Lecture - 8

Marine Hydrodynamics

1. Stokes Theorem:

Stokes theorem states that the circulation round any closed curve C drawn in a fluid is equal to the surface integrals of the normal component of spin taken over any surface, provided the surface lies wholly in the fluid.

i.e $$\Gamma = \int_C \vec{q}\cdot d\vec{r} = \iint_S \operatorname{curl} \vec{q}\, ds \quad \ldots\ldots (1.1)$$

or $$\Gamma = \iint_S \vec{\omega}\cdot\vec{n}\, ds.$$

<u>Corollary</u>: for ir-rotational motion, the circulation

$\Gamma = 0$ as $$\Gamma = \int_C \vec{q}\cdot d\vec{r} = \iint_S \nabla\times\vec{q}\, ds$$

for ir-rotational fluid $\nabla\times\vec{q} = 0$

$\Rightarrow \Gamma = 0.$

2. Kelvin Circulation Theorem:

Kelvin Theorem states that the circulation round any closed curve is invariant in an inviscid fluid, provided that the body forces are conservative and the pressure is single valued function of density only.

$\Rightarrow$ with above condition: $$\boxed{\frac{d\Gamma}{dt} = 0}$$

(proof:- take as homework)

Green's Theorem:.

If ϕ and ϕ' are both single valued and continuously differentiable scalar point function such that $\nabla\phi$ and $\nabla\phi'$ are also continuously differentiable, then:.

$$\iiint_V (\nabla\phi . \nabla\phi')\, dv = -\iint_S \phi \frac{\partial\phi'}{\partial n} ds - \iiint_V \phi \nabla^2\phi' dv \quad (3.1a)$$

$$= -\iint_S \phi' \frac{\partial\phi}{\partial n} ds - \iiint_V \phi' \nabla^2\phi\, dv. \quad \ldots (3.1b)$$

if you combine (3.1a) and (3.1b) we get

$$-\iint_S \phi \frac{\partial\phi'}{\partial n} ds - \iiint_V \phi \nabla^2\phi' dv = -\iint_S \phi' \frac{\partial\phi}{\partial n} ds - \iiint_V \phi' \nabla^2\phi\, dv$$

$$\Rightarrow \iiint_V \phi \nabla^2\phi' dv - \iiint_V \phi' \nabla^2\phi\, dv = \iint_S \phi' \frac{\partial\phi}{\partial n} ds - \iint_S \phi \frac{\partial\phi'}{\partial n} ds$$

$$\Rightarrow \boxed{\iiint_V [\phi \nabla^2\phi' - \phi' \nabla^2\phi]\, dv = \iint_S \left[\phi' \frac{\partial\phi}{\partial n} - \phi \frac{\partial\phi'}{\partial n}\right] ds} \quad (3.2)$$

proof:- we use the identity

$$\nabla . (\phi \vec{a}) = \vec{a} . (\nabla\phi) + \phi (\nabla . \vec{a}) \quad \ldots (3.3)$$

Now take $\vec{a} = \nabla\phi'$ we get from (3.3)

$$\nabla . (\phi \nabla\phi') = \nabla\phi' . \nabla\phi + \phi (\nabla . \nabla\phi')$$

plying to the volume $\vec{V}$ enclosed by the surface (3)
$\vec{S}$, we get

$$\iiint_V \nabla\cdot(\phi\nabla\phi')\,dv = \iiint_V \nabla\phi'\cdot\nabla\phi\,dv + \iiint_V \phi\cdot\nabla^2\phi'\,dv$$

now applying gauss divergent theorem for L.H.S, we get

$$-\iint_S (\phi\nabla\phi')\cdot\vec{n}\,ds = \iiint_V \nabla\phi'\cdot\nabla\phi\,dv + \iiint_V \phi\nabla^2\phi'\,dv$$

$$\Rightarrow \boxed{\iiint_V \nabla\phi\cdot\nabla\phi'\,dv = -\iint_S \phi\frac{\partial\phi}{\partial n}\,ds - \iiint_V \phi\cdot\nabla^2\phi'\,dv} \rightarrow (3\cdot4)$$

4. Some important deduction from Green's Theorem

a) Let ϕ' is a constant $= A$, then $\nabla^2\phi' = 0$ every where, if ϕ be the velocity potential we have

$$\Rightarrow \iint_S \frac{\partial\phi}{\partial n} = 0$$

which shows that total ~~fluid~~ flow of liquid into any closed region at any instant is zero.

b). Let $\phi = \phi' =$ a velocity potential. then $\nabla^2\phi$ and $\nabla^2\phi'$ both goes to zero. then from (3.4) we get

$$\iiint_V (\nabla\phi)^2\,dv = -\iint_S \phi\frac{\partial\phi}{\partial n}\,ds$$

Now kinetic energy of liquid inside s may be written as

$$T = \frac{1}{2}\rho \iiint_V (\nabla\phi)^2 \, dv$$

$$\Rightarrow \boxed{T = -\frac{\rho}{2} \iint_S \phi \cdot \frac{\partial \phi}{\partial n} \, ds}$$

c). Let $\frac{\partial \phi}{\partial n} = 0$ on the boundary

$$\Rightarrow \iiint_V (\nabla\phi)^2 \, dv = 0$$

$$\Rightarrow \iiint_V \vec{q}^{\,2} \cdot dv = 0$$

since q^2 is +ve definite $\Rightarrow \vec{q} = 0$. at everywhere in v. i.e, the liquid is at rest. Thus irrotational motion is not possible in a liquid bounded by fixed rigid boundary. If a ~~closed~~ vessel which moves in an unbounded fluid, if suddenly brought to rest, the liquid is also brought to rest.

d). also, one can find out if ϕ and ϕ' both are velocity potential then

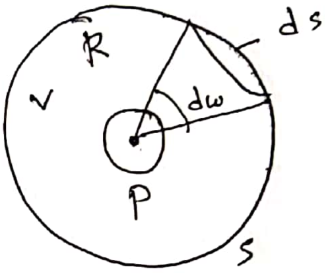
$$\boxed{\iint_S \phi \frac{\partial \phi'}{\partial n} \, ds = \iint \phi' \frac{\partial \phi}{\partial n} \, ds}$$

This is an important findings and we discuss the same latter.

Mean value of a velocity potential over a spherical surface [understanding the meaning of Laplacian].

If a region lying wholly in the liquid be bounded by a spherical surface, the mean value of the velocity potential over the surface ~~over the~~ is equal to its value at the centre of the sphere.

Let ϕ_r and ϕ_m denotes the value of the velocity potential ~~over the surface~~ at P and mean value over the surface.



Now
$$\phi_m = \frac{1}{4\pi r^2} \iint_S \phi . ds$$

$$\Rightarrow \frac{\partial \phi_m}{\partial r} = \frac{\partial}{\partial r} \iint \frac{1}{4\pi r^2} \phi \, ds \quad ..... (5.1)$$

Now assume that $d\omega$ is the solid angle subtended at the centre of a sphere of radius r such that

$$\boxed{\frac{ds}{d\omega} = r^2}$$

then ~~as~~ (5.1) gives.

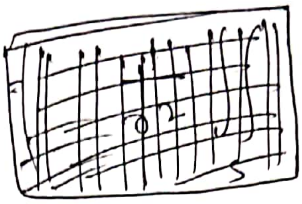
$$\frac{\partial \phi_m}{\partial r} = \frac{1}{4\pi} \frac{\partial}{\partial r} \iint_R \frac{1}{\cancel{r^2}} \phi . \cancel{r^2} . d\omega$$

$$\Rightarrow \boxed{\frac{\partial \phi_m}{\partial r} = \frac{1}{4\pi} \iint_R \frac{\partial \phi}{\partial r} d\omega} \longrightarrow (5.2)$$

Therefore, $\dfrac{\partial \phi_m}{\partial r} = \dfrac{1}{4\pi} \displaystyle\iint_R \dfrac{\partial \phi}{\partial r}\, d\omega$

Now for sphere

$$\iint_R \frac{\partial \phi}{\partial n}\, d\omega = \iint_R \frac{\partial \phi}{\partial r} \cdot d\omega$$



$$\therefore \iint \frac{\partial \phi}{\partial r} \cdot d\omega = \iint_R (\nabla \phi) \cdot n\, d\omega$$

$$= \frac{1}{r^2} \iint_S (\nabla \phi) \cdot \vec{m} \cdot ds$$

$$= \frac{1}{r^2} \iint_V \mathrm{div}(\nabla \phi)\, dv$$

$$= \frac{1}{r^2} \cdot \iint \nabla^2 \phi\, dv$$

$$= 0$$

$\Rightarrow \dfrac{\partial \phi_m}{\partial r} = 0 \Rightarrow \phi_m = \text{Constant} \rightarrow (5 \cdot 3)$

which shows that ϕ_m is independent of the radius r, and hence the mean 's' approaches to a point P, $\phi_m = \phi$, thus this mean value is the value of ϕ at the centre.

6. If $f(r) = \frac{1}{r}$ is velocity potential ? where $r^2 = x^2 + y^2 + z^2$

$$f(r) = \frac{1}{r} \quad \ldots\ldots (6.1)$$

Now $r^2 = x^2 + y^2 + z^2$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\Rightarrow \nabla\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial y} + \hat{k}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\cdot\left(\frac{\partial r}{\partial z}\right)$$

$$\Rightarrow \nabla\left(\frac{1}{r}\right) = \hat{i}(-)\frac{1}{r^2}\cdot\frac{x}{r} + (-)\frac{1}{r^2}\cdot\frac{y}{r}\hat{j} + (-)\frac{1}{r^2}\cdot\frac{z}{r}\hat{k}$$

$$= \frac{-1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Now $\nabla^2\left(\frac{1}{r}\right) = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\frac{x\hat{i}}{r^3} + \frac{y\hat{j}}{r^3} + \frac{z\hat{k}}{r^3}\right]$

$$\Rightarrow \nabla^2\left(\frac{1}{r}\right) = -\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$

$$\cancel{\Rightarrow -\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^3} + x\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right) + \frac{1}{r^3} + \frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)}$$

$$-\nabla^2\left(\frac{1}{r}\right) = \frac{3}{r^3} + x\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\cdot\frac{\partial r}{\partial x} + y\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\frac{\partial r}{\partial y} + z\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\frac{\partial r}{\partial z}$$

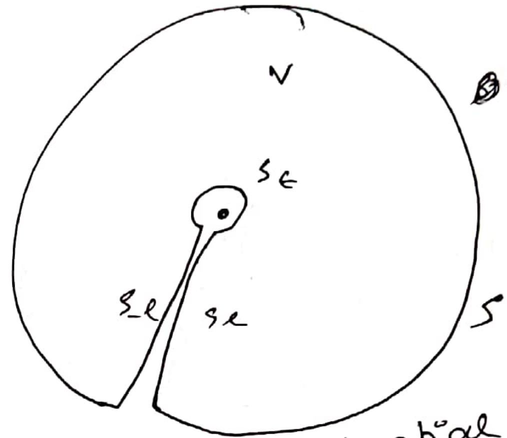
$$\Rightarrow -\nabla^2\left(\frac{1}{r}\right) = \frac{3}{r^3} - \left[\frac{3x^2}{r^5} + \frac{3y^2}{r^5} + \frac{3z^2}{r^5}\right]$$

$$\Rightarrow -\nabla^2\left(\frac{1}{r}\right) = \frac{3}{r^3} - \frac{3}{r^3} \quad \left[\because x^2+y^2+z^2 = r^2\right]$$

$$\Rightarrow \nabla^2\left(\frac{1}{r}\right) = 0 \quad \text{for } r \neq 0$$

$$= \alpha \quad \text{for } r = 0$$

$$\boxed{\therefore \frac{1}{r} \text{ is velocity potential apart from } r=0}$$



8. Let us consider the velocity potential ϕ and $\phi' = \frac{1}{4\pi r}$, then considering the above volume V which is surrounded by the surface S and S_ϵ, S_l, S_{-l}, Applying green's theorem (3.2) we get

$$\frac{1}{4\pi}\iint_{S+S_\epsilon}\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]ds' = 0$$

$$\Rightarrow \frac{1}{4\pi}\iint_{S}\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]ds$$

$$= -\frac{1}{4\pi}\iint_{S_\epsilon}\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]ds$$

In the limit $\epsilon \to 0$ from the result (5.3), we know that value of ϕ is same as value at the centre of the sphere and may be taken out of the integral, ∴ from the 1st integral we have

$$\frac{1}{4\pi}\iint \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS_\epsilon = \frac{\phi}{4\pi}\iint_{S_\epsilon} \frac{\partial}{\partial r}\left(\frac{1}{r}\right) dS_\epsilon$$

Now $dS_\epsilon = 4\pi r^2$

and $\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2}$

$$\Rightarrow \frac{1}{4\pi}\iint \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS_\epsilon = -\phi(x,y,z).$$

we argue: the second term of the integral $\to 0$ as $\epsilon \to 0$ as it is weaker singularity. Hence we finally get a very important results as:

$$\phi(x,y,z) = -\frac{1}{4\pi}\iint_S \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] ds \longrightarrow (8.1)$$

The equation (8.1) is heavily used in the domain of ocean engineering for ~~the appli~~ solving various ~~past~~ hydrodynamic problems. for example find the force acting on a body moving in unbounded fluid, in propeller hydrodynamics etc. More we discuss in the latter stage of this course.