3. Consider initial value problem

$$y' = x(y+x) - 2, \quad y(0) = 2.$$

Use Euler's method & Euler-Cauchy method with step size $h = 0.3$, $h = .2$ and $h = .15$ to compute $y(0.6)$ (upto 5 decimal places).

$$y_{n+1} = y_n + h\, y_n'$$

$$= y_n + h\, f(x_n, y_n)$$

$$= y_n + h\,[x_n(y_n + x_n) - 2]$$

$$= (1 + h x_n) y_n + h x_n^2 - 2h$$

$\underline{\underline{h = .3}}$

$$y_1 = y(.3) = (1 + h x_0) y_0 + h x_0^2 - 2.(.3)$$

$$= 2 - .6 = 1.4$$

$$y_2 = y(.6) = (1 + h x_1) y_1 + h x_1^2 - 2h$$

$$= (1 + h^2)\, 1.4 + h^3 - 2h$$

$$= (1 + .09)\, 1.4 + .0027 - .6$$

$$= 1.09 \times 1.4 + 0.027 - .6 = 1.526 + 0.0027 - 0.6$$

$$= 0.953$$

(4) Apply Euler-Cauchy method with step length h to the problem

$$y' = -y, \quad y(0) = 1$$

(a) Determine an explicit expression for y_n

(b) for which value of h, the sequence $\{y_n\}_0^\infty$ bounded.

— <u>Cauchy - Euler Method</u>

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n+h, y_n+k_1)$$

$$k_2 = hf(x_n+h, y_n+hf(x_n,y_n))$$

$$y' = y \quad , \quad f(x,y) = -y$$

$$k_1 = -h y_n$$

$$k_2 = -h(y_n + k_1) = -h(y_n - h y_n)$$

$$= -h(1-h) y_n$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$= y_n + \frac{1}{2}[-hy_n - h(1-h)y_n]$$

$$= y_n[1 - \frac{1}{2}h - \frac{1}{2}h(1-h)]$$

$$= y_n[1 - \frac{h}{2} - \frac{h}{2} + \frac{h^2}{2}]$$

$$y_{n+1} = y_n[1 - h + \frac{h^2}{2}]$$

$$y_1 = [1 - h + \frac{h^2}{2}] y_0 = [1 - h + \frac{h^2}{2}]$$

$y_2 = [1-h+\frac{h^2}{2}]\, y_1 = [1-h+\frac{h^2}{2}]^2$

$\vdots$

$y_n = [1-h+\frac{h^2}{2}]^n$

(5) $\{y_n\}_{n=0}^{\infty}$ will remain bounded iff

$$|1-h+\frac{h^2}{2}| \le 1$$

$$2-2h+h^2 \le 2$$

$$h^2-2h+1+1 \le 2$$

$$(h-1)^2+1 \le 2$$

$$(h-1)^2 \le 1$$

so if $(h-1) \le 1$ then $(h-1)^2 \le 1$

or $h \le 2$

for $0 < h \le 2$ the sequence $\{y_n\}_{n=0}^{\infty}$ is bounded.

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(8) Solve the differential equation

$y' = x + y \quad y(0) = 1$ by Euler's method

(Take $h = .1$)

$f(x,y) = x+y$

$y_{n+1} = y_n + h f(x_n, y_n)$

$= y_n + h(x_n + y_n)$

$= (1+h)\, y_n + h x_n$