Lecture - 4

Marine Hydrodynamics

Ex-1 : For a two dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian Co-ordinate by

$u = x + y + 2t, \quad v = 2y + t.$

find its Lagrange co-ordinate as a function of initial positions x_0, y_0 and the time t.

Solution : $\therefore$ $u = \frac{dx}{dt} = x + y + 2t \quad \ldots\ldots (1)$

$v = \frac{dy}{dt} = 2y + t \quad \ldots\ldots (2)$

from (2) we get

$$\frac{dy}{dt} - 2y = t \quad \ldots\ldots (3)$$

integrating factor $= e^{\int p(t)\, dt} = e^{\int -2\, dt} = e^{-2t}$

since $p(t) = -2$ here

then $y e^{-2t} = \int e^{-2t} \cdot t \, dt + c$

$\Rightarrow y e^{-2t} = -\frac{t}{2} e^{-2t} + \frac{1}{2} \int e^{-2t} \, dt + c$

$\Rightarrow y e^{-2t} = -\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + c$

$\Rightarrow y = c' e^{2t} - \frac{1}{4}(2t+1) \quad \ldots\ldots (4)$

substituting (4) in (1) we get

$$\frac{dx}{dt} = x + ce^{2t} - \frac{1}{4}(2t+1) + 2t$$

$$\Rightarrow \frac{dx}{dt} - x = ce^{2t} - \frac{1}{4}\left(2t+1\right) + 2t$$

in this case $p(t) = -1$

$\therefore$ integrating factor $= e^{\int -1\, dt} = e^{-t}$

$$\Rightarrow xe^{-t} = \int e^{-t}\left[ce^{2t} - \frac{1}{4}(2t+1) + 2t\right]dt + D$$

$$\Rightarrow xe^{-t} = c\int e^{t}\, dt - \frac{1}{4}\int (2t+1)e^{-t}\, dt + \int e^{-t}\cdot 2t\, dt + D$$

$$\Rightarrow xe^{-t} = ce^{t} + \frac{1}{4}e^{-t} - \frac{1}{4}\int 2te^{-t}\, dt + \int e^{-t}\, 2t\, dt$$

$$\Rightarrow xe^{-t} = ce^{t} + \frac{1}{4}e^{-t} + \frac{3}{2}\int te^{-t}\, dt$$

$$\Rightarrow xe^{-t} = ce^{t} + \frac{1}{4}e^{-t} + \frac{3}{2}\left[-te^{-t} + \int e^{-t}\, dt\right] + D$$

$$\Rightarrow xe^{-t} = ce^{t} + \frac{1}{4}e^{-t} + \frac{3}{2}\left[-te^{-t} - e^{-t}\right] + D$$

$$\Rightarrow x = Ce^{2t} + \frac{1}{4} + \frac{3}{2}\left[-t-1\right] + De^{t}$$

$$\Rightarrow x = Ce^{2t} + De^{t} - \frac{1}{4}\left[6t+5\right]$$

$\therefore$ the path line

$$x = Ce^{2t} + De^{t} - \frac{1}{4}(6t+5)$$

$$y = Ce^{2t} - \frac{1}{4}(2t+1)$$

Now value of C & D can be determined from the initial condition at $t=0$, $x_0 = x_0$ & $y = y_0$

$$\Rightarrow x_0 = C + D - \frac{5}{4} \quad \ldots\ldots (5)$$

$$y_0 = C - \frac{1}{4} \Rightarrow C = y_0 + \frac{1}{4}$$

$$\Rightarrow x_0 = y_0 + \frac{1}{4} + D - \frac{5}{4}$$

$$\Rightarrow D = x_0 - y_0 + 1$$

$$\therefore x = \left(y_0 + \frac{1}{4}\right)e^{2t} + (x_0 - y_0 + 1)e^{t} - \frac{1}{4}(6t+5)$$

$$y = \left(y_0 + \frac{1}{4}\right)e^{2t} - \frac{1}{4}(2t+1)$$

Ans

Ex-2 : The velocity vector $\vec{q}$ is given by $\vec{q} = x\hat{i} - y\hat{j}$, ~~do~~ is it a fluid flow? if so, determine the stream line.

Solution. Let us find out the value of $\operatorname{div}(q)$

Now $\operatorname{div} q = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}x - \hat{j}y + \hat{k}0\right)$

$$= \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y) + 0$$

$$= 1 - 1 = 0$$

$\because \operatorname{div} q = 0 \Rightarrow$ it is a fluid flow.

Now for stream line equation

$$\vec{q} \times dr = 0$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ x & -y & 0 \\ dx & dy & dz \end{vmatrix} = 0$$

$$\Rightarrow x\,dy + y\,dx = 0$$

$$\Rightarrow d(xy) = 0$$

$$\Rightarrow \cancel{xy}\ \boxed{xy = \text{constant}}$$

$\therefore$ It is rectangular hyperbola (Answer)

Ex- 3 :. Find the equation of stream line and path line if $u = \frac{x}{1+t}$, $v = \frac{y}{1+t}$, $w = \frac{z}{1+t}$

Solution :. Equation of the streamline are given by :-

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

taking 1st two component :.

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \quad \log x = \log y + \log A$$

$$\Rightarrow \quad x = Ay \longrightarrow (1)$$

integrating (1) & (3) we get

$$\log x = \log z + \log B$$

$$\Rightarrow \quad x = Bz \longrightarrow (2)$$

Hence the stream line is given by (1) & (2).

Now & for path line

$$u = \frac{dx}{dt} = \frac{x}{1+t} \Rightarrow \frac{dx}{x} = \frac{dt}{1+t} \Rightarrow x = C(1+t)$$

Similarly
$$\left.\begin{aligned} y &= D(1+t) \\ z &= E(1+t) \end{aligned}\right\} (4)$$

(4) gives the equation of path lines.

Q-4: If $u = yz + t$, $v = xz - t$, $w = xy$. Determine the accelaration at the point $(2, 1, 3)$ at $t = 0.5$ second.

Solution: We know

$$\vec{q} = (yz+t)\hat{i} + (xz-t)\hat{j} + xy\hat{k}$$

Accelaration

$$\frac{D\vec{q}}{Dt} = \frac{\partial q}{\partial t} + (\vec{q}.\nabla)\vec{q}$$

$$= \frac{\partial q}{\partial t} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z}$$

$$\Rightarrow f_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$f_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$f_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$\therefore f_x = 1 + (xz-t).z + xy.y$$

$$\Rightarrow f_x = 1 + xz^2 + xy^2 - tz$$

Similarly

$$f_y = -1 + yz^2 + x^2y + tz$$

$$f_z = y^2z + x^2z - (y-x)t$$

at $(2,1,3)$ & $t = 0.5$, $f_x = 19.5\ m/s^2$

Similarly find f_y and f_z

Show that $\phi = \frac{1}{2} a(x^2 + y^2 - 2z^2)$ represents the velocity potential for fluid flow.

Solution :- Here :-

$$\frac{\partial \phi}{\partial x} = ax$$

$$\frac{\partial^2 \phi}{\partial x^2} = a$$

Similarly

$$\frac{\partial^2 \phi}{\partial y^2} = a$$

$$\frac{\partial^2 \phi}{\partial z^2} = -2a$$

$$\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$\therefore$ ϕ represents the velocity potential.

Ex:-6 : Show that $u = -\dfrac{2xyz}{(x^2+y^2)^2}$

$$v = \frac{(x^2 - y^2) z}{(x^2+y^2)^2}$$

$$w = \frac{y}{x^2+y^2}$$

represents ir-rotational flow.

Hint. 1st you have to show that $\operatorname{div} \vec{q} = 0$

i.e $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

for ir-rotational flow:

$$\nabla \times \vec{q} = 0$$

i.e $\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$

Now $\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{(x^2-y^2)}{(x^2+y^2)^2} = 0$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{2xz(3y^2-x^2)}{(x^2+y^2)^2} - \frac{2xz(3y^2-x^2)}{(x^2+y^2)^2} = 0$$

$\therefore \nabla \times \vec{q} = 0 \quad \therefore$ the flow is ir-rotational.

Home work:

1. Show that $\phi = x f(r)$ is a possible form for the velocity potential of an incompressible liquid motion. if $q \to 0$ as $r \to \infty$, show that ~~$(r^2 + 3x^2) r^{-8}$ = constant~~ $= c$

surface of constant speed are $(r^2 + 3x^2) r^{-8} = \text{const.}$

2. Show that $\frac{x^2}{a^2} \tan^2 t + \frac{y^2}{b^2} \cot^2 t = 1$ a possible form of boundary surface.