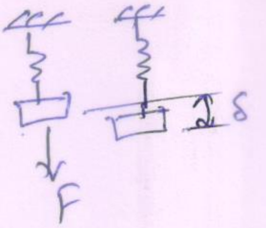
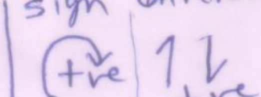
Matrix method for structural analysis

Stiffness and flexibility



sign convention



Stiffness(k) is force required for creating unit displacement. It is commonly denoted by "K" $= \frac{F}{\delta}$

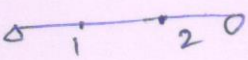
Flexibility is displacement caused by unit force, denoted by "f"

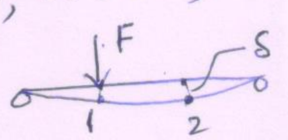
Flexibility is reciprocal of stiffness.

$f = \frac{\delta}{F}$. Hence flexibility is reciprocal of stiffness.

$$f = \frac{1}{k}$$

Consider an object (e.g., a beam) shown below, with two nodes (1,2)





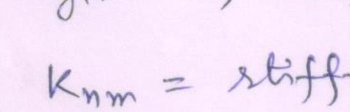
F is applied at 1 and the deflection is measured at 2.

Here, stiffness experienced at 2 when load is applied at 1 is given by $K_{21} = \frac{F}{\delta}$ or f_{21} (flexibility) $= \frac{\delta}{F}$

$\therefore K_{nm}$ = stiffness at n^{th} degree of freedom (which is deflection at 2 in above) due to force applied at m^{th} degree of freedom (which is vertical direction at 1 in above figure). Similarly deflection δ in above figure should be written as δ_{21}, i.e., deflection at 2 when load applied at 1.

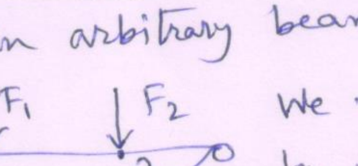
Maxwell's reciprocal theorem

Consider an arbitrary beam (can be any arbitrary object)



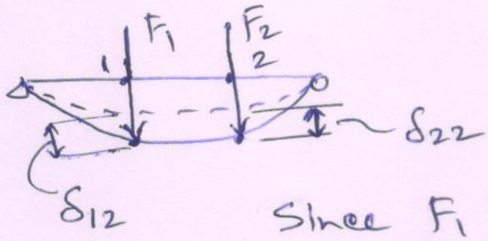
We want to find the total work done by the external forces for this Problem. We can do it in two alternate sequences, which are ① first apply F_1 gradually, then apply F_2, or ② first apply F_2 gradually and then F_1.

$\therefore$ case ①



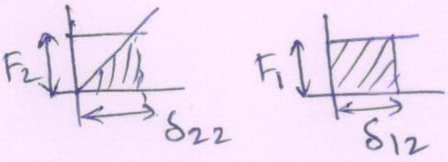
work done $= \frac{1}{2} F_1 \delta_{11}$

Now apply F_2 at node 2.



work done $= \frac{1}{2}F_2\delta_{22} + F_1\delta_{12}$

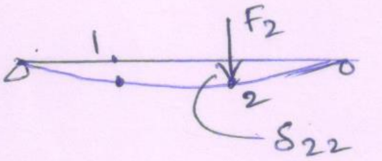
Since F_1 is already applied, only deflection δ_{12} is happening, $\frac{1}{2}$ is not multiplied in the second term.



$\therefore$ total work done

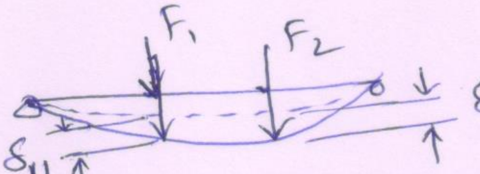
$$= \frac{1}{2}F_1\delta_{11} + \frac{1}{2}F_2\delta_{22} + F_1\delta_{12}$$

case ②



work done $= \frac{1}{2}F_2\delta_{22}$ when F_2 is first applied.

Now F_1 is applied gradually.



work done $= \frac{1}{2}F_1\delta_{11} + \delta_{21}\cdot F_2$

$\therefore$ total work done $= \frac{1}{2}F_2\delta_{22} + \frac{1}{2}F_1\delta_{11} + F_2\delta_{21}$

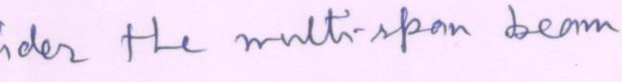
Since the total work done must be the same in these two cases, we get $\frac{1}{2}F_1\delta_{11} + \frac{1}{2}F_2\delta_{22} + F_1\delta_{12} = \frac{1}{2}F_2\delta_{22} + \frac{1}{2}F_1\delta_{11} + F_2\delta_{21}$

or $F_1\delta_{12} = F_2\delta_{21} \Rightarrow \frac{F_1}{\delta_{21}} = \frac{F_2}{\delta_{12}}$ or $K_{21} = K_{12}$

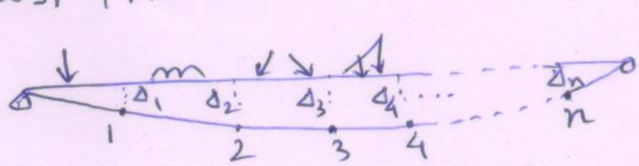
Remember that the above can be written if F, δ are linearly related, which requires that the material should be linear, elastic, isotropic and follows Hooke's law, i.e, in such cases we get $K_{21} = K_{12}$ or $f_{21} = f_{12}$. This is the reciprocal theorem.

You have already solved statically indeterminate problems. In simple terms, what we did was, (I) remove the right number of constraints / boundary condition to convert it into a statically determinant problems, (II) apply unknown forces (reactions) to each of those constraints, (III) Write the equations of consistent deformation/constraint condition/boundary conditions using the known (external) and unknown (reactions) forces. Thus we get "n" number of linear simultaneous equations. Then we solve them using various techniques to get the unknowns. You know that a set of linear simultaneous equation can be solve using matrix algebra, such as Gauss elimination method etc. This is what we do in matrix method of structural analysis. We can write those equations either by using flexibility or by stiffness. First we study the flexibility based matrix method.

Consider the multi-span beam as shown below:



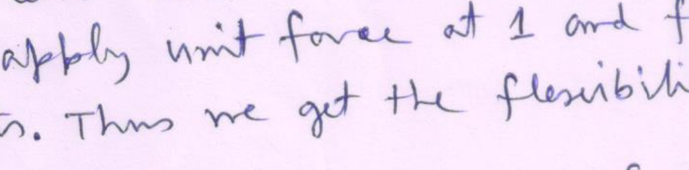
This is statically indeterminate. To convert it into a statically determinate problem, we can remove all intermediate supports keeping only those at the ends. This will make it a simply supported beam.



All support positions are marked by integers.

We assume that "n" number of supports are removed. The deflection at these locations can be calculated since it is now determinate problem. These deflections are $\Delta_1, \Delta_2, \cdots$

Now let's assume that all external loads are removed and we apply unit force at 1 and find the deflections at $1, 2, \cdots$ points. Thus we get the flexibility at various points.

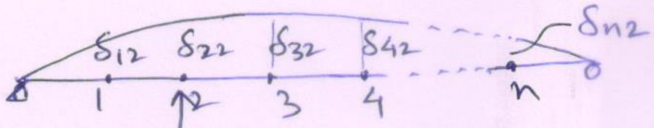


This is also determinate problem and we can calculate the deflections $\delta_{11}, \delta_{21}, \delta_{31}, \cdots$

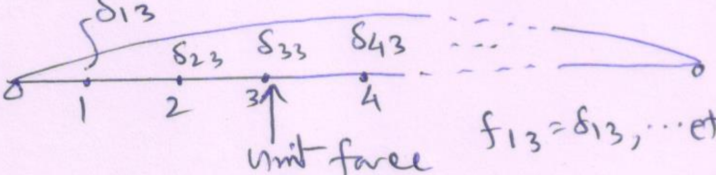
From this we get the flexibility, $f_{11} = \frac{\delta_{11}}{1}$, $f_{21} = \frac{\delta_{21}}{1}$, etc.

We can do the same exercise for other points like 2, 3, 4, ..., n.

$\therefore f_{12} = \delta_{12}, \delta_{22} = f_{22}$ etc.



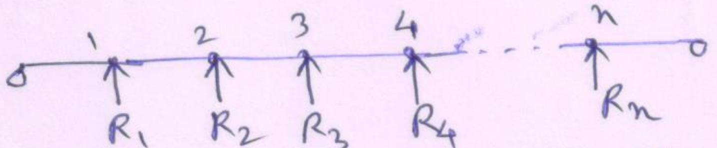
etc.



$f_{13} = \delta_{13}, \cdots$ etc.

Thus all flexibility values can be found out.

Now, since we have removed "n" supports, we need to apply unknown reactions to each of them.



Since we know that the deflection at all support points will be = 0, we can write the expression for the total deflection (consistent deformation).

For example, at ①, $R_1 f_{11} + R_2 f_{12} + R_3 f_{13} + \cdots + R_n f_{1n} - \Delta_1 = 0$

i.e., we are superimposing all deflection at ①.

Similarly total deflection at ②

$$R_1 f_{21} + R_2 f_{22} + R_3 f_{23} + \cdots + R_n f_{2n} - \Delta_2 = 0$$

at ③, $R_1 f_{31} + R_2 f_{32} + R_3 f_{33} + \cdots + R_n f_{3n} - \Delta_3 = 0$

In matrix form we can write, $[f]_{n \times n} \{R\}_{n \times 1} = \{\Delta\}_{n \times 1}$

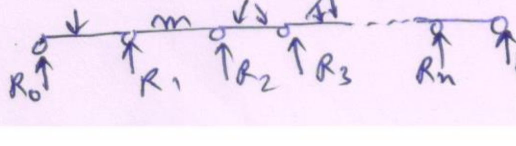
From Reciprocal theorem we know that $f_{ij} = f_{ji}$

thus $[f]$ is a symmetric matrix.

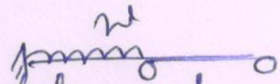
$\therefore$ the unknown forces are $\{R\} = [f]^{-1} \{\Delta\}$

Thus once the unknowns are solved, the end support reactions can be found using the equations of statics.

Now the SF & BM at any location in the beam can be obtained.

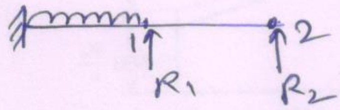


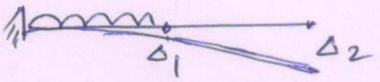
Solve the following problem using flexibility based matrix method.



EI This is statically indeterminate with degree of indeterminacy = 2.

So, we need to remove 2 constraints to make it determinate, and apply 2 unknown reactions R_1, R_2.

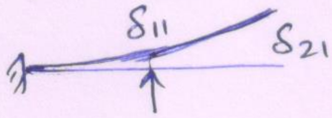




From previous studies, we know

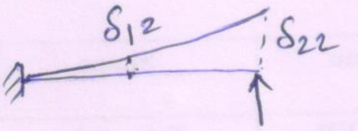
$$\Delta_1 = \frac{wl^4}{8EI}, \quad \Delta_2 = \frac{wl^4}{8EI} + \frac{wl^3}{6EI}\cdot l = \frac{7}{24}\frac{wl^4}{EI}$$

Now find the flexibilities at the two points.



$$\delta_{11} = \frac{1.l^3}{3EI} \quad \therefore f_{11} = \frac{l^3}{3EI}$$

$$\delta_{21} = \frac{1.l^3}{3EI} + \frac{1.l^2}{2EI}\cdot l = \frac{5}{6}\frac{l^3}{EI}$$



$$\therefore f_{21} = \frac{5}{6}\frac{l^3}{EI}$$

From reciprocal theorem we know that $f_{12} = f_{21}$

$$\therefore \delta_{22} = \frac{1.(2l)^3}{3EI} = \frac{8}{3}\frac{l^3}{EI}, \quad \therefore f_{22} = \frac{8}{3}\frac{l^3}{EI}$$

$\therefore$ we can write the equations ~~for~~ each of the removed constraint.

$$\text{node } 1 \Rightarrow \quad \frac{l^3}{3EI}R_1 + \frac{5}{6}\frac{l^3}{EI}R_2 - \frac{wl^4}{8EI} = 0$$

$$\text{node } 2 \Rightarrow \quad \frac{5}{6}\frac{l^3}{EI}R_1 + \frac{8}{3}\frac{l^3}{EI}R_2 - \frac{7}{24}\frac{wl^4}{EI} = 0$$

$$\sim \begin{bmatrix} \frac{l^3}{3EI} & \frac{5}{6}\frac{l^3}{EI} \\ \frac{5}{6}\frac{l^3}{EI} & \frac{8}{3}\frac{l^3}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \frac{wl^4}{8EI} \\ \frac{7}{24}\frac{wl^4}{EI} \end{Bmatrix}$$

$$\sim \begin{bmatrix} \frac{1}{3} & \frac{5}{6} \\ \frac{5}{6} & \frac{8}{3} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \frac{wl}{8} \\ \frac{7}{24}\frac{wl}{1} \end{Bmatrix}$$

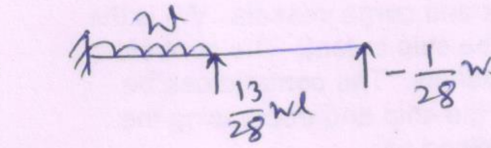
Note that the forces R_1, R_2 are unknown here, which is why it is also called Force~~s~~ based matrix method.

multiply $\left(-3 \times \frac{5}{6}\right)$ to the 1st row and subtract from 2nd row (Gauss elimination method), the equation become

$$\begin{bmatrix} 1/3 & 5/6 \\ 0 & 7/12 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} wl/8 \\ -wl/48 \end{Bmatrix}$$

$\therefore R_2 = -\frac{wl}{28}$. For R_1, the 1st equation is $\frac{R_1}{3} + \frac{5}{6}R_2 = \frac{wl}{8}$

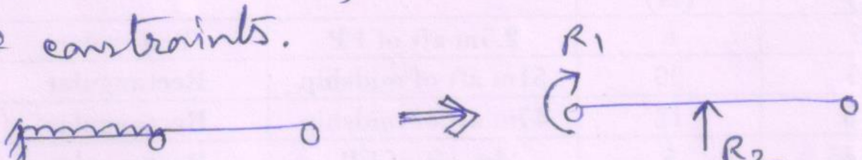
or $\frac{R_1}{3} - \frac{5}{6} \times \frac{wl}{28} = \frac{wl}{8} \Rightarrow R_1 = \frac{13}{28} wl$

$\therefore$ 

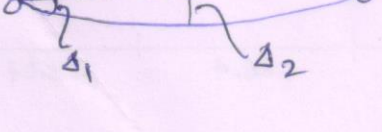
Now we can find SF, BM at any point in the beam.

We can solve the problem in a different way.

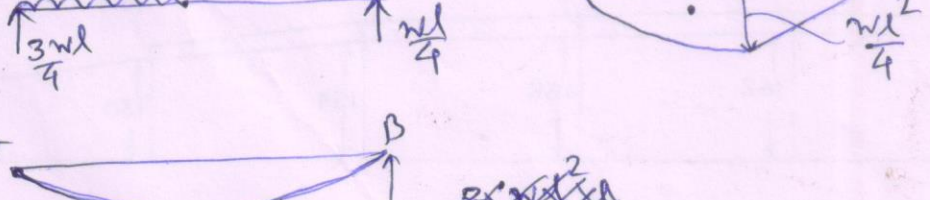
Let us assume that instead of making it a cantilever, we make it a simply supported beam, i.e., after removing 2 off constraints.



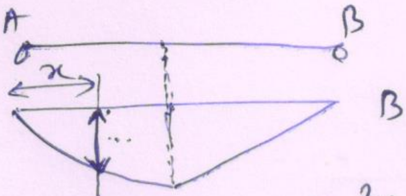
i.e., here the bending constraint at the fixed end and the intermediate support have been removed.

$\therefore$ 

To find Δ_1, Δ_2 we use the Moment-area method.



$=$ moment of area of $\frac{M}{EI}$ diagram w.r.t. "B".



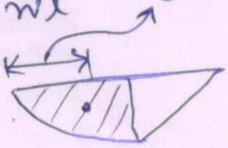
$\left(\frac{3}{4}wlx - \frac{wx^2}{2}\right)$

BM diagram.

$\therefore$ location of COG of this left half BM diagram

$$= \frac{\int_0^l \left(\frac{3}{4}wlx - wx^2/2\right)x\,dx}{\int_0^l \left(\frac{3}{4}wlx - \frac{wx^2}{2}\right)dx} =$$

$$= \frac{\left[\frac{3}{4}wl\frac{x^3}{3} - \frac{w}{2}\frac{x^4}{4}\right]_0^l}{\left[\frac{3}{4}wl\frac{x^2}{2} - \frac{w}{2}\frac{x^3}{3}\right]_0^l} = \frac{\frac{3}{4}\times\frac{wl^4}{3} - \frac{wl^4}{8}}{\frac{3}{8}wl^3 - \frac{w}{6}l^3} = \frac{\frac{wl^4}{8}}{\frac{5}{24}wl^3} = \frac{3}{5}l$$



total area of the left half $= \frac{5}{24}wl^3$

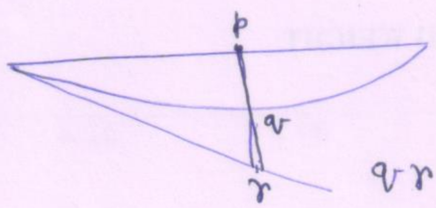
$\therefore$

$$= \left[\left[\frac{5}{24}wl^3\times\left(l+\frac{2l}{5}\right)\right] + \left[\frac{1}{2}\times\frac{wl^2}{4}\times l\times\frac{2}{3}l\right]\right]\times\frac{1}{EI}$$

$$= \left[\frac{5}{24}\times\frac{7}{5} + \frac{1}{12}\right]\frac{wl^4}{EI} = \frac{9}{24}\frac{wl^4}{EI} = \frac{3}{8}\frac{wl^4}{EI}$$

$$\therefore \Delta_1 = \frac{3}{8}\frac{wl^4}{EI}\times\frac{1}{2l} = \frac{3}{16}\frac{wl^3}{EI}$$

For Δ_2,

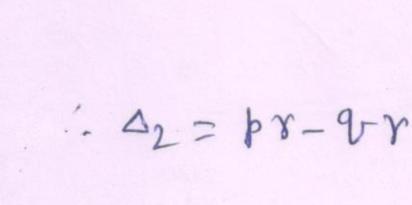


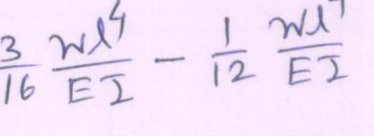
$$pr = \frac{3}{16}\frac{wl^3}{EI}\times l = \frac{3}{16}\frac{wl^4}{EI}$$

$$qr = \frac{1}{EI}\left(\frac{5}{24}wl^3\times\frac{2}{5}l\right) = \frac{1}{12}\frac{wl^4}{EI}$$

$$\therefore \Delta_2 = pr - qr = \frac{3}{16}\frac{wl^4}{EI} - \frac{1}{12}\frac{wl^4}{EI} = \left(\frac{9-4}{48}\right)\frac{wl^4}{EI} = \frac{5}{48}\frac{wl^4}{EI}$$

Now, find the flexibilities:

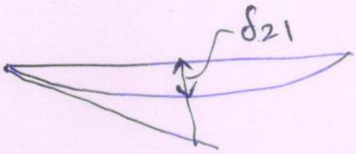


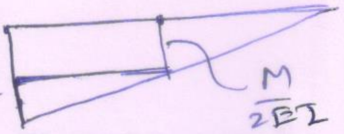


$$\delta_{11} = \left(\frac{M}{EI}\times\frac{1}{2}\times 2l\right)\times\frac{2}{3}\times\frac{2l}{2l}$$

$$\therefore \delta_{11} = \frac{2}{3}\frac{l}{EI} \quad [\because M = 1]$$

$$\therefore f_{11} = \frac{2}{3}\frac{l}{EI}$$



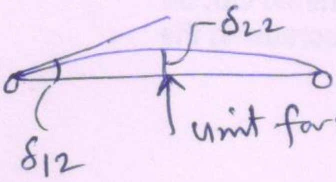


$$\therefore \delta_{21} = \frac{2}{3}\frac{Ml}{EI}\times l - \left(\frac{M}{2EI}l\cdot\frac{l}{2} + \frac{1}{\cancel{2}}\cdot\frac{M}{2EI}l\cdot\frac{\cancel{2}}{3}l\right)$$

$$= \frac{2}{3}\frac{Ml^2}{EI} - \left(\frac{1}{4}+\frac{1}{6}\right)\frac{Ml^2}{EI} = \left(\frac{2}{3}-\frac{5}{12}\right)\frac{Ml^2}{EI} = \frac{1}{4}\frac{Ml^2}{EI}$$

with $M = 1$ (unit) $\delta_{21} = \frac{1}{4}\frac{l^2}{EI}$

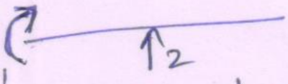
$$\therefore f_{21} = \frac{l^2}{4EI}$$



we know that $f_{ij} = f_{ji}$

$$\therefore f_{12} = f_{21} = \frac{l^2}{4EI}$$

$$f_{22} = \frac{(2l)^3}{48EI} = \frac{l^3}{6EI}$$

Note that here δ_{12} is anti-clockwise. Infact when unknowns were introduced, it was taken as  so, we took moment at 1 clockwise. But in the given case it should have been counter-clockwise. 

$\therefore$ we can write the equations as:

at (1) $\frac{2}{3}\frac{l}{EI}R_1 - \frac{l^2}{4EI}R_2 = -\frac{3}{16}\frac{wl^3}{EI}$

at (2) $\frac{l^2}{4EI}R_1 - \frac{l^3}{6\cancel{48}EI}R_2 = -\frac{5}{48}\frac{wl^4}{EI}$

$$\begin{bmatrix} 2l/3 & -l^2/4 \\ -l^2/4 & l^3/6\cancel{8} \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \frac{3wl^3}{16} \\ +\frac{5wl^4}{48} \end{Bmatrix}.$$

-1 multiplied to lower row to show that it is symmetric. It was unnecessary.

$$\therefore \begin{bmatrix} 2/3 & -l/4 \\ -1/4 & l/6\cancel{8} \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \frac{3}{16}wl^2 \\ +\frac{5}{48}wl^2 \end{Bmatrix}$$

multiply 3/8 to the 1st row and add to 2nd row, we get:

$$\begin{bmatrix} 2/3 & -l/4 \\ 0 & +\frac{7}{96}l \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \frac{3}{16}wl^2 \\ +\frac{13}{384}wl^2 \end{Bmatrix} \Rightarrow R_2 = +\frac{13}{\cancel{384}_4}\times\frac{\cancel{96}}{7}wl$$

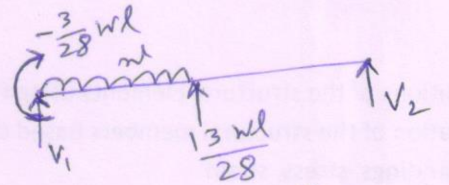
$$= \frac{13}{28}wl$$

Now, $\frac{2}{3}R_1 - \frac{l}{4}R_2 = -\frac{3}{16}wl^2$ or $\frac{2}{3}R_1 - \frac{l}{4}\times\frac{13}{28}wl = -\frac{3}{16}wl^2$

or $\frac{2}{3}R_1 = \left(-\frac{3}{16} + \frac{13}{112}\right)wl^2 = \frac{-21+13}{112} = \frac{-8}{112}wl^2$

$\therefore R_1 = -\frac{8\times 3}{2\times \cancel{112}_{14}}wl^2 = -\frac{3}{28}wl^2$

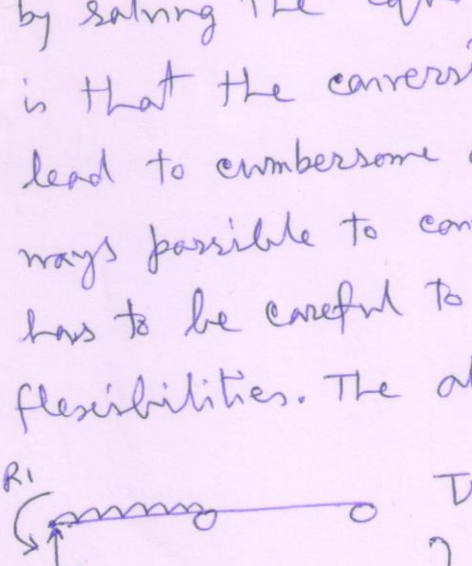
The -ve sign indicate that the moment at the support is counter-clockwise.



Now using equation of statics, other unknown V_1, V_2 can be calculated. You can use the previous results to check if R_1, R_2 are matching or not.

Advantage of flexibility/force based matrix method is that the unknowns are the forces which are directly found out by solving the equation $[f]\{R\}=\{\Delta\}$. However, disadvantage is that the conversion to statically determinate problem can lead to cumbersome algebra. Since there are many different ways possible to convert it into a determinate problem, user has to be careful to choose the one with easier estimation of flexibilities. The above problem can also be converted as:

Try to solve it on your own.



solve these with flexibility based approach.