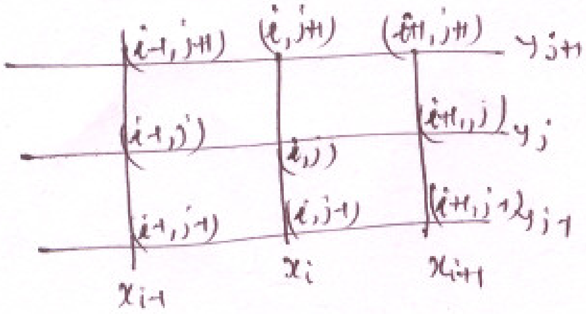
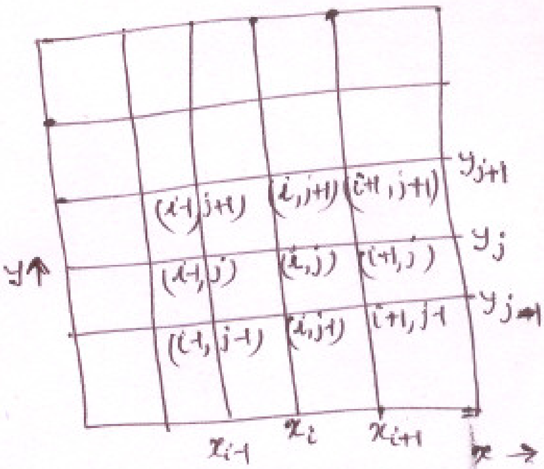
Elliptic PDE

$\nabla^2 u = u_{xx} + u_{yy}$





① Explicit method (five point)

$$u_{xx} + u_{yy} = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$$

② five point diagonal

$$u_{xx} + u_{yy} = \frac{1}{2h^2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{2h^2}\Big[u(x-h, y+h) + u(x+h, y+h) - 4u(x,y) + u(x-h, y-h) + u(x+h, y-h)\Big]$$

$$= (u_{xx} + u_{yy}) + \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial y^4} + 6\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2}\right) + O(h^6).$$

Parabolic PDE

Dufort - Frankel method

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{——} (1)$$ One dimensional diffusion eq.

$$\left.\frac{\partial u}{\partial t}\right|_{(x_i,t_j)} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i,t_j)}$$

$$\frac{u_i^{j+1} - u_i^{j-1}}{2\Delta t} = \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{\Delta x^2} \quad \text{——} (2)$$

Now $y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \cdots$

$y(x-h) = y(x) - h y'(x) + \frac{h^2}{2!} y''(x) + \cdots$

$y(x+h) + y(x-h) = 2y(x) + 2\frac{h^2}{2!} y''(x)$

$$\frac{1}{2}\left[y(x+h) + y(x-h)\right] = y(x) + O(h^2)$$

or $$y(x) = \frac{1}{2}\left[y(x+h) + y(x-h)\right] + O(h^2)$$

So $$u_i^j = \frac{1}{2}\left(u_i^{j+1} + u_i^{j-1}\right) + O((\Delta t)^2) \quad \text{—} (3)$$

Now use (3) in (2) we get

$$\frac{u_i^{j+1} - u_i^{j-1}}{2\Delta t} = \frac{1}{(\Delta x)^2}\left[u_{i-1}^j - \left(u_i^{j+1} + u_i^{j-1}\right) + u_{i+1}^j\right]$$

$$u_i^{j+1} - u_i^{j-1} = \frac{2\Delta t}{(\Delta x)^2}\left[u_{i-1}^j - \left(u_i^{j+1} + u_i^{j-1}\right) + u_{i+1}^j\right]$$

Take $r = \frac{\Delta t}{\Delta x^2}$ mesh ratio parameter

$$(1+2r)u_i^{j+1} - (1-2r)u_i^{j-1} - 2r\,u_{i+1}^j - 2r\,u_{i-1}^j = 0$$

or

$$u_t - u_{xx} = \begin{bmatrix} 0 & 1+2r & 0 \\ -2r & 0 & -2r \\ 0 & -1+2r & 0 \end{bmatrix} u_{ij}$$

or

$$u_i^{j+1} = \frac{1-2r}{1+2r} u_i^{j-1} + \frac{2r}{1+2r} u_{i+1}^{j} + \frac{2r}{1+2r} u_{i-1}^{j}$$

$j = 1, 2, \ldots$

for $j = 1$

$$u_i^2 = \frac{1-2r}{1+2r} u_i^0 + \frac{2r}{1+2r} u_{i+1}^1 + \frac{2r}{1+2r} u_{i-1}^1$$

So we require u_i^1's which may be calculate from some other method, e.g., explicit method.

Ex Solve the heat conduction equation (Iyengar P21)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the initial and boundary conditions

$$u(x,0) = \sin \pi x, \quad 0 \le x \le 1$$

$$u(0,t) = u(1,t) = 0$$

using the following methods

(i) the Schmidt method

(ii) the Laasonen method

(iii) the Crank-Nicolson method

(iv) the Dufort-Frankel method

for $\Delta x = 1/3$ $(h = 1/3)$ and $\Delta t = 1/36$ $(k = 1/36)$.

Parabolic PDE

Unsteady Heat flow Equation or Diffusion Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{c\rho}{K}\frac{\partial u}{\partial t} \qquad \text{— (1)}$$

with boundary conditions

B.C. $\quad u(0,t) = C_1 \;, \quad u(L,t) = C_2 \qquad \text{— (2)}$

or $\quad A_1 u(0,t) + B_1 \frac{\partial u}{\partial x}(0,t) = F_1(t)$

$\quad A_2 u(L,t) + B_2 \frac{\partial u}{\partial x}(L,t) = F_2(t)$

and

I.C. $\quad u(x,t)\big|_{t=0} = f(x) \qquad \text{— (3)}$

Schmidt method

Explicit Method: Using central difference approximation for $\frac{\partial^2 u}{\partial x^2}$ and forward for $\frac{\partial u}{\partial t}$ we get- $\quad u(x_i,t_j) = u_i^j$

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{c\rho}{K}\frac{\partial u}{\partial t}(x_i,t_j)$$

$$\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\Delta x^2} = \frac{c\rho}{K}\,\frac{u_i^{j+1} - u_i^j}{\Delta t}$$

$$\frac{K\Delta t}{c\rho\Delta x^2}\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) + u_i^j = u_i^{j+1}$$

or $$u_i^{j+1} = \frac{K\Delta t}{c\rho\Delta x^2}\left(u_{i+1}^j + u_{i-1}^j\right) + \left(1 - 2\frac{K\Delta t}{c\rho\Delta x^2}\right)u_i^j \qquad \text{— (4)}$$

If the ratio is chosen so that

$$r = \frac{K\Delta t}{c\rho\Delta x^2} = \frac{1}{2} \qquad \text{— (5)}$$

then (4) becomes

$$u_i^{j+1} = \frac{1}{2}\left(u_{i+1}^j + u_{i-1}^j\right) \qquad \text{— (6)}$$

Ex A large flat steel plate is 2 cm thick. If the initial temperature (°C) with in the plate are given as a function of distance from one face by the equation

$$u(x,t)\big|_{t=0} = 100 \sin \frac{\pi x}{2}.$$

find the temperature as a function of x and t if both faces are maintained at $0°C$.

i.e. B.C. $\quad u(0,t) = u(2,t) = 0$

I.C. $\quad u(x,0) = 100 \sin \frac{\pi x}{2}$

for steel $\quad K = 0.13\ \text{cal/sec. cm. °C}$

$c = 0.11\ \text{cal/g. °C}$

$\rho = 7.8\ \text{g/cm}^3$

Let $\Delta x = 0.25$, gives eight subdiv. of x-axis.

$$\frac{K\Delta t}{c\rho \Delta x^2} = \frac{1}{2} \Rightarrow \Delta t = 0.206\ \text{sec}$$

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8$

$0\ \ 1/4\ \ 2/4\ \ 3/4\ \ 4/4\ \ 5/4\ \ 6/4\ \ 7/4\ \ 8/4 = 2$

$t_0 = 0, \quad t_1 = \Delta t = .206, \quad t_2 = .412, \quad t_3 = .619, \ldots$

for $t=0$ from ⑥

$$u_i^1 = \frac{1}{2}\left(u_{i+1}^0 + u_{i-1}^0\right)$$

Now for $x = x_1$ i.e., $i = 1$

$$u_1^1 = \frac{1}{2}\left(u_2^0 + u_0^0\right)$$

↑ boundary condition

Similarly for

$i=7 \quad u_7^1 = \frac{1}{2}\left(u_8^0 + u_6^0\right)$

↑ boundary condition

Numerical Solution to heat flow example

t=	x=0	x=.25	x=.50	x=.75	x=1.0	x=1.25
0	0	38.3	70.7	92.4	100	92.4
.206	0	35.35	65.35	85.35	92.4	85.35
.412	0	32.68	60.35	78.88	85.35	78.88
.619	0	30.18	55.78	77.06	77.08	72.86
.825	0	27.89	51.52	67.33	72.86	67.33
1.031	0	25.76	47.61	62.19	67.33	62.19
	0					

For $j=0$

$$u_i^1 = \frac{1}{2}\left(u_{i+1}^0 + u_{i-1}^0\right)$$

x_1 $\quad i=1$ $\quad u_1^1 = \frac{1}{2}\left(u_2^0 + u_0^0\right)$ $\longrightarrow$ b.c.

x_2 $\quad i=2$ $\quad u_2^1 = \frac{1}{2}\left(u_3^0 + u_1^0\right)$

x_3 $\quad i=3$ $\quad u_3^1 = \frac{1}{2}\left(u_4^0 + u_2^0\right)$

$\vdots$

Then u_i^1's are known, Next

for $j=1$

$$u_i^2 = \frac{1}{2}\left(u_{i+1}^1 + u_{i-1}^1\right)$$

x_1 $\quad i=1$ $\quad u_i^2 = \frac{1}{2}\left(u_2^1 + u_1^1\right)$

$\vdots$

Crank-Nicolson Method (Implicit Method)

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i, t_{j+1/2})} = \frac{c\rho}{k} \left.\frac{\partial u}{\partial t}\right|_{(x_i, t_{j+1/2})}$$

Central

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2)$$

Central with spacing $h/2$

$$y'(x) = \frac{y(x+h/2) - y(x-h/2)}{h} + O(h^2)$$

$$y(x+h/2) = y(x) + h/2\, y'(x) + O(h^2)$$
$$= y(x) + \frac{1}{2}(y(x+h) - y(x)) + O(h^2)$$
$$= \frac{1}{2}[y(x+h) + y(x)] + O(h^2)$$

$$\frac{1}{2}\Big[\underbrace{\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i, t_{j+1})} + \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i, t_j)}}_{\text{Central difference Approximation}} \Big] = \frac{c\rho}{k} \underbrace{\left.\frac{\partial u}{\partial t}\right|_{(x_i, t_{j+1/2})}}_{\text{Central difference Approximation with } \Delta t/2 \text{ as spacing}}$$

$$\frac{1}{2}\left[\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{\Delta x^2} + \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{\Delta x^2} \right]$$
$$= \frac{c\rho}{k}\left[\frac{u_i^{j+1} - u_i^j}{\Delta t} \right]$$

$$\frac{k\Delta t}{2c\rho\Delta x^2}\left[u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} + u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j} \right]$$
$$= u_i^{j+1} - u_i^j$$

Take $r = \dfrac{k\Delta t}{c\rho\Delta x^2}$

$$r[u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} + u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}] = 2u_i^{j+1} - 2u_i^j$$

$$r u_{i+1}^{j+1} - (2+2r)u_i^{j+1} + r u_{i-1}^{j+1} = -r u_{i+1}^j + (2r-2)u_i^j - r u_{i-1}^j$$

Multiply by -1 on both sides we get

$$-r\,u_{i-1}^{j+1} + (2+2r)\,u_i^{j+1} - r\,u_{i+1}^{j+1} = r\,u_{i-1}^{j} + (2-2r)\,u_i^{j} + r\,u_{i+1}^{j}$$

r is called mesh ratio parameter

The advantage of the Crank-Nicolson method is that (7) it is stable for any value of r, although small values are more accurate. Values much larger than unity are not desirable.

Ex $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{D}\dfrac{\partial u}{\partial t}$ $\quad$ I.C. $u(x,0) = 2.0$

B.C $\quad u(0,t) = 0.0$

$D = 0.119\ \mathrm{cm^2/sec}$ $\quad u(20,t) = 10.0$

Take $\dfrac{D\Delta t}{(\Delta x)^2} = 1$, & $\Delta x = 4$

	$i=1$	$i=2$	$i=3$	$i=4$	
x_0	x_1	x_2	x_3	x_4	x_5
0	4	8	12	16	20

then $\Delta t = 134.4$ sec.

for $r=1$ $\quad u_{i+1}^{j+1} - 4u_i^{j+1} + u_{i-1}^{j+1} = -u_{i+1}^{j} - u_{i-1}^{j}$

for $j=0$ $\quad$ or $\quad -u_{i-1}^{j+1} + 4u_i^{j+1} - u_{i+1}^{j+1} = u_{i-1}^{j} + u_{i+1}^{j}$

$i=1 \qquad -0 + 4u_1 - u_2 = 0 + 2$

$i=2 \qquad -u_1 + 4u_2 - u_3 = 2. + 2$

$i=3 \qquad -u_2 + 4u_3 - u_4 = 2. + 2$

$i=4 \qquad -u_3 + 4u_4 - 10 = 2. + 10.$

or

$$4u_1 - u_2 = 2$$
$$-u_1 + 4u_2 - u_3 = 4$$
$$-u_2 + 4u_3 - u_4 = 4$$
$$-u_3 + 4u_4 = 22$$

$$\begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array}$$
$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 22 \end{bmatrix}$$

time in sec	$x=4$	$x=8$	$x=12$	$x=16$
134.4	.986	2.019	3.072	5.992
268.8	1.070	2.363	4.305	6.555
403.2	1.276	2.061	4.762	6.962
537.6	1.471	3.165	5.115	7.159

Convergence of Explicit Method:

$$\frac{\partial U}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 U}{\partial x^2} \quad \text{--- (1)}$$

Explicit method

$$u_i^{j+1} = \frac{k\Delta t}{c\rho\Delta x^2}\left(u_{i+1}^j + u_{i-1}^j\right) + \left(1 - \frac{2k\Delta t}{c\rho\Delta x^2}\right)u_i^j$$

Let $r = \frac{k\Delta t}{c\rho\Delta x^2}$ then

$$u_i^{j+1} = r\left(u_{i+1}^j + u_{i-1}^j\right) + (1-2r)u_i^j \quad \text{--- (2)}$$

u - Numerical solution, U - exact solution

Error $e = U - u$ or $u = U - e$

Substituting u in (2) we get

$$U_i^{j+1} - e_i^{j+1} = r\left(U_{i+1}^j - e_{i+1}^j + U_{i-1}^j - e_{i-1}^j\right) + (1-2r)\left(U_i^j - e_i^j\right)$$

$$-e_i^{j+1} = r\left(U_{i+1}^j + U_{i-1}^j\right) - r\left(e_{i+1}^j + e_{i-1}^j\right) + (1-2r)U_i^j - (1-2r)e_i^j - U_i^{j+1}$$

$$e_i^{j+1} = r\left(e_{i+1}^j + e_{i-1}^j\right) - r\left(U_{i+1}^j + U_{i-1}^j\right) + (1-2r)e_i^j - (1-2r)U_i^j + U_i^{j+1}$$

$$e_i^{j+1} = r(e_{i+1}^j + e_{i-1}^j) + (1-2r)e_i^j - r(U_{i+1}^j + U_{i-1}^j) - (1-2r)U_i^j + U_i^{j+1} \quad \text{③}$$ ⑦

Now

$$U_{i+1}^j = U_i^j + \left(\frac{\partial U}{\partial x}\right)_{i,j} \Delta x + \frac{(\Delta x)^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)(\xi_1, t_j) \qquad x_i < \xi_1 < x_{i+1}$$

$$U_{i-1}^j = U_i^j - \left(\frac{\partial U}{\partial x}\right)_{i,j} \Delta x + \frac{(\Delta x)^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)(\xi_2, t_j) \qquad x_{i-1} < \xi_2 < x_i$$

$$U_{i+1}^j + U_{i-1}^j = 2U_i^j + (\Delta x)^2 \left(\frac{\partial^2 U}{\partial x^2}\right)(\xi, t_j) \qquad x_{i-1} < \xi < x_{i+1}$$

$$U_i^{j+1} = U_i^j + \left(\frac{\partial U}{\partial t}\right)(x_i, \eta)\Delta t \qquad t_j < \eta < t_{j+1}$$

Then from ③

$$e_i^{j+1} = r(e_{i+1}^j + e_{i-1}^j) + (1-2r)e_i^j - \cancel{2rU_i^j} - (\Delta x)^2 r\left(\frac{\partial^2 U}{\partial x^2}\right)(\xi, t_j) - \cancel{(1-2r)U_i^j} + \cancel{U_i^j} + \Delta t\left(\frac{\partial U}{\partial t}\right)(x_i, \eta)$$

or

$$e_i^{j+1} = r(e_{i+1}^j + e_{i-1}^j) + (1-2r)e_i^j - \frac{k\Delta t}{c\rho}\left(\frac{\partial^2 U}{\partial x^2}\right)(\xi, t_j) - \Delta t\left(\frac{\partial U}{\partial t}\right)(x_i, \eta)$$

$$e_i^{j+1} = r(e_{i+1}^j + e_{i-1}^j) + (1-2r)e_i^j - \Delta t\left[\frac{k}{c\rho}\frac{\partial^2 U}{\partial x^2}(\xi, t_j) - \left(\frac{\partial U}{\partial t}\right)(x_i, \eta)\right] \quad \text{④}$$

Now take $0 < r \le \frac{1}{2}$ ($2r \le 1$ or $1-2r \ge 0$)

$$|e_i^{j+1}| \le r(|e_{i+1}^j| + |e_{i-1}^j|) + (1-2r)|e_i^j| + M\Delta t \quad \text{⑤}$$

let $\exists M$ s.t. $\left|\frac{k}{c\rho}\frac{\partial^2 U}{\partial x^2}(\xi, t_j) - \left(\frac{\partial U}{\partial t}\right)(x_i, \eta)\right| < M$

Take $E^j = \max_i |e_i^j|$

from ⑤

$$|e_i^{j+1}| \le r(E^j + E^j) + (1-2r)E^j + M\Delta t$$

$|e_i^{j+1}| \le E^j + M\Delta t$

This is true for each i as r.h.s is independent of i ⑧

$\max_i |e_i^{j+1}| \le E^j + M\Delta t$.

$$E^{j+1} \le E^j + M\Delta t \le E^{j-1} + 2M\Delta t \ldots$$

$$\le E^{j-j} + (j+1)M\Delta t$$

$$= E^0 + M(j+1)\Delta t = E^0 + M t_{j+1}$$

E^0 is error at $t=0$, so $E^0 = 0$ since initial conditions are given

So $E^{j+1} \le M\, t_{j+1}$

$$r = \frac{k\Delta t}{c\rho \Delta x^2} \le \frac{1}{2} \quad \text{or} \quad \Delta t \le \frac{1}{2}\frac{c\rho}{k}(\Delta x)^2$$

Now, as $\Delta x \to 0$, $\Delta t \to 0$ and

$$\frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2}(\xi, t_j) - \frac{\partial u}{\partial t}(x_i, \eta) \to \left(\frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right)_{i,j} = 0$$

i.e. $M \to 0$ and thus ~~the~~ error goes to zero and the explicit method is convergent for $r \le \frac{1}{2}$.

Mr..

Laasonen method:

$$\frac{\partial^2 u}{\partial x^2} = \frac{c\rho}{k}\frac{\partial u}{\partial t}$$

$$\frac{u_{i+1}^{j} - 2u_i^j + u_{i-1}^{j}}{\Delta x^2} = \frac{c\rho}{k}\,\frac{u_i^j - u_i^{j-1}}{\Delta t}$$

$$\frac{k\Delta t}{c\rho\Delta x^2}\left(u_{i+1}^{j} - 2u_i^j + u_{i-1}^{j}\right) = u_i^j - u_i^{j-1}$$

$$r = \frac{k\Delta t}{c\rho\Delta x^2}$$

$$r\left(u_{i+1}^{j} - 2u_i^j + u_{i-1}^{j}\right) = u_i^j - u_i^{j-1}$$

$$-r\left(u_{i+1}^{j} - 2u_i^j + u_{i-1}^{j}\right) = u_i^{j-1} - u_i^j$$

$$-r u_{i-1}^{j} + (1+2r)u_i^j - r u_{i+1}^{j} = u_i^{j-1}$$

or, $$-r u_{i-1}^{j+1} + (1+2r)u_i^{j+1} - r u_{i+1}^{j+1} = u_i^j$$

for $j=0$

$$-r u_{i-1}^{1} + (1+2r)u_i^1 - r u_{i+1}^{1} = u_i^0$$

Initial cond. $u_i^0 = u(x_i, t)\big|_{t=0}$

$i=1$

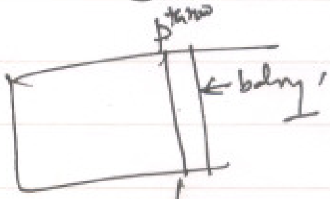
$$-r \underbrace{u_0^1}_{BC} + (1+2r)u_1^1 - r u_2^1 = \underbrace{u_1^0}_{IC}$$

$$(1+2r)u_1^1 - r u_2^1 = u_1^0 + r u_0^1$$

$i=2$

$$-r u_1^1 + (1+2r)u_2^1 - r u_3^1 = u_2^0$$

⋮



for p no. of rows

$i = p$ (last value of i)

$$-r u_{p-1}^{1} + (1+2r)u_p^1 - r u_{p+1}^{1} = u_p^0$$

$$-r u_{p-1}^{1} + (1+2r)u_p^1 = u_p^0 + r u_{p+1}^{1}$$

Derivative B.C. for Explicit method

$$u_i^{j+1} = r(u_{i+1}^j + u_{i-1}^j) + (1-2r)u_i^j \quad \text{—— } (*)$$

Suppose we have boundary condition of the type

$$\alpha_1 u(0,t) + \beta_1 \frac{\partial u}{\partial x}(0,t) = f_1(t) \quad \text{—— } (1)$$

and

$$\alpha_2 u(b,t) + \beta_2 \frac{\partial u}{\partial x}(b,t) = f_2(t) \quad \text{—— } (2)$$

From $(*)$ for $i=0$

$$u_0^{j+1} = r(u_1^j + u_L^j) + (1-2r)u_0^j \quad \text{—— } (3)$$

Now from b.c. (1)

$$\alpha_1 u_0^j + \beta_1 \frac{\partial u}{\partial x}(0,t_j) = f_1(t_j)$$

or

$$\alpha_1 u_0^j + \beta_1 \frac{u_1^j - u_L^j}{2\Delta x} = f_1(t_j) \quad \text{—— } (4)$$

Then eliminating u_L^j from (3) & (4) we get appropriate discretization.

Similarly for $i=N$ from $(*)$ (where total N number of subdivisions are there for x-axis).

$$u_N^{j+1} = r(u_R^j + u_{N-1}^j) + (1-2r)u_N^j \quad \text{—— } (5)$$

Now from b.c (2)

$$\alpha_2 u_N^j + \beta_2 \frac{\partial u}{\partial x}(N,j) = f_2(t_j)$$

$$\alpha_2 u_N^j + \beta_2 \frac{u_R^j - u_{N-1}^j}{2\Delta x} = f_2(t_j) \quad \text{—— } (6)$$

Now eliminating u_R from (5) & (6) we get appropriate discretization.

Derivative boundary condition (Explicit method)

$$u_i^{j+1} = r(u_{i+1}^j + u_{i-1}^j) + (1-2r)u_i \quad \text{——— } (+)$$

$i = 0$

$$u_0^{j+1} = r(u_1^j + u_L^j) + (1-2r)u_0^j \quad \text{——— } (*)$$

Now from the derivative B.C. at $x=0$

$$A_1 u(0,t) + B_1 \frac{\partial u}{\partial x}(0,t) = f_1(t)$$

$$A_1 u_0^j + B_1 \frac{\partial u}{\partial x}(0, t_j) = f_1(t_j)$$

$$f'(0) = \frac{f_1 - f_{-1}}{2(\Delta x)}$$

$$A_1 u_0^j + B_1 \frac{u_1^j - u_L^j}{2\Delta x} = F_1(t_j) \quad \text{——— } (**)$$

So from $(*)$ & $(**)$ u_L^j can be eliminated.

Similarly for other B.C. at $x = L$

$$A_2 u(b,t) + B_2 \frac{\partial u}{\partial x}(b,t) = f_2(t) \quad \text{——— } (1)$$

From $(+)$ for $i = b$

$$u_b^{j+1} = r(u_R^j + u_{b-1}^j) + (1-2r)u_b^j \quad \text{——— } (2)$$

Now from (1)

$$f'(b) = \frac{f_{b+1} - f_{b-1}}{2\Delta x} = \frac{f_R - f_{b-1}}{2\Delta x}$$

$$A_2 u_b^j + B_2 \frac{(u_R^j - u_b^j)}{2\Delta x} = f_2(t_j) \quad \text{——— } (3)$$

So from (2) and (3) u_R^j can be eliminated.