Problem on iteration method

Ex Solve the bvp $u'' = \frac{3}{2} u^2$, $u(0) = 4$, $u(1) = 1$ with $h = 1/3$.

Solⁿ

x_0	x_1	x_2	x_3
0	$1/3$	$2/3$	$3/3$

$u_0 = 4, \quad u_3 = 1$

$$h^2 u_j'' = h^2 \frac{3}{2} u_j^2$$

$$-u_{j-1} + 2u_j - u_{j+1} = -h^2 \frac{3}{2} u_j^2 = -\frac{1}{6} u_j^2$$

for $j = 1$

$$-u_0 + 2u_1 - u_2 = -\frac{1}{6} u_1^2$$

or

$$2u_1 - u_2 = -\frac{1}{6} u_1^2 + 4 \quad — (1)$$

for $j = 2$

$$-u_1 + 2u_2 - u_3 = -\frac{1}{6} u_2^2$$

or

$$-u_1 + 2u_2 = -\frac{1}{6} u_2^2 + 1 \quad — (2)$$

Then from iteration method (1) & (2) can be written as

$$2u_1^{[r+1]} - u_2^{[r+1]} = -\frac{1}{6}\left(u_1^{[r]}\right)^2 + 4 \quad — (1)'$$

$$-u_1^{[r+1]} + 2u_2^{[r+1]} = -\frac{1}{6}\left(u_2^{[r]}\right)^2 + 1 \quad — (2)'$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1^{[r+1]} \\ u_2^{[r+1]} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}\left(u_1^{[r]}\right)^2 + 4 \\ -\frac{1}{6}\left(u_2^{[r]}\right)^2 + 1 \end{bmatrix} \quad (2)$$

$$\quad (3)$$

Now Take initial guess for u_1 & u_2 as

$u_1^{[0]} = 2, \quad u_2^{[0]} = 1.5$

Then from (3)

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1^{[1]} \\ u_2^{[1]} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 + 24 \\ -2.25 + 6 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 20 \\ 3.75 \end{bmatrix}$$

Solving this $\begin{pmatrix} u_1^{[1]} \\ u_2^{[1]} \end{pmatrix}$ is available, then proceed for $\begin{pmatrix} u_1^{[2]} \\ u_2^{[2]} \end{pmatrix}$ and so on.

Prob. on N-R method

Ex Solve the bvp $u'' = \frac{3}{2}u^2$, $u(0)=4$, $u(1)=1$ with $h=\frac{1}{3}$.

from (1) & (2)

$$2u_1 - u_2 = -\frac{1}{6}u_1^2 + 4 \quad \text{— (1)}$$

$$-u_1 + 2u_2 = -\frac{1}{6}u_2^2 + 1 \quad \text{— (2)}$$

or:

$$12u_1 - 6u_2 = -u_1^2 + 24 \quad \text{— (1)}'$$

$$-6u_1 + 12u_2 = -u_2^2 + 6 \quad \text{— (2)}'$$

Take

$$f_1(u_1,u_2) = 12u_1 - 6u_2 + u_1^2 - 24 = 0$$

$$f_2(u_1,u_2) = -6u_1 + 12u_2 + u_2^2 - 6 = 0$$

$$J = \frac{\partial(f_1,f_2)}{\partial(u_1,u_2)} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix}$$

$$\frac{\partial f}{\partial u_1} = 12 + 2u_1 \qquad \frac{\partial f_1}{\partial u_2} = -6$$

$$\frac{\partial f_2}{\partial u_1} = -6 \qquad \frac{\partial f_2}{\partial u_2} = 12 + 2u_2$$

$$J\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = -\begin{pmatrix} f_1(u_1,u_2) \\ f_2(u_1,u_2) \end{pmatrix}$$

$$\begin{bmatrix} 12+2u_1^{[r]} & -6 \\ -6 & 12+2u_2^{[r]} \end{bmatrix} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{bmatrix} 12u_1^{[r]} - 6u_2^{[r]} + \left(u_1^{[r]}\right)^2 - 24 \\ -6u_1^{[r]} + 12u_2^{[r]} + \left(u_2^{[r]}\right)^2 - 6 \end{bmatrix} \qquad (4)$$

~~Then~~ Take $u_1^{(0)} = 2, \quad u_2^{(0)} = 1.5$ (3)

$$u_1^{(1)} = u_1^{(0)} + \Delta u_1$$
$$u_2^{(1)} = u_2^{(0)} + \Delta u_2$$

From (3)

$$\begin{bmatrix} 16 & -6 \\ -6 & 15 \end{bmatrix} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{bmatrix} \cancel{24} - 9 + 4 - \cancel{24} \\ -\cancel{12} + \cancel{18} + 2.25 - \cancel{6} \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 2.25 \end{bmatrix}$$

Solve this then $\Delta u_1, \Delta u_2$ is available.