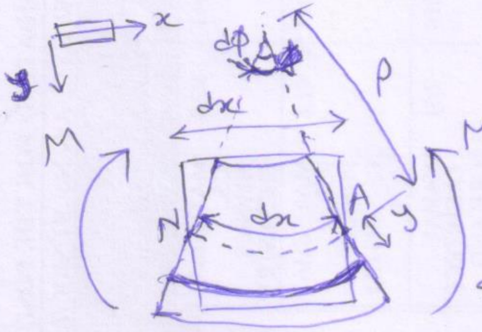
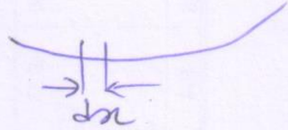
Moment-Area method / Area-Moment theorem

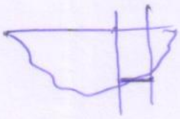
This method uses the moment-curvature relation of a simple beam.

The assumptions of a simple beam:

1. Length of beam >> height and width
2. Deflection (and slope) is small.
3. Material is linear isotropic, follows Hook's law.
4. A plane cross-section perpendicular to the beam axis remains plane and perpendicular before and after bending.
5. Do you remember any other points? ...(eg., section undergoes pure bending)







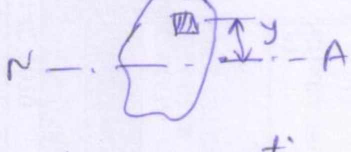
For a small dx, BM is taken as constant.

For a fibre at y from the NA, we can write

$$\text{strain}\ (\epsilon) = \frac{(\rho+y)\,d\phi - \rho\, d\phi}{\rho\, d\phi} = \frac{y\, d\phi}{\rho\, d\phi} = \frac{y}{\rho}$$

$$\text{or}\ \frac{\sigma}{E} = \frac{y}{\rho} \quad \text{or}\ \frac{\sigma}{y} = \frac{E}{\rho}$$

On the cross-section of the beam having pure bending sum of all moments acting due to normal stress over small areas = total moment acting.



beam X-section

If σ = normal stress acting on an area dA,

$$\therefore \int_A y\,\sigma\, dA = M \quad \text{or} \int_A y\,\frac{Ey}{\rho}\, dA = M$$

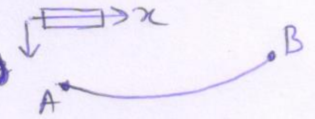
$$\text{or}\ M = \frac{E}{\rho}\int y^2\, dA = \frac{EI}{\rho} \quad \text{or}\ \underline{\underline{\frac{M}{EI} = \frac{1}{\rho}}}$$ moment curvature relation

Now, $\frac{1}{\rho} = \dfrac{\left(\frac{d^2y}{dx^2}\right)}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$. Since, slope $\left(\frac{dy}{dx}\right)$ is small, we can ignore $\left(\frac{dy}{dx}\right)^2$.

Thus we get $\dfrac{M}{EI} = \dfrac{d^2y}{dx^2}$

If a problem of beam bending is statically determinate, we can get its shear force diagram (SFD) and bending moment diagram (BMD) easily.

Now, we take an elastic curve (deformed shape of a beam, longitudinal axis)

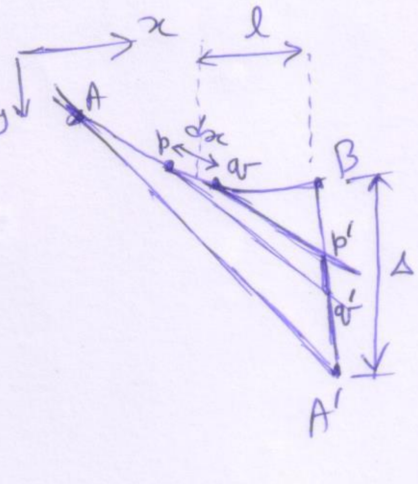


we have, $\frac{d^2y}{dx^2} = \frac{M}{EI}$

$$\int d\left(\frac{dy}{dx}\right) = \int \frac{M}{EI}\, dx$$

$$\therefore \left(\frac{dy}{dx}\right)_A - \left(\frac{dy}{dx}\right)_B = \int_A^B \frac{M}{EI}\, dx$$

∴ The difference of slope between two points = Area under the $\frac{M}{EI}$ diagram between those points.



Now, take Δ shown below

Δ = vertical distance of B from tangent drawn at A = BA'

To find out Δ, we take two points p, q on the elastic curve such that pq = dx = small distance

The tangents drawn at p and q intersect the vertical line BA' at p' and q'.

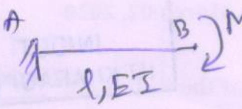
$$\therefore p'q' = (\text{difference of slope at } p \text{ and } q) \times l$$

$$= \frac{M}{EI} dx \times l = \frac{Ml}{EI} dx$$

$$\therefore \Delta = \int_{A'}^{B} p'q' = \int_{x(A)}^{x(B)} \frac{M}{EI} dx\, l = \text{1st moment of } \frac{M}{EI} \text{ diagram about } B.$$

∴ Vertical distance of a point with respect to tangent drawn at another = 1st moment of $\frac{M}{EI}$ diagram between the points with respect to the point whose distance is to be found.

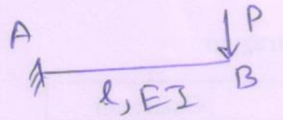
Let us take these problems to explain the above method.
Results of these problems are often used in structural engineering.

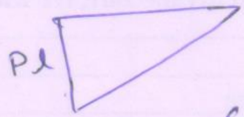


At A, slope, deflection = 0

$$\therefore \theta_B = \frac{Ml}{EI} = (\theta_B - \theta_A), \quad \theta_A = 0$$

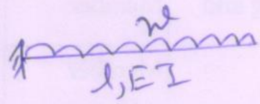
$$\delta = \underbrace{\left(\frac{Ml}{EI}\right)}_{\text{area}}\underbrace{\left(\frac{l}{2}\right)}_{\text{arm}} = \frac{Ml^2}{2EI}$$

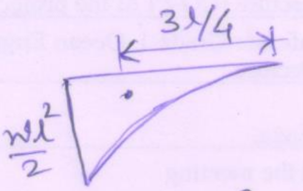




$$\theta_B - \theta_A = \theta_B = \frac{1}{2}\cdot\frac{Pl}{EI}\cdot l = \frac{Pl^2}{2EI}$$

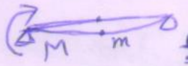
$$\delta_B = \underbrace{\left(\frac{Pl^2}{2EI}\right)}_{\text{area}} \times \underbrace{\left(\frac{2}{3}l\right)}_{\text{arm w.r.t. B}} = \frac{Pl^3}{3EI}$$



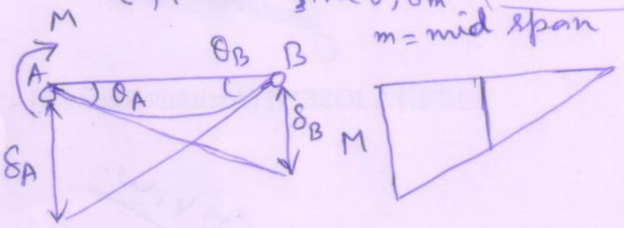


$$\theta_B - \theta_A = \theta_B = \left(\frac{wl^2}{2}\times\frac{l}{EI}\right)\times\frac{1}{3} = \frac{wl^3}{6EI}$$

$$\boxed{\delta_B = \frac{wl^3}{6EI}\times\frac{3l}{4} = \frac{wl^4}{8EI}}$$



find θ, δ_m
m = mid span

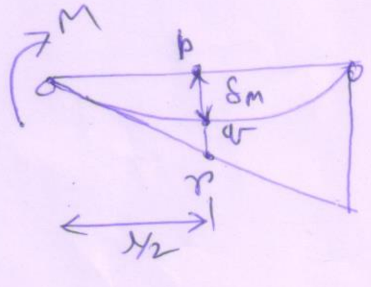


$$\delta_B = \left(\frac{1}{2}\cdot\frac{M}{EI}\cdot l\cdot\frac{2}{3}l\right) = \frac{Ml^2}{3EI}$$

$$\therefore \theta_A = \frac{\delta_B}{l} = \frac{Ml}{3EI}$$

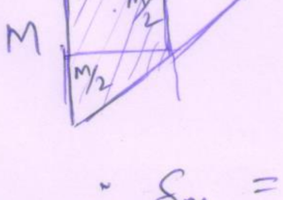
similarly, $\delta_A = \frac{1}{2}\frac{Ml}{EI}\frac{1}{3}l = \frac{Ml^2}{6EI}$, $\therefore \theta_B = \frac{\delta_A}{l} = \frac{Ml}{6EI}$

$$\therefore \theta_A = 2\theta_B$$



mid span displacement $= \delta_M$

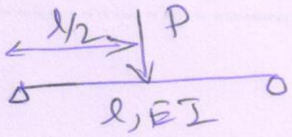
$$\delta_M = pq = pr - qr = \frac{Ml^2}{3EI}\times\frac{1}{2} - \left(\underbrace{\frac{M}{2EI}\cdot\frac{l}{2}\cdot\frac{l}{4}}_{\text{rectangle}} + \underbrace{\frac{1}{2}\frac{M}{2}\frac{l}{2}\frac{2}{3}\frac{l}{2EI}}_{\text{triangle}}\right)$$

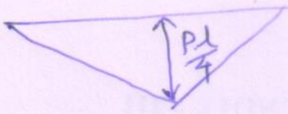


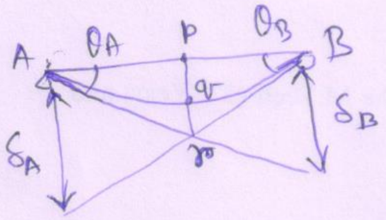
$$= \frac{Ml^2}{6EI} - \left(\frac{Ml^2}{16EI} + \frac{Ml^2}{24EI}\right) = \cancel{\frac{5}{48}}$$

$$\therefore \delta_M = \frac{Ml^2}{6EI} - \frac{5}{48}\frac{Ml^2}{EI} = \frac{Ml^2}{16EI}$$

(mid span deflection)





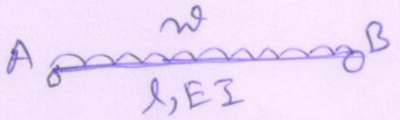


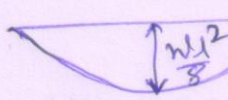
$$\delta_B = \left(\frac{1}{2}\cdot\frac{Pl}{4}\cdot l\right)\frac{l}{2}\cdot\frac{1}{EI} = \frac{Pl^3}{16EI}$$

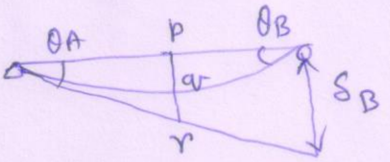
$$\therefore \theta_A = \theta_B = \frac{\delta_B}{l} = \frac{Pl^2}{16EI}$$

mid span deflection $(\delta_m) = pq = pr - qr$

$$= \frac{1}{2}\cdot\frac{Pl^3}{16EI} - \left(\frac{1}{2}\cdot\frac{Pl}{4EI}\cdot\frac{l}{2}\cdot\frac{1}{3}\cdot\frac{l}{2}\right) = \frac{Pl^3}{32EI} - \frac{Pl^3}{96EI} = \frac{Pl^3}{48EI}$$



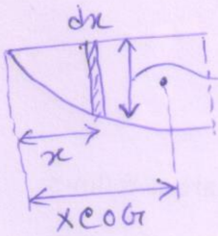




$$\delta_B = \left(\frac{2}{3}\times\frac{wl^2}{8}\times l\right)\times\frac{l}{2}\times\frac{1}{EI} = \frac{wl^4}{24EI}$$

$$\therefore \theta_A = \theta_B = \frac{wl^3}{24EI}$$

$$\delta_M = pr - qr = \frac{wl^4}{48EI} - \frac{2}{3}\times\frac{wl^2}{8}\times\frac{l}{2}\times\delta\times\frac{1}{EI} = \frac{wl^4}{48EI} - \frac{wl^3\delta}{24EI}$$

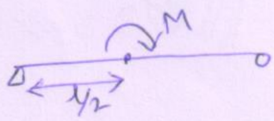


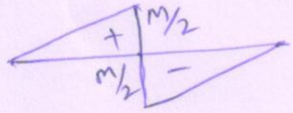
$$\left(\frac{wl}{2}x - \frac{wx^2}{2}\right) \qquad \therefore xCOG = \frac{\int_0^{l/2} x\left(\frac{wlx}{2} - \frac{wx^2}{2}\right)dx}{\frac{2}{3}\cdot\frac{wl^2}{8}\cdot\frac{l}{2}}$$

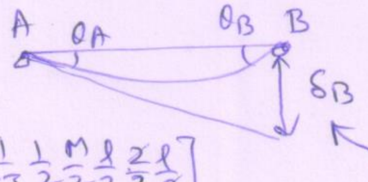
$$= \frac{\left[\frac{wlx^3}{6} - \frac{wx^4}{8}\right]_0^{l/2}}{\frac{wl^3}{24}} = \frac{\frac{wl^4}{48} - \frac{wl^4}{128}}{\frac{wl^3}{24}} = \frac{5l}{384}\cdot\frac{24}{1} = \frac{5l}{16}$$

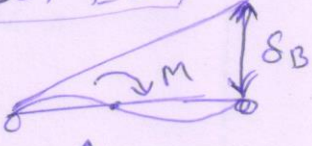
$$\therefore \delta = \frac{l}{2} - \frac{5l}{16} = \frac{3l}{16}$$

$$\therefore \delta_M = \frac{wl^4}{48EI} - \frac{wl^3}{24EI}\cdot\frac{3l}{16} = \frac{wl^4}{48EI} - \frac{3wl^4}{384EI} = \frac{5}{384}\frac{wl^4}{EI}$$









which one is correct?

$$\theta_A = \theta_B = \frac{\delta_B}{l} = \frac{1}{l}\left[\frac{1}{EI}\cdot\frac{1}{2}\frac{M}{2}\frac{l}{2}\left(\frac{l}{2}+\frac{1}{3}\frac{l}{2}\right) - \frac{1}{EI}\frac{1}{2}\frac{M}{2}\frac{l}{2}\frac{2}{3}\frac{l}{2}\right]$$

$$= \frac{Ml}{EI}\left[\frac{1}{8}\cdot\frac{1}{2}\cdot\frac{4}{3} - \frac{1}{24}\right] = \frac{Ml}{24EI}$$

Find δ_M yourselves