The concept of stability and convergence can be understood by analysing the numerical solutions of linear equation $u' = \lambda u$ corresponding to non-linear equation $u' = f(x,u)$. This linear form $u' = \lambda u$ is called test function.

Consider IVP (linear

$$u' = \lambda u \quad , \quad u(t_0) = \eta_0 \qquad \text{— (1)}$$

exact solution is $u(t) = \eta_0 e^{\lambda t}$ — (2)

$$\frac{u'}{u} = \lambda$$

$$\ln u = \lambda t$$

$$u = ce^{\lambda t}$$

$$\text{or } u = \eta_0 e^{\lambda t}$$

$$u(t_{j+1}) = \eta_0 e^{\lambda t_{j+1}}$$

$$u(t_j) = \eta_0 e^{\lambda t_j}$$

$$\frac{u(t_{j+1})}{u(t_j)} = e^{\lambda(t_{j+1} - t_j)} = e^{\lambda h}$$

$$\text{or} \quad u(t_{j+1}) = e^{\lambda h} u(t_j) \qquad \text{— (3)}$$

(*) A numerical method can be obtained by writing an approximation to $e^{\lambda t}$ say

(i) $e^{\lambda h} \simeq 1 + \lambda h$

(ii) $e^{\lambda h} \simeq 1 + \lambda h + \frac{\lambda^2 h^2}{2}$

(iii) $e^{\lambda h} \simeq 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}$

(iv) $e^{\lambda h} \simeq 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}$ — (4)

(v) $e^{\lambda h} \simeq \dfrac{1 + \lambda h/2}{1 - \lambda h/2}$

Consider the problem $u' = \lambda u$ i.e, $f(x,u) = \lambda u$

Euler method

$$u_{j+1} = u_j + hf \quad = u_j + h\lambda u_j = (1+\lambda h)u_j$$

Taylor's second order method

$$u' = f = \lambda u$$

$$u'' = \frac{d}{dx}u' = \lambda u' = \lambda f = \lambda \cdot \lambda u = \lambda^2 u$$

$$u''' = \lambda^2 u' = \lambda \cdot \lambda u = \lambda^3 u$$

$$u_{j+1} = u_j + hu_j' + \frac{h^2}{2!}u_j''$$

$$= u_j + h\lambda u_j + \frac{h^2}{2!}\lambda^2 u_j$$

$$= \left[1 + \lambda h + \frac{(\lambda h)^2}{2!}\right]u_j$$

Similarly for higher order methods.

So a numerical method can be obtained by writing an approximation to $e^{\lambda h}$ say

(i) $e^{\lambda h} \simeq 1 + \lambda h$ Euler method

(ii) $e^{\lambda h} \simeq 1 + \lambda h + \frac{\lambda^2 h^2}{2!}$ Taylor's second order method

Stability

Let these approximations for $e^{\lambda h}$ be denoted by $E(\lambda h)$ then the methods will be written as

$$u_{j+1} = E(\lambda h)\, u_j \qquad \text{— (5)}$$

and the exact solution will satisfy

$$u(t_{j+1}) = E(\lambda h)\, u(t_j) + T_j \qquad \text{— (6)}$$

Let $\epsilon_j = u_j - u(t_j)$ then $u_j = \epsilon_j + u(t_j)$

and from (5) we get

$$\epsilon_{j+1} + u(t_{j+1}) = E(\lambda h)\,(\epsilon_j + u(t_j))$$

$$\epsilon_{j+1} = -u(t_{j+1}) + E(\lambda h)\,\epsilon_j + E(\lambda h)\, u(t_j)$$

from (3) $u(t_{j+1}) = e^{\lambda h} u(t_j)$

Then

$$\epsilon_{j+1} = -e^{\lambda h} u(t_j) + E(\lambda h)\, u(t_j) + E(\lambda h)\,\epsilon_j$$

$$= \underbrace{\left[E(\lambda h) - e^{\lambda h}\right] u(t_j)}_{\text{Truncation error}} + \underbrace{E(\lambda h)\,\epsilon_j}_{\text{Propogation of the error}}$$

Next for $\epsilon_{j+2} = [E(\lambda h) - e^{\lambda h}]\, u(t_{j+1}) + E(\lambda h)\,\epsilon_{j+1}$

Now $u(t_{j+1}) = e^{\lambda h} u(t_j)$ so

$$\epsilon_{j+2} = (E(\lambda h) - e^{\lambda h})\, e^{\lambda h} u(t_j) + E(\lambda h)\,[E(\lambda h) - e^{\lambda h}]\, u(t_j) + E^2(\lambda h)\,\epsilon_j$$

$$\boxed{\epsilon_{j+2} = \underbrace{[E^2(\lambda h) - e^{2\lambda h}]\, u(t_j)}_{\text{Truncation error}} + \underbrace{E^2(\lambda h)\,\epsilon_j}_{\text{Propogation of error}}}$$

Stability

for first term on r.h.s.

$$u(t_{j+1}) = e^{\lambda h} u(t_j)$$

$$u(t_{j+2}) = e^{\lambda h} u(t_{j+1}) = e^{2\lambda h} u(t_j)$$

and numerical sol.

$$u_{j+1} = E(\lambda h)\, u_j$$

$$u_{j+2} = E(\lambda h)\, u_{j+1} = E^2(\lambda h)\, u_j$$

Thus Truncation error $= \left[E^2(\lambda h) - e^{2\lambda h}\right] u(t_j)$

Now at $j+k$ step.

$$\boxed{\epsilon_{j+k} = \underbrace{\left(E^k(\lambda h) - e^{k\lambda h}\right) u(t_j)}_{\text{Truncation error}} + \underbrace{E^k(\lambda h)\, \epsilon_j}_{\text{propagation of the error.}}}$$

Defⁿ Stable A method is stable if cumulative effect of all errors, including round off. errors is bounded, independent of mesh divisions.

Absolutely Stable If

$$|E(\lambda h)| < 1$$ then method is called absolutely stable.

Relatively Stable If $E(\lambda h) < e^{\lambda h}$ then the method is called relatively stable.

Now (i) If $E(\lambda h) = 1+\lambda h < e^{\lambda h}$ for $\lambda > 0$

and hence for $\lambda>0$ it is relatively stable (always).

for. Absolute Stability

$$|E(\lambda h)| < 1$$

$$|1+\lambda h| < 1 \quad \text{or} \quad -1 < 1+\lambda h < 1$$

$$\text{or} \quad -2 < \lambda h < 0$$

Thus for $\lambda h \in (-2, 0)$ the method is absolutely stable and this interval is called interval of stability.

(ii) $E(\lambda h) = 1+\lambda h + \frac{\lambda^2 h^2}{2}$

Again for $\lambda > 0$ $E(\lambda h) < e^{\lambda h}$ ~~thus~~ relatively stable for $\lambda > 0$ the method is relatively stable

for Absolute Stability

$$|E(\lambda h)| < 1$$

$$\left|1+\lambda h + \frac{\lambda^2 h^2}{2}\right| < 1$$

or $\left|\frac{1}{2}[(1+\lambda h)^2 + 1]\right| < 1$

$1+\lambda h + \frac{\lambda^2 h^2}{2}$

$\frac{1}{2}(\lambda^2 h^2 + 2\lambda h + 2)$

$= \frac{1}{2}[(1+\lambda h)^2 + 1]$

or

$$-1 < \frac{1}{2}[(1+\lambda h)^2 + 1] < 1$$

$$-2 < (1+\lambda h)^2 + 1 < 2$$

The left hand side inequality is always satisfied for R.h.S.

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$$(1+\lambda h)^2 + 1 < 2$$

or $(1+\lambda h)^2 < 1$

$$-1 < (1+\lambda h) < 1$$

$$-2 < \lambda h < 0$$

So interval of stability is $(-2, 0)$.

(iii) Similarly for

$$E(\lambda) = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}$$

$\lambda h = -2.5$ $\quad \frac{6.25}{2}$

$(1 - 2.5 + 3.125 - 2.6042)$

$= -.979$

λh	0	-1	-2	-2.5
$E(\lambda h)$	1			-.979

So $\lambda h \in (-2.5, 0)$.

(iv) Similarly for

$$E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}$$

λh	0	-1	-2	-2.2	-2.6	-3.0
$E(\lambda h)$	1	0.3750	0.333	0.4212	.7547	1.375

(v) $E(\lambda h) = \dfrac{1 + \lambda h/2}{1 - \lambda h/2}$

we want $|E(\lambda)| < 1$ or $\left|\dfrac{1+\lambda h/2}{1-\lambda h/2}\right| < 1$

or $|1 + \lambda h/2| < |1 - \lambda h/2|$ — (*)

for $\lambda < 0$ inequality (*) is always satisfied, so interval of absolute stability $(-\infty, 0)$. Such method

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are called unconditionally stable.

Stability

Euler Method

$$y_{n+1} = y_n + h f_n$$

Apply Euler method for $y' = \lambda y$

$$y_{n+1} = y_n + \lambda h y_n$$

$$y_{n+1} = (1 + \lambda h) y_n$$

$$E(\lambda h) = 1 + \lambda h$$

2nd Order R-K method

$$y_{n+1} = y_n + \frac{1}{4}(k_1 + 3k_2)$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + 2/3 h,\ y_n + 2/3 k_1)$$

Apply R-K method for $y' = \lambda y$

$$k_1 = h \lambda y_n$$

$$k_2 = h\lambda(y_n + 2/3 k_1)$$

$$= h\lambda(y_n + 2/3 h\lambda y_n)$$

$$= [h\lambda + 2/3 (h\lambda)^2] y_n$$

$$y_{n+1} = y_n + \left[\frac{1}{4} h\lambda + \frac{3}{4}(h\lambda) + \frac{\cancel{3}}{\cancel{4}_2} \cdot \frac{\cancel{2}}{\cancel{3}}(h\lambda)^2\right] y_n$$

$$= \left[1 + h\lambda + \frac{h^2\lambda^2}{2}\right] y_n$$

$$E(\lambda h) = 1 + h\lambda + \frac{h^2\lambda^2}{2}$$

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4th order R-K method

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + h/2,\ y_n + \frac{1}{2}k_1)$

$k_3 = hf(x_n + h/2,\ y_n + \frac{1}{2}k_2)$

$k_4 = hf(x_n + h,\ y_n + k_3)$

Apply this method to $y' = \lambda y$

$k_1 = h\lambda y_n$

$$k_2 = h\lambda(y_n + \frac{1}{2}k_1)$$
$$= h\lambda[y_n + \frac{1}{2}h\lambda y_n]$$
$$= [h\lambda + \frac{1}{2}(h\lambda)^2]y_n$$

$$k_3 = h\lambda(y_n + \frac{1}{2}k_2)$$
$$= h\lambda[y_n + \frac{1}{2}(h\lambda + \frac{1}{2}(\lambda h)^2)y_n]$$
$$= h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{4}(h\lambda)^3]y_n$$

$$k_4 = h\lambda(y_n + k_3)$$
$$= h\lambda[y_n + (h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{4}(h\lambda)^3)y_n]$$
$$= [h\lambda + (h\lambda)^2 + \frac{1}{2}(h\lambda)^3 + \frac{1}{4}(h\lambda)^4]y_n$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= y_n + \frac{1}{6}[h\lambda + 2\{h\lambda + \frac{1}{2}(h\lambda)^2\} + 2\{(h\lambda) + \frac{1}{2}(h\lambda)^2 + \frac{1}{4}(h\lambda)^3\} + \{h\lambda + (h\lambda)^2 + \frac{1}{2}(h\lambda)^3 + \frac{1}{4}(h\lambda)^4\}]$$

$$y_{n+1} = [1 + h\lambda + \frac{1}{6}(h\lambda)^2(1+1+1) + \frac{1}{6}(h\lambda)^3(\frac{1}{2}+\frac{1}{2}) + \frac{1}{6\cdot 4}(h\lambda)^4]\, y_n$$

$$y_{n+1} = [1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} + \frac{(h\lambda)^4}{24}]\, y_n$$

$$E(\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} + \frac{(h\lambda)^4}{24}$$