$$= \frac{\partial}{\partial t}\{\rho\,(1-\cos\theta)\,\delta r.\, r\,\delta\theta.\, r\sin\theta\,\delta\phi\} \text{ per unit time} \quad \ldots (2)$$

From the equation of continuity, we have

$$\frac{\partial}{\partial t}\{\rho\,(1-\cos\theta)\,\delta r.\, r\,\delta\theta.\, r\sin\theta\,\delta\phi\}$$

$$= -\frac{\partial}{r\,\partial\theta}\{\rho u\, r\sin\theta\,\delta\phi\,(1-\cos\theta)\,\delta r\}\, r\,\delta\theta$$

$$-\frac{\partial}{r\sin\theta\,\delta\phi}\{\rho v\, r\,\delta\theta\,\delta r\,(1-\cos\theta)\}\, r\sin\theta\,\delta\phi$$

or $$\frac{\partial\rho}{\partial t}\{(1-\cos\theta)\,\delta r\, r\,\delta\theta\, r\sin\theta\,\delta\phi\}$$

$$+\frac{\partial}{r\,\partial\theta}\{\rho u\sin\theta\,(1-\cos\theta)\, r^2\,\delta r\,\delta\theta\,\delta\phi$$

$$+\frac{\partial}{r\sin\theta\,\partial\phi}\{\rho v\}\, r^2\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi = 0$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r\sin\theta\,(1-\cos\theta)}\frac{\partial}{\partial\theta}\{\rho u\sin\theta\,(1-\cos\theta)\}$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho v) = 0$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r\sin\theta\,(1-\cos\theta)}\left[\sin\theta\,(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u)\right.$$

$$\left.+\rho u\,\{\cos\theta\,(1-\cos\theta)+\sin^2\theta\}\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho v) = 0$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho u)+\frac{\rho u\,(1+\cos\theta-2\cos^2\theta)}{r\sin\theta\,(1-\cos\theta)}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho v) = 0$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho u)+\frac{\rho u\,(1+2\cos\theta)}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho v) = 0$$

or $$r\sin\theta\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\phi}(\rho v)+\sin\theta\frac{\partial}{\partial\theta}(\rho u)+\rho u\,(1+2\cos\theta) = 0.$$

Proved.

Exercises

1. The velocities at a point in a fluid in the Eulerian system are given by $u = x+y+z+t$, $v = 2(x+y+z)+t$, $w = 3(x+y+z)+t$. Find the displacements of a fluid particle in the Lagrangian system. Also, determine the velocity of fluid particle at (x_0, y_0, z_0) initially.
2. If $u = yz+t$, $v = xz-t$ and $w = xy$, determine the acceleration at the point (2, 1, 3) at $t = 0\cdot5$ sec.

Hint :

Since $\mathbf{q} = (yz + t)\,\mathbf{i} + (xz - t)\,\mathbf{j} + xy\,\mathbf{k},$

let $\mathbf{f}$ *be the acceleration of a fluid particle, then*

$$\mathbf{f} = \frac{\partial \mathbf{q}}{\partial t} + u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z}$$

$$\mathbf{f} = (\mathbf{i} - \mathbf{j}) + (yz + t)(z\mathbf{j} + y\mathbf{k}) + (xz - t)(z\mathbf{i} + x\mathbf{k}) + xy\,(y\mathbf{i} + x\mathbf{j})$$

$$\mathbf{f} = (1 + xz^2 + xy^2 - tz)\,\mathbf{i} + (-1 + yz^2 + x^2y + zt)\,\mathbf{j} + (y^2z + x^2z + yt - xt)\,\mathbf{k}$$

$$f_x = 1 + xz^2 + xy^2 - tz,$$

$$f_y = -1 + yz^2 + x^2y + zt,$$

$$f_z = y^2z + x^2z + (y - x)\,t,$$

At $(2, 1, 3)$ *and* $t = 0\cdot5$; $f_x = 19\cdot5\ m/sec^2$ *etc.* **Ans.**

3. **The velocity components in spherical polar coordinates** (r, θ, ϕ) **of a flow are**

$$q_r = (r^2/t^2)\sin\phi,\ q_\theta = (r/t)\sin\theta\cos\phi,\ q_\phi = (r/t)\cot\theta\,\mathrm{cosec}\,\phi.$$

Determine the components of acceleration of a fluid particle.

4. **Show that the equation of continuity reduces to Laplace's equation when the liquid is incompressible and the motion is irrotational. Does the velocity distribution** $\mathbf{q} = 5x\mathbf{i} + 5y\mathbf{j} - 10z\mathbf{k}$ **satisfy the law of conservation of mass for incompressible flow ?**

5. **For the plane flow of an incompressible fluid, the velocity component in the** x**-direction is** $u = ax^2 + by$. **Find the velocity component** v **in the** y**-direction, assuming (i)** $u = 0$ **at** $y = 0$, **and (ii)** $v = ct$ **at** $y = 0$.

6. **In a two dimensional flow, the** x**-component of velocity distribution is given by** $u = x^2 - 4xy - 3y$. **Determine the** y**-component for velocity.**

Hint. $\dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x} = -2x + 4y$

$\Rightarrow\ v = 2y^2 - 2xy + f(x).$

7. **A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis, show that the equation of continuity is**

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \omega)}{\partial \theta} = 0$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

8. Each particle of a mass of liquid moves in a plane through the axis of Z; show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v \sin\theta) = 0.$$

9. Homogeneous liquid moves so that the path of any particle P lies in the plane POX where OX is a fixed axis. Prove that if $OP = r$ and the $\angle XOP = \theta$, the equation of continuity may be written as

$$\frac{\partial}{\partial r}(ur^2) - \frac{\partial}{\partial \mu}(vr \sin\theta) = 0$$

where u, v are the component velocities along and perpendicular to OP in the plane POX and $\mu = \cos\theta$.

10. Does the two-dimensional incompressible flow given by

$$u(x, y) = A\left[\frac{x^2 + y^2 - B^2}{(x^2 + y^2 - B^2)^2 + 4B^2y^2}\right],$$

$$v(x, y) = 2A\left[\frac{xy}{(x^2 + y^2 - B^2) + 4B^2y^2}\right], w = 0$$

where A and B are arbitrary constants, satisfy the continuity equation.

11. Does the three-dimensional incompressible flow given by

$$u(x, y, z) = \frac{Ax}{(x^2 + y^2 + z^2)^{3/2}}, \quad v(x, y, z) = \frac{Ay}{(x^2 + y^2 + z^2)^{3/2}},$$

$$w(x, y, z) = \frac{Az}{(x^2 + y^2 + z^2)^{3/2}}$$

satisfy the equation of continuity? A is an arbitrary constant.

12. Show that the velocity field $q_r = 0$, $q_\theta = Ar + \frac{B}{r}$, $q_z = 0$, satisfy the equation of motion $\frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right) = 0$, where A and B are constants.

Hint : $\dfrac{dq_\theta}{dr} = A - \dfrac{B}{r^2}, \dfrac{d^2 q_\theta}{dr^2} = \dfrac{2B}{r^3}$

$$\frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right) = \frac{2B}{r^3} + \frac{d}{dr}\left(A + \frac{B}{r^3}\right) = \frac{2B}{r^3} - \frac{2B}{r^3} = 0.$$

Proved

13. Does the two-dimensional incompressible flow satisfies the continuity equation ?

$$q_r(r, \theta) = A\left(\frac{1}{r^2} - 1\right)\cos\theta, \; q_\theta(r, \theta) = A\left(\frac{1}{r^2} + 1\right)\sin\theta, \; r > 0$$

where A is an arbitrary non-zero constant.

Hint. *The equation of continuity in spherical polar coordinates is given by*

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho q_\theta \sin\theta) = 0.$$

or $$\frac{A}{r^2}\frac{\partial}{\partial r}\left[(1 - r^2)\cos\theta\right] + \frac{A}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\left(\frac{1}{r^2} + 1\right)\sin^2\theta\right] = 0.$$

or $$\frac{A}{r^2}(-2r\cos\theta) + \frac{A}{r\sin\theta}\left(\frac{1}{r^2} + 1\right)2\sin\theta\cos\theta = 0$$

or $$-\frac{2A}{r}\cos\theta + \frac{2A}{r^3}\cos\theta + \frac{2A}{r}\cos\theta = 0.$$

or $(2A/r^3)\cos\theta \neq 0 \Rightarrow$ *that the velocity components does not satisfy the continuity equation.*

14. Determine the constant l, m and n in order that the velocity

$$\mathbf{q} = \{(x + lr)\,\mathbf{i} + (y + mr)\,\mathbf{j} + (z + nr)\,\mathbf{k}\}/\{r(x + r)\},$$

where $r = \sqrt{(x^2 + y^2 + z^2)}$ may satisfy the equation of continuity for a liquid.

Hint. *The equation of continuity is*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{x + lr}{r(x + r)}\right\} = \frac{1 + l(x/r)}{r(x + r)} + (x + lr)$$

$\times$ [illegible]

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial r}\left\{\frac{y + mr}{r(x + r)}\right\} = \frac{1 + m(y/r)}{r(x + r)} + (y + mr)$$

The necessary and sufficient condition for (4) to hold is

$$\nabla \times \mathbf{q} = 0$$

or
$$\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0. \quad \dots (6)$$

or
$$\xi \mathbf{i} + \eta \mathbf{j} + \zeta \mathbf{k} = 0$$

where $\xi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$ etc. are known as the *spin components*.

If the relation (6) exists then the flow is said to be *irrotational*, otherwise the motion is said to be *rotational*. In other words, when the motion is irrotational, the velocity vector is the gradient of a scalar function ϕ *i.e.*, the necessary condition of the existence of a velocity potential in a fluid is irrotational motion. Potential flow is the irrotational flow of an inviscid or perfect fluid.

Ex. 22. *Determine whether the motion specified by*

$$\mathbf{q} = \frac{A(x\mathbf{i} - y\mathbf{j})}{x^2 + y^2}, \ (A = constant)$$

is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines. Also, show that the motion is of potential kind. Find the velocity potential.

Solution. We know that $\nabla \cdot \mathbf{q} = 0$. ... (1)

or
$$A\left\{-\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\} = 0$$

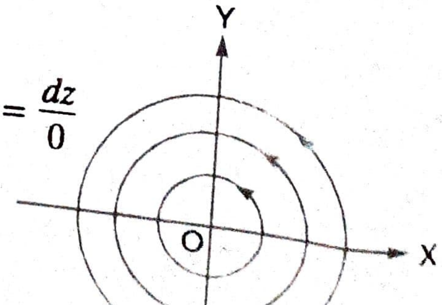
or
$$A\left\{\frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right\} = 0$$

which is evident. Thus the equation of continuity for an incompressible fluid is satisfied and hence is a possible motion.

The equation of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or
$$\frac{dx}{-\left(\frac{Ay}{x^2+y^2}\right)} = \frac{dy}{\left(\frac{Ax}{x^2+y^2}\right)} = \frac{dz}{0}$$



or
$$x\,dx + y\,dy = 0, \quad dz = 0.$$

By integrating, we have

$$x^2 + y^2 = \text{constant}, \ z = \text{constant}.$$

Thus the streamlines are circles whose centres are on Z-axis, their planes being perpendicular to the axis.

$$\nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{Ay}{x^2+y^2} & \dfrac{Ax}{x^2+y^2} & 0 \end{vmatrix}$$

or $$\nabla \times \mathbf{q} = \mathbf{k}\left[\frac{\partial}{\partial x}\left\{\frac{Ax}{x^2+y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{Ay}{x^2+y^2}\right\}\right],$$

or $$\nabla \times \mathbf{q} = \mathbf{k}A\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] = 0.$$

Thus the flow is of potential kind, so we can determine $\phi(x, y, z)$ such that $\mathbf{q} = -\nabla\phi$.

or $$\frac{\partial \phi}{\partial x} = -u = \frac{Ay}{x^2+y^2}, \qquad \ldots (4)$$

$$\frac{\partial \phi}{\partial y} = -v = -\frac{Ax}{x^2+y^2}, \qquad \ldots (5)$$

$$\frac{\partial \phi}{\partial z} = -w = 0 \qquad \ldots (6)$$

which shows that ϕ is independent of z, hence $\phi = \phi(x, y)$.

Integrating the relation (4), we have

$$\phi(x, y) = f(y) + A\tan^{-1}\left(\frac{x}{y}\right) \Rightarrow \frac{\partial \phi}{\partial y} = f'(y) - \frac{Ax}{x^2+y^2}$$

Using the relation (5), we get

$$f'(y) = 0 \Rightarrow f(y) = \text{constant}.$$

Therefore, $\phi(x, y) = A\tan^{-1}\left(\dfrac{x}{y}\right)$. **Ans.**

Ex. 23. *Show that the velocity potential*

$$\phi = \tfrac{1}{2}a(x^2 + y^2 - 2z^2),$$

satisfies the Laplace equation. Also determine the streamlines.

Solution. Let ϕ be the velocity potential for the velocity field, then

$$\mathbf{q} = -\nabla\phi = -\tfrac{1}{2}a\,\nabla\,(x^2+y^2-2z^2),$$

or
$$\mathbf{q} = -\tfrac{1}{2}a\,(2x\mathbf{i}+2y\mathbf{j}-4z\mathbf{k}).$$

Taking divergence of both the sides, we have

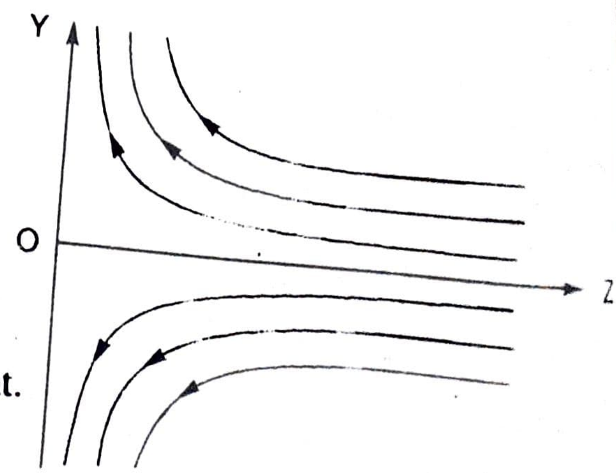
$$\nabla^2\phi = \nabla\cdot\mathbf{q} = -\tfrac{1}{2}a\,\nabla\cdot\,(2x\mathbf{i}+2y\mathbf{j}-4z\mathbf{k}).$$

or
$$\nabla^2\phi = -\tfrac{1}{2}a\,(2+2-4) = 0.$$

Hence Laplace equation is satisfied.

The equation of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$



or
$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az}$$
$$(1) \qquad (2) \qquad (3)$$

From (2) and (3), we have

$$\log y + \tfrac{1}{2}\log z = \log C,$$

where C is an integration constant.

or
$$y^2 z = C,$$

which represents a cubical hyperbola. **Ans.**

Ex. 24. *Show that*

$$u = -\frac{2xyz}{(x^2+y^2)^2}, \quad v = \frac{(x^2-y^2)\,z}{(x^2+y^2)^2}, \quad w = \frac{y}{x^2+y^2},$$

are the velocity components of a possible liquid motion. Is this motion irrotational ?

Solution. The condition for the possible liquid motion is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$2yz.\frac{3x^2-y^2}{(x^2+y^2)^3} + 2yz.\frac{y^2-3x^2}{(x^2+y^2)^3} + 0 = 0$$

which is an identity. Hence (u, v, w) are the velocity components of a possible liquid motion.

Again condition for irrotational motion is given by

Thus $$\mathbf{q} = \frac{C}{3r^3}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right),$$

and $$\mathbf{q}^2 = \mathbf{q}\cdot\mathbf{q} = \frac{C^2}{9r^6}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right)\cdot\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right)$$

$$\mathbf{q}^2 = \frac{C^2}{9r^6}\left(1 - \frac{6x\mathbf{r}.\mathbf{i}}{r^2} + \frac{9x^2\mathbf{r}^2}{r^4}\right).$$

Hence q^2 = constant gives the surfaces of constant speed as

$(r^2 + 3x^2)\, r^{-8}$ = constant. **Proved.**

Exercises

1. Find the velocity vector $\mathbf{q}$ for the following velocity potential ϕ

(a) $\phi = c\,(x^2 - y^2)$ (b) $\phi = (x^2 + y^2)/(x - y)$

(c) $\phi = A \sin^{-1}(y/x)$ (d) $\phi = cxy$.

2. At a point in an incompressible fluid having spherical polar coordinates (r, θ, ϕ), the velocity components are

$$[2Ar^{-3}\cos\theta, Ar^{-2}\sin\theta, 0],$$

where A is the constant. Show that the velocity is of the potential kind. Find the velocity potential and the equations of the stream-line.

Hint. *Let q_r, q_θ, q_ϕ be the components of the fluid particle. The velocity $\mathbf{q}$ is of the potential kind if it satisfies the condition*

$$\nabla \times \mathbf{q} = 0.$$

$$\nabla \times \mathbf{q} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ q_r & q_\theta & q_\phi \end{vmatrix}$$

$$= \frac{1}{r^2 \sin^2\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 2Ar^{-3}\cos\theta & Ar^{-2}\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin^2\theta}\left[-\frac{\partial}{\partial\phi}(Ar^{-2}\sin\theta)\,\hat{r} + \frac{\partial}{\partial\phi}(2Ar^{-3}\cos\theta)\,r\,\hat{\theta}\right.$$

Let $F(r, \theta, \phi)$ *is the velocity potential, then*

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta + \frac{\partial F}{\partial \phi} d\phi$$

$$dF = -(2Ar^{-3} \cos\theta)\, dr - (Ar^{-2} \sin\theta)\, d\theta$$

$$dF = d(Ar^{-2} \cos\theta)$$

$$F = Ar^{-2} \cos\theta.$$

The stream lines are given by

$$\frac{dr}{q_r} = \frac{r\, d\theta}{q_\theta} = \frac{r \sin\theta\, d\phi}{q_\phi}$$

3. Show that the variable ellipsoid

$$\frac{x^2}{a^2 k^2 t^4} + kt^2 \left\{ \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \right\} = 1,$$

is a possible form for the boundary surface of a liquid at any time t.

4. Show that the variable ellipsoid

$$\frac{x^2}{a^2 e^{-t} \cos(t + \frac{1}{4}\pi)} + \frac{y^2}{b^2 e^t \sin(t + \frac{1}{4}\pi)} + \frac{z^2}{c^2 \sec 2t} = 1,$$

is a possible form of boundary surface of a liquid for any time t and determine the velocity components of any particle on this boundary.

5. Show that a surface of the form

$$ax^4 + by^4 + cz^4 - \mu(t) = 0$$

is a possible form of a boundary surface of a homogeneous liquid at time t, the velocity potential of the liquid motion being $\phi = (\beta - \gamma)x^2 + (\gamma - \alpha)y^2 + (\alpha - \beta)z^2$, where $\mu, \alpha, \beta, \gamma$ are given functions of time and a, b, c are suitable functions of time.

6. In the steady motion of homogeneous liquid if the surfaces $f_1 = a_1, f_2 = a_2$ define the streamlines, prove that the most general values of the velocity components u, v, w are

$$F(f_1 f_2) \frac{\partial(f_1, f_2)}{\partial(y, z)},\ F(f_1, f_2) \frac{\partial(f_1, f_2)}{\partial(z, x)},\ F(f_1, f_2) \frac{\partial(f_1, f_2)}{\partial(x, y)}$$

7 is a possible form of the boundary surface of a liquid.

Prove that $\left(\dfrac{x^2}{a^2}\right) \phi(t) + \left(\dfrac{y^2}{b^2}\right) \dfrac{1}{\phi(t)} = 1$