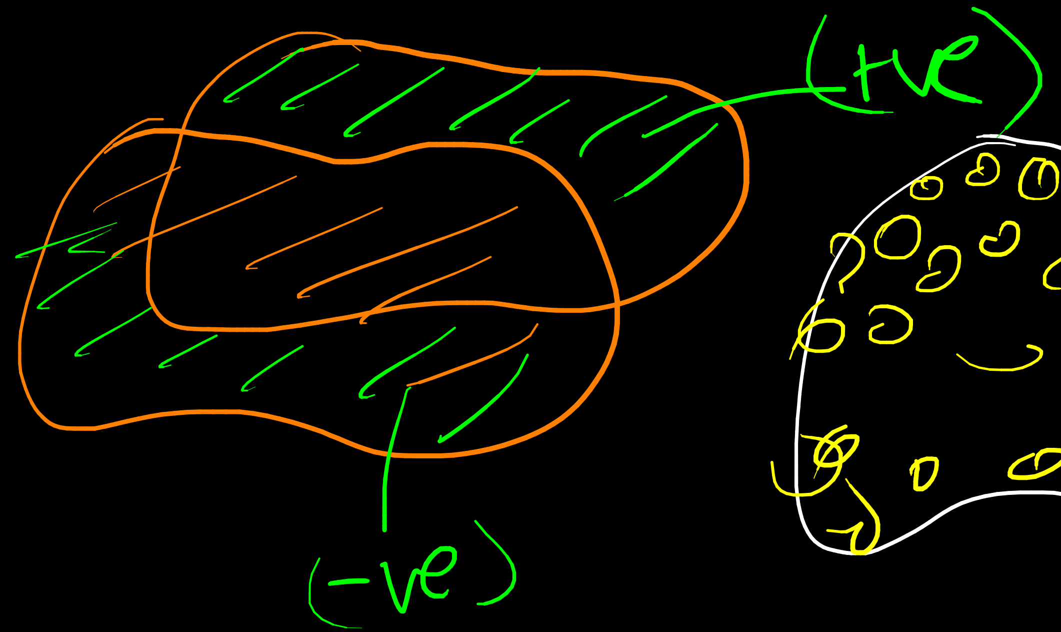
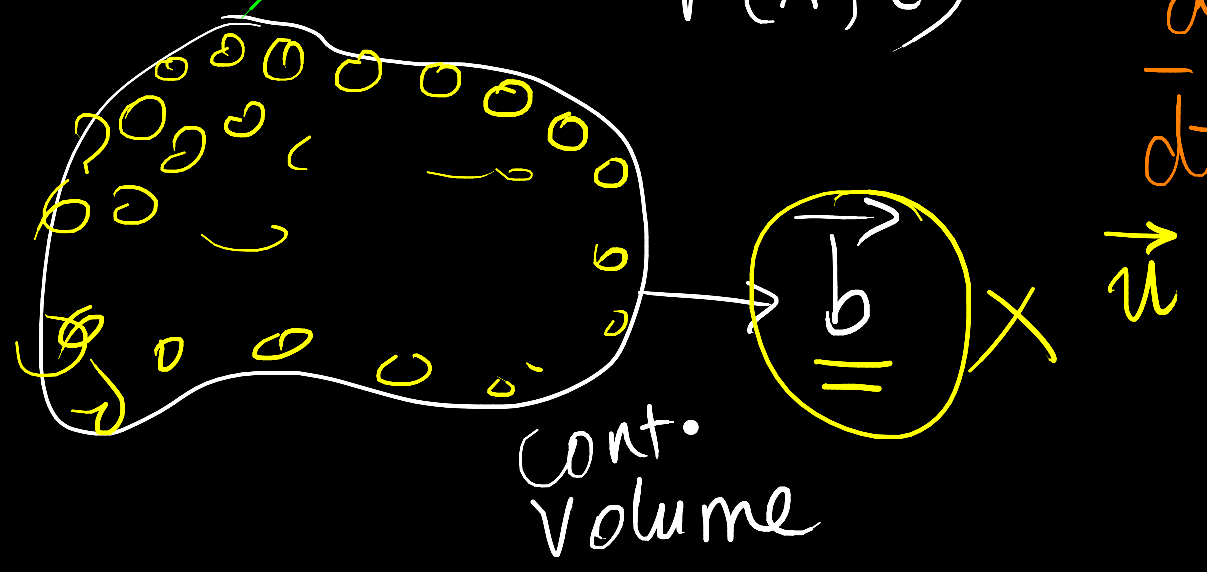


$F(\vec{x}, t)$



$$\frac{d}{dt}\int_{\forall} F(\vec{x}, t)\, d\forall = \int_{V} \frac{\partial F}{\partial t}(\vec{x}, t)\, d\forall + \int_{A} F\, \vec{b}\cdot\vec{n}\, dA$$

Conservation of mass

$F(\vec{x}, t) \qquad \rho(\vec{x}, t)$

$$\frac{d}{dt}\int_{\forall} \rho\, d\forall = \frac{d}{dt} m_{cv}$$

$$0 = \frac{d}{dt} m_{cv} = \int_{\forall} \frac{\partial \rho}{\partial t}\, d\forall + \int \rho\, \vec{u}\cdot\vec{n}\, dA$$

Control mass

no particle leaves or enters

$$\boxed{\int_{\forall} \frac{\partial \rho}{\partial t}\, d\forall + \int_{A} \rho\, \vec{u}\cdot\vec{n}\, dA = 0}$$

We must know $\underline{\vec{u}}$ over the area A

Generic control volume

Doesn't conform w/ control mass

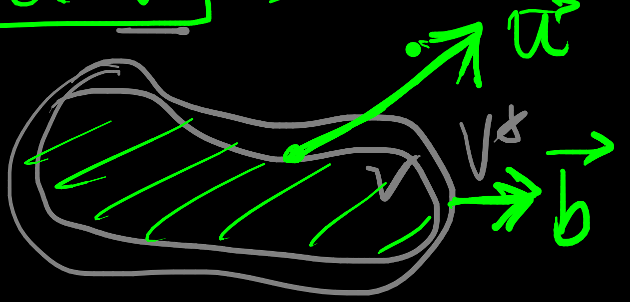
Doesn't move $\vec{u}$

(V^*, A^*) $\vec{b}$

$$\boxed{\frac{d}{dt}\int_{V^*} \rho \, dV} = \int_{V^*} \frac{\partial \rho \, dV}{\partial t} + \int_{A^*} \rho \, \vec{b} \cdot \underline{\vec{n}} \, dA$$

$$0 = \int_{V} \frac{\partial \rho \, dV}{\partial t} + \boxed{\int \rho \, \vec{u} \cdot \underline{\vec{n}} \, dA}$$

Choose $\boxed{V \text{ and } V^*}$ to coincide

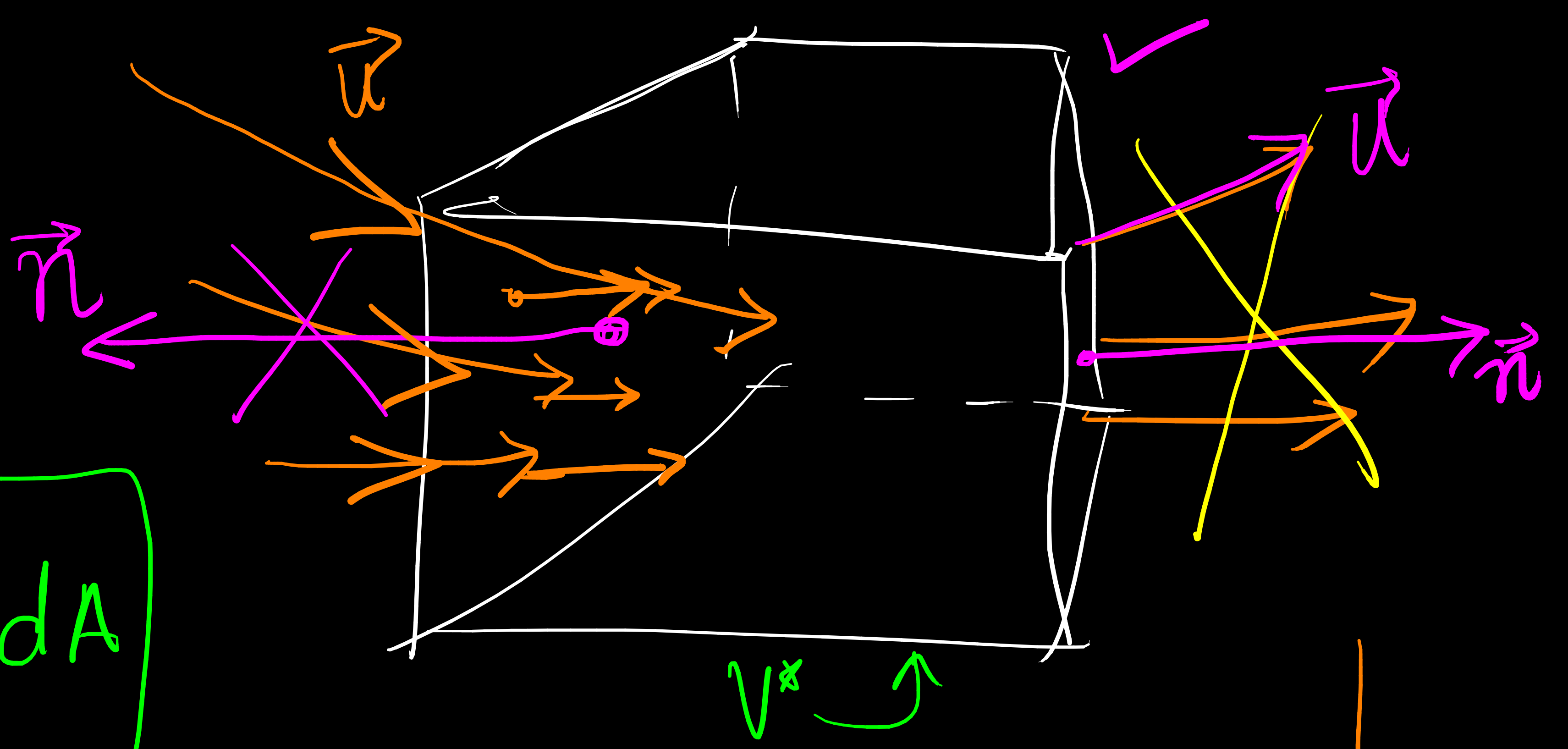


① and ② are identical

Areas coincide

$\frac{d}{dt}\int \rho \, d\forall = -\rho a A$

$\boxed{\vec{b} = 0}$



$$\boxed{\frac{d}{dt}\int_{\forall^*} \rho \, d\forall = -\int \rho \vec{u} \cdot \vec{n} \, dA}$$

$(\vec{u} \cdot \vec{n} = |u||n| \cos\theta) \rightarrow (-b)$

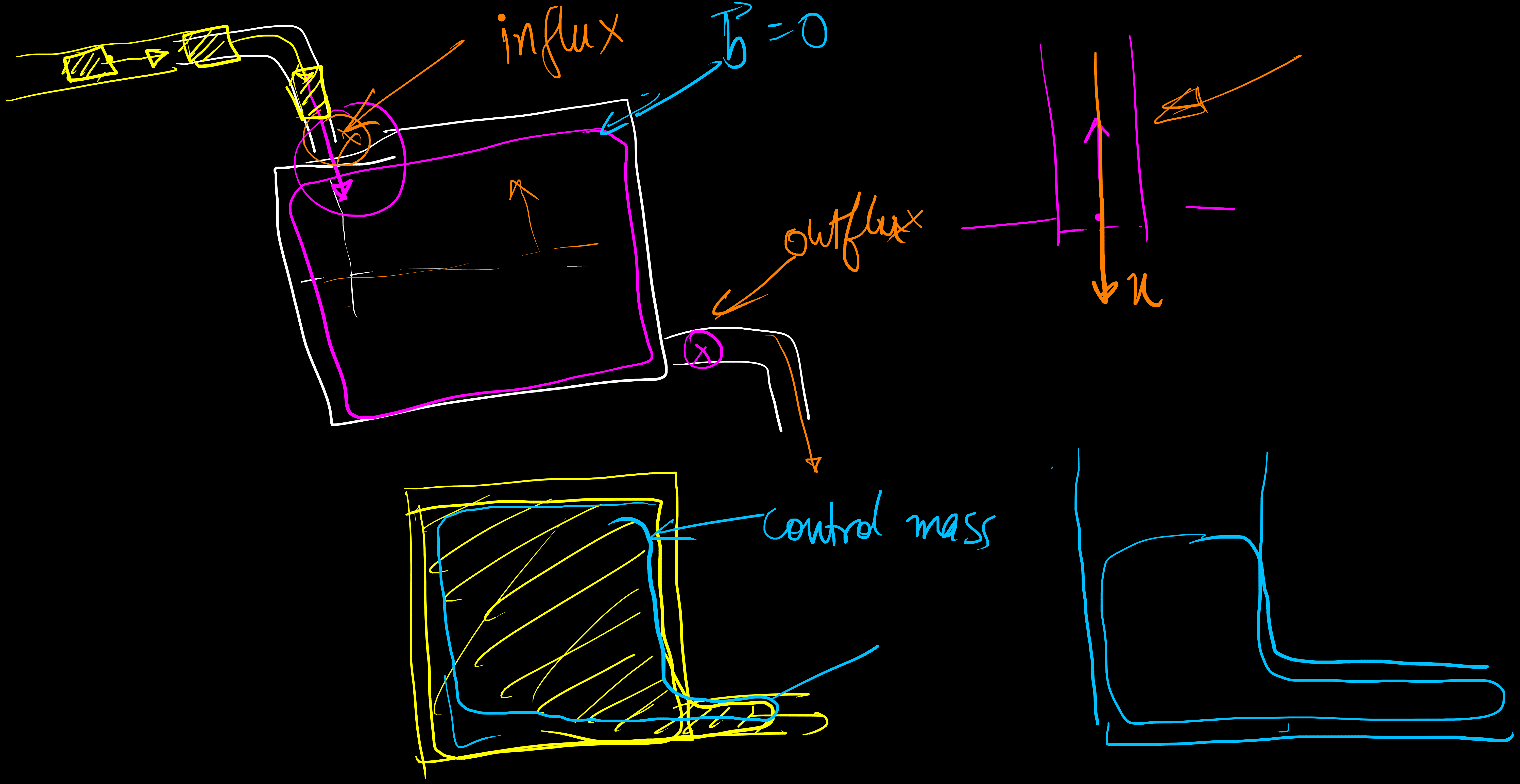
$= -\int \rho (-b) dA = \boxed{\rho b A}$

$\boxed{\vec{u} \cdot \vec{n} \ +ve} = a$

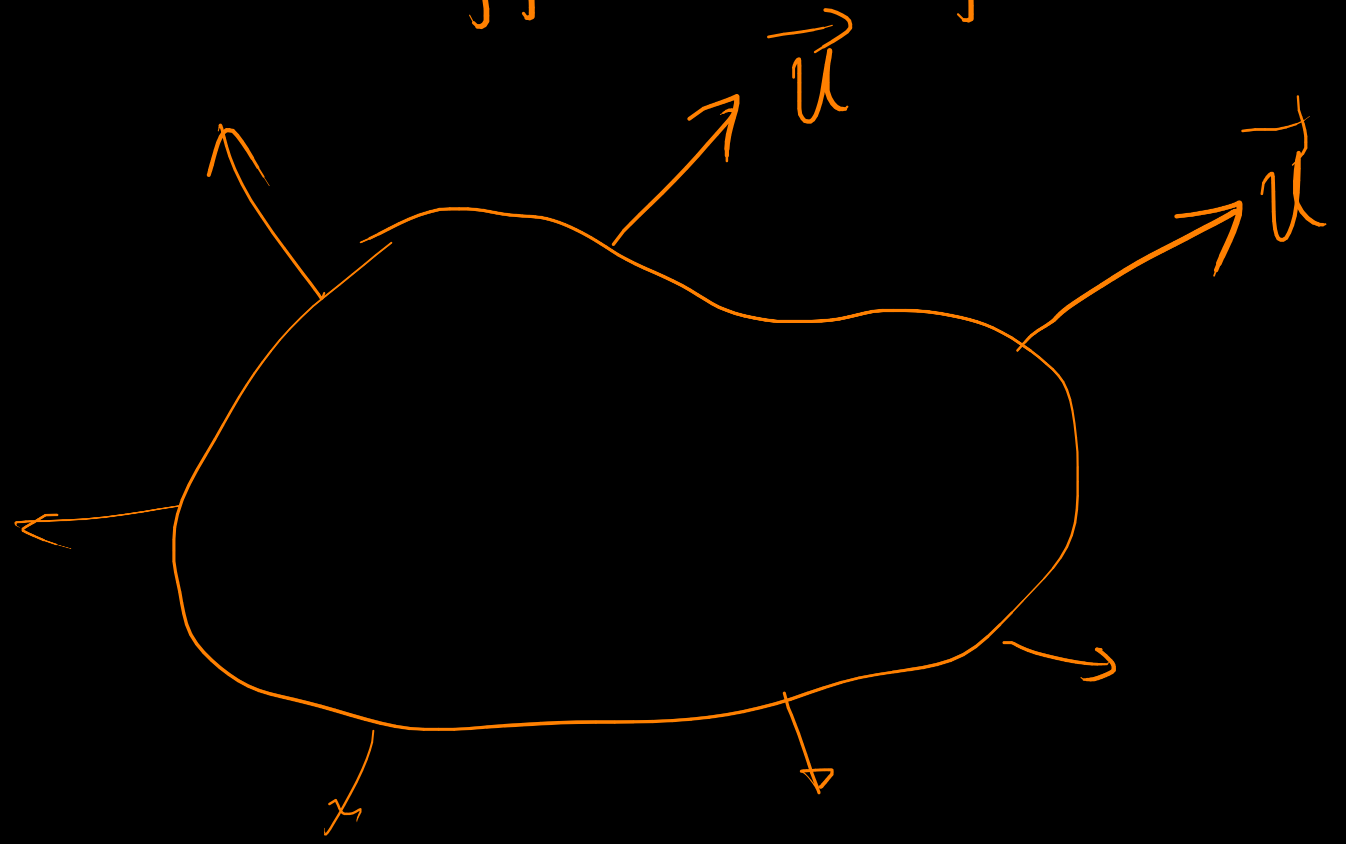
$-\int \rho a dA \ \boxed{-\rho a A}$

[Rate of increase of mass of system] = [Total influx of mass through the surface]

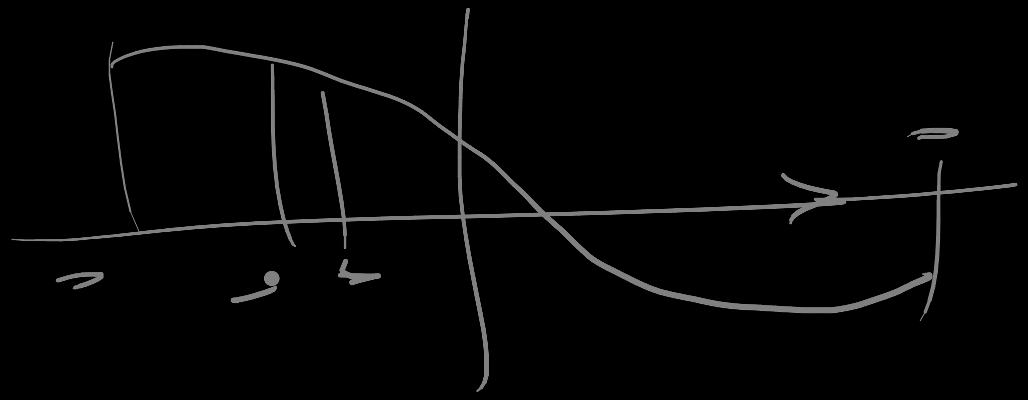
$-(\vec{u} \cdot \vec{n} \, dA) \rightarrow$ Rate of loss of vol. from Cont. vol


influx
$\vec{b}=0$
outflux
u
control mass

Differential formulation



$$\int_A \vec{\phi}\cdot\vec{n}\, dA = \int_V \nabla\cdot\vec{\phi}\, d\forall$$



$$\int_{\forall} \frac{\partial \rho}{\partial t}\, d\forall + \int_A \rho\, \vec{u}\cdot\vec{n}\, dA = 0$$

$$\int_V \frac{\partial \rho}{\partial t}\, d\forall + \int_V \nabla\cdot(\rho\vec{u})\, d\forall = 0$$

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right] d\forall = 0$$

Any arbitrary control volume

$$\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

$$\nabla\cdot(\rho\vec{u}) = \frac{\partial}{\partial x_i}(\rho u_i) = u_i\frac{\partial \rho}{\partial x_i} + \rho\frac{\partial u_i}{\partial x_i} = \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u}$$

$$\nabla\cdot(\rho\vec{u}) = \underbrace{(\vec{u}\cdot\nabla)}_{\text{operator}}\rho + \rho\underbrace{\nabla\cdot\vec{u}}_{\text{scalar}}$$

$$\frac{d}{dt}$$

$$\frac{D\rho}{Dt} = \left\{\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho\right\}$$

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0 \Rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{u} = 0$$

$$\frac{d}{dt}\int_{V^*} \rho \, dV = -\int_{A^*} \rho \, \vec{u}\cdot\vec{n} \, dA + \int_{A^*} \rho \, \vec{b}\cdot\vec{n} \, dA$$

$$\frac{d}{dt}\int_{V^*} \rho \, dV = \int \rho \left(\vec{b}-\vec{u}\right)\cdot\vec{n} \, dA$$

when $\vec{b} = \vec{u}$?

$$\boxed{\frac{d}{dt}\int_{V^*} \rho \, dV = 0}$$

$$\boxed{\int_{V_0} \frac{\partial \rho}{\partial t} \, dV + \int \rho \, \vec{u}\cdot\vec{n} \, dA = 0}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{u} = 0$$

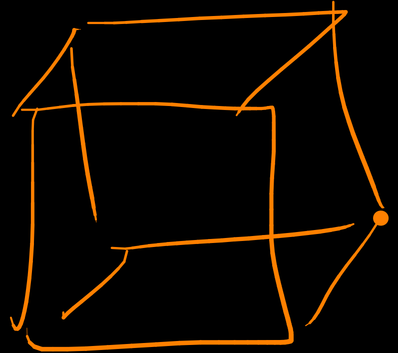
$$\boxed{\nabla\cdot\vec{u} = 0}$$

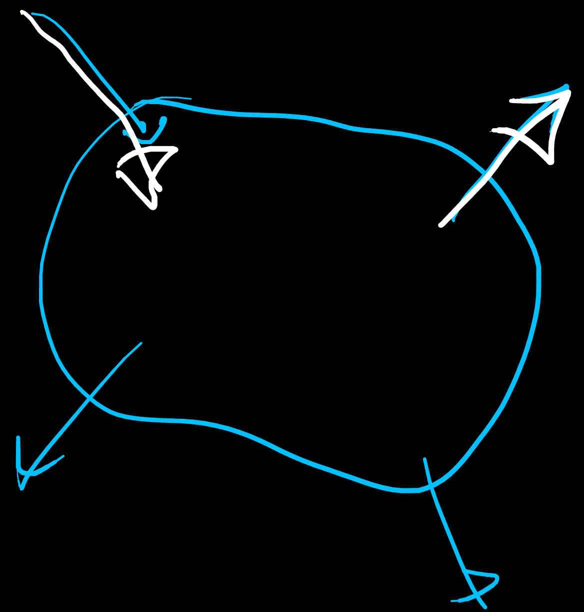
(a) constant density

incompressible element incompressible flows

Incompressibility

$$\boxed{\frac{1}{\Delta V}\frac{d\,\Delta V}{dt} = \nabla\cdot\vec{u}}$$



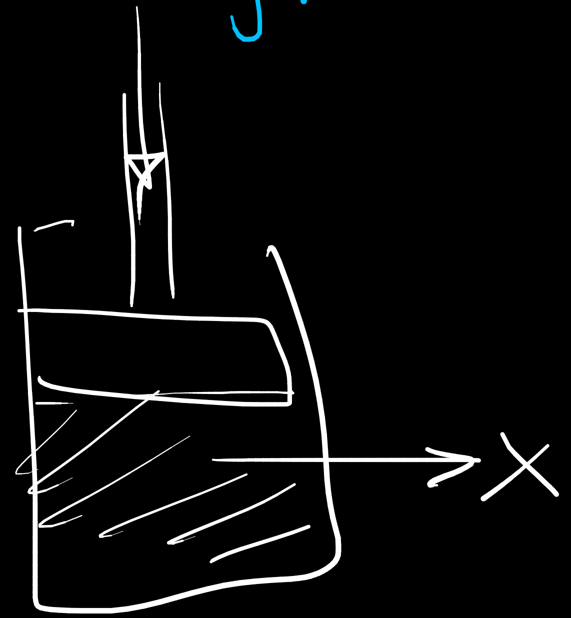


$$\int_A \vec{u}\cdot\vec{n}\,dA = 0$$

$$\int \nabla\cdot\vec{u}\,dV$$

incompressible fluid:

$\frac{d\rho}{dp}$ ← physical property

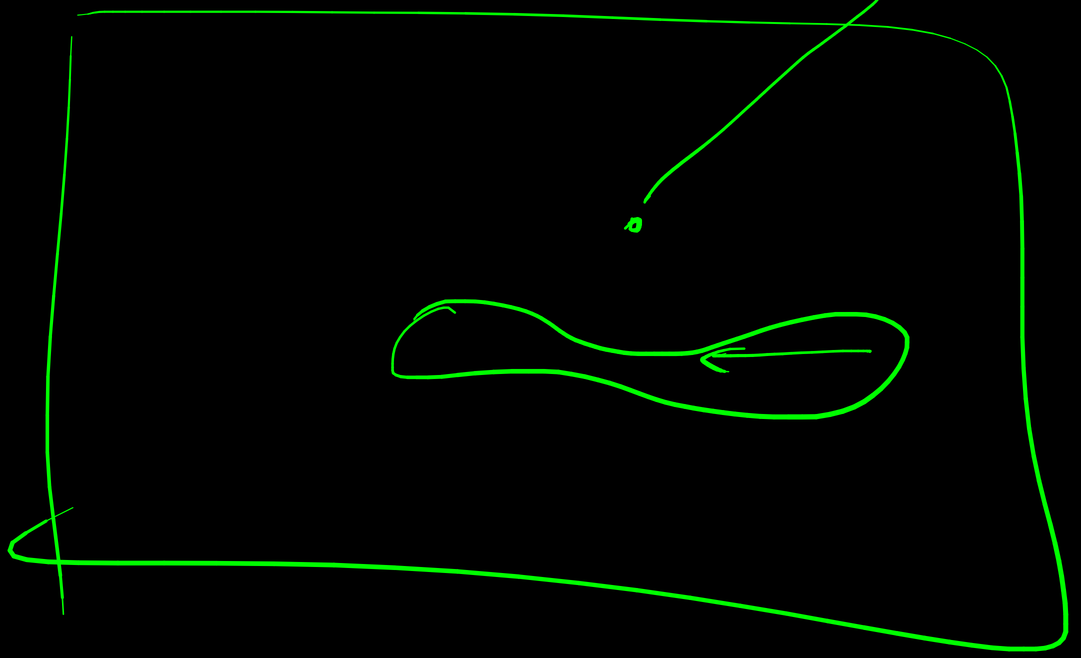


$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \cancel{\boxed{\nabla\cdot\vec{u}}} = 0}$$

$$\boxed{\nabla\cdot\vec{u} = 0} \quad \text{(Incompressible flow)}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = 0 \Rightarrow \boxed{\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{\vec{u}\cdot\nabla\rho}{\rho}} = 0$$

Variable density incompressible flow



$$\nabla\cdot\vec{u} = 0 \quad ; \quad \vec{u} = \nabla\times\vec{\Psi}$$

Helmholtz decomposition

$$\vec{\Psi} = \chi\nabla\psi \quad (2\ \text{scalars})$$

$$\vec{u} = \nabla\times(\chi\nabla\psi)$$

$$u_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\chi\frac{\partial\psi}{\partial x_k}$$

$$= \varepsilon_{ijk}\frac{\partial\chi}{\partial x_j}\frac{\partial\psi}{\partial x_k} + \varepsilon_{ijk}\chi\frac{\partial^2\psi}{\partial x_j\partial x_k}$$

$$u_i = (\nabla\chi\times\nabla\psi)_i + \left[\cancel{\varepsilon_{ijk}\frac{\partial^2\psi}{\partial x_j\partial x_k}\chi}\right] \to 0$$

$$\underline{\underline{j,k}} \quad \left(\frac{\partial^2\psi}{\partial x_j\partial x_k}\right)$$

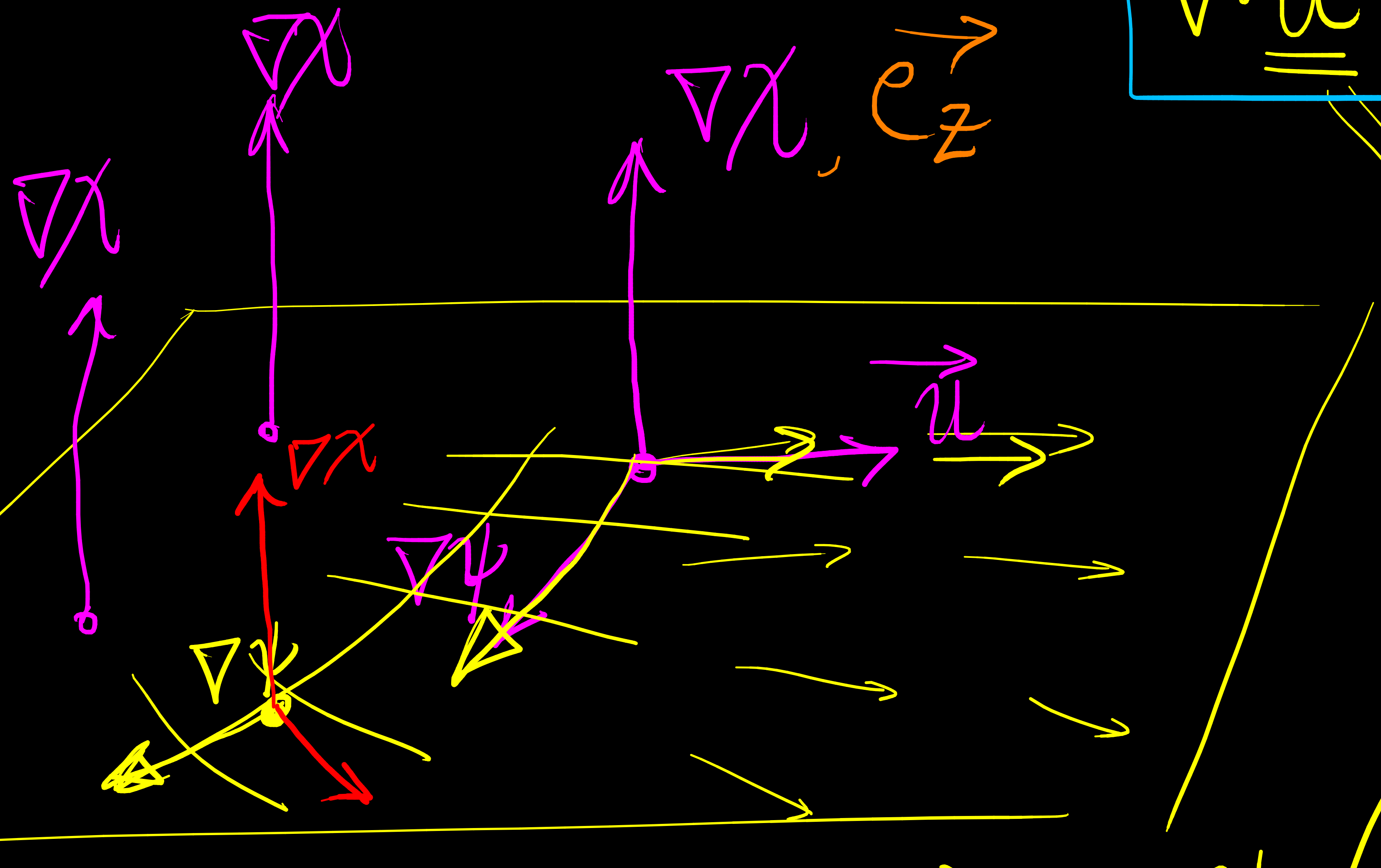
$$\boxed{\vec{u} = \nabla\chi\times\nabla\psi}$$

$\vec{u}$ is a plane $\perp$ $\nabla\chi$ $\nabla\psi$

$$\chi = z \qquad \overset{2D}{\nabla\chi = \vec{e}_z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

$$= -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$



$$\boxed{\nabla \cdot \vec{u} = 0}$$

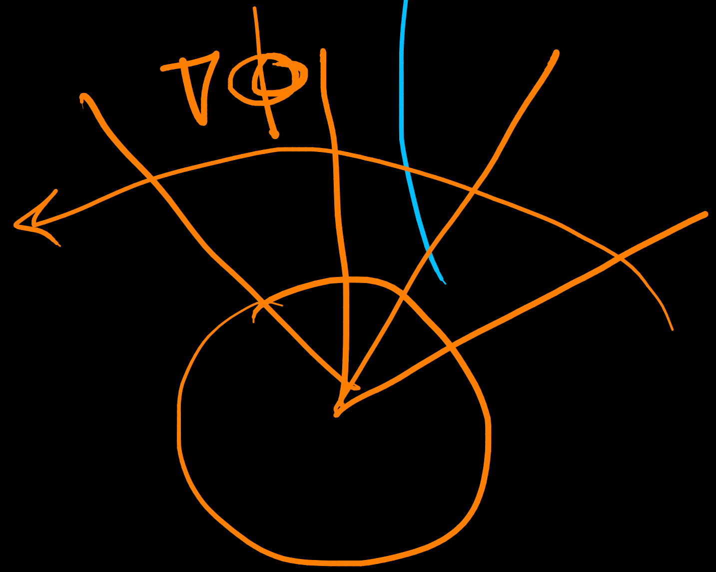
$$\vec{u} = \vec{e}_z \times \nabla\psi$$

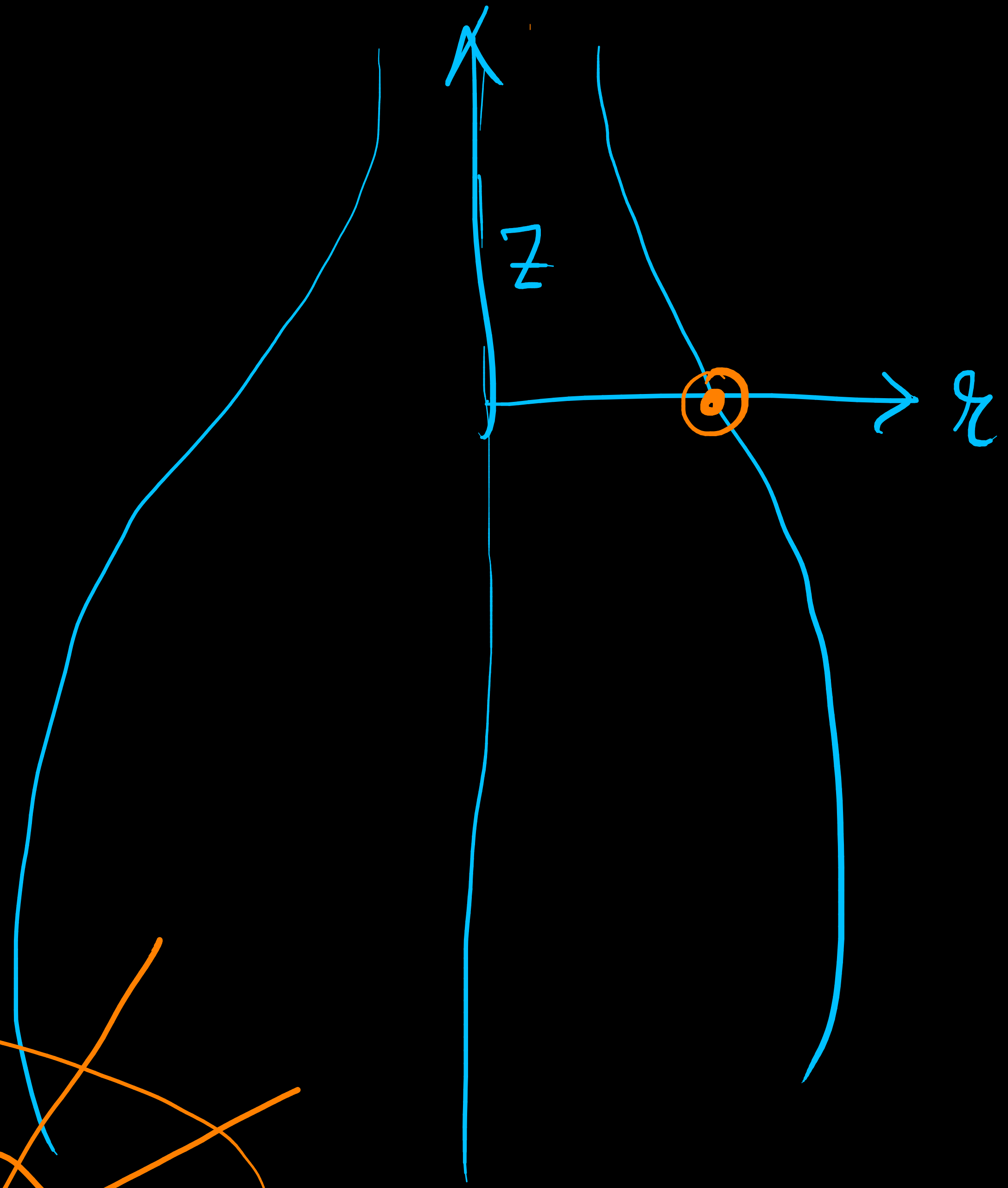
$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & 0 \end{vmatrix} = i\left(-\frac{\partial \psi}{\partial y}\right) + j\left(\frac{\partial \psi}{\partial x}\right)$$

$$\boxed{\begin{aligned} u &= -\frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \psi}{\partial x} \end{aligned}}$$

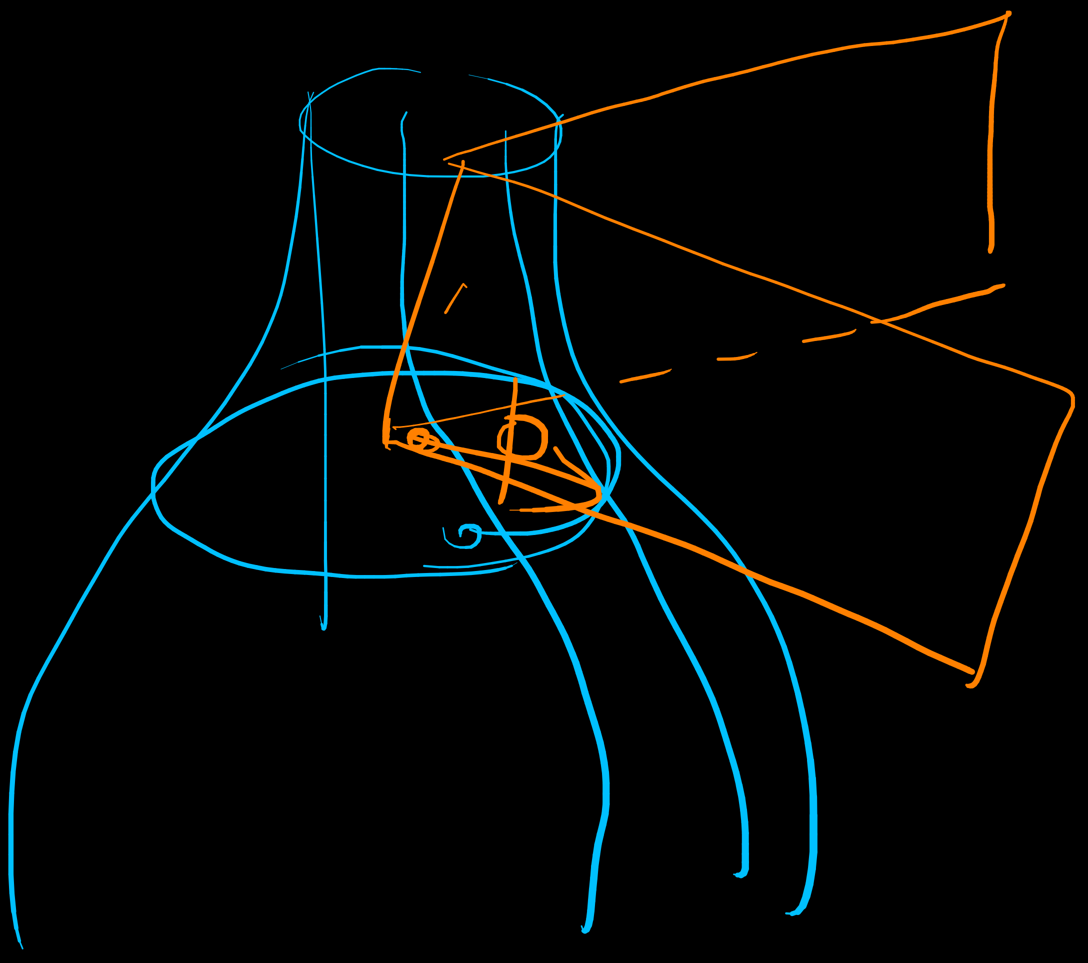
$\phi =$

$\chi = \phi$





axisymm



$(\psi) \leftarrow (u_r, u_z)$