Fourier Series.

Consider $f(x)$, a real-valued function, that is integrable on an integral of length $2L$, which will be the period of the Fourier Series.

Then, $f(x)$ can be written as fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where,
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Period of $f(x) = 2L$.

Q1:- Use the Fourier Series

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

to deduce the value of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3}.$$

Soln:-

Integrating the Fourier Series from $0 \to \pi/2$:

$$\int_0^{\pi/2} x^2\,dx = \int_0^{\pi/2} \frac{\pi^2}{3}\,dx + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^{\pi/2} \cos nx\,dx.$$

$$\Rightarrow \frac{\pi^3}{24} = \frac{\pi^2}{3}\left(\frac{\pi}{2} - 0\right) + 4\sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{(-1)^n}{n^2} \sin\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow \frac{\pi^3}{24} = \frac{\pi^3}{6} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi}{2}\right)$$

$$\because \sin\left(\frac{n\pi}{2}\right) = 0, 1, 0, -1, \ldots \quad \text{for } n = 0, 1, 2, \ldots$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi}{2}\right) = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} \quad -(1)$$

And

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi}{2}\right) = \frac{1}{4}\left(\frac{\pi^3}{6} - \frac{\pi^3}{24}\right) = \frac{\pi^3}{32}$$

from (1).

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \frac{\pi^3}{32}$$

Q2:- Given the Fourier Series

$$t^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt).$$

deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Soln. Applying Parseval's Theorem to this series,

$$\int_{-\pi}^{\pi} (t^2)^2 \, dt = \int_{-\pi}^{\pi} \left(\frac{\pi^2}{3}\right)^2 dt + 16 \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \frac{1}{n^4} \cos^2 nt \, dt.$$

$$\Rightarrow \frac{2}{5}\pi^5 = 2\pi\left(\frac{\pi^2}{3}\right)^2 + \pi \sum_{n=1}^{\infty} \frac{16}{n^4}.$$

$$\Rightarrow \frac{2}{5}\pi^5 = \frac{2}{9}\pi^5 + \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\Rightarrow 16 \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4\left(\frac{2}{5} - \frac{2}{9}\right)$$

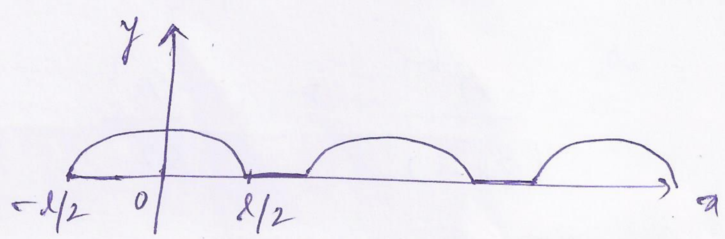
$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Q3:- Expand the function $f(x)$, defined by

$$f(x) = \begin{cases} \cos\left(\frac{\pi x}{l}\right), & \text{for } 0 \le x \le l/2 \\ 0, & \text{for } \frac{l}{2} \le x \le l \end{cases}$$

in Cosine Series.

Soln.



8. Period of $f(x) = 2l$.

for $l/2 < x \le l$, we have $f(x) = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^{l/2} \cos\left(\frac{\pi x}{l}\right) dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l}\int_0^{l/2} \cos\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right) dx$$

Put $\frac{\pi x}{l} = t$.

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} \cos t \cos nt \, dt$$

$$a_n = \frac{1}{\pi}\left[\frac{\sin(n+1)t}{n+1} + \frac{\sin(n-1)t}{n-1}\right]_0^{\pi/2} \quad (n > 1)$$

$a_n = 0$ if n-odd.

$$a_n = \frac{-2(-1)^{n/2}}{\pi(n^2-1)}, \quad n\text{-even}.$$

$b_n = 0 \quad \forall n$.

Thus, we have

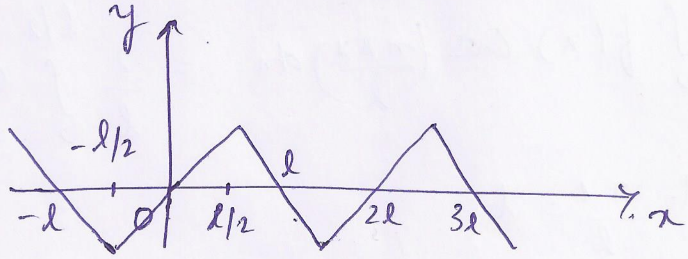
$$\frac{1}{\pi} + \frac{1}{2}\cos\left(\frac{\pi x}{l}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos\left(\frac{2n\pi x}{l}\right) = \begin{cases} \cos\frac{\pi x}{l}, & 0 \le x \le l/2 \\ 0, & \frac{l}{2} < x \le l \end{cases}$$

Ques4:- Expand the function $f(x)$, defined by

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le l/2 \\ l-x & \text{for } l/2 < x \le l \end{cases}$$

in the series.

Soln:



period of $f(x) = 2l$.

$a_n = 0 \quad \forall n.$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx + \frac{2}{l}\int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$(n = 1, 2, \ldots)$

Setting $\frac{\pi x}{l} = t$, we obtain

$$b_n = \frac{2l}{\pi^2}\int_0^{\pi/2} t\sin nt\,dt + \frac{2l}{\pi^2}\int_{\pi/2}^{\pi}(\pi - t)\sin nt\,dt$$

$$b_n = \frac{4l}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore \frac{4l}{\pi^2}\left(\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} - \cdots\right) = \begin{cases} x & \text{for } 0 \le x \le l/2 \\ l-x & \text{for } \frac{l}{2} < x \le l. \end{cases}$$

Ans 5: Show that the Fourier Series for

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad -\pi < x < \pi$$

can be integrated from 0 to x when $-\pi \le x \le \pi$ and obtain a converging series.

$$x^3 - \pi^2 x = 12\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^3}.$$

Soln:

$$\int_0^x t^2\,dt = \frac{\pi^2}{3}\int_0^x dt + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\int_0^x \cos nt\,dt$$

$$\left[\frac{t^3}{3}\right]_0^x = \frac{\pi^2}{3}\left[t\right]_0^x + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\left[\sin nt\right]_0^x$$

$$\Rightarrow \quad x^3 - \pi^2 x = 12\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^3}.$$

If $F(x) = x^3 - \pi^2 x$, then $F(-\pi) = F(\pi)$.

Series of $F(x)$ is uniformly and absolutely convergent to $F(x)$ for $-\pi \le x \le \pi$.

Ans 6: Differentiate the series

$$\cos x = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n \sin 2nx}{4n^2 - 1}.$$

and investigate the possibility of the newly formed series converging to the function $\sin x$.

Sol:- By termwise differentiation we have the series

$$\frac{16}{\pi} \sum_{n=1}^{\infty} \frac{n^2 \cos 2nx}{4n^2 - 1}.$$

$$\lim_{n \to \infty} \frac{16}{\pi} \cdot \frac{n^2 \cos 2nx}{4n^2-1} \neq 0.$$

$\therefore$ This new series is divergent and cannot be the convergence of $\sin x$.

If the function f is replaced by f' with $L = \pi$, then we ~~have~~ are assured that the series corresponding to f' converges.

If f' is periodic with a period 2π and PWS on $-\pi \le x \le \pi$, then the corresponding fourier series

$$\frac{a_0'}{2} + \sum_{n=1}^{\infty} (a_n' \cos nx + b_n' \sin nx)$$

where

$$a_n' = \frac{1}{\pi} \int_{-\pi}^{\pi} f' \cos nx \, dx, \quad n \in \mathbb{N}_0$$

$$b_n' = \frac{1}{\pi} \int_{-\pi}^{\pi} f' \sin nx \, dx, \quad n \in \mathbb{N}$$

converges to

$$\frac{f'(x+) + f'(x-)}{2}$$

If we add that $f(-\pi) = f(\pi)$ and make f a continous function with f' PWS, then both f' and f are PWC. Coefficients

$$a_0' = 0, \quad a_n' = nb_n, \quad b_n' = -na_n$$

have been determined. The derivative f' is continous where f'' exists. for the values of x where f'' exists.

$$f'(x) = f'(x+) = f'(x-)$$

&

$$\frac{f'(x+) + f'(x-)}{2} = f'(x)$$

or

$$f'(x) = \sum_{n=1}^{\infty} (nb_n \cos nx - na_n \sin nx)$$

The following theorem contains the results.

Ques 7: find the fourier series of the function defined as

$$f(x) = \begin{cases} x+\pi & \text{for } 0 \le x \le \pi \\ -x-\pi & \text{for } -\pi \le x < 0 \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x)$$

Soln: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (x+\pi)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \begin{cases} \dfrac{-4}{n^2\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(\frac{-\cos nx}{n}\right) - (+1)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \begin{cases} \dfrac{4}{n}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots +$$

$$b_1 \sin x + b_2 \sin 2x + \dots$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots\right] +$$

$$4\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \dots\right].$$

Ques 8 :- Find the fourier half-range cosine series of the function

$$f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2 \end{cases}$$

Sol:

$$f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2 \end{cases}$$

Let $f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{c} + a_2 \cos\frac{2\pi t}{c} + a_3 \cos\frac{3\pi t}{c} + \dots$

$+ b_1 \sin\frac{\pi t}{c} + b_2 \sin\frac{2\pi t}{c} + b_3 \sin\frac{3\pi t}{c} + \dots \quad (1)$

Here, $c = 2$, because it is half range series.

Hence, $a_0 = \frac{2}{c}\int_0^c f(t)\,dt$

$$= \frac{2}{c}\int_0^1 2t\,dt + \frac{2}{2}\int_1^2 2(2-t)\,dt$$

$$= 2.$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\frac{n\pi t}{c}\,dt$$

$$= \frac{2}{2}\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \frac{2}{2}\int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt$$

$$a_n = \frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$f(t) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]\cos\frac{n\pi t}{2}$$

Ques 9: Obtain the complex from of the fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

Soln:

$$C_0 = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0.\,e^{-inx}\,dx + \int_0^{\pi} 1.\,e^{-inx}\,dx\right]$$

$$= \begin{cases} \frac{1}{in\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left[\frac{e^{ix}}{1} + \frac{e^{i3x}}{3} + \cdots\right] + \frac{1}{i\pi}\left[\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \cdots\right]$$

$$= \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

Ques 10:- Prove that for $0 < x < \pi$

a) $x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right]$

b) $x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right]$

Deduce from a) and b) respectively that

c) $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ d) $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^4}{945}$.

Soln a) Half range cosine series

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx$$

$$a_n = \begin{cases} \frac{-4}{n^2}, & n - \text{even}. \\ 0, & n - \text{odd} \end{cases}$$

Hence, $x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \cdots\right]$

By Parseval's formula

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi-x)^2\,dx = \frac{a_0^2}{2} + \Sigma a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} \left(\pi^2x^2 - 2\pi x^3 + x^4\right)dx = \cancel{\frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]}$$

$$\frac{1}{2}\left(\frac{\pi^4}{9}\right) + 16\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right]$$

$$\Rightarrow \quad \frac{2}{\pi}\left[\frac{\pi^5}{3} - \frac{2\pi^5}{4} + \frac{\pi^5}{5}\right] = \frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\frac{\pi^4}{15} = \frac{\pi^4}{18} + \sum_{n=1}^{\infty}\frac{1}{n^4} \quad \text{or} \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

b) Half range sine series

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$b_n = \begin{cases} \dfrac{8}{n^3\pi}, & n\text{-odd} \\ 0, & n\text{-even} \end{cases}$$

$$\therefore \quad x(\pi-x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right]$$

By Parsevel's formula

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi - x^2)\, dx = \sum b_n^2$$

$$\frac{\pi^4}{15} = \frac{64}{\pi^2}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots\right]$$

$$\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots$$

let $S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots\right) +$

$$\left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \cdots\right)$$

$$S = \frac{\pi^6}{960} + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \cdots\right) = \frac{\pi^6}{960} + \frac{S}{64}$$

$$S - \frac{S}{64} = \frac{\pi^6}{960}$$

$$\Rightarrow \frac{63}{64} S = \frac{\pi^6}{960}$$

$$S = \frac{\pi^6}{960} \times \frac{64}{63} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

Que: 11 Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the fourier expression of $f(x)$.

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$.

Sol^n Let $f(x) = x + x^2 = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} [x + x^2]\,dx$$

$$\Rightarrow a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$\Rightarrow a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\,dx$$

$$\Rightarrow b_n = -\frac{2}{n}(-1)^n$$

So, $f(x) = x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right] - 2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right]$.

Hence, $f(x)$ is valid for all values of x between $-\pi$ and π but not at the end points $-\pi$ and π due to open interval.

$$f(-\pi) = \frac{1}{2}\left[f(-\pi - 0) + f(-\pi + 0)\right]$$

{ $f(x)$ is periodic with period 2π.

$$= \frac{1}{2}\left[f(\pi - 0) + f(-\pi + 0)\right]$$

$$= \frac{1}{2}\left[(\pi + \pi^2) + ((-\pi) + (-\pi)^2)\right]$$

$$= \pi^2$$

$$f(-\pi) = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] - 0$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots}$$

Ques 8. Find the Fourier Series expansion for $f(x) = x + \frac{x^2}{4}$; $-\pi \le x \le \pi$.

Soln: Let: $x + \frac{x^2}{4} = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots +$
$+ b_1 \sin x + b_2 \sin 2x + \cdots$

where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(x + \frac{x^2}{4}\right) dx$$

$$\Rightarrow a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(x + \frac{x^2}{4}\right) \cos nx\, dx$$

$$\Rightarrow a_n = \frac{(-1)^n}{n^2}$$

$a_1 = -1$; $a_2 = \frac{1}{4}$; $a_3 = -\frac{1}{9}$; ……

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(x + \frac{x^2}{4}\right) \sin nx\, dx$$

$$\Rightarrow b_n = \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx\, dx}_{\text{Even function}} + \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x^2}{4} \sin nx\, dx}_{\text{odd function}}$$

$$\Rightarrow b_n = \frac{2}{\pi} \int_{0}^{\pi} x \sin nx\, dx + 0$$

$$\Rightarrow b_n = -\frac{2(-1)^n}{n}$$

$b_1 = \frac{2}{1}$; $b_2 = -1$; $b_3 = \frac{2}{3}$; ……

Hence, Fourier Series is —

$$\boxed{f(x) = \frac{\pi^2}{12} - \cos x + \frac{1}{4} \cos 2x - \frac{1}{9} \cos 3x + \cdots + 2 \sin x - \sin 3x + \frac{2}{3} \sin 4x + \cdots}$$

Que: 13. Expand the function $f(x) = x\sin x$, as a Fourier Series in the interval $-\pi \le x \le \pi$. Hence deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots = \frac{\pi - 2}{4}$.

Solⁿ

$$f(x) = x.\sin x$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$\Rightarrow a_0 = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x.\cos nx\,dx$$

$$\Rightarrow a_n = \frac{2(-1)^{n+1}}{n^2-1}.$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$\Rightarrow a_1 = -\frac{1}{2}$$

$\left\{ \because x\sin x.\sin nx \text{ is an odd function.} \right.$

$$b_n = 0$$

Hence, $$f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}.\cos nx$$

$$= 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{(n+1)(n-1)}\cos nx.$$

$$\Rightarrow x\sin x = 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{1.3}\cos 2x + \frac{1}{2.4}\cos 3x + \cdots\right]$$

$\underline{x = \frac{\pi}{2}} \Rightarrow$

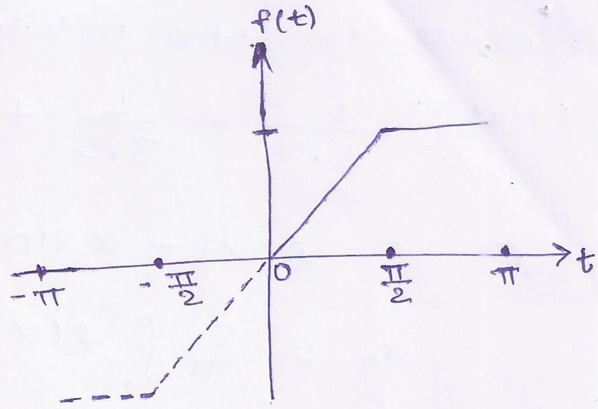
$$\Rightarrow \frac{\pi}{2} = 1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots\right]$$

$$\Rightarrow \frac{\pi}{4} - \frac{1}{2} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots$$

$$\Rightarrow \boxed{\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots}$$

Que:14 Represent the following function by a Fourier sine series :

$$f(t) = \begin{cases} t & ; 0 < t \le \frac{\pi}{2} \\ \frac{\pi}{2} & ; \frac{\pi}{2} < t \le \pi \end{cases}$$



Sol^n

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t) \sin nt \, dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2} t \sin nt \, dt$$

$$+ \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{\pi}{2} \sin nt \, dt$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2} \cdot \frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right]$$

$$\Rightarrow b_1 = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[-\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right]$$

$$\Rightarrow b_2 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$\Rightarrow b_3 = -\frac{2}{9\pi} + \frac{1}{3}$$

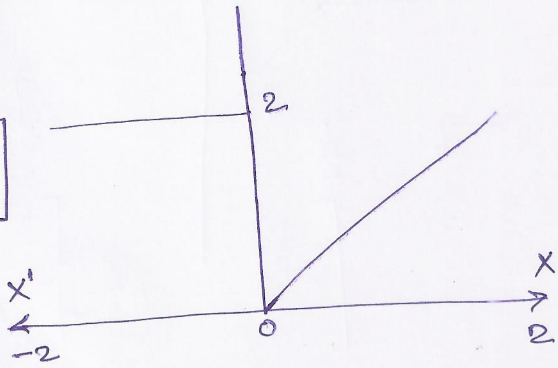
$$\boxed{f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \cdots}$$

Que:15. Find the Fourier Series corresponding to the function $f(x)$ defined in $(-2, 2)$ as follows—

$$f(x) = \begin{cases} 2 & \text{in } -2 \le x \le 0 \\ x & \text{in } 0 < x < 2 \end{cases}$$

Sol^n: Here the interval is $(-2, 2)$ and $c = 2$.

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx$$



$$\Rightarrow a_0 = \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right]$$

$$\Rightarrow a_0 = 3$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

$$\Rightarrow a_n = \begin{cases} -\frac{4}{n^2\pi^2} & ;\text{ when } n \text{ is odd} \\ 0 & ;\text{ when } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$\Rightarrow b_n = -\frac{2}{n\pi}$$

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + a_3\cos\frac{3\pi x}{c} + \cdots$$
$$\cdots + b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + b_3\sin\frac{3\pi x}{c} + \cdots$$

$$\Rightarrow \boxed{f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right\} - \frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right\}.}$$