Wave Equation

Consider one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{—} ①$$

$$\text{—} ②$$

b.c $y(0,t) = C_1 \qquad y(L,t) = C_2$

I.C $y(x,0) = f(x) \qquad \left.\frac{\partial y}{\partial t}\right|_{t=0} = g(x) \qquad \text{—} ③$

Use central difference approximation for ①

$$y(x_i, t_j) = y_i^j$$

$$\frac{\partial^2 y}{\partial t^2}(x_i, t_j) = c^2 \frac{\partial^2 y}{\partial x^2}(x_i, t_j)$$

$$\frac{y_i^{j+1} - 2y_i^j + y_i^{j+1}}{(\Delta t)^2} = c^2 \frac{y_{i+1}^j - 2y_i^j + y_{i-1}^j}{\Delta x^2}$$

$$y_i^{j+1} - 2y_i^j + y_i^{j-1} = \frac{c^2 \Delta t^2}{\Delta x^2}\left(y_{i+1}^j - 2y_i^j + y_{i-1}^j\right)$$

or $$y_i^{j+1} = \frac{c^2 \Delta t^2}{\Delta x^2}\left(y_{i+1}^j + y_{i-1}^j\right) - y_i^{j-1} + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) y_i^j \qquad \text{—} ④$$

and from ③ $$\frac{\partial y}{\partial t}(x_i, t_0) = g(x_i)$$

$$\frac{y_i^1 - y_i^{-1}}{2\Delta t} = g(x_i)$$

$$y_i^{-1} = y_i^1 - 2\Delta t\, g(x_i) \qquad \text{—} ⑤$$

Now from ④ for $j = 0$

$$y_i^1 = \frac{c^2 \Delta t^2}{\Delta x^2}\left(y_{i+1}^0 + y_{i-1}^0\right) - y_i^{-1} + 2\left(1 - \frac{c^2 \Delta t^2}{\Delta x^2}\right) y_i^0 \qquad \text{—} ⑥$$

Now putting y_i^{-1} from ⑤ we get

$$\frac{\partial F}{\partial(x+ct)} = \frac{\partial F}{\partial x} \Big/ \frac{\partial(x+ct)}{\partial x} = \frac{\partial F}{\partial x}$$

$$cF' - cG' = g(x)$$

$$F' - G' = \frac{1}{c} g(x) \qquad x-ct < a < x+ct.$$

$$F - G = \int_a^x \frac{1}{c} g(\tau)\, d\tau$$

$$F + G = f(x)$$

$$F = \frac{1}{2} f(x) + \frac{1}{2c} \int_a^x g(\tau)\, d\tau$$

$$G(x) = +\frac{1}{2} f(x) - \frac{1}{2c} \int_a^x g(\tau)\, d\tau$$

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c} \int_a^{x+ct} g(\tau)\, d\tau - \frac{1}{2c} \int_a^{x-ct} g(\tau)\, d\tau$$

$$\int_{x-ct}^{a} + \int_a^{x+ct}.$$

$$y_i^1 = \frac{c^2\Delta t^2}{\Delta x^2}(y_{i+1}^0 + y_{i-1}^0) - y_i^1 + 2\Delta t\, g(x_i)$$

$$y_i^1 = \frac{1}{2}\frac{c^2\Delta t^2}{\Delta x^2}(y_{i+1}^0 + y_{i-1}^0) + g(x_i)\Delta t + \left(1 - \frac{c^2\Delta t^2}{\Delta x^2}\right)y_i^0 \; \underline{+ \; 2\left(1 - \frac{c^2\Delta t^2}{\Delta x^2}\right) y_i^0} \xrightarrow{\quad} ⑦$$

So difference equations ④ with ⑦ can be used to solve wave equation ① - ③.

We take $\frac{c^2\Delta t^2}{\Delta x^2} = 1$. Infact if $\frac{c^2\Delta t^2}{\Delta x^2} > 1$ then we cannot be sure of convergence and stability also sets a limit of unity to the ratio.

Note: It is surprising to find that if one uses a value of $\frac{c^2\Delta t^2}{\Delta x^2}$ ~~of~~ less than one, the results are less accurate while with the ratio equal to one we get better result (infact exact for $g = 0$).

For $\frac{c^2\Delta t^2}{\Delta x^2} = 1$ ④ & ⑦ can be written as

$$y_i^{j+1} = (y_{i+1}^j + y_{i-1}^j) - y_i^{j-1} \qquad ⑧$$

$$y_i^1 = \frac{1}{2}(y_{i+1}^0 + y_{i-1}^0) + g(x_i)\Delta t \qquad ⑨$$

Comparison to the D'Alembert Solution

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ⑩$$

let $y(x,t) = F(x+ct) + G(x-ct)$ ⑪

where F and G are two arbitrary functions

$$\frac{\partial y}{\partial t} = \frac{\partial F}{\partial(x+ct)} \cdot \frac{\partial(x+ct)}{\partial t} + \frac{\partial G}{\partial(x-ct)} \cdot \frac{\partial(x-ct)}{\partial t}$$

$$= cF' - cG' = c(F' - G')$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 F'' + c^2 G'' = c^2(F'' + G'') \qquad i)$$

$$\frac{\partial y}{\partial x} = \frac{\partial F}{\partial(x+ct)} \cdot \frac{\partial(x+ct)}{\partial x} + \frac{\partial G}{\partial(x-ct)} \cdot \frac{\partial(x-ct)}{\partial x}$$

$$= F' + G'$$

$$\frac{\partial^2 y}{\partial x^2} = F'' + G'' \qquad \text{— (ii)}$$

Thus from (i) & (ii) $y(x,t)$ in ⑪ satisfy ⑩.
Now next we want to find these arbitrary functions

$$y(x,0) = f(x)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x)$$

$$y(x,0) = f(x) \Rightarrow F(x) + G(x) = f(x) \qquad \text{— (iii)}$$

$$\frac{\partial y}{\partial t}(x,t) = \frac{\partial F}{\partial(x+ct)} \cdot c - \frac{\partial G}{\partial(x-ct)} \cdot c$$

and $\dfrac{\partial F}{\partial(x+ct)} = \dfrac{\partial F}{\partial x} \Big/ \dfrac{\partial(x+ct)}{\partial x} = \dfrac{\partial F}{\partial x}$

and. $\dfrac{\partial G}{\partial(x-ct)} = \dfrac{\partial G}{\partial x} \Big/ \dfrac{\partial(x-ct)}{\partial x} = \dfrac{\partial G}{\partial x}$

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = g(x) \Rightarrow$$

$$c\left.\frac{\partial F}{\partial x}\right|_{t=0} - c\left.\frac{\partial G}{\partial x}\right|_{t=0} = g(x)$$

$$c.\frac{dF(x)}{dx} - c\frac{dG(x)}{dx} = g(x) \qquad \text{— (iv)}$$

Now integrating (iv) we get $\qquad x-ct < a < x+ct$

$$cF(x) - cG(x) = \int_a^x g(t)\,dt$$

$$\text{or} \quad F(x) - G(x) = \frac{1}{c}\int_a^x g(t)\,dt \qquad \text{— (v)}$$

Now from (iii) & (v)

$$2 \text{ [illegible]} = f(x) + \frac{1}{c}\int_a^x g(t)\,dt$$

$$\text{or} \quad F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_a^x g(t)\,dt$$

Now from (~~*~~) (iii) $G(x) = f(x) - F(x)$

$$G(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_a^x g(\tau)\, d\tau$$

Thus

$$y(x,t) = F(x+ct) + G(x-ct)$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_a^{x+ct} g(\tau)\, d\tau - \frac{1}{2c}\int_a^{x-ct} g(\tau)\, d\tau$$

$$\boxed{y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau}$$

Now

$$y_i^j = y(x_i, t_j) = F(x_i + ct_j) + G(x_i - ct_j)$$

$$\frac{c^2 \Delta t^2}{\Delta x^2} = 1 \implies \Delta x = c\Delta t$$

$$x_i = x_0 + i\Delta x \quad , \quad t_j = t_0 + j\Delta t$$

$$ct_j = ct_0 + jc\Delta t$$

for $t_0 = 0$ (initial value)

$$ct_j = jc\Delta t = j\Delta x$$

$$y_i^j = F(x_0 + i\Delta x + j\Delta x) + G(x_0 + i\Delta x - j\Delta x)$$

$$y_i^j = F(x_0 + (i+j)\Delta x) + G(x_0 + (i-j)\Delta x)$$

Now we see R.H.S. of ⑧

$$y_{i-1}^j + y_{i+1}^j - y_i^{j-1} = F(x_0 \cancel{+(i-1+j)\Delta x}) + G(x_0 + (i-1-j)\Delta x)$$

$$+ F(x_0 + (i+1+j)\Delta x) + G(x_0 + \cancel{(i+1-j)\Delta x})$$

$$- F(\cancel{x_0 + (i+j-1)\Delta x}) - G(x_0 + \cancel{(i-j+1)\Delta x})$$

$$= F(x_0 + (i + \overline{j+1})\Delta x) + G(x_0 + (i - (j+1))\Delta x)$$

$$= y_i^{j+1}$$

This shows that except the discretization at $t=0$ this provides exact solution

Now we come back to ③ (i.e. discretization at t=0)

$$\frac{\partial y}{\partial t}(x_i, t_0) = g(x_i) \leftarrow \text{initial velocity}$$

$$\underbrace{\frac{y_i^1 - y_i^{-1}}{2\Delta t}}_{\uparrow} = g(x_i)$$

average velocity from t_{-1} to t_1 = initial velocity

In general this is not true that initial velocity is same as average velocity.

Now

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$$

$$y(x,t_1) = y(x,\Delta t) = \frac{1}{2}\left[f(x+c\Delta t) + f(x-c\Delta t)\right] + \frac{1}{2c}\int_{x-c\Delta t}^{x+c\Delta t} g(\tau)\,d\tau$$

$$y(x,\Delta t) = \frac{1}{2}\left[f(x+\Delta x) + f(x-\Delta x)\right] + \frac{1}{2c}\int_{x-\Delta x}^{x+\Delta x} g(\tau)\,d\tau$$

$$y_i^1 = y(x_i,\Delta t) = \frac{1}{2}\left[f(x_{i+1}) + f(x_{i-1})\right] + \frac{1}{2c}\int_{x_{i-1}}^{x_{i+1}} g(\tau)\,d\tau$$

$y(x,0) = f(x) \Rightarrow f(x_i) = y_i^0$

$$y_i^1 = \frac{1}{2}\left[y_{i+1}^0 + y_{i-1}^0\right] + \frac{1}{2c}\int_{x_{i-1}}^{x_{i+1}} g(\tau)\,d\tau \quad \text{———} ⑫$$

Now from ⑨

$$y_i^1 = \frac{1}{2}\left[y_{i+1}^0 + y_{i-1}^0\right] + g(x_i)\Delta t \quad \longrightarrow \boxed{⑨}$$

In case $g(x) = 0$ i.e. $\frac{\partial y}{\partial t}\Big|_{t=0} = 0$ Then both ⑨ & ⑫ are same and hence will provide exact solution except for round off error.

If $g(x) \neq 0$ Then ⑫ can be used and then one has to use some quadrature formula for the integral term and then error will be committed which will be propagated throughout but if $g(x)$ is exactly integrable then we can use (12) without committing any error.

Hyperbolic Case

Ex Find the solution of initial boundary value problem

$$u_{tt} = u_{xx}$$

s.t. IC : $u(x,0) = \sin\pi x, \quad 0 \le x \le 1$

$u_t(x,0) = 0 \quad 0 \le x \le 1$

and BC : $u(0,t) = u(1,t) = 0, \quad t > 0$

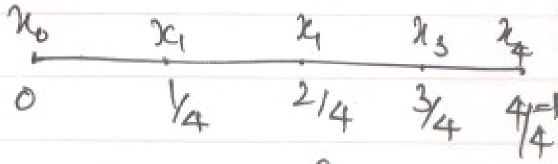
by using (i) explicit method (ii) the implicit scheme

$$\frac{1}{k^2}\delta_t^2 u_m^n = \frac{1}{h^2}\delta_x^2\left[\theta u_m^{n+1} + (1-2\theta)u_m^n + \theta u_m^{n-1}\right]$$

$0 \le \theta \le 1$

for $\theta = 1/2$.

Take $h = 1/4$, $r = \frac{k}{h} = 3/4$



$k = rh = 3/4 \cdot 1/4 = \frac{3}{16}$

$x_m = mh, \; m = 0,1,2,3,4$

$h = 1/4$

$t_n = nk, \quad n = 0,1,2,\ldots$

The IC. gives. $u(x,0) = \sin\pi x$

$u_m^0 = \sin\pi x_m \qquad m = 1,2,3,$

$u_t(x,0) = 0$

$u_t(x_m,0) = 0$

$$\frac{u_m^{1} - u_m^{-1}}{2k} = 0 \Rightarrow \boxed{u_m^{-1} = u_m^{1}}$$

BC $\quad u(0,t) = u(1,t) = 0$

$$u_0^n = u_4^n = 0$$

Explicit Scheme

$$u_{tt}\Big|_{(x_m,t_n)} = u_{xx}\Big|_{(x_m,t_n)}$$

$$\frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{k^2} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}$$

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \frac{k^2}{h^2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)$$

Take $r = k/h$

$$u_m^{n+1} = r^2\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) + 2u_m^n - u_m^{n-1}$$

$$\boxed{u_m^{n+1} = r^2\left(u_{m+1}^n + u_{m-1}^n\right) + 2(1-r^2)\,u_m^n - u_m^{n-1}}$$

for $r = 3/4$

$$\boxed{u_m^{n+1} = \frac{9}{16}\left(u_{m+1}^n + u_{m-1}^n\right) + \frac{7}{8}u_m^n - u_m^{n-1}}$$

for $n=0$

$\left(1-\frac{9}{16}\right) = \frac{7\times}{16\,8}$

$$u_m^1 = \frac{9}{16}\left(u_{m+1}^0 + u_{m-1}^0\right) + \frac{7}{8}u_m^0 - u_m^{-1}$$

$$= \frac{9}{16}\left(u_{m+1}^0 + u_{m-1}^0\right) + \frac{7}{8}u_m^0 - u_m^1$$

or

$$2u_m^1 = \frac{9}{16}\left(u_{m+1}^0 + u_{m-1}^0\right) + \frac{7}{8}u_m^0$$

$$\boxed{u_m^1 = \frac{9}{32}\left(u_{m+1}^0 + u_{m-1}^0\right) + \frac{7}{16}u_m^0}$$

for $m=1$

$$u_1^1 = \frac{9}{32}\left(u_2^0 + u_0^0\right) + \frac{7}{16}u_1^0$$

$$= \frac{9}{32}\left(\sin \cancel{2}\pi/\cancel{4}2 + 0\right) + \frac{7}{16}\sin \pi/4$$

$$= \frac{9}{32} + \frac{7}{16\times}\frac{1}{\sqrt{2}} = .5906092\,16$$

$u_2^1 = \cdot 03525$

$u_3^1 = \quad = \cdot 590609216$

$1+\frac{9}{16}$

(ii) <u>Implicit method</u>

$$u_{tt}\Big|_{(x_m,t_n)} = u_{xx}\Big|_{(x_m,t_n)}$$

$$\frac{1}{k^2}\delta_t^2 u_m^n = \frac{1}{h^2}\delta_x^2 u_m^n$$

$$= \frac{1}{h^2}\delta_x^2\left[\theta u_m^{n+1} + (1-2\theta)u_m^n + \theta u_m^{n-1}\right]$$

for $\theta = 1/2$ $\quad = \frac{1}{h^2}\delta_x^2\left[\frac{1}{2}u_m^{n+1} + \frac{1}{2}u_m^{n-1}\right]$

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \frac{k^2}{2h^2}\left[u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1} + u_{m+1}^{n-1} - 2u_m^{n-1} + u_{m-1}^{n-1}\right]$$

$r = k/h = 3/4$

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \frac{9}{32}\left[u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1} + u_{m+1}^{n-1} - 2u_m^{n-1} + u_{m-1}^{n-1}\right]$$

$$u_m^{n+1} - \frac{9}{32}u_{m+1}^{n+1} - \frac{9}{32}u_{m-1}^{n+1} + \frac{18}{32}u_m^{n+1} = 2u_m^n - u_m^{n-1} + \frac{9}{32}u_{m+1}^{n-1} - \frac{\overset{9}{\cancel{18}}}{\cancel{32}\,16}u_m^{n-1} + \frac{9}{32}u_{m-1}^{n-1}$$

$$\frac{25}{16}u_m^{n+1} - \frac{9}{32}u_{m+1}^{n+1} - \frac{9}{32}u_{m-1}^{n+1}$$

$$= 2u_m^n + \frac{9}{32}u_{m-1}^{n-1} - \frac{25}{16}u_m^{n-1} + \frac{9}{32}u_{m+1}^{n-1}$$

$$-\frac{9}{32}u_{m-1}^{n+1} + \frac{25}{16}u_m^{n+1} - \frac{9}{32}u_{m+1}^{n+1}$$

$$= 2u_m^n + \frac{9}{32}u_{m-1}^{n-1} - \frac{25}{16}u_m^{n-1} + \frac{9}{32}u_{m+1}^{n-1}$$

<u>for $n=0$</u>

$$-\frac{9}{32}u_{m-1}^1 + \frac{25}{16}u_m^1 - \frac{9}{32}u_{m+1}^1 = 2u_m^0 + \frac{9}{32}u_{m-1}^{-1} - \frac{25}{16}u_m^{-1} + \frac{9}{32}u_{m+1}^{-1}$$

Now $u_m^{-1} = u_m^1$

$$\left(-\frac{9}{32}u_{m-1}^1 - \frac{9}{32}u_{m-1}^1\right) + \left(\frac{25}{16}u_m^1 + \frac{25}{16}u_m^1\right) - \left(\frac{9}{32} + \frac{9}{32}\right)u_{m+1}^1$$

$$= 2u_m^0$$

$$\boxed{-\frac{9}{16}u_{m-1}^1 + \frac{25}{8}u_m^1 - \frac{9}{16}u_{m+1}^1 = 2u_m^0}$$

for $m=1$

$$-\frac{9}{16}\overset{\nearrow 0\,\text{BC}}{u_0^1} + \frac{25}{8}u_1^1 - \frac{9}{16}u_2^1 = 2u_1^0$$

$$\frac{25}{8}u_1^1 - \frac{9}{16}u_2^1 = 2u_1^0 \qquad \text{——} ①$$

<u>$m=2$</u>

$$-\frac{9}{16}u_1^1 + \frac{25}{8}u_2^1 - \frac{9}{16}u_3^1 = 2u_2^0 \qquad \text{——} ②$$

<u>$m=3$</u>

$$-\frac{9}{16}u_2^1 + \frac{25}{8}u_3^1 - \frac{9}{16}\overset{\nearrow 0\,\text{B-C}}{u_4^1} = 2u_3^0$$

$$-\frac{9}{16}u_2^1 + \frac{25}{8}u_3^1 = 2u_3^0 \qquad \text{——} ③$$

$$\begin{bmatrix} \frac{25}{8} & -\frac{9}{16} & 0 \\ -9/16 & \frac{25}{8} & -\frac{9}{16} \\ 0 & -\frac{9}{16} & \frac{25}{8} \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 2u_1^0 \\ 2u_2^0 \\ 2u_3^0 \end{bmatrix}$$

After solving we get

$$u_1^1 = u_3^1 = 0.60709, \quad u_2^1 = 0.85855$$