PART – II

Laplace Transform (contd.), Fourier Series & Fourier Integral

Debapriya Biswas

Laplace Transform

1. **Differential equations with variable co-efficients**

We know that

$$\mathcal{L}\{t f(t)\} = -F'(s) = -\frac{d}{ds}[F(s)] \longrightarrow (1)$$

$\therefore$ with $f(t) = y' = \frac{dy}{dt}$ &

$$\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{y'\} = sY - y(0)$$

& subsequent product differentiation, we obtain

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds} \quad \left[\because \frac{dy(0)}{ds} = 0\right] \longrightarrow (2)$$

Similarly, by (1). with $f(t) = y''$, we find

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}[s^2 Y - sy(0) - y'(0)]$$

$$= -2sY - s^2\frac{dY}{ds} + y(0) \longrightarrow (3)$$

Hence, if a d.eqn has co-efficients such as $(at+b)$, we get a first-order d.eqn for Y, which is sometimes simpler than the given eqn. But if the latter has co-efficients (at^2+bt+c) we get, by two applications of eqn (1), a second-order d.eqn for Y, & this shows

shows that the Laplace Transform method works well only for very special eqns with variable co-efficients.

Eg. 1) Laguerre's d. eqn, Laguerre polynomials:

Laguerre's d. eqn is

$$ty'' + (1-t)y' + ny = 0 \longrightarrow (4)$$

We determine a solution of (4) with $n = 0, 1, 2, \cdots$

From eqns (2) & (3), we get

$$\mathcal{L}\{ty''\} + \mathcal{L}\{(1-t)y'\} + n\mathcal{L}\{y\} = 0.$$

$$\Rightarrow \left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0)$$

$$-\left[-Y - s\frac{dY}{ds}\right] + nY = 0,$$

which on simplification gives,

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

On separating variables, using partial fraction, integrating (with the constant of integration taken zero) & taking exponentials, we get

$$\frac{dY}{Y} = -\frac{(n+1-s)}{(s-s^2)}\,ds$$

$$= \left\{\frac{-s + (n+1)}{s(s-1)}\right\}$$

$$\therefore \frac{dY}{Y} = \left[\frac{n}{s-1} - \frac{(n+1)}{s}\right] ds$$

(ms)

$$\ln Y = n\ln(s-1) - (n+1)\ln s$$

$$\Rightarrow \ln Y = \ln(s-1)^n - \ln s^{n+1}$$

$$\Rightarrow Y = \frac{(s-1)^n}{s^{n+1}} \longrightarrow (*)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

We write $l_n = \mathcal{L}^{-1}[Y(s)]$ & show that

$$l_0 = 1,\ l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right),$$ (Rodrigues's formula)

$$\therefore \mathcal{L}\{l_n(t)\} = \mathcal{L}\left\{\frac{e^t}{n!}\frac{d^n}{dt^n}\{t^n e^{-t}\}\right\} = \frac{(s-1)^n}{s^{n+1}}$$

These are polynomials $\longrightarrow (5)$
because the exponential terms cancel if we perform the indicated differentiation. They are called Laguerre polynomials, & are usually denoted by L_n, but we conform to our convention of reserving capital letters for transforms.

We prove (5),

we know $\mathcal{L}\{t^n e^{-t}\} = \dfrac{n!}{(s+1)^{n+1}}$

[By first shifting theorem $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$.

Here, $f(t) = t^n e^{-t}$

Now, using $\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$. $\longrightarrow (6)$

By (6), since the derivatives are zero at 0.

$$\therefore \mathcal{L}\left\{\left(t^n e^{-t}\right)^{(n)}\right\} = s^n \mathcal{L}(f) = \frac{n!\, s^n}{(s+1)^{n+1}}$$

Now, we make another shift & divide by $n!$ to get (5) & (*).

$$\therefore \mathcal{L}\left\{\frac{e^t}{n!} \frac{d^n}{dt^n}\{t^n e^{-t}\}\right\} = \frac{(s-1)^n}{s^{n+1}}.$$

ie, $\mathcal{L}\{l_n\} = \frac{(s-1)^n}{s^{n+1}} = Y.$

Eg. 2) **Bessel functions:**

We define a Bessel function of order n by

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)} \left\{1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \cdots\right\} \longrightarrow (i)$$

. Some important properties are:

① $J_{-n}(t) = (-1)^n J_n(t)$, if n is a positive integer.

② $J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t)$.

③ $\frac{d}{dt}\{t^n J_n(t)\} = t^n J_{n-1}(t)$.

If $n = 0$, we have $J_0'(t) = -J_1(t)$.

(4) $e^{\frac{1}{2}t(u-1/u)} = \sum_{n=-\infty}^{\infty} J_n(t)\, u^n .$

This is called the generating function for the Bessel functions.

$J_n(t)$ satisfies Bessel's d.eqⁿ

$$t^2 y''(t) + t\, y'(t) + (t^2 - n^2)\, y(t) = 0, \quad n = 0, 1, 2, \dots$$

It is convenient to define

$J_n(it) = i^{-n} I_n(t)$, where $\left\{ \text{'}i\text{' is a complex no.} \right\}$

$I_n(t)$ is called the modified Bessel function of order n.

Bessel functions:

Q1) Find $\mathcal{L}\{J_0(t)\}$, where $J_0(t)$ is the

a) Bessel function of order zero.

b) Use the result of (a) to find $\mathcal{L}\{J_0(at)\}$.

Solⁿ:- Method 1 / (Using series)

Letting $n=0$ in eqⁿ (i), we find that

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 . 4^2 . 6^2} + \cdots$$

Then $\mathcal{L}\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2} \cdot \frac{2!}{s^3} + \frac{1}{2^2 4^2} \cdot \frac{4!}{s^5}$

$$- \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{6!}{s^7} + \cdots$$

$$\therefore \mathcal{L}\{J_0(t)\} = \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^4}\right) - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{s^6}\right) + \cdots\right]$$

$$= \frac{1}{s}\left[\left(1+\frac{1}{s^2}\right)^{-1/2}\right] = \frac{1}{\sqrt{s^2+1}}$$, using the Binomial Theorem

Method 2 / (Using d.eqns):

The function $J_0(t)$ satisfies the d.eqn

$$t J_0''(t) + J_0'(t) + t J_0(t) = 0 \rightarrow (1)$$

(with $n=0$). Taking the Laplace transform of both sides of (1), & using $J_0(0)=1$, $J_0'(0)=0$, $y = \mathcal{L}\{J_0(t)\}$, we have

$$\mathcal{L}\{t J_0''(t)\} + \mathcal{L}\{J_0'(t)\} + \mathcal{L}\{t J_0(t)\} = 0.$$

$$\Rightarrow -\frac{d}{ds}\{s^2 Y - s(1) - 0] + \{sY - 1] - \frac{dY}{ds} = 0.$$

$$\Rightarrow -\left[2sY + s^2\frac{dY}{ds} - \cancel{1}\right] + \left[sY - \cancel{1}\right] - \frac{dY}{ds} = 0$$

$$\Rightarrow -(s^2+1)\frac{dY}{ds} - sY = 0 \Rightarrow \frac{dY}{ds} = -\frac{sY}{(s^2+1)}$$

$$\Rightarrow \frac{dY}{Y} = -\frac{s\,ds}{(s^2+1)}.$$ On integration, we have

$$\ln Y = -\frac{1}{2}\frac{d(s^2+1)}{(s^2+1)} + \ln c.$$

$\Rightarrow Y(s) = \dfrac{c}{\sqrt{s^2+1}}$, to find c?

Now, $\underset{s\to\infty}{Lt}\, sY(s) = \underset{s\to\infty}{Lt} \dfrac{cs}{\sqrt{s^2+1}} = \underset{s\to\infty}{Lt} \dfrac{c}{\sqrt{1+1/s^2}} = c$

& $\underset{t\to 0}{Lt}\, J_0(t) = 1$

[Here, $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$]

Thus by the initial-value theorem, we have $c = 1$ & so $\mathcal{L}\{J_0(t)\} = \dfrac{1}{\sqrt{s^2+1}}$.

(b) By Problem 1,

$$\mathcal{L}\{J_0(at)\} = \frac{1}{a}\cdot\frac{1}{\sqrt{(s/a)^2+1}} = \frac{1}{\sqrt{s^2+a^2}}.$$

[as if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f(at)\} = \dfrac{1}{a} F(s/a)$]

Eg. 3/ Find $\mathcal{L}\{J_1(t)\}$, where $J_1(t)$ is Bessel's function of order one.

Sol^n:- We have, $J_0'(t) = -J_1(t)$ [see property 3 for Bessel's function]

Hence,

$$\mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_0'(t)\} = -\{s\,\mathcal{L}\{J_0(t)\} - J_0(0)\}$$

$$= -\{s\,\mathcal{L}\{J_0(t)\} - 1\}$$

$$= 1 - \frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}.$$

Eg. 4/ Show that $\int_0^\infty J_0(t)\,dt = 1$.

Sol^n:- We have $\mathcal{L}\{J_0(t)\} = \int_0^\infty e^{-st} J_0(t)\,dt$

$$= \frac{1}{\sqrt{s^2+1}}.$$

Then letting $s \to 0^+$, we find

$$\int_0^\infty J_0(t)\,dt = 1.$$

---

2. <u>Integral equations</u> :-

Convolution also helps in solving certain <u>integral equations</u>, that is, equations in which the unknown function $y(t)$ appears under the integral (& perhaps also outside of it). This concerns only very special ones (those whose integral is of the form of a convolution).

eg, 1) Solve the integral eq^n

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$$

(which is a Volterra Integral eq^n of the second kind)

Sol^n :- Here,

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau.$$

1st step :- Eq^n in terms of convolution.

We see that the given eq^n can be written as

$$y = t + y * \sin t$$

2nd step :- Application of the convolution theorem

We write $Y = \mathcal{L}(y)$.

By the convolution theorem,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{y * \sin t\}$$

$$\Rightarrow Y(s) = 1/s^2 + Y(s)\cdot\frac{1}{(s^2+1)}$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4},$$

3rd step :- Taking the inverse transform

This gives the solution

$$y(t) = t + \frac{1}{6}t^3 = t + t^3/6.$$

We may check this by substitution & evaluating the integral by repeated integration by parts.

eg. 2) Solve
$$y(t) - \int_0^t (1+\tau)\, y(t-\tau)\, d\tau = 1 - \sinh t .$$

Sol^n :- By the help of convolution ~~theorem~~, we can write
$$y - (1+t) * y = 1 - \sinh t$$
writing $Y = \mathcal{L}(y)$, we obtain by using the Convolution theorem & then taking common denominators.
$$\mathcal{L}\{y\} - \mathcal{L}\{(1+t) * y\} = \mathcal{L}\{1 - \sinh t\}$$
$$\Rightarrow Y - \mathcal{L}(1+t)\, \mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{\sinh t\}$$
$$\Rightarrow Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{(s^2-1)} .$$
$$\Rightarrow Y(s) \cdot \left(\frac{s^2 - s - 1}{s^{\cancel{2}}}\right) = \frac{(s^2 - 1 - s)}{\cancel{s}(s^2-1)} .$$
$$\Rightarrow Y(s) = \left(\frac{s}{s^2-1}\right)$$
$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}$$
$$= \cosh t .$$

3. Systems of D. eq^{ns} :

The Laplace transform method may also be used for solving systems of d. eq^{ns}.

We shall explain this in terms of typical applications.

For a first-order linear system,

$$\left.\begin{aligned} y_1' &= a_{11} y_1 + a_{12} y_2 + g_1(t) \\ y_2' &= a_{21} y_1 + a_{22} y_2 + g_2(t) \end{aligned}\right\} \rightarrow (1)$$

writing $Y_1 = \mathcal{L}\{y_1\}$, $Y_2 = \mathcal{L}\{y_2\}$, $G_1 = \mathcal{L}\{g_1\}$, $G_2 = \mathcal{L}\{g_2\}$, we obtain from (1), the subsidiary equations.

$$sY_1 - y_1(0) = a_{11} Y_1 + a_{12} Y_2 + G_1(s)$$

$$sY_2 - y_2(0) = a_{21} Y_1 + a_{22} Y_2 + G_2(s).$$

or, by collecting the Y_1- & Y_2- terms,

$$\left.\begin{aligned} (a_{11} - s) Y_1 + a_{12} Y_2 &= -y_1(0) - G_1(s) \\ a_{21} Y_1 + (a_{22} - s) Y_2 &= -y_2(0) - G_2(s) \end{aligned}\right\} \rightarrow (2)$$

This must be solved algebraically for $Y_1(s)$ & $Y_2(s)$. The solution of the given system is then obtained if we take the inverse $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$

Simultaneous ordinary d. eqns

e.g. 1) Solve :- $\begin{cases} \dfrac{dx}{dt} = 2x - 3y \\ \dfrac{dy}{dt} = y - 2x \end{cases}$,

subject to $x(0) = 8$, $y(0) = 3$.

Soln :- Taking the Laplace Transform, we have if

$$\mathcal{L}\{x\} = X, \quad \mathcal{L}\{y\} = Y.$$

$$\mathcal{L}\{x'\} = \mathcal{L}\{2x\} - 3\mathcal{L}\{y\}, \quad \mathcal{L}\{y'\} = \mathcal{L}\{y\} - 2\mathcal{L}\{x\}.$$

$$\therefore \left.\begin{array}{l} sX - 8 = 2X - 3Y \\ sY - 3 = Y - 2X \end{array}\right] \Rightarrow \begin{cases} s\mathcal{L}\{x\} - x(0) = 2X - 3Y \\ s\mathcal{L}\{y\} - y(0) = Y - 2X. \end{cases}$$

or,
$$(s-2)X + 3Y = 8 \longrightarrow (1)$$
$$2X + (s-1)Y = 3 \longrightarrow (2)$$

Solving (1) & (2) simultaneously,

$$X = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)}$$

$$= \frac{5}{s+1} + \frac{3}{s-4} \quad \left[\text{Using partial fractions}\right]$$

$$Y = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)}$$

$$= \frac{5}{s+1} - \frac{2}{s-4},$$ then $x = \mathcal{L}^{-1}\{X\} = 5e^{-t} + 3e^{4t}$

$y = \mathcal{L}^{-1}\{Y\} = 5e^{-t} - 2e^{4t}$

e.g. 2) Solve :-

$$x'' + y' + 3x = 15e^{-t}$$
$$y'' - 4x' + 3y = 15\sin 2t,$$

subject to $x(0) = 35$, $x'(0) = -48$,
$y(0) = 27$, $y'(0) = -55$.

Solⁿ :- ∴ General solution is

$$x = \mathcal{L}^{-1}\{X\} = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t$$

$$y = \mathcal{L}^{-1}\{Y\} = 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t.$$

Fourier Series

1. Periodic functions :-

A function $f(x)$ is called periodic if it is defined for all real x & if there is some positive number p such that

$$f(x+p) = f(x), \ \forall x \quad \longrightarrow (1)$$

This number 'p' is called a period of $f(x)$. The graph of such a function is obtained by periodic repetition of its graph in any interval of length p. Periodic phenomena &