Scalars, Vectors & Tensors

<u>Scalars</u> — magnitude

<u>Vectors</u> — magnitude + direction

<u>Tensors</u> — magnitude — Areal, Forcing } Stress

$\underline{p, T, \rho}$

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3$$
$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1, 2, 3

x y z

i, j, k

Basis vectors

$\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$

$\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$

$$\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3 \quad \text{[Vector notation]}$$

$$= \sum_{i=1}^{3} a_i \vec{e}_i \quad \text{[Index notation]}$$

$$\vec{a} = a_i \vec{e}_i \quad \text{[Summation is implicit]}$$

Einstein notation

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i = a_i b_i$$

$$\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3$$

$$\vec{b} = b_1 \vec{e}_1 + b_2 \vec{e}_2 + b_3 \vec{e}_3$$

Scalar $\leftarrow$

$$\boxed{\nabla \cdot \vec{a}} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$$

$$= \sum_{i=1}^{3} \frac{\partial a_i}{\partial x_i} = \frac{\partial a_i}{\partial x_i}$$

$$\nabla = \frac{\partial}{\partial x_1} \vec{e}_1 + \frac{\partial}{\partial x_2} \vec{e}_2 + \frac{\partial}{\partial x_3} \vec{e}_3$$

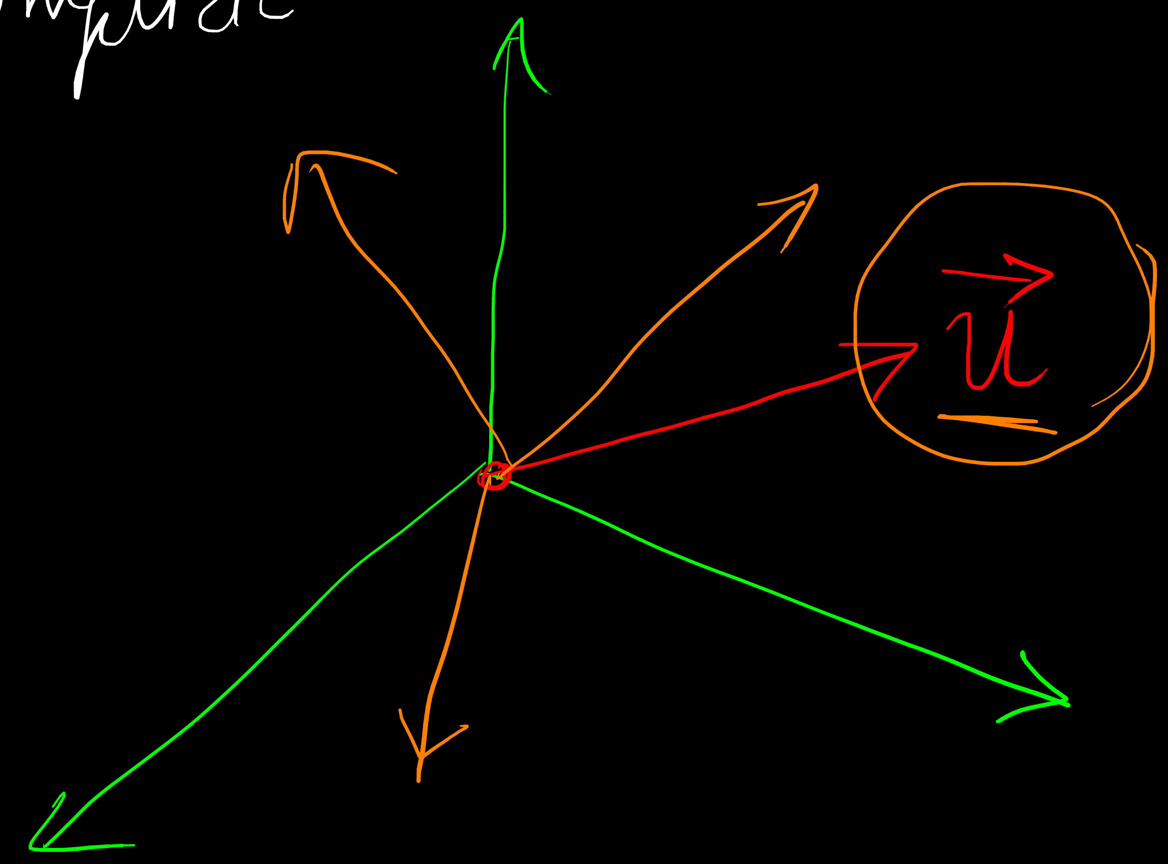
Gradient ?

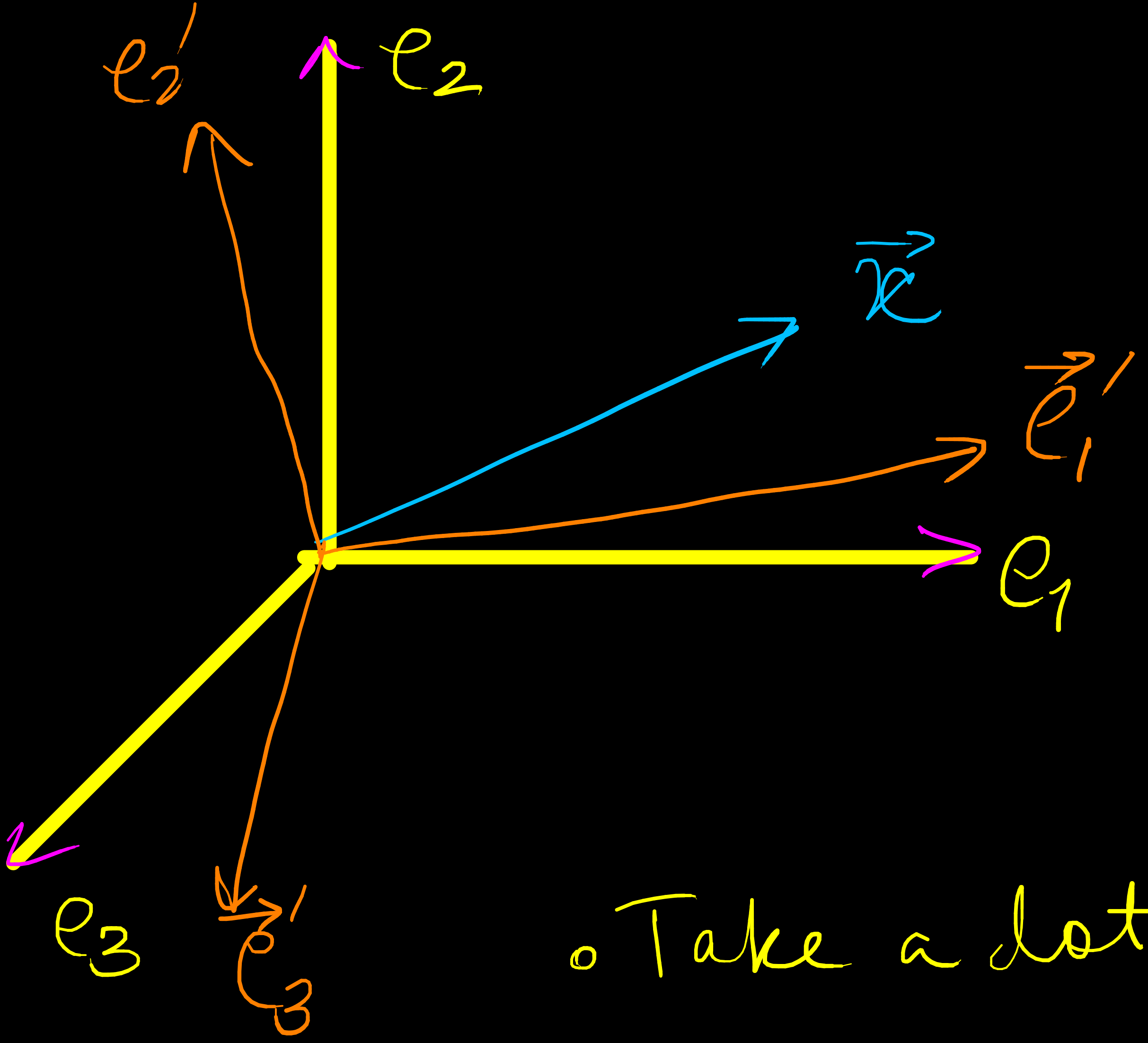
$$\underline{\nabla \phi} = \frac{\partial \phi}{\partial x_1}\vec{e}_1 + \frac{\partial \phi}{\partial x_2}\vec{e}_2 + \frac{\partial \phi}{\partial x_3}\vec{e}_3 = \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i}\vec{e}_i$$

$$\nabla\phi = \frac{\partial \phi}{\partial x_i} e_i$$ } Summation is implicit

Rotation of axis

How does component change
vector / pseudo-vector





$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$
$$= x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3'\vec{e}_3'$$

$$\underbrace{x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3'\vec{e}_3'} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$

Isolate ~~$\vec{e}_1'$~~ component of $\vec{e}_1'$; x_1'

- Take a dot prod w/ $\vec{e}_1'$

$$x_1'(\vec{e}_1'\cdot\vec{e}_1') + x_2'\underbrace{(\vec{e}_2'\cdot\vec{e}_1')}_{0} + x_3'\underbrace{\vec{e}_3'\cdot\vec{e}_1'}_{0} = x_1\vec{e}_1\cdot\vec{e}_1' + x_2\vec{e}_2\cdot\vec{e}_1' + x_3\vec{e}_3\cdot\vec{e}_1'$$

$$x_1' = x_1\boxed{\vec{e}_1\cdot\vec{e}_1'} + x_2\boxed{\vec{e}_2\cdot\vec{e}_1'} + x_3\boxed{\vec{e}_3\cdot\vec{e}_1'}$$

$$\boxed{x_1' = \sum x_i\vec{e}_i\cdot\vec{e}_1'}$$

$$x_1' = x_i\vec{e}_i\cdot\vec{e}_1'$$

$$x_2' = x_i \vec{e}_i \cdot \vec{e}_2'$$

$$x_3' = x_i \vec{e}_i \cdot \vec{e}_3'$$

What is this?

direction cosines

Rotation matrix

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \vec{e}_1' \cdot \vec{e}_1 & \vec{e}_1' \cdot \vec{e}_2 & \vec{e}_1' \cdot \vec{e}_3 \\ \vec{e}_2' \cdot e_1 & \vec{e}_2' \cdot \vec{e}_2 & \vec{e}_2' \cdot e_3 \\ \vec{e}_3' \, e_1 & e_3' \cdot \vec{e}_2 & \vec{e}_3' \cdot e_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_{ij} = e_i \cdot e_j'$$

$$\boxed{x_j' = x_i C_{ij}}$$

component of the vector must satisfy this rotation law

$$= x_i e_i \cdot e_j'$$

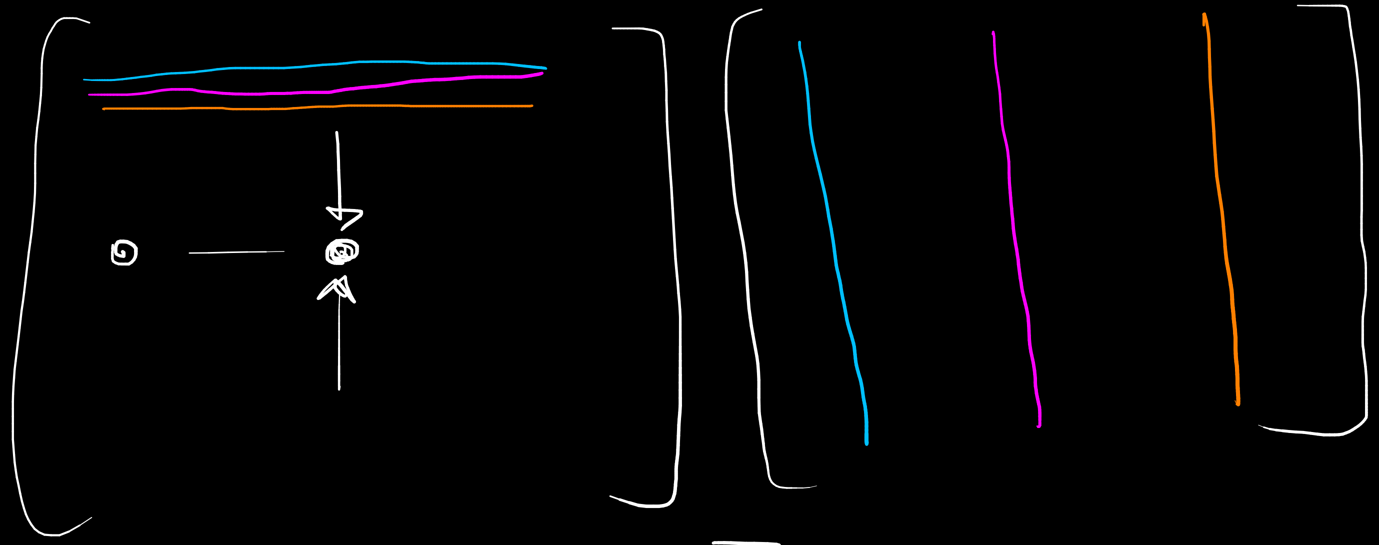
$$x_j' = x_1 e_1 \cdot e_j' + x_2 e_2 \cdot e_j' + x_3 e_3 \cdot e_j'$$

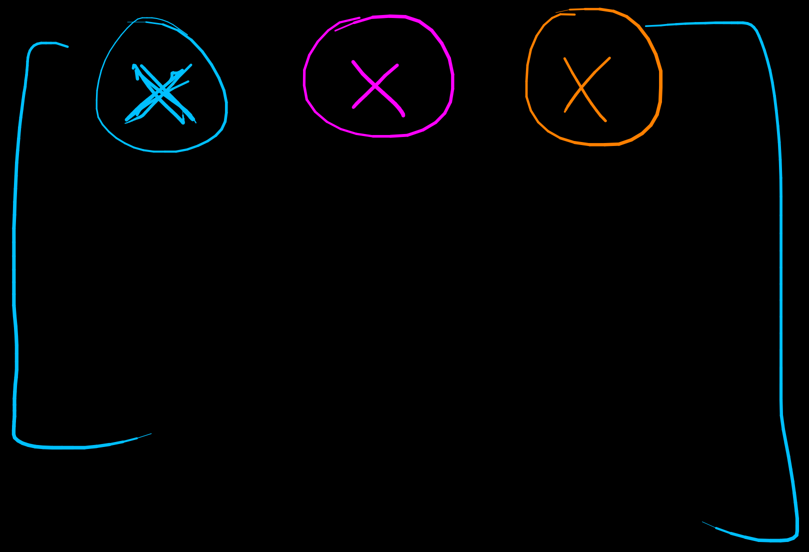
$j = 1$

$C_{ij} \longrightarrow$ Matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{\text{Rotation}} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Multiplication of matrices





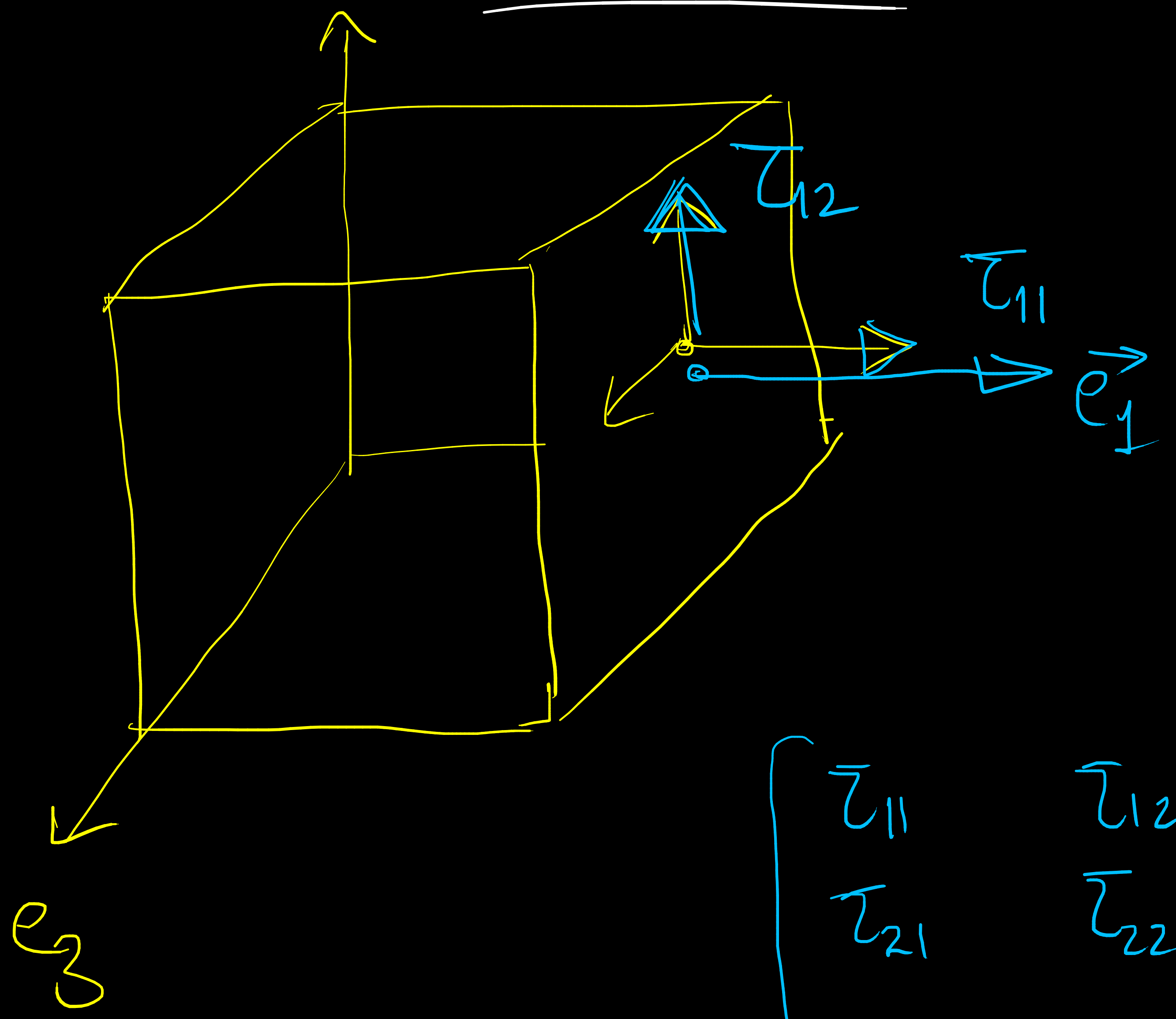
$$\bar{\bar{A}} = \bar{\bar{B}} \cdot \bar{\bar{C}}$$

$$\boxed{\bar{\bar{A}}_{ij} = \sum_{k=1}^{3} B_{ik} C_{kj}}$$

$$A_{ij} = B_{ik} C_{kj}$$

(summation implicit)

Tensors



→ normal / shear force ?

→ Area vector

τ_{11}: area vector, direction of force

τ_{12}: area, force

$$\left.\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}\right\} \text{State of stress}$$

→ $\boxed{\text{Rotation law ??}}$

Mohr's circle

$$\left.\begin{matrix} \tau_{11}' \\ \tau_{22}' \\ \tau_{12}' \end{matrix}\right\} \boxed{\begin{matrix} \tau_{11} & \cos\theta \\ \tau_{22} & \sin\theta \\ \tau_{12} & \end{matrix}}$$

Homework : Find out rotation law for a tensor

τ_{ij}

$$\boxed{C_{ij} = \vec{e}_i \cdot \vec{e}_j{}'}$$

$$\tau'_{mn} = \sum_{i=1}^{3}\sum_{j=1}^{3} C_{im} C_{jn} \tau_{ij}$$

$$\tau'_{mn} = C_{im} C_{jn} \tau_{ij}$$

$$= C^T_{mi} \tau_{ij} C_{jn}$$

$$\boxed{\bar{\bar{\tau}}' = \bar{\bar{C}}^T \cdot \bar{\bar{\tau}} \cdot \bar{\bar{C}}}$$

$$\mu \times \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$$

Heat conductivity

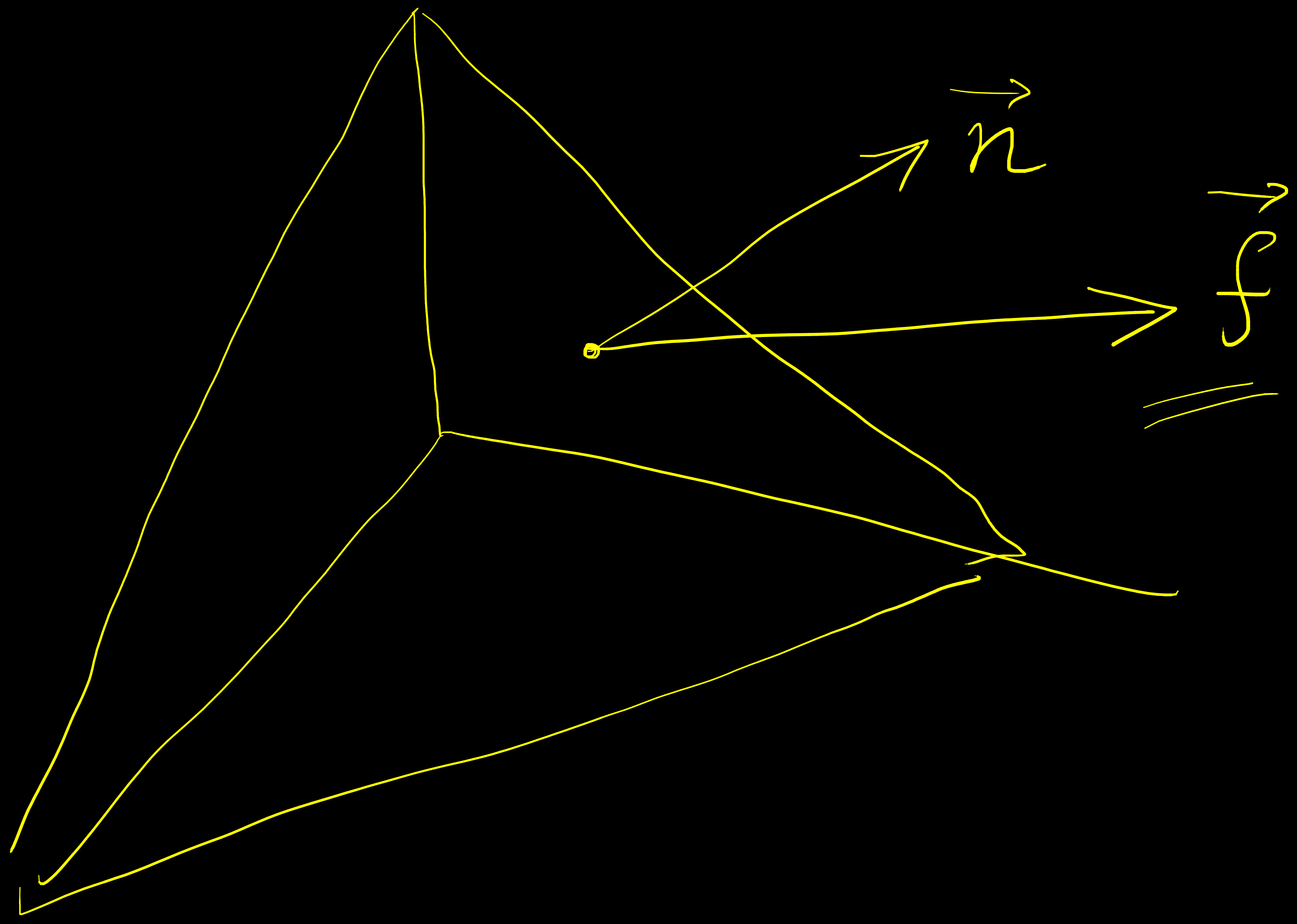
Contraction

$$\operatorname{trace}(A_{ij}) = \underline{\underline{A_{ii}}}$$

$$\operatorname{trace}(\bar{\bar{\delta}}) = \,? = \underline{\underline{3}}$$

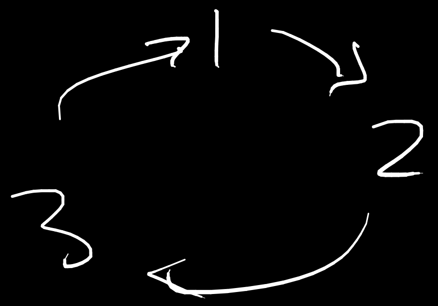
$$\bar{\bar{\delta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



$\vec{f} = \bar{\bar{\tau}}$?

$\boxed{\times}$



Alternating tensor

$$\boxed{\varepsilon_{ijk}} = \begin{cases} 1 & ijk \text{ cyclic} \\ -1 & ijk \text{ anticyclic} \\ 0 & i, j, k \text{ repeat} \end{cases}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\varepsilon_{123} = 1$ $\quad$ $\varepsilon_{312} = 1$

$\varepsilon_{231} = 1$

$\varepsilon_{213} = -1$

$\varepsilon_{113} = 0$

Cross product

$$(\vec{a} \times \vec{b})_k = \varepsilon_{ijk} \, a_i \, b_j$$

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{e}_1 \underline{(a_2 b_3 - a_3 b_2)}$$

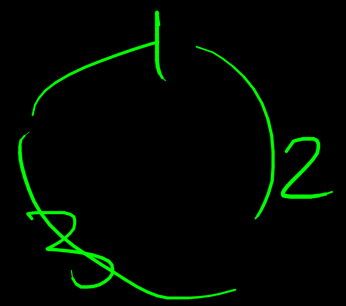
component: "1"

$k=1 \quad (\vec{a}\times\vec{b})_1 = \varepsilon_{ij1}\, a_i\, b_j$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij1}\, a_i b_j = \sum_{j=1}^{3} \varepsilon_{2j1}\, a_2 b_j + \sum_{j=1}^{3} \varepsilon_{3j1}\, a_3 b_j$$

$$= \varepsilon_{231}\, a_2 b_3 + \varepsilon_{321}\, a_3 b_2$$

$$= \underline{a_2 b_3 - a_3 b_2}$$



- Do it for 2 and 3rd components of the cross prod

- $\underline{\varepsilon - \delta}$ relationship

$$\boxed{\varepsilon_{ijk}\, \varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}}$$