$$\lim_{n\to\infty} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi}{l} x\right) dx = 0$$

$$\lim_{n\to\infty} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx = 0,$$

This is called Riemann's theorem.

## 13. Gibbs Phenomenon :-

To discuss the Gibbs phenomenon, let us consider the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 \le x < \pi . \end{cases}$$

The graph of $f(x)$ is given in Fig 1.

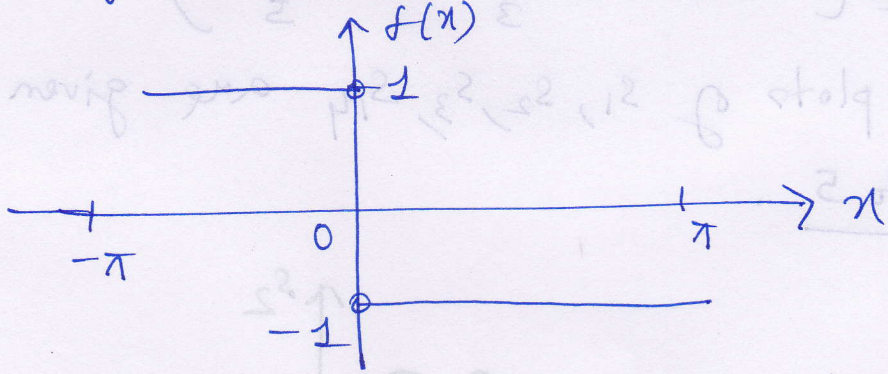


Fig 1 :- Graph of $f(x)$

The function is odd. Therefore we have a sine series. We have —

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{odd} \cdot \underbrace{\sin(nx)}_{odd} dx \quad (= even)$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx)\, dx .$$

$= -\frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = \frac{2}{\pi}\left[\cos nx\right]_0^{\pi} = \frac{2}{\pi}\left[\cos n\pi - 1\right]$.

$= \frac{2}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ -\left(\frac{4}{n\pi}\right), & n \text{ odd}. \end{cases}$

$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ $\quad \left[ = \sum_{n=odd}\left(-\frac{4}{n\pi}\right)\sin(nx) = -\frac{4}{\pi}\sum_{n=odd}\left\{\frac{\sin(nx)}{n}\right\}. \right.$

$= -\frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

We denote the partial sums of the series as $s_1 = -\frac{4}{\pi}\sin x$, $s_2 = -\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3}\right)$,

$s_3 = -\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right) + \cdots$.

The typical plots of $s_1, s_2, s_3, s_4$ are given in Fig 2 to 5.

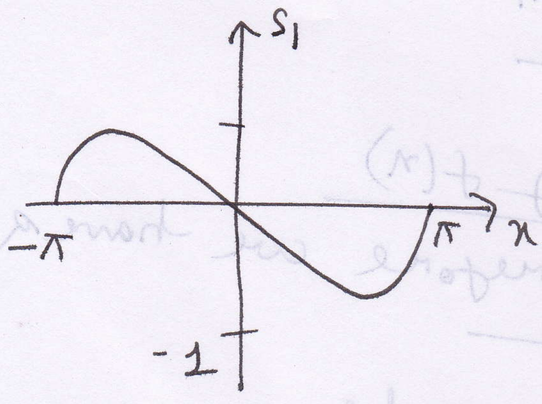


Fig 2:- (Typical plot of $s_1(x)$)

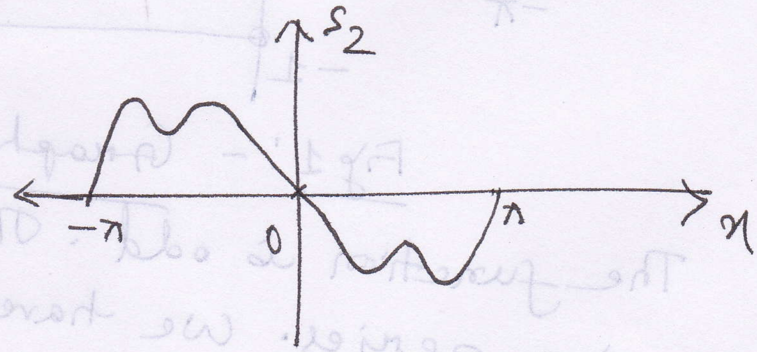


Fig 3 :- Typical plot of $s_2(x)$

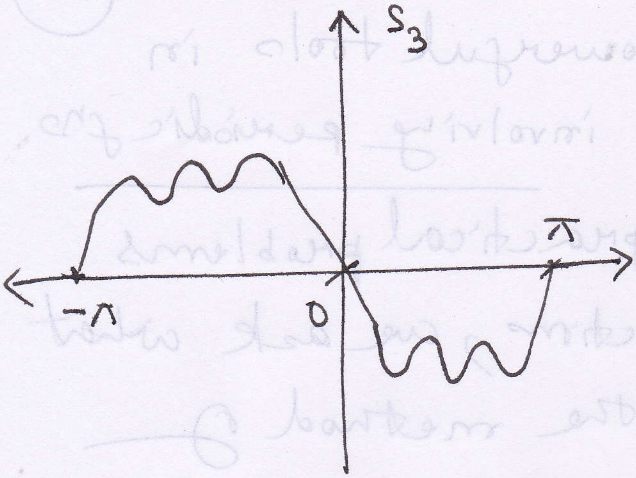


Fig 4 :- (Typical plot of $S_3(x)$)

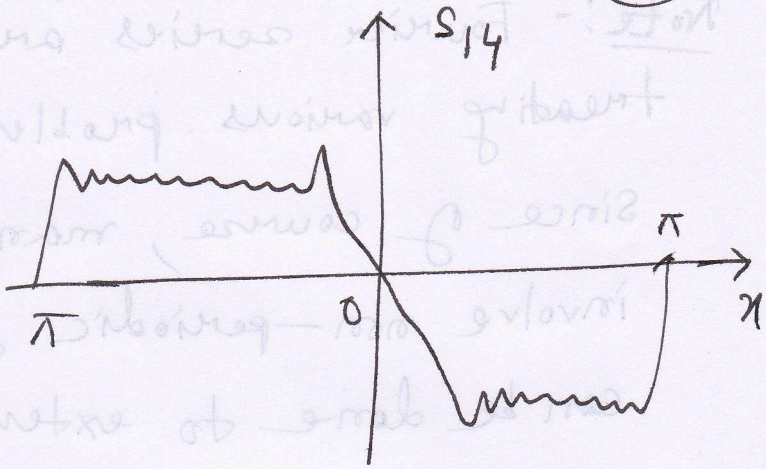


Fig 5 :- (Typical plot of $S_{14}(x)$)

It can be observed that the graph of $S_{14}(x)$ displays spikes near the discontinuities at $x = -\pi, 0$ & $\pi$.

This oscillatory behaviour of the partial sums $S_n$ for large $n$, about the true value near a point of discontinuity does not smoothen out even for very large $n$. This behaviour of the Fourier series near a point of discontinuity is called the Gibbs phenomenon.

**Note**:- Fourier series are powerful tools in treating various problems involving periodic $f^{ns}$. Since of course, many practical problems involve non-periodic functions, we ask what can be done to extend the method of Fourier series to such functions.

First we begin with a special function $f_L(x)$ of period $2L$ & see what happens to its Fourier series if we let $L \to \infty$. Then we consider the Fourier series of an arbitrary function $f_L$ of period $2L$ & again let $L \to \infty$.

**Eg-1)** **Square wave**:-

Let us consider the periodic square wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1 \\ 1 & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < L, \end{cases}$$

& $f(x) = \lim\limits_{L \to \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise}. \end{cases}$

Fig 1 shows this function for $2L = 4, 8, 16$ as well as the non-periodic function $f(x)$, which we obtain from $f_L$ if we let $L \to \infty$, we now explore what happens to the Fourier co-efficients of $f_L$ as $L$ increases.

Since $f_L$ is even (why?), $b_n = 0$ for all $n$. For $a_n$, the Euler formulas give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \quad a_n = \frac{1}{L}\int_{-1}^{1} \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^1 \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$= \frac{2}{L}\cdot\frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^1 = \frac{2}{L}\cdot\frac{\sin(n\pi/L)}{(n\pi/L)},$$

This sequence of Fourier co-efficients is called the amplitude spectrum of $f_L$ because $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$

$\therefore |a_n| \to 0$ as $L \to \infty$

Wave form $f_L(x)$

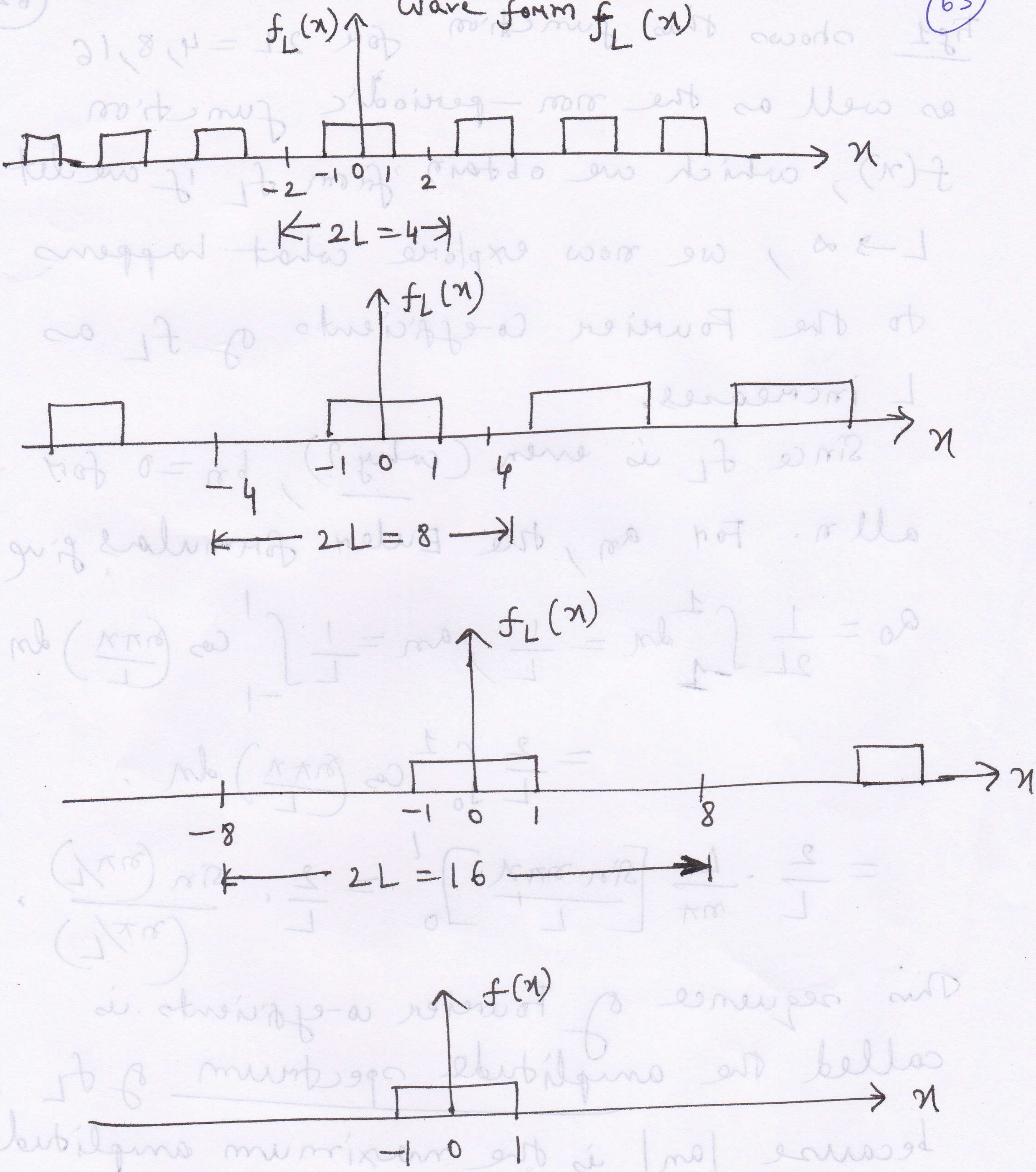


Fig 1 :- Wave forms

# 14. From Fourier Series to the Fourier Integral

We now consider any periodic function $f_L(x)$, of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x), \quad \omega_n = \frac{n\pi}{L}$$

& find out what happens if we let $L \to \infty$. Together with our example, the present calculation will suggest that we should expect an integral (instead of a series) involving $\cos \omega x$ & $\sin \omega x$ with $\omega$ no longer restricted to integer multiples, $\omega = \omega_n = \frac{n\pi}{L}$ of $\pi/L$ but taking all values.

We shall also see, what form such an integral might have.

If we insert $a_n$ & $b_n$, & denote the variable of integration by $v$, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos \omega_n x \int_{-L}^{L} f_L(v) \cos \omega_n(v)$$

$$+ \sin \omega_n x \int_{-L}^{L} f_L(v) \sin \omega_n v\, dv\Big]$$

We now let

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \pi/L.$$

Then $\frac{1}{L} = \frac{\Delta\omega}{\pi}$, and we may write the Fourier series in the form

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos \omega_n x)\,\Delta\omega \int_{-L}^{L} f_L(v)\cos(\omega_n v)\,dv$$

$$+ (\sin \omega_n x)\,\Delta\omega \int_{-L}^{L} f_L(v)\sin(\omega_n v)\,dv\Big]$$

This representation is valid $\longrightarrow$ (1) for any fixed L arbitrarily large, but finite.

We now let $L \to \infty$ & assume that the resulting non-periodic function

$$f(x) = \underset{L\to\infty}{Lt}\, f_L(x) \longrightarrow (2)$$

is absolutely integrable on the $x$-axis, that is the following (finite!) limits exist:-

$$\lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx$$

(written as $\int_{-\infty}^{\infty} |f(x)\,dx$). $\longrightarrow$ (2)

Then $1/L \to 0$ & if $\int_{-\infty}^{\infty} f_L(v)\,dv$ exists then the value of the first term on the R.H.S of (1) approaches zero. Also, $\Delta\omega = \pi/L \to 0$ & it seems plausible that the infinite series in (1) becomes an integral from 0 to $\infty$, which represents $f(x)$, namely

i.e., $\lim_{\Delta\omega\to 0}\sum_{n=1}^{\infty} F(\omega_n)\,\Delta\omega$ is suggestive of the definition of the integral $\int_0^{\infty} F(\alpha)\,d\omega$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos\omega x\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv + \sin\omega x\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv\right]d\omega$$

$\longrightarrow$ (3)

If we introduce the notations

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv,$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)\,dv,$$

$\longrightarrow$ (4)

we can write this in the form

$$f(x) = \int_0^{\infty}\boxed{A(\omega)\cos\omega x + B(\omega)\sin\omega x}\,d\omega$$

$\longrightarrow$ (5)

This is called a representation of $f(x)$ by a Fourier integral.

It is clear that our naive approach merely suggests the representation (5), but by no means establishes it, in fact, the limit of the series in (1) as $\Delta\omega$ approaches zero is not the defn of the integral (3)

Sufficient condns for the validity of (5) are as follows.

—

Th-1/ (Fourier Integral) 

If $f(x)$ is piecewise continuous in every finite interval & has a R.H. Derivative & a L.H. Derivative at every point & if the integral ② exists, then $f(x)$ can be represented by a Fourier integral ⑤ with A & B given by ④. At a point where $f(x)$ is discontinuous, the value of the Fourier integral equals the average of the left- & right-hand limits of $f(x)$ at that point.

· Equivalent forms of Fourier Integral

Theorem :-

Fourier's integral theorem can also be written in the forms

$$f(x) = \frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(x-u)}\,du\,d\alpha$$

(complex form of Fourier Integral)

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\alpha)\, e^{-i\alpha u}\, d\alpha,$$

$$c(\alpha) = \int_{-\infty}^{\infty} f(u)\, e^{i\alpha x}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du,$$

where it is understood that if $f(x)$ is not continuous at $x$, the left side must be replaced by

$$\frac{f(x+0)+f(x-0)}{2}.$$

These results can be simplified somewhat if $f(x)$ is either odd or even function.

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\alpha x\, d\alpha \int_0^{\infty} f(u)\sin\alpha u\, du \text{ if } f(x) \text{ is odd}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\alpha x\, d\alpha \int_0^{\infty} f(u)\cos\alpha u\, du \text{ if } f(x) \text{ is even}$$

Note :- Fourier integrals are used mainly in solving ODeq$^n$ & PD eq$^n$s. However, they also help in evaluating integrals as well.

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## 15. Fourier Cosine & Sine Integrals :-

For an even or odd function, the Fourier integral becomes simpler, just as in the case of Fourier series.

Indeed if $f(x)$ is an even function, then $B(\omega) = 0$ in eq$^n$ (4) &

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\,dv.$$

The Fourier integral (5) then reduces to the Fourier Cosine integral

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega \quad (\underline{f \text{ even}}).$$

Similarly, if $f(x)$ is odd, then in (4), we have

$$A(\omega) = 0 \quad \& \quad B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin(\omega v)\,dv.$$

The Fourier integral (5) then reduces to the Fourier sine integral

$$f(x) = \int_0^\infty B(\omega)\sin(\omega x)\,d\omega \quad (\underline{f \text{ odd}})$$

**Eg. 1)** Find the Fourier cosine & sine integrals of $f(x) = e^{-kx}$ $(x>0,\ k>0)$.

**Ans:-** (a)

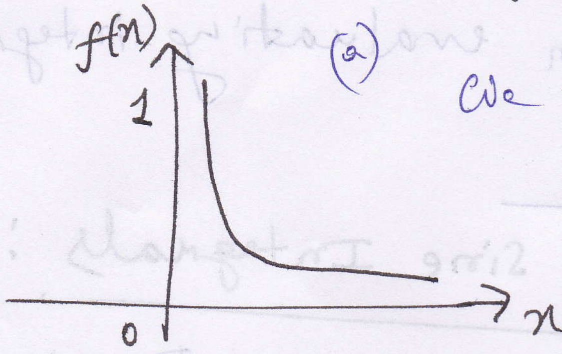



Fig: $f(x)$

We have, $A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\,dv$

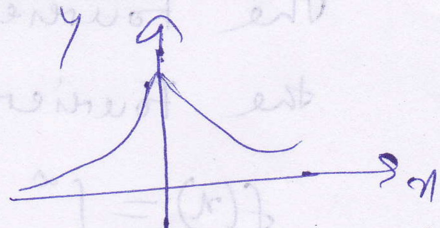
$= \frac{2}{\pi}\int_0^\infty e^{-kv}\cos(\omega v)\,dv.$

Now, integrating by parts, we obtain

$$\int e^{-kv}\cos(\omega v)\,dv = \frac{-k}{(k^2+\omega^2)}\, e^{-kv}\left(-\frac{\omega}{k}\sin\omega v + \cos\omega v\right)$$

If $v=0$, the expression on the right equals $\frac{-k}{(k^2+\omega^2)}$, if $v$ approaches infinity, it approaches zero because of the exponential factor.

Thus, $A(\omega) = \frac{2k/\pi}{(k^2+\omega^2)}$




Cosine integral

$\therefore$ the Fourier cosine integral representation is

$$f(x) = e^{-kx} = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega$$

$$= \frac{2k}{\pi}\int_0^\infty \frac{\cos\omega x}{(k^2+\omega^2)}\,d\omega \quad (x>0,\ k>0)$$

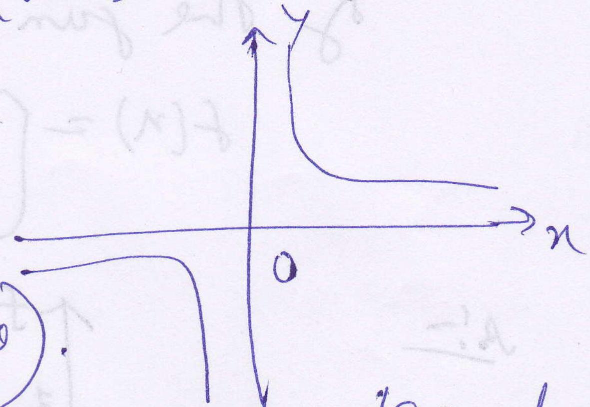
$$\Rightarrow \int_0^\infty \frac{\cos\omega x}{(k^2+\omega^2)}\,d\omega = \frac{\pi}{2k}e^{-kx} \quad (x>0, k>0) \longrightarrow (1)$$

(b) Similarly, we have

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin(\omega v)\,dv$$

$$= \frac{2}{\pi}\int_0^\infty e^{-kv}\sin(\omega v)\,dv$$

Integration by parts, we have



$$\int e^{-kv}\sin(\omega v)\,dv$$

$$= -\frac{\omega}{(k^2+\omega^2)}\,e^{-kv}\left(k/\omega \sin\omega v + \cos\omega v\right).$$

sine Integral

This equals

$-\frac{\omega}{k^2+\omega^2}$ if $v = 0$ & approaches to 0 as $v \to \infty$.

Thus, $B(\omega) = \dfrac{2\omega/\pi}{k^2+\omega^2}$

∴ The Fourier sine integral representation

$$f(x) = e^{-kx} = \int_0^\infty B(\omega)\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\omega\sin(\omega x)}{(k^2+\omega^2)}\,d\omega$$

$$\Rightarrow \int_0^\infty \frac{\omega\sin(\omega x)}{k^2+\omega^2}\,d\omega = \frac{\pi}{2}e^{-kx} \quad (x>0,\ k>0) \longrightarrow (2)$$

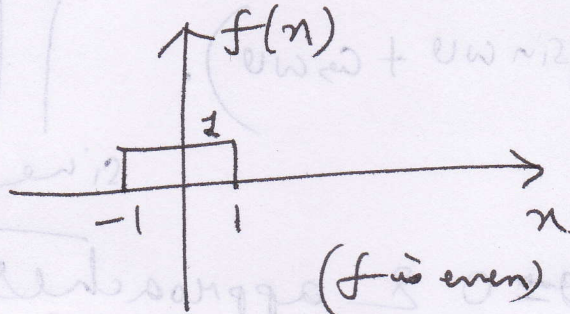
The above integrals (i.e., (1) & (2)) are called the Laplace integrals.

Eg:2/ Single pulse, sine integral

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$

A:-




(f is even)

We have, $A(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv$

$$= \frac{1}{\pi}\int_{-1}^{1} \cos(\omega v)\,dv = \left[\frac{\sin(\omega v)}{\pi\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega},$$

$$B(\omega) = \frac{1}{\pi}\int_{-1}^{1} \sin(\omega v)\,dv = 0.$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos\omega x \sin\omega}{\omega}\,d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\cos\omega x \sin\omega}{\omega}\,d\omega = \frac{\pi}{2} f(x) \rightarrow (1)$$

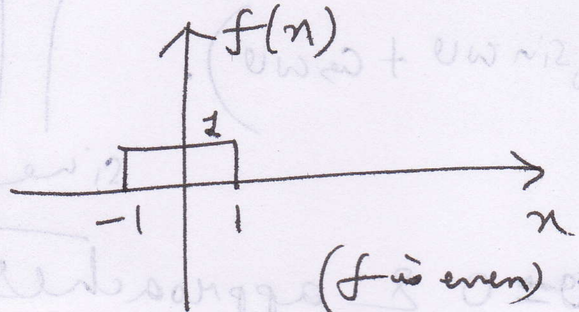
The average of the left & right-hand limits of $f(x)$ at $x = 1$ is equal to $\dfrac{(1+0)}{2} = \dfrac{1}{2}$.

Eg:2/ Single pulse, sine integral

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$

A:–




(f is even)

We have, $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv$

$$= \frac{1}{\pi}\int_{-1}^{1} \cos(\omega v)\,dv = \left[\frac{\sin(\omega v)}{\pi\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega},$$

$$B(\omega) = \frac{1}{\pi}\int_{-1}^{1} \sin(\omega v)\,dv = 0.$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos\omega x \sin\omega}{\omega}\,d\omega$$

$$\Rightarrow \int_{0}^{\infty} \frac{\cos\omega x \sin\omega}{\omega}\,d\omega = \frac{\pi}{2} f(x) \longrightarrow (1)$$

The average of the left & right-hand limits of $f(x)$ at $x = 1$ is equal to $\frac{(1+0)}{2} = \frac{1}{2}$.

$\therefore$ from (1) & Th-1 (Fourier integral), we get

$$\int_0^\infty \frac{\cos\omega x \sin\omega}{\omega}\, d\omega = \begin{cases} \frac{\pi}{2}\cdot 1 = \frac{\pi}{2} & \text{if } 0 \le x < 1 \\ \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4} & \text{if } x = 1 \\ \frac{\pi}{2}\cdot 0 = 0, & \text{if } x > 1. \end{cases}$$

This integral is called Dirichlet's discontinuous factor. Let $x = 0$, then

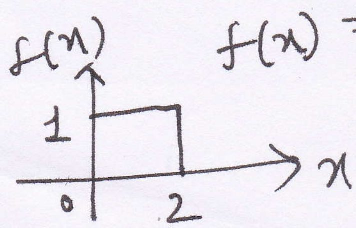
$$\int_0^\infty \frac{\sin\omega}{\omega}\, d\omega = \pi/2.$$

We see that this integral is the limit of the so-called sine integral,

$$Si(u) = \int_0^u \frac{\sin\omega}{\omega}\, d\omega \quad \text{as } u \to \infty.$$

[Note:- Periodic functions are represented by a Fourier series & non-periodic functions are represented in the form of an integral i.e., by Fourier integral.]

Eg.2) Find the Fourier integral representation of the piece-wise continuous function




$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

Hint:- $A(\alpha) = \frac{\sin 2\alpha}{\alpha}$

$B(\alpha) = \frac{1 - \cos 2\alpha}{\alpha}$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha \cos[\alpha(x-1)]}{\alpha}\, d\alpha$$