Elliptic PDE

Five point diagonal formula

$$\left[u_{xx}+u_{yy}\right]_{(x_i,y_j)} = \frac{1}{2h^2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{bmatrix} u_{ij}$$

$$= \frac{1}{2h^2}\left[u_{i-1,j+1}+u_{i+1,j+1}-4u_{ij}+u_{i-1,j-1}+u_{i+1,j-1}\right]$$

$$\frac{1}{2h^2}\left[u(x-h,y+h)+u(x+h,y+h)-4u(x,y)+u(x-h,y-h)+u(x+h,y-h)\right]$$

$$= u_{xx}+u_{yy}+\frac{h^4}{12}\left[\frac{\partial^4 u}{\partial y^4}+6\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial x^4}\right)+O(h^6)$$

Ex: Solve the Poisson equation

$$u_{xx}+u_{yy} = -\left[2+\pi^2 x(1-x)\right]\cos(\pi y)$$

on the domain $R=\left\{(x,y): 0<x<1,\ 0<y<\tfrac{1}{2}\right\}$. The Dirichlet boundary conditions are given by

$$u(0,y) = u(1,y) = 0$$

$$u(x,0) = x(1-x),\quad u(x,\tfrac{1}{2}) = 0.$$

Use five point formula and five point diagonal formula. Take $\Delta x = 1/4$ and $\Delta y = 1/4$. The exact solution of the problem is given by

$$u(x,y) = x(1-x)\cos(\pi y).$$

Find absolute errors, $|u(x_i,y_j)-u_{ij}|$, for both the methods.