Partial Differential Equations.

Solution Using Laplace and Fourier Transform Techniques.

Let $z = f(x,y)$ be a function of two variables x and y. We know the partial derivatives of z w.r.t x & y as

$z_x, \; z_y, \; z_{xx}, \; z_{xy}, \; z_{yy}$, etc..

Formation of p.d.es:

<u>Ex1</u>: Obtain the p.d.e whose solution is given by the surface $z = f(x^2-y^2)$.

<u>Sol</u>: $p := \frac{\partial z}{\partial x} = 2x \cdot f'(x^2-y^2)$

$q := \frac{\partial z}{\partial y} = -2y \cdot f'(x^2-y^2)$

$\therefore \quad p/q = -x/y \Rightarrow yp + xq = 0$

is the p.d.e whose solution is $z = f(x^2-y^2)$.

<u>Ex2</u>: S.T. $z = f(x+at) + g(x-at)$ is the solution of $z_{tt} = a^2 z_{xx}$.

Sol: $p = f' + g'$; $q = af' - ag'$

$r := z_{xx} = f'' + g''$; $t := z_{tt} = a^2(f''+g'')$

$\Rightarrow a^2 z_{xx} = a^2 z_{tt}$.

Classification of 2^{nd} order linear p.d.e:

Consider $A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x$
$+ E u_y + F u = G(x,y)$ —— (1).

where $A, B, \ldots, G$ are functions of x & y.

Equation (1) is classified as the parabolic or Hyperbolic or Elliptic p.d.e as below.

For all x, y in the domain of the p.d.e

if $B^2 - AC = 0$: p.d.e is PARABOLIC
if $B^2 - AC > 0$: p.d.e is Hyperbolic
if $B^2 - AC < 0$: p.d.e is ELLIPTIC.

(For more details, see any Standard Text book on p.d.es).

Ex: Classify the p.d.e $u_{xx} + 2x u_{xy} + (1-y^2) u_{yy} = 0$

Sol: Here $A = 1$, $B = x$, $C = 1-y^2$.

$B^2 - AC = x^2 + y^2 - 1$.

For all (x,y), on $x^2+y^2 = 1$, the p.d.e is parabolic.

For all (x,y) inside $x^2+y^2 = 1$, the p.d.e is elliptic
For all (x,y) outside $x^2+y^2 = 1$, the p.d.e is hyperbolic

Ex 2: One dimensional Heat Conduction problem.

$T = T(x, y)$: Temperature in the thin rod.

k : Thermal conductivity of the material.

$$\frac{\partial T}{\partial t} = k \cdot \frac{\partial^2 T}{\partial x^2} \quad \text{—— (2)}$$

$A = k, \quad B = 0, \quad C = 0; \quad B^2 - AC = 0.$

(2) is an Initial Boundary value problem.
It needs 1 initial and 2 boundary conditions to have a particular solution.

Ex 3: Wave propagation in a thin String.

$u = u(x, y)$: Displacement of the particle in the String.

c : wave constant.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (3)}$$

$A = 1, \quad B = 0, \quad C = -c^2; \quad B^2 - AC > 0.$

(3) is an initial boundary value problem (ibvp).
This needs 2 initial and 2 boundary conditions to have a particular solution.
It is given by $u(x,t) = f(x+ct) + f(x-ct)$.

Ex 4: Potential Problems: Laplace equation.

$\varphi = \varphi(x, y)$ is the potential function.

Laplace equation: $\dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \varphi}{\partial y^2} = 0.$ — (4)

Poisson equation: $\dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \varphi}{\partial y^2} = f(x, y).$ — (5)

$A = 1$, $B = 0$, $C = 0$, $B^2 - AC < 0$.

This is a pure boundary value problem.

This requires 2 boundary conditions w.r.t 'x' and 2 boundary conditions w.r.t 'y' to have a particular solution.

Laplace transform of partial derivatives.

Let $y = y(x,t)$, $t \geq 0$.

A) $\mathcal{L}\left[\frac{\partial y}{\partial t}\right] = s\,\tilde{y}(x,s) - y(x,0)$.

B) $\mathcal{L}\left[\frac{\partial y}{\partial x}\right] = \frac{d}{dx}\tilde{y}(x,s)$.

C) $\mathcal{L}\left[\frac{\partial^2 y}{\partial t^2}\right] = p^2\tilde{y}(x,p) - p\,y(x,0) - \frac{\partial y}{\partial t}(x,0)$.

D) $\mathcal{L}\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d^2}{dx^2}\tilde{y}(x,s)$.

Here $\mathcal{L}[y(x,t)] = \tilde{y}(x, s \text{ or } p)$.

Fourier Transform of partial derivatives.

$z = z(x,t)$; $\mathcal{F}[z(x,t)] = \tilde{Z}(s,t)$.

Fourier transform w.r.t 'x'.

E)
$$\mathcal{F}\left[\frac{\partial^2 z}{\partial x^2}\right] = -s^2\,\mathcal{F}[z(x,t)]; \quad \left(\begin{matrix}-\infty < x < \infty \\ t > 0\end{matrix}\right)$$

Provided both $z(x,t)$ and $\frac{\partial z}{\partial x}(x,t) \to 0$ as $|x| \to \infty$.

F) $\mathcal{F}_s\left[\frac{\partial^2 z}{\partial x^2}\right] = s\,\tilde{Z}(0,t) - s^2\,\mathcal{F}_s[z(x,t)]$, provided

(Fourier Sine Transform). $\left(\begin{matrix}0 < x < \infty \\ t > 0\end{matrix}\right)$

both $z(x,t)$ and $\frac{\partial z}{\partial x}(x,t) \to 0$ as $x \to \infty$.

G) $\mathcal{F}_c\left[\frac{\partial^2 z}{\partial x^2}\right] = -\frac{\partial z}{\partial x}(0,t) - s^2\,\mathcal{F}_c[z(x,t)]$, $0 < x < \infty$, $t > 0$

Provided $z(x,t) \to 0$ as $x \to \infty$. (Fourier Cosine Transform).

*Note: For semi_infinite domains $(0 < x < \infty)$, result (F) is useful for Dirichlet problems and result (G) is useful for the Neumann problems.

Ex 5: Solve $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$; $t > 0$, $x \geq 0$;

$y(x,0) = 6e^{-3x}$, $y(x,t)$ is bounded for all $t > 0$, $x > 0$.

Solution: Let us use the Laplace transform tech:

Let $\mathcal{L}[y(x,t)] = \tilde{y}(x,p)$.

$$\mathcal{L}\left[\frac{\partial y}{\partial x}\right] = \mathcal{L}\left[2\cdot\frac{\partial y}{\partial t}\right] + \mathcal{L}[y(x,t)]$$

$$\Rightarrow \frac{d\tilde{y}(x,p)}{dx} = 2\cdot[p\tilde{y}(x,p) - y(x,0)] + \tilde{y}(x,p)$$

or $\tilde{y}' - (2p+1)\tilde{y} = -12e^{-3x}$.

$$\Rightarrow \tilde{y} = ce^{(2p+1)x} + \frac{6}{p+2}e^{-3x}$$

where c is arb. constr.

Given $y(x,t)$ is bounded for all $t > 0$ & $x > 0$

$\Rightarrow$ $\tilde{y}(x,p)$ should also be bdd for $\forall p$ & $x \geq 0$.

$\Rightarrow$ $c = 0$.

$$\therefore \tilde{y}(x,p) = \frac{6}{p+2}e^{-3x}$$

On inversion, we get

$$\mathcal{L}^{-1}[\tilde{y}(x,p)] = y(x,t) = 6e^{-2t-3x}.$$

1. Solution of heat conduction problem by Laplace Transform:

A semi-infinite solid $x > 0$ is initially at temperature zero. At time $t = 0$, a constant temperature u_0 is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid and at any time $t > 0$.

Sol:

$u(x,t)$: temp at any point x and at any t.

4a)

Now solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $x > 0, t \geq 0$ $\longrightarrow (1)$

s.t :

$u = 0$ when $t = 0$ $\forall x \geq 0$ I.C $\longrightarrow (2)$

$u = u_0$ when $x = 0$, $\forall t$ $\longrightarrow (3)$ B.C.

u is finite $\forall x$ & $\forall t$ $\longrightarrow (4)$.

Multiplying eq (1) by e^{-st} and integrate w.r.t 't' from 0 to ∞,

$$\int_0^\infty \frac{\partial u}{\partial t} e^{-st} . dt = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} . e^{-st} dt$$

$$\text{or } \underbrace{e^{-st} . u \Big|_0^\infty}_{0} - \int_0^\infty -s . e^{-st} . u . dt = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} . e^{-st} . dt.$$

or $\quad s\bar{u} = c^2 \cdot \dfrac{d^2 \bar{u}}{dx^2} \longrightarrow (5)$

Similarly apply L-T on eq (3), we get

$$\int_0^\infty u \cdot e^{-st} \cdot dt = \int_0^\infty u_0 \cdot e^{-st} \cdot dt. \qquad \underline{\text{at } x=0}$$

$$\bar{u} = \frac{u_0}{s}. \longrightarrow (6)$$

or

$$\text{Sol. of (5)} \Rightarrow \quad \bar{u} = A e^{\frac{\sqrt{s}}{c} x} + B e^{-\frac{\sqrt{s}}{c} x}. \longrightarrow \underline{(7)}$$

u is finite at $x \to \infty$

$$\Rightarrow \quad A = 0.$$

$$(7) \Rightarrow \quad \bar{u} = B e^{-\frac{\sqrt{s}}{c} x}.$$

$$(6) \Rightarrow \quad \frac{u_0}{s} = B \cdot e^{-\frac{\sqrt{s}}{c} \cdot 0} = B.$$

$$\therefore \quad \bar{u} = \frac{u_0}{s} e^{-\frac{\sqrt{s}}{c} x}$$

Inverting this we get

$$u(x,t) = u_0 \left(1 - \operatorname{erf} \frac{x}{2c\sqrt{t}}\right).$$

Show that $\quad L\left[1 - \operatorname{erf} \dfrac{x}{2c\sqrt{t}}\right] = \dfrac{1}{s} e^{-\frac{\sqrt{s}}{c} x}.$

Solution of wave equation by Laplace Transforms:

Ex: An infinitely long string having one end at $x=0$ is initially at rest along the x-axis. The end $x=0$ is given a transverse displacement $f(x)$ when $t \geq 0$. Find the displacement of any point of the string at any time.

Sol: Let $y(x,t)$ be the displacement, then

the wave equation is:

(8) $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

S.t:

$$y(x,0) = 0 \longrightarrow (2) \quad \text{I.C.}$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \longrightarrow (3) \quad \text{I.C}$$

$$y(0,t) = f(t) \longrightarrow (4) \quad \text{B.C}$$

$$y(x,t) \text{ is bounded.} \longrightarrow (5). \text{ B.C.}$$

Now $L[(1)] \Rightarrow$

$$s^2 \tilde{y} - s\,y(x,0) - \frac{\partial y}{\partial t}(x,0) = c^2 \frac{d^2 \tilde{y}}{dx^2} \longrightarrow (6)$$

$y(x,0) = 0$, $\frac{\partial y}{\partial t}(x,0) = 0 \Rightarrow$

$$s^2 \tilde{y} = c^2 \frac{d^2 \tilde{y}}{dx^2}$$

or

$$\frac{d^2 \tilde{y}}{dx^2} = \left(\frac{s}{c}\right)^2 \tilde{y}. \longrightarrow (7)$$

$$\Rightarrow \tilde{y} = A e^{\frac{sx}{c}} + B e^{-\frac{sx}{c}} \longrightarrow (8)$$

L.T of (4) $\Rightarrow$ $\tilde{y}(0, s) = \tilde{f}(s)$ at $x = 0$.

(5) $\Rightarrow$ $A = 0$.

$\therefore \quad \tilde{y} = B e^{-\frac{sx}{c}}$

Using above condition, $\tilde{y} = \tilde{f}(s) e^{-\frac{sx}{c}}$.

Now $y = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{(t - \frac{x}{c})s} \cdot \tilde{f}(s)\, ds$.

<u>Complex inversion formula</u>

$\therefore \; y(x,t) = f(t - \frac{x}{c})$.