Stability of finite difference methods

We now examine the stability of finite difference formulas. Take a simple 2nd order differential eqⁿ with significant first derivative

$$u'' + k u' = 0 \qquad \text{--- (1)}$$

where k is a constant & $k >> 1$. We consider the following three different f.d. formulas

$$u_k'' + k u_k' = 0$$

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + k\,\frac{u_{j+1} - u_{j-1}}{2h} = 0 \qquad \text{Central difference --- (2)}$$

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + k\,\frac{u_{j+1} - u_j}{h} = 0 \qquad \text{forward --- (3)}$$

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + k\,\frac{u_j - u_{j-1}}{h} = 0 \qquad \text{Backward --- (4)}$$

Analytical solⁿ of (1)

$$u(x) = A_1 + B_1 e^{-kx}$$

$$(D^2 + kD) = 0$$

$$D(D+k) = 0$$

$$D = 0,\ D = -k$$

$$\frac{du'}{u'} = -k\,dx$$

$$\ln u'/B = -kx \quad \Rightarrow \quad u' = B e^{-kx}$$

Now integrate once more

$$u = A_1 + B/(-k)\, e^{-kx}$$

$$u = A_1 + B_1 e^{-kx}$$

$$\boxed{u = A_1 + B_1 e^{-kx}} \qquad \text{--- (5)}$$

Now put $u_j = A\xi^j$ in (2)

$$\frac{1}{h^2}A\left(\xi^{j+1} - 2\xi^j + \xi^{j-1}\right) + \frac{kA}{2h}\left(\xi^{j+1} - \xi^{j-1}\right) = 0$$

divide by ξ^{j-1} we get

$$\frac{1}{h^2}\left(\xi^2 - 2\xi + 1\right) + \frac{k}{2h}\left(\xi^2 - 1\right) = 0$$

$$2(\xi^2-2\xi+1)+kh(\xi^2-1)=0$$

$$2(\xi-1)^2+kh(\xi-1)(\xi+1)=0$$

$$(\xi-1)[2(\xi-1)+kh(\xi+1)]=0$$

$$(\xi-1)[(2+kh)\xi-(-kh+2)]=0$$

$$(\xi-1)[(2+kh)\xi-(2-kh)]=0$$

$$\xi=1,\quad \xi=\frac{2-kh}{2+kh}$$

then soln of ② will be written as

$$u_j=A_1+B_1\left(\frac{2-kh}{2+kh}\right)^j \qquad \text{——— ⑥}$$

Each of three representation (2), (3) and ④ has A_1 as solution, so we examine how close are the non-constant components of their solution to e^{-kx}. We expect that the finite difference solutions also behaves monotonically as e^{-kx} for $k>0$ and $k<0$.

For $k>0$ Consider ⑥

Now $\frac{2-kh}{2+kh}<1$

thus $\left(\frac{2-kh}{2+kh}\right)^j \downarrow$ as $j\uparrow$

e^{-kx} for $k>0$ $\downarrow$ as $x\uparrow$

Take $2-kh>0$ so that $(-1)^j$ should not come. ($kh<2$ or $h<2/k$)

For $k<0$. Take $k=-p$ $\quad p>0$

$$\frac{2-kh}{2+kh}=\frac{2+ph}{2-ph}$$

for $ph<2$ or $h<2/p$ $\Rightarrow$ so that $(-1)^j$ should not come.

e^{-kx} for $k<0$ $\uparrow$ as $x\uparrow$

then $\left(\frac{2-kh}{2+kh}\right)^j=\left(\frac{2+ph}{2-ph}\right)^j$ $\uparrow$ as $j\uparrow$

For k very large

$$\frac{2-kh}{2+kh} = \frac{2/k - h}{\frac{2}{k} + h}$$

as $k \to \infty$, $\frac{2-kh}{2+kh} \to (-1)$

and $\left(\frac{2-kh}{2+kh}\right)^j \to (-1)^j$

and the solution (error) oscillates. There fore stability condition will make scheme (2) computationally infeasible.

Next consider the scheme (3)

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + k\,\frac{u_{j+1} - u_j}{h} = 0$$

put $u_j = A\xi^j$

$$\frac{\xi^{j+1} - 2\xi^j + \xi^{j-1}}{h^2} + \frac{k}{h}\left(\xi^{j+1} - \xi^j\right) = 0$$

$$(\xi^2 - 2\xi + 1) + kh\,(\xi^2 - \xi) = 0$$

$$(\xi-1)^2 + kh\,\xi(\xi-1) = 0$$

$$(\xi-1)\left[\xi - 1 + kh\xi\right] = 0$$

$$(\xi-1)\left[(1+kh)\xi - 1\right] = 0$$

$$\xi = 1, \quad \xi = \frac{1}{1+kh}$$

$$\boxed{u_j = A_1 + B_1\left(\frac{1}{1+kh}\right)^j}$$

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$$y_j = A_1 + B_1 \left(\frac{1}{1+kh}\right)^j$$

Now for $k > 0$, $1+kh > 1$ and $\frac{1}{1+kh} < 1$

and $\left(\frac{1}{1+kh}\right)^j \downarrow$ as $j \uparrow$

e^{-kx} $k>0$
$\downarrow$ as $x \uparrow$

So always stable.

for $k < 0$ Take $k = -p$, $p > 0$

e^{-kx} $k<0$
$\uparrow$ as $x \uparrow$

$$1+kh = 1-ph$$

$$\frac{1}{1+kh} = \frac{1}{1-ph}$$

$1-ph>0$

Take $ph < 1$, $h < 1/p$ then $\frac{1}{1+kh} = \frac{1}{1-ph} > 1$

$1-ph<1$

and $\left(\frac{1}{1+kh}\right)^j = \left(\frac{1}{1-ph}\right)^j \uparrow$ as $j \uparrow$.

for k very large & $k < 0$ $\quad k \to -\infty$
$k = -p$, $p \to \infty$

$\frac{1}{1-ph} \downarrow 0$ as $p \uparrow \infty$ but $e^{-kx} = e^{px} \uparrow \infty$ as $p \uparrow \infty$

So difference scheme is infeasible.

Stability of FDM

Next we consider the scheme (4)

$$\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}+k\,\frac{u_j-u_{j-1}}{h}=0$$

$u_j = A\xi^j$

$$\frac{\xi^{j+1}-2\xi^j+\xi^{j-1}}{h^2}+\frac{k}{h}\left(\xi^j-\xi^{j-1}\right)=0$$

$$(\xi^2-2\xi+1)+kh(\xi-1)=0$$

$$(\xi-1)^2+kh(\xi-1)=0$$

$$(\xi-1)[\xi-1+kh]=0$$

$$\xi=1,\ 1-kh$$

$$\boxed{u_j = A_1 + B_1(1-kh)^j}$$

For $k>0$ $\quad 1-kh>0$
$kh<1$ or $h<1/k$ $\qquad e^{-kx}\quad k>0$
$e^{-kx}\downarrow$ as $x\uparrow$

then u_j behaves like e^{-kx}

So $h<1/k$ is the stability condition

for $k<0$ $\qquad 1-kh>1$ and $(1-kh)^j\uparrow$ as $j\uparrow$

always stable

$e^{-kx}\quad k<0$
$e^{-kx}\uparrow$ as $x\uparrow$

for large k and $k>0$

$$1-kh\to-\infty \text{ as } k\to\infty$$

but u_j has term $(1-kh)^j\to(-1)^j\infty$

Thus u_j oscillates and therefore the scheme is infeasible for this case.

Stability of FDM

Hence, for stability it is necessary that different difference approximations for the first order term (i.e; for u') must be used depending on sign of k. We may use approximation

Backward

$$u'(x_j) = \begin{cases} \dfrac{u_j - u_{j-1}}{h} & k > 0 \\ \dfrac{u_{j+1} - u_j}{h} & k < 0 \end{cases}$$

The one sided difference scheme is unconditionally stable. However, it suffers from disadvantage that it is only first-order accurate.