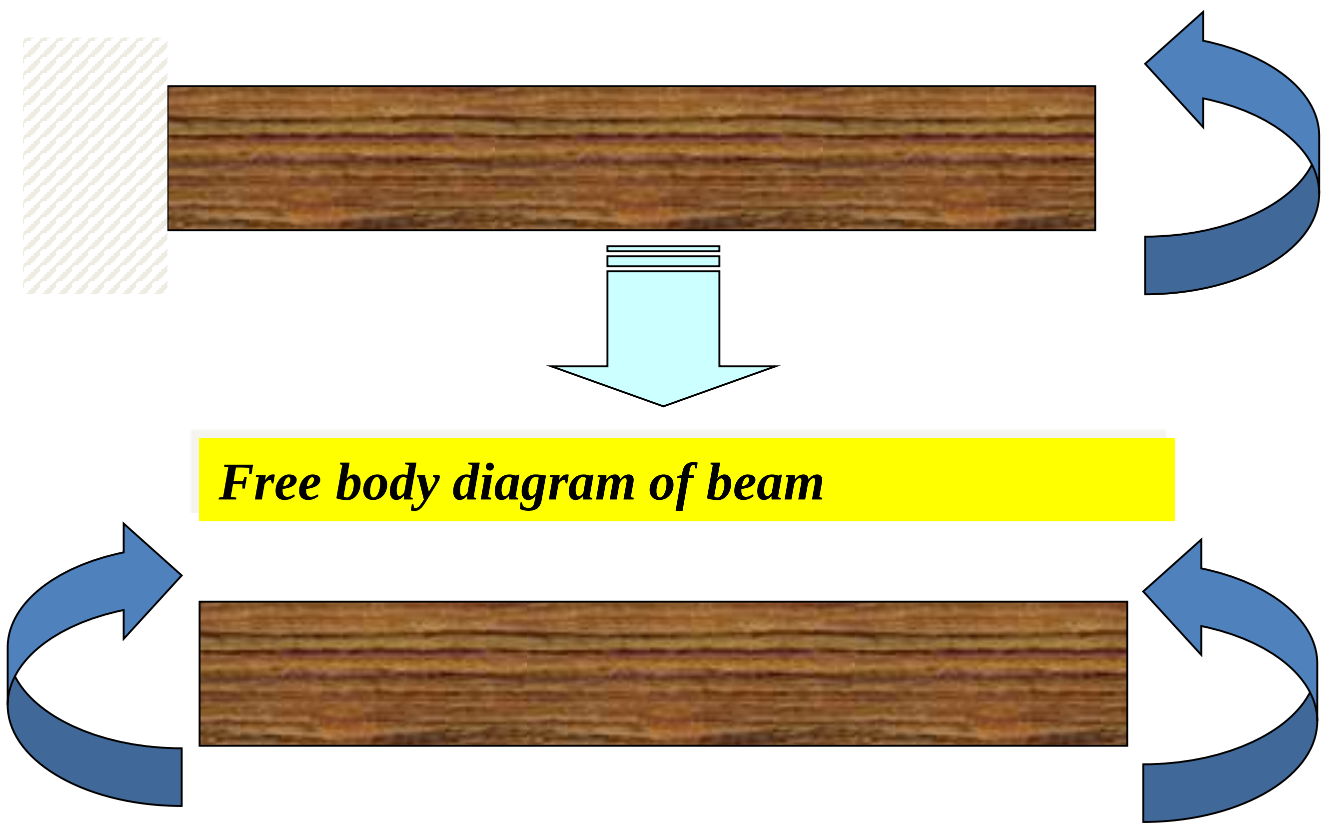
# Deflection of Beams

# Bending of a beam (Revision)

- Consider a straight beam of length L fixed at one end and a moment acting at the free end. A moment has been considered to keep the free body diagram simple

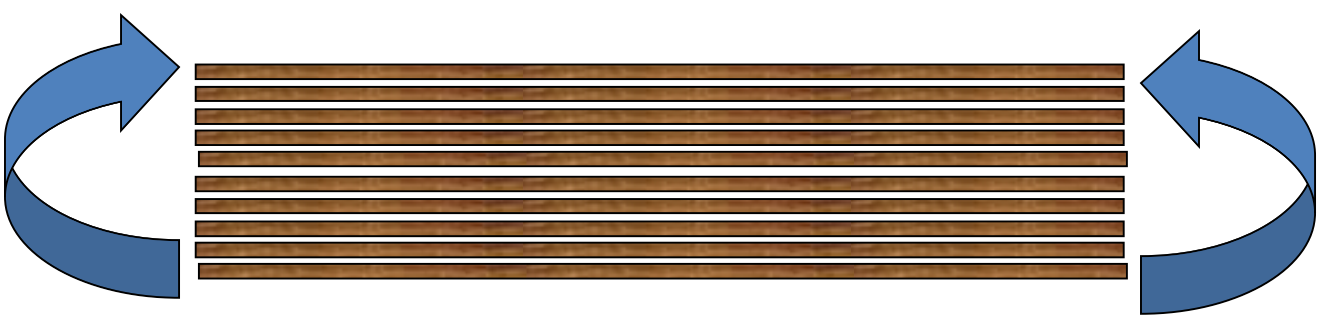
# Bending of a beam (Revision)

- Free Body Diagram of the beam



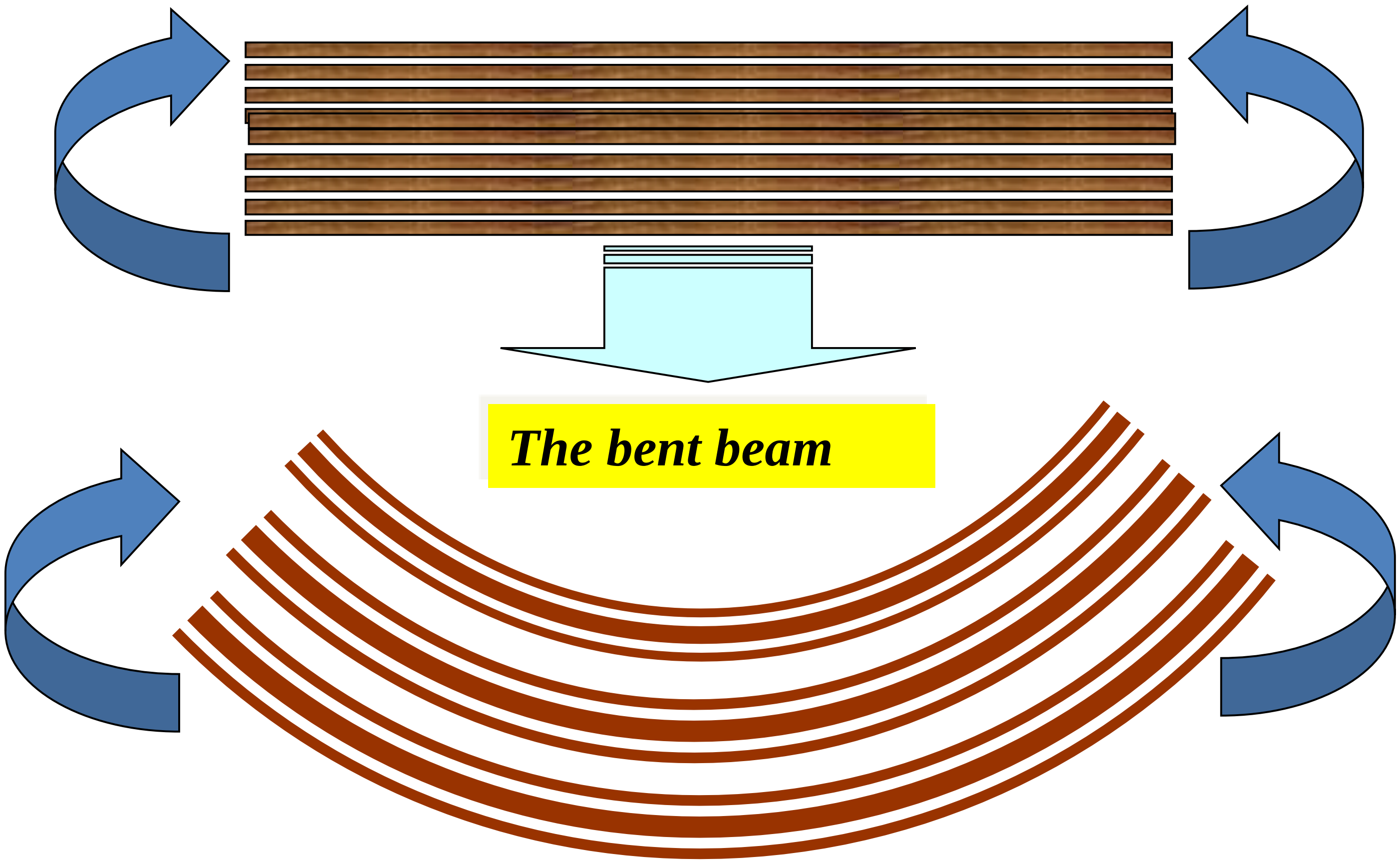
# Bending of a beam (Revision)

- We now consider the beam to be made of layers of thin beams (like a laminate or 3 ply or 5 ply plywood) all of length L.
- Actually it is made of many fibres
- This is how it looks like initially



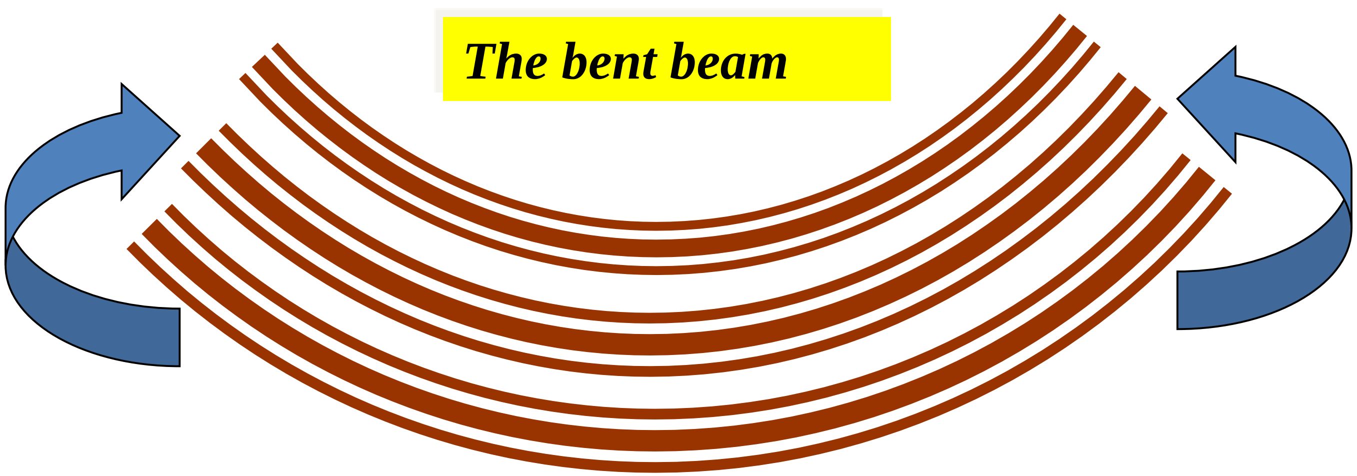
# Bending of a beam (Revision)

- After bending this is what it looks like
- To keep things simple only three fibres or plies are shown



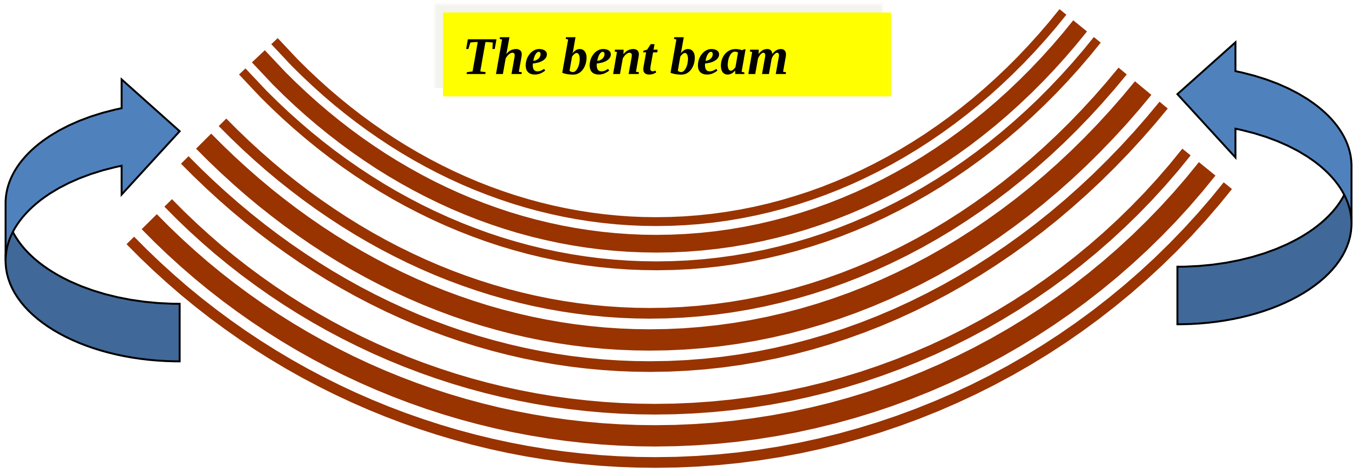
# Bending of a beam (Revision)

- *For the bent beam, it is obvious the top fibers contract to a length less than L, while the bottom fibers expand to a length greater than L*
- ***So there is a fiber in between which retains its length!***



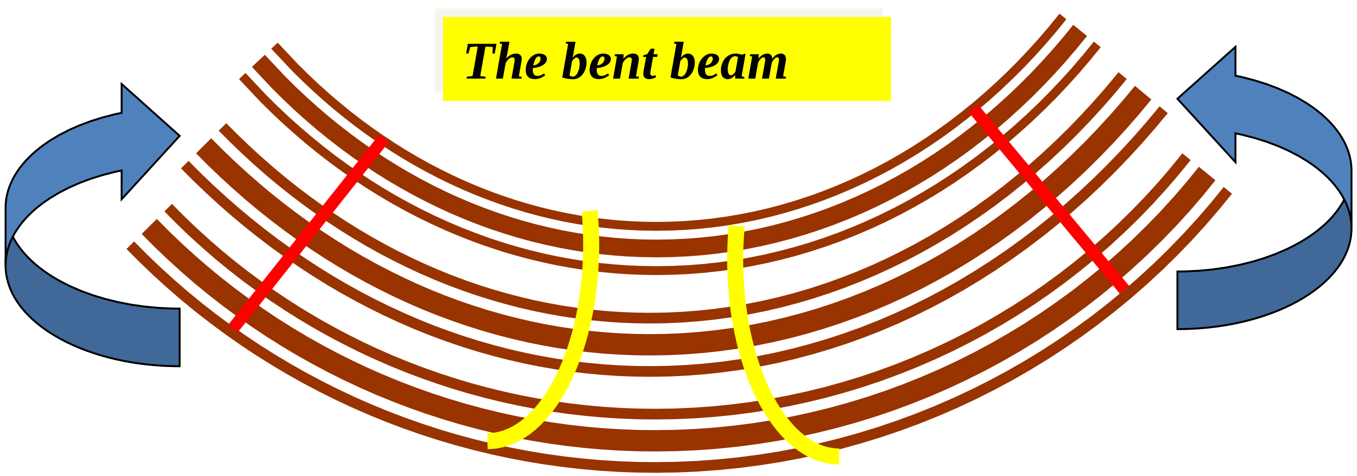
# Bending of a beam (Revision)

- *We can call this fiber or ply as a neutral fiber since it neither contracts or expands (and is hence neutral)*
- ***The plane in which this fiber lies is called the neutral plane. In 2 D analysis we call it neutral axis***



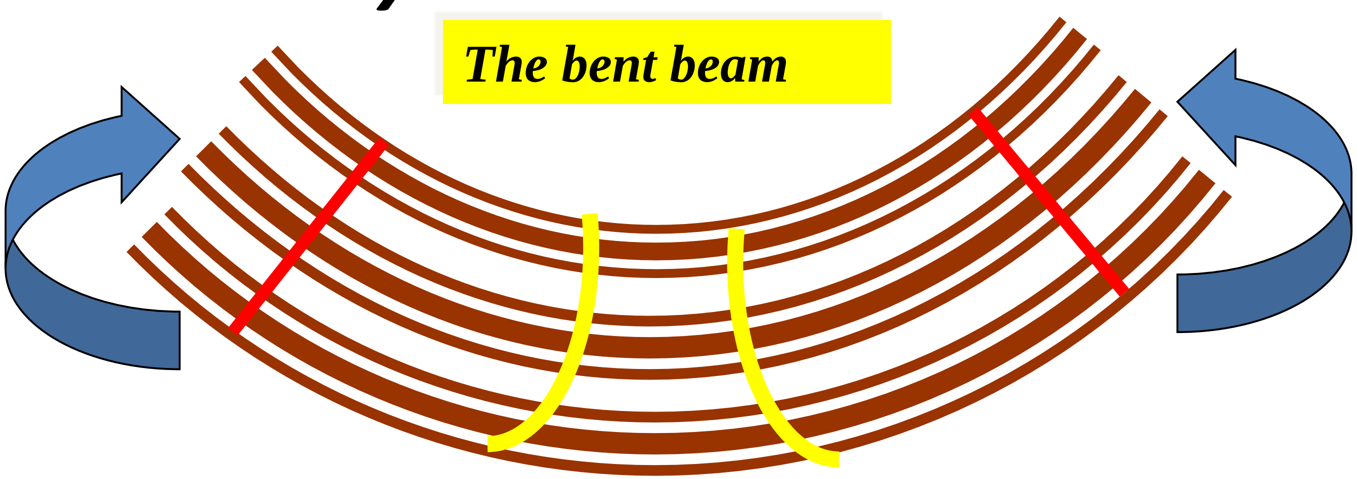
# Bending of a beam (Revision)

- *We will add another assumption, reasonably valid for small amounts of bending.*
- ***A transverse plane section of the beam remains plane after bending (red).*** *It does not look like the yellow planes.*



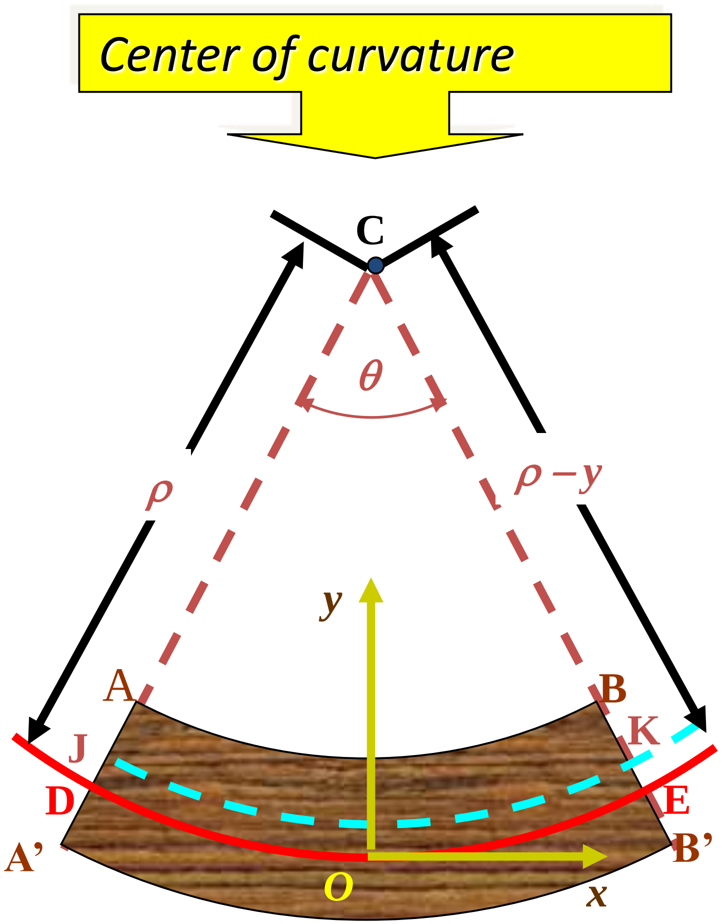
# Bending of a beam (Revision)

- *What is the consequence of this assumption ?*
- ***All the partial circles formed by the fibres or plies must be concentric!***
- ***This will let us set up a nice little coordinate system for our next analysis.***



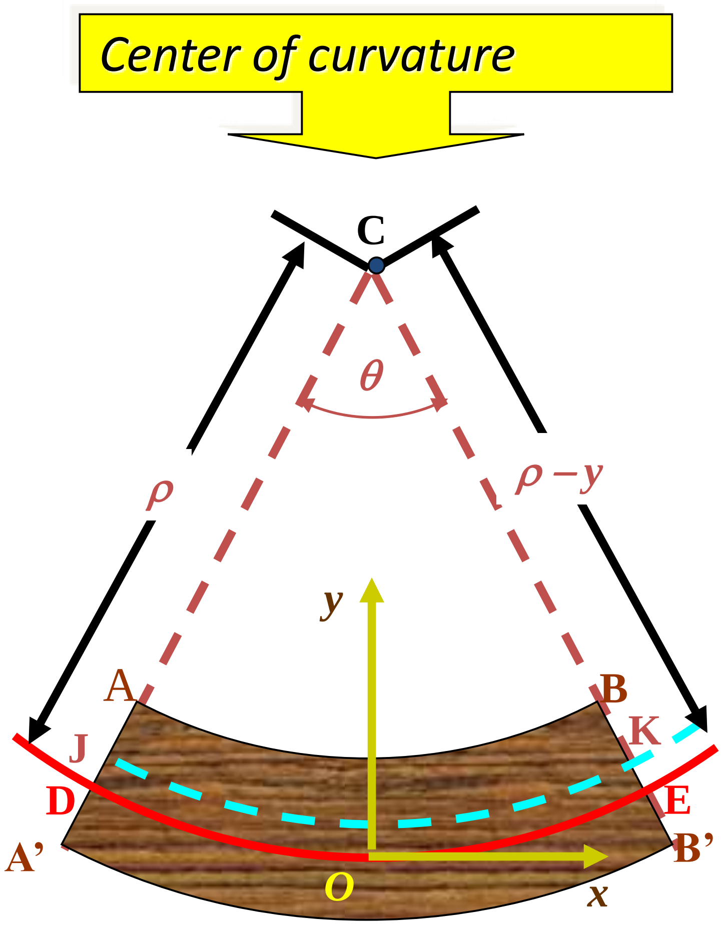
# Setting up a coordinate system

- All fibres share a common center of curvature C
- We choose that point on the neutral fibre directly below this center of curvature as our origin and the tangent to the neutral fibre as our x axis. OC is the radius of the circle formed by the neutral axis = $\rho$.
- Positive is towards right
- We a looking at a small segment of the bent beam spanning an angle $\theta$.



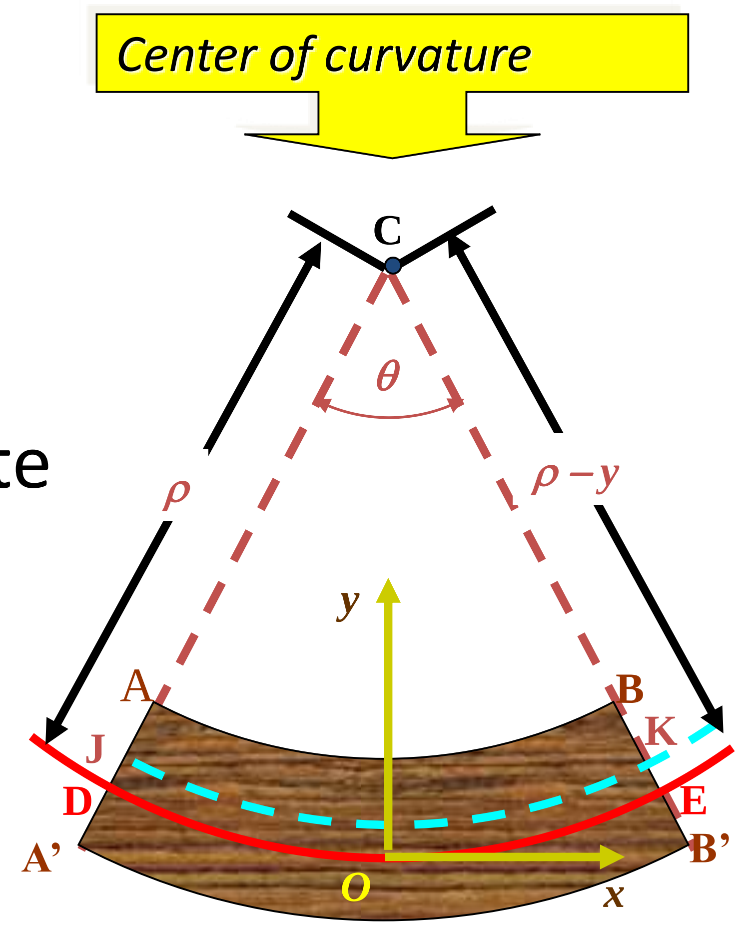
# Coordinate system is not fixed (yet)

- There is a catch however!
- We know the directions of x and y , but we do not know which fibre is the neutral fibre, and hence we do not know the origin. In other words $\rho$ is unknown! So we have a coordinate system with given orientation but we do not know where to fix it. We have a photograph but not the nail from where to hang it!



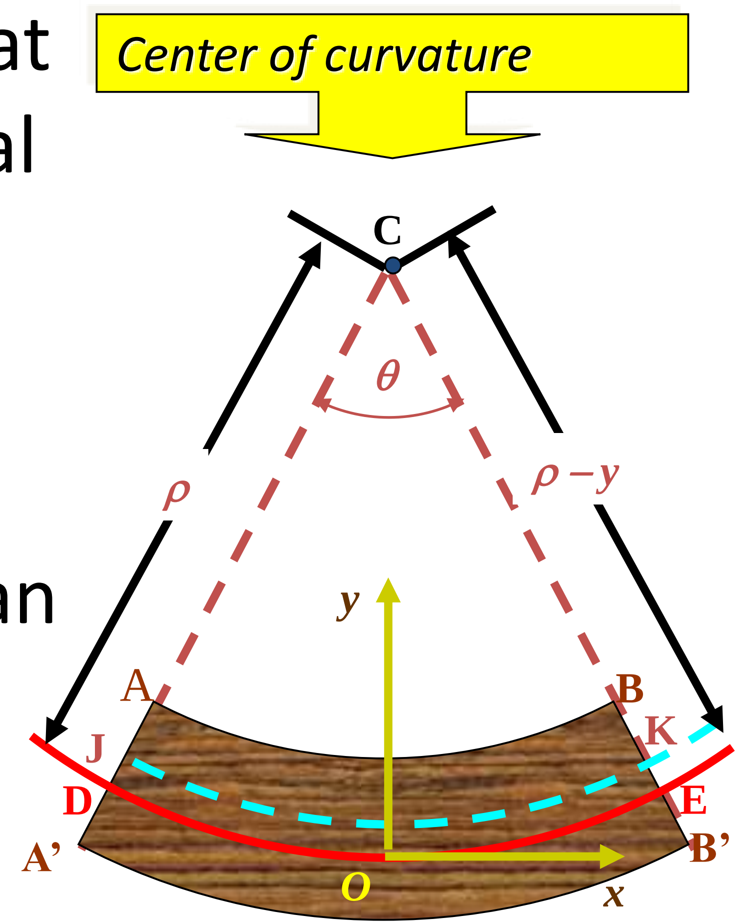
# Remedy

- We will not let this stop us.
- We will try to see if the force equilibrium equations help
- With that aim in mind we will proceed in a direction opposite to what we normally do.
- We will find out elongations, then strain, then stress and then force and moments.



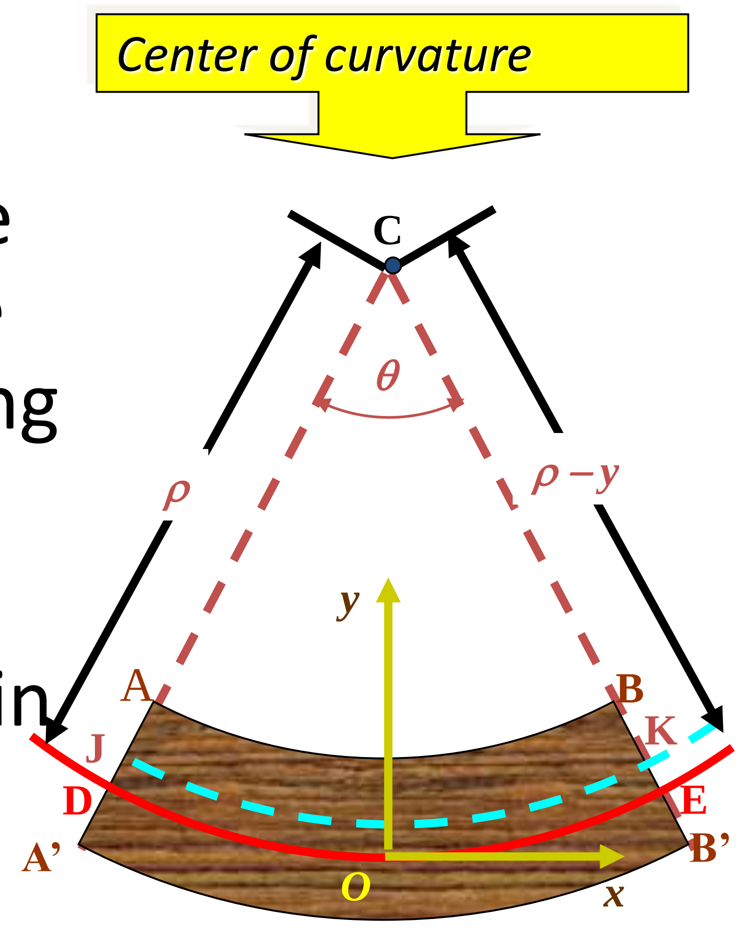
# Elongation of a fibre

- We will consider a fibre JK, at a distance y from the neutral fibre, in its bent state.
- What will be the radius of this fibre in the bent state ?
- We look at the figure and can see it is ρ -y



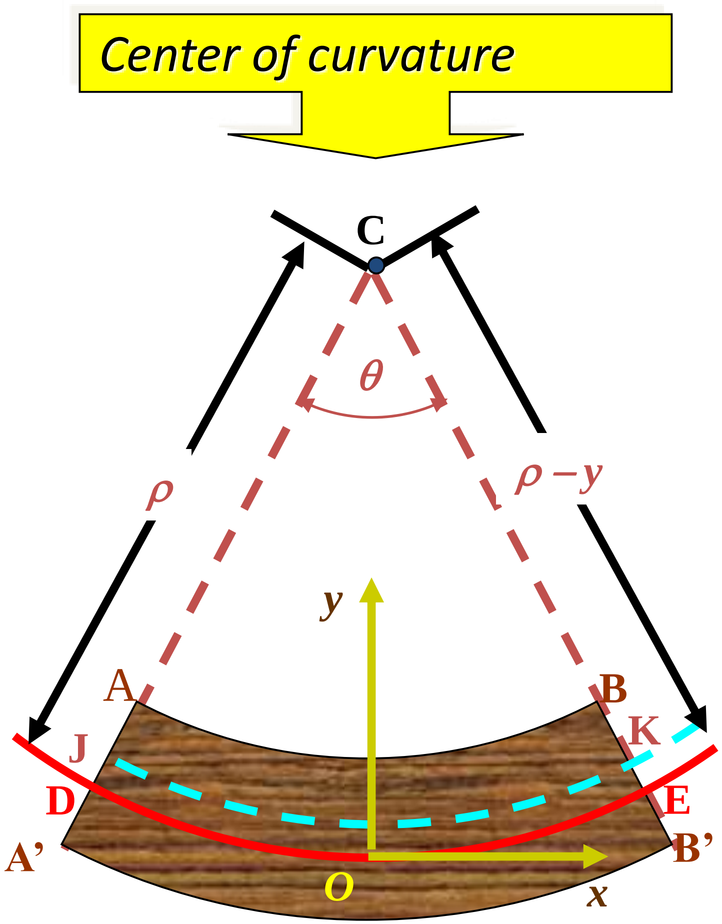
# Elongation of a fibre

- All fibres had the same length when straight. Let us say that the length of all the fibres of the segment of the beam, which is now spanning an angle θ, was L.
- The neutral fibre was also having a length L therefore in the unbent state.



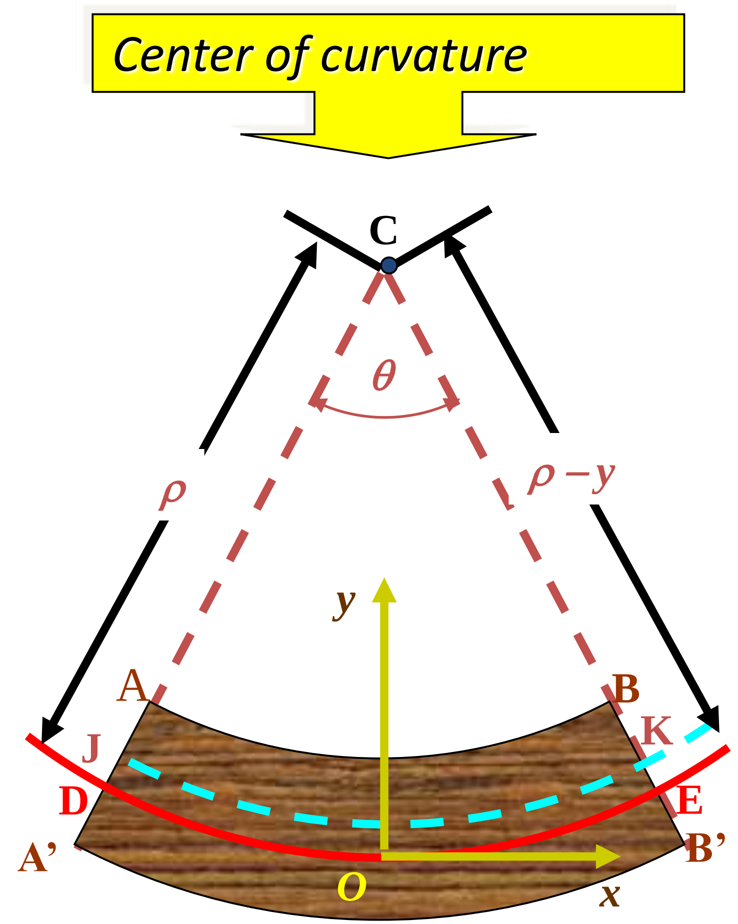
# Elongation of a fibre

- The neutral fibre was also having a length L in the unbent state means that in the bent state also it must be having the same length L, since it is **neutral**!
- Since it is bent into a circle of radius $\theta$ spanning an angle $\theta$ , its length in bent state is $\rho\theta$.
- Hence L= $\rho\theta$



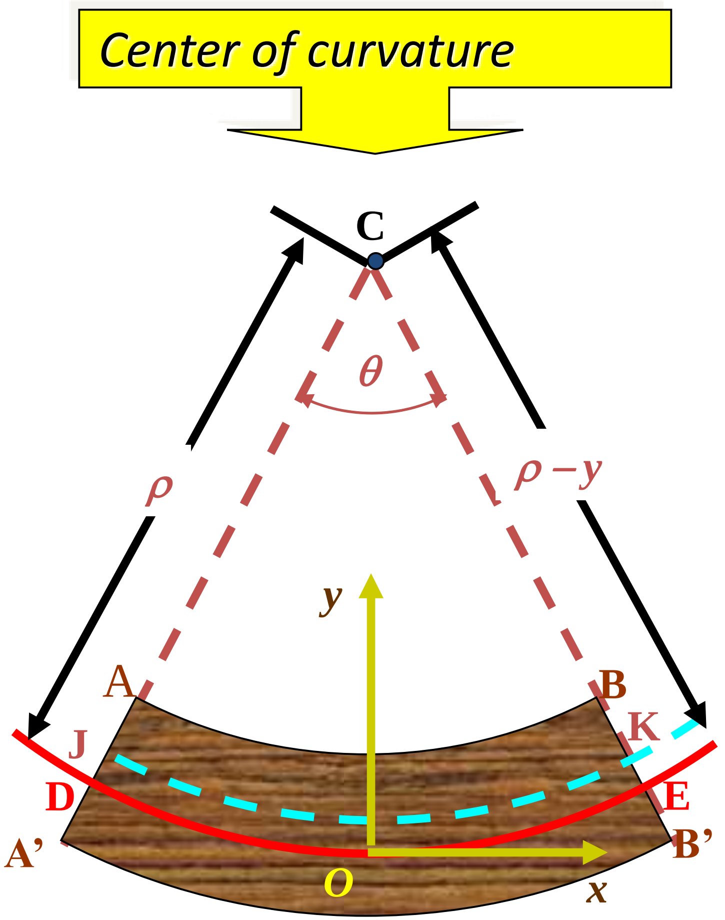
# Elongation of a fibre

- The neutral fibre has a length $L = \rho\theta$.
- Since it is the neutral fibre its length was the same before bending and was hence still $L = \rho\theta$.
- All other fibres had the same length as the neutral fibre before bending.
- **So before bending all the fibres had the length $L = \rho\theta$.**



# Elongation of a fibre

- A look at the figure tells us that after bending the fibre JK which is y distance away from the neutral fibre has a radius ρ-y
- Hence its length is (ρ-y)θ
- Before bending its length was L= ρθ.
- So change in length is **δ = –yθ** !
- Negative sign is because of contraction
- Without even knowing the origin we have been able to figure this out
- We will procced to extract more information based on this important factoid.

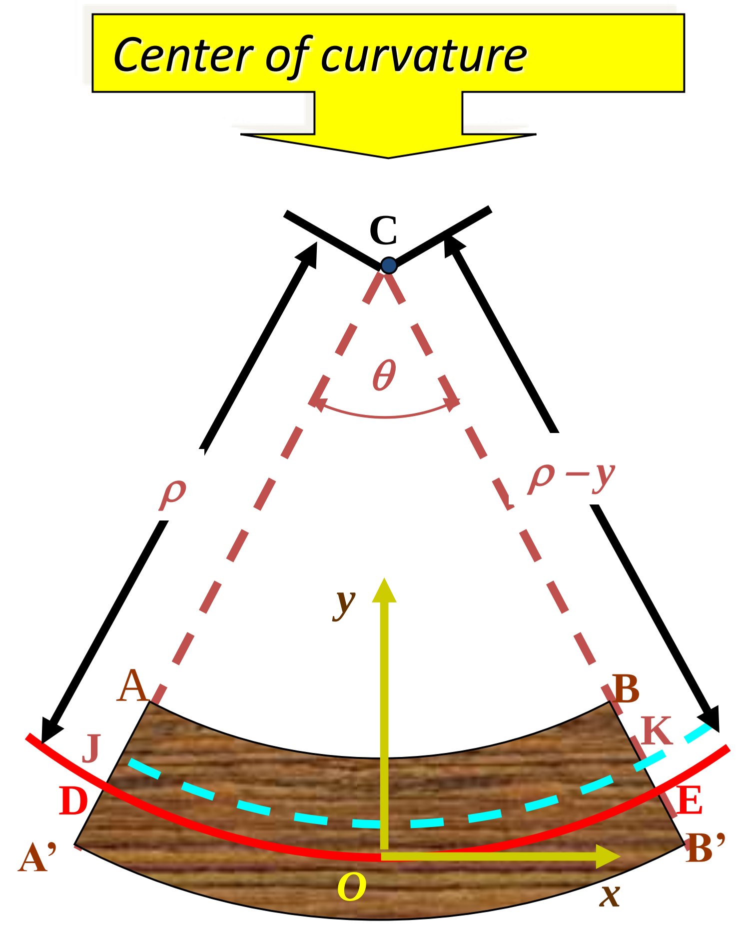


# Strain in a fibre

- Change in length is **θ = −yθ**.
- Original length is L = ρθ
- If we assume that θ is small, that is we are looking at a small segment of the beam, we can now say

$$\varepsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta} = -\frac{y}{\rho}$$

- Not bad at all considering we still do not know the origin.



# Strain in a fibre

- Let us organize our findings about strain
- **Strain varies linearly with distance from reference axis (Plane section remains plane)**

$$\varepsilon_x = -\frac{y}{\rho}$$

- Hence if c is the distance of the most extreme fiber (top most and/ or bottom most, note that c will be negative for the bottom most fiber ) the maximum strain is

$$\varepsilon_m = -\frac{c}{\rho}$$

- Therefore we can also say

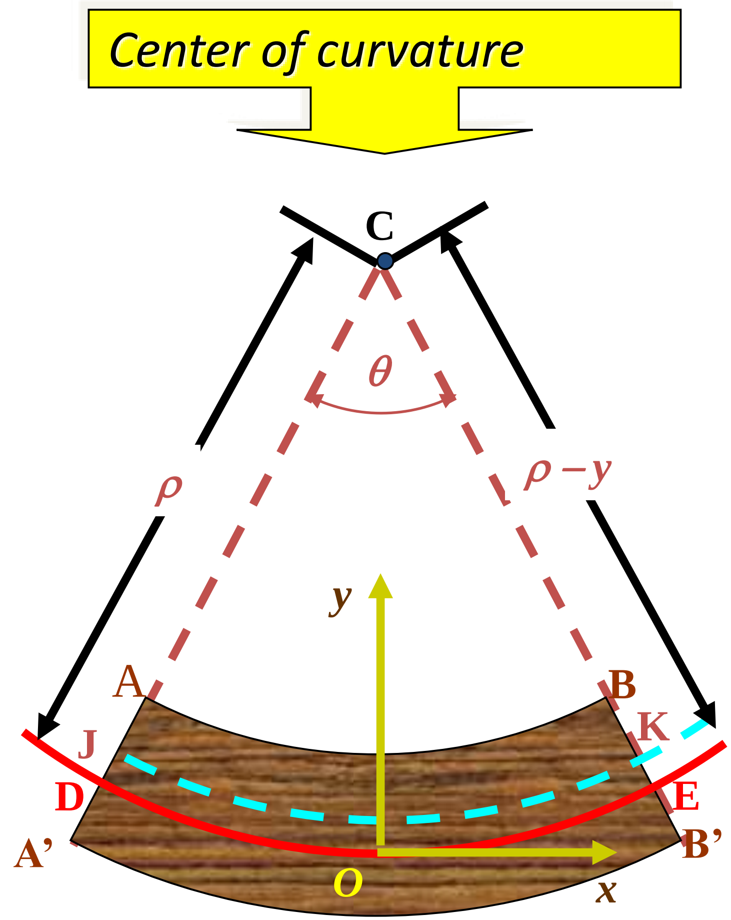
$$\varepsilon = \varepsilon_m \frac{y}{c}$$

*Why did we derive this new expression?* ***Because for a particular section there is only one unique maximum strain, which is therefore a constant for a given load***. *This little fact will help us when we perform any integration, involving strain, over the section.*

# Stress in a fibre

- Strain is $\varepsilon_x = -\dfrac{y}{\rho}$
- Hooke's law states $\sigma_x = E\varepsilon_x$
- We thus squeeze more juice out of our orange while still blindfolded

$$\sigma_x = -E\frac{y}{\rho}$$



# Stress in a fibre

- Let us now organize our findings about stress
- **Stress varies linearly with distance from reference axis**

$$\sigma_x = -E\frac{y}{\rho}$$

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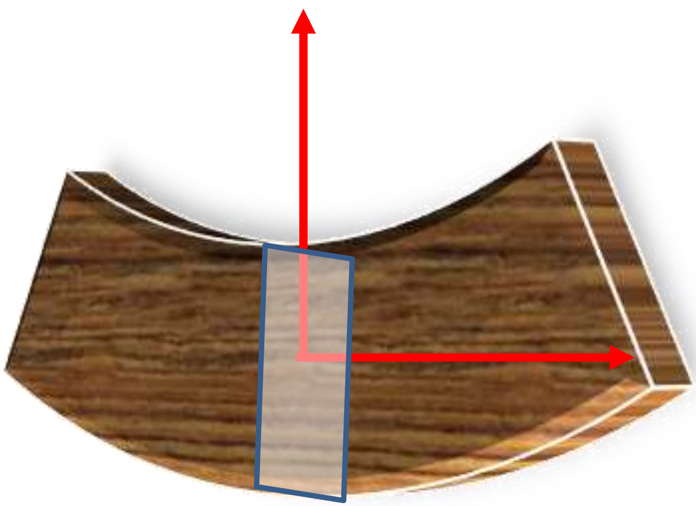
- Therefore we can also say

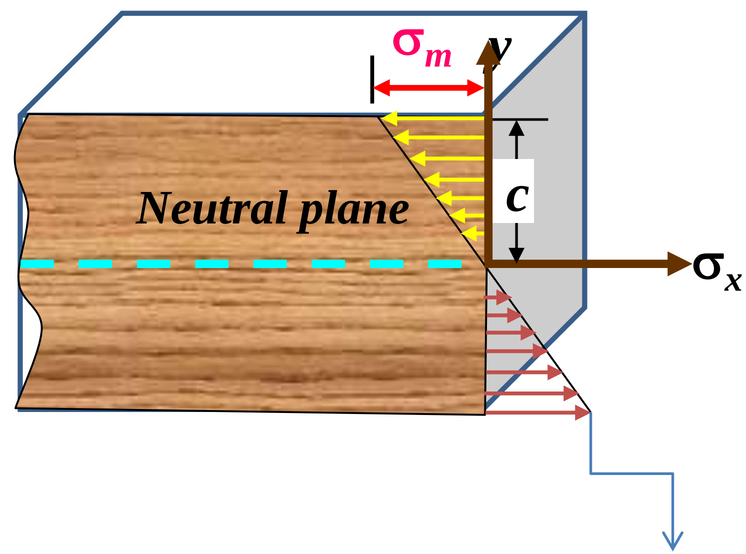
$$\sigma = \sigma_m \frac{y}{c}$$

***For a particular section there is only one unique maximum stress, which is therefore a constant for a given load****. This little fact will help us when we integrate* ***stress*** *over the section to find force.*

# Internal force at a section

- We now take a transverse section at the origin

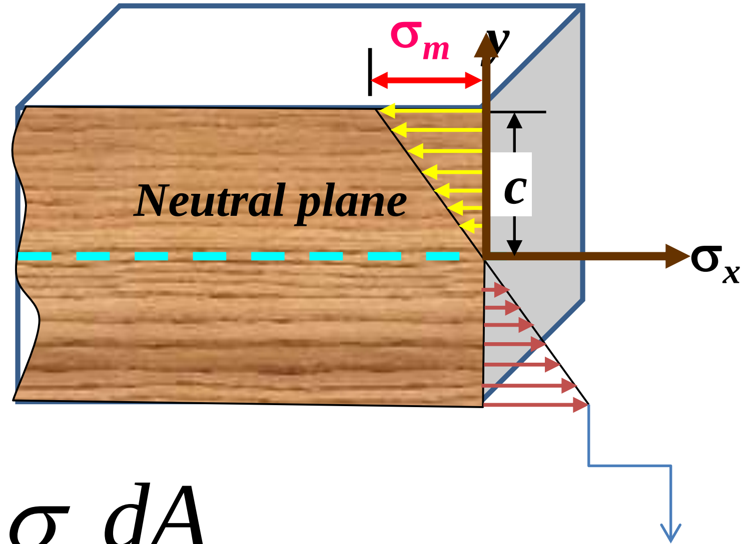





$$E\varepsilon_x = \frac{y}{c}\left(E\varepsilon_m\right)$$

# Equilibrium of forces at a section

- Since the beam is NOT subjected to any external axial force (when that happens we call it a beam column




- Sum of all forces along x is $\int_A \sigma_x dA$

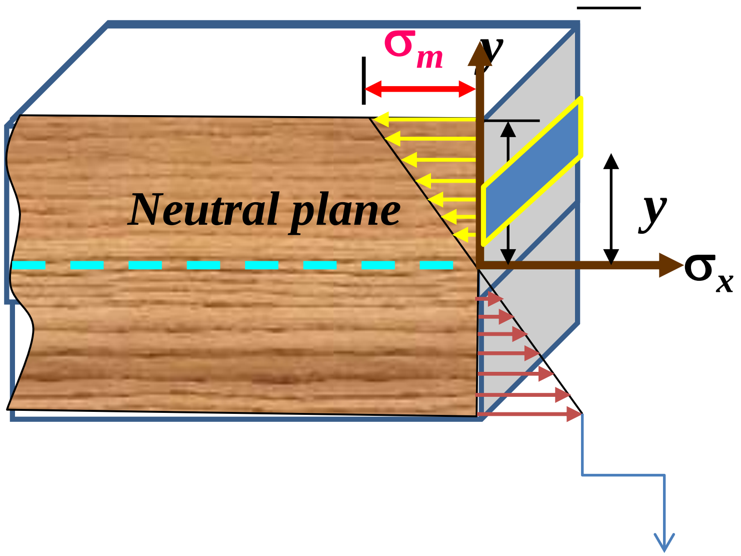
$$\sigma_x = \frac{\sigma_m}{c} y$$

- And for equilibrium that must be zero

$$\int_A \sigma_x dA = 0$$

# Equilibrium of forces at a section

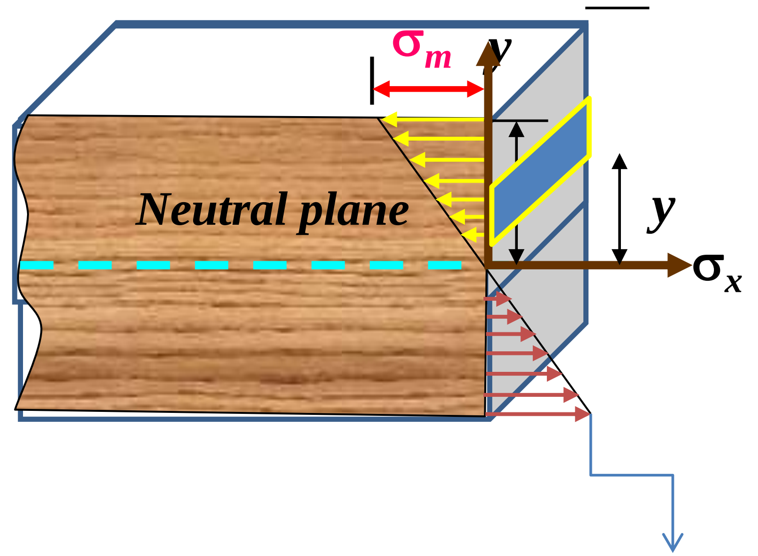
- Consider the yellow bordered rectangle in the cross section at a distance y from the origin. It has a height dy and base length b, which may or may not be a function of y (think of a beam with a trapezoidal cross section)
- This is the area dA=b(y)dx




$$\sigma_x = \frac{\sigma_m}{c} y$$

# Equilibrium of forces at a section

- Let us simplify the integral with what we know. First we use our formula derived earlier for stress



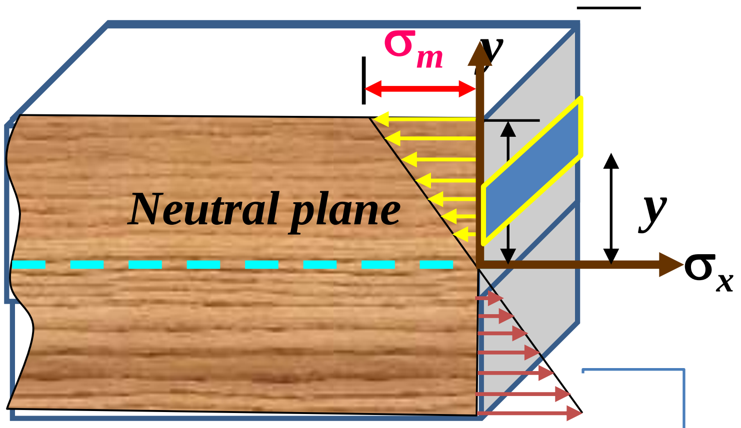

$$\sigma_x = \frac{\sigma_m}{c} y$$

$$\int_A \sigma_x dA = 0$$

$$\Rightarrow \int_A \frac{\sigma_m}{c} y dA = 0$$

# Location of origin

- But at any section, maximum stress is a constant. Also c, the distance of the most extreme fibre is a constant

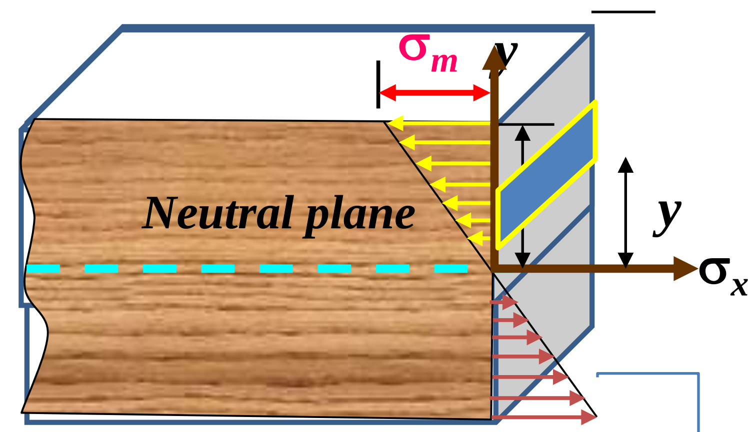



$$\sigma_x = \frac{\sigma_m}{c} y$$

$$\int_A \sigma_x dA = 0 \Rightarrow \frac{\sigma_m}{c}\int_A y dA = 0 \Rightarrow \int_A y dA = 0$$

# Location of origin

- Does this look familiar ?
- Yes. Recall the definition of centroid of an area in the y direction



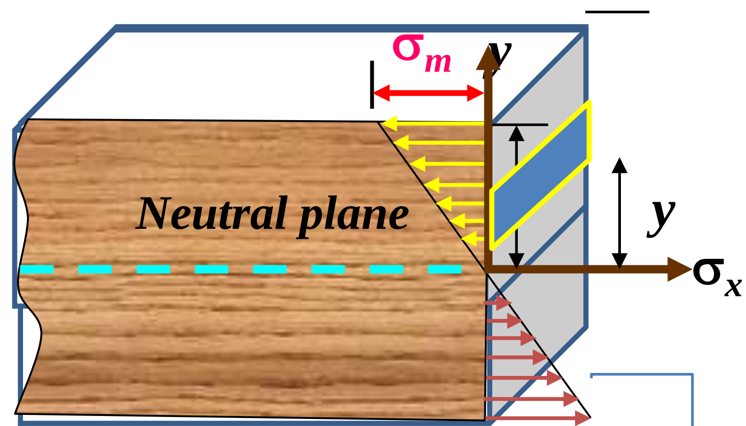

$$\sigma_x = \frac{\sigma_m}{c} y$$

$$y_C = \frac{\int_A y\,dA}{A}$$

# Location of origin

- But here $\int_A y\,dA = 0$

- So for the centroid of this cross section

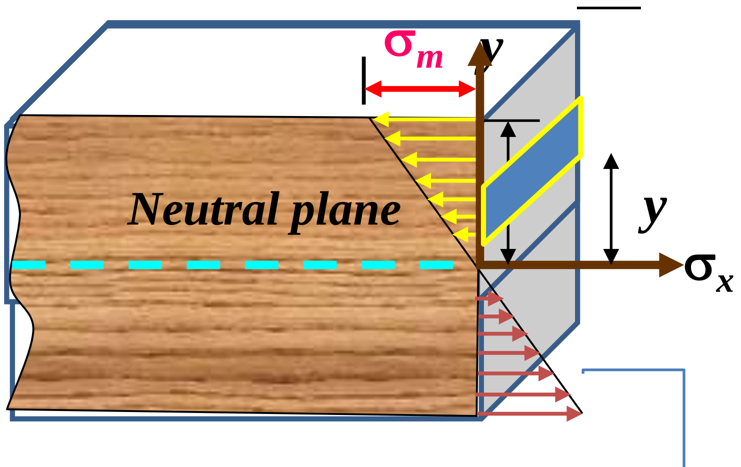



$$\sigma_x = \frac{\sigma_m}{c} y$$

$$y_C = \frac{\int_A y\,dA}{A} = 0$$

# Location of origin

- What does this mean in simple English ?
- The centroid is at the origin
- Recasting this statement  we get
- **The origin (that we had chosen) is the centroid of the cross section**
- We have found the origin and can now safely hang our painting and go about doing other things!




$$\sigma_x = \frac{\sigma_m}{c} y$$

***Given a cross section, the origin can always be found without bothering about stress or strain or load and so we can start our analysis for a section by simply locating its centroid.***

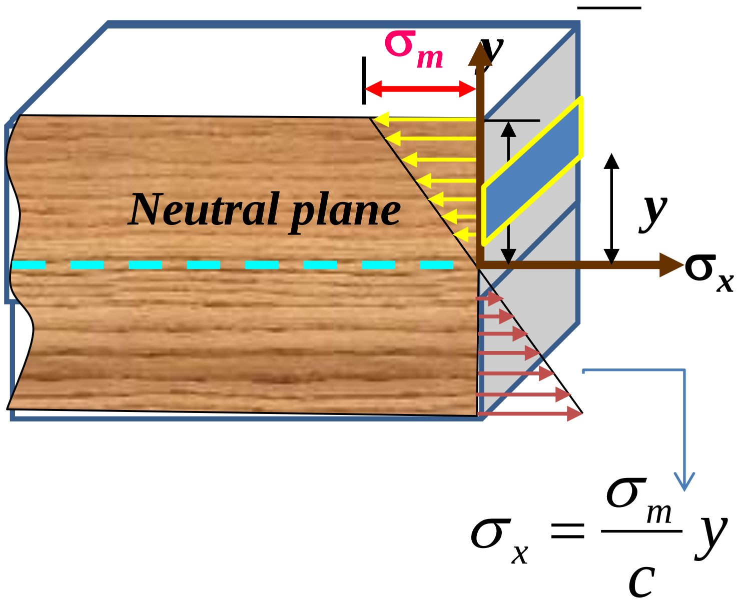
# Other things (Moment equilibrium)

- The force in that thin yellow bordered rectangle at a distance y from our new found origin is

$$\sigma_x dA$$

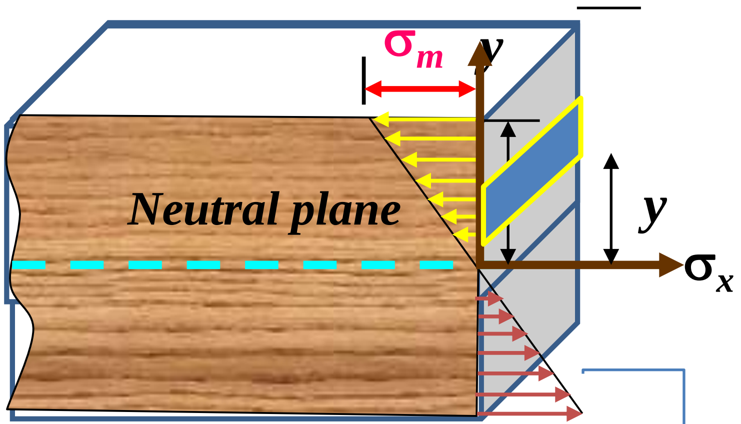
- The moment exerted about the z axis (which passes through the origin) is therefore

$$y\sigma_x dA$$



# Other things (Moment equilibrium)

- Given a beam, the external load and support conditions, we can, from static equilibrium analysis find out the moment at any section. Let us say that moment is M(x)
- The sum of the moments of internal forces about the z axis must be equal to this M(x)




$$\sigma_x = \frac{\sigma_m}{c} y$$

$$\int_A y\sigma_x(y)\, dA = M(x)$$

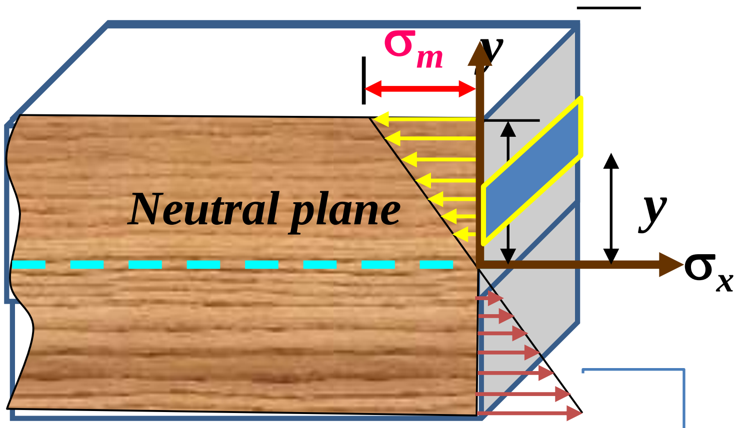
# Moment equilibrium

- Using our expression for stress

$$\sigma_x = \frac{y}{c}\sigma_m$$

- We get, once again recalling that maximum stress and c are constants for a particular section

$$\int_A y\left(\frac{y}{c}\sigma_m\right)dA = M \Rightarrow \frac{\sigma_m}{c}\int_A y^2 dA = M$$



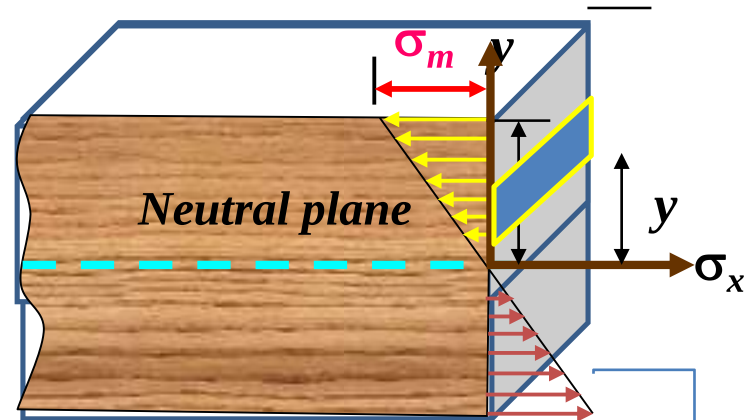

$$\sigma_x = \frac{\sigma_m}{c}y$$

# Moment equilibrium

- Does this term below look familiar?

$$\int_A y^2 dA$$

- Yes. It is a geometrical property of an area called the second moment of area about the x axis called $I_{yy}$ which can be obtained once again without bothering about forces or stresses and strains
- So once again, we can calculate and find this value before starting our analysis.




$$\sigma_x = \frac{\sigma_m}{c} y$$

# Moment equilibrium

- Our expression for moment equilibrium now becomes

$$\frac{\sigma_m}{c} I_{yy} = M \Rightarrow \sigma_m = \frac{Mc}{I_{yy}}$$

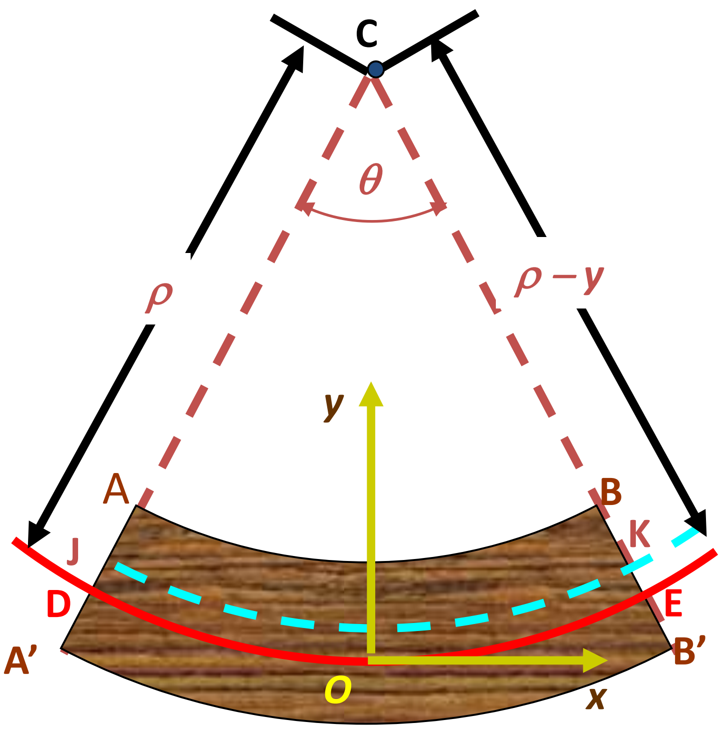
- We use our expression for stress $\sigma_x = \sigma_m \frac{y}{c} = \frac{Mc}{I_{yy}} \frac{y}{c} = \frac{My}{I_{yy}}$
- We finally get $\sigma_x(y) = \frac{M(x)y}{I_{yy}}$

Given a cross section, we can find Iyy about the centroid from simple geometry and M(x) from static equilibrium analysis. So we can now find the stress at any point of a beam with *one axis of symmetry loaded along that axis*.

**Note that compressive stress is taken as positive and curving upwards has been considered as positive curvature.**

# Deflection of beams

- We go back to our old picture

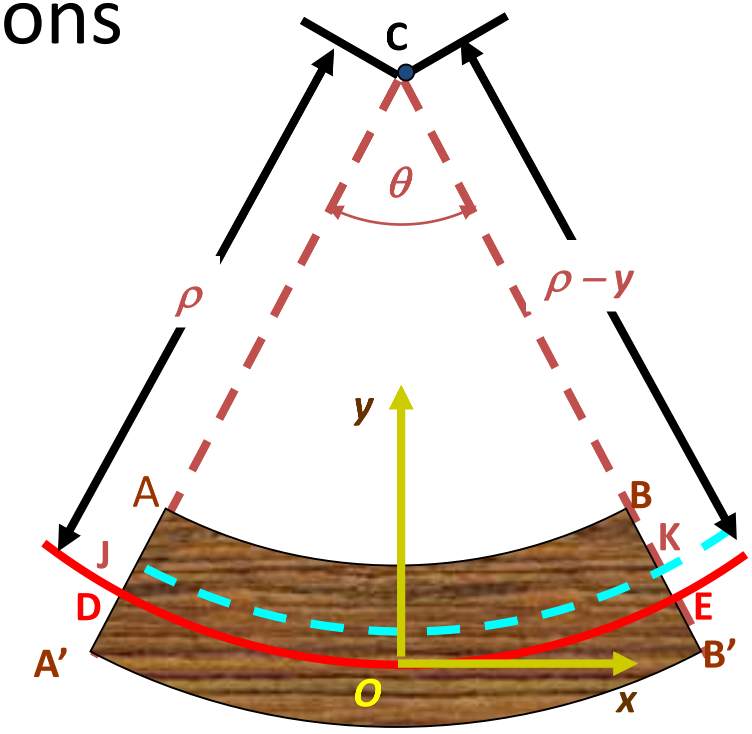


# Deflection of beams

- We also recall our expressions for strain and stress

$$\varepsilon_x(y) = -\frac{y}{\rho(x)}$$

$$\sigma_x(y) = \frac{M(x)\,y}{I_{yy}(x)}$$



# Deflection of beams

- We use Hooke's law $\varepsilon_x = E\sigma_x$
- We get $E(x)\dfrac{y}{\rho} = \dfrac{M(x)\,y}{I_{yy}(x)}$
- We get radius of curvature as

$$\frac{1}{\rho(x)} = \frac{M(x)}{E(x)\,I_{yy}(x)}$$

# Deflection of beams

- We next recall the expression for radius of curvature from calculus.
- We are using v here to avoid confusion with y
- This v is the y coordinate of any point on the neutral fibre of the beam

$$\frac{1}{\rho} = \frac{\dfrac{d^2 v}{dx^2}}{\left[1 + \left(\dfrac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$$

# Deflection of beams

- If we are dealing with small deflections (v), then the slope is also small. Measured in radians it will be much lesser than 1. Hence

$$\boxed{\frac{1}{\rho} \approx \frac{d^2 v}{dx^2}}$$

# Deflection of beams

- We now equate the two expressions

$$\frac{d^2v}{dx^2} = \frac{M(x)}{E(x)I_{yy}(x)}$$

- We can now integrate this expression twice
- We will get two constants which we can find out by using the geometrical constraints

# Deflection of beams

- Examples of boundary conditions to be used to fid the constants
- Cantilever beam fixed at x=0

$$\boxed{\begin{aligned} &y(0)=0 \\ &\frac{dy}{dx}(0)=0 \end{aligned}}$$

# Deflection of beams

- Examples of boundary conditions to be used to fid the constants
- Simply supported beam with pin and roller support at the two ends

$$\boxed{\begin{gathered} y(0) = 0 \\ y(L) = 0 \end{gathered}}$$