(9) Inverse Fourier Transform:

We know the Fourier transform to be

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx,$$

then we define <u>inverse Fourier transform</u> of $\hat{f}(w)$ to be

$$\mathcal{F}^{-1}(\hat{f}) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw \quad ---- (K_1)$$

Then we can check that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dx = f(x)$$

(by (B) of Theorem - 2 i.e. Fourier Integral Formula)

Similarly, we define <u>inverse Fourier cosine transform</u>

$$\mathcal{F}_c^{-1}\left\{\hat{f}_c(w)\right\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(w) \cos wx \, dw$$

$$----- (K_2)$$

for an even function f

From (D_1), we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(v) \cos wv \, dv \cos wx \, dw$$

i.e. $f(x) = \mathcal{F}_c^{-1}\left\{\mathcal{F}_c\left\{f(x)\right\}\right\}$

We define inverse Fourier sine transform

$$\mathcal{F}_s^{-1}\left\{\mathcal{F}_s(w)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w) \sin xw \, dw$$

From (D_2), we see that

$$\mathcal{F}_s^{-1}\left\{\mathcal{F}_s\left\{f(x)\right\}\right\} = f(x)$$

Applications :

(1) Solve for $Y(x)$ the integral equation

$$\int_0^\infty Y(x) \sin xt \, dx = \begin{cases} 1 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 0 & t \geq 2 \end{cases}$$

Answer :

Write v in place of x. Then this equation becomes

$$\int_0^\infty Y(v) \sin vt \, dv = \begin{cases} 1 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 0 & t \geq 2 \end{cases}$$

$$\Rightarrow \sqrt{\frac{\pi}{2}}\,\mathcal{F}_s\left\{Y(x)\right\} = \begin{cases} 1 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 0 & t \geq 2 \end{cases}$$

We apply $\mathcal{F}_s^{-1}$

$$\Rightarrow \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty Y(v) \sin vt \, dv \cdot \sin xt \, dt$$

$$= \int_0^1 \sin xt \, dt + \int_1^2 2 \sin xt \, dt$$

$$\Rightarrow \frac{\pi}{2} \cdot Y(x) = -\frac{1}{x} \left([\cos xt]_0^1 + [2 \cos xt]_1^2 \right)$$

$$\Rightarrow \frac{\pi}{2} \cdot Y(x) = \frac{1}{x} \left(1 + \cos x - 2 \cos 2x \right)$$

$$\Rightarrow Y(x) = \frac{2}{\pi x} \left(1 + \cos x - 2 \cos 2x \right)$$

(2) Let $f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$

Find Fourier sine transformation of $f(x)$

show that $\int_0^\infty \left(\frac{1 - \cos x}{x} \right)^2 dx = \frac{\pi}{2}$

Answer:

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin wx \, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w} \cdot \left[-\cos wx \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w} \cdot (1 - \cos w)$$

By Parseval's identity for Fourier sine transform,

$$\int_0^\infty |\mathcal{F}_s(f)|^2 dw = \int_0^\infty |f(x)|^2 dx$$

$$\Rightarrow \int_0^\infty \frac{2}{\pi} \cdot \frac{1}{w^2}(1-\cos w)^2\, dw = \int_0^1 dx = 1$$

$$\Rightarrow \int_0^\infty \left(\frac{1-\cos w}{w}\right)^2 dw = \frac{\pi}{2}$$

(3) Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Evaluate $\displaystyle\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\, dx$

Answer:

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}(1-x^2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{e^{-iwx}}{-iw}(1-x^2)\right]_{-1}^{1} - \frac{1}{iw}\int_{-1}^{1} 2x \cdot e^{-iwx} dx \right)$$

$$= \frac{-1}{\sqrt{2\pi}} \cdot \frac{1}{iw} \left(\left[\frac{-1}{+iw} 2x \cdot e^{-iwx}\right]_{-1}^{1} + \int_{-1}^{1} \frac{2}{iw} e^{-iwx} dx \right)$$

$$= \frac{+2}{\sqrt{2\pi}\, iw} \left(\frac{2}{iw} \cdot \cos w + \frac{1}{(iw)^2} \cdot \left(e^{-iw} - e^{iw}\right) \right)$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^2}\left(2\cos w + \frac{2}{iw}(-i\sin w)\right)$$

$$= -2\sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^3}\left(w\cos w - \sin w\right)$$

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos wx\, dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty} f(x) \sin wx\, dx$$

$$= -2\sqrt{\frac{2}{\pi}}\, \frac{1}{w^3} \cdot \left(w\cos w - \sin w\right)$$

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos wx\, dx = -2\sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^3}\left(w\cos w - \sin w\right)$$

$$\therefore \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx\, dx = -2\sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^3}\left(w\cos w - \sin w\right)$$

as $f(x)$ is an even function

$$\therefore \int_{0}^{\infty} f(x) \cos wx\, dx = -2 \cdot \frac{w\cos w - \sin w}{w^3}$$

From (θ_1) we get for an even function $f(x)$,

$$f(x) = \int_{0}^{\infty} A(w) \cos wx\, dw, \text{ where}$$

$$A(w) = \frac{2}{\pi} \int_{0}^{\infty} f(v) \cos wv\, dv$$

In our example as $f(x)$ is an even function, so

$$\cancel{A}\ A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv = -\frac{4}{\pi} \cdot \frac{w\cos w - \sin w}{w^3}$$

$$\therefore f(x) = -\frac{4}{\pi}\int_0^\infty \frac{w\cos w - \sin w}{w^3}\cos wx\, dw$$

Take $x = \frac{1}{2}$, then

$$1 - \left(\frac{1}{2}\right)^2 = \frac{4}{\pi}\int_0^\infty \frac{w\cos w - \sin w}{w^3}\cos\frac{w}{2}\, dw$$

$$\therefore \int_0^\infty \frac{w\cos w - \sin w}{w^3}\cos\frac{w}{2}\, dw = -\frac{3\pi}{16}$$

To obtain real Fourier Integral form from complex Fourier Integral:

Complex form of Fourier Integral formula is given by (B),

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)\, e^{-iwv}\, dv\right] e^{iwx}\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)(\cos wv - i\sin wv)\, dv\right](\cos wx + i\sin wx)\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} f(v)\,\big(\cos wv \cos wx + \sin wv \sin wx +$$
$$i\,(\sin wx \cos wv - \sin wv \cos wx)\big)\,dv\Big]\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \cos w(x-v)\,dv\Big]\,dw +$$
$$\frac{i}{2\pi}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \sin w(x-v)\,dv\Big]\,dw$$

$\int_{-\infty}^{\infty} f(v) \sin w(x-v)\,dv$ is an odd function in w (as $\sin w(x-v)$ is an odd function of w), so

$$\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \sin w(x-v)\,dv\Big]\,dw = 0$$

Similarly, $\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \cos w(x-v)\,dv\Big]\,dw$

$$= 2\int_{0}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \cos w(x-v)\,dv\Big]\,dw$$

$$\therefore f(x) = \frac{1}{\pi}\int_{0}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \cos w(x-v)\,dv\Big]\,dw$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \cos wv\,dv\Big] \cos xw\,dw +$$
$$\frac{1}{\pi}\int_{0}^{\infty}\Big[\int_{-\infty}^{\infty} f(v) \sin wv\,dv\Big] \sin xw\,dw,$$

Which is (c)

Some More Examples:

1) Show that $\int_0^\infty \frac{\cos wx}{w^2+1}\,dw = \frac{\pi}{2} e^{-x},\ x \geq 0$

Answer: We know that for an even function $f(x) = e^{-x}$,

$$f(x) = \int_0^\infty A(w) \cos xw \, dw \text{ where}$$

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv.$$

Here $A(w) = \frac{2}{\pi}\int_0^\infty e^{-v} \cos wv \, dv = \frac{2}{\pi} \cdot \frac{1}{w^2+1}$.

$$\therefore f(x) = \int_0^\infty \frac{2}{\pi} \cdot \frac{\cos xw}{1+w^2}\, dw,$$

$$\therefore \int_0^\infty \frac{\cos wx}{1+w^2}\, dw = \frac{\pi}{2} \cdot e^{-x}$$

2) Solve the integral equation $\int_0^\infty f(x) \cos wx \, dw$

$$= \begin{cases} 1-w, & 0 \leq w \leq 1 \\ 0, & w > 1. \end{cases}$$

~~Ans~~ Answer: Applying inverse Fourier cosine transform, we get

$$\mathcal{F}_c^{-1}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \cos wx \, dx\right\} = \frac{2}{\pi}\int_0^1 (1-w) \cos vw \, dw$$

$$\Rightarrow f(v) = \frac{2}{\pi}\left(\left[\frac{\sin vw}{v}(1-w)\right]_0^1 + \int_0^1 \frac{\sin vw}{v}\, dw\right)$$

$$= \frac{2}{\pi} \cdot \frac{1}{v^2}(1-\cos v).$$

Self-Reciprocity :

We study the Fourier transformation of the function $f(x) := e^{-ax^2}$ $(a>0)$

Its Fourier tranform is

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx - ax^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x+\frac{iw}{2a})^2 - \frac{w^2}{4a}} dx$$

Put $y = x + \frac{iw}{2a}$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-ay^2} dy$$

$$= \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

If $a = 1/2$, then $\hat{f}(w) = e^{-\frac{w^2}{2}}$, i.e. $\mathcal{F}\{f(x)\} = \hat{f}(w)$

Such a function is said to be self-reciprocal under the Fourier transformation.

Finite Fourier Sine and Cosine Transform:

These transforms are mainly used in solving boundary value and initial value probems. From Half-Range Expansion, we know if a function $f(x)$ is a piecewise continuous function (with left and right hand derivatives at each point) on a finite interval $0 < x < L$, then we can extend this to first an odd function on $[-L, L]$, then on the real axis $(-\infty, \infty)$ as a periodic function of period $2L$. Then we can obtain the Fourier series of this odd function. This way we can write $f(x)$ (defined on $[0, L]$) as a series of Sine functions, ie.

$$f(x) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(v) \sin\left(\frac{n\pi}{L} v\right) dv \right) \sin\left(\frac{n\pi}{L}\right) x \quad \text{------} \quad (L1)$$

So, we define finite Fourier sine transform of $f(x)$ to be

$$\hat{f}_s(n) = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{----} \quad (M_1)$$

$$n = 1, 2, 3 \ldots$$

We define inverse finite Fourier sine transform (in view of (L1) to be

$$\mathcal{F}_s^{-1}\left\{\hat{f}_s(n)\right\} = \frac{2}{L} \sum_{n=1}^{\infty} \hat{f}_s(n) \sin\left(\frac{n\pi x}{L}\right) \text{---} \quad (N1)$$

Using Half-Range Expansion, we can write a piecewise continuous function $f(x)$ on $[0, L]$ (similar to (L1)) as a series of cosine functions as

$$f(x) = \frac{1}{L}\int_0^L f(v)\,dv + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(v)\cos\left(\frac{n\pi}{L}v\right)dv\right)\cos\left(\frac{n\pi}{L}x\right) \quad \cdots\cdots (L_2)$$

So, we define finite Fourier cosine transform of $f(x)$ to be

$$\hat{f}_c(n) = \int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad \cdots\cdots (M_2)$$

$$n = 0, 1, 2, \ldots$$

In view of (L_2), we define inverse finite Fourier cosine transform to be

$$\mathcal{F}_c^{-1}\left\{\hat{f}_c(x)\right\} = \frac{1}{L}\hat{f}_c(0) + \frac{2}{L}\sum_{n=1}^{\infty}\hat{f}_c(n)\cos\left(\frac{n\pi}{L}x\right) \quad \cdots\cdots (N_2)$$

Example:

Find the finite Fourier sine transform and finite Fourier cosine transform of the function

$$f(x) = x^2, \quad 0 < x < l$$

Answer:

$$\hat{f}_s(n) = \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

$$= \int_0^l x^2 \sin\left(\frac{n\pi}{l}x\right) dx$$

$$= \frac{-\left[x^2 \cos \frac{n\pi}{l} x\right]_0^l}{\frac{n\pi}{l}} + \frac{2l}{n\pi}\int_0^l x \cos\left(\frac{n\pi}{l}x\right) dx$$

$$= -\frac{l^2 \cos(n\pi)}{\frac{n\pi}{l}} + \frac{2l}{n\pi}\left(\left[\frac{x \sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^l - \int_0^l \frac{l}{n\pi} \sin \frac{n\pi}{l} x \, dx\right)$$

$$= -\frac{l^3}{n\pi}(-1)^n + \frac{2l}{n\pi} \cdot \frac{l}{n\pi}\left(+ \frac{\left[\cos \frac{n\pi}{l}\right]_0^l}{\frac{n\pi}{l}}\right)$$

$$= -(-1)^n \frac{l^3}{n\pi} + \frac{2l^3}{(n\pi)^3}(\cos n\pi - 1)$$

Similarly, finite Fourier cosine transform of $f(x)$ is ~~$\hat{f}_c$ (~~ $\hat{f}_c(n) = \int_0^l f(x)\cos\left(\frac{n\pi}{l}x\right)dx$

$$= \int_0^l x^2 \cos\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{2l^3}{n^2\pi^2}\left(\cos n\pi - 1\right)$$

<u>Some more example:-</u>

Show that $\mathcal{F}\left\{e^{-ax}u(x-b)\right\} = \frac{1}{\sqrt{2\pi}}\frac{1}{a+iw}e^{-b(a+iw)}$, $a>0$

where $u(x-b) := \begin{cases} 0 & \text{if } x<b \\ 1 & \text{if } x>b. \end{cases}$

<u>Answer:-</u> $\mathcal{F}\left\{e^{-ax}u(x-b)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx}e^{-ax}u(x-b)dx$

$$= \frac{1}{\sqrt{2\pi}}\int_b^{\infty} e^{-x(a+iw)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{a+iw}\left[e^{-x(a+iw)}\right]_{\infty}^{b}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{a+iw}\cdot e^{-b(a+iw)}.$$

Gibbs' Phenomenon:-

Gibbs' Phenomenon was discovered by Henry Wibraham and rediscovered by J. Willard Gibbs. It is the particular manner in which the Fourier series as well as the Fourier integral of a piecewise continuously differentiable function behaves at a jump discontinuity.

Let $F(\omega) := \mathcal{F}(f)$ be the Fourier transform of f. Then by Fourier integral formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega.$$

Let $$f_\lambda(x) := \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} F(\omega).\, e^{i\omega x}\, d\omega,$$

As $\lambda \to \infty$, $f_\lambda(x)$ will tend to $f(x)$ pointwise for all x if f is continuous at x. And, if a is a point of jump discontinuity,

$$\frac{\lim\limits_{x \to a-} f(x) + \lim\limits_{x \to a+} f(x)}{2} = \frac{1}{\sqrt{2\pi}} \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} F(\omega)\, e^{i\omega x}\, d\omega.$$

For x on the left (or right) of a, the difference $f_\lambda(x) - f(x)$ oscillates above and below the value 0 as $x \to a$. This $f_\lambda(x) - f(x)$ attains a maximum value at some point, say $x = x_\lambda$. The value $f_\lambda(x_\lambda) - f(x_\lambda)$ is called the overshoot.

As $|\lambda| \to \infty$, the overshoot $f_\lambda(x_\lambda) - f(x_\lambda)$ does not go to 0, but tends to a finite limit. The existence of this non-zero finite limiting value for the overshoot is known as the Gibbs' Phenomenon.

Discrete and Fast Fourier Transforms:-

In interpolation, we have learnt how to approximate a function f by another suitable function when the function is given only in terms of values at finitely many points. If this happens for a function whose Fourier transform we want to find, then that needs a large amount of equally spaced data which occur in telecommunication, time series analysis, various simulation problems. In this situation, while we deal with sampled values rather than functions, we replace the Fourier transform by the discrete Fourier transform.

Let $f(x)$ be periodic of period 2π (say).

Let $f(x)$ have been sampled at the nodes

$$x_k = \frac{2\pi k}{N}, \quad 0 \le k \le N-1$$

Then the complex trigonometric polynomial

$$q(x) = \sum_{n=0}^{N-1} c_n e^{inx_n}$$

interpolates $f(x)$ at the nodes. We need to find the co-efficients $c_0, \ldots, c_{N-1}$ so that

$$f_k = f(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}. \quad \text{---- (01)}$$

We multiply the last equation by e^{-imx_k} and sum over k from 0 to $N-1$. So,

$$\sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} c_n e^{i(n-m)x_k}$$

$$= \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)x_k}.$$

For $n \neq m$, $\displaystyle\sum_{k=0}^{N-1} e^{i(n-m)x_k} = \sum_{k=0}^{N-1} e^{i(n-m)\frac{2\pi k}{N}}$

$$= \frac{1 - \left(e^{\frac{i(n-m)2\pi}{N}}\right)^N}{1 - e^{\frac{i(n-m)2\pi}{N}}}$$

$$= 0.$$

$n = m \Rightarrow \displaystyle\sum_{k=0}^{N-1} e^{i(n-m)\frac{2\pi k}{N}} = N.$

$$\therefore c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-inx_k}.$$

Discrete Fourier transform of the given signal $f = [f_0, -, f_{N-1}]^T$ is the vector $\hat{f} = [\hat{f}_0, -, \hat{f}_{N-1}]^T$ with components

$$\hat{f}_n = N c_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}, \quad n = 0, 1, --, N-1.$$

$\therefore \hat{f} = F_N f$, where $F_N = [e_{nk}]$ with the entries $e_{nk} = e^{-\frac{2\pi i k}{N} n} = w^{nk}$, where $w = e^{-\frac{2\pi i}{N}}$.

For large N, computing these $\hat{f}_n$ involves a huge number of operations. This difficulty can be overcome by the fast Fourier transform. Here one choose $N = 2^p$ and uses this special form of Fourier transform to break down the given problem into smaller problems. Let $N = 2M$.

Then, $w_N^2 = w_{2M}^2 = w_M$.

We denote $f_{ev} = [f_0, f_2, -, f_{N-2}]^T$ and $f_{od} = [f_1, f_3, -, f_{N-1}]^T$

Write $\hat{f}_{ev} = [\hat{f}_{ev,0}, \hat{f}_{ev,1}, - , \hat{f}_{ev,\frac{N-2}{2}}] = F_M f_{ev}$

and $\hat{f}_{od} = [\hat{f}_{od,0}, \hat{f}_{od,1}, -, \hat{f}_{od,\frac{N-2}{2}}] = \cancel{F_N f_{od}}\ F_M f_{od}$

Then one can observe that

$$\left.\begin{aligned} \hat{f}_n &= \hat{f}_{ev,n} + w_N^n \hat{f}_{od} \\ \hat{f}_{n+M} &= \hat{f}_{ev,n} - w_N^n \hat{f}_{od,n} \quad , n = 0, 1, -, M-1. \end{aligned}\right\} \text{--(02)}$$

For $N = 2^p$, this breakdown can be repeated $(p-1)$-times to finally arrive at $\frac{N}{2}$ problems of size 2.

Example: $N = 4$. Let the sample values be $f = [0, 1, 4, 9]^T$.

Then, $w = e^{-\frac{2\pi i}{4}} = -i$.

$$\therefore \hat{f} = F_4 f = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^{3} & w^6 & w^9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}.$$

$$\hat{f}_{ev} = F_2 f_{ev} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$

and

$$\hat{f}_{od} = F_2 f_{od} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \end{bmatrix}.$$

$$\therefore \hat{f}_0 = \hat{f}_{ev,0} + w_4^0 \hat{f}_{od,0} = 14,$$

$$\hat{f}_1 = \hat{f}_{ev,1} + w_4^1 \hat{f}_{od,1} = -4+8i$$

$$\hat{f}_2 = \hat{f}_{ev,0} - w_4^0 \hat{f}_{od,0} = -6$$

$$\hat{f}_3 = \hat{f}_{ev,1} - w_4^1 \hat{f}_{od,1} = -4-8i.$$

This agrees with the previous formula (02).