Elliptic PDE

Laplace Equation (five point formula)

$$\nabla^2 u = 0 \qquad \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i, y_j)} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{(x_i, y_j)} = 0$$

$$\nabla^2 u\Big|_{(x_i, y_j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

Take $\Delta x = \Delta y = h$ then

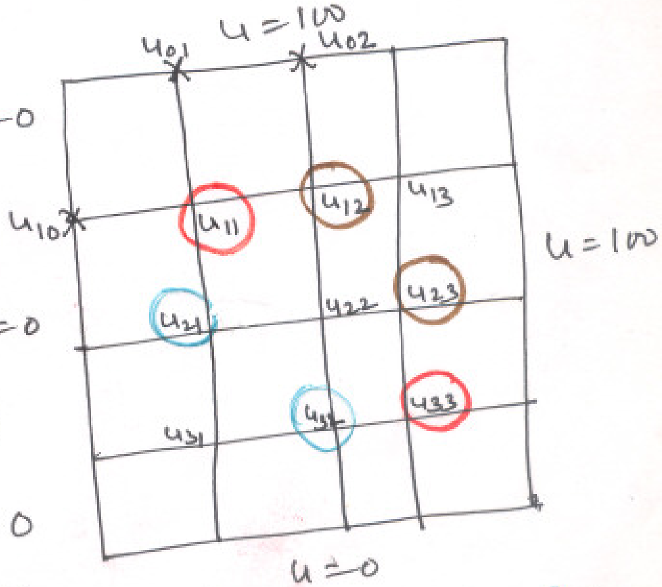
$$\nabla^2 u\Big|_{(x_i, y_j)} = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} - 4u_{ij} + u_{i,j-1} + u_{i,j+1}\right]$$

$$= \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{ij}$$

Ex steady state temperature on square plate

$$\nabla^2 u = 0$$

$$u_{i-1,j} + u_{i+1,j} - 4u_{ij} + u_{i,j-1} + u_{i,j+1} = 0$$



$i=1$

$j=1$: $u_{01} + u_{21} - 4u_{11} + u_{10} + u_{12} = 0$

$$100 + u_{21} - 4u_{11} + 0 + u_{12} = 0 \quad \text{— (1)}$$

$j=2$

$$u_{02} + u_{22} - 4u_{12} + u_{11} + u_{13} = 0$$

$$100 + u_{22} - 4u_{12} + u_{11} + u_{13} = 0 \quad \text{— (2)}$$

$$\left\{\begin{aligned} u_{11} &= u_{33} \\ u_{12} &= u_{23} \\ u_{21} &= u_{32} \end{aligned}\right\}$$

Similarly

$i=1, j=3$: $u_{12} + 100 + 100 + u_{23} - 4u_{13} = 0$ — (3)

$i=2, j=1$: $0 + u_{11} + u_{22} + u_{31} - 4u_{21} = 0$ — (4)

$u_{21} + u_{12} + u_{23} + u_{32} - 4u_{22} = 0$ — (5) $i=2, j=2$

$u_{22} + u_{13} + 100 + u_{33} - 4u_{23} = 0$ — (6) $i=2, j=3$

$0 + u_{21} + u_{32} + 0 - 4u_{31} = 0$ — (7) $i=3$ $j=1$

$u_{31} + u_{22} + u_{33} + 0 - 4u_{32} = 0$ — (8) $i=3, j=2$

$u_{32} + u_{23} + 100 + 0 - 4u_{33} = 0$ — (9) $i=3, j=3$

Now $u_{11} = u_{33}$, $u_{12} = u_{23}$, $u_{21} = u_{32}$

So we have only 6 variables so these 9 equations will reduce to 6 equations only and they will be as follows

$$
\begin{array}{c}
\begin{array}{cccccc} u_{11} & u_{12} & u_{13} & u_{21} & u_{22} & u_{31} \end{array} \\
\begin{bmatrix}
-4 & 1 & 0 & 1 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 \\
0 & 2 & -4 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 1 \\
0 & 2 & 0 & 2 & -4 & 0 \\
0 & 0 & 0 & 2 & 0 & -4
\end{bmatrix}
\end{array}
\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{31} \end{bmatrix}
=
\begin{bmatrix} -100 \\ -100 \\ -200 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Using Gauss-Elimination.

$u_{11} = u_{33} = 50.0$

$u_{12} = u_{23} = 71.43$

$u_{13} = 85.75$

$u_{21} = u_{32} = 28.57$

$u_{22} = 50.0$

$u_{31} = 14.29$

for Gauss - Seidel Method

$$u_{11} = \frac{1}{4}(100 + u_{12} + u_{21})$$

$$u_{12} = \frac{1}{4}(100 + u_{11} + u_{13} + u_{22})$$

$$u_{13} = \frac{1}{4}(200 + 2u_{12})$$

$$u_{21} = \frac{1}{4}(0 + u_{11} + u_{22} + u_{31})$$

$$u_{22} = \frac{1}{4}(0 + 2u_{12} + 2u_{21})$$

$$u_{31} = \frac{1}{4}(0 + 2u_{21})$$

When we use Gauss-Seidel Method to solve the system of equations then this is called Liebmann's Method.

One can use Relaxation method.

Poisson Eqn

$\nabla^2\phi = 2$, $\phi = 0$ on boundary of 6×8 unit^2 rectangular region

$h = k = 2$ unit

$$0 + 0 + \phi_{12} + \phi_{21} - 4\phi_{11} + 8 = 0$$

$$\phi_{11} + 0 + 0 + \phi_{22} - 4\phi_{12} + 8 = 0$$

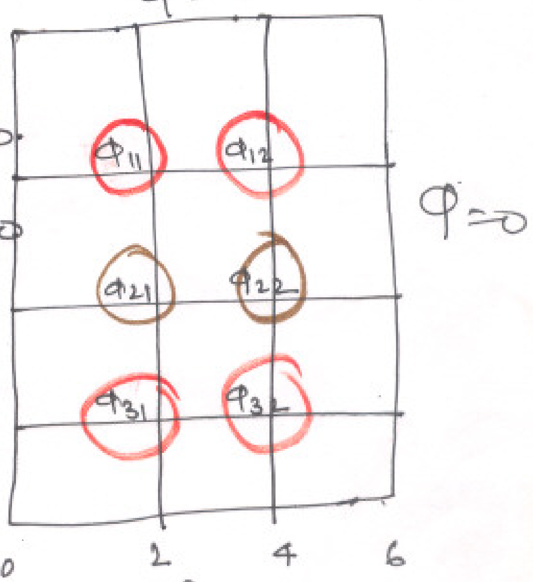
$$0 + \phi_{11} + \phi_{22} + \phi_{31} - 4\phi_{21} + 8 = 0$$

$$\phi_{21} + \phi_{12} + 0 + \phi_{32} - 4\phi_{22} + 8 = 0$$

$$0 + \phi_{21} + \phi_{32} + 0 - 4\phi_{31} + 8 = 0$$

$$\phi_{31} + \phi_{22} + 0 + 0 - 4\phi_{32} + 8 = 0$$

$\phi = 0$



$$\phi_{11} = \phi_{12} = \phi_{31} = \phi_{32}$$

$$\phi_{21} = \phi_{22}$$

$\phi = 0$

So we get only two equations in two unknowns.

$$\phi_{21} - 3\phi_{11} + \theta = 0 \qquad \text{— (1)}$$

$$2\phi_{11} - 3\phi_{21} + \theta = 0 \qquad \text{— (2)}$$

$$\phi_{11} = 4.56, \quad \phi_{21} = 5.72.$$

Derivative boundary conditions

$$\nabla^2 u = -\frac{Q}{K}, \qquad Q = 10 \text{ cal/cm}^3\text{sec}, \quad K = 0.16 \text{ cal/sec.cm}^2.°C/cm$$

Region: 8×8 unit^2 (1 unit Thick)

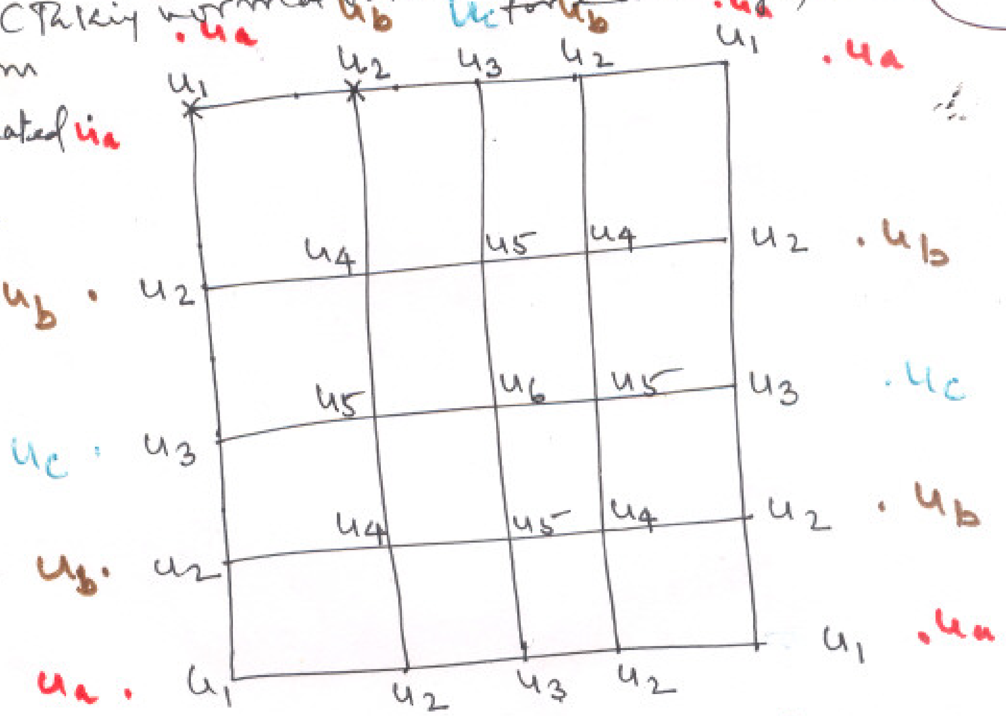
B.C

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 1.5 \ °C/cm \qquad h = 2$$

(Taking normal outward derivative) for each edges.

Top and bottom faces are insulated so no heat is lost and heat is lost from edges of the plate.

From all the edges / sides of heat loss is same.



$$\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{ij} = -\frac{Q}{K}$$

$$\frac{1}{2^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{ij} = -\frac{Q}{K}$$

$$\frac{1}{2^2}\left[u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}\right] = -\frac{Q}{K}$$

$$\frac{1}{2^2}(u_a + u_a + u_2 + u_2 - 4u_1) = -\frac{10}{0.16} \quad -(1)$$

$$\frac{1}{2^2}(u_1 + u_b + u_3 + u_4 - 4u_2) = -\frac{10}{0.16} \quad -(2)$$

$$\frac{1}{2^2}(u_2 + u_c + u_2 + u_5 - 4u_3) = -\frac{10}{0.16} \quad -(3)$$

$$\frac{1}{2^2}(u_2 + u_2 + u_5 + u_5 - 4u_4) = -\frac{10}{0.16} \quad -(4)$$

$$\frac{1}{2^2}(u_4 + u_3 + u_4 + u_6 - 4u_5) = -\frac{10}{0.16} \quad -(5)$$

$$\frac{1}{2^2}(u_5 + u_5 + u_5 + u_5 - 4u_6) = -\frac{10}{0.16}. \quad -(6)$$

$$\left(\frac{\partial u}{\partial y}\right)_1 = \frac{u_2 - u_a}{2.(2)} = 1.5 \quad -(7) \qquad \underset{\text{Central}}{y'} = \frac{y(x+h) - y(x-h)}{2h}$$

$$\left(\frac{\partial u}{\partial y}\right)_2 = \frac{u_4 - u_b}{2.(2)} = 1.5 \quad -(8)$$

$$\left(\frac{\partial u}{\partial y}\right)_3 = \frac{u_5 - u_c}{2(2)} = 1.5 \quad -(9)$$

from (7) ~~&~~ (8) & (9) we have

$$u_a = u_2 - 6$$

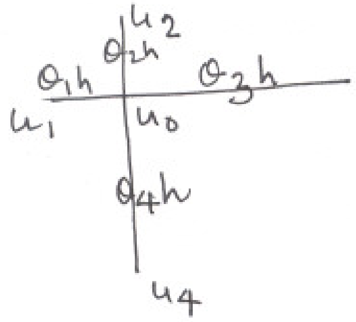
$$u_b = u_4 - 6$$

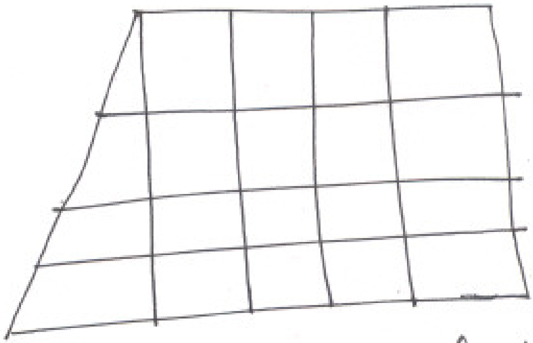
$$u_c = u_5 - 6$$

So u_a, u_b & u_c will be eliminated and we will get 6 equations in 6 unknowns. We can use Gauss-elimination, Gauss-Seidel (called Liebmann's method) or relaxation method

Irregular Region

$\nabla^2 u = u_{xx} + u_{yy}$





We represent each distances by $\theta_i h$, where θ_i is the fraction of the standard spacing h

Along the line 1 − 0 − 3

$$V_{1-0} = \left(\frac{\partial u}{\partial x}\right)_{1-0} = \frac{u_0 - u_1}{\theta_1 h} \quad , V_{0-3} = \left(\frac{\partial u}{\partial x}\right)_{0-3} = \frac{u_3 - u_0}{\theta_3 h}$$

forward approximation

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{1-0-3} = \frac{V_{0-3} - V_{1-0}}{\frac{1}{2}(\theta_1 + \theta_3)h} =$$

average of the two distances

forward approximation

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{1-0-3} = \frac{2}{h(\theta_1+\theta_3)}\left[\frac{u_3 - u_0}{\theta_3 h} - \frac{u_0 - u_1}{\theta_1 h}\right]$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{1-0-3} = \frac{2}{h^2}\left[\frac{u_1 - u_0}{\theta_1(\theta_1+\theta_3)} + \frac{u_3 - u_0}{\theta_3(\theta_1+\theta_3)}\right] \quad \text{— (1)}$$

Similarly

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{2-0-4} = \frac{2}{h^2}\left[\frac{u_2 - u_0}{\theta_2(\theta_2+\theta_4)} + \frac{u_4 - u_0}{\theta_4(\theta_2+\theta_4)}\right]$$

Thus

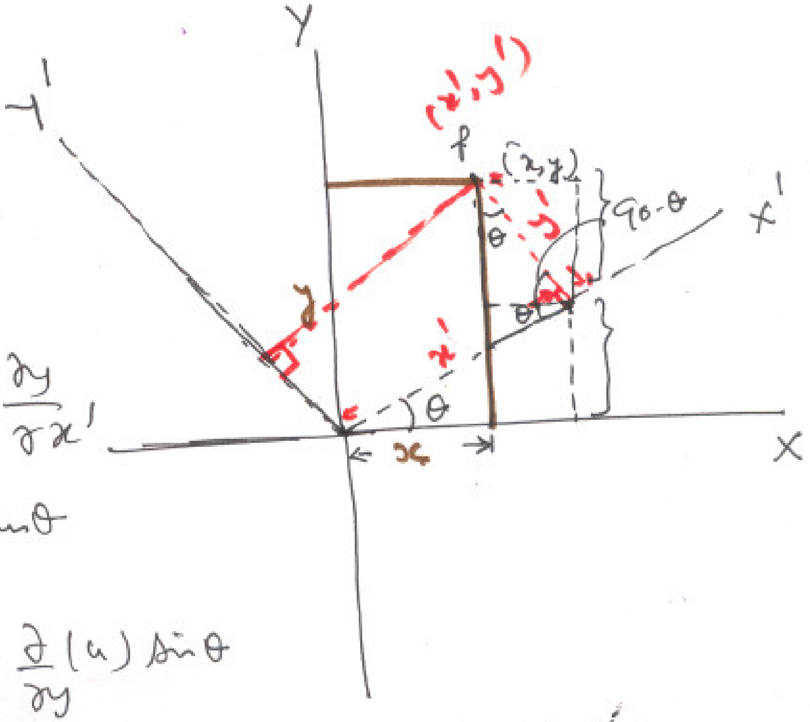
$$\nabla^2 u = u_{xx} + u_{yy} = \frac{2}{h^2}\left[\frac{u_1 - u_0}{\theta_1(\theta_1+\theta_3)} + \frac{u_3 - u_0}{\theta_3(\theta_1+\theta_3)} + \frac{u_2 - u_0}{\theta_2(\theta_2+\theta_4)} + \frac{u_4 - u_0}{\theta_4(\theta_2+\theta_4)}\right]$$

$$= \frac{2}{h^2}\left[\frac{u_1}{\theta_1(\theta_1+\theta_3)} + \frac{u_2}{\theta_2(\theta_2+\theta_4)} + \frac{u_3}{\theta_3(\theta_1+\theta_3)} + \frac{u_4}{\theta_4(\theta_2+\theta_4)} - u_0\left\{\frac{1\cdot\cancel{(\theta_1+\theta_3)}}{\cancel{(\theta_1+\theta_3)}\cdot\theta_1\theta_3} + \frac{1\cdot\cancel{(\theta_2+\theta_4)}}{\cancel{(\theta_2+\theta_4)}\theta_2\theta_4}\right\}\right]$$

$$\nabla^2 u = \frac{2}{h^2}\left[\frac{u_1}{\theta_1(\theta_1+\theta_3)} + \frac{u_2}{\theta_2(\theta_2+\theta_4)} + \frac{u_3}{\theta_3(\theta_1+\theta_3)} + \frac{u_4}{\theta_4(\theta_2+\theta_4)} - u_0\left(\frac{1}{\theta_1\theta_3} + \frac{1}{\theta_2\theta_4}\right)\right]$$

Laplacian Operator in non-rectangular region

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$



$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

$$u(x, y) = u(x', y')$$

$$\frac{\partial u}{\partial x'} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x'}$$

$$= u_x\cos\theta + u_y\sin\theta$$

$$\frac{\partial}{\partial x'}(u) = \frac{\partial}{\partial x}(u)\cdot\cos\theta + \frac{\partial}{\partial y}(u)\sin\theta$$

$$\frac{\partial}{\partial x'} = \cos\theta\cdot\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$\frac{\partial^2 u}{\partial x'^2} = \left(\frac{\partial}{\partial x'}\left(\frac{\partial u}{\partial x'}\right)\right.$$

$$= \cos\theta\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x'}\right) + \sin\theta\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x'}\right)$$

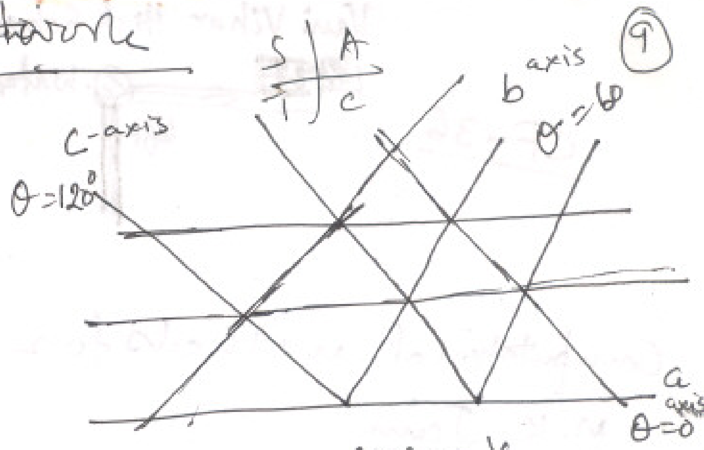
$$= \cos\theta\frac{\partial}{\partial x}\left(u_x\cos\theta + u_y\sin\theta\right) + \sin\theta\frac{\partial}{\partial y}\left(u_x\cos\theta + u_y\sin\theta\right)$$

$$= \cos\theta\left[\left(u_{xx}\cos\theta + u_{yx}\sin\theta\right)\right] + \sin\theta\left[\left(u_{xy}\cos\theta + u_{yy}\sin\theta\right)\right]$$

$$= u_{xx}\cos^2\theta + u_{yx}\sin\theta\cos\theta + u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta$$

$$\frac{\partial^2 u}{\partial x'^2} = u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta \quad \text{—— (I)}$$

For an equispaced triangular Network



$\underline{\theta = 0^\circ}$

$$\frac{\partial^2 u}{\partial a^2} = u_{xx}$$

$\sin 30 = 1/2$

$\cos 30 = \sqrt{3}/2$

$\underline{\theta = 60^\circ}$

$$\frac{\partial^2 u}{\partial b^2} = \frac{1}{4} u_{xx} + \frac{\sqrt{3}}{2} u_{xy} + \frac{3}{4} u_{yy}$$

$$\begin{cases} \cos 60 = \dfrac{1}{2} \\ \sin 60 = \dfrac{\sqrt{3}}{2} \end{cases}$$

$\underline{\theta = 120^\circ}$

$$\frac{\sqrt{3}}{2} = \sin 120 = \sin(180-60) = \sin 60 = \frac{\sqrt{3}}{2}$$

$$\frac{\partial^2 u}{\partial c^2} = \frac{1}{4} u_{xx} - \frac{\sqrt{3}}{2} u_{xy} + \frac{3}{4} u_{yy}$$

$$-\frac{1}{2} = \cos 120 = \cos(180-60) = -\cos 60 = -\frac{1}{2}$$

Then $$\frac{\partial^2 u}{\partial a^2} + \frac{\partial^2 u}{\partial b^2} + \frac{\partial^2 u}{\partial c^2} = \frac{3}{2}(u_{xx} + u_{yy})$$

$$= \frac{3}{2} \nabla^2 u$$

or $$\nabla^2 u = \frac{2}{3}\left(\frac{\partial^2 u}{\partial a^2} + \frac{\partial^2 u}{\partial b^2} + \frac{\partial^2 u}{\partial c^2}\right)$$

$$= \frac{2}{3h^2}\Big[u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1} - 6u_{ij,k}\Big]$$