11.1 Miscellaneous Example Problems

11.1.1 Problem 1

Using the convolution theorem prove that

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad [m,n>0].$$

Solution: Let $f(t) = t^{m-1},\ g(t) = t^{n-1}$, then

$$(f*g)(t) = \int_0^t \tau^{m-1}(t-\tau)^{n-1}\mathrm{d}\tau,$$

Substituting $\tau = ut$ so that $\mathrm{d}\tau = t\,\mathrm{d}u$ we obtain

$$(f*g)(t) = \int_0^1 t^{m-1}u^{m-1}t^{n-1}(1-u)^{n-1}t\mathrm{d}u$$

We simplify the above expression to get

$$(f*g)(t) = t^{m+n-1}\int_0^1 u^{m-1}(1-u)^{n-1}\mathrm{d}u = t^{m+n-1}B(m,n)$$

Taking Laplace transform and using convolution property, we find

$$L[t^{m+n-1}B(m,n)] = L[f(t)]\cdot L[g(t)] = L[t^{m-1}]\cdot L[t^{n-1}] = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}$$

Taking inverse Laplace transform,

$$t^{m+n-1}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$$

Hence, we get the desired result as

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

11.1.2 Problem 2

Show that

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}.$$

Solution: We know

$$L[\sin t] = \frac{1}{s^2+1}$$

Therefore, we get

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s = \frac{\pi}{2} - \tan^{-1} s.$$

Taking limit as $s \to 0$ (see remarks below for details) we find

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}.$$

Remark 1: *If f is a piecewise continuous function and $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s)$ converges uniformly for all $s \in E$, then $F(s)$ is a continuous function on E, that is, for $s \to s_0 \in E$,*

$$\lim_{s\to s_0} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s_0) = \int_0^\infty \lim_{s\to s_0} e^{-st} f(t)\,\mathrm{d}t.$$

Remark 2: *Recall that the integral $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is said to converge uniformly for s in some domain Ω if for any $\epsilon > 0$ there exists some number τ_0 such that if $\tau \geq \tau_0$ then*

$$\left|\int_\tau^\infty e^{-st} f(t)\,\mathrm{d}t\right| < \epsilon$$

for all s in Ω.

11.1.3 Problem 3

Using Laplace transform, evaluate the following integral

$$\int_{-\infty}^\infty \frac{x \sin xt}{x^2 + a^2}\,dx$$

Solution: Let

$$f(t) = \int_0^\infty \frac{x \sin xt}{x^2+a^2}\,\mathrm{d}x$$

Taking Laplace transform, we get

$$F(s) = \int_0^\infty \frac{x}{x^2+a^2}\,\frac{x}{x^2+s^2}\,\mathrm{d}x$$

Using the method of partial fractions we obtain

$$F(s) = \int_0^\infty \frac{1}{s^2 - a^2} \left(\frac{s^2}{x^2 + s^2} - \frac{a^2}{x^2 + a^2} \right) \, \mathrm{d}x$$

Evaluating the above integrals we have

$$F(s) = \frac{1}{s^2 - a^2} \left[s \tan^{-1} \left(\frac{x}{s} \right) - a \tan^{-1} \left(\frac{x}{a} \right) \right]_0^\infty$$

On simplification we obtain

$$F(s) = \frac{1}{2} \frac{\pi}{s + a}$$

Taking inverse Laplace transform we find

$$f(t) = \frac{1}{2} \pi e^{-at}$$

Hence the value of the given integral

$$\int_{-\infty}^\infty \frac{x \sin xt}{x^2 + a^2} \, \mathrm{d}x = 2 \int_0^\infty \frac{x \sin xt}{x^2 + a^2} \, \mathrm{d}x = \pi e^{-at}.$$

11.1.4 Problem 4

Evaluate $\displaystyle\int_0^\infty \frac{\cos tx}{x^2 + 1} \, \mathrm{d}x, \quad t > 0.$

Solution: Let

$$f(t) = \int_0^\infty \frac{\cos tx}{x^2 + 1} \, \mathrm{d}x.$$

Taking Laplace transform on both sides,

$$\begin{aligned} L[f(t)] &= \int_0^\infty \frac{s}{(x^2 + 1)(s^2 + x^2)} \, \mathrm{d}x \\ &= \frac{s}{s^2 - 1} \int_0^\infty \left(\frac{1}{x^2 + 1} - \frac{1}{s^2 + x^2} \right) \, \mathrm{d}x \\ &= \frac{s}{s^2 - 1} \left[\tan^{-1} x - \frac{1}{s} \tan^{-1} \left(\frac{1}{s} \right) \right]_0^\infty \\ &= \frac{s}{s^2 - 1} \left(\frac{\pi}{2} - \frac{\pi}{2s} \right) = \frac{\pi}{2} \frac{1}{s + 1}. \end{aligned}$$

Taking inverse Laplace transform on both sides,

$$f(t) = \frac{\pi}{2} e^{-t}.$$

11.1.5 Problem 5

Evaluate $\int_0^\infty e^{-x^2} \mathrm{d}x.$

Solution: Let

$$g(t) = \int_0^\infty e^{-tx^2} \, \mathrm{d}x$$

Now taking Laplace on both sides,

$$L[g(t)] = \int_0^\infty \frac{1}{s + x^2} \, \mathrm{d}x = \frac{1}{\sqrt{s}} \arctan\left(\frac{x}{\sqrt{s}}\right) \Big|_0^\infty = \frac{1}{\sqrt{s}} \frac{\pi}{2}$$

Taking inverse Laplace transform we obtain

$$g(t) = \frac{\pi}{2} L^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{\pi}{2} \frac{1}{\sqrt{\pi}\sqrt{t}}.$$

Hence for $t = 1$ we get

$$\int_0^\infty e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Example Let

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi. \end{cases}$$

Express the given function $f(t)$ as a difference of two unit step functions and hence find its Laplace transform.

Solution $f(t) = \sin t\ [H(t) - H(t-\pi)]$

$$\boxed{L\left[f(t-a)H(t-a)\right] = e^{-as}F(s)}$$

$$= L[\sin t\, H(t)] - L[\sin(t)\, H(t-\pi)]$$

$$= L[\sin t\, H(t)] + L[\sin(t-\pi)\, H(t-\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

Example Find the Laplace transform of $C(t) = \int_t^{\infty} \frac{\cos x}{x} dx$

Solution Let $f(t) = \int_t^{\infty} \frac{\cos x}{x}\, dx$ Note $\lim_{t\to\infty} f(t) = 0$

$$f'(t) = -\frac{\cos t}{t} \implies t f'(t) = -\cos t$$

$$L[f'(t)] = sL[f(t)] - f(0), \quad Re(s) > \alpha.$$

$$L[t f(t)] = -\frac{d}{ds} F(s)$$

$$-\frac{d}{ds}[s\, F(s) - f(0)] = -\frac{s}{s^2+1}$$

$$sF(s) = \frac{1}{2}\ln(s^2+1) + c \implies \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{1}{2}\ln(s^2+1) + c = c$$

Using final value theorem: $\lim_{t\to\infty} f(t) = c \Rightarrow c = 0$

$$sF(s) = \frac{1}{2}\ln(s^2+1) \Rightarrow F(s) = \frac{1}{2s}\ln(s^2+1)$$