Lecture Note :- 17

Marine Hydrodynamics

1. Superposition of Plane waves.

The plane progressive wave moving in a +ve x-direction is written as

$$\eta = a\cos(kx - \omega t) \quad \cdots\cdots \quad (1.1)$$

Similarly

$$\eta = a\cos(kx + \omega t) \quad \cdots\cdots \quad (1.2)$$

corresponds to plane progressive wave moving in the negative x-directions.

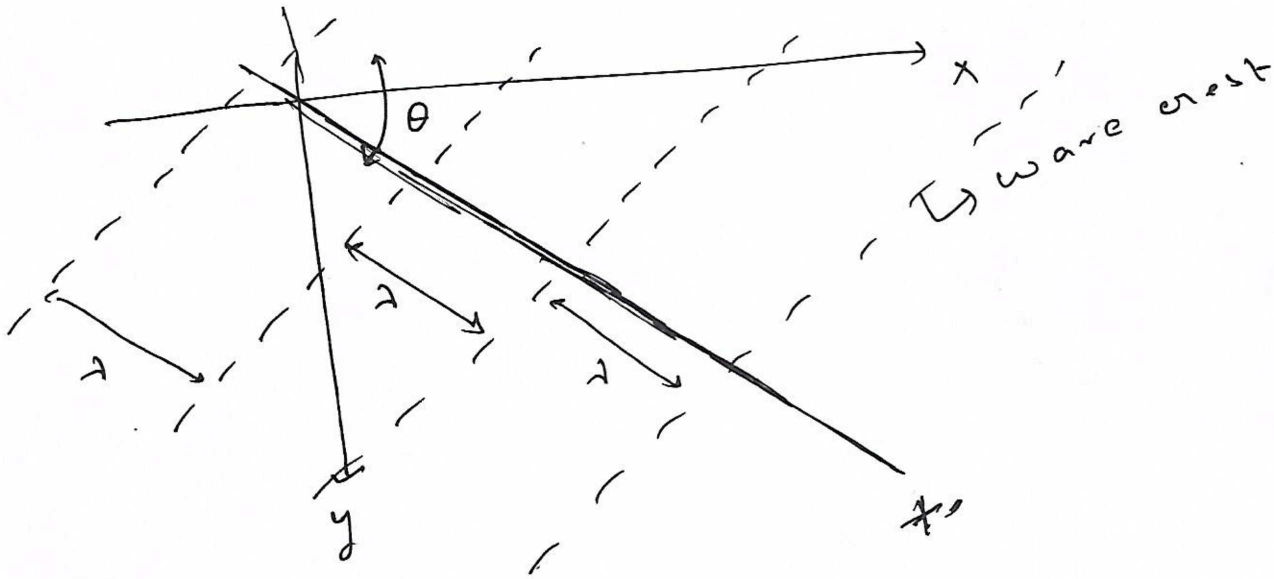

Figure 1.1 :

More generally, a plane wave can move any arbitrary direction. In figure 1.1 a plane wave is moving in the direction of x', which has an angle "θ" with positive x as axis.

since $\quad x' = x\cos\theta + y\sin\theta$

the appropriate generalized form of (1.2) is given by:

$$\eta = a\cos(kx\cos\theta + ky\sin\theta - \omega t) \quad \cdots (1.3)$$

The equation (1.1) and (1.2) may obtained by taking $\theta = 0$ & $\theta = \pi$

In the linearized theory, solution may be superposed without violating the boundary conditions or Laplace Equation. Thus, considerable scope for further generalization is provided by the superposition of plane-wave solutions. The simplest example is a standing wave formed by adding two identical plane wave moving in opposite directions. The sum of these two wave system is

$$\eta = a\cos(kx - \omega t) + a\cos(kx + \omega t) \quad \cdots (1.2a)$$

$$= 2a\cos kx \cos\omega t. \quad \cdots (1.3)$$

and velocity potential for deep water is

$$\phi = -\frac{2ag}{\omega} e^{kz} \cos kx \sin\omega t \quad \cdots (1.4)$$

The standing wave (1.4) is sinusoidal in time, for fixed position x, and vice versa. This wave motion is oscillatory but not progressive. It is typical of sloshing motion in closed containers such as swimming pool, tanks and the wells of some drilling ships.

standing waves are also of physical relevance if the plane waves are incident upon a perfectly reflecting vertical wall, say at $x = 0$, and solution of (1.3) and (1.4) corresponds to the two dimensional case where the wave crest are parallel to the wall. If the fluid domain is $x < 0$, the 1st term in (1.2a) is for incident wave 2nd term in (1.2a) is for reflected waves.

2. Solution : for this particular problem : Show that for full reflection, the wave elevation of the reflected wave is $\eta = a\cos(kx + \omega t)$.

The solution is very simple for reflected wave, the wave is travelled in -ve x. direction

Hence the profile of reflected wave would be

$$\eta^R = a' \cos(kx + \omega t), \quad \text{where } a' \text{ is} \qquad (2.1)$$

unknown.

then the velocity potential for reflected wave potential would be

$$\phi^R = \frac{a' g}{\omega} e^{kz} \sin(kx + \omega t) \rightarrow (2.2)$$

Now for full reflection:

$$\frac{\partial \phi^R}{\partial x} = -\frac{\partial \phi^I}{\partial x} \quad \ldots\ldots \quad (2.3)$$

which means

$$(\nabla \phi^R) . n_x = (\nabla \phi^I) . (-n_x) \Big|_{\text{at } x=0.}$$

~~Substitute~~ where $\phi_I = \frac{ag}{\omega} e^{kz} \sin(kx - \omega t)$

Substitute everything in (2.3) we get $a' = a$

$\therefore$ the reflected wave elevation $\eta = a\cos(kx + \omega t)$

3. More formal approach

The above problem may be solved in more formal way as follows:-

Suppose ϕ^D be the velocity potential of reflected / diffracted wave, then ϕ^D must satisfied the following

(i) $\nabla^2 \phi^D = 0$ for entire fluid domain.

with the boundary condition —

(ii) $\phi^D_{tt} + g\phi^D_z = 0$ at $z=0$ [Linearised free surface boundary condition]

(iii) $\frac{\partial \phi^D}{\partial z} = 0$ at $z = -h$ [bottom boundary condition]

if ϕ^I be the velocity potential of the incident wave, then

(iv) $\frac{\partial \phi^D}{\partial z} = -\frac{\partial \phi^I}{\partial z}$ at $x=0$ [rigid wall condition]

Now, ϕ^D can be solved by using the governing equation (i) with boundary condition (iii) – (iv). At this moment, we do not discuss the solution technique as our main ~~og~~ goal is to understand the concept of group velocity. To do that, it is very essential to get an idea about linear superposition of waves and that is why the above discussion is made..

4. Group velocity

Let us ~~cons~~ consider the two dimensional case, we begin ~~to~~ by forming a discrete sum of waves of the form:

$$\eta = \sum_{n=1}^{N} a_n \cos(k_n x - \omega_n t) \quad \cdots (4.1)$$

Now in equation (4.1), each waves travel with a different phase velocity, it results a continuously changing wave pattern. Nevertheless, if we consider with nearly equal / narrow band of the components waves with nearly equal wave length and also the direction, a characteristic of the resulting distribution is that the waves travels in a group. The group velocity v_g can be derived from a dynamic analysis of energy flux as discussed before

However, a simpler approach to this subject followes from a purely kinematic study of the group of waves formed by two nearly equal plane waves.

For simplicity, let us take two nearly equal waves with same amplitude, i.e, let us say –

$$\left.\begin{aligned} \eta_1 &= a\cos(k_1 x - \omega_1 t) \quad \text{and} \\ \eta_2 &= a\cos(k_2 x - \omega_2 t) \end{aligned}\right\} \quad (4.2)$$

then $\eta_1 + \eta_2 = a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t)$ ... (4.3)

applying the formula $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

in (4.3) we get

$$\eta_1 + \eta_2 = 2a\cos\left[(k_1+k_2)x - (\omega_1+\omega_2)t\right] \times \cos\left[(k_1-k_2)x - (\omega_1-\omega_2)t\right] \quad \ldots (4.4)$$

or $\eta_1 + \eta_2 = 2a\cos\left((k_1+k_2)x - (\omega_1+\omega_2)x\right)\cos\left(dk.x - d\omega.t\right)$ ... (4.5)

- assume that since k_1 is nearly equal to k_2 and ω_1 is nearly equal to ω_2 then we can write

$k_1 - k_2 = dk$ and $\omega_1 - \omega_2 = d\omega$.

Comparing the expression of (4.5) with (4.2) we can conclude that the wave is propagating with a variable amplitude of ~~2a cos~~ $2a\cos((k_1+k)x - (\omega_1+\omega_2)t)$

in the +ve x direction with the velocity $\frac{d\omega}{dk}$

[Since we know the velocity of the wave propagation is $\frac{\omega}{K}$, here, we get $\omega = d\omega$ & $k = dk$]

$\therefore$ we get the group velocity $\boxed{v_g = \frac{d\omega}{dk}}$ ----(4.6)

This type of wave motion is illustrated in the figure (4.1), which shows a group of carrier wave enclosed by a slowly-varying envelope. ~~The~~

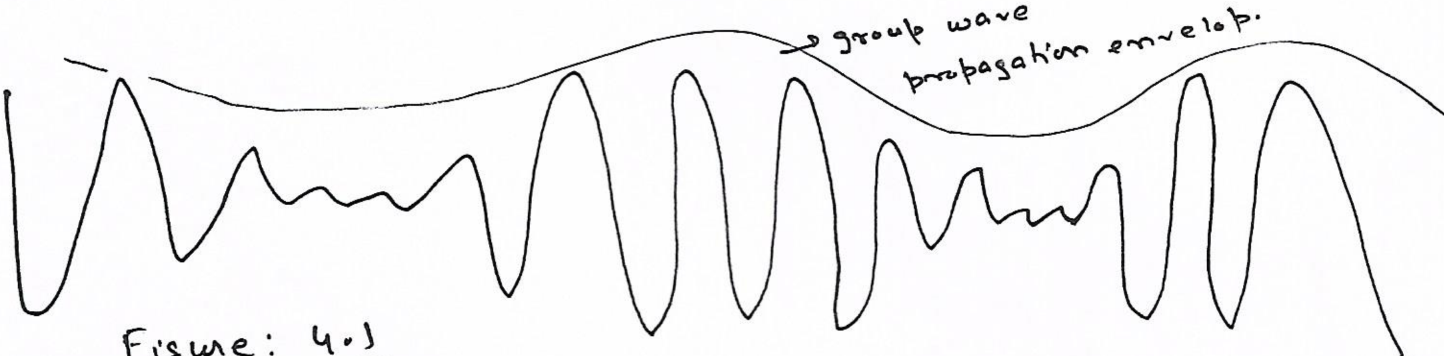

Figure: 4.1

a wave group resulting from the superposition of two nearly equal waves. The individual waves travel with the phase velocity, while the envelop travels with group velocity.

One has to note that, in order to get the group velocity v_g in (4.6) as ~~Sto~~ $dk \to 0$, x should be large enough so that $dk.x$ is finite, similarly as $d\omega \to 0$, t should be large enough so that $d\omega.t$ is finite. Then only, under this circumstances, v_g will approach the finite limit $\boxed{v_g = \frac{d\omega}{dk}}$

In general, the phase and the group velocity differ unless $\frac{\omega}{k} = \frac{d\omega}{dk}$, which will occur, only if the frequency and wave number are directly proportion and the phase velocity is constant. This exceptions occurs when $kh \ll 1$, i.e in shallow water region. thats why, in shallow water $\boxed{v_g = C}$ .... (4.7).

Similarly at deep water case

$$\omega^2 = gk$$

$$\Rightarrow \quad 2\omega\, d\omega = g\, dk$$

$$\text{or} \quad 2 \cdot \frac{d\omega}{dk} = \frac{g}{\omega} = C \quad \left[\begin{array}{l} \because \omega^2 = gk \\ \frac{\omega}{k} = \frac{g}{\omega} \\ \therefore C = \frac{g}{\omega} \end{array} \right]$$

$$\text{or} \quad 2 \cdot v_g = C$$

$$\text{or} \quad \boxed{v_g = \frac{1}{2} C} \quad \ldots (4.8).$$

The similar expression we get ~~for~~ earlier with different approach. Now for finite depth

$$\omega^2 = gk \tanh(kh) \quad \ldots \ldots (4.9)$$

$$\therefore \quad 2\omega\, d\omega = g\, dk \tanh(kh) + gk\, \mathrm{sech}^2(kh) \cdot kh$$

$$\text{or} \quad 2 \cdot \frac{d\omega}{dk} = \frac{g}{\omega} \left[\tanh(kh) + \frac{kh}{\cosh^2(kh)} \right.$$

$$\text{or} \quad \frac{d\omega}{dk} = \frac{g \tanh(kh)}{\omega} \left[\frac{1}{2} + \frac{kh}{2 \sinh(kh) \cosh(kh)} \right.$$

$\rightarrow$ (4.10)

Now from (4.9) we get

$$\omega^2 = gk \tanh(kh)$$

or $$\frac{\omega}{k} = \frac{g}{\omega} \tanh(kh)$$

$$\Rightarrow \quad c = \frac{g}{\omega} \tanh(kh) \quad \cdots\cdots (4.11)$$

Now substituting (4.11) into (4.10) we get

$$\boxed{\frac{d\omega}{dk} = c\left[\frac{1}{2} + \frac{kh}{\sinh 2(kh)}\right]} \quad \cdots\cdots (4.1$$

Since : $2\sinh(kh)\cosh(kh) = \sinh 2(kh)$.

Similar expression can be obtained from energy equation also.

Problem :- in deep water, if the time period of a wave is 10 sec, what would be the group velocity?

Soln :- Here $T = 10$ sec. Now for deep water

$\lambda = 1.56 \cdot T^2 \Rightarrow \lambda = 1.56 \times 10^2 = 156$ m.

$$\therefore k = \frac{2\pi}{\lambda} = \frac{2 \times 3.14}{156} = 0.04$$

$$\& \quad \omega = \frac{2\pi}{T} = \frac{2 \times 3.14}{10} = 0.628$$

$$\therefore \text{group velocity } v_g = \frac{1}{2} \cdot \frac{\omega}{k} = \frac{1}{2} \frac{0.628}{0.04}$$

$$\Rightarrow \boxed{v_g = 7.85 \text{ m/sec}}$$

(unit of ω = rad/sec, unit of k = rad/m.)