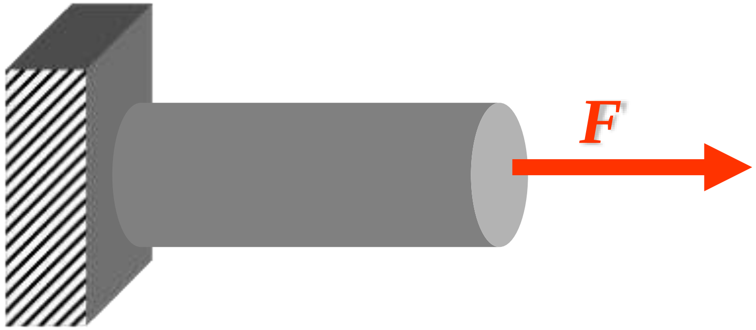
Energy Methods

Strain Energy for various cases

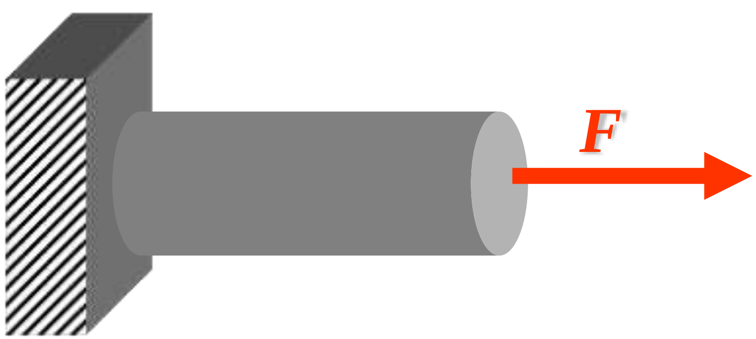
Strain Energy in Axially Loaded Rods

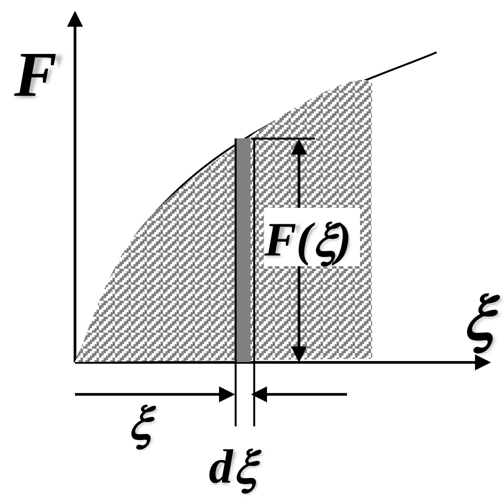
- We consider a rod of length L and area of cross section A subjected to an axial load F
- The axial load increases slowly from 0 to the final value F, thus ensuring that at every stage equilibrium is maintained



Strain Energy in Axially Loaded Rods

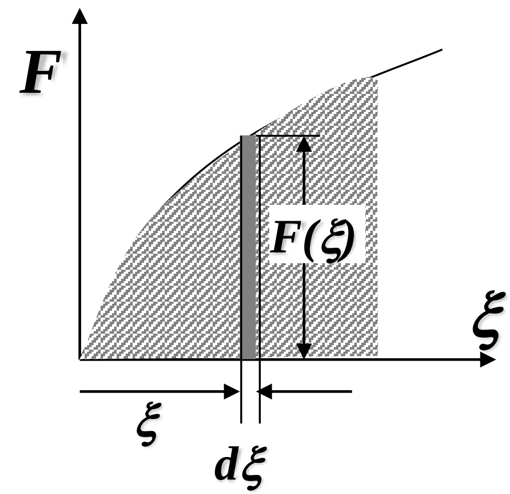
- The force need not increase linearly, but it needs to increase very slowly.
- It increases from zero to F, while the rod increases in length from 0 to x





Strain Energy in Axially Loaded Rods

- If the force increases as described (the description for such an increase is quasistatically increasing), then all the work done (area under the curve) has to go into the rod, with no scope of dissipation.
- In other words as internal energy
- Also called in this case as strain energy U



$$\boxed{U(x) = \int_{o}^{x} F d\xi}$$

Strain Energy in Axially Loaded Rods

- The rod is uniform in terms of material properties and cross section
- We will do some preliminary simple derivations
- Volume of the rod $V = LA$
- Stress in the rod $\sigma = \dfrac{F}{A}$
- Elongation of the rod $\xi = \varepsilon L$
- Strain in the rod $\varepsilon = \dfrac{\xi}{L}$

Strain Energy in Axially Loaded Rods

- We do the following manipulations to the work energy balance equation

$$\frac{U\left(x\right)}{V}=\int_{o}^{x}\frac{F}{A}\frac{d\xi}{L}=\int_{o}^{x}\sigma d\left(\frac{\xi}{L}\right)=\int_{o}^{x}\sigma d\varepsilon$$

- We can now define a new term energy density

$$u=\frac{U}{V}=\int_{o}^{x}\sigma d\varepsilon$$

Strain Energy in Axially Loaded Rods

- We do the following manipulations to the work energy balance equation. The strain in the upper limit is the final strain when F increases to its final value from 0

$$\frac{U(x)}{V} = \int_o^x \frac{F}{A}\frac{d\xi}{L} = \int_o^x \sigma d\left(\frac{\xi}{L}\right) = \int_o^\varepsilon \sigma d\varepsilon$$

- We can now define a new term energy density

$$u = \frac{U}{V} = \int_o^\varepsilon \sigma d\varepsilon$$

Strain Energy in Axially Loaded Rods

- This definition is however restrictive, because we started with the assumption of uniform cross section
- The definition can be extended for a rod of arbitrary cross section by a simple modification

$$u = \frac{dU}{dV} = \int_{o}^{\varepsilon} \sigma \, d\varepsilon$$

Strain Energy in Axially Loaded Rods

- Within proportional limits Hooke's law holds good

$$\sigma = E\varepsilon$$

- Hence if E is constant

$$u = \int_o^{\varepsilon} \sigma d\varepsilon = \int_o^{\varepsilon} E\varepsilon d\varepsilon = \frac{E\varepsilon^2}{2} = \frac{E}{2}\left(\frac{\sigma}{E}\right)^2 = \frac{\sigma^2}{2E}$$

- Or

$$u = \int_o^{\varepsilon} \sigma d\varepsilon = \int_o^{\sigma} \sigma d\left(\frac{\sigma}{E}\right) = \int_o^{\sigma} \frac{\sigma}{E} d\sigma = \frac{\sigma^2}{2E}$$

Strain Energy in Axially Loaded Rods

- Coming back to a rod of arbitrary cross section and varying material properties
- Stress at any point $\sigma(x) = \dfrac{P(x)}{A(x)}$
- Hence, strain energy density

$$u = \frac{dU}{dV} = \frac{P(x)^2}{2E(x)A(x)^2}$$

Strain Energy in Axially Loaded Rods

- Hence Strain Energy for a rod of arbitrary cross section and varying material properties

$$U = \int_V \frac{P(x)^2}{2E(x)A(x)^2} dv$$

$$= \int_0^L \frac{P(x)^2}{2E(x)A(x)^2} A(x)dx = \int_0^L \frac{P(x)^2}{2E(x)A(x)} dx$$

Strain Energy in Axially Loaded Rods

- Strain Energy for a uniform rod with a single force P acting at one end

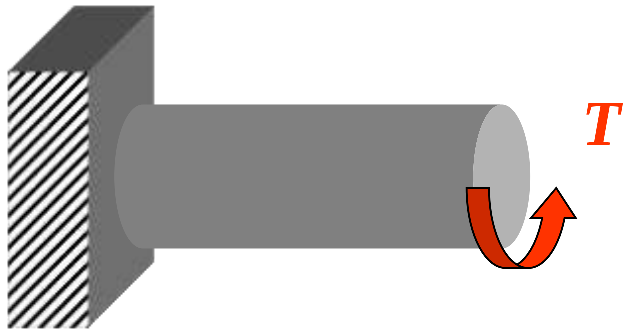
$$U = \int_V \frac{P}{2EA^2}\,dv$$

$$= \int_0^L \frac{P^2}{2EA^2}\,A\,dx = \int_0^L \frac{P^2}{2EA}\,dx$$

$$\Rightarrow U = \frac{P^2 L}{2EA}$$

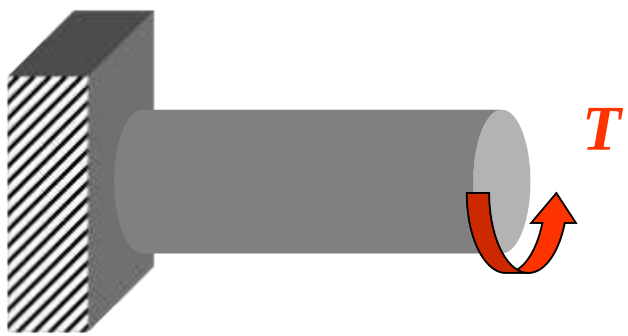
Strain Energy in Torsion

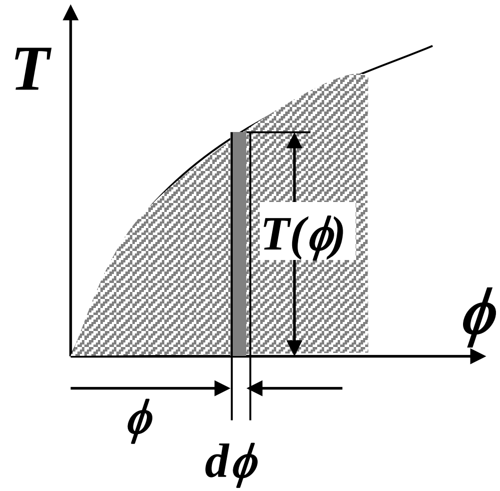
- We consider a rod of length L and second moment of area of cross section J subjected to an axial torque T.
- The torque increases slowly from 0 to the final value T, thus ensuring that at every stage equilibrium is maintained



Strain Energy in Torsion

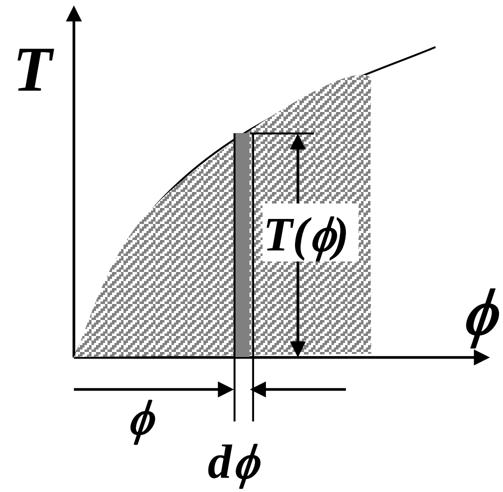
- The torque need not increase linearly, but it needs to increase very slowly.
- It increases from zero to T, while the rod twists from 0 to ϕ





Strain Energy in Torsion

- For a quasistatically increasing torque, all the work done (area under the curve) has to go into the rod, with no scope of dissipation.



- Hence the strain energy U is

$$\boxed{U\left(x\right)=\int_{o}^{\phi} T d\theta}$$

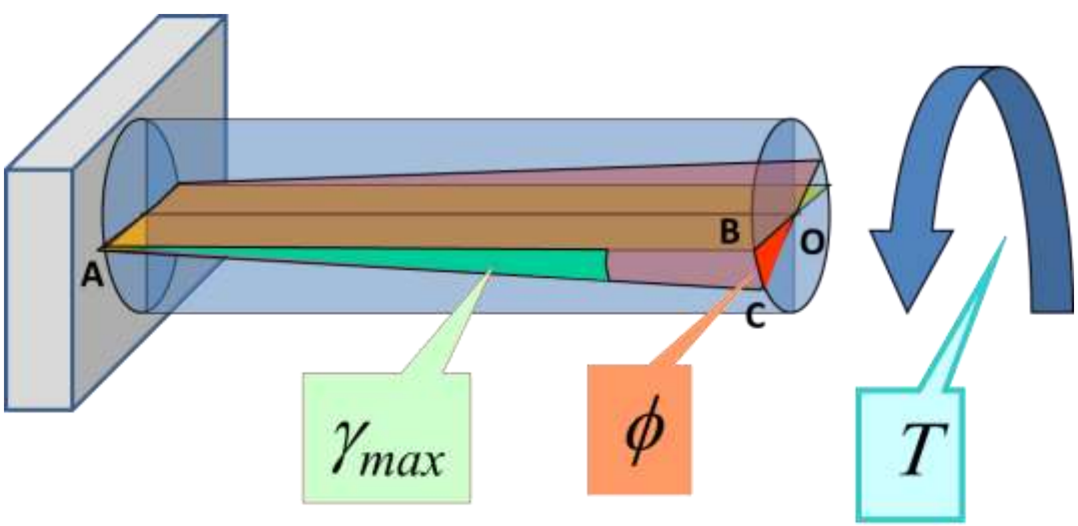
Strain Energy in Torsion

- We need to keep in mind that for simple torsion i.e. the kind of torsion we have studied and will be studying the rod always has a circular cross section
- We will need to revisit some basic concepts of torsion once again to modify the internal energy expression that we have derived

$$\boxed{U(x) = \int_{o}^{\phi} T\,d\theta}$$

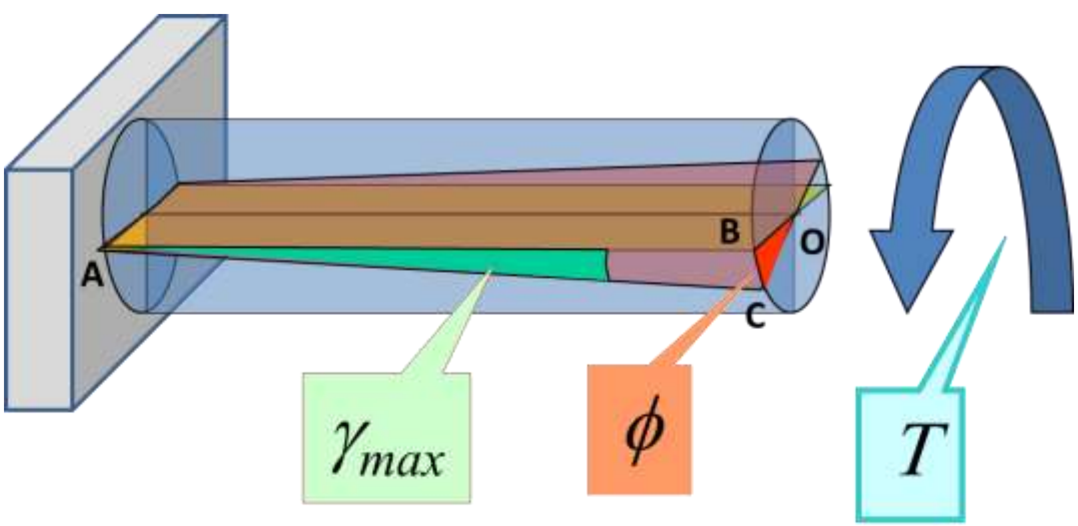
Strain Energy in Torsion

- **Shear stress and strain in torsion**
- We consider a rectangular section that was initially horizontal containing the points A and B which formed a horizontal line on the circumference
- OB was a radial line in the horizontal plane and also part of this rectangular section



Strain Energy in Torsion

- **Shear stress and strain in torsion**
- After twisting due to an external torque OB transformed to OC and remained a straight radial line
- After twisting due to an external torque AB transformed to AC and remained a straight line

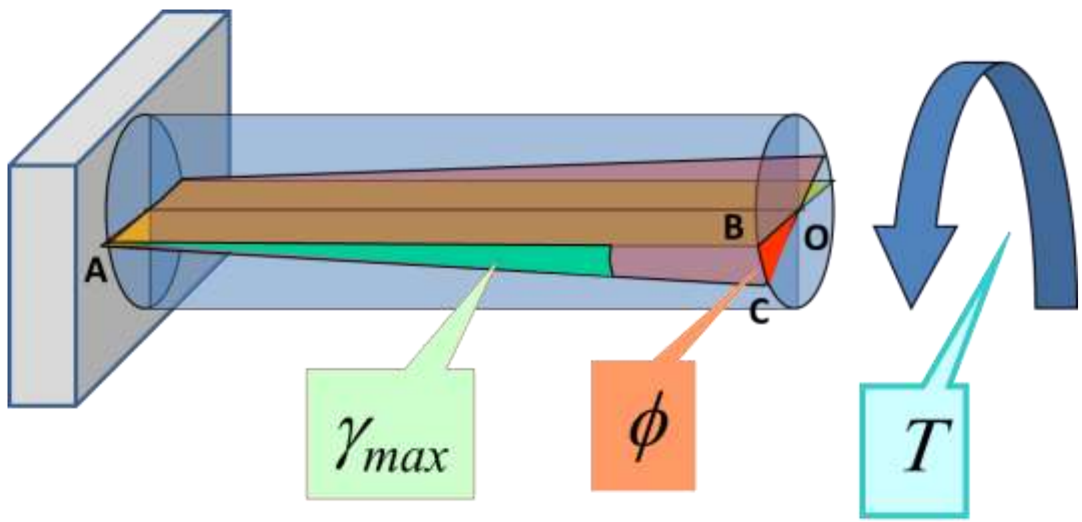


Strain Energy in Torsion

- **Shear stress and strain in torsion**
- Let γ_{max} be the shear strain at the surface
- Because AB, AC, OB and OC are all straight lines we can write

$$L\gamma_{\max} = r\phi$$

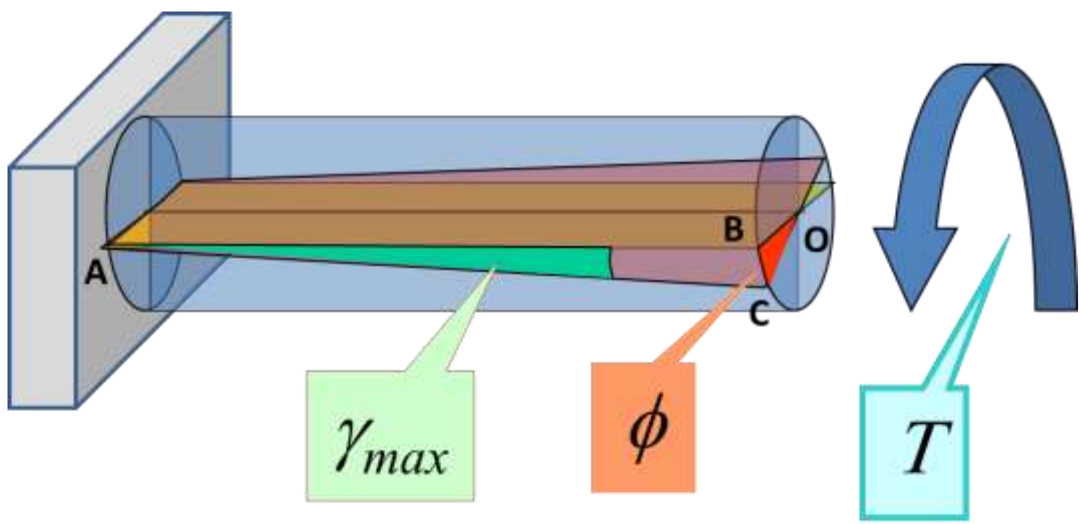
$$\Rightarrow \gamma_{\max} = \frac{r\phi}{L}$$



Strain Energy in Torsion

- **Shear stress and strain in torsion**
- Since radial straight lines remain straight and radial after the twist we can get an expression for γ even for an interior point at a radial distance ρ at the face at the free end as

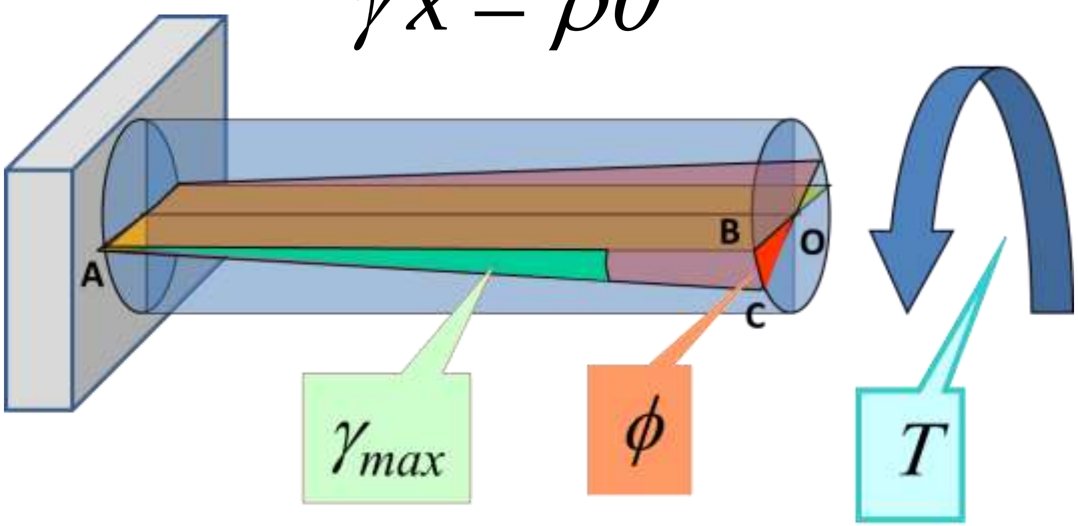
$$\gamma L = \rho\phi$$



Strain Energy in Torsion

- **Shear stress and strain in torsion**
- Because of the linearity we can conclude for any point at a distance x from the fixed end and at a radial distance ρ, the relation between the shear strain γ and the twist θ will be

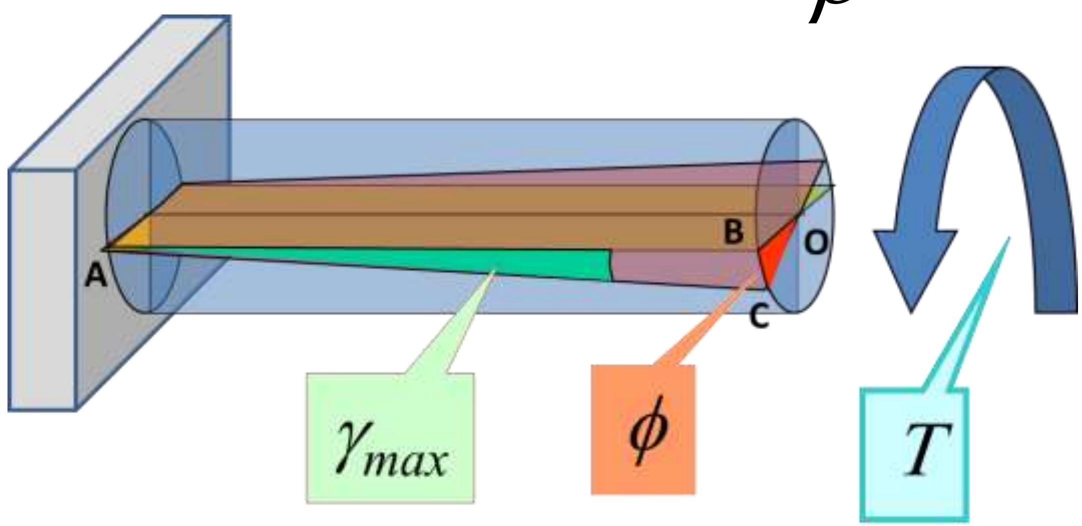
$$\gamma x = \rho\theta$$



Strain Energy in Torsion

- **Shear stress in torsion**
- For an infinitesimal distance dx along the length undergoing an infinitesimal twist we can write for the infinitesimal incremental shear strain

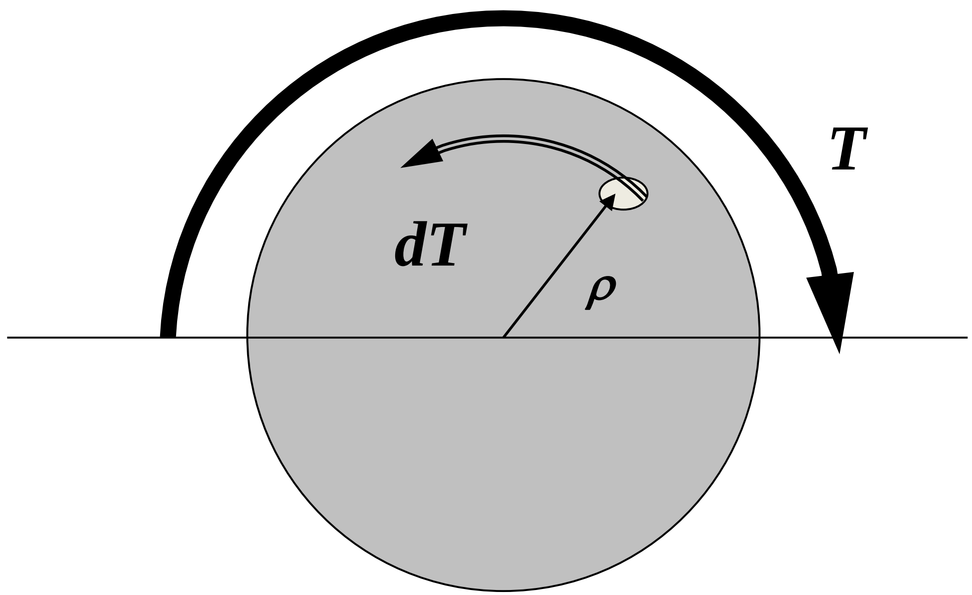
$$d\gamma dx = \rho d\theta \Rightarrow d\theta = \frac{1}{\rho} d\gamma dx$$



Strain Energy in Torsion

- **Shear stress and strain in torsion**
- We now look at a transverse section whose area is A
- For equilibrium

$$T = \int_A \rho(\tau dA)$$



Strain Energy in Torsion

- We now define the strain energy density for shear

$$u = \frac{dU}{dV} = \int_0^{\gamma} \tau \, d\gamma$$

$$\because \tau = G\gamma \Rightarrow \gamma = \frac{\tau}{G}$$

$$\therefore u = \frac{dU}{dV} = \int_0^{\gamma} G\gamma \, d\gamma = \frac{G\gamma^2}{2} = \frac{\tau^2}{2G}$$

Strain Energy in Torsion

- We now apply these factoids to find the strain energy in torsion

$$U = \int u dV$$

$$= \int \frac{\tau^2}{2G} dV = \int \frac{1}{2G(x)} \left\{ \frac{T(x)\rho(x)}{J(x)} \right\}^2 dV$$

$$= \int \frac{T(x)^2 \rho(x)^2}{2G(x) J(x)^2} dV$$

Strain Energy in Torsion

- Further simplification

$$U = \int_0^x \int_A \frac{T(x)^2 \rho(x)^2}{2G(x)J(x)^2} dA dx$$

$$= \int_0^x \frac{T(x)^2}{2G(x)J(x)^2} \left(\int \rho(x)^2 dA \right) dx$$

Strain Energy in Torsion

- Recall the definition of polar moment of area

$$J = \int \rho(x)^2 \, dA$$

- Hence

$$U = \int_0^x \frac{T(x)^2}{2G(x)J(x)^2} \int_A \left(\int \rho(x)^2 \, dA \right) dx$$

$$= \int_0^x \frac{T(x)^2}{2G(x)J(x)^2} J(x) \, dx$$

Strain Energy in Torsion

- Further simplification

$$U = \int_0^x \frac{T(x)^2}{2G(x)J(x)^2} J(x)\,dx$$

$$\Rightarrow U = \int \frac{T(x)^2}{2G(x)J(x)} dx$$

- This is analogous to the expression for strain energy for axial loading

Strain Energy in Torsion

- Thus strain energy for a general rod in torsion

$$U = \int \frac{T(x)^2}{2G(x)J(x)}dx$$

- For a single applied torque acting on a uniform rod of length L

$$U = \frac{T^2 L}{2GJ}$$

Strain Energy in Bending of Beams

- We consider a beam of length L acted upon by a force that, as stated in the other cases, increases slowly from 0 to the final value, thus ensuring that at every stage equilibrium is maintained.
- The force results in an internal moment M(x), where x is the distance of a point on the beam from the chosen origin

Strain Energy in Bending of Beams

- Normal stress in a transverse section of the beam is known to us in terms of the moment M(x) and the geometrical parameters of the beam
- Hence we can directly start from the strain energy density

$$\sigma = \frac{My}{I}$$

Strain Energy in Bending of Beams

- The strain energy density will be

$$\boxed{u = \frac{dU}{dV} = \frac{\sigma_x{}^2}{2E} = \frac{\left\{\dfrac{M(x)\,y}{I(x)}\right\}^2}{2E(x)}}$$

Strain Energy in Bending of Beams

- The strain energy density will be

$$\boxed{u = \frac{dU}{dV} = \frac{{\sigma_x}^2}{2E} = \frac{\left\{\dfrac{M(x)\,y}{I(x)}\right\}^2}{2E(x)}}$$

Strain Energy in Bending of Beams

- Total strain energy

$$U = \int_V u dV = \iint_V \frac{\left\{ \frac{M(x) y}{I(x)} \right\}^2}{2E(x)} dA dx$$

$$\Rightarrow U = \int_0^L \left\{ \frac{M^2(x)}{2E(x) I^2(x)} \int_A y^2 dA \right\} dx$$

Strain Energy in Bending of Beams

- We recall the definition of second moment of area

$$I = \int_A y^2 dA$$

- Hence

$$U = \int_0^L \left\{ \frac{M^2(x)}{2E(x)I^2(x)} I(x) \right\} dx$$

$$\Rightarrow U = \int_0^L \frac{M^2(x)}{2E(x)I(x)} dx$$

Strain Energy in Bending of Beams

- In case of beam bending we rarely encounter cases of uniform bending moment, but we do encounter beams of uniform material property and cross section
- For such simple cases

$$U = \frac{1}{2EI}\int_0^L M^2(x)\,dx$$

Strain Energy in Bending of Beams

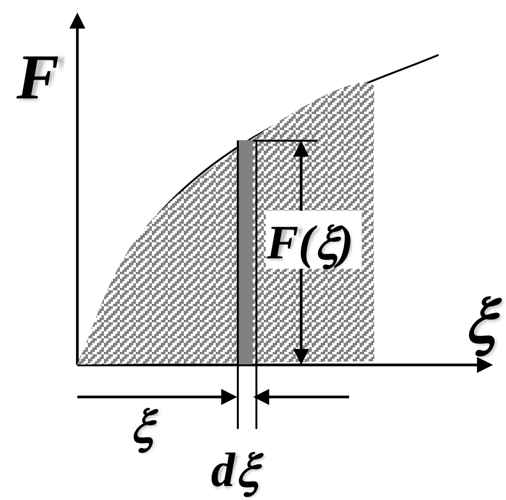
- In case of beam bending we rarely encounter cases of uniform bending moment, but we do encounter beams of uniform material property and cross section
- For such simple cases

$$U = \frac{1}{2EI}\int_0^L M^2(x)\,dx$$

Complementary Strain Energy

- In all these cases we have considered the work done by a generalized force due to the generalized displacement and equated that to the internal strain energy.
- This energy is equal to the area under the curve

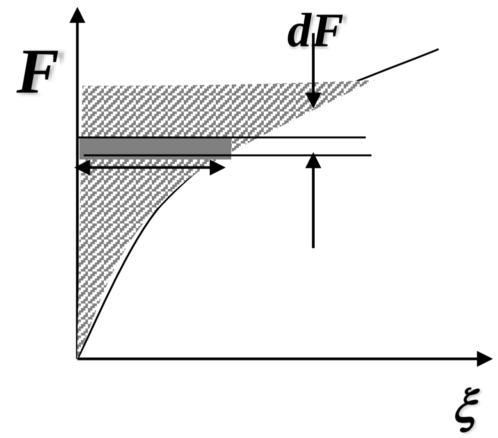
$$U = \int_0^x F(\xi)\, d\xi$$



Complementary Strain Energy

- However if we consider the area enclosed by the curve along the F axis we get a different area and a different energy
- This energy is known as complementary strain energy and may not be equal to strain energy

$$U_c = \int_0^X \xi(F)\, dF$$

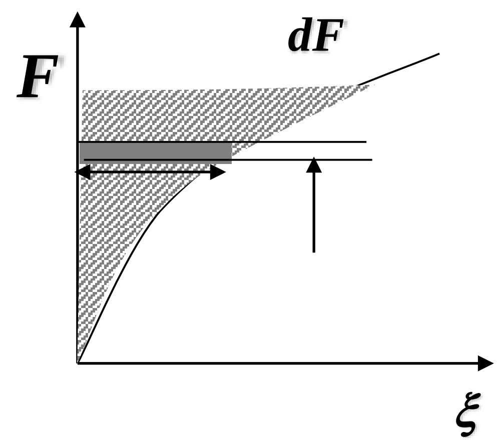


Complementary Strain Energy

- When material properties and generalized stiffnesses are linear, σ vs ε curve is a straight line as is the F vs x curve
- Hence both these areas are equal
- U_c is used in energy methods for non linear cases

$$U = U_c$$

$$u = \int \sigma d\varepsilon = \int \varepsilon d\sigma = u_c$$



Applications to a beam problem

- Consider a cantilever beam of length L and acted upon at the free end by a vertical force P

$$M\left(x\right)=P\left(L-x\right)\Rightarrow U=\frac{1}{2EI}\int_{0}^{L}P\left(L-x\right)^{2}dx$$

Applications to a beam problem

- The strain energy is given by

$$U = \frac{1}{2EI}\int_0^L \left\{P\left(L-x\right)\right\}^2 dx$$

$$= -\frac{P^2\left(L-x\right)^3}{6EI}\Bigg|_0^L = \frac{P^2L^3}{6EI}$$

Applications to a beam problem

- If we assume that the force was increased linearly, with tip deflection v, from 0 to the final value P, then the work done by the force is

$$W = \frac{1}{2} P v(L)$$

Applications to a beam problem

- Equating work done by the external force with the strain energy generated

$$\frac{1}{2}Pv(L)=\frac{P^2L^3}{6EI}$$

$$\Rightarrow v(L)=\frac{PL^3}{3EI}$$