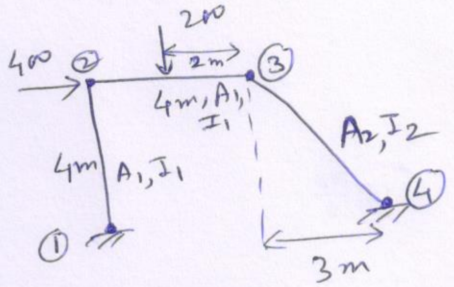


$E = 2 \times 10^7 \ kN/m^2$

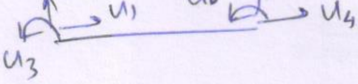
$I_1 = 12 \times 10^{-5} \ m^4$

$I_2 = 15 \times 10^{-5} \ m^4$

$A_1 = 0.03 \ m^2, \ A_2 = 0.035 \ m^2$

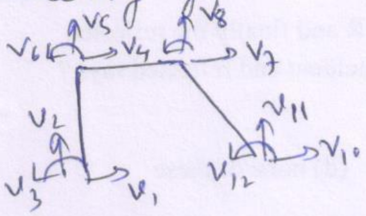
4 off nodes are connected by 4 off elements. The nodes are ①, ②, ③, ④

The stiffness matrix of an element →



$$\underset{6\times1}{\{f\}} = \underset{6\times6}{\begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ & & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l^2} \\ & & & \frac{EA}{l} & 0 & 0 \\ & \text{Symmetric} & & & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ & & & & & \frac{4EI}{l} \end{bmatrix}} \underset{6\times1}{\{u\}} \quad \text{or} \ \{f\} = [K_e]\{u\}$$

Considering global DOF^s



$\therefore$ the equation will be $\underset{12\times1}{\{F\}} = \underset{12\times12}{[K_G]} \underset{12\times1}{\{v\}}$

Since nodes ① & ④ are fixed, their DOF^s can be removed.

$\therefore$ then $\underset{6\times1}{\{F\}} = \underset{6\times6}{[K]} \underset{6\times1}{\{v\}}$, where $\{F\} = \begin{Bmatrix} F_4 \\ F_5 \\ \vdots \\ F_9 \end{Bmatrix}$, $\{v\} = \begin{Bmatrix} v_4 \\ v_5 \\ \vdots \\ v_9 \end{Bmatrix}$

Considering element ①-②,

stiffness matrix of the element $= \begin{bmatrix} 15\times10^4 & 0 & 0 & -15\times10^4 & 0 & 0 \\ & 450 & 900 & 0 & -450 & 900 \\ & & 2400 & 0 & -900 & 1200 \\ & & & 15\times10^4 & 0 & 0 \\ & & & & 450 & -900 \\ & & & & & 2400 \end{bmatrix}$

This element is rotated by $+90°$ if we consider basic element to be horizontal.

$\therefore$ Transformation matrix, $T = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Since $\theta = 90^\circ$, $[T] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = [T^{1-2}]$

To convert the stiffness matrix in the global system, we use

$[K_G^{1-2}] = [T]^T [K_e^{1-2}] [T]$

After multiplication,

$$[K_G^{1-2}] = \begin{array}{c} \begin{array}{cccccc} \cancel{1} & \cancel{2} & \cancel{3} & 4 & 5 & 6 \leftarrow \text{global DOF} \end{array} \\ \left[\begin{array}{cccccc} 450 & 0 & -900 & -450 & 0 & -900 \\ & 15\times10^4 & 0 & 0 & -15\times10^4 & 0 \\ & & 2400 & 900 & 0 & 1200 \\ & & & 450 & 0 & 900 \\ & \text{Symmetric} & & & 15\times10^4 & 0 \\ & & & & & 2400 \end{array}\right] \begin{array}{c} \cancel{1} \\ \cancel{2} \\ \cancel{3} \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

global DOF

Here, 1, 2, 3 doesn't exist due to support fixity

Hence this 3×3 matrix will be placed in the global stiffness matrix

$$\underbrace{\begin{bmatrix} & \\ & \boxed{3\times3} \end{bmatrix}}_{6\times6}$$

For element ②-③, no transformation is required since it is horizontal.

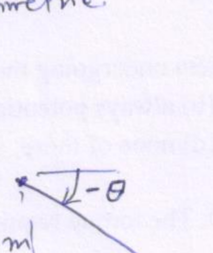
$$\therefore [K_G^{2-3}] = [K_e^{2-3}] = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \leftarrow \text{global DOFs} \end{array} \\ \left[\begin{array}{cccccc} 15\times10^4 & 0 & 0 & -15\times10^4 & 0 & 0 \\ & 450 & 900 & 0 & -450 & 900 \\ & & 2400 & 0 & -900 & 1200 \\ & & & 15\times10^4 & 0 & 0 \\ & \text{Symmetric} & & & 450 & -900 \\ & & & & & 2400 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \end{array}$$

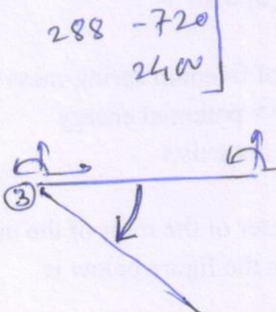
global DOFs

For element ③-④, local stiffness matrix →

$$[K_e^{3-4}] = \begin{bmatrix} 14\times10^4 & 0 & 0 & -14\times10^4 & 0 & 0 \\ & 288 & 720 & 0 & -288 & 720 \\ & & 2400 & 0 & -720 & 1200 \\ & & & 14\times10^4 & 0 & 0 \\ & \text{Symmetric} & & & 288 & -720 \\ & & & & & 2400 \end{bmatrix}$$

Element ③-④ →





Transformation matrix $[T] = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & -0.8 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore [T]^T [K_e^{3-4}][T]$ gives

$$[K_G^{3-4}] = \begin{array}{c} \\ \left[\begin{array}{cccccc} 7 & 8 & 9 & \cancel{10} & \cancel{11} & \cancel{12} \\ 50584 & -67062 & 576 & -50584 & 67062 & 576 \\ & 89704 & 432 & 67062 & -89704 & 432 \\ & & 2400 & 576 & 432 & 1200 \\ & & & 50584 & -67062 & -576 \\ & \text{Symmetric} & & & 89704 & -432 \\ & & & & & 2400 \end{array}\right] \begin{array}{c} \\ 7 \\ 8 \\ 9 \\ \cancel{10} \\ \cancel{11} \\ \cancel{12} \end{array} \end{array}$$

Here, 10, 11, 12 does not exist due to support fixity.

Hence $[\ ]_{3\times3}$ matrix will be located in the global matrix $[\ ]_{6\times6}$

$\therefore$ Adding all 3 elements we get

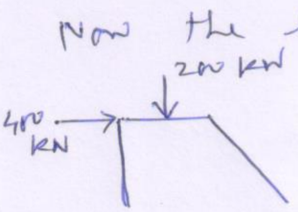
global DOFs

$$\begin{array}{c|cccccc} & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 4 & 150450 & 0 & 900 & -150000 & 0 & 0 \\ 5 & & 150450 & 900 & 0 & -450 & 900 \\ 6 & & & 4800 & 0 & -900 & 1200 \\ 7 & & & & 200584 & -67062 & 576 \\ 8 & & & & & 90154 & -468 \\ 9 & & & & & & 4800 \end{array} \begin{Bmatrix} v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{Bmatrix} = \{F\} = \begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} \quad \text{— (1)}$$

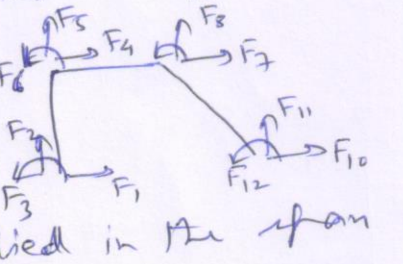
global DOFs

↑ Global stiffness matrix of the problem to be evaluated.

Now the load vector to be evaluated. The 400 kN load is applied directly on the node. Hence

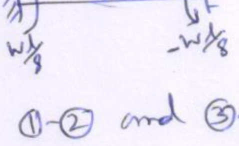


$F_4 = 400$ kN

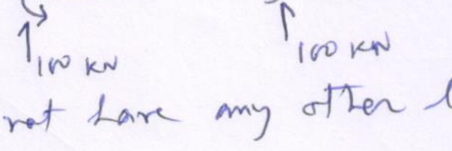


But 200 kN load is applied in the span (2)-(3).

$\therefore$

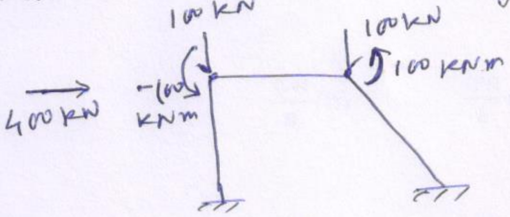


$\equiv$



Span (1)-(2) and (3)-(4) do not have any other loads.

Hence for joint equilibrium / equilibrium at DOFs, the following loads must be acting on the structure.



These loads together with support reaction, of ②-③ will be in equilibrium.

$$\therefore \{F\} = \begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \begin{Bmatrix} 400 \\ -100 \\ -100 \\ 0 \\ -100 \\ 100 \end{Bmatrix}.$$

Solving eq. ①, we get
$$\{v\} = \begin{Bmatrix} 0.2876 \\ 0.0001 \\ -0.0395 \\ 0.2856 \\ 0.2107 \\ 0.0169 \end{Bmatrix}$$

Now inorder to find the end forces of a member due to nodal displacements, we use the relation $[K_e]\{u\}=\{f\}$. Since $\{u\}=[T]\{v\}$, we get $[K_e][T]\{v\}=\{f\}$

For member ①-②,
$$\{f\}_{\text{due to nodal displacements}} = \left[K'^{\,1-2}_e\right][T]\begin{Bmatrix} v_1=0 \\ v_2=0 \\ v_3=0 \\ v_4=0.2876 \\ v_5=0.0001 \\ v_6=-0.0395 \end{Bmatrix} = \begin{Bmatrix} -15.11 \\ 93.87 \\ 211.44 \\ 15.11 \\ -93.87 \\ 164.04 \end{Bmatrix}$$

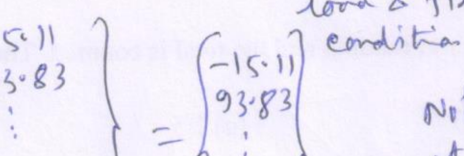
$\therefore$ Final end forces for member ①-② $= \{f\}_{\text{due to span load \& fixed condition}} + \{f\}_{\text{due to nodal displacements}}$

$$\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\text{due to FE condition}} + \begin{Bmatrix} -15.11 \\ 93.83 \\ \vdots \\ 164.04 \end{Bmatrix} = \begin{Bmatrix} -15.11 \\ 93.83 \\ \vdots \\ 164.04 \end{Bmatrix}$$

Note: 400 kN load is applied at the node, not on the span. So, no end forces develop due to this.

For element ②-③, end forces due to fixed end condition

$$\begin{Bmatrix} 0 \\ 100 \\ 100 \\ 0 \\ 100 \\ -100 \end{Bmatrix}$$



Hence, end forces of 2-3 is given by

$$\{f\}_{\substack{total\\2-3}} = \begin{Bmatrix} 0 \\ 100 \\ 100 \\ 0 \\ 100 \\ -100 \end{Bmatrix} + \left[K_e^{2-3}\right]_{6\times6} \left[T^{2-3}\right]_{6\times6} \begin{Bmatrix} 0.2876 \\ 0.0001 \\ -0.0395 \\ 0.2852 \\ 0.2107 \\ 0.0169 \end{Bmatrix} = \begin{Bmatrix} 300 \\ -15 \\ -164 \\ -300 \\ 215 \\ -296 \end{Bmatrix}.$$

For member 3-4, there is no span loadings. Hence, the total end forces given by

$$\{f\}_{\substack{total\\3-4}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \left[K_e^{3-4}\right]_{6\times6} \left[T^{3-4}\right]_{6\times6} \begin{Bmatrix} 0.2856 \\ 0.2107 \\ 0.0169 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{6\times1} = \begin{Bmatrix} 392 \\ 114 \\ 296 \\ -392 \\ -114 \\ 275.8 \end{Bmatrix}.$$

→ we donot require this [T] matrix since the element is horizontal; i.e., $\left[K_e^{2-3}\right]\left[T^{2-3}\right] = \left[K_e^{2-3}\right]$

there will be no change after multiplication.

Once end forces are known, we can get SF and BM at any arbitrary point. For eg., member 2-3