Adams Methods

Backward differences

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla^{r+1} f(x) = \nabla^r f(x) - \nabla^r f(x-h)$$

$$\nabla^2 f(x) = \nabla \nabla f(x)$$
$$= \nabla (f(x) - f(x-h))$$
$$= \nabla f(x) - \nabla f(x-h)$$

$$\nabla f(x_k) = f(x_k) - f(x_{k-1})$$
$$= (x_k - x_{k-1}) f[x_k, x_{k-1}]$$
$$= h f[x_k, x_{k-1}]$$

$$f[x_k, x_{k-1}] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$f(x_k) - f(x_{k-1}) = (x_k - x_{k-1}) f[x_k, x_{k-1}]$$

and in general

$$\nabla^r f(x_k) = r!\, h^r f[\underbrace{x_k, x_{k-1}, \ldots x_{k-r}}_{(r+1)\text{ points}}] \quad (1)$$

$$f(x) = f(x_n) + (x - x_n) f[x_n, x_{n-1}] + (x-x_n)(x-x_{n-1}) f[x_n, x_{n-1}, x_{n-2}]$$
$$+ \cdots + (x-x_n)(x-x_{n-1}) \cdots (x-x_1) f[x_n, x_{n-1}, \ldots x_0] + E(x) \quad (2)$$

$$E(x) = (x - x_n)(x - x_{n-1}) \cdots (x - x_0) \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\min\{x_0, x_1, \ldots x_n, x\} < \xi < \max\{x_0, x_1, \ldots x_n, x\}$$

Now using (1) we can write (2) as

$$f(x) = f_n + (x - x_n)\frac{\nabla f_n}{h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 f_n}{2!\, h^2} +$$
$$+ (x-x_n)(x-x_{n-1})(x-x_{n-2})\frac{\nabla^3 f_n}{3!\, h^3} + \cdots$$
$$+ (x-x_n)(x-x_{n-1}) \cdots (x-x_1)\frac{\nabla^2 f_n}{n!\, h^n} + E(x).$$

Now write $s = \dfrac{x - x_n}{h}$ or $\boxed{x = x_n + hs}$ (3)

or $(x - x_n) = hs$

$$f(x) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \cdots + \frac{s(s+1)\cdots(s+n-1)}{n!}\nabla^n f_n + E_s$$

$$E(x) = (x-x_n)(x-x_{n-1})\cdots(x-x_0)\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$= h^{n+1}\frac{s(s+1)\cdots(s+n)}{(n+1)!}f^{(n+1)}(\xi)$$

$$= \frac{h^{n+1}}{(n+1)!}s(s+1)\cdots(s+n)f^{(n+1)}(\xi).$$

$x - x_n = hs$
$x - x_{n-1} = h(s+1)$
$\vdots$
$x - x_0 = x - x_{n-n} = h(s+n)$

$$y' = f(x, y(x))$$

Now integrate from $x_n \to x_{n+1}$

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(t, y(t))\,dt \qquad (*)$$

<u>Adams-Bashforth methods</u> Let $P_p(t)$ denote the polynomial of degree $\le p$ which interpolates $y'(t)$ at $x_{n-p}, \ldots, x_n$. The most convenient form for $P_p(t)$ will be the Newton Backward difference formula expanded about x_n.

$$P_p(t) = y_n' + \frac{(t-x_n)}{h}\nabla y_n' + \frac{(t-x_n)(t-x_{n-1})}{2!\,h^2}\nabla^2 y_n' + \cdots + \frac{(t-x_n)(t-x_{n-1})\cdots(t-x_{n-p+1})}{p!\,h^p}\nabla^p y_n'$$

and the error $E_p(t)$ is given by

$$E_p(t) = \frac{(t-x_n)(t-x_{n-1})\cdots(t-x_{n-p})}{(p+1)!}\left(y'\right)^{(p+1)}(\xi)$$

③

$$E_p(t) = \frac{(t-x_{n-p}) \cdots (t-x_n)}{(p+1)!} y^{(p+2)}(\xi)$$

$$x_{n-p} \le \xi \le x_{n+1}$$

provided $x_{n-p} \le t \le x_{n+1}$ and $y(x)$ is $(p+2)$ times differentiable

Now

$$\int_{x_n}^{x_{n+1}} P_p(t)\,dt = \sum_{j=0}^{p} \frac{1}{j!\,h^j} \nabla^j y_n' \int_{x_n}^{x_{n+1}} (t-x_n) \cdots (t-x_{n+1-j})\,dt$$

$$= \sum_{j=0}^{p} \frac{1}{j!\,\cancel{h^j}} \nabla^j y_n' \int_0^1 \cancel{h^j}\, s(s+1) \cdots (s+j-1) \cdot h\,ds$$

$$= h \sum_{j=0}^{p} \nabla^j y_n' \frac{1}{j!} \int_0^1 s(s+1) \cdots (s+j-1)\,ds$$

$t - x_n = hs$

$dt = h\,ds$

$t - x_{n-0} = hs$

$t - x_{n-1} = h(s+1)$

$\vdots$

$t - x_{n-(j-1)} = h(s+j-1)$

$r_0 = \int_0^1 ds$

$r_1 = \int_0^1 s\,ds$

$r_2 = \int_0^1 s(s+1)\,ds$

Take

$$\boxed{r_j = \frac{1}{j!} \int_0^1 s(s+1) \cdots (s+j-1)\,ds}$$

$$\int_{x_n}^{x_{n+1}} P_p(t)\,dt = h \sum_{j=0}^{p} r_j \nabla^j y_n' \qquad \text{——— (1)}$$

Now the error

$$T_{n+1}(y) = \int_{x_n}^{x_{n+1}} \frac{(t-x_n)(t-x_{n-1}) \cdots (t-x_{n-p})}{(p+1)!} y^{(p+2)}(\xi_n)\,dt$$

$$= \frac{y^{(p+2)}(\xi)}{(p+1)!} \int_0^1 h^{p+1} s(s+1) \cdots (s+p)\, h\,ds$$

$$= h^{p+2} y^{(p+2)}(\xi) \frac{1}{(p+1)!} \int_0^1 s(s+1) \cdots (s+p)\,ds$$

$$T_{n+1}(y) = r_{p+1} h^{p+2} y^{(p+2)}(\xi)$$

Now putting this interpolation for y' in ⊛ we get

$$\boxed{y_{n+1} = y_n + h \sum_{j=0}^{p} r_j \nabla^j y_n' + r_{p+1} h^{p+2} y^{(p+2)}(\xi)}$$

and corresponding numerical method is given by

$$y_{n+1} = y_n + h \sum_{j=0}^{p} r_j \nabla^j y_n' \qquad n \geq p$$

In the formula $y_j' = f(x_j, y_j)$, $\nabla y_j' = y_j' - y_{j-1}'$

Adam-Bashforth methods

for $p = 0$ (order 1)

$$y_{n+1} = y_n + h y_n' + \frac{1}{2} h^2 y''(\xi) \quad \text{Euler Method.}$$

for $p = 1$ (order 2)

$$y_{n+1} = y_n + \frac{h}{2}(3y_n' - y_{n-1}') + \frac{5}{12} h^3 y'''(\xi)$$

for $p = 2$ (order 3)

$$y_{n+1} = y_n + \frac{h}{12}[23 y_n' - 16 y_{n-1}' + 5 y_{n-2}'] + \frac{3}{8} h^4 y^{(4)}(\xi)$$

for $p = 3$ (Order 4)

$$y_{n+1} = y_n + \frac{h}{24}[55 y_n' - 59 y_{n-1}' + 37 y_{n-2}' - 9 y_{n-3}'] + \frac{251}{720} h^5 y^{(5)}(\xi).$$

r_0	r_1	r_2	r_3	r_4	r_5
1	1/2	5/12	3/8	251/720	95/288

(4)+

Adams - Moulton methods

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(t, y(t))\,dt$$

Now interpolate y' at $(p+1)$ points $x_{n+1}, \dots, x_{n-(p-1)} = x_{n-p+1}$

$$p_p(t) = y'_{n+1} + \frac{t-x_{n+1}}{h}\nabla y'_{n+1} + \frac{(t-x_{n+1})(t-x_n)}{2!\,h^2}\nabla^2 y'_{n+1}$$

$$+ \frac{(t-x_{n+1})\cdots(t-x_{n-p+2})}{p!}\nabla^p y'_{n+1} + \frac{(t-x_{n+1})(t-x_n)\cdots(t-x_{n-p+1})}{(p+1)!}(y')^{p+1}(\xi)$$

Then,

$$\int_{x_n}^{x_{n+1}} p_p(t)\,dt = \sum_{j=0}^{p} \frac{1}{j!\,h^j}\nabla^j y'_{n+1}\int_{x_n}^{x_{n+1}} (t-x_{n+1})\cdots(t-x_{n-j+2})\,dt \quad \text{(*)}$$

Now taking $t - x_n = hs \quad dt = h\,ds$

$$t - x_{n+1} = t-(x_n+h) = t - x_n - h = h(s-1)$$

$$t - x_n = hs$$

$$t - x_{n-1} = h(s+1) \cdots$$

{p terms: x_{n+1}, x_{n+0}, $x_{n-(p-2)}$

$$\int_{x_n}^{x_{n+1}} (t-x_{n+1})\cdots(t-x_{n-j+2})\,dt =$$

$$= \int_0^1 h^j (s-1)s(s+1)\cdots(s-j+2)\,h\,ds \quad \text{(**)}$$

So

$$\int_{x_n}^{x_{n+1}} p_p(t)\,dt = h\sum_{j=0}^{p}\frac{1}{j!}\nabla^j y'_{n+1}\int_0^1 (s-1)s(s+1)\cdots(s-j+2)\,ds$$

$$= h\sum_{j=0}^{p}\delta_j\,\nabla^j y'_{n+1}$$

$$\delta_j = \frac{1}{j!}\int_0^1 (s-1)s(s+1)\cdots(s-j+2)\,ds$$

$$\delta_0 = \int_0^1 ds = 1, \quad \delta_1 = \frac{1}{1!}\int_0^1 (s-1)\,ds, \quad \delta_2 = \frac{1}{2!}\int_0^1 (s-1)\cdot s\,ds \quad \cdots$$

Adams-Moulton methods: Again we use the integral formula

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(t, y(t))\, dt \qquad \text{——} \quad (*)$$

$x_{n-(p-1)} \cdots x_n, x_{n+1}$

Now interpolate $y'(t) = f(t, y(t))$ at $p+1$ points $x_{n+1}, \ldots, x_{n-p+1}$ for $p \geq 0$ and following exactly same process we get

$$y_{n+1} = y_n + h \sum_{j=0}^{p} \delta_j \nabla^j y'_{n+1} + \delta_{p+1} h^{p+2} y^{(p+2)}(\xi_n) \qquad \text{——} \quad (1)$$

$x_{n-p+1} \leq \xi_n \leq x_{n+1}$. The coefficients δ_j are defined by

$$\delta_j = \frac{1}{j!} \int_0^1 (s-1)s(s+1) \cdots (s+j-2)\, ds \qquad j \geq 1$$

with $\delta_0 = 1$. The numerical method associated with (1) is given by

$$y_{n+1} = y_n + h \sum_{j=0}^{p} \delta_j \nabla^j y'_{n+1} \qquad n \geq p-1$$

with $y'_j = f(x_j, y_j)$ This is implicit method

Adams-Moulton formulas

$p=0$	$y_{n+1} = y_n + h y'_{n+1} - \frac{1}{2} h^2 y''(\xi_n)$
$p=1$	$y_{n+1} = y_n + \frac{h}{2}(y'_{n+1} + y'_n) - \frac{1}{12} y'''(\xi_n)$ Trapezoidal method.
$p=2$	$y_{n+1} = y_n + \frac{h}{12}(5y'_{n+1} + 8y'_n - y'_{n-1}) - \frac{1}{24} h^4 y^{(4)}(\xi_n)$
$p=3$	$y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}] - \frac{19}{720} h^5 y^{(5)}(\xi_n)$

Milne-Simpson Method (Integration is from $x_{n-1} \to x_{n+1}$ instead $x_n \to x_{n+1}$)

For this method we integrate $y' = f(x,y)$ from x_{n-1} to x_{n+1}

$$Y_{n+1} = Y_{n-1} + \int_{x_{n-1}}^{x_n} y'(t)\,dt$$

Then interpolate $y'(t)$ at $(p+1)$ points $x_{n+1}, \ldots x_{n-(p+1)}$ and from (4)$^+$, (*) can be written as

$$\int_{x_{n-1}}^{x_{n+1}} P_p(t) = \sum_{j=0}^{p} \frac{1}{j!\,h^j} \nabla^j Y_{n+1} \int_{x_{n-1}}^{x_{n+1}} (t-x_{n+1})(t-x_n)\cdots(t-x_{n-j+2})\,dt$$

writing $t - x_n = hs$, $t = x_{n-1}$, $s = -1$

$$x_{n-1} - x_n = hs$$
$$-h = hs$$

so then integral in (**)

$$\int_{x_{n-1}}^{x_{n+1}} (t-x_{n+1})\cdots(t-x_{n-j+2})\,dt = h^{j+1}\int_{-1}^{1} (s-1)(s)(s+1)\cdots(s-j+2)\,ds$$

$$Y_{n+1} = Y_{n-1} + h\sum_{j=0}^{p} \delta_j \nabla^j Y'_{n+1}$$

$$\delta_0 = 2,\quad \delta_1 = -2,\quad \delta_2 = -1/3,\quad \delta_3 = 0,\ \delta_4 = \frac{-1}{90}$$

Method of order 1

$$Y_{n+1} = Y_{n-1} + 2h\,Y'_{n+1}$$

Method of Order 2

$$Y_{n+1} = Y_{n-1} + 2h\,Y'_n$$

Order 3

$$Y_{n+1} = Y_{n-1} + h\left[\frac{1}{3}Y'_{n+1} + \frac{4}{3}Y'_n + \frac{1}{3}Y'_{n-1}\right]$$

or $$Y_{n+1} = Y_{n-1} + \frac{h}{3}\left[Y'_{n+1} + 4Y'_n + Y'_{n-1}\right]$$

Order 4 method

$$y_{n+1} = y_{n-1} + h\left[\frac{29}{90}y'_{n+1} + \frac{124}{90}y'_n + \frac{24}{90}y'_{n-1} + \frac{4}{90}y'_{n-2} - \frac{1}{90}y'_{n-3}\right]$$

or $$y_{n+1} = y_{n-1} + \frac{1h}{90}\left[29y'_{n+1} + 124y'_n + 24y'_{n-1} + 4y'_{n-2} - y'_{n-3}\right]$$

$$y' = f(x, y(x))$$

Now integrate this from $x_n \to x_{n+1}$

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(t, y(t))\,dt \equiv \int_{x_n}^{x_{n+1}} y'(t)\,dt \quad \text{— (1)}$$

<u>Adams-Bashforth methods</u>: Let $p_p(t)$ denote the polynomial of degree $\leq p$ which interpolates $y'(t)$ at $x_{n-p}, \ldots, x_n$. The most convenient form for $p_p(t)$ will be the Newton's backward difference formula expanded about x_n.

Take the points as $x_n, x_{n-1} \ldots x_{n-p}$ ($(p+1)$ no. of pts so polynomial will be of degree $\leq p$.)

then

$$p_p(t) = y_n' + \frac{t - x_n}{h} \nabla y_n' + \frac{(t-x_n)(t-x_{n-1})}{2!\,h^2} \nabla^2 y_n' + \cdots + \frac{(t-x_n)\cdots(t-x_{n-p+1})}{p!\,h^p} \nabla^p y_n' \quad \text{— (2)}$$

and the error $E_p(t)$ is given by

$$E_p(t) = \frac{(t-x_n)(t-x_{n-1})\cdots(t-x_{n-p})}{(p+1)!} (y')^{(p+1)}(\xi)$$

$$x_{n-p} < \xi < x_n$$

$$E_p(t) = \frac{(t-x_n)(t-x_{n-1})\cdots(t-x_{n-p})}{(p+1)} y^{(p+2)}(\xi) \quad \text{— (3)}$$

provided $x_{n-p} \leq t \leq x_{n+1}$ and $y(t)$ is $(p+2)$ times differentiable.

Now integrating ② from $x_n \to x_{n+1}$ we get

$$\int_{x_n}^{x_{n+1}} p_p(t)\,dt = y_n' \int_{x_n}^{x_{n+1}} dt + \frac{\nabla y_n'}{h}\int_{x_n}^{x_{n+1}} (t-x_n)\,dt + \frac{\nabla^2 y_n'}{2!h^2}\int_{x_n}^{x_{n+1}} (t-x_n)(t-x_{n-1})\,dt$$

Take $t - x_n = hs$ $\quad t \to x_n \quad s \to 0$

$dt = h\,ds$ $\quad t \to x_{n+1} \quad s \to 1$

$t - x_{n-1} = t - x_n + h = h(s+1)$

$$\int_{x_n}^{x_{n+1}} p_p(t)\,dt = h\,y_n' \int_0^1 ds + \frac{\nabla y_n'}{h}\int_0^1 hs \cdot h\,ds$$

$$+ \frac{\nabla^2 y_n'}{2!h^2}\int_0^1 hs \cdot h(s+1) \cdot h\,ds + \cdots$$

$$= h y_n' \int_0^1 ds + h\nabla y_n' \int_0^1 1 \cdot s\,ds + h\nabla^2 y_n' \frac{1}{2!}\int_0^1 1 \cdot s(s+1)\,ds + \cdots$$

Define $\gamma_0 = \int_0^1 ds$, $\gamma_1 = \int_0^1 1 \cdot s\,ds$

$$\gamma_2 = \frac{1}{2!}\int_0^1 1 \cdot s(s+1)\,ds, \quad \cdots$$

$$\gamma_j = \frac{1}{j!}\int_0^1 1 \cdot s(s+1)\cdots(s+j-1)\,ds$$

Then $$\int_{x_n}^{x_{n+1}} p_p(t)\,dt = h\sum_{j=0}^{p} \gamma_j \nabla^j y_n' \qquad \text{—— (4)}$$

Now the error

$$T_{n+1}(y) = \int_{x_n}^{x_{n+1}} \frac{(t-x_n)(t-x_{n-1})\cdots(t-x_{n-p})}{(p+1)!} y^{(p+2)}(\xi)\,dt$$

$$T_{n+1} = \int_0^1 hs\, h(s+1) \cdots h(s+p)\, h.\, ds\, \frac{y^{p+2}(\xi)}{(p+1)!}$$

$$= h^{(p+2)} y^{(p+2)}(\xi) \frac{1}{(p+1)!} \int_0^1 1.\, s(s+1) \cdots (s+p) ds$$

$$\boxed{T_{n+1} = h^{p+r} \gamma_{p+1} y^{(p+2)}(\xi)} \quad \text{—— (5)}$$

Now putting (4) & (5) in (1) we get

$$Y_{n+1} = Y_n + h \sum_{j=0}^{p} \gamma_j \nabla^j Y_n' + \gamma_{p+1} h^{p+2} y^{(p+2)}(\xi) \quad \text{—— (6)}$$

and corresponding Numerical method is given by

$$y_{n+1} = y_n + h \sum_{j=0}^{p} \gamma_j \nabla^j y_n' \quad \text{—— (7)}$$

and is called Adam Bashforth methods.
In the formula $y_j' = f(x_j, y_j)$

$$\nabla y_j' = y_j' - y_{j-1}'$$

(1) for $p=0$ (order 1 method)

$$Y_{n+1} = Y_n + hY_n' + h^2\gamma_2 Y''(\xi)$$

$$\gamma_1 = \int_0^1 s\, ds = \frac{1}{2}$$

$$Y_{n+1} = Y_n + hY_n' + \frac{1}{2}h^2 y''(\xi)$$ — This is Euler method

Adam Bashforth method

Order 1 ($p=0$)

$$y_{n+1} = y_n + h y_n' + \frac{1}{2} h^2 y''(\xi) \qquad \text{Euler Method}$$

Order 2 ($p = 1$)

$$y_{n+1} = y_n + h[r_0 y_n' + r_1 \nabla y_n'] + r_2 h^3 y'''(\xi)$$

$r_0 = 1, \quad r_1 = \frac{1}{2}, \quad r_2 = \frac{5}{12}$

$$y_{n+1} = y_n + h[y_n' + \frac{1}{2}(y_n' - y_{n-1}')] + \frac{5}{12} h^3 y'''(\xi)$$

$$y_{n+1} = y_n + \frac{h}{2}[3y_n' - y_{n-1}'] + \frac{5}{12} h^3 y'''(\xi)$$

Order 3 ($p = 2$)

$$y_{n+1} = y_n + h[r_0 y_n' + r_1 \nabla y_n' + r_2 \nabla^2 y_n'] + r_3 h^4 y^{(4)}(\xi)$$

$r_0 = 1, \; r_1 = \frac{1}{2}, \; r_2 = \frac{5}{12}, \; r_3 = \frac{3}{8} \qquad \nabla y_n' = y_n' - y_{n-1}'$

$$\nabla^2 y_n' = y_n' - 2y_{n-1}' + y_{n-2}'$$

$$y_{n+1} = y_n + h[y_n' + \frac{1}{2}(y_n' - y_{n-1}') + \frac{5}{12}(y_n' - 2y_{n-1}' + y_{n-2}')] + \frac{3}{8} h^4 y^{(4)}(\xi)$$

$$y_{n+1} = y_n + h[(1 + \frac{1}{2} + \frac{5}{12}) y_n' - (\frac{1}{2} + \frac{10}{12}) y_{n-1}' + \frac{5}{12} y_{n-2}'] + \frac{3}{8} h^4 y^{(4)}(\xi)$$

$$y_{n+1} = y_n + \frac{h}{12}[23y_n' - 16y_{n-1}' + 5y_{n-2}'] + \frac{3}{8} h^4 y^{(4)}(\xi)$$