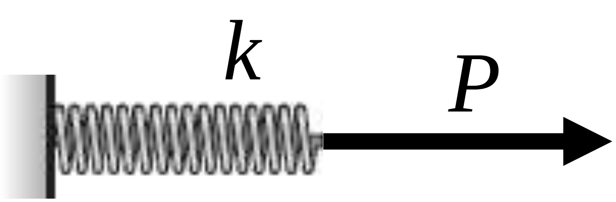
Energy Methods

Strain Energy

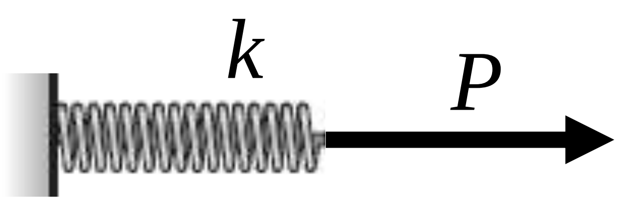
Spring pulled by a force P

- We will consider a spring which has extended by an amount x very slowly over a long period of time under the action of a force P which was gradually increased from zero to its final value P very slowly over that long time.
- Essentially it means that the spring was almost at equilibrium at every stage of expansion



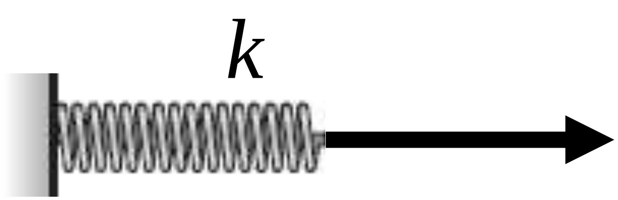
Spring pulled by a force P

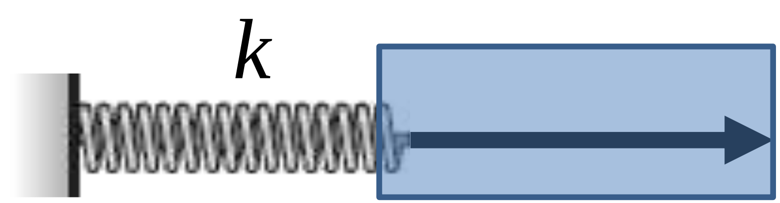
- The spring is uniform and linear. So the stiffness does not vary across the length. Also the force developed due to an extension x is kx and not kx^3 or $kx^{0.5}$ or ksin(x) or some other function of x.



Spring pulled by a force P

- Consider that the spring was extended by an amount ξ and comes to an equilibrium with the external force. So the force felt by the spring is k ξ. We are not talking f the external force here, but what we will see if we cut the spring just before the free end.





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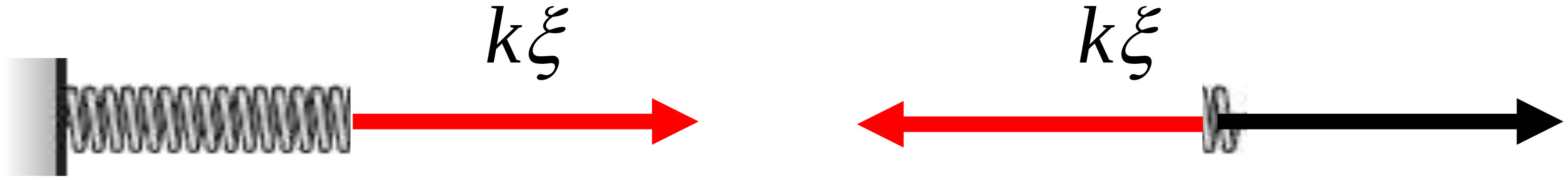
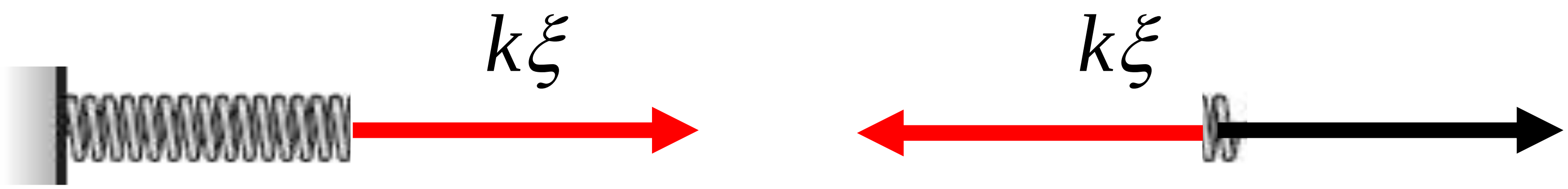


Spring pulled by a force P

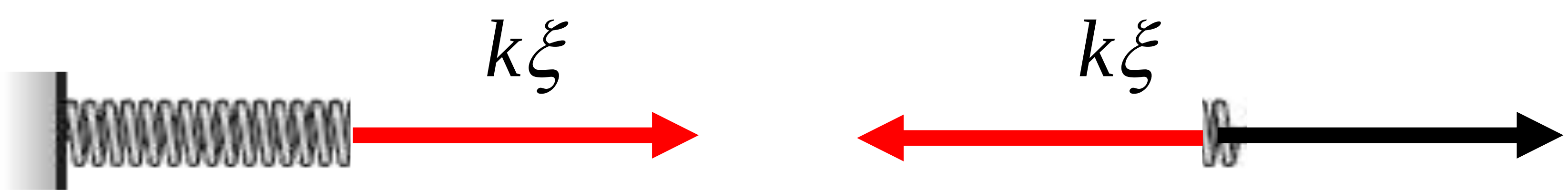
- Now the spring is further extended by a very small amount dξ.
- Since this amount is very small, we may assume that the force still remains kξ.
- The work done will therefore be (kξ)dξ.



Spring pulled by a force P

- So what will be the total work done by this internal force once the final extension x is achieved?
- We can find that out by integrating from 0 to x.
- Do keep in mind that x has nothing to do with the length of the spring but only the extension

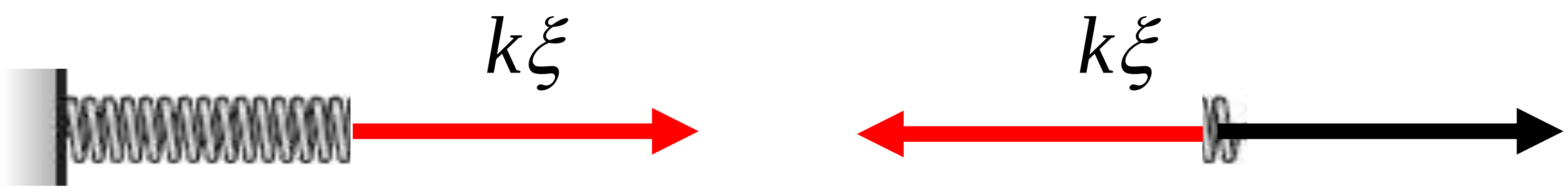
$$\int_0^x k\xi d\xi = \frac{1}{2}kx^2$$



Spring pulled by a force P

- This is the energy that is now stored inside the spring.
- It is a very simple type of strain energy

$$U = \int_0^x k\xi d\xi = \frac{1}{2}kx^2$$

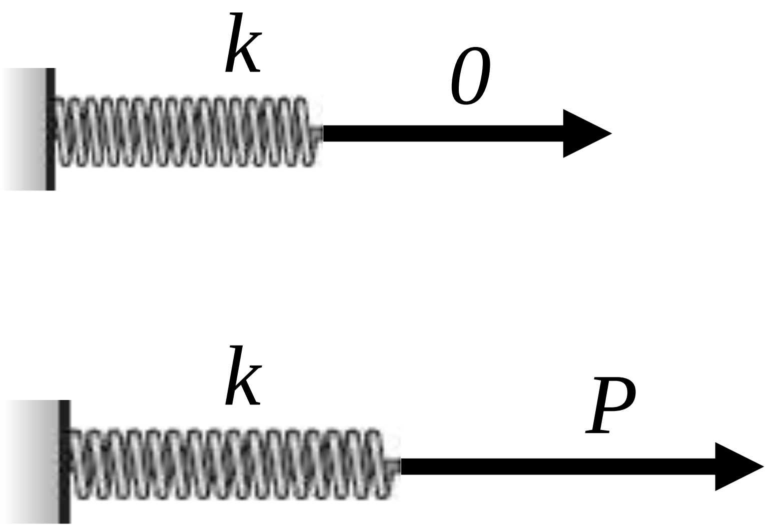


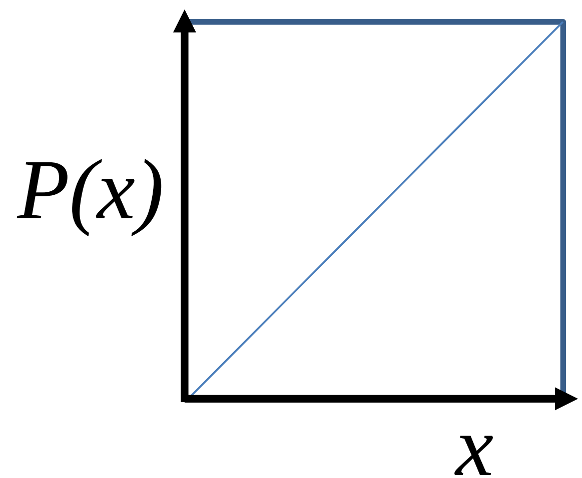
Spring pulled by a force P

- We now look at the other player in this game – the force P
- Keep in mind that the force increased from 0 to its final value P.
- Also the force at each incremental deformation dξ was equal to (kξ). So we can safely say that P was increasing linearly

Spring pulled by a force P

- So if we plot a graph of P vs x we will get a straight line

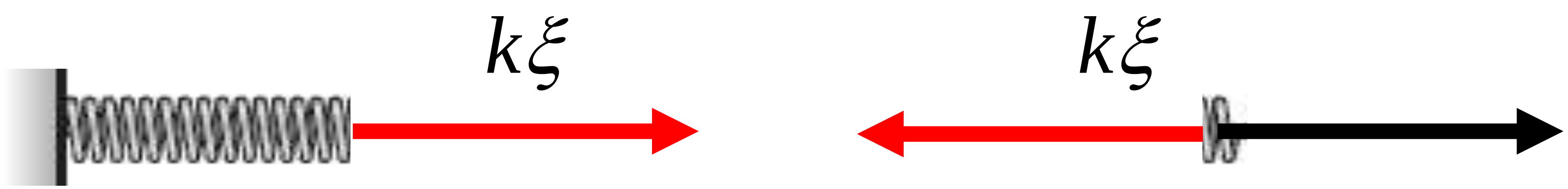




Spring pulled by a force P

- From the point of view of the external force therefore, work done is

$$W = \frac{1}{2} P x$$



Spring pulled by a force P

- Now we will do something that will appear very trivial.
- We will equate the work done with the strain energy
- The results appear very very trivial and not worth the derivations.

$$W = P$$

$$\Rightarrow \frac{1}{2}Px = \frac{1}{2}kx^2$$

$$\Rightarrow P = kx$$

$$\Rightarrow x = \frac{P}{k}$$

Spring pulled by a force P

- We will now modify the strain energy expression by using the force relation

$$P = kx \Rightarrow x = \frac{P}{k}$$

- Substituting in the strain energy expression

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{P}{k}\right) = \frac{1}{2}\frac{P^2}{k}$$

Spring pulled by a force P

- We use this new expression and equate work done with strain energy

$$U = W \Rightarrow \frac{1}{2}\frac{P^2}{k} = \frac{1}{2}Px$$

$$\Rightarrow x = \frac{P}{k}$$

Spring pulled by a force P

- This still looks trivial
- However note that now our strain energy expression contains only the external force and a property of the spring (stiffness) only

$$U = \frac{1}{2}\frac{P^2}{k}$$

- Both of these are known to us, when we face the problem of finding the deformation of a spring pulled by a force

Spring pulled by a force P

- In the work done expression, we have only one unknown

$$Q = \frac{1}{2} P x$$

- So when we equate the two we have an equation with one unknown only, which is good

$$\frac{1}{2} P x = \frac{1}{2} \frac{P^2}{k}$$

Spring pulled by a force P

- Still, the problem we have tackled being simple does not really explain how useful this approach can be.
- So we will solve a tougher problem by skipping derivations which will be explained in details later.

Spring pulled by a force P

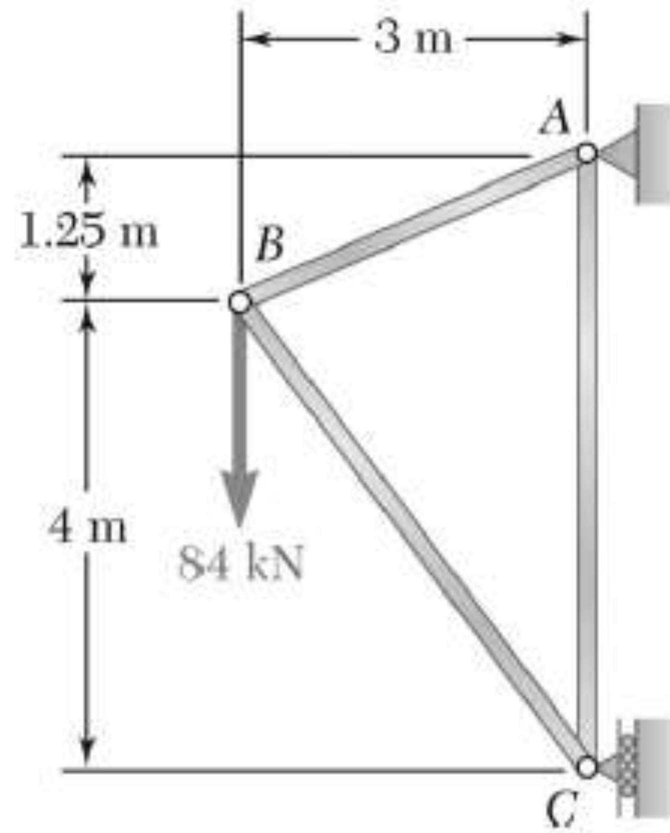
- We have already discussed how a rod of length L and area of cross section A acted upon by an axial force P can be treated as a spring.
- The stiffness of such a rod was derived as

$$k = \frac{EA}{L}$$

- We will now consider a problem involving a truss keeping in mind that a truss is made up of rods carrying only axial forces.

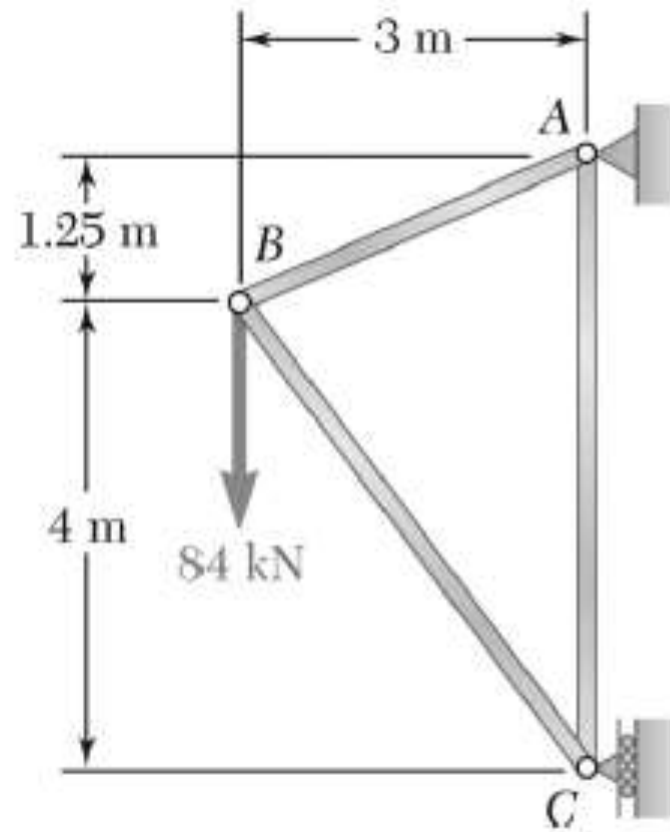
Truss acted upon by a force

- We wish to find out the downward deflection of the point B.
- We will assume that the rods are made of the same material with modulus of elasticity E and have the same area of cross section A.



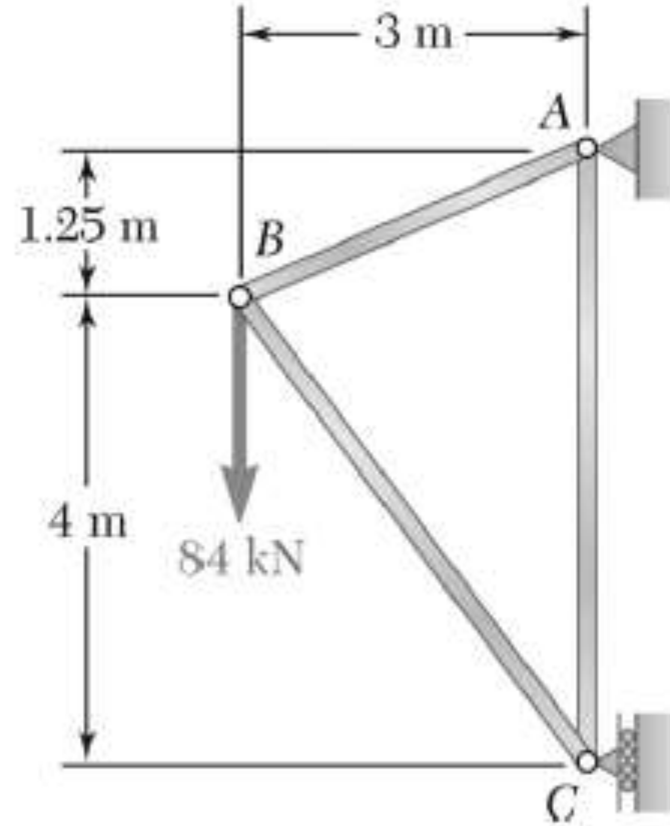
Truss acted upon by a force

- Using basic knowledge of trusses we can solve for the forces in each member
- Using geometry we can find out the unknown lengths.



Truss acted upon by a force

- The solutions are as follows

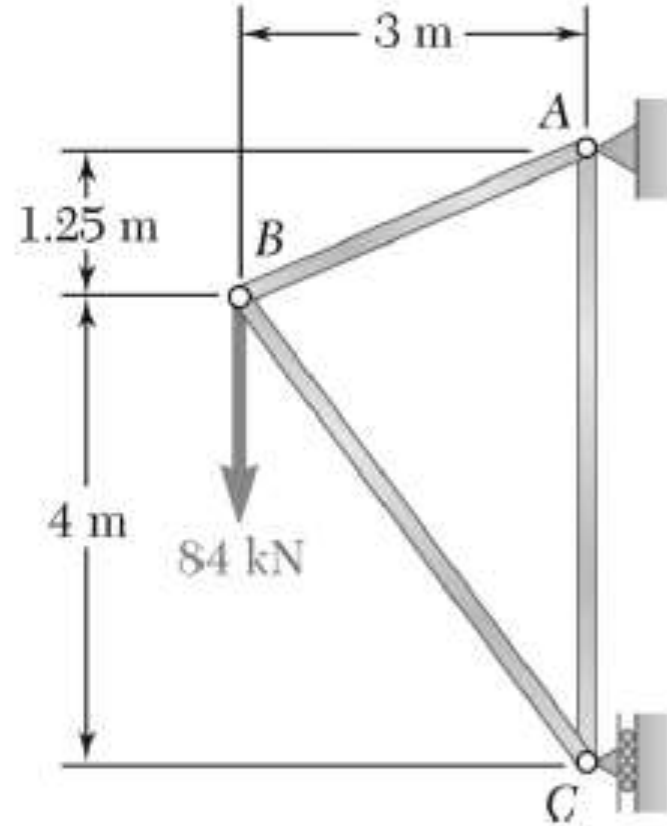


$$L_{AB} = 3.25m, L_{BC} = 5m, L_{CA} = 5.25m$$

$$F_{AB} = 52kN, F_{BC} = -80kN, F_{CA} = 64kN$$

Truss acted upon by a force

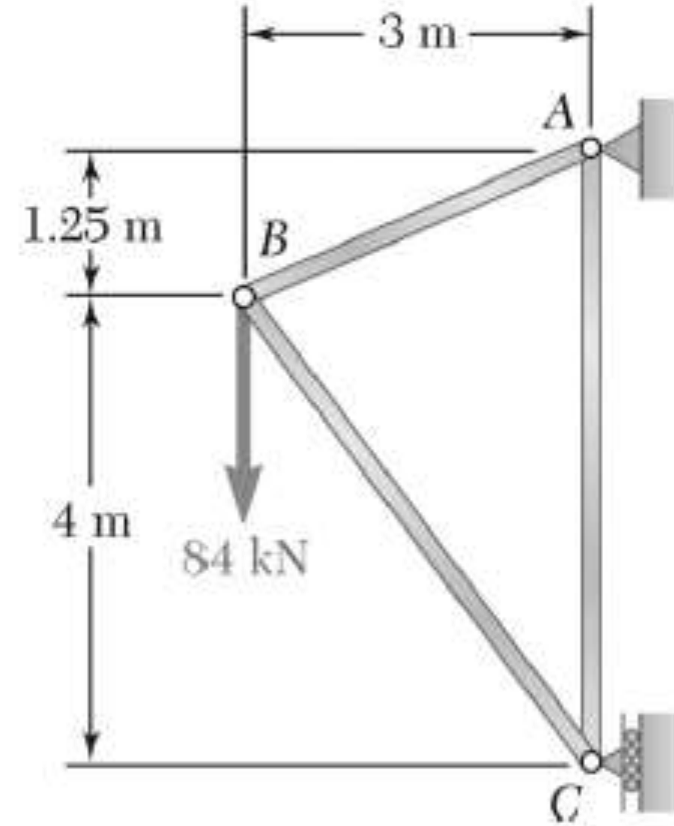
- We will treat the external force as an unknown force P as a variable for the time being.
- Hence



$$F_{AB} = \frac{52}{84}P, F_{BC} = -\frac{80}{84}P, F_{CA} = \frac{64}{84}P$$

Truss acted upon by a force

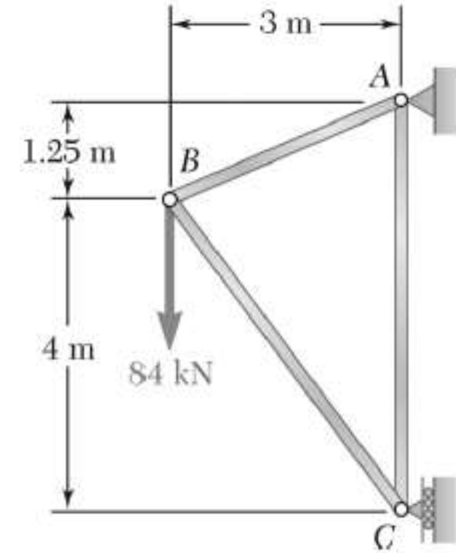
- Now force P is acting vertically. Hence the work done by the force will involve only the vertical deflection at P. The horizontal deflection, if any will not contribute to the work done



- Hence

$$W = \frac{1}{2} P \delta_{B,v}$$

Truss acted upon by a force



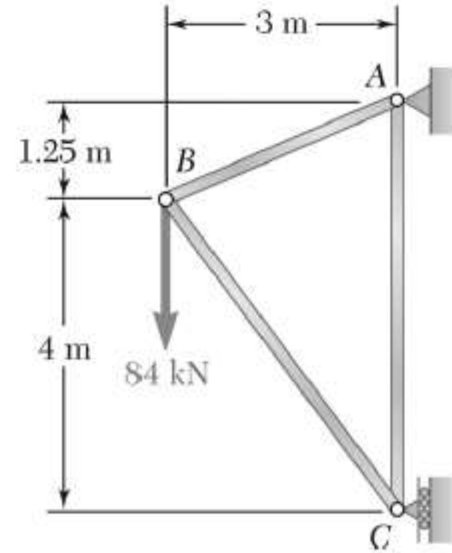
- The equivalent stiffness of each member AB,BC, CA for axial loading will be

$$k_{AB} = \frac{EA}{L_{AB}}, k_{BC} = \frac{EA}{L_{BC}}, k_{CA} = \frac{EA}{L_{CA}}$$

- We will now apply the strain energy expression for a spring

$$U = \frac{P^2}{2k}$$

Truss acted upon by a force



- The strain energies of each member AB,BC, CA due to the forces developed in them will be

$$U_{AB} = \frac{F_{AB}^{\ 2}}{2\frac{EA}{L_{AB}}}, U_{BC} = \frac{F_{BC}^{\ 2}}{2\frac{EA}{L_{BC}}}, U_{CA} = \frac{F_{CA}^{\ 2}}{2\frac{EA}{L_{CA}}}$$

Truss acted upon by a force

- The total strain energy of the truss will therefore be

$$F_{AB} = \frac{52}{84}P, F_{BC} = -\frac{80}{84}P, F_{CA} = \frac{64}{84}P$$

$$U = \frac{F_{AB}{}^2 L_{AB}}{2EA} + \frac{F_{BC}{}^2 L_{BC}}{2EA} + \frac{F_{CA}{}^2 L_{CA}}{2EA}$$

$$= \frac{P}{EA}\left(\frac{169L_{AB}}{882} + \frac{400L_{BC}}{882} + \frac{64L_{CA}}{882}\right)$$

Truss acted upon by a force

- Since this strain energy comes from the work done by the force P, hence

$$\frac{1}{2}P\delta_{B,v} = \frac{P}{EA}\left(\frac{169L_{AB}}{882} + \frac{400L_{BC}}{882} + \frac{64L_{CA}}{882}\right)$$

$$\Rightarrow \delta_{B,v} = \frac{1}{EA}\left(\frac{169L_{AB}}{441} + \frac{400L_{BC}}{441} + \frac{64L_{CA}}{441}\right)$$

- Everything on the RHS is known to us and hence we can calculate the deflection

Truss acted upon by a force

- Note that only one force has been considered
- Also only deflection in the direction of that force has been obtained.
- However given the simplicity of the theory even this can be considered a big benefit
- We are not having to calculate individual deformations and then apply geometrical conditions to find the deflection in the vertical direction.
- With improved energy methods even multiple forces and deflections in any directions can be handled.