Reduction of higher order differential equation to the system of 1st order differential equation:

$$xy'' - y' + 4x^2 y = 0, \qquad y(1) = 1, \quad y'(1) = 2 \quad \text{--- (1)}$$

Put $u_1 = y$

$u_2 = y'$

then $u_1' = u_2$

and $u_2' = y'' = \frac{1}{x}[y' - 4x^2 y] = \frac{1}{x}[u_2 - 4x^2 u_1]$

So, the bvp (1) is reduced to

$$u_1' = u_2 \qquad u_1(1) = 1$$
$$u_2' = \frac{1}{x}[u_2 - 4x^2 u_1] \qquad u_2(1) = 2$$

or $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} f_1(x, u_1, u_2) \\ f_2(x, u_1, u_2) \end{pmatrix}$, where $f_1(x, u_1, u_2) = u_2$

$f_2(x, u_1, u_2) = \frac{1}{x}[u_2 - 4x^2 u_1]$ --- (2)

$u_1(1) = 1$

$u_2(1) = 2$

Take $\bar{u} = (u_1, u_2)^T$, $\bar{F} = (f_1, f_2)^T$ then (2) can be written as

$$\bar{u}' = \bar{F}(x, \bar{u})$$
$$\bar{u}(1) = (1, 2)^T$$

Ex Use Taylor series method of order two for step by step integration of the differential equations

$$y' = xz + 1, \quad y(0) = 0$$
$$z' = -xy, \quad z(0) = 1$$

with $h = 0.1$, and $0 \le x \le 0.2$

Solⁿ Second order Taylor series for IVP can be written as

$0 \quad x_1 = h = .1 \quad x_2 = 2h = .2$

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' \quad \text{--- (1)}$$

and

$$z_{n+1} = z_n + h z_n' + \frac{h^2}{2} z_n'' \quad \text{--- (2)}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(xz+1) = xz' + z = -x^2 y + z$$

$$y'' = -x^2 y + z$$

$$z'' = \frac{d}{dx} z'(x) = -xy' - y = -x(xz+1) - y$$

$$z'' = -x^2 z - x - y$$

from (1)

$$y_{n+1} = y_n + h x_n z_n + h + \frac{h^2}{2!}\left(-x_n^2 y_n + z_n\right)$$

$$y_{n+1} = \left(1 - \frac{h^2 x_n^2}{2}\right) y_n + \left(h x_n + \frac{h^2}{2}\right) z_n + h \quad \text{--- (1)'}$$

Similarly for z_{n+1}

$$z_{n+1} = z_n + h(-x_n y_n) + \frac{h^2}{2}\left(-x_n^2 z_n - x_n - y_n\right)$$

$$z_{n+1} = \left(-h x_n - \frac{h^2}{2}\right) y_n + \left(1 - \frac{h^2}{2} x_n^2\right) z_n - \frac{h^2 x_n}{2} \quad \text{--- (2)'}$$

with $h = 0.1$ we obtain

$n=0$

from (1)' & (2)' $\quad y_1 = 0 + \frac{(.1)^2}{2} + 0.1 = .105$

$z_1 = 1$

for $n=1$

$$y_2 = \left[1 - \frac{(0.1)^2 (0.1)^2}{2}\right] 0.105 + \left(0.1 \times 0.1 + \frac{(0.1)^2}{2}\right) \cdot 1 + 0.1$$

$$= .219475$$

$$z_2 = \left(-(0.1)^2 - \frac{(0.1)^2}{2}\right) .105 + \left(1 - \frac{(0.1)^4}{2}\right) - \frac{(0.1)^2}{2} \times 0.1$$

$$= .997875$$

So required values are

$y_1 = 0.105 \quad | \quad y_2 = .219475$

$z_1 = 1.0 \quad | \quad z_2 = 0.997875$

Ex $\quad y' = xz + 1 \qquad y(0) = 0$

$\quad z' = -xy \qquad z(0) = 1$

Apply Euler method.

$$y_{n+1} = y_n + h f(x_n, y_n, z_n)$$

$$z_{n+1} = z_n + h g(x_n, y_n, z_n)$$

$$y_{n+1} = y_n + h(x_n z_n + 1)$$

$$z_{n+1} = z_n + h(-x_n y_n)$$

System of differential equations

$$y_{n+1} = y_n + h x_n z_{n+1}$$
$$z_{n+1} = z_n - h x_n y_n.$$

$$\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & h x_n \\ -h x_n & 1 \end{bmatrix} \begin{bmatrix} y_n \\ z_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Ex find the solution of the system of equations

$$y' = u \qquad y(0) = 1$$
$$u' = -4y - 2u \qquad u(0) = 1$$

by the Runge-Kutta method of 4th order with $h = 0.1$. find $y(0.2)$ and $u(0.2)$

$$y' = f(x, y, u)$$
$$u' = g(x, y, u)$$

Solⁿ

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 3k_3 + k_4]$$
$$u_{n+1} = u_n + \frac{1}{6}[l_1 + 2l_2 + 3l_3 + l_4]$$

$$k_1 = h f(x_n, y_n, u_n)$$
$$k_2 = h f(x_n + \tfrac{1}{2}h, \; y_n + \tfrac{1}{2}k_1, \; u_n + \tfrac{1}{2}l_1)$$
$$k_3 = h f(x_n + \tfrac{1}{2}h, \; y_n + \tfrac{1}{2}k_2, \; u_n + \tfrac{1}{2}l_2)$$
$$k_4 = h f(x_n + h, \; y_n + k_3, \; u_n + l_3)$$

$$u' = g(x, y, u) = -4y - 2u$$

Similarly $l_1 = h g(x_n, y_n, u_n)$

$$l_2 = h g(x_n + \tfrac{1}{2}h, \; y_n + \tfrac{1}{2}k_1, \; u_n + \tfrac{1}{2}l_1)$$
$$l_3 = h g(x_n + \tfrac{1}{2}h, \; y_n + \tfrac{1}{2}k_2, \; u_n + \tfrac{1}{2}l_2)$$
$$l_4 = h g(x_n + h, \; y_n + k_3, \; u_n + l_3)$$

System of differential equations $g(x,y,u) = -4y - 2u$, $f(x,y,u) = u$

$$\boxed{K_1 = h u_n}, \quad \boxed{l_1 = h(-4y_n - 2u_n)}$$

$$K_2 = h f(x_n + h/2,\ y_n + \tfrac{1}{2}K_1,\ u_n + \tfrac{1}{2}l_1)$$
$$= h[u_n + \tfrac{1}{2}l_1]$$
$$= h[u_n + \tfrac{1}{2}(-4hy_n - 2hu_n)]$$

$$\boxed{K_2 = -2h^2 y_n + (h - h^2)u_n}$$

$$l_2 = h g(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}K_1,\ u_n + \tfrac{1}{2}l_1)$$
$$= h[-4(y_n + \tfrac{1}{2}K_1) - 2(u_n + \tfrac{1}{2}l_1)]$$
$$= h[-4y_n - 2K_1 - 2u_n - l_1]$$
$$= h[-4y_n - 2hu_n - 2u_n + h(4y_n + 2u_n)]$$
$$= h[4(h-1)y_n - 2u_n]$$

$$\boxed{l_2 = 4h(h-1)y_n - 2hu_n}$$

$$K_3 = h f(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}K_2,\ u_n + \tfrac{1}{2}l_2)$$
$$= h(u_n + \tfrac{1}{2}l_2)$$

$$= h[u_n + 2h(h-1)y_n - hu_n]$$

$$\boxed{K_3 = (h - h^2)u_n + 2h^2(h-1)y_n}$$

<u>System of differential equations</u>

$g(x,y,u) = -4y - 2u$

$$l_3 = hg\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}k_2,\ u_n + \tfrac{1}{2}l_2\right)$$

$$= h\left[-4\left(y_n + \tfrac{1}{2}k_2\right) - 2\left(u_n + \tfrac{1}{2}l_2\right)\right]$$

$$= h\left[-4y_n - 2k_2 - 2u_n - l_2\right]$$

$$= h\left[-4y_n - \left\{-4h^2 y_n + 2(h - h^2)u_n\right\} - 2u_n - 4h(h-1)y_n + 2hu_n\right]$$

$$\boxed{l_3 = (-4h + 4h^2)y_n + (-2h + 2h^3)u_n}$$

Similarly

$$k_4 = (-4h^2 + 4h^3)y_n + (h - 2h^2 + 2h^4)u_n$$

$$l_4 = (-4h + 8h^2 - 8h^4)y_n + (-2h + 4h^3 - 4h^4)u_n$$

and

$$y_{n+1} = \left(1 - 2h^2 + \tfrac{4}{3}h^3\right)y_n + \left(h - h^2 + \tfrac{1}{3}h^4\right)u_n$$

$$u_{n+1} = \left(-4h + 4h^2 - \tfrac{4}{3}h^4\right)y_n + \left(1 - 2h + \tfrac{4}{3}h^3 - \tfrac{2}{3}h^4\right)u_n$$

$$\begin{bmatrix} y_{n+1} \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} 1 - 2h^2 + 4/3\,h^3 & h - h^2 + 1/3\,h^4 \\ -4h + 4h^2 - 4/3\,h^4 & 1 - 2h + 4/3\,h^3 - 2/3\,h^4 \end{bmatrix} \begin{bmatrix} y_n \\ u_n \end{bmatrix}$$

for $n = 0$, $h = 0.1$, $y_0 = 1$, $u_0 = 1$

$$\begin{bmatrix} y_{n+1} \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} .98133 & 0.09003 \\ -.36013 & 0.80127 \end{bmatrix} \begin{bmatrix} y_n \\ u_n \end{bmatrix}$$

for n=0

$$\begin{bmatrix} y_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} .98133 & .09003 \\ -.36013 & .80127 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$y_1 = 1.07136$, $\quad u_1 = .44114$

Similarly

n=1

$y_2 = 1.09108 \qquad u_2 = -0.03236$

Fourth Order Runge-Kutta method:

Ex Use the classical Runge-kutta method of 4th order to find the numerical solution at $x = 0.8$ for

$$y' = \sqrt{x+y}, \quad y(0.4) = 0.41.$$

Assume the steplength $h = 0.2$

Soln

$$\begin{array}{ccc} .4 & .6 & .8 \\ x_0 & x_1 & x_2 \end{array}$$

$$y_{j+1} = y_j + \tfrac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_j, y_j)$$

$$k_2 = hf(x_j + \tfrac{1}{2}h, \; y_j + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_j + \tfrac{1}{2}h, \; y_j + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_j + h, \; y_j + k_3)$$

for $j = 0$ $\quad h = 0.2$

$x_0 = 0.4, \quad y_0 = 0.41$

$$k_1 = hf(x_0, y_0) = 0.2(0.4 + 0.41)^{1/2} = 0.18$$

$$k_2 = hf(x_0 + h/2, \; y_0 + \tfrac{1}{2}k_1)$$

$$= 0.2\left[\underline{0.4 + 0.1} + 0.41 + \tfrac{1}{2}(0.18)\right]^{1/2}$$

$$k_2 = 0.2$$

$$k_3 = hf(x_0 + h/2, \; y_0 + \tfrac{1}{2}k_2)$$

$$= 0.2\left[0.4 + 0.1 + 0.4 + \tfrac{1}{2}(0.2)\right]^{1/2}$$

$$= 0.2009975$$

4th order R-K method

$K_4 = hf(x_0+h,\ y_0+k_3)$

$= 0.2\,[0.4+0.2 + 0.41 + 0.2009975]^2$

$K_4 = .2200906$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 0.41 + \frac{1}{6}[0.18 + 2\times .2 + 2\times .2009975 + .2200906]$

$= 0.41 + .2003476 = .6103476$

for $j = 1$ $\quad x_1 = 0.6,\ y_1 = .6103476$

$k_1 = 0.2200315$

$k_2 = 0.2383579$

$k_3 = 0.2391256$

$k_4 = 0.2568636$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= .6103476 + 0.2386436$

$= .8489912$

$$\boxed{y(.8) = .8489912}$$