13 ①

The general k-step method for solution of $u' = f(x,u)$, $u(t_0) = u_0$ may be written as

$$u_{j+1} = a_1 u_j + a_2 u_{j-1} + \dots + a_k u_{j-k+1} + h(b_0 u'_{j+1} + b_1 u'_j + \dots + b_k u'_{j-k+1})$$

or

$$u_{j+1} = \sum_{i=1}^{k} a_i u_{j-i+1} + h \sum_{i=0}^{k} b_i u'_{j-i+1} \qquad \text{— ①}$$

Symbolically, we can write ① as

$$\rho(E)\, u_{j-k+1} - h\sigma(E)\, u'_{j-k+1} = 0, \text{ where} \qquad \text{②}$$

$$\left.\begin{aligned} \rho(\xi) &= \xi^k - a_1 \xi^{k-1} - a_2 \xi^{k-2} \dots - a_k \\ \sigma(\xi) &= b_0 \xi^k + b_1 \xi^{k-1} + \dots + b_k \end{aligned}\right\} \text{for k-step method } \rho(\xi) \text{ is of degree } \underline{\underline{k}}$$

$$E f(x_i) = f(x_i + h)$$

$$E^n f(x_i) = f(x_i + nh)$$

$$\rho(E) = E^k - a_1 E^{k-1} - a_2 E^{k-2} \dots - a_k E^0$$

$$\sigma(E) = b_0 E^k + b_1 E^{k-1} + \dots + b_k E^0$$

$$\rho(E) u_{j-k+1} = (E^k - a_1 E^{k-1} - a_2 E^{k-2} - \dots - a_k E^0)\, u_{j-k+1}$$

$$= u_{j-\cancel{k}+1+\cancel{k}} - a_1 u_{j-\cancel{k}+\cancel{1}+\cancel{k}-\cancel{1}} - \dots - a_k u_{j-k+1}$$

$$= u_{j+1} - a_1 u_j - a_2 u_{j-1} \dots - a_k u_{j-k+1}$$

Similarly

$$\sigma(E) u'_{j-k+1} = (b_0 E^k + b_1 E^{k-1} + \dots + b_k E^0)\, u'_{j-k+1}$$

$$\sigma(E)\, u'_{j-k+1} = b_0 u'_{j+1} + b_1 u'_{j-k+1+k-1} + \cdots + b_k u'_{j-k+1}$$

$$\sigma(E) = b_0 u'_{j+1} + b_1 u'_j + \cdots + b_k u'_{j-k+1}.$$

Thus (1) can be written as

$$\boxed{\rho(E)\, u_{j-k+1} - h\sigma(E)\, u'_{j-k+1} = 0}$$

Now if $\rho(\xi)$ is given then find $\sigma(\xi)$ if the order of the method is known

We use the relation for a method of order p

$$\rho(\xi) - (\log \xi)\, \sigma(\xi) = C_{p+1}(\xi-1)^{p+1} + O(\xi-1)^{p+2} \quad \text{--- (3)}$$

Note: If an implicit method is to be obtained, then $\rho(\xi)$ and $\sigma(\xi)$ are of same degree. If an explicit method is to be obtained then $\rho(\xi)$ is one degree higher than $\sigma(\xi)$.

We rewrite (3)

$$\frac{\rho(\xi)}{\log \xi} - \sigma(\xi) = C_{p+1}\frac{(\xi-1)^{p+1}}{\log \xi} + O\left(\frac{(\xi-1)^{p+2}}{\log \xi}\right)$$

$$\log \xi = \log[1+(\xi-1)] = (\xi-1) - \frac{1}{2}(\xi-1)^2 + \frac{1}{3}(\xi-1)^3 - \cdots$$

So $$\frac{\rho(\xi)}{\log \xi} - \sigma(\xi) = C_{p+1}(\xi-1)^{p} + O(\xi-1)^{p+1}$$

Ex If $\rho(\xi) = \xi(\xi-1)$ then find $\sigma(\xi)$

Now we consider

$$\frac{\rho(\xi)}{\log \xi} = \frac{\xi(\xi-1)}{\log[1+(\xi-1)]} = \frac{(\xi-1)+(\xi-1)^2}{\log[1+(\xi-1)]}$$

$$\frac{\rho(\xi)}{\log\xi} = \frac{(\xi-1)+(\xi-1)^2}{[(\xi-1)-\frac{1}{2}(\xi-1)^2+\frac{1}{3}(\xi-1)^3 - \cdots]}$$

$$= [1+(\xi-1)]\left[1-\frac{1}{2}(\xi-1)+\frac{1}{3}(\xi-1)^2 - \cdots\right]^{-1}$$

$$= 1+\frac{3}{2}(\xi-1)+\frac{5}{12}(\xi-1)^2 + O((\xi-1)^3)$$

Now for method of order 2

$$\sigma(\xi) = 1+\frac{3}{2}(\xi-1) = 1+\frac{3}{2}\xi - 3/2 = \frac{3}{2}\xi - 1/2$$

and for method of order 3

$$\sigma(\xi) = 1+\frac{3}{2}(\xi-1)+\frac{5}{12}(\xi-1)^2$$

for method of order 2

$$\rho(\xi) = \xi(\xi-1) = \xi^2 - \xi$$

$$\sigma(\xi) = \frac{3}{2}\xi - \frac{1}{2}$$

Then the method will be

$$\rho(E)u_{j-2+1} - h\sigma(E)u'_{j-2+1} = 0$$

$$\rho(E)u_{j-1} - h\sigma(E)u'_{j-1} = 0$$

$$[1+(\xi-1)]\left[1-\left(\frac{1}{2}(\xi-1)-\frac{1}{3}(\xi-1)^2+\text{---}\right)\right]^{-1}$$

$$= [1+(\xi-1)]$$

$$\left[1+\frac{1}{2}(\xi-1)-\frac{1}{3}(\xi-1)^2+O(\xi-1)^3\right.$$

$$\left.+\left(\frac{1}{2}(\xi-1)-1/3(\xi-1)^2+\text{---}\right)^2+\cdot\cdot\right]$$

$$= [1+(\xi-1)]\left[1+\frac{1}{2}(\xi-1)-\frac{1}{3}(\xi-1)^2+\frac{1}{4}(\xi-1)^2+O(\xi-1)^3\right]$$

$$= 1+\frac{1}{2}(\xi-1)-\frac{1}{3}(\xi-1)^2+\frac{1}{4}(\xi-1)^2+(\xi-1)+\frac{1}{2}(\xi-1)^2$$

$$+O(\xi-1)^3$$

$$= 1+\frac{3}{2}(\xi-1)+\left[-\frac{1}{3}+\frac{1}{4}+\frac{1}{2}\right](\xi-1)^2+O(\xi-1)^3$$

$$-\frac{1}{3}+\frac{1}{4}+\frac{1}{2}$$

$$= \frac{1}{12}(-4+3+6)$$

$$= 5/12$$

$$= 1+\frac{3}{2}(\xi-1)+\frac{5}{12}(\xi-1)^2+O(\xi-1)^3$$

$$\rho(E)\, u_{j-1} = (E^2 - E)\, u_{j-1}$$
$$= u_{j-1+2} - u_{j-1+1} = u_{j+1} - u_j$$
$$\sigma(E)\, u'_{j-1} = \left(\tfrac{3}{2}E - \tfrac{1}{2}E^0\right) u'_{j-1}$$
$$= \tfrac{3}{2}u'_j - \tfrac{1}{2}u'_{j-1}$$

$$\rho(E)\, u_{j-1} - h\,\sigma(E)\, u'_{j-1} = 0$$
$$u_{j+1} - u_j - h\tfrac{1}{2}(3u'_j - u'_{j-1}) = 0$$
$$u_{j+1} = u_j + \tfrac{h}{2}(3u'_j - u'_{j-1})$$

which is 2nd order Adam-Bashforth method.

Adams - Moulton Methods

$\rho(\xi) = \xi^{k-1}(\xi - 1)$, then find $\sigma(\xi)$ of degree k

(In other words find an implicit method of order k)

For $k = 2$

$$\rho(\xi) = \xi(\xi-1) = \xi^2 - 1 = (\xi-1) + (\xi-1)^2$$

$$\frac{\rho(\xi)}{\log \xi} = 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2 + O(\xi-1)^3$$

Since we want an implicit method so degree of $\rho(\xi)$ and $\sigma(\xi)$ will be same and thus

$$\sigma(\xi) = 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2$$

$$\sigma(\xi) = \frac{5}{12}\xi^2 + \frac{2}{3}\xi - \frac{1}{12}$$

and the method is given by

$$\rho(E)\,u_{j-1} - h\,\sigma(E)\,u'_{j-1} = 0$$

$$(E^2 - E)\,u_{j-1} - \frac{h}{12}\left[5E^2 + 8E - 1\right]u'_{j-1} = 0$$

$$u_{j+1} - u_j - \frac{h}{12}\left[5u'_{j+1} + 8u'_j - u'_{j-1}\right] = 0$$

or

$$\boxed{u_{j+1} = u_j + \frac{h}{12}\left[5u'_{j+1} + 8u'_j - u'_{j-1}\right]}$$

A general k-step method for solution of IVP : $u' = f(t,u)$, $u(t_0) = u_0$ may be written as (1)

$$u_{j+1} = a_1 u_j + a_2 u_{j-1} + \cdots + a_k u_{j-k+1} + h(b_0 u'_{j+1} + b_1 u'_j + \cdots + b_k u'_{j-k+1})$$

Or,

$$u_{j+1} = \sum_{i=1}^{k} a_i u_{j-i+1} + h \sum_{i=0}^{k} b_i u'_{j-i+1} \quad \text{——} (*)$$

and Local ~~truncated~~ truncation error can be written as

$$T_{j+1} = u(t_{j+1}) - \sum_{i=1}^{k} a_i u(t_{j-i+1}) - h \sum_{i=0}^{k} b_i u'(t_{j-i+1})$$

—— (1)

Now

$$u(t_{j+1}) = u(t_j) + h u'(t_j) + \frac{h^2}{2!} u''(t_j) + \frac{h^3}{3!} u'''(t_j) + \cdots \quad \text{——(i)}$$

$$u(t_{j-i+1}) = u(t_j + (1-i)h)$$

$$= u(t_j) + (1-i) h u'(t_j) + \frac{(1-i)^2}{2!} h^2 u''(t_j) + \cdots \quad \text{——(ii)}$$

and

$$u'(t_{j-i+1}) = u'(t_j) + (1-i) h u''(t_j) + \frac{(1-i)^2}{2!} h^2 u'''(t_j) + \cdots \quad \text{——(iii)}$$

Substituting (i), (ii) & (iii) in (1) we get

$$T_{j+1} = c_0 u(t_j) + c_1 h u'(t_j) + c_2 h^2 u''(t_j) + \cdots + c_p h^p u^{(p)}(t_j) + O(h^{p+1})$$

—— (2)

where

$$c_0 = 1 - \sum_{i=1}^{k} a_i \qquad \text{— (3)}$$

$$c_1 = \left[1 - \sum_{i=1}^{k} (1-i) a_i\right] - \sum_{i=0}^{k} b_i \qquad \text{— (4)}$$

Similarly the c_q for $q = 1, 2, 3, \dots$ can be written as

$$c_q = \frac{1}{q!}\left[1 - \sum_{i=1}^{k} a_i (1-i)^q\right] - \frac{1}{(q-1)!} \sum_{i=0}^{k} b_i (1-i)^{q-1}$$

$$q = 1, 2, \dots$$

Def^{n} Order of a method: The linear multistep method (*) is said to be of order p if in (2)

$$T_{j+1} = c_0 u(t_j) + c_1 h u'(t_j) + \dots + c_p h^p u^{(p)}(t_j) + c_{p+1} h^{p+1} u^{(p+1)}(t_j) + \dots$$

$$c_0 = c_1 = c_2 = \dots = c_p = 0 \text{ and } c_{p+1} \neq 0$$

Def^{n} The multistep method (*) is said to consistent if it has order $p \geq 1$, i.e., $c_0 = c_1 = 0$.

$$\rho(\xi) = \xi^k - a_1 \xi^{k-1} - a_2 \xi^{k-2} - \dots - a_k \quad \text{— (i)}$$

$$\sigma(\xi) = b_0 \xi^k + b_1 \xi^{k-1} + \dots + b_k \quad \text{— (ii)}$$

from (3) $\quad c_0 = 0 \Rightarrow 1 - \sum_{i=1}^{k} a_i = 0 \Rightarrow \rho(1) = 0$

from (4), $c_1 = 0 \Rightarrow 1 - \sum_{i=1}^{k} (1-i) a_i - \sum_{i=0}^{k} b_i = 0$

or $\quad 1 + \sum_{i=1}^{k} (i-1) a_i - \sigma(1) = 0 \qquad \text{— (1)}$

or

$$\left[1 + a_2 + 2a_3 + \cdots + (k-1)a_k\right] - (b_0 + b_1 + \cdots b_k) = 0$$

$$1 + [a_1 + 2a_2 + 3a_3 + \cdots + ka_k] - [a_1 + a_2 + \cdots + a_k] - \sigma(1) = 0$$

— (2)

Now

$$\rho(\xi) = \xi^k - a_1\xi^{k-1} - a_2\xi^{k-2} - \cdots - a_k$$

$$\rho'(\xi) = k\xi^{k-1} - (k-1)a_1\xi^{k-2} - (k-2)a_2\xi^{k-3} - \cdots - a_{k-1}$$

$$\rho'(1) = k - (k-1)a_1 - (k-2)a_2 - \cdots - a_{k-1}^{*} - ka_{k-1} - ka_k + ka_{k-1}^{*} + k^{*}a_k$$

$$= k[1 - a_1 - a_2 - \cdots - a_{k-1} - a_k] + [a_1 + 2a_2 + 3a_3 + \cdots + (k-1)a_{k-1} + ka_k]$$

$$\rho'(1) = [a_1 + 2a_2 + 3a_3 + \cdots + ka_k] \quad \text{— (3)}$$

Now from (2) & (3)

$$\rho'(1) - \sigma(1) = 0 \text{ or } \rho'(1) = \sigma(1)$$

So if the method is consistent then

$$\rho(1) = 0 \;\&\; \rho'(1) = \sigma(1)$$

Consider general p MSM <u>Consistency condition</u>

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}) \quad \text{—} (1)$$

Now apply this on $y' = \lambda y$.

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h\lambda \sum_{j=-1}^{p} b_j y_{n-j}$$

Now put $y_n = \xi^n$

$$\xi^{n+1} = \sum_{j=0}^{p} a_j \xi^{n-j} + h\lambda \sum_{j=-1}^{p} b_j \xi^{n-j}$$

$$\xi^{n+1} = (a_0 \xi^n + a_1 \xi^{n-1} + \cdots + a_p \xi^{n-p}) + \lambda h (b_{-1} \xi^{n+1} + b_0 \xi^n + b_1 \xi^{n-1} + \cdots b_p \xi^{n-p})$$

or

$$(\xi^{n+1} - a_0 \xi^n - a_1 \xi^{n-1} - \cdots - a_p \xi^{n-p}) - \lambda h [b_{-1} \xi^{n+1} + b_0 \xi^n + b_1 \xi^{n-1} + \cdots + b_p \xi^{n-p}] = 0$$

Now divide by ξ^{n-p}

$$(\xi^{p+1} - a_0 \xi^p - a_1 \xi^{p-1} - \cdots - a_p) - \lambda h [b_{-1} \xi^{p+1} + b_0 \xi^p + b_1 \xi^{p-1} + \cdots + b_p] = 0 \quad (2)$$

Define $\rho(\xi) = \xi^{p+1} - a_0 \xi^p - a_1 \xi^{p-1} \cdots - a_p$

$$\sigma(\xi) = b_{-1} \xi^{p+1} + b_0 \xi^p + b_1 \xi^{p-1} + \cdots + b_p$$

Then (2) can be written as

$$\boxed{\rho(\xi) - \lambda h \, \sigma(\xi) = 0} \quad (3)$$

This is called characteristic equation.

Consistency Condition

Consistency condition: A multistep method of the form ① is consistent if

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1)$$

$$\rho(1) = 0 \Rightarrow 1 = \sum_{j=0}^{p} a_j$$

$$\rho'(\xi) = (p+1)\xi^{p} - p a_0 \xi^{p-1} - \cdots - a_{p-1}$$

$$\rho'(1) = (p+1) - p a_0 - (p-1)a_1 + (p-2)a_2 \cdots \mp a_{p-1}$$

$- p a_{p-1}$

$- p a_p$

$+ p a_{p-1} + p a_p$

$$= p(1 - a_0 - a_1 - a_2 - \cdots - a_p)$$

$$+ 1 + [a_1 + 2a_2 + \cdots + (p-1)a_{p-1} + p a_p]$$

$$= 1 + \sum_{j=0}^{p} j a_j$$

$$\sigma(1) = \sum_{j=-1}^{p} b_j$$

$$\rho'(1) = \sigma(1) \Rightarrow 1 + \sum_{j=0}^{p} j a_j = \sum_{j=-1}^{p} b_j$$

or $$\boxed{1 = -\sum_{j=0}^{p} j a_j + \sum_{j=-1}^{p} b_j}$$

Consistency Conditions

Defn Root Condition : A ~~linear~~ multistep method of the form

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}) \quad \text{—} \; (*)$$

is said to satisfy the root condition if the roots of the equation $\rho(\xi) = 0$ lie inside the unit circle in the complex plane, and are simple if they lie on the unit circle.

Convergence of multistep methods:

Defn The linear multistep method (*) is said to be convergent if

$$\lim_{h \to 0} u_j = u(t_j), \qquad 0 \le j \le N$$

provided the rounding errors arising from all initial conditions tend to zero.

Theorem: The linear multistep method of the form (*) is convergent if and only if the method is consistent and satisfies the root condition.

Stability of Multistep method

Let us examine the error for special equation $y' = \lambda y$. The computed solution satisfies

$$y_{n+1} = y_{n-l} + h \sum_{m=0}^{k+1} r_m f_{n-m+1} \quad \text{--- (1)}$$

ignoring the round off errors. The true solution (exact solution) satisfies

$$y(t_{n+1}) = y(t_{n-l}) + h \sum_{m=0}^{k+1} r_m f(t_{n-m+1}, y(t_{n-m+1})) + T_{n+1}(Y) \quad \text{--- (2)}$$

where $T_{n+1}(Y)$ is local truncation error. Subtracting (2) from (1) and taking

$$\epsilon_n = y_n - y(t_n) \qquad f = \lambda y$$

$$\epsilon_{n+1} = \epsilon_{n-l} + h\lambda \sum_{m=0}^{k+1} r_m \epsilon_{n-m+1} - T_{n+1}(Y) \quad \text{--- (3)}$$

Now consider the homogeneous equation

$$\epsilon_{n+1} = \epsilon_{n-l} + h\lambda \sum_{m=0}^{k+1} r_m \epsilon_{n-m+1} \quad \text{--- (4)}$$

Now we seek solution of (4) as

$$\epsilon_n = A\xi^n$$

$$A\xi^{n+1} = A\xi^{n-l} + h\lambda A \sum_{m=0}^{k+1} r_m \xi^{n-m+1}$$

$$\xi^{n+1} = \xi^{n-l} + h\lambda [r_0 \xi^{n+1} + r_1 \xi^n + \cdots + r_{k+1}\xi^{n-k}]$$

Now divide by ξ^{n-k} we get

$$\xi^{k+1} = \xi^{k-l} + h\lambda [r_0 \xi^{k+1} + r_1 \xi^k + \cdots + r_{k+1}]$$

(x) To find the solution of non-homogeneous difference equation (3), i.e.

$$\epsilon_{n+1} = \epsilon_{n-l} + h\lambda \sum_{m=0}^{k+1} \gamma_m \epsilon_{n-m+1} - T_{n+1}$$

So we find particular solution and then general solution will be sum of particular solution and general solution of homogeneous equation.

Since T_{n+1} does not depend on ϵ_n so it is a constant

let $\epsilon_n = \beta$ is particular solution then

$$\beta = \beta + h\lambda \sum_{m=0}^{k+1} \gamma_m \beta - T_{n+1} .$$

$$\text{or} \quad h\lambda\beta \sum_{m=0}^{k+1} \gamma_m - T_{n+1} = 0$$

$$h\lambda\sigma(1)\beta = T_{n+1} \qquad \sum_{m=0}^{k+1} \gamma_m = \sigma(1)$$

$$\text{or } \beta = \frac{T_{n+1}}{h\lambda\sigma(1)} = \frac{T_{n+1}}{h\lambda\rho'(1)}$$

As $\rho'(1) = \sigma(1)$ since method is consistent.

and hence general solution of homogeneous difference equation is given by

$$\boxed{\epsilon_n = A_0\xi_0^n + A_1\xi_1^n + \cdots + A_k\xi_k^n + \frac{T_{n+1}}{h\lambda\rho'(1)}}$$

Stability of multistep method

$$\underbrace{\xi^{k+1} - \xi^{k-l}}_{} - h\lambda\left[r_0\xi^{k+1} + - + r_k\right] = 0$$

$$\boxed{\rho(\xi) - h\lambda\sigma(\xi) = 0} \leftarrow \text{Characteristic Equation} \quad \text{—— (5)}$$

Here k and l are non-negative integer such that $k-l \geq 0$ and thus (5) is an algebraic relation of degree $k+1$ so it has $(k+1)$ roots say $\xi_0, \dots \xi_k$ then general solution of (4) can be written as

$$\epsilon_n = A_0\xi_0^n + - + A_k\xi_k^n \quad \text{—— (6)}$$

If all the roots $\xi_0, \dots \xi_k$ are distinct then ϵ_n will be of the form (6). If $\xi_1 = \xi_2$ i.e. ξ_1 is double root then $A_1\xi_1^n + A_2\xi_2^n$ will be replaced by

$$(A_1 + nA_2)\xi_1^n$$

Now from (6) it is obvious that these roots will affect the growth of ϵ_n. If $|\xi| > 1$, then the error ϵ_n will grow without bound or oscillate without bound as n becomes large. A similar effect is also evident if root of (5), of multiplicity greater than one, has modulus one. We will say that method given by (1) is stable if $|\epsilon_n| < \infty$ as $n \to \infty$. (*)

Absolute Stable! $|\xi_i| \leq 1$, $i = 0(1)k$ and roots of unit magnitude are simple.

Region of Absolute Stability! The region of absolute stability is defined to be set of points in $h\lambda$ plane for which the method is absolutely stable.

Stability of multistep method

Since the method is consistent so $\rho(1)=0 \Rightarrow 1$ is always a root of $\rho(\xi)=0$

Def^n A linear k-step multistep method of the form

$$y_{n+1} = \sum_{j=0}^{k-1} a_j y_{n-j} + h \sum_{j=-1}^{k-1} b_j f(x_{n-j}, y_{n-j})$$

is said to be

(i) Stable if $|\xi_j| < 1$, for $j \neq 1$

(ii) unstable if $|\xi_j| > 1$ for some j or if there is multiple root of $\rho(\xi)=0$ of modulus unity

(iii) Conditionally stable if ξ_j are simple and if more than one of these roots have modulus unity.

Now consider the characteristic polynomial

$$\rho(\xi) - \lambda h \sigma(\xi) = 0 \quad \text{——— (1)}$$

which is a polynomial of degree k

where $\epsilon_n = A\xi^n$ ——— (2)

From (1) it is clear that for $h \to 0$ characteristic polynomial equation will reduce to $\rho(\xi)=0$ Thus $\rho(\xi)=0$ is called reduced characteristic polynomial equation.

Now let k roots of (1) are $\xi_{1h}, \xi_{2h}, \ldots \xi_{kh}$ (all the roots of (1) will depend on h)

and let $\xi_1 \ldots \xi_k$ be the roots of $\rho(\xi)=0$

Since as $h \to 0$ $\therefore$ roots of characteristic equation becomes roots of $\rho(\xi)=0$ Thus (*)

$$\xi_{ih} = \xi_i \left[1 + \bar{h} k_i + O(|\bar{h}|^2)\right] \quad \text{where } \bar{h} = \lambda h, \; i = 1(1)k$$

the coefficient k_i are called growth parameters.

Stability of multistep method.

Now from characteristic equation

$$\rho(\xi_{ih}) - \lambda h\, \sigma(\xi_{ih}) = 0$$

$$\rho\left[\xi_i(1+\bar{h}k_i + O(|\bar{h}|^2))\right] - \bar{h}\,\sigma\left[\xi_i(1+\bar{h}k_i + O(|\bar{h}|^2))\right] = 0$$

Now expanding using Taylor series we get

$$\rho(\xi_i) + \bar{h}k_i\xi_i\rho'(\xi_i) - \bar{h}\,\sigma(\xi_i) + O(|\bar{h}|^2) = 0$$

$\rho(\xi_i) = 0$ as ξ_i are roots of $\rho(\xi) = 0$ implies

$$\bar{h}k_i\xi_i\rho'(\xi_i) - \bar{h}\,\sigma(\xi_i) = 0 \quad \text{ignoring } O(|\bar{h}|^2) \text{ terms}$$

$$k_i = \frac{\sigma(\xi_i)}{\xi_i\rho'(\xi_i)} \qquad \text{—— (3)}$$

Now from (*)

$$\xi_{ih}^n = \xi_i^n\left[1+\bar{h}k_i + O(|\bar{h}|^2)\right]^n$$

$$\xi_{ih}^n \simeq \xi_i^n \exp(n\bar{h}k_i) \qquad \text{—— (4)}$$

$i = 1(1)k$

Now for ξ_{1h}. $\rho(1) = 0$ and $\sigma(1) = \rho'(1)$

Take $\xi_1 = 1$

$$k_1 = \frac{\sigma(1)}{\rho'(1)} = \frac{\rho'(1)}{\rho'(1)} = 1$$

$$\xi_{1h}^n = \xi_1^n\left[1+\bar{h} + O(|\bar{h}|^2)\right]^n \rightarrow 1+n\bar{h}+O(|\bar{h}|^2)$$

$$\boxed{\xi_{1h}^n \simeq \xi_1^n e^{n\bar{h}} + O(|\bar{h}|^2)} \longrightarrow (5)$$

We have applied multistep method on $y' = \lambda y$ which has solution $y = e^{\lambda x}$ (and $y_n = e^{\lambda x_n} = e^{\lambda x_0} \cdot e^{n\bar{h}}$) $y_n = e^{\lambda x_0} \cdot e^{n\bar{h}}$

Therefore the root ξ_{1h} approximates the solution of the differential equation $y' = \lambda y$. This is called the principal root and the remaining $(K-1)$ roots are called extraneous roots which arise because a first order differential equation is replaced by k^{th} order difference equation.

As we have seen that ϵ_n is written as

$$\epsilon_n = A_1 \xi_{1h} + \underbrace{A_2 \xi_{2h} + \cdots + A_K \xi_{Kh}}_{\text{Extraneous roots}} + \frac{T_{n+1}}{h\lambda f'(1)}$$

principal root

So behaviour of ϵ_n will depend on extraneous roots behaviour.

Def^n Absolutely Stable: A multistep method of the form

$$y_{n+1} = \sum_{j=0}^{K-1} a_j y_{n-j} + h \sum_{j=-1}^{K-1} b_j f(x_{n-j}, y_{n-j})$$

when applied to the test equation $y' = \lambda y, \lambda < 0$ is said to be absolutely stable if

$$|\xi_{ih}| \leq 1, \quad i = 1(1)k$$

(roots of unit magnitude is simple).

Def^n A A-stable linear multistep method when applied to the differential equation $y' = \lambda y$, where λ is a complex constant with negative real part is said to be A-stable if all the solutions $y_n \to 0$ as $n \to \infty$.

Stability of multistep method

$y' = \lambda y$, $y(0) = y_0$, Sol $y = y_0 e^{\lambda x}$

Ex Consider the mid point method $y_{n+1} = y_{n-1} + 2h f_n$

$$y_{n+1} = y_{n-1} + 2h\lambda y_n \qquad n \geq 1 \quad \text{——} \quad (1)$$

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Substituting $y_n = \xi^n$

$$\xi^{n+1} = \xi^{n-1} + 2h\lambda \xi^n$$

$$\xi^2 = 1 + 2h\lambda \xi$$

$$\xi^2 - 2\lambda h \xi - 1 = 0 \quad \text{——} \quad (2)$$

$$\xi = \frac{2\lambda h \pm \sqrt{4\lambda^2 h^2 + 4.1.}}{2} = \lambda h \pm \sqrt{\lambda^2 h^2 + 1}$$

$$\xi = \lambda h \pm \sqrt{1 + \lambda^2 h^2} \quad \text{——} \quad (3)$$

Let

$$g(\omega) = \omega + \sqrt{1+\omega^2}$$

from (3)

$$\xi_1 = \lambda h + \sqrt{1+\lambda^2 h^2}, \qquad \xi_2 = \lambda h - \sqrt{1+\lambda^2 h^2}$$

$$= g(\lambda h) \qquad\qquad = -\left[(-\lambda h) + \sqrt{1+(-\lambda h)^2}\right]$$

$$= -g(-\lambda h)$$

$$\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{2!} = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2} = -\frac{1}{8}$$

$$g(\omega) = \omega + (1+\omega^2)^{1/2}$$

$$= \omega + \left(1 + \frac{1}{2}\omega^2 - \frac{1}{8}\omega^4 + \cdots\right)$$

$$= \left(1 + \omega + \frac{\omega^2}{2} + \frac{\omega^3}{6} + \frac{\omega^4}{24} + \cdots\right)$$

$$- \left[\frac{\omega^3}{6} + \frac{\omega^4}{24} + \cdots\right] + \left(-\frac{1}{8}\omega^4 + \cdots\right)$$

$$= e^{\omega} + \omega^3 B(\omega)$$

where $B(\omega)$ is bounded and continuous about zero.

The solution of (1) can be written

$$y_n = c_1 \xi_1^n + c_2 \xi_2^n$$

$$\xi_1^n = [g(\lambda h)]^n = \left[e^{\lambda h} + (\lambda h)^3 B(\lambda h)\right]^n$$

$$= e^{\lambda h n}\left[1 + (\lambda h)^3 e^{-\lambda h} B(\lambda h)\right]^n$$

Stability of multistep method

Take $\bar{h} = \lambda h$

$$\xi_1^n = e^{n\bar{h}}\left[1 + O(\bar{h}^2)\right] \text{ and } \quad \text{—— (i)}$$

$$\xi_2 = -g(-\bar{h})$$

$$= -\left[e^{-\bar{h}} + (-\bar{h})^3 B(-\bar{h})\right] = -e^{-\bar{h}}\left[1 + O(\bar{h}^3)\right]$$

$$\xi_2^n = (-1)^n e^{-n\bar{h}}\left[1 + O(\bar{h}^2)\right] \quad \text{—— (ii)}$$

Thus sol^n of (1) is written as

$$y_n = C_1 \underbrace{e^{n\bar{h}}\left[1 + O(\bar{h}^2)\right]}_{\mathrm{I}} + C_2 \underbrace{(-1)^n e^{-n\bar{h}}\left[1 + O(\bar{h}^2)\right]}_{\mathrm{II}} \quad \text{—— (4)}$$

Case (i) $\lambda > 0$, $\bar{h} = \lambda h$: principal root $\xi_{1h} = e^{\bar{h}}\left[1 + O(\bar{h}^2)\right]$ and extraneous root $\xi_{2h} = -e^{-\bar{h}}\left[1 + O(\bar{h}^2)\right]$

The principal root ξ_{1h}^n behaves like exact solution and extraneous solution (II sol^n) which is given by $(-1)^n e^{-n\bar{h}}\left[1 + O(\bar{h}^2)\right]$ goes to zero as n increases.

Case (ii) $\lambda < 0$: Principal root ξ_{1h}^n behaves like exact sol^n but extraneous solution

$$(-1)^n e^{-n\bar{h}}\left[1 + O(\bar{h}^2)\right]$$

magnitude becomes large as n increases.

Thus mid point method for this case i.e, ($\bar{h} < 0$) is unstable.

<u>Stability of multistep method</u>

<u>Case (iii)</u> for $\bar{h} = 0$ characteristic equation will be the reduced characteristic equation, i.e.,

$$\rho(\xi) = 0 \qquad \rho(\xi) = \xi^2 - 1 = 0 \Rightarrow \xi = \pm 1$$

Thus the method is absolutely stable

<u>Def^n</u> <u>Weakly stable</u>: The linear multistep method of the form

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j})$$

is said to be weakly stable if there is more than one simple root of the polynomial $\rho(\xi) = 0$ on the unit circle (and strongly stable if not).

Thus mid point method ~~if~~ for $\bar{h} = 0$ is also weakly stable.

<u>Case (iv)</u> <u>λ is pure imaginary</u> Let $\lambda = ip$

$\bar{h} = \lambda h = iph$ then characteristic equation will be

Stability of multistep method

Case (iv) λ is pure imaginary let $\lambda = ip$

$\bar{h} = \lambda h = iph$

$$(\xi^2 - 1) = 2\bar{h}\xi$$

$$\left(\xi - \frac{1}{\xi}\right) = 2\bar{h}$$

$\xi = e^{i\theta}$ $\quad 2i\sin\theta = 2iph$

$$ph = \sin\theta$$

$0 < |ph| \leq 1$ will be the interval of stability

Ex Consider following Milne-Simpson method $y' = \lambda y$

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n+1} + 4y'_n + y'_{n-1}\right] \quad (*)$$

$f(x_{n+1}, y_{n+1}) + 4f(x_n, y_n) + f(x_{n-1}, y_{n-1})$

$\lambda y_{n+1} + 4\lambda y_n + \lambda y_{n-1}$

put $y_n = \xi^n$

$$\xi^{n+1} = \xi^{n-1} + \frac{h}{3}\left[\lambda \xi^{n+1} + 4\lambda\xi^n + \lambda\xi^{n-1}\right]$$

$$\xi^2 = 1 + \frac{\lambda h}{3}\left[\xi^2 + 4\xi + 1\right]$$

$$(\xi^2 - 1) - \frac{\lambda h}{3}(\xi^2 + 4\xi + 1) = 0$$

$$\rho(\xi) - \lambda h\,\sigma(\xi) = 0$$

$$\xi^2\left(1 - \frac{\lambda h}{3}\right) - \frac{4\lambda h}{3}\xi - \left(1 + \frac{\lambda h}{3}\right) = 0 \quad (1)$$

$$\xi_{ih} = \xi_i\left[1 + k_i\bar{h} + O(\bar{h}^2)\right]$$

$\rho(\xi) = (\xi^2 - 1)$, $\quad \sigma(\xi) = \frac{1}{3}(\xi^2 + 4\xi + 1)$

Stability of multistep method

$\rho(\xi) = 0 \Rightarrow \xi^2 - 1 = 0 \Rightarrow \xi = 1, -1$

$$k_i = \frac{\sigma(\xi_i)}{\xi_i \, \rho'(\xi_i)}$$

for $\xi = 1$

$$\xi_{1h} = \xi_1 [1 + k_1 \bar{h} + O(\bar{h}^2)]$$

$$k_1 = \frac{\sigma(1)}{1.\,\rho'(1)} = \frac{2}{2} = 1$$

$\sigma(1) = 2$
$\rho'(\xi) = 2\xi$
$\rho'(1) = 2$

$$\xi_{1h} = [1 + \bar{h} + O(\bar{h}^2)]$$
$$\simeq e^{\bar{h}} + O(\bar{h}^2) = e^{\bar{h}}[1 + O(\bar{h}^2)]$$

for $\xi = -1$

$\sigma(\xi) = \frac{1}{3}(\xi^2 + 4\xi + 1)$
$\sigma(-1) = \frac{1}{3}(1 - 4 + 1)$
$= -2/3$
$\rho'(\xi_2) = -2$

$$\xi_{2h} = \xi_2 [1 + k_2 \bar{h} + O(\bar{h}^2)]$$

$$k_2 = \frac{\sigma(\xi_2)}{\xi_2 \, \rho'(\xi_2)}$$

$$k_2 = \frac{-2}{3 \cdot (-1) \cdot (-2)} = -\frac{1}{3}$$

$$\xi_{2h} = -[1 - \tfrac{1}{3}\bar{h} + O(\bar{h}^2)]$$
$$\simeq -e^{-\bar{h}/3} + O(\bar{h}^2) = -e^{-\bar{h}/3}[1 + O(\bar{h}^2)]$$

Thus soln of (*) will be

$$y_n = \underbrace{C_1 e^{n\bar{h}}}_{\mathrm{I}} + \underbrace{C_2 (-1)^n e^{-n\bar{h}/3}}_{\mathrm{II}} + O(\bar{h}^2)$$

Case (i) $\lambda > 0$ $\bar{h} = \lambda h$, principal solution will be close to exact soln and extraneous soln (II soln) will die for $n \uparrow \infty$ Thus it is stable case

$$ph = \frac{3\sin\theta}{2+\cos\theta}$$

$$|ph| = \left|\frac{3\sin\theta}{2+\cos\theta}\right|$$

$$\leq \frac{3}{1} = 3$$

$$-1 \leq \cos\theta \leq 1$$

$$1 \leq 2+\cos\theta \leq 3$$

$$\frac{1}{3} \leq \frac{1}{2+\cos\theta} \leq 1$$

$$0 < |ph| \leq 3$$

Interval of absolute stability

Order	Adams-Bashforth	Adams-Moulton
1	$(-2, 0)$	$(-\infty, 0)$
2	$(-1, 0)$	$(-\infty, 0)$
3	$(-0.5, 0)$	$(-6, 0)$
4	$(-0.3, 0)$	$(-3, 0)$
5	$(-0.2, 0)$	$(-1.9, 0)$

Stability of multistep method

Case (ii) $\lambda < 0$, Principal soln will approach to exact soln while Extraneous soln (IInd soln) will increase as $n \uparrow \infty$ Thus in this case method is unstable.

Case (iii) $\bar{h} = 0$ Characteristic equation will be reduced characteristic eq. $\rho(\xi) = 0$

$$\Rightarrow \xi^2 - 1 = 0 \Rightarrow \xi = \pm 1$$

Thus method is absolutely stable.

Case (iv) λ is purely imaginary $\lambda = ip$, p is real.

Characteristic eq

$$\xi^2 - 1 - \frac{\lambda h}{3}(\xi^2 + 4\xi + 1) = 0$$

$$(\xi^2 - 1) = \frac{\lambda h}{3}(\xi^2 + 4\xi + 1)$$

$$(\xi^2 - 1) = \frac{iph}{3}(\xi^2 + 4\xi + 1)$$

$$\text{or } ph = \frac{3(\xi^2 - 1)}{i(\xi^2 + 4\xi + 1)}$$

$$ph = \frac{3(\xi - 1/\xi)}{i\left(\xi + 4 + \frac{1}{\xi}\right)}$$

put $\xi = e^{i\theta}$

$$ph = \frac{3 \cancel{6} i \sin\theta}{\cancel{2} i(\cos\theta + 2)} \times \frac{i}{i} = \frac{3\sin\theta}{(2 + \cos\theta)}$$

$$0 < |ph| \le \frac{3}{1}$$

$$\boxed{0 < |ph| \le 3}$$

$$\xi - \frac{1}{\xi} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} + 4 = 2\cos\theta + 4$$

$$-1 \le \sin\theta \le 1$$

$$-1 \le \cos\theta \le 1$$

$$1 \le 2 + \cos\theta \le 3$$

$$\frac{1}{3} \le \frac{1}{2 + \cos\theta} \le 1$$

Stability of MSM

Relative Stability A linear multistep method of the form

$$y_{n+1} = \sum_{j=0}^{K-1} a_j y_{n-j} + h \sum_{j=-1}^{K-1} b_j f(x_{n-j}, y_{n-j})$$

is said to be relatively stable if

$$|\xi_{ih}| \le |\xi_{1h}| \quad i = 2, 3, \ldots k$$

The region of relatively stability is defined to be the set of points in the λh-plane for which the method is relatively stable.

Eg Consider 2nd order Adams-Bashforth method

$$y_{n+1} = y_n + \frac{h}{2}(3y_n' - y_{n-1}')$$

for $y' = \lambda y$

$$y_{n+1} = y_n + \frac{h\lambda}{2}(3y_n - y_{n-1})$$

put $y_n = \xi^n$

$$\xi^{n+1} = \xi^n + \frac{h\lambda}{2}(3\xi^n - \xi^{n-1})$$

divide by ξ^{n-1}

$$\xi^2 = \xi + \frac{h\lambda}{2}(3\xi - 1)$$

$$(\xi^2 - \xi) - \frac{h\lambda}{2}(3\xi - 1) = 0 \qquad \bar{h} = \lambda h$$

$$\xi^2 - \left(1 + \frac{3}{2}\bar{h}\right)\xi + \frac{\bar{h}}{2} = 0 \qquad \text{Characteristic eq.}$$

$$\xi = \frac{\left(1 + \frac{3}{2}\bar{h}\right) \pm \sqrt{\left(1 + \frac{3}{2}\bar{h}\right)^2 - 4 \cdot \frac{\bar{h}}{2}}}{2}$$

$$= \frac{\frac{1}{2}(2 + 3\bar{h}) \pm \sqrt{\frac{(2 + 3\bar{h})^2}{4} - \frac{4\bar{h}}{2}}}{2}$$

$$= \frac{(2 + 3\bar{h}) \pm \sqrt{(2 + 3\bar{h})^2 - 8\bar{h}}}{4}$$

Stability MSM.

$n = \frac{1}{2}$

$\frac{n-1}{} = t - \frac{1}{2}$

$$\xi = \frac{1}{4}\left[(2+3\bar{h}) \pm \sqrt{4+9\bar{h}^2+12\bar{h}-8\bar{h}}\right]$$

$$= \frac{1}{4}\left[(2+3\bar{h}) \pm \sqrt{9\bar{h}^2+4\bar{h}+4}\right]$$

$$\xi_{1h} = \frac{1}{4}\left[2+3\bar{h}+(4+4\bar{h}+9\bar{h}^2)^{1/2}\right]$$

$$= \frac{1}{2}+\frac{3}{4}\bar{h}+\frac{2}{4}\left(1+\bar{h}+\frac{9}{4}\bar{h}^2\right)^{1/2}$$

$$= \frac{1}{2}+\frac{3}{4}\bar{h}+\frac{1}{2}\left(1+\bar{h}+\frac{9}{4}\bar{h}^2\right)^{1/2}$$

$$= \frac{1}{2}+\frac{3}{4}\bar{h}+\frac{1}{2}\left[1+\frac{1}{2}\left(\bar{h}+\frac{9}{4}\bar{h}^2\right)-\frac{1}{4\cdot 2}\left(\bar{h}+\frac{9}{4}\bar{h}^2\right)^2 + - \right]$$

$$= 1+\bar{h}+\frac{9}{16}\bar{h}^2 - \frac{1}{16}\left(\bar{h}^2+\frac{81}{16}\bar{h}^4+\frac{18}{4}\bar{h}^3\right)+-$$

$$= 1+\bar{h}+\frac{\bar{h}^2}{2}+O(\bar{h}^3) = e^{\bar{h}}+O(h^3)$$

Similarly $2\left(1+\bar{h}+\frac{9}{4}\bar{h}^2\right)^{1/2} = 2\left[\left(1+\bar{h}+\frac{9}{4}\bar{h}^2\right)-\frac{1}{8}\left(\bar{h}+\frac{9}{4}\bar{h}\right)^2 + \frac{3}{48}\left(\bar{h}+\frac{9}{4}\bar{h}\right)^3+\cdots\right.$

$$\xi_{2h} = \frac{1}{4}\left[(2+3\bar{h}) - (4+4\bar{h}+9\bar{h}^2)^{1/2}\right]$$

$$\simeq \frac{\bar{h}}{2}-\frac{\bar{h}^2}{2}+\frac{\bar{h}^3}{4}+O(\bar{h}^4)$$

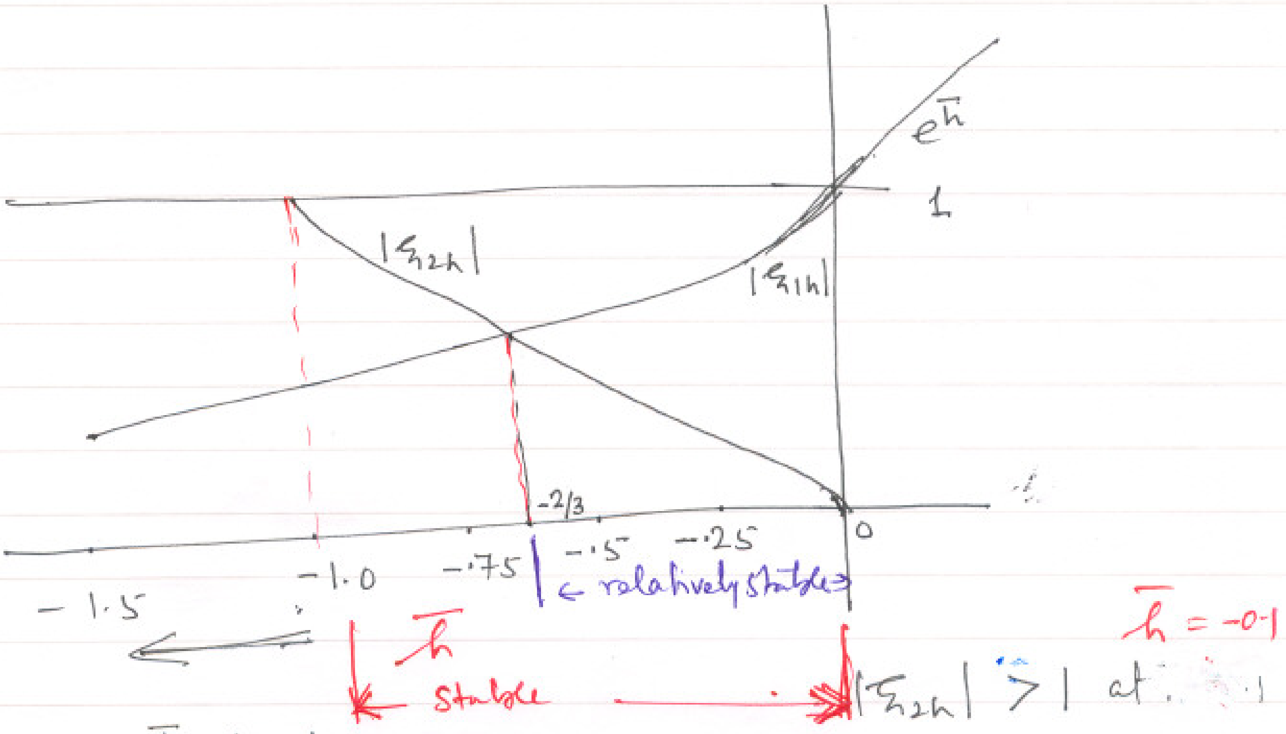
$$= \frac{\bar{h}}{2}\left(1-\bar{h}+\frac{\bar{h}^2}{2}\right)+O(\bar{h}^4)$$

$$\simeq \frac{\bar{h}}{2}e^{-\bar{h}}+O(\bar{h}^4)$$

Thus soln

$$y_n = C_1 \xi_{1h}^n + C_2 \xi_{2h}^n$$

The roots $|\xi_{1h}|$ & $|\xi_{2h}|$ are plotted below



1. Unstable $\bar h \leq -1.0$
2. Absolutely stable $-1.0 < \bar h < 0$

$\xi_{2h} = -1.359$ at $\bar h = -0.1$

$|\xi_{2h}| = 1.359$

3. $|\xi_{2h}| < |\xi_{1h}|$ for $-2/3 < \bar h < 0$

So method is relatively stable for $-2/3 < \bar h < 0$