Fourier Integral &

Fourier Transform

Fourier Transform

Fourier transform, Fourier cosine transform, Fourier sine transforms are Integral transforms.
Recall Integral transform.

Fourier Transform: Here $K(s,t) = e^{-ist}$, $-\infty < t < \infty$, $s \in (-\infty, \infty)$

We denote it by

$$\mathcal{F}\{f(x)\} = \hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} f(x)\,dx \quad \text{-----} (A1)$$

Fourier Cosine Transform:

If $f(x)$ is an even function, then

$$\mathcal{F}_c(f) = \hat{f}_c(w) := \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos wx\, f(x)\,dx \quad \text{-----} (A_2)$$

Fourier Sine Transform:

If $f(x)$ is an odd function, then

$$\mathcal{F}_s(f) = \hat{f}_s(w) := \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin wx\, dx \quad \text{-----} (A_3)$$

Many linear boundary value and initial value problems in applied mathematics, mathematical physics and engineering science can be effectively solved by the use of Fourier transform, Fourier Cosine transforms or Fourier sine transforms. These transforms are very useful for solving differential or integral equations. For first, these transforms replace differential or integral equations by simple algebraic equations.

Second, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary and initial value problems.

Example :

(1) Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Answer :

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \left.\frac{e^{-iwx}}{-iw}\right|_{-1}^{1}$$

$$= \frac{+\,i}{\sqrt{2\pi}\, w} \left(e^{-iw} - e^{iw}\right)$$

$$= \frac{i}{\sqrt{2\pi}\, w} \left(-2i \sin w\right) = \sqrt{\frac{2}{\pi}}\, \frac{\sin w}{w}$$

(2) Find the Fourier transform of $e^{-a|x|}$, $a > 0$

Answer :

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} e^{-(a+iw)x}\, dx + \int_{-\infty}^{0} e^{(a-iw)x}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\lim_{M_1 \to \infty} \int_{0}^{M_1} e^{-(a+iw)x}\, dx + \lim_{M_2 \to \infty} \int_{-M_2}^{0} e^{(a-iw)x}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\lim_{M_1 \to \infty} \left[\frac{e^{-(a+iw)x}}{-(a+iw)} \right]_0^{M_1} + \lim_{M_2 \to \infty} \left[\frac{e^{(a-iw)x}}{a-iw} \right]_{-M_2}^{0} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+iw} + \frac{1}{a-iw} \right] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+w^2} \text{ as}$$

$$\lim_{M_1 \to \infty} e^{-(a+iw)M_1} = \lim_{M_1 \to \infty} e^{-aM_1} \left(\cos wM_1 + i \sin wM_1 \right)$$

$$= 0$$

(3) Find Fourier cosine transform of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } 1 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$$

Answer : We get Fourier cosine transform for even functions, i.e. here $f(-x) = f(x) \; \forall x > 0$

$$\therefore \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^1 \cos wx \, dx + \int_1^2 - \cos wx \, dx \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w} \left(\left[\sin wx \right]_0^1 - \left[\sin wx \right]_1^2 \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w} \left(-\sin 2w + 2 \sin w \right)$$

Example:- (4) Show that

$$\mathcal{F}_s\{\operatorname{erfc}(ax)\} = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{w}\left[1 - e^{-\frac{w^2}{4a^2}}\right]$$

Answer:-

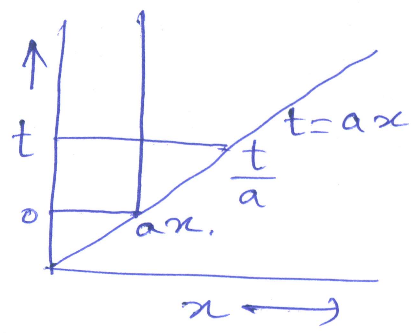
$$\mathcal{F}_s\{\operatorname{erfc}(ax)\}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \operatorname{erfc}(ax)\sin wx\,dx.$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^\infty \sin wx \int_{ax}^\infty e^{-t^2}\,dt$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^\infty e^{-t^2}\,dt\int_0^{t/a}\sin wx\,dx$$ (by interchanging the order of integration

as



$$= \frac{2\sqrt{2}}{\pi w}\int_0^\infty e^{-t^2}\left\{1 - \cos\left(\frac{wt}{a}\right)\right\}dt$$

$$= \frac{2\sqrt{2}}{\pi w}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}e^{-\frac{w^2}{4a^2}}\right]$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{w}\left[1 - e^{-\frac{w^2}{4a^2}}\right]$$

~~(as $\frac{d}{dw}\int e^{-t^2}\cos$~~

(as $\frac{d}{dw}\left(\int_0^\infty e^{-t^2}\cos\frac{wt}{a}\,dt\right)$

$= -\int_0^\infty e^{-t^2}\frac{t}{a}\sin\frac{wt}{a}\,dt$ and

then integration by parts)

Existence of Fourier, Fourier Cosine, Fourier Sine transforms :

Absolutely Integrable Functions :

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Then if the following integral exists,

$$\lim_{a \to -\infty} \int_a^0 |f(x)|\, dx + \lim_{b \to \infty} \int_0^b |f(x)|\, dx$$

We write this as $\int_{-\infty}^{\infty} |f(x)|\, dx$ and say f is absolutely integrable on the real axis.

Theorem 1 (Existence of the Fourier transform) :

If $f(x)$ is absolutely integrable and piecewise continuous on every finite interval, then the Fourier transform, Fourier Cosine and Fourier Sine transform exist.

Relation with Fourier Series :

If a function f is given in an interval $[-a, a]$, piecewise continuous on $[-a, a]$, $f(x)$ has a left hand and right hand derivative at each point of the interval, then we can extend f for each real number by making the new extended function periodic of period $2a$ and we get the Fourier series

$$f(x) = a_0 + \sum \left(a_n \cos \frac{n}{a} x + b_n \sin n\frac{\pi}{a} x\right),$$

where a_0, a_n, b_n are given by Euler's formula.

But, here given function f is defined for $[-a, a]$. If $f(x)$ is defined for each real number $-\infty < x < \infty$, then what can we expect?

First, write the Fourier series in complex form -

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{a}}, \text{ where}$$

$$c_n = \frac{1}{2a} \int_{-a}^{a} f(v) e^{-\frac{in\pi v}{a}} dv$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2a} \left[\int_{-a}^{a} f(v) e^{-\frac{in\pi v}{a}} dv \right] \cdot e^{\frac{in\pi x}{a}}.$$

As f is defined on $(-\infty, \infty)$, so we take $a \to \infty$

Let, $w_n = \frac{n\pi}{a}$. Let, $\Delta w := w_{n+1} - w_n = \frac{(n+1)\pi}{a} - \frac{n\pi}{a}$

$$= \frac{\pi}{a}$$

$$\therefore f(x) = \sum_{n=\infty}^{\infty} \frac{\Delta w}{2\pi} \left[\int_{-a}^{a} f(v) e^{-iv w_n} dv \right] e^{ix w_n}$$

As $a \to \infty$, we have $w_n = \frac{n\pi}{a}$ become closer together and form a dense set in $\{\frac{\pi}{a} x \mid x \in \mathbb{R}\}$. We can expect w_n becomes a continuous variable and Δw becomes dw. So, we expect to get an integral instead of a series. So, we can expect to get,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) e^{-iwv} dv \right] e^{iwx} dw \quad \text{---- (B)}$$

This is known as celebrated <u>Fourier Integral Formula</u>.

Though the above arangements do not constitute a rigorous proof, the formula is correct and valid for functions with the following properties. ~~Though the above properties~~

<u>Theorem 2</u> : (Fourier Integral)

If $f(x)$ is piecewise continuous (the interval can be divided into finitely many subintervals in each of which f is defined and continuous and has finite limits as x approaches either ~~and~~ end points of such a subinterval from the interior) in every finite interval and has a left and right hand derivatives at every point and if f is absolutely integrable on the real axis, then $f(x)$ can be represented by a Fourier integral.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) e^{-iwv} dv \right] e^{iwx} dw \quad \text{---- (B)}$$

At a point where $f(x)$ is discontinuous, the value of the Fourier integral equals the average of the left hand

and right hand limits of $f(x)$ at that point.

Other Form of Fourier Integral Formula :

$$f(x) = \int_0^\infty \left(\left[\frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv \right] \cos wx + \left[\frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv \right] \sin wx \right) dw$$

$$= \int_0^\infty (A(w) \cos wx + B(w) \sin wx) \, dw, \quad \text{----- (C)}$$

where $A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv,$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$$

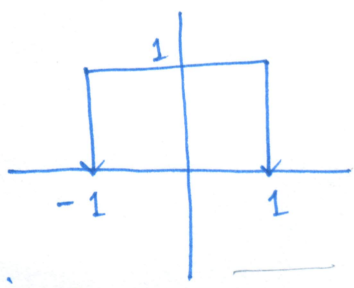
Note :

$\sqrt{\frac{\pi}{2}}\, A(w)$ and $\sqrt{\frac{\pi}{2}}\, B(w)$ are Fourier cosine and sine transforms respectively.

Example :

(1) Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$



Here, $A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$

$$= \frac{1}{\pi} \int_{-1}^{1} \cos wv \, dv = \frac{2 \sin w}{w \pi}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv = \frac{1}{\pi}\int_{-1}^{1}\sin wv\,dv$$

$$= \frac{-1}{\pi w}\Big[\cos wv\Big]_{-1}^{1}$$

$$= 0$$

$\therefore$ Fourier integral representation is

$$f(x) = \int_0^{\infty} \frac{2\sin w}{\pi w}\cdot \cos wx\,dw$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\cos wx\sin w}{w}\,dw$$

$$\therefore \int_0^{\infty}\frac{\cos wx\sin w}{w}\,dw = \begin{cases}\pi/2 & \text{if } 0 \le x < 1\\ \pi/4 & \text{if } x = 1\\ 0 & \text{if } x > 1\end{cases}$$

This integral is called Dirichlet's discontinuous factor.

At $x = 0$,

$$\therefore \int_0^{\infty}\frac{\sin w}{w}\,dw = \pi/2$$

Here $\displaystyle\int_0^{\infty}\frac{\sin w}{w}\,dw = \lim_{a\to\infty}\int_0^{a}\frac{\sin w}{w}\,dw$

(2) Show that $\displaystyle\int_0^{\infty}\frac{\cos xw + w\sin xw}{1+w^2}\,dw = \begin{cases}0 & \text{if } x<0\\ \pi/2 & \text{if } x=0\\ \pi e^{-x} & \text{if } x>0\end{cases}$

Solution :

Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$

Then, $A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$

$$= \frac{1}{\pi} \int_{0}^{\infty} \pi e^{-v} \cos wv \, dv$$

$$= \int_{0}^{\infty} e^{-v} \cos wv \, dv$$

$$= \frac{1}{1+w^2} \left[e^{-v} (w \cdot \sin wv - \cos wv) \right]_0^{\infty}$$

$$= \frac{1}{1+w^2}$$

Similarly,

$$B(w) = \frac{w}{1+w^2}$$

$$\therefore f(x) = \int_0^{\infty} \frac{\cos wx + w \sin wx}{1+w^2} \, dw$$

Fourier Cosine Integral, Fourier Sine Integral :

We have seen earlier that for an even periodic function $f(x)$, $\infty < x < \infty$ $(f(-x) = f(x))$ of period $2L$ the general Fourier series

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

reduces to $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$, where

$a_0 = \frac{1}{L} \int_0^L f(x)\, dx$, $a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx.$

~~Simult~~ Similarly, for an even function $f(x)$ $(-\infty < x < \infty)$, we have $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \sin vw \, dv = 0$ and

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos vw \, dw = \frac{2}{\pi}\int_0^{\infty} f(v) \cos vw \, dv$$

$\therefore$ $f(x) = \int_0^{\infty} (A(w) \cos wx + B(w) \sin wx) \, dw$

reduces to $f(x) = \int_0^{\infty} A(w) \cos wx \, dw$, where

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(v) \cos wv \, dv \quad \text{------} \quad (D1)$$

This is called <u>Fourier cosine integral</u>.

Similarly, if the function $f(x)$ $(-\infty < x < \infty)$ is an odd function, then $A(w) = 0$ and (C) reduces to $f(x) = \int_0^{\infty} B(w) \sin wx \, dw$, where $B(w) = \frac{2}{\pi}\int_0^{\infty} f(v) \sin wv \, dv$ ------- (D 2)

(D2) is called <u>Fourier Sine integral</u>.

Beside differential equations, these representations also help in evaluating integrals.

<u>Example:</u>

<u>(1) Laplace Integrals:</u>

To derive Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$; $x > 0$, $k > 0$

Answer: Fourier Cosine integral:

From (D1),

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv$$

$$= -\frac{2}{\pi}\frac{k}{k^2+w^2}\left[e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right)\right]_0^\infty$$

$$= \frac{2}{\pi}\frac{k}{k^2+w^2} \text{ as } e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right) \to 0 \text{ as } v\to\infty$$

$$\therefore e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw \quad (x>0, k>0)$$

$$\therefore \int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw = \frac{\pi}{2k}e^{-kx} \quad \text{-------} (i)$$

Here we extend e^{-kx} by $e^{-k(-x)}$, $x<0$

From (D2), we get,
by extending e^{-kx} for negative numbers to f by

$$f(x) = -e^{-k(-x)}, \; x<0$$

$$\int_0^\infty \frac{\sin wx}{k^2+w^2}\, dw = \frac{\pi}{2}e^{-kx} \quad \text{--------} (ii)$$

$$(x>0, \; k>0)$$

(i) and (ii) are called Laplace integrals.

(2) Represent $f(x)$ as Fourier Cosine integral,
where $f(x) = \dfrac{1}{1+x^2}$, $x>0$

Answer:

Again extend $f(x)$ on negative numbers by

$$f(x) = \frac{1}{1+(-x)^2}, \quad x < 0$$

Then,
$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\cos wv}{1+v^2}\, dv$$
$$= \frac{2}{\pi}\cdot\frac{\pi}{2}\cdot e^{-w} \text{ (by (i) in the previous page)}$$
$$= e^{-w}$$

$$\therefore\ f(x) = \int_0^\infty A(w)\cos wx\, dw = \int_0^\infty e^{-w}\cos xw\, dw$$

(3) Find the Fourier sine integral of

$$f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Answer:

Extend f to an odd function by defining

$$f(x) = -f(-x) \text{ for } x < 0$$
$$= \begin{cases} -e^{+x} & \text{if } -1 < x < 0 \\ 0 & \text{if } x < -1 \end{cases}$$

$$\therefore\ B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\, dv$$
$$= \frac{2}{\pi}\int_0^1 e^{-v}\sin wv\, dv$$
$$= \frac{-1}{1+w^2}\left[e^{-v}(\sin wv + w\cos wv)\right]_0^1 \cdot \frac{2}{\pi}$$

$$= \frac{w}{1+w^2} \cdot \frac{2}{\pi} \left[w - e^{-1} (\sin w + w \cos w) \right]$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{w}{1+w^2} \left[w - \frac{\sin w + w \cos w}{e} \right] \sin xw \, dw$$

Properties of Fourier, Fourier cosine & Fourier sine Transforms:

(1) Linearity:

Fourier, Fourier cosine and Fourier sine transforms are linear operations, that is, for any functions f and g whose Fourier (or Fourier cosine or Fourier sine) transforms exist, then for any constants a and b, the Fourier (or Fourier cosine or Fourier sine) transforms exist and

(a) $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$ ---- (E1)

(b) $\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$, ---- (E2)

(c) $\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$ ----- (E3)

(2) Transforms of Derivatives:

(a) Let $f(x)$ be continuous on the x-axis and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}(f'(x)) = iw\,\mathcal{F}(f(x)) \quad \text{------} \quad (F_1)$$

(b) Let $f(x)$ be continuous and absolutely integrable on the real axis, let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Then,

(i) $\mathcal{F}_c(f'(x)) = w\,\mathcal{F}_s(f(x)) - \sqrt{\frac{2}{\pi}}\, f(0)$ ----- $[F_2]$

(ii) $\mathcal{F}_s(f'(x)) = -w\,\mathcal{F}_c(f(x))$ ----- $[F_3]$

<u>Proof:</u>

(a) $$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[f(x) e^{-iwx} \right]_{-\infty}^{\infty} + iw \int_{-\infty}^{\infty} f(x) \cdot e^{-iwx}\, dx \right)$$

$$= 0 + iw\,\mathcal{F}(f(x)) \text{ as } f(x) \longrightarrow 0 \text{ as } |x| \to \infty$$

(b) (i) $$\mathcal{F}_c(f'(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\left[f(x) \cos wx \right]_0^{\infty} + w \int_0^{\infty} f(x) \sin wx\, dx \right)$$

$$= -\sqrt{\frac{2}{\pi}}\, f(0) + w\,\mathcal{F}_s(f(x))$$

Similarly, (ii) $\mathcal{F}_s(f'(x)) = -w\,\mathcal{F}_c(f(x))$

We also get,

$$\mathcal{F}_c(f''(x)) = w\,\mathcal{F}_s(f'(x)) - \sqrt{\frac{2}{\pi}}\, f'(0)$$

[When f' and f'' satisfy the respective assumptions for f, f']

or, $\mathcal{F}_c\left\{f''(x)\right\} = -w^2 \mathcal{F}_c\left\{f(x)\right\} - \sqrt{\frac{2}{\pi}} f'(0)$ ------ (G1)

and similarly,

$\mathcal{F}_s\left\{f''(x)\right\} = -w^2 \mathcal{F}_s\left\{f(x)\right\} + \sqrt{\frac{2}{\pi}}\, w\, f(0).$ ------- (G2)

and

$\mathcal{F}\left\{f^{(n)}(x)\right\} = (iw)^n \mathcal{F}\left\{f(x)\right\}$ ------ (G3)

Example: Find the Fourier cosine transform $\mathcal{F}_c\left(e^{-ax}\right)$ of $f(x) = e^{-ax}$, where $a > 0$

Answer:

$$\left(e^{-ax}\right)'' = a^2 e^{-ax}$$

$$\mathcal{F}_c\left\{\left(e^{-ax}\right)''\right\} = -w^2 \mathcal{F}_c\left\{e^{-ax}\right\} - \sqrt{\frac{2}{\pi}}\left(e^{-ax}\right)'_{\text{at } x=0}$$

$$\Rightarrow a^2 \mathcal{F}_c\left\{e^{-ax}\right\} = -w^2 \mathcal{F}_c\left\{e^{-ax}\right\} + \sqrt{\frac{2}{\pi}}\, a$$

$$\Rightarrow \mathcal{F}_c\left\{e^{-ax}\right\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+w^2}, \quad (a > 0)$$

(3) Convolution :

The convolution $f * g$ of functions f and g is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp$$

$$= \int_{-\infty}^{\infty} f(x-p)\, g(p)\, dp$$

(a) Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded and absolutely integrable on the real axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\; \mathcal{F}(f) \cdot \mathcal{F}(g) \text{ ---- } (H_1)$$

(b) If $\mathcal{F}_c(f(x)) = F_c(w)$ and $\mathcal{F}_c(g(x)) = G_c(w)$, then

$$\int_0^{\infty} F_c(w)\, G_c(w) \cdot \cos wx\, dw = \frac{1}{2}\int_0^{\infty} f(v)\left[g(x+v) + g(|x-v|)\right] dv \text{ ---- } (H_2)$$

(c) If $\mathcal{F}_s(f(x)] = F_s(w)$ and $\mathcal{F}_s(g(x)) = G_s(w)$, then

$$\int_0^{\infty} F_s(w)\, G_s(w) \cdot \cos wx\, dw = \frac{1}{2}\int_0^{\infty} f(v)\left[g(x+v) + g(v-x)\right] dv \text{ ---- } (H_3)$$

Proof of (H_1):

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp\, e^{-iwx}\, dx$$

Interchanging the order of integration, we get

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, e^{-iwx}\, dx\, dp$$

put $x-p=q$, then $x=p+q$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(q) \cdot e^{-iw(p+q)}\, dq\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{-iwp}\, dp \int_{-\infty}^{\infty} g(q)\, e^{-iwq}\, dq$$

$$= \frac{1}{\sqrt{2\pi}} \mathcal{F}(f) \cdot \mathcal{F}(g).$$

The convolution has the following algebraic properties–

a) $f*g = g*f$ (Commutative)

b) $f*(g*h) = (f*g)*h$ (Associative)

c) $(\alpha f + \beta g)*h = \alpha(f*h) + \beta(g*h)$ (Distributive)

Example:– If $a>0$ and $b>0$, show that

$$\int_0^\infty \frac{x^2\,dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}.$$

Proof:– We have computed the Fourier cosine transform of e^{-ax}. Similarly, we can compute the Fourier sine transform.

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\,\frac{w}{a^2+w^2} =: F_s(w)$$

$$\mathcal{F}_s\{e^{-bx}\} = \sqrt{\frac{2}{\pi}}\cdot\frac{w}{b^2+w^2} =: G_s(w)$$

In $(\#_3)$ by putting $x=0$, we get

$$\int_0^\infty F_s(w)\,G_s(w)\,dw = \tfrac{1}{2}\cdot 2\int_0^\infty g(v)\,f(v)\,dv$$

$$\Rightarrow \int_0^\infty \frac{w^2\,dw}{(a^2+w^2)(b^2+w^2)} = \frac{\pi}{2}\int_0^\infty e^{-(a+b)v}\,dv = \frac{\pi}{2(a+b)}.$$

$$\Rightarrow \int_0^\infty \frac{x^2\,dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}.$$

(4) If $f(x)$ is piecewise continuously differentiable and absolutely integrable, then

(i) $F(w) = \mathcal{F}\{f(x)\}$ is bounded for $-\infty < w < \infty$

(ii) $F(w)$ is continuous for $-\infty < w < \infty$

(5) Riemann - ~~Leb~~ Lebesgue Lemma :

If $F(w) = \mathcal{F}\{f(x)\}$; then

$$\lim_{|w| \to \infty} |F(w)| = 0$$

(6) General Parseval's identity :

If $\mathcal{F}\{f(x)\} = F(w)$ and $\mathcal{F}\{g(x)\} = G(w)$,

then

$$\int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx = \int_{-\infty}^{\infty} F(w)\, \overline{G(w)}\, dw \quad \text{----} \; (I_1)$$

If $\mathcal{F}_c\{f(x)\} = F_c(w)$ and $\mathcal{F}_c\{g(x)\} = G_c(w)$,

then

$$\int_{0}^{\infty} F_c(w)\, G_c(w)\, dw = \int_{0}^{\infty} f(x)\, g(x)\, dx \quad \text{----} \; (I_2)$$

Substituting $g(x) = \overline{f(x)}$,

$$\int_0^\infty |F_c(w)|^2 \, dw = \int_0^\infty |f(x)|^2 \, dx \quad \text{----} (J_1)$$

This is Parseval's relation for the Fourier cosine transform.

If $\mathcal{F}_s\{f(x)\} = F_s(w)$ and $\mathcal{F}_s\{g(x)\} = G_s(w)$,

then

$$\int_0^\infty F_s(w) \, G_s(w) \, dw = \int_0^\infty f(x) \, g(x) \, dx$$

(follows from (H_3) by putting $x = 0$) ---- (I_3)

Replacing $g(x)$ by $\overline{f(x)}$, we get

$$\int_0^\infty |F_s(w)|^2 \, dw = \int_0^\infty |f(x)|^2 \, dx \text{ as } \cdots\cdots (J_2)$$

$$G_s(w) = \overline{F_s(w)}$$

This is Parseval's relation for the Fourier sine transform.

Example :

Evaluate $\int_0^{\infty} \frac{x^2\, dx}{(x^2+1)^2}$ by use of Parseval's identity.

Answer :

First we find the Fourier sine transform of e^{-x}, $x > 0$

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin wx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \frac{1}{1+w^2} \left[e^{-v} (\sin wv + w \cos wv) \right]_0^{\infty}$$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{w}{1+w^2}$$

$$\therefore F_s(w) = -\sqrt{\frac{2}{\pi}} \cdot \frac{w}{1+w^2}$$

$\therefore$ By Parseval's identity,

$$\int_0^{\infty} |F_s(w)|^2 \, dw = \int_0^{\infty} |f(x)|^2 \, dx$$

$$\therefore \frac{2}{\pi} \int_0^{\infty} \frac{w^2}{(1+w^2)^2} \, dw = \int_0^{\infty} e^{-2x} \, dx = -\frac{1}{2} \left[e^{-2x} \right]_0^{\infty}$$

$$= -\frac{1}{2} \left[e^{-2x} \right]_0^{\infty} = \frac{1}{2} \cdot 1$$

$$\therefore \int_0^\infty \frac{x^2}{(1+x^2)^2}\, dx = \frac{\pi}{4}$$

7. Scaling :

(a) $\mathcal{F}\{f(ax)\} = \frac{1}{|a|} F\left(\frac{w}{a}\right)$, where $F(w) = \mathcal{F}\{f(x)\}$,

(b) $\mathcal{F}_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{w}{a}\right)$, $a > 0$

(c) $\mathcal{F}_s\{f(ax)\} = \frac{1}{a} \cdot F_s\left(\frac{w}{a}\right)$, $a > 0$

8. Conjugation :

(a) $\mathcal{F}\{\overline{f(-x)}\} = \overline{\mathcal{F}\{f(x)\}}$

(b) $\mathcal{F}_c\{\overline{f(x)}\} = \overline{\mathcal{F}_c\{f(x)\}}$

(c) $\mathcal{F}_s\{\overline{f(x)}\} = \overline{\mathcal{F}_s\{f(x)\}}$