Multistep methods

For $h>0$ define nodes $x_n = x_0 + nh$, $n \geq 0$, the general form of multistep methods to be considered is

$y' = f$

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}) \quad \text{--- (1)}$$

$n = p, p+1, \ldots$

the coefficients $a_0, a_1 \ldots a_p, b_0, \ldots b_p$ are constants, and $p \geq 0$. If either $a_p \neq 0$, or $b_p \neq 0$, the method is called a $p+1$ step method because $(p+1)$ previous values are used to compute y_{n+1}. Eq ① can be written as

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_p y_{n-p} + h\left[b_{-1} f(x_{n+1}, y_{n+1})\right] + h\left[b_0 f(x_n, y_n) + b_1 f(x_{n-1}, y_{n-1}) + \cdots + b_p f(x_{n-p}, y_{n-p})\right] \quad \text{--- (2)}$$

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + \cdots + a_p y_{n-p} + h\left[b_{-1} f(x_{n+1}, y_{n+1})\right] + h\left[b_0 y'_n + b_1 y'_{n-1} + \cdots + b_p y'_{n-p}\right] \quad \text{--- (3)}$$

Examples ① Midpoint-method

$y'_n = f(x_n, y_n)$

$$\frac{y_{n+1} - y_{n-1}}{2h} = f(x_n, y_n)$$

$$y_{n+1} = y_{n-1} + 2h f(x_n, y_n)$$

② Trapezoidal method

$y' = f(x, y)$

Integrate from $x_n \rightarrow x_{n+1}$

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx \quad \text{--- (4)}$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\left[f(a)+f(b)\right]$$

$$\int_{x_n}^{x_{n+1}} f(x,y(x))\,dx = \frac{h}{2}\left[f(x_n,y(x_n)) + f(x_{n+1},y(x_{n+1}))\right]$$

Then trapezoid method is given by

$$\boxed{y_{n+1} - y_n = \frac{h}{2}\left[f(x_n,y_n) + f(x_{n+1},y_{n+1})\right] \qquad n \geq 0}$$

This is an implicit one step method.

③ Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{1}{3}\cdot\left(\frac{b-a}{2}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$y' = f(x,y)$$

Integrate from x_{n-1} to x_{n+1}

$$\frac{x_{n+1}+x_{n-1}}{2}$$

$$y_{n+1} = y_{n-1} + \int_{x_{n-1}}^{x_{n+1}} f(x,y(x))\,dx.$$

$$\frac{x_n+\cancel{h}+x_n-\cancel{h}}{2} = \frac{2x_n}{2} = x_n$$

$$= y_{n-1} + \frac{1}{3}\cdot\frac{\cancel{2}h}{\cancel{2}}\left[f(x_{n-1},y_{n-1}) + 4f(x_n,y_n) + f(x_{n+1},y_{n+1})\right]$$

$$\boxed{y_{n+1} = y_{n-1} + \frac{h}{3}\left[f(x_{n-1},y_{n-1}) + 4f(x_n,y_n) + f(x_{n+1},y_{n+1})\right]}$$

Consider the general multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}) \qquad (5)$$

$n = p, p+1, \ldots$

For any differentiable $Y(x)$, define the truncation error for integrating $Y'(x)$ by

$$T_{n+1}(Y) = Y(x_{n+1}) - \left[\sum_{j=0}^{p} a_j Y(x_{n-j}) + h \sum_{j=-1}^{p} b_j Y'(x_{n-j}) \right]$$

$n \geq p$ — (6)

Define the function $\tau_{n+1}(Y)$ by

$$\tau_{n+1}(Y) = \frac{1}{h} T_{n+1}(Y) \qquad (7)$$

$$\tau(h) = \max_{x_p \leq x_n \leq b} |\tau_{n+1}(Y)|$$

In order to prove the convergence of the approximate solution $\{ y_n \mid x_0 \leq x_n \leq b \}$ of (5) to the solution $Y(x)$ of

$$y' = f(x,y), \quad y(x_0) = y_0$$

it is necessary that

$$\tau(h) \to 0 \quad \text{as } h \to 0$$

This is often called the consistency condition for the method (5). We say that the speed of the convergence is m where m is the largest value s.t.

$$\tau(h) = O(h^m).$$

Theorem: Let $m \geq 1$ be a given integer. In order that (4)

$$\tau(h) = \max_{x_p \leq x_n \leq b} |T_{n+1}(Y)| \to 0 \text{ as } h \to 0$$

holds for all continuously differentiable function $Y(x)$, that is, that the method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j})$$

$$n \geq p$$

be consistent, it is necessary - and sufficient that

$$\sum_{j=0}^{p} a_j = 1, \quad -\sum_{j=0}^{p} j a_j + \sum_{j=-1}^{p} b_j = 1 \quad \text{— (8)}$$

and for

$$\tau(h) = O(h^m)$$

to be valid for all functions $Y(x)$ that are $(m+1)$ times continuously differentiable, it is necessary and sufficient that (8) holds and that

$$\sum_{j=0}^{p} (-j)^i a_j + i \sum_{j=-1}^{p} (-j)^{i-1} b_j = 1 \quad i = 2, \ldots m$$

— (9)

Proof

$$T_{n+1}(Y) = Y(x_{n+1}) - \left[\sum_{j=0}^{p} a_j Y(x_{n-j}) + h \sum_{j=-1}^{p} b_j Y'(x_{n-j})\right] \quad \text{— (*)}$$

$$\begin{aligned} T_{n+1}(\alpha Y + \beta W) &= [\alpha Y(x_{n+1}) + \beta W(x_{n+1})] \\ &\quad - \sum_{j=0}^{p} a_j (\alpha Y(x_{n-j}) + \beta W(x_{n-j})) \\ &\quad + h \sum_{j=-1}^{p} b_j (\alpha Y'(x_{n-j}) + \beta W'(x_{n-j})) \\ &= \alpha T_{n+1}(Y) + \beta T_{n+1}(W). \end{aligned}$$

Thus

$$T_{n+1}(\alpha Y+\beta W) = \alpha T_{n+1}(Y) + \beta T_{n+1}(W)$$

for all constants α, β and all differentiable functions Y, W.

Now

$$Y(x) = Y(x_n + (x - x_n))$$

$$= Y(x_n) + (x-x_n)Y'(x_n) + \frac{(x-x_n)^2}{2!}Y''(x_n) + \cdots + \frac{(x-x_n)^m}{m!}Y^{(m)}(x_n) \quad (10)$$

$$+ R_{m+1}(x)$$

$$\boxed{R_{m+1}(x) = \frac{1}{n!}\int_{x_n}^{x}(x-t)^n Y^{(n+1)}(t)\,dt = \frac{(x-x_n)^{n+1}}{(n+1)!}Y^{(n+1)}(\xi_n)}$$

ξ_n lies between x_n & x.

$$Y(x) = \sum_{i=0}^{m}\frac{(x-x_n)^i}{i!}Y^{(i)}(x_n) + R_{m+1}(x) \quad (11)$$

Now substituting (11) in (*) and using linearity property we get

Apply the operator T_{n+1} on both sides of (11)

$$T_{n+1}(Y) = \sum_{i=0}^{m}\frac{1}{i!}Y^{(i)}(x_n)\,T_{n+1}[(x-x_n)^i] + T_{n+1}(R_{m+1}(x)) \quad (12)$$

It is necessary to calculate $T_{n+1}((x-x_n)^i)$ $\underline{i \geq 0}$

for $\underline{i=0}$

from (*) $\rightarrow$ $T_{n+1}(1) = 1 - \sum_{j=0}^{p} a_j$

$\boxed{\text{let } T_{n+1}(1) = c_0}$

$$c_0 = 1 - \sum_{j=0}^{p} a_j \quad \text{(i)}$$

for $i \geq 1$

$$T_{n+1}((x-x_n)^i)$$

$$T_{n+1}(Y) = Y(x_{n+1}) - \left[\sum_{j=0}^{p} a_j Y(x_{n-j}) + h\sum_{j=-1}^{p} b_j Y'(x_{n-j})\right]$$

⑥

let $p(x) = (x - x_n)^i$, $\quad p'(x) = i(x - x_n)^{i-1}$

$$T_{n+1}(p(x)) = p(x_{n+1}) - \left[\sum_{j=0}^{p} a_j\, p(x_{n-j}) + h \sum_{j=-1}^{p} b_j\, p'(x_{n-j})\right]$$

$$p(x_{n+1}) = (x_{n+1} - x_n)^i = h^i$$

$x_{n+1} - x_n = h$

$$p(x_{n-j}) = (x_{n-j} - x_n)^i = (-jh)^i = (-j)^i h^i$$

$x_{n-j} = x_0 + (n-j)h$
$x_n = x_0 + nh$
$x_{n-j} - x_n$
$= [(n-j) - n]h$
$= (-jh)$

$$p'(x_{n-j}) = i(x_{n-j} - x_n)^{i-1} = i(-jh)^{i-1} = i \cdot (-j)^{i-1} h^{i-1}$$

$$\boxed{\text{let } T_{n+1}((x - x_n)^i) = c_i h^i \qquad i \geq 1}$$

Then

$$c_i h^i = h^i \left[1 - \left\{\sum_{j=0}^{p} a_j (-j)^i + \sum_{j=-1}^{p} b_j\, i\, (-j)^{i-1}\right\}\right]$$

or

$$\boxed{c_i = 1 - \left[\sum_{j=0}^{p} (-j)^i a_j + i \sum_{j=-1}^{p} (-j)^{i-1} b_j\right] \qquad \underline{\underline{i \geq 1}}} \quad (11)$$

This is useful for writing coefficients of h^i for a multistep method [illegible]

This gives (from (12))

$$T_{n+1}(y) = \sum_{i=0}^{m} \frac{1}{i!} y^{(i)}(x_n)\, c_i h^i + T_{n+1}(R_{m+1}) \quad (13)$$

And if we write remainder $R_{m+1}(x)$ as

$$R_{m+1}(x) = \frac{1}{(m+1)!}(x - x_n)^{m+1} y^{(m+1)}(x_n) + \cdots$$

$$T(R_{m+1}(x)) = \frac{1}{(m+1)!} y^{(m+1)}(x_n)\, T[(x - x_n)^{m+1}] + \cdots$$

$$T[(x-x_n)^{m+1}] = C_{m+1} h^{m+1}$$

where C_{m+1} is given by (11), then

$$T(R_{m+1}(x)) = \frac{C_{m+1}}{(m+1)!} h^{m+1} y^{(m+1)}(x_n) + O(h^{m+2}) \quad \text{--- (14)}$$

To obtain consistency condition we need

$$\tau(h) = O(h)$$

since we want

$$\tau(h) = \max_{x_p \le x_n \le b} |T_{n+1}(Y)| \to 0 \quad \text{as } h \to 0$$

$$\text{so } \tau(h) = O(h) \quad \text{or} \quad T_{n+1}(Y) = O(h^2)$$

Then from (13) $\quad C_0 = 0 \;\&\; C_1 = 0$

$$\Rightarrow C_0 = 0 \Rightarrow 1 - \sum_{j=0}^{p} a_j = 0 \qquad \Rightarrow \sum_{j=0}^{p} a_j = 1$$

$$C_1 = 0 \Rightarrow 1 - \sum_{j=0}^{p} (-j) a_j + \sum_{j=-1}^{p} (-j)^0 b_j = 0$$

or

$$\Rightarrow -\sum_{j=0}^{p} j a_j + \sum_{j=-1}^{p} b_j = 1$$

To obtain

$$\tau(h) = O(h^m) \quad \text{or} \quad T_{n+1}(Y) = O(h^{m+1})$$

again from (13) $\quad C_0 = 0 = C_1 = \cdots = C_m$ or $C_i = 0,\ i = 0(1)m$

gives

$$\sum_{j=0}^{p} (-j)^i a_j + i \sum_{j=-1}^{p} (-j)^{i-1} b_j = 1$$

$$i = 1, 2, \ldots m$$

Theorem: Consider solving the IVP

$$y' = f(x,y), \quad y(x_0) = Y_0, \quad x_0 \le x \le b$$

using multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}). \qquad (1)$$

$n \geq p$

$n = p, p+1, \ldots$

Let the initial error satisfy

$$\eta(h) = \max_{0 \le i \le p} |Y(x_i) - y_i| \to 0 \quad \text{as } h \to 0.$$

Assume the method is consistent, i.e., it satisfies

$$\tau(h) = \max_{x_p \le x_n \le b} |T_{n+1}(Y)| \to 0 \quad \text{as } h \to 0.$$

And finally, assume that the coefficients a_j are all non-negative

$$a_j \geq 0, \quad j = 0, 1, \ldots p$$

then the method (1) is convergent, and

$$\max_{x_0 \le x_n \le b} |Y(x_n) - y_n| \le c_1 \eta(h) + c_2 \tau(h)$$

for suitable constants c_1 & c_2. If the method (1) is of order m, and the initial error satisfy $\eta(h) = O(h^m)$, then the speed of the convergence of the method (1) is $O(h^m)$.

Note: To obtain a rate of convergence of $O(h^m)$ for the method (1), it is necessary that each step have an error

$$T_{n+1}(Y) = O(h^{m+1})$$

sufficient.

But the initial values $y_0, y_1, \ldots y_p$ need to be computed only with an accuracy of $O(h^m)$, since $\eta(h) = O(h^m)$ is

Proof

$$T_{n+1}(Y) = Y(x_{n+1}) - \left[\sum_{j=0}^{p} a_j Y(x_{n-j}) + h \sum_{j=-1}^{p} b_j Y'(x_{n-j})\right] \quad n \geq p$$

$$\tau_{n+1}(Y) = \frac{1}{h} T_{n+1}(Y)$$

so
$$Y(x_{n+1}) = \sum_{j=0}^{p} a_j Y(x_{n-j}) + h \sum_{j=-1}^{p} b_j Y'(x_{n-j}) + h\tau_{n+1}(Y) \quad n \geq p \quad (1)$$

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}) \quad n = p, p+1, \ldots \quad (2)$$

Take $e_i = Y(x_i) - y_i$

substracting (2) from (1) we get

$$e_{n+1} = \sum_{j=0}^{p} a_j e_{n-j} + h \sum_{j=-1}^{p} b_j \left[f(x_{n-j}, Y(x_{n-j})) - f(x_{n-j}, y_{n-j})\right] + h\tau_{n+1} \quad (3)$$

Now if f satisfies Lipschitz condition then

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$$

Then from (3)

$$|e_{n+1}| \leq \sum_{j=0}^{p} a_j |e_{n-j}| + hK \sum_{j=-1}^{p} |b_j||e_{n-j}| + h\tau(h) \quad (4)$$

since $a_j \geq 0$ $\underline{|a_j| = a_j}$

Let
$$f_n = \max_{0 \leq i \leq n} |e_i|, \quad n = 0, 1, 2, \ldots, \underline{N}(\underline{h})$$

$$|e_{n+1}| \leq \sum_{j=0}^{p} a_j f_n + hK \sum_{j=-1}^{p} |b_j| f_{n+1} + h\tau(h) \quad (5)$$

for $j = -1$. $|e_{n+1}|$ is present in second sum.

$\sum_{j=0}^{p} a_j = 1$, $\boxed{-\sum_{j=0}^{p} j a_j + \sum_{j=-1}^{p} b_j = 1}$

from (5)

$$|e_{n+1}| \le f_n + hc f_{n+1} + hT(h) \quad \text{where} \quad c = K\sum_{j=-1}^{p}|b_j| \qquad \text{— (6)}$$

(6) is true for all n (rather for all $|e_{n+1}|$ and so it is true for $\max\limits_{0\le i\le n+1}|e_i|$ which is one of $|e_i|$, $i=0(1)n+1$

thus

$$\max_{0\le i\le n+1}|e_i| \le f_n + hc f_{n+1} + hT(h)$$

or

$$f_{n+1} \le f_n + hc f_{n+1} + hT(h)$$

or

$$f_{n+1} \le \frac{f_n}{1-hc} + \frac{h}{1-hc}T(h)$$

provided $1-hc>0$

~~for $hc<1/2$ ($1-hc>0$)~~

~~This is possible as $h\to 0$~~

Take $hc<1/2$ (This is possible by taking h small enough)

$$f_{n+1} \le \frac{f_n}{1-hc} + \frac{h}{1-hc}T(h) \qquad \text{— (7)}$$

for $hc<1/2$, $1-hc>1/2$ or $\frac{1}{1-hc}<2$

& Consider

$$\begin{aligned}(1-hc)(1+2hc) &= 1-hc+2hc(1-hc)\\ &= 1-hc+2hc-2(hc)^2\\ &= 1+hc-2(hc)^2\\ &= 1+hc(1-2hc)\\ &\ge 1\end{aligned}$$

$hc<1/2$
$\Rightarrow 2hc<1$
$\Rightarrow 1-2hc>0$
$c>0,\ h>0$
$\Rightarrow hc(1-2hc)>0$

$$\Rightarrow \frac{1}{1-hc} \le (1+2hc)$$

then from (7)

$$f_{n+1} \le (1+2hc)f_n + 2hT(h) \qquad \text{— (8)}$$

from (8)

$$f_n \le (1+2hc) f_{n-1} + 2h\tau(h)$$
$$\le (1+2hc)[(1+2hc) f_{n-2} + 2h\tau(h)] + 2h\tau(h)$$
$$= (1+2hc)^2 f_{n-2} + 2h\tau(h)(1+2hc) + 2h\tau(h)$$
$$\le (1+2hc)^2[f_{n-3}(1+2hc) + 2h\tau(h)] + 2h\tau(h)(1+2hc) + 2h\tau(h)$$
$$= (1+2hc)^3 f_{n-3} + 2h\tau(h)[1+(1+2hc)+(1+2hc)^2]$$
$$\vdots$$
$$\le (1+2hc)^{n-p} f_{n-\overline{n-p}} + 2h\tau(h)[1+(1+2hc)+(1+2hc)^2 + \cdots + (1+2hc)^{n-p-1}]$$
$$= (1+2hc)^{n-p} f_p + 2h\tau(h)\left[\frac{(1+2hc)^{n-p}-1}{\cancel{1}+2hc-\cancel{1}}\right]$$

Note that $f_p = \eta(h)$

$f_p = \max_{0\le i\le p} |e_i|$

$$= (1+2hc)^{n-p}\eta(h) + \frac{2h}{2hc}\tau(h)[(1+2hc)^{n-p}-1]$$
$$= (1+2hc)^{n-p}\eta(h) + \tau(h)\left[\frac{(1+2hc)^{n-p}-1}{c}\right]$$

So $$f_n \le (1+2hc)^{n-p}\eta(h) + \tau(h)\left[\frac{(1+2hc)^{n-p}-1}{c}\right] \quad \text{—— (9)}$$

Lemma for any real x
$$1+x \le e^x \quad \text{—— (i)}$$
and for any $x \ge -1$
$$0 \le (1+x)^m \le e^{mx} \quad \text{—— (ii)}$$

Pf Using Taylor expansion $e^x = 1 + x + \frac{x^2}{2} e^{\xi}$, ξ lies between 0 & x

Since $\frac{x^2}{2} e^{\xi} \ge 0 \Rightarrow e^x \ge 1+x$

Now from (ii) (12)

$$(1+2hc)^{n-p} \le e^{(n-p)2hc} \le e^{2hcn} = e^{2c\,nh} = e^{2c(x_n-x_0)} \le e^{2c(b-x_0)}$$

Then from (9)

$$f_n \le e^{2c(b-x_0)}\eta(h) + \tau(h)\left[\frac{e^{2c(b-x_0)}-1}{c}\right] \quad \text{——— (10)}$$

$x_0 \le x_n \le b$

$$f_n = \max_{0\le i\le n}|e_i| = \max_{0\le i\le n}|Y(x_i) - y_i|$$

(10) is true for all n thus

$$\max_{x_0\le x_i\le b}|Y(x_i) - y_i| \le e^{2c(b-x_0)}\eta(h) + \left[\frac{e^{2c(b-x_0)}-1}{c}\right]\tau(h)$$

$$= c_1\,\eta(h) + c_2\,\tau(h).$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}$$

$$\begin{bmatrix}1&2\\0&-2\end{bmatrix} - \begin{bmatrix}1&2\\0&1\end{bmatrix} \to \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\left[\begin{array}{cc|cc}1&2&1&0\\3&4&0&1\end{array}\right] \xrightarrow{R_2\leftarrow R_2-3R_1} \left[\begin{array}{cc|cc}1&2&1&0\\0&-2&-3&1\end{array}\right] \xrightarrow{R_2\leftarrow R_2\times -\frac{1}{2}} \left[\begin{array}{cc|cc}1&2&1&0\\0&1&+3/2&-1/2\end{array}\right]$$

$$\xrightarrow{R_1\leftarrow R_1-2R_2} \left[\begin{array}{cc|cc}1&0&-2&1\\0&1&3/2&-1/2\end{array}\right]$$

$$\begin{bmatrix}-2&1\\3/2&-1/2\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} = \begin{bmatrix}-2+3&-4+4\\3/2-3/2&3-2\end{bmatrix}$$

$$= \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$C_2 \leftarrow C_2-2C_1 \left[\begin{array}{cc|cc}1&0&1&-2\\0&1&3/2&-1/2-3\end{array}\right]$$

$$= \left[\begin{array}{cc|cc}1&0&1&-2\\0&1&3/2&-7/2\end{array}\right] \times$$

$(1,2,3)\ (0\ 1\ 0)\ (0,0,1)$

$$\begin{bmatrix}1&2&3\\0&1&0\\0&0&1\end{bmatrix} \quad 1 \neq 0 \ \checkmark$$

$(1,2,3)\ (2,3,4)\ (0,0,1)$

$$\begin{bmatrix}1&2&3\\2&3&4\\0&0&1\end{bmatrix} \quad 3-4=-1 \neq 0 \ \checkmark$$