Transform Calculus

(MA-20101)

Solutions- 1

1. i) Let the Laplace transform of $f(t) = \frac{1}{\sqrt{t}}$ be $F(s)$. Then,

$$F(s) = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}dt = \int_0^\infty e^{-st}t^{-1/2}dt.$$

Let $st = u$. So, $s\ dt = du$. Therefore,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-u}(\tfrac{u}{s})^{-1/2}\tfrac{du}{s} \\ &= s^{-1/2}\int_0^\infty e^{-u}u^{\frac{1}{2}-1}du \\ &= s^{-1/2}\Gamma(1/2) \\ &= \sqrt{\tfrac{\pi}{s}}. \end{aligned}$$

ii) Sufficient condition for the existence of Laplace transform:-

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies

$$|f(t)| \leq Me^{kt}$$

for some constant M and $k(> 0)$ and for all $t \geq 0$, then the Laplace transform of f exists for all $s > k$.

The function $\frac{1}{\sqrt{t}}$ is not piecewise continuous on $[0, \infty)$ as $\lim\limits_{t\to 0}\frac{1}{\sqrt{t}} = \infty$, ie is not finite. By i), we have seen that the Laplace transform of $\frac{1}{\sqrt{t}}$ exists.

2. i) Here $F(s) = \int_0^\infty e^{-st}f(t)dt$. Let the Laplace transform of $f(ct)$ be $G(s)$. Then $G(s) = \int_0^\infty e^{-st}f(ct)dt$.

Let $ct = u$. So $c\ dt = du$. Therefore

$$\begin{aligned} G(s) &= \int_0^\infty e^{-s\frac{u}{c}} \frac{f(u)}{c} du \\ &= \frac{1}{c}\int_0^\infty e^{-(\frac{s}{c})u} f(u) du \\ &= \frac{1}{c}F(\frac{s}{c}). \end{aligned}$$

ii) The Laplace transform of $\cos t$ is $\frac{s}{s^2+1}$. So, the Laplace transform of $\cos \omega t$ is

$$\frac{1}{\omega}\frac{\frac{s}{\omega}}{(\frac{s}{\omega})^2+1} = \frac{s}{s^2+\omega^2}.$$

3. Let the Laplace transform of $f(t)$ be $\mathcal{L}(f)$.

i) We have

$$\begin{aligned} f(t) &= \cos^2(\tfrac{1}{2}\pi t) \\ &= \tfrac{1}{2}(1+\cos \pi t). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L}(f) &= \tfrac{1}{2}\mathcal{L}(1) + \tfrac{1}{2}\mathcal{L}(\cos \pi t) \text{ (by linearity)} \\ &= \tfrac{1}{2}\tfrac{1}{s} + \tfrac{1}{2}\tfrac{s}{s^2+\pi^2} \\ &= \tfrac{1}{2}\tfrac{s^2+s^2+\pi^2}{s(s^2+\pi^2)} \\ &= \tfrac{1}{2}\tfrac{2s^2+\pi^2}{s(s^2+\pi^2)}. \end{aligned}$$

ii) $\mathcal{L}(t^3 e^{-3t}) = F(s+3)$, (by First Shifting Theorem) where $F(s)$ is the Laplace transform of t^3. We know that $F(s) = \frac{3!}{s^4}$.

So $\mathcal{L}(t^3 e^{-3t}) = F(s+3) = \frac{6}{(s+3)^4}$.

iii) $\mathcal{L}(e^{-\frac{t}{2}}u(t-2)) = F(s+\frac{1}{2})$, where $F(s)$ is the Laplace transform of $u(t-2)$ (by First Shifting Theorem).

$F(s) = \frac{e^{-2s}}{s}$. So $\mathcal{L}(e^{-\frac{t}{2}}u(t-2)) = F(s+\frac{1}{2}) = \frac{e^{-2(s+\frac{1}{2})}}{s+\frac{1}{2}} = \frac{e^{-2s-1}}{s+\frac{1}{2}}$.

iv) $\mathcal{L}((t-a)^n)u(t-a)) = e^{-as}F(s)$ (by Second Shifting Theorem), where $F(s) = \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.
$\therefore \mathcal{L}((t-a)^n\ u(t-a)) = \frac{n!\ e^{-as}}{s^{n+1}}$.

4. Let the Laplace transform of $f(t)$ be $F(s)$.

$$\begin{aligned}F(s) &= \int_0^\infty e^{-st} f(t)dt\\ &= \int_0^p e^{-st} f(t)dt + \int_p^{2p} e^{-st} f(t)dt + \cdots + \int_{(n-1)p}^{np} e^{-st} f(t)dt + \cdots\\ &= (1 + e^{-ps} + \cdots + e^{-(n-1)ps} + \cdots)\left(\int_0^p e^{-st} f(t)dt\right)\\ &= \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)dt.\end{aligned}$$

5. Let the Laplace transform of $f(t)$ be $\mathcal{L}(f)$.

   i) Here $f(t) = t\cos\omega t$,
   $\therefore f'(t) = \cos\omega t - \omega t \sin\omega t$,
   $\therefore f''(t) = -2\omega\sin\omega t - \omega^2 t\cos\omega t$.

   By the result of Laplace transform of derivative, we know that
   $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$.
   $\therefore \mathcal{L}(-2\omega\sin\omega t - \omega^2 t\cos\omega t) = s^2\mathcal{L}(t\cos\omega t) - s.0 - 1$
   $\therefore -2\omega\mathcal{L}(\sin\omega t) - \omega^2\mathcal{L}(t\cos\omega t) = s^2\mathcal{L}(t\cos\omega t) - 1$
   $\therefore (s^2+\omega^2)\mathcal{L}(t\cos\omega t) = 1 - 2\omega\mathcal{L}(\sin\omega t)$
   $\therefore (s^2+\omega^2)\mathcal{L}(t\cos\omega t) = 1 - \frac{2\omega^2}{s^2+\omega^2}$
   $\therefore \mathcal{L}(t\cos\omega t) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$.

   ii) Here $f(t) = t\sinh\omega t$
   $\therefore f'(t) = \sinh\omega t + t\omega\cosh\omega t$
   $\therefore f''(t) = 2\omega\cosh\omega t + \omega^2 t\sinh\omega t$.
   By the result of Laplace transform of derivative, we have
   $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$
   $\therefore \mathcal{L}(2\omega\cosh\omega t + \omega^2 t\sinh\omega t) = s^2\mathcal{L}(t\sinh\omega t) - s.0 - 0$
   $\therefore 2\omega\mathcal{L}(\cosh\omega t) + \omega^2\mathcal{L}(t\sinh\omega t) = s^2\mathcal{L}(t\sinh\omega t)$
   $\therefore (s^2-\omega^2)\mathcal{L}(t\sinh\omega t) = 2\frac{\omega s}{s^2-\omega^2}$
   $\therefore \mathcal{L}(t\sinh\omega t) = \frac{2\omega s}{(s^2-\omega^2)^2}$.

6. Let $\mathcal{L}(f)$ denote the Laplace transform of $f(t)$.

$$\begin{aligned}\mathcal{L}(f') &= \int_0^\infty e^{-st} f'(t)dt\\ &= \int_0^a e^{-st} f'(t)dt + \int_a^\infty e^{-st} f'(t)dt\\ &= [e^{-st}f(t)]_0^a + s\int_0^a e^{-st} f(t)dt + [e^{-st}f(t)]_a^\infty + s\int_a^\infty e^{-st} f(t)dt\\ &= e^{-sa}f(a-0) - f(0) + s\int_0^a e^{-st} f(t)dt - e^{-sa}f(a+0) + s\int_a^\infty e^{-st} f(t)dt\\ &= s\mathcal{L}(f) - f(0) - e^{-sa}[f(a+0) - f(a-0)].\end{aligned}$$

7. Let the inverse Laplace function of $F(s)$ be $\mathcal{L}^{-1}(F)$.

$$\begin{aligned}
\mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\Big(\tfrac{A_1}{(s-a)^m} + \tfrac{A_2}{(s-a)^{m-1}} + \cdots + \tfrac{A_m}{s-a} + \tfrac{B_1}{s-b_1} + \cdots + \tfrac{B_n}{s-b_n}\Big) \\
&= \mathcal{L}^{-1}\Big(\tfrac{A_1}{(s-a)^m}\Big) + \mathcal{L}^{-1}\Big(\tfrac{A_2}{(s-a)^{m-1}}\Big) + \cdots + \mathcal{L}^{-1}\Big(\tfrac{A_m}{s-a}\Big) + \mathcal{L}^{-1}\Big(\tfrac{B_1}{s-b_1}\Big) + \cdots + \mathcal{L}^{-1}\Big(\tfrac{B_n}{s-b_n}\Big) \\
&= e^{at}\Big(A_1\tfrac{t^{m-1}}{(m-1)!} + A_2\tfrac{t^{m-2}}{(m-2)!} + \cdots + A_m\Big) + B_1e^{b_1t} + \cdots + B_ne^{b_nt}.
\end{aligned}$$

8. i)

$$f(t) = t^2(1-u(t-2)) + 4t\ u(t-2).$$

ii)

$$f(t) = \sin t(1-u(t-\pi)) + \sin 2t(u(t-\pi)-u(t-2\pi)) + \sin 3t\ u(t-2\pi).$$

9. Let $\mathcal{L}(f)$ be the Laplace transform of $f(t)$.

i) $f(t) = 5(1-u(t-7))$.

$$\begin{aligned}
\mathcal{L}(f) &= \mathcal{L}(5(1-u(t-7))) \\
&= \mathcal{L}(5) - 5\mathcal{L}(u(t-7)) \text{ by linearty} \\
&= \tfrac{5}{s} - 5\tfrac{e^{-7s}}{s}.
\end{aligned}$$

ii) $f(t) = \sin t(u(t-\frac{\pi}{2}) - u(t-\pi))$.

$$\begin{aligned}
\mathcal{L}(f) &= \mathcal{L}(\sin t(u(t-\tfrac{\pi}{2}) - u(t-\pi))) \\
&= \mathcal{L}(\sin t\ u(t-\tfrac{\pi}{2})) - \mathcal{L}(\sin t\ u(t-\pi)) \\
&= \mathcal{L}(\sin(t-\tfrac{\pi}{2}+\tfrac{\pi}{2})u(t-\tfrac{\pi}{2})) - \mathcal{L}(\sin(t-\pi+\pi)u(t\pi)) \\
&= \mathcal{L}((\sin(t-\tfrac{\pi}{2})\cos\tfrac{\pi}{2} + \cos(t-\tfrac{\pi}{2})\sin\tfrac{\pi}{2})u(t-\tfrac{\pi}{2})) - \mathcal{L}((\sin(t-\pi)\cos\pi \\
&= +\cos(t-\pi)\sin\pi)u(t-\pi)) \\
&= \mathcal{L}(\cos(t-\tfrac{\pi}{2})u(t-\tfrac{\pi}{2})) + \mathcal{L}(\sin(t-\pi)u(t-\pi)) \\
&= e^{-\frac{\pi}{2}s}\mathcal{L}(\cos\ t) + e^{-\pi s}\mathcal{L}(\sin t) \\
&= e^{-\frac{\pi}{2}s}\tfrac{s}{s^2+1} + e^{-\pi s}\tfrac{1}{s^2+1}.
\end{aligned}$$

10. Let $\mathcal{L}(f)$ be the Laplace transform of $f(t)$.

i) We have

$$\begin{aligned}
\mathcal{L}(\sin 3t) &= \tfrac{3}{s^2+9} \\
\therefore \mathcal{L}(t\sin 3t) &= -\tfrac{d}{ds}(\tfrac{3}{s^2+9}) \text{ by differentiation of Laplace transform} \\
\therefore \mathcal{L}(t\sin 3t) &= \tfrac{6s}{(s^2+9)^2} \\
\therefore \mathcal{L}(t^2\sin 3t) &= -\tfrac{d}{ds}(\tfrac{6s}{(s^2+9)^2}) \text{ by differentiation of Laplace transform} \\
\therefore \mathcal{L}(t^2\sin 3t) &= \tfrac{18s^2-54}{(s^2+9)^3}.
\end{aligned}$$

ii) Let $f(t) = \sin t$.
By integration of Laplace transform, we have $\mathcal{L}(\frac{f(t)}{t}) = \int_s^\infty F(\tilde{s})d\tilde{s}$.
Here $F(s) = \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$. So,

$$\begin{aligned}
\mathcal{L}(\tfrac{\sin t}{t}) &= \int_s^\infty \frac{a}{\tilde{s}^2+a^2}d\tilde{s} \\
&= \int_{\frac{s}{a}}^\infty \frac{ds'}{s'^2+1} \\
&= [\tan^{-1} s']_{\frac{s}{a}}^\infty \\
&= \frac{\pi}{2} - \tan^{-1}(\tfrac{s}{a}) \\
&= \tan^{-1}(\tfrac{a}{s}).
\end{aligned}$$

iii) We have

$$\begin{aligned}
&\mathcal{L}(4t * e^{-2t}) \\
= \;& \mathcal{L}(4t).\mathcal{L}(e^{-2t}) \text{ by Convolution Theorem} \\
= \;& 4\mathcal{L}(t).\mathcal{L}(e^{-2t}) \\
= \;& \frac{4}{s^2}\frac{1}{s+2} \\
= \;& \frac{4}{s^2(s+2)}.
\end{aligned}$$

11. i) We prove this by induction. From the result on the differentiation of the Laplace transform we know that

$$\mathcal{L}(tf(t)) = -F'(s), \text{ where } \mathcal{L}(f) \text{ is the Laplace transform of } f.$$

Let's assume that $\mathcal{L}(t^{n-1}f(t)) = (-1)^{n-1}F^{(n-1)}(s)$. Then

$$\begin{aligned}
\mathcal{L}(t^n f(t)) &= \mathcal{L}(t.t^{n-1}f(t)) \\
&= -((-1)^{n-1}F^{(n-1)}(s))' \\
&= (-1)^n F^{(n)}(s).
\end{aligned}$$

ii) We have

$$\begin{aligned}
\mathcal{L}(t^n e^{kt}) &= (-1)^n\big(\mathcal{L}(e^{kt})\big)^{(n)} \\
&= (-1)^n\big(\tfrac{1}{s-k}\big)^{(n)} \\
&= (-1)^{2n}\frac{n!}{(s-k)^{(n+1)}} \\
&= \frac{n!}{(s-k)^{n-1}}.
\end{aligned}$$

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