Canonical form for Elliptic Equation $B^2 - 4AC < 0$ so the characteristic equations can be written as

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

and

$$\frac{dy}{dx} = \frac{+B + \sqrt{B^2 - 4AC}}{2A}$$

gives us complex conjugate co-ordinates, say ξ and η. Now we make another transformation from (ξ, η) to (α, β) so that

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2i}$$

which gives us the required canonical form.

To illustrate the procedure we consider the following example

$$u_{xx} + x^2 u_{yy} = 0$$

so $B^2 - 4AC = -4x^2 < 0$, hence pde is elliptic.

The characteristic equations are

$$\frac{dy}{dx} = \frac{\sqrt{-4x^2}}{2} \left(= \frac{B - \sqrt{B^2 - 4AC}}{2A} \right) = -ix$$

and $$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{\sqrt{-4x^2}}{2} = ix$$

Integration of these equations yields

$$iy + \frac{x^2}{2} = c_1 , \quad -iy + \frac{x^2}{2} = c_2$$

Hence we may assume that

$$\xi = \frac{1}{2} x^2 + iy, \qquad \eta = \frac{1}{2} x^2 - iy.$$

Now introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2} , \quad \beta = \frac{\xi - \eta}{2i}$$

we obtain $\quad \alpha = \frac{x^2}{2} , \quad \beta = y$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = x^2$$

$$\bar{B} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y = 0$$

$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 = x^2$$

$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y = 1$$

$$\bar{E} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y = 0$$

$$\bar{F} = 0, \quad \bar{G} = 0$$

Thus the required canonical equation is

$$x^2 u_{\alpha\alpha} + x^2 u_{\beta\beta} + u_\alpha = 0 \text{ or } u_{\alpha\alpha} + u_{\beta\beta} = \frac{-u_\alpha}{2\alpha}.$$