Solution set

(A) Q.1.
$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, \quad y > 0$$
$$u(x,0) = f(x)$$
u and u_x vanish as $|x| \to \infty$
and u is bounded as $y \to \infty$.

Sol: $$\tilde{u}(s,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-isx} dx$$

$$\tilde{u}_{yy} - s^2 \tilde{u} = 0$$
$$\Rightarrow \tilde{u} = A(s) e^{sy} + B(s) e^{-sy}$$
$$\tilde{u}(s,0) = \tilde{f}(s) \text{ and } \tilde{u}(s,y) \to 0 \text{ as } y \to \infty$$
$$\Rightarrow A(s) = 0 \text{ for } s > 0$$
$$\text{and } B(s) = 0 \text{ for } s < 0$$
$$\therefore \quad \tilde{u}(s,y) = \tilde{f}(s) e^{-|s|y}$$

Inverting the function,
$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-|s|y} e^{-is\xi} d\xi \right] e^{isx} ds$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) d\xi \int_{-\infty}^{\infty} e^{-s[i(\xi-x)] - |s|y} ds$$
$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi-x)^2 + y^2} d\xi$$

$$\left[\because \int_{-\infty}^{\infty} e^{-s[i(\xi-x)] - |s|y} ds = \frac{2y}{(\xi-x)^2 + y^2} \right]$$

Q2 $\quad u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \; y > 0$

$u_y(x,0) = g(x), \quad -\infty < x < \infty$

u is bounded as $y \to \infty$

u and u_x vanish as $|x| \to \infty$

sol: Let $v(x,y) = u_y(x,y)$

then $u(x,y) = \int_0^y v(x,y)\,dy$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}(u_{xx} + u_{yy}) = 0$$

also $v(x,0) = u_y(x,0) = g(x)$

This translates to a Dirichlet problem in $v(x,y)$

$\therefore$ The solution is

$$v(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)\,d\xi}{(\xi - x)^2 + y^2}$$

or
$$u(x,y) = \frac{1}{\pi} \int_0^y \eta \int_{-\infty}^{\infty} \frac{g(\xi)\,d\xi}{(\xi - x)^2 + \eta^2} \cdot d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi)\,d\xi \int_0^y \frac{2\eta\,d\eta}{(\xi - x)^2 + \eta^2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) \log\left[(x-\xi)^2 + y^2\right] d\xi + C$$

Q2 $u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \; y > 0$

$u_y(x,0) = g(x), \quad -\infty < x < \infty$

u is bounded as $y \to \infty$

u and u_x vanish as $|x| \to \infty$

sol: Let $v(x,y) = u_y(x,y)$

then $u(x,y) = \int_0^y v(x,y)\,dy$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}(u_{xx} + u_{yy}) = 0$$

also $v(x,0) = u_y(x,0) = g(x)$

This translates to a Dirichlet problem in $V(x,y)$

$\therefore$ The solution is

$$v(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)\,d\xi}{(\xi - x)^2 + y^2}$$

$$\text{or } u(x,y) = \frac{1}{\pi} \int_0^y \eta \int_{-\infty}^{\infty} \frac{g(\xi)\,d\xi}{(\xi - x)^2 + \eta^2} \cdot d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi)\,d\xi \int_0^y \frac{2\eta\,d\eta}{(\xi - x)^2 + \eta^2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) \log[(x-\xi)^2 + y^2]\,d\xi + C$$

$\nabla \phi = 0, \quad y > 0$

s.t. $\frac{\partial \phi}{\partial x}$ and $\phi \to 0$ as $\sqrt{x^2+y^2} \to \infty$

$$\phi(x,0) \cancel{\phi(x,0)} = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Sol: Using Fourier transform in x

$$\frac{\partial^2 \tilde{\phi}(s,y)}{\partial y^2} = s^2 \tilde{\phi}(s,y)$$

$$\Rightarrow \tilde{\phi} = c \cdot e^{-|s|y} \quad \text{——— } (*)$$

$$\phi(x,0) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 0 \end{cases}$$

$$\mathcal{F}[\phi(x,0)] = \int_{-1}^{1} 1 \cdot e^{-isx}\, dx = 2\frac{\sin s}{s} = \tilde{\phi}(x,0)$$

$$(*) \Rightarrow c = \frac{2\sin s}{s}$$

$$\therefore \tilde{\phi} = 2\frac{\sin s}{s} e^{-|s|y}$$

Using the Convolution, for inversion

$$F * G = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau) \cdot G(x) e^{-ix(t-\tau)}\, d\tau\, dx$$

$$= \int_{-\infty}^{\infty} G(x) e^{-ixt} \int_{-\infty}^{\infty} f(\tau) e^{ix\tau}\, d\tau\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x) \cdot F(x)\, e^{-ixt}\, dx = \frac{1}{2\pi}\mathcal{F}(F \cdot G)$$

Now, $\mathcal{F}[e^{-|s|y}] = \frac{y}{s^2+\omega^2}$

& $\tilde{\phi}(x,0) = 2\frac{\sin s}{s}$

$$\therefore \phi(x,y) = \frac{1}{\pi} \cdot y \int_{-1}^{1} \frac{1}{(x-\tau)^2+y^2}\, d\tau = \frac{y}{\pi}\left[\tan^{-1}\left(\frac{x-1}{y}\right) + \tan^{-1}\left(\frac{x+1}{y}\right)\right]$$

(B) Q1. (a)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x>0,\ t>0 \quad \text{—— (1)}$$

s.t. $u(x,0) = 0, \quad \forall x$ —— (2)

$u(0,t) = u_0, \quad \forall t$ —— (3)

u is finite $\forall x$ and $\forall t$ —— (4)

Multiplying eqn (1) by e^{-st} and integrating w.r.t t from 0 to ∞

$$\int_0^\infty \frac{\partial u}{\partial t} e^{-st} dt = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st} dt$$

or $$e^{-st} u \Big|_0^\infty - \int_0^\infty (-s) e^{-st} u \, dt = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st} dt$$

or $$s\bar{u} = c^2 \frac{d^2}{dx^2}\bar{u} \quad \text{—— (5)}$$

Similarly applying L.T. on eqn (3), we get

$$\int_0^\infty u . e^{-st} dt = \int_0^\infty u_0 e^{-st} dt \quad \text{at } x=0$$

$$\Rightarrow \bar{u} = \frac{u_0}{s} \quad \text{—— (6)}$$

Sol. of (5) $\Rightarrow$ $\bar{u} = A e^{\frac{\sqrt{s}}{c}x} + B e^{-\frac{\sqrt{s}}{c}x}$ —— (7)

u is finite at $x \to \infty$

$\Rightarrow A = 0$

eqn (7) $\Rightarrow$ $\bar{u} = B e^{-\frac{\sqrt{s}}{c}x}$

eqn (6) $\Rightarrow$ $\frac{u_0}{s} = B e^{-\frac{\sqrt{s}}{c}.0} = B$

$\therefore \bar{u} = \frac{u_0}{s} e^{-\frac{\sqrt{s}}{c}x}$

Inverting this we get

$$u(x,t) = u_0\left(1 - \operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right)\right)$$

b) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0<x<\infty, \quad t>0$$

st. $u(x,0) = 0, \quad \forall x$

$u(0,t) = 1, \quad \forall t$

$\lim_{x\to\infty} u(x,t) = 0, \quad \forall t$

Sol: Taking LT in time $t>0$

$$s\bar{u}(x,s) - u(x,0) = \frac{d^2\bar{u}}{dx^2}$$

or $$\frac{d^2\bar{u}}{dx^2} = s\bar{u}$$

$$\Rightarrow \quad \tilde{u}(x,s) = Ae^{-x\sqrt{s}} + Be^{x\sqrt{s}}$$

Since $\lim\limits_{x\to\infty} u(x,t) = 0 \Rightarrow \tilde{u}(x,s) \to 0$ as $x \to \infty$

$$\Rightarrow \quad B = 0$$

$$\therefore \tilde{u}(x,s) = Ae^{-x\sqrt{s}}$$

B.C. 1. $\Rightarrow \quad \tilde{u}(0,s) = \int_0^\infty 1 \cdot e^{-st}\,dt = \frac{1}{s}$

$$\Rightarrow \quad A = \frac{1}{s}$$

$$\therefore \tilde{u}(x,s) = \frac{1}{s}e^{-x\sqrt{s}}$$

and $L^{-1}[\hat{u}(x,s)] = u(x,t) = L^{-1}\left[\frac{1}{s}e^{-x\sqrt{s}}\right]$

$$= \operatorname{erfc}\left(\frac{1}{2}\frac{x}{\sqrt{t}}\right)$$

Q2. $u_t = u_{xx}$ st $u = 0$, when $x = 0$, $t > 0$

$$u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}, \quad t = 0$$

$u(x,t)$ is bounded.

sol: $$\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

or $$\frac{\partial}{\partial t} \bar{u}(s,t) = -s^2 \bar{u}(s) + s\,u(0) \quad [\because u = 0, \text{ when } x = 0]$$

or $$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \Rightarrow \bar{u} = A e^{-s^2 t}$$

$$\therefore \bar{u}(s,t) = A e^{-s^2 t} = \int_0^\infty u(x,t) \sin sx \, dx$$

$$\bar{u}(s,0) = \int_0^\infty u(x,0) \sin sx \, dx$$

$$= \int_0^1 1.\sin sx \, dx = \frac{1 - \cos s}{s}$$

$$\therefore \bar{u}(s,0) = A = \frac{1 - \cos s}{s}$$

$$\therefore \bar{u}(s,t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

or $$u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos s}{s} e^{-s^2 t} \sin sx \, ds$$

Q3. $$u_t = k u_{xx}, \quad 0 < x < \infty, \quad t > 0$$

st: $u(x,0) = 0$, $\forall x$

$\frac{\partial u}{\partial t}(0,t) = -u_0$, $\forall t$

$u(x,t)$ is bounded.

Sol: $$F_c\left[\frac{\partial u}{\partial t}\right] = F_c[k u_{xx}]$$

or $$\frac{d\bar{u}}{dt} = k\left[-s^2 \bar{u} - \frac{\partial u(0,t)}{\partial x}\right]$$

or $$\frac{d\bar{u}}{dt} + k s^2 \bar{u} = k u_0$$

or $\bar{u} = \left(k u_0 \frac{e^{ks^2t}}{ks^2} + c \right) e^{-ks^2t}$

or $\bar{u}(s,t) = \frac{u_0}{s^2} + c e^{-ks^2t}$

$$u(x,0) = 0, \quad \forall\ x \geq 0$$

$$\Rightarrow \bar{u}(s,0) = 0$$

$\therefore \quad 0 = \frac{u_0}{s^2} + c \Rightarrow c = \frac{-u_0}{s^2}$

$\therefore \quad \bar{u}(s,t) = \frac{u_0}{s^2}\left(1 - e^{-ks^2t}\right)$

and $\quad u(x,t) = \frac{2}{\pi} u_0 \int_0^\infty \frac{1 - e^{-ks^2t}}{s^2} \cos sx \, ds$

Q.4) $\quad u_t = k u_{xx}, \quad -\infty < x < \infty, \quad t > 0$

$u(x,0) = f(x), \quad -\infty < x < \infty$

$u(x,t) \longrightarrow 0 \quad \text{as } |x| \to \infty$

where $u(x,t)$ – temperature distribution and is bounded and k is a constant of diffusivity

sol: $\quad \tilde{u}(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} u(x,t)\, dx$

$$\Rightarrow \frac{\partial \tilde{u}(s,t)}{\partial t} + k s^2 \tilde{u} = 0$$

and $\quad \tilde{u}(s,0) = \tilde{f}(s)$

$$\Rightarrow \tilde{u}(s,t) = \tilde{f}(s) e^{-s^2kt}$$

on inversion

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(s) e^{-s^2kt} e^{isx} \, ds$$

Using the convolution theorem

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi$$

where $g(x)$ is the inversion of $\tilde{g}(s) = e^{-s^2kt}$

and so $g(x) = \mathcal{F}^{-1}\left[e^{-s^2kt}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-s^2kt}\, e^{isx}\, ds$

$$= \frac{1}{\sqrt{2kt}}\cdot e^{-x^2/4kt}$$

$$\therefore \quad u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\xi)\cdot \exp\left[-\frac{(x-\xi)^2}{4kt}\right] d\xi \quad \text{—— } (*)$$

Put $\frac{\xi - x}{2\sqrt{kt}} = \eta \Rightarrow d\xi = 2\sqrt{kt}\, d\eta$

and $$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2\sqrt{kt}\,\eta)\, e^{-\eta^2}\, d\eta \quad \text{—— } (**)$$

$(*)$ or $(**)$ is called the Poisson integral representation of the temperature distribution

Case 1: Take $f(x) = \begin{cases} 0, & x < 0 \\ a, & x > 0 \end{cases}$

$$= a\, H(x)$$

$$\therefore \quad u(x,t) = \frac{a}{2\sqrt{\pi kt}}\int_{0}^{\infty} \exp\left[\frac{-(x-\xi)^2}{4kt}\right] d\xi$$

Put $\eta = \frac{\xi - x}{2\sqrt{kt}}$

$$u(x,t) = \frac{a}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{kt}}}^{\infty} e^{-\eta^2}\, d\eta$$

$$= \frac{a}{\sqrt{\pi}}\left[\int_{-\frac{x}{2\sqrt{kt}}}^{0} e^{-\eta^2} d\eta + \int_{0}^{\infty} e^{-\eta^2} d\eta\right]$$

$$= \frac{a}{\sqrt{\pi}}\left[\int_{0}^{x/2\sqrt{kt}} e^{-\eta^2} d\eta + \frac{\sqrt{\pi}}{2}\right]$$

$$= \frac{a}{2}\left[1 + \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)\right]$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-\eta^2} d\eta$

(c) BL

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{———} \quad (1)$$

s.t

$$u(x,0) = 0 \quad \text{———} \quad (2)$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{———} \quad (3) \qquad \Big\} \text{ I.Cs}$$

$$u(0,t) = f(t) \quad \text{———} \quad (4)$$

$$u(x,t) \text{ is bounded} \quad \text{———} \quad (5) \qquad \Big\} \text{ B.Cs.}$$

Sol: $\mathcal{L}[(1)] \Rightarrow s^2\tilde{u} - s\,u(x,0) - \frac{\partial u}{\partial t}(x,0) = c^2 \frac{d^2\tilde{u}}{dx^2}$ ——— (6)

$u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 0 \Rightarrow$

$$s^2\tilde{u} = c^2 \frac{d^2\tilde{u}}{dx^2}$$

or $\frac{d^2\tilde{u}}{dx^2} = \left(\frac{s}{c}\right)^2 \tilde{u}$ ——— (7)

$\Rightarrow \tilde{u} = A e^{\frac{sx}{c}} + B e^{-\frac{sx}{c}}$ ——— (8)

L.T. of (4) $\Rightarrow \tilde{u}(0,s) = \tilde{f}(s)$ at $x = 0$

(5) $\Rightarrow A = 0$

$\therefore \tilde{u} = B e^{-\frac{sx}{c}}$

Using above condition, $\tilde{u} = \tilde{f}(s)\, e^{-\frac{sx}{c}}$

Now, $u = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\left(t - \frac{x}{c}\right)s} \cdot \tilde{f}(s)\, ds$

Complex inversion formula

$\therefore u(x,t) = f\left(t - \frac{x}{c}\right)$

Q.27. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t \geq 0$

s.t. $u(x,0) = f(x)$, $\frac{\partial u(x,0)}{\partial t} = g(x)$ } I.C

u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$

sol: $\mathcal{F}[u_{tt} = c^2 u_{xx}]$

$\Rightarrow \frac{d^2\tilde{u}}{dt^2} = c^2(-s^2\tilde{u})$

$\therefore \tilde{u}(s,t) = A e^{icst} + B e^{-icst}$ —— (1)

$u(x,0) = f(x)$ & $\frac{\partial u}{\partial t}(x,0) = g(x)$

$\Rightarrow \tilde{u}(s,0) = \tilde{f}(s)$ & $\frac{\partial \tilde{u}}{\partial t}(s,0) = \tilde{g}(s)$

(1) $\Rightarrow$ $\tilde{u}(s,0) = A + B = \tilde{f}(s)$ —— (2)

$\frac{d\tilde{u}}{dt}(s,0) = ics(A-B) = g(s)$ —— (3)

(2), (3) $\Rightarrow$ $A = \frac{1}{2}\left[\tilde{f}(s) + \frac{\tilde{g}(s)}{ics}\right]$

$B = \frac{1}{2}\left[\tilde{f}(s) - \frac{\tilde{g}(s)}{ics}\right]$

$\therefore \tilde{u}(s,t) = \frac{1}{2}\left[\tilde{f}(s) + \frac{\tilde{g}(s)}{ics}\right] e^{icst} + \frac{1}{2}\left[\tilde{f}(s) - \frac{\tilde{g}(s)}{ics}\right] e^{-icst}$ —— (4)

By inversion

$u(x,t) = \frac{1}{2}\left[f(x-ct) - \frac{1}{c}\int_c^{x-ct} g(\theta)\,d\theta\right]$

$+ \frac{1}{2}\left[f(x+ct) + \frac{1}{c}\int_c^{x+ct} g(\theta)\,d\theta\right]$

$\because \mathcal{F}\left[\int_\alpha^x f(t)\,dt\right] = \frac{\tilde{f}(s)}{-is}$