In this last lesson of this module we demonstrate the potential of Laplace transform for solving partial differential equations. If one of the independent variables in partial differential equations ranges from 0 to ∞ then Laplace transform may be used to solve partial differential equations.

15.1 Solving Partial Differential Equations

Working steps are more or less similar to what we had for solving ordinary differential equations. We take the Laplace transform with respect to the variable that ranges from 0 to ∞. This will convert the partial differential equation into an ordinary differential equation. Then, the transformed ordinary differential equation must be solved considering the given conditions.

Denoting the Laplace transform of unknown variable $u(x,t)$ with respect to t by $U(x,s)$ and using the definition of Laplace transform we have

$$U(x,s) = L[u(x,t)] = \int_0^\infty e^{-st}u(x,t)\,\mathrm{d}t$$

Then, for the first order derivatives, we have

$$(i)\;\; L\left[\frac{\partial u}{\partial x}\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\,\mathrm{d}t = \frac{d}{dx}\int_0^\infty e^{-st}u(x,t)\,\mathrm{d}t = \frac{dU}{dx}$$

$$\begin{aligned}(ii)\;\; L\left[\frac{\partial u}{\partial t}\right] &= \int_0^\infty e^{-st}\frac{\partial u}{\partial t}\,\mathrm{d}t = e^{-st}u\Big|_0^\infty - \int_0^\infty u(-s)e^{-st}\,\mathrm{d}t \\ &= -u(x,0) + s\int_0^\infty ue^{-st}\,\mathrm{d}t \\ \Rightarrow\;\; L\left[\frac{\partial u}{\partial t}\right] &= -u(x,0) + sU(x,s)\end{aligned}$$

$$(iii)\;\; L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2U}{dx^2}$$

$$(iv)\;\; L\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0)$$

$$(v)\;\; L\left[\frac{\partial^2 u}{\partial x\partial t}\right] = s\frac{d}{dx}U(x,s) - \frac{d}{dx}u(x,0)$$

Remark: *In order to derive the above results, besides the assumptions of piecewise continuity and exponential order of $u(x,t)$ with respect to t, we have also used the following assumptions: (i) The differentiation under integral sign is valid and (ii) The limit of the Laplace transform is the Laplace transform of the limit, i.e.,* $\lim_{x\to x_0} L\left[u(x,t)\right] = L\left[\lim_{x\to x_0} u(x,t)\right]$.

15.2 Example Problems

15.2.1 Problem 1

Solve the following initial boundary value problem

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad u(x,0) = x,\ u(0,t) = t$$

Solution: Taking Laplace transform

$$\frac{d}{dx}U(x,s) = sU(x,s) - u(x,0)$$

Using the initial values we get

$$\frac{d}{dx}U(x,s) - sU(x,s) = -x$$

The integrating factor is

$$I.F. = e^{-\int s\,\mathrm{d}x} = e^{-sx}$$

Hence, the solution can be written as

$$U(x,s)e^{-sx} = -\int xe^{-sx}\,\mathrm{d}x + c$$

On integration by parts we find

$$U(x,s)e^{-sx} = -x\frac{e^{-sx}}{-s} - \int \frac{e^{-sx}}{s}\,\mathrm{d}x + c$$

Simplify, the above expression we have

$$U(x,s) = \frac{x}{s} + \frac{1}{s^2} + ce^{sx}$$

Using given boundary condition we find

$$\frac{1}{s^2} = \frac{1}{s^2} + c \cdot 1 \quad \Rightarrow \quad c = 0$$

With this we obtain

$$U(x,s) = \frac{x}{s} + \frac{1}{s^2}$$

Taking inverse Laplace transform, we find the desired solution as

$$u(x,t) = x + t$$

15.2.2 Problem 2

Solve the following partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x, \quad x > 0,\, t > 0$$

with the following initial and boundary condition

$$u(x,0) = 0,\, x > 0 \;\; \textit{and} \;\; u(0,t) = 0,\, t > 0$$

Solution: Taking Laplace transform with respect to t we have

$$sU(x,s) - u(x,0) + x\frac{d}{dx}U(x,s) = \frac{x}{s}, \quad s > 0$$

Using the given initial value we find

$$\frac{d}{dx}U(x,s) + \frac{s}{x}U(x,s) = \frac{1}{s}$$

Its integrating factor is x^s and therefore the solution can be written as

$$U(x,s)x^s = \int \frac{1}{s}x^s \,\mathrm{d}x + c \quad \Rightarrow \quad U(x,s) = \frac{1}{s(s+1)}x + \frac{c}{x^s}$$

Boundary condition provides

$$u(0,t) = 0 \quad \Rightarrow \quad U(0,s) = 0, \quad \Rightarrow \quad c = 0$$

Thus we have

$$U(x,s) = \frac{x}{s(s+1)} = x\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

Taking inverse Laplace transform we find the desired solution as

$$u(x,t) = x\left[1 - e^{-t}\right]$$

15.2.3 Problem 3

Solve the following heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0,\, t > 0$$

with the initial and boundary conditions

$$u(x,0) = 1,\, u(0,t) = 0,\, \lim_{x\to\infty} u(x,t) = 1$$

Solution: Taking Laplace transform we find

$$sU(x,s) - u(x,0) = \frac{d^2}{dx^2}U(x,s)$$

Using the given initial condition we have

$$\frac{d^2}{dx^2}U(x,s) - sU(x,s) = -1$$

Its solution is given as

$$U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{1}{s}$$

The given boundary conditions give

$$\lim_{x\to\infty} U(x,s) = \frac{1}{s} \quad \Rightarrow \quad c_1 = 0$$

and

$$U(0,s) = 0 \quad \Rightarrow \quad c_1 + c_2 + \frac{1}{s} = 0 \quad \Rightarrow \quad c_2 = -\frac{1}{s}$$

Hence, we have

$$U(x,s) = -\frac{1}{s}e^{-\sqrt{s}x} + \frac{1}{s}$$

Taking inverse Laplace transform we find the desired solution as

$$u(x,t) = 1 - L^{-1}\left[\frac{1}{s}e^{-\sqrt{s}x}\right] = 1 - \left[1 - \text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] = \text{erf}\left(\frac{x}{2\sqrt{t}}\right)$$

15.2.4 Problem 4

Solve the one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad x > 0,\ t > 0$$

with the initial conditions

$$y(x,0) = 0,\ y_t(x,0) = 0$$

and boundary conditions

$$y(0,t) = \sin \omega t,\ \lim_{x\to\infty} y(x,t) = 0$$

Solution: Taking Laplace transform we get

$$s^2 Y(x,s) - sy(x,0) - y_t(x,0) - a^2 \frac{d^2}{dx^2} Y(x,s) = 0$$

With the given initial condition we have the resulting differential equation

$$\frac{d^2 Y}{dx^2} - \frac{s^2}{a^2} = 0$$

Its general solution is given as

$$Y(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x}$$

The given boundary conditions provides

$$\lim_{x\to\infty} Y(x,s) = 0 \quad \Rightarrow \quad c_1 = 0,$$

and

$$Y(0,s) = \frac{\omega}{s^2+\omega^2} \quad \Rightarrow \quad c_2 = \frac{\omega}{s^2+\omega^2}$$

Thus we have

$$Y(x,s) = \frac{\omega}{s^2+\omega^2} e^{-\frac{s}{a}x}$$

Taking inverse Laplace transform we obtain

$$y(x,t) = \sin\left[\omega\left(t - \frac{x}{a}\right)\right] H\left(t - \frac{x}{a}\right).$$