Advanced Numerical Techniques

Gaussian Elimination: Consider a tridiagonal system of equations

$AX = B$ where

$$A = \begin{pmatrix} d_1 & u_1 & & & \\ l_2 & \ddots & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & u_{N-1} \\ 0 & & & l_N & d_N \end{pmatrix}$$

i.e., $A = (a_{ij})$

$a_{ii} = d_i$, $i = 1(1)N$; $a_{i,i-1} = l_i$, $i = 2(1)N$

$a_{i,i+1} = u_i$, $i = 1(1)N-1$

$X = (x_1 \cdots x_N)^T$, $B = (b_1 \cdots b_N)^T$

$$\begin{pmatrix} d_1 & u_1 & 0 & 0 & 0 & \\ l_2 & d_2 & u_2 & 0 & 0 & 0 \\ 0 & l_3 & d_3 & u_3 & 0 & \\ & & l_K & d_K & u_K & \\ & & & & \ddots & \\ & 0 & & & l_N & d_N \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_K \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_K \\ \vdots \\ b_N \end{pmatrix}$$

$R_2 \leftarrow R_2 - \frac{R_1}{d_1} * l_2$ To make $l_2 \to 0$

$d_2 \leftarrow d_2 - \frac{u_1}{d_1} * l_2$

$b_2 \leftarrow b_2 - \frac{b_1}{d_1} * l_2$

Now the new d_2 and b_2 are available and l_2 is reduced to zero.

Next to reduce $l_3 \to 0$ we do the following

$R_3 \leftarrow R_3 - \frac{R_2}{d_2} * l_3$

Then $d_3 \leftarrow d_3 - \frac{u_2}{d_2} * l_3$

$b_3 \leftarrow b_3 - \frac{b_2}{d_2} * l_3$

Proceeding this way for k^{th} row we get

$$d_k \leftarrow d_k - \frac{u_{k-1}}{d_{k-1}} * l_k \qquad k = 2(1)N$$

$$b_k \leftarrow b_k - \frac{b_{k-1}}{d_{k-1}} * l_k \qquad k = 2(1)N$$

Now after these operations the system is reduced to upper triangular system and we will apply backward substitution

$$\begin{pmatrix} d_1 & u_1 & & & \\ & \ddots & & & \\ & & d_k & u_k & \\ & 0 & & \ddots & \\ & & & d_{N-1} & u_{N-1} \\ & & & 0 & d_N \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} = \begin{pmatrix} b_1 \\ b_k \\ b_{N-1} \\ b_N \end{pmatrix}$$

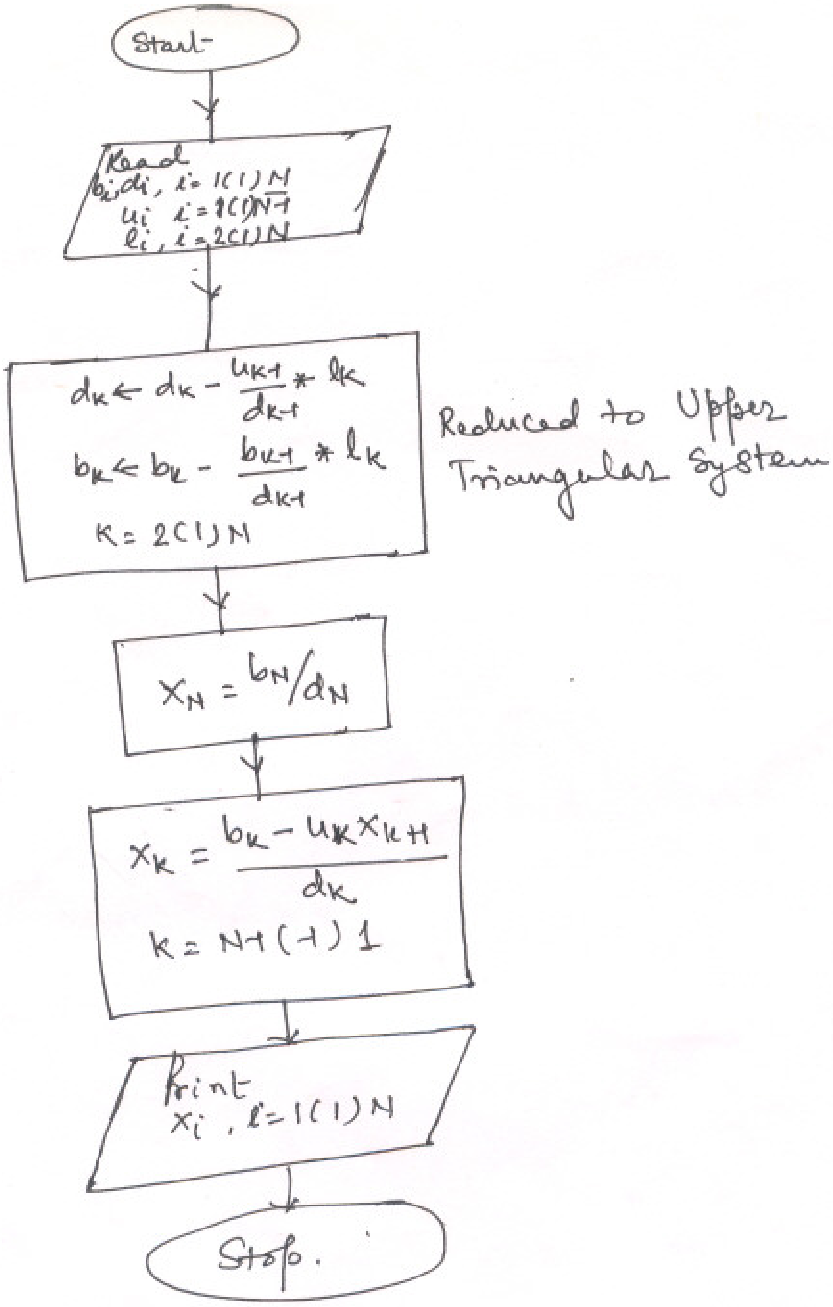
$$d_N x_N = b_N \implies \boxed{x_N = b_N / d_N}$$

$$d_{N-1} x_{N-1} + u_{N-1} x_N = b_{N-1}$$

$$x_{N-1} = \frac{b_{N-1} - u_{N-1} x_N}{d_{N-1}}$$

for k^{th} component

$$\boxed{x_k = \frac{b_k - u_k x_{k+1}}{d_k}, \quad k = N-1(-1)1}$$


Start
Read
bi, di, i = 1(1)N
ui i = 1(1)N-1
li, i = 2(1)N
dk ← dk - uk-1/dk-1 * lk
bk ← bk - bk-1/dk-1 * lk
k = 2(1)N
Reduced to Upper Triangular System
xN = bN/dN
xk = (bk - uk xk+1)/dk
k = N-1(-1)1
Print
xi, i = 1(1)N
Stop.

Gauss-Jordan Elimination

Using the Gauss-Jordan elimination solve the following system of equations

(1)
$$\begin{aligned} 3x - y + z &= -2 \\ x + 5y + 2z &= 6 \\ 2x + 3y + z &= 0 \end{aligned}$$

(2)
$$\begin{aligned} 3x + 4y - 7z &= -7 \\ x - 2y + z &= 1 \end{aligned}$$

(3)
$$\begin{aligned} 2w - 4x + 3y - z &= 3 \\ w - 2x + 5y - 3z &= 6 \\ 3w - 6x - y - z &= 0 \end{aligned}$$

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(1) Apply the Gauss-Seidel iteration to the system

$$\left.\begin{aligned} 10x + y + z &= 6 \\ x + 10y + z &= 6 \\ x + y + 10z &= 6 \end{aligned}\right\} \quad (*)$$

Starting from (a) 0, 0, 0 (b) 10, 10, 10. Compare and comment.

(2) Apply Gauss-Seidel and Jacobi iterations to the system (*) starting from 1, 1, 1. Compare and comment.

__Note__ For sparse systems (having many zero coefficients) iterative methods are better than direct method since direct method will be more laborious and need much storage.

(3) Apply Jacobi method to the following system

$W = x = 87.5$

$y = z = 62.5$

$$w - 0.25x - 0.25y = 50$$
$$-0.25w + x - 0.25z = 50$$
$$-0.25w + y - 0.25z = 25$$
$$-0.25x - 0.25y + z = 25$$

Start from $w_0 = 100$, $x_0 = 100$, $y_0 = 100$, $z_0 = 100$.