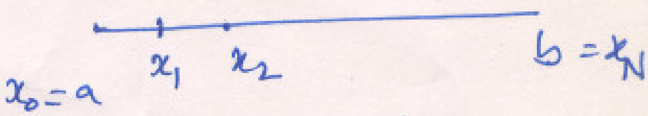
2

1

$$A(x)y''(x) + B(x)y'(x) + C(x)y(x) = f(x) \quad \text{—(1)}$$

$$y(a) = \alpha, \quad y(b) = \beta \qquad a \le x \le b \quad \text{—(2)}$$



$$x_k = x_0 + kh$$

$$h = \frac{b-a}{N}$$

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(iv)}(\xi) \quad \text{—(3)}$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(iv)}(\xi) \quad \text{—(4)}$$

$$x-h < \xi < x+h$$

(3) + (4)

$$y(x+h) + y(x-h) = 2y(x) + 2\frac{h^2}{2}y''(x) + 2\frac{h^4}{4!}y^{(iv)}(\xi)$$

$$y(x+h) + y(x-h) - 2y(x) = h^2y''(x) + \frac{2h^4}{4\cdot 3\cdot 2}y^{(iv)}(\xi)$$

$$h^2y''(x) = y(x+h) + y(x-h) - 2y(x) - \frac{h^4}{12}y^{(iv)}(\xi) \quad \text{—(5)}$$

(3) − (4)

$$y(x+h) - y(x-h) = 2hy'(x) + 2\frac{h^3}{6}y'''(x)$$

or

$$2hy'(x) = y(x+h) - y(x-h) - \frac{2h^3}{6}y'''(x) \quad \text{—(6)}$$

$$h^2A_ky_k'' + h^2B_ky_k' + h^2C_ky_k = h^2f_k$$

$$h^2A_ky_k'' + 2hy_k'\cdot B_k\cdot\frac{h}{2} + h^2C_ky_k = h^2f_k$$

$$A_k\left[y_{k+1} + y_{k-1} - 2y_k - \frac{h^4}{12}y^{(iv)}(\xi_k)\right] + \frac{h}{2}B_k\left[y_{k+1} - y_{k-1} - \frac{2h^3}{6}y_k'''\right] + h^2C_ky_k = h^2f_k$$

(2)

$$(A_k - h/2\, B_k)\, y_{k-1} - (2A_k - h^2 c_k)\, y_k + (A_k + h/2\, B_k)\, y_{k+1} = h^2 f_k + \frac{h^4}{12} A_k\, y^{(iv)}(\xi_k) + \frac{h^4}{6} B_k\, y_k'''$$

$$-(A_k - h/2\, B_k)\, y_{k-1} + (2A_k - h^2 c_k)\, y_k - (A_k + h/2\, B_k)\, y_{k+1} = -h^2 f_k - \frac{h^4}{12}\left[A_k\, y^{(iv)}(\xi_k) + 2B_k y_k'''\right]$$

$$K = 1(1)\overline{N-1} \qquad (7)$$

Hence the method will be given as

$$-(A_k - h/2\, B_k)\, y_{k-1} + (2A_k - h^2 c_k)\, y_k - (A_k + h/2\, B_k)\, y_{k+1} = -h^2 f_k \qquad (8)$$

$$K = 2(1)\overline{N-2}$$

for $K=1$

$$-(A_1 - h/2\, B_1)\, y_0 + (2A_1 - h^2 c_1)\, y_1 - (A_1 + h/2\, B_1)\, y_2 = -h^2 f_1$$

or $$(2A_1 - h^2 c_1)\, y_1 - (A_1 + h/2\, B_1)\, y_2 = -h^2 f_1 + (A_1 - h/2\, B_1)\alpha \qquad (9)$$

Similarly for $K = N-1$

$$-(A_{N-1} - h/2\, B_{N-1})\, y_{N-2} + (2A_{N-1} - h^2 c_{N-1})\, y_{N-1} = -h^2 f_{N-1} + (A_{N-1} + h/2\, B_{N-1})\beta \qquad (10)$$

Thus we have $(N-1)\times(N-1)$ system of equations in $(N-1)$ unknowns $y_1 \cdots y_{N-1}$

$$DY = B$$

where D is a tridiagonal matrix with

$d_{ii} = (2A_i - h^2 c_i)$, $d_{i\,i-1} = -(A_i - h/2\, B_i)$

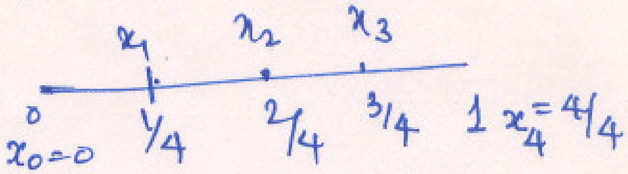
$d_{i,i+1} = -(A_i + h/2\, B_i)$, $Y = (y_1 \cdots y_{N-1})^T$

$B = (b_1 \cdots b_{N-1})^T$, $b_i = -h^2 f_i$, $b_1 = -h^2 f_1 + (A_1 - \frac{h B_1}{2})\alpha$

$b_{N-1} = -h^2 f_{N-1} + (A_1 + \frac{h}{2} B_{N-1})\beta$

Ex $y'' = 2, \quad y(0) = 0, \quad y(1) = 1$

exact solution $y(x) = x^2$



$h = \frac{1}{4}$

$$h^2 y_k'' = 2h^2$$

$$y_k'' = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

$$y_{k+1} - 2y_k + y_{k-1} = 2h^2$$

$y_0 = 0, \; y_4 = 1$

$$-y_{k-1} + 2y_k - y_{k+1} = -2h^2$$

$\underline{K=1}$ $\quad 2y_1 - y_2 = -2h^2 = -2 \cdot \frac{1}{16} = -\frac{1}{8}$

$\underline{K=2}$ $\quad -y_1 + 2y_2 - y_3 = -2h^2 = -1/8$

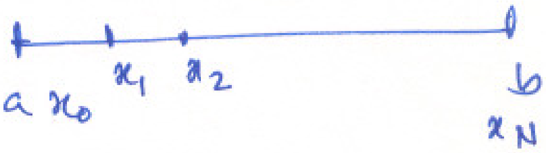
$\underline{K=3}$ $\quad -y_2 + 2y_3 = -2h^2 + 1 = \frac{-1}{8} + 1 = -7/8$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1/8 \\ -1/8 \\ -7/8 \end{pmatrix}$$

④

Another Boundary condition

$$Ay'' + By' + Cy = f(x) \qquad y'(a) = \alpha, \; y(b) = \beta$$



$$-(A_k - h/2\, B_k)\, y_{k-1} + (2A_k - h^2 C_k)\, y_k - (A_k + h/2\, B_k)\, y_{k+1} = -h^2 f_k$$

for $K=0$

$$-(A_0 - h/2\, B_0)\, y_{-1} + (2A_0 - h^2 C_0)\, y_0 - (A_0 + h/2\, B_0)\, y_1 = -h^2 f_0 \qquad (*)$$

and from $y'(0) = \alpha \Rightarrow y_0' = \alpha$

$$y_k' = \frac{y_{k+1} - y_{k-1}}{2h}$$

$$\Rightarrow \frac{y_1 - y_{-1}}{2h} = \alpha$$

$$y_1 - y_{-1} = 2h\alpha$$

$$y_{-1} = y_1 - 2h\alpha \qquad (*)$$

Now putting value of y_{-1} in $(*)$ we get

$$-(A_0 - h/2\, B_0)\, y_1 + 2h\alpha\,(A_0 - h/2\, B_0) + (2A_0 - h^2 C_0)\, y_0 - (A_0 + h/2\, B_0)\, y_1 = -h^2 f_0$$

$$-2A_0\, y_1 + (2A_0 - h^2 C_0)\, y_0 = -h^2 f_0 - 2h\alpha\,(A_0 - h/2\, B_0)$$

or

$$(2A_0 - h^2 C_0)\, y_0 - 2A_0\, y_1 = -h^2 f_0 - 2h\alpha\,(A_0 - h/2\, B_0) \qquad (1)$$

$$-(A_k - h/2\, B_k)\, y_{k-1} + (2A_k - h^2 C_k)\, y_k - (A_k + h/2\, B_k)\, y_{k+1} = -h^2 f_k \qquad (2)$$

$$K = 1(1)(N-2)$$

$$-(A_{N-1} - h/2\, B_{N-1})\, y_{N-2} + (2A_{N-1} - h^2 C_{N-1})\, y_{N-1} = -h^2 f_{N-1} + (A_{N-1} + h/2\, B_{N-1})\beta \qquad (3)$$

$y'' = 2 \qquad y'(0) = 0, \; y(1) = 1$

$y = x^2$

$y' = 2x$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$0 \quad 1/4 \quad 2/4 \quad 3/4 \quad 1 = 4/4$

$h = 1/4$

$h^2 y_k'' = 2h^2$

$-y_{k-1} + 2y_k - y_{k+1} = -2h^2$

K=0

$-y_{-1} + 2y_0 - y_1 = -2h^2$ — (1)

from $y'(0) = 0$ or $y_0' = 0$

$y_k' = \dfrac{y_{k+1} - y_{k-1}}{2h}$

$\dfrac{y_1 - y_{-1}}{2h} = 0 \Rightarrow y_{-1} = y_1$

putting this in (1) we get

$-2y_1 + 2y_0 = -2.h^2$ — (2)

K=1 $\quad -y_0 + 2y_1 - y_2 = -2h^2$ — (3)

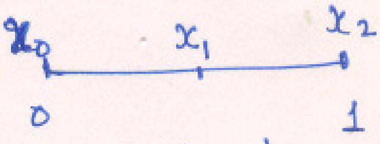
K=2 $\quad -y_1 + 2y_2 - y_3 = -2h^2$ — (4)

K=3 $\quad -y_2 + 2y_3 = -2h^2 + 1$ — (5)

Now (2)-(5) can be written as

$$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2h^2 \\ -2h^2 \\ -2h^2 \\ -2h^2 + 1 \end{pmatrix}$$

$y'' = 2 \qquad y'(0) = 0, \quad y'(1) = 2$



$h = \frac{1}{2}$

$$-y_{k-1} + 2y_k - y_{k+1} = -2h^2$$

<u>K=0</u>

$$2y_0 - 2y_1 = -2h^2 \quad \text{--- (1)}$$

<u>K=1</u>

$$-y_0 + 2y_1 - y_2 = -2h^2 \quad \text{--- (2)}$$

<u>K=2</u>

$$-y_1 + 2y_2 - \underline{\underline{y_3}} = -2h^2$$

$$-y_1 + 2y_2 - y_r = -2h^2 \quad \text{--- } (*)$$

from $y_2' = 2$ $\qquad y_k' = \frac{y_{k+1} - y_{k-1}}{2h}$

$$\frac{y_3 - y_1}{2h} = 2$$

$$y_r = y_3 = y_1 + 4h \quad \text{--- } (**)$$

putting value of y_r in $(*)$ we get

$$-y_1 + 2y_2 - y_1 - 4h = -2h^2$$

$$-2y_1 + 2y_2 = 4h - 2h^2 \quad \text{--- (3)}$$

Representing (1) - (3) in matrix form we get

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2h^2 \\ -2h^2 \\ 4h-2h^2 \end{pmatrix} \quad \text{--- (4)}$$

$$DY = B$$

Here the matrix D is singular, so (4) does not have unique solution and the method fails.

Solution of the bvp $y'' = 2,\ y'(0) = 0,\ y'(1) = 2$ is given by

$y(x) = x^2 + c$, where c is any constant so the bvp does not have unique solution. This is the reason that the method fails.