PROBABILITY & STATISTICS(MA20104, SEC 4)

BUDDHANANDA BANERJEE

Contents

Date: Updated on January 16, 2021, Weekly classes:Mon(8-10am), Tue(12-1pm) .

1. Syllabus

Probability.

(1) Probability: Classical, relative frequency and axiomatic definitions of probability, addition rule and conditional probability, multiplication rule, total probability, Bayes' Theorem and independence.
(2) Random Variables: Discrete, continuous and mixed random variables, probability mass, probability density and cumulative distribution functions, mathematical expectation, moments, moment generating function, Chebyshev's inequality.
(3) Special Distributions: Discrete uniform, binomial, geometric, negative binomial, hypergeometric, Poisson, uniform, exponential, gamma, normal, beta, lognormal, Weibull, Laplace, Cauchy, Pareto distributions. Functions of a Random Variable.
(4) Joint Distributions: Joint, marginal and conditional distributions, product moments, correlation, independence of random variables, bivariate normal distribution, simple, multiple and partial correlation, regression.
(5) Sampling Distributions: Law of large numbers, Central Limit Theorem, distributions of the sample mean and the sample variance for a normal population, Chi-Square, t and F distributions.

Statistics.

(1) Estimation: The method of moments and the method of maximum likelihood estimation, properties of best estimates, confidence intervals for the mean(s) and variance(s) of normal populations.
(2) Testing of Hypotheses: Null and alternative hypotheses, the critical and acceptance regions, two types of error, power of the test, the most powerful test and Neyman-Pearson Fundamental Lemma, standard tests for one and two sample problems for normal populations.

2. Books

(1) 1. An Introduction to Probability and Statistics by V.K. Rohatgi & A.K. Md. E. Saleh
(2) Probability and Statistical Inference by Hogg, R. V., Tanis, E. A. & Zimmerman D. L.
(3) Probability and Statistics in Engineering by W.W. Hines, D.C. Montgomery, D.M. Goldsman, C.M. Borror
(4) Introduction to Probability and Statistics for Engineers and Scientists by S.M. Ross
(5) Introduction to Probability and Statistics by J.S. Milton & J.C. Arnold.
(6) Introduction to Probability Theory and Statistical Inference by H.J. Larson
(7) Probability and Statistics for Engineers and Scientists by R.E. Walpole, R.H. Myers, S.L. Myers, Keying Ye
(8) Modern Mathematical Statistics by E.J. Dudewicz & S.N. Mishra
(9) Introduction to the Theory of Statistics by A.M. Mood, F.A. Graybill and D.C. Boes

3. Evaluation

- Continuous evaluation

4. Probability : Definition & laws

Definition 1. Random experiment: A random experiment is a physical phenomena which satisfies the followings.

(1) It has more than one outcomes.
(2) The outcome of a particular trial is not known in advance.
(3) It can be repeated countably many times in in *identical* condition.

Example 2. (a) Tossing a coin, (b) Rolling a die and (c) Arranging 52 cards etc.

Definition 3. Sample space: A set which is collection of all possible outcomes of a random experiment is known as sample space for the experiment and it is denoted by Ω or S.

Example 4. For the above examples the sample spaces are
(a) $\{H, T\}$, (b) $\{1, 2, 3, 4, 5, 6\}$, (c) $\{\pi | \pi$ is any permutation of 52 cards$\}$ respectively

Definition 5. Classical definition of probability: If the sample space (Ω) of a random experiment is a $finite$ set and $A \subseteq \Omega$ the probability of A is defined as

$$P(A) = \frac{|A|}{|\Omega|}$$

under the assumption that all outcomes are equally likely. Here $|\cdot|$ denotes the cardinality of a set.

Exercise 6. Consider the equation $a_1 + a_2 + \cdots + a_r = n$ where $r < n$. Suppose a computer provides an integral solution of it at random such that each $a_i \in \mathbb{N} \cup \{0\}$ for any solution. Find the probability that each $a_i \in \mathbb{N}$ only, for a solution.

Definition 7. Frequency definition of probability: If the sample space (Ω)of a random experiment is a *countable* set and $A \subseteq \Omega$ the probability of A is defined as

$$P(A) = \lim_{n \uparrow \infty} \frac{|A_n|}{|\Omega_n|}$$

where $\lim_{n \uparrow \infty} A_n = A$ and $\lim_{n \uparrow \infty} \Omega_n = \Omega$.

Exercise 8. What is the probability that a randomly chosen number from $\mathbb{N}$ will be a even number?

Exercise 9. What is the probability that a randomly chosen number from $\mathbb{N}$ will be a \$k-digit\$ number? Can you conclude an answer with frequency definition of probability?

Definition 10. Algebra: A collection $\mathcal{A}$ of the subsets of Ω is called an algebra if

(1) $\Omega \in \mathcal{A}$
(2) any $A \subseteq \Omega$ and $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$ [closed under complementation]
(3) any $A, B \subseteq \Omega$ and $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$ [closed under finite union]

Definition 11. σ-algebra or σ-field: An algebra $\mathcal{A}$ of the subsets of Ω is called an σ-algebra/ field if $\{A_i\} \subseteq \Omega$ and $\{A_i\} \in \mathcal{A}$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$ [closed countable union].

Example 12. (a) $\mathcal{A} = \{\emptyset, \Omega\}$, (b) $\mathcal{A} = \{\emptyset, \Omega, A, A^c\}$, (c) $\mathcal{A} = 2^{\Omega}$ [power set]

Definition 13. Axiomatic definition of probability (Kolmogorov): If $\mathcal{A}$ is σ-algebra of the subsets of a non-empty set Ω then the probability (P) is defined to be a function $P : \mathcal{A} \mapsto [0, 1]$ which satisfies,

(1) $P(\Omega) = 1$,
(2) $P(A) \geq 0$ for any $A \in \mathcal{A}$,
(3) $\{A_i\} \in \mathcal{A}$ implies $P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ if $A_i \cap A_j = \emptyset \quad \forall i \neq j$.

Example 14. What is the probability that a randomly chosen number from $\Omega = [0, 1]$ will be

(1) a rational number?
(2) less than 0.4? Can you conclude an answer with classical / frequency definition of probability?

Definition 15. Probability space: $(\Omega, \mathcal{A}, P)$ is known as a probability space.

Definition 16. Event: For a given Probability space $(\Omega, \mathcal{A}, P)$ if $A \subseteq \Omega$ and $A \in \mathcal{A}$ then A is called an event.

Definition 17. Conditional Probability: For a given Probability space $(\Omega, \mathcal{A}, P)$ if A and$B$ are two events such that $P(B) > 0$ then the conditional probability of A given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 18. Independent events: For a given probability space $(\Omega, \mathcal{A}, P)$ the two events A and B are called *independent* if $P(A|B) = P(A)$, which implies

$$P(A \cap B) = P(A)P(B).$$

Remark 19. If the probability function P is changed to some P_1 on the same $(\Omega, \mathcal{A})$ then events A and B may not be independent any more.

Definition 20. Pairwise independence: For a given probability space $(\Omega, \mathcal{A}, P)$ consider a sequence of events $\{A_i\}$. This sequence of events are called pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j) \quad \forall i \neq j.$$

Definition 21. Mutual independence: For a given probability space $(\Omega, \mathcal{A}, P)$ consider a sequence of events $\{A_i\}$. This sequence of events are called mutually independent if

$$P(\cap_{i_1, i_2 \cdots i_k} A_i) = \Pi_{i_1, i_2, \cdots i_k} P(A_i) \quad \forall i_1 \neq i_2, \neq \cdots \neq i_k. \text{for any} \quad k \in \mathbb{N}$$

Exercise 22. Give an example to show that pairwise independence does not imply mutual independence.

Definition 23. Mutually exclusive events: For a given probability space $(\Omega, \mathcal{A}, P)$ a sequence of events $\{A_i\}$ are called mutually exclusive if $A_i \cap A_j = \emptyset \quad \forall i \neq j$

Definition 24. Mutually exhaustive events: For a given probability space $(\Omega, \mathcal{A}, P)$ a sequence of events $\{A_i\}$ are called mutually exhaustive if $\cup_{i=1}^{\infty} A_i = \Omega$

Remark 25. Mutually exhaustive or exclusive events **does not depend the probability function**.

Definition 26. Partition: For a given probability space $(\Omega, \mathcal{A}, P)$ a sequence of events $\{A_i\}$ are called a partition of Ω if $\{A_i\}$ are **mutually exclusive and exhaustive.**

Exercise 27. Prove the following properties:

(1) $P(A^c) = 1 - P(A)$
(2) $P(\emptyset) = 0$
(3) If $A \subseteq B$ then $P(A) \leq P(B)$
(4) $1 - P(\cup_{i=1}^{\infty} A_i) = P(\cap_{i=1}^{\infty} A_i^c)$
(5) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(6) $P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$
(7) $P\left(\cup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} S_k$ where

$$S_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k})$$

Exercise 28. Suppose n letters are put in n envelops distinct by addresses. What is the probability that no letter will reach to the correct address. What is the limiting probability as $n \uparrow \infty$?

Theorem 29. ***Bayes Theorem:*** *Let $A_1, A_2, \cdots A_k$ is a partition of Ω and $(\Omega, \mathcal{A}, P)$ be a probability space with $P(A_i) > 0 \ \ \forall i$ and $P(B) > 0$ for some $B \subseteq \Omega$. Then*

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}.$$

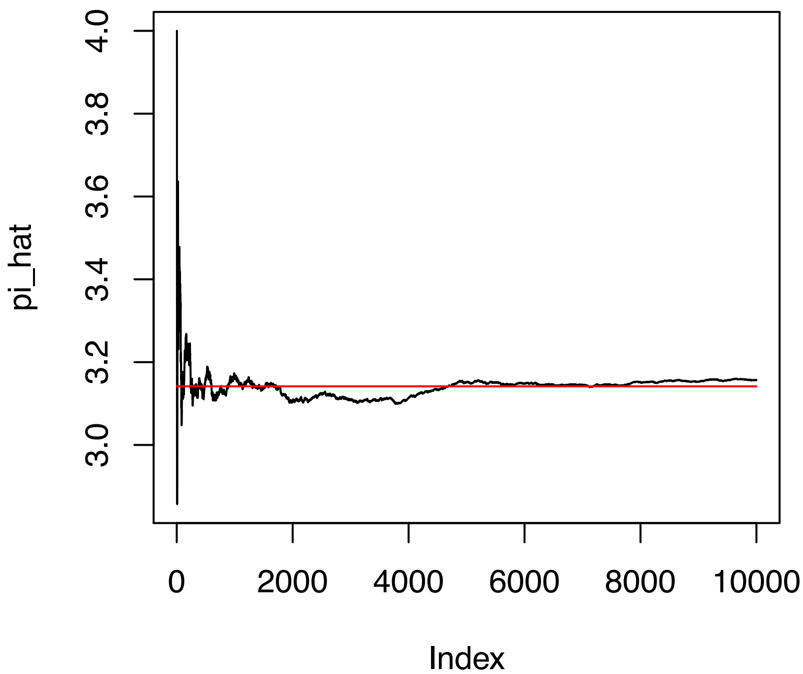
Exercise 30. There are three drawers in a table. The first drawer contains two gold coins. The second drawer contains a gold and a silver coin. The third one contains two silver coins. Now a drawer is chosen at random and a coin is also chose randomly. It is found that the a gold coin has been selected then what is the probability that the second drawer was chosen?

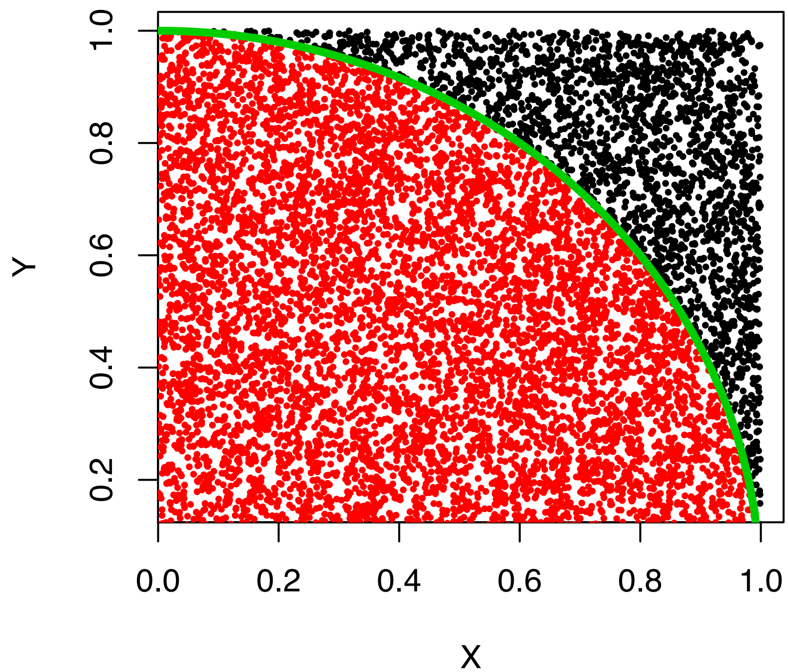
Exercise 31. Consider the quadratic equation $u^2 - \sqrt{Y}u + X = 0$, where (X, Y) is a random point chosen uniformly from a unit square. What is the probability that the equation will have a real root?

Exercise 32. Give a randomized algorithm to approximate value of π.

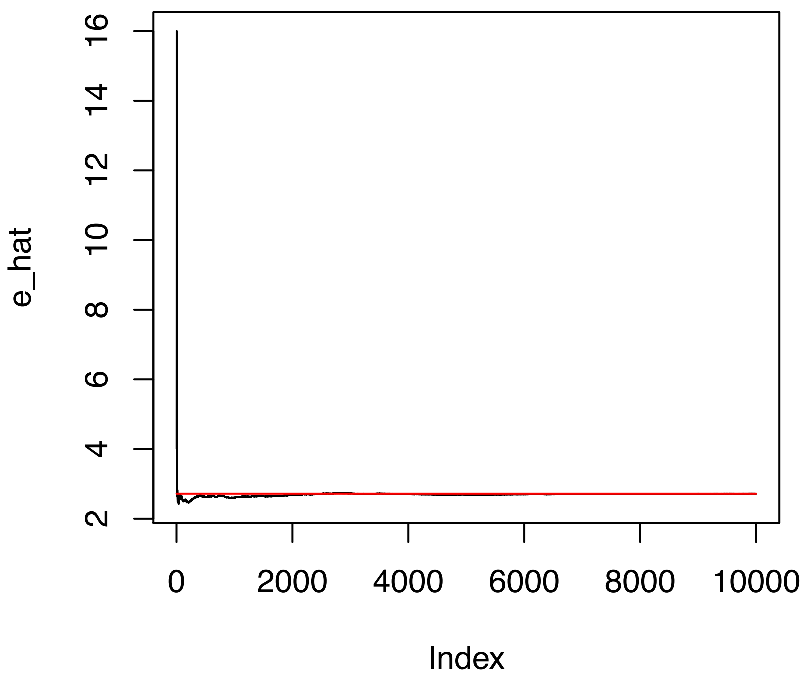
Exercise 33. Give a randomized algorithm to approximate value of e.

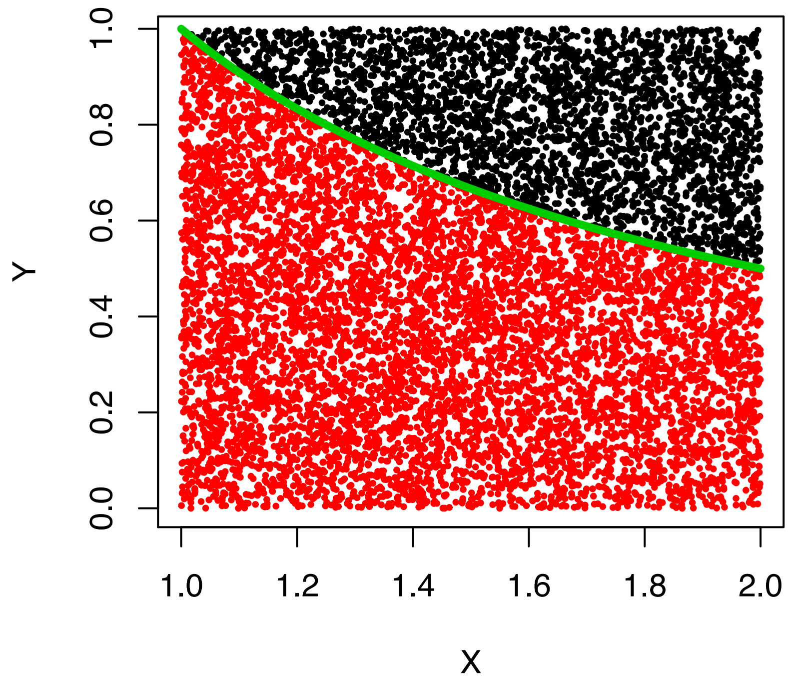
```r
# Date:26 July 2019
## Pi estimation
## Range of X
a1<-0
b1<-1
## Range of Y
a2<-0
b2<-1
itrn<- 10000 # iteration number
x<-runif(itrn,a1,b1) # generate X
y<-runif(itrn,a2,b2) # generate Y
pi_true<-rep(pi,itrn) # True value of pi
pi_hat<-array(0,dim=c(itrn))
count<-array(0,dim=c(itrn))
xx<-seq(a1,b1,by=0.01) # Sequence on [0,1]

for(i in 1 : itrn){
  count [i]<- (x[i]^2+y[i]^2<1) # (X,Y) in circle or not
  pi_hat[i]<-4*(sum(count)/i) # How many in cirecle amonge 'i' trials
}
# Plot
plot(pi_hat, type = 'l')
lines(pi_true, col=2)

s0<- which (count==0) # points out of circle
s1<- which (count==1) # points in circle
plot(y[s0]~x[s0], pch = 20, cex = 0.5, xlab="X", ylab = "Y")
lines (y[s1]~x[s1], col=2, type='p', pch = 20, cex = 0.5)
lines(sqrt(1-(xx)^2)~xx , col=3, lwd=4) # equation of circle
```


pi_hat
4.0
3.8
3.6
3.4
3.2
3.0
0
2000
4000
6000
8000
10000
Index


1.0
0.8
0.6
0.4
0.2
Y
0.0
0.2
0.4
0.6
0.8
1.0
X

```r
# e estimation
# # Date:26 July 2019
a1<-1
b1<-2
a2<-0
b2<-1
itrn<- 10000
x<-runif(itrn,a1,b1)
y<-runif(itrn,a2,b2)
e_true<-rep(exp(1),itrn)
e_hat<-array(0,dim=c(itrn))
count<-array(0,dim=c(itrn))
xx<-seq(a1,b1,by=0.01)
for(i in 1 : itrn){
  count [i]<- (x[i]*y[i]<1)
  area<-(sum(count)/i)
  e_hat[i]<- 2^(1/area)

}
plot(e_hat, type = 'l')
lines(e_true, col=2)
s0<- which (count==0)
s1<- which (count==1)
plot(y[s1]~x[s1], col=2, pch = 20, cex = 0.5, xlab="X", ylab = "Y")
lines (y[s0]~x[s0], type='p', pch = 20, cex = 0.5)
lines((1/(xx))~xx , col=3, lwd=4)
```


e_hat
2
4
6
8
10
12
14
16
0
2000
4000
6000
8000
10000
Index


1.0
0.8
0.6
0.4
0.2
0.0
Y
1.0
1.2
1.4
1.6
1.8
2.0
X

5. Random Variable and it's moments

Definition 34. Random variable: Let $(\Omega, \mathcal{A}, P)$ be a probability space. Then a function $X : \Omega \to \mathbb{R}$ is called a random variable if

$$X^{-1}((-\infty, x]) \equiv \{\omega | X(\omega) \leq x\} \in \mathcal{A} \ \ \forall x \in \mathbb{R}$$

Remark 35. **Random variable is a deterministic function which has nothing random in it.**

Remark 36. Consider a function $g : \mathbb{R} \to \mathbb{R}$ and X is a random variable on $(\Omega, \mathcal{A}, P)$. Then $Y = g(X)$ is also a random variable. It implies that $X^{-1}(g^{-1}(-\infty, x]) \in \mathcal{A}$ for any $x \in \mathbb{R}$. It means $Q((-\infty, x]) = P(Y \in (-\infty, x]) = P(X \in g^{-1}((-\infty, x]))$, which is known as **push forward** of probability.

Definition 37. Vector Valued Random variable: $(X_1(\omega), X_2(\omega), \cdots, X_k(\omega))$ is a vector valued random variable where $\omega \in \Omega$.

Remark 38. Let X and Y be random variables. Then,

- $aX + bY$ is a random variable for all$a, b \in \mathbb{R}$.
- $\max\{X, Y\}$ and $\min\{X, Y\}$ are random variables.
- XY is a random variable.
- Provided that $P(Y(\omega) = 0) = 0$ for each $\omega \in \Omega$, then X/Y is a random variable.

Definition 39. Cumulative distribution function (c.d.f.): Cumulative distribution function of a random variable X is a function $F : \mathbb{R} \to [0, 1]$ defined as

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= P(X^{-1}(-\infty, x]) \\ &= P(\{\omega | X(\omega) \in (-\infty, x]\}) \ \ \forall x \in \mathbb{R}. \end{aligned}$$

Remark 40. Cumulative distribution function uniquely identifies a random variable.

Exercise 41. Prove the properties of a c.d.f. :

(1) $F(-\infty) = \lim_{x \downarrow -\infty} F(x) = 0$
(2) $F(\infty) = \lim_{x \uparrow \infty} F(x) = 1$
(3) $F(a) \leq F(b) \quad \forall a \leq b \in \mathbb{R}$ [non-decreasing]
(4) $F(a) = \lim_{x \downarrow a} F(x) \quad \forall a \in \mathbb{R}$ [right-continuous]

Definition 42. Discrete valued random variable: For a given probability space $(\Omega, \mathcal{A}, P)$ a random variable X is said to be a discrete valued random variable if $S = \{X(\omega) | \omega \in \Omega\}$ is a **finite or countably infinite** set and $X^{-1}(s_i) \in \mathcal{A}$ for all $s_i \in S$.

Remark 43. There can be finitely or countably many jump discontinuities in a c.d.f. of a random variable. The sum of the magnitude of jumps is one, which is the total probability.

Definition 44. Probability mass function(p.m.f.): If X is a discrete valued random variable ina a given probability space $(\Omega, \mathcal{A}, P)$ then a non-negative function

$f(x) := P(X = x)$ on $\mathbb{R}$ is called a probability mass function or discrete density function of the random variable X. Probability mass function has the following properties

- $f(x) \geq 0 \ \ \forall x \in \mathbb{R}$.
- $S = \{x | f(x) > 0\}$ is finite or a countably infinite set.
- $\sum\limits_{s \in S} f(s) = 1$

Definition 45. Continuous valued random variable: For a given probability space $(\Omega, \mathcal{A}, P)$ a random variable X is said to be a continuous valued random variable $P(X = x) = 0 \forall x \in \mathbb{R}$.

Definition 46. Probability density function(p.d.f.): If X is a continuous valued random variable in a given probability space $(\Omega, \mathcal{A}, P)$ with c.d.f $F(\cdot)$ then a non-negative function $f : \mathbb{R} \to [0, \infty)$ is called **a probability density function** of X if

$$P(X \in A) = \int_x f(x) \mathbf{1}_{\{x \in A\}} dx$$

Remark 47. In particular for $A = (-\infty, x]$ for any $x \in \mathbb{R}$ then

$$f(x) = \frac{d}{dx} F(x) = \int_{-\infty}^{x} f(t) dt.$$

Definition 48. Expectation: The expectation of a random variable X with c.d.f $F_X(\cdot)$ is defined as $E(X) = \int x dF_X(x)$ where,

$$\int x dF_X(x) = \begin{cases} \sum_x x f(x), & \text{if } \sum_x |x| f(x) < \infty \text{ for discrete } X, \\ \int_x x f(x), & \text{if } \int_x |x| f(x) < \infty \text{ for continuous } X. \end{cases}$$

Exercise 49. Find the expectation of the random variables with the following densities

(a) $f(x) = \frac{1}{\pi(1+x^2)}$ when $x \in \mathbb{R}$

(b) $f(x) = \frac{1}{|x|(1+|x|)}$ when $x \in S = \{(-1)^n n | n \in \mathbb{N}\}$

Definition 50. Moment generating function: The moment generating function (**m.g.f.**) of a random variable X is defined as

$$M_X(t) = E(e^{tX}) \text{ if } E(e^{tX}) < \infty \ \forall t \in (-\epsilon, \epsilon) \text{ for some } \epsilon > 0$$

- Cumulative distribution function (c.d.f) and uniquely identify the probability distribution of a random variable.
- Moment generating function (m.g.f.) if exists then uniquely identifies the probability distribution of a random variable.
- Probability density function identifies the probability distribution of a random variable up to some length or volume zero set. So it is not unique in general.

Remark 51. Probability mass function can be considered as discrete density function with respect to count measure. If X is a discrete valued random variable with $P(X \in S) = 1$, where S is a countable set, then a non-negative function f is called a probability mass function or discrete density function of the random variable X if $P(X \leq x) = \sum_{s \in S} f(s)\mathbf{1}_{\{s \leq x\}} \quad \forall x \in \mathbb{R}$.

5.1. **Moments of a random variable.**

Definition 52. Raw moment: Let X be a discrete valued random variable with p.m.f $f(\cdot)$ such that $\sum_x |x^r| f(x) < \infty$. Then the r^{th} order raw moment of X is defined as

$$\mu_r^{'} = E(X^r) = \sum_x x^r f(x)$$

Definition 53. Central moment: Let X be a discrete valued random variable with p.m.f $f(\cdot)$ such that $\sum_x |(x-\mu_x)^r| f(x) < \infty$. Then the r^{th} order raw moment of X is defined as

$$\mu_r = E(X-\mu_x)^r = \sum_x (x-\mu_x)^r f(x).$$

- Mean of random variable X is $\mu_1^{'} = E(X) = \sum_x x f(x) = \mu_x$.
- Variance of random variable X is $\mu_2 = E(X-\mu_x)^2 = \sum_x (x-\mu_x)^2 f(x) = Var(X)$.

Exercise 54. Prove that:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$$

Exercise 55. Show that $g(a) = \frac{1}{n}\sum_{i=1}^{n}(x_i-a)^2$ is minimum if $a = \bar{x}$.

Theorem 56. *If X is a non-negative integer-valued random variable with finite expectation then*

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k)$$

Moments from Moment generating function: Let X be a discrete valued random with moment generating function

$$M_X(t) = E(e^{tx}) = \sum_{k=0}^{\infty}\frac{t^k E(X^k)}{k!}$$

Then one can obtain the kth order raw moment from m.g.f. by

$$\frac{\partial^k}{\partial t^k} M_X(t)|_{t=0} = \mu_k^{'}$$

Exercise 57. Prove the following inequalities:

- **Markov's Inequity:** If X is a non-negative valued random variable then
$$P(X > t) \leq \frac{E(X)}{t} \quad \forall t > 0$$
- **Chebyshev's Inequity:** $P(|X-\mu_x| > \epsilon) \leq \frac{E(X-\mu_x)^2}{\epsilon^2}$

Definition 58. For a discrete probability distribution, a median is by definition any real number m that satisfies the inequalities

$$\mathrm{P}(X \leq m) \geq \frac{1}{2} \text{ and } \mathrm{P}(X \geq m) \geq \frac{1}{2}$$

and for a continuous probability distribution,

$$\mathrm{P}(X \leq m) \geq \frac{1}{2} = \mathrm{P}(X \geq m) = \frac{1}{2}\,.$$

Exercise 59. Graphically show that $g(a) = \frac{1}{n}\sum_{i=1}^{n} |x_i - a|$ is minimum if $a =$ *median* of $\{x_1, \cdots, x_n\}$.

6. Modeling with Random Variables

Definition 60. Independent Random Variables: Two random variables X and Y are said to be independently distributed if

$$P((X,Y) \in A \times B)) = P(X \in A)P(Y \in B) \text{ for any } A, B \in \mathcal{B}(\mathbb{R})$$

This implies
(a)$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$ for any $x, y \in \mathbb{R}$
(b)$f(x,y) = f_x(x)f_y(y)$

Definition 61. Identically distributed random variables: Two random variables X and Y are said to be identically distributed if

$$P(X \in (-\infty, a]) = P(Y \in (-\infty, a]) \text{ for any } a \in \mathbb{R}.$$

Remark 62. Two random variables X and Y are **independently and identically distribute (i.i.d.)** random variables if the above two definitions hold.

Definition 63. Uniform Distribution[0,1]: A random variable X is said to have uniform distribution over $[0,1]$ if

$$P(X \in A) = \frac{length(A)}{length([0,1])} = length(A)$$

for any interval $A \subseteq [0,1]$. If $X \sim U[0,1]$ then the p.d.f. is

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and the c.d.f. is

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x \in [0,1] \\ 1, & \text{if } x > 1 \end{cases}$$

6.1. **Examples of discrete random variables:**

- **Discrete uniform:** [random sampling with replacement from a finite population]

$$P(X = s) = \begin{cases} \frac{1}{k}, & \text{if } s \in \mathcal{S} = \{s_1, s_2, \cdots s_k\} \\ 0, & \text{otherwise} \end{cases}$$

- **Bernoulli (p):** [binary $\{0,1\}$ random variable]

$$P(X = x) = \begin{cases} p^x(1-p)^{1-x}, & \text{if } x \in \mathcal{S} = \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$

- **Binomial (n,p):** [sum of n i.i.d. Bernoulli(p)]

$$P(X = x) = \begin{cases} \binom{n}{x} p^x(1-p)^{n-x}, & \text{if } x \in \mathcal{S} = \{0, 1 \cdots, n\} \\ 0, & \text{otherwise} \end{cases}$$

- **Geometric (p):** [number of failures preceding to the first success]

$$P(X = x) = \begin{cases} p(1-p)^x, & \text{if } x \in \mathcal{S} = \{0\} \cup \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

- **Negative Binomial (r,p)** [sum of r i.i.d. geometric(p)]

$$P(X=x) = \begin{cases} \binom{x+r-1}{r-1} p^r (1-p)^x, & \text{if } x \in \mathcal{S} = \{0,1,2\cdots\} \\ 0, & \text{otherwise} \end{cases}$$

- **Poisson (λ):** [limiting distribution of $bin(n,p_n)$ when $n \uparrow \infty$, $p_n \downarrow 0$ but $np_n \to \lambda > 0$]

$$P(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x \in \mathcal{S} = \{0\} \cup \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

- **Hyper-geometric:** [random sampling without replacement from a finite population divided into two categories]

$$P(X=x) = \begin{cases} \frac{\binom{n_1}{r_1}\binom{n_2}{r_2}}{\binom{n_1+n_2}{r}}, & \text{if } r_1 = 0,1,\cdots \min\{n_1,r\}; r_2 = 0,1,\cdots \min\{n_2,r\}; r = r_1+r_2, \\ 0, & \text{otherwise} \end{cases}$$

Theorem 64. *Let $\{X_n\}$ be a sequence of random variables with corresponding m.g.f.s as$M_{X_n}(t)$ such that $\lim_{n\uparrow\infty} M_{X_n}(t) = M_Y(t)$ for some random variable Y. Then*

$$\lim_{n\uparrow\infty} F_{X_n}(a) = F_Y(a)$$

for all such $a \in \mathbb{R}$, where $F_Y(a)$ is continuous. We say X_n converges in distribution to Y. [Proof is not included in the syllabus]

6.2. **Examples of continuous random variables:**

Exercise 65. Let $X \sim U[0,1]$. Find the c.d.f and p.d.f

(a)$Y_1 = cX$ where $c > 0$. [this is known as $U[0,c]$]
(b)$Y_2 = a + (b-a)X$ where $b > a$.[this is known as $U[a,b]$]
(c) Show that $E(Y_2) = \frac{b+a}{2}$ and $Var(Y_2) = \frac{(b-a)^2}{12}$.

Exercise 66. Let $X \sim U[0,1]$. Find the c.d.f and p.d.f.
(a)$Z = \frac{-1}{\lambda}\log(1-X)$ where $\lambda > 0$[it is known as $exponential(\lambda)$ distribution]
(b)Show that $E(Z) = \frac{1}{\lambda}$ and $Var(Z) = \frac{1}{\lambda^2}$.

- A positive valued random Y variable is said to follow *Gamma distribution* with shape parameter $\alpha(>0)$ and scale parameter $\lambda(>0)$ if it has p.d.f.

$$f(y) = \frac{\lambda^\alpha e^{-\lambda y} y^{\alpha-1}}{\Gamma(\alpha)} \mathbf{1}_{\{y>0\}}$$

Remark 67. Let $X_1 \sim Gamma(a,\lambda)$ and $X_2 \sim Gamma(b,\lambda)$ are independently distributed then $Y = X_1 + X_2 \sim Gamma(a+b,\lambda)$. Use MGF.

Remark 68. If $X \sim Gamma(a,\lambda)$, then $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$

- Let $X_1 \sim G(a,\lambda)$ and $X_2 \sim G(b,\lambda)$ are independently distributed then $Y = \frac{X_1}{X_1+X_2}$ is said to follow$Beta(a,b)$ and the p.d.f. of $Y$is given by

$$f(y) = \begin{cases} \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}, & \text{if } y \in [0,1], a > 0, b > 0 \\ 0, & \text{otherwise} \end{cases}$$

Remark. Let $Y \sim B(a,b)$ then, $E(Y) = \frac{a}{a+b}$ and $Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}$

- A random variable X is is said to follow $Cauchy(\mu, \sigma)$ if it has the p.d.f.

$$f(x) = \frac{\sigma}{\pi(\sigma^2 + (y-\mu)^2)}$$

and has c.d.f.

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{y-\mu}{\sigma}\right)$$

- A random variable X is is said to follow $Normal(\mu, \sigma^2)$ if it has the p.d.f.

$$f(x) = \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}}$$

where $x, \mu \in \mathbb{R}$ and $\sigma > 0$.

NOTE:

(1) $N(0,1)$ is also known as standard normal distribution.
(2) p.d.f of standard normal distribution is denoted by $\phi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$
(3) c.d.f of standard normal distribution is denoted by $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} dt$
(4) If $Z \sim N(0,1)$ then$Y = \mu + \sigma Z \sim N(\mu, \sigma^2)$ with $E(Y) = \mu$ & $Var(Y) = \sigma^2$
(5) The p.d.f of Y can be written as $f(y) = \frac{1}{\sigma}\phi(\frac{y-\mu}{\sigma})$
(6) $\phi(\cdot)$ is a symmetric function around zero i.e. $\phi(z) = \phi(-z)$
(7) $\Phi(-z) = 1 - \Phi(z)$
(8) $P(Z < 1.64) = 0.95$ and $P(Z < 1.96) = 0.975$.

- **Chi-squared Distribution:** If $Z \sim N(0,1)$ then random variable$Y = Z^2$ is said to follow χ_1^2 i.e. chi-squared distribution with one degree of freedom.

Remark 69. Y following χ_1^2 has same p.d.f of $Gamma(1/2, 1/2)$ distribution.

Remark 70. If Z_i be i.i.d.$N(0,1)$ then random variable $Y = \sum_{i=1}^{n} Z_i^2$ follows χ_n^2 i.e. chi-squared distribution with$n$ degree of freedom which is equivalent to $Gamma(n/2, 1/2)$ distribution.

Remark 71. Let $Y \sim \chi_n^2$ then show that $E(Y) = n$ and $Var(Y) = 2n$

- $\mathbf{t-distribution}$: If $Z \sim N(0,1)$ and $Y \sim \chi_k^2$ are independently distributed random variables the

$$X = \frac{Z}{\sqrt{Y/k}} \sim t_k, \text{ i.e. t-distribution with k degrees freedom.}$$

- $\mathbf{F-distribution}$:If $Y_1 \sim \chi_{k_1}^2$ and $Y_2 \sim \chi_{k_2}^2$ are independently distributed random variables the

$$X = \frac{Y_1/k_1}{Y_2/k_2} \sim F_{k_1,k_2}, \text{ i.e. F-distribution with } k_1, k_2 \text{ degrees freedom.}$$

Theorem 72. *Let ϕ be a strictly monotone function on $I = (a,b)$ with the range $\phi(I)$ and differentiable inverse function $\phi^{-1}(\cdot)$ on $\phi(I)$. Also assume that X be a continuous valued random variable with p.d.f $f_X(x) = 0$ if $x \notin I$. Then $Y = \phi(X)$ has density $g(\cdot)$ on $\phi(I)$ as*

$$g(y) = f(\phi^{-1}(y))\left|\frac{d}{dy}\phi^{-1}(y)\right|.$$

Exercise 73. Let $X \sim bin(n,p)$. Show that $M_X(t) = (pe^t + 1 - p)^n$

Exercise 74. Let $Y \sim pois(\lambda)$. Show that $M_Y(t) = e^{-\lambda(1-e^t)}$

Exercise 75. $X \sim bin(n, p_n)$ such that $n \uparrow \infty$, $p_n \downarrow 0$ and $np_n \to \lambda > 0$. Show that X_n converges in distribution to Y, where $Y \sim pois(\lambda)$

Exercise 76. Let $Y \sim pois(\lambda)$. Show that $E(Y) = Var(Y) = E(Y-\lambda)^3 = \lambda$

Exercise 77. Let $X \sim N(\mu, \sigma^2)$ then find the density function of $Y = e^X$. [Y is said to follow $lognormal(\mu, \sigma^2)$]

Exercise 78. Let $Y \sim lognormal(\mu, \sigma^2)$ find $E(X)$ and $Var(X)$.

Exercise 79. Let $X \sim N(0,1)$ then find the density function of$Y = X^2$.

Exercise 80. Let $X \sim N(\mu, \sigma^2)$ then them MGF of X is $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$

Exercise 81. Let $X \sim exp(\lambda)$. Show that$Y = [X]$ has geometric distribution.

Exercise 82. Let $X \sim geo(p)$. Show that X has **memory less property** i.e. $P(X > m+n | X > m) = P(X > n) = q^n$ where $m, n \in \mathbb{N}$

Exercise 83. Let $X \sim exp(\lambda)$. Show that X has **memory less** property i.e.$P(X > t+s | X > t) = P(X > s)$ where $s, t \in \mathbb{R}$

Exercise 84. Let X be a continuous values random variable. Find the distribution of $Y = F(X)$.

Exercise 85. Let $(\frac{X}{\lambda})^k \sim exp(1)$ for $\lambda > 0$,$k > 0$, find the p.d.f. X. [X is said to follow Weibull distribution]

Exercise 86. Let $1 - (\frac{\lambda}{X})^k \sim U(0,1)$ for $X > \lambda$,$k > 0$, find the p.d.f. X. [X is said to follow Pareto distribution]

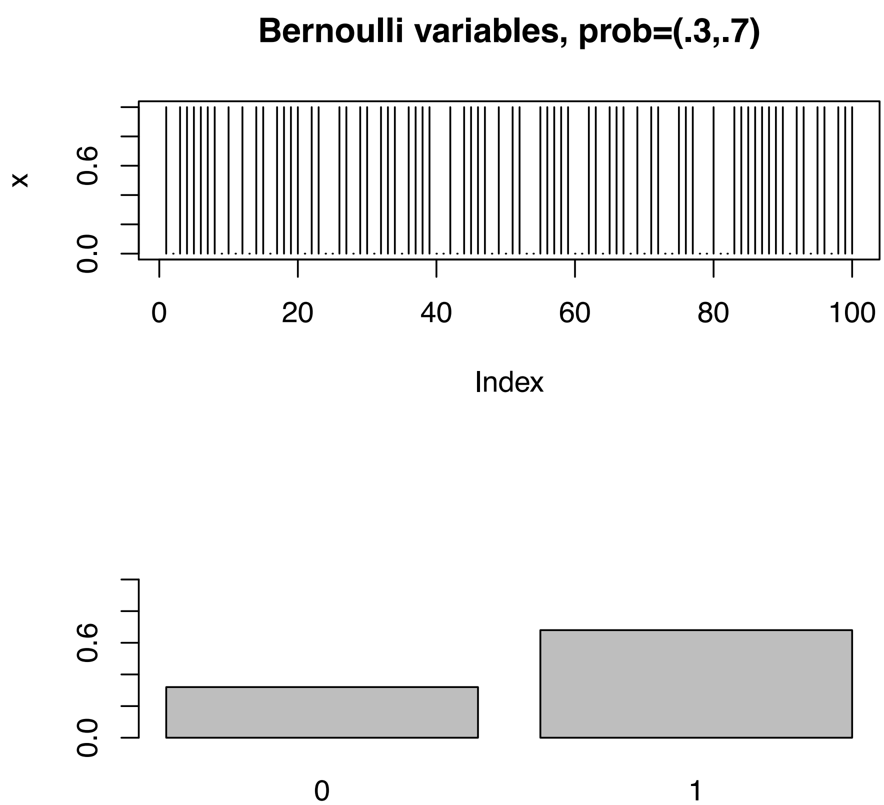
Exercise 87. Let $X \sim bin(n_1, p)$ and $Y \sim bin(n_2, p)$ independently . Use MGF to show that $Z = X + Y \sim bin(n_1 + n_2, p)$

Exercise 88. Let $X \sim pois(\lambda_1)$ and $Y \sim pois(\lambda_2)$ independently . Use MGF to show that $Z = X + Y \sim pois(\lambda_1 + \lambda_2)$

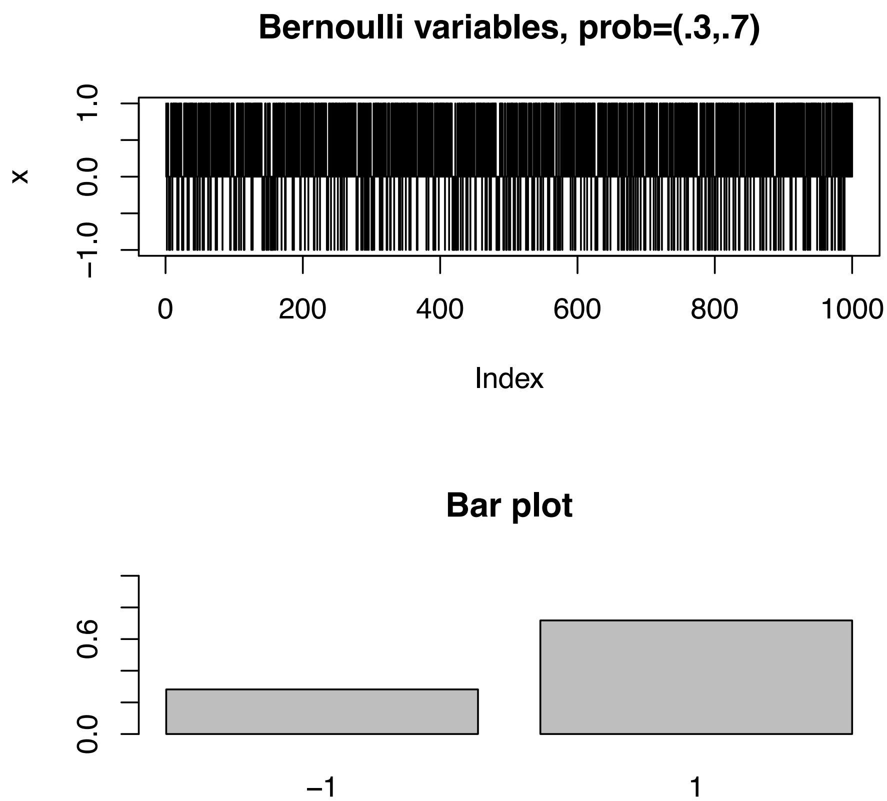
Exercise 89. Let $X \sim gamma(\alpha_1, \lambda)$ and $Y \sim bin(\alpha_2, \lambda)$ independently . Use MGF to show that $Z = X + Y \sim bin(\alpha_1 + \alpha_2, \lambda)$

Exercise 90. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ independently . Use MGF to show that $Z = X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

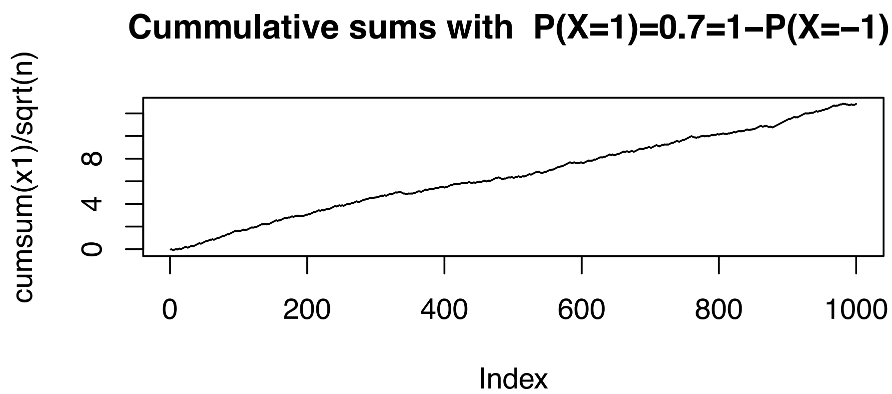
```r
# Bernoulli distribution of parameter p=0.7
n <- 100
x <- sample(c(0,1), n, replace=T, prob=c(.3,.7))
par(mfrow=c(2,1))
plot(x, type='h',main="Bernoulli variables, prob=(.3,.7)")
barplot(table(x)/n, ylim = c(0,1))
```

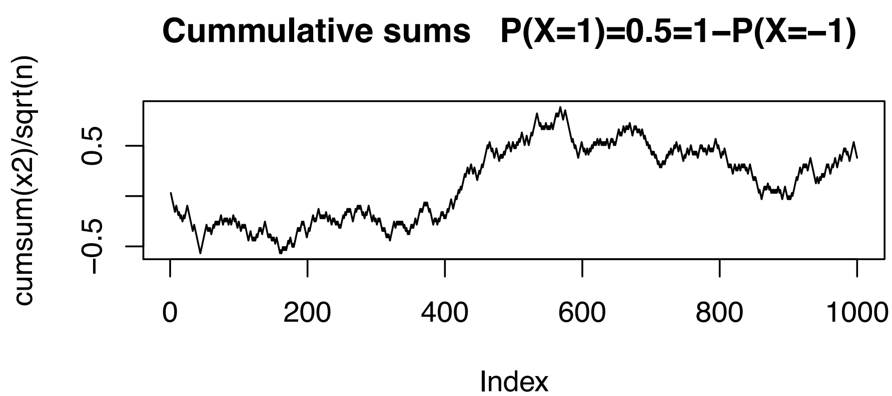



```r
# Bernoulli distribution of parameter p=0.8 and X =-1 and +1
n <- 1000
x <- sample(c(-1,1), n, replace=T, prob=c(.3,.7))
par(mfrow=c(2,1))
plot(x, type='h', main="Bernoulli variables, prob=(.3,.7)")
barplot(table(x)/n, ylim = c(0,1), main = "Bar plot")
```

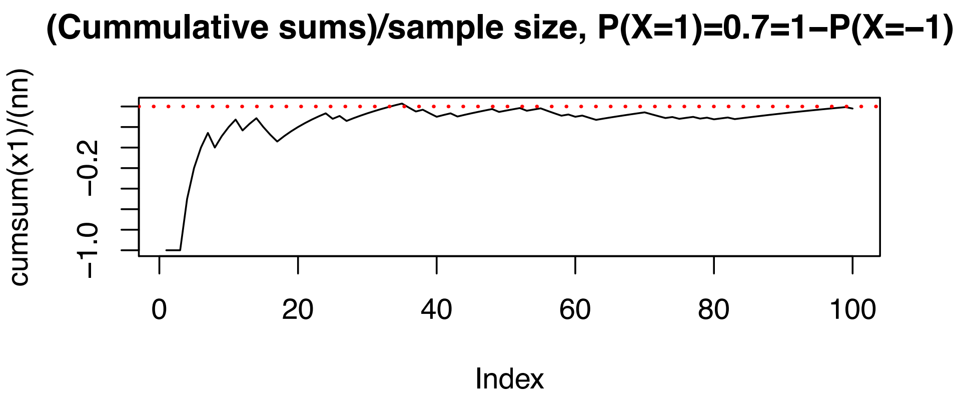


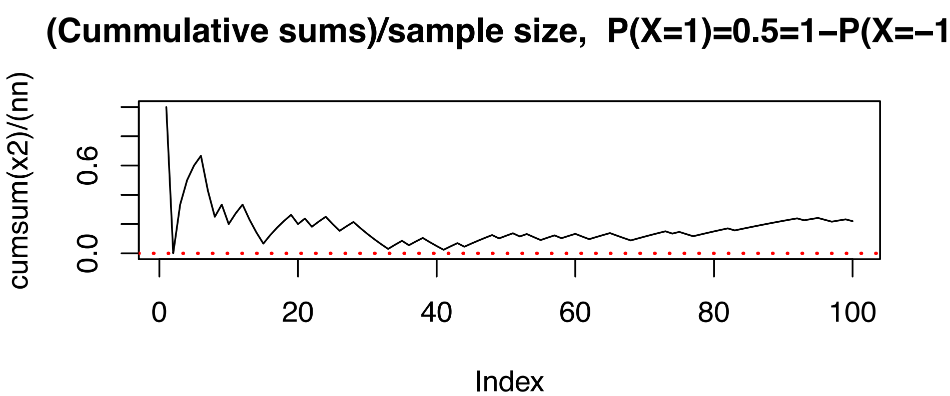

```
# Cummulative sums  X =-1 and +1 with scaling 1/sqrt(n)
n <- 1000
x1 <- sample(c(-1,1), n, replace=T, prob=c(.3,.7))
x2 <- sample(c(-1,1), n, replace=T, prob=c(.5,.5))
par(mfrow=c(2,1))
plot(cumsum(x1)/sqrt(n), type='l',main="Cummulative sums with  P(X=1)=0.7=1-P(X=-1) ")
plot(cumsum(x2)/sqrt(n), type='l',main="Cummulative sums   P(X=1)=0.5=1-P(X=-1) ")
```



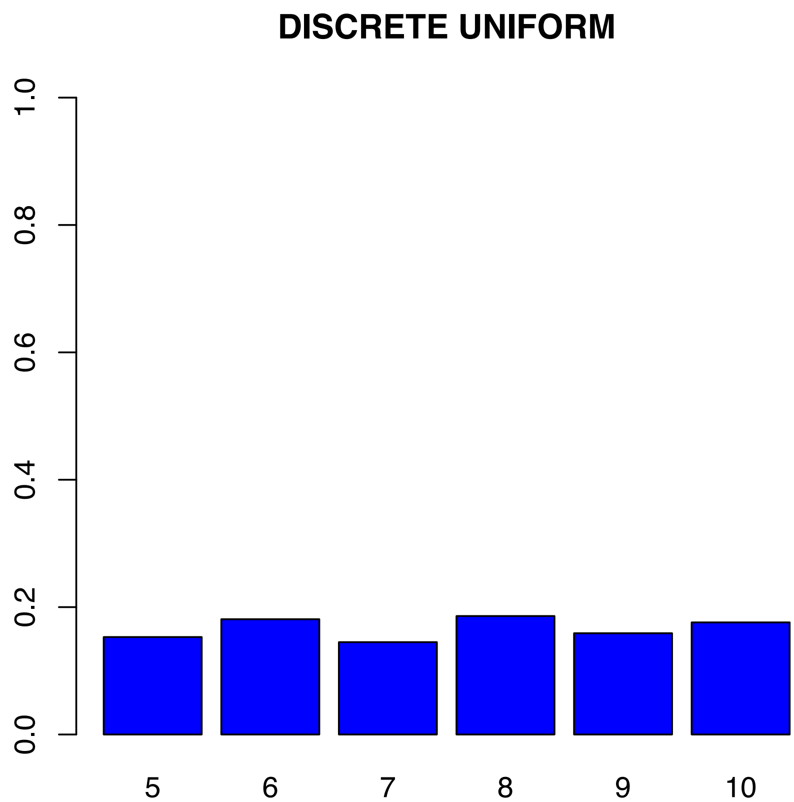



```r
# Cummulative sums  X =-1 and +1 with scaling 1/(n)
n <- 100
nn<-((1:n))
x1 <- sample(c(-1,1), n, replace=T, prob=c(.3,.7))
x2 <- sample(c(-1,1), n, replace=T, prob=c(.5,.5))
par(mfrow=c(2,1))
plot(cumsum(x1)/(nn), type='l',
     main="(Cummulative sums)/sample size, P(X=1)=0.7=1-P(X=-1)  ")
abline(h=0.4, col=2,lwd=2, lty=3)
plot(cumsum(x2)/(nn), type='l',
     main="(Cummulative sums)/sample size,  P(X=1)=0.5=1-P(X=-1) ")
abline(h=0.0, col=2,lwd=2, lty=3)
```

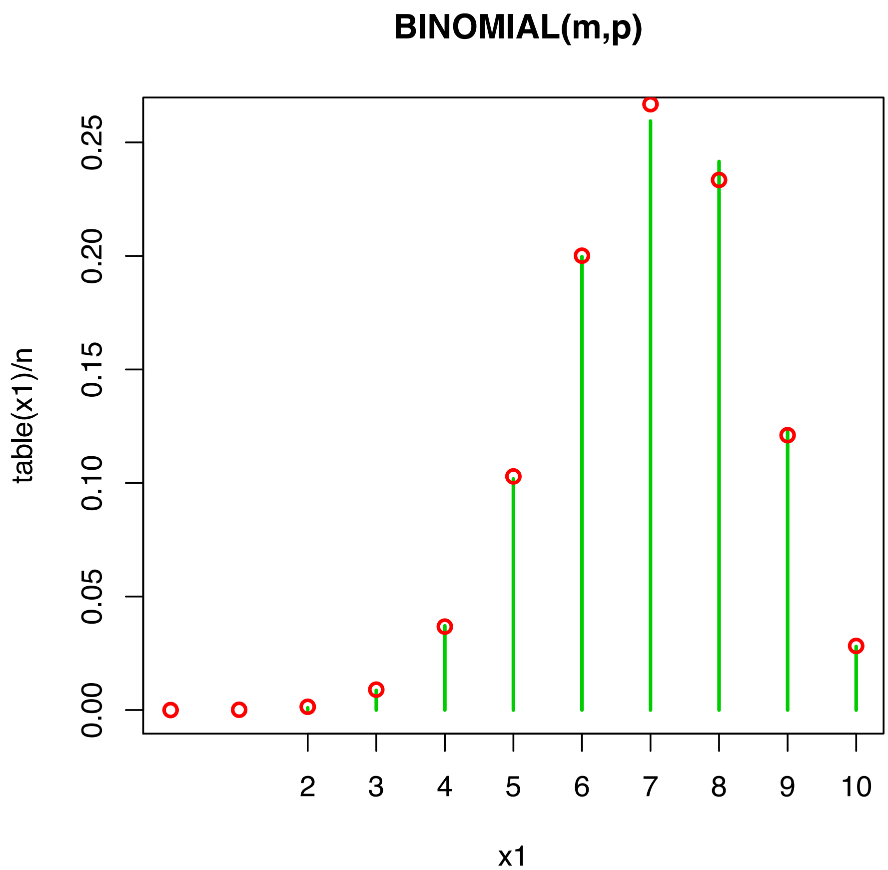




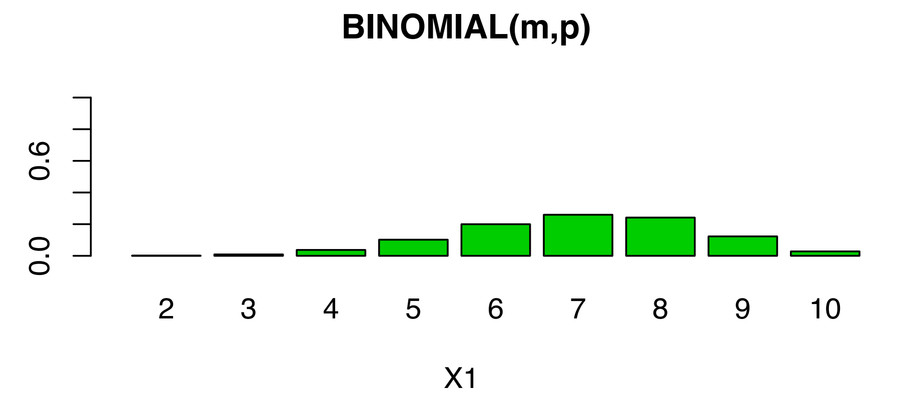

```
####################################################
#   DISCRETE UNIFORM
####################################################
n<- 1000 # sample size
sp<- 5:10 # sample space
x<-sample(sp, n, replace=T)   # data
par(mfrow=c(1,1))
barplot(table(x)/n, col=4, ylim=c(0,1), main = " DISCRETE UNIFORM ")
```

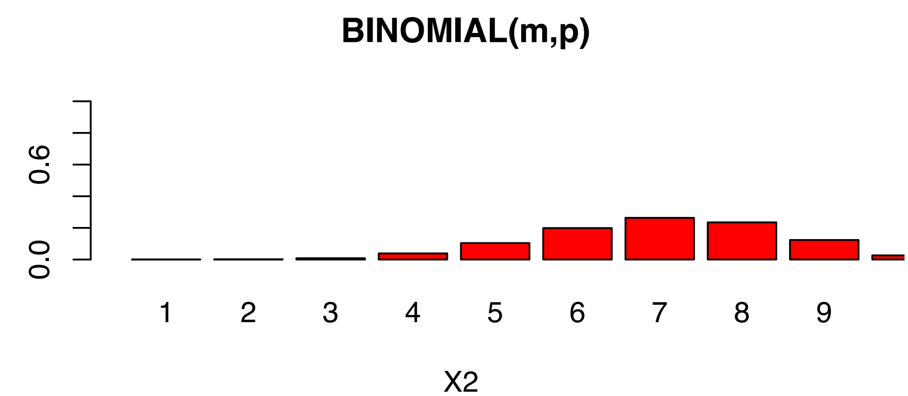



```
##################################################
#   BINOMIAL(m,p)
#It is the total number of  successrs  in a series of  m Bernoulli events.
##################################################
n<- 10000 # sample size
m<-10
p<-0.7
x1<-rbinom(n,m,p)   # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, xlim=c(0,m), main = " BINOMIAL(m,p)")
lines(dbinom(0:m, m,p)~c(0:m), type='p', col=2, lwd=2)
```

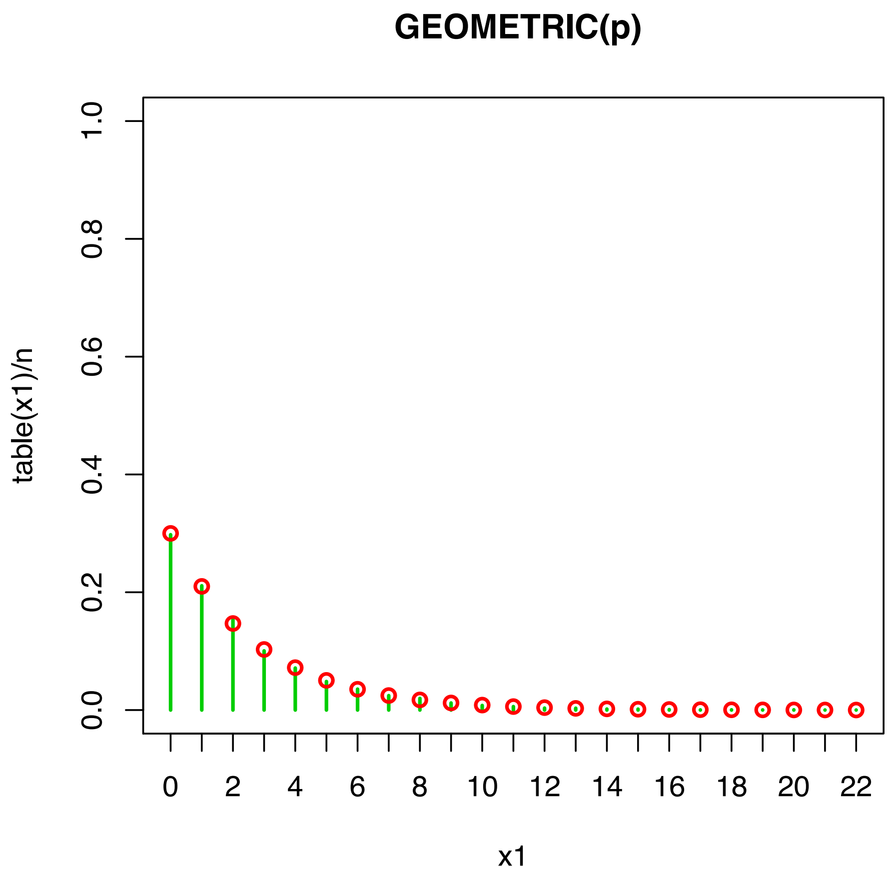



```
# Binomoal from Uniform(0,1)
x2 <- array(0,dim=c(n))
for (i in 1:n) {
  x2[i] <- sum(runif(m)<p)
}
par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, ylim=c(0,1), xlim=c(0,m), xlab='X1', main = " BINOMIAL(m,p)")
barplot(table(x2)/n, col=2, ylim=c(0,1), xlim=c(0,m),xlab='X2', main = " BINOMIAL(m,p)")
```

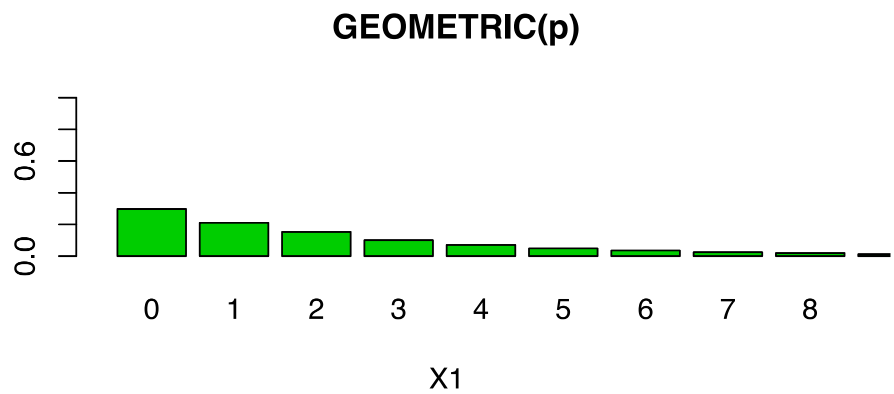


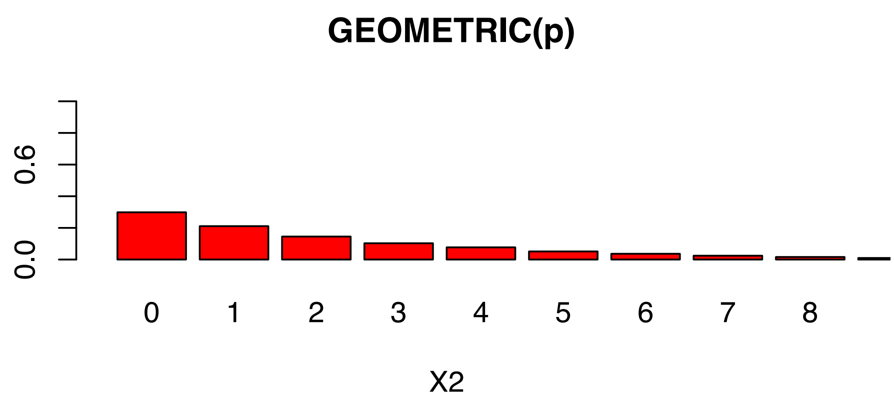



```
#####################################################
#   GEOMETRIC(p)
#It is the number of failures  before a success in a series of Bernoulli event.
#####################################################
n<- 10000 # sample size
p<-0.3
x1<-rgeom(n,p)   # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, ylim=c(0,1), main = " GEOMETRIC(p)")
lines(dgeom(0:max(x1),p)~c(0:max(x1)), type='p', col=2, lwd=2)
```

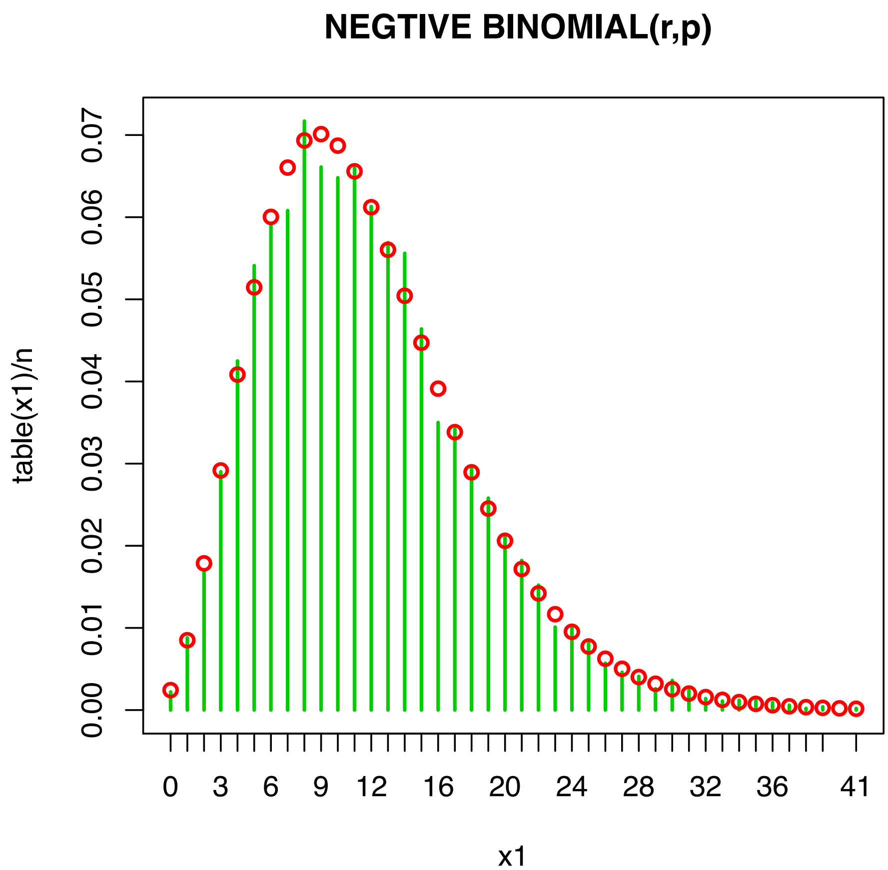



```
# Geometric from Uniform(0,1)
x2 <- array(0,dim=c(n))
for(i in 1 :n){
count<-0
s<-0
while (s==0) {

  count=count+1
  s<-(runif(1)<p)
}
x2[i]<-count-1
}
par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, ylim=c(0,1), xlim=c(0,m), xlab='X1', main = "  GEOMETRIC(p)")
barplot(table(x2)/n, col=2, ylim=c(0,1), xlim=c(0,m), xlab='X2',main = " GEOMETRIC(p)")
```

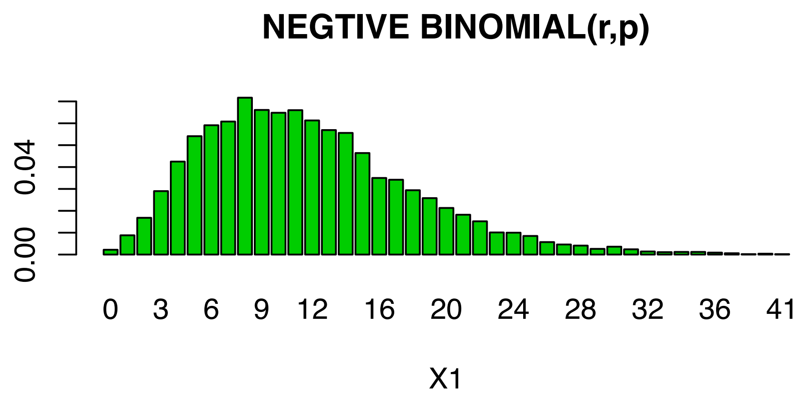


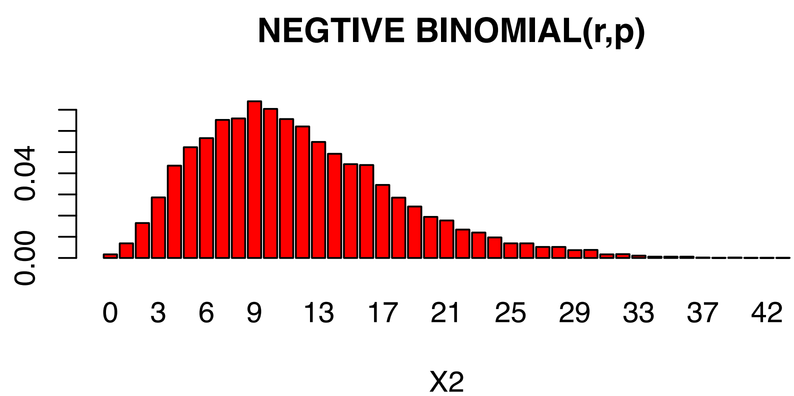



```
#####################################################
#NEGTIVE BINOMIAL(r,p)
#It is the number of  failures  before a success in a series of Bernoulli events.
#####################################################
n<- 10000 # sample size
r<-5
p<-0.3
x1<-rnbinom(n,r,p)   # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, main = " NEGTIVE BINOMIAL(r,p)")
lines(dnbinom(0:max(x1),r,p)~c(0:max(x1)), type='p', col=2, lwd=2)
```

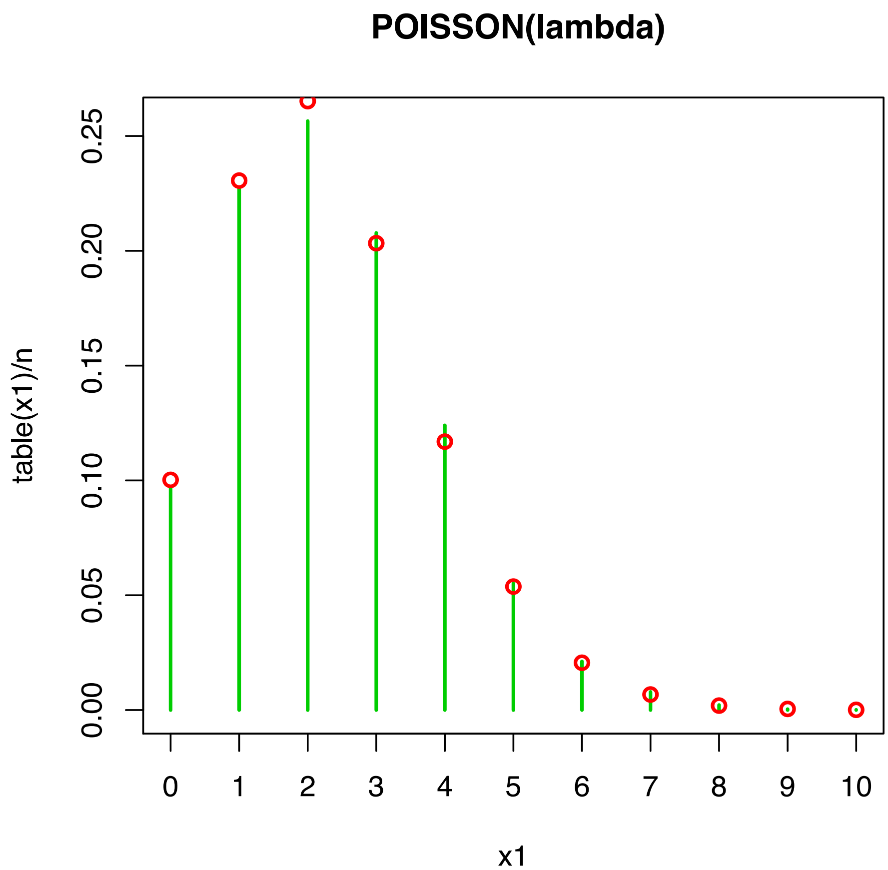



```
# NEGTIVE BINOMIAL(r,p) as a sum of r independent  GEOMETRIC(p)

x2 <- array(0,dim=c(n))
for(i in 1 : n){
  x2[i]<-sum(rgeom(r,p))
}

par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, xlab='X1', main = "  NEGTIVE BINOMIAL(r,p)")
barplot(table(x2)/n, col=2, xlab='X2',  main = " NEGTIVE BINOMIAL(r,p)")
```

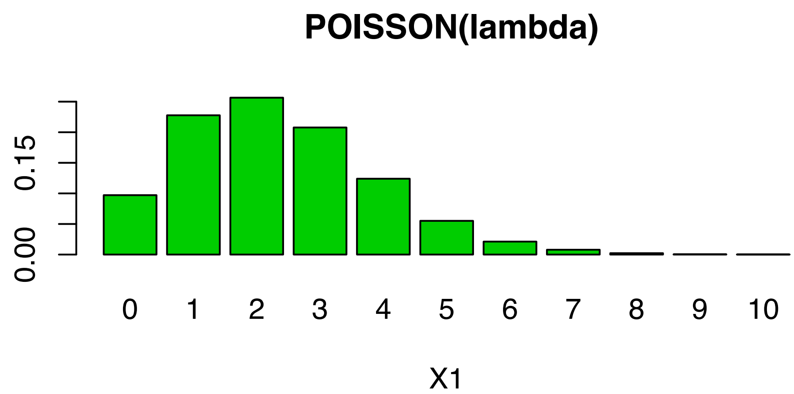


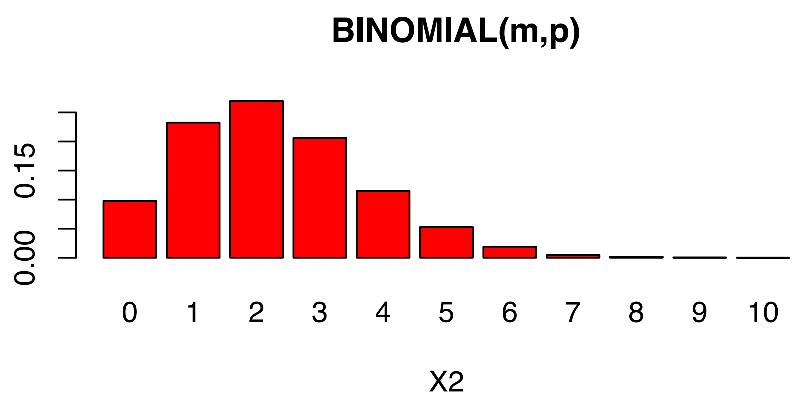



```
#####################################################
#POISSON(lambda)
# Binomal(m,p), m is large and p is small sych thet p*m converges to lambda
#####################################################
n<- 10000 # sample size
lambda<-2.3
x1<-rpois(n, 2.3)   # data
par(mfrow=c(1,1))
plot(table(x1)/n, col=3, main = " POISSON(lambda)")
lines(dpois(0:max(x1),lambda)~c(0:max(x1)), type='p', col=2, lwd=2)
```

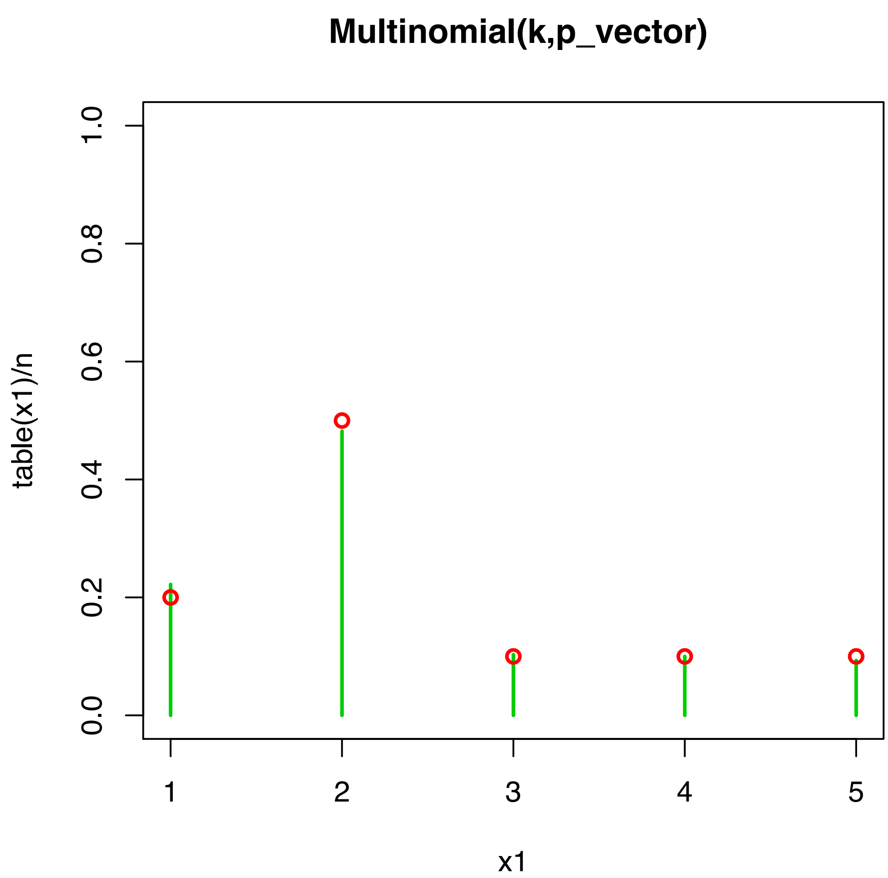



```
# Poisson as  a limit of binomial
n<- 10000 # sample size
m<-100
p<-.023
x2<-rbinom(n,m,p)   # data
par(mfrow=c(2,1))
barplot(table(x1)/n, col=3, xlab='X1', main = " POISSON(lambda)")
barplot(table(x2)/n, col=2, xlab='X2',  main = "  BINOMIAL(m,p)")
```

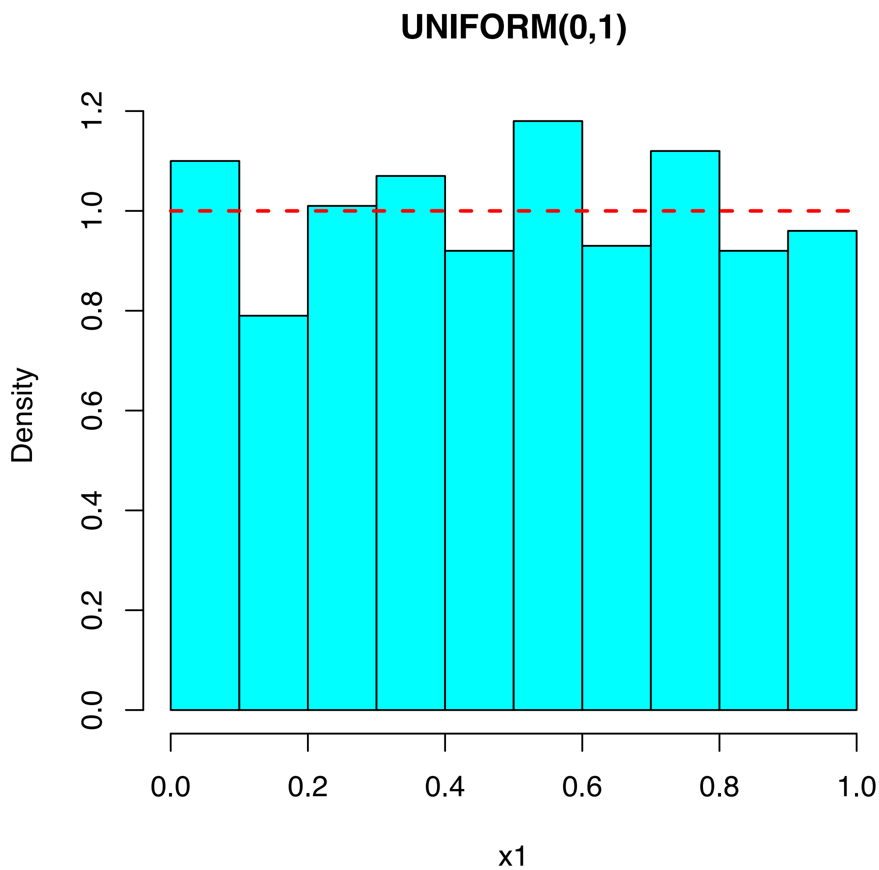




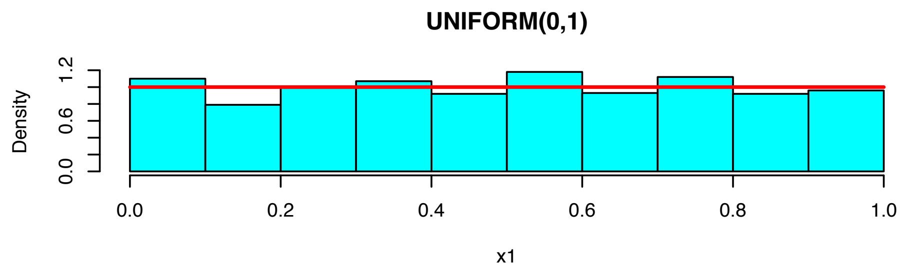

```
cat("\newpage")
##
## ewpage
###############################
#Multinomial(k,p_vector)
###############################
k <- 5  # categories
n <- 1000 # sample size
p <- c(.2,.5,.1,.1,.1) # probbility vector
x1 <- sample(1:k, n, replace=T, prob=p) # data
print(table(x1)/n )
## x1
##     1     2     3     4     5
## 0.222 0.482 0.103 0.100 0.093
par(mfrow=c(1,1))
plot(table(x1)/n,ylim=c(0,1), col=3, main = " Multinomial(k,p_vector)")
lines(p, type='p', col=2, lwd=2)
```

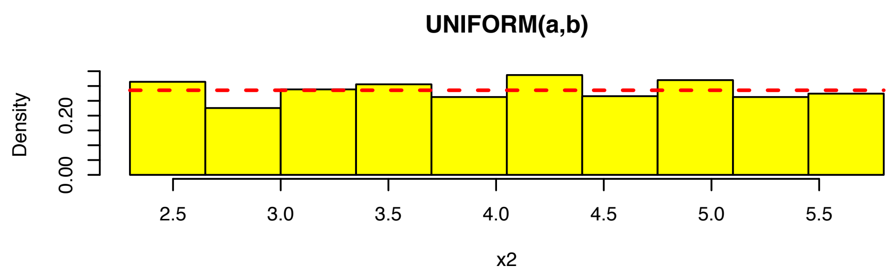


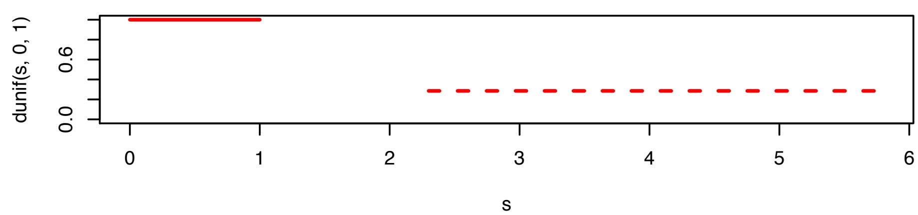

```
#########################################################
#COUNTINUOUS DISTRIBUTIONS
#########################################################

##############################
#UNIFORM(a,b)
##############################
a<-0
b<-1
n<- 1000 #Sample size
x1<-runif(n, a, b)
s<-seq(a,b,length=11)
par(mfrow=c(1,1))
hist(x1, col=5,probability = T, breaks =s, main='UNIFORM(0,1)')
lines(dunif(s,a,b)~s, col=2, lwd=2, lty=2)
```



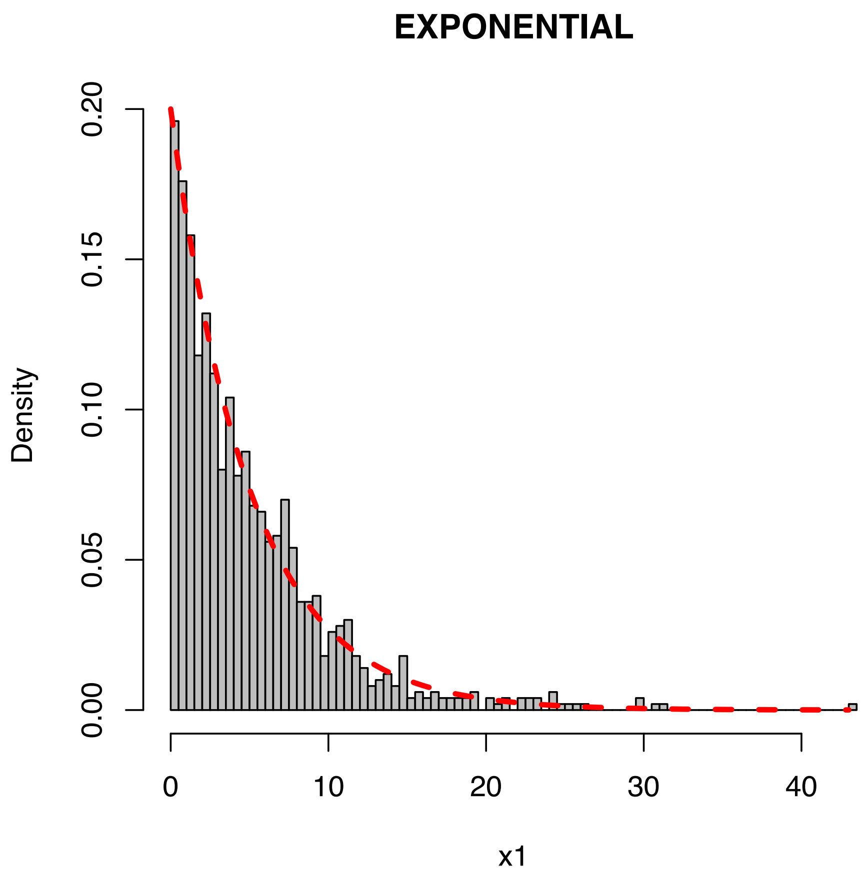

```
a<-2.3
b<-5.8

x2<-a+(b-a)*x1
ss<-a+s*(b-a)


par(mfrow=c(3,1))
hist(x1, col=5,probability = T, breaks =s, main='UNIFORM(0,1)')
lines(dunif(s,0,1)~s, col=2, lwd=2, lty=1)

hist(x2, col=7,probability = T, breaks =ss, main='UNIFORM(a,b)')
lines(dunif(ss,a,b)~ss, col=2, lwd=2, lty=2)

plot(dunif(s,0,1)~s, xlim=c(0,b),ylim=c(0,1), col=2, lwd=2, type='l')
lines(dunif(ss,a,b)~ss, col=2, lwd=2, lty=2)
```



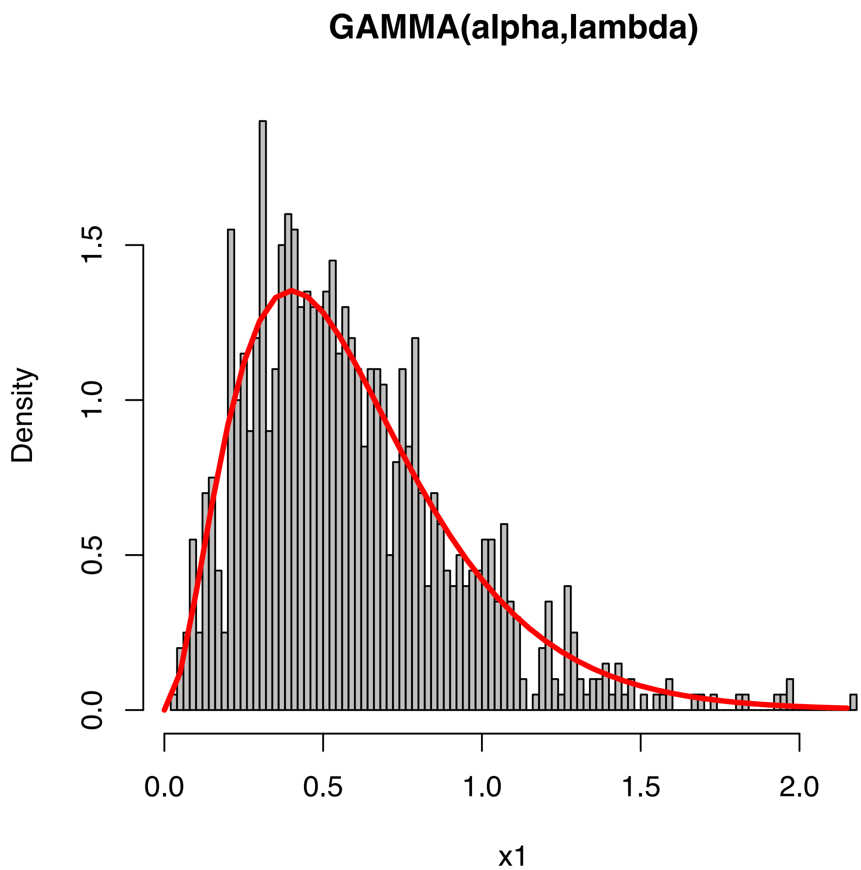




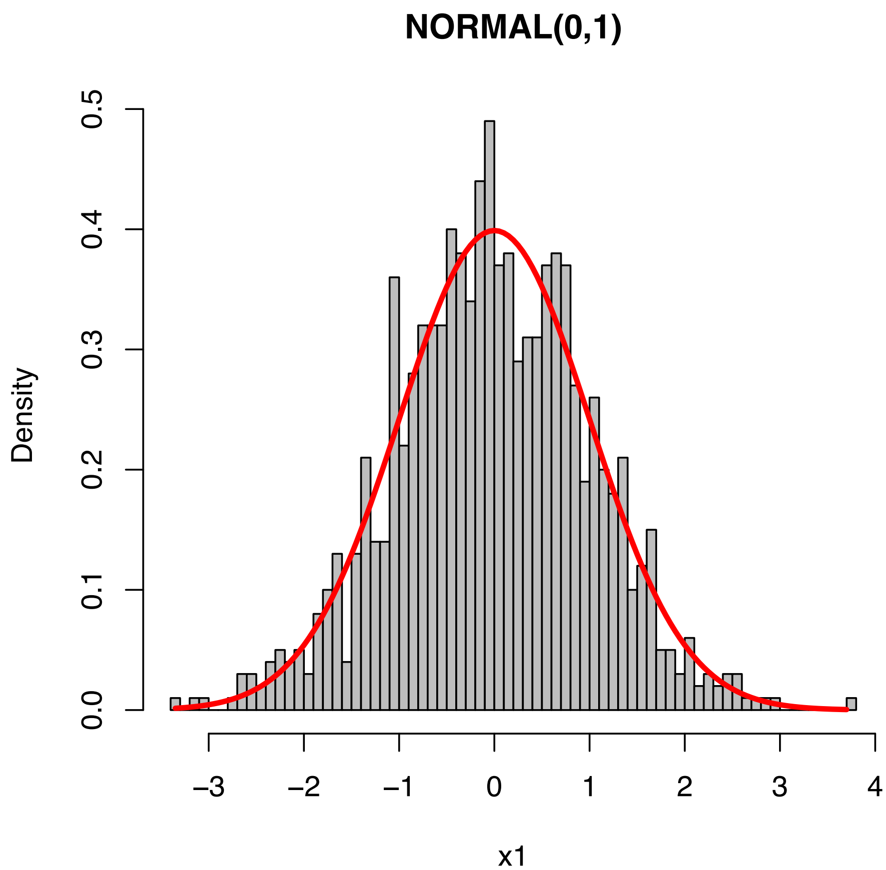

```
##############################
#EXPONENTIAL (LAMBDA)
##############################
lambda<-.2 #rate
mu=1
n<-1000 #sample size
x1 <- rexp(n, lambda)
s<-seq(0,max(x1),by=0.5)
par(mfrow=c(1,1))
hist(x1,probability = T,breaks = 100, col=8, main='EXPONENTIAL')
lines(dexp(s,lambda)~s, col=2, lwd=3, lty=2)
```



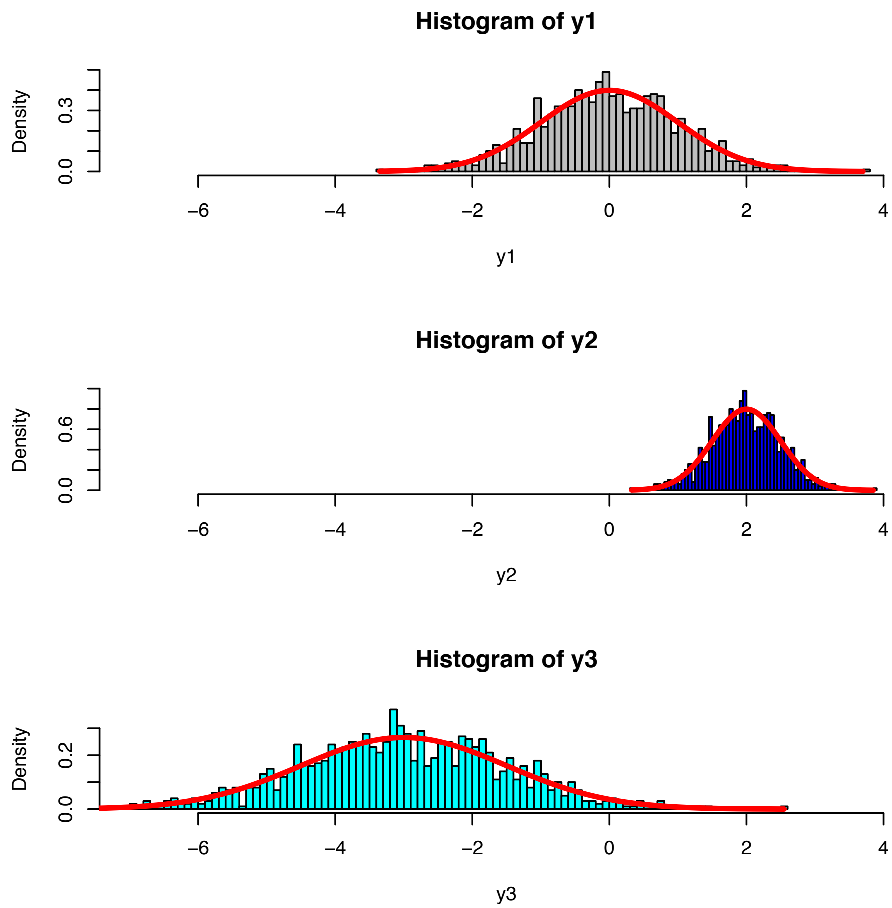

```
##############################
#GAMMA(alpha,lambda)
###############################
alpha<-3 # shape
lambda<-.2 # rate

n<-1000 #sample size
x1 <- rgamma(n, shape=alpha, scale=lambda)
s<-seq(0,max(x1),by=0.05)
par(mfrow=c(1,1))
hist(x1,probability = T,breaks = 100, col=8, main='GAMMA(alpha,lambda)')
lines(dgamma(s, shape=alpha, scale=lambda)~s, col=2, lwd=3)
```



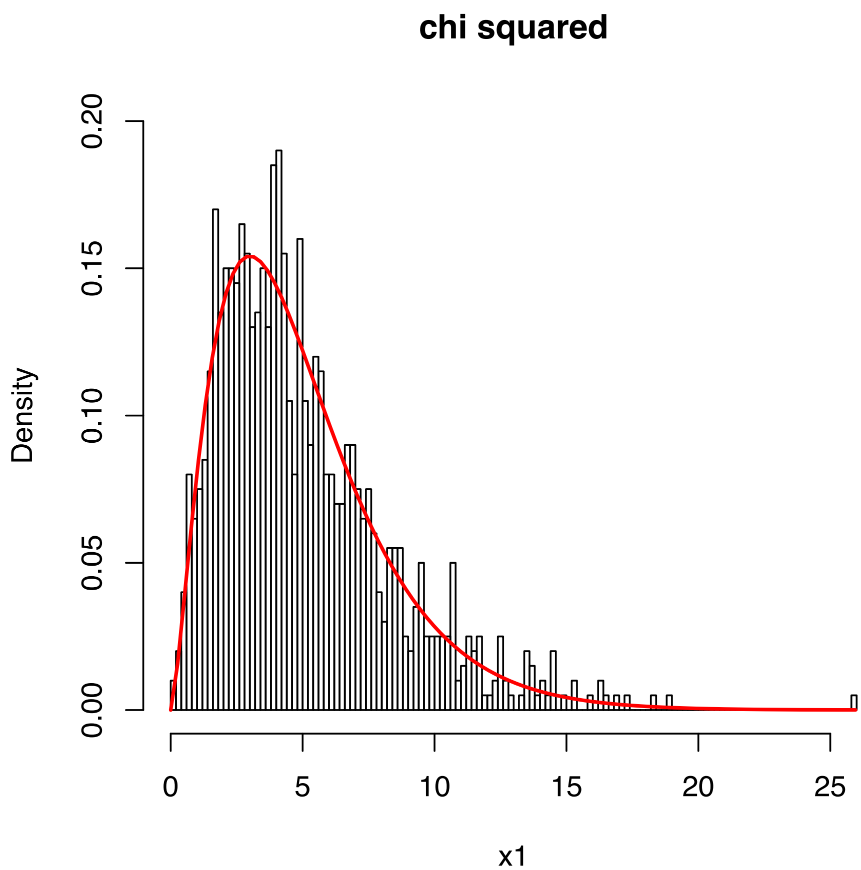

```
################################
# NORMAL(mu, sigma)
################################

mu=0  # location
sigma=1 # scale

n<-1000
x1<-rnorm(n, mu,sigma)
s<-seq(min(x1),max(x1),by=0.05)

par(mfrow=c(1,1))
hist(x1,probability = T,breaks = 100, col=8, main='NORMAL(0,1)')
lines(dnorm(s, mean=0, sd=1)~s, col=2, lwd=3)
```



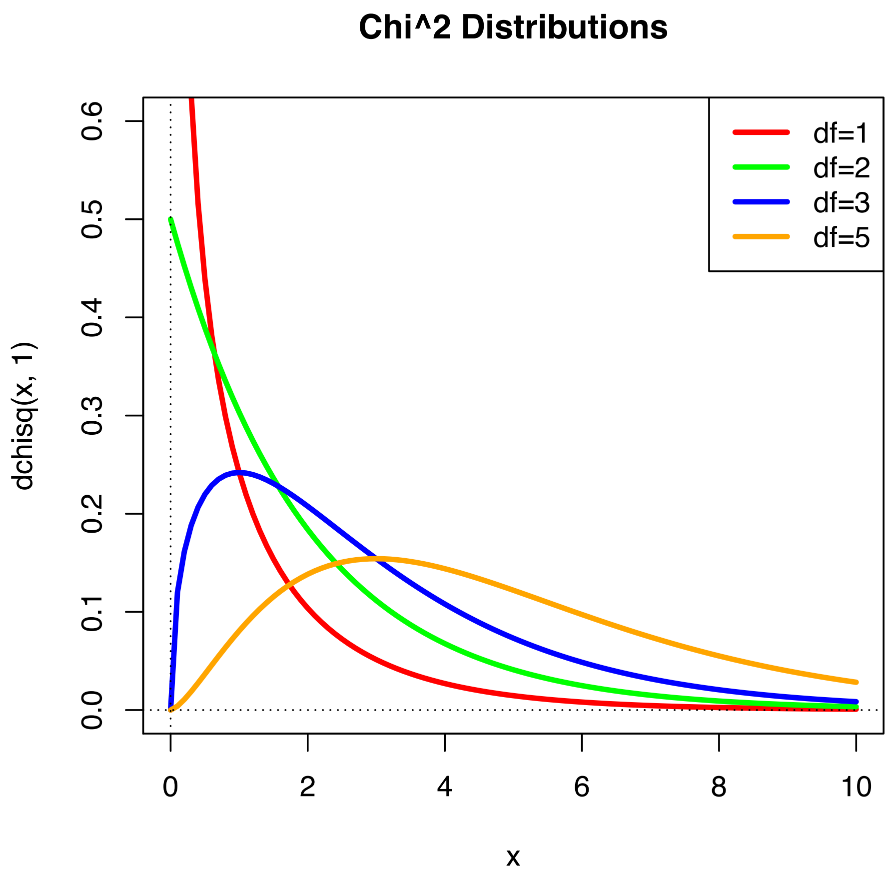

```
y1<-x1
s1<-s
y2<-2+0.5*x1
s2<-2+0.5*s
y3<- -3+1.5*x1
s3<- -3+1.5*s

par(mfrow=c(3,1))
hist(y1,probability = T,breaks = 100, col=8, xlim=c(-7,4))
lines(dnorm(s1, mean=0, sd=1)~s1, col=2, lwd=3)

hist(y2,probability = T,breaks = 100, col=4, xlim=c(-7,4))
lines(dnorm(s2, mean=2, sd=0.5)~s2, col=2, lwd=3)
hist(y3,probability = T,breaks = 100, col=5, xlim=c(-7,4))
lines(dnorm(s3, mean= -3, sd=1.5)~s3, col=2, lwd=3)
```



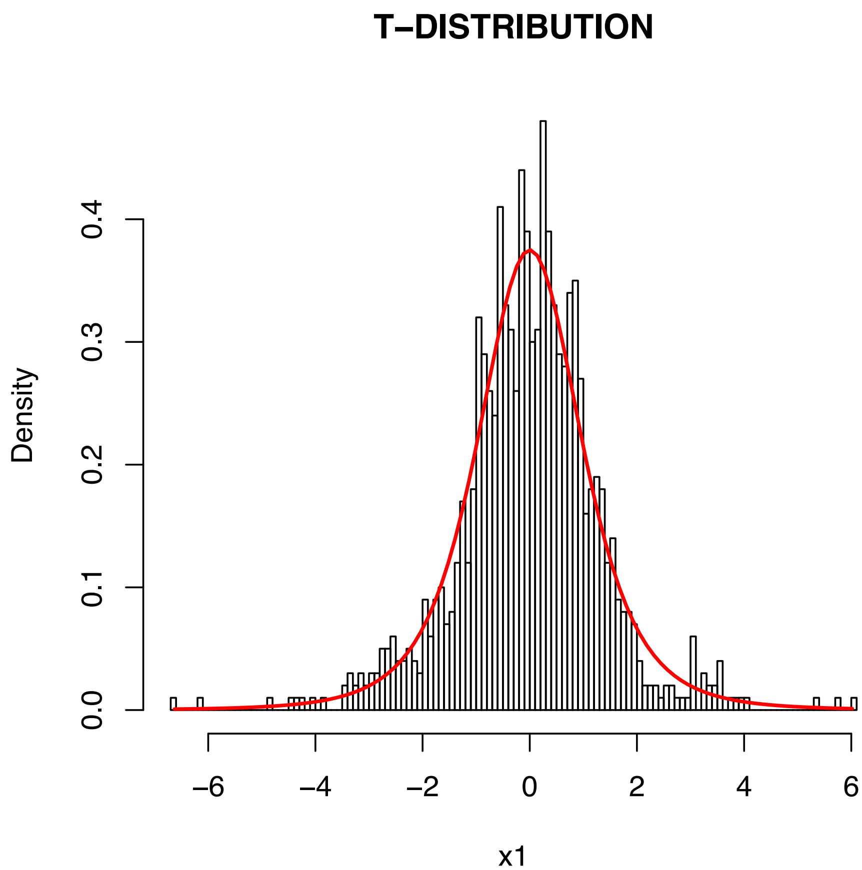

```
cat("mean(Y1)=", mean(y1), "sd(Y1)=",sd(y1),"\n")
## mean(Y1)= -0.01903163 sd(Y1)= 1.016756
cat("mean(Y2)=", mean(y2), "sd(Y2)=",sd(y2),"\n")
## mean(Y2)= 1.990484 sd(Y2)= 0.5083779
cat("mean(Y3)=", mean(y3), "sd(Y2)=",sd(y3),"\n")
## mean(Y3)= -3.028547 sd(Y2)= 1.525134
############################
#Chi^2- distribution
############################
df<-5
x1<-rchisq(n = 1000,df=df)
par(mfrow=c(1,1))
s<-seq(0,max(x1),length=100)
hist(x1, probability = T, breaks=100,  ylim=c(0,.2), main="chi squared")
lines( dchisq(s,df)~s, col='red', lwd=2 )
```



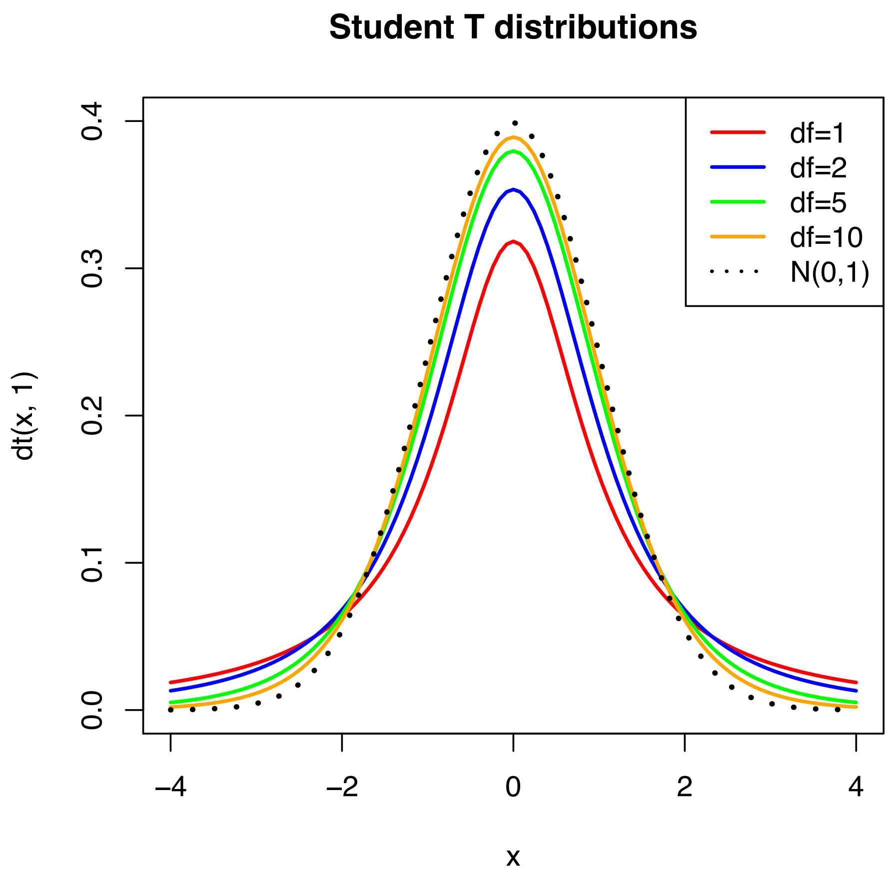

```
par(mfrow=c(1,1))
curve(dchisq(x,1), xlim=c(0,10), ylim=c(0,.6), col='red', lwd=3)
curve(dchisq(x,2), add=T, col='green', lwd=3)
curve(dchisq(x,3), add=T, col='blue', lwd=3)
curve(dchisq(x,5), add=T, col='orange', lwd=3)
abline(h=0,lty=3)
abline(v=0,lty=3)
legend(par('usr')[2], par('usr')[4], xjust=1,
       c('df=1', 'df=2', 'df=3', 'df=5'),
       lwd=3,
       lty=1,
       col=c('red', 'green', 'blue', 'orange')
)
title(main='Chi^2 Distributions')
```



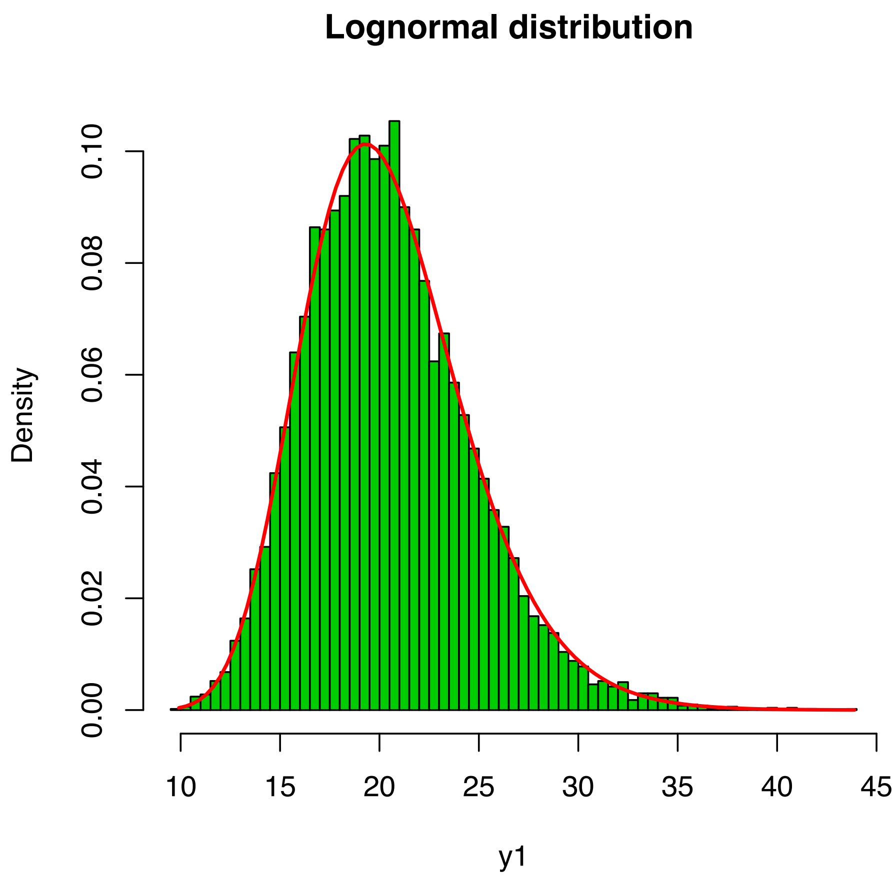

```
##################################
#T-DISTRIBUTION
##################################
df<-4
x1<-rt(n = 1000,df=df)
par(mfrow=c(1,1))
s<-seq(min(x1),max(x1),length=100)
hist(x1, probability = T, breaks=100, main="T-DISTRIBUTION" )
lines( dt(s,df )~s, col='red', lwd=2 )
```




```
par(mfrow=c(1,1))
curve( dt(x,1), xlim=c(-4,4), ylim=c(0,.4), col='red', lwd=2 )
curve( dt(x,2), add=T, col='blue', lwd=2 )
curve( dt(x,5), add=T, col='green', lwd=2 )
curve( dt(x,10), add=T, col='orange', lwd=2 )
curve( dnorm(x), add=T, lwd=3, lty=3 )
title(main="Student T distributions")
legend(par('usr')[2], par('usr')[4], xjust=1,
       c('df=1', 'df=2', 'df=5', 'df=10', 'N(0,1)'),
       lwd=c(2,2,2,2,2),
       lty=c(1,1,1,1,3),
       col=c('red', 'blue', 'green', 'orange', par("fg")))
```




```
######################################
# Lognormal distribution
#######################################


mu=3
sigma=.2

n<-10000

x1<-rnorm(n, mean=mu,sd=sigma)
y1<-exp(x1) # rlnorm(n,mu, sigma)
s<-seq(min(y1),max(y1), length=100)

hist(y1,breaks = 100, probability = T, col=3, main = "Lognormal distribution ")
lines(dlnorm(s,mu,sigma)~s, col=2, lwd=2)
```




```
######################################
#Cauchy distribution
#######################################

n <- 100                              # sample size
alpha <- runif(n, -pi/2, pi/2)    # Direction of the arrow
x <- tan(alpha)                       # Arrow impact
plot.new()
plot.window(xlim=c(-5, 5), ylim=c(-1.5, 1.5))
segments( 0, -1,      # Position of the Bowman
          x, 0    )   # Impact
d <- density(x)
lines(d$x, 5*d$y, col="red", lwd=3 )
box()
abline(h=0, col=3)
title(main="The bowman's distribution (Cauchy)")
```

The bowman's distribution (Cauchy)