7.3 Linearity of Inverse Laplace Transform

If $F_1(s)$ and $F_2(s)$ are the Laplace transforms of the function $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}[a_1F_1(s) + a_2F_2(s)] = a_1L^{-1}[F_1(s)] + L^{-1}[F_2(s)] = a_1f_1(t) + a_2f_2(t)$$

where a_1 and a_2 are constants.

7.4 Example Problems

7.4.1 Problem 1

Find the inverse Laplace transform of

$$F(s) = \frac{6}{2s-3} + \frac{8-6s}{16s^2+9}$$

Solution: Using linearity of the inverse Laplace transform we have

$$f(t) = 6L^{-1}\left[\frac{1}{2s-3}\right] + 8L^{-1}\left[\frac{1}{16s^2+9}\right] - 6L^{-1}\left[\frac{s}{16s^2+9}\right]$$

Rewriting the above expression as

$$f(t) = 3L^{-1}\left[\frac{1}{s-(3/2)}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2+(9/16)}\right] - \frac{3}{8}L^{-1}\left[\frac{s}{s^2+(9/16)}\right]$$

Using the result

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

and taking the inverse transform we obtain

$$f(t) = 3e^{3t/2} + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}.$$

7.4.2 Problem 2

Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s+1}{s^3+s}$$

Solution: We use the method of partial fractions to write F in a form where we can use the table of Laplace transform. We factor the denominator as $s(s^2+1)$ and write

$$\frac{s^2+s+1}{s^3+s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Putting the right hand side over a common denominator and equating the numerators we get $A(s^2+1)+s(Bs+C) = s^2+s+1$. Expanding and equating coefficients we obtain $A+B=1$, $C=1$, $A=1$, and thus $B=0$. In other words,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

By linearity of the inverse Laplace transform we get

$$L^{-1}\left[\frac{s^2+s+1}{s^3+s}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] = 1+\sin t.$$

7.5 First Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)] = f(t),$ *then* $L^{-1}[F(s-a)] = e^{at}f(t)$

7.6 Example Problems

7.6.1 Problem 1

Evaluate $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

Solution: Rewriting the given expression as

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = L^{-1}\left[\frac{1}{(s-(-1))^2}\right]$$

Applying the first shifting property of the inverse Laplace transform

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

Thus we obtain

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t}.$$

7.6.2 Problem 2

Find $L^{-1}\left[\dfrac{1}{s^2+4s+8}\right]$.

Solution: First we complete the square to make the denominator $(s+2)^2+4$. Next we find

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$L^{-1}\left[\frac{1}{s^2+4s+8}\right] = L^{-1}\left[\frac{1}{(s+2)^2+4}\right] = \frac{1}{2}\,e^{-2t}\sin(2t).$$

7.7 Second Shifting Property of Inverse Laplace Transform

If $L^{-1}[F(s)] = f(t),$ *then* $L^{-1}\left[e^{-as}F(s)\right] = f(t-a)\,H(t-a)$

7.8 Example Problems

7.8.1 Problem 1

Find the inverse Laplace transform of

$$F(s) = \frac{e^{-s}}{s(s^2+1)}$$

Solution: First we compute the inverse Laplace transform

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{s}{(s^2+1)}\right]$$

Using linearity of the inverse transform we get

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{s}{(s^2+1)}\right] = 1-\cos t$$

We now find

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = L^{-1}\left[e^{-s}L[1-\cos t]\right]$$

Using the second shifting theorem we obtain

$$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = \big[1-\cos(t-1)\big]\, H(t-1).$$

7.8.2 Problem 2

Find the inverse Laplace transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+4} + \frac{e^{-2s}}{s^2+4} + \frac{e^{-3s}}{(s+2)^2}$$

Solution: First we find that

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t$$

and using the first shifting property

$$L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

By linearity we have

$$f(t) = L^{-1}\left[\frac{e^{-s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-2s}}{s^2+4}\right] + L^{-1}\left[\frac{e^{-3s}}{(s+2)^2}\right]$$

Putting it all together and using the second shifting theorem we get

$$f(t) = \frac{1}{2}\sin 2(t-1)\, H(t-1) + \frac{1}{2}\sin 2(t-2)\, H(t-2) + e^{-2(t-3)}\,(t-3)\, H(t-3)$$

7.9 Convolution

The convolution of two given functions $f(t)$ and $g(t)$ is written as $f * g$ and is defined by the integral

$$(f*g)(t) := \int_0^t f(\tau)g(t-\tau)\, d\tau. \tag{7.1}$$

As you can see, the convolution of two functions of t is another function of t.

7.10 Example Problems

7.10.1 Problem 1

Find the convolution of $f(t) = e^t$ *and* $g(t) = t$ *for* $t \geq 0$.

Solution: By the definition we have

$$(f * g)(t) = \int_0^t e^{\tau}(t - \tau)\, d\tau$$

Integrating by parts, we obtain

$$(f * g)(t) = e^t - t - 1.$$

7.10.2 Problem 2

Find the convolution of $f(t) = \sin(\omega t)$ *and* $g(t) = \cos(\omega t)$ *for* $t \geq 0$.

Solution: By the definition of convolution we have

$$(f * g)(t) = \int_0^t \sin(\omega\tau)\, \cos\big(\omega(t - \tau)\big)\, d\tau.$$

We apply the identity $\cos(\theta)\sin(\psi) = \frac{1}{2}\big(\sin(\theta + \psi) - \sin(\theta - \psi)\big)$ to get

$$(f * g)(t) = \int_0^t \frac{1}{2}\left(\sin(\omega t) + \sin(2\omega\tau - \omega t)\right)\, d\tau$$

On integration we obtain

$$(f * g)(t) = \left[\frac{1}{2}\,\tau\sin(\omega t) - \frac{1}{4\omega}\,\cos(2\omega\tau - \omega t)\right]_{\tau=0}^{t} = \frac{1}{2}\,t\sin(\omega t).$$

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t.

7.11 Properties of Convolution

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions, then

(i) $f * g = g * f,\quad$ [symmetry]

(ii) $c(f * g) = cf * g = f * cg,\quad$ [c=constant]

(iii) $f * (g * h) = (f * g) * h,\quad$ [associative property]

(iv) $f * (g + h) = f * g + f * h,\quad$ [distributive property]

Proof: We give proof of (i) and all others can be done similarly. By the definition of convolution we have

$$f * g = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$$

Substituting $t - \tau = u \Rightarrow -d\tau = du$ we get

$$f * g = -\int_t^0 f(t-u)g(u)\mathrm{d}u = \int_0^t f(t-u)g(u)\mathrm{d}u = g * f$$

This completes the proof. ■

The most interesting property for us, and the main result of this lesson is the following theorem.

7.12 Convolution Theorem

If f and g are piecewise continuous on $[0, \infty)$ and of exponential order α, then

$$L[(f * g)(t)] = L[f(t)]L[g(t)].$$

Proof: From the definition,

$$L[(f * g)(t)] = \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\mathrm{d}t,\ \ [\text{Re(s)}> \alpha\ ]$$

Changing the order of integration,

$$L[(f * g)(t)] = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)g(t-\tau)\mathrm{d}t\mathrm{d}\tau,$$

We now put $t - \tau = u \Rightarrow dt = du$ and get,

$$\begin{aligned} L[(f * g)(t)] &= \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f(\tau)g(u)\mathrm{d}u\mathrm{d}\tau \\ &= \int_0^\infty e^{-s\tau} f(\tau)\mathrm{d}\tau \int_0^\infty e^{-su} g(u)\mathrm{d}u = L[f(t)]L[g(t)] \end{aligned}$$

This completes the proof. ■