9.1 Complex form of Fourier Series

It is often convenient to work with complex form of Fourier series. In deed, the complex form of Fourier series has applications in the field of signal processing which is of great interest to many electrical engineers.

Given the Fourier series of a function $f(x)$ as

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right], \quad -\pi < x < \pi \tag{9.1}$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \quad n = 0, 1, 2\ldots$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x, \quad n = 1, 2\ldots$$

We know from Euler's formula

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2} \qquad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$

Substituting these values of $\cos(nx)$ and $\sin(nx)$ into the equation (9.1) we obtain

$$\begin{aligned} f &\sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i}\right] \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}\right] \end{aligned}$$

Let us define new coefficients as

$$c_n = \frac{1}{2}(a_n - ib_n), \qquad k_n = \frac{1}{2}(a_n + ib_n) \tag{9.2}$$

Note that $c_0 = a_0/2$ because $b_0 = 0$. Then the Fourier series becomes

$$f \sim c_0 + \sum_{n=1}^{\infty}\left[c_n e^{inx} + k_n e^{-inx}\right] \tag{9.3}$$

where the coefficients are given as

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[\cos(nx) - i\sin(nx)\right]\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x$$

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[\cos(nx) + i\sin(nx)\right]\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,\mathrm{d}x$$

From the above calculation we get $k_n = c_{-n}$. Substituting the value of k_n into the Fourier series (9.3) we have

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{9.4}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x, \quad n = 0, \pm 1, \pm 2, \ldots \tag{9.5}$$

The series on the right side of equation (9.4) is called complex form of the Fourier series.

For a function of period $2L$ defined in $[-L, L]$, the complex form of the Fourier series can analogously be derived to have

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}\,\mathrm{d}x, \quad n = 0, \pm 1, \pm 2, \ldots$$

9.2 Example Problems

9.2.1 Problem 1

Find the complex Fourier series of

$$f(x) = e^x \textit{ if } -\pi < x < \pi \textit{ and } f(x + 2\pi) = f(x)$$

Solution: We calculate the coefficients c_n as

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\frac{e^{(1-in)x}}{1-in}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}\left[e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right] \end{aligned}$$

Substituting $e^{\pm in\pi} = \cos n\pi \pm i\sin n\pi = (-1)^n$ we get

$$c_n = \frac{1}{\pi}\frac{1+in}{(1-in)(1+in)}(-1)^n\sinh\pi = \frac{1}{\pi}\frac{1+in}{(1+n^2)}(-1)^n\sinh\pi$$

Then, the Fourier is given as

$$f \sim \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1+in}{(1+n^2)} e^{inx}.$$

9.2.2 Problem 2

Determine the complex Fourier series representation of

$$f(x) = x \text{ if } -l < x < l \text{ and } f(x+2l) = f(x)$$

.

Solution: The complex Fourier series representation of a function $f(x)$ is given as

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}} \, \mathrm{d}x = \frac{1}{2l} \int_{-l}^{l} x e^{\frac{-in\pi x}{l}} \, \mathrm{d}x$$

For $n \neq 0$, integrating by parts we get

$$c_n = \frac{1}{2l} \left[\left(x e^{\frac{-in\pi x}{l}} \frac{-l}{in\pi} \right) \Big|_{-l}^{l} + \frac{l}{in\pi} \int_{-l}^{l} e^{\frac{-in\pi x}{l}} \, \mathrm{d}x \right],$$

Further application of integration by parts simplifies to

$$c_n = \frac{1}{2l} \left(-\frac{l^2}{in\pi} e^{-in\pi} - \frac{l^2}{in\pi} e^{in\pi} \right) - \frac{l^2}{(in\pi)^2} \underbrace{e^{\frac{-in\pi x}{l}} \Big|_{-l}^{l}}_{=0},$$

Finally, it simplifies to

$$c_n = \frac{(-1)^n il}{n\pi}, \quad n = \pm 1, \pm 2, \ldots$$

Now c_0 can be calculated as

$$c_0 = \frac{1}{2l} \int_{-l}^{l} x \, \mathrm{d}x = 0$$

Therefore, the Fourier series is given as

$$f \sim \frac{il}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n}{n} e^{\frac{in\pi x}{l}}$$

9.2.3 Problem 3

Show that Parseval's identity for the complex form of Fourier series takes the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,\mathrm{d}x=\sum_{n=-\infty}^{\infty}|c_n|^2$$

Solution: For the real form of Fourier series the Parseval's identity is given as

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,\mathrm{d}x \tag{9.6}$$

We know that

$$c_0=\frac{a_0}{2},\qquad c_n=\frac{1}{2}(a_n-ib_n),\qquad c_{-n}=\frac{1}{2}(a_n+ib_n)$$

We can deduce that

$$|c_n|^2=\frac{1}{4}(a_n^2+b_n^2),\qquad |c_{-n}|^2=\frac{1}{4}(a_n^2+b_n^2) \tag{9.7}$$

Diving the equation (9.6) by 2 and then splitting the second term as

$$\frac{a_0^2}{4}+\frac{1}{4}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)+\frac{1}{4}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,\mathrm{d}x$$

Using the relations (9.7) we obtain

$$c_0^2+\sum_{n=1}^{\infty}|c_n|^2+\sum_{n=1}^{\infty}|c_{-n}|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,\mathrm{d}x$$

This can be rewritten as

$$\sum_{n=-\infty}^{\infty}|c_n|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,\mathrm{d}x$$

9.2.4 Problem 4

Given the Fourier series

$$e^x\sim\frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1+in}{(1+n^2)}e^{inx}.$$

deduce the value of

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$$

Solution: From the given series we clearly have

$$c_n = (-1)^n \frac{e^{\pi} - e^{-\pi}}{2\pi} \frac{1+in}{(1+n^2)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

These coefficients can be simplified

$$|c_n|^2 = \frac{\left(e^{\pi} - e^{-\pi}\right)^2}{4\pi^2} \frac{(1+n^2)}{(1+n^2)^2} = \frac{\left(e^{\pi} - e^{-\pi}\right)^2}{4\pi^2} \frac{1}{(1+n^2)}$$

A simple calculation gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(x)\}^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2x} \, \mathrm{d}x = \frac{e^{2\pi} - e^{-2\pi}}{4\pi}$$

Thus, by Parseval's identity we have

$$\frac{e^{2\pi} - e^{-2\pi}}{4\pi} = \frac{\left(e^{\pi} - e^{-\pi}\right)^2}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2)}$$

Therefore, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2)} = \frac{\pi\left(e^{\pi} + e^{-\pi}\right)}{(e^{\pi} - e^{-\pi})} = \pi \cot h\pi.$$