Single step methods

Consider the IVP

$$y' = f(x,y), \quad y(x_0) = y_0 \qquad \text{——(1)}$$

The methods for solution of initial value problem(1) can be classified mainly in two types.
(i) Single Step methods (ii) Multistep methods

(i) Single step methods: The solution at any ~~step~~ point is obtained by using the solution at previous point. Thus a general single step method can be written as

$$y_{n+1} = y_n + h\,\phi(x_{n+1}, x_n, y_n, y_{n+1}; h) \qquad \text{——(2)}$$

where ϕ, dependents on f also, is called increment function.

Explicit method If ϕ is independent of u_{n+1}, then the method is called explicit and (2) may be written as

$$y_{n+1} = y_n + h\,\phi(x_n, u_n, h)$$

Implicit method: If ϕ depends on u_{n+1} then the method is called implicit.

Local truncation error or Discretization error:
The exact solution $u(x_n)$ satisfies the equation

$$u(x_{n+1}) = u(x_n) + h\,\phi(x_{n+1}, x_n, u(x_{n+1}), u(x_n), h) + T_{n+1}$$

where T_{n+1} is called the local truncation error or discretization error of the method. Therefore the truncation error is given by

$$T_{n+1} = u(x_{n+1}) - u(x_n) - h\,\phi(x_{n+1}, x_n, u(x_{n+1}), u(x_n); h)$$

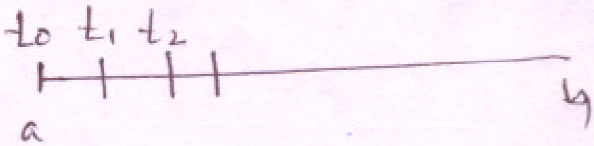
<u>Order of the method</u>: The order of a method is the largest integer p for which

$$\left|\frac{1}{h} T_{n+1}\right| = O(h^p)$$

<u>Taylor series method</u>: Consider the IVP

$$u' = f(t,u), \quad u(t_0) = \eta_0 \quad t \in [a,b]$$

$$t_j = t_0 + jh, \quad h = \frac{b-a}{N}$$



$$u(t_{j+1}) = u(t_j + h)$$
$$= u(t_j) + h u'(t_j) + \frac{h^2}{2!} u''(t_j) + \frac{h^3}{3!} u'''(t_j) + \cdots + \frac{1}{p!} h^p u^{(p)}(t_j) + T_{j+1}$$

$$T_{j+1} = \frac{h^{p+1}}{(p+1)!} u^{(p+1)}(t_j + \theta h) \quad \text{where } 0 < \theta < 1$$

Now denote $u_j = u(t_j)$

$$u_{j+1} = u_j + h u_j' + \frac{h^2}{2!} u_j'' + \cdots + \frac{h^p}{p!} u_j^{(p)} + \frac{h^{p+1}}{(p+1)!} u^{(p+1)}(t_j + \theta h) \quad \text{——— (1)}$$

<u>Now define</u>

$$h\,\phi(t_j, u_j, h) = h u_j' + \frac{h^2}{2!} u_j'' + \cdots + \frac{h^p}{p!} u_j^{(p)}$$

then (1) can be written as

$$u_{j+1} = u_j + h\,\phi(t_j, u_j, h) + T_{j+1} \quad \text{--- (2)}$$

where

$$T_{j+1} = \frac{h^{p+1}}{(p+1)!}\, u^{(p+1)}(t_j + \theta h) \quad \text{for some } 0 < \theta < 1 \quad \text{--- (3)}$$

Thus the method (2) is called Taylor series method of order p.

Euler method for $p = 1$

$$u_{j+1} = u_j + h u_j' = u_j + h f(t_j, u_j)$$

Note: To use Taylor series method we require $u', u'', u''', u^{(iv)}$ and so on.

$$u' = f(t, u)$$

$$u'' = \frac{d}{dt} f(t, u) = f_t + f_u \cdot u' = f_t + f_u \cdot f$$

$$u''' = \frac{d}{dt} u'' = \frac{d}{dt}\left[f_t(t,u) + f_u(t,u) \cdot f(t,u)\right]$$

$$= f_{tt} + f_{tu} u' + (f_{ut} + f_{uu} u') f + f_u (f_t + f_u u')$$

if $f_{tu} = f_{ut}$

$$= f_{tt} + f_{tu} f + f_{tu} f + f_{uu} \cdot f^2 + f_t f_u + f_u^2 f$$

$$u''' = f_{tt} + 2 f_{tu} f + f_u (f_t + f \cdot f_u) + f_{uu} f^2$$

$\vdots$

The error in (2) is given by

$$T_{j+1} = \frac{h^{p+1}}{(p+1)!} u^{(p+1)}(t_j+\theta h), \quad \text{for some } 0<\theta<1$$

The number of terms to be included in (2) is fixed by permissible error. If the error is ϵ and $f^{(p)}(t_j+\theta h)$ is bounded then

$$|T_{j+1}| = \frac{h^{p+1}}{(p+1)!} \left| u^{(p+1)}(t_j+\theta h)\right| < \epsilon$$

or $$\frac{h^{p+1}}{(p+1)!} |f^{(p)}(t_j+\theta h)| < \epsilon$$

or $$h^{p+1} |f^{(p)}(t_j+\theta h)| < \epsilon \cdot (p+1)! \quad \text{(4)}$$

Inequality (4) will determine p as ϵ, h are known.

Example: Solve the differential equation

$$y' = t+y \qquad y(0)=1, \quad t\in[0,1]$$

by Taylor's series method. Determine the number of terms to be included in Taylor's series to obtain an accuracy of 10^{-10}.

Sol^{n}

$$u_{j+1} = u_j + h u_j' + \frac{h^2}{2!} u_j'' + \frac{h^3}{3!} u_j''' + \cdots + \frac{h^p}{p!} u_j^{(p)} + \frac{h^{p+1}}{(p+1)!} u_j^{(p+1)}(t_j+\theta h)$$

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2!} y_j'' + \frac{h^3}{3!} y_j''' + \cdots + \frac{h^p}{p!} y_j^{(p)} + \frac{h^{p+1}}{(p+1)!} y_j^{(p+1)}(t_j+\theta h)$$

$$y' = t + y \qquad y(0) = 1 \Rightarrow y'(0) = y(0) = 1$$
$$y'' = 1 + y' \qquad y''(0) = 1 + y'(0) = 2$$
$$y''' = y'' \qquad y'''(0) = y''(0) = 2$$
$$y^{(4)} = y''' \qquad y^{(4)}(0) = y'''(0) = y''(0) = 2$$
$$\text{so } y^{(r+1)} = y^{(r)} \qquad y^{(r+1)}(0) = y^{(r)}(0) = 2$$
$$r = 3, 4, 5, \ldots$$

$$y(t) = y(0+t)$$
$$= y(0) + t\,y'(0) + \frac{t^2}{2!}y''(0) + \cdots \frac{t^p}{p!}y^{(p)}(0) + \frac{t^{p+1}}{(p+1)!}y^{(p+1)}(\xi)$$
$$0 < \xi < t$$

$$y(t) = 1 + t + 2\frac{t^2}{2!} + 2\frac{t^3}{3!} + \cdots + 2\frac{t^p}{p!} + \frac{t^{p+1}}{(p+1)!}y^{(p+1)}(\xi)$$

$$y^{(p+1)}(\xi) = f^{(p)}(\xi) \qquad f^{(p)}(\xi) = f^{(p)}(\xi, y(\xi))$$

$$f = t + y$$
$$f' = 1 + y' = 1 + t + y$$
$$f'' = 1 + y' = 1 + t + y$$
$$\vdots$$
$$f^{(p)} = 1 + t + y$$
$$= 1 + t + 1 + t + \frac{2t^2}{2!} + \frac{2t^3}{3!} + \cdots + \frac{2t^p}{p!}$$
$$= 2\left[1 + t + \frac{t^2}{2!} + \cdots + \frac{t^p}{p!}\right]$$

$$|f^{(p)}(\xi)| = 2\left|1 + t + \frac{t^2}{2!} + \cdots + \frac{t^p}{p+1}\right|$$
$$\le 2\left|1 + t + \frac{t^2}{2!} + \cdots\right|$$
$$= 2e^t$$

so $\text{error} = \left| \dfrac{t^{p+1}}{(p+1)!} y^{(p+1)}(z) \right| < 10^{-10}$

So we want

$$\max \left| \frac{t^{p+1}}{(p+1)!} y^{(p+1)}(z) \right| < 10^{-10}$$

$$\frac{2e}{(p+1)!} < 10^{-10}$$

for $p = 15$ $\quad \dfrac{2 \times 3}{15!} = .4508 \times 10^{-11}$

$p \simeq 15$