# Boundary Value Problems

$$u'' = f(x,u), \qquad a \le x \le b$$

with b.c.

(1.) **B.C of 1st kind**

$$u(a) = r_1, \quad u(b) = r_2$$

(2) 2nd Kind $u'(a) = r_1, \quad u'(b) = r_2$

(3) Third Kind (Mixed kind)

$$a_0 u(a) - a_1 u'(a) = r_1$$
$$b_0 u(b) + b_1 u'(b) = r_2$$

where $a_0, a_1, b_0, b_1$ are constants such that

$$a_0 a_1 \geq 0, \qquad |a_0| + |a_1| \neq 0$$
$$b_0 b_1 \geq 0, \qquad |b_0| + |b_1| \neq 0 \text{ and } |a_0| + |b_0| \neq 0$$

**Shooting Method**

Consider linear 2nd order differential eq of the form

$$-u'' + p(x)u' + q(x)u = r(x) \qquad a < x < b \qquad —(1)$$

subject to boundary conditions. We assume that $p(x), q(x) > 0$ and $r(x)$ is continuous on $[a,b]$ so that bvp has unique solution.

The general solution of (1) can be written as

$$u(x) = u_0(x) + \mu_1 u_1(x) + \mu_2 u_2(x) \qquad —(2)$$

where $u_0(x)$ is a particular solution of the non-homogeneous equation (1) and $u_1(x)$ and $u_2(x)$ are two linearly independent solution of corresponding homogeneous equation

$$-u'' + p(x)u' + q(x)u = 0$$

BVP



Thus we have

$$-u_0'' + p(x)\, u_0' + q(x)\, u_0 = r(x) \qquad \text{——} \quad (3)$$

$$-u_1'' + p(x)\, u_1' + q(x)\, u_1 = 0 \qquad \text{——} \quad (4)$$

and

$$-u_2'' + p(x)\, u_2' + q(x)\, u_2 = 0 \qquad \text{——} \quad (5)$$

Now first we discuss B.C. of 1st kind

i.e.,
$$u(a) = r_1 \qquad u(b) = r_2 \qquad \text{——} \quad (6)$$

Case(i) Let $r_1 \neq 0$ then we choose

$$u_0(a) = u_1(a) = u_2(a) = r_1 \qquad \text{——} \quad (7)$$

and
$$u_0'(a) = 0, \quad u_1'(a) = 1 \quad u_2'(a) = 0 \qquad \text{——} \quad (8)$$

other choices of linearly independent values can be taken.

Now we consider following three initial value problems

I
$$-u_0'' + p(x)\, u_0' + q(x)\, u_0 = r(x)$$
$$u_0(a) = r_1, \quad u_0'(a) = 0$$

II
$$-u_1'' + p(x)\, u_1' + q(x)\, u_1 = 0$$
$$u_1(a) = r_1, \quad u_1'(a) = 1$$

III
$$-u_2'' + p(x)\, u_2' + q(x)\, u_2 = 0$$
$$u_2(a) = r_1 \quad u_2'(a) = 0$$

We solve all these IVPs. using some initial value methods with same step length and obtain $u_0(b)$, $u_1(b)$ and $u_2(b)$.

BVP

The general soln $u(x)$ of non-homogeneous eq (1) is given by

$$u(x) = u_0(x) + \mu_1 u_1(x) + \mu_2 u_2(x) \qquad \text{---} (9)$$

Now at $x=a$ $u(a) = r_1$ implies (from (9))

$$r_1 = u(a) = u_0(a) + \mu_1 u_1(a) + \mu_2 u_2(a) \qquad \text{---} (10)$$

$$r_1 = r_1 + \mu_1 r_1 + \mu_2 r_1$$

or $$\mu_1 + \mu_2 = 0 \qquad \text{---} (i)$$

and at $x=b$

$$r_2 = u(b) = u_0(b) + \mu_1 u_1(b) + \mu_2 u_2(b)$$

from (i) $\boxed{\mu_1 = -\mu_2}$

$$r_2 = u_0(b) - \mu_2 u_1(b) + \mu_2 u_2(b)$$

$$r_2 - u_0(b) = \mu_2 (u_2(b) - u_1(b))$$

or $\boxed{\mu_2 = \dfrac{r_2 - u_0(b)}{u_2(b) - u_1(b)}}$ provided $u_2(b) \neq u_1(b)$

So $\mu_1$ & $\mu_2$ are known then $u(x)$ is known.

<u>Case 2</u> $u(a) = r_1 = 0$ $u(b) = r_2$

Then we take following set of initial conditions

$$u_0(a) = 0 \quad | \quad u_1(a) = 1 \quad | \quad u_2(a) = 0$$
$$u_0'(a) = 0 \quad | \quad u_1'(a) = 0 \quad | \quad u_2'(a) = 1$$

Then we solve three IVPs

I $-u_0'' + p(x)\, u_0' + q(x)\, u_0 = r(x)$

$u_0(a) = 0, \quad u_0'(a) = 0$

II $-u_1'' + p(x)\, u_1' + q(x)\, u_1 = 0$

$u_1(a) = 1, \quad u_1'(a) = 0$

III $-u_2'' + p(x)\, u_2' + q(x)\, u_2 = 0$

$u_2(a) = 0, \quad u_2'(a) = 1$

Solve these IVPs for same step length

The general soln

$$u(x) = u_0 + \mu_1 u_1(x) + \mu_2 u_2(x)$$

at $x = a$

$$0 = u(a) = u_0(a) + \mu_1 u_1(a) + \mu_2 u_2(a)$$

$$0 = 0 + \mu_1 + 0 \Rightarrow \mu_1 = 0$$

at $x = b$

$$r_2 = u(b) = u_0(b) + \mu_1 u_1(b) + \mu_2 u_2(b)$$

$$r_2 - u_0(b) = \mu_2 u_2(b)$$

$$\mu_2 = \frac{r_2 - u_0(b)}{u_2(b)} \quad \text{provided } u_2(b) \neq 0.$$

Boundary conditions of 2nd Kind

$$u'(a) = r_1, \quad u'(b) = r_2 \qquad \text{——} \quad (1)$$

Here since $u'(a)$ is given so we guess value for $u(a)$.

Case (1) $r_1 \neq 0$

$$u_0'(a) = u_1'(a) = u_2'(a) = r_1$$

$$u_0(a) = 0, \quad u_1(a) = 1, \quad u_2(a) = 0$$

and solve following three IVPs

I

$$-u_0'' + p(x)\, u_0' + q(x)\, u_0 = r(x)$$

$$u_0(a) = 0, \quad u_0'(a) = r_1$$

II

$$-u_1'' + p(x)\, u_1' + q(x)\, u_1 = 0$$

$$u_1(a) = 1, \quad u_1'(a) = r_1$$

III

$$-u_2'' + p(x)\, u_2' + q(x)\, u_2 = 0$$

$$u_2(a) = 0, \quad u_2'(a) = r_1$$

Now the general soln will be written as

$$u(x) = u_0(x) + \mu_1 u_1(x) + \mu_2 u_2(x)$$

at $x=a$ and at $x=b$ provides

$$\mu_1 + \mu_2 = 0 \qquad \text{——} \quad (i)$$

$$\mu_2 = \frac{r_2 - u_0'(b)}{u_2'(b) - u_1'(b)} \qquad u_1'(b) \neq u_2'(b).$$

Case (2) $r_1 = 0$ the we take following set of initial values

$$u_0(a) = 0 \qquad u_1(a) = 1 \qquad u_2(a) = 0$$
$$u_0'(a) = 0 \qquad u_1'(a) = 0 \qquad u_2'(a) = 1$$

$$u(x) = u_0(x) + \mu_1 u_1(x) + \mu_2 u_2(x)$$

$$\mu_2 = 0,$$

$$\mu_1 = \frac{r_2 - u_0'(b)}{u_1'(b)} \qquad u_1'(b) \neq 0$$

Boundary condition of 3rd Kind

$$a_0 u(a) - a_1 u'(a) = r_1 \qquad \text{—(i)}$$
$$b_0 u(b) + b_1 u'(b) = r_2 \qquad \text{—(ii)}$$

we consider following set of initial conditions

$$u_0(a) = 0 \quad\Big|\quad u_1(a) = 1 \quad\Big|\quad u_2(a) = 0$$
$$u_0'(a) = 0 \quad\Big|\quad u_0'(a) = 0 \quad\Big|\quad u_2'(a) = 1$$

and general solⁿ is written by

$$u(x) = u_0(x) + \mu_1 u_1(x) + \mu_2 u_2(x) \qquad \text{—(1)}$$

Now at $x = a$ & at $x = b$ (1) satisfies (i) & (ii) which will give two equations in $\mu_1$ & $\mu_2$ so $\mu_1$ & $\mu_2$ can be determined.

## Shooting method

**Prob** Using the shooting method, solve the bvp

$$u'' = u + 1, \quad 0 < x < 1 \quad \text{——(1)}$$

$$u(0) = 0, \quad u(1) = e - 1 \quad \text{——(2)}$$

**Soln**

We will consider three ivp's as follows

I $\quad u_0'' = u_0 + 1, \quad u_0(0) = 0, \quad u_0'(0) = 0 \quad$ ——(3)

II $\quad u_1'' = u_1 \quad u_1(0) = 1 \quad u_1'(0) = 0 \quad$ ——(4)

III $\quad u_2'' = u_2 \quad u_2(0) = 0 \quad u_0'(0) = 1 \quad$ ——(5)

We first write (3) (4) & (5) as following systems

$$u_0'' = u_0 + 1 \qquad u_0(0) = 0 \Rightarrow Y_0(0) = 0, \quad u_0'(0) = 0 \Rightarrow Z_0(0) = 0$$

$Y_0 = u_0, \quad Z_0 = u_0'$

$$u_0 = Y_0, \quad u_0' = Z_0 \quad \Rightarrow Y_0' = Z_0$$

$$Z_0' = u_0'' = u_0 + 1 \quad \Rightarrow Z_0' = Y_0 + 1$$

$$\begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}' = \begin{pmatrix} Z_0 \\ 1 + Y_0 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} Y_0(0) \\ Z_0(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{——(6)}$$

Similarly for (4) & (5) we get

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix}' = \begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} \qquad \begin{pmatrix} Y_1(0) \\ Z_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{——(7)}$$

where $Y_1 = u_1$, $Z_1 = u_1'$

$$\begin{pmatrix} Y_2 \\ Z_2 \end{pmatrix}' = \begin{pmatrix} Y_2 \\ Z_2 \end{pmatrix} \qquad \begin{pmatrix} Y_2(0) \\ Z_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{——(8)}$$

$Y_2 = u_2$

$Z_2 = u_2'$

Shooting method

Euler-Cauchy method

$$u_{n+1} = u_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_n, u_n), \quad k_2 = hf(x_n + h, u_n + k_1)$$

$= + l_1$

For 1st system (6)

$$y_0' = z_0 \qquad y_0(0) = 0 \qquad y_0' = f(x, y_0, z_0)$$

$$z_0' = 1 + y_0 \qquad z_0(0) = 0 \qquad z_0' = g(x, y_0, z_0)$$

$$f = z_0, \quad g = 1 + y_0$$

$$f = z_0 \qquad \begin{pmatrix} k_1 = hf(x_n, y_{0n}, z_{0n}) \\ k_2 = hf(x_n + h, y_{0n} + k_1, z_{0n} + l_1) \end{pmatrix}$$

$$k_1 = h\, z_{0,n}$$

$$k_2 = hf(x_n + h, y_{0,n} + k_1, z_{0,n} + l_1)$$

$$= h(z_{0,n} + l_1)$$

$$= h[z_{0n} + h(1 + y_{0,n})]$$

$$= h z_{0n} + h^2(1 + y_{0,n})$$

$$g = 1 + y_0 \qquad l_1 = hg(x_n, y_{0n}, z_{0n}), \quad l_2 = hg(x_n, y_{0n} + k_1, z_{0n} + l_1)$$

$$l_1 = h(1 + y_{0,n})$$

$$l_2 = hg(x_n + h, y_{0,n} + k_1, z_{0,n} + l_1)$$

$$= h(1 + y_{0,n} + k_1)$$

$$= h(1 + y_{0,n} + h z_{0,n})$$

$$y_{0,n+1} = y_{0,n} + \frac{1}{2}[h z_{0,n} + h z_{0,n} + h^2(1 + y_{0,n})]$$

$$= y_{0,n}\left(1 + \frac{h^2}{2}\right) + z_{0,n}\left(\frac{h}{2} + \frac{h}{2}\right) + \frac{h^2}{2}$$

$$y_{0,n+1} = \left(1 + \frac{h^2}{2}\right) y_{0,n} + h\, z_{0,n} + h^2/2 \qquad (9)$$

$$z_{0,n+1} = z_{0,n} + \frac{1}{2}(l_1 + l_2)$$

$$z_{0,n+1} = z_{0,n} + \frac{1}{2}[h(1 + y_{0,n}) + h(1 + y_{0n} + h z_{0,n})]$$

## Shooting method

$$z_{0,n+1} = z_{0n} + \frac{1}{2}\left[ y_{0,n}(2h) + z_{0,n}(h^2) + 2h \right]$$

$$= z_{0,n} + h y_{0n} + \frac{h^2}{2} z_{0,n} + h$$

$$z_{0,n+1} = h y_{0,n} + \left(1 + \frac{h^2}{2}\right) z_{0,n} + h \qquad \text{—— (10)}$$

Now from (9) & (10)

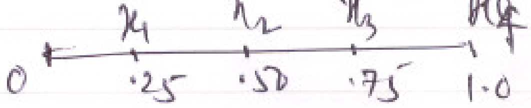
$$\begin{bmatrix} y_{0,n+1} \\ z_{0,n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1+\frac{h^2}{2} & h \\ h & 1+h^2/2 \end{bmatrix}}_{B(h)} \begin{bmatrix} y_{0,n} \\ z_{0,n} \end{bmatrix} + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} \qquad \text{—— (11)}$$

with $y_{0,0} = 0$, $z_{0,0} = 0$ as initial conditions.

The system (2) & (3) immediately can be written as

$$\begin{bmatrix} y_{1,n+1} \\ z_{1,n+1} \end{bmatrix} = \begin{bmatrix} B(h) \end{bmatrix} \begin{bmatrix} y_{1,n} \\ z_{1,n} \end{bmatrix} \qquad y_{1,0} = 1,\ z_{1,0} = 0$$

$$\begin{bmatrix} y_{2,n+1} \\ z_{2,n+1} \end{bmatrix} = B(h) \begin{bmatrix} y_{2n} \\ z_{2n} \end{bmatrix} \qquad y_{2,0} = 0 \quad z_{2,0} = 1$$




Take $h = 0.25$

$$\begin{bmatrix} y_{0,n+1} \\ z_{0,n+1} \end{bmatrix} = \begin{bmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{bmatrix} \begin{bmatrix} y_{0,n} \\ z_{0,n} \end{bmatrix} + \begin{bmatrix} .03125 \\ .25 \end{bmatrix}$$

$y_{0,0} = 0$, $z_{0,0} = 0$

$u_0(.25) = y_{01} = 0.03125$ $\qquad u_0'(0.25) = z_{0,1} = 0.25$

$u_0(0.50) = y_{02} = 0.12598$ $\qquad u_0'(0.50) = z_{02} = 0.51563$

$u_0(0.75) = y_{03} = 0.29007$ $\qquad u_0'(0.75) = z_{03} = .81324$

$u_0(1.0) = y_{04} = 0.53369$ $\qquad u_0'(1.0) = z_{04} = 1.16117$

Next for II IVP we have

$$\begin{bmatrix} y_{1\,n+1} \\ z_{1\,n+1} \end{bmatrix} = \begin{bmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{bmatrix} \begin{bmatrix} y_{1n} \\ z_{1n} \end{bmatrix}$$

with $y_{10} = 1$, $z_{10} = 0$

$u_1(0.25) = y_{11} = 1.03125$ $\qquad u_1'(.25) = z_{11} = 0.25$

$u_1(0.50) = y_{12} = 1.12598$ $\qquad u_1'(.50) = z_{12} = 0.51563$

$u_1(0.75) = y_{13} = 1.29107$ $\qquad u_1'(.75) = z_{13} = .81324$

$u_1(1.0) = y_{14} = 1.53369$ $\qquad u_1'(1.0) = z_{14} = 1.16117.$

Next for III IVP we have

$$\begin{bmatrix} y_{2\,n+1} \\ z_{2\,n+1} \end{bmatrix} = \begin{bmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{bmatrix} \begin{bmatrix} y_{2n} \\ z_{2n} \end{bmatrix}$$

$y_{20} = 0$, $z_{20} = 1$

$u_2(.25) = y_{21} = .25$ $\qquad u_2'(.25) = z_{21} = 1.03125$

$u_2(.50) = y_{22} = 0.51563$ $\qquad u_2'(.50) = z_{22} = 1.12598$

$u_2(.75) = y_{23} = 0.81324$ $\qquad u_2'(0.75) = z_{23} = 1.29107$

$u_2(1.0) = y_{24} = 1.16117$ $\qquad u_2'(1.0) = z_{24} = 1.53369$

$$u(x) = u_0(x) + M_1 u_1(x) + M_2 u_2(x)$$

$u(0) = 0$, $u(1) = e - 1$

$u(0) = 0 \Rightarrow \quad u_0(0) + M_1 u_1(0) + M_2 u_2(0) = 0$

$$= 0 + M_1 1 + M_2 . 0 = 0 \Rightarrow M_1 = 0$$

Shooting method

$u(1) = e-1 \Rightarrow$

$$e-1 = u_0(1) + \mu_1 u_1(1) + \mu_2 u_2(1)$$

but $\mu_1 = 0 \Rightarrow$

$$(e-1) = u_0(1) + \mu_2 u_2(1)$$

or

$$\mu_2 = \frac{(e-1) - u_0(1)}{u_2(1)} = \frac{e-1-.53369}{1.16117}$$

$$= 1.02017$$

Thus

$$\boxed{u(x) = u_0(x) + 1.02017\, u_2(x)}$$

| $x_n$ | $u(x_n)$ |
|---|---|
| 0.25 | 0.20629 |
| 0.50 | 0.65201 |
| 0.75 | 1.11971 |
| 1.0 | 1.71828 |

$u(1) = e-1$