Nonlinear second order differential equation

$$u'' = f(x, u) \quad \text{—(1)}$$

$$a_0 u(a) - a_1 u'(a) = \gamma_1 \quad -2(a)$$

$$b_0 u(b) + b_1 u'(b) = \gamma_2 \quad 2(b)$$

Let

$$u_1 = u_0 + h u_0' + h^2 (\beta_0 u_0'' + \beta_1 u_1'') \quad \text{—(3)}$$

We basically want $O(h^4)$ approximation for u_0'

Let

$$T_0 = u_1 - u_0 - h u_0' - h^2 (\beta_0 u_0'' + \beta_1 u_1'')$$

Use Taylor expansion to expand u_1 about $x = x_0$

$$T_0 = \cancel{u_0} + \cancel{h u_0'} + \frac{h^2}{2!} u_0'' + \frac{h^3}{3!} u_0''' + \frac{h^4}{4!} u_0^{(iv)} + \cdots$$

$$- \cancel{u_0} - \cancel{h u_0'} - h^2 \left[\beta_0 u_0'' + \beta_1 \left\{ u_0'' + h u_0''' + \frac{h^2}{2} u_0^{(iv)} + \cdots \right\}\right]$$

$$= h^2 u_0'' \left(\frac{1}{2} - \beta_0 - \beta_1\right) + \left(\frac{1}{6} - \beta_1\right) h^3 u_0''' + \left(\frac{1}{24} - \frac{1}{2}\beta_1\right) h^4 u_0^{(4)} + \cdots$$

Now we make $O(h^2)$, and $O(h^3)$ terms to zero so we get

$$\beta_0 + \beta_1 = 1/2$$

$$\beta_1 = 1/6 \Rightarrow \beta_0 = 1/3$$

So we may write,

$$u_1 = u_0 + h u_0' + \frac{h^2}{6}(2 u_0'' + u_1'') + O(h^4)$$

or
$$h u_0' = u_1 - u_0 - \frac{h^2}{6}\left(2f(x_0, u_0) + f(x_1, y_1)\right)$$ ②

$$h u_0' = u_1 - u_0 - \frac{h^2}{6}(2f_0 + f_1)$$

Now the b.c. at $x = a$ is

$$a_0 u(a) - a_1 u'(a) = \gamma_1$$

or
$$a_0 h u_0 - a_1 h u_0' = h\gamma_1$$

$$a_0 h u_0 - \left[u_1 - u_0 - \frac{h^2}{6}(2f_0 + f_1)\right] = h\gamma_1$$

or

$$(h a_0 + a_1) u_0 - a_1 u_1 + \frac{a_1 h^2}{6}(2f_0 + f_1) = h\gamma_1$$

(A)

<u>Similarly</u> we write

$$h u_N' = -u_{N-1} + u_N + \frac{h^2}{6}[f_{N-1} + 2f_N]$$

and then use bc at $x = b$

$$b_0 u(b) + b_1 u'(b) = \gamma_2$$

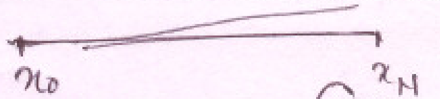
$$b_0 h u_N + b_1 h u_N' = h\gamma_2$$

and hence we get

$$-b_1 u_{N-1} + (h b_0 + b_1) u_N + \frac{b_1 h^2}{6}(f_{N-1} + 2f_N) = h\gamma_2$$

(B).

Iterative method



$$u'' = f(x,u), \quad a \le x \le b \qquad \text{— (1)}$$

$$a_0 u(a) - a_1 u'(a) = \gamma_1 \qquad \text{— 2(a)}$$

$$b_0 u(b) + b_1 u'(b) = \gamma_2 \qquad \text{— 2(b)}$$

then the difference scheme is given by

$$(ha_0 + a_1) u_0 - a_1 u_1 + \frac{a_1 h^2}{6}(2f_0 + f_1) = h\gamma_1 \qquad \text{— (I)}$$

$$-u_{k-1} + 2u_k - u_{k+1} + h^2 f_k = 0 \qquad \text{— (II)}$$

$$k = 2(1)\overline{N-1}$$

$$-b_1 u_{N-1} + (hb_0 + b_1) u_N + \frac{b_1 h^2}{6}(f_{N-1} + 2f_N) = h\gamma_2 \qquad \text{— (III)}$$

for iteration method

$$(ha_0 + a_1) u_0^{[r+1]} - a_1 u_1^{[r+1]} + \frac{a_1 h^2}{6}\left(2f(x_0, u_0^{[r]}) + f(x_1, u_1^{[r]})\right) = h\gamma_1 \qquad \text{— (iv)}$$

$$-u_{k-1}^{[r+1]} + 2u_k^{[r+1]} - u_{k+1}^{[r+1]} + h^2 f(x_k, u_k^{[r]}) = 0 \qquad \text{— (v)}$$

$$k = 2(1)\overline{N-1}$$

$$-b_1 u_{N-1}^{[r+1]} + (hb_0 + b_1) u_N^{[r+1]} + \frac{b_1 h^2}{6}\left[f(x_{N-1}, u_{N-1}^{[r]}) + 2f(x_N, u_N^{[r]})\right] = h\gamma_2 \qquad \text{— (VI)}$$

Equations (iv) – (vi) can be written as

$$A u^{[r+1]} = -\frac{h^2}{6} B F^{[r]} + C$$

where

$$A = \begin{pmatrix} ha_0 + a_1 & -a_1 & 0 & \cdots & & & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \cdots 0 \\ 0 & \cdots & & & 0 & -b_1 & (hb_0 + b_1) \end{pmatrix}$$

$$B = \begin{pmatrix} 2a_1 & a_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 \;\; 6 & 0 & \cdots 0 \\ & & & \ddots & \\ & & & b_1 & 2b_1 \end{pmatrix}$$

$$f^{[r]} = \begin{pmatrix} f(x_0, u_0^{[r]}) \\ \vdots \\ \vdots \\ f(x_N, u_N^{[r]}) \end{pmatrix}, \quad C = \begin{pmatrix} hr_1 \\ 0 \\ \vdots \\ 0 \\ hr_2 \end{pmatrix}$$