$\tau = \mu \left(\frac{\partial u}{\partial y}\right)$ $\qquad \frac{m/s}{m}$ $\qquad \frac{1}{s}$

$Pa = \mu \left[\frac{1}{s}\right]$

$\mu = Pa \cdot s$.

Dimensions & Units

Pressure	Pa	N/m²
Temp.	K	°C
Density	kg/m³	
Viscosity	Pa·s	

SI

Système Internationale

- mass — kg — pound lb
- length — m — in ft

- Gimli Glider
- Mars orbiter

Dimension? — $\boxed{\text{Basis}}$ of the quantity

length :	$M^0 L^1 T^0 \theta^0$
mass :	$M^1 L^0 T^0 \theta^0$
velocity :	$M^0 L^1 T^{-1} \theta^0$
density :	$M^1 L^{-3} T^0 \theta^0$
Force :	$M^1 L^1 T^{-2} \theta^0$
Pressure :	$M^1 L^{-1} T^{-2} \theta^0$
Area :	$M^0 L^2 T^0 \theta^0$
Viscosity :	$M^1 L^{-1} T^{-1} \theta^0$ (Pa·s)

M : mass
L : length
T : time
θ : Temperature

$\boxed{\text{Dimensions}}$

- Use?
  1. Dimensions independent of units!
  2. Homogeneity. All terms in an expressions must have same dimensions

$$p + \frac{1}{2}\rho U^2 + \boxed{\rho g z} = \text{const.}$$

$\boxed{M^1 L^{-1} T^{-2}}$

ρ: $M^1 L^{-3} T^0$ $\qquad (M^0 L^1 T^{-1})^2$

U^2: $M^0 L^2 T^{-2}$

$\boxed{M L^{-1} T^{-2}}$

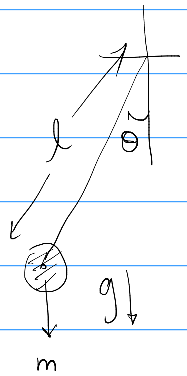
ρ: $M^1 L^{-3} T^0$

g: $M^0 L^1 T^{-2}$

z: $M^0 L^1 T^0$

$M^1 L^{-1} T^{-2}$

Dimensional Analysis



Parameters	Dimensions
length of string	$M^0L^1T^0$
mass of bob	$M^1L^0T^0$
Gravity	$M^0L^1T^{-2}$
Time of oscillation	$M^0L^0T^1$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$ mx

- All terms must have same dimension

$M^0L^0T^1$

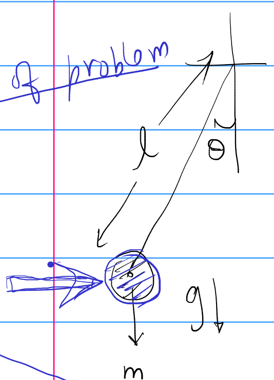
$$\sqrt{\frac{M^0L^1T^0}{M^0L^1T^{-2}}} = \sqrt{M^0L^0T^2} = M^0L^0T^1$$

Nature $\not\rightarrow$ our rules — mathematics, units, physics } understanding

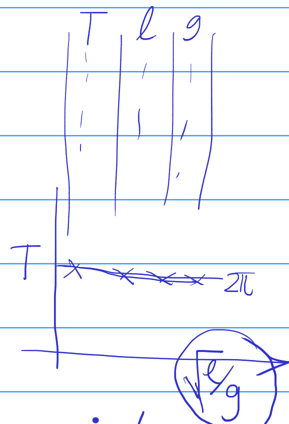
devoid of units

dimensionless

Physics of problem



Parameters	Dimensions
length of string	$M^0L^1T^0$
mass of bob	$M^1L^0T^0$
Gravity	$M^0L^1T^{-2}$
Time of oscillation	$M^0L^0T^1$

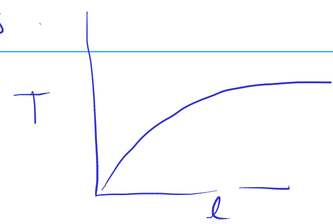


→ construct multiple terms, no dimensions

$$\left(\frac{T}{\sqrt{l/g}}\right) = M^0L^0T^0 = \text{const.}$$

Q. Can I construct a dimensionless term containing mass

$$\boxed{m^a l^b g^c T^d}$$



Formalize

n terms   5 terms

Basis   L   velocity   mass - - pressure , viscosity

$$\begin{pmatrix} M \\ L \\ T \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \checkmark \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

How many dimensionless terms can we make?

$$\begin{array}{ccc} & l_1 & l_2 \\ M: & 0 & 0 \\ L: & 1 & 1 \\ T: & 0 & 0 \end{array} \Bigg\} \quad \text{only } 1 \quad \frac{l_1}{l_2}, \frac{l_2}{l_1}$$

$\underline{n=3, \; b=1}$ $n-b = 3-1 = 2$

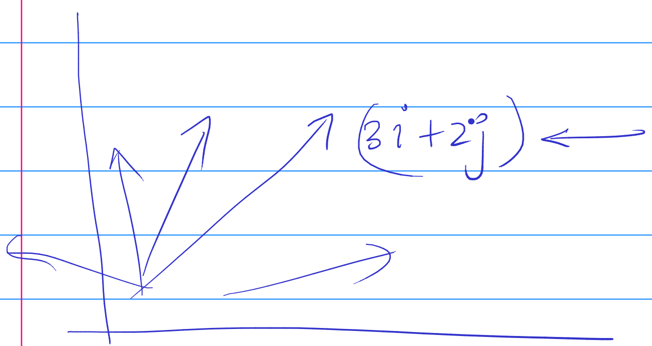
$$\begin{array}{ccc} \rightarrow l_1 & l_2 & l_3 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & 0 & 0 \\ & 1 & 1 \\ & 0 & 0 \end{array} \Bigg\} \quad 2 \quad \frac{l_2}{l_1} \quad \frac{l_3}{l_1}$$

l basis

$$\begin{array}{cccc} & l & \text{Area} & \text{volume} \\ M & 0 & 0 & 0 \\ L & 1 & 2 & 3 \\ T & 0 & 0 & 0 \end{array} \Bigg\} \quad 2 \quad \overbrace{\frac{\text{Area}}{l^2}, \quad \frac{Vol}{l^3}}^{l \text{ basis}}$$

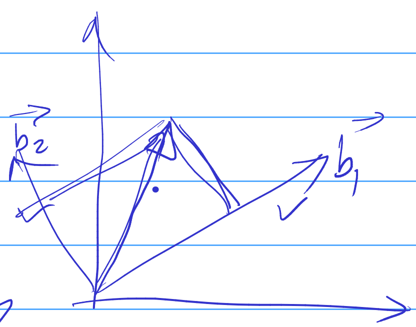
Area × l $\frac{Vol}{A^3 \times l}$

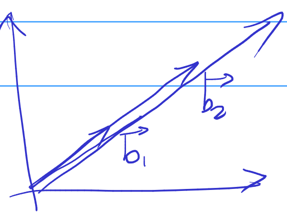
2 $\left(\frac{l}{\sqrt{A}}\right), \left(\frac{Vol}{A^{3/2}}\right)$



$(3\hat{i} + 2\hat{j}) \longleftarrow (\hat{i}, \hat{j})$

$(3\hat{i} + 2\hat{j})$





Rank of a matrix

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

non-zero rows
non-zero columns
size of smallest submatrix whose is non zero determinant

rank of this matrix

Basis vectors

I should be able to construct

$\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ as Lco of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left(-\frac{1}{2}\right)\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

n terms 5 terms

	L	velocity	mass	pressure	viscosity
$\begin{pmatrix} M \\ L \\ T \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

$$\begin{array}{c|ccc|c|c} & L & V & m & p & \mu \\ \hline M & 0 & 0 & 1 & 1 & 1 \\ L & 1 & 1 & 0 & -1 & -1 \\ T & 0 & -1 & 0 & -2 & -1 \end{array}$$

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = 1(-1) \cancel{=} -1$$

(l, v, m)

$$p = L^a V^b m^c$$

$$\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$p = L^{-3} V^2 m^1$$

$a = -3$
$b = 2$
$c = 1$

$$p = \frac{mv^2}{l^3} = \rho v^2$$

dimensionless term

$$\Pi_1 = p \cdot l^a V^b m^c$$

$$= (M^1 L^{-1} T^{-2})(M^0 L^a T^0)(M^0 L^b T^{-b})(M^c L^0 T^0)$$

$$\boxed{\Pi_1 = M^{1+c} L^{-1+a+b+0} T^{-2+0-b}}$$

Each exponent has to be zero

$1+c=0$
$a+b-1=0$
$-2-b=0$

$b=-2$
$a=3$
$c=-1$

$$\Pi_1 = p\, l^3 V^{-2} m^{-1}$$

$$= \frac{p}{(mV^2/l^3)} = \frac{p}{\rho V^2}$$

$$\Pi_2 = \mu \times (l^a V^b m^c)$$

$$= (M^1 L^{-1} T^{-1})(M^0 L^a T^0)(M^0 L^b T^{-b})(M^c L^0 T^0)$$

$\mu = c \times (\cdot\ \ )$

$\Rightarrow$ $a = +2$
$b = -1$
$c = -1$

$$\Pi_2 = \boxed{\mu\, l^{-2} V m}$$

1. → Find all relevant parameters of the problem
2. → ? Basis vector : Dimension matrix
   → which combination of terms to use to make all other terms dimensionless
3. Rank of the dimension matrix
4. n, r → dimensionless terms : $(n-r)$
5. Find out all $\Pi_1, \Pi_2 \cdots \Pi_{n-r}$

   Buckingham π-theorem

6. Physics $\Rightarrow$ $\Pi_1, \Pi_2 \cdots \Pi_{n-r}$

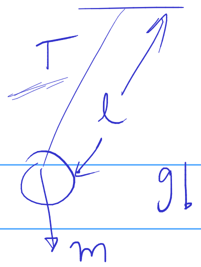
$$\phi(\Pi_1, \Pi_2 \cdots \Pi_{n-r}) = 0$$

Implicit relationship

Explicit

Experiments

①



	T	m	l	g
M	0	1	0	0
L	0	0	1	1
T	1	0	0	−2

- Rank of dimensional matrix? rank = 3
- m, l, g ⟶ rank = 3

$\left.\begin{matrix} n = 4 \\ r = 3 \end{matrix}\right\}$ π_1 $4-3=1$

$$\pi_1 = T\, m^a\, l^b\, g^c$$
$$= (M^0 L^0 T^1)(M^1 L^0 T^0)^a (M^0 L^b T^0)(M^0 L^c T^{-2c})$$

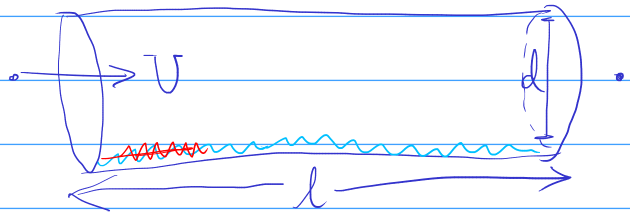
$$\pi_1 = M^{0+a}\, L^{0+b+c}\, T^{1-2c}$$

$$\left.\begin{matrix} 0 = 0 + a \\ 0 = b + c \\ 0 = 1 - 2c \end{matrix}\right\} \quad \begin{matrix} a = 0 \\ b = -c \\ c = 1/2 \end{matrix} \quad \Rightarrow b = -1/2$$

$$\pi_1 = T\, m^0\, l^{-1/2}\, g^{1/2}$$

$$\pi_1 = T\sqrt{\frac{g}{l}} = \underline{const}$$

Pressure drop in a pipe flow



Parameters

- $\boxed{\Delta p}$ pressure loss
- l length of pipe
- U velocity
- d diameter
- μ viscosity
- ρ density
- ε pipe roughness

$$\begin{array}{c|ccccccc} & \Delta p & l & d & \varepsilon & U & \rho & \mu \\ \hline M & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ L & -1 & 1 & 1 & 1 & 1 & -3 & -1 \\ T & -2 & 0 & 0 & 0 & -1 & 0 & -1 \end{array}$$

$$\begin{array}{cccc} & l & U & \rho \end{array}$$

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ 0 & -1 & 0 \end{pmatrix} = -1$$

(l, U, ρ)

number = 7
rank = 3
dimensionless terms = 7 − 3 = 4

$\Pi_1 = d/l$
$\Pi_2 = \varepsilon/l$
$\Pi_3 = \Delta p \, l^a U^b \rho^c \leftarrow$
$\Pi_4 = \mu \, l^a U^b \rho^c \leftarrow$

$\Pi_3 = \Delta p/(\rho U^2)$
$\Pi_4 = \mu/(\rho U l)$

$\Pi_1 = \dfrac{d}{l}$ = pipe aspect ratio

$\Pi_2 = \varepsilon/l = \left(\dfrac{\varepsilon}{d}\right) \cdot \dfrac{d}{l} \rightarrow (RR) \cdot (AR)$ relative roughness

$\Pi_3 = \dfrac{\Delta p}{\rho U^2}$ = dimensionless pressure

$\Pi_4 = \dfrac{\mu}{\rho U l} = \left(\dfrac{\mu}{\rho U d}\right)\left(\dfrac{d}{l}\right) = \dfrac{1}{Re} \cdot (AR)$

Poiseuille

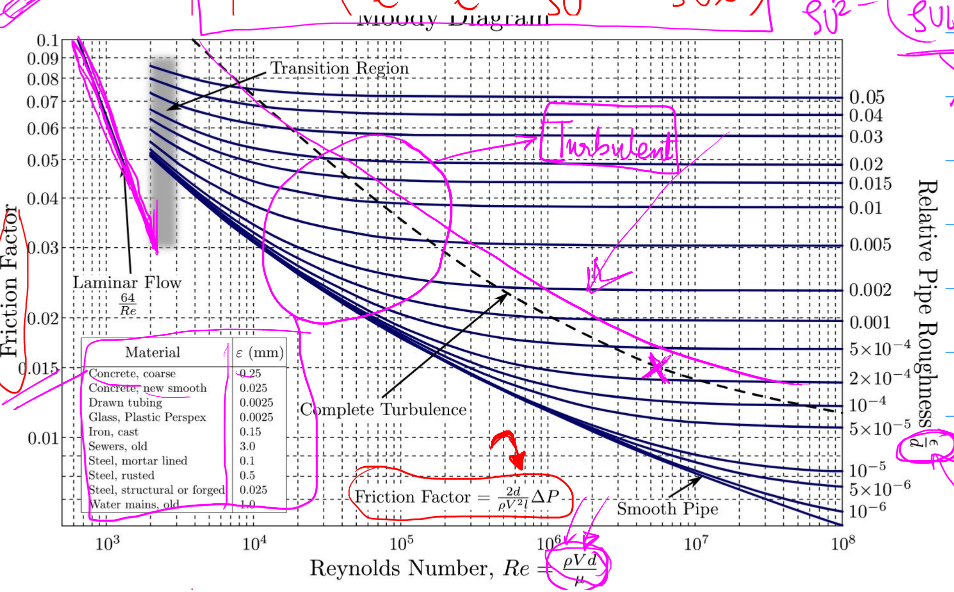
$$\begin{array}{cccc} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \end{array}$$

$$\boxed{\phi = \left(\dfrac{d}{l}, \dfrac{\varepsilon}{l}, \dfrac{\Delta p}{\rho U^2}, \dfrac{\mu}{\rho U l}\right)}$$

$$\dfrac{\Delta p}{\rho U^2} = \left(\dfrac{\mu}{\rho U l}\right)^a \left(\dfrac{\varepsilon}{l}\right)^b \times \left(\dfrac{2 d \Delta p}{\rho U^2 l}\right) = \Pi_3 \times \Pi_1$$

$\dfrac{\varepsilon}{d} = \dfrac{\Pi_2}{\Pi_1}$

$\dfrac{\rho U d}{\mu} = \dfrac{1}{\Pi_4} \times \Pi_1$



Material	ε (mm)
Concrete, coarse	0.25
Concrete, new smooth	0.025
Drawn tubing	0.0025
Glass, Plastic Perspex	0.0025
Iron, cast	0.15
Sewers, old	3.0
Steel, mortar lined	0.1
Steel, rusted	0.5
Steel, structural or forged	0.025
Water mains, old	1.0

DA

1 m   1 mm
1 cm   10 μm