Multistep Methods

Ex Find the solution at $x = 0.3$ for the differential equation (1)

$$y' = x - y^2, \quad y(0) = 1$$

using

$$y_{n+1} = y_n + \frac{h}{2}(3y_n' - y_{n-1}')$$

Adams-Bashforth method of order 2 ($p = 1$), $n = 1, 2 ..$

with $h = 0.1$.

Determine the starting values using second order Runge-Kutta method.

Sol$^n$ We need $y(x_1)$ at $x = x_1$ for starting the computation. This value is determined with the help of the second order Runge-Kutta method.

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + h, y_n + K_1)$$

$$y' = x - y^2, \quad y_0 = 1$$

$$K_1 = 0.1(0 - 1) = -0.1$$

$$K_2 = 0.1(0.1 - (1 - .0.1)^2) = -.071$$

$$y_1 = 1 + \frac{1}{2}(-.1 - .071) = 0.9145$$

$$y_1' = 0.1 - (.9145)^2 = -0.73631$$

Using Adams-Bashforth method, we now obtain

$$y_2 = y_1 + \frac{.1}{2}(3y_1' - y_0')$$

$$= .9145 + \frac{0.1}{2}(.3 \times (-).73631 + 1)$$

$$= .05405$$

$$y_2' = 0.2 - (.05405)^2 = -.52940$$

$$y_3 = y_2 + \frac{0.1}{2}(3y_2' - y_1')$$

$$= .05405 + \frac{.1}{2}(3 \times (-0.52940) + 0.73631)$$

$$= .01146$$

Ex. Derive a fourth order method of the form

$$y_{n+1} = a y_{n-2} + h(b y_n' + c y_{n-1}' + d y_{n-2}' + e y_{n-3}')$$

for the solution of $y' = f(x, y)$. Find truncation error.

$$T_{n+1}(Y) = y(x_{n+1}) - a y(x_{n-2}) - h[b y'(x_n) + c y'(x_{n-1}) + d y'(x_{n-2}) + e y'(x_{n-3})]$$

$$= C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + C_3 h^3 y'''(x_n) + C_4 h^4 y^{(4)}(x_n) + C_5 h^5 y^{(5)}(x_n) + \cdots$$

For fourth order method.

$$\frac{1}{h} T_{n+1}(Y) = O(h^4)$$

or

$$T_{n+1}(Y) = O(h^5)$$

$$C_0 = 1 - a = 0$$

$$C_1 = 1 + 2a - (b + c + d + e)$$

$$C_2 = \frac{1}{2}(1 - 4a) + (c + 2d + 3e) = 0$$

$$C_3 = \frac{1}{6}(1 + 8a) - \frac{1}{2}(c + 4d + 9e) = 0$$

$$C_4 = \frac{1}{24}(1-a) + \frac{1}{6}(c+8d+27e) = 0$$

$a = 1$, $b = 21/8$, $c = -9/8$, $d = 15/8$, $e = -3/8$

$$y_{n+1} = y_{n-2} + \frac{h}{8}\left(21y_n' - 9y_{n-1}' + 15y_{n-2}' - 3y_{n-3}'\right)$$

with Truncation error

$$T_{n+1} = \frac{81}{240} h^5 y^{(5)}(\xi)$$

$$x_{n-3} < \xi < x_{n+1}$$

<u>Ex</u> Consider an implicit two-step method

$$y_{n+1} - (1+a)y_n + a y_{n-1} = \frac{h}{12}\left[(5+a)y_{n+1}' + 8(1-a)y_n' - (1+5a)y_{n-1}'\right]$$

where $-1 \le a \le 1$

Show that the order of the two step method is 3 if $a \neq -1$ and is 4 if $a = -1$.

<u>Soln</u>

$$T_{n+1} = y(x_{n+1}) - (1+a)y(x_n) + a y(x_{n-1}) - \frac{h}{12}\left[(5+a)y'(x_{n+1}) + 8(1-a)y'(x_n) - (1+5a)y'(x_{n-1})\right]$$

$$= C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + C_3 h^3 y'''(x_n) + C_4 h^4 y^{(4)}(x_n) + \cdots$$

$C_0 = 0$, $C_1 = 0$

$C_2 = 0$, $C_3 = 0$

$$C_4 = -\frac{1}{24}(1+a)$$

Hence truncation error is

$$T_{n+1} = -\frac{1}{24}(1+a)h^4 f^{(4)}(x_n) + O(h^5).$$

Thus, 2-step method has order 3 if $a \neq -1$ and order 4 if $a = -1$.

Ex Determine the constants $\alpha, \beta, \gamma$ so that the difference approximation

$$y_{n+2} - y_{n-2} + \alpha(y_{n+1} - y_{n-1}) = h[\beta(f_{n+1} + f_{n-1}) + \gamma f_n]$$

for $y' = f(x,y)$ will have the order of approximation 6.

Soln The truncation error of the method is given by

$$T_{n+1} = y(x_{n+2}) - y(x_{n-2}) + \alpha(y(x_{n+1}) - y(x_{n-1})) - h[\beta(y'(x_{n+1}) + y'(x_{n-1})) + \gamma y'(x_n)]$$

$$= C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + C_3 h^3 y'''(x_n) + C_4 h^4 y^{(4)}(x_n) + C_5 h^5 y^{(5)}(x_n) + C_6 h^6 y^{(6)}(x_n) + C_7 h^7 y^{(7)}(x_n) + \cdots$$

$C_0 = 0$

$C_1 = 4 + 2\alpha - 2\beta - \gamma$

$C_2 = 0,$

$C_3 = \frac{1}{6}(16 + 2\alpha) - \beta$

$C_4 = 0$

$C_5 = \frac{1}{120}(64 + 2\alpha) - \frac{1}{12}\beta$

$C_6 = 0$

$$C_7 = \frac{1}{5040}(256 + 2\alpha) - \frac{1}{360}\beta$$

Setting $C_i = 0$, $i = 1, 3, 5$, we obtain

$$\alpha = 28, \quad \beta = 12, \quad \gamma = 36, \quad \text{and}$$

$$C_7 = \frac{1}{35}$$

Then the sixth order method is

$$y_{n+2} + 28y_{n+1} - 28y_{n-1} - y_{n-2} = h(12f_{n+1} + 36f_n + 12f_{n-1})$$

with truncation error

$$T_{n+1} = \frac{1}{35}h^7 y^{(7)}(x_n) + O(h^8) \ldots$$

Ex One method for the solution of the differential equation $y' = f(y)$ with $y(0) = y_0$ is the implicit mid point method

$$y_{n+1} = y_n + hf\left(\frac{1}{2}(y_n + y_{n+1})\right)$$

Find the local truncation error.

Soln

$$T_{n+1} = y(x_{n+1}) - y(x_n) - hf\left(\frac{1}{2}(y(x_n) + y(x_{n+1}))\right)$$

$$= y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + \cdots - y(x_n) - hf\left(y(x_n) + \frac{1}{2}hy'(x_n) + \frac{1}{4}h^2 y''(x_n) + \cdots\right)$$

$$= hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n)$$
$$- h\left[f_n + \left(\frac{1}{2}hy_n' + \frac{1}{4}h^2 y_n'' + \cdots\right)f_y + \frac{1}{2}\left(\frac{1}{2}hy_n' + \frac{1}{4}h^2 y_n'' + \cdots\right)^2 f_{yy} + \cdots\right]$$

$$T_{n+1} = -\frac{1}{24} h^3 f_n \left(2 f_y^2 - f f_{yy}\right)_{x_n} + O(h^4)$$ ⑥

## Adams - Moulton method



Ex for IVP $u' = t^2 + u^2, \quad u(1) = 2$ find an estimate for $u(1.2)$ using Adams-Moulton 3rd order method with $h = 0.1$.

Sol$^n$ $h = 0.1, \quad t_0 = 1 \qquad t_1 = 1.1, \; t_2 = 1.2$

$$y_{n+1} = y_n + \frac{h}{12}[5f_{n+1} + 8f_n - f_{n-1}]$$

for $n+2$

$$y_{n+2} = y_{n+1} + \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n]$$

$$u_{n+2} = u_{n+1} + \frac{h}{12}[5(t_{n+2}^2 + u_{n+2}^2) + 8(t_{n+1}^2 + u_{n+1}^2) - (t_n^2 + u_n^2)]$$

for $n=0$

$$u_2 = u_1 + \frac{h}{12}[5(t_2^2 + u_2^2) + 8(t_1^2 + u_1^2) - (t_0^2 + u_0^2)] \qquad (*)$$

Now we calculate $u_1$ using some other method say Taylor's method

$$u_1 = u_0 + hu_0' + \frac{h^2}{2!}u_0'' + \frac{h^3}{3!}u_0'''$$

$$u_0 = 2, \quad u_0' = t_0^2 + u_0^2 = 5$$

$$u_0'' = 2t_0 + 2u_0u_0' = 22$$

$$u_0''' = 2 + 2u_0u_0'' + 2(u_0')^2 = 140$$

$$u(1.1) \equiv u_1 = 2 + (0.1)5 + \frac{(0.1)^2}{2}.22 + \frac{(0.1)^3}{6}140$$

$$= 2.633333$$

Now putting value of $u_1$ in $(*)$ we get a non-linear

Adams- Moulton



equation in $u_2$ and solve using Newton-Raphson method.

$$u_2 = .041667\, u_2^2 + 3.194629$$

Take $f(u_2) = .041667\, u_2^2 - u_2 + 3.194629$

$$f'(u_2) = .083334\, u_2 - 1$$

$$u_2^{(r+1)} = u_2^{(r)} - \frac{f(u_2^{(r)})}{f'(u_2^{(r)})}, \quad r = 0, 1, 2 \ldots$$

Take $u_2^{(0)} = u_1 = 2.633333$

$$u_2^{(1)} = u_2^{(0)} - \frac{f(u_2^{(0)})}{f'(u_2^{(0)})} = 3.722602$$

$$u_2^{(2)} = u_2^{(1)} - \frac{f(u_2^{(1)})}{f'(u_2^{(1)})} = 3.794275$$

$$u_2^{(3)} = u_2^{(2)} - \frac{f(u_2^{(2)})}{f'(u_2^{(2)})} = 3.794508$$

Now stop the process as $f(u_2^{(3)}) = .0000001$

$u(1.2) = u_2 = u_2^{(3)} = 3.794508$