Fluid Mechanics

Assignment-4

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1. An incompressible, inviscid fluid flows into a horizontal round tube through its porous wall. The tube is closed at the left end and the flow discharges from the tube to the atmosphere at the right end. For simplicity, consider the x component of velocity in the tube uniform across any cross section. The density of the fluid is ρ, the tube diameter and length are D and L, respectively, and the uniform inflow velocity is v_0. The flow is steady.

   (i) Obtain an algebraic expression for the x component of acceleration of a fluid particle located at position x, in terms of v_0, x, and D.
   (ii) Find an expression for the pressure gradient, $\frac{\partial p}{\partial x}$, at position x.
   (iii) Obtain an expression for the gage pressure at $x = 0$.

   Solution:

   Applying conservation of mass:

$$0 = \cancelto{0}{\frac{\partial}{\partial t}} \int_{CV} \rho d\cancel{V} + \int_{CS} \overrightarrow{V} \cdot d\overrightarrow{A}$$

$$\overrightarrow{a}_x = \cancelto{0}{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + \cancelto{0}{v}\frac{\partial u}{\partial y} + \cancelto{0}{w}\frac{\partial u}{\partial z}$$

   The conservation of momentum (Euler's equation) along x direction takes the form:

$$-\frac{\partial p}{\partial x} = \rho a_x$$

   The control surface comprises of the curved surface of the tube along with the right end circular surface through which fluid flow is entering to or leaving from the C.V.

   Note: The fluid velocity at the boundaries are v_0 and u at the curved and the right-end, respectively. However, inside the tube $v \approx w \approx 0$, i.e. $\overrightarrow{V} = u\,\hat{i}$ (unidirectional flow).

   Hence,

$$\int_{CS} \overrightarrow{V} \cdot d\overrightarrow{A} = -v_0(\pi D x) + u(\pi D^2/4) = 0$$

$$\text{or, } u(x) = 4v_0 x/D.$$

Also,

$$\boxed{a_x = (4v_0 x/D) \times (4v_0/D) = 16v_0^2 x/D^2}.$$

Thus the Euler's equation gives

$$\boxed{\frac{\partial p}{\partial x} = -\rho a_x = -16v_0^2 x/D^2}.$$

Under the assumption of unidirectional flow, the pressure inside the tube is only a function of x, i.e. $p = p(x)$,

$$dp = \frac{\partial p}{\partial x} dx.$$

Integrating the above equation,

$$\int_0^L dp = \int_0^L (-16v_0^2 x/D^2)\, dx$$
$$\text{or, } p(x = L) - p(x = 0) = \frac{-8\rho v_0^2 L^2}{D^2}.$$

Now, it is given that the fluid discharges at the atmospheric pressure at the right end, i.e. $p(x = L) = p_{\text{atm}}$.

$\therefore$ The gauge pressure at $x = 0$ is

$$\boxed{p(x = L) = \frac{8\rho v_0^2 L^2}{D^2}}.$$

2. The equation of conservation of mass can be derived from the Reynolds Transport Theorem. If it is given that there is no generation or consumption of mass in the system, then which among the below statements is true in this context: (symbols have their usual meanings)

   (a) The equation

$$\int_{V(t)} \frac{\partial \rho(\overrightarrow{x}, t)}{\partial t} d\not{V} + \int_{A(t)} \rho(\overrightarrow{x}, t)\overrightarrow{u}(\overrightarrow{x}, t) \cdot \hat{n} dA = 0$$

   is applicable when the control volume moves with velocity of fluid. The control volume can be deformable in this case.

   (b) The equation

$$\frac{d}{dt}\int_{V(t)} \rho(\overrightarrow{x}, t)\, d\not{V} + \int_{A(t)} \rho(\overrightarrow{x}, t)(\overrightarrow{u}(\overrightarrow{x}, t) - \overrightarrow{b}(\overrightarrow{x}, t)) \cdot \hat{n} dA = 0$$

   is applicable only when the control volume and fluid both move with a velocity $\overrightarrow{b}$.

   (c) Both (a) and (b) are correct.

   (d) Both (a) and (b) are wrong.

Solution: Correct Answer: **D**

Explanation: The equation

$$\int_{V(t)} \frac{\partial \rho(\overrightarrow{x}, t)}{\partial t} dV + \int_{A(t)} \rho(\overrightarrow{x}, t)\overrightarrow{u}(\overrightarrow{x}, t) \cdot \hat{n} dA = 0$$

is applicable only for fixed control volume.

For the control volume to be deformable, the volume integral in this equation must allow the volume elements to distort with time. Thus the time derivative must be applied after integration. In that case, the equation should read

$$\frac{d}{dt}\int_{V(t)} \rho(\overrightarrow{x}, t)\, dV + \int_{A(t)} \rho(\overrightarrow{x}, t)\overrightarrow{u}(\overrightarrow{x}, t) \cdot \hat{n} dA = 0$$

The equation

$$\frac{d}{dt}\int_{V(t)} \rho(\overrightarrow{x}, t)\, d\not{V} + \int_{A(t)} \rho(\overrightarrow{x}, t)(\overrightarrow{u}(\overrightarrow{x}, t) - \overrightarrow{b}(\overrightarrow{x}, t)) \cdot \hat{n} dA = 0$$

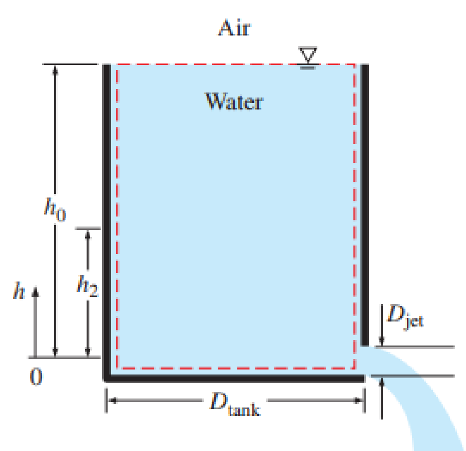
is applicable when the control volume moves with an arbitrary velocity $\overrightarrow{b}$. It is not necessary for the control volume to have the same velocity as that of the fluid (i.e. $\overrightarrow{u}(\overrightarrow{x}, t) \neq \overrightarrow{b}(\overrightarrow{x}, t)$).

3. A 4-ft-high, 3-ft-diameter cylindrical water tank whose top is open to the atmosphere is initially filled with water. Now the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.5 in streams out. The average velocity of the jet is approximated as $v = \sqrt{2gh}$, where h is the height of water in the tank measured from the center of the hole and g is the gravitational acceleration. Determine how long it takes for the water level in the tank to drop to 2 ft from the bottom.

   You may assume the following: (i) Water is a nearly incompressible substance. (ii) The distance between the bottom of the tank and the center of the hole is negligible compared to the total water height. (iii) The gravitational acceleration is 32.2 ft/s^2.

   Solution:

   We take the volume occupied by water as the control volume. The size of the control volume decreases in this case as the water level drops, and thus this is a variable control volume. (We could also treat this as a fixed control volume that consists of the interior volume of the tank by disregarding the air that replaces the space vacated by the water.) This is obviously an unsteady-flow problem since the properties (such as the amount of mass) within the control volume change with time.



   The conservation of mass relation for a control volume undergoing any process is given in rate form as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt} \tag{1}$$

   During this process no mass enters the control volume ($\dot{m}_{in} = 0$), and the mass flow rate of discharged water is

$$\dot{m}_{out} = (\rho \mathcal{V} A)_{out} = \rho\sqrt{2gh}A_{jet}, \tag{2}$$

where $A_{jet} = \pi D_{jet}^2/4$. is the cross-sectional area of the jet, which is constant. Noting that the density of water is constant, the mass of water in the tank at any time is $m_{CV} = \rho \cancel{V} = \rho A_{tank} h$, where $A_{tank} = \pi D_{tank}^2/4$ is the base area of the cylindrical tank. Hence the mass balance equation can be written as

$$-\rho\sqrt{2gh}A_{jet} = \frac{d(\rho A_{tank} h)}{dt} \tag{3}$$

Canceling the densities and other common terms and separating the variables give

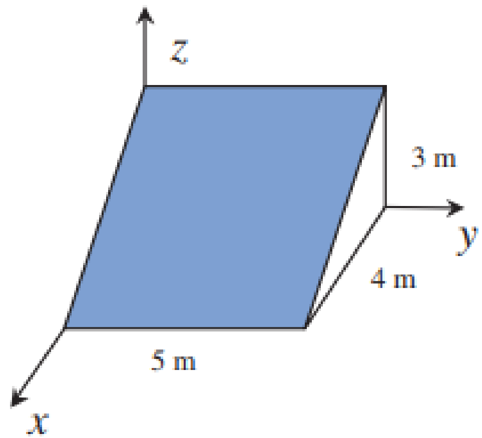
$$dt = -\frac{D_{tank}^2}{D_{jet}^2}\frac{dh}{\sqrt{2gh}} \tag{4}$$

Integrating from $t = 0$ at which $h = h_0$ to $t = t$ at which $h = h_2$ gives

$$\int_0^t dt = -\frac{D_{tank}^2}{D_{jet}^2}\frac{1}{\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}}$$
$$or, t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{tank}}{D_{jet}}\right)^2$$

Substituting numeric values, the time of discharge is determined to be 757 s.

4. The shaded area shown is in a flow where the velocity field is given by $\overrightarrow{V} = ax\hat{\mathbf{i}} + by\hat{\mathbf{j}}$; $a = b = 1\,\mathrm{s}^{-1}$, and the coordinates are measured in meters. Evaluate the (a) volume flow rate and (b) the momentum flux through the shaded area ($\rho = 1\,\mathrm{kg/m^3}$).

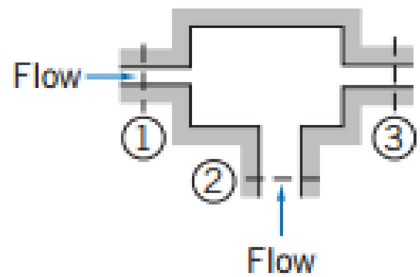


Answer: (a) Volume flow rate $Q = 30\,\mathrm{m}^3$ /s; (b) Momentum flux $= 80\hat{\mathbf{i}} + 75\hat{\mathbf{j}}$ N.

5. In the incompressible flow through the device shown, velocities may be considered uniform over the inlet and outlet sections. The following conditions are known: $A_1 = 0.1\,\mathrm{m}^2$, $A_2 = 0.2\,\mathrm{m}^2$, $A_3 = 0.15\,\mathrm{m}^2$, $V_1 = 10\,e^{-t/2}\,\mathrm{m\,s^{-1}}$, $V_2 = 2\cos(2\pi t)\,\mathrm{m\,s^{-1}}$.

   (a) Obtain an expression for the velocity at section 3.

   (b) What is the total mean volumetric flow at section 3 ?
   Hints: 'Total' indicates upto infinite amount of time. The mean of a sinusoidal function has to be utilized.

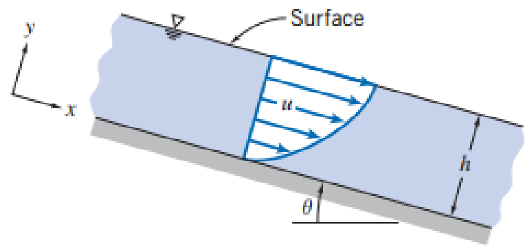


Answer:

(a) $V_3 = 6.67\,e^{-t/2} + 2.67\cos(2\pi t)\,\mathrm{m\,s^{-1}}$;

(b) $Q = 2\,\mathrm{m^3/s}$

6. Oil flows steadily in a thin layer down an inclined plane. The velocity profile is $u = \dfrac{\rho g \sin(\theta)}{\mu}\left[hy - \dfrac{y^2}{2}\right]$. Express the mass flow rate per unit width in terms of $\rho, \mu, g, \theta,$ and h.

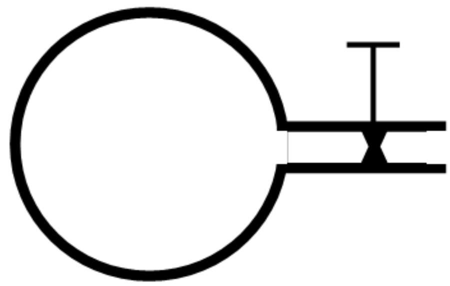


Answer: Mass flow rate per unit width $= \dfrac{\rho^2 g \sin(\theta) w h^3}{3\mu}$

7. A cylindrical tank, 0.3 m in diameter, drains through a hole in its bottom. At the instant when the water depth is 0.6 m, the flow rate from the tank is observed to be 4 kg/s. Determine the rate of change of water level at this instant.

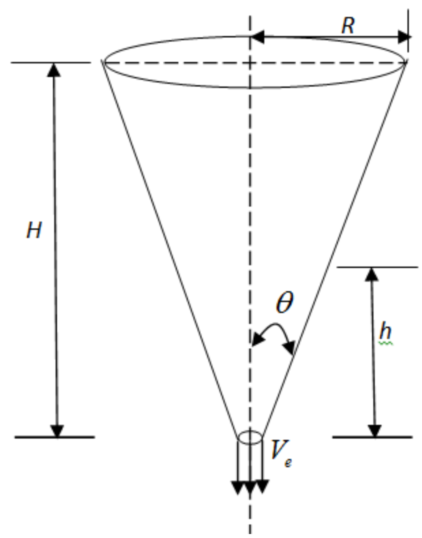
   Answer: The water level is falling at $56.6\,\mathrm{mm\,s^{-1}}$

8. A spherical tank of 0.4 m^3 volume contains compressed air. A valve is opened and air escapes with a velocity of $250\,\mathrm{m\,s^{-1}}$ through an opening of $100\,\mathrm{mm^2}$ area. Air temperature passing through the opening is 220^0C and the absolute pressure is 300 kPa. Find the rate of change of density of the air in the tank at this moment.



Answer: $\dfrac{d\rho_{\text{tank}}}{dt} = -0.258\,\mathrm{m}^3\ /\mathrm{s}$

9. A conical flask contains water to height $H = 36.8$ mm, where the flask diameter is $D = 2R = 29.4$ mm. Water drains out through a smoothly rounded hole of diameter $d = 7.35$ mm at the apex of the cone. The flow speed at the exit is approximately $V = \sqrt{2gy}$, where y is the height of the liquid free surface above the hole. A stream of water flows into the top of the flask at constant volume flow rate, $Q = 3.75\times10^{-7}\mathrm{m}^3/\mathrm{hr}$. Find the volume flow rate from the bottom of the flask. Evaluate the direction and rate of change of water surface level in the flask at this instant.

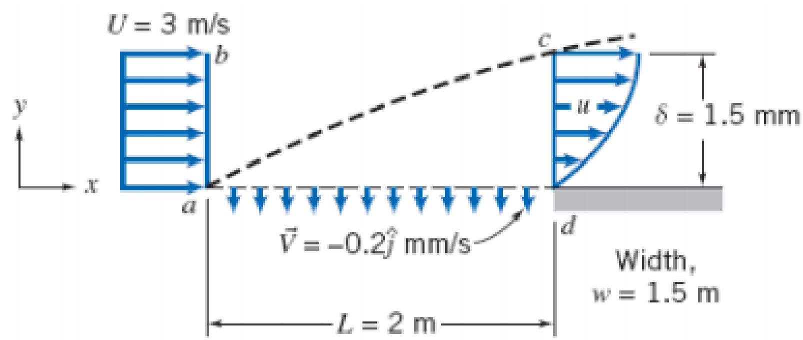


Answer: $\dfrac{dy}{dt} = -0.0532\,\mathrm{m\,s^{-1}}$

10. Water flows steadily past a porous flat plate. Constant suction is applied along the porous section (ad in figure). The velocity profile at section cd is

$$\frac{u}{U_\infty} = 3\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^{3/2}$$

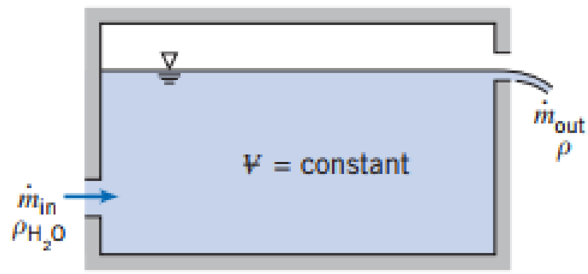
Evaluate the mass flow rate across section bc.



Answer:

$\dot{m}_{bc} = 1.42\,\mathrm{kg\,s}$

11. A tank of fixed volume contains brine with initial density, ρ_i, greater than water. Pure water enters the tank steadily and mixes thoroughly with the brine in the tank. The liquid level in the tank remains constant. Derive expressions for (a) the rate of change of density of the liquid mixture in the tank and (b) the time required for the density to reach the value ρ_f, where $\rho_i > \rho_f > \rho_{\mathrm{H_2O}}$.

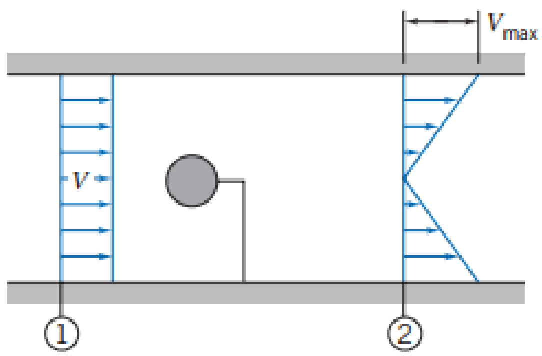


Answer:

(a) $\dfrac{d\rho}{dt} = -\dfrac{(\rho - \rho_{\mathrm{H_2O}})VA}{\forall}$

(b) $t = -\dfrac{\forall}{VA} \ln\left(\dfrac{\rho_f - \rho_{\mathrm{H_2O}}}{\rho_i - \rho_{\mathrm{H_2O}}}\right)$

12. A small round object is tested in a 0.75 m diameter wind tunnel. The pressure is uniform across sections 1 and 2. The upstream pressure is 30 mm H_2O (gage), the downstream pressure is 15 mm H_2O (gage), and the mean air speed is 12.5 m/s. The velocity profile at section 2 is linear; it varies from zero at the tunnel centerline to a maximum at the tunnel wall. Calculate the mass flow rate in the wind tunnel.



Answer:

$$\dot{m}_{flow} = \frac{\rho V_1 \pi D_1^2}{4}$$