Lecture - 6

Marine Hydrodynamics

Ex:- 1: Find the pressure if the velocity field is given by for a inviscid fluid as:-

$$u(x,y) = \frac{A(x^2-y^2)}{(x^2+y^2)^2}, \quad v(x,y) = \frac{2Axy}{(x^2+y^2)^2}$$

Now from equation of motion, we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \dots (1)$$

$$\& \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \dots (2)$$

Now

$$\frac{\partial u}{\partial x} = \frac{A(x^2+y^2)^2 2x - A(x^2-y^2)2(x^2+y^2).2x}{(x^2+y^2)^4}$$

$$= \frac{A.(2x)(x^2+y^2)\left[(x^2+y^2) - 2(x^2-y^2)\right]}{(x^2+y^2)^4}$$

$$= \frac{A2x(x^2+y^2)\left[x^2+y^2-2x^2+2y^2\right]}{(x^2+y^2)^4}$$

$$= \frac{2Ax(3y^2-x^2)}{(x^2+y^2)^3}$$

Similarly

$$\frac{\partial u}{\partial y} = -\frac{2Ay(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial x} = \frac{2Ay(y^2 - 3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{2Ax(x^2+y^2)}{(x^2+y^2)^3}$$

<u>Now.</u> $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \Rightarrow$

$$\frac{A(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2Ax(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2Axy}{(x^2+y^2)^2}\cdot\frac{2Ay(3x^2-y^2)}{(x^2+y^2)^3} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{2Ax}{(x^2+y^2)^5}\left[(x^2-y^2)(3y^2-x^2) - 2y^2(3x^2-y^2)\right] = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{2Ax}{(x^2+y^2)^5}\left[3x^2y^2 - 3y^4 - x^4 + x^2y^2 - 6x^2y^2 + 2y^4\right] = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow -\frac{2Ax}{(x^2+y^2)^5}\cdot\left[x^4 + 2x^2y^2 + y^4\right] = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{2Ax(x^2+y^2)^{\cancel{2}}}{(x^2+y^2)^{\cancel{5}\,3}} = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\therefore \frac{\partial p}{\partial x} = \frac{2Ax}{(x^2+y^2)}$$

Similarly :.

$$\frac{2A^2 y}{(x^2+y^2)^3} = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Now. $$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

or $$dp = \frac{2A^2\rho x}{(x^2+y^2)^3}dx + \frac{2A^2\rho y}{(x^2+y^2)^3}dy$$

$$= \frac{A^2\rho\left[2x\,dx + 2y\,dy\right]}{(x^2+y^2)^3}$$

or $$dp = A^2\rho \cdot \frac{d(x^2+y^2)}{(x^2+y^2)^3}$$

$\therefore$ $$p = \frac{-A^2\rho}{2(x^2+y^2)^2} + C$$

or $\Rightarrow$ $$\boxed{p = -\frac{A^2\rho}{2}\cdot\frac{1}{(x^2+y^2)^2} + C}$$

1. <u>Two dimensional motion</u> : concept of stream function.

we understand the concept of stream line, it states that the line $\parallel^l$ to the velocity field. and hence $\vec{q} \times d\vec{s} = 0$ and from that we have arrived that for streamline:.

$$\begin{vmatrix} i & j & k \\ dx & dy & dz \\ u & v & w \end{vmatrix} = 0$$

$$\Rightarrow \quad \hat{i}(w\,dy - v\,dz) + \hat{j}(u\,dz - w\,dx) + \hat{k}(v\,dx - u\,dy) = 0$$

$\Rightarrow$ which implies.

$w\,dy - v\,dz = 0 \quad \ldots (1.1) \rightarrow x\text{- plane}$

$u\,dz - w\,dx = 0 \quad \ldots (1.2) \; - y\text{- plane}$

$v\,dx - u\,dy = 0 \quad \ldots (1.3) \; - z\text{- plane}$

<u>Now</u>, suppose we consider the 2D flow [i.e. in xy plane] we get the equation of of a

Stream line $= v\,dx - u\,dy = 0 \quad \ldots (1.4)$

<u>Now</u>, from (1.4) we understand that any there is no fluid velocity flux perpendicular to the stream line. This is straight forward.

<u>suppose</u> we try to apply the same concept along a arbitrary path ~~fo~~ from A to B. like

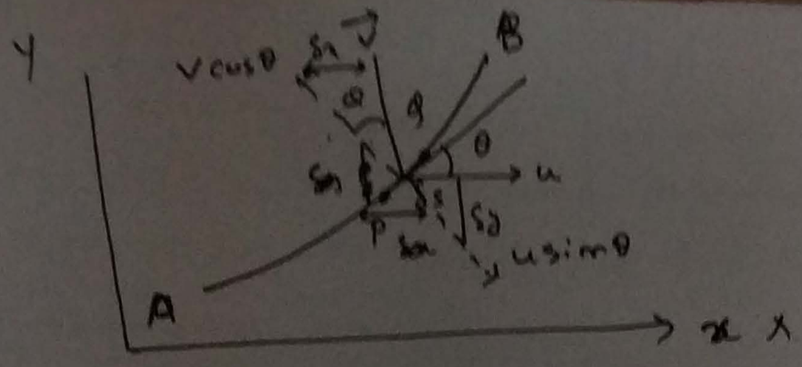


Figure - 1.1.

~~for~~ it mentioned in Figure 1.1, in that case, ~~for~~ for the small section δs, if we assume the potential difference in $\delta\psi$, then

$$\delta\psi = (v\cos\theta - u\sin\theta)\,\delta s \quad \dots (1.5)$$

Now from Figure 1.1: $\cos\theta = \frac{\partial x}{\partial s}$, $\sin\theta = \frac{\partial y}{\partial s}$

$$\Rightarrow d\,\delta\psi = v\,dx - u\,dy \quad \dots (1.6)$$

in case of stream line $\delta\psi = 0$ but for this case, we claim that ψ must be perfect differential as u and v ~~both~~ satisfy the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

~~Because~~ Because :. if $d\psi$ is perfect differential, then

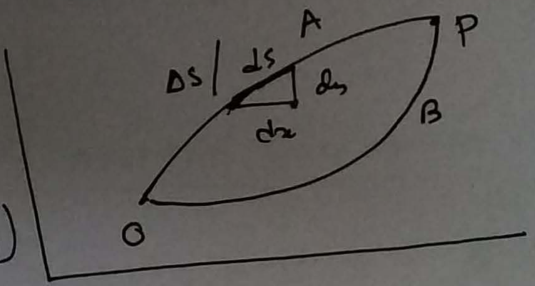
$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \quad \dots (1.7)$$

comparing $\frac{\partial\psi}{\partial x} = v$, $\frac{\partial\psi}{\partial y} = -u$ which automatically satisfies $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

With this analogy, one can say that, for 2D flow, if the fluid is inviscid and homogeneous, there exists a stream function $\psi(x, y, t)$ for both rotational or ir-rotational flow, where:

$$\boxed{\frac{\partial \psi(x,y,t)}{\partial x} = v, \qquad -\frac{\partial \psi(x,y,t)}{\partial y} = u}$$

2. Physical interpretation of stream function



Flow across the element ΔS or ds of OAP from O to P $\rho(v\,dx - u\,dy)$ per unit time.

$\therefore$ total net flow $= \rho \int_O^P (v\,dx - u\,dy)$ · per unit time.

Similarly, total net force across OBP also

$$\rho \int_O^P (v\,dx - u\,dy).$$

Now, path A & B are arbitrary except O & P, so at any instant, for arbitrary paths between O & P, mass flow across OAP must be same that of the OBP

$$\therefore \int_{O,\ \text{path OAP}}^{P} (v\,dx - u\,dy) = \int_{O,\ \text{path OBP}}^{P} (v\,dx - u\,dy)$$

$\Rightarrow$ the integral $\int_O^P (v\,dx - u\,dy)$ only depends on the end point, not the path.

which implies.

$$\int_{O(x_0,y_0)}^{P(x_1,y_1)} (v\,dx - u\,dy) = \text{a function of } (x,y;t) = \psi(x,y;t)$$

and

$$\boxed{\int_{O(x_0,y_0)}^{P(x_1,y_1)} (v\,dx - u\,dy) = \psi(x_1,y_1) - \psi(x_0,y_0)}$$

in-case $\widehat{OP}$ forms a closed curve $\psi(x,y) = 0$

3. velocity potential and stream function

For ir-rotational flow, velocity potential $\phi(x,y)$ exists,

and we know that $\frac{\partial \phi}{\partial x} = u$, $\frac{\partial \phi}{\partial y} = v$.

Now, for any 2-D ~~flow~~ inviscid, homogeneous fluid, we have stream function $\psi(x,y)$ such that :.

$$\frac{\partial \psi}{\partial x} = v, \quad \frac{\partial \psi}{\partial y} = -u.$$

which simply tells :.

$$\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

Now, if we change the sign convension, it is similar to say

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \& \quad \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad (3.1)$$

Now, it is known from complex analysis that, if z denotes the complex variable $z = x + iy$, and w the function $w(z) = \bar{u}(x,y) + i\,v(x,y)$

then $w = f(z)$ implies that w has a definite differential co-efficient if and only if

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

from this analogy, one can conclude that,

since $\phi(x,y)$ and $\psi(x,y)$ satisfies the relation

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \Big| \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

there exists a complex potential $w(z)$ such that

$$\boxed{w(z) = \phi(x,y) + i\,\psi(x,y)}$$

where $\phi(x,y)$ is the velocity potential and $\psi(x,y)$ is the stream function.

Ex:- 1: Find the stream function $\psi(x,y;t)$ for the given velocity potential $u = Ut$, $v = x$.

here $\dfrac{\partial \phi}{\partial x} = Ut$, $\dfrac{\partial \phi}{\partial y} = x$.

Now $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$

$\Rightarrow \dfrac{\partial \psi}{\partial y} = Ut \Rightarrow \psi(x,y,t) = Uyt + g(x,t) \rightarrow (1)$

Differentiating (1) w. r. t. x we get

$$\frac{\partial \psi}{\partial x} = g'(x,t)$$

Now $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

$\Rightarrow \quad -x = g'(x,t)$

integrating we get $\quad g(x,t) = -\frac{1}{2}x^2 + f(t)$

$\therefore \quad \psi = Uyt - \frac{1}{2}x^2 + f(t)$

~~Potential~~ Now,

$\frac{\partial \phi}{\partial x} = Ut \quad \Rightarrow \quad \phi(x,y,t) = Uxt + f(y,t)$

differentiating w. r. to. y we get

$\frac{\partial \phi}{\partial y} = f'(y,t) \quad \Rightarrow \quad f'(y,t) = x$

$\Rightarrow \quad f = \cancel{\text{constant}}\ xy$.

$\therefore \quad \phi(x,y,t) = Uxt + c.$

Now omitting constant and time varying part

we get

$$\phi + i\psi = Uxt + i\left(Uyt - \frac{1}{2}x^2\right)$$

$$= Ut(x+iy) - \frac{1}{2}ix^2$$

need to check

Lecture 7

Marine Hydrodynamics

Ex:-1 . If $\phi = A(x^2-y^2)$ represents possible flow phenomenon. Determine the stream function.

Soln:- $\phi(x,y) = A(x^2-y^2)$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = 2Ax \quad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = +\frac{\partial\psi}{\partial x} = +2Ay \quad \ldots (2)$$

integrating (1) we get

$$\psi(x,y) = 2Axy + f(x)$$

differentiating with respect to x we get

$$\frac{\partial\psi}{\partial x} = 2Ay + f'(x)$$

$$\Rightarrow \quad 2Ay = 2Ay + f'(x) \quad \text{[from (2)]}$$

$$\Rightarrow \quad f'(x) = 0 \Rightarrow f(x) = \text{Constant.}$$

Hence stream function $\psi(x,y) = 2Axy + C.$

Complex potential :-

$$w = \phi + i\psi$$

$$= A(x^2-y^2) + 2Aixy$$

$$= A[x^2 + 2ixy + (iy)^2]$$

$$= A(x+iy)^2$$

$$\boxed{w = Az^2}$$

Alternative method:

$$\phi(x, y) = A(x^2 - y^2)$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = 2Ax$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = 2Ay$$

Now since $\psi(x, y)$ is perfect differential.

$$\cancel{d\phi = \frac{\partial \phi}{\partial}}$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$\Rightarrow d\psi = 2Ay\,dx + 2Ax\,dy$$

$$\Rightarrow d\psi = 2A\,d(xy)$$

$$\Rightarrow \boxed{\psi = 2Axy + C}$$

Ex. 2 The velocity component $u = 2xy$ and $v = a^2 + x^2 - y^2$ show that a velocity potential exists and find out the velocity potential.

Suppose $\frac{\partial \phi}{\partial x} = u = 2xy$ $\quad \frac{\partial \phi}{\partial y} = v = a^2 + x^2 - y^2$

Now $\frac{\partial^2 \phi}{\partial x^2} = 2y$ $\quad \frac{\partial^2 \phi}{\partial y^2} = -2y$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

w $\therefore$ since velocity potential is perfectly ~~differentiable~~ differentiable $\therefore$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= 2xy\, dx + (a^2 + x^2 - y^2)\, dy$$

$$= a^2 dy - y^2 dy + 2xy\, dx + x^2 dy$$

$$= a^2 dy - y^2 dy + d(x^2 y)$$

$$\therefore \phi = a^2 y - \frac{1}{3} y^3 + x^2 y + c$$

Similarly $\quad d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$

$$= -\frac{\partial \phi}{\partial y} dx + \frac{\partial \phi}{\partial x} dy$$

$$= -(a^2 + x^2 - y^2)\, dx + (2xy)\, dy$$

$$= -a^2 dx - x^2 dx + y^2 dx + 2xy\, dy$$

$$\psi = -a^2 x - \frac{1}{3} x^3 + \cancel{d} \; xy^2$$

$\therefore$

Home work

Find the complex potential

Elementary Flows in 2-D plane.

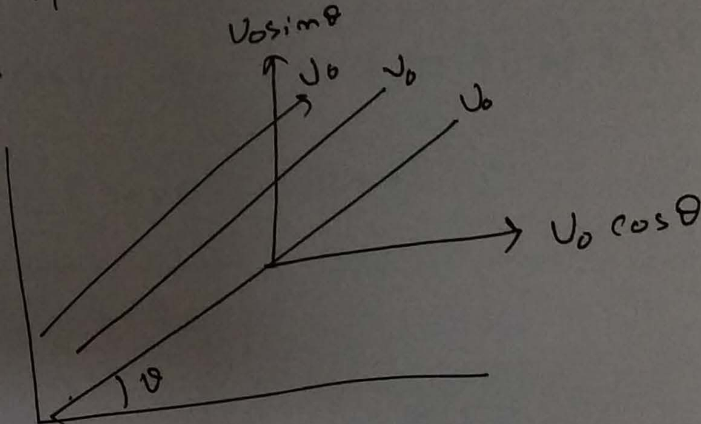
i) Uniform flow : $\frac{\partial \phi}{\partial x} = U_0, \quad \frac{\partial \phi}{\partial y} = 0$

$$\phi(x) = U_0 x + c_1$$

$$\psi(x) = U_0 y + c_2$$

Complex potential $\boxed{w(z) = U_0 z}$

(ii) Flow of a uniform stream with an angle θ with x axis



$\therefore$ ~~ue~~ $\frac{\partial \phi}{\partial x} = U_0 \cos\theta \quad , \quad \frac{\partial \phi}{\partial y} = U_0 \sin\theta$

$$\therefore \quad \phi(x, y) = U_0 x \cos\theta + f(y)$$

$$\frac{\partial \phi}{\partial y} = f'(y)$$

$$\Rightarrow \quad U_0 \sin\theta = f'(y)$$

$$\Rightarrow \quad f(y) = U_0 y \sin\theta$$

$$\therefore \quad \phi(x, y) = U_0 (x \cos\theta + y \sin\theta)$$

Similarly $\psi(x, y) = U_0 (y \cos\theta - x \sin\theta)$

Complex potential:

$$w(z) = \phi + i\psi$$

$$= U_0(x\cos\theta + y\sin\theta) + iU_0(y\cos\theta - x\sin\theta)$$

$$= U_0(x\cos\theta + iy\cos\theta) +$$

$$U_0(y\sin\theta \mp ix\sin\theta)$$

$$\therefore w(z) = U_0 z\cos\theta - iU_0(x+iy)\sin\theta$$

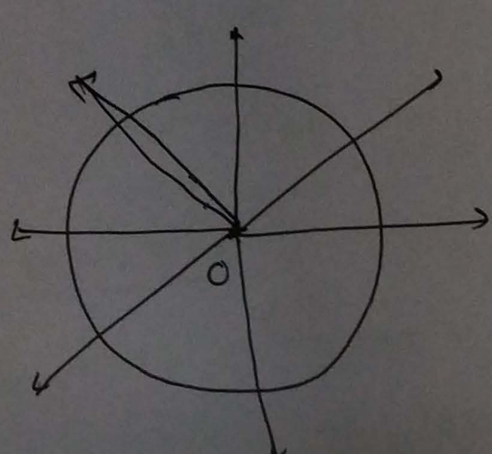
$$= U_0 z(\cos\theta - i\sin\theta)$$

$$\boxed{\therefore w(z) = U_0 z e^{-i\theta}}$$

2. Source or Sink

Note that, in polar co-ordinate, (r, θ)

$$v_r = \frac{\partial\phi}{\partial r}, \quad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$



Consider a source of a strength 'm' at origin O.
Let q_r be the radial velocity and q_θ be the angular velocity. Since the flow is radial and symmetric, the flux across a circle of radius r is $2\pi r\rho q_r$, also, since the strength of the source is m, ∴ flow across any small surrounding is $2\pi\rho m$ / m

then $2\pi\rho m = 2\pi r\rho q_r$

$$\Rightarrow q_r = \frac{m}{r}$$

now $q_r = \dfrac{\partial\phi}{\partial r} \Rightarrow \dfrac{m}{r} = \dfrac{\partial\phi}{\partial r}$

$$\Rightarrow \phi = m\log r$$

Since $\dfrac{1}{r}\cdot\dfrac{\partial\phi}{\partial\theta} = 0$ 'ϕ' is not function of θ.

Now, Note that :-

$$\boxed{\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}} \qquad \boxed{\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}}$$

$$\Rightarrow \frac{1}{r}\cdot\frac{\partial\psi}{\partial\theta} = \frac{m}{r}$$

$$\Rightarrow \psi = m\theta \qquad \left[\because \frac{\partial\psi}{\partial r} = 0\right]$$

∴ complex potential

$$w(z) = m\log r + im\theta$$

$$= m[\log r + i\theta]$$

$$w(z) = m\left[\log r + \log(e^{i\theta})\right]$$

$$w(z) = m\left[\log r.e^{i\theta}\right]$$

$$\Rightarrow \boxed{w(z) = m\log z}$$

$\therefore$ Complex potential for source $\Rightarrow$ $w(z) = m\log z$

$\therefore$ Complex potential for sink $\Rightarrow$ $w(z) = -m\log z$

Note that, in this case source is situated at origin, if the origin situated at 'a' then

$$w(z) = m\log(z-a) \quad \text{for source}$$

$$w(z) = -m\log(z-a) \quad \text{for sink}$$

in case of Dipole the complex potential can be written as

$$\boxed{w(z) = \frac{\mu}{z}}$$

where μ is the strength of the dipole/ doublet.

$\therefore$ Complex potential of a ~~a~~ combination of a source ~~of~~ at 'a' and sink at '−a' is

$$\boxed{w(z) = m\log(z-a) - m\log(z+a)}$$

complex potential of ~~a~~ combination of a uniform flow and a dipole

$$\boxed{w(z) = U_0 z + \frac{\mu}{z}}$$

Ex:- suppose complex potential ~~ϕ~~ of a 2-D flow is given by $f(z) = z^2$

Find :- (i) velocity potential

(ii) Stagnation point

(iii) Equation for stream line.

Solution :- we know

$$w(z) = \phi(x,y) + i\psi(x,y)$$

Now $w(z) = (x + iy)^2$

$$= x^2 + (iy)^2 + 2ixy$$

$$= (x^2 - y^2) + i(2xy)$$

$$\Rightarrow \phi(x,y) = x^2 - y^2, \quad \psi(x,y) = 2xy$$

Now we note that $\boxed{\frac{dw}{dz} = u - iv}$

now for stagnation point $u=0, v=0 \Rightarrow \frac{dw}{dz} = 0$

Now $\frac{dw}{dz} = 2z = 0 \Rightarrow z = 0 \Rightarrow r \cdot e^{i\theta} = 0$

since $e^{i\theta} \neq 0$, $r = 0$

$\Rightarrow$ (only possible at origin)

now equation of stream line is ψ = constant

$\Rightarrow$ $\boxed{xy = \text{Constant}}$

which is a rectangular hyperbola.

Ex: 2 Find the equation for stream line, flow velocity for $w(z) = \frac{Ua^2}{z}$

now $w(z) = \frac{Ua}{x+iy}$

or $w(z) = \frac{Ua\,(x-iy)}{(x+iy)\,(x-iy)}$

or $w(z) = \frac{Uax}{x^2+y^2} - i\,\frac{Uay}{x^2+y^2}$

$\therefore$ $\phi = \frac{Uax}{x^2+y^2}$ $\psi = -\frac{Uay}{x^2+y^2}$

Now for equation of stream-line

$-Uay = \frac{1}{k}(x^2+y^2)$

$\Rightarrow$ $\boxed{x^2+y^2+kUay = 0}$

Now $u = \frac{\partial\phi}{\partial x} = \frac{(x^2+y^2)Ua - Uax(2x)}{(x^2+y^2)^2}$

$= \frac{Ua(x^2-2x^2)+Uay^2}{(x^2+y^2)^2}$

$= \frac{Ua(y^2-x^2)}{(x^2+y^2)^2}$

Similarly find v