LU Decomposition and Solution of Algebraic Equations

$$AX = B \quad , \quad |A| \neq 0 \qquad \text{— ①}$$

If $A = LU$ where L is lower triangular matrix and U is Upper triangular matrix

Then $$LUX = B \qquad \text{— ②}$$

Take $UX = y$ then — ③

$$Ly = B \qquad \text{— ④}$$

first solve $Ly = B$ get y then solve (3) to get X.

How to get LU decomposition

Take $A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix}$,

Let $A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}}$

1st row

$d = 3$ $\quad e = 5$ $\quad f = 2$

2nd row

$ad = 0$
$\Rightarrow a = 0$
$(\because d = 3)$

$ae + g = 8$
$\Rightarrow g = 8$

$h = 2$

3rd row

$bd = 6$
$d = 3 \Rightarrow b = 2$

$be + cg = 2$
$2 \cdot 5 + c \cdot 8 = 2$
$\Rightarrow c = -1$

$bf + ch + i = 8$
$2 \cdot 2 + (-1) \cdot 2 + i = 8$
$\Rightarrow i = 6$

Thus

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}}$$

Now Solve

$$AX = (8, -7, 26)^T$$

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

$$LUX = B$$

Take $UX = Y$

$$LY = B$$

Now first solve $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

Then solve $Ux = y$

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 4 \\ -1 \\ 1/2 \end{bmatrix}$$

Note Gauss-elimination require $2n^3/3$ operation (multiplications and divisions) while LU decomposition method require only $n^3/3$ (half of G-E).