In this lesson some properties of the Fourier coefficients will be given. We will mainly derive two important inequalities related to Fourier series, in particular, Bessel's inequality and Parseval's identity. One of the applications of Parseval's identity for summing certain infinite series will be discussed.

8.1 Theorem (Bessel's Inequality)

If f be a piecewise continuous function in $[-\pi, \pi]$, then

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, \mathrm{d}x$$

where $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ are Fourier coefficients of f.

Proof: Clearly, we have

$$\int_{-\pi}^{\pi} \left[f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \right]^2 \mathrm{d}x \geq 0$$

Expanding the integrands we get

$$\int_{-\pi}^{\pi} f^2(x)\, \mathrm{d}x + \frac{a_0^2}{2}\pi + \int_{-\pi}^{\pi} \left[\sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \right]^2 \mathrm{d}x - a_0 \int_{-\pi}^{\pi} f(x)\, \mathrm{d}x$$
$$- 2 \int_{-\pi}^{\pi} f(x) \left[\sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \right] \mathrm{d}x + a_0 \int_{-\pi}^{\pi} \left[\sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \right] \mathrm{d}x \geq 0$$

Using the orthogonality of the trigonometric system and definition of Fourier coefficients we get

$$\int_{-\pi}^{\pi} f^2(x)\, \mathrm{d}x + \frac{a_0^2}{2}\pi + \pi \sum_{k=1}^{n} \left(a_k^2 + b_k^2\right) - a_0^2\pi - 2\pi \sum_{k=1}^{n} \left(a_k^2 + b_k^2\right) + 0 \geq 0$$

This can be further simplified

$$\int_{-\pi}^{\pi} f^2(x)\, \mathrm{d}x - \frac{a_0^2}{2}\pi - \pi \sum_{k=1}^{n} \left(a_k^2 + b_k^2\right) \geq 0$$

This implies

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} \left(a_k^2 + b_k^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, \mathrm{d}x$$

Passing the limit $n \to \infty$, we get the required Bessel's inequality. ∎

Indeed the above Bessel's inequality turns into an equality named Parseval's identity. However, for the sake of simplicity of proof we state the following theorem for more restrictive function but the result holds under less restrictive conditions (only piecewise continuity) same as in Theorem 8.1.

8.2 Theorem (Parseval's Identity)

If f is a continuous function in $[-\pi, \pi]$ and one sided derivatives exit then we have the equality

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x \tag{8.1}$$

where $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ are Fourier coefficients of f.

Proof: From the Dirichlet's convergence theorem for $x \in (-\pi, \pi)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

Integrating by $f(x)$ and integrating term by term from $-\pi$ to π we obtain

$$\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{a_0}{2} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\right)$$

Using the definition of Fourier coefficients we get

$$\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{\pi a_0^2}{2} + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

Dividing by π we obtain the required identity. ∎

Remark: *As stated earlier Parseval's identity can be proved for piecewise continuous functions. Further, for a piecewise continuous function on $[-L, L]$ we can get Parseval's identity just by replacing π by L in (8.1).*

8.3 Example Problems

8.3.1 Problem 1

Consider the Fourier cosine series of $f(x) = x$:

$$x \sim 1 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}[\cos(n\pi) - 1] \cos \frac{n\pi x}{2}$$

a) Write Parseval's identity corresponding to the above Fourier series

b) Determine from a) the sum of the series

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

Solution: a) We first find the Fourier coefficient and the period of the Fourier series just by comparing the given series with the standard Fourier series

$$a_0 = 2, \quad a_n = \frac{4}{\pi^2 n^2}[\cos(n\pi) - 1],\, n = 1, 2 \dots, \quad b_n = 0$$
$$\text{period } = 2L = 4 \Rightarrow L = 2$$

Writing Parseval's identity as

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

This implies

$$\frac{1}{2}\int_{-2}^{2} x^2\,\mathrm{d}x = \frac{4}{2} + \sum_{n=1}^{\infty} \frac{16}{\pi^4 n^4}(\cos(n\pi) - 1)^2$$

This can be simplified to give

$$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

Then we obtain

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

b) Let

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

This series can be rewritten as

$$\begin{aligned} S &= \Big(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\Big) + \Big(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\Big) \\ &= \frac{\pi^4}{96} + \frac{1}{2^4} S \end{aligned}$$

Then we have the required sum as $S = \dfrac{\pi^4}{90}$.

8.3.2 Problem 2

Find the Fourier series of x^2, $-\pi < x < \pi$ and use it along with Parseval's theorem to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Solution: Since $f(x) = x^2$ is an even function, so $b_n = 0$. The Fourier coefficients a_n will be given as

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\,\mathrm{d}x = \frac{2}{\pi} \int_0 \pi x^2 \cos(nx)\,\mathrm{d}x$$

This can be further simplified for $n \neq 0$ to

$$a_n = \frac{2}{\pi} \left[0 - \frac{2}{n} \int_0^{\pi} x \sin(nx)\,\mathrm{d}x \right] = \frac{4}{n^2}(-1)^n$$

The coefficient a_0 can be evaluated separately as

$$a_0 = \frac{2}{\pi} \int_0 \pi x^2\,\mathrm{d}x = \frac{2\pi^2}{3}$$

The the Fourier series of $f(x) = x^2$ will be given as

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

Now by parseval's theorem we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

Using $\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 \,\mathrm{d}x = \frac{2\pi^4}{5}$ we get

$$\frac{4\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{2\pi^4}{5}$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Now using the idea of splitting of the series from the Example 8.3.1 (b), we have

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \sum_{n=1}^{\infty} \frac{1}{n^4} - \frac{1}{16}\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{15}{16}\sum_{n=1}^{\infty} \frac{1}{n^4}$$

Substituting the value of $\sum_{k=1}^{\infty} \frac{1}{n^4}$ in the above equation we get the required sum.

8.3.3 Problem 3

Given the Fourier series

$$\cos\left(\frac{x}{2}\right) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2-1)}\cos(nx)$$

deduce the value of

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}.$$

Solution: By Parseval's theorem for

$$a_0 = \frac{4}{\pi},\ a_n = \frac{4}{\pi}\frac{(-1)^{n+1}}{(4n^2-1)},\ f(x) = \cos(x/2)$$

we have

$$\frac{1}{2}\frac{16}{\pi^2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(x/2)\,\mathrm{d}x = 1$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}.$$