Lecture - 14

Marine Hydrodynamics

1. Trajectories :

$$u = \frac{dx}{dt} = a\omega e^{kz} \cos(kx - \omega t)$$

$$\therefore \quad x = \int a\omega e^{kz} \cos(kx - \omega t) + c_1$$

or
$$\boxed{x = -a e^{kz} \sin(kx - \omega t) + c_1} \rightarrow (1.1)$$

Similarly

$$w = \frac{dz}{dt} = a\omega e^{kz} \sin(kx - \omega t)$$

$$\therefore \quad z = \int a\omega e^{kz} \sin(kx - \omega t) + c_2$$

or
$$z = a e^{kz} \cos(kx - \omega t) + c_2 \rightarrow (1.2)$$

Now from (1.1) & (1.2) we get

$$\sin(kx - \omega t) = -\frac{x - c_1}{a e^{kz}} \quad \cdots (1.3)$$

$$\cos(kx - \omega t) = \frac{z - c_2}{a e^{kz}} \quad \cdots (1.4)$$

Now
$$\sin^2(kx - \omega t) + \cos^2(kx - \omega t) = 1 \Rightarrow$$

$$\boxed{(x - c_1)^2 + (z - c_2)^2 = a^2 e^{2kz}} \quad \cdots \rightarrow (1.5)$$

If we remember the co-ordinate system:

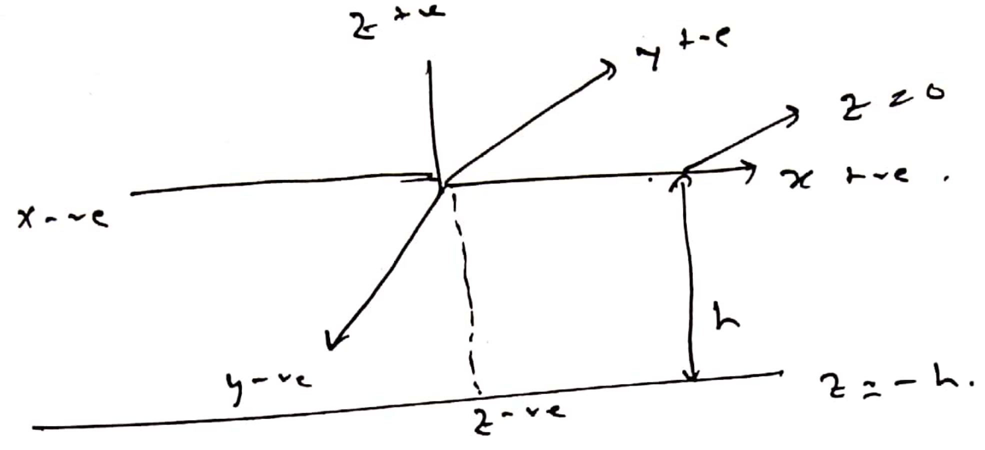


Figure 1.1

if we understand the figure 1.1 we know that at free surface, $z = 0$, $\therefore$ at free surface, the particle trajectory takes the form

$$\boxed{(x - c_1)^2 + (z - c_2)^2 = a^2} \longrightarrow (1.6)$$

$\Rightarrow$ at free surface water particle is moving in circular fashion with the radius 'a'.

Now, the water particle ~~moving~~ trajectory is a function of e^{2kz} also, then at any depth $z = -h'$, the particle trajectory is

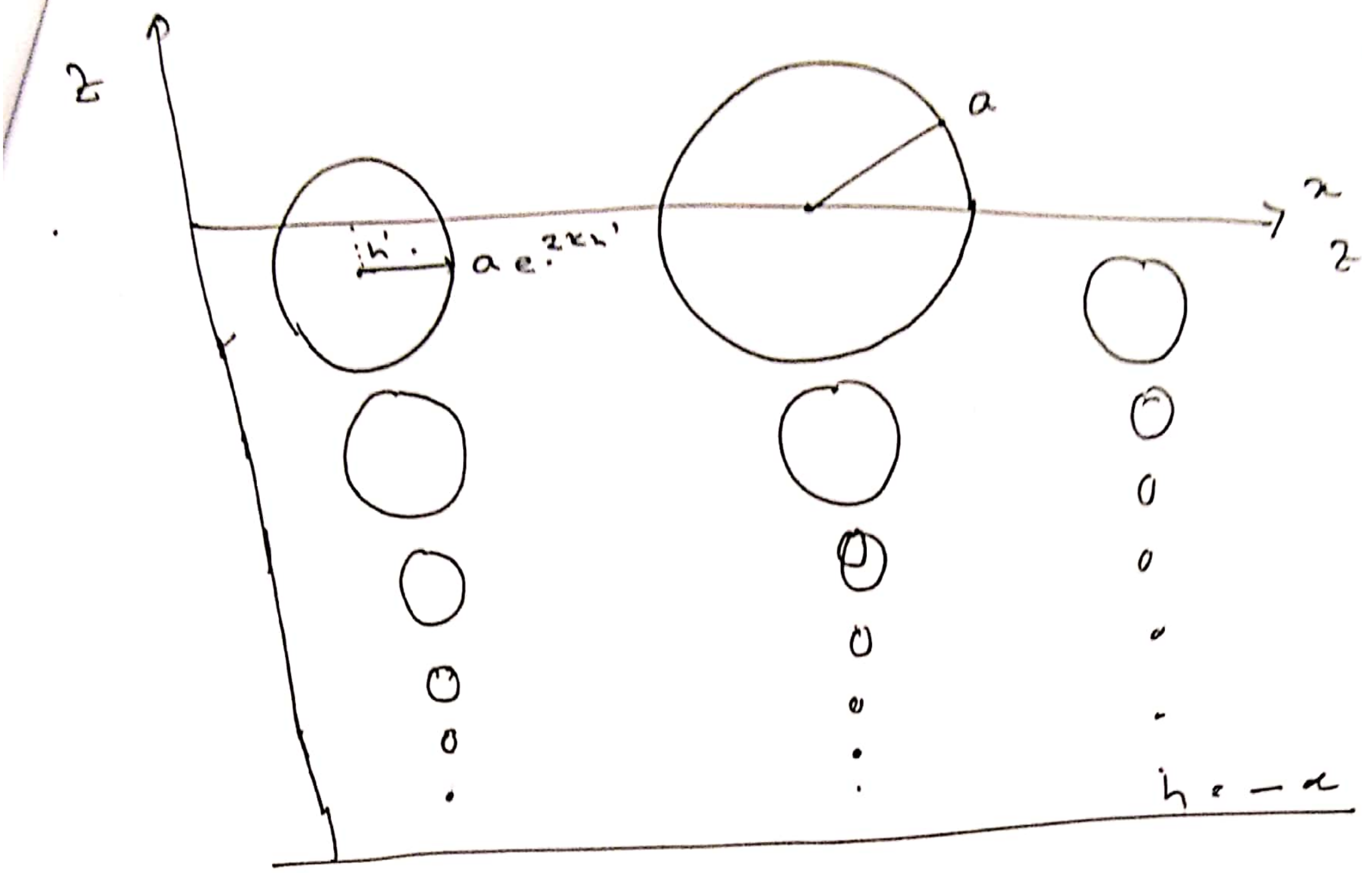
$$(x - c_1)^2 + (z - c_2)^2 = a^2 . e^{-2kh'} \longrightarrow (1.7)$$

Now, the value of $e^{-2kh'}$ is always less than 1.

$\therefore$ radius of circle (1.7) is less than (1.6).

and at $h \rightarrow -\alpha$, radius becomes zero.

Therefore, the trajectory of the waterparticle along 'z' axis may be demonstrated as



<u>Range of deep water:</u> normally $h = 100$ m can be consider as deep water range.

as $\frac{1}{e^{2k \times 100}} \approx 0$ as any practical value of k. for example, even for $T = 10$ sec, which is quite practical wave as $\lambda = 1.56 \times 10^2$

$\Rightarrow \lambda = 1.56 \times 100 \text{ m} = 156$ m.

$$\therefore \frac{1}{e^{2k \times 100}} = \frac{1}{e^{\frac{2 \times 100 \times 3.14}{156}}} = \frac{1}{e^{\frac{6.28}{1.56}}}$$

$$\approx \frac{1}{e^4} \approx \frac{1}{55}$$

$$\approx \underline{0.01}$$

<u>Normally</u> we take $h \geq 100 - 150$ m as deep water case.

effect of frequency in particle trajectory

again from (1.6), we know that the particle trajectories ~~of~~ on free surface is

$$(x - c_1)^2 + (z - c_2)^2 = a^2 \longrightarrow (2.1)$$

Now, the linearized free surface condition gives

$$\phi_{tt} + g\phi_z = 0 \quad \text{at} \quad z = 0 \longrightarrow (2.2)$$

Assuming $\phi(x, z, t)$ is time harmonic, then $\phi(x, z, t)$ can be approximated as

$$\phi(x, z, t) = \phi(x, z) e^{i\omega t} \longrightarrow (2.3)$$

Substitute (2.3) in (2.2) we get

$$-\omega^2 \phi + g\phi_z = 0 \longrightarrow (2.4)$$

Now in case of $\omega \to 0$ [low frequency]

we get $\boxed{\phi_z = 0} \longrightarrow (2.5)$

and in case of $\omega \to \alpha$, (2.4) gives

$$\phi - \frac{g}{\omega^2}\phi_z = 0$$

$$\Rightarrow \boxed{\phi = 0} \longrightarrow (2.6)$$

This two are extreem situation. ~~i~~ $\omega \to 0 \Rightarrow T \to \alpha$

it means at very high time period, $\phi_z = 0$, i.e

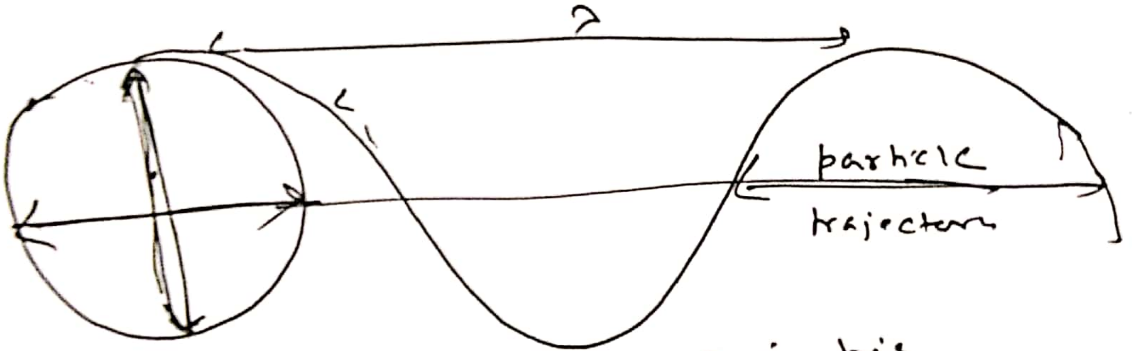
$$\frac{\partial \phi}{\partial z} = 0 \Rightarrow (\nabla \phi).n_z = 0$$

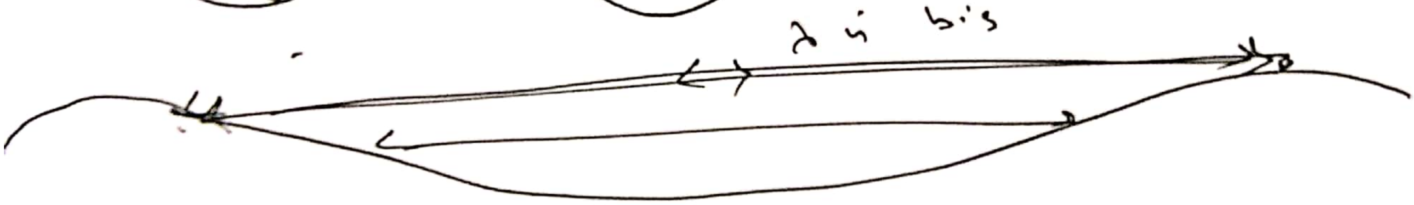
$\Rightarrow$ no velocity ~~along~~ component along z direction is zero, which is same as uniform flow. i.e in this case, we do not see any wave.

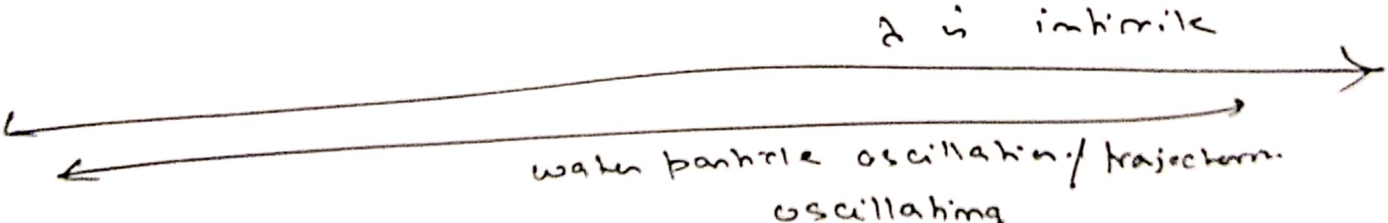
now for $\omega \to \infty \Rightarrow T \to 0$, i.e for very short time period, $\phi = 0$, i.e Now if $\phi = 0$; then $\nabla\phi = 0$ i.e in this case also, no wave profile is possible as particle velocity is zero.

How to look into this phenomenon???

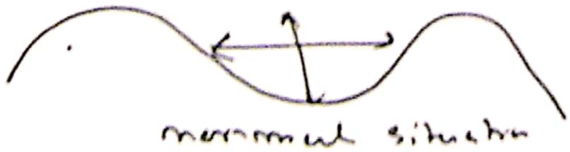
we know $T \to \infty \Rightarrow \lambda \to \infty$, i.e in case of $\omega \to 0$, the wave length $\to \infty$. now if we stretch the following picture lets see what happened

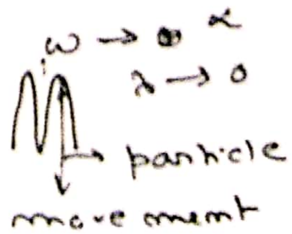






as if particles are only ~~oscillating~~ oscillating horizontally direction, similar argument, in case of high frequency,





one can visualize that, water particle is only having the verticle oscillation.

From the above ~~to~~ picture, we can get a very interesting observation, in case of both two extreme situation, water particle can not have both verticle and horizontal velocity, and also we get that in this two situation, there are no waves. Hence, to get the wave, particle must have both the velocity component. This is a very important findings and helped us to understand the propagation of wave energy.

3. particle trajectory for shallow water case

The general expression for ~~shallow water situation~~. ϕ for any arbitrary depth 'h' is

$$\phi = \frac{ag}{\omega} \frac{\cosh k(h+z)}{\cosh(kh)} \sin(kx - \omega t)$$

$$u = \frac{\partial \phi}{\partial x} = \frac{agk}{\omega} \frac{\cosh k(h+z)}{\cosh(kh)} \cos(kx - \omega t)$$

$$x = \int \frac{agk}{\omega} \frac{\cosh k(h+z)}{\cosh(kh)} \cos(kx - \omega t)\, dt + C_1 \quad \rightarrow (3.1)$$

$$x = -\frac{agk}{\omega^2} \frac{\cosh k(h+z)}{\cosh(kh)} \sin(kx - \omega t) + C_1 \quad \rightarrow (3.2)$$

$$w = \frac{\partial \phi}{\partial z} = \frac{agk}{\omega} \cdot \frac{\sinh k(h+z)}{\cosh(kh)} \sin(kx - \omega t)$$

$$\therefore \quad z = \int \frac{agk}{\omega} \frac{\sinh k(h+z)}{\cosh(kh)} \sin(kx - \omega t)\, dt + c_2$$

$$z = \frac{ag\cancel{\omega}k}{\omega^2} \frac{\sinh k(h+z)}{\cosh(kh)} \cos(kx - \omega t) + c_2$$

<u>assume</u>

$$\frac{agk \cosh(kh+kz)}{\omega^2 \cosh(kh)} = a \rightarrow (3.3)$$

<u>and</u>

$$\frac{agk}{\omega^2} \frac{\sinh(kh+kz)}{\cosh(kh)} = b \rightarrow (3.4)$$

<u>we get</u>

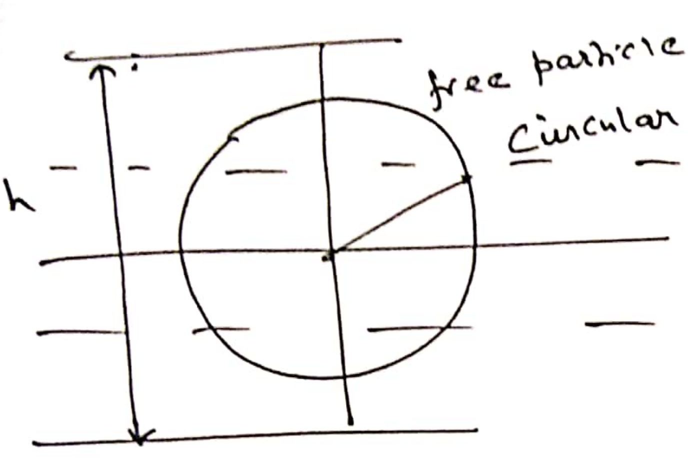
$$\boxed{\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1} \quad \ldots\ (3.5)$$

(3.5) can obtain by setting $c_1, c_2 = 0$ and substitute (3.3) and (3.4) in (3.1) and (3.2) respectively and use the identity

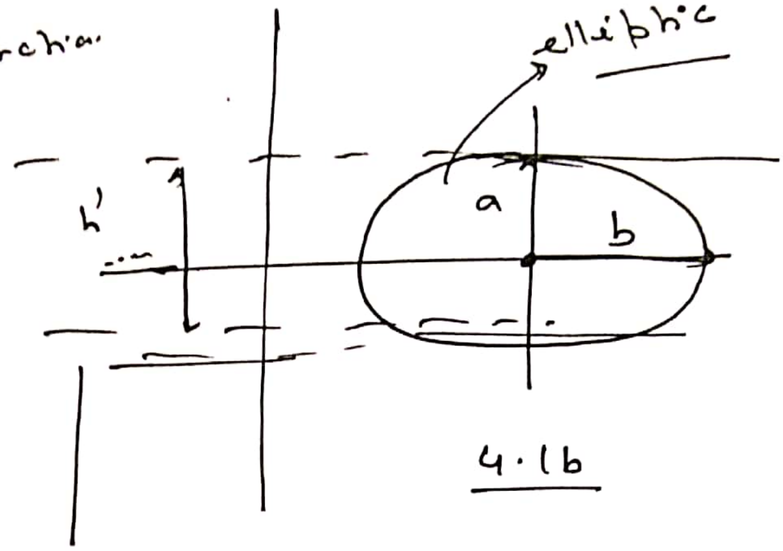
$$\cos^2(kx - \omega t) + \sin^2(kx - \omega t) = 1$$

$\therefore$ the particle trajectory for shallow water may be demonstrated as

How to look at this from physical sense :



4.1a



4.1b

subpressing the depth to h'

Figure (4.1a) and (4.1b) may be the possible explaination of the phenomenom. for deep water, the particle is allowed ~~to~~ to rotate in circlular fashom. however, as the depth reduces, the particle is ~~b~~ no longer able to rotate ~~circularly~~ ~~Ero~~ circularly, and take elliptic path, if you further reduce depth, the aspect ratio $\frac{b}{a}$ will further increase, and at some point of time, at a particular threshold depth h, $\frac{b}{a}$ increases so largely that particle is no longer able to rotate even in elliptic fashon and broaks. ~~Tho~~ This is known as wave breaking. we get to this topic ~~led~~ in latter lecture.