Picard's Successive approximation :

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

Integrate it from $x_0 \to x$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\, dx$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)\, dx$$

$$y(0) = y_0$$

$k \geq 0$
$$y^{(k+1)}(x) = y_0 + \int_{x_0}^{x} f[x, y^{(k)}(x)]\, dx$$

Eg $\quad y' = x^2 - y \quad , \quad y(0) = 1 \quad$ exact sol^n

$$\underline{y = 2 - 2x + n^2 - e^{-x}}$$

$$y^{(k+1)} = 1 + \int_{0}^{x} f[x, y^{(k)}(x)]\, dx$$

$$y^{(k+1)} = 1 + \int_{0}^{x} [x^2 - y^k_{(x)}]\, dx$$

$$y^{(1)} = 1 + \int_{0}^{x} (x^2 - 1)\, dx$$

$$y^{(1)} = 1 + \frac{x^3}{3} - x \qquad = 1 - x + \frac{x^3}{3}$$

$$y^{(2)} = 1 + \int_{0}^{x} [x^2 - y^{(1)}_{(x)}]\, dx$$

$$= 1 + \int_{0}^{x} \left[x^2 - 1 + x - \frac{x^3}{3}\right]\, dx$$

$$= 1 + \frac{x^3}{3} - x + \frac{x^2}{2} - \frac{x^4}{3 \cdot 4}$$

$$= 1 - x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{12}$$

Picard's Successive Approximation

While Picard's method is of great theoretical importance, the explicit evaluation of integrals in

$$y^{(k+1)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(k)}(x)) \, dx$$

is often impracticable.

For the problem

$$y' = \cos(x+y), \quad y(0) = 1$$

$$y^{(0)}(x) = 1, \qquad y^{(1)}(x) = 1 - \sin 1 + \sin(x+1)$$

and the second iteration would involve the evaluation of the form

$$y^{(2)}(x) = 1 + \int_{0}^{x} \cos[x + 1 - \sin 1 + \sin(x+1)] \, dx.$$

Runge - Kutta Methods:

$$y' = f(t,y), \qquad y(t_0) = y_0, \qquad t \in [t_0; b] \quad ①$$

Now from Taylor series expansion

$$y(t_{n+1}) = y(t_n) + h y'(\xi_n). \qquad ②$$

where $\xi_n = t_n + \theta_n h, \quad 0 < \theta_n < 1$

Euler's method with spacing $h/2$ is

given by

$$y(t_n + h/2) \simeq y(t_n) + \frac{h}{2} f(t, y_n) \qquad \Big| \quad y(t_{n+1}) = y(t_n) + h f(t, y_n)$$

$$③$$

Now from ②

$$y(t_{n+1}) = y(t_n) + h f(\xi_n, y(\xi_n))$$

where
$\xi_n = t_n + \theta_n h$

Take $\theta_n = \frac{1}{2}$

$\xi_n = t_n + h/2$

$$= y(t_n) + h f(t_n + h/2, \; y(t_n + h/2))$$

as $\theta_n = \frac{1}{2}$

$$\simeq y(t_n) + h f(t_n + h/2, \; y(t_n) + h/2 f(t_n, y_n))$$

from Euler's Method

So

$$y_{n+1} \simeq y_n + h f(t_n + h/2, \; y_n + h/2 f(t_n, y_n)) \quad \text{from } ③$$

$$④$$

Alternate method

Now from ②

$$y(t_{n+1}) = y(t_n) + h y'(\xi_n)$$

$$\xi_n = t_n + \theta_n h = t_n + \frac{1}{2} h$$

where $\theta_n = \frac{1}{2}$

$$y(t_{n+1}) = y(t_n) + h y'(t_n + h/2)$$

$$⑤$$

Now $y'(t_n + h/2) = \frac{1}{2} [y'(t_{n+1}) + y'(t_n)]$

$$y(x + \frac{h}{2}) = y(x) + \frac{h}{2} y'(x) + \frac{h^2}{4 \cdot 2!} y''(x)$$

$$y(x + \frac{h}{2}) = y(x) + \frac{h}{2} [y(x+h) - y(x) + O(h^2)] + O(h^2)$$

$$y(x + h/2) = \frac{1}{2} (y(x+h) + y(x)) + O(h^2)$$

then ⑤ can be written as

$$y(t_{n+1}) = y(t_n) + \frac{h}{2} [y'(t_{n+1}) + y'(t_n)]$$

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] \quad \textcircled{2}$$

or $$y(t_{n+1}) = y(t_n) + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right]$$

from Euler's method.

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}\left[f(t_n, y_n) + f(t_n^{th}, y_n + hf(t_n, y_n))\right]$$
$$\textcircled{6}$$

Now from $\textcircled{5}$ & $\textcircled{6}$ we can write

$$\boxed{y_{n+1} = y_n + h \times (\text{average slope})}$$

This is the idea of Runge - Kutta approach. In general we find slop at t_n and several other points, average these slopes, multiply by h and add the result to y_n. $\quad\times$

Derivation of RK Methods

All RK methods will be written in the form

$$\boxed{y_{n+1} = y_n + h F(x_n, y_n, h; f) \qquad n \geq 0} \quad \textcircled{1}$$

At this point it is intuitive that we want
$$F(x, y, h; f) \simeq y'(x) = f(x, Y(x)) \quad \leftarrow$$
$$\underline{Y - \text{exact soln.}}$$

for all values of h.

Define Truncation error for $\textcircled{1}$ by
$$T_{n+1}(Y) = Y(x_{n+1}) - Y(x_n) - h F(x_n, Y(x_n); h; f)$$
$$n \geq 0$$
$$\textcircled{2}$$

and define $\tau_{n+1}(Y)$ as
$$T_{n+1}(Y) = h \tau_{n+1}(Y) \quad\longrightarrow \textcircled{3}$$

then from $\textcircled{2}$ & $\textcircled{3}$
$$h \tau_{n+1}(Y) = Y(x_{n+1}) - Y(x_n) - h F(x_n, Y(x_n), h; f)$$

ar.

$$Y(x_{n+1}) = Y(x_n) + h F(x_n, Y(x_n), h; f) + h \tau_{n+1}(Y) \quad \text{③}$$

$$\underset{h \geq 0}{\underline{\qquad}} \quad \text{④}$$

Now we suppose general form of F as

$$F(x, Y(x), h; f) = r_1 f(x, y) + r_2 f(x + \alpha h, y + \beta h f(x, y)) \quad \text{⑤}$$

Now

$$F(x, Y(x), h; f) = r_1 f(x, y) + r_2 f(x + \alpha h, y + \beta h f(x, y))$$

$$= r_1 f + r_2 \Big[f(x, y) + \alpha h f_x + \beta h f \cdot f_y + \frac{\alpha^2 h^2}{2!} f_{xx}$$

$$+ \frac{(\beta h f)^2}{2!} f_{yy} + \frac{2 \alpha h \cdot \beta h f}{2!} \cdot f_{xy} \Big] + O(h^3)$$

$$= r_1 f + r_2 \Big[f + h(\alpha f_x + \beta f f_y) + h^2 \Big(\frac{\alpha^2}{2} f_{xx} + \frac{\beta^2}{2} f f_{yy}$$

$$+ \alpha \beta f f_{xy} \Big) \Big] + O(h^3)$$

$$\boxed{f(x_0 + h, y + k) = f(x, y) + h f_x + k f_y + \frac{1}{2!} \big[h^2 f_{xx} + 2 h k f_{xy} + k^2 f_{yy} \big] + \bullet \cdots}$$

$$F(x, Y(x), h; f) = r_1 f + r_2 \Big[f + h(\alpha f_x + \beta f_y f) + h^2 \Big(\frac{\alpha^2}{2} f_{xx} + \frac{\beta^2}{2} f f_{yy}$$

$$+ \alpha \beta f f_{xy} \Big) + O(h^3)$$

$$\underset{\longleftarrow}{\qquad} \quad \text{⑥}$$

Now $\quad Y'(x) = f(x, Y(x))$

$$y'' = \frac{d}{dx} f(x, Y(x)) = f_x + f_y Y'(x)$$

$$y'' = f_x + f_y f \qquad \text{⑦}$$

$$y''' = \frac{d}{dx} \big[f_x + f_y f \big]$$

$$= (f_{xx} + f_{xy} Y'(x)) + \big[(f_{yx} f + f_{yy} Y' f) + (f_y f_x + f_y f_y Y') \big]$$

$$= f_{xx} + f_{xy} f + f_{yx} f + f_{yy} f^2 + f_y f_x + f_y^2 f$$

$$y''' = f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y f_x + f_y^2 f \quad \text{⑧}$$

Now from truncation error expression ②

$$T_{n+1}(Y) = Y(x_{n+1}) - Y(x_n) - h F(x_n, Y(x_n); h; f)$$

$$= Y(x_n) + h Y'(x_n) + \frac{h^2}{2} Y''(x_n) + \frac{h^3}{6} Y'''(x_n)$$

$$- Y(x_n) - h F(x_n, Y(x_n), h; f)$$

Now putting expansion of $F(x, Y; h; f)$ and Y', Y''

Y'' from ⑥, ⑦ & ⑧ we get –

$$T_{n+1}(Y) = h Y'(x_n) + \frac{h^2}{2} Y''(x_n) + \frac{h^3}{6} Y'''(x_n)$$

$$- h F(x_n, Y(x_n), h; f)$$

$$= h f + \frac{h^2}{2}(f_x + f_y f) + \frac{h^3}{6}(f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y f_x + f_y^2 f)$$

$$- h[(r_1 f + r_2 f) + h r_2(\alpha f_x + \beta f_y \, f)$$

$$+ h^2 r_2 (\frac{\alpha^2}{2} f_{xx} + \frac{\beta^2}{2} f^2 f_{yy} + \alpha\beta f_{xy} f)$$

$$+ O(h^3)]$$

$$T_{n+1}(Y) = h(1 - r_1 - r_2) f + h^2 \left[(\frac{1}{2} - \alpha r_2) f_x + (\frac{1}{2} - \beta r_2) f f_y\right]$$

$$+ h^3\left[(\frac{1}{6} - \frac{\alpha^2 r_2}{2}) f_{xx} + (\frac{1}{3} - \alpha\beta r_2) f_{xy} f\right.$$

$$+ (\frac{1}{6} - r_2 \frac{\beta^2}{2}) f_{yy} f^2 + \frac{1}{6} f_y f_x + \frac{1}{6} f_y^2 f\right]$$

$$+ O(h^4)$$

here all derivatives are evaluated at (x_n, Y_n).

⑨

Now to make $T_{n+1}(Y)$ of $O(h^3)$ we must have coefficients of h & h^2 to zero. Thus

$$r_1 + r_2 = 1, \qquad \alpha\, r_2 = \frac{1}{2}, \qquad \beta r_2 = \frac{1}{2}$$

⑩

So there are 4 equations.   constants in the three

So we write

$$r_1 = 1 - r_2 \qquad \alpha = \beta = \frac{1}{2r_2} \qquad \longrightarrow \quad (11)$$

with r_2 arbitrary

Take $\underline{r_2 = \frac{1}{2}}$ we get $\qquad \Rightarrow \quad r_1 = \frac{1}{2}, \quad \alpha = \beta = 1$

$$F(x, Y(x), h, f) = r_1 f(x, y) + r_2 f(x + \alpha h, y + \beta h f(x,y))$$
$$= \frac{1}{2} f(x, y) + \frac{1}{2} f(x + h, y + h f(x,y))$$

Then the method will be

$$y_{n+1} = y_n + h F(x_n, y_n, h; f)$$
$$= y_n + \frac{h}{2} \left[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n)) \right]$$

$\underline{\text{and with}}$ $r_2 = 1 \quad \Rightarrow \quad r_1 = 0, \qquad \alpha = \beta = \frac{1}{2}$

$$\Rightarrow \quad F(x, Y(x), h; f) = r_1 f(x, y) + r_2 f(x + \alpha h, y + h \beta f(x,y))$$
$$= \qquad f\left(x + \frac{h}{2}, y + \frac{h}{2} f(x, y)\right)$$

Then the method will be

$$y_{n+1} = y_n + h F(x_n, y_n, h; f)$$
$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

This is called Modified Euler method.

Now we can not make $O(h^3)$ terms to ⑥
zero but we can minimize this so that
the error $T_{n+1}(Y)$ can be minimized. We
will minimize this for arbitrary f i.e, we
choose r_2 such that $O(h^3)$ term is minimum
for arbitrary f.

$$T_{n+1}(Y) = C(f, r_2) h^3 + O(h^4) \quad —— ⑫$$

where

$$C(f, r_2) = \left(\frac{1}{6} - \frac{\alpha^2 r_2}{2}\right) f_{xx} + \left(\frac{1}{3} - \alpha\beta r_2\right) f_{xy} f$$
$$+ \left(\frac{1}{6} - \frac{r_2 \beta^2}{2}\right) f_{yy} f^2 + \frac{1}{6} f_y f_x$$
$$+ \frac{1}{6} f_y^2 f \qquad \text{i}$$

From ⑪ $\alpha = \beta = \frac{1}{2r_2}$ so

$$C(f, r_2) = \left(\frac{1}{6} - \frac{1}{8r_2}\right) f_{xx} + \left(\frac{1}{3} - \frac{1}{4r_2}\right) f_{xy} f$$
$$+ \left(\frac{1}{6} - \frac{1}{8r_2}\right) f_{yy} f^2 + \frac{1}{6} f_y f_x$$
$$+ \frac{1}{6} f_y^2 f.$$

Define
$$C_1(f) = \left(f_{xx}, f_{xy} f, f_{yy} f^2, f_x f_y, f_y^2 f\right)^T$$
$$C_2(r_2) = \left[\left(\frac{1}{6} - \frac{1}{8r_2}\right), \left(\frac{1}{3} - \frac{1}{4r_2}\right), \left(\frac{1}{6} - \frac{1}{8r_2}\right), \frac{1}{6}, \frac{1}{6}\right]$$

$$\langle C_1(f), (C_2(r_2))\rangle$$

$$= C(f, r_2) \quad [\text{by usual definition}]$$

$$= C_2^T C_1$$

$$y^T x = \langle \vec{x}, y \rangle = \sum_{i=1}^{n} x_i y_i$$
$$\text{for real } x, y$$

Cauchy-Schwarz inequality

$$\left|\sum x_i y_i\right| \leq \left(\sum x_i^2\right)^{1/2} \cdot \left(\sum y_i^2\right)^{1/2}$$

Cauchy-Schwarz inequality

$$\left|\sum x_i y_i\right| = |\langle x, y \rangle| \leq \|x\|_2 \|y\|_2$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$$|C(f, r_2)| = |\langle C_1(f), C_2(r_2)\rangle| \leq \left(\sum x_i^2\right)^{1/2} \cdot \left(\sum y_i^2\right)^{1/2}$$

$$\leq \|C_1(f)\|_2 \cdot \|C_2(r_2)\|_2$$

$$= \left[f_{xx}^2 + f_{xy}^2 f^2 + f_{yy}^2 f^4 + f_x^2 f_1^2 + f_y^2 f^2\right]^{1/2}$$

$$\cdot \underbrace{\left[2\left(\frac{1}{6} - \frac{1}{8r_2}\right)^2 + \left(\frac{1}{3} - \frac{1}{4r_2}\right)^2 + \frac{1}{36} + \frac{1}{36}\right]^{1/2}}_{h(r_2)}$$

Now we will minimize $\|C_2(r_2)\|_2$

$$h(r_2) = \|C_2(r_2)\|_2 = \left[2\left(\frac{1}{6} - \frac{1}{8r_2}\right)^2 + \left(\frac{1}{3} - \frac{1}{4r_2}\right)^2 + \frac{1}{18}\right]^{1/2}$$

$$h'(r_2) = \frac{1}{2\sqrt{\left[2\left(\frac{1}{6} - \frac{1}{8r_2}\right)^2 + \left(\frac{1}{3} - \frac{1}{4r_2}\right)^2 + \frac{1}{18}\right]}} \left[\frac{1}{2} \frac{\frac{1}{6} - \frac{1}{8r_2}}{r_2^2} + \frac{1}{2} \frac{\frac{1}{3} - \frac{1}{4r_2}}{r_2^2}\right]$$

$$h'(r_2) = 0 \quad \Rightarrow \quad r_2 = \frac{63}{84} = \frac{3}{4}$$

for $r_2 = 3/4$ $\quad h(r_2) = \frac{1}{\sqrt{18}}$ $\quad$ and resulting

2nd order method will be given by

$$r_1 = 1 - r_2 = 1 - 3/4 = \frac{1}{4}, \quad \alpha = \beta = \frac{1}{2 \cdot 3} A^2 = \frac{2}{3}$$

$$\therefore \; Y_{n+1} = Y_n + h f(x_n, y_n, h; f) \qquad \textcircled{8}$$

$$f(x, y, h, f) = r_1 f(x, y) + r_2 f(x + \alpha h, \, y + \beta h f(x, y))$$

$$\boxed{y_{n+1} = y_n + \frac{h}{4} f(x_n, y_n) + \frac{3}{4} h f\left(x_n + \frac{2}{3} h, \, y_n + \frac{2}{3} h f(x_n, y_n)\right)}$$

This is an optimal method of 2nd order $\textcircled{13}$
(RK) in the sense of $C(r_2, f)$ is minimized
for r_2:

Take $\quad K_1 = h f(x_n, y_n)$

$$K_2 = h f\left(x_n + \frac{2}{3} h, \; y_n + \frac{2}{3} K_1\right)$$

Then $\quad \boxed{y_{n+1} = y_n + \frac{1}{4}(K_1 + 3 K_2)}$

$\textcircled{1}$ R-K method of 2nd order $\quad r_2 = \frac{1}{2}, \; r_1 = \frac{1}{2}, \alpha = \beta = 1$

$$y_{n+1} = y_n + \frac{1}{2}\left[h f(x_n, y_n) + h f(x_n + h, \, y_n + h f(x_n, y_n))\right]$$

$$K_1 = h f(x_n, y_n)$$

$$K_2 = h f(x_n + h, \, y_n + K_1)$$

$$\boxed{y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)}$$

$\textcircled{2}$ with $r_2 = 1, \; r_2 = 0, \; \alpha = \beta = \frac{1}{2}$

$$y_{n+1} = y_n + h f\left(x_n + \frac{1}{2}, \, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$$K_1 = h f(x_n, y_n), \quad K_2 = h f\left(x_n + \frac{h}{2}, \, y_n + \frac{1}{2} K_1\right)$$

$$\boxed{y_{n+1} = y_n + K_2} \; - \text{Modified Euler method} -$$