ASSIGNMENT – 5

Numerical Solutions of Ordinary and Partial Differential Equations

1. Derive the Crank-Nicolson method. Use it to solve the parabolic partial differential

   equation $u_t = u_{xx}$, $x \in (0,1), t \in (0,\infty)$

   with initial condition $u(x,0) = 2x$, boundary conditions $u_x(0,t) = 0$ and

   $u_x(1,t) = 1$. Use the central difference approximation for the boundary conditions.

   Take $h = k = 0.5$. Mention the value of $u(0.5, 0.5)$.

2. Using the Crank-Nicolson method with $h = 1/2$ and the mesh ratio parameter $r = 1/3$

   find the solution of $u_t = u_{xx}$ with

   Initial condition $u(x,0) = \cos\dfrac{\pi x}{2}$, $-1 \le x \le 1, t = 0;$

   boundary conditions $u(-1,t) = u(1,t) = 0$, $t > 0$

   at the first time step (i.e. $t = k$).

3. Use the Crank-Nicolson method and the central difference for the boundary condition to

   solve the B.V.P. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, 0 < x < 1,$

   $$u(x,0) = 2, 0 \le x \le 1,$$

   $$u(0,t) = 2, t \ge 0,$$

   $$\frac{\partial u}{\partial t}(1,t) = -u(1,t), t \ge 0,$$

   With step length h = 1/3 and λ = 1/3. Integrate upto two time steps.

4. Use the explicit method to solve the wave equation

$$u_{tt} = u_{xx},\ 0 < x < 1, t > 0$$

with boundary and initial conditions

$$u(0,t) = -\sin t,\ u(1,t) = \sin(1-t),\ u(x,0) = \sin x, \quad u_t(x,0) = -\cos(x).$$

Take step length along *x*-axis and *t*-axis as 1/5 and 1 respectively. Find solution for $t = 2$.

5. Using standard 5-point formula, derive the system of algebraic equations at the nodal points for the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2, \quad -1 < x < 1, -1 < y < 1,$ $u = 2$ at $x = -1$ & $x = 1$; $u = 1$ at $y = -1$ & $y = 1$. Take $h = k = 1/2$. Setup the Gauss-Seidel iteration for the system of equations.

6. Use the explicit method

$$u_m^{n+1} = 2(1-p^2)u_m^n + p^2(u_{m-1}^n + u_{m+1}^n) - u_m^{n-1}$$

to find the solution of the below pde at the second time step $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ with $u(x,0) = \frac{1}{10}x^2, \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < 1$

and $\frac{\partial u}{\partial x}(0,t) = \frac{1}{5}t, u(1,t) = \frac{1}{10}(1+t)^2, \quad t > 0$.

Use $h = 1/2, k = 0.1; x \in [0,1]$ and use central difference approximation for the derivatives in the initial and boundary conditions.

7. Use the implicit scheme

$$\delta_t^2 u_m^n = r^2 \delta_x^2 [\theta u_m^{n+1} + (1-2\theta)u_m^n + \theta u_m^{n-1}]$$

with $\theta = 1/2$ and other symbols have their usual meanings, to solve the hyperbolic equation

$$u_{tt} = u_{xx}$$

with initial conditions $u(x,0) = \sin x$ and $u_t(x,0) = -\frac{1}{5}\cos x$

And the boundary conditions $u(0,t) = -\sin(\frac{t}{5})$ and $u(1,t) = \sin(1-\frac{t}{5})$.

Take $h = k = 0.25$. Solve for the first time level.

8. Use the explicit method to solve the wave equation $u_{tt} = \frac{1}{25} u_{xx}$, $0 < x < 1, t > 0$ with boundary and initial conditions $u(0,t) = -\sin(t/5)$, $u(1,t) = \sin(1 - t/5)$, $u(x,0) = \sin(x)$, $u_t(x,0) = -\frac{1}{5}\cos(x)$. Take step length along x-axis and t-axis as 1/5 and 1 respectively. Find solution for $t = 2$.

9. Using standard 5-point formula, derive the system of algebraic equations at the nodal points for the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8xy$, $-1 < x < 1, -1 < y < 1$, $u = 2$ at $x = -1$ & $x = 1$, $u = 1$ at $y = -1$ & $y = 1$, with $h = k = 1/2$. Setup the Gauss-Seidel iteration for the system of equations.

10. The torsion of an elastic beam of square cross section requires the solution of the BVP

$$u_{xx} + u_{yy} + 2 = 0, \quad (x,y) \in (-1,1) \times (-1,1)$$

with $u = 0$ on the boundary of the square. First write the discretization scheme using a step length $h = k = 0.5$. Now use symmetry of the problem to reduce the number of unknowns. Solve the equation by a direct method to find $u(0,0)$.

11. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in $0 \le x, y \le 1$ with $u(x,y) = e^{3x}\cos 3y$ on the boundary using the standard 5-point formula with $h = k = 1/3$. Use Gauss-Seidel iteration to solve the system of equations .

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