Transform Calculus

(MA-20101)

Solutions- 2

1. We do this by induction.
$(1 * 1)(t) = \int_0^t 1(\tau)1(t-\tau)d\tau = \int_0^t d\tau = t$. We assume that $1 * \cdots * 1(n-1 \text{ times}) = \frac{t^{n-2}}{(n-2)!}$.
$$\begin{aligned} \therefore 1 * \cdots * 1(n \text{ times}) &= \tfrac{t^{n-2}}{(n-2)!} * 1 \\ &= \int_0^t \tfrac{\tau^{n-2}}{(n-2)!} 1(t-\tau)d\tau \\ &= \int_0^t \tfrac{\tau^{n-2}}{(n-2)!} d\tau \\ &= \tfrac{t^{n-1}}{(n-1)!}. \end{aligned}$$

2. We have
$$\begin{aligned} \mathcal{L}\big(\int_0^t \sin u \cos(t-u)du\big) &= \mathcal{L}(\sin t * \cos t) \\ &= \mathcal{L}(\sin t).\mathcal{L}(\cos t)(\text{ by Convolution theorem}) \\ &= \tfrac{1}{s^2+1}.\tfrac{s}{s^2+1} \\ &= \tfrac{s}{(s^2+1)^2}. \end{aligned}$$
On the other hand,
$$\begin{aligned} \mathcal{L}(\tfrac{1}{2}t\sin t) &= -\tfrac{1}{2}\big(\mathcal{L}(\sin t)\big)' \\ &= -\tfrac{1}{2}\big(\tfrac{1}{s^2+1}\big)' \\ &= \tfrac{1}{2}\tfrac{2s}{(s+1)^2} \\ &= \tfrac{s}{(s^2+1)^2}. \end{aligned}$$
By uniqueness of inverse Laplace transform, $\int_0^t \sin u \cos(t-u)du = \frac{1}{2}t\sin t$.

3. i) We have a partial fraction representation
$\frac{s-1}{(s+3)(s^2+2s+2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+2}$
$\therefore s-1 = A(s^2+2s+2) + (Bs+C)(s+3)$
$\therefore s-1 = (A+B)s^2 + (2A+3B+C)s + (2A+3C)$

Equating the coeffcients of each power of s on both sides gives the three equations

(a) $A+B=0$, (b) $2A+3B+C=1$, (c) $2A+3C=-1$.

$$\therefore A=-4/5,\ B=4/5,\ C=1/5.$$

$$\begin{aligned}
\therefore \tfrac{s-1}{(s+3)(s^2+2s+2)} &= \tfrac{-\frac{4}{5}}{s+3}+\tfrac{\frac{4}{5}s+\frac{1}{5}}{s^2+2s+2}\\
\therefore \mathcal{L}^{-1}\Big(\tfrac{s-1}{(s+3)(s^2+2s+2)}\Big) &= \mathcal{L}^{-1}\Big(\tfrac{-\frac{4}{5}}{s+3}+\tfrac{1}{5}\cdot\tfrac{4s+1}{s^2+2s+2}\Big)\\
&= -\tfrac{4}{5}\mathcal{L}^{-1}\big(\tfrac{1}{s+3}\big)+\tfrac{4}{5}.\mathcal{L}^{-1}\big(\tfrac{s+\frac{1}{4}}{(s+1)^2+1)}\big)\\
&= -\tfrac{4}{5}e^{-3t}+\tfrac{4}{5}\mathcal{L}^{-1}\Big(\tfrac{s+1}{(s+1)^2+1}\Big)-\tfrac{4}{5}.\tfrac{3}{4}\mathcal{L}^{-1}\Big(\tfrac{1}{(s+1)^2+1}\Big)\\
&= -\tfrac{4}{5}e^{-3t}+\tfrac{4}{5}e^{-t}\cos t-\tfrac{3}{5}e^{-t}\sin t.
\end{aligned}$$

ii)

$$\frac{se^{-2s}}{s^2+3s+2}=\frac{s}{(s+2)(s+1)}e^{-2s}.$$

We have a partial fraction representation
$\frac{s}{(s+2)(s+1)}=\frac{A}{s+2}+\frac{B}{s+1}$
$\therefore \frac{s}{(s+2)(s+1)}=\frac{(A+B)s+(A+2B)}{(s+2)(s+1)}$
Equating the coeffcients of each power of s on both sides gives the two equations
$A+B=1,\ A+2B=0.$
$\therefore A=2,\ B=-1.$
$\therefore \frac{s}{(s+2)(s+1)}=\frac{2}{s+2}-\frac{1}{s+1}.$

$$\begin{aligned}
\mathcal{L}^{-1}\Big(\tfrac{se^{-2s}}{(s+2)(s+1)}\Big) &= \mathcal{L}^{-1}\Big(\tfrac{2}{s+2}e^{-2s}-\tfrac{1}{s+1}e^{-2s}\Big)\\
&= \mathcal{L}^{-1}\Big(\tfrac{2}{s+2}e^{-2s}\Big)-\mathcal{L}^{-1}\Big(\tfrac{1}{s+1}e^{-2s}\Big)\\
&= 2\ u(t-2)e^{-2(t-2)}-u(t-2)e^{-1(t-2)}\\
&= 2e^{-2t+4}\ u(t-2)-e^{-t+2}\ u(t-2)
\end{aligned}$$

$$\therefore f(t)=\begin{cases}2e^{-2t+4}-e^{-t+2}, \text{ if } t>2\\ 0, \text{ if } t<2.\end{cases}$$

iii) We have

$$\frac{1}{s^2+6s+10}=\frac{1}{(s+3)^2+1}=\mathcal{L}(e^{-3t}\sin t)(\text{ by First Shifting Theorem}).$$

The,

$$\begin{aligned}
\tfrac{2s+6}{(s^2+6s+10)^2} &= -\tfrac{d}{ds}\Big(-\tfrac{1}{s^2+6s+10}\Big)\\
&= \mathcal{L}(te^{-3t}\sin t)
\end{aligned}$$

So, the inverse Laplace transform is $te^{-3t}\sin t$.

iv) Let $F(s) = \ln\left(\frac{s+2}{s+1}\right) = \ln(s+2) - \ln(s+1)$.
$\therefore F'(s) = \frac{1}{s+2} - \frac{1}{s+1}$.
$\therefore \mathcal{L}^{-1}(F'(s)) = e^{-2t} - e^{-t}$
$\therefore \mathcal{L}^{-1}(F'(s)) = -t.\frac{e^{-t}-e^{-2t}}{t}$
$\therefore \mathcal{L}^{-1}(F(s)) = \frac{e^{-t}-e^{-2t}}{t}$.

v) We have

$$\begin{aligned}
\mathcal{L}^{-1}\Big(\frac{1}{(s+3)(s-1)}\Big) &= \mathcal{L}^{-1}\Big(\frac{1}{s+3}\Big).\mathcal{L}^{-1}\Big(\frac{1}{s-1}\Big) \\
&= e^{-3t} * e^{t} \\
&= \int_0^t e^{-3\tau} e^{t-\tau} d\tau \\
&= \int_0^t e^{t-4\tau} d\tau \\
&= \frac{e^t}{-4}[e^{-4\tau}]_0^t \\
&= \frac{1}{4}(e^t - e^{-3t}).
\end{aligned}$$

vi) We have

$$\begin{aligned}
\mathcal{L}^{-1}\Big(\frac{s}{(s^2-a^2)^2}\Big) &= \mathcal{L}^{-1}\Big(\frac{1}{2}.\frac{2s}{(s^2-a^2)^2}\Big) \\
&= -\frac{1}{2}\frac{d}{ds}\Big(\frac{1}{(s^2-a^2)}\Big) \\
&= -\frac{1}{2a}\frac{d}{ds}\Big(\mathcal{L}(\sinh at)\Big) \\
&= -\frac{d}{ds}\Big(\mathcal{L}\big(\frac{\sinh at}{2a}\big)\Big) \\
\therefore \mathcal{L}^{-1}\Big(\frac{s}{(s^2-a^2)^2}\Big) &= \frac{t\sinh at}{2a}.
\end{aligned}$$

4. Let the Laplace transform of $f(t)$ be $\mathcal{L}(f)$.

i) We have

$$\begin{aligned}
\mathcal{L}\Big(\int_0^t\{\int_0^{t_1} f(\tau)d\tau\}dt_1\Big) &= \frac{1}{s}\mathcal{L}\Big(\int_0^{t_1} f(\tau)d\tau\Big) \\
&= \frac{1}{s}\Big(\frac{1}{s}\mathcal{L}(f(t))\Big) \\
&= \frac{1}{s^2}F(s).
\end{aligned}$$

Therefore the inverse Laplace transform of $\frac{F(s)}{s^2}$ is $\int_0^t\{\int_0^{t_1} f(\tau)d\tau\}dt_1$.

ii) We have

$$\begin{aligned}
\mathcal{L}(t^2 f''(t)) &= -\mathcal{L}(tf'')' \\
&= \mathcal{L}(f''(t))'' \\
&= (s^2\mathcal{L}(f) - s\ f(0) - f'(0))'' \\
&= (s^2F(s) - s\ f(0) - f'(0))'' \\
&= \big(2s\ F(s) + s^2F'(s)) - f(0)\big)' \\
&= 2\ F(s) + 4s\ F'(s) + s^2F''(s) \\
&= s^2F''(s) + 4s\ F'(s) + 2\ F(s).
\end{aligned}$$

5. Let the Laplace transform of $f(t)$ be $\mathcal{L}(f)$.

i) We have

$$\begin{aligned}
\mathcal{L}(\mathcal{J}_0(t)) &= \mathcal{L}(1-\tfrac{t^2}{2^2}+\tfrac{t^4}{2^24^2}-\tfrac{t^6}{2^24^26^2}+\cdots\cdots) \\
&= \mathcal{L}(1)-\mathcal{L}(\tfrac{t^2}{2^2})+\mathcal{L}(\tfrac{t^4}{2^24^2})-\mathcal{L}(\tfrac{t^6}{2^24^26^2})+\cdots \\
&= \tfrac{1}{s}-\tfrac{2!}{2^2s^3}+\tfrac{4!}{2^24^2s^5}-\tfrac{6!}{2^24^26^2s^7}+\cdots\cdots \\
&= \tfrac{1}{s}.\big(1-\tfrac{1}{2}\tfrac{1}{s^2}+\tfrac{1.3}{2.4}\tfrac{1}{s^4}-\tfrac{1.3.5}{2.4.6}\tfrac{1}{s^6}+\cdots\cdots\big) \\
&= \tfrac{1}{s}\big(1+\tfrac{1}{s^2}\big)^{-\frac{1}{2}} \\
&= \tfrac{1}{\sqrt{s^2+1}}.
\end{aligned}$$

Then by Problem 2, the Laplace transform of $\mathcal{J}_0(at)$ is $\frac{1}{a}.\frac{1}{\sqrt{(\frac{s}{a})^2+1}} = \frac{1}{\sqrt{s^2+a^2}}$.

ii) Using series, we have

$$\sin\sqrt{t} = t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \frac{t^{\frac{7}{2}}}{7!} + \cdots .$$

The Laplace transform is

$$\begin{aligned}
\mathcal{L}(\sin\sqrt{t}) &= \mathcal{L}(t^{\frac{1}{2}})-\tfrac{1}{3!}\mathcal{L}(t^{\frac{3}{2}})+\tfrac{1}{5!}\mathcal{L}(t^{\frac{5}{2}})-\tfrac{1}{7!}\mathcal{L}(t^{\frac{7}{2}})+\cdots \\
&= \tfrac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}}-\tfrac{\Gamma(\frac{5}{2})}{3!s^{\frac{5}{2}}}+\tfrac{\Gamma(\frac{7}{2})}{5!s^{\frac{7}{2}}}-\tfrac{\Gamma(\frac{9}{2})}{7!s^{\frac{9}{2}}}+\cdots \\
&= \tfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\{1-\tfrac{1}{2^2s}+\tfrac{(\frac{1}{2^2s})^2}{2!}-\tfrac{(\frac{1}{2^2s})^3}{3!}\} \\
&= \tfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{2^2s}}
\end{aligned}$$

6. Let the Laplace transform of $f(t)$ be $\mathcal{L}(f)$.

i) We have

$$\begin{aligned}
\int_0^\infty t\ e^{-st}\cos t dt &= \mathcal{L}(t\ \ \cos t) \\
&= -\tfrac{d}{ds}\big(\mathcal{L}(\cos t)\big) \\
&= -\tfrac{d}{ds}\big(\tfrac{s}{s^2+1}\big) \\
&= \tfrac{s^2-1}{(s^2+1)^2}
\end{aligned}$$

$\therefore \int_0^\infty t\ e^{-2t}\cos t dt = [\frac{s^2-1}{(s^2+1)^2}]_{s=2} = \frac{3}{25}$.

ii) We have

$$\begin{aligned}
\int_0^\infty e^{-st}\Big(\tfrac{e^{-3t}-e^{-6t}}{t}\Big) &= \mathcal{L}\Big(\tfrac{e^{-3t}-e^{-6t}}{t}\Big) \\
&= \int_s^\infty \mathcal{L}(e^{-3t}-e^{-6t})d\tilde{s} \\
&= \int_s^\infty \Big(\tfrac{1}{\tilde{s}+3}-\tfrac{1}{\tilde{s}+6}\Big)d\tilde{s} \\
&= [\ln(\tilde{s}+3)]_s^\infty - [\ln(\tilde{s}+6)]_s^\infty \\
&= \ln\left(\tfrac{s+6}{s+3}\right)
\end{aligned}$$

Taking the limit as $s \to 0+$, we get $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t} = \ln 2$.

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