

# Transform Calculus

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## Operator

An operator or a transformation when applied to a function produce another function.

$$\phi y = y^2 \rightarrow \phi \text{ is a squaring operator}$$

$$\phi y = Dy \rightarrow \phi \text{ is a derivative operator}$$

## Linear operator

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$$

## Definition of Integral transform

Let  $K(s, t)$  be a function of  $s$  and  $t$ , where  $s$  is a parameter (may be real or complex) independent of  $t$ . The function  $f(s)$  defined by the integral (assumed to be convergent)

$$f(s) = \int_{-\infty}^{\infty} K(s, t) F(t) dt$$

is called the integral transform of the  $f^n$ .  $F(t)$  and is denoted by  $T\{F(t)\}$ .  $K(s, t)$  is called the kernel of the transformation.

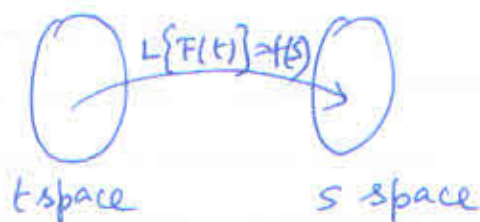
## Definition of Laplace transform

If the kernel  $K(s, t)$  is defined as

$$K(s, t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-st} & \text{for } t \geq 0 \end{cases}$$

then  $f(s) = \int_0^{\infty} e^{-st} F(t) dt$ . The  $f^n$ .  $f(s)$  is called L.T. of the  $f^n$ .  $F(t)$  and is denoted by  $L\{F(t)\}$ .

Another way of looking at the Laplace transform is as a mapping from points in the  $t$  domain to points in the  $s$  domain.



### Theorem

The L.T. is a linear transformation

$$\text{i.e. } L\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$$

Proof:

$$\begin{aligned} & L\{a_1 F_1(t) + a_2 F_2(t)\} \\ &= \int_0^{\infty} e^{-st} \{a_1 F_1(t) + a_2 F_2(t)\} dt \\ &= a_1 \int_0^{\infty} e^{-st} F_1(t) dt + a_2 \int_0^{\infty} e^{-st} F_2(t) dt \\ &= a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\} \end{aligned}$$

Ex If  $F(t) = 1$  for  $t \geq 0$ , then find  $L\{F(t)\}$

Sol<sup>n</sup>:

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} \cdot 1 \, dt \\ &= \lim_{\tau \rightarrow \infty} \left( \frac{e^{-st}}{-s} \right)_0^{\tau} \\ &= \lim_{\tau \rightarrow \infty} \left( \frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) \\ &= \frac{1}{s} \end{aligned}$$

provided  $s > 0$  (if  $s$  is real). So  $L(1) = \frac{1}{s}$ ,  $s > 0$   
 If  $s \leq 0$ , then the integral will diverge and there will be no L.T.

Ex Attempt to find  $L\left\{\frac{1}{t^2}\right\}$

$$\begin{aligned}\text{Sol}^n: L\left\{\frac{1}{t^2}\right\} &= \int_0^{\infty} \frac{e^{-st}}{t^2} dt \\ &= \int_0^1 \frac{e^{-st}}{t^2} dt + \int_1^{\infty} \frac{e^{-st}}{t^2} dt\end{aligned}$$

When  $0 \leq t \leq 1$ ,  $e^{-st} \geq e^{-s}$  if  $s > 0$

$$\therefore \int_0^{\infty} \frac{e^{-st}}{t^2} dt \geq \int_0^1 \frac{e^{-s}}{t^2} dt + \int_1^{\infty} \frac{e^{-st}}{t^2} dt$$

$$\text{i.e. } \int_0^{\infty} \frac{e^{-st}}{t^2} dt \geq e^{-s} \int_0^1 \frac{dt}{t^2} + \int_1^{\infty} \frac{e^{-st}}{t^2} dt$$

But  $\int_0^1 \frac{1}{t^2} dt$  diverges and hence  $L\left\{\frac{1}{t^2}\right\}$  fails to converge. As a result  $\frac{1}{t^2}$  fails to have a L.T.

Ex For the  $f^n$ .  $f(t) = e^{t^2}$

$$\text{Sol}^n: \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{-st} e^{t^2} dt = \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{t^2 - st} dt = \infty$$

for any choice of the variable  $s$ , since the integrand grows without bound as  $\tau \rightarrow \infty$ .

Definition of jump discontinuity

A  $f^n$ .  $f$  has a jump discontinuity at a point  $t_0$  if both the limits  $\lim_{t \rightarrow t_0^-} f(t) = f(t_0^-)$  and  $\lim_{t \rightarrow t_0^+} f(t) = f(t_0^+)$

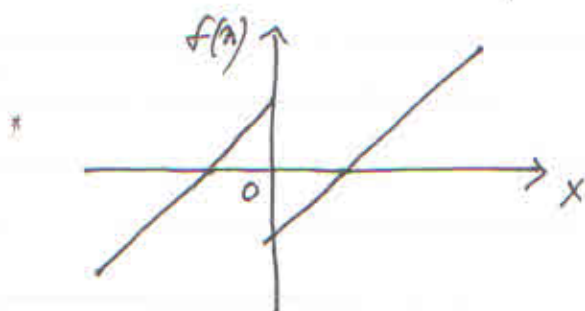
exist (as finite numbers) and  $f(t_0^-) \neq f(t_0^+)$ .





Ex

$$f(x) = \begin{cases} x+1 & \text{if } x \leq 0 \\ x-1 & \text{if } x > 0 \end{cases}$$



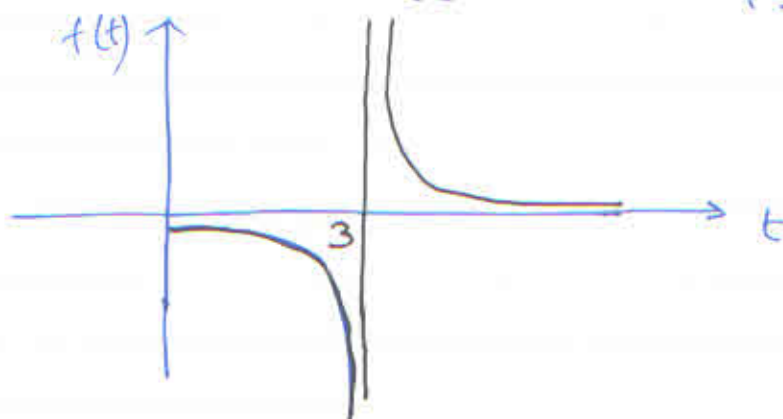
$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x+1 = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x-1 = -1$$

Ex

The f<sup>n</sup>.  $f(t) = \frac{1}{t-3}$

has a discontinuity at  $t=3$  but it is not a jump discontinuity since neither  $\lim_{t \rightarrow 3^-} f(t)$  nor  $\lim_{t \rightarrow 3^+} f(t)$  exist.

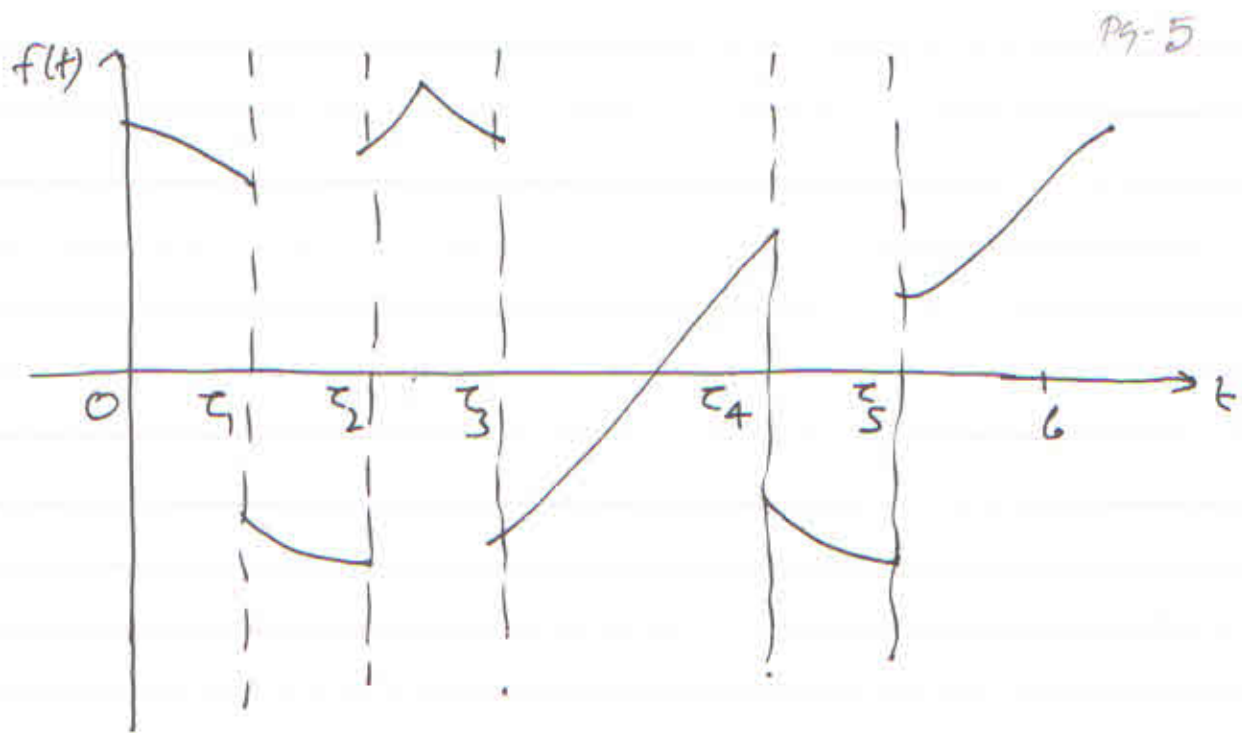


Def<sup>n</sup>. of piecewise continuous function

A f<sup>n</sup>.  $f$  is PWC on the interval  $[0, b)$  if

(i)  $\lim_{t \rightarrow 0^+} f(t) = f(0^+)$  exist and (ii)  $f$  is continuous

on every finite interval  $(0, b)$  except possibly at a finite number of points  $\tau_1, \tau_2, \dots, \tau_n$  in  $(0, b)$  at which  $f$  has a jump discontinuity.



### Exponential order

A f<sup>n</sup>. f has exponential order  $\alpha$  if there exist constants  $M > 0$  and  $\alpha$  such that for some  $t_0 \geq 0$

$$|f(t)| \leq M e^{\alpha t}, \quad t \geq t_0$$

$$\left[ \text{or } \lim_{t \rightarrow \infty} \frac{f(t)}{e^{\alpha t}} \text{ is finite} \right]$$

The f<sup>n</sup>.  $t^n$  has exponential order  $\alpha$  for any  $\alpha > 0$  and any  $n \in \mathbb{N}$  (i.e.  $n = 1, 2, 3, \dots$ )

Because  $\lim_{t \rightarrow \infty} \frac{t^n}{e^{\alpha t}} = 0 \quad \alpha > 0$

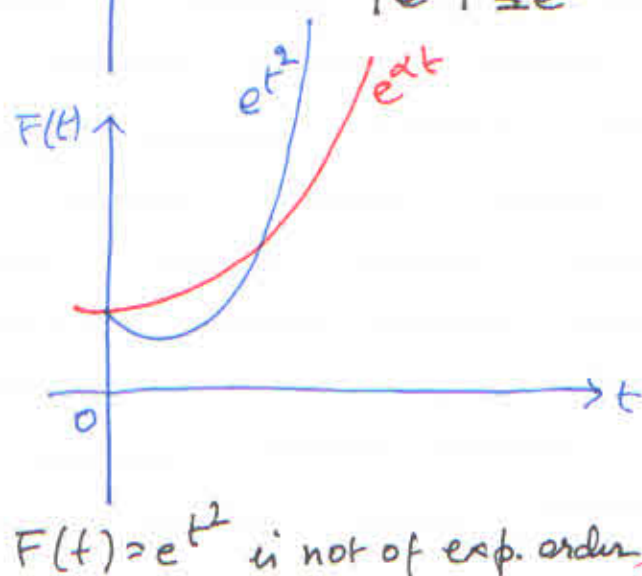
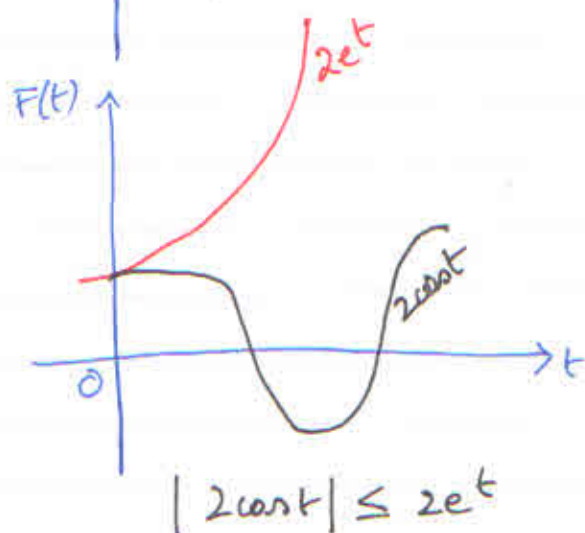
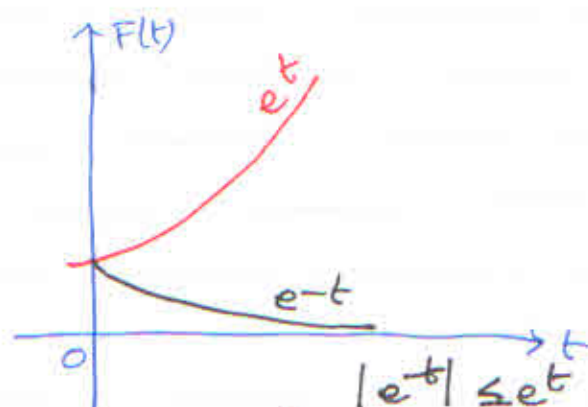
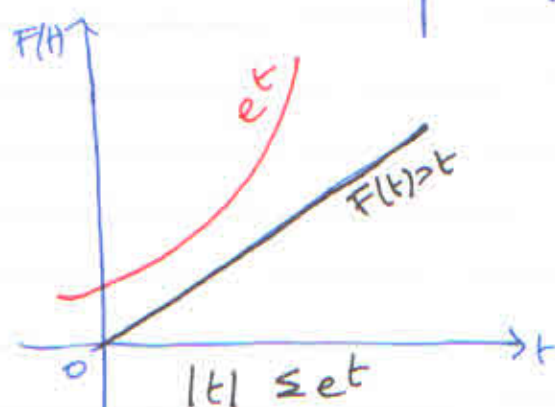
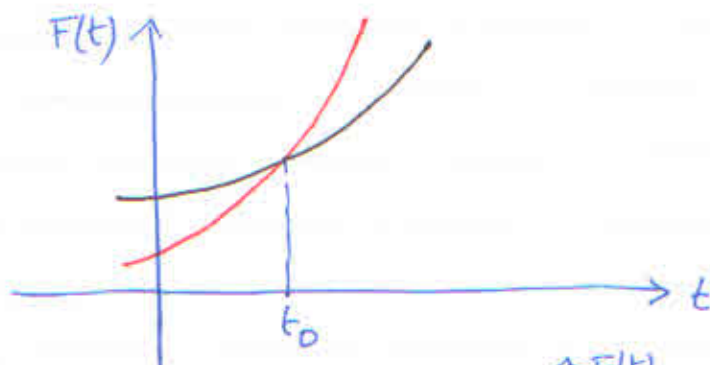
$$= \lim_{t \rightarrow \infty} \frac{t^n}{e^{\alpha t}} \quad \left[ \text{form } \frac{\infty}{\infty} \right]$$

$$= \lim_{t \rightarrow \infty} \frac{n!}{\alpha^n e^{\alpha t}} = \frac{n!}{\infty} = 0$$

If  $f$  is an increasing function, then the condition

$|f(t)| \leq Me^{\alpha t}$ ,  $t > t_0$  simply states that the graph of  $f$  on the interval  $(t_0, \infty)$  does not grow faster than the graph of the exponential  $f^{\sim} Me^{\alpha t}$ , where  $\alpha$  is a positive constant.

Function with black curve is of exponential order.



$F(t) = e^{t^2}$  is not of exp. order.

Definition: Function of class A

A f<sup>n</sup>.  $f(t)$  is said to be of class A if (i) it is piecewise continuous over every finite interval in the range  $t \geq 0$  (ii)  $f(t)$  is of exponential order.

Existence of Laplace transform (Sufficient condition)

If  $F(t)$  is a f<sup>n</sup>. of class A, then L.T. of  $F(t)$  exists  
or

If  $F(t)$  is piecewise continuous on  $[0, \infty)$  and of exponential order  $\alpha$ , then the L.T.  $L\{F(t)\}$  exists for  $\text{Re}(s) > \alpha$  and converges absolutely.

Proof:  $L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$

$$\begin{aligned} \left| \int_0^{\tau} e^{-st} F(t) dt \right| &\leq \int_0^{\tau} |e^{-st} F(t)| dt \\ &\leq \int_0^{\tau} e^{-st} M e^{\alpha t} dt \\ &= M \int_0^{\tau} e^{-(s-\alpha)t} dt \\ &= \frac{M e^{-(s-\alpha)t}}{-(s-\alpha)} \bigg|_0^{\tau} \\ &= \frac{M}{s-\alpha} - M \frac{e^{-(s-\alpha)\tau}}{s-\alpha} \end{aligned}$$

Letting  $\tau \rightarrow \infty$   $[\text{Re}(s) > \alpha]$

$$\int_0^{\infty} |e^{-st} F(t)| dt \leq \frac{M}{s-\alpha}$$



By the above theorem, piecewise continuous  $f^n$ 's. on  $[0, \infty)$  having exponential order belong to  $L$ . However, there are functions in  $L$  that do not satisfy one or both of these conditions.

Ex  $f(t) = \frac{1}{\sqrt{t}}$

It is not PWC on  $[0, \infty)$  since  $f(t) \rightarrow \infty$  as  $t \rightarrow 0^+$  i.e.  $t=0$  is not a point of jump discontinuity. But still we can compute  $\mathcal{L}\left[\frac{1}{\sqrt{t}}\right]$ .

$$\begin{aligned} \mathcal{L}\left[\frac{1}{\sqrt{t}}\right] &= \int_0^{\infty} e^{-st} \frac{1}{\sqrt{t}} dt \\ &= \int_0^{\infty} e^{-st} t^{-\frac{1}{2}} dt \\ &= \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-x} x^{\frac{1}{2}-1} dx \quad \begin{array}{l} \text{Put } st = x \\ \therefore dt = \frac{dx}{s} \end{array} \\ &= \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-x} \Gamma\left(\frac{1}{2}\right) \\ &= \sqrt{\frac{\pi}{s}}, \quad s > 0 \end{aligned}$$

### Uniqueness

If the L.T. of a given  $f^n$  exists, it is uniquely determined.

### Behaviour of $f(s)$ as $s \rightarrow \infty$

If  $F$  is a PNL on  $[0, \infty)$  and has exponential order  $\alpha$ , then  $f(s) = L\{F(t)\} \rightarrow 0$  as  $\text{Re}(s) \rightarrow \infty$

By the existence theorem

$$\left| \int_0^{\infty} e^{-st} F(t) dt \right| \leq \frac{M}{s-\alpha} \quad [\operatorname{Re}(s) > \alpha]$$

and letting  $s \rightarrow \infty$  gives the result.

Remark: Any  $f^n$ .  $f(s)$  without the behaviour  $f(s) \rightarrow 0$  as  $\operatorname{Re}(s) \rightarrow \infty$  say  $s^2$ ,  $\frac{e^s}{s}$  cannot be the L.T. of any  $f^n$ .  $f$  (of class A).

Laplace transform of some elementary functions

Ex Find the L.T. of

- (i) 1 (ii)  $t$  (iii)  $t^n$ ,  $n$  +ve integer  
 (iv)  $e^{at}$  (v)  $\sin at$  (vi)  $\cos at$  (vii)  $\sinh at$  (viii)  $\cosh at$

Sol<sup>n</sup>: (i)  $L[1] = \int_0^{\infty} e^{-st} \cdot 1 dt = \left( \frac{e^{-st}}{-s} \right)_0^{\infty} = \frac{1}{s}, s > 0$

(ii)  $L[t] = \int_0^{\infty} e^{-st} t dt$   
 $= \left( -\frac{t}{s} e^{-st} \right)_0^{\infty} - \int_0^{\infty} -\frac{1}{s} e^{-st} dt$   
 $= \left( -\frac{t}{s} e^{-st} \right)_0^{\infty} + \left[ -\frac{1}{s^2} e^{-st} \right]_0^{\infty}$   
 $= \frac{1}{s^2}$

(iii)  $L[t^n] = \int_0^{\infty} e^{-st} t^n dt$ ,  $n$  +ve integer  
 $= \left[ -\frac{t^n}{s} e^{-st} \right]_0^{\infty} + \int_0^{\infty} \frac{n t^{n-1}}{s} e^{-st} dt$   
 $= \frac{n}{s} L(t^{n-1})$

If  $n=2$ ,  $L[t^2] = \frac{2}{s^3} L[t] = \frac{2}{s^3}$

If  $n=3$ , - - -

By induction  $L[t^n] = \frac{n!}{s^{n+1}}$

Aliter

$$L[t^n] = \int_0^\infty e^{-st} t^n dt$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} \quad st = x$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1} dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{if } s > 0 \text{ \& } n+1 > 0$$

In particular if  $n$  is a +ve integer, then  $\Gamma(n+1) = n!$   
 so that  $L[t^n] = \frac{n!}{s^{n+1}}$ ,  $s > 0$ . Similarly, other  
 L.Ts can be found.

Table 1

	$F(t)$	$L\{F(t)\}$
1.	1	$\frac{1}{s}, s > 0$
2.	$t^n$ ( $n$ +ve int)	$\frac{n!}{s^{n+1}}, s > 0$
3.	$e^{at}$	$\frac{1}{s-a}, s > a$
4.	$\sin at$	$\frac{a}{s^2+a^2}, s > 0$
5.	$\cos at$	$\frac{s}{s^2+a^2}, s > 0$
6.	$\sinh at$	$\frac{a}{s^2-a^2}, s >  a $
7.	$\cosh at$	$\frac{s}{s^2-a^2}, s >  a $

Few examplesEx Find  $L\{\sin \omega t\}$ 

Sol<sup>n</sup>.  $L\{\sin \omega t\} = L\left\{\frac{1}{2} \sin 2t\right\} = \frac{1}{2} L\{\sin 2t\}$

$$= \frac{1}{2} \frac{2}{s^2 + 4} = \frac{1}{s^2 + 4}, \quad s > 0$$

Ex Find  $L\{F(t)\}$  where  $F(t) = \begin{cases} 0 & 0 < t < 1 \\ t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$ 

Sol<sup>n</sup>:  $L\{F(t)\} = \int_0^1 e^{-st} \cdot 0 \, dt + \int_1^2 e^{-st} t \, dt + \int_2^\infty e^{-st} \cdot 0 \, dt$

$$= \int_1^2 e^{-st} \cdot t \, dt$$

$$= -\left[t \frac{e^{-st}}{s}\right]_1^2 - \left[\frac{e^{-st}}{s^2}\right]_1^2$$

$$= -\frac{2}{s} e^{-2s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}$$

$$= \left(\frac{1}{s} + \frac{1}{s^2}\right) e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s}\right) e^{-2s}$$

Ex Find  $L\{F(t)\}$  where  $F(t) = \begin{cases} 4 & 0 < t < 1 \\ 3 & t > 1 \end{cases}$ 

Sol<sup>n</sup>.  $L\{F(t)\} = \int_0^\infty e^{-st} F(t) \, dt$

$$= \int_0^1 e^{-st} \cdot 4 \, dt + \int_1^\infty e^{-st} \cdot 3 \, dt$$

$$= \left[-\frac{4}{s} e^{-st}\right]_0^1 + \left[-\frac{3}{s} e^{-st}\right]_1^\infty$$

$$= \frac{1}{s} (4 - e^{-s}), \quad s > 0$$



Ex. Find  $L\{\sin^3 2t\}$

Sol<sup>n</sup>:  $\sin 3t = 3\sin t - 4\sin^3 t$   
 $\therefore \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$   
 and so  $\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$$\begin{aligned}\therefore L\{F(t)\} &= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\} \\ &= \frac{3}{4}\left[\frac{2}{s^2+4}\right] - \frac{1}{4}\left[\frac{6}{s^2+36}\right], s>0 \\ &= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right] \\ &= \frac{48}{(s^2+4)(s^2+36)}\end{aligned}$$

Ex Find  $L\{\sin at \sin bt\}$

Sol<sup>n</sup>:  $F(t) = \frac{1}{2}(2\sin at \sin bt)$   
 $= \frac{1}{2}[\cos(at-bt) - \cos(at+bt)]$   
 $= \frac{1}{2}\cos(a-b)t - \frac{1}{2}\cos(a+b)t$

$$\begin{aligned}\therefore L\{F(t)\} &= \frac{1}{2}L\{\cos(a-b)t\} - \frac{1}{2}L\{\cos(a+b)t\} \\ &= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2}\right] - \frac{1}{2}\left[\frac{s}{s^2+(a+b)^2}\right], s>0 \\ &= \frac{2abs}{\{s^2+(a-b)^2\}\{s^2+(a+b)^2\}}, s>0\end{aligned}$$

Ex Find  $L\{e^{at} \cos bt\}$  and  $L\{e^{at} \sin bt\}$

Sol<sup>n</sup>: Let  $F(t) = e^{(a+ib)t}$

$$L\{F(t)\} = \frac{1}{s - (a+ib)} = \frac{1}{s-a-ib}$$

$$= \frac{(s-a) + ib}{(s-a)^2 + b^2}$$

Again  $e^{(a+ib)t} = e^{at} [\cos bt + i \sin bt]$

$$= e^{at} \cos bt + i e^{at} \sin bt$$

$$\therefore L\{F(t)\} = L\{e^{at} \cos bt + i e^{at} \sin bt\}$$

$$= L\{e^{at} \cos bt\} + i L\{e^{at} \sin bt\}$$

$$\therefore L\{e^{at} \cos bt\} + i L\{e^{at} \sin bt\} = \frac{(s-a) + ib}{(s-a)^2 + b^2}$$

Comparing real and imaginary parts,

$$L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$$

$$L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

### Infinite series

For an infinite series  $\sum_{n=0}^{\infty} a_n t^n$ , in general, it is not possible to obtain the L.T. of the series by taking the transform term by term.

### Theorem

If  $F(t) = \sum_{n=0}^{\infty} a_n t^n$  converges for  $t \geq 0$  with  $|a_n| \leq \frac{K \alpha^n}{n!}$

for all  $n$  sufficiently large and  $\alpha > 0$ ,  $K > 0$ , then

$$L\{F(t)\} = \sum_{n=0}^{\infty} a_n L(t^n) = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}} \quad (\text{Re } s > \alpha)$$

Ex Find  $L\{\sin \sqrt{t}\}$

$$\begin{aligned} \text{Sol}^n: \quad L\{\sin \sqrt{t}\} &= L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \dots\right\} \\ &= L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \dots\right\} \\ &= L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{1}{5!} L\{t^{5/2}\} - \dots \\ &= \frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{1}{3!} \frac{\Gamma(\frac{5}{2})}{s^{5/2}} + \frac{1}{5!} \frac{\Gamma(\frac{7}{2})}{s^{7/2}} - \dots \\ &= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6} \frac{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{120} \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \dots \\ &= \frac{\sqrt{\pi}}{2s^{3/2}} \left[1 - \frac{1}{4s} + \frac{1}{2!} \left(\frac{1}{4s}\right)^2 - \frac{1}{3!} \left(\frac{1}{4s}\right)^3 + \dots\right] \\ &= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}} \end{aligned}$$

## Elementary properties of Laplace Transform

### Theorem 1

First translation (or shifting) theorem

If  $L\{F(t)\} = f(s)$ ,  $s > \alpha$

then  $L\{e^{at} F(t)\} = f(s-a)$ ,  $s > \alpha + a$

Proof:  $f(s) = L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$

$$\therefore f(s-a) = \int_0^{\infty} e^{-(s-a)t} F(t) dt$$

$$= \int_0^{\infty} e^{-st} e^{at} F(t) dt$$

$$= L\{e^{at} F(t)\}$$

Ex Find the L.T. of  $F(t) = t^3 e^{-3t}$

Sol<sup>n</sup>:  $L\{t^3\} = f(s) = \frac{3!}{s^4} = \frac{6}{s^4}$

$$L\{F(t)\} = L\{t^3 e^{-3t}\} = f(s-a) = \frac{6}{(s+3)^4}$$

Ex Find  $L\{e^t \sin^2 t\}$

Sol<sup>n</sup>:  $L\{\sin^2 t\} = L\left\{\frac{1}{2}(1 - \cos 2t)\right\} = \frac{1}{2} \left[ \frac{1}{s} - \frac{s}{s^2 + 2^2} \right]$

$$= \frac{2}{s(s^2 + 4)} = f(s)$$

$$L\{e^t \sin^2 t\} = f(s-1) = \frac{2}{(s-1)\{(s-1)^2 + 4\}}$$

$$= \frac{2}{(s-1)(s^2 - 2s + 5)}$$



Ex Find L.T. of  $t \sin at$  and  $t \cos at$

Sol<sup>n</sup>:  $L\{t\} = \frac{1}{s^2} = f(s)$

$$L\{te^{iat}\} = L\{t \cos at\} + iL\{t \sin at\}$$

$$\begin{aligned} \text{Also } L\{te^{iat}\} &= \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{[(s-ia)(s+ia)]^2} \\ &= \frac{(s^2 - a^2) + i(2as)}{(s^2 + a^2)^2} \end{aligned}$$

$$\therefore L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2} ; L\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2}$$

Ex Find the L.T. of  $F(t) = \sinh 3t \cos^2 t$

Sol<sup>n</sup>:  $L\{\cos^2 t\} = \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\}$

$$= \frac{1}{2} \left[ \frac{1}{s} + \frac{s}{s^2 + 4} \right] = \frac{s^2 + 2}{s(s^2 + 4)}, \quad s > 0$$

$$\begin{aligned} \therefore L\{\sinh 3t \cos^2 t\} &= L\left\{ \frac{e^{3t} - e^{-3t}}{2} \cos^2 t \right\} \\ &= \frac{1}{2}L\{e^{3t} \cos^2 t\} - \frac{1}{2}L\{e^{-3t} \cos^2 t\} \\ &= \frac{1}{2} \left[ \frac{(s-3)^2 + 2}{(s-3)[(s-3)^2 + 4]} - \frac{(s+3)^2 + 2}{(s+3)[(s+3)^2 + 4]} \right] \\ &= \frac{1}{2} \left[ \frac{s^2 - 6s + 11}{(s-3)(s^2 - 6s + 13)} - \frac{s^2 + 6s + 11}{(s+3)(s^2 + 6s + 13)} \right] \end{aligned}$$

## Theorem 2

Second translation (or shifting) theorem:

If  $L\{F(t)\} = f(s)$  and  $G$  is a f<sup>n</sup>. defined by

$$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

then  $L\{G(t)\} = e^{-as} f(s)$ 

Proof:  $L\{G(t)\} = \int_0^{\infty} e^{-st} G(t) dt$

$$= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} F(t-a) dt$$

$$= \int_a^{\infty} e^{-st} F(t-a) dt$$

$$= \int_0^{\infty} e^{-s(a+x)} F(x) dx \quad \begin{matrix} t-a=x \\ dt=dx \end{matrix}$$

$$= e^{-sa} \int_0^{\infty} e^{-sx} F(x) dx$$

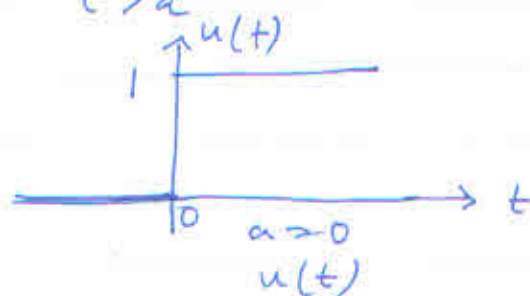
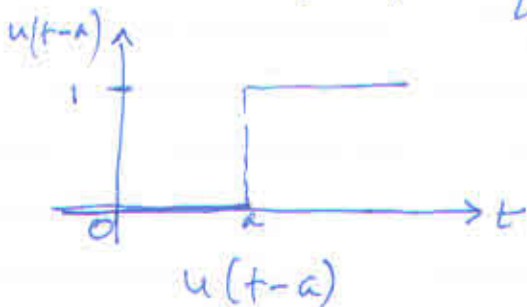
$$= e^{-sa} \int_0^{\infty} e^{-st} F(t) dt$$

$$= e^{-sa} L\{F(t)\}$$

$$= e^{-as} f(s)$$

Unit step f<sup>n</sup>. or Heaviside's unit f<sup>n</sup>.

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \quad a \geq 0$$



Alternative statement of 2nd shifting theorem

If  $F(t)$  has the transform  $f(s)$ , then the shifted  $t^2$ .

$$F(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ F(t-a) & \text{if } t > a \end{cases}$$

has the transform  $e^{-as} f(s)$  i.e.

$$L\{F(t-a)u(t-a)\} = e^{-as} F(s)$$

Laplace transform of unit step function

$$\begin{aligned} L\{u(t-a)\} &= \int_0^{\infty} e^{-st} u(t-a) dt \\ &= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} \cdot 1 dt \\ &= -\frac{1}{s} e^{-st} \Big|_a^{\infty} = \frac{e^{-as}}{s} \end{aligned}$$

Ex Find  $L\{G(t)\}$  where  $G(t) = \begin{cases} e^{t-a}, & t > a \\ 0 & t < a \end{cases}$

Sol<sup>n</sup>: Let  $F(t) = e^t \therefore L\{F(t)\} = \frac{1}{s-1}, s > 1$   
 $\quad \quad \quad = f(s)$

$$G(t) = \begin{cases} F(t-a) = e^{t-a} & t > a \\ 0 & t < a \end{cases}$$

$$\therefore L\{G(t)\} = e^{-as} f(s) = \frac{e^{-as}}{s-1}, s > 1$$

Ex Find  $L\{F(t)\}$  where

$$F(t) = \begin{cases} \cos(t - \frac{2\pi}{3}), & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$$

Sol<sup>n</sup>: 1st method (from definition)

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} F(t) dt \\ &= \int_0^{\frac{2\pi}{3}} e^{-st} \cdot 0 dt + \int_{\frac{2\pi}{3}}^{\infty} e^{-st} \cos(t - \frac{2\pi}{3}) dt \\ &= \int_{\frac{2\pi}{3}}^{\infty} e^{-st} \cos(t - \frac{2\pi}{3}) dt \\ t - \frac{2\pi}{3} &= x \\ &= \int_0^{\infty} e^{-s(x + \frac{2\pi}{3})} \cos x dx \\ &= e^{-s\frac{2\pi}{3}} \int_0^{\infty} e^{-sx} \cos x dx \\ &= e^{-s\frac{2\pi}{3}} \int_0^{\infty} e^{-st} \cos t dt \\ &= e^{-s\frac{2\pi}{3}} L\{\cos t\} = e^{-\frac{2}{3}s\pi} \frac{s}{s^2+1}, \quad s > 0 \end{aligned}$$

2nd method (by 2nd shifting theorem)

$$\text{Let } \varphi(t) = \cos t \quad F(t) = \begin{cases} \varphi(t - \frac{2\pi}{3}) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$$

$$L\{\varphi(t)\} = \frac{s}{s^2+1} = f(s)$$

$$\therefore L\{F(t)\} = e^{-\frac{2\pi s}{3}} f(s) = e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}$$



## Theorem 3

Change of scale property

If  $L\{F(t)\} = f(s)$ , then  $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$ ,  $a > 0$ 

Proof:

$$L\{F(at)\}$$

$$= \int_0^{\infty} e^{-st} F(at) dt$$

$$at = x$$

$$dt = \frac{1}{a} dx$$

$$= \frac{1}{a} \int_0^{\infty} e^{-s\left(\frac{x}{a}\right)} F(x) dx$$

$$= \frac{1}{a} \int_0^{\infty} e^{-\left(\frac{s}{a}\right)t} F(t) dt$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right)$$

Ex Find  $L\{\cos 5t\}$

Sol<sup>n</sup>:  $L\{\cos t\} = \frac{s}{s^2+1} = f(s)$   $s > 0$

$$L\{\cos 5t\} = \frac{1}{5} f\left(\frac{s}{5}\right) = \frac{1}{5} \frac{\frac{s}{5}}{\left(\frac{s}{5}\right)^2 + 1} = \frac{s}{s^2 + 25}, s > 0$$

Ex Find  $L\{\sinh 3t\}$

Sol<sup>n</sup>:  $L\{\sinh t\} = \frac{1}{s^2-1} = f(s)$

$$L\{\sinh 3t\} = \frac{1}{3} f\left(\frac{s}{3}\right) = \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1} = \frac{3}{s^2 - 9}$$

Laplace transform of derivatives of  $F(t)$ Theorem 4 (or PNL)

Let  $F(t)$  be continuous  $\forall t \geq 0$  and be of exponential order  $a$  and if  $F'(t)$  is of class A, then L.T. of  $F'(t)$  exists when  $s > a$  and

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Proof Case I

In case  $F'(t)$  is continuous  $\forall t \geq 0$ , then

$$\begin{aligned} L\{F'(t)\} &= \int_0^{\infty} e^{-st} F'(t) dt \\ &= [e^{-st} F(t)]_0^{\infty} + s \int_0^{\infty} e^{-st} F(t) dt \\ &= \lim_{t \rightarrow \infty} e^{-st} F(t) - F(0) + sL\{F(t)\} \end{aligned}$$

Now  $|F(t)| \leq M e^{at} \quad \forall t \geq 0$  and for some const.  $a$  and  $M$

$$\begin{aligned} \therefore |e^{-st} F(t)| &= e^{-st} |F(t)| \leq e^{-st} M e^{at} \\ &= M e^{-(s-a)t} \rightarrow 0 \text{ as } t \rightarrow \infty \\ &\quad \text{if } s > a \end{aligned}$$

$$\therefore L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Case II In case  $F'(t)$  is PNL, the integral may be broken as the sum of integrals in different ranges from 0 to  $\infty$  such that  $F'(t)$  is continuous in each of such parts. Then proceeding as case-I, we get the result.

## Generalized result

## Theorem 5

Let  $F(t)$  and its derivatives  $F'(t), F''(t), \dots, F^{(n-1)}(t)$  be continuous f'n.  $\forall t \geq 0$  and be of exponential order and if  $F^{(n)}(t)$  is of class A, then L.T. of  $F^{(n)}(t)$  exists when  $s > a$  and is given by

$$L\{F^{(n)}(t)\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0)$$

[Important —  $L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0)$ ]

Ex Find  $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Sol<sup>n</sup>:  $F(t) = \sin\sqrt{t}$   
 $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$   $F(0) = 0$

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = sL\{\sin\sqrt{t}\} = s \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}} \text{ [Done already]}$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}$$

## Theorem 6

Laplace transform of integrals

If  $F(t)$  belongs to class A  $\forall t \geq 0$ , then

$$L\left\{\int_0^t F(x) dx\right\} = \frac{1}{s} L\{F(t)\}$$

Proof: Let  $G(t) = \int_0^t F(x) dx$   $\therefore G(0) = 0$

$$G'(t) = \frac{d}{dt} \left( \int_0^t F(x) dx \right) = F(t)$$

$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$

$$\text{i.e. } L\{F(t)\} = sL\{G(t)\} - 0 = sL\{G(t)\}$$

$$\Rightarrow \frac{1}{s} f(s) = L\{G(t)\} = L\left\{\int_0^t F(x) dx\right\}$$

## Multiplication by powers of $t$

### Theorem 7

If  $L\{F(t)\} = f(s)$ , then  $L\{tF(t)\} = -f'(s)$

Proof:  $f(s) = L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$

$$\begin{aligned} \frac{d}{ds} f(s) &= \frac{d}{ds} \int_0^{\infty} e^{-st} F(t) dt \\ &= \int_0^{\infty} \frac{\partial}{\partial s} \{e^{-st} F(t)\} dt \\ &= \int_0^{\infty} -t e^{-st} F(t) dt \\ &= - \int_0^{\infty} e^{-st} \{t F(t)\} dt \\ &= -L\{t F(t)\} \end{aligned}$$

### Generalised result

#### Theorem 8

If  $L\{F(t)\} = f(s)$ , then  $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

### Division by $t$

#### Theorem 9

If  $L\{F(t)\} = f(s)$ , then  $L\{\frac{1}{t} F(t)\} = \int_s^{\infty} f(x) dx$   
provided  $\lim_{t \rightarrow 0} \{\frac{1}{t} F(t)\}$  exist.

Proof: Let  $G(t) = \frac{1}{t} F(t)$   $\therefore F(t) = t G(t)$

$$L\{F(t)\} = L\{t G(t)\} = -\frac{d}{ds} L\{G(t)\}$$

$$\Rightarrow f(s) = -\frac{d}{ds} L\{G(t)\}$$

$$- [L\{G(t)\}]_s^{\infty} = \int_s^{\infty} f(s) ds$$

$$\Rightarrow -\lim_{s \rightarrow \infty} L\{G(t)\} + L\{G(t)\} = \int_s^{\infty} f(s) ds$$

$$\Rightarrow 0 + L\{G(t)\} = \int_s^{\infty} f(s) ds \quad \Rightarrow L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(x) dx$$



Ex Find  $L\{t \cos at\}$

Sol<sup>n</sup>:  $L\{\cos at\} = \frac{s}{s^2 + a^2}, s > 0$

$$\begin{aligned} L\{t \cos at\} &= -\frac{d}{ds} L\{\cos at\} \\ &= -\frac{d}{ds} \left( \frac{s}{s^2 + a^2} \right) = \frac{s^2 - a^2}{(s^2 + a^2)^2} \end{aligned}$$

Ex Find  $L\{t^2 \sin at\}$

Sol<sup>n</sup>:  $L\{\sin at\} = \frac{a}{s^2 + a^2}$

$$\begin{aligned} L\{t^2 \sin at\} &= (-1)^2 \frac{d^2}{ds^2} L\{\sin at\} \\ &= \frac{d^2}{ds^2} \left\{ \frac{a}{s^2 + a^2} \right\} \\ &= \frac{d}{ds} \left\{ -\frac{2as}{(s^2 + a^2)^2} \right\} \\ &= \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}, s > 0 \end{aligned}$$

Ex Use L.T. to prove that  $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

Sol<sup>n</sup>: Let  $F(t) = \sin t$   $f(s) = \frac{1}{s^2 + 1}$

$$\begin{aligned} L\left\{\frac{1}{t} \sin t\right\} &= \int_0^\infty e^{-st} \frac{\sin t}{t} dt \\ &= \int_s^\infty f(x) dx \\ &= \int_s^\infty \frac{1}{x^2 + 1} dx \\ &= \left[ \tan^{-1} x \right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s \end{aligned}$$

Taking limit as  $s \rightarrow 0$ ,  $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

Ex Use L.T. to prove

$$\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t} dt = \ln \frac{b}{a}$$

Sol<sup>n</sup>: Let  $F(t) = e^{-at} - e^{-bt}$

$$f(s) = L\{F(t)\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(x) dx$$

$$\therefore \int_0^{\infty} e^{-st} \frac{(e^{-at} - e^{-bt})}{t} dt = \int_s^{\infty} \left[ \frac{1}{x+a} - \frac{1}{x+b} \right] dx$$

$$= \lim_{x \rightarrow \infty} \left[ \ln(x+a) - \ln(x+b) \right]_s^x$$

$$= \lim_{x \rightarrow \infty} \left[ \ln \frac{x+a}{x+b} - \ln \frac{s+a}{s+b} \right]$$

$$= \lim_{x \rightarrow \infty} \ln \frac{1 + \frac{a}{x}}{1 + \frac{b}{x}} - \ln \frac{s+a}{s+b} = \ln \frac{s+b}{s+a}$$

Taking limit as  $s \rightarrow 0$

$$\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t} dt = \ln \frac{b}{a}$$

Ex Evaluate  $\int_0^{\infty} t e^{-3t} \sin t dt$  by L.T.

$$\text{Sol}^n: L\{t \sin t\} = -\frac{d}{ds} L\{\sin t\}$$

$$\Rightarrow \int_0^{\infty} e^{-st} t \sin t dt = -\frac{d}{ds} \left( \frac{1}{s^2+1} \right) = \frac{2s}{(s^2+1)^2}$$

Putting  $s=3$ ,

$$\int_0^{\infty} t e^{-3t} \sin t dt = \frac{3}{50}$$