Exercise 1.1

Given $T = 20^\circ C$

$\alpha = 90^\circ$

$\sigma = 0.073$ N/m

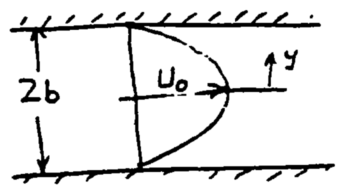
$R = 1.5$ mm $= 1.5\times10^{-3}$ m

Therefore

$$h = \frac{2\sigma \sin\alpha}{\rho g R} \qquad \text{(From Example 1.1)}$$

$$= 2(0.073)(1)/(1000)(9.81)(1.5\times10^{-3}) = 0.00992 \text{ m} = 0.99 \text{ cm}$$

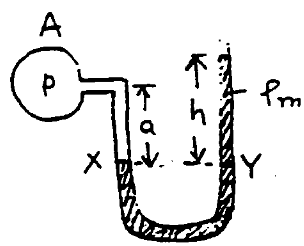
Exercise 1.2



$$u = U_o(1 - y^2/b^2)$$

$$\tau = \mu\frac{du}{dy} = -\mu\frac{2U_o y}{b^2} = -\frac{\mu U_o}{b}$$

Exercise 1.3



$$p_X = p + \rho g a$$

$$p_Y = p_{atm} + \rho_m g h$$

Since $p_X = p_Y$, we get $p + \rho g a = p_{atm} + \rho_m g h$

which gives $p - p_{atm} = \rho_m g h - \rho g a$

Exercise 1.4

Given $m = 2$ kg

$T_1 = 50^\circ C = 323$ K

$p_1 = 3\times10^5$ Pa

$p_2 = 8\times10^5$ Pa

Therefore

$$v_1 = mRT_1/p_1 = 2(287)(323)/(3\times10^5) = 0.618 \text{ m}^3$$

For isothermal process

$$v_2 = p_1 v_1 / p_2 = 3(0.618)/8 = 0.232 \text{ m}^3$$

For isentropic process

$$v_2 = v_1 (p_1/p_2)^{\frac{1}{\gamma}} = 0.618(3/8)^{\frac{1}{1.4}} = 0.307 \text{ m}^3$$

Exercise 1.5

Since $$T = T_o + Kz$$

we have

$$\frac{dp}{dz} = -g\rho = -g(p/RT) = -gp/[R(T_o + Kz)]$$

Therefore

$$\frac{dp}{p} = -\frac{g}{R}\frac{dz}{T_o + Kz}$$

Defining $y \equiv T_o + Kz$ (so that $K\,dz = dy$) and integrating,

$$\Big[\log p\Big]_{p_o}^{p} = -\frac{g}{R}\int_{T_o}^{y}\frac{\frac{1}{K}dy}{y} = -\frac{g}{KR}\Big[\log y\Big]_{T_o}^{y} = -\frac{g}{KR}\Big[\log(T_o + Kz)\Big]_{z=0}^{z}$$

Then

$$\log(p/p_o) = -g/(KR)\log[(T_o + Kz)/T_o] = \log\left(\frac{T_o + Kz}{T_o}\right)^{-g/RK}$$

Therefore

$$\frac{p}{p_o} = \left[\frac{T_o}{T_o + Kz}\right]^{\frac{g}{RK}}$$

Exercise 1.6

Since $T = 15 - 0.001z$, temperature decreases at the rate

$$\frac{dT}{dz} = -0.001\,^\circ\text{C/m} = -1.0\,^\circ\text{C/km}$$

Because this is less than the adiabatic decrease of -10° C/km, the atmosphere is *stable*.

Exercise 1.7

From the first of Eqs. (1.18), $ds = de/T + (p/T)\,dv$. This suggests that we write $s = s(e, v)$ so $ds = (\partial s/\partial e)_v\, de + (\partial s/\partial v)_e\, dv$. Comparing with the line above, $(\partial s/\partial e)_v = 1/T$, $(\partial s/\partial e)_v = p/T$. Cross-differentiating,

$$\frac{\partial^2 s}{\partial e \partial v} = \left[\frac{\partial}{\partial v}\left(\frac{1}{T}\right)\right]_e = \left[\frac{\partial}{\partial e}\left(\frac{p}{T}\right)\right]_v.$$

Now if $e(T, v) = e(T)$ only, then $T(e, v) = T(e)$ only and likewise for $1/T$. Then

$$\left[\frac{\partial}{\partial v}\left(\frac{1}{T}\right)\right]_e = 0.$$

Integrating the right hand side, $p/T = f_1(v)$.

From the second of Eqs. (1.18), $ds = dh/T - (v/T)dp$. This suggests that we write $s = s(h, p)$ so $ds = (\partial s/\partial h)_p\, dh + (\partial s/\partial p)_h\, dp$. Comparing with the line above,

$$\left(\frac{\partial s}{\partial h}\right)_p = \frac{1}{T}, \quad \left(\frac{\partial s}{\partial p}\right)_h = -\frac{v}{T}.$$

Cross-differentiating,

$$\frac{\partial^2 s}{\partial h \partial p} = \left[\frac{\partial}{\partial p}\left(\frac{1}{T}\right)\right]_h = -\left[\frac{\partial}{\partial h}\left(\frac{v}{T}\right)\right]_p.$$

Now if $h(T, p) = h(T)$ only, then $T(h, p) = T(h)$ only and likewise for $1/T$. Then

$$\frac{\partial}{\partial p}\left[\frac{1}{T}\right]_h = 0.$$

Integrating the right hand side, $v/T = f_2(p)$. Putting both results together, $T = p/f_1(v) = v/f_2(p)$ for all p and v. Then $vf_1(v) = pf_2(p) = k$, yielding finally $pv = kT$.

Exercise 1.8

Eqs. (1.18) give $T\,ds = de + p\,dv$ and $T\,ds = dh - v\,dp$. A reversible adiabatic process is isentropic, so $ds = 0$. For a perfect gas with constant specific heats, $de = c_v\,dT, dh = c_p\,dT$. Eqs. (18) reduce to $c_v\,dT = -p\,dv, c_p\,dT = v\,dp$. Dividing the two equations, $c_p/c_v = \gamma = -v\,dp/p\,dv$. Then,

$$\frac{dp}{p} = -\gamma\frac{dv}{v} = \gamma\frac{d\rho}{\rho},$$

since

$$\rho = \frac{1}{v}, \qquad \frac{dv}{v} = -\frac{d\rho}{\rho}.$$

Integrating, $\ln p = \gamma \ln \rho + \ln \text{const.}$ or $p = \text{const.}\rho^{\gamma}$. (1.25)

$$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)^{1/\gamma}, \quad p_1 = \rho_1 R T_1, \ \ p_2 = \rho_2 R T_2 \qquad \frac{p_1}{p_2} = \frac{\rho_1}{\rho_2}\frac{T_1}{T_2}$$

$$\frac{p_1}{p_2}\left(\frac{T_1}{T_2}\right)^{-1} = \left(\frac{p_1}{p_2}\right)^{1/\gamma} \qquad 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma}$$

$$\left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma} = \frac{T_1}{T_2}. \tag{1.26}$$

Exercise 1.9

Since no work is done and no heat is transferred out of the enclosure, the final energy is the sum of the energies in the two compartments. Since $E = mC_vT$ and $E_1 + E_2 = E_f$, $m = \rho V$, $\rho_1 V_1 C_{v_1} T_1 + \rho_2 V_2 C_{v_2} T_2 = \rho_1 V_1 C_{v_1} T_f + \rho_2 V_2 C_{v_2} T_f$

$$T_f = \frac{m_1 C_{v_1} T_1 + m_2 C_{v_2} T_2}{m_1 C_{v_1} + m_2 C_{v_2}}, \quad m_{1,2} = \rho_{1,2} V_{1,2}.$$

Here C_v is the mass specific heat.

Exercise 1.10

Now let C_v be the molar specific heat. Then, since the total energy is unchanged, $n_1 C_{v_1} T_1 + n_2 C_{v_2} T_2 = (n_1 + n_2) C_{v_f} T_f$. Here C_{v_f} is the molar specific heat of the mixture of gases from compartments 1 and 2: $(n_1 + n_2) C_{v_f} = n_1 C_{v_1} + n_2 C_{v_2}$. Thus

$$T_f = \frac{n_1 C_{v_1} T_1 + n_2 C_{v_2} T_2}{n_1 C_{v_1} + n_2 C_{v_2}}.$$

Exercise 1.11

From the first law of thermodynamics, with $Q = 0$, $\Delta E =$ Work $= WL$. For a perfect gas with constant specific heats $E = C_v T$ so $E_2 - E_1 = C_v(T_2 - T_1) = WL$. Then $T_2 = T_1 + WL/C_v$. For a perfect gas $pV/T =$ const. so $p_1 V_1/T_1 = p_2 V_2/T_2$. Now $V_2 = V_1 - AL$ and $p_2 = p_1 + W/A$. Then

$$\frac{p_1 V_1}{T_1} = \frac{(p_1 + W/A)(V_1 - AL)}{T_1 + WL/C_v}.$$

Solve for L.

$$\frac{p_1 V_1}{T_1}\left(T_1 + \frac{WL}{C_v}\right) = \left(p_1 + \frac{W}{A}\right)(V_1 - AL),$$

$$L\left[\frac{p_1 V_1}{T_1}\frac{W}{C_v} + \left(p_1 + \frac{W}{A}\right)A\right] = \left(p_1 + \frac{W}{A}\right)V_1 - p_1 V_1 = \frac{W}{A}V_1,$$

$$L = \frac{WV_1/A}{(p_1 V_1/T_1)(W/C_v) + p_1 A + W}.$$

Exercise 2.1

To show $(\underset{\sim}{a} \times \underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$ (1)

Let $\underset{\sim}{d} = \underset{\sim}{b} \times \underset{\sim}{c}$. Then m component of left side of (1) is

$$(a \times d)_m = \varepsilon_{pqm} a_p d_q = \varepsilon_{pqm} a_p [\varepsilon_{ijq} \; b_i c_j] = - \varepsilon_{ijq} \varepsilon_{qpm} b_i c_j a_p$$

$$= - (\delta_{ip} \delta_{jm} - \delta_{im} \delta_{pj}) b_i c_j a_p = - b_p c_m a_p + b_m c_p a_p$$

This is the m component of right side of (1).

Exercise 2.2

$\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are coplanar if $(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c} = 0$. This requires

$$\varepsilon_{ijk} a_i b_j c_k = 0$$

Exercise 2.3

(i) $\delta_{ij} \delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(ii) $\varepsilon_{pqr} \varepsilon_{pqr} = \varepsilon_{pqr} \varepsilon_{rpq} = \delta_{pp} \delta_{qq} - \delta_{pq} \delta_{qp} = 3(3) - \delta_{pp}$

$$= 9 - 3 = 6$$

(iii) $\varepsilon_{pqi} \varepsilon_{pqj} = \varepsilon_{ipq} \varepsilon_{pqj} = - \varepsilon_{ipq} \varepsilon_{qpj}$

$$= - (\delta_{ip} \delta_{pj} - \delta_{ij} \delta_{pp}) = - \delta_{ij} + 3 \delta_{ij} = 2 \delta_{ij}$$

Exercise 2.4

To show $\underset{\sim}{C} \cdot \underset{\sim}{C}^T = \underset{\sim}{C}^T \cdot \underset{\sim}{C} = \underset{\sim}{\delta}$. The original and transformed coordinates (primed) are related by

$$x_j = C_{ji} x'_i \qquad (1)$$

$$x'_j = C_{ij} x_i \qquad (2)$$

From (2) $x'_i = C_{mi} x_m$. Then (1) becomes

$$x_j = C_{ji} C_{mi} x_m$$

which can be written as

$$x_m \delta_{mj} = C_{ji} C_{mi} x_m$$

Therefore $\delta_{mj} = C_{ji} C_{mi} = C_{mi} C^T_{ij} = (\underset{\sim}{C} \cdot \underset{\sim}{C}^T)_{mj}$. In vector notation, this is $\underset{\sim}{\delta} = \underset{\sim}{C} \cdot \underset{\sim}{C}^T$.

Exercise 2.5

(i) First prove I_1 is invariant. From transformation rule (2.12), the components of $\underset{\sim}{A}$ in rotated coordinates are

$$A'_{mn} = C_{im} C_{jn} A_{ij}$$

Then $I'_1 = A'_{mn} = C_{im} C_{jm} A_{ij} = C_{im} C^T_{mj} A_{ij}$

$= \delta_{ij} A_{ij}$, using the result of Exercise 4

$= A_{ii} = I_1$

This shows that the value of I_1 is same in the two coordinate systems.

(ii) To prove that I_2 is invariant, first show that $A_{mn} A_{nm}$ is invariant.

$$A'_{mn} A'_{nm} = (C_{im} C_{jn} A_{ij})(C_{pn} C_{qm} A_{pq}) = (C_{im} C_{qm})(C_{jn} C_{pn}) A_{ij} A_{pq}$$

$$= \delta_{iq} \delta_{jp} A_{ij} A_{pq} = A_{qj} A_{jq} = A_{mn} A_{mn}$$

This shows that $A_{mn} A_{nm}$ is invariant. In fact, all contracted products of the form $A_{ij} A_{jk} \ldots A_{mi}$, being scalars, are invariants.

Now $I_2 = \frac{1}{2}(I_1^2 - A_{mn} A_{nm})$. For simplicity let us check this ~~This can be checked~~ in two dimensions, for which

$$\frac{1}{2}(I_1^2 - A_{mn} A_{nm}) = \frac{1}{2}[(A_{11} + A_{22})^2 - A_{11} A_{11} - A_{12} A_{21} - A_{21} A_{12} - A_{22} A_{22}]$$

$$= A_{11} A_{22} - A_{12} A_{21} = I_2$$

Since I_1 and $A_{mn}A_{nm}$ are both invariants, so is I_2.

(iii) To prove that I_3 is invariant, first prove that

$$I_3 = A_{ij}A_{jk}A_{ki} + I_2 A_{mm} - I_1 A_{pq}A_{qp}$$

Since the rhs is invariant, so is I_3.

Exercise 2.6

Using the transformation rule (1.8) for vectors

$$u'_m = C_{im}u_i \qquad \text{and} \qquad v'_n = C_{jn}v_j$$

Multiplying,

$$u'_m v'_n = C_{im}C_{jn}u_i v_j$$

This agrees with the transformation rule (2.12) for second-order tensors.

Exercise 2.7

In transformed coordinates

$\delta'_{mn} = C_{im}C_{jn}\delta_{ij} = C_{im}C_{in} = \delta_{mn}$, since $\underset{\sim}{\delta} = \underset{\sim}{C}\cdot\underset{\sim}{C}^T$ from Exercise 2.5.

Exercise 2.8

Let the scalar function be $\psi(r, \phi, z)$ in the cylindrical coordinate system described below.

$$\operatorname{grad} \psi = \lim_{V \to 0} \frac{1}{V} \int_{A=\partial V} d\mathbf{A}\, \psi.$$

CYLINDRICAL COORDINATES

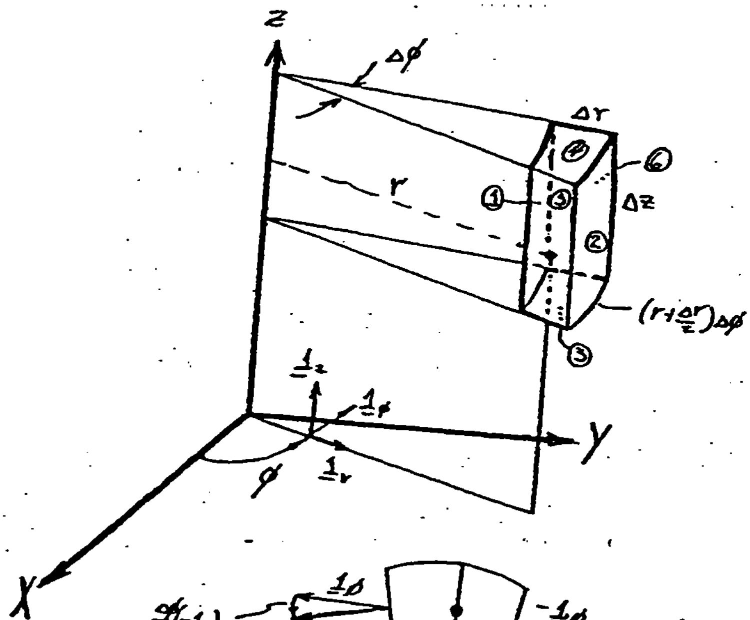
point (r, ϕ, z) at center of volume element.

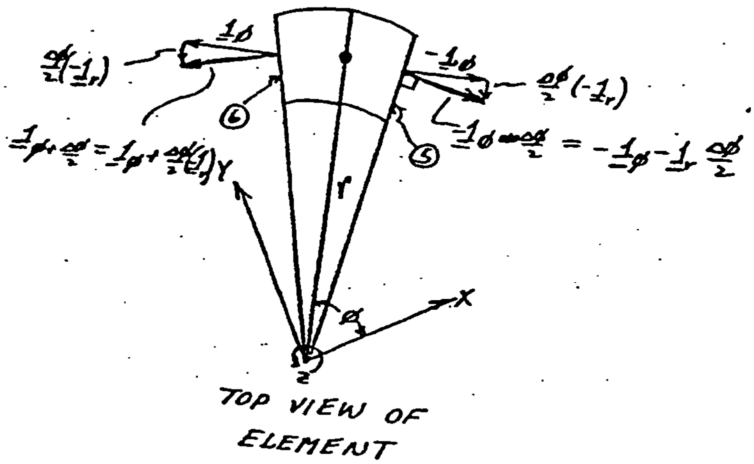
line element

$$\mathbf{dl} \equiv \mathbf{dr} = \mathbf{1}_r\, dr + \mathbf{1}_\phi r\, d\phi + \mathbf{1}_z\, dz$$

volume element

$$\Delta V = (r \Delta\phi)(\Delta r)(\Delta z).$$

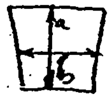




SURFACE ELEMENTS

area $(ab) \times$ normal $\mathbf{1}_i$

$$
\begin{aligned}
\Delta\mathbf{a}_1 &= \left(r - \frac{\Delta r}{2}\right)\Delta\phi\Delta z(-\mathbf{1}_r)\\
\Delta\mathbf{a}_2 &= \left(r + \frac{\Delta r}{2}\right)\Delta\phi\Delta z(\mathbf{1}_r)\\
\Delta\mathbf{a}_3 &= (r\Delta\phi\Delta r)(-\mathbf{1}_z)\\
\Delta\mathbf{a}_4 &= (r\Delta\phi\Delta r)(\mathbf{1}_z)\\
\Delta\mathbf{a}_5 &= (\Delta r\Delta z)\left(-\mathbf{1}_\phi - \mathbf{1}_r\frac{\Delta\phi}{2}\right)\\
\Delta\mathbf{a}_6 &= (\Delta r\Delta z)\left(\mathbf{1}_\phi - \mathbf{1}_r\frac{\Delta\phi}{2}\right).
\end{aligned}
$$



Unit vectors $\mathbf{1}_r$, $\mathbf{1}_\phi$, $\mathbf{k}$ are evaluated at the center of the volume element. ($\mathbf{k} \equiv \mathbf{1}_z$)

$$
\begin{aligned}
\operatorname{grad}\psi = \lim_{\substack{\Delta r\to 0\\ \Delta\phi\to 0\\ \Delta z\to 0}} \Bigg\{ \frac{1}{r\Delta\phi\Delta r\Delta z}\Bigg[&\mathbf{1}_r\left(r + \frac{\Delta r}{2}\right)\Delta\phi\Delta z\,\psi\left(r + \frac{\Delta r}{2}, \phi, z\right)\\
&- \mathbf{1}_r\left(r - \frac{\Delta r}{2}\right)\Delta\phi\Delta z\,\psi\left(r - \frac{\Delta r}{2}, \phi, z\right)\\
&+ \left(\mathbf{1}_\phi - \frac{\Delta\phi}{2}\mathbf{1}_r\right)\Delta r\Delta z\,\psi\left(r, \phi + \frac{\Delta\phi}{2}, z\right)\\
&- \left(\mathbf{1}_\phi + \frac{\Delta\phi}{2}\mathbf{1}_r\right)\Delta r\Delta z\,\psi\left(r, \phi - \frac{\Delta\phi}{2}, z\right)\\
&+ \mathbf{k}\Delta r\cdot r\Delta\phi\,\psi\left(r, \phi, z + \frac{\Delta z}{2}\right) - \mathbf{k}\Delta r\cdot r\Delta\phi\,\psi\left(r, \phi, z - \frac{\Delta z}{2}\right)\Bigg]\Bigg\}.
\end{aligned}
$$

Here the mean value theorem has been used and ψ has been evaluated at the center of each of the six surfaces. Higher order terms will vanish in the limiting processes. Taking the limit of the differences,

$$
\begin{aligned}
\operatorname{grad}\psi &= \mathbf{1}_r\frac{1}{r}\frac{\partial}{\partial r}(r\psi) + \frac{\mathbf{1}_\phi}{r}\frac{\partial\psi}{\partial\phi} - \mathbf{1}_r\left(\frac{1}{2}\frac{\psi}{r} + \frac{1}{2}\frac{\psi}{r}\right) + \mathbf{k}\frac{\partial\psi}{\partial z}\\
\operatorname{grad}\psi &= \mathbf{1}_r\frac{\partial\psi}{\partial r} + \frac{\mathbf{1}_\phi}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{k}\frac{\partial\psi}{\partial z}.
\end{aligned}
$$

Exercise 2.9

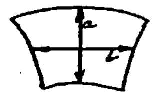
Let the vector function be $\mathbf{F}(r,\theta,\phi)$ in the spherical coordinates described below.

$$\operatorname{div}\mathbf{F} = \lim_{V\to 0}\frac{1}{V}\int_{A=\partial V} d\mathbf{A}\cdot\mathbf{F}.$$

SPHERICAL POLAR COORDINATES
point (r,θ,ϕ) at center of volume element.

line element

$$\mathbf{dl} \equiv \mathbf{dr} = \mathbf{1}_r\,\mathbf{dr} + \mathbf{1}_\theta r\,d\theta + \mathbf{1}_\phi r\sin\theta\,d\phi,$$



volume element

$$\Delta V = (r\Delta\theta)(r\sin\theta\Delta\phi)\Delta r.$$

SURFACE ELEMENTS
area (ab) $\times$ normal $\mathbf{1}_i$:

$$\Delta\mathbf{a}_1 = \left(r+\frac{\Delta r}{2}\right)\Delta\theta\left(r+\frac{\Delta r}{2}\right)\sin\theta\,\Delta\phi(\mathbf{1}_r)$$

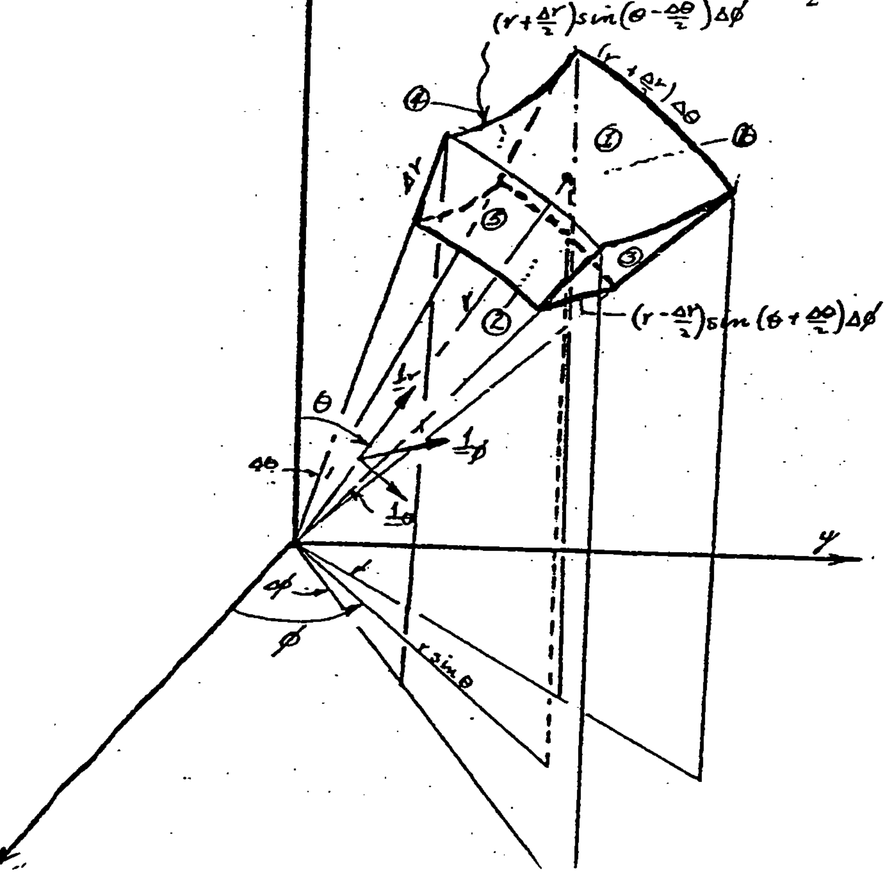
$$\Delta\mathbf{a}_2 = \left(r-\frac{\Delta r}{2}\right)\Delta\theta\left(r-\frac{\Delta r}{2}\right)\sin\theta\,\Delta\phi(-\mathbf{1}_r)$$

$$\Delta\mathbf{a}_3 = \left[r\sin\left(\theta+\frac{\Delta\theta}{2}\right)\Delta\phi\Delta r\right]\left[\mathbf{1}_\theta+\frac{\Delta\theta}{2}(-\mathbf{1}_r)\right]$$

$$\Delta\mathbf{a}_4 = \left[r\sin\left(\theta-\frac{\Delta\theta}{2}\right)\Delta\phi\Delta r\right]\left[-\mathbf{1}_\theta+\frac{\Delta\theta}{2}(-\mathbf{1}_r)\right]$$

$$\Delta\mathbf{a}_5 = [r\Delta\theta\Delta r]\left[-\mathbf{1}_\phi+\sin\theta\frac{\Delta\phi}{2}(-\mathbf{1}_r)+\cos\theta\frac{\Delta\phi}{2}(-\mathbf{1}_\theta)\right]$$

$$\Delta\mathbf{a}_6 = [r\Delta\theta\Delta r]\left[\mathbf{1}_\phi+\sin\theta\frac{\Delta\phi}{2}(-\mathbf{1}_r)+\cos\theta\frac{\Delta\phi}{2}(-\mathbf{1}_\theta)\right]$$



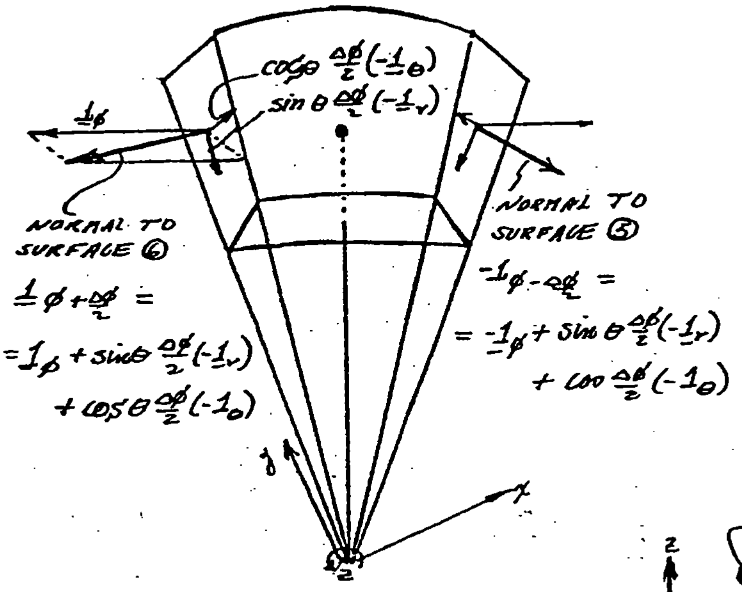
This volume element is bounded by the following six surfaces:

$$\left.\begin{array}{ll} (1) & r + \dfrac{\Delta r}{2} = \text{const.} \\ (2) & r - \dfrac{\Delta r}{2} = \text{const.} \end{array}\right\} \text{spheres}$$

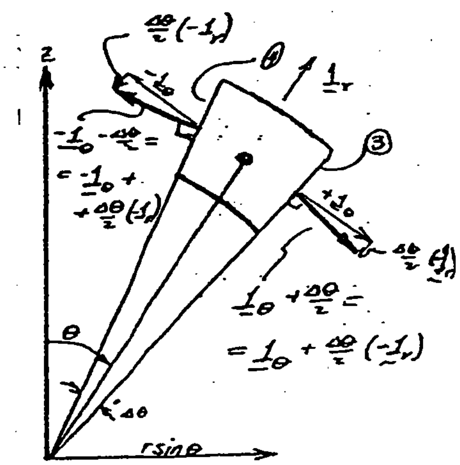
$$\left.\begin{array}{ll} (3) & \theta + \dfrac{\Delta \theta}{2} = \text{const.} \\ (4) & \theta - \dfrac{\Delta \theta}{2} = \text{const.} \end{array}\right\} \text{cones}$$

$$\left.\begin{array}{ll} (5) & \phi - \dfrac{\Delta \phi}{2} = \text{const.} \\ (6) & \phi + \dfrac{\Delta \phi}{2} = \text{const.} \end{array}\right\} \text{planes}$$

Unit vectors are defined at the center of ΔV.



TOP VIEW OF ELEMENT
SHOWING NORMALS



SIDE VIEW OF ELEMENT
SHOWING NORMALS

Using the mean value theorem to evaluate each of the six surface integrals,

$$
\begin{aligned}
\operatorname{div}\mathbf{F} = \lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0\\ \Delta\phi\to 0}} & \left\{\frac{1}{r^2\sin\theta\Delta\theta\Delta\phi\Delta r}\left[\left(r+\frac{\Delta r}{2}\right)^2\sin\theta\Delta\theta\Delta\phi F_r\left(r+\frac{\Delta r}{2},\theta,\phi\right)\right.\right. \\
& -\left(r-\frac{\Delta r}{2}\right)^2\sin\theta\Delta\theta\Delta\phi F_r\left(r-\frac{\Delta\theta}{2},\theta,\phi\right) \\
& +r\sin\left(\theta+\frac{\Delta\theta}{2}\right)\Delta\phi\Delta r\left(\mathbf{1}_\theta-\mathbf{1}_r\frac{\Delta\theta}{2}\right)\cdot\mathbf{F}\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) \\
& +r\sin\left(\theta-\frac{\Delta\theta}{2}\right)\Delta\phi\Delta r\left(-\mathbf{1}_\theta-\mathbf{1}_r\frac{\Delta\theta}{2}\right)\cdot\mathbf{F}\left(r,\theta-\frac{\Delta\theta}{2},\phi\right) \\
& +r\Delta\theta\Delta r\left(\mathbf{1}_\phi-\mathbf{1}_r\sin\theta\frac{\Delta\phi}{2}-\mathbf{1}_\theta\cos\theta\frac{\Delta\phi}{2}\right)\cdot\mathbf{F}\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right) \\
& \left.\left.+r\Delta\theta\Delta r\left(-\mathbf{1}_\phi-\mathbf{1}_r\sin\theta\frac{\Delta\phi}{2}-\mathbf{1}_\theta\cos\theta\frac{\Delta\phi}{2}\right)\cdot\mathbf{F}\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\right]\right\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbf{F}\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) = {} & F_r\left(r,\theta+\frac{\Delta\theta}{2},\phi\right)\mathbf{1}_r\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) \\
& + F_\theta\left(r,\theta+\frac{\Delta\theta}{2},\phi\right)\mathbf{1}_\theta\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) \\
& + F_\phi\left(r,\theta+\frac{\Delta\theta}{2},\phi\right)\mathbf{1}_\phi\left(r,\theta+\frac{\Delta\theta}{2},\phi\right).
\end{aligned}
$$

In terms of constant unit vectors **i**, **j**, **k**,

$$
\begin{aligned}
\mathbf{1}_r(r,\theta,\phi) &= \mathbf{i}\cos\phi\sin\theta+\mathbf{j}\sin\phi\sin\theta+\mathbf{k}\cos\theta, \\
\mathbf{1}_\theta(r,\theta,\phi) &= \mathbf{i}\cos\phi\cos\theta+\mathbf{j}\sin\phi\cos\theta-\mathbf{k}\sin\theta, \\
\mathbf{1}_\phi(r,\theta,\phi) &= -\mathbf{i}\sin\phi+\mathbf{j}\cos\phi.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathbf{1}_r\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) &= \mathbf{i}\cos\phi\sin\left(\theta+\frac{\Delta\theta}{2}\right)+\mathbf{j}\sin\phi\sin\left(\theta+\frac{\Delta\theta}{2}\right)+k\cos\left(\theta+\frac{\Delta\theta}{2}\right) \\
&= \mathbf{i}\cos\phi\sin\theta+\mathbf{j}\sin\phi\sin\theta+\mathbf{k}\cos\theta \\
&\qquad +\frac{\Delta\theta}{2}(\mathbf{i}\cos\phi\cos\theta+\mathbf{j}\sin\phi\cos\theta-\mathbf{k}\sin\theta)=\mathbf{1}_r-\frac{\Delta\theta}{2}\mathbf{1}_\theta, \\
\mathbf{1}_\theta\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) &= \mathbf{1}_\theta-\frac{\Delta\theta}{2}\mathbf{1}_r, \\
\mathbf{1}_\phi\left(r,\theta+\frac{\Delta\theta}{2},\phi\right) &= \mathbf{1}_\phi.
\end{aligned}
$$

Also,

$$\mathbf{1}_r\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)=\mathbf{1}_r+\frac{\Delta\phi}{2}[-\mathbf{i}\sin\phi\sin\theta+\mathbf{j}\cos\phi\sin\theta]=\mathbf{1}_r+\frac{\Delta\phi}{2}\sin\theta\mathbf{1}_\phi,$$

$$\mathbf{1}_\theta\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)=\mathbf{1}_\theta+\frac{\Delta\phi}{2}[-\mathbf{i}\sin\phi\cos\theta+\mathbf{j}\cos\phi\cos\theta]=\mathbf{1}_\theta+\frac{\Delta\phi}{2}\cos\theta\mathbf{1}_\phi,$$

$$\mathbf{1}_\phi\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)=\mathbf{1}_\phi+\frac{\Delta\phi}{2}[-\mathbf{i}\cos\phi-\mathbf{j}\sin\phi]=\mathbf{1}_\phi+\frac{\Delta\phi}{2}[-\sin\theta\mathbf{1}_r-\cos\theta\mathbf{1}_\theta].$$

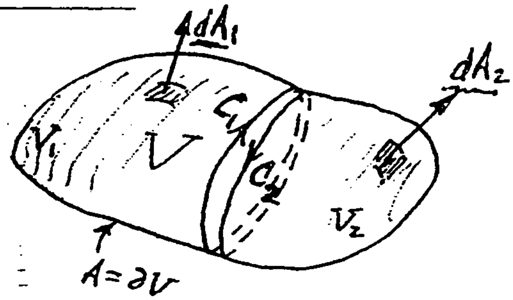
For the evaluations on the $\theta-\Delta\theta/2$ and $\phi-\Delta\phi/2$ surfaces, we can use the results derived above and change the sign of $\Delta\theta/2$ and $\Delta\phi/2$, respectively. Putting all these results into the first expression above,

$$\begin{aligned}
\operatorname{div}\mathbf{F}=\lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0\\ \Delta\phi\to 0}}&\left\{\frac{1}{r^2\sin\theta\Delta\theta\Delta\phi\Delta r}\left[\left(r+\frac{\Delta r}{2}\right)^2\sin\theta\Delta\theta\Delta\phi F_r\left(r+\frac{\Delta r}{2},\theta,\phi\right)\right.\right.\\
&-\left(r-\frac{\Delta r}{2}\right)^2\sin\theta\Delta\theta\Delta\phi F_r\left(r-\frac{\Delta r}{2},\theta,\phi\right)\\
&+r\sin\left(\theta+\frac{\Delta\theta}{2}\right)\Delta\phi\Delta r\left(\mathbf{1}_\theta-\frac{\Delta\theta}{2}\mathbf{1}_r\right)\cdot\left(F_r\left(r,\theta+\frac{\Delta\theta}{2},\phi\right)\right.\\
&\quad\times\left(\mathbf{1}_r+\frac{\Delta\theta}{2}\mathbf{1}_\theta\right)+F_\theta\left(r,\theta+\frac{\Delta\theta}{2},\phi\right)\left.\left(\mathbf{1}_\theta-\frac{\Delta\theta}{2}\mathbf{1}_r\right)\right)\\
&+r\sin\left(\theta-\frac{\Delta\theta}{2}\right)\Delta\phi\Delta r\left(-\mathbf{1}_\theta-\frac{\Delta\theta}{2}\mathbf{1}_r\right)\cdot\left(F_r\left(r,\theta-\frac{\Delta\theta}{2},\phi\right)\right.\\
&\quad\times\left(\mathbf{1}_r-\frac{\Delta\theta}{2}\mathbf{1}_\theta\right)+F_\theta\left(r,\theta-\frac{\Delta\theta}{2},\phi\right)\left.\left(\mathbf{1}_\theta+\frac{\Delta\theta}{2}\mathbf{1}_r\right)\right)\\
&+r\Delta\theta\Delta r\left(\mathbf{1}_\phi-\frac{\Delta\phi}{2}\sin\theta\mathbf{1}_r-\frac{\Delta\phi}{2}\cos\theta\mathbf{1}_\theta\right)\cdot\left(F_r\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)\right.\\
&\quad\times\left(\mathbf{1}_r+\frac{\Delta\phi}{2}\sin\theta\mathbf{1}_\phi\right)+F_\theta\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)\left(\mathbf{1}_\theta+\frac{\Delta\phi}{2}\cos\theta\mathbf{1}_\phi\right)\\
&+F_\phi\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)\left.\left(\mathbf{1}_\phi-\frac{\Delta\phi}{2}\sin\theta\mathbf{1}_r-\frac{\Delta\phi}{2}\cos\theta\mathbf{1}_\theta\right)\right)\\
&+r\Delta\theta\Delta r\left(-\mathbf{1}_\phi-\mathbf{1}_r\sin\theta\frac{\Delta\phi}{2}-\mathbf{1}_\theta\cos\theta\frac{\Delta\phi}{2}\right)\cdot\left(F_r\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\right.\\
&\quad\times\left(\mathbf{1}_r-\frac{\Delta\phi}{2}\sin\theta\mathbf{1}_\phi\right)+F_\theta\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\left(\mathbf{1}_\theta-\frac{\Delta\phi}{2}\cos\theta\mathbf{1}_\phi\right)\\
&\left.\left.+F_\phi\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\left.\left(\mathbf{1}_\phi+\frac{\Delta\phi}{2}\sin\theta\mathbf{1}_r+\frac{\Delta\phi}{2}\cos\theta\mathbf{1}_\theta\right)\right)\right]\right\}.
\end{aligned}$$

Taking the limit of the differences

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) - \frac{F_r}{r} + \frac{F_r}{r} \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} - \frac{F_r}{r} + \frac{F_r}{r} - \frac{\cos\theta F_\theta}{r\sin\theta} + \frac{\cos\theta F_\theta}{r\sin\theta} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.
\end{aligned}$$

Exercise 2.10



Start with the divergence theorem

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_{A=\partial V} d\mathbf{A} \cdot \mathbf{F}$$

for any vector **F**. Let $\mathbf{F} = \text{curl}\, \mathbf{u}$.

$$\int_V \text{div}(\text{curl}\, \mathbf{u})\, dV = \int_{A=\partial V} d\mathbf{A} \cdot \text{curl}\, \mathbf{u}.$$

Now split the volume into two parts:

$$V = V_1 + V_2, \quad A_1 = \partial V_1, \quad A_2 = \partial V_2$$

$$\int_V \text{div}(\text{curl}\, \mathbf{u})\, dV = \int_{A_1=\partial V_1} d\mathbf{A} \cdot \text{curl}\, \mathbf{u} + \int_{A_2=\partial V_2} d\mathbf{A} \cdot \text{curl}\, \mathbf{u}.$$

A_1 and A_2 have a part of their surfaces that bound the volume V_1 and on those portions $d\mathbf{A}_1$, and $d\mathbf{A}_2$ are oriented normal outwards from V as shown in the sketch. In addition A_1 and A_2 have a *common* boundary surface on the cut. On that common boundary $d\mathbf{A}_1 = -d\mathbf{A}_2$. Thus the sum of the two integrals on the cut surface $= 0$. The remaining portions of A_1 and A_2 are open surfaces such that they sum to $A = \partial V$ and are each in turn bounded by a closed path $C_1 = \partial A_1$ and $C_2 = \partial A_2$, respectively. Using Stokes' theorem,

$$\int_V \text{div}(\text{curl}\, \mathbf{u})\, dV = \oint_{C_1=\partial A_1} \mathbf{u} \cdot d\mathbf{r} + \oint_{C_2=\partial A_2} \mathbf{u} \cdot d\mathbf{r}.$$

We note that C_1 and C_2 are in fact the same contour traversed in opposite directions (by the right hand rule):

$$C_2 = -C_1 \quad \text{so} \quad \oint_{C_1} + \oint_{C_2} = 0 \text{ and } \int_V \text{div}(\text{curl}\, \mathbf{u})\, dV = 0$$

for any volume V. For a continuous integrands, we must then have

$$\text{div}(\text{curl}\, \mathbf{u}) = 0.$$

Exercise 2.11

Stokes' theorem is

$$\int_A \operatorname{curl} \mathbf{F} \cdot d\mathbf{A} = \int_{C=\partial A} \mathbf{F} \cdot d\mathbf{r}$$

for any vector $\mathbf{F}$. Let $\mathbf{F} = \operatorname{grad} \phi$. Now $(\operatorname{grad} \phi) \cdot d\mathbf{r} = d\phi$, so

$$\int_A \operatorname{curl}(\operatorname{grad} \phi) \cdot d\mathbf{A} = \oint_{C=\partial A} d\phi = 0$$

for all single-valued ϕ.

Exercise 3.1

Given u = y and v = x. The streamlines are given by dx/u = dy/v, that is by dx/y = dy/x. Therefore xdx - ydy = 0. Integrating we get $x^2 - y^2$ = const.

Exercise 3.2

Obviously a stream function defined by

$$\rho R u_R = \frac{\partial \psi}{\partial x} \qquad \rho R u_x = -\frac{\partial \psi}{\partial R}$$

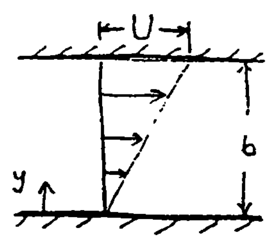
satisfies the equation of continuity.

Exercise 3.3

$u = ay$, $v = 0$. On a circle of radius $= 1$, $\mathbf{ds} = 1d\theta \mathbf{1}_\theta(\theta)$, $\mathbf{1}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$. $\mathbf{u} = \mathbf{i}ay$. $\oint \mathbf{u}\cdot\mathbf{ds} = -\int_0^{2\pi} ay\sin\theta d\theta$. On a circle of radius $= 1$, $y = 1\sin\theta$. $\oint \mathbf{u}\cdot\mathbf{ds} = -a\int_0^{2\pi}\sin^2\theta d\theta = -\pi a$. By Stokes' theorem, $\oint \mathbf{u}\cdot\mathbf{dr} = \int_A(\nabla\times\mathbf{u})\cdot\mathbf{dA}$. Here $\nabla\times\mathbf{u} = -\mathbf{k}a$, $\mathbf{dA} = rdrd\theta\mathbf{k}$. $\int_A(\nabla\times\mathbf{u})\cdot\mathbf{dA} = -a\int_0^{2\pi}d\theta\int_0^1 dr\cdot r = -\pi a$.

Exercise 3.4

Given u = Uy/b. Strain rates and vorticity are



$$e_{xx} = \frac{\partial u}{\partial x} = 0$$

$$e_{yy} = \frac{\partial v}{\partial y} = 0$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = U/2b \qquad \underset{\sim}{e} = \begin{bmatrix} 0 & U/2b \\ U/2b & 0 \end{bmatrix}$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -U/b$$

Stream function is given by

$$u = Uy/b = \frac{\partial \psi}{\partial y}$$

Integration gives $\qquad \psi = Uy^2/2b + \text{const}$

Exercise 3.5

Vorticity and stream function are related by

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

For $\psi = Uy^2/2b$, we get

$$\omega_z = 0 - \frac{U}{2b}(2) = -U/b$$

Exercise 3.6

Given $\qquad u = x/(1+t) \qquad v = 2y/(2+t) \qquad (1)$

Streamlines: $dx/u = dy/v$ gives

$$(1+t)dx/x = (2+t)dy/2y$$

Instantaneous streamlines are obtained by integration *at constant t*, giving

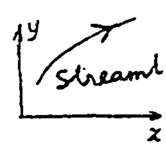
$$(1+t)\log x = (1+t/2)\log y + \log f(t)$$

where we have written $\log f(t)$ for the constant of integration. The streamlines are given by

$$\log(x^{1+t}) = \log(y^{1+t/2}) + \log f(t) = \log[f(t)y^{1+t/2}]$$

or

$$x^{1+t} = f(t)\, y^{1+t/2}$$



Pathlines: To obtain particle paths, we have to follow the motion of each particle. This means that we have to solve

$$d\underset{\sim}{x}/dt = \underset{\sim}{u}(\underset{\sim}{x}, t) \tag{2}$$

subject to $\underset{\sim}{x} = \underset{\sim}{x}_o$ at $t = 0$. The x-component of (2) gives

$$dx/dt = u = x/(1 + t) \quad \longrightarrow \quad dx/x = dt/(1 + t)$$

Integrating,

$$\log x = \log (1 + t) + \log f(x_o) = \log[f(x_o)(1 + t)]$$

where $\log f(x_o)$ is the constant of integration, the particle tag x_o being constant on pathlines. We get

$$x = f(x_o)(1 + t)$$

Initial condition $x = x_o$ at $t = 0$ gives $f(x_o) = x_o$. Therefore the x-component of the path line is given by

$$\boxed{x = x_o(1 + t)} \tag{3}$$

As a check, differentiation of (3) at constant x_o gives

$$u = \left(\frac{\partial x}{\partial t}\right)_{x_o} = x_o = x/(1 + t)$$

which agrees with (1). Similarly the y-component of (2) gives

$$dy/dt = v = 2y/(2 + t) \quad \longrightarrow \quad dy/y = 2\, dt/(2 + t)$$

Integrating, $\log y = 2 \log(2 + t) + 2 \log g(y_o)$

or $y = g^2(y_o)\,(2 + t)^2$

Initial condition $y = y_o$ at $t = 0$ gives $g^2(y_o) = y_o/4$. Therefore

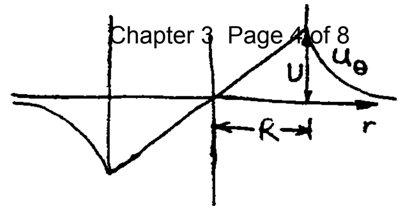
$$\boxed{y = \frac{y_o}{4}(2 + t)^2} \tag{4}$$

As a check, differentiation of (4) at constant y_o gives

$$v = dy/dt = (y_o/4)2(2 + t) = [y/(2 + t)^2]2(2 + t) = 2y/(2 + t)$$

which agrees with (1).

<u>**Exercise 3.7**</u>



For r ≤ R

$$u_\theta = Cr = Ur/R = \frac{\partial\psi}{\partial r}$$

$$\therefore\ \psi = Ur^2/2R + \text{const} = Ur^2/2R,\ \text{taking } \psi = 0 \text{ at } r = 0$$

$$\therefore\ \psi_{r=R} = UR/2$$

For r ≥ R

$$u_\theta = C/r = UR/r = \frac{\partial\psi}{\partial r}$$

$$\therefore\ \psi = UR\log r + \text{const} = UR\log(r/R) + \text{const}$$

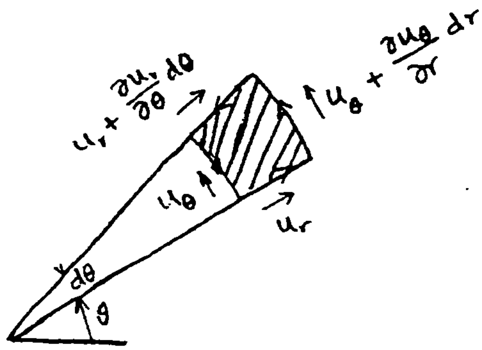
Boundary condition $\psi = UR/2$ at $r = R$ gives

$$UR/2 = UR\log(1) + \text{const}$$

Using the above constant, we get

$$\psi = UR\log(r/R) + UR/2$$

Exercise 3.8



Application of Stokes theorem $\int \underset{\sim}{\omega}\cdot d\underset{\sim}{A} = \int \underset{\sim}{u}\cdot d\underset{\sim}{s}$ gives

$$\omega_z\, r d\theta\, dr = u_r dr + \left(u_\theta + \frac{\partial u_\theta}{\partial r}dr\right)(r + dr)d\theta$$

$$- \left(u_r + \frac{\partial u_r}{\partial\theta}d\theta\right)dr - u_\theta r d\theta$$

$$= u_\theta dr d\theta + r\frac{\partial u_\theta}{\partial\theta}dr d\theta + \frac{\partial u_\theta}{\partial r}d\theta \underbrace{\cancel{(dr)^2}}_{\simeq 0 \text{ (higher order)}} - \frac{\partial u_r}{\partial\theta}d\theta\, dr$$

Division by $r d\theta\, dr$ gives

$$\omega_z = \frac{1}{r}\left[u_\theta + r\frac{\partial u_\theta}{\partial r} - \frac{\partial u_r}{\partial\theta}\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right]$$

Exercise 3.9

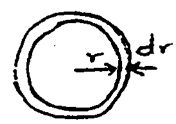
Given
$$u = 2xy^2 + 2xz^2$$
$$v = x^2y$$
$$w = x^2z$$

Take a sphere $x^2 + y^2 + z^2 = a^2$, and apply

$$\int \nabla \cdot \mathbf{u}\, dV = \int \mathbf{u} \cdot d\mathbf{A} \qquad (1)$$

Now $\nabla \cdot \mathbf{u} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = (2y^2 + 2z^2) + x^2 + x^2$

$$= 2(x^2 + y^2 + z^2) = 2r^2$$



The two sides of (1) are as follows:

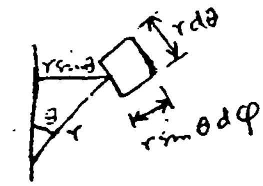
$$\text{LHS} = \int \nabla \cdot \mathbf{u}\, dV = \int 2r^2 dV = 2\int_0^a r^2(4\pi r^2 dr) = 8\pi[r^5/5]_0^a = 8\pi a^5/5$$

RHS = $\int \mathbf{u} \cdot d\mathbf{A}$. To evaluate the integral, note that the area vector dA is

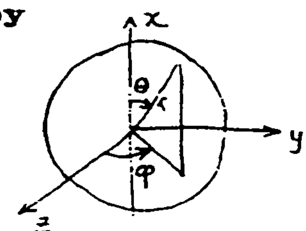
$$d\mathbf{A} = \mathbf{i}dA_x + \mathbf{j}dA_y + \mathbf{k}dA_z = \mathbf{i}dA\cos\theta + \mathbf{j}dA\sin\theta\,\sin\varphi + \mathbf{k}dA\sin\theta\cos\varphi$$

where the area magnitude is

$$dA = rd\theta[r\sin\theta\, d\varphi]$$



Also, the Cartesian and spherical polar coordinates are related by



$$x = r\cos\theta$$
$$y = [r\sin\theta]\,\sin\varphi \qquad (2)$$
$$z = [r\sin\theta]\,\cos\varphi$$

Therefore

$$\mathbf{u} \cdot d\mathbf{A} = udA_x + vdA_y + wdA_z = dA(u\cos\theta + v\sin\theta\,\sin\varphi + w\sin\theta\,\cos\varphi)$$

$$= rd\theta(r\sin\theta\, d\varphi)[\{2xy^2 + 2xz^2\}\cos\theta + x^2y\sin\theta\sin\varphi + x^2z\sin\theta\cos\varphi]$$

Substituting for x, y and z from (2), this simplifies to

$$\underset{\sim}{u}\cdot d\underset{\sim}{A} = 3r^5\sin^3\theta\cos^2\theta\, d\theta\, d\varphi$$

Therefore the RHS of (1), evaluated at r = a, is

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\underset{\sim}{u}\cdot d\underset{\sim}{A} = 3a^5(2\pi)\int_{\theta=0}^{\pi}\sin^2\theta(1-\sin^2\theta)\sin\theta\, d\theta$$

$$= 6a^5\int_{-1}^{1}(1-x^2)x^2dx, \qquad \text{setting } x = \cos\theta$$

$$= 8\pi a^5/5 = \text{LHS}$$

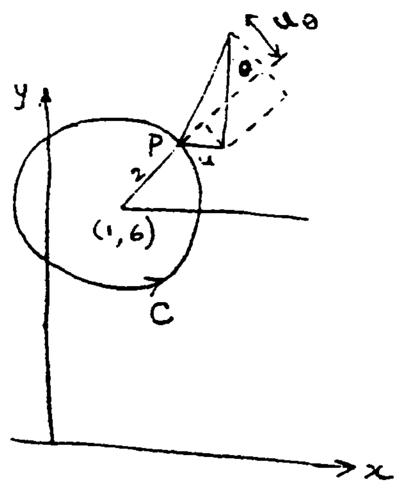
Exercise 3.10

$$\underset{\sim}{\omega} = \underset{\sim}{\nabla}\times\underset{\sim}{u} \qquad \Rightarrow \qquad \omega_m = \varepsilon_{ijm}\frac{\partial u_j}{\partial x_i}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{\omega} = \frac{\partial\omega_m}{\partial x_m} = \frac{\partial}{\partial x_m}\left(\varepsilon_{ijm}\frac{\partial u_j}{\partial x_i}\right)$$

$= \varepsilon_{ijm}\dfrac{\partial^2 u_j}{\partial x_m\partial x_i}$ = 0 by symmetry of the derivative in m and i, and antisymmetry of ε_{ijm} in m and i.

Exercise 3.11



Given $\quad u = 3x + y$

$\qquad\quad v = 2x - 3y$

Use polar coordinates centered at (1,6).
For point P,

$$x = 1 + 2\cos\theta$$
$$y = 6 + 2\sin\theta$$

Velocity component tangential to circle is

$$u_\theta = v\cos\theta - u\sin\theta = (3x + y)\cos\theta - (2x - 3y)\sin\theta$$

$$= [3(1 + 2\cos\theta) + 6 + 2\sin\theta]\cos\theta$$

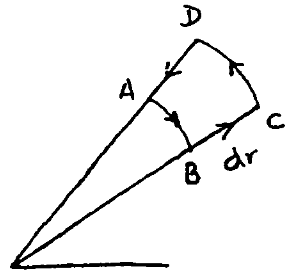
$$- [2(1 + 2\cos\theta) - 3(6 + 2\sin\theta)]\sin\theta$$

$$= 1 + 3\cos 2\theta - 6\sin 2\theta - 16\cos\theta - 9\sin\theta$$

$$\Gamma = \int_C \underset{\sim}{u}\cdot d\underset{\sim}{s} = \int_0^{2\pi} u_\theta (2d\theta)$$

$$= 2\int_0^{2\pi} (1 + 3\cos 2\theta - 6\sin 2\theta - 16\cos\theta - 9\sin\theta)d\theta = 2(2\pi) = 4\pi$$

Exercise 3.12



Given $u_\theta = \omega_o r$, $\quad u_r = 0$

Vorticity $\omega = 2\omega_o$

Circulation around ABCD is

$$d\Gamma = \Gamma_{CD} + \Gamma_{AB} = [\omega_o(r + dr)](r + dr)d\theta - (\omega_o r)(r d\theta)$$

$$= (\omega_o r + \omega_o dr)(r d\theta + dr d\theta) - (\omega_o r)(r d\theta)$$

$$= 2\omega_o(dr\ r d\theta) = \text{vorticity X area}$$

Exercise 3.13

To prove $\quad \underset{\sim}{a} = \dfrac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{\nabla}(\tfrac{1}{2}q^2) + \underset{\sim}{\omega}\times\underset{\sim}{u}$

we have to show that $\quad \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u} = \underset{\sim}{\nabla}(\tfrac{1}{2}q^2) + \underset{\sim}{\omega}\times\underset{\sim}{u}$

We have $\quad u_j u_{i,j} = u_j(u_{i,j} - u_{j,i}) + u_j u_{j,i} \qquad (1)$

Now

$$(\underset{\sim}{\omega}\times\underset{\sim}{u})_i = \varepsilon_{ijk}\,\omega_j u_k = \varepsilon_{ijk}[\varepsilon_{mnj} u_{n,m}]u_k = \varepsilon_{ijk}\varepsilon_{mnj} u_{n,m} u_k$$

$$= -\varepsilon_{mnj}\varepsilon_{jik}\ u_{n,m}u_k = -(\delta_{mi}\delta_{nk} - \delta_{mk}\delta_{in})u_{n,m}u_k$$

$$= (-u_{k,i} + u_{i,k})u_k = u_j(u_{i,j} - u_{j,i})$$

Then (1) becomes $\quad u_j u_{i,j} = (\underset{\sim}{\omega}\times\underset{\sim}{u})_i + \tfrac{1}{2}(u_j^2)_{,i} = (\underset{\sim}{\omega}\times\underset{\sim}{u})_i + \dfrac{\partial}{\partial x_i}(\tfrac{1}{2}q^2)$

Exercise 3.14

In indicial notation, the definition $\underset{\sim}{u} = -\underset{\sim}{k} \times \underset{\sim}{\nabla}\psi$ is written as

$$u_i = (-\underset{\sim}{k} \times \underset{\sim}{\nabla}\psi)_i = -\varepsilon_{ijm} k_j \frac{\partial \psi}{\partial x_m} \qquad (1)$$

Since vector $\underset{\sim}{k}$ is perpendicular to the plane of flow on the $x_1 x_2$-plane, components of vector $\underset{\sim}{k}$ are $k_1 = k_2 = 0$, and $k_3 = 1$. In the sum over the index j in (1), it can therefore only take the value j = 3. Equation (1) then gives

$$u_1 = -\overset{-1}{\varepsilon_{132}}\, k_3 \frac{\partial \psi}{\partial x_2} = \frac{\partial \psi}{\partial x_2}$$

and

$$u_2 = -\underset{1}{\varepsilon_{231}}\, k_3 \frac{\partial \psi}{\partial x_1} = -\frac{\partial \psi}{\partial x_1}$$

which agree with equations (34).

Exercise 4.1

Given
$$u = u(x,t)$$
$$v = 0$$
$$w = 0$$
$$\rho = \rho_o(2 - \cos\omega t)$$
$$u(0,t) = U$$

Use continuity equation
$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underset{\sim}{u} = 0$$

or
$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} = 0$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{\rho_o \omega \sin\omega t}{\rho_o(2-\cos\omega t)} = -\frac{\omega \sin\omega t}{2-\cos\omega t}$$

$$u = -\left(\frac{\omega \sin\omega t}{2-\cos\omega t}\right)x + f(t)$$

Initial condition gives $\quad u(0,t) = f(t) = U$

$$\therefore u = U - \left(\frac{\omega \sin\omega t}{2-\cos\omega t}\right)x$$

Exercise 4.2

Integral form of conservation of mass for a fixed volume $\mathcal{V}$ is

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} = 0 \qquad (1)$$

Using equation (4.5)

$$\frac{D}{Dt}\int_{\mathcal{V}} F(\underset{\sim}{x},t)\,d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial F}{\partial t}\,d\mathcal{V} + \int_A F\underset{\sim}{u}\cdot d\underset{\sim}{A}$$

equation (1) becomes

$$\frac{D}{Dt}\int \rho \, d\mathcal{V} = \int \frac{\partial \rho}{\partial t}\,d\mathcal{V} + \int \rho \underset{\sim}{u}\cdot d\underset{\sim}{A} = 0$$

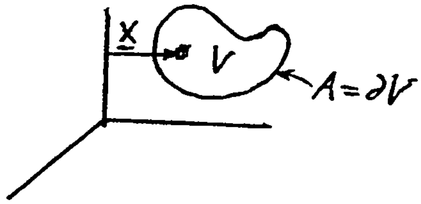
Using Gauss's theorem to transform the second integral above, we get

$$\frac{D}{Dt}\int \rho \, d\mathcal{V} = \int \frac{\partial \rho}{\partial t}\,d\mathcal{V} + \int \underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{u})\,d\mathcal{V}$$

which leads to

Exercise 4.3

From the problem statement we write conservation of angular momentum as



$$\int_V \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \rho\mathbf{g}\right] dV$$
$$= \int_{A=\partial V} \mathbf{x} \times (\mathbf{n}\cdot\boldsymbol{\tau})\, dA = \int_{A=\partial V} \mathbf{x} \times (d\mathbf{A}\cdot\boldsymbol{\tau}).$$

In Cartesian tensor notation, the right hand side $= \int_{A=\partial V} \epsilon_{ijk} x_j dA_m \tau_{mk}$. The divergence theorem can be applied to transform the closed surface integral to a volume integral, yielding

$$= \int_V \epsilon_{ijk} \frac{\partial}{\partial x_m}(x_j \tau_{mk}) dV = \int_V \epsilon_{ijk} \left(\frac{\partial x_j}{\partial x_m}\tau_{mk} + x_j \frac{\partial \tau_{mk}}{\partial x_m}\right) dV.$$

In vector notation, the second term is $\int_V \mathbf{x} \times \nabla\cdot\boldsymbol{\tau}\, dV$. We also note that $\partial x_j/\partial x_m = \delta_{jm}$. Now we have

$$\int_V \mathbf{x} \times \left[\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \rho\mathbf{g} - \nabla\cdot\boldsymbol{\tau}\right] dV = \int_V \mathbf{T}_x\, dV,$$

where the vector $\mathbf{T}_x = \epsilon_{ijk}\tau_{jk}$. The left hand side $= 0$ by Eq. (4.17), conservation of momentum. Then $\mathbf{T}_x = 0$. Each of its components is of the form $\tau_{ij} - \tau_{ji} = 0$.

Exercise 4.4

We are required to start with

$$\underset{\sim}{F} = \frac{d\underset{\sim}{M}}{dt} + \dot{\underset{\sim}{M}}^{out} \qquad (1)$$

and derive

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

Equation (1) can be written as

$$\int \rho g_i\, dV + \int \tau_{ij}\, dA_j = \frac{d}{dt}\int \rho u_i\, dV + \int (\rho u_j\, dA_j) u_i$$

or $\int\left(\rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) dV = \int \frac{\partial}{\partial t}(\rho u_i)dV + \int \frac{\partial}{\partial x_j}(\rho u_i u_j)dV$

$$\rho g_i + \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j)$$

$$= \rho\frac{\partial u_i}{\partial t} + \underbrace{u_i\frac{\partial \rho}{\partial t} + u_i\frac{\partial}{\partial x_j}(\rho u_j)}_{\text{zero by continuity}} + \rho u_j\frac{\partial u_i}{\partial x_j}$$

$$= \rho\frac{Du_i}{Dt}$$

Exercise 4.5

We are required to show that

$$\Phi = 2\mu e_{ij}e_{ij} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})^2 = 2\mu[e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})\delta_{ij}]^2$$

Completing the square,

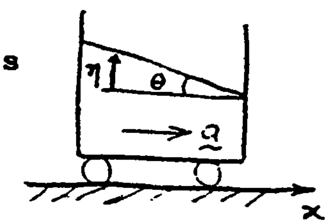
$$\text{RHS} = 2\mu[e_{ij}e_{ij} + \frac{1}{9}(\nabla\cdot\mathbf{u})^2\underbrace{\delta_{ij}\delta_{ij}}_{3} - \frac{2}{3}e_{ij}(\nabla\cdot\mathbf{u})\delta_{ij}]$$

$$= 2\mu[e_{ij}e_{ij} + \frac{1}{3}(\nabla\cdot\mathbf{u})^2 - \frac{2}{3}\underbrace{e_{ii}}_{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_i}+\frac{\partial u_i}{\partial x_i}\right) = \nabla\cdot\mathbf{u}}(\nabla\cdot\mathbf{u})]$$

$$= 2\mu[e_{ij}e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})^2]$$

Exercise 4.6

Since there is no friction, the equation of motion is

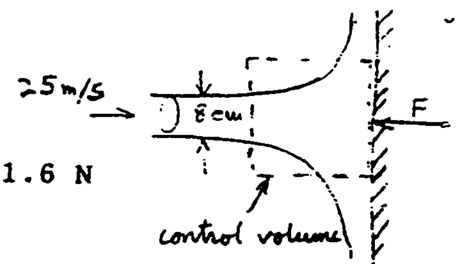


$$a = -\frac{1}{\rho}\frac{dp}{dx} = -g\frac{d\eta}{dx} = +g\tan\theta, \text{ since } \frac{d\eta}{dx} < 0.$$

which gives $\tan\theta = a/g$

Exercise 4.7

The force on the plate is



$$F = \dot{m}U = \rho A U^2 = 1000\left[\frac{\pi}{4}(0.08)^2\right](25)^2 = 3141.6 \text{ N}$$

The average pressure is

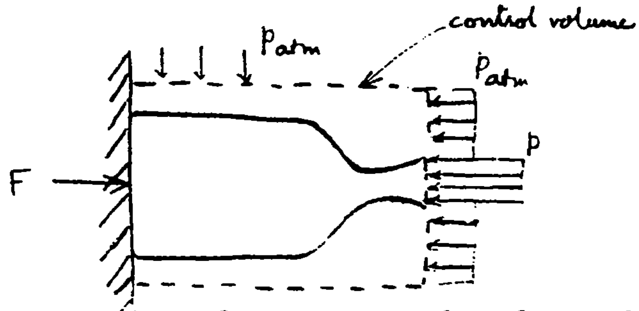
$$\bar{p} = \frac{F}{A_{plate}} = \frac{F}{20\left[\frac{\pi}{4}(.08)^2\right]} = \frac{3141.6}{20\left[\frac{\pi}{4}(.08)^2\right]} = 31{,}250\ N/m^2$$

Stagnation pressure is $p_{stag} = \frac{1}{2}\rho U^2 = \frac{1}{2}1000(25)^2 = 312{,}500\ N/m^2$

Therefore

$$\frac{\bar{p}}{p_{stag}} = \frac{1}{10}$$

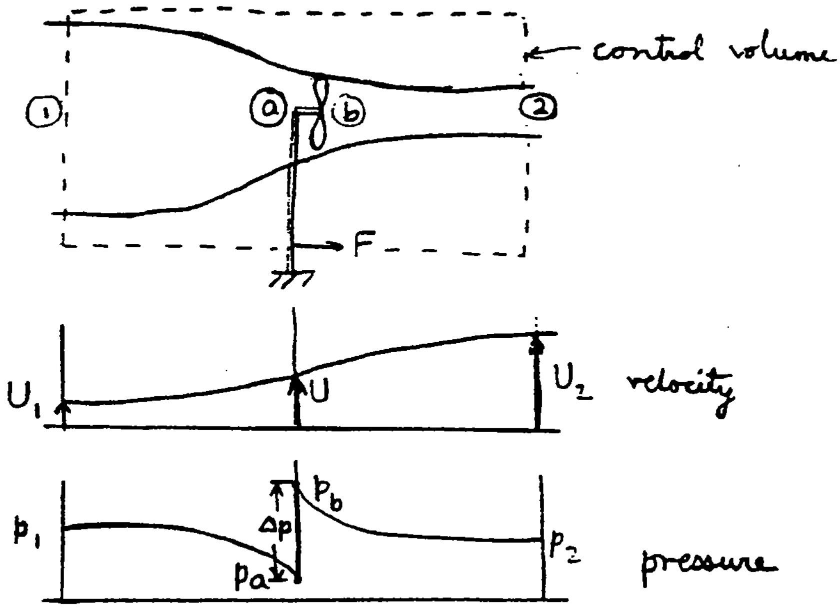
Exercise 4.8



Application of momentum principle to control volume shown gives

$$F - A(p - p_{atm}) = \dot{m}U = \rho AU^2$$

Exercise 4.9



Apply Bernoulli equation between points 1 and a, and again between b and 2. This gives

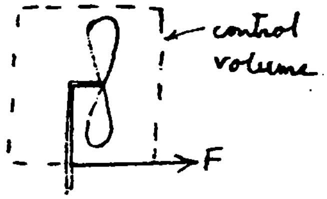
$$p_1/\rho + U_1^2/2 = p_a/\rho + U_a^2/2$$

$$p_2/\rho + U_2^2/2 = p_b/\rho + U_b^2/2$$

Since $p_1 = p_2$ and $U_a = U_b$, we get

$$\Delta p = p_b - p_a = \frac{\rho}{2}(U_2^2 - U_1^2)$$

Now apply momentum principle to a control volume around the propeller. If F is the force applied by the propeller, then



$$F + Ap_a - Ap_b = 0$$

$$F = A\,\Delta p = \tfrac{1}{2}\rho A(U_2^2 - U_1^2)$$

To find U, take a control volume shown at the beginning of the problem. Momentum principle gives

$$F = \dot{m}(U_2 - U_1)$$

or

$$\tfrac{1}{2}\rho A(U_2^2 - U_1^2) = \rho AU(U_2 - U_1)$$

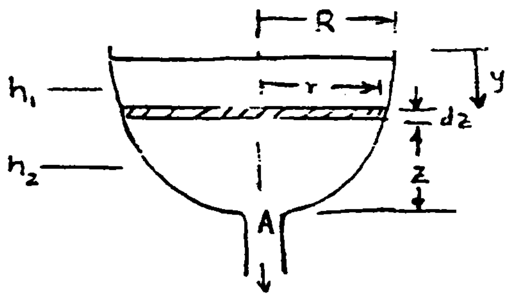
$$\therefore \quad U = \frac{1}{2}(U_1 + U_2)$$

Exercise 4.10

The rate of flow through the orifice is

$$A\sqrt{2gz} = -\pi r^2 \frac{dz}{dt} \qquad \left(\text{note}: \frac{dz}{dt} < 0\right) \qquad (1)$$

From geometry shown



$$r^2 + y^2 = R^2$$

or $$r^2 + (R - z)^2 = R^2$$

This gives $$r^2 = 2Rz - z^2$$

Then (1) becomes

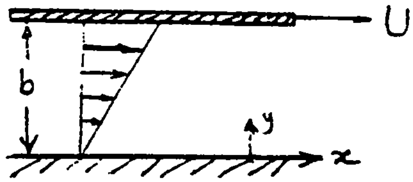
$$A\sqrt{2gz} = -\pi(2Rz - z^2)\frac{dz}{dt}$$

Integrating

$$t = -\frac{\pi}{A\sqrt{2g}}\int_{h_1}^{h_2}(2R\sqrt{z} - z^{3/2})\,dz = -\frac{\pi}{A\sqrt{2g}}\left[2R\frac{2}{3}z^{3/2} - \frac{2}{5}z^{5/2}\right]_{h_1}^{h_2}$$

$$t = \frac{2\pi}{A\sqrt{2g}}\left[\frac{2R}{3}\left(h_1^{3/2} - h_2^{3/2}\right) - \frac{1}{5}\left(h_1^{5/2} - h_2^{5/2}\right)\right]$$

Exercise 4.11



Velocity distribution

y-momentum equation is $0 = -dp/dy - \rho g$

$$\therefore\ p = p_o - \rho g y \text{ (hydrostatic)}$$

Integration of x-momentum equation gives $0 = \nu \dfrac{d^2u}{dy^2}$

$$u = Ay + B$$

BC gives B = 0, and U = Ab $\longrightarrow$ A = U/b. Therefore

$$u = Uy/b$$

Strain rate

Components of $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ are

$$e_{11} = 0 = e_{22}$$

$$e_{12} = e_{21} = \frac{1}{2}\frac{du}{dy} = U/2b$$

$$\underset{\sim}{e} = \begin{bmatrix} 0 & \frac{U}{2b} \\ \frac{U}{2b} & 0 \end{bmatrix}$$

Stress tensor

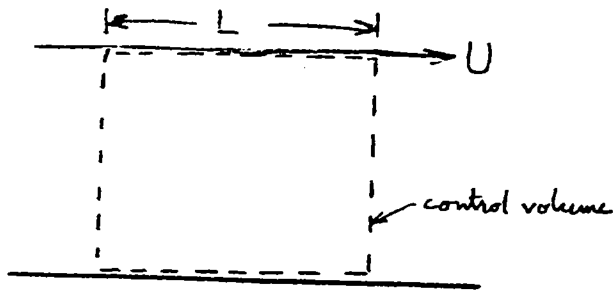
Components of $\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}$ are

$$\underset{\sim}{\tau} = \begin{bmatrix} -p & -p + \frac{\mu U}{b} \\ -p + \frac{\mu U}{b} & -p \end{bmatrix}$$

Viscous dissipation

$$\phi = 2\mu e_{ij}\, e_{ij} = 2\mu[e_{12}\, e_{12} + e_{21}\, e_{21}] = 4\mu e_{12}^2 = \mu U^2/b^2$$

Energy balance



$$E = u_i^2/2$$

Equation (4.58) is

$$\cancel{\frac{d}{dt}\int E\,dV} + \cancel{\int E\,\underset{\sim}{u}\cdot d\underset{\sim}{A}} = \cancel{\int \rho\,\underset{\sim}{g}\cdot\underset{\sim}{u}\,dV} + \int u_i \tau_{ij}\,dA_j + \cancel{\int p(\underset{\sim}{\nabla}\cdot\underset{\sim}{u})\,dV} - \int \phi\,dV$$

The terms crossed out are obviously zero. The balance is

$$\int u_i \tau_{ij}\,dA_j = \int \phi\,dV$$

Pressure terms cancel out, so that

$$\text{LHS} = [u_1 \tau_{12}\,L]_{\text{upper plate}} = U(\mu U/b)L = \mu U^2 L/b$$

$$\text{RHS} = \frac{\mu U^2}{b}\cdot L$$

This verifies the energy balance.

Exercise 4.12

(a)
$$\int_{A=\partial V}\rho\mathbf{u}\cdot d\mathbf{A}=\int_0^1 dy\int_0^1 dz\,4x^2y\Big|_{x=1}-\int_0^1 dy\int_0^1 dz\,4x^2y\Big|_{x=0}$$
$$+\int_0^1 dz\int_0^1 dx\,xyz\Big|_{y=1}-\int_0^1 dz\int_0^1 dx\,xyz\Big|_{y=0}$$
$$+\int_0^1 dx\int_0^1 dy\,yz^2\Big|_{z=1}-\int_0^1 dx\int_0^1 dy\,yz^2\Big|_{z=0}$$
$$=2+0+\tfrac{1}{4}+0+\tfrac{1}{2}+0=\tfrac{11}{4}.$$

(b) $\nabla\cdot\mathbf{u}=\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=8xy+xz+2yz$

$$\int_V \nabla\cdot\mathbf{u}\,dV=\int_0^1 dx\int_0^1 dy\int_0^1 dz\,(8xy+xz+2yz)$$
$$=8\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{2}\cdot\tfrac{1}{2}+2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}=\tfrac{11}{4}.$$

Exercise 4.13

Calculate the vorticity

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{k}{2\pi}\right) = 0 \quad \text{if } r \neq 0.$$

ω_z is undefined if $r = 0$. Thus for any contour not enclosing the origin,

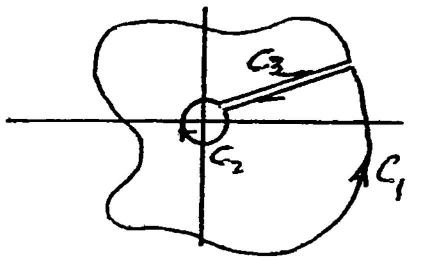
$$\Gamma = \oint_{C=\partial A} \mathbf{u} \cdot d\mathbf{s} = \int_A (\nabla \times \mathbf{u}) \cdot d\mathbf{A} = 0$$

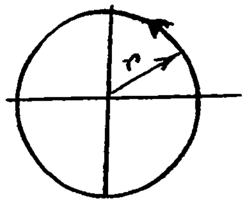
because $\nabla \times \mathbf{u} = \boldsymbol{\omega} = 0$ at every point in A. Thus for every contour C not enclosing the origin, $\Gamma = 0$. Now calculate the circulation about a circle of radius$= r$ about 0.

$$\oint \mathbf{u} \cdot d\mathbf{s} = \int_0^{2\pi} u_\theta r d\theta = \int_0^{2\pi} \frac{k}{2\pi r} r\, d\theta = k.$$

Consider a contour as follows:

C_1 an arbitrarily shaped contour enclosing the orgin traversed counterclockwise
C_2 a circle enclosing the origin traversed clockwise
C_3 the connection $\rightleftharpoons$ between C_1 and C_2.





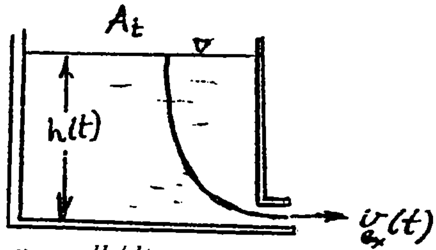
$\oint_{C_1+C_2+C_3=\partial A} \mathbf{u} \cdot d\mathbf{s} = \int_A (\nabla \times \mathbf{u}) \cdot d\mathbf{A} = 0$ since $C_1 + C_2 + C_3$ bounds an area excluding the origin and $\nabla \times \mathbf{u} = 0$ everywhere in that area. $\int_{C_3} \mathbf{u} \cdot d\mathbf{s} = 0$ because C_3 is the same path traversed in opposite directions. $\int_{C_2} \mathbf{u} \cdot d\mathbf{s} = -k$ because it is traversed clockwise. Then $\oint_{C_1} \mathbf{u} \cdot d\mathbf{s} = k$ for all C_1 enclosing the origin.

Exercise 4.14

Assume one dimensional flow in the water so that at each station z, $\rho v(z)A(z) = \rho v_0 A_0$ and $\rho =$ constant. From Bernoulli's equation $p/\rho + (1/2)v^2 + gz =$ const. on a streamline in the fluid column. Neglecting surface tension, $p =$ const. $= p_{atm}$ across the air–water interface. Then $p =$ const. $= p_{atm}$ throughout the water column. Consider a streamline downwards from $z = 0$ to $z < 0$:

$$\tfrac{1}{2}v^2 + gz = \tfrac{1}{2}v_0^2 \quad \text{so} \quad v = (v_0^2 - 2gz)^{1/2} \text{ and } A(z) = A_0 \frac{v_0}{(v_0^2 - 2gz)^{1/2}}.$$

Exercise 4.15



From Bernoulli's theorem on a streamline

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz = \text{const.}$$

At both the top surface and at the exit, $p = p_{\text{atm}}$, at the top, $v = -dh/dt$.

$$\frac{p_{\text{atm}}}{\rho} + \frac{1}{2}\left(-\frac{dh}{dt}\right)^2 + gh = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2}v_{\text{ex}}^2 \quad (z = 0 \text{ at exit}).$$

From mass conservation,

$$A_t\left(-\frac{dh}{dt}\right) = A_{\text{ex}}v_{\text{ex}}$$

so

$$v_{\text{ex}}^2 = \left(\frac{A_t}{A_{\text{ex}}}\right)^2\left(\frac{dh}{dt}\right)^2$$

$$\frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(1 - \frac{A_t^2}{A_{\text{ex}}^2}\right) = -gh(t)$$

$$\frac{1}{h}\left(\frac{dh}{dt}\right)^2 = \frac{2g}{A_t^2/A_{\text{ex}}^2 - 1}$$

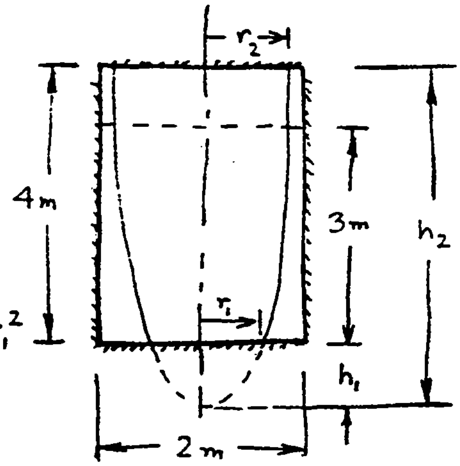
$$h^{-1/2}\frac{dh}{dt} = -\left(\frac{2g}{A_t^2/A_{\text{ex}}^2 - 1}\right)^{1/2} \quad \text{(negative root since } dh/dt < 0\text{)}.$$

Integrate:

$$2h^{1/2} = 2h_0^{1/2} - \left(\frac{2g}{A_t^2/A_{\text{ex}}^2 - 1}\right)^{1/2} t \quad (\text{const. } h_0 \text{ evaluated at } t = 0).$$

Then $h = 0$ when

$$t = t_f = \sqrt{\frac{2h_0}{g}}\left(\frac{A_t^2}{A_{\text{ex}}^2} - 1\right)^{1/2}.$$



Exercise 5.1

Angular velocity ω = 40 rad/s

$h = \omega^2 r^2/2g$

$h_1 = \omega^2 r_1^2/2g = (40)^2 r_1^2/2(9.81) = 81.5 r_1^2$ (1)

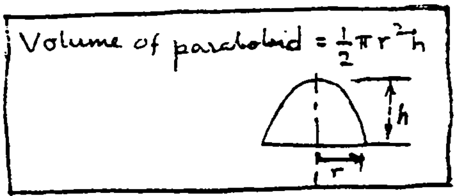
$h_2 = 4 + h_1 = \omega^2 r_2^2/2g = 81.5 r_2^2$ (2)

Subtracting (2) - (1), we get

$$4 = 81.5(r_2^2 - r_1^2) \longrightarrow \qquad r_2^2 - r_1^2 = 0.0491 \text{ m}^2 \qquad (3)$$

Now volume of air within the tank does not change, so

$$\frac{\pi}{4}(2)^2(4 - 3) = \frac{1}{2}\pi r_2^2 h_2 - \frac{1}{2}\pi r_1^2 h_1$$



or
$$2 = h_2 r_2^2 - h_1 r_1^2$$
$$= 81.5(r_2^4 - r_1^4) = 81.5(r_2^2 + r_1^2)(r_2^2 - r_1^2)$$

where we have used (1) and (2) to replace h_1 and h_2. Using (3), the above becomes

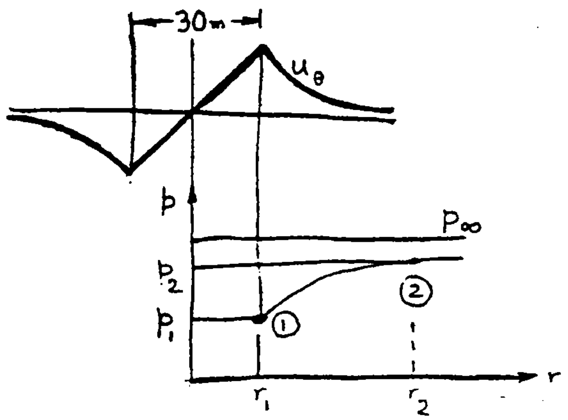
$$r_2^2 + r_1^2 = 2/(81.5)(r_2^2 - r_1^2) = 2/(81.5)(0.0491) = 0.5 \qquad (4)$$

Subtracting (4) - (3), we get $2r_1^2 = 0.5 - 0.0491 = 0.45$. Thus

$$r_1 = 0.475 \text{ m}$$

so that Area uncovered $= \pi r_1^2 = 0.71 \text{ m}^2$

Exercise 5.2



p_1 = - 2000 N/m^2

p_2 = - 500 N/m^2

ρ = 1.18 kg/m^3 at 25° C

$V = 25$ m/s

(a) Applying Bernoulli equation between infinity and edge of core, we get

$$U_1 = \sqrt{2(p_\infty - p_1)/\rho} = \sqrt{2(2000)/1.18} = 58.2 \text{ m/s}$$

$$\therefore \quad \Gamma = 2\pi r_1 U_1 = 2\pi(15)(58.2) = 5485 \text{ m}^2/\text{s}$$

(b) Apply Bernoulli equation between points 2 and 1:

$$p_1 + \tfrac{1}{2}\rho U_1^2 = p_2 + \tfrac{1}{2}\rho U_2^2$$

$$U_2 = \sqrt{2(p_1 - p_2)/\rho + U_1^2} = \sqrt{2(-2000 + 500)/1.18 + 58.2^2}$$

$$= 29.07 \text{ m/s}$$

Since circulation outside core is constant, $r_1 U_1 = r_2 U_2$. So

$$r_2 = r_1 U_1 / U_2 = 30.0 \text{ m}$$

$$\text{Time required} = (r_2 - r_1)/V = (30 - 15)/25 = 0.6 \text{ s}$$

__Exercise 5.3__

Given

$$u_R = 0$$
$$u_\varphi = aRx$$
$$u_x = 0$$

(a) From Appendix B the vorticity components are

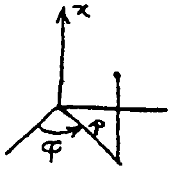
$$\omega_R = \frac{1}{R}\cancel{\frac{\partial u_x}{\partial \varphi}} - \frac{\partial u_\varphi}{\partial x} = -aR$$

$$\omega_\varphi = \cancel{\frac{\partial u_R}{\partial x}} - \cancel{\frac{\partial u_x}{\partial R}} = 0$$

$$\omega_x = \frac{1}{R}\frac{\partial}{\partial R}(Ru_\varphi) - \frac{1}{R}\cancel{\frac{\partial u_R}{\partial \varphi}} = \frac{1}{R}\frac{\partial}{\partial R}(aR^2x) = 2ax$$

(b)

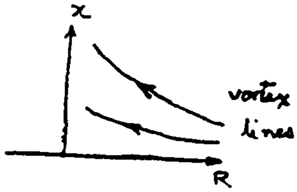
$$\nabla \cdot \omega = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega_R}{\partial R}\right) + \frac{1}{R}\frac{\partial \omega_\varphi}{\partial \varphi} + \frac{\partial \omega_x}{\partial x}$$



$$= \frac{1}{R}\frac{\partial}{\partial R}(-\ aR^2) + 0 + \frac{\partial}{\partial x}(2ax) = -\ 2a + 2a = 0$$

(c) Vortex lines are given by

$$\frac{dx}{\omega_x} = \frac{dR}{\omega_R} \longrightarrow \frac{dx}{2ax} = \frac{dR}{-aR}$$

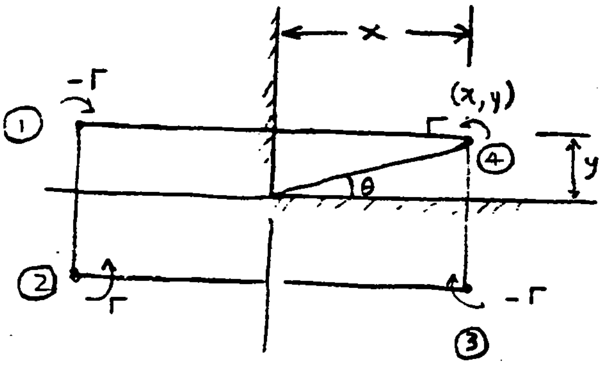


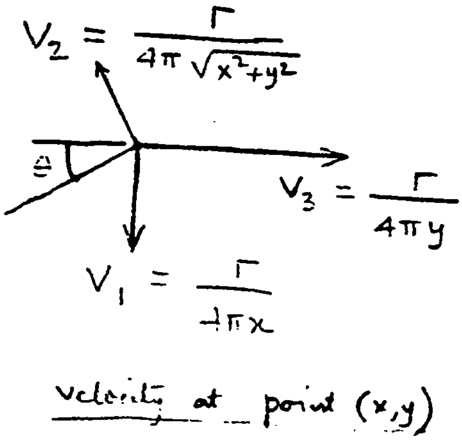
Integrating

$$\tfrac{1}{2}\log x = -\ \log R + \text{constant} \longrightarrow \qquad xR^2 = \text{constant}$$

Since u_φ is the only nonzero component of velocity, streamlines are circles around the x-axis. They cut the xR plane at po

Exercise 5.4





The components of the net velocity at point (x,y) due to the three image vortices are

$$u = \sum V_x = V_3 - V_2\sin\theta = \frac{\Gamma}{4\pi y} - \frac{\Gamma}{4\pi\sqrt{x^2+y^2}}\cdot\frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{\Gamma}{4\pi}\left[\frac{1}{y} - \frac{y}{x^2+y^2}\right] = \frac{\Gamma}{4\pi}\frac{x^2}{y(x^2+y^2)}$$

$$v = \sum V_y = -V_1 + V_2\cos\theta = -\frac{\Gamma}{4\pi x} + \frac{\Gamma}{4\pi\sqrt{x^2+y^2}}\,\frac{x}{\sqrt{x^2+y^2}}$$

$$= -\frac{\Gamma}{4\pi}\left[\frac{1}{x} - \frac{x}{x^2+y^2}\right] = -\frac{\Gamma}{4\pi}\frac{y^2}{x(x^2+y^2)}$$

Path lines are given by

$$dx/dt = u(t) \qquad\qquad dy/dt = v(t)$$

Therefore

$$\frac{dx}{dt} = \frac{\Gamma}{4\pi}\left[\frac{x^2}{y(x^2+y^2)}\right]$$

$$\frac{dy}{dt} = -\frac{\Gamma}{4\pi}\left[\frac{y^2}{x(x^2+y^2)}\right]$$

Dividing we get

$$dy/dx = -\,y^3/x^3 \qquad \longrightarrow \qquad dy/y^3 = -\,dx/x^3$$

Integration gives

$$1/x^2 + 1/y^2 = F(t)$$

Initial condition gives $1/x_o^2 + 1/y_o^2 = F(t)$, where (x_o, y_o) are the initial coordinates of the vortex. Thus $F(t)$ has to be a constant.

<u>Exercise 5.5</u>

Equation of motion in rotating coordinates is

$$\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho}\underset{\sim}{\nabla}p + \underset{\sim}{g} - 2\underset{\sim}{\Omega} \times \underset{\sim}{u}$$

In Section 5.4, we took the dot product of each term with $d\underset{\sim}{x}$. Under barotropic and inviscid conditions, and in the presence of conservative body forces, we showed that

$$\frac{D}{Dt}(\Gamma) = 0$$

The extra term here is the Coriolis force. Taking its dot product with element $d\underset{\sim}{x}$ parallel to a circuit C, we get

$$-(2\underset{\sim}{\Omega} \times \underset{\sim}{u})\cdot d\underset{\sim}{x} = -\,2(\underset{\sim}{\Omega} \times \underset{\sim}{u})_i\, dx_i = -\,2\,\varepsilon_{ijk}\Omega_j\, u_k dx_i = -\,2\Omega_j\,\varepsilon_{ijk}\, u_k dx_i$$

$$= 2\Omega_j\,\varepsilon_{jik}\, dx_i u_k = 2\,\Omega_j\,(d\underset{\sim}{x} \times \underset{\sim}{u})_j = -\,2\Omega\cdot(\underset{\sim}{u} \times d\underset{\sim}{x})$$

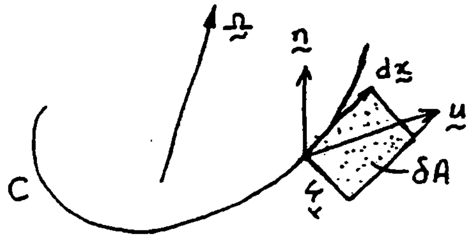
$$= -\,2\underset{\sim}{\Omega}\cdot\underset{\sim}{n}\; u_\perp\; dx \tag{1}$$

where $\underset{\sim}{n}$ is the unit vector perpendicular to the plane of $\underset{\sim}{u}$ and $d\underset{\sim}{x}$, and $u_\perp$ is the component of $\underset{\sim}{u}$ perpendicular to $d\underset{\sim}{x}$. In an interval of time dt, the perpendicular component will locally stretch out the area enclosed by C through an amount

$$\delta A = dx\ dL$$

where

$$dL = u_\perp\ dt$$



Since $\underset{\sim}{n}$ is normal to δA, the vector form of the area element is

$$\delta \underset{\sim}{A} = \underset{\sim}{n}\ dx\ dL$$

The rate of change of area element is

$$\frac{D}{Dt}(\delta \underset{\sim}{A}) = \underset{\sim}{n}\ dx\ u_\perp$$

Equation (1) therefore becomes

$$-(2\underset{\sim}{\Omega} \times \underset{\sim}{u}) \cdot d\underset{\sim}{x} = -2\underset{\sim}{\Omega} \cdot \frac{D}{Dt}(\delta \underset{\sim}{A})$$

Integrating around the entire circuit,

$$-\int_C (2\underset{\sim}{\Omega} \times \underset{\sim}{u}) \cdot d\underset{\sim}{x} = -\int_C 2\underset{\sim}{\Omega} \cdot \frac{D}{Dt}(\delta \underset{\sim}{A}) = -\frac{D}{Dt}\int_C (2\underset{\sim}{\Omega} \cdot d\underset{\sim}{A})$$

Including other terms considered in Section 5.4, we have proved that

$$\frac{D}{Dt}(\Gamma_a) = \frac{D}{Dt}\int_C (\underset{\sim}{\omega} + 2\underset{\sim}{\Omega}) \cdot d\underset{\sim}{A} = 0 \qquad (2)$$

This means that in the presence of $\underset{\sim}{\Omega}$ an increase of area will decrease the normal component of relative vorticity $\underset{\sim}{\omega}$.

Equation (2) actually follows directly from Kelvin's circulation theorem. The absolute vorticity is $\underset{\sim}{\omega} + 2\underset{\sim}{\Omega}$.

Equation (2) then follows since the absolute circulation must vanish.

Exercise 5.6

Consider two vortices of similar sense of rotation, as shown in Fig 1. The radii and speeds of two vortices are equal. As in Fig 5.12 of the text, the motion at A is the resultant of V_B, V_C, and V_D, while the motion at C is the resultant of V_A, V_B, and V_D. Comparing the velocity components on A and C, it is clear that the net resultant is an enlargement of the forward vortex, and contraction of the backward vortex, as indicated by the vertical arrows in Fig 2a. Figs 2b and 2c show that the rear vortex, due to its greater horizontal speed, passes through the forward vortex. Fig 2c shows that the more advanced vortex grows, and the rear vortex contracts, so that the first is decelerated and the second is accelerated. This results in Fig 2d, which is identical to Fig 1a, and the entire process starts all over again.

Fig 1

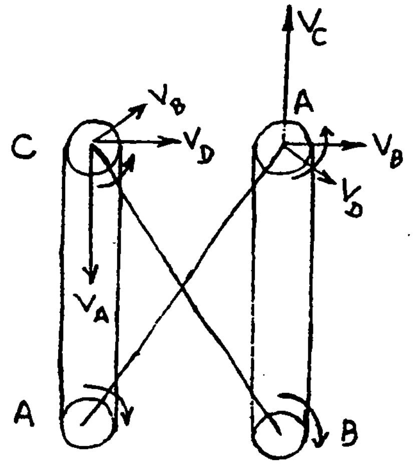
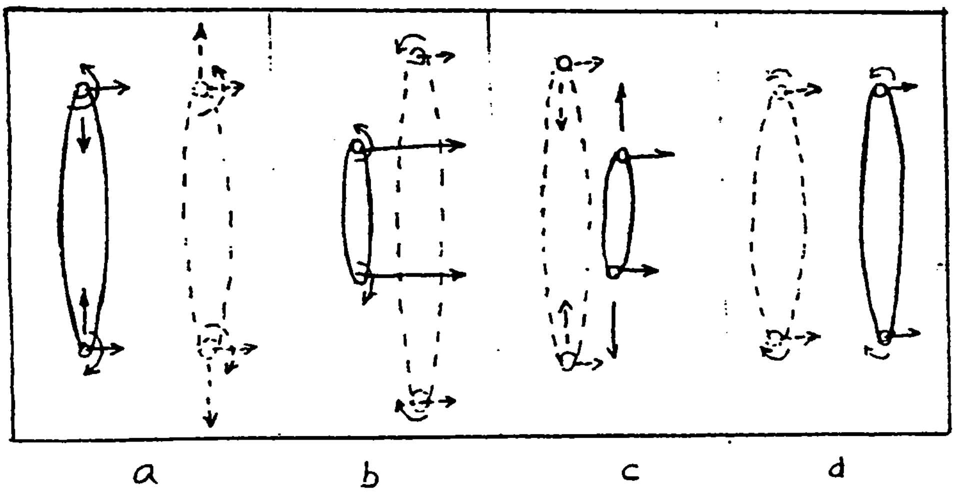


Fig 2



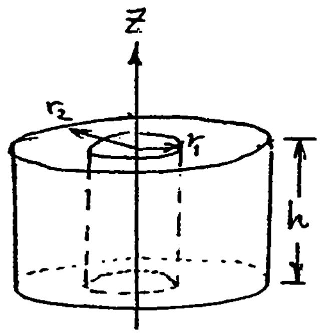
Exercise 5.7

For an irrotational vortical flow with circulation Γ, $u_\theta = \Gamma/2\pi r$. The volumetric flow rate is

$$Q = \int_{r_1}^{r_2} dr \int_0^h dz\, v_\theta = \frac{h\Gamma}{2\pi} \ln \frac{r_2}{r_1},$$

$$\text{Kinetic energy} = \int_0^h dz \int_{r_1}^{r_2} dr \int_0^{2\pi} d\theta \cdot r \cdot \frac{1}{2}\rho u_\theta^2,$$

$$= \frac{\rho}{2}\frac{\Gamma^2}{4\pi^2} \cdot h \cdot 2\pi \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\rho\Gamma^2 h}{4\pi} \ln\frac{r_2}{r_1} = \frac{1}{2}\rho\Gamma Q.$$



Exercise 5.8

a) $\nabla \times \mathbf{u} = \boldsymbol{\omega} = 2\boldsymbol{\Omega} = \text{const.} \qquad \int_A \boldsymbol{\omega} \cdot d\mathbf{A} = 2\Omega\pi R^2,$

b) $\int_A \boldsymbol{\omega} \cdot d\mathbf{A} = \oint_{C=\partial A} \mathbf{u} \cdot ds = \int_0^{2\pi} u_\theta \cdot R d\theta = 0,$

since $u_\theta(r = R) = 0$ by no slip.

Exercise 5.9

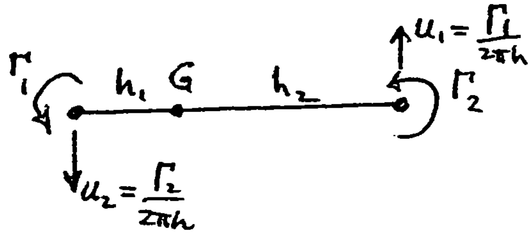
$$u_2 h_1 = u_1 h_2$$

$$\frac{\Gamma_2 h_1}{2\pi h} = \frac{\Gamma_1 h_2}{2\pi h} \quad \text{so} \quad \frac{h_1}{h_2} = \frac{\Gamma_1}{\Gamma_2}.$$

$$h_1\Gamma_2 = (h - h_2)\Gamma_2 = h_2\Gamma_1, \qquad h_2(\Gamma_1 + \Gamma_2) = h\Gamma_2$$

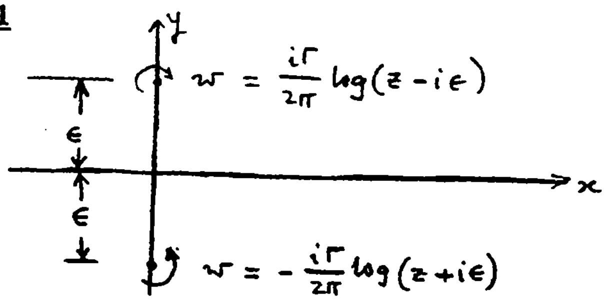
so

$$h_2 = \frac{h\Gamma_2}{\Gamma_1 + \Gamma_2}, \quad \text{and} \quad h_1 = h\left(1 - \frac{\Gamma_2}{\Gamma_1 + \Gamma_2}\right) = \frac{h\Gamma_1}{\Gamma_1 + \Gamma_2}.$$



$h = h_1 + h_2$

Exercise 6.1



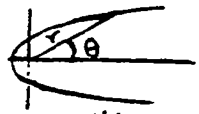
Complex potential for the vortex pair is

$$w = \frac{i\Gamma}{2\pi}[\log(z - i\epsilon) - \log(z + i\epsilon)] = \frac{i\Gamma}{2\pi}\log\frac{z - i\epsilon}{z + i\epsilon} \simeq \frac{i\Gamma}{2\pi}\log\left(1 - \frac{2i\epsilon}{z}\right)$$

$$\simeq \frac{i\Gamma}{2\pi}\left[-\frac{2i\epsilon}{z}\right], \text{ since } \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$= \mu/z, \qquad \text{where } \mu \equiv \Gamma\epsilon/\pi$$

Exercise 6.2



For half body $w = Uz + \frac{m}{2\pi}\log z \longrightarrow \psi = Ur\sin\theta + \frac{m}{2\pi}\theta$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{2\pi r} + U\cos\theta \quad \text{and} \quad u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta$$

$$\therefore q^2 = u_r^2 + u_\theta^2 = U^2 + \frac{m^2}{4\pi^2 r^2} + \frac{mU}{\pi r}\cos\theta$$

$$C_p = (p - p_\infty)/\tfrac{1}{2}\rho U^2 = 1 - q^2/U^2 = 1 - \frac{1}{U^2}\left[U^2 + \frac{m^2}{4\pi^2 r^2} + \frac{mU}{\pi r}\cos\theta\right]$$

$$= -\frac{m}{2\pi U}\left[\frac{m}{2\pi U r^2} + \frac{2\cos\theta}{r}\right]$$

From Section 6.8, the radial coordinate on the body is given by

$$r = m(\pi - \theta)/2\pi U\sin\theta \longrightarrow m/2\pi U = r\sin\theta/(\pi - \theta)$$

Therefore

$$C_p = -\frac{r\sin\theta}{\pi - \theta}\left[\frac{r\sin\theta}{(\pi - \theta)r^2} + \frac{2\cos\theta}{r}\right]$$

$$= -\frac{\sin\theta}{\pi - \theta}\left[\frac{\sin\theta}{\pi - \theta} + 2\cos\theta\right]$$

$$\text{Drag } D = \int_0^{2\pi} C_p \, d\theta = -\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{(\pi-\theta)^2} - 2\int_0^{2\pi} \frac{\sin\theta\cos\theta \, d\theta}{\pi-\theta}$$

Evaluation of the integrals give D = 0

Exercise 6.3

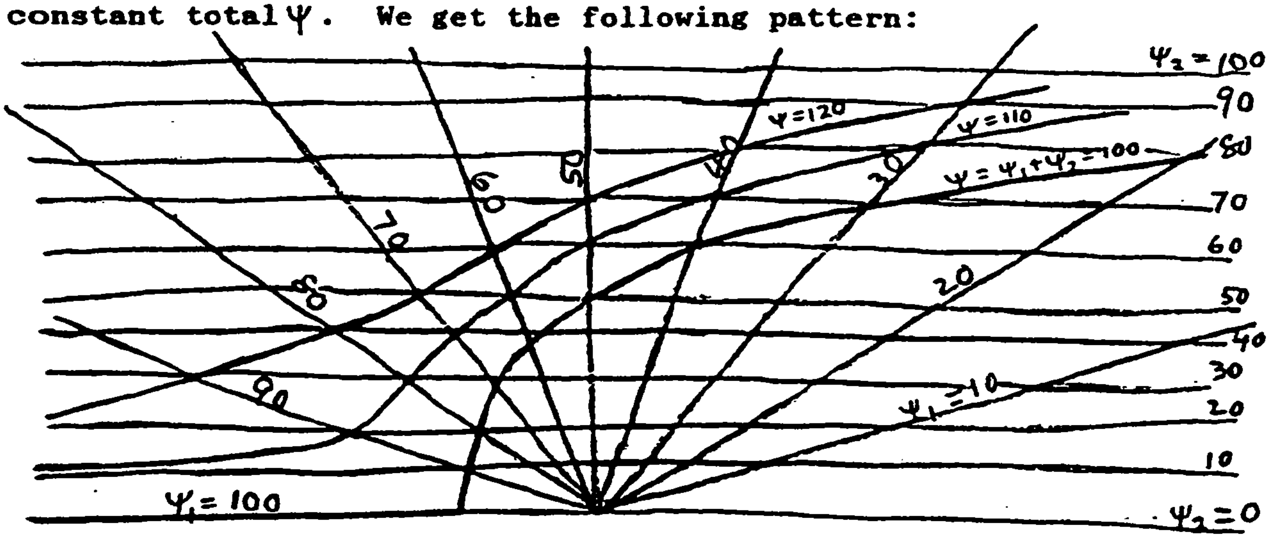
Given $m = 200 \text{ m}^2/\text{s}$ — 1 represents source

$U = 10 \text{ m/s}$ — 2 represents stream

$$\psi_1 = \frac{m}{2\pi}\theta \longrightarrow \Delta\psi_1 = \frac{m}{2\pi}\Delta\theta = \frac{m}{20} = 10 \text{ m}^2/\text{s}$$

$$\psi_2 = Uy \longrightarrow \Delta\psi_2 = U\Delta y = 10 \text{ m}^2/\text{s}$$

With the above intervals for stream functions, draw the streamlines for the source and the stream, and connect values of constant total ψ. We get the following pattern:



Solution for Exercise 6.4.

From Eq. (6.23), for a source/sink at the origin, $\psi = m\theta/(2\pi), \theta = \tan^{-1}(y/x), m = \pm$ for a source/sink. With reference to the figure, by superposition of the streamfunctions,

$\psi(x,y) = Uy + [m/(2\pi)]\cdot\{\tan^{-1}[y/(x+a)] - \tan^{-1}[y/(x-a)]\}$ for a uniform flow in the x-direction (U), a source (m) at $x = -a$ and a sink ($-m$) at $x=a$. Combining the angles,

$\psi(x,y) = Uy + [m/(2\pi)]\cdot\tan^{-1}\{[y/(x+a) + y/(x-a)]/[1 + y^2/(x^2 - a^2)]\}$

$= Uy - [m/(2\pi)]\cdot\tan^{-1}[2ay/(x^2 + y^2 - a^2)]$.

Differentiating to obtain the velocity components,

$u = \partial\psi/\partial y = U - (m/2\pi)\cdot[2a(x^2 + y^2 - a^2) - 4ay^2]/[(x^2 + y^2 - a^2)^2 + 4a^2y^2]$

$v = -\partial\psi/\partial x = -(m/\pi)\cdot[2axy]/[(x^2 + y^2 - a^2)^2 + (2ay)^2]$.

To obtain the stagnation points, we locate those places where $u = v = 0$. Now $u = 0$ on $y = 0$ ($x = 0$ is outside the domain of interest). Then $u = 0$ where $y = 0$ and $0 = U - (m/2\pi)\cdot(2ax^2 - 2a^3)/(x^2 - a^2)^2$. Then $(x^2 - a^2)^2 = [(ma)/(\pi U)]\cdot(x^2 - a^2)$. Solving, $x \equiv x_s = \pm a[1 + m/(aU\pi)]^{1/2}$ are the stagnation points. We note from the form of ψ that if $y \to -y, \psi \to -\psi$. Thus the shape $\psi = 0$ is symmetric (with respect to reflection in the x-axis). The stagnation points $x = \pm x_s, y = 0$ are the maximum upstream and downstream locations of the $\psi = 0$ surface, and are symmetrically located with respect to the x-axis. We thus expect the maximum width of the $\psi = 0$ shape to occur at the midpoint, or $x=0$. Denote the width by $y = h$. Then at the point $(0,h)$ on $\psi = 0$, $\psi = 0 = Uh - [m/(2\pi)]\cdot\tan^{-1}[2ah/(h^2 - a^2)]$. Taking the tangent, $\tan(2\pi Uh/m) = 2ah/(h^2 - a^2)$. From here, the relation cited in the text may be derived after considerable algebra, as follows. We use the identity $\tan(2\Delta) = (2\tan\Delta)/(1 - \tan^2\Delta)$ and let $\Delta = \pi Uh/m$. We obtain $[2\tan(\pi Uh/m)]/[1 - \tan^2(\pi Uh/m)] = 2ah/(h^2 - a^2)$. This gives a quadratic equation for $\tan\Delta$, *viz.*: $\tan^2\Delta + [(h^2 - a^2)/(ah)]\cdot\tan\Delta - 1 = 0$. Solving, $\tan\Delta = -[(h^2 - a^2)/(2ah)] \pm (1/2)\{[(h^2 - a^2)/(ah)]^2 + 4\}^{1/2}$. For positive results, the + sign must be chosen. The right hand side reduces to simply $a/h = \tan\Delta$. Thus $h/a = \cot\Delta = \cot(\pi Uh/m) = \cot[(\pi Ua/m)\cdot h/a]$.

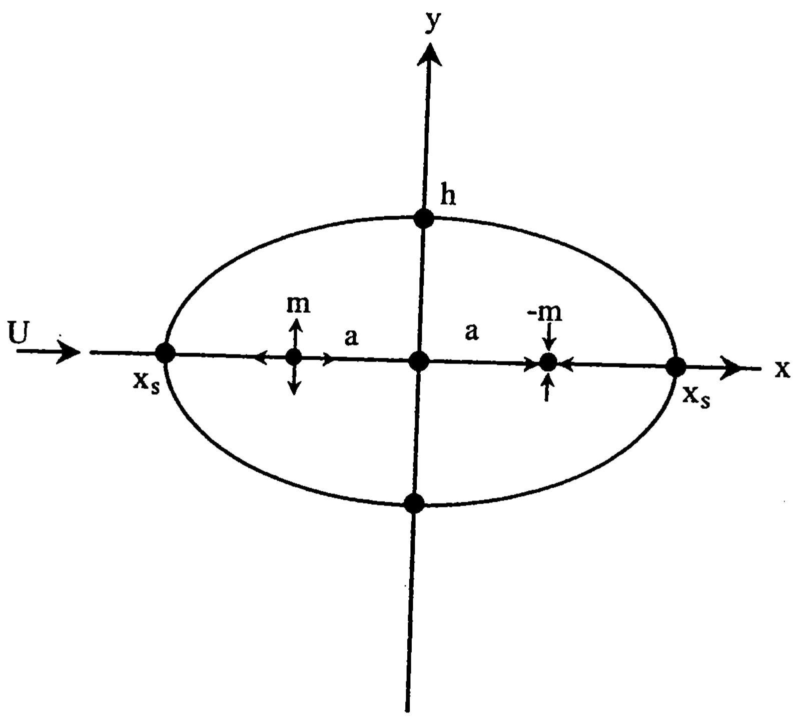


Figure for Exercise 6.4

Exercise 6.6

For a point source of strength Q (m^3/s), the tangential velocity is zero by symmetry. The radial velocity times area $4\pi r^2$ equals Q, so that $u_r = Q/4\pi r^2$. Therefore

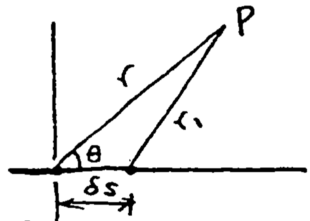
$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} = Q/4\pi r^2$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} = 0$$

Integrating u_θ, we see that $\phi = \phi(r)$ and $\psi = \psi(\theta)$. Integrating u_r we get

$$\phi = -Q/4\pi r \qquad \text{and} \qquad \psi = -Q\cos\theta/4\pi$$

<u>Exercise 6.7</u>



Potential due to a source-sink pair is

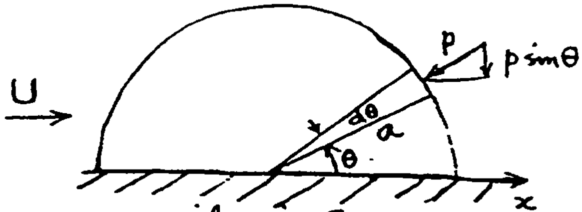
$$\phi = -Q/4\pi r + Q/4\pi r_1 = \frac{Q}{4\pi}\,\frac{r - r_1}{r r_1}$$

As $\delta s \longrightarrow 0$, $r - r_1 \longrightarrow \delta s\cos\theta$ and $rr_1 \longrightarrow r^2$. Then

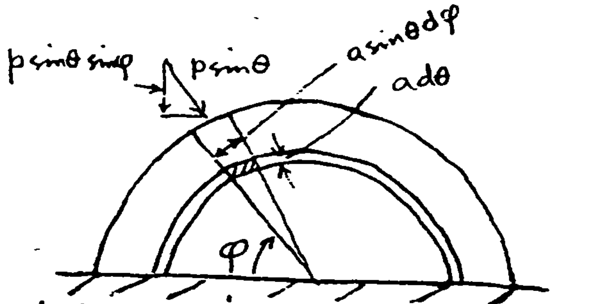
$$\phi \rightarrow \frac{Q}{4\pi}\frac{\delta s\cos\theta}{r^2} = \frac{m\cos\theta}{r^2}$$

where $m = \lim Q\,\delta s/4\pi$. The derivation for stream function is similar.

<u>Exercise 6.8</u>



side view



end view (looking into the direction of U)

Take a spherical coordinate element. Then $dA = (a\,d\theta)(a\sin\theta\,d\varphi)$. From geometry shown, component of downward pressure force on the element is

$$dL = (p\sin\theta\,\sin\varphi)dA = a^2 p\sin^2\theta\,\sin\varphi\,d\theta\,d\varphi$$

Total downward force due to the flow is

$$L = a^2\int_0^\pi p\sin^2\theta\,d\theta \int_0^\pi \sin\varphi\,d\varphi = a^2\tfrac{1}{2}\rho U^2 \int_0^\pi C_p \sin^2\theta\,d\theta \quad [2]$$

since value of the φ-integral is 2. From equation (6.79), the pressure coefficient for a sphere is $C_p = 1 - 9\sin^2\theta/4$. Thus

$$L = \rho a^2 U^2 \int_0^{\pi} \left[\sin^2\theta - \frac{9}{4}\sin^4\theta\right] d\theta = -11\pi\rho a^2 U^2/32$$

In order to stay in place, the weight of the hemisphere must be greater than the sum of weight of displaced air and the aerodynamic lift. This requires

$$\frac{2}{3}\pi a^3 g \rho_h > \frac{2}{3}\pi a^3 g \rho + 11\pi\rho a^2 U^2/32$$

or

$$\rho_h > \rho\left[1 + 33U^2/64ag\right]$$

Exercise 6.9

From Blasius theorem (6.45), the drag and lift are given by

$$D - iL = \frac{1}{2} i\rho \oint \left(\frac{dw}{dz}\right)^2 dz$$

For circular cylinder with circulation, the complex velocity is

$$w = U(z + a^2/z) + \frac{i\Gamma}{2\pi}\log z$$

from which

$$\frac{dw}{dz} = U(1 - a^2/z^2) + i\Gamma/2\pi z$$

$$D - iL = \frac{1}{2} i\rho \oint [U(1 - a^2/z^2) + i\Gamma/2\pi z]^2 dz$$

$$= \frac{1}{2} i\rho \oint [\cancel{U^2(1 - a^2/z^2)^2} - \cancel{\Gamma^2/4\pi^2 z^2} + 2U(1 - a^2/z^2)(i\Gamma/2\pi z)]dz$$

Retaining only the term that has a residue (that is, term of the form $\frac{1}{z}$)

$$D - iL = \frac{1}{2} i\rho \oint 2U(i\Gamma/2\pi z)dz = -\frac{\rho U\Gamma}{2\pi}\oint \frac{dz}{z} = -\frac{\rho U\Gamma}{2\pi}(2\pi i)$$

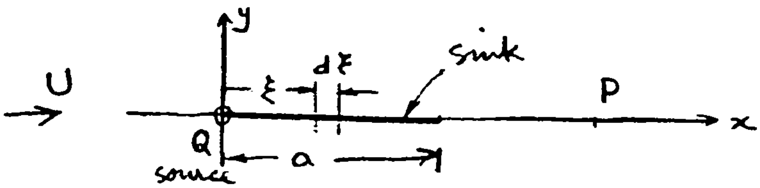
since the sum of the residue is 1. This gives

$$D - iL = -i\rho U\Gamma \longrightarrow D = 0 \text{ and } L = \rho U\Gamma$$

Exercise 6.10

Obviously the stagnation points are on the x-axis. To locate them, consider the cases of points to the right and left of the line sink.

Case 1: Stagnation point P to the right of line sink:



Velocity (directed to the positive x-axis) at P is

u_P = due to source + due to sink + due to stream

$$= Q/4\pi x^2 - \int_0^a \frac{k\,d\xi}{4\pi(x-\xi)^2} + U$$

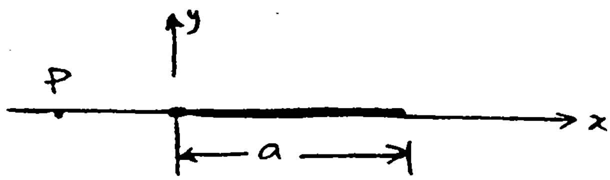
Performing the integration we get

$$u_P = U + \frac{Q}{4\pi}\left[\frac{1}{x^2} - \frac{1}{x(x-a)}\right] = 0 \text{ if P is a stagnation point}$$

This gives

$$\frac{x^2}{a^2}\left(\frac{x}{a} - 1\right) = \frac{Q}{4\pi U a^2}$$

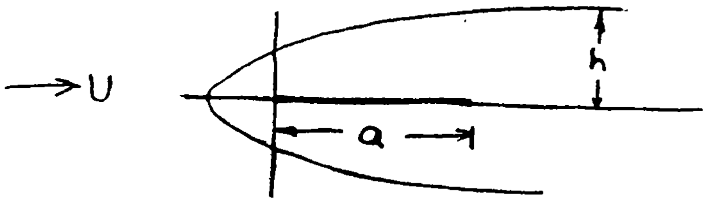
Case 2: P to the left of line sink



Now $u_P = -\,Q/4\pi x^2 + \dfrac{Q}{4\pi a}\displaystyle\int \frac{d\xi}{(x+\xi)^2} + U$

Setting $u_P = 0$ gives $$\frac{x^2}{a^2}\left(\frac{x}{a} + 1\right) = \frac{Q}{4\pi U a^2}$$

Exercise 6.11



The flow becomes uniform with speed U far downstream of the body. Let h be the asymptotic radius of the half body. Then a mass balance gives

$$U[\pi h^2] = \text{flow out of the line source} = ka$$

from which

$$h = \sqrt{ak/\pi U}$$

Exercise 6.12

$$\phi = \frac{\mu x}{x^2+y^2}, \quad \mu > 0 \qquad u = \frac{\partial \phi}{\partial x} = \frac{\mu}{x^2+y^2} - \frac{2\mu x^2}{(x^2+y^2)^2},$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{2\mu xy}{(x^2+y^2)^2} \qquad \frac{u}{\mu} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Thus $u > 0$ when $y > x$ and $u < 0$ when $y < x$.

$$\frac{v}{\mu} = -\frac{2xy}{(x^2+y^2)^2}.$$

$v < 0$ in first quadrant; $v > 0$ in second quadrant.

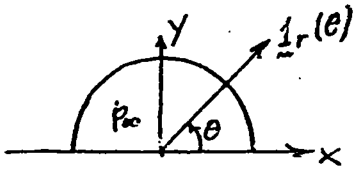
Exercise 6.13

We assume that the flow is the top half of flow over a circular cylinder given in Eq. (6.35); $\phi(r,\theta) = U(r + a^2/r)\cos\theta$, where U is the windspeed far upstream and a is the radius of the cylinder. On the surface of the body $u_r = \partial\phi/\partial r\big|_{r=a} = 0$, $u_\theta = (1/r)\partial\phi/\partial\theta\big|_{r=a} = -2U\sin\theta$.

Applying Bernoulli's equation to a streamline on the surface of the body,

$$p_\infty + \tfrac{1}{2}\rho U^2 = p + \tfrac{1}{2}\rho u_\theta^2 = p + \tfrac{1}{2}\rho U^2(4\sin^2\theta).$$

Then $p_\infty - p = \frac{1}{2}\rho U^2(4\sin^2\theta - 1)$.



This must be integrated over the half cylinder $d\mathbf{A} = a\,d\theta\,\mathbf{1}_r(\theta) = a\,d\theta\,(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$ per unit depth.

$$\mathbf{F} = \int (p_\infty - p)\,d\mathbf{A} = \frac{1}{2}\rho U^2 a \int_0^\pi (4\sin^2\theta - 1)(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)\,d\theta$$

x-component

$$\int_0^\pi \sin^2\theta\cos\theta\,d\theta = 0, \qquad \int_0^\pi \cos\theta\,d\theta = 0$$

y-component

$$\int_0^\pi [4(1-\cos^2\theta) - 1]\sin\theta\,d\theta = -3\cos\theta\big|_0^\pi + \tfrac{4}{3}\cos^3\theta\big|_0^\pi$$

$$= 6 - \tfrac{8}{3}$$

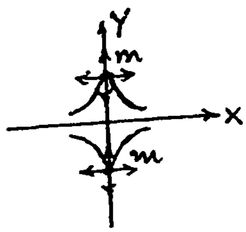
$$\mathbf{F} = \mathbf{j}\tfrac{5}{3}\rho U^2 a \text{ per unit depth.}$$

$\rho = 1.23\,\text{kg/m}^3$, $U = 40\,\text{m/s}$, $a = 3\,\text{m}$, $\mathbf{F} = \mathbf{j} \times 9840\,\text{N/m}$ depth of building.

Exercise 6.14

Replace the plane by the superposition of two sources at $y = \pm a$. In terms of the complex variable z,

$$w(z) = \phi + i\psi = m\ln(z - ia) + m\ln(z + ia).$$



Using polar coordinates, $z = re^{i\theta}$, $r = (x^2 + y^2)^{1/2}$, $\theta = \tan^{-1} y/x$,

$$\psi = \mathrm{Im}\{w\} = m\tan^{-1}\left(\frac{y-a}{x}\right) + m\tan^{-1}\left(\frac{y+a}{x}\right).$$

$\psi = 0$ on $y = 0$ for all x.

$$v = -\frac{\partial\psi}{\partial x} = 0 \quad \text{on } y = 0,$$

$$u = \frac{\partial\psi}{\partial y} = m\left\{\frac{1/x}{1 + ((y-a)/x)^2} + \frac{1/x}{1 + ((y+a)/x)^2}\right\}$$

$$= m\left[\frac{x}{x^2 + (y-a)^2} + \frac{x}{x^2 + (y+a)^2}\right],$$

$$u(y = 0) = \frac{2mx}{x^2 + a^2}.$$

From Bernoulli's theorem $p + \frac{1}{2}\rho v^2 = p_\infty + \frac{1}{2}\rho v_\infty^2$. At ∞, $v_\infty = 0$ so $p - p_\infty = -\frac{1}{2}\rho v^2$. On $y = 0$,

$$p - p_\infty = -\frac{\rho}{2}\frac{4m^2x^2}{(x^2 + a^2)^2}$$

To find force/depth,

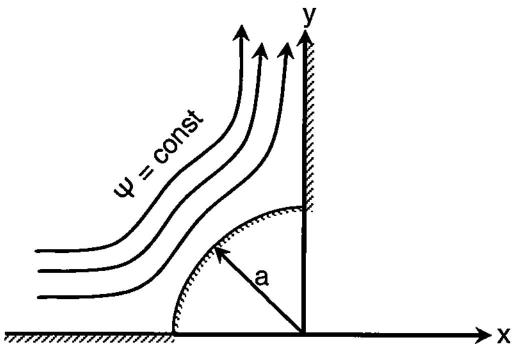
$$F = -\int_{-\infty}^{\infty} \frac{2\rho m^2 x^2}{(x^2 + a^2)^2}\,dx = -4\rho m^2 \int_0^{\infty} \frac{x^2\,dx}{(x^2 + a^2)^2}$$

$$= -4\rho m^2\left[-\frac{x}{2(x^2 + a^2)} + \frac{1}{2a}\tan^{-1}\frac{x}{a}\right]_0^{\infty} = -4\rho m^2\left(\frac{\pi}{2}\cdot\frac{1}{2a}\right)$$

$$= -\rho m^2\pi/a \quad \text{(downward force/depth on wall).}$$

Exercise 6.15

Consider a two-dimensional, constant density potential flow over a circular cylinder of radius $r = a$ with axis coincident with a right angle corner, as shown in the figure below. Solve for the streamfunction and velocity components.



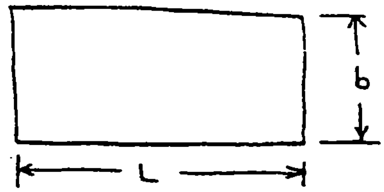
We have seen at the end of Section 6.13 that the mapping $w = z^2$ transforms flow over a plane surface into flow in a right angle corner. Apply this to flow over a circular cylinder as described in Eq. (6.34), say with unit speed. This gives the complex potential, $w = \phi + i\psi = z^2 + a^4 / z^2$. Using polar coordinates, $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, $z^2 = r^2 e^{2i\theta}$. Then the complex potential for the desired flow is

$w = \phi + i\psi = r^2 \cos 2\theta + a^4 r^{-2} \cos 2\theta + i(r^2 \sin 2\theta - a^4 r^{-2} \sin 2\theta)$. The streamfunction is the imaginary part of the complex potential,

$\psi = r^2 \sin 2\theta - a^4 r^{-2} \sin 2\theta$. We see that the streamfunction is zero on $r = a$ and also on all coordinate planes. The velocity components are

$u_r = r^{-1} \partial\psi / \partial\theta = 2\cos 2\theta (r - a^4 r^{-3})$; $u_\theta = -\partial\psi / \partial r = -2\sin 2\theta (r + a^4 r^{-3})$.

Exercise 7.1



Given $\phi = A\cos(m\pi x/L)\cos(n\pi y/b)\cosh k(z+H)\,e^{-i\omega t}$

$$u = \frac{\partial\phi}{\partial x} = -A(m\pi/L)\sin(m\pi x/L)\cos(n\pi y/b)\cosh k(z+H)\,e^{-i\omega t}$$

= 0 at x = 0 and x = L. So BC is satisfied.

$$v = \frac{\partial\phi}{\partial y} = -A(n\pi/b)\cos(m\pi x/L)\sin(n\pi y/b)\cosh k(z+H)\,e^{-i\omega t}$$

= 0 at y = 0 and y = b. So BC is satisfied.

The Laplacian is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -A(m\pi/L)^2\cos(m\pi x/L)\cos(n\pi y/b)\cosh k(z+H)\,e^{-i\omega t}$$
$$- A(n\pi/b)^2\cos(m\pi x/L)\cos(n\pi y/b)\cosh k(z+H)\,e^{-i\omega t}$$
$$+ Ak^2\cos(m\pi x/L)\cos(n\pi y/b)\cosh k(z+H)\,e^{-i\omega t}$$

This is zero if

$$(m\pi/L)^2 + (n\pi/b)^2 = k^2$$

The free surface boundary conditions are

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z} \qquad \text{and} \qquad \frac{\partial\phi}{\partial t} = -g\eta \qquad \text{at } z = 0$$

Combining the two conditions, we get

$$\frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial\eta}{\partial t} = -g\frac{\partial\phi}{\partial z} \qquad \text{at } z = 0$$

This requires

$$A\cos(m\pi x/L)\cos(n\pi y/b)\cosh k(z+H)(-i\omega)^2 e^{-i\omega t}\Big|_{z=0}$$

$$= -gA\cos(m\pi x/L)\cos(n\pi y/b)\,[k\sinh(z+H)]\,e^{-i\omega t}\Big|_{z=0}$$

or
$$\cosh kH(-\omega^2) = -gk\sinh kH$$

or
$$\omega^2 = gk\tanh kH$$

Exercise 7.2

Given $L = 30$ km $\quad b = 2$ km $\quad H = 100$ m

For $m = 1$, $n = 0$ mode, the result of Exercise 7.1 gives

$$k = m\pi/L = \pi/30\times10^3 = 1.047\times10^{-4}\ \text{m}^{-1}$$

$$\omega = \sqrt{gk\tanh kH} = \sqrt{(9.81)(1.047\times10^{-4})\tanh[(1.047\times10^{-4})(100)]}$$

$$= 3.27\times10^{-3}\ \text{s}^{-1}$$

$$\text{Period} = \frac{2\pi}{\omega} = 32\ \text{min}$$

Exercise 7.3

For surface gravity waves in the presence of surface tension, the dispersion relation is

$$\omega = \sqrt{k(g+\sigma k^2/\rho)\tanh kH} = \sqrt{(\sigma k^3/\rho)\tanh kH}\text{ , neglecting gravity}$$

$$\therefore\quad c_g = \frac{\partial\omega}{\partial k} = \sqrt{\frac{\sigma}{\rho}}\left\{k^{3/2}H\,\text{sech}^2kH/[2\sqrt{\tanh kH}] + \frac{3}{2}\sqrt{k\tanh kH}\right\}$$

The deep water approximation $kH \gg 1$ gives

$$\text{sech}^2kH \longrightarrow 0 \qquad\text{and}\qquad \tanh kH \longrightarrow 1$$

Then
$$c_g = \frac{\partial\omega}{\partial k} = \sqrt{\sigma/\rho}\,\{\tfrac{3}{2}\sqrt{k}\} = \tfrac{3}{2}\sqrt{\sigma k/\rho} \qquad (1)$$

For pure capillary waves in deep water, the phase speed is

$$c = \sqrt{(g/k+\sigma k/\rho)\tanh kH} \simeq \sqrt{\sigma k/\rho}$$

Then (1) reduces to $\qquad c_g = 3c/2$

Exercise 7.4

From (7.65), dispersion relation of surface gravity waves with surface tension σ is

$$\omega = \sqrt{k(g + \sigma k^2/\rho)\tanh kH}$$
$$\simeq \sqrt{k(g + \sigma k^2/\rho)} \quad \text{for deep water}$$

$$\therefore \; c_g = \frac{\partial\omega}{\partial k} = \frac{1}{2\sqrt{kg + \sigma k^3/\rho}}\left(g + \frac{3\sigma k^2}{\rho}\right) = \frac{1}{2}\sqrt{\frac{g}{k}}\,\frac{1 + 3\sigma k^2/\rho g}{\sqrt{1 + \sigma k^2/\rho g}}$$

Setting $\partial c_g/\partial k = 0$ for minimum c_g gives (letting $x = \sigma k^2/\rho g$)

$$3x^2 + 6x - 1 = 0 \qquad \longrightarrow \qquad x = 2/\sqrt{3} - 1 = 0.1547$$

The corresponding wavenumber is

$$k = \sqrt{\rho g(0.1547)/\sigma} = \sqrt{1000(9.81)(0.1547)/0.074} = 143.2 \text{ m}^{-1}$$

$$\therefore \; c_{g_{min}} = \frac{1}{2}\sqrt{\frac{g}{k}}\,\frac{1+3x}{\sqrt{1+x}} = \frac{1}{2}\sqrt{\frac{9.81}{143.2}}\,\frac{1+3\times 0.1547}{\sqrt{1+0.1547}} = 0.166 \text{ m/s}$$

Exercise 7.5

Given $\Delta T = 10^\circ C$

$H = 100$ m

$\alpha = 0.0025^\circ C^{-1}$

This is a case of an internal wave at the interface between a shallow layer overlying an infinitely deep fluid, discussed in Section 7.17.

Reduced gravity $\quad g' = g\alpha\Delta T = 9.81(0.0025)10 = 0.245 \text{ m/s}^2$

From (7.126) $\quad c = \sqrt{g'H} = \sqrt{(0.245)(100)} = 4.95$ m/s

Exercise 7.6

Given $N = 0.02 \text{ s}^{-1}$

$\omega = 0.01 \text{ s}^{-1}$

$$k = 2\pi/\lambda = 2 \text{ m}^{-1}$$
$$\hat{w} = 0.01 \text{ m/s}$$
$$\rho_o = 800 \text{ kg/m}^3$$

From (7.152) $\omega = N\cos\theta \longrightarrow \cos\theta = \omega/N = 0.01/0.02 = 0.5$

$$\therefore \quad \theta = 60^\circ$$

Energy flux:

Vertical wavenumber $m = k\tan\theta = 2(\tan 60^\circ) = 3.464 \text{ m}^{-1}$

From (7.171) the energy flux is

$$F = [\rho_o \omega m \hat{w}^2/2k^2][\underset{\sim}{i}_x m/k - \underset{\sim}{i}_z]$$
$$= \left[(800)(0.01)(3.464)(0.01)^2/\{2(2)^2\}\right]\left[1.732\underset{\sim}{i}_x - \underset{\sim}{i}_z\right]$$
$$= 3.464 \times 10^{-4} \; [1.732\underset{\sim}{i}_x - \underset{\sim}{i}_z] \text{ W/m}^2$$

Magnitude $F = 3.464 \times 10^{-4}\sqrt{1.732^2 + 1^2} = 6.92 \times 10^{-4} \text{ W/m}^2$

Exercise 7.7

From Section 15, the velocity potentials above and below the interface are

$$\phi_1 = \frac{i\omega a}{k} e^{-kz} e^{i(kx-\omega t)}$$

$$\phi_2 = -\frac{i\omega a}{k} e^{kz} e^{i(kx-\omega t)}$$

The velocity components are therefore

$$u_1 = \frac{\partial \phi_1}{\partial x} = \frac{i\omega a}{k}(ik)\, e^{-kz} e^{i(kx-\omega t)} = -\omega a\, e^{-kz}\cos(kx - \omega t)$$

$$u_2 = \frac{\partial \phi_2}{\partial x} = -\frac{i\omega a}{k}(ik)\, e^{kz} e^{i(kx-\omega t)} = \omega a\, e^{kz}\cos(kx - \omega t)$$

$$w_1 = \frac{\partial \phi_1}{\partial z} = \frac{i\omega a}{k}(-k)\, e^{-kz} e^{i(kx-\omega t)} = \omega a\, e^{-kz}\sin(kx - \omega t)$$

$$w_2 = \frac{\partial\phi_2}{\partial z} = -\frac{i\omega a}{k}(k)\ e^{kz}\ e^{i(kx-\omega t)} = \omega a\ e^{kz}\ \sin(kx - \omega t)$$

where we have taken real parts. The kinetic energy is therefore

$$KE = \tfrac{1}{2}\rho_1\int_0^{\infty}(u_1^2 + w_1^2)dz + \tfrac{1}{2}\rho_2\int_{-\infty}^{0}(u_2^2 + w_2^2)dz$$

$$= \tfrac{1}{2}\rho_1\omega^2 a^2\int_0^{\infty} e^{-2kz}\,dz + \tfrac{1}{2}\rho_2\omega^2 a^2\int_{-\infty}^{0} e^{2kz}\,dz =$$

$$= (\rho_2+\rho_1)\frac{\omega^2 a^2}{4k} = (\rho_2-\rho_1)ga^2/4$$

where we have used the dispersion relation $\omega^2 = gk(\rho_2-\rho_1)/(\rho_2+\rho_1)$

Exercise 7.8

Equations (7.116)-(7.119) are

$$A = -(ia/2)(\omega/k + g/\omega)$$

$$B = (ia/2)(\omega/k - g/\omega)$$

$$C = -(ia/2)(\omega/k + g/\omega) - (ia/2)(\omega/k - g/\omega)\ e^{2kH}$$

$$b = (iC/\omega)k\ e^{-kH}$$

Applying BC (7.111)

$$\rho_1\frac{\partial\phi_1}{\partial t} + \rho_1 g\zeta = \rho_2\frac{\partial\phi_2}{\partial t} + \rho_2 g\zeta \qquad \text{at } z = -H$$

we get

$$\rho_1(-i\omega)(Ae^{-kH} + Be^{kH}) + \rho_1 gb = \rho_2(-i\omega)Ce^{-kH} + \rho_2 gb$$

Substituting for b in terms of C, this becomes

$$-\rho_1\omega(Ae^{-kH} + Be^{kH}) = Ce^{-kH}\left[\frac{gk}{\omega}(\rho_2-\rho_1) - \rho_2\omega\right]$$

Substituting for A, B and C, this becomes (writing $x = \omega^2/gk$)

$$\rho_1[-(x+1)e^{-kH} + (x-1)e^{kH}] = \left[(x+1)e^{-kH} + (x-1)e^{kH}\right]\left[\frac{\rho_2-\rho_1}{x} - \rho_2\right]$$

Rearranging, this gives

$$(x - 1)\left\{x\left[2\rho_1 \sinh kH + 2\rho_2 \cosh kH\right] - 2(\rho_2 - \rho_1)\sinh kH\right\} = 0$$

Exercise 7.9

The solution is of the form

$$p(r,t) = \tfrac{1}{r}[f(r-ct) + g(r+ct)]$$

with initial conditions

$$p(r,0) = e^{-r} = \tfrac{1}{r}[f(r) + g(r)],$$
$$\frac{\partial p}{\partial t}(r,0) = 0 = \frac{c}{r}[-f'(r) + g'(r)].$$

Thus $f' = g'$ so $f = g = re^{-r}/2$. Then

$$p(r,t) = \frac{1}{2r}\left[(r-ct)e^{-(r-ct)} + (r+ct)e^{-(r+ct)}\right].$$

Exercise 8.1

Given $$P = f(d, U, \omega, c, \rho, \mu)$$

Dimension of power is $[P] = [W]/T = FL/T = (MLT^{-2})L/T = ML^2/T^3$.
The dimensional matrix is

	P	d	U	ω	c	ρ	μ
M	1	0	0	0	0	1	1
L	2	1	1	0	1	−3	−1
T	−3	0	−1	−1	−1	0	−1

We find

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ -3 & 0 & -1 \end{vmatrix} = 1(-1 - 0) = -1 \neq 0$$

Rank = 3, variables = 7. $\therefore$ # of nondimensional numbers = 4.
We choose U, d and ρ as repeating variables.

Let $\Pi_1 = U^a d^b \rho^c P$

$\therefore M^0 L^0 T^0 = (LT^{-1})^a (L)^b (ML^{-3})^c ML^2T^{-3} = M^{1+c} L^{a+b-3c-2} T^{-a-3}$

Solving we get $a = -3,\ b = -2,\ c = -1$

Thus $$\Pi_1 = P/\rho U^3 d^2$$

Let $\Pi_2 = U^a d^b \rho^c \omega$

$\therefore M^0 L^0 T^0 = (LT^{-1})^a (L)^b (ML^{-3})^c T^{-1} = M^c L^{a+b-3c} T^{-a-1}$

Solving we get $a = -1,\ b = 1,\ c = 0$

Thus $$\Pi_2 = d\omega/U$$

Let $\Pi_3 = U^a d^b \rho^c c \longrightarrow \Pi_3 = U/c$

Let $= U^a d^b \rho^c \mu \longrightarrow \Pi_4 = \rho U d/\mu$

Therefore

$$P/\rho U^3 d^2 = f(d\omega/U,\ U/c,\ \rho Ud/\mu)$$

At low speeds the power output is likely to depend most strongly on the size and angular speed on the propeller, and viscous effects; so $P/\rho U^3 d^2$ is likely to be determined mostly by $d\omega/U$ and $\rho Ud/\mu$. At higher speeds compressibility effects are likely to be more important than viscous effects, so $d\omega/U$ and U/c are most important.

Exercise 8.2

Given $L_m/L_p = 1/25$

$p_m = 200$ kPa

$T_m = 300\ \text{K} \longrightarrow \rho_m = p_m/RT_m = 2\times10^5/(287)(300) = 2.32\ \text{kg/m}^3$

$U_p = 30$ km/h

$U_m = ?$

$D_m/D_p = ?$

Since the submarine would not operate near the sea surface, gravity is unimportant. Therefore C_D depends only on Re, which should be duplicated in the model test.

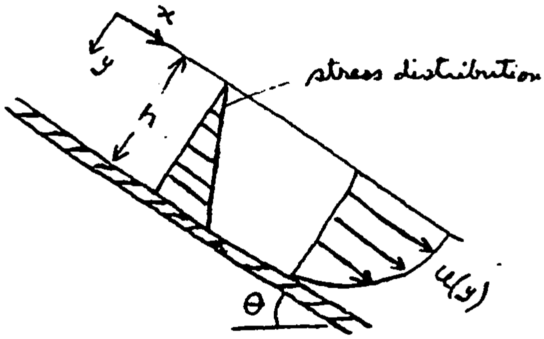
Equating $U_m L_m/\nu_m = U_p L_p/\nu_p$ we get

$$U_m = U_p(L_p/L_m)(\nu_m/\nu_p) = 30(1/25)(1.5\times10^{-5}/10^{-6}) = 18\ \text{km/h} = 5\ \text{m/s}$$

Equating C_D, $D_m/\rho_m U_m^2 L_m^2 = D_p/\rho_p U_p^2 L_p^2$

$$D_m/D_p = (\rho_m/\rho_p)(U_m/U_p)^2(L_m/L_p)^2 = (2.32/1000)(18/30)^2(1/25)^2$$
$$= 1.336\times10^{-6}$$

Exercise 9.1



x-momentum equation gives $0 = g\sin\theta + \nu \dfrac{d^2u}{dy^2}$

Integrating $u = -\dfrac{g\sin\theta}{2\nu}y^2 + Ay + B$

BC1: Zero shear stress at free surface requires du/dy = 0 at y = 0. This gives

$$A = 0$$

BC2: u = 0 at y = h. This gives

$$B = \frac{g\sin\theta\, h^2}{2\nu}$$

Therefore

$$u = \frac{g\sin\theta}{2\nu}(h^2 - y^2)$$

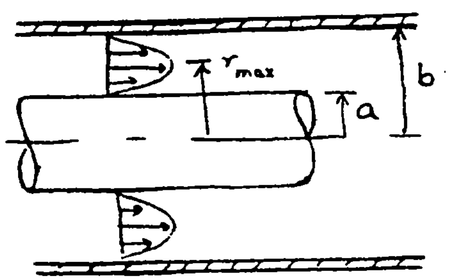
Volume rate of flow per unit width is

$$Q = \int_0^h u\, dy = \frac{g\sin\theta}{2\nu}\int_0^h (h^2 - y^2)\, dy = \frac{g\sin\theta\, h^3}{3\nu}$$

Magnitude of shear stress is

$$\tau = \mu\frac{du}{dy} = \mu\left(-\frac{g\sin\theta}{\nu}y\right) = \rho g y \sin\theta = \rho g h \sin\theta \quad \text{at the wall}$$

Exercise 9.2



Use cylindrical coordinates, with r denoting radial direction. The x-momentum equation is

$$\frac{1}{\mu}\frac{dp}{dx} = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

where dp/dx is a negative constant. Integrating

$$\frac{du}{dr} = \frac{r}{2\mu}\frac{dp}{dx} + \frac{A}{r}$$

$$u = \frac{r^2}{4\mu}\frac{dp}{dx} + A \log r + B$$

BC1: $u = 0$ at $r = a$

BC2: $u = 0$ at $r = b$

These give

$$A = -\frac{1}{4\mu}\frac{dp}{dx}\frac{b^2 - a^2}{\log(b/a)}$$

$$B = \frac{1}{4\mu}\frac{dp}{dx}\left[\frac{b^2 - a^2}{\log(b/a)}\log a - a^2\right]$$

Then

$$u = \frac{1}{4\mu}\frac{dp}{dx}\left[(r^2 - a^2) - \frac{b^2 - a^2}{\log(b/a)}\log\frac{r}{a}\right]$$

Shear stress is

$$\tau = \mu\frac{du}{dr} = \frac{r}{2}\frac{dp}{dx} - \frac{1}{4}\frac{dp}{dx}\frac{b^2 - a^2}{r\log(b/a)}$$

Velocity is maximum where $\tau = 0$. This gives

$$0 = \frac{r_{max}}{2}\frac{dp}{dx} - \frac{1}{4}\frac{dp}{dx}\frac{b^2 - a^2}{r_{max}\log(b/a)} \longrightarrow r_{max} = \sqrt{\frac{b^2 - a^2}{2\log(b/a)}}$$

Volume rate of flow is

$$Q = \int_a^b u(2\pi r\, dr) = \frac{\pi}{8\mu}\frac{dp}{dx}\left[\frac{(b^2 - a^2)^2}{\log(b/a)} + a^4 - b^4\right]$$

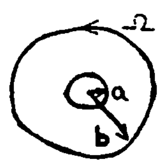
__Exercise 9.3__

From Equation (9.14)

$$u_\theta = \frac{1}{1 - a^2/b^2}\left\{(\Omega - 0)r + \frac{a^2}{r}(0 - \Omega)\right\}$$

$$= \frac{b^2\Omega}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)$$

From Appendix B the shear stress is



$$\tau_{r\theta} = 2\mu\left[\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right] = \mu r\frac{d}{dr}\left(\frac{u_\theta}{r}\right) = \frac{\mu r b^2\Omega}{b^2-a^2}\frac{d}{dr}\left(1-\frac{a^2}{r^2}\right)$$

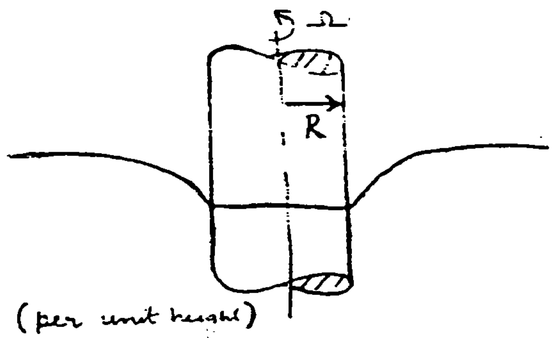
$$= 2\mu a^2 b^2\Omega / \left[(b^2 - a^2)r^2\right]$$

Torque on inner cylinder $= \left(\tau_{r\theta}\right)_{r=a}$ **(area)(radius)**

$$= \frac{2\mu a^2 b^2\Omega}{(b^2-a^2)a^2}(2\pi a h)a = \frac{4\pi\mu a^2 b^2 h\Omega}{b^2-a^2}.$$

It is easy to see that torque on the outer cylinder is the same.

<u>Exercise 9.4</u>



$u_\theta = \Omega R^2/r$

$\therefore \tau_{r\theta} = \mu r \frac{d}{dr}\left(\frac{u_\theta}{r}\right) = -2\mu\Omega R^2/r^2$

Work done $= \left|2\pi R\,\tau_{r\theta} u_\theta\right|_{r=R} = 4\pi\mu\Omega^2 R^2$ (per unit height)

Viscous dissipation is $\phi = 2\mu e_{ij}^2$. **In cylindrical coordinates, with velocity distribution** $u_\theta = \Omega R^2/r$, **it reduces to**

$$\phi = \mu\left[\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right]^2 = 4\mu\Omega^2 R^4/r^4$$

Integrating over the entire region, per unit height

Total dissipation $= \int_R^\infty \phi\, 2\pi r\, dr = 8\pi\mu\Omega^2 R^4\int_R^\infty r^{-3}\, dr = 4\pi\mu\Omega^2 R^2$

= Work done

<u>Exercise 9.5</u>

The solution to the problem is very similar to the decay of line vortex, treated in Section 9.9 in detail. Using cylindrical coordinates, the governing equation is

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)\right]$$

BC1: $u_\theta(r,0) = 0$

BC2: $u_\theta(r,\infty) = \Gamma/2\pi r$

BC3: $u_\theta(\infty,t) = 0$

We get similarity solution by defining

$$u' = \frac{u_\theta}{\Gamma/2\pi r}$$

which must be of the form $u' = f(r,t,\nu)$. Dimensional analysis shows that the nondimensional coordinate is of the form

$$\eta^2 = r^2/4\nu t$$

The governing set is then

$$f'' + f' = 0$$

subject to

$$f(0) = 1$$
$$f(\infty) = 0$$

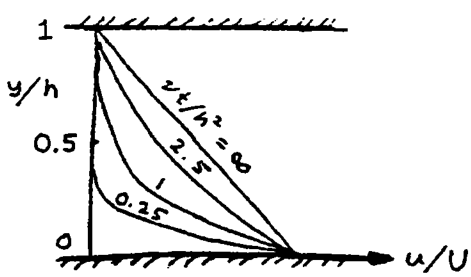
The solution is

$$f = e^{-\eta}$$

that is

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-r^2/4\nu t}$$

<u>Exercise 9.6</u>



The differential equation is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

with BC1: $u(0,t) = U$

BC2: $u(h,t) = 0$

IC: $u(y,0) = 0$

The solution of (1) may be found by first transforming the dependent variable from u to

$$q(y,t) = U(1 - y/h) - u$$

which represents the deviation of the velocity from the final linear distribution. Clearly q satisfies the same differential equation, and the homogeneous boundary conditions

$$q(0,t) = q(h,t) = 0$$

and the initial condition

$$q(y,0) = U(1 - y/h)$$

A solution for q satisfying the two boundary conditions is

$$q = \sum_{n=1}^{\infty} A_n \exp\left(- n^2\pi^2 \frac{\nu t}{h^2}\right) \sin(n\pi y/h)$$

The values of the constants A_n are now found by applying the initial condition, which gives

$$\sum_{n=1}^{\infty} A_n \sin(n\pi y/h) = U(1 - y/h)$$

so that

$$A_n = \frac{2}{h}\int_0^h U(1 - y/h)\sin(n\pi y/h)\,dy = 2U/\pi n$$

The velocity distribution is then

$$u(y,t) = U(1 - y/h) - \frac{2U}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\exp\left(- n^2\pi^2\frac{\nu t}{h^2}\right)\sin(n\pi y/h)$$

The velocity profiles for different values of $\nu t/h^2$, sketched above, show that the effect of the stationary upper boundary is initially negligible. Gradually it influences the entire region by viscous [redacted] diffusion.

Exercise 9.7

The plates are infinite in x and z so $\partial/\partial x = 0$, $\partial/\partial z = 0$. Liquids are incompressible so mass conservation ($\nabla \cdot \mathbf{u} = 0$) reduces to $\partial v/\partial y = 0$ with $v = 0$ on $y = 0, h$ so $v = 0$ everywhere. Momentum conservation in each fluid reduces to

$$\frac{d}{dy}\left(\mu \frac{du}{dy}\right) = 0.$$

With μ = const. in each fluid, $u = c_1 y + c_2$ in fluid 1, and $u = c_3 y + c_4$ in fluid 2. The boundary conditions are:

$$y = h \ : \ u = U, \quad y = 0 \ : \ u = 0, \quad y = d \ : \ u_1 = u_2$$

and

$$\mu_1(\partial u_1/\partial y)\big|_d = \mu_2(\partial u_2/\partial y)\big|_d$$

since both tangential velocity and stress are continuous at the interface. Then we have $U = c_1 h + c_2$, $c_4 = 0$, $c_1 d + c_2 = c_3 d$, and $\mu_1 c_1 = \mu_2 c_3$. Solving,

$$c_3 = \frac{\mu_1}{\mu_2} c_1,$$
$$\left(\frac{\mu_1}{\mu_2} - 1\right) c_1 = \frac{c_2}{d},$$
$$c_2 = \left(\frac{\mu_1}{\mu_2} - 1\right) d c_1.$$

Then

$$U = c_1 \left[h + d\left(\frac{\mu_1}{\mu_2} - 1\right)\right]$$

so

$$c_1 = \frac{U}{[h + d(\mu_1/\mu_2 - 1)]},$$
$$c_2 = \frac{(\mu_1/\mu_2 - 1)\, dU}{[h + d((\mu_1/\mu_2) - 1)]},$$
$$c_3 = \frac{(\mu_1/\mu_2)U}{[h + d(\mu_1/\mu_2 - 1)]}.$$

Now we can write
Fluid 1:

$$u = \frac{Uy + (\mu_1/\mu_2 - 1)\, dU}{[h + d(\mu_1/\mu_2 - 1)]}$$

Fluid 2:

$$u = \frac{(\mu_1/\mu_2)Uy}{[h + d(\mu_1/\mu_2 - 1)]}$$

Exercise 9.8

We must consider flow inside the sphere as well as outside the sphere. The dimensionless variables are u_r/U, u_θ/U, r/a, $\psi/(Ua)$, $(p - p_\infty)/(\mu U/a)$, $\tau/(\mu U/a)$ represented by the symbols u_r, u_θ, r, ψ, p, τ, respectively. The spherical surface is $r = 1$. For the flow outside the sphere, $r \geq 1$,

$$\psi_o = \sin^2\theta(Ar^4 + Br^2 + Cr + D/r).$$

To satisfy uniform flow at infinity we must take $A = 0$, $B = 1/2$.

Inside the sphere, the flow is similarly represented by

$$\psi_i = \sin^2\theta(\bar{A}r^4 + \bar{B}r^2 + \bar{C}r + \bar{D}/r), \quad r \leq 1.$$

Now $\bar{D} = 0$ to avoid a singular streamfunction at $r = 0$. From the definition of the streamfunction,

$$u_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

Outside flow:

$$u_{r_o} = -2\cos\theta\left(\frac{1}{2} + \frac{C}{r} + \frac{D}{r^3}\right),$$
$$u_{\theta_o} = \sin\theta\left(1 + \frac{C}{r} - \frac{D}{r^3}\right).$$

Inside flow:

$$u_{r_i} = -2\cos\theta(\bar{A}r^2 + \bar{B} + \bar{C}/r),$$
$$u_{\theta_i} = \sin\theta(4\bar{A}r^2 + 2\bar{B} + \bar{C}/r).$$

For a finite velocity inside the sphere, $\bar{C} = 0$. There are four boundary conditions at $r = 1$ to determine the four remaining constants.

$$u_{r_o}(r = 1) = 0 = \tfrac{1}{2} + C + D, \tag{1}$$
$$u_{r_i}(r = 1) = 0 = \bar{A} + \bar{B}, \tag{2}$$
$$u_{\theta_o}(r = 1) = u_{\theta_i}(r = 1) : 1 + C - D = 4\bar{A} + 2\bar{B}, \tag{3}$$

and stress continuity $(\boldsymbol{\tau} - p\mathbf{I}) \cdot \mathbf{1}_r$ is continuous at $r = 1$. The θ component is

$$\mu\left(\frac{\partial u_{\theta_o}}{\partial r} + \frac{1}{r}\frac{\partial u_{r_o}}{\partial\theta} - \frac{u_{\theta_o}}{r}\right)_{r=1} = \mu'\left(\frac{\partial u_{\theta_i}}{\partial r} + \frac{1}{r}\frac{\partial u_{r_i}}{\partial\theta} - \frac{u_{\theta_i}}{r}\right)_{r=1}$$

$$\begin{aligned}
&\mu(3D - C + 2D + 2C + 1 + D - C - 1)\sin\theta \\
&\quad = \mu'(-\bar{C} + 8\bar{A} + 2\bar{B} + 2\bar{A} - \bar{C} - 2\bar{B} - 4\bar{A})\sin\theta \\
&6D\mu = 6\bar{A}\mu'.
\end{aligned} \tag{4}$$

Solving Eqs. 1, 2, 3, 4. we obtain

$$C = -\frac{1}{4}\frac{3\mu' + 2\mu}{\mu' + \mu}, \quad D = \frac{1}{4}\frac{\mu'}{\mu + \mu'}, \quad \bar{A} = \frac{1}{4}\frac{\mu}{\mu + \mu'}, \quad \bar{B} = -\frac{1}{4}\frac{\mu}{\mu + \mu'}.$$

Integrating around the body surface, the (dimensionless) force on the sphere is $\mathbf{F}_B = -\int_{S_1}(pI - \tau)\cdot \mathbf{d}S_1$ where S_1 is the surface $r = 1$. The force is made dimensionless here by $(\mu U/a)\cdot a^2 = \mu U a$. The surface element is $d\mathbf{S}_1 = r\,d\theta\, r\sin\theta\, d\phi|_{r=1}\mathbf{1}_r(\theta,\phi)$. (Note that in the text, this surface area elemented is called "$d\mathbf{A}_1$.")

$$\tau \cdot d\mathbf{S}_1 = \left\{\left[\mathbf{1}_r \cdot 2\frac{\partial u_r}{\partial r} + \mathbf{1}_\theta\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\right]^{r^2}\right\}_{r=1} \sin\theta\, d\theta\, d\phi.$$

The θ-component is

$$6D\sin^2\theta\, d\theta\, d\phi\, \mathbf{1}_\theta = \mathbf{1}_\theta \cdot \frac{3}{2}\frac{\mu'}{\mu + \mu'}\sin^2\theta\, d\theta\, d\phi.$$

The r-component includes a pressure contribution for which we must solve. For the outside flow,

$$\psi_o = \sin^2\theta\left[\frac{r^2}{2} + \frac{1}{4}\frac{\mu'}{\mu + \mu'}\cdot\frac{1}{r} - \frac{1}{4}\frac{(3\mu' + 2\mu)}{\mu' + \mu}r\right]$$

$$-\nabla p = \nabla \times \nabla \times \mathbf{u}$$

in our dimensionless formulation. This gives

$$\begin{aligned}
-\frac{\partial p}{\partial r} &= \frac{1}{r^2\sin\theta}\left[2\sin\theta\cos\theta\left(1 + \frac{1}{2}\frac{\mu'}{\mu + \mu'}\frac{1}{r^3}\right)\right. \\
&\quad \left. - \left(\frac{1}{2} + \frac{1}{4}\frac{\mu'}{\mu + \mu'}\frac{1}{r^3} - \frac{1}{4}\frac{3\mu' + 2\mu}{\mu' + \mu}\frac{1}{r}\right)4\sin\theta\cos\theta\right] \\
&= \frac{3\mu' + 2\mu}{\mu' + \mu}\frac{\cos\theta}{r^3}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{r}\frac{\partial p}{\partial \theta} &= \frac{\sin^2\theta}{r\sin\theta}\left(-\frac{3}{2}\frac{\mu'}{\mu + \mu'}\frac{1}{r^4}\right) \\
&\quad - \frac{2\sin\theta}{r}\left[\frac{\partial}{\partial r}\left(\frac{1}{2} + \frac{1}{4}\frac{\mu'}{\mu + \mu'}\frac{1}{r^3} - \frac{1}{4}\frac{3\mu' + 2\mu}{\mu' + \mu}\frac{1}{r}\right)\right] \\
-\frac{\partial p}{\partial \theta} &= \frac{3\mu' + 2\mu}{\mu' + \mu}\frac{\sin\theta}{2r^2}.
\end{aligned}$$

Integrating for p,

$$p = \frac{3\mu' + 2\mu}{\mu' + \mu}\frac{\cos\theta}{2r^2} \quad [p(\infty) = 0].$$

The r-component of the integrand is

$$\left(-p + 2\frac{\partial u_r}{\partial r}\right) r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{1}_r\Big|_{r=1} = -\frac{3}{2}\cos\theta\frac{\mu' + 2\mu}{\mu' + \mu}\sin\theta\, d\theta\, d\phi\, \mathbf{1}_r.$$

The force on the liquid sphere is then

$$\mathbf{F}_B = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta \left[\underbrace{-\frac{3}{2}\frac{\mu' + 2\mu}{\mu' + \mu}\cos\theta\, \mathbf{1}_r}_{\text{normal stress}} + \underbrace{\frac{3}{2}\frac{\mu'}{\mu + \mu'}\sin\theta\, \mathbf{1}_\theta}_{\text{shear stress}} \right]$$

The unit vectors on the surface of a sphere are funtions of the polar and azimuthal angles, θ, ϕ, respectively. They must be put in terms of unit vectors that do not vary with position **i**, **j**, **k**. Using the spherical coordinate system defined for Exercise 2.9 in which the z-axis is the axis of symmetry from which the polar angle θ is measured, and the azimuthal angle ϕ is measured from the x axis,

$$\mathbf{1}_r(\theta, \phi) = \mathbf{i}\cos\phi\sin\theta + \mathbf{j}\sin\phi\sin\theta + \mathbf{k}\cos\theta,$$
$$\mathbf{1}_\theta(\theta, \phi) = \mathbf{i}\cos\phi\cos\theta + \mathbf{j}\sin\phi\cos\theta - \mathbf{k}\sin\theta.$$

Integrating first over ϕ, the **i** and **j** components integrate to 0. (The flow at infinity is in the $-z$ direction, so we would expect the force to be in that direction.)

Finally,

$$\mathbf{F}_B = 2\pi\mathbf{k} \int_0^{\pi} d\theta \left[-\frac{3}{2}\frac{\mu' + 2\mu}{\mu' + \mu}\sin\theta\cos^2\theta - \frac{3}{2}\frac{\mu'}{\mu' + \mu}\sin^3\theta \right]$$

Carrying out the trigonometric integrals,

$$\mathbf{F}_B = -2\pi\mathbf{k} \left(\underbrace{\frac{\mu' + 2\mu}{\mu' + \mu}}_{\text{normal stress}} + \underbrace{\frac{2\mu'}{\mu' + \mu}}_{\text{shear stress}} \right) = -2\pi\mathbf{k}\frac{3\mu' + 2\mu}{\mu' + \mu} = -6\pi\mathbf{k}\frac{3\mu' + 2\mu}{3\mu' + 3\mu},$$

in dimensionless form. In dimensional form the drag becomes

$$D = 6\pi\mu U a \frac{3\mu' + 2\mu}{3\mu' + 3\mu} = D_s \frac{3\mu' + 2\mu}{3\mu' + 3\mu},$$

where D_s is the Stokes drag for a solid particle.

Exercise 9.9

Mass and momentum conservation are expressed in terms of the dimensionless variables r/a, $\mathbf{u}/U$, $(p-p_\infty)/(\mu U/a)$, $\psi/(Ua)$ and $\mathrm{Re} = Ua/\nu$. Without change of notation, the dimensionless equations are, $\nabla \cdot \mathbf{u} = 0$ and $\mathrm{Re}\left(\partial \mathbf{u}/\partial t + \nabla u^2/2 - \mathbf{u}\times\nabla\times\mathbf{u}\right) + \nabla p = -\nabla\times\nabla\times\mathbf{u}$. In the limit $\mathrm{Re}\to\infty$ we neglect the acceleration (fluid inertia). Take curl to eliminate p: $\nabla\times\nabla\times\nabla\times\mathbf{u} = O(\mathrm{Re}) \to 0$. The flow is in an (r,θ) plane ($z=$ const.) so $\mathbf{u} = -\mathbf{k}\times\nabla\psi$. Then

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r},$$

$$\begin{aligned}
\nabla\times(\mathbf{k}\times\nabla\psi) &= \mathbf{k}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right],\\
\nabla\times\nabla\times(\mathbf{k}\times\nabla\psi) &= \mathbf{1}_r\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]\\
&\quad - \mathbf{1}_\theta\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right],\\
-\nabla\times\nabla\times\nabla\times(\mathbf{k}\times\nabla\psi) &= \mathbf{k}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)\right]\right.\\
&\quad \left. + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]\right\} = 0.
\end{aligned}$$

For large r, $u_r \sim \cos\theta$, $u_\theta \sim -\sin\theta$ so we expect $\psi \sim \sin\theta$. Thus we try $\psi(r,\theta) = f(r)\sin\theta$.

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{df}{dr}\right)\sin\theta - \frac{f}{r^2}\sin\theta\right)\right]\\
&\quad + \frac{1}{r^3}\frac{\partial}{\partial r}\left(r\frac{df}{dr}\right)(-\sin\theta) + \frac{f}{r^4}\sin\theta = 0.
\end{aligned}$$

Divide by $\sin\theta$ and recognize an equidimensional D.E. with solution of the form $f(r) = r^m$. Substituting r^m and dividing by r^{m-4},

$$m^2(m-2)^2 - (m-2)^2 - m^2 + 1 = 0.$$

Two roots may be seen immediately, $m = 1$, $m = -1$. Expand and refactor

$$(m-1)(m+1)(m^2-4m+3) = (m-1)(m+1)(m-3)(m-1) = 0.$$

So $m = 1$ is a double root corresponding to the solutions $r, r\ln r$. Now $f(r) = c_1 r + c_2 r\ln r + c_3 r^3 + c_4 r^{-1}$. r^3 grows too fast to match with an outer solution as $r\to\infty$ so we must choose $c_3 = 0$.

$$\psi(r,\theta) = \sin\theta(c_1 r + c_2 r\ln r + c_4/r).$$

The boundary conditions are at $r = 1$, $u_r = 0$, $u_\theta = 0$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\bigg|_{r=1} = \cos\theta(c_1 + c_4) = 0, \qquad c_4 = -c_1,$$
$$u_\theta = -\frac{\partial\psi}{\partial r}\bigg|_{r=1} = \sin\theta(c_1 + c_2 - c_4) = 0, \qquad c_2 = -2c_1.$$

$\psi(r,\theta) = c_1 \sin\theta(r - 2r\ln r - 1/r)$. c_1 is determined by matching with a solution valid for large r. For large r, $\psi \sim \sin\theta \cdot r\ln r$ so $u_r \sim \cos\theta \ln r$, $u_\theta \sim \sin\theta \ln r$. $\mathbf{u} \cdot \nabla\mathbf{u} \sim \ln r/r$ and viscous stresses $\sim \nabla^2 u \sim 1/r^2$. Terms neglected $\sim$ terms retained when $\mathrm{Re}(\ln r/r) \sim 1/r^2$ or $r\ln r \sim 1/\mathrm{Re}$. This solution is valid for $r\ln r \ll 1/\mathrm{Re}$.

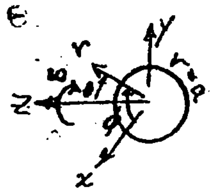
Exercise 9.10

For very small Re we want to solve $\nabla p = -\mu\nabla \times \nabla \times \mathbf{u}$ in spherical coordinates corresponding to boundary conditions representing rotation about a diameter (say the z-axis). Since the flow is driven by the rotation of the sphere, assume $u_r = 0$, $u_\theta = 0$, $\mathbf{u} = u_\phi \mathbf{1}_\phi$ only. By symmetry we would expect $\partial p/\partial\phi = 0$ and $\partial u_\phi/\partial\phi = 0$. Then $\nabla \cdot \mathbf{u} = 0$ is satisfied identically. Now

$$\nabla \times \mathbf{u} = \frac{\mathbf{1}_r}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\phi \sin\theta) - \frac{\mathbf{1}_\theta}{r}\frac{\partial}{\partial r}(r u_\phi),$$

$$\nabla \times \nabla \times \mathbf{u} = -\frac{\mathbf{1}_\phi}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial(r u_\phi)}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\phi\sin\theta)\right)\right]$$

$$= 0 = -\frac{1}{\mu}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}\mathbf{1}_\phi.$$



We seek to solve

$$\frac{\partial^2}{\partial r^2}(r u_\phi) + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\phi\sin\theta)\right] = 0,$$

with boundary conditions $u_\phi = \omega a \sin\theta$ on $r = a$ (no slip)

$$u_\phi \to 0 \quad \text{as } r \to \infty.$$

This suggests a solution of the form $u_\phi(r,\theta) = \omega\sin\theta F(r)$. Substituting and dividing by $\sin\theta$, $rF'' + 2F' - 2F/r = 0$ or $r^2F'' + 2rF' - 2F = 0$. This has a solution of the form $F = r^m$. Substituting and dividing by r^m: $(m-1)m + 2m-2 = 0$. $m^2+m-2 = 0$, $(m+2)(m-1) = 0$, $m = +1, -2$. $u_\phi = \omega\sin\theta(c_1 r + c_2/r^2)$. $u_\phi \to 0$ as $r \to \infty$ so $c_1 = 0$. At $r = a$: $\omega\sin\theta c_2/a^2 = \omega a\sin\theta$ so $c_2 = a^3$. Then $u_\phi = (\omega a^3/r^2)\sin\theta$. The shear stress

$$\tau_{\phi r} = 2\mu\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right),$$

$$\tau_{\phi r} = \mu r\omega a^3\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right) = -\left.\frac{3\mu\omega a^3\sin\theta}{r^3}\right|_{r=a} = -3\mu\omega\sin\theta \text{ on surface.}$$

$$\text{Torque on sphere} = \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, a^2 \sin\theta}_{dA} \underbrace{(-3\mu\omega\sin\theta)}_{\tau_{\phi r}} \underbrace{a\sin\theta}_{\text{momentarm}}$$

$$= -6\pi\,\mu\omega a^3 \int_0^{\pi} \underbrace{d\theta \sin^3\theta}_{4/3} = -8\pi\,\mu\omega a^2,$$

where the $-$ sign arises because the torque opposes the rotation (which is in the positive ϕ direction).

Exercise 10.1

We are required to solve $ff'' + f''' = 0$
subject to $f'(\infty) = 1$, $f(0) = 0$, and $f'(0) = 0$

To solve the problem by Runge-Kutta integration scheme, we need to reduce it to a set of 3 first order equations by defining

$$g(\eta) = f' \qquad \text{and} \qquad h(\eta) = f''$$

Then the problem reduces to

$$df/d\eta = g$$

$$dg/d\eta = h \tag{1}$$

$$dh/d\eta = -fh$$

subject to

$$f(0) = g(0) = 0 \tag{2}$$

$$g(\infty) = 1 \tag{3}$$

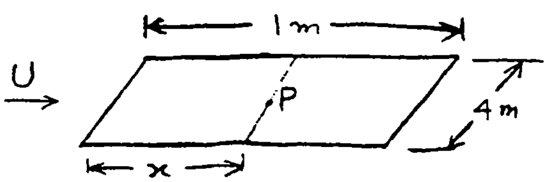
<u>Procedure</u>: Starting at $\eta = 0$, integrate (1) over successive steps $\Delta\eta$ by Runge-Kutta method, which is a standard computer library subroutine. Since we have initial conditions (at $\eta = 0$) for f and g only, we need to search for h(0) that will result in $g(\infty) = 1$. This can be done by trial and error. Because of the boundary-layer nature of the solution, condition (3) can be replaced by g = 1 at $\eta = 10$ (say). The details of the procedure, including the computer algorithm, is given by Carnahan, Luther and Wilkes (1969, page 407) *Applied Numerical Methods*, Wiley.

Exercise 10.2

$\nu = 2.29 \times 10^{-6}$ m^2/s

$\rho = 800$ kg/m^3

U = 0.5 m/s



$Re_L = UL/\nu = (0.5)(1)/2.29\times10^{-6} = 2.18\times10^5 < Re_{cr} = 10^6$
Therefore the flow is laminar everywhere.

<u>At x = 0.5 m</u>

$Re_x = Ux/\nu = 1.09\times10^5$. From equation (10.35): $\delta/x = 4.9/\sqrt{Re_x}$

$\therefore\ \delta = 4.9x/\sqrt{Re_x} = 4.9(0.5)/\sqrt{1.09\times10^5} = 0.742$ cm

From (10.36): $\tau_o = 0.332\rho U^2/\sqrt{Re_x} = 0.332(800)(0.5)^2/\sqrt{1.09\times10^5}$

$= 0.2\ N/m^2$

<u>At x = 1 m</u>

$\delta = 4.9L/\sqrt{Re_L} = 4.9(1)/\sqrt{2.18\times10^5} = 1.05$ cm

$\tau_o = 0.332\rho U^2/\sqrt{Re_L} = 0.142\ N/m^2$

<u>Total drag</u>

From (10.38): $C_D = 1.33/\sqrt{Re_L} = 2.849\times10^{-3}$

$D = C_D(\frac{1}{2}\rho U^2)(bL) = (2.849\times10^{-3})(0.5)(800)(0.5)^2(4)(1) = 1.13$ N

<u>Exercise 10.3</u>

Given $\rho = 1.167\ kg/m^3$
$\nu = 1.5\times10^{-5}\ m^2/s$
$U = 6$ m/s
$u/U = 0.456$
$x = 15$ cm
$y, v = ?$

<u>Calculate y</u>

From Fig 10.5: $\eta = y\sqrt{U/\nu x} = 1.4$ when $u/U = 0.456$

$\therefore\ y = 1.4\sqrt{\nu x/U} = 1.4\sqrt{(1.5\times10^{-5})(0.15)/6} = 8.57\times10^{-4} = 0.857$ mm

<u>Calculate v</u>

From Fig 10.6: $v\sqrt{x/\nu U} = 0.16$ when $\eta = 1.4$

$\therefore\ v = 0.16\sqrt{\nu U/x} = 0.16\sqrt{(1.5\times10^{-5})(6)/0.15} = 0.39\times10^{-2}$ m/s

Exercise 10.4

Given $\qquad u/U = \sin(\pi y/2\delta)$

Momentum integral equation is (10.42):

$$\frac{d}{dx}\int_0^\delta (U - u)u\,dy = \frac{\tau_o}{\rho}$$

$$\text{RHS} = \frac{\tau_o}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_o = \nu\pi U/2\delta$$

$$\text{LHS} = \frac{d}{dx}\int_0^\delta \left(U - U\sin\frac{\pi y}{2\delta}\right)U\sin\frac{\pi y}{2\delta}\,dy = \frac{U^2}{2\pi}(4-\pi)\frac{d\delta}{dx}$$

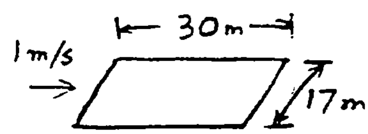
Momentum equation gives $\qquad \dfrac{U^2}{2\pi}(4-\pi)\dfrac{d\delta}{dx} = \dfrac{\nu\pi U}{2\delta}$

Integration gives

$$\delta = \sqrt{\frac{2\pi^2}{4-\pi}}\sqrt{\frac{\nu x}{U}} = 4.795\sqrt{\frac{\nu x}{U}}$$

$$C_f = \frac{\tau_o}{\frac{1}{2}\rho U^2} = \frac{\nu\pi}{\delta U} = 0.655/\sqrt{Re_x}$$

Exercise 10.5



$Re_L = UL/\nu = (1)(30)/10^{-6} = 3\times10^7 > Re_{cr} = 10^6$

From Fig 10.9: $\qquad C_D \equiv D/\frac{1}{2}\rho U^2 A = 0.00\cancel{5}\ 2.5$

$\therefore\ D = \frac{1}{2}\rho U^2 A C_D = \frac{1}{2}(1000)(1)^2(30)(17)(\overset{.0025}{0.005}) = \overset{2550}{1275}$ N

Exercise 10.6

Fall velocity $\quad U = \sqrt{2gh} = \sqrt{2(9.81)(2.5)} = 7.0$ m/s

At steady state the drag on parachute equals the load, so that D = 80(9.81) N. Parachute cross sectional area is

$A = D/C_D\frac{1}{2}\rho U^2 = 80(9.81)/(2.3)(0.5)(1.167)(7)^2 = 11.93\ m^2 = \pi d^2/4$

$$d = \sqrt{4(11.93)/\pi} = 3.9 \text{ m}$$

Exercise 10.7

We have to determine the roots of

$$x^2 - (3 + 2\epsilon)x + 2 + \epsilon = 0 \qquad (1)$$

for small ϵ. For $\epsilon = 0$, (1) becomes $x^2 - 3x + 2 = (x - 2)(x - 1) = 0$, whose roots are $x = 1$ and $x = 2$. For small ϵ, we assume that each root has the perturbation expansion of the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \qquad (2)$$

Substituting (2) into (1), we get

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots)^2 - (3+2\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots) + 2 + \epsilon = 0 \qquad (3)$$

We now collect terms of the same order in ϵ. Using the binomial theorem to expand the first term, we have

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots)^2 = x_0^2 + 2x_0(\epsilon x_1 + \epsilon^2 x_2 + \ldots) + (\epsilon x_1 + \epsilon^2 x_2 + \ldots)^2$$

$$= x_0^2 + 2\epsilon x_0 x_1 + 2\epsilon^2 x_0 x_2 + \epsilon^2 x_1^2 + 2\epsilon^3 x_1 x_2 + \epsilon^4 x_2^2 + \ldots$$

$$= x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0 x_2 + x_1^2) + \ldots \qquad (4)$$

where we have retained terms up to ϵ^2.

Similarly, multiplying the factors in the second term in (3), we have

$$(3 + 2\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots) = 3x_0 + 3\epsilon x_1 + 3\epsilon^2 x_2 + 2\epsilon x_0 + 2\epsilon^2 x_1 + 2\epsilon^3 x_2$$

$$= 3x_0 + \epsilon(3x_1 + 2x_0) + \epsilon^2(3x_2 + 2x_1) + \ldots \qquad (5)$$

Substituting (4) and (5) into (3), we have

$$x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(2x_0 x_2 + x_1^2) - 3x_0 - \epsilon(3x_1 + 2x_0) - \epsilon^2(3x_2 + 2x_1) + 2 + \epsilon \ \ldots = 0$$

Collecting terms of like powers of ϵ yields

$$(x_0^2 - 3x_0 + 2) + \epsilon(2x_0x_1 - 3x_1 - 2x_0 + 1) + \epsilon^2(2x_0x_2 + x_1^2 - 3x_2 - 2x_1) + \ldots = 0 \qquad (6)$$

Equating coefficients of each power of to zero, we get

$$x_0^2 - 3x_0 + 2 = 0 \qquad (7)$$

$$2x_0x_1 - 3x_1 - 2x_0 + 1 = 0 \qquad (8)$$

$$2x_0x_2 + x_1^2 - 3x_2 - 2x_1 = 0 \qquad (9)$$

The above set has to be solved for x_0, x_1, and x_2. Solution of (7) is

$$x_0 = 1,\ 2$$

With x_0 known, (8) can be solved for x_1, and (9) can be solved for x_2. Corresponding to $x_0 = 1$ and $x_0 = 2$, we get

$$x_1 = -1,\ 3$$
$$x_2 = 3,\ -3$$

Substituting into the expansion (2), the to approximate roots are

$$x = 1 - \epsilon + 3\epsilon^2 + \ldots$$

and

$$x = 2 + 3\epsilon - 3\epsilon^2 + \ldots$$

<u>Exercise</u> <u>10.8</u>

Consider $$\epsilon y'' - (2x + 1)y' + 2y = 0 \qquad (1)$$
with $$y(0) = \alpha \qquad (2)$$
$$y(1) = \beta \qquad (3)$$

As $\epsilon \longrightarrow 0$, the order of the differential equation reduces to one. This means that one of the boundary conditions is not satisfied by the outer solution, and a boundary layer is necessary to satisfy both boundary conditions. In the present case the outer solution satisfies condition $y(0) = \alpha$, but not $y(1) = \beta$. We shall proceed with the assumption that a boundary layer is necessary at $x = 1$. If our attempt at producing matchable expansions fails, we shall conclude that the boundary layer should be at the other end.

We try an expansion

$$y = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots$$

Substitution of which into (1) gives

$$\epsilon[y_0'' + \epsilon y_1'' + \epsilon^2 y_2''] - (2x + 1)[y_0' + \epsilon y_1' + \epsilon^2 y_2'] + 2[y_0 + \epsilon y_1 + \epsilon^2 y_2] = 0$$

Equating coefficients of ϵ^0 on both sides, we get

$$-(2x + 1)y_0' + 2y_0 = 0, \qquad \text{with } y_0(0) = \alpha$$

or $$dy_0/y_0 = 2dx/(2x + 1)$$

or $$\log y_0 = \log(2x + 1) + \log A, \qquad \text{where } \log A \text{ is const}$$

or $$y_0 = A(2x + 1)$$

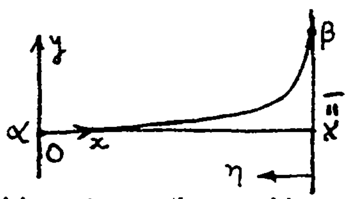
BC $y_0(0) = \alpha$ gives $A = \alpha$. Therefore the first term of the outer solution is

$$y_0 = \alpha(2x + 1)$$

This, however does not satisfy the BC at $x = 1$.

Having gotten some insight into the problem, let us define the natural distance in the boundary layer near $x = 1$ by

$$\eta = \frac{1 - x}{\epsilon} \longrightarrow dx = -\epsilon d\eta \qquad (4)$$



The transformation (4) stretches distances, and highest order derivative remains in the differential equation within the boundary layer.

Following Section 10.14, we now write a composition expansion

$$y(x) = [y_0(x) + \epsilon y_1(x) + \ldots] + \{\hat{y}_0(\eta) + \epsilon \hat{y}_1(\eta) + \ldots\} \tag{5}$$

where [ ] is the outer solution, and { } is the *correction* required within the boundary layer so that the solution is valid everywhere. All terms in { } go to zero as $\eta \longrightarrow \infty$.

Substitution of (5) into (1) gives

$$[\epsilon y_0'' - (2x + 1)y_0' + 2y_0] + \epsilon[\epsilon y_1'' - (2x + 1)y_1' + 2y_1]$$

$$+ \{\frac{\epsilon}{\epsilon^2}\hat{y}_0'' + (3 - 2\epsilon\eta)\frac{1}{\epsilon}\hat{y}_0' + 2\hat{y}_0\} + \epsilon\{\frac{\epsilon}{\epsilon^2}\hat{y}_1'' + (3 - 2\epsilon\eta)\frac{1}{\epsilon}\hat{y}_1' + 2\hat{y}_1\} + \ldots = 0$$

Arranging in increasing powers of ϵ, we get

$$-(2x + 1)y_0' + 2y_0 + \epsilon[y_0'' - (2x + 1)y_1' + 2y_1 + ..] + \epsilon^2[\ldots]$$

$$+ \frac{1}{\epsilon}[\hat{y}_0'' + 3\hat{y}_0'] + [-2\eta\hat{y}_0' + 2\hat{y}_0 + \hat{y}_1'' + 3\hat{y}_1'] + \ldots = 0 \tag{6}$$

As explained in Section 10.14, under the simultaneous limit $\epsilon \longrightarrow 0$, with first x held fixed and then η held fixed, we can set coefficients of like powers of ϵ in (6) to zero, with boundary layer terms treated *separately*. To the lowest order, this gives

$$-(2x + 1)y_0' + 2y_0 = 0 \tag{7}$$

$$\hat{y}_0'' + 3\hat{y}' = 0 \tag{8}$$

Solutions of (7) and (8) are respectively

$$y_0(x) = A(2x + 1) \tag{9}$$

and

$$\hat{y}_0(\eta) = B\exp(-3\eta) \tag{10}$$

With the assumed composition (5), the BCs (2) and (3) become

$$y_0(0) + \cancel{\hat{y}_0(\infty)} + \epsilon[y_1(0) + \cancel{\hat{y}_1(\infty)}] + \ldots = \alpha$$

$$y_0(1) + \hat{y}_0(0) + \epsilon[y_1(1) + \hat{y}_1(0)] + \ldots = \beta$$

To the lowest order, these become

$$y_0(0) = \alpha \tag{11}$$

$$y_0(1) + \hat{y}_0(0) = \beta \tag{12}$$

Use of BC (11) in (9) gives $A = \alpha$. So outer solution is

$$\boxed{y_0 = \alpha(2x + 1)} \tag{13}$$

This gives $y_0(1) = 3\alpha$, and BC (12) becomes

$$\hat{y}_0(0) = \beta - 3\alpha \tag{14}$$

Use of BC (14) in (10) gives

$$B = \beta - 3\alpha$$

so that (10) becomes

$$\hat{y}_0 = (\beta - 3\alpha)\exp[-3(1 - x)/\epsilon]$$

The lowest order composite solution is therefore

$$y = y_0 + \hat{y}_0 = \alpha(2x + 1) + (\beta - 3\alpha)\exp[-3(1 - x)/\epsilon] + \ldots$$

Exercise 10.9

Choose for dimensionless variables u/U, v/v_0, $(p - p_\infty)/\rho U^2$, and normalize x and y by the same length. For steady, constant density flow in two dimensions,

$$\partial u/\partial x + (v_0/U)\partial v/\partial y = 0 \quad \text{(mass conservation)}$$

$$u\frac{\partial u}{\partial x} + \frac{v_0}{U}v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

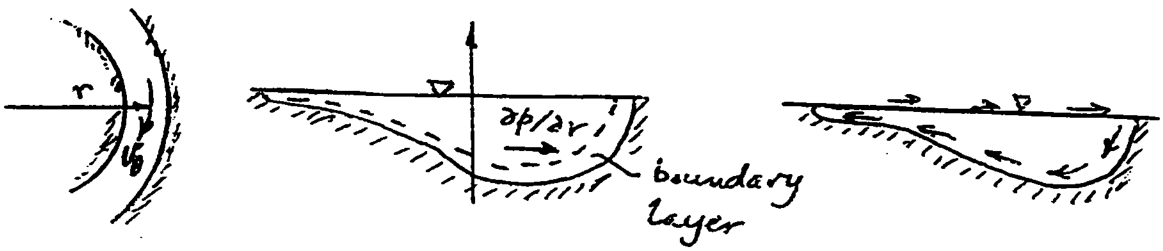
Given $v_0/U = o(1) \to 0$. Outside a boundary layer, we have $u = 1$, $p = 0$ for the outer flow. Since v is decoupled from u by virtue of suction, the y-momentum equation gives $\partial p/\partial y = o(1) \to 0$. The boundary layer stretching must restore the viscous stress and balance it against the dominant convective acceleration. This requires us to define the boundary layer variable by $\tilde{y} = y \cdot (v_0/U)\,\mathrm{Re}$ which is a large asymptotic stretching of the y coordinate. Then mass conservation becomes $\partial u/\partial x + (v_0/U)\cdot(v_0/U)\,\mathrm{Re}\,\partial v/\partial\tilde{y} = 0$ where the second term is dominant. Then $\partial v/\partial\tilde{y} \approx 0$. $v = \text{const.} = 1$ (dimensionless). Writing the x-momentum equation in the boundary layer,

$$u\frac{\partial u}{\partial x} + \frac{v_0}{U}\cdot\frac{v_0}{U}\,\mathrm{Re}\,v\frac{\partial u}{\partial\tilde{y}} + \frac{\partial p}{\partial x} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{v_0^2}{U^2}\mathrm{Re}^2\frac{\partial^2 u}{\partial\tilde{y}^2}\right).$$

Here $v = 1$ and $(v_0^2/U^2)\,\mathrm{Re} \gg 1$. The dominant part of this equation is $\partial u/\partial\tilde{y} = \partial^2 u/\partial\tilde{y}^2$. Integrating, $u = c_1 e^{\tilde{y}} + c_2$ with $u(0) = 0$, $u(-\infty) = 1$ so $u = 1 - e^{\tilde{y}}$. Note $v_0 < 0$ so $\tilde{y} < 0$.

Exercise 10.10

The Reynolds number based on channel width or radius of curvature is very large. Thus the boundary layer is thin compared with the width or depth. Then most of the flow is inviscid. The primary flow around the bend is governed by $\partial p/\partial r = \rho v_\theta^2/r$. v_θ decreases rapidly to zero in the thin boundary layer near the bottom. Thus $\partial p/\partial r$, which is balanced by $\rho v_{\theta\,inv}^2/r$, is *not balanced* by $\rho v_{\theta\,b.l.}^2/r$ since $v_{\theta\,b.l.} < v_{\theta\,inv}$. $\partial p/\partial r$ is too large in the bottom boundary layer and p increases too rapidly with r for fluid particles to be in equilibrium. This creates an unbalanced pressure force tending to move the fluid inwards in the bottom boundary layer. This requires a downward secondary flow on the outer bank and sets up the secondary flow pattern shown. In the Mississippi River, the water transports silt. The silt is scoured from the outer bank and deposited on the inner bank accounting for the cross-section shown.



Exercise 10.11

The boundary layer must always decay into the domain. Since this is a singular perturbation problem and the boundary layer is thin, the first two terms will dominate the third in the boundary layer. Then the location of the boundary layer is determined by the sign of the ratio of coefficients of the dominant terms in the balance. This is positive near $x = 1$ indicating that the boundary layer will decay inwards from $x = 1$. Thus the boundary layer is on the left and the main region problem, $(\cos x)\,df/dx + (\sin x)f = O(\epsilon) \to 0$ must satisfy the boundary condition at $x = 2$. The solution is $f(x) = c_1 \cos x$ subject to $f(2) = \cos 2$. Then $c_1 = 1$ and $f(x) = \cos x$ outside the boundary layer. To balance the first two terms, we introduce the stretched coordinate $\xi = (x-1)/\epsilon$ so $x = 1 + \epsilon\xi$ in the boundary layer. The equation in the boundary layer becomes $d^2f/d\xi^2 + (\cos 1)\,df/d\xi = O(\epsilon) \to 0$. Integrating this equation $df/d\xi + (\cos 1)f = c_2$ and $f(\xi) = c_3 e^{-\xi \cos 1} + c_2/\cos 1$. The boundary conditions are $f(0) = 0$, $f(\xi \to \infty) = f(x \to 1)$ (matching). Then $c_2 = \cos^2 1$, $c_3 = -\cos 1$ so $f_{BL} = \cos 1 - \cos 1 e^{-\xi \cos 1}$. A uniformly valid solution can be constructed by adding the boundary layer and main region solutions and subtracting the common overlap ($\cos 1$). Thus we can write $f(x) = \cos x - \cos 1 e^{-[(x-1)/\epsilon]\cos 1} + O(\epsilon)$ which reproduces each of the solutions in the appropriate regions.

Exercise 10.12

For two dimensional, ρ = const., steady flow in the limit $\mathrm{Re} = U_1 x/\nu \to \infty$, $\partial u/\partial x + \partial v/\partial y = 0$ and, with a thin shear layer downstream for large Re, $\partial p/\partial y = 0$. Since p = const. outside the shear layer, p = const. everywhere. Then $u(\partial u/\partial x) + v(\partial u/\partial y) = \nu(\partial^2 u/\partial y^2)$ within the shear layer subject to $u(y \to \infty) = U_1$, $u(y \to -\infty) = U_2$. In terms of the streamfunction $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$,

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3}.$$

Sufficiently far downstream that an initial condition is forgotten, the similarity formulation $\psi(x, y) = \sqrt{\nu U_1 x} f(\eta)$, $\eta = (y/x)\sqrt{U_1 x/\nu}$ is obtained. This is the universal dimensionless similarity formulation for the laminar boundary layer equations. Substituting, we obtain $f''' + (1/2)ff'' = 0$, $f'(\infty) = 1$, $f'(-\infty) = U_2/U_1$. We have one too few boundary conditions. We don't know the value of ψ or f at $\eta = 0$. Consider the special case $U_2/U_1 = 1 - \epsilon$. Then let $f = f^{(0)} + \epsilon f^{(1)} + O(\epsilon^2)$. The equation for $f^{(0)}$ is $f^{(0)'''} + \frac{1}{2}f^{(0)}f^{(0)''} = 0$ subject to $f^{(0)'}(\infty) = 1$, $f^{(0)'}(-\infty) = 1$. The solution is $f^{(0)'} = 1$ or $f^{(0)} = \eta + c$. Here c cannot be determined. To next order $f^{(1)'''} + (1/2)f^{(0)}f^{(1)''} = 0$, $f^{(1)}(\infty) = 0$, $f^{(1)}(-\infty) = -1$.

Let

$$\eta + c \equiv \hat\eta. \qquad f^{(1)'''} + \tfrac{1}{2}\hat\eta f^{(1)''} = 0, \qquad f^{(1)''} = Ae^{-\hat\eta^2/4},$$

$$f^{(1)'} = -A\int_{\hat\eta}^{\infty} d\eta\, e^{-\eta^2/4} + B \qquad f^{(1)'}(\infty) = 0$$

so $B = 0$.

$$f^{(1)'}(-\infty) = -A\int_{-\infty}^{\infty} e^{-\eta^2/4}\,d\eta = -1, \quad A = 1/(2\sqrt{\pi})$$

$$f^{(1)'} = -\frac{1}{2\sqrt{\pi}}\int_{\hat\eta}^{\infty} e^{-\eta^2/4}\,d\eta$$

so that

$$u = U_1\left[1 - \frac{\epsilon}{2\sqrt{\pi}}\int_{\hat\eta}^{\infty} e^{-\eta^2/4}d\eta\right],$$

where the ambiguity remains in the origin of η, the value of c, which cannot be determined.

Exercise 10.13

Eq. (10.104) is

$$F\frac{d^2F}{df^2}+\left(\frac{dF}{df}\right)^2+\frac{1}{2}f\frac{dF}{df}=0$$

with boundary conditions $F(\infty)=1$, $F(-\infty)=U_2/U_1$. If $U_2/U_1=1-\epsilon$, $\epsilon\ll 1$, then $F(f)$ is everywhere near 1. We could let $F=1-\epsilon G(f)$ or equivalently just recognize that F is near 1 and derivatives of F are $O(\epsilon)$. Then Eq. (104) reduces to

$$\frac{d^2F}{df^2}+\frac{1}{2}f\frac{dF}{df}=O(\epsilon)\to 0,\qquad \frac{F''}{F'}=-\frac{1}{2}f,$$

$$\ln F'=\ln c_1-\frac{f^2}{4},\qquad F'=c_1e^{-f^2/4},\qquad F=c_2+c_1\int_{-\infty}^{f}e^{-f^2/4}\,df,$$

$$F(-\infty)=\frac{U_2}{U_1}=c_2,\qquad F(\infty)=1=\frac{U_2}{U_1}+c_1\underbrace{\int_{-\infty}^{\infty}e^{-t^2/4}\,dt}_{2\sqrt{\pi}},$$

$$c_1=\frac{1-U_2/U_1}{2\sqrt{\pi}}=\frac{\epsilon}{2\sqrt{\pi}},$$

$$F(f)=\frac{U_2}{U_1}+\frac{1-U_2/U_1}{2\sqrt{\pi}}\int_{-\infty}^{f}e^{-f^2/4}\,df=1-\epsilon+\frac{\epsilon}{2\sqrt{\pi}}\int_{-\infty}^{f}e^{-t^2/4}\,dt+O(\epsilon^2).$$

11.1 Show that the stability condition for the explicit scheme (11.10) is the condition (11.26).

Solution

The stability of the explicit scheme

$$T_i^{n+1} = T_i^n - \alpha\left(T_{i+1}^n - T_{i-1}^n\right) + \beta\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)$$

with $\alpha = u\dfrac{\Delta t}{2\Delta x}$, $\beta = D\dfrac{\Delta t}{\Delta x^2}$, may be examined using the von Neumann method. The condition for stability is discussed in the text, and is given by $\left|g^{n+1}/g^n\right| \le 1$, where

$$\begin{aligned} g^{n+1}/g^n &= (\alpha+\beta)e^{-i\pi k\Delta x} + (1-2\beta) + (\beta-\alpha)e^{i\pi k\Delta x} \\ &= (\alpha+\beta)(\cos\theta - i\sin\theta) + (1-2\beta) + (\beta-\alpha)(\cos\theta + i\sin\theta) \\ &= (1-2\beta) + 2\beta\cos\theta - i2\alpha\sin\theta \\ &= 1 - 2\beta(1-\cos\theta) - i2\alpha\sin\theta \\ &= 1 - 4\beta\sin^2(\theta/2) - i2\alpha\sin\theta \end{aligned}$$

for any value of the wavenumber k, where $\theta = \pi k\Delta x$. Therefore, the stability condition reduces to

$$\left(1 - 4\beta\sin^2(\theta/2)\right)^2 + (2\alpha\sin\theta)^2 \le 1 .$$

11.2 For the heat conduction equation $\dfrac{\partial T}{\partial t} - D\dfrac{\partial^2 T}{\partial x^2} = 0$, one of the discretized forms is

$$-sT_{j+1}^{n+1} + (1+2s)T_j^{n+1} - sT_{j-1}^{n+1} = T_j^n$$

where $s = D\dfrac{\Delta t}{\Delta x^2}$. Show that this implicit algorithm is always stable.

Solution

For this implicit scheme, the evolution of error satisfies

$$-s\xi_{j+1}^{n+1} + (1+2s)\xi_j^{n+1} - s\xi_{j-1}^{n+1} = \xi_j^n \qquad (1)$$

Consider a component of the error in the Fourier space

$$\xi_j^n = g^n(k)e^{i\pi kx_j},$$

where k is the wavenumber in Fourier space, and g^n represents the function g at time $t = t_n$. The component at the next time level has a similar form

$$\xi_j^{n+1} = g^{n+1}(k)e^{i\pi kx_j} .$$

Substituting the above two expressions into the error equation (1), we have,

$$g^{n+1}\left[-se^{i\pi kx_{j+1}} + (1+2s)e^{i\pi kx_j} - se^{i\pi kx_{j-1}}\right] = g^n e^{i\pi kx_j},$$

or

$$\begin{aligned} g^{n+1}/g^{n} &= \frac{1}{\left[-se^{i\pi k\Delta x}+(1+2s)-se^{-i\pi k\Delta x}\right]} \\ &= \frac{1}{1+2s-2s\cos\theta} \\ &= \frac{1}{1+2s(1-\cos\theta)} \\ &= \frac{1}{1+4s\sin^2(\theta/2)} \end{aligned}$$

where $\theta=\pi k\Delta x$. Since $4s\sin^2(\theta/2)$ is always positive, thus $\left|g^{n+1}/g^{n}\right|\le 1$, the scheme is unconditionally stable.

11.3 An insulated rod initially has a temperature of $T(x,0)=0°C$ $(0\le x\le 1)$. At $t=0$ hot reservoirs $(T=100°C)$ are brought into contact with the two ends, A $(x=0)$ and B $(x=1)$: $T(0,t)=T(1,t)=100°C$. Numerically find the temperature $T(x,t)$ of any point in the rod. The governing equation of the problem is the heat conduction equation $\frac{\partial T}{\partial t}-D\frac{\partial^2 T}{\partial x^2}=0$. The exact solution to this problem is

$$T^*\left(x_j,t_n\right)=100-\sum_{m=1}^{NM}\frac{400}{(2m-1)\pi}\sin\left[(2m-1)\pi x_j\right]\exp\left[-D(2m-1)^2\pi^2 t_n\right]$$

where NM is the number of terms used in the approximation.

(a). Try to solve the problem with the explicit FTCS (forward time, central space) scheme. Use the parameter $s=D\frac{\Delta t}{\Delta x^2}$ =0.5 and 0.6 to test the stability of the scheme.

(b). Solve the problem with a stable explicit or implicit scheme. Test the rate of convergence numerically, using the error at $x=0.5$.

<u>Solution</u>

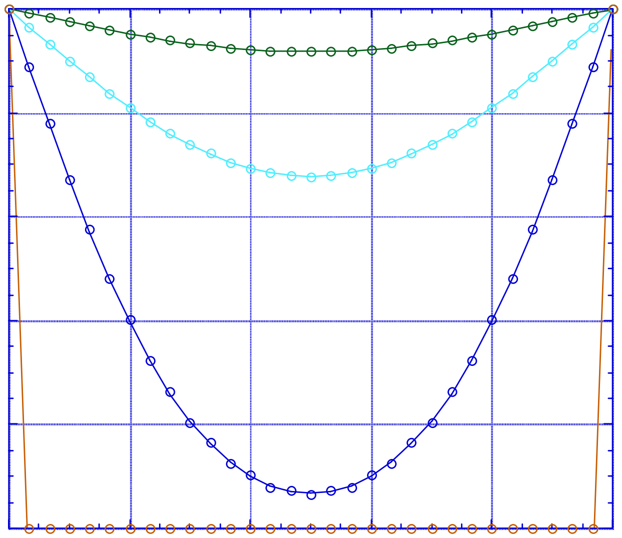
(a). The FTCS scheme can be written as

$$T_i^{n+1}=T_i^n+s\left(T_{i+1}^n-2T_i^n+T_{i-1}^n\right)$$

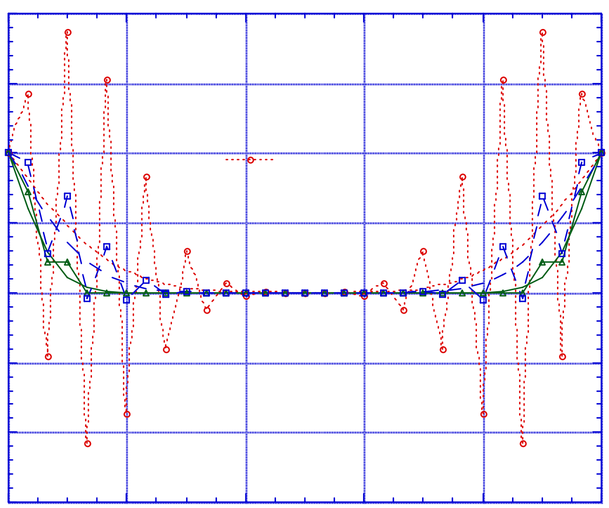
where $s=D\frac{\Delta t}{\Delta x^2}$. In the program, a uniform grid spacing and constant time step are used, and their values are $\Delta x=1/30$ (with 31 grid points in the domain) and $\Delta t=1/500$ (with 500 time steps reaching the final time of 1 second), respectively.

The exact solution is evaluated with 10 terms in the summation. It is found that 5 terms would give almost the same solution.

The case with s=0.5 is first computed and the results are shown in figure 1. The temperature distributions at four different times are plotted. It is shown that the solution is stable, and agrees with the exact solution.



The results for the case with s=0.6 are presented in figures 2. It is observed that the solution is not stable. Oscillations with ever larger amplitudes occur. Eventually the solution blows up. The stability condition for this explicit scheme is $s \le 0.5$.



(b) Using the FTCS scheme, the rate of the convergence is numerically evaluated. In the evaluation, the time step is fixed at $\Delta t = 10^{-7}$, and the grid spacing is reduced from $\Delta x = 0.1$ to 0.00625. These setting are chosen such that, when the constant $D = 1$, the stability condition, $s \leq 0.5$, is always satisfied. The differences between the numerical solution and the exact solution (error) at $x = 0.5$ and $t = 0.1$ are listed in the table below and are plotted in figure 3 as a function of grid spacing Δx. It is observed that the rate of the convergence (the slope of the curve fitted line) is approximately 2.0.

Table 1. Convergence test.

Δx	$Error = \lvert T_{num} - T_{exact} \rvert$
0.10000	1.2062E-02
0.050000	2.4300E-03
0.025000	5.7286E-04
0.012500	2.5716E-04
0.0062500	5.6250E-05

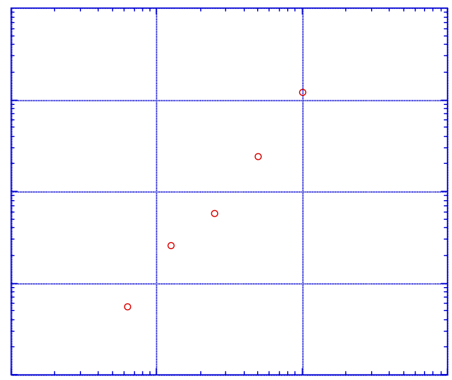


Figure 3. Convergence of the numerical solution.

The FORTRAN source code for this program is listed below.

```
c*****************************************************************
c
c    program solves the unsteady heat conduction problem
c    using an explicit FTCS scheme
c
c*****************************************************************
      program FTCS
      implicit none
      integer nx,nt,i,n,m
      real s,dx,dt
      parameter (nt=500, nx=30)
      parameter (s=0.5)

      real*8 t(0:nt,0:nx),ts(0:nt,0:nx)
      real*8 pi,time,x,D
c
      open(unit=12,file='output')
c
      pi = 3.14159265358979d0
      dt = 1.0/nt
      dx = 1.0/nx
c
c     initial conditions
      do i=1,nx-1
        t(0,i) = 0
      enddo
c
c     boundary conditions
      do n=0,nt
        t(n,0) = 100
        t(n,nx) = 100
```

```
      enddo
c
c     numerical solution with FTCS
      do n=0,nt-1
        do i=1,nx-1
          t(n+1,i) = t(n,i) + s*(t(n,i-1)-2.0*t(n,i)+t(n,i+1))
        enddo
      enddo
c
c     exact solution
      D = s*dx**2/dt
      do n=1,nt
        time = n*dt
        do i=0,nx
          x = i*dx
          ts(n,i) = 100.0
          do m=1,10
            ts(n,i) = ts(n,i) - (400.d0/(2*m-1)/pi)*
     &                sin((2*m-1)*pi*x)*exp(-D*((2*m-1)*pi)**2*time)
          enddo
        enddo
      enddo

c
c     print-out
      do n=50,nt,50
        do i=0,nx
          write(12,'(5e12.5)')
     &      n*dt,i*dx,t(n,i),ts(n,i),abs(t(n,i)-ts(n,i))
        enddo
      enddo
c
      stop
      end
```

11.4 Derive the weak form, Galerkin form, and the matrix form of the following strong problem:

Given functions $D(x)$, $f(x)$, and constants g, h, find $u(x)$ such that

$$[D(x)u_{,x}]_{,x} + f(x) = 0 \quad \text{on} \quad \Omega = (0,1),$$

with $u(0) = g$ and $-u_{,x}(1) = h$.

Write a computer program solving this problem using piecewise linear shape functions. You may set $D = 1$, $g = 1$, $h = 1$ and $f(x) = \sin(2\pi x)$. Check your numerical result with the exact solution.

<u>Solution</u>

Define the functional space and the variational space for the trial solutions,

$$S = \{u(x) \mid u \in H^1, u(0) = g\} \text{ and } V = \{w(x) \mid w \in H^1, w(0) = 0\},$$

respectively. Multiply the governing equation by a function in the variational space ($w \in V$), and integrate the product over the domain $\Omega = (0,1)$,

$$\int_0^1 \left([D(x)u_{,x}]_{,x} + f \right) w dx = 0 \ .$$

Integrating by parts, we have

$$\begin{aligned} & -\int_0^1 \left(D u_{,x} w_{,x} \right) dx + \left[D u_{,x} w \right]_0^1 + \int_0^1 f(x) w dx \\ & = -\int_0^1 \left(D u_{,x} w_{,x} \right) dx - D(1) h w(1) + \int_0^1 f(x) w dx \\ & = 0 \end{aligned}$$

where the boundary conditions, $-u_{,x}(1) = h$ and $w(0) = 0$, are applied. Therefore, the weak form can be stated as:

> Given functions $D(x)$, $f(x)$, and constant h, find $u \in S$ such that for all $w \in V$,
> $$\int_0^1 \left(D u_{,x} w_{,x} \right) dx = \int_0^1 f(x) w dx - h D(1) w(1) \ .$$

We can construct finite-dimensional approximations to S and V, which are denoted by S^h and V^h, respectively. We can also define a new function $g^h(x)$ such that $g^h(0) = g$. The Galerkin form of the problem can be expressed as,

> Find $u^h = v^h + g^h$, where $v^h \in V^h$, such that for all $w^h \in V^h$,
> $$a\left(w^h, v^h\right) = -hD(1) w^h(1) + \left(w^h, f\right) - a\left(w^h, g^h\right),$$
> where $a(w,v) = \int_0^1 \left(D w_{,x} v_{,x} \right) dx$ and $(w, f) = \int_0^1 (wf) dx$.

Next, we construct the finite-dimensional variational space V^h explicitly using the shape functions, $N_A(x)$, $A = 1,2,...,n$,

$$w^h = \sum_{A=1}^{n} c_A N_A(x) \ .$$

The function g^h can be expressed with an additional shape function $N_0(x)$ ($N_0(0) = 1$),

$$g^h(x) = g N_0(x) \text{ , such that } g^h(0) = g \ .$$

With these definitions, the approximate solution can be written as

$$v^h(x) = \sum_{A=1}^{n} d_A N_A(x).$$

Substitution of the approximate solution into the Galerkin formulation yields

$$a\left(\sum_{A=1}^{n} c_A N_A, \sum_{B=1}^{n} d_B N_B \right) = -hD(1) \sum_{A=1}^{n} c_A N_A(1) + \left(\sum_{A=1}^{n} c_A N_A, f \right) - a\left(\sum_{A=1}^{n} c_A N_A, g N_0 \right) .$$

Since the coefficients c_A are arbitrary, the above equation reduces to

$$\sum_{B=1}^{n} d_B a\left(N_A, N_B\right) = -hD(1) N_A(1) + \left(N_A, f\right) - g a\left(N_A, N_0\right) \text{ for } A=1,2,\ldots,n.$$

Therefore, the matrix form of the problem can be written as

$$\mathbf{Kd} = \mathbf{F} \ ,$$

where

$$\mathbf{K} = [K_{AB}],\ \mathbf{F} = \{F_A\},\ \mathbf{d} = \{d_B\}$$

$$K_{AB} = \int_0^1 \left(DN_{B,x}N_{A,x}\right)dx,\quad F_A = -hD(1)N_A(1) + \int_0^1 \left(N_A f\right)dx - g\int_0^1 \left(DN_{0,x}N_{A,x}\right)dx \ .$$

Set $D=1$, $g=1$, $h=1$ and $f(x)=\sin(2\pi x)$, the exact solution for this problem is found to be

$$u(x) = \frac{\sin(2\pi x)}{4\pi^2} - \left(1 + \frac{1}{2\pi}\right)x + 1 .$$

In the numerical solution, we have the stiffness matrix,

$$K_{AB} = \int_0^1 \left(N_{B,x}N_{A,x}\right)dx ,$$

More explicitly, with piece-wise linear shape functions, most of the elements in the stiffness matrix are zero, except three non-zero ones (per equation),

$$K_{AA} = \int_0^1 N_{A,x}^2 dx = \int_{x_{A-1}}^{x_A} N_{A,x}^2 dx + \int_{x_A}^{x_{A+1}} N_{A,x}^2 dx = \int_{x_{A-1}}^{x_A} \frac{1}{h^2}\,dx + \int_{x_A}^{x_{A+1}} \frac{1}{h^2}\,dx = \frac{2}{h} ,$$

$$K_{A-1,A} = K_{A+1,A} = \int_{x_{A-1}}^{x_A} N_{A,x}N_{A-1,x}dx = \int_{x_{A-1}}^{x_A} \frac{1}{h}\left(-\frac{1}{h}\right)dx = -\frac{1}{h} ,$$

and $$K_{nn} = \int_{x_{n-1}}^{x_n} N_{A,x}^2 dx = \int_{x_{A-1}}^{x_A} \frac{1}{h^2}\,dx = \frac{1}{h} \ .$$

Therefore, the stiffness matrix is evaluated as,

$$\mathbf{K} = \begin{bmatrix} 2/h & -1/h & 0 & \ldots & 0 & 0 \\ -1/h & 2/h & -1/h & \ldots & 0 & 0 \\ 0 & -1/h & 2/h & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 2/h & -1/h \\ 0 & 0 & 0 & \ldots & -1/h & 1/h \end{bmatrix} .$$

It is a tridiagonal matrix.

In evaluating the force vector, we first interpolate the function by $f(x) = \sum_{B=0}^{n} f_B N_B(x)$, thus

$$
\begin{aligned}
F_A &= -N_A(1) + \int_0^1 f(x) N_A dx - \int_0^1 \left(N_{0,x} N_{A,x}\right) dx \\
&= -\delta_{An} + \sum_{B=0}^{n} f_B \int_0^1 N_B N_A dx + \frac{1}{h}\delta_{A1} \\
&= \begin{cases} \left(\frac{1}{6} f_{A-1} + \frac{2}{3} f_A + \frac{1}{6} f_{A+1}\right) h + \frac{1}{h}, & A = 1 \\ \left(\frac{1}{6} f_{A-1} + \frac{2}{3} f_A + \frac{1}{6} f_{A+1}\right) h, & A = 2, \ldots, n-1 \\ -1 + \left(\frac{1}{6} f_{A-1} + \frac{1}{3} f_A\right) h, & A = n \end{cases}
\end{aligned}
$$

The program for the numerical solution is attached below. The numerical solution uses $n = 50$ uniform elements. The comparison of the numerical solution with the exact one is presented in figure 4. The agreement is almost perfect.

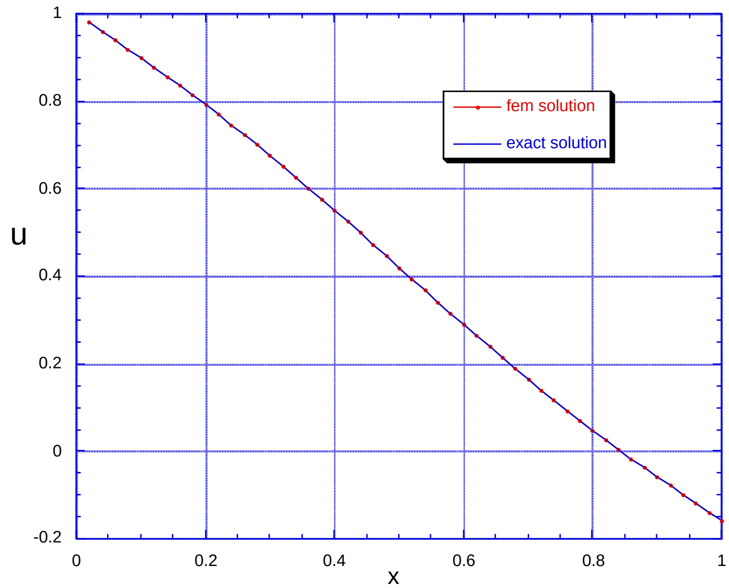


Figure 4. Comparison of FEM solution with the exact solution.

The FORTRAN program for this problem is attached below.

```
c
c     solving a tridiagonal system with TDMA scheme
c       a(i)*v(i)=c(i)*v(i-1)+b(i)*v(i+1)+d(i)
c
      program FEM1D
      implicit none
      integer n
      parameter (n=50)

      real*8 a(n),b(n),c(n),d(n),v(n),vexact(n)

      integer i,j
      real*8 h,pi,p(n),q(n),t

      pi = 3.141592654
```

```
      h = 1.d0/n
c
c     load the stiffness matrix
      do i=1,n
         a(i) = 2.d0/h
         b(i) = 1.d0/h
         c(i) = 1.d0/h
      enddo
      a(n) = 1.d0/h
      b(n) = 0.d0
      c(1) = 0.d0
c
c     load the force vector
      t = 2.d0*pi*h
      do i=1,n-1
         d(i) = h/6.d0*(sin(t*(i-1))+4.d0*sin(t*i)+sin(t*(i+1)))
      enddo
      d(1) = d(1)+1.d0/h
      d(n) = -1.d0+h/6.d0*(sin(t*(n-1))+2.d0*sin(t*n))
c
c     TDMA solver
      p(1) = b(1)/a(1)
      q(1) = d(1)/a(1)
      do i=2,n
         t = a(i)-c(i)*p(i-1)
         p(i) = b(i)/t
         q(i) = (d(i)+c(i)*q(i-1))/t
      enddo
      v(n) = q(n)
      do i=n-1,1,-1
         v(i) = p(i)*v(i+1)+q(i)
      enddo
c
c     exact solution
      do i=1,n
         t = i*h
         vexact(i) = sin(2.d0*pi*t)/(4.d0*pi**2)-(1+1/(2*pi))*t+1.d0
      enddo
c
c     print results
      do i=1,n
           write(12,'(3(e12.5,A))') i*h,',',v(i),',',vexact(i)
      enddo
c
      stop
      end
```

11.5 Solve numerically the steady convective transport equation, $u\dfrac{\partial T}{\partial x} = D\dfrac{\partial^2 T}{\partial x^2}$, for $0 \le x \le 1$, with two boundary conditions $T(0) = 0$ and $T(1) = 1$, where u and D are two constants,

(a) using the central finite difference scheme in equation (11.91), and compare with the exact solution;

(b) using the upwind scheme (11.93); and compare with the exact solution.

Solution

(a) When the central difference scheme is used,

$$0.5R_{cell}\left(T_{j+1}-T_{j-1}\right)=\left(T_{j+1}-2T_j+T_{j-1}\right),$$

where the cell Peclet number $R_{cell}=u\Delta x/D$ and the grid spacing $\Delta x=1/n$. Or

$$\left(1-0.5R_{cell}\right)T_{j+1}-2T_j+\left(1+0.5R_{cell}\right)T_{j-1}=0,$$

with $T_0=0$ and $T_n=1$.

The numerical solution of the problem is presented in the figure 5. In the solution the grid space is selected as $\Delta x=0.05$ $(n=20)$. Three values of the Peclet number are used, $R_{cell}=1.0,\ 2.0,\ 3.0$, in the computation. The scheme generates oscillatory solutions when $R_{cell}>2.0$, near the boundary. The FORTRAN code for this case is attached below.

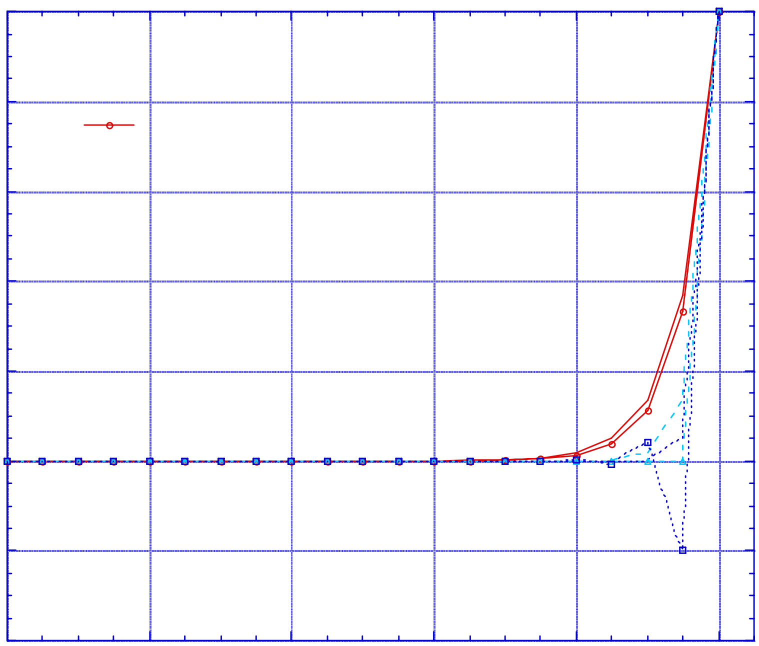


Figure 5. Numerical solution with the central difference scheme.

(b) When the first order upwind scheme is used,

$$R_{cell}\left(T_j-T_{j-1}\right)=\left(T_{j+1}-2T_j+T_{j-1}\right),$$

or

$$T_{j+1}-\left(2+R_{cell}\right)T_j+\left(1+R_{cell}\right)T_{j-1}=0,$$

with $T_0=0$ and $T_n=1$.

The solution of the problem is presented in the figure 6. In the solution the grid space is selected as $\Delta x = 0.05$ $(n = 20)$. Two values of the Peclet number are used, $R_{cell} = 1.0,\ 3.0$. The scheme is stable, and the solution does not generate oscillations in the boundary layer. However, the solution is quite diffusive.

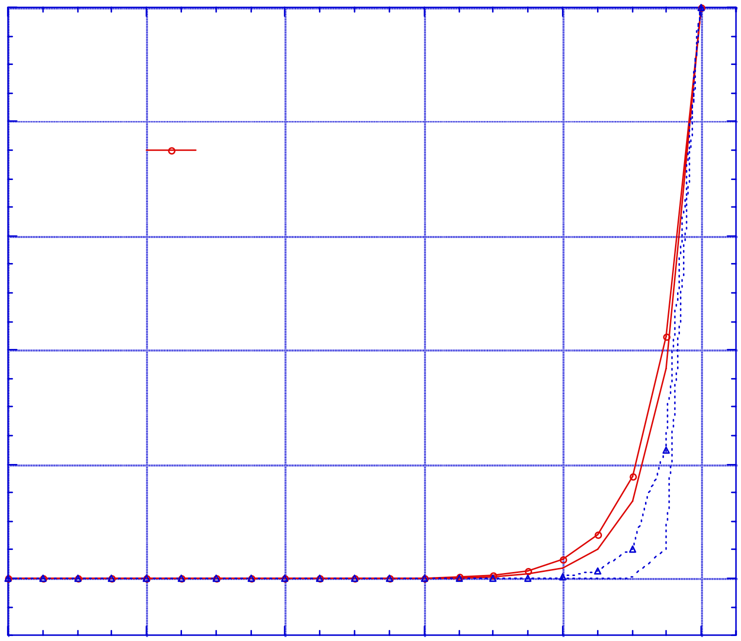


Figure 6. Numerical solution with the first order upwind scheme.

```
c
c     solving a tridiagonal system with TDMA scheme
c       a(i)*v(i)=c(i)*v(i-1)+b(i)*v(i+1)+d(i)
c
      program hw5a
      implicit none
      integer n
      parameter (n=20)

      real*8 a(0:n),b(0:n),c(0:n),d(0:n),v(0:n)

      integer i
      real*8 h,p(0:n),q(0:n),t,rcell

      rcell = 3.d0
      h=1.d0/n
c
c     stiffness matrix
      do i=1,n
         a(i) = -2.d0
         b(i) = -(1.d0-0.5*rcell)
         c(i) = -(1.d0+0.5*rcell)
```

```
      enddo
      a(0) = 1.d0
      b(0) = 0.d0
      c(0) = 0.d0
      a(n) = 1.d0
      b(n) = 0.d0
      c(n) = 0.d0
c
c     force vector
      do i=0,n-1
         d(i) = 0.d0
      enddo
      d(n) = 1.d0
c
c     TDMA solver
      p(0) = b(0)/a(0)
      q(0) = d(0)/a(0)
      do i=1,n
         t = a(i)-c(i)*p(i-1)
         p(i) = b(i)/t
         q(i) = (d(i)+c(i)*q(i-1))/t
      enddo
      v(n) = q(n)
      do i=n-1,0,-1
         v(i) = p(i)*v(i+1)+q(i)
      enddo
c
c     print results
        do i=0,n
           write(12,'(2(e12.5,A))') i*h,',',v(i)
           write(*,'(2(e12.5,A))') i*h,',',v(i)
      enddo
c
      stop
      end
```

6. Code the explicit MacCormack scheme with the FF/BB arrangement for the driven cavity flow problem as described in Section 5. Compute the flow field at Re=100 and 400, and explore effects of Mach number and the stability condition (11.110).

Sample c-code:

```
//
// Authors:     Howard H. Hu and Andy E. Perrin
// Explicit scheme using FF/BB MacCormack method for a driven cavity problem
//

#include <stdlib.h>
#include <string.h>
#include <stdio.h>
#include <math.h>
```

```
#define MIN(i,j)  ((i)<(j) ? (i) : (j))

//globals
int    it,it0,ntime,nprint;
double  curtime;
int    ivar=0;
int next=0;

int    nx, ny, node;
double  Re,M,L,H,dx,dy,dt;
double  *roe_n, *roeU_n, *roeV_n, *roe_s, *roeU_s, *roeV_s, *u, *v;
int     nbd;
double  c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11;
double  *var, umin,umax;

void Initial( void)
{
    int i,k;

    nx = 101;
    ny = 101;
    Re = 400.0;
    M = 0.05;
    L = 1.0;
    H = 1.0;
    dx = L/(nx-1);
    dy = H/(ny-1);
    dt = 0.8*M*dx/sqrt(3.0);
    ntime = (int)(80.0/dt);
    nprint = 25;

    printf("dt=%13f  ntime=%i\n",dt,ntime);

    c1 = dt/dx;
    c2 = dt/dy;
    c3 = dt/(dx*M*M);
    c4 = dt/(dy*M*M);
    c5 = 4.0*dt/(3.0*Re*dx*dx);
    c6 = dt/(Re*dy*dy);
    c7 = dt/(Re*dx*dx);
    c8 = 4.0*dt/(3.0*Re*dy*dy);
    c9 = dt/(12.0*Re*dx*dy);
    c10 = (-2.0)*(c5+c6);
    c11 = (-2.0)*(c7+c8);

    node = nx*ny;
    roe_n  = calloc(node,sizeof(double));
    roeU_n = calloc(node,sizeof(double));
    roeV_n = calloc(node,sizeof(double));
    roe_s  = calloc(node,sizeof(double));
    roeU_s = calloc(node,sizeof(double));
    roeV_s = calloc(node,sizeof(double));
    u      = calloc(node,sizeof(double));
    v      = calloc(node,sizeof(double));
    var    = calloc(node,sizeof(double));
```

```
    for(i=0;i<nx;i++)
    {
    k = ny*i+(ny-1);
    u[k]  = 1.0;
    roeU_n[k] = u[k];
    roeU_s[k] = u[k];
    }

    it = 0;
    it0 =0;
    curtime = 0.0;

}

void NextIter( void)
{
    int i,j,k,k1,k2,k3,k4;

    //  node ordering
    //                k1=(i,j+1)
    //   k4=(i-1,j)  k= (i,j)     k3=(i+1,j)
    //                k2=(i,j-1)

    it0 = it0+1;
    curtime = curtime + dt;

    //Predictor
    for(i=1;i<nx-1;i++)
    {
    for(j=1;j<ny-1;j++)
    {
        k = i*ny+j;
        k1 = k+1;
        k2 = k-1;
        k3 = k+ny;
        k4 = k-ny;

        roe_s[k] = roe_n[k] - c1*(roeU_n[k3]-roeU_n[k])
                            - c2*(roeV_n[k1]-roeV_n[k]);

        roeU_s[k] = roeU_n[k]
                  - c1*(roeU_n[k3]*u[k3] - roeU_n[k]*u[k])
                  - c2*(roeV_n[k1]*u[k1] - roeV_n[k]*u[k])
                  - c3*(roe_n[k3]-roe_n[k])
                  + c10*u[k] + c5*(u[k3]+u[k4]) + c6*(u[k1]+u[k2])
                  + c9*(v[k3+1]+v[k4-1]-v[k3-1]-v[k4+1]);

        roeV_s[k] = roeV_n[k]
                  - c1*(roeU_n[k3]*v[k3]-roeU_n[k]*v[k])
                  - c2*(roeV_n[k1]*v[k1]-roeV_n[k]*v[k])
                  - c4*(roe_n[k1]-roe_n[k])
                  + c11*v[k] + c7*(v[k3]+v[k4]) + c8*(v[k1]+v[k2])
                  + c9*(u[k3+1]+u[k4-1]-u[k4+1]-u[k3-1]);

    }
    }
```

```
//BC
for(j=1;j<ny-1;j++)
{
k=j;
roe_s[k] = roe_n[k]
         -0.5*c1*(-roeU_n[k+2*ny]+4.0*roeU_n[k+ny]-3.0*roeU_n[k])
         -0.5*c2*(roeV_n[k+1]-roeV_n[k-1]);
k = (nx-1)*ny+j;
roe_s[k] = roe_n[k]
         -0.5*(-c1)*(-roeU_n[k-2*ny]+4.0*roeU_n[k-ny]-3.0*roeU_n[k])
         -0.5*c2*(roeV_n[k+1]-roeV_n[k-1]);
}
for(i=0;i<nx;i++)
{
k = i*ny;
roe_s[k] = roe_n[k]
         -0.5*c1*(roeU_n[k+ny]-roeU_n[k-ny])
         -0.5*c2*(-roeV_n[k+2]+4.0*roeV_n[k+1]-3.0*roeV_n[k]);
k = i*ny+(ny-1);
roe_s[k] = roe_n[k]
         -0.5*c1*(roeU_n[k+ny]-roeU_n[k-ny])
         -0.5*(-c2)*(-roeV_n[k-2]+4.0*roeV_n[k-1]-3.0*roeV_n[k]);
}

for(k=0;k<node;k++)
{
u[k] = roeU_s[k]/(1.0+roe_s[k]);
v[k] = roeV_s[k]/(1.0+roe_s[k]);
}

//Corrector
for(i=1;i<nx-1;i++)
{
for(j=1;j<ny-1;j++)
{
    k = i*ny+j;
    k1 = k+1;
    k2 = k-1;
    k3 = k+ny;
    k4 = k-ny;

    roe_n[k] = 0.5*( (roe_n[k]+roe_s[k]) - c1*(roeU_s[k]-roeU_s[k4])
                                          - c2*(roeV_s[k]-roeV_s[k2])  );

    roeU_n[k] = 0.5*( (roeU_n[k]+roeU_s[k])
              - c1*(roeU_s[k]*u[k] - roeU_s[k4]*u[k4])
              - c2*(roeV_s[k]*u[k] - roeV_s[k2]*u[k2])
              - c3*(roe_s[k]-roe_s[k4])
              + c10*u[k] + c5*(u[k3]+u[k4]) + c6*(u[k1]+u[k2])
              + c9*(v[k3+1]+v[k4-1]-v[k3-1]-v[k4+1])  );

    roeV_n[k] = 0.5*( (roeV_n[k] + roeV_s[k])
              - c1*(roeU_s[k]*v[k] - roeU_s[k4]*v[k4])
              - c2*(roeV_s[k]*v[k] - roeV_s[k2]*v[k2])
              - c4*(roe_s[k]-roe_s[k2])
              + c11*v[k] + c7*(v[k3]+v[k4]) + c8*(v[k1]+v[k2])
              + c9*(u[k3+1]+u[k4-1]-u[k4+1]-u[k3-1])  );
```

```
    }
    } //end for i

    //BC
    for(j=1;j<ny-1;j++)
    {
    k = j;
    roe_n[k] = 0.5*(roe_n[k]+roe_s[k]
             - 0.5*c1*(-roeU_s[k+2*ny]+4.0*roeU_s[k+ny]-3.0*roeU_s[k])
             - 0.5*c2*(roeV_s[k+1]-roeV_s[k-1]) );
    k = (nx-1)*ny+j;
    roe_n[k] = 0.5*(roe_n[k]+roe_s[k]
             - 0.5*(-c1)*(-roeU_s[k-2*ny]+4.0*roeU_s[k-ny]-3.0*roeU_s[k])
             - 0.5*c2*(roeV_s[k+1]-roeV_s[k-1]) );
    }
    for(i=0;i<nx;i++)
    {
    k = i*ny;
    roe_n[k] = 0.5*(roe_n[k]+roe_s[k]
             - 0.5*c1*(roeU_s[k+ny]-roeU_s[k-ny])
             - 0.5*c2*(-roeV_s[k+2]+4.0*roeV_s[k+1]-3.0*roeV_s[k]) );
    k = i*ny+ny-1;
    roe_n[k] = 0.5*(roe_n[k]+roe_s[k]
             - 0.5*c1*(roeU_s[k+ny]-roeU_s[k-ny])
             - 0.5*(-c2)*(-roeV_s[k-2]+4.0*roeV_s[k-1]-3.0*roeV_s[k]) );
    }

    for(k=0;k<node;k++)
    {
    u[k] = roeU_n[k]/(1.0+roe_n[k]);
    v[k] = roeV_n[k]/(1.0+roe_n[k]);
    }

}


int main (int argc, char *argv[])
{
int  i;

    Initial();
    ivar = -1;
    for (it=0; it<ntime; it=it+nprint)
        for (i=0; i<nprint; i++) NextIter();

    free(roe_n);
    free(roeU_n);
    free(roeV_n);
    free(roe_s);
    free(roeU_s);
    free(roeV_s);
    free(u);
    free(v);
    free(var);
    return 0;
}
```

Exercise 12.1

The odd mode solution is

$$W = A \sin q_o z + B \sinh qz + C \sinh q^*z \qquad (1)$$

where, from Section 12.3, the six roots of q are

$$\pm iq_o = = \pm iK(s-1)^{1/2}$$

$$\pm q = \pm K[1 + s(1 + i\sqrt{3})/2]^{1/2} \qquad (2)$$

$$\pm q^* = \pm K[1 + s(1 - i\sqrt{3})/2]^{1/2}$$

with $$s \equiv (Ra/K^4)^{1/3} \qquad (3)$$

BC: $$W = DW = (D^2 - K^2)^2 W = 0 \qquad \text{at } z = \pm 1/2 \qquad (4)$$

From (1) we find

$$DW = A\, q_o \cos q_o z + Bq \cosh qz + Cq^* \cosh q^*z$$

$$D^2 W = - Aq_o^2 \sin q_o z + Bq^2 \sinh qz + Cq^{*2} \sinh q^*z \qquad (5)$$

$$D^4 W = Aq_o^4 \sin q_o z + Bq^4 \sinh qz + Cq^{*4} \sinh q^*z$$

From (5) we find

$$(D^2 - K^2)^2 W = \left(D^4 - 2K^2 D^2 + K^4\right) W$$

$$= A(q_o^2 + K^2)^2 \sin q_o z + B(q^2 - K^2)^2 \sinh qz + C(q^{*2} - K^2)^2 \sinh q^*z$$

Application of the BCs (4), either at z = 1/2 or z = -1/2, gives

$$\begin{bmatrix} \sin(q_o/2) & \sinh(q/2) & \sinh(q^*/2) \\ q_o \cos(q_o/2) & q \cosh(q/2) & q^* \cosh(q^*/2) \\ (q_o^2 + K^2)^2 \sin\frac{q_o}{2} & (q^2 - K^2)^2 \sinh\frac{q}{2} & (q^{*2} - K^2)^2 \sinh\frac{q^*}{2} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = 0$$

For nontrivial solutions, the determinant of the matrix has to be zero. Diving by the first row of the determinant, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ q_o\cot(q_o/2) & q\coth(q/2) & q^*\coth(q^*/2) \\ (q_o^2+K^2)^2 & (q^2-K^2)^2 & (q^{*2}-K^2)^2 \end{vmatrix} = 0 \qquad (6)$$

Now from (2) and (3) we get

$$q_o^2 = K^2(s-1)$$

$$(q_o^2+K^2)^2 = [K^2(s-1)+K^2]^2 = K^4s^2$$

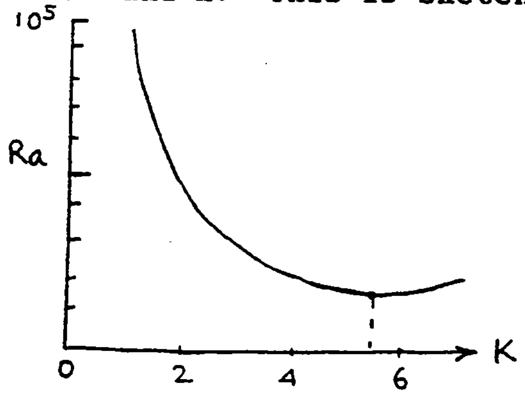
$$(q^2-K^2)^2 = \{K^2[1+\tfrac{1}{2}s(1+i\sqrt{3})] - K^2\}^2 = \frac{K^4}{4}[s(1+i\sqrt{3})]^2$$

$$(q^{*2}-K^2) = \frac{K^4}{4}[s(1-i\sqrt{3})]^2$$

Equation (6) then becomes, canceling a factor K^4s^2 from the bottom row

$$\begin{vmatrix} 1 & 1 & 1 \\ q_o\cot(q_o/2) & q\coth(q/2) & q^*\coth(q^*/2) \\ 1 & (1+i\sqrt{3})^2/4 & (1-i\sqrt{3})^2/4 \end{vmatrix} = 0 \qquad (7)$$

Expanding the determinant, and using definitions (2) and (3), we get a relation between Ra and K. This is sketched below



This has a minimum of Ra_{cr} = 17,610 at K = 5.36. From Section 12.2, the critical Ra for even modes is 1708, signifying that even modes are more unstable.

Exercise 12.2

From (37), the perturbation equations are

$$\frac{\partial u_r}{\partial t} - \frac{2V}{r}u_\theta = -\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right)$$

$$\frac{\partial u_\theta}{\partial t} + \left(\frac{dV}{dr} + \frac{V}{r}\right)u_r = \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right) \tag{1}$$

$$\frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

where the constant density has been absorbed into p. Substituting

$$u_r = u(r)\, e^{\sigma t} \cos kz \qquad u_z = w(r)\, e^{\sigma t} \sin kz$$

$$u_\theta = v(r)\, e^{\sigma t} \cos kz \qquad p = \hat{p}(r)\, e^{\sigma t} \cos kz$$

equations (1) become

$$\nu(DD^* - k^2 - \frac{\sigma}{\nu})u + 2\frac{V}{r}v = \frac{d\hat{p}}{dr} \tag{1}$$

$$\nu(DD^* - k^2 - \frac{\sigma}{\nu})v - (D^*V)u = 0 \tag{2}$$

$$\nu(D^*D - k^2 - \frac{\sigma}{\nu})w = -k\hat{p} \tag{3}$$

$$D^*u = -kw \tag{4}$$

where

$$\nabla^2 = \left(\frac{d}{dr} + \frac{1}{r}\right)\frac{d}{dr} - k^2 = D^*D - k^2 = DD^* + \frac{1}{r^2} - k^2 \tag{5}$$

and

$$D = \frac{d}{dr} \quad \text{and} \quad D^* = \frac{d}{dr} + \frac{1}{r}$$

Eliminating w between (3) and (4), we have

$$\frac{\nu}{k^2}\left(D^*D - k^2 - \frac{\sigma}{\nu}\right)D^* u = \hat{p}$$

Inserting this expression for $\hat{p}$ into (1), we get after some algebra

$$\frac{\nu}{k^2}(DD^* - k^2 - \frac{\sigma}{\nu})(DD^* - k^2)u = 2\frac{V}{r}v \tag{6}$$

We now have a pair of equations (2) and (6) relating u and v.

Measuring r in units of the radius R_2 of the outer cylinder and writing

$$k^2 = K^2/R_2^2 \qquad \text{and} \qquad \omega = \sigma R_2^2/\nu$$

equations (2) and (6) become

$$(D^*D - K^2 - \omega)(DD^* - K^2)u = K^2\frac{2B}{\nu}\left(\frac{1}{r^2} + \frac{AR_2^2}{B}\right)v$$

and $$(DD^* - K^2 - \omega)v = \frac{2A}{\nu}R_2^2\, u$$

where V(r) has the particular form

$$V = Ar + B/r$$

given by (36) in the text.

It is convenient to make the transformation

$$\frac{2AR_2^2}{\nu}u \longrightarrow u$$

Then the equations take the more convenient forms

$$(DD^* - K^2 - \omega)(DD^* - K^2)u = -Ta\; K^2\left(\frac{1}{r^2} - \varkappa\right)v$$

$$(DD^* - K^2 - \omega)v = u$$

where

$$Ta = -\frac{4AB}{\nu^2}R_2^2 = \frac{4\Omega_1^2 R_1^4}{\nu^2}\,\frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)^2}$$

$$\varkappa = -\frac{AR_2^2}{B} = \frac{1-\mu/\eta^2}{1-\mu}$$

$$\mu = \Omega_2/\Omega_1$$

$$\eta = R_1/R_2$$

$$A = -\Omega_1 \eta^2 \frac{1-\mu/\eta^2}{1-\eta^2}$$

$$B = \Omega_1 \frac{R_1^2(1-\mu)}{1-\eta^2}$$

BC: No slip conditions at the walls require

$$u = v = 0 \qquad \text{and} \qquad Du = 0 \text{ for } r = 1 \text{ and } \eta$$

where the last of the three boundary conditions is equivalent to $w = 0$ [see equation (4)].

Now consider the narrow gap approximation, when the gap $R_2 - R_1$ between the two cylinders is small compared to their mean radius $\frac{1}{2}(R_2 + R_1)$. Then we need not distinguish between D and D* in equations (6) and (2). We can also replace $(A + B/r^2)$ which occurs on the right-hand side of equation (6) by

$$\Omega_1 [1 - (1-\mu)\frac{r-R_1}{R_2-R_1}]$$

It is convenient to measure radial distances from the surface of the inner cylinder in the unit $d = R_2 - R_1$. Thus, letting

$$x = (r - R_1)/d, \qquad k = K/d, \qquad \text{and} \qquad \omega = \sigma d^2/\nu$$

we have to consider the equations

$$(D^2 - K^2 - \omega)(D^2 - K^2)u = \frac{2\Omega_1 d^4}{\nu} K^2[1 - (1-\mu)x]v$$

$$(D^2 - K^2 - \omega)v = \frac{2Ad^4}{\nu} u$$

By further transformation $\quad u \longrightarrow \dfrac{2\Omega_1 d^2 K^4}{\nu} u$
the equations become

$$(D^2 - K^2 - \omega)(D^2 - K^2)u = (1 + \alpha x)v$$

$$(D^2 - K^2 - \omega)v = -\,\mathrm{Ta}\, K^2 u$$

where, now,

$$\mathrm{Ta} = -\frac{4A\Omega_1}{\nu^2} d^4$$

and

$$\alpha = -(1-\mu)$$

Exercise 12.3

From Exercise 12.2, the marginal state ($\omega = 0$) satisfies

$$(D^2 - K^2)^2 u = (1 + \alpha x)v \tag{1}$$

$$(D^2 - K^2)v = - Ta\, K^2 u \tag{2}$$

together with the BCs

$$u = Du = v = 0 \qquad \text{at } x = 0 \text{ and } 1 \tag{3}$$

Since v is required to vanish at x = 0 and 1, we expand it in a sine series of the form

$$v = \sum_{m=1}^{\infty} C_m \sin m\pi x \tag{4}$$

Having chosen v in this manner, we next *solve* the equation

$$(D^2 - K^2)^2 u = (1 + \alpha x)\sum_{m=1}^{\infty} C_m \sin m\pi x \tag{5}$$

obtained by inserting (4) in (1), and arrange that the solution satisfies the four remaining boundary conditions on u. With u determined in this fashion and v given by (4), equation (2) will lead to an equation for Ta.

The solution of equation (5) is straightforward. The general solution can be written in the form

$$u = \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2 + K^2)^2}\{A_1^{(m)} \cosh Kx + B_1^{(m)} \sinh Kx + A_2^{(m)} x \cosh Kx$$
$$+ B_2^{(m)} x \sinh Kx + (1 + \alpha x)\sin m\pi x + \frac{4\alpha m\pi}{m^2\pi^2 + K^2}\cos m\pi x\} \tag{6}$$

where the constants of integration $A_1^{(m)}$, $A_2^{(m)}$, $B_1^{(m)}$, and $B_2^{(m)}$ are to be determined by the boundary condition u = Du = 0 at x = 0 and 1. These latter conditions lead to the equations

$$A_1^{(m)} = - \frac{4m\pi\alpha}{m^2\pi^2 + K^2}, \qquad KB_1^{(m)} + A_2^{(m)} = - m\pi$$

$$A_1^{(m)} \cosh K + B_1^{(m)} \sinh K + A_2^{(m)} \cosh K + B_2^{(m)} \sinh K$$
$$= (-1)^{m+1} \frac{4m\pi\alpha}{m^2\pi^2 + K^2} \tag{7}$$

$$A_1^{(m)} K \sinh K + B_1^{(m)} K \cosh K + A_2^{(m)} (\cosh K + K \sinh A) +$$
$$+ B_2^{(m)} (\sinh K + K \cosh K) = (-1)^{m+1} (1 + \alpha) m\pi$$

On solving these equations, we find that

$$
\begin{aligned}
A_1^{(m)} &= - \frac{4\alpha m\pi}{m^2\pi^2+K^2} \\
B_1^{(m)} &= \frac{m\pi}{\Delta} \{K + \beta_m(\sinh K + K \cosh K) - \gamma_m \sinh K\} \\
A_2^{(m)} &= - \frac{m\pi}{\Delta} \{\sinh^2 K + \beta_m K(\sinh K + K \cosh K) - \gamma_m K \sinh K\} \\
B_2^{(m)} &= \frac{m\pi}{\Delta} \{(\sinh K \cosh K - K) + \beta_m K^2 \sinh K - \gamma_m (K \cosh K - \sinh K)\}
\end{aligned} \tag{8}
$$

where
$$\Delta = \sinh^2 K - K^2$$
$$\beta_m = \frac{4\alpha}{m^2\pi^2+K^2} [(-1)^{m+1} + \cosh K]$$
$$\gamma_m = (-1)^{m+1} (1+\alpha) + \frac{4\alpha}{m^2\pi^2+K^2} K \sinh K$$

Now substituting for v and u from equations (4) and (6) into (2), we get

$$
\begin{aligned}
\sum_{n=1}^{\infty} C_n (n^2\pi^2 + K^2) \sin n\pi x &= \mathrm{Ta}\, K^2 \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2+K^2)^2} \{A_1^{(m)} \cosh Kx + \\
&+ B_1^{(m)} \sinh Kx + A_2^{(m)} x \cosh Kx + B_2^{(m)} x \sinh Kx + (1 + \alpha x) \sin m\pi x \\
&+ \frac{4\alpha m\pi}{m^2\pi^2+K^2} \cos m\pi x\}
\end{aligned} \tag{9}
$$

Multiplying equation (9) by sin $n\pi x$ and integrating over the range of x, we obtain a system of linear homogeneous equations for the constants $C_m/(m^2\pi^2+K^2)^2$. The requirement that these constants are not all zero leads to the equation

$$
\begin{aligned}
\Bigg\| \frac{n\pi}{(n^2\pi^2+K^2)} \Big\{ &[1 + (-1)^{n+1} \cosh K] A_1^{(m)} + [(-1)^{n+1} \sinh K] B_1^{(m)} + \\
&+ (-1)^{n+1} [\cosh K - \frac{2K}{n^2\pi^2+K^2} \sinh K] A_2^{(m)} + \\
&+ [(-1)^{n+1} \sinh K - \frac{2K}{n^2\pi^2+K^2} \{1 + (-1)^{n+1} \cosh K\}] B_2^{(m)} \Big\} + \\
&+ \alpha X_{nm} + \tfrac{1}{2}\delta_{nm} - \tfrac{1}{2}(n^2\pi^2 + K^2)^3 \frac{\delta_{nm}}{K^2 \mathrm{Ta}} \Bigg\| = 0
\end{aligned} \tag{10}
$$

where

$$X_{nm} = \begin{cases} 0 \text{ if } m + n \text{ is even and } m \neq n \\ 1/4 \text{ if } m = n \\ \dfrac{4nm}{n^2-m^2}\left\{\dfrac{2}{m^2\pi^2+K^2} - \dfrac{1}{\pi^2(n^2-m^2)}\right\} \text{ if } m + n \text{ is odd} \end{cases}$$

On using the first two equations of (7), equation (10) simplifies to

$$\left\| \frac{n\pi}{n^2\pi^2+K^2}\left\{\frac{4m\pi\alpha}{m^2\pi^2+K^2}[(-1)^{m+n}-1] - \right.\right.$$
$$- \frac{2K}{n^2\pi^2+K^2}[(-1)^{n+1}(A_2^{(m)}\sinh K + B_2^{(m)}\cosh K + B_2^{(m)}] +$$
$$\left.+ \alpha X_{nm} + \frac{1}{2}\delta_{nm} - \frac{1}{2}(n^2\pi^2 + K^2)^3\frac{\delta_{nm}}{K^2 Ta}\right\| = 0 \qquad (11)$$

On substituting for the constants $A_2^{(m)}$ and $B_2^{(m)}$ given in (8), we find that (11) simplifies greatly and we are left with

$$\left\| \frac{4mn\pi^2\alpha}{(n^2\pi^2+K^2)(m^2\pi^2+K^2)}[(-1)^{m+n} - 1] - \right.$$
$$- \frac{2Kmn\pi^2}{(n^2\pi^2+K^2)^2(\sinh^2 K - K^2)}\{(\sinh K\cosh K - K)[1 + (1+\alpha)(-1)^{m+n}] +$$
$$+ (\sinh K - K\cosh K)[(-1)^{n+1} + (1+\alpha)(-1)^{m+1}] -$$
$$- \frac{4K\alpha\sinh K}{(m^2\pi^2+K^2)}[\sinh K + K(-1)^{m+1}][(-1)^{m+n} - 1]\} +$$
$$\left.+ \frac{1}{2}\delta_{nm} + \alpha X_{nm} - \frac{1}{2}(n^2\pi^2 + K^2)^3\frac{\delta_{nm}}{K^2 Ta}\right\| = 0 \qquad (12)$$

A first approximation to the solution of equation (12) is obtained by setting the (1,1)-element of the determinant to zero. We find

$$\frac{1}{2}(\pi^2 + K^2)^3\frac{1}{TaK^2} = \frac{1}{4}\alpha + \frac{1}{2} -$$
$$- \frac{2K\pi^2(2+\alpha)}{(\pi^2+K^2)^2(\sinh^2 K - K^2)}[(\sinh K\cosh K - K) + (\sinh K - K\cosh K)]$$

which simplifies to

$$Ta = \frac{2}{2+\alpha}\,\frac{(\pi^2+K^2)^3}{K^2\left\{1 - 16K\pi^2\cosh^2\frac{K}{2}/\left[(\pi^2+K^2)^2(\sinh K + K)\right]\right\}} \qquad (13)$$

A plot of $Ta(2+\alpha)$ as a function of K from this solution shows that the minimum value of Taylor number is

$$Ta_{cr} = 3430/(2+\alpha), \quad \text{reached at } K_{cr} = 3.12$$

Exercise 12.4

When the system is given an upward acceleration a, the effective downward body force is $g' = g + a$ (d'Alembert's principle). The rest of the arguments are given in the problem itself.

Exercise 12.5

(i) Since the stream function is $\Psi = \phi \exp[ik(x - ct)]$, the cross stream velocity is

$$v = -\frac{\partial \Psi}{\partial x} = -\phi\, ik \exp[ik(x-ct)] = \hat{v} \exp[ik(x-ct)]$$

so that

$$\hat{v} = -ik\phi$$

Since $\hat{v}$ and ϕ are proportional, the equations and boundary conditions governing the instability in terms of these variables are identical.

(ii) Equation (91) is

$$(U - c)(\hat{v}'' - k^2\hat{v}) - U''\hat{v} = 0 \tag{91}$$

where y-derivatives are indicated by prime ('). Taking the complex conjugate of (91), we get

$$(U - c^*)(\hat{v}^{*''} - k\,\hat{v}^*) - U''\hat{v}^* = 0$$

This equation is identical to (91), except that c^* replaces c and $\hat{v}^*$ replaces $\hat{v}$. The BC's on $\hat{v}$ and $\hat{v}^*$ are also identical, namely $\hat{v} = \hat{v}^* = 0$ at y_1 and y_2. Thus, if $\hat{v}$ is an eigenfunction with eigenvalue c for some k, then $\hat{v}^*$ is an eigenfunction with eigenvalue $c^* = c_r - ic_i$ for the same k. This property is only possible since (91) does not involve i. This is discussed further in Section 9.

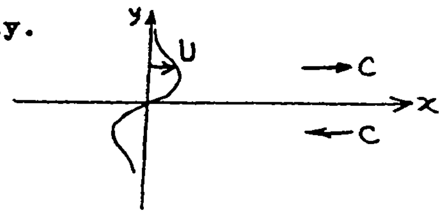
(iii) Take an antisymmetric background current $U(y) = -U(-y)$, so that all derivatives of U change sign when y is oppositely directed. That is, $U'(y) = -U'(-y)$ and $U''(y) = -U''(-y)$. Letting $y \longrightarrow -y$, (91) becomes

$$(-U - c)(\hat{v}'' - k^2\hat{v}) + U''\hat{v} = 0$$

or

$$(U + c)(\hat{v}'' - k^2\hat{v}) - U''\hat{v} = 0 \qquad (92)$$

All terms in (91) and (92) are the same, except that c is replaced by -c. Thus if c is an eigenvalue, then - c is also an eigenvalue for the same k. This means that to each propagation in the positive x direction, there is a corresponding propagation in the negative x direction. This is obvious from the following sketch of the antisymmetric basic flow, and from the observation that the two directions of propagation are "similar" with respect to the basic velocity.



(iv) Now consider a symmetric flow $U(y) = U(-y)$, so that $U''(y) = U''(-y)$. Letting $y \longrightarrow -y$, (91) becomes

$$(U - c)(\hat{v}'' - k^2\hat{v}) - U''\hat{v} = 0$$

so that $\hat{v}(-y)$ satisfies (91) with the same eigenvalue c. By adding and subtracting the two solutions, we get symmetric and antisymmetric solutions

$$S(y) = \hat{v}(y) + \hat{v}(-y) = S(-y)$$
$$A(y) = \hat{v}(y) - \hat{v}(-y) = -A(-y)$$

Obviously S and A satisfy the differential equation, which we indicate by [S-eqn] = 0 and [A-eqn] = 0. We therefore have

$$A[S\text{-eqn}] - S[A\text{-eqn}] = 0 \qquad (93)$$

Using (91), this gives

$$A\left[S'' - k^2 S - \frac{U''S}{U-c}\right] - S\left[A'' - k^2 A - \frac{U''A}{U-c}\right] = 0$$

Canceling terms, we get $\quad S\,A'' - A\,S'' = 0$

which can be written as $\quad (S\,A' - A\,S')' = 0$

Integrating, we get $\quad S\,A' - A\,S' = \text{constant}$

Evaluating the constant term at the wall, it turns out to be zero. Therefore

$$S'/S = A'/A$$

Integrating, we get $\quad \log S = \log A + \log a$

where a is the constant of integration. Therefore

$$S = aA \tag{94}$$

Since the symmetric and antisymmetric functions cannot be proportional, either S or A must be zero. [Both zero is the trivial case. Incidentally, (94) does not mean that both S and A have to be zero if either is zero; the arguments starting from (93) don't hold if either S or A is zero.]

The unstable waves on a symmetric jet are then either sinuous or sausage-like, but not a combination of both.

<u>Exercise 12.6</u>

From (54), multiplying the perturbation equations for the Kevin-Helmholtz instability by u, v, and $g^2\rho/\rho_0^2N^2$, we get

$$\{u_t + Uu_x + wU_z + p_x/\rho_0 = 0\}\ u$$

$$\{w_t + Uw_x + \rho g/\rho_0 + p_z/\rho_0 = 0\}\ w$$

$$\{\rho_t + U\rho_x - \rho_0 wN^2/g = 0\}\ (g^2\rho/\rho_0^2N^2)$$

where we have indicated that the first equation is multiplied by u, the second equation by w, and the third equation by $g^2\rho/\rho_0 N^2$. The three equations then give

$$\tfrac{1}{2}(u^2)_t + U(\tfrac{1}{2}u^2)_x + uwU_z + up_x/\rho_0 = 0$$

$$\tfrac{1}{2}(w^2)_t + U(\tfrac{1}{2}w^2)_x + gw\rho/\rho_o + wp_z/\rho_o = 0$$

$$(g^2\rho/\rho_o^2 N^2)(\rho_t + U\rho_x) - gw\rho/\rho_o = 0$$

Adding the three equations, we get

$$\tfrac{1}{2}(u^2 + w^2 + g^2\rho^2/\rho_o^2 N^2)_t + \tfrac{1}{2}U(u^2 + w^2 + g^2\rho^2/\rho_o^2 N^2)_x +$$

$$uwU_z + [(pu)_x + (pw)_z]/\rho_o - p(u_x + w_z)/\rho_o = 0$$

On global integration ~~with~~ over an integral number x-wavelengths, the second term vanishes. Integral of the fourth term, being $\rho_o^{-1}\int \nabla\cdot(p\mathbf{v})\,dV$, transforms into a surface integral $\rho_o^{-1}\int p\mathbf{v}\cdot d\mathbf{A}$, which vanishes because of no flow across channel walls and equal influx and outflux across open boundaries of the control volume (Figure 11.25). The last term vanishes because of the continuity equation. This gives

$$\frac{1}{2}\frac{d}{dt}(u^2 + w^2 + g^2\rho^2/\rho_o^2 N^2)\,dV = -\int uwU_z\,dV$$

which shows that the rate of change of the sum of kinetic and potential energies equals the energy extracted by the interaction of the Reynolds stress and the mean shear. Of course, in order for the perturbations to extract energy from mean shear, the phases of u and w have to be such that the average uw is negative if U_z is positive. This is typical of shear instabilities.

Exercise 13.1

Given
$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau)\, d\tau$$

Decomposing into real and imaginary parts
$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\cos\omega\tau - i\sin\omega\tau]R(\tau)\, d\tau$$

Since $\sin\omega\tau$ is odd and $R(\tau)$ is even in τ, the imaginary part integrates to zero, leaving the real part
$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\omega\tau\, R(\tau)\, d\tau$$

This also shows the symmetry $S(\omega) = S(-\omega)$.

Exercise 13.2

For $u(t) = U_0\cos\omega t + \bar{U}$, the mean is obviously $\bar{U}$. Defining $\tau \equiv \omega t$, the mean square is
$$\overline{u^2} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left(U_0\cos\omega t + \bar{U}\right)^2 dt = \frac{1}{2\pi}\int_0^{2\pi}\left(U_0\cos\tau + \bar{U}\right)^2 d\tau$$
$$= \frac{1}{2\pi}\int_0^{2\pi}(U_0\cos^2\tau + \bar{U}^2 + 2U_0\bar{U}\cos\tau)\, d\tau = U_0^2/2 + \bar{U}^2 + 0$$
$$u_{rms} = \sqrt{U_0^2/2 + \bar{U}^2}$$
$$u_{SD} = \left[\frac{1}{2\pi}\int_0^{2\pi}U_0^2\cos^2\tau\, d\tau\right]^{\frac{1}{2}} = U_0/\sqrt{2}$$

Exercise 13.3

Auto-correlation is $R(\tau) = \overline{u(t)\, u(t+\tau)}$. If $u = U\cos\omega t$, then
$$\frac{R(\tau)}{U^2} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\cos\omega t\,\cos\omega(t+\tau)\, dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\cos t\,\cos(t+\tau)\, dt, \text{ letting } \omega t \longrightarrow t \text{ and } \omega\tau \longrightarrow \tau$$
$$= \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}[\cos\tau + \cancel{\cos(2t+\tau)}]\, dt = \frac{1}{4\pi}\cos\tau\,(2\pi) = \frac{1}{2}\cos\tau$$

which shows that $R(\tau)$ is a periodic function.

Exercise 13.4

Given

$$u = \cos\omega t$$
$$v = \cos(\omega t + \phi)$$

The zero-lag cross-correlation is

$$C(0) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} u(t)\ v(t)\ dt$$

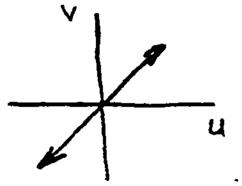
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos t\ \cos(t + \phi)\ dt, \text{ letting } \omega t \longrightarrow t$$

$$= \frac{1}{4\pi}\left[\int_0^{2\pi} \cos\phi\ dt + \cancel{\int_0^{2\pi} \cos(2t + \phi)\, dt}\right]$$

$$= \frac{1}{2}\cos\phi$$

For $\phi = 0$

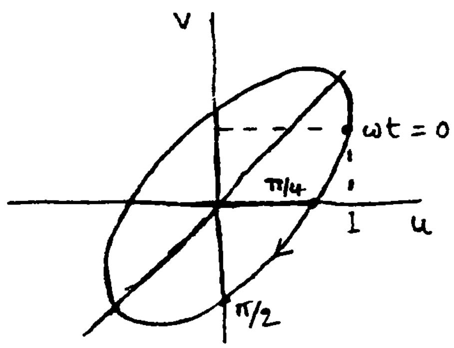
$$u = \cos\omega t$$
$$v = \cos\omega t$$



For $\phi = \pi/4$

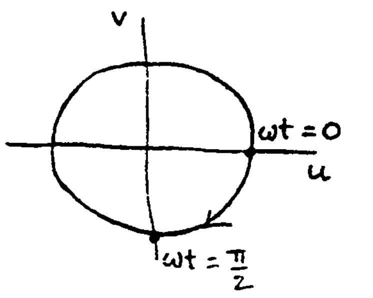
$$u = \cos\omega t$$
$$v = \cos(\omega t + \frac{\pi}{4})$$

ωt	u	v
0	1	$1/\sqrt{2}$
$\pi/4$	$1/\sqrt{2}$	0
$\pi/2$	0	$-1/\sqrt{2}$



For $\phi = \pi/2$

$$u = \cos\omega t$$
$$v = -\sin\omega t$$



Exercise 13.5

Given $w_{rms} = 1$ m/s
$T'_{rms} = 0.1^{o}C$
$r = 0.5$

$$Q = \rho C_p \overline{wT'} = \rho C_p (0.5 w_{rms} T'_{rms}) = (1.2)(1012)(0.5)(1)(0.1) = 60.7\ W/m^2$$

Exercise 13.6

$$\text{Power} = mC_p \Delta T/\Delta t = (10)(4200)(1)/3600 = 11.67\ W$$

$$\text{Dissipation}\quad \varepsilon = \text{Power}/m = 11.67/10 = 1.167\ m^2/s^3$$

$$\therefore \eta = (\nu^3/\epsilon)^{1/4} = (10^{-18}/1.667)^{1/4} = 3.04\times 10^{-5}\ m = 0.0304mm$$

Exercise 13.7

Given $D = 0.2$ m
$dp/dx = 8\ N/m^2$ per m

We know $dp/dx = 2\tau_o/R$. Thus

$$\tau_o = \frac{D}{4}\frac{dp}{dx} = (0.2)(8)/4 = 0.4\ N/m^2$$
$$u_* = \sqrt{\tau_o/\rho} = \sqrt{0.4/1000} = 0.02\ m/s$$

From Section 12 the viscous sublayer thickness is

$$\delta_\nu \simeq 5\nu/u_* = 5(10^{-6})/0.02 = 2.5\times 10^{-4}\ m = 0.25\ mm$$

Exercise 13.8

Given $U = \dfrac{u_*}{k}\log y + C$

In the viscous sublayer the velocity profile is linear: $U = u_*^2 y/\nu$.
At the edge of the viscous sublayer of thickness $\delta_\nu \simeq 10.7\nu/u$, the velocity is

$$U_\delta = u_*^2 \delta_\nu/\nu = \frac{u_*^2}{\nu}\left(\frac{10.7\nu}{u_*}\right) = 10.7u_*$$

Fitting this condition at the edge of the logarithmic distribution gives

$$10.7u_* = \frac{u_*}{k}\log\delta_\nu + C \longrightarrow C = 10.7u_* - \frac{u_*}{k}\log\frac{10.7\nu}{u_*}$$

With this constant, the logarithmic distribution becomes

$$U = \frac{u_*}{k}\log y + 10.7u_* - \frac{u_*}{k}\log\frac{10.7\nu}{u_*} = \frac{u_*}{k}\log\frac{yu_*}{10.7\nu} + 10.7u_*$$

$$\therefore\ \frac{U}{u_*} = \frac{1}{k}\log\frac{yu_*}{\nu} + \frac{1}{k}\log\frac{1}{10.7} + 10.7 = \frac{1}{k}\log\frac{yu_*}{\nu} - 5.8 + 10.7$$

$$= \frac{1}{k}\log\frac{yu_*}{\nu} + 5$$

Exercise 13.9

Given

$$\tau_0 = 0.1\ \mathrm{N/m^2}$$
$$Q = 200\ \mathrm{W/m^2}$$

Then

$$u_* = \sqrt{\tau_0/\rho} = \sqrt{0.1/1.2} = 0.28867\ \mathrm{m/s}$$

$$\overline{wT'} = Q/\rho C_p = 200/(1.2)(1012) = 0.1647\ \mathrm{m\,{}^\circ C\,s^{-1}}$$

$$L_M = u_*^3/k\alpha g\,\overline{wT'} = (0.28867)^3/(0.41)(3.38\times10^{-3})(9.81)(0.1647)$$
$$= 10.7\ \mathrm{m}$$

Exercise 13.10

Given

$$r(\tau) = e^{-\tau^2/t_c^2}$$
$$t_c = 1\ \mathrm{s}$$
$$u_{rms} = 1\ \mathrm{m/s}$$

$$\mathcal{J} = \int_0^\infty e^{-\tau^2/t_c^2}\,d\tau = \frac{\sqrt{\pi}}{2}t_c = 0.89\ \mathrm{s}$$

From (12.89) the eddy viscosity for large times is of order

$$K_e \sim \overline{u^2}\mathcal{J} = (1)(0.89) = 0.89\ \mathrm{m^2/s}$$

Exercise 13.11

For a wake under conditions of self-preserving flows,

$$\frac{U_\infty - U(x, y)}{U_\infty - U_c(x)} = f\left(\frac{y}{\delta(x)}\right).$$

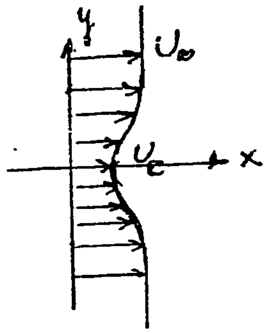
In a wake the expression for momentum flux that is preserved downstream is $M = \rho\int_{-\infty}^{\infty} U(U_\infty - U)dy =$ constant in a wake. Substituting for U and $U_\infty - U$ into the integral,

$$M = \rho\delta\int_{-\infty}^{\infty}\left[U_\infty - (U_\infty - U_c)f\left(\frac{y}{\delta}\right)\right]\left[(U_\infty - U_c)f\left(\frac{y}{\delta}\right)\right]d\left(\frac{y}{\delta}\right) = \text{const.}$$

$$= \rho\delta U_\infty(U_\infty - U_c)\underbrace{\int_{-\infty}^{\infty} f\left(\frac{y}{\delta}\right)d\left(\frac{y}{\delta}\right)}_{c_1} - \rho\delta(U_\infty - U_c)^2\underbrace{\int_{-\infty}^{\infty} f^2\left(\frac{y}{\delta}\right)d\left(\frac{y}{\delta}\right)}_{c_2}$$

$$M = \rho\delta c_1 U_\infty(U_\infty - U_c) - \rho\delta c_2(U_\infty - U_c)^2.$$

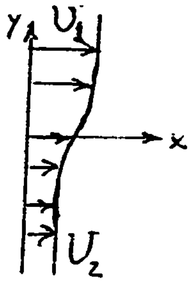
As $x \to \infty$, $U_\infty - U_c \to 0$ so $(U_\infty - U_c)^2 \ll (U_\infty - U_c)$. Then, for large x, $\delta(x)(U_\infty - U_c(x)) =$ const. so $U_\infty - U_c \sim 1/\delta(x)$. Far downstream, $U_\infty - U_c = F(x, \rho, M)$ only. From these quantities, only one dimensionless parameter is available, $(U_\infty - U_c)\sqrt{\rho x/M} =$ const. Then $U_\infty - U_c \sim 1/\sqrt{x} \sim 1/\delta(x)$ so $\delta(x) \sim \sqrt{x}$.



For a shear layer, we follow the arguments of Townsend (1976), pp. 195–198. We start with the x-momentum equation for the mean flow (U, V)

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial x}(\overline{u^2} - \overline{v^2}) + \frac{\partial}{\partial y}(\overline{uv}) = 0,$$

where $U_{1,2}$ are constant with x and $\partial P/\partial x \approx 0$. Fluctuations are denoted by (u, v). For self-preserving flow $U = U_1 + u_0 f(y/\delta)$, $\overline{u^2} = q_0^2 g_1(y/\delta)$, $\overline{v^2} = q_0^2 g_2(y/\delta)$, $\overline{uv} = q_0^2 g_{12}(y/\delta)$. Mass conservation for the mean flow is $\partial U/\partial x + \partial V/\partial y = 0$. In the following, let $\eta = y/\delta(x)$.



Integrating,

$$V = -\int \frac{\partial U}{\partial x} dy = -u_0 \int f'(\eta) \frac{\partial \eta}{\partial x} \cdot d\eta \cdot \delta$$
$$= u_0 \int \eta f' d\eta \cdot \frac{d\delta}{dx} \qquad \left(\frac{\partial \eta}{\partial x} = -\frac{\eta}{\delta}\frac{d\delta}{dx}\right)$$
$$= u_0 \left(\eta f - \int f d\eta\right) \frac{d\delta}{dx}$$

$$\frac{\partial U}{\partial x} = \frac{du_0}{dx} f(\eta) - u_0 f' \frac{\eta}{\delta}\frac{d\delta}{dx}$$
$$U\frac{\partial U}{\partial x} = [U_1 + u_0 f(\eta)] \left[\frac{du_0}{dx} f - u_0 f' \frac{\eta}{\delta}\frac{d\delta}{dx}\right]$$
$$= U_1 \frac{du_0}{dx} f - u_0 U_1 f' \frac{\eta}{\delta}\frac{d\delta}{dx} + u_0 \frac{du_0}{dx} f^2 - u_0^2 \frac{1}{\delta}\frac{d\delta}{dx} \eta f f'$$
$$\frac{\partial U}{\partial y} = u_0 f'(\eta) \frac{1}{\delta},$$
$$V = -\int \frac{\partial U}{\partial x} dy = -\frac{du_0}{dx} \int f\, d\eta \cdot \delta + u_0 \int f' \frac{\eta}{\delta}\frac{d\delta}{dx} d\eta \cdot \delta$$
$$= -\frac{du_0}{dx} \delta \int f\, d\eta + u_0 \frac{d\delta}{dx}\left(\eta f - \int f\, d\eta\right)$$
$$= -\frac{d}{dx}(\delta u_0) \int f\, d\eta + u_0 \frac{d\delta}{dx} \eta f,$$
$$V\frac{\partial U}{\partial y} = -\frac{u_0}{\delta} f' \frac{d}{dx}(\delta u_0) \int f\, d\eta + u_0^2 \frac{1}{\delta}\frac{d\delta}{dx} \eta f f',$$
$$U\frac{\partial U}{dx} + V\frac{\partial U}{dy} = U_1 \frac{du_0}{dx} f - u_0 U_1 f' \frac{\eta}{\delta}\frac{d\delta}{dx} + u_0 \frac{du_0}{dx} f^2 - \frac{u_0}{\delta}\frac{d}{dx}(\delta u_0) f' \int f\, d\eta$$
$$= -\frac{\partial}{\partial x}(\overline{u^2} - \overline{v^2}) + \frac{\partial}{\partial y}(\overline{uv})$$
$$= -\frac{\partial}{\partial x}\left[q_0^2 g_1\left(\frac{y}{\delta}\right) - q_0^2 g_2\left(\frac{y}{\delta}\right)\right] + q_0^2 g_{12}' \frac{1}{\delta}$$
$$= -(g_1 - g_2)\frac{dq_0^2}{dx} + q_0^2(g_1' - g_2')\frac{\eta}{\delta}\frac{d\delta}{dx} + q_0^2 g_{12}' \frac{1}{\delta}.$$

For self-preserving flows, all x-dependence must cancel. Then $u_0(x) = $ const., $q_0^2(x) = $ const., and

$$\frac{1}{\delta}\frac{d\delta}{dx} \sim \frac{1}{\delta} \quad \text{so} \quad \frac{d\delta}{dx} \sim \text{const. or } \delta \sim x.$$

Exercise 14.1

Given $v = 2$ m/s

$$f = 2\Omega \sin 45^\circ = 2(0.73\times10^{-4})\sin 45^\circ = 1.03\times10^{-4}\ \mathrm{s}^{-1}$$

By geostrophy $fv = \frac{1}{\rho}\,dp/dx = g\,d\eta/dx$, so the sea surface slope is

$$d\eta/dx = fv/g = (1.03\times10^{-4})(2)/9.81 = 2.1\times10^{-5}$$

which is equivalent to an eastward surface rise of 2.1 cm per km.

Exercise 14.2

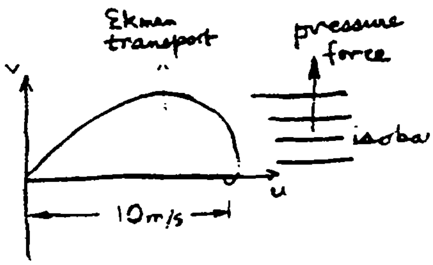
Given Ω = 10 rpm = 20π rad/min = 1.047 s^{-1}. The Coriolis frequency is

$$f = 2\Omega = 2.09\ \mathrm{s}^{-1}$$

From Section 7, the Ekman layer thickness is

$$\delta = \sqrt{2\nu/f} = \sqrt{2(10^{-6})/2.09} = 10^{-3}\ \mathrm{m} = 1\ \mathrm{mm}$$

Exercise 14.3



Given $\nu = 10\ \mathrm{m^2/s}$
$f = 1.03\times10^{-4}\ \mathrm{s}^{-1}$ (at 45° N)
$U = 10$ m/s

From Section 7, the Ekman layer height is

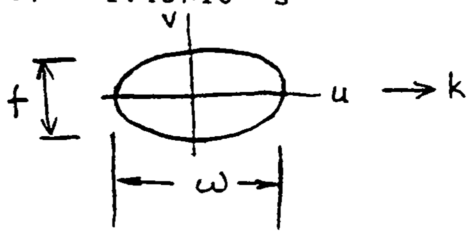
$$\delta = \sqrt{2\nu/f} = \sqrt{2(10)/1.03\times10^{-4}} = 440.7\ \mathrm{m}$$

and the transport perpendicular to the geostrophic stream is

$$\text{Transport} = \tfrac{1}{2}U\delta = (0.5)(10)(440.7) = 2{,}203\ \mathrm{m^2/s}$$

Exercise 14.4

Given $f = 1.03\times10^{-4}\ \mathrm{s}^{-1}$ (at 45° N)
$\omega = 2$ rev/day $= 4\pi/(24)(60)(60) = 1.45\times10^{-4}\ \mathrm{s}^{-1}$
$H = 4$ km



From Section 12, the axis ratio of rotational gravity waves is

$$\text{Axis ratio} = f/\omega = 1.03/1.45 = 0.708$$

$$\text{Phase speed } c = \sqrt{gH} = \sqrt{(9.81)(4000)} = 198 \text{ m/s}$$

The group speed is the same because it is a shallow water wave.

$$\text{Wavelength} = (\text{speed})(\text{period}) = (198)(12)(3600) = 8553.6 \text{ km}$$

The large wavelength shows that the neglect of coastal boundary is unrealistic.

Exercise 14.5

Given $f = -2\Omega\sin 30^{\circ} = -0.73\times10^{-4}\ \text{s}^{-1}$

$H = 50$ m

$\Delta\rho = 2$ kg/m^3

This is the case of a shallow layer of lighter water, overlying a deep sea. From equation (7.126), the internal gravity wave speed in this case is

$$c = \sqrt{g'H}$$

where g' = reduced gravity = $g\Delta\rho/\rho_0 = (9.81)(2)/1000 = 0.0196$ m/s^2

$$c = \sqrt{g'H} = \sqrt{(0.0196)(50)} = 0.99 \text{ m/s}$$

The waves propagate southward along the west coast of Australia. The decay scale perpendicular to coast is the Rossby radius

$$\Lambda = c/f = 0.99/(0.73\times10^{-4}) = 1.36\times10^{4} \text{ m} = 13.6 \text{ km}$$

Exercise 14.6

Given $$m^2 = k^2(N^2 - \omega^2)/(\omega^2 - f^2) \tag{1}$$

Rearranging $$\omega^2 = f^2 + k^2(N^2 - \omega^2)/m^2 \tag{1$'$}$$

Differentiating with respect to k gives

$$2\omega\frac{\partial\omega}{\partial k} = \frac{1}{m^2}\left[2k(N^2 - \omega^2) + k^2\left(-2\omega\frac{\partial\omega}{\partial k}\right)\right]$$

Using $c_{gx} = \frac{\partial\omega}{\partial k}$, this gives $m^2 2\omega c_{gx} = 2k(N^2 - \omega^2) - 2\omega k^2 c_{gx}$

or
$$2\omega c_{gx}(m^2 + k^2) = 2k(N^2 - \omega^2)$$

or
$$c_{gx} = k(N^2 - \omega^2)/[\omega(m^2 + k^2)]$$

Eliminating ω by (1), this reduces to

$$c_{gx} = km^2(N^2 - f^2)/[(m^2 + k^2)^{3/2}(m^2 f^2 + k^2 N^2)^{1/2}] \qquad (2)$$

Similarly differentiating (1)′ with respect to m, we get

$$2\omega \frac{\partial\omega}{\partial m} = k^2(-2\omega\frac{\partial\omega}{\partial m})/m^2 - 2k^2(N^2 - \omega^2)/m^3$$

Using $c_{gz} = \frac{\partial\omega}{\partial m}$, this reduces to

$$c_{gz} = -k^2(N^2 - \omega^2)/[m\omega(m^2 + k^2)$$

On eliminating ω by (1), this reduces to

$$c_{gz} = -k^2 m(N^2 - f^2)/[(m^2 + k^2)^{3/2}(m^2 f^2 + k^2 N^2)^{1/2}] \qquad (3)$$

To show that $\underset{\sim}{c}_g$ and $\underset{\sim}{c}$ are perpendicular, we show that their dot product is zero:

$$\underset{\sim}{c}_g \cdot \underset{\sim}{c} = (\underset{\sim}{i}c_{gx} + \underset{\sim}{j}c_{gz})\cdot(\underset{\sim}{i}\frac{\omega}{k} + \underset{\sim}{j}\frac{\omega}{m}) = 0$$

where (2) and (3) have been used. Since the sign of $c_z = \omega/m$ is opposite to that of c_{gz} (see equation 3), it follows that the vertical components of $\underset{\sim}{c}_g$ and $\underset{\sim}{c}$ are oppositely directed.

Finally, $\underset{\sim}{c}_g$ and $\underset{\sim}{u}$ are parallel because

$$c_{gx}/c_{gz} = -m/k = u/w \qquad (4)$$

where equation (111) of the text has been used for the ratio u/w.

<u>Exercise 14.7</u>

Given $f = 1.03\times10^{-4}\ s^{-1}$ (at 45°N)

$H = 10$ km

$\Delta T = 50^\circ C$

At temperature T, the thermal expansion coefficient is $\alpha = 1/T$. Therefore the buoyancy frequency is given by

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} = g\alpha\frac{dT}{dz} = \frac{g}{T}\frac{dT}{dz} = (9.81/300)(50/10^4) = 1.635 \times 10^{-4}\ s^{-2}$$

$$\therefore\ N = 0.01279\ s^{-1}$$

From equation (13.71), the internal gravity wave speed corresponding to the n-th normal mode is $c_n = NH/n\pi$. Therefore

$$c_1 = NH/\pi = (0.01279)(10^4)/\pi = 40.71\ m/s$$

From Section 15, the westward speed of non-dispersive Rossby wave (corresponding to n = 1 mode) is

$$c_x = -\beta c_1^2/f^2 = -(2\times10^{-11})(40.71)^2/(1.03\times10^{-4})^2 = -3.12\ m/s$$

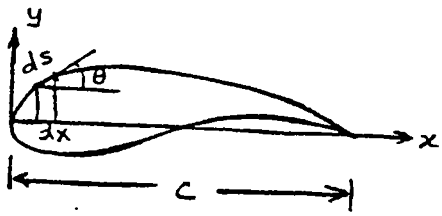
Exercise 14.8

For $R_0 \ll 1$ and $E \ll 1$, the Taylor–Proudman theorem gives $2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p/\rho$ and $\nabla \cdot \mathbf{u} = 0$, where p includes the hydrostatic pressure. The boundary conditions $w = 0$ on $z = 0, L$ give $w = 0$ everywhere and $\partial \mathbf{u}/\partial z = 0$. Thus $\mathbf{u} = \mathbf{i}u(x, y) + \mathbf{j}v(x, y)$. Since all streamlines are in z = const. planes, $\mathbf{u} = -\mathbf{k} \times \nabla\psi$ in terms of the streamfunction ψ. Since $\boldsymbol{\Omega} = \mathbf{k}\Omega$, the momentum equation becomes

$$2\Omega\mathbf{k} \times (-\mathbf{k} \times \nabla\psi) = -\nabla p/\rho$$
$$-2\Omega[\mathbf{k}\underbrace{(\mathbf{k}\cdot\nabla\psi)}_{=0} - \nabla\psi\underbrace{(\mathbf{k}\cdot\mathbf{k})}_{=1}] = -\nabla p/\rho$$
$$-2\rho\Omega\nabla\psi = \nabla p.$$

Since all streamlines are in z = const. planes, $p + 2\rho\Omega\psi$ = const. in z = const. planes. Since ψ = const. on streamlines, p = const. on streamlines.

Exercise 15.1



u ⟶ upper
l ⟶ lower

$$dF_y = -(p_u ds)\cos\theta = -p_u dx$$

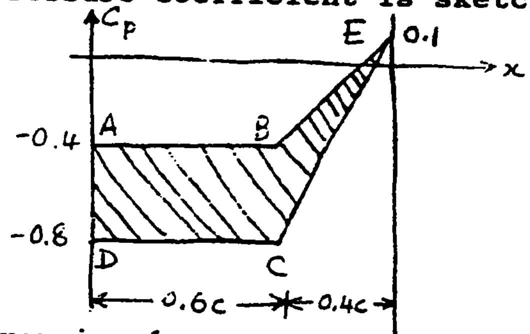
$$\therefore \quad F_y = -\int_0^c p_u\,dx + \int_0^c p_l\,dx = -\int_0^c (p_u - p_\infty)dx + \int_0^c (p_l - p_\infty)dx$$

$$\therefore \quad C_y \equiv F_y / \tfrac{1}{2}\rho U^2 c = -\frac{1}{c}\int_0^c C_{pu}\,dx + \frac{1}{c}\int_0^c C_{pl}\,dx = \frac{1}{c}\int_0^c C_{pl}\,dx + \frac{1}{c}\int_c^0 C_{pu}\,dx$$

$$= \oint C_p\, d\left(\frac{x}{c}\right).$$

Exercise 15.2

The distribution of pressure coefficient is sketched below.

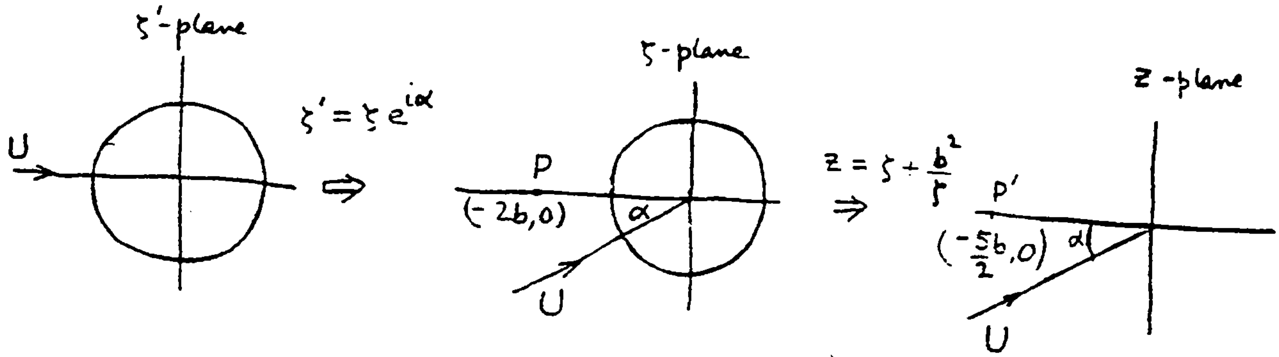


Using the result of Exercise 1,

$$C_y = \oint C_p\, d\left(\frac{x}{c}\right) = ABCD + BCE = (0.4)(0.6) + \frac{1}{2}(0.4)(0.4) = 0.32$$

$$C_L = C_y \cos 4^\circ = 0.32$$

Exercise 15.3



(i) From equation (14.11): $\Gamma = 4\pi U b \sin\alpha$

Take the ζ'-plane, with U coming in along the real axis. Then

$$\zeta = \zeta' e^{i\alpha}$$

rotates the ζ'-plane anticlockwise to give ζ-plane. Complex potential is

$$w = U(\zeta' + b^2/\zeta') + \frac{i\Gamma}{2\pi}\log\zeta'$$

$$= U(\zeta e^{-i\alpha} + b^2 e^{i\alpha}/\zeta) + \frac{i\Gamma}{2\pi}\log[\zeta e^{-i\alpha}]$$

$$= U(\zeta e^{-i\alpha} + e^{i\alpha} b^2/\zeta) + \frac{i\Gamma}{2\pi}[\log\zeta - i\alpha]$$

(ii) Complex velocity is

$$\frac{dw}{d\zeta} = U(e^{-i\alpha} - e^{i\alpha} b^2/\zeta^2) + i\Gamma/2\pi\zeta$$

At point P,

$$\left.\frac{dw}{d\zeta}\right|_P = U\left[e^{-i\alpha} - e^{i\alpha} b^2/(-2b)^2\right] + i\Gamma/2\pi(-2b)$$

$$= U(\cos\alpha - i\sin\alpha - \frac{1}{4}\cos\alpha - \frac{i}{4}\sin\alpha) - i(4\pi U b\sin\alpha)/4\pi b$$

$$= U(\frac{3}{4}\cos\alpha - \frac{5i}{4}\sin\alpha) - iU\sin\alpha = \frac{3}{4}U\cos\alpha - \frac{i9}{4}U\sin\alpha = u - iv$$

$\therefore \quad u = \frac{3}{4}U\cos\alpha, \; v = \frac{9}{4}U\sin\alpha$ at P on ζ-plane

(iii) $z = \zeta + b^2/\zeta$ gives the transformed point

$$z_{P'} = -2b + b^2/(-2b) = -5b/2$$

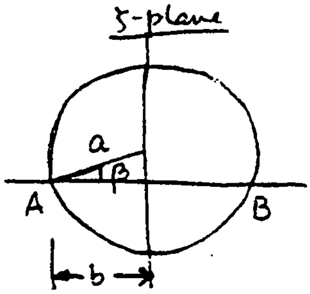
(iv) Complex velocity at $(-5b/2,0)$ in z-plane is

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = \frac{dw}{d\zeta}\frac{\zeta^2}{\zeta^2 - b^2} = (\frac{3}{4}U\cos\alpha - \frac{i9}{4}U\sin\alpha)\left[\frac{(-2b)^2}{(-2b)^2 - b^2}\right]$$

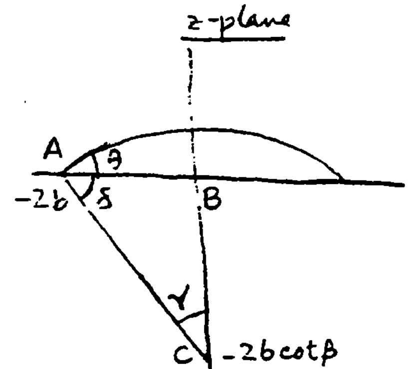
$$= U\cos\alpha - i3U\sin\alpha$$

where we have put $\zeta = (-2b,0)$ since that is the corresponding point in the ζ-plane.

Exercise 15.4



$$z = \zeta + \frac{b^2}{\zeta}$$



From triangle ABC: $\tan\gamma = 2b/(2b\cot 2\beta) = \tan 2\beta$

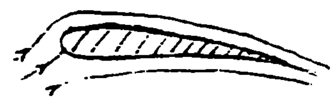
$$\therefore\ \gamma = 2\beta$$

$$\therefore\ \delta = \frac{\pi}{2} - \gamma = \frac{\pi}{2} - 2\beta$$

$$\therefore\ \theta = \frac{\pi}{2} - \delta = \frac{\pi}{2} - \frac{\pi}{2} + 2\beta = 2\beta$$

Exercise 15.5

Use computer to plot



Exercise 15.6

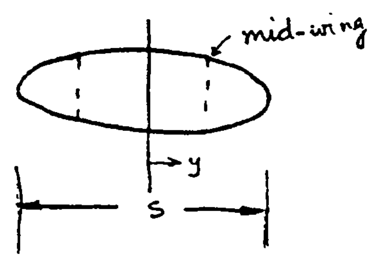
From (14.12), the lift coefficient of Zhukhovski airfoil is

$$C_L = 2\pi(\alpha + \beta)$$

It is given that $C_L = 0.3$ for $\alpha = 0$. Then $\beta = 0.3/2\pi$. At $\alpha = 5^\circ$ the lift coefficient is

$$C_L = 2\pi(\alpha + \beta) = 2\pi(5\pi/180 + 0.3/2\pi) = 0.848$$

Exercise 15.7



Given $L = 80{,}000$ N

$U = 300$ km/hour $= 83.3$ m/s

$s = 20$ m

D_i, Γ = ?

(i) An untwisted wing of elliptic area is expected to have an elliptic circulation distribution. From Section 14.11, the induced drag is

$$D_i = 2L^2/\rho U^2 \pi s^2 = 2(8\times 10^4)^2/(1.2)(83.3)^2\pi(20)^2 = 1.222 \text{ N}$$

(ii) From Section 14.11, the maximum circulation Γ_0 is given by

$$D_i = \frac{\pi}{8}\rho\Gamma_0^2 \longrightarrow \quad \Gamma_0 = \sqrt{8D_i/\pi\rho} = \sqrt{8(1222)/(\pi)(1.2)} = 50.9 \text{ m}^2/\text{s}$$

$$\Gamma \text{ at mid-wing} = \Gamma_0\left[1 - (2y/s)^2\right]^{1/2} = 50.9\left[1 - (1/2)^2\right]^{1/2} = 44.1 \text{ m}^2/\text{s}$$

Exercise 15.8

From $\Gamma = \Gamma_0(1 - 4y^2/s^2) \longrightarrow \quad \frac{d\Gamma}{dy} = -8\Gamma_0 y/s^2$

From equation (14.13) the downwash at y_1 is

$$w(y_1) = \frac{1}{4\pi}\int_{-s/2}^{s/2}\frac{d\Gamma}{dy}\frac{dy}{y_1 - y} = \frac{8\Gamma_0}{4\pi s^2}\int_{-s/2}^{s/2}\frac{y\,dy}{y - y_1}$$

$$= \frac{2\Gamma_0}{\pi s^2}\Big[y + y_1 \log(y - y_1)\Big]_{-s/2}^{s/2} = \frac{2\Gamma_0}{\pi s^2}\left[s + y_1 \log\frac{y_1 - s/2}{y_1 + s/2}\right]$$

This varies along the span. In contrast the downwash for elliptic distribution is $w = \Gamma_0/2s$, which is independent of y_1.

Exercise 16.1

$$A/A^* = (\rho^* c^*/\rho u) = (\rho^*/\rho_o)(\rho_o/\rho)(c^*/c)(c/u)$$

$$= (\rho^*/\rho_o)(\rho_o/\rho)\sqrt{(T^*/T)}(1/M) = (\rho^*/\rho_o)(\rho_o/\rho)\sqrt{(T^*/T_o)}\sqrt{(T_o/T)}(1/M)$$

$$= \left[1 + \frac{\gamma-1}{2}\underset{1}{\underset{\shortparallel}{M^{*2}}}\right]^{-\frac{1}{\gamma-1}}\left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{1}{\gamma-1}}\left[1 + \frac{\gamma-1}{2}\underset{1}{\underset{\shortparallel}{M^{*2}}}\right]^{-\frac{1}{2}}\left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{1}{2}}\frac{1}{M}$$

where we have used the relations given in Section 4. On simplifying this gives

$$A/A^* = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}$$

Exercise 16.2

From (15.35), the entropy change across a normal shock is

$$(S_2 - S_1)/C_v = \log\left\{\left[1 + \frac{2\gamma}{\gamma+1}(M^2 - 1)\right]\left[[(\gamma - 1)M^2 + 2]/[(\gamma + 1)M^2]\right]^\gamma\right\}$$

Now $[(\gamma - 1)M^2 + 2]/[(\gamma + 1)M^2] = [(\gamma + 1)M^2 - 2M^2 + 2]/[(\gamma + 1)M^2]$

$$= 1 - [2(M^2 - 1)]/[(\gamma + 1)M^2] \equiv 1 - \delta, \text{ where } \delta << 1$$

$$\therefore (S_2 - S_1)/C_v = \log\left\{(1 + \epsilon)(1 - \delta)^\gamma\right\} = \log(1 + \epsilon) + \gamma\log(1 - \delta)$$

$$= \epsilon - \epsilon^2/2 + \epsilon^3/3 + \dots \quad + \gamma(-\delta - \delta^2/2 - \delta^3/3 - \dots)$$

$$= (\epsilon - \gamma\delta) - \frac{1}{2}(\epsilon^2 + \gamma\delta^2) + \frac{1}{3}(\epsilon^3 - \gamma\delta^3) + \dots$$

In this

$$\epsilon - \gamma\delta = \frac{2\gamma}{\gamma+1}(M^2 - 1) - \frac{\gamma 2(M^2-1)}{(\gamma+1)M^2} = \frac{2\gamma(M^2-1)}{(\gamma+1)}\left[1 - \frac{1}{M^2}\right]$$

$$= 2\gamma(M^2 - 1)^2/[M^2(\gamma + 1)]$$

$$\frac{1}{2}(\epsilon^2 + \gamma\delta^2) = \frac{1}{2}\left[\frac{4\gamma^2(M^2-1)^2}{(\gamma+1)^2} + \frac{\gamma 4(M^2-1)^2}{(\gamma+1)^2 M^4}\right] = \frac{2\gamma(M^2-1)^2}{(\gamma+1)}\left[\gamma + \frac{1}{M^4}\right]$$

$$\frac{1}{3}(\epsilon^3 - \gamma\delta^3) = \frac{1}{3}\frac{8\gamma^3(M^2-1)^3}{(\gamma+1)^3} - \frac{\gamma 8(M^2-1)^3}{3(\gamma+1)^3 M^6} \simeq \frac{8\gamma(M^2-1)^3}{3(\gamma+1)^3}\left(\gamma^2 - \frac{1}{M^6}\right)$$

Then

$$(S_2 - S_1)C_v = \frac{2\gamma(M^2-1)^2}{M^2(\gamma+1)} - \frac{2\gamma(M^2-1)^2}{\gamma+1}\left[\gamma + \frac{1}{M^4}\right] + \frac{8\gamma(M^2-1)^3}{3(\gamma+1)^3}\left[\gamma^2 - \frac{1}{M^6}\right]$$

On simplification this becomes

$$(S_2 - S_1)/C_v = \frac{2\gamma(M^2-1)^3}{\gamma+1}\left\{\frac{3(\gamma+1)(-\gamma M^2+1) + 4(\gamma^2 M^4 - M^{-2})}{3M^4(\gamma+1)^2}\right\}$$

Clearly we can put $M \simeq 1$ in { }. Then

$$(S_2 - S_1)/C_v \simeq \frac{2\gamma(M^2-1)^3}{\gamma+1}\left\{\frac{3(\gamma+1)(-\gamma+1) + 4(\gamma^2-1)}{3(\gamma+1)^3}\right\}$$

which simplifies to

$$(S_2 - S_1)/C_v = \frac{2\gamma(\gamma-1)}{3(\gamma+1)^2}(M^2-1)^3$$

Exercise 16.3

Since $h_0 = h + u^2/2$, u is maximum when h = 0, which implies T = 0.

$$u_{max} = \sqrt{2(h_0 - h)} = \sqrt{2C_p T_0}$$

$$M_{max} = u_{max}/c = \infty$$

$$c = 0$$

Exercise 16.4

Given

$$u_1 = 250 \text{ m/s}$$
$$T_1 = 300 \text{ K}$$
$$p_1 = 200 \text{ kPa}$$
$$u_2 = 300 \text{ m/s}$$
$$p_2 = 150 \text{ kPa}$$

$$c_1 = \sqrt{\gamma R T_1} = \sqrt{1.4(287)(300)} = 347.2 \text{ m/s}$$

$$M_1 = u_1/c_1 = 0.72$$

From Table 16.1: $p_1/p_{01} = 0.708$, $T_1/T_{01} = 0.9061$

$$p_{01} = 200/0.708 = 282.49 \text{ kPa}$$

$$T_{01} = 300/0.9061 = 331.09 \text{ K} = T_{02} = T_0 \text{ (say)}$$

$$h_0 = C_p T_0 = 1005(331.09) = 332{,}745 \text{ J/kg} = h_2 + u_2^2/2$$

$$\therefore h_2 = 332{,}745 - (300)^2/2 = 287{,}745 \text{ J/kg}$$

$$T_2 = h_2/C_p = 287{,}745/1005 = 286.3 \text{ K}$$

$$c_2 = \sqrt{\gamma R T_2} = \sqrt{1.4(287)(286.3)} = 339.16 \text{ m/s}$$

$$M_2 = u_2/c_2 = 300/339.16 = 0.8845$$

Locating this value of M in Table 15.1, we get $p_2/p_{02} = 0.6041$

$$p_{02} = p_2/0.6041 = 150/0.6041 = 248.3 \text{ kPa}$$

Loss of stagnation pressure and gain of entropy are

$$p_{01} - p_{02} = 282.49 - 248.3 = 34.187 \text{ kPa}$$

$$S_2 - S_1 = S_{02} - S_{01} = \cancel{C_p \log(T_{02}/T_{01})} - R\log(p_{02}/p_{01}) = -R\log(p_{02}/p_{01})$$

$$= -287\log(248.3/282.5) = 37.03 \text{ m}^2/\text{s}^2\text{K}$$

Exercise 16.5

We transform the propagating shock to a stationary shock as follows:

Given $p_2 = 700$ kPa
$p_1 = 101$ kPa
$T_1 = 300$ K

(The upstream values p_1 and T_1 are assumed to be atmospheric.)

$$p_2/p_1 = 700/101 = 7$$

Locating this value of pressure ratio in Table 15.2, we get

$$M_1 = 2.48$$
$$M_2 = 0.515$$
$$T_2/T_1 = 2.118$$

Then

$$c_1 = \sqrt{\gamma R T_1} = \sqrt{1.4(287)(300)} = 347.2 \text{ m/s}$$

$$u_1 = M_1 c_1 = (2.48)(347.2) = 861 \text{ m/s}$$

$$T_2 = 300(2.118) = 635.4 \text{ K}$$

$$c_2 = \sqrt{1.4(287)(635.4)} = 505.28 \text{ m/s}$$

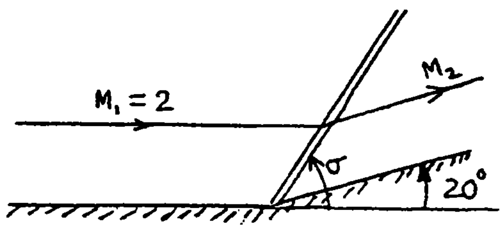
$$u_2 = 0.515(505.28) = 260.2 \text{ m/s}$$

Velocity downstream of shock = $u_1 - u_2 = 861 - 260.2 = 601$ m/s

Exercise 16.6

From Fig 15.15, no deflection angle of 60° is possible. But a deflection angle of 40° is possible.

Exercise 16.7



From Fig 15.15, the shock angle is $\sigma = 53°$

$\therefore \quad M_{n1} = M_1 \sin\sigma = 2 \sin(53°) = 1.6$

$\therefore \quad M_{n2} = 0.668$ (From Table 15.2)

Since $M_{n2} = M_2 \sin(\sigma - \delta)$, we get $M_2 = 0.668/\sin(53° - 20°) = 1.227$

By weak shock theory

From Section 15.9, we get

$M_1^2\sin^2\sigma - 1 = \dfrac{M_1^2(\gamma+1)}{2\sqrt{M_1^2-1}}\delta = 2^2(2.4)(20\pi/180)/[2\sqrt{(2^2-1)}] = 0.9674$

$\therefore \quad \sin^2\sigma = 1.9674/2^2 = 0.4918 \longrightarrow \sigma = 44.5^\circ$

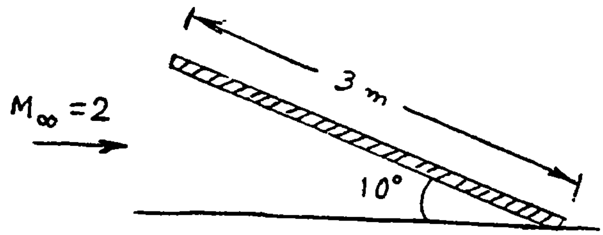
$M_{n1} = M_1\sin\sigma = 2\sin(44.5^\circ) = 1.4$

$\therefore \quad M_{n2} = 0.74$ (From Table 15.2)

$M_{n2} = M_2\sin(\sigma-\delta) \longrightarrow M_2 = 0.74/\sin(44.5^\circ - 20^\circ) = 1.784$

Error = (1.784 - 1.227)/1.227 = 45%

Exercise 16.8



From equation (15.45), the lift and drag forces are

$L = 2\alpha\gamma M_\infty^2 p_\infty b/\sqrt{M_\infty^2 - 1} = 2(10\pi/180)(1.4)(2)^2(101\text{x}1000)(3)/\sqrt{2^2 - 1}$
$= 342{,}000$ N

$D = \alpha L = (10\pi/180)L = 59.7$ N

Exercise 16.9

$$\text{Mass/time} = \dot{m} = \rho u A = \frac{p}{RT} u A = \frac{p}{p_0}\left(\frac{T_0}{T}\right)^{1/2} \frac{u}{\sqrt{\gamma R T}} \sqrt{\frac{\gamma}{R}} A \text{ (perfect gas)}$$
$$= \sqrt{\frac{\gamma}{R}} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{2} - \frac{\gamma}{\gamma - 1}} \cdot M \cdot A,$$

from Eqs. (16.20), (16.21).

Differentiate with respect to M for extremum of $\dot{m}$:

$$\frac{d\dot{m}}{dM} = 0 = \sqrt{\frac{\gamma}{R}} \cdot A \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-(\gamma+1)/2(\gamma-1)}$$
$$- \sqrt{\frac{\gamma}{R}} \cdot A.M \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-((\gamma+1)/(2(\gamma-))-1} \cdot \frac{\gamma + 1}{2(\gamma - 1)} \cdot (\gamma - 1) M.$$

Multiply by reciprocal of first term:

$$0 = 1 - \frac{(\gamma - 1) M^2}{1 + [(\gamma - 1)/2] M^2} \cdot \frac{\gamma + 1}{2(\gamma - 1)}, \qquad 1 + \frac{\gamma - 1}{2} M^2 = \frac{\gamma + 1}{2} M^2.$$

Solve: $M^2 = 1$ for max $\dot{m}$. Substitute into $\dot{m}$

$$\dot{m}_{\max} = \sqrt{\frac{\gamma}{R}} \left(\frac{\gamma + 1}{2}\right)^{-(\gamma+1)/2(\gamma-1)} \cdot A.$$

Exercise 16.10

From Section 8,

$$\frac{M_2}{M_1} = \frac{1+\gamma M_2^2}{1+\gamma M_1^2 - f}\left[\frac{1+[(\gamma-1)/2]M_1^2+q}{1+[(\gamma-1)/2]M_2^2}\right]^{1/2}$$

For maximum values of f or q, the flow is choked at the exit so that $M_2 = 1$.

(a) $f = 0$, $\quad \dfrac{1}{M_1} = \dfrac{1+\gamma}{1+\gamma M_1^2}\left[\dfrac{1+[(\gamma-1)/2]M_1^2+q}{(\gamma+1)/2}\right]^{1/2}$

Solve for q. First, clear the square root.

$$\left[\frac{1+\gamma M_1^2}{(\gamma+1)M_1}\right]^2 = \left(1+\frac{\gamma-1}{2}M_1^2+q\right)\Big/\left(\frac{\gamma+1}{2}\right),$$

$$q = \frac{\gamma+1}{2}\left[\frac{1+\gamma M_1^2}{(\gamma+1)M_1}\right]^2 - \left(1+\frac{\gamma-1}{2}M_1^2\right).$$

Note $q = 0$ when $M_1 = 1$. This can be simplified slightly to

$$q = \frac{1}{2(\gamma+1)}\left[\frac{1+\gamma M_1^2}{M_1}\right]^2 - \left(1+\frac{\gamma-1}{2}M_1^2\right).$$

(b) $q = 0, M_2 = 1$

$$\frac{1}{M_1} = \frac{\gamma+1}{1+\gamma M_1^2 - f}\left[\frac{1+[(\gamma-1)/2]M_1^2}{(\gamma+1)/2}\right]^{1/2},$$

$$1+\gamma M_1^2 - f = (\gamma+1)M_1\left(\frac{2}{\gamma+1}\right)^{1/2}\left(1+\frac{\gamma-1}{2}M_1^2\right)^{1/2},$$

$$f = 1+\gamma M_1^2 - [2(\gamma+1)]^{1/2}\left(1+\frac{\gamma-1}{2}M_1^2\right)^{1/2}.$$

Note $f = 0$ when $M_1 = 1$.

Exercise 16.11

From thin airfoil theory,

$$\frac{p - p_\infty}{p_\infty} = \frac{\gamma M_\infty^2 \delta}{\sqrt{M_\infty^2 - 1}},$$

where δ is the slope of the streamline at the body.

$$y_u = t \sin\frac{\pi x}{c}, \qquad \frac{dy_u}{dx} = \frac{t\pi}{c}\cos\frac{\pi x}{c} = \tan\delta = \delta$$

since $\delta \ll 1$.

$$\frac{p_u - p_\infty}{p_\infty} = \frac{\gamma M_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{t\pi}{c}\cos\frac{\pi x}{c}$$

$$y_L = 0 \quad \text{so} \quad \frac{p_L - p_\infty}{p_\infty} = 0$$

$$\text{Drag} = D = \int (p_u - p_L)\, d\mathbf{S}\cdot\mathbf{i}, \quad d\mathbf{S}\cdot\mathbf{i} = \frac{dx}{\cos\delta}\sin\delta = dx\tan\delta = \delta\, dx$$

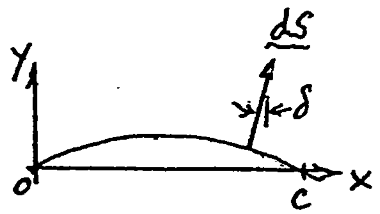
$$D = \frac{c}{\pi}\int_0^c dx \frac{\pi}{c}\frac{\gamma M_\infty^2 p_\infty}{\sqrt{M_\infty^2 - 1}}\left(\frac{t\pi}{c}\cos\frac{\pi x}{c}\right)^2$$

$$= \frac{c}{\pi}\int_0^\pi \frac{\gamma M_\infty^2 p_\infty}{\sqrt{M_\infty^2 - 1}}\left(\frac{t\pi}{c}\right)^2 \cos^2\xi\, d\xi, \quad \text{where } \xi = \frac{\pi x}{c}, \quad \int_0^\infty \cos^2\xi\, d\xi = \frac{\pi}{2},$$

$$D = \frac{c}{2}\frac{\gamma M_\infty^2 p_\infty}{\sqrt{M_\infty^2 - 1}}\left(\frac{t\pi}{c}\right)^2.$$

$$\text{Lift} = L = \int (p_L - p_u)\, d\mathbf{S}\cdot\mathbf{j}, \qquad d\mathbf{S}\cdot\mathbf{j} = \frac{dx}{\cos\delta}\cdot\cos\delta = dx$$

$$L = -\int_0^c \frac{\gamma M_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{t\pi}{c}\cos\frac{\pi x}{c}\, dx = 0.$$



The reason for these results is that $p > p_\infty$ on top for $0 < x < c/2$ and $p < p_\infty$ on top for $c/2 < x < c$ with an average value of p_∞. Thus no lift results. However with a higher pressure on the front half and a lower pressure on the back half, a drag is produced.

Exercise 16.12

For a volume V containing only fluid particles bounded by a surface $A = \partial V$, conservation of momentum may be written as

$$\frac{d}{dt}\int_V \rho\mathbf{u}\, dV = -\int_{A=\partial V} [\rho\mathbf{u}\mathbf{u} + p\mathbf{I} - \boldsymbol{\tau}] \cdot d\mathbf{A} + \int_V \rho\mathbf{g}\, dV.$$

Apply this to a small "pillbox" like that shown in Fig. 4.22. From mass conservation $d/dt \int_V \rho\, dV = -\int_{A=\partial V} \rho\mathbf{u} \cdot d\mathbf{A}$. Let the endfaces 1 and 2 remain on opposite sides of the shock surface but let $V \to 0$ as the distance between A_1 and $A_2 \to 0$. Then $A_{sides} \to 0$ as well and $\int_{A_1} \rho\mathbf{u} \cdot d\mathbf{A}_1 + \int_{A_2} \rho\mathbf{u} \cdot d\mathbf{A}_2 = 0$. A_1 and A_2 are equal and opposite in direction. Thus $\rho_1\mathbf{u}_1 \cdot \mathbf{n} = \rho_2\mathbf{u}_2 \cdot \mathbf{n}$. Similarly, momentum conservation reduces to

$$[\rho\mathbf{u}\mathbf{u} + p\mathbf{I} - \boldsymbol{\tau}]_1 \cdot \mathbf{n} = [\rho\mathbf{u}\mathbf{u} + pI - \boldsymbol{\tau}]_2 \cdot \mathbf{n}.$$

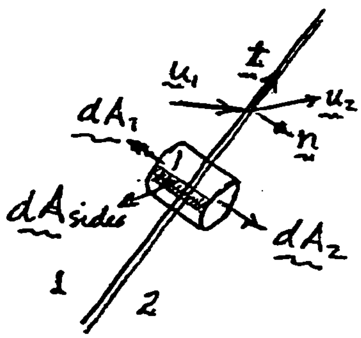
We also require that the endfaces A_1 and A_2 are outside of the structure so that $\boldsymbol{\tau}_1 = 0$, $\boldsymbol{\tau}_2 = 0$. In between (within the structure) $\boldsymbol{\tau}$ may be large. Then

$$\mathbf{u}_1(\rho\mathbf{u})_1 \cdot \mathbf{n} + p_1\mathbf{n} = \mathbf{u}_2(\rho\mathbf{u})_2 \cdot \mathbf{n} + p_2\mathbf{n}.$$

Take $\mathbf{t}$ (tangential component, $\mathbf{t} \cdot \mathbf{n} = 0$).

$$\mathbf{t} \cdot \mathbf{u}_1[(\rho\mathbf{u})_1 \cdot \mathbf{n}] = \mathbf{t} \cdot \mathbf{u}_2[(\rho\mathbf{u})_2 \cdot \mathbf{n}].$$

Using mass conservation across the shock, we obtain $\mathbf{t} \cdot \mathbf{u}_1 = \mathbf{t} \cdot \mathbf{u}_2$. Tangential component of $\mathbf{u}$ is unchanged across a shock.



Problem 1(a).

From Equation (17.17),

$$\dot{Q} = -\frac{\pi a^4}{8\mu}\left(\frac{dp}{dx}\right)$$

Therefore, $\dfrac{dp}{dx} = \dfrac{8\mu}{\pi a^4}\dot{Q}$.

Integrating,

$$p(L) = p(0) - \frac{8\mu}{\pi}\dot{Q}\int_0^L \frac{dx}{[a(x)]^4},$$

where $p(0)$ is the pressure at the entrance of the tube.

Problem 1(b). Again, $\dot{Q} = -\dfrac{\pi a^4}{8\mu}\left(\dfrac{dp}{dx}\right)$. Here,

$a(x) = a_0 + \dfrac{\alpha p}{2}$. Therefore, $\left(\dfrac{da}{dx}\right) = \dfrac{\alpha}{2}\left(\dfrac{dp}{dx}\right)$.

Then,

$\dot{Q} = -\dfrac{\pi a^4}{4\mu\alpha}\left(\dfrac{da}{dx}\right)$. Note that $\dot{Q}$ is constant in this flow. Therefore, integrating

$\dot{Q}x = -\dfrac{\pi}{20\mu\alpha}a^5 + \text{Constant}.$

Values of $a(x)$ at $x = 0$ and $x = L$ are given as $a(0)$ and $a(L)$.

When $x = 0, a = a(0)$, which implies Constant $= \frac{\pi}{20\mu\alpha}[a(0)]^5$. Therefore,

$$\dot{Q}x - \frac{\pi}{20\mu\alpha}[a(0)]^5 = -\frac{\pi}{20\mu\alpha}a^5.$$

When $x = L, a = a(L)$. Therefore,

$$\dot{Q}L - \frac{\pi}{20\mu\alpha}[a(0)]^5 = -\frac{\pi}{20\mu\alpha}[a(L)]^5.$$

or,

$$\begin{aligned} \frac{20\mu\alpha L}{\pi}\dot{Q} &= [a(0)]^5 - [a(L)]^5 \\ \dot{Q} &= \frac{\pi}{20\mu\alpha L}\left\{[a(0)]^5 - [a(L)]^5\right\} \end{aligned}$$

Problem 2.

$\Delta p = f(L, a, \rho, \mu, \omega, U)$

$\Delta p, L, a, \rho, \mu, \omega, U;\ n = 7$, dimensional parameters.

Primary dimensions: $M, L, t; r = 3$, primary dimensions. Now,

$[\Delta p] = ML^{-1}t^{-2},\ [L] = L,\ [a] = L,\ [\rho] = ML^{-3},\ [\mu] = ML^{-1}t^{-1},\ [\omega] = t^{-1},\ [U] = Lt^{-1}$.

Selecting repeating parameters: $a,\ \rho,\ U,\ m = r = 3$ repeating parameters.

Then, $n - m = 4$ dimensionless groups will result.

Setting up the dimensional equations, we have:

$$\begin{aligned} \Pi_1 &\equiv a^{C_1}\rho^{C_2}U^{C_3}L^{C_4} \\ &\equiv L^{C_1}(ML^{-3})^{C_2}(Lt^{-1})^{C_3}L^{C_4} \end{aligned}$$

Therefore,

$$\begin{aligned} L:&\quad C_1 - 3C_2 + C_3 + C_4 &= 0 \\ M:&\quad C_2 &= 0 \\ t:&\quad -3C_3 &= 0 \end{aligned}$$

Therefore, $C_2 = 0 = C_3$ and $C_1 = -C_4$. Thus,

$$\Pi_1 \equiv \left(\frac{L}{a}\right)^{C_4}. \tag{1}$$

Next,

$$\begin{aligned} \Pi_2 &\equiv a^{C_5}\rho^{C_6}U^{C_7}\mu^{C_8} \\ &\equiv L^{C_5}(ML^{-3})^{C_6}(Lt^{-1})^{C_7}\left(ML^{-1}t^{-1}\right)^{C_8} \end{aligned}$$

Therefore,

$$\begin{aligned} L:&\quad C_5 - 3C_6 + C_7 - C_8 &= 0 \\ M:&\quad C_6 + C_8 &= 0 \\ t:&\quad -C_7 - C_8 &= 0 \end{aligned}$$

Therefore, $C_6 = -C_8, C_7 = -C_8$, and $C_5 = -C_8$. Thus,

$$\Pi_2 \equiv \left(\frac{\mu}{a\rho U}\right)^{C_8}. \tag{2}$$

Next,

$$\begin{aligned}\Pi_3 &\equiv a^{C_9}\rho^{C_{10}}U^{C_{11}}\omega^{C_8}\\ &\equiv L^{C_9}(ML^{-3})^{C_{10}}(Lt^{-1})^{C_{11}}\left(t^{-1}\right)^{C_{12}}\end{aligned}$$

Therefore,

$$\begin{aligned} L: &\quad C_9 - 3C_{10} + C_{11} &= 0\\ M: &\quad C_{10} &= 0\\ t: &\quad -C_{11} - C_{12} &= 0\end{aligned}$$

Therefore, $C_{10} = 0, C_{11} = -C_{12}$, and $C_9 = -C_{12}$. Thus,

$$\Pi_3 \equiv \left(\frac{a\omega}{U}\right)^{C_9}. \tag{3}$$

Next,

$$\begin{aligned}\Pi_4 &\equiv a^{C_{13}}\rho^{C_{14}}U^{C_{15}}\Delta p^1\\ &\equiv L^{C_{13}}(ML^{-3})^{C_{14}}(Lt^{-1})^{C_{15}}\left(ML^{-1}t^{-2}\right)^1\end{aligned}$$

Therefore,

$$\begin{aligned} L: &\quad C_{13} - 3C_{14} + C_{15} - 1 &= 0\\ M: &\quad C_{14} + 1 &= 0\\ t: &\quad -C_{15} - 2 &= 0\end{aligned}$$

Therefore, $C_{15} = -2, C_{14} = -1$, and $C_{13} = 0$. Thus,

$$\Pi_4 \equiv \left(\frac{\Delta p}{\rho U^2}\right). \tag{4}$$

From 1-4, we get

$$\frac{\Delta p}{\rho U^2} \equiv C\left(\frac{L}{a}\right)^{C_4}\left(\frac{\mu}{a\rho U}\right)^{C_8}\left(\frac{a\omega}{U}\right)^{C_9}. \tag{5}$$

where, C, C_4, C_8, C_9 are constants.

Therefore, we may write

$$\frac{\Delta p}{\rho U^2} = C_1\left(\frac{L}{a}\right)^{C_2}(Re)^{C_3}(St)^{C_4}. \tag{6}$$

where, C_1, C_2, C_3, C_4 are constants.

Problem 3.

Morgan and Young invoke the following assumptions: (1). A collar-like stenosis may be approximated by a smooth, rigid, axisymmetric constriction in a long straight tube, (2). The effect of the stenosis geometry is dominant and any influence of wall distensibility is negligible, (3). Blood can be treated as a Newtonian fluid at the flow rates encountered in the large arteries where stenosis commonly occur, (4). Blood flow is laminar, and (5). Steady flow assumption is acceptable although the arterial blood flow is pulsatile.
The stenosis is shown in the following figure: The dimensional variables are designated by

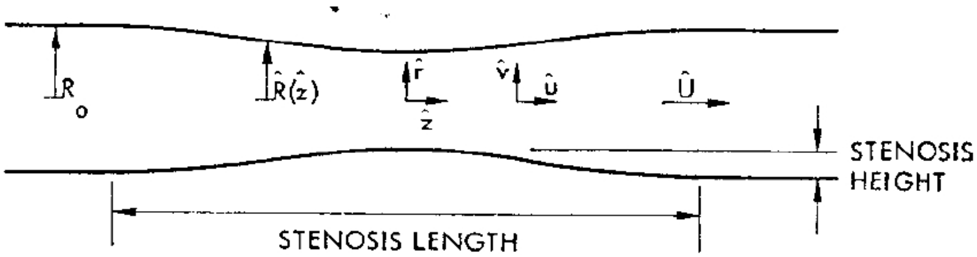


Figure 1: Axisymmetric stenosis.

$\hat{}$. The axial coordinate and velocity are $\hat{z}$ and $\hat{u}$. $\hat{U}$ is the centerline velocity. The radial coordinate is $\hat{r}$ and the corresponding velocity component is $\hat{v}$. The dimensionless variables are: $r = \hat{r}/R_0$, $z = \hat{z}/R_0$, $R = \hat{R}/R_0$, $u = \hat{u}/\bar{U}_0$, $v = \hat{v}/\bar{U}_0$, $U = \hat{U}/\bar{U}_0$, and $p = \hat{p}/\rho\bar{U}_0{}^2$, where, $\bar{U}_0$ is the average velocity in the unobstructed tube, and $\hat{p}$ is the pressure and ρ is the density.
The usual continuity, axial momentum, and radial momentum equations for axisymmetric flow in a tube are considered. The dimensionless momentum equations involve the Reynolds number, $Re = 2R_0\bar{U}_0\rho/\mu$, where μ is the fluid viscosity. By integrating the axial momentum equation over the cross section of the tube and by invoking the no slip condition, $u = v = 0$ at the wall of the vessel, the integral momentum equation is developed. Next, by multiplying the axial momentum equation by ru, and integrating over the cross section, the integral energy equation is developed. It is now noted that in these two equations, the terms due to the viscous component of the normal stress in the axial direction $(\partial^2 u/\partial z^2)$ are negligible.
Next, an important observation is made in regard to the pressure gradient $\partial p/\partial z$. It is noted that the pressure gradient $\partial p/\partial z$ is independent of the radial coordinate r for flow in a straight tube, whereas for flow in a constricted tube the pressure gradient will, in general, vary over the cross section. However, if the integral energy equation is multiplied by R^2 the resulting integral involving the pressure gradient is equal to the corresponding integral in the integral momentum equation for two special cases: (i) $\partial p/\partial z$ is independent of r; (ii) the velocity u is independent of r. Since these two conditions are approached in a constriction, i.e. in the gradually varying initial portion of the constriction the pressure gradient is nearly constant and in the rapidly converging portion the velocity profile tends to become flattened, we may let,

$$\int_0^R r\frac{\partial p}{\partial z}dr \approx R^2 \int_0^R ru\frac{\partial p}{\partial z}dr. \tag{1}$$

This approximation would enable the elimination of the pressure gradient term in both the integral momentum and integral energy equations. Then it is possible to combine the integral momentum and energy equations into a single equation in terms of axial velocity:

$$\frac{1}{2}R^2\frac{\partial}{\partial z}\int_0^R ru^3dr - \frac{\partial}{\partial z}\int_0^R ru^2dr = -\frac{2}{Re}\left[R^2\int_0^R r\left(\frac{\partial u}{\partial r}\right)^2 dr + R\left(\frac{\partial u}{\partial r}\right)\bigg|_R\right]. \quad (2)$$

The equation (2) is subject to (i) $u = U$ at $r = 0$; (ii) $u = 0$ at $r = R$; (iii) $\partial u/\partial r = 0$ at $r = 0$; (iv) $\partial^3 u/\partial r^3 = 0$ at $r = 0$; and, (v) the condition that the net flow through any cross section must be the same for any incompressible fluid which may be expressed by:

$$\int_0^R r\ u\ dr = \frac{1}{2}. \quad (3)$$

In the above, the condition (iii) is derived from a consideration of the forces on a cylindrical element having its axis along the tube centerline. If the pressure and the inertial forces are to be finite as the radius of the element approaches zero, the viscous force, which is proportional to $\partial u/\partial r$, must approach zero. The condition (iv) is developed by eliminating the pressure between the axial momentum and radial momentum equations and considering the resulting equation as r approaches zero.

Next, it is noted that at high Reynolds numbers, the profile must allow a thin region of high shear near the wall in the converging section with a relatively flat profile in the core. To accommodate these requirements, Morgan and Young construct a polynomial fit which permits the shear near the wall to become large while maintaining a flat core flow. This fit is given by,

$$u = U \text{ for } 0 \leq \frac{r}{R} \leq \lambda, and, u = a + b\left(\frac{r}{R}\right)^2 + c\left(\frac{r}{R}\right)^4 \text{ for } \lambda \leq \frac{r}{R} \leq 1, \quad (4)$$

where a, b, and c are unknown coefficients and λ is the value of (r/R) at the juncture separating the flat and polynomial parts of the profile. The unknown coefficients are determined from the no slip condition along with two compatibility conditions $u = U$ and $\partial u/\partial r = 0$ at $(r/R) = \lambda$. The constraint (v) enables expressing λ in terms of R and U. Thus the polynomial fit profile for u is entirely in terms of U, r, and R. The profile is now introduced into the equation (2). The resulting first order, non-linear ordinary differential equation is numerically solved by assuming that Poiseuille flow prevails far upstream of the stenosis. The solution provides the desired velocity profiles. These are plotted by Morgan and Young. The wall shear stress is evaluated from

$$\hat{\tau}_w = \mu\left(\frac{\partial \hat{u}}{\partial \hat{r}}\right)\bigg|_{\hat{R}}\left[1 + \left(\hat{R}'\right)^2\right], \quad (5)$$

where $\hat{R}'$ is the slope $d\hat{R}/d\hat{z}$ of the wall. The results are included in the paper by Morgan and Young.

Problem 4.

In a pure pressure-gravity flow, the effects of friction are negligible compared with the effects due to changes of the external pressure and the elevation in the gravity field. Lengthwise alterations of speed, pressure, area, etc., are brought about by changes in p_e and z. Changes in p_e may be brought about by : (i) active muscle tone; (ii) elastic constrictions, or sphincters, as where veins pass from the abdominal cavity to the thoracic cavity, and in the pulmonary system; (iii) weights, (iv) pressurizing cuffs, and (v) clamps etc.,. Changes in z are important because (i) in the vertical position of a human being, the hydrostatic pressure exceeds the venous and arterial pressure levels;(ii) during aircraft maneuvers; and, (iiI) during certain phases of space flight when effective g may be greatly increased.
Retaining only the terms in $d(p_e + \rho g z)$ in the table of influence coefficients for the tube law,

$$-\mathcal{P} \approx \alpha^{-n} - 1, \;\; \text{and,} \;\; n = \frac{3}{2}, \tag{1}$$

the governing equations of a pure pressure-gravity flow are written as,

$$\left(1 - S^2\right) \frac{1}{\alpha} \frac{d\alpha}{dx} = -\frac{2}{3} \alpha^{\frac{3}{2}} \frac{d\Pi}{dx}, \tag{2}$$

and,

$$\left(1 - S^2\right) \frac{1}{S^2} \frac{dS^2}{dx} = +\frac{1}{3} \alpha^{\frac{3}{2}} \frac{d\Pi}{dx}, \tag{3}$$

where,

$$\Pi = \frac{(p_e + \rho g z)}{K_p}. \tag{4}$$

Dividing equation (3) by equation (2) and integrating, for a pure pressure-gravity flow we have,

$$\frac{\alpha}{\alpha^*} = \frac{A}{A^*} = \left(\frac{u}{u^*}\right)^{-1} = S^{-4}, \tag{5}$$

where, * denotes the value of the quantity at $S = 1$. For the tube law given by equation (1), the wave speed is known to be,

$$c = \left[\frac{n K_p \alpha^{-n}}{\rho}\right]^{\frac{1}{2}}, \; n = \frac{3}{2}. \tag{6}$$

From equations(5) and (6),

$$\frac{c}{c^*} = \left(\frac{\alpha}{\alpha^*}\right)^{-\frac{3}{4}} = S^3. \tag{7}$$

Also, from Bernoulli's theorem,

$$\frac{(p - p^*)}{\frac{1}{2}\rho c^{*2}} = 1 - S^8 + \frac{g\,(z^* - z)}{c^{*2}}. \tag{8}$$

With equation (8), and by the elimination of α from equation (3) using equation (5), Shapiro develops,

$$\frac{1}{3} \alpha^{*\frac{3}{2}} d\Pi = S^4 (1 - S^2) dS^2. \tag{9}$$

Integration of equation (9) between limits Π and Π^* corresponding to S and 1, gives

$$\frac{1}{3}\alpha^{*\frac{3}{2}}\left(\Pi - \Pi^*\right) = \frac{1}{3}\left(S^6 - 1\right) - \frac{1}{4}\left(S^8 - 1\right). \tag{10}$$

Shapiro provides graphs of equation(10). The graphs show that increasing values of Π drive S towards unity and decreasing values of Π drive S away from unity. From these, it is concluded that: (i) Choking occurs when the value of Π continually increases with either $S < 1$ or $S > 1$; (ii) Continuous transition through $S = 1$ may be achieved by means of an increase in Π until $S = 1$ is reached, with $d\Pi/dx$ becoming zero exactly at $S = 1$ followed then by a decrease of Π.

Problem 5.

To determine the volume flux for the flow with the Power-law model, consider an annular volume element of length L and thickness dr in the flow, as shown in the figure below:

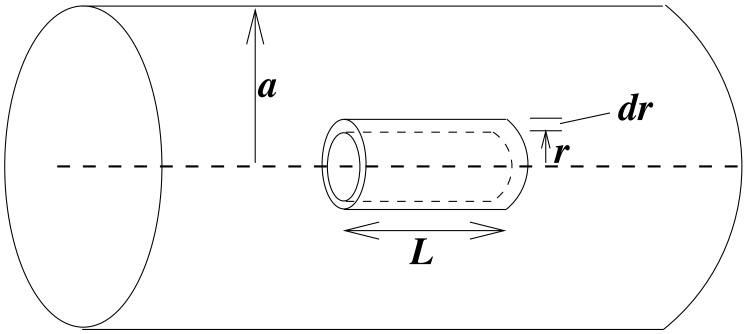


Figure 2: Representative volume element.

Let there be a constant pressure gradient, $-\Delta p/L$, across the element of length L.

$$\text{Force due to pressure gradient on the annular element } = +\frac{\Delta p}{L}(2\pi r L)\, dr. \tag{1}$$

This must be balanced by shear force which is given by,

$$\text{Shear force } = (2\pi dr)\, L\frac{d}{dr}(r\tau). \tag{2}$$

Therefore,

$$\frac{d}{dr}(r\tau) = \frac{\Delta p}{L} r \tag{3}$$

$$\tau = \frac{\Delta p}{L}\frac{r}{2} + \frac{c}{2} \tag{4}$$

where c is a constant. The stress is finite at axis $r = 0$. Therefore $c = 0$ and hence, from 4,

$$\tau = \frac{\Delta p}{L}\frac{r}{2}. \tag{5}$$

For a power-law fluid,

$$\tau = \mu\dot{\gamma}^n. \tag{6}$$

$$\text{Therefore } \dot{\gamma} = \left(\frac{\Delta p}{L}\frac{r}{2\mu}\right)^{\frac{1}{n}}. \tag{7}$$

In this problem, since velocity is a function of r only,

$$\dot{\gamma} = -\frac{du}{dr}. \tag{8}$$

where $u(r)$ is velocity component in r direction.
From equations 7 and 8,

$$\frac{du}{dr} = -\left(\frac{\Delta p}{L}\frac{r}{2\mu}\right)^{\frac{1}{n}}. \tag{9}$$

The no-slip condition at the wall requires that

$$u = 0 \text{ at } r = a. \tag{10}$$

By integrating equation 9 and using equation 10, one obtains

$$u = \left(\frac{\Delta p}{2\mu L}\right)^{1/n} \frac{n}{n+1}\left(a^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right). \tag{11}$$

Now, the flux for flow, Q, is given by

$$Q = \int_0^R 2\pi r u dr. \tag{12}$$

With integration by parts and invoking the no-slip conditions at $r = a$,

$$Q = \pi \int_0^a r^2 \left(-\frac{du}{dr}\right) dr. \tag{13}$$

With equation 11,

$$Q = \pi \int_0^a r^2 \left(\frac{\Delta p}{L}\frac{r}{2\mu}\right)^{1/n} dr \tag{14}$$

$$= \left(\frac{\Delta p}{2\mu L}\right)^{1/n} \frac{n\pi}{3n+1} a^{\frac{3n+1}{n}} \tag{15}$$

When $n = 1$,

$$Q = \left(\frac{\Delta p}{2\mu L}\right)\frac{\pi}{4}a^4 \tag{16}$$

$$= \left(\frac{\pi a^4}{8\mu}\right)\frac{\Delta p}{L} a^{\frac{3n+1}{n}}$$

which agrees with equations (17.17) and is the Poiseuille formula.

Problem 6.

To determine the volume flux for the flow with Herschel-Bulkley model, we first note that

$$\begin{aligned} \tau &= \mu\dot{\gamma}^n + \tau_0 \quad , \quad \tau \geq \tau_0 \\ \text{and} \qquad \dot{\gamma} &= 0 \qquad\qquad , \quad \tau < \tau_0 \end{aligned} \tag{1}$$

There is yield stress and as a consequence the flow region includes plug flow in the core as shown in figure Let the radius of the plug flow region be r_p.

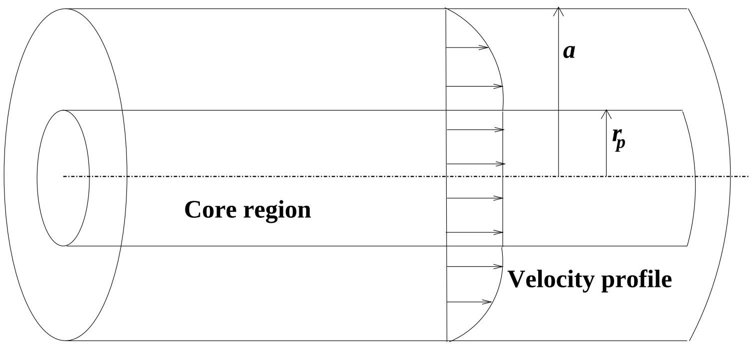


Figure 3: Representative volume element.

$$\text{For } \; r \leq r_p, \qquad \tau(r) = \tau_0 = \text{constant} \tag{2}$$

$$\text{Therefore in the core, } \; \dot{\gamma} = 0. \tag{3}$$

$$\text{Therefore in the core, } \; \frac{du}{dr} = 0. \tag{4}$$

Therefore,

$$\begin{aligned} u &= \text{constant} && (5) \\ &= u_p \text{ (say)} && (6) \\ & && (7) \end{aligned}$$

For a constant pressure gradient, $-(\Delta p/L)$, across an element of length L in the core of region, from a force balance,

$$2\pi r_p L \tau_0 = \frac{\Delta p}{L}\pi {r_p}^2 L \tag{8}$$

$$\text{Thus,} \qquad r_p = \frac{2\tau_0}{(\Delta p/L)} \tag{9}$$

$$\tau_0 = \frac{\Delta p}{L}\frac{r_p}{2} \tag{10}$$

Outside the core region, with $u = u(r)$, just as in problem 5,

$$\frac{du}{dr} = -\left(\frac{\Delta p}{2\mu L}\right)^{\frac{1}{n}} (r - r_p)^{\frac{1}{n}} . \tag{11}$$

By integration,

$$u = -\left(\frac{\Delta p}{2\mu L}\right)^{1/n} \frac{n}{n+1} \left(r - r_p\right)^{\frac{n+1}{n}} + C. \tag{12}$$

where C is a constant of integration.
Now, at $r = a$, $u = 0$ (no-slip condition). Therefore,

$$u = \left(\frac{\Delta p}{2\mu L}\right)^{1/n} \frac{n}{n+1} \left((a - r_p)^{\frac{n+1}{n}} - (r - r_p)^{\frac{n+1}{n}}\right). \tag{13}$$

We can also calculate the plug velocity by setting $r = r_p$ in equation 12. Thus,

$$u_p = \left(\frac{\Delta p}{2\mu L}\right)^{1/n} \frac{n}{n+1} \left(a - r_p\right)^{\frac{n+1}{n}}. \tag{14}$$

The volume flux may be calculated from,

$$Q = \pi {r_p}^2 u_p + \int_{r_p}^{a} 2\pi r u dr. \tag{15}$$

With equations 13 and 14, we can evaluate equation 15.

$$\text{Let} \quad \xi = r_p/a. \tag{16}$$

Then, after integration and considerable algebra,

$$\begin{aligned} Q \quad = \quad & \left(\frac{\Delta p}{2\mu L}\right)^{1/n} \frac{n\pi}{n+1} a^{\frac{3n+1}{n}} \left[\xi^2 (1-\xi)^{\frac{n+1}{n}} + (1+\xi)(1-\xi)^{\frac{2n+1}{n}} \right. \\ & \left. - \frac{2n}{3n+1}(1-\xi)^{\frac{3n+1}{n}} - \frac{2n}{2n+1}\xi(1-\xi)^{\frac{2n+1}{n}}\right] \end{aligned} \tag{17}$$

When $\tau_0 = 0$, the Herschel-Bulkley model reduces to the Power-law model. Therefore, with $\tau_0 = 0$ and $\xi = 0$, equation 17 reduces to the result of problem 5.