Revisiting Statically Determinate systems on the way to Statically Determinate systems

We will begin with problems in one dimension.

That is where we have dealt with one unknown reaction (constraint) only.

What is a statically determinate system ?

- A simple answer, within the context of what we learn in first year mechanics, is when the number of unknown reactions (constraints) is equal to the number of equations of static equilibrium that we can obtain from the system. Hence number of (generalized) FORCE equilibrium equations must be equal to number of unknown reactions. Why GENERALIZED ? Because this definition will then also consider Moments of a force and Moment reactions.

Relation with dimensions

- A one dimensional problem will have only one equilibrium equation. Rotation requires at least two dimensions. In a one dimensional system only possible motion is translation along one given direction. Hence the only possible reaction required to make the system static is also a force in that direction (actually in the opposite or negative with respect to that direction).

A one dimensionsal problem

- We have a rod with one end fixed to a wall and a force P pulling it at the free end. We can figure out the reaction at the wall simply by looking at the problem but we will take a more formal, long winded approach, which will become inevitable as the problems become more complicated.

A one dimensionsal problem

- We start with the free body diagram and replace the constraint (the fixed joint at the wall) with its effect, namely the reaction force R.

A one dimensionsal problem

- We now apply the equation of equilibrium and get P+R=0, which tells us that R=-P. The problem is solved and it should be because it is statically determinate.

A one dimensionsal problem

- We now consider a case where the bar is fixed at both the ends. The force P acts at the same point. The drawing has been changed slightly only to ensure the clarity of the picture.

A one dimensionsal problem

- We draw the free body diagram as before. Since there are fixed joints at both ends we will get one more reaction force Q

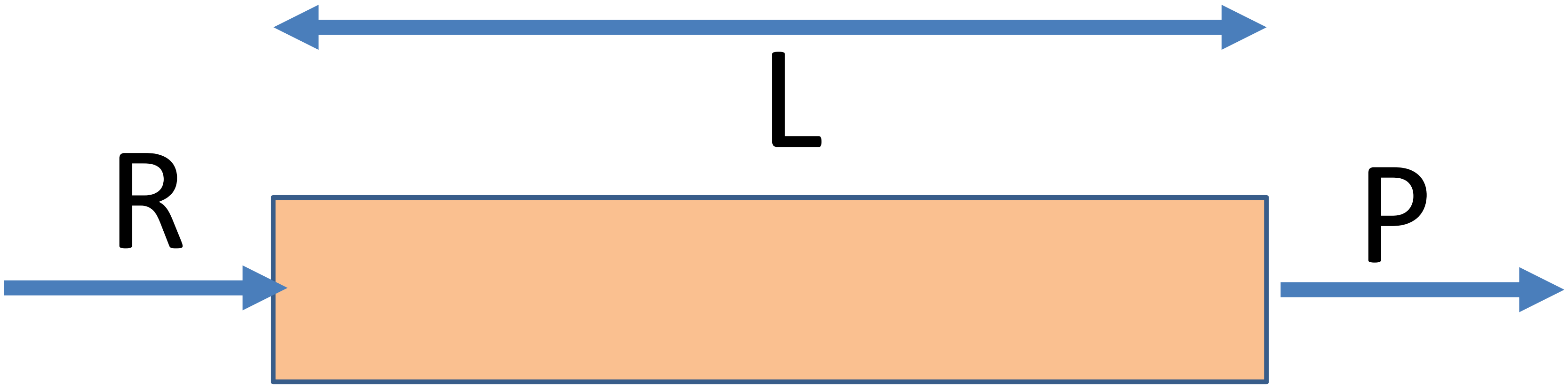
A one dimensionsal problem

- We apply the equation of equilibrium and get P+Q+R=0, which tells us that R+Q=-P. So we know the sum of the two unknown reactions but we do not know the value of either P or Q. We cannot proceed further because we have no more equilibrium equations in the bank. Hence this problem is statically indeterminate

A one dimensionsal problem

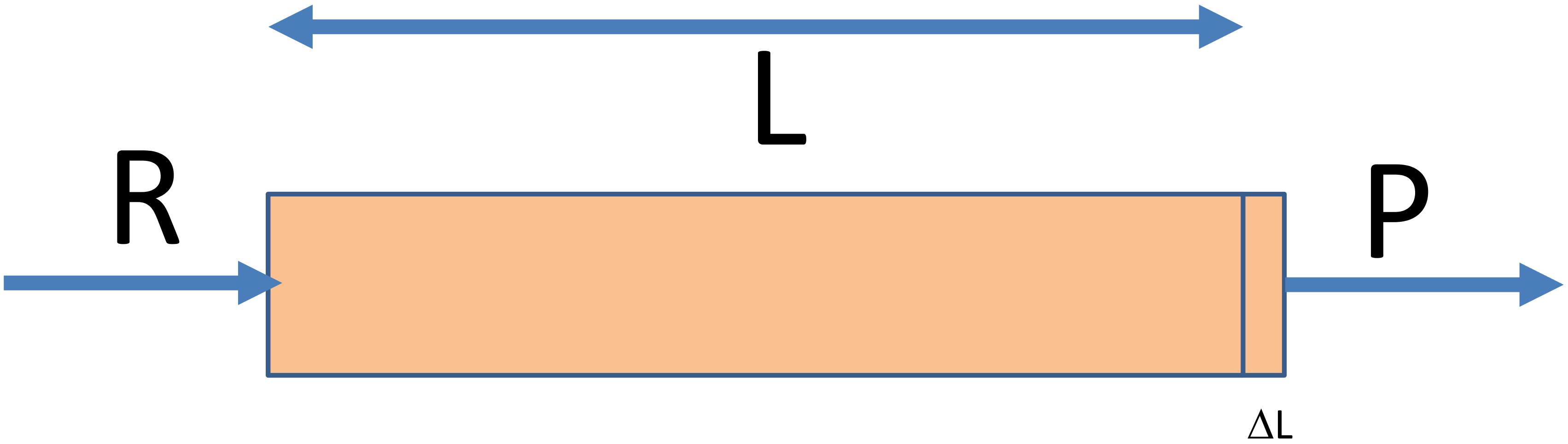
- We need something more. To get this something more we will solve a few complicated versions of the original problem.

Change in length



- The something more will come from the change in geometry due to deformation. In one dimension, there is only one relevant geometrical feature, which is the length of the bar L.

Change in length



- We will focus on finding the deformation ΔL

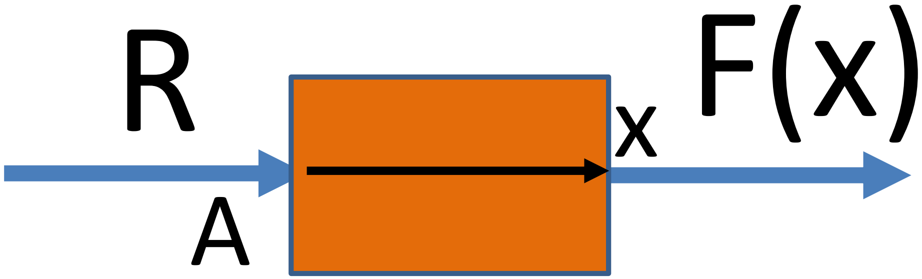
Find the internal force

- We first label the two ends of the rod as A and B and set up a coordinate system. A is the origin with the direction from A to B being positive.

Take a section

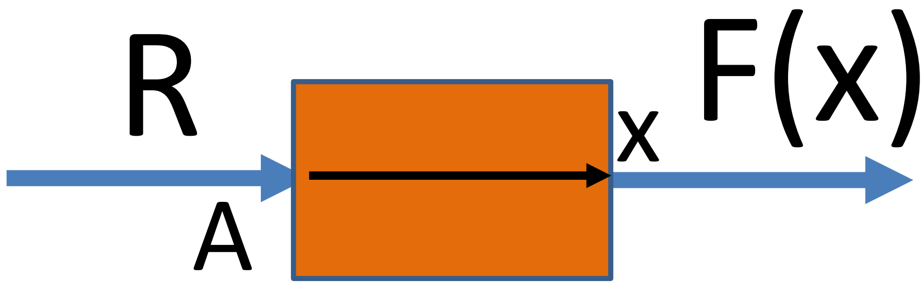
- We take a section at a distance x from the origin.

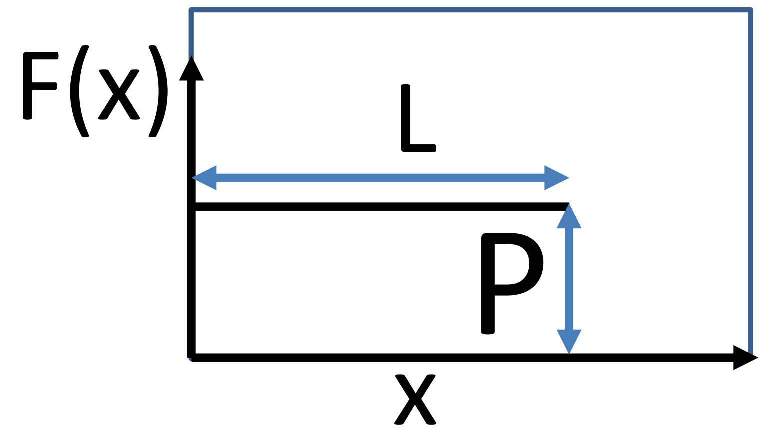
Take a section



- We take a section at a distance x from the origin. That is we cut the rod at a distance x. What we will be seeing is a force perpendicular to the exposed surface, which may or may not be dependent on x. We will call it F(x) to indicate that it is a function of x (remember that a constant value can also be considered a special case of a function of x).

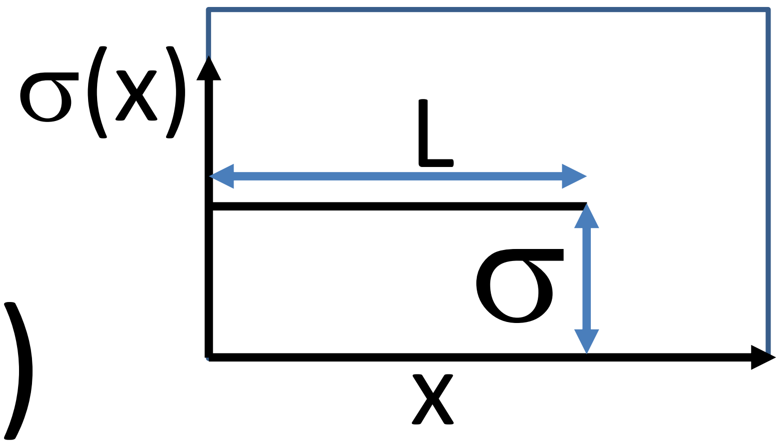
Take a section





- Simple equilibrium of forces tells us that F(x)=-R. Since we already know that R=-P, hence F(x)=P. We also draw a graph of our results.

Revisiting stress

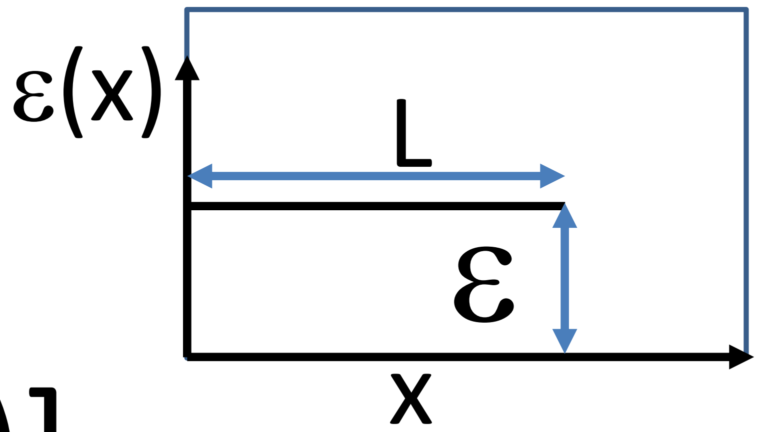


$$\sigma(x) = F(x)/A(x)$$

- It is now easy to find the stress at any x as long as we know the area of cross section at that x, which is A(x). Once again A(x) may or may not be a constant. And we quickly draw a graph of $\sigma(x)$ for our particular problem where A is a constant and hence $\sigma(x) = P/A$.

Revisiting Hooke's law

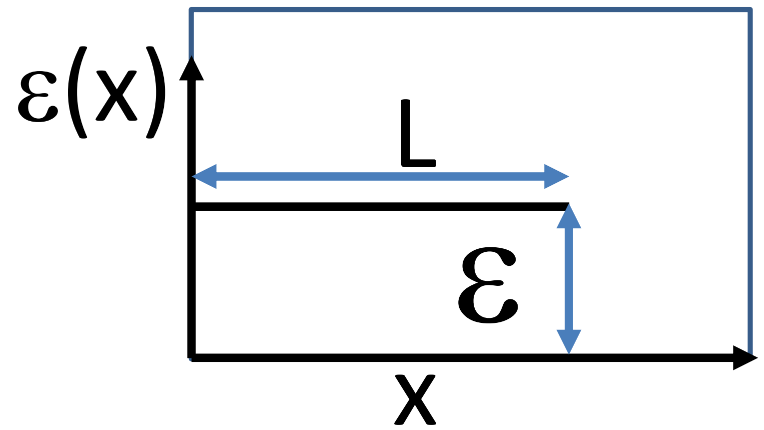
$$\varepsilon(x) = \sigma(x)/E(x)$$
$$= F(x)/[E(x)A(x)]$$



- We can now use Hooke's law to find strain at any x provided we know the modulus of elasticity E at that x, which is E(x). Once again E(x) may or may not be a constant (the rod can be made of two different materials). And we quickly draw a graph of $\varepsilon(x)$ for our particular problem where E is a constant and hence $\varepsilon(x) = P/[EA]$.

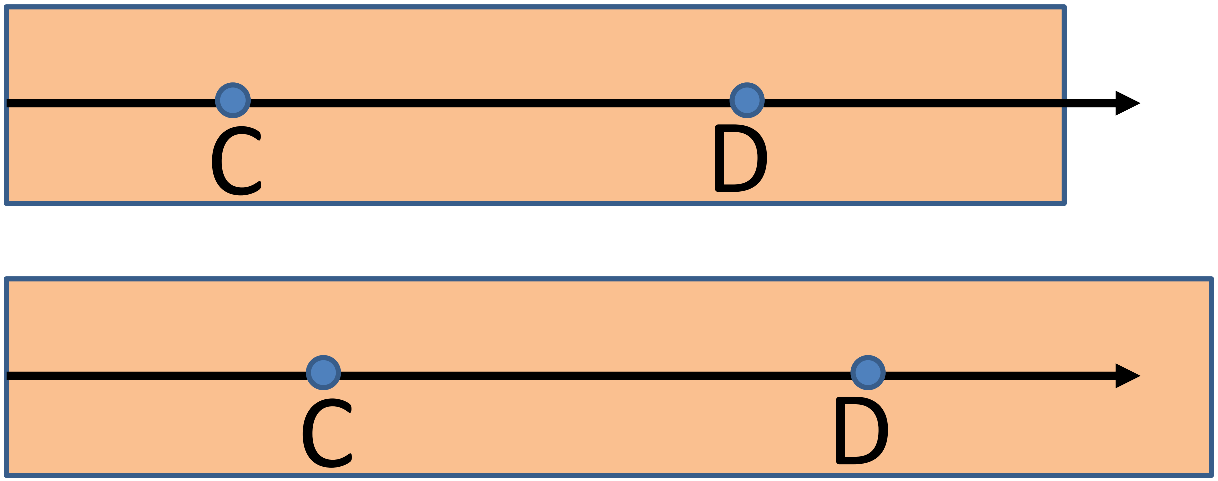
Revisiting definition of strain

$$\varepsilon(x) = du(x)/dx$$



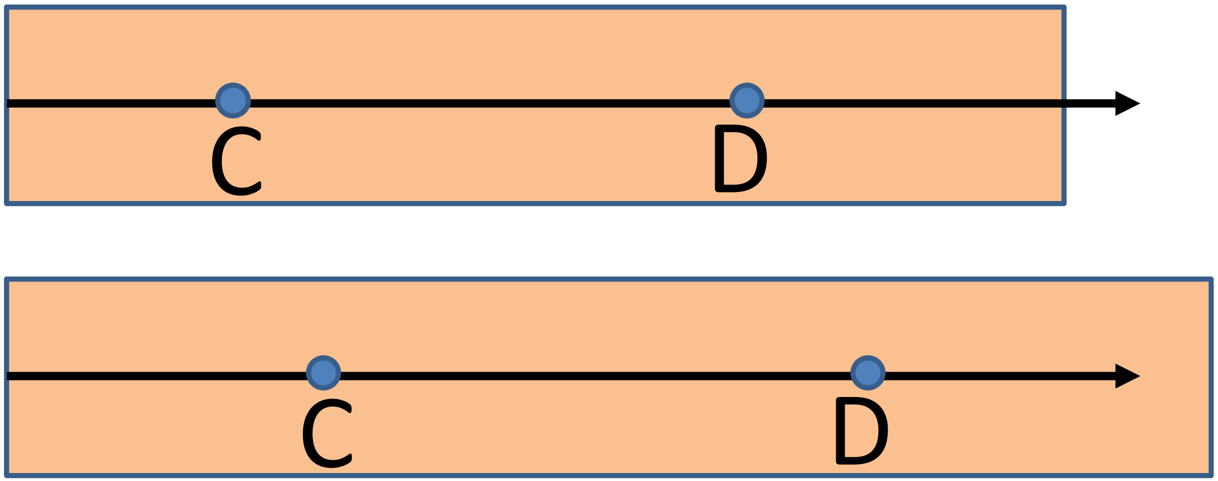
- We may recall that u(x) is a function defining the displacement of a point at x on the rod due to the application of the external force P. Strain is defined in terms of that displacement or to be more precise the gradient or rate of change of that displacement with respect to x.

Explanation of u



- Consider two points on the rod C and D whose coordinates are x_C and x_D. After being pulled by the force P, C will be displaced by an amount u_C and D will be displaced by an amount u_D. This amount MAY NOT be the same for C and D. It depends on what are the external forces , where they are applied as well as the material and dimensions of the rod. This will become clear when we consider a rod with different materials and/or different cross sections at different x.

Explanation of u



- A rough estimate of strain is then = $(u_D - u_C)/(x_D - x_C)$
- When C and D are very close together it becomes du/dx

Equating the two expressions for strain

- $\varepsilon(x) = du(x)/dx = F(x)/[E(x)A(x)]$

For our simple case

$\varepsilon(x) = du(x)/dx = F/[EA]$

Equating the two expressions for strain

$$\frac{du}{dx} = \frac{F(x)}{E(x)A(x)} \Rightarrow u(x) = \int_o^x \frac{F(x)}{E(x)A(x)} dx$$

For our simple case F, E, A are constants

$$\frac{du}{dx} = \frac{F}{EA} \Rightarrow u(x) = \int_o^x \frac{F}{EA} dx = \frac{Fx}{EA}$$

Total elongation

For our simple case the total elongation or change of length of the rod will be simply the displacement of the point B

$$u\left(x_B\right)=u\left(L\right)=\frac{FL}{EA}$$

This is a very familiar high school formula. But the relatively complicated derivation will pay off when the problems are complicated.

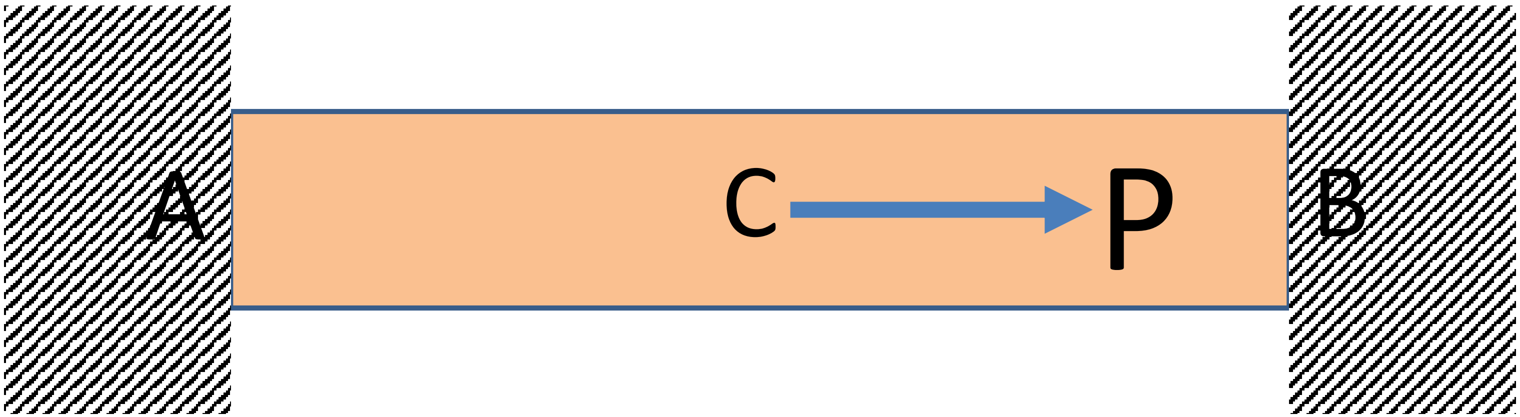
How does this help with indeterminate problems ?

- To answer this question we will add a simple twist to our initial simple question. What is the force Q required at the free end so that P produces zero elongation at that end ?

Back to the original problem

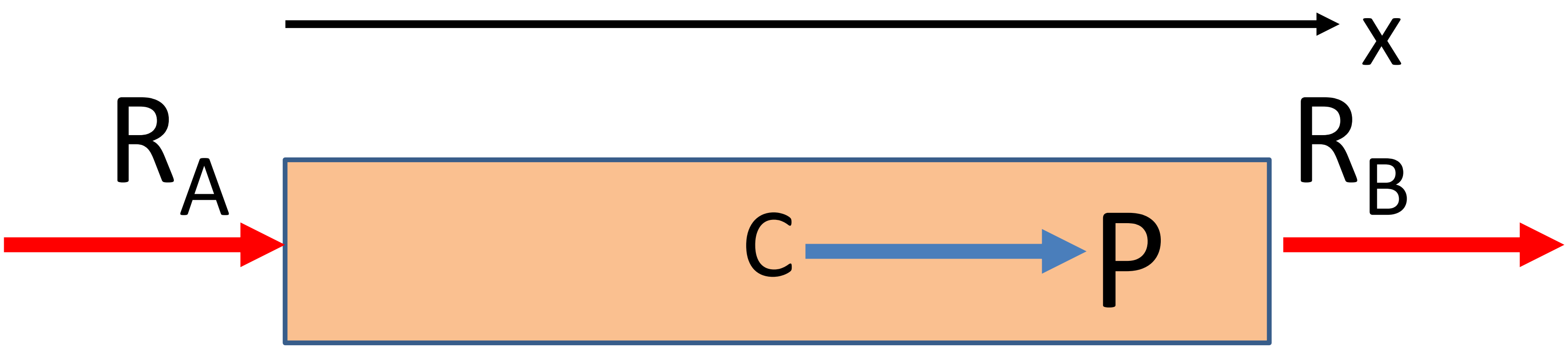
- What does the second wall do ? It produces a reaction Q that ensures that there is no displacement at the second wall.
  Now ask the question. Is the problem in the previous slide the same but with some phrases being different ?

A more realistic problem



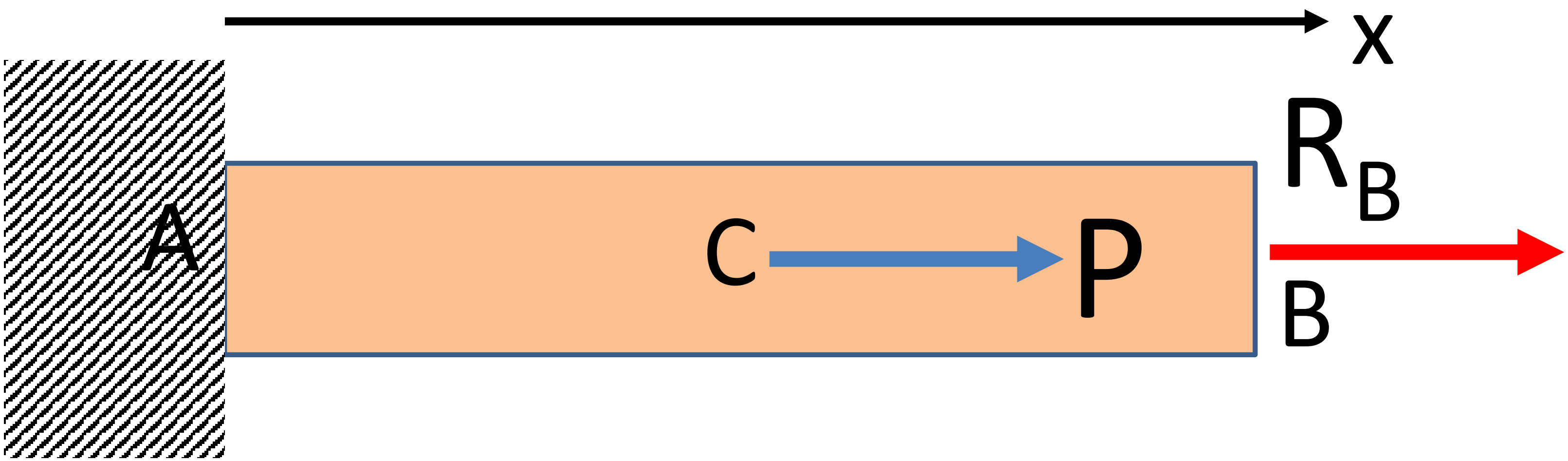
- The force P is now acting at the midpoint of the rod, which is fixed at both ends. The task ahead is to find out the reactions at A and B.

Start with FBD



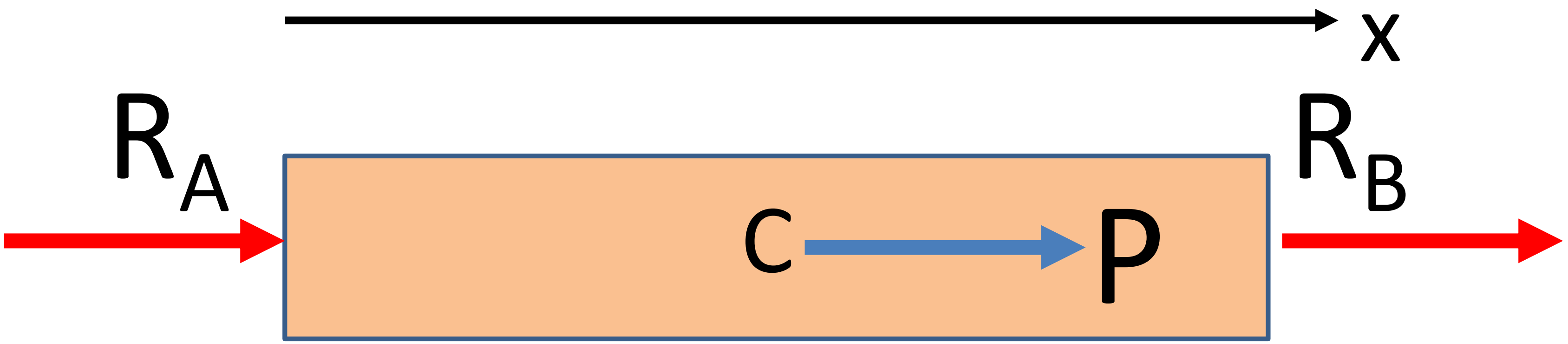
- First set up a coordinate system. In this case the origin can be at A and the +ve direction will be from A to B
- The reactions will be R_A and R_B. The directions shown have intentionally been kept positive, although they are counterintuitive. You would certainly have liked at least one of them to have a direction opposite to P. But as we work with complicated problems we will find this has advantages. Once we get the answers the signs will tell us the directions.

The alternative problem



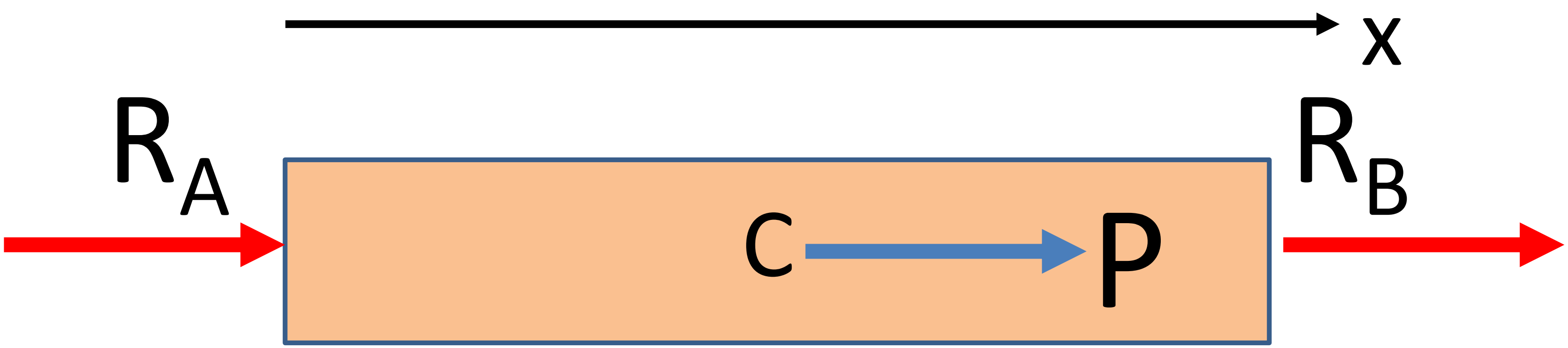
- We will try to solve the following problem. What should be R_B so that the displacement at point B is zero ?

FBD of the alternative problem



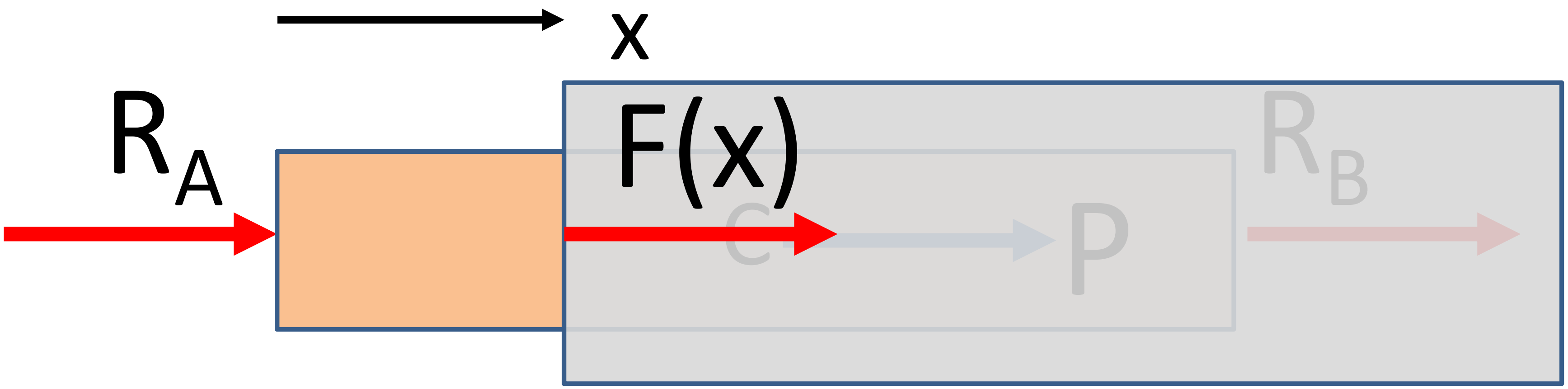
- The free body diagram comes out to be the same

Critical Points and Domains



- A critical point is a point where there is a sudden change - in forces, in dimensions, in material properties, or there is new constraint. Here we have three critical points A, C and B and hence two domains – A to C and C to B.
- We will need to take a section in every domain

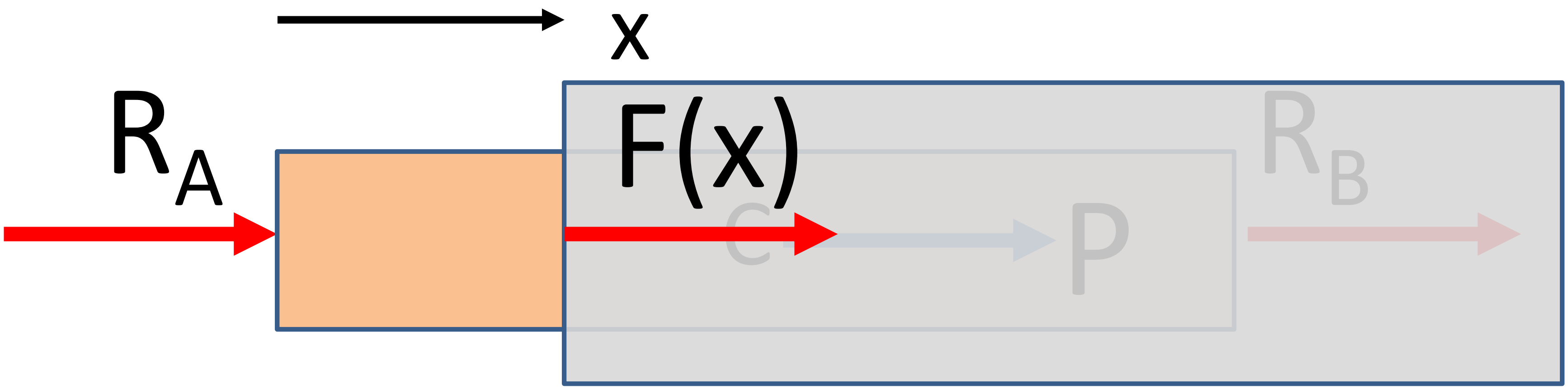
Domain AC: Force



- Once we cut the section we will see an internal force F(x) at the cut. Using equilibrium (ONLY FOR THE SECTION UPTO x) we will get

$$R_A + F(x) = 0 \Rightarrow F(x) = -R_A$$

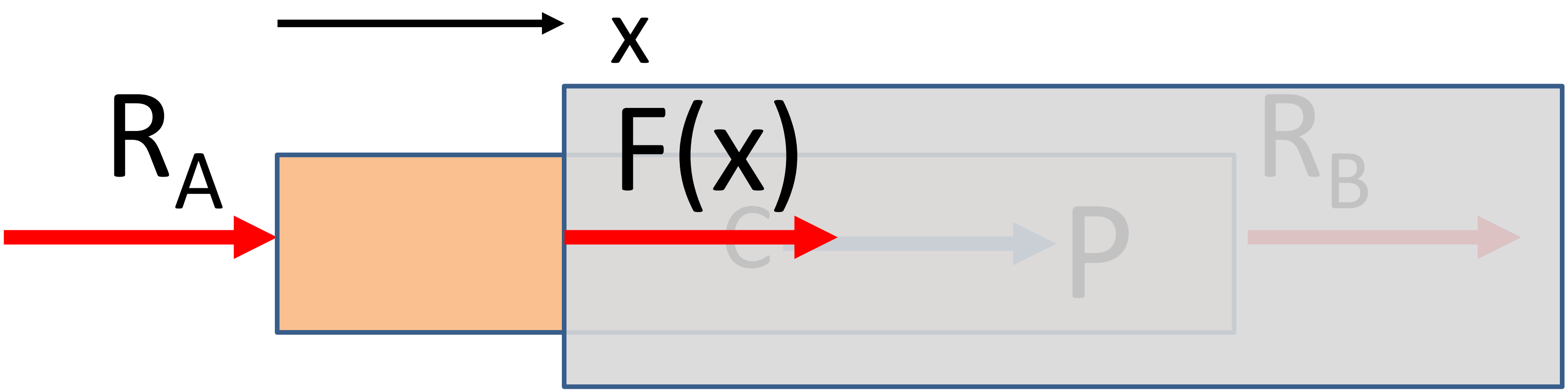
Domain AC: Stress



- Area of cross section is A for the entire rod. So

$$\sigma\left(x\right)=\frac{F\left(x\right)}{A}=-\frac{R_A}{A}$$

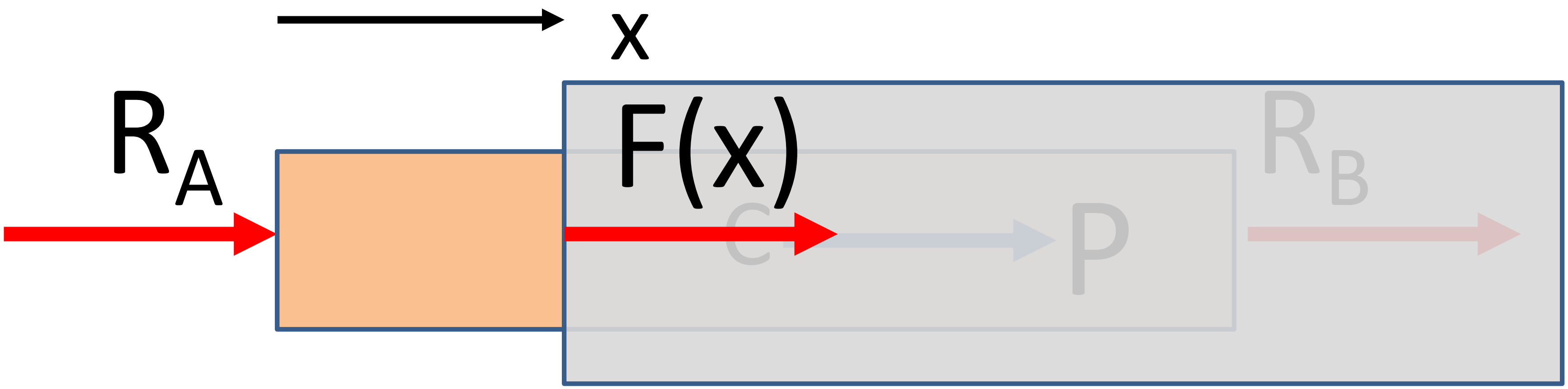
Domain AC: Strain



- Modulus of elasticity is E for the entire rod. So

$$\varepsilon(x) = \frac{\sigma(x)}{E} = -\frac{R_A}{EA}$$

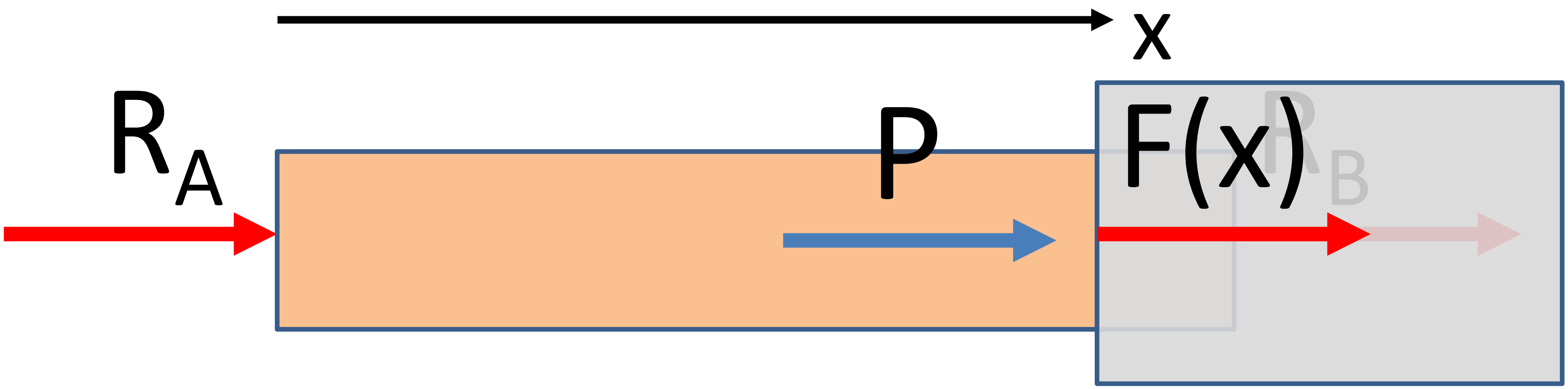
Domain AC: Displacement



- Hence

$$u(x) = \int_0^x \varepsilon(x)\,dx = -\int_0^x \frac{R_A}{EA}\,dx = -\frac{R_A x}{EA}$$

Domain CB: Force

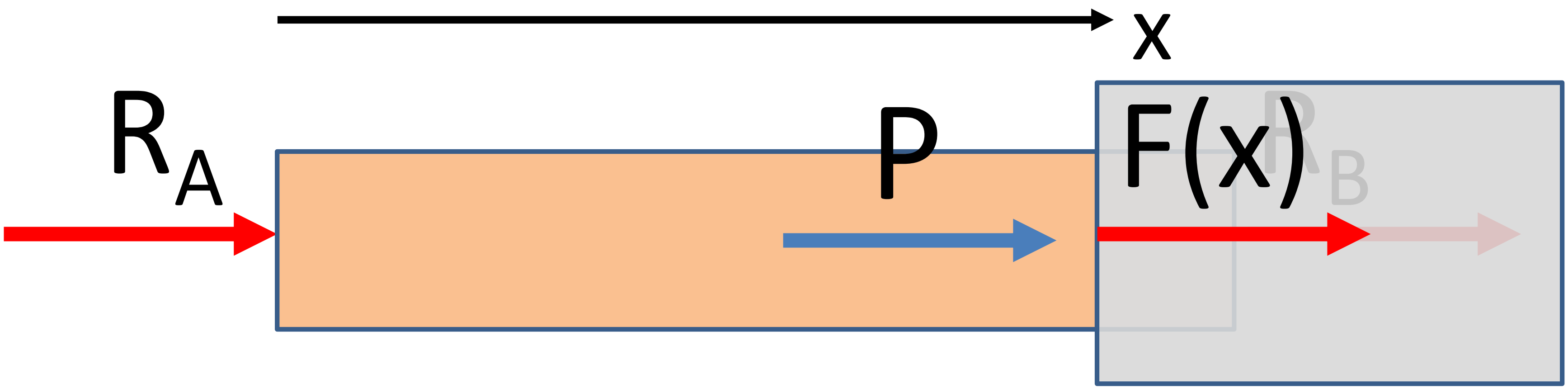


- Once we cut the section between C and B we will see an internal force F(x) at the cut.
- Note that in this domain we can now see P, but R_B still remains hidden. Also origin and coordinate system remain unchanged.
- Using equilibrium we get

$$R_A + P + F(x) = 0$$

$$\Rightarrow F(x) = -R_A - P$$

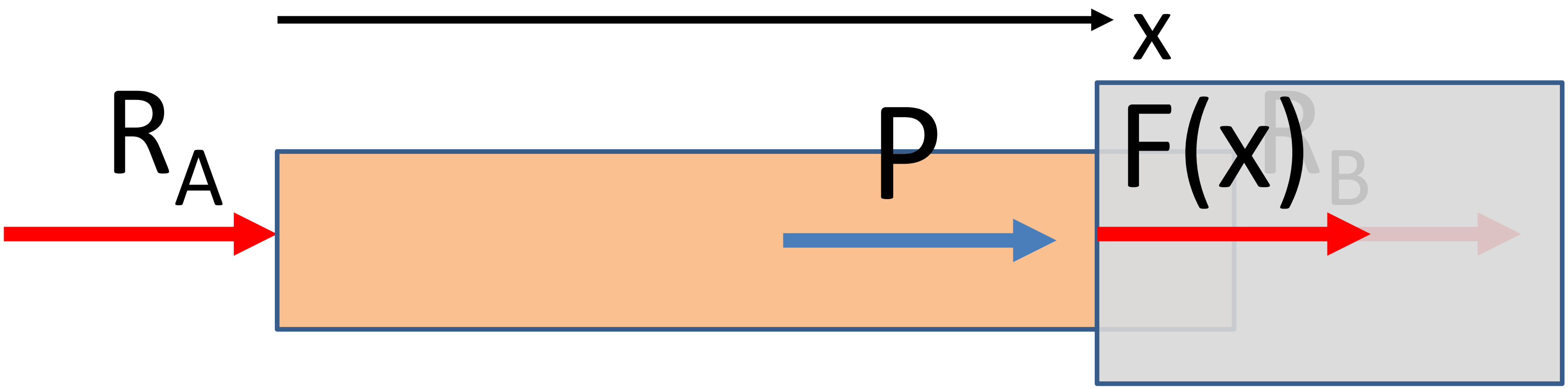
Domain CB: Stress



- Area of cross section is A for the entire rod. So

$$\sigma(x)=\frac{F(x)}{A}=-\frac{R_A+P}{A}$$

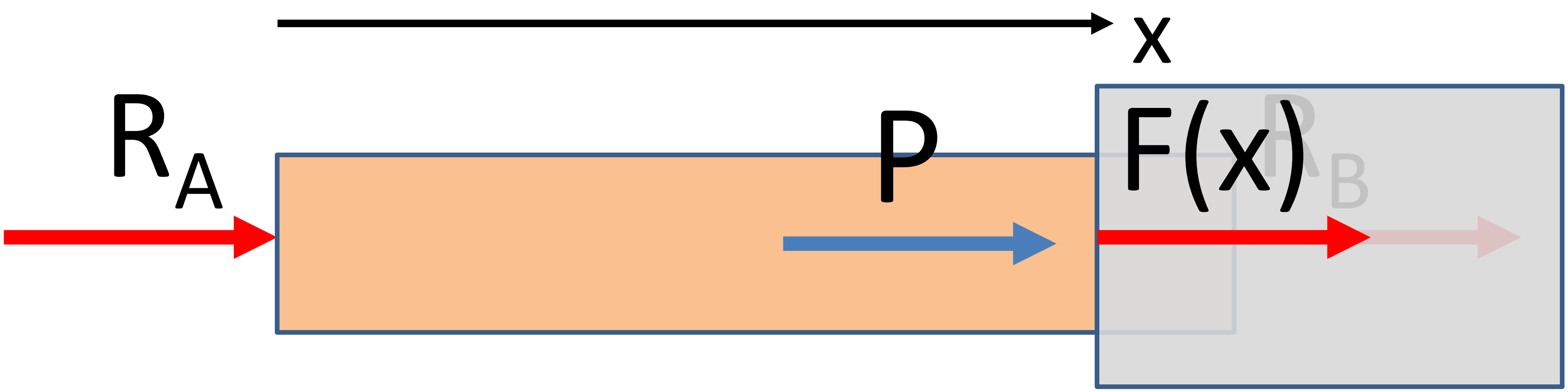
Domain CB: Strain



- Modulus of elasticity is E for the entire rod. So

$$\varepsilon(x) = \frac{\sigma(x)}{E} = -\frac{R_A + P}{EA}$$

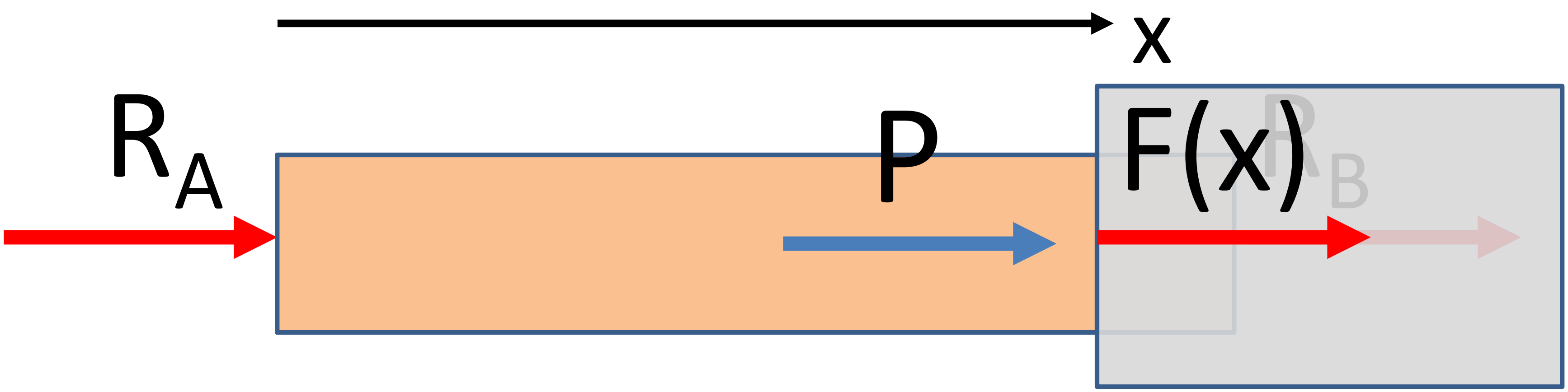
Domain CB: Displacement



- Here we need to understand that the domain of integration must span both AC and CB since integration is **from 0 to x and now x spans both domains. We will need to split the interval of integration into two intervals and use the expressions for ε(x) derived for the domains AC and CB.**

$$u(x) = \int_0^x \varepsilon(x)\,dx = \int_0^{x_C} \varepsilon(x)\,dx + \int_{x_C}^{x} \varepsilon(x)\,dx$$

Domain CB: Displacement

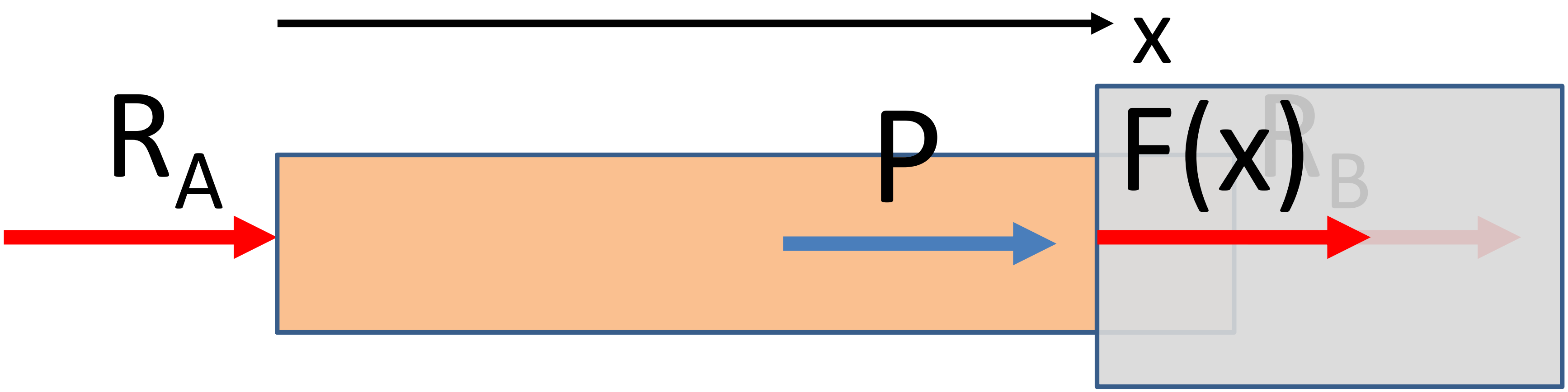


- We can now calculate the displacement at x in domain CB. Since C is the midpoint $x_c = L/2$

$$u(x) = -\int_0^{L/2} \frac{R_A}{EA}\,dx - \int_{L/2}^{x} \frac{R_A + P}{EA}\,dx$$

$$= -\frac{R_A L}{2EA} - \frac{(R_A + P)(x - L/2)}{EA}$$

Displacement at B

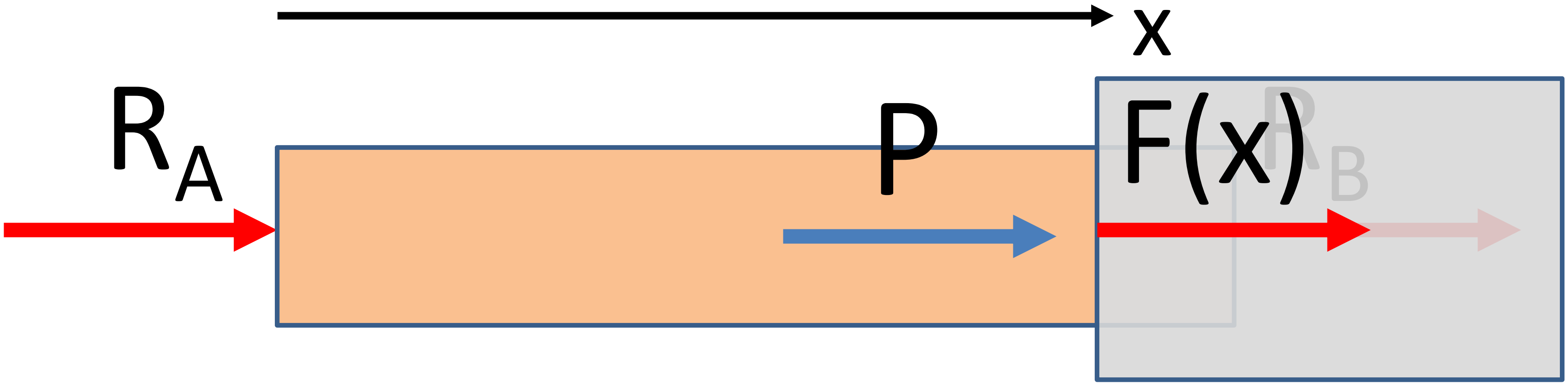


- We can now calculate the displacement at B, $x_B = L$

$$u(L) = -\int_0^{L/2} \frac{R_A}{EA}dx - \int_{L/2}^{L} \frac{R_A + P}{EA}dx$$

$$= -\frac{R_A L}{2EA} - \frac{(R_A + P)(L - L/2)}{EA} = -\frac{(2R_A + P)L}{2EA}$$

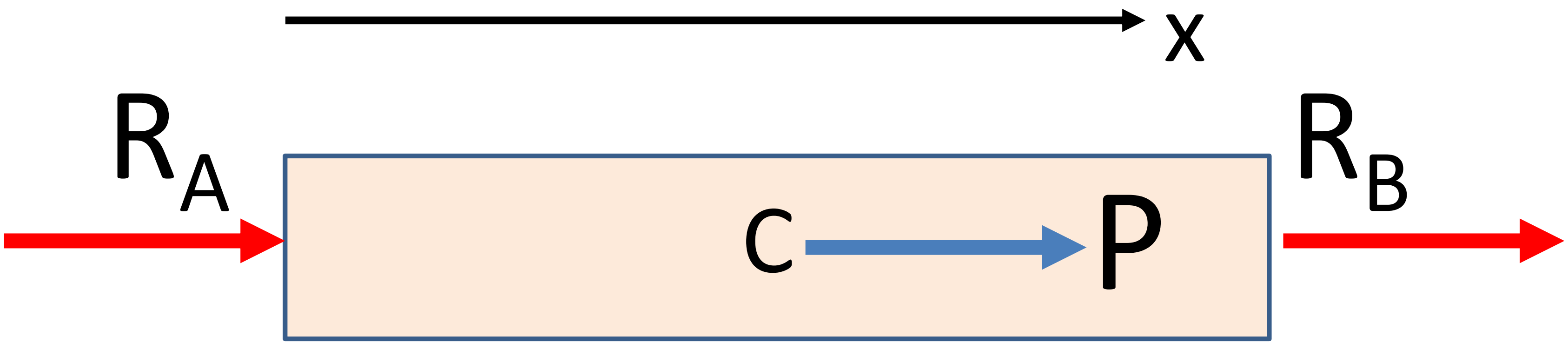
Displacement at B



- But we wish that $u_B = 0$

$$u(L) = -\frac{(2R_A + P)L}{2EA} = 0 \Rightarrow R_A = -\frac{P}{2}$$

Equilibrium for the whole bar

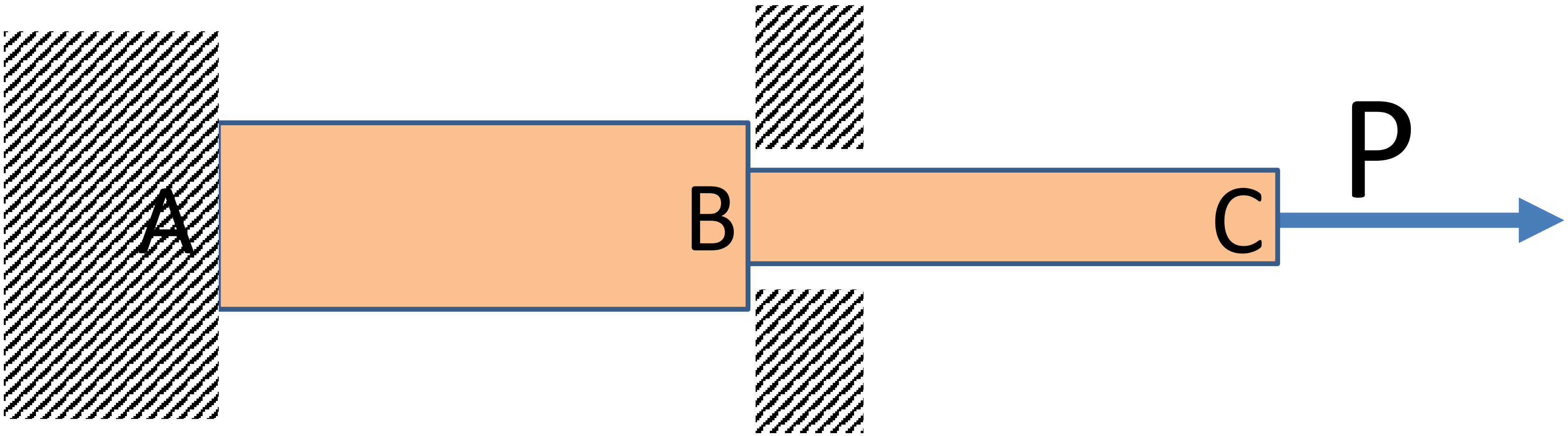


- Force equilibrium gives us $R_A + P + R_B = 0$
- But we already know

$$R_A = -\frac{P}{2}$$

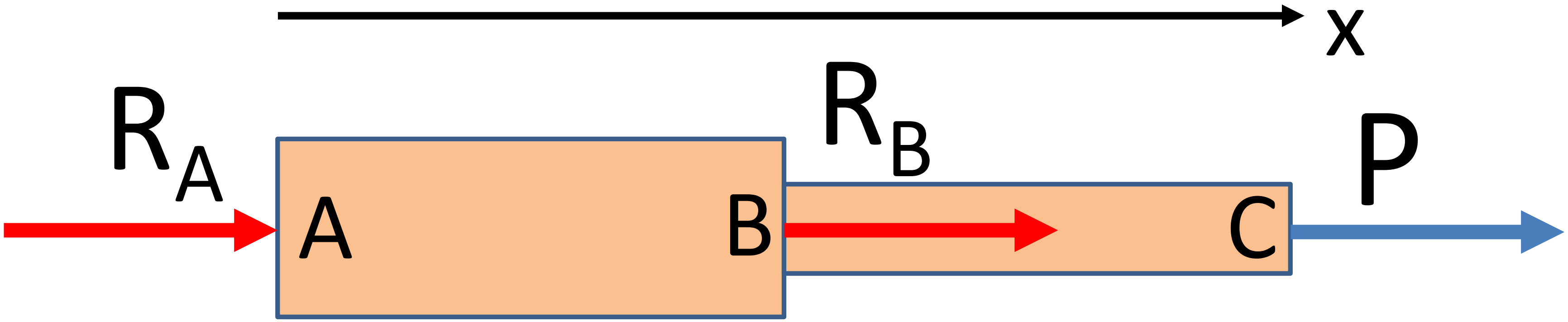
$$\Rightarrow -\frac{P}{2} + P + R_B = 0 \Rightarrow R_B = -\frac{P}{2}$$

A more complicated problem



- The force P is acting at the free end C of the rod, which is fixed to a wall at end A. At the midpoint B the rod passes through a hole in another wall, ensuring that point B does not move. Length of AB=length of BC=L/2. Area of cross section for AB = 2a. Area of cross section for BC = a. Modulus of elasticity for both segments is E. We are required to find the reactions.

Start with FBD



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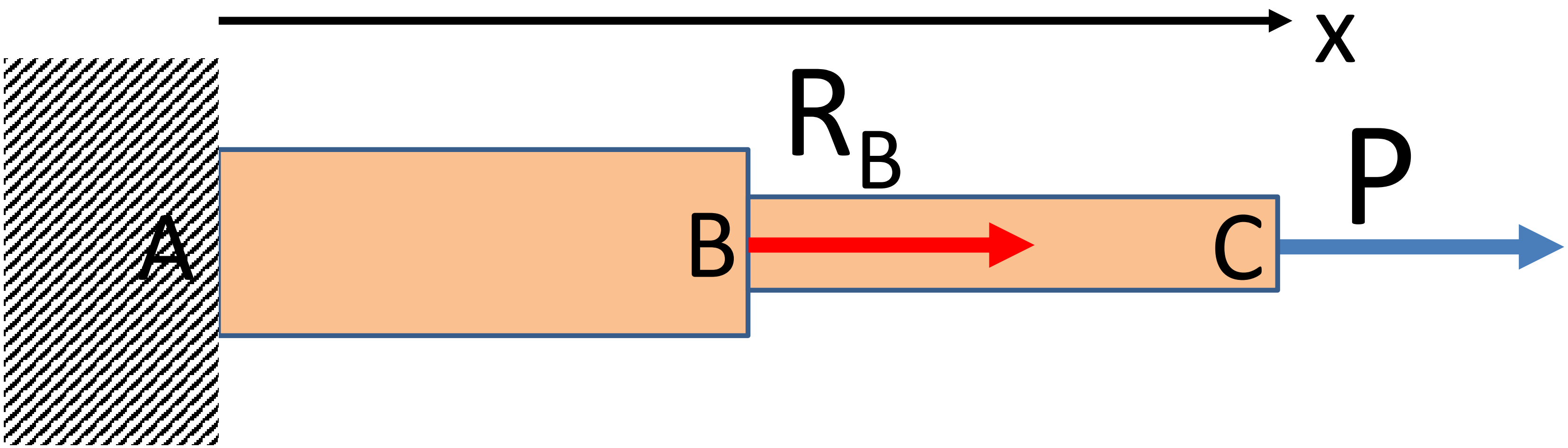
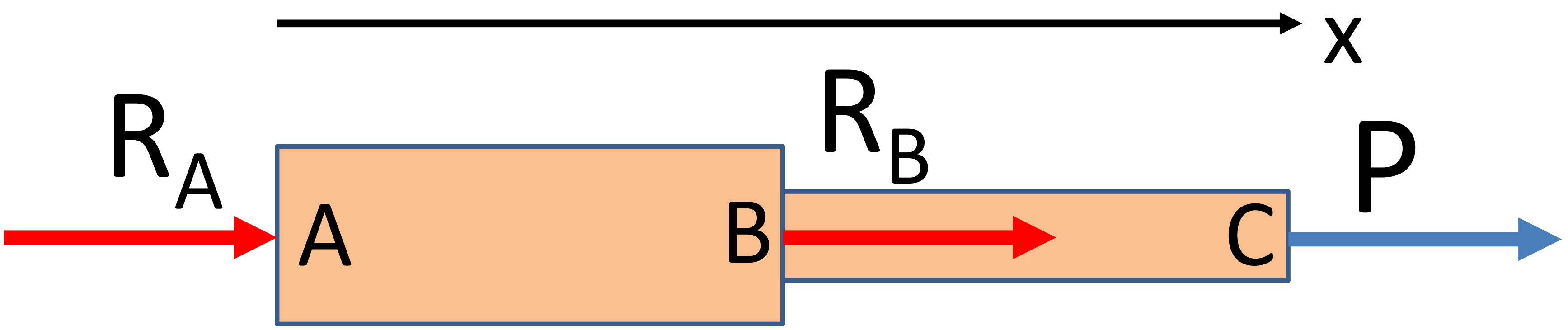
- The reactions will be R_A and R_B. The directions shown have intentionally been kept positive, although they are counterintuitive. You would certainly have liked at least one of them to have a direction opposite to P. But as we work with complicated problems we will find this has advantages. Once we get the answers the signs will tell us the directions.

The alternative problem



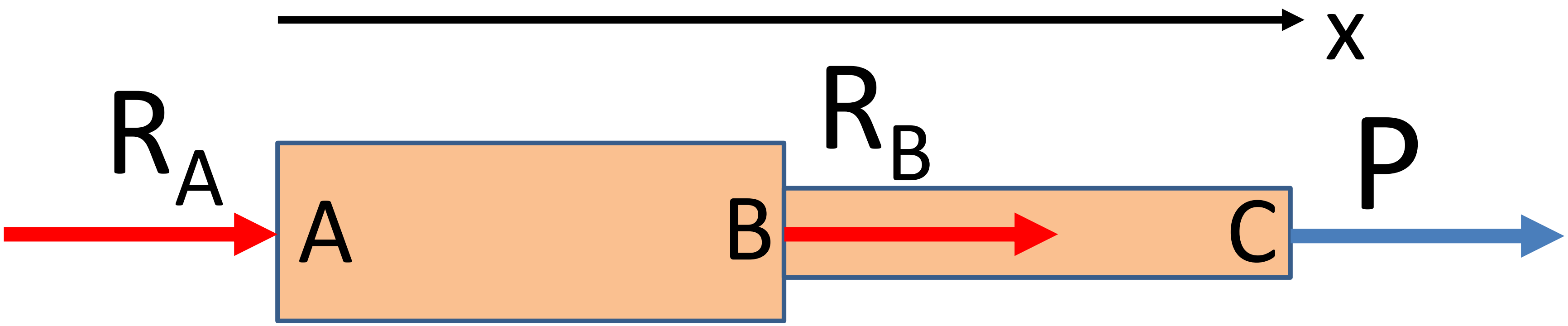
- We will try to solve the following problem. What should be R_B so that the displacement at point B is zero ?

FBD of the alternative problem



- The free body diagram comes out to be the same

Critical Points and Domains



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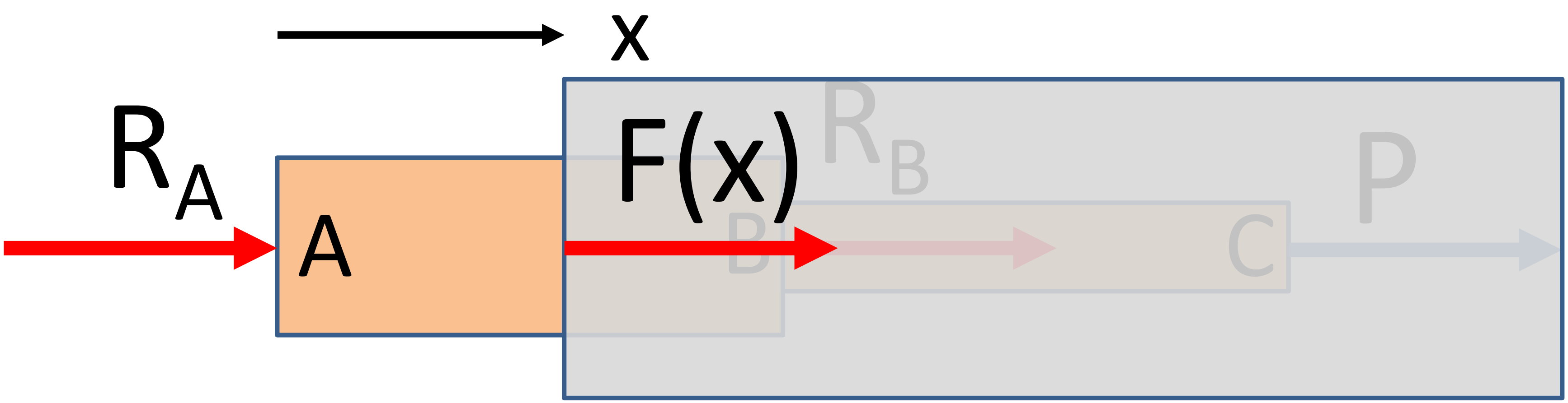
- We will need to take a section in every domain

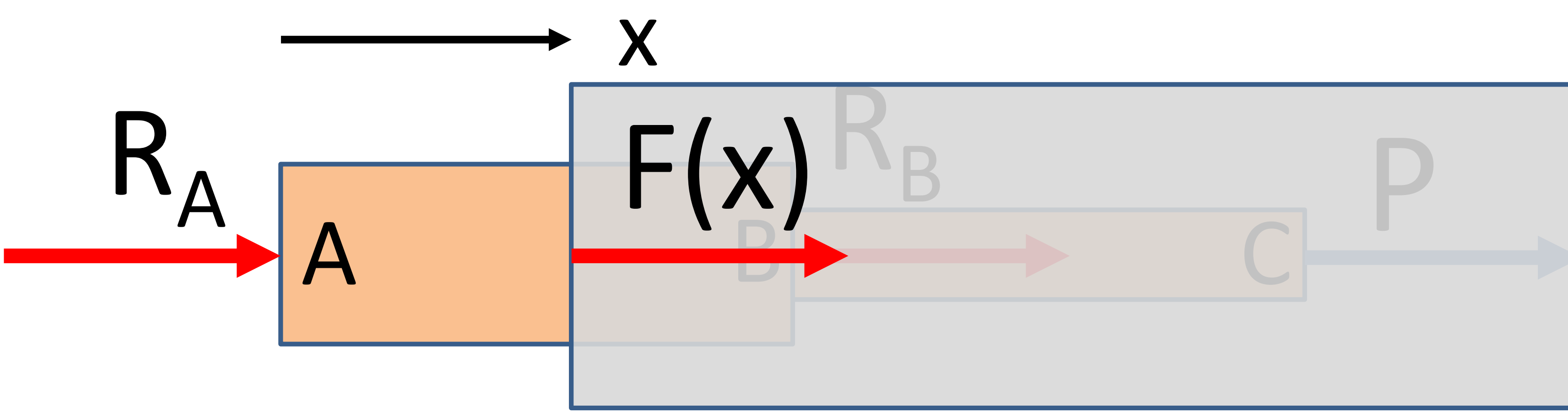
Domain AB: Force



- Once we cut the section we will see an internal force F(x) at the cut. Using equilibrium (ONLY FOR THE SECTION UPTO x) we will get

$$R_A + F\left(x\right) = 0 \Rightarrow F\left(x\right) = -R_A$$

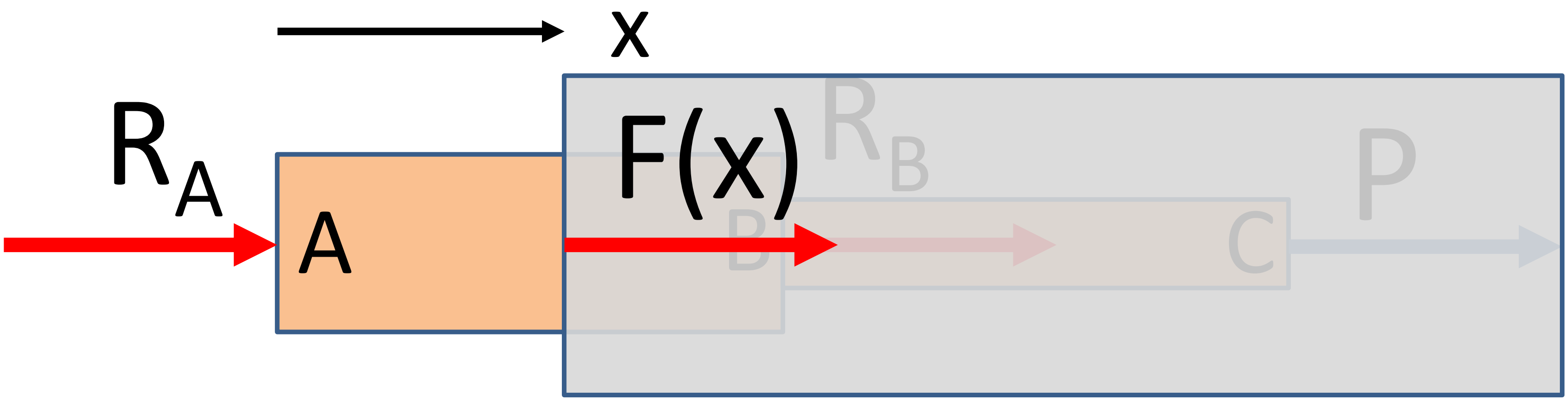
Domain AB: Stress



- Area of cross section is 2a from A to B. So

$$\sigma(x) = \frac{F(x)}{2a} = -\frac{R_A}{2a}$$

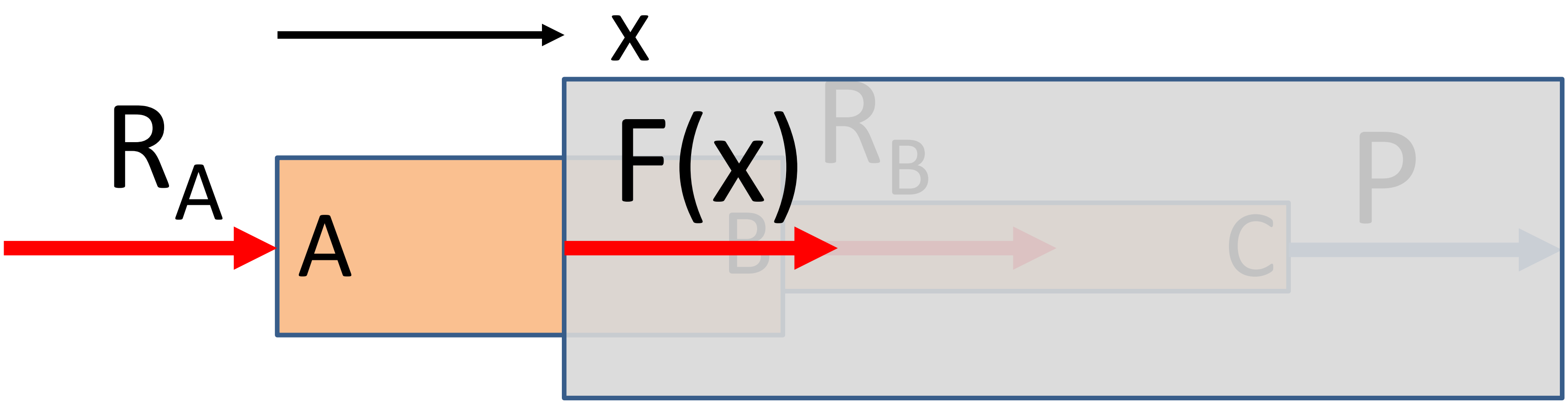
Domain AB: Strain



- Modulus of elasticity is E for the entire rod. So

$$\varepsilon(x) = \frac{\sigma(x)}{E} = -\frac{R_A}{2Ea}$$

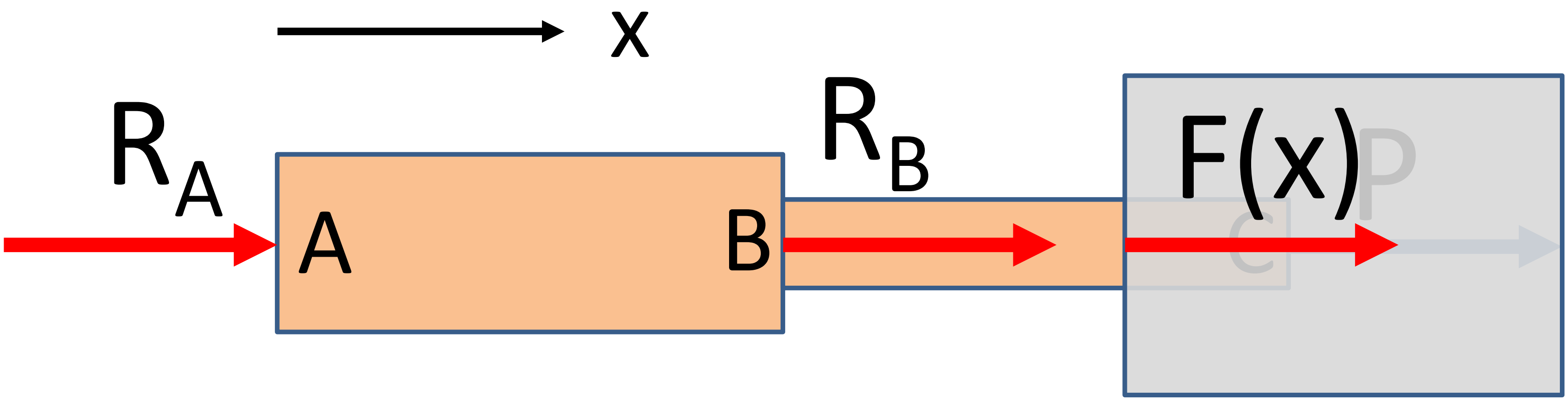
Domain AB: Displacement



- Hence

$$u(x) = \int_0^x \varepsilon(x)\,dx = -\int_0^x \frac{R_A}{2Ea}\,dx = -\frac{R_A x}{2Ea}$$

Domain BC: Force

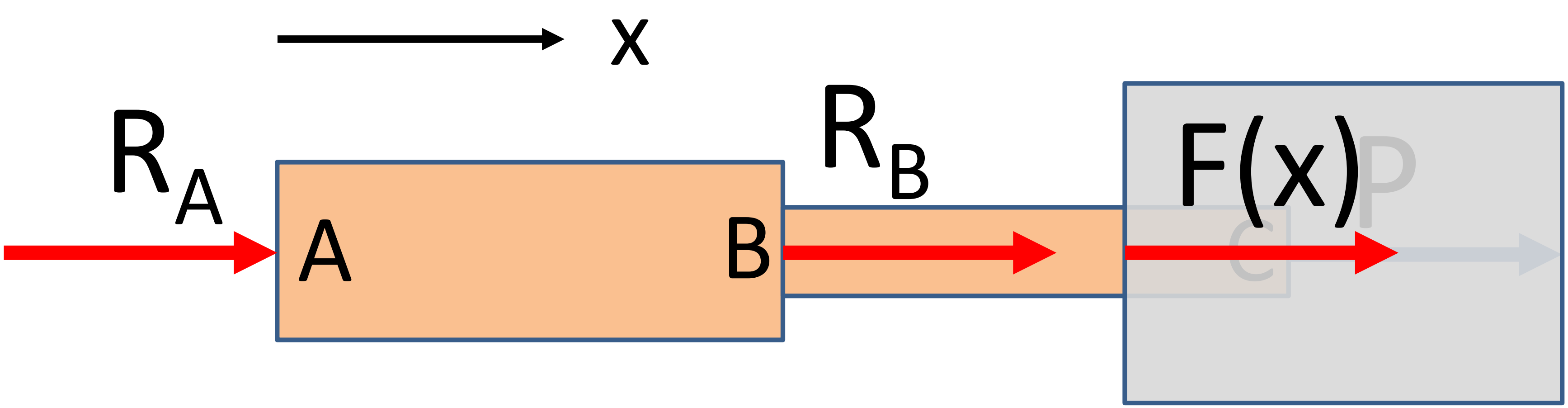


- Once we cut the section between B and C we will see an internal force F(x) at the cut.
- Note that in this domain we can now see R_B, but P still remains hidden. Also origin and coordinate system remain unchanged.
- Using equilibrium we get

$$R_A + R_B + F(x) = 0$$

$$\Rightarrow F(x) = -R_A - R_B$$

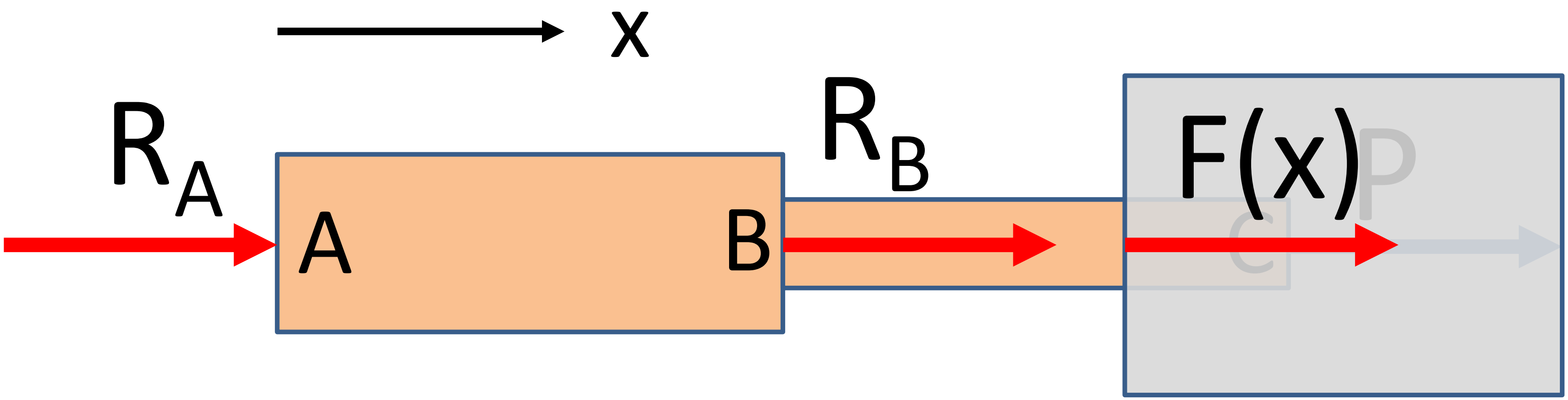
Domain BC: Stress



- Area of cross section is a from B to C. So

$$\sigma(x) = \frac{F(x)}{A} = -\frac{R_A + R_B}{a}$$

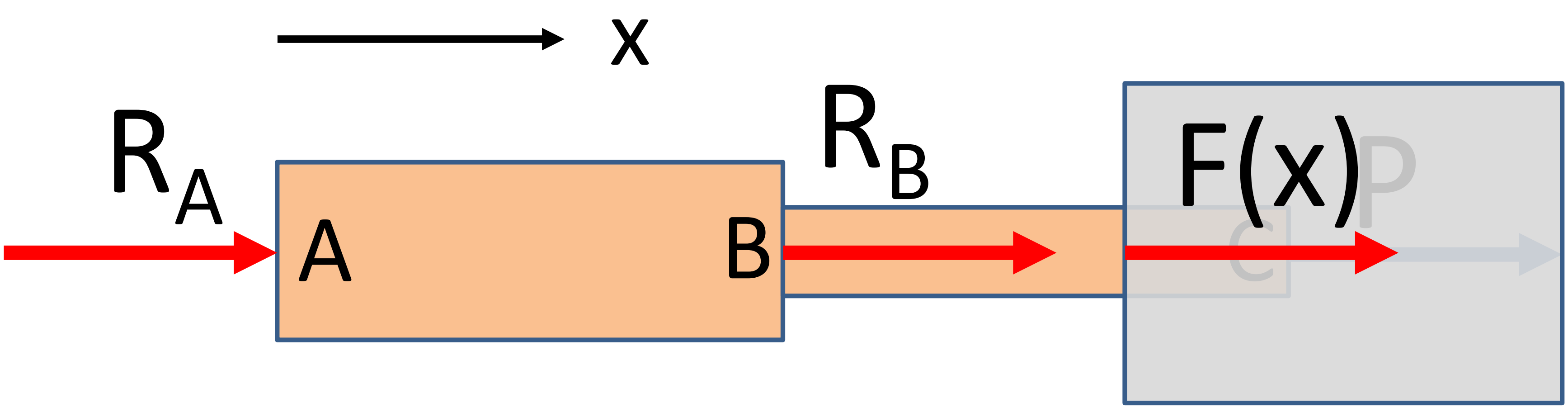
Domain BC: Strain



- Modulus of elasticity is E for the entire rod. So

$$\varepsilon(x) = \frac{\sigma(x)}{E} = -\frac{R_A + R_B}{Ea}$$

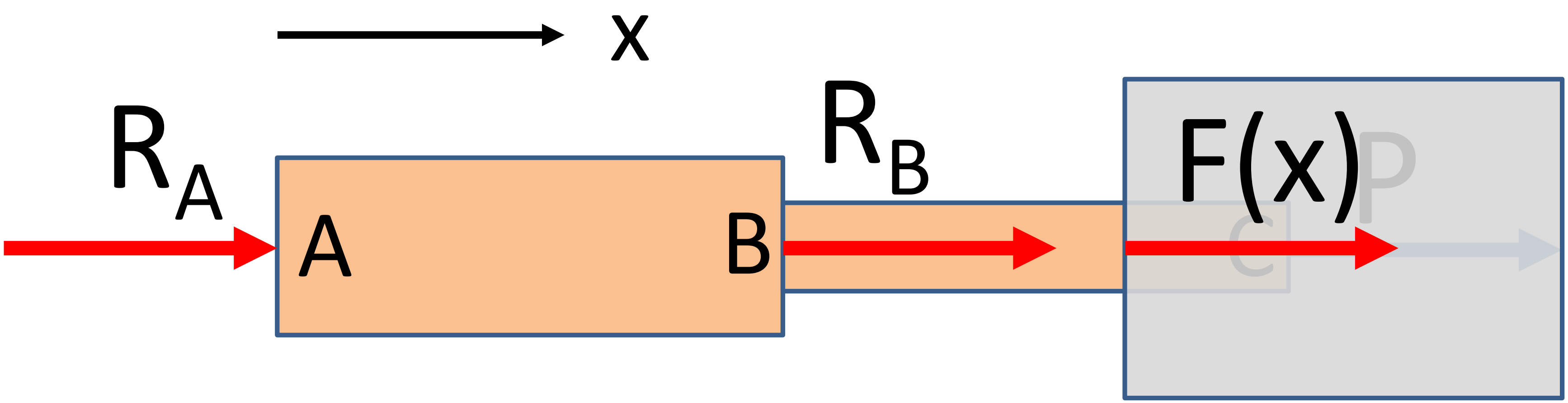
Domain BC: Displacement



- Here we need to understand that the domain of integration must span both AC and CB since integration is **from 0 to x and now x spans both domains. We will need to split the interval of integration into two intervals and use the expressions for ε(x) derived for the domains AB and BC.**

$$u(x) = \int_0^x \varepsilon(x)\,dx = \int_0^{x_B} \varepsilon(x)\,dx + \int_{x_B}^{x} \varepsilon(x)\,dx$$

Domain BC: Displacement

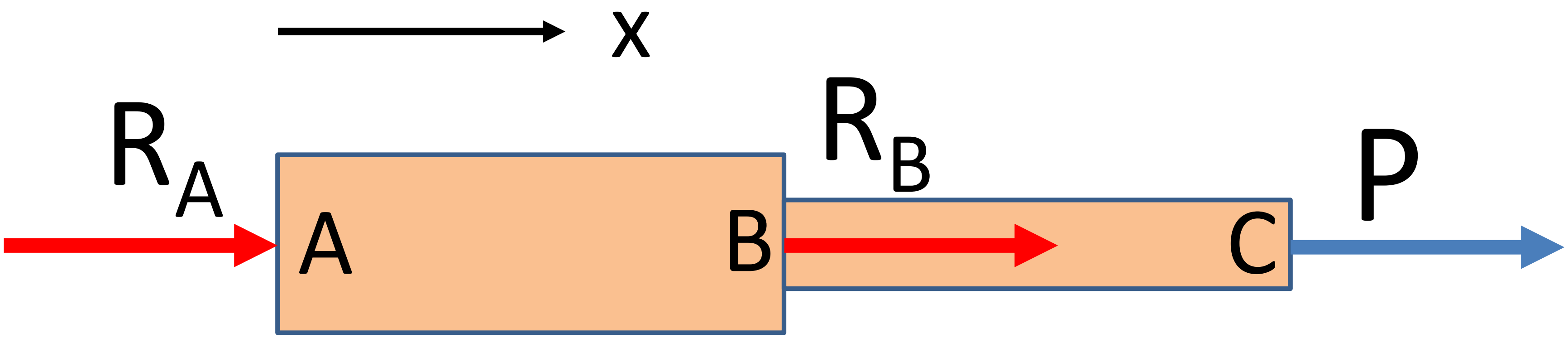


- We can now calculate the displacement at x in domain BC. Since B is the midpoint $x_B = L/2$

$$u(x) = -\int_0^{L/2} \frac{R_A}{2Ea}\,dx - \int_{L/2}^{x} \frac{R_A + R_B}{Ea}\,dx$$

$$= -\frac{R_A L}{4Ea} - \frac{(R_A + R_B)(x - L/2)}{Ea}$$

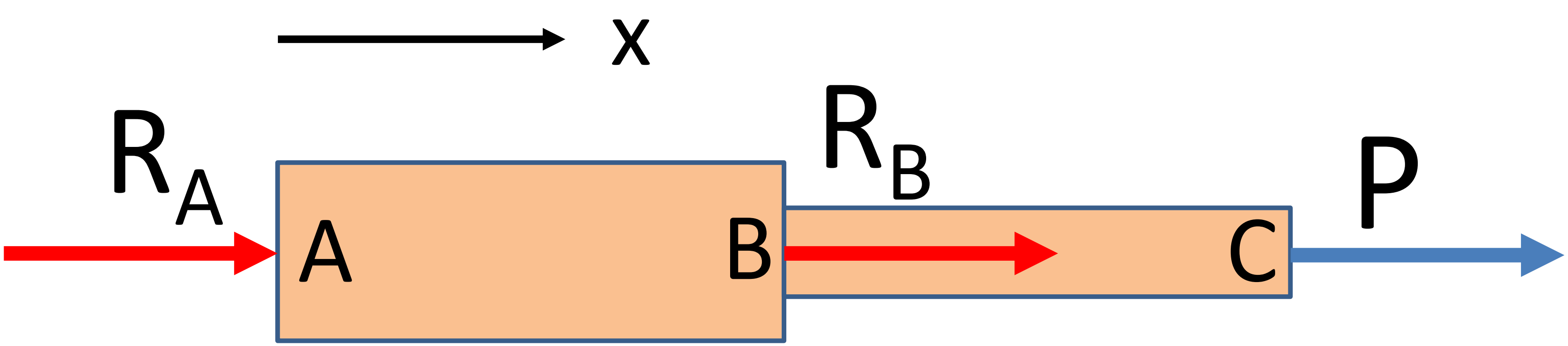
Displacement at B



- We can now calculate the displacement at B, $x_B = L/2$ by using any of the two expressions for u(x). First we use the expression for domain AB

$$u\left(\frac{L}{2}\right) = -\int_0^{L/2} \frac{R_A}{2Ea}\,dx = -\frac{R_A L}{4Ea}$$

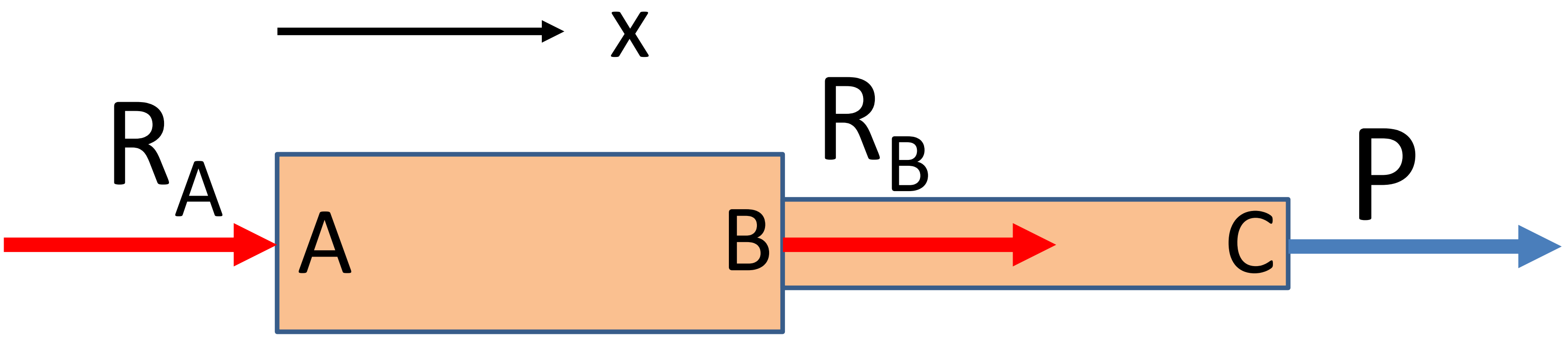
Displacement at B



- Next we use the expression for domain BC
- As it MUST be, both answers are same

$$u\left(\frac{L}{2}\right) = -\int_0^{L/2} \frac{R_A}{2Ea}dx - \int_{L/2}^{L/2} \frac{R_A + R_B}{Ea}dx = -\frac{R_A L}{4Ea}$$

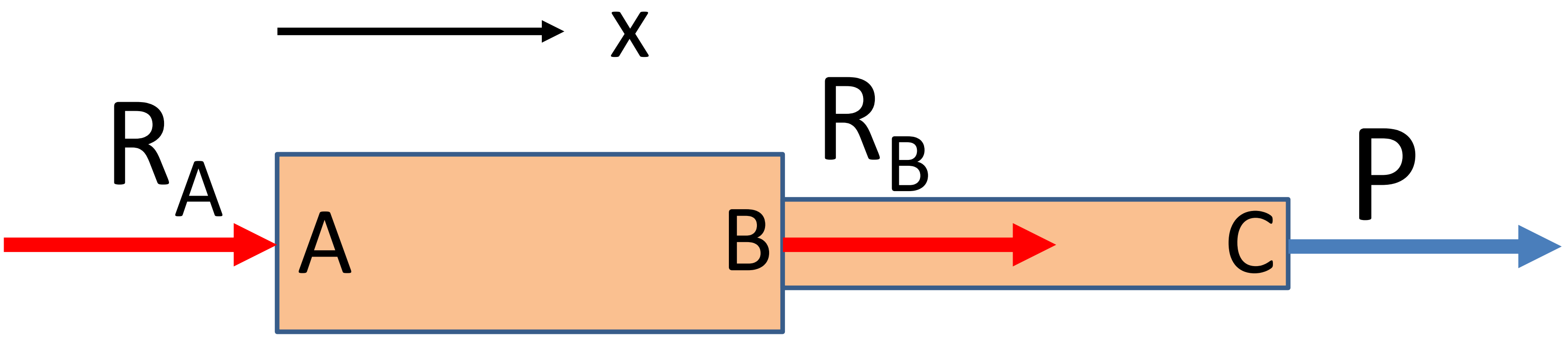
Displacement at B



- But we wish that $u_B = 0$

$$u\left(\frac{L}{2}\right) = -\frac{R_A L}{4Ea} = 0 \Rightarrow R_A = 0$$

Equilibrium for the whole bar

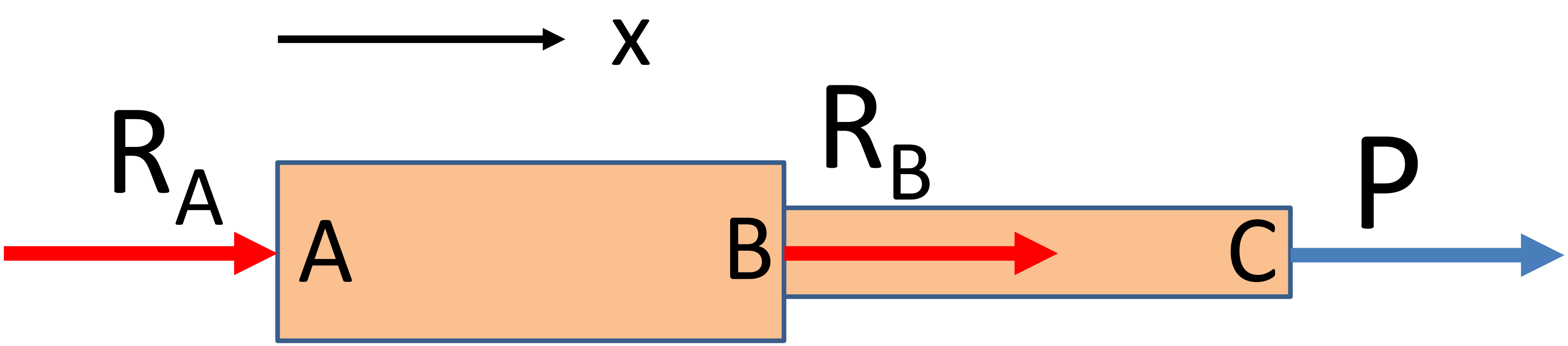


- Force equilibrium gives us $R_A + R_B + P = 0$
- But we already know

$$R_A = 0$$

$$\Rightarrow 0 + P + R_B = 0 \Rightarrow R_B = -P$$

Displacement at C

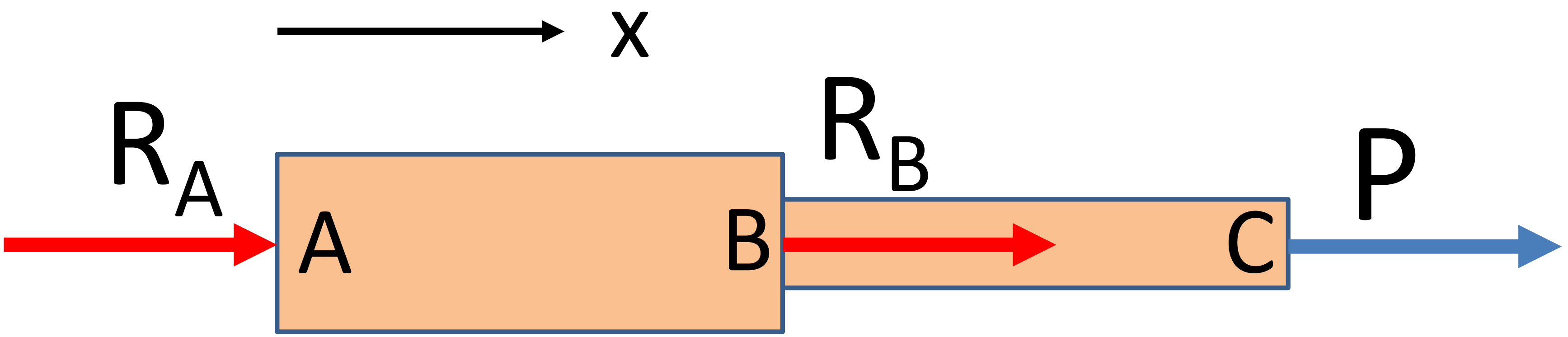


- We have to use the expression for domain BC and use $x_c=L$

$$u(L) = -\int_0^{L/2} \frac{R_A}{2Ea}dx - \int_{L/2}^{L} \frac{R_A+R_B}{Ea}dx$$

$$= -\frac{R_A L}{2Ea} - \frac{(R_A+R_B)L}{2Ea} = -\frac{R_B L}{2Ea}$$

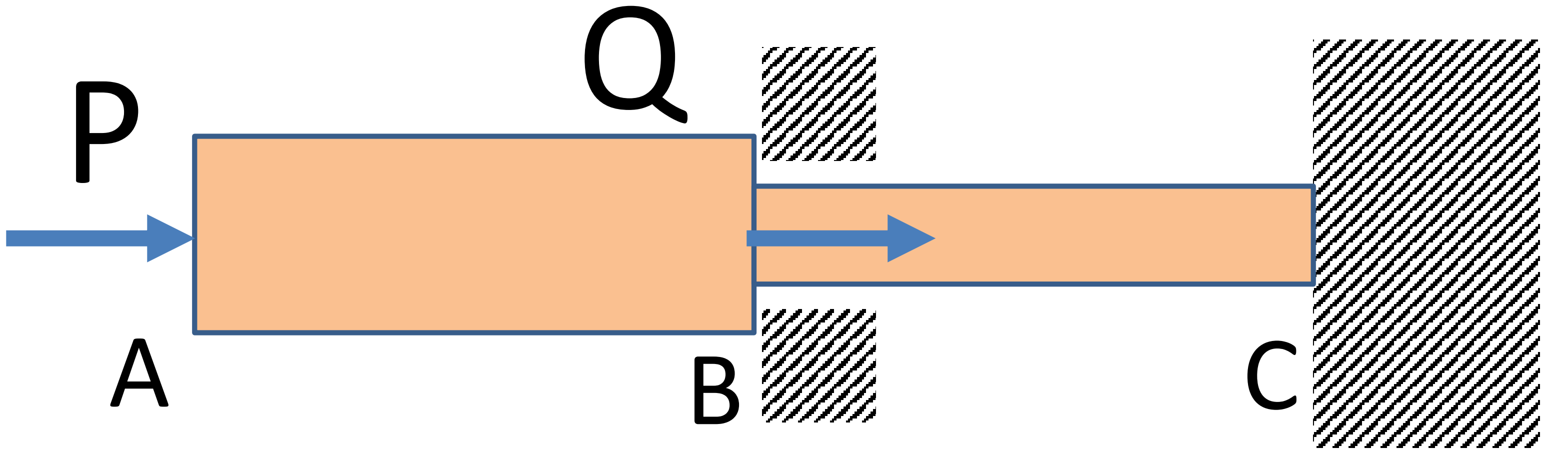
Displacement at C



- We can also solve the problem by recognizing that at C also there is no movement. Hence

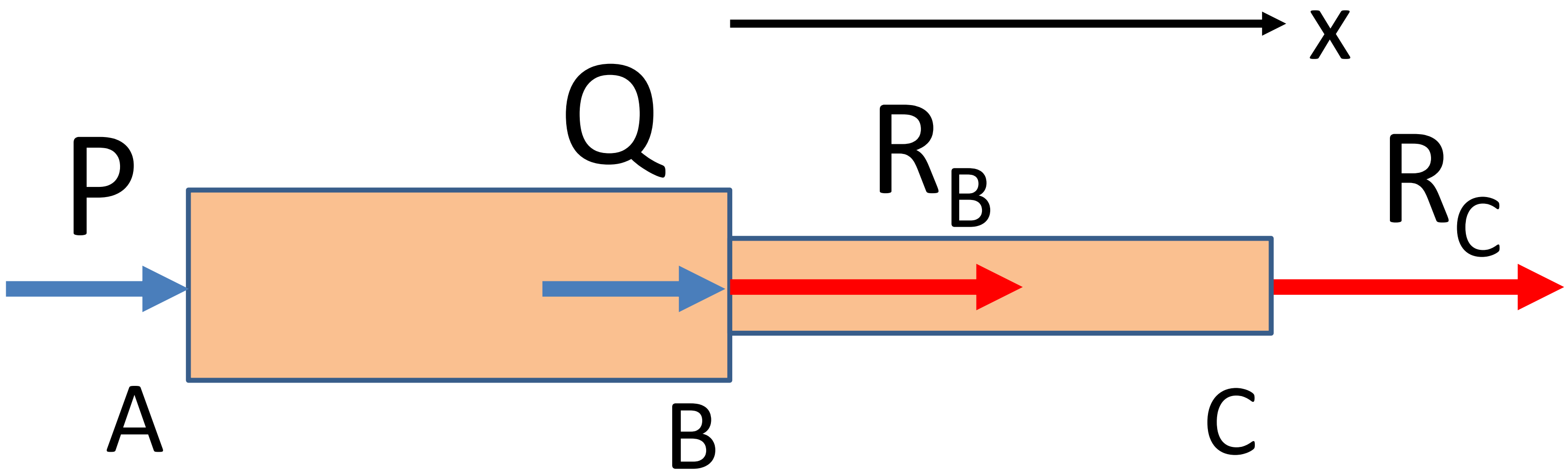
$$u(L) = -\frac{R_B L}{2Ea} = 0 \Rightarrow R_B = 0$$

Another complicated problem



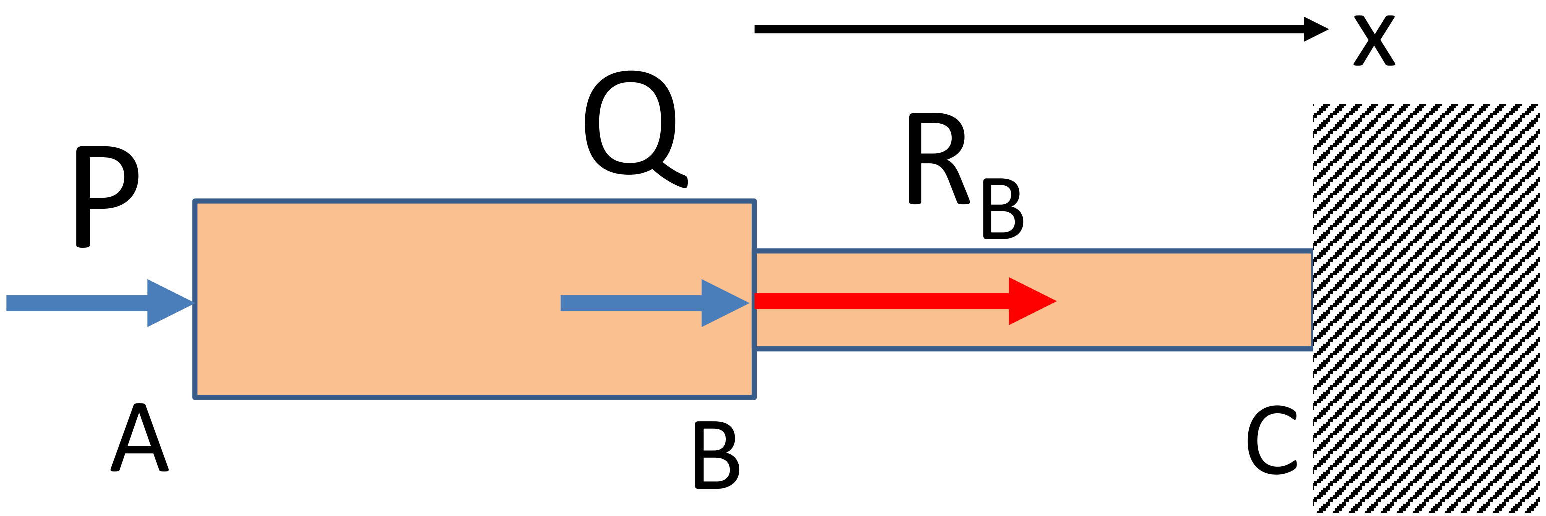
- The force P is acting at the free end A of the rod, which is fixed to a wall at end C. At the midpoint B the rod passes through a hole in another wall, ensuring that point B does not move. A force Q acts at B. Length of AB=length of BC=L/2. Area of cross section for AB = 2a. Area of cross section for BC = a. Modulus of elasticity for AB = E. Modulus of elasticity for AB = 2E. We are required to find the reactions.

Start with FBD



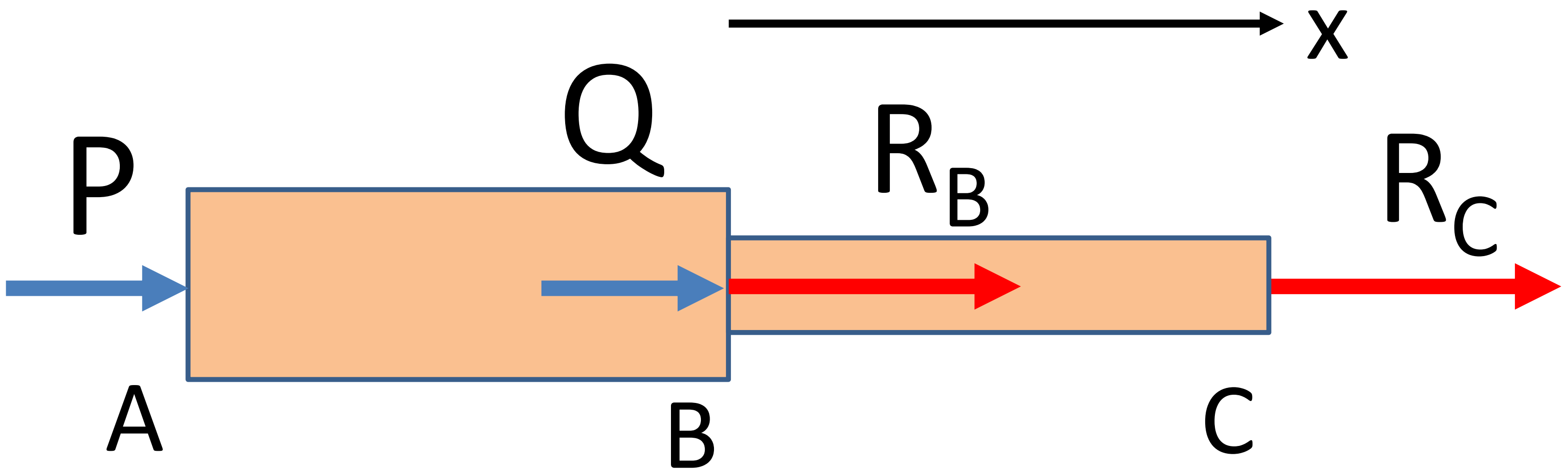
- First set up a coordinate system. In this case the origin can be at B (not A, since A will be moving) and the +ve direction will be from B to C
- The reactions will be R_B and R_C. The directions shown have intentionally been kept positive, although they are counterintuitive. You would certainly have liked at least one of them to have a direction opposite to P. But as we work with complicated problems we will find this has advantages. Once we get the answers the signs will tell us the directions.

The alternative problem



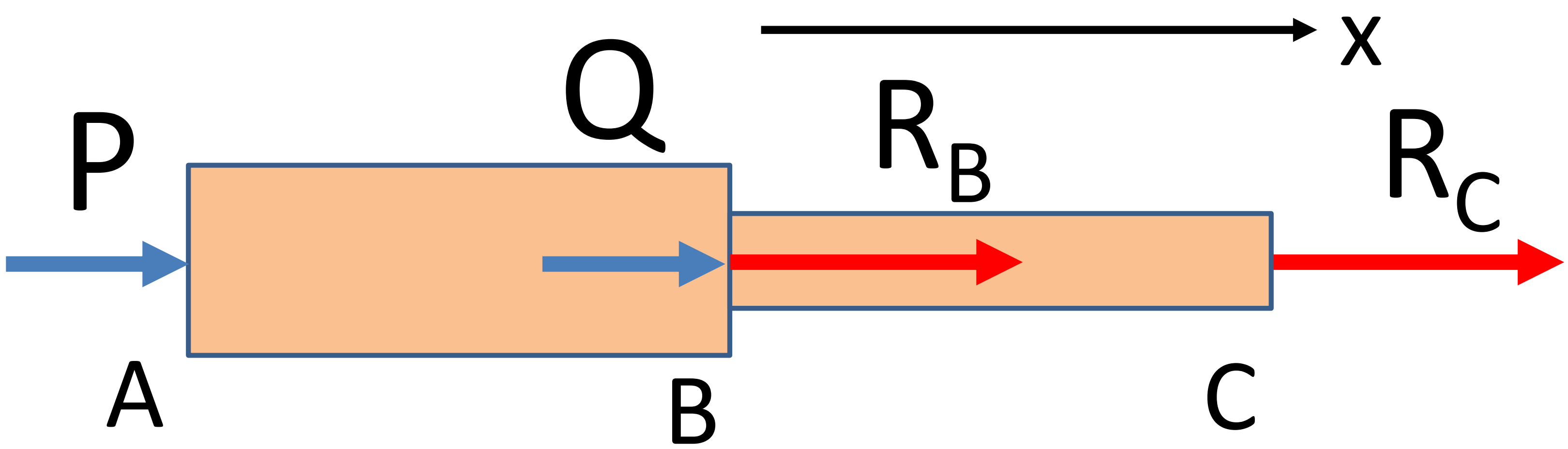
- We will try to solve the following problem. What should be R_C so that the displacement at point C is zero ?

FBD of the alternative problem



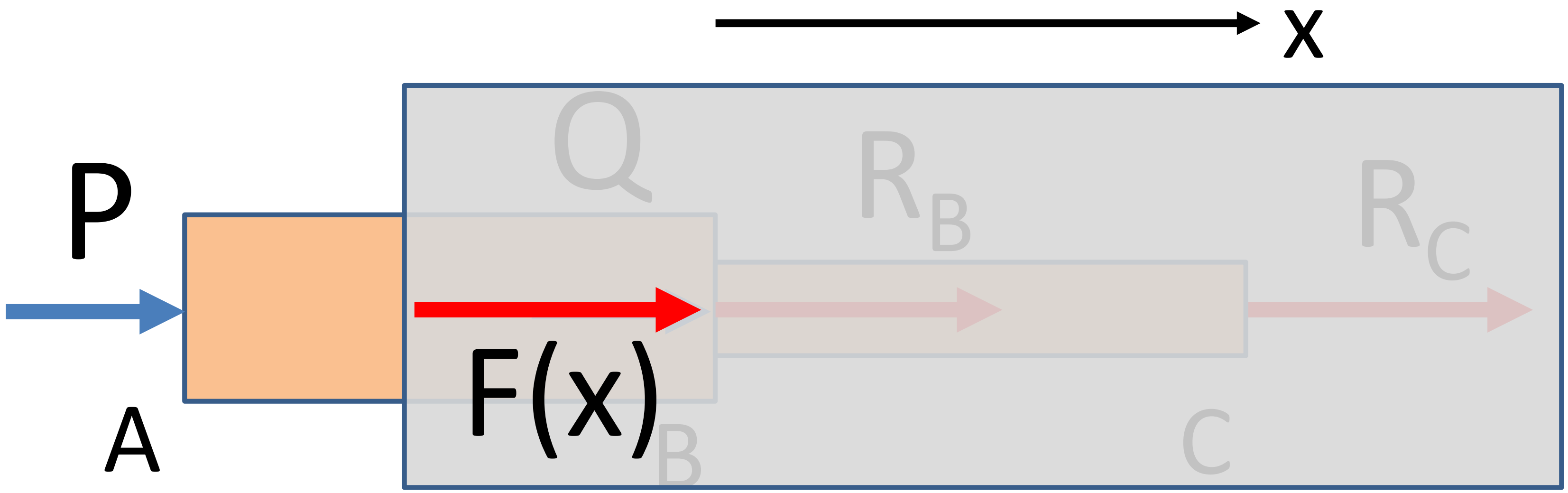
- The free body diagram comes out to be the same

Critical Points and Domains



- A critical point is a point where there is a sudden change - in forces, in dimensions, in material properties, or there is new constraint. Here we have three critical points A, B and C and hence two domains – A to B and B to C.
- We will need to take a section in every domain

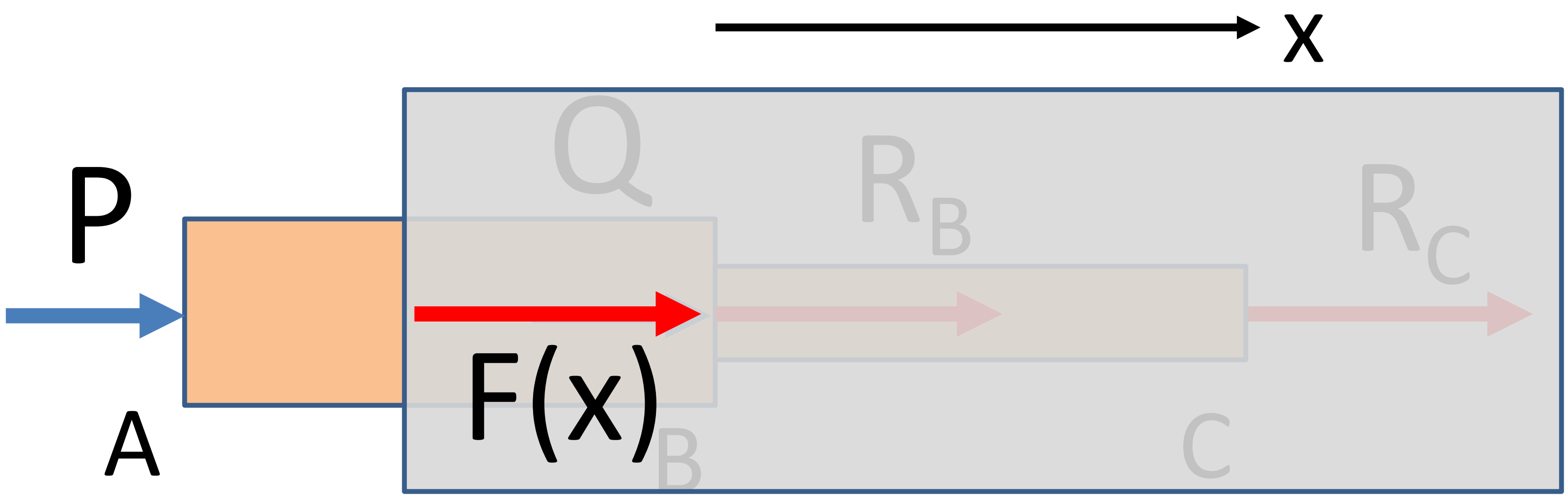
Domain AB: Force



- Once we cut the section we will see an internal force F(x) at the cut. Using equilibrium (ONLY FOR THE SECTION UPTO x) we will get

$$P+F(x)=0 \Rightarrow F(x)=-P$$

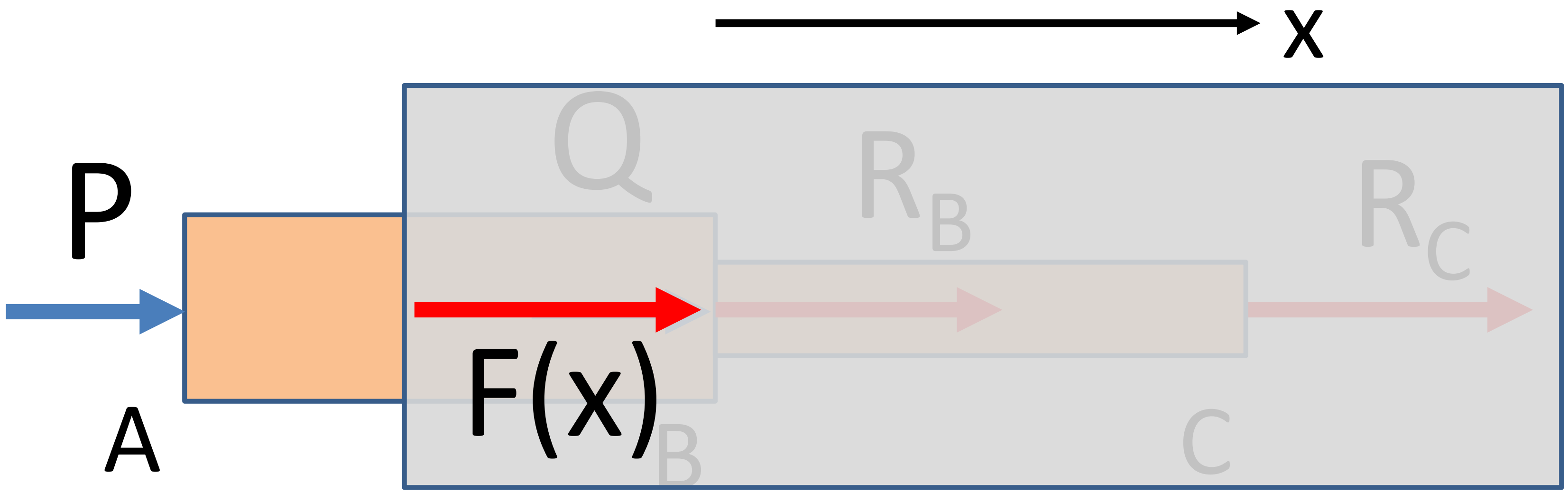
Domain AB: Stress



- Area of cross section is 2a from A to B. So

$$\sigma(x) = \frac{F(x)}{2a} = -\frac{P}{2a}$$

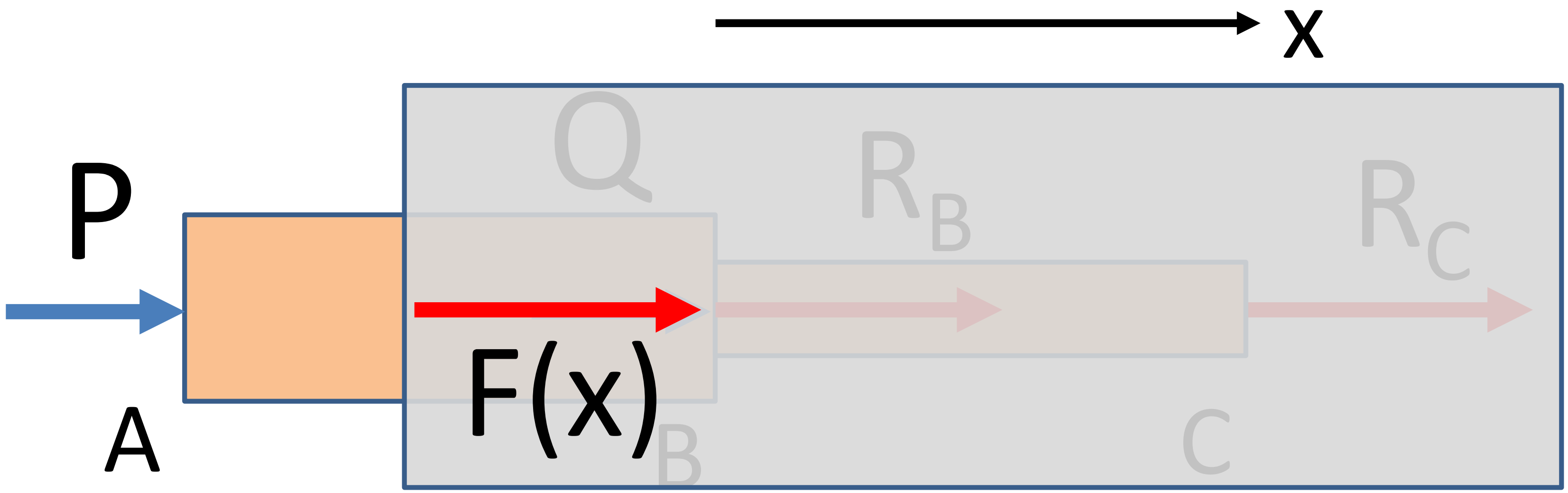
Domain AB: Strain



- Modulus of elasticity is E for AB. So

$$\varepsilon(x) = \frac{\sigma(x)}{E} = -\frac{P}{2Ea}$$

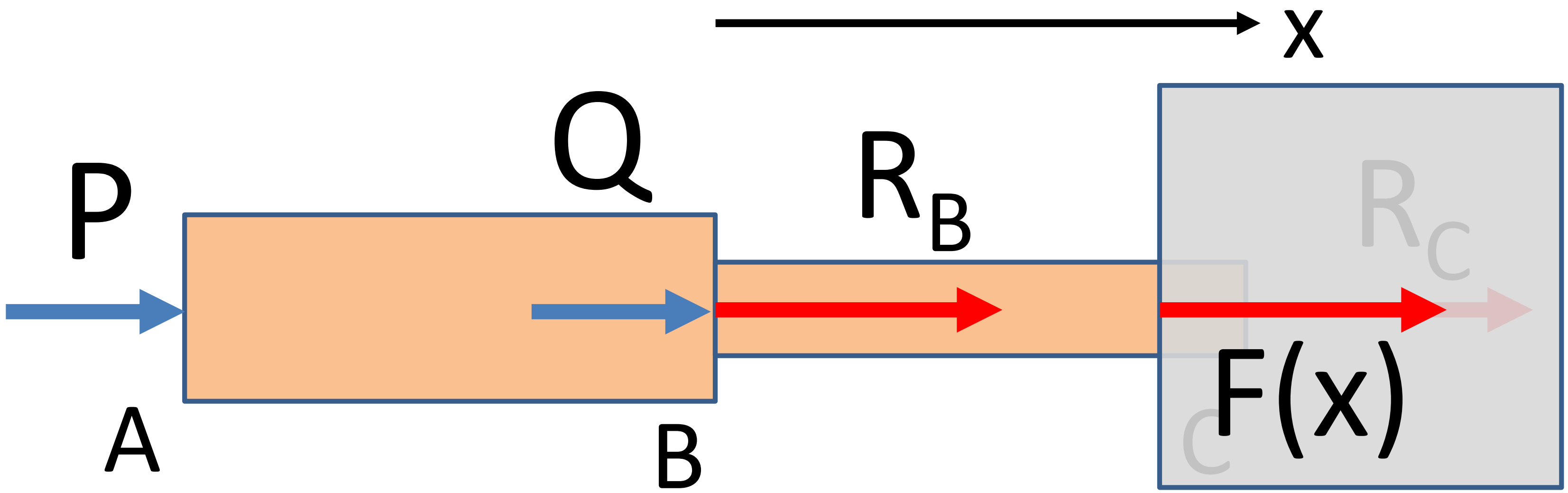
Domain AB: Displacement



- Here x_A=-L/2. Not that we are moving from the origin upto our x only, (which is on the negative side of the origin but the sign is automatically taken care of), and not necessarily upto A. Hence

$$u(x) = \int_0^x \varepsilon(x)\,dx = -\int_{-L/2}^{x} \frac{P}{2Ea}\,dx = -\frac{Px}{2Ea}$$

Domain BC: Force

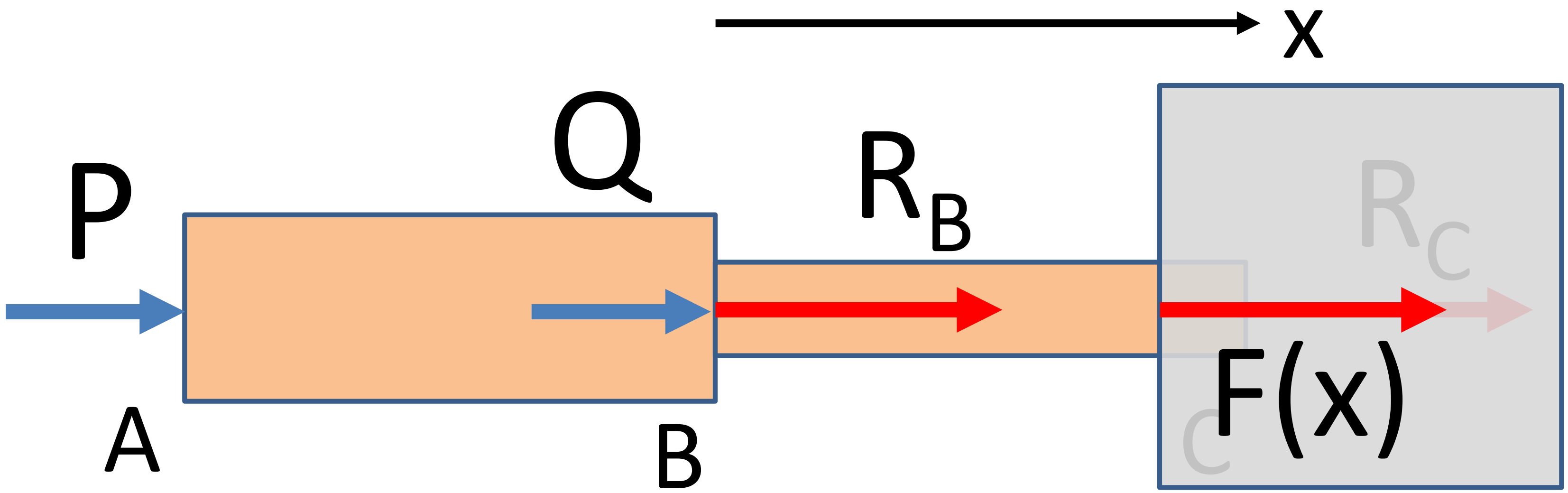


- Once we cut the section between B and C we will see an internal force F(x) at the cut.
- Note that in this domain we can now see Q and R_B, but R_C still remains hidden. Also origin and coordinate system remain unchanged.
- Using equilibrium we get

$$P + Q + R_B + F\left(x\right) = 0$$

$$\Rightarrow F\left(x\right) = -\left(P + Q + R_B\right)$$

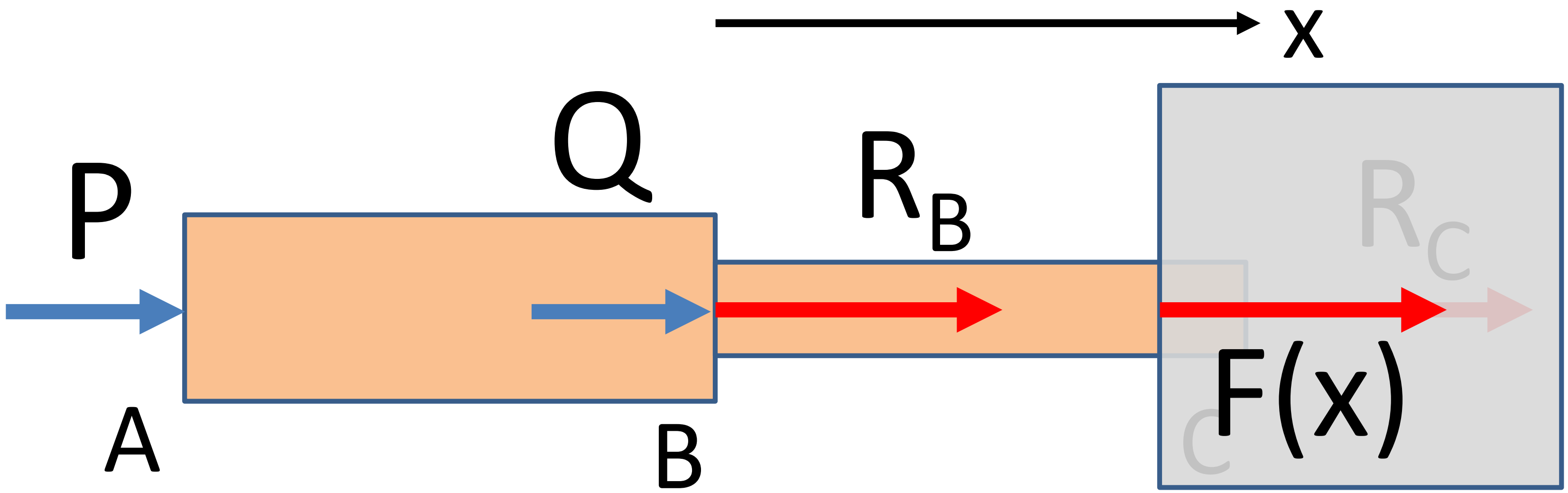
Domain BC: Stress



- Area of cross section is a from B to C. So

$$\sigma(x) = \frac{F(x)}{A} = -\frac{P+Q+R_B}{a}$$

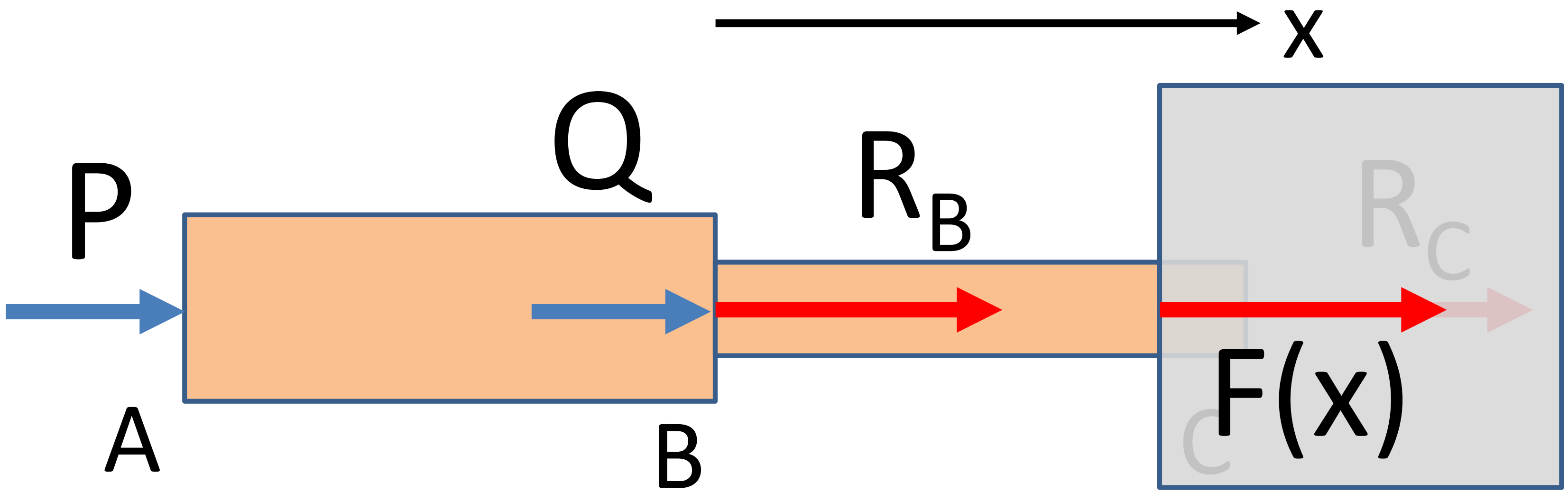
Domain BC: Strain



- Modulus of elasticity is 2E from B to C. So

$$\varepsilon(x) = \frac{\sigma(x)}{2E} = -\frac{P+Q+R_B}{2Ea}$$

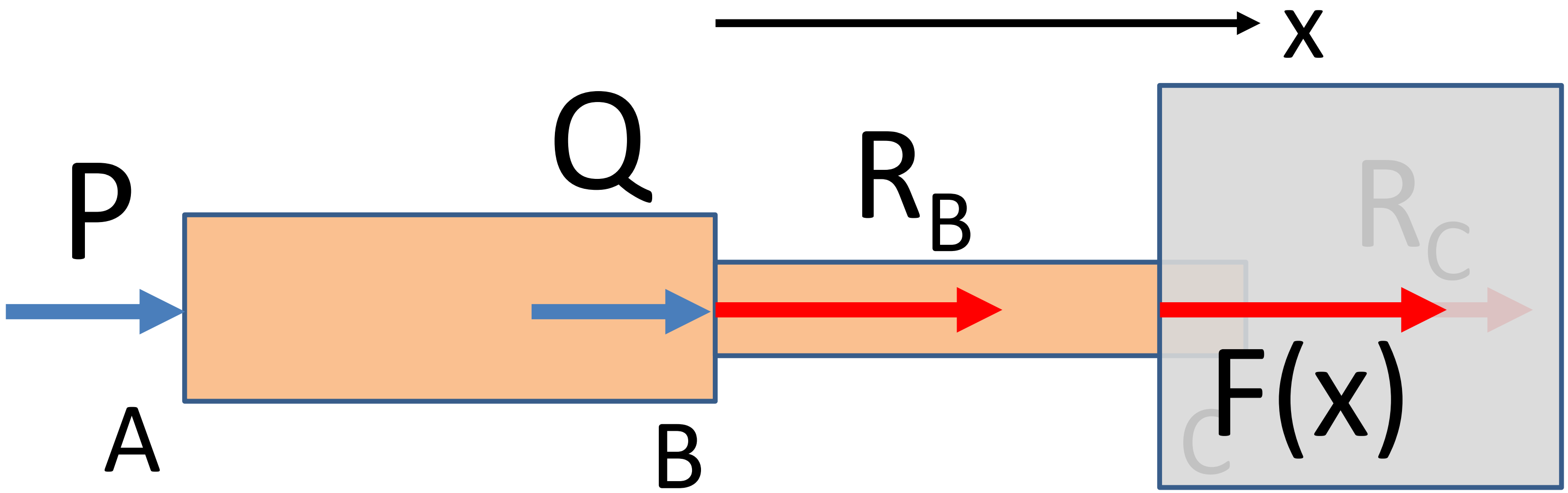
Domain BC: Displacement



- Here we need to understand **that the domain of integration does not need to span AB** since integration is **from 0 to x and now x spans only BC. We will not need to split the interval of integration into two intervals.**

$$u(x) = \int_{x_B}^{x} \varepsilon(x)\,dx = \int_{0}^{x} \varepsilon(x)\,dx$$

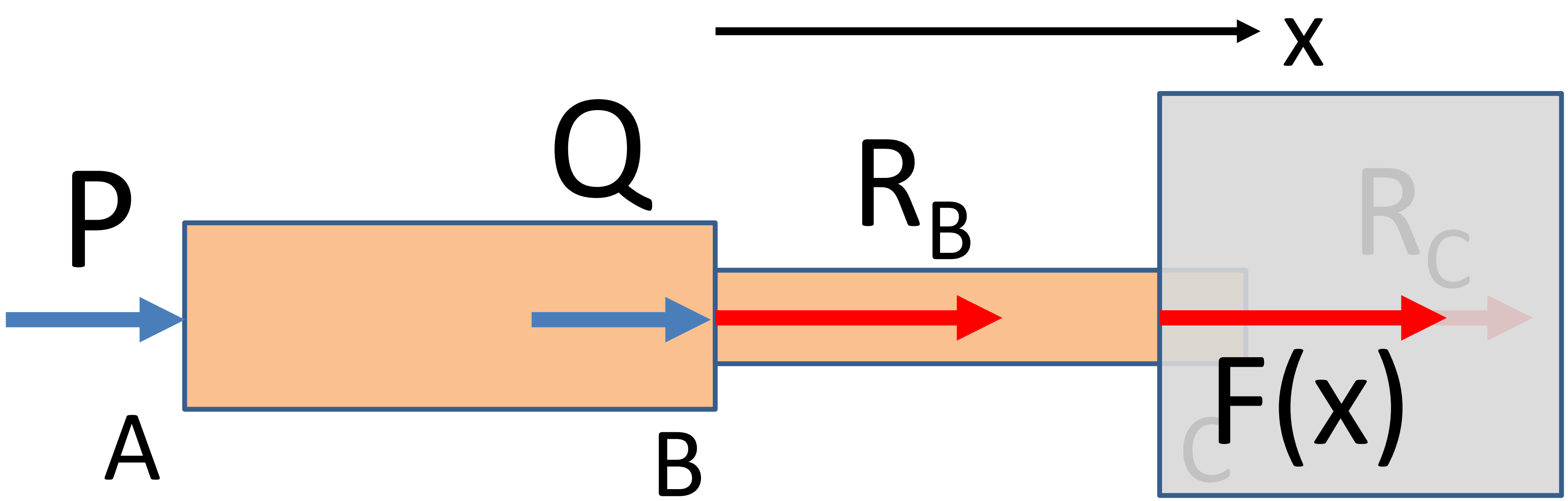
Domain BC: Displacement



- We can now calculate the displacement at x in domain BC. Since $x_B = 0$

$$u(x) = -\int_0^x \frac{P+Q+R_B}{2Ea}\,dx = -\frac{(P+Q+R_B)\,x}{2Ea}$$

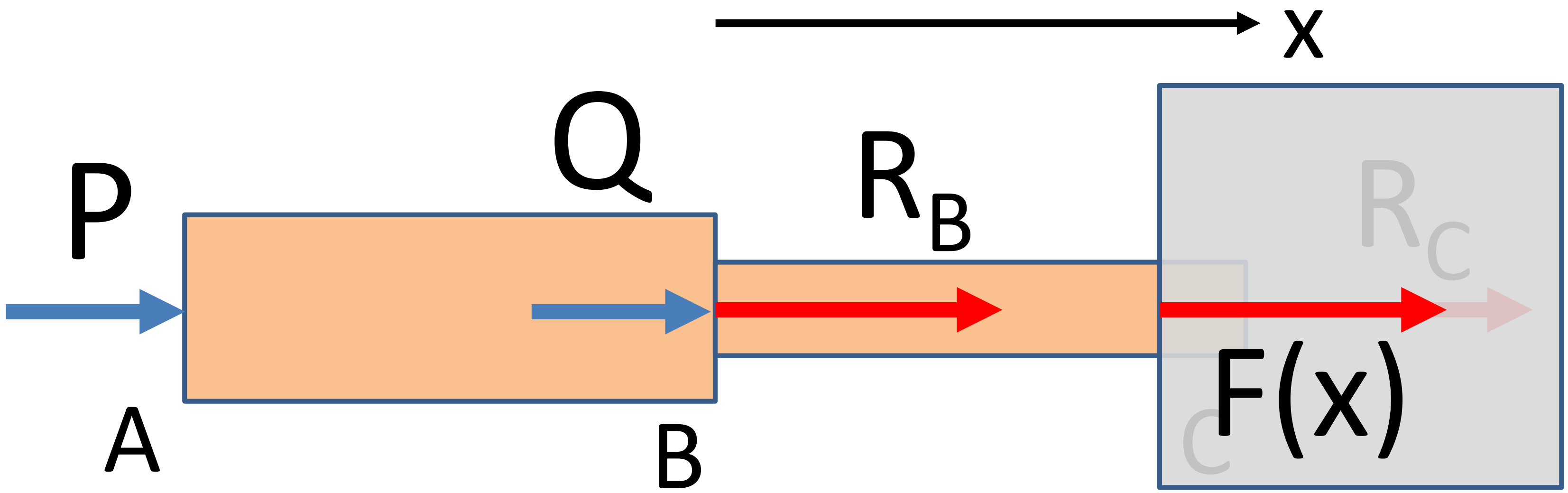
Displacement at B



- We can now calculate the displacement at B, where x = 0, by using any of the two expressions for u(x).
- Either way we get

$$u(0) = -\frac{Px}{2Ea} = -\frac{(P+Q+R_B)x}{2Ea} = 0$$

Displacement at B

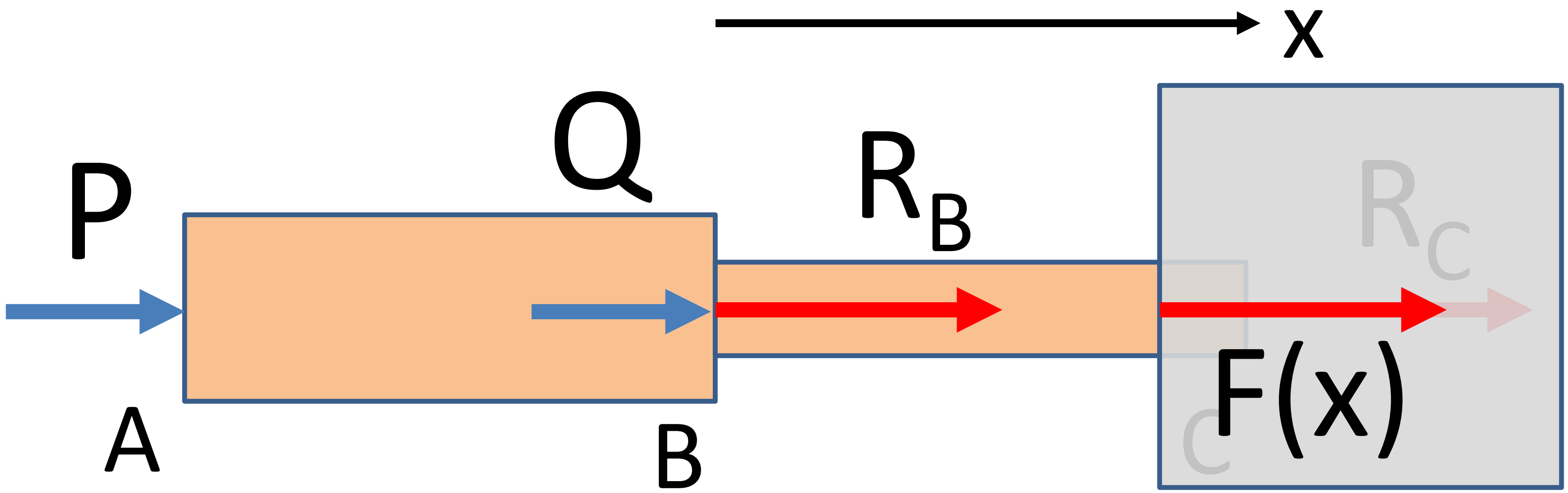


- We can now calculate the displacement at B, where x = 0, by using any of the two expressions for u(x).
- Either way we get

$$u(0) = -\frac{Px}{2Ea} = 0$$

$$u(0) = -\frac{(P+Q+R_B)x}{2Ea} = 0$$

Displacement at B

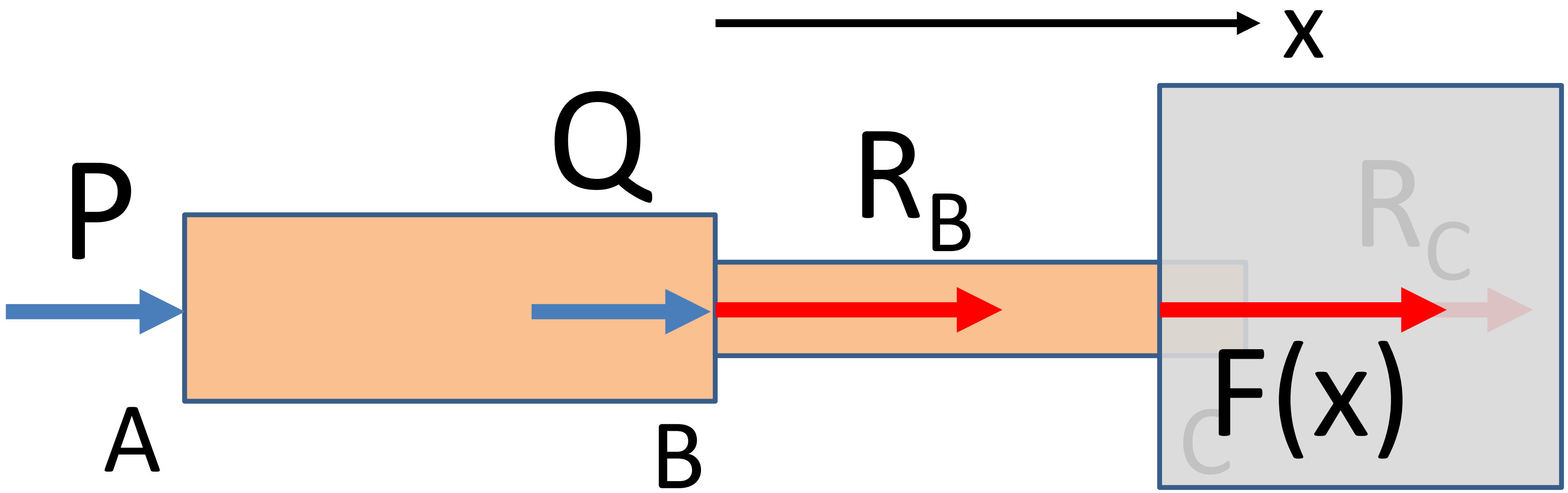


- Equating these two zeros would lead to the wrong conclusion shown below.

$$u(0) = -\frac{Px}{2Ea} = -\frac{(P+Q+R_B)x}{2Ea} \Rightarrow Q + R_B = 0$$

- We should only conclude that since the constraint is getting satisfied we are not getting useful information from x=0

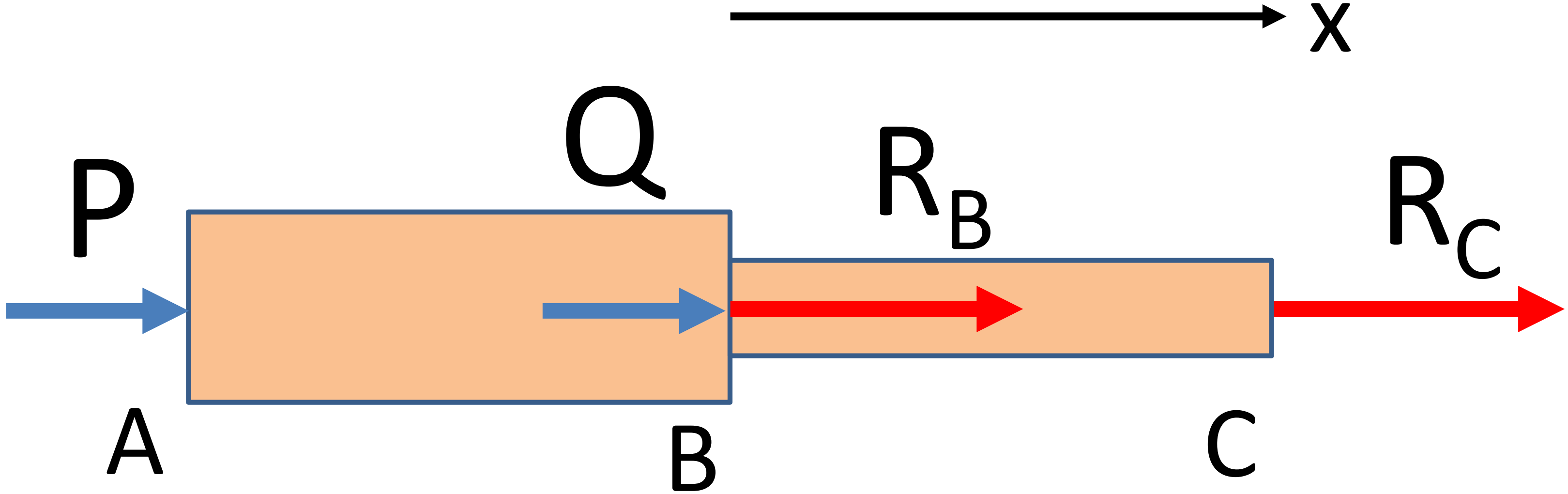
Displacement at C



- We therefore look at C, where x = L/2. We get

$$u\left(\frac{L}{2}\right) = -\frac{\left(P+Q+R_B\right)L}{4Ea} = 0 \Rightarrow P+Q+R_B = 0$$

Equilibrium for the whole bar



- Force equilibrium gives us $P + Q + R_B + R_C = 0$
- But we already $P + Q + R_B = 0$
- Hence

$$R_C = 0, R_B = -\left(P + Q\right)$$