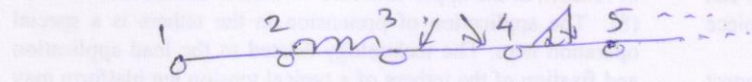
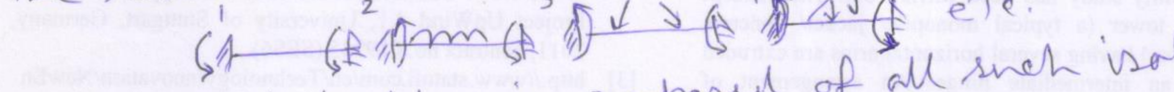
In stiffness/displacement based formulation of matrix methods, the displacements are taken as unknowns and stiffness at various points are used. When stiffness and displacements are multiplied, we get force. Hence, in this method, equations are written for equilibrium consideration at a point joint.

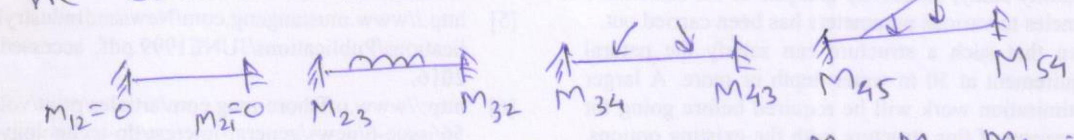


We can now introduce the idea of an element, which is in this case, the span of a beam with ends are fixed to isolate from the rest.



The actual problem is composed of all such isolated spans.

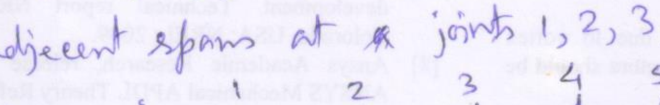
First we estimate the individual fixed end moments.



$M_1 = M_{12}$ $\quad M_2 = M_{21} - M_{23}$ $\quad M_3 = M_{32} - M_{34}$ $\quad M_4 = M_{43} - M_{45}$ - etc.

$M_1, M_2, M_3,$ etc. are the resultant of moment between two adjacent spans at joints 1, 2, 3, 4, ... etc. (applied loads)

Now the beam is



We assume the rotations of the joints 1, 2, 3, 4 ... as $\delta_1, \delta_2, \delta_3, \delta_4, \ldots$

∴ Now the equilibrium of joint 1 can be written as

$$K_{11}\delta_1 + K_{12}\delta_2 + K_{13}\delta_3 + K_{14}\delta_4 + \cdots = M_1$$

Note that this part is unnecessary as $\delta_3, \delta_4, \ldots$ do not affect equilibrium at 1, since they are totally isolated from 1

Joint 2: $K_{21}\delta_1 + K_{22}\delta_2 + K_{23}\delta_3 + K_{24}\delta_4 + K_{25}\delta_5 + \cdots = M_2$

this part is unnecessary.

Joint 3: $K_{31}\delta_1 + K_{32}\delta_2 + K_{33}\delta_3 + K_{34}\delta_4 + K_{35}\delta_5 + \cdots = M_3$

here again, at joint 3, there is no influence of $\delta_1, \delta_5, \delta_6, \ldots$

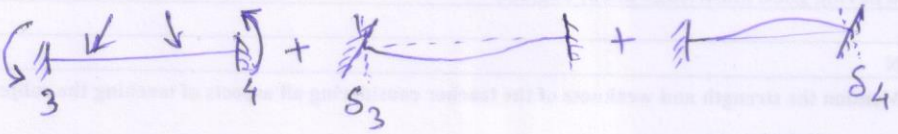
Thus we get "n" equations with "n" unknown

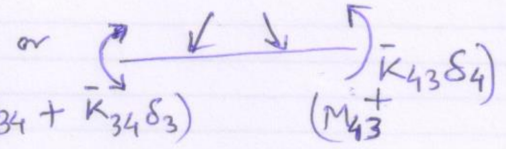
So, the equilibrium equation becomes

$$[K]\{\delta\} = \{M\}$$

We can solve it by standard methods like Gauss elimination.

Once the displacements are known, we take each span to get their end forces, i.e.,

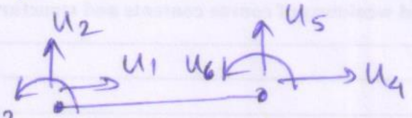




Since the end forces are known, the same can be calculated at any point in the span.

$\bar{K}_{34}, \bar{K}_{43}$ are stiffness of isolated span.

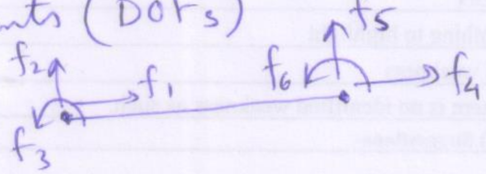
In the above example, only the rotational degree of freedom has been considered. Let us now consider a simple 2D beam elements having 2 nodes at the ends. Each node is having 3 degrees of freedom (DOF)



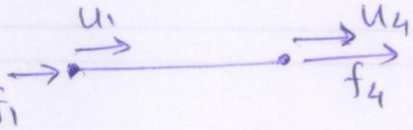
Here we take $\curvearrowright +$ to be consistent with most of the text books $\uparrow + \downarrow$

here, $\{u\}$ = nodal displacements (DOFs)

Forces acting in each DOF



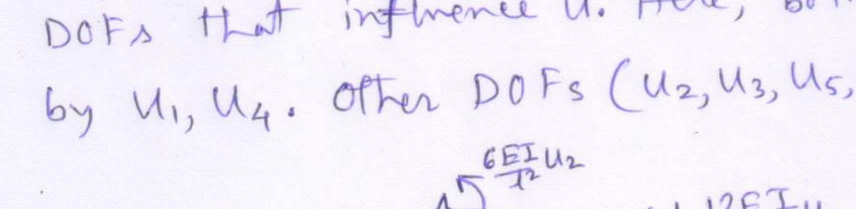
The equilibrium of the beam is:



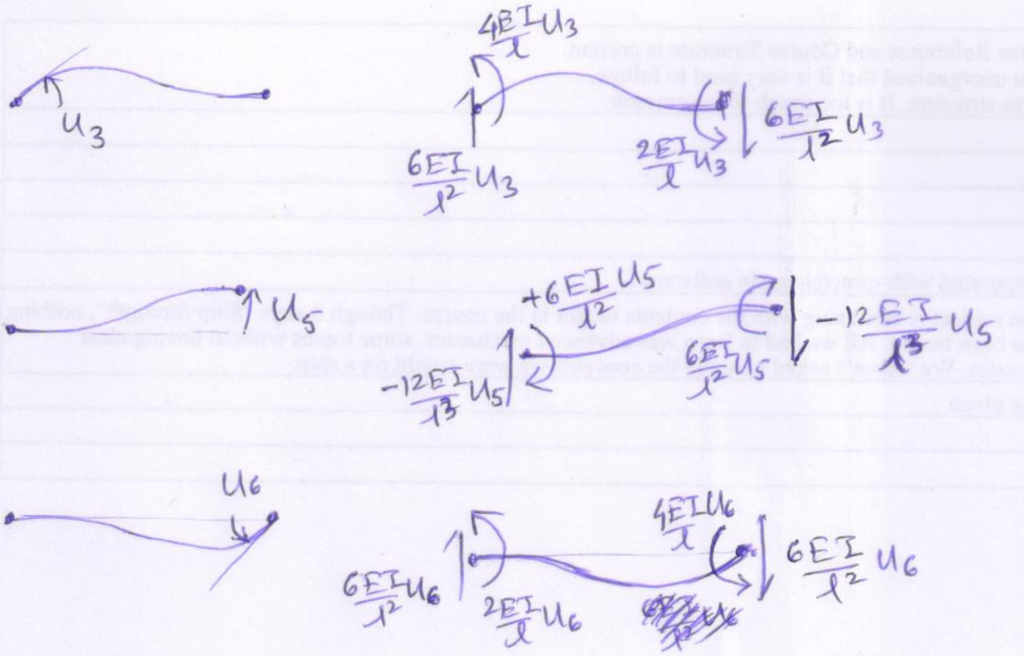
$$f_1 = \frac{EA}{l}u_1 - \frac{EA}{l}u_4$$

$$f_4 = \frac{EA}{l}u_4 - \frac{EA}{l}u_1$$

So, equilibrium at a DOF must include effects of all DOFs that influence it. Here, both f_1, f_4 are influenced by u_1, u_4. Other DOFs (u_2, u_3, u_5, u_6) do not affect f_1, f_4.



So, we see that u_2 will affect f_2, f_3, f_5, f_6



Equilibrium of f_2 can be written as;

$$f_2 = \frac{12EI}{l^3}u_2 + \frac{6EI}{l^2}u_3 - \frac{12EI}{l^3}u_5 + \frac{6EI}{l^2}u_6$$

similarly, $$f_3 = \frac{6EI}{l^2}u_2 + \frac{4EI}{l}u_3 - \frac{6EI}{l^2}u_5 + \frac{2EI}{l}u_6$$

–ve sign because it is clockwise

$$f_5 = -\frac{12EI}{l^3}u_2 - \frac{6EI}{l^2}u_3 + \frac{12EI}{l^3}u_5 - \frac{6EI}{l^2}u_6 \quad \text{(upward +ve)}$$

$$f_6 = \frac{6EI}{l^2}u_2 + \frac{2EI}{l}u_3 - \frac{6EI}{l^2}u_5 + \frac{4EI}{l}u_6$$

∴ we get,
$$\underbrace{\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix}}_{6\times1} = \underbrace{\begin{bmatrix} EA/l & 0 & 0 & -EA/l & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -EA/l & 0 & 0 & EA/l & 0 & 0 \\ 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}}_{6\times6} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}}_{6\times1}$$

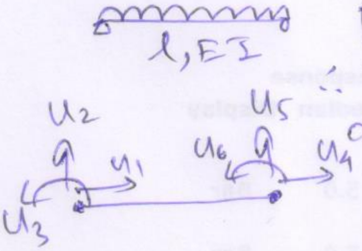
Element Stiffness matrix

$\{f\} = [K_e]\{u\}$, $[K_e]$ is symmetric, $K_{ij} = K_{ji}$

At the moment, it is only an elemental stiffness matrix, not a structure with ~~boundation~~ boundary condition, i.e, it is singular. After imposing BC's, it becomes appropriate structure.

The size of the stiffness matrix depends on the total number degrees of freedom.

Solve the following problem:

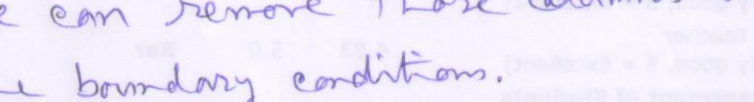


$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

Here, u_1, u_2, u_4, u_5 are restrained, corresponding equations are not required, those rows can be removed. Also see that all elements in 1st column are multiplied with u_1, similarly all elements in 2nd column are multiplied with u_2, and so on. We can remove those columns as well. Thus we impose the boundary conditions.

$$\therefore \begin{Bmatrix} f_3 \\ f_6 \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_6 \end{Bmatrix}$$

considering the fixed end moments



these moments the supports are applying on the beam.

$\therefore$ The moment the beam is applying on the support/surrounding are the opposite of the previous.

$= \quad -\frac{wl^2}{12} \qquad \frac{wl^2}{12}$ These moment will cause deformations in the structural system.

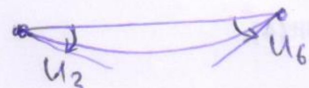
$$\therefore \begin{Bmatrix} f_3 \\ f_6 \end{Bmatrix} = \begin{Bmatrix} -\frac{wl^2}{12} \\ \frac{wl^2}{12} \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_6 \end{Bmatrix}$$

multiply $-\frac{1}{2}$ to the 1st row and add to second row:

$$\begin{Bmatrix} -\frac{wl^2}{12} \\ \frac{wl^2}{8} \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ 0 & \frac{3EI}{l} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_6 \end{Bmatrix}$$

$$\therefore u_6 = \frac{wl^2}{8}\cdot\frac{l}{3EI} = \frac{wl^3}{24EI}\ ; \quad u_3 = -\frac{wl^3}{24EI}$$

* check these values by other method

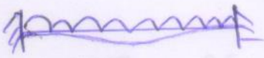


u_3 = clockwise = −ve

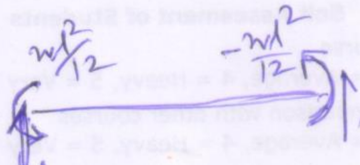
$\therefore$ element end forces due to these displacements u_3, u_6:

$$\{f\}_{6\times1} = [K]_{6\times6}\begin{Bmatrix}0\\0\\-\frac{wl^3}{24EI}\\0\\0\\\frac{wl^3}{24EI}\end{Bmatrix}_{6\times1} = \begin{Bmatrix}0\\-\frac{wl}{4}+\frac{wl}{4}\\-\frac{wl^2}{12}\\0\\0\\\frac{wl^2}{12}\end{Bmatrix} = \begin{Bmatrix}0\\0\\-\frac{wl^2}{12}\\0\\0\\\frac{wl^2}{12}\end{Bmatrix} = \{f\}_{disp}$$

These has to be super imposed with fixed end forces, because the final forces = Fixed end forces + force due to end displacements. (here u_3, u_6)



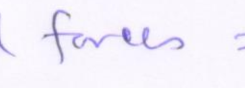
Fixed end forces =



$$= \begin{Bmatrix}0\\ wl/2\\ \frac{wl^2}{12}\\ 0\\ wl/2\\ -\frac{wl^2}{12}\end{Bmatrix} = \{f\}_{FE}$$

$$\therefore \text{total end forces} = \{f\}_{FE} + \{f\}_{disp} = \begin{Bmatrix}0\\ wl/2\\ 0\\ 0\\ wl/2\\ 0\end{Bmatrix}$$

which is correct for a simply supported beam.



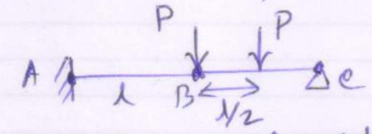
Hence, the total process can be summed up as:

1. Find out fixed end forces $\{f\}_{FE}$
2. Find out how much displacements happening at the ends, $\{u\}$

3. Find out how much end forces will be created due to such displacements $\{f\}_{disp}$

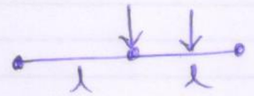
4. Add up together algebraically to get the final forces at the ends $\{f\} = \{f\}_{FE} + \{f\}_{disp}$.

Consider the following problem:



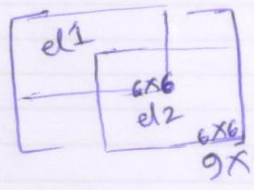
we want to find the displacement at B.

This problem should be considered as composed of 2 elements.

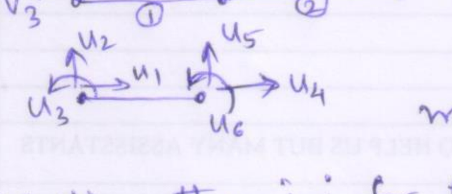


For a problem, we define the DOFs in the global co-ordinates; while for an individual element, we call it local DOFs in local coordinate system.

So, total 9 DOFs exists for this structure. Size of the stiffness matrix = 9×9.

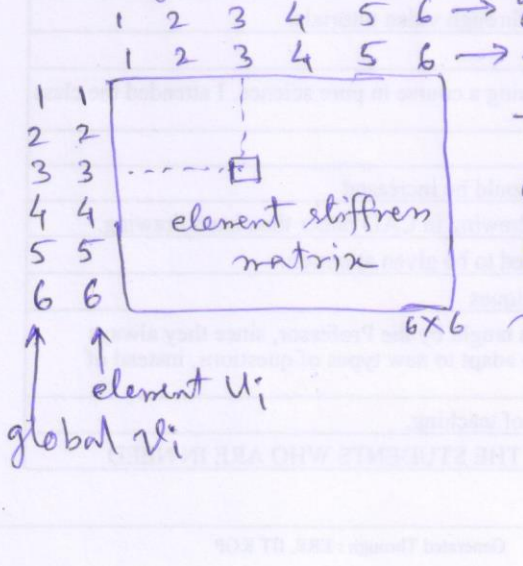


So, two element stiffness matrix of 6×6 size will sit in the global stiffness matrix with an overlap. This overlap will correspond to the connectivity between the elements.



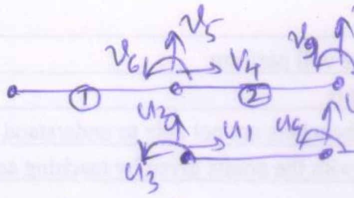
Comparing the 1st element of the structure with the element stiffness matrix, we can see directly $v_1 = u_1$, $v_2 = u_2$, … $v_6 = u_6$.

∴ Writing the i, j for the element stiffness matrix, we get

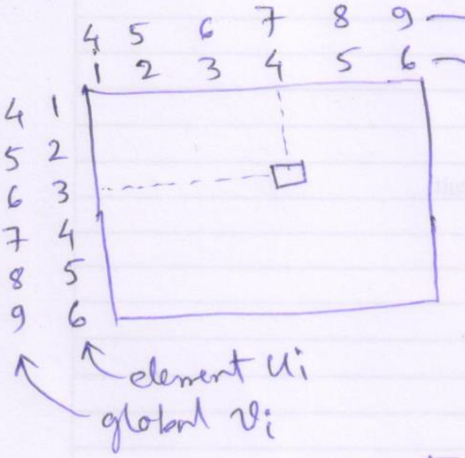


This shows that for an element $\square_{3,3}$ of the element stiffness matrix will be placed at (3,3) in the global stiffness matrix, and so on….

For the 2nd element

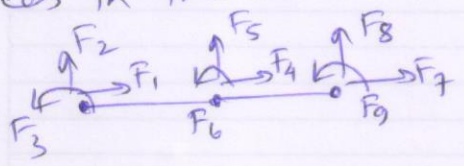


So using the same element stiffness matrix, we can directly compare and find that $u_1 = v_4,\ u_2 = v_5,\ u_3 = v_6,\ u_4 = v_7,$
$u_5 = v_8,\ u_6 = v_9$



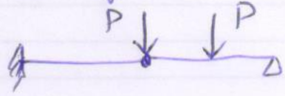
Hence the element $\square_{(3,4)}$ from local system will be placed at $(6,7)$ in the global. Thus the whole 9×9 stiffness matrix can be formulated.

Forces in the structure (global) is represented by $\{F\}$



$$\{F\} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \\ \uparrow \text{global} \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \to \text{global} \\ \left[\begin{array}{ccccccccc}
\frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & 0 & 0 \\
0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} & 0 & 0 & 0 \\
-\frac{EA}{l} & 0 & 0 & \left(\frac{EA}{l}+\frac{EA}{l}\right) & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\
0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \left(2\times\frac{12EI}{l^3}\right) & \left(-\frac{6EI}{l^2}+\frac{6EI}{l^2}\right) & & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\
0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & \left(-\frac{6EI}{l^2}+\frac{6EI}{l^2}\right) & \left(\frac{4EI}{l}\times 2\right) & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\
0 & 0 & 0 & -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & \\
0 & 0 & 0 & 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\
0 & 0 & 0 & 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l}
\end{array}\right] \end{array} \{v\}$$

considering the boundary conditions of the given problem

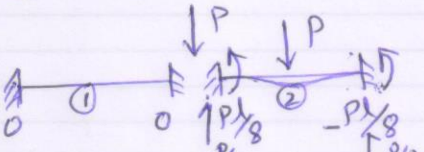


the stiffness matrix and the equation becomes:

v_1, v_2, v_3, v_7, v_8 are restrained, only v_4, v_5, v_6, v_9 exists.

$$\therefore \begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_9 \end{Bmatrix} = \begin{bmatrix} \frac{2EA}{l} & 0 & 0 & 0 \\ 0 & \frac{24EI}{l^3} & 0 & \frac{6EI}{l^2} \\ 0 & 0 & \frac{8EI}{l} & \frac{2EI}{l} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} v_4 \\ v_5 \\ v_6 \\ v_9 \end{Bmatrix}$$

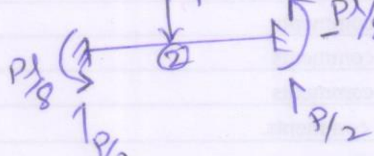
$\{F\}$ is the externally applied forces to be obtained from fixed end condition.



span ① has no load. Load P is applied at node 2 directly.

$\therefore F_4 = 0$ (no axial load), $F_5 = -P - \frac{P}{2} = -\frac{3P}{2}$

$F_6 = -\frac{Pl}{8}$ ($\because -1 \times$ Fixed end force)



Fixed end forces

$F_9 = \frac{Pl}{8}$

$$\therefore \{F\} = \begin{Bmatrix} 0 \\ -\frac{3}{2}P \\ -\frac{Pl}{8} \\ \frac{Pl}{8} \end{Bmatrix} = \begin{bmatrix} \frac{2EA}{l} & 0 & 0 & 0 \\ 0 & \frac{24EI}{l^3} & 0 & \frac{6EI}{l^2} \\ 0 & 0 & \frac{8EI}{l} & \frac{2EI}{l} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} v_4 \\ v_5 \\ v_6 \\ v_9 \end{Bmatrix}$$

multiply $-\frac{l}{4}$ to the 2nd row and add to last row $\Rightarrow$

$$\begin{bmatrix} \frac{2EA}{l} & 0 & 0 & 0 \\ 0 & \frac{24EI}{l^3} & 0 & \frac{6EI}{l^2} \\ 0 & 0 & \frac{8EI}{l} & \frac{2EI}{l} \\ 0 & 0 & \frac{2EI}{l} & \frac{5}{2}\frac{EI}{l} \end{bmatrix} \begin{Bmatrix} v_4 \\ v_5 \\ v_6 \\ v_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\frac{3}{2}P \\ -\frac{Pl}{8} \\ \frac{Pl}{2} \end{Bmatrix}$$

multiply $-\frac{1}{4}$ to the 3rd row and add to last row ⇒

$$\begin{bmatrix} \frac{2EA}{l} & 0 & 0 & 0 \\ 0 & \frac{24EI}{l^3} & 0 & \frac{6EI}{l^2} \\ 0 & 0 & \frac{8EI}{l} & \frac{2EI}{l} \\ 0 & 0 & 0 & \frac{2EI}{l} \end{bmatrix} \begin{Bmatrix} v \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\frac{3}{2}P \\ -\frac{Pl}{8} \\ \frac{17}{32}Pl \end{Bmatrix}$$

$$\therefore \; v_9 = \frac{17}{32} Pl \times \frac{l}{2EI} = \frac{17 Pl^2}{64 EI}, \quad v_4 = 0$$

$$\frac{8EI}{l} v_6 + \frac{2EI}{l} v_9 = -\frac{Pl}{8} \quad \text{or} \quad \frac{8EI}{l} v_6 + \frac{2E\cancel{I}}{\cancel{l}} \cdot \frac{17 Pl^{\cancel{2}}}{64 E\cancel{I}} = -\frac{Pl}{8}$$

$$\text{or} \quad \frac{8EI}{l} v_6 = -\frac{Pl}{8} - \frac{17}{32} Pl = -\frac{21}{32} Pl$$

$$\therefore \; v_6 = -\frac{21}{256} \frac{Pl^2}{EI}$$

for v_5,

$$\frac{24EI}{l^3} v_5 + \frac{6E\cancel{I}}{l^{\cancel{2}}} \cdot \frac{17 Pl^{\cancel{2}}}{64 E\cancel{I}} = -\frac{3}{2} P$$

$$\frac{24EI}{l^3} v_5 = -\frac{3}{2}P - \frac{51}{32}P = -\left(\frac{48+51}{32}\right)P = -\frac{99}{32}P$$

$$v_5 = -\frac{99}{768} \frac{Pl^3}{EI}$$

If we want to find the end forces for element 1,

end forces due to displacements :

$$\{f\}_{FF} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \quad \underset{6\times1}{\{f\}_{disp}} = \underset{6\times6}{\left[K_{e\_1} \right]} \underset{6\times1}{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{99}{768}\frac{Pl^3}{EI} \\ -\frac{21}{256}\frac{Pl^2}{EI} \end{Bmatrix}} \quad \begin{matrix} u_1 = v_1 \\ u_2 = v_2 \\ \vdots \\ \vdots \\ u_6 = v_6 \end{matrix}$$

$\therefore$ final end forces for element 1

$$\{f\}_{end\ force\_1} = \{f\}_{FE} + \{f\}_{disp}$$

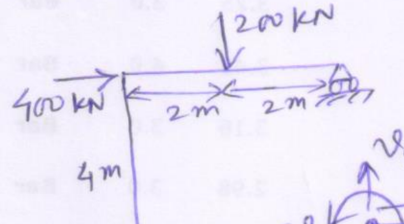
For the 2nd element:

$$\{f\}_{FE} = \begin{Bmatrix} 0 \\ P/2 \\ Pl/8 \\ 0 \\ P/2 \\ -Pl/8 \end{Bmatrix} ; \quad \{f\}_{\substack{disp \\ 6\times1}} = \Big[Ke_2 \Big]_{6\times6} \begin{Bmatrix} 0 \\ -\frac{99}{768}\frac{Pl^3}{EI} \\ -\frac{21}{256}\frac{Pl^2}{EI} \\ 0 \\ 0 \\ \frac{17}{64}\frac{Pl^2}{EI} \end{Bmatrix}$$

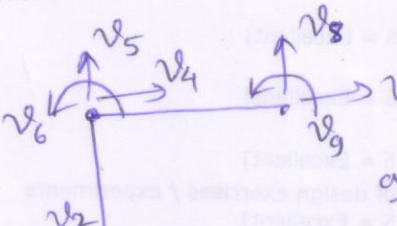
<u>using</u>
$u_1 = v_4$
$u_2 = v_5$
$u_3 = v_6$
$\vdots$
$u_6 = v_9$

$$\{f\}_{total} = \{f\}_{FE} + \{f\}_{disp}$$

Now consider the following problem:



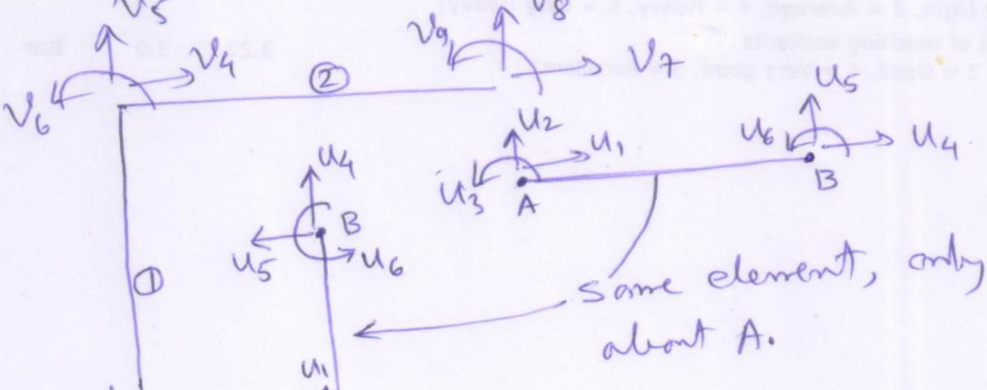
given, $E = 2\times10^7\ kN/m^2$, $I = 12\times10^{-5}\ m^4$, $A = 0{\cdot}03\ m^2$



9 DOFs of the whole structure.

$\therefore$ size of global stiffness matrix $= 9\times9$.

$$\{F\}_{9\times1} = [K]_{9\times9}\{v\}_{9\times1}$$



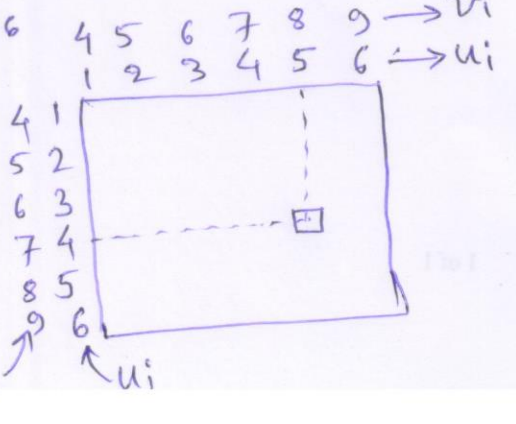
Same element, only rotated by 90° (anticlockwise) about A.

Comparing element ② with the basic element, we can see that

$v_4 = u_1,\ v_5 = u_2,\ v_6 = u_3,\ \ldots,\ v_9 = u_6$

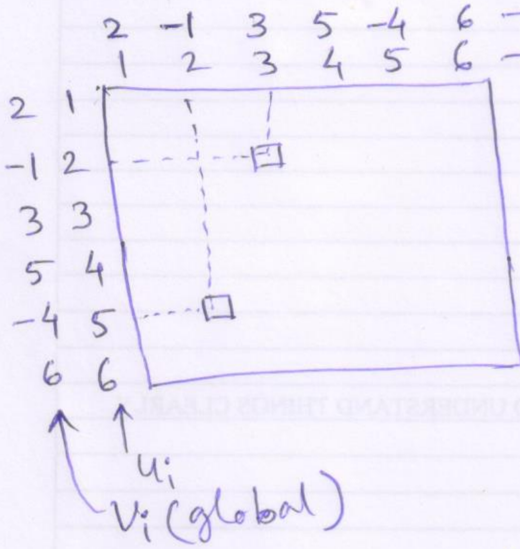
Hence the elements will be placed as

$\therefore$ element $\square(4,5)$ from local matrix will be placed at $(7,8)$ in global.



Comparing element ① of structure to the basic element (vertical)

we see $v_1 = -u_2$, $v_2 = u_1$, $v_3 = u_3$, $v_4 = -u_5$, $v_5 = u_4$, $v_6 = u_6$



$\rightarrow v_i$ (global) Here, sign is used on v.

$\rightarrow u_i$ (local) we could also use in on u instead of v.

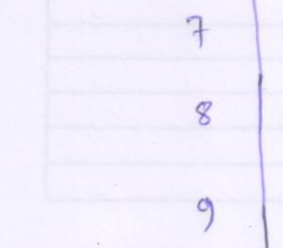
Here, the element $\square(2,3)$ of local will be placed at $(1,3)$ with a "−ve" sign.

element $\square(5,2)$ of local stiffness matrix will be placed at $(4,1)$ in global. No "−ve" sign will be used here.

With these the equation becomes

$$
\{F\}_{9\times1} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \rightarrow v_i \\
\begin{bmatrix}
\frac{12EI}{l^3} & 0 & -\frac{6EI}{l^2} & -\frac{12EI}{l^3} & 0 & -\frac{6EI}{l^2} & 0 & 0 & 0 \\
0 & \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 & 0 & 0 \\
-\frac{6EI}{l^2} & 0 & \frac{4EI}{l} & \frac{6EI}{l^2} & 0 & \frac{2EI}{l} & 0 & 0 & 0 \\
-\frac{12EI}{l^3} & 0 & \frac{6EI}{l^2} & \left(\frac{12EI}{l^3}+\frac{EA}{l}\right) & 0 & \frac{6EI}{l^2} & -\frac{EA}{l} & 0 & 0 \\
0 & -\frac{EA}{l} & 0 & 0 & \left(\frac{12EI}{l^3}+\frac{EA}{l}\right) & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\
-\frac{6EI}{l^2} & 0 & \frac{2EI}{l} & \frac{6EI}{l^2} & \frac{6EI}{l^2} & \frac{8EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\
0 & 0 & 0 & -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\
0 & 0 & 0 & 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l}
\end{bmatrix}_{9\times9}
\end{array}
\begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_9 \end{Bmatrix}_{9\times1}
$$

global v_i



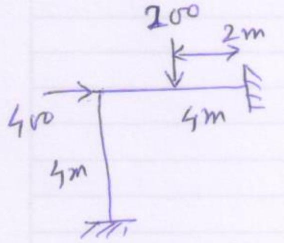
Imposing boundary conditions,
v_1, v_2, v_3, v_8 do not exists.
active DOFs are v_4, v_5, v_6, v_7, v_9

$\therefore$ the reduced stiffness matrix (after imposing the boundary conditions):

$$\therefore \begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_9 \end{Bmatrix} = \begin{bmatrix} \left(\frac{12EI}{l^3}+\frac{EA}{l}\right) & 0 & \frac{6EI}{l^2} & -\frac{EA}{l} & 0 \\ 0 & \left(\frac{12EI}{l^3}+\frac{EA}{l}\right) & \frac{6EI}{l^2} & 0 & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{6EI}{l^2} & \frac{8EI}{l} & 0 & \frac{2EI}{l} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_9 \end{Bmatrix}$$

This is a 5X5 matrix, diffieult to solve by hand calculation.

Let us consider the following problem.

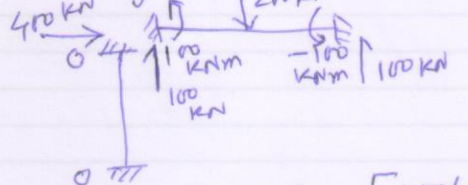


This is the same above problem with different boundary condition - it will have a 3X3 matrix since only v_4, v_5, v_6 are the active DOFs.

Hence we can write:

$$\begin{Bmatrix} F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} \left(\frac{12EI}{l^3}+\frac{EA}{l}\right) & 0 & \frac{6EI}{l^2} \\ 0 & \left(\frac{12EI}{l^3}+\frac{EA}{l}\right) & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{6EI}{l^2} & \frac{8EI}{l} \end{bmatrix} \begin{Bmatrix} v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

considering the fixed end forces



$\therefore$ here, $F_4 = 400$ kN, $F_5 = -100$ kN

$F_6 = -100$ kNm

$$\begin{Bmatrix} 400 \\ -100 \\ -100 \end{Bmatrix} = \begin{bmatrix} 150450 & 0 & 900 \\ 0 & 150450 & 900 \\ 900 & 900 & 4800 \end{bmatrix} \begin{Bmatrix} v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

Note: Fixed end forces are those applied by support to the beam. But we need to take those going to the support; so FE force X -1 is taken.

solving, we get $\{v\} = \begin{Bmatrix} 2.79\times10^{-3} \text{ m} \\ -5.36\times10^{-4} \text{ m} \\ -0.0215 \text{ rad.} \end{Bmatrix}$

To find the member end forces:

member ① - vertical

$$\{f\}_{total} = \{f\}_{FE} + \{f\}_{disp} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} K_{e\_1} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -5.36\times10^{-4} \\ -2.79\times10^{-3} \\ -0.0215 \end{Bmatrix}$$

since, $v_4 = -u_5$, $v_5 = u_4$, $v_6 = u_6$

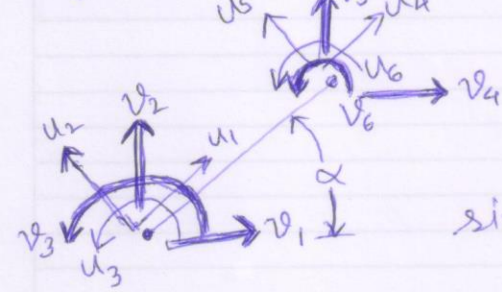
member ② – horizontal

$$\{f\}_{total} = \begin{Bmatrix} 0 \\ 100 \\ 100 \\ 0 \\ 100 \\ -100 \end{Bmatrix} + \begin{bmatrix} & & \\ & Ke\_2 & \\ & & \end{bmatrix} \begin{Bmatrix} 2.79\times10^{-3} \\ -5.36\times10^{-4} \\ -0.0215 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

↑ fixed end forces

Workout these forces and find the location of maximum bending moment.

In this problem it was easy to compare the DOFs of the local element and global structural. But the orientation is arbitrary, visual comparison won't work any more and a general method is required to transform from local to global and vice-versa.



we can write, $u_1 = v_1 \cos\alpha + v_2 \sin\alpha$

$u_2 = v_2 \cos\alpha - v_1 \sin\alpha$

$u_3 = v_3$

similarly, $u_4 = v_4 \cos\alpha + v_5 \sin\alpha$

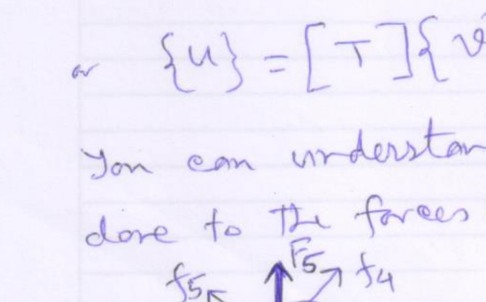
$u_5 = v_5 \cos\alpha - v_1 \sin\alpha$

$u_6 = v_6$

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

or $\{u\} = [T]\{v\}$, $[T]$ is the transformation matrix.

You can understand that the same transformation can be done to the forces (nodal) as well.



$$\therefore \{f\} = [T]\{F\}$$

Previously when we compared the DOFs visually, we ~~made~~ placed the stiffness matrix of an element into the global system $\{F\} = [K_G]\{v\}$, such that element end forces become a part of the structure in global system.

We know that for an element, the equilibrium equations are

$$\{f\} = [K_e]\{u\}$$

Using the transformation matrix, to convert into the global co-ordinates we may write

$$[T]\{F\} = [K_e][T]\{v\}$$

$$\text{or } \{F\} = [T]^{-1}[K_e][T]\{v\}$$

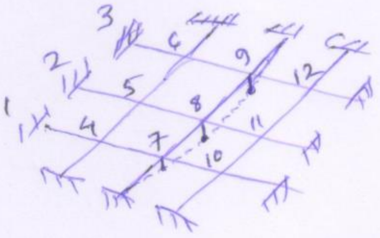
We can show that $[T]^{-1} = [T]^T$ (work this out yourselves)

$$\therefore \{F\} = [T]^T[K_e][T]\{v\}$$

$\therefore$ the matrix $[T]^T[K_e][T]$ is the element stiffness matrix converted into the global coordinate system. Thus we no longer need to write the local and global DOFs over the matrix and compare, we can get the element's place from the global DOFs only.

Refer the example of a frame that is solved and attached.

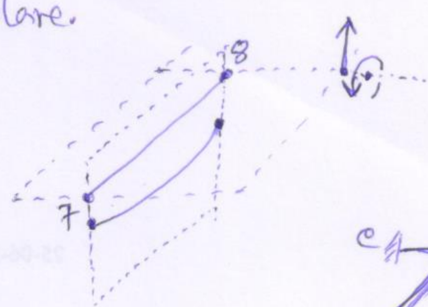
The arrangement of crossing stiffners are called "grillage". For example the floor arrangement in the bottom of a ship. All such members will be on a plane as shown below.



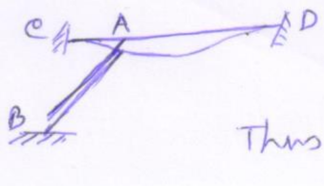
It is assumed that each such element can only deform on its vertical plane.

e.g. point 8 can move upward or downward, (element 78 or 89 etc.) but it cannot move sideways (i.e, in horizontal plane).

isometric view - arrangement of crisscrossing beams on a horizontal plane.

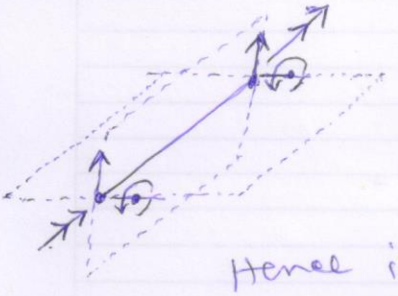


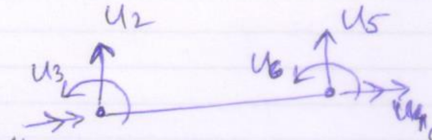
Further, since the ends are connect to another beams, the element can twist as the supporting beam bends as shown.



as CD deforms, slope will be created at A, which will cause twist in AB.

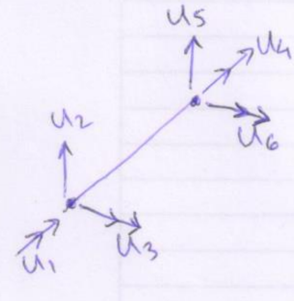
Thus a grillage element has 3 DOFs, at each end.





Right hand cork-screw can be used for the direction of the twist.

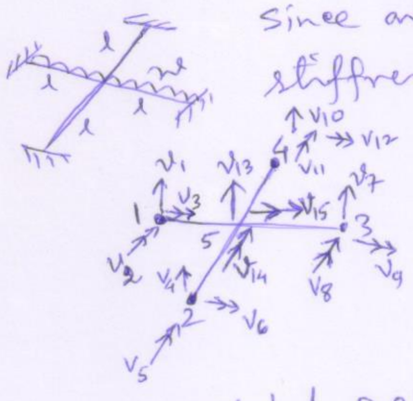
Hence in comparison to a simple beam, the axial DOF is replaced by the twist. The equilibrium eqn can be written as



$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{bmatrix} \frac{GJ}{l} & 0 & 0 & -\frac{GJ}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{GJ}{l} & 0 & 0 & \frac{GJ}{l} & 0 & 0 \\ 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

u_1, u_4 = total angle of twist, f_1, f_4 = twisting moment.

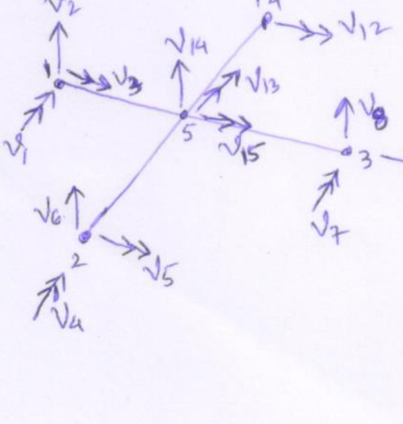
Solve the following problem; use direct comparison of the DOFs.



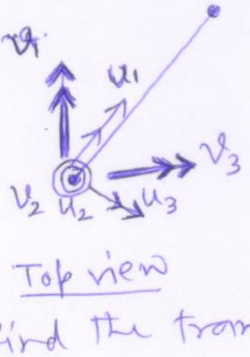
Since only 3 global DOFs are active, the reduced stiffness matrix will be 3×3.

$$\begin{Bmatrix} F_{13} \\ F_{14} \\ F_{15} \end{Bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{Bmatrix} v_{13} \\ v_{14} \\ v_{15} \end{Bmatrix}$$

Name the global DOFs the way you like. As an example



you can see that now it'll be easier to compare. We can also get it done using transformation matrix.



note that in the top view, u_2, v_2 are vertical DOFs, shown as ⊙

Top view

For your own interest, you can find the transformation matrix. Follow the same procedure explained earlier.