PDE

6 1

Def^n The order of a partial differential equation is the order of highest derivative occurring in equation.

An equation of following type

$$f(x, y, z, t, \ldots, u_x, u_y, u_z, u_t \ldots u_{xx} u_{yy} \ldots, u_{xy}, \ldots) = 0$$

is called a general PDE. — (*)

Ex
1. $u_t = k(u_{xx} + u_{yy} + u_{zz})$
2. $u_{xx} + u_{yy} + u_{zz} = 0$
3. $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$

} 2nd order PDE

4. $u_t = u u_{xxx} + \sin x$ — Third order PDE.

Def^n If the function f is linear in u_x, u_y, u_z and in other partial derivatives then (*) is called linear PDE.

Next we consider the following linear PDE.

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad (1)$$

where $A, B, C, \ldots$ are functions of x and y. Now if

$B^2 - 4AC < 0$ the (1) is called Elliptic

$B^2 - 4AC = 0$ ——. Parabolic

$B^2 - 4AC > 0$ ——. Hyperbolic PDE

and $B^2 - 4AC$ is called discriminant.

Canonical Form

Consider most general transformation of independent variables x and y of equation ① to new variable ξ, η where

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) \qquad \text{——— (2)}$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 = (\xi_x \eta_y - \xi_y \eta_x) \qquad \text{——— (3)}$$

Using chain rule of partial differentiation the partial derivatives becomes

$$u = u(\xi,\eta)$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}$$

Substituting these in ① we get

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}u = \bar{G} \qquad (4)$$

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$\bar{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$$\bar{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$$\bar{F} = \hat{F}, \quad \bar{G} = G.$$

It may be noted that the transformed equation (4) has the same form as that of original equation (1).

It can be ... that under the transformation ②, the equation ① Take the following forms:

(i) $u_{\xi\xi} - u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

or $u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the hyperbolic case

(ii) $u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u_\xi, u_\eta)$ in elliptic case

(iii) $u_{\xi\xi} = \phi(\xi, \eta, u, u_\xi, u_\eta)$

or $u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the parabolic case

It can easily be seen that

$$\bar{B}^2 - 4\bar{A}\bar{C} = (\xi_x \eta_y - \xi_y \eta_x)^2 (B^2 - 4AC) \quad \text{——} \ (5)$$

Cannonical form for Hyperbolic Equation:

Since $\bar{B}^2 - 4\bar{A}\bar{C} > 0$ for hyperbolic case

We set $\bar{A} = 0$, $\bar{C} = 0$ i.e.,

Ps. $\bar{A} = 0$, $\bar{C} = 0$ says that in the canonical form $u_{\xi\xi}$, $u_{\eta\eta}$ coefficients are zero.

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \quad \text{——} \ (6)$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \quad \text{——} \ (7)$$

from (6)

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\frac{\xi_x}{\xi_y} + C = 0 \quad \text{——} \ (8)$$

and from (7)

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0 \quad \text{——} \ (9)$$

Solving (8) & (9) we get

$$\frac{\xi_x}{\xi_y} = \frac{-B+\sqrt{B^2-4AC}}{2A}$$ Only +ve solⁿ (10)

$$\frac{\eta_x}{\eta_y} = \frac{-B-\sqrt{B^2-4AC}}{2A}$$ only −ve solⁿ (11)

We have considered only one solution for each otherwise we will end up with same two coordinates.

(10) & (11) are called characteristic equations.

Here we have two family of curves

$\xi(x,y) = C_1$ & $\eta(x,y) = C_2$ — (12)

from $\xi(x,y) = C_1$

$$d\xi = \xi_x dx + \xi_y dy = 0$$

or

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right) \quad (13)$$

and from $\eta(x,y) = C_2$

$$d\eta = \eta_x dx + \eta_y dy = 0$$

implies

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B-\sqrt{B^2-4AC}}{2A}\right) \quad (14)$$

Integrating (13) & (14) we get the equations of family of curves $\xi(x,y) = C_1$ & $\eta(x,y) = C_2$ which are called characteristics of PDE (1).

PDE

Ex $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$

$A = 3,\ B = 10,\ C = 3$

$B^2 - 4AC = 100 - 4 \cdot 3 \cdot 3 = 100 - 36 = 64 > 0$

Hence this is Hyperbolic PDE. and Characteristic equations

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right)$$

and

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right)$$

$$\frac{dy}{dx} = -\left(\frac{-10 + 8}{6}\right) = \frac{2}{6} = 1/3 \quad \text{--- (1)}$$

and

$$\frac{dy}{dx} = -\left(\frac{-10 - 8}{6}\right) = 3 \quad \text{--- (2)}$$

from (1) $y = \frac{1}{3}x + c_1$ --- (3)

$c_1 = y - x/3$

$$\boxed{\xi(x,y) = y - x/3 = c_1}$$

from (2) $y = 3x + c_2$ --- (4)

$c_2 = y - 3x$

$$\boxed{\eta(x,y) = y - 3x = c_2}$$

Characteristics

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 3\left(-\tfrac{1}{3}\right)^2 + 10\left(-\tfrac{1}{3}\right)(1) + 3(1)$$

$$= +\frac{1}{3} - \frac{10}{3} + 3$$

$$= -\frac{9}{3} + 3$$

$$= -3 + 3 = 0$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$= 3.9 + 10(-3)(1) + 3.(1)^2$$

$$= 27 - 30 + 3 = 0$$

$$\bar{C} = 0$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2.\cancel{3}.(-1/\cancel{3})(-3) + 10\left[(-1/3)(1) + (1)(-3)\right]$$

$$+ 2 3.(1)(1)$$

$$= +6 - 10\left[\frac{1+9}{3}\right] + 6$$

$$= 12 - \frac{100}{3} = \frac{36-100}{3} = \frac{-64}{3}$$

Hence the required canonical form is

$$\frac{64}{3}u_{\xi\eta} = 0 \quad \text{or} \quad u_{\xi\eta} = 0$$

Canonical form for Parabolic Eq

$\bar{B}^2 - 4\bar{A}\bar{C} = 0$ which can be true if $\bar{B} = 0$ and $\bar{A}$ or $\bar{C}$ is equal to zero. Let $\bar{A} = 0$

$$\bar{A} = 0 \Rightarrow \bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

$$\frac{\xi_x}{\xi_y} = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

Since $\bar{B}^2 - 4\bar{A}\bar{C} = 0 \Rightarrow B^2 - 4AC = 0$

Thus

$$\frac{\xi_x}{\xi_y} = \frac{-B}{2A}$$

and from the curve $\xi(x,y) = c_1$

$d\xi = \xi_x dx + \xi_y dy = 0$ implies

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B}{2A} \qquad \text{———} \quad (1)$$

$\bar{A} = 0 \Rightarrow \bar{B} = 0$ which can be seen as follows (*)

Now from (1)

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = \frac{B}{2A}$$ which gives the family of curve $\xi(x,y) = c_1$ and $\eta(x,y)$ can be taken such that $\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$

Ex $x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} = e^x$

$A = x^2, \quad B = -2xy, \quad C = y^2$

$B^2 - 4CA = 4x^2y^2 - 4x^2y^2 = 0$

Hence given PDE is parabolic every where.

The characteristic equation is

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{-\cancel{2}xy}{\cancel{2}x^2} = \frac{-y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$\ln y = -\ln x + \ln c$ or $\ln xy = \ln c$

$\boxed{xy = c}$ characteristic

Then $\xi = xy = c$, Now Take $\eta = y$

So.

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y$$

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$
$$= \cancel{x^2y^2} - \cancel{2x^2y^2} + \cancel{y^2x^2} = 0$$

$$\bar{B} = 0$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$
$$= x^2\cdot 0 - 2xy(0)\cdot 1 + y^2\cdot(1)^2$$
$$= y^2$$

$$\bar{D} = -2xy$$

$$\bar{E} = 0,\ \bar{F} = 0,\ \bar{G} = e^x$$

Hence the transformed eq. is

$$y^2 u_{\eta\eta} - 2xy\, u_\xi = e^x$$

or $\boxed{\eta^2 u_{\eta\eta} = 2\xi u_\xi + e^{\xi/\eta}}$

* $\underline{\bar{A}=0 \Rightarrow \bar{B}=0}$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \quad \text{— (1)}$$

$$B^2 - 4AC = 0 \Rightarrow B^2 = 4AC$$
$$B = 2\sqrt{AC}$$

Then (1) can be written as

$$\bar{B} = 2A\xi_x\eta_x + 2\sqrt{AC}\,(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$\bar{B} = 2(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y)(\sqrt{A}\,\eta_x + \sqrt{C}\,\eta_y) \quad \text{— (2)}$$

and we have

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} = -\frac{\cancel{2}\sqrt{AC}}{\cancel{2}A} = -\sqrt{\frac{C}{A}}$$

$$\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y = 0$$

Thus from (2) we get

$$\bar{B} = 0$$

So $\bar{A} = 0 \Rightarrow \bar{B} = 0$

We therefore chose ξ in such a way that both $\bar{A}$ and $\bar{B}$ are zero. Then η can be chosen any way we like as long as it is not parallel to ξ-coordinate. In other words, we choose η such that the Jacobian of the transform is not zero.