Solution Model of Assignment 3

Q1) The error function is defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx.$$

Then show that $\mathcal{L}\{\operatorname{erf}(\sqrt{t})\} = \frac{1}{s\sqrt{s+1}}$.

Sol^n:- We have,

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\, dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^5}{3!} + \dots\right) dx\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7 \cdot 3!} + \dots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \dots\right]$$

Taking Laplace transform of both the sides, we get

$$\mathcal{L}\{\operatorname{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!\, s^{7/2}} - \frac{\Gamma(9/2)}{7 \cdot 3!\, s^{9/2}} + \dots\right]$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{s^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{1}{s^3} + \dots\right]$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s(s+1)^{1/2}}.$$

Q2) Evaluate $\int_0^t J_0(u)\, J_0(t-u)\, du$.

Sol^n:- Let $\mathcal{L}\{f(t)\} = \mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}} = F(s)$.

Then $\int_0^t f(u) f(t-u)\, du = \mathcal{L}^{-1}\{F(s) \cdot F(s)\}$

or, $\int_0^t J_0(u)\, J_0(t-u)\, du = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}\sqrt{s^2+1}}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

Q3) Using the Laplace Transform technique, solve the following o.d.eq^n:

(i) $y''(t) + y(t) = 0$, $y(0) = 1$, $y'(0) = 0$.

Sol^n:- The given d.eq^n is $y''(t) + y(t) = 0$

Taking the Laplace Transform of both sides, we get

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 0$$

$\left[\text{Here, } \mathcal{L}\{y(t)\} = Y(s)\right]$

$$\Rightarrow \left[s^2 \mathcal{L}\{y\} - s\underbrace{y(0)}_{(=1)} - \underbrace{y'(0)}_{(=0)}\right] + \mathcal{L}\{y\} = 0.$$

$$\Rightarrow s^2 Y(s) - s \cdot 1 + Y(s) = 0$$

$$\Rightarrow (s^2+1)\, Y(s) = s \Rightarrow Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t,$$

which is the required solution.

(ii) $y''(t)+4y'(t)+4y(t)=\sin(\omega t),\ t>0,\ y(0)=y_0$
$y'(0)=y_1$.

Sol^n:- Taking the Laplace transform of both sides of the given equation, we get

$$\mathcal{L}\{y''(t)\}+4\mathcal{L}\{y'(t)\}+4\mathcal{L}\{y(t)\}=\mathcal{L}\{\sin(\omega t)\}$$

$$\Rightarrow \left[s^2\mathcal{L}\{y\}-s\underbrace{y(0)}_{(=y_0)}-\underbrace{y'(0)}_{(=y_1)}\right]+4\left[s\mathcal{L}\{y\}-\underbrace{y(0)}_{(=y_0)}\right]+4\mathcal{L}\{y\}=\frac{\omega}{s^2+\omega^2}$$

$\left[\text{here } \mathcal{L}\{y(t)\}=Y(s)\right]$

$$\Rightarrow s^2Y(s)-sy_0-y_1+4sY(s)-4y_0+4Y(s)=\frac{\omega}{(s^2+\omega^2)}$$

$$\Rightarrow \underbrace{(s^2+4s+4)}_{=(s+2)^2}Y(s)=sy_0+y_1+4y_0+\frac{\omega}{(s^2+\omega^2)}$$

$$\Rightarrow Y(s)=\frac{sy_0}{(s+2)^2}+\frac{(y_1+4y_0)}{(s+2)^2}+\frac{\omega}{(s+2)^2(s^2+\omega^2)}$$

$$\Rightarrow y(t)=y_0\mathcal{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2}\right\}+(y_1+4y_0)\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$+\omega\mathcal{L}^{-1}\left\{\frac{1}{(4+\omega^2)(s+2)^2}+\frac{4}{(4+\omega^2)^2(s+2)}+\frac{-4s+4-\omega^2}{(4+\omega^2)^2(s^2+\omega^2)}\right\}.$$

[By Partial Fraction]

$$=y_0e^{-2t}\mathcal{L}^{-1}\left\{\frac{s-2}{s^2}\right\}+(y_1+4y_0)\cdot e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$+\frac{\omega e^{-2t}}{(4+\omega^2)}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{4\omega e^{-2t}}{(4+\omega^2)^2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{4\omega}{(4+\omega^2)^2}.$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}+\frac{(4-\omega^2)\omega}{(4+\omega^2)^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\}.$$

$$= e^{-2t}\left[y_0(1-2t) + (y_1+4y_0)t + \frac{\omega}{(4+\omega^2)}t + \frac{4\omega}{(4+\omega^2)^2}\right]$$

$$-\frac{4\omega}{(4+\omega^2)^2}\cos\omega t + \frac{(4-\omega^2)}{(4+\omega^2)^2}\sin\omega t.$$

This is the required solution.

(iii) $y''(t) + 9y(t) = \cos(2t)$, $y(0) = 1$, $y(\pi/2) = -1$.

Soln:- Taking Laplace Transform for both sides of the given equation, we have

$$\mathcal{L}\{y''(t)\} + 9\mathcal{L}\{y(t)\} = \mathcal{L}\{\cos 2t\}$$

[here $\mathcal{L}\{y(t)\} = Y(s)$]

$$\text{or, } s^2\mathcal{L}\{y(t)\} - s\underbrace{y(0)}_{=1} - y'(0) + 9\mathcal{L}\{y(t)\} = \frac{s}{s^2+4}$$

$$\text{or, } s^2Y(s) - s - y'(0) + 9Y(s) = \frac{s}{s^2+4}$$

$$\text{or, } (s^2+9)Y(s) - s - A = \frac{s}{(s^2+4)}, \text{ where } y'(0) = A.$$

$$\text{or, } Y(s) = \frac{s+A}{(s^2+9)} + \frac{s}{(s^2+9)(s^2+4)}$$

$$= \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

Con,
$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] + A\,\mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right]$$
$$+ \frac{1}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right].$$
$$= \cos(3t) + \frac{A}{3}\sin(3t) + \frac{1}{5}\cos(2t) - \frac{1}{5}\cos(3t)$$
$$= \frac{4}{5}\cos(3t) + \frac{A}{3}\sin(3t) + \frac{1}{5}\cos(2t).$$

Now, it is given that $y(\pi/2) = -1$

$$\Rightarrow -1 = \frac{4}{5}\cos\left(\frac{3\pi}{2}\right) + \frac{A}{3}\sin(3\pi/2) + \frac{1}{5}\cos(\pi)$$
$$\Rightarrow -1 = -\frac{A}{3} - \frac{1}{5} \Rightarrow A = \frac{12}{5}.$$

Hence, the required solution will be

$$y = \frac{4}{5}\cos(3t) + \frac{4}{5}\sin(3t) + \frac{1}{5}\cos(2t).$$

(iv) $y''(t) + y(t) = \sin(t)\sin(2t),\ t>0,\ y(0) = 1,\ y'(0) = 0.$

Soln:- The given equation is

$$y''(t) + y(t) = \sin t \sin(2t) = \frac{1}{2}(\cos t - \cos 3t)$$
$$\therefore \mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{1}{2}\mathcal{L}\{\cos t\} - \frac{1}{2}\mathcal{L}\{\cos 3t\}$$
$$\Rightarrow s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{(=1)} - \underbrace{y'(0)}_{(=0)} + \mathcal{L}\{y\}$$
$$= \frac{s}{2(s^2+1)} - \frac{s}{2(s^2+9)}.$$

$$\Rightarrow (s^2+1)\, \mathcal{L}\{y\} - s = \frac{s}{2(s^2+1)} - \frac{s}{2(s^2+9)}$$ [$\mathcal{L}\{y($ $= \ldots$

$$\Rightarrow \mathcal{L}\{y\} = \frac{s}{(s^2+1)} + \frac{s}{2(s^2+1)^2} - \frac{s}{2(s^2+1)(s^2+9)}$$

$$\Rightarrow Y(s) = \frac{s}{(s^2+1)} - \frac{1}{4}\left[\frac{d}{ds}\frac{1}{(s^2+1)}\right] - \frac{s}{16(s^2+1)} + \frac{s}{16(s^2+9)}$$

[by Partial fraction]

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{d}{ds}\frac{1}{(s^2+1)}\right\}$$
$$- \frac{1}{16}\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= \cos t + \frac{t}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{16}\cos t + \frac{1}{16}\cos(3t)$$

$$\Rightarrow y(t) = \frac{15}{16}\cos t + \frac{t}{4}\sin t + \frac{1}{16}\cos(3t).$$

This is the required solution.

(v) $y''(t) - t\,y'(t) + y(t) = 1,\; y(0) = 1,\; y'(0) = 2.$

Soln :- Taking the Laplace transform of both sides of the given equation, we get

$$\mathcal{L}\{y''\} - \mathcal{L}\{ty'\} + \mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$\Rightarrow s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{(=1)} - \underbrace{y'(0)}_{(=2)} + \frac{d}{ds}\left[\mathcal{L}\{y'\}\right] + \mathcal{L}\{y\} = 1/s.$$

$\Rightarrow s^2 \mathcal{L}\{y\} - s - 2 + \frac{d}{ds}[s\,\mathcal{L}\{y\} - \underset{(=1)}{y(0)}] + \mathcal{L}\{y\} = \frac{1}{s}$

$\Rightarrow s^2 z - s - 2 + \frac{d}{ds}(sz - 1) + z = \frac{1}{s}$, where

$z = \mathcal{L}\{y\}$

$\Rightarrow s\frac{dz}{ds} + (s^2+2)z = s + 2 + \frac{1}{s}.$

$\Rightarrow \frac{dz}{ds} + \left(s + \frac{2}{s}\right)z = 1 + \frac{2}{s} + \frac{1}{s^2}$, which is linear.

$\therefore I \cdot F = e^{\int (s + 2/s)\,ds} = e^{\left(\frac{1}{2}s^2 + 2\log s\right)}$

$= s^2 e^{s^2/2}.$

$\therefore s^2 e^{s^2/2} z = c_1 + \int \left(1 + 2/s + 2/s^2\right) s^2 e^{s^2/2}\,ds$

$= c_1 + \int (s^2 + 2s + 1) e^{s^2/2}\,ds$

$= c_1 + \int (s^2+1) e^{s^2/2}\,ds + 2\int s e^{s^2/2}\,ds$

$= c_1 + \int (2v+1) e^{v} \cdot \frac{dv}{\sqrt{2v}} + 2\int \sqrt{2v}\, e^{v} \frac{dv}{\sqrt{2v}}$

Putting $\frac{s^2}{2} = v$, so that

$s\,ds = dv$ or, $ds = \frac{dv}{\sqrt{2v}}$

$= c_1 + \int \sqrt{2v}\, e^{v}\,dv + \int \frac{e^{v}}{\sqrt{2v}}\,dv + 2\int e^{v}\,dv$

$= c_1 + \sqrt{2v}\, e^{v} + 2e^{v} = c_1 + s e^{s^2/2} + 2e^{s^2/2}.$

$$\Rightarrow z = \mathcal{L}\{y\} = \frac{c_1}{s^2} e^{-s^2/2} + \frac{1}{s} + \frac{2}{s^2}$$

$$= \frac{c_1}{s^2}\left[1 - s^2/2 + s^4/4\cdot 2! \cdots\right] + \frac{1}{s} + \cdots$$

$$= \frac{(2+c_1)}{s^2} - \frac{c_1}{2} + \frac{c_1}{8}s^2 - \cdots + \frac{1}{s}$$

$$\therefore y = (2+c_1)\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{c_1}{2}\mathcal{L}^{-1}\{1\} + \frac{c_1}{8}\mathcal{L}^{-1}\{s^2\} + \cdots + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= (2+c_1)t + 1, \text{ since } \mathcal{L}^{-1}\{s^n\} = 0, n = 0,1,2 \ldots$$

But since $y'(0) = 2$.

$\therefore 2 = c_1 + 2$ or, $c_1 = 0$.

Hence, $y = 2t + 1$.

This is the required solution.

Q3) vi) $t y''(t) + y'(t) + 4t y(t) = 0$, $y(0) = 3$, $y'(0) = 0$.

Sol^n:- Taking the Laplace transform of both sides of the given equation, we get —

$$\mathcal{L}[ty''] + \mathcal{L}\{y'\} + 4\mathcal{L}[ty] = 0$$

$$\text{or, } -\frac{d}{ds}\mathcal{L}[y''] + \mathcal{L}\{y'\} + 4(-1)\frac{d}{ds}\mathcal{L}[y] = 0.$$

$$\Rightarrow -\frac{d}{ds}\left[s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{(=3)} - \underbrace{y'(0)}_{(=0)}\right] + \left[s\mathcal{L}\{y\} - \underbrace{y(0)}_{(=3)}\right] - 4\frac{d}{ds}\mathcal{L}\{y\} = 0$$

$$\Rightarrow -\frac{d}{ds}\left(s^2 Z - 3s\right) + \left(sZ - 3\right) - 4\frac{dZ}{ds} = 0,$$

where $Z = \mathcal{L}\{y\}$.

$$\Rightarrow -\left(s^2+4\right)\frac{dZ}{ds} - sZ = 0.$$

$$\Rightarrow \frac{dZ}{Z} + \frac{s}{(s^2+4)}\, ds = 0.$$

Integrating, we get

$$\ln Z + \frac{1}{2}\ln\left(s^2+4\right) = \ln c_1$$

$$\Rightarrow Z = \frac{c_1}{\sqrt{s^2+4}}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{c_1}{\sqrt{s^2+4}}$$

$$\Rightarrow y = c_1\, \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s^2+4}}\right]$$

$$\Rightarrow y = c_1\, J_0(2t).$$

since $y(0) = 3$,

$$\therefore 3 = c_1 J_0(0) = c_1 \qquad [\because J_0(0) = 1]$$

$$\therefore y = 3J_0(2t).$$

This is the required solution.

Q4) Solve :- $\left(x''(t) - x(t)\right) + 5y'(t) = t$

$-2x'(t) + \left(y''(t) - 4y(t)\right) = -2$,

if $x(0) = 0 = x'(0) = y(0) = y'(0)$.

Soln :- Taking Laplace transform of both sides of the two equations, we get

$$\mathcal{L}[x''] - \mathcal{L}\{x\} + 5\mathcal{L}\{y'\} = \mathcal{L}[t]$$

& $$-2\mathcal{L}[x'] + \mathcal{L}\{y''\} - 4\mathcal{L}\{y\} = -2\mathcal{L}\{1\}$$

or, $$s^2X - s\underbrace{x(0)}_{(=0)} - \underbrace{x'(0)}_{(=0)} - X + 5\left[sY - \underbrace{y(0)}_{(=0)}\right] = \frac{1}{s^2}$$ $\left[\mathcal{L}\{x\} = X,\ \mathcal{L}\{y\} = Y.\right.$

& $$-2\left[sX - \underbrace{x(0)}_{(=0)}\right] + \left(s^2Y - s\underbrace{y(0)}_{(=0)} - \underbrace{y'(0)}_{(=0)}\right) - 4Y = -\frac{2}{s},$$

or, $$(s^2 - 1)X - 5sY = 1/s^2$$

& $$-2sX + (s^2 - 4)Y = -\frac{2}{s}.$$

Solving for X & Y, we get

$$X = \frac{11s^2 - 4}{s(s^2+1)(s^2+4)} = -\frac{1}{s^2} + \frac{5}{s^2+1} - \frac{4}{s^2+4}$$ [By partial fraction]

& $$Y = \frac{-2s^2 + 4}{s(s^2+1)(s^2+4)} = \frac{1}{s} - \frac{2s}{(s^2+1)} + \frac{s}{(s^2+4)}$$

$$\therefore x = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 5\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= -t + 5\sin t - 2\sin(2t).$$

& $$y = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = 1 - 2\cos t + \cos 2t.$$

5) Solve :-
$$x'(t) + y'(t) = t$$
$$x''(t) - y(t) = e^{-t}$$
if $x(0) = 3$, $x'(0) = -2$, $y(0) = 0$.

Soln :- Taking Laplace Transform of both sides of the given equations, we get

$$\mathcal{L}\{x'\} + \mathcal{L}\{y'\} = \mathcal{L}\{t\}$$
$$\& \quad \mathcal{L}\{x''\} - \mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$\left[\begin{array}{l}\text{Let } \mathcal{L}\{x\} = X \\ \mathcal{L}\{y\} = Y\end{array}\right]$

or, $$sX - \underbrace{x(0)}_{(=3)} + sY - \underbrace{y(0)}_{(=0)} = \frac{1}{s^2}$$
$$\& \quad s^2 X - s\underbrace{x(0)}_{(=3)} - \underbrace{x'(0)}_{(=-2)} - Y = \frac{1}{s+1}$$

or, $$sX + sY = 3 + \frac{1}{s^2}$$
$$\& \quad s^2 X - Y = 3s - 2 + \frac{1}{(s+1)}$$

Solving for X & Y, we get —

$$X = \frac{3s^2+1}{s^3(1+s^2)} + \frac{3s}{(1+s^2)} - \frac{2}{(1+s^2)} - \frac{1}{(s+1)(s^2+1)}$$

$$= \frac{1}{s^3}\left(1 + 2s^2 - \frac{2s^4}{1+s^2}\right) + \frac{3s}{1+s^2} - \frac{2}{(1+s^2)} + \frac{1}{2(s+1)}$$
$$- \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}.$$

$$= \frac{2}{s} + \frac{1}{s^3} + \frac{1}{2(s+1)} + \frac{s}{2(1+s^2)} - \frac{3}{2(s^2+1)}$$
— (i)

& $Y = \dfrac{1}{s(s+1)(s^2+1)} + \dfrac{2}{(s^2+1)}$

$= \dfrac{1}{s} - \dfrac{1}{2(s+1)} - \dfrac{s}{2(s^2+1)} - \dfrac{1}{2(s^2+1)} + \dfrac{2}{(s^2+1)}$

$= \dfrac{1}{s} - \dfrac{1}{2(s+1)} - \dfrac{s}{2(s^2+1)} + \dfrac{3}{2(s^2+1)} \longrightarrow (ii)$

Taking inverse transforms of (i) & (ii), we get

$x = 2\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s^3}\right\} + \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\}$

$+ \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} - \dfrac{3}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}$

$= 2 + \dfrac{1}{2}t^2 + \dfrac{1}{2}e^{-t} + \dfrac{1}{2}\cos t - \dfrac{3}{2}\sin t.$

& $y = \mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} - \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} - \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\}$

$+ \dfrac{3}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}$

$= 1 - \dfrac{1}{2}e^{-t} - \dfrac{1}{2}\cos t + \dfrac{3}{2}\sin t.$

Q6) Solve the integral equation using Laplace transform technique :

(i) $y(t) = t + 2\int_0^t \cos(t-\tau)\, y(\tau)\, d\tau.$

( ) :- The given eqn is also expressible as

$$y(t) = t + 2\cos t * y(t)$$

Taking Laplace transform $[\mathcal{L}\{y(t)\} = Y(s)]$

$$Y(s) = \frac{1}{s^2} + \frac{2s}{(s^2+1)} \cdot Y(s)$$

$$\Rightarrow Y(s)\left\{1 - \frac{2s}{(s^2+1)}\right] = \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{(s^2+1)}{s^2(s-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{s^2(s-1)^2}.$$

Taking inverse Laplace Transform,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2(s-1)^2}\right\} \rightarrow (i)$$

Now, $\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = e^t \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right] = te^t \rightarrow (ii)$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)^2}\right] = \int_0^t ue^u\,du = \left[e^u \cdot u - \int e^u\,du\right]_0^t$$

$$= \left[ue^u - e^u\right]_0^t = \left[(u-1)e^u\right]_0^t = e^t(t-1) + 1$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)^2}\right] = \int_0^t \left[e^u(u-1) + 1\right] du$$

$$= \int_0^t ue^u\,du - \int_0^t e^u\,du + \int_0^t du$$

$$= e^t(t-1) + 1 - (e^t - 1) + t = e^t(t-2) + t + 2. \rightarrow (iii)$$

writing (i) with the help of (ii) & (iii),

$$y(t) = te^t + e^t(t-2) + t + 2$$

$$= 2e^t(t-1) + 2 + t.$$

(ii) $y(t) = 1 + \int_0^t y(\tau)\sin(t-\tau)\,d\tau.$

Solⁿ:- The given equation is expressible as

$$y(t) = 1 + y(t) * \sin t.$$

Taking Laplace Transform, $[\mathcal{L}\{y(t)\} = Y(s)]$

$$Y(s) = \frac{1}{s} + Y(s)\cdot\frac{1}{(s^2+1)}.$$

$$\Rightarrow Y(s) = \frac{(s^2+1)}{s^2\cdot s} = \frac{1}{s} + \frac{1}{s^3}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$\Rightarrow y(t) = 1 + t^2/2.\ \left[\text{as } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}\right].$$

(iii) $\int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau = 1 + t + t^2.$

Solⁿ:- The given eqⁿ can be written as

$$y(t) * t^{-1/2} = 1 + t + t^2$$

Taking Laplace Transform,

$$Y(s)\cdot\frac{\Gamma(1/2)}{s^{1/2}} = \frac{1}{s} + \frac{1}{s^2} + \frac{2!}{s^3},\ \text{for } \mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\Rightarrow \sqrt{\pi}\,Y(s) = \frac{1}{s^{1/2}} + \frac{1}{s^{3/2}} + \frac{2}{s^{5/2}}$$

$$\Rightarrow \sqrt{\pi}.\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}} + \frac{1}{s^{3/2}} + \frac{2}{s^{5/2}}\right\}.$$

$$\Rightarrow \sqrt{\pi}\,y(t) = \frac{t^{-1/2}}{\Gamma(1/2)} + \frac{t^{1/2}}{\Gamma(3/2)} + \frac{2t^{3/2}}{\Gamma(5/2)}$$

$$\Rightarrow y(t) = \frac{1}{\pi}\left[t^{-1/2} + 2t^{1/2} + 2\cdot\frac{2}{3}\cdot\frac{2}{1}\, t^{3/2}\right] = \frac{1}{\pi}\left[t^{-1/2} + 2t^{1/2} + \frac{8}{3}t^{3/2}\right]$$

6) iii) $\int_0^t y(\tau)\, y(t-\tau)\, d\tau = 16 \sin(4t)$.

Sol^n:- The given eq^n is also expressible as

$$y(t) * y(t) = 16 \sin(4t).$$

Taking Laplace transform of both sides, we find that

if $\mathcal{L}\{y(t)\} = Y(s)$, then

$$\Rightarrow \mathcal{L}\{y * y\} = \mathcal{L}\{16 \sin(4t)\}$$

$$\Rightarrow Y \cdot Y = 16 \cdot \frac{4}{(s^2+16)} = \frac{64}{(s^2+16)}$$

$$\Rightarrow Y = \pm \frac{8}{\sqrt{s^2+4^2}}.$$

$$\Rightarrow \mathcal{L}^{-1}\{Y\} = \pm 8\, \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+4^2}}\right\}$$

$$\Rightarrow y(t) = \pm 8 J_0(4t).$$

Q7) Find $\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}}}{s}\right\}$ & hence deduce that

$$\mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left[\frac{x}{2\sqrt{t}}\right].$$

Sol^n:- Let $F(s) = e^{-\sqrt{s}} = \mathcal{L}\{f(t)\}$.

Then $f(t) = \mathcal{L}^{-1}\{e^{-\sqrt{s}}\}$

$$= \mathcal{L}^{-1}\left[1 - s^{1/2} + \frac{s}{2!} - \frac{s^{3/2}}{3!} + \frac{s^2}{4!} - \frac{s^{5/2}}{5!} + \dots\right]$$

$\longrightarrow$ (i)

since $\mathcal{L}^{-1}\left[s^{(n+1/2)}\right] = \mathcal{L}^{-1}\left\{\frac{1}{s^{-n-(3/2)+1}}\right\}$

$$= \frac{t^{-n-3/2}}{\Gamma(-n-3/2)} = \frac{t^{-n-3/2}}{\Gamma(-n-\frac{1}{2}-1)}$$

$$= \frac{(-1)^{n+1}}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \left(\frac{2n+1}{2}\right) \cdot t^{-n-(3/2)}$$

& $\mathcal{L}^{-1}[s^n] = 0$, if n is a positive integer.

$\therefore$ from (i),

$$f(t) = \frac{(-1)\, t^{3/2}}{\sqrt{\pi}} \cdot \frac{1}{2} - \frac{1}{3!}\frac{(-1)^2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{3}{2}\, t^{-5/2}$$

$$- \frac{1}{5!}\frac{(-1)^3}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot t^{-7/2} + \ldots$$

$$= \frac{1}{2t^{3/2}\sqrt{\pi}}\left[1 - \frac{1}{4t} + \frac{1}{2!}\left(\frac{1}{4t}\right)^2 - \frac{1}{3!}\left(\frac{1}{4t}\right)^3 + \ldots\right]$$

or, $$f(t) = \frac{1}{2t^{3/2}\sqrt{\pi}}\, e^{-1/4t} \longrightarrow \text{(ii)}$$

Since $\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(x)\,dx$

$$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}}}{s}\right\} = \int_0^t \frac{1}{2\sqrt{\pi}\, x^{3/2}}\, e^{-1/4x}\, dx$$

$$= -\frac{2}{\sqrt{\pi}}\int_{\infty}^{1/(2\sqrt{t})} e^{-y^2}\,dy, \quad \left[\text{Let } x = \frac{1}{4y^2},\ dx = \frac{-dy}{2y^3}\right.$$

$$= \frac{2}{\sqrt{\pi}} \int_{1/(2\sqrt{t})}^{\infty} e^{-y^2} dy \quad = \text{erfc}\left\{\frac{1}{2\sqrt{t}}\right\}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s}}}{s}\right] = \text{erfc}\left\{\frac{1}{2\sqrt{t}}\right\}.$$

We know that $\text{erfc}(t) = 1 - \text{erf}(t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$

$$= \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-x^2} dx.$$

$\text{erfc}(t)$
$= \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-y^2} dy$
is called complementary error function.

$\therefore$ from above, we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}}}{s}\right\} = \text{erfc}\left\{\frac{1}{2\sqrt{t}}\right\}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{(x^2 s)}}}{x^2 s}\right\} = \frac{1}{x^2}\, \text{erfc}\left\{\frac{1}{2\sqrt{(t/x^2)}}\right\}$$

by change of scale property

$$\mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} = \text{erfc}\left\{\frac{x}{2\sqrt{t}}\right\}.$$

Q8) Solve using Laplace transform techniques

$$y''(t) + y(t) = 4\,\delta(t - 2\pi),$$

subject to the conditions:

(i) $y(0) = 1,\ y'(0) = 0.$

Sol^n:- Taking Laplace Transform

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = 4\,\mathcal{L}\{\delta(t - 2\pi)\}$$

$$\left[s^2 Y(s) - s\underset{(=1)}{y(0)} - \underset{(=0)}{y'(0)}\right] + Y(s) = 4e^{-2\pi s}$$

$$= \left[(s^2+1)Y(s) - s\right]$$

$$\Rightarrow Y(s) = \frac{s}{(s^2+1)} + \frac{4e^{-2\pi s}}{(s^2+1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{(s^2+1)}\right\}$$

$$= \cos t + 4\sin(t-2\pi)\,u(t-2\pi)$$

$$\therefore y(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \geq 2\pi. \end{cases}$$

$\boxed{\mathcal{L}\{y(t)\} = Y(s)}$

(ii) $y(0) = 0$, $y'(0) = 0$.

In this case, we have

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = 4e^{-2\pi s}$$

$$\Rightarrow s^2 Y(s) + Y(s) = 4e^{-2\pi s}$$

$$\Rightarrow Y(s) = \frac{4e^{-2\pi s}}{(s^2+1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{(s^2+1)}\right\}$$

$$= 4\sin(t-2\pi)\,u(t-2\pi).$$

$$= \begin{cases} 0, & 0 \le t < 2\pi \\ 4\sin t, & t \geq 2\pi. \end{cases}$$

Q9) Prove that

$$\mathcal{L}\{Si(t)\} = \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right).$$

Sol^n :- Let $f(t) = \int_0^t \frac{\sin u}{u}\, du.$

Then $f(0) = 0$, & $f'(t) = \frac{\sin t}{t}$

$\Rightarrow t f'(t) = \sin t$

Taking the Laplace Transform,

$\mathcal{L}\{t f'(t)\} = \mathcal{L}\{\sin t\}$

$\Rightarrow -\frac{d}{ds}\{sF(s) - f(0)\} = \frac{1}{(s^2+1)}$ $\quad\left[\because \mathcal{L}\{f(t)\} = F(s)\right.$

$\Rightarrow \frac{d}{ds}\{sF(s)\} = -\frac{1}{(s^2+1)}$

(on $\int$)

$sF(s) = -\tan^{-1} s + c$

By the initial value theorem,

$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t) = f(0) = 0$

so that $0 = -\pi/2 + c \Rightarrow c = \pi/2$

$\therefore sF(s) = \pi/2 - \tan^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right)$

$\Rightarrow F(s) = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$

Q10) Solve the integral eq^n

$y(t) = \frac{1}{2}\sin(2t) + \int_0^t y(\tau)\, y(t-\tau)\, d\tau.$

The integral eqn can be written as

$$y(t) = \frac{1}{2}\sin 2t + y(t) * y(t)$$

Taking the Laplace transform, using the convolution theorem, we find

$$Y(s) = \frac{1}{(s^2+4)} + \{Y(s)\}^2$$

or, $$(Y(s))^2 - Y(s) + \frac{1}{(s^2+4)} = 0.$$

solving, we get $$Y(s) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{4}{(s^2+4)}}$$

$$= \frac{1}{2} \pm \frac{1}{2}\frac{s}{\sqrt{s^2+4}}.$$

Thus, $$Y(s) = \frac{1}{2}\left\{\frac{\sqrt{s^2+4}+s}{\sqrt{s^2+4}}\right\} \rightarrow (i)$$

& $$Y(s) = \frac{1}{2}\left\{\frac{\sqrt{s^2+4}-s}{\sqrt{s^2+4}}\right\} \rightarrow (ii)$$

From (ii), we find the solution

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{2}\left(\frac{\sqrt{s^2+4}-s}{\sqrt{s^2+4}}\right)\right] = J_1(2t) \rightarrow (iii)$$

eqn (i), can be written as

$$Y(s) = -\frac{1}{2}\left(\frac{\sqrt{s^2+4}-s}{\sqrt{s^2+4}} - 2\right) = 1 - \frac{1}{2}\left(\frac{\sqrt{s^2+4}-s}{\sqrt{s^2+4}}\right)$$

Hence, a second solution is $y(t) = \delta(t) - J_1(2t)$, $\rightarrow$ (iv)

where $\delta(t)$ is the Dirac delta function.

The soln (iii) is continuous & bounded for $t \geq 0$.

2) Find $\mathcal{L}\{J_1(t)\}$, where $J_1(t)$ is Bessel's function of order one.

Sol^n :- We know, $J_0'(t) = -J_1(t)$.

Hence, $\mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_0'(t)\}$

$$= -\left[s\,\mathcal{L}\{J_0(t)\} - 1\right]$$

$$= 1 - \frac{s}{\sqrt{s^2+1}} \qquad \left[\begin{array}{l}\text{as}\\ \mathcal{L}\{J_0(t)\} = \dfrac{1}{\sqrt{s^2+1}}\end{array}\right]$$

$$= \frac{\left(\sqrt{s^2+1} - s\right)}{\left(\sqrt{s^2+1}\right)}$$

—x—