③ Consider initial value problem

$$y' = x(y+x) - 2, \qquad y(0) = 2.$$

Use Euler's method & Euler-Cauchy method. with step size $h = 0.3, h = .2$
and $h = .15$ to compute $y(0.6)$ (upto 5 decimal places).

$$y_{n+1} = y_n + h\, y_n'$$
$$= y_n + h\, f(x_n, y_n)$$
$$= y_n + h\, [x_n(y_n + x_n) - 2]$$
$$= (1 + h x_n) y_n + h x_n^2 - 2 \cdot h$$

$\underline{h = .3}$

$$y_1 = y(.3) = (1 + h x_0) y_0 + h x_0^2 - 2 \cdot (.3)$$
$$= 2 - .6 \quad = 1.4$$

$$y_2 = y(.6) = (1 + h x_1) y_1 + h x_1^2 - 2h$$
$$= (1 + h^2) 1.4 + h^3 - 2h$$
$$= (1 + .09) 1.4 + .0027 - .6$$
$$= 1.09 \times 1.4 + 0.027 - .6 = 1.526 + 0.0027 - 0.6$$
$$= 0.953$$

$= y(.2)$

④ Apply Euler-Cauchy method with step length h
to the problem

$$y' = -y, \quad y(0) = 1$$

ⓐ Determine an explicit expression for y_n

ⓑ For which value of h, the sequence $\{y_n\}_0^\infty$
bounded.

— **Cauchy - Euler Method**

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2)$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + h, y_n + k_1)$$

$$k_2 = h f(x_n + h, y_n + h f(x_n, y_n))$$

$$y' = y \quad , \quad f(x, y) = -y$$

$$k_1 = -h \, y_n$$

$$k_2 = -h(y_n + k_1) = -h(y_n \oplus h \, y_n)$$
$$= -h(1-h) y_n$$

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2)$$
$$= y_n + \tfrac{1}{2}[-hy_n - h(1-h)y_n]$$
$$= y_n[1 - \tfrac{1}{2}h - \tfrac{1}{2}h(1-h)]$$
$$= y_n[1 - \tfrac{h}{2} - \tfrac{h}{2} + \tfrac{h^2}{2}]$$

$$y_{n+1} = y_n[1 - h + \tfrac{h^2}{2}]$$

$$y_1 = [1 - h + \tfrac{h^2}{2}] \, y_0 = [1 - h + \tfrac{h^2}{2}]$$

$$y_2 = [1 - h + \tfrac{h^2}{2}] y_1 = [1 - h + \tfrac{h^2}{2}]^2$$

$$\vdots$$

$$y_n = [1 - h + \tfrac{h^2}{2}]^n$$

⑥ $\{y_n\}_{n=0}^{\infty}$ will remain bounded iff

$$\left| 1 - h + \tfrac{h^2}{2} \right| \leq 1$$

$$2 - 2h + h^2 \leq 2$$

$$h^2 - 2h + 1 + 1 \leq 2$$

$$(h-1)^2 + 1 \leq 2$$

$$(h-1)^2 \leq 1$$

so if $(h-1) \leq 1$ then $(h-1)^2 \leq 1$

or $h \leq 2$

for $0 < h \leq 2$ the sequence $\{y_n\}_{n=0}^{\infty}$ is bounded.

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⑤ Solve the differential equation:

$$y' = x + y \qquad y(0) = 1 \qquad \text{by Euler's method}$$

(Take $h = .1$)

$$f(x,y) = x + y$$

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$= y_n + h(x_n + y_n)$$

$$= (1 + h) y_n + h\, x_n$$