1.1 Introduction to Fourier Series

Before we start discussion on Fourier transform it is very important to discuss Fourier series firstly because it gives a pathway to understand Fourier transform. Fourier series has a wide range of applications, viz. in analysis of current flow, sound waves, image analysis and many more. They are also used to solve differential equations. In a general sense, we use Fourier series to represent a periodic functions. Indeed, not only periodic functions but also to represent and approximate functions defined on a finite interval.

1.2 Periodic Functions

If a function f is periodic with period $T > 0$ then $f(t) = f(t + T)$, $-\infty < t < \infty$. The smallest of T, for which the equality $f(t) = f(t + T)$ is true, is called fundamental period of $f(t)$. However, if T is the period of a function f then nT, n is any natural number, is also a period of f. Some familiar periodic functions are $\sin x$, $\cos x$, $\tan x$ etc.

1.2.1 Properties of Periodic Functions

We consider two important properties of periodic function. These properties will be used to discuss the Fourier series.

1. It should be noted that the sum, difference, product and quotient of two functions is also a periodic function. Consider for example:

$$
\begin{aligned}
f(x) &= \underbrace{\sin x}_{} + \underbrace{\sin 2x}_{} + \underbrace{\cos 3x}_{} \\
\text{period:} &\quad 2\pi \quad \frac{2\pi}{2} = \pi \quad \frac{2\pi}{3}
\end{aligned}
$$

   Period of f = common period of $(\sin x, \sin 2x, \cos 3x)$ = 2π

   One can also confirms the period of the function $f(x)$ as

$$
\begin{aligned}
f(x + 2\pi) &= \sin(x + 2\pi) + \sin(2x + 2\pi) + \cos(3x + 2\pi) \\
&= \sin(x) + \sin(2x) + \cos(3x) = f(x)
\end{aligned}
$$

2. If a function is integrable on any interval of length T, then it is integrable on any other intervals of the same length and the value of the integral is the same, that is,

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_b^{b+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x \text{ for any value of } a \text{ and } b$$

This property has been depicted in Figure 1.2.1.

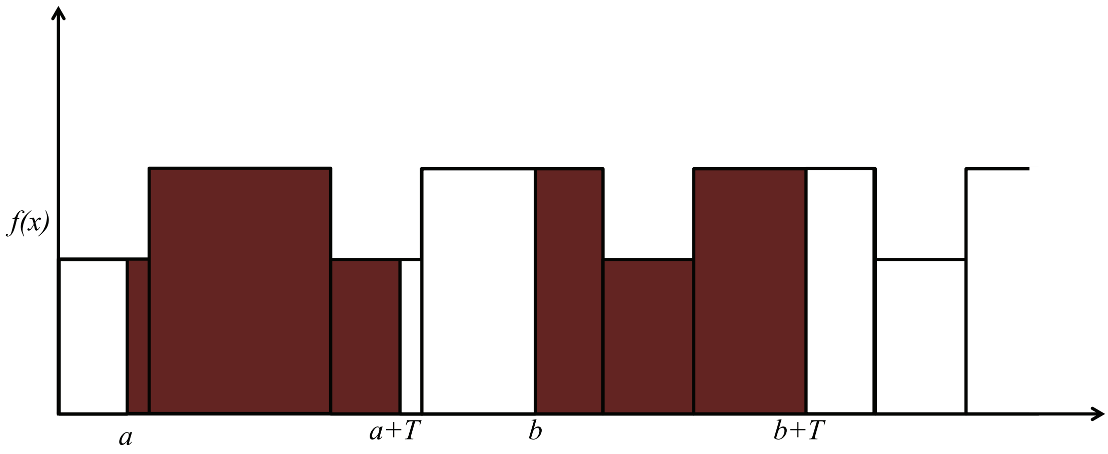

Figure 1.1: Area showing integral of a typical periodic function

1.3 Trigonometric Polynomials and Series

- Trigonometric polynomial of order n is defined as

$$S_n(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right)$$

Here a_n and b_n are some constants. Since the sum of the periodic functions again represents a periodic function. Therefore S_n will be a periodic function. What will be the period of the function S_n? The period can be identified simply by looking at the common period of the functions involved in the sum as

$$\text{Period of } S_n(x) = \text{common period of} \left(\cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \ldots, \sin \frac{n\pi x}{l}, \cos \frac{n\pi x}{l} \right)$$
$$= 2\pi/(\pi/l) = 2l.$$

- The infinite trigonometric series

$$S(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{\pi k x}{l} + b_k \sin \frac{\pi k x}{l} \right),$$

if it converges, also represents a function of period $2l$.

Now the question aries whether any function of period $T = 2l$ can be represented as the sum of a trigonometric series? The answer to this question is affirmative and it is possible for a very wide class of periodic functions. In the next lesson we will see how to obtain the constants a_n and b_n in order this trigonometric series to represent a given periodic function.

Remark 1: *Though sine and cosine functions are quite simple in nature but their sum function may be quite complex. One can see the plot of* $\sin x + \sin 2x + \cos 3x$ *in Figure 1.3. However, the function has a period* 2π *which is a common period of* $\sin x, \sin 2x, \cos 3x$.

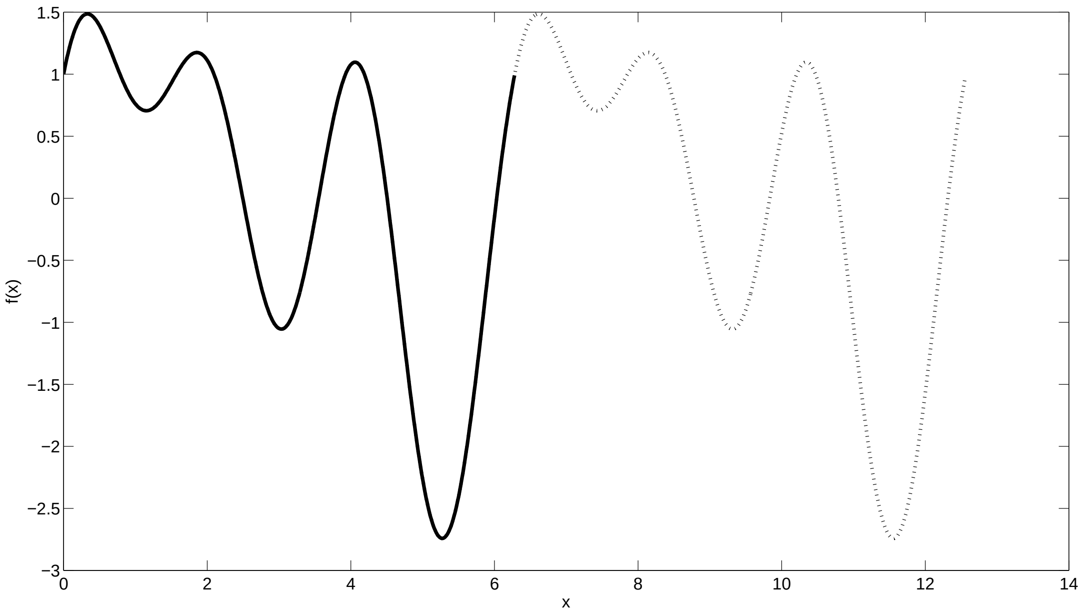

Figure 1.2: Plot of a trigonometric polynomial $f(x) = \sin x + \sin 2x + \cos 3x$

1.4 Orthogonality Property of Trigonometric System

We call two functions $\phi(x)$ and $\psi(x)$ to be orthogonal on the interval $[a, b]$ if

$$\int_a^b \phi(x)\psi(x)\,dx = 0$$

With this definition we can say that the basic trigonometric system viz.

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

is orthogonal on the interval $[-\pi, \pi]$ or $[0, 2\pi]$. In particular, we shall prove that any two distinct functions are orthogonal.

To show the orthogonality we take different possible combination as:

For any integer $n \neq 0$**:** We have the following integrals to show the orthogonality of the function 1 with any member of sine or cosine family

$$\int_{-\pi}^{\pi} 1 \cdot \cos(nx)\,\mathrm{d}x = \frac{\sin(nx)}{n}\bigg|_{-\pi}^{\pi} = 0, \qquad \int_{-\pi}^{\pi} 1 \cdot \sin(nx)\,\mathrm{d}x = -\frac{\cos(nx)}{n}\bigg|_{-\pi}^{\pi} = 0$$

We have also the following useful results

$$\int_{-\pi}^{\pi} \cos^2(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2} = \pi, \qquad \int_{-\pi}^{\pi} \sin^2(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2} = \pi$$

For any integer m **and** n **(m** $\neq$ **n):** Now we show that any two different members of the same family (sine or cosine) are orthogonal. For the cosine family we have

$$\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x]\ \mathrm{d}x = 0$$

and for the sine family we have

$$\int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x]\ \mathrm{d}x = 0$$

For any integer m **and** n**:** Here we show that any two members of the two different family (sine and cosine) are orthogonal

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x = 0$$

Note that the integrand is an odd function and therefore the integral is zero.

The above result can be summarized in a more general setting in the following theorem.

1.4.1 Theorem

The trigonometric system

$$1,\ \cos\frac{\pi x}{l},\ \sin\frac{\pi x}{l},\ \cos\frac{2\pi x}{l},\ \sin\frac{2\pi x}{l},\ldots$$

is orthogonal on the interval $[-l, l]$ *or* $[a, a+2l]$*, where* a *is any real number.*

Proof: Note that the common period of the trigonometric system

$$1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l}, \ldots$$

is $2l$. Similar to the evaluation of the integral appeared above to show orthogonality of the basic trigonometric system, we have the following results:

a) $$\int_{-l}^{l} \cos\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = \int_{a}^{a+2l} \cos\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ l & \text{if } m = n \neq 0 \end{cases}$$

b) $$\int_{-l}^{l} \sin\frac{m\pi x}{l}\,\sin\frac{n\pi x}{l}\,\mathrm{d}x = \int_{a}^{a+2l} \sin\frac{m\pi x}{l}\,\sin\frac{n\pi x}{l}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ l & \text{if } m = n \neq 0 \end{cases}$$

c) $$\int_{-l}^{l} \sin\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = \int_{a}^{a+2l} \sin\frac{m\pi x}{l}\,\cos\frac{n\pi x}{l}\,\mathrm{d}x = 0$$

This completes the proof of the above theorem. ■

To summarize, the value of the integral over length of period of integrand is equal to zero if the integrand is a product of two different members of trigonometric system. If the integrand is product of two same member from sine or cosine family then the value of the integral will be half of the interval length on which the integral is performed. These results will be used to establish Fourier series of a function of period $2l$ defined on the interval $[-l, l]$ or $[a, a+2l]$. It should be noted that for $l = \pi$ we obtain results for standard trigonometric system of common period 2π.

In this lesson we shall introduce Fourier series of a piecewise continuous periodic function. First we construct Fourier series of periodic functions of standard period 2π and then the idea will be extended for a function of arbitrary period.

2.1 Piecewise Continuous Functions

A function f is piecewise continuous on $[a, b]$ if there are points

$$a < t_1 < t_2 < \ldots < t_n < b$$

such that f is continuous on each open sub-interval (a, t_1), (t_j, t_{j+1}) and (t_n, b) and all the following one sided limits exist and are finite

$$\lim_{t\to a+} f(t),\ \lim_{t\to t_j-} f(t),\ \lim_{t\to t_j+} f(t),\ \text{and}\ \lim_{t\to b-} f(t),\ j = 1, 2, \ldots, n$$

This mean that f is continuous on $[a, b]$ except possibly at finitely many points, at each of which f has finite one sided limits. It should be clear that all continuous functions are obviously piecewise continuous.

2.1.1 Example 1

Consider the function

$$f(x) = \begin{cases} 3, & \textit{for } x = -\pi; \\ x^2, & \textit{for } -\pi < x < 1; \\ 1 - x^2, & \textit{for } 1 \leq x < 2; \\ 2, & \textit{for } 2 \leq x \leq \pi. \end{cases}$$

At each point of discontinuity the function has finite one sided limits from both sides. At the end points $x = -\pi$ and π right and left sided limits exist, respectively. Therefore, the function is piecewise continuous.

2.1.2 Example 2

A simple example that is not piecewise continuous includes

$$f(x) = \begin{cases} 0, & x = 0; \\ x^{-n}, & x \in (0, 1],\ n > 0\,. \end{cases}$$

Note that f is continuous everywhere except at $x = 0$. The function f is also not piecewise continuous on $[0, 1]$ because $\lim_{x=0+} f(x) = \infty$.

An important property of piecewise continuous functions is boundedness and integrability over closed interval. A piecewise continuous function on a closed interval is bounded and integrable on the interval. Moreover, if f_1 and f_2 are two piecewise continuous functions then their product, $f_1 f_2$, and linear combination, $c_1 f_1 + c_2 f_2$, are also piecewise continuous.

2.2 Fourier Series of a 2π Periodic Function

Let f be a periodic piecewise continuous function on $[-\pi, \pi]$ and has the following trigonometric series expansion

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] \tag{2.1}$$

The aim is to determine the coefficients $a_k, k = 0, 1, 2, \ldots$ and $b_k, k = 1, 2, \ldots$. First we assume that the above series can be integrated term by term and its integral is equal to the integral of the function f over $[-\pi, \pi]$, that is,

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{a_0}{2}\,\mathrm{d}x + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)\,\mathrm{d}x + b_k \int_{-\pi}^{\pi} \sin(kx)\,\mathrm{d}x \right)$$

This implies

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x.$$

Multiplying the series by $\cos(nx)$, integrating over $[-\pi, \pi]$ and assuming its value equal to the integral of $f(x)\cos(nx)$ over $[-\pi, \pi]$, we get

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = 0 + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(nx)\cos(kx)\,\mathrm{d}x + b_k \int_{-\pi}^{\pi} \cos(nx)\sin(kx)\,\mathrm{d}x \right)$$

Note that the first term on the right hand side is zero because $\int_{-\pi}^{\pi} \cos(kx)\,\mathrm{d}x = 0$. Further, using the orthogonality of the trigonometric system we obtain

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x$$

Similarly, by multiplying the series by $\sin(nx)$ and repeating the above steps we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x$$

The coefficients $a_n,\ n = 0, 1, 2, \dots$ and $b_n,\ n = 1, 2, \dots$ are called **Fourier coefficients** and the trigonometric series (2.1) is called the **Fourier series** of $f(x)$. Note that by writing the constant $a_0/2$ instead of a_0, one can use a single formula of a_n to calculate a_0.

Remark 1: *In the series (2.1) we can not, in general, replace $\sim$ by $=$ sign as clear from the determination of the coefficients. In the process we have set two integrals equal which does not imply that the function $f(x)$ is equal to the trigonometric series. Later we will discuss conditions under which equality holds true.*

Remark 2: **(Uniqueness of Fourier Series)** *If we alter the value of the function f at a finite number of points then the integral defining Fourier coefficients are unchanged. Thus function which differ at finite number of points have exactly the same Fourier series. In other words we can say that if f, g are piecewise continuous functions and Fourier series of f and g are identical, then $f(x) = g(x)$ except at a finite number of points.*

2.3 Fourier Series of a $2l$ Periodic Function

Let $f(x)$ be piecewise continuous function defined in $[-l, l]$ and it is $2l$ periodic. The Fourier series corresponding to $f(x)$ is given as

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_n \cos\frac{k\pi x}{l} + b_n \sin\frac{k\pi x}{l}\right] \tag{2.2}$$

where the Fourier coefficients, derived exactly in the similar manner as in the previous case, are given as

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}\,\mathrm{d}x, \quad k = 0, 1, 2, \dots$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x \quad k = 1, 2, \dots$$

In must be noted that just for simplicity we will be discussing Fourier series of 2π periodic function. However all discussions are valid for a function of an arbitrary period.

Remark 3: *It should be noted that piecewise continuity of a function is sufficient for the existence of Fourier series. If a function is piecewise continuous then it is always possible to calculate Fourier coefficients. Now the question arises whether the Fourier series of a function f converges and represents f or not. For the convergence we need additional conditions on the function f to ensure that the series converges to the desired values. These issues on convergence will be taken in the next lesson.*

2.4 Example Problems

2.4.1 Problem 1

Find the Fourier series to represent the function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0; \\ x, & 0 < x < \pi. \end{cases}$$

Solution: The Fourier series of the given function will represent a 2π periodic function and the series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

with

$$a_0 = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi\,\mathrm{d}x + \int_{0}^{\pi} x\,\mathrm{d}x\right] = -\frac{\pi}{2}$$

and the coefficients a_n, $n = 1, 2, \ldots$ as

$$\begin{aligned} a_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x\right] \\ =& -\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\left\{x\frac{\sin(nx)}{n}\right\}_{0}^{\pi} - \int_{0}^{\pi}\frac{\sin(nx)}{n}\,\mathrm{d}x\right] \end{aligned}$$

It can be further simplified to give

$$a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ is even;} \\ -\frac{2}{n^2\pi}, & n \text{ is odd.} \end{cases}$$

Similarly b_n, $n = 1, 2, \ldots$ can be calculated as

$$\begin{aligned} b_n =& \frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi \sin(nx)\,\mathrm{d}x + \int_0^{\pi} x\sin(nx)\,\mathrm{d}x\right] \\ =& \left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\left\{x\frac{\cos(nx)}{n}\right\}_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,\mathrm{d}x\right] \end{aligned}$$

After simplification we get

$$b_n = \frac{1}{n}\left[1 - 2(-1)^n\right] = \begin{cases} -\frac{1}{n}, & n \text{ is even;} \\ \frac{3}{n}, & n \text{ is odd.} \end{cases}$$

Substituting the values of a_n and b_n, we get

$$f(x) \sim -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left[3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \ldots\right].$$

Remark 4: *Let a function is defined on the interval $[-l, l]$. It should be noted that the periodicity of the function is not required for developing Fourier series. However, the Fourier series, if it converges, defines a $2l$-periodic function on $\mathbb{R}$. Therefore, this is sometimes convenient to think the given function as $2l$-periodic defined on $\mathbb{R}$.*

2.4.2 Problem 2

Expand $f(x) = |\sin x|$ in a Fourier series.

Solution: There are two possibilities to work out this problem. This may be treated as a function of period π and we can work in the interval $(0, \pi)$ or we treat this function as of period 2π and work in the interval $(-\pi, \pi)$.

Case I: First we treat the function $|\sin x|$ as π periodic we have $2l = \pi \Rightarrow l = \frac{\pi}{2}$. The coefficient a_0 is given as

$$a_0 = \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{4}{\pi}.$$

The other coefficient a_n, $n = 1, 2, \ldots$ are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\cos(2nx)\,\mathrm{d}x = \frac{1}{l}\int_0^{\pi}\left[\sin(2n+1)x - \sin(2n-1)x\right]\,\mathrm{d}x$$

It can be further simplified to have

$$a_n = \frac{1}{\pi}\left[-\frac{\cos(2n+1)x}{2n+1}\Big|_0^\pi + \frac{\cos(2n-1)x}{2n-1}\Big|_0^\pi\right] = \frac{1}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1}\right] = -\frac{4}{\pi(4n^2-1)}$$

Now we compute the coefficients b_n, $n = 1, 2, \ldots$ as

$$\begin{aligned} b_n = \frac{2}{\pi}\int_0^\pi \sin x \sin(2nx)\,\mathrm{d}x &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left[\cos(2n-1)x - \cos(2n+1)x\right]\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[-\frac{\sin(2n-1)x}{2n-1}\Big|_0^\pi + \frac{\sin(2n+1)x}{2n+1}\Big|_0^\pi\right] = 0 \end{aligned}$$

Hence the Fourier series is given by

$$f(x) \sim \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)}\cos(2nx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}, \quad 0 \le x \le 1$$

Case II: If we treat $f(x)$ as 2π periodic then

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi \left[\sin(n+1)x - \sin(n-1)x\right]\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1}\Big|_0^\pi + \frac{\cos(n-1)x}{n-1}\Big|_0^\pi\right]\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{-(-1)^{n+1}+1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right] \end{aligned}$$

Thus, for $n \neq 1$ we have

$$a_n = \begin{cases} 0, & \text{when } n \text{ is odd;} \\ -\frac{1}{\pi}\frac{4}{n^2-1}, & \text{when } n \text{ is even} \end{cases}$$

The coefficient a_1 needs to calculated separately as

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi \sin 2x\,\mathrm{d}x = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]\Big|_0^\pi = \frac{1}{2\pi}\left[-1+1\right] = 0$$

Clearly, the coefficients b_n's are zero because

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\sin x|\sin(nx)}_{\text{odd function}}\,\mathrm{d}x = 0$$

The Fourier series can be written as

$$f(x) \sim \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right] = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}.$$

Therefore we ended up with the same series.

Remark 5: *If we develop the Fourier series of a function considering its period as any integer multiple of its fundamental period, we shall end up with the same Fourier series.*

Remark 6: *Note that in the above example the given function is an even function and therefore the Fourier series is simpler as we have seen that the coefficient* b_n *is zero in this case. The determination of the Fourier series of a given function becomes simpler if the function is odd or even. More detail of this we shall see in the next Lesson.*

We have seen that piecewise continuity of a function is sufficient for the existence of the Fourier series. We have not yet discussed the convergence of the Fourier series. Convergence of the Fourier series is a very important topic to be explored in this lesson.

In order to motivate the discussion on convergence, let us construct the Fourier series of the function

$$f(x) = \begin{cases} -\cos x, & -\pi/2 \leq x < 0; \\ \cos x, & 0 \leq x \leq \pi/2. \end{cases} \qquad f(x+\pi) = f(x).$$

In this case the function is an odd function and therefore $a_n = 0$, $n = 0, 1, 2, \ldots$. We compute the Fourier coefficient b_n by

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin(2nx)\,\mathrm{d}x = \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin(2nx)\,\mathrm{d}x = \frac{8}{\pi}\frac{n}{(4n^2-1)}$$

The Fourier series is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(2nx) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n\sin(2nx)}{4n^2-1}.$$

Note that the Fourier series at $x = 0$ converges to 0. So the Fourier series of f does not converge to the value of the function at $x = 0$.

With this example we pose the following questions in connection to the convergence of the Fourier series

1. Does the Fourier series of a function $f(x)$ converges at a point $x \in [-L, L]$.
2. If the series converges at a point x, is the sum of the series equal to $f(x)$.

The answers of these questions are in the negative because

1. There are Lebesgue integrable functions on $[-L, L]$ whose Fourier series diverge everywhere on $[-L, L]$.
2. There are continuous functions whose Fourier series diverge at a countable number of points.
3. We have already seen in the above examples that the Fourier series converges at a point but the sum is not equal to the the value of the function at that point.

We need some additional conditions to ensure that the Fourier series of a function $f(x)$ converges and it converges to the function $f(x)$. Though, we have several notions of convergence like pointwise, uniform, mean square, etc. we first stick to the most common notion of convergence, that is, pointwise convergence. Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges pointwise to f on $[a, b]$ if for each $x \in [a, b]$ we have $\lim_{m\to\infty} f_m(x) = f(x)$. A more formal definition of pointwise convergence will be given later.

3.1 Convergence Theorem (Dirichlet's Theorem, Sufficient Conditions)

Theorem Statement: *Let f be a* ***piecewise continuous function*** *on $[-L, L]$ and the* ***one sided derivatives*** *of f, that is,*

$$\lim_{h\to 0+} \frac{f(x+h) - f(x+)}{h} \text{ in } x \in [-L, L) \quad \& \quad \lim_{h\to 0+} \frac{f(x-) - f(x-h)}{h} \text{ in } x \in (-L, L] \tag{3.1}$$

exist (and are finite), then for each $x \in (-L, L)$ the Fourier series converges and we have

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{k\pi x}{L} + b_n \sin\frac{k\pi x}{L} \right]$$

At both endpoints $x = \pm L$ the series converges to $[f(L-) + f((-L)+)]/2$, thus we have

$$\frac{f(L-) + f((-L)+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (-1)^n a_n$$

Remark 1: *If the function is continuous at a point x, that is, $f(x+) = f(x-)$ then we have*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_n \cos\frac{k\pi x}{L} + b_n \sin\frac{k\pi x}{L} \right] \tag{3.2}$$

In other words, if f is continuous with $f(-L) = f(L)$ and one sided derivatives (3.1) exist then equality (3.2) holds for all x.

Remark 2: *In the above theorem condition on f are sufficient conditions. One may replace these conditions (piecewise continuity and one sided derivatives) by slightly more restrictive conditions of piecewise smoothness. A function is said to be* ***piecewise smooth*** *on $[-L, L]$ if it is piecewise continuous and has a piecewise continuous derivative. The* *difference between the two similar restrictions on f will be clear from the example of the function*

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

It can easily easily be shown that derivative of the function exist everywhere and thus the function has one sided derivatives and satisfy the conditions of the convergence Theorem (3.1). However the function is not piecewise smooth because the $\lim_{x\to 0} f'(x)$ does not exist as

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

If a function is piecewise smooth then it can easily be shown that left and right derivatives exist. Let f be a piecewise smooth function on $[-L, L]$ then $\lim_{x\to a\pm} f'(x)$ exists for all $a \in [-L, L]$. This implies

$$\lim_{x\to a+} f'(x) = \lim_{x\to a+} \left(\lim_{h\to 0+} \frac{f(x+h) - f(x)}{h} \right)$$

Interchanging the two limits on the right hand side we obtain

$$\lim_{x\to a+} f'(x) = \lim_{h\to 0+} \left(\lim_{x\to a+} \frac{f(x+h) - f(x)}{h} \right) = \lim_{h\to 0+} \frac{f(a+h) - f(a+)}{h}$$

Similarly one can shown the existence of left derivative. This example confirms that piecewise smoothness is stronger condition than piecewise continuity with existence of one sided derivatives.

3.2 Different Notions of Convergence

3.2.1 Mean Square Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. Let f be defined on $[a, b]$. We say that the sequence $\{f_m\}_{m=1}^{\infty}$ converges in the mean square sense to f on $[a, b]$ if

$$\lim_{m\to\infty} \int_a^b |f(x) - f_m(x)|^2 \, \mathrm{d}x = 0$$

3.2.2 Pointwise Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a,b]$ and let f be defined on $[a,b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges pointwise to f on $[a,b]$ if for each $x \in [a,b]$ we have $\lim_{m\to\infty} f_m(x) = f(x)$. That is, for each $x \in [a,b]$ and $\varepsilon > 0$ there is a natural number $N(\varepsilon, x)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon, x)$$

3.2.3 Uniform Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a,b]$ and let f be defined on $[a,b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges uniformly to f on $[a,b]$ if for each $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon), \text{ and for all } x \in [a,b]$$

There is one more interesting fact about the uniform convergence. If $\{f_m\}_{m=1}^{\infty}$ is a sequence of continuous functions which converge uniformly to a function to f on $[a,b]$, then f is continuous.

3.2.4 Example 1

Let $u_n = x^n$ on $[0,1)$. Clearly, the sequence $\{u_n\}_{n=1}^{\infty}$ converges pointwise to 0, that is, for fixed $x \in [0,1)$ we have $\lim\limits_{n\to\infty} u_n = 0$. But it does not converge uniformly to 0 as we shall show that for given ε there does not exist a natural number N independent of x such that $|u_n - 0| < \varepsilon$. Suppose that the series converges uniformly, then for a given ε with

$$|u_n - 0| < \varepsilon, \tag{3.3}$$

we seek for a natural number $N(\varepsilon)$ such that relation (3.3) holds for $n > N$. Note that relation (3.3) holds true if

$$x^n < \varepsilon \Longleftrightarrow n > \frac{\ln \varepsilon}{\ln x}$$

It should be evident now that for given x and ε one can define

$$N := \left[\frac{\ln \varepsilon}{\ln x}\right], \qquad \textit{where } [\,] \textit{ gives integer rounded towards infinity}$$

It once again confirms pointwise convergence. However if x is not fixed then $\ln\varepsilon/\ln x$ grows without bounds for $x \in [0,1)$. Hence it is not possible to find N which depends only on ε and therefore the sequence u_n does not converge uniformly to 0.

3.2.5 Example 2

Let $u_n = \dfrac{x^n}{n}$ on $[0,1)$. This sequence converges uniformly and of course pointwise to 0. For given $\varepsilon > 0$ take $n > N := \left[\frac{1}{\varepsilon}\right]$ then noting $\left[\frac{1}{\varepsilon}\right] > \frac{1}{\varepsilon}$ we have $|u_n - 0| < x^n/n < 1/n < \varepsilon$ for all $n > N$ Hence the sequence u_n converges uniformly.

4.1 Different Notions of Convergence

4.1.1 Mean Square Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$. Let f be defined on $[a, b]$. We say that the sequence $\{f_m\}_{m=1}^{\infty}$ converges in the mean square sense to f on $[a, b]$ if

$$\lim_{m\to\infty} \int_a^b |f(x) - f_m(x)|^2 \, \mathrm{d}x = 0$$

4.1.2 Pointwise Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let f be defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges pointwise to f on $[a, b]$ if for each $x \in [a, b]$ we have $\lim_{m\to\infty} f_m(x) = f(x)$. That is, for each $x \in [a, b]$ and $\varepsilon > 0$ there is a natural number $N(\varepsilon, x)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon, x)$$

4.1.3 Uniform Convergence

Let $\{f_m\}_{m=1}^{\infty}$ be sequence of functions defined on $[a, b]$ and let f be defined on $[a, b]$. We say that $\{f_m\}_{m=1}^{\infty}$ converges uniformly to f on $[a, b]$ if for each $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N(\varepsilon), \text{ and for all } x \in [a, b]$$

There is one more interesting fact about the uniform convergence. If $\{f_m\}_{m=1}^{\infty}$ is a sequence of continuous functions which converge uniformly to a function to f on $[a, b]$, then f is continuous.

4.1.4 Example 1

Let $u_n = x^n$ on $[0, 1)$. Clearly, the sequence $\{u_n\}_{n=1}^{\infty}$ converges pointwise to 0, that is, for fixed $x \in [0, 1)$ we have $\lim\limits_{n\to\infty} u_n = 0$. But it does not converge uniformly to 0 as we shall *show that for given ε there does not exist a natural number N independent of x such that $|u_n - 0| < \varepsilon$. Suppose that the series converges uniformly, then for a given ε with*

$$|u_n - 0| < \varepsilon, \tag{4.1}$$

we seek for a natural number $N(\varepsilon)$ such that relation (4.1) holds for $n > N$. Note that relation (4.1) holds true if

$$x^n < \varepsilon \Longleftrightarrow n > \frac{\ln \varepsilon}{\ln x}$$

It should be evident now that for given x and ε one can define

$$N := \left[\frac{\ln \varepsilon}{\ln x}\right], \quad \textit{where } [\,] \textit{ gives integer rounded towards infinity}$$

It once again confirms pointwise convergence. However if x is not fixed then $\ln \varepsilon / \ln x$ grows without bounds for $x \in [0, 1)$. Hence it is not possible to find N which depends only on ε and therefore the sequence u_n does not converge uniformly to 0.

4.1.5 Example 2

Let $u_n = \dfrac{x^n}{n}$ on $[0, 1)$. This sequence converges uniformly and of course pointwise to 0. For given $\varepsilon > 0$ take $n > N := \left[\frac{1}{\varepsilon}\right]$ then noting $\left[\frac{1}{\varepsilon}\right] > \frac{1}{\varepsilon}$ we have $|u_n - 0| < x^n/n < 1/n < \varepsilon$ for all $n > N$ Hence the sequence u_n converges uniformly.

Now we discuss these three types of convergence for the Fourier series of a function.

- Let f be a **piecewise continuous function** on $[-\pi, \pi]$ then the Fourier series of f convergence to f in the mean square sense. That is

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \left[\frac{a_0}{2} + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)\right]\right|^2 \mathrm{d}x = 0$$

- Let f be a **piecewise continuous function** on $[-\pi, \pi]$ and the appropriate **one sided derivatives** of f at each point in $[-\pi, \pi]$ exists then for each $x \in [-\pi, \pi]$ the Fourier series of f converges pointwise to the value $(f(x-) + f(x+))/2$.

- If f **is continuous** on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and f' **is piecewise continuous** on $[-\pi, \pi]$, then the Fourier series of f converges uniformly (and also absolutely) to f on $[-\pi, \pi]$.

4.2 Best Trigonometric Polynomial Approximation

An interesting property of the partial sums of a Fourier series is that among all trigonometric polynomials of degree N, the partial sum of Fourier Series yield the best approximation of f in the mean square sense. This result has been summarized in the following lemma.

4.2.1 Lemma

Let f be piecewise continuous function on $[-\pi, \pi]$ and let the mean square error is defined by the following function

$$E(c_0, \ldots, c_N, d_1, \ldots, d_N) = \int_{-\pi}^{\pi} \left| f - \left[\frac{c_0}{2} + \sum_{k=1}^{N} (c_k \cos kx + d_k \sin kx) \right] \right|^2 \mathrm{d}x$$

then $E(a_0, \ldots, a_N, b_1, \ldots, b_N) \leq E(c_0, \ldots, c_N, d_1, \ldots, d_N)$ for any real numbers $c_0, c_1, \ldots, c_N$ and $d_1, d_2, \ldots, d_N$. Note that a_k and b_k are the Fourier coefficients of f.

4.3 Example Problems

4.3.1 Problem 1

Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0; \\ x, & 0 < x < \pi. \end{cases}$$

Find the sum of the Fourier series for all point in $[-\pi, \pi]$.
Solution: At $x = 0$, the Fourier series will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{0 + (-\pi)}{2} = -\frac{\pi}{2}$$

Again, $x = \pm\pi$ are another points of discontinuity and the value of the series at these point will be

$$\frac{f(\pi-) + f((-\pi)+)}{2} = \frac{\pi + (-\pi)}{2} = 0;$$

At all other points the series will converge to functional value $f(x)$.

4.3.2 Problem 2

Let the Fourier series of the function $f(x) = x + x^2$, $-\pi < x < \pi$ *be given by*

$$x + x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{4}{n^2} \cos nx - \frac{2}{n} \sin nx \right]$$

Find the sum of the Fourier series for all point in $[-\pi, \pi]$. *Applying the result on convergence of the Fourier series find the value of*

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \qquad \textit{and} \qquad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Solution: Clearly the required series may be obtained by substituting $x = \pm\pi$ and $x = 0$. At the points of discontinuity $x = \pm\pi$ the series converges to

$$\frac{f(\pi-) + f((-\pi)+)}{2} = \frac{(\pi + \pi^2) + (-\pi + \pi^2)}{2} = \pi^2;$$

Substituting $x = \pm\pi$ into the series we get

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{(2n)} \frac{4}{n^2} = \pi^2 \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

At the point $x = 0$ is a point of continuity and therefore the series will converge to 0. Substituting $x = 0$ into the series we obtain

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{(n)} \frac{4}{n^2} = 0 \Longrightarrow \sum_{n=1}^{\infty} (-1)^{(1+n)} \frac{1}{n^2} = \frac{\pi^2}{12}.$$

In this chapter, we start discussion on even and odd function. As mentioned earlier if the function is odd or even then the Fourier series takes a rather simple form of containing sine or cosine terms only. Then we discuss a very important topic of developing a desired Fourier series (sine or cosine) of a function defined on a finite interval by extending the given function as odd or even function.

5.1 Even and Odd Functions

A function is said to be an even about the point a if $f(a-x) = f(a+x)$ for all x and odd about the point a if $f(a-x) = -f(a+x)$ for all x. Further, note the following properties of even and odd functions:

a) The product of two even or two odd functions is again an even function.

b) The product of and even function and an odd function is an odd function.

Using these properties we have the following results for the Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\begin{cases} \displaystyle\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x, & \text{when } f \text{ is even function about } 0 \\ 0, & \text{when } f \text{ is odd function about } 0 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\begin{cases} 0, & \text{when } f \text{ is even function about } 0 \\ \displaystyle\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x, & \text{when } f \text{ is odd function about } 0 \end{cases}$$

From these observation we have the following results

5.1.1 Proposition

Assume that f is a piecewise continuous function on $[-\pi, \pi]$. Then

a) *If f is an even function then the Fourier series takes the simple form*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \qquad \textit{with} \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x, n = 0, 1, 2, \ldots.$$

Such a series is called a cosine series.

b) *If f is an odd function then the Fourier series of f has the form*

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \qquad \textit{with} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x, n = 1, 2, \ldots.$$

Such a series is called a sine series.

5.2 Example Problems

5.2.1 Problem 1

Obtain the Fourier series to represent the function $f(x)$

$$f(x) = \begin{cases} x, & \textit{when } 0 \le x \le \pi \\ 2\pi - x, & \textit{when } \pi < x \le 2\pi \end{cases}$$

Solution: The given function is an even function about $x = \pi$ and therefore

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,\mathrm{d}x = 0.$$

The coefficient a_0 will be calculated as

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi} x\,\mathrm{d}x + \int_{pi}^{2\pi} (2\pi - x)\,\mathrm{d}x\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

The other coefficients a_n are given as

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi} x\cos(nx)\,\mathrm{d}x + \int_{pi}^{2\pi} (2\pi - x)\cos(nx)\,\mathrm{d}x\right]$$

It can be further simplified as

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{when } n \text{ is even} \\ -\frac{4}{n^2\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Therefore, the Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \text{ where } 0 \le x \le 2\pi. \tag{5.1}$$

In this case as the function is continuous and f' is piecewise continuous, the series converges uniformly to $f(x)$ and we can write the equality (5.1).

5.2.2 Problem 2

Determine the Fourier Series of $f(x) = x^2$ on [-π, π] and hence find the value of the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution: The function $f(x) = x^2$ is even on the interval [-π, π] and therefore b_n=0 for all n. The coefficient a_0 is given as

$$a_0 = \frac{1}{\pi}\int_{\pi}^{\pi} x^2 \, \mathrm{d}x = \frac{x^3}{3\pi}\Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}.$$

The other coefficients can be calculated by the general formula as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx) \, \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(nx) \, \mathrm{d}x = \frac{2}{\pi}\left[x^2 \frac{\sin(nx)}{n}\Big|_{0}^{\pi} - \frac{1}{n}\int_{0}^{\pi} 2x \sin(nx) \, \mathrm{d}x\right]$$

Again integrating by parts we obtain

$$a_n = \frac{4}{n\pi}\left[x\frac{\cos(nx)}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi} \frac{\cos(nx)}{n} \, \mathrm{d}x\right] = \frac{4}{n\pi}\left[\frac{\pi(-1)^n}{n} - 0\right] = \frac{4(-1)^n}{n^2}$$

Therefore the Fourier series is given as

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx) \quad \text{for} \quad x \in [-\pi, \pi]. \tag{5.2}$$

If we substitute $x = 0$ in the equation (5.2) we get

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

If we now substitute $x = \pi$ in the equation (5.2) we get

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{2n}}{n^2} \Rightarrow \frac{1}{4}\frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

6.1 Half Range Series

Suppose that $f(x)$ is a function defined on $(0, l]$. Suppose we want to express $f(x)$ in the cosine or sine series. This can be done by extending $f(x)$ to be an even or an odd function on $[-l, l]$. Note that there exists an infinite number of ways to express the function in the interval $[-l, 0]$. Among all possible extension of f there are two, even and odd extensions, that lead to simple and useful series:

a) If we want to express $f(x)$ in cosine series then we extend $f(x)$ as an even function in the interval $[-l, l]$.

b) On the other hand, if we want to express $f(x)$ in sine series then we extend $f(x)$ as an odd function in $[-l, l]$.

We summarize the above discussion in the following proposition

6.1.1 Proposition

Let f be a piecewise continuous function defined on $[0, l]$. The series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \textit{with} \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

is called half range cosine series of f. Similarly, the series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \textit{with} \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

is called half range sine series of f.

Remark: *Note that we can develop a Fourier series of a function f defined in $[0, l]$ and it will, in general, contain all sine and cosine terms. This series, if converges, will represent a l-periodic function. The idea of half range Fourier series is entirely different where we extend the function f as per our desire to have sine or cosine series. The half range series of the function f will represent a $2l$-periodic function.*

6.2 Example Problems

6.2.1 Problem 1

Obtain the half range sine series for e^x in $0 < x < 1$.

Solution: Since we are developing sine series of f we need to compute b_n as

$$
\begin{aligned}
b_n =& \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = 2\int_0^1 e^x \sin n\pi x = 2\left[e^x \sin n\pi x\big|_0^1 - n\pi\int_0^1 e^x\cos n\pi x\, dx\right]\mathrm{d}x \\
=& 2\left[-n\pi\{e^x\cos n\pi x\big|_0^1 + n\pi\int_0^1 e^x \sin n\pi x\,\mathrm{d}x\}\right] = -2n\pi(e(-1)^n - 1) - n^2\pi^2 b_n
\end{aligned}
$$

Taking second term on the right side to the left side and after simplification we get

$$
b_n = \frac{2n\pi\left[1 - e(-1)^n\right]}{1 + n^2\pi^2}
$$

Therefore, the sine series of f is given as

$$
e^x = 2\pi\sum_{n=1}^{\infty}\frac{n\left[1 - e(-1)^n\right]}{1 + n^2\pi^2}\sin n\pi x \quad \text{for} \quad 0 < x < 1
$$

6.2.2 Problem 2

Let $f(x) = \sin\frac{\pi x}{l}$ on $(0, l)$. Find Fourier cosine series in the range $0 < x < l$.

Solution: Sine we want to find cosine series of the function f we compute the coefficients a_n as

$$
a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x = \frac{1}{l}\int_0^l\left[\sin\frac{(n+1)\pi x}{l} + \sin\frac{(1-n)\pi x}{l}\right]\mathrm{d}x
$$

For $n \neq 1$ we can can compute the integrals to get

$$
a_n = \frac{1}{l}\left[-\frac{\cos\frac{(n+1)\pi x}{l}}{\frac{(n+1)\pi}{l}} + \frac{\cos\frac{(1-n)\pi x}{l}}{\frac{(n-1)\pi}{l}}\right]_0^l = \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right]
$$

It can be further simplified as

$$
a_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\frac{4}{\pi(n+1)(n-1)}, & \text{when } n \text{ is even} \end{cases}
$$

The coefficient a_1 needs to calculated separately as

$$a_1 = \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,\mathrm{d}x = \frac{1}{l}\left[\cos\frac{2\pi x}{l}\,\frac{l}{2\pi}\right]_0^l = \frac{1}{2\pi}(1-1) = 0$$

The Fourier cosine series of f is given as

$$\sin\frac{\pi x}{l} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3} + \frac{\cos\frac{4\pi x}{l}}{3\cdot 5} + \frac{\cos\frac{6\pi x}{l}}{5\cdot 7} + \dots\right]$$

6.2.3 Problem 3

Expand $f(x) = x,\ \ 0 < x < 2$ in a (i) sine series and (ii) cosine series.

Solution: (i) To get sine series we calculate b_n as

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

Integrating by parts we obtain

$$b_n = \left[x\cos\frac{n\pi x}{2}\left(-\frac{2}{n\pi}\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,\mathrm{d}x = -\frac{4}{n\pi}\cos n\pi.$$

Then for $0 < x < 2$ we have the Fourier sine series

$$x = -\frac{4}{\pi}\sum\nolimits_{n=1}^{\infty}\frac{\cos n\pi}{n}\sin\frac{n\pi x}{2} = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \dots\right).$$

(ii) Now we express $f(x) = x$ in cosine series. We need to calculate a_n for $n \neq 0$ as

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,\mathrm{d}x = \left[x\sin\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)\right]_0^2 - \int_0^2 \sin\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)\,\mathrm{d}x$$

After simplifications we obtain

$$a_n = \frac{2}{n\pi}\left(\frac{2}{n\pi}\right)\left[\cos\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n^2\pi^2}\left(\cos n\pi - 1\right) = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

The coefficient a_0 is given as

$$a_0 = \int_0^2 x\,\mathrm{d}x = 2$$

Then the Fourier sine series of $f(x) = x$ for $0 < x < 2$ is given as

$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \dots\right).$$

It is interesting to note that the given function $f(x) = x, \;\; 0 < x < 2$ is represented by two entirely different series. One contains only sine terms while the other contains only cosine terms.

Note that we have used series equal to the given function because the series converges for each $x \in (0, 2)$ to the function value. It should also be pointed out that one can deduce sum of several series by putting different values of $x \in (0, 2)$ in the above sine and cosine series.

In this lesson we discuss differentiation and integration of the Fourier series of a function. We can get some idea of the complexity of the new series if looking at the terms of the series. In the case of differentiation we get terms like $n\sin(nx)$ and $n\cos(nx)$, where presence of n as product makes the magnitude of the terms larger then the original and therefore convergence of the new series becomes more difficult. This is exactly other way round in the case of integration where n appears in division and new terms become smaller in magnitude and thus we expect better convergence in this case. We shall deal these two case separately in next sections.

7.1 Differentiation

We first discuss term by term differentiation of the Fourier series. Let f be a piecewise continuous with the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{7.1}$$

Can we differentiate term by term the Fourier series of a function f in order to obtain the Fourier series of f'? In other words, is it true that

$$f'(x) \sim \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)]? \tag{7.2}$$

In general the answer to this question is no.

Let us consider the Fourier series of $f(x) = x$ in $[-\pi, \pi]$. This is an odd function and therefore Fourier series will be $x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$. If we differentiate the series term by term we get $\sum_{n=1}^{\infty} 2(-1)^{n+1} \cos(nx)$. Note that this is not the Fourier series of $f'(x) = 1$ since the Fourier series of $f(x) = 1$ is simply 1.

We consider one more simple example to illustrate this fact. Consider the half range sine series for $\cos x$ in $(0, \pi)$

$$\cos x \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n\sin(2nx)}{(4n^2-1)}$$

If we differentiate this series term by term then we obtain the series

$$\frac{16}{\pi} \sum_{n=1}^{\infty} \frac{n^2 \cos(2nx)}{(4n^2-1)}$$

This series can not be the Fourier series of $-\sin x$ because it diverges as

$$\lim_{n\to\infty} \frac{16}{\pi} \frac{n^2 \cos(2nx)}{(4n^2-1)} \neq 0$$

For the term by term differentiation we have the following result

7.1.1 Theorem

If f is continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, f' is piecewise continuous on $[-\pi, \pi]$, and if

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

(in fact in this case we can replace $\sim$ by $=$) is the Fourier series of f, then the Fourier series of f' is given by

$$f'(x) \sim \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)] .$$

Moreover, at a point x, we have

$$\frac{f'(x+) + f'(x-)}{2} = \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)] .$$

If f' is continuous at a point x then

$$f'(x) = \sum_{n=1}^{\infty} [-na_n \sin(nx) + nb_n \cos(nx)] .$$

Proof: Since f' is piecewise continuous and this is sufficient condition for the existence of Fourier series of f'. So we can write Fourier series of as

$$f'(x) \sim \frac{\bar{a}_0}{2} + \sum_{n=1}^{\infty} \left[\bar{a}_n \cos(nx) + \bar{b}_n \sin(nx)\right] \tag{7.3}$$

where

$$\bar{a}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos(nx)\,\mathrm{d}x, \quad \bar{b}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin(nx)\,\mathrm{d}x$$

Now we simplify coefficients $\bar{a}_n$ and $\bar{b}_n$ and write them in terms of a_n and b_n. Using the condition $f(-\pi) = f(\pi)$, we can easily show that

$$\bar{a}_0 = 0, \quad \bar{a}_n = nb_n, \quad \bar{b}_n = -na_n$$

Now the Fourier series of f' (7.3) reduces to

$$f'(x) \sim \sum_{n=1}^{\infty} [nb_n \cos(nx) - na_n \sin(nx)]$$

Convergence of this series to $\dfrac{f'(x+) + f'(x-)}{2}$ or $f'(x)$ is a direct consequence of convergence theorem of Fourier series. ∎

7.2 Integration

In general, for an infinite series uniform convergence is required to integrate the series term by term. In the case of Fourier series we do not even have to assume the convergence of the Fourier series to be integrated. However, integration term by term of a Fourier series does not, in general, lead to a Fourier series. The main results can be summarize as:

7.2.1 Theorem

Let f be piecewise continuous function and have the following Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{7.4}$$

Then no matter whether this series converges or not we have for each $x \in [-\pi, \pi]$,

$$\int_{-\pi}^{x} f(t)dt = \frac{a_0(x+\pi)}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \Big(\cos(nx) - \cos n\pi \Big) \right] \tag{7.5}$$

and the series on the right hand side converges uniformly to the function on the left.

Proof: We define

$$g(x) = \int_{-\pi}^{x} f(t)dt - \frac{a_0}{2}x$$

Since f is piecewise continuous function, it is easy to prove that g is continuous. Also

$$g'(x) = f(x) - \frac{a_0}{2} \tag{7.6}$$

at each point of continuity of f. This implies that g' is piecewise continuous and further we see that

$$g(-\pi) = \frac{a_0\pi}{2}$$

and

$$g(\pi) = \int_{-\pi}^{\pi} f(t)dt - \frac{a_0}{2}\pi = \pi a_0 - \frac{a_0}{2}\pi = \frac{a_0 \pi}{2}$$

Hence, the Fourier series of the function g converges uniformly to g on $[-\pi, \pi]$. Thus we have

$$g(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} [\alpha_n \cos(nx) + \beta_n \sin(nx)]$$

Using Theorem 7.1.1 we have the following result for the Fourier series of g' as

$$g'(x) \sim \sum_{n=1}^{\infty} [-n\alpha_n \sin(nx) + n\beta_n \cos(nx)]$$

Fourier series of f and the relation (7.6) gives

$$g'(x) = f(x) - \frac{a_0}{2} \sim \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Now comparing the last two equations we get

$$n\beta_n = a_n \qquad -n\alpha_n = b_n \quad n = 1, 2, \ldots$$

Substituting these values in the Fourier series of g we obtain

$$g(x) = \int_{-\pi}^{x} f(t)dt - \frac{a_0}{2}x = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \cos(nx) \right]$$

We can rewrite this to get

$$\int_{-\pi}^{x} f(t)dt = \frac{a_0}{2}x + \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \cos(nx) \right] \tag{7.7}$$

To obtain α_0 we set $x = \pi$ in the above equation

$$\alpha_0 = a_0\pi + \sum_{n=1}^{\infty} \frac{2b_n}{n} \cos(n\pi)$$

Substituting α_0 in the equation (7.7) we obtain the required result (7.5). ■

Remark 1: *Note that the series on the right hand side of (7.5) is not a Fourier series due to presence of x.*

Remark 2: *The above Theorem on integration can be established in a more general sense as:*
If f be piecewise continuous function in $-\pi \le x \le \pi$ and if

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

is its Fourier series then no matter whether this series converges or not, it is true that

$$\int_a^x f(t)dt = \frac{a_0}{2} \int_a^x a_0 \,\mathrm{d}x + \sum_{n=1}^{\infty} \int_a^x [a_n \cos(nx) + b_n \sin(nx)] \,\mathrm{d}x$$

where $-\pi \le a \le x \le \pi$ and the series on the right hand side of converges uniformly in x to the function on the left for any fixed value of a.

In this lesson some properties of the Fourier coefficients will be given. We will mainly derive two important inequalities related to Fourier series, in particular, Bessel's inequality and Parseval's identity. One of the applications of Parseval's identity for summing certain infinite series will be discussed.

8.1 Theorem (Bessel's Inequality)

If f be a piecewise continuous function in $[-\pi,\pi]$, then

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)\leq\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,\mathrm{d}x$$

where $a_0,a_1,\ldots$ and $b_1,b_2,\ldots$ are Fourier coefficients of f.

Proof: Clearly, we have

$$\int_{-\pi}^{\pi}\left[f(x)-\frac{a_0}{2}-\sum_{k=1}^{n}\left[a_k\cos(kx)+b_k\sin(kx)\right]\right]^2\,\mathrm{d}x\geq 0$$

Expanding the integrands we get

$$\int_{-\pi}^{\pi}f^2(x)\,\mathrm{d}x+\frac{a_0^2}{2}\pi+\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}\left[a_k\cos(kx)+b_k\sin(kx)\right]\right]^2\,\mathrm{d}x-a_0\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x$$
$$-2\int_{-\pi}^{\pi}f(x)\left[\sum_{k=1}^{n}\left[a_k\cos(kx)+b_k\sin(kx)\right]\right]\,\mathrm{d}x+a_0\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}\left[a_k\cos(kx)+b_k\sin(kx)\right]\right]\,\mathrm{d}x\geq 0$$

Using the orthogonality of the trigonometric system and definition of Fourier coefficients we get

$$\int_{-\pi}^{\pi}f^2(x)\,\mathrm{d}x+\frac{a_0^2}{2}\pi+\pi\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)-a_0^2\pi-2\pi\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)+0\geq 0$$

This can be further simplified

$$\int_{-\pi}^{\pi}f^2(x)\,\mathrm{d}x-\frac{a_0^2}{2}\pi-\pi\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)\geq 0$$

This implies

$$\frac{a_0^2}{2}+\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)\leq\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,\mathrm{d}x$$

Passing the limit $n \to \infty$, we get the required Bessel's inequality. ∎

Indeed the above Bessel's inequality turns into an equality named Parseval's identity. However, for the sake of simplicity of proof we state the following theorem for more restrictive function but the result holds under less restrictive conditions (only piecewise continuity) same as in Theorem 8.1.

8.2 Theorem (Parseval's Identity)

If f is a continuous function in $[-\pi, \pi]$ and one sided derivatives exit then we have the equality

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x \tag{8.1}$$

where $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ are Fourier coefficients of f.

Proof: From the Dirichlet's convergence theorem for $x \in (-\pi, \pi)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

Integrating by $f(x)$ and integrating term by term from $-\pi$ to π we obtain

$$\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{a_0}{2} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \right)$$

Using the definition of Fourier coefficients we get

$$\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{\pi a_0^2}{2} + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

Dividing by π we obtain the required identity. ∎

Remark: *As stated earlier Parseval's identity can be proved for piecewise continuous functions. Further, for a piecewise continuous function on $[-L, L]$ we can get Parseval's identity just by replacing π by L in (8.1).*

8.3 Example Problems

8.3.1 Problem 1

Consider the Fourier cosine series of $f(x) = x$:

$$x \sim 1 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}[\cos(n\pi) - 1] \cos \frac{n\pi x}{2}$$

a) Write Parseval's identity corresponding to the above Fourier series

b) Determine from a) the sum of the series

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

Solution: a) We first find the Fourier coefficient and the period of the Fourier series just by comparing the given series with the standard Fourier series

$$a_0 = 2, \quad a_n = \frac{4}{\pi^2 n^2}[\cos(n\pi) - 1],\ n = 1, 2 \dots, \quad b_n = 0$$
$$\text{period } = 2L = 4 \Rightarrow L = 2$$

Writing Parseval's identity as

$$\frac{1}{L} \int_{-L}^{L} f^2(x)\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

This implies

$$\frac{1}{2} \int_{-2}^{2} x^2\,\mathrm{d}x = \frac{4}{2} + \sum_{n=1}^{\infty} \frac{16}{\pi^4 n^4}(\cos(n\pi) - 1)^2$$

This can be simplified to give

$$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

Then we obtain

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

b) Let

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

This series can be rewritten as

$$
\begin{aligned}
S &= \Big(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\Big)+\Big(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\dots\Big)\\
&= \frac{\pi^4}{96}+\frac{1}{2^4}S
\end{aligned}
$$

Then we have the required sum as $S = \dfrac{\pi^4}{90}$.

8.3.2 Problem 2

Find the Fourier series of x^2, $-\pi < x < \pi$ and use it along with Parseval's theorem to show that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Solution: Since $f(x) = x^2$ is an even function, so $b_n = 0$. The Fourier coefficients a_n will be given as

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0 \pi x^2\cos(nx)\,\mathrm{d}x$$

This can be further simplified for $n \neq 0$ to

$$a_n = \frac{2}{\pi}\left[0 - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,\mathrm{d}x\right] = \frac{4}{n^2}(-1)^n$$

The coefficient a_0 can be evaluated separately as

$$a_0 = \frac{2}{\pi}\int_0 \pi x^2\,\mathrm{d}x = \frac{2\pi^2}{3}$$

The the Fourier series of $f(x) = x^2$ will be given as

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

Now by parseval's theorem we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Using $\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 \,\mathrm{d}x = \frac{2\pi^4}{5}$ we get

$$\frac{4\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{2\pi^4}{5}$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Now using the idea of splitting of the series from the Example 8.3.1 (b), we have

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \sum_{n=1}^{\infty} \frac{1}{n^4} - \frac{1}{16}\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{15}{16}\sum_{n=1}^{\infty} \frac{1}{n^4}$$

Substituting the value of $\sum_{k=1}^{\infty} \frac{1}{n^4}$ in the above equation we get the required sum.

8.3.3 Problem 3

Given the Fourier series

$$\cos\left(\frac{x}{2}\right) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2-1)} \cos(nx)$$

deduce the value of

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}.$$

Solution: By Parseval's theorem for

$$a_0 = \frac{4}{\pi},\ a_n = \frac{4}{\pi}\frac{(-1)^{n+1}}{(4n^2-1)},\ f(x) = \cos(x/2)$$

we have

$$\frac{1}{2}\frac{16}{\pi^2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(x/2)\,\mathrm{d}x = 1$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}.$$