

$\vec{b}$  can vary locally

$$\frac{d}{dt} \int_V F dV = \int_V \frac{\partial F}{\partial t} dV + \int_A F \vec{b} \cdot \vec{n} dA$$

$$\vec{b} = \vec{u}$$

Chunk of fluid moves with the fluid

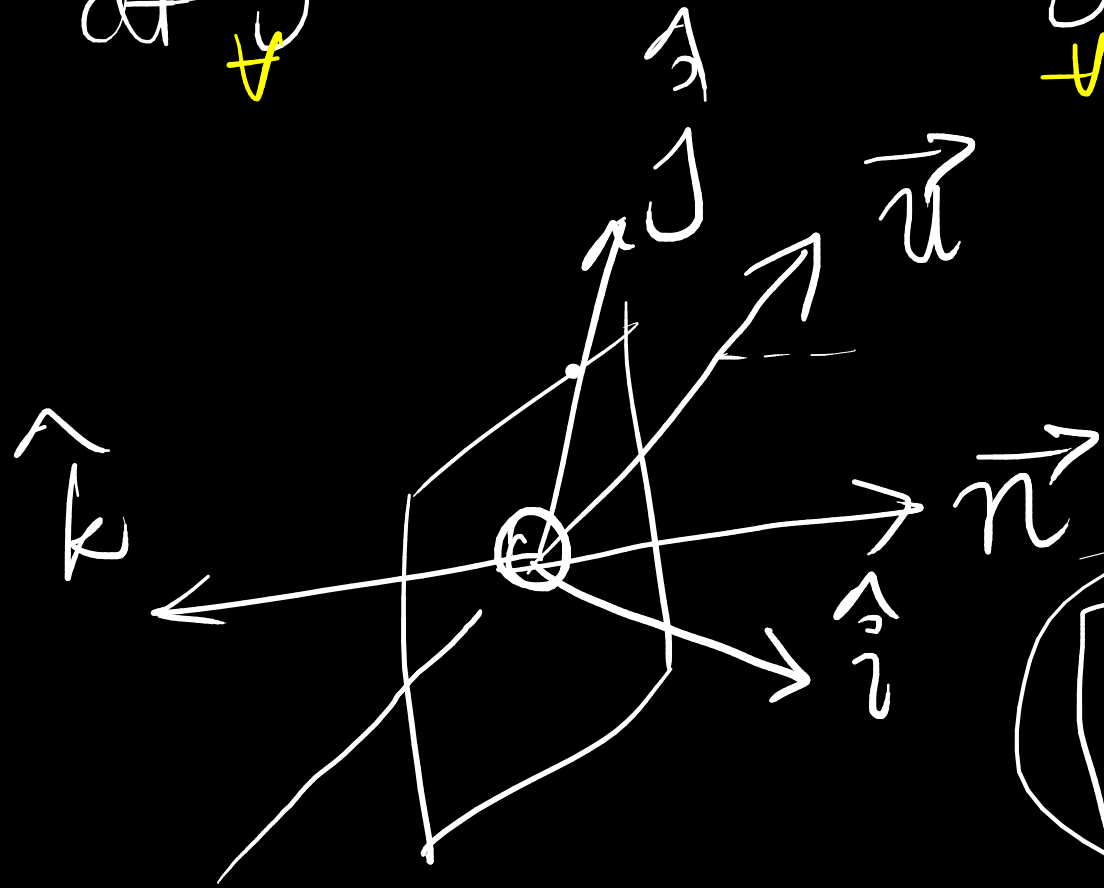
$$\frac{d}{dt} \int_V F dV = \int_V \frac{\partial F}{\partial t} dV + \int_A F \vec{u} \cdot \vec{n} dA$$

( $\vec{u}$ )  $\nearrow$  momentum/vol  $\rho \vec{u}$

$$\left| \frac{d}{dt} \int_V \rho \vec{u} dV = \int_V \frac{\partial (\rho \vec{u})}{\partial t} dV + \int_A \rho \vec{u} \vec{u} \cdot \vec{n} dA \right.$$

$$\frac{d}{dt} \int_V \rho u dV = \int_V \frac{\partial}{\partial t} (\rho u) + \int_A \rho u \underbrace{\vec{u} \cdot \vec{n} dA}_{\text{flux}}$$

$\downarrow$   
 quantity carried by  $\vec{u}$



$$\vec{u} \cdot \vec{n}$$

$$\left\{ \begin{array}{l} \int \vec{u} \cdot \hat{i} \\ \int \vec{u} \cdot \hat{j} \\ \int \vec{u} \cdot \hat{k} \end{array} \right\} \begin{array}{l} x \text{ mom/vol} \\ y \text{ } \\ z \text{ } \end{array}$$

Cause of the rate of ch-mom

$$\int_V \frac{\partial}{\partial t} (\rho \vec{u}) dV + \int_A \rho \vec{u} \cdot \vec{n} dA = \Sigma F = F_{\text{body}} + F_{\text{surf}}$$

Body force

→ Acts on total volume

Gravity  
Electromagnetic  
Magnetic  
Thermal.

$$\int_V \rho \vec{g} dV$$

$$F_{b,g} =$$

$$\int_V \rho \vec{g} dV$$

○ (x)

○ (y)

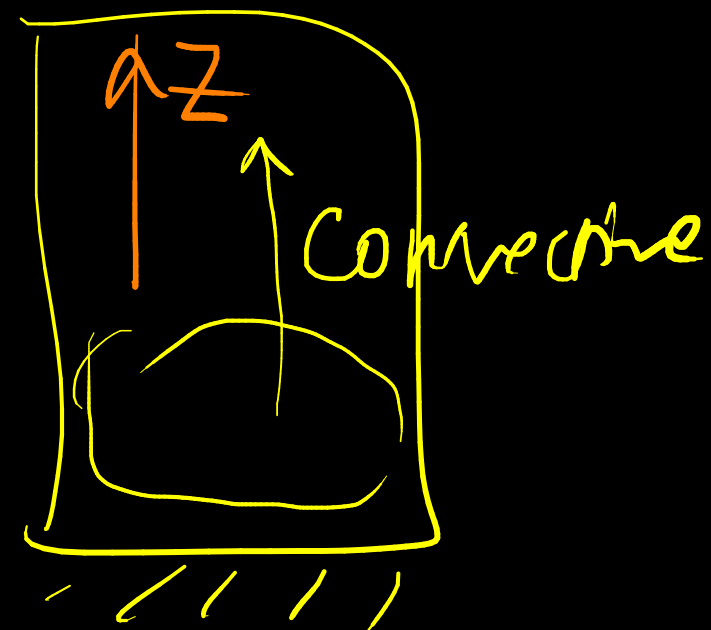
$$= \int_V \rho g dV$$

Conservative

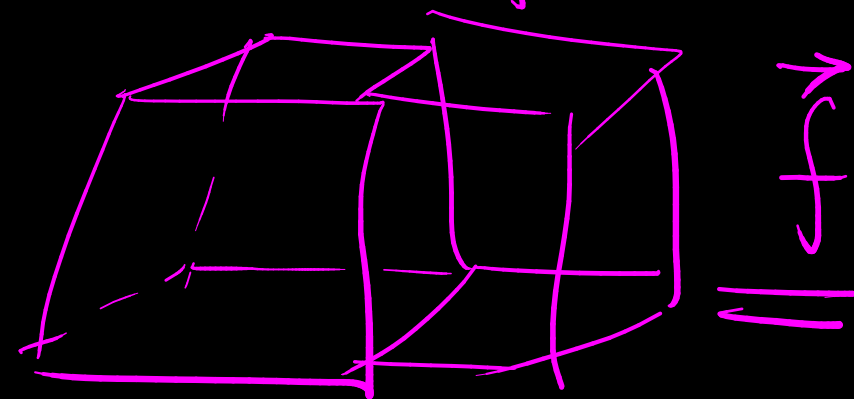
non conservative

$$\vec{F} = -\nabla \phi$$

$$\rho \vec{g} = -\nabla \phi \Rightarrow \phi = -\rho g z$$



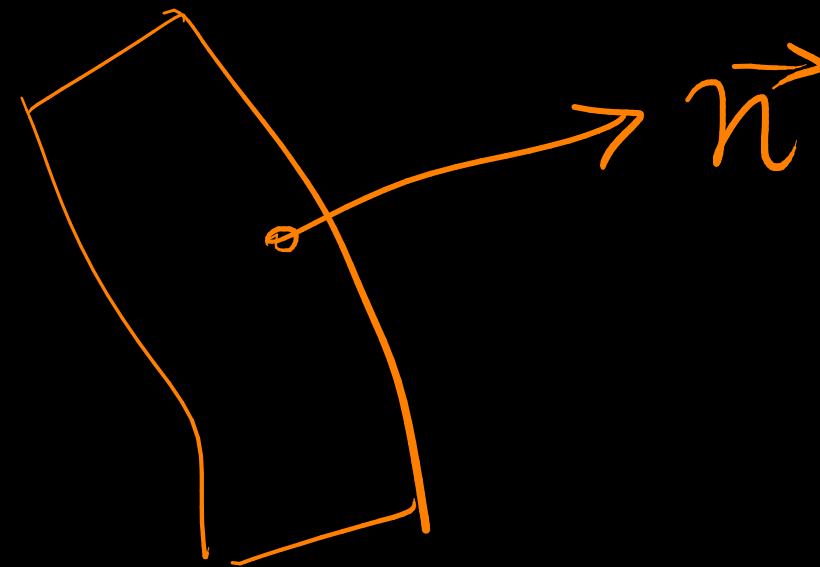
Surface forces



traction  
vector

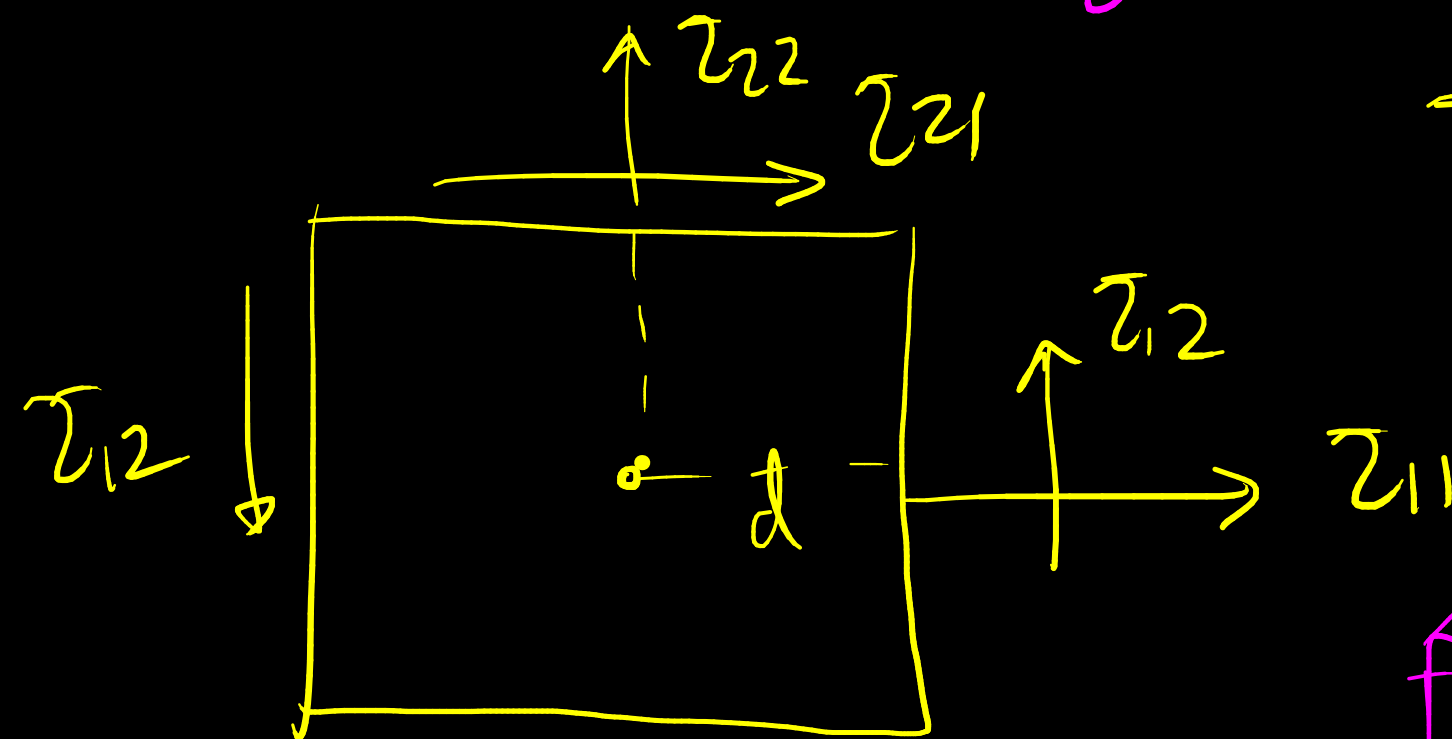
$$\vec{f}_s = \underline{\underline{\tau}} \cdot \vec{n}$$

$$\int_A \vec{f} dA$$



$$f_j = \tau_{ij} n_i$$

Stress tensor is symmetric



$$\tau_{12} - \tau_{21} = \underbrace{\text{net moment}}_d = 0$$

$$\tau_{12} = \tau_{21}$$

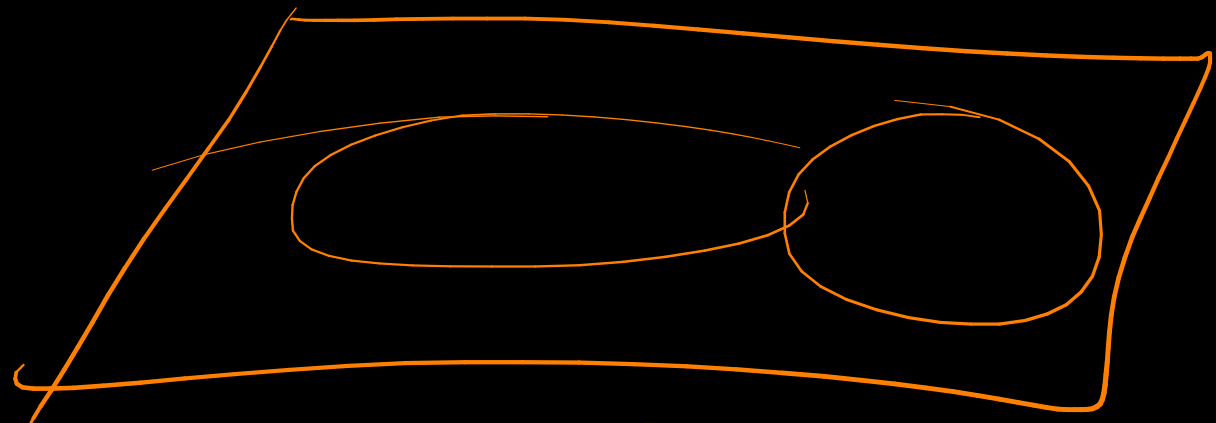
For equilibrium

$$// \int_V \frac{\partial}{\partial t} (\rho \vec{u}) dV + \int_A \rho \vec{u} \cdot \vec{n} dA = \int_V \rho \vec{g} dV + \int_A \vec{F} dA$$

interfacial  
forces

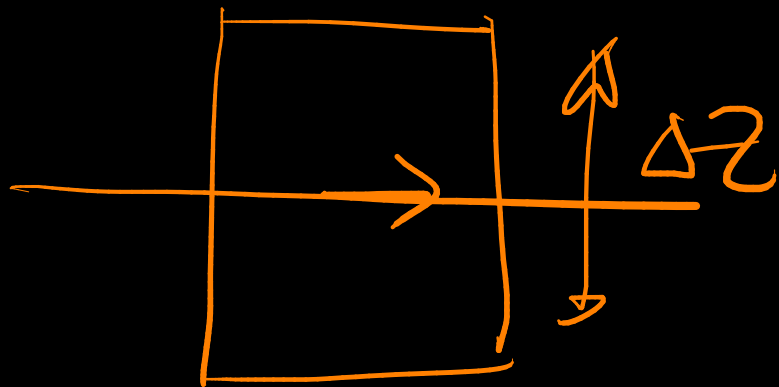
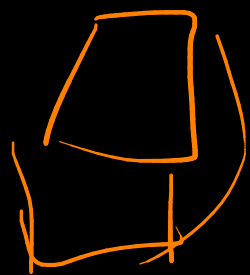
surface

Boundary condition



(1)

(2)



$$\boxed{\int_V \frac{\partial(\rho \vec{u})}{\partial t} dV} + \int_V \rho \vec{u} \cdot \vec{u} \cdot \vec{n} dA = \int_V \rho \vec{g} dV + \int_A \vec{f} dA$$

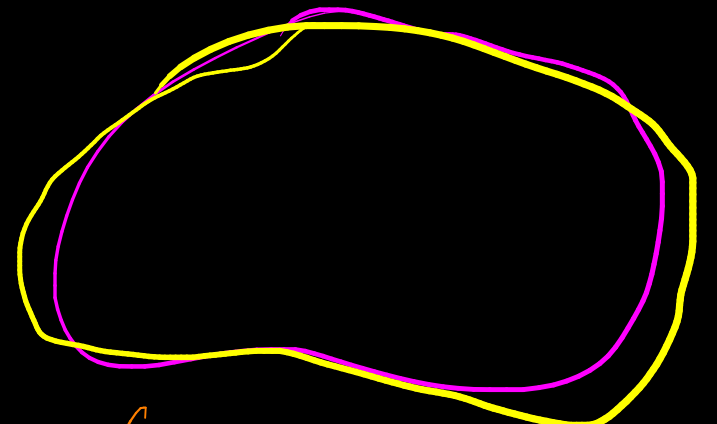
→ Any general CV moving w/  $\underline{\vec{b}}$

$$\frac{d}{dt} \int_{V^*} \rho \vec{u} dV =$$

$$\int_{V^*} \frac{\partial(\rho \vec{u})}{\partial t} dV +$$

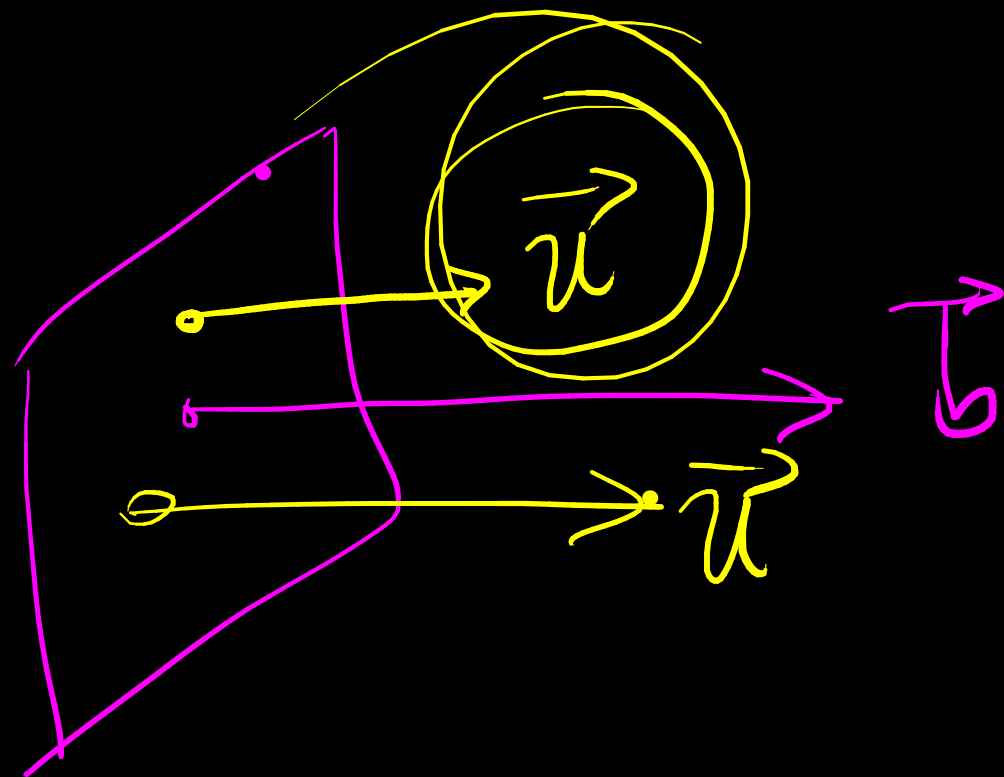
$$\int_A \rho \vec{u} \cdot \vec{b} \cdot \vec{n} dA$$

$$\int \frac{\partial(\rho \vec{u})}{\partial t} dV$$



$$\frac{d}{dt} \int \rho \vec{u} dV - \int \rho \vec{u} \cdot \vec{b} \cdot \vec{n} dA + \int \rho \vec{u} \cdot \vec{u} \cdot \vec{n} dA = \int \rho \vec{g} dV + \int \vec{f} dA$$

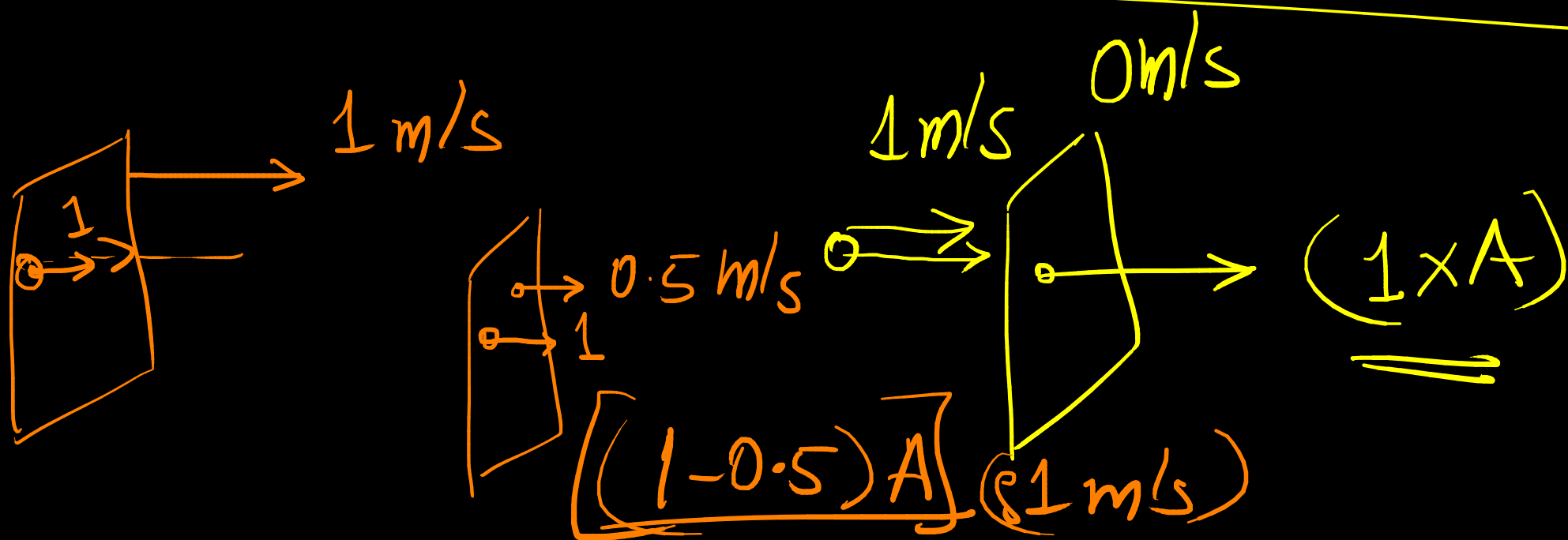
$$\frac{d}{dt} \int \rho \vec{u} dV + \int \rho \vec{u} \cdot (\vec{u} - \vec{b}) \cdot \vec{n} dA = \int \rho \vec{g} dV + \int \vec{f} dA$$

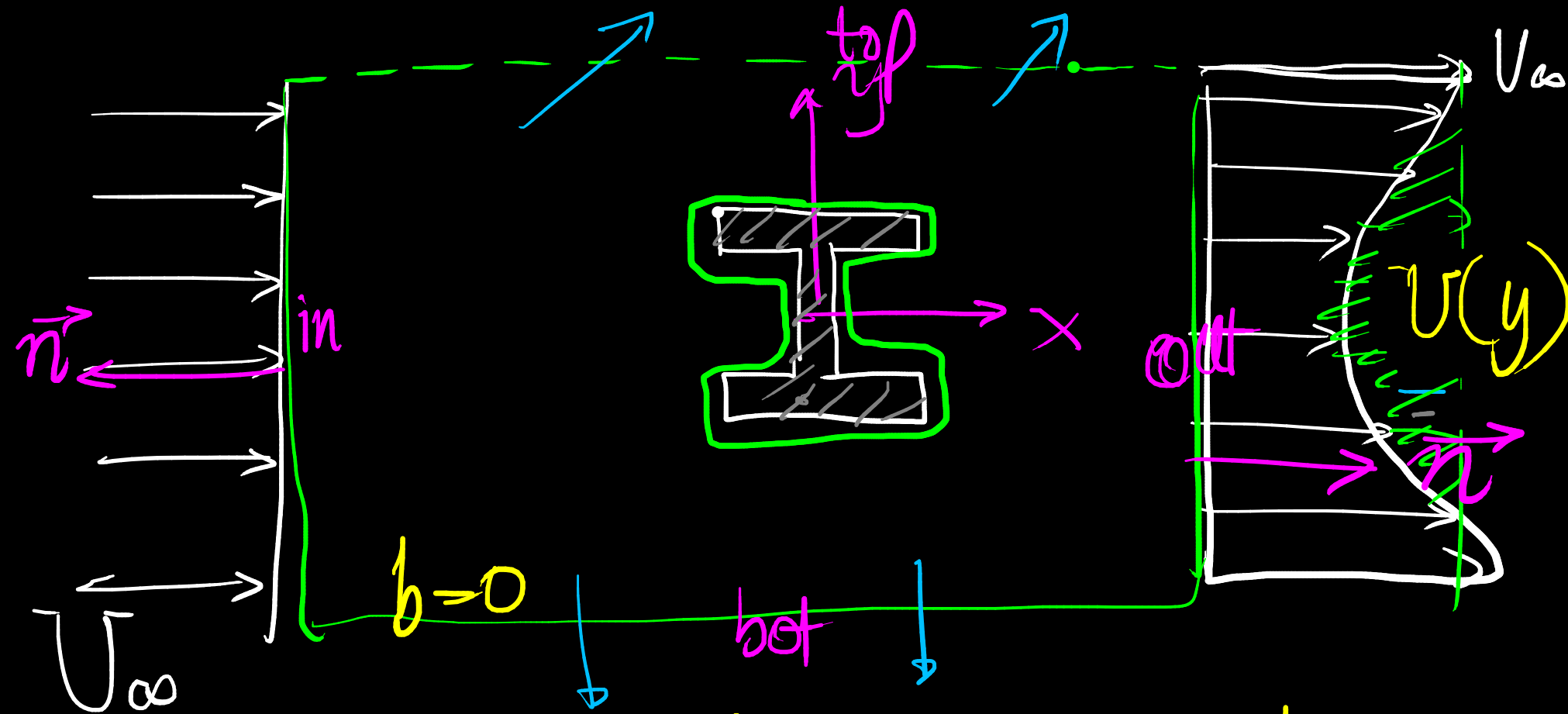


$$(\vec{u} - \vec{b}) \cdot \vec{n} dA$$

Relative flux  
to moving CV

no flux if flux crosses at all





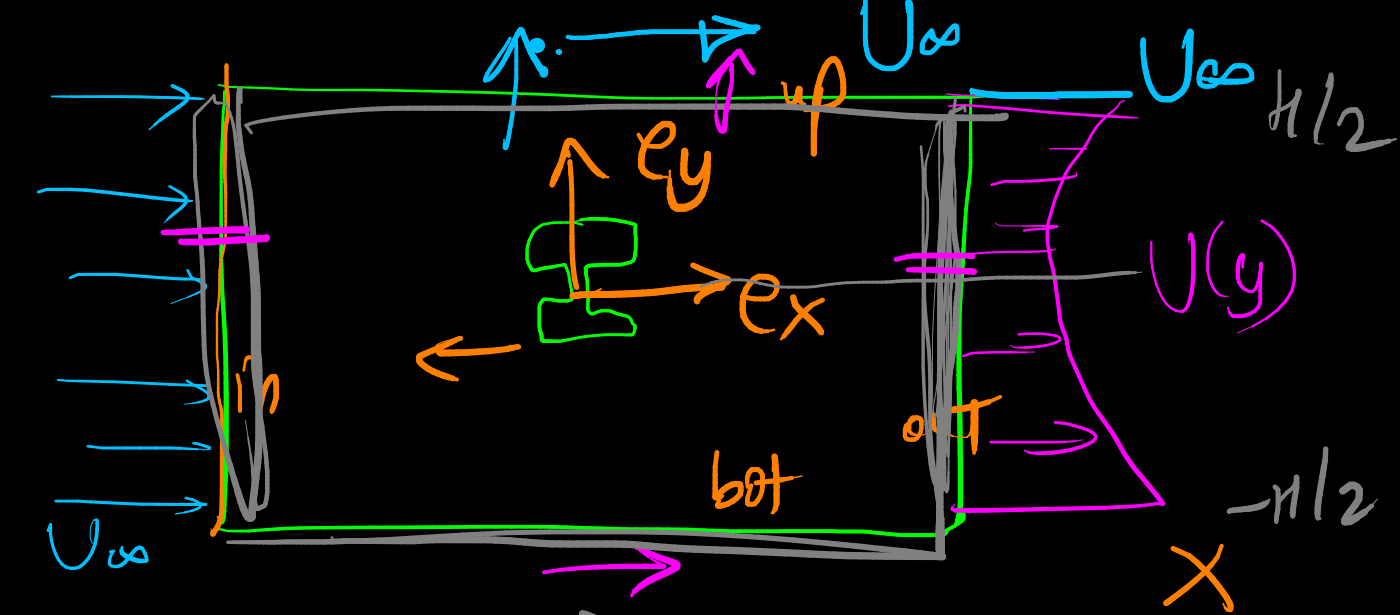
- Find out expr for drag acting on body.

Step 1: Mass conservation

$$\int \rho \vec{u} \cdot \vec{n} dA = 0$$

$$\begin{aligned} & \int_{in} \rho \vec{u} \cdot \vec{n} dA + \int_{out} \rho \vec{u} \cdot \vec{n} dA + \int_{up} \rho \vec{u} \cdot \vec{n} dA + \int_{bot} \rho \vec{u} \cdot \vec{n} dA = 0 \\ & - \int \rho U_{\infty} dA + \int \rho U(y) dA + \left[ \int_{up} \rho \vec{u} \cdot \vec{n} dA + \int_{bot} \rho \vec{u} \cdot \vec{n} dA \right] = 0 \end{aligned}$$





$$\boxed{\int_S \vec{u} \cdot \vec{n} dA + \int_{\text{bot}} \rho \vec{u} \cdot \vec{n} dA = \int_S \rho (U_\infty - U(y)) dA}$$

+ve

$$+ \int_S \rho U_\infty^2 dA - \int_S \rho U(y)^2 dA$$

$$+ U_\infty \times \int_S \rho (U_\infty + U(y)) dA = +F_D$$

in:  $\vec{u} = U_\infty \vec{e}_x$

$$\int_S \rho \vec{u} \cdot \vec{n} dA = \int_S \rho U_\infty (-U_\infty) dA$$

$$= \int_S \rho U(y) (U(y)) dA$$

up bot:

$$\int_S \rho U_\infty \vec{u} \cdot \vec{n} dA + \int_{\text{bot}} \rho U_\infty \vec{u} \cdot \vec{n} dA,$$

$$= U_\infty \left[ \int_{\text{top}} \rho \vec{u} \cdot \vec{n} dA + \int_{\text{bot}} \rho \vec{u} \cdot \vec{n} dA \right]$$

$$l \int_S \rho U_\infty^2 dA - l \int_S \rho U(y)^2 dA$$

$$+ \int_S \rho (-U_\infty^2 + U_\infty U(y)) dy l = F_D$$

$$\boxed{\frac{F_D}{l} = \int_{-h/2}^{h/2} \rho U(y) [U_\infty - U(y)] dy}$$



$$\left[ U_0 \vec{e}_x + v \vec{e}_y \right] \leftarrow$$

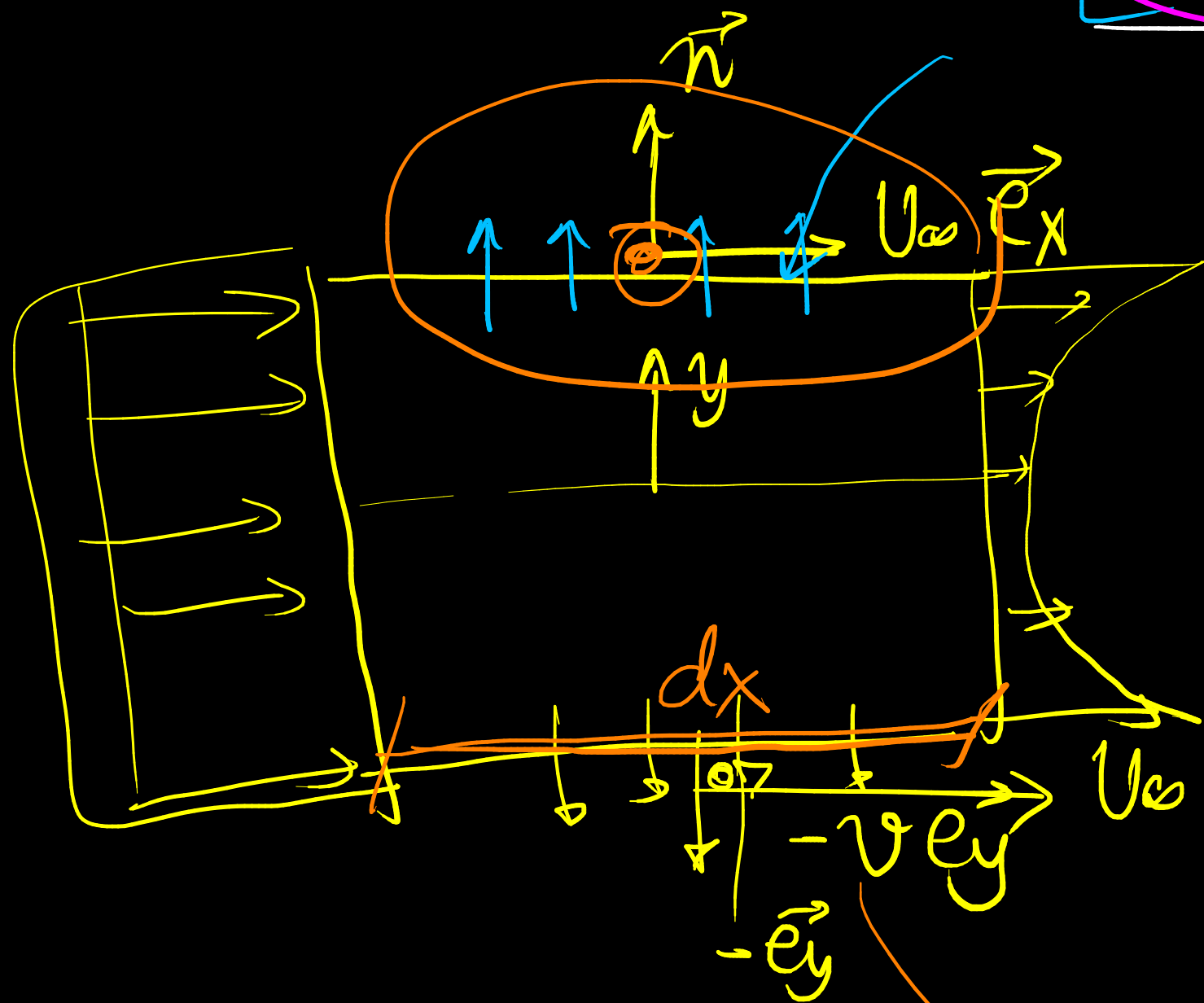
$$\int \boxed{v \, dx} \, l$$

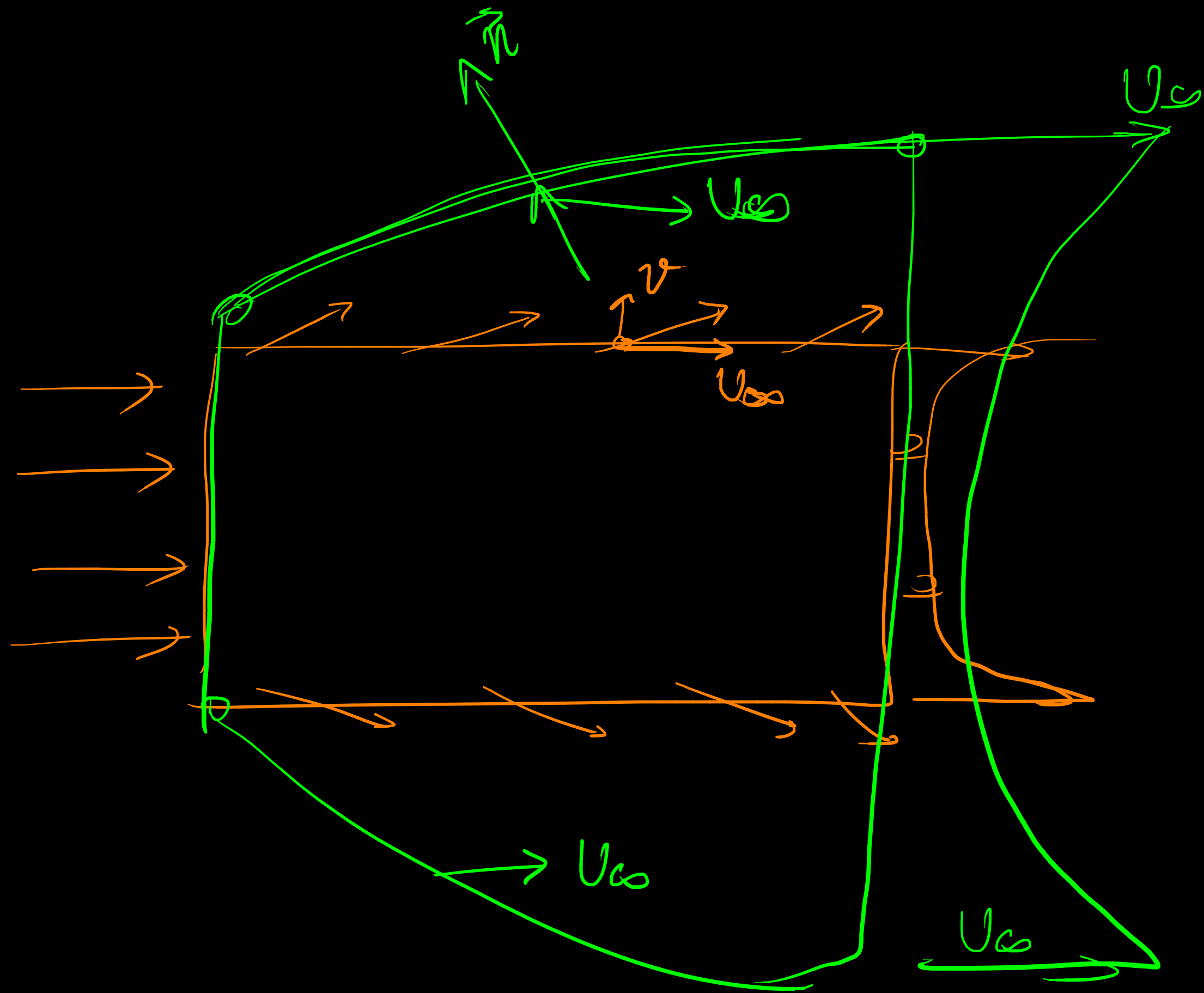
$$\int \delta U_{\infty} v dx \quad l$$

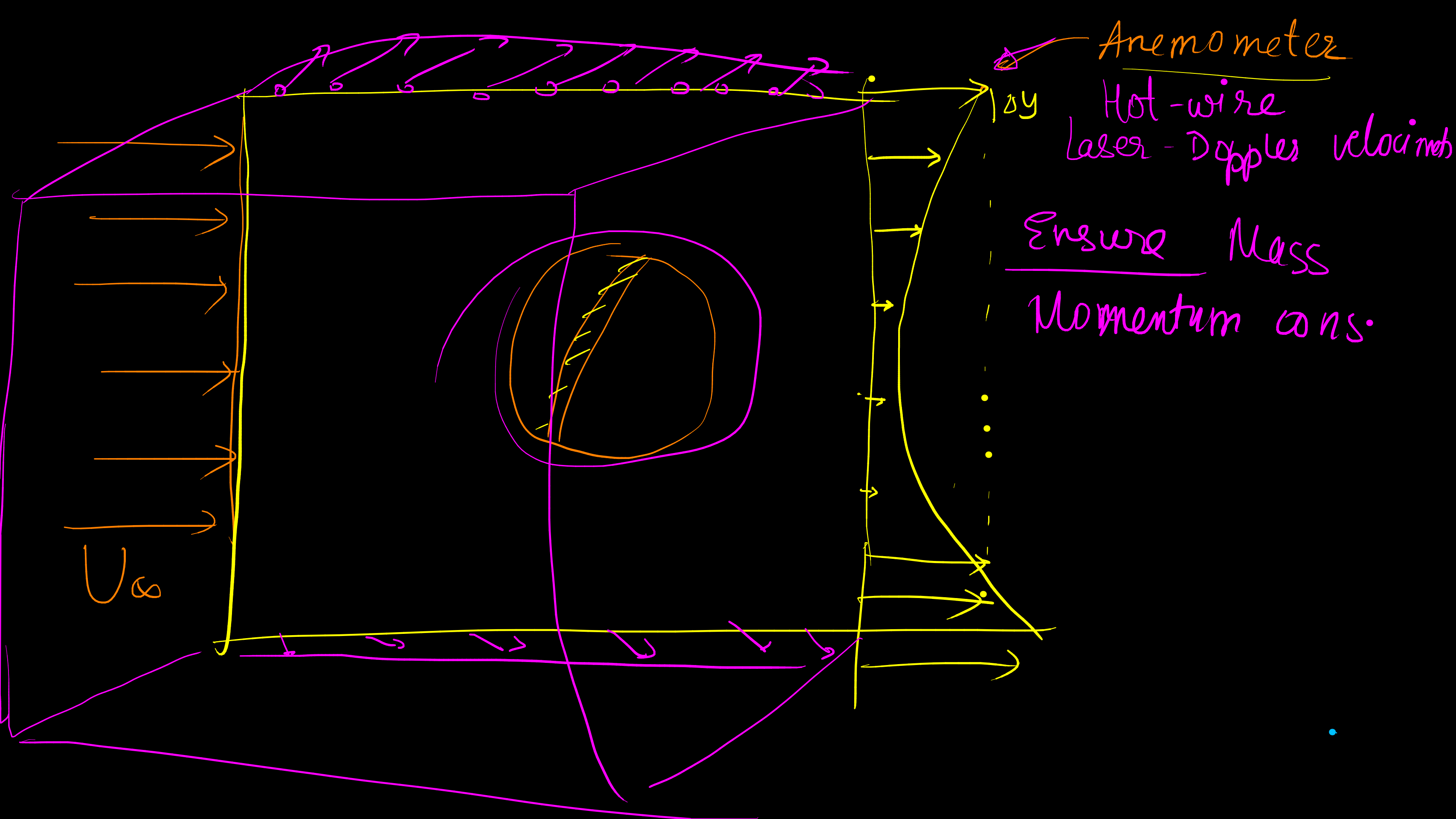
$$\int \sin x \, dx$$

$$(V_{\infty} \vec{e}_x - v \vec{e}_y)$$

$$\int -8v^2 v dx$$



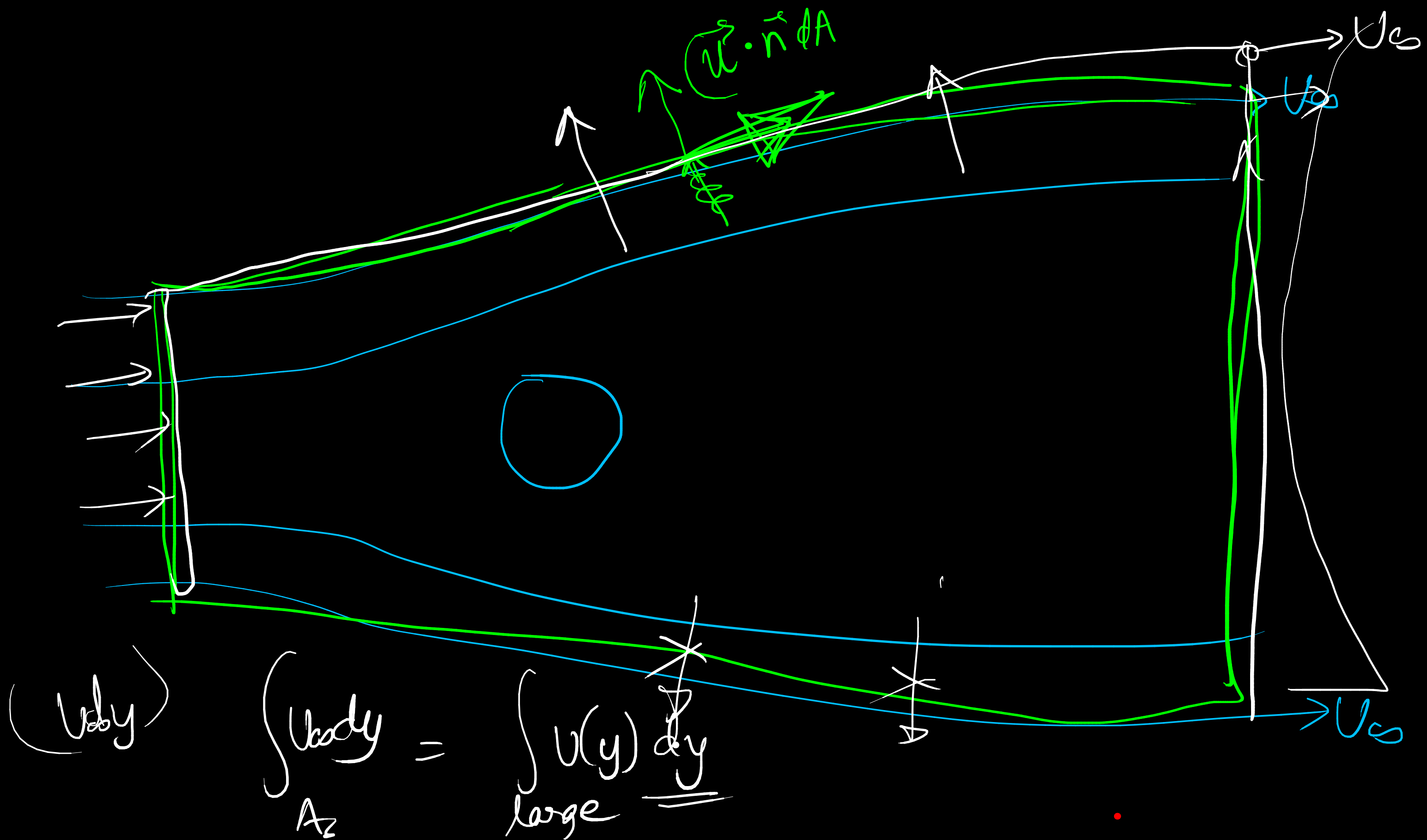


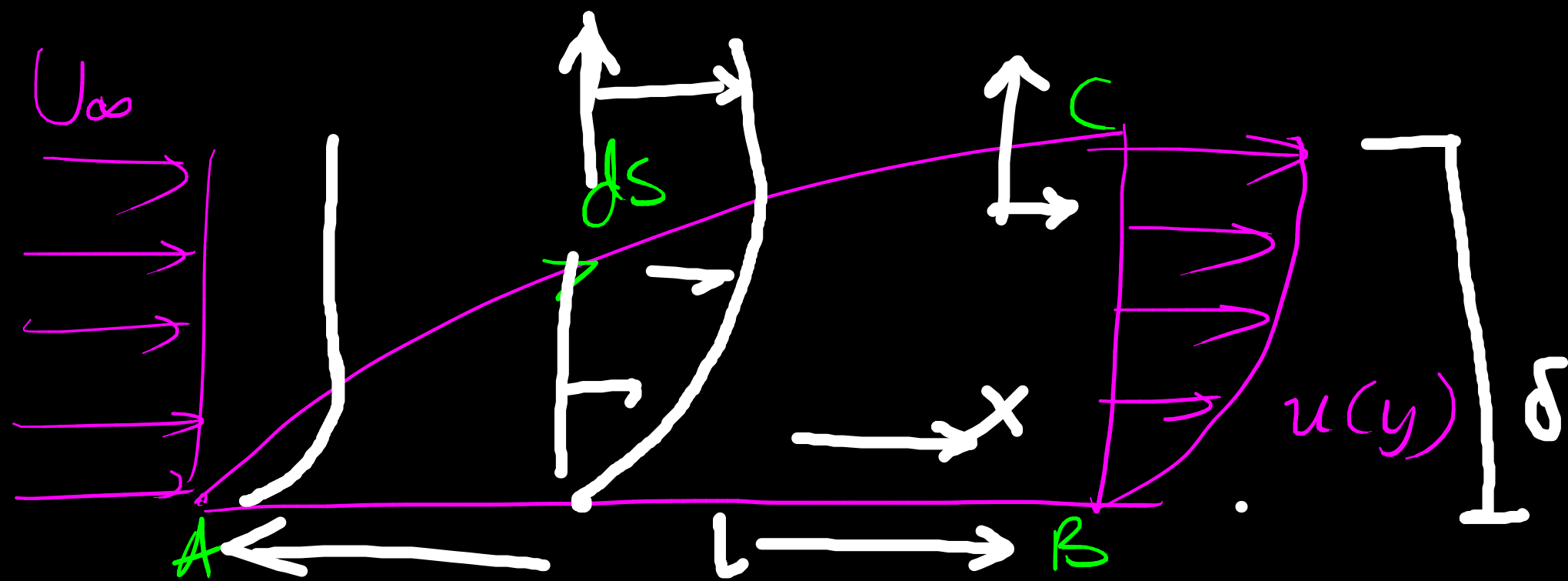


Anemometer

Hot-wire  
laser-Doppler velocimetry

Ensure Mass  
Momentum cons.





$$\int_{AC} \rho \mathbf{u} \cdot \vec{n} ds = - \int_{BC} \rho u dy$$

$$F_x = - \int \rho u^2 dy - \int \rho U_{\infty} \vec{u} \cdot \vec{n} ds$$

$$= - \int \rho u^2 dy + \int \rho U_{\infty} u dy$$

$$= \int \rho u (U_{\infty} - u) dy = \int_0^h \rho \frac{u}{U_{\infty}} \left( 1 - \frac{u}{U_{\infty}} \right) U_{\infty}^2 dy$$

$$F_x = \int_0^1 \rho U_\infty^2 \delta \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) d\left(\frac{y}{\delta}\right) \quad \frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

$$= \rho U_\infty^2 \delta \int_0^1 (2\bar{y} - \bar{y}^2)(1 - 2\bar{y} + \bar{y}^2) d\bar{y}$$

$$= \rho U_\infty^2 \delta \left[ \int_0^1 (2\bar{y} - \bar{y}^2 - 4\bar{y}^2 + 2\bar{y}^3 + 2\bar{y}^3 - \bar{y}^4) d\bar{y} \right]$$

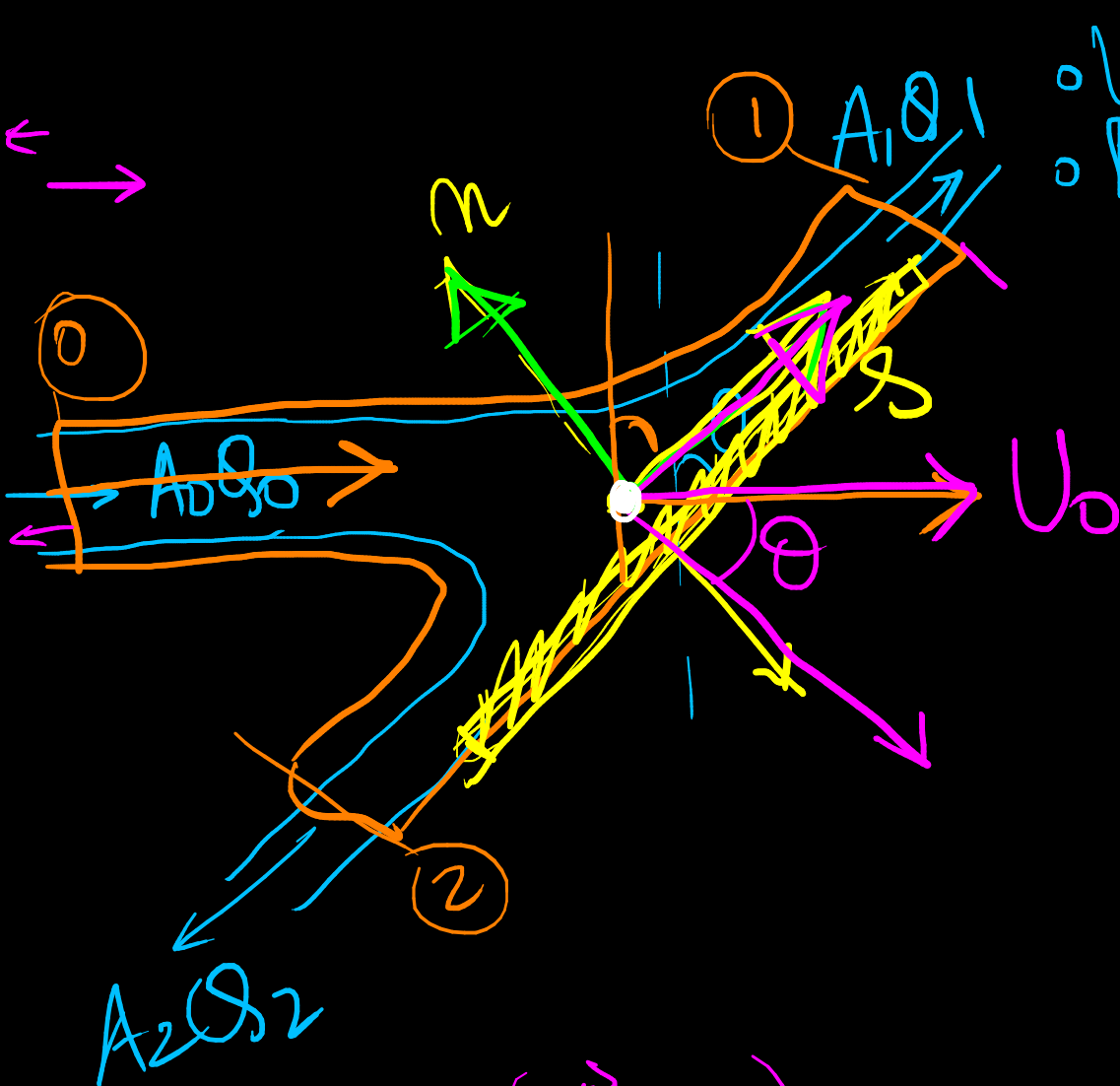
$$= \rho U_\infty^2 \delta \left[ \int_0^1 (2\bar{y} - 5\bar{y}^2 + 4\bar{y}^3 - \bar{y}^4) d\bar{y} \right]$$

$$= \rho U_\infty^2 \delta \left[ 1 - \frac{5}{3} + 1 - \frac{1}{6} \right] \left( \frac{1}{3} - \frac{1}{5} \right)$$

$$\boxed{F_x = \frac{2}{15} \rho U_\infty^2 \delta}$$







- Velocity uniform
  - Flow is inviscid / incompressible
- Momentum:

Vector form

$$\frac{d}{dt} \int \rho \vec{u} dV + \int \rho \vec{u} (\vec{u} - \vec{b}) \cdot \vec{n} dA = F_s \vec{e}_s + F_n \vec{e}_n$$

$$\int \rho \vec{u} \cdot \vec{n} dA$$

$$\textcircled{0}: \int \rho (U_0 \sin \theta \vec{e}_s - U_0 \cos \theta \vec{e}_n) (-U_0) dA = -\rho [U_0 \sin \theta \vec{e}_s - U_0 \cos \theta \vec{e}_n] U_0 A_0$$

$$\textcircled{1} \quad \underline{\rho U_1 \vec{e}_s} [\underline{U_1 A_1}]$$

$$\textcircled{2} \quad [-\rho U_2] \vec{e}_s [U_2 A_2]$$

$$\rho U_1 Q_1 \vec{e}_s - \rho U_2 Q_2 \vec{e}_s - \rho Q_0 U_0 \sin \theta \vec{e}_s + \rho Q_0 U_0 \cos \theta \vec{e}_n$$

$$= F_s \vec{e}_s + F_n \vec{e}_n$$

$$F_s = \rho U_1 Q_1 - \rho U_2 Q_2 - \rho Q_0 U_0 \sin \theta$$

$$F_n = \rho Q_0 U_0 \cos \theta$$

Cons. mass ( $\vec{b} = 0$ )

$$\frac{d}{dt} \int \rho dV + \int \rho (\vec{u} - \vec{b}) \cdot \vec{n} dA = 0$$

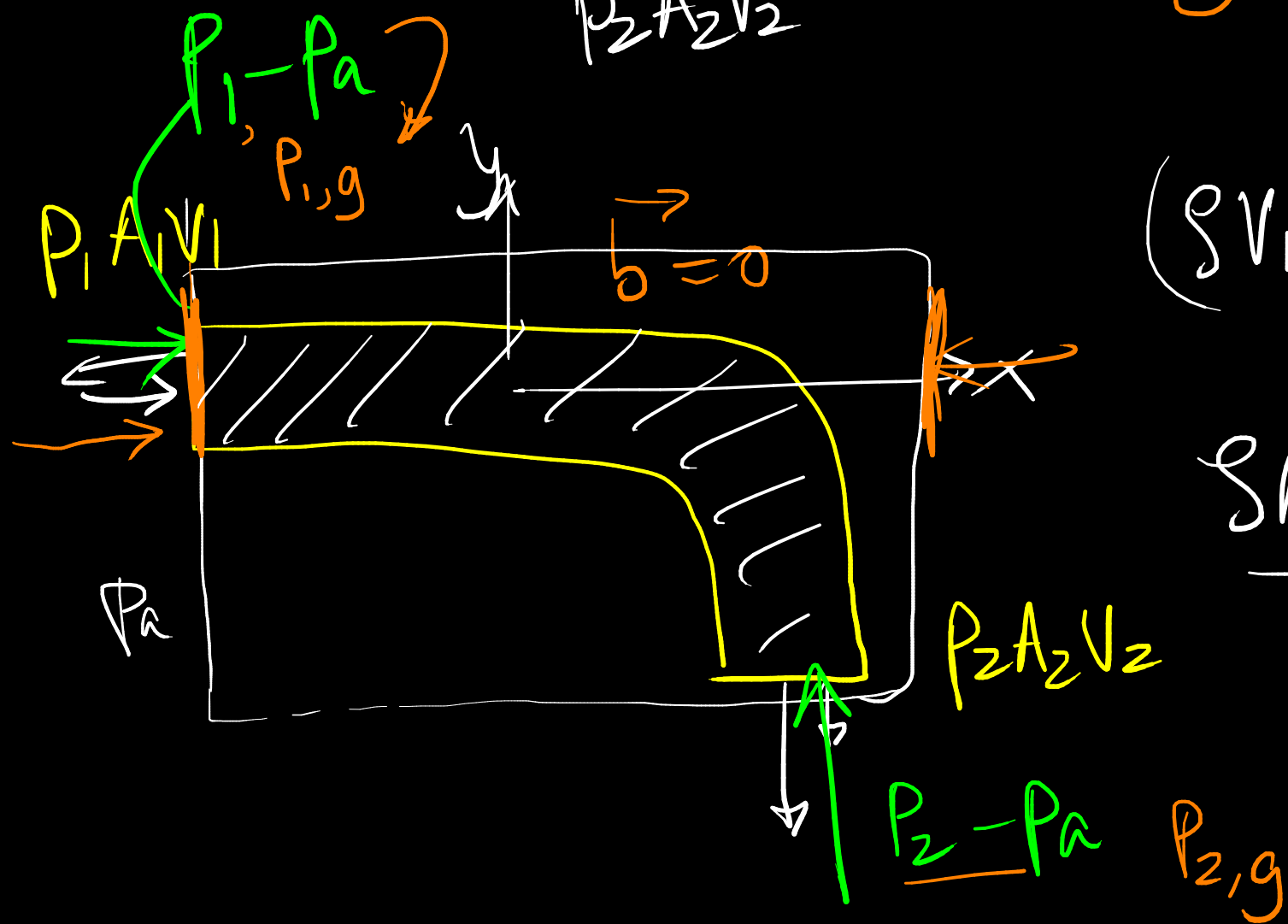
$$\int \rho \vec{u} \cdot \vec{n} dA = 0$$

$$-\rho Q_0 + \rho Q_1 + \rho Q_2 = 0$$

$$Q_1 + Q_2 = Q_0$$

$$A_1 V_1 + A_2 V_2 = A_0 V_0$$

A hand-drawn diagram of a thermodynamic cycle on a P-V diagram. The cycle is a closed loop with two isotherms (blue and red) and two adiabats (black). The cycle is labeled with  $P_1, A_1, V_1$  at the top left and  $P_2, A_2, V_2$  at the bottom right. A green vertical rectangle is drawn inside the cycle, and a blue arrow labeled  $W$  points downwards from the center of the cycle.



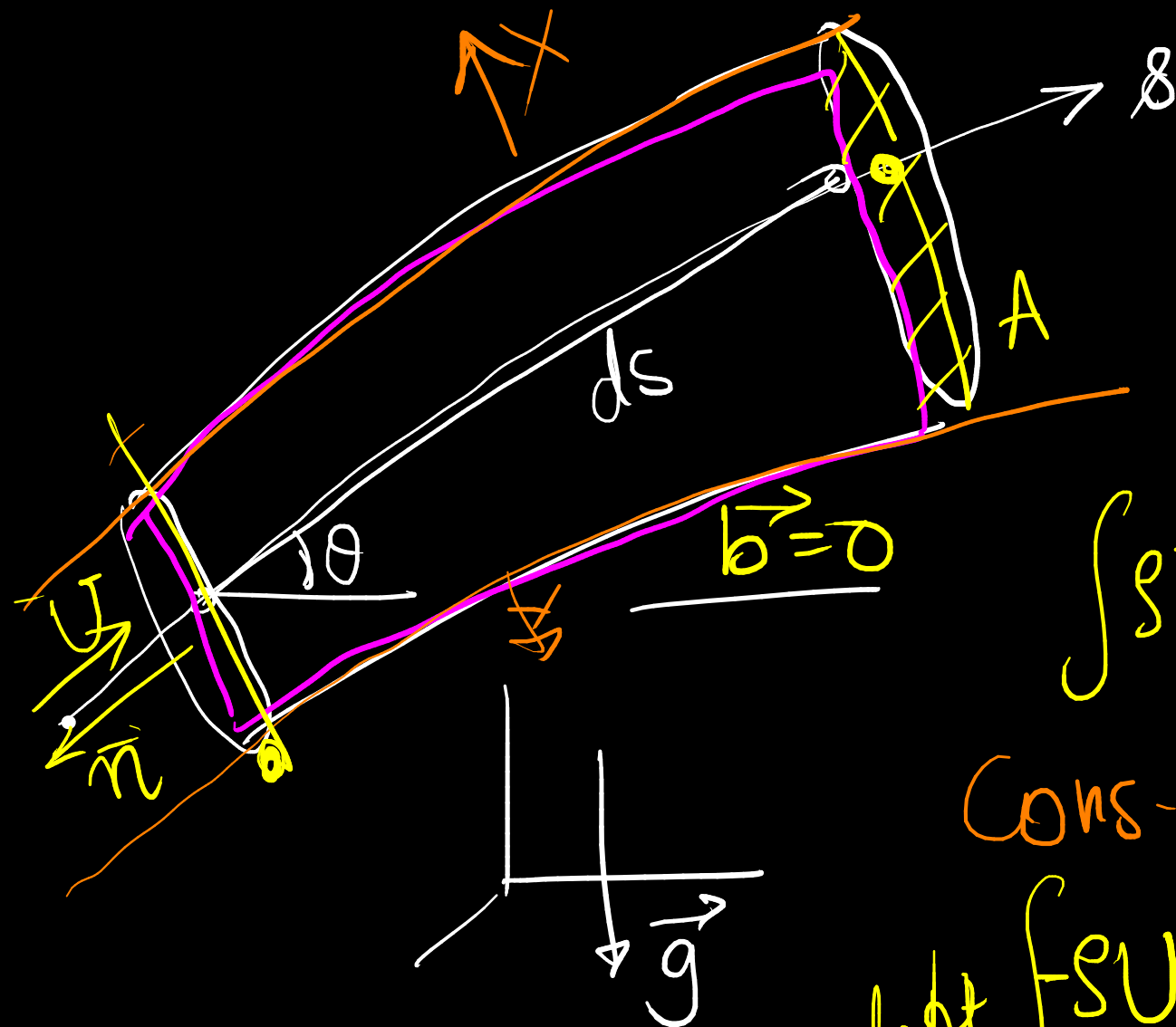
$$F_{\text{water, el.}}$$

$$\frac{d}{dt} \oint \vec{s} \cdot \vec{u} \, d\psi + \int \vec{s} \cdot \vec{u} \, \vec{u} \cdot \vec{n} \, dA = \vec{F}_{cv}$$

$$\oint \vec{s} \cdot \vec{u} \, \vec{u} \cdot \vec{n} \, dA = \vec{F}_{cv}$$

$$(SV_1 \vec{e}_x)(-V_1 A_1) + SV_2(-\vec{e}_y)(-V_2 A_2) = \vec{F}_{cv}$$

$$\begin{aligned} \underline{\rho A_2 V_2^2 (\vec{e}_y)} - \underline{\rho A_1 V_1^2 (\vec{e}_x)} &= \vec{F} \\ &= \underline{P_{1,g} A_1 \vec{e}_x + P_{2,g} A_2 \vec{e}_y} \\ &\quad + \cancel{F} + V \rho_w g (-\vec{e}_y) \end{aligned}$$



$$P_s + \frac{1}{2} u^2 + g z = \text{const.}$$

$$\frac{d}{dt} \int s dA + \int s (\vec{u} - \vec{b}) \cdot \vec{n} dA = 0 \Rightarrow \boxed{\int s \vec{u} \cdot \vec{n} dA = 0}$$

$$\int s \vec{u} \cdot \vec{n} dA = \int s \vec{g} dA + \int -p \vec{n} dA$$

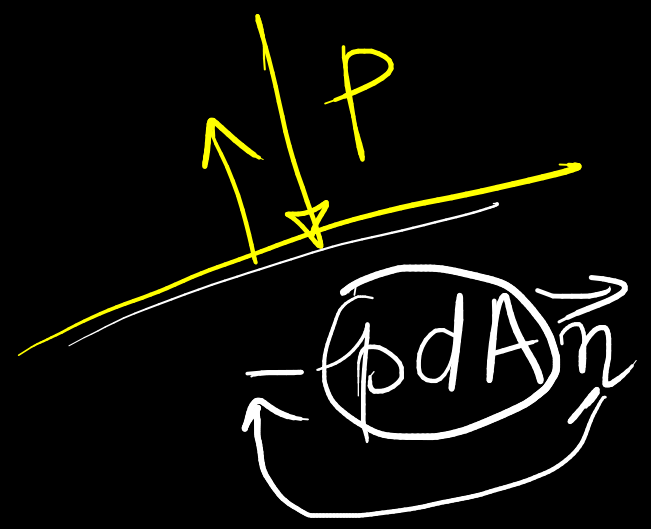
Cons. of mass

left face  $\int s u dA = -s u A$

$$\int s \left( u + \frac{\partial u}{\partial s} ds \right) dA \quad \begin{matrix} \text{A(s+ds)} \\ \downarrow \end{matrix}$$

$$= s \left( u + \frac{\partial u}{\partial s} ds \right) \left( A + \frac{\partial A}{\partial s} ds \right)$$

$$s \left( u + \frac{\partial u}{\partial s} ds \right) \left( A + \frac{\partial A}{\partial s} ds \right) - s u A = 0$$

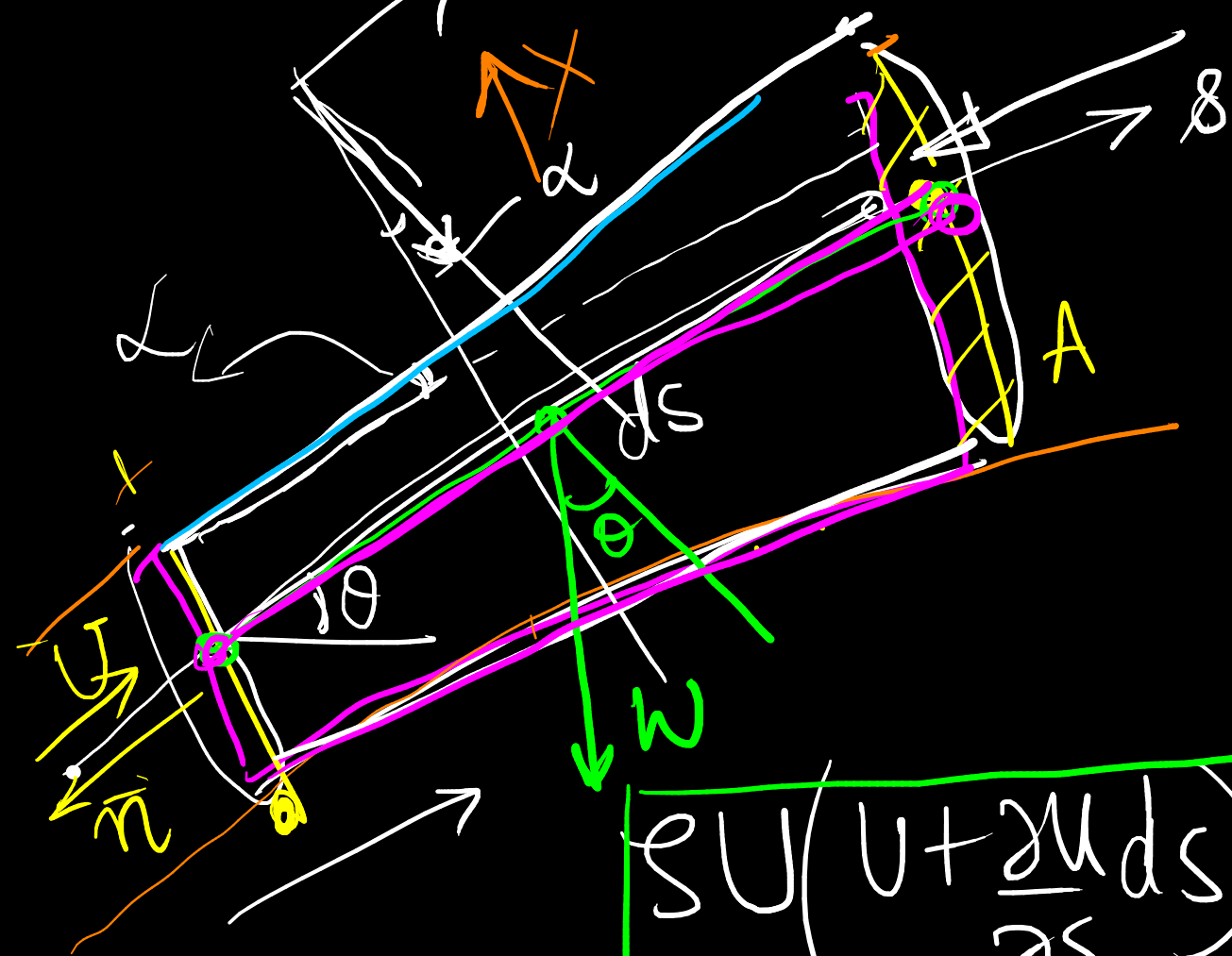
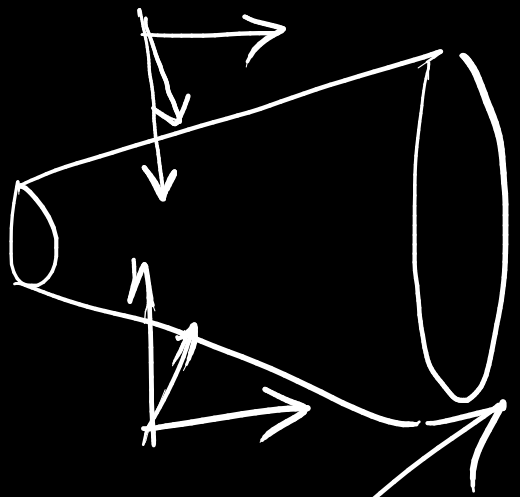




$$\vec{F}_{w,el}$$

$$= \rho A_2 V_2^2 (\vec{e}_y) - \rho A_1 V_1^2 \vec{e}_x - p_{1g} A_1 \vec{e}_x - p_{2g} A_2 \vec{e}_y + \rho_w g h_w \vec{e}_y$$

$$\vec{F}_{el,w} = -\vec{F}_{w,el}$$



$$\int \rho \vec{u} \cdot \vec{n} dA = \int \rho \vec{g} dV + \int -p \vec{n} dA$$

$$\int \rho U (-U) dA = -\rho U^2 A$$

$$\int \rho \left( U + \frac{\partial U}{\partial s} ds \right) \left( U + \frac{\partial U}{\partial s} ds \right) dA$$

$$= \rho \left( U + \frac{\partial U}{\partial s} ds \right)^2 \left( A + \frac{\partial A}{\partial s} ds \right)$$

$$= \rho \left( U + \frac{\partial U}{\partial s} ds \right) (U A)$$

$$\rho U \left( U + \frac{\partial U}{\partial s} ds \right) A - \rho U^2 A$$

$$= F_w + F_{pn} + F_{pp}$$

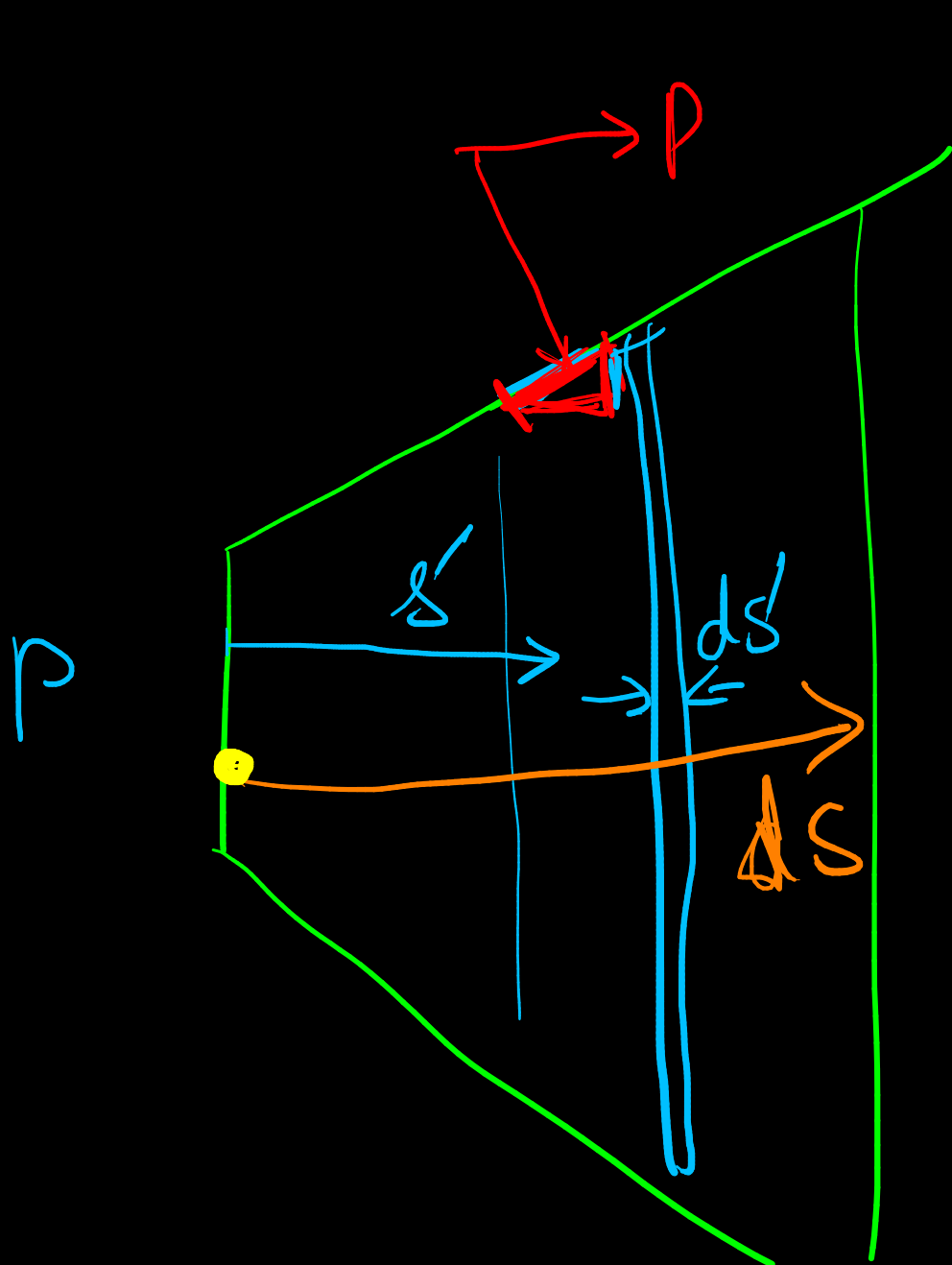
$$F_w = -\rho g ds \left( A + \frac{\partial A}{\partial s} \frac{ds}{2} \right) \sin \theta$$

$$\rho U A = \rho \left( U + \frac{\partial U}{\partial s} ds \right) \left( A + \frac{\partial A}{\partial s} ds \right)$$

$F_{pn}$

$$\int p dA = pA$$

$$-\int \left( p + \frac{\partial p}{\partial s} ds \right) dA = - \left( p + \frac{\partial p}{\partial s} ds \right) \left( A + \frac{\partial A}{\partial s} ds \right)$$



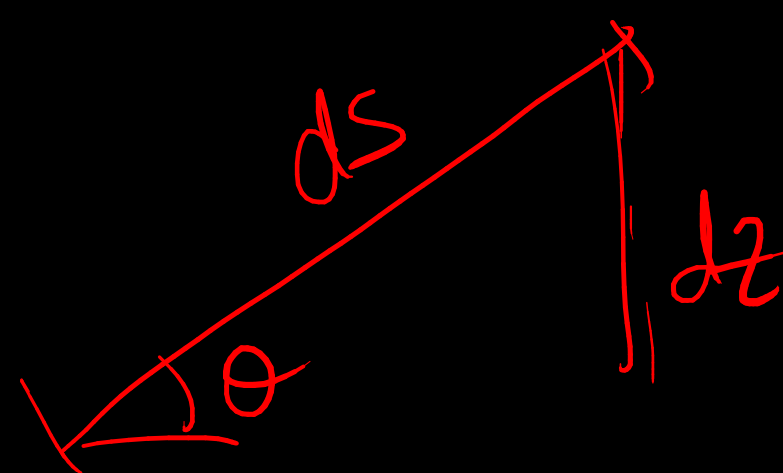
$$\int \underbrace{p}_{s'} \times \underbrace{dA}_{p_{s'}} = \left( p + \frac{\partial p}{\partial s'} s' \right)$$

$$ds' = ds$$

$$\frac{A_{s'+ds'} - A_{s'}}{ds} = \frac{\partial A}{\partial s'} ds'$$

$$\int_0^{ds} \left( p + \frac{\partial p}{\partial s'} s' \right) \left( \frac{\partial A}{\partial s'} ds' \right) = \int_0^{ds} p \frac{\partial A}{\partial s'} ds' + \int_0^{ds} \frac{\partial p}{\partial s'} \frac{\partial A}{\partial s'} s' ds'$$

$$= p \frac{\partial A}{\partial s'} ds + \frac{\partial p}{\partial s'} \frac{\partial A}{\partial s'} \frac{ds^2}{2}$$



$$\left( p + \frac{\partial p}{\partial s} \frac{ds}{2} \right) \frac{\partial A}{\partial s} ds = \left( p + \frac{\partial p}{\partial s} \frac{ds}{2} \right) \frac{\partial A}{\partial s} ds$$

$$\rho U \left( U + \frac{\partial U}{\partial s} ds \right) A - \rho U^2 A = \boxed{-\rho g \sin \theta \left( A + \frac{\partial A}{\partial s} \frac{ds}{2} \right) ds}$$

$$+ pA - \left( p + \frac{\partial p}{\partial s} ds \right) \left( A + \frac{\partial A}{\partial s} ds \right)$$

$$+ \left( p + \frac{\partial p}{\partial s} \frac{ds}{2} \right) \left( \frac{\partial A}{\partial s} ds \right)$$

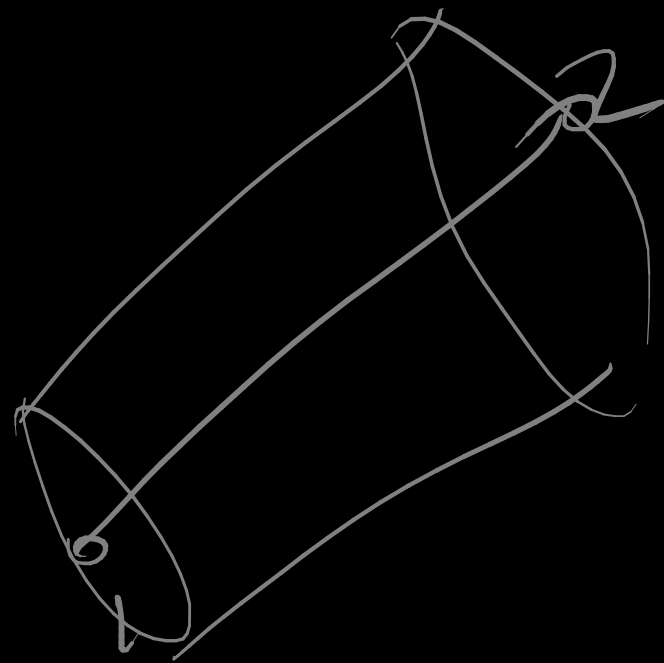
$$\rho U \frac{\partial U}{\partial s} ds A = -p \frac{\partial A}{\partial s} ds - \frac{\partial p}{\partial s} ds A - \frac{\partial p}{\partial s} \frac{\partial A}{\partial s} ds^2 - \rho g A dz - \rho g \frac{\partial A}{\partial s} \frac{ds}{2} dz$$



$ds \rightarrow 0$

$$\left( p \frac{\partial A}{\partial s} ds + \cancel{\frac{\partial A^2}{\partial s} \frac{dp}{ds} \frac{ds^2}{2}} \right)$$

$$\rho U \frac{\partial U}{\partial s} ds A = - \frac{\partial p}{\partial s} ds A - \cancel{p \frac{\partial A}{\partial s} ds} - \underline{\rho g A dz}$$



$$\int_1^2 U \frac{\partial U}{\partial s} ds$$

$$\int_1^2 \cancel{\left( \frac{1}{2} \frac{\partial p}{\partial s} ds + p \frac{\partial A}{\partial s} ds \right)}$$

$$= \int_1^2 - \frac{1}{\rho} \frac{\partial p}{\partial s} ds$$

$$- \int_1^2 g dz$$

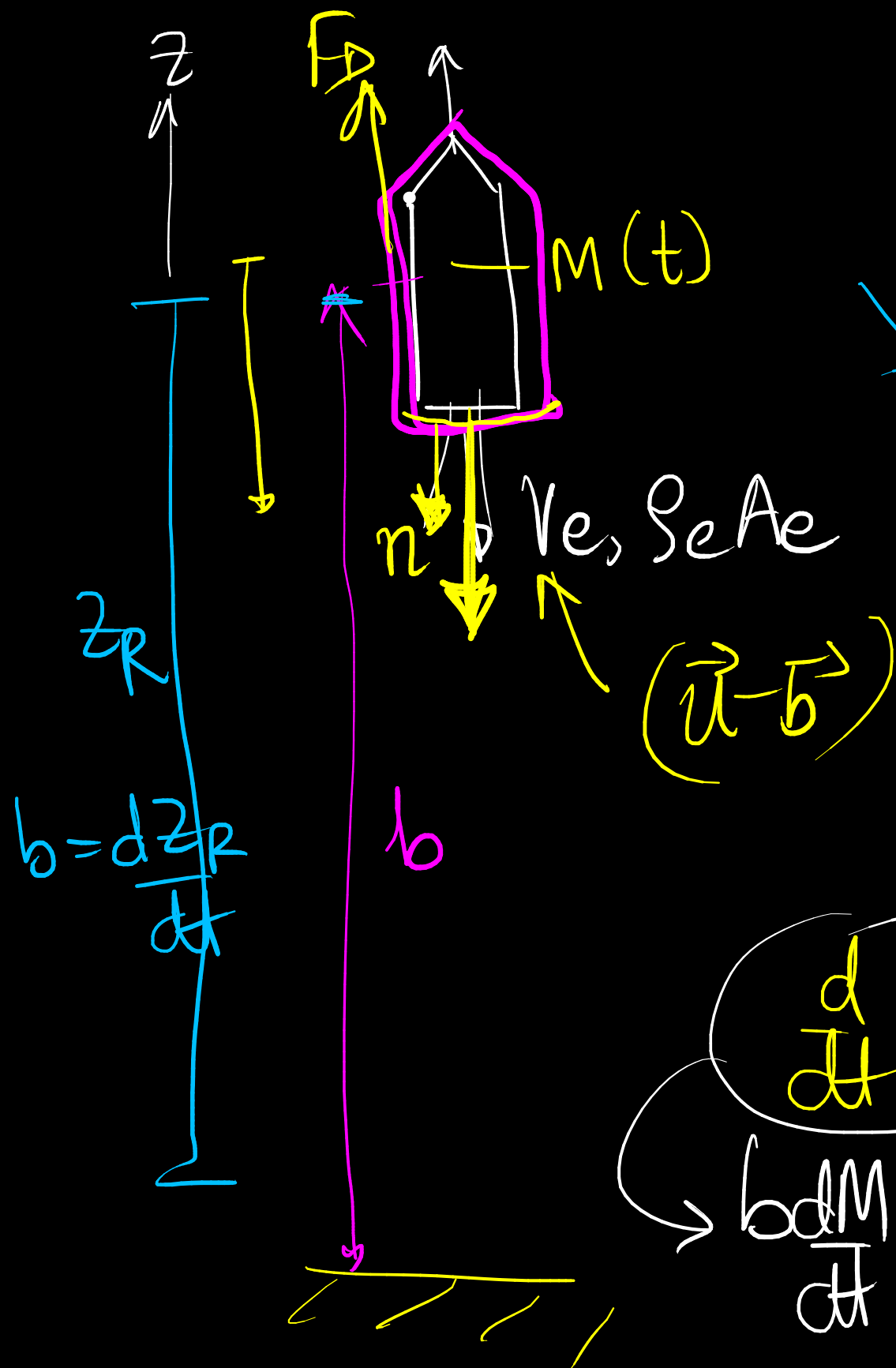
$$\left[ \frac{U^2}{2} \right]_1^2 =$$

$$- \frac{1}{\rho} (p_2 - p_1)$$

$$- g (z_2 - z_1)$$

$$\boxed{\frac{U^2}{2} + \frac{p}{\rho} + gz = \text{constant}}$$

← Bernoulli eqn.



DE, motion

$$\vec{b} = b \vec{e}_z$$

$$\vec{v}_{e,r} = \vec{v}_{e,a} - \vec{v}_z$$

$$\vec{v}_e = \vec{v}_{e,a} - \vec{b}$$

$$\vec{v}_{e,a} = \vec{v}_e + \vec{b}$$

$$= -v_e \vec{e}_z + b \vec{e}_z = (b - v_e) \vec{e}_z$$

cons. mass

$$\frac{d}{dt}(M) + \int \rho(\vec{u} - \vec{b}) \cdot \vec{n} dA = 0$$

$$\frac{dM}{dt} + \rho v_e A_e = 0$$

$$\frac{d}{dt}(Mb) + \int \rho u_z (\vec{u} - \vec{b}) \cdot \vec{n} dA = -Mg + F_D$$

$$\frac{d}{dt}(Mb) +$$

$$\rho(b - v_e) v_e A_e = -Mg + F_D$$

$$\begin{aligned} b \frac{dM}{dt} + M \frac{db}{dt} &= M \frac{db}{dt} - \cancel{b \rho v_e A_e} + \cancel{\rho b v_e A_e} - \rho v_e^2 A_e \\ &= M \frac{db}{dt} - \rho v_e^2 A_e = -Mg + F_D \end{aligned}$$

CV: Force  $F$

$$\frac{d}{dt} \int \rho dV + \int \rho \vec{u} \cdot \vec{n} dA = 0$$

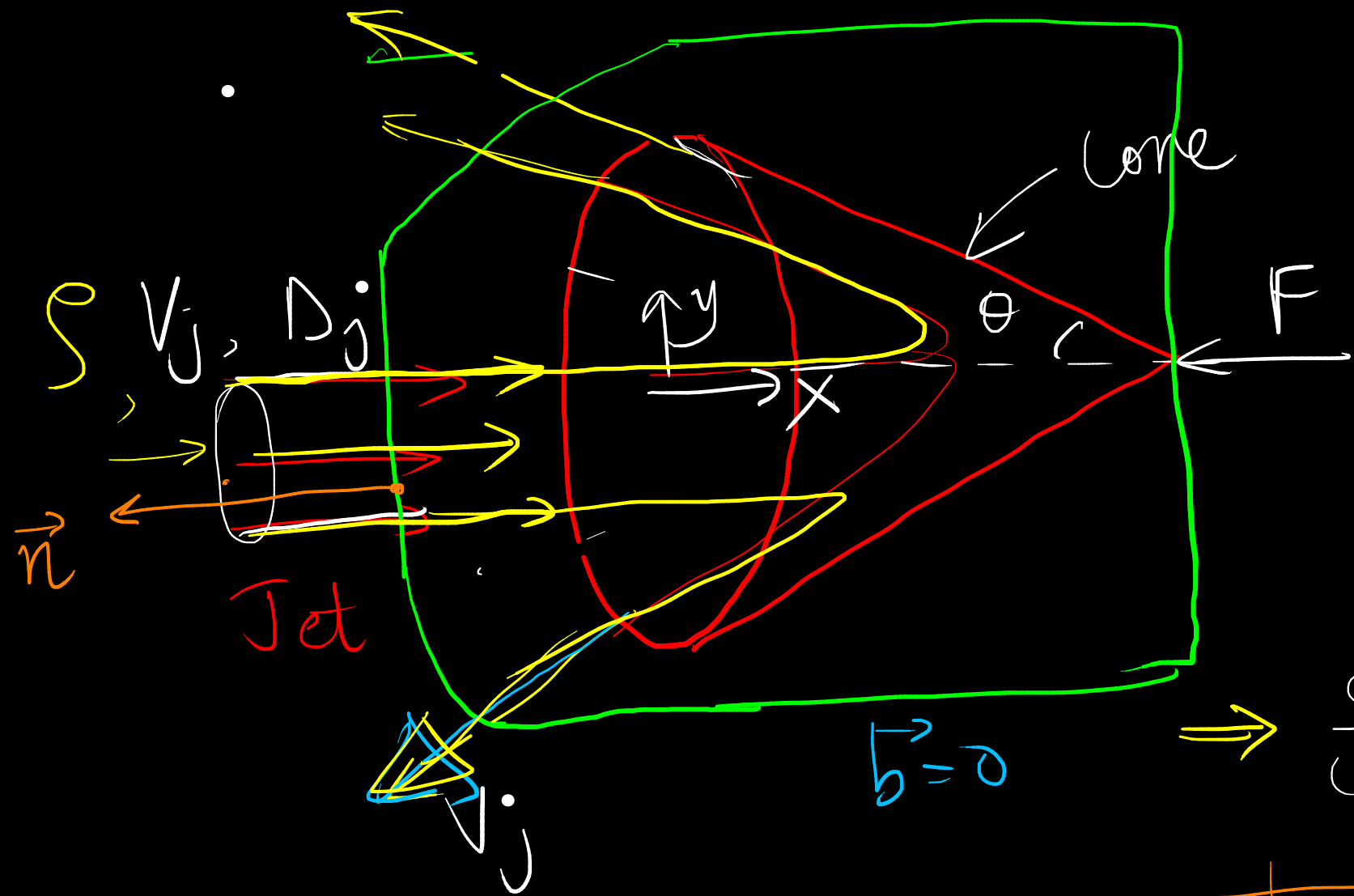
$$\int_{in} S \underbrace{\vec{u} \cdot \vec{n}}_{\text{in}} dA + \int_{ext} S \vec{u} \cdot \vec{n} dA = 0$$

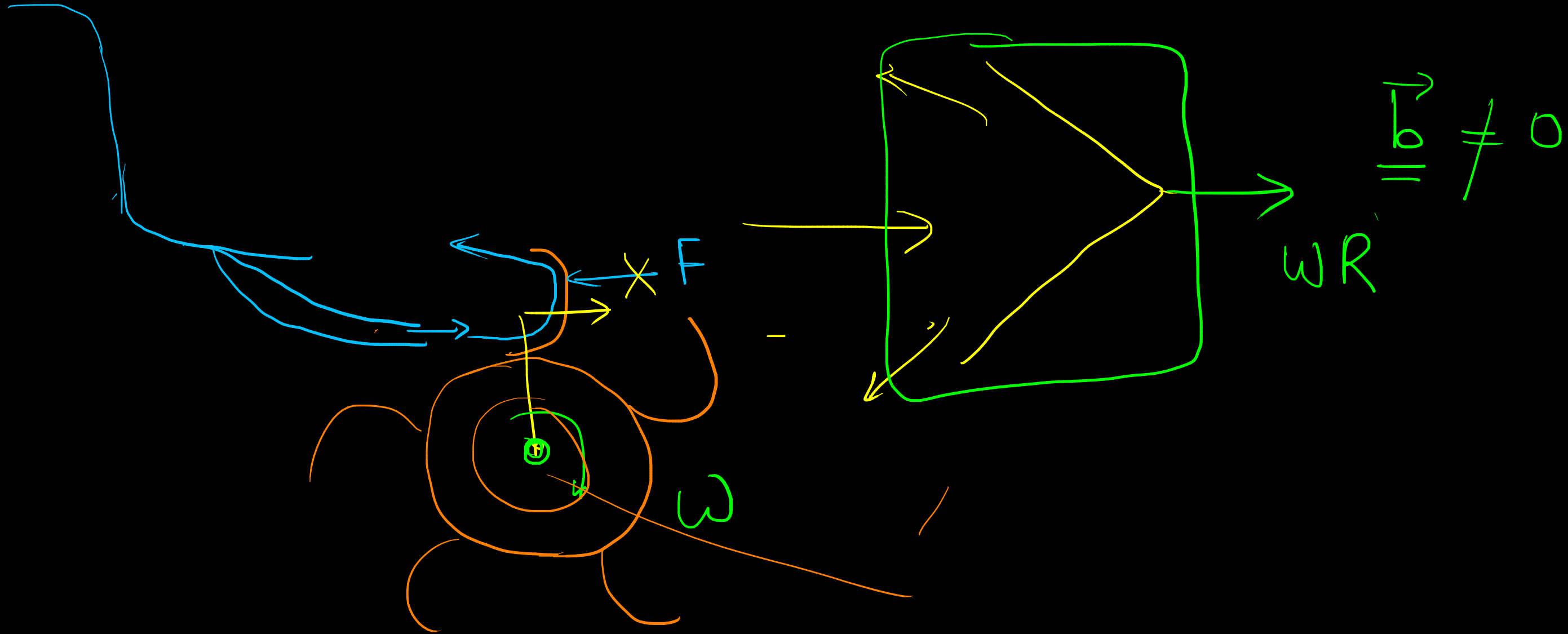
$$-S V_j A_j + \dot{m}_{\text{exit}} = 0$$

$$\frac{d}{dt} \int \rho \vec{u} dV + \int \rho \vec{u} \cdot \vec{n} dA = -F \vec{e}_x$$

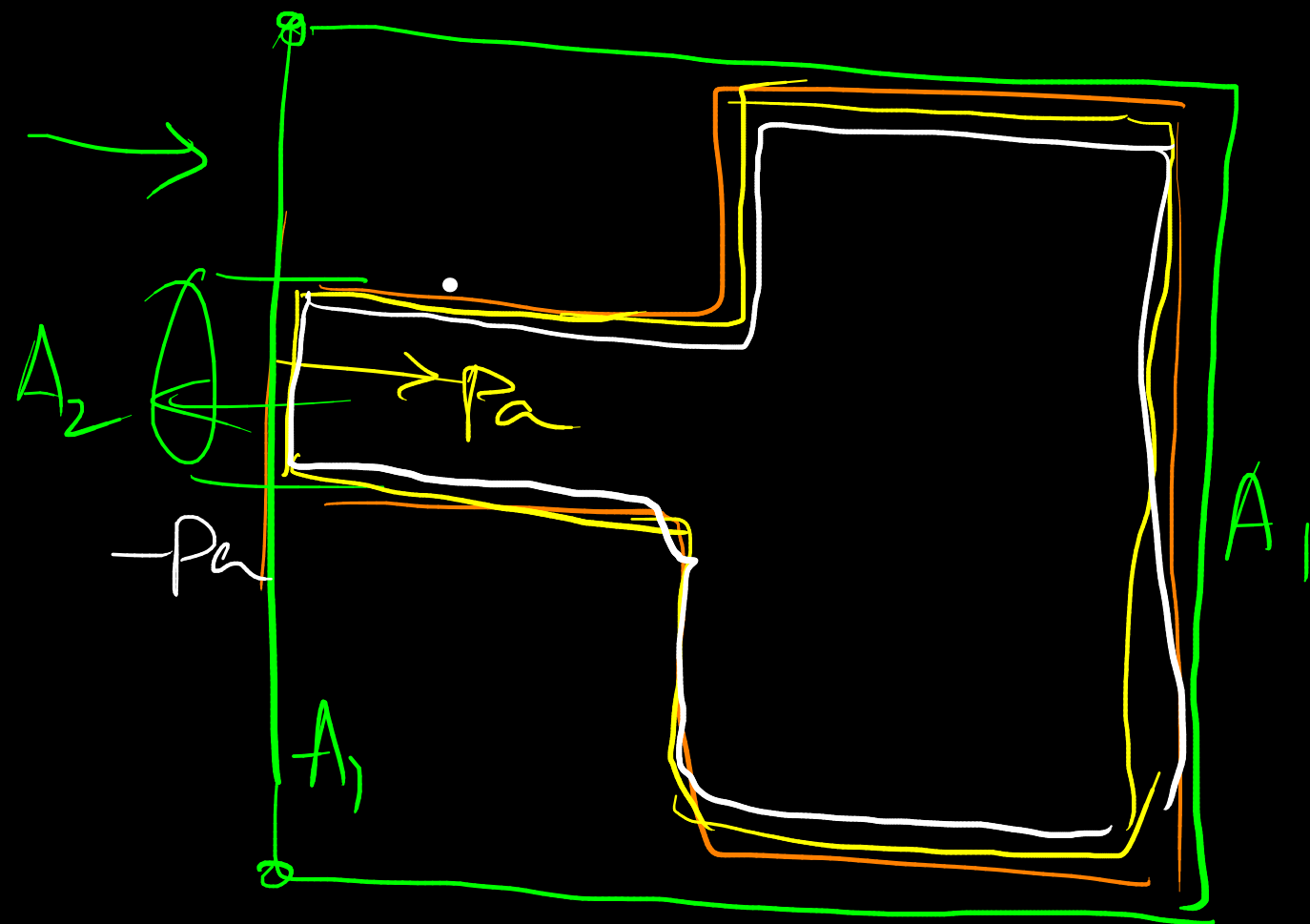
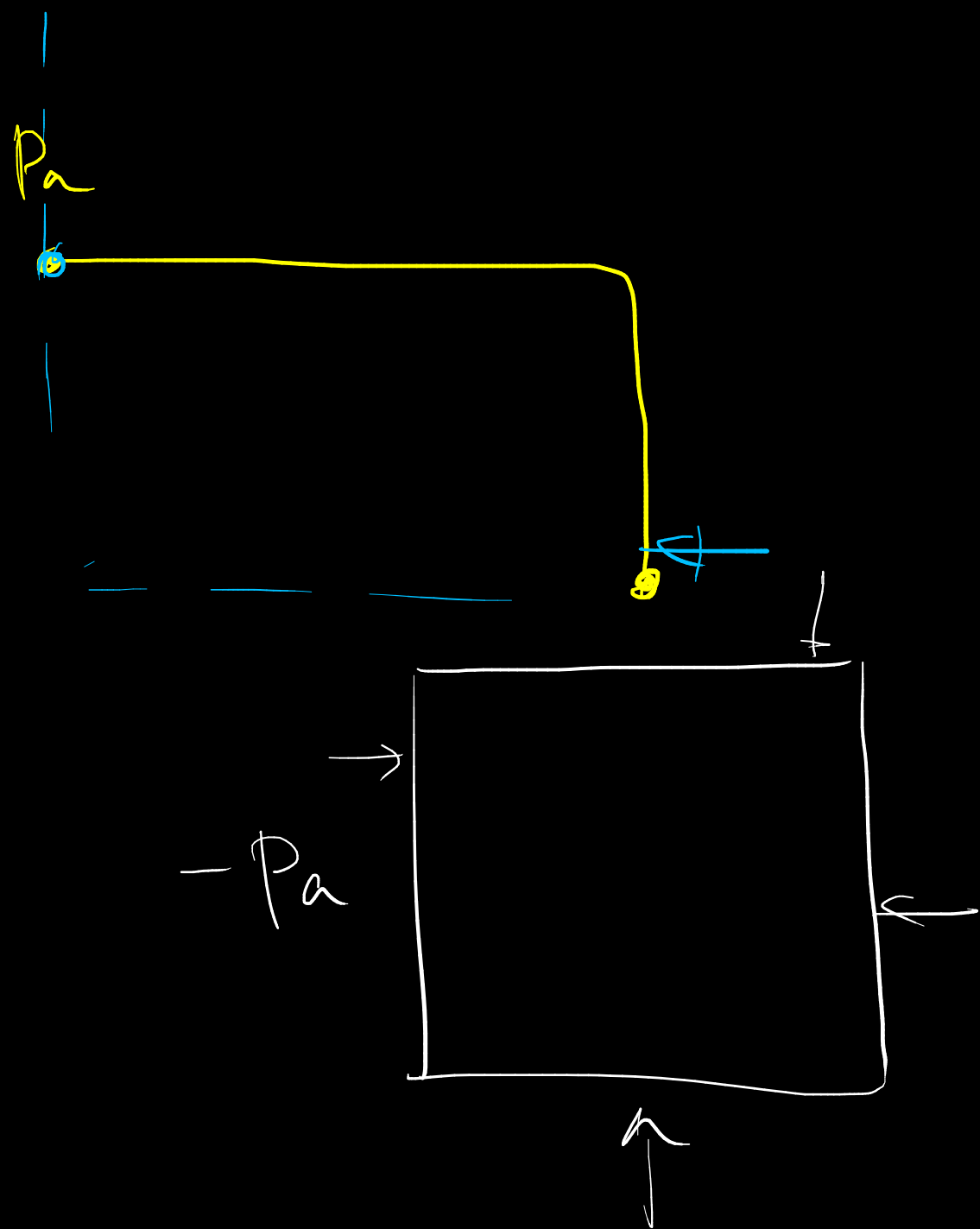
$$S V_j^2 A_j (1 + \cos \theta) = F \leftarrow$$

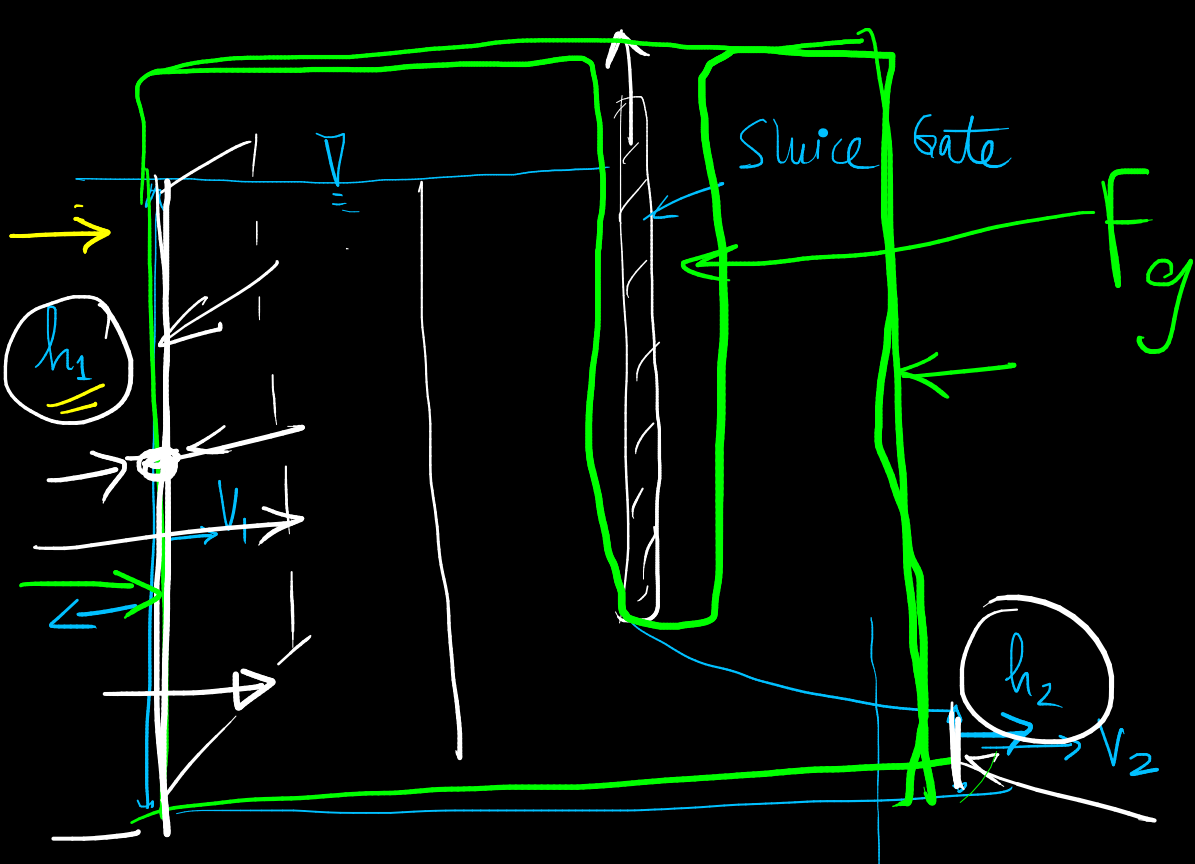
$$\int \rho u \vec{u} \cdot \vec{n} dA = -F$$











CV:  $F_g$   $P_e$   $P_r$   $F_g$   $-F_{cv}$   $F$   $F_{cv} + F = 0$   
Stationary  $F = F_{cv}$   
 Statics for gate

$$\frac{d}{dt} \int \rho dV + \int \rho \vec{u} \cdot \vec{n} dA = 0$$

$$F_{gf} - F = 0 \quad \times$$

$$-\rho V_1 h_1 w + \rho V_2 h_2 w = 0$$

$$V_1 h_1 = V_2 h_2$$

$$\frac{V_1}{V_2} = \frac{h_2}{h_1}$$

$$\frac{h_1}{h_2} = \frac{V_2}{V_1}$$

$$\frac{d}{dt} \int \rho \vec{u} dV + \int \rho \vec{u} \cdot \vec{n} dA = F + F_p \quad \text{Hydrostatics}$$

$$-\rho V_1^2 h_1 w + \rho V_2^2 h_2 w = F_{cv} + \frac{1}{2} h_1 w \rho g h_1 - \frac{1}{2} h_2 w \rho g h_2$$

$$F = \rho V_2^2 h_2 w \left[ 1 - \frac{V_2^2}{V_1^2} \frac{h_1}{h_2} \right] + \frac{1}{2} h_2 w \rho g h_2 \left( 1 - \frac{h_1^2}{h_2^2} \right)$$

$$F = \rho V_2^2 h_2 w \left[ 1 - \frac{V_2^3}{V_1^3} \right] + \frac{1}{2} w \rho g h_2^2 \left( 1 - \frac{h_2^2}{V_1^2} \right)$$

$$\frac{\partial F}{\partial h} =$$

$V_1^2 \ll g h_1 \iff$  Pressure dominates momentum flux

