Non-linear differential equation $u'' = f(x, u, u')$

$$u_{j-1} - 2u_j + u_{j+1} = h^2 f(x_j, u_j, u_j') \quad \text{—} (1)$$

Now here we require approximation for u_j', we may take one of them

$$u_j' = \begin{cases} \dfrac{u_{j+1} - u_{j-1}}{2h} + O(h^2) & \text{Central} \\ \dfrac{u_{j+1} - u_j}{h} + O(h) & \text{forward} \\ \dfrac{u_j - u_{j-1}}{h} + O(h) & \text{Backward.} \end{cases}$$

The method will be finally $O(h^2)$ accurate if central one is used otherwise $O(h)$ accurate

Ex
$$u'' = u' + 1$$
$$u(0) = 1, \quad u(1) = 2(e-1)$$

$$u_{j-1} - 2u_j + u_{j+1} = h^2 f(x_j, u_j, u_j')$$

or
$$u_{j-1} - 2u_j + u_{j+1} = h^2 (u_j' + 1)$$
$$= h^2 \left(\frac{u_{j+1} - u_{j-1}}{2h} + 1 \right)$$
$$= \frac{h}{2} (u_{j+1} - u_{j-1}) + h^2$$

or
$$\boxed{\left(1 + \frac{h}{2}\right) u_{j-1} - 2u_j + \left(1 - \frac{h}{2}\right) u_{j-1} = h^2}$$

Note that the coefficient matrix is not a tridiagonal matrix.

Take $h = 1/3$

$x_1 \quad x_2 \quad x_3$

$0 \quad 1/3 \quad 2/3 \quad 1 = 3/3$

x_0

$j = 1$

$$(7/6)\,u_0 - 2u_1 + 5/6\,u_2 = 1/9$$
$$7/6\,u_1 - 2u_2 + (5/6)\,u_3 = 1/9$$

$u_0 = 1, \; u_3 = 2(e-1)$

So we get

$$-36u_1 + 15u_2 = -19$$
$$21u_1 - 36u_2 = 32 - 30e = -49.54045$$