Shooting method

Nonlinear 2nd order differential eq

$$y'' = f(x, y, y') \quad a < x < b \qquad (1)$$

with boundary condition of

1. first kind $\quad y(a) = r_1 \quad y(b) = r_2$
2. 2nd kind $\quad y'(a) = r_1 \quad y'(b) = r_2$
3. 3rd kind $\quad a_0 y(a) - a_1 y'(a) = r_1$
   $\quad b_0 y(b) + b_1 y'(b) = r_2$

(1) B.C. of 1st kind

$$y'' = f(x, y, y')$$
$$y(a) = r_1, \quad y(b) = r_2$$

we solve IVP
$$\left.\begin{array}{l} y'' = f(x, y) \text{ with} \\ y(a) = r_1 \quad y'(a) = s \end{array}\right\} \qquad (2)$$

Then the solution of (2) will depend on s, i.e. $y \equiv y(x, s)$ and this solution $y(x, s)$ must satisfy the b.c. at $x = b$ i.e.

$$y(b, s) = r_2$$

Define $\phi(s) = y(b, s) - r_2$

Now we find s s.t. $\phi(s) = 0$

(2) B C of 2nd kind $\quad y'(a) = r_1, \quad y'(b) = r_2$

Solving IVP $\quad y'' = f(x, y, y') \quad y'(a) = r_1, \; y(a) = s$

Then again solution y will depend on s, i.e. $y(x) \equiv y(x, s)$ and this must satisfy the b.c. at $x = b$ i.e. $y'(b, s) = r_2$

or take $\phi(s) = y'(b, s) - r_2$ and we find s s.t. $\phi(s) = 0$

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(3) B.C. of 3rd Kind $a_0 y(a) - a_1 y'(a) = r_1$ — (i)

$b_0 y(b) + b_1 y'(b) = r_2$ — (ii)

For this case either we take $y(a) = s$ or $y'(a) = s$ and the other is fixed by (i). Let $y'(a) = s$ then from (i)

$$a_0 y(a) = r_1 + a_1 y'(a) = r_1 + a_1 s$$

$$y(a) = \frac{1}{a_0}(r_1 + a_1 s)$$

and we solve the IVP

$y'' = f(x, y)$ with IV $y'(a) = s$,

$y(a) = \frac{1}{a_0}(r_1 + a_1 s)$

and the solution $y \equiv y(x, s)$ and this must satisfy

$$b_0 y(b, s) + b_1 y'(b, s) = r_2$$

or

$$\phi(s) = b_0 y(b, s) + b_1 y'(b, s) - r_2$$

Then we find s s.t. $\phi(s) = 0$

Ex $y'' = 2yy'$ $0 < x < 1$ — (1)

$y(0) = 0.5$, $y(1) = 1$ — (2)

Consider IVP $y'' = 2yy'$ $y(0) = 0.5$, $y'(0) = s$

we take some initial value of s say $s^{(0)}$ and solve IVP and take another value of s say $s^{(1)}$ and solve the IVP with $s^{(1)}$ and using $s^{(0)}$ and $s^{(1)}$ we get $s^{(2)}$ by secant method

$$s^{(k+1)} = s^{(k)} - \left[\frac{s^{(k)} - s^{k-1}}{\phi(s^k) - \phi(s^{k-1})}\right]\phi(s^{(k)})$$

$k = 1, 2, \ldots$

and continue the process till we get appropriate s

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for which $\phi(s) = 0$ (or close to zero up to certain decimal place.). (8)

Now solve IVP — Iterate until tolerance is less than 0.005

$$y'' = 2yy' \quad \text{—} (3)$$

$$y(0) = 0.5, \quad y'(0) = 0.5$$

$$\boxed{s^{(0)} = 0.5}$$

use Taylors method of order 2 ($y_0 = 0.5$, $y_0' = 0.5$)

Take $h = 0.25$

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n'''$$

$$y_{n+1}' = y_n' + h y_n'' + \frac{h^2}{2!} y_n'''$$

$y_{n+1} =$

$$y'' = 2yy' \qquad y''' = 2yy'' + 2y'^2$$

$$= 2y \cdot 2yy' + 2y'^2$$

$$y''' = 4y^2y' + 2y'^2$$

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{\cancel{2!}} \cdot \cancel{2} y_n y_n' + \frac{h^3}{3!}\left(4y_n^2 y_n' + 2y_n'^2\right)$$

$$\boxed{y_{n+1} = y_n + h y_n' + h^2 y_n y_n' + \frac{h^3}{3}\left(y_n'^2 + 2y_n^2 y_n'\right)} \quad (4)$$

$y_{n+1}' = y_n' + h y_n'' + \frac{h^2}{2!} y_n'''$

$$y_{n+1}' = y_n' + 2h y_n y_n' + \frac{h^2}{2!}\left[4y_n^2 y_n' + 2y_n'^2\right]$$

$$\boxed{y_{n+1}' = y_n' + 2h y_n y_n' + h^2\left[2y_n^2 y_n' + y_n'^2\right]} \quad (5)$$

Take $h = 0.25$

$y(.25) = y_1 = 0.64323$ $\qquad$ $y'(.25) = y_1' = 0.65625$

$y(.50) = y_2 = 0.83075$ $\qquad$ $y'(.5) = y_2' = .92017$

$y(.75) = y_3 = 1.13674$ $\qquad$ $y'(.75) = y_3' = 1.45289$

$y(1.0) = y_4 = 1.62699$ $\qquad$ $y'(1.0) = y_4' = 2.63844$

$$\phi(s^{(0)}) = y(1, s^{(0)}) - 1.0 = .62699$$

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Next we take $s^{(1)} = 0.1$, i.e. we solve IVP
$$y'' = 2yy', \quad y(0) = 0.5 \quad y'(0) = s^{(1)} = 0.1$$

$y(0.25) = y_1 = .52044$ $\qquad$ $y'(.25) = y_1' = 0.12075$

$y(.50) = y_2 = 0.56534$ $\qquad$ $y'(.5) = y_2' = .16036$

$y(.75) = y_3 = .61407$ $\qquad$ $y'(.75) = y_3' = .22437$

$y(1.0) = y_4 = .67991$ $\qquad$ $y'(1.0) = y_4' = .30698$

$$\phi(s^{(1)}) = y(1, s^{(1)}) - 1.0 = -0.32009$$

Now we use Secant method to get $s^{(2)}$

Using $s^{(2)}$.

$$s^{(2)} = s^{(1)} - \left[\frac{s^{(1)} - s^{(0)}}{\phi(s^{(1)}) - \phi(s^{0})}\right]\phi(s^{(1)})$$

$$= 0.1 - \left[\frac{0.1 - 0.5}{-.32009 - 0.62699}\right](-0.32009)$$

$$= 0.23519$$

Next we solve IVP $y'' = 2yy'$; $y(0) = 0.5$, $y'(0) = s^{(2)} = 0.23519$

and obtain following:

$y(0.25) = y_1 = 0.56705$; $\quad y'(0.25) = y_1' = 0.30479$

$y(0.50) = y_2 = 0.65555$; $\quad y'(0.50) = y_2' = 0.40926$

$y(0.75) = y_3 = 0.77734$; $\quad y'(0.75) = y_3' = 0.57586$

$y(1.0) = y_4 = 0.95464$; $\quad y'(1.0) = y_4' = 0.86390$

Then $\phi(s^{(2)}) = y(1, s^{(2)}) - 1.0 = .95464 - 1$
$$= -.04536$$

Again using secant method, we obtain

$$s^{(3)} = s^{(2)} - \left[\frac{s^{(2)} - s^{(1)}}{\phi(s^{(2)}) - \phi(s^{(1)})}\right]\phi(s^{(2)})$$

$$= 0.23519 - \left[\frac{0.23519 - 0.1}{-0.04536 + 0.32009}\right](-0.04536)$$

$$= 0.25751$$

$y(0.25) = y_1 = 0.57344$; $y'(0.25) = y_1' = 0.33408$

$y(0.50) = y_2 = 0.67066$; $y'(0.50) = y_2' = 0.45068$

$y(0.75) = y_3 = 0.80536$; $y'(0.75) = y_3' = 0.63969$

$y(1.0) = y_4 = 1.00394$; $y'(1.0) = y_4' = 0.97472$

$$\phi(s^{(3)}) = y(1, s^{(3)}) - 1 = 1.00394 - 1$$

$$= .00394 < .005$$

So we stops the process.

<u>Newton-Raphson Method</u>:

$$u'' = f(x, u, u') \quad \text{——(1)} \qquad \begin{aligned} a_0 u(a) - a_1 u'(a) &= r_1 \quad \text{(i)} \\ b_0 u(b) + b_1 u'(b) &= r_2 \quad \text{(ii)} \end{aligned}$$

Now take. $u'(a) = s$ then from (i)

$$u(a) = \frac{1}{a_0}(r_1 + a_1 u'(a)) = \frac{a_1 s + r_1}{a_0}$$

Now solve the IVP

$$\left.\begin{aligned} u'' &= f(x, u, u') \\ u(a) &= \frac{a_1 s + r_1}{a_0}, \quad u'(a) = s \end{aligned}\right\} \text{——(I)}$$

Now after solving (I) we get solution $u(x,s)$ which must satisfy b.c. at $x=b$ i.e.,

$$b_0 u(b,s) + b_1 u'(b,s) = r_2 \quad \text{——} \ (2)$$

Hence the problem is to find s. s.t.

$$\phi(s) = b_0 u(b,s) + b_1 u'(b,s) - r_2 = 0 \quad \text{—} \ (3)$$

Now if we use Newton-Raphson method to solve (3) then

$$s^{(k+1)} = s^{(k)} - \frac{\phi(s^k)}{\phi'(s^k)}, \quad k = 0,1,2,\ldots$$

Now to determine $\phi'(s^k)$ we use the following method

Let $u_s = u(x,s)$, $u_s' = u'(x,s)$, $u_s'' = u''(x,s)$

$\left[' - \frac{d}{dx}\right]$ then

$u'' = f(x,u,u')$ will be written as

$$u_s'' = f(x, u_s, u_s') \quad \text{——} \ (4)$$

$$u_s(a) = \frac{a_1 s + r_1}{a_0}, \quad u_s'(a) = s \quad \text{——} \ (5)$$

Differentiating (4) partially w.r.t. s we get

$$\frac{\partial}{\partial s}(u_s'') = \frac{\partial}{\partial s} f(x, u_s, u_s')$$

(as x is independent of s)

$$= \frac{\partial f}{\partial x}\cdot \overbrace{\frac{\partial x}{\partial s}}^{\to 0} + \frac{\partial f}{\partial u_s}\cdot\frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u_s'}\cdot\frac{\partial u_s'}{\partial s}$$

$$\frac{\partial}{\partial s}(u_s'') = \frac{\partial f}{\partial u_s}\,\frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u_s'}\cdot\frac{\partial u_s'}{\partial s} \quad \text{——} \ (6)$$

Now differentiating (5) w.r.t. s

$$\frac{\partial}{\partial s} u_s(a) = \frac{a_1}{a_0} \quad ; \quad \frac{\partial}{\partial s}\left(u'_s(a)\right) = 1 \qquad \text{—— (7)}$$

Let $v = \frac{\partial}{\partial s}(u_s) = \frac{\partial u_s}{\partial s}$

$$v' = \frac{\partial}{\partial x} v = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial s} u_s\right) = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial x} u_s\right)$$

$$= \frac{\partial}{\partial s}\left(u'_s\right)$$

$$v'' = \frac{\partial}{\partial x}\left(v'\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial s} u'_s\right) = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial x} u'_s\right)$$

$$= \frac{\partial}{\partial s}\left(u''_s\right)$$

Then from (6)

$$\frac{\partial}{\partial s}\left(u''_s\right) = \frac{\partial f}{\partial u_s} \cdot \frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u'_s} \cdot \frac{\partial u'_s}{\partial s}$$

$$\left.\begin{aligned} v'' &= \frac{\partial f}{\partial u_s} v + \frac{\partial f}{\partial u'_s} v' && \text{—— (8)} \\ \text{and from (7)} \quad v(a) &= \frac{a_1}{a_0}, \quad v'(a) = 1 && \text{—— (9)} \end{aligned}\right\} \text{(II)}$$

Now we solve IVP (I) and (II) simultaneously and then $v(b)$ and $v'(b)$ are available. Then from (3)

$$\phi(s) = b_0 u_s(b, s) + b_1 u'_s(b, s) - r_2$$

$$\frac{d\phi(s)}{ds} = b_0 \frac{\partial u_s(b,s)}{\partial s} + b_1 \frac{\partial}{\partial s} u'_s(b, s)$$

$$= b_0 v(b) + b_1 v'(b)$$

Thus $\phi'(s_k)$ is available.

<u>B.C. of 1st Kind</u> $u(a) = r_1$, $u(b) = r_2$

i.e., $a_1 = 0$, $a_0 = 1$ & $b_0 = 1$, $b_1 = 0$

$$\boxed{\frac{d\phi(s)}{ds} = v(b),}$$

Shooting method (Newton-Raphson method)

$$v'' = \frac{\partial f}{\partial u_s}(x, u_s, u_s')\, v + \frac{\partial f}{\partial u_s'}(x, u_s, u_s')\, v'$$

$$v(a) = a_1/a_0, \quad v'(a) = 1 \qquad \left[\begin{array}{l} a_0 u(a) - a_1 u'(a) = r_1 \\ b_0 u(b) + b_1 u'(b) = r_2 \end{array}\right]$$

Ex Use shooting method to solve the bvp

$$u'' = 2uu' \qquad 0 < x < 1$$

$$u(0) = 0.5, \quad u(1) = 1.$$

Use Runge-Kutta method

$$K_1 = \frac{h^2}{2} f(x_j, u_j, u_j')$$

$$K_2 = \frac{h^2}{2} f\left(x_j + \frac{2}{3}h,\ u_j + \frac{2}{3}hu_j' + \frac{2}{3}K_1,\ u_j' + \frac{4}{3h}K_1\right)$$

$$u_{j+1} = u_j + hu_j' + \frac{1}{2}(K_1 + K_2)$$

$$u_{j+1}' = u_j' + \frac{1}{2h}(K_1 + 3K_2)$$

with $h = 0.25$ to solve corresponding IVP. Use Newton-Raphson method, assuming the starting values of the slope at $x=0$ as $s^{(0)} = u'(0) = 0.3$.

IVP
$$\left.\begin{array}{l} u'' = 2uu' \\ u(0) = 0.5, \quad u'(0) = s^{(0)} = 0.3 \end{array}\right\} \text{IVP (I)}$$

the IVP occurring in the application of the N-R method is given by

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$$v'' = \frac{\partial f}{\partial u_s} v + \frac{\partial f}{\partial u_s'} v'$$ $\quad f = 2uu'$

$$v(a) = \frac{a_1}{a_0}, \quad v'(a) = 1$$ $\quad \frac{\partial f}{\partial u} = 2u', \; \frac{\partial f}{\partial u'} = 2u$

$$v'' = 2u'v + 2uv' = 2(u'v + uv')$$ IVP II

$$v(0) = 0, \quad v'(0) = 1$$

IVP I

$$k_1 = \frac{h^2}{2} f(x_j, u_j, u_j') = \frac{h^2}{\cancel{2}} \cdot \cancel{2} u_j u_j' = h^2 u_j u_j'$$

$$k_2 = \frac{h^2}{2} f\left(x_j + \tfrac{2}{3}h, u_j + \tfrac{2}{3}hu_j' + \tfrac{2}{3}k_1, u_j' + \tfrac{4}{3h}k_1\right)$$

$$= \frac{h^2}{2} \cdot 2\left(u_j + \tfrac{2}{3}hu_j' + \tfrac{2}{3}k_1\right) \cdot \left(u_j' + \tfrac{4}{3h}k_1\right)$$

$$u_{j+1} = u_j + hu_j' + \frac{1}{2}(k_1 + k_1)$$

$$u_{j+1}' = u_j' + \frac{1}{2h}(k_1 + 3k_2)$$

IVP II

$$k_1^* = \frac{h^2}{\cancel{2}} \cdot \cancel{2}(u_j' v_j + u_j v_j') = h^2(u_j' v_j + u_j v_j')$$

$$k_2^* = \frac{h^2}{\cancel{2}} \cdot \cancel{2}\left[u_j'\left(v_j + \tfrac{2}{3}hv_j' + \tfrac{2}{3}k_1^*\right) + u_j\left(v_j' + \tfrac{4}{3h}k_1^*\right)\right]$$

$$k_2^* = h^2\left[u_j'\left(v_j + \tfrac{2}{3}hv_j' + \tfrac{2}{3}k_1^*\right) + u_j\left(v_j' + \tfrac{4}{3h}k_1^*\right)\right]$$

$$v_{j+1} = v_j + hv_j' + \frac{1}{2}(k_1^* + k_2^*)$$

$$v_{j+1}' = v_j' + \frac{1}{2h}(k_1^* + 3k_2^*)$$

first iteration $h=0.25$

$j=0$ $\quad u_0=0.5,\ u_0'=0.3,\ v_0=0,\ v_0'=1,\ h=0.25$

$k_1=.009375,\ k_2=0.01297$ $\quad u_1=0.58577,\ u_1'=0.39177$

$k_1^*=0.03125,\ k_2^*=0.03997,\ v_1=0.28561$ $\quad v_1'=1.30232$

$j=1$

$k_1=0.01434,\ k_2=0.01933$ $\quad u_2=0.70055,\ u_2'=0.53643$

$k_1^*=0.05467,\ k_2^*=0.07155$ $\quad v_2=0.67430,\ v_2'=1.84096$

$\vdots$

$j=3$

$k_1=0.04225,\ k_2=0.06441$ $\quad u_4=1.11222$ $\quad u_4'=1.25429$

$k_1^*=0.21722$ $\quad k_2^*\ 0.31035$ $\quad v_4=2.24160$ $\quad v_4'=5.18338$

$$\phi(s^{0)})=u(b)-1.0=u_4-1.0$$
$$=1.11222-1=0.11222$$

and $\dfrac{d\phi}{ds}(s^0)=v(b)=v_4=2.24160$

N-R- method gives the next approximation

$$s^{(1)}=s^{0)}-\frac{\phi(s^0)}{\phi'(s^0)}=0.3-\frac{0.11222}{2.24160}$$
$$=0.24994$$

Second iteration

for $j=0$ $\quad u_0 = 0.5, \quad u_0' = 0.24994, \quad v_0 = 0, v_0' = 1$

⋮

$j=3$ $\quad k_1 = 0.03138 \quad k_2 = 0.04599 \quad u_4 = 0.99414, \quad u_4' = 0.96780$

$k_1^* = 0.10511, \quad k_2^* = 0.25718 \quad v_4 = 2.13686 \quad v_4' = 4.65418$

$$\phi(s^{(1)}) = u(b) - 1.0 = u_4 - 1.0 = 0.99414 - 1.0$$
$$= -0.00586$$

$$\frac{d\phi(s^{(1)})}{ds} = v(b) = 2.13686$$

$$s^{(2)} = s^{(1)} - \frac{\phi(s^{(1)})}{\phi'(s^{(1)})} = 0.24994 - \frac{0.00586}{2.13686}$$
$$= 0.25268.$$