

Runge-Kutta Methods

1. Second order Runge-Kutta (R-K) methods

$$k_1 = h f(t_j, u_j)$$

$$\text{for } u' = f(x, u) \\ u(0) = u_0$$

$$k_2 = h f(t_j + c_2 h, u_j + a_{21} k_1)$$

$$u_{j+1} = u_j + w_1 k_1 + w_2 k_2$$

c_2	a_{21}
	$w_1 \quad w_2$

1	1
	$\frac{1}{2} \quad \frac{1}{2}$

Euler-Cauchy

$\frac{1}{2}$	$\frac{1}{2}$
	0 1

Modified
Euler-Cauchy

$\frac{2}{3}$	$\frac{2}{3}$
	$\frac{1}{4} \quad \frac{3}{4}$

Optimal

2. Third Order R-K methods

$$u_{j+1} = u_j + w_1 k_1 + w_2 k_2 + w_3 k_3$$

$$k_1 = h f(t_j, u_j)$$

$$k_2 = h f(t_j + c_2 h, u_j + a_{21} k_1)$$

$$k_3 = h f(t_j + c_3 h, u_j + a_{31} k_1 + a_{32} k_2)$$

c_2	a_{21}		
c_3	a_{31}	a_{32}	
	w_1	w_2	w_3

v_2	v_2		
1	-1	2	
	$1/6$	$4/6$	$1/6$

Classical

v_2	v_2		
$3/4$	0	$3/4$	
	$2/9$	$3/9$	$4/9$

Nearly optimal

$2/3$	$2/3$		
$2/3$	0	$2/3$	
	$2/8$	$3/8$	$3/8$

Nystrom

v_2	v_3		
$2/3$	0	$2/3$	
	$1/4$	0	$3/4$

Heun

3. Fourth order R-K methods

$$u_{j+1} = u_j + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4$$

$$k_1 = h f(t_j, u_j)$$

$$k_2 = h f(t_j + c_2 h, u_j + a_{21} k_1)$$

$$k_3 = h f(t_j + c_3 h, u_j + a_{31} k_1 + a_{32} k_2)$$

$$k_4 = h f(t_j + c_4 h, u_j + a_{41} k_1 + a_{42} k_2 + a_{43} k_3)$$

c_2	a_{21}			
c_3	a_{31}	a_{32}		
c_4	a_{41}	a_{42}	a_{43}	
	w_1	w_2	w_3	w_4

y_2	y_2			
y_2	0	y_2		
1	0	0	1	
	$1/6$	$2/6$	$2/6$	$1/6$

Classical

y_3	y_3			
$2/3$	$-y_3$	1		
1	1	-1	1	
	$1/8$	$3/8$	$3/8$	$1/8$

Kutta