e.g. 2) Solve :-

$$x'' + y' + 3x = 15e^{-t}$$

$$y'' - 4x' + 3y = 15 \sin 2t,$$

subject to $x(0) = 35$, $x'(0) = -48$,

$y(0) = 27$, $y'(0) = -55$.

Sol^n :- ∴ General solution is

$$x = \mathcal{L}^{-1}\{X\} = 30 \cos t - 15 \sin 3t + 3e^{-t} + 2 \cos 2t$$

$$y = \mathcal{L}^{-1}\{Y\} = 30 \cos 3t - 60 \sin t - 3e^{-t} + \sin 2t.$$

Fourier Series

1. Periodic functions :-

A function $f(x)$ is called periodic if it is defined for all real x & if there is some positive number p such that

$$f(x+p) = f(x), \ \forall x \quad \longrightarrow (1)$$

This number 'p' is called a period of $f(x)$. The graph of such a function is obtained by periodic repetition of its graph in any interval of length p. Periodic phenomena &

Periodic phenomena & functions have many applications.

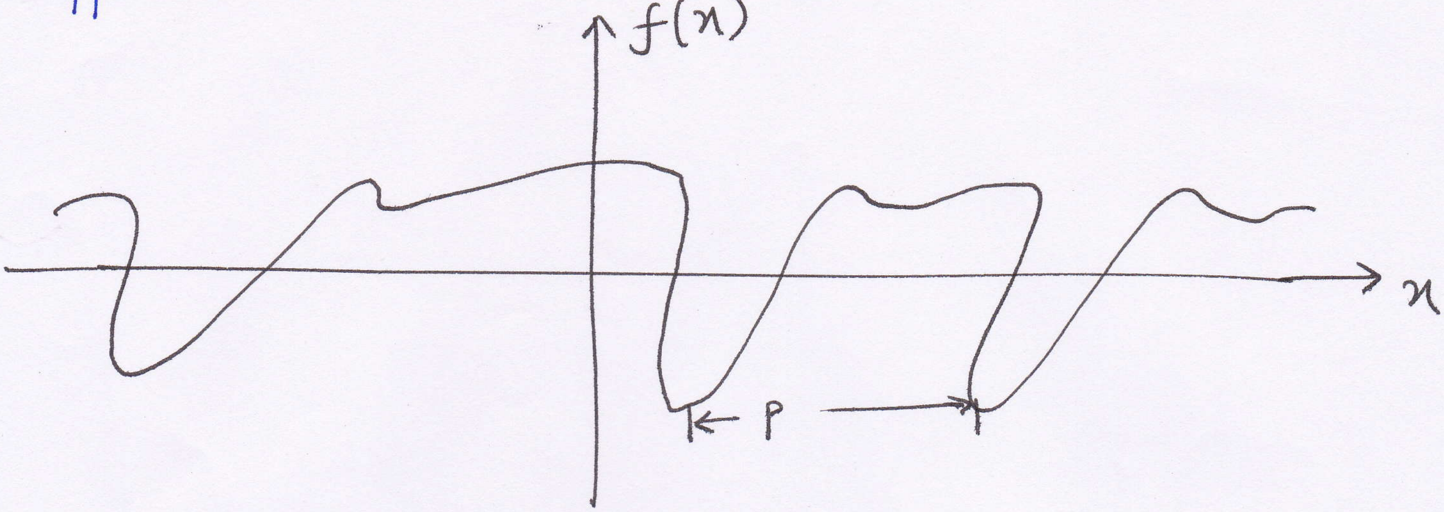


Fig 1 :- Periodic function

Familiar periodic functions are the sine & cosine functions. We note that the function $f = c =$ constant is also a periodic function in the sense of the defn, because it satisfies ① for every positive p. [as $f(x+p) = c = f(x)$]

Examples of functions that are not periodic are x, x^2, x^3, e^x, $\cosh x$ & $\ln x$ to mention just a few.

From ①, we have —

$$f(x+2p) = f\left[(x+p)+p\right] = f(x+p) = f(x)$$

etc., & for any integer n,

$$f(x+np) = f(x), \quad \forall x \quad \text{②}$$

Hence, $2p, 3p, 4p, \dots$ are also periods of $f(x)$.

Also, if $f(x)$ & $g(x)$ have period P, then the function $h(x) = a f(x) + b g(x)$ (a, b constant) also has the period P.

If a periodic function $f(x)$ has a smallest period P $(P>0)$, this is often called the <u>fundamental period</u> of $f(x)$.

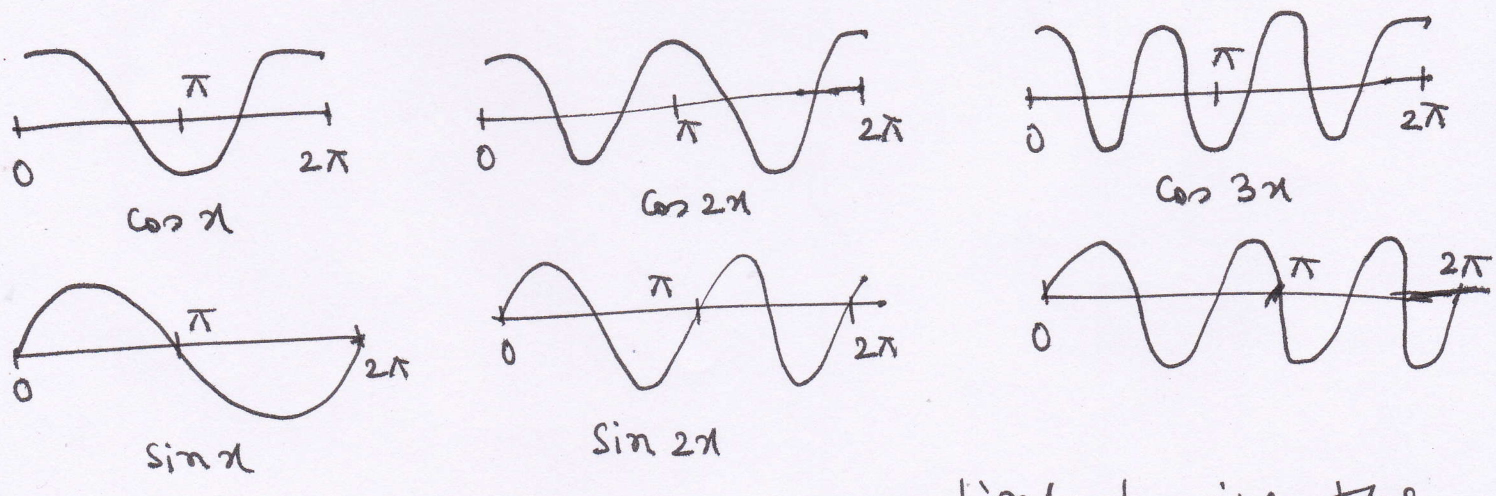
For $\cos x$ & $\sin x$ the fundamental period is 2π, for $\cos 2x$ & $\sin 2x$ it is π, & so on.

A function without fundamental period is $f =$ constant.

2. <u>Trigonometric Series</u>:

Our problem will be the representation of various functions of period $p = 2\pi$ in terms of the simple functions:

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \quad \rightarrow (3)$$



<u>Fig 2</u>:- Cosine & sine functions having the period 2π

These functions have the period 2π. Fig. 2 shows the first few of them.

The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots,$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are real constants.

Such a series is called a trigonometric series, & the a_n & b_n are called the co-efficients of the series.

Using the summation sign, we may write this series as

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (5)$$

The set of functions (3) from which we have made up the series (5) is often called the trigonometric system, to have a short name for it.

We see that each term of the series (5) has the period 2π. Hence, if the series (5) converges, its sum will be a function of period 2π.

The point is that trigonometric series can be used for representing any practically important periodic function f simpler or complicated, of any period P. (This series will then be called the Fourier series of f).

---

3. Fourier Series:

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine & sine functions. These series are trigonometric series whose co-efficients are determined from $f(x)$ by the "Euler formulas" which we shall derive first.

---

4. Applications:

Fourier series consitute a very important tool in solving problems that involve O.D. eqns & P.D. eqns.

The theory of Fourier series is rather complicated but the application of these series is simple. Fourier series are, in a certain sense, more universal than Taylor series because many discontinuous periodic functions of practical interest can be can be developed in Fourier series, but, of course do not have Taylor series representations.

Fourier integrals & Fourier transforms which extend the ideas & techniques of Fourier series to non-periodic functions defined for all x.

---

5. Euler formulas for the Fourier Co-efficients:

Let us assume that $f(x)$ is a periodic function of period 2π & is integrable over a period.

Let us further assume that $f(x)$ can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \quad \rightarrow (1)$$

that is, we assume that this series converges & has $f(x)$ as its sum. Given such a function $f(x)$, we want to determine the co-efficients a_n & b_n of the corresponding series (1).

- **Determination of the constant term a_0:**

Integrating on both sides of (1) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right] dx$$

If term-by-term integration of the series is allowed, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right)$$

$$= 2\pi a_0 + 0$$

[$\because$ all the other integrals on the R.H.S are zero by integration]

Hence, our first result is

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx \longrightarrow (2)$$

- Determination of the co-efficients a_n of the cosine terms.

Similarly, we multiply (1) by $\cos mx$, where m is any fixed positive integer, & integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \cos mx\,dx = \int_{-\pi}^{\pi} \left\{\left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right] \cos mx\,dx\right\}. \longrightarrow (3)$$

Integrating term by term, we see that the R.H.S becomes

$$a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx\right].$$

The first integral is zero,

& $$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x \, dx.$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin (n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin (n-m)x \, dx.$$

Integration shows that the four terms on the right are zero, except for the last term in the first line, which equals π when $n=m$. Since in (3), this term is multiplied by a_m, the right side in (3) equals $a_m\pi$. Our second result is

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \, dx, \quad m = 1, 2, \dots \rightarrow (4)$$

<u>Determination of the co-efficients b_n of the sine terms</u>

We finally multiply (1) by $\sin mx$, where m is any fixed positive integer, & then integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)\right] \sin mx \, dx. \rightarrow (5)$$

Integrating term by term, we see that the R.H.S becomes —

$$a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \right].$$

The first integral is zero. The next integral is of the kind considered before, & is zero for all $n = 1, 2, \ldots$. For the last integral, we obtain

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos (n-m) x \, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos (n+m) x \, dx$$

The last term is zero.

The first term on the right is zero when $n \neq m$ & is π when $n = m$. Since in (5), this term is multiplied by b_m; the right side in (5) is equal to $b_m \pi$, & our last result is

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx, \quad m = 1, 2 \ldots \qquad (6)$$

• Summary :- From eqns (2), (4) & (6), writing n in place of m, we have the so-called Euler formulas

(7)

(a) $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

(b) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \ldots$$

(c) $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \ldots$$

These numbers given by (7) are called the Fourier co-efficients of $f(x)$.

The trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (8)$$

with co-efficients given in (7) is called the Fourier series of $f(x)$ (regardless of convergence).

Eg. 1) Rectangular wave :-

Find the Fourier co-efficients of the periodic function $f(x)$, given by

$$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0, \\ k, & \text{if } 0 < x < \pi, \end{cases} \quad \& \ f(x+2\pi) = f(x).$$

[Note :- Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits etc.]

Soln :- [The value of $f(x)$ at a single point does not affect the integral, hence we can leave $f(x)$ undefined at $x=0$ & $x = \pm\pi$.]

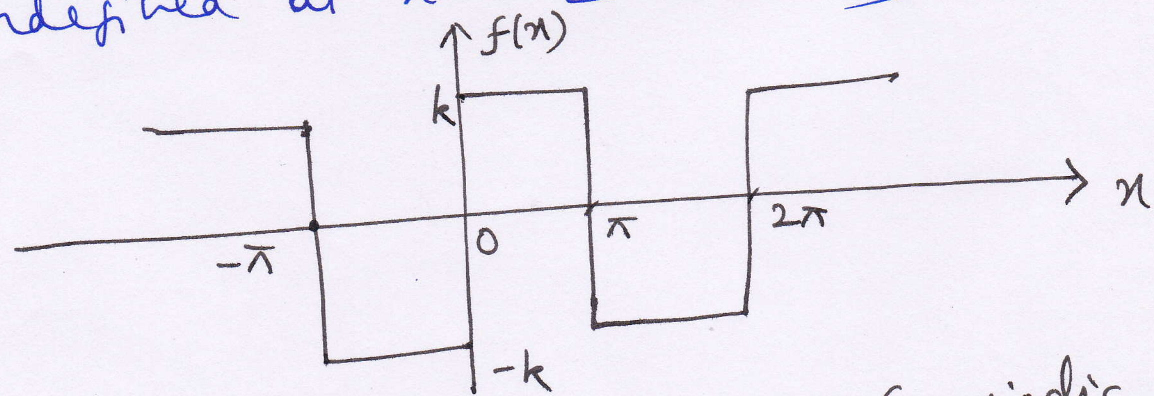


Fig 1(a): The given function $f(x)$ (periodic square wave)

Solⁿ:- From eqⁿ (7a), we obtain

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= -k\pi + k\pi = \boxed{0}$$

This can also be seen without integration, since the area under the curve of $f(x)$ betⁿ $-\pi$ & π is zero.

From eqⁿ (7b), $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[k\frac{\sin nx}{n}\right]_{0}^{\pi} = \boxed{0},$$

because $\sin nx = 0$ at $-\pi, 0, \& \pi$ for all $n = 1, 2, \ldots$

Similarly, from eqⁿ (7c), we obtain —

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\right]_{-\pi}^{0} - \left[k\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$\therefore b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right]$$

$$= \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos n\pi = \begin{cases} -1 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$ & thus $1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$

Hence, the Fourier co-efficients b_n of our function are

$b_1 = \frac{4k}{\pi}$, $b_2 = 0$, $b_3 = \frac{4k}{3\pi}$, $b_4 = 0$, $b_5 = \frac{4k}{5\pi}$, ...

Since the a_n are zero, the Fourier series of $f(x)$ is
$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \rightarrow (*)$$

Furthermore, setting $x = \pi/2$ in $(*)$, we have
$$f(\pi/2) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

Thus, $1 - 1/3 + \frac{1}{5} - 1/7 + \cdots = \pi/4$.

[$\because$ $0 < x < \pi$, $f(x) = k$
$\therefore f(\pi/2) = k$]

This is a famous result by Leibnitz (obtained in 1673 from geometrical considerations). It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points.

6. Orthogonality of the Trigonometric system

The trigonometric system —

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$$

is orthogonal on the interval $-\pi \le x \le \pi$ (& hence on any interval of length 2π, because of periodicity).

By def^n, this means that the integral of the product of any two different of these functions over that interval is zero, in formulas, for any integers m & $n \neq m$, we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad (m \neq n)$$

& $$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad (m \neq n),$$

& for any integers m & n (including $m = n$) we have

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 .$$

This is the most important property of the trigonometric system, the key in deriving the Euler formulas.

7. Convergence & Sum of Fourier series :-

The class of functions that can be represented by Fourier series is surprisingly large & general.

Corresponding sufficient condn covering almost any conceivable application are as follows :-

(Dirichlet Theorem)

Th-1 / (Representation by a Fourier series)

If a periodic function $f(x)$ with period 2π is piece-wise continuous in the interval in the interval $-\pi \le x \le \pi$ & has a left-hand derivative & right-hand derivative at each point of that interval, then the Fourier series ⑧ of $f(x)$ [with co-efficients ⑦ (see earlier)] is convergent. Its sum is $f(x)$ except at a point x_0 at which $f(x)$ is discontinuous & the sum of the series is the average of the left- & right-hand limits of $f(x)$ at x_0.

Proof:- We prove convergence for a continuous function $f(x)$ having continuous first & second derivatives.

Integrating the equation —

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2, \ldots$$

by parts, we get

$$a_n = \underbrace{\left[\frac{f(x)\sin nx}{n\pi}\right]_{-\pi}^{\pi}}_{(=0)} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx.$$

$$\therefore\ a_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx.$$

Another integration by parts gives

$$a_n = \left[\frac{f'(x)\cos nx}{n^2\pi}\right]_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx.$$

The first term on the right is zero because of the periodicity & continuity of $f'(x)$.

[as $\frac{f'(\pi)\cos n\pi}{n^2\pi} - \frac{f'(-\pi)\cos n\pi}{n^2\pi} = 0$. & $f'(-\pi) = f'(-\pi+2\pi) = f'(\pi)$]

Since $f''(x)$ is continuous in the interval of integration, we have —

$$|f''(x)| < M \quad (\because \text{it is bounded})$$

for an appropriate constant M.

Furthermore, $|\cos nx| \leq 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx\, dx\right|$$

$$< \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M\, dx$$

$$= \frac{2M}{n^2}$$

for all n.

Slly, $|b_n| < \frac{2M}{n^2}$,

Hence, the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding terms of the series.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

$$\Rightarrow |f(x)| \leq |a_0| + \sum_{n=1}^{\infty}(|a_n| + |b_n|)$$

$$\therefore |f(x)| < |a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent.

[by p-series test $\sum_{n=1} \frac{1}{n^p}$ convergent, $p > 1$]

$\therefore |f(x)|$ is convergent $\Rightarrow f(x)$ is convergent. Hence, the Fourier series converges & the proof is complete. The proof of convergence in the case of a piece-wise continuous function $f(x)$ & the proof that under the assumptions in the theorem the Fourier series (8) with co-efficients (7) represents $f(x)$ are substantially more complicated.

Eg. 2) Convergence at a jump as indicated by Th-1:

The square wave in Eg. 1) has a jump at $x=0$. Its left-hand limit there is $(-k)$ & its right-hand limit is k (at 0). Hence, the average of these two limits is 0. The Fourier series (9) of the square wave is

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

does indeed converge to this value when $x=0$ because then all its terms are 0. Similarly, for the other jumps.

This is in agreement with Th-1.

• **Note :-** A Fourier series of a given function $f(x)$ of period 2π is a series of the form (8) with co-efficients given by the Euler formulas (7). Th-1 gives conditions that are sufficient for this series to converge & at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand & right-hand limits of $f(x)$ at that point.

---

8. Functions of any period $p = 2L$:-

The functions considered so far had period 2π, for simplicity. Of course, in applications, periodic functions will generally have other periods. But we show that the transition from period $p = 2\pi$ to period $p = 2L$ is quite simple. It amounts to a stretch (or, contraction) of scale on the axis.

Th-2 / If a function $f(x)$ of period $P = 2L$ has a Fourier series, we claim that this series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \longrightarrow (1)$$

with the Fourier co-efficients of $f(x)$ given by the Euler formulas

(a) $a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx.$

(b) $a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$

(c) $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$

$\longrightarrow (2)$

The series in eqn (1) with arbitrary co-efficients is called a trigonometric series & Th-2 can be extended to any period P.

Proof:- Eqns (1) & (2) follow by a change of scale, say, by setting

$v = \dfrac{\pi x}{L}$. Then $x = \dfrac{Lv}{\pi}$. Also, $x = \pm L$ corresponds to $v = \pm\pi$.

Thus f, regarded as a function of v that we call $g(v)$.

Now, $f(x) = g(v)$, has period 2π.

Accordingly, by (8) & (7), with v instead of x, this 2π-periodic function $g(v)$ has the Fourier series. (31)

$$g(v) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nv + b_n \sin nv \right) \longrightarrow (3)$$

with co-efficients

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\, dv,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv\, dv,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv\, dv \longrightarrow (4)$$

Since $v = \frac{\pi x}{L}$ & $g(v) = f(x)$, formula (3) gives (1). In (4), we introduce

$x = \frac{Lv}{\pi}$ as variable of integration.

Then the limits of integration $v = \pm \pi$ become $x = \pm L$. Also, $v = \frac{\pi x}{L}$ implies

$$dv = \frac{\pi\, dx}{L}.$$

Thus, $\frac{dv}{2\pi} = \frac{dx}{2L}$ in a_0 $\left[\because a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \right]$

Similarly, $\frac{dv}{\pi} = \frac{dx}{L}$ in a_n & b_n. $\left[\because a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \ n = 1, 2, \ldots \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \ n = 1, 2, \ldots \right]$

Hence, (4) gives (2).

Interval of integration :-

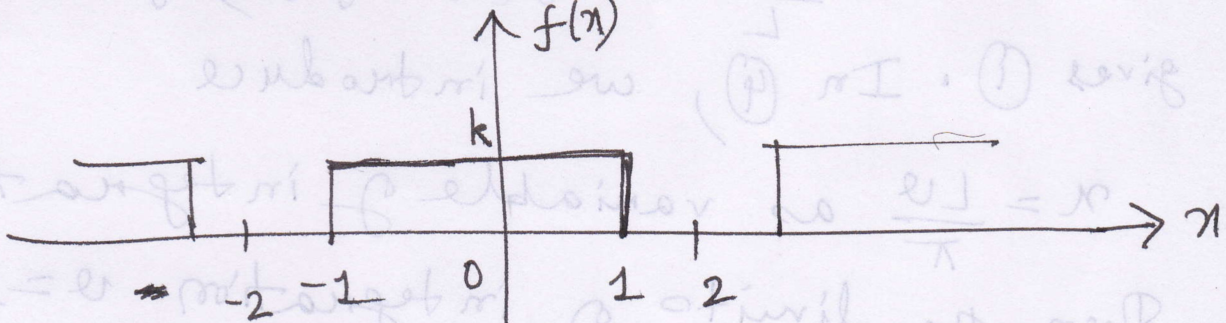
In (2), we may replace the interval of integration by any interval of length $p = 2L$, for example, by the interval $0 \le x \le 2L$.

Eg. 1) **Periodic square wave:**

Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2, \end{cases}$$

$P = 2L = 4, \quad L = 2.$



Sol^n :- From eq^n (2a) & (2b) (see earlier), we obtain

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = k/2 .$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right) d$$

$\therefore \; a_n = \dfrac{2k}{n\pi} \sin\left(\dfrac{n\pi}{2}\right)$.

Thus $a_n = 0$ if n is even

& $a_n = \dfrac{2k}{n\pi}$ if $n = 1, 5, 9, \ldots$

& $a_n = \dfrac{-2k}{n\pi}$ if $n = 3, 7, 11, \ldots$

From (2c), we find that

$b_n = 0$, for $n = 1, 2, \ldots$ (as odd f^n).

Hence, the result is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + 0 \cdot \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= k/2 + \frac{2k}{\pi}\left(\cos \frac{\pi}{2}x - \frac{1}{3}\cos \frac{3\pi}{2}x + \frac{1}{5}\cos \frac{5\pi}{2}x - \cdots\right)$$

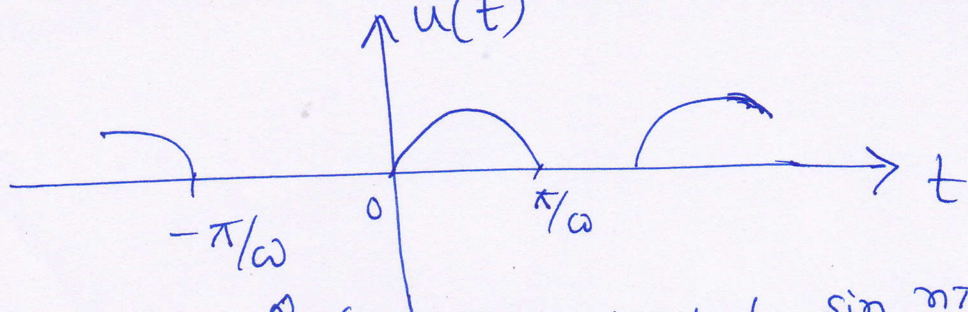
Eg. 2/ **Half-wave rectifier:**

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0, & \text{if } -L < t < 0 \\ E \sin \omega t, & \text{if } 0 < t < L, \end{cases}$$

$P = 2L = 2\pi/\omega$, $L = \pi/\omega$.

Sol^n :-



$$u(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$= \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right)$$

---

9. **Even & odd functions :**
(Half - Range Expansions)

The function in the previous example (e.g.: periodic square wave) was even & had only cosine terms in its Fourier series, no sine terms. This is typical. In fact, unnecessary work (& corresponding sources of errors) in determining Fourier coefficients can be avoided if a function is even or odd.

. **Even & odd Functions :-**

A function $y = g(x)$ is even if
$$g(-x) = g(x), \ \forall x.$$
The graph of such a function is symmetric w.r.t the y-axis.

A function $h(x)$ is odd if $h(-x) = -h(x), \forall x$.

The function $\cos nx$ is even, while $\sin nx$ is odd.

($h(x)$ is anti-symmetric, it shows symmetry w.r.t the origin).

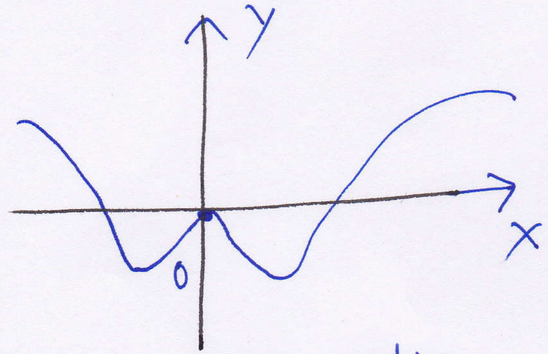


Fig 1: Even function

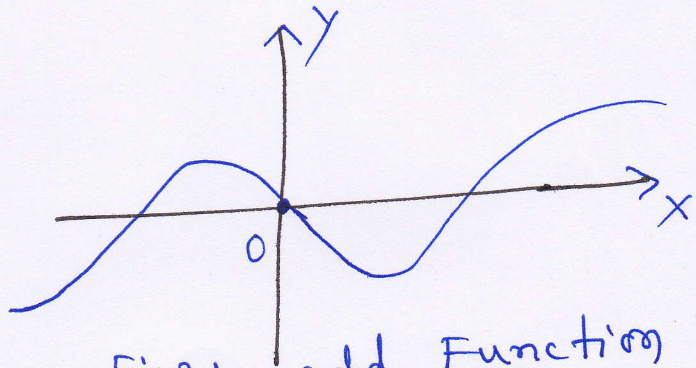


Fig 2:- odd Function

. Note:- 1) If $g(x)$ is an even function, then

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx \longrightarrow (1) \quad (g \text{ even})$$

2) If $h(x)$ is an odd function, then

$$\int_{-L}^{L} h(x)\,dx = 0 \longrightarrow (2) \quad (h \text{ odd})$$

3) The product of an even and an odd function is odd.

Proof:- Eqns (1) & (2) are obvious from the graphs of g & h.

Now, $q = gh$ with even g & odd h because

$$q(-x) = g(-x)\,h(-x) = g(x)\,(-h(x))$$

$$= -g(x)\,h(x) = -q(x)$$

$\therefore$ $q(x)$ is odd.

Hence, if $f(x)$ is even, then $f(x)\sin\left(\frac{n\pi}{L}x\right)$ is odd, so (2) implies that $b_n = 0$ in (2c) (last section done earlier).

Similarly, if $f(x)$ is odd, so is $f(x)\cos\left(\frac{n\pi x}{L}\right)$ & $a_n = 0$, $a_0 = 0$ in (2a), (2b).

From this & (1), we have

$$\left[f(x) = a_0 + \sum\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)\right]$$

Th-1 / Fourier Cosine Series, Fourier Sine Series

The Fourier series of an even function of period $2L$ is a "Fourier Cosine Series"

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \quad (f \text{ even}) \longrightarrow (3)$$

with co-efficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx, \quad n = 1, 2, \cdots \longrightarrow (4)$$

The Fourier series of an odd function of period $2L$ is a "Fourier Sine series"

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \quad (f \text{ odd}) \longrightarrow (5)$$

with co-efficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \longrightarrow (6)$$

- The case of period 2π :-

In this case Th-1 gives for an even function simply

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad (f \text{ even}) \quad \rightarrow (3^*)$$

with co-efficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2, \ldots \quad \rightarrow (4^*)$$

Similarly, for an odd 2π-periodic function, we simply have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad (f \text{ odd}) \quad \rightarrow (5^*)$$

with co-efficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots \quad \rightarrow (6^*).$$

e.g., $f(x)$ in Eg-1 (Rectangular wave, Fourier series) is odd & therefore, is represented by a Fourier sine series.

Th-2 / (sum of functions)

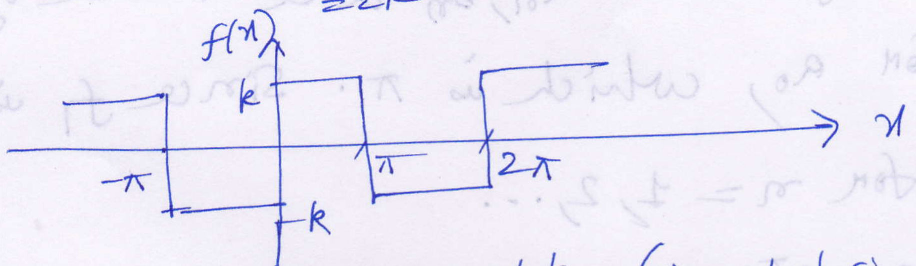
The Fourier co-efficients of a sum $(f_1 + f_2)$ are the sums of the corresponding Fourier co-efficients of f_1 & f_2.

The Fourier co-efficients of cf are c times the corresponding Fourier co-efficients of f.

Eg. 1) Rectangular Pulse :-

The function $f^*(x)$ in Fig. 3 is the sum of the function $f(x)$ in Eg. 1 $\left[\text{ie, } f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}\right]$ & the constant k.

$$\therefore f^*(x) = f_1 + f_2 = \begin{cases} \underset{(=0)}{-k+k} & \text{if } -\pi < x < 0 \\ \underset{=2k}{k+k} & \text{if } 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x).$$



Hence, $f^*(x) = k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

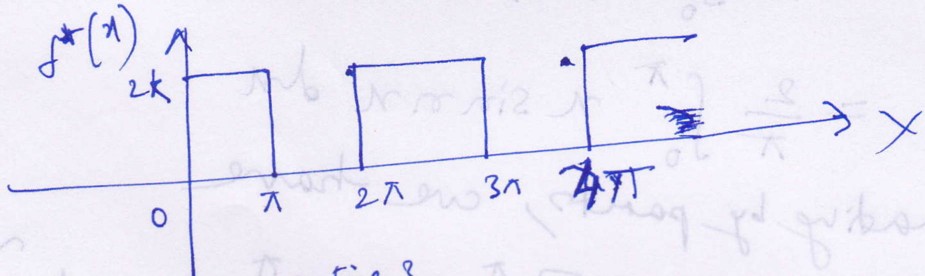


Fig 3

Eg. 2) Sawtooth wave :-

Find the Fourier series of the function

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi$$

$$\& \quad f(x+2\pi) = f(x).$$

Sol^n :-

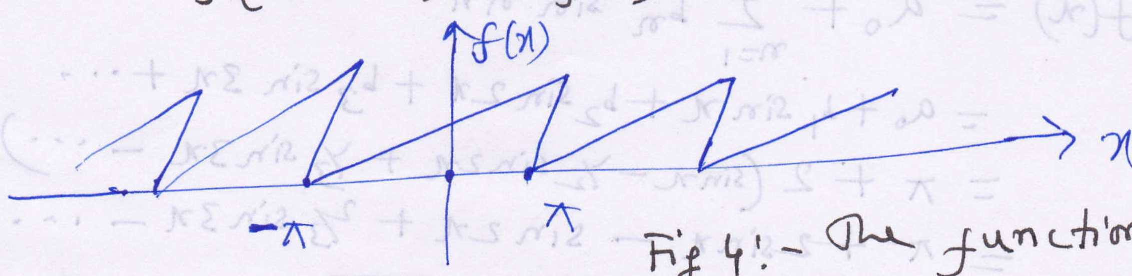


Fig 4 :- The function $f(x)$.

We may write $f = f_1 + f_2$, where $f_1 = x$ & $f_2 = \pi$
(odd) (even)

The Fourier co-efficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Th-2 (sum of functions), the Fourier co-efficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$, for $n = 1, 2, \ldots$

& $$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

Integrating by parts, we have

$$b_n = \frac{2}{\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}$$

$$= -\frac{2}{n}\cos n\pi .$$

Hence, $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4, \ldots$

& the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= a_0 + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$= \pi + 2\left(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \cdots\right)$$

$$= \pi + 2\sin x - \sin 2x + \tfrac{2}{3}\sin 3x - \cdots$$

10. Half - Range Expansions :-

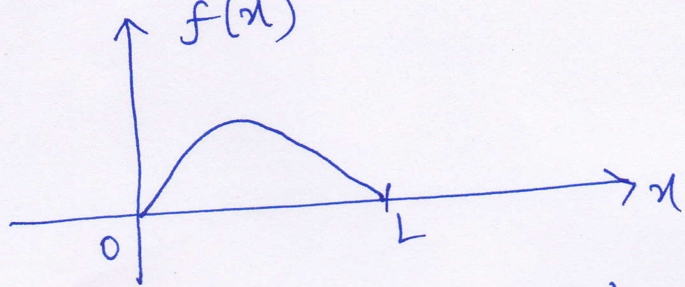
This concerns a practically useful simple idea. In applications, we often want to employ a Fourier series for a function f that is given only on some interval, say, $0 \le x \le L$ as in Fig 1(a). This function f can be the displacement of a violin (violin) string of (undistorted) length L or the temperature in a metal bar of length L, & so on.

Now, the key idea is as follows.

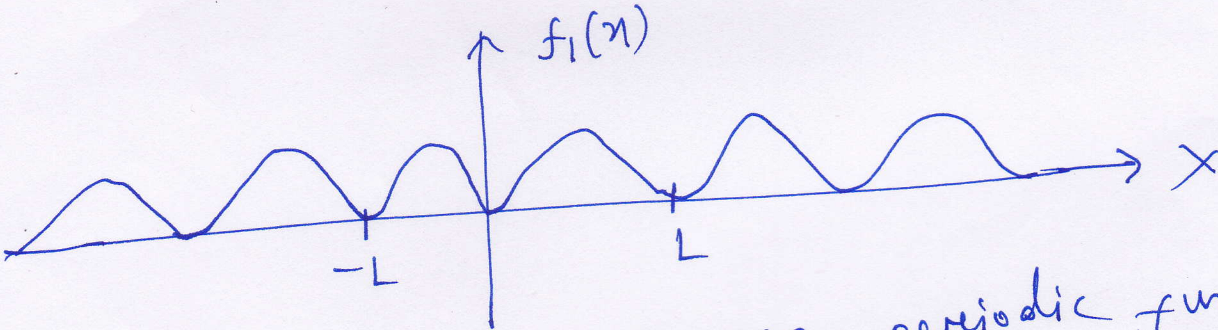
For our function f we can calculate Fourier co-efficients from 4 or 6 in Th-1 (see Fourier cosine series). And we have a choice. If we use 4, we get a Fourier cosine series 3. This series represents the even periodic extension f_1 of f in Fig1(b).

If in a practical problem we think that using 6 is better, we get a Fourier sine series 5. This series represents the odd periodic extension f_2 of f in Fig 1(c).

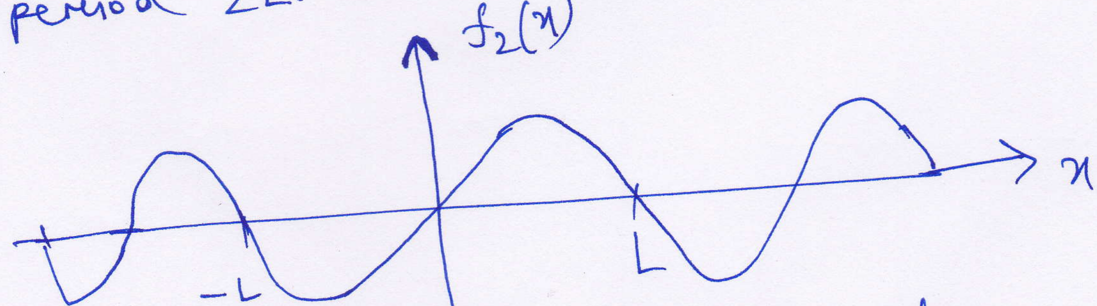
Both extensions have period $2L$. This motivates the name <u>half-range expansions</u>: f is given (& of physical interest) only on half the range, half the interval of periodicity of length $2L$.



(a) The given function $f(x)$



(b) $f(x)$ extended as an even periodic function of period $2L$.



(c) $f(x)$ extended as an odd periodic function of period $2L$.

Fig 1 :- (a) Function $f(x)$ given on an interval $0 \le x \le L$.

(b) Even extension to the full "range" (interval) $-L \le x \le L$ (heavy curve) & the periodic extension of period $2L$ to the x-axis.

(c) Odd extension to $-L \le x \le L$ (heavy curve) & the periodic extension of period $2L$ to the x-axis.

Eg-1) "Triangle" & its half range expansions:

Find the two half-range expansions of the function (Fig 2)

$$f(x) = \begin{cases} \frac{2k}{L} x, & \text{if } 0 < x < L/2 \\ \frac{2k}{L}(L-x), & \text{if } L/2 < x < L. \end{cases}$$

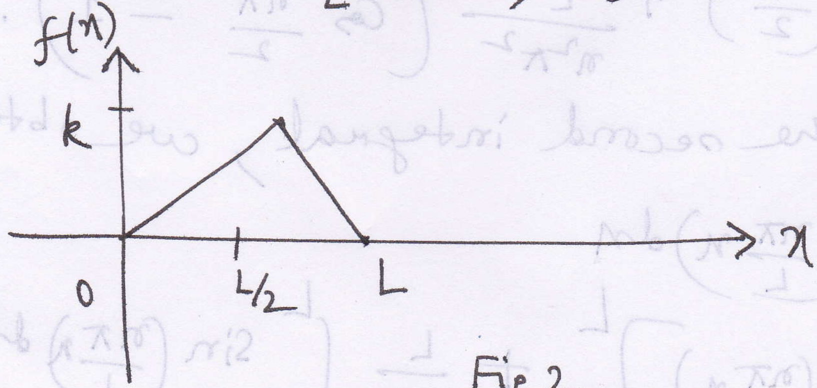


Fig 2

Sol^n:– (a) Even periodic extension

From eq^n (4), ie, $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$, we get

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx,$$

$n = 1, 2, \cdots,$

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\,dx\right]$$

$$= k/2.$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\left(\frac{n\pi}{L}x\right)dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)dx\right]$$

We consider a_n. For the first integral, we obtain by integration by parts —

$$\int_0^{L/2} x\cos\left(\frac{n\pi}{L}x\right)dx = \left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\right]_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Similarly, for the second integral, we obtain —

$$\int_{L/2}^{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \left[\frac{L}{n\pi}(L-x)\sin\left(\frac{n\pi}{L}x\right)\right]_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\left(\frac{n\pi}{L}x\right)dx$$

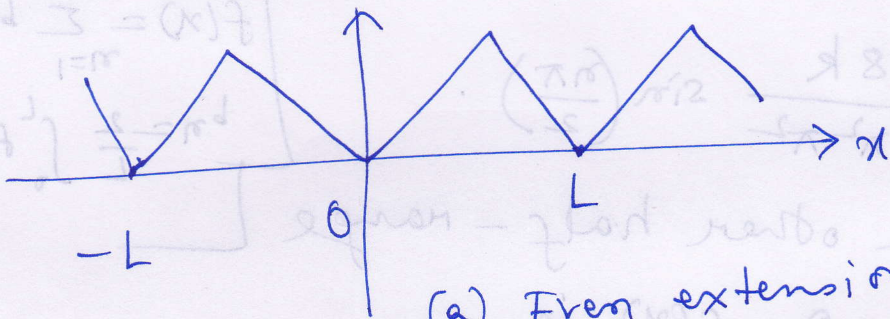
$$= 0 - \frac{L}{n\pi}(L - L/2)\sin n\pi/2 - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos n\pi/2\right).$$

We insert these two results into the formula for a_n. The sine terms cancel & so does a factor L^2. This gives

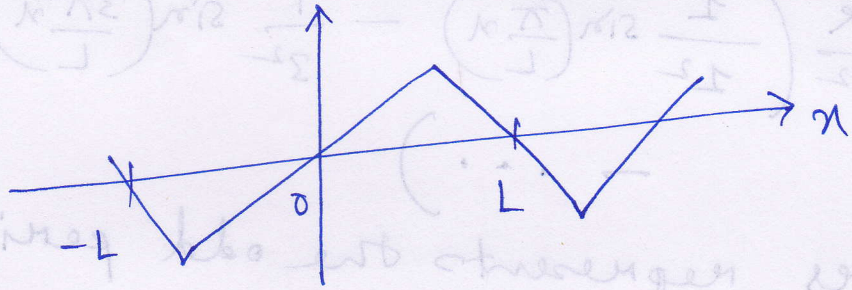
$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus $a_2 = -\frac{16k}{2^2\pi^2}$, $a_6 = -\frac{16k}{6^2\pi^2}$,

$a_{10} = -\frac{16k}{10^2\pi^2}$ etc. [At $n = 1, 3, 5, \ldots$, $a_n = 0$]



(a) Even extension



(b) odd extension

Fig 3 :- Periodic extension of $f(x)$ in Fig 2.

& $a_n = 0$ if $n \neq 2, 6, 10, 14, \ldots$

Hence, the first half-range expansion of $f(x)$ is

$$f(x) = k/2 - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$, shown in Fig 3(a).

(b) Odd periodic extension :

Similarly, from (6), we obtain

[ie, from
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x,$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$
]

$$b_n = \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

Hence, the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\left(\frac{\pi}{L}x\right) - \frac{1}{3^2}\sin\left(\frac{3\pi}{L}x\right) + \frac{1}{5^2}\sin\left(\frac{5\pi}{L}x\right) - \cdots\right)$$

This series represents the odd periodic extension of $f(x)$ of period $2L$, shown in Fig 3(b).

Note :- Half range series, are, as the name implies, series defined over half of the normal range. That is, for standard trigonometric Fourier series, the function $f(x)$ is defined only in $[0, \pi]$ instead of $[-\pi, \pi]$.

The value that $f(x)$ takes in the other half of the interval $[-\pi, 0]$ is free to be defined.

If we take $f(x) = f(-x)$, ie, $f(x)$ is even, then the Fourier series for $f(x)$ can be entirely expressed in terms of even functions ie, cosine entirely.

If on the other hand, $f(x) = -f(-x)$ ie, $f(x)$ is an odd function, then the Fourier series is correspondingly odd & consists only of sine terms.

We are not defining the same function as two different Fourier series, for $f(x)$ is different, at least over half the range.

11. Complex Fourier Series :

The Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

can be written in complex form, which sometimes simplifies calculations.

This is done by the Euler formula, with nx instead of x, that is,

$$e^{inx} = \cos nx + i \sin nx \longrightarrow (2)$$

$$e^{-inx} = \cos nx - i \sin nx \longrightarrow (3)$$

By addition of these two formulas (2) & (3), & division by 2, we get

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}) \longrightarrow (4)$$

Subtraction & division by $(2i)$ gives —

$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx}) \longrightarrow (5)$$

From this, using $\frac{1}{i} = -i$, we have in (1),

$$a_n \cos nx + b_n \sin nx = \frac{1}{2} a_n (e^{inx} + e^{-inx})$$

$$+ \frac{1}{2i} b_n (e^{inx} - e^{-inx})$$

$$= \frac{1}{2}(a_n - i b_n) e^{inx} + \frac{1}{2}(a_n + i b_n) e^{-inx}$$

We insert this into (i), writing

$$a_0 = c_0, \quad \frac{a_n - ib_n}{2} = c_n, \quad \& \quad \frac{a_n + ib_n}{2} = k_n.$$

Then (i) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + k_n e^{-inx} \right) \longrightarrow (6)$$

For the co-efficients $c_1, c_2, \dots$ & $k_1, k_2, \dots$, we obtain from (2'), (3) & the Euler formulas (6),

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx - i\sin nx\right) dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx,$$

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx + i\sin nx\right) dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx \longrightarrow (7)$$

Finally, we can combine the two formulas (7) into one by the trick of writing $k_n = c_{-n}$. Then (6), (7) together with (6a) $\left[\text{ie., } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\right]$ give

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\longrightarrow (8)$$

This is the so-called <u>complex form of the Fourier series</u>, or, more briefly, the <u>complex Fourier series</u>, of $f(x)$. The c_n are called the complex <u>Fourier co-efficients of $f(x)$</u>. For a function of period $2L$, our reasoning gives the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L},$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx, \quad \longrightarrow (9)$$

This complex form of Fourier series is true when $f(x)$ is continuous at x & the Dirichlet's conditions are satisfied.

If $f(x)$ is discontinuous at x, the L.H.S of $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$ is replaced by $\frac{f(x+0) + f(x-0)}{2}$

ie, $f(x) \longmapsto \frac{f(x+0) + f(x-0)}{2}$.

Eg. 1) Complex Fourier Series :-

Find the complex Fourier series of

$f(x) = e^x$ if $-\pi < x < \pi$ &

$f(x+2\pi) = f(x)$, & obtain

from it the usual Fourier series.

Sol^n :- Since $\sin n\pi = 0$ for integer n, we have

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = \cos n\pi = (-1)^n.$$

With this we obtain from (8) by integration

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cdot e^{-inx}\, dx = \frac{1}{2\pi}\cdot\frac{1}{(1-in)}\left[e^{x-inx}\right]_{x=-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-in)}\left[e^{\pi-in\pi} - e^{-\pi+in\pi}\right]$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-in)}\left(e^{\pi}-e^{-\pi}\right)(-1)^n.$$

On the right,

$$\frac{1}{(1-in)} = \frac{1+in}{(1-in)(1+in)} = \frac{1+in}{1+n^2} \quad \&$$

$$e^{\pi} - e^{-\pi} = 2\sinh\pi,$$

$$\therefore c_n = \frac{1}{\pi}\sinh(\pi)(-1)^n\,\frac{(1+in)}{(1+n^2)}.$$

Hence, the complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\therefore e^x = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(1+in)}{(1+n^2)} e^{inx} \quad (-\pi < x < \pi) \longrightarrow (10)$$

From this let us derive the real Fourier series. Using (2) & $i^2 = -1$, we have in (10).

$$(1+in)\, e^{inx} = (1+in)(\cos nx + i \sin nx)$$
$$= (\cos nx - n \sin nx) + i\,(n \cos nx + \sin nx)$$

Now, (10) also has a corresponding term with $(-n)$ instead of n. Since $\cos(-nx) = \cos nx$ & $\sin(-nx) = -\sin nx$, we obtain in this term

$$(1-in)\, e^{-inx} = (1-in)(\cos nx - i \sin nx)$$
$$= (\cos nx - n \sin nx) - i\,(n \cos nx + \sin nx).$$

If we add these two expressions, the imaginary parts cancel

Hence, their sum is

$$2(\cos nx - n\sin nx), \quad n = 1, 2, \ldots$$

For $n = 0$, we get 1 (not 2) because there is only one term. Hence, the real Fourier series is —

$$e^x = 2\,\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(\cos nx - n\sin nx)}{(1+n^2)},$$

$$+\frac{\sinh\pi}{\pi} \qquad n = 1, 2, \ldots$$

$$= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) + \cdots\right],$$

where $-\pi < x < \pi$.

11. Properties of Fourier series :-

In this section we shall be concerned with the integration & differentiation of Fourier series.

Hence, their sum is

$$2(\cos nx - n \sin nx), \quad n = 1, 2, \ldots$$

For $n=0$, we get 1 (not 2) because there is only one term. Hence, the real Fourier series is —

$$e^x = \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n (\cos nx - n \sin nx)}{(1+n^2)}, \quad n = 1, 2, \ldots$$

$$+ \frac{\sinh \pi}{\pi}$$

$$= \frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2 \sin 2x) - \cdots \right],$$

where $-\pi < x < \pi$.

11. Properties of Fourier series :-

In this section we shall be concerned with the integration & differentiation of Fourier series.

Th-1/ If f is continuous on $[-\pi, \pi]$ & piecewise differentiable in $(-\pi, \pi)$ which means that the derivative f' is piecewise continuous on $[-\pi, \pi]$, & if $f(x)$ has the Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx],$$

then the Fourier series of the derivative of $f(x)$ is given by

$$f'(x) = \sum_{n=1}^{\infty} [-n a_n \sin nx + n b_n \cos nx].$$

Th-2/ If f is piecewise continuous on the interval $[-\pi, \pi]$ & has the Fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx],$$

then for each $x \in [-\pi, \pi]$,

$$\int_{-\pi}^{x} f(t)\,dt = \frac{1}{2} a_0 (x+\pi) + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin nx - \frac{b_n}{n} (\cos nx - \cos n\pi)\right]$$

& the function on the right converges uniformly to the function on the left.

Th-3/ If $f(t)$ & $g(t)$ are continuous in $(-\pi, \pi)$ & provided

$$\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty \ \& \ \int_{-\pi}^{\pi} |g(t)|^2 dt < \infty,$$

if a_n, b_n are the Fourier Co-efficients of $f(t)$ & α_n, β_n those of $g(t)$, then

$$\int_{-\pi}^{\pi} f(t) g(t)\, dt = \frac{1}{2}\pi a_0 \alpha_0 + \pi \sum_{n=1}^{\infty} (\alpha_n a_n + \beta_n b_n).$$

Proof:– Since

$$f(t) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

$$\& \ g(t) = \alpha_0/2 + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt),$$

we can write

$$f(t) g(t) = \frac{a_0}{2} g(t) + \sum_{n=1}^{\infty} (a_n g(t) \cos nt + b_n g(t) \sin nt).$$

Integrating this series from $(-\pi)$ to π, gives

$$\int_{-\pi}^{\pi} f(t) g(t)\, dt = \frac{1}{2} a_0 \int_{-\pi}^{\pi} g(t)\, dt + \sum_{n=1}^{\infty} \left\{ a_n \int_{-\pi}^{\pi} g(t) \cos nt\, dt + b_n \int_{-\pi}^{\pi} g(t) \sin nt\, dt \right\},$$

provided the Fourier series for $f(t) g(t)$ is uniformly convergent, enabling the summation & integration operations to be interchanged. This follows from the Cauchy – Schwarz inequality.

$$\int_{-\pi}^{\pi} |f(t)g(t)\,dt| \le \left(\int_{-\pi}^{\pi} |f(t)|^2 dt\right)^{1/2} \left(\int_{-\pi}^{\pi} |g(t)|^2 dt\right)^{1/2} < \infty.$$

However, we know that —

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt = \alpha_n$$

$$\& \ \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt = \beta_n,$$

& that $\dfrac{\alpha_0}{2} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} g(t)\,dt$

so this implies

$$\int_{-\pi}^{\pi} f(t)g(t)\,dt = \frac{1}{2}\pi a_0\alpha_0 + \pi\sum_{n=1}^{\infty}(\alpha_n a_n + \beta_n b_n).$$

If we put $f(t) = g(t)$ in the above result, the following important theorem follows.

Th-4 (Parseval)

If $f(t)$ is continuous in the range $(-\pi, \pi)$, is square integrable (i.e., $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$) & has the Fourier Co-efficients a_n, b_n then

$$\int_{-\pi}^{\pi} [f(t)]^2 dt = \frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

$$= \pi \left[a_0^2/2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right].$$

[Note :- Parseval's theorem states that the energy of a signal expressed as a waveform is proportional to the sum of the squares of its Fourier Co-efficients.

Eg. 1) Given the Fourier series

$$t^2 = \pi^2/3 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt).$$

Deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Sol^n :- Applying Parseval's theorem to this series, the L.H.S becomes

$$\int_{-\pi}^{\pi} [f(t)]^2 dt = \int_{-\pi}^{\pi} (t^2)^2 dt = \left[\frac{t^5}{5}\right]_{-\pi}^{\pi} = \frac{2}{5}\pi^5.$$

The R.H.S becomes —

$$\frac{1}{2\pi}\left(\frac{2\pi^2}{3}\right)^2 + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$= \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}.$$

Here, $f(t)=t^2$, $a_0/2 = \pi^2/3$

$\Rightarrow a_0 = 2\pi^2/3$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$\text{R.H.S} = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

$$= \pi\left[\frac{4\pi^4}{2\times 9} + \sum_{n=1}^{\infty}\frac{16(-1)^{2n}}{n^4}\right]$$

Equating these leads to

$$\frac{2}{5}\pi^5 = \frac{2}{9}\pi^5 + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \pi^4\left(\frac{2}{5}-\frac{2}{9}\right) \quad (= 8/45)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

12. Parseval's identity for Fourier Series:

Parseval's identity states that

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right),$$

where a_n & b_n are given by

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

An important consequence is that —

$$\lim_{n\to\infty}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx = 0$$

$$\lim_{n\to\infty}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = 0,$$

This is called Riemann's theorem.

13. Gibbs Phenomenon :-

To discuss the Gibbs phenomenon, let us consider the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 \le x < \pi. \end{cases}$$

The graph of $f(x)$ is given in Fig 1.

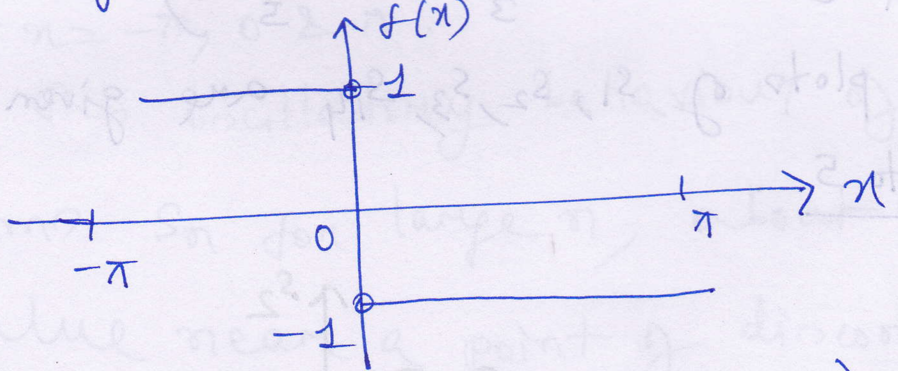


Fig 1 :- Graph of $f(x)$

The function is odd. Therefore we have a sine series. We have —

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}} \cdot \underbrace{\sin(nx)}_{\text{odd}}\, dx \quad (= \text{even})$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx.$$