PROBABILITY & STATISTICS(MA20104, SEC 4)

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Contents

Date: Updated on April 11, 2021, Weekly classes:Mon(8-10am), Tue(12-1pm) .

1. Syllabus

Probability.

(1) Probability: Classical, relative frequency and axiomatic definitions of probability, addition rule and conditional probability, multiplication rule, total probability, Bayes' Theorem and independence.
(2) Random Variables: Discrete, continuous and mixed random variables, probability mass, probability density and cumulative distribution functions, mathematical expectation, moments, moment generating function, Chebyshev's inequality.
(3) Special Distributions: Discrete uniform, binomial, geometric, negative binomial, hypergeometric, Poisson, uniform, exponential, gamma, normal, beta, lognormal, Weibull, Laplace, Cauchy, Pareto distributions. Functions of a Random Variable.
(4) Joint Distributions: Joint, marginal and conditional distributions, product moments, correlation, independence of random variables, bivariate normal distribution, simple, multiple and partial correlation, regression.
(5) Sampling Distributions: Law of large numbers, Central Limit Theorem, distributions of the sample mean and the sample variance for a normal population, Chi-Square, t and F distributions.

Statistics.

(1) Estimation: The method of moments and the method of maximum likelihood estimation, properties of best estimates, confidence intervals for the mean(s) and variance(s) of normal populations.
(2) Testing of Hypotheses: Null and alternative hypotheses, the critical and acceptance regions, two types of error, power of the test, the most powerful test and Neyman-Pearson Fundamental Lemma, standard tests for one and two sample problems for normal populations.

2. Books

(1) 1. An Introduction to Probability and Statistics by V.K. Rohatgi & A.K. Md. E. Saleh
(2) Probability and Statistical Inference by Hogg, R. V., Tanis, E. A. & Zimmerman D. L.
(3) Probability and Statistics in Engineering by W.W. Hines, D.C. Montgomery, D.M. Goldsman, C.M. Borror
(4) Introduction to Probability and Statistics for Engineers and Scientists by S.M. Ross
(5) Introduction to Probability and Statistics by J.S. Milton & J.C. Arnold.
(6) Introduction to Probability Theory and Statistical Inference by H.J. Larson
(7) Probability and Statistics for Engineers and Scientists by R.E. Walpole, R.H. Myers, S.L. Myers, Keying Ye
(8) Modern Mathematical Statistics by E.J. Dudewicz & S.N. Mishra
(9) Introduction to the Theory of Statistics by A.M. Mood, F.A. Graybill and D.C. Boes

3. Evaluation

- Continuous evaluation

4. Probability : Definition & laws

Definition 1. Random experiment: A random experiment is a physical phenomena which satisfies the followings.

(1) It has more than one outcomes.
(2) The outcome of a particular trial is not known in advance.
(3) It can be repeated countably many times in in *identical* condition.

Example 2. (a) Tossing a coin, (b) Rolling a die and (c) Arranging 52 cards etc.

Definition 3. Sample space: A set which is collection of all possible outcomes of a random experiment is known as sample space for the experiment and it is denoted by Ω or S.

Example 4. For the above examples the sample spaces are
(a) $\{H, T\}$, (b) $\{1, 2, 3, 4, 5, 6\}$, (c) $\{\pi|\pi$ is any permutation of 52 cards$\}$ respectively

Definition 5. Classical definition of probability: If the sample space (Ω) of a random experiment is a $finite$ set and $A \subseteq \Omega$ the probability of A is defined as

$$P(A) = \frac{|A|}{|\Omega|}$$

under the assumption that all outcomes are equally likely. Here $|\cdot|$ denotes the cardinality of a set.

Exercise 6. Consider the equation $a_1 + a_2 + \cdots + a_r = n$ where $r < n$. Suppose a computer provides an integral solution of it at random such that each $a_i \in \mathbb{N} \cup \{0\}$ for any solution. Find the probability that each $a_i \in \mathbb{N}$ only, for a solution.

Definition 7. Frequency definition of probability: If the sample space $(\Omega)$of a random experiment is a *countable* set and $A \subseteq \Omega$ the probability of A is defined as

$$P(A) = \lim_{n\uparrow\infty} \frac{|A_n|}{|\Omega_n|}$$

where $\lim_{n\uparrow\infty} A_n = A$ and $\lim_{n\uparrow\infty} \Omega_n = \Omega$.

Exercise 8. What is the probability that a randomly chosen number from $\mathbb{N}$ will be a even number?

Exercise 9. What is the probability that a randomly chosen number from $\mathbb{N}$ will be a \$k-digit\$ number? Can you conclude an answer with frequency definition of probability?

Definition 10. Algebra: A collection $\mathcal{A}$ of the subsets of Ω is called an algebra if

(1) $\Omega \in \mathcal{A}$
(2) any $A \subseteq \Omega$ and $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$ [closed under complementation]
(3) any $A, B \subseteq \Omega$ and $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$ [closed under finite union]

Definition 11. σ-algebra or σ-field: An algebra $\mathcal{A}$ of the subsets of Ω is called an σ-algebra/ field if $\{A_i\} \subseteq \Omega$ and $\{A_i\} \in \mathcal{A}$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$ [closed countable union].

Example 12. (a) $\mathcal{A} = \{\emptyset, \Omega\}$, (b) $\mathcal{A} = \{\emptyset, \Omega, A, A^c\}$, (c) $\mathcal{A} = 2^{\Omega}$ [power set]

Definition 13. Axiomatic definition of probability (Kolmogorov): If $\mathcal{A}$ is σ-algebra of the subsets of a non-empty set Ω then the probability (P) is defined to be a function $P : \mathcal{A} \mapsto [0, 1]$ which satisfies,

(1) $P(\Omega) = 1$,
(2) $P(A) \geq 0$ for any $A \in \mathcal{A}$,
(3) $\{A_i\} \in \mathcal{A}$ implies $P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ if $A_i \cap A_j = \emptyset \quad \forall i \neq j$.

Example 14. What is the probability that a randomly chosen number from $\Omega = [0, 1]$ will be

(1) a rational number?
(2) less than 0.4? Can you conclude an answer with classical / frequency definition of probability?

Definition 15. Probability space: $(\Omega, \mathcal{A}, P)$ is known as a probability space.

Definition 16. Event: For a given Probability space $(\Omega, \mathcal{A}, P)$ if $A \subseteq \Omega$ and $A \in \mathcal{A}$ then A is called an event.

Definition 17. Conditional Probability: For a given Probability space $(\Omega, \mathcal{A}, P)$ if A and$B$ are two events such that $P(B) > 0$ then the conditional probability of A given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 18. Independent events: For a given probability space $(\Omega, \mathcal{A}, P)$ the two events A and B are called *independent* if $P(A|B) = P(A)$, which implies

$$P(A \cap B) = P(A)P(B).$$

Remark 19. If the probability function P is changed to some P_1 on the same $(\Omega, \mathcal{A})$ then events A and B may not be independent any more.

Definition 20. Pairwise independence: For a given probability space $(\Omega, \mathcal{A}, P)$ consider a sequence of events $\{A_i\}$. This sequence of events are called pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j) \quad \forall i \neq j.$$

Definition 21. Mutual independence: For a given probability space $(\Omega, \mathcal{A}, P)$ consider a sequence of events $\{A_i\}$. This sequence of events are called mutually independent if

$$P(\cap_{i_1, i_2 \cdots i_k} A_i) = \Pi_{i_1, i_2, \cdots i_k} P(A_i) \quad \forall i_1 \neq i_2, \neq \cdots \neq i_k. \text{for any} \quad k \in \mathbb{N}$$

Exercise 22. Give an example to show that pairwise independence does not imply mutual independence.

Definition 23. Mutually exclusive events: For a given probability space $(\Omega, \mathcal{A}, P)$ a sequence of events $\{A_i\}$ are called mutually exclusive if $A_i \cap A_j = \emptyset \quad \forall i \neq j$

Definition 24. Mutually exhaustive events: For a given probability space $(\Omega, \mathcal{A}, P)$ a sequence of events $\{A_i\}$ are called mutually exhaustive if $\cup_{i=1}^{\infty} A_i = \Omega$

Remark 25. Mutually exhaustive or exclusive events **does not depend the probability function**.

Definition 26. Partition: For a given probability space $(\Omega, \mathcal{A}, P)$ a sequence of events $\{A_i\}$ are called a partition of Ω if $\{A_i\}$ are **mutually exclusive and exhaustive.**

Exercise 27. Prove the following properties:

(1) $P(A^c) = 1 - P(A)$
(2) $P(\emptyset) = 0$
(3) If $A \subseteq B$ then $P(A) \leq P(B)$
(4) $1 - P(\cup_{i=1}^{\infty} A_i) = P(\cap_{i=1}^{\infty} A_i^c)$
(5) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(6) $P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$
(7) $P(\cup_{i=1}^{n} A_i) = \sum_{k=1}^{n} (-1)^{k-1} S_k$ where

$$S_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k})$$

Exercise 28. Suppose n letters are put in n envelops distinct by addresses. What is the probability that no letter will reach to the correct address. What is the limiting probability as $n \uparrow \infty$?

Theorem 29. ***Bayes Theorem:*** *Let $A_1, A_2, \cdots A_k$ is a partition of Ω and $(\Omega, \mathcal{A}, P)$ be a probability space with $P(A_i) > 0 \ \ \forall i$ and $P(B) > 0$ for some $B \subseteq \Omega$. Then*

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}.$$

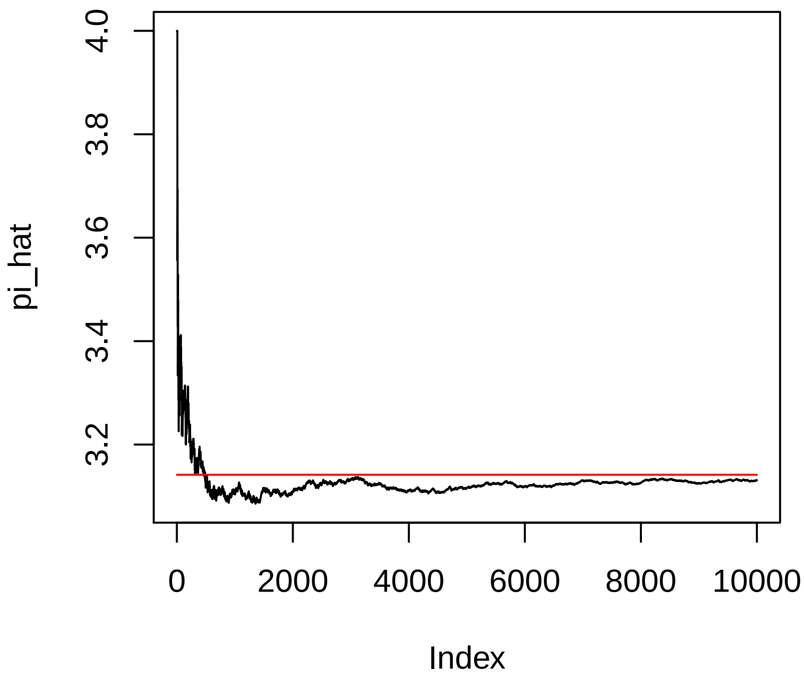
Exercise 30. There are three drawers in a table. The first drawer contains two gold coins. The second drawer contains a gold and a silver coin. The third one contains two silver coins. Now a drawer is chosen at random and a coin is also chose randomly. It is found that the a gold coin has been selected then what is the probability that the second drawer was chosen?

Exercise 31. Consider the quadratic equation $u^2 - \sqrt{Y}u + X = 0$, where (X, Y) is a random point chosen uniformly from a unit square. What is the probability that the equation will have a real root?

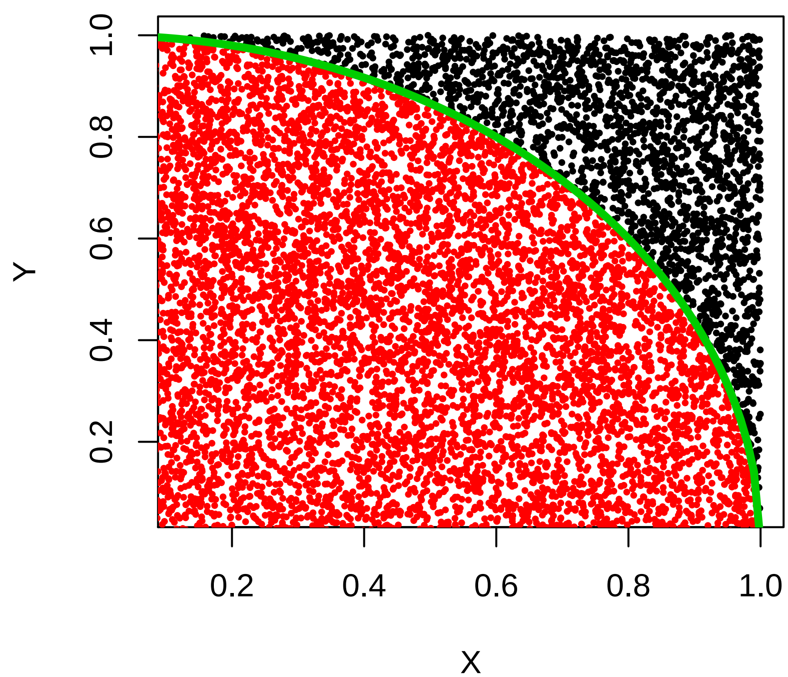
Exercise 32. Give a randomized algorithm to approximate value of π.

Exercise 33. Give a randomized algorithm to approximate value of e.

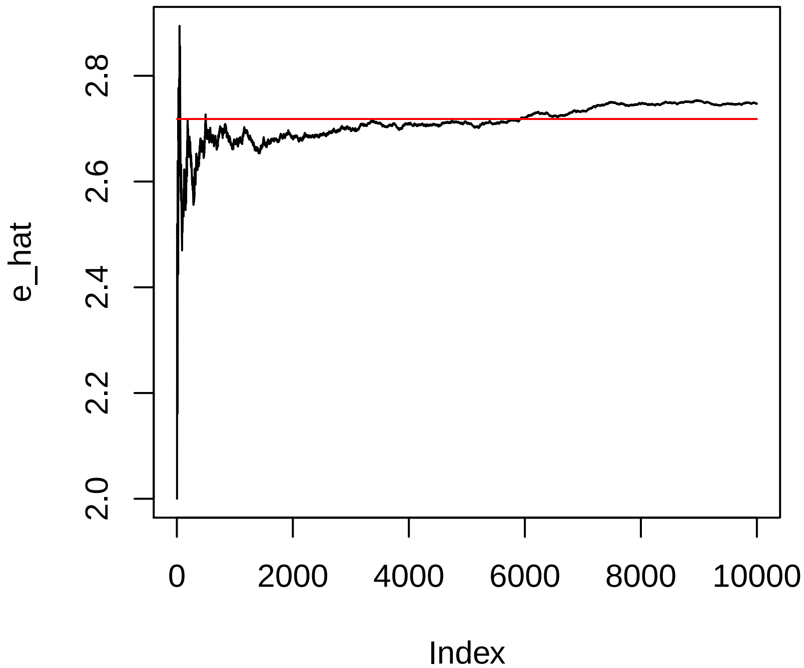
```
# Date:26 July 2019
## Pi estimation
## Range of X
a1<-0
b1<-1
## Range of Y
a2<-0
b2<-1
itrn<- 10000 # iteration number
x<-runif(itrn,a1,b1) # generate X
y<-runif(itrn,a2,b2) # generate Y
pi_true<-rep(pi,itrn) # True value of pi
pi_hat<-array(0,dim=c(itrn))
count<-array(0,dim=c(itrn))
xx<-seq(a1,b1,by=0.01) # Sequence on [0,1]

for(i in 1 : itrn){
  count [i]<- (x[i]^2+y[i]^2<1) # (X,Y) in circle or not
  pi_hat[i]<-4*(sum(count)/i) # How many in cirecle amonge 'i' trials
}
# Plot
plot(pi_hat, type = 'l')
lines(pi_true, col=2)

s0<- which (count==0) # points out of circle
s1<- which (count==1) # points in circle
plot(y[s0]~x[s0], pch = 20, cex = 0.5, xlab="X", ylab = "Y")
lines (y[s1]~x[s1], col=2, type='p', pch = 20, cex = 0.5)
lines(sqrt(1-(xx)^2)~xx , col=3, lwd=4) # equation of circle
```



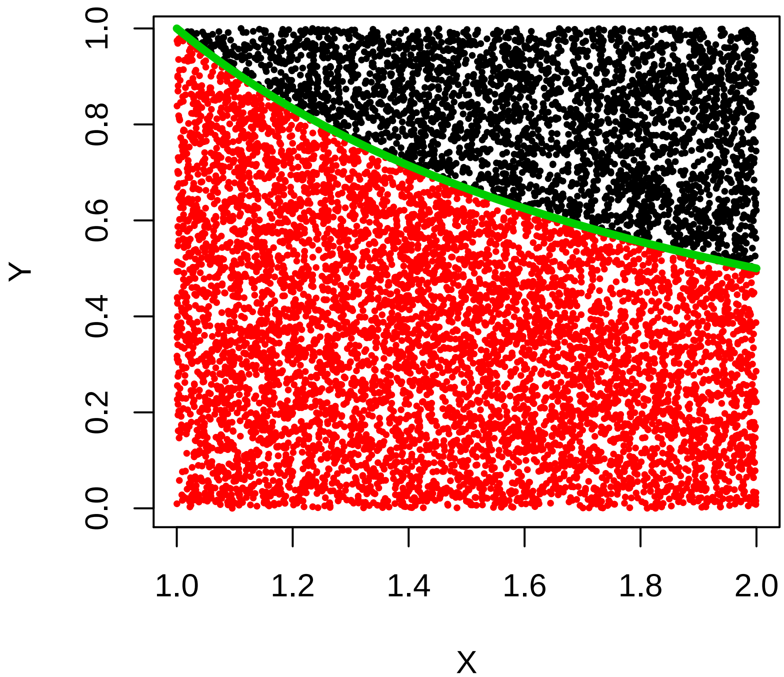
pi_hat
4.0
3.8
3.6
3.4
3.2
0
2000
4000
6000
8000
10000
Index


1.0
0.8
0.6
0.4
0.2
Y
0.2
0.4
0.6
0.8
1.0
X

```r
# e estimation
# # Date:26 July 2019
a1<-1
b1<-2
a2<-0
b2<-1
itrn<- 10000
x<-runif(itrn,a1,b1)
y<-runif(itrn,a2,b2)
e_true<-rep(exp(1),itrn)
e_hat<-array(0,dim=c(itrn))
count<-array(0,dim=c(itrn))
xx<-seq(a1,b1,by=0.01)
for(i in 1 : itrn){
  count [i]<- (x[i]*y[i]<1)
  area<-(sum(count)/i)
  e_hat[i]<- 2^(1/area)

}
plot(e_hat, type = 'l')
lines(e_true, col=2)
s0<- which (count==0)
s1<- which (count==1)
plot(y[s1]~x[s1], col=2, pch = 20, cex = 0.5, xlab="X", ylab = "Y")
lines (y[s0]~x[s0], type='p', pch = 20, cex = 0.5)
lines((1/(xx))~xx , col=3, lwd=4)
```



e_hat
2.0
2.2
2.4
2.6
2.8
0
2000
4000
6000
8000
10000
Index


Y
1.0
0.8
0.6
0.4
0.2
0.0
1.0
1.2
1.4
1.6
1.8
2.0
X

5. Random Variable and it's moments

Definition 34. Random variable: Let $(\Omega, \mathcal{A}, P)$ be a probability space. Then a function $X : \Omega \to \mathbb{R}$ is called a random variable if

$$X^{-1}((-\infty, x]) \equiv \{\omega | X(\omega) \le x\} \in \mathcal{A} \ \ \forall x \in \mathbb{R}$$

Remark 35. **Random variable is a deterministic function which has nothing random in it.**

Remark 36. Consider a function $g : \mathbb{R} \to \mathbb{R}$ and X is a random variable on $(\Omega, \mathcal{A}, P)$. Then $Y = g(X)$ is also a random variable. It implies that $X^{-1}(g^{-1}(-\infty, x]) \in \mathcal{A}$ for any $x \in \mathbb{R}$. It means $Q((-\infty, x]) = P(Y \in (-\infty, x]) = P(X \in g^{-1}((-\infty, x]))$, which is known as **push forward** of probability.

Definition 37. Vector Valued Random variable: $(X_1(\omega), X_2(\omega), \cdots, X_k(\omega))$ is a vector valued random variable where $\omega \in \Omega$.

Remark 38. Let X and Y be random variables. Then,

- $aX + bY$ is a random variable for all$a, b \in \mathbb{R}$.
- $\max\{X, Y\}$ and $\min\{X, Y\}$ are random variables.
- XY is a random variable.
- Provided that $P(Y(\omega) = 0) = 0$ for each $\omega \in \Omega$, then X/Y is a random variable.

Definition 39. Cumulative distribution function (c.d.f.): Cumulative distribution function of a random variable X is a function $F : \mathbb{R} \to [0, 1]$ defined as

$$\begin{aligned} F(x) &= P(X \le x) \\ &= P(X^{-1}(-\infty, x]) \\ &= P(\{\omega | X(\omega) \in (-\infty, x]\}) \ \ \forall x \in \mathbb{R}. \end{aligned}$$

Remark 40. Cumulative distribution function uniquely identifies a random variable.

Exercise 41. Prove the properties of a c.d.f. :

(1) $F(-\infty) = \lim_{x \downarrow -\infty} F(x) = 0$
(2) $F(\infty) = \lim_{x \uparrow \infty} F(x) = 1$
(3) $F(a) \le F(b) \quad \forall a \le b \in \mathbb{R}$ [non-decreasing]
(4) $F(a) = \lim_{x \downarrow a} F(x) \quad \forall a \in \mathbb{R}$ [right-continuous]

Definition 42. Discrete valued random variable: For a given probability space $(\Omega, \mathcal{A}, P)$ a random variable X is said to be a discrete valued random variable if $S = \{X(\omega) | \omega \in \Omega\}$ is a **finite or countably infinite** set and $X^{-1}(s_i) \in \mathcal{A}$ for all $s_i \in S$.

Remark 43. There can be finitely or countably many jump discontinuities in a c.d.f. of a random variable. The sum of the magnitude of jumps is one, which is the total probability.

Definition 44. Probability mass function(p.m.f.): If X is a discrete valued random variable ina a given probability space $(\Omega, \mathcal{A}, P)$ then a non-negative function

$f(x) := P(X = x)$ on $\mathbb{R}$ is called a probability mass function or discrete density function of the random variable X. Probability mass function has the following properties

- $f(x) \geq 0 \quad \forall x \in \mathbb{R}$.
- $S = \{x | f(x) > 0\}$ is finite or a countably infinite set.
- $\sum\limits_{s \in S} f(s) = 1$

Definition 45. Continuous valued random variable: For a given probability space $(\Omega, \mathcal{A}, P)$ a random variable X is said to be a continuous valued random variable $P(X = x) = 0 \forall x \in \mathbb{R}$.

Definition 46. Probability density function(p.d.f.): If X is a continuous valued random variable in a given probability space $(\Omega, \mathcal{A}, P)$ with c.d.f $F(\cdot)$ then a non-negative function $f : \mathbb{R} \to [0, \infty)$ is called **a probability density function** of X if

$$P(X \in A) = \int_x f(x) \mathbf{1}_{\{x \in A\}} dx$$

Remark 47. In particular for $A = (-\infty, x]$ for any $x \in \mathbb{R}$ then

$$f(x) = \frac{d}{dx} F(x) = \int_{-\infty}^{x} f(t) dt.$$

Definition 48. Expectation: The expectation of a random variable X with c.d.f $F_X(\cdot)$ is defined as $E(X) = \int x dF_X(x)$ where,

$$\int x dF_X(x) = \begin{cases} \sum_x x f(x), & \text{if } \sum_x |x| f(x) < \infty \text{ for discrete } X, \\ \int_x x f(x), & \text{if } \int_x |x| f(x) < \infty \text{ for continuous } X. \end{cases}$$

Exercise 49. Find the expectation of the random variables with the following densities

(a) $f(x) = \frac{1}{\pi(1+x^2)}$ when $x \in \mathbb{R}$

(b) $f(x) = \frac{1}{|x|(1+|x|)}$ when $x \in S = \{(-1)^n n | n \in \mathbb{N}\}$

Definition 50. Moment generating function: The moment generating function (**m.g.f.**) of a random variable X is defined as

$$M_X(t) = E(e^{tX}) \text{ if } E(e^{tX}) < \infty \; \forall t \in (-\epsilon, \epsilon) \text{ for some } \epsilon > 0$$

- Cumulative distribution function (c.d.f) and uniquely identify the probability distribution of a random variable.
- Moment generating function (m.g.f.) if exists then uniquely identifies the probability distribution of a random variable.
- Probability density function identifies the probability distribution of a random variable up to some length or volume zero set. So it is not unique in general.

Remark 51. Probability mass function can be considered as discrete density function with respect to count measure. If X is a discrete valued random variable with $P(X \in S) = 1$, where S is a countable set, then a non-negative function f is called a probability mass function or discrete density function of the random variable X if $P(X \leq x) = \sum_{s \in S} f(s)\mathbf{1}_{\{s \leq x\}} \quad \forall x \in \mathbb{R}$.

5.1. **Moments of a random variable.**

Definition 52. Raw moment: Let X be a discrete valued random variable with p.m.f $f(\cdot)$ such that $\sum_x |x^r|f(x) < \infty$. Then the r^{th} order raw moment of X is defined as

$$\mu_r^{'} = E(X^r) = \sum_x x^r f(x)$$

Definition 53. Central moment: Let X be a discrete valued random variable with p.m.f $f(\cdot)$ such that $\sum_x |(x-\mu_x)^r|f(x) < \infty$. Then the r^{th} order raw moment of X is defined as

$$\mu_r = E(X-\mu_x)^r = \sum_x (x-\mu_x)^r f(x).$$

- Mean of random variable X is $\mu_1^{'} = E(X) = \sum_x xf(x) = \mu_x$.
- Variance of random variable X is $\mu_2 = E(X-\mu_x)^2 = \sum_x (x-\mu_x)^2 f(x) = Var(X)$.

Exercise 54. Prove that:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$$

Exercise 55. Show that $g(a) = \frac{1}{n}\sum_{i=1}^{n}(x_i-a)^2$ is minimum if $a = \bar{x}$.

Theorem 56. *If X is a non-negative integer-valued random variable with finite expectation then*

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k)$$

Moments from Moment generating function: Let X be a discrete valued random with moment generating function

$$M_X(t) = E(e^{tx}) = \sum_{k=0}^{\infty} \frac{t^k E(X^k)}{k!}$$

Then one can obtain the kth order raw moment from m.g.f. by

$$\frac{\partial^k}{\partial t^k} M_X(t)|_{t=0} = \mu_k^{'}$$

Exercise 57. Prove the following inequalities:

- **Markov's Inequity:** If X is a non-negative valued random variable then
$$P(X > t) \leq \frac{E(X)}{t} \quad \forall t > 0$$
- **Chebyshev's Inequity:** $P(|X-\mu_x| > \epsilon) \leq \frac{E(X-\mu_x)^2}{\epsilon^2}$

Definition 58. For a discrete probability distribution, a median is by definition any real number m that satisfies the inequalities

$$\mathrm{P}(X \leq m) \geq \frac{1}{2} \text{ and } \mathrm{P}(X \geq m) \geq \frac{1}{2}$$

and for a continuous probability distribution,

$$\mathrm{P}(X \leq m) \geq \frac{1}{2} = \mathrm{P}(X \geq m) = \frac{1}{2}\,.$$

Exercise 59. Graphically show that $g(a) = \frac{1}{n}\sum_{i=1}^{n}|x_i - a|$ is minimum if $a =$ *median* of $\{x_1, \cdots, x_n\}$.

6. Modeling with Random Variables

Definition 60. Independent Random Variables: Two random variables X and Y are said to be independently distributed if

$$P((X,Y) \in A \times B)) = P(X \in A)P(Y \in B) \text{ for any } A, B \in \mathcal{B}(\mathbb{R})$$

This implies
(a)$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$ for any $x, y \in \mathbb{R}$
(b)$f(x,y) = f_x(x)f_y(y)$

Definition 61. Identically distributed random variables: Two random variables X and Y are said to be identically distributed if

$$P(X \in (-\infty, a]) = P(Y \in (-\infty, a]) \text{ for any } a \in \mathbb{R}.$$

Remark 62. Two random variables X and Y are **independently and identically distribute (i.i.d.)** random variables if the above two definitions hold.

Definition 63. Uniform Distribution[0,1]: A random variable X is said to have uniform distribution over $[0,1]$ if

$$P(X \in A) = \frac{length(A)}{length([0,1])} = length(A)$$

for any interval $A \subseteq [0,1]$. If $X \sim U[0,1]$ then the p.d.f. is

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and the c.d.f. is

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x \in [0,1] \\ 1, & \text{if } x > 1 \end{cases}$$

6.1. Examples of discrete random variables:

- **Discrete uniform:** [random sampling with replacement from a finite population]

$$P(X = s) = \begin{cases} \frac{1}{k}, & \text{if } s \in \mathcal{S} = \{s_1, s_2, \cdots s_k\} \\ 0, & \text{otherwise} \end{cases}$$

- **Bernoulli (p):** [binary $\{0,1\}$ random variable]

$$P(X = x) = \begin{cases} p^x(1-p)^{1-x}, & \text{if } x \in \mathcal{S} = \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$

- **Binomial (n,p):** [sum of n i.i.d. Bernoulli(p)]

$$P(X = x) = \begin{cases} \binom{n}{x}\, p^x(1-p)^{n-x}, & \text{if } x \in \mathcal{S} = \{0, 1\cdots, n\} \\ 0, & \text{otherwise} \end{cases}$$

- **Geometric (p):** [number of failures preceding to the first success]

$$P(X = x) = \begin{cases} p(1-p)^x, & \text{if } x \in \mathcal{S} = \{0\} \cup \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

- **Negative Binomial (r,p)** [sum of r i.i.d. geometric(p)]

$$P(X=x)=\begin{cases}\binom{x+r-1}{r-1}p^r(1-p)^x, & \text{if } x\in\mathcal{S}=\{0,1,2\cdots\}\\ 0, & \text{otherwise}\end{cases}$$

- **Poisson (λ):** [limiting distribution of $bin(n,p_n)$ when $n\uparrow\infty$, $p_n\downarrow 0$ but $np_n\to\lambda>0$]

$$P(X=x)=\begin{cases}\frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x\in\mathcal{S}=\{0\}\cup\mathbb{N}\\ 0, & \text{otherwise}\end{cases}$$

- **Hyper-geometric:** [random sampling without replacement from a finite population divided into two categories]

$$P(X=x)=\begin{cases}\frac{\binom{n_1}{r_1}\binom{n_2}{r_2}}{\binom{n_1+n_2}{r}}, & \text{if } r_1=0,1,\cdots\min\{n_1,r\}; r_2=0,1,\cdots\min\{n_2,r\}; r=r_1+r_2,\\ 0, & \text{otherwise}\end{cases}$$

Theorem 64. *Let $\{X_n\}$ be a sequence of random variables with corresponding m.g.f.s as$M_{X_n}(t)$ such that $\lim_{n\uparrow\infty}M_{X_n}(t)=M_Y(t)$ for some random variable Y. Then*

$$\lim_{n\uparrow\infty}F_{X_n}(a)=F_Y(a)$$

for all such $a\in\mathbb{R}$, where $F_Y(a)$ is continuous. We say X_n converges in distribution to Y. [Proof is not included in the syllabus]

6.2. **Examples of continuous random variables:**

Exercise 65. Let $X\sim U[0,1]$. Find the c.d.f and p.d.f

(a)$Y_1=cX$ where $c>0$. [this is known as $U[0,c]$]
(b)$Y_2=a+(b-a)X$ where $b>a$.[this is known as $U[a,b]$]
(c) Show that $E(Y_2)=\frac{b+a}{2}$ and $Var(Y_2)=\frac{(b-a)^2}{12}$.

Exercise 66. Let $X\sim U[0,1]$. Find the c.d.f and p.d.f.
(a)$Z=\frac{-1}{\lambda}\log(1-X)$ where $\lambda>0$[it is known as $exponential(\lambda)$ distribution]
(b)Show that $E(Z)=\frac{1}{\lambda}$ and $Var(Z)=\frac{1}{\lambda^2}$.

- A positive valued random Y variable is said to follow *Gamma distribution* with shape parameter $\alpha(>0)$ and scale parameter $\lambda(>0)$ if it has p.d.f.

$$f(y)=\frac{\lambda^\alpha e^{-\lambda y}y^{\alpha-1}}{\Gamma(\alpha)}\mathbf{1}_{\{y>0\}}$$

Remark 67. Let $X_1\sim Gamma(a,\lambda)$ and $X_2\sim Gamma(b,\lambda)$ are independently distributed then $Y=X_1+X_2\sim Gamma(a+b,\lambda)$. Use MGF.

Remark 68. If $X\sim Gamma(a,\lambda)$, then $E(X)=\alpha/\lambda$ and $Var(X)=\alpha/\lambda^2$

- Let $X_1\sim G(a,\lambda)$ and $X_2\sim G(b,\lambda)$ are independently distributed then $Y=\frac{X_1}{X_1+X_2}$ is said to follow$Beta(a,b)$ and the p.d.f. of $Y$is given by

$$f(y)=\begin{cases}\frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}, & \text{if } y\in[0,1], a>0, b>0\\ 0, & \text{otherwise}\end{cases}$$

Remark. Let $Y\sim B(a,b)$ then, $E(Y)=\frac{a}{a+b}$ and $Var(Y)=\frac{ab}{(a+b)^2(a+b+1)}$

- A random variable X is is said to follow $Cauchy(\mu, \sigma)$ if it has the p.d.f.

$$f(x) = \frac{\sigma}{\pi(\sigma^2 + (y-\mu)^2)}$$

and has c.d.f.

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{y-\mu}{\sigma}\right)$$

- A random variable X is is said to follow $Normal(\mu, \sigma^2)$ if it has the p.d.f.

$$f(x) = \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}}$$

where $x, \mu \in \mathbb{R}$ and $\sigma > 0$.

NOTE:

(1) $N(0,1)$ is also known as standard normal distribution.
(2) p.d.f of standard normal distribution is denoted by $\phi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$
(3) c.d.f of standard normal distribution is denoted by $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} dt$
(4) If $Z \sim N(0,1)$ then$Y = \mu + \sigma Z \sim N(\mu, \sigma^2)$ with $E(Y) = \mu$ & $Var(Y) = \sigma^2$
(5) The p.d.f of Y can be written as $f(y) = \frac{1}{\sigma}\phi(\frac{y-\mu}{\sigma})$
(6) $\phi(\cdot)$ is a symmetric function around zero i.e. $\phi(z) = \phi(-z)$
(7) $\Phi(-z) = 1 - \Phi(z)$
(8) $P(Z < 1.64) = 0.95$ and $P(Z < 1.96) = 0.975$.

- **Chi-squared Distribution:** If $Z \sim N(0,1)$ then random variable$Y = Z^2$ is said to follow χ_1^2 i.e. chi-squared distribution with one degree of freedom.

Remark 69. Y following χ_1^2 has same p.d.f of $Gamma(1/2, 1/2)$ distribution.

Remark 70. If Z_i be i.i.d.$N(0,1)$ then random variable $Y = \sum_{i=1}^{n} Z_i^2$ follows χ_n^2 i.e. chi-squared distribution with$n$ degree of freedom which is equivalent to $Gamma(n/2, 1/2)$ distribution.

Remark 71. Let $Y \sim \chi_n^2$ then show that $E(Y) = n$ and $Var(Y) = 2n$

- $\mathbf{t-distribution}$: If $Z \sim N(0,1)$ and $Y \sim \chi_k^2$ are independently distributed random variables the

$$X = \frac{Z}{\sqrt{Y/k}} \sim t_k, \text{ i.e. t-distribution with k degrees freedom.}$$

- $\mathbf{F-distribution}$:If $Y_1 \sim \chi_{k_1}^2$ and $Y_2 \sim \chi_{k_2}^2$ are independently distributed random variables the

$$X = \frac{Y_1/k_1}{Y_2/k_2} \sim F_{k_1,k_2}, \text{ i.e. F-distribution with } k_1, k_2 \text{ degrees freedom.}$$

Theorem 72. *Let ϕ be a strictly monotone function on $I = (a,b)$ with the range $\phi(I)$ and differentiable inverse function $\phi^{-1}(\cdot)$ on $\phi(I)$. Also assume that X be a continuous valued random variable with p.d.f $f_X(x) = 0$ if $x \notin I$. Then $Y = \phi(X)$ has density $g(\cdot)$ on $\phi(I)$ as*

$$g(y) = f(\phi^{-1}(y))\left|\frac{d}{dy}\phi^{-1}(y)\right|.$$

Exercise 73. Let $X \sim bin(n,p)$. Show that $M_X(t) = (pe^t + 1 - p)^n$

Exercise 74. Let $Y \sim pois(\lambda)$. Show that $M_Y(t) = e^{-\lambda(1-e^t)}$

Exercise 75. $X \sim bin(n, p_n)$ such that $n \uparrow \infty$, $p_n \downarrow 0$ and $np_n \to \lambda > 0$. Show that X_n converges in distribution to Y, where $Y \sim pois(\lambda)$

Exercise 76. Let $Y \sim pois(\lambda)$. Show that $E(Y) = Var(Y) = E(Y-\lambda)^3 = \lambda$

Exercise 77. Let $X \sim N(\mu, \sigma^2)$ then find the density function of $Y = e^X$. [Y is said to follow $lognormal(\mu, \sigma^2)$]

Exercise 78. Let $Y \sim lognormal(\mu, \sigma^2)$ find $E(X)$ and $Var(X)$.

Exercise 79. Let $X \sim N(0,1)$ then find the density function of$Y = X^2$.

Exercise 80. Let $X \sim N(\mu, \sigma^2)$ then them MGF of X is $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$

Exercise 81. Let $X \sim exp(\lambda)$. Show that$Y = [X]$ has geometric distribution.

Exercise 82. Let $X \sim geo(p)$. Show that X has **memory less property** i.e. $P(X > m+n | X > m) = P(X > n) = q^n$ where $m, n \in \mathbb{N}$

Exercise 83. Let $X \sim exp(\lambda)$. Show that X has **memory less** property i.e.$P(X > t+s | X > t) = P(X > s)$ where $s, t \in \mathbb{R}$

Exercise 84. Let X be a continuous values random variable. Find the distribution of $Y = F(X)$.

Exercise 85. Let $(\frac{X}{\lambda})^k \sim exp(1)$ for $\lambda > 0$,$k > 0$, find the p.d.f. X. [X is said to follow Weibull distribution]

Exercise 86. Let $1 - (\frac{\lambda}{X})^k \sim U(0,1)$ for $X > \lambda$,$k > 0$, find the p.d.f. X. [X is said to follow Pareto distribution]

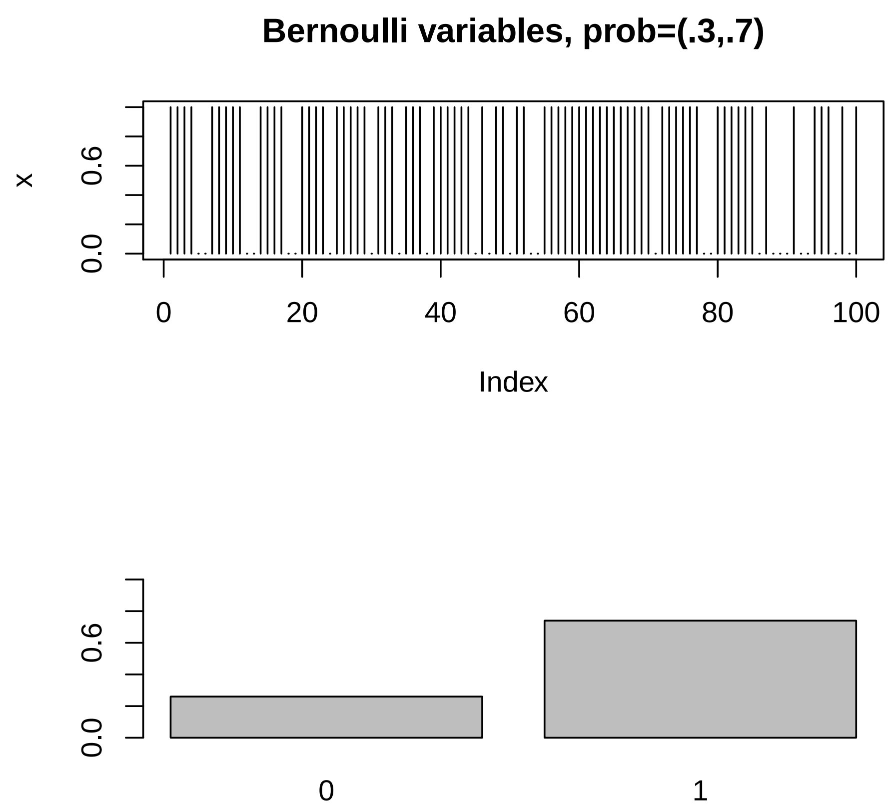
Exercise 87. Let $X \sim bin(n_1, p)$ and $Y \sim bin(n_2, p)$ independently . Use MGF to show that $Z = X + Y \sim bin(n_1 + n_2, p)$

Exercise 88. Let $X \sim pois(\lambda_1)$ and $Y \sim pois(\lambda_2)$ independently . Use MGF to show that $Z = X + Y \sim pois(\lambda_1 + \lambda_2)$

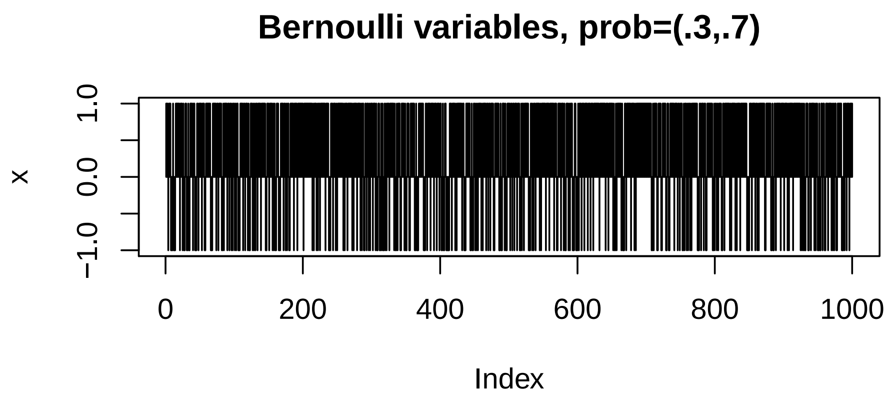
Exercise 89. Let $X \sim gamma(\alpha_1, \lambda)$ and $Y \sim bin(\alpha_2, \lambda)$ independently . Use MGF to show that $Z = X + Y \sim bin(\alpha_1 + \alpha_2, \lambda)$

Exercise 90. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ independently . Use MGF to show that $Z = X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

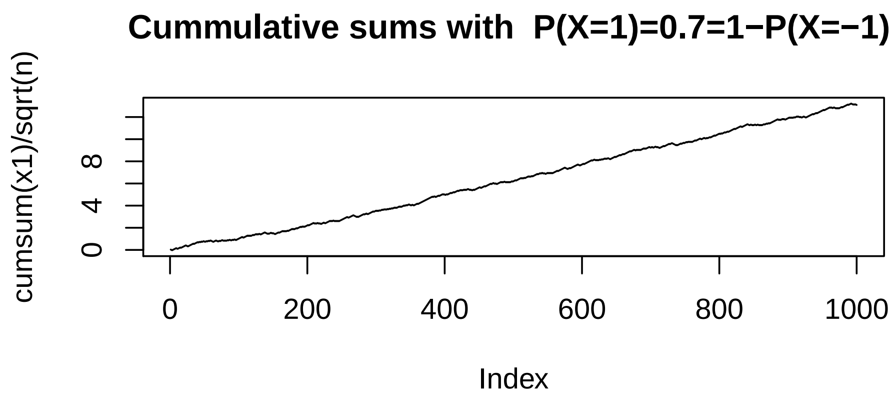
```r
# Bernoulli distribution of parameter p=0.7
n <- 100
x <- sample(c(0,1), n, replace=T, prob=c(.3,.7))
par(mfrow=c(2,1))
plot(x, type='h',main="Bernoulli variables, prob=(.3,.7)")
barplot(table(x)/n, ylim = c(0,1))
```

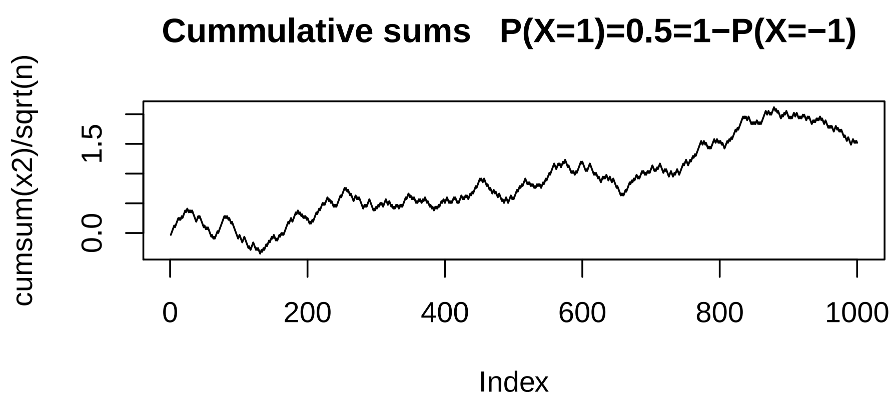



```r
# Bernoulli distribution of parameter p=0.8 and X =-1 and +1
n <- 1000
x <- sample(c(-1,1), n, replace=T, prob=c(.3,.7))
par(mfrow=c(2,1))
plot(x, type='h', main="Bernoulli variables, prob=(.3,.7)")
barplot(table(x)/n, ylim = c(0,1), main = "Bar plot")
```

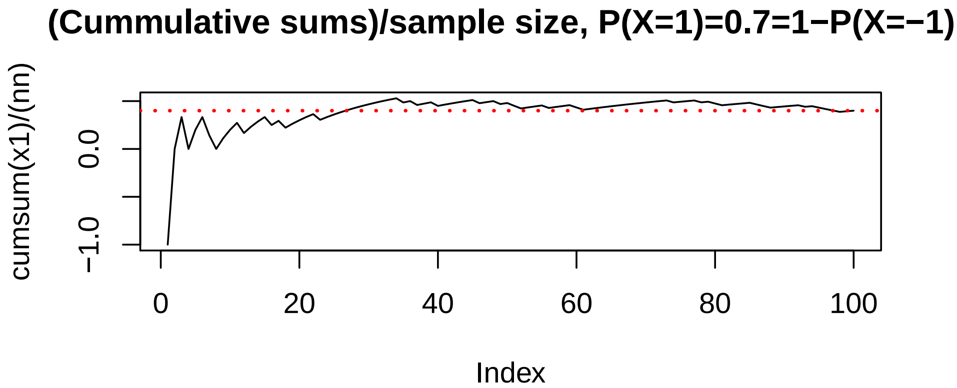


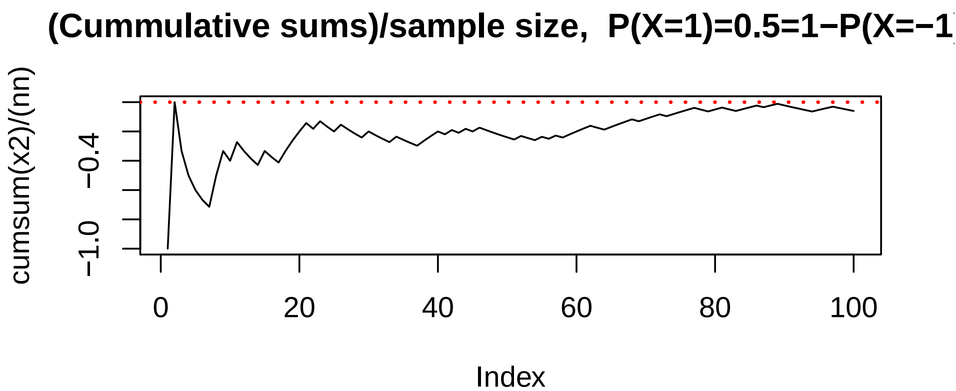

```
# Cummulative sums  X =-1 and +1 with scaling 1/sqrt(n)
n <- 1000
x1 <- sample(c(-1,1), n, replace=T, prob=c(.3,.7))
x2 <- sample(c(-1,1), n, replace=T, prob=c(.5,.5))
par(mfrow=c(2,1))
plot(cumsum(x1)/sqrt(n), type='l',main="Cummulative sums with  P(X=1)=0.7=1-P(X=-1) ")
plot(cumsum(x2)/sqrt(n), type='l',main="Cummulative sums   P(X=1)=0.5=1-P(X=-1) ")
```



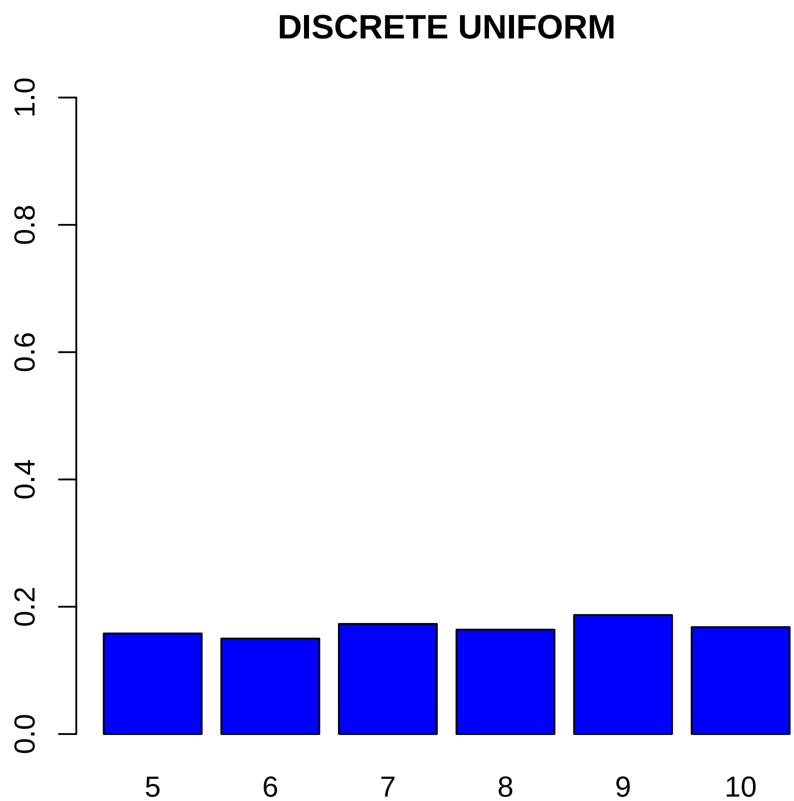



```
# Cummulative sums  X =-1 and +1 with scaling 1/(n)
n <- 100
nn<-((1:n))
x1 <- sample(c(-1,1), n, replace=T, prob=c(.3,.7))
x2 <- sample(c(-1,1), n, replace=T, prob=c(.5,.5))
par(mfrow=c(2,1))
plot(cumsum(x1)/(nn), type='l',
     main="(Cummulative sums)/sample size, P(X=1)=0.7=1-P(X=-1)  ")
abline(h=0.4, col=2,lwd=2, lty=3)
plot(cumsum(x2)/(nn), type='l',
     main="(Cummulative sums)/sample size,  P(X=1)=0.5=1-P(X=-1) ")
abline(h=0.0, col=2,lwd=2, lty=3)
```

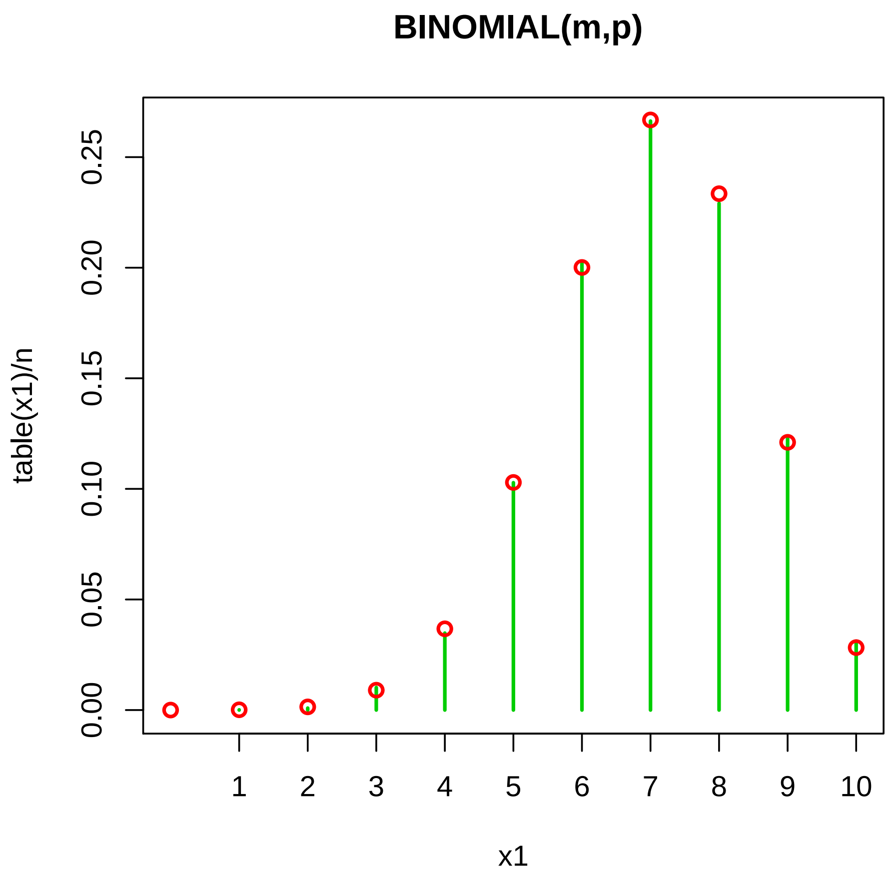




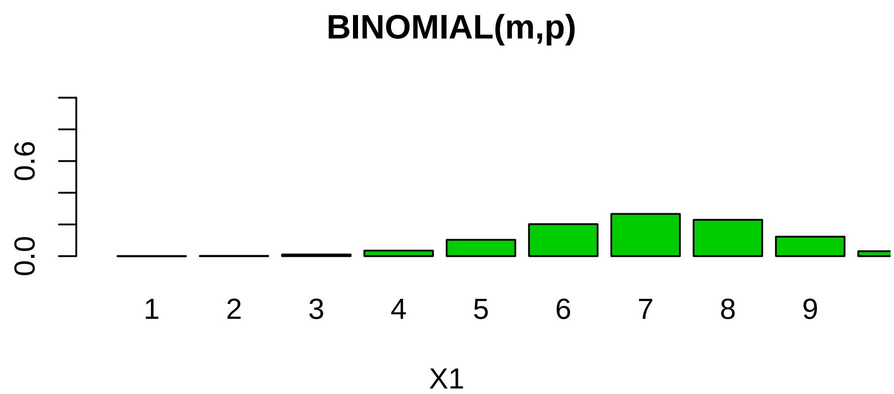

```
####################################################
#   DISCRETE UNIFORM
####################################################
n<- 1000 # sample size
sp<- 5:10 # sample space
x<-sample(sp, n, replace=T)   # data
par(mfrow=c(1,1))
barplot(table(x)/n, col=4, ylim=c(0,1), main = " DISCRETE UNIFORM ")
```

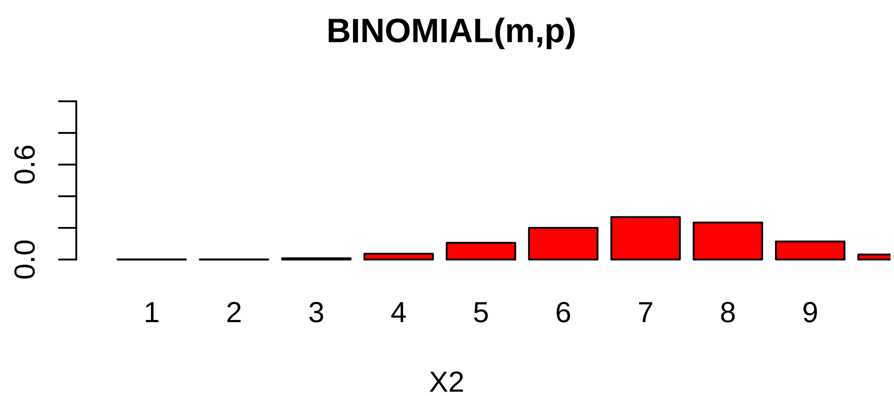



```
####################################################
#   BINOMIAL(m,p)
#It is the total number of  successrs  in a series of  m Bernoulli events.
####################################################
n<- 10000 # sample size
m<-10
p<-0.7
x1<-rbinom(n,m,p)   # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, xlim=c(0,m), main = " BINOMIAL(m,p)")
lines(dbinom(0:m, m,p)~c(0:m), type='p', col=2, lwd=2)
```

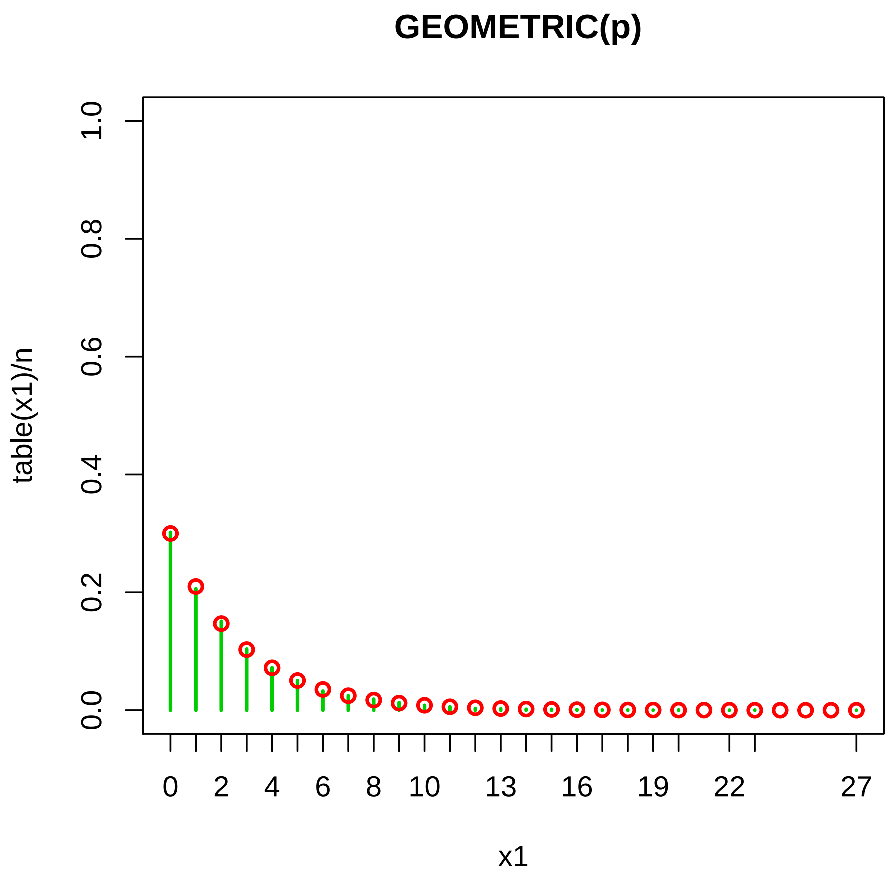



```
# Binomoal from Uniform(0,1)
x2 <- array(0,dim=c(n))
for (i in 1:n) {
  x2[i] <- sum(runif(m)<p)
}
par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, ylim=c(0,1), xlim=c(0,m), xlab='X1', main = " BINOMIAL(m,p)")
barplot(table(x2)/n, col=2, ylim=c(0,1), xlim=c(0,m),xlab='X2', main = " BINOMIAL(m,p)")
```

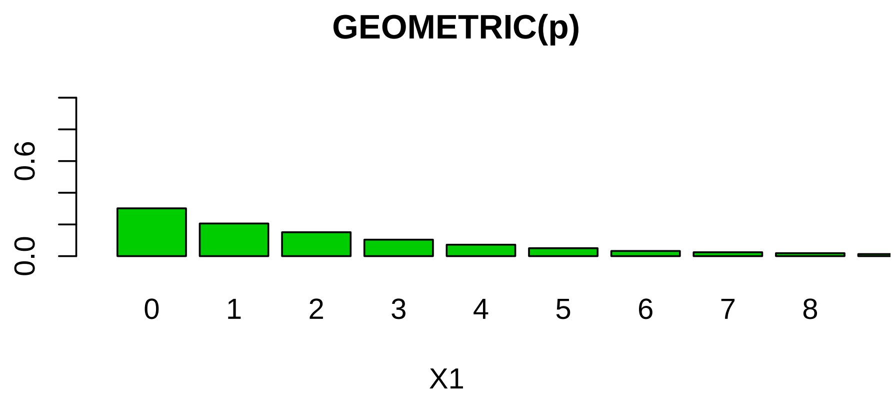


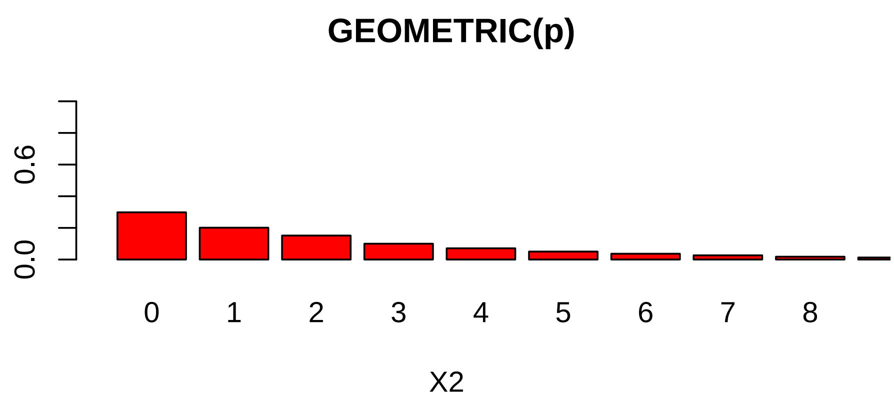



```
#####################################################
#   GEOMETRIC(p)
#It is the number of failures  before a success in a series of Bernoulli event.
#####################################################
n<- 10000 # sample size
p<-0.3
x1<-rgeom(n,p)    # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, ylim=c(0,1), main = " GEOMETRIC(p)")
lines(dgeom(0:max(x1),p)~c(0:max(x1)), type='p', col=2, lwd=2)
```

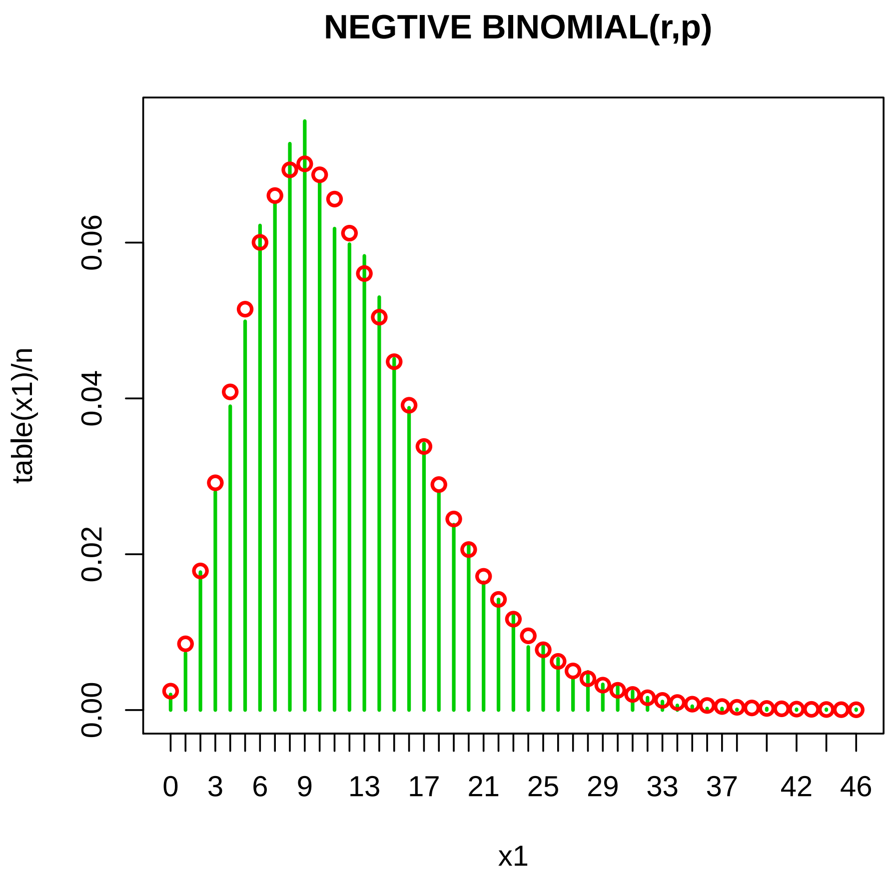



```
# Geometric from Uniform(0,1)
x2 <- array(0,dim=c(n))
for(i in 1 :n){
count<-0
s<-0
while (s==0) {

  count=count+1
  s<-(runif(1)<p)
}
x2[i]<-count-1
}
par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, ylim=c(0,1), xlim=c(0,m), xlab='X1', main = "  GEOMETRIC(p)")
barplot(table(x2)/n, col=2, ylim=c(0,1), xlim=c(0,m), xlab='X2',main = " GEOMETRIC(p)")
```

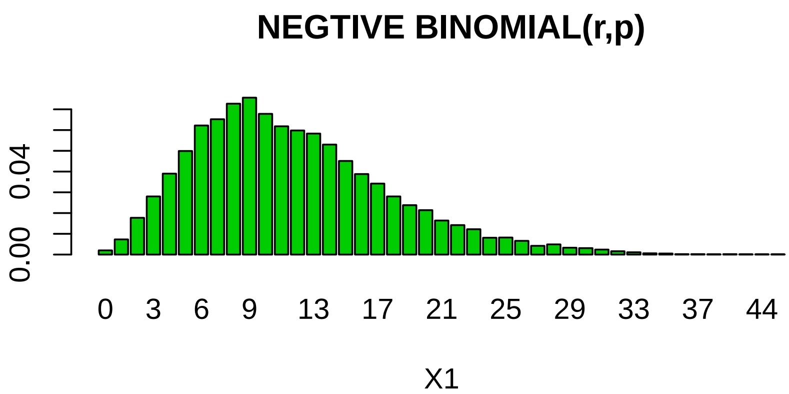


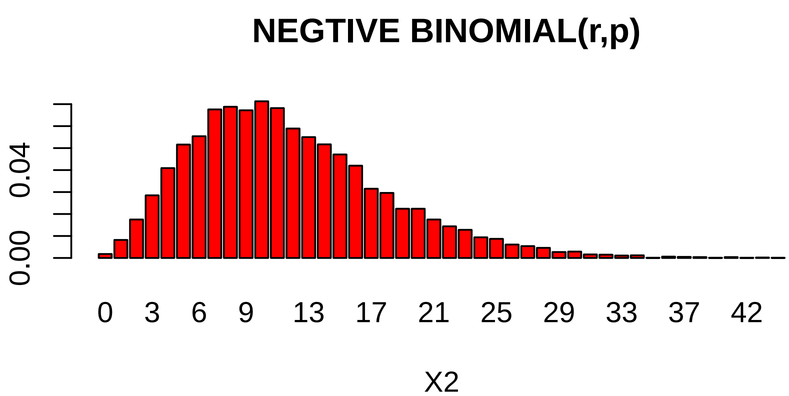



```
####################################################
#NEGTIVE BINOMIAL(r,p)
#It is the number of  failures  before a success in a series of Bernoulli events.
####################################################
n<- 10000 # sample size
r<-5
p<-0.3
x1<-rnbinom(n,r,p)   # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, main = " NEGTIVE BINOMIAL(r,p)")
lines(dnbinom(0:max(x1),r,p)~c(0:max(x1)), type='p', col=2, lwd=2)
```

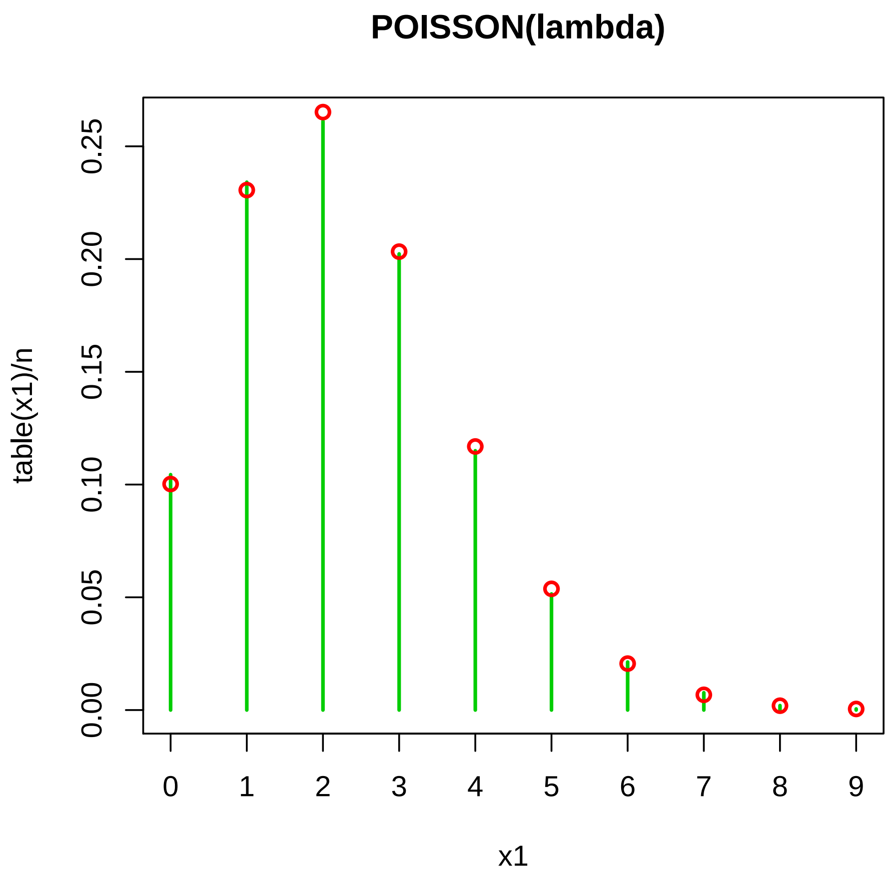



```r
# NEGTIVE BINOMIAL(r,p) as a sum of r independent  GEOMETRIC(p)

x2 <- array(0,dim=c(n))
for(i in 1 : n){
  x2[i]<-sum(rgeom(r,p))
}

par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, xlab='X1', main = "  NEGTIVE BINOMIAL(r,p)")
barplot(table(x2)/n, col=2, xlab='X2',  main = " NEGTIVE BINOMIAL(r,p)")
```

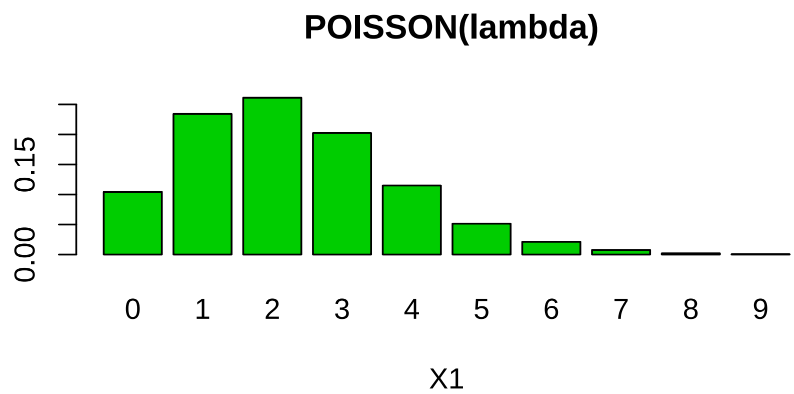


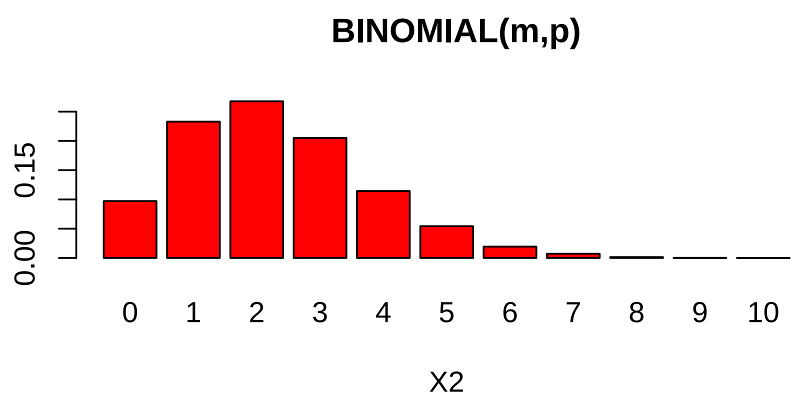



```
#####################################################
#POISSON(lambda)
# Binomial(m,p), m is large and p is small sych thet p*m converges to lambda
#####################################################
n<- 10000 # sample size
lambda<-2.3
x1<-rpois(n, 2.3)   # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, main = " POISSON(lambda)")
lines(dpois(0:max(x1),lambda)~c(0:max(x1)), type='p', col=2, lwd=2)
```

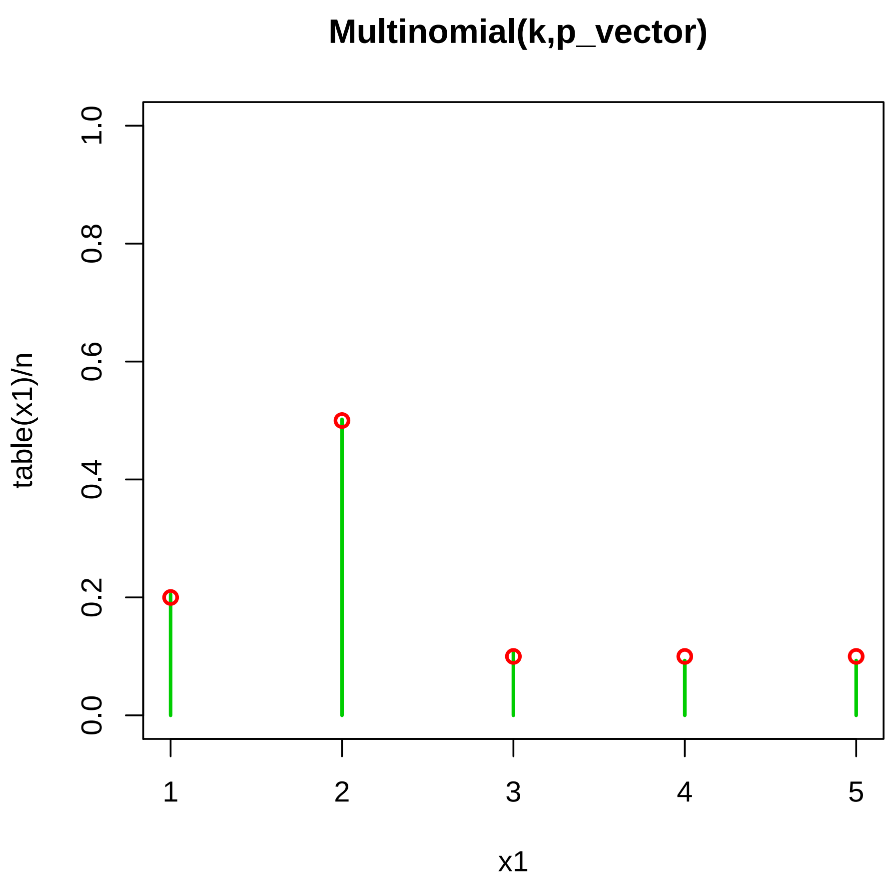



```
# Poisson as  a limit of binomial
n<- 10000 # sample size
m<-100
p<-.023
x2<-rbinom(n,m,p)   # data
par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, xlab='X1', main = " POISSON(lambda)")
barplot(table(x2)/n, col=2, xlab='X2',  main = "  BINOMIAL(m,p)")
```

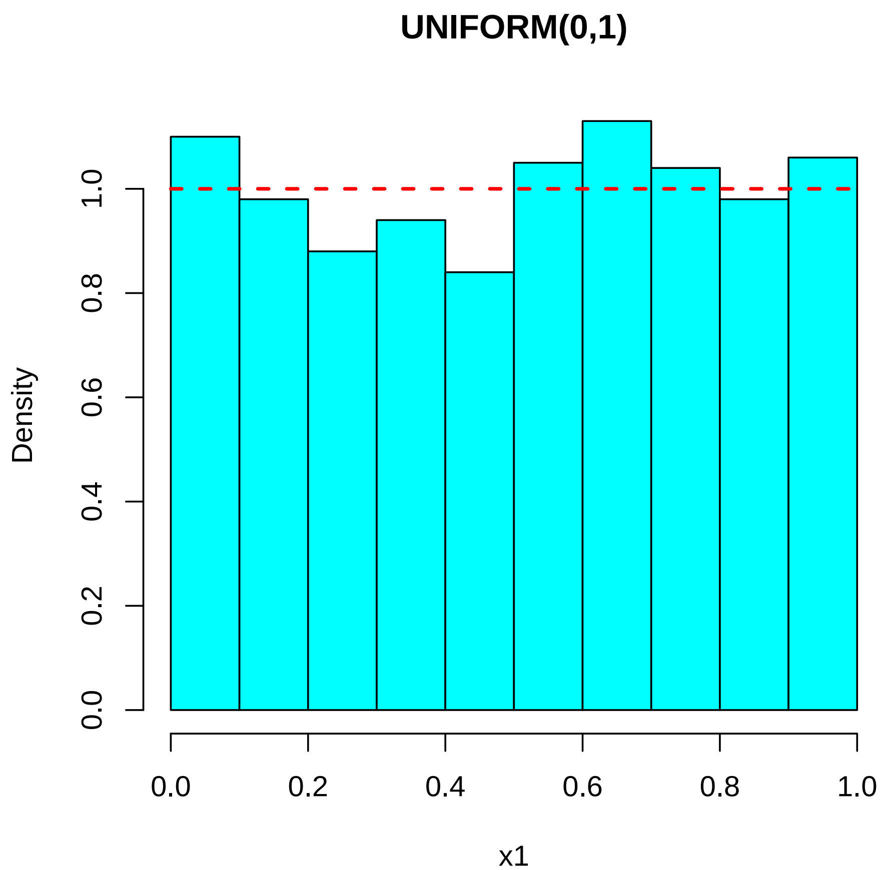




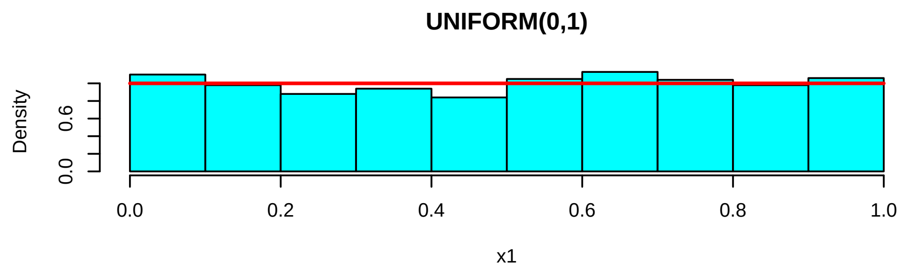

```
cat("\newpage")
##
## ewpage
##############################
#Multinomial(k,p_vector)
##############################
k <- 5  # categories
n <- 1000 # sample size
p <- c(.2,.5,.1,.1,.1) # probbility vector
x1 <- sample(1:k, n, replace=T, prob=p) # data
print(table(x1)/n )
## x1
##     1     2     3     4     5
## 0.205 0.502 0.107 0.093 0.093
par(mfrow=c(1,1))
plot(table(x1)/n,ylim=c(0,1), col=3, main = " Multinomial(k,p_vector)")
lines(p, type='p', col=2, lwd=2)
```

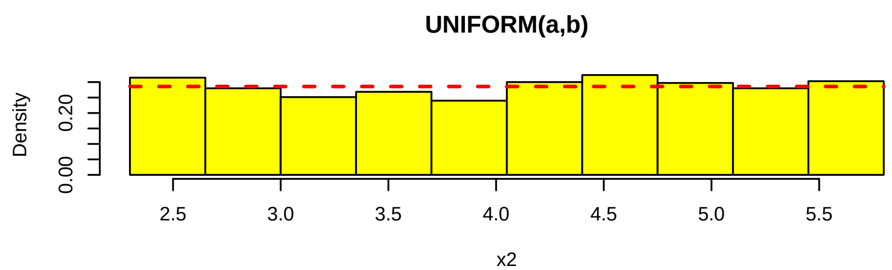


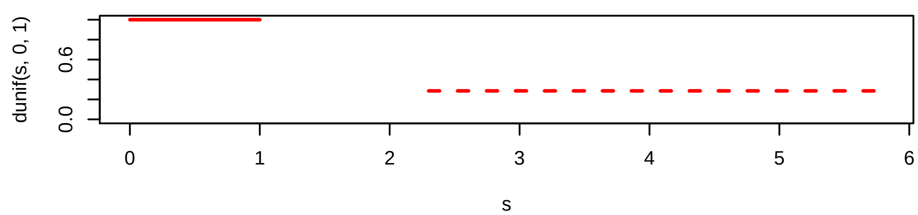

```
##########################################################
#COUNTINUOUS DISTRIBUTIONS
##########################################################

##############################
#UNIFORM(a,b)
##############################
a<-0
b<-1
n<- 1000 #Sample size
x1<-runif(n, a, b)
s<-seq(a,b,length=11)
par(mfrow=c(1,1))
hist(x1, col=5,probability = T, breaks =s, main='UNIFORM(0,1)')
lines(dunif(s,a,b)~s, col=2, lwd=2, lty=2)
```



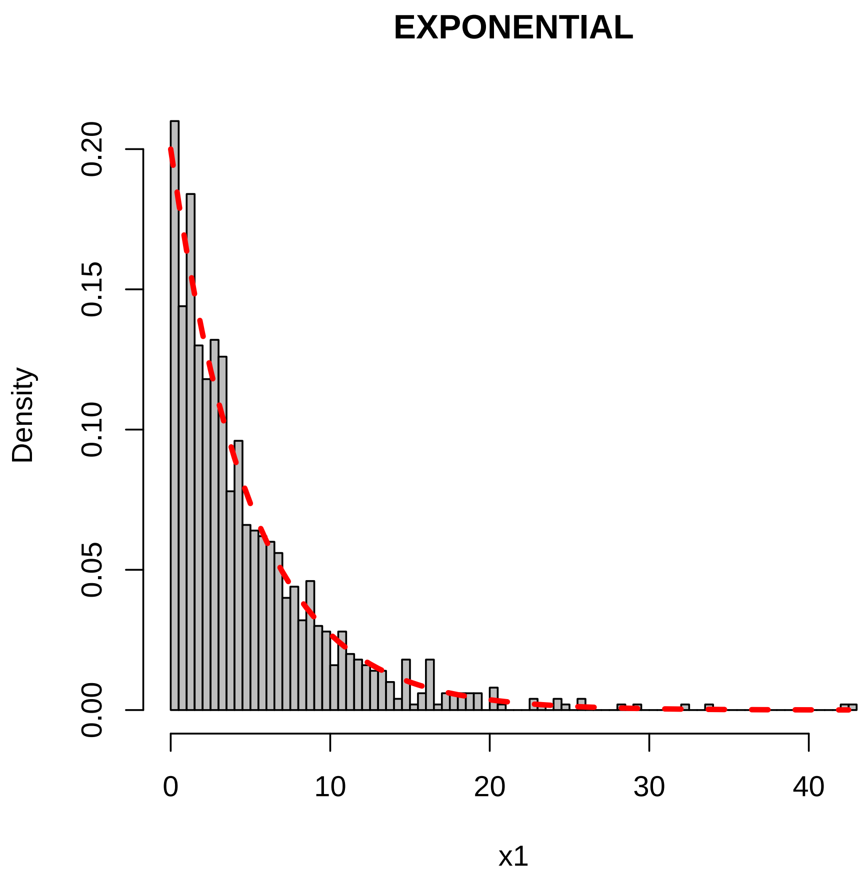

```
a<-2.3
b<-5.8

x2<-a+(b-a)*x1
ss<-a+s*(b-a)


par(mfrow=c(3,1))
hist(x1, col=5,probability = T, breaks =s, main='UNIFORM(0,1)')
lines(dunif(s,0,1)~s, col=2, lwd=2, lty=1)

hist(x2, col=7,probability = T, breaks =ss, main='UNIFORM(a,b)')
lines(dunif(ss,a,b)~ss, col=2, lwd=2, lty=2)

plot(dunif(s,0,1)~s, xlim=c(0,b),ylim=c(0,1), col=2, lwd=2, type='l')
lines(dunif(ss,a,b)~ss, col=2, lwd=2, lty=2)
```



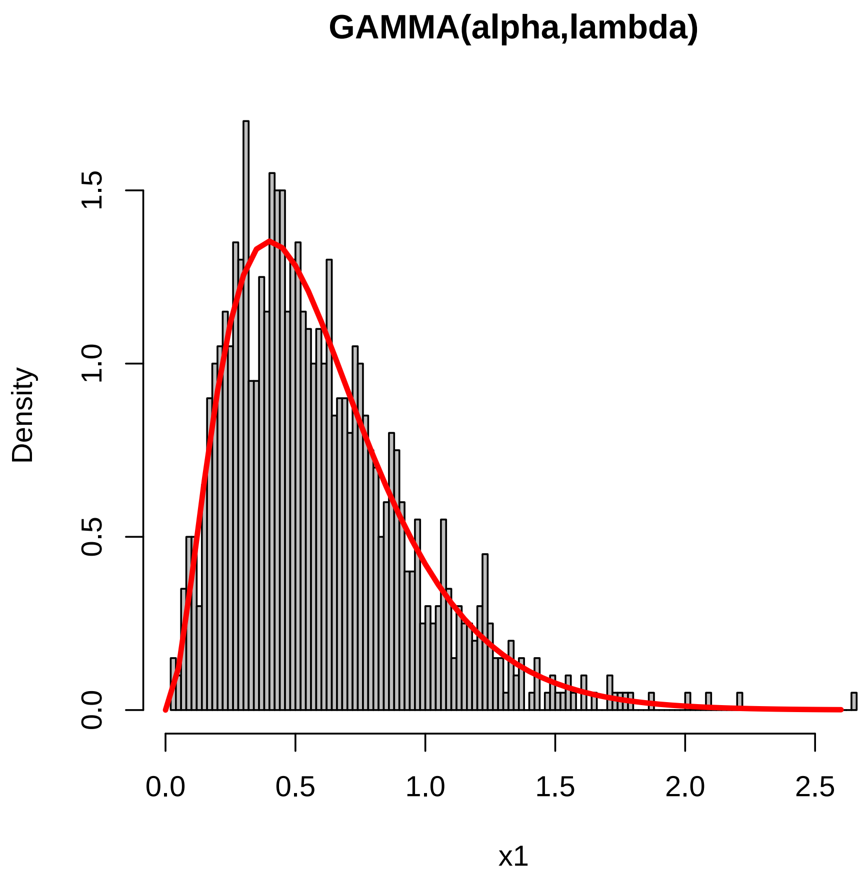




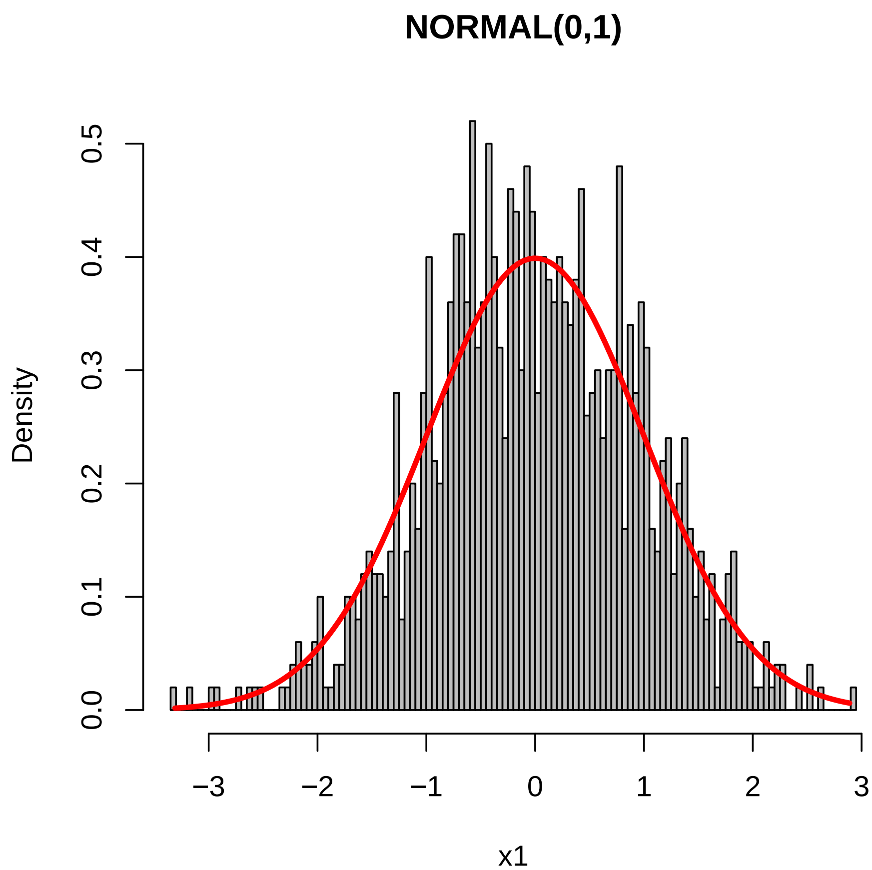

```
############################
#EXPONENTIAL (LAMBDA)
############################
lambda<-.2 #rate
mu=1
n<-1000 #sample size
x1 <- rexp(n, lambda)
s<-seq(0,max(x1),by=0.5)
par(mfrow=c(1,1))
hist(x1,probability = T,breaks = 100, col=8, main='EXPONENTIAL')
lines(dexp(s,lambda)~s, col=2, lwd=3, lty=2)
```



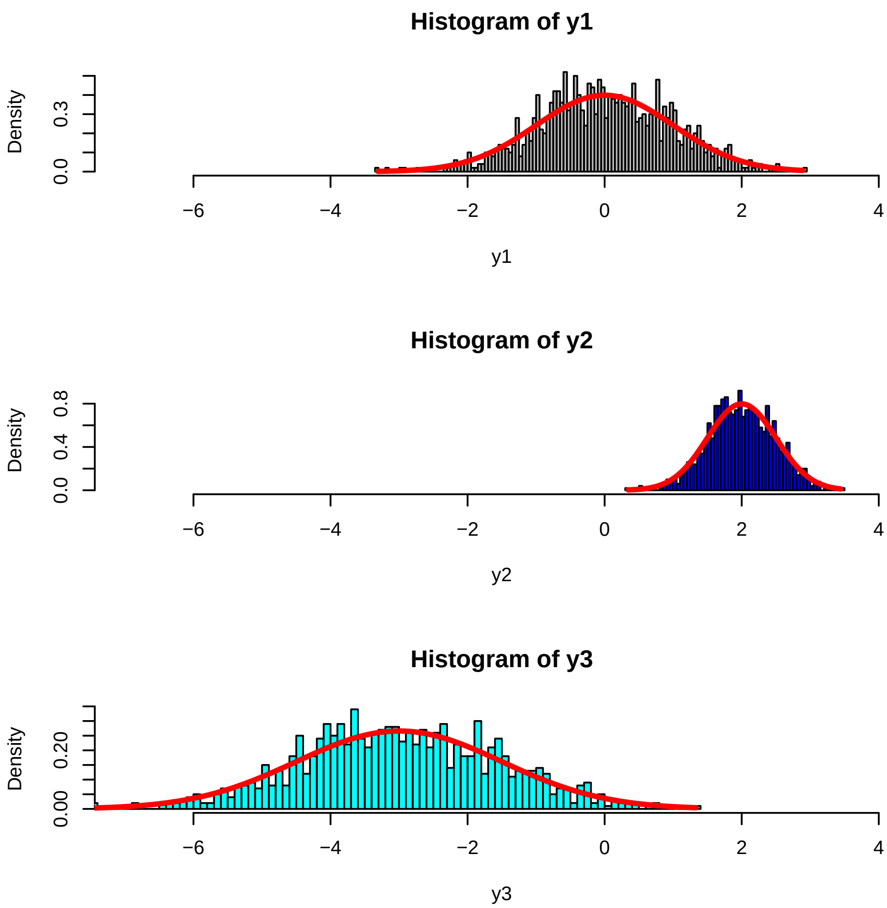

```
###############################
#GAMMA(alpha,lambda)
################################
alpha<-3 # shape
lambda<-.2 # rate

n<-1000 #sample size
x1 <- rgamma(n, shape=alpha, scale=lambda)
s<-seq(0,max(x1),by=0.05)
par(mfrow=c(1,1))
hist(x1,probability = T,breaks = 100, col=8, main='GAMMA(alpha,lambda)')
lines(dgamma(s, shape=alpha, scale=lambda)~s, col=2, lwd=3)
```



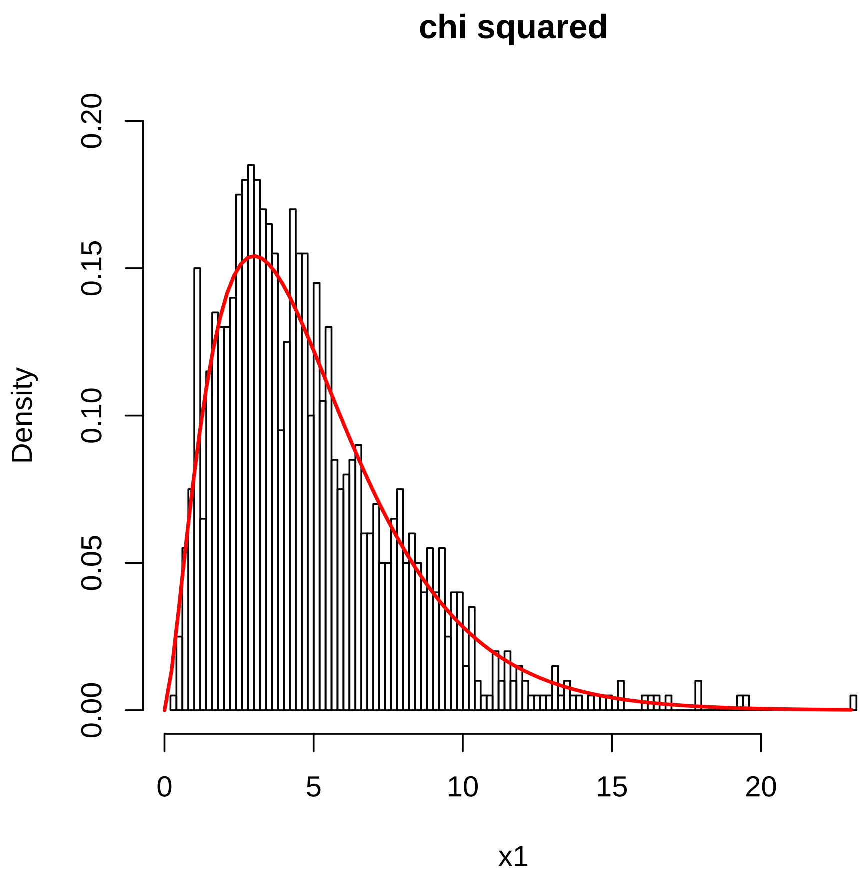

```
##################################
# NORMAL(mu, sigma)
##################################

mu=0  # location
sigma=1 # scale

n<-1000
x1<-rnorm(n, mu,sigma)
s<-seq(min(x1),max(x1),by=0.05)

par(mfrow=c(1,1))
hist(x1,probability = T,breaks = 100, col=8, main='NORMAL(0,1)')
lines(dnorm(s, mean=0, sd=1)~s, col=2, lwd=3)
```



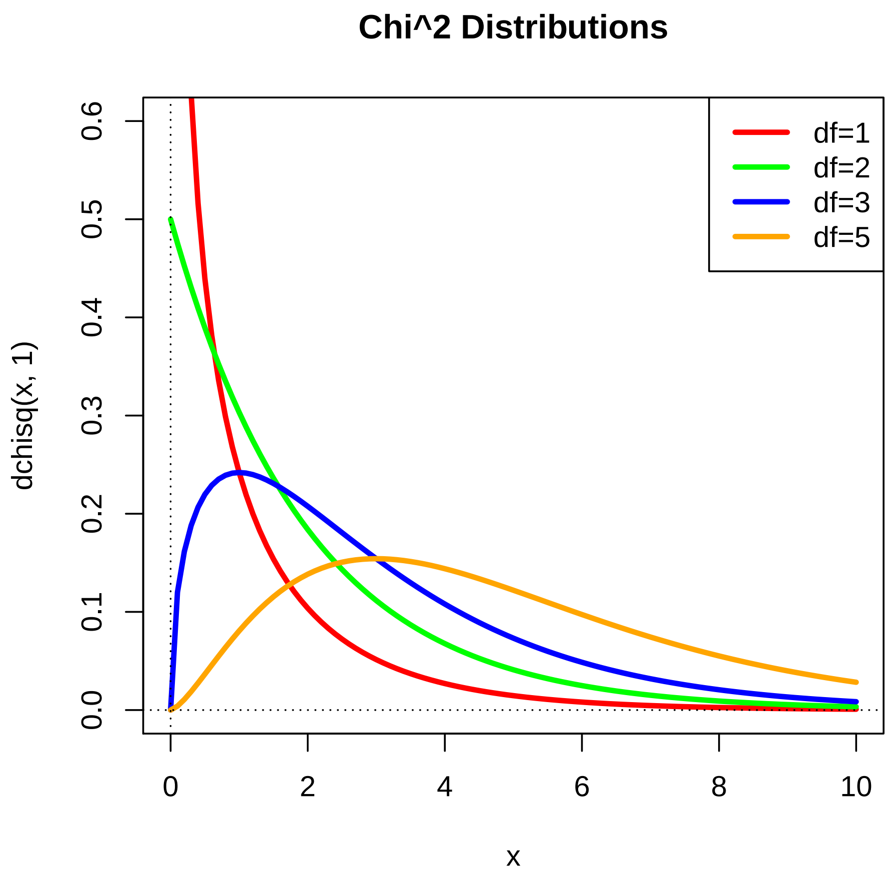

```
y1<-x1
s1<-s
y2<-2+0.5*x1
s2<-2+0.5*s
y3<- -3+1.5*x1
s3<- -3+1.5*s

par(mfrow=c(3,1))
hist(y1,probability = T,breaks = 100, col=8, xlim=c(-7,4))
lines(dnorm(s1, mean=0, sd=1)~s1, col=2, lwd=3)

hist(y2,probability = T,breaks = 100, col=4, xlim=c(-7,4))
lines(dnorm(s2, mean=2, sd=0.5)~s2, col=2, lwd=3)
hist(y3,probability = T,breaks = 100, col=5, xlim=c(-7,4))
lines(dnorm(s3, mean= -3, sd=1.5)~s3, col=2, lwd=3)
```



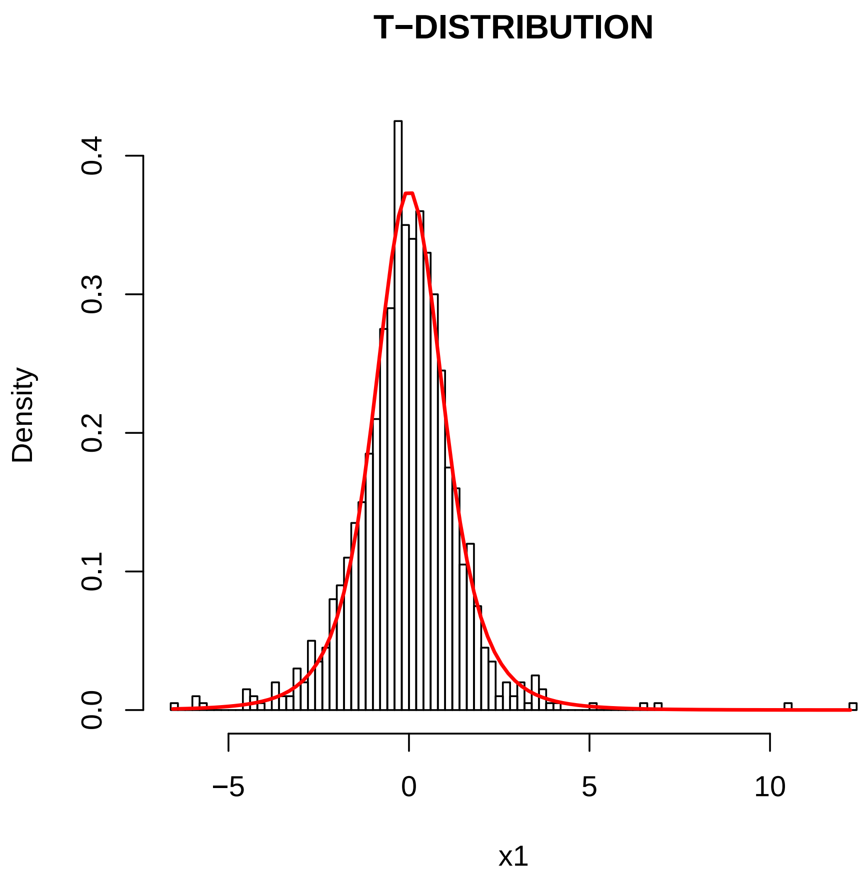

```
cat("mean(Y1)=", mean(y1), "sd(Y1)=",sd(y1),"\n")
## mean(Y1)= -0.004828409 sd(Y1)= 0.9698561
cat("mean(Y2)=", mean(y2), "sd(Y2)=",sd(y2),"\n")
## mean(Y2)= 1.997586 sd(Y2)= 0.484928
cat("mean(Y3)=", mean(y3), "sd(Y2)=",sd(y3),"\n")
## mean(Y3)= -3.007243 sd(Y2)= 1.454784
############################
#Chi^2- distribution
############################
df<-5
x1<-rchisq(n = 1000,df=df)
par(mfrow=c(1,1))
s<-seq(0,max(x1),length=100)
hist(x1, probability = T, breaks=100,  ylim=c(0,.2), main="chi squared")
lines( dchisq(s,df)~s, col='red', lwd=2 )
```



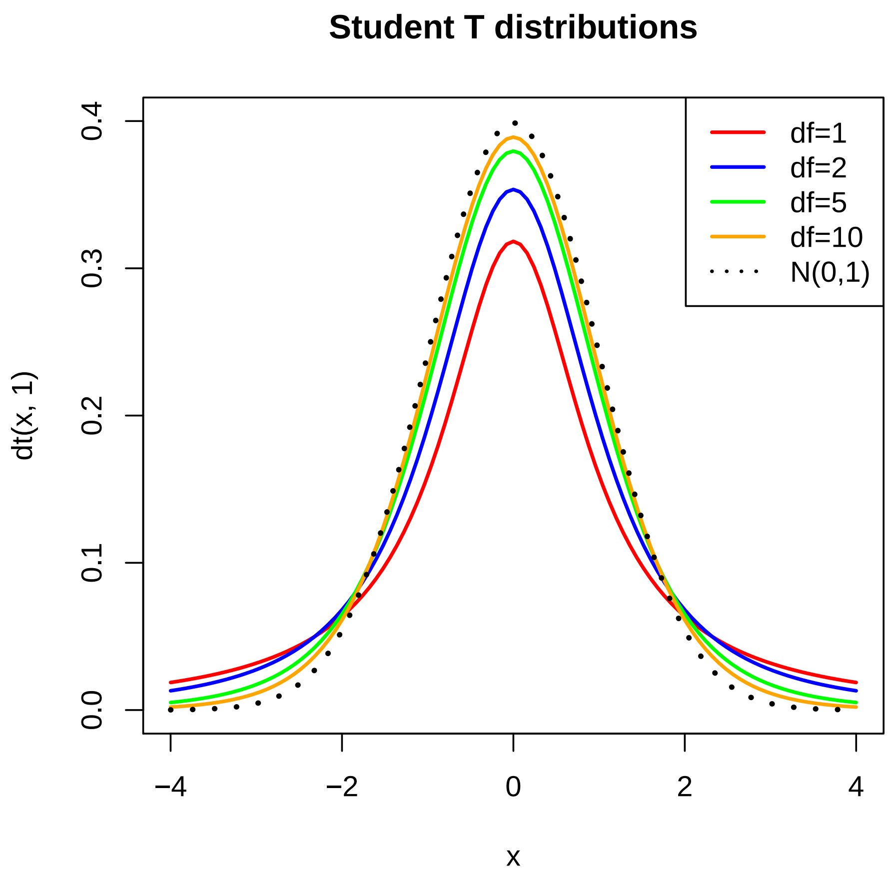

```
par(mfrow=c(1,1))
curve(dchisq(x,1), xlim=c(0,10), ylim=c(0,.6), col='red', lwd=3)
curve(dchisq(x,2), add=T, col='green', lwd=3)
curve(dchisq(x,3), add=T, col='blue', lwd=3)
curve(dchisq(x,5), add=T, col='orange', lwd=3)
abline(h=0,lty=3)
abline(v=0,lty=3)
legend(par('usr')[2], par('usr')[4], xjust=1,
       c('df=1', 'df=2', 'df=3', 'df=5'),
       lwd=3,
       lty=1,
       col=c('red', 'green', 'blue', 'orange')
)
title(main='Chi^2 Distributions')
```



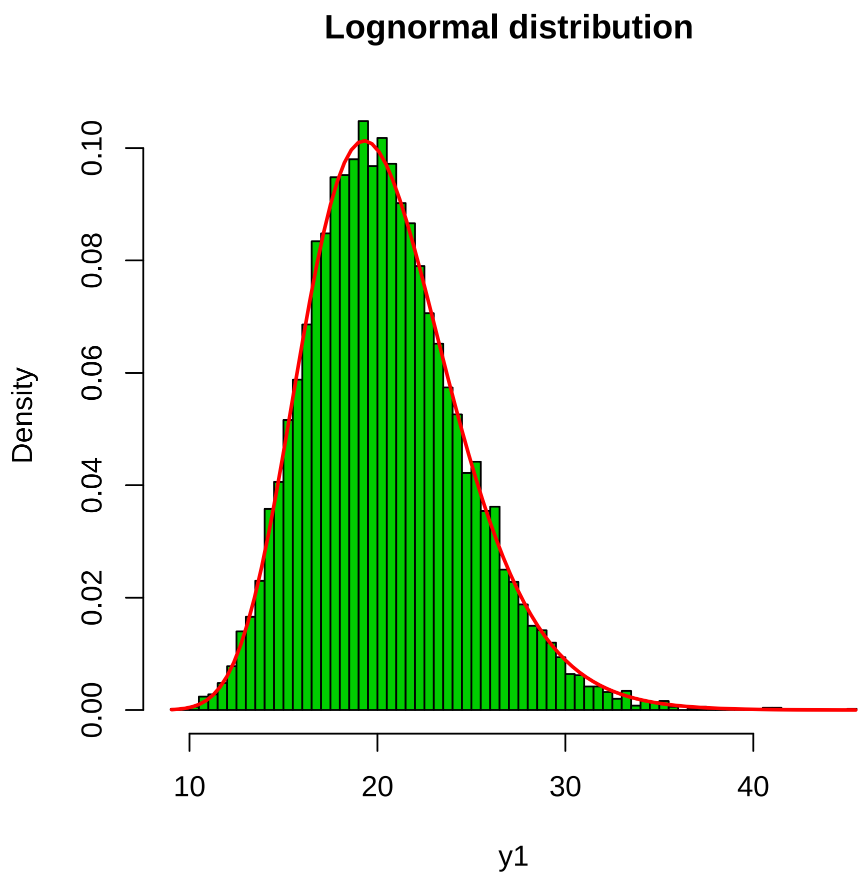

```
##################################
#T-DISTRIBUTION
##################################
df<-4
x1<-rt(n = 1000,df=df)
par(mfrow=c(1,1))
s<-seq(min(x1),max(x1),length=100)
hist(x1, probability = T, breaks=100, main="T-DISTRIBUTION" )
lines( dt(s,df )~s, col='red', lwd=2 )
```



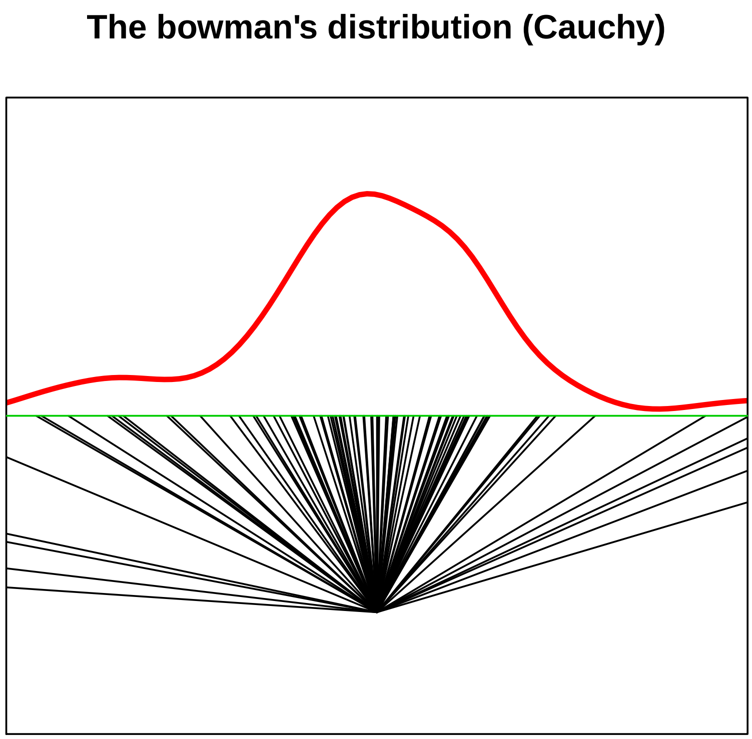

```
par(mfrow=c(1,1))
curve( dt(x,1), xlim=c(-4,4), ylim=c(0,.4), col='red', lwd=2 )
curve( dt(x,2), add=T, col='blue', lwd=2 )
curve( dt(x,5), add=T, col='green', lwd=2 )
curve( dt(x,10), add=T, col='orange', lwd=2 )
curve( dnorm(x), add=T, lwd=3, lty=3 )
title(main="Student T distributions")
legend(par('usr')[2], par('usr')[4], xjust=1,
       c('df=1', 'df=2', 'df=5', 'df=10', 'N(0,1)'),
       lwd=c(2,2,2,2,2),
       lty=c(1,1,1,1,3),
       col=c('red', 'blue', 'green', 'orange', par("fg")))
```




```
####################################
# Lognormal distribution
#####################################


mu=3
sigma=.2

n<-10000

x1<-rnorm(n, mean=mu,sd=sigma)
y1<-exp(x1) # rlnorm(n,mu, sigma)
s<-seq(min(y1),max(y1), length=100)

hist(y1,breaks = 100, probability = T, col=3, main = "Lognormal distribution ")
lines(dlnorm(s,mu,sigma)~s, col=2, lwd=2)
```




```
#####################################
#Cauchy distribution
######################################

n <- 100                        # sample size
alpha <- runif(n, -pi/2, pi/2)    # Direction of the arrow
x <- tan(alpha)                   # Arrow impact
plot.new()
plot.window(xlim=c(-5, 5), ylim=c(-1.5, 1.5))
segments( 0, -1,      # Position of the Bowman
          x, 0    )    # Impact
d <- density(x)
lines(d$x, 5*d$y, col="red", lwd=3 )
box()
abline(h=0, col=3)
title(main="The bowman's distribution (Cauchy)")
```



7. Joint and conditional distributions

Definition 91. Vector Valued Random variable: $(X_1(\omega), X_2(\omega), \cdots, X_k(\omega))$ is a vector valued random variable where $\omega \in \Omega$.

Definition 92. Let (X, Y) be a pair of random variable with joint distribution function F on same probability space$(\Omega, \mathcal{A}, P)$ then

$$\begin{aligned} F(x,y) &= P(X \le x, Y \le y) \\ &= P(\{\omega | \omega \in \Omega, X(\omega) \in (-\infty, x], Y(\omega) \in (-\infty, y]\}) \end{aligned} \tag{7.1}$$

Properties:
(a) $\lim_{x\downarrow-\infty, y\downarrow-\infty} F(x,y) = 0$
(b) $\lim_{x\uparrow\infty, y\uparrow\infty} F(x,y) = 1$
(c) $\lim_{y\uparrow\infty} F(x,y) = F_X(x)$ [Marginal distribution of X]
(d) $\lim_{x\uparrow\infty} F(x,y) = F_Y(y)$ [Marginal distribution of Y]
(e) $F(a < X \le b, c < Y \le d) = F(b,d) - F(a,d) - F(b,c) + F(a,c) \ge 0$ [Non decreasing]
(f) $F(a < X \le b, c < Y \le d) = (F_X(b) - F_X(a))(F_Y(d) - F_Y(c))$ iff X and Y are independent.
(g) Joint p.d.f of (X, Y) is $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$ such that $\int_x \int_y f(u,v)dudv = 1$
(h) Marginal densities: $\int_x f(u,y)du = f_Y(y)$ and $\int_y f(x,v)dv = f_X(x)$.
(i) $f(x,y) = f_X(x) f_Y(y)$ iff X and Y are independent.
(j) Conditional density of $Y|X = x$ is $f_{Y|x}(y|x) = \frac{f(x,y)}{f_X(x)}$ if $f_X(x) > 0$
(k) Conditional exception (regression) of $g(Y)$ given $X = x$ i.e. $E(g(Y)|X = x) = \int_{y|x} g(y) f_{Y|x}(y|x)dy$ is a function of x.

Definition 93. A non negative function $P(X = x, Y = y) = f(x,y)$ is said to be the the joint p.m.f of (X, Y) if they have the following properties
(a) $\sum_x \sum_y f(x,y) = 1$
(b) $\sum_y f(x,y) = f_X(x)$ i.e. marginal p.m.f. of X.
(c) $\sum_x f(x,y) = f_Y(y)$ i.e. marginal p.m.f. of Y.

7.1. **Laws of expectation.** Assume that (X, Y) has joint p.m.f. $f(x,y)$ with $\sum_x |x|^2 f_X(x) < \infty$ and $\sum_y |y|^2 f_Y(y) < \infty$ then
(1) $E(\alpha X) = \alpha E(X)$
(2) $E(X + Y) = E(X) + E(Y)$ NOTE: Independence NOT required.
(3) $E(XY) = E(X)E(Y)$ if$X$ and Y are independent but the reverse is not true in general.
(4) **Product moment** $Cov(X,Y) = E(X - \mu_x)(Y - \mu_y) = E(XY) - \mu_x \mu_y$.
(5)$Cov(X,X) = Var(X) = \sigma_x^2$
(6)$Var(aX) = a^2 Var(X)$
(7)$Var(aX + bY) = a^2 \; Var(X) + b^2 \; Var(Y) + 2ab \;\; Cov(X,Y)$
(8)$E(Y) = E_X E_{Y|X}(Y|X = x)$
(9)$Var(Y) = E_X V_{Y|X}(Y|X = x) + V_X E_{Y|X}(Y|X = x)$
(10) If $P(X \ge Y) = 1$ then $E(X) \ge E(Y)$
(11) If X has MGF $M_X(t)$ then $Y = a + bX$ has MGF $M_Y(t) = e^{at} M_X(bt)$
(12) If two independent random variables X has MGF $M_X(t)$ and Y has MGF $M_Y(t)$ then $Z = X + Y$ has MGF $M_Z(t) = M_X(t) M_Y(t)$

Definition 94. $Corr(X,Y) = \rho(X,Y) = \frac{E(XY)-E(X)E(Y)}{\sqrt{(Var(X)Var(Y)}}$

Remark 95. Correlation can measure **linear dependency** between two random variables.

Exercise 96. Show that $|\rho(X,Y)| \leq 1$.

Exercise 97. Let $X_1 \sim bin(n_1, p)$ and $X_2 \sim bin(n_2, p)$ be independently distributed. Find the conditional distribution of X_1 when it is given $X_1 + X_2 = k$.

Exercise 98. Let $X_1 \sim pois(\lambda_1)$ and $X_2 \sim pois(\lambda_2)$ be independently distributed. Find the conditional distribution of X_1 when it is given $X_1 + X_2 = k$.

Exercise 99. Let $P \sim U[0,1]$ and $Y|P = p \sim bin(n,p)$. Find the marginal distribution of Y.

Exercise 100. Let $\Lambda \sim exponential(1)$ and $Y|\Lambda = \lambda \sim pois(\lambda)$. Find the marginal distribution of Y.

Exercise 101. Let $P \sim B(a,b)$ and $Y|P = p \sim bin(n,p)$. Find the marginal distribution of Y.

Definition 102. If (X,Y) has joint density function $f(x,y)$ as

$$f(x,y) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}}{2\pi\sigma_x, \sigma_y\sqrt{1-\rho^2}}$$

then (X,Y) is said to follow Bivariate normal$(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

Exercise 103. If (X,Y) follow Bivariate normal$(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ the show that $Y|X = x$ follows $N(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x), (1-\rho^2)\sigma_y^2)$

Theorem 104. *Let (X,Y) be continuous random variables with density function $f(x,y)$ then the density function of $(U,V) = (U(X,Y), V(X,Y))$ is*

$$g(u,v) = f(x(u,v), y(u,v)) \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\|$$

where $\left\| \frac{\partial(x,y)}{\partial(u,v)} \right\|$ *stands for the absolute value of the determinant of the Jacobian matrix*$\frac{\partial(x,y)}{\partial(u,v)}$.

Exercise 105. Let X and Y be i.i.d. $N(0,1)$ random variables. Find the joint density of $U = X+Y$ and $V = X-Y$. Are (U,V) independently distributed?

Exercise 106. Let X and Y be i.i.d. $N(0,1)$ random variables. Find the p.d.f. of (r,θ) where $X = r\cos\theta$ and $Y = r\sin\theta$.

Exercise 107. Give an algorithm to generate X, Y which are i.i.d. $N(0,1)$ from U, V which are i.i.d. $U(0,1)$.

Exercise 108. Let X and Y be i.i.d. $N(0,1)$ random variables. Find the p.d.f. of $U = X/Y$.

Exercise 109. Show that t_1 and $C(0,1)$ are the same distribution. [Alternative statement: Let X and Y be i.i.d. $N(0,1)$ random variables. Then $U = X/|Y|$ has $C(0,1)$ distribution.]

Exercise 110. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\mathrm{N}(\mu, \sigma^2)$ population. Then show that

(1) $\overline{X}$ has a $\mathrm{N}(\mu, \sigma^2/n)$ distribution,
(2) $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ has a $\mathrm{N}(0,1)$ distribution,
(3) $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2-distribution with $n-1$ degrees of freedom
(4) $\overline{X}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ are independent random variables,
(5) $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ has a t-distribution with $n-1$ degrees of freedom

FIGURE 7.1. Bivariate normal joint p.d.f.

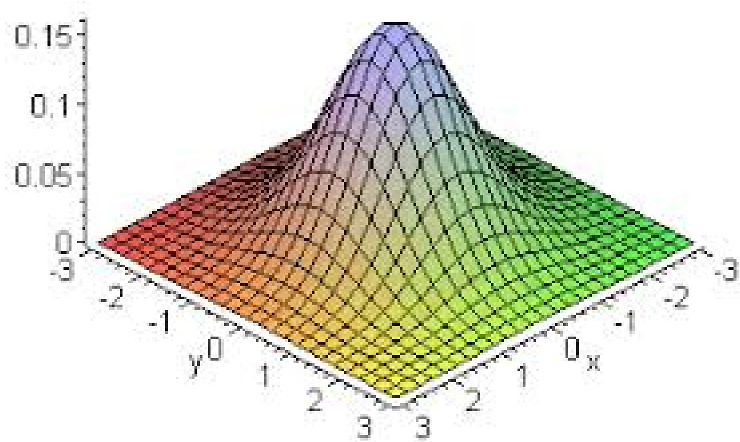
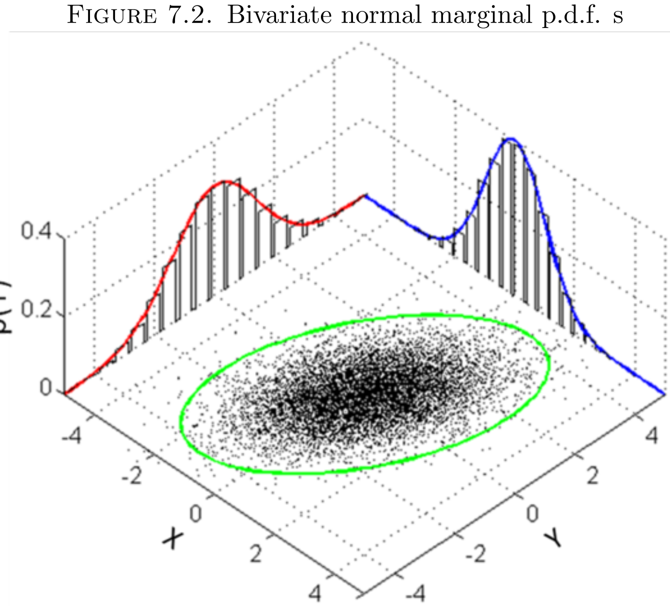
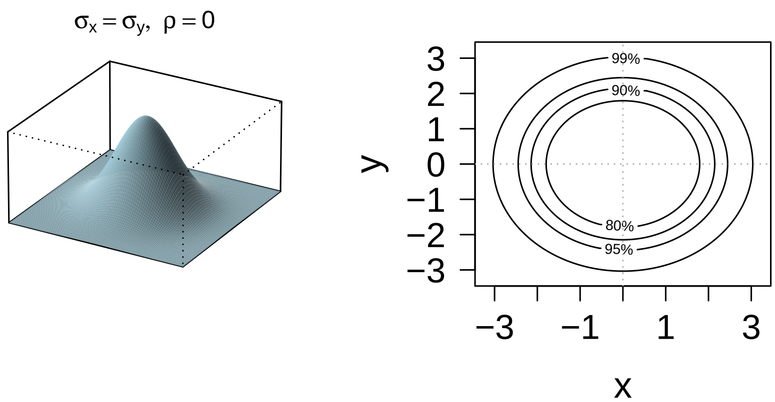
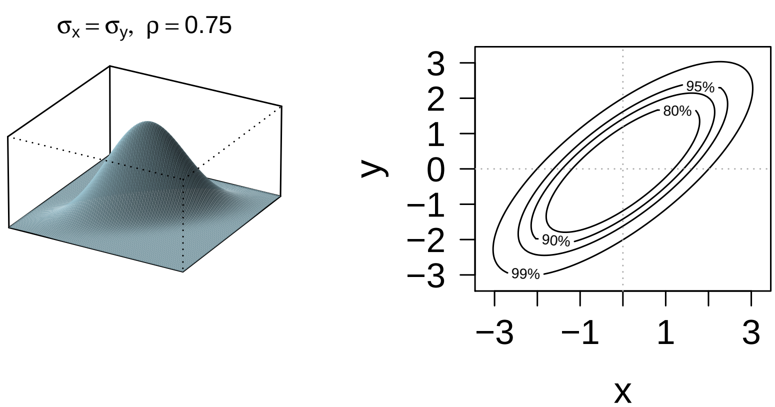
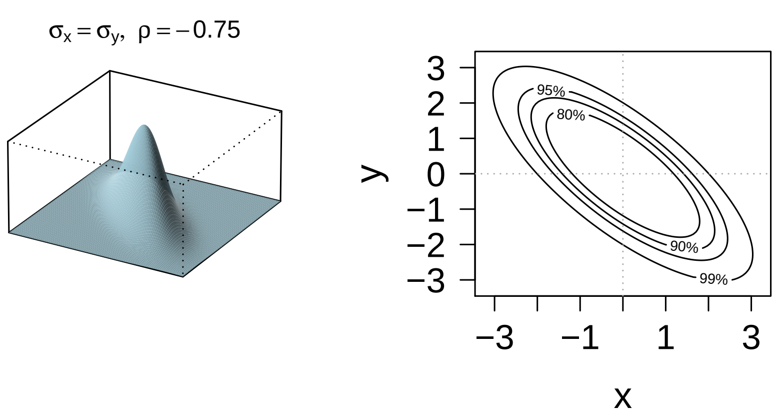


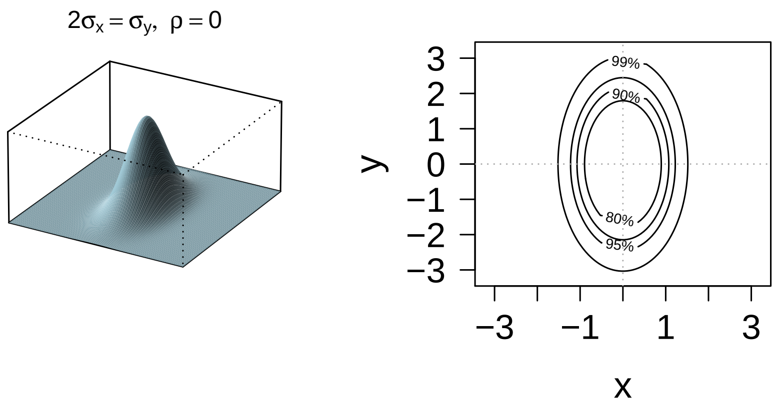
FIGURE 7.2. Bivariate normal marginal p.d.f. s

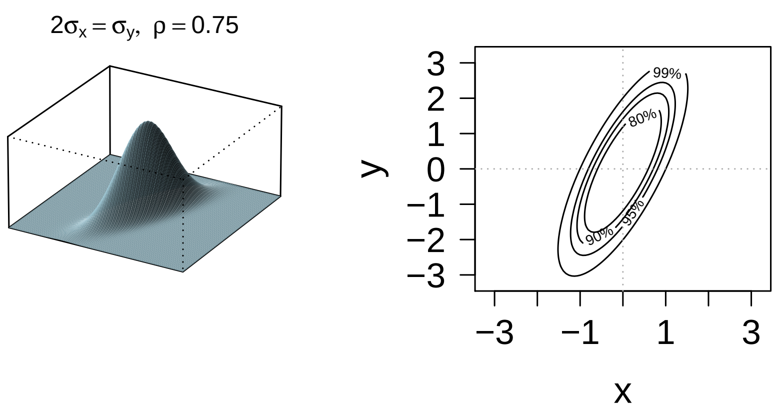



$\sigma_x = \sigma_y,\ \rho = 0$
99%
90%
80%
95%
y
x
3 2 1 0 −1 −2 −3
−3 −1 1 3


$\sigma_x = \sigma_y,\ \rho = 0.75$
95%
80%
90%
99%
y
x
3 2 1 0 −1 −2 −3
−3 −1 1 3


$\sigma_x = \sigma_y,\ \rho = -0.75$
95%
80%
90%
99%
y
x
3 2 1 0 −1 −2 −3
−3 −1 1 3


$2\sigma_x = \sigma_y,\ \rho = 0$
99%
90%
80%
95%
3 2 1 0 −1 −2 −3
−3 −1 1 3
x
y


$2\sigma_x = \sigma_y,\ \rho = 0.75$
99%
80%
90%
95%
3 2 1 0 −1 −2 −3
−3 −1 1 3
x
y

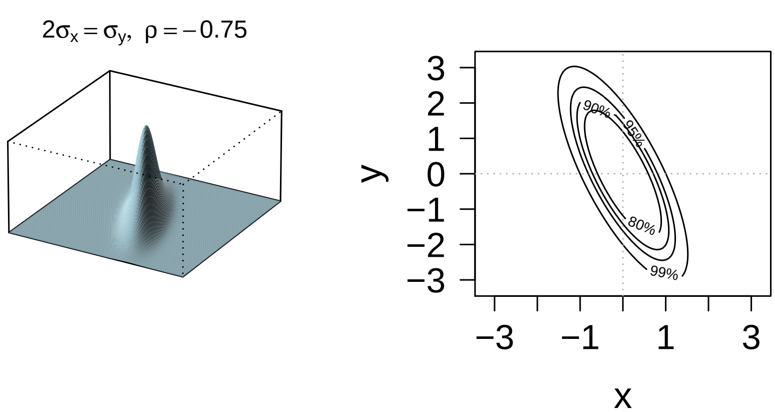

$2\sigma_x = \sigma_y,\ \rho = -0.75$
90%
95%
80%
99%
3 2 1 0 −1 −2 −3
−3 −1 1 3
x
y

FIGURE 7.3. Conditional distribution

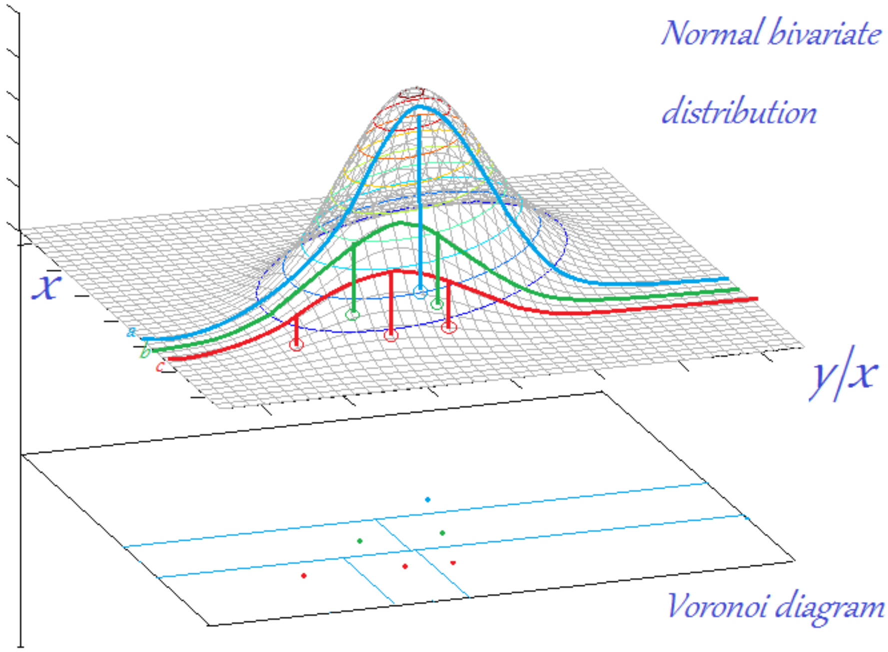
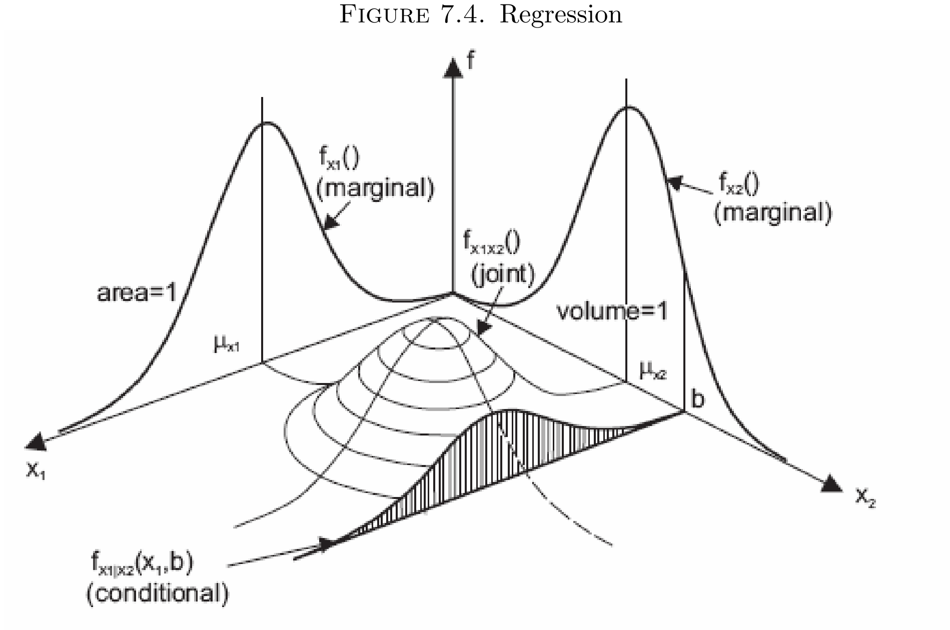
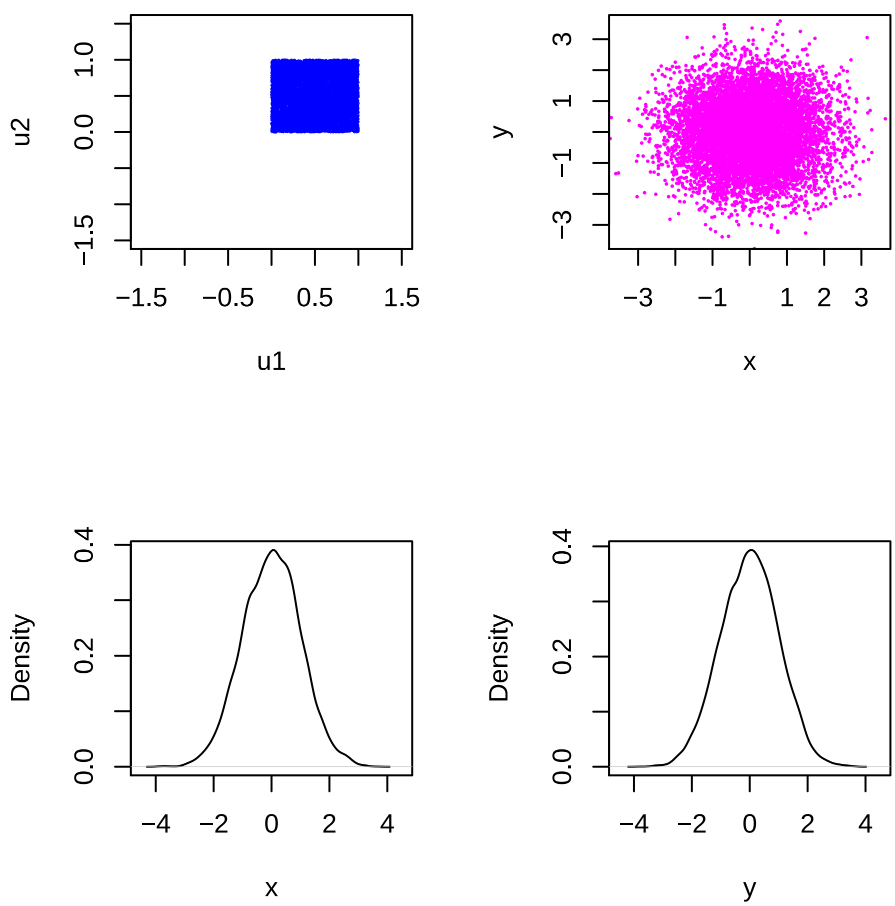


FIGURE 7.4. Regression



```
###################################
#Box-Muller method
###################################
set.seed(123)
n<-9000
u1<- runif(n,min = 0,max = 1)
u2<- runif(n,min = 0,max = 1)
x<-sqrt(-2*log(u1))*cos(2*pi*u2)
y<-sqrt(-2*log(u1))*sin(2*pi*u2)
par(mfrow=c(2,2))
plot(u2~u1,xlim=c(-1.5,1.5), ylim=c(-1.5,1.5), type="p", pch = 20, cex = 0.1, col=4)
plot(y~x, xlim=c(-3.5,3.5), ylim=c(-3.5,3.5), type="p",  pch = 20, cex = 0.2, col=6)
plot(density(x), xlim=c(-4.5,4.5),main = "", xlab = "x")
plot(density(y), xlim=c(-4.5,4.5),main = "", xlab= "y")
```



8. Law of Large Numbers

Definition 111. Convergence in probability: A sequence of random variables $X_n\ \forall n \in N$ on probability space $(\Omega, \mathcal{F}, P)$ is said to converge in probability to X on $(\Omega, \mathcal{F}, P)$ if

$$\lim_{n\uparrow\infty} P(\{\omega | |X_n(\omega) - X(\omega)| < \epsilon\}) = 1 \forall\ \epsilon > 0$$

Theorem 112. ***Weak law of large number (WLLN):*** *Let $\{X_n\}$ be a sequence of i.i.d random variables with finite variance then*

$$\lim_{n\uparrow\infty} P(|\bar{X} - \mu_x| \leq \epsilon) = 1$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\epsilon > 0$.

Definition 113. Convergence in distribution: A sequence of random variables $X_n\ \forall n \in N$ on probability space$(\Omega_n, \mathcal{F}_n, P_n)$ is said to converge in distribution to X on $(\Omega, \mathcal{F}, P)$ if

$$\lim_{n\uparrow\infty} F_{X_n}(a) = F_Y(a)$$

for all such $a \in \mathbb{R}$, where $F_Y(a)$ is continuous.

Remark 114. $X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X$ **but the reverse implications are not true in general**. [Proof is not in syllabus]

Remark 115. If $X_n \xrightarrow{p} X$ then $h(X_n) \xrightarrow{p} h(X)$ for any continuous function h. [Proof is not in syllabus]

Theorem 116. ***Continuity theorem:*** *Let $\{X_n\}$ be a sequence of random variables with corresponding characteristic functions as $E(e^{itX}) = \phi_{X_n}(t)$ such that $\lim\limits_{n\uparrow\infty} \phi_{X_n}(t) = \phi_Y(t)$ for some random variable Y. Then*

$$\lim_{n\uparrow\infty} F_{X_n}(a) = F_Y(a)$$

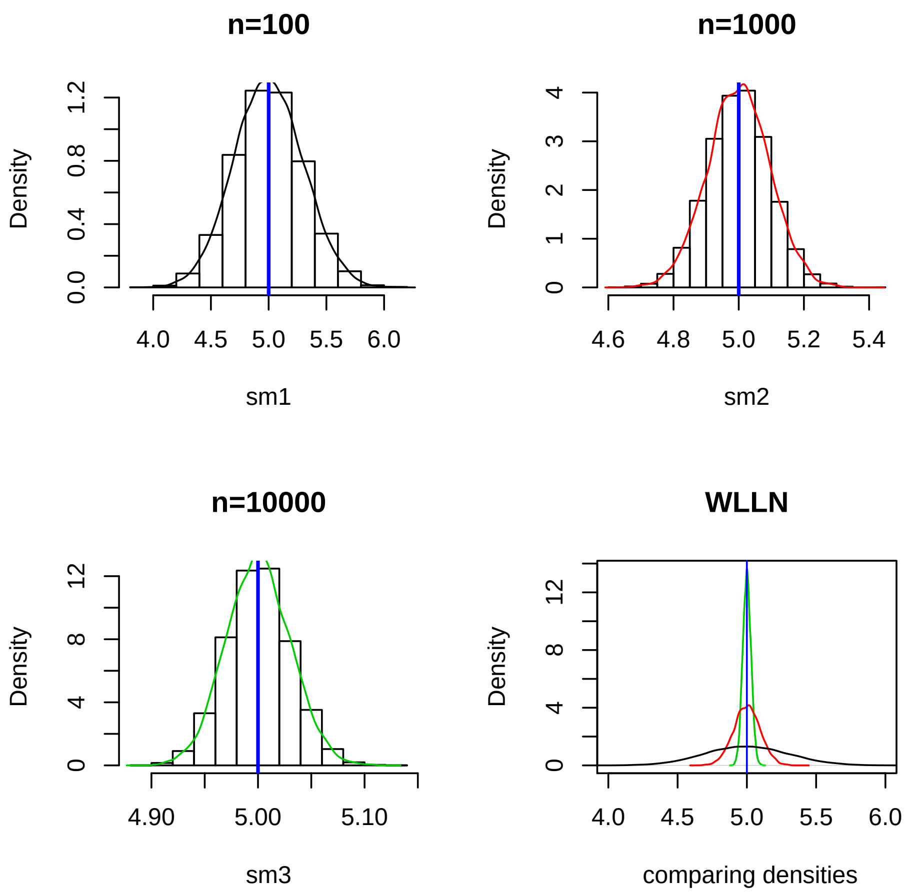
for all such $a \in \mathbb{R}$, where $F_Y(a)$ is continuous. We say X_n converges in distribution to Y. [We can have the above result with MGF, if it exists. Proof is not in syllabus]

Theorem 117. ***Central Limit Theorem(CLT)****: IfX_is be i.i.d. random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Define $S_n = \sum_{i=1}^{n} X_i = n\bar{X}$ and $T_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$. Then*

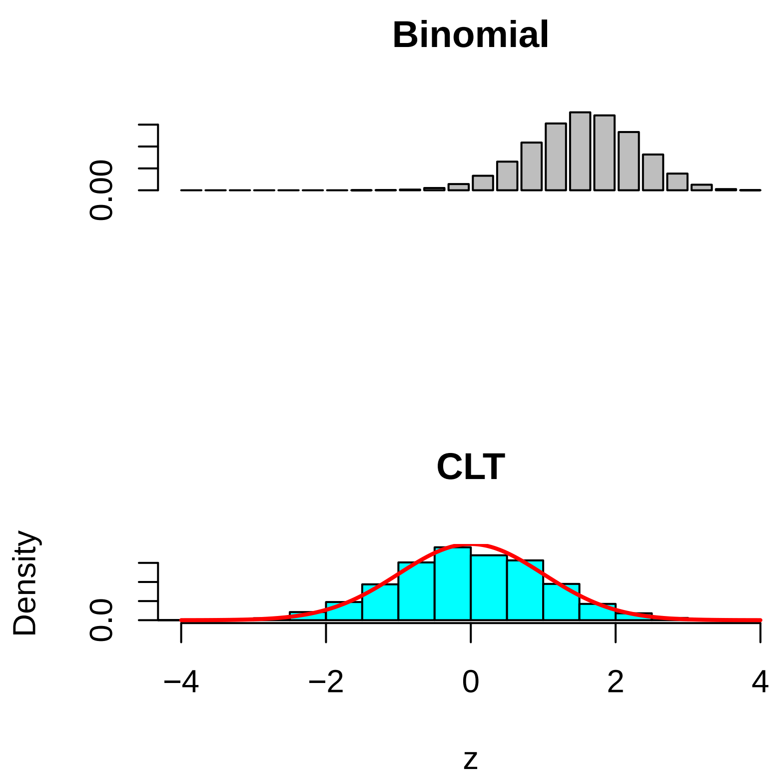
$$\lim_{n\uparrow\infty} P(T_n \leq t) = \Phi(t)$$

Exercise 118. Find the value of $\lim_{n\uparrow\infty} e^{-n}\sum_{k=0}^{n} \frac{n^k}{k!}$.

```
itrn<-10000
sm1<-array(0,dim = c(itrn))
sm2<-array(0,dim = c(itrn))
sm3<-array(0,dim = c(itrn))

for(i in 1 : itrn){
n<-100
x<-rnorm(n = n, mean = 5,sd = 3)
sm1[i]<-(mean(x))

n<-1000
x<-rnorm(n = n, mean = 5,sd = 3)
sm2[i]<-(mean(x))

n<-10000
x<-rnorm(n = n, mean = 5,sd = 3)
sm3[i]<-(mean(x))

}
par(mfrow=c(2,2))
hist(sm1, probability = T, main = "n=100")
lines(density(sm1),col=1)
abline(v=5, col=4, lwd=2)

hist(sm2, probability = T, main = "n=1000")
lines(density(sm2),col=2)
abline(v=5, col=4, lwd=2)

hist(sm3, probability = T, main = "n=10000")
lines(density(sm3),col=3)
abline(v=5, col=4, lwd=2)


plot(density(sm3), col=3,xlim=c(4,6),main = "WLLN", xlab = " comparing densities")
lines(density(sm2),col=2)
lines(density(sm1),col=1)
abline(v=5, col=4, lwd=1)
```

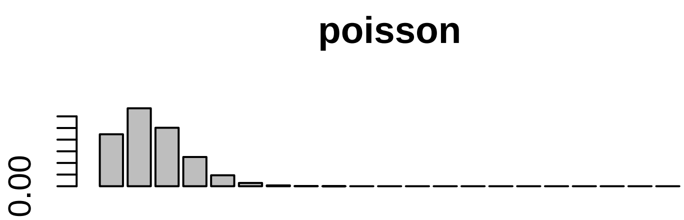

n=100
Density
0.0
0.4
0.8
1.2
4.0
4.5
5.0
5.5
6.0
sm1
n=1000
Density
0
1
2
3
4
4.6
4.8
5.0
5.2
5.4
sm2
n=10000
Density
0
4
8
12
4.90
5.00
5.10
sm3
WLLN
Density
0
4
8
12
4.0
4.5
5.0
5.5
6.0
comparing densities

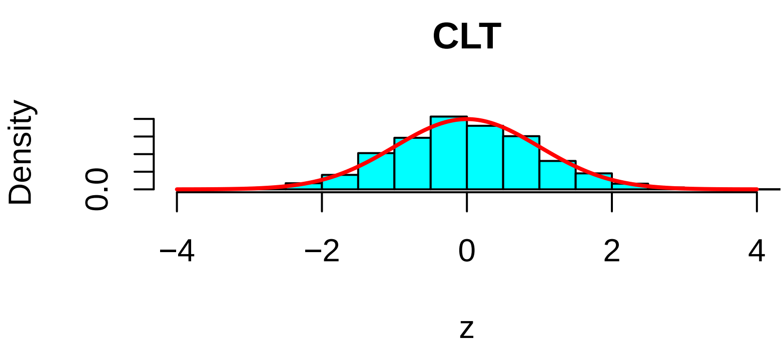
```
# Binomial CLT
bin_clt<-function(itrn,ss,n,p){
  z<-array(0,dim=c(itrn))
  for(i in 1 : itrn){
    x<-rbinom(ss,n,p)
    z[i]<-sqrt(ss)*(mean(x)-n*p)/sqrt(n*p*(1-p))
  }
  par(mfrow=c(2,1))
  barplot(dbinom(0:n,n,p), main = "Binomial")
  hist(z, probability = T, xlim = c(-4,4) ,col=5 ,main = "CLT" )
  s<-seq(-4,4,by=0.01)
  lines(dnorm(s,0,1)~s , col=2, lwd=2)
}
ss<-100 # sample size
itrn<-10000 #iteration
n=23
p=0.7
bin_clt(itrn,ss,n,p)
```




```
#Poisson CLT
poiss_clt<-function(itrn,ss,lambda){
```

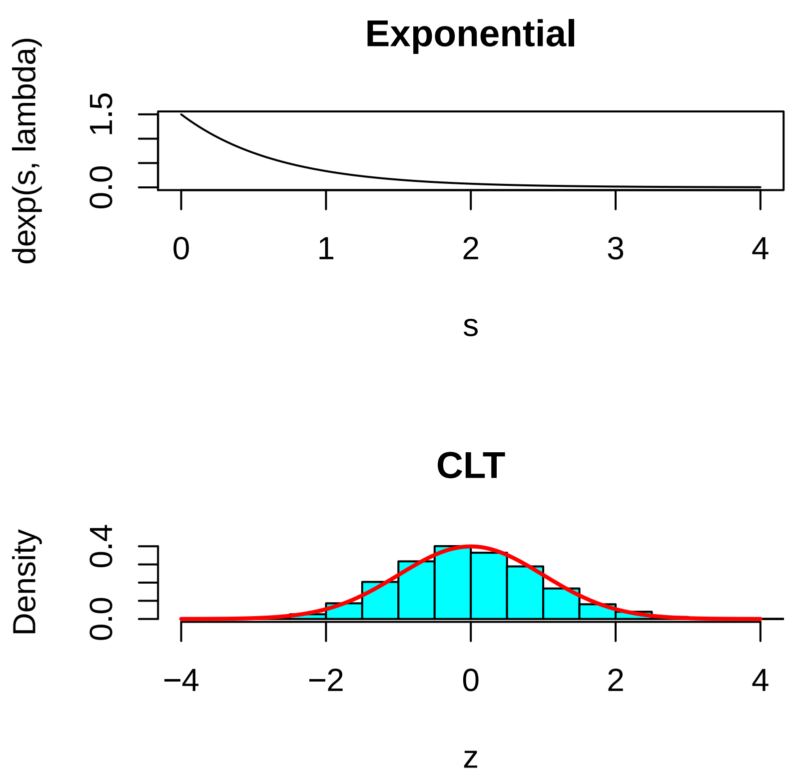
```
  z<-array(0,dim=c(itrn))
  for(i in 1 : itrn){
    x<-rpois(ss,lambda)
    z[i]<-sqrt(ss)*(mean(x)-lambda)/sqrt(lambda)

  }
  par(mfrow=c(2,1))
  barplot(dpois(0:20,lambda), main = "poisson")
  hist(z, probability = T, xlim = c(-4,4) ,col=5 ,main = "CLT" )
  s<-seq(-4,4,by=0.01)
  lines(dnorm(s,0,1)~s , col=2, lwd=2)
}
ss<-150 # sample size
itrn<-10000 #iteration
lambda= 1.5
poiss_clt(itrn,ss,lambda)
```






```
#exponential CLT
exp_clt<-function(itrn,ss,lambda){
  z<-array(0,dim=c(itrn))
  for(i in 1 : itrn){
```

```
    x<-rexp(ss,lambda)
    z[i]<-sqrt(ss)*(mean(x)-1/lambda)/sqrt(1/lambda^2)
  }
  par(mfrow=c(2,1))
  s<-seq(0,4, by=0.01)
  plot(dexp(s,lambda)~s, main = "Exponential", type="l")
  hist(z, probability = T, xlim = c(-4,4) ,col=5 ,main = "CLT" )
  s<-seq(-4,4,by=0.01)
  lines(dnorm(s,0,1)~s , col=2, lwd=2)
}
ss<-150 # sample size
itrn<-10000 #iteration
lambda= 1.5
exp_clt(itrn,ss,lambda)
```



9. Estimation

Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be the observed/ realized values of a set of i.i.d. random variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ where$X_i \overset{iid}{\sim} f_\theta$ for some $\theta \in \Theta$. Here a family of distributions is denoted by

$$\mathcal{F} = \{f(x|\theta)|\theta \in \Theta\} \text{ or } \{F(x|\theta)|\theta \in \Theta\}$$

Parametric Estimation: In a parametric inference problem it is assumed that the family of the distribution is known but the particular value of the parameter is unknown. We estimate the value of the parameter θ as a function of the observations $\mathbf{x}$. The ultimate goal is to approximate the p.d.ff_θ or F_θ through the estimation of θ itself. Parametric estimation has two aspects, namely, (a) **Point estimation** and (b) **Interval estimation** .[We will learn it after Testing]

In point estimation we will learn
(a) Definition of an estimator
(b) Good properties of an estimator
(c) Methods of estimation (MME and MLE)

Definition 119. Statistic: A statistic is a function of random variables and it is free from any unknown parameter. Being a (measurable) function,$T(\mathbf{X})$ say , of random variables it is also a random variable.

Definition 120. Estimator: If the statistic $T(\mathbf{X})$ is used to estimate a parametric function $g(\theta)$ then $T(X)$ is said to be {an estimator of $g(\theta)$. And a realized value of it for $\mathbf{X} = \mathbf{x}$ i.e. $T(\mathbf{x})$ is know as **an estimate** of θ. We often abuse the notation as $g(\hat{\theta}) = T(\mathbf{x})$ and $g(\hat{\theta}) = T(\mathbf{X})$ which are understood from the context.

Definition 121. Unbiased estimator: An estimator $T(\mathbf{X})$ is said to be an unbiased estimator of a parametric function $g(\theta)$ if $E(T(\mathbf{X}) - g(\theta)) = 0 \ \forall\ \theta \in \Theta$.

Remark 122. It does not require$T(\mathbf{x}) = g(\theta)$ to be hold or it may hold with probability zero.

Definition 123. Bias: The bias of an estimator $T(\mathbf{X})$ while estimating a parametric function $g(\theta)$ is $B_{g(\theta)}(T(\mathbf{X})) = E(T(\mathbf{X}) - g(\theta)) \ \forall\ \theta \in \Theta$.

Definition 124. Asymptotically unbiased estimator: Denoting $T_n = T(X_1, X_2, \cdots, X_n)$ an estimator T_n is said to be asymptotically unbiased of $g(\theta)$ if

$$\lim_{n\to\infty} B_{g(\theta)}(T_n) = \lim_{n\to\infty} E(T_n - g(\theta)) = 0$$

Definition 125. Consistent estimator: An estimator T_n is said to be consistent estimator $g(\theta)$ if $T_n \xrightarrow{P} g(\theta)$ i.e.

$$\lim_{n\to\infty} P(|T_n - g(\theta)| < \epsilon) = 1 \ \forall\ \theta \in \Theta, \epsilon > 0$$

Definition 126. Mean squared error (MSE): The MSE of an estimator $T(\mathbf{X})$ while estimating a parametric function $g(\theta)$ is

$$MSE_{g(\theta)}(T(\mathbf{X})) = E[(T(\mathbf{X}) - g(\theta))^2] \ \forall\ \theta \in \Theta.$$

Exercise 127. Show that $MSE_{g(\theta)}(T(\mathbf{X})) = Var(T(\mathbf{X})) + B^2_{g(\theta)}(T(\mathbf{X}))$

Exercise 128. If $MSE_{g(\theta)}(T_n(\mathbf{X})) \downarrow 0$ as $n \uparrow \infty$ then show that $(T_n(\mathbf{X}))$ is a consistent estimator.

Remark 129. Asymptotic unbiasedness and consistency are large sample properties and both are based on L_1 norm. . MSE is defined based on L_2 norm.

Exercise 130. Let $(X_1, X_2, \cdots, X_n)$ be i.i.d random variables with $E(X) = \mu$ and $Var(X) = \sigma^2$. and define $T_n(\mathbf{X}) = \bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $S_1^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ and $S_2^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$. Show that
(a) $T_n(\mathbf{X})$ is an unbiased estimator of μ.
(b) S_1^2 is a biased estimator of σ^2
(c) S_2^2 is an asymptotically unbiased estimator of σ^2

Exercise 131. Let $(X_1, X_2, \cdots, X_n) \overset{iid}{\sim} N(\mu, \sigma^2)$. Show that $MSE(S_2^2) < MSE(S_1^2)$. Note: Unbiased estimator need not have minimum MSE.

Definition 132. Method of Moment for Estimation (MME): Consider $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be the observed/ realized values of a set of i.i.d. random variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ where $X_i \overset{iid}{\sim} f_\theta$ for some $\theta \in \Theta$. Then
Step 1: Computer theoretical moments from the p.d.f.
Step 2: Computer empirical moments from the data.
Step 3: Construct k equations if you have k unknown parameters.
Step 4: Solve the equations for the parameters.

Remark 133. We can not use MME to estimate the parameters of $C(\mu, \sigma)$, because the moments does not exists for Cauchy distribution.

Definition 134. Maximum Likelihood Estimator: Consider $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be the observed/ realized values of a set of i.i.d. random variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ where $X_i \overset{iid}{\sim} f_\theta$ for some $\theta \in \Theta$. Then the joint p.d.f. of $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is a function of $\mathbf{x}$ when the parameter value is fixed i.e.

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n f(x_i, \theta)$$

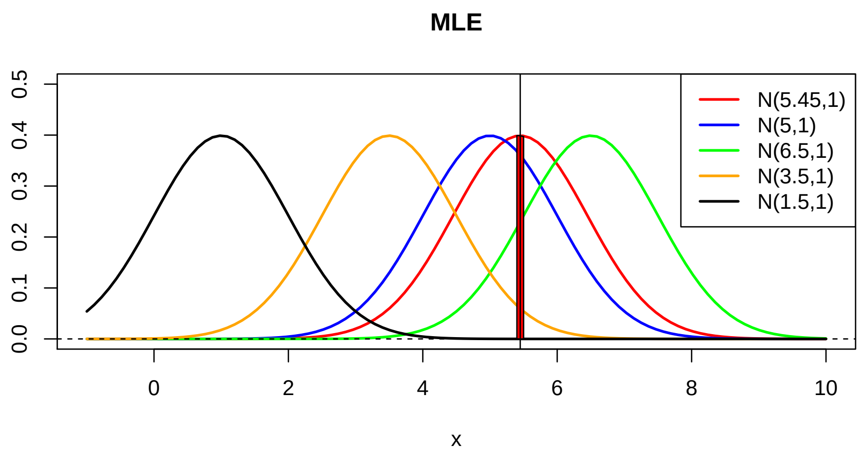
and the likelihood of a function of parameter for a given set of data $\mathbf{X} = \mathbf{x}$ i.e.

$$\ell(\theta|\mathbf{x}) = \prod_{i=1}^n f(x_i, \theta).$$

Hence the maximum likelihood estimator of θ is

$$\hat{\theta}_{mle} = \arg\max_{\theta \in \Theta} \ell(\theta|\mathbf{x}) = \arg\max_{\theta \in \Theta} \log \ell(\theta|\mathbf{x})$$

NOTE: Finding the maxima through differentiation is possible **only of** ℓ is a smoothly differentiable function w.r.t θ. Otherwise it has to be maximized by some other methods. **Differentiation is not the only way of finding maxima or minima.**



Exercise 135. $(X_1, X_2, \cdots, X_n) \overset{iid}{\sim} U(0, \theta)$. where $\theta \in \Theta = (0, \infty)$.
(a)Find the MLE of θ.
(b) Is it an unbiased estimator ?
(c) Find the MSE.

Exercise 136. Let $(X_1, X_2, \cdots, X_n) \overset{iid}{\sim} N(\mu, \sigma^2)$.
(a) Find the MME and MLE of μ and σ^2. Are they same ?
(b) Are they unbiased estimators?

Exercise 137. Let $(X_1, X_2, \cdots, X_n) \overset{iid}{\sim} Gamma(\alpha, \lambda)$.
(a) Find the MME of (α, λ)?
(b) Find MLE of (α, λ) by by an iterative method of solution.
NOTE: You may use the MME as an initial value of iteration to obtain the MLE.

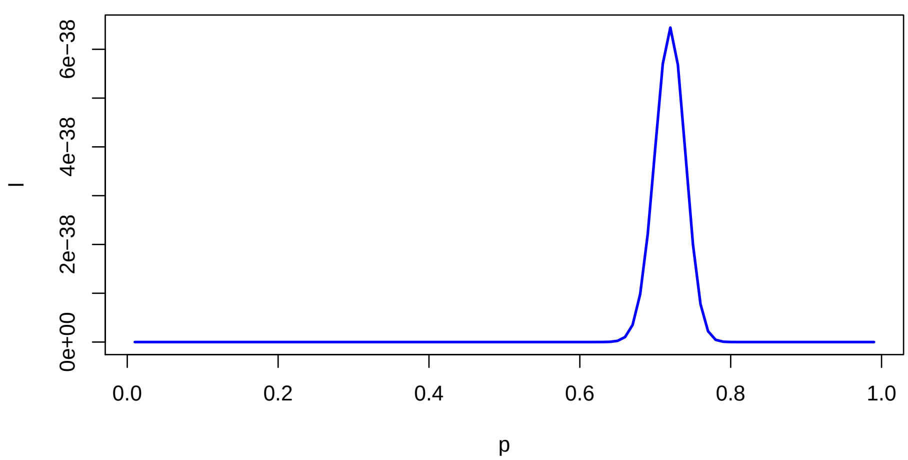
Propertied of MLE:
(a) MLE need not be unique.
(b) MLE need not be an unbiased estimator.
(c) MLE is always a consistent estimator.
(d) MLE is asymptotically normally distributed up to some location and scale when some regularity condition satisfied like
(1) Range of the random variable is free from parameter.
(2) Likelihood is smoothly differentiable for up to 3rd order and corresponding
expectations exists.

Exercise 138. $(X_1, X_2, \cdots, X_n) \overset{iid}{\sim} U(\theta - 0.5, \theta + 0.5)$ where $\theta \in \Theta = (-\infty, \infty)$.
(a)Find the MLE of θ.
(b) Is it unique?
(c) Is it consistent? Find the MSE.

```r
# MLE of binomial parameter
set.seed(12)
n<-10  # size of binomial
x<- sort(rbinom (50, n, 0.7))  # sample  given
print(x)
```

```
##  [1]  4  5  5  5  5  6  6  6  6  6  6  6  6  6  6  6  7  7  7  7  7  7  7
## [24]  7  7  7  7  7  8  8  8  8  8  8  8  8  8  8  8  8  8  8  9  9  9  9
## [47]  9  9 10 10
```

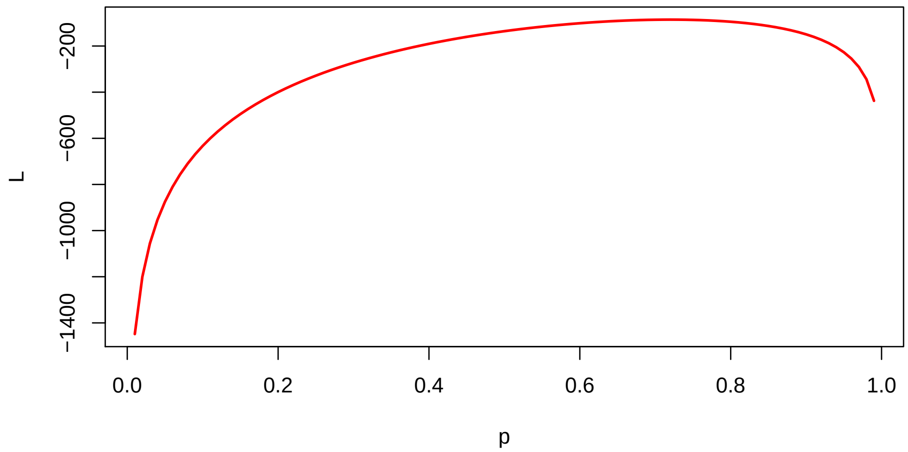
```r
# MLE finding
p<-seq(0.01,0.99,by = 0.01)
l<-array(0,dim=c(length(p)))
for (i in 1 : length(p)){
  l[i]<-prod(dbinom(x,n,p[i]))  # product of likelihood
}

plot(l~p, type='l', col=4, lwd=2)
```




```r
mle1<-p[which(l==max(l))]
print(mle1)
```

```
## [1] 0.72
```

```r
L<-array(0,dim=c(length(p)))
for (i in 1 : length(p)){
  L[i]<-sum(log(dbinom(x,n,p[i])))  #sum of log likelihood
}

plot(L~p,type='l', col=2, lwd=2)
```




```
mle2<-p[which(L==max(L))]
print(mle2)
## [1] 0.72
```

Definition 139. Interval Estimation: Consider a pair of statistic $(L(\mathbf{X}), U(\mathbf{X}))$ such that for a parameter θ ,

$$P_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})]) = 1 - \alpha$$

Then a $100(1-\alpha)\%$ confidence interval of θ is considered to be $[L(\mathbf{X}), U(\mathbf{X})]$.

Example 140. If $X_1, X_2, \ldots, X_n$ are i.i.d random variables with $N(\mu, \sigma^2)$ distribution with known value of σ^2. Then a $100(1-\alpha)\%$ CI of μ is

$$\left[L(\mathbf{X}) = \overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, U(\mathbf{X}) = \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right]$$

Example 141. If $X_1, X_2, \ldots, X_n$ are i.i.d random variables with $N(\mu, \sigma^2)$ distribution . Then a $100(1-\alpha)\%$ CI of μ is

$$\left[L(\mathbf{X}) = \overline{X} - \frac{\hat{\sigma_u}}{\sqrt{n}} \tau_{\alpha/2,n-1}, U(\mathbf{X}) = \overline{X} + \frac{\hat{\sigma_u}}{\sqrt{n}} \tau_{\alpha/2,n-1}\right]$$

$\hat{\sigma}_u^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$ is an unbiased estimator of unknown variance and a $100(1-\alpha)\%$ CI of σ^2 is

$$\left[L(\mathbf{X}) = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\chi^2_{\alpha/2,(n-1)}}, U(\mathbf{X}) = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\chi^2_{1-\alpha/2,(n-1)}}\right]$$

10. Testing of Hypothesis

Definition 142. Hypothesis: A hypothesis in parametric inference is a statement about the population parameter. It has two categories. A **null hypothesis** (H_0) specifies a subset Θ_0 in the parameter space Θ. If Θ_a is a singleton set then it called a **simple null**, otherwise a **composite null.** On the other hand an **alternative hypothesis** (H_1) specifies another subset $\Theta_a \subset \Theta$ which is disjoint to Θ_0.

Definition 143. Test Rule: A test rule is a statistical procedure, based on the distribution of the test statistic, which will reject the null hypothesis in favour of the alternative hypothesis.

Definition 144. Rejection Region or Critical region: A rejection Region or critical region is a subset $C \subset \mathbb{R}^n$ such that $\mathbf{X} \in C \Leftrightarrow T(\mathbf{X})$ will reject the null hypothesis.

Definition 145. Level-α test: For any $\alpha \in (0,1)$, a test is said to be level-α test if

$$\sup_{\theta \in \Theta_0} P_\theta(\mathbf{X} \in C) \leq \alpha.$$

Definition 146. Size-α test: For any $\alpha \in (0,1)$, a test is said to be size-α test if

$$\sup_{\theta \in \Theta_0} P_\theta(\mathbf{X} \in C) = \alpha.$$

Definition 147. Power-function: A power function is a function

$$P_\theta(\mathbf{X} \in C) : \Theta_a \to [0,1]$$

Remark 148. More than one tests with same level can be compared in terms of power functions. A test procedure with more power than the other with same level can be considered a better test.

Definition 149. Type-I error: The event $\mathbf{X} \in C$ when $\theta \in \Theta_0$ is known as Type-I error.

Definition 150. Type-II error: The event $\mathbf{X} \in C^c$ when $\theta \in \Theta_a$ is known as Type-II error. Power is 1-P(Type-II error).

Lemma 151. ***Neyman-Pearson Lemma (1933):*** *To test $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ reject H_0 in favour of H_1 at level/ size α if*

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x}|\theta_0)}{f(\mathbf{x}|\theta_1)} \leq \xi \quad \textit{such that} \quad P_{\theta_0}(\Lambda(\mathbf{X}) \leq \xi) = \alpha$$

How to perform a test ??

Step1: Estimate the parameter for which the testing to be done.
Step2: Estimate the unknown parameters if any.
Step3: Construct the test statistic and obtain its value.
Step4: Obtain the exact or asymptotic distribution of the test statistic under the null hypothesis.
Step5: Depending on the alternative hypothesis (H_1) and level (α) decide the cut-off value or rejection condition.
Step6: Compare the observed value of test statistic (from Step 4) and the cut off value (from Step 5) to conclude the test. You may use **p-value** also.

Exercise 152. Let $X_1, ..., X_n \sim N(\mu, \sigma^2)$ Perform a test at size 0.05 for
(a)$H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$. when σ^2 is known
(b)$H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$. when σ^2 is unknown
(a)$H_0 : \sigma^2 = \sigma_0^2$ vs $H_1 : \sigma^2 \neq \sigma_0^2$ when μ is unknown

```
library("TeachingDemos")
n<-10
mu_true<-10.5
sd_true<-1.2
x<-rnorm(10,mu_true,sd_true) # generate data
########################
print(x)
##  [1] 10.179138 10.261073 10.657347 10.674960 10.934478 11.308777 12.986443
##  [8]  9.850766  9.215409 10.053052
cat("Unbiased estimate of mean =",mean(x), "\n")
## Unbiased estimate of mean = 10.61214
cat("Unbiased estimate of variance =",var(x), "\n")
## Unbiased estimate of variance = 1.042931
alpha<-0.05
## (a)H_0: mu = 10  vs H_1: mu not equal to 10  when sigma^2 = (1.2)^2 is known
za<-z.test(x,mu = 10,stdev =  sd_true ,alternative =c("two.sided"),conf.level = (1-alpha))
print(za)
##
##  One Sample z-test
##
## data:  x
## z = 1.6131, n = 10.00000, Std. Dev. = 1.20000, Std. Dev. of the
## sample mean = 0.37947, p-value = 0.1067
## alternative hypothesis: true mean is not equal to 10
## 95 percent confidence interval:
##   9.86839 11.35590
## sample estimates:
## mean of x
##  10.61214
##(b)H_0: mu = 10  vs H_1: mu not equal to 10  when sigma^2  is unknown
```

```
 ta<-t.test(x, mu = 10,alternative =c("two.sided"),conf.level = (1-alpha))
print(ta)
##
##  One Sample t-test
##
## data:  x
## t = 1.8955, df = 9, p-value = 0.09054
## alternative hypothesis: true mean is not equal to 10
## 95 percent confidence interval:
##   9.881593 11.342695
## sample estimates:
## mean of x
##  10.61214
##(c)H_0: sigma^2 = 1  vs H_0: sigma^2 neq 1  when mu is unknown
 va<-sigma.test(x, sigma = 1,alternative = "two.sided", conf.level = (1-alpha))
print(va)
##
##  One sample Chi-squared test for variance
##
## data:  x
## X-squared = 9.3864, df = 9, p-value = 0.8048
## alternative hypothesis: true variance is not equal to 1
## 95 percent confidence interval:
##  0.4934285 3.4759337
## sample estimates:
## var of x
## 1.042931
```

Exercise 153. Let $X_1, ..., X_n \sim N(\mu_1, \sigma^2)$ (iid) and $Y_1, ..., Y_m \sim N(\mu_2, \sigma^2)$ (iid) are independent. Perform a test at size 0.05 for $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 \neq \mu_2$.

Exercise 154. Let $(X_i, Y_i) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho),\ i = 1, 2, ..., n$. Perform a test at size 0.05 for $H_0 : \mu_x = \mu_y$ vs $H_1 : \mu_x \neq \mu_y$. (this is known as paired-T test).

Exercise 155. Let $X_1, X_2 \ldots, X_n \sim Bernoulli(p)$ Perform a test for $H_0 : p = 0.5$ vs $H_1 : p = 0.5$ at size 0.05.

Exercise 156. Let $X_1, ..., X_n \sim N(\mu_1, \sigma_1^2)$ (iid) and $Y_1, ..., Y_m \sim N(\mu_2, \sigma_2^2)$ (iid) are independent. Perform a test at size 0.05 for $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$.

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