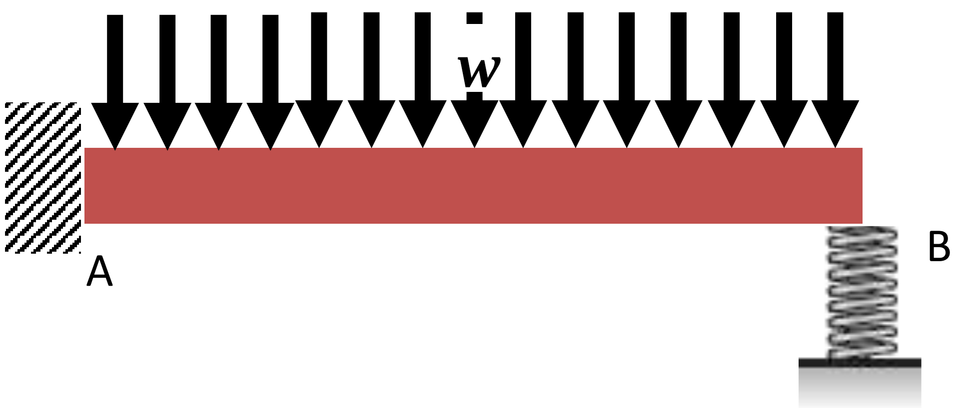
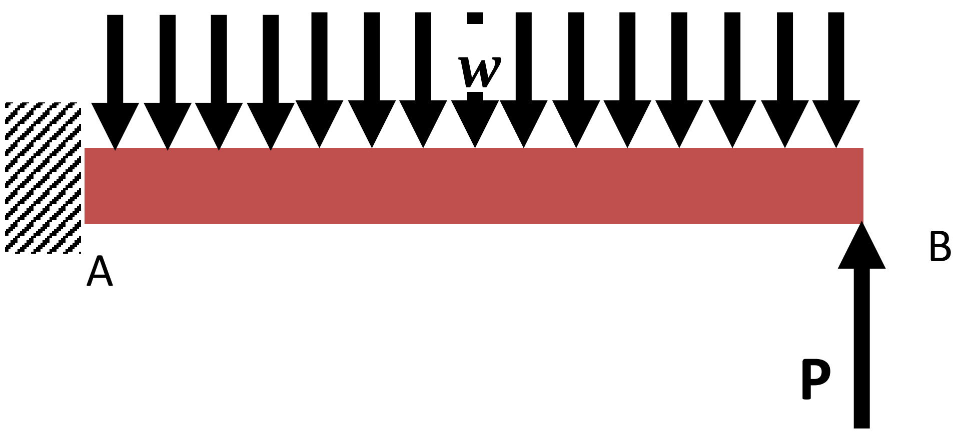
Beam with spring support

- Cantilever with a spring at one end



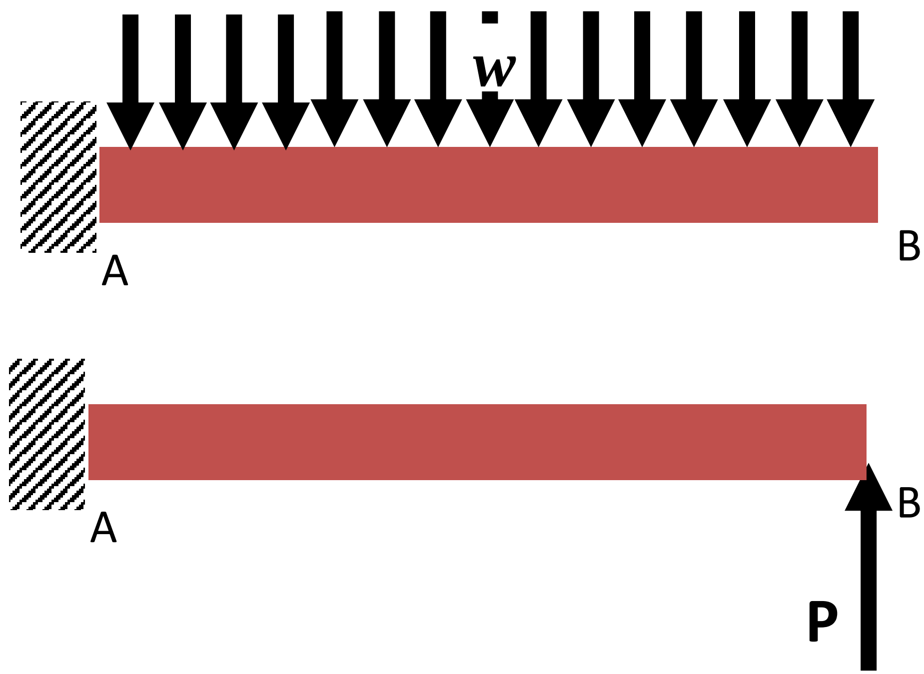
Beam with spring support

- Replace the spring with a force



Beam with spring support

- We will use superposition
- We will split the problem into two problems we have solved before



Beam with spring support

- For the first problem the solutions are as shown below (recall the beam AB of the hinged beam problem)

$$EIv' = -\frac{wx^3}{6} + wLx^2 - \frac{3wL^2x}{2}$$

$$EIv = -\frac{wx^4}{24} + wL\frac{x^3}{3} - \frac{3wL^2x^2}{4}$$

Beam with spring support

- For the second problem the solutions are as shown below (we have to substitute –P for P the cantilever with upward point load at tip solved earlier)

$$EIv' = PLx - \frac{Px^2}{2}$$

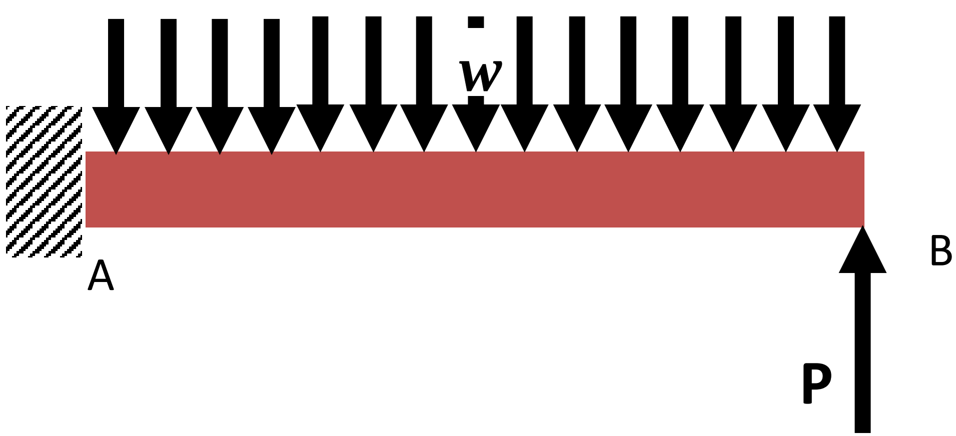
$$EIv = PL\frac{x^2}{2} - \frac{Px^3}{6}$$

Beam with spring support

- Adding the two solutions will give us the solution for the problem shown below

$$EIv' = -\frac{wx^3}{6} + wLx^2 - \frac{3wL^2x}{2} + PLx - \frac{Px^2}{2}$$

$$EIv = -\frac{wx^4}{24} + wL\frac{x^3}{3} - \frac{3wL^2x^2}{4} + PL\frac{x^2}{2} - \frac{Px^3}{6}$$

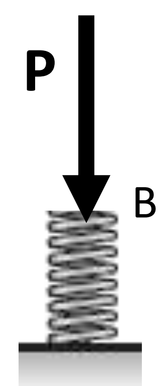


Beam with spring support

- Let us now look at the spring alone
- The spring is being compressed by the force P
- Hence the spring gets compressed by an amount Δ
- This must also be the same as the deflection of the tip of the beam. Hence

$$\Delta = v(x = L) = v(L)$$

$$\Rightarrow P = k\Delta = kv(L)$$



Beam with spring support

- The deflection at the tip is

$$EIv(L) = -\frac{wL^4}{24} + \frac{wL^4}{3} - \frac{3wL^4}{4} + \frac{PL^3}{2} - \frac{PL^3}{6}$$

$$\Rightarrow v(L) = \frac{PL^3}{3EI} - \frac{11wL^4}{24EI}$$

- The compression of the spring is

$$v(L) = \mathit{\Delta} = \frac{P}{k}$$

Beam with spring support

- Equating the deflection of the tip to the compression of the spring will let us solve for P

$$v(L) = \Delta$$

$$\Rightarrow \frac{PL^3}{3EI} - \frac{11wL^4}{24EI} = \frac{P}{k}$$

$$\Rightarrow P\left(\frac{L^3}{3EI} - \frac{1}{k}\right) = \frac{11wL^4}{24EI}$$

$$\Rightarrow P = \frac{11wL^4k}{8L^3k - 24EI}$$

Beam with spring support

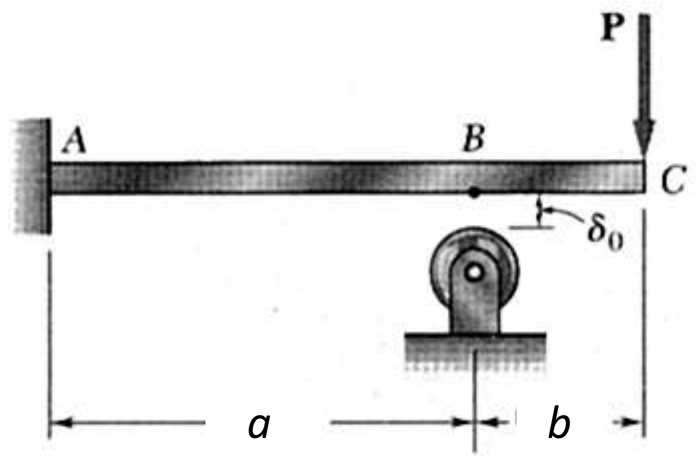
- We can now substitute this expression for P and get the solution for the entire problem

$$v' = \frac{w}{EI}\left\{-\frac{x^3}{6} + \frac{L}{16}\left(\frac{5k - 48\frac{EI}{L^3}}{k - 3\frac{EI}{L^3}}\right)x^2 - \frac{L^2}{8}\left(\frac{k - 36\frac{EI}{L^3}}{k - 3\frac{EI}{L^3}}\right)x\right\}$$

$$v = \frac{w}{EI}\left\{-\frac{x^4}{24} + \frac{L}{48}\left(\frac{5k - 48\frac{EI}{L^3}}{k - 3\frac{EI}{L^3}}\right)x^3 - \frac{L^2}{16}\left(\frac{k - 36\frac{EI}{L^3}}{k - 3\frac{EI}{L^3}}\right)x^2\right\}$$

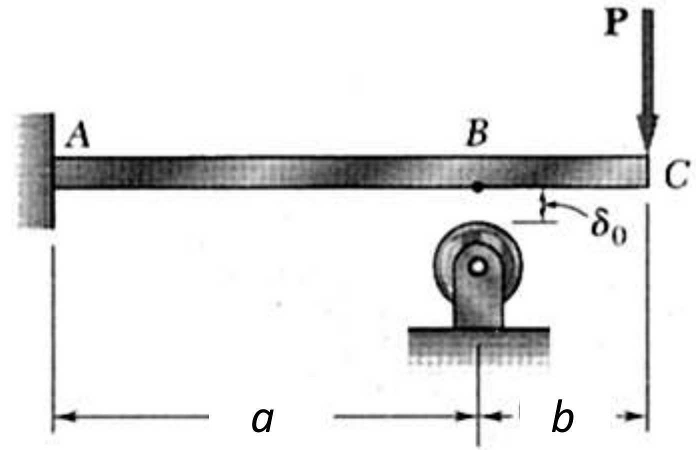
Beam with a gapped support

- *A gap δ_0 is present between the cantilever beam ABC and the support at B at B under no load. Find the deflection at C after the load P is applied.*



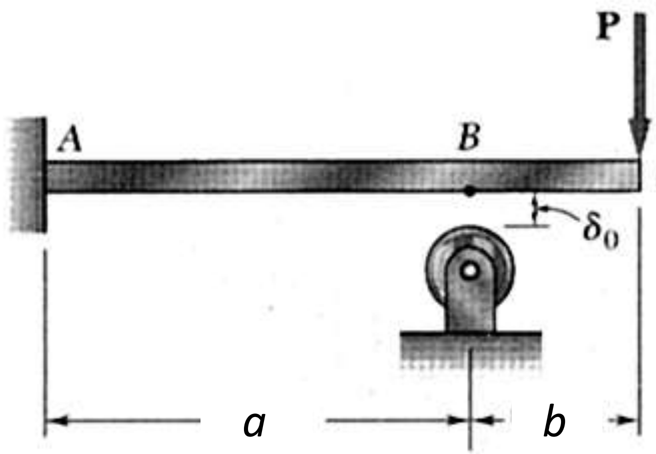
Beam with a gapped support

- *This is again a problem in two parts. If P is not large enough to literally bridge the gap, then there are no worries. Deflection at C will be the usual tip deflection for a cantilever. We can also, if required calculate the deflection at B due to P and subtract that from δ_0 to obtain the new gap at B.*



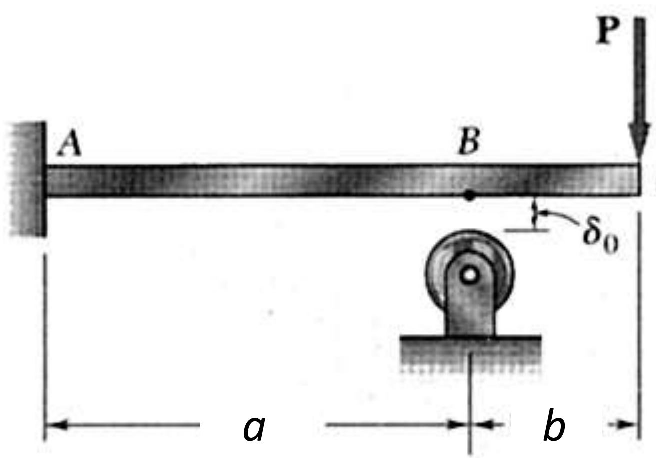
Beam with a gapped support

- *We will consider the case when P is large enough to cause B to come in contact with the support.*



Beam with a gapped support

- *For this we need to solve the first problem anyway, because we will need to know the deflection at C when the beam just touches B without causing any contact force.*



Beam with a gapped support

- *We also need to understand the phenomena clearly. The force at the tip increases slowly from 0 to a value Q, when there is bare minimum contact at B. Then the force increases slowly to a value P till the final equilibrium shape is reached.*
- *We are not dealing with quick instantaneous loading*
- *The theories that we usually use in this subject are not valid for such rapid loading. In all problems we have solved so far in mechanics, the inherent assumption is that the load increases very slowly to its final value that is given in the text of the problem*

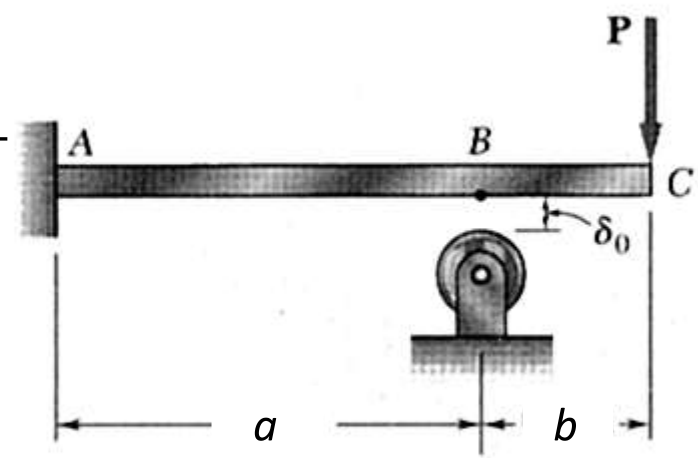
Beam with a gapped support

- *For a cantilever beam with a load Q at the tip we know*

$$EIv(x) = -Q(a+b)\frac{x^2}{2} + \frac{Qx^3}{6}$$

- *At B, x=a*. Hence

$$v(a) = -Q(a+b)\frac{a^2}{2EI} + \frac{Qa^3}{6EI}$$

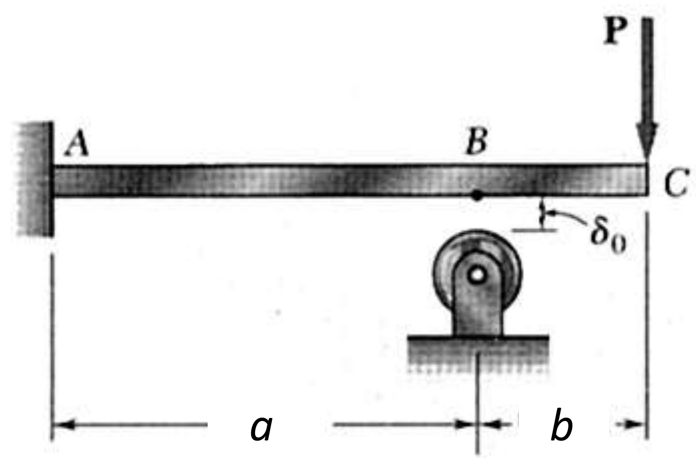


Beam with a gapped support

- Since $v(a) = -\delta_0$
- We can find the force Q

$$-Q(a+b)\frac{a^2}{2EI} + \frac{Qa^3}{6EI} = -\delta_0$$

$$\Rightarrow Q = \frac{6EI\delta_0}{a^2(2a+3b)}$$



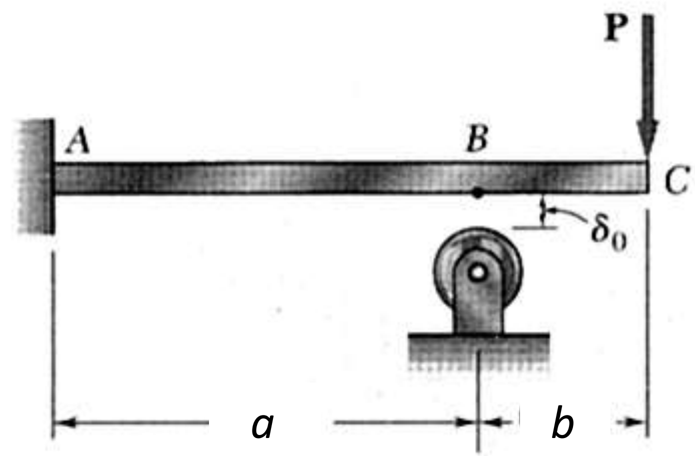
Beam with a gapped support

- So deflection at C due to the action of this force Q is

$$\delta_{C,Q} = -\frac{Q(a+b)^3}{3EI}, Q = \frac{6EI\delta_0}{a^2(2a+3b)}$$

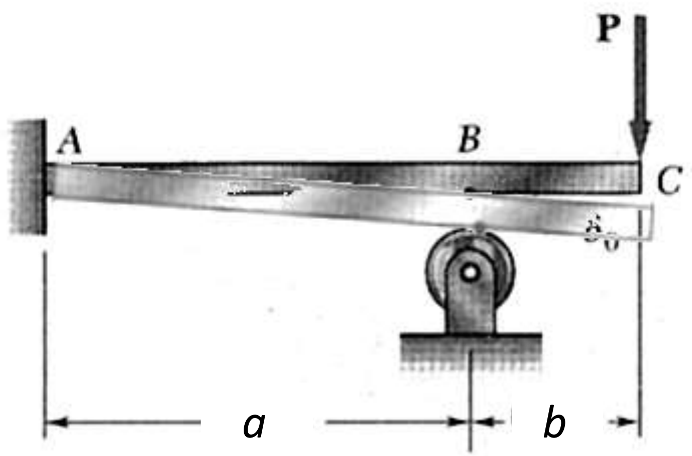
$$\Rightarrow \delta_{C,Q} = -\frac{6EI\delta_0}{a^2(2a+3b)} \frac{(a+b)^3}{3EI}$$

$$\Rightarrow \delta_{C,Q} = -\frac{2\delta_0(a+b)^3}{a^2(2a+3b)}$$



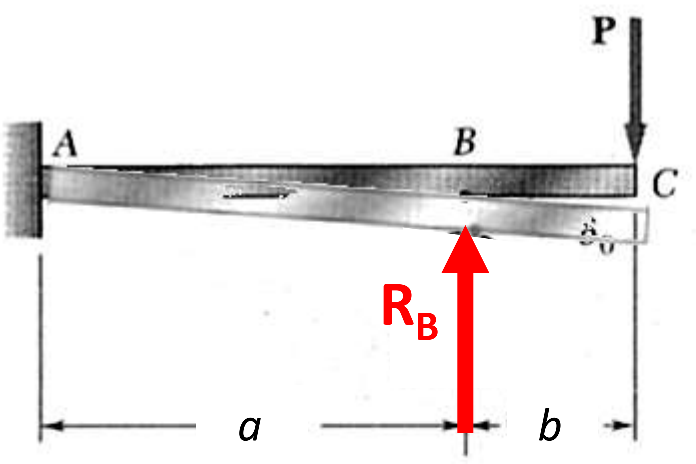
Beam with a gapped support

- Let us now consider what happens when the force goes beyond Q to a value P=P'+Q



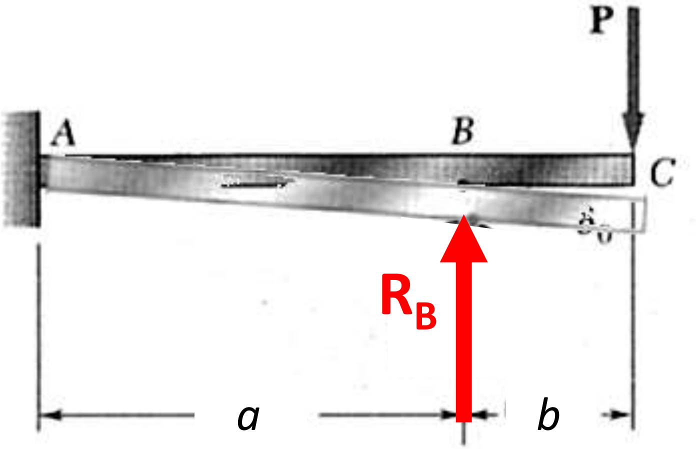
Beam with a gapped support

- We now re draw the diagram of the beam in its final state replacing the support by a reaction R_B



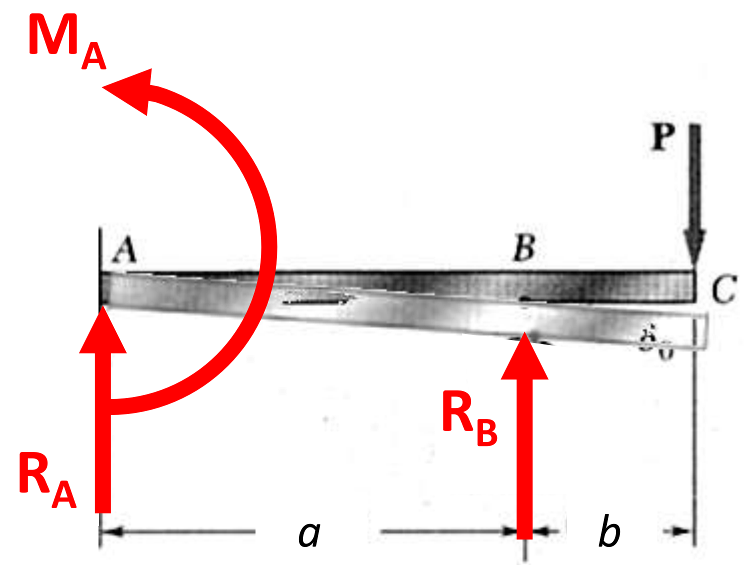
Beam with a gapped support

- We can split this problem again into two simpler problems
- A cantilever beam loaded at the tip C by a force P=P’, where P’ is the extra force added after deformation by Q
- A cantilever beam with overhang BC loaded at B by a force R_B



Beam with a gapped support

- However we will choose to do this by the usual process.
- So we start by drawing the FBD.

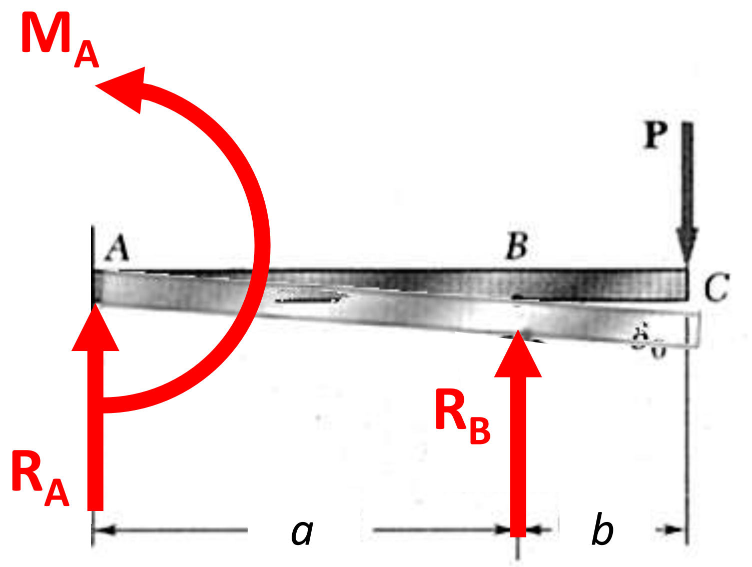


Beam with a gapped support

- We write down the equations of static equilibrium.
- Since there are 3 unknown reactions and 2 equations the problem is statically indeterminate

$$R_A + R_B = P$$

$$M_A + R_B a = P(a+b)$$



Beam with a gapped support

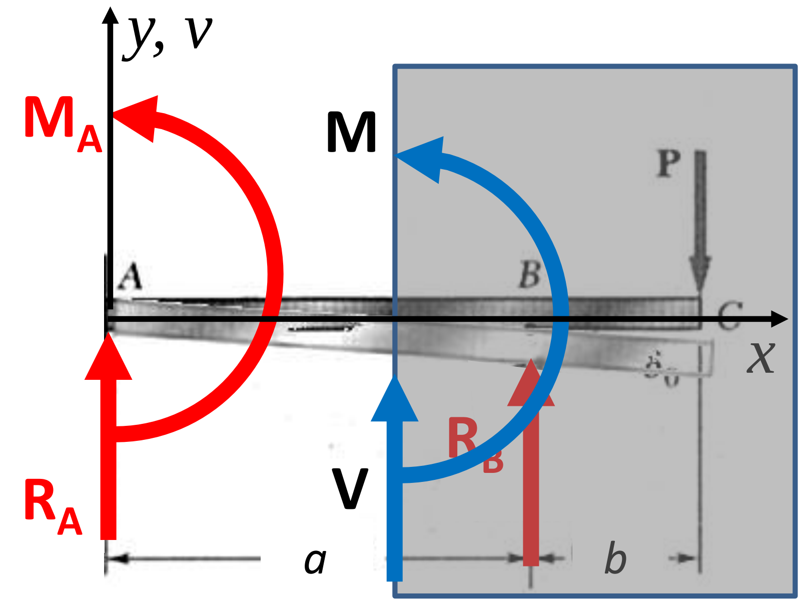
- There are two domains AB and BC.
- We consider AB first

$$R_A + V = 0 \Rightarrow V = -R_A$$

$$M_A + M + Vx = 0$$

$$\Rightarrow M = -M_A - Vx$$

$$\Rightarrow M = -M_A + R_A x$$



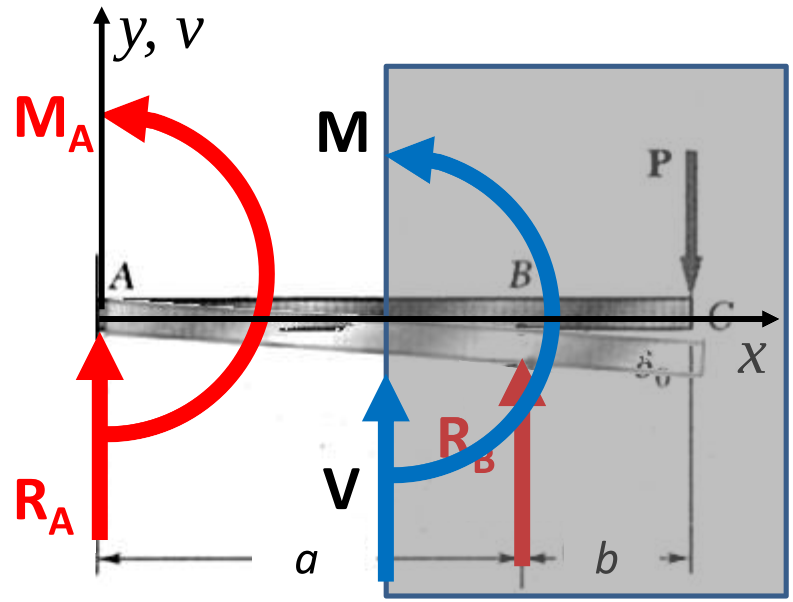
Beam with a gapped support

- Apply flexure equation

$$EIv'' = R_A x - M_A$$

$$\Rightarrow EIv' = R_A \frac{x^2}{2} - M_A x + C_1$$

$$\Rightarrow EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} + C_1 x + C_2$$



Beam with a gapped support

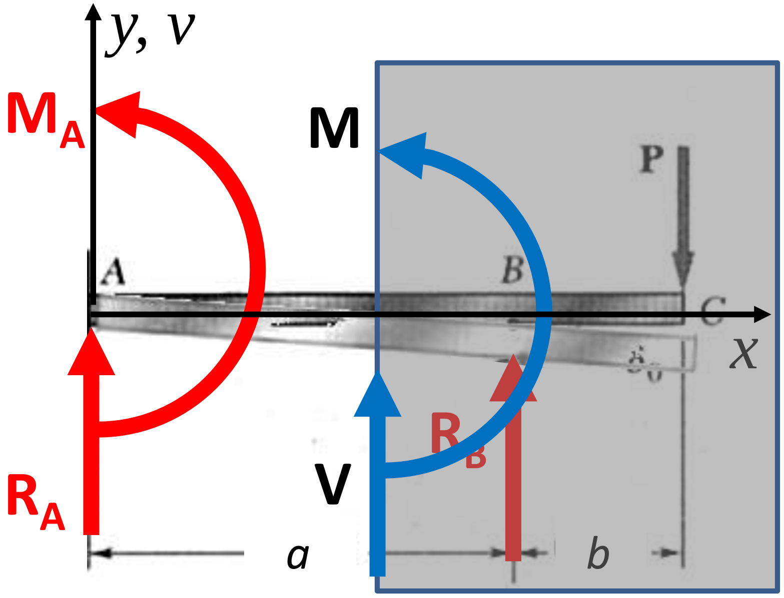
- Apply boundary conditions for this domain by considering the slope and deflection at A

$$v'(0)=0, v(0)=0 \Rightarrow C_1=0, C_2=0$$

- The solution is

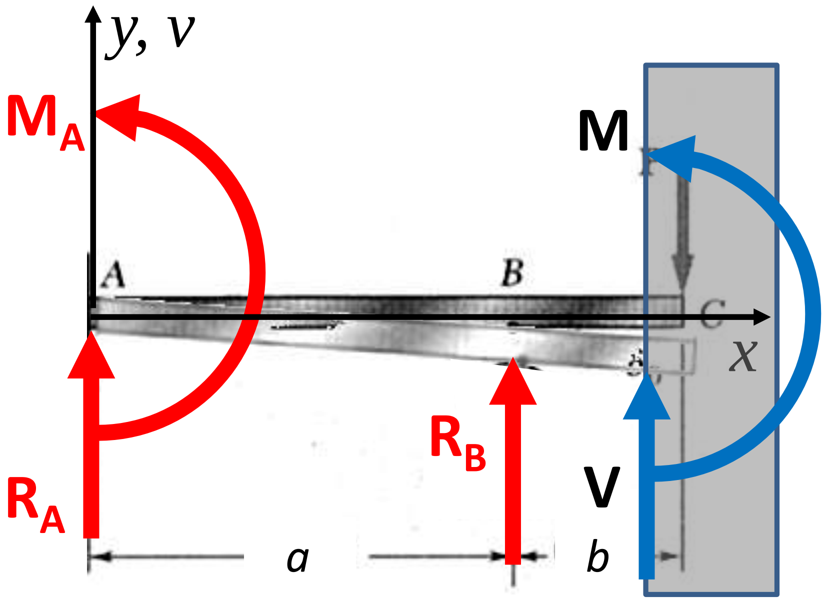
$$EIv' = R_A \frac{x^2}{2} - M_A x$$

$$EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2}$$



Beam with a gapped support

- We consider BC next



$$R_A + V + R_B = 0$$

$$\Rightarrow V = -R_A - R_B$$

$$M_A + R_B a + M + Vx = 0$$

$$\Rightarrow M = -M_A - R_B a - Vx$$

$$\Rightarrow M = -M_A - R_B a + \left(R_A + R_B\right)x$$

Beam with a gapped support

- Apply flexure equation

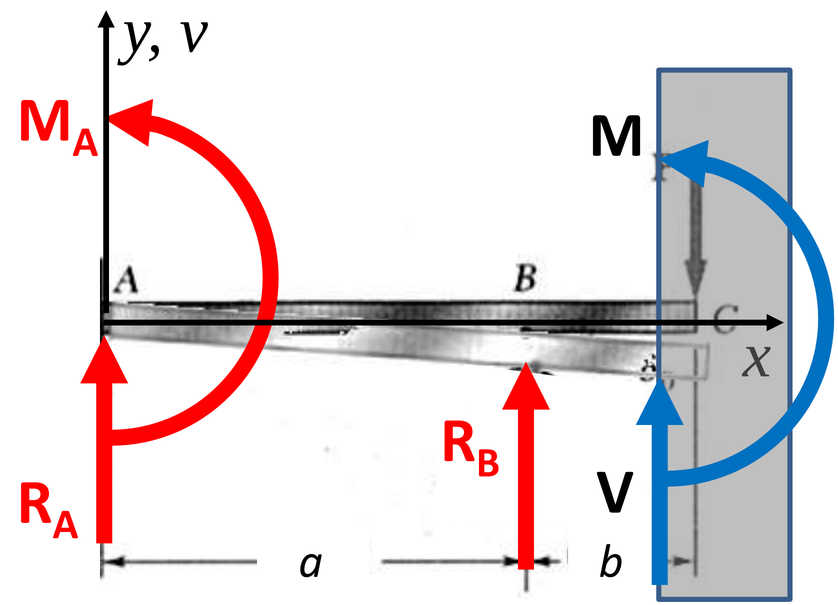
$$EIv'' = \left(R_A + R_B\right)x - M_A - R_B a$$

$$\Rightarrow EIv' = \left(R_A + R_B\right)\frac{x^2}{2}$$

$$-\left(M_A + R_B a\right)x + D_1$$

$$\Rightarrow EIv = \left(R_A + R_B\right)\frac{x^3}{6}$$

$$-\left(M_A + R_B a\right)\frac{x^2}{2} + D_1 x + D_2$$

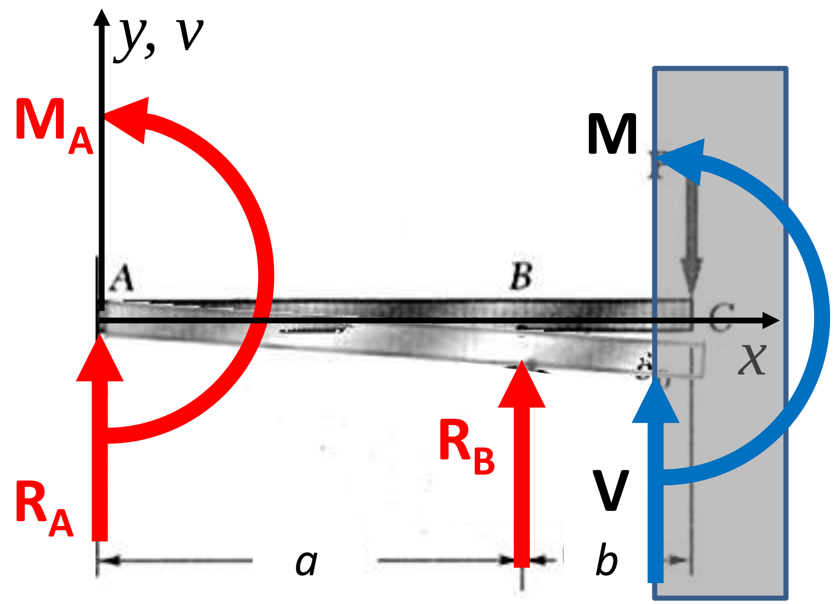


Beam with a gapped support

- Apply boundary conditions for this domain by considering the slope and deflection at B

$$v'(a-) = v'(a+)$$

$$v(a-) = v(a+)$$

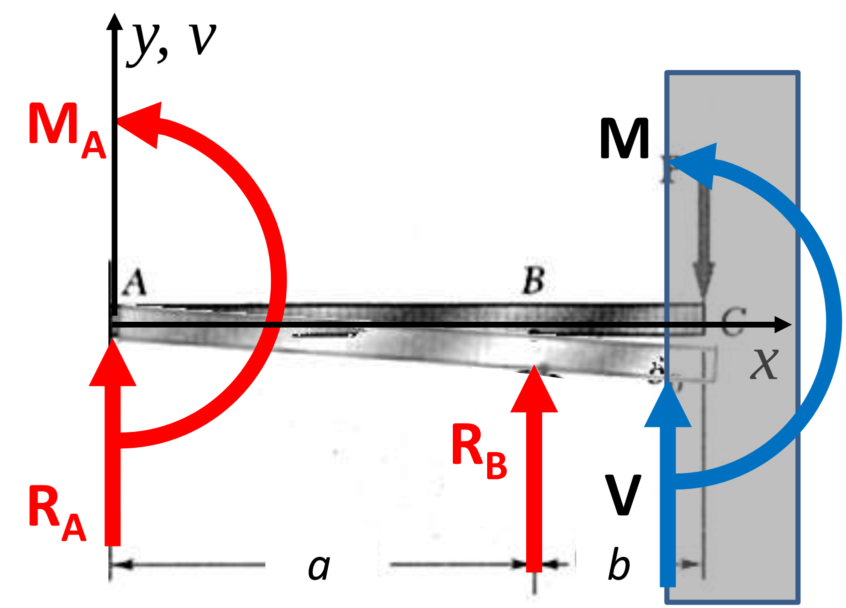


Beam with a gapped support

- Considering the slope at B

$$R_A \frac{a^2}{2} - M_A a = (R_A + R_B)\frac{a^2}{2} - (M_A + R_B a)a + D_1$$

$$\Rightarrow D_1 = R_B \frac{a^2}{2}$$



Beam with a gapped support

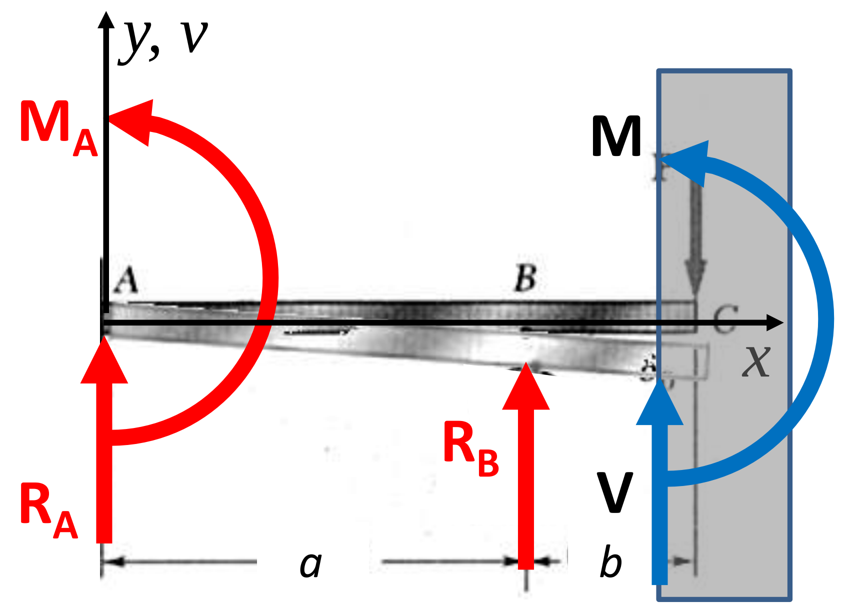
- Considering the deflection at B

$$R_A \frac{a^3}{6} - M_A \frac{a^2}{2} = (R_A + R_B)\frac{a^3}{6}$$

$$-(M_A + R_B a)\frac{a^2}{2} + D_1 a + D_2$$

$$\Rightarrow D_1 a + D_2 = R_B \frac{a^3}{3}$$

$$\Rightarrow D_2 = -R_B \frac{a^3}{6}$$



Beam with a gapped support

- Solution

$$EIv' = \left(R_A + R_B\right)\frac{x^2}{2} - \left(M_A + R_B a\right)x + R_B\frac{a^2}{2}$$

$$EIv = \left(R_A + R_B\right)\frac{x^3}{6} - \left(M_A + R_B a\right)\frac{x^2}{2} + R_B\frac{a^2}{2}x - R_B\frac{a^3}{6}$$

Beam with a gapped support

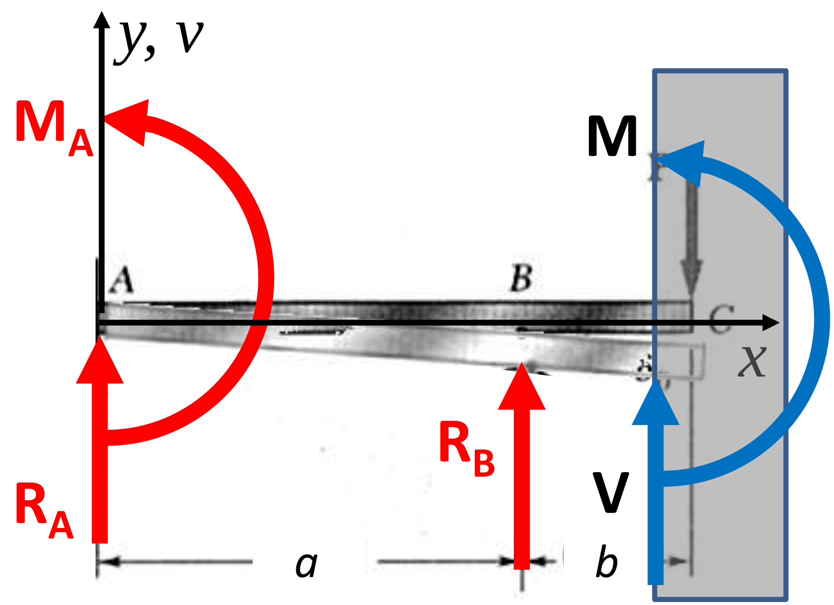
- Considering the deflection at B

$$R_A \frac{a^3}{6EI} - M_A \frac{a^2}{2EI} = \delta_0$$

- Static equilibrium equations

$$R_A + R_B = P$$

$$M_A + R_B a = P\left(a+b\right)$$



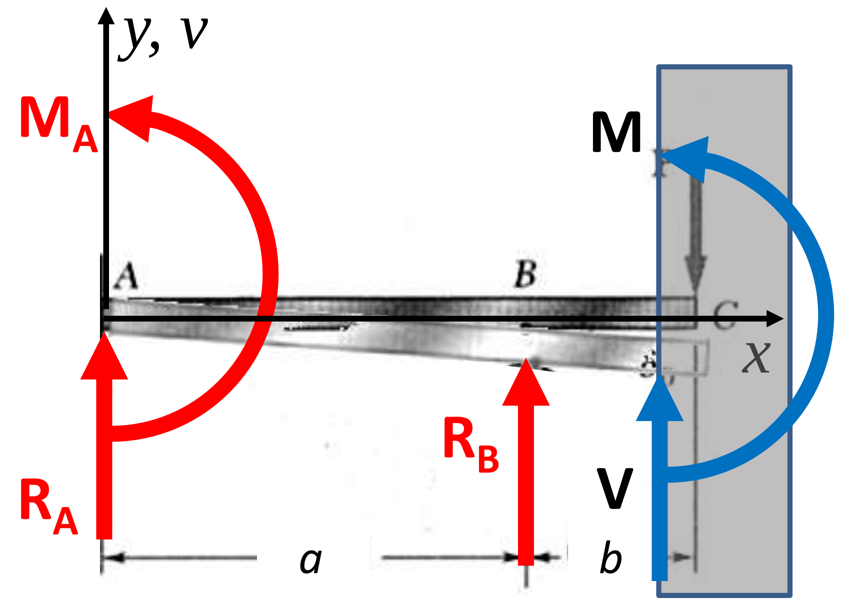
Beam with a gapped support

- We can solve for the unknown reactions

$$R_A = -\frac{3}{2}\left(\frac{2EI}{a^3}\delta_0 + P\frac{b}{a}\right)$$

$$R_B = P + \frac{3}{2}\left(\frac{2EI}{a^3}\delta_0 + P\frac{b}{a}\right)$$

$$M_A = -\left(3\frac{EI}{a^2}\delta_0 - \frac{1}{2}Pb\right)$$



Beam with a gapped support

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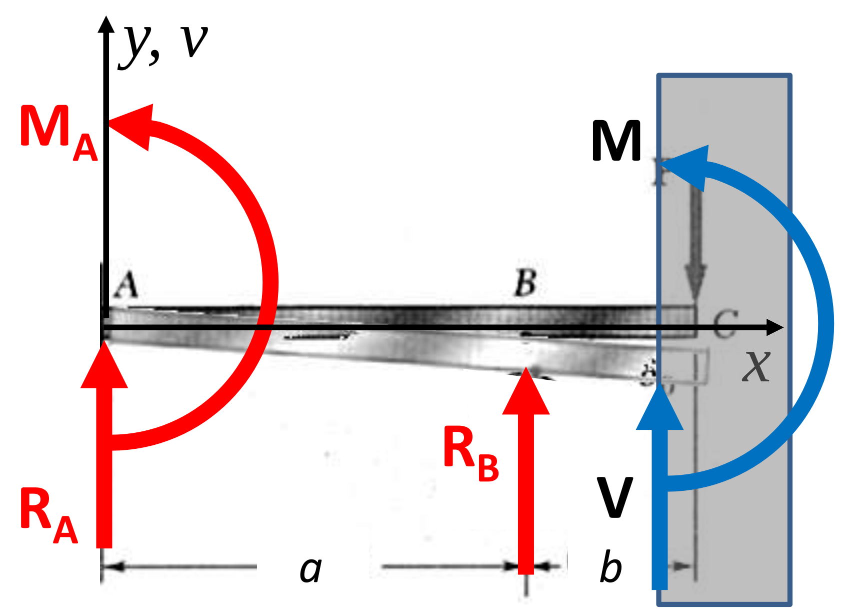
$$M_A = -\left(3\frac{EI}{a^2}\delta_0 - \frac{1}{2}Pb\right)$$



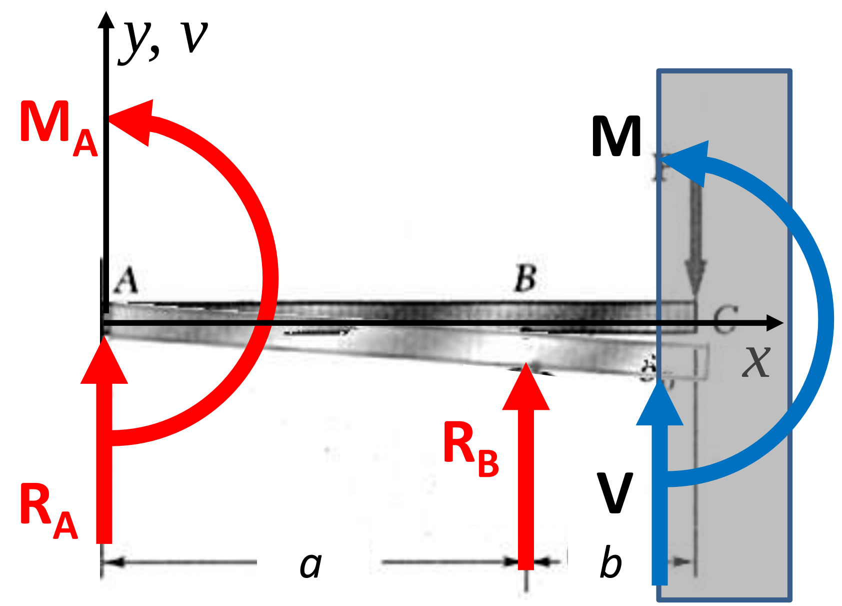
Beam with a gapped support

- We can solve for the unknown reactions

$$R_A = -\frac{3}{2}\left(\frac{2EI}{a^3}\delta_0 + P\frac{b}{a}\right)$$

$$R_B = P + \frac{3}{2}\left(\frac{2EI}{a^3}\delta_0 + P\frac{b}{a}\right)$$

$$M_A = -\left(3\frac{EI}{a^2}\delta_0 - \frac{1}{2}Pb\right)$$



Beam with a gapped support

- Deflection at C after contact due to a force P'=P-Q is therefore

$$\delta_{C,P'} = P'a\frac{(2a+3b)^2}{12EI} - P'\frac{(a+b)^3}{3EI} + \frac{(2a+3b)}{2a}\delta_0$$

$$= (P-Q)\left\{a\frac{(2a+3b)^2}{12EI} - \frac{(a+b)^3}{3EI}\right\} + \frac{(2a+3b)}{2a}\delta_0$$

Beam with a gapped support

- Deflection at C after contact due to a force P-Q

$$\because Q = \frac{6EI\delta_0}{a^2\left(2a+3b\right)}$$

$$\delta_{C,P'} = P\left\{a\frac{\left(2a+3b\right)^2}{12EI} - \frac{\left(a+b\right)^3}{3EI}\right\} + \frac{2\delta_0\left(a+b\right)^3}{a^2\left(2a+3b\right)}$$

Beam with a gapped support

- Total deflection at C after contact is therefore

$$\delta_{C,P} = P\left\{a\frac{\left(2a+3b\right)^2}{12EI} - \frac{\left(a+b\right)^3}{3EI}\right\} + \frac{2\delta_0\left(a+b\right)^3}{a^2\left(2a+3b\right)} + \delta_{C,Q}$$

$$\delta_{C,Q} = -\frac{2\delta_0\left(a+b\right)^3}{a^2\left(2a+3b\right)}$$

$$\therefore \delta_{C,P} = P\left\{\frac{a\left(2a+3b\right)^2 - 4\left(a+b\right)^3}{12EI}\right\} = P\frac{b^2\left(4b-3a\right)}{12EI}$$

Beam on Beam

L shaped beam

L shaped rod