Lecture - 1

Marine Hydrodynamics

1. Some basic mathematical understanding and useful results.

a) How to define a vector using $(\hat{i}, \hat{j}, \hat{k})$. / unit vector

$$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$$

where a, b, c are known as scalar.

b) Meaning of 'dot' product.

It is known also as scalar product as final answer would be scalar.

Suppose,
$$\vec{v_1} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{v_2} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

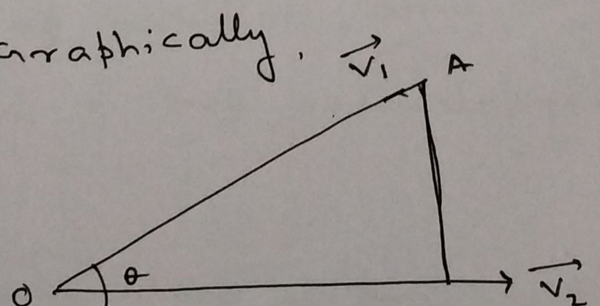
Then $\vec{v_1} \cdot \vec{v_2} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

also we know

$$\vec{v_1} \cdot \vec{v_2} = |v_1| . |v_2| \cos\theta.$$

$$\frac{\vec{v_1} \cdot \vec{v_2}}{|v_2|} = |v_1| \cos\theta.$$

Now $\frac{\vec{v_2}}{|v_2|}$ is a unit vector along the direction of v_2. Graphically.

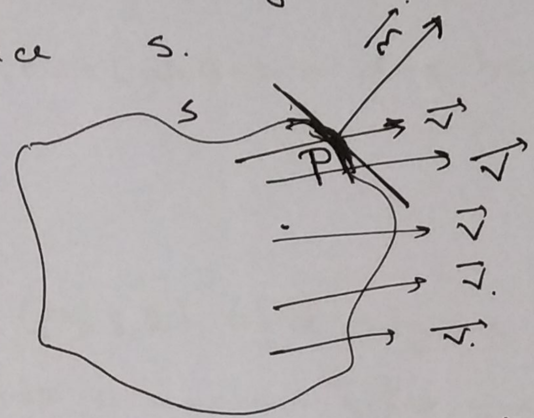


Now $OC = |v_1| \cos\theta$.

which means, $\dfrac{\vec{v_1} \cdot \vec{v_2}}{|v_2|}$ implies projection of $\vec{v_1}$ in the direction of $\vec{v_2}$.

* How to relate the same in our context?

suppose $\vec{v}$ is the velocity of a fluid flow inside a surface S.



Now, let us assume that, at point P, the unit normal vector to the surface $\hat{n}$.

Then what is the meaning of $\vec{v} \cdot \hat{n}$?

It is simple, it means component of velocity normal to the surface at P is $\vec{v} \cdot \hat{n}$.

2. <u>Scalar Field</u>:

Suppose that to each point (x, y, z) of a region D in space, there corresponds a ~~num~~ number (scalar) $\phi(x, y, z)$. Then ϕ is called a scalar function of position, and we say that a scalar field ϕ has been defined on D.

<u>Example</u>:

1> The temperature at any point within or on the earth surface at a certain time defines a scalar field.

2) More useful example :- pressure field.

For example, hydrostatic pressure at any point on the fluid can be written as :-

$$\phi(x, y, z) \approx p(x, y, z) = -\rho g z.$$

where $\rho \equiv$ density of the fluid

$g \equiv$ accelaration due to gravity.

3). <u>Vector field</u>

Suppose to each (x, y, z) of a region D in space there corresponds a vector $\vec{V}(x, y, z)$. Then $\vec{V}$ is called a vector function of position, and we say that a vector field $\vec{V}$ has been defined on D.

<u>Relevant examples</u> :

i) Velocity field

ii) Force field.

4. Relate the velocity field with scalar field.

<u>Example</u> : The force due to hydrostatic pressure may be written as

$$F(\bar{x}) = \iint_S p\,\hat{n}\,ds$$

$$= -\iint_S (\rho g z) \cdot n \cdot ds.$$

5. Definition of gradient, divergence and curl.

i) Vector differential operator $\equiv \nabla$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

5.1 : Gradient

Suppose $\phi(x, y, z)$ be a scalar function. if $\phi(x, y, z)$ is differentiable at each point, then, the gradient of ϕ, written as $\nabla\phi$ or grad ϕ is defined as :-

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

Remember '$\nabla\phi$' is the vector field.

Some Physical understanding :-

Normally, we call $\phi(x, y, z)$ as scalar potential. Now, if $\phi(x, y, z)$ represent a surface, then $\frac{\nabla\phi}{|\nabla\phi|}$ is the unit normal to that surface.

Just to show that, take a surface

$$\phi(x, y, z) = x^2 + y^2 + z^2 - a^2$$

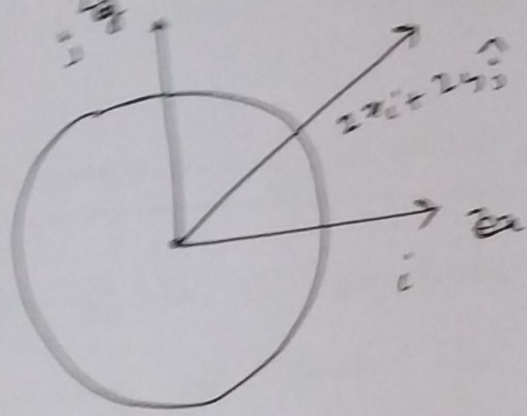
then $\frac{\partial \phi}{\partial x} = 2x$, $\frac{\partial \phi}{\partial y} = 2y$, $\frac{\partial \phi}{\partial z} = 2z$

Where

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}.2x + \hat{j}\,2y + \hat{k}\,2z$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$



Clearly one can understand that it is $\perp^r$ to the circle, for 2D, hence it $\perp^r$ to sphere also

Proof :- Before we prove this, let us recall one relationship :-

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}.dy + \frac{\partial\phi}{\partial z}.dz$$

$$\therefore \frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}.\frac{dx}{dt} + \frac{\partial\phi}{\partial y}.\frac{dy}{dt} + \frac{\partial\phi}{\partial z}.\frac{dz}{dt}$$

Now :- Since $\phi(x, y, z)$ is a surface, $\frac{d\phi}{dt} = 0$

$$\Rightarrow \frac{\partial\phi}{\partial x}.\dot{x} + \frac{\partial\phi}{\partial y}.\dot{y} + \frac{\partial\phi}{\partial z}.\dot{z} = 0$$

$$\Rightarrow \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right) = 0$$

$$\Rightarrow (\nabla\phi).\dot{r} = 0$$

$\Rightarrow$ $\underline{\nabla\phi \perp^r \dot{r}}$ as dot product is zero.

but '$\dot{r}$' is the velocity field, $\therefore$ $\dot{r}$ is tangent to the surface.

$\therefore$ $\nabla\phi$ is $\perp^r$ to the surface.

$\therefore$ $\dfrac{\nabla\phi}{|\nabla\phi|}$ is unit normal to the surface ϕ.

Some more meaning in this context

~~'ϕ'~~ $\phi(x,y,z)$ is a scalar field, we can add any meaning to that, it could be pressure field or ~~any th~~ anything. Now, let us assume, $\phi(x,y,z)$ represent a scalar potential for velocity, which means $\nabla\phi$ represents some velocity. then:

$$\frac{d\phi}{dn} = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dn} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{dn} + \frac{\partial\phi}{\partial z}\cdot\frac{dz}{dn}$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(n_x\hat{i} + n_y\hat{j} + n_z\hat{k}\right)$$

$$= (\nabla\phi)\cdot\hat{n}$$

meaning ???

It means, ~~fluid~~ component of fluid velocity in the direction of $\hat{n}$.

(very important)!!!!!

Lecture 2

Marine Hydrodynamics

1. Divergence

Suppose $\vec{v}(x, y, z) = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$ and differentiable at each point. Then the divergence of $\vec{v}$, written as $\nabla . \vec{v}$ or div $\vec{v}$ is defined as

$$\nabla . \vec{v} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial k} \hat{k} \right) . \left(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Clearly $\nabla . \vec{v}$ is a scalar.

* How to understand physically the meaning of div $(\vec{v})$? (Home work) / discuss latter.

2. Curl

Suppose, $\vec{v}(x, y, z) = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$ is a differentiable vector field. Then curl or rotation of $\vec{v}$, written $\nabla \times \vec{v}$ or curl $\vec{v}$ is defined as:

$$\nabla \times \vec{v} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \times \left(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \hat{k}$$

Clearly, $\nabla \times \vec{v}$ is a vector.

3). Divergence Theorems.

suppose 'v' is the volume bounded by a closed surface 's' and $\vec{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla \cdot \vec{A} \, dv = \iint_S \vec{A} \cdot \hat{n} \, ds$$

where $\vec{n}$ is the positive outward drawn normal to s.

4). Green's third identity. / theorem.

$$\iiint_V \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi \right] dv$$

$$= \iint \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] ds.$$

where ϕ and ψ are the scalar potentials.

5). Stokes Theorem.

suppose 's' is an open, two sided surface bounded by a closed, non intersecting curve C (simple closed curve), and suppose $\vec{A}$ is a vector function of position with continuous derivatives, Then

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \vec{n} \, ds.$$

Marine Hydrodynamics

Marine
↳ wave
↳ Body in waves

Fluid is inviscid
Ideal fluid
Ir-rotational
In-compressible
Homogeneous.

Now, we understand each of the above one by one, however, to start with, how we can sea a fluid in general. i.e, how it differs with respect to the rigid body system. How the conservation laws holds for ~~this~~ fluids.

For example: consider a drop of ink, with time it's spreaded over the paper, then in this case, how the mass is conserved?

From our knowledge we know ~~volume~~

mass $m = \rho . V$ [where $\rho \equiv$ density, $V \equiv$ volume].

Now at time $t = t_0$ assume the density of the ink ρ_0 and volume $V_0 = dx_0 dy_0 dz_0$

$\therefore$ m at $t_0 = \rho_0 dx_0 dy_0 dz_0$

Similarly: at $t = t_1$, assume $\rho = \rho_1$,

$V = dx_1 dy_1 dz_1$ $\therefore$ m at $t_1 = \rho_1 dx_1 dy_1 dz_1$

since both are same as mass is conserved:.

$$\iiint_{V_0} \rho_0 \, dx_0 \, dy_0 \, dz_0 = \iiint_{V_1} \rho_1 \, dx_1 \, dy_1 \, dz_1$$

Now, if we express the same in some uniform co-ordinate. we may write.

$$\iiint_{V} \rho_0 |J_0| \, da \, db \, dc = \iiint_{V} \rho_1 |J_1| \, da \, db \, dc$$

$$\Rightarrow \boxed{\rho_0 J_0 = \rho_1 J_1}$$ → mass conservation equation.

This is easy to understand, This is known as continuity Equation using Lagrangian approach.

Now, one has to understand from the above, that, over time, fluid particle also gets dostented. which implies, the property of fluid not only the function of time, but also, it is a function of space. Therefore, a so called total derivative of :: $\frac{d}{dt}$ is not sufficient to express the rate of change of fluid properties, because

$$\frac{d}{dt} \equiv \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

However, we need

$$\frac{D}{Dt} \equiv \lim_{\Delta t \to 0} \frac{f(x+\Delta x, t+\Delta t) - f(x,t)}{\Delta t}$$

$$f(x+\Delta x, t+\Delta t) = f(x,t) + \Delta x\, f_x(\bar{x},t) + \Delta t \frac{\partial f}{\partial t} + \Delta y\, f_y(\bar{x},t) + \Delta z\, f_z(\bar{x},t)$$

$$\Rightarrow \frac{f(x+\Delta x, t+\Delta t) - f(x,t)}{\Delta t} = \frac{\partial f}{\partial t} + \underbrace{\frac{\Delta x}{\Delta t}}_{\bar{u}} \nabla f$$

taking $\Delta t \to 0$ we get

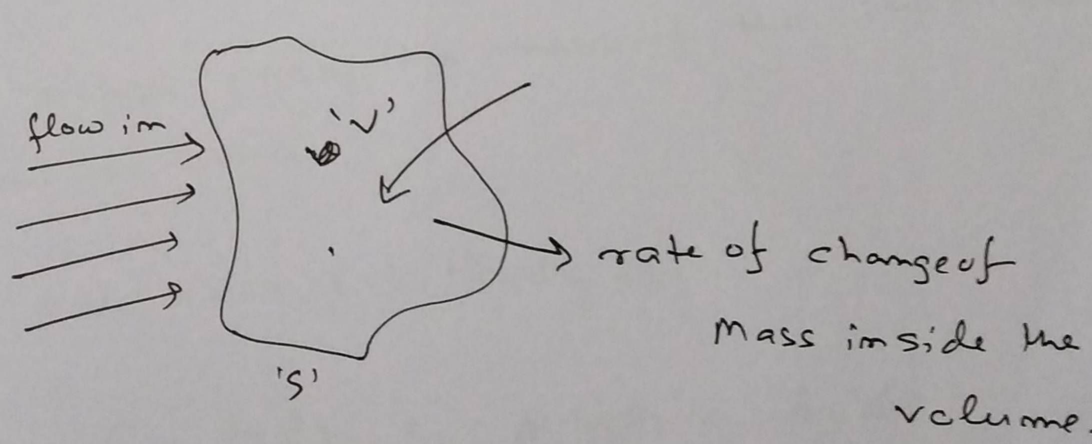
$$\frac{DF}{Dt} = \frac{\partial f}{\partial t} + u \cdot \nabla f$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla) \longrightarrow \text{Material derivative.}$$

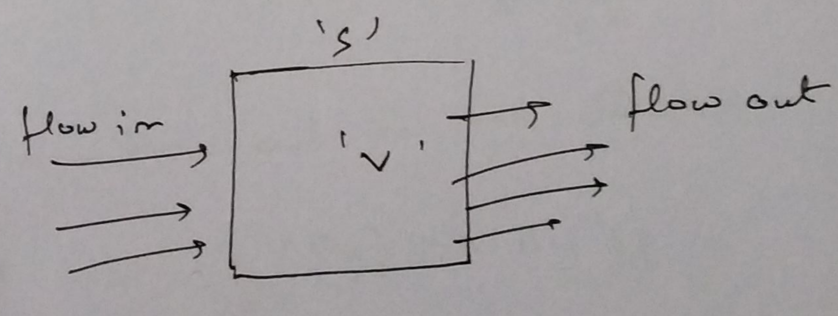
Another approach to see mass conservation property

consider the following surface 's' of a volume 'v.'

Picture 1



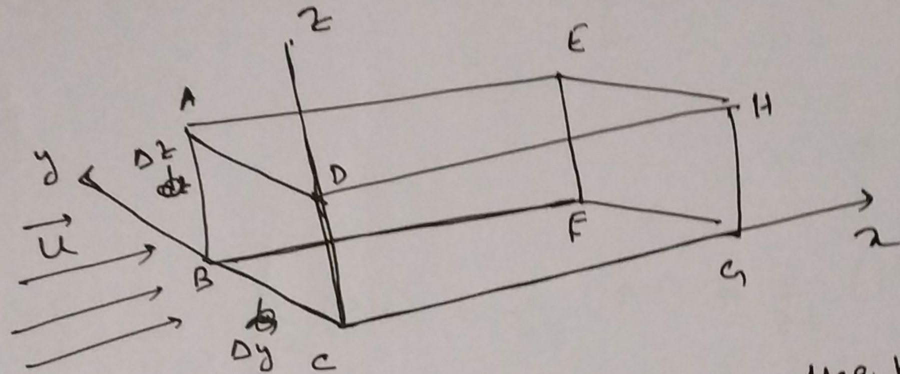
Picture 2



in both the picture, instead of fluid particle, we have taken a volume 'v' that enclosed by some surface 's'

In this case, the idea is to consider a domain and check if the conservation of mass is maintained in that region. This concept is known as Eulerian approach.

Let us consider the following box



Now, consider the surface 'ABCD', the per unit time, the ~~volum~~ mass of the fluid entered

$= \cancel{\rho u\, dy\, dz} \quad \rho u\, \Delta y\, \Delta z$

Now, at surface EFGH, the mass of the fluid ~~out~~ exit ~~into~~ from the surface

$= \rho u(x+\Delta x, y, z, t)\cdot dy\, dz$

$= \rho u\, dy\, dz + \frac{\partial(u\rho)}{\partial x}\Delta x.\, dy\, dz$

$\therefore$ flow in − flow out in the direction of x

$= -\frac{\partial}{\partial x}(\rho u)\,\Delta x\,\Delta y\,\Delta z$

Similarly

for y direction $= -\frac{\partial}{\partial y}(\rho u)\,\Delta x\,\Delta y\,\Delta z$

for z direction $= -\frac{\partial}{\partial z}(\rho u)\,\Delta x\,\Delta y\,\Delta z$

Now according to the theory

rate of change of mass inside the box

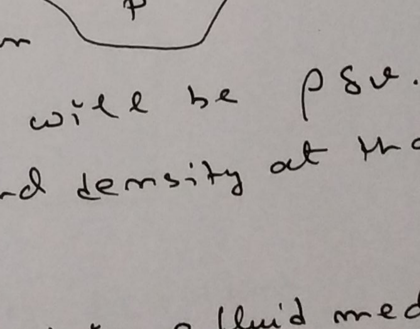
= flow in − flow out.

$$\Rightarrow \frac{\partial}{\partial t}(\rho) \Delta x \Delta y \Delta z = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho u) + \frac{\partial}{\partial z}(\rho u)\right] \times \Delta x \Delta y \Delta z$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho u) + \frac{\partial}{\partial z}(\rho u) = 0$$

$$\boxed{\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0}$$ vector form.

More general approach



Consider a fluid of infinitesimal volume δv, density ρ. Then the mass of the element at any position will be $\rho \delta v$. if P at any arbitrary time t will be δv and ρ be the volume and density at that particular time instant at P.

Consider a closed surface 's' in a fluid medium containing a volume V fixed in space. Let $\hat{n}$ be the unit normal (out ward drawn) at a surface element δs. If $\vec{u}$ be the fluid velocity at the element δs, then the normal component of $\vec{u}$ measured out ward from V will be given by $\vec{u}.\hat{n}$.

$\because$ Rate of mass flow across δs / unit time $= \rho.\vec{u}.\hat{n}$.

$\therefore$ Total rate of mass flow out of v δs is $\int_s \rho\,\vec{u}.\hat{n}\,ds$

$\therefore$ Total rate of mass flow into v is $-\int_s \hat{n}.(\rho\vec{u})\,ds$

$$= -\int_v \nabla.(\rho\vec{u})\,dv \quad [\text{by Gauss Theorem}]$$

Also rate of change of mass within v

$$= \frac{\partial}{\partial t}\int_v \rho\,dv \Rightarrow \int_v \frac{\partial \rho}{\partial t}.\,dv.$$

$\therefore$ By principle of mass conservation

$$\int_v \frac{\partial \rho}{\partial t}\,dv = -\int_v \nabla.(\rho\vec{u})\,dv$$

$$= \int_v \left[\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{u})\right] dv = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{u}) = 0}$$

This is known as continuity equation based on Eulerian approach.

[Home work :- Submit the continuity equation with respect to spherical polar co-ordinate system]

Lecture 3

Marine Hydrodynamics

1) Material derivatives: Symbol $\frac{D}{Dt}$

Definitions for total derivative may be expressed as

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \quad \ldots\ldots (1.1)$$

However, this holds well for one dimensional solid object which may be considered as a particle. Now for fluid, the arbitrary function 'f' is not only the function of 't', it is function of space variable also. Therefore, in such case, the r.h.s. of the equation takes the form

$$\lim_{\Delta t \to 0} \frac{f(\vec{r} + \Delta\vec{r},\ t + \Delta t) - f(\vec{r}, t)}{\Delta t} = \frac{Df}{Dt} \longrightarrow (1.2)$$

where $\vec{r}$ denote the space variable, typically,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}.$$

Now using Taylor series and taking the 1st order component

$$f(\vec{r} + \Delta\vec{r},\ t + \Delta t) = f(r, t) + \Delta\vec{r}\cdot\nabla f + \Delta t\,\frac{\partial f}{\partial t}$$

or $\lim_{\Delta t \to 0}$

$$\frac{f(\vec{r} + \Delta r,\ t + \Delta t) - f(r, t)}{\Delta t} = \frac{dr}{dt}\cdot\nabla f + \frac{\partial f}{\partial t}$$

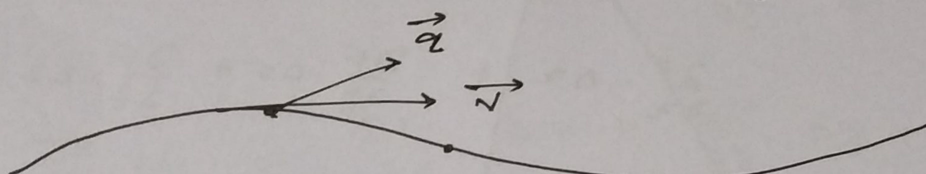
$$\Rightarrow \boxed{\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{v}\cdot\nabla) f} \longrightarrow (1.3)$$

<u>for Cartisian system</u>

$$\frac{DF}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}.$$

<u>where</u> $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

2) Concept of Boundary surface



Let us assume $F(x, y, z, t) = 0$ be a boundary surface, $\vec{V}$ be the velocity of the boundary surface, and $\vec{q}$ be the velocity of the fluid particle.

In order to stay the fluid particle $\vec{q}$ on the boundary surface $F(x, y, z, t) = 0$ if normal velocity of the boundary surface is equal to the normal velocity of the fluid particle.

<u>Then</u> $\vec{V} \cdot \vec{n} = \vec{q} \cdot \vec{n} \longrightarrow (2.1)$

<u>now</u>, since $F(x, y, z, t) = 0$ is a boundary surface

$\frac{\nabla F}{|\nabla F|}$ is the unit normal to the surface ~~F(x,y,z,t)=0~~. $F(x, y, z, t) = 0$

$\therefore$ from (2.1) we get

$$\vec{V} \cdot \frac{\nabla F}{|\nabla F|} = \vec{q} \cdot \frac{\nabla F}{|\nabla F|}$$

or $\vec{V} \cdot \nabla F = \vec{q} \cdot \nabla F \quad \ldots\ldots (2.2)$

Now since $F(x, y, z, t) = 0$ then

$$F(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) = 0$$

Now using Taylor's series expamssion, we get

$$F(x + \Delta x, y + \Delta y, \cancel{z + \Delta z}, t + \Delta t) = \cancel{F(x, y, z, t)} + \frac{\partial F}{\partial x} \cdot \Delta x + \frac{\partial F}{\partial y} \Delta y + \frac{\partial F}{\partial z} \Delta z + \frac{\partial F}{\partial t} \Delta t$$

$$\Rightarrow -\frac{\partial F}{\partial t} \Delta t = \frac{\partial F}{\partial x} \cdot \Delta x + \frac{\partial F}{\partial y} \Delta y + \frac{\partial F}{\partial z} \Delta z$$

$$\Rightarrow -\frac{\partial F}{\partial t} \cdot \Delta t = \nabla F \cdot \Delta r \quad \left[\because \Delta r = \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k}\right]$$

$$\Rightarrow -\frac{\partial F}{\partial t} = \vec{v} \cdot \nabla F \quad \left[\because \frac{\Delta r}{\Delta t} = \vec{v}\right] \rightarrow (2.3)$$

Substituting (2.3) into (2.2) we get

$$-\frac{\partial F}{\partial t} = \vec{q} \cdot \nabla F$$

$$\Rightarrow \frac{\partial F}{\partial t} + \vec{q} \cdot \nabla F = 0$$

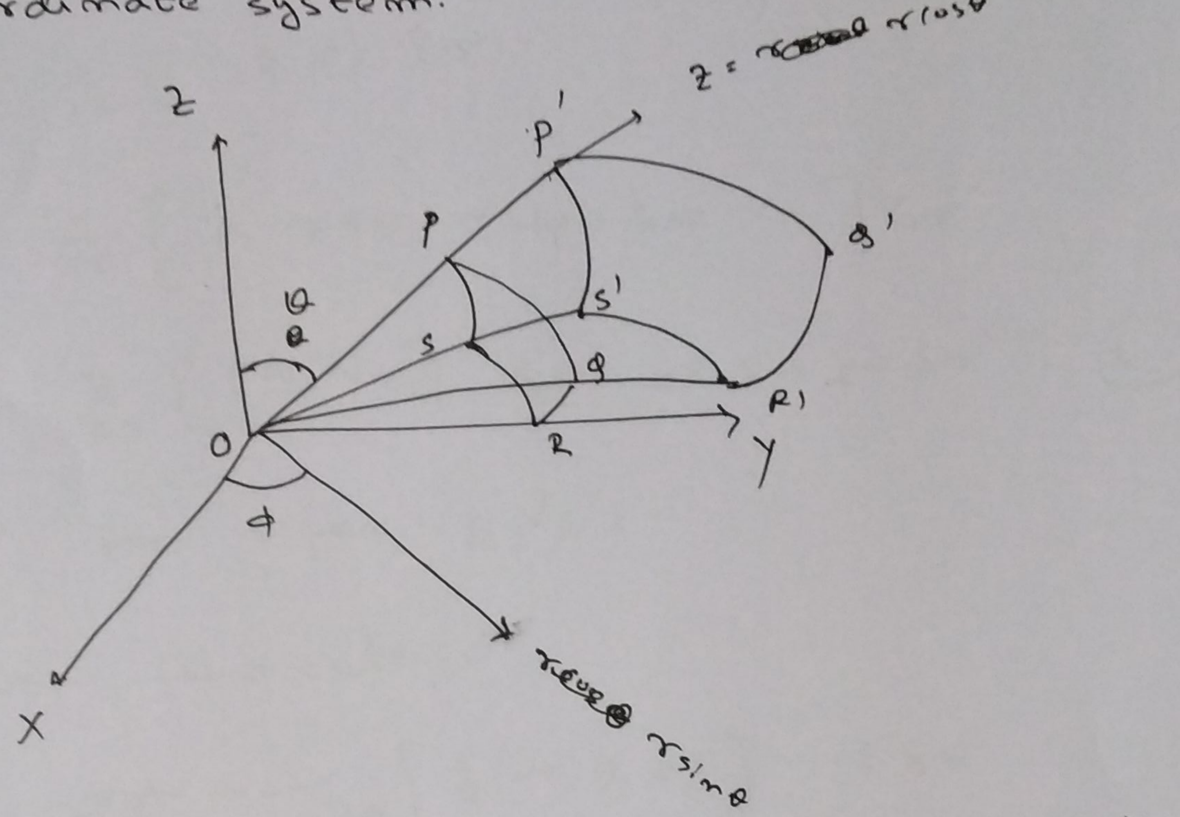
or

$$\boxed{\frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = 0} \rightarrow (2.4)$$

~~Same time~~ The equation of every boundary surface must satisfy the differential equation (2.4). If the surface is at rest, then $\because \frac{\partial F}{\partial t} = 0$

$$\Rightarrow \boxed{u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + v \frac{\partial F}{\partial y} = 0} \rightarrow (2.5)$$

Equation of continuity for spherical polar co-ordinate system.



Let ρ be the density of the fluid at the point $P(r, \theta, \phi)$. Now it is well understand the area of surface

$PQRS = r\delta\theta \cdot r\sin\theta\,\delta\phi$

$PP'S'S = \delta r \cdot r\sin\theta\,\delta\phi$

$SS'R'R = \delta r \cdot r\delta\theta$

if the component of velocity along $r, \theta, \phi = v_r, v_\theta, v_\phi$,

Now, the mass flow along $PQRS$

$= (\rho \cdot r\delta\theta \cdot r\sin\theta\,\delta\phi) \cdot v_r = f(r, \theta, \phi)$

Similarly ~~flow~~ mass flow out of surface $P'Q'R'S'$

$\cancel{= \frac{\partial}{\partial r}(\rho r\,\delta\theta\, r\sin\theta\, v_r)\,\delta\phi\,\delta\theta}$

$\cancel{= \frac{\partial}{\partial r}(\rho r^2 v_r)}$

$= f(r+\delta r, \theta, \phi) = f(r, \theta, \phi) - \frac{\partial}{\partial r} f(r, \theta, \phi) \cdot \delta r$

$\therefore$ flow in - flow out

$$= -\frac{\partial}{\partial r} f(r,\theta,\phi).\ \delta r$$

$$= -\frac{\partial}{\partial r}\left[\rho.\ r\delta\theta.\ r\sin\theta\ \delta\phi\ v_r\right]\delta r$$

$$= -\frac{\partial}{\partial r}\left[\rho r^2 v_r\right]\ \sin\theta\ \delta\theta\ \delta\phi\ \delta r \quad \ldots (3.1)$$

Similarly for s.face $PP'S'S$

flow in - flow out

$$= -\frac{1}{r\partial\theta}\left[g(r,\theta,\phi)\right].\ r\delta\theta$$

$$= -\frac{1}{r\partial\theta}\left[\delta r . r\sin\theta\ \delta\phi . \rho\ v_\theta . r\right]\delta\theta$$

$$= -\frac{1}{r\partial\theta}\left[\rho r^2 \sin\theta\ v_\theta\right]\ \delta r\ \delta\phi\ \delta\theta$$

$$= -\frac{1}{r\partial\theta}\left[\rho\ \sin\theta\ v_\theta\right]\ r^2\ \delta r\ \delta\phi\ \delta\theta \quad \ldots (3.2)$$

Similarly for ϕ face $SS'R'R$

$$\text{flow in - flow out} = -\frac{1}{r\sin\theta\,\partial\phi}\left[h(r,\theta,\phi)\right] r\sin\theta\ \delta\phi$$

$$= -\frac{1}{r\sin\theta\,\partial\phi}\left[\rho.\ \delta r.\ r\delta\theta\ v_\phi\right]\ r\sin\theta\ \delta\phi$$

$$= -\frac{1}{r\sin\theta\,\partial\phi}\left[\rho v_\phi\right]\ r^2\sin\theta\ \delta r\ \delta\theta\ \delta\phi \quad \ldots (3.3)$$

Now total rate of change of mass inside the volume $PP'Q'R$

$$= \frac{\partial}{\partial t}\left[\rho\, \delta r .\, r .\, \delta\theta .\, r \sin\theta\, \delta\phi\right]$$

$$= \frac{\partial}{\partial t}\left[\rho\right] r^2 \delta r .\, \delta\theta .\, \delta\phi$$

$$= \frac{\partial \rho}{\partial t} .\, r^2 \sin\theta\, \delta r .\, \delta\theta .\, \delta\phi \quad \ldots \quad (3.4)$$

Equating the total flow in − flow out with (3.4) we get

$$\frac{\partial \rho}{\partial t} r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi = -\frac{\partial}{\partial r}\left[\rho r^2 v_r\right] \sin\theta\, \delta r\, \delta\theta\, \delta\phi - \frac{1}{r\,\partial\theta}\left[\rho \sin\theta\, v_\theta\right] r^2\, \delta r\, \delta\theta\, \delta\phi - \frac{1}{r\sin\theta\, \partial\phi}\left[\rho v_\phi\right] r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi$$

$$\Rightarrow r^2 \sin\theta \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left[\rho r^2 v_r\right]\sin\theta + \frac{\partial}{r\,\partial\theta}\left[\rho \sin\theta\, v_\theta\right] r^2 + \frac{1}{r\sin\theta\, \partial\phi}\left[\rho v_\phi\right] r^2 \sin\theta = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\rho \sin\theta\, v_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\rho v_\phi\right] = 0}$$

$\rightarrow (3.5)$

4. Some useful informations

i) Continuity equation in vector form.

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{q}) = 0 \qquad (4.1)$$

ii) in case of in-compressible fluid, $\frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \quad \nabla . (\rho \vec{q}) = 0 \quad \text{or} \quad \mathrm{div}(\rho \vec{q}) = 0$$

in case of fluid is homogeneous, ρ = constant

$$\Rightarrow \quad \rho . \mathrm{div}\, \vec{q} = 0$$

$$\boxed{\Rightarrow \quad \mathrm{div}\, \vec{q} = 0}$$ important !!! → (4.2)

(ii) In cartesian co-ordinate system.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

in case of steady homogeneous fluid

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \rightarrow (4.3)$$

~~doo, so~~

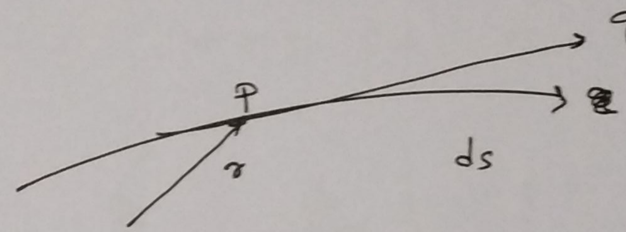
5. Some basic definition and meaning.

i) Path line :- The curve describe in space by a moving fluid element is known as trajectory or path line.

The path line are given by $\vec{q} = \frac{d\bar{r}}{dt}$. In cartesian co-ord.

$$\frac{dx}{dt} = u(x, y, z, t), \quad \frac{dy}{dt} = v(x, y, z, t), \quad \frac{dz}{dt} = w(x, y, z, t)$$

(vi) Streamline: A streamline is a continuous line of flow drawn in the fluid so that tangent at every point is the direction of the fluid velocity at the point at given instant.



consider ds be an element of the streamline passing through P(r) at a certain instant of time. Let the $\vec{q}$ be the velocity vector at P, then by definition $ds \parallel \vec{q}$

$$\Rightarrow ds \times \vec{q} = 0$$

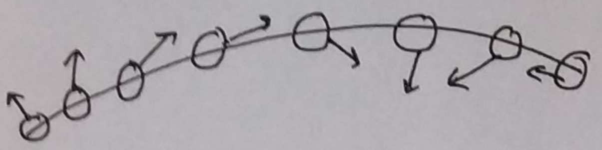
$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix} = 0$$

$$\Rightarrow (w\,dy - v\,dz)\hat{i} + (u\,dz - w\,dx)\hat{j} + (v\,dx - u\,dy)\hat{k} = 0$$

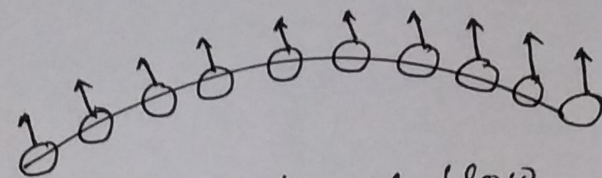
Now each component of $\hat{i}, \hat{j}, \hat{k}$ must be zero $\Rightarrow w\,dy - v\,dz = 0,\ u\,dz - w\,dx = 0,\ v\,dx - u\,dy = 0$

$$\Rightarrow \boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}} \quad (5.1)$$

(iii) rotational and ir-rotational flow :



rotational flow



ir-rotational flow

Mathematically, if $\vec{q}$ be the velocity of fluid particle, then for rotational flow $\nabla \times \vec{q}$ or curl (q) is non zero, i.e $\nabla \times q \neq 0$ or curl $\vec{q} \neq 0$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \neq 0$$

in that case

$$\left.\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad (5.2)$$

Now suppose, for ir-rotational flow, $\nabla \times \vec{q} = 0$ also from vector we know that curl(grad ϕ) $= 0$ where 'ϕ' is a scalar function, using this information, one can take $\vec{q} = \nabla \phi$.

then, the flow automatically becomes ir-rotational i.e. $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$, $w = \frac{\partial \phi}{\partial z}$.

substitute this in our continuity equation we get from (4.2) and (4.3) that

$\nabla^2 \phi = 0$ .... (5.3) or $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \rightarrow (5.4)$

This equ^n is known as Laplace equation !!!