## 5.1 Laplace Transform of Derivatives

Before we state the derivative theorem, it should be noted that this results is the key aspect for its application of solving differential equations.

### 5.1.1 Derivative Theorem

*Suppose $f$ is continuous on $[0, \infty)$ and is of exponential order $\alpha$ and that $f^{'}$ is piecewise continuous on $[0, \infty)$. Then*

$$L[f^{'}(t)] = sL[f(t)] - f(0), \quad \textit{Re(s)} > \alpha.$$

**Proof:** Consider the following integral

$$\int_0^R f^{'}(t)e^{-st}\,\mathrm{d}t$$

Note that the above integral exist because $f'$ is piece-wise continuous. Integrating by parts, we get

$$\int_0^R f^{'}(t)e^{-st}\,\mathrm{d}t = f(t)e^{-st}\Big|_0^R - \int_0^R f(t)e^{-st}(-s)\,\mathrm{d}t$$

This can be further rewritten as

$$\int_0^R f^{'}(t)e^{-st}\,\mathrm{d}t = f(R)e^{-sR} - f(0) + s\int_0^R f(t)e^{-st}\,\mathrm{d}t$$

By the definition of Laplace transform, we have

$$L[f^{'}(t)] = \lim_{R\to\infty}\int_0^R f^{'}(t)e^{-st}\,\mathrm{d}t = \lim_{R\to\infty}\left(f(R)e^{-sR} - f(0) + s\int_0^R f(t)e^{-st}\,\mathrm{d}t\right)$$

Using the fact that $f$ is of exponential order ($\lim_{R\to\infty} f(R)e^{-sR} = 0$), we get

$$L[f^{'}(t)] = -f(0) + sL[f(t)], \quad \text{Re(s)} > \alpha.$$

This completes the proof. ∎

**Remark 1:** *Suppose $f(t)$ is not continuous at $t = 0$, then the results of the above theorem takes the following form*

$$L[f^{'}(t)] = -f(0+0) + sL[f(t)]$$

**Remark 2:** *An interesting feature of the derivative theorem is that $L[f^{'}(t)]$ exists without the requirement of $f^{'}$ to be of exponential order. Recall the existence of Laplace transform of $f(t) = 2te^{t^2}\cos\left(e^{t^2}\right)$ which is obvious now by the derivative theorem because*

$$f(t) = \left(\sin\left(e^{t^2}\right)\right)^{'}.$$

**Remark 3:** *The derivative theorem can be generalized as*

$$\begin{aligned} L[f^{''}(t)] &= -f^{'}(0) + sL[f^{'}(t)] \\ &= -f^{'}(0) + s\left\{-f(0) + sL[f(t)]\right\} = s^2L[f(t)] - sf(0) - f^{'}(0). \end{aligned}$$

*In general, for $n$th derivative we have*

$$L[f^{(n)}(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f^{'}(0) - ... - f^{n-1}(0).$$

## 5.2 Example Problems

### 5.2.1 Problem 1

*Determine* $L[\sin^2\omega t]$.

**Solution:** Let us assume that

$$f(t) = \sin^2\omega t$$

Now we compute the derivative of $f$ as

$$f^{'}(t) = 2\sin\omega t\cos\omega t\omega = \omega\sin 2\omega t.$$

Using the derivative theorem we have

$$L[f^{'}(t)] = -f(0) + sL[f(t)]$$

Substituting the function $f(t)$ and its derivative we find

$$L[\omega \sin 2\omega t] = sL[\sin^2 \omega t] - 0$$

Therefore, we have

$$L[\sin^2 \omega t] = \frac{\omega}{s}\left(\frac{2\omega}{s^2 + 4\omega^2}\right)$$

### 5.2.2 Problem 2

*Using derivative theorem, find $L[t^n]$.*

**Solution:** Let

$$f(t) = t^n.$$

Then

$$f^{'}(t) = nt^{n-1}, \;\; f^{''}(t) = n(n-1)t^{n-2}, \ldots, \;\; f^n(t) = n!.$$

From derivative theorem we have

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f^{'}(0) - ... - f^{n-1}(0).$$

Therefore, we find

$$L[n!] = s^n L[t^n] \Rightarrow L[t^n] = \frac{n!}{s^{n+1}}.$$

### 5.2.3 Problem 3

*Using derivative theorem, find $L[\sin kt]$.*

**Solution:** Let $f(t) = \sin kt$ and therefore we have

$$f'(t) = k \cos kt \quad \text{and} \quad f''(t) = -k^2 \sin kt$$

Substituting in the derivative theorem

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

yields

$$L[-k^2 \sin kt] = s^2 L[\sin kt] - 0 - k$$

On simplifications we get

$$L[\sin kt] = \frac{k}{s^2 + k^2}$$

### 5.2.4 Problem 4

*Using $L[t^2] = 2/s^3$ and derivative theorem, find $L[t^5]$.*

**Solution:** Let $f(t) = t^5$ so that $f'(t) = 5t^4, \quad f''(t) = 20t^3 \quad f'''(t) = 60t^2$. The derivative theorem for third derivative reads as

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

This implies

$$L[60t^2] = s^3 L[f(t)] \quad \Rightarrow \quad L[f(t)] = \frac{120}{s^6}.$$

### 5.2.5 Problem 5

*Using the Laplace transform of $L\left[\sin\sqrt{t}\right]$ and applying the derivative theorem, find the Laplace transform of the function*

$$\frac{\cos\sqrt{t}}{\sqrt{t}}$$

**Solution:** We know that

$$L\left[\sin\sqrt{t}\right] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

Let $f(t) = \sin\sqrt{t}$, then we have

$$f(0) = 0 \text{ and } f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Substitution of $f(t)$ in the derivative theorem

$$L[f'(t)] = sL[f(t)] - f(0)$$

yields

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = s\frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

Thus, we get

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

## 5.3 Laplace Transform of Integrals

### 5.3.1 Theorem

*Suppose $f(t)$ is piecewise continuous on $[0,\infty)$ and the function*

$$g(t) = \int_0^t f(u)\,\mathrm{d}u$$

*is of exponential order. Then*

$$L[g(t)] = \frac{1}{s}F(s).$$

**Proof:** Clearly $g(0) = 0$ and $g^{'}(t) = f(t)$. Note that $g(t)$ is piecewise continuous and is of exponential order as well as $g^{'}(t) = f(t)$ is piecewise continuous. Then, we get using the derivative theorem

$$L[g^{'}(t)] = sL[g(t)] - g(0)$$

Since $g(0) = 0$ we obtain the desired result as

$$L[g(t)] = \frac{1}{s}L[f(t)]$$

This completes the proof. ■

## 5.4 Example Problems

### 5.4.1 Problem 1

*Given that*

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{1+s^2}\,\mathrm{d}s.$$

*Find the Laplace transform of the integral*

$$\int_0^t \frac{\sin u}{u}\,\mathrm{d}u.$$

**Solution:** Direct application of the above result gives

$$L\left[\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right] = \frac{1}{s}L\left[\frac{\sin t}{t}\right]$$
$$= \frac{1}{s}\int_s^\infty \frac{1}{1+s^2}\,\mathrm{d}s = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

Thus, we have

$$L\left[\int_0^t \frac{\sin u}{u}\,\mathrm{d}u\right] = \frac{1}{s}\cot^{-1} s$$

### 5.4.2 Problem 2

*Find Laplace transform of the following integral*

$$\int_0^t u^n e^{-au} du$$

**Solution:** With the application of the first shifting theorem we know that

$$L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$$

It follows from the above result on Laplace transform of integrals

$$L\left[\int_0^t u^n e^{-au} du\right] = \frac{1}{s}L[t^n e^{-at}] = \frac{n!}{s(s+a)^{n+1}}.$$

## 5.5 Multiplication by $t^n$

### 5.5.1 Theorem

*If $F(s)$ is the Laplace transform of $f(t)$, i.e., $L[f(t)] = F(s)$ then,*

$$L[tf(t)] = -\frac{d}{ds}F(s)$$

*and in general the following result holds*

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s).$$

**Proof:** By definition we know

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Using Leibnitz rule for differentiation under integral sign we obtain

$$\frac{dF(s)}{ds} = \int_0^\infty (-t)e^{-st} f(t)\,\mathrm{d}t$$

Thus we get

$$\frac{dF(s)}{ds} = -L[tf(t)]$$

Repeated differentiation under integral sign gives the general rule. ∎

Applicability of the above result will now be demonstrated by some examples.

## 5.6 Example Problems

### 5.6.1 Problem 1

*Find Laplace transform of the function* $t^2 \cos at$.

**Solution:** We know from Laplace transform of elementary functions that

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

Direct application of the above rule gives

$$L\left[t^2 \cos at\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2 + a^2}\right) = \frac{d}{ds}\left(\frac{s^2 + a^2 - 2s^2}{\left(s^2 + a^2\right)^2}\right) = \frac{d}{ds}\left(\frac{a^2 - s^2}{\left(s^2 + a^2\right)^2}\right)$$

On simplifications we find

$$L\left[t^2 \cos at\right] = \frac{2s\left(s^2 - 3a^2\right)}{\left(s^2 + a^2\right)^3}$$

### 5.6.2 Problem 2

*Evaluate (i)* $L[te^{-t}]$ *(ii)* $L[t^2e^{-t}]$ *(iii)* $L[t^ke^{-t}]$

**Solution:** (i) We know that

$$L[e^{-t}] = \frac{1}{s+1}$$

Using the above mentioned rule we find

$$L[te^{-t}] = -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

(ii) Applying the same idea once again, we obtain

$$L[t^2e^{-t}] = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

(iii) Similarly, we can further generalize this result as

$$L[t^ke^{-t}] = \frac{k!}{(s+1)^{k+1}}$$

### 5.6.3 Problem 3

*Find the Laplace transform of* $f(t) = (t^2 - 3t + 2)\sin t$

**Solution:** Using linearity of the Laplace transform we have

$$L[f(t)] = L[t^2 \sin t] - 3L[t \sin t] + 2L[\sin t] \tag{5.1}$$

Since we know

$$L[\sin t] = \frac{1}{1+s^2}$$

then

$$L[t \sin t] = -\frac{d}{ds}\frac{1}{1+s^2} = \frac{2s}{(1+s^2)^2}$$

and

$$L[t^2 \sin t] = -\frac{d}{ds}\frac{2s}{(1+s^2)^2} = \frac{2(1+s^2)^2 - 8s^2(1+s^2)}{(1+s^2)^4} = \frac{6s^2-2}{(1+s^2)^3}$$

Substituting the above values in the equation (5.1), we find

$$L[f(t)] = \frac{6s^2-2}{(1+s^2)^3} - \frac{6s}{(1+s^2)^2} + \frac{2}{1+s^2}$$

Further simplifications lead to

$$L[f(t)] = \frac{6s^2 - 2 - 6s(1+s^2) + 2(1+s^2)^2}{(1+s^2)^3}$$

Finally, we obtain

$$L[f(t)] = \frac{(2s^4 - 6s^3 + 10s^2 - 6s)}{(s^6 + 3s^4 + 3s^2 + 1)}$$

## 5.7 Division by $t$

### 5.7.1 Theorem

*If $f$ is piecewise continuous on $[0, \infty)$ and is of exponential order $\alpha$ such that*

$$\lim_{t \to 0+} \frac{f(t)}{t}$$

*exists, then,*

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u, \quad [s > \alpha]$$

**Proof:** This can easily be proved by letting $g(t) = \dfrac{f(t)}{t}$ so that $f(t) = tg(t)$.

Hence,

$$F(s) = L[f(t)] = L[tg(t)] = -\frac{d}{ds}L[g(t)]$$

Integrating with respect to $s$ we get,

$$-L[g(t)]\Big|_s^\infty = \int_s^\infty F(s)\,\mathrm{d}s.$$

Since $g(t)$ is piecewise continuous and of exponential order, it follows that $\lim_{s\to\infty} L[g(t)] \to 0$. Thus we have

$$L[g(t)] = \int_s^\infty F(s)\,\mathrm{d}s.$$

This completes the proof. ■

**Remark:** *It should be noted that the condition $\lim_{t\to 0+} [f(t)/t]$ is very important because without this condition the function $g(t)$ may not be piecewise continuous on $[0, \infty)$. Thus without this condition we can not use the fact $\lim_{s\to\infty} L[g(t)] \to 0$.*

### 5.7.2 Corollary

*If* $L[f(t)] = F(s)$ *then* $\displaystyle\int_0^\infty \frac{f(t)}{t}\, dt = \int_0^\infty F(s)\, ds$*, provided that the integrals converge.*

**Proof:** We know that

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,\mathrm{d}u$$

Using the definition of Laplace transform we get

$$\int_0^\infty e^{-st}\frac{f(t)}{t}\,\mathrm{d}t = \int_s^\infty F(u)\,\mathrm{d}u$$

Taking limit $s \to 0$ in above two integrals we obtain

$$\int_0^\infty \frac{f(t)}{t}\,\mathrm{d}t = \int_0^\infty F(u)\,\mathrm{d}u$$

This completes the proof. ■

## 5.8 Example Problems

### 5.8.1 Problem 1

*Find the Laplace transform of the function*

$$f(t) = \frac{\sin at}{t}$$

**Solution:** We know,

$$L[\sin at] = \frac{a}{s^2+a^2} \quad \text{and} \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,\mathrm{d}s$$

On integrating we get,

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,\mathrm{d}s = \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

Thus we have

$$L\left[\frac{\sin at}{t}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

### 5.8.2 Problem 2

*Find the Laplace transform of the function*

$$f(t) = \frac{2\sin t \sinh t}{t}$$

**Solution:** Using Division by $t$ property of the Laplace transform we get

$$L[f(t)] = \int_s^\infty L\left[\sin t\left(e^t - e^{-t}\right)\right]\,\mathrm{d}s \tag{5.2}$$

Now we evaluate $L\left[\sin t\left(e^t - e^{-t}\right)\right]$ using linearity of the Laplace transform as

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = L\left[e^t \sin t\right] - L\left[e^{-t}\sin t\right]$$

Applying the first shifting theorem we obtain

$$L\left[\sin t\left(e^t - e^{-t}\right)\right] = \frac{1}{1+(s-1)^2} - \frac{1}{1+(s+1)^2}$$

Substituting this value in the equation (5.2) we find

$$L[f(t)] = \int_s^\infty \left[\frac{1}{1+(s-1)^2} - \frac{1}{1+(s+1)^2}\right]\mathrm{d}s$$

On integrating, we have

$$\begin{aligned} L[f(t)] =& \tan^{-1}(s-1)\Big|_s^\infty - \tan^{-1}(s+1)\Big|_s^\infty \\ =& \frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1) \end{aligned}$$

On cancellation of $\pi/2$ we get

$$L[f(t)] = \tan^{-1}(s+1) - \tan^{-1}(s-1)$$

This can be further simplified to obtain

$$L[f(t)] = \tan^{-1}\left(\frac{2}{s^2}\right)$$