If $f(x)$ is defined on a finite interval $[-l, l]$ and is piecewise continuous then we can construct a Fourier series corresponding to the function f and this series will represent the function on this interval if the function satisfies some additional conditions discussed before. Furthermore, if f is periodic then we may be able to represent the function by its Fourier series on the entire real line. Now suppose the function is not periodic and is defined on the entire real line. Then we do not have any possibility to represent the function by the Fourier series. However, we may still be able to represent the function in terms of sine and cosines using an integral, called Fourier integral, instead of a summation. In this lesson we discuss a representation of a non-periodic function by letting $l \to \infty$ in the Fourier series of a function defined on $[-l, l]$.

10.1 Fourier Integral Representation of a Function

Consider any function $f(x)$ defined on $[-l, l]$ that can be represented by a Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right). \tag{10.1}$$

For a more general case we can replace left hand side of the above equation by the average value $(f(x+) + f(x-))/2$. We now see what will happen if we let $l \to \infty$. It should be mentioned that as l approaches to ∞ the function $f(x)$ becomes non-periodic defined on the real axis. Substituting a_n and b_n in the equation (10.1) we get

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)\,\mathrm{d}u + \frac{1}{l}\sum_{n=1}^{\infty}\left(\int_{-l}^{l} f(u)\cos\frac{n\pi u}{l}\,\mathrm{d}u\,\cos\frac{n\pi x}{l} + \int_{-l}^{l} f(u)\sin\frac{n\pi u}{l}\,\mathrm{d}u\,\sin\frac{n\pi x}{l}\right)$$

Using the identity $\cos x\cos y + \sin x \sin y = \cos(x-y)$, we get

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)\,\mathrm{d}u + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(u)\cos\frac{n\pi}{l}(u-x)\,\mathrm{d}u \tag{10.2}$$

If we assume that $\displaystyle\int_{-\infty}^{\infty} |f(u)|\,\mathrm{d}u$ converges, the first term on the right hand side approaches to 0 as $l \to \infty$ since $\displaystyle\left|\frac{1}{2l}\int_{-l}^{l} f(u)\,\mathrm{d}u\right| \le \frac{1}{2l}\int_{-\infty}^{\infty} |f(u)|\,\mathrm{d}u$.

Letting $l \to \infty$ in equation (10.2), we get

$$f(x) = \lim_{l\to\infty}\frac{1}{l}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\frac{n\pi}{l}(u-x)\,\mathrm{d}u = \lim_{l\to\infty}\frac{\pi}{l}\sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\frac{n\pi}{l}(u-x)\,\mathrm{d}u$$

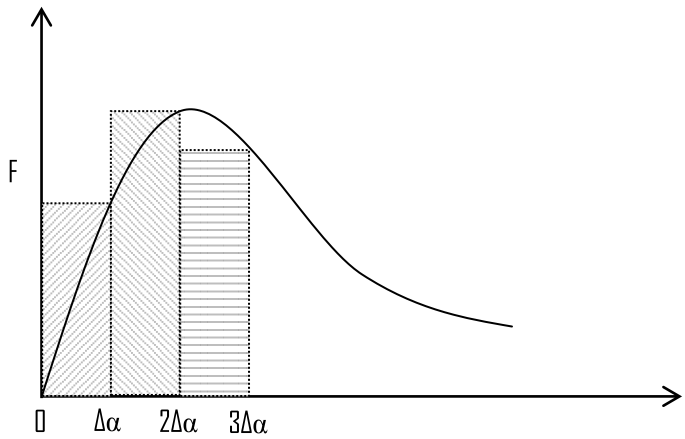

Figure 10.1: Sum of area of trapezoid as area under curve in the limiting case

For simplifications, we define

$$\Delta\alpha = \frac{\pi}{l} \quad \text{and} \quad F(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha(u-x)\,\mathrm{d}u$$

With these definitions and noting $\Delta\alpha \to 0$ as $l \to \infty$, we have

$$f(x) = \lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty}\Delta\alpha F(n\Delta\alpha)$$

Refereing Figure 10.1, we can write this limit of the sum in the form of improper integral as

$$f(x) = \int_0^{\infty} F(\alpha)\mathrm{d}\alpha = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(u-x)\,\mathrm{d}u\,\mathrm{d}\alpha$$

This is called *Fourier Integral Representation* of f on the real line. Equivalently, this can be rewritten as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty} f(u)\cos\alpha u\,\mathrm{d}u\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(u)\sin\alpha u\,\mathrm{d}u\right)\sin\alpha x\right]\,\mathrm{d}\alpha$$

It is often convenient to write

$$f(x) = \int_0^{\infty}\left[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right]\,\mathrm{d}\alpha$$

where the Fourier Integral Coefficients are

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,\mathrm{d}u \quad \text{and} \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,\mathrm{d}u$$

Remark *It should be mentioned that above derivation is not rigorous proof of convergence of the Fourier Integral to the function. This is just to give some idea of transition form Fourier series to Fourier Integral. Nevertheless we summarize the convergence result, without proof, in the next theorem. In addition to all conditions required for the convergence of Fourier series we need one more condition, namely, absolute integrability of f. Further, note that Fourier integral representation of $f(x)$ is entirely analogous to a Fourier series representation of a function on finite interval $\left(\sum_{n=1}^{\infty}\cdots\right.$, is replaced with $\left.\int_0^\infty \cdots\, du\right)$.*

10.1.1 Theorem

Assume that f is piecewise smooth on every finite interval on the x axis (or piecewise continuous and one sided derivatives exist) and let f be absolutely integrable over entire real axis. Then for each x on the entire axis we have

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u)\cos\alpha(u-x)\,du = \frac{f(x+)+f(x-)}{2}$$

As in the convergence of Fourier series if f is continuous and all other conditions are satisfied then the Fourier integral converges

10.2 Example Problems

10.2.1 Problem 1

Let a be a real constant and the function f is defined as

$$f(x) = \begin{cases} 0, & x < 0; \\ x, & 0 < x < a; \\ 0, & x > a. \end{cases}$$

i) Find the Fourier integral representation of f. ii) Determine the convergence of the integral at $x = a$. iii) Find the value of the integral $\displaystyle\int_0^\infty \frac{1-\cos\alpha}{\alpha^2}\,d\alpha$.

Solution: *i)* The integral representation of f is

$$f(x) \sim \int_0^\infty \left[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right]\,\mathrm{d}\alpha \tag{10.3}$$

where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,\mathrm{d}u = \frac{1}{\pi}\int_0^a u\cos\alpha u\,\mathrm{d}u = \frac{1}{\pi}\left[\left(\frac{u\sin\alpha u}{\alpha}\right)\Big|_0^a - \int_0^a \frac{\sin\alpha u}{\alpha}\,\mathrm{d}u\right]$$

$$= \frac{1}{\pi}\left[\frac{a\sin\alpha a}{\alpha} + \frac{(\cos\alpha a - 1)}{\alpha^2}\right] = \frac{1}{\pi}\left[\frac{\cos\alpha a + \alpha a\sin\alpha a - 1}{\alpha^2}\right]$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,\mathrm{d}u = \frac{1}{\pi}\int_0^a u\sin\alpha u\,\mathrm{d}u = \frac{1}{\pi}\left[\left(\frac{-u\cos\alpha u}{\alpha}\right)\Big|_0^a + \int_0^a \frac{\cos\alpha u}{\alpha}\,\mathrm{d}u\right]$$

$$= \frac{1}{\pi}\left[\frac{-a\cos\alpha a}{\alpha} + \frac{\sin\alpha a}{\alpha^2}\right] = \frac{1}{\pi}\left[\frac{\sin\alpha a - \alpha a\cos\alpha a}{\alpha^2}\right]$$

Replacing $A(\alpha)$ and $B(\alpha)$ in equation (10.3), we have

$$f(x) \sim \frac{1}{\pi}\int_0^{\infty}\left[\left(\frac{\cos\alpha a + \alpha a\sin\alpha a - 1}{\alpha^2}\right)\cos\alpha x + \left(\frac{\sin\alpha a - \alpha a\cos\alpha a}{\alpha^2}\right)\sin\alpha x\right]\,\mathrm{d}\alpha$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\cos\alpha(a-x) + \alpha a\sin\alpha(a-x) - \cos\alpha x}{\alpha^2}\,\mathrm{d}\alpha$$

ii) The function is not defined at $x = a$. The value of the Fourier integral at $x = a$ is given as

$$\frac{1}{\pi}\int_0^{\infty}\frac{1-\cos\alpha a}{\alpha^2}\,\mathrm{d}\alpha = \frac{f(a+) + f(a-)}{2} = \frac{0+a}{2} = \frac{a}{2}$$

iii) Substituting $a = 1$ in the above integral we get

$$\frac{1}{\pi}\int_0^{\infty}\frac{1-\cos\alpha}{\alpha^2}\,\mathrm{d}\alpha = \frac{1}{2} \implies \int_0^{\infty}\frac{1-\cos\alpha}{\alpha^2}\,\mathrm{d}\alpha = \frac{\pi}{2}$$

10.2.2 Problem 2

Determine the Fourier integral representing

$$f(x) = \begin{cases} 1, & 0 < x < 2; \\ 0, & x < 0 \text{ and } x > 2. \end{cases}$$

Further, find the value of the integral $\displaystyle\int_0^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha.$

Solution: The Fourier integral representation of f is

$$f(x) \sim \int_0^{\infty}\left[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right]\,\mathrm{d}\alpha \tag{10.4}$$

where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,\mathrm{d}u = \frac{1}{\pi}\int_{0}^{2}\cos\alpha u\,\mathrm{d}u = \frac{1}{\pi}\frac{\sin\alpha u}{\alpha}\bigg|_0^2 = \frac{1}{\pi}\frac{\sin 2\alpha}{\alpha}$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,\mathrm{d}u = \frac{1}{\pi}\int_{0}^{2}\sin\alpha u\,\mathrm{d}u = \frac{1}{\pi}\frac{-\cos\alpha u}{\alpha}\bigg|_0^2 = \frac{1}{\pi}\frac{(1-\cos 2\alpha)}{\alpha}$$

Then, substituting calculated values of $A(\alpha)$ and $B(\alpha)$ in equation (10.4), we obtain

$$\begin{aligned} f(x) \sim& \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin 2\alpha}{\alpha}\cos\alpha x + \frac{(1-\cos 2\alpha)}{\alpha}\sin\alpha x\right]\mathrm{d}\alpha \\ =& \frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha(2-x)+\sin\alpha x}{\alpha}\,\mathrm{d}\alpha \end{aligned}$$

To find the value of the given integral we substitute $x = 1$ in the above Fourier integral and use convergence theorem to get

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin\alpha}{\alpha}\,\mathrm{d}\alpha = f(1) = 1$$

This gives the value of the desired integral as

$$\int_0^{\infty}\frac{\sin\alpha}{\alpha}\,\mathrm{d}\alpha = \frac{\pi}{2}$$