In this lesson we discuss application of Laplace transform for solving integral equations, integro-differential equations and simultaneous differential equations.

14.1 Integral Equation

An equation of the form

$$f(t) = g(t) + \int_0^t K(t,u) f(u)\,\mathrm{d}u,$$

or

$$g(t) = \int_0^t K(t,u) f(u)\,\mathrm{d}u$$

are known as the integral equations, where $f(t)$ is the unknown function. When the kernel $K(t,u)$ is of the particular form $K(t,u) = K(t-u)$ then the equations can be solved using Laplace transforms. We apply the Laplace transform to the first equation to obtain

$$F(s) = G(s) + K(s)F(s),$$

where $F(s)$, $G(s)$, and $K(s)$ are the Laplace transforms of $f(t)$, $g(t)$, and $K(t)$ respectively. Solving for $F(s)$, we find

$$F(s) = \frac{G(s)}{1-K(s)}.$$

To find $f(t)$ we now need to find the inverse Laplace transform of $F(s)$. Similar steps can be followed to solve the integral equation of second type mentioned above.

14.2 Example Problems

14.2.1 Problem 1

Solve the following integral equation

$$f(t) = e^{-t} + \int_0^t \sin(t-u) f(u)\,\mathrm{d}u.$$

Solution: Applying Laplace transform on both sides and using convolution theorem we get,

$$L[f(t)] = \frac{1}{s+1} + L[\sin t]L[f(t)]$$

On simplifications, we obtain

$$L[f(t)]\left[1-\frac{1}{s^2+1}\right]=\frac{1}{s+1}$$

This further implies

$$L[f(t)]=\frac{s^2+1}{s^2(s+1)}$$

Partial fractions leads to

$$L[f(t)]=\frac{2}{s+1}+\frac{1}{s^2}-\frac{1}{s}$$

Taking inverse Laplace transform we obtain the desired solution as

$$f(t)=2e^{-t}+t-1$$

14.2.2 Problem 2

Solve the differential equation

$$x(t)=e^{-t}+\int_0^t \sinh(t-\tau)x(\tau)\,d\tau.$$

Solution: We apply Laplace transform to obtain

$$X(s)=\frac{1}{s+1}+\frac{1}{s^2-1}X(s),$$

or

$$X(s)=\frac{\frac{1}{s+1}}{1-\frac{1}{s^2-1}}=\frac{s-1}{s^2-2}=\frac{s}{s^2-2}-\frac{1}{s^2-2}.$$

It is not difficult to take inverse Laplace transform to find

$$x(t)=\cosh(\sqrt{2}\,t)-\frac{1}{\sqrt{2}}\sinh(\sqrt{2}\,t).$$

14.2.3 Problem 3

Solve the following integral equation for $x(t)$

$$t^2=\int_0^t e^{\tau}x(\tau)\,d\tau,$$

Solution: We apply the Laplace transform and the shifting property to get

$$\frac{2}{s^3} = \frac{1}{s}\,L[e^t x(t)] = \frac{1}{s}\,X(s-1),$$

where $X(s) = L[x(t)]$. Thus, we have

$$X(s-1) = \frac{2}{s^2} \qquad \text{or} \qquad X(s) = \frac{2}{(s+1)^2}.$$

We use the shifting property again to obtain

$$x(t) = 2e^{-t}t.$$

14.3 Integro-Differential Equations

In addition to the integral we have a differential term in the integro differential equations. The idea of solving ordinary differential equations and integral equations are now combined. We demonstrate the procedure with the help of the following example.

14.3.1 Example

Solve

$$\frac{dy}{dt} + 4y + 13\int_0^t y(u)\,\mathrm{d}u = 3e^{-2t}\sin 3t, \qquad y(0) = 3.$$

Solution: Taking Laplace transform and using its appropriate properties we obtain,

$$sY(s) - y(0) + 4Y(s) + 13\frac{Y(s)}{s} = 3\frac{3}{(s+2)^2+9}.$$

Collecting terms of $Y(s)$ we get

$$\frac{s^2+4s+13}{s}Y(s) = \frac{9}{(s+2)^2+9} + 3$$

On simplification we have

$$Y(s) = \frac{9s}{\left[(s+2)^2+9\right]^2} + \frac{3s}{(s+2)^2+9}$$

Taking inverse Laplace transform and using shifting theorem we get

$$y(t) = e^{-2t}L^{-1}\left[\frac{9(s-2)}{(s^2+9)^2} + \frac{3(s-2)}{s^2+9}\right].$$

We now break the functions into the known forms as

$$\begin{aligned} y(t) =& e^{-2t}L^{-1}\left[\frac{9s}{(s^2+9)^2} - \frac{18}{(s^2+9)^2} + \frac{3s}{s^2+9} + \frac{1}{s^2+9} - \frac{7}{s^2+9}\right] \\ =& e^{-2t}L^{-1}\left[\frac{9s}{(s^2+9)^2} + \frac{s^2-9}{(s^2+9)^2} + \frac{3s}{s^2+9} - \frac{7}{s^2+9}\right] \end{aligned}$$

Using the the following basic inverse transforms

$$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at, \quad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right] = t\sin at, \quad L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at.$$

We find the desired solution as

$$y(t) = e^{-2t}\left[\frac{3}{2}t\sin 3t + t\cos 3t + 3\cos 3t - \frac{7}{3}\sin 3t\right]$$

14.4 Simultaneous Differential Equations

At the end we show with the help of an example the application of Laplace transform for solving simultaneous differential equations.

14.4.1 Example

Solve

$$\frac{dx}{dt} = 2x - 3y, \quad \frac{dy}{dt} = y - 2x$$

subject to the initial conditions

$$x(0) = 8, \quad y(0) = 3.$$

Solution: Taking Laplace transform on both sides we get

$$sX(s) - x(0) = 2X(s) - 3Y(s)$$

and

$$sY(s) - y(0) = Y(s) - 2X(s)$$

Collecting terms of $X(s)$ and $Y(s)$ we have the following equations

$$(s-2)X(s) + 3Y(s) = 8 \tag{14.1}$$

$$2X(s) + (s-1)Y(s) = 3 \tag{14.2}$$

Eliminating $Y(s)$ we obtain

$$[(s-1)(s-2) - 6]\, X(s) = 8(s-1) - 9$$

On simplifications we receive

$$X(s) = \frac{8s-17}{(s-4)(s+1)}$$

Partial fractions lead to

$$X(s) = \frac{5}{s+1} + \frac{3}{s-4},$$

Taking inverse Laplace transform both sides we get

$$x(t) = 5e^{-t} + 3e^{4t}$$

Now we solve the above equations (14.1) and for (14.2) $Y(s)$

$$[6 - (s-1)(s-2)]\, Y(s) = 16 - 3(s-2)$$

On simplifications we get

$$Y(s) = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s-4)(s+1)}$$

Using the method of partial fractions we obtain

$$Y(s) = \frac{5}{s+1} - \frac{2}{s-4}$$

Taking inverse transform we get

$$y(t) = 5e^{-t} - 2e^{4t}.$$