D.S. Senanayake College - Colombo 07
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Third Term Test, November 2023

Combined Maths I සංයුක්ත ගණිතය I	Grade 13 13 ශ්‍රේණිය	*Three hours and ten minutes* *පැය තුනයි විනාඩි දහයයි*

Index No. :	

Instruction:

- ★ This Question paper consists of two parts.
  Part A (Questions 1 -10) and **Part B** (Questions 11 -17).
- ★ **Part A**
  Answer all questions. Write your answer to question in the space provided.
- ★ **Part B**
  Answer any 5 Questions.
- ★ At the end of the time allotted, tie the answers of the two parts together so that part A is on top of Part B before handing them over to the supervisor.
- ★ You are permitted to remove only Part B of the question paper from the examination hall.

For Examiners' Use only

(10) Combined Mathematics I		
Part	**Question No.**	**Marks**
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
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	17	
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In Numbers	
In Words	

Code Numbers

Marking Examiner	
Checked by : 1.	
2.	
Supervised by :	

Part A

01. Using the Principle of mathematical induction prove that $\sum_{r=1}^{n}(3r+1)=\frac{n}{2}(3n+5)$ for all $n\in\mathbb{Z}^{+}$.

02. Sketch the graphs of $y=|2x-3|-3$ and $y=3-|x|$ in the same diagram Hence or otherwise find all the real values of x satisfying the inequality $|2x-1|+|x|\leq 2$.

03. Let $a \in \mathbb{R}$, Write down the expansion of $(1+ax)^5$ in ascending powers of x up to the term including x^2. Hence, find the values of 'a' for which coefficient of x^2 in the expansion of $(1+x)^2(1+ax)^5$ is 21.

04. Shade in an Argand diagram the region consisting of points that represent the complex numbers Z satisfying the inequalities $\left|\overline{Z} - 2 + 2\sqrt{3}\, i\right| \le 2$ and $Arg(\overline{Z} + 4) \ge -\dfrac{\pi}{3}$.

Find the greatest value of $\left|\overline{Z}\right|$ for the complex numbers Z represented by the point in this in this shaded region.

05. Show that $\lim\limits_{x \to 0} \dfrac{2x\sin 2x + \cos 2x - 1}{\sqrt{1+x^2} - \sqrt{1-x^2}} = 2$

06. Show that the equation of the normal to the curve $\dfrac{x^2}{3} + y^2 = 1$ at the point $P(\sqrt{3}\cos\theta,\ \sin\theta)$ is $\sqrt{3}x\sin\theta - y\cos\theta = 2\sin\theta\cos\theta$ for $0 < \theta < \dfrac{\pi}{3}$. Given that the normal at P meets the coordinate axes at A and B. The area of ΔOAB is $\dfrac{1}{2}$ square units, find the value of θ.

07. Using $\dfrac{d}{dx}\left(\dfrac{x}{1+x^2}\right)=\dfrac{2}{(1+x^2)^2}-\dfrac{1}{(1+x^2)}$, show that $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}dx=\frac{1}{8}(\pi+2)$

The region enclosed by the curves $y=\dfrac{4}{1+x^2}$; $x=0$; $x=1$ and $y=0$ is rotated through 2π radians.

Show that the volume of the solid generated is $2\pi(\pi+2)$

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08. Let $\lambda\in\mathbb{R}$, Find the equation of the straight line with gradient $\dfrac{1}{\sqrt{3}}$ and passes through $P\equiv(2\lambda-1,\ \sqrt{3}\,\lambda)$.

If it intersects the coordinate axes at A and B and AB = 6 find the possible values of λ.

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09. Find the equation of the circle which touches the x axis, passes through (2, 2) and the centre lies on the y axis. Determine the position of the point (- 2, 3) about the above circle.

10. Let $x > 0$, solve the equation $2\tan^{-1}\left(\dfrac{x}{3}\right) + \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$ for x.

Part - B

Answer 5 questions only.

11. a) Let p and q are positive constants, show that the roots of $x^2-(p+2q)x+q^2=0$ are real and distinct. If α and $\beta(<\alpha)$ are the roots of the above equation express $(\alpha-q)(q-\beta)$ in forms of p and q and deduce that $\alpha>q$ and $\beta<q$, show that $\alpha-\beta=\sqrt{p(p+4q)}$ .Show that the equation whose roots are $(\alpha-q)$ and $(\beta-q)$ is $x^2-\sqrt{p(p+4q)}\,x+pq=0$.

b) Let $f(x)=x^3-(a+b)x^2+b(a+1)x-ab$ where $a,b\in\mathbb{R}$ constants $b\neq 0$ show that $(x-a)$ is a factor of $f(x)$ for all $a,b\in\mathbb{R}$.
Given that the remainder is ab when $f(x)$ is divided by $(x-b)$ show that $b=2a$.
If $(x-2)$ is a factor of $f'(x)$ and it is not a factor of $f''(x)$, Find the values of a and b. Hence express $f(x)$ as a product of factors. Find the range of values of x for which $f(x)>0$.
Where $f'(x)$ and $f''(x)$ are derivatives of $f(x)$ and $f'(x)$ respectively with respect to x.

12. a) Find the number of permutations that can be done by taking four letters at a time from the letters of the word 'CHEMISTRY'.
Among them how many permutations are.
(i) beginning with T.
(ii) ending with E.
(iii) including all the vowels.
(iv) including all the vowels and they do not lie next to each other.

b) For $r\in\mathbb{Z}^+$, find the values of λ and μ such that $(r+1)\equiv\lambda(r+4)-\mu$. The r[th] term U_r of an infinite sequence is given by $U_r=\dfrac{3^r(r+1)}{(r+4)!}$, find $f(r)$ such that $U_r=f(r)-f(r+1)$.

Prove that $\displaystyle\sum_{r=1}^{n}U_r=\frac{1}{8}-\frac{3^{n+1}}{(n+4)!}$. If $W_r=U_{2r-1}+U_{2r}$, Find $\displaystyle\sum_{r=1}^{n}W_r$ in terms of n.

13. a) Let $A=\begin{pmatrix}1 & 0 & -2\\ 2 & -1 & 0\end{pmatrix}$ and $B=\begin{pmatrix}a & 3 & 0\\ 1 & b & -1\end{pmatrix}$ be two matrices where a and b are two positive integers.

Given that $AB^T=C$, show that $C=\begin{pmatrix}a & 3\\ 2a-3 & 2-b\end{pmatrix}$.

If C is a singular matrix show that $0<a\le 2$. Hence show that $a=1$ and $b=5$.

Let D = C + I, find D^{-1} and deduce that $D^3=D$ and find D^{2023}.

Write down the simultaneous equations $4x+6y=11$
$x+2y=3$ in the form $D\begin{pmatrix}x\\ y\end{pmatrix}=P$.
Where P is a 2×1 matrix, Hence find the values of x and y.

b) Let $Z, \omega \in \mathbb{C}, \omega \neq 0$ show that $|Z|^2 = Z\overline{Z}$, hence show that $\left|\frac{Z}{\omega}-1\right|^2 = 1+\left|\frac{Z}{\omega}\right|^2 - 2\,\mathrm{Re}\left(\frac{Z}{\omega}\right)$

Given that $|Z+\omega| = |Z-\omega|$ and $|Z| = k|\omega|$ where $k \in \mathbb{R}^+$, show that $\mathrm{Re}\left(\frac{Z}{\omega}\right) = 0$ and deduce that $|Z+\omega|^2 = |Z|^2 + |\omega|^2$ and that $Z = ki\omega$ and give a geometric interpretation for it. Where the points representing Z, ω and 0 in the argand diagram are non collinear.

(c) Show that $(2+\sqrt{3}+i) = 4\cos\frac{\pi}{12}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$. Hence deduce a similar expression for $(2+\sqrt{3}+i)$.

Show that $(2+\sqrt{3}+i)^6 = 2^{12}\left(\cos^6\frac{\pi}{12}\right)i$ and deduce that $(2+\sqrt{3}+i)^6 + (2+\sqrt{3}-i)^6$ is purely real and find its value.

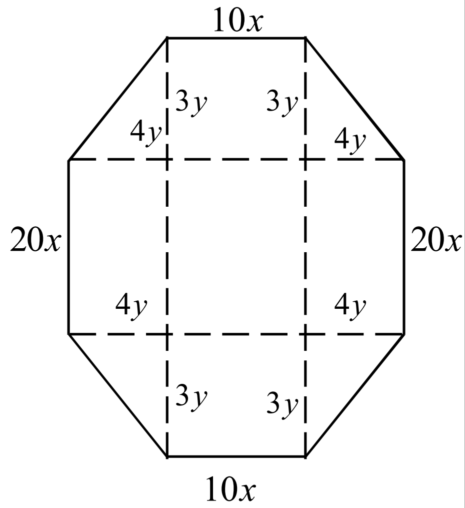
14. a) Let $f(x) = \frac{2(1-2x)}{(x+1)^3},\ x \neq -1$.

If $f'(x)$ is the derivative of $f(x)$, show that $f'(x) = \frac{2(4x-5)}{(x+1)^4},\ x \neq -1$.

Given that $f''(x) = \frac{24(3-x)}{(x+1)^5},\ x \neq -1$. Sketch $y = f(x)$ by clearly indicating turning points, points of inflection and asymptotes.

Sketch $y = |f(x)|$ in a separate diagram hence find the number of real roots for $4|f(x)| = 1$.

b) Given that the perimeter of the octagon shown in the diagram is 2440 cm show that the area A cm² is given by $A = 24y^2 + 220xy + 200x^2$. Hence find the value of x and y when area is maximum justify your answer.



15. a) Find the values of the constants A, B, C and D such that

$x^2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$ for all $x \in \mathbb{R}$. Hence write down $\frac{x^2}{(x^2-1)(x^2+1)}$ in partial fraction and find $\int \frac{x^2}{(x^2-1)(x^2+1)}\,dx$ using the substitution $t^4 = \frac{(1+x^4)}{x^4}$ find $\int \frac{1}{(1+x^4)^{\frac{1}{4}}}\,dx$.

b) Find the constants α and β such that $x^2 - x + 1 = (x-\alpha)^2 + \beta$. Hence by using the substitution $\theta = \tan^{-1}\left(\dfrac{x-\alpha}{\sqrt{\beta}}\right)$ and find $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2-x+1}}dx$.

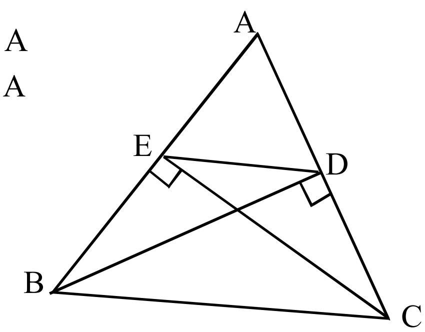
Using the above substitution show that $\displaystyle\int_0^1 \sqrt{x^2-x+1}\,dx = \frac{3}{4}\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \sec^3\theta d\theta$.

Using integration by parts prove that $\displaystyle\int_0^1 \sqrt{x^2-x+1}\,dx = \frac{1}{2} + \frac{3}{8}\ln 3$

Let $I = \displaystyle\int_0^1 \frac{\sin^2\left(\frac{\pi}{2}x\right)}{\sqrt{x^2-x+1}}dx$, using $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$ show that $I = \dfrac{1}{2} + \dfrac{3}{8}\ln 3$

16. Let $P \equiv (x_1, y_1)$ and l be the straight line given by $ax + by + c = 0$. Show that the coordinates of any point on the line through the point P and parallel to l are given by $(x_1 + b\lambda, y_1 - a\lambda)$ where $\lambda \in \mathbb{R}$.
Let l_1 and l_2 be two straight lines given by $4x - 3y + a = 0$ and $x + y + 2a = 0$ respectively. Show that l_1 and l_2 intersect at $A' \equiv (-a, -a)$.
Also, Find that the equations of the bisectors of the angle between l_1 and l_2.
Show that the two points $A = (a, 2a)$, $B = (2a, 4a)$ lie on the same side of the line $l_1 \equiv 4x - 3y + a = 0$ for $a > 0$.
Find the equations of the circles S_1, S_2 in terms of 'a' touching the line l_1 and having A and B as their centres respectively.
Show that the two circles do not intersect and lie outside to each other.

17. a) Write down the sine rule with usual notation for any ΔABC. The area of the acute angled triangle ABC be Δ, show that $\Delta = \dfrac{1}{2}bc\sin A$ with usual notation. Write down another two expressions for Δ.
BD and CE the altitudes of the acute angle ΔABC shown in the diagram. Find $A\hat{E}D$ and $A\hat{D}E$ of ΔAED in terms of B and C.
By using the sine rule for the above triangle show that $DE = a\cos A$
Deduce that the perimeter of ΔAED is given by $(a+b+c)\cos A$ and show that the area of ΔADE is given by $\Delta\cos^2 A$.



b) Show that $\cot 70^\circ + 4\cos 70^\circ = \sqrt{3}$ and find the general solution of $\cos x + \sqrt{3}\sin x = \cot 70^\circ + 4\cos 70^\circ$