Cantilever with point load at free end

- This is the easiest problem to solve. The BMD is simple and so are the equations obtained from boundary conditions.

Cantilever with point load at free end

- Draw the FBD
- At the fixed end, since neither rotation or translation is permitted there will be both a force and a moment as reactions

Cantilever with point load at free end

- Write the equilibrium equations. Here moments are being taken about A.

$$R_A = P, M_A = PL$$

Cantilever with point load at free end

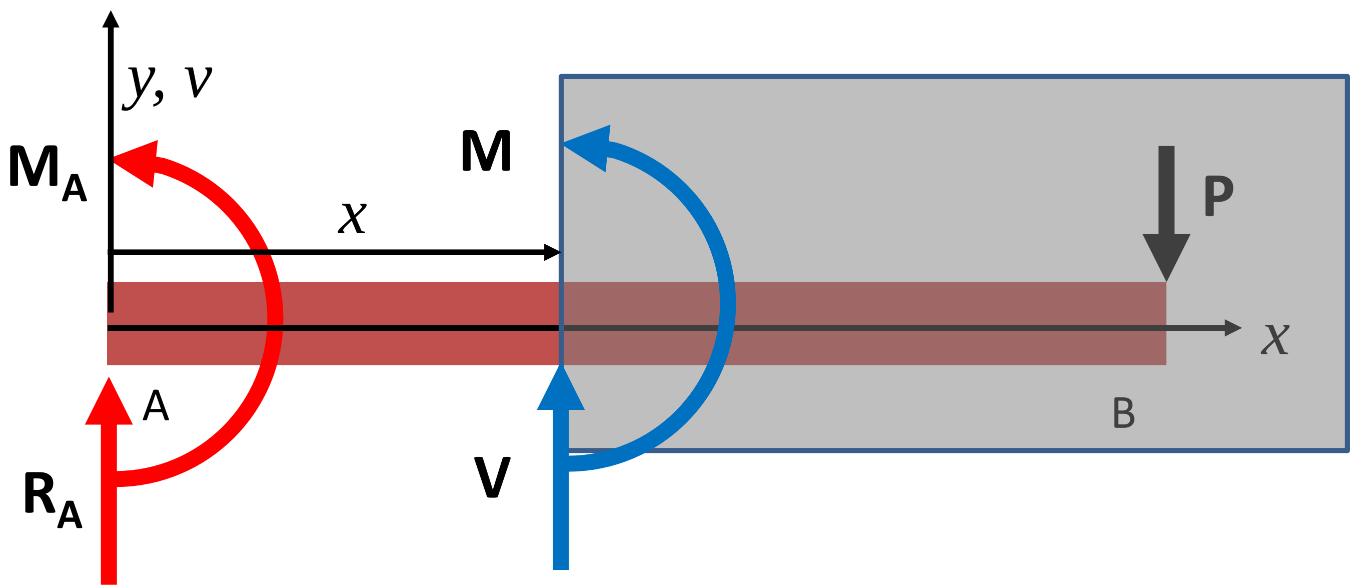
- Set up a coordinate system
- We choose A to be the origin, x as positive from A to B and y,v as positive upwards

Cantilever with point load at free end

- There is only one domain to be considered here – AB
- We need to take one section only

Cantilever with point load at free end

- Section is taken at distance x from A
- In case of a beam, which can bend, a vertical shear force and a bending moment will show up at the cut as internal (generalized) forces.

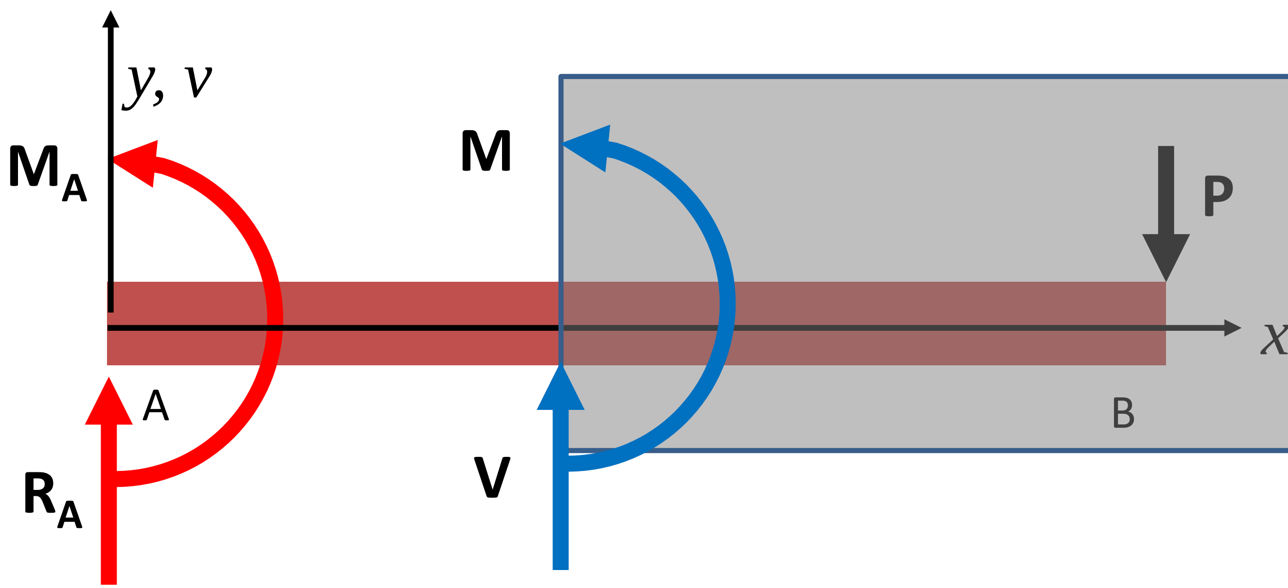


Cantilever with point load at free end

- Solve equilibrium equations

$$V + R_A = 0 \Rightarrow V(x) = -R_A = -P$$

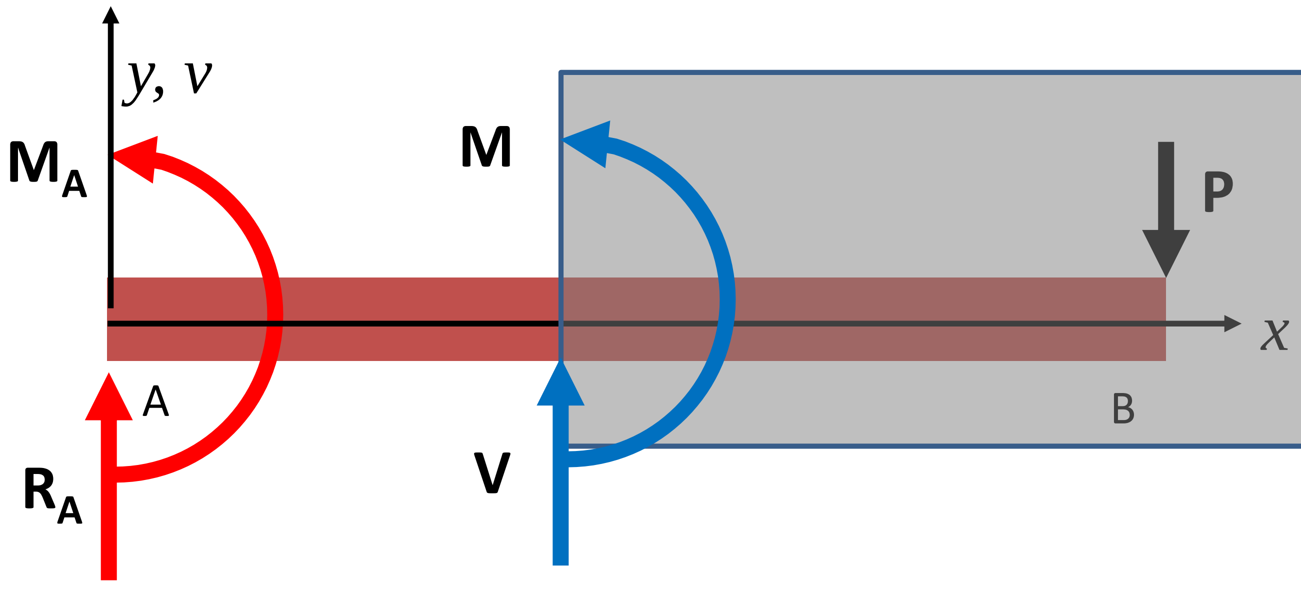
$$M + M_A + Vx = 0 \Rightarrow M(x) = -PL + Px$$



Cantilever with point load at free end

- Use flexure equation

$$EIv'' = -PL + Px$$

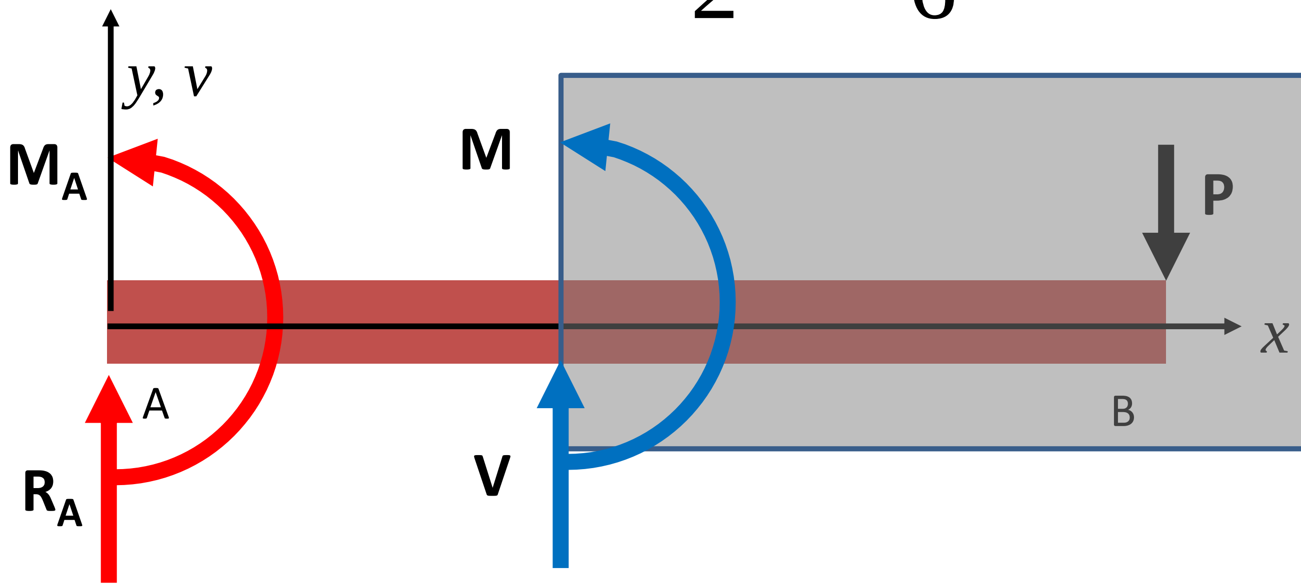


Cantilever with point load at free end

- Integrate twice to get

$$EIv' = -PLx + \frac{Px^2}{2} + C_1$$

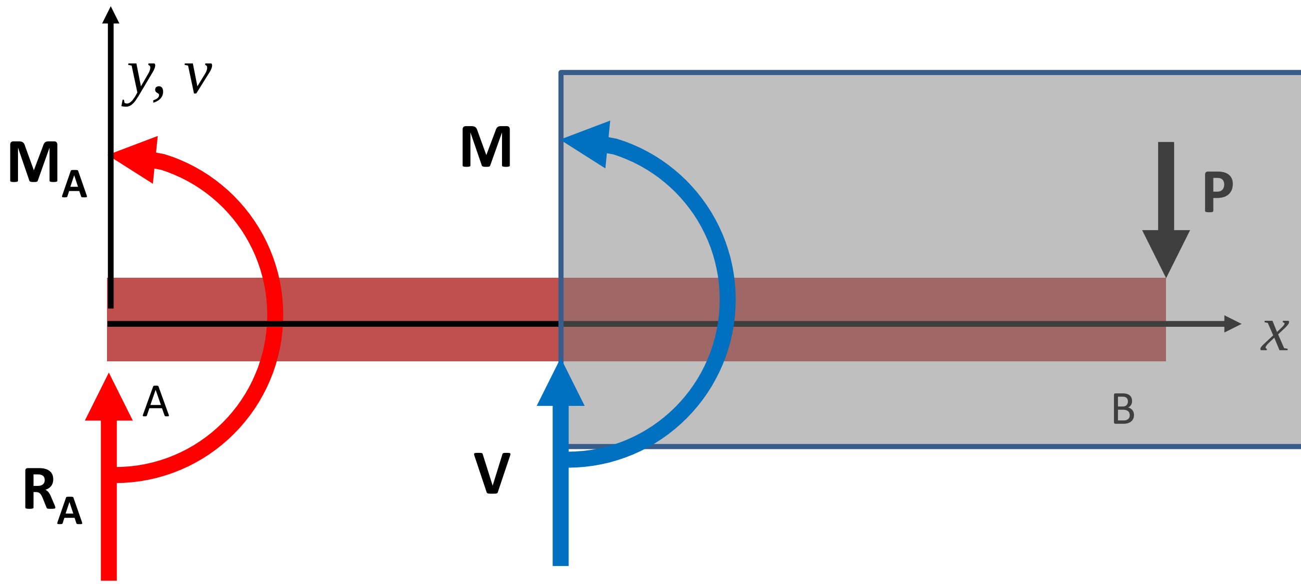
$$EIv = -PL\frac{x^2}{2} + \frac{Px^3}{6} + C_1 x + C_2$$



Cantilever with point load at free end

- The boundary conditions are both slope and deflection at the fixed end, i.e. origin are zero

$$v'(0) = 0, v(0) = 0$$

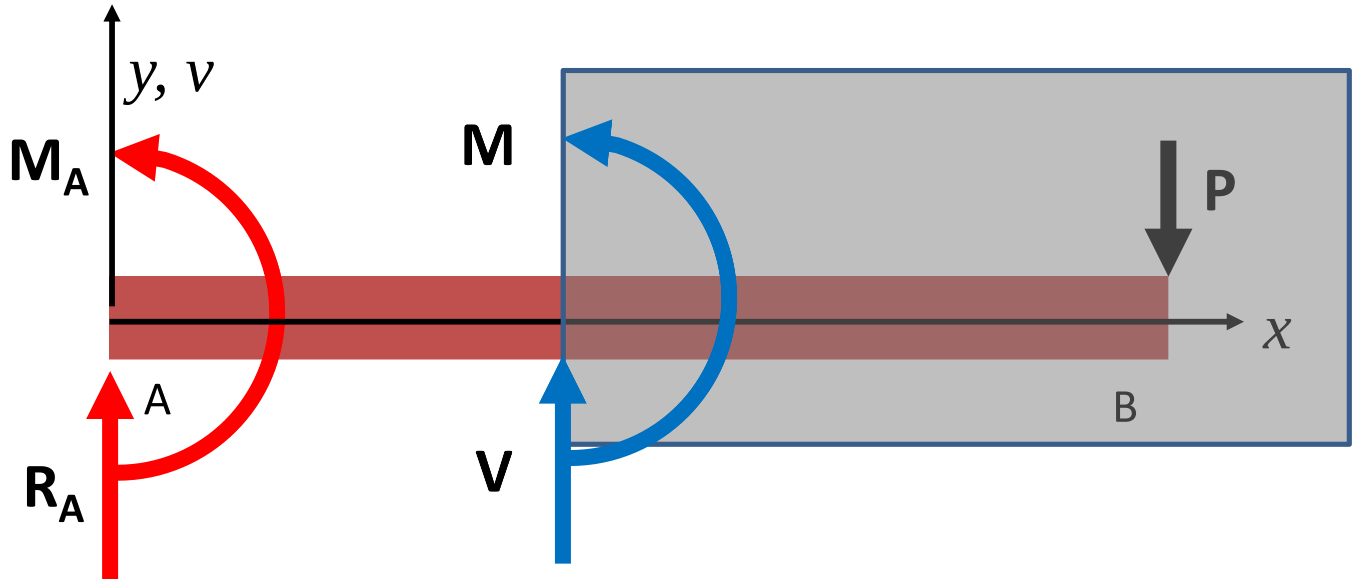


Cantilever with point load at free end

- Applying boundary conditions (BCs) we get

$$EIv'(0) = -PL.0 + \frac{P.0^2}{2} + C_1 = 0 \Rightarrow C_1 = 0$$

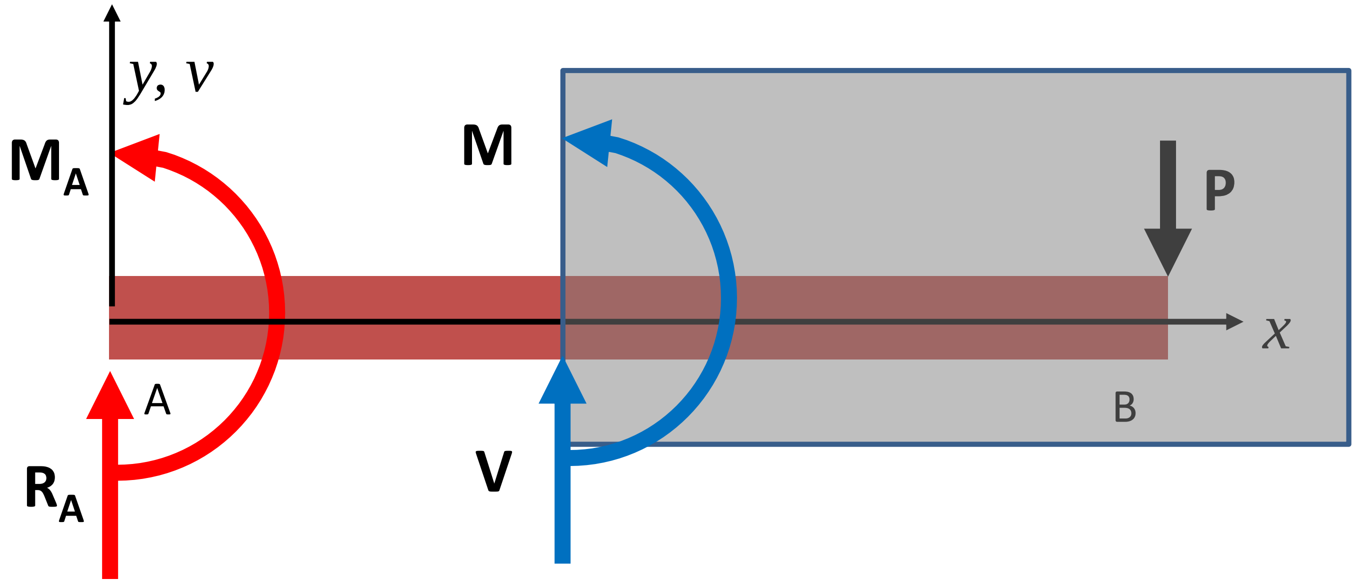
$$EIv(0) = -PL\frac{0^2}{2} + \frac{P.0^3}{6} + 0.0 + C_2 \Rightarrow C_2 = 0$$



Cantilever with point load at free end

- Thus the deflection curve is

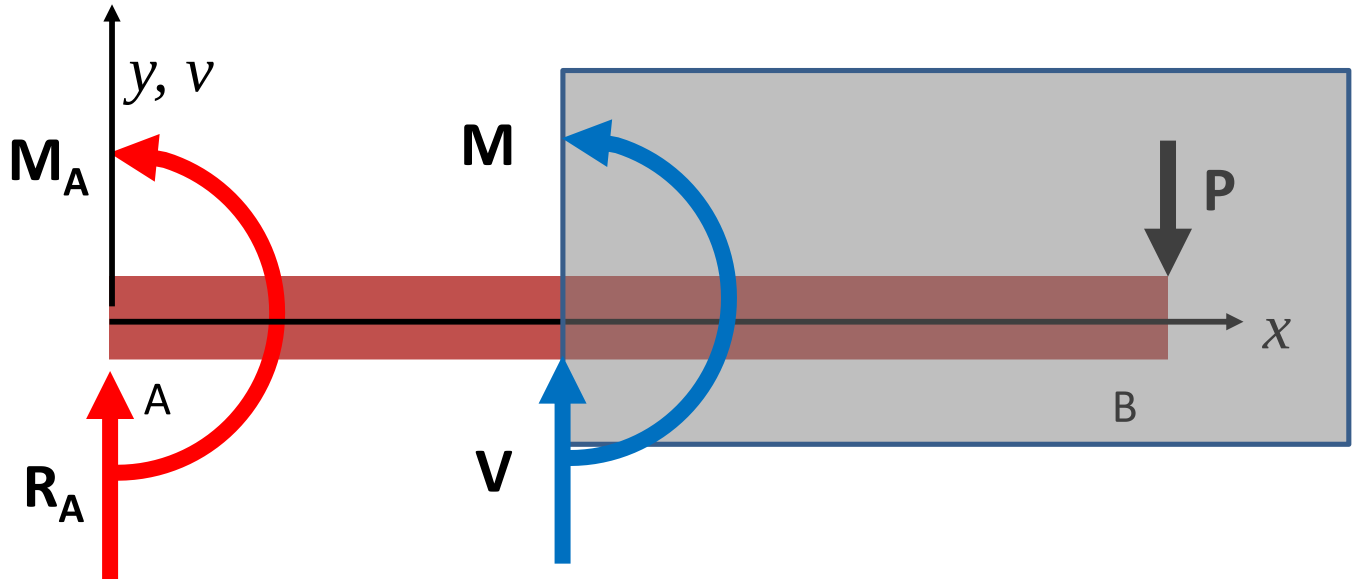
$$v(x) = -\frac{PLx^2}{2EI} + \frac{Px^3}{6EI}$$



Cantilever with point load at free end

- Some useful information

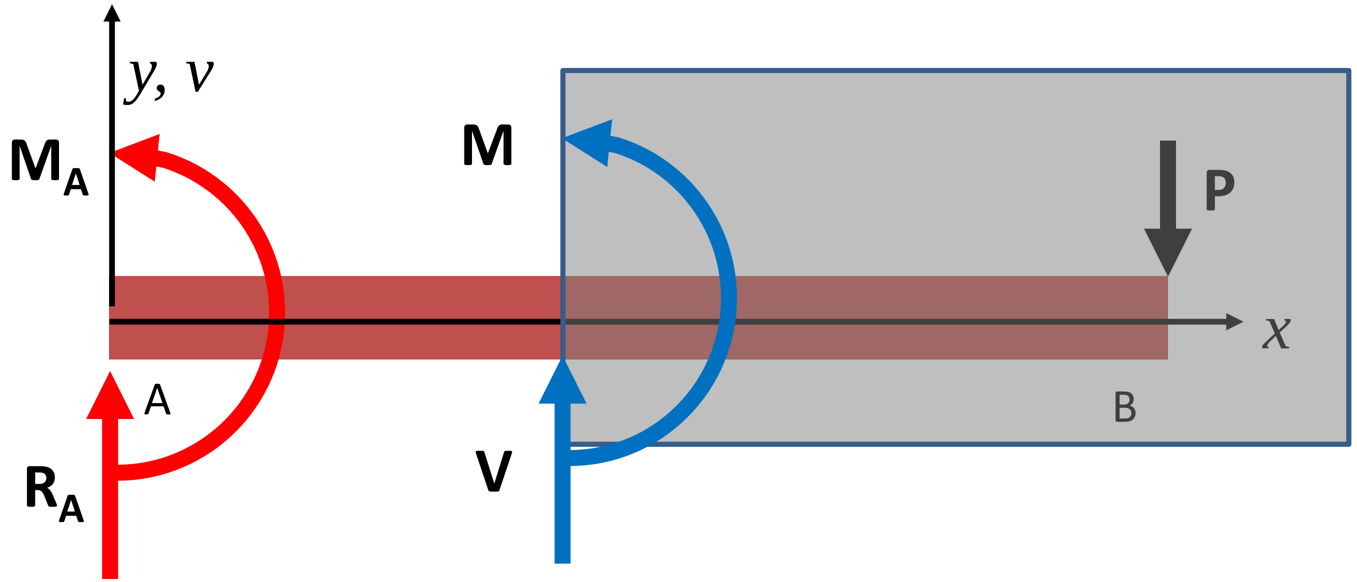
$$v(L) = -\frac{PL^3}{2EI} + \frac{PL^3}{6EI} = \frac{PL^3}{3EI}$$



Cantilever with point load at free end

- Some useful information

$$v'(L) = -\frac{PL^2}{EI} + \frac{PL^2}{2EI} = -\frac{PL^2}{2EI}$$



Cantilever with point load at free end

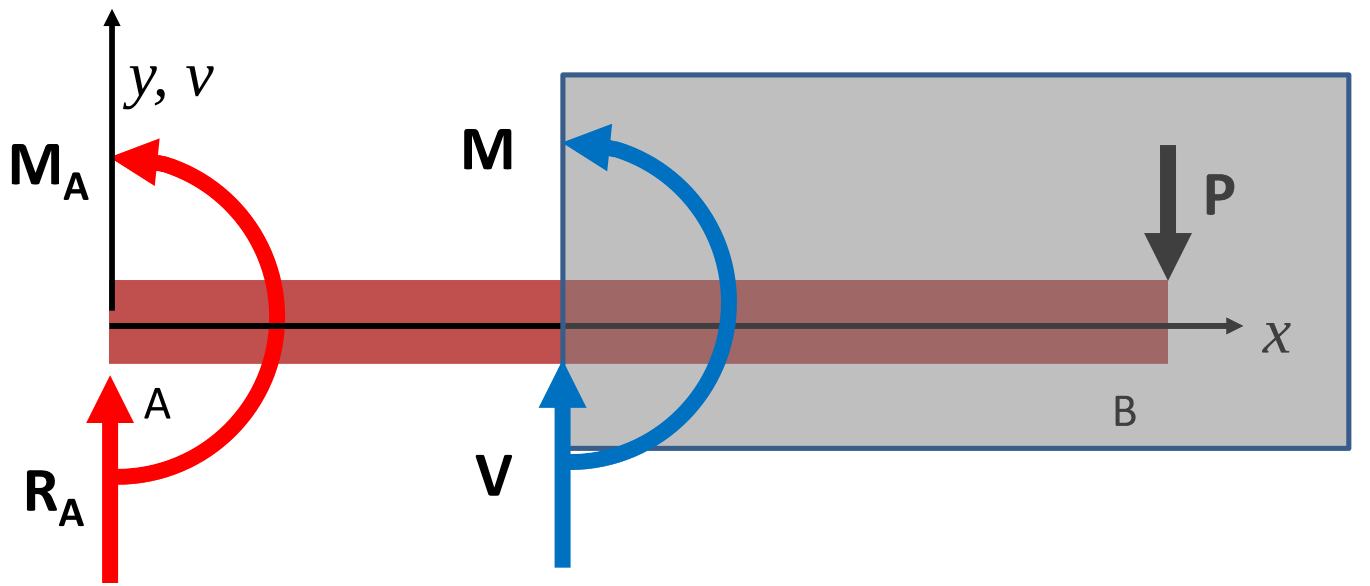
- The extrema for v is at x=0,2L
- The minimum is at x=0
- Since v''<0 within AB, maximum deflection is at x=L
- Maximum slope is also at x=L

$$v'(x) = 0 \Rightarrow -\frac{PLx}{EI} + \frac{Px^2}{2EI} = 0$$

$$\Rightarrow x = 0, x = 2L$$

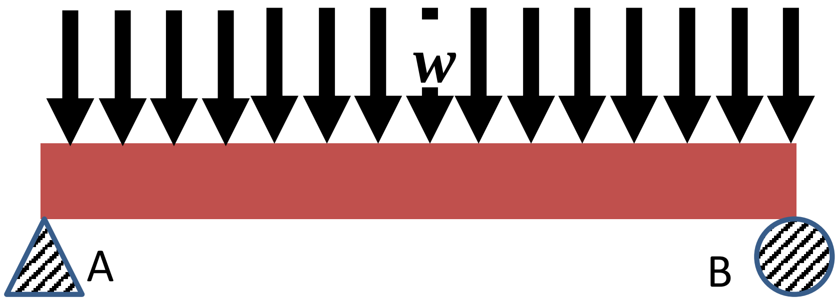
$$v''(x) = -\frac{P(L-x)}{EI} \leq 0$$

$$v''(0) = 0 \Rightarrow x = L$$



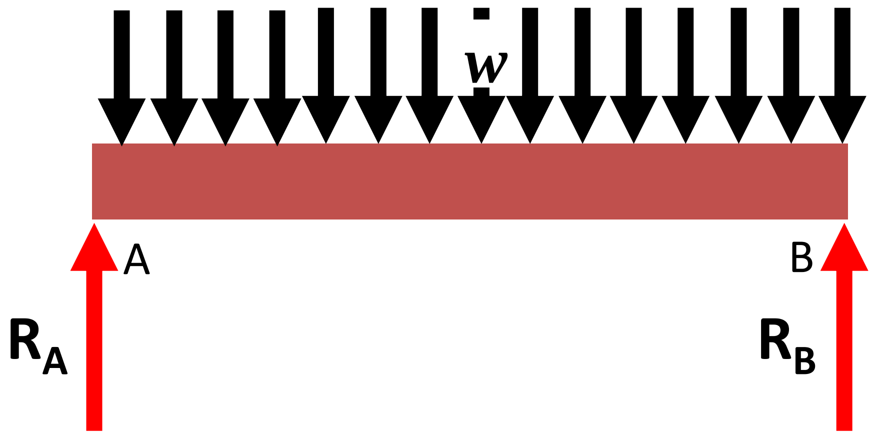
Simply supported beam with uniformly distributed load

- This is another easy problem to solve. The BMD is simple and so are the equations obtained from boundary conditions.



Simply supported beam with uniformly distributed load

- Draw the FBD
- At both ends, since pin (or roller) permits rotation but no (vertical) translation there will be only a force as reaction at each end

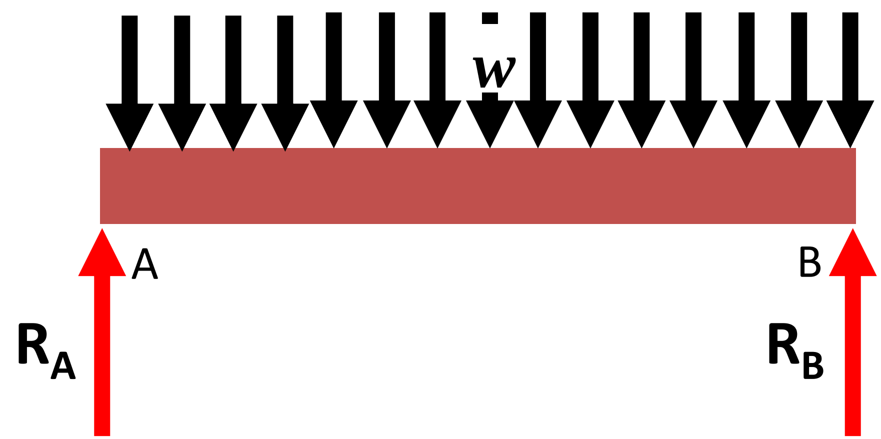


Simply supported beam with uniformly distributed load

- Write the equilibrium equations. Here moments are being taken about A.

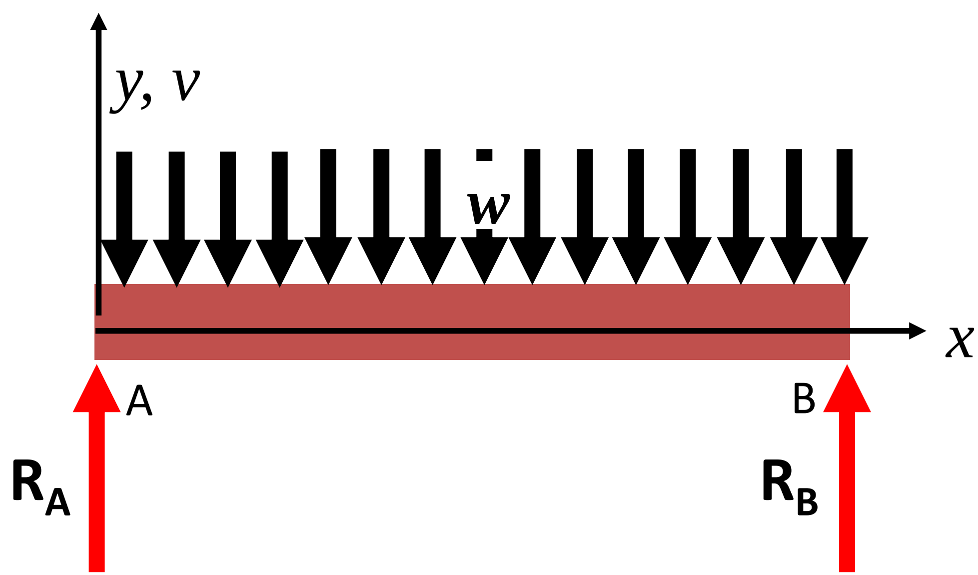
$$R_A + R_B = \int_0^L w dx = wL, R_B = \int_0^L x(w dx) = \frac{wL^2}{2}$$

$$\therefore R_A = R_B = \frac{wL}{2}$$



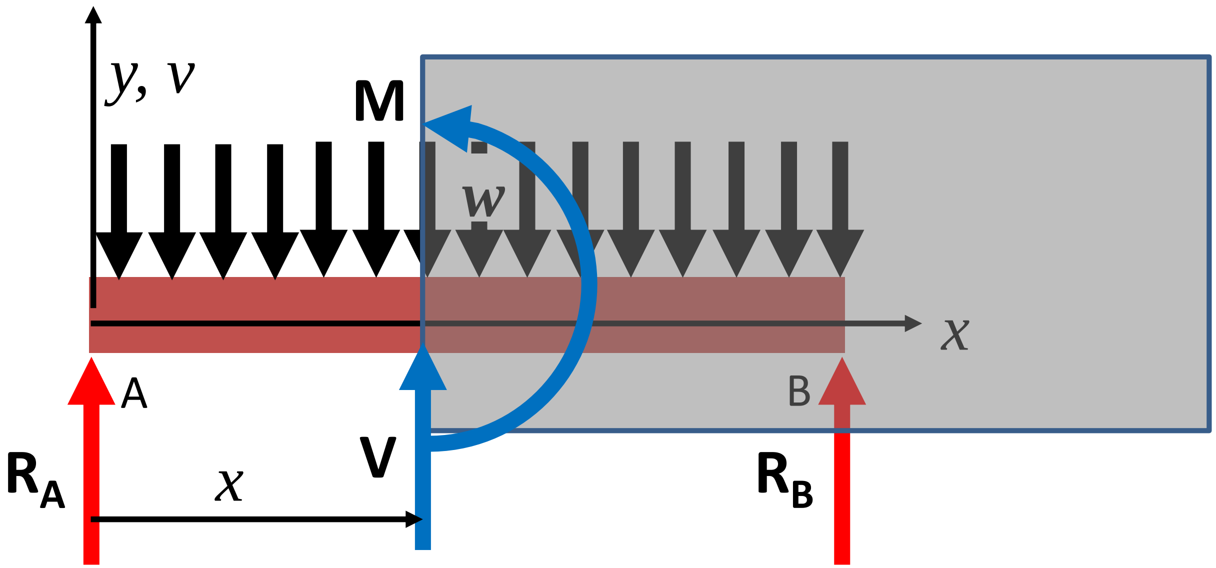
Simply supported beam with uniformly distributed load

- Set up a coordinate system
- We choose A to be the origin, x as positive from A to B and y,v as positive upwards
- There is only one domain to be considered here – AB
- We need to take one section only between A and B



Simply supported beam with uniformly distributed load

- Section is taken at distance x from A. For this section, while integrating for forces and moments, since the integral will be from 0 to x. Since the limit involves x we will be using a different variable ξ under the integral sign

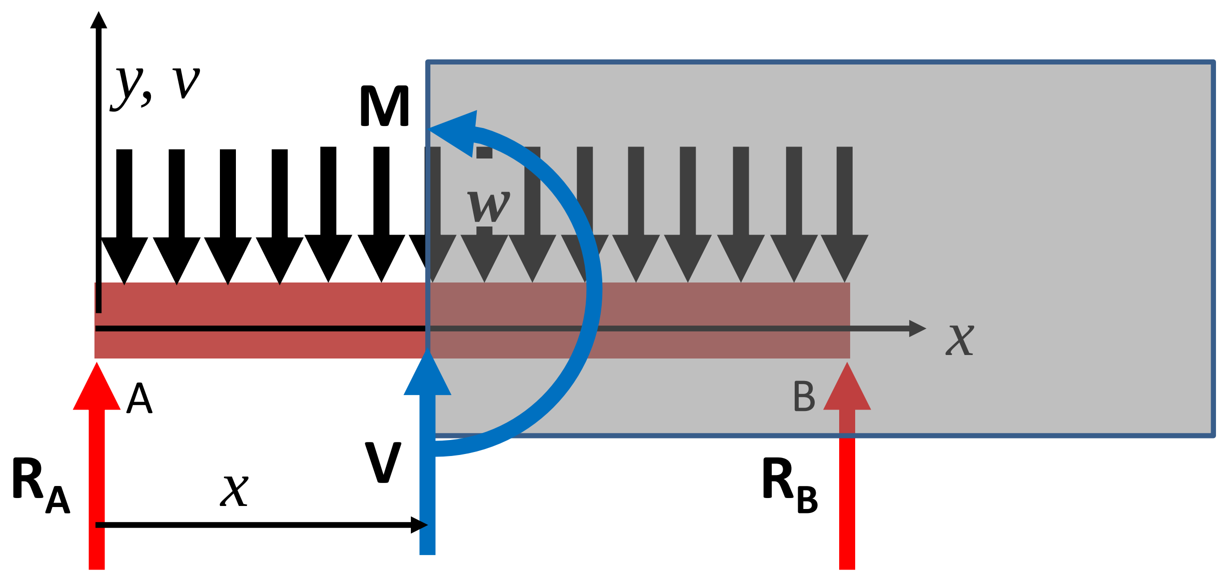


Simply supported beam with uniformly distributed load

- Solve equilibrium equations

$$V + R_A - \int_0^x w d\xi = 0 \Rightarrow V(x) = w\left(x - \frac{L}{2}\right)$$

$$M + Vx - \int_0^x \xi \left(w d\xi\right) = 0 \Rightarrow M(x) = \frac{wx}{2}\left(L - x\right)$$

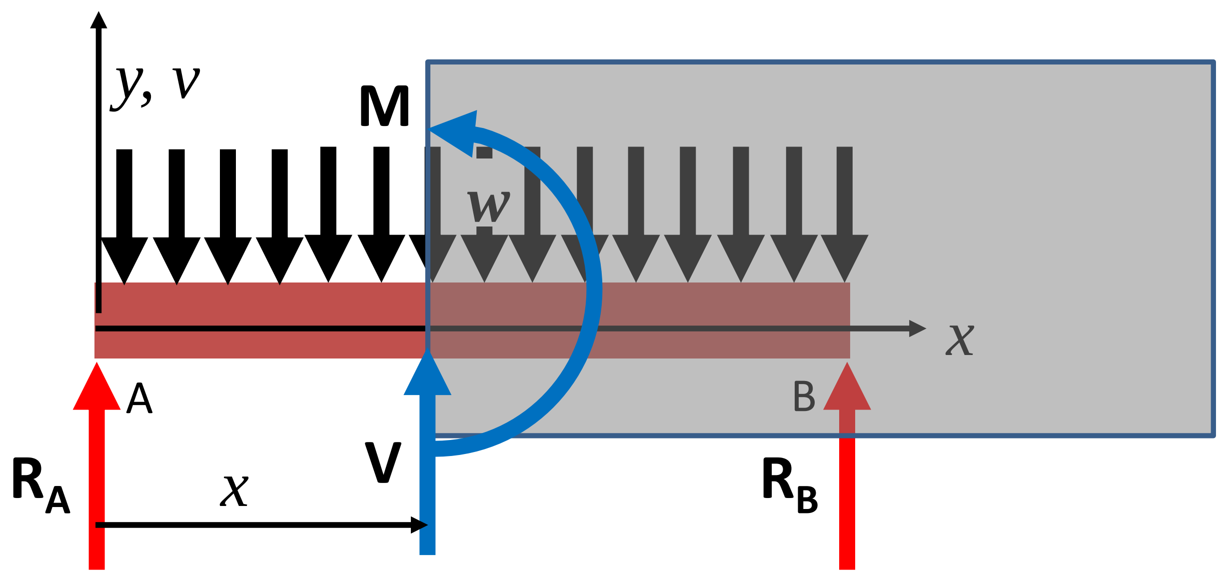


Simply supported beam with uniformly distributed load

- Solve the flexure equation

$$EIv'' = \frac{wx}{2}(L-x) \Rightarrow EIv' = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1$$

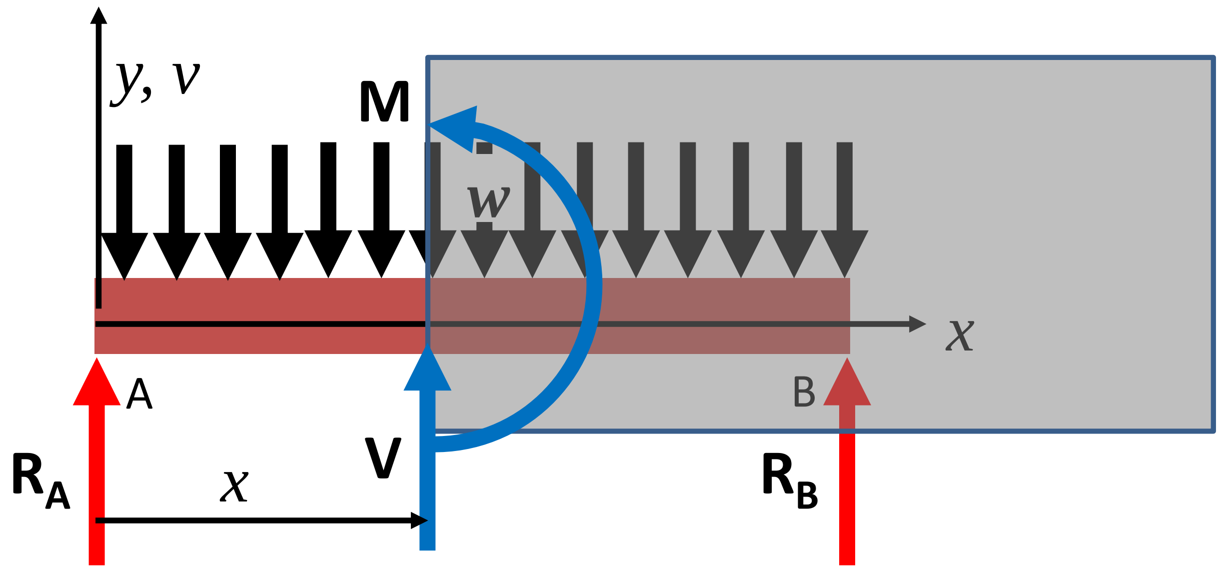
$$\Rightarrow EIv = \frac{wLx^3}{12} - \frac{wx^4}{24} + C_1 x + C_2$$



Simply supported beam with uniformly distributed load

- The boundary conditions are deflection at the two ends are zero

$$v(0)=0, v(L)=0$$

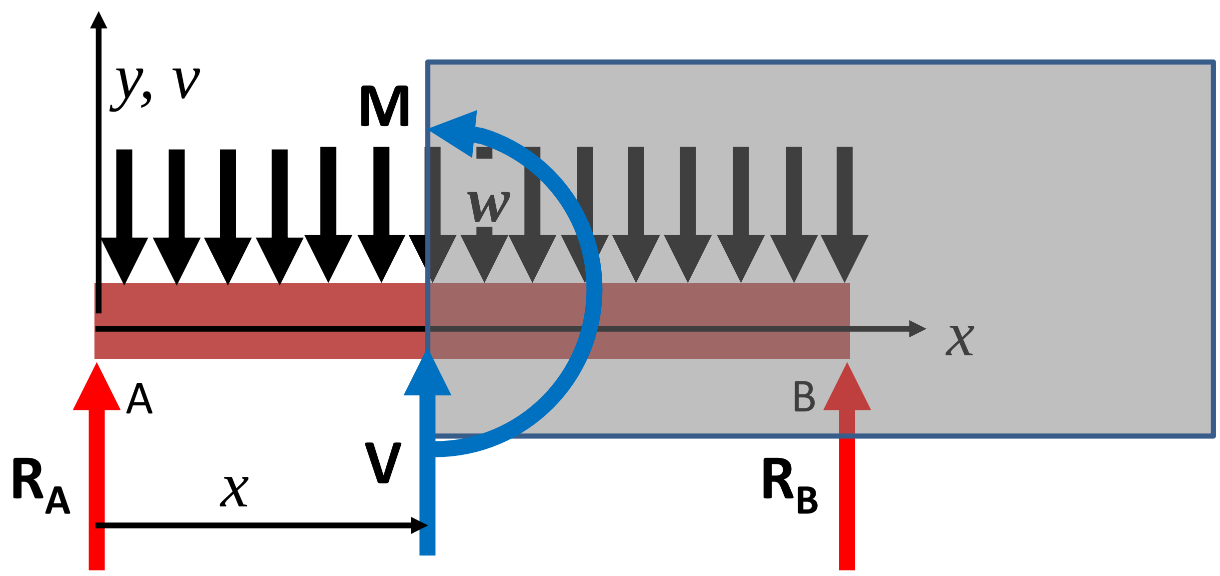


Simply supported beam with uniformly distributed load

- Applying boundary conditions (BCs) we get

$$v(0)=0 \Rightarrow C_2=0$$

$$v(L)=0 \Rightarrow \frac{wL^4}{12}-\frac{wL^4}{24}+C_1L=0 \Rightarrow C_1=-\frac{wL^3}{24}$$

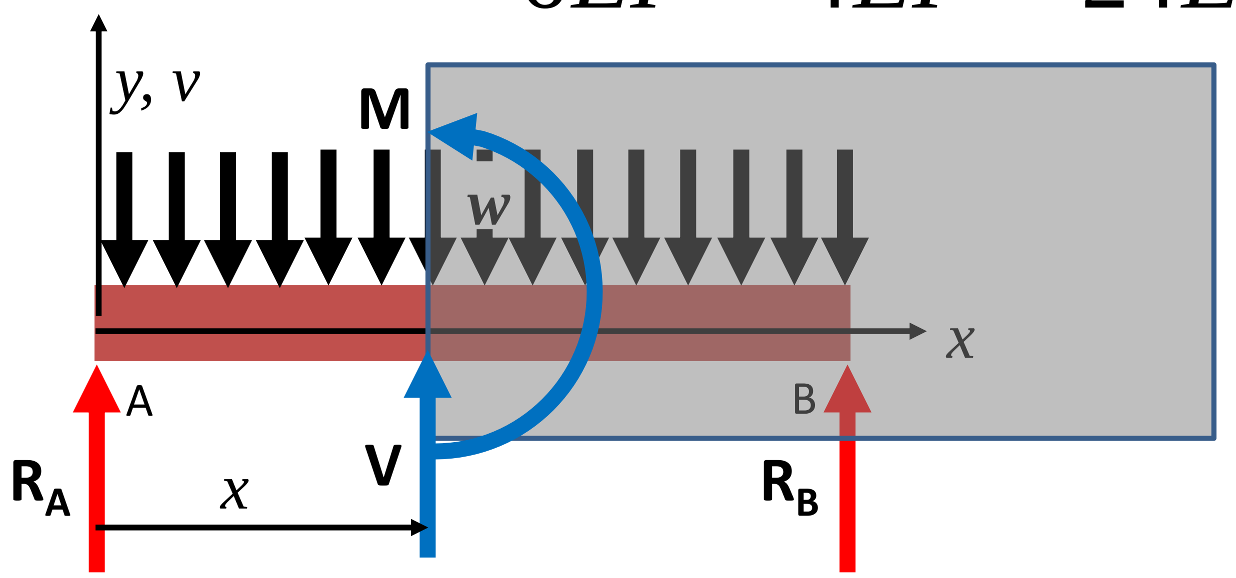


Simply supported beam with uniformly distributed load

- Thus the equation of the deflection curve and its gradient are

$$v(x) = -\frac{wx^4}{24EI} + \frac{wLx^3}{12EI} - \frac{wL^3x}{24EI}$$

$$v'(x) = -\frac{wx^3}{6EI} + \frac{wLx^2}{4EI} - \frac{wL^3}{24EI}$$

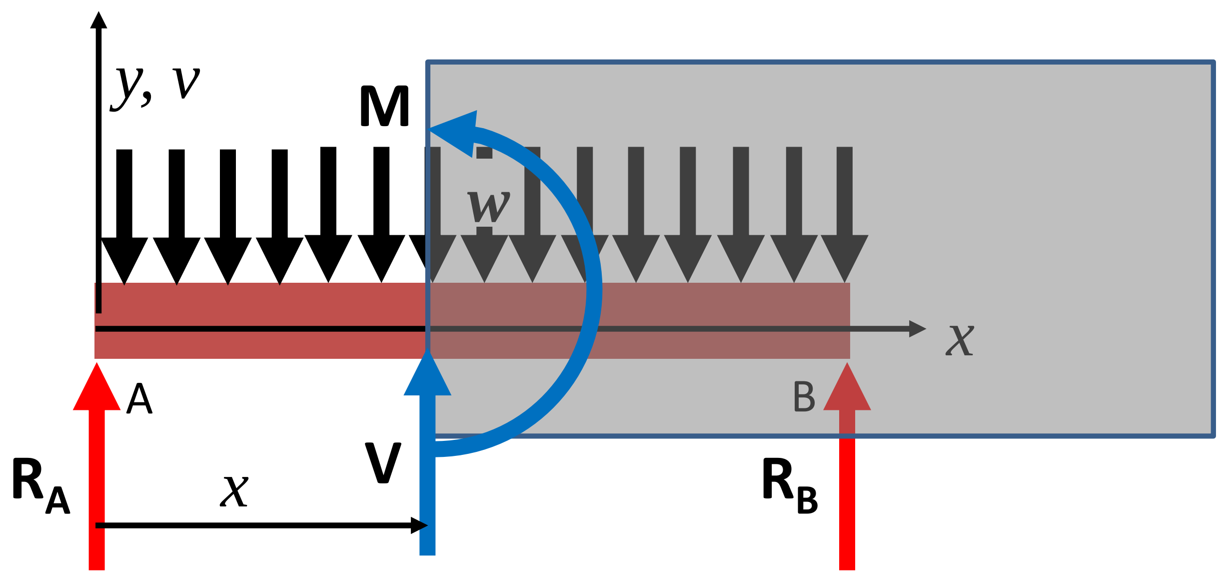


Simply supported beam with uniformly distributed load

- Slope is zero at ?

$$v'(x)=0 \Rightarrow -\frac{wx^3}{6EI}+\frac{wLx^2}{4EI}-\frac{wL^3}{24EI}=0$$

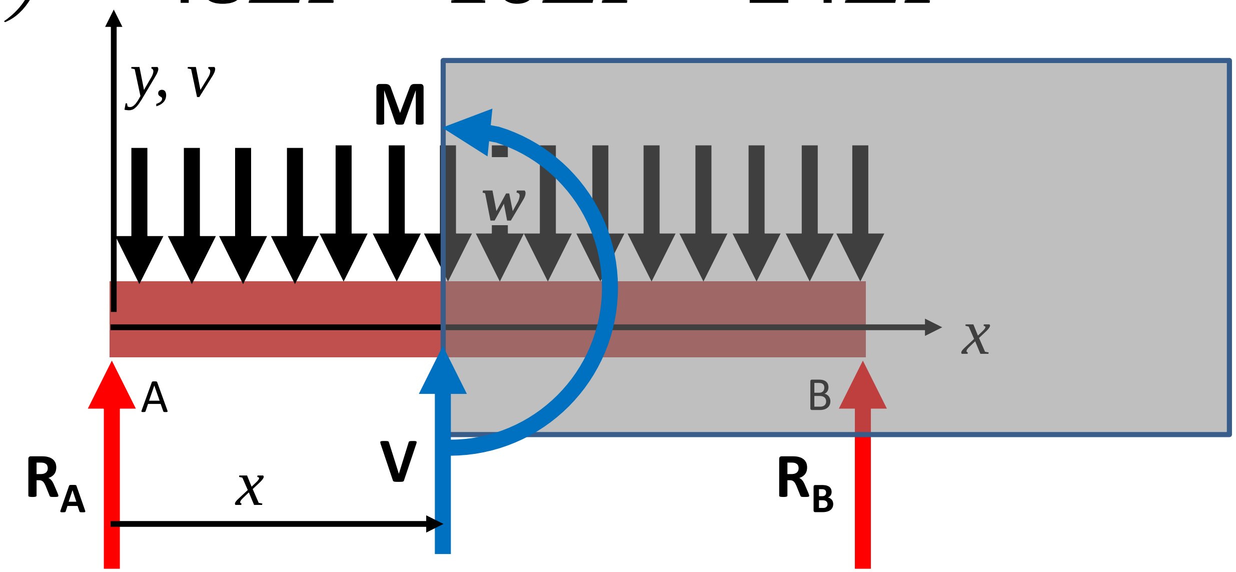
$$-4x^3+6Lx^2-L^3=0$$



Simply supported beam with uniformly distributed load

- Solving this equation is not easy
- We can say from symmetry considerations that deflection must be maximum at midpoint and verify by checking the slope at midpoint

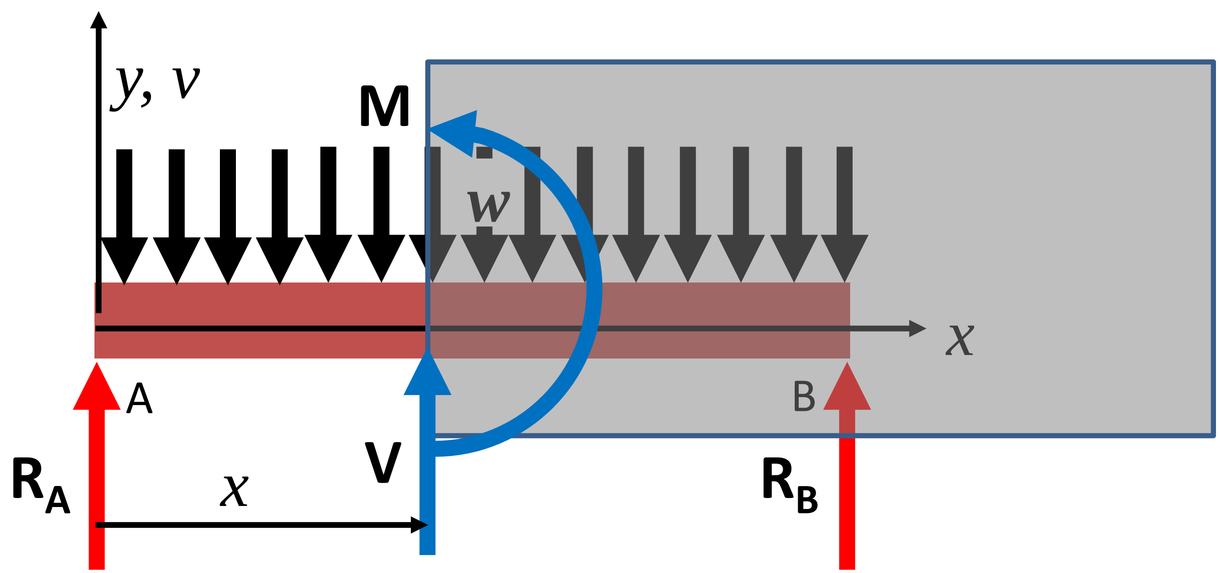
$$v'\left(\frac{L}{2}\right)=\frac{wL^3}{48EI}+\frac{wL^3}{16EI}-\frac{wL^3}{24EI}=0$$



Simply supported beam with uniformly distributed load

- Maximum deflection (at midpoint) is

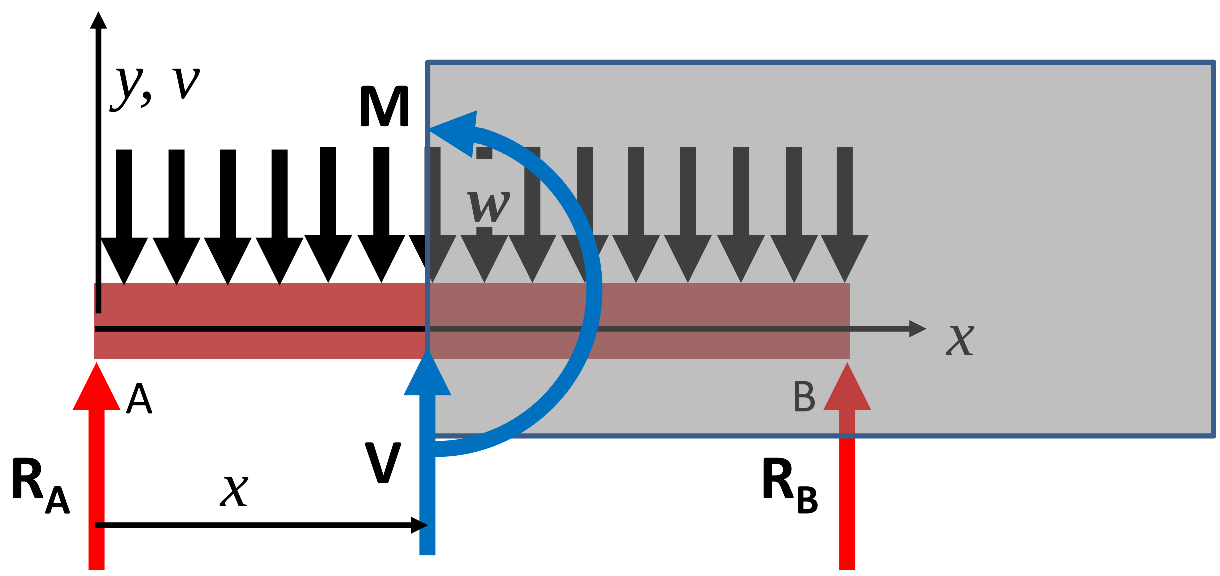
$$v\left(\frac{L}{2}\right) = -\frac{wL^4}{384EI} + \frac{wL^4}{96EI} - \frac{wL^4}{48EI} = -\frac{5wL^4}{384EI}$$



Simply supported beam with uniformly distributed load

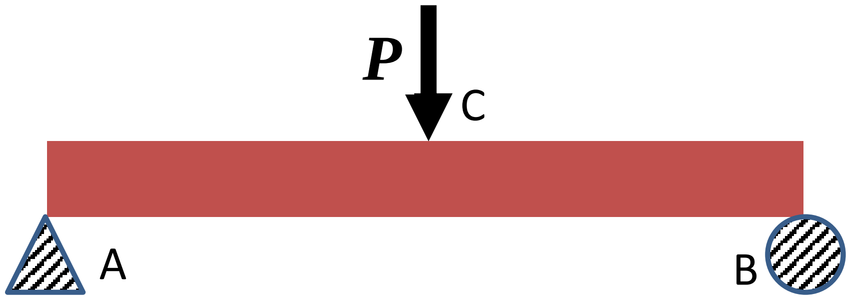
- Slopes (gradients) at the supports are equal in magnitude but opposite in sign and are

$$v'(0) = -v'(L) = -\frac{wL^3}{24EI}$$



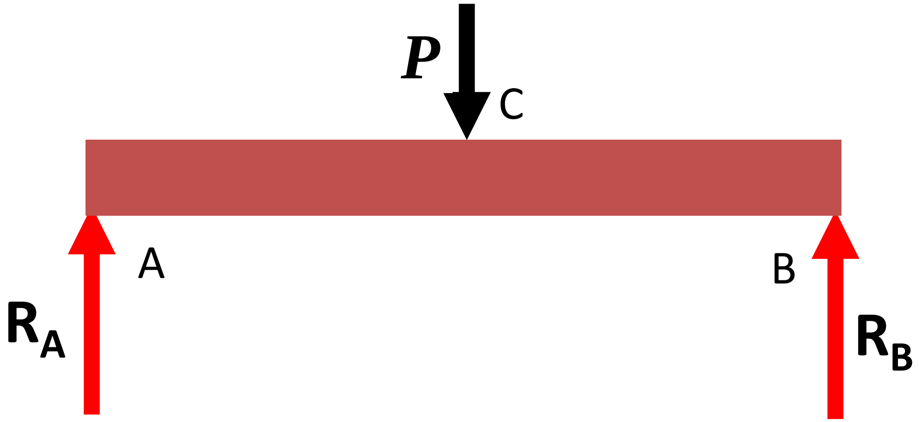
Simply supported beam with point load in the middle

- This is a slightly complicated problem although it does not look that way. The BMD is has a discontinuity at the midpoint C.
- Hence two domains are required for analysis and additional boundary conditions at the midpoint also need to be considered.



Simply supported beam with point load in the middle

- Draw the FBD
- At both ends, since pin (or roller) permits rotation but no (vertical) translation there will be only a force as reaction at each end

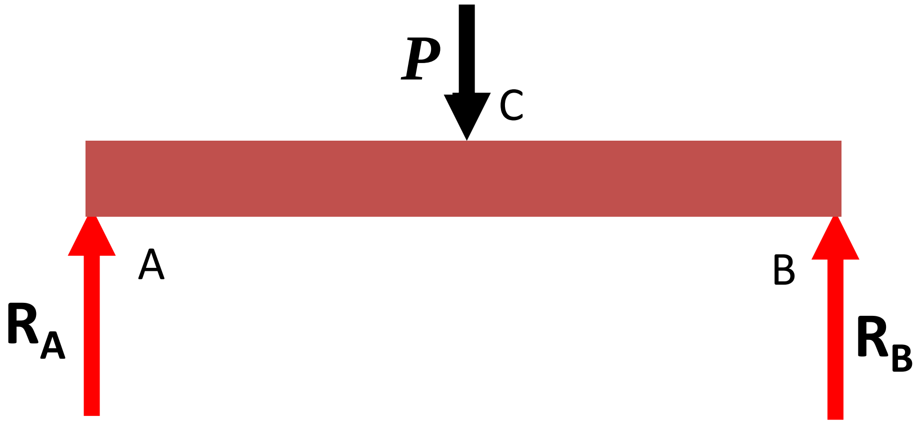


Simply supported beam with point load in the middle

- Write the equilibrium equations. Here moments are being taken about A.

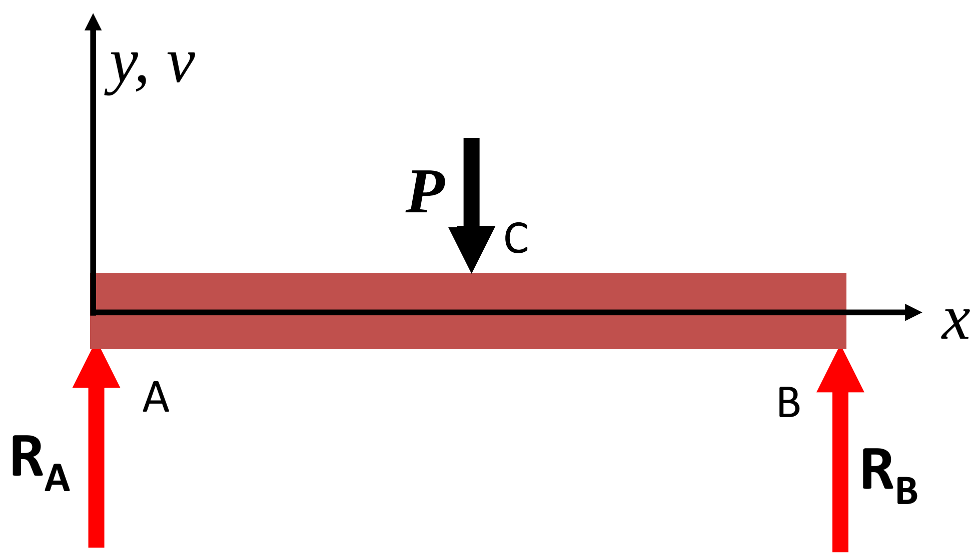
$$R_A + R_B = P, R_B = \frac{P}{2}$$

$$\therefore R_A = R_B = \frac{P}{2}$$



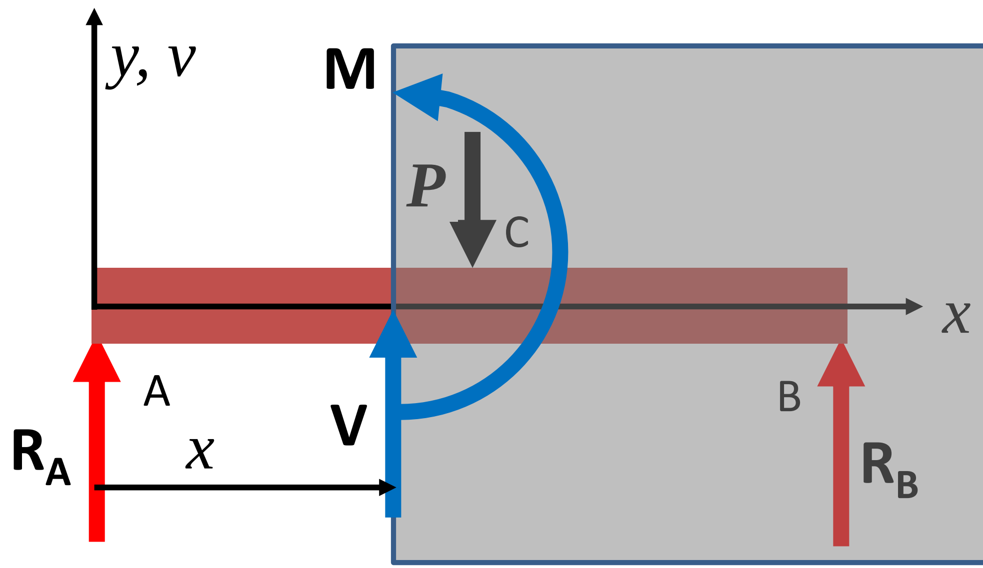
Simply supported beam with point load in the middle

- Set up a coordinate system
- We choose A to be the origin, x as positive from A to B and y,v as positive upwards
- There are 2 domains to be considered here – AC and CB



Simply supported beam with point load in the middle

- Domain AC. Section is taken at distance x from A

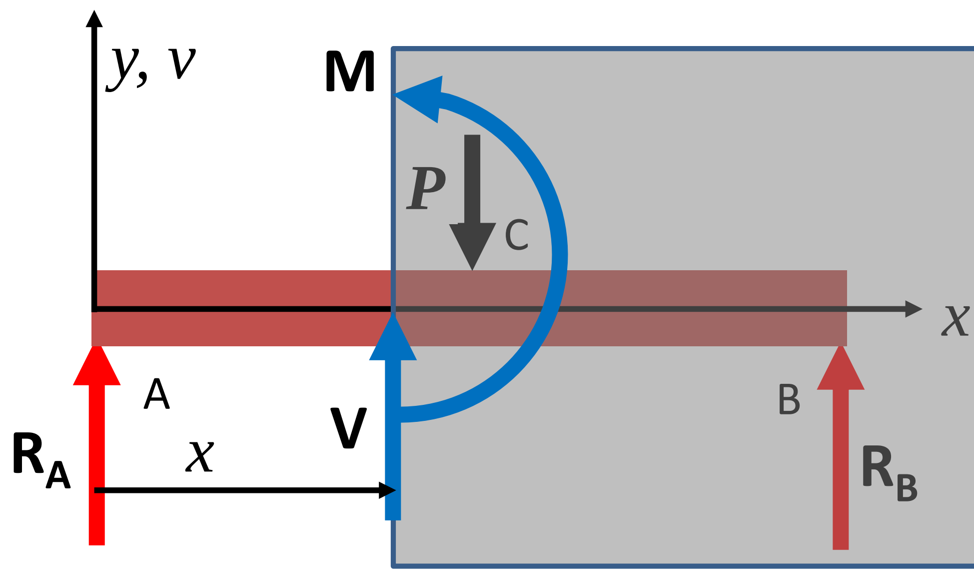


Simply supported beam with point load in the middle

- Solve equilibrium equations

$$V + R_A = 0 \Rightarrow V\left(x\right) = -R_A = -\frac{P}{2}$$

$$M + Vx = 0 \Rightarrow M\left(x\right) = -Vx = \frac{Px}{2}$$

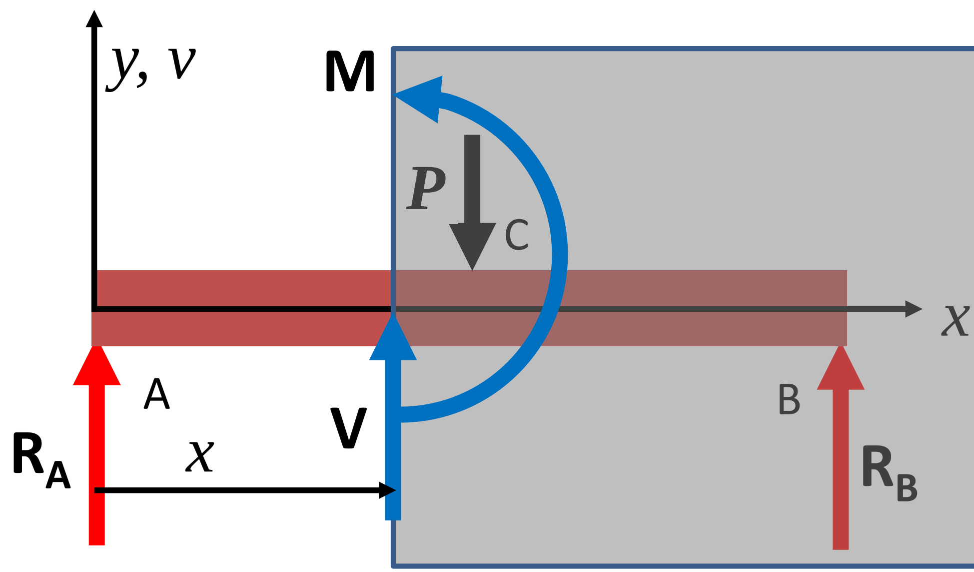


Simply supported beam with point load in the middle

- Solve the flexure equation

$$EIv'' = \frac{Px}{2} \Rightarrow EIv'(x) = \frac{Px^2}{4} + C_1$$

$$\Rightarrow EIv(x) = \frac{Px^3}{12} + C_1 x + C_2$$

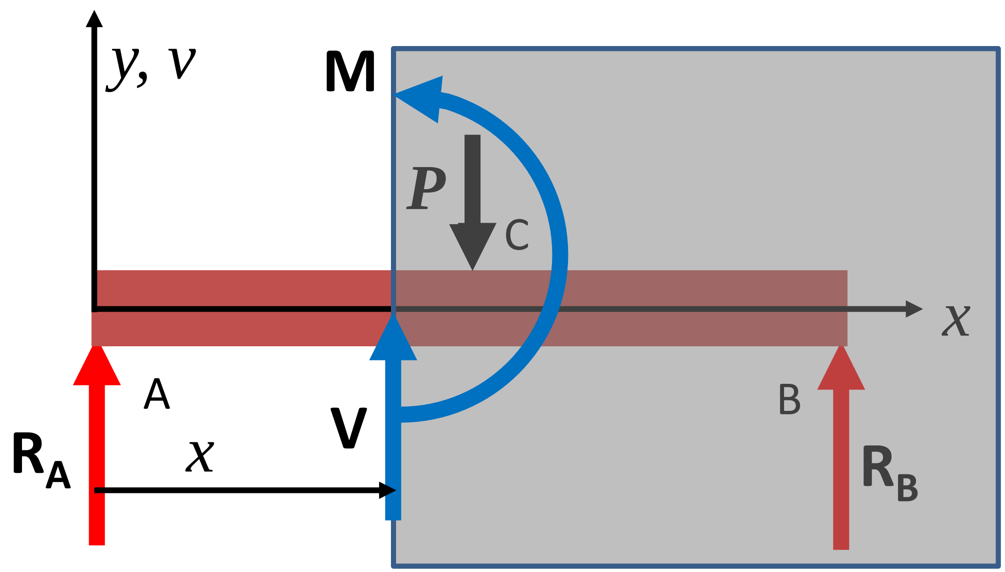


Simply supported beam with point load in the middle

- Try applying boundary conditions. Only boundary A is in this domain. Hence the only BC applicable is

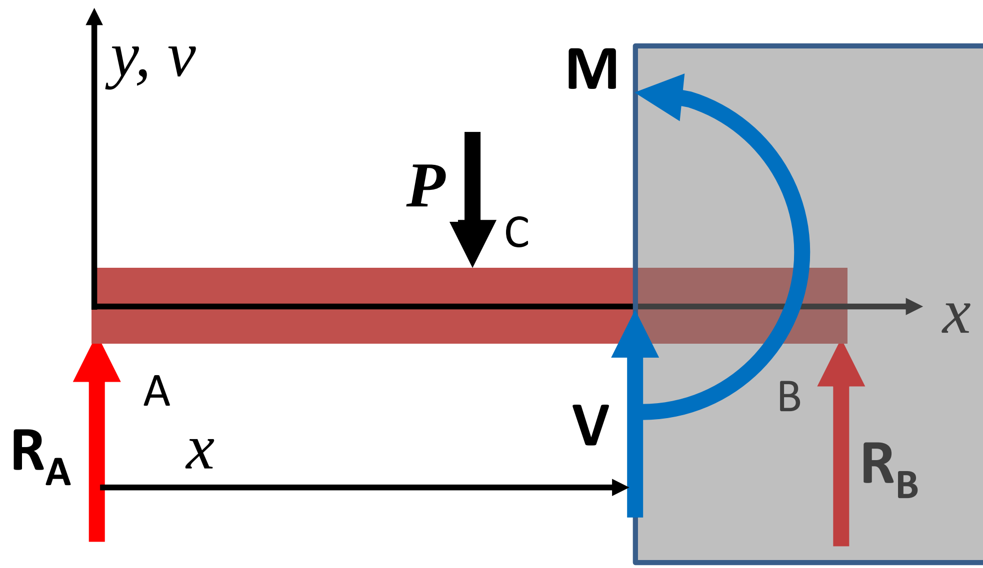
$$v(0) = 0 \Rightarrow C_2 = 0$$

- We cannot find C_1



Simply supported beam with point load in the middle

- Domain CB. Section is taken at distance x from A

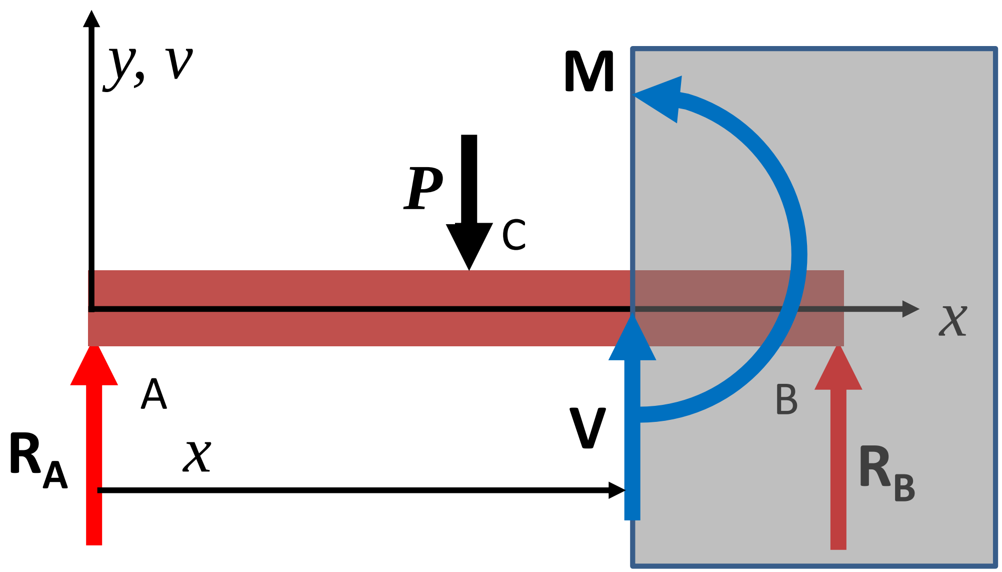


Simply supported beam with point load in the middle

- Solve equilibrium equations

$$V + R_A = P \Rightarrow V(x) = P - R_A = \frac{P}{2}$$

$$M + Vx = \frac{PL}{2} \Rightarrow M(x) = \frac{PL}{2} - Vx = \frac{P(L-x)}{2}$$

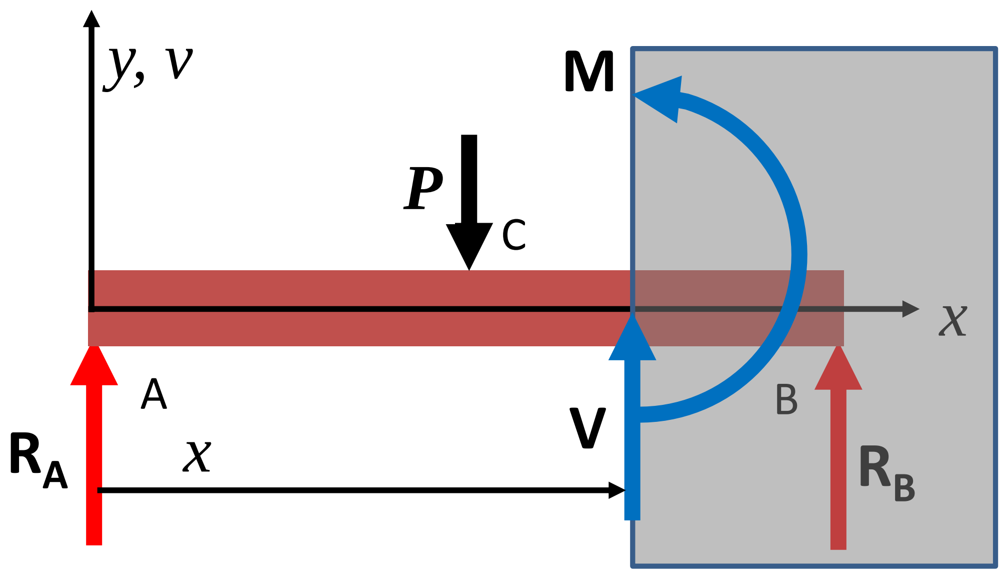


Simply supported beam with point load in the middle

- Solve the flexure equation

$$EIv'' = \frac{P(L-x)}{2} \Rightarrow EIv'(x) = \frac{PLx}{2} - \frac{Px^2}{4} + D_1$$

$$\Rightarrow EIv(x) = \frac{PLx^2}{4} - \frac{Px^3}{12} + D_1 x + D_2$$

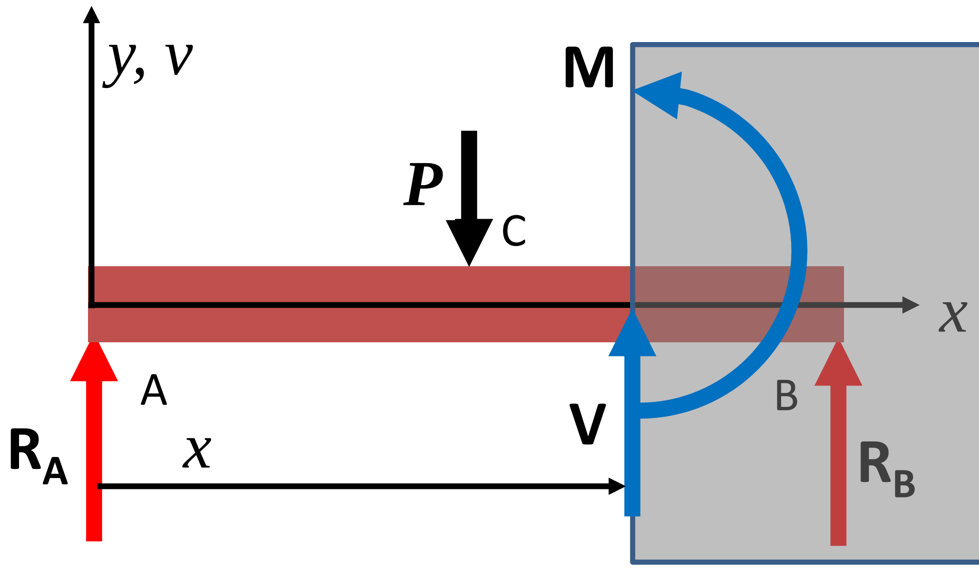


Simply supported beam with point load in the middle

- Try applying boundary conditions. Only boundary B is in this domain. Hence the only BC applicable is

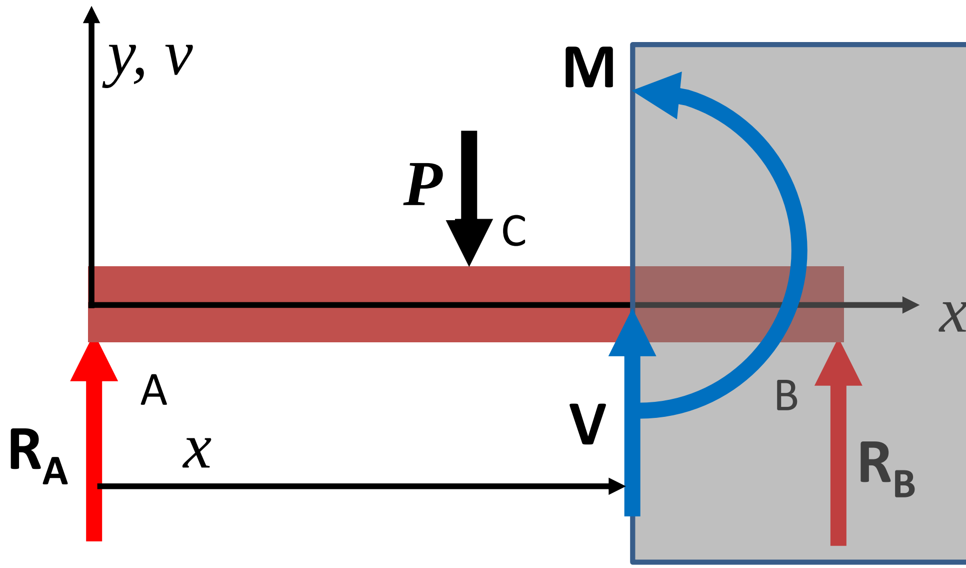
$$v(L) = 0 \Rightarrow \frac{PL^3}{6} + D_1 L + D_2 = 0$$

- We cannot find $D_1$and/or D_2 and only get a relation.



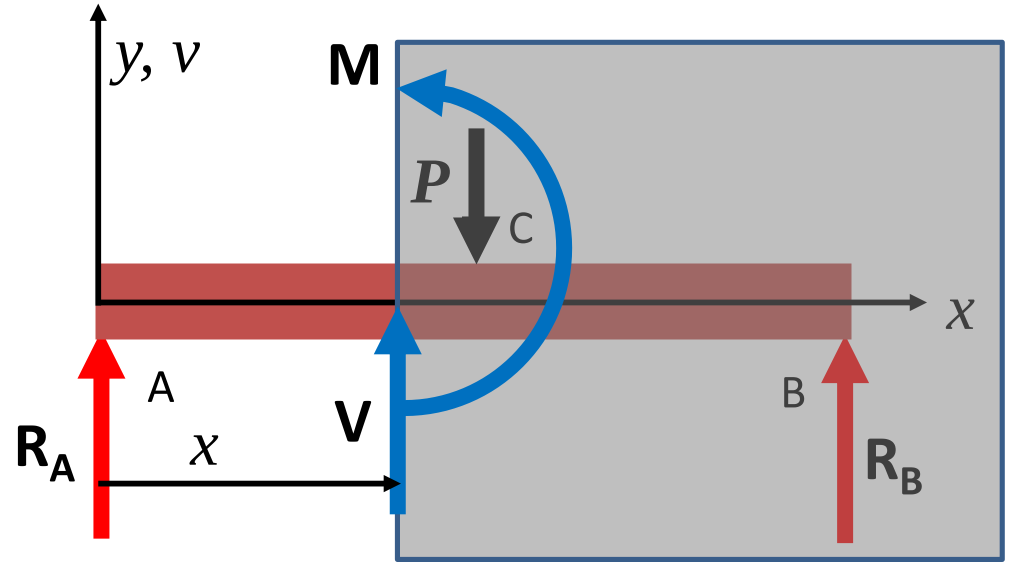
Simply supported beam with point load in the middle

- We need to consider that the beam cannot break or have a kink at C. Hence slope and deflection obtained at C from expressions obtained by analyzing domains AC and BC must match.
- We must therefore match BCs at domain boundaries as well whenever there is a change in the nature of external load, i.e. wherever a new load appears.



Simply supported beam with point load in the middle

- Slope and deflection at L/2 in domain AC. Remember that C2 has already been found to be 0.

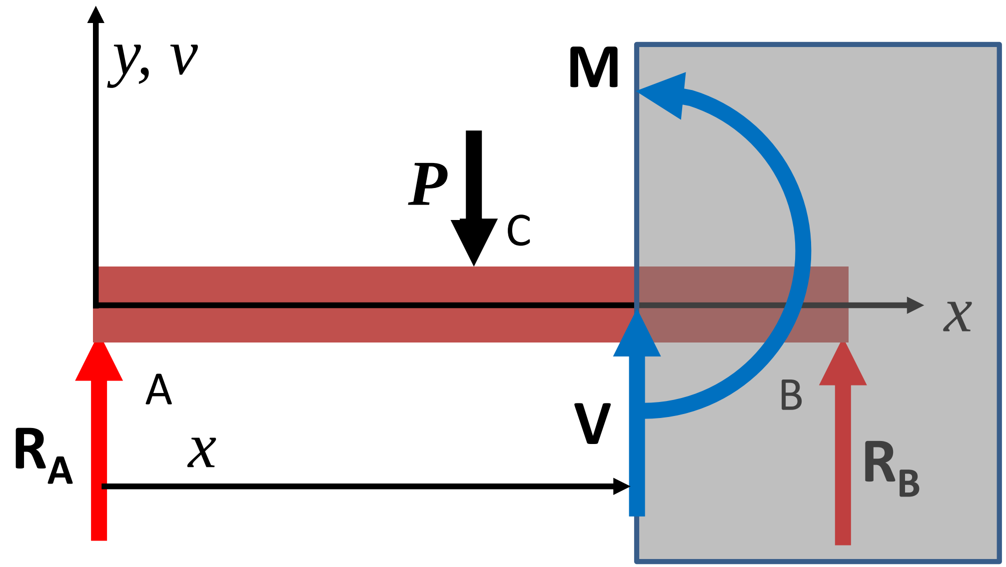


$$EIv'\left(\frac{L}{2}\right)=\frac{PL^2}{16}+C_1$$

$$\Rightarrow EIv\left(\frac{L}{2}\right)=\frac{PL^3}{96}+C_1\frac{L}{2}$$

Simply supported beam with point load in the middle

- Slope and deflection at L/2 in domain CB.



$$EIv'\left(\frac{L}{2}\right)=\frac{3PL^2}{16}+D_1$$

$$\Rightarrow EIv\left(\frac{L}{2}\right)=\frac{5PL^3}{96}+D_1\frac{L}{2}+D_2$$

Simply supported beam with point load in the middle

- Matching slopes and deflections at C

$$EIv'\left(\frac{L}{2}-\right)=EIv'\left(\frac{L}{2}+\right),EIv\left(\frac{L}{2}-\right)=EIv\left(\frac{L}{2}+\right)$$

$$\frac{PL^2}{16}+C_1=\frac{3PL^2}{16}+D_1$$

$$\frac{PL^3}{96}+C_1\frac{L}{2}=\frac{5PL^3}{96}+D_1\frac{L}{2}+D_2$$

Simply supported beam with point load in the middle

- We now have 3 equations (one from v(L) =0) for the 3 unknowns left and can hence solve for all the unknowns

$$\frac{PL^3}{6} + D_1 L + D_2 = 0$$

$$\frac{PL^2}{16} + C_1 = \frac{3PL^2}{16} + D_1$$

$$\frac{PL^3}{96} + C_1 \frac{L}{2} = \frac{5PL^3}{96} + D_1 \frac{L}{2} + D_2$$

Simply supported beam with point load in the middle

- We now have 3 equations for the 3 unknowns left and can hence solve for all the unknowns

$$C_1 = -\frac{PL^2}{16}, D_1 = -\frac{3PL^2}{16}, D_2 = \frac{PL^3}{48}$$

Simply supported beam with point load in the middle

- Useful information
- Maximum deflection (at C) is

$$v\left(\frac{L}{2}\right) = -\frac{PL^3}{48EI}$$

- Slope at the endpoints are

$$v'(0) = -v'(L) = -\frac{PL^2}{16EI}$$