11.1 Fourier Sine and Cosine Integrals

Consider the Fourier integral representation of a function f as

$$f(x) \sim \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,\mathrm{d}\alpha$$

where the Fourier Integral Coefficients are

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,\mathrm{d}u \quad \text{and} \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,\mathrm{d}u$$

If the function f is an even function, the integral of $A(\alpha)$ has an even integrand. Therefore we can simplify the integral to

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty f(u)\cos\alpha u\,\mathrm{d}u$$

Since the integrand of the integral in $B(\alpha)$ is odd and therefore $B(\alpha) = 0$. Thus for even function f we have

$$f(x) \sim \int_0^\infty A(\alpha)\cos\alpha x\,\mathrm{d}\alpha$$

Similarly, for an odd function f we have

$$A(\alpha) = 0 \quad \text{and} \quad B(\alpha) = \frac{2}{\pi}\int_0^\infty f(u)\sin\alpha u\,\mathrm{d}u$$

and

$$f(x) \sim \int_0^\infty B(\alpha)\sin\alpha x\,\mathrm{d}\alpha$$

Remark: *Similar to half range Fourier series, we can represent a function defined for all real $x > 0$ by Fourier sine or Fourier cosine integral by extending the function as an odd function or as an even function over the entire real axis, respectively.*

We summarize the above results in the following theorem:

11.1.1 Theorem

Assume that f is piecewise smooth function on every finite interval on the positive x-axis and let f be absolutely integrable over 0 to ∞. Then f may be represented by either:
a) Fourier cosine integral

$$f(x) \sim \int_0^\infty A(\alpha)\cos\alpha x,\, d\alpha \quad 0 < x < \infty,$$

where

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty f(u)\cos\alpha u\,du$$

b) Fourier sine integral

$$f(x) \sim \int_0^\infty B(\alpha)\sin\alpha x\,d\alpha \quad 0 < x < \infty$$

where

$$B(\alpha) = \frac{2}{\pi}\int_0^\infty f(u)\sin\alpha u\,du$$

Moreover, the above Fourier cosine and sine integrals converge to $[f(x+) + f(x-)]/2$.

11.2 Example Problems

11.2.1 Problem 1

For the function

$$f = \begin{cases} 0, & -\infty < x < -\pi; \\ -1, & -\pi < x < 0; \\ 1, & 0 < x < \pi; \\ 0, & \pi < x < \infty. \end{cases}$$

determine the Fourier integral. To what value does the integral converge at $x = -\pi$*?*

Solution: Since the given function is an odd function we can directly put $A(\alpha) = 0$ and evaluate $B(\alpha)$ as

$$B(\alpha) = \frac{2}{\pi}\int_0^\infty f(u)\sin\alpha u\,\mathrm{d}u = \frac{2}{\pi}\int_0^\pi \sin\alpha u\,\mathrm{d}u = \frac{2}{\pi\alpha}(1-\cos\alpha\pi)$$

Therefore, the Fourier integral representation is

$$f(x) \sim \frac{2}{\pi}\int_0^\infty \frac{1-\cos\alpha\pi}{\alpha}\sin\alpha x\,\mathrm{d}\alpha$$

The function is not defined at $x = -\pi$ and therefore the Fourier integral at $x = -\pi$ will converge to the average value $\frac{0-1}{2} = -\frac{1}{2}$.

11.2.2 Problem 2

Find a Fourier sine and cosine integral representation of

$$f = \begin{cases} 1, & 0 < x < \pi; \\ 0, & \pi < x < \infty. \end{cases}$$

Hence evaluate

$$\int_0^\infty \frac{\sin \pi\alpha \cos \pi\alpha}{\alpha}\, d\alpha \quad \textit{and} \quad \int_0^\infty \frac{(1-\cos \pi\alpha)\sin \pi\alpha}{\alpha}\, d\alpha$$

Solution: Fourier sine representation is given as

$$f(x) \sim \int_0^\infty B(\alpha) \sin \alpha x \,\mathrm{d}\alpha$$

where

$$B(\alpha) = \frac{2}{\pi}\int_0^\infty f(u) \sin \alpha u \,\mathrm{d}u = \frac{2}{\pi}\int_0^\pi \sin \alpha u \,\mathrm{d}u = \frac{2}{\pi}\frac{(1-\cos \pi\alpha)}{\alpha}$$

Therefore

$$f(x) \sim \frac{2}{\pi}\int_0^\infty \frac{(1-\cos \pi\alpha)}{\alpha} \sin \alpha x \,\mathrm{d}\alpha$$

Using convergence theorem, we have

$$\frac{2}{\pi}\int_0^\infty \frac{(1-\cos \pi\alpha)}{\alpha} \sin \alpha x \,\mathrm{d}\alpha = \begin{cases} 0, & x > \pi; \\ 1/2, & x = \pi; \\ 1, & 0 < x < \pi. \end{cases}$$

To get the desired integral we substitute $x = \pi$ in the above integral

$$\frac{2}{\pi}\int_0^\infty \frac{(1-\cos \pi\alpha)}{\alpha} \sin \pi\alpha \,\mathrm{d}\alpha = \frac{1}{2} \quad \Longrightarrow \quad \int_0^\infty \frac{(1-\cos \pi\alpha)}{\alpha} \sin \pi\alpha \,\mathrm{d}\alpha = \frac{\pi}{4}$$

For the Fourier cosine representation we evaluate

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty f(u) \cos \alpha u \,\mathrm{d}u = \frac{2}{\pi}\int_0^\pi \cos \alpha u \,\mathrm{d}u = \frac{2}{\pi}\frac{\sin \pi\alpha}{\alpha}$$

Thus, the Fourier cosine integral representation is given as

$$f(x) \sim \frac{2}{\pi}\int_0^\infty \frac{\sin \pi\alpha \cos \alpha x}{\alpha} \,\mathrm{d}\alpha$$

Applying convergence theorem we have

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \pi\alpha \cos \alpha x}{\alpha}\, \mathrm{d}\alpha = \begin{cases} 0, & x > \pi; \\ 1/2, & x = \pi; \\ 1, & 0 < x < \pi. \end{cases}$$

To get the required integral we now substitute $x = \pi$ into the above integral

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \pi\alpha \cos \pi\alpha}{\alpha}\, \mathrm{d}\alpha = \frac{1}{2} \quad \Longrightarrow \quad \int_0^\infty \frac{\sin \pi\alpha \cos \pi\alpha}{\alpha}\, \mathrm{d}\alpha = \frac{\pi}{4}$$

In this lesson we introduce Fourier cosine and sine transforms. Evaluation and properties of Fourier cosine and sine transform will be discussed. The parseval's identities for Fourier cosine and sine transform will be given.

11.3 Fourier Cosine and Sine Transform

Consider the Fourier cosine integral representation of a function f as

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u)\cos\alpha u\,\mathrm{d}u\cos\alpha x\,\mathrm{d}\alpha = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\alpha u\,\mathrm{d}u\right)\cos\alpha x\,\mathrm{d}\alpha$$

In this integration representation, we set

$$\hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\alpha u\,\mathrm{d}u \tag{11.1}$$

and then

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\alpha)\cos\alpha x\,\mathrm{d}\alpha \tag{11.2}$$

The function $\hat{f}_c(\alpha)$ as given by (11.1) is known as the *Fourier cosine transform* of $f(x)$ in $0 < x < \infty$. We shall denote Fourier cosine transform by $F_c(f)$. The function $f(x)$ as given by (11.2) is called *inverse Fourier cosine transform* of $\hat{f}_c(\alpha)$. It is denoted by $F_c^{-1}(\hat{f}_c)$.

Similarly we define *Fourier sine* and *inverse Fourier sine* transform by

$$F_s(f) = \hat{f}_s(\alpha) := \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\sin\alpha u\,\mathrm{d}u \text{ and } F_s^{-1}(\hat{f}) = f(x) := \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\alpha)\sin\alpha x\,\mathrm{d}\alpha$$

11.4 Properties

We mention here some important properties of Fourier cosine and sine transform that will be used in the application to solving differential equations.

1. Linearity: Let f and g are piecewise continuous and absolutely integrable functions. Then for constants a and b we have

$$F_c(af + bg) = aF_c(f) + bF_c(g) \quad \text{and} \quad F_s(af + bg) = aF_s(f) + bF_c(g)$$

Note that these properties are obvious and can be proved just using linearity of the integrals.

2. Transform of Derivatives: Let $f(x)$ be continuous and absolutely integrable on $x-$axis. Let $f'(x)$ be piecewise continuous and on each finite interval on $[0, \infty]$ and $f(x) \to 0$ as $x \to \infty$. Then,

$$F_c\{f'(x)\} = \alpha F_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f(0) \text{ and } F_s\{f'(x)\} = -\alpha F_c\{f(x)\}$$

Proof: By the definition of Fourier cosine transform we have

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \alpha x \,\mathrm{d}x$$

Integrating by parts we get

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \left[(f(x) \cos \alpha x) \Big|_0^\infty + \alpha \int_0^\infty f(x) \sin \alpha x \,\mathrm{d}x \right]$$

Using the definition of Fourier sine integral we obtain

$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}} f(0) + \alpha\, F_s\{f(x)\}.$$

Similarly the other result for Fourier sine transform can be obtained.

Remark: *The above results can easily be extended to the second order derivatives to have*

$$F_c\{f''(x)\} = -\alpha^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0) \;\textit{ and }\; F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\, \alpha f(0) - \alpha^2 F_s\{f(x)\}$$

Note that here we have assumed continuity of f and f' and piecewise continuity of f''. Further, we also assumed that f and f' both goes to 0 as x approaches to ∞.

3. Parseval's Identities: For Fourier sine and cosine transform we have the following identities

i) $\displaystyle\int_0^\infty \hat{f}_c(\alpha)\hat{g}_c(\alpha)\,\mathrm{d}\alpha = \int_0^\infty f(x)g(x)\,\mathrm{d}x$ *ii)* $\displaystyle\int_0^\infty \left[\hat{f}_c(\alpha)\right]^2 \mathrm{d}\alpha = \int_0^\infty [f(x)]^2$

iii) $\displaystyle\int_0^\infty \hat{f}_s(\alpha)\hat{g}_s(\alpha)\,\mathrm{d}\alpha = \int_0^\infty f(x)g(x)\,\mathrm{d}x$ *iv)* $\displaystyle\int_0^\infty \left[\hat{f}_s(\alpha)\right]^2 \mathrm{d}\alpha = \int_0^\infty [f(x)]^2$

Proof: We prove the first identity and rest can be proved similarly. We take the right hand side of the identity and use the definition of the inverse cosine transform to get

$$\int_0^\infty f(x)g(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\int_0^\infty \hat{g}_c(\alpha)\cos(\alpha x)\,\mathrm{d}\alpha\,\mathrm{d}x$$

Changing the order of integration we obtain

$$\int_0^\infty f(x)g(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{g}_c(\alpha)\int_0^\infty f(x)\cos(\alpha x)\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty \hat{f}_c(\alpha)\hat{g}_c(\alpha)\,\mathrm{d}\alpha$$

11.5 Example Problems

11.5.1 Problem 1

Find the Fourier sine transform of e^{-x}, $x > 0$. *Hence show that*

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi e^{-m}}{2},\ m > 0$$

Solution: Using the definition of Fourier sine transform

$$F_s\{e^{-x}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\alpha x\,\mathrm{d}x$$

Let us denote the integral on the right hand side by i and evaluate it by integrating by parts as

$$I = \int_0^\infty e^{-x}\sin\alpha x\,\mathrm{d}x = -e^{-x}\sin\alpha x\Big|_0^\infty + \alpha\int_0^\infty e^{-x}\cos\alpha x\,\mathrm{d}x = \alpha\int_0^\infty e^{-x}\cos\alpha x\,\mathrm{d}x$$

Again integrating by parts

$$I = \alpha\left[-e^{-x}\cos\alpha x\Big|_0^\infty - \alpha\int_0^\infty e^{-x}\sin\alpha x\,\mathrm{d}x\right] = \alpha\left[1-\alpha I\right]$$

This implies

$$I = \frac{\alpha}{1+\alpha^2}$$

Finally substituting the value of I to the expression of Fourier sine transform above we get

$$F_s\{e^{-x}\} = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{1+\alpha^2}\right)$$

Taking inverse Fourier transform

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\alpha) \sin \alpha x \, \mathrm{d}\alpha = \frac{2}{\pi} \int_0^\infty \frac{\alpha}{1+\alpha^2} \sin \alpha x \, \mathrm{d}\alpha$$

Changing x to m and α to x we obtain

$$\int_0^\infty \frac{x}{1+x^2} \sin(xm) \, \mathrm{d}x = \frac{\pi}{2} e^{-m}$$

11.5.2 Problem 2

Find the Fourier cosine transform of e^{-x^2}, $x > 0$.

Solution: By the definition of Fourier cosine transform we have

$$F_c\{e^{-x^2}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos(\alpha u) \, \mathrm{d}u = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u^2} \cos(\alpha u) \, \mathrm{d}u$$

Let us denote the integral on the right hand side by I and differentiate it with respect to α as

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha} \int_0^\infty e^{-u^2} \cos(\alpha u) \, \mathrm{d}u = -\int_0^\infty e^{-u^2} \sin(\alpha u) u \, \mathrm{d}u$$

Integrating by parts we get

$$\frac{dI}{d\alpha} = \frac{1}{2} \left[e^{-u^2} \sin(\alpha u) \right]_0^\infty - \frac{\alpha}{2} \int_0^\infty e^{-u^2} \cos(\alpha u) \, \mathrm{d}u = -\frac{\alpha}{2} I$$

This implies

$$I = c \, e^{-\alpha^2/4}$$

Using $I(0) = \frac{\sqrt{\pi}}{2}$, we evaluate the constant $c = \frac{\sqrt{\pi}}{2}$. Then we have

$$I = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$$

Therefore the desired Fourier cosine transform is given as

$$F_c\{e^{-x^2}\} = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4} = \frac{1}{\sqrt{2}} e^{-\alpha^2/4}$$

11.5.3 Problem 3

Using Parseval's identities, prove that

i) $\displaystyle\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$ *ii)* $\displaystyle\int_0^\infty \frac{t^2}{(t^2+1)}\,dt = \frac{\pi}{4}$

Solution: *i)* For the first part let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. It can easily be shown that

$$F_c\{f\} = \hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\alpha x\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\alpha^2}$$

$$F_c\{g\} = \hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\cos\alpha x\,\mathrm{d}x = \sqrt{\frac{2}{\pi}}\frac{b}{b^2+\alpha^2}$$

Using Parseval's identity we get

$$\int_0^\infty \hat{f}_c(\alpha)\hat{g}_c(\alpha)\,\mathrm{d}\alpha = \int_0^\infty f(x)g(x)\,\mathrm{d}x \Rightarrow \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+\alpha^2}\,\frac{b}{b^2+\alpha^2}\,\mathrm{d}\alpha = \int_0^\infty e^{-(a+b)x}\,\mathrm{d}x$$

This can be further simplified as

$$\frac{2ab}{\pi}\int_0^\infty \frac{\mathrm{d}\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)}\,\mathrm{d}\alpha = -\frac{e^{-(a+b)x}}{a+b}\Big|_0^\infty$$

Thus we get

$$\int_0^\infty \frac{\mathrm{d}\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)} = \frac{\pi}{ab(a+b)}$$

ii) For the second part we use Fourier sine transform of e^{-x} as

$$F_s\{e^{-x}\} = \hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{1+\alpha^2}.$$

Using Parseval's identity we obtain

$$\frac{2}{\pi}\int_0^\infty \frac{\alpha^2}{(1+\alpha^2)^2}\,\mathrm{d}\alpha = \int_0 \infty \left(e^{-x}\right)^2\,\mathrm{d}x = \frac{1}{2}$$

Hence we have the desired results

$$\int_0^\infty \frac{\alpha^2}{(1+\alpha^2)^2}\,\mathrm{d}\alpha = \frac{\pi}{2}.$$