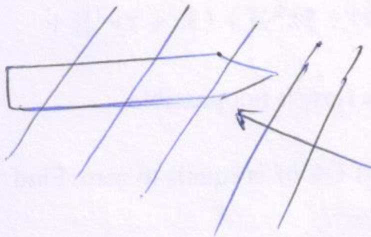
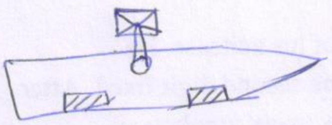
Torsion

Torsion in hull girder appears due to waves and load distribution.

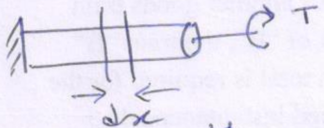


If angle of attack $\neq 0$, variation of buoyancy may cause torsion.



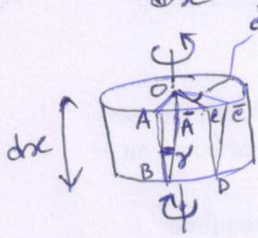
Load distribution may also cause torsional moment, e.g. during installation of a subsea module.

Basics of torsion



circular cross-section, fixed at one end. Plane section remains plane before and after twist.

radius $= r$



Suppose we have marked the lines OA, AB, OC, CD on the small piece of the rod. The length of the piece $= dx$

After twist, A moves to $\bar{A}$, C moves to $\bar{C}$.

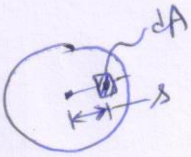
$\therefore \angle AO\bar{A} = \angle CO\bar{C} = d\phi =$ angle of twist

$\therefore A\bar{A} = r\,d\phi = \gamma\,dx$, where $\gamma =$ shear strain $= \frac{\tau}{G}$

$G =$ shear modulus

$\therefore r\frac{d\phi}{dx} = \gamma = \frac{\tau}{G}$ or $\boxed{\tau = Gr\theta}$, where $\theta = \frac{d\phi}{dx} =$ angle of twist per unit length of the rod.

NOTE: r is any radius



Considering τ (shear stress) acting on dA (a small area)

we get torque $= T = \int_A \tau s\,dA = \int_A G s \theta s\,dA = G\theta\int_A s^2 dA$

$$\boxed{T = G\theta J}$$

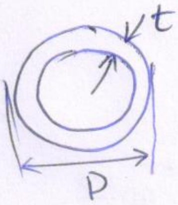
$J = \int_A s^2 dA =$ Polar moment of inertia.

In general, J is also called Torsional constant.

$GJ =$ Torsional ~~constant~~ rigidity,

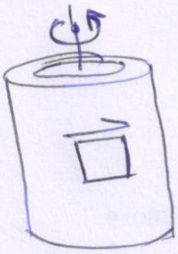
<u>Thin walled closed section</u>

<u>Work out</u>



Show that for the tubular section, $J = \frac{\pi}{4} D^3 t$

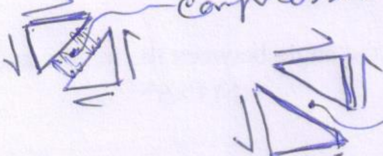
We know that power transmission shafts are often made of tubes.



Torsional moment induce shear force on an element in the cross-section as shown.

Considering this element,  is the shear stress diagram.

From this, we see that shear stress cause tensile/compressive stress as well diagonally, i.e.,



Hence, large torque may cause buckling in the shaft. Typicall $t/r > 1/60$ to avoid danger of buckling at normal working condition.

If the cross-section deforms in its own plane, i.e., part of the section leaves its original pane, we call it warping.

We assume that our rod is free to warp.

If we consider an arbitrary cross-section,

Total torque $(T) = \int r \tau t \, ds$

s = length along perimiter

From our shear strength class, we have seen that in a closed section, τt = constant.

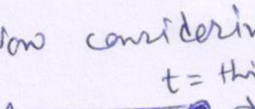
$\therefore T = \tau t \int_s r \, ds = 2 \tau t A$; A = total cross-sectional area enclosed.

As we see that with respect to a point P inside the area, the shaded area $= \frac{1}{2} r \, ds$

$\therefore dA = \frac{1}{2} r \, ds$ or $2 dA = r \, ds$

$\therefore \int 2 dA = \int r \, ds \Rightarrow 2A = \int r \, ds$

Now considering the strain energy stored due to torsional deformation



t = thickness (may vary in X-section)

ϕ = total angle of twist at the free end.

$\therefore$ total external work done $= \frac{1}{2} T \phi$

Strain energy stored in per unit volume of material $= \frac{\tau^2}{2G}$

$\therefore$ total energy stored in the shaft/rod $= \int \frac{\tau^2}{2G} t\, l\, ds$

where, l = total length, t = thickness at a location,

ds = elemental length along periphery.

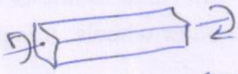
Here, we can assume thickness t at a location on ~~that~~ periphery is same along length.

$$\therefore \frac{1}{2} T \phi = \oint_s \frac{\tau^2}{2G} t\, l\, ds = \frac{\tau t l}{2G} \oint_s \tau\, ds$$

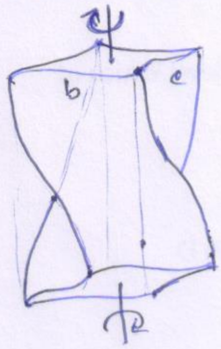
$$\text{or } T \frac{\phi}{l} = \frac{\tau t}{G} \oint_s \tau\, ds = \frac{T}{2AG} \oint_s \tau\, ds$$

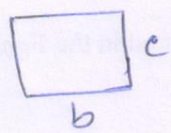
$$\text{or } \theta = \frac{1}{2A} \oint_s \frac{\tau}{G} ds \Rightarrow \oint_s \frac{\tau}{G} ds = 2A\theta$$

This walled open section

Let us now consider rectangular shaft 

St. Venant (1855) summarized the following results



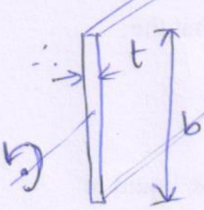


$$b > c \quad \tau_{max} = \frac{T}{\alpha b c^2}$$

$$\theta = \frac{T}{\beta b c^3 G}$$

b/c	α	β
1	0.208	0.141
2	0.246	0.229
10	0.313	0.313
⋮		
∞	0.333 ($\frac{1}{3}$)	0.333 ($\frac{1}{3}$)

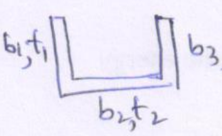
So, we see from his results that, $\theta = \frac{T}{\frac{1}{3} b t^3 G}$, where $c = t$ for thin plate



$\therefore$ for thin plate type twisting element,

$$\theta = \frac{T}{G J_t} \quad \text{where, } J_t = \text{torsional constant} = \frac{1}{3} b t^3$$

This results is used for thin walled open section. Such as,



For this section, $J_t = \sum_{i=1}^{3} \frac{1}{3} b_i t_i^3$

Let us consider a tubular shaft

radius = r, thickness = t

$\because$ We have seen previously that

$$J_t = \frac{\pi}{4} D^3 t = 2\pi r^3 t$$

$\therefore$ for this section, the angle of twist per unit length (θ) =

$$\theta_1 = \frac{T}{GJ_t} = \frac{T}{2\pi r^3 t G}$$

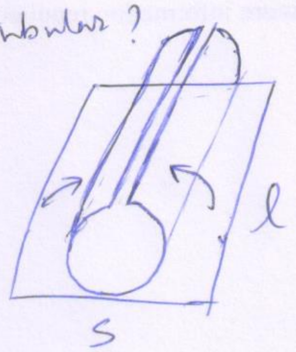
Let us assume that we have cut the circular section as shown,

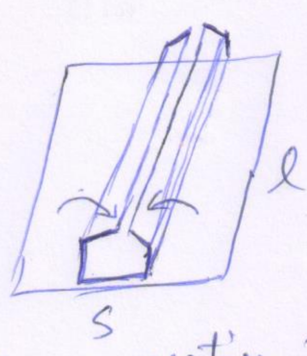
$\therefore$ Now, $J_t = \frac{1}{3} b t^3 = \frac{1}{3}(2\pi r t^3)$

$$\therefore \theta_2 = \frac{3T}{2\pi r t^3 G} \qquad \therefore \frac{\theta_1}{\theta_2} = \frac{2\pi r t^{\cancel{3}\,2} G}{3T} \cdot \frac{T}{2\pi r^3 \cancel{t} G} = \frac{t^2}{3r^2}$$

If $\frac{t}{r} = \frac{1}{10}$, $\frac{\theta_1}{\theta_2} = \frac{1}{300}$ or $\theta_2 = 300\,\theta_1$ i.e., tube with a cut is 300 times more flexible in twist for $t/r = 0.1$ compared to the tube intact. Hence, closed sections are stronger in twist.

Let us assume that we have a metal sheet plate of $S \times l$ size and this will be cold deformed to make a shaft. ~~what~~ which geometry for cross-section will be preferred? square or tubular?





We know that for a closed section, $\tau = \frac{T}{2At}$

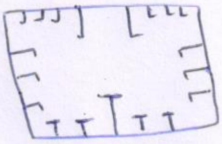
Since, both are made of same plate, t = same.

$$\therefore \frac{\tau_c}{\tau_{sq}} = \frac{A_{sq}}{A_c} = \cancel{\frac{\pi R^2}{a^2}} = \frac{a^2}{\pi R^2}$$ since the plate is same, perimeter is also same, i.e., $2\pi R = 4a$ or $\frac{R}{a} = \frac{2}{\pi}$

$$\therefore \frac{\tau_c}{\tau_{sq}} = \boxed{\cancel{\frac{\pi \times 4}{\pi^2} = \frac{4}{\pi} = 1.273}} = \frac{a^2}{\pi R^2} = \frac{1}{\pi}\frac{\pi^2}{4} = \frac{\pi}{4} = 0.785$$

$\therefore$ $\tau_c = 0.785\,\tau_{sq}$, i.e., shear stress in square shaft is larger than circular shaft.

Hence, it is preferable to make a circular/tubular shaft.



For our applications, if we have closed cross-section,
How to find torsional constant (J_t)?

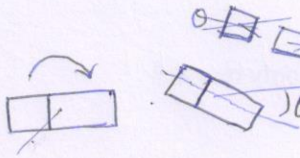
We have seen previously that $\oint \frac{\tau}{G} ds = 2A\theta$ and $J_t = \frac{T}{G\theta}$

or $\therefore \oint \tau ds = 2AG\theta$ or $\oint \frac{\tau t}{t} ds = 2AG\theta$ or $\tau t \oint \frac{ds}{t} = 2AG\theta$
[since τt = constant in a section]

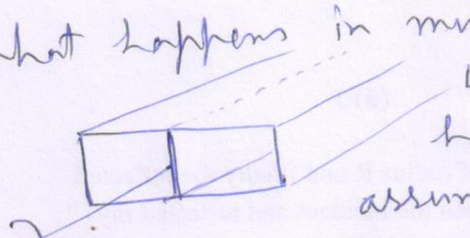
Also we have seen, for closed section, $T = 2\tau t A$

$\therefore \frac{T}{2A} \oint \frac{ds}{t} = 2AG\theta$ or $\frac{T}{G\theta} = \frac{4A^2}{\oint \frac{ds}{t}}$

$\therefore$ Total torsional constant $= J_t = \underbrace{\frac{4A^2}{\oint \frac{ds}{t}}}_{\text{for closed section}} + \underbrace{\sum_{i=1}^{n} \frac{d_i t_i^3}{3}}_{\text{for all other small members (stiffeners)}}$



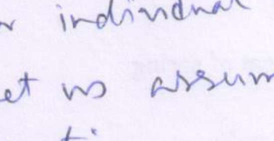
What happens in multi-cell sections?



Like we did in shear in multicell section, here, the approach is the same, i.e., we assume to cut-open the sections and apply shear flow (τt) to enclose the cut.

If the angle of twist per unit length of combined section $= \theta$, then for individual cells also twists by an angle θ.

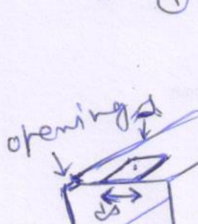
Let us assume, q_1 and q_2 are the shear flow acting in the sections. $q_1 = \tau t$ in cell 1, $q_2 = \tau t$ in cell 2.



For an enclosed section, $\boxed{T = 2\tau t A}$

$\therefore$ we can write total torque in both sections combined

$$= T = 2q_1 A_1 + 2q_2 A_2 \quad \text{——— (1)}$$



From shear strength class, we know that gap created

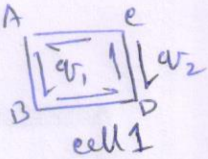
$$= \text{gap} = \oint \gamma ds = \oint \frac{\tau}{G} ds = 2A\theta$$

also, $\text{gap} = \oint \frac{\tau}{G} ds = \oint \frac{\tau t}{Gt} ds = \frac{\tau t}{G} \oint \frac{ds}{t} = \frac{q}{G} \oint \frac{ds}{t}$

Now, we can write 2 equations for the gaps, one for cell 1, other for cell 2.

For gap in cell 1

$$\frac{q_1}{G}\oint_{①}\frac{ds}{t} - \frac{q_2}{G}\int_{eD}\frac{ds}{t} = 2A_1\theta \quad \text{(ii)}$$



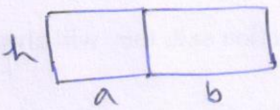
Similarly for gap in cell 2

$$\frac{q_2}{G}\oint\frac{ds}{t} - \frac{q_1}{G}\int_{eD}\frac{ds}{t} = 2A_2\theta \quad \text{(iii)}$$

We can solve (ii) and (iii) and put in (i) to get final answers.

Let us consider the following dimensions,

thickness = t for all.



$$\therefore \quad 2q_1ah + 2q_2bh = T \quad \text{(a)}$$

$$\frac{q_1}{Gt}(2a+2h) - \frac{q_2}{Gt}h = 2ah\theta \quad \text{(b)}$$

$$\frac{q_2}{Gt}(2b+2h) - \frac{q_1}{Gt}h = 2bh\theta \quad \text{(c)}$$

$$A_1q_1 - A_2q_2 = 2ah\theta$$

$$B_1q_2 - B_2q_1 = 2bh\theta$$

or

$$B_2\cancel{A_1}q_1 - B_2A_2q_2 = 2ahB_2\theta$$

$$+ \quad A_1B_1q_2 - A_1\cancel{B_2}q_1 = 2bhA_1\theta$$

$$(A_1B_1 - A_2B_2)q_2 = 2h\theta(aB_2 - bA_1)$$

$$\therefore \quad q_2 = \left(\frac{aB_2 - bA_1}{A_1B_1 - A_2B_2}\right)2h\theta$$

$$A_1B_1q_1 - A_2B_1q_2 = 2ahB_1\theta$$

$$+ \quad A_2B_1q_2 - A_2B_2q_1 = 2bhA_2\theta$$

$$(A_1B_1 - A_2B_2)q_1 = 2h\theta(aB_1 - bA_2)$$

$$q_1 = \frac{(aB_1 - bA_2)2h\theta}{A_1B_1 - A_2B_2}$$

Using eq. (a),

$$2ah \cdot 2h\theta\frac{(aB_2 - bA_1)}{A_1B_1 - A_2B_2} + 2bh \cdot 2h\theta\frac{(aB_1 - bA_2)}{A_1B_1 - A_2B_2} = T$$

$$\therefore \quad 4h^2\theta(a^2B_2 - abA_1 + abB_1 - b^2A_2) = T(A_1B_1 - A_2B_2)$$

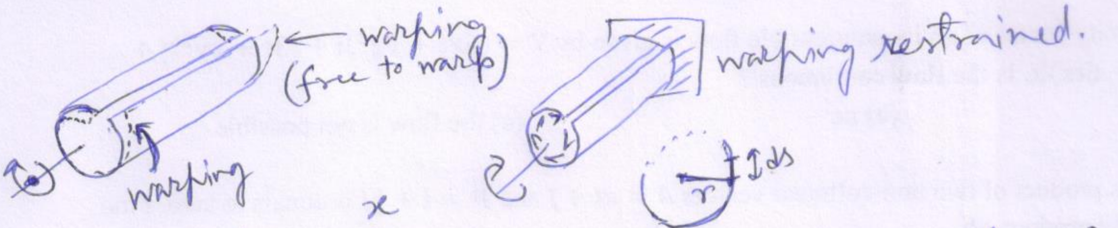
$$\therefore \quad \theta = \frac{T(A_1B_1 - A_2B_2)}{4h^2(a^2B_2 - b^2A_2 + abB_1 - abA_1)}$$

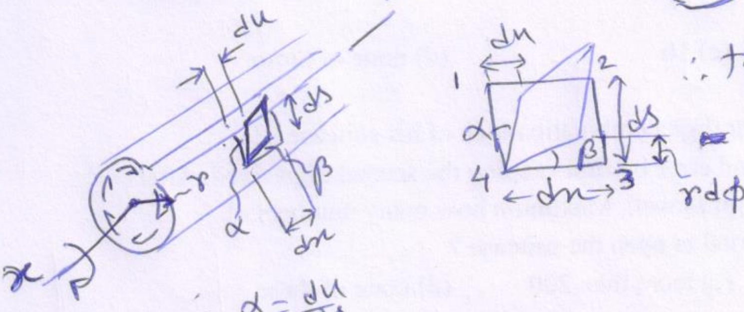
Using this expression of θ, we can get q_1 and q_2

$$\therefore \quad q_1 = \frac{(aB_1 - bA_2)}{2h(a^2B_2 - b^2A_2 + abB_1 - abA_1)}, \quad q_2 = - -$$

<u>Work out</u> — put the values of A_1, B_1, A_2, B_2 and find q_1 and q_2.

So far we have seen sections which are free to warp. Now let us see what happens if warping is restrained.





$\therefore$ total shear strain $(\gamma) = \alpha + \beta$

r = distance of element from the line of twist (radial distance)

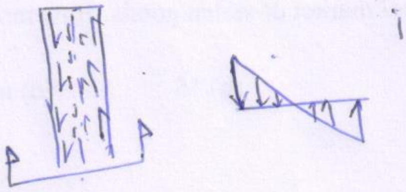
$d\phi$ = angle of twist betwen faces 1-4 and 2-3.

$\therefore \alpha = \frac{du}{ds}$ $\qquad$ $\therefore \beta = \frac{r\,d\phi}{dx}$

du = displacement along x-axis

$$\therefore \gamma = \frac{du}{ds} + \frac{r\,d\phi}{dx} = \frac{\tau}{G}$$

Since the shear stress distribution in an open due to torque has zero mean, i.e., over the thickness, no



shear strain will be appeared; i.e., $\gamma = \frac{\tau}{G} = 0$

we have $\gamma = \frac{du}{ds} + r\frac{d\phi}{dx} = 0$

$\therefore$ for thin walled open section,

$$du = \left(-r\frac{d\phi}{dx}\right) ds$$

$$\therefore \quad u = \int_s -r\frac{d\phi}{dx}\,ds + u_0$$

If we take the initial displacement along length $u_0 = 0$ (i.e., we measure from u_0), then we can write $u = -\int_s r\frac{d\phi}{dx}\,ds = -\frac{d\phi}{dx}\int_s r\,ds = -\theta\omega$

independent of s

where, $\frac{d\phi}{dx}$ = angle of twist per unit length $= \theta$

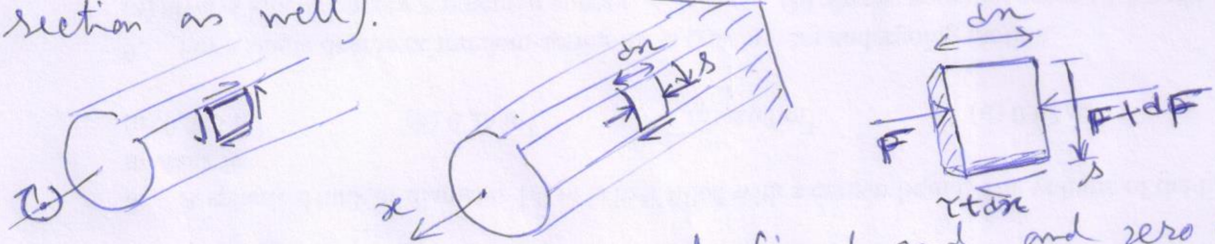
$\omega = \int_s r\,ds$ $\qquad$ [In some text book, $\int r\,ds$ is taken as 2ω]

So, now we have the displacement along the length. If this is restrained at the end(s), normal stress on the section will be developed.

$\therefore$ Due to restraint, longitudinal strain $\epsilon_x = \frac{du}{dx} = -\omega\frac{d\theta}{dx}$ (ω is taken as constant-sectional property along length x)

$\therefore$ Normal stress developed $= \sigma_x = E\epsilon_x = -E\omega \frac{d\theta}{dx}$

Just like in a beam, this σ_x varies in the section. Hence this will induce longitudinal shear stress (and complementary shear stress in the section as well).



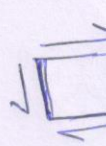
Normal stress is maximum at fixed end and zero at free end, i.e., σ is larger towards fixed end.

From equilibrium, $dF + \tau t\, dx = 0$. [τ is taken along +ve x-axis]

$$\therefore \tau t = -\frac{dF}{dx} = -\frac{d}{dx}\left[\int_{A_s} \sigma\, dA\right] = -\frac{d}{dx}\left[-\int_A E\omega \frac{d\theta}{dx}\cdot dA\right]$$

$$= \int_{A_s} E\omega \frac{d^2\theta}{dx^2}\, dA = E\frac{d^2\theta}{dx^2}\int_{A_s} \omega\, dA$$

$$\therefore \tau t = E\frac{d^2\theta}{dx^2}\int \omega\, dA = \text{Shear flow (along thickness) in section.}$$

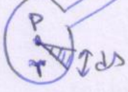


complementary shear will cause Twisting moment in section. which is torque due to warping restraint.

$$T_\omega = \int_A \tau t\, r\, ds$$

$$= \int\left(E\frac{d^2\theta}{dx^2}\int\omega\, dA\right) r\, ds = E\frac{d^2\theta}{dx^2}\iint(\omega\, dA)\, r\, ds$$

Previously we have used $\int r\, ds = \omega$



$$\therefore r\, ds = d\omega$$

$$\therefore T_\omega = E\frac{d^2\theta}{dx^2}\iint \underbrace{(\omega\, dA)}_{1st}\, \underbrace{d\omega}_{2nd}$$

Using integration by parts, $= E\frac{d^2\theta}{dx^2}\left[\left(\int\omega\, dA\right).\omega - \int \omega\, dA.\omega\right]_A$

$$= E\frac{d^2\theta}{dx^2}\left[\omega\int\omega\, dA - \int\omega^2\, dA\right]_A$$

while estimating ω (sectorial co-ordinate), we can select the point P in such a way that $\int\omega\, dA = 0$. Such a point P is the Principal Pole.

Corresponding $\int\omega^2\, dA = J_\omega$ = Principal sectorial moment of inertia.

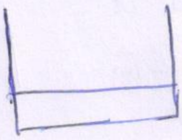
$$\therefore T_\omega = -E J_\omega \frac{d^2\theta}{dx^2}$$

Hence the total external torque has two components we can say,

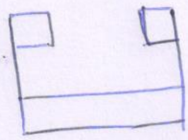
i.e., $T = T_s + T_\omega$ = St. Venant Torque + Torque due to warping restraint.

$$= \underbrace{G J_t \theta}_{\text{Torsional rigidity}} - \underbrace{E J_\omega \frac{d^2\theta}{dx^2}}_{\text{warping rigidity}}$$

If end restraints do not exist, only St. Venant component will be present. In case of a ship, ends are restraints as they are enclosed. Hence both component will exist.



We have already seen how to estimate J_t for a section like this. Due to warping restraint, the free ends experience large ~~stress~~, and the plates may buckle also.



The boxes added at the free ends are called "Torsion box" of a ship. This boxes increases the value of J_t and also ~~hot~~ helps to perform better against normal warping stress.

$\therefore$ We have, $T = G J_t \theta - E J_\omega \frac{d^2\theta}{dx^2}$

or $\frac{d^2\theta}{dx^2} - K^2\theta = -\frac{K^2 T}{G J_t}$; where, $K^2 = \frac{G J_t}{E J_\omega}$

If arbitrary torque is acting along the length of the member, we can express $T = \frac{T_0}{2} + \sum_{n=1}^{\infty}(T_{nc}\cos p_n x + T_{ns}\sin p_n x)$; [Refer maths class on Fourier series]

The solution of θ is given as

$$\theta = A_0 + A_1 \sinh Kx + A_2 \cosh Kx + \sum_{n=1}^{\infty}(\alpha_n \cos p_n x + \beta_n \sin p_n x)$$

where, $p_n = \frac{\pi n}{L}$, $\alpha_n = \frac{T_{cn}}{G J_t (p_n^3 + K^2 p_n)}$, $\beta_n = -\frac{T_{sn}}{G J_t (p_n^3 + K^2 p_n)}$

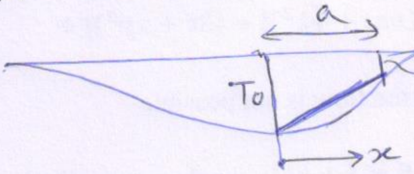
So, once θ is known for a given problem, σ, τ_t, T_ω etc. can be calculated.

Somtime, a simplified approach may be adopted as shown

Actual torsional moment distribution in hull

In stead of actual variation, a straight line or parabolic variation may be assumed (origin at mid-ship)

A, A' are location where hull is constrained thereafter.



$$T = T_0\left[1-(1-\alpha)\frac{x}{a}\right] \quad \text{St. line variation (linear)}$$

or

$$T = T_0\left[1-(1-\alpha)\left(\frac{x}{a}\right)^2\right] \quad \text{Parabolic variation.}$$

α is some constant.

The boundary conditions are; at $x=0$, we take angle of twist $\phi = 0$ and ~~rate of~~ angle of twist per unit length is minimum,

i.e., $\frac{d\phi}{dx}$ = minimum or $\frac{d^2\phi}{dx^2} = 0$

at $x = a$, $T = T_e(1-f)$; f = degree of constraint, i.e., for fully constrained, $f = 1$

$\therefore$ at $x = a$, $\theta = \frac{d\phi}{dx} = \frac{T_e(1-f)}{GJ_t}$

$\therefore$ From the equation $T = GJ_t\theta - EJ_\omega \frac{d^2\theta}{dx^2}$

or $T = GJ_t\frac{d\phi}{dx} - EJ_\omega\frac{d^3\phi}{dx^3}$ $\left[\because \theta = \frac{d\phi}{dx}\right]$

The solution will be ~~ϕ~~ as given below for linear variation of Torque,

$$\phi = \frac{T_0 a}{GJ_t}\left[\frac{x}{a} - \frac{1}{2}(1-\alpha)\left(\frac{x}{a}\right)^2 - f\alpha\frac{\sinh kx}{ka\cosh(kx)} - \frac{1-\alpha}{(ka)^2\eta}\right]$$

where, α = some constant in the Torque expression

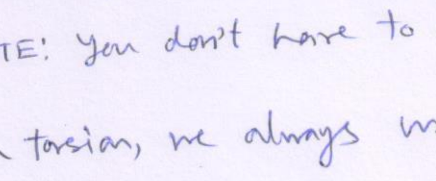
$$\eta = 1 + \tanh(ka)\sinh(kx) - \cosh(kx)$$

From ϕ, all other useful parameters can be obtained.

NOTE: You don't have to memorize the expression of ϕ

In torsion, we always use the sectorial properties of a section.

with respect to an arbitrary pole "P", the sectorial coordinate is defined as



"O" → local origin

$$\omega = \int r\,ds$$

If ds is such that it cause anti-clockwise rotation about P, then $\omega = -ve$, otherwise $\omega = +ve$.

If the point P (Pole) is the shear center of the section, then the sectorial properties are the Principal sectorial properties. This is similar to the centroid and principal axis of a section.

If P is the shear center (or Principal Pole), the $\int_A \omega \, dA = 0$

Sectorial moment of inertia $J_\omega = \int_A \omega^2 dA$

In order to find the shear center, we can use Principal co-ordinates and an arbitrary Pole.

step-1 Calculate ω' with respect to an arbitrary Pole and arbitrary origin.

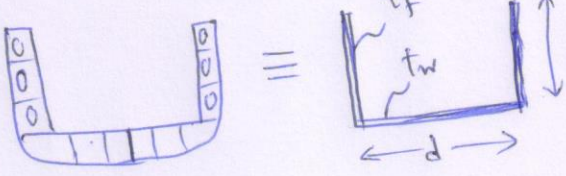
step-2 Calculate location of shear center from the arbitrary Pole using the equations $e_x = -\dfrac{\int_A \omega' y \, dA}{I_x}$, $e_y = \dfrac{-\int_A \omega' x \, dA}{I_y}$ | Note: e_x, e_y are from the Pole

I_x, I_y are Principal moment of inertias

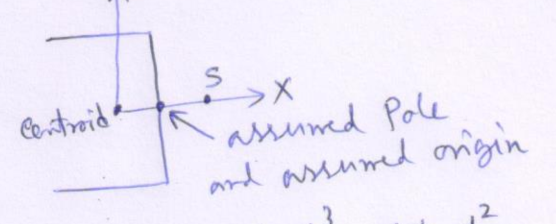
step-3 Calculate ω with respect to Shear center or Principal Pole.

step-4 Calculate S_ω, J_ω, ω, ... etc.

A ship is often idealized as a channel section. This of course depends on the type of ship.



thickness of flange (t_f) and web (t_w) are determined as equivalent of the actual ship.

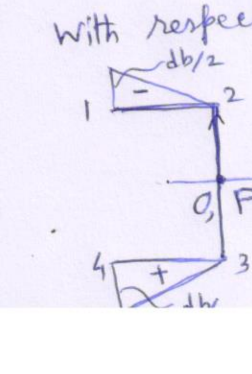


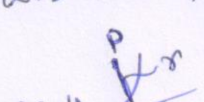
To find shear center (S), we see that $e_y = 0$ due to symmetry.

$$e_x = -\frac{\int_A \omega' y \, dA}{I_x}$$

$$I_x = \frac{t_w d^3}{12} + b t_f \frac{d^2}{4} \times 2 = \frac{t_w d^3}{12} + \frac{b t_f d^2}{2}$$

With respect to assumed Pole and origin, ω' is drawn,





$\omega' = \int r \, ds$

Along 2-3, $r = 0$, $\therefore \omega' = 0$

Along 1-2 and ~~3-4~~ 3-4, $r = d/2$

$\therefore \omega' = \dfrac{d}{2}\int_0^s ds = \dfrac{d}{2} s$.

Along 2-1, it is ↺ about P ⇒ −ve

Along 3-4, it is ↻ about P ⇒ +ve

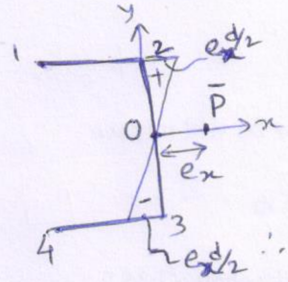
$\leftarrow s = -ve$

$$\therefore \quad \frac{-\int \omega' y\, dA}{I_x} = \frac{-1}{I_x}\left[\int_0^{-b} -\frac{d}{2} s \cdot \frac{d}{2} \cdot t_f\, ds\right] + \frac{-1}{I_x}\left[\int_0^{-b} \frac{d}{2} s \frac{d}{2} t_f\, ds\right]$$

$\nwarrow s = +ve$, but $y = \frac{d}{2}$ is $-ve$

$$= \frac{-1}{I_x}\left[-\frac{t_f d^2}{4}\frac{b^2}{2} - \frac{t_f d^2}{4}\frac{b^2}{2}\right]$$

$$e_x = \frac{t_f d^2 b^2}{4 I_x} = \frac{t_f d^2 b^2 \times 12}{4(t_w d^3 + 6 b t_f d^2)} = \frac{3 b^2 t_f}{(t_w d + 6 b t_f)}$$



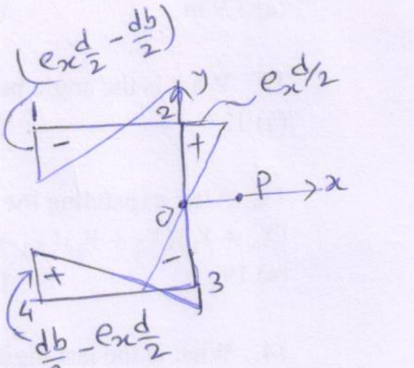
Now evaluate ω with respect to $\bar{P}$

Along 1-2 and 3-4, $r = \frac{d}{2}$ and $-\frac{d}{2}$

$\therefore$ Along 2-3, now $r = e_x$,

$\therefore$ along 0-2, $\omega = \int r\, ds = e_x s$

along 0-3, $\omega = -e_x s$



along 2-1, $\omega = e_x \frac{d}{2} - \int_0^s \frac{d}{2}\, ds = e_x \frac{d}{2} - \frac{d}{2} s$

along 3-4, $\omega = -e_x \frac{d}{2} + \frac{d}{2} s$

Now the ω is the Principal sectorial diagram.

We can now calculate J_ω or any other terms such as, Sectorial static moment $= \int_A \omega\, dA$

Sectorial Linear moment $= \int_A \omega y\, dA$ and $\int_A \omega x\, dA$

Sectorial moment of inertia $= J_\omega = \int \omega^2\, dA$

For calculating J_ω, along 0-2 and 0-3

$$\int \omega^2 dA = 2\int_0^{d/2} (e_x s)^2 t_w\, ds = 2 e_x^2 t_w \left[\frac{s^3}{3}\right]_0^{d/2} = 2 e_x^2 t_w \frac{d^3}{24} = \frac{e_x^2 t_w d^3}{12}$$

along 2-1 and 3-4,

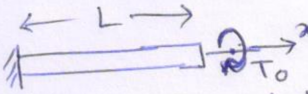
$$\int \omega^2 dA = 2\int_0^b \left(e_x \frac{d}{2} - \frac{d}{2} s\right)^2 t_f\, ds = 2\left[e_x^2 \frac{d^2}{4} s + \frac{d^2}{4}\frac{s^3}{3} - \frac{e_x d^2 s^2}{2}\right]_0^b t_f$$

$$= 2\left[\frac{e_x^2 d^2 b}{4} + \frac{d^2 b^3}{12} - \frac{e_x d^2 b^2}{4}\right] t_f$$

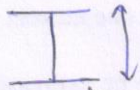
$$\therefore J_\omega = \frac{e_x^2 t_w d^3}{12} + \frac{e_x^2 d^2 b t_f}{2} + \frac{d^2 b^3 t_f}{6} - \frac{e_x d^2 b^2 t_f}{2}$$

Thus for the given channel section, which can represent a strip, J_ω can be estimated and warping torque can be obtained.

Solve the following problem.



An I section beam fixed at one end experiencing torque T_0 at the free end. Find the maximum normal stress, shear stress etc. Given,



$h = 200\text{ mm}$
all thickness $= 10\text{ mm}$.
$b = 100\text{ mm}$

The general equation of torsion $\frac{d^2\theta}{dx^2} - k^2\theta = -\frac{k^2 T}{GJ_t}$; $k = \frac{GJ_t}{EJ_w}$

Here, $T = T_0$ (acting at the end).

$\therefore$ Particular solution is $\theta = \frac{T_0}{GJ_t}$

total solution = Complementary + Particular

$$\theta = A_1 \sinh kx + A_2 \cosh kx + \frac{T_0}{GJ_t}$$

The boundary conditions are at $x=0$, longitudinal displacement along $x=0$
i.e., $u=0$

also, at $x=0$, $\theta = 0$ [θ = angle of twist per unit length]

$$\therefore \quad 0 = A_2 + \frac{T_0}{GJ_t} \Rightarrow A_2 = \frac{T_0}{GJ_t}$$

Further, at the free end, normal stress σ must be $=0$
i.e., at $x=L$, $\sigma = 0$

Now, we know that $\sigma = -E\omega\frac{d\theta}{dx} = 0$

E, ω are properties of material and section, and so $\neq 0$

$$\therefore \frac{d\theta}{dx} = 0 \Rightarrow A_1 k\cosh(kL) + A_2 k\sinh(kL) = 0$$

$$\therefore A_1 = \frac{T_0}{GJ_t}\tanh(kL)$$

$$\therefore \theta = \frac{T_0}{GJ_t}\left[1 + \tanh kL \cdot \sinh kx - \cosh kx\right] = \text{angle of twist per unit length} = \frac{d\phi}{dx}$$

$\therefore$ Total angle of twist between the fixed end and free end =

$$= \phi = \int_0^L \theta\, dx = \frac{T_0 L}{GJ_t}\left[1 - \frac{\tanh(kL)}{kL}\right]$$ ← Work this out

At the ~~free~~ fixed end $x=0$, Since, warping resistance exists there, maximum normal stress will be developed there.

$$\therefore \sigma_{max} = \sigma\,(\text{at } x=0) = -E\omega\left(\frac{d\theta}{dx}\right)_{x=0}$$ ← work this out

$$= -\frac{ET_0}{GJ_t}\omega k\tanh(kL)$$ ← note that it also depends on ω.

St. Venant torque $= GJ_t\theta = T_0[1+\tanh(kL)\cdot\sinh(kx) - \cosh(kx)] = T_{st}$

Warping torque $= EJ_w\frac{d^2\theta}{dx^2} = T_0[\tanh(kL)\sinh(kx) - \cosh(kx)] = T_w$

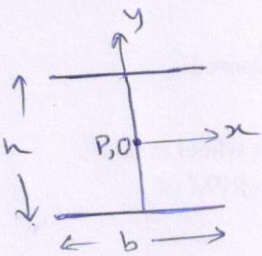
~~Max~~. Shear stress due to St. Venant torque $= \frac{T_{st}\cdot t}{J_t}$; t = thickness at the concerned location

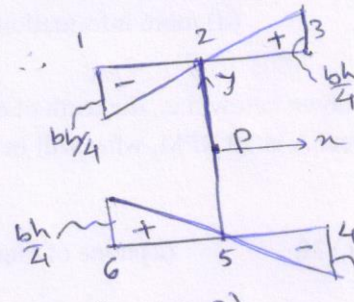
where $J_t = \sum \frac{b_i t_i^3}{3}$

~~Max~~. Warping shear stress $= \tau = \frac{E}{t}\frac{d^2\theta}{dx^2}\int \omega\, dA$

or $\tau = \frac{T_w S_w}{J_w t}$, where $S_w = \int \omega\, dA$ at a location.

Since the section is I, which is symmetric section, shear center will be at the centroid, = Principal Pole.





along 2-5, $r = 0$ $\therefore \omega = 0$

along 2-3, $\omega = \int_0^s \frac{h}{2}\,ds = +\frac{h}{2}s$ $\therefore$ (PL)

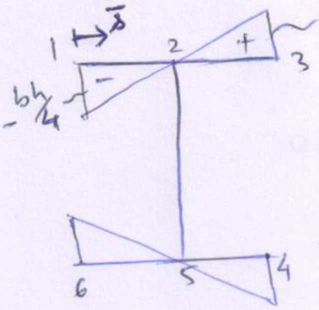
along 2-1, $\omega = -\frac{h}{2}s$, (P↺)

along 5-4, $\omega = -\frac{h}{2}s$, (P), along 5-6, $\omega = \frac{h}{2}s$ (PL)

Along any of the flanges, (eg. 1-2-3)

$\therefore S_w = \int \omega\, dA = \cancel{\int \omega \cdot t\, d\bar{s} = \int\left(\frac{h}{2}\bar{s}\right)t\,d\bar{s} = \frac{h t \bar{s}^2}{4}}$

we measure from the free end 1.



$S_w = \int \omega\, dA$ where, $\omega = \cancel{+\frac{bh}{4}} - \frac{h}{2}\left(\frac{b}{2} - \bar{s}\right)$

at $\bar{s} = 0$, $\omega = -\frac{bh}{4}$, at $\bar{s} = b$, $\omega = \frac{bh}{4}$

$$\therefore S_w = \int_0^{\bar{s}} -\frac{h}{2}\left(\frac{b}{2} - \bar{s}\right)t\,d\bar{s} = \left[\frac{ht}{2}\right]\left[\frac{\bar{s}^2}{2} - \frac{b\bar{s}}{2}\right]_0^{\bar{s}} = \frac{ht}{4}\left(\bar{s}^2 - b\bar{s}\right)$$

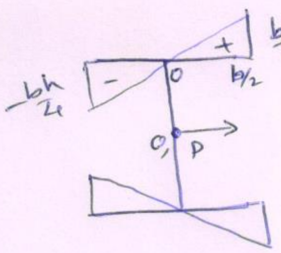
$\therefore S_w(max)$ is at $\bar{s} = b/2$ $\therefore S_{w(max)} = -\frac{htb^2}{16}$

$\therefore$ Maximum warping shear stress appear at 2 (or 5) $\Rightarrow \tau_{w,max} = \frac{T_w S_{w(max)}}{J_w t}$

Note that for the whole area, $\int_A \omega\, dA = 0$. But at a certain location (w.r.t. Principal Pole) the value exists and measured from free end for shear stress calculation.

Similarly, $\sigma = -E\omega\frac{d\theta}{dx}$ $\therefore$ σ_{max} in the section will appear at the point of ω_{max} i.e, at points 1, 3, 4 & 6. ($\omega_{max} = \frac{bh}{4}$)

Also for above calculations, we need to evaluate $J_w = \int_A \omega^2\, dA$



$$\int_A \omega^2 \, dA = 4\int_0^{b/2} \left(\frac{h}{2} s\right)^2 t \, ds = \int_0^{b/2} h^2 s^2 t \, ds$$

$$= h^2 t \left[\frac{s^3}{3}\right]_0^{b/2} = \frac{b^3 h^2 t}{24} = J_\omega$$

Torsional constant (St. Venant) $= J_t = \dfrac{(2b+h)t^3}{3}$

Workout the values of σ_{max}, τ_{max} due to warping restraint.

Note: The torsional equation $T = G J_t \theta - E J_\omega \dfrac{d^2\theta}{dx^2}$

If the section is free to warp, then $E J_\omega \dfrac{d^2\theta}{dx^2}$ does not appear.

If warping restraint exists, then normal stress develops, which produces longitudinal shear stress and that creates complementary shear stress in the section. This complementary shear cause additional torsional moment in the section.

You can refer the book "Torsion and Shear stress in Ships" by M. Shama for most of the stuff written above.