Transform Calculus Assignment 04

Q.1) $y' + 6y + 5\int_0^t y(\tau)\,d\tau = 1+t, \qquad y(0)=1$

Soln: Applying Laplace transform on both sides

$$sY(s) - y(0) + 6Y(s) + \frac{5}{s}Y(s) = \frac{1}{s}+\frac{1}{s^2}$$

$$\Rightarrow sY(s) - 1 + 6Y(s) + \frac{5}{s}Y(s) = \frac{1}{s}+\frac{1}{s^2}$$

$$\Rightarrow Y(s)\left(s+6+\frac{5}{s}\right) = 1+\frac{1}{s}+\frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{s^2+s+1}{s^2} \times \frac{s}{s^2+6s+5} = \frac{s^2+s+1}{s(s^2+6s+5)}$$

$$\Rightarrow Y(s) = \frac{1}{s+5} - \frac{1}{4(s+1)} + \frac{1}{20(s+5)} + \frac{1}{5s}$$

Again, applying inverse Laplace transform

$$y(t) = e^{-5t} - \frac{1}{4}e^{-t} + \frac{1}{20}e^{-5t} + \frac{1}{5}$$

$$\therefore\ y(t) = \frac{1}{5} - \frac{1}{4}e^{-t} + \frac{21}{20}e^{-5t}$$

Q.2) Soln: In terms of the unit step function, $f(t)$ can be written as

$$f(t) = k\left[u(t-a) - u(t-b)\right]$$

The given ODE is

$$y' + 5y + 4\int_0^t y(\tau)\,d\tau = f(t), \qquad y(0)=2$$

Applying Laplace transform on both sides

$$sY(s) - y(0) + 5Y(s) + \frac{4}{s}Y(s) = \mathcal{L}[f(t)]$$

$$\Rightarrow sY(s) - 2 + 5Y(s) + \frac{4}{s}Y(s) = \frac{k}{s}\left(e^{-as} - e^{-bs}\right)$$

$$\Rightarrow (s^2+5s+4)Y(s) = 2s + k\left(e^{-as}-e^{-bs}\right)$$

$$\Rightarrow Y(s) = \frac{2s}{(s+4)(s+1)} + \frac{k(e^{-as}-e^{-bs})}{(s+1)(s+4)}$$

$$= \frac{2}{3}\left[\frac{4}{s+4} - \frac{1}{s+1}\right] + \frac{k}{3}\left[\frac{1}{s+1} - \frac{1}{s+4}\right](e^{-as}-e^{-bs})$$

Now, applying inverse Laplace transform

$$y(t) = \frac{2}{3}\left[4e^{-4t} - e^{-t}\right] + \frac{k}{3}\Big[\left\{e^{-(t-a)} - e^{-4(t-a)}\right\}u(t-a)$$
$$-\left\{e^{-(t-b)} - e^{-4(t-b)}\right\}u(t-b)\Big]$$

$$\left[\because\ \mathcal{L}^{-1}\left[e^{-as}F(s)\right] = f(t-a)u(t-a)\right]$$

$y(t)$ can also be written as

$$y(t) = \begin{cases} \frac{2}{3}(4e^{-4t} - e^{-t}), & 0 \le t < a \\ \frac{2}{3}(4e^{-4t} - e^{-t}) + \frac{k}{3}\left[e^{-(t-a)} - e^{-4(t-a)}\right], & a \le t < b \\ \frac{2}{3}(4e^{-4t} - e^{-t}) + \frac{k}{3}\left[\left\{e^{-(t-a)} - e^{-4(t-a)}\right\} - \left\{e^{-(t-b)} - e^{-4(t-b)}\right\}\right], & t \ge b \end{cases}$$

Q.3) soln: $y'' + 6y' + 9y = 8te^{2t}, \quad y(0) = 0, \ y'(0) = -1$

Applying Laplace transform on both sides

$$s^2Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 9Y(s) = \frac{8}{(s-2)^2}$$

$$\Rightarrow s^2Y(s) + 1 + 6sY(s) + 9Y(s) = \frac{8}{(s-2)^2}$$

$$\Rightarrow (s^2+6s+9)Y(s) = \frac{8}{(s+2)^2} - 1$$

$$\Rightarrow Y(s) = \frac{8}{(s+2)^2(s+3)^2} - \frac{1}{(s+3)^2}$$

Using partial fractions and applying inverse Laplace transform

$$y(t) = \frac{1}{125}\left[(16-85t)e^{-3t} + (40t-16)e^{2t}\right]$$

soln: $y'' + 9y = \sin 3t, \quad y(0)=0, \quad y'(0)=0$

Applying Laplace transform on both sides

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{3}{s^2+9}$$

$$\Rightarrow (s^2+9)Y(s) = \frac{3}{s^2+9}$$

$$\Rightarrow Y(s) = \frac{3}{(s^2+9)^2} = \frac{1}{3}\left(\frac{3}{s^2+9}\right)\left(\frac{3}{s^2+9}\right)$$

$$= \frac{1}{3}F(s)G(s)$$

where $f(t) = \mathcal{L}^{-1}[F(s)] = \sin 3t = g(t)$

Now, using Convolution theorem

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{3}F(s)G(s)\right] = \frac{1}{3}\int_0^t \sin 3\tau \sin 3(t-\tau)\,d\tau$$

$$= \frac{1}{6}\int_0^t \left[\cos 3(2\tau - t) - \cos 3t\right] d\tau$$

$$= \frac{1}{6}\left[\frac{1}{6}\sin 3(2\tau - t) - \tau\cos 3t\right]_0^t$$

$$= \frac{1}{6}\left[\frac{1}{6}(\sin 3t + \sin 3t) - t\cos 3t\right]$$

$$= \frac{1}{18}\left[\sin 3t - 3t\cos 3t\right]$$

Q.5) soln: $y'' + 4y = -8t^2, \quad y(0)=3, \quad y(\frac{\pi}{4})=0$

Applying Laplace transform on both sides

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = -8\cdot\frac{2}{s^3}$$

$$\Rightarrow (s^2+4)Y(s) = \frac{-16}{s^3} + \alpha + 3s \qquad [\text{Taking } y'(0)=\alpha]$$

$$\Rightarrow Y(s) = \frac{-16}{s^3(s^2+4)} + \frac{\alpha}{s^2+4} + \frac{3s}{s^2+4}$$

$$= -4\left(\frac{2}{s^3}\right)\left(\frac{2}{s^2+4}\right) + \frac{\alpha}{s^2+4} + \frac{3s}{s^2+4}$$

ow, applying inverse Laplace transform and the Convolution theorem

$$y(t) = -4\int_0^t \tau^2 \sin 2(t-\tau)\,d\tau + \frac{\alpha}{2}\sin 2t + 3\cos 2t$$

$$= -4\left[\left\{\frac{\tau^2}{2}\cos 2(t-2)\right\}_0^t - \int_0^t \tau \cos 2(t-\tau)\,d\tau\right] + \frac{\alpha}{2}\sin 2t + 3\cos 2t$$

$$= -4\left[\frac{t^2}{2} - \left\{-\frac{\tau}{2}\sin 2(t-\tau)\right\}_0^t - \int_0^t \frac{\sin 2(t-\tau)}{2}\,d\tau\right] + \frac{\alpha}{2}\sin 2t + 3\cos 2t$$

$$= -2t^2 + \left\{\cos 2(t-\tau)\right\}_0^t + \frac{\alpha}{2}\sin 2t + 3\cos 2t$$

$$= -2t^2 + 1 + \frac{\alpha}{2}\sin 2t + 3\cos 2t$$

Using the condition at the other boundary

$$y\left(\frac{\pi}{4}\right) = 0 \Rightarrow -2\frac{\pi^2}{16} + 1 + \frac{\alpha}{2} = 0 \Rightarrow \alpha = 2\left(\frac{\pi^2}{8} - 1\right)$$

$$\therefore\ y(t) = 1 - 2t^2 + \left(\frac{\pi^2}{8} - 1\right)\sin 2t + 3\cos 2t$$

Q.6> soln: $y'' + ty' - 2y = 6 - t,\quad y(0) = 0,\quad y'(0) = 1$

Applying Laplace transform on both sides

$$s^2 Y(s) - sy(0) - y'(0) + \left[-\frac{d}{ds}\{sY(s) - y(0)\}\right] - 2Y(s) = \frac{6}{s} - \frac{1}{s^2}$$

$$\because \left\{\mathcal{L}[tf'(t)] = -\frac{d}{ds}[F(s)]\right\}$$

$$\therefore\ s^2Y(s) - [sY'(s) + Y(s)] - 2Y(s) = 1 + \frac{6}{s} - \frac{1}{s^2}$$

$$\Rightarrow\quad Y'(s) + \left[\frac{3}{s} - s\right]Y(s) = -\left(\frac{1}{s} + \frac{6}{s^2} - \frac{1}{s^3}\right)$$

$$\text{I.F.} = e^{\int\left(\frac{3}{s} - s\right)ds} = s^3 e^{-s^2/2}$$

$$\therefore\ s^3 e^{-s^2/2}\,Y(s) = \int e^{-s^2/2}\,ds - 6\int s e^{-s^2/2}\,ds - \int s^2 e^{-s^2/2}\,ds + C$$

$$\Rightarrow s^3 e^{-s^2/2} Y(s) = \int e^{-s^2/2} ds + 6 e^{-s^2/2} - \left[-s e^{-s^2/2} + \int e^{-s^2/2} ds\right] + c$$

$$= (6+s) e^{-s^2/2} + c$$

$$\therefore \; Y(s) = \frac{6}{s^3} + \frac{1}{s^2} + \frac{c}{s^3} e^{s^2/2}$$

Since, $y(t)$ satisfies the condition of the existence theorem, we require that $\lim_{s \to \infty} Y(s) = 0$

This gives, $c = 0$

$$\therefore \; Y(s) = \frac{6}{s^3} + \frac{1}{s^2}$$

Again, taking the inverse Laplace transform

$$y(t) = 3t^2 + t$$

Q.7) soln:

$$y'' + y = 2t, \quad y(\tfrac{\pi}{4}) = \tfrac{\pi}{2}, \quad y'(\tfrac{\pi}{4}) = 2 - \sqrt{2}$$

Setting $t = \tilde{t} + \frac{\pi}{4}$ or $\tilde{t} = t - \frac{\pi}{4}$,

$$y(t) = \tilde{y}(\tilde{t})$$

$$\Rightarrow y(\tfrac{\pi}{4}) = \tfrac{\pi}{2} = \tilde{y}(0)$$

$$\& \quad y'(\tfrac{\pi}{4}) = 2 - \sqrt{2} = \tilde{y}'(0)$$

The equation becomes

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{\pi}{4})$$

Applying Laplace transform on both sides

$$s^2 \tilde{Y}(s) - s\tilde{y}(0) - \tilde{y}'(0) + \tilde{Y}(s) = \frac{2}{s^2} + \frac{\pi}{2s}$$

$$\Rightarrow \quad \tilde{Y}(s) = \frac{2}{s^2(s^2+1)} + \frac{\pi/2}{s(s^2+1)} + \frac{\tilde{y}(0)s}{s^2+1} + \frac{\tilde{y}'(0)}{s^2+1}$$

Applying inverse Laplace transform

$$\Rightarrow \tilde{y}(\tilde{t}) = 2(\tilde{t} - \sin\tilde{t}) + \frac{\pi}{2}(1 - \cos\tilde{t})$$

$$+ \frac{1}{2}\pi \cos\tilde{t} + (2 - \sqrt{2}) \sin\tilde{t}$$

$y(t) = 2t - \sin t + \cos t \qquad [\because \tilde{t} = t - \pi/4]$

3.8) soln: $\quad y'' + 3y' + 2y = r(t), \qquad y(0) = 0, \ y'(0) = 0$

Applying Laplace transform on both sides

$$s^2 Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = R(s)$$

$$\Rightarrow (s^2 + 3s + 2)Y(s) = R(s)$$

a) $\quad R(s) = \mathcal{L}[u(t-1) - u(t-2)] = (e^{-s} - e^{-2s})\frac{1}{s}$

so, $\quad Y(s) = (e^{-s} - e^{-2s})\frac{1}{s}\cdot\frac{1}{(s+1)}\cdot\frac{1}{(s+2)}$

$$\cancel{y(t)} = \left[\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right](e^{-s} - e^{-2s})$$

$$\therefore \quad y(t) = \left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right]u(t-1)$$

$$- \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u(t-2)$$

$$= \begin{cases} 0, & 0 \le t < 1 \\ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}, & 1 \le t < 2 \\ -e^{-(t-1)} + e^{-(t-2)} + \frac{1}{2}e^{-2(t-1)} - \frac{1}{2}e^{-2(t-2)}, & t \ge 2 \end{cases}$$

b) $\quad R(s) = \mathcal{L}[\delta(t-1)] = e^{-s}$

so, $\quad Y(s) = \dfrac{e^{-s}}{(s+1)(s+2)} = \left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right)e^{-s}$

$$\therefore \quad y(t) = \left(e^{-(t-1)} - e^{-2(t-1)}\right)u(t-1)$$

$$= \begin{cases} 0, & 0 \le t < 1 \\ e^{-(t-1)} - e^{-2(t-1)}, & t \ge 1 \end{cases}$$

Soln: $\left.\begin{array}{l} x'' + kx + k(x-y) = 0 \\ y'' + ky + k(y-x) = 0 \end{array}\right\}$ $x(0)=1, \quad y(0)=1$
$x'(0) = \sqrt{3k}, \quad y'(0) = -\sqrt{3k}$

Let $\frac{d}{dt} = D$.

$$\therefore (D^2 + 2k)x - ky = 0$$
$$(D^2 + 2k)y - kx = 0$$

Applying Laplace, transform in the two equations

$$s^2X(s) - sx(0) - x'(0) + 2kX(s) - kY(s) = 0$$

$$\& \; s^2Y(s) - sy(0) - y'(0) + 2kY(s) - kX(s) = 0$$

$$\Rightarrow \quad (s^2 + 2k)X(s) - kY(s) = s + \sqrt{3k}$$

$$\& \; (s^2 + 2k)Y(s) - kX(s) = s - \sqrt{3k}$$

Solving for $X(s)$ and $Y(s)$

$$X(s) = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}$$

$$\& \; Y(s) = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}$$

Applying inverse Laplace transform

$$x(t) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$

$$y(t) = \cos\sqrt{k}t - \sin\sqrt{3k}t$$

$$f(t) = t + e^{-2t} + \int_0^t f(\tau) e^{2(t-\tau)} d\tau$$

Q.10) soln: Let us say $\mathcal{L}[f(t)] = F(s)$

and $f * g = \int_0^t f(\tau) g(t-\tau) d\tau$, where $g(t) = e^{2t}$

Applying Laplace transform in the given eqn.

$$F(s) = \frac{1}{s^2} + \frac{1}{s+2} + F(s) \cdot \frac{1}{s-2}$$

$$\Rightarrow \left(1 - \frac{1}{s-2}\right) F(s) = \frac{1}{s^2} + \frac{1}{s+2}$$

$$\Rightarrow \frac{s-3}{s-2} F(s) = \frac{1}{s^2} + \frac{1}{s+2}$$

$$\Rightarrow F(s) = \left(\frac{s-2}{s-3}\right)\left(\frac{1}{s^2} + \frac{1}{s+2}\right)$$

$$= \frac{1}{45}\left[14 \cdot \frac{1}{s-3} - \frac{5}{s} + \frac{30}{s^2} + \frac{36}{s+2}\right]$$

$$\therefore \quad f(t) = \frac{1}{45}\left[14e^{3t} - 5 + 30t + 36e^{-2t}\right]$$