Stability by fourier series method (von Neumann's method)

In fourier series $\sum a_n \cos(n\pi x/l)$ & $\sum b_n \sin(n\pi x/l)$ can be written in exponential form as

$$\sum A_n e^{in\pi x/l} \qquad x \le l$$

Now we write u_{ij} by $u_{p,q} = u(x_p, t_q)$
$= u(ph, qk)$

and l is last value of x
so $l = Nh$

and $A_n e^{in\pi x/l} = A_n e^{in\pi ph/Nh} = A_n e^{i\frac{n\pi}{Nh}ph}$

take $n\pi/Nh = \beta_n$ then

$$A_n e^{in\pi x/l} = A_n e^{i\beta_n ph}$$

Now for two variable $u(ph, qk)$ we write

$u(ph, qk) = e^{i\beta_n ph} \cdot e^{\alpha qk}$ where α in general a complex no.

leaving constant term

Take

$\xi = e^{\alpha k}$ then

$$\boxed{u(ph, qk) = e^{i\beta_n ph} \cdot \xi^{q}}$$

Now we need to investigate the propagation of this term as t increases, i.e. how ξ^q behaves.

A necessary and sufficient condition for stability is $\boxed{|\xi| \le 1}$

Eg Investigate the stability of the scheme

$$\frac{1}{k}(u_{p,q+1} - u_{p,q}) = \frac{1}{h^2}(u_{p-1,q+1} - 2u_{p,q+1} + u_{p+1,q+1}) \quad \text{—(1)}$$

approximating $u_t = u_{xx}$ at (ph, qk)

Substitute $u_{p,q} = e^{i\beta ph}\xi^q$ into the difference scheme (1)

from (1) $(r = k/h^2)$

$$u_{p,q+1} - u_{p,q} = r(u_{p-1,q+1} - 2u_{p,q+1} + u_{p+1,q+1})$$

$$e^{i\beta ph}\xi^{q+1} - e^{i\beta ph}\xi^q = r(e^{i\beta(p-1)h}\xi^{q+1} - 2e^{i\beta ph}\xi^{q+1} + e^{i\beta(p+1)h}\xi^{q+1})$$

Divide by $e^{i\beta ph}\xi^q$

$$\xi - 1 = r\xi(e^{-i\beta h} - 2 + e^{i\beta h})$$

$$= r\xi(2\cos\beta h - 2)$$

$$= 2r\xi(\cos\beta h - 1)$$

$$= -4r\xi\sin^2(\beta h/2)$$

$$\xi[1 + 4r\sin^2(\beta h/2)] = 1$$

$$\xi = \frac{1}{1 + 4r\sin^2(\beta h/2)}$$

for $r > 0$, $0 < \xi \le 1$ for all β. Therefore the equation is unconditionally stable.

Ex Investigate the stability of explicit scheme for wave eq

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \frac{(u_{p+1,q} - 2u_{p,q} + u_{p-1,q})}{h^2}$$

or $r = k/h$

$$u_{p,q+1} - 2u_{p,q} + u_{p,q-1} = r^2(u_{p+1,q} - 2u_{p,q} + u_{p-1,q})$$

put $u_{p,q} = e^{i\beta p h}\xi^q$

$$e^{i\beta p h}\xi^{q+1} - 2e^{i\beta p h}\xi^q + e^{i\beta p h}\xi^{q-1} =$$
$$= r^2[e^{i\beta(p+1)h}\xi^q - 2e^{i\beta p h}\xi^q + e^{i\beta(p-1)h}\xi^q]$$

Divide by $e^{i\beta p h}\xi^{q-1}$

$$\xi^2 - 2\xi + 1 = r^2\xi[e^{i\beta h} - 2 + e^{-i\beta h}]$$

$$\xi^2 - 2\xi + 1 = 2r^2\xi[\cos\beta h - 1] = -4r^2\xi\sin^2(\beta h/2)$$

$$\xi^2 - 2\xi[1 - 2r^2\sin^2(\beta h/2)] + 1 = 0$$

Take $A = 1 - 2r^2\sin^2(\beta h/2)$

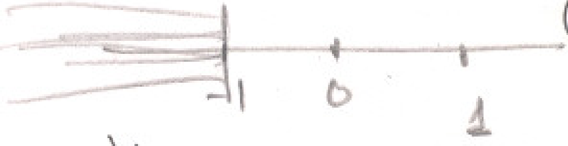
$$\xi^2 - 2A\xi + 1 = 0$$

$$\xi = \frac{2A \pm \sqrt{4A^2 - 4}}{2} = A \pm \sqrt{A^2 - 1} = A \pm (A^2-1)^{1/2}$$

$$\xi_1 = A + (A^2-1)^{1/2}, \qquad \xi_2 = A - (A^2-1)^{1/2}$$

from $A = 1 - 2r^2\sin^2(\beta h/2)$

As β, h, r are real $A \le 1$



Case (i) if $A < -1$

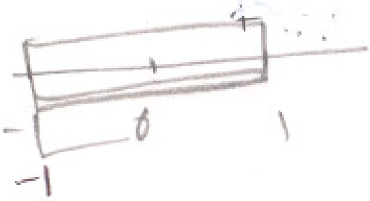
Let $A = -p, \quad p > 1$

$$\xi_2 = A - (A^2-1)^{1/2} = -p - (p^2-1)^{1/2}$$

$$|\xi_2| = p + (p^2-1)^{1/2} > 1 \quad \text{as } p > 1 \text{ and } p^2 - 1 > 0$$

Thus this leads to instability.

Case (ii) $-1 \leq A \leq 1$ so $A^2 \leq 1 \Rightarrow (A^2-1) \leq 0 \qquad 1 - A^2 \geq 0$



$$\xi_1 = A + (A^2-1)^{1/2}, \quad \xi_2 = A - (A^2-1)^{1/2}$$

$$= A + [i^2(1-A^2)]^{1/2} \qquad \xi_2 = A - [i^2(1-A^2)]^{1/2}$$

$$\xi_1 = A + i(1-A^2)^{1/2} \qquad \xi_2 = A - i(1-A^2)^{1/2}$$

$$|\xi_1| = |\xi_2| = \left[A^2 + (1-A^2)\right]^{1/2} = 1$$

Thus this leads to stability.

Hence the scheme is stable for $-1 \leq A \leq 1$

or

$$-1 \leq 1 - 2r^2 \sin^2(\beta h/2) \leq 1$$

$$1 - 2r^2 \sin^2 \beta h/2 \leq 1 \Rightarrow -r^2 \sin^2(\beta h/2) \leq 0$$

$$\Rightarrow r^2 \sin^2(\beta h/2) \geq 0$$ which does not provide any extra condition as it is always true

for

$$-1 \leq 1 - 2r^2 \sin^2(\beta h/2)$$

$$\Rightarrow 2r^2 \sin^2(\beta h/2) \leq 2$$

or $$r^2 \sin^2(\beta h/2) \leq 1$$

which gives $\underline{\underline{r \leq 1}}$ $\left(r = k/h \text{ mesh ratio}\right)$

which is necessary and sufficient condition for stability of explicit scheme.

Implicit Scheme for Wave Eq.

$$u_{tt} = u_{xx} \qquad \frac{1}{k^2}\left(u_i^{j+1} - 2u_i^j + u_i^{j-1}\right) = \frac{1}{h^2}\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right)$$

$$\frac{1}{k^2}\delta_t^2 u_i^j = \frac{1}{h^2}\delta_x^2 u_i^j \qquad \text{—— (1)}$$

Now if we replace u_i^j on r.h.s of (1) by weighted sum

$$\theta u_i^{j+1} + (1-2\theta) u_i^j + \theta u_i^{j-1} \qquad 0 \leq \theta \leq 1$$

we get the modified scheme as

$$\frac{1}{k^2}\delta_t^2 u_i^j = \frac{1}{h^2}\delta_x^2\left(\theta u_i^{j+1} + (1-2\theta) u_i^j + \theta u_i^{j-1}\right)$$

for $\theta = 1/2$ we get implicit scheme which is as follows.