Transform Calculus

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Operator

An operator or a transformation when applied to a function produce another function.

$$\phi y = y^2 \quad \rightarrow \phi \text{ is a squaring operator}$$
$$\phi y = Dy \quad \rightarrow \phi \text{ is a derivative operator}$$

Linear operator

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$
$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$$

Definition of Integral transform

Let $K(S,t)$ be a function of S and t, where S is a parameter (may be real or complex) independent of t. The function $f(S)$ defined by the integral (assumed to be convergent)

$$f(S) = \int_{-\infty}^{\infty} K(S,t)\, F(t)\, dt$$

is called the integral transform of the f^n. $F(t)$ and is denoted by $T\{F(t)\}$. $K(S,t)$ is called the kernel of the transformation.

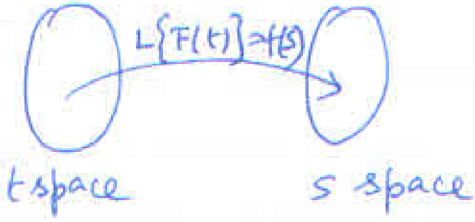
Definition of Laplace transform

If the kernel $K(S,t)$ is defined as

$$K(S,t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-St} & \text{for } t \geq 0 \end{cases}$$

then $f(S) = \int_0^\infty e^{-St} F(t)\, dt$. The f^n. $f(S)$ is called L.T. of the f^n. $F(t)$ and is denoted by $L\{F(t)\}$.

Another way of looking at the Laplace transform is as a mapping from points in the t domain to points in the s domain.



Theorem

The L.T. is a linear transformation

i.e. $L\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$

proof: $L\{a_1 F_1(t) + a_2 F_2(t)\}$

$$= \int_0^\infty e^{-st}\{a_1 F_1(t) + a_2 F_2(t)\}\, dt$$

$$= a_1 \int_0^\infty e^{-st} F_1(t)\, dt + a_2 \int_0^\infty e^{-st} F_2(t)\, dt$$

$$= a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$$

Ex If $F(t) = 1$ for $t \geq 0$, then find $L\{F(t)\}$

Sol^n: $L\{F(t)\} = \int_0^\infty e^{-st} \cdot 1 \, dt$

$$= \lim_{\tau \to \infty} \left(\frac{e^{-st}}{-s} \right)_0^\tau$$

$$= \lim_{\tau \to \infty} \left(\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right)$$

$$= \frac{1}{s}$$

provided $s > 0$ (if s is real). So $L(1) = \frac{1}{s}$, $s > 0$

If $s \leq 0$, then the integral will diverge and there will be no. L.T.

Ex Attempt to find $L\{\frac{1}{t^2}\}$

Solⁿ: $L\{\frac{1}{t^2}\} = \int_0^\infty \frac{e^{-st}}{t^2}\,dt$

$$= \int_0^1 \frac{e^{-st}}{t^2}\,dt + \int_1^\infty \frac{e^{-st}}{t^2}\,dt$$

When $0 \le t \le 1$, $e^{-st} \geq e^{-s}$ if $s > 0$

$$\therefore \int_0^\infty \frac{e^{-st}}{t^2}\,dt \geq \int_0^1 \frac{e^{-s}}{t^2}\,dt + \int_1^\infty \frac{e^{-st}}{t^2}\,dt$$

i.e. $$\int_0^\infty \frac{e^{-st}}{t^2}\,dt \geq e^{-s}\int_0^1 \frac{dt}{t^2} + \int_1^\infty \frac{e^{-st}}{t^2}\,dt$$

But $\int_0^1 \frac{1}{t^2}\,dt$ diverges and hence $L\{\frac{1}{t^2}\}$ fails to converge. As a result $\frac{1}{t^2}$ fails to have a L.T.

Ex For the f^n. $f(t) = e^{t^2}$

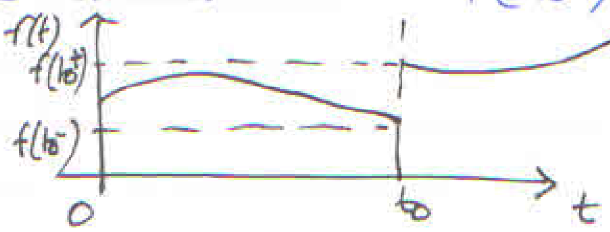
Solⁿ: $$\lim_{\tau\to\infty} \int_0^\tau e^{-st} e^{t^2}\,dt = \lim_{\tau\to\infty} \int_0^\tau e^{t^2 - st}\,dt = \infty$$

for any choice of the variable s, since the integrand grows without bound as $\tau \to \infty$.

Definition of jump discontinuity

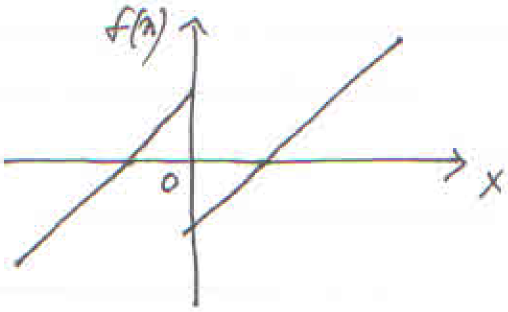
A f^n. f has a jump discontinuity at a point t_0 if both the limits $\lim_{t\to t_0^-} f(t) = f(t_0^-)$ and $\lim_{t\to t_0^+} f(t) = f(t_0^+)$

exist (as finite numbers) and $f(t_0^-) \neq f(t_0^+)$.



Ex $\quad f(x) = x+1 \quad \text{if } x \le 0$

$\quad\quad\quad\;\; = x-1 \quad \text{if } x > 0$

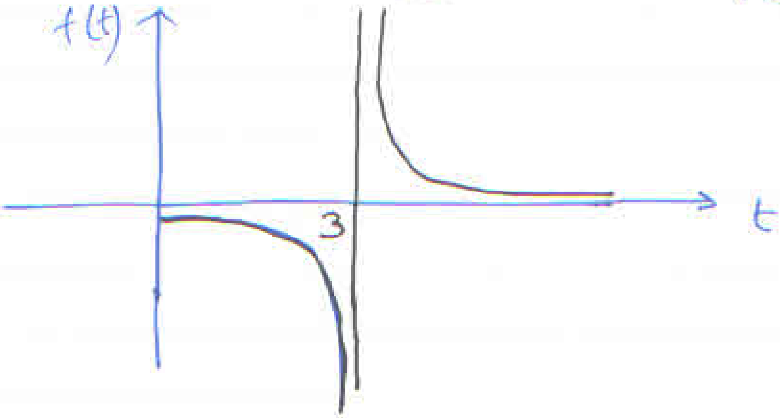


$\lim_{x\to 0-} f(x) = \lim_{x\to 0} x+1 = 1$

$\lim_{x\to 0+} f(x) = \lim_{x\to 0} x-1 = -1$

Ex $\quad$ The f^n. $\quad f(t) = \dfrac{1}{t-3}$

has a discontinuity at $t=3$ but it is not a jump discontinuity since neither $\lim_{t\to 3^-} f(t)$ nor $\lim_{t\to 3^+} f(t)$ exist.

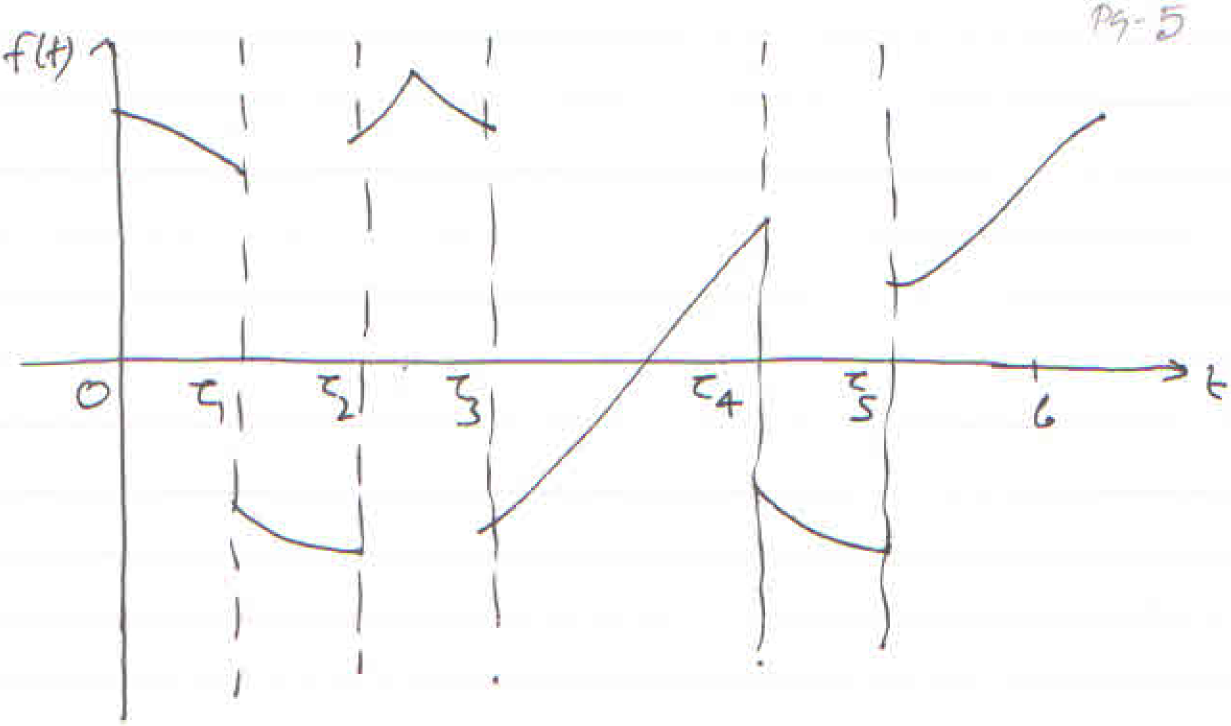


Def^n. of piecewise continuous function

A f^n. f is PWC on the interval $[0,\infty)$ if

(i) $\lim_{t\to 0^+} f(t) = f(0^+)$ exist and (ii) f is continuous

on every finite interval $(0,b)$ except possibly at a finite number of points $\tau_1, \tau_2, \dots \tau_n$ in $(0,b)$ at which f has a jump discontinuity.



Exponential order

A f^n. f has exponential order α if there exist constants $M > 0$ and α such that for some $t_0 \geq 0$

$$|f(t)| \leq M e^{\alpha t}, \quad t \geq t_0$$

$$\left[\text{or } \lim_{t \to \infty} \frac{f(t)}{e^{\alpha t}} \text{ is finite} \right]$$

The f^n. t^n has exponential order α for any $\alpha > 0$ and any $n \leq \mathbb{N}$ (i.e. $n = 1, 2, 3, \ldots$)

Because $\lim_{t \to \infty} e^{-\alpha t} t^n \qquad \alpha > 0$

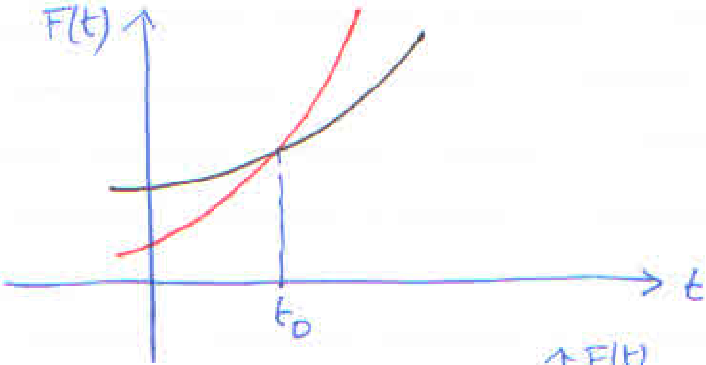
$$= \lim_{t \to \infty} \frac{t^n}{e^{\alpha t}} \quad \left[\text{form } \frac{\infty}{\infty} \right]$$

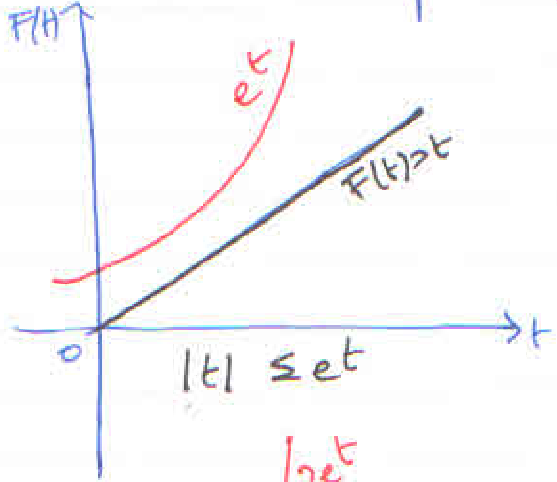
$$= \lim_{t \to \infty} \frac{n!}{\alpha^n e^{\alpha t}} = \frac{n!}{\infty} = 0$$

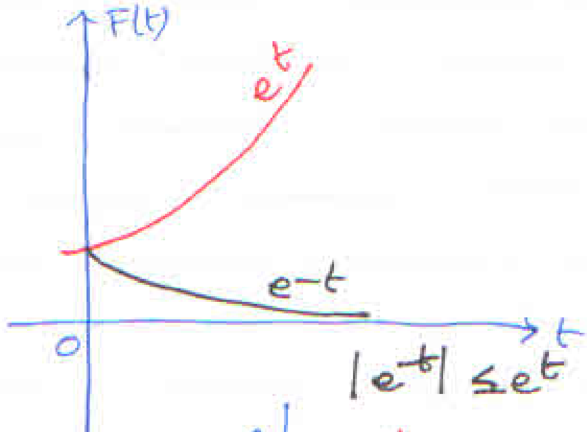
If f is an increasing function, then the condition

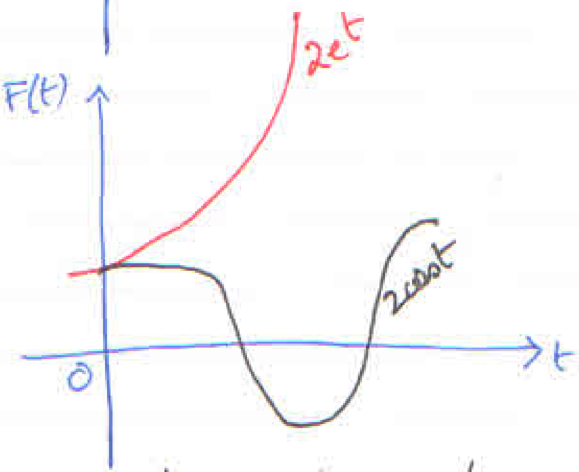
$|f(t)| \leq Me^{\alpha t}$, $t > t_0$ simply states that the graph of f on the interval (t_0, ∞) does not grow faster than the graph of the exponential f^n. $Me^{\alpha t}$, where α is a positive constant.

Functions with black curve is of exponential order.

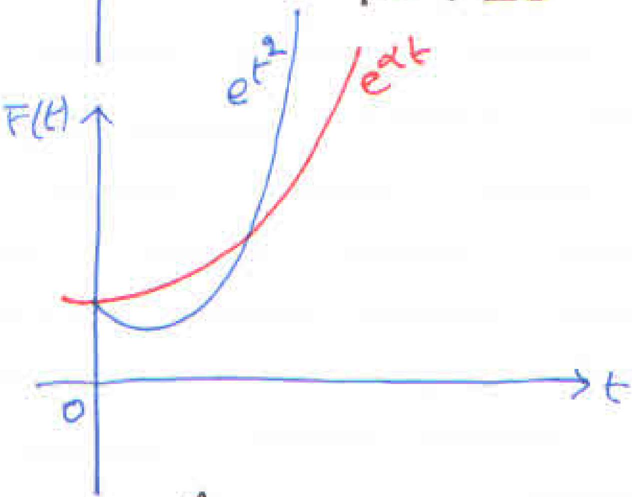








$|2\cos t| \leq 2e^t$



$F(t) = e^{t^2}$ is not of exp. order

Definition: Function of class A

A f^n. $f(t)$ is said to be of class A if (i) it is piecewise continuous over every finite interval in the range $t \geq 0$ (ii) $f(t)$ is of exponential order.

Existence of Laplace transform (Sufficient condition)

If $F(t)$ is a f^n. of class A, then L.T. of $F(t)$ exists

OR

If $F(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then the L.T. $L\{F(t)\}$ exists for $Re(S) > \alpha$ and converges absolutely.

Proof: $L\{F(t)\} = \int_0^\infty e^{-St} F(t)\, dt$

$$\left| \int_0^\tau e^{-St} F(t)\, dt \right| \leq \int_0^\tau \left| e^{-St} F(t) \right| dt$$

$$\leq \int_0^\tau e^{-St} M e^{\alpha t}\, dt$$

$$= M \int_0^\tau e^{-(S-\alpha)t}\, dt$$

$$= \left. \frac{M e^{-(S-\alpha)t}}{-(S-\alpha)} \right|_0^\tau$$

$$= \frac{M}{S-\alpha} - M \frac{e^{-(S-\alpha)\tau}}{S-\alpha}$$

Letting $\tau \to \infty$ $[Re(S) > \alpha]$

$$\int_0^\infty \left| e^{-St} F(t) \right| dt \leq \frac{M}{S-\alpha}$$

By the above theorem, piecewise continuous f^ns. on $[0,\infty)$ having exponential order belong to L. However, there are functions in L that do not satisfy one or both of these conditions.

Ex $\qquad f(t) = \frac{1}{\sqrt{t}}$

It is not PWC on $[0,\infty)$ since $f(t)\to\infty$ as $t\to 0^+$ i.e. $t=0$ is not a point of jump discontinuity. But still we can compute $L\left[\frac{1}{\sqrt{t}}\right]$.

$$L\left[\frac{1}{\sqrt{t}}\right] = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt$$

$$= \int_0^\infty e^{-st}t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{\frac{1}{2}-1}\,dx \qquad \text{Put } st=x \quad \therefore dt=\frac{dx}{s}$$

$$= \frac{1}{\sqrt{s}}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \sqrt{\frac{\pi}{s}}\,, \qquad s>0$$

Uniqueness

If the L.T. of a given f^n. exists, it is uniquely determined.

Behaviour of $f(s)$ as $s\to\infty$

If F is PWC on $[0,\infty)$ and has exponential order α, then $f(s) = L\{F(t)\}\to 0$ as $Re(s)\to\infty$

By the existence theorem

$$\left| \int_0^\infty e^{-st} F(t)\,dt \right| \le \frac{M}{s-\alpha} \qquad [\,Re(s) > \alpha\,]$$

and letting $s \to \infty$ gives the result.

Remark: Any f^n. $f(s)$ without the behaviour $f(s) \to 0$ as $Re(s) \to \infty$ say s^2, $\frac{e^s}{s}$ cannot be the L.T. of any f^n. f (of class A).

Laplace transform of some elementary functions

Ex Find the L.T. of
(i) 1 (ii) t (iii) t^n, n +ve integer
(iv) e^{at} (v) $\sin at$ (vi) $\cos at$ (vii) $\sinh at$ (viii) $\cosh at$

Sol^n: (i) $L[1] = \int_0^\infty e^{-st} \cdot 1\,dt = \left(\frac{e^{-st}}{-s} \right)_0^\infty = \frac{1}{s}$, $s > 0$

(ii) $L[t] = \int_0^\infty e^{-st}\, t\,dt$

$$= \left(-\frac{t}{s} e^{-st} \right)_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st}\,dt$$

$$= \left(-\frac{t}{s} e^{-st} \right)_0^\infty + \left[-\frac{1}{s^2} e^{-st} \right]_0^\infty$$

$$= \frac{1}{s^2}$$

(iii) $L[t^n] = \int_0^\infty e^{-st}\, t^n\,dt$, n +ve integer

$$= \left[-\frac{t^n}{s} e^{-st} \right]_0^\infty + \int_0^\infty \frac{n\,t^{n-1}}{s} e^{-st}\,dt$$

$$= \frac{n}{s} L(t^{n-1})$$

If $n=2$, $L[t^2] = \frac{2}{s} L[t] = \frac{2}{s^3}$

If $n=3$, - - - .

By induction $L[t^n] = \frac{n!}{s^{n+1}}$

Aliter

$$L[t^n] = \int_0^\infty e^{-st} t^n \, dt$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} \qquad st = x$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1} \, dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{if } s>0 \text{ \& } n+1>0$$

In particular if n is a +ve integer, then $\Gamma(n+1) = n!$ so that $L[t^n] = \frac{n!}{s^{n+1}}$, $s>0$. Similarly, other L.Ts can be found.

Table 1

	$F(t)$	$L\{F(t)\}$
1.	1	$\frac{1}{s}$, $s>0$
2.	t^n (n +ve int)	$\frac{n!}{s^{n+1}}$, $s>0$
3	e^{at}	$\frac{1}{s-a}$, $s>a$
4.	$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
5.	$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
6.	$\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
7.	$\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$

Few examples

Ex Find $L\{\sin t \cos t\}$

Sol^n. $L\{\sin t\cos t\} = L\{\frac{1}{2}\sin 2t\} = \frac{1}{2}L\{\sin 2t\}$

$= \frac{1}{2}\frac{2}{s^2+4} = \frac{1}{s^2+4}, \quad s>0$

Ex Find $L\{F(t)\}$ where $F(t) = \begin{cases} 0 & 0<t<1 \\ t & 1<t<2 \\ 0 & t>2 \end{cases}$

Sol^n: $L\{F(t)\} = \int_0^1 e^{-st}.0\,dt + \int_1^2 e^{-st}\,t\,dt + \int_2^\infty e^{-st}.0\,dt$

$= \int_1^2 e^{-st}.t\,dt$

$= -\left[t\,\frac{e^{-st}}{s}\right]_1^2 - \left[\frac{e^{-st}}{s^2}\right]_1^2$

$= -\frac{2}{s}e^{-2s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}$

$= \left(\frac{1}{s}+\frac{1}{s^2}\right)e^{-s} - \left(\frac{1}{s^2}+\frac{2}{s}\right)e^{-2s}$

Ex Find $L\{F(t)\}$ where $F(t) = \begin{cases} 4 & 0<t<1 \\ 3 & t>1 \end{cases}$

Sol^n: $L\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$

$= \int_0^1 e^{-st}.4\,dt + \int_1^\infty e^{-st}.3\,dt$

$= \left[-\frac{4}{s}e^{-st}\right]_0^1 + \left[-\frac{3}{s}e^{-st}\right]_1^\infty$

$= \frac{1}{s}\left(4-e^{-s}\right), \quad s>0$

Ex. Find $L\{\sin^3 2t\}$

Sol^n: $\sin 3t = 3\sin t - 4\sin^3 t$

$\therefore \sin^3 t = \frac{3}{4}\sin t + \frac{1}{4}\sin 3t$

and so $\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$$\therefore L\{F(t)\} = \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$

$$= \frac{3}{4}\left[\frac{2}{s^2+4}\right] - \frac{1}{4}\left[\frac{6}{s^2+36}\right], \quad s>0$$

$$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

$$= \frac{48}{(s^2+4)(s^2+36)}$$

Ex Find $L\{\sin at \sin bt\}$

Sol^n: $F(t) = \frac{1}{2}(2\sin at \sin bt)$

$$= \frac{1}{2}\left[\cos(at-bt) - \cos(at+bt)\right]$$

$$= \frac{1}{2}\cos(a-b)t - \frac{1}{2}\cos(a+b)t$$

$$\therefore L\{F(t)\} = \frac{1}{2}L\{\cos(a-b)t\} - \frac{1}{2}L\{\cos(a+b)t\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2}\right] - \frac{1}{2}\left[\frac{s}{s^2+(a+b)^2}\right], \quad s>0$$

$$= \frac{2abs}{\{s^2+(a-b)^2\}\{s^2+(a+b)^2\}}, \quad s>0$$

Ex Find $L\{e^{at}\cos bt\}$ and $L\{e^{at}\sin bt\}$

Sol^n: Let $F(t) = e^{(a+ib)t}$

$$L\{F(t)\} = \frac{1}{s-(a+ib)} = \frac{1}{s-a-ib}$$

$$= \frac{(s-a)+ib}{(s-a)^2+b^2}$$

Again $e^{(a+ib)t} = e^{at}[\cos bt + i\sin bt]$

$$= e^{at}\cos bt + i e^{at}\sin bt$$

$$\therefore L\{F(t)\} = L\{e^{at}\cos bt + i e^{at}\sin bt\}$$

$$= L\{e^{at}\cos bt\} + iL\{e^{at}\sin bt\}$$

$$\therefore L\{e^{at}\cos bt\} + iL\{e^{at}\sin bt\} = \frac{(s-a)+ib}{(s-a)^2+b^2}$$

Comparing real and imaginary parts,

$$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

Infinite series

For an infinite series $\sum_{n=0}^{\infty} a_n t^n$, in general, it is not possible to obtain the L.T. of the series by taking the transform term by term.

Theorem

If $F(t) = \sum_{n=0}^{\infty} a_n t^n$ converges for $t \geq 0$ with $|a_n| \leq \frac{K\alpha^n}{n!}$

for all n sufficiently large and $\alpha > 0$, $K > 0$, then

$$L\{F(t)\} = \sum_{n=0}^{\infty} a_n L(t^n) = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}} \quad (\text{Re}\, s > \alpha)$$

Ex Find $L\{\sin\sqrt{t}\}$

Sol^n: $L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right\}$

$$= L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right\}$$

$$= L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \cdots$$

$$= \frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(\frac{5}{2})}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(\frac{7}{2})}{s^{7/2}} - \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$$

Elementary properties of Laplace Transform

Theorem 1

First translation (or shifting) theorem

If $L\{F(t)\} = f(s)$, $s > \alpha$

then $L\{e^{at} F(t)\} = f(s-a)$, $s > \alpha + a$

Proof: $f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$\therefore f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)\,dt$$

$$= \int_0^\infty e^{-st} e^{at} F(t)\,dt$$

$$= L\{e^{at} F(t)\}$$

Ex Find the L.T. of $F(t) = t^3 e^{-3t}$

Sol^n: $L\{t^3\} = f(s) = \frac{3!}{s^4} = \frac{6}{s^4}$

$$L\{F(t)\} = L\{t^3 e^{-3t}\} = F(s-a) = \frac{6}{(s+3)^4}$$

Ex Find $L\{e^t \sin^2 t\}$

Sol^n: $L\{\sin^2 t\} = L\{\frac{1}{2}(1-\cos 2t)\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$

$$= \frac{2}{s(s^2+4)} = f(s)$$

$$L\{e^t \sin^2 t\} = f(s-1) = \frac{2}{(s-1)\{(s-1)^2+4\}}$$

$$= \frac{2}{(s-1)(s^2-2s+5)}$$

Ex Find L.T. of $t\sin at$ and $t\cos at$

Sol^n: $L\{t\} = \frac{1}{s^2} = f(s)$

$$L\{te^{iat}\} = L\{t\cos at\} + iL\{t\sin at\}$$

Also $$L\{te^{iat}\} = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$$

$$= \frac{(s^2-a^2)+i(2as)}{(s^2+a^2)^2}$$

$$\therefore L\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2} \; ; \; L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

Ex Find the L.T. of $F(t) = \sinh 3t \cos^2 t$

Sol^n: $L\{\cos^2 t\} = \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\}$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = \frac{s^2+2}{s(s^2+4)}, \quad s>0$$

$$\therefore L\{\sinh 3t \cos^2 t\} = L\left\{\frac{e^{3t}-e^{-3t}}{2}\cos^2 t\right\}$$

$$= \frac{1}{2}L\{e^{3t}\cos^2 t\} - \frac{1}{2}L\{e^{-3t}\cos^2 t\}$$

$$= \frac{1}{2}\left[\frac{(s-3)^2+2}{(s-3)[(s-3)^2+4]} - \frac{(s+3)^2+2}{(s+3)[(s+3)^2+4]}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-6s+11}{(s-3)(s^2-6s+13)} - \frac{s^2+6s+11}{(s+3)(s^2+6s+13)}\right]$$

Theorem 2

Second translation (or shifting) theorem:

If $L\{F(t)\} = f(s)$ and G is a f^n. defined by

$$G(t) = \begin{cases} F(t-a), & t > a \\ \infty, & t < a \end{cases}$$

then $L\{G(t)\} = e^{-as} f(s)$

Proof: $L\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt$

$$= \int_0^a e^{-st} \cdot 0 \,dt + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= \int_0^\infty e^{-s(a+x)} F(x)\,dx \qquad \begin{matrix} t-a=x \\ dt=dx \end{matrix}$$

$$= e^{-sa} \int_0^\infty e^{-sx} F(x)\,dx$$

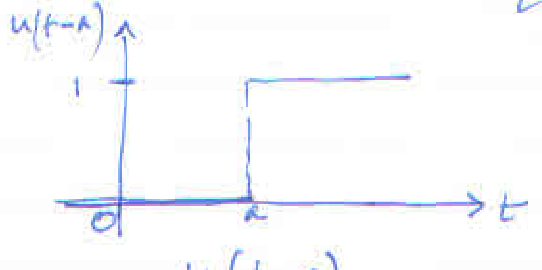
$$= e^{-sa} \int_0^\infty e^{-st} F(t)\,dt$$

$$= e^{-sa} L\{F(t)\}$$

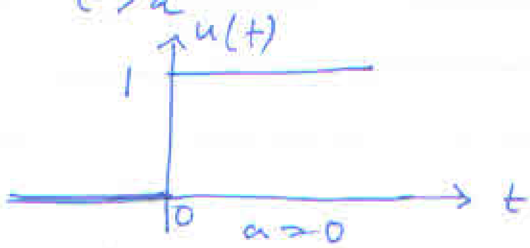
$$= e^{-as} f(s)$$

Unit step f^n. or Heaviside's unit f^n.

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \qquad a \geq 0$$



$u(t-a)$



$a=0$

$u(t)$

Alternative statement of 2nd shifting theorem

If $F(t)$ has the transform $f(s)$, then the shifted f^n.

$$F(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ F(t-a) & \text{if } t>a \end{cases}$$

has the transform $e^{-as} f(s)$ i.e.

$$L\{F(t-a)u(t-a)\} = e^{-as} F(s)$$

Laplace transform of unit step function

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_a^\infty = \frac{e^{-as}}{s}$$

Ex Find $L\{G(t)\}$ where $G(t) = \begin{cases} e^{t-a}, & t>a \\ 0 & t<a \end{cases}$

Soln: Let $F(t) = e^t$ $\therefore L\{F(t)\} = \frac{1}{s-1}$, $s>1$

$$= f(s)$$

$$G(t) = \begin{cases} F(t-a) = e^{t-a} & t>a \\ 0 & t<a \end{cases}$$

$$\therefore L\{G(t)\} = e^{-as} f(s) = \frac{e^{-as}}{s-1}, \quad s>1$$

Ex Find $L\{F(t)\}$ where

$$F(t) = \begin{cases} \cos\left(t - \frac{2}{3}\pi\right), & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$$

Sol^n: <u>1st method</u> (from definition)

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^{\frac{2\pi}{3}} e^{-st}\, 0\, dt + \int_{\frac{2\pi}{3}}^\infty e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt$$

$$= \int_{\frac{2\pi}{3}}^\infty e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt$$

$t - \frac{2\pi}{3} = x$

$$= \int_0^\infty e^{-s\left(x + \frac{2\pi}{3}\right)} \cos x\, dx$$

$$= e^{-s\frac{2\pi}{3}} \int_0^\infty e^{-sx} \cos x\, dx$$

$$= e^{-s\frac{2\pi}{3}} \int_0^\infty e^{-st} \cos t\, dt$$

$$= e^{-s\frac{2\pi}{3}} L\{\cos t\} = e^{-\frac{2}{3}s\pi} \frac{s}{s^2+1}, \quad s > 0$$

<u>2nd method</u> (by 2nd shifting theorem)

Let $\varphi(t) = \cos t$ $\quad F(t) = \begin{cases} \varphi\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$

$$L\{\varphi(t)\} = \frac{s}{s^2+1} = f(s)$$

$$\therefore L\{F(t)\} = e^{-\frac{2\pi s}{3}} f(s) = e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}$$

Theorem 3

Change of scale property

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$, $a > 0$

Proof!

$$L\{F(at)\}$$

$$= \int_0^\infty e^{-st} F(at)\, dt \qquad at = x,\quad dt = \frac{1}{a} dx$$

$$= \frac{1}{a}\int_0^\infty e^{-s\left(\frac{x}{a}\right)} F(x)\, dx$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} F(t)\, dt$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right)$$

Ex Find $L\{\cos 5t\}$

Sol^n: $L\{\cos t\} = \frac{s}{s^2+1} = f(s) \qquad s > 0$

$$L\{\cos 5t\} = \frac{1}{5} f\left(\frac{s}{5}\right) = \frac{1}{5} \frac{\frac{s}{5}}{\left(\frac{s}{5}\right)^2 + 1} = \frac{s}{s^2+25}, \quad s > 0$$

Ex Find $L\{\sinh 3t\}$

Sol^n: $L\{\sinh 3t\} = \frac{1}{s^2-1} = f(s)$

$$L\{\sinh 3t\} = \frac{1}{3} f\left(\frac{s}{3}\right) = \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1} = \frac{3}{s^2-9}$$

Laplace transform of derivatives of $F(t)$

Theorem 4

Let $F(t)$ be continuous (or PWC) $\forall\, t \geq 0$ and be of exponential order a and if $F'(t)$ is of class A, then L.T. of $F'(t)$ exists when $s > a$ and

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Proof Case I

In case $F'(t)$ is continuous $\forall\, t \geq 0$, then

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt$$

$$= \left[e^{-st} F(t)\right]_0^\infty + s\int_0^\infty e^{-st} F(t)\,dt$$

$$= \lim_{t \to \infty} e^{-st} F(t) - F(0) + sL\{F(t)\}$$

Now $|F(t)| \leq Me^{at}$ $\forall\, t \geq 0$ and for some const. a and M

$$\therefore \left|e^{-st} F(t)\right| = e^{-st} |F(t)| \leq e^{-st} Me^{at}$$

$$= Me^{-(s-a)t} \to 0 \text{ as } t \to \infty \text{ if } s > a$$

$$\therefore L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Case II In case $F'(t)$ is PWC, the integral may be broken as the sum of integrals in different ranges from 0 to ∞ such that $F'(t)$ is continuous in each of such parts. Then proceeding as case-I, we get the result.

Generalized result

Theorem 5

Let $F(t)$ and its derivatives $F'(t), F''(t), \cdots F^{n-1}(t)$ be continuous f^ns. $\forall\, t \geq 0$ and be of exponential order and if $F^n(t)$ is of class A, then L.T. of $F^n(t)$ exists when $s > a$ and is given by

$$L\{F^n(t)\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) \cdots - F^{n-1}(0)$$

[Important — $L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0)$]

Ex Find $L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Sol^n:
$$F(t) = \sin\sqrt{t}$$
$$F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}} \qquad F(0) = 0$$

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = sL\{\sin\sqrt{t}\} = s\,\frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-\frac{1}{4s}} \text{ [Done already]}$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\,e^{-\frac{1}{4s}}$$

Theorem 6

Laplace transform of integrals

If $F(t)$ belongs to class A $\forall\, t \geq 0$, then

$$L\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{s}L\{F(t)\}$$

Proof: Let $G(t) = \int_0^t F(x)\,dx \quad \therefore G(0) = 0$

$$G'(t) = \frac{d}{dt}\left(\int_0^t F(x)\,dx\right) = F(t)$$

$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$

i.e. $L\{F(t)\} = sL\{G(t)\} - 0 = sL\{G(t)\}$

$$\Rightarrow \frac{1}{s}f(s) = L\{G(t)\} = L\left\{\int_0^t F(x)\,dx\right\}$$

Multiplication by powers of t

Theorem 7

If $L\{F(t)\} = f(s)$, then $L\{t F(t)\} = -f'(s)$

Proof: $f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$\frac{d}{ds} f(s) = \frac{d}{ds} \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} F(t)\}\,dt$$

$$= \int_0^\infty -t e^{-st} F(t)\,dt$$

$$= -\int_0^\infty e^{-st}\{t F(t)\}\,dt$$

$$= -L\{t F(t)\}$$

Generalised result

Theorem 8

If $L\{F(t)\} = f(s)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

Division by t

Theorem 9

If $L\{F(t)\} = f(s)$, then $L\{\frac{1}{t} F(t)\} = \int_s^\infty f(x)\,dx$

provided $\lim_{t\to 0} \{\frac{1}{t} F(t)\}$ exist.

Proof: Let $G(t) = \frac{1}{t} F(t)$ $\therefore F(t) = t G(t)$

$L\{F(t)\} = L\{t G(t)\} = -\frac{d}{ds} L\{G(t)\}$

$\Rightarrow f(s) = -\frac{d}{ds} L\{G(t)\}$

$-[L\{G(t)\}]_s^\infty = \int_s^\infty f(s)\,ds$

$\Rightarrow -\lim_{s\to\infty} L\{G(t)\} + L\{G(t)\} = \int_s^\infty f(s)\,ds$

$\Rightarrow 0 + L\{G(t)\} = \int_s^\infty f(s)ds$ $\Rightarrow L\{\frac{F(t)}{t}\} = \int_s^\infty f(x)\,dx$

Ex Find $L\{t\cos at\}$

Sol^n: $L\{\cos at\} = \frac{s}{s^2+a^2}, \quad s>0$

$$L\{t\cos at\} = -\frac{d}{ds} L\{\cos at\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

Ex Find $L\{t^2\sin at\}$

Sol^n: $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L\{t^2\sin at\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin at\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{a}{s^2+a^2}\right\}$$

$$= \frac{d}{ds}\left\{-\frac{2as}{(s^2+a^2)^2}\right\}$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}, \quad s>0$$

Ex Use L.T. to prove that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Sol^n: Let $F(t) = \sin t \qquad f(s) = \frac{1}{s^2+1}$

$$L\left\{\frac{1}{t}\sin t\right\} = \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt$$

$$= \int_s^\infty f(x)\,dx$$

$$= \int_s^\infty \frac{1}{x^2+1}\,dx$$

$$= \left[\tan^{-1}x\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s$$

Taking limit as $s \to 0$, $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Ex Use L.T. to prove

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \ln \frac{b}{a}$$

Sol^n: Let $F(t) = e^{-at} - e^{-bt}$

$f(s) = L\{F(t)\} = \frac{1}{s+a} - \frac{1}{s+b}$

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\, dx$$

$$\therefore \int_0^\infty e^{-st} \frac{(e^{-at} - e^{-bt})}{t}\, dt = \int_s^\infty \left[\frac{1}{x+a} - \frac{1}{x+b}\right] dx$$

$$= \lim_{X\to\infty} \Big[\ln(x+a) - \ln(x+b)\Big]_s^X$$

$$= \lim_{X\to\infty} \left[\ln \frac{X+a}{X+b} - \ln \frac{s+a}{s+b}\right]$$

$$= \lim_{X\to\infty} \ln \frac{1+\frac{a}{X}}{1+\frac{b}{X}} - \ln \frac{s+a}{s+b} = \ln \frac{s+b}{s+a}$$

Taking limit as $s \to 0$

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \ln \frac{b}{a}$$

Ex Evaluate $\int_0^\infty t e^{-3t} \sin t\, dt$ by L.T.

Sol^n: $L\{t \sin t\} = -\frac{d}{ds} L\{\sin t\}$

$$\Rightarrow \int_0^\infty e^{-st}\, t \sin t\, dt = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

Putting $s = 3$,

$$\int_0^\infty t e^{-3t} \sin t\, dt = \frac{3}{50}$$