In this lesson we shall first present complex form of Fourier integral. We then introduce Fourier sine and cosine integral. The convergence of these integrals with its application to evaluate integrals will be discussed. In this lesson will be very useful to introduce Fourier transforms.

12.1 The Exponential Fourier Integral

It is often convenient to introduce complex form of Fourier integral. In fact, using complex form of Fourier integral we shall introduce Fourier transform, sometimes referred as Fourier exponential transform, in the next lesson. We start with the following Fourier integral

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(u) \cos \alpha(u-x) \, \mathrm{d}u \, \mathrm{d}\alpha \tag{12.1}$$

Note that the integral

$$\int_{-\infty}^\infty f(u) \cos \alpha(u-x) \, \mathrm{d}u$$

is an even function of α and therefore the integral (12.1) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(u) \cos \alpha(u-x) \, \mathrm{d}u \, \mathrm{d}\alpha \tag{12.2}$$

Also, note that the integral

$$\int_{-\infty}^\infty f(u) \sin \alpha(u-x) \, \mathrm{d}u$$

is an odd function of α and therefore we have the following result

$$\frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(u) \sin \alpha(u-x) \, \mathrm{d}u \, \mathrm{d}\alpha = 0 \tag{12.3}$$

Multiplying the equation (12.3) by i and adding into the equation (12.2) we obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(u) \left[\cos \alpha(u-x) + i \sin \alpha(u-x)\right] \, \mathrm{d}u \, \mathrm{d}\alpha \tag{12.4}$$

This may be rewritten as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(u) e^{i\alpha(u-x)} \, \mathrm{d}u \, \mathrm{d}\alpha \tag{12.5}$$

If we subtract the equation (12.3) after multiplying by i from the equation (12.2) we obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(u) e^{-i\alpha(u-x)} \, \mathrm{d}u \, \mathrm{d}\alpha \tag{12.6}$$

Either (12.5) or (12.6) are exponential form of the Fourier integral.

12.1.1 Example

Compute the complex Fourier integral representation of $f(x) = e^{-a|x|}$.

Solution: The complex integral representation of f is given as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\alpha(u-x)} \, \mathrm{d}u \, \mathrm{d}\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \int_{-\infty}^{\infty} f(u) e^{i\alpha u} \, \mathrm{d}u \, \mathrm{d}\alpha \tag{12.7}$$

We first compute the inner integral

$$\int_{-\infty}^{\infty} f(u) e^{i\alpha u} \, \mathrm{d}u = \int_{-\infty}^{0} e^{au} e^{i\alpha u} \, \mathrm{d}u + \int_{0}^{\infty} e^{-au} e^{i\alpha u} \, \mathrm{d}u = \left[\frac{e^{(a+i\alpha)u}}{a + i\alpha} \right]_{-\infty}^{0} + \left[-\frac{e^{-(a-i\alpha)u}}{a - i\alpha} \right]_{0}^{\infty}$$

This can be further simplified

$$\int_{-\infty}^{\infty} f(u) e^{-i\alpha u} \, \mathrm{d}u = \left(\frac{1}{a + i\alpha} + \frac{1}{a - i\alpha} \right) = \frac{2a}{a^2 + \alpha^2}$$

Then the complex Fourier integral representation of f is

$$f(x) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \alpha^2} e^{-i\alpha x} \, \mathrm{d}\alpha \tag{12.8}$$

In this lesson we describe Fourier transform. We shall connect Fourier series with the Fourier transform through Fourier integral. Several interesting properties of the Fourier Transform such as linearity, shifting, scaling etc. will be discussed.

12.2 Fourier Transform

Consider the Fourier integral defined in earlier lessons as

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u)\cos\alpha(u-x)du\right]d\alpha$$

Since the inner integral is an even function of α we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(u)\cos\alpha(u-x)du\right]d\alpha \tag{12.9}$$

Further note that

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)\sin\alpha(u-x)du\,d\alpha \tag{12.10}$$

as the integral

$$\int_{-\infty}^\infty f(u)\sin\alpha(u-x)du\,d\alpha$$

is an odd function of α. Multiplying the equation (12.10) by the imaginary unit i and adding to the equation (12.9), we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(u)e^{i\alpha(u-x)}du\,d\alpha \tag{12.11}$$

This is the complex Fourier integral representation of f on the real line. Now we split the exponential integrands and the pre-factor $1/(2\pi)$ as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(u)e^{i\alpha u}du\right]e^{-i\alpha x}d\alpha \tag{12.12}$$

The term in the parentheses is what we will the *Fourier transform* of f. Thus the Fourier transform of f, denoted by $\hat{f}(\alpha)$, is defined as

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(u)e^{i\alpha u}du$$

Now the equation (12.12) can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha \tag{12.13}$$

The function $f(x)$ in equation (12.13) is called the inverse Fourier transform of $\hat{f}(\alpha)$. We shall use F for Fourier transformation and F^{-1} for inverse Fourier transformation in this lesson.

Remark: *It should be noted that there are a number of alternative forms for the Fourier transform. Different forms deals with a different pre-factor and power of exponential. For example we can also define Fourier and inverse Fourier transform in the following manner.*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha \quad \textit{where} \quad \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

or

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha \quad \textit{where} \quad \hat{f}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

or

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha \quad \textit{where} \quad \hat{f}(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

We shall remain with our original form because it is easy to remember because of the same pre-factor in front of both forward and inverse transforms.

13.1 Properties of Fourier Transform

We now list a number of properties of the Fourier transform that are useful in their manipulation.

1. Linearity: Let f and g are piecewise continuous and absolutely integrable functions. Then for constants a and b we have

$$F(af + bg) = aF(f) + bF(g)$$

Proof: Similar to the Fourier sine and cosine transform this property is obvious and can be proved just using linearity of the Fourier integral.

2. Change of Scale Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$F[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right), \quad a \neq 0$$

Proof: By the definition of Fourier transform we get

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{i\alpha x}dx$$

Substituting $ax = t$ so that $adx = dt$, we have

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\alpha\frac{t}{a}}\frac{dt}{a} = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right).$$

3. Shifting Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$F[f(x-a)] = e^{i\alpha a}F[f(x)]$$

Proof: By definition, we have

$$\begin{aligned} F[f(x-a)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(x-a)e^{i\alpha x}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(t)e^{i\alpha(t+a)}dt = e^{i\alpha a}\hat{f}(\alpha) \end{aligned}$$

3. Duality Property: If $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$ then

$$F[\hat{f}(x)] = f(-\alpha)$$

Proof: By definition of the inverse Fourier transform, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha$$

Renaming x to α and α to x, we have

$$f(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{-i\alpha x} dx$$

Replacing α to $-\alpha$, we obtain

$$f(-\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{i\alpha x} dx = F[\hat{f}(x)].$$

13.2 Fourier Transforms of Derivatives

13.2.1 Theorem

If $f(x)$ is continuously differential and $f(x) \to 0$ as $|x| \to \infty$, then

$$F[f'(x)] = (-i\alpha) F[f(x)] = (-i\alpha)\hat{f}(\alpha).$$

Proof: By the definition of Fourier transform we have

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{i\alpha x} dx$$

Integrating by parts we obtain

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}} \left\{ \left[f(x) e^{i\alpha x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) e^{i\alpha x} (i\alpha) dx \right\}.$$

Since $f(x) \to 0$ as $|x| \to \infty$, we get

$$F[f'(x)] = -i\alpha \hat{f}(\alpha).$$

This proves the result. ■

Note that the above result can be generalized. If $f(x)$ is continuously n-times differentiable and $f^k(x) \to 0$ as $|x| \to \infty$ for $k = 1, 2, ..., n-1$, then the Fourier transform of nth derivative is

$$F[f^n(x)] = (-i\alpha)^n \hat{f}(\alpha).$$

13.3 Convolution for Fourier Transforms

13.3.1 Theorem

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is $\sqrt{2\pi}$ times the product of the Fourier transforms of $f(x)$ and $g(x)$, i.e.,

$$F[f * g] = \sqrt{2\pi} F(f)F(g).$$

Proof: By definition, we have

$$F[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y \right) e^{i\alpha x}\mathrm{d}x$$

Changing the order of integration we obtain

$$F[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x-y)e^{i\alpha x}\mathrm{d}x\,\mathrm{d}y$$

By substituting $x - y = t \Rightarrow dx = dt$ we get

$$F[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(t)e^{i\alpha(y+t)}\mathrm{d}t\,\mathrm{d}y$$

Splitting the integrals we get

$$F[f * g] = \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{i\alpha y}\,\mathrm{d}y \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)e^{i\alpha t}\,\mathrm{d}t \right)$$

Finally we have the following result

$$F[f * g] = \sqrt{2\pi}\, F[f]F[g] = \sqrt{2\pi}\, \hat{f}(\alpha)\hat{g}(\alpha)$$

This proves the result. ∎

The above result is sometimes written by taking the inverse transform on both the sides as

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha)\hat{g}(\alpha)e^{-i\alpha x}\,\mathrm{d}\alpha$$

or

$$\int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y = \int_{-\infty}^{\infty} \hat{f}(\alpha)\hat{g}(\alpha)e^{-i\alpha x}\,\mathrm{d}\alpha$$

13.4 Parseval's Identity for Fourier Transforms

13.4.1 Theorem

If $\hat{f}(\alpha)$ and $\hat{g}(\alpha)$ are the Fourier transforms of the $f(x)$ and $g(x)$ respectively, then

$$(i) \int_{-\infty}^{\infty} \hat{f}(\alpha)\overline{\hat{g}(\alpha)}\,\mathrm{d}\alpha = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x \quad (ii) \int_{-\infty}^{\infty} |\hat{f}(\alpha)|^2\,\mathrm{d}\alpha = \int_{-\infty}^{\infty} |f(\alpha)|^2\,\mathrm{d}\alpha.$$

Proof: (i) Use of the inversion formula for Fourier transform gives

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{\hat{g}(\alpha)e^{i\alpha x}}\,\mathrm{d}\alpha\right)\,\mathrm{d}x$$

Changing the order of integration we have

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{\hat{g}(\alpha)e^{i\alpha x}}\,\mathrm{d}x\,\mathrm{d}\alpha$$

Using the definition of Fourier transform we get

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \overline{\hat{g}(\alpha)}\hat{f}(\alpha)\,\mathrm{d}\alpha.$$

(ii) Taking $f(x) = g(x)$ we get,

$$\int_{-\infty}^{\infty} \hat{f}(\alpha)\overline{\hat{f}(\alpha)}\,\mathrm{d}\alpha = \int_{-\infty}^{\infty} f(x)\overline{f(x)}\,\mathrm{d}x$$

This implies $$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |f(\hat{\alpha})|^2\,\mathrm{d}\alpha.$$

13.5 Example Problems

13.5.1 Problem 1

Find the Fourier transform of the following function

$$X_{[-a,a]}(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a. \end{cases} \tag{13.1}$$

Solution: By the definition of Fourier transform, we have

$$F\left[X_{[-a,a]}(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X_{[-a,a]}(x) e^{i\alpha x} dx$$

Using the given value of given function we get

$$\begin{aligned} F\left[X_{[-a,a]}(x)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha}(e^{i\alpha a} - e^{-i\alpha a}) \\ &= \frac{2}{\sqrt{2\pi}} \left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i\alpha} \right) = \frac{2}{\sqrt{2\pi}} \left(\frac{\sin(\alpha a)}{\alpha} \right). \end{aligned}$$

13.5.2 Problem 2

Find the Fourier transform of e^{-ax^2}.

Solution: Using the definition of the Fourier Transform

$$F(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{i\alpha x} dx$$

Further simplifications leads to

$$\begin{aligned} F\left[e^{-ax^2}\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{[-a(x-\frac{i\alpha}{2a})^2 - \frac{\alpha^2}{4a}]} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{4a}} \int_{-\infty}^{\infty} e^{-ay^2} dy = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}} \end{aligned}$$

If $a = 1/2$ then $F\left[e^{-\frac{1}{2}x^2}\right] = e^{-\frac{\alpha^2}{2}}$. This shows $F[f(x)] = f(\alpha)$ such function is said to be self-reciprocal under the Fourier transformation.

13.5.3 Problem 3

Find the inverse Fourier transform of $\hat{f}(\alpha) = e^{-|\alpha|y}$, *where* $y \in (0, \infty)$.

Solution: By the definition of inverse Fourier transform

$$\begin{aligned} F^{-1}\left[\hat{f}(\alpha)\right] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\alpha|y} e^{-i\alpha x} d\alpha \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\alpha y} e^{-i\alpha x} d\alpha + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\alpha y} e^{-i\alpha x} d\alpha \end{aligned}$$

Combining the two exponentials in the integrands

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(y-ix)\alpha} d\alpha + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(y+ix)\alpha} d\alpha$$

Now we can integrate the above two integrals to get

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(y-ix)\alpha}}{(y-ix)}\right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(y+ix)\alpha}}{-(y+ix)}\right]_{0}^{\infty}$$

Noting $\lim_{\alpha\to-\infty} e^{(y-ix)\alpha} = 0$ and $\lim_{\alpha\to\infty} e^{-(y+ix)\alpha} = 0$, we obtain

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \frac{1}{y-ix} + \frac{1}{\sqrt{2\pi}} \frac{1}{y+ix}$$

This can be further simplified to give

$$F^{-1}\left[\hat{f}(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \frac{y+ix+y-ix}{(y-ix)(y+ix)}$$

Hence we get

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{y}{(x^2+y^2)}.$$

13.1 Example Problems

13.1.1 Problem 1

Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & \textit{when } |x| < a \\ 0, & \textit{when } |x| > a \end{cases}$$

and hence evaluate

$$(i) \int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} \,\mathrm{d}\alpha, \quad (ii) \int_{0}^{\infty} \frac{\sin \alpha a}{\alpha} \,\mathrm{d}\alpha \;\; \textit{and} \;\; (iii) \int_{0}^{\infty} \frac{\sin^2 x}{x^2} \,\mathrm{d}x.$$

Solution: (i) Let $\hat{f}(\alpha)$ be the Fourier transform of $f(x)$. Then, by the definition of Fourier transform

$$\begin{aligned} \hat{f}(\alpha) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\alpha x} dx \\ =& \frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} \left(e^{i\alpha a} - e^{-i\alpha a} \right) dx \end{aligned}$$

This gives

$$\hat{f}(\alpha) = \frac{2}{\sqrt{2\pi}} \frac{\sin a\alpha}{\alpha}$$

From the definition of inverse Fourier transform we also know that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} \,\mathrm{d}\alpha$$

This implies that

$$\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} \,\mathrm{d}\alpha = \sqrt{2\pi} f(x) = \begin{cases} \sqrt{2\pi}, & \text{when } |x| < a \\ 0, & \text{when } |x| > a \end{cases}$$

Substituting $\hat{f}(\alpha)$ in the above equation we get

$$\int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{\sin a\alpha}{\alpha} (\cos \alpha x - i \sin \alpha x) \,\mathrm{d}\alpha = \begin{cases} \sqrt{2\pi}, & \text{when } |x| < a \\ 0, & \text{when } |x| > a \end{cases}$$

We now split the left hand side into real and imaginary parts to get

$$\int_{-\infty}^{\infty} \frac{\sin a\alpha \cos x\alpha}{\alpha} \,\mathrm{d}\alpha - i \int_{-\infty}^{\infty} \frac{\sin \alpha a \sin \alpha x}{\alpha} \,\mathrm{d}\alpha = \begin{cases} \pi, & \text{when } |x| < a \\ 0, & \text{when } |x| > a \end{cases}$$

Equating real part on both sides we get the desired result as

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} \,\mathrm{d}\alpha = \begin{cases} \pi, & \text{when } |x| < a \\ 0, & \text{when } |x| > a \end{cases}$$

(ii) If we set $x = 0$ and $a = 1$ in the above results, we get

$$\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha} \,\mathrm{d}\alpha = \pi, \quad \text{Since } |x| < a$$

Since the integrand is an even function, we get the the desired results

$$\int_{0}^{\infty} \frac{\sin \alpha}{\alpha} = \frac{\pi}{2}$$

(ii) We now apply Parseval's identity for Fourier transform

$$\int_{-\infty}^{\infty} |\hat{f}(\alpha)|^2 \,\mathrm{d}\alpha = \int_{-\infty}^{\infty} |f(\alpha)|^2 \,\mathrm{d}\alpha$$

Substituting the function $f(x)$ and its Fourier transform we get

$$\int_{-\infty}^{\infty} \frac{4}{2\pi} \frac{\sin^2 a\alpha}{\alpha^2} \,\mathrm{d}\alpha = \int_{-a}^{a} \mathrm{d}\alpha = 2a$$

This implies

$$\int_{-\infty}^{\infty} \frac{\sin^2 a\alpha}{\alpha^2} \,\mathrm{d}\alpha = \pi a$$

Since the integrand is an even function we have the desired result as

$$\int_{0}^{\infty} \frac{\sin^2 a\alpha}{\alpha^2} \,\mathrm{d}\alpha = \frac{\pi}{2} a$$

13.1.2 Problem 2

Evaluate the Fourier transform of the rectangular pulse function

$$\Pi(t) = \begin{cases} 1, & \textit{if } |t| < 1/2; \\ 0, & \textit{otherwise.} \end{cases}$$

Apply the convolution theorem to evaluate the Fourier transform of the triangular pulse function

$$\Lambda(t) = \begin{cases} 1 - |t|, & \textit{if } |t| < 1; \\ 0, & \textit{otherwise.} \end{cases}$$

Solution: It is well known result that $\Lambda = \Pi * \Pi$. It can easily be sheen by observing

$$(\Pi * \Pi)(t) = \int_{-\infty}^{\infty} \Pi(y)\Pi(t-y)dy = \begin{cases} \int_{-1/2}^{t+1/2} 1 \cdot 1dy, & \text{if } -1 < t < 0; \\ \int_{t-1/2}^{1/2} 1 \cdot 1dy, & \text{if } 0 < t < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, we have

$$(\Pi * \Pi)(t) = \int_{-\infty}^{\infty} \Pi(y)\Pi(t-y)dy = \begin{cases} 1+t, & \text{if } -1 < t < 0; \\ 1-t, & \text{if } 0 < t < 1; \\ 0 & \text{otherwise.} \end{cases} = \Lambda(t)$$

Using $a = 1/2$ in the previous example we have

$$F(\Pi) = \frac{2}{\sqrt{2\pi}} \frac{\sin(\alpha/2)}{\alpha}$$

Now using convolution result we get

$$F[\Lambda(t)] = F[(\Pi * \Pi)(t)] = \sqrt{2\pi} F(\Pi) F(\Pi) = \frac{4}{\sqrt{2\pi}} \frac{\sin^2(\alpha/2)}{\alpha^2}.$$

In this lesson we provide some miscellaneous examples of Fourier transforms. One of the major applications of Fourier transforms for solving partial differential equations will not be discussed in this module. However, we shall highlights some other applications like evaluating special integrals and the idea of solving ordinary differential equations.

13.2 Example Problems

13.2.1 Problem 1

Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2, & \textit{when } |x|<1 \\ 0, & \textit{when } |x|>1 \end{cases}$$

and hence evaluate

$$\int_0^\infty \frac{-x\cos x+\sin x}{x^3}\cos\frac{x}{2}\,\mathrm{d}x.$$

Solution: Using the definition of Fourier transform we get

$$\begin{aligned}\hat{f}(\alpha) = F\left[f(x)\right] =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,\mathrm{d}x \\ =& \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{i\alpha x}\left(1-x^2\right)\,\mathrm{d}x\end{aligned}$$

Integrating by parts we obtain

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\frac{e^{i\alpha x}}{i\alpha}(1-x^2)\Big|_{-1}^{1} - \int_{-1}^{1}\frac{e^{i\alpha x}}{i\alpha}(-2x)\,\mathrm{d}x$$

Again, the application of integration by parts gives

$$\hat{f}(\alpha) = \frac{2}{\sqrt{2\pi}}\left[\frac{e^{i\alpha x}}{(i\alpha)^2}x\Big|_{-1}^{1} - \int_{-1}^{1}\frac{e^{i\alpha x}}{(i\alpha)^2}\,\mathrm{d}x\right]$$

Further simplifications leads to

$$\begin{aligned}\hat{f}(\alpha) =& \frac{2}{\sqrt{2\pi}}\left[-\frac{1}{\alpha^2}\left(e^{i\alpha}+e^{-i\alpha}-\frac{e^{i\alpha x}}{i\alpha}\Big|_{-1}^{1}\right)\right] \\ =& -\frac{1}{\sqrt{2\pi}}\frac{2}{\alpha^2}\left[e^{i\alpha}+e^{-i\alpha}-\frac{e^{i\alpha}}{i\alpha}+\frac{e^{-i\alpha}}{i\alpha}\right]\end{aligned}$$

Using Euler's equality we obtain

$$\begin{aligned}\hat{f}(\alpha) =& -\frac{1}{\sqrt{2\pi}}\frac{4}{\alpha^2}\left[\cos\alpha - \frac{\sin\alpha}{\alpha}\right]\\ =& \frac{1}{\sqrt{2\pi}}\frac{4}{\alpha^3}\left[-\alpha\cos\alpha + \sin\alpha\right]\end{aligned}$$

We know from the Fourier inversion formula that

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\alpha)e^{-i\alpha x}\,\mathrm{d}\alpha$$

This implies

$$f(x) = \frac{4}{2\pi}\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha + \sin\alpha}{\alpha^3}e^{-i\alpha x}\,\mathrm{d}\alpha$$

Equating real parts, on both sides we get

$$\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha + \sin\alpha}{\alpha^3}\cos\alpha x\,\mathrm{d}\alpha = \frac{\pi}{2}f(x)$$

Substituting the value of the function we obtain

$$\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha + \sin\alpha}{\alpha^3}\cos\alpha x\,\mathrm{d}\alpha = \begin{cases}\frac{\pi}{2}(1-x^2), \text{ when } |x|<1\\ 0, \text{ when } |x|>1\end{cases}$$

Substitution $x = 1/2$ gives

$$\int_{-\infty}^{\infty}\frac{-\alpha\cos\alpha + \sin\alpha}{\alpha^3}\cos\frac{\alpha}{2}\,\mathrm{d}\alpha = \frac{\pi}{2}\left(1 - \frac{1}{4}\right),$$

This implies

$$2\int_{0}^{\infty}\frac{-\alpha\cos\alpha + \sin\alpha}{\alpha^3}\cos\frac{\alpha}{2}\,\mathrm{d}\alpha = \frac{3\pi}{8}$$

Hence we get the desired result as

$$\int_{0}^{\infty}\frac{-\alpha\cos\alpha + \sin\alpha}{\alpha^3}\cos\frac{\alpha}{2}\,\mathrm{d}\alpha = \frac{3\pi}{16}$$

13.2.2 Problem 2

Find the Fourier transformation of the function $f(t) = e^{-at}H(t), a > 0$ *where*

$$H(t) = \begin{cases} 0, & \textit{when } t < 0 \\ 1, & \textit{when } t \geq 0 \end{cases}$$

Solution: Using the definition of Fourier transform

$$\begin{aligned} F\left[f(t)\right] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} \, \mathrm{d}t \\ =& \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-at} e^{i\alpha t} \, \mathrm{d}t \end{aligned}$$

Solving integral leads to

$$F\left[f(t)\right] = \frac{1}{\sqrt{2\pi}} \frac{e^{(-a+i\alpha)t}}{(-a+i\alpha)} \Big|_0^{\infty}$$

Since we know that

$$\lim_{t\to\infty} e^{-at} e^{i\alpha t} = \lim_{t\to\infty} e^{-at} \left(\cos \alpha t + i \sin \alpha t\right) = 0$$

We get the required transform as

$$F\left[f(t)\right] = -\frac{1}{\sqrt{2\pi}} \frac{1}{(-a+i\alpha)} = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-i\alpha}\right).$$

13.2.3 Problem 3

Find the Fourier transform of Dirac-Delta function $\delta(t-a)$.

Solution: Recall that the Dirac-Delta function can be thought as

$$\delta(t-a) = \lim_{\epsilon\to 0} \delta_\epsilon(t-a) = \begin{cases} 0, & \text{when } t < a, \quad a > 0 \\ \frac{1}{\epsilon}, & \text{when } a \leq t \leq a+\epsilon \\ 0, & \text{when } t > a+\epsilon \end{cases}$$

Applying the definition of Fourier transform we get

$$\begin{aligned} F\left[\delta(t-a)\right] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t-a) e^{i\alpha t} \, \mathrm{d}t \\ =& \frac{1}{\sqrt{2\pi}} \int_{a}^{a+\epsilon} \lim_{\epsilon\to 0} \frac{1}{\epsilon} e^{i\alpha t} \, \mathrm{d}t \end{aligned}$$

On integrating we obtain

$$\begin{aligned}
F\left[\delta(t-a)\right] &= \lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{\epsilon}\frac{e^{i\alpha t}}{i\alpha}\Big|_{a}^{a+\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi}}\frac{1}{\epsilon}\frac{1}{i\alpha}\left(e^{i\alpha(a+\epsilon)}-e^{i\alpha a}\right) \\
&= \frac{1}{\sqrt{2\pi}}e^{i\alpha a}\lim_{\epsilon\to 0}\frac{e^{i\alpha\epsilon}-1}{i\alpha\epsilon} = \frac{1}{\sqrt{2\pi}}e^{i\alpha a}
\end{aligned}$$

With this results we deduce that $F^{-1}(1)=\sqrt{2\pi}\delta(t)$.

13.2.4 Problem 4

Find the Fourier transform of

$$f(t)=e^{-a|t|}, \quad -\infty<t<\infty,\ a>0.$$

Solution: Using the definition of Fourier transform we have

$$\begin{aligned}
F\left[e^{-a|t|}\right] &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{at}e^{i\alpha t}\,\mathrm{d}t+\int_{0}^{\infty} e^{-at}e^{i\alpha t}\,\mathrm{d}t\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{(a+i\alpha)t}}{a+i\alpha}\Big|_{-\infty}^{0}+\frac{e^{(-a+i\alpha)t}}{-a+i\alpha}\Big|_{0}^{\infty}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+i\alpha}+(-1)\frac{1}{-a+i\alpha}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+i\alpha}+\frac{1}{a-i\alpha}\right] = \frac{1}{\sqrt{2\pi}}\frac{2a}{a^2+\alpha^2}.
\end{aligned}$$

13.2.5 Problem 5

Find the inverse Fourier transform of $\hat{f}(\alpha)=\dfrac{1}{2\pi(a-i\alpha)^2}$.

Solution: Writing the given function as a product of two functions as

$$F^{-1}\left[\hat{f}(\alpha)\right]=F^{-1}\left[\frac{1}{\sqrt{2\pi}(a-i\alpha)}\,\frac{1}{\sqrt{2\pi}(a-i\alpha)}\right]$$

Application of convolution theorem gives

$$f(t)=\frac{1}{\sqrt{2\pi}}F^{-1}\left[\frac{1}{\sqrt{2\pi}(a-i\alpha)}\right]*F^{-1}\left[\frac{1}{\sqrt{2\pi}(a-i\alpha)}\right]=\frac{1}{\sqrt{2\pi}}\left[e^{-at}H(t)*e^{-at}H(t)\right]$$

Evaluating the convolution

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax} H(x) e^{-a(t-x)} H(t-x) dx = \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(x) H(t-x) dx$$

Note that $H(x)H(t-x) = 0$ when $x < 0$ or when $t - x < 0$, i.e.,

$$H(x)H(t-x) = \begin{cases} 1, & \text{if } 0 < x < t; \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$$f(t) = \frac{e^{at}}{\sqrt{2\pi}} \int_0^t dx = \begin{cases} \frac{te^{-at}}{\sqrt{2\pi}}, & \text{if } t > 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Thus we get

$$f(t) = \frac{te^{-at}}{\sqrt{2\pi}} H(t).$$

13.2.6 Problem 6

Using Fourier transform, find the solution of the differential equation

$$y' - 2y = H(t)e^{-2t}, \quad -\infty < t < \infty, \quad y \to 0 \text{ as } |t| \to \infty$$

Solution: Taking Fourier transform on both sides we get

$$F\left[y'\right] - 2F[y] = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{-2 + i\alpha} \right)$$

Aplying the property of Fourier transform of derivatives we get

$$-i\alpha\hat{y} - 2\hat{y} = -\frac{1}{\sqrt{2\pi}} \left(\frac{1}{-2 + i\alpha} \right)$$

Simple algebraic calculation gives the value of transformed variable as

$$\hat{y} = -\frac{1}{\sqrt{2\pi}} \frac{1}{4 + \alpha^2}$$

Taking inverse Fourier transform we get the desired solution as $y = -\frac{1}{4} e^{-2|t|}$.

Summary: Fourier Transform

Fourier cosine and inverse Fourier cosine transform

$$\boxed{F_c(f) = \hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \alpha u \, \mathrm{d}u} \quad \boxed{F_c^{-1}(\hat{f}) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\alpha) \cos \alpha x \, \mathrm{d}\alpha}$$

Fourier sine and inverse Fourier sine transform

$$\boxed{F_s(f) = \hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \alpha u \, \mathrm{d}u} \quad \boxed{F_s^{-1}(\hat{f}) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\alpha) \sin \alpha x \, \mathrm{d}\alpha}$$

Derivative formula: Assuming that f and f' both goes to 0 as x approaches to ∞

$$\boxed{F_c\{f''(x)\} = -\alpha^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)} \quad \boxed{F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}} \alpha f(0) - \alpha^2 F_s\{f(x)\}}$$

Fourier transform

$$\boxed{F(f) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(u) e^{i\alpha u} \, \mathrm{d}u} \quad \boxed{F^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(\alpha) e^{-i\alpha x} \, \mathrm{d}\alpha}$$

Derivative formula-1: Assuming that f goes to 0 as $|x|$ approaches to ∞

$$\boxed{F\{f'(x)\} = -i\alpha F\{f(x)\}}$$

Derivative formula-2: Assuming that f and f' both go to 0 as $|x|$ approaches to ∞

$$\boxed{F\{f''(x)\} = -\alpha^2 F\{f(x)\}}$$

Convolution property

$$\boxed{(f * g)(x) = \int_{-\infty}^\infty f(y) g(x - y) \mathrm{d}y} \quad \boxed{F\{(f * g)\} = \sqrt{2\pi} F\{f\} F\{g\}}$$