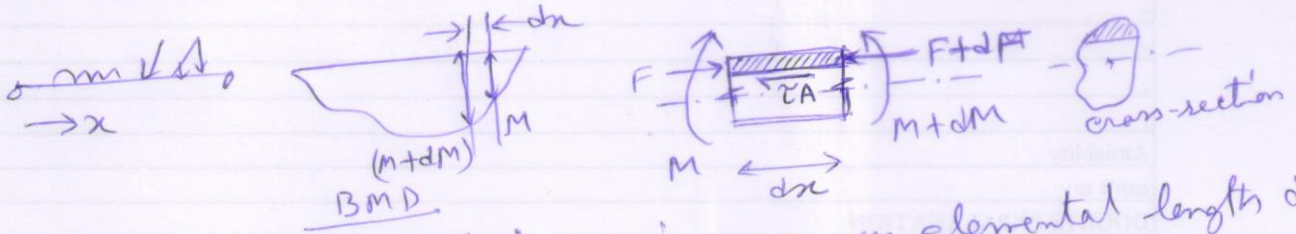
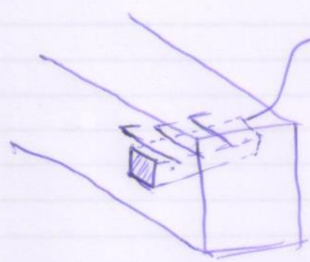
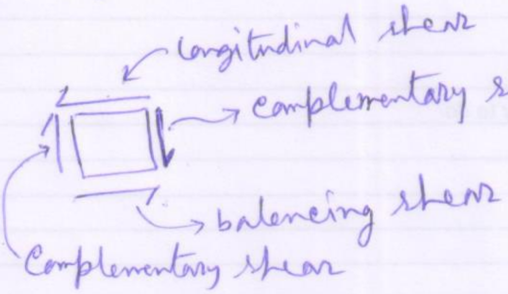
Shear

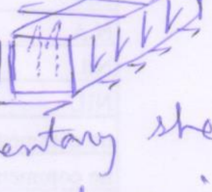
When a beam bends, it generates normal stress and shear stress in the section. Consider the beam in the example below.



Since the bending moment is varying over an elemental length dx, the normal stress also varies (Normal stress due to bending). This difference in normal stress require shear stress (τA) acting along the x-direction (longitudinal direction) for equilibrium of the upper part (~~shaded~~ hatched in fig.). This is the longitudinal shear stress.

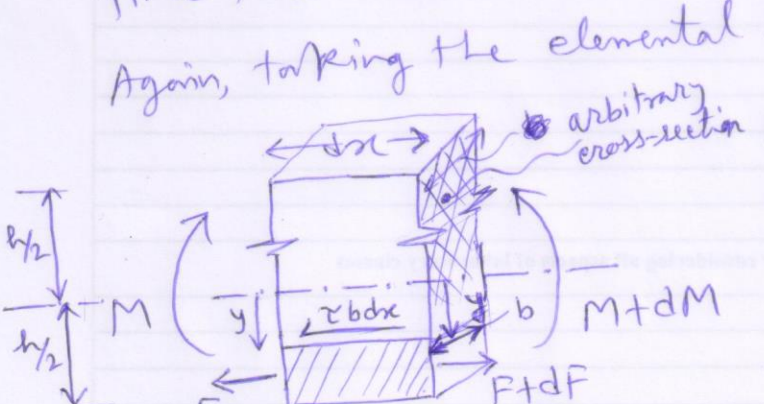




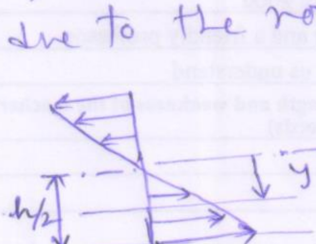


If we take a cross-section, we see the complementary shear.
For equilibrium, longitudinal shear = complementary shear, i.e., finding longitudinal shear is the general approach to find the complementary shear/transverse shear.

Again, taking the elemental length "dx", we get

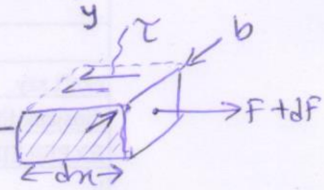


F, $F+dF$ are the resultant force due to the normal stress.



$$\therefore F = \int_y^{h/2} \sigma \, dA$$

$$\therefore F = \int_y^{h/2} \frac{M}{I} y \, dA, \quad F + dF = \int_y^{h/2} \frac{(M+dM)}{I} y \, dA$$



b = width at location where shear to be found

$h/2$ = location of extreme fibre.

For the shear force, we can write,

$$F + \tau b\,dx = (F + dF)$$

$$\text{or} \quad \tau b\,dx = \int_y^{h/2} \frac{M+dM}{I}\, y\, dA - \int_y^{h/2} \frac{M}{I}\, y\, dA$$

M, $M+dM$ are not function of A, so can be taken out.

$$\therefore \tau b\,dx = \frac{dM}{I}\int_y^{h/2} y\,dA \quad \text{or} \quad \tau b = \frac{dM}{dx}\frac{1}{I}\int_y^{h/2} y\,dA$$

$$\text{or} \quad \tau = \frac{VQ}{Ib}; \quad \text{where, } \frac{dM}{dx} = V = \text{Vertical shear force in the section.}$$

$$Q = \int_y^{h/2} y\,dA$$

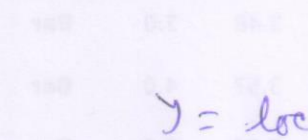
y = location where we want to find shear

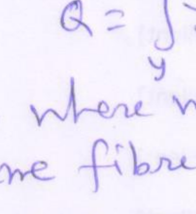
$h/2$ = extreme fibre

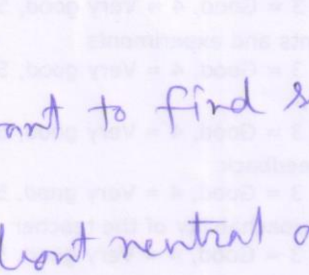
$\therefore Q$ = 1st moment of area about neutral axis from the location of interest to the extreme end.

Note that we assume prismatic but arbitrary cross-section.

Instead of an arbitrary cross-section, let us now consider a rectangular cross-section.







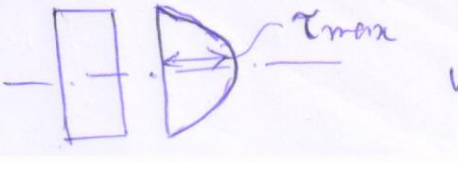
rectangular cross-section

$$\tau = \frac{VQ}{Ib}; \quad \text{for a certain location, } V, I, b \text{ are constant}$$

$$Q = \int_y^{h/2} y\,dA = \int_y^{h/2} y\, b\,dy = b\left[\frac{y^2}{2}\right]_y^{h/2} = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$\therefore \tau = \frac{V}{Ib}\cdot\frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \rightarrow \text{parabolic variation}$$

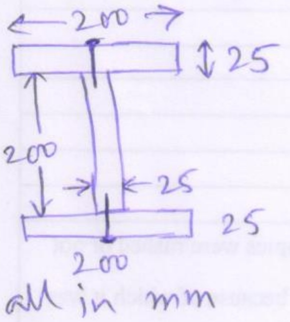
$$\tau_{max} = \tau(\text{at } y=0) = \frac{V}{2I}\left(\frac{h^2}{4}\right)$$



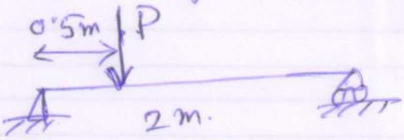
$$\text{using } I = \frac{bh^3}{12}, \quad \tau_{max} = \frac{3}{2}\frac{V}{bh} = 1.5\,\tau_{avg}$$

Note; V = shear force (vertical) in the section to be found from the shear force diagrams. $\frac{V}{bh} = \tau_{avg}$ = average shear stress.

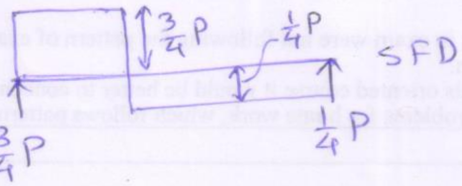
Example: An I-beam is made out of three rectangular wooden planks by driving nails. It will support a point load as shown. Find the required spacing of the nails. Given, allowable normal stress in bending = $7 \, N/mm^2$, allowable shear force in nails = 3 KN

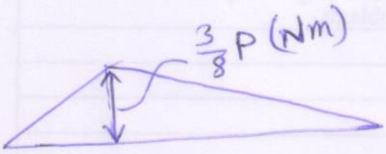


all in mm



First find the SFD of the problem and also BMD.





BMD

N.A. will pass through the middle of the section. So, the

$$\text{centroidal MOI} = \frac{25 \times 200^3}{12} + 2\times\frac{200 \times 25^3}{12} + \left[(200 \times 25) \times 112.5^2\right] \times 2$$

$$= 143.75 \times 10^6 \; mm^4$$

Given that allowable normal stress in bending = $7 \, N/mm^2$

$\times 1000$ (since moment was in Nm)

$$\therefore \frac{M}{I} y = \sigma \quad \text{or} \quad \frac{\frac{3}{8} P \times 125}{143.75 \times 10^6} = 7 \, N/mm^2$$

$$\therefore P = \cancel{35.6 \times 10} \; 21466 \; N.$$

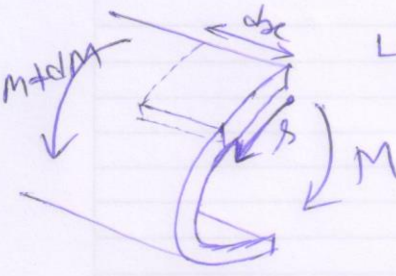
Due to bending, there will be longitudinal shear force. This force has to be carried by the nails. If "s" be the required spacing, $\therefore \tau b s = 3000 N$. or $s = \frac{3000}{25\tau}$; here, $b = 25$ mm

$$\text{or } s = \frac{3000 . I . b}{25 . VQ} = \frac{3000 \times 143.75 \times 10^6}{\left(\frac{3}{4} \times \cancel{35.6 \times 10^6} \; 21466\right) \times (200 \times 25 \times 112.5)} = 47.62 \text{ mm}$$

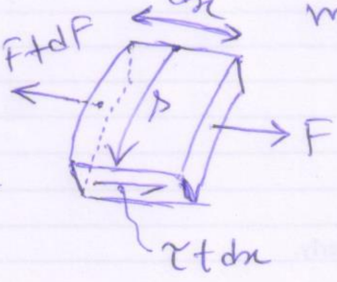
$\therefore$ Drive nails @ 45 mm. Note that vertical shear force is taken as $\frac{3}{4}$P.

In case of ships, we deal with thin walled section, i.e., thickness is much smaller compared to other dimensions of the ship.

Shear in thin walled section can be obtained in similar way.



Let us assume a cantilever beam



we measure "s" from the free end.

$\therefore F + dF = F + \tau t\,dx$

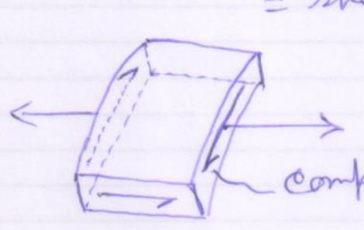
Let, vertical shear force $= \frac{dM}{dx} = V$

or $\int \frac{(M+dM)}{I} y\,dA = \int \frac{M}{I} y\,dA + \tau t\,dx$

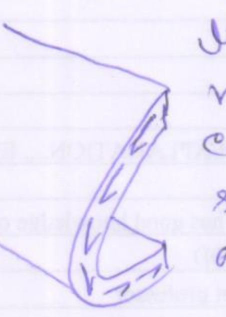
or $\frac{dM}{I}\int y\,dA = \tau t\,dx$ or $\tau t = \frac{dM}{dx}\cdot\frac{1}{I}\int y\,dA$

or τt = commonly known as shear flow $= \frac{VQ}{I}$

= shear force per unit length



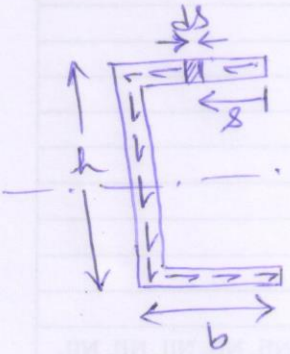
complementary shear



It can be worked out that complementary shear is distributed as show here.

This is the shear flow (τt) in thin walled section.

Now let us consider a channel section:

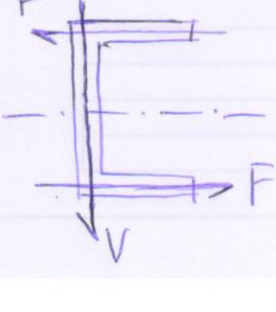


we measure "s" from the free end and find the shear flow there.

$\therefore \tau t = \frac{VQ}{I} = \frac{V}{I}\int y\,dA = \frac{V}{I}\int_0^s \frac{h}{2}\,t\,ds = \frac{Vhts}{2I}$

$\therefore$ total force acting along the flange

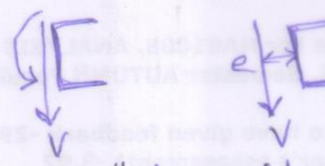
$$F = \int_0^b \tau t\,ds = \int_0^b \frac{Vhts}{2I}\,ds = \frac{Vht}{2I}\left[\frac{s^2}{2}\right]_0^b$$



$$= \frac{Vhtb^2}{4I}$$

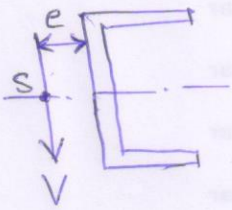
Work out: show that total force in web = V

∴ we see that the vertical bending is causing not only the vertical force (total) in web = V, but a couple that tries to twist the section = Fh



This is equivalent to apply the vertical force V at a distance "e" from the web, such that

$$\therefore\ Ve = Fh = \frac{Vh^2tb^2}{4I} \qquad \therefore\ e = \frac{b^2h^2t}{4I}$$



The point on the horizontal axis at a distance "e" is called shear centre (S). Since the horizontal axis is an axis of symmetry, "S" will lie on it.

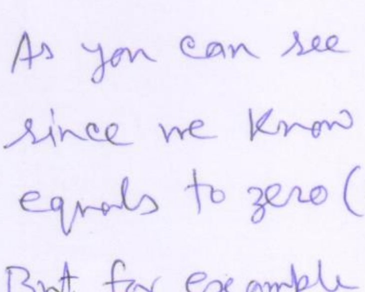
Note that b, h, t, I all are sectional properties. Hence shear centre is also a property of the section.

Two important facts about shear centre – ① If force/vertical force is applied along shear centre, there will be only bending and no torsion. (ii) Under a twisting moment, a section twists about its shear centre.

Try to understand these two points. Hint: for ① consider FBD of the whole section considering reaction also. for ② twisting/rotation takes place about that point where moment = 0.

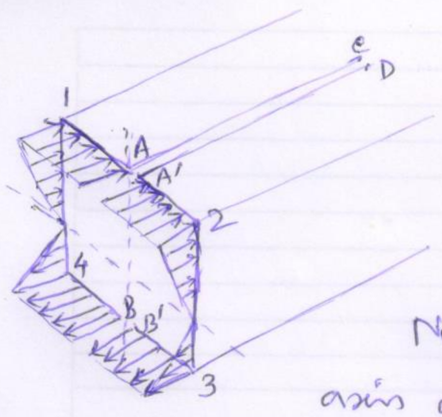
As you can see in above that we start from a free end since we know that shear stress at the free end must be equals to zero (0).

But, for example in case of a ship which is rectangular type closed section, there is no free end. Such sections are taken as indeterminate problem. Typically there will be a vertical axis of symmetry.



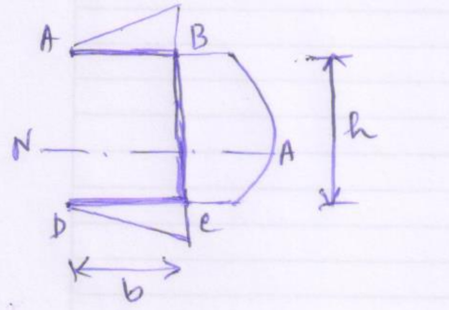
Let us consider the distribution of normal stress due to bending.

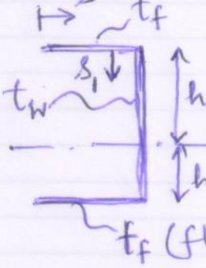
We split the section along vertical line AB:



You can understand that the resultant longitudinal force in A14B and A'23B' are zero, hence there is no action along AC or A'D for relative movement. Hence shear stress at A, A', B, B' must be zero.

Note that it is possible only when AB is an axis of symmetry. So, now the shear stress can be obtained by taking one half of the section and this is similar to a channel section.





shear flow along AB:

$$\tau t = \frac{V}{I}\int h_1 t_f\, ds = \frac{V h_1 t_f\, s}{I}$$

$\therefore$ τt varies linearly along AB.

similarly, along DC, τt varies linearly: $\tau t = \dfrac{V h_2 t_f\, s}{I}$

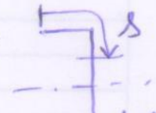
Along BC, $\tau t = \dfrac{VQ}{I}$, $\quad Q = \displaystyle\int_0^{s} y\, dA = \underbrace{(b t_f h_1)}_{\text{this part is for flange}} + \int_0^{s_1} (h_1 - s_1)\, t_w\, ds_1$

$$= b t_f h_1 + h_1 t_w s_1 - t_w \frac{s_1^2}{2}$$

$$\therefore \tau t = \frac{V}{I}\left(b t_f h_1 + h_1 t_w s_1 - t_w \frac{s_1^2}{2}\right)$$

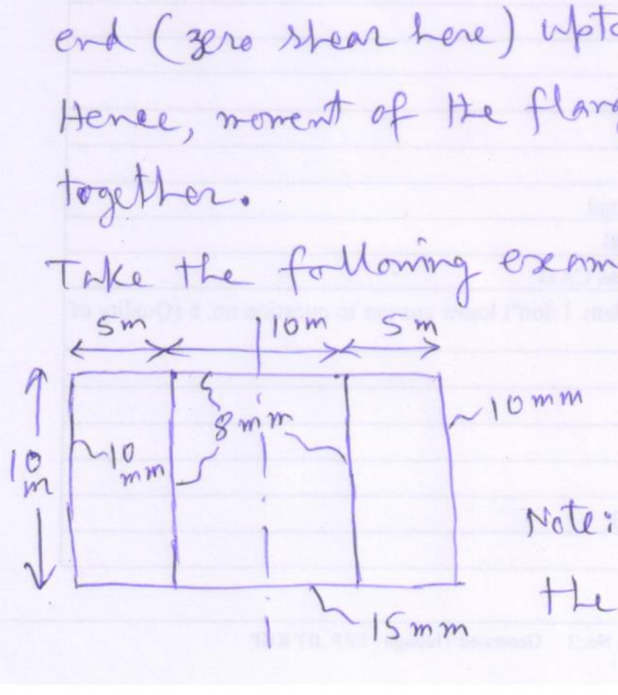
this τt is maximum at $s_1 = h_1$, i.e., at the neutral axis.

Note that $Q = \displaystyle\int_0^{s} y\, dA \rightarrow$ is the 1st moment of area from free end (zero shear here) upto the point of interest.



Hence, moment of the flange (constant) and web are added together.

Take the following example:



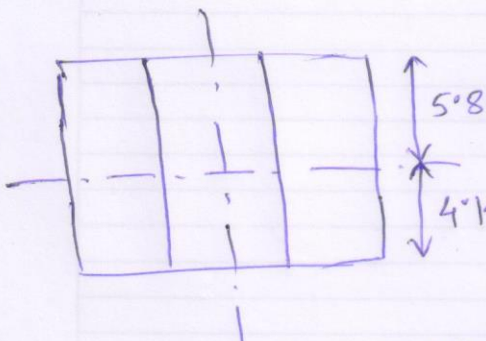
The cross-section of a ship is shown. It carries vertical shear force of 15 MN. Draw the shear flow diagram.

Note: there are 2 longitudinal bulkheads. The cross-section has a vertical axis of symmetry.

total area in the cross-section $= (20\times0{\cdot}008) + 2(10\times0{\cdot}01) + 2(10\times0{\cdot}08) + 20\times0{\cdot}015 = 0{\cdot}82\ m^2$

location of NA from bottom plate $= \dfrac{(20\times0{\cdot}008\times10) + (10\times0{\cdot}008\times5)\times2 + (10\times0{\cdot}01\times5)\times2}{0{\cdot}82} = 4{\cdot}14\ m$

moment of bottom plate ignored.



$\therefore$ Moment of inertia $= \Big[(20\times0{\cdot}008\times5{\cdot}86^2) + (20\times0{\cdot}015\times4{\cdot}14^2) + \left(0{\cdot}01\times\frac{10^3}{12} + 10\times0{\cdot}01\times0{\cdot}86^2\right)\times2 + \left(0{\cdot}008\times\frac{10^3}{12} + 10\times0{\cdot}008\times0{\cdot}86^2\right)\times2\Big]$

MOI of top and bottom plate about their own axis is ignored

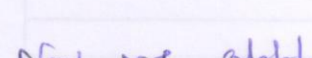
$= 13{\cdot}9\ m^4$

$\tau t = \dfrac{VQ}{I} = \dfrac{15\,MN}{13{\cdot}9\ m^4}\cdot Q = 1{\cdot}079\,Q\ ;\quad Q = \int_0^s y\,dA$

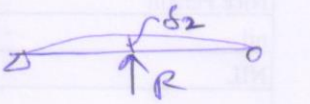
Since the section has a vertical axis of symmetry, we can assume that point A, F are the locations of zero (0) shear force, and we take the right half of the section for calculation. The other half is just a mirror image of the right half.

Also, note that there is a closed loop BCDE. This is an indeterminate problem as we do not know how the shear will be distributed just by using $\tau t = \frac{VQ}{I}$.

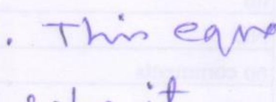
Earlier we have solved statically indeterminate problem by using the method of consistent deformation.



first we remove B and the problem become a determinate problem.

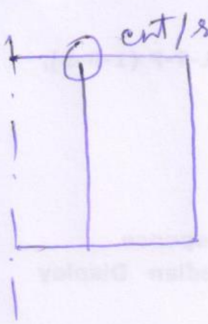


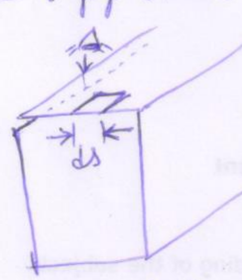
Now we apply unknown force R at B



From given condition, we write $\delta_1 - \delta_2 = 0$. This equation gives us the value of R, and thus we solve it.

So, in case of the closed loop, we can take similar approach. We assume that there is a slit / cut in the loop, so $\tau = 0$ at free end. Then estimate the value of gap created by the applied loads.





total gap created by this $= \oint \gamma\, ds = \oint \frac{\tau}{G} ds = \frac{1}{G}\oint \tau\, ds = \frac{V}{IG}\oint \frac{Q}{t} ds$

Now we apply an unknown twisting moment in the section to close the gap. Twisting moment in a section causes shear stress / shear flow (τt). It can be shown that in an arbitrary section (thin walled), τt due to torque remains constant. (This is shown little later).

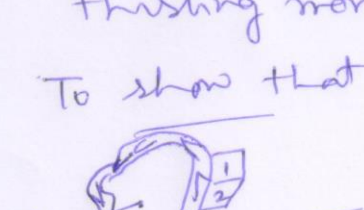
$\therefore$ gap created by twisting moment $= \frac{1}{G}\oint \tau\, ds = \frac{1}{G}\oint \frac{\tau t}{t} ds$

$= \frac{(\tau t)}{G}\oint \frac{ds}{t}$ ($\because \tau t =$ constant here)

Then we use $\underbrace{\frac{1}{G}\oint \tau\, ds}_{= \frac{V}{IG}\oint \frac{Q}{t} ds} = \frac{(\tau t)}{G}\oint \frac{ds}{t}$ to get (τt) due to the twisting moment.

To show that (τt) is constant for a torque, take

let us consider 2 adjecent element

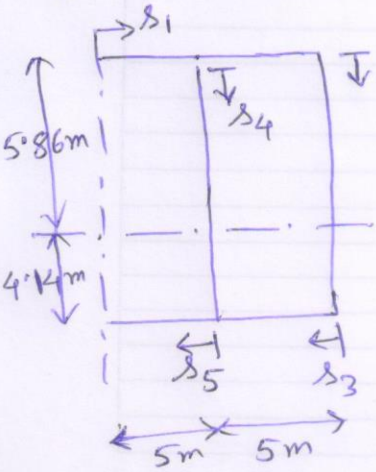


force "f" is common to both

For equilibrium of these 2 elements, the force marked by "f" acting in between 1 and 2 must be same. Hence $\tau_1 t_1 = \tau_2 t_2$ for the equilibrium of each elements.

Thus τt remains constant along the section encountering a twisting moment T.

Going back to the problem,



Due to the cut assumed, shear stress = 0 at the top of the vertical plate.

since, $\tau t = 1.079\, Q$; $Q = \int y\, dA$, let us evaluate Q along the lines.

$$Q_1 = \int_0^{s_1} 5.86 \times 0.008\, ds_1 = 0.047\, s_1$$

$$Q_1(s_1 = 10) = 0.47\, m^3, \quad \tau t(s_1 = 10) = 0.507\ MN/m$$

$$Q_2 = 0.47 + \int_0^{s_2} (5.86 - s_2) \times 0.01 \times ds_2 = 0.47 + 0.0586\, s_2 - 0.005\, s_2^2$$

$$Q_2(s_2 = 5.86) = 0.6417\, m^3, \quad Q_2(s_2 = 10) = 0.556\, m^3$$

$$\therefore \tau t(s_2 = 5.86) = 0.692\ MN/m, \quad \tau t(s_2 = 10) = 0.6\ MN/m$$

$$Q_3 = 0.556 + \int_0^{s_3} (-4.14) \times 0.015\, ds_3 = 0.556 - 0.0621\, s_3$$

$$Q_3(s_3 = 5) = 0.245, \quad \tau t(s_3 = 5) = 0.264\ MN/m.$$

$$Q_4 = 0 + \int_0^{s_4} (5.86 - s_4) \times 0.008 \times ds_4 = 0.047\, s_4 - 0.004\, s_4^2$$

$$Q_4(s_4 = 5.86) = 0.138\, m^3, \quad Q_4(s_4 = 10) = 0.07\, m^3$$

$$\tau t(s_4 = 5.86) = 0.149\ MN/m, \quad \tau t(s_4 = 10) = 0.075\ MN/m$$

$$Q_5 = (0.07 + 0.245) + \int_0^{s_5} (-4.14) \times 0.015 \times ds_5 = 0.315 - 0.0621\, s_5$$

$Q_5(s_5 = 5) = 0.0045 \rightarrow$ this should be actually $= 0$. Due to approximations made before, some small values appeared, we can ignore it.

Now since we have to use the expression $\frac{V}{IG} \oint \frac{Q}{t} ds = \frac{\tau t}{G} \oint \frac{ds}{t}$

Let us find $\oint \frac{Q}{t} ds$ for the loop.

evaluation of $\frac{V}{IG}\oint \frac{Q}{t} ds$

$$\therefore \int \frac{Q_1}{t_1} ds_1 = \int_5^{10} \frac{0.047 s_1}{0.008} ds_1 = \left[5.875 \frac{s_1^2}{2}\right]_5^{10} = 220.31$$

$$\int \frac{Q_2}{t_2} ds_2 = \int_0^{10} \frac{0.47 + 0.0586 s_2 - 0.005 s_2^2}{0.01} ds_2 = 596.3$$

$$\int \frac{Q_3}{t_3} ds_3 = \int_0^{5} \frac{0.556 - 0.0621 s_3}{0.015} ds_3 = 133.58$$

$$\int \frac{Q_4}{t_4} ds_4 = \int_0^{10} \frac{0.047 s_4 - 0.004 s_4^2}{0.008} ds_4 = 127.1$$

$$\therefore \oint \frac{Q}{t} ds = (220.3 + 596.3 + 133.58 - 127.1)$$

-ve sign since Q_4 is estimated in opposite direction of the loop.

$$= 823.2$$

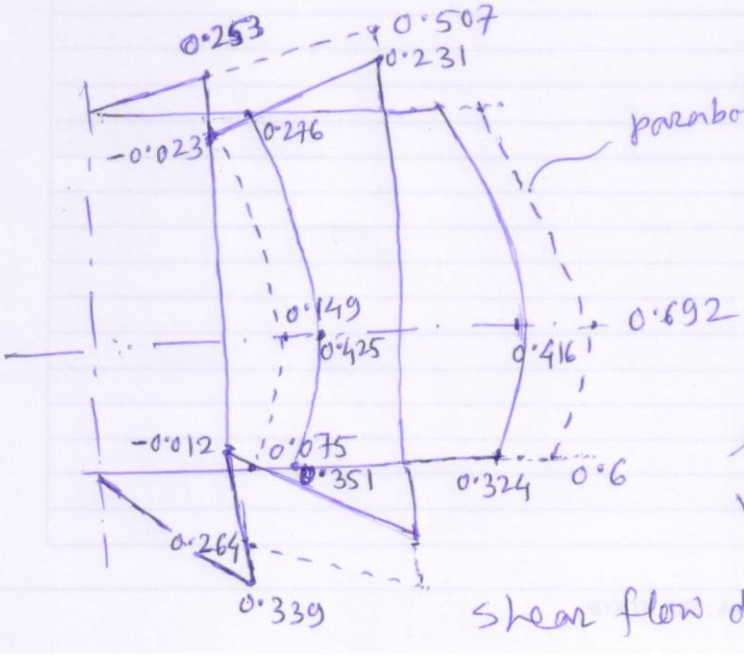
$$\therefore \text{gap} = \frac{V}{IG}\oint \frac{Q}{t} ds = \frac{1.079}{G} \times 823.2$$

to close the gap, apply torque in opposite direction

$$\therefore \text{gap due to torque} = \frac{(\tau t)}{G}\oint \frac{ds}{t} = \frac{(\tau t)}{G}\left[\frac{5}{0.008} + \frac{10}{0.008} + \frac{5}{0.015} + \frac{10}{0.01}\right]$$

$$= \frac{(\tau t)}{G} \cdot 3208.3$$

$\therefore$ we get, $(\tau t) = 0.276$ MN/m



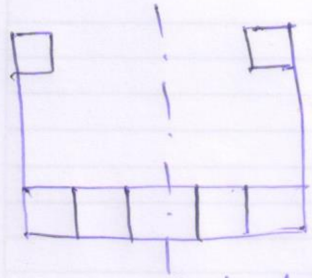
----- shear flow after introducing a cut

——— shear flow including shear due to torque. This is the actual shear flow.

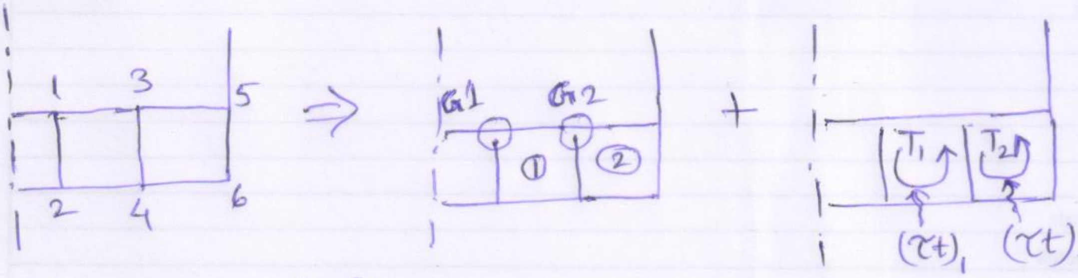
Shear flow diagrams

Shear force at any point can be obtained from the shear flow value by dividing with thickness.

Sometime we also deal with multi-cell sections.



Shear stress in multi-cell section are also indeterminate problems. The solution technique is similar to what we have already seen. So here, the first step is to convert the problem into determinate problem by assuming cut in the loop.



Later in Torsion you will see that $\tau t = \frac{T}{2A}$ for a closed section

we have seen before, gap created in a section with cut

is $= \cancel{\oint \frac{\tau}{G} ds} = \oint \gamma\, ds = \oint \frac{\tau}{G} ds$

Now, $\tau = \frac{VQ}{It}$. $\therefore$ gap $= \oint \frac{VQ}{ItG} ds = \frac{V}{IG} \oint \frac{Q}{t} ds$ = gap due to applied shear.

Now to close the gap we apply torque (or shear flow $= \tau t$) in the section. Since τt due to torque in a thin walled section is constant, we have, gap $= \oint \frac{\tau}{G} ds = \frac{(\tau t)}{G} \cdot \oint \frac{ds}{t}$

$\therefore$ here, for cell ①, we can write for gap G1

~~G1~~ $$\frac{V}{IG} \oint_{\text{cell1}} \frac{Q}{t} ds + \frac{(\tau t)_1}{G} \oint_{\text{cell1}} \frac{ds}{t} + \frac{(\tau t)_2}{G} \oint_{\text{cell2}} \frac{ds}{t} = 0 \quad \text{— (I)}$$

only for side 34 of cell 2 for G1

Note that G1 will be affect by not only $(\tau t)_1$, but also $(\tau t)_2$ since $(\tau t)_2$ is acting in their common side.

similarly for G2, we can write:

$$\frac{V}{IG} \oint_{\text{cell 2}} \frac{Q}{t} ds + \frac{(\tau t)_1}{G} \oint_{\text{cell 1}} \frac{ds}{t} + \frac{(\tau t)_2}{G} \oint_{\text{cell 2}} \frac{ds}{2} \quad \text{— (II)}$$

only side 34 will be considered since this will affect G2

$\therefore$ Here, $(\tau t)_1$ and $(\tau t)_2$ are the only unknowns which can be found by solving eqn (I) and (II). All others are known. Rest of the job is same as before.