Lecture - 18

1. Mass Transport :

One of the most interesting non-linear feature of plane progressive waves is the occurance of second order mean drift ~~force~~ of the fluid particle, in the same direction as the wave propagation. This effect can be calculated most easily for the infinite depth case.

The existence of a net flux follows because the horizontal velocity $u = a\omega e^{kz}\cos(kx - \omega t)$ is equal in magnitude and opposite in sign beneath the crest and trough, at points of equal depth z below the mean free surface. Since u is +ve beneath the crest, where the total elevation of the fluid is greater, the total horizontal flux near/beneath the crest will exceed that beneath the trough, and on the avarage a net mass transport will ~~occo~~ occur.

The orbital motion of a particular fluid particle can be computed in terms of the Lagrangian co-ordinates $[x_0(t), z_0(t)]$ which define the position of a particle. These must be satisfy the relations:

$$\left.\begin{aligned} \frac{dx_0}{dt} &= u(x_0, z_0, t) \\ \frac{dz_0}{dt} &= v(x_0, z_0, t) \end{aligned}\right\} \rightarrow (1.1)$$

If x_0 and z_0 differ by a small amount, from the fixed position (x, y), the taylor series gives

$$\frac{dx_0}{dt} = u(x, z, t) + (x - x_0)\frac{du}{dx} +$$

$$(z - z_0)\frac{\partial u}{\partial z} + \cdots \qquad (1.2)$$

Similarly

$$\frac{dz_0}{dt} = \omega(x, z, t) + (x - x_0)\frac{d\omega}{dx} +$$

$$(z - z_0)\frac{d\omega}{dz} + \cdots \qquad (1.3)$$

Now

$$u = \frac{dx}{dt} = a\omega e^{kz}\cos(kx - \omega t) \cdots (1.4)$$

$$\therefore \frac{\partial u}{\partial x} = -ak\omega e^{kz}\sin(kx - \omega t) \cdots (1.5)$$

integrating (1.4) we get

$$(x - x_0) = -ae^{kz}\sin(kx - \omega t) \cdots (1.6)$$

$$\boxed{\cancel{\therefore (x - x_0)\frac{du}{dx} = a^2\omega k e^{2kz}\sin(kx - \omega t)}}$$

$$\therefore (x - x_0)\frac{du}{dx} = a^2k\omega e^{2kz}\sin^2(kx - \omega t) \cdots (1.7)$$

Now we have

$$\frac{\partial u}{\partial z} = ak\omega e^{kz}\cos(kx - \omega t) \cdots (1.8)$$

Also we have

$$\omega = \frac{dz}{dt} = a\omega e^{kz}\sin(kx - \omega t) \cdots (1.9)$$

Then integrating (1.9) we get

$$(z - z_0) = a e^{kz} \cos(kx - \omega t)$$

$$\therefore (z - z_0) \frac{du}{dz} = a^2 k \omega e^{2kz} \cos^2(kx - \omega t)$$

$$\therefore (x - x_0) \frac{\partial u}{\partial x} + (z - z_0) \frac{\partial u}{\partial z} = a^2 k \omega e^{2kz} \quad \text{.. (1.10)}$$

Substitute (1.10) into (1.2) we get

$$\frac{dx_0}{dt} = u(x, z, t) + \underline{\underline{\omega a^2 k e^{2kz}}}.$$

or $$\frac{dx_0}{dt} = a\omega e^{kz} \cos(kx - \omega t) + \underline{\underline{\omega a^2 k e^{2kz}}}$$

which shows the horizontal component has steady drift, this is known as stokes' drift.

However: we can get $\frac{dz_0}{dt} = \omega a e^{kz} \sin(kx - \omega t)$.

i.e. vertical component does not experience the same.

The presence of a mean drift is obvious from the observation of small vessels floating in waves. In some cases, longshore current also associate with this phenomenon, In shallow water, viscous effect significantly modify the mean drift flow.

2. Two-dimensional Ship Waves :

Let us now consider the problem of wave generation by moving vessel and the associated wave resistance. We take the simplest case, the 2-D motion by a moving body. ~~[illegible]~~

Since our approach will be based on energy conservation, we are going to ignore the detail flow near the body and focus attention on the wave system far downstream. If the body moves with constant velocity U in calm water, the only waves that can exist down stream move with the phase velocity $V_p = U$. Any other waves would either overtake the body or drop further behind, in an unsteady manner. Since phase velocity is fixed, i.e. $V_p = U$. Then for deep water case →

~~[illegible]~~ $\omega^2 = gk \Rightarrow \frac{\omega^2}{k^2} = \frac{g}{k}$.

$\therefore U^2 = \frac{g}{k} \Rightarrow k = \frac{g}{U^2}$ $\left[\text{Since } \frac{\omega}{k} = V_p = U\right]$

$\rightarrow (2.1)$

$\therefore \lambda = \frac{2\pi}{k} = \frac{2\pi U^2}{g}$ ... (2.2)

The waves generated by the body contain energy that must be imparted to the fluid as work done by the body on the fluid.

Thus the body experience a drag force D due to its wave resistance. The purpose of this discussion is relate 'D' to the wave amplitude "a" far down stream.

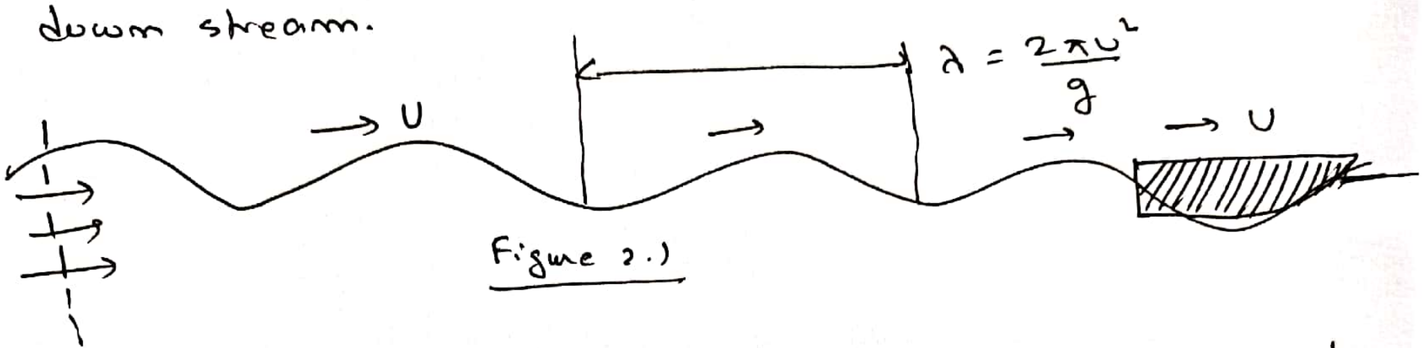

Figure 2.1

Figure 2.1 shows a illustration in a fixed frame of reference. It shows that the body and wave move with velocity U in +ve x direction, and thus there is a +ve flux of energy across the fixed control surface downstream.

Since the control surface is fixed and the body is moving, the length of the fluid region between the two will increase with velocity U, and the total energy in this region will increase ~~with velocity~~. at a rate equal to the product of U and the energy density $\frac{1}{2}\rho g a^2$. The energy input necessary to balance this increase results in part from the work done at a rate DU by the body, in oposition to the wave drag D. Energy also enters the fluid region across the control surface down stream, at a rate equal to the product of the energy density and group velocity $\frac{1}{2}U$. From energy conservation, it follows that:

$$\frac{1}{2}\rho g a^2 U = DU + \frac{1}{4}\rho g a^2 U$$

$$\Rightarrow \boxed{D = \frac{1}{4}\rho g a^2} \longrightarrow (2.3)$$

Thus, if a two dimensional body generated waves of amplitude a because of its steady motion on the free surface, the associated wave drag is $\frac{1}{4}\rho g a^2$.

3. Body Response in regular waves

A subject of great interest to ocean engineers and naval architect is the effect suffered by a floating or submarged vessel in the presence of ocean waves.

This subject is complex and very intence, however, at this level, we only study the basic understanding. Here, we mainly discuss about the component of various wave forces that occur ~~inelsoao~~ because of small amplitude regular progressive waves.

A vessel can have six-degree of fredom motion as sketched below:-

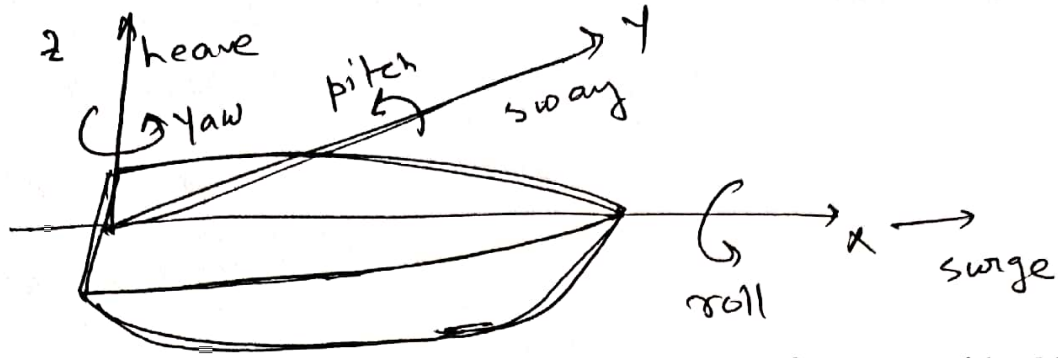

Fig-3.1: Definition sketch of body motion in six-degree of freedom.

Now, there will be two types of wave can be created due to the action of ~~wave~~ incident wave and the body.

(i) Scattering wave:- this wave is basically when the incident wave approaches to the body and some part diffracted back, the resultant wave may be denoted by scattering wave. suppose, ϕ^I be the incident potential and ϕ^D be the diffracted potential, then '$\phi_I + \phi_D$' is called the scattering potential.

(ii) Radiated wave: suppose under the action of incident wave, the body gets ~~ex~~ excited and starts ~~os~~ oscillating. Then the wave created due this oscillation is called radiated wave.

The force due to scattering wave is called the exciting force, the force due to radiated wave is called radiation force.

Now, since there are six-degree of freedom, the body may oscillate in 6 modes, if ~~generated If we assume,~~ the incident wave potential is denoted as ϕ_0 and diffracted wave potential is denoted by ϕ_7, then, under the linear superposition principle, the total wave potential ϕ may be written as

$$\phi = Re\left[\sum_{j=1}^{6} \xi_j \phi_j(x,y,z) + a\{\phi_0 + \phi_7\}\right] e^{i\omega t} \quad \text{---} (3.1)$$

where ϕ_j denotes the radiation potential for unit amplitude of body motion, ξ_j denote the radiation wave amplitude for j^{th} mode.

a is the incident wave amplitude.

Now, the velocity potential ϕ satisfy the followings:-

i) Laplace equ^n [Governing Equ^n]

ii) Free surface boundary condition
$[\phi_{tt} + g\phi_z = 0]$ at $z=0$

iii) Bottom boundary condition

~~(iv)~~ (iv) radiation condition at ∞

apart from this, the ϕ must satisfy the kinematic body boundary condition, i.e normal velocity of the body must be equal to the normal velocity of the fluid particle.

since the ϕ_j represents ~~[illegible]~~ j^{th} $[j=1,6]$ the velocity potential for unit amplitude motion of the body, then one can assume the body motion in j^{th} mode is

$$x_j = 1 . e^{i\omega t}$$

$$\therefore \dot{x}_j = i\omega e^{i\omega t} \quad \ldots \quad (3.2)$$

then it follows that

$$\left.\begin{aligned} \frac{\partial \phi_j}{\partial n} &= i\omega \vec{n}_j , \quad j = 1, 2, 3 \\ &= i\omega (\vec{r} \times \vec{n})_{j-3} , \quad j = 4, 5, 6 \end{aligned}\right\} (3.3a)$$

on the wetted surface SB. and $\vec{n}$ is the unit normal vector on the body surface, directed into the body, $\vec{r}$ is the position vector. The forced motion potential ϕ_j, $j=1,\ldots 6$ are known as radiation problem.

The remaining potential represents by the ϕ_0 and ϕ_7. for this two, ~~since~~ the appropriate boundary condition on the body surface is :-

$$\frac{\partial \phi_0}{\partial n} + \frac{\partial \phi_7}{\partial n} = 0 \quad \text{on} \quad S_B$$

or $$\frac{\partial \phi_7}{\partial n} = -\frac{\partial \phi_0}{\partial n} . \quad \rightarrow (3.3b)$$

This problem is known as diffraction problem.

After solving the radiation and diffraction problem, the total dynamic pressure can be calculated by

$$p = -\rho \frac{\partial \Phi}{\partial t}$$

$$= -\rho \, Re \left[\sum_{j=1}^{6} \xi_j \phi_j + a(\phi_0 + \phi_7) \right] i\omega e^{i\omega t} \quad \rightarrow (3.4)$$

The force F and moment M can be determined as

$$\begin{pmatrix} F \\ M \end{pmatrix} = -\rho \, Re \sum_{j=1}^{6} i\omega \xi_j e^{i\omega t} \iint_{S_B} \begin{pmatrix} \vec{n} \\ \vec{r} \times \vec{n} \end{pmatrix} \phi_j \, ds$$

$$-\rho \, Re \, i\omega a e^{i\omega t} \iint_{S_B} \begin{pmatrix} \vec{n} \\ \vec{r} \times \vec{n} \end{pmatrix} (\phi_0 + \phi_7) \, ds . \quad \cdots\cdots (3.5)$$

The 1st part of equation (3.5) is known as radiation force and 2nd part of the (3.5) is known as exciting force.

4. Added mass and Damping

The exciting force is responsible for the body to oscillate, however, the radiation force occurs due to the oscillation of the body. In general, there is always a time lag between the excitation force acted on the body and the re-action of the body. Because of that the component of radiation force can be further splitted in two component.

Suppose the time harmonic exciting force is denoted as: ~~F_e ... $\cos$~~

$$F_e = \xi_e \cos(\omega t). \quad \ldots (4.1)$$

Then the radiation force due to F_e can be written as $F_R = \xi_R \cos(\omega t + \delta)$ where δ is the phase difference, ω is angular frequency, ξ_e and ξ_R are the amplitude ~~and~~ of exciting and radiation force respectively. Now

$$F_R = \xi_R \cos\omega t \cos\delta + \xi_R \sin\omega t \sin\delta$$

$$\text{or } F_R = (\xi_R \cos\delta)\cos\omega t + (\xi_R \sin\delta)\sin\omega t$$

$$\text{or } F_R = A\cos\omega t + B\sin\omega t \quad \ldots (4.2)$$

Hence we can say that radiation force can be further splitted into two component. one is along the direction of accelaration (component with $\cos\omega t$)

~~and~~

and the another is along velocity (the component along $\sin\omega t$). The component along acclaration is called "added mass" and the component along velocity is called "damping" force respectively.

~~Equation of~~

5. Equation of motion: (single degree of freedom)

Now from Newton equation of motion, we know

$$M\ddot{x} = F^T \quad \ldots \quad (4.3)$$

where $F^T \equiv$ total force, $M =$ mass of the body.

the F^T can be further splitted into three component

(i) exciting force : F^e

(ii) Radiation force : $-A\ddot{x} - B\dot{x}$

(iii) restoring force : $-Cx$ where 'c' is called the restoring co-efficient (similar to mass-spring system).

Then equation (4.3) may be written as

$$M\ddot{x} = F^e - A\ddot{x} - B\dot{x} - Cx$$

or $$(M + A)\ddot{x} + B\dot{x} + Cx = F^e \quad \ldots \quad (4.4)$$

Equation (4.4) is general equation of motion for single degree of freedom body ~~mo~~ oscillating in water under the action of wave.