Ex Find $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$ by L.T.

Sol^n: $L\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1}s$ [Done already]

$\therefore \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}s$

Putting $s=1$

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \frac{\pi}{4}$$

Ex Find L.T. of $\int_0^t \left(\frac{1-e^{-2x}}{x}\right)dx$

Sol^n: If $L\{F(t)\} = f(s)$, then

(i) $L\left[\int_0^t F(x)\,dx\right] = \frac{f(s)}{s}$

(ii) $\int_s^\infty f(x)\,dx = L\left[\frac{F(t)}{t}\right]$

$$L\{1-e^{-2t}\} = \frac{1}{s} - \frac{1}{s+2}$$

$$L\left\{\frac{1-e^{-2t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+2}\right)ds$$

$$= \{\ln s - \ln(s+2)\}_s^\infty$$

$$= -\left[\ln\left(1+\frac{2}{s}\right)\right]_s^\infty$$

$$= -\left[\ln 1 - \ln\left(1+\frac{2}{s}\right)\right]$$

$$= \ln\left(1+\frac{2}{s}\right)$$

$$\therefore L\int_0^t \left[\frac{1-e^{-2x}}{x}\right]dx = \frac{1}{s}\ln\left(1+\frac{2}{s}\right)$$

Ex Find $L\{H(t)\}$ where $H(t)$ is defined as

$$H(t) = \begin{cases} t+1 & 0 \le t \le 2 \\ 3 & t > 2 \end{cases}$$

and determine $L\{H'(t)\}$

Sol^n: $L\{H(t)\} = \int_0^\infty e^{-st} H(t)\, dt$

$$= \int_0^2 e^{-st} H(t)\, dt + \int_2^\infty e^{-st} H(t)\, dt$$

$$= \int_0^2 e^{-st} (t+1)\, dt + \int_2^\infty e^{-st} \cdot 3\, dt$$

$$= \int_0^2 t e^{-st}\, dt + \left[\frac{e^{-st}}{-s}\right]_0^2 + \left[\frac{3e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{1}{s^2}\left[s+1-e^{-2s}\right] = h(s) \quad \text{[after simplification]}$$

$H(0) = (t+1)_{t=0} = 1$

$L\{H'(t)\} = s\,h(s) - H(0) = \dfrac{1-e^{-2s}}{s}$

Theorem 10

Initial value theorem

Let $F(t)$ be continuous $\forall\, t \ge 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A,

then $\lim\limits_{t\to 0} F(t) = \lim\limits_{s\to\infty} s f(s)$

Proof! By the theorem of L.T. of derivative of $F(t)$

$$L\{F'(t)\} = s f(s) - F(0)$$

$$\Rightarrow \int_0^\infty e^{-st} F'(t)\, dt = s f(s) - F(0)$$

Making $s \to \infty$, LHS is 0 [Behaviour of $f(s)$ as $s \to \infty$]

$\therefore \lim\limits_{s\to\infty} s f(s) = F(0) = \lim\limits_{t\to 0} F(t)$

Theorem 11

Final value theorem

Let $F(t)$ be continuous $\forall\ t \geq 0$ and be of exponential order and if $F'(t)$ is of class A, then

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sf(s)$$

Proof: $L\{F'(t)\} = sf(s) - F(0)$

or, $\int_0^\infty e^{-st} F'(t)\,dt = sf(s) - F(0)$

Making $s \to 0$,

$$\lim_{s\to 0} sf(s) - F(0) = \lim_{s\to 0} \int_0^\infty e^{-st} F'(t)\,dt$$

$$= \int_0^\infty \left(\lim_{s\to 0} e^{-st}\right) F'(t)\,dt$$

$$= \int_0^\infty F'(t)\,dt$$

$$= \left[F(t)\right]_0^\infty$$

$$= \lim_{t\to\infty} F(t) - F(0)$$

$$\lim_{s\to 0} sf(s) = \lim_{t\to\infty} F(t)$$

Periodic functions

Definition

If $F(t)$ is a f^n. that obeys the rule

$$F(t) = F(t+nT) \quad n=1,2,3,\ldots$$

for some real T for all values of t, then $F(t)$ is called a periodic f^n. with period T.

Theorem 12

Let $F(t)$ be a periodic f^n. with period $T>0$ i.e. $F(t) = F(t+nT)$, then

$$L\{F(t)\} = \frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$$

Proof: $L\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$

$$= \int_0^T e^{-st}F(t)\,dt + \int_T^{2T} e^{-st}F(t)\,dt + \ldots$$

$t = u+T$
$t = u+2T$
in the 2nd, 3rd integral

$$= \int_0^T e^{-st}F(t)\,dt + \int_0^T e^{-s(u+T)}F(u+T)\,du + \ldots$$

$$= \int_0^T e^{-su}F(u)\,du + e^{-sT}\int_0^T e^{-su}F(u)\,du + \ldots$$

$$= \left[1+e^{-sT}+e^{-2sT}+\ldots\right]\int_0^T e^{-su}F(u)\,du$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-su}F(u)\,du$$

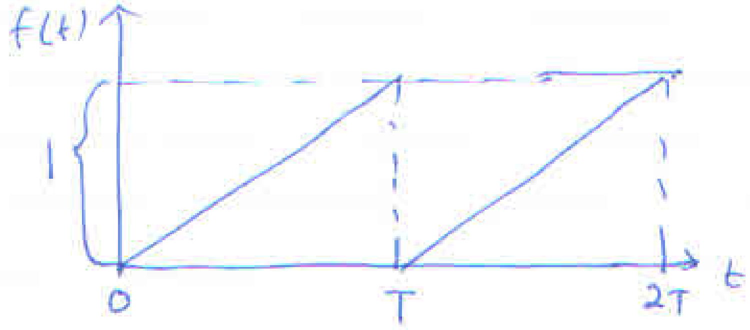
$$= \frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$$

Problems on periodic f^ns:

Ex Find the L.T. of the saw-tooth wave f^n.

$$f(t) = \frac{t}{T} \text{ of period } T, \quad 0 \le t < T \text{ i.e. } f(t+T) = f(t)$$

Sol^n:



$$L\{F(t)\} = \frac{1}{1-e^{-ST}} \int_0^T e^{-St} \frac{t}{T}\, dt$$

$$= \frac{1}{T(1-e^{-ST})} \int_0^T e^{-St}\, t\, dt$$

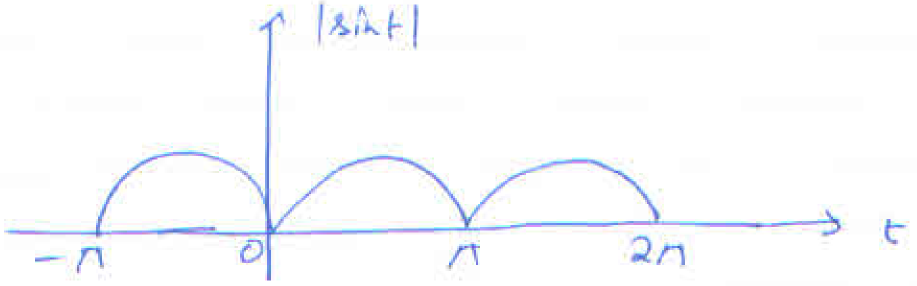
$$= \frac{1}{T(1-e^{-ST})} \left[\frac{te^{-St}}{-S}\Big|_0^T + \frac{1}{S}\int_0^T e^{-St}\, dt \right]$$

$$= \frac{1}{T(1-e^{-ST})} \left[\frac{Te^{-ST}}{-S} - \frac{1}{S^2}\left(e^{-ST}-1\right) \right]$$

$$= \frac{1}{S^2 T} - \frac{e^{-ST}}{S(1-e^{-ST})}, \quad S>0$$

Ex Find $L\left[|\sin t|\right]$

Sol^n:



$T=\pi$

$$L\left[|\sin t|\right]$$
$$= \frac{1}{1-e^{-s\pi}} \int_0^\pi e^{-st} \sin t \, dt$$
$$= \frac{1}{1-e^{-s\pi}} I_1$$

$$I_1 = \int_0^\pi e^{-st} \sin t \, dt$$
$$= \left[-e^{-st}\cos t\right]_0^\pi - s\int_0^\pi e^{-st}\cos t\,dt$$
$$= -\left[-e^{-s\pi}-1\right] - s\left[e^{-st}\sin t\right]_0^\pi - s^2 \underbrace{\int_0^\pi e^{-st}\sin t\,dt}_{I_1}$$

$$\therefore (1+s^2)\, I_1 = 1 + e^{-\pi s}$$
$$I_1 = \frac{1+e^{-\pi s}}{1+s^2}$$
$$\therefore L\left[|\sin t|\right] = \frac{1}{1-e^{-s\pi}} \frac{1+e^{-s\pi}}{1+s^2}$$

Laplace transform of some special functions

1. Sine integral function

$$S_i(t) = \int_0^t \frac{\sin x}{x}\,dx$$

Solⁿ: $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{x^2+1}\,dx$ $\left[\begin{array}{l} L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(s)}{s} \\ L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx \end{array}\right]$

$$= \left[\tan^{-1}x\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\frac{1}{s}$$

$$\therefore L\left\{\int_0^t \frac{\sin x}{x}\,dx\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

2. Cosine integral function

$$C_i(t) = \int_t^\infty \frac{\cos x}{x}\,dx$$

Solⁿ: Let $F(t) = \int_t^\infty \frac{\cos x}{x}\,dx$

$$\boxed{\begin{array}{l} \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx \\ = \int_a^b \frac{\partial}{\partial t} f(x,t)\,dx \\ \quad + f(b,t)\,b'(t) - f(a,t)\,a'(t) \end{array}}$$

$$F'(t) = -\frac{\cos t}{t}$$

$$\therefore tF'(t) = -\cos t$$

$$\therefore L\{tF'(t)\} = -L\{\cos t\}$$

$$\therefore -\frac{d}{ds}\{s f(s) - F(0)\} = -\frac{s}{s^2+1}$$

$$\therefore \frac{d}{ds}\{s f(s)\} = \frac{s}{s^2+1} \quad [\because F(0) \text{ is constant}]$$

$$\therefore s f(s) = \frac{1}{2}\ln(s^2+1) + c$$

By final value theorem, $\lim_{s\to 0} s f(s) = \lim_{t\to\infty} F(t) = 0 \quad \therefore c = 0$

$$\therefore s f(s) = \frac{1}{2}\ln(s^2+1)$$

$$\therefore f(s) = \frac{\ln(s^2+1)}{2s}$$

3. Exponential integral f^n.

$$E(t) = \int_t^{\infty} \frac{e^{-x}}{x}\, dx$$

Let $F(t) = \int_t^{\infty} \frac{e^{-x}}{x}\, dx$

$\therefore F'(t) = -\frac{e^{-t}}{t}$ $\quad \therefore tF'(t) = -e^{-t}$

$\therefore L\{tF'(t)\} = -L\{e^{-t}\}$

$\Rightarrow -\frac{d}{ds}\{s\,f(s) - F(0)\} = -\frac{1}{s+1}$

$\Rightarrow \frac{d}{ds}\{s\,f(s)\} = \frac{1}{s+1}$ $\qquad F(0) = \text{const.}$

Integrating,

$$s\,f(s) = \ln(s+1) + c$$

$\therefore \lim_{s \to 0} s\,f(s) = \lim_{s \to 0} \ln(s+1) + c = 0 + c = c$

By final value theorem

$$\lim_{s \to 0} s\,f(s) = \lim_{t \to \infty} F(t)$$

Now $\lim_{t \to \infty} F(t) = 0$

$\therefore c = 0$

$s\,f(s) = \ln(s+1)$

$\therefore f(s) = \frac{\ln(s+1)}{s}$

4. Error function

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx$$

To find $L\{\mathrm{erf}(t)\}$ and $L\{\mathrm{erf}\sqrt{t}\}$.

$$\mathrm{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\, dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \cdots\right]$$

$$L\{\mathrm{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{\Gamma(\frac{5}{2})}{3\, s^{5/2}} + \frac{\Gamma(\frac{7}{2})}{5\; 2!\; s^{7/2}} - \cdots\right]$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1.3}{2.4}\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\frac{1}{s^3} + \cdots\right]$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-1/2}$$

$$= \frac{1}{s(s+1)^{1/2}}$$

$$\mathrm{erf}(0) = 0$$

$$\mathrm{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2}\, dx = \frac{2}{\sqrt{\pi}}\, \frac{\sqrt{\pi}}{2} = 1$$

$$\operatorname{erf_c}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-x^2}dx - \int_0^t e^{-x^2}dx \right]$$

$$= 1 - \operatorname{erf}(t)$$

$$L\{\operatorname{erf}(t)\} = L\left\{ \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\,du \right\}$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} L\left\{ e^{-u^2} \right\}$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^\infty e^{-su} e^{-u^2}\,du$$

$$= \frac{2}{s\sqrt{\pi}} e^{+\frac{s^2}{4}} \int_0^\infty e^{-\left(u+\frac{s}{2}\right)^2} du$$

Put $x = u + \frac{s}{2}$

$$\therefore L\{\operatorname{erf} t\} = \frac{2}{s\sqrt{\pi}} e^{\frac{s^2}{4}} \int_{s/2}^\infty e^{-x^2}\,dx$$

$$= \frac{1}{s} e^{\frac{s^2}{4}} \operatorname{erf_c}\left(\frac{s}{2}\right)$$

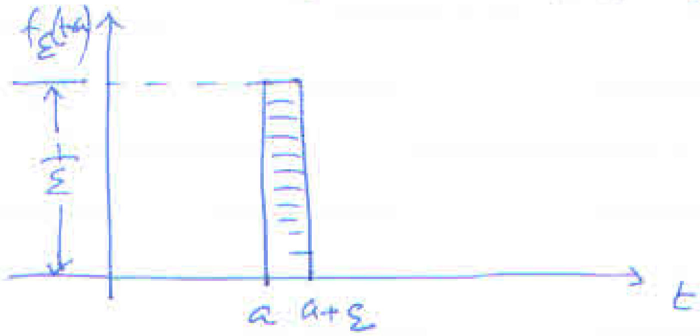
5. Unit step function or Heaviside's unit function

Already done

6. Unit impulse f^n. or Dirac delta f^n.

In mechanics, the impulse of a force $f(t)$ over a time interval $a \le t \le a+\varepsilon$ is defined to be the integral of $f(t)$ from a to $a+\varepsilon$. Of particular practical interest is the case of a very short ε (limit of $\varepsilon \to 0$) i.e. the impulse of a force acting only for an instant. To handle this case, we consider the f^n.

$$f_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & \text{if } a \le t \le a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



$\therefore$ Impulse $I_\varepsilon = \int_0^\infty f_\varepsilon(t-a)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1$

We can represent $f_\varepsilon(t-a)$ in terms of two unit step f^ns.

$$f_\varepsilon(t-a) = \frac{1}{\varepsilon}\left[u(t-a) - u\{t-(a+\varepsilon)\}\right]$$

$$\therefore L\{f_\varepsilon(t-a)\} = \frac{1}{\varepsilon s}\left[e^{-as} - e^{-(a+\varepsilon)s}\right] = e^{-as}\,\frac{1-e^{-\varepsilon s}}{\varepsilon s} \quad \text{(I)}$$

The limit of $f_\varepsilon(t-a)$ as $\varepsilon \to 0$ $(\varepsilon > 0)$ is denoted by $\delta(t-a)$ i.e.

$$\delta(t-a) = \lim_{\varepsilon \to 0} f_\varepsilon(t-a)$$

$\delta(t-a)$ is called the Dirac Delta function.

The quotient in I has the limit 1 as $\varepsilon \to 0$ by L'Hospital's rule. Hence the right side of (I) has the limit e^{-as}. This suggests defining the L.T. of $\delta(t-a)$ by the limit in I i.e.

$$L\{\delta(t-a)\} = e^{-as}$$

We note that $\delta(t-a)$ is not a f^n. in the ordinary sense as used in Calculus.

In $f_\varepsilon(t-a)$ and I_ε with $\varepsilon \to 0$

imply

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}$$

and $\int_0^\infty \delta(t-a)\,dt = 1$

An ordinary f^n. that is everywhere 0 except at a single pt. must have the integral 0.

Inverse Laplace transform

Definition

If $F(t)$ has the L.T. $f(s)$ i.e.

$$L\{F(t)\} = f(s)$$

then the inverse L.T. is defined by

$$L^{-1}\{f(s)\} = F(t)$$

Null f^n.

If $N(t)$ is a f^n. of t such that $\int_0^t N(t)\,dt = 0$ then $N(t)$ is called a null f^n.

Lerch's theorem

If $F_1(t)$ and $F_2(t)$ are two f^ns. having the same L.T. $f(S)$, then $F_1(t) - F_2(t) = N(t)$ where $N(t)$ is a null f^n. $\forall\ t > 0$

i.e. an inverse L.T. is unique except for the addition of a null f^n.

Ex Find $L^{-1}\left\{\frac{s}{s^2+2} + \frac{6s}{s^2-16} + \frac{3}{s-3}\right\}$

$$= L^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} + 6L^{-1}\left\{\frac{s}{s^2-4^2}\right\} + 3L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \cos\sqrt{2}\,t + 6\cosh 4t + 3e^{3t}$$

Ex Find $L^{-1}\left\{\frac{5}{s^2} + \left(\frac{\sqrt{s}-1}{s}\right)^2 - \frac{7}{3s+2}\right\}$

Sol^n:
$$= L^{-1}\left\{\frac{5}{s^2} + \frac{s-2\sqrt{s}+1}{s^2} - \frac{7}{3}\,\frac{1}{s+\frac{2}{3}}\right\}$$

$$= 6L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{1}{s^{3/2}}\right\} - \frac{7}{3}L^{-1}\left\{\frac{1}{s+\frac{2}{3}}\right\}$$

$$= 6t + 1 - 2\,\frac{t^{\frac{3}{2}-1}}{\Gamma\left(\frac{3}{2}\right)} - \frac{7}{3}e^{-\frac{2t}{3}}$$

$$= 6t + 1 - 4\sqrt{\frac{t}{\pi}} - \frac{7}{3}e^{-\frac{2t}{3}}$$

Ex Find $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\}$

Sol^n:
$$= L^{-1}\left\{\frac{3s-2}{(s-2)^2+16}\right\}$$

$$= L^{-1}\left\{\frac{3(s-2)}{(s-2)^2+16} + \frac{4}{(s-2)^2+16}\right\}$$

$$= 3e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + 4e^{2t}\sin 4t$$

Ex Find $L^{-1}\left\{\frac{s-1}{(s+3)(s^2+2s+2)}\right\}$

Sol^n:

$$= L^{-1}\left\{-\frac{4}{5(s+3)} + \frac{4s+1}{5(s^2+2s+2)}\right\}$$

$$= -\frac{4}{5}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{5}L^{-1}\left\{\frac{4(s+1)-3}{(s+1)^2+1}\right\}$$

$$= -\frac{4}{5}e^{-3t} + \frac{e^{-t}}{5}\left[L^{-1}\left\{\frac{4s}{s^2+1} - \frac{3}{s^2+1}\right\}\right]$$

$$= -\frac{4}{5}e^{-3t} + \frac{e^{-t}}{5}(4\cos t - 3\sin t)$$

Ex Find $L^{-1}\ \frac{s}{(s+1)^5}$

Sol^n:

$$L^{-1}\left\{\frac{(s+1)-1}{(s+1)^5}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s-1}{s^5}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{s^4}\right\} - e^{-t}L^{-1}\left\{\frac{1}{s^5}\right\}$$

$$= e^{-t}\frac{t^3}{3!} - e^{-t}\frac{t^4}{4!}$$

$$= \frac{e^{-t}}{24}(4t^3 - t^4)$$

Ex Find $L^{-1} \dfrac{e^{-7s}}{(s-3)^3}$

Sol^n: $L^{-1} \dfrac{1}{(s-3)^3} = \frac{1}{2} t^2 e^{3t}$ [1st shifting theorem]

$$L^{-1} \frac{e^{-7s}}{(s-3)^3} = \begin{cases} \frac{1}{2}(t-7)^2 e^{3(t-7)}, & t > 7 \\ 0, & 0 \le t \le 7 \end{cases}$$

$$= \frac{1}{2} H(t-7)(t-7)^2 e^{3(t-7)}$$

Ex Find $L^{-1}\left\{\dfrac{s^2}{s^2+1}\right\}$

Sol^n: $\dfrac{s^2}{s^2+1} = 1 - \dfrac{1}{s^2+1}$

$$\therefore L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = L^{-1}\{1\} - L^{-1}\left\{\frac{1}{s^2+1}\right\} = \delta(t) - \sin t$$

Ex If $L^{-1}\left\{\dfrac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$, find $L^{-1}\left\{\dfrac{9s^2-1}{(9s^2+1)^2}\right\}$

Sol^n: $L^{-1}\left\{\dfrac{a^2s^2-1}{(a^2s^2+1)^2}\right\} = \dfrac{1}{a}\dfrac{t}{a}\cos\dfrac{t}{a}$

Put $a=3$, $L^{-1}\left\{\dfrac{9s^2-1}{(9s^2+1)^2}\right\} = \dfrac{t}{9}\cos\left(\dfrac{t}{3}\right)$

Ex Prove that

$$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\frac{t\sqrt{3}}{2}$$

Sol^n:

$$= L^{-1}\left\{\frac{s}{(s^2+1)^2-s^2}\right\}$$

$$= L^{-1}\left\{\frac{s}{(s^2+s+1)(s^2-s+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{2(s^2-s+1)} - \frac{2}{2(s^2+s+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{2\left[(s-\frac{1}{2})^2+\frac{3}{4}\right]} - \frac{1}{2\left[(s+\frac{1}{2})^2+\frac{3}{4}\right]}\right\}$$

$$= \frac{e^{t/2}}{2}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\} - \frac{e^{-\frac{t}{2}}}{2}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}$$

$$= \frac{e^{t/2}}{2}\frac{2}{\sqrt{3}}\sin\frac{t\sqrt{3}}{2} - \frac{e^{-\frac{t}{2}}}{2}\frac{2}{\sqrt{3}}\sin\frac{t\sqrt{3}}{2}$$

$$= \frac{2}{\sqrt{3}}\sin\frac{t\sqrt{3}}{2}\quad\frac{e^{t/2}-e^{-t/2}}{2}$$

$$= \frac{2}{\sqrt{3}}\sin\frac{t\sqrt{3}}{2}\sinh\frac{t}{2}$$

Convolution

Definition

Let $F(t)$ and $G(t)$ be two f^ns. of class A. Then the convolution of the two f^ns. $F(t)$ and $G(t)$ denoted by $F*G$ is defined by the relation

$$F*G = \int_0^t F(x)\,G(t-x)\,dx$$

Properties

(i) $F*G$ is commutative

i.e. $F*G = G*F$

Proof:
$$\begin{aligned} F*G &= \int_0^t F(x)\,G(t-x)\,dx \\ &= -\int_t^0 F(t-y)\,G(y)\,dy \qquad \text{Putting } t-x=y \\ &= \int_0^t G(y)\,F(t-y)\,dy \\ &= G*F \end{aligned}$$

(ii) Associative property

i.e. $(F*G)*H = F*(G*H)$

(iii) Distributive property w.r.t. addition

i.e. $F*(G+H) = F*G + F*H$

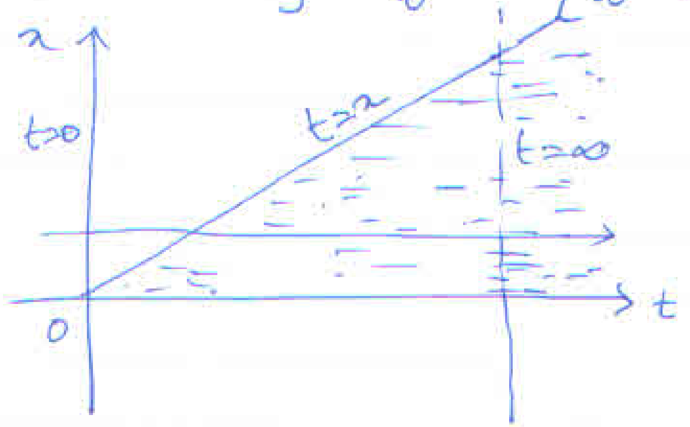
Convolution theorem

Let $F(t)$ and $G(t)$ be two functions of class A and let $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$.

Then $L^{-1}\{f(s)g(s)\} = \int_0^t F(x)G(t-x)\,dx = F*G$

i.e. $L\{F*G\} = f(s)g(s) = L\{F(t)\}L\{G(t)\}$

Proof: $L\{F(t)*G(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t F(x)G(t-x)\,dx\right\}dt$



Changing the order of integration

$$L\{F(t)*G(t)\} = \int_0^\infty \int_{t=x}^\infty e^{-st} F(x)\,G(t-x)\,dt\,dx$$

$$= \int_0^\infty F(x)\left\{\int_x^\infty e^{-st}G(t-x)\,dt\right\}dx$$

Let $u = t-x$ $\therefore \int_x^\infty e^{-st}G(t-x)\,dt$

$$= \int_0^\infty e^{-s(u+x)}G(u)\,du$$

$$= e^{-sx}\int_0^\infty e^{-su}G(u)\,du$$

$$= e^{-sx}g(s)$$

$$L\{F*G\} = \int_0^\infty F(x)e^{-sx}g(s)\,dx = g(s)f(s)$$
$$= f(s)g(s)$$

Ex Find the value of $\cos t * \sin t$

Sol^n: $\cos t * \sin t = \int_0^t \cos x \sin(t-x)\,dx$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\sin(t-x)\cos x = \frac{1}{2}\left[\sin t + \sin(t-2x)\right]$

$$\therefore \cos t * \sin t = \frac{1}{2}\int_0^t \left[\sin t + \sin(t-2x)\right] dx$$
$$= \frac{1}{2}\sin t\,[x]_0^t + \frac{1}{4}\left[\cos(t-2x)\right]_0^t$$
$$= \frac{1}{2} t\sin t + \frac{1}{4}\left[\cos(-t) - \cos t\right]$$
$$= \frac{1}{2} t \sin t$$

Ex Find the value of $e^t * t$

Sol^n: $e^t * t = \int_0^t e^x (t-x)\,dx$
$$= te^x\Big|_0^t - (xe^x - e^x)\Big|_0^t$$
$$= e^t - t - 1$$

Ex Find $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$ by convolution theorem

Sol^n: $L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \qquad L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$L\{\cos t\}\,L\{\sin t\} = \frac{s}{(s^2+1)^2}$

$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cos t * \sin t = \frac{1}{2} t \sin t$

Ex Find $L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right]$

Solⁿ: 1st method $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$

$$\therefore L[e^t \operatorname{erf}\sqrt{t}] = \frac{1}{\sqrt{s}\,(s-1)}$$

$$\therefore L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = e^t \operatorname{erf}\sqrt{t}$$

2nd method (By convolution theorem)

Let $f(s) = \frac{1}{\sqrt{s}}$ $\qquad g(s) = \frac{1}{s-1}$

$G(t) = e^t$

$L\{t^{-1/2}\} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$ $\qquad \therefore L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} = F(t)$

By convolution theorem

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(x)\,G(t-x)\,dx$$

$$= \int_0^t \frac{1}{\sqrt{\pi x}}\, e^{(t-x)}\,dx$$

$$= \frac{e^t}{\sqrt{\pi}} \int_0^t \frac{e^{-x}}{\sqrt{x}}\,dx$$

Putting $x = u^2$

$$= \frac{e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{e^{-u^2}}{u} \cdot 2u\,du$$

$$= e^t \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\,du$$

$$= e^t \operatorname{erf}\sqrt{t}$$

Ex Apply convolution theorem to prove that

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m>0, n>0$$

Solⁿ: Let $F(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx$

Let $F_1(t) = t^{m-1}$ $\quad F_2(t) = t^{n-1}$

$$F(t) = \int_0^t F_1(x)\,F_2(t-x)\,dx = F_1 * F_2$$

$$L\{F(t)\} = L\{F_1 * F_2\}$$

$$= L\{F_1(t)\}\cdot L\{F_2(t)\}$$

$$= L\{t^{m-1}\}\cdot L\{t^{n-1}\}$$

$$= \frac{\Gamma(m)}{s^m}\cdot\frac{\Gamma(n)}{s^n} = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}$$

$$\therefore F(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx = L^{-1}\left\{\frac{\Gamma(m)\Gamma(n)}{s^{m+n}}\right\}$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\,t^{m+n-1}$$

Put $t=1$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Applications

I Evaluation of improper integrals

Evaluate $\int_0^\infty e^{-x^2}\,dx$

Solⁿ: Let $F(t) = \int_0^\infty e^{-tx^2}\,dx$

$$L\{F(t)\} = \int_0^\infty e^{-st}\left\{\int_0^\infty e^{-tx^2}\,dx\right\}dt$$

$$= \int_0^\infty \left\{\int_0^\infty e^{-st}e^{-tx^2}\,dt\right\}dx$$

$$= \int_0^\infty \left[L\{e^{-tx^2}\}\right]dx$$

$$= \int_0^\infty \frac{dx}{s+x^2} \qquad \left[L\{e^{at}\} = \frac{1}{s-a}\right]$$

$$= \left[\frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\right]_0^\infty$$

$$= \frac{\pi}{2\sqrt{s}}$$

$$F(t) = \frac{\pi}{2}L^{-1}\left[\frac{1}{\sqrt{s}}\right]$$

$$= \frac{\pi}{2}\frac{1}{\sqrt{\pi t}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

$$\int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

Put $t=1$ $\quad\therefore \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Ex Evaluate $\int_0^\infty \cos x^2 \, dx$

Sol^n: Let $F(t) = \int_0^\infty \cos tx^2 \, dx$

$$L\{F(t)\} = \int_0^\infty e^{-st}\left\{\int_0^\infty \cos tx^2 \, dx\right\} dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cos tx^2 \, dt\right] dx$$

$$= \int_0^\infty L(\cos tx^2) \, dx$$

$$= \int_0^\infty \frac{s}{s^2+x^4} \, dx$$

Put $x^2 = s\tan\theta$ i.e. $x = \sqrt{s}\sqrt{\tan\theta}$ $dx = \dfrac{s\sec^2\theta \, d\theta}{2\sqrt{(s\tan\theta)}}$

$$I = \frac{1}{2\sqrt{s}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}}$$

$$= \frac{1}{2\sqrt{s}}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta \, d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{2\Gamma(1)}$$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}{2}$$

$$= \frac{1}{4\sqrt{s}} \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2s}}$$

$$\left[\begin{array}{l} \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})} \\ p > -1, \; q > -1 \\ \Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi} \\ 0 < n < 1 \end{array}\right]$$

$$\therefore F(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left\{\frac{1}{s^{1/2}}\right\}$$

$$= \frac{\pi}{2\sqrt{2}} \frac{t^{1/2-1}}{\Gamma(\frac{1}{2})} = \frac{\pi}{2\sqrt{2}} \frac{1}{\sqrt{\pi t}} = \frac{1}{2}\sqrt{\frac{\pi}{2t}}$$

Put $t=1$, $\int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

E1 Evaluate $\int_0^\infty \frac{x\sin x}{1+x^2}\,dx$

Sol^n: $F(t) = \int_0^\infty \frac{x\sin tx}{1+x^2}\,dx$

$$L[F(t)] = \int_0^\infty e^{-st}\int_0^\infty \frac{x\sin tx}{1+x^2}\,dx\,dt$$

$$= \int_0^\infty \frac{x\,dx}{1+x^2} L\{\sin xt\}$$

$$= \int_0^\infty \frac{x^2}{(1+x^2)(x^2+s^2)}\,dx$$

$$= \int_0^\infty \frac{x^2+1-1}{(x^2+1)(x^2+s^2)}\,dx$$

$$= \int_0^\infty \frac{dx}{x^2+s^2} - \int_0^\infty \frac{dx}{(x^2+1)(x^2+s^2)}$$

$$= \left[\frac{1}{s}\tan^{-1}\frac{x}{s}\right]_0^\infty - \frac{1}{s^2-1}\left[\tan^{-1}x - \frac{1}{s}\tan^{-1}\frac{x}{s}\right]_0^\infty$$

$$= \frac{1}{s}\frac{\pi}{2} - \frac{1}{s^2-1}\left[\frac{\pi}{2} - \frac{1}{s}\frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{s} - \frac{1}{s^2-1} + \frac{1}{s(s^2-1)}\right]$$

$$= \frac{\pi}{2}\,\frac{1}{s+1}$$

$\therefore\ F(t) = \frac{\pi}{2}e^{-t}$

i.e. $\int_0^\infty \frac{x\sin tx}{1+x^2}\,dx = \frac{\pi}{2}e^{-t}$

Putting $t=1$, $\int_0^\infty \frac{x\sin x}{1+x^2}\,dx = \frac{\pi}{2e}$

II Solution of ordinary differential equations

(a) Sol^n. of linear ODE with constant coefficients

(i) First order ODE

Ex Use L.T. to solve

$$\frac{dy}{dt} + 3y = 13\sin 2t \qquad y(0)=6$$

Sol^n: Taking L.T. of both sides,

$$L\left\{\frac{dy}{dt}\right\} + 3L\{y\} = 13\,L\{\sin 2t\}$$

$$\Rightarrow\quad sY(s) - y(0) + 3L\{y\} = 13\,\frac{2}{s^2+4}$$

$$\Rightarrow\quad sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4}$$

$$\Rightarrow\quad (s+3)\,Y(s) = 6 + \frac{26}{s^2+4}$$

$$\Rightarrow\quad Y(s) = \frac{6}{s+3} + \frac{26}{(s^2+4)(s+3)}$$

$$= \frac{6s^2+50}{(s+3)(s^2+4)}$$

$$= \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$= \frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}$$

Taking inverse L.T.

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

(ii) Second order ODE

Ex Solve $\frac{d^2y}{dt^2} + y = 0$, $y(0) = 1$, $\frac{dy}{dt}\Big|_{t=0} = 0$

Solⁿ: $L\{y''\} + L\{y\} = 0$

$\Rightarrow s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = 0$

$\Rightarrow s^2 L\{y\} - s + L\{y\} = 0$

$\Rightarrow L\{y\} = \frac{s}{s^2+1}$

$\therefore y = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$

Ex Solve $(D^2 - 2D + 2)y = 0$ $y = Dy = 1$ at $t = 0$

Solⁿ: $s^2 L\{y\} - s\,y(0) - y'(0) - 2[sL\{y\} - y(0)] + 2L\{y\} = 0$

$\Rightarrow s^2 L\{y\} - s - 1 - 2sL\{y\} + 2 + 2L\{y\} = 0$

$\Rightarrow L\{y\} = \frac{s-1}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1}$

$y = L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}$

$= e^t L^{-1}\left\{\frac{s}{s^2+1}\right\}$

$\therefore y = e^t \cos t$

Ex Solve $(D^2+1)y = t\cos 2t \quad y(0)=0,\ y'(0)=0$

Sol^n: $L\{y''\} + L\{y\} = L\{t\cos 2t\}$

$$\Rightarrow s^2 L\{y\} - sy(0) - y'(0) + L\{y\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$\Rightarrow L\{y\} = \frac{s^2-4}{(s^2+1)(s^2+4)^2}$$

$$= -\frac{5}{9(s^2+1)} + \frac{5}{9(s^2+4)} + \frac{8}{3(s^2+4)^2}$$

$$y = -\frac{5}{9}L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{5}{9}L^{-1}\left\{\frac{1}{s^2+4}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\int_0^t \frac{1}{2}\sin 2x\,\frac{1}{2}\sin 2(t-x)\,dx$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\int_0^t \left\{\cos 2(t-2x) - \cos 2t\right\}dx$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\left[-\frac{1}{4}\sin 2(t-2x) - x\cos 2t\right]_0^t$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{12}\sin 2t - \frac{t}{3}\cos 2t + \frac{1}{12}\sin 2t$$

$$= -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{t}{3}\cos 2t$$