Lecture - 12

Marine Hydrodynamics

1. A linear progressive wave looks like as follows:

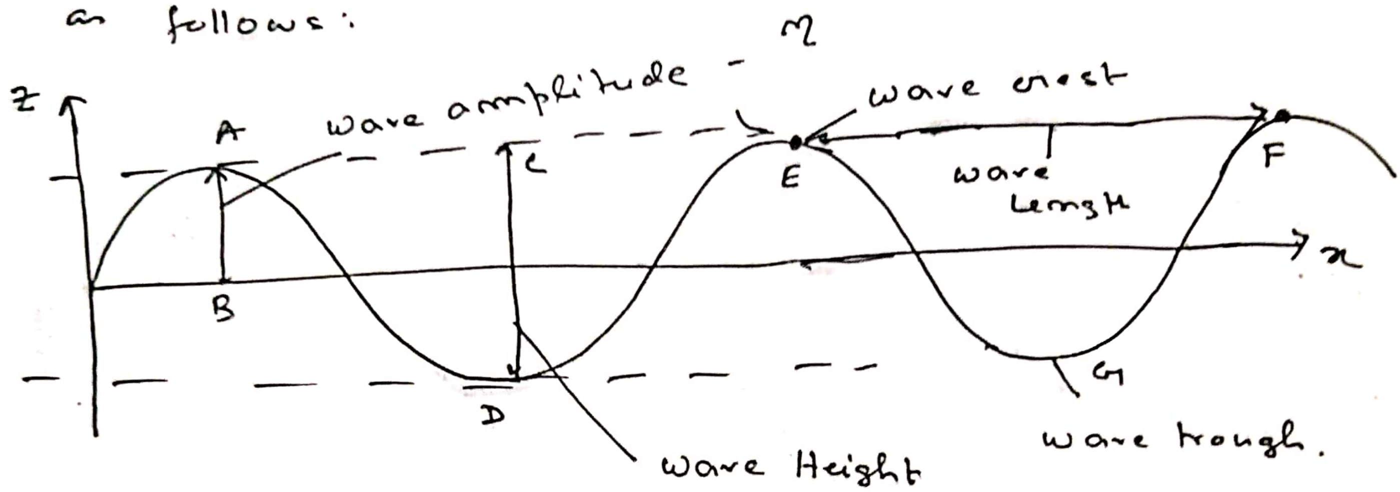

Figure - 1.1

Figure 1.1 is a snapshot of wave, i.e at a particular time t, we have taken a snap shot. ~~along~~ Therefore it is a wave pattern along the x axis.

From the above figure :. let us define the

AB = wave amplitude $= a \,|\, \zeta_a \,|\, \eta_a$

CD = wave height $= H \Rightarrow H = 2a$

EF = wave length $= \lambda$

A, E, F $\equiv$ wave crest

D, G = wave trough.

Now: consider the following circular motion.

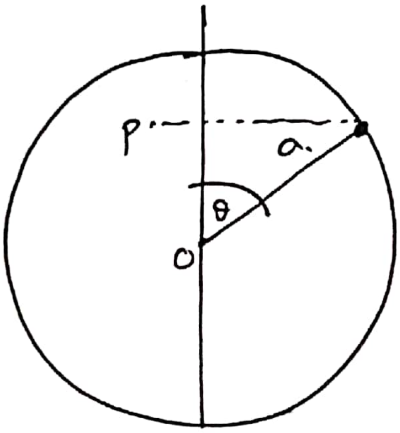

Figure:- 1.2

$$OP = a\cos\theta$$

now assume $\frac{d\theta}{dx} = K$

$\Rightarrow \theta = Kx$

$\therefore$ $OP = a\cos(Kx)$

Now, in this equation, K is called wave number.

if 'λ' be the wave length (similar to time period T), then 'K' and '~~a~~ λ' are related by

$$K = \frac{2\pi}{\lambda} \quad \text{.....} \quad (1.1)$$

(1.1) is the relationship with wave number and wave length.

Second ~~pit~~ picture [fixed in space]

if we consider the motion of a buoy floating we get the following picture:

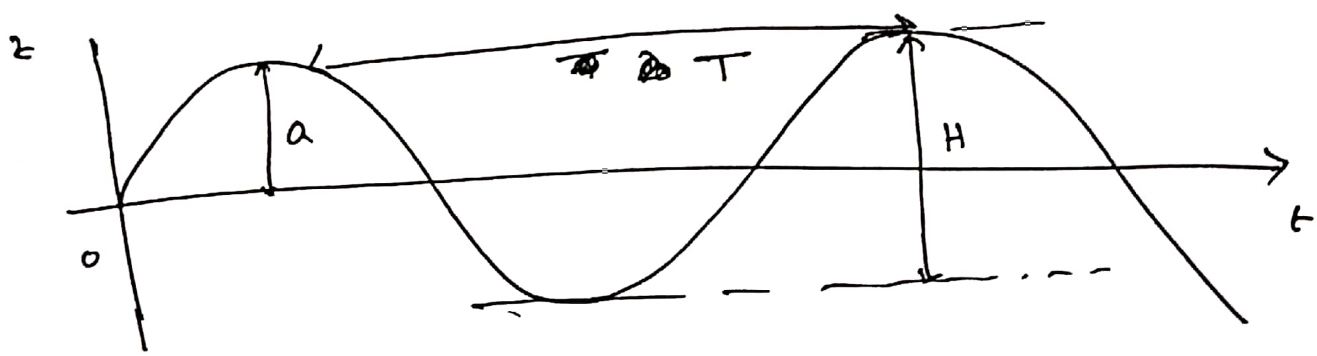

Figure 1.3

According to figure 1.3, the time period T and angular frequency ω is related as

$$\omega = \frac{2\pi}{T} \quad \text{....} \quad (1.2)$$

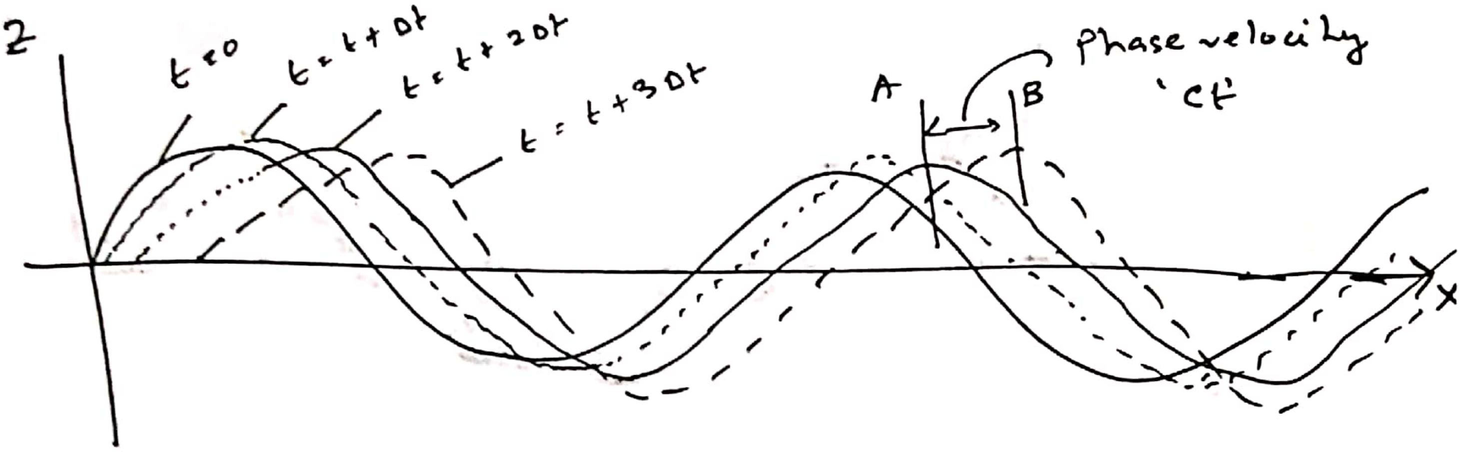

Fig. 2.1

Now, if you ~~see~~ observe the figure 2.1, This is how a wave propagate in the x axis. now, at each $t = 0$, $t + \Delta t$, $t + 2\Delta t$ and so on, we have a sinusoidal wave as

$$\eta = a \cos kX \quad \ldots (2.1)$$

now, however, at each time of Δt, the initial position shifts. now assume, if the velocity of this shift $= c$, then, for time t, the wave profile moves with a distance ct.

Now take $X = x - ct$.

then equation (2.1) takes the form

$$\eta = a \cos k(x - ct)$$

or

$$\eta = a \cos(kx - k.c.t) \quad \ldots (2.2)$$

Now where c is called the phase velocity, or the velocity at which the wave propagates.

now, it clearly $\boxed{c = \frac{\lambda}{T}}$ ... (2.3)

Therefore, using (2.3), we get

$$\eta = a\cos\left(kx - \frac{k\lambda}{T}t\right) \quad \dots \quad (2.4)$$

Now, we know $k = \frac{2\pi}{\lambda}$, then using this gives

$$\eta = a\cos\left(kx - \frac{2\pi}{T}t\right) \quad \dots\dots \quad (2.5)$$

now, we know $\omega = \frac{2\pi}{T}$, substitute this we get the ~~plan~~ expression for a plane progressive wave ~~as~~ propagate in +ve x direction as.

$$\boxed{\eta = a\cos(kx - \omega t)} \quad \dots \quad (2.6)$$

most general expression for (2.6) is

$$\boxed{\eta = a\cos(kx - \omega t + \epsilon)} \quad \dots \quad (2.7)$$

where 'ϵ' is called random phase angle, most of our study, we set $\epsilon = 0$.

from (2.6) we can easily derive that the equation of a progressive wave in -ve direction as

$$\eta = a\cos(kx + \omega t). \quad \dots \quad (2.8)$$

3. Standing wave :

Now, it is again trivial to understand linear superposition of (2.6) and (2.8) will give the expression for standing wave, which is

$$\eta = a\cos(kx + \omega t) + a\cos(kx - \omega t)$$

$$\text{or } \eta = 2a\cos kx\cos\omega t \Rightarrow \boxed{\eta = H\cos kx\cos\omega t}$$

we shall come back to standing wave latter, before we learn more about progressive wave.

4. Introduction of linear potential theory:

Let us now concentrate our attention to small amplitude potential theory. The term small amplitude does not mean that the wave amplitude is small. it ~~me~~ means the wave slope is small.

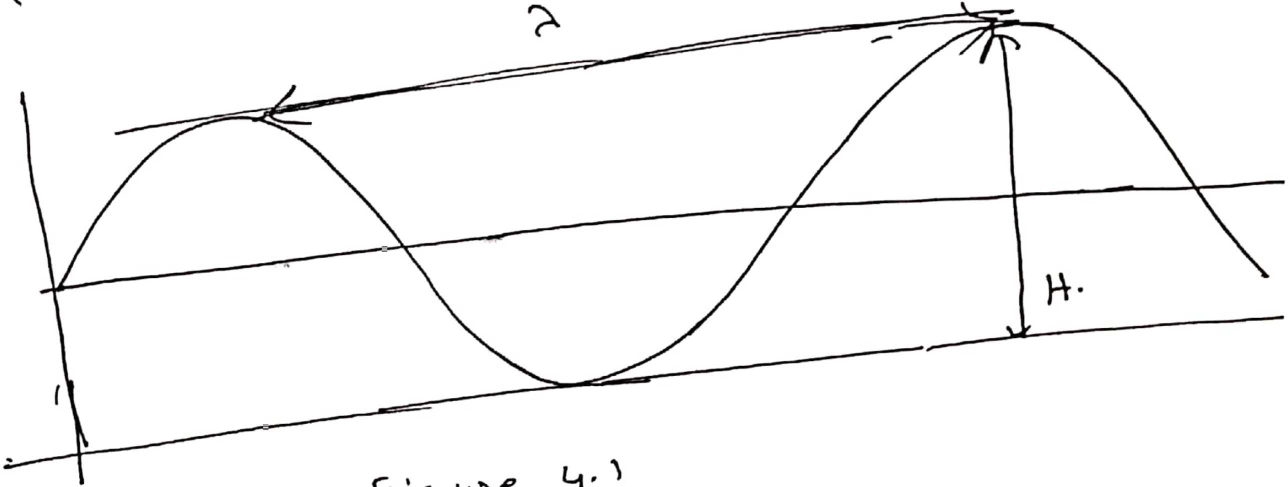

Figure 4.1

The wave slope is measured as the ratio of wave hight to wave length.

i.e. $\boxed{\text{wave slope} = \frac{H}{\lambda}}$ ..... (4.1)

For practical situation, if $\frac{H}{\lambda} \le \frac{1}{25}$, we can say the we are in the region of small amplitude ~~p~~ linear wave theory, if

$\frac{1}{25} \le \frac{H}{\lambda} \le \frac{1}{75}$ it is intermediate.

when $H/\lambda > \frac{1}{1[illegible]}$ — ~~we~~ linear theory won't work.

Let us consider the small amplitude wave, assume ϕ be the velocity potential of the fluid particle. we are interested to find out the expression for ϕ. Now $\phi(x,y,z,t)$ must satisfy certain condition. Under the assumption of potential theory, i.e fluid particle is inviscid, incompressible, homogeneous and flow is ir-rotational, ϕ must satisfy the laplace eqn.

$$\nabla^2 \phi = 0 \quad \text{in fluid domain } \Omega.$$

also it satisfies two free surface condition

i) kinematic boundary condition: i.e fluid particle must ~~stay~~ attached to the boundary surface, which is basically the free surface.

(ii) Dynamic free surface condition. i.e there must be a dynamic pressure equilibrium at free surface.

(iii) bottom boundary condition. i.e at bottom, there is no flow $\perp^r$ to bottom surface

Before we solving for ϕ, it is interesting to find out the mathematical expression for the boundary.

5. Kinematic free surface condition:-

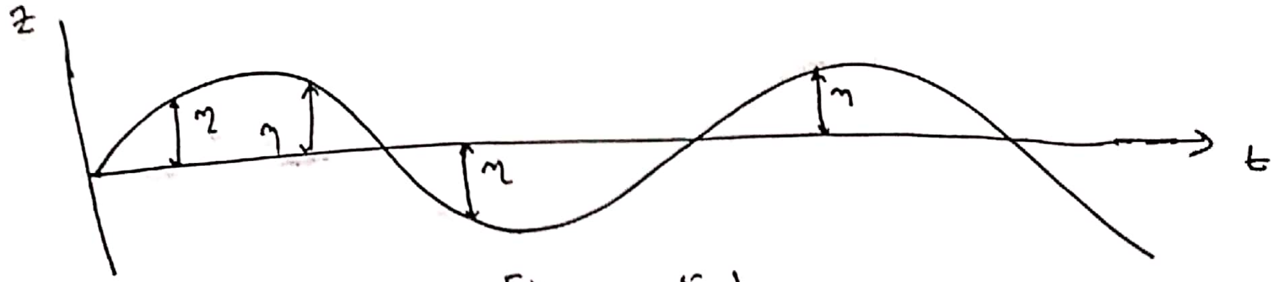

Figure - 5.1

From the above figure (Figure 5.1), it may be noted that the z is function of $\eta(x,t)$ only, then we can write

$$z = \eta(\vec{x}, t) \longrightarrow (5.1)$$

$\therefore$ we can get the boundary surface

$$F(x, z, t) = 0 \quad \text{as} \quad F = z - \eta(\vec{x}, t).$$

Now if $\vec{V}$ be the velocity of the boundary surface and $\vec{q}$ be the velocity of the ~~boundary~~ fluid particle, the water particle will stick with boundary surface, if

$$\vec{V} \cdot \vec{n} = \vec{q} \cdot \vec{n}.$$

$$\Rightarrow \vec{V} \cdot \frac{\nabla F}{|\nabla F|} = \vec{q} \cdot \frac{\nabla F}{|\nabla F|}$$

$$\Rightarrow \vec{V} \cdot \nabla F = \vec{q} \cdot \nabla F. \quad \dots (5.1)$$

we can say $F(x, z, t) = 0 \Rightarrow F(\vec{r}, t) = 0$

$$F(r + \delta r, t + \delta t) = F(r, t) + \nabla F \cdot \delta r + \frac{\partial F}{\partial t} \cdot \delta t$$

$F(r + \delta r, t + \delta t) = 0$ as $F(r, t) = 0$

$\therefore \quad -\frac{\partial F}{\partial t} \cdot \delta t = \nabla F \cdot \delta r$

$\Rightarrow \quad -\frac{\partial F}{\partial t} = \nabla F \cdot \frac{\delta r}{\delta t}$

$\therefore$ at limiting case $-\frac{\partial F}{\partial t} = v \cdot \nabla F \quad \ldots (5.2)$

substituting (5.2) in (5.1) we get

$$-\frac{\partial F}{\partial t} = q \cdot \nabla F$$

or $$\frac{\partial F}{\partial t} + q \cdot \nabla F = 0. \quad \ldots (5.3)$$

or $$\frac{DF}{Dt} = 0 \quad \ldots \quad (5.4)$$

<u>Now</u> substitu $F = z - \eta(\vec{r}, t)$ in (5.3) we get

$$\frac{\partial}{\partial t}\left[z - \eta(r,t)\right] + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\left(z - \eta(r,t)\right) = 0$$

$$\Rightarrow -\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y} + w = 0$$

$$\Rightarrow \boxed{w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}} \quad (5.5)$$

<u>Now</u> under potential theory $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$ and $w = \frac{\partial \phi}{\partial z}$. substituting in (5.5) we get

$$\boxed{\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \eta}{\partial y}} \text{ at } z = \eta$$