2) + Be^{-x} \cdot (-1) + \cos x$ (05)

$\left(\frac{dy}{dx}\right)\Big|_{x=0} = 0$

$-2A - B - 1 = 0$

$2A + B = 1$ — (2)

(2) − (1) (05)

$A = 1 \quad B = -1$

— 15

$y = e^{-2x} - e^{-x} + \sin x$

$\frac{dy}{dx} = -2e^{-2x} + e^{-x} + \cos x$ (05)

$\frac{d^2y}{dx^2} = 4e^{-2x} + e^{-x} \cdot (-1) - \sin x$

$= 4e^{-2x} - e^{-x} - \sin x$ (05)

$$4e^{-2x} - e^{-x} - \sin x$$
$$+ C(-2e^{-2x} + e^{-x} + \cos x) \quad (05)$$
$$+ D(e^{-2x} - e^{-x} + \sin x) = 3\cos x + \sin x$$

$$e^{-2x}(4 - 2C + D) + e^{-x}(-1 + C - D)$$
$$+ \sin x(-1 + D) + \cos x(C)$$
$$= 3\cos x + \sin x$$

$\sin x$;
$-1 + D = 1$
$D = 2$ //

$\cos x$; (05)
$C = 3$ //

20

$$\therefore \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3\cos x + \sin x$$

$$\left(\frac{dy}{dx}\right)_{x=\pi} = -2e^{-2\pi} + e^{-\pi} + \cos \pi$$
$$= e^{-\pi} - 2e^{-2\pi} - 1 \;// \quad (05)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=\pi} = 4e^{-2\pi} - e^{-\pi} - \sin(\pi)$$
$$= 4e^{-2\pi} - e^{-\pi} \;// \quad (05)$$

10

b) i) $\sqrt{y+x} + \sqrt{y-x} = C$

$$\frac{1}{2\sqrt{y+x}}\left(\frac{dy}{dx} + 1\right) + \frac{1}{2\sqrt{y-x}}\left(\frac{dy}{dx} - 1\right) = 0 \quad (05)$$

$$\frac{dy}{dx} = \frac{\sqrt{y+x} - \sqrt{y-x}}{\sqrt{y+x} + \sqrt{y-x}} \quad (05)$$

$$= \frac{\left(\sqrt{y+x} - \sqrt{y-x}\right)^2}{\cancel{y} + x - \cancel{y} + x} \quad (05)$$

$$= \frac{y + \cancel{x} + y - \cancel{x} - 2\sqrt{y^2 - x^2}}{2x}$$

$$= \frac{y}{x} - \sqrt{\frac{y^2 - x^2}{x^2}} \quad (05)$$

$$= \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1} \;//$$

20

ii) $y = x e^{-1/x}$

$$\frac{dy}{dx} = x \cdot e^{-1/x} \cdot \left(\frac{1}{x^2}\right) + e^{-1/x} \quad (05)$$

$$x^2\frac{dy}{dx} = x e^{-1/x} + x^2 e^{-1/x}$$

$$x^2\frac{dy}{dx} = y + xy \quad (05)$$

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{dy}{dx} + x\frac{dy}{dx} + y \quad (05)$$

$$\times x \quad x^3\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - x\frac{dy}{dx} - xy = 0$$

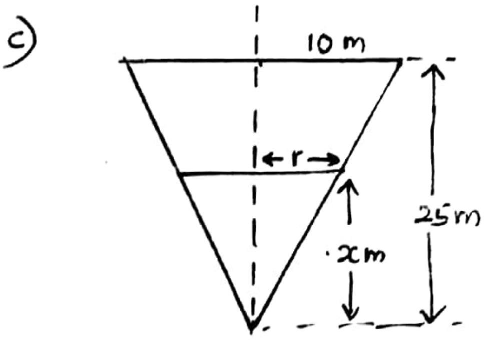
$$x^3\frac{d^2y}{dx^2} + y + \cancel{xy} - x\frac{dy}{dx} - \cancel{xy} = 0 \quad (05)$$

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + \cancel{\frac{dy}{dx}} - x\frac{d^2y}{dx^2} - \cancel{\frac{dy}{dx}} = 0 \quad (05)$$

$$x^3\frac{d^3y}{dx^3} + x(3x - 1)\frac{d^2y}{dx^2} = 0 \;//$$

30

c)



$$\frac{10}{r} = \frac{25}{x}$$

$$r = \frac{2x}{5} \quad (05)$$

Volume of the water

$$V = \frac{1}{3}\pi r^2 x = \frac{1}{3}\pi\left(\frac{2x}{5}\right)^2 x \quad (10)$$

$$= \frac{1}{3}\pi \frac{4x^3}{25} = \frac{4\pi x^3}{75}$$

$$\frac{dx}{dt} \times 100\% = -t\% \quad (05)$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} \quad (05)$$

$$= \frac{d}{dx}\frac{4\pi x^3}{75} \times (-t\%) \quad (05)$$

$$= \frac{4\pi}{75} \times 3x^2 \times (-t\%)$$

$$= \frac{4}{25} \times \frac{22}{7} \times (-t\%)\, x^2 \quad (05)$$

$$= \frac{-88x^2 t\%}{175} \;//$$

(14) a) $y = \dfrac{6(x^2-8x+1)}{(x-2)^3}$

$$\frac{dy}{dx} = \frac{6\left[(x-2)^3(2x-8) - (x^2-8x+1)3(x-2)^2\right]}{(x-2)^6} \quad (10)$$

$$= \frac{6(x-2)^2\left[2(x-2)(x-4) - 3(x^2-8x+1)\right]}{(x-2)^6}$$

$$= \frac{-6}{(x-2)^4}\left[x^2-12x-13\right] \quad (05)$$

$$f'(x) = \frac{-6(x-13)(x+1)}{(x-2)^4}$$

$$f''(x) = \frac{-6\left[(x-2)^4(2x-12) - (x^2-12x-13)4(x-2)^3\right]}{(x-2)^8}$$

$$= \frac{-6}{(x-2)^5}\left[2(x-2)(x-6) - (x^2-12x-13)4\right]$$

$$= \frac{+12(x^2-16x-38)}{(x-2)^5} \quad (10)$$

$$x = \frac{16 \pm \sqrt{408}}{2} = \frac{16 \pm 2\sqrt{102}}{2}$$

$$= 8 \pm \sqrt{102} \quad (\sqrt{102} \simeq 10)$$

$x = 18$ or $x = -2$ (05)

$$\therefore f''(x) = \frac{12(x-18)(x+2)}{(x-2)^5}$$

At the turning points $f'(x) = 0$

$x = -1$ or $x = 13$

When $\dfrac{dy}{dx} = 0$ $\quad x = -1, 13$ (05)

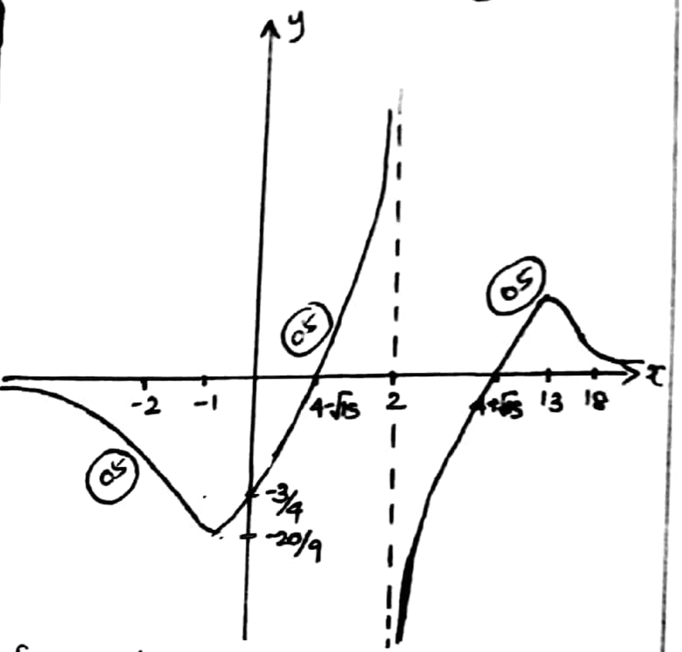
$1 \quad 2 \quad 13 \rightarrow \mathbb{R}$

	$-\infty < x < -1$	$-1 < x < 2$	$2 < x < 13$	$13 < x < \infty$
Sign of dy/dx	$-$ (05)	$+$ (05)	$+$ (05)	$-$ (05)
	fun is decreasing	fun is increasing	fun is increasing	fun is decreasing

$\left(-1, -\dfrac{20}{9}\right)$ $\qquad$ $\left(13, \dfrac{36}{121}\right)$

$$\lim_{x \to \pm\infty} f(x) \begin{cases} \lim_{x \to +\infty} & y \to 0^+ \\ \lim_{x \to -\infty} & y \to 0^- \end{cases} \quad (05)$$

when $x = 0$ $\quad y = -3/4$ (05)



for $x \neq 2$

$$f''(x) = \frac{12(x-18)(x+2)}{(x-2)^5}$$

$-2 \quad 2 \quad 18 \rightarrow \mathbb{R}$

$f''(x) = 0 \Leftrightarrow x = 18, -2$ (05)

	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < 18$	$18 < x < \infty$
Sign of $f''(x)$	$-$	$+$	$-$	$+$
Concavity	Concave down	Concave up	Concave down	Concave up

(10)

Vertical asymptote : $\lim_{x \to \pm\infty} f(x) = 0$

$\therefore y = 0$

$\lim_{x \to 2^-} f(x) \to \infty$ and $\lim_{x \to 2^+} f(x) \to -\infty$ (05)

Vertical asymptote : $x = 2$

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b) $V = x(8-2x)^2$ (05)

$$\frac{dV}{dx} = x\cdot 2(8-2x)\cdot(-2) + (8-2x)^2\cdot 1 \quad (10)$$

$$= (8-2x)[-4x+8-2x]$$

$$= (8-2x)(8-6x)$$

$$= +12(x-4)(x-4/3) \quad (05)$$

when $\frac{dV}{dx}=0$ $x=4$ or $x=4/3$ (05)

$x = 4$ #

	$0<x<4/3$	$4/3<x<4$
Sign of $\frac{dV}{dx}$	$+$	$-$

(10)

i.e. when $x = 4/3$ V gets maximum. (05)

$$\therefore V_{max} = \frac{4}{3}\left[8 - 2\cdot\frac{4}{3}\right]^2 \quad (10)$$

$$= \frac{4}{3}\cdot\frac{(16)^2}{9} = \frac{1024}{27}\ cm^3$$

—50

(15) a) i) $\int_{-1}^{1} x(1-x)^{2019}\,dx$

$(1-x) = t$ (05) $x=-1 \rightarrow t=2$

$-\frac{dx}{dt} = 1$ $x = 1 \rightarrow t = 0$ (05)

$dx = -dt$

$$-\int_0^2 (1-t)t^{2019}\,dt \ (05) = -\int_0^2 t^{2019}\,dt + \int_0^2 t^{2020}\,dt$$

$$= -\left[\frac{t^{2020}}{2020}\right]_0^2 + \left[\frac{t^{2021}}{2021}\right]_0^2 \quad (05)$$

$$= \frac{2^{2021}}{2021} - \frac{2^{2020}}{2020} \quad (05)$$

$$= 2^{2020}\left[\frac{2}{2021} - \frac{1}{2020}\right]$$

$$= \frac{2019\cdot 2^{2020}}{(2021)\cdot(2020)} \quad (05)$$

—30

ii) $\int_0^{\pi/2} x^4\sqrt[3]{\frac{x^3}{x^3+1}}\,dx$

$$= \int_0^{\pi/2} \frac{x^5}{(x^3+1)^{1/3}}\,dx = \int_0^{\pi/2} x^5(x^3+1)^{-1/3}\,dx$$

$$= \frac{1}{3}\int_0^{\pi/2} x^3\cdot\left[3x^2(x^3+1)^{-1/3}\right]dx$$

$$= \frac{1}{3}\int_0^{\pi/2} x^3\frac{d}{dx}\left[\frac{(x^3+1)^{2/3}}{2/3}\right]dx \quad (05)$$

$$= \frac{1}{3}\left[\frac{3}{2}x^3(x^3+1)^{2/3}\right]_0^{\pi/2} \ (05) - \frac{1}{3}\int_0^{\pi/2}\frac{3}{2}(x^3+1)^{2/3}\cdot 3x^2\,dx$$

$$= \frac{1}{3}\left[\frac{3}{2}x^3(x^3+1)^{2/3}\right]_0^{\pi/2} \ (05) - \frac{1}{2}\int_0^{\pi/2}\frac{d}{dx}\left(\frac{(x^3+1)^{5/3}}{5/3}\right)dx$$

$$= \left[\frac{(x^3+1)^{2/3}x^3}{2}\right]_0^{\pi/2} - \frac{1}{2}\left[\frac{(x^3+1)^{5/3}}{5/3}\cdot 1\right]_0^{\pi/2} \ (05) + \underbrace{\frac{1}{2}\int_0^{\pi/2}\frac{(x^3+1)^{5/3}}{5/3}\cdot 0\,dx}_{=0}$$

$$= \frac{(\pi^3+8)^{2/3}}{16}\cdot\frac{\pi^3}{8} - \frac{3(\pi^3+8)^{5/3}}{80} + \frac{3}{10} \quad (05)$$

$$\cancel{= \frac{(\pi^3+8)^{2/3}(\pi^3-12)}{320} + \frac{3}{10}}$$

$$= \frac{(\pi^3-12)(\pi^3+8)^{2/3} + 96}{320}$$

—25

b) $y_n = \cos^n x$

$$\frac{dy_n}{dx} = n\cos^{n-1}x\cdot(-\sin x) \quad (05)$$

$$\frac{d^2y_n}{dx^2} = -n\left[\cos^{n-1}x\cdot\cos x + \sin x\cdot(n-1)\cos^{n-2}x(-\sin x)\right] \quad (05)$$

$$= -n\left[\cos^n x - \sin^2 x\,(n-1)\cos^{n-2}x\right]$$

$$= -n\left[\cos^n x - (1-\cos^2 x)\cos^{n-2}x\,(n-1)\right]$$

$$= -n\left[\cos^n x + (n-1)\cos^n x - (n-1)\cos^{n-2}x\right]$$

$$= -n\left[y_n + (n-1)y_n - (n-1)y_{n-2}\right]$$

$$= -n^2 y_n + n(n-1)\,y_{n-2} \quad 05$$

$$I_n = \int_0^{\pi} e^{-x} y_n\,dx$$

$$I_n = -\int_0^{\pi} y_n\,(e^{-x})\,dx \quad 05$$

$$I_n = \left[-y_n e^{-x}\right]_0^{\pi} - \int_0^{\pi} e^{-x}\frac{dy_n}{dx}\,dx$$

$$= -\left[(y_n)_{\pi} e^{-\pi} - (y_n)_0 e^{0}\right] \quad 05$$

$$+ \int_0^{\pi} \frac{dy_n}{dx}\cdot\frac{d}{dx}(e^{-x})\,dx \quad 05$$

$y_n = \cos^n x$

when $x = \pi$ $(y_n)_{\pi} = \cos^n \pi$

(when n is even)

$$= (\cos\pi)^n = 1 \quad 05$$

$$(y_n)_{\pi} = 1$$

$$(y_n)_0 = \cos^n 0 = (\cos 0)^n = 1$$

(when n is even) 05

$$I_n = -\left[e^{-\pi} - 1\right] + \int_0^{\pi} \frac{dy_n}{dx}\frac{d}{dx}(e^{-x})\,dx$$

$$I_n = 1 - e^{-\pi} + \underbrace{\left[\frac{dy_n}{dx}e^{-x}\right]_0^{\pi}}_{=0} \quad 05$$

$$- \int_0^{\pi} e^{-x}\frac{d^2y_n}{dx^2}\,dx$$

when $\dfrac{dy_n}{dx} = -n\cos^{n-1}x\sin x$

when $x = \pi$, $x = 0$ $\left(\dfrac{dy_n}{dx}\right) = 0$ 05

$$I_n = -\int_0^{\pi} e^{-x}\frac{d^2y_n}{dx^2}\,dx + 1 - e^{-\pi}$$

$$= 1 - e^{-\pi} - \int_0^{\pi} e^{-x}\frac{d^2y_n}{dx^2}\,dx \quad 05$$

$$I_n = 1 - e^{-\pi} + \int_0^{\pi} e^{-x}\left(\frac{d^2y_n}{dx^2}\right)dx$$

$$= 1 - e^{-\pi} + \int_0^{\pi} e^{-x}\left[-n^2y_n + n(n-1)y_{n-2}\right]dx \quad 05$$

$$I_n = 1 - e^{-\pi} - n^2\underbrace{\int_0^{\pi} e^{-x}y_n\,dx}_{I_n} + n(n-1)\int_0^{\pi} e^{-x}y_{n-2}\,dx \quad 05$$

$$I_n(1+n^2) = 1 - e^{-\pi} + n(n-1)I_{n-2}$$

$$n(n-1)I_{n-2} = (n^2+1)I_n - (1-e^{-\pi})$$

$n=2$ $\quad 2I_0 = 5I_2 - (1-e^{-\pi})$ — (1)

$n=4$ $\quad 12I_2 = 17I_4 - (1-e^{-\pi})$ — (2) 05

(2) × 5 + (1) × 12

$$60I_2 - 85I_4 + 24I_0 - 60I_2$$

$$= -5(1-e^{-\pi}) - 12(1-e^{-\pi}) \quad 05$$

$$-17(1-e^{-\pi}) = -85I_4 + 24I_0 \quad (3)$$

$$I_0 = \int_0^{\pi} e^{-x}y_0\,dx$$

$$= \int_0^{\pi} e^{-x}dx = -\left[e^{-x}\right]_0^{\pi}$$

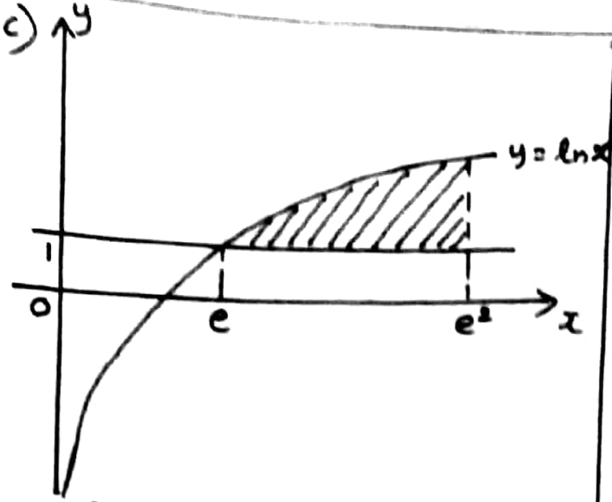
$$= -\left[e^{-\pi} - 1\right] \quad 05$$

$$= 1 - e^{-\pi}$$

(3) $\Rightarrow$ $\quad 85I_4 = 24(1-e^{-\pi}) + 17(1-e^{-\pi})$

$$I_4 = \frac{41}{85}(1-e^{-\pi}) \quad 05$$

c)



dx

$$A = \int_e^{e^2} \ln x\,dx - [e^2 - e] \times 1 \quad (05)$$

$$= \int_e^{e^2} \ln x \frac{d(x)}{dx}\,dx - e^2 + e$$

$$= [x \ln x]_e^{e^2} - \int_e^{e^2} x \cdot \frac{1}{x}\,dx \quad (05)$$

$$- e^2 - e$$

$$= [e^2 \ln e^2 - e \ln e]$$

$$- e^2 + e \quad - e^2 + e \quad (05)$$

$$= 2e^2 - e - 2e^2 + 2e = e$$

$$A = e \simeq 2.718 //$$

15

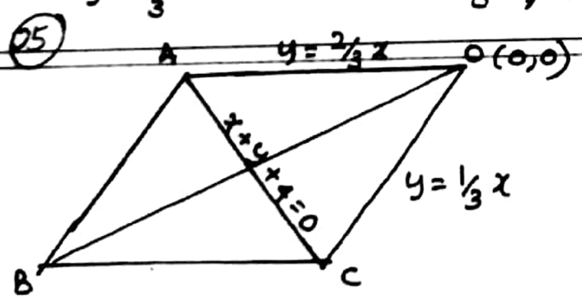
(16) $2x^2 - 9xy + 9y^2 = 0$

$(x - 3y)(2x - 3y) = 0$

$x - 3y = 0$ or $2x - 3y = 0$

$y = \frac{1}{3}x$ or $y = \frac{2}{3}x //$

(25)



$A \equiv \left(-\frac{12}{5}, -\frac{8}{5}\right)$ $\quad C \equiv (-3, -1)$

eq^n of AB

$y + \frac{8}{5} = \frac{1}{3}\left(x + \frac{12}{5}\right)$

$y = \frac{1}{3}x - \frac{4}{5} \quad -(1) \quad (10)$

$15y - 5x + 12 = 0$

eq^n of BC

$y + 1 = \frac{2}{3}(x + 3) \quad (5) \quad (5)$

$y = \frac{2}{3}x + 1 // \quad -(2) \quad 3y - 2x - 1 = 0$

$(1) = (2) \quad \frac{1}{3}x - \frac{4}{5} = \frac{2}{3}x + 1$

$-\frac{9}{5} = \frac{1}{3}x$

$x = -27/5$

$y = -13/5$

$\therefore B \equiv \left(-\frac{27}{5}, -\frac{13}{5}\right)$

eq^n of OB (05)

$y = \frac{13}{27}x // \quad 27y - 13x = 0$

$(OC)^2 = \sqrt{9+1} = \sqrt{10} \quad (05)$

OC perpendicular distance from A

$$= \left| \frac{-3(8/5) + \frac{12}{5}}{\sqrt{10}} \right|$$

$$= \left| \frac{12}{5\sqrt{10}} \right| \quad (5)$$

Area of triangle OAC

$= \frac{1}{2}\sqrt{10} \times \frac{12}{5\sqrt{10}} = \frac{6}{5}$ square units.

Area of OABC parallelogram

$= 2 \times \frac{6}{5} = \frac{12}{5}$ square units // (5)

$27y - 13x + \lambda(x + y + 4) = 0 \quad (5)$

$l_1 \equiv (27+\lambda)y + (\lambda - 13)x + 4\lambda = 0$

$l_2 \equiv 2x - y - 5 = 0$

$m_{l_1} = \frac{13-\lambda}{27+\lambda} \quad m_{l_2} = 2$

$$\tan 45^\circ = \left| \frac{\frac{13-\lambda}{27+\lambda} - 2}{1 + 2\frac{(13-\lambda)}{27+\lambda}} \right| \quad (5)$$

$$1 = \left|\frac{13-\lambda-54-2\lambda}{27+\lambda+26-2\lambda}\right|$$

$$1 = \left|\frac{-41-3\lambda}{53-\lambda}\right|$$

(+) $53-\lambda = -41-3\lambda$

$2\lambda = -94$

$\lambda = -47$

(−) $53-\lambda = 41+3\lambda$

$4\lambda = 53-41 = 12$

$\lambda = 3$

$\lambda = -47 \quad \frac{-41-3\lambda}{53-\lambda} = \frac{-41+141}{53+47}$ (5)

$= \frac{100}{100} = 1$

$\lambda = 3 \quad \frac{-41-9}{53-3} = \frac{-50}{50} = -1$ # (5)

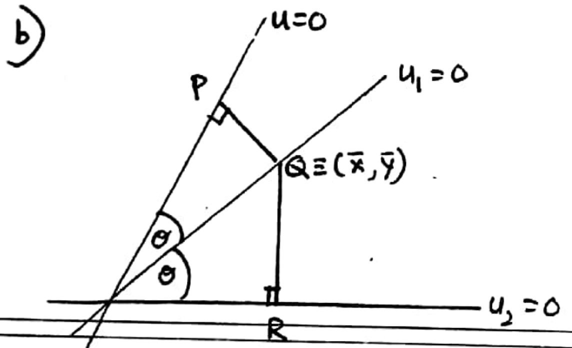
$\therefore \lambda = -47$

$(27-47)y + (-47-13)x + 4\times(-47) = 0$

$-20y - 60x - 4\times 47 = 0$

$5y + 15x + 47 = 0$ // (10)

b)



$u = u_1 + \lambda u_2 = 0$ (5)

$u = l_1x + m_1y + n_1 + \lambda(l_2x + m_2y + n_2) = 0$

$u \equiv (l_1+\lambda l_2)x + (m_1+m_2\lambda)y + n_1 + n_2 = 0$

Let $Q = (\bar{x},\bar{y})$ on $u_1 = 0$

$\therefore l_1\bar{x} + m_1\bar{y} + n_1 = 0$

Also $QP = QR$

$$\left|\frac{(l_1+l_2\lambda)\bar{x} + (m_1+m_2\lambda)\bar{y} + (n_1+\lambda n_2)}{\sqrt{(l_1+l_2\lambda)^2 + (m_1+m_2\lambda)^2}}\right|$$

$$= \left|\frac{l_2\bar{x} + m_2\bar{y} + n_2}{\sqrt{l_2^2+m_2^2}}\right|$$

Since $l_1\bar{x} + m_1\bar{y} + n_1 = 0$

$$\frac{\lambda|(l_2\bar{x} + m_2\bar{y} + n_2)|}{(l_1+l_2\lambda)^2 + (m_1+m_2\lambda)^2}$$

$$= \frac{|l_2\bar{x} + m_2\bar{y} + n_2|}{\sqrt{l_2^2+m_2^2}}$$

$\therefore \lambda\sqrt{l_2^2+m_2^2} = \sqrt{(l_1+l_2\lambda)^2 + (m_1+m_2\lambda)^2}$ (5)

$\lambda^2(l_2^2+m_2^2) = l_1^2 + 2l_1l_2\lambda + l_2^2\lambda^2 + m_1^2 + 2m_1m_2\lambda + m_2^2\lambda^2$

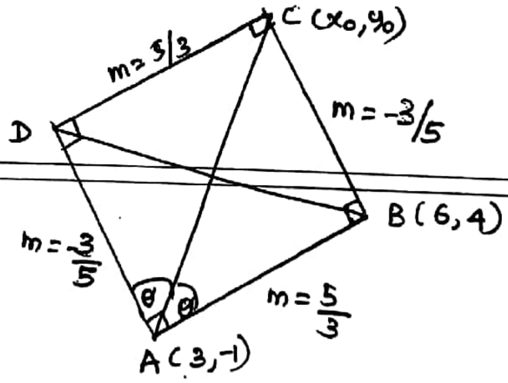
$-[l_1^2+m_1^2] = \lambda[2l_1l_2 + 2m_1m_2]$

$$\lambda = \frac{-(l_1^2+m_1^2)}{2(l_1l_2+m_1m_2)}$$

$\therefore$ eqⁿ

$$l_1x + m_1y + n_1 - \frac{(l_1^2+m_1^2)[l_2x+m_2y+n_2]}{2(l_1l_2+m_1m_2)} = 0$$

$2(l_1l_2+m_1m_2)u_1 + (l_1^2+m_1^2)u_2 = 0$ //



AD ⇒

$$\frac{y+1}{x-3} = -\frac{3}{5}$$

$5y + 5 = -3x + 9$

$5y + 3x - 4 = 0$

AB ⇒ $y + 1 = \frac{5}{3}(x-3)$

$3y - 5x + 18 = 0$ // (5)

using above proved eq^n

$2(l_1 l_2 + m_1 m_2)u_1 - (l_1^2 + m_1^2)u_2 = 0$

$AD = 5y + 3x - 4 = 0$ (5)

$AC = y - mx + 1 + 3m$ ← $y + 1 = m(x-3)$ (5)

$2(-3m + 1\times 5)(y - mx + 1 + 3m) - (m^2+1)(5y + 3x - 4) = 0$ (5)

$y(10 - 6m - 5(m^2+1)) + x[6m^2 - 10m - 3m^2 - 3] + (1+3m)(10-6m) + 4(m^2+1) = 0$

$y(-5m^2 - 6m + 5) + x(3m^2 - 10m - 3) + (1+3m)(10-6m) + 4(m^2+1) = 0$ $\perp AB$

$AB \Rightarrow 3y - 5x + 18 = 0$

(5)

Comparing the ratio of coefficients;

$$\frac{-5m^2 - 6m + 5}{3} = \frac{3m^2 - 10m - 3}{-5}$$ (5)

$25m^2 + 30m - 25 = 9m^2 - 30m - 9$

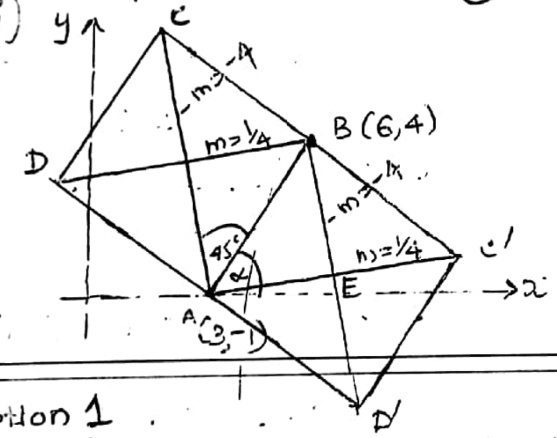
$16m^2 + 60m - 16 = 0$

$4m^2 + 15m - 4 = 0$

$(m - \frac{1}{4})(m + 4) = 0$

$m = 1/4$ or $m = -4$ (5)

i)



Option 1

When the point C lies the left of AB.

$\tan\alpha = 5/3 \quad \alpha = \tan^{-1}(5/3)$

$\left.\begin{array}{l}\alpha > \pi/4 \\ B\hat{A}C = 45^\circ\end{array}\right\}$ $E\hat{A}C > \pi/2 \quad \therefore m_{AC} = -4$

$m_{BD} = 1/4$

$(\because m_{AC} \times m_{BD} = -1)$

$\therefore$ Equation of AC

$y + 1 = -4(x-3)$

$y + 4x - 11 = 0$ (5)

Eq^n of BD

$y - 4 = \frac{1}{4}(x-6)$ (5)

$4y - x - 10 = 0$ ←

(see the last page)

(17) a)

$$L.H.S. \quad \frac{1}{1+2\cos(\pi/3+\theta)} + \frac{1}{1+2\cos(\pi/3-\theta)}$$

$$= \frac{1}{1+\cos\theta - \sqrt{3}\sin\theta} + \frac{1}{1+\cos\theta + \sqrt{3}\sin\theta}$$ (5)

$$= \frac{1+\cos\theta+\sqrt{3}\sin\theta + 1 + \cos\theta - \sqrt{3}\sin\theta}{(1+\cos\theta - \sqrt{3}\sin\theta)(1+\cos\theta+\sqrt{3}\sin\theta)}$$ (5)

$$= \frac{2(1+\cos\theta)}{1+\cos^2\theta + 2\cos\theta - 3\sin^2\theta}$$

$$= \frac{2(1+\cos\theta)}{2(2\cos\theta - 1)(1+\cos\theta)}$$ (5)

$$= \frac{1}{2\cos\theta - 1} = R.H.S.$$

b) $L.H.S. = \sin^2\alpha + \sin^2\beta - \sin^2\gamma$

$= \frac{1}{2}(2\sin^2\alpha + 2\sin^2\beta - 2\sin^2\gamma)$

$= \frac{1}{2}[1 - \cos 2\alpha + 1 - \cos 2\beta - 2\sin^2\gamma]$ (5)

$= \frac{1}{2}[2(1-\sin^2\gamma) - (\cos 2\alpha + \cos 2\beta)]$

$= \frac{1}{2}[2\cos^2\gamma - 2\cos(\alpha+\beta)\cos(\alpha-\beta)]$ (5)

$= \frac{1}{2}[2\cos^2\gamma + 2\cos\gamma\cos(\alpha-\beta)]$ (5)

$= \cos\gamma[\cos\gamma + \cos(\alpha-\beta)]$

$= \cos\gamma[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$= 2\cos\gamma\sin\alpha\sin\beta$ (5)

$= 2\sin\alpha\sin\beta\cos\gamma = R.H.S.$

Theory (5)

c) $a^3\cos(B-C) = a^2 k\sin A\cos(B-C)$ (5)

$= a^2 k\sin(B+C)\cos(B-C)$

$= \frac{a^2 k}{2}[\sin 2B + \sin 2C]$

$= a^2 k\sin B\cos B + a^2 k\sin C\cos C$

$= a^2 b\cos B + a^2 c\cos C$ — (1) (5)

Similarly

$b^3\cos(C-A) = b^2 c\cos C + b^2 a\cos A$ — (2) (5)

$c^3\cos(A-B) = c^2 a\cos A + c^2 b\cos B$ — (3) (5)

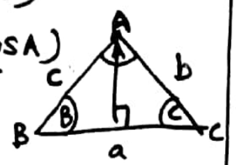
① + ② + ③ ⇒

$a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B)$ 5

$= a^2b\cos B + a^2c\cos C + b^2c\cos C$
$\quad + b^2a\cos A + c^2a\cos A + c^2b\cos B$

$= ab(\underbrace{a\cos B + b\cos A}_{c})$
$\quad + bc(\underbrace{b\cos C + c\cos B}_{a})$
$\quad + ac(\underbrace{a\cos C + c\cos A}_{b})$

$= 3abc$ // 5

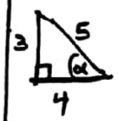


d) $f(\theta) = 4\cos^2\theta + 6\sin\theta\cos\theta + 12\sin^2\theta$

$= 2(1+\cos 2\theta) + 3\sin 2\theta + 6(1-\cos 2\theta)$ 5

$= 8 - 4\cos 2\theta + 3\sin 2\theta$

$= 8 - 5\left(\frac{4}{5}\cos 2\theta - \frac{3}{5}\sin 2\theta\right)$



$= 8 - 5(\cos 2\theta\cos\alpha - \sin 2\theta\sin\alpha)$

$f(\theta) = 8 - 5\cos(2\theta+\alpha)$ 5

$a = 8 \quad b = 5 \quad \alpha = \sin^{-1} 3/5$

$\theta = 0 \Rightarrow f(\theta) = 8 - 5\cos\alpha$
$= 4$ 5

$\theta = \pi \Rightarrow f(\theta) = 8 - 5\cos(2\pi+\alpha)$
$= 8 - 5\cos\alpha$
$= 4$ 5

min point

$f(\theta) = 3$

$8 - 5\cos(2\theta+\alpha) = 3$

$\cos(2\theta+\alpha) = 1$ 5

$2\theta + \alpha = 2n\pi \;;\; n \in \mathbb{Z}$

$\theta = n\pi - \alpha/2$

$\theta = \pi - \alpha/2 \,,\; \theta \in [0,\pi]$ 5

max point,

$f(\theta) = 13$

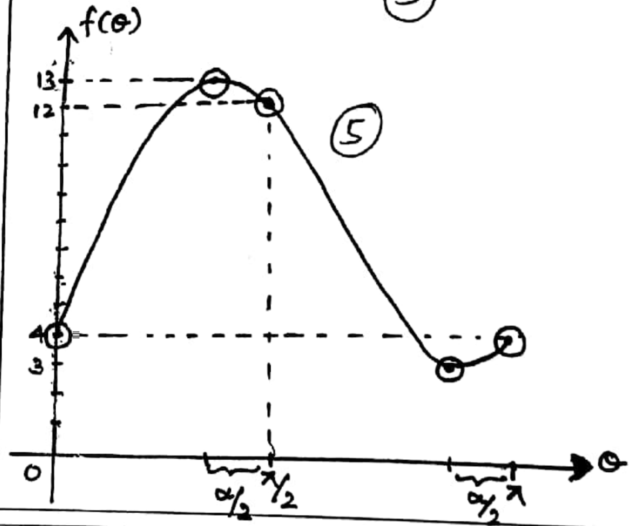
$8 - 5\cos(2\theta+\alpha) = 13$ 5

$\cos(2\theta+\alpha) = -1$

$2\theta + \alpha = 2n\pi \pm \pi \,,\; n \in \mathbb{Z}$

$\theta = n\pi \pm \frac{\pi}{2} - \frac{\alpha}{2}$

$\theta = \frac{\pi}{2} - \frac{\alpha}{2} \,,\; \theta \in [0,\pi]$ 5



5

$-1 \le \cos(2\theta+\alpha) \le 1$ 5

$-5 \le 5\cos(2\theta+\alpha) \le 5$

$5 \ge -5\cos(2\theta+\alpha) \ge -5$

$13 \ge 8 - 5\cos(2\theta+\alpha) \ge 3$

$3 \le f(\theta) \le 13$ 5

$\theta = \pi/2 \Rightarrow f(\theta) = 8 - 5\cos(\pi+\alpha)$
$= 8 + 5\cos\alpha$
$= \underline{12}$ 5

i) $k = 3$ and $k = 13$ 5

ii) $3 < k < 4$ and $4 < k < 13$ 5

iii) $k = 4$ 5

iv) $k < 3$ and $k > 13$ 5

Continue 16 part b)

Option 2

When the point C lies the right of AB. (C')

$m_{AC'} = 1/4$

Equation of AC'

$y+1 = \frac{1}{4}(x-3)$

$4y - x + 7 = 0$ // (05)

$m_{BD'} = -4$

Equation of BD'

$y - 4 = -4(x-6)$

$y + 4x - 28 = 0$ // (05)

ii) Equation of AD = $5y + 3x - 4 = 0$.

Equation of DC.

AD = u_2 . BD = u_1

AD = $5y + 3x - 4 = 0$ BD = $4y - x - 10 = 0$

$l_2 = 3$ $m_2 = 5$ $l_1 = -1$ $m_1 = 4$

Using above proved equation

$2(l_1 l_2 + m_1 m_2)u_1 - (l_1^2 + m_1^2)u_2 = 0$.

$2(-3+20)(4y-x-10) - (1+16)(5y+3x-4) = 0$

$34(4y-x-10) - 17(5y+3x-4) = 0$

$2(4y-x-10) - 5y - 3x + 4 = 0$

$8y - 2x - 20 - 5y - 3x + 4 = 0$

~~$3y - 5x - 16 = 0$ //~~ (05)

Equation D'C'

$u_1 = BD' = y + 4x - 28 = 0$ $u_2 = AD' = 5y + 3x - 4 = 0$.

$l_1 = 4$ $m_1 = 1$ $l_2 = 3$ $m_2 = 5$.

$2(l_1 l_2 + m_1 m_2)u_1 - (l_1^2 + m_1^2)u_2 = 0$.

$2(12+5)(y+4x-28) - (16+1)(5y+3x-4) = 0$.

$2(y+4x-28) - 5y - 3x + 4 = 0$.

$2y + 8x - 56 - 5y - 3x + 4 = 0$.

$-3y + 5x - 52 = 0$.

$3y - 5x + 52 = 0$ // (05)