Complex roots

Complex roots: For a polynomial with real coefficients the complex roots occur as conjugate pair.

Let $\xi_1 = \alpha + i\beta = re^{i\theta}$ and $\xi_2 = \alpha - i\beta = e^{-i\theta}$, where $r = \sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$ be the complex roots of homogeneous equation

$$\epsilon_{n+1} = \epsilon_{n-l} + h\lambda \sum_{m=0}^{K+1} \gamma_m \epsilon_{n-m+1}$$ (referred to eq (4) on page 1)

Corresponding to

$$y_{n+1} = y_{n-l} + h \sum_{m=0}^{K+1} \gamma_m f_{n-m+1}$$ (refered as eq (1) on page 1)

is given by

$$\epsilon_n = (A_0 \cos n\theta + A_1 \sin n\theta)|\xi_1|^n + A_3 \xi_3^n + \cdots + A_k \xi_k^n$$

[instead of

$$\epsilon_n = A_0 \xi_0^n + A_1 \xi_1^n + \cdots + A_k \xi_k^n$$
for real roots only]

Now the problem is, that given nth degree polynomial

$$Q(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

determine whether or not all solutions to $Q(z) = 0$ lie inside the unit circle $\{z : |z| < 1\}$. Here we use trick, or, bilinear

Transformation (Möbius transformation, or Conformal map)

$$z = f(\xi) = \frac{\xi - 1}{\xi + 1} \quad \text{——— (1)}$$

This map has the following properties:

$$f: \begin{array}{l} \{|\xi| = 1\} \to \{\mathrm{Re}(z) = 0\} \\ \{|\xi| < 1\} \to \{\mathrm{Re}(z) < 0\} \\ \{|\xi| > 1\} \to \{\mathrm{Re}(z) > 0\} \end{array}$$

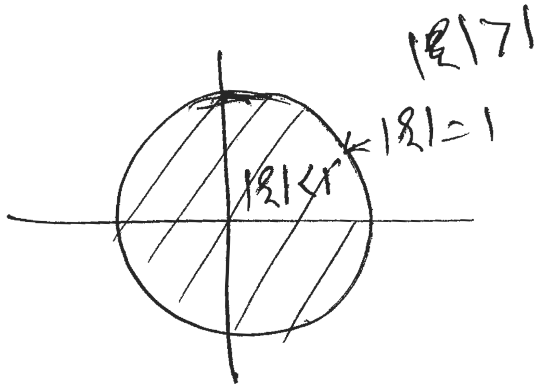
as well as a nice inverse

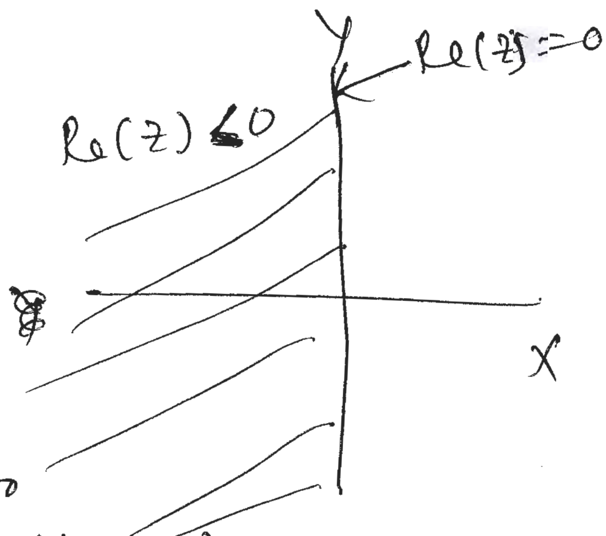
$$\xi = \frac{1 + z}{1 - z} \quad \text{—— (2)}$$

<u>Note</u>

From (1) at $\xi = 1$ $\qquad z = 0$ and

$\xi = -1$ $\qquad z \to -\infty$





Boundary $|\xi| = 1$ goes to

Hurwitz Criterion

Let

$$p(z) = a_0 z^k + a_1 z^{k-1} + \cdots + a_k$$

and

$$D = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & a_0 & a_2 & \cdots & a_{2k-4} \\ \vdots & & & & \\ 0 & \cdots & 0 & & a_k \end{vmatrix}_{k \times k}$$

where $a_j \geq 0$ for $j = 1, 2, \cdots k$ and $a_j = 0$ for $j > k$. Then, the real parts of $p(z)$ are negative if and only if the leading principal minors of D are positive.

Example(s) Let $p(z) = a_0 z^6 + a_1 z^5 + a_2 z^4 + a_4 z^2 + a_5 z + a_6$

then Hurwitz matrix D is given by

$$D = \begin{vmatrix} a_1 & a_3 & a_5 & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & a_6 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 & 0 \\ 0 & 0 & a_0 & a_2 & a_4 & a_6 \end{vmatrix}_{6 \times 6}$$

Example (2) let $p(z) = a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5$ (4)

then the Hurwitz matrix D is given by

$$D = \begin{vmatrix} a_1 & a_3 & a_5 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & 0 \\ 0 & a_0 & a_2 & a_4 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 \end{vmatrix}_{5\times 5}$$

then leading principal minors of order 1, 2, 3, 4, 5 are

$\Delta_1 = a_1$ $\quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

$\Delta_5 = D$