Transform Calculus

(MA-20101)

Assignment-1

1. i) Show that the Laplace transform of the function $\frac{1}{\sqrt{t}}$ is $\sqrt{\frac{\pi}{s}}$.

   ii) State the sufficient condition for the function $f(t),\ t \geq 0$ to have the Laplace transform. Give one example to show that these conditions are not the necessary conditions for $f(t)$ to have Laplace transform.

2. i) If the Laplace transform of the function $f(t)$ is $F(s)$ and c is a positive constant, show that the Laplace transform of the function $f(ct)$ is $\frac{F(s/c)}{c}$.

   ii) Find the Laplace transform of $\cos(\omega t)$ (You can use that the Laplace transform of $\cos t$ is $\frac{s}{s^2+1}$).

3. Find the Laplace transform of the following functions—

   i) $\cos^2(\frac{1}{2}\pi t)$ ii) $t^3 e^{-3t}$

   iii) $e^{-t/2}u(t-2)$ iv) $(t-a)^n u(t-a)$.

4. Show that the Laplace transform of a piecewise continuous function $f(t)$ with period p is

$$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt \ (s > 0).$$

5. Prove that the Laplace transform of

   i) $t \cos \omega t$ is $\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$,

   ii) $t \sinh \omega t$ is $\frac{2\omega s}{(s^2-\omega^2)^2}$.

   (Hint: Use the results of Laplace transform of derivatives).

6. If f is continuous, except for an ordinary discontinuity (finite jump) at some $t = a(> 0)$, satisfies the growth restriction, $f'(t)$ is piecewise continuous on every finite interval in $[0, \infty)$ except at $t = a$. Then show that the Laplace transform of $f'(t)$ is
$$sF(s) - f(0) - [f(a+0) - f(a-0)]e^{-as}$$
(where $F(s)$ is Laplace transform of $f(t)$.)

7. If $F(s) = \frac{A_1}{(s-a)^m} + \frac{A_2}{(s-a)^{m-1}} + \cdots + \frac{A_m}{s-a} + \frac{B_1}{s-b_1} + \frac{B_2}{s-b_2} + \cdots + \frac{B_n}{s-b_n}$, then find the inverse Laplace transform of $F(s)$.

8. Express the following functions in terms of Heaviside unit step functions:-

   i) $f(t) = \begin{cases} t^2 & \text{if } 0 < t < 2 \\ 4t & \text{if } t > 2 \end{cases}$

   ii) $f(t) = \begin{cases} sin\ t & \text{if } 0 < t < \pi \\ sin\ 2t & \text{if } \pi < t < 2\pi \\ sin\ 3t & \text{if } t > 2\pi \end{cases}$

9. Find the Laplace transform of the following functions–

   i) $f(t) = \begin{cases} 5 & \text{if } 0 < t < 7 \\ 0 & \text{if } t \geq 7 \end{cases}$

   ii) $f(t) = \begin{cases} \sin t & \text{if } \pi/2 < t < \pi \\ 0 & \text{if } t \leq \pi/2,\ t \geq \pi \end{cases}$

10. Find the Laplace transforms of the following functions–

    i) $t^2 \sin 3t$,

    ii) $\frac{\sin\ at}{t}$,

    iii) $4t * e^{-2t}$.

11. i) If the Laplace transform of $f(t)$ is $F(s)$, prove that the Laplace transform of $t^n f(t)$ is $(-1)^n F^{(n)}(s)$ (where $F^{(n)}(s)$ is the n-th derivative of $F(s)$).

    ii) Find the Laplace transform $t^n e^{kt}$.

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