Implicit Runge-Kutta Methods

Method

$$u_{j+1} = u_j + W_1 K_1 \quad — ①$$

$$y' = f(t, y)$$

$$K_1 = h f(t_j + c_1 h, u_j + a_{11} K_1) \quad - ②$$

find W_1, c_1 and a_{11}

$$u(t_{j+1}) = u(t_j) + h u'(t_j) + \frac{h^2}{2} u''(t_j) + \cdots$$

$$= u(t_j) + h f(t_j, u(t_j)) + \frac{h^2}{2} (f_t + f f_u)_{t_j} + O(h^3) \quad — ③$$

and

$$K_1 = h f(t_j + c_1 h, u_j + a_{11} K_1)$$

$$K_1 = h[f(t_j, u_j) + (c_1 h f_t + a_{11} K_1 f_u + \cdots)_{t_j}]$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h a_{11} K_1 f_u + O(h^3)$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h a_{11} f_u \cdot h f(t_j + c_1 h, u_j + a_{11} K_1) + O(h^3)$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h^2 a_{11} f_u [f(t_j, u_j) + c_1 h f_t + a_{11} K_1 f_u] + O(h^3)$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h^2 a_{11} f_u f + h^3 [a_{11} c_1 f_t f_u + a_{11} f_u f] + O(h^3)$$

$$K_1 = hf + h^2 [c_1 f_t + a_{11} f_u f] + O(h^3) \quad — ④$$

Now putting expressions for $u(t_{j+1})$ and k_1 from ③ & ④ in ①, we get:

$$u(t_{j+1}) = u(t_j) + W_1 k_1$$

$$u(t_j) + hf + \frac{h^2}{2}(f_t + f f_u)_{t_j} + O(h^3)$$

$$= u(t_j) + hW_1 f + h^2 [W_1 c_1 f_t + W_1 a_{11} f_u f]_{t_j} + O(h^3)$$

This implies

$$W_1 = 1, \quad W_1 c_1 = 1/2, \quad W_1 a_{11} = 1/2$$

$$\Rightarrow W_1 = 1, \quad c_1 = 1/2, \quad a_{11} = 1/2$$

Thus second order implicit-Runge-Kutta method is given by

$$u_{j+1} = u_j + k_1$$

$$k_1 = hf\left(t_j + \frac{h}{2}, u_j + \frac{1}{2}k_1\right)$$

Fourth Order implicit Runge-Kutta Method

$$u_{j+1} = u_j + W_1 k_1 + W_2 k_2 \qquad \text{— ⑤}$$

$$k_1 = hf(t_j + c_1 h, u_j + a_{11} k_1 + a_{12} k_2)$$

$$k_2 = hf(t_j + c_2 h, u_j + a_{21} k_1 + a_{22} k_2)$$

Find $W_1, W_2, c_1, c_2, a_{11}, a_{12}, a_{21}, a_{22}$.

Now use Taylor expansions for u_{j+1}, k_1 and k_2

$$u_{j+1} = u(t_{j+1})$$
$$= u(t_j) + h\,u'(t_j) + \frac{h^2}{2} u''(t_j) + \frac{h^3}{3!} u'''(t_j) + \frac{h^4}{4!} u^{(4)}(t_j) + O(h^5)$$

and k_1 & k_2 may be written as

$$K_1 = hA_1 + h^2 B_1 + h^3 C_1 + O(h^4)$$
$$K_2 = hA_2 + h^2 B_2 + h^3 C_2 + O(h^4)$$

Put all these expressions in (5) and compare the coefficients of h, h^2, h^3, h^4 we get all the values as

$$W_1 = \frac{1}{2}, \quad W_2 = \frac{1}{2}, \quad c_1 = \frac{3-\sqrt{3}}{6}, \quad c_2 = \frac{3+\sqrt{3}}{6}$$

$$a_{11} = 1/4 \quad a_{12} = \frac{3-2\sqrt{3}}{12}, \quad a_{21} = \frac{3+2\sqrt{3}}{12}, \quad a_{22} = 1/4$$

and the method will be as follows:

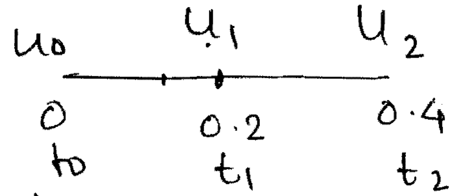
$$u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2)$$
$$K_1 = h f\left(t_j + \frac{3-\sqrt{3}}{6} h,\ u_j + \frac{k_1}{4} + \frac{3-2\sqrt{3}}{12} k_2\right)$$
$$K_2 = h f\left(t_j + \frac{3+\sqrt{3}}{6} h,\ u_j + \frac{3+2\sqrt{3}}{12} k_1 + \frac{1}{4} K_2\right)$$

Prob: Solve the initial value problem

$$u' = -2tu^2, \quad u(0) = 1$$

with $h = 0.2$ on the interval $[0, 0.4]$. Use the second order implicit Runge-Kutta method.

Sol^n



$$u_{j+1} = u_j + K_1, \quad j = 0, 1$$

$$K_1 = h f\left(t_j + \frac{h}{2}, u_j + \frac{1}{2}K_1\right)$$

So $$K_1 = -h(2t_j + h)\left(u_j + \frac{1}{2}K_1\right)^2 \quad \text{--- (1)}$$

This is an implicit equation in K_1 so we use Newton-Raphson method

Take $$f(K_1) = K_1 + h(2t_j + h)\left(u_j + \frac{1}{2}K_1\right)^2$$

as $h = 0.2$

$$f(K_1) = K_1 + 0.2(2t_j + 0.2)\left(u_j + \frac{1}{2}K_1\right)^2$$

$$\left(\frac{df}{dK_1} =\right) f'(K_1) = 1 + 0.2(2t_j + 0.2)\left(u_j + \frac{1}{2}K_1\right)$$

Then from NR method

$$K_1^{(s+1)} = K_1^{(s)} - \frac{f(K_1^{(s)})}{f'(K_1^{(s)})}, \quad s = 0, 1, 2, \ldots$$

Take $K_1^{(0)} = h f(t_j, u_j), \quad j = 0, 1$

for $j=0$, $t_0=0$, $u_0=1$, $k_1^{(0)} = hf(t_0, u_0^2)$

$$= -h\, 2t_0 u_0^2$$

$$= 0 \quad \text{as } t_0 = 0$$

$f(k_1^{(0)}) = 0.04$, $f'(k_1) = 1.04$

$k_1^{(1)} = -0.03846150$

$f(k_1^{(1)}) = 0.00001483$, $f'(k_1^{(1)}) = 1.03923077$

$k_1^{(2)} = -0.03847567$

Take $k_1 = k_1^{(2)}$

and $u(0.2) (= u_1) = u_0 + k_1 = 1 - 0.3847567$

$u(0.2) = 0.96152433$

Similarly for $j=1$ find k_1 and u_2

$k_1 = k_1^{(3)} = -0.09973420$

$u(0.4) (= u_2) = u_1 + k_1$

$$= 0.96152433 + (-0.0997342)$$

$$= 0.86179013$$