In this lesson we introduce the concept of inverse Laplace transform and discuss some of its important properties that will be helpful to evaluate inverse Transform of some complicated functions. As mention in the beginning of this module that the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the transformed domain we will like to get back to the time domain and therefore we need to introduce the concept of inverse Laplace transform. Further, we introduce the convolution property of the Laplace transform. We shall start with the definition of convolution followed by an important theorem on Laplace transform of convolution. Convolution theorem plays an important role for finding inverse Laplace transform of complicated functions and therefore very useful for solving differential equations.

7.1 Inverse Laplace Transform

If $F(s) = L[f(t)]$ for some function $f(t)$. We define the *inverse Laplace transform* as

$$L^{-1}[F(s)] = f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself as it requires complex numbers and path integrals. The easiest way of computing the inverse is using table of Laplace transform. For example,

$$L[\sin wt] = \frac{w}{s^2 + w^2}$$

This implies

$$L^{-1}\left[\frac{w}{s^2 + w^2}\right] = \sin wt,\ t \geq 0$$

and similarly

$$L[\cos wt] = \frac{s}{s^2 + w^2} \quad \Rightarrow \quad L^{-1}\left[\frac{s}{s^2 + w^2}\right] = \cos wt,\ t \geq 0$$

7.2 Uniqueness of Inverse Laplace Transform

If we have a function $F(s)$, to be able to find $f(t)$ such that $L[f(t)] = F(s)$, we need to first know if such a function is unique.

Consider

$$g(t) = \begin{cases} 1 & \text{when } t = 1 \\ \sin(t) & \text{when otherwise} \end{cases}$$

$$L[g(t)] = \frac{1}{s^2+1} = L[\sin t]$$

Thus we have two different functions $g(t)$ and $\sin t$ whose Laplace transform are same. However note that the given two functions are different at a point of discontinuity. Thanks to the following theorem where we have uniqueness for continuous functions:

7.2.1 Theorem (Lerch's Theorem)

If f and g are continuous and are of exponential order, and if $F(s) = G(s)$ for all $s > s_0$ then $f(t) = g(t)$ for all $t > 0$.

Proof: If $F(s) = G(s)$ for all $s > s_0$ then,

$$\begin{aligned}
&\int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^\infty e^{-st} g(t)\mathrm{d}t, \;\; \forall s > s_0 \\
&\Rightarrow \int_0^\infty e^{-st}[f(t) - g(t)]\mathrm{d}t = 0, \;\; \forall s > s_0 \\
&\Rightarrow f(t) - g(t) \equiv 0, \;\; \forall t > 0. \\
&\Rightarrow f(t) = g(t), \;\; \forall t > 0.
\end{aligned}$$

This completes the proof. ■

Remark: *The uniqueness theorem holds for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function or the function $g(t)$ defined above. Since the Laplace integral however does not "see" values at the discontinuities. So in this case we can only conclude that $f(t) = g(t)$ outside of discontinuities.*

We now state some important properties of the inverse Laplace transform. Though, these properties are the same as we have listed for the Laplace transform, we repeat them without proof for the sake of completeness and apply them to evaluate inverse Laplace transform of some functions.

$$\int_0^\infty e^{-st}\, h(t)\, dt \;= 0 \Longrightarrow h(t) = 0\ ?$$

Lemma: If $h(t)$ is continuous on $[0, 1]$ and $\int_0^1 h(t)t^n\, dt = 0$ for $n = 0,1,2, \dots$ then $h(t) = 0$

Since $h(t)$ is continuous function then we can find a polynomial P_ϵ such that

$$|h(t) - P_\epsilon(t)| < \epsilon, \qquad \forall t$$

$$\int_0^1 h(t)\, t^n\, dt = 0 \Longrightarrow \int_0^1 h(t)P_\epsilon(t)\, dt = 0 \;\overset{\epsilon \to 0}{\Longrightarrow}\; \int_0^1 \underbrace{h(t)\, h(t)}_{\geq 0}\, dt = 0 \;\Longrightarrow\; h(t) = 0$$

$$\int_0^\infty e^{-st}\, h(t)\, dt \;= 0, s > \alpha$$

Fix $s = s_0 > \alpha$ Substitute $u = e^{-t}$ Take $s = s_0 + n + 1$

$$0 = \int_0^1 u^n\;\; u^{s_0} h(-\ln u)\, du \quad \Longrightarrow \quad u^{s_0} h(-\ln u) = 0 \quad \Longrightarrow \quad h(t) = 0$$