Lecture-16

Marine Hydrodynamics

1. Wave energy :

1.1 Now, Kinetic Energy is defined by $K.E = \frac{1}{2} m v^2$

where v = velocity of the particle
m = mass of the particle.

in case of water / fluid $\vec{V} = u\hat{i} + w\hat{k}$

[neglect 'y' direction]

then $v^2 = u^2 + w^2$.

and $$m = \int_{-h}^{0}\int_{0}^{\lambda} \rho \, dx\, dz \quad \cdots (1.1)$$

Equation (1.1) is written in terms of linear wave theory, where it is assumed that slope is very small and upper limit of 'z' can be approximated by $z=0$ instead of $z = 0$.

$$\therefore K.E = \frac{1}{2}\int_{-h}^{0}\int_{0}^{\lambda} \rho\,(u^2 + w^2)\,dx\,dz. \quad \cdots (1.2)$$

Now, from lecture note 15 we know that

$$u^2 + w^2 = a^2\omega^2 e^{2kz} \quad \cdots\cdots (1.3)$$

please note that, in (1.2) the left hand side 'w' is the velocity component in 'z' direction. However, right hand side 'ω' is basically angular frequency and should be read as "omega". [just to avoid confusion].

Substituting (1.3) in (1.2) we get

$$K.E = \frac{1}{2}\int_{-h}^{0}\int_{0}^{\lambda} a^2\omega^2 e^{2kz}\,dx\,dz \qquad \ldots (1.4)$$

Also it may be noted that equation (1.3) is obtained under deep water approximation where we take $\phi = \frac{ag}{\omega} e^{kz}\sin(kx - \omega t)$.

The general expression of (1.4), however, is bit complex, but readers can derive that expression.

Now from (1.4) we get

$$K.E = \frac{1}{2}\rho\int_{-h}^{0}\int_{0}^{\lambda} a^2\omega^2 e^{2kz}\,dx\,dz$$

$$= \frac{\rho}{2}a^2\omega^2\lambda\int_{-h}^{0} e^{2kz}\,dz$$

$$= \frac{\rho}{2}a^2\omega^2\lambda\left[\frac{e^{2kz}}{2k}\right]_{-h}^{0}$$

$$= \frac{\rho}{2}a^2\omega^2\lambda\left[\frac{1}{2k} - \frac{1}{2k\,e^{2kh}}\right]$$

Now in case of deep water $h \rightarrow \infty$ and,

therefore $\frac{1}{2ke^{2kh}} \rightarrow 0$, and finally we get

$$K.E = \frac{\rho}{4} \rho \frac{a^2 \omega^2}{k} \lambda$$

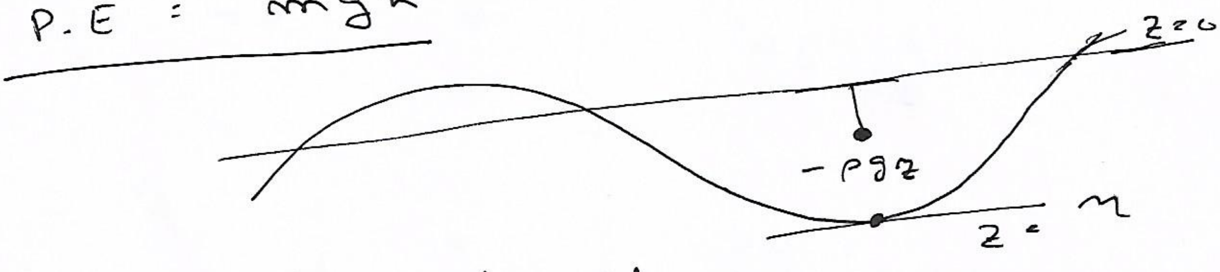
or $\frac{K.E}{\lambda} = \frac{\rho}{4} a^2 \frac{\omega^2}{k}$ [using $\omega^2 = gk$, we get $\frac{\omega^2}{k} = g$]

or $$\boxed{\frac{\overline{K.E.}}{\lambda} = \cancel{\frac{\rho}{2}}\, \frac{1}{4} \rho g a^2} \quad \ldots\ldots (1.5)$$

Equation (1.5) tell that energy / unit wave length $= \frac{1}{4}\rho g a^2$, this definition can be easily extended for 3D as

$\frac{1}{4}\rho g a^2$ = kinetatic energy of unit surface area ~~of sea area~~ / horizontal sea surface area.

1.2, Now, potential energy (P.E) of a ~~system~~ particle of mass 'm' can be defined as $P.E = mgh$



taking 'z' positive upward, in case of wave, the expression would be

$$P.E = -\int_0^{\lambda}\int_{\eta}^{0} \rho g z \, dx\, dz$$

or $$P.E = -\int_0^{\lambda}\int_{\eta}^{0} \rho g z \, dx \, dz$$

$$= -\int_0^{\lambda} \rho g \left[\frac{1}{2}z^2\right]_{\eta}^{0} dx$$

or $$P.E = +\frac{1}{2}\rho g \int_0^{\lambda} \eta^2 \, dx$$

taking $\eta = a\cos(kx - \omega t)$, we get

$$P.E = +\frac{1}{2}\rho g \int_0^{\lambda} a^2 \cos^2(kx - \omega t)\, dx$$

or $$P.E = \frac{1}{2}\rho g a^2 . \frac{\lambda}{2} \left[\because \int_0^{\lambda} \cos^2(kx - \omega t)\, dx = \frac{\lambda}{2}\right]$$

or $$P.E = \frac{1}{4}\rho g a^2 . \lambda$$

or $$\boxed{\frac{P.E}{\lambda} = \frac{1}{4}\rho g a^2} \quad \ldots\ldots (1.6)$$

Similarly P.E / unit wave length = $\frac{1}{4}\rho g a^2$

or, in 3D, P.E / unit surface area = $\frac{1}{4}\rho g a^2$

Adding (1.5) and (1.6) we get ~~Kinetic~~ total wave energy / unit surface area and that is given as

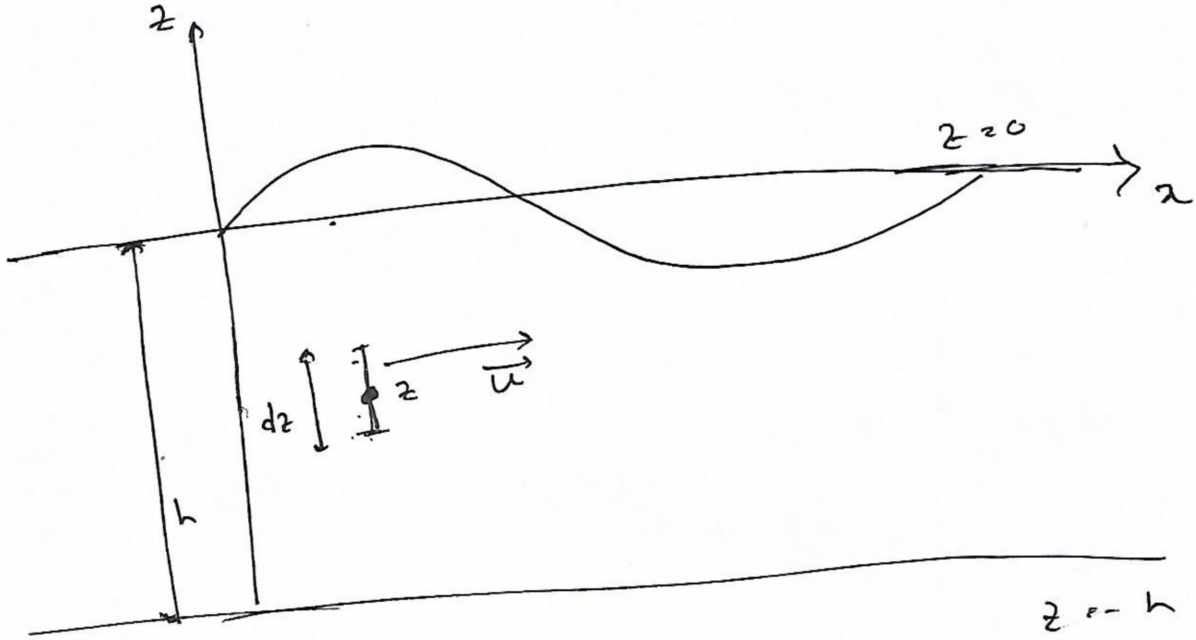
$$T.E = \frac{1}{4}\rho g a^2 + \frac{1}{4}\rho g a^2 = \frac{1}{2}\rho g a^2$$

∴ total wave energy / unit surface area

so $$\boxed{T.E = \frac{1}{2}\rho g a^2}$$ ... (1.7)

or $$\boxed{T.E = \frac{1}{8}\rho g H^2}$$ $\left[\because a = \frac{1}{2}H\right]$

2. Energ transport / power



we know the definition of work done $\vec{W}$ would be

$$\vec{W} = \text{force} \times \text{displacement}.$$

Now at ~~a depth~~ an arbitrary depth z, assume that the dynamic pressure = p. and horizontal velocity of fluid particle is u. then, the force acting on a small segment dz in horizontal direction is

$dF = p.1.dz.$ and displacement

$dx = u.dt.$

$\therefore$ work done $= dF \cdot dx = p \cdot u \cdot dz \cdot dx$

$\therefore$ under the assumption of linearity, the ~~total work done~~ ~~avg~~ average work done

$$\overline{W} = \frac{1}{T}\int_0^T \int_{-h}^0 p \cdot u \cdot dz \cdot dt$$

Now $\phi = \frac{ag}{\omega} e^{kz} \sin(kx - \omega t)$

$$\therefore \frac{\partial \phi}{\partial t} = -ag\, e^{kz} \cos(kx - \omega t)$$

$$\therefore p = -\rho \frac{\partial \phi}{\partial t} = a\rho g e^{kz} \cos(kx - \omega t) \quad \ldots (2.1)$$

Similarly $u = \frac{\partial \phi}{\partial x} = a\omega\, e^{kz} \cos(kx - \omega t) \quad \ldots (2.2)$

$$\therefore \overline{W} = \frac{1}{T}\int_0^T \int_{-h}^0 \rho a^2 g \omega\, e^{2kz} \cos^2(kx - \omega t)\, dz\, dt$$

$$= \frac{1}{T}\int_{-h}^0 \rho a^2 g \omega e^{2kz} \left[\int_0^T \cos^2(kx - \omega t)\, dt \right] dz$$

$$= \frac{1}{2}\int_{-h}^0 \rho a^2 g \omega\, e^{2kz}\, dz$$

$$= \frac{1}{2} \rho a^2 g \omega \left[\frac{e^{2kz}}{2k} \right]_{-h}^0$$

$$= \frac{\rho}{2} a^2 g \cdot \frac{\omega}{2k} \quad \left[\begin{array}{l} \text{as } e^{-2kh} \to 0 \\ \text{as } h \to \infty \end{array} \right]$$

or $\overline{W} = \frac{1}{4} \rho a^2 g . \left(\frac{\omega}{k}\right)$ .... (2·1)

but we know the phase velocity

$$C = \frac{\lambda}{T} = \frac{(2\pi/k)}{(2\pi/\omega)} = \frac{\omega}{k} \quad \cdots (2\cdot2)$$

Substituting (2·2) into (2·1) we get

$$\overline{W} = \frac{1}{4} \rho g a^2 c \quad \cdots\cdots (2\cdot3)$$

Now we know $E = \frac{1}{2} \rho g a^2$ ... (2·4)

substituting (2·4) in (2·3) we get

$$\boxed{\overline{W} = \frac{1}{2} . E . C} \quad \cdots\cdots (2\cdot5)$$

Alternative definition of work done

we also know that $W = E . C_g$ where

E = energy and C_g is the velocity of the energy. Now at this moment, let as call at group velocity and denoted as cg.

then $\boxed{\overline{W} = E . C_g}$ ... (2·6) may be the alternative definition of work done. Now equating (2·6) and (2·5) we get

$$\boxed{C_g = \frac{1}{2} C} \quad \cdots\cdots (2\cdot7)$$

(2·7) is very important relationship. Here we get to know that under deep water condition, the velocity at which the energy travels is equals to the half of the phase velocity.

3. Relation between 'c_g' & 'c' in general.

~~if you one CB~~

take $\phi = \frac{ag}{\omega} \frac{\cosh(kh + kz)}{\cosh(kh)} \sin(kx - \omega$

and get the expression for p, u, and and then find the work done $\overline{W}$ from

$$\boxed{\overline{W} = \frac{1}{T} \int_0^T \int_{-h}^0 p \cdot u \, dz \cdot dt} \quad \ldots (3.1)$$

we get the general relationship

$$\boxed{c_g = \frac{c}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)} \quad \ldots (3.2)$$

from (3·2) taking $h \to \infty$ we get $c_g = \frac{1}{2} c$.

from (3·2) taking $h \to 0$

$$c_g = \frac{c}{2}\left[1 + \frac{2kh}{2kh}\right] \quad \left[\begin{array}{l} \text{at } h \to 0 \\ \sinh x = x \end{array}\right.$$

$$\text{or} \quad \boxed{c_g = c} \quad \ldots (3.3)$$

so for deep water case $c_g = \frac{1}{2} c$ → (3·4)

for shallow water case $c_g = c$

Find the expression of (3·2)