Newton Raphson method for system of Non-linear equations

$$\left.\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}\right\} \quad — (1)$$

Take (α, β) be the solution of the system (1) then

$$0 = f(\alpha, \beta) = f(x_0 + \alpha - x_0,\ y_0 + \beta - y_0)$$

$$= f(x_0, y_0) + (\alpha - x_0) f_x + (\beta - y_0) f_y$$

So $\quad (\alpha - x_0) f_x + (\beta - y_0) f_y = -f(x_0, y_0) \quad — (2)$

Similarly for $g(x,y)$

$$(\alpha - x_0) g_x + (\beta - y_0) g_y = -g(x_0, y_0) \quad — (3)$$

Let $\alpha - x_0 = \Delta x \qquad \beta - y_0 = \Delta y$

$$\Delta x\, f_x + \Delta y\, f_y = -f(x_0, y_0) \quad — (4)$$

$$\Delta x\, g_x + \Delta y\, g_y = -g(x_0, y_0) \quad — (5)$$

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -f \\ -g \end{pmatrix}$$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}^{-1} \begin{pmatrix} -f \\ -g \end{pmatrix}$$

or $\quad \begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix} = -\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}^{-1} \begin{pmatrix} f \\ g \end{pmatrix}$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}^{-1} \begin{pmatrix} f \\ g \end{pmatrix} \quad — (6)$$

N-R method for non-linear system

let $J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \frac{\partial(f, g)}{\partial(x, y)}$ Jacobian of (f, g) with respect to (x, y).

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - J^{-1} \begin{pmatrix} f \\ g \end{pmatrix}\Bigg|_{(x_n, y_n)} \quad \text{———} \quad (7)$$

Newton-Raphson method for system of Equations

$$F(X) = 0 \qquad F \doteq (f_1, \ldots f_n)$$
$$X = (x_1, \ldots x_n)$$

$$f_1(x_1, \ldots x_n) = 0$$
$$f_2(x_1, \ldots x_n) = 0$$
$$\vdots$$
$$f_n(x_1, \ldots x_n) = 0$$

$$X^{(n+1)} = X^{(n)} - J_n^{-1} F(X^{(n)})$$
$$X^{(n+1)} - X^{(n)} = - J_n^{-1} f(x^{(n)})$$
$$\Delta X = -J_n^{-1} f(x^{(n)})$$
$$J_n \Delta X = -f(x^{(n)})$$

$$J = \text{Jacobian Matrix} = \frac{\partial F}{\partial X} = \frac{\partial(f_1, \ldots f_n)}{\partial(x_1, \ldots x_n)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

$$J_n = J(x^{(n)})$$

Ex Solve the system of equations

$$\left.\begin{aligned} f_1(x,y) &= x^2 - y - 1 = 0 \\ f_2(x,y) &= (x-2)^2 + (y-.5)^2 - 1 = 0 \end{aligned}\right\} \quad (1)$$

$$J(x,y) = \begin{pmatrix} \partial f_1/\partial x & \frac{\partial f_1}{\partial y} \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 2(x-2) & 2(y-.5) \end{pmatrix}$$

$$J(x,y) = \begin{pmatrix} 2x & -1 \\ 2x-4 & 2y-1 \end{pmatrix}$$

det J

$$|J| = 2x(2y-1) + 2x - 4$$
$$= 4xy - \cancel{2x} + \cancel{2x} - 4$$
$$= 4xy - 4$$

Then Newton - Raphson method becomes

$$x^{(n+1)} = x^{(n)} - J_n^{-1} F(x^{(n)})$$

$$\begin{pmatrix} x^{(n+1)} \\ y^{(n+1)} \end{pmatrix} = \begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix} - J_n^{-1} \begin{pmatrix} f_1(x^{(n)}, y^{(n)}) \\ f_2(x^{(n)}, y^{(n)}) \end{pmatrix} \longleftarrow \text{A}$$

$$J_n^{-1} = J^{-1}(x^{(n)}, y^{(n)})$$

$$J^{-1} = \frac{\text{Adj } J}{|J|} = \frac{1}{4xy-4} \text{Adj } J$$

Cofactor Matrix $= \begin{pmatrix} 2y-1 & -(2x-4) \\ +1 & 2x \end{pmatrix}$

$$\text{Adj } J = \begin{pmatrix} 2y-1 & 1 \\ -(2x-4) & 2x \end{pmatrix}$$

$$J^{-1} = \frac{1}{4xy-4} \begin{pmatrix} 2y-1 & 1 \\ -(2x-4) & 2x \end{pmatrix}$$

Let $x^{(0)} = (x^{(0)}, y^{(0)}) = (0,0)$

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{(-4)} \begin{pmatrix} -1 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} f_1(x^{(0)}, y^{(0)}) \\ f_2(x^{(0)}, y^{(0)}) \end{pmatrix}$$

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = +\frac{1}{4}\begin{pmatrix} -1 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 13/4 \end{pmatrix}$$

$$f_2(x^{(0)}, y^{(0)}) = 4 + \frac{1}{4} - 1$$
$$= 3 + \frac{1}{4} = \frac{13}{4}$$

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 + 13/4 \\ -4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 17/4 \\ -4 \end{pmatrix} = \begin{pmatrix} 17/16 \\ -1 \end{pmatrix}$$

The system (1) has two roots

$r_1 = (1.546343, 1.391176)$

$r_2 = (1.067346, .139227)$

This system (1) could be solved as follows also.

$$\begin{pmatrix} x^{(n+1)} \\ y^{(n+1)} \end{pmatrix} - \begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix} = -J_n^{-1}\begin{pmatrix} f_1(x^{(n)}, y^{(n)}) \\ f_2(x^{(n)}, y^{(n)}) \end{pmatrix}$$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = -J_n^{-1}\begin{pmatrix} f_1(x^{(n)}, y^{(n)}) \\ f_2(x^{(n)}, y^{(n)}) \end{pmatrix}$$

$$J_n\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = -\begin{pmatrix} f_1(x^{(n)}, y^{(n)}) \\ f_2(x^{(n)}, y^{(n)}) \end{pmatrix}$$

which is of the form

$$A\,\Delta X = B$$

Solve this for ΔX, then the next value will be $X^{(0)} + \Delta X = X^{(1)}$ and so on.

Now consider $y'' = f(x,y)$ $y(0) = A,\ y(1) = B$ — (1)

then the second order
method for (1) is as follows

Second Order

$$\left.\begin{aligned} &2y_1 - y_2 + h^2 f(x_1, y_1) - A = 0 \\ &-y_{K-1} + 2y_K - y_{K+1} + h^2 f(x_K, y_K) = 0 \\ &\qquad K = 2(1)\overline{N-2} \\ &-y_{N-2} + 2y_{N-1} + h^2 f(x_{N-1}, y_{N-1}) - B = 0 \end{aligned}\right\} \quad (2)$$

or.

$$\left.\begin{aligned} &g_1(y_1, y_2, \dots y_{N-1}) = 0 \\ &g_2(y_1, y_2, \dots y_{N-1}) = 0 \\ &\vdots \\ &g_{N-1}(y_1, y_2, \dots y_{N-1}) = 0 \end{aligned}\right\} \quad — (3)$$

$$G(Y) = 0 \quad — (4)$$

$$G = (g_1, \dots g_{N-1})$$

$$G(y_1, \dots y_{N-1}) = 0$$

$$\|$$

$$(g_1(y_1 \dots y_{N-1}), \dots g_{N-1}(y_1 \dots y_{N-1})) = (0, \dots 0)$$

$$Y^{(n+1)} = Y^{(n)} - J_n^{-1}(y_1 \dots y_n)\, G(Y^{(n)})$$

$$\Delta Y = -J_n^{-1} G(Y^{(n)})$$

$$J_n \Delta Y = -G(Y^{(n)}) \quad — (5)$$

$$g_1(y_1, \dots y_{N-1}) \equiv 2y_1 - y_2 + h^2 f(x_1, y_1) - A = 0$$

$$g_K(y_1 \dots y_{N-1}) \equiv -y_{K-1} + 2y_K - y_{K+1} + h^2 f(x_K, y_K) = 0$$

$$K = 2(1)\overline{N-2}$$

$$g_{N-1}(y_1, \dots y_{N-1}) \equiv -y_{N-1} + 2y_{N-1} + h^2 f(x_{N-1}, y_{N-1}) - B = 0$$

$$J(y_1 \dots y_n) = \begin{pmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \frac{\partial g_1}{\partial y_3} & \cdots & & & \cdots & \partial g_1/\partial y_{N-1} \\ \vdots & & & & & & & \\ \partial g_k/\partial y_1 & \partial g_k/\partial y_2 & \cdots & \frac{\partial g_k}{\partial y_{k-1}} & \frac{\partial g_k}{\partial y_k} & \frac{\partial g_k}{\partial y_{k+1}} & \cdots & \partial g_k/\partial y_{N-1} \\ & & & & & & & \\ \partial g_{N-1}/\partial y_1 & & & & & & \frac{\partial g_{N-1}}{\partial y_{N-2}} & \partial g_{N-1}/\partial y_{N-1} \end{pmatrix}$$

$$J = \begin{pmatrix} 2 + h^2 \partial f_1/\partial y_1 & -1 & 0 & \text{———} & 0 \\ & & & & \\ & -1 & 2 + h^2 \partial f_k/\partial y_k & -1 & O \\ O & & & \ddots & \\ & & & -1 & 2 + h^2 \partial f_{N-1}/\partial y_{N-1} \end{pmatrix}$$

$$J = \begin{pmatrix} 2 & -1 & 0 \text{———} 0 \\ -1 & 2 & -1 \quad O \\ O & & \\ & -1 & 2 \end{pmatrix} + h^2 \begin{pmatrix} \partial f_1/\partial y_1 & & O \\ & \ddots & \\ O & & \partial f_{N-1}/\partial y_{N-1} \end{pmatrix}$$

Now the system (2) can be written as

$$DY + h^2 F(Y) - Q = 0 \quad \text{———} \quad (6)$$

$D = (d_{ij})$ $\quad d_{ii} = 2, \quad d_{i,i\pm1} = -1$

$F(Y) = (f_1, \dots f_{N-1})$, $\quad Q = (A, 0 \dots 0, B)$

$$G(Y) = DY + h^2 F(Y) - Q = 0 \quad \text{--- (7)}$$

Now using Newton-Raphson method for system (2) is as follows

$$J_n \Delta Y = -G(Y^{(n)}) \quad \text{--- (8)}$$

Let $Y^{(0)} = (y_1^{(0)}, \ldots y_{N+1}^{(0)})$ so next we get $Y^{(1)} = (y_1^{(1)}, \ldots y_{N+1}^{(1)})$

$$J_0 \Delta Y = -G(Y^{(0)}) \quad \text{--- (9)}$$

Solving this system of algebraic equations we get ΔY and then $Y^{(1)} = Y^{(0)} + \Delta Y$. Now proceeding this way we can get improved values of $Y^{(n)}$ and hence the non-linear system of equations (2) is solved.

Note

Equation (8) is as follows

$$\left(D + h^2 F(Y)\right)_n \Delta Y = -G(Y^{(n)}) \quad \text{--- (10)}$$

where

$$F(Y) = \begin{pmatrix} \partial f_1/\partial y_1 & & 0 \\ & \ddots & \\ 0 & & \partial f_{N+1}/\partial y_{N+1} \end{pmatrix}$$

$$(D^{T}|T|)_{mn} = \frac{h^4}{12}\sum_{h=1}^{N-1} d_{m,n} = \frac{h^4}{12}\cdot N4 \cdot \frac{1}{2h^2}$$

$$= \frac{N4}{24}\cdot h^2$$

Thus $\boxed{\|E\|_\infty = O(h^2)}$

Ex Solve the bvp

$$y'' = -2 + \sinh y \qquad y(0) = 0,\ y(1) = 0$$

$N = 4$

$h = 1/4 = .25$

$K=1 \quad g_1(y_1,y_2,y_3) = 2y_1 - y_2 + h^2 \sinh y_1 = 0$

$$\boxed{-y_{K-1} + 2y_K - y_{K+1} + h^2 \sinh y_K = 0}$$

$K = 2,\ g_2(y_1,y_2,y_3) = -y_1 + 2y_2 - y_3 + h^2 \sinh y_2 = 0$

$K = 3,\ g_3(y_1,y_2,y_3) = -y_2 + 2y_3 + h^2 \sinh y_3 = 0$

$$g_1(y_1,y_2,y_3) = 0$$

$$g_2(y_1,y_2,y_3) = 0$$

$$g_3(y_1,y_2,y_3) = 0$$

$$G(Y) = 0$$

$$\Delta Y = -J_n^{-1} G(Y^{(n)})$$

or

$$J_n \Delta Y = -G(Y^{(n)})$$

$$J = \begin{pmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \partial g_1/\partial y_3 \\ \partial g_2/\partial y_1 & \partial g_2/\partial y_2 & \partial g_2/\partial y_3 \\ \partial g_3/\partial y_1 & \partial g_3/\partial y_2 & \partial g_3/\partial y_3 \end{pmatrix}$$

$$J = \begin{pmatrix} 2 + h^2 \frac{\partial}{\partial y_1}\sinh y_1 & -1 & 0 \\ -1 & 2 + h^2 \frac{\partial}{\partial y_2}\sinh y_2 & -1 \\ 0 & -1 & 2 + h^2\frac{\partial}{\partial y_3}\sinh y_3 \end{pmatrix}$$

$$J = D + h^2 \begin{pmatrix} \partial/\partial y_1 \sinh y_1 & & 0 \\ & \frac{\partial}{\partial y_2} \sinh y_2 & \\ 0 & & \partial/\partial y_3 \sinh y_3 \end{pmatrix}$$

$$J = D + h^2 \bar{F}(Y) \qquad \bar{F}(Y) = \mathrm{diag}\left(\frac{\partial}{\partial y_1}\sinh y_1, \frac{\partial}{\partial y_2}\sinh y_2, \frac{\partial}{\partial y_3}\sinh y_3\right)$$

$$\left(D + h^2 \bar{F}(Y)\right)_n \Delta Y = -G(Y^{(n)})$$

To get $\Delta Y_{(1)}$

$$\left(D + h^2 \bar{F}(Y)\right)_{(0)} \Delta Y = -G(Y^{(0)})$$

How to get $y^{(0)}$ (initial guess)

Consider $y'' = -2 \qquad y(0) = 0, \quad y(1) = 0$

$y' = -2x + c_1$

$y = -\cancel{2}\frac{x^2}{\cancel{2}} + c_1 x + c_2 = -x^2 + c_1 x + c_2$

$y(0) = 0 \Rightarrow 0 = c_2, \quad y(1) = 0 \Rightarrow 0 = -1 + c_1 \Rightarrow c_1 = 1$

$\Rightarrow y = -x^2 + x$ is the solution

$y = x(1-x)$ is the solution of linear problem.

Set of non-linear equations (System of non-linear equations).

$$f(x,y) = 0$$
$$g(x,y) = 0$$

Let $x = r$, $y = s$ be solution of the above system of equations then

$$0 = f(r,s) = f(x_1 + r - x_1, y_1 + s - y_1)$$
$$0 = g(r,s) = f(x_1 + r - x_1, y_1 + s - y_1)$$

where (x_1, y_1) is initial guess (say).

$$0 = f(x_1 + r - x_1, y_1 + s - y_1) = f(x_1, y_1) + (r - x_1)\frac{\partial f}{\partial x}(x_1, y_1) + (s - y_1)\frac{\partial f}{\partial y}(x_1, y_1)$$

$$0 = g(x_1 + r - x_1, y_1 + s - y_1) = g(x_1, y_1) + (r - x_1)\frac{\partial g}{\partial x}(x_1, y_1) + (s - y_1)\frac{\partial g}{\partial y}(x_1, y_1)$$

$$(r - x_1)\frac{\partial f}{\partial x} + (s - y_1)\frac{\partial f}{\partial y} = -f$$

$$(r - x_1)\frac{\partial g}{\partial x} + (s - y_1)\frac{\partial g}{\partial y} = -g.$$

$$\begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} \begin{bmatrix} r - x_1 \\ s - y_1 \end{bmatrix} = \begin{bmatrix} -f \\ -g \end{bmatrix}$$

$$\epsilon_1^{(x)} = r - x_1 = \frac{\begin{vmatrix} -f & \partial f/\partial y \\ -g & \partial f/\partial y \end{vmatrix}}{\begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix}}, \quad \underset{\epsilon_1^{(y)}}{\overset{\shortparallel}{s - y_1}} = \frac{\begin{vmatrix} \partial f/\partial x & -f \\ \partial g/\partial x & -g \end{vmatrix}}{\begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix}}$$

Next improved value $x_1 + \epsilon_1^{(x)}$ & $y_1 + \epsilon_1^{(y)}$.

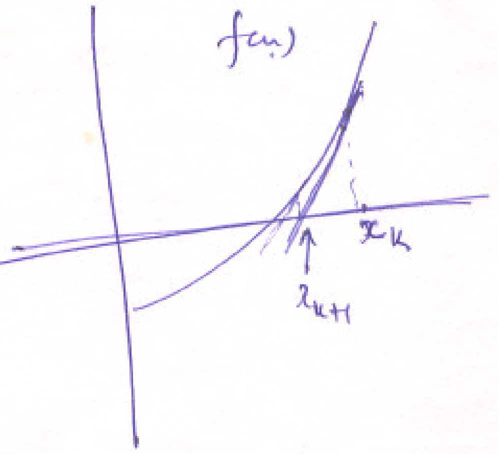
Take initial guess $x_1 = 1$, $y_1 = -1.7$

$$f(x,y) = 4 - x^2 - y^2 = 0 \qquad \left| \quad \begin{matrix} x^2 + y^2 = 4 \\ e^x + y = 1 \end{matrix} \right\}$$
$$g(x,y) = 1 - e^x - y = 0$$

Def^n: Order of a method: A method is said to be of order p if p is the largest number for which there exists a finite constant c such that

$$|x_{n+1} - \alpha| \leq c\, |x_n - \alpha|^p$$

Newton - Raphson method:



Equation of tangent at x_k

$$y - f(x_k) = f'(x_k)(x - x_k)$$

Now this cuts x-axis at $x = x_{k+1}$ then

$$0 - f(x_k) = f'(x_k)(x_{k+1} - x_k)$$

or $$(x_{k+1} - x_k) = -\frac{f(x_k)}{f'(x_k)}$$ provided $f'(x_k) \neq 0$.

or $$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Another Way to derive Newton- Raphson method

Let x_k be the approximation to the root at some stage and $x_k + \Delta x$ will be the actual root then

$$0 = f(x_k + \Delta x) = f(x_k) + \Delta x\, f'(x_k) + \cdots$$

or $$\Delta x = -\frac{f(x_k)}{f'(x_k)}$$ $\Delta x = x_{k+1} - x_k$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$