Mesh or Grid: Divide the interval $[a,b]$ into finite number of sub intervals as follows

a=x0_____x1____x2____x3________________xn=b

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

$x_0, x_1, \ldots x_n$ are called mesh points or grid points, and $h_j = x_j - x_{j-1}$ is mesh spacing. this is variable mesh spacing if $h_j = h \ \forall j$ then the mesh is called uniform mesh.

$$u(x_{j+1}) = u_{j+1} = u(x_j + h) = u_j + hu_j' + \frac{h^2}{2!}u_j'' + \frac{h^3}{3!}u_j''' + \frac{h^4}{4!}u_j^{(4)}(\xi_1) \quad -(1)$$

$$x_j < \xi_1 < x_{j+1}$$

$$u_{j-1} = u_j - hu_j' + \frac{h^2}{2!}u_j'' - \frac{h^3}{3!}u_j''' + \frac{h^4}{4!}u^{iv}(\xi_2) \quad -(2)$$

$$x_{j-1} < \xi_2 < x_j$$

From (1)

$$u_{j+1} = u_j + hu_j' + \frac{h^2}{2}u''(\xi_1), \quad x_j < \xi_1 < x_{j+1}$$

$$hu_j' = u_{j+1} - u_j - \frac{h^2}{2}u''(\xi_1)$$

or $$u_j' = \frac{u_{j+1} - u_j}{h} - \frac{h}{2}u''(\xi_1)$$

So if we write

$$\boxed{u_j' \simeq \frac{u_{j+1} - u_j}{h}} \quad -(3)$$

then the error committed in this will be $-hu''(\xi_1)$ or we say that we commit error of $O(h)$ provided u'' is bounded

$\left[f(x) = O(g(x)) \right.$ means

$\left. \left| \frac{f(x)}{g(x)} \right| \le M \text{ for some } M \right]$

This approximation (3) is called forward approximation for u'. Similarly Backward approximation for u' can be obtained from (2) which is as follows

$$u_j' = \frac{u_j - u_{j-1}}{h} + \frac{h}{2} u''(\xi_2), \quad x_{j-1} < \xi_2 < x_j$$

or

$$\boxed{u_j' \simeq \frac{u_j - u_{j-1}}{h}} \quad \text{— (4)}$$

Again from (1) & (2)

$$u_{j+1} - u_{j-1} = 2h u_j' + 2\frac{h^3}{6} u'''(\xi)$$

$$x_{j-1} < \xi < x_{j+1}$$

or

$$u_j' = \frac{u_{j+1} - u_{j-1}}{2h} - \frac{h^2}{6} u'''(\xi)$$

or

$$\boxed{u_j' \simeq \frac{u_{j+1} - u_{j-1}}{2h}}$$

Here approximation for u_j' is of $O(h^2)$. This is called central difference approximation.

Next: if we add (1) & (2) we get
We rewrite (1) & (2) as follows:

$$u_{j+1} = u_j + h u_j' + \frac{h^2}{2!} u_j'' + \frac{h^3}{3!} u_j''' + \frac{h^4}{4!} u^{(4)}(\xi)$$

$$u_{j-1} = u_j - h u_j' + \frac{h^2}{2!} u_j'' - \frac{h^3}{3!} u_j''' + \frac{h^4}{4!} u^{(4)}(\xi)$$

$$x_{j-1} < \xi < x_{j+1}$$

$$u_{j+1} + u_{j-1} = 2u_j + h^2 u_j'' + \frac{2h^4}{4!} u^{(4)}(\xi)$$

or $$h^2 u_j'' = u_{j+1} + u_{j-1} - 2u_j - \frac{2h^4}{4.3.\cancel{2}} u^{(4)}(\xi)$$

or $$u_j'' = \frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} - \frac{h^2}{12} u^{(4)}(\xi)$$

$x_{j-1} < \xi < x_{j+1}$

Thus $$\boxed{u_j'' \simeq \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}}$$ with error of order $O(h^2)$.

Euler's method

Consider IVP $\quad y' = f(x,y)\,;\ y(x_0) = y_0 \quad$ — (1)

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2!} y''(\xi)\,, \quad x_j < \xi < x_{j+1}$$

$$y_{j+1} = y_j + h f(x_j, y(x_j)) + t(h) \quad \text{———}$$

$$\boxed{t(h) = \frac{h^2}{2} y''(\xi)}$$ Local truncation error.

Then Euler's method is given by

$$y_{j+1} = y_j + h f(x_j, y(x_j)) \quad \text{———} (*)$$

$$\boxed{y_{j+1} = y_j + h f_j}$$

Backward Euler's method

$$y_{j-1} = y_j - h y_j' + \frac{h^2}{2!} y''(\xi_1)\,, \quad x_{j-1} < \xi_1 < x_j$$

or

$$y_j = y_{j-1} + h y_j' - \frac{h^2}{2} y''(\xi_1)$$

Now for $j = j+1$

$$y_{j+1} = y_j + h y_{j+1}' - \frac{h^2}{2} y''(\xi_2)\,, \quad x_j < \xi_2 < x_{j+1}$$

$$y_{j+1} = y_j + h f(x_{j+1}, y_{j+1}) - \frac{h^2}{2} y''(\xi_2)$$

$$y_{j+1} = y_j + h f(x_{j+1}, y_{j+1}) + t(h)$$

$$\boxed{t(h) = -\frac{h^2}{2} y''(\xi_2)}$$

So Backward Euler method is given by

$$y_{j+1} = y_j + h f(x_{j+1}, y_{j+1}) \quad \text{——— } (**)$$

which is an implicit equation in y_{j+1} so it is non-linear algebraic equation in y_{j+1}

Write (**) as

$$G(y_{j+1}) = y_{j+1} - y_j - h f(x_{j+1}, y_{j+1})$$

then apply Newton-Raphson method

$$y_{j+1}^{(n+1)} = y_{j+1}^{(n)} - \frac{G(y_{j+1}^{(n)})}{G'(y_{j+1}^{(n)})} \qquad n = 0, 1, 2, \ldots$$

Take $y_{j+1}^{(0)} = y_j$

Mid-point Method

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2!} y_j'' + \frac{h^3}{3!} y'''(\xi)$$

$$y_{j-1} = y_j - h y_j' + \frac{h^2}{2!} y_j'' - \frac{h^3}{3!} y'''(\xi)$$

$$y_{j+1} - y_{j-1} = 2h y_j' + 2\frac{h^3}{6} y'''(\xi)$$

$$y_{j+1} = y_{j-1} + 2h f(x_j, y_j) + \frac{h^3}{3} y'''(\xi)$$

$$y_{j+1} = y_{j-1} + 2h f(x_j, y_j) + t(h)$$

$$t(h) = \frac{h^3}{3} y'''(\xi)$$

So mid point method is given by

$$\boxed{y_{j+1} = y_{j-1} + 2h f(x_j, y_j)}$$ with local truncation error $t(h) = h^3/3 \, y'''(\xi)$.

Euler & ~~Euler~~ Backward Euler method

Midpoint method

$$y_{j+1} = y_{j-1} + 2h\, f(x_j, y_j)$$

$j=1$; $y_2 = y_0 + 2h\, f(x_1, y_1)$ ← y_1 is not known

$j=2$; $y_3 = y_1 + 2h\, f(x_2, y_2)$

$j=3$; $y_4 = y_2 + 2h\, f(x_3, y_3)$

$\vdots$

$j=N-1$, $y_N = y_{N-2} + 2h\, f(x_{N-1}, y_{N-1})$

So y_1 has to be calculated by some other method
we may use Taylor's method

$$y_1 = y_0 + h y_0' + h^2 y_0'' \; (+\, O(h^3)) \qquad (2)$$

y_0 is know, $y_0' = f(x_0, y_0)$ so y_0' is known

for y_0'': $y' = f(x, y)$

$$y'' = \frac{d}{dx} f(x, y(x)) = f_x + f_y \cdot \frac{dy}{dx}$$
$$= f_x + f_y\, y'$$
$$= f_x + f\, f_y$$

$$y_0'' = f_x(0, y(0)) + f(0, y_0)\, f_y(0, y_0)$$

Putting all these back in (2) we get y_1 and then all other values can be calculated.

Backward Euler Method

Ex Solve the IVP $u' = -2xu^2$, $u(0) = 1$ with $h = 0.2$ on the interval $[0, 0.4]$ using Backward Euler method.

Backward Euler method.

$$u_{j+1} = u_j + h f(x_{j+1}, u_{j+1}) \quad — (1)$$

$$u_{j+1} = u_j + h(-2x_{j+1} u_{j+1}^2)$$

or

$$u_{j+1} = u_j - 2h x_{j+1} u_{j+1}^2 \quad — (2)$$

$$F(u_{j+1}) = u_{j+1} - u_j - h f(x_{j+1}, u_{j+1})$$

$$u_{j+1}^{(n+1)} = u_{j+1}^{(n)} - \frac{F(u_{j+1}^{n})}{F'(u_{j+1}^{(n)})}, \quad n = 0, 1, 2, \dots \quad — (3)$$

from (2) $F(u_{j+1}) = u_{j+1} - u_j + 2h x_{j+1} u_{j+1}^2$

$$F'(u_{j+1}) = 1 + 4h x_{j+1} u_{j+1}$$

for $h = .2$

$$F(u_{j+1}) = u_{j+1} - u_j + 0.4\, x_{j+1} u_{j+1}^2 \quad — (4)$$

$$F'(u_{j+1}) = 1 + 0.8\, x_{j+1} u_{j+1} \quad — (5)$$

Take $u_{j+1}^{(0)} = u_j$

x_1 x_2
0 0.2 0.4

~~But~~ for $j = 0$ $u_1^{(0)} = u_0 = 1$

$$u_1^{(1)} = u_1^{(0)} - \frac{F(u_1^{(0)})}{F'(u_1^{0})}$$

$u_1^{(0)} = u_0 = 1$

from 4

$\underline{j=0}$ $\quad f(u_1) = u_1 - u_0 + .4 \cdot x_1 u_1^2$

$= u_1 - 1 + .4 . .2 \times u_1^2$

$= u_1 - 1 + .08\, u_1^2 \quad \text{—— } 6$

$f(u_1^{(0)}) = 1 - 1 + .08\, u_1^2 \quad \text{—— } 7$

$= .08\, u_1^{(0)2} = 0.08$

$f'(u_1) = 1 + .8\, x_1 u_1$

$f'(u_1^{(0)}) = 1 + .8 \times .2 \times u_1^{(0)}$

$f'(u_1^{(0)}) = 1 + .16 = 1.16$

$u_1^{(1)} = u_1^{(0)} - \dfrac{f(u_1^{(0)})}{f'(u_1^{(0)})} = 1 - \dfrac{.08}{1.16}$

$= 1 - \dfrac{8}{116}$

$= 1 - .068965517$

$= .931034483$

$\boxed{u_1^{(1)} = .931034483}$

$u_1^{(2)} = u_1^{(1)} - \dfrac{f(u_1^{(1)})}{f'(u_1^{(1)})}$

from 6

$f(u_1^{(1)}) = u_1^{(1)} - 1 + .08\, u_1^{(1)2}$

$$f(u_1^{(1)}) = .931034483 - 1 + .08 \times .866825208$$

$$= .931034483 - 1 + .069346016$$

$$= .000380499$$

from (7)

$$f'(u_1^{(1)}) = 1 + .8 \times .2\, u_1^{(1)2}$$

$$= 1 + .16\, u_1^{(1)2} \qquad = 1 + .16 \times (.931034483)^2$$

$$= 1.138692033$$

$$u_1^{(2)} = u_1^{(1)} - \frac{f(u_1^{(1)})}{f'(u_1^{(1)})}$$

$$= .931034483 - \frac{.000380499}{1.138692033}$$

$$u_1^{(2)} = .93070332$$

Similarly $u_1^{(3)} = .93070331$.

$$f(u_1^{(2)}) = .14 \times 10^{-7}, \quad f'(u_1^{(2)}) = 1.14891253$$

Thus $\boxed{u(0.2) = u_1 = .93070331}$

$u_{j+1}^{(0)} = u_j \rightarrow u_2^{(0)} = u_1$

Next For $j = 1$ from

$$u_{j+1}^{(n+1)} = u_{j+1}^{(n)} - \frac{f(u_{j+1}^{(n)})}{f'(u_{j+1}^{(n)})}$$

$$u_2^{(1)} = u_2^{(0)} - \frac{f(u_2^{(0)})}{f'(u_2^{(0)})}$$

$$f(u_2^{(0)}) = .13859338, \quad f'(u_2^{(0)}) = 1.29782507$$

$$u_2^{(1)} = .8239143 6$$

Next: find $u_2^{(2)} = .82247043$

$$\boxed{u_2^{(3)} = .82247016}$$

Runge - Kutta Method

$$u' = f(x, u)$$

$$\int_{x_j}^{x_{j+1}} u' dx = \int_{x_j}^{x_{j+1}} f(x, u)\, dx$$

Mean value of Integral Calculus

$$= f(x_j + \theta h, u(x_j + \theta h)) \cdot \int_{x_j}^{x_{j+1}} dx$$

$$= h f(x_j + \theta h, u(x_j + \theta h))$$

$$u(x_{j+1}) - u(x_j) = h f(x_j + \theta h, u(x_j + \theta h)) \quad (1)$$

$$0 < \theta < 1$$

Case 1 $\theta = 0$

$u_{j+1} = u_j + h f(x_j, u_j)$ which is Euler method

Case 2 $\theta = 1$

$u_{j+1} = u_j + h f(x_{j+1}, u_{j+1})$ which is Backward Euler method

Now in Backward Euler method if we approximate u_{j+1} in $f(x_{j+1}, u_{j+1})$ by Euler that then

$$u_{j+1} = u_j + h f(x_{j+1}, u_j + h f(x_j, u_j)) \quad (2)$$

Now take $k_1 = h f(x_j, u_j)$

$k_2 = h f(x_{j+1}, u_j + k_1)$

then (2) can be written as

R-K method

$$\boxed{u_{j+1} = u_j + K_2} \longleftarrow 3$$

$$K_2 = hf(x_{j+1}, u_j + K_1)$$

$$K_1 = hf(x_j, u_j)$$

Case $\theta = 1/2$

$$u(x_{j+1}) = u(x_j) + hf(x_j + h/2, u(x_j + h/2))$$

But $x_j + h/2$ is not a nodal point so we write

$$u(x_j + h/2) = u_j + h/2\, u_j' = u_j + h/2 f(x_j, u_j)$$

Then $\boxed{u_{j+1} = u_j + hf(x_j + h/2, u_j + h/2 f(x_j, u_j))}$

Take $K_1 = hf_j$

$$K_2 = hf(x_j + h/2, u_j + 1/2 K_1)$$

$$\boxed{u_{j+1} = u_j + K_2}$$

Euler Cauchy Method

$$u_{j+1} = u_j + hu_j' + \frac{h^2}{2}u_j''$$

~~First we prove the following~~

$$u(x_j + h/2) \simeq \frac{1}{2}[u(x_j) + u(x_{j+1})]$$

$$u(x_j + h/2) = u(x_j) + \frac{h}{2}u'(x_j) + \frac{h^2}{8}u''(\xi)$$

$$= u_j + \frac{1}{2}\left[u_{j+1} - u_j - \frac{h^2}{2}u''(\xi_1)\right] + \frac{h^2}{8}u''(\xi)$$

$$= \frac{1}{2}[u_{j+1} + u_j] + \frac{h^2}{8}[u''(\xi) - 2u''(\xi_1)]$$

So

$$u(x_j + h/2) \simeq \frac{1}{2}[u_{j+1} + u_j]$$

Now

$$u'(x_j + h/2) = \frac{1}{2}[u'(x_j) + u'(x_{j+1})]$$

$$= \frac{1}{2}[f(x_j, u_j) + f(x_{j+1}, u_{j+1})]$$

Now use Euler's method

$$u'(x_j + h/2) = \frac{1}{2}[f(x_j, u_j) + f(x_{j+1}, u_j + hf(x_j, u_j))]$$

or $f(x_j + h/2) = \frac{1}{2}[f(x_j, u_j) + f(x_{j+1}, u_j + hf(x_j, u_j))]$ (*)

Now for $\theta = 1/2$ from (1)

$u_{j+1} - u_j = hf(x_j + \theta h, u(x_j + \theta h))$ for $\theta = 1/2$ it is given

by $u_{j+1} - u_j = h\, \underbrace{f(x_j + \frac{1}{2}h, u(x_j + h/2))}_{u'(x_j + h/2)}$

$$= h \cdot \frac{1}{2}[f(x_j, u_j) + f(x_{j+1}, u_j + hf(x_j, u_j))]$$

So

$$u_{j+1} = u_j + \frac{h}{2}[f(x_j, u_j) + f(x_{j+1}, u_j + hf_j)]$$

Take $K_1 = hf_j$, $K_2 = hf(x_{j+1}, u_j + K_1)$

$$\boxed{u_{j+1} = u_j + \frac{1}{2}[K_1 + K_2]}$$

which is Euler-Cauchy Method.

RK method

$u_{j+1} = u_j + h \times (\text{avarage slop})$