

(13)

The general k-step method for solution of $u' = f(u, u')$
 $u(t_0) = u_0$ may be written as

$$u_{j+1} = a_1 u_j + a_2 u_{j-1} + \dots + a_k u_{j-k+1} \\ + h(b_0 u_{j+1}' + b_1 u_j' + \dots + b_k u_{j-k+1}')$$

or

$$u_{j+1} = \sum_{i=1}^k a_i u_{j-i+1} + h \sum_{i=0}^k b_i u_{j-i+1}' \quad \dots \quad (1)$$

Symbolically, we can write (1) as

$$\rho(E) u_{j-k+1} - h \sigma(E) u_{j-k+1}' = 0, \text{ where } \dots \quad (2)$$

$$\rho(\xi) = \xi^k - a_1 \xi^{k-1} - a_2 \xi^{k-2} - \dots - a_k \quad \left\{ \begin{array}{l} \text{for k-step} \\ \text{method} \end{array} \right.$$

$$\sigma(\xi) = b_0 \xi^k + b_1 \xi^{k-1} + \dots + b_k \quad \left\{ \begin{array}{l} \text{if } (2) \text{ is of} \\ \text{degree } k \end{array} \right.$$

$$E f(x_i) = f(x_i + h)$$

$$E^n f(x_i) = f(x_i + nh)$$

$$\rho(E) = E^k - a_1 E^{k-1} - a_2 E^{k-2} - \dots - a_k E^0$$

$$\sigma(E) = b_0 E^k + b_1 E^{k-1} + \dots + b_k E^0$$

$$\rho(E) u_{j-k+1} = (E^k - a_1 E^{k-1} - a_2 E^{k-2} - \dots - a_k E^0) u_{j-k+1}$$

$$= u_{j-k+1+k} - a_1 u_{j-k+1+k-1} - \dots - a_k u_{j-k+1}$$

$$= u_{j+1} - a_1 u_j - a_2 u_{j-1} - \dots - a_k u_{j-k+1}$$

Similarly

$$\sigma(E) u_{j-k+1}' = (b_0 E^k + b_1 E^{k-1} + \dots + b_k E^0) u_{j-k+1}'$$

$$\sigma(E) u'_{j-n+1} = b_0 u'_{j+1} + b_1 u'_{j-n+1+n} + \dots + b_k u'_{j-n+k}$$

$$\sigma(E) = b_0 u'_{j+1} + b_1 u'_{j} + \dots + b_k u'_{j-n+k}.$$

Thus ① can be written as

$$f(E) u'_{j-n+1} - h\sigma(E) u'_{j-n+1} = 0$$

Now if $\phi(z)$ is given then find $\sigma(z)$ if the order of the method is known

We use the relation for a method of order p

$$f(z) - (\log \xi) \sigma(z) = c_{p+1} (z-1)^{p+1} + O((z-1)^{p+2}) \quad \text{--- } ③$$

~~assuming $\sigma(z)$ is known~~

Note: If an implicit method is to be obtained, then $f(z)$ and $\sigma(z)$ are of same degree. If an explicit method is to be obtained then $f(z)$ is one degree higher than $\sigma(z)$.

We rewrite ③

$$\frac{f(z)}{\log \xi} - \sigma(z) = c_{p+1} \frac{(z-1)^{p+1}}{\log \xi} + O \left(\frac{(z-1)^{p+2}}{\log \xi} \right)$$

$$\log \xi = \log [1 + (z-1)] = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \dots$$

$$\text{So } \frac{f(z)}{\log \xi} - \sigma(z) = c_{p+1} (z-1)^p + O((z-1)^{p+1})$$

If $f(z) = \zeta(z-1)$ then find $\delta(z)$

Now we consider

$$\frac{f(z)}{\log z} = \frac{\zeta(z-1)}{\log[1+(z-1)]} = \frac{(z-1)+(z-1)^2}{\log[1+(z-1)]}$$

$$\begin{aligned}\frac{f(z)}{\log z} &= \frac{(z-1)+(z-1)^2}{[(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{3}(z-1)^3-\dots]} \\ &= [1+(z-1)][1-\frac{1}{2}(z-1)+\frac{1}{3}(z-1)^2-\dots]^{-1} \\ &= 1 + \frac{3}{2}(z-1) + \frac{5}{12}(z-1)^2 + O((z-1)^3)\end{aligned}$$

Now for method of order 2.

$$\sigma(z) = 1 + \frac{3}{2}(z-1) = 1 + \frac{3}{2}z - \frac{3}{2}h = \frac{3}{2}z - \frac{3}{2}h$$

and for method of order 3

$$\delta(z) = 1 + \frac{3}{2}(z-1) + \frac{5}{12}(z-1)^2$$

for method of order 2.

$$f(z) = \zeta(z-1) = z^2 - h$$

$$\sigma(z) = \frac{3}{2}z - \frac{1}{2}$$

Then the method will be

$$f(E)u_{j-2H} - h\sigma(E)u_{j-2H}' = 0$$

$$f(E)u_{j+1} - h\sigma(E)u_{j+1}' = 0$$

(3)

$$[1 + (\xi - 1)] \left[1 - \left[\frac{1}{2}(\xi - 1) - \frac{1}{3}(\xi - 1)^2 + \dots \right] \right]^{-1}$$

$$= [1 + (\xi - 1)] \left[1 + \frac{1}{2}(\xi - 1) - \frac{1}{3}(\xi - 1)^2 + O(\xi - 1)^3 \right. \\ \left. + \left(\frac{1}{2}(\xi - 1) - \frac{1}{3}(\xi - 1)^2 + \dots \right)^2 + \dots \right]$$

$$= [1 + (\xi - 1)] \left[1 + \frac{1}{2}(\xi - 1) - \frac{1}{3}(\xi - 1)^2 + \frac{1}{4}(\xi - 1)^2 + O(\xi - 1)^3 \right]$$

$$= 1 + \frac{1}{2}(\xi - 1) - \frac{1}{3}(\xi - 1)^2 + \frac{1}{4}(\xi - 1)^2 + (\xi - 1) + \frac{1}{2}(\xi - 1)^2 \\ + O(\xi - 1)^3$$

$$= 1 + \frac{3}{2}(\xi - 1) + \left[-\frac{1}{3} + \frac{1}{4} + \frac{1}{2} \right] (\xi - 1)^2 + O(\xi - 1)^3 \\ = -\frac{1}{3} + \frac{1}{4} + \frac{1}{2} \\ = \frac{1}{12}(-4 + 3 + 6) \\ = \frac{5}{12}$$

$$\boxed{=} \\ = 1 + \frac{3}{2}(\xi - 1) + \frac{5}{12}(\xi - 1)^2 + O(\xi - 1)^3$$

$$\rho(E) u_{j+1} = (E^2 - E) u_{j+1}$$

$$= u_{j-1+2} - u_{j-1+1} = u_{j+1} - u_j$$

$$\sigma(E) u_{j+1}^1 = \left(\frac{3}{2}E - \frac{1}{2}E^0\right) u_{j+1}^1$$

$$= \frac{3}{2}u_j^1 - \frac{1}{2}u_{j+1}^1$$

$$\rho(E) u_{j+1} - h \sigma(E) u_{j+1}^1 = 0$$

$$u_{j+1} - h \frac{1}{2} (3u_j^1 - u_{j+1}^1) = 0$$

$$u_{j+1} = u_j + h \frac{1}{2} (3u_j^1 - u_{j+1}^1)$$

which is 2nd order Adam-Basforth method.

Adams - Moulton Methods

$f(\xi) = \xi^{k+1}(\xi-1)$, then find $\sigma(\xi)$ of degree k
 (in other words find an implicit method of
 order k)

For $k=2$

$$f(\xi) = \xi(\xi-1) = \xi^2 - 1 = (\xi-1) + (\xi-1)^2$$

$$\frac{f(\xi)}{\log \xi} = 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2 + O(\xi-1)^3$$

Since we want an implicit method so
 degree of $f(\xi)$ and $\sigma(\xi)$ will be same
 and thus

$$\sigma(\xi) = 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2$$

$$\sigma(\xi) = \frac{5}{12}\xi^2 + \frac{8}{3}\xi - \frac{1}{12}$$

and the method is given by

$$f(E)u_{j+1} - h\sigma(E)u_{j+1}' = 0$$

$$(E^2 - E)u_{j+1} - \frac{h}{12}[5E^2 + 8E - 1]u_{j+1}' = 0$$

$$u_{j+1} - u_j - \frac{h}{12}[5u_{j+1}' + 8u_j' - u_{j+1}'] = 0$$

or

$$\boxed{u_{j+1} = u_j + \frac{h}{12}[5u_{j+1}' + 8u_j' - u_{j+1}']}$$

Consider general p^{MSM} Consistency condition (1)

$$Y_{n+1} = \sum_{j=0}^p a_j Y_{n-j} + h \sum_{j=-1}^p b_j f(Y_{n-j}, Y_{n-j}) \quad \rightarrow (1)$$

Now apply this on $y^t = \lambda y$.

$$Y_{n+1} = \sum_{j=0}^p a_j Y_{n-j} + h\lambda \sum_{j=-1}^p b_j Y_{n-j}$$

Now put $Y_n = \xi^n$

$$\xi^{n+1} = \sum_{j=0}^p a_j \xi^{n-j} + h\lambda \sum_{j=-1}^p b_j \xi^{n-j}$$

$$\xi^{n+1} = (a_0 \xi^n + a_1 \xi^{n-1} + \dots + a_p \xi^{n-p})$$

$$+ \lambda h (b_{-1} \xi^{n+1} + b_0 \xi^n + b_1 \xi^{n-1} + \dots + b_p \xi^{n-p})$$

$$\left(\xi^{n+1} - a_0 \xi^n - a_1 \xi^{n-1} - \dots - a_p \xi^{n-p} \right) \\ - \lambda h [b_{-1} \xi^{n+1} + b_0 \xi^n + b_1 \xi^{n-1} + \dots + b_p \xi^{n-p}] = 0$$

Now divide by ξ^{n-p}

$$(\xi^{p+1} - a_0 \xi^p - a_1 \xi^{p-1} - \dots - a_p)$$

$$- \lambda h [b_{-1} \xi^{p+1} + b_0 \xi^p + b_1 \xi^{p-1} + \dots + b_p]$$

$$\text{Define } \varphi(\xi) = \xi^{p+1} - a_0 \xi^p - a_1 \xi^{p-1} - \dots - a_p \quad = 0 \quad (2)$$

$$\sigma(\xi) = b_{-1} \xi^{p+1} + b_0 \xi^p + b_1 \xi^{p-1} + \dots + b_p$$

Then (2) can be written as

$$\boxed{\varphi(\xi) - \lambda h \sigma(\xi) = 0}$$

This is called characteristic equation. (3)

Consistency Condition

Consistency condition: A multistep method of the form (1) is consistent if

$$\varphi(1) = 0 \text{ and } \varphi'(1) = \sigma(1) \quad \checkmark$$

$$\varphi(1) = 0 \Rightarrow 1 = \sum_{j=0}^p a_j$$

$$\varphi'(1) = (p+1) \sum_{j=0}^p - p a_0 \sum_{j=1}^{p-1} - \dots - a_{p-1}$$

$$= (p+1) - p a_0 - (p-1)a_1 - \dots - a_{p-1}$$

$$\varphi'(1) = (p+1) - p a_0 - (p-1)a_1 + (p-2)a_2 - \dots + a_{p-1} - p a_0 p_1$$

$$= p(1 - a_0 - a_1 - a_2 - \dots - a_p)$$

$$+ 1 + [a_1 + 2a_2 + \dots + (p-1)a_{p-1} + p a_p]$$

$$\therefore + p a_{p-1} + p a_p$$

$$= 1 + \sum_{j=0}^p j a_j$$

$$\sigma(1) = \sum_{j=-1}^p b_j$$

$$\varphi'(1) = \sigma(1) \Rightarrow 1 + \sum_{j=0}^p j a_j = \sum_{j=-1}^p b_j$$

$$\text{or } 1 = - \sum_{j=0}^p j a_j + \sum_{j=-1}^p b_j$$

Consistency conditions

(3)

Def^w Root Condition: A linear multistep method of the form

$$y_{n+1} = \sum_{j=0}^p a_j y_{n-j} + h \sum_{j=-1}^p b_j f(x_{n-j}, y_{n-j}) \quad \text{--- } \star$$

is said to satisfy the root condition if the roots of the equation $f(z) = 0$ lie inside the unit circle in the complex plane, and are simple if they lie on the unit circle.

Convergence of multistep methods:

Def^w The linear multistep method \star is said to be convergent if

$$\lim_{h \rightarrow 0} u_j = u(t_j), \quad \text{as } j \leq N$$

provided the rounding errors arising from all initial conditions tend to zero.

Theorem: The linear multistep method of the form \star is convergent if and only if the method is consistent and satisfies the root condition.

Stability of Multistep method

(1)

Let us examine the error for special equation $y' = \lambda y$. The computed solution satisfies

$$Y_{n+1} = Y_{n-l} + h \sum_{m=0}^{k+l} r_m f_{n-m+l} \quad \text{--- (1)}$$

ignoring the round off errors. The true solution (exact solution) satisfies

$$y(t_{n+1}) = y(t_{n-l}) + h \sum_{m=0}^{k+l} r_m f(t_{n-m+l}, y(t_{n-m+l})) + T_{n+1}(y) \quad \text{--- (2)}$$

where $T_{n+1}(y)$ is local truncation error. Substituting (2) from (1) and taking

$$E_n = y_n - y(t_n) \quad f = \lambda y$$

$$E_{n+1} = E_{n-l} + h \lambda \sum_{m=0}^{k+l} r_m E_{n-m+l} - T_{n+1}(y) \quad \text{--- (3)}$$

Now consider the homogeneous equation

$$E_{n+1} = E_{n-l} + h \lambda \sum_{m=0}^{k+l} r_m E_{n-m+l} \quad \text{--- (4)}$$

Now we seek solution of (4)

$$E_n = A \xi^n$$

$$A \xi^{n+1} = A \xi^{n-l} + h \lambda A \sum_{m=0}^{k+l} r_m \xi^{n-m+l}$$

$$\xi^{n+1} = \xi^{n-l} + h \lambda [r_0 \xi^{n+1} + r_1 \xi^n + \dots + r_{k+l} \xi^{n+k+l}]$$

now divide by ξ^{n-k} we get

$$\xi^{k+l} = \xi^{k-l} + h \lambda [r_0 \xi^{k+l} + r_1 \xi^k + \dots + r_{k+l}]$$

To find the solution of non-homogeneous difference equation (3), i.e.

$$E_{n+1} = E_{n-1} + h\lambda \sum_{m=0}^{k+1} r_m E_{n-m+1} - T_{n+1}$$

So we find particular solution and then general solution will be sum of particular solution and general solution of homogeneous equation.

Since T_{n+1} does not depend on E_n so it is a constant

Let $E_n = \beta$ is particular solution then

$$\beta = \beta + h\lambda \sum_{m=0}^{k+1} r_m \beta - T_{n+1}$$

$$\text{or } h\lambda \beta \sum_{m=0}^{k+1} r_m - T_{n+1} = 0$$

$$\sum_{m=0}^{k+1} r_m = \sigma(1)$$

$$h\lambda \sigma(1)\beta = T_{n+1}$$

$$\text{or } \beta = \frac{T_{n+1}}{h\lambda \sigma(1)} = \frac{T_{n+1}}{h\lambda \rho'(1)}$$

As $\rho'(1) = \sigma(1)$
since method
is consistent.

and hence general solution of
homogeneous difference equation

is given by

$$\boxed{E_n = A_0 \sum_0^n + A_1 \sum_1^n + \dots + A_k \sum_k^n + \frac{T_{n+1}}{h\lambda \rho'(1)}}$$

Stability of multistep method

(2)

$$\sum_{l=0}^{k+l} - \sum_{l=0}^{k-l} - h\lambda [r_0 \xi^{k+l} + \dots + r_n] = 0$$

$$P(\xi) - h\lambda \sigma(\xi) = 0 \quad \leftarrow \text{Characteristic Equation}$$

(5)

Here k and l are non-negative integers such that $k-l \geq 0$ and thus (5) is an algebraic relation of degree $k+l$ so it has $(k+l)$ roots say ξ_0, \dots, ξ_k . Then general solution of (4) can be written as

$$E_n = A_0 \xi_0^n + \dots + A_k \xi_k^n \quad (6)$$

If all the roots ξ_0, \dots, ξ_k are distinct then E_n will be of the form (6). If $\xi_1 = \xi_2$ i.e. ξ_1 is double root then $A_1 \xi_1^n + A_2 \xi_2^n$ will be replaced by $(A_1 + n A_2) \xi_1^n$.

Now from (6) it is obvious that these roots will affect the growth of E_n . If $|\xi| > 1$, then the error E_n will grow without bound or oscillate without bound as n becomes large. At similar effect is also evident if one root of (5), of multiplicity greater than one, has modulus one. We will say that method given by (1) is stable if $|E_n| < \infty$ as $n \rightarrow \infty$. *

Absolute Stable: $|\xi_i| \leq 1, i=0 \dots k$ and roots of unit magnitude are simple.

Region of Absolute Stability: The region of absolute stability is defined to be set of points in $h\lambda$ plane for which the method is absolutely stable.

Stability of multistep method

(3)

Def^w Since the method is consistent so $f(1) = 0 \Rightarrow 1$ is always a root of $\ell(z) = 0$. A linear multistep method of the form

K-step

$$y_{n+1} = \sum_{j=0}^{K-1} a_j y_{n-j} + h \sum_{j=-1}^{K-1} b_j f(x_{n-j}, y_{n-j})$$

is said to be

- (i) Stable if $|z_j| < 1$, for $j \neq 1$
- (ii) unstable if $|z_j| > 1$ for some j or if there is multiple root of $\ell(z) = 0$ of modulus unity
- (iii) conditionally stable if z_j are simple and if more than one of these roots have modulus unity.

Now consider the characteristic polynomial

$$\ell(z) - \lambda h \sigma(z) = 0 \quad \text{--- (1)}$$

which is a polynomial of degree $\leq K$

where $E_n = A z^n \quad \text{--- (2)}$

From (1) it is clear that for $h \rightarrow 0$ characteristic polynomial will reduce to $\ell(z) = 0$. Thus $\ell(z) = 0$ is called reduced characteristic polynomial.

Now let K roots of (1) are $\xi_{1h}, \xi_{2h}, \dots, \xi_{Kh}$
 (all the roots of (1) will depend on h)
 and let ξ_1, \dots, ξ_K be the roots of $\ell(z) = 0$
 Since as $h \rightarrow 0 \rightarrow$ roots of characteristic equation becomes roots of $\ell(z) = 0$. Thus

$$\xi_{ih} = \xi_i [1 + \bar{h} k_i + o(\bar{h}^2)] \quad \text{where } \bar{h} = \lambda h$$

The coefficient k_i are called growth parameters.

Stability of multistep method

(4)

Now from characteristic equation

$$f(z_{ih}) - \lambda h \sigma(z_{ih}) = 0$$

$$f[z_i(1 + \bar{h} k_i + O(\bar{h}^2))] - \bar{h} \sigma[z_i(1 + \bar{h} k_i + O(\bar{h}^2))] = 0$$

Now expanding using Taylor series we get

$$f(z_i) + \bar{h} k_i z_i f'(z_i) - \bar{h} \sigma(z_i) + O(\bar{h}^2) = 0$$

$f(z_i) = 0$ as z_i are roots of $f(z) = 0$ implies

$$\bar{h} k_i z_i f'(z_i) - \bar{h} \sigma(z_i) = 0 \quad \text{ignoring } O(\bar{h}^2) \text{ terms}$$

$$k_i = \frac{\sigma(z_i)}{z_i f'(z_i)} \quad \text{--- (3)}$$

Now from (1)

$$z_{ih} = z_i^n [1 + \bar{h} k_i + O(\bar{h}^2)]^n$$

$$z_{ih} \cong z_i^n e^{n\bar{h} k_i} \quad \text{--- (4)}$$

Now for z_{1h} . $f(1) = 0$ and $\sigma(1) = f'(1)$

$$k_1 = \frac{\sigma(1)}{f'(1)} = \frac{f'(1)}{f'(1)} = 1 \quad \text{Take } z_i = 1$$

$$z_{1h} = z_i^n [1 + \bar{h} + O(\bar{h}^2)]^n$$

$$z_{1h} \cong z_i^n e^{n\bar{h}} + O(\bar{h}^2) \quad \rightarrow \quad 1 + n\bar{h} + O(\bar{h}^2)$$

We have applied multistep method on $y' = y$
which has solution $y = e^{\lambda x}$ (as $y_n = e^{\lambda x_n} = e^{\lambda x_0 + \lambda(n-0)}$)

Therefore the root z_{1h} approximates the solution
of the differential equation $y' = \lambda y$. This is

Stability of multistep method

called the principal root and the remaining $(K-1)$ roots are called extraneous roots which arise because a first order differential equation is replaced by K th order difference equation.

As we have seen that E_n is written as

$$E_n = A_0 \xi_{1h} + A_1 \xi_{2h} + \dots + A_{K-1} \xi_{Kh} + \frac{T_{n+1}}{h \lambda + f(1)}$$

↑
 principal
 root

Extraneous roots

So behaviour of E_n will depend on extraneous roots behaviour.

Defⁿ Absolutely Stable:

the form

$$y_{n+1} = \sum_{j=0}^{K-1} a_j y_{n-j} + h \sum_{j=-1}^{K-1} b_j f(t_{n+j}, y_{n+j})$$

when applied to the test equation $y' = \lambda y, \lambda < 0$ is said to be absolutely stable if

$$|\xi_{ih}| \leq 1, i=1(1)K$$

(roots of unit magnitude is stable).

Defⁿ A-stable multistep method when applied to the differential equation $y' = \lambda y$, where λ is a complex constant with negative real part is said to be A-stable if all the solutions $y_n \rightarrow 0$ as $n \rightarrow \infty$.

Stability of multistep method

$$y' = \lambda y \quad y(0) = y_0$$

Ex Consider the mid point method $y_{n+1} = y_n + 2h f_n$ — (1)

$$y_{n+1} = y_n + 2h \lambda y_n$$

$$n \geq 1$$

$$\text{Substituting } y_n = \xi^n$$

$$\xi^{n+1} = \xi^n + 2h \lambda \xi^n$$

$$\xi^2 = 1 + 2h \lambda \xi$$

$$\xi^2 - 2\lambda h \xi - 1 = 0$$

$$\xi = \frac{2\lambda h \pm \sqrt{4\lambda^2 h^2 + 4 \cdot 1}}{2} = \lambda h \pm \sqrt{\lambda^2 h^2 + 1}$$

$$\xi = \lambda h \pm \sqrt{1 + \lambda^2 h^2}$$

— (2)

— (3)

(Let)

$$g(\omega) = \omega + \sqrt{1 + \omega^2}$$

$$\begin{aligned} \text{from (3)} \quad \xi_1 &= \lambda h + \sqrt{1 + \lambda^2 h^2}, \quad \xi_2 = \lambda h - \sqrt{1 + \lambda^2 h^2} \\ &= g(\lambda h) \\ &= -[(-\lambda h) + \sqrt{1 + (-\lambda h)^2}] \\ &= -g(-\lambda h) \end{aligned}$$

$$g(\omega) = \omega + (1 + \omega^2)^{1/2}$$

$$= \omega + \left(1 + \frac{1}{2}\omega^2 - \frac{1}{8}\omega^4 + \dots\right)$$

$$= \left(1 + \omega + \frac{\omega^2}{2} + \frac{\omega^3}{6} + \frac{\omega^4}{24} + \dots\right)$$

$$- \left[\frac{\omega^3}{6} + \frac{\omega^4}{24} + \dots\right] + \left(-\frac{1}{8}\omega^4 + \dots\right)$$

$$= e^\omega + \omega^3 B(\omega)$$

$$\frac{y_2 \cdot (y_2 - 1)}{2!} = \frac{y_2 \cdot (-1)}{2} = -\frac{1}{2} y_2$$

where $B(\omega)$ is bounded
and continuous about zero.

The solution of (1) can be written

$$y_n = C_1 \xi_1^n + C_2 \xi_2^n$$

$$\begin{aligned} \xi_1^n &= [g(\lambda h)]^n = \\ &= e^{\lambda h n} \left[1 + (\lambda h)^3 e^{-\lambda h} B(\lambda h)\right]^n \end{aligned}$$

$$[e^{\lambda h} + (\lambda h)^3 B(\lambda h)]^n$$

(7)

Take $\bar{h} = \lambda h$ Stability of multistep method

$$\xi_1^h = e^{nh} [1 + O(h^2)] \text{ and} \quad \text{--- (i)}$$

$$\xi_2 = -g(-\bar{h})$$

$$= -[e^{-\bar{h}} + (\bar{h})^3 B(-\bar{h})] = -e^{-\bar{h}} [1 + O(\bar{h}^3)]$$

$$\xi_2^h = (-1)^n e^{-n\bar{h}} [1 + O(\bar{h}^2)] \quad \text{--- (ii)}$$

Thus soln of (1) is written as

$$y_n = C_1 e^{nh} \underset{\text{I}}{[1 + O(h^2)]} + C_2 (-1)^n e^{-n\bar{h}} \underset{\text{II}}{[1 + O(\bar{h}^2)]} \quad \text{--- (4)}$$

Case(i) $\lambda > 0$ principal root $\xi_{1h} = e^{\bar{h}} [1 + O(\bar{h}^2)]$ and
 $\bar{h} = \lambda h$ extraneous root $\xi_{2h} = -e^{-\bar{h}} [1 + O(\bar{h}^2)]$

The principal root ξ_{1h}^n behaves like exact solution and extraneous solution (II) which is given by $(-1)^n e^{-n\bar{h}} [1 + O(\bar{h}^2)]$ goes to zero as n increases.

Case(ii) $\lambda < 0$ principal root ξ_{1h}^n behaves like exact soln but extraneous solution $(-1)^n e^{-n\bar{h}} [1 + O(\bar{h}^2)]$ magnitude becomes large as n increases.

Thus wind point method for this case i.e., ($\bar{h} < 0$) is unstable.

Stability of multistep method

8

Case(iii) for $\bar{h} = 0$ characteristic equation will be the reduced characteristic equation, i.e.,

$$\ell(\xi) = 0 \quad \ell(\xi) = \xi^2 - 1 = 0 \Rightarrow \xi = \pm 1$$

Thus the method is absolutely stable 

Def^h weakly stable: The linear multistep method of the form

$$y_{n+1} = \sum_{j=0}^p a_j y_{n-j} + h \sum_{j=1}^p b_j f^{(h)}_{n-j}, y_{n-j}$$

is said to be weakly stable if there is more than one simple root of the polynomial $\ell(\xi) = 0$ on the unit circle (and strongly stable if not).

Thus mid point method for $\bar{h} = 0$ is ^{abs} weakly stable.

Case(iv) λ is pure imaginary let $\lambda = i\phi$

$\bar{h} = \lambda h = i\phi h$ then characteristic equation

will be

Stability of multistep method

(9)

Case (ii) x is pure imaginary let $\lambda = i\beta$

$$\bar{h} = \lambda h = i\beta h$$

$$(\xi^2 - 1) = 2\bar{h}\xi$$

$$(\xi - \frac{1}{\xi}) = 2\bar{h}$$

$$\xi = e^{i\theta} \quad 2i\sin\theta = 2i\beta h$$

$$\beta h = \sin\theta$$

$0 < |\beta h| \leq 1$ will be the interval of stability

Ex Consider following Milne-Simpson method $y' = \lambda y$

$$y_{n+1} = y_n + \frac{h}{3} [y'_n + 4y'_m + y'_{m-1}] \quad (R)$$

$$\text{put } y_n = \xi^n \quad f(x_{n+1}, y_{n+1}) + 4f(x_m, y_m) + f(x_{m-1}, y_{m-1})$$

$$\xi^{n+1} = \xi^n + \frac{h}{3} [\lambda \xi^{n+1} + 4\lambda \xi^n + \lambda \xi^{n-1}]$$

$$\xi^2 = 1 + \frac{\lambda h}{3} [\xi^2 + 4\xi + 1]$$

$$(\xi^2 - 1) - \frac{\lambda h}{3} (\xi^2 + 4\xi + 1) = 0$$

$$\tau(\xi) - \lambda h \sigma(\xi) = 0$$

$$\xi^2 (1 - \frac{\lambda h}{3}) - 4\frac{\lambda h}{3} \xi - (1 + \frac{\lambda h}{3}) = 0 \quad (1)$$

$$\therefore \xi_{in} = \xi_i [1 + K_i \bar{h} + O(\bar{h}^2)]$$

$$\tau(\xi) = (\xi^2 - 1), \quad \sigma(\xi) = \frac{1}{3} (\xi^2 + 4\xi + 1)$$

Stability of multistep method

(1b)

$$\varphi(\xi) = 0 \Rightarrow \xi^2 - 1 = 0 \Rightarrow \xi = 1, -1$$

$$k_i = \frac{\sigma(\xi_i)}{\xi_i \varphi'(\xi_i)}$$

for $\xi = 1$

$$\xi_{1h} = \xi_1 [1 + k_1 \bar{h} + O(\bar{h}^2)]$$

$$k_1 = \frac{\sigma(1)}{1 \cdot \varphi'(1)} = \frac{2}{2} = 1$$

$$\begin{aligned}\sigma(1) &= 2 \\ \varphi'(1) &= 2 \\ \varphi''(1) &= 2\end{aligned}$$

$$\xi_{1h} = [1 + \bar{h} + O(\bar{h}^2)] \approx e^{\bar{h}} + O(\bar{h}^2) = e^{\bar{h}} [1 + O(\bar{h}^2)]$$

for $\xi = -1$

$$\xi_{2h} = \xi_2 [1 + k_2 \bar{h} + O(\bar{h}^2)]$$

$$k_2 = \frac{\sigma(-1)}{\xi_2 \varphi'(-1)}$$

$$k_2 = \frac{-2}{3 \cdot (-1) \cdot (-2)} = -\frac{1}{3}$$

$$\sigma(\xi) = \frac{1}{3} (\xi^2 + 4\xi + 1)$$

$$\sigma(-1) = \frac{1}{3} (1 - 4 + 1)$$

$$\therefore = -\gamma_3$$

$$\varphi'(-1) = -2$$

$$\xi_{2h} = -[1 - \frac{1}{3} \bar{h} + O(\bar{h}^2)]$$

$$= -e^{-\bar{h}/3} [1 + O(\bar{h}^2)]$$

Thus, both of ① will be

$$y_n = C_1 \underbrace{e^{n\bar{h}}}_I + C_2 (-1)^n \underbrace{e^{-n\bar{h}/3}}_I + O(\bar{h}^2)$$

Case i) $\lambda > 0$ $\bar{h} = \lambda h$, principal solution will be close to exact soln and extraneous soln (II soln) will die " for $n \uparrow \infty$ Thus it is stable case

$$p_h = \frac{3 \sin \theta}{2 + h \sin \theta}$$

$$-5 \leq h \sin \theta \leq 1$$

$$|p_h| = \left| \frac{3 \sin \theta}{2 + h \sin \theta} \right|$$

$$1 \leq 2 + h \sin \theta \leq 3$$

$$\leq \frac{3}{1} = 3$$

$$\frac{1}{3} \leq \frac{1}{2 + h \sin \theta} \leq 1$$

$$0 < |p_h| \leq 3$$

Interval of absolute stability -

order	Adams-Basforth	Adams-Moulton
1	(-2, 0)	(-\infty, 0)
2	(-1, 0)	(-\infty, 0)
3	(-0.5, 0)	(-6, 0)
4	(-0.3, 0)	(-3, 0)
5	(-0.1, 0)	(-1.9, 0)

Stability of multistep method

(11)

Case(ii) $\lambda < 0$, Principal root will approach to exact root while Extraneous root (2nd root) will increase as $n \uparrow \infty$. Thus in this case method is unstable.

Case(iii) $\bar{h} \Rightarrow$ Characteristic equation will be reduced characteristic eq. $f(\xi) = 0$
 $\Rightarrow \xi^2 - 1 = 0 \Rightarrow \xi = \pm 1$

Thus method is absolutely stable.

Case(iv) λ is purely imaginary $\lambda = i\beta$, β is real.
 Characteristic eq

$$\xi^2 - 1 - \frac{\lambda h}{3} (\xi^2 + 4\xi + 1) = 0$$

$$(\xi^2 - 1) = \frac{\lambda h}{3} (\xi^2 + 4\xi + 1)$$

$$(\xi^2 - 1) = \frac{i\beta h}{3} (\xi^2 + 4\xi + 1)$$

$$\text{or } \beta h = \frac{3(\xi^2 - 1)}{i(\xi^2 + 4\xi + 1)}$$

$$\beta h = \frac{3(\xi - 1/\xi)}{i(\xi + 4 + 1/\xi)}$$

$$\text{but } \xi = e^{i\theta}$$

$$\beta h = \frac{3i \sin \theta}{2i(6\sin \theta + 2)} \times \frac{i}{i} = \frac{3 \sin \theta}{(2 + 6\sin \theta)}$$

$$-6 \leq |\beta h| \leq \frac{3}{1} \quad \boxed{0 < |\beta h| \leq 3}$$

$$\xi - \frac{1}{\xi} = e^{i\theta} - e^{-i\theta}$$

$$= 2i \sin \theta$$

$$e^{i\theta} + e^{-i\theta} + 4 \\ = 2 \cos \theta + 4$$

$$-1 \leq \sin \theta \leq 1$$

$$-1 \leq \cos \theta \leq 1$$

$$1 \leq 2 + \cos \theta \leq 3$$

$$\frac{1}{3} \leq \frac{1}{2 + \cos \theta} \leq 1$$

Stability of MSM

Relative Stability A linear multistep method of the form

$$y_{n+1} = \sum_{j=0}^{K-1} a_j y_{n+j} + h \sum_{j=1}^{K-1} b_j f(x_{n+j}, y_{n+j})$$

is said to be relatively stable if

$$|\xi_{ih}| \leq |\xi_{1h}| \quad i = 2, 3, \dots, K$$

The region of relatively stability is defined to be the set of points in λh -plane for which the method is relatively stable.

Ex Consider 2nd order Adams-Basforth method

$$y_{n+1} = y_n + \frac{h}{2} (3y'_n - y'_{n-1})$$

for $y' \rightarrow y$

$$y_{n+1} = y_n + \frac{h\lambda}{2} (3y_n - y_{n-1})$$

$$\text{put } y_n = \xi^n \\ \xi^{n+1} = \xi^n + \frac{h\lambda}{2} (3\xi^n - \xi^{n-1})$$

divide by ξ^{n+1}

$$\xi = \xi + \frac{h\lambda}{2} (3\xi - 1)$$

$$(\xi^2 - \xi) - \frac{h\lambda}{2} (3\xi - 1) = 0$$

$$\xi^2 - \left(1 + \frac{3}{2}\bar{h}\right)\xi + \frac{\bar{h}}{2} = 0$$

$$\bar{h} = \lambda h$$

Characteristic eq.

$$\xi = \frac{\left(1 + \frac{3}{2}\bar{h}\right) \pm \sqrt{\left(1 + \frac{3}{2}\bar{h}\right)^2 - \frac{3}{4}\cdot\frac{\bar{h}}{2}}}{2}$$

$$= \frac{1}{2} \frac{(2 + 3\bar{h}) \pm \sqrt{\frac{(2 + 3\bar{h})^2}{4} - \frac{4\bar{h}}{2}}}{2}$$

$$= \frac{(2 + 3\bar{h}) \pm \sqrt{(2 + 3\bar{h})^2 - 8\bar{h}}}{4}$$

Stability MSM.

$$\xi = \frac{1}{4} \left[(2+3\bar{h}) \pm \sqrt{4+9\bar{h}^2+12\bar{h}-8\bar{h}} \right]$$

$$= \frac{1}{4} \left[(2+3\bar{h}) \pm \sqrt{9\bar{h}^2+4\bar{h}+4} \right]$$

$$n = k$$

$$n-1 = t - \xi$$

$$\xi_{1h} = \frac{1}{4} \left[2 + 3\bar{h} + (4 + 4\bar{h} + 9\bar{h}^2)^{\frac{1}{2}} \right]$$

$$= \frac{1}{2} + \frac{3}{4}\bar{h} + \frac{1}{4}(1 + \bar{h} + \frac{9}{4}\bar{h}^2)^{\frac{1}{2}}$$

$$= \frac{1}{2} + \frac{3}{4}\bar{h} + \frac{1}{2}(1 + \bar{h} + \frac{9}{4}\bar{h}^2)^{\frac{1}{2}}$$

$$= \frac{1}{2} + \frac{3}{4}\bar{h} + \frac{1}{2} \left[1 + \frac{1}{2}(\bar{h} + \frac{9}{4}\bar{h}^2) - \frac{1}{4 \cdot 2} (\bar{h} + \frac{9}{4}\bar{h}^2)^2 \right. \\ \left. + \dots \right]$$

$$= 1 + \bar{h} + \frac{9}{16}\bar{h}^2 - \frac{1}{16}(\bar{h}^2 + \frac{81}{16}\bar{h}^4 + \frac{18}{4}\bar{h}^3) + \dots$$

$$= 1 + \bar{h} + \frac{\bar{h}^2}{2} + O(\bar{h}^3) = e^{\bar{h}} + O(\bar{h}^3)$$

Similarly $2(1 + \bar{h} + \frac{9}{4}\bar{h}^2)^{\frac{1}{2}} = 2 \left[(1 + \bar{h} + \frac{9}{4}\bar{h}^2) - \frac{1}{8}(\bar{h} + \frac{9}{4}\bar{h}^2)^3 + \dots \right]$

$$\xi_{2h} = \frac{1}{4} \left[(2+3\bar{h}) - \frac{1}{8} (4+4\bar{h}+9\bar{h}^2)^{\frac{1}{2}} \right]$$

$$\approx \frac{\bar{h}}{2} - \frac{\bar{h}^2}{2} + \frac{\bar{h}^3}{4} + O(\bar{h}^4)$$

$$= \frac{\bar{h}}{2} \left(1 - \bar{h} + \frac{\bar{h}^2}{2} \right) + O(\bar{h}^4)$$

$$\approx \frac{\bar{h}}{2} e^{-\bar{h}} + O(\bar{h}^4)$$

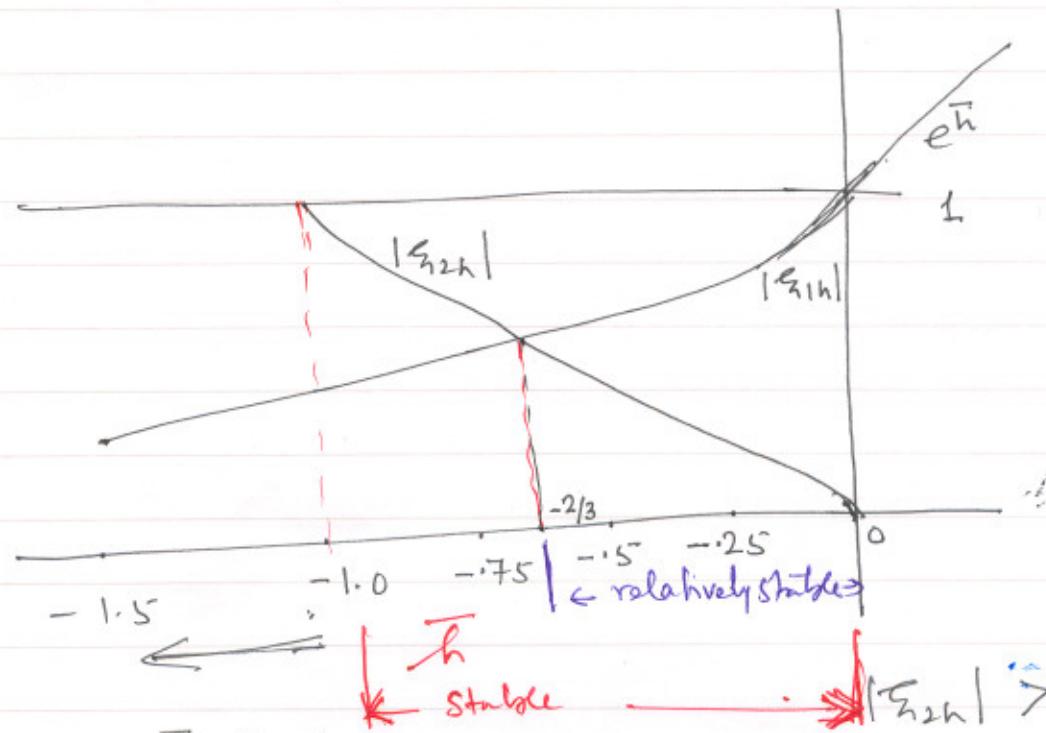
Stability MSM

(14)

Thus Soln

$$y_n = C_1 \xi_{1h}^n + C_2 \xi_{2h}^n$$

The roots $|\xi_{1h}|$ & $|\xi_{2h}|$ are plotted below



1. Unstable $\bar{h} \leq -1.0$

2. Absolutely Stable $-1.0 < \bar{h} < 0$

3. $|\xi_{2h}| < |\xi_{1h}|$ for $-\frac{2}{3} < \bar{h} < 0$

so method is relatively stable for $-\frac{2}{3} < \bar{h} < 0$

$$\begin{aligned} \bar{h} &= -0.1 \\ \xi_{2h} &= -1.359 \text{ at } \bar{h} = -0.1 \\ |\xi_{2h}| &= 1.359 \end{aligned}$$