In this lesson we shall introduce Fourier series of a piecewise continuous periodic function. First we construct Fourier series of periodic functions of standard period 2π and then the idea will be extended for a function of arbitrary period.

2.1 Piecewise Continuous Functions

A function f is piecewise continuous on $[a, b]$ if there are points

$$a < t_1 < t_2 < \ldots < t_n < b$$

such that f is continuous on each open sub-interval (a, t_1), (t_j, t_{j+1}) and (t_n, b) and all the following one sided limits exist and are finite

$$\lim_{t \to a+} f(t), \ \lim_{t \to t_j -} f(t), \ \lim_{t \to t_j +} f(t), \text{ and } \lim_{t \to b-} f(t), \ j = 1, 2, \ldots, n$$

This mean that f is continuous on $[a, b]$ except possibly at finitely many points, at each of which f has finite one sided limits. It should be clear that all continuous functions are obviously piecewise continuous.

2.1.1 Example 1

Consider the function

$$f(x) = \begin{cases} 3, & \text{for } x = -\pi; \\ x^2, & \text{for } -\pi < x < 1; \\ 1 - x^2, & \text{for } 1 \le x < 2; \\ 2, & \text{for } 2 \le x \le \pi. \end{cases}$$

At each point of discontinuity the function has finite one sided limits from both sides. At the end points $x = -\pi$ and π right and left sided limits exist, respectively. Therefore, the function is piecewise continuous.

2.1.2 Example 2

A simple example that is not piecewise continuous includes

$$f(x) = \begin{cases} 0, & x = 0; \\ x^{-n}, & x \in (0, 1], \ n > 0\,. \end{cases}$$

Note that f is continuous everywhere except at $x = 0$. The function f is also not piecewise continuous on $[0, 1]$ because $\lim_{x=0+} f(x) = \infty$.

An important property of piecewise continuous functions is boundedness and integrability over closed interval. A piecewise continuous function on a closed interval is bounded and integrable on the interval. Moreover, if f_1 and f_2 are two piecewise continuous functions then their product, $f_1 f_2$, and linear combination, $c_1 f_1 + c_2 f_2$, are also piecewise continuous.

2.2 Fourier Series of a 2π Periodic Function

Let f be a periodic piecewise continuous function on $[-\pi, \pi]$ and has the following trigonometric series expansion

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] \tag{2.1}$$

The aim is to determine the coefficients $a_k, k = 0, 1, 2, \ldots$ and $b_k, k = 1, 2, \ldots$. First we assume that the above series can be integrated term by term and its integral is equal to the integral of the function f over $[-\pi, \pi]$, that is,

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{a_0}{2}\,\mathrm{d}x + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)\,\mathrm{d}x + b_k \int_{-\pi}^{\pi} \sin(kx)\,\mathrm{d}x \right)$$

This implies

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x.$$

Multiplying the series by $\cos(nx)$, integrating over $[-\pi, \pi]$ and assuming its value equal to the integral of $f(x)\cos(nx)$ over $[-\pi, \pi]$, we get

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = 0 + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(nx)\cos(kx)\,\mathrm{d}x + b_k \int_{-\pi}^{\pi} \cos(nx)\sin(kx)\,\mathrm{d}x \right)$$

Note that the first term on the right hand side is zero because $\int_{-\pi}^{\pi} \cos(kx)\,\mathrm{d}x = 0$. Further, using the orthogonality of the trigonometric system we obtain

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x$$

Similarly, by multiplying the series by $\sin(nx)$ and repeating the above steps we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x$$

The coefficients a_n, $n = 0, 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$ are called **Fourier coefficients** and the trigonometric series (2.1) is called the **Fourier series** of $f(x)$. Note that by writing the constant $a_0/2$ instead of a_0, one can use a single formula of a_n to calculate a_0.

Remark 1: *In the series (2.1) we can not, in general, replace $\sim$ by $=$ sign as clear from the determination of the coefficients. In the process we have set two integrals equal which does not imply that the function $f(x)$ is equal to the trigonometric series. Later we will discuss conditions under which equality holds true.*

Remark 2: **(Uniqueness of Fourier Series)** *If we alter the value of the function f at a finite number of points then the integral defining Fourier coefficients are unchanged. Thus function which differ at finite number of points have exactly the same Fourier series. In other words we can say that if f, g are piecewise continuous functions and Fourier series of f and g are identical, then $f(x) = g(x)$ except at a finite number of points.*

2.3 Fourier Series of a $2l$ Periodic Function

Let $f(x)$ be piecewise continuous function defined in $[-l, l]$ and it is $2l$ periodic. The Fourier series corresponding to $f(x)$ is given as

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_n \cos\frac{k\pi x}{l} + b_n \sin\frac{k\pi x}{l}\right] \tag{2.2}$$

where the Fourier coefficients, derived exactly in the similar manner as in the previous case, are given as

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}\,\mathrm{d}x, \quad k = 0, 1, 2, \ldots$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x \quad k = 1, 2, \ldots$$

In must be noted that just for simplicity we will be discussing Fourier series of 2π periodic function. However all discussions are valid for a function of an arbitrary period.

Remark 3: *It should be noted that piecewise continuity of a function is sufficient for the existence of Fourier series. If a function is piecewise continuous then it is always possible to calculate Fourier coefficients. Now the question arises whether the Fourier series of a function f converges and represents f or not. For the convergence we need additional conditions on the function f to ensure that the series converges to the desired values. These issues on convergence will be taken in the next lesson.*

2.4 Example Problems

2.4.1 Problem 1

Find the Fourier series to represent the function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0; \\ x, & 0 < x < \pi. \end{cases}$$

Solution: The Fourier series of the given function will represent a 2π periodic function and the series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

with

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi\,\mathrm{d}x + \int_{0}^{\pi} x\,\mathrm{d}x\right] = -\frac{\pi}{2}$$

and the coefficients a_n, $n = 1, 2, \ldots$ as

$$\begin{aligned} a_n =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x\right] \\ =& -\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\left\{x\frac{\sin(nx)}{n}\right\}_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin(nx)}{n}\,\mathrm{d}x\right] \end{aligned}$$

It can be further simplified to give

$$a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ is even;} \\ -\frac{2}{n^2\pi}, & n \text{ is odd.} \end{cases}$$

Similarly $b_n,\ n = 1, 2, \ldots$ can be calculated as

$$\begin{aligned} b_n =& \frac{1}{\pi}\left[-\int_{-\pi}^{0} \pi \sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} x \sin(nx)\,\mathrm{d}x\right] \\ =& \left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\left\{x\frac{\cos(nx)}{n}\right\}_{0}^{\pi} + \int_{0}^{\pi} \frac{\cos(nx)}{n}\,\mathrm{d}x\right] \end{aligned}$$

After simplification we get

$$b_n = \frac{1}{n}\left[1 - 2(-1)^n\right] = \begin{cases} -\frac{1}{n}, & n \text{ is even;} \\ \frac{3}{n}, & n \text{ is odd.} \end{cases}$$

Substituting the values of a_n and b_n, we get

$$f(x) \sim -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left[3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \ldots\right].$$

Remark 4: *Let a function is defined on the interval $[-l, l]$. It should be noted that the periodicity of the function is not required for developing Fourier series. However, the Fourier series, if it converges, defines a $2l$-periodic function on $\mathbb{R}$. Therefore, this is sometimes convenient to think the given function as $2l$-periodic defined on $\mathbb{R}$.*

2.4.2 Problem 2

Expand $f(x) = |\sin x|$ in a Fourier series.

Solution: There are two possibilities to work out this problem. This may be treated as a function of period π and we can work in the interval $(0, \pi)$ or we treat this function as of period 2π and work in the interval $(-\pi, \pi)$.

Case I: First we treat the function $|\sin x|$ as π periodic we have $2l = \pi \Rightarrow l = \frac{\pi}{2}$. The coefficient a_0 is given as

$$a_0 = \frac{1}{\frac{\pi}{2}}\int_{0}^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} \sin x\,\mathrm{d}x = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{4}{\pi}.$$

The other coefficient $a_n,\ n = 1, 2, \ldots$ are given by

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos(2nx)\,\mathrm{d}x = \frac{1}{l}\int_{0}^{\pi} \left[\sin(2n+1)x - \sin(2n-1)x\right]\,\mathrm{d}x$$

It can be further simplified to have

$$a_n = \frac{1}{\pi}\left[-\frac{\cos(2n+1)x}{2n+1}\Big|_0^\pi + \frac{\cos(2n-1)x}{2n-1}\Big|_0^\pi\right] = \frac{1}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1}\right] = -\frac{4}{\pi(4n^2-1)}$$

Now we compute the coefficients b_n, $n = 1, 2, \dots$ as

$$\begin{aligned} b_n = \frac{2}{\pi}\int_0^\pi \sin x \sin(2nx)\, \mathrm{d}x &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left[\cos(2n-1)x - \cos(2n+1)x\right]\, \mathrm{d}x \\ &= \frac{1}{\pi}\left[-\frac{\sin(2n-1)x}{2n-1}\Big|_0^\pi + \frac{\sin(2n+1)x}{2n+1}\Big|_0^\pi\right] = 0 \end{aligned}$$

Hence the Fourier series is given by

$$f(x) \sim \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)}\cos(2nx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}, \quad 0 \le x \le 1$$

Case II: If we treat $f(x)$ as 2π periodic then

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos(nx)\, \mathrm{d}x = \frac{1}{\pi}\int_0^\pi \left[\sin(n+1)x - \sin(n-1)x\right]\, \mathrm{d}x \\ &= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1}\Big|_0^\pi + \frac{\cos(n-1)x}{n-1}\Big|_0^\pi\right]\, \mathrm{d}x = \frac{1}{\pi}\left[\frac{-(-1)^{n+1}+1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right] \end{aligned}$$

Thus, for $n \neq 1$ we have

$$a_n = \begin{cases} 0, & \text{when } n \text{ is odd;} \\ -\frac{1}{\pi}\frac{4}{n^2-1}, & \text{when } n \text{ is even} \end{cases}$$

The coefficient a_1 needs to calculated separately as

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\, \mathrm{d}x = \frac{1}{\pi}\int_0^\pi \sin 2x\, \mathrm{d}x = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]\Big|_0^\pi = \frac{1}{2\pi}\left[-1+1\right] = 0$$

Clearly, the coefficients b_n's are zero because

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\sin x|\sin(nx)}_{\text{odd function}}\, \mathrm{d}x = 0$$

The Fourier series can be written as

$$f(x) \sim \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \dots\right] = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}.$$

Therefore we ended up with the same series.

Remark 5: *If we develop the Fourier series of a function considering its period as any integer multiple of its fundamental period, we shall end up with the same Fourier series.*

Remark 6: *Note that in the above example the given function is an even function and therefore the Fourier series is simpler as we have seen that the coefficient b_n is zero in this case. The determination of the Fourier series of a given function becomes simpler if the function is odd or even. More detail of this we shall see in the next Lesson.*