Lecture - 10

Marine Hydrodynamics

Ex 1.: Show that the velocity induced by two sources of equal strength at all points on a plane perpendicular to the line joing the two sources is always parallel to the plane. By qualitative argument, extend this fact to explain how the interaction problem of two geometries with centerline symmetry moving $\parallel^{le}$ to each other and in same direction can reduce to the problem of solving of one geometry moving $\parallel^{le}$ to an infinite plane wall.

Ans: Definition of velocity induced by any point P, due to a source at Q is grad $[\phi(P, Q)]$

or grad $\left(\frac{1}{4\pi r}\right)$, where $r = |P - Q|$

Now Consider two point Q_1 and Q_2 is placed at $Q_1(x, y, z)$ and $(x, y, -z)$ i.e they are symmetric about $z = 0$.

Now, if any point on $z = 0$ is $(\xi, \eta, 0)$

Then ~~$\vec{r}$~~ with respect to Q_1

Q_1: $\vec{r_1} = (x-\xi)\hat{i} + (y-\eta)\hat{j} + z\hat{k}$

Q_2: $\vec{r_2} = (x-\xi)\hat{i} + (y-\eta)\hat{j} - z\hat{k}$

Now $r^2 = (x-\xi)^2 + (y-\eta)^2 + z^2$

$\therefore 2r\frac{dr}{dx} = 2(x-\xi)$ or $\frac{dr}{dx} = \frac{(x-\xi)}{r}$

similarly $\frac{dr}{dy} = \frac{(y-\eta)}{r}$, $\frac{dr}{dz} = \frac{z}{r}$

$\therefore$ ~~grad~~ $\text{grad}\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$\therefore$ for Q_1: $\text{grad}\left(\frac{1}{r_1}\right) = -\frac{1}{r^2}\frac{(x-\xi)}{r}\hat{i} - \frac{1}{r^2}\frac{(y-\eta)}{r}\hat{j} + \frac{1}{r^2}\frac{z}{r}\hat{k}$

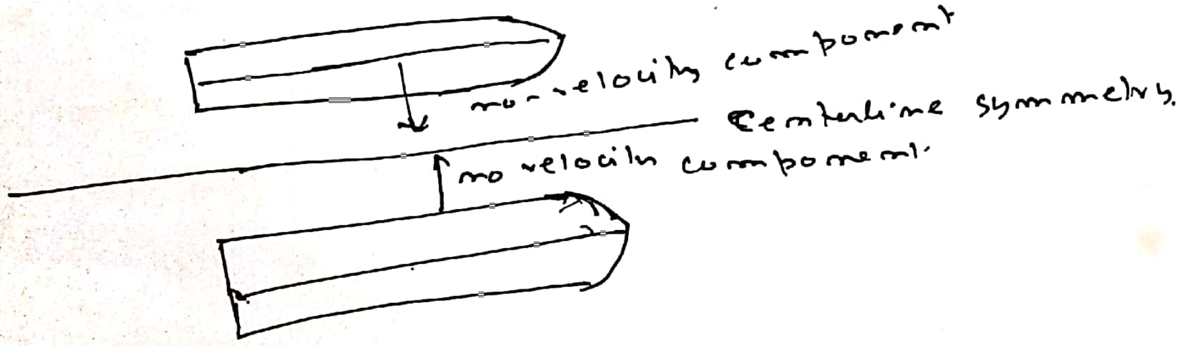
for Q_2: $\text{grad}\left(\frac{1}{r_2}\right) = -\frac{\hat{i}}{r^2}\frac{(x-\xi)}{r} - \frac{\hat{j}}{r^2}\frac{(y-\eta)}{r} + \frac{\hat{k}}{r^2}\frac{z}{r}$

$\therefore \text{grad}\left(\frac{1}{r_1}\right) + \text{grad}\left(\frac{1}{r_2}\right) = \left(-\frac{2(x-\xi)}{r^3}\right)\hat{i} + \left(-\frac{2(y-\eta)}{r^3}\right)\hat{j}$

$\Rightarrow$ there is no velocity component along $\hat{k}$ direction

$\Rightarrow$ the ~~v~~ velocity is always $\parallel^{l}$ to 'z' plane.

(ii)



Now, according to the previous problem, if the two ships are moving in same direction, then there will be no velocity $\perp^r$ to each other, as sources, that ~~are~~ distributed over the body may be arranged in a way that source of one ship may be the image point of the other ship with respect to the centre line.

Now, if a ship passing through a rigid infinite wall, the problem may be placed as :

$$\nabla^2 \phi = 0 \quad \text{with} \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{at wall}$$

which is exactly the same when two ships moving ~~in parallel~~ parallely in same direction.

Ex 2 :. The force on a body in an ideal fluid is given by :.

$$F = -\rho \frac{d}{dt} \iint_{S_B} \phi \vec{n}\, ds - \rho \iint_{S_C} \left[\frac{\partial \phi}{\partial n} \nabla \phi - \vec{n}\, \nabla \phi . \nabla \phi \right] d. \longrightarrow (1)$$

where S_B is the body surface and S_C is a surface completely surronding the body. For a body moving parallel to an infinitely long plane wall, S_C can be taken as the sum of plane wall plus a surface at infinity. From the previous problem and using the arguments of part (a), show that the two geometries as mentioned in part (a) and moving steadily will always pull each other.

Ans: : Now, if two ships are moving steadily $\parallel^{le}$ in same direction, then, from the previous problem, this can be model as a ship is moving steadily near a wall, then in the force term

$$F = -\rho \frac{d}{dt} \iint_{S_B} \phi \vec{n} - \rho \iint_{S_c} \left[\frac{\partial \phi}{\partial n} \nabla \phi - \hat{n} \cdot \nabla \phi \nabla \phi \right] ds$$

now for steady flow 1st term $\rightarrow 0$ ∴

∴ F is further reduced to

$$F = -\rho \iint_{S_c} \left[\frac{\partial \phi}{\partial n} \nabla \phi - \hat{n} \cdot \nabla \phi \cdot \nabla \phi \right] ds$$

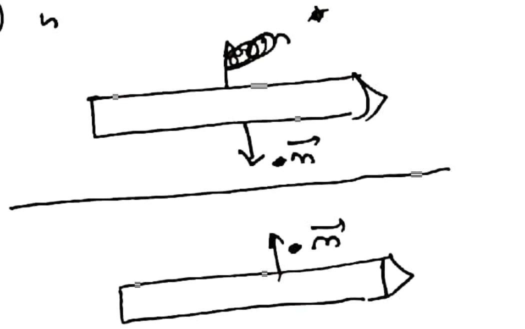
now here, S_c = the wall + infinity.

now at wall, $\frac{\partial \phi}{\partial n} = 0$ and at infinity $\nabla \phi \rightarrow 0$

∴ F is further reduced to

$$F = \rho \iint_{S_{wall}} (\nabla \phi \cdot \nabla \phi) \vec{n} \, ds.$$

now, the term $(\nabla\phi).(\nabla\phi)$ is a quadratic term and this is always positive, and $\vec{m}$ is directed to the wall. Therefore the direction of the force is always directed to the wall



$\Rightarrow$ Two ships will pull each other.

3.a : The laplace equation for the cylindrical co-ord system for 2D flow may be written as :

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2\phi}{\partial\theta}\right)=0 \quad \cdots (1)$$

Show that $\phi = f(r)\cos\theta$ will be solution of

(1) if $$r^2\frac{d^2f}{dr^2}+r\frac{df}{\partial r}-f=0 \quad \cdots (2)$$

moreover if $r=a$, $-\frac{\partial\phi}{\partial r}=U\cos\theta$, $r\to\alpha$, $-\frac{\partial\phi}{\partial r}\to 0$

Show that, the complex potential will take the form : $w(z)=\frac{Ua^2}{z}$

Ans: $\phi(r,\theta) = f(r)\cos\theta$

$$\frac{\partial\phi}{\partial r} = f'(r)\cos\theta$$

$$\therefore \frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)=\frac{\partial}{\partial r}\left[r f'(r)\cos\theta\right]$$
$$= f'(r)\cos\theta + r f''(r)\cos\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{1}{r}\left[f'(r)\cos\theta + f''(r)\cos\theta\right]$$

Now $\quad \frac{\partial\phi}{\partial\theta} = -f(r)\sin\theta$

$$\frac{\partial^2\phi}{\partial\theta^2} = -f(r)\cos\theta$$

$$\therefore \frac{1}{r^2}\left[\frac{\partial\phi}{\partial\theta^2}\right] = -\frac{1}{r^2} f(r)\cos\theta$$

Substituting we get:

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial\theta}\right)\right] + \frac{1}{r^2}\left(\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$$

$$\Rightarrow \frac{1}{r}f'(r)\cos\theta + f''(r)\cos\theta - \frac{1}{r^2}f(r)\cos\theta = 0$$

$$\Rightarrow \left[r^2 f''(r) + rf'(r) - f(r)\right]\cos\theta = 0$$

or $\quad \boxed{r^2 f''(r) + rf'(r) - f(r) = 0} \longrightarrow$ 2 proved

assume ~~$r = \log z$~~ $z = \log r$, then $\frac{df}{dr} = \frac{df}{dz}\cdot\frac{dz}{dr}$

$$\Rightarrow \frac{dz}{dz} = \frac{1}{r} \qquad \Rightarrow \frac{df}{dr} = \frac{1}{r}\cdot\frac{df}{dz}$$

$$\Rightarrow r\frac{df}{dr} = \frac{df}{dz} \quad \ldots \text{(3)}$$

Similarly, differentiating 3 we get

$$r\frac{d^2f}{dr^2} + \frac{df}{dr} = \frac{d^2f}{dz^2}\cdot\frac{1}{r}$$

Substituting we get :-

$$\frac{d^2f}{dz^2} \cancel{+ \frac{df}{d\theta}} = r^2\frac{d^2f}{dr^2} + r\frac{df}{dz}$$

Substituting in (2) we get

$$\frac{d^2f}{dz^2} - \frac{df}{dz} + \frac{df}{dz} - f = 0$$

$$\Rightarrow \frac{d^2f}{dz^2} - f = 0 \quad \ldots\ldots (4)$$

Solving (4) we get

$$f(r) = Ae^{z} + Be^{-z}$$

$$= Ar + \frac{B}{r}$$

$$\therefore \phi(r,\theta) = \left(Ar + \frac{B}{r}\right)\cos\theta.$$

using boundary condition we get

$$-U\cos\theta = \left(A - \frac{B}{a^2}\right)\cos\theta$$

$$\Rightarrow A - \frac{B}{a^2} = U \quad \ldots\ldots (1)$$

and $-\frac{\partial\phi}{\partial r} \to 0$ as $r \to \infty \to A = 0$

$$\therefore B = Ua^2$$

$$\therefore \phi(r,\theta) = \frac{Ua^2}{r}\cos\theta$$

Let ψ be the stream function. (8)

then $\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $-\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial r}$

$$\Rightarrow \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{Ua}{r^2}\cos\theta$$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} = -\frac{Ua}{r}\cos\theta$$

$$\therefore \psi = -\frac{Ua}{r}\sin\theta + g(r)$$

$$\frac{\partial \psi}{\partial r} = \frac{Ua}{r^2}\sin\theta + g'(r) = -\frac{Ua}{r^2}\sin\theta$$

$$\Rightarrow g'(r) = 0 \quad \therefore g(r) = \text{const} = 0 \text{ (say)}$$

$$\therefore \phi(r,\theta) = \frac{Ua^2}{r}\cos\theta, \quad \psi(r,\theta) = -\frac{Ua}{r}\sin\theta$$

$$\therefore \text{ \& } w = \phi + i\psi$$

$$= \frac{Ua^2}{r}\cos\theta - i\frac{Ua^2}{r}\sin\theta$$

$$= \frac{Ua^2}{r}[\cos\theta - i\sin\theta]$$

$$= \frac{Ua^2}{r}\cdot e^{-i\theta}$$

$$= \frac{Ua^2}{re^{i\theta}} = \frac{Ua^2}{z}$$

$$\therefore \boxed{w = \frac{Ua^2}{z}}$$

proved.

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1. The complex potential of flow past a 2D fixed cylinder may be written as $w(z) = Uz + \frac{a^2U}{z}$

Draw a sketch of stream line past a cylinder.

Find the position of stagnation point. Also show the maximum tangential velocity at the surface of cylinder is equal to the twice of the uniform flow, i.e. $q = 2|U|$

- Ans :

$$w(z) = Uz + \frac{a^2U}{z}$$

Now $\frac{dw}{dz} = 0 \Rightarrow U - \frac{a^2U}{z^2} = 0$

$\Rightarrow z^2 = a^2 \Rightarrow z = \pm a$

$\therefore$ stagnation point $= (\pm a, 0)$.

- Now $\frac{dw}{dz} = \cancel{ue^{i\theta}}\ u - iv$

$\Rightarrow$ ~~we~~ $u - iv = U - \frac{a^2U}{z^2}$

or $u - iv = U - \frac{a^2U}{\cancel{(x+iy)}\ r^2 e^{2i\theta}}$

or $u - iv = U - \frac{a^2U}{r^2} e^{-2i\theta}$

~~or~~ now at surface of cylinder $r = a$

$\Rightarrow u - iv = U(1 - e^{-2i\theta})$

$\therefore q = |u - iv| = |U||1 - e^{-2i\theta}| = \underline{2|U|}$.

$\because$ max of $|1 - e^{-2i\theta}| = 1$

as ~~q~~ max $|1 - e^{-2i\theta}| = 2$