Solution: FM Assignment 1

1. Use dimensional analysis to find the parametric dependence of the scale height H in a static isothermal atmosphere at temperature T_o composed of a perfect gas with average molecular weight M_w when the gravitational acceleration is g.

   Solution:

   As given in the problem, the parameter list must include H, T, M_w, and g. Here there is no velocity parameter, and there is no need for a second specification of a thermodynamic variable since a static pressure gradient prevails. However, the universal gas constant R must be included to help relate the thermal variable T to the mechanical ones. We write the dimensional matrix corresponding to H, T, M_w, g and R as $[M\ L\ T\ \theta]^T$:

   $$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

   Here the rank of the matrix is 4 and number of dimensionless group is 1.

   $$H = T^a M_w^b \mathrm{g}^c \mathrm{R}^d$$

   We now express this in the dimensions of each term:

   $$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 2 \\ -2 \\ -1 \end{bmatrix}$$

   Solving for a, b, c, d yields

   $$a = 1;\ b = -1;\ c = -1;\ d = 1$$

   And we get

   $$H = K \cdot \frac{RT}{M_w \mathrm{g}}$$

   where K indicates a dimensionless constant.

2. Use dimensional analysis to determine the energy E released in an intense point blast if the blast-wave propagation distance D into an undisturbed atmosphere of density ρ is known as a function of time t following the energy release (Taylor, **1950**; see Fig. 1).

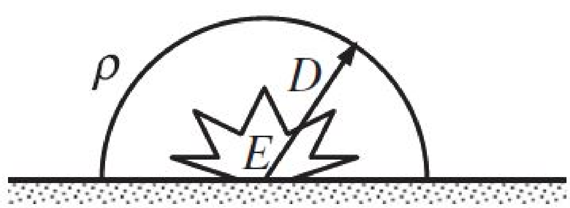

Figure 1: In an atmosphere with undisturbed density ρ, a point release of energy E produces a hemispherical blast wave that travels a distance D in time t.

Solution:

We first write the dimensional matrix corresponding to E, D, ρ and t:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$$

Here the rank of the matrix is 3 and number of dimensionless group is 1. Hence, the the only Π term becomes

$$\Pi_1 = E \cdot D^a \rho^b t^c$$

We can write the above equation into a dimensional matrix form. Since the Π term is dimensionless, it will be represented by the zero column vector.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+b \\ 2-a-3b \\ -2+c \end{bmatrix}$$

Which yield values of

$$a = -5;\ b = -1;\ c = 2$$

and subsequently

$$\Pi_1 = E \cdot D^{-5} \rho^{-1} t^2$$

$$\boxed{E = K \cdot \frac{D^5 \rho}{t^2}}$$

Here K indicates a dimensionless constant and is not determined by the dimensional analysis.

3. Use dimensional analysis to determine how the average light intensity S (Watts/m2) scattered from an isolated particle depends on the incident light intensity I (Watts/m2), the wavelength of the light λ (m), the volume of the particle V (m3), the index of refraction of the particle n_s (dimensionless), and the distance d (m) from the particle to the observation point. Can the resulting dimensionless relationship be simplified to better determine parametric effects when $\lambda \gg V^{1/3}$?

   Solution:

   For the given problem n becomes 6, and we express the dimensional matrix of the quantities S, I, λ, V, n_s and d in the form of

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 3 & 0 & 1 \\ -3 & -3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   The rank of the matrix is 2 and the number of dimensionless groups become 6. By inspection one can deduce the dimensionless quantities as

$$\Pi_1 = \frac{S}{I} \tag{1}$$
$$\Pi_2 = \frac{d}{\lambda} \tag{2}$$
$$\Pi_3 = \frac{V}{\lambda^3} \tag{3}$$
$$\Pi_4 = n_s \tag{4}$$

Therefore, the dimensionless relationship is:

$$\frac{S}{I} = \Phi_1 \left(\frac{d}{\lambda}, \frac{V}{\lambda^3}, n_s \right)$$

The light scattering from the particle must conserve energy and this implies: $4\pi d^2 S = const$, hence $S \propto 1/d^2$. So the results simplify to:

$$\frac{S}{I} = \left(\frac{\lambda}{d} \right)^2 \Phi_2 \left(\frac{V}{\lambda^3}, n_s \right)$$

Further, when λ is large compared to the size of the scatterer, the scattered field amplitude will be produced from the dipole moment induced in the scatterer by the incident field, and this scattered field amplitude will be proportional to V. Thus, S, which is proportional to field amplitude squared, will be proportional to V^2. The simplified relation becomes

$$\frac{S}{I} = \left(\frac{\lambda}{d} \right)^2 \left(\frac{V}{\lambda^3} \right)^2 \Phi_3 (n_s)$$

No, further simplification is possible and the function Φ_3 will be determined experimentally.

4. The surface force F_j per unit volume on a fluid element is the vector derivative, $\partial/\partial x_i$, of the stress tensor T_{ij}. Determine the three components of the vector F_j.

Solution:

F_j is the surface force, given as $\Delta V \frac{\partial T_{ij}}{\partial x_i}$. According to the problem $F_j = \frac{\partial T_{ij}}{\partial x_i}$.
The three components of the vector F_j becomes:

$$\begin{aligned} F_1 &= \Delta V \frac{\partial T_{i1}}{\partial x_i} \\ &= \Delta V \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{21}}{\partial x_2} + \frac{\partial T_{31}}{\partial x_3} \right) \end{aligned}$$

$$\begin{aligned} F_2 &= \Delta V \frac{\partial T_{i2}}{\partial x_i} \\ &= \Delta V \left(\frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{32}}{\partial x_3} \right) \end{aligned}$$

$$\begin{aligned} F_3 &= \Delta V \frac{\partial T_{i3}}{\partial x_i} \\ &= \Delta V \left(\frac{\partial T_{13}}{\partial x_1} + \frac{\partial T_{23}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \right) \end{aligned}$$

5. For two three-dimensional vectors with Cartesian components a_i and b_i, prove the Cauchy-Schwartz inequality: $(a_i b_i)^2 \leq (a_i)^2 (b_i)^2$

   Solution:

$$(a_i b_i)^2 \leq (a_i)^2 (b_i)^2$$

   LHS is recognized to be $(\mathbf{a} \cdot \mathbf{b})^2$, which is $a^2 b^2 \cos^2\theta$. Here, a and b are the magnitudes of the respective vectors. RHS is $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) = a^2 b^2$. Since $\cos^2\theta \leq 1$, the original statement is true.

6. For two three-dimensional vectors with Cartesian components a_i and b_i, prove the triangle inequality: $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$.

   Solution:

   To prove:

$$|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$$

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \cos\theta$$

   Since

$$\cos\theta \leq 1,$$

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

   Multiplying 2 and adding $|\mathbf{a}|^2 + |\mathbf{b}|^2$ to both the sides will not affect the truth of the inequality.

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|$$

   The left side is the square of the magnitude of sum of two vectors, and the right side is collapsible into $(|\mathbf{a}| + |\mathbf{b}|)^2$, which becomes

$$|\mathbf{a} + \mathbf{b}|^2 \leq (|\mathbf{a}| + |\mathbf{b}|)^2$$

   Both the sides when square rooted, cannot yield negative values, allowing to write

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

   which proves the triangle inequality.

7. Using Cartesian coordinates where the position vector is $\mathbf{x} = (x_1, x_2, x_3)$ and the fluid velocity is $\mathbf{u} = (u_1, u_2, u_3)$, write out the three components of the vector: $(\mathbf{u} \cdot \nabla)\mathbf{u} = u_i \frac{\partial u_j}{\partial x_i}$.

$$(\mathbf{u} \cdot \nabla)\mathbf{u}$$

   The operator $\mathbf{u} \cdot \nabla$ can be written in index notation as

$$\mathbf{u} \cdot \nabla = u_i \frac{\partial}{\partial x_i} = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}$$

$$((\mathbf{u} \cdot \nabla)\mathbf{u})_1 = u_1 \frac{\partial}{\partial x_1} u_1 + u_2 \frac{\partial}{\partial x_2} u_1 + u_3 \frac{\partial}{\partial x_3} u_1$$
$$((\mathbf{u} \cdot \nabla)\mathbf{u})_2 = u_1 \frac{\partial}{\partial x_1} u_2 + u_2 \frac{\partial}{\partial x_2} u_2 + u_3 \frac{\partial}{\partial x_3} u_2$$
$$((\mathbf{u} \cdot \nabla)\mathbf{u})_3 = u_1 \frac{\partial}{\partial x_1} u_3 + u_2 \frac{\partial}{\partial x_2} u_3 + u_3 \frac{\partial}{\partial x_3} u_3$$

8. Show that the condition for the vectors **a**, **b**, and **c** to be coplanar is $\epsilon_{ijk}a_ib_jc_k = 0$

   Solution:

   The condition for coplanarity of three vectors is

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$$

   The k_{th} component of $\mathbf{a} \times \mathbf{b}$ is denoted in the index notation by

$$(\mathbf{a} \times \mathbf{b})_k = \epsilon_{ijk}a_ib_j$$

   So, the first condition can be expressed as a dot product with vector **c**, which must be zero.

$$\epsilon_{ijk}a_ib_jc_k = 0$$

9. Prove the following relationships: $\delta_{ij}\delta_{ij} = 3$, $\epsilon_{pqr}\epsilon_{pqr} = 6$, $\epsilon_{pqi}\epsilon_{pqj} = 2\delta_{ij}$

   Solution:

   We may expand the implicit summation over i:

$$\delta_{1j}\delta_{1j} + \delta_{2j}\delta_{2j} + \delta_{3j}\delta_{3j}$$

   Out of the many terms from further expansion of implicit summation, we know that a term would survive if the indices of δ are the same. That brings us one term from each of the three terms listed.

$$1 + 1 + 1 = 3$$

   We can loosely imagine all the terms to be expanded. We know that since the indices are the same for both levi-civita tensors, there will not be any negative terms. There are only $3! = 6$ permutations where i, j, and k are distinct (to yield non-zero terms).

$$\therefore \epsilon_{pqr}\epsilon_{pqr} = 6$$

$$\begin{aligned}
\epsilon_{pqr}\epsilon_{pqr} &= \epsilon_{pqr}\epsilon_{rpq} \\
&= \delta_{pp}\delta_{qq} - \delta_{pq}\delta_{qp} \\
&= 3(3) - \delta_{pp} = 9 - 3 = 6
\end{aligned}$$

$$\epsilon_{pqi}\epsilon_{pqj} = 2\delta_{ij}$$

$$\sum_p \sum_q \epsilon_{pqi}\epsilon_{pqj}$$

$$\sum_q \left(\epsilon_{1qi}\epsilon_{1qj} + \epsilon_{2qi}\epsilon_{2qj} + \epsilon_{3qi}\epsilon_{3qj}\right)$$

Since q would be a fixed number per term (generated after further expansion), i or j would somehow make a term zero unless they were equal (since there are two alternating tensors per term, one with i and the other with j). Else the entire thing would be zero.

q is present in all the three partially expanded terms, so one of them must always be zero, and the others nonzero (provided that $i = j$)

The non-zero partially expanded terms would yield 3 each, when $i = j$.

So, the given expression is evaluated to be 6 when $i = j$ and zero otherwise. This matches the definition of $2\delta_{ij}$ which would be $2\delta_{ii} = 2 \times 3 = 6$.

10. Prove that $\nabla \cdot \nabla \times \mathbf{u} = 0$ for any arbitrary vector function $\mathbf{u}$ regardless of the coordinate system.

Solution:

To prove:

$$\nabla \cdot \nabla \times \mathbf{u} = 0$$

We know:

$$(\nabla \times \mathbf{u})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k$$

So, in the index notation:

$$\nabla \cdot \nabla \times \mathbf{u} = \frac{\partial}{\partial x_i} \left(\epsilon_{ijk} \frac{\partial}{x_j} u_k \right)$$

Expanding it, keeping in mind that mixed partial derivatives are equal, and that all indices of the alternating tensor must be unique.

$$\begin{aligned}
=&\epsilon_{123} \frac{\partial}{\partial x_1 \partial x_2} u_3 + \epsilon_{132} \frac{\partial}{\partial x_1 \partial x_3} u_2 + \epsilon_{231} \frac{\partial}{\partial x_2 \partial x_3} u_1 + \\
&\epsilon_{213} \frac{\partial}{\partial x_1 \partial x_2} u_3 + \epsilon_{312} \frac{\partial}{\partial x_1 \partial x_3} u_2 + \epsilon_{321} \frac{\partial}{\partial x_2 \partial x_3} u_1 \\
=&0
\end{aligned}$$

The terms cancel each other out by the virtue of the presence of both the cyclic and anticyclic varieties of similar partial derivatives. This particular fact is true regardless of the value of $\mathbf{u}$'s components, which may depend on the choice of frame.

11. Determine the divergence and curl of $\mathbf{u} = a\frac{\mathbf{X}}{x^3}$ and $\mathbf{u} = \mathbf{b} \times \frac{\mathbf{X}}{x^2}$ where $\sqrt{x_1^2 + x_2^2 + x_3^2} = x = \sqrt{|\mathbf{X}|^2}$

Given:

$$\mathbf{u} = a\frac{\mathbf{X}}{x^3}$$

Divergence:

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= a\, \nabla \cdot \left(\frac{\mathbf{X}}{x^3} \right) \\
&= a \left(\frac{1}{x^3} \nabla \cdot \mathbf{X} + \mathbf{X} \cdot \nabla \frac{1}{x^3} \right) \\
&= a \left(\frac{3}{x^3} + \mathbf{X} \cdot \nabla \frac{1}{x^3} \right)
\end{aligned}$$

(Since $\nabla \cdot \mathbf{X} = 3$)

The second term, in index notation(x_i represents components of $\mathbf{X}$; $\sqrt{x_1^2+x_2^2+x_3^2} = x$):

$$\begin{aligned}\mathbf{X}\cdot\nabla\frac{1}{x^3} &= x_i\frac{\partial}{\partial x_i}\frac{1}{(x_1^2+x_2^2+x_3^2)^{3/2}}\\ &= x_i\frac{-3x_i}{(x_1^2+x_2^2+x_3^2)^{5/2}}\\ &= \frac{-3x^2}{x^5}\\ &= \frac{-3}{x^3}\end{aligned}$$

So,

$$\nabla\cdot\mathbf{u} = \left(\frac{3}{x^3}-\frac{3}{x^3}\right) = 0$$

Curl:

$$\begin{aligned}\nabla\times\mathbf{u} &= a\ \nabla\times\left(\frac{\mathbf{X}}{x^3}\right)\\ &= a\left(\frac{1}{x^3}\nabla\times\mathbf{X}+\nabla\frac{1}{x^3}\times\mathbf{X}\right)\end{aligned}$$

For the first term:

$$\begin{aligned}\nabla\times\mathbf{X} &= \begin{vmatrix} e_1 & e_2 & e_3\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ x_1 & x_2 & x_3\end{vmatrix}\\ &= 0\end{aligned}$$

Second term: (using the gradient result from the divergence case)

$$\begin{aligned}\nabla\frac{1}{x^3}\times\mathbf{X} &= \epsilon_{ijk}\frac{-3}{(x_1^2+x_2^2+x_3^2)^{5/2}}x_jx_k\\ &= 0\end{aligned}$$

(a vector cross itself is zero)

Given:

$$\mathbf{u} = \mathbf{b}\times\frac{\mathbf{X}}{x^2}$$

$$u_k = \epsilon_{ijk}\frac{b_ix_j}{x_1^2+x_2^2+x_3^2}$$

Divergence:

$$\begin{aligned}\nabla\cdot\mathbf{b}\times\frac{\mathbf{X}}{x^2} &= \frac{\mathbf{X}}{x^2}\cdot(\nabla\times\mathbf{b}) - \mathbf{b}\cdot\nabla\times\frac{\mathbf{X}}{x^2}\\ &= \frac{x_k}{x^2}\epsilon_{ijk}\frac{\partial}{\partial x_j}b_k - b\cdot 0\\ &= \frac{x_k}{x^2}\epsilon_{ijk}\frac{\partial}{\partial x_j}b_k\end{aligned}$$

This would be zero had $\mathbf{b}$ been a constant vector.

Curl:

Let $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} + b_3\mathbf{e_3}$

$$\begin{aligned}
\nabla \times \mathbf{u} &= \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{b_2x_3 - b_3x_2}{x^2} & \frac{b_3x_1 - b_1x_3}{x^2} & \frac{b_1x_2 - b_2x_1}{x^2} \end{vmatrix} \\
&= \frac{2}{x^4}[(b_1x_1^2 + b_2x_1x_2 + b_3x_1x_3)\mathbf{e_1} + (b_2x_2^2 + b_1x_2x_3 + b_3x_2x_1)\mathbf{e_2} + (b_3x_3^2 + b_2x_1x_3 + b_1x_2x_3)\mathbf{e_3}]
\end{aligned}$$