Picard's Successive approximation :

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

Integrate it from $x_0 \to x$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

$y^{(0)} = y_0$

$k \geq 0$ $$y^{(k+1)}(x) = y_0 + \int_{x_0}^{x} f[x, y^{(k)}(x)]\,dx$$

Ex $y' = x^2 - y \quad , \quad y(0) = 1$

Exact Sol^n

$$y = 2 - 2x + x^2 - e^{-x}$$

$$y^{(k+1)} = 1 + \int_0^x f[x, y^{(k)}(x)]\,dx$$

$$y^{(k+1)} = 1 + \int_0^x [x^2 - y^{k}(x)]\,dx$$

$$y^{(1)} = 1 + \int_0^x (x^2 - 1)\,dx$$

$$y^{(1)} = 1 + \frac{x^3}{3} - x = 1 - x + \frac{x^3}{3}$$

$$y^{(2)} = 1 + \int_0^x [x^2 - y^{(1)}(x)]\,dx$$

$$= 1 + \int_0^x [x^2 - 1 + x - \frac{x^3}{3}]\,dx$$

$$= 1 + \frac{x^3}{3} - x + \frac{x^2}{2} - \frac{x^4}{3 \cdot 4}$$

$$= 1 - x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{12}$$

Picard's Successive Approximation

While Picard's method is of great theoretical importance, the explicit evaluation of integrals in

$$y^{(k+1)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(k)}(x))\,dx$$

is often impracticable.

For the problem

$$y' = \cos(x+y), \quad y(0) = 1$$

$$y^{(0)}(x) = 1, \quad y^{(1)}(x) = 1 - \sin 1 + \sin(x+1)$$

and the second iteration would involve the evaluation of the form

$$y^{(2)}(x) = 1 + \int_0^x \cos[x + 1 - \sin 1 + \sin(x+1)]\,dx.$$

Runge - Kutta Methods :

$$y' = f(t,y), \quad y(t_0) = y_0, \quad t \in [t_0; b] \quad \text{--- (1)}$$

Now from Taylor series expansion

$$y(t_{n+1}) = y(t_n) + h\, y'(\xi_n). \quad \text{--- (2)}$$

where $\xi_n = t_n + \theta_n h, \; 0 < \theta_n < 1$

Euler's method with spacing $h/2$ is given by

$$y(t_n + h/2) \simeq y(t_n) + \frac{h}{2} f(t, y_n) \quad \text{--- (3)}$$

$| \; y(t_{n+1}) = y(t_n) + h f(t, y_n)$

Now from (2)

$$y(t_{n+1}) = y(t_n) + h f(\xi_n, y(\xi_n))$$

where
$\xi_n = t_n + \theta_n h$
Take $\theta_n = 1/2$
$\xi_n = t_n + h/2$

$$= y(t_n) + h f(t_n + h/2, y(t_n + h/2))$$

as $\theta_n = 1/2$

$$\simeq y(t_n) + h f(t_n + h/2, \underbrace{y(t_n) + h/2 f(t_n, y_n)}_{\text{from Euler's Method}})$$

So

$$y_{n+1} \simeq y_n + h f(t_n + h/2, y_n + h/2 f(t_n, y_n)) \quad \text{from (3)} \quad \text{--- (4)}$$

Alternate method

Now from (2)

$$y(t_{n+1}) = y(t_n) + h y'(\xi_n)$$

$$\xi_n = t_n + \theta_n h = t_n + \frac{1}{2} h$$

where $\theta_n = 1/2$

$$y(t_{n+1}) = y(t_n) + h y'(t_n + h/2) \quad \text{--- (5)}$$

Now $y'(t_n + h/2) = \frac{1}{2}[y'(t_{n+1}) + y'(t_n)]$

$| \; y(x + \frac{h}{2}) = y(x) + \frac{h}{2} y'(x) + \frac{h^2}{4\,2!} y''(x)$

$| \; y(x + \frac{h}{2}) = y(x) + \frac{h}{2}[y(x+h) - y(x) + O(h^2)] + O(h^2)$

$| \; y(x + h/2) = \frac{1}{2}(y(x+h) + y(x)) + O(h^2)$

then (5) can be written as

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}[y'(t_{n+1}) + y'(t_n)]$$

$$y(t_{n+1}) = y(t_n) + h/2\,[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \quad (2)$$

or $$y(t_{n+1}) = y(t_n) + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$$

from Euler's method.

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}[f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))] \quad (6)$$

Now from (5) & (6) we can write

$$\boxed{y_{n+1} = y_n + h \times (\text{average slop})}$$

This is the idea of Runge - Kutta approach. In general we find slop at t_n and several other points, average these slopes, multiply by h and add the result to y_n.

Derivation of RK Methods All RK methods will be written in the form

$$\boxed{y_{n+1} = y_n + hF(x_n, y_n, h; f) \qquad n \geq 0} \quad (1)$$

At this point it is intitutive that we want

$$F(x, y, h; f) \simeq Y'(x) = f(x, Y(x))$$

Y - exact soln.

for all values of h.

Define truncation error for (1) by

$$T_{n+1}(Y) = Y(x_{n+1}) - Y(x_n) - hF(x_n, Y(x_n); h; f) \qquad n \geq 0 \quad (2)$$

and define $\tau_{n+1}(Y)$ as

$$T_{n+1}(Y) = h\,\tau_{n+1}(Y) \quad (3)$$

then from (2) & (3)

$$h\,\tau_{n+1}(Y) = Y(x_{n+1}) - Y(x_n) - hF(x_n, Y(x_n), h; f)$$

or.
$$Y(x_{n+1}) = Y(x_n) + hF(x_n, Y(x_n), h; f) + h\,T_{n+1}(Y) \quad (3)$$

$h \geq 0$ — (4)

Now we suppose general form of F as

$$F(x, Y(x), h; f) = r_1 f(x,y) + r_2 f(x+\alpha h, y+\beta h f(x,y)) \quad (5)$$

Now

$$F(x, Y(x), h; f) = r_1 f(x,y) + r_2 f(x+\alpha h, y+\beta h f(x,y))$$

$$= r_1 f + r_2 \Big[f(x,y) + \alpha h f_x + \beta h f \cdot f_y + \frac{\alpha^2 h^2}{2!} f_{xx} + \frac{(\beta h f)^2}{2!} f_{yy} + \frac{2\alpha h \cdot \beta h f}{2!} f_{xy}\Big] + O(h^3)$$

$$= r_1 f + r_2 \Big[f + h(\alpha f_x + \beta f f_y) + h^2\Big(\frac{\alpha^2}{2} f_{xx} + \frac{\beta^2}{2} f^2 f_{yy} + \alpha\beta f f_{xy}\Big)\Big] + O(h^3)$$

$$\boxed{f(x_0+h, y+k) = f(x,y) + h f_x + k f_y + \frac{1}{2!}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right] + \cdots}$$

$$F(x, Y(x), h; f) = r_1 f + r_2 \Big[f + h(\alpha f_x + \beta f_y f) + h^2\Big(\frac{\alpha^2}{2} f_{xx} + \frac{\beta^2}{2} f^2 f_{yy} + \alpha\beta f f_{xy}\Big) + O(h^3) \quad (6)$$

Now $Y'(x) = f(x, Y(x))$

$$Y'' = \frac{d}{dx} f(x, Y(x)) = f_x + f_y Y'(x)$$

$$Y'' = f_x + f_y f \quad (7)$$

$$Y''' = \frac{d}{dx}\left[f_x + f_y f\right]$$

$$= (f_{xx} + f_{xy} Y'(x)) + \left[(f_{yx} f + f_{yy} Y' f) + (f_y f_x + f_y f_y Y')\right]$$

$$= f_{xx} + \underline{f_{xy} f} + \underline{f_{yx} f} + f_{yy} f^2 + f_y f_x + f_y^2 f$$

$$Y''' = f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y f_x + f_y^2 f \quad (8)$$

Now from truncation error ~~expression~~ (2)

$$T_{n+1}(Y) = Y(x_{n+1}) - Y(x_n) - hF(x_n, Y(x_n); h; f)$$

$$= \cancel{Y(x_n)} + hY'(x_n) + \frac{h^2}{2}Y''(x_n) + \frac{h^3}{6}Y'''(x_n)$$

$$- \cancel{Y(x_n)} - hF(x_n, Y(x_n), h; f)$$

Now putting expansion of $F(x, Y; h; f)$ and Y', Y'' Y''' from (6), (7) & (8) we get—

$$T_{n+1}(Y) = hY'(x_n) + \frac{h^2}{2}Y''(x_n) + \frac{h^3}{6}Y'''(x_n) - hF(x_n, Y(x_n), h; f)$$

$$= hf + \frac{h^2}{2}(f_x + f_y f) + \frac{h^3}{6}(f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_y f_x + f_y^2 f)$$

$$- h\left[(r_1 f + r_2 f) + hr_2(\alpha f_x + \beta f_y f) + h^2 r_2\left(\frac{\alpha^2}{2}f_{xx} + \frac{\beta^2}{2}f^2 f_{yy} + \alpha\beta f_{xy}f\right) + O(h^3)\right]$$

$$T_{n+1}(Y) = h(1 - r_1 - r_2)f + h^2\left[\left(\tfrac{1}{2} - \alpha r_2\right)f_x + \left(\tfrac{1}{2} - \beta r_2\right)ff_y\right]$$

$$+ h^3\left[\left(\frac{1}{6} - \frac{\alpha^2 r_2}{2}\right)f_{xx} + \left(\frac{1}{3} - \alpha\beta r_2\right)f_{xy}f + \left(\frac{1}{6} - r_2\frac{\beta^2}{2}\right)f_{yy}f^2 + \frac{1}{6}f_y f_x + \frac{1}{6}f_y^2 f\right]$$

$$+ O(h^4)$$

here all derivatives are evaluated at (x_n, y_n). (9)

Now to make $T_{n+1}(Y)$ of $O(h^3)$ we must have coefficients of h & h^2 ~~be~~ zero. Thus

$$r_1 + r_2 = 1, \quad \alpha r_2 = \frac{1}{2}, \quad \beta r_2 = \frac{1}{2} \qquad (10)$$

So there are 4 constants with three equations.

So we write

$$r_1 = 1 - r_2 \qquad \alpha = \beta = \frac{1}{2r_2} \quad \text{——— (11)}$$

with r_2 arbitrary

Take $r_2 = \frac{1}{2}$ we get $\Rightarrow r_1 = 1/2, \quad \alpha = \beta = 1$

$$F(x, y(x), h, f) = r_1 f(x,y) + r_2 f(x+\alpha h, y + \beta h f(x,y))$$
$$= \frac{1}{2} f(x,y) + \frac{1}{2} f(x+h, y + hf(x,y))$$

Then the method will be

$$y_{n+1} = y_n + hF(x_n, y_n, h; f)$$
$$= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]$$

and with $r_2 = 1 \Rightarrow r_1 = 0, \quad \alpha = \beta = \frac{1}{2}$

$$F(x, y(x), h; f) = r_1 f(x,y) + r_2 f(x+\alpha h, y + h\beta f(x,y))$$
$$= f\left(x + \frac{h}{2}, y + \frac{h}{2} f(x,y)\right)$$

then the method will be

$$y_{n+1} = y_n + hF(x_n, y_n, h; f)$$
$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + h/2\, f(x_n, y_n)\right)$$

This is called Modified Euler method.

Now we can not make $O(h^3)$ terms to zero but we can minimize this so that the error $T_{n+1}(Y)$ can be minimized. We will minimize this for arbitrary f i.e, we choose r_2 such that $O(h^3)$ term is minimum for arbitrary f.

$$T_{n+1}(Y) = C(f, r_2)\, h^3 + O(h^4) \quad \text{— (12)}$$

where

$$C(f, r_2) = \left(\frac{1}{6} - \frac{\alpha^2 r_2}{2}\right) f_{xx} + \left(\frac{1}{3} - \alpha\beta r_2\right) f_{xy} f + \left(\frac{1}{6} - \frac{r_2 \beta^2}{2}\right) f_{yy} f^2 + \frac{1}{6} f_y f_x + \frac{1}{6} f_y^2 f$$

From (11) $\alpha = \beta = \frac{1}{2r_2}$ so

$$C(f, r_2) = \left(\frac{1}{6} - \frac{1}{8r_2}\right) f_{xx} + \left(\frac{1}{3} - \frac{1}{4r_2}\right) f_{xy} f + \left(\frac{1}{6} - \frac{1}{8r_2}\right) f_{yy} f^2 + \frac{1}{6} f_y f_x + \frac{1}{6} f_y^2 f.$$

Define

$$C_1(f) = \left(f_{xx},\ f_{xy} f,\ f_{yy} f^2,\ f_x f_y,\ f_y^2 f\right)^T$$

$$C_2(r_2) = \left[\left(\frac{1}{6} - \frac{1}{8r_2}\right), \left(\frac{1}{3} - \frac{1}{4r_2}\right), \left(\frac{1}{6} - \frac{1}{8r_2}\right), \frac{1}{6}, \frac{1}{6}\right)$$

$\langle C_1(f), (C_2(r_2)) \rangle$

$= C(f, r_2)$ [by usual definition]

$= C_2^T C_1$

$y^T x = \langle \vec{x}, y \rangle = \sum_{i=1}^{n} x_i y_i$

for real x, y

Cauchy-Schwartz inequality

Cauchy-Schwartz inequality

$|\sum x_i y_i| \le (\sum x_i^2)^{1/2} (\sum y_i^2)^{1/2}$ $|\sum x_i y_i| = |\langle x, y \rangle| \le \|x\|_2 \|y\|_2$

$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

$|C(f, r_2)| = |\langle C_1(f), C_2(r_2) \rangle| \le (\sum x_i^2)^{1/2} \cdot (\sum y_i^2)^{1/2}$

$\le \|C_1(f)\|_2 \cdot \|C_2(r_2)\|_2$

$= [f_{xx}^2 + f_{xy}^2 f^2 + f_{yy}^2 f^4 + f_x^2 f_y^2 + f_y^2 f^2]^{1/2}$

$\cdot \underbrace{\left[2\left(\frac{1}{6} - \frac{1}{8r_2}\right)^2 + \left(\frac{1}{3} - \frac{1}{4r_2}\right)^2 + \frac{1}{36} + \frac{1}{36}\right]^{1/2}}_{h(r_2)}$

Now we will minimize $\|C_2(r_2)\|_2$

$h(r_2) = \|C_2(r_2)\|_2 = \left[2\left(\frac{1}{6} - \frac{1}{8r_2}\right)^2 + \left(\frac{1}{3} - \frac{1}{4r_2}\right)^2 + \frac{1}{18}\right]^{1/2}$

$h'(r_2) = \dfrac{1}{2\sqrt{\left[2\left(\frac{1}{6} - \frac{1}{8r_2}\right)^2 + \left(\frac{1}{3} - \frac{1}{4r_2}\right)^2 + \frac{1}{18}\right]}} \left[\frac{1}{2} \frac{\frac{1}{6} - \frac{1}{8r_2}}{r_2^2} + \frac{1}{2} \frac{\frac{1}{3} - \frac{1}{4r_2}}{r_2^2}\right]$

$h'(r_2) = 0 \Rightarrow r_2 = \frac{\cancel{6}\,3}{\cancel{8}\,4} = \frac{3}{4}$

for $r_2 = 3/4$ $h(r_2) = \frac{1}{\sqrt{18}}$ and resulting 2nd order method will be given by

$r_1 = 1 - r_2 = 1 - 3/4 = \frac{1}{4}$, $\alpha = \beta = \frac{1}{\cancel{2} \cdot 3} 4^2 = \frac{2}{3}$

$$y_{n+1} = y_n + h f(x_n, y_n, h; f) \qquad (8)$$

$$f(x,y,h,f) = r_1 f(x,y) + r_2 f(x+\alpha h, y+\beta h f(x,y))$$

$$\boxed{y_{n+1} = y_n + \frac{h}{4} f(x_n,y_n) + \frac{3}{4} h f\left(x_n+\frac{2}{3}h, y_n + \frac{2}{3} h f(x_n,y_n)\right)} \qquad (13)$$

This is an optimal method of 2nd order (RK) in the sense of $C(r_2, f)$ is minimized for r_2:

Take $\quad K_1 = h f(x_n, y_n)$

$\quad K_2 = h f(x_n + \frac{2}{3}h, y_n + \frac{2}{3} K_1)$

Then $\quad \boxed{y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_2)}$

(1) R-K method of 2nd Order $r_2 = 1/2$, $r_1 = 1/2$, $\alpha = \beta = 1$

$$y_{n+1} = y_n + \tfrac{1}{2}\left[h f(x_n, y_n) + h f(x_n + h, y_n + h f(x_n, y_n))\right]$$

$K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + h, y_n + K_1)$

$$\boxed{y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)}$$

(2) with $r_2 = 1$, $r_2 = 0$, $\alpha = \beta = 1/2$

$$y_{n+1} = y_n + h f(x_n + h/2, y_n + \tfrac{h}{2} f(x_n, y_n))$$

$K_1 = h f(x_n, y_n)$, $K_2 = h f(x_n + h/2, y_n + \frac{1}{2} K_1)$

$\boxed{y_{n+1} = y_n + K_2}$ — Modified Euler method -