Lecture - 15

Marine Hydrodynamics

1. **Dynamic pressure**: the total pressure p_T may be written as:

$$p_T = -\rho \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi) . (\nabla \phi) + gz \right] \quad \ldots (1.1)$$

Now
$$\phi = \frac{ag}{\omega} e^{kz} \sin(kx - \omega t) \quad \ldots (1.2)$$

[For simplicity, considering deep water case]

then from (1.2)

$$\frac{\partial \phi}{\partial t} = -ag e^{kz} \cos(kx - \omega t) \quad \ldots (1.2)$$

$$\nabla \phi = \left(i \frac{\partial}{\partial x} + k \frac{\partial}{\partial z} \right) \phi \qquad [\text{2D- case}]$$

$$= i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial z}$$

$$\therefore \nabla \phi . \nabla \phi = \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2$$

$$= \left[a\omega e^{kz} \cos(kx - \omega t) \right]^2 + \left[a\omega e^{kz} \sin(kx - \omega t) \right]^2$$

See previous lecture note for expression of $u = \frac{\partial \phi}{\partial x}$ & $w = \frac{\partial \phi}{\partial z}$

$$= \cancel{a^2\omega^2 e^{2kz}} \; a^2 \omega^2 e^{2kz} \quad \ldots (1.3)$$

$\therefore$ from equation (1.1), we get

$$p = -\rho \left[-ag e^{kz} \cos(kx - \omega t) + \frac{1}{2} a^2 \omega^2 2 e^{kz} + gz \right] \quad \ldots (1.4)$$

Now in equation there are 3 components:

(i) Hydrostatic part: $\rightarrow gz$

dynamic pressure (ii) First order Dynamic pressure ~~wave loads~~: $\rightarrow -age^{kz}\cos(kx-\omega t)$

(iii) 2nd order drift forces $\frac{1}{2}a^2\omega^2e^{2kz}$.

2. Total pressure distribution due to wave under wave crest and trough.

under the linear theory, the 2nd order component may be ignored and then equation (1.4) takes the form

$$p = -\rho\left[-age^{kz}\cos(kx-\omega t) + gz\right] \quad \ldots (1.5)$$

$$\text{or } p = \rho\left[age^{kz}\cos(kx-\omega t) - gz\right] \quad \ldots (1.6)$$

Figure (2.1) shows the schematic diagram of pressure profile along z-axis

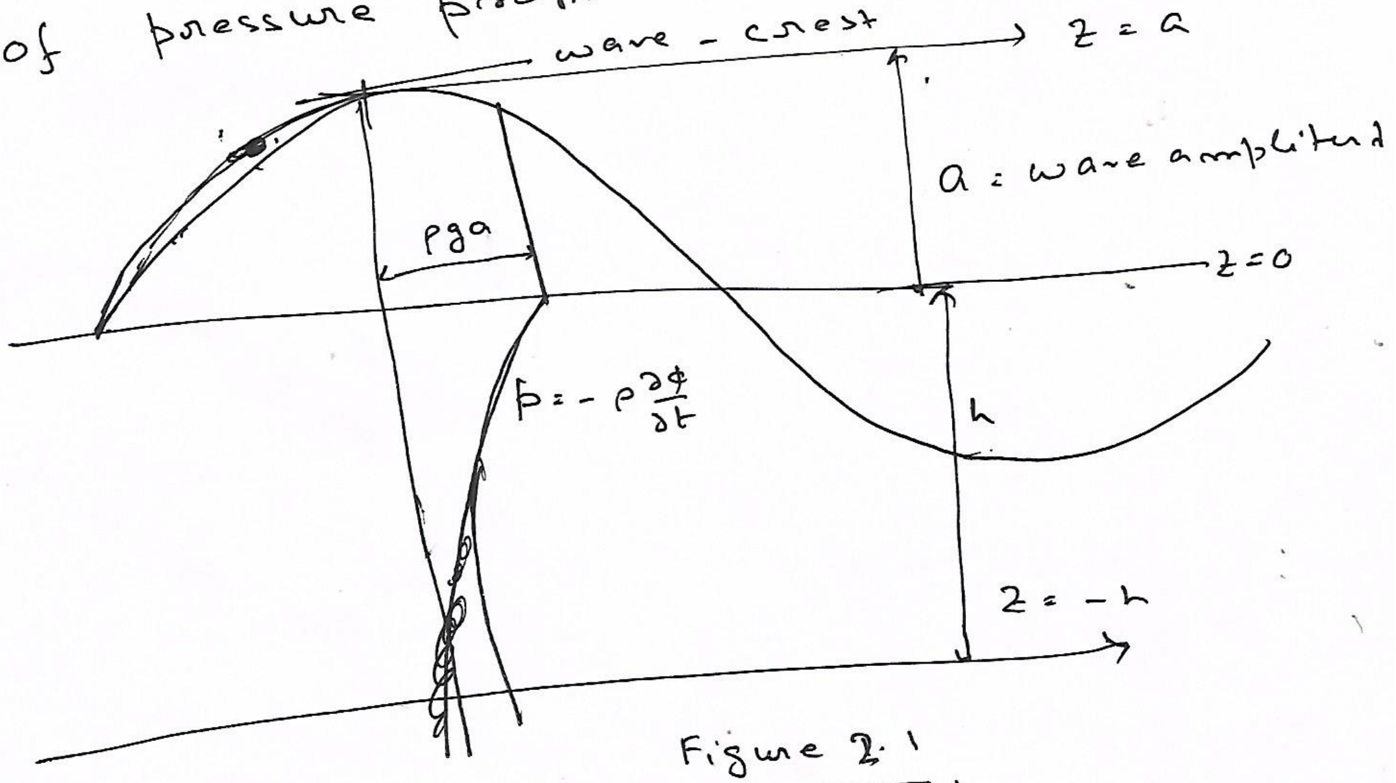

Figure 2.1

Now, the dynamic pressure $-\rho\frac{\partial\phi}{\partial t}$ is driven by e^{kz} as z decreases from $z=0$ to $z=-h$.

That is why the term $-\rho\frac{\partial\phi}{\partial t}$ decreases exponentially as depth decreases linearly from $z=0$ to $z=-h$.

the dynamic pressure at $z=0$,

$$p_d = -\rho\frac{\partial\phi}{\partial t}$$

$$= +\rho a g e^{kz}\Big|_{z=0} \quad \left[\cos(kx-\omega t)=1 \text{ at wave crest}\right]$$

$$= +\rho a g$$

$\therefore$ at $z=0$, the dynamic pressure is $\rho a g$.

Now, under the assumption of linearity, the dynamic pressure at $z=0$ should be equal to $z=a$ also. [at wave crest $|\eta| = a$]

$\therefore$ the figure (2.1) is justified.

2.1 (b) Hydrostatic pressure distribution at wave crest

Figure 2.2 ~~figures~~ shows the distribution of the hydrostatic force from $z=a$ to $z=-h$, the graph is varying linearly from $z=a$ to $z=-h$, and thus looks like as given in (2.

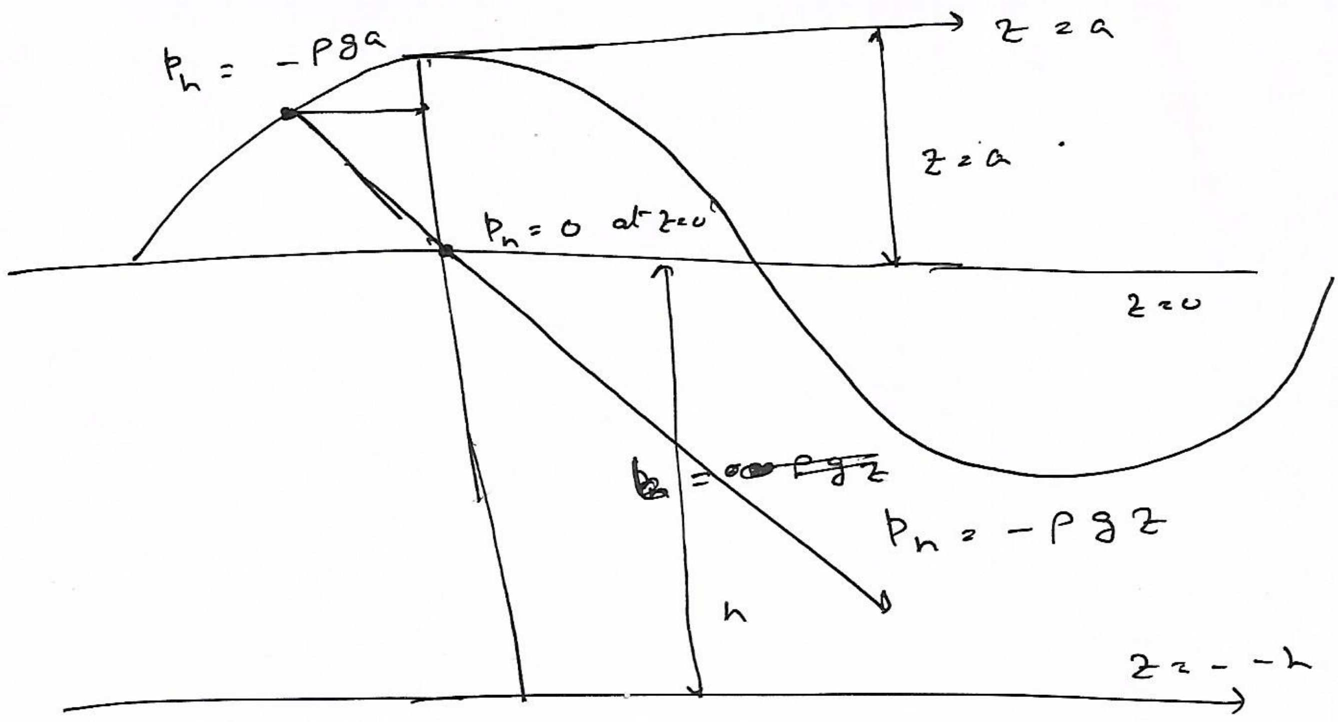

Figure 2.2

From the expression of hydrostatic pressure,
$p_h = 0$ at $z = 0$, $p_h = \rho g h$ at $z = -h$,
$p_h = -\rho g a$ at $z = a$.

Now, this two expression is automatically satisfied the dynamic free surface boundary condition as p_T at $z = a$ [which is free surface at wave crest]

$$p_T = -\rho \left[\frac{\partial \phi}{\partial t} + g z \right] \Big|_{z = a}$$

$$= -\rho \left[ag - ag \right] = 0$$

$\therefore$ $p_T = 0$ at wave crest.

which, confirms that, under the assumption of the linear theory, dynamic free surface boundary condition is automatically satisfied at wave crest.

3. Total pressure under the wave trough

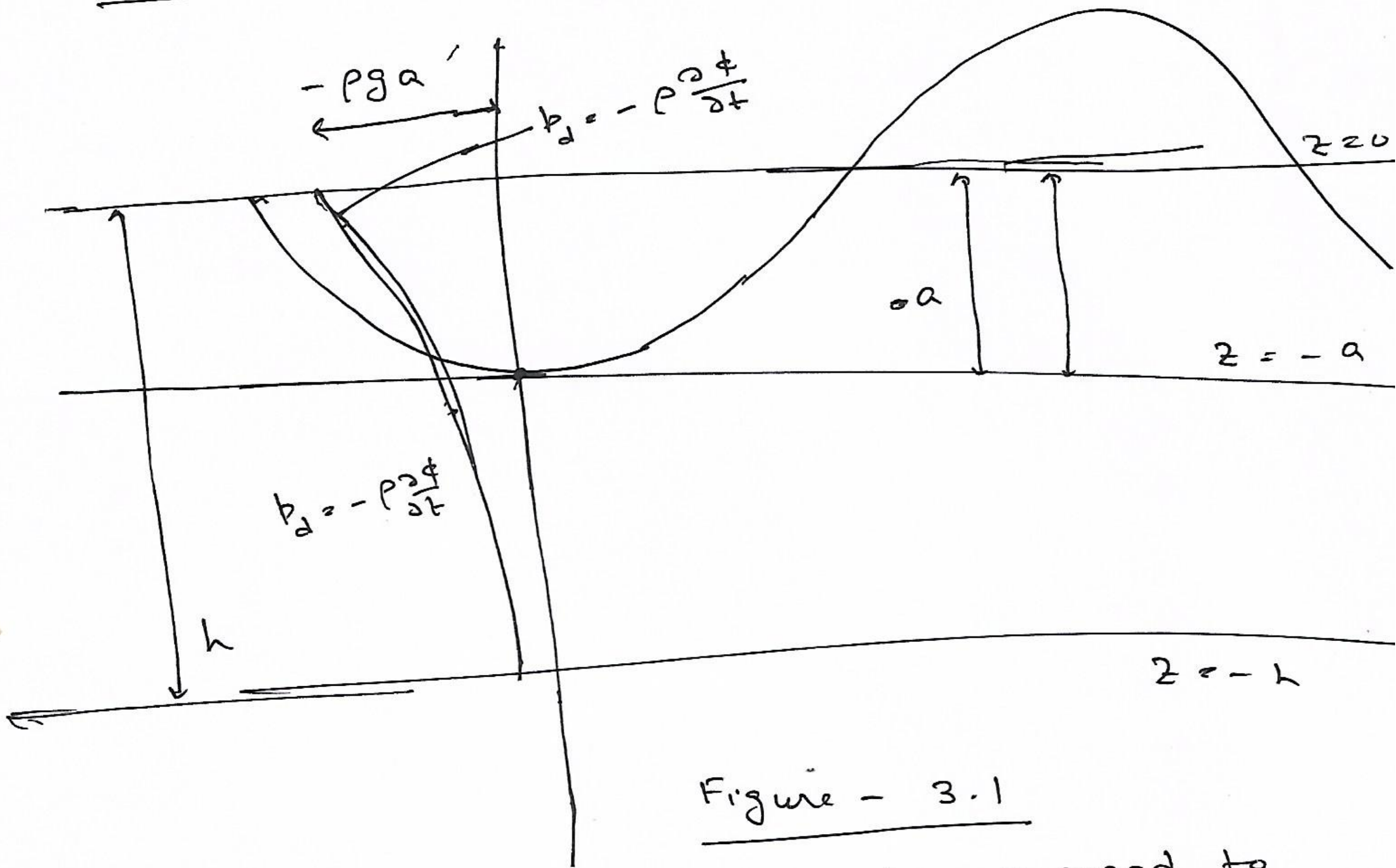

Figure - 3.1

In case of wave trough, we need to remember that here, $\eta = -a$ as $\cos(kx-\omega t) = -1$ at wave trough. In that case the dynamic pressure

$$p = -\rho\frac{\partial\phi}{\partial t} = -\rho\left[-a g e^{kz}\cos(kx-\omega t)\right]$$

$$= -\rho a g \quad \left[\text{as } \cos(kx-\omega t) = -1\right]$$

$\therefore$ the dynamic pressure at $z=0$ is $-\rho a g$ and then it decreases exponentially from $z=0$ to $z=-h$. Therefore the figure 3.1 is justified.

3.1 Hydrostatic pressure variation at wave trough.

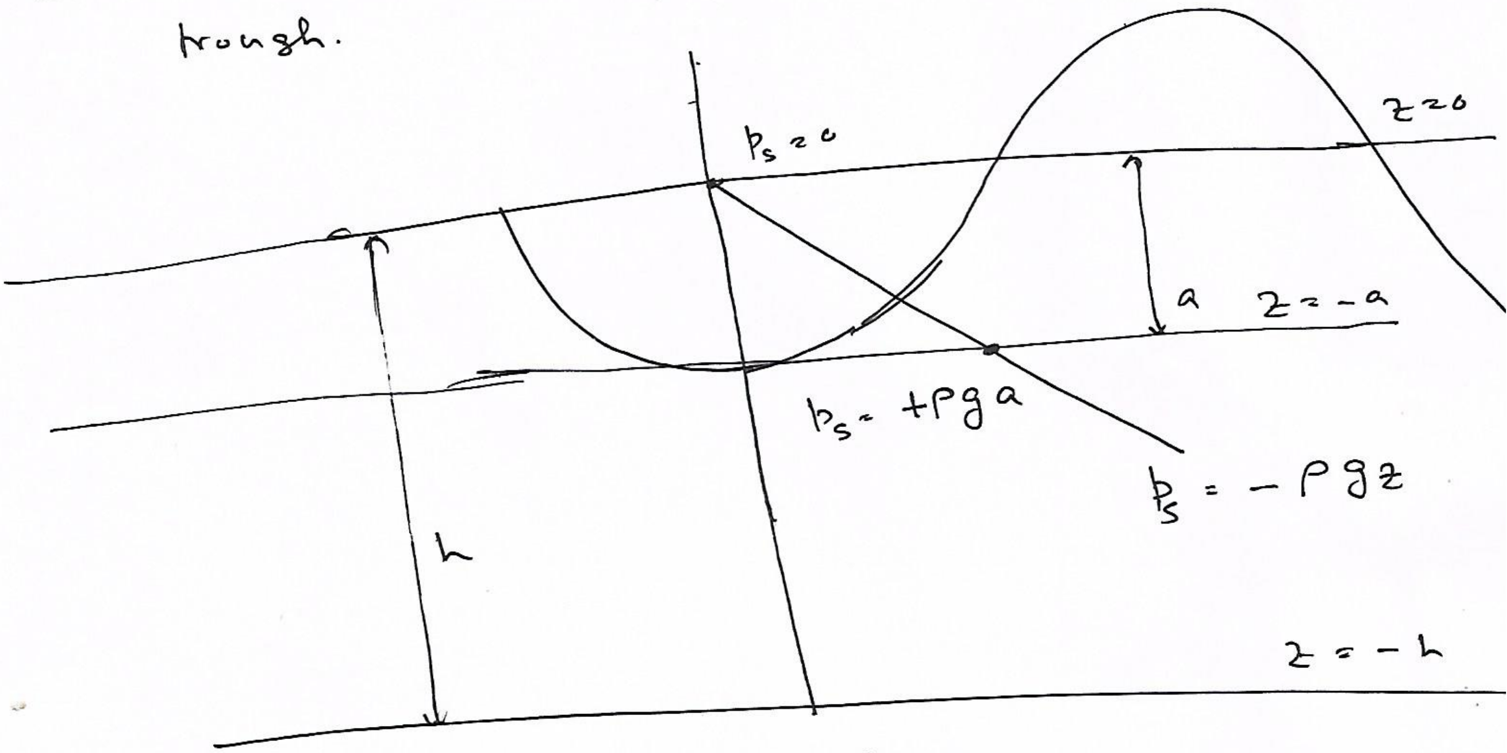

Figure 3.2

Now, ~~Figure~~ the hydrostatic pressure $p_s = 0$ at $z = 0$, $p_s = \rho g h$ at $z = -h$ and $p_s = \rho g a$ at $z = -a$. and the ~~static~~ hydrostatic pressure p_s is varying linearly from $z = 0$ to $z = -h$. Thus the figure 3.2 is justified.

Now: interestingly, one can note from figure (3.1) and (3.2), qualitatively, at $z = -a$
~~totala~~ Total pressure

$$p_T = -\rho \left[\frac{\partial \phi}{\partial t} + g z \right]_{at\ z = -a} \neq 0$$

since $p = -\rho g z = \rho g a$ at $z = -a$ but

$$-\rho \frac{\partial \phi}{\partial t}\Big|_{z=-a} \neq -\rho g a$$

as it is the value at $z = 0$ and then it is decreases exponentially from $z = 0$ to $z = -a$.

This is an anomaly of the linear theory. The dynamic free surface condition is satisfied at $z = a$, i.e at wave crest but it is not zero at wave trough, i.e at $z = -a$.

The possible reason would be: to drop the quadratic term $\frac{1}{2}a^2\omega^2 e^{2kz}$ from the total ~~dyn~~ pressure equation.

Now, collecting these concept, the pressure variation (total pressure) at wave crest and wave trough may be schematically given as follows (only qualatative) (see Fig. 3.3)

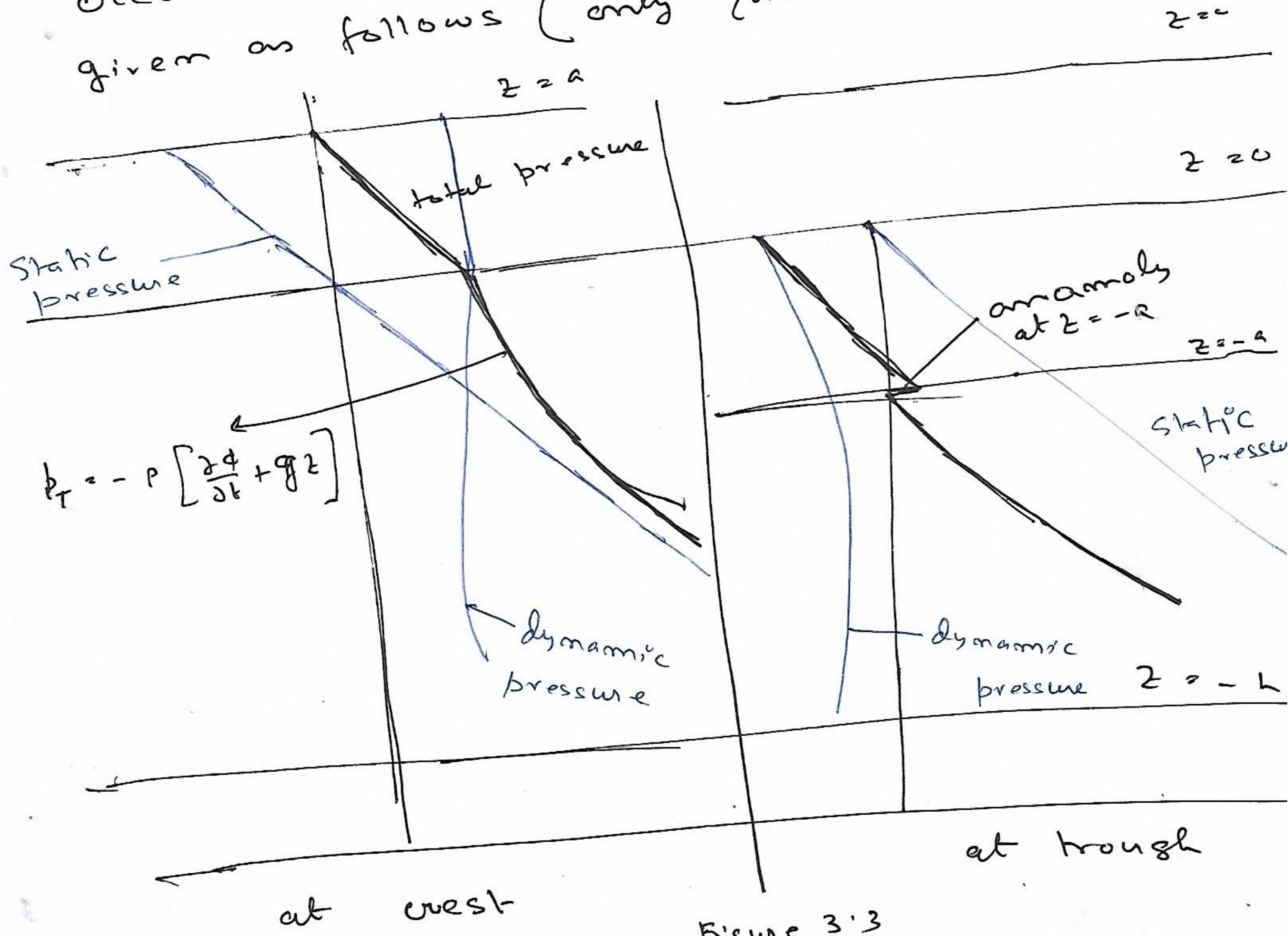

Figure 3.3

4. Froud- Krylov Force on submarged structure

we know that, if $\eta = a\cos(kx - \omega t)$ is the equation of the wave elevation. Then velocity potential of the fluid particle can be written as

$$\phi = \frac{ag}{\omega} e^{kz} \sin(kx - \omega t)$$

[Deep water case]

and the dynamic pressure due to wave may be written as

$$p_d = -\rho \frac{\partial \phi}{\partial t} = -\rho \left[-a g e^{kz} \cos(kx - \omega t) \right]$$

or $$\boxed{p_d = \rho a g e^{kz} \cos(kx - \omega t)} \quad \ldots (4.1)$$

where 'z' varies from $z = -h$ to $z = 0$

∴ if 'S_F' be the surface of the submarged portion of the body, then the total force due to p_d is called Froud-Krylov or F-K force, and the expression of this force is

$$\boxed{F_k = \iint_{S_F} p . n \, dS_F} \quad \ldots (4.2)$$

or

$$\boxed{F_k = \iint \left[\rho a g e^{kz} \cos(kx - \omega t) \right] . \vec{n} \, ds.} \quad (4.3)$$

(4.3) represents the F-K force on the body at any instant of time 't' due to wave elevat $\eta = a\cos(kx - \omega t)$