been achieved over the airfoil, with the starting vortex somewhere downstream. The fluid elements that initially made up curve C_1 in Figure 4.21*a* have moved downstream and now make up curve C_2, which is the complete circuit *abcda* shown in Figure 4.21*b*. Thus, from Kelvin's theorem, the circulation Γ_2 around curve C_2 (which encloses both the airfoil and the starting vortex) is the same as that around curve C_1, namely, zero. $\Gamma_2 = \Gamma_1 = 0$. Now let us subdivide C_2 into two loops by making the cut *bd*, thus forming curves C_3 (circuit *bcdb*) and C_4 (circuit *abda*). Curve C_3 encloses the starting vortex, and curve C_4 encloses the airfoil. The circulation Γ_3 around curve C_3 is due to the starting vortex; by inspecting Figure 4.21*b*, we see that Γ_3 is in the counterclockwise direction (i.e., a negative value). The circulation around curve C_4 enclosing the airfoil is Γ_4. Since the cut *bd* is common to both C_3 and C_4, the sum of the circulations around C_3 and C_4 is simply equal to the circulation around C_2:

$$\Gamma_3 + \Gamma_4 = \Gamma_2$$

However, we have already established that $\Gamma_2 = 0$. Hence,

$$\Gamma_4 = -\Gamma_3$$

that is, the circulation around the airfoil is equal and opposite to the circulation around the starting vortex.

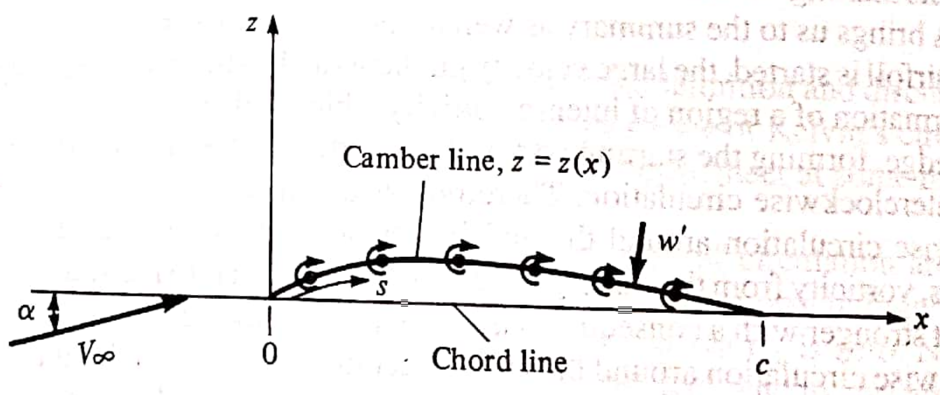
This brings us to the summary as well as the crux of this section. As the flow over an airfoil is started, the large velocity gradients at the sharp trailing edge result in the formation of a region of intense vorticity which rolls up downstream of the trailing edge, forming the starting vortex. This starting vortex has associated with it a counterclockwise circulation. Therefore, as an equal-and-opposite reaction, a clockwise circulation around the airfoil is generated. As the starting process continues, vorticity from the trailing edge is constantly fed into the starting vortex, making it stronger with a consequent larger counterclockwise circulation. In turn, the clockwise circulation around the airfoil becomes stronger, making the flow at the trailing edge more closely approach the Kutta condition, thus weakening the vorticity shed from the trailing edge. Finally, the starting vortex builds up to just the right strength such that the equal-and-opposite clockwise circulation around the airfoil leads to smooth flow from the trailing edge (the Kutta condition is exactly satisfied). When this happens, the vorticity shed from the trailing edge becomes zero, the starting vortex no longer grows in strength, and a steady circulation exists around the airfoil.

4.7 CLASSICAL THIN AIRFOIL THEORY: THE SYMMETRIC AIRFOIL

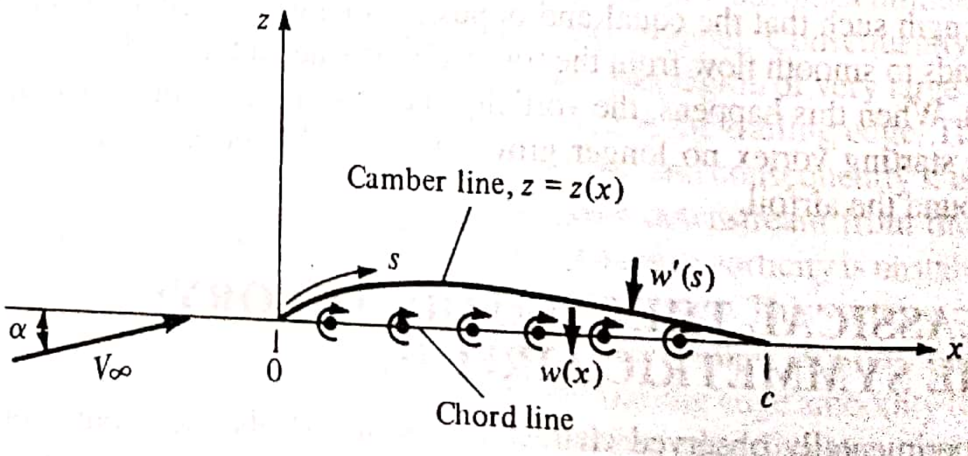
Some experimentally observed characteristics of airfoils and a philosophy for the theoretical prediction of these characteristics have been discussed in the preceding sections. Referring to our chapter road map in Figure 4.7, we have now completed the central branch. In this section, we move to the right-hand branch of Figure 4.7, namely, a quantitative development of thin airfoil theory. The basic equations necessary for the calculation of airfoil lift and moments are established in this section, with an application to symmetric airfoils. The case of cambered airfoils will be treated in Section 4.8.

For the time being, we deal with *thin* airfoils; for such a case, the airfoil can be simulated by a vortex sheet placed along the camber line, as discussed in Section 4.4. Our purpose is to calculate the variation of $\gamma(s)$ such that the camber line becomes a streamline of the flow and such that the Kutta condition is satisfied at the trailing edge; that is, $\gamma(\text{TE}) = 0$ [see Equation (4.10)]. Once we have found the particular $\gamma(s)$ that satisfies these conditions, then the total circulation Γ around the airfoil is found by integrating $\gamma(s)$ from the leading edge to the trailing edge. In turn, the lift is calculated from Γ via the Kutta-Joukowski theorem.

Consider a vortex sheet placed on the camber line of an airfoil, as sketched in Figure 4.22*a*. The freestream velocity is V_∞, and the airfoil is at the angle of attack α. The x axis is oriented along the chord line, and the z axis is perpendicular to the chord. The distance measured along the camber line is denoted by s. The shape of the camber line is given by $z = z(x)$. The chord length is c. In Figure 4.22*a*, w' is the component of velocity normal to the camber line induced by the vortex sheet; $w' = w'(s)$. For a thin airfoil, we rationalized in Section 4.4 that the distribution



(*a*) Vortex sheet on the camber line



(*b*) Vortex sheet on the chord line

Figure 4.22 Placement of the vortex sheet for thin airfoil analysis.

of a vortex sheet over the surface of the airfoil, when viewed from a distance, looks almost the same as a vortex sheet placed on the camber line. Let us stand back once again and view Figure 4.22*a* from a distance. If the airfoil is thin, the camber line is close to the chord line, and viewed from a distance, the vortex sheet appears to fall approximately on the chord line. Therefore, once again, let us reorient our thinking and place the vortex sheet on the chord line, as sketched in Figure 4.22*b*. Here, $\gamma = \gamma(x)$. We still wish the camber line to be a streamline of the flow, and $\gamma = \gamma(x)$ is calculated to satisfy this condition as well as the Kutta condition $\gamma(c) = 0$. That is, the strength of the vortex sheet on the chord line is determined such that the camber line (not the chord line) is a streamline.

For the camber line to be a streamline, the component of velocity normal to the camber line must be zero at all points along the camber line. The velocity at any point in the flow is the sum of the uniform freestream velocity and the velocity induced by the vortex sheet. Let $V_{\infty,n}$ be the component of the freestream velocity normal to the camber line. Thus, for the camber line to be a streamline,

$$V_{\infty,n} + w'(s) = 0 \tag{4.12}$$

at every point along the camber line.

An expression for $V_{\infty,n}$ in Equation (4.12) is obtained by the inspection of Figure 4.23. At any point P on the camber line, where the slope of the camber line is dz/dx, the geometry of Figure 4.23 yields

$$V_{\infty,n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right] \tag{4.13}$$

For a thin airfoil at small angle of attack, both α and $\tan^{-1}(-dz/dx)$ are small values. Using the approximation that $\sin\theta \approx \tan\theta \approx \theta$ for small θ, where θ is in

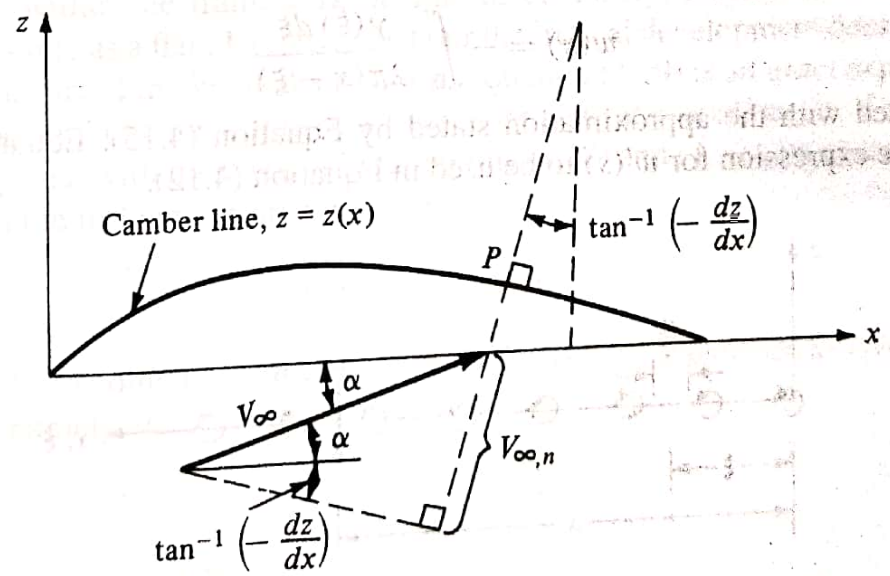


Figure 4.23 Determination of the component of freestream velocity normal to the camber line.

radians, Equation (4.13) reduces to

$$V_{\infty,n} = V_\infty \left(\alpha - \frac{dz}{dx} \right) \tag{4.14}$$

Equation (4.14) gives the expression for $V_{\infty,n}$ to be used in Equation (4.12). Keep in mind that, in Equation (4.14), α is in radians.

Returning to Equation (4.12), let us develop an expression for $w'(s)$ in terms of the strength of the vortex sheet. Refer again to Figure 4.22*b*. Here, the vortex sheet is along the chord line, and $w'(s)$ is the component of velocity normal to the camber line induced by the vortex sheet. Let $w(x)$ denote the component of velocity normal to the *chord line* induced by the vortex sheet, as also shown in Figure 4.22*b*. If the airfoil is thin, the camber line is close to the chord line, and it is consistent with thin airfoil theory to make the approximation that

$$w'(s) \approx w(x) \tag{4.15}$$

An expression for $w(x)$ in terms of the strength of the vortex sheet is easily obtained from Equation (4.1), as follows. Consider Figure 4.24, which shows the vortex sheet along the chord line. We wish to calculate the value of $w(x)$ at the location x. Consider an elemental vortex of strength $\gamma\, d\xi$ located at a distance ξ from the origin along the chord line, as shown in Figure 4.24. The strength of the vortex sheet γ varies with the distance along the chord; that is, $\gamma = \gamma(\xi)$. The velocity dw at point x induced by the elemental vortex at point ξ is given by Equation (4.1) as

$$dw = -\frac{\gamma(\xi)\, d\xi}{2\pi(x - \xi)} \tag{4.16}$$

In turn, the velocity $w(x)$ induced at point x by *all* the elemental vortices along the chord line is obtained by integrating Equation (4.16) from the leading edge ($\xi = 0$) to the trailing edge ($\xi = c$):

$$w(x) = -\int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x - \xi)} \tag{4.17}$$

Combined with the approximation stated by Equation (4.15), Equation (4.17) gives the expression for $w'(s)$ to be used in Equation (4.12).

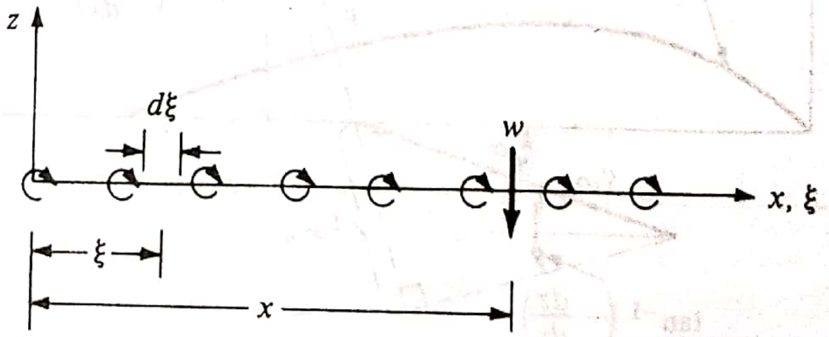


Figure 4.24 Calculation of the induced velocity at the chord line.

Recall that Equation (4.12) is the boundary condition necessary for the camber line to be a streamline. Substituting Equations (4.14), (4.15), and (4.17) into (4.12), we obtain

$$V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} = 0$$

or

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)} \tag{4.18}$$

the *fundamental equation of thin airfoil theory;* it is simply a statement that the camber line is a streamline of the flow.

Note that Equation (4.18) is written at a given point x on the chord line, and that dz/dx is evaluated at that point x. The variable ξ is simply a dummy variable of integration which varies from 0 to c along the chord line, as shown in Figure 4.24. The vortex strength $\gamma = \gamma(\xi)$ is a variable along the chord line. For a given airfoil at a given angle of attack, both α and dz/dx are known values in Equation (4.18). Indeed, the only unknown in Equation (4.18) is the vortex strength $\gamma(\xi)$. Hence, Equation (4.18) is an integral equation, the solution of which yields the variation of $\gamma(\xi)$ such that the camber line is a streamline of the flow. The central problem of thin airfoil theory is to solve Equation (4.18) for $\gamma(\xi)$, subject to the Kutta condition, namely, $\gamma(c) = 0$.

In this section, we treat the case of a symmetric airfoil. As stated in Section 4.2, a symmetric airfoil has no camber; the camber line is coincident with the chord line. Hence, for this case, $dz/dx = 0$, and Equation (4.18) becomes

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\alpha \tag{4.19}$$

In essence, within the framework of thin airfoil theory, a symmetric airfoil is treated the same as a flat plate; note that our theoretical development does not account for the airfoil thickness distribution. Equation (4.19) is an *exact* expression for the inviscid, incompressible flow over a flat plate at a small angle of attack.

To help deal with the integral in Equations (4.18) and (4.19), let us transform ξ into θ via the following transformation:

$$\xi = \frac{c}{2}(1 - \cos\theta) \tag{4.20}$$

Since x is a fixed point in Equations (4.18) and (4.19), it corresponds to a particular value of θ, namely, θ_0, such that

$$x = \frac{c}{2}(1 - \cos\theta_0) \tag{4.21}$$

Also, from Equation (4.20),

$$d\xi = \frac{c}{2}\sin\theta\,d\theta \tag{4.22}$$

Substituting Equations (4.20) to (4.22) into (4.19), and noting that the limits of integration become $\theta = 0$ at the leading edge (where $\xi = 0$) and $\theta = \pi$ at the trailing edge (where $\xi = c$), we obtain

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty\alpha \tag{4.23}$$

A rigorous solution of Equation (4.23) for $\gamma(\theta)$ can be obtained from the mathematical theory of integral equations, which is beyond the scope of this book. Instead, we simply state that the solution is

$$\boxed{\gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}} \tag{4.24}$$

We can verify this solution by substituting Equation (4.24) into (4.23) yielding

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = \frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{(1+\cos\theta)\, d\theta}{\cos\theta - \cos\theta_0} \tag{4.25}$$

The following standard integral appears frequently in airfoil theory and is derived in Appendix E of Reference 9:

$$\int_0^\pi \frac{\cos n\theta\, d\theta}{\cos\theta - \cos\theta_0} = \frac{\pi \sin n\theta_0}{\sin\theta_0} \tag{4.26}$$

Using Equation (4.26) in the right-hand side of Equation (4.25), we find that

$$\frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{(1+\cos\theta)\, d\theta}{\cos\theta - \cos\theta_0} = \frac{V_\infty\alpha}{\pi}\left(\int_0^\pi \frac{d\theta}{\cos\theta - \cos\theta_0} + \int_0^\pi \frac{\cos\theta\, d\theta}{\cos\theta - \cos\theta_0}\right)$$

$$= \frac{V_\infty\alpha}{\pi}(0+\pi) = V_\infty\alpha \tag{4.27}$$

Substituting Equation (4.27) into (4.25), we have

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty\alpha$$

which is identical to Equation (4.23). Hence, we have shown that Equation (4.24) is indeed the solution to Equation (4.23). Also, note that at the trailing edge, where $\theta = \pi$, Equation (4.24) yields

$$\gamma(\pi) = 2\alpha V_\infty \frac{0}{0}$$

which is an indeterminant form. However, using L'Hospital's rule on Equation (4.24),

$$\gamma(\pi) = 2\alpha V_\infty \frac{-\sin\pi}{\cos\pi} = 0$$

Thus, Equation (4.24) also satisfies the Kutta condition.

We are now in a position to calculate the lift coefficient for a thin, symmetric airfoil. The total circulation around the airfoil is

$$\Gamma = \int_0^c \gamma(\xi)\, d\xi \tag{4.28}$$

Using Equations (4.20) and (4.22), Equation (4.28) transforms to

$$\Gamma = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\, d\theta \tag{4.29}$$

Substituting Equation (4.24) into (4.29), we obtain

$$\Gamma = \alpha c V_\infty \int_0^\pi (1 + \cos\theta)\, d\theta = \pi\alpha c V_\infty \tag{4.30}$$

Substituting Equation (4.30) into the Kutta-Joukowski theorem, we find that the lift per unit span is

$$L' = \rho_\infty V_\infty \Gamma = \pi\alpha c \rho_\infty V_\infty^2 \tag{4.31}$$

The lift coefficient is

$$c_l = \frac{L'}{q_\infty S} \tag{4.32}$$

where

$$S = c(1)$$

Substituting Equation (4.31) into (4.32), we have

$$c_l = \frac{\pi\alpha c\rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_\infty^2 c(1)}$$

or

$$\boxed{c_l = 2\pi\alpha} \tag{4.33}$$

and

$$\boxed{\text{Lift slope} = \frac{dc_l}{d\alpha} = 2\pi} \tag{4.34}$$

Equations (4.33) and (4.34) are important results; they state the theoretical result that the lift coefficient is *linearly proportional to angle of attack,* which is supported by the experimental results discussed in Section 4.3. They also state that the theoretical lift slope is equal to 2π rad^{-1}, which is 0.11 degree^{-1}. The experimental lift coefficient data for an NACA 0012 symmetric airfoil are given in Figure 4.25; note that Equation (4.33) accurately predicts c_l over a large range of angle of attack. (The NACA 0012 airfoil section is commonly used on airplane tails and helicopter blades.)

The moment about the leading edge can be calculated as follows. Consider the elemental vortex of strength $\gamma(\xi)\, d\xi$ located a distance ξ from the leading edge, as sketched in Figure 4.26. The circulation associated with this elemental vortex is $d\Gamma = \gamma(\xi)\, d\xi$. In turn, the increment of lift dL contributed by the elemental vortex is $dL = \rho_\infty V_\infty\, d\Gamma$. This increment of lift creates a moment

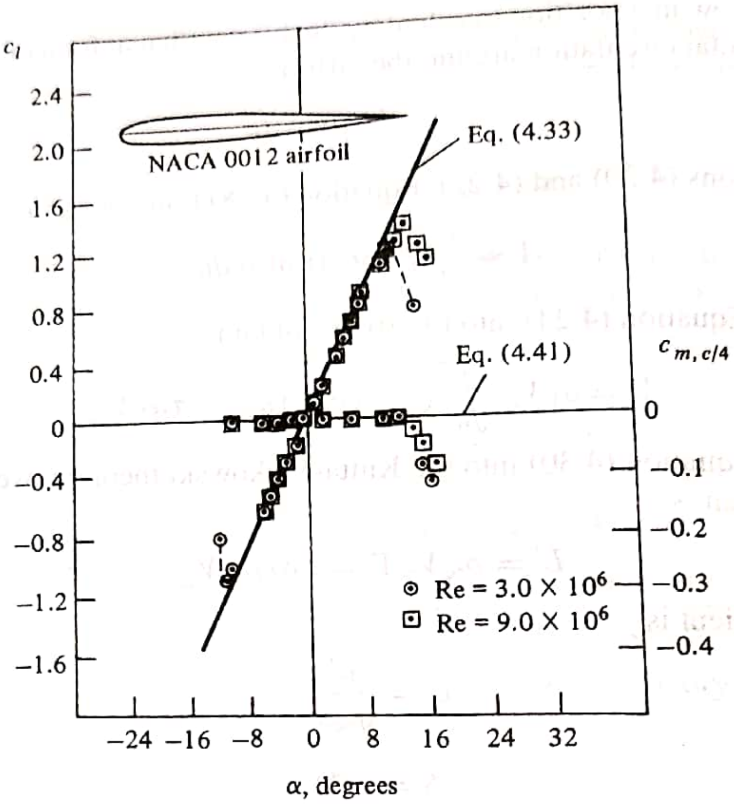


Figure 4.25 Comparison between theory and experiment for the lift and moment coefficients for an NACA 0012 airfoil. *(Source: Abbott and von Doenhoff, Reference 11.)*

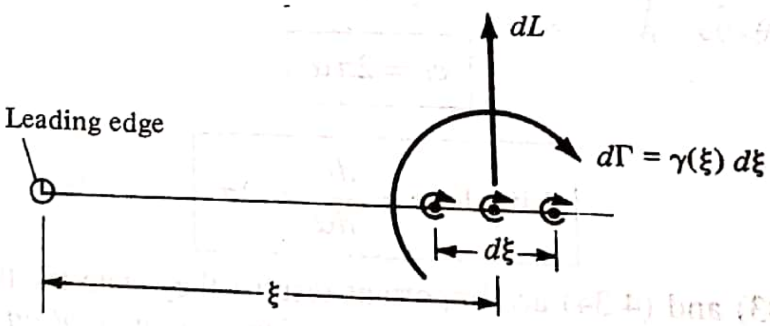


Figure 4.26 Calculation of moments about the leading edge.

about the leading edge $dM = -\xi(dL)$. The total moment about the leading edge (LE) (per unit span) due to the entire vortex sheet is therefore

$$M'_{\mathrm{LE}} = -\int_0^c \xi(dL) = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi)\, d\xi \tag{4.35}$$

Transforming Equation (4.35) via Equations (4.20) and (4.22), and performing the integration, we obtain (the details are left for Problem 4.4):

$$M'_{\mathrm{LE}} = -q_\infty c^2 \frac{\pi\alpha}{2} \tag{4.36}$$

The moment coefficient is

$$c_{m,\mathrm{le}} = \frac{M'_{\mathrm{LE}}}{q_\infty S c}$$

where $S = c(1)$. Hence,

$$c_{m,\mathrm{le}} = \frac{M'_{\mathrm{LE}}}{q_\infty c^2} = -\frac{\pi\alpha}{2} \tag{4.37}$$

However, from Equation (4.33),

$$\pi\alpha = \frac{c_l}{2} \tag{4.38}$$

Combining Equations (4.37) and (4.38), we obtain

$$\boxed{c_{m,\mathrm{le}} = -\frac{c_l}{4}} \tag{4.39}$$

From Equation (1.22), the moment coefficient about the quarter-chord point is

$$c_{m,c/4} = c_{m,\mathrm{le}} + \frac{c_l}{4} \tag{4.40}$$

Combining Equations (4.39) and (4.40), we have

$$\boxed{c_{m,c/4} = 0} \tag{4.41}$$

In Section 1.6, a definition is given for the center of pressure as that point about which the moments are zero. Clearly, Equation (4.41) demonstrates the theoretical result that the *center of pressure is at the quarter-chord point for a symmetric airfoil.*

By the definition given in Section 4.3, that point on an airfoil where moments are independent of angle of attack is called the aerodynamic center. From Equation (4.41), the moment about the quarter chord is zero for all values of α. Hence, for a symmetric airfoil, we have the theoretical result that the *quarter-chord point is both the center of pressure and the aerodynamic center.*

The theoretical result for $c_{m,c/4} = 0$ in Equation (4.41) is supported by the experimental data given in Figure 4.25. Also, note that the experimental value of $c_{m,c/4}$ is constant over a wide range of α, thus demonstrating that the real aerodynamic center is essentially at the quarter chord.

Let us summarize the above results. The essence of thin airfoil theory is to find a distribution of vortex sheet strength along the chord line that will make the camber line a streamline of the flow while satisfying the Kutta condition $\gamma(\mathrm{TE}) = 0$. Such a vortex distribution is obtained by solving Equation (4.18) for $\gamma(\xi)$, or in terms of the transformed independent variable θ, solving Equation (4.23) for $\gamma(\theta)$ [recall that Equation (4.23) is written for a symmetric airfoil]. The resulting vortex distribution $\gamma(\theta)$ for a symmetric airfoil is given by Equation (4.24). In turn, this vortex distribution, when inserted into the Kutta-Joukowski theorem, gives the following important theoretical results for a symmetric airfoil:

1. $c_l = 2\pi\alpha$.
2. Lift slope $= 2\pi$.
3. The center of pressure and the aerodynamic center are both located at the quarter-chord point.

LE 4.4

Consider a thin flat plate at 5 deg. angle of attack. Calculate the: (*a*) lift coefficient, (*b*) moment coefficient about the leading edge, (*c*) moment coefficient about the quarter-chord point, and (*d*) moment coefficient about the trailing edge.

■ Solution

Recall that the results obtained in Section 4.7, although couched in terms of a thin symmetric airfoil, apply in particular to a flat plate with zero thickness.

(*a*) From Equation (4.33),

$$c_\ell = 2\pi\alpha$$

where α is in radians

$$\alpha = \frac{5}{57.3} = 0.0873 \text{ rad}$$

$$c_\ell = 2\pi(0.0873) = \boxed{0.5485}$$

(*b*) From Equation (4.39)

$$c_{m,\ell e} = -\frac{c_\ell}{4} = -\frac{0.5485}{4} = \boxed{-0.137}$$

(*c*) From Equation (4.41)

$$c_{m,c/4} = \boxed{0}$$

(*d*) Figure 4.27 is a sketch of the force and moment system on the plate. We place the lift at the quarter-chord point, along with the moment about the quarter-chord point.

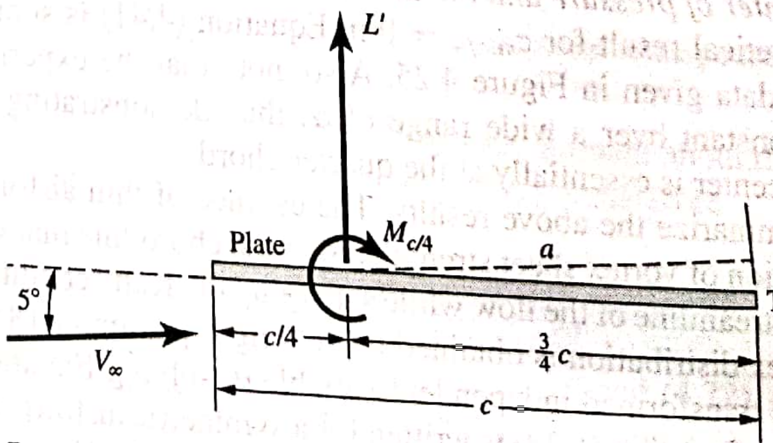


Figure 4.27 Flat plate at 5-degree angle of attack.

This represents the force and moment system on the plate. Recall from the discussion in Section 1.6 that the force and moment system can be represented by the lift acting through any point on the plate, and giving the moment about that point. Here, for convenience, we place the lift at the quarter-chord point.

The lift acts perpendicular to V_∞. (Part of the statement of the Kutta-Joukowski theorem given by Equation (3.140) is that the direction of the force associated with the circulation Γ is perpendicular to V_∞.) From Figure 4.27, the moment arm from L' to the trailing edge is the length a, where

$$a = \left(\tfrac{3}{4}c\right)\cos\alpha = \left(\tfrac{3}{4}c\right)\cos 5^\circ$$

One of the assumptions of thin airfoil theory is that the angle of attack is small, and hence we can assume that $\cos\alpha \approx 1$. Therefore, the moment arm from the point of action of the lift to the trailing edge is reasonably given by $\frac{3}{4}c$. (Note that, in the previous Figure 4.26, the assumption of small α is already implicit because the moment arm is drawn parallel to the plate.)

Examining Figure 4.27, the moment about the trailing edge is

$$M'_{\text{te}} = \left(\frac{3}{4}c\right)L' + M'_{c/4}$$

$$c_{m,\text{te}} = \frac{M'_{\text{te}}}{q_\infty c^2} = \left(\frac{3}{4}c\right)\frac{L'}{q_\infty c^2} + \frac{M'_{c/4}}{q_\infty c^2}$$

$$c_{m,\text{te}} = \frac{3}{4}c_\ell + c_{m,c/4}$$

Since $c_{m,c/4} = 0$ we have

$$c_{m,\text{te}} = \frac{3}{4}c_\ell$$

$$c_{m,\text{te}} = \frac{3}{4}(0.5485) = \boxed{0.411}$$

4.8 THE CAMBERED AIRFOIL

Thin airfoil theory for a cambered airfoil is a generalization of the method for a symmetric airfoil discussed in Section 4.7. To treat the cambered airfoil, return to Equation (4.18):

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right) \tag{4.18}$$

For a cambered airfoil, dz/dx is finite, and this makes the analysis more elaborate than in the case of a symmetric airfoil, where $dz/dx = 0$. Once again, let us transform Equation (4.18) via Equations (4.20) to (4.22), obtaining

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta - \cos\theta_0} = V_\infty\left(\alpha - \frac{dz}{dx}\right) \tag{4.42}$$