Lecture - 5

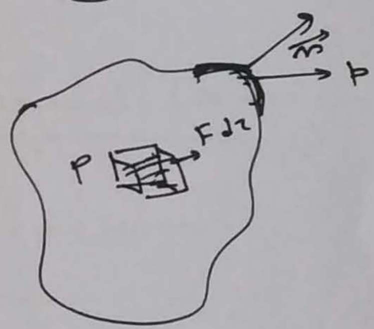
Marine Hydrodynamics

1) Conservation of Momentum. for inviscid fluid.

Let ρ be the density of the fluid particle. P within a closed surface and $d\tau$ be the volume enclosing them P. The mass of the element $= \rho\, d\tau$ will always remain constant.

Let $\vec{q}$ be the velocity of the fluid particle. then

$$\text{momentum } M = \int \vec{q}\,.\,\rho\,.\, d\tau \quad \cdots\cdots \quad (1.1)$$



Now
$$\frac{DM}{Dt} = \frac{D}{Dt}\int \vec{q}\,\rho\, d\tau$$

$$= \int \frac{D}{Dt}(\vec{q}\,\rho)\, d\tau$$

$$= \int \frac{Dq}{Dt} + \int \vec{q}\,\frac{D\rho}{Dt}$$

$$= \int \frac{D\vec{q}}{Dt} \quad \left[\text{as } \frac{D\rho}{Dt} \to 0\right].$$

Since mass remain constant.

Now Let F be the external force per unit mass.

$$\therefore \text{ Total force } = \int F\,\rho\, d\tau \quad \cdots\cdots \quad (1.2)$$

if p be the pressure at a point on the surface. in $\vec{n}$ be the outward drawn normal, then total force acting on the particle due to the pressure of the surrounding fluid $= -\int_S p.\, n\, ds$

Hence, applying Gauss Theorem

$$F = -\int \nabla p \, d\tau. \quad \ldots \quad (1.3)$$

Equating (1.1), (1.2), (1.3) we get

$$\int_{\tau} \rho \frac{Dq}{Dt} \, d\tau = -\int_{\tau} \nabla p \, d\tau + \int_{\tau} F\rho \, d\tau$$

$$\Rightarrow \quad \rho \frac{Dq}{Dt} = - \nabla p + F\rho$$

$$\Rightarrow \quad \boxed{\frac{Dq}{Dt} = -\frac{1}{\rho} \nabla p + F}$$

General form.

This is known as Euler Equation of motion.

in vector form.

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} = -\frac{1}{\rho} \nabla p + \vec{F}$$

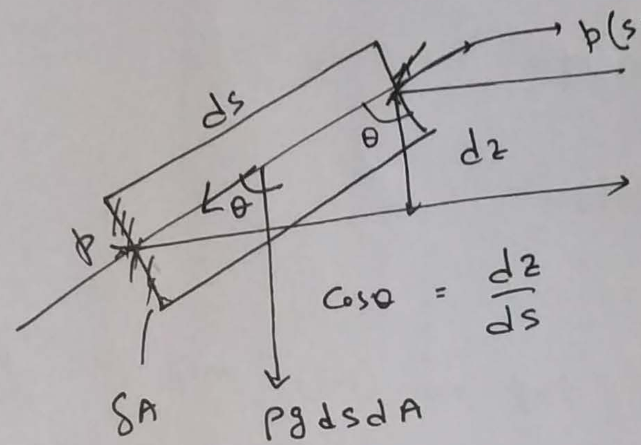
in cartisian co-ordinate :-

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z.$$

2. Euler's Equation of motion along a stream line.



Consider an elementary section of a stream tube. Let ds be the length of the tube. Mass of the fluid particle moves along a stream line, in the positive direction s $\rho\, dA\, ds$. The force acting on the ~~elem~~ element are two types ... (i) Body Force and

(ii) surface force due to ~~hydrostatic~~ pressure. diff.

Now, the body force $F = (\rho F_s) . dA . ds$.

Now resultant pressure force

$$= p . dA - p(s+ds) . dA$$

$$= p . dA - p . dA - \frac{\partial p}{\partial s} . ds . dA$$

$$= -\frac{\partial p}{\partial s} . ds . dA.$$

Now we know from Euler Equation

$$\rho \frac{Dq}{Dt} = \rho F_s - \frac{\partial p}{\partial s}$$

$$\Rightarrow \frac{Dq}{Dt} = F_s - \frac{1}{\rho}\frac{\partial p}{\partial s} \quad \ldots (2.1)$$

Now, consider the body force due to the pull of gravity then

$$\rho F_s \, ds \, dA = -\rho \, ds \, dA . \, g \cos\theta$$

$$F_s = -g . \frac{\partial z}{\partial s}$$

$$\left[\text{Since } \frac{dz}{ds} = \cos\theta \right]$$

Substituting it in (2.1) we get

$$\frac{Dq}{Dt} = -g \frac{\partial z}{\partial s} - \frac{1}{\rho} . \frac{\partial p}{\partial s}$$

Now, $\Rightarrow \frac{\partial \vec{q}}{\partial t} + (\vec{q} . \nabla) \vec{q} = -g \frac{\partial z}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s}$

$$\Rightarrow \frac{\partial \vec{q}}{\partial t} + \vec{q} \frac{\partial \vec{q}}{\partial s} = -g \frac{\partial z}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s}$$

$$\left[\text{Here } \nabla \equiv \frac{\partial}{\partial s} \right]$$

for steady flow, $\frac{\partial \vec{q}}{\partial t} = 0$

$$\Rightarrow \vec{q} . \frac{\partial q}{\partial s} = -g \frac{\partial z}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s} + c$$

Now, $\vec{q}, \vec{p}, \vec{z}$ are function of s only, therefore, by integration, we get

$$c + \frac{1}{2} \vec{q}^2 = -gz - \int \frac{\partial p}{\rho}$$

$$\Rightarrow \boxed{\int \frac{dp}{\rho} + \frac{1}{2} \vec{q}^2 + gz = C} \quad (2.2)$$

which is an alternative form of Euler equation of motion along a stream line.

if you consider 'ρ' is constant for homogeneous fluid, we get

$$\boxed{\frac{p}{\rho} + gz + \frac{1}{2}q^2 = \text{constant}} \rightarrow (2.4)$$

From (2.4), it may be seen that, the 2nd component is a ~~kinetic~~ potential energy component, ~~where~~ then in general (2.4) can be further ~~writ~~ written as

$$\boxed{\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = \text{constant}} \rightarrow (2.5)$$

(2.5) ~~no~~ is known as Bernoulli's equation for steady flow.

(3) Bernoulli's Equation for unsteady flow, the fluid is assumed to be incompressible, inviscid, and ir-rotational.

we know the Euler Equation for inviscid fluid as

$$\frac{dq}{dt} + (\vec{q}.\nabla)\vec{q} = -\frac{1}{\rho}\nabla p + \vec{F} \quad \ldots (3.1)$$

Now, if the flow is ir-rotational then one can take $\vec{q} = \nabla\phi \quad \ldots\ldots (3.2)$.

Now, if we consider the body force is comming by the action of gravity, then

$$\vec{F} = \cancel{-\rho g}\,(0,\ 0,\ \cancel{\rho g}\ -\rho g)$$

substituting in (3.1) we get

$$\nabla\left(\frac{\partial \phi}{\partial t}\right) + (\nabla\phi.\nabla)(\nabla\phi) = -\frac{1}{\rho}\nabla p - \nabla(\rho g \hat{z}) \rightarrow (3.3)$$

with some mathematical manipulation and taking $\nabla^2\phi = 0$, the second term may be written as

$$(\nabla\phi.\nabla)(\nabla\phi) = \frac{1}{2}\nabla.(\nabla\phi)^2$$

putting everything on (3.3), we get

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = -\frac{p}{\rho} - \rho g z$$

$$\Rightarrow \boxed{p = -\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g z\right]} \quad (3.3)$$

(3.3) is known as ~~unsteady~~. Barnoulli equation for ussteady flow. which we often use for our solution.

proof of $(\nabla\phi.\nabla)(\nabla\phi) = \frac{1}{2}\nabla(\nabla\phi)^2$

use the results from vector calculas

$$\nabla(\vec{a}.\vec{b}) = (\vec{b}.\nabla)\vec{a} + (\vec{a}.\nabla)\vec{b} + \vec{b}\times(\nabla\times\vec{a}) + \vec{a}\times(\nabla\times\vec{b})$$

now put $\vec{a} = \vec{q}$, $\vec{b} = \vec{q}$ we have

$$\nabla(\vec{q}.\vec{q}) = (\vec{q}.\nabla)\vec{q} + (\vec{q}.\nabla)\vec{q} + \vec{q}\times(\nabla\times\vec{q}) + \vec{q}\times(\nabla\times\vec{q})$$

now, for ir. rotational flow $\nabla\times\vec{q} = 0$

$$\Rightarrow \nabla(\vec{q}.\vec{q}) = 2(\vec{q}.\nabla)\vec{q}$$

$$\nabla \cdot (\phi A) = \nabla \phi \cdot A + \phi \cdot (\nabla \cdot A)$$

$$\vec{q} \cdot \nabla \vec{q} = \nabla \cdot \vec{q}$$

$$\nabla (\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} + (\vec{A} \cdot \nabla) \cdot \vec{B} + B \cancelto{0}{(\nabla \times A)} + A \times \cancelto{0}{(\nabla \times B)}$$

$$\nabla (\vec{A} \cdot \vec{B}) = (\vec{q} \cdot \nabla) \vec{q} + (\vec{q} \cdot \nabla) \vec{q}$$

$$= 2 (\vec{q} \cdot \nabla) \vec{q}$$

$$= (\vec{q} \cdot \nabla) \vec{q} = \frac{1}{2} \nabla (\vec{q} \cdot \vec{q})$$

$$\Rightarrow (\vec{q}\cdot\nabla)\vec{q} = \frac{1}{2}\nabla(\vec{q}\cdot\vec{q})$$

now substituting $\vec{q} = \nabla\phi$ gives

$$(\nabla\phi\cdot\nabla)\nabla\phi = \frac{1}{2}\nabla(\nabla\phi)^2$$

(4) The importance of equation (3·3) in the domain of marine hydrodynamics.

It is interesting to note that, the most of the fluid mechanics problem we deal with, we normally ignore the $\frac{\partial q}{\partial t}$ term and solve the problem with equation (2·5) which simply tells the pressure of a particular point is dominated by q^2, i.e the quadratic part of the fluid particle velocity, not the 1st order time varying component.

However, in the domain of marine hydrodynamics, we normally ignore the $\frac{1}{2}(\nabla\phi)^2$ term and pressure at any point we measure using the formula

$$\boxed{p = -\rho\left[\frac{\partial\phi}{\partial t} + gz\right]} \quad (3\cdot 4)$$

which simply imply, that in what extend, we deviate from traditional fluid mechanics problems.

Euler Equation in various forms.

i) vector form

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} = -\frac{1}{\rho}\nabla p + \vec{F}$$

ii) Cartesian form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z$$

iii) in polar form (~~spherical~~ Cylindrical polar co-ordinate system)

$$\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} = F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial \theta}$$

$$\frac{Dq_z}{Dt} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z}$

(iv) In spherical co-ordinate system:

$$\frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{Dq_\theta}{Dt} + \frac{q_r q_\theta - q_\phi^2 \cot\theta}{r} = F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial\theta}$$

$$\frac{Dq_\phi}{Dt} + \frac{q_r q_\phi + q_\theta q_\phi \cot\theta}{r} = F_\phi - \frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial\theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Ex:- 1). Find the pressure if the velocity field is given for a inviscid fluid as:

$$u(x, y) = \frac{A(x^2 - y^2)}{(x^2+y^2)^2}$$

$$v(x, y) = \frac{2Axy}{(x^2+y^2)^2}$$

Ex : 2> Find the pressure field if the velocity of a fluid motion is given by

$$u = A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}, \quad v = 0$$

$$w = A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}$$