Applications to Partial Differential Equations

Solution to Heat Equation

Fourier cosine and inverse Fourier cosine transform

$$\boxed{F_c(f) = \hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \alpha u \, \mathrm{d}u} \quad \boxed{F_c^{-1}(\hat{f}) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\alpha) \cos \alpha x \, \mathrm{d}\alpha}$$

Fourier sine and inverse Fourier sine transform

$$\boxed{F_s(f) = \hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \alpha u \, \mathrm{d}u} \quad \boxed{F_s^{-1}(\hat{f}) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\alpha) \sin \alpha x \, \mathrm{d}\alpha}$$

Derivative formula: Assuming that f and f' both goes to 0 as x approaches to ∞

$$\boxed{F_c\{f''(x)\} = -\alpha^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)} \quad \boxed{F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \alpha f(0) - \alpha^2 F_s\{f(x)\}}$$

Fourier transform

$$\boxed{F(f) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{i\alpha u} \, \mathrm{d}u} \quad \boxed{F^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} \, \mathrm{d}\alpha}$$

Derivative formula: Assuming that f and f' both goes to 0 as $|x|$ approaches to ∞

$$\boxed{F\{f''(x)\} = -\alpha^2 F\{f(x)\}}$$

Convolution property

$$\boxed{(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \mathrm{d}y} \quad \boxed{F\{(f * g)\} = \sqrt{2\pi} F\{f\} F\{g\}}$$

Problem: $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t};\ -\infty < x < \infty, t > 0$

BCs: $u(x,t)$ and $u_x(x,t)$ both $\to 0$ as $|x| \to \infty$

ICs: $u(x,0) = f(x), -\infty < x < \infty.$

Solution: Taking Fourier transform with respect to x

$$-k\,\alpha^2\,\hat{u}(\alpha,t) = \frac{d\hat{u}}{dt} \quad \Rightarrow \quad \frac{d\hat{u}}{dt} + k\alpha^2\hat{u}(\alpha,t) = 0$$

Note that BCs are already used.

$$\frac{d\hat{u}}{dt} + k\alpha^2\hat{u}(\alpha,t) = 0 \qquad \Longrightarrow \hat{u}(\alpha,t) = ce^{-k\alpha^2 t}$$

The Fourier transform of the initial condition $u(x,0) = f(x)$ gives:

$$\hat{u}(\alpha,0) = \hat{f}(\alpha)$$

We use this condition to get c as

$$\hat{f}(\alpha) = c \quad \Rightarrow \hat{u}(\alpha,t) = \hat{f}(\alpha)e^{-k\alpha^2 t}$$

Taking inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\alpha)e^{-k\alpha^2 t}e^{-i\alpha x}d\alpha$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-k\alpha^2 t} e^{-i\alpha x} d\alpha$$

Note that we would like to have $f(x)$ in the solution but not $\hat{f}(\alpha)$.

Product form $\hat{f}(\alpha)e^{-k\alpha^2 t}$ suggest that we can use convolution theorem.

Recall the convolution theorem: $F\{f * g\} = \sqrt{2\pi}\, \hat{f}(\alpha)\, \hat{g}(\alpha)$

Let $e^{-k\alpha^2 t}$ be the Fourier transform of $g(x)$. Then

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k\alpha^2 t}\, e^{-i\alpha x} d\alpha$$

Consider the Integral: $\quad I = \int_{-\infty}^{\infty} e^{-ax^2 - 2bx}\, dx$

$$I = \int_{-\infty}^{\infty} e^{-ax^2-2bx}\, dx = \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)^2+\frac{b^2}{a}}\, dx$$

$$= e^{\left(\frac{b^2}{a}\right)} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}\,x+\frac{b}{\sqrt{a}}\right)^2}\, dx$$

Substitute $\sqrt{a}x + \frac{b}{\sqrt{a}} = t \Rightarrow dx = \frac{dt}{\sqrt{a}}$

$$I = e^{\left(\frac{b^2}{a}\right)} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{a}} = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\left(\frac{b^2}{a}\right)}$$

$$\Longrightarrow \int_{-\infty}^{\infty} e^{-ax^2-2bx}\, dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\left(\frac{b^2}{a}\right)}$$

$$I = \int_{-\infty}^{\infty} e^{-ax^2-2bx}\,dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\left(\frac{b^2}{a}\right)}$$

Let $a = kt$ and $b = \dfrac{ix}{2}$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-kt\alpha^2-ix\alpha}\,d\alpha = \frac{\sqrt{\pi}}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}$$

Convolution Theorem:

$$\sqrt{2\pi}\, F^{-1}\left[\hat{f}(\alpha)\, \hat{g}(\alpha)\right] = f * g$$

$$\int_{-\infty}^{\infty} \hat{f}(\alpha)\, \hat{g}(\alpha)\, e^{-i\alpha x}\, d\alpha = f * g$$

Recall: $g(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-k\alpha^2 t}\, e^{-i\alpha x} d\alpha \quad \Rightarrow g(x) = \dfrac{1}{\sqrt{2\pi}} \dfrac{\sqrt{\pi}}{\sqrt{kt}} e^{-\frac{x^2}{4kt}} = \dfrac{1}{\sqrt{2kt}} e^{-\frac{x^2}{4kt}}$

To summarize: $u(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-k\alpha^2 t} e^{-i\alpha x} d\alpha = \dfrac{1}{\sqrt{2\pi}} f * g$ (convolution)

$$\hat{g}(\alpha) = e^{-k\alpha^2 t} \qquad\qquad g(x) = \frac{1}{\sqrt{2kt}} e^{-\frac{x^2}{4kt}}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\beta)g(x-\beta)\ d\beta$$

$$g(x) = \frac{1}{\sqrt{2kt}}e^{-\frac{x^2}{4kt}}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\beta)\frac{1}{\sqrt{2kt}}e^{-\left(\frac{(x-\beta)^2}{4kt}\right)}\ d\beta$$

Substituting $\quad z = -\frac{(x-\beta)}{\sqrt{4kt}} \Rightarrow\ dz = \frac{d\beta}{\sqrt{4kt}}$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f\left(x + \sqrt{4kt}\, z\right) e^{-z^2}\ dz$$

Example: $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \qquad 0 < x < \infty, t > 0.$

BCs: $u(0,t) = u_0, t \geq 0$ $\qquad$ u and $\dfrac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$

ICs: $u(x,0) = 0, 0 < x < \infty$

Solution: Since u is specified at $x = 0$ and $0 < x < \infty$, the Fourier sine transform is applicable

Taking Fourier sine transform,

$$\sqrt{\frac{2}{\pi}}\, k \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin(\alpha x)\, dx = \frac{d}{dt}\hat{u}_s(\alpha, t)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\, k \left[\frac{\partial u}{\partial x} \sin(\alpha x) \Big|_0^\infty - \alpha \int_0^\infty \frac{\partial u}{\partial x} \cos(\alpha x)\; dx \right] = \frac{d}{dt}\hat{u}_s$$

$$\Rightarrow k\sqrt{\frac{2}{\pi}}\left[-\alpha\left(u\cos(\alpha x)\right)\Big|_0^\infty+\int_0^\infty u\sin(\alpha x)\;(\alpha)\,dx\right]=\frac{d}{dt}\hat{u}_s$$

$$\Rightarrow k\sqrt{\frac{2}{\pi}}\left[\alpha\,u(0)-\alpha^2\int_0^\infty u\sin(\alpha x)\,dx\right]\;=\frac{d}{dt}\hat{u}_s$$

$$\Rightarrow k\alpha\sqrt{\frac{2}{\pi}}u_0-k\alpha^2\hat{u}_s(\alpha,t)\;=\frac{d}{dt}\hat{u}_s$$

$$\boxed{F_s\{f''(x)\}=\sqrt{\frac{2}{\pi}}\alpha f(0)-\alpha^2F_s\{f(x)\}}$$

$$\Rightarrow \frac{d}{dt}\hat{u}_s+k\alpha^2\hat{u}_s(\alpha,t)=\sqrt{\frac{2}{\pi}}\,k\,\alpha\,u_0$$

I.F.: $e^{k\alpha^2t}$

$$\hat{u}_s\ e^{k\alpha^2 t} = \sqrt{\frac{2}{\pi}} \int k\ \alpha\ u_0\ e^{k\alpha^2 t}\ dt + c$$

$$\hat{u}_s = \left(\sqrt{\frac{2}{\pi}} \frac{1}{\alpha} u_0 \int k\ \alpha^2 e^{k\alpha^2 t} dt \right) e^{-k\alpha^2 t} + c e^{-k\alpha^2 t}$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{1}{\alpha} u_0 e^{k\alpha^2 t} e^{-k\alpha^2 t} + c e^{-k\alpha^2 t}$$

$$\hat{u}_s = \sqrt{\frac{2}{\pi}}\ \frac{u_0}{\alpha} + c e^{-k\alpha^2 t}$$

Initial Condition: $u(x, 0) = 0\ \Rightarrow\ \hat{u}_s(\alpha, 0) = 0$

$$\Rightarrow \hat{u}_s(\alpha, 0) = 0 = \sqrt{\frac{2}{\pi}}\,\frac{u_0}{\alpha} + \mathrm{c} \quad \Rightarrow \mathrm{c} = -\sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha}$$

$$\Rightarrow \hat{u}_s(\alpha, t) = \sqrt{\frac{2}{\pi}}\,\frac{u_0}{\alpha}\,\left(1 - e^{-k\alpha^2 t}\right)$$

Taking inverse sine transform:

$$u(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_s\,(\alpha, t)\sin(\alpha x)\,d\alpha$$

$$= \frac{2}{\pi}u_0\int_0^\infty \frac{\sin(\alpha x)}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)\,d\alpha$$

Example: $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$ subject to the conditions

ICs: $u(x,0) = 0, \qquad x \geq 0$

BCs: $u_x(0,t) = -\mu \text{ (constant)}, \qquad t > 0.$

u and $\frac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$.

Solution: Since u_x is specified at $x = 0$, the Fourier cosine transform is applicable to this problem

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = kF_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = kF_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} \Rightarrow \frac{d}{dt}\hat{u}_c = k\left[-\sqrt{\frac{2}{\pi}}u_x(0,t) - \alpha^2 F_c\{u\}\right] \Rightarrow \frac{d\hat{u}_c}{dt} + k\alpha^2\hat{u}_c = \sqrt{\frac{2}{\pi}}\,k\mu$$

Integrating factor: $e^{K\alpha^2 t}$

$$\Longrightarrow \hat{u}_c e^{k\alpha^2 t} = \sqrt{\frac{2}{\pi}}\int k\,\mu\, e^{k\alpha^2 t}\,dt + c$$

$$\Rightarrow \hat{u}_c e^{k\alpha^2 t} = \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}e^{k\alpha^2 t} + c$$

Since, $u(x,0)=0 \Rightarrow \hat{u}_c(x,0)=0 \Rightarrow 0=\sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}+c$

$$\Rightarrow \hat{u}_c=\sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}+ce^{-k\alpha^2 t} = \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}\left(1-e^{-k\alpha^2 t}\right)$$

Taking inverse Fourier Transform:

$$u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_c(\alpha,t)\cos(\alpha x)\,d\alpha$$

$$=\frac{2}{\pi}\mu\int_0^\infty \frac{\cos(\alpha x)}{\alpha^2}\left(1-e^{-k\alpha^2 t}\right)d\alpha$$

Solution of Wave Equations

Problem: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty.$

ICs: $u(x,0) = f(x), \qquad -\infty < x < \infty, \qquad u_t(x,0) = 0, \qquad -\infty < x < \infty$

BCs: u and $\dfrac{\partial u}{\partial x}$ both tends to zero as $|x| \to \infty$

Solution: Taking Fourier transform of PDE, we have

$$\frac{d^2\hat{u}(\alpha,t)}{dt^2} = c^2\left(-\alpha^2\hat{u}(\alpha,t)\right)$$

$$\Rightarrow \frac{d^2\hat{u}}{dt^2} + c^2\alpha^2\hat{u}(\alpha,t) = 0$$

It's general solution $\hat{u}(\alpha, t) = c_1 \cos(c\alpha t) + c_2 \sin(c\alpha t)$

Fourier transform of initial condition,

$$u(x,0) = f(x) \Rightarrow \hat{u}(\alpha, 0) = \hat{f}(\alpha) \quad \Rightarrow c_1 = \hat{f}(\alpha)$$

$$u_t(x,0) = 0 \Rightarrow \frac{d\hat{u}(\alpha, 0)}{dt} = 0$$

$$\frac{d\hat{u}}{dt} = -c_1 \sin(c\alpha t)(c\alpha) + c_2 \cos(c\alpha t)(c\alpha)$$

$$\Rightarrow 0 = c_2\, c\alpha \Rightarrow c_2 = 0$$

$$\Rightarrow \hat{u}(\alpha, t) = \hat{f}(\alpha) \cos(c\alpha t)$$

Taking inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) \cos(c\alpha t) e^{-i\alpha x} d\alpha$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) \left\{ \frac{e^{ic\alpha t} + e^{-ic\alpha t}}{2} \right\} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha(x-ct)} d + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha(x+ct)} d\alpha \right]$$

$$= \frac{1}{2} [f(x-ct) + f(x+ct)]$$

This is known as **D'Alembert's solution** of the wave equation.

Problem: $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 < x < \infty, t > 0.$$

ICs: $u(x,0) = f(x), \qquad u_t(x,0) = g(x)$

BCs: $u(0,t) = 0 \qquad u$ and $\dfrac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$.

Solution: Taking Fourier Sine transform of PDE, we have

$$\frac{d^2 \hat{u}_s(\alpha,t)}{dt^2} = c^2 \left[\sqrt{\frac{2}{\pi}} \alpha \, u(0,t) - \alpha^2 \hat{u}_s(\alpha,t) \right]$$

$$\Rightarrow \frac{d^2\hat{u}_s(\alpha,t)}{dt^2} + \alpha^2 c^2\, \hat{u}_s(\alpha,t) = 0$$

Its general solution: $\hat{u}_s(\alpha,t) = c_1\cos(c\alpha t) + c_2\sin(c\alpha t)$

Initial conditions: $u(x,0) = f(x) \quad \Rightarrow \hat{u}_s(\alpha,0) = \hat{f}_s(\alpha) \quad \Rightarrow c_1 = \hat{f}_s(\alpha)$

$$u_t(x,0) = g(x) \quad \Rightarrow \frac{d\hat{u}_s(\alpha,0)}{dt} = \hat{g}_s(\alpha)$$

$$\frac{d\hat{u}_s}{dt} = -c_1\sin(c\alpha t)(c\alpha) + c_2\cos(c\alpha t)(c\alpha)$$

$$\Rightarrow \hat{g}_s(\alpha) = c_2(c\alpha)$$

$$\Rightarrow \hat{u}_s(\alpha,t) = \hat{f}_s(\alpha)\cos(c\alpha t) + \frac{\hat{g}_s(\alpha)}{c\alpha}\sin(c\alpha t)$$

$$\hat{u}_s(\alpha,t)=\widehat{f_s}(\alpha)\cos(c\alpha t)+\frac{\hat{g}_s(\alpha)}{c\alpha}\sin(c\alpha t) \qquad \Longrightarrow u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_s\,(\alpha,t)\sin(\alpha x)\,d\alpha$$

$$\Longrightarrow u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty [\widehat{f_s}(\alpha)\cos(c\alpha t)\sin\alpha x+\frac{\hat{g}_s(\alpha)}{c\alpha}\sin(c\alpha t)\sin(\alpha x)]d\alpha$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\widehat{f_s}(\alpha)}{2}[\sin(x+ct)\alpha+\sin(x-ct)\alpha]\,d\alpha+\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\hat{g}_s(\alpha)}{2c\alpha}[\cos(x-ct)\alpha-\cos(x+ct)\alpha]d\alpha$$

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\hat{g}_s(\alpha)}{2c\alpha}[\cos(x-ct)\alpha-\cos(x+ct)\alpha]d\alpha$$

Since $g(u)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{g}_s(\alpha)\ \sin(\alpha u)\,d\alpha \quad \Rightarrow \int_{x-ct}^{x+ct} g(u)\,du=\sqrt{\frac{2}{\pi}}\int_{x-ct}^{x+ct}\int_0^\infty \hat{g}_s(\alpha)\sin(\alpha u)\,d\alpha\,du$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{g}_s(\alpha)\int_{x-ct}^{x+ct}\sin(\alpha u)\ du\,d\alpha$$

$$\Rightarrow \int_{x-ct}^{x+ct} g\,(u)du=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{g}_s\,(\alpha)\left[-\frac{\cos\alpha u}{\alpha}\right]_{x-ct}^{x+ct} d\alpha\ =\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\hat{g}_s(\alpha)}{\alpha}[\cos(x-ct)\alpha-\cos(x+ct)\alpha]d\alpha$$

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\hat{g}_s(\alpha)}{2c\alpha}[\cos(x-ct)\alpha-\cos(x+ct)\alpha]d\alpha$$

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g\,(u)du.$$

Solution of Laplace Equation

Problem: $u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, y > 0$

Bcs: $u(x,0) = f(x), -\infty < x < \infty$

u is bounded as $y \to \infty$;

u and $\dfrac{\partial u}{\partial x}$ both tend to zero as $|x| \to \infty$

Solution: Taking Fourier transform with respect to x

$$-\alpha^2\, \hat{u}(\alpha, y) + \frac{d^2}{dy^2}\hat{u}(\alpha, y) = 0$$

Its solution: $\hat{u}(\alpha, y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y}$

$\hat{u}(\alpha, y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y}$

Since u is bounded as $y \to \infty$ $\Rightarrow \hat{u}(\alpha, y)$ must be bounded as $y \to \infty$

$\Rightarrow c_1 = 0$ for $\alpha > 0$, $\quad c_2 = 0$ if $\alpha < 0$.

Hence for any α: $\quad \hat{u}(\alpha, y) = ce^{-|\alpha|y}$

Using BC: $\quad \hat{u}(\alpha, 0) = \hat{f}(\alpha) \Rightarrow c = \hat{f}(\alpha)$

$\Rightarrow \hat{u}(\alpha, y) = \hat{f}(\alpha)e^{-|\alpha|y}$

$$\hat{u}(\alpha, y) = \hat{f}(\alpha)e^{-|\alpha|y} \Longrightarrow \quad u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}\,(\alpha)\, e^{-|\alpha|y}\, e^{-i\alpha x}\, d\alpha \;= F^{-1}\left[\hat{f}(\alpha)\, e^{-|\alpha|y}\right]$$

It does not look good to have solution in terms of $\hat{f}\,(\alpha)$. Let $g(x) = F^{-1}\{e^{-|\alpha|y}\}$.

Then, by convolution theorem: $F\{f * g\} = \sqrt{2\pi}\,\hat{f}\,(\alpha)\,\hat{g}(\alpha)$

$$\Rightarrow F^{-1}\{\hat{f}\,(\alpha)\hat{g}(\alpha)\} = \frac{1}{\sqrt{2\pi}}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\,(\beta) g(x - \beta) d\beta$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\alpha|y} e^{-i\alpha x}\, d\alpha = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\alpha y} \cos \alpha x \;\, d\alpha$$

$$\Longrightarrow g(x) = \sqrt{\frac{2}{\pi}}\underbrace{\int_0^\infty e^{-\alpha y}\ \cos(\alpha x)\, d\alpha}_{I}$$

$$\text{Let } I = \int_0^\infty e^{-\alpha y} \cos(\alpha x)\, d\alpha$$

$$\Longrightarrow I = \frac{e^{-\alpha y}}{-y}\cos(\alpha x)\Big|_0^\infty - \int_0^\infty \frac{e^{-\alpha y}}{-y}(-\sin(\alpha x))\, x\, d\alpha$$

$$= \frac{1}{y} - \frac{x}{y}\int_0^\infty e^{-\alpha y} \sin\,(\alpha x)\, d\alpha, \qquad y > 0$$

$$= \frac{1}{y} - \frac{x}{y}\left[\frac{e^{-\alpha y}}{-y}\sin(\alpha x)\Big|_0^\infty - \int_0^\infty \frac{e^{-\alpha y}}{-y}\cos(\alpha x)\, x\, d\alpha\right]$$

$$\mathrm{I} = \frac{1}{y} - \frac{x}{y}\left[\frac{e^{-\alpha y}}{-y}\sin(\alpha x)\Big|_0^\infty - \int_0^\infty \frac{e^{-\alpha y}}{-y}\cos(\alpha x)\, x\, d\alpha\right] = \frac{1}{y} - \frac{x}{y}\frac{x}{y} I$$

$$g(x) = \sqrt{\frac{2}{\pi}}\ I$$

$$\Rightarrow \mathrm{I} = \frac{1}{y}\frac{y^2}{x^2+y^2} = \frac{y}{x^2+y^2} \Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\left(\frac{y}{x^2+y^2}\right)$$

$$u(x,y) = F^{-1}\left[\hat{f}(\alpha)\, e^{-|\alpha| y}\right]$$

$$g(x) = F^{-1}\{e^{-|\alpha| y}\}.$$

$$\Rightarrow u(x,y) = F^{-1}\{\hat{f}(\alpha)e^{-|\alpha| y}\} = \frac{1}{\sqrt{2\pi}} f * g = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(\beta)\frac{y}{(x-\beta)^2+y^2}\, d\beta$$

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\beta)\frac{y}{(x-\beta)^2+y^2}\, d\beta$$

This solution is a well-known **Poisson integral** formula.

Problem: Solve two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad -\infty < x < \infty;\ 0 < y < \infty$$

Subject to the conditions: $u(x,0) = f(x)$, $\quad \frac{\partial u}{\partial y} = 0$ at $y = 0$

u and u_x both vanish as $|x| \to \infty$

Solution: Taking Fourier transform:

$$\frac{d^2}{dy^2}\hat{u}(\alpha, y) - \alpha^2 \hat{u}(\alpha, y) = 0$$

Its solution: $\hat{u}(\alpha, y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y}$

Now, $u(x,0) = f(x) \Rightarrow \hat{u}(\alpha, 0) = \hat{f}(\alpha) \Rightarrow \hat{f}(\alpha) = c_1 + c_2$

$$u_y = 0 \Rightarrow \frac{d}{dy}\hat{u}(\alpha, 0) = 0 = \{\alpha c_1 e^{\alpha y} - c_2 \alpha e^{-\alpha y}\}_{y=0} \quad \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$$

Hence $\hat{f}(\alpha) = c_1 + c_2 \Rightarrow \; c_1 = c_2 = \dfrac{\hat{f}(\alpha)}{2}$ Solution: $\hat{u}(\alpha, y) = \dfrac{\hat{f}(\alpha)}{2}[e^{\alpha y} + e^{-\alpha y}]$

Taking inverse Fourier transform,

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(\alpha)}{2} (e^{\alpha y} + e^{-\alpha y}) e^{-i\alpha x} d\alpha$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(\alpha)}{2} \left(e^{-i\alpha(x - iy)} + e^{-i\alpha(x + iy)}\right) d\alpha$$

$$= \frac{1}{2} [f(x - iy) + f(x + iy)]$$