Transform Calculus
Assignment- 7 Autumn, 2020

A. Solve the following elliptic partial differential equations using the Fourier transform technique

1. Dirichlet problem in the upper half plane $y > 0$.
   Solve
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ -\infty < x < \infty, \ y > 0$$
   s.t. $u(x,0) = f(x), \ -\infty < x < \infty$;
   both u and $\frac{\partial u}{\partial x}$ vanish as $|x| \to \infty$;
   and u is bounded as $y \to \infty$.
   ($u(x,y)$ is the potential function)

2. Neumann's problem in the upper half plane $y > 0$.
   Solve
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ -\infty < x < \infty, \ y > 0$$
   s.t. $\frac{\partial u}{\partial y}(x,0) = g(x), \ -\infty < x < \infty$;
   u is bounded $y \to \infty$;
   both u and $\frac{\partial u}{\partial x}$ are bounded as $|x| \to \infty$.

3. Solve
$$\nabla^2 \phi = 0, \quad y > 0$$
   s.t. both $\phi(x,y)$ and $\frac{\partial \phi}{\partial x}(x,y) \to 0$ as $\sqrt{x^2+y^2} \to \infty$;
$$\phi(x,y) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

B. Solve the following parabolic partial differential equations

1. Solve the following heat conduction problem using the Laplace transform technique. $u(x,t)$ denotes the temperature at the location x at any time t.

   (a) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$
   subject to
      i. $u(x,0) = 0, \ \forall \ x$
      ii. $u(0,t) = u_0, \ \forall \ t$
      iii. u is finite $\forall \ x$ and $\forall \ t$.

(b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$
subject to

i. $u(x, 0) = 0, \ \forall \ x$
ii. $u(0, t) = 1, \ \forall \ t$
iii. $\lim_{x \to \infty} u(x, t) = 0, \ \forall \ t.$

2. Solve 1-D heat conduction problem given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$

subject to $u(0, t) = 0, \ \forall \ t$; $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$;
and $u(x, t)$ is bounded $\forall \ x$ and $\forall \ t$
using the Fourier sine transformation technique.

3. Solve

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$

subject to

(a) $\frac{\partial u}{\partial t}(0, t) = u_0, \ \forall \ t$
(b) $u(x, 0) = 0, \ \forall \ x$
(c) $u(x, t)$ is bounded $\forall \ x$ and $\forall \ t$

using the Fourier cosine transform technique.

4. Solve the 1-D heat conduction problem

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

subject to

(a) $u(x, 0) = f(x), \ \forall \ x$
(b) $u(x, t) \to 0$ as $|x| \to \infty$
(c) $u(x, t)$ is bounded $\forall \ x$ and $\forall \ t$

using the Fourier transform technique.
Take $f(x) = \begin{cases} 0, & x < 1 \\ a, & x > 1 \end{cases}$ and obtain the particular solution.

C. Solve the following hyperbolic partial differential equations

1. Solve the 1-D wave propagation equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

subject to

(a) $u(x,0) = 0, \ \frac{\partial u}{\partial t}(x,0) = 0, \quad \forall\ x;$
(b) $u(0,t) = f(t), \quad \forall\ t;$
(c) $u(x,t)$ is bounded $\forall\ x$ and $\forall\ t$

using the Laplace transform technique.

2. Solve the 1-D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

subject to

(a) $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x), \forall\ x;$
(b) both $u(x,t)$ and $\frac{\partial u}{\partial x}(x,t) \to 0$ as $|x| \to \infty$

using the Fourier transform technique.