Lecture - 13

Marine Hydrodynamics

6. Dynamic free surface boundary condition

Dynamic free surface condition implies the pressure of water and air at free surface must be same.

which implies

$$\frac{p_w}{\rho} - \left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + g\eta\right] = \frac{p_a}{\rho}$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + g\eta = 0 \quad \text{at} \quad z = \eta$$

..... (6.1)

7. <u>Linearization</u> :

dropping the quadratic term from (5.6) we get

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}, \quad \text{---} \quad \text{at} \quad z = 0 \rightarrow (7.1)$$

dropping the quadratic term from (6.1) gives

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \ldots \quad \text{at } z = 0$$

... (7.2)

(7.1) and (7.2) gives the linear free surface kinematic condition (KFSC) and Dynamic free surface condition (DFSC) respectively.

~~Fu the~~ <u>Further</u> in case of linear model.

(7.1) and (7.2) may be combined as follows :-

Differentiating (7.2) with respect to 't' we get

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \eta}{\partial t} = 0 \quad \text{at} \quad z = 0 \qquad \ldots (7.3)$$

Substituting (7.1) to (7.3) we get

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = 0} \qquad \ldots (7.4)$$

(7.4) represents the linearized free surface boundary condition.

8. <u>Solution for the velocity potential ϕ</u>

Let us assume the trial solution for ϕ as

$\phi_I(x, y, z, t) = P(z) \sin(Kx - \omega t)$ in which $P(z)$ is unknown. ϕ_I must satisfy

(i) Laplace equation

(ii) KFSBC (Linear)

(iii) DFSBC (Linear)

(iv) Bottom boundary condition.

<u>Now</u> $\nabla^2 \phi_I = 0$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \ldots (8.1)$$

<u>Now</u>, ~~we assu~~ wave doesn't propagates in the direction of ~~z~~ y

$\Rightarrow$ (8.1) reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

where, in (8.2), the expression for 'c' is not known.

Now dynamic free-surface condition gives

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = 0$$

$$\Rightarrow -c\omega \cosh k(h+z) \overset{\cos}{\cancel{\sin}} (kx - \omega t) + ga\cos(kx - \omega t) = 0$$

at $z = 0$

$$\Rightarrow -c\omega \cosh(kh) \cos(kx - \omega t) + ga\cos(kx - \omega t) = 0$$

$$\Rightarrow -c\omega \cosh(kh) + ga = 0$$

$$\Rightarrow c = \frac{ag}{\omega} \cdot \frac{1}{\cosh(kh)}$$

$$\Rightarrow \boxed{\phi_I = \frac{ag}{\omega} \frac{\cosh k(h+z)}{\cosh(kh)} \sin(kx - \omega t)} \quad \text{— (8.3)}$$

(8.3) gives you the expression for the velocity potential of fluid particle.

Now from kinematic free surface condition, we get

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at } z = 0$$

$$\Rightarrow \frac{agk}{\omega} \frac{\sinh k(h+z)}{\cosh(kh)} \sin(kx - \omega t)$$

$$= a\omega \sin(kx - \omega t)$$

at $z = 0$

$$\Rightarrow \quad -k^2 P(z)\sin(kx-\omega t) + \frac{d^2P}{dz^2}\sin(kx-\omega t) = 0$$

$$\Rightarrow \quad \frac{d^2P}{dz^2} - k^2 P(z) = 0$$

$$\Rightarrow \quad P(z) = c_1 e^{kz} + c_2 e^{-kz}$$

$$\therefore \quad P(z) = (c_1 e^{kz} + c_2 e^{-kz}) \sin(kx-\omega t).$$

Now bottom boundary condition give.

$$\frac{\partial \phi_I}{\partial z} = 0 \quad \text{for} \quad z = -h$$

$$\Rightarrow \quad k\left(c_1 e^{kz} - c_2 e^{-kz}\right)\Big|_{z=-h} = 0$$

$$\Rightarrow \quad c_1 e^{-kh} - c_2 e^{kh} = 0$$

$$\Rightarrow \quad c_1 e^{-kh} = c_2 e^{kh} = \frac{c}{2} \quad [\text{say}]$$

$$\therefore \quad c_1 = \frac{c}{2} e^{kh}$$

$$c_2 = \frac{c}{2} e^{-kh}$$

$$\Rightarrow \quad \phi_I = \frac{c}{2}\left[e^{kh} e^{kz} + e^{-kh} \cdot e^{-kz}\right] \sin(kx-\omega t)$$

$$\Rightarrow \quad \phi_I = c\left[\frac{e^{k(h+z)} + e^{-k(h+z)}}{2}\right] \sin(kx-\omega t)$$

$$\Rightarrow \quad \phi_I = c \cdot \cosh k(h+z) \sin(kx-\omega t) \quad \ldots\ldots (8.2)$$

$$\Rightarrow \frac{gk}{\omega} \frac{\sin(kh)}{\cos(kh)} = \omega \quad \text{at } z=0$$

$$\Rightarrow \boxed{\omega^2 = gk \tanh(kh)} \quad \text{at } z=0 \quad \ldots\ldots (8.4)$$

(8.4) represents the dispersion relation. This is extreemly important relation in water wave.

Now we already know that, the phase velocity

$$c = \frac{\lambda}{T} \quad \text{now, take } \left.\begin{array}{l} \lambda = \frac{2\pi}{k} \\ \text{and } T = \frac{2\pi}{\omega} \end{array}\right\} (8.5)$$

$$\Rightarrow c = \frac{\omega}{k} \quad \text{using } (8.5)$$

Now, (8.4) may be re-written as

$$\frac{\omega^2}{k^2} = \frac{g}{k} \tanh(kh)$$

$$\text{or} \quad \frac{\omega}{k} = \frac{g}{\omega} \tanh(kh)$$

$$\text{or} \quad c = \frac{g}{\omega} \tanh(kh) \quad \ldots (8.6)$$

Now in (8.6), g and tanh(kh) is constant,

∴ one can easily derive

$$\boxed{c \propto \frac{1}{\omega}} \quad \text{or} \quad \boxed{c \propto T} \quad \rightarrow (8.7)$$

it means, since the phase velocity is directly proportion to time period, it means if the time period is large, wave phase velocity is more

This implies that tsunami travel ~~much~~ much faster in ~~comparision~~ compare to other wind generated wave.

9. Deep and Shallow water approximation.

Now from dispersion relation, we get

$$\omega^2 = gk \tanh(kh)$$

in case of deep water $\tanh(kh) \rightarrow 1$

$$\Rightarrow \omega^2 = gk \quad \ldots\ldots \quad (9.1)$$

Now substitute $\omega = \frac{2\pi}{T}$ and $k = \frac{2\pi}{\lambda}$ we get

$$\frac{4\pi^2}{T^2} = g \cdot \frac{2\pi}{\lambda}$$

$$\Rightarrow 2\pi\lambda = gT^2$$

$$\Rightarrow \lambda = \frac{g}{2\pi} T^2$$

Now substitute $g = 9.81\ m/s^2$, $2\pi = 2 \times 3.14 = 6.28$

we get $\boxed{\lambda = 1.56\, T^2}$ ... (9.2)

$$\Rightarrow \boxed{\lambda \propto T^2}$$

which implies wave length is proportion to square of the time period. In deep water region, it is equal to $\boxed{1.56 \times T^2}$

Again $\omega^2 = gk \tanh(kh)$

for shallow water region, when $h \to 0$, ~~tanh~~

$\tanh(kh) \to kh$.

$$\Rightarrow \quad \omega^2 = gk^2 . h$$

$$\Rightarrow \quad \frac{\omega^2}{k^2} = gh$$

or $$\boxed{c = \sqrt{gh}}$$

Now, interestingly, in shallow water region, phase velocity is independent of wave frequency.

$\Rightarrow$ all frequency waves travells in same phase velocity, which is ~~so~~ true for other mode of wave [light wave / sound wave etc].

10. expression of the velocity potential in case of deep water. :

$$\phi_I = \frac{ag}{\omega} \frac{\cosh(kh + kz)}{\cosh(kh)} \sin(kx - \omega t)$$

Now $\cosh x = \frac{1}{2}\left[e^{x} + e^{-x}\right]$

$$\therefore \cosh k(z+h) = \frac{1}{2}\left[e^{k(z+h)} + e^{-k(z+h)}\right]$$

$$= \frac{1}{2}\left[e^{kz}.e^{kh} + e^{-kz}.e^{-kh}\right]$$

Now for $h \to \infty$ $e^{-kz}.e^{kh}$ is ignorable and $\cosh k(h+z) = \frac{1}{2} e^{k(z+h)}$

Similarly $\cosh(kh) = \frac{1}{2}\left[e^{kh} + e^{-kh}\right] \simeq \frac{1}{2} e^{kh}$.

using this approximation, we get

$$\boxed{\phi_I = \frac{ag}{\omega} e^{kz} \sin(kx - \omega t)} \quad \ldots\ldots (10.1)$$

11. <u>water particle kinematics</u>:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left[\frac{ag}{\omega} e^{kz} \sin(kx - \omega t)\right]$$

$$\boxed{u = \frac{agk}{\omega} e^{kz} \cos(kx - \omega t)} \quad \text{-- } (11.1)$$

$$\boxed{w = \frac{\partial \phi}{\partial z} = \frac{agk}{\omega} e^{kz} \sin(kx - \omega t)} \longrightarrow (11.2)$$

<u>substitute</u> $\omega^2 = gk \Rightarrow \frac{\omega}{k} = \frac{g}{\omega}$ in (11.1) & (11.2)

<u>we get</u>:

$$\boxed{\begin{aligned} u &= a\omega e^{kz} \cos(kx - \omega t) \\ w &= a\omega e^{kz} \sin(kx - \omega t) \end{aligned}} \quad \ldots\ldots (11.3)$$

Accelaration

$$\dot{u} = a\omega k e^{kz} \overset{\cos}{\cancel{\sin}}(kx - \omega t)$$

$$\dot{w} = a\omega k e^{kz} \cos(kx - \omega t).$$

for any time instant :