Introduction to Plate bending

Previously, we have seen the problem of beam bending. We have the relations such as:

Shear $V = \frac{dM}{dx}$, external load $p = \frac{dV}{dx} = \frac{d^2M}{dx^2}$

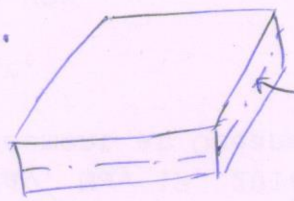
Moment-curvature relation, $M = -EI\frac{d^2y}{dx^2}$

we may combine these into, $p = -EI\frac{d^4y}{dx^4}$

Similar relations can be obtained for a plate element.

Just like beams, we have assumptions for plate as well.

① Deflection of mid surface is small compared to thickness of the plate.



② Mid plane remains unstrained subsequent to bending. (like neutral plane of beam)

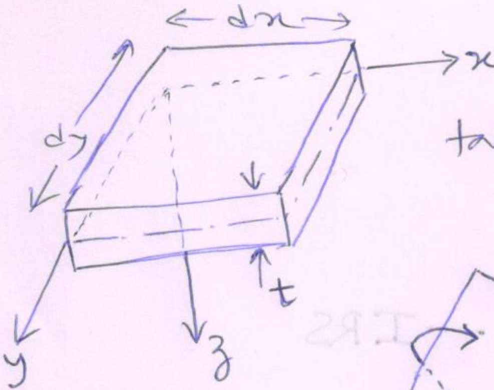
③ Plane section initially normal to the mid surface remain plane and normal to the surface after bending. This means vertical shear strain γ_{xz}, γ_{yz} are negligible.

④ The stress normal to mid plane (σ_z) is small compared to other stress components and neglected.

⑤ Material is linear, isotropic.

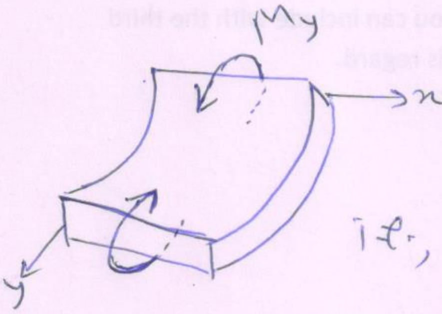
⑥ Plate thickness is small compared to other dimensions. This is required for ③.

These assumptions known as Kirchhoff hypothesis. These fundamental assumptions are the basis of the small-deflection theory or classical theory for isotropic, homogenous, elastic thin plate.



Deflections in x, y, z directions are taken as u, v, w.

this x indicates that the plane on which the moment is acting has perpendicular that is parallel to the x-axis.

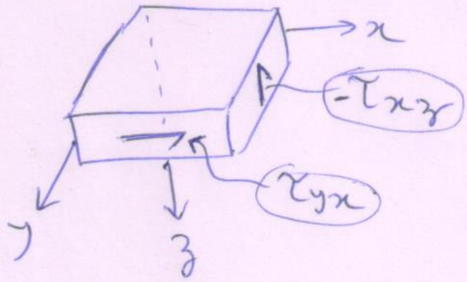


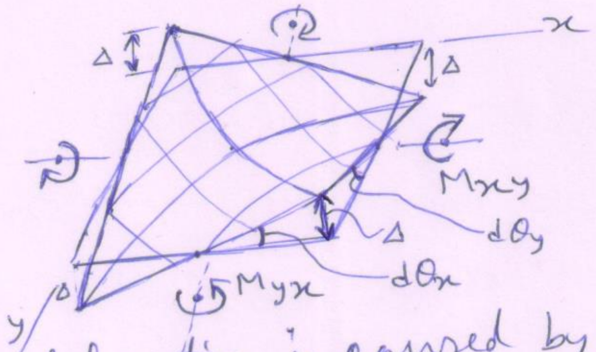
All forces acting at the ends of the plate are taken as uniformly distributed.

i.e., M_x = uniformly distributed moment on the edge dx (length $= dx$)

The force/displacement variables are denoted by 2

E.g., σ_{xz} – this means σ is acting on a plane that has perpendicular parallel to the x-axis, and the direction of σ is parallel to the z-axis.





Such a deformation is caused by twisting moment. M_{xy} = twisting moment acting per unit length along "dy" edge length.

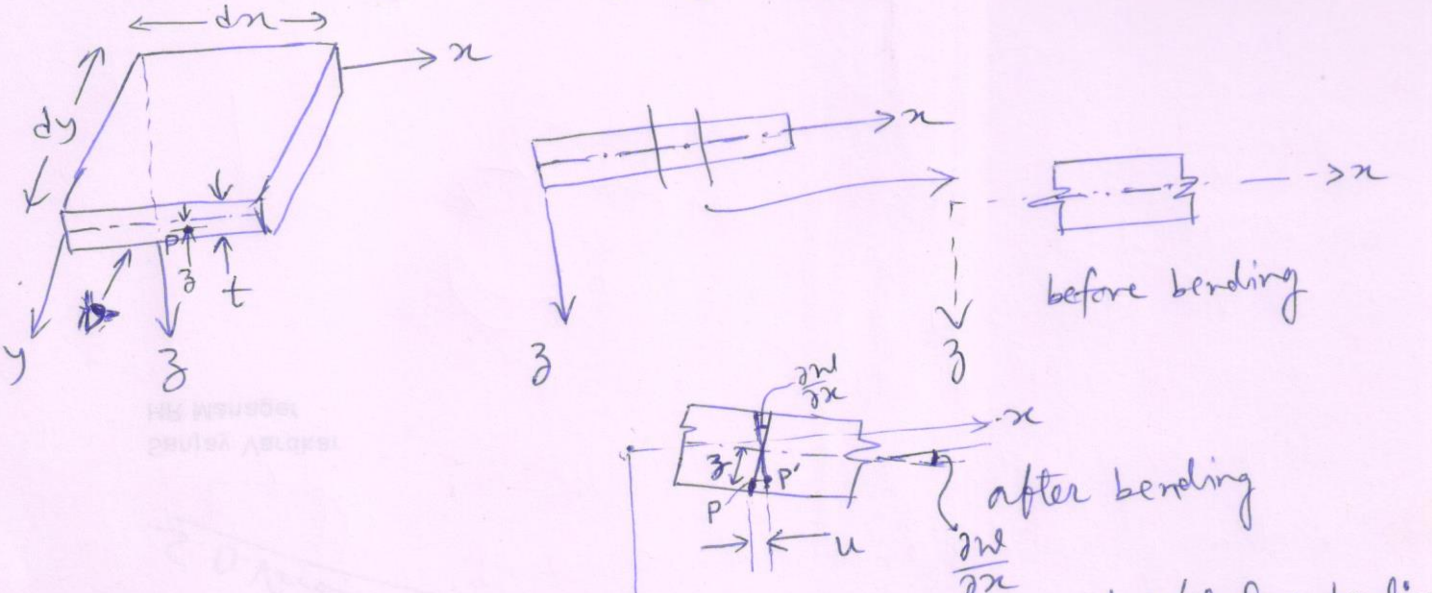
We can write, $\Delta = d\theta_x \cdot \frac{dx}{2} = d\theta_y \cdot \frac{dy}{2}$

or $\frac{d\theta_x}{dy} = \frac{d\theta_y}{dx}$

If we define, $\theta_x = \frac{dw}{dx}$, $\theta_y = \frac{dw}{dy}$. Hence we get, $\frac{d^2w}{dxdy} = \frac{d^2w}{dydx}$

In fact, we should write this in partial derivative form,

i.e., for plate bending $\frac{\partial w}{\partial x \partial y} = \frac{\partial w}{\partial x}$



The point P was at P' before bending.

$\therefore u = -z\frac{\partial w}{\partial x}$; since u is in $-ve$ direction of x-axis.

we get similar results from assumption no. 3;

i.e., γ_{xz} and γ_{yz} are negligible.

From basics, $\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$; $\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$ [basics of solid mechanics]

$\therefore \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$ or $u = -\frac{\partial w}{\partial x} z + u_0$, since w is taken as mid-surface deflection $= f(x, y)$.

$\frac{\partial w}{\partial x}$ is a function of x, y only.

u_0 = initial displacement in x. If we start measuring from u_0, i.e., $u_0 = 0$, then we get

$$u = -z\frac{\partial w}{\partial x}$$

Similarly from $\gamma_{yz} = 0$ gives us $v = -z\frac{\partial w}{\partial y}$

Now, by definition, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. Inserting u, v we get

$$\gamma_{xy} = \underbrace{-z\frac{\partial^2 w}{\partial x \partial y}}_{\text{for } v} \underbrace{- z\frac{\partial^2 w}{\partial y \partial x}}_{\text{for } u} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Hence we get the strain-displacement relations

$$\epsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \quad \epsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

~~Next~~, Note – for information, from solid mechanics, we can write $\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \cdots$, $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left[\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \cdots\right.$

Here, we ignore all higher order terms by assuming that the deflections are small.

Now, the stress-strain relation

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_y), \quad \sigma_y = \frac{E}{1-\mu^2}(\epsilon_y + \mu\epsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$

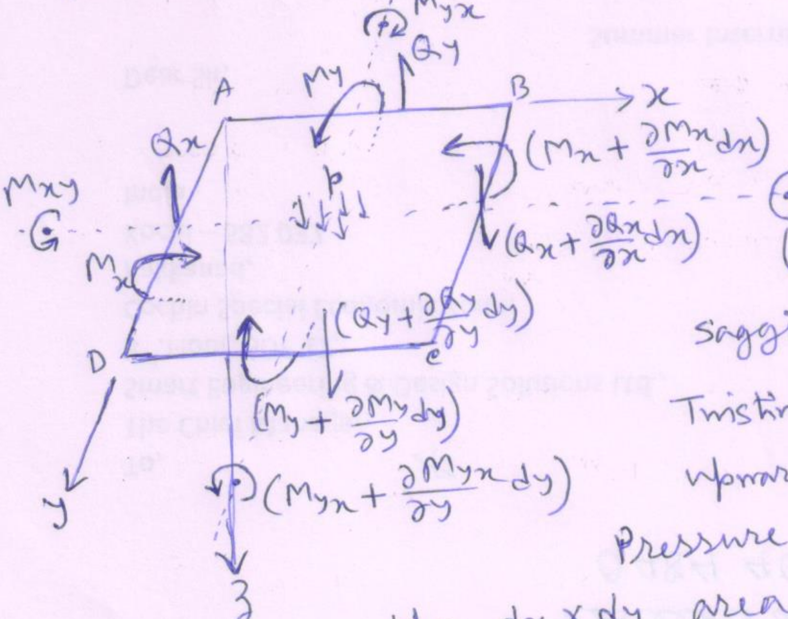
$$G = \frac{E}{2(1+\mu)}$$

$\therefore$ stress-displacement relations

$$\sigma_x = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right), \quad \sigma_y = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy} = -\frac{Ez}{1+\mu}\frac{\partial^2 w}{\partial x \partial y}$$

Now we consider the equilibrium of the element



$\bar{x} \times \bar{y} = \bar{z}$ is the +ve direction (Right hand thumb and fingers)

Sagging bending moments are +ve

Twisting moment causing point C moving upward is +ve. Similarly, shear also.

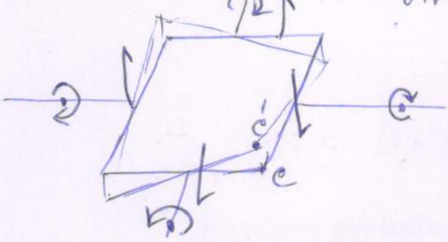
Pressure p uniformly distributed in the $dx \times dy$ area.

All moments, shears are uniformly distributed along the edge. For eg, total bending moment acting on AD = $M_x\,dy$

total shear acting on AB = $Q_y . dx$

These are valid since dx, dy are small.

If you have confusion regarding sign, we shall discuss it in the class. Try to think in this line --- what moment and shear are required to take c to c'?



Let us now write the equilibrium equations →

Taking sum of all vertical forces = 0,

$$p.\,dy\,dx + \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx + \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy - Q_y\,dx - Q_x\,dy = 0$$

$$\text{or} \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \quad \text{——— (i)}$$

Taking sum of all moment about x-axis = 0,

$$\left(M_y + \frac{\partial M_y}{\partial y}dy\right)dx - M_y\,dx - \underbrace{\left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx\,dy}_{\text{arm of force}} + \underbrace{Q_x\,dy.\frac{dy}{2}}_{\text{arm of force}} -$$

$$- M_{xy}\,dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}dx\right)dy - \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy\frac{dy}{2} - p\,dy\,dx\frac{dy}{2} = 0$$

$$\text{or} \quad \frac{\partial M_y}{\partial y}dy\,dx - Q_y\,dx\,dy - \frac{\partial Q_y}{\partial y}dx\,dy\,dy - \frac{\partial Q_x}{\partial x}dx\,dy\frac{dy}{2} + \frac{\partial M_{xy}}{\partial x}dx\,dy - p\,dx\,dy\frac{dy}{2} = 0$$

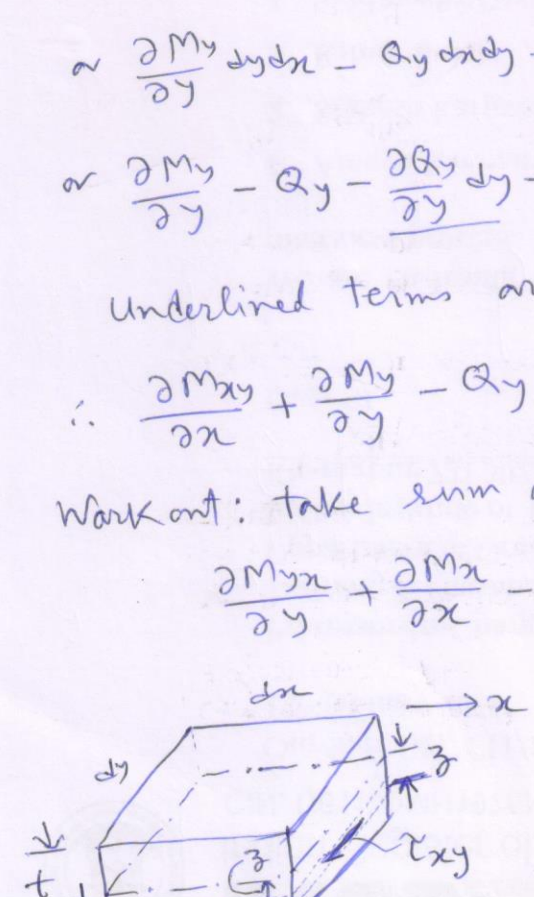
$$\text{or} \quad \frac{\partial M_y}{\partial y} - Q_y - \underline{\frac{\partial Q_y}{\partial y}dy} - \underline{\frac{\partial Q_x}{\partial x}\frac{dy}{2}} + \frac{\partial M_{xy}}{\partial x} - \underline{p\frac{dy}{2}} = 0$$

Underlined terms are very small and can be ignored.

$$\therefore \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \quad \text{— (ii)}$$

Work out: take sum of all moments about y-axis = 0, find

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \quad \text{—— (iii)}$$



We know that twisting moment generates shear stress on the surface of a section. $\therefore$ M_{xy}, M_{yx} will create shear stress τ_{xy} and τ_{yx} such that

$$M_{xy} = \int_{-t/2}^{t/2} \tau_{xy}.\,z\,dz\,, \quad M_{yx} = \int_{-t/2}^{t/2} \tau_{yx}\,z\,dz$$

Note that if $\tau_{xy} \neq \tau_{yx}$, then there will be a twist of the element along vertical direction. This is not possible, as it will cause stress in the mid-surface.

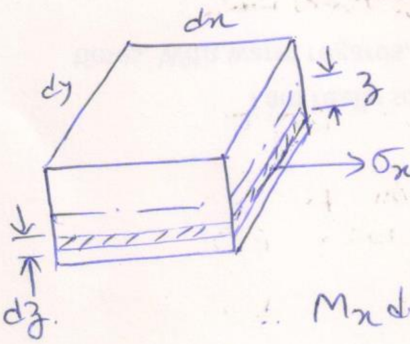
not possible (similar to assumption 2)

Hence, it must be $\tau_{xy} = \tau_{yx}$, (i.e., $M_{xy} = M_{yx}$) and they balance each other.

Now, eqs. (I), (II) and (III) can be combined to obtain,

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p. \quad \text{—(IV)}$$

This is similar to $\frac{d^2M}{dx^2} = p$ in case of a simple beam.



We can write the expression of the total moment acting on an edge in terms of stress and deflection.

$$\therefore \; M_x dy = \int_{-t/2}^{t/2} \sigma_x z \, dz \, dy = \int_{-t/2}^{t/2} \frac{-Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) z \, dz \, dy$$

$$M_x = -\frac{E}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)\int_{-t/2}^{t/2} z^2 dz = -\frac{Et^3}{12(1-\mu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$= -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right), \text{ where, } D = \frac{Et^3}{12(1-\mu^2)} = \text{Flexural rigidity of plate (thin) bending.}$$

If we consider a beam of unit width, thickness $= t$ in bending, its flexural-rigidity $= EI = \frac{E \cdot 1 \cdot t^3}{12} = \frac{Et^3}{12}$

Since, $\mu < 1$, $D > EI$, i.e, plate is more rigid in bending than a beam.

Work out: other relations in the same way,

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right), \quad M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

From (II) & (III), $Q_x = \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

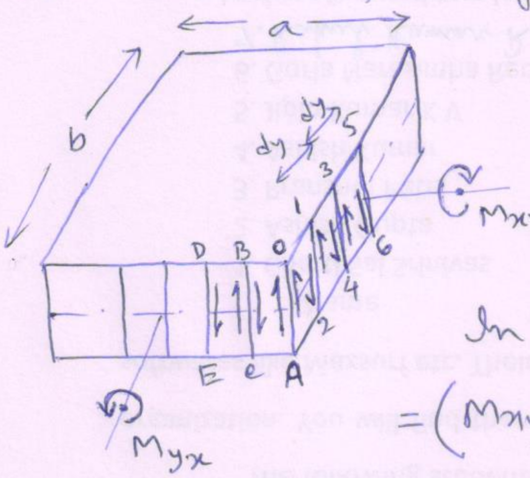
The expressions of M_x, M_y, M_{xy} can be inserted in (iv)

to get $\dfrac{\partial^4 w}{\partial x^4} + 2\dfrac{\partial^4 w}{\partial x^2 \partial y^2} + \dfrac{\partial^4 w}{\partial y^4} = \dfrac{p}{D}$

or $\nabla^4 w = \dfrac{p}{D}$, where $\nabla^4 = \dfrac{\partial^4}{\partial x^4} + \dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$

This is the basic equation of Kirchoff's plate ~~theory~~ equation or Thin plate bending equation.

In order to see the role of the twisting moment along an edge, let's take the following.



Twisting moment acting along the edges per unit length

In 3465, twisting moment acting $= M_{xy}\,dy$

If dy is small, twisting moment can be represented by a pair of forces $= \dfrac{M_{xy}\,dy}{dy} = M_{xy}$

In the adjacent 1243, twisting moment

$$= \left(M_{xy} + \frac{\partial M_{xy}}{\partial y} dy\right) dy$$

$\therefore$ the magnitude of the pair of force $= M_{xy} + \dfrac{\partial M_{xy}}{\partial y} dy$

Hence net force on 34 $= \dfrac{\partial M_{xy}}{\partial y} dy$

This force can be added with the shear force to get the total vertical force on the edge $dy = \bar{V}_x = Q_x\,dy + \dfrac{\partial M_{xy}}{\partial y} dy$

$\therefore$ shear force along dy per unit length $= V_x = Q_x + \dfrac{\partial M_{xy}}{\partial y}$

Similarly, we can get $V_y = Q_y + \dfrac{\partial M_{xy}}{\partial x}$

$$\text{or} \quad V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$

For simply supported case,
it is interesting to note that at the corner OA, the shear force due to twisting moment do not cancel each other, rather a concentrated force is developed $= 2\left(M_{xy} + \dfrac{\partial M_{xy}}{\partial y} dy\right) \simeq 2M_{xy}$

$\therefore$ corner force $= 2M_{xy} = -2D(1-\mu)\dfrac{\partial^2 w}{\partial x \partial y}$; (at $x=a$, $y=b$)

Note that such a force at the corner is developed if the edge is restrained, i.e, for eg, edges of the plate are simply supported and span loading exists.

If edges are not restrained.

For simply supported case, this corner deflection is restrained, giving rise to concentrated forces.

Additional corner forces for plate having various edge conditions may be determined similarly. For eg., when two adjacent plate edges are fixed or free, corner force $(F_c) = 0$ since no twisting moment exists along the edges in these boundary conditions.

The bending moments at the edges is zero for simply supported condition.

The boundary conditions are-

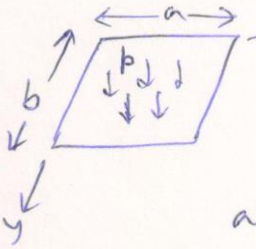
clamped / fixed $\Rightarrow$ implies $w = 0$, $\frac{\partial w}{\partial x} = 0$ (slope and deflection is zero at the edge)

Simply supported edge $\Rightarrow$ $w = 0$, $M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) = 0$

Free edge $\Rightarrow$ all moments and shear $= 0$ at that edge

$$\therefore \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} = 0, \quad \frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2} = 0$$

Let us consider a rectangular plate having simply supported edges, carrying pressure $p = q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$ in the span.



the basic equation is $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}$

Instead of trying to solve it directly, we assume that the solution (or deflection w) will follow the loading, i.e., $w = c \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$, [c = constant]

Now, inserting w in the basic equation, we get

$$c = \frac{1}{\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2} \cdot \frac{q_0}{D}$$

and we formulate $w = \frac{1}{\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2} \frac{q_0}{D} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$

Now we can find M_x, M_y, M_{xy}, Q_x, Q_y etc. at any (x, y)

From symmetry we know that maximum deflection will occur at $x = a/2$, $y = b/2$

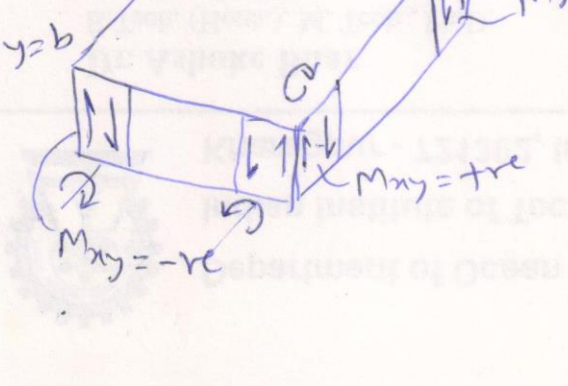
$$\therefore w_{max} = w(x = a/2, y = b/2) = \frac{1}{\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2} \frac{q_0}{D}$$

Workout: Find out the corner forces at $x = a$, $y = b$.

Find the bending stress (σ) at $x = a/2$, $y = b/2$, $z = t/4$

note: (bending stress max is on the surface of mid span) i.e, $x = a/2$, $y = b/2$, $z = t/2$

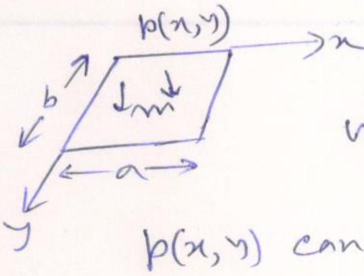
For corner forces, hint about directions



at $(x,y) = (a, 0)$, M_{xy} should be $-$ve to get force similar to (a, b)

similarly for $(x, y) = (0, b)$

Navier's solution of Thin plate bending.



This method uses the previous solution by using Fourier series of arbitrary load $p(x,y)$

$p(x,y)$ can be expressed as, $p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

where $A_{m'n'} = \frac{4}{ab}\int_0^a\int_0^b p(x,y) \sin\frac{m'\pi x}{a} \sin\frac{n'\pi y}{b}\, dx dy$

By using previous solution, we assume the final solution is the superposition of many single solutions,

$$\therefore w = \frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

For example, if we have uniformly distributed pressure q_0 over the entire area,

$$p(x,y) = q_0$$

$$\therefore a_{mn} = \frac{4q_0}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx dy = \frac{16}{\pi^2 mn} q_0 \quad \left[\text{for both } m,n = 1,3,\dots\right]$$

$$= 0 \quad \left[\text{for any or both } m,n \text{ are} = 2,4,\dots\right]$$

$$\therefore w = \frac{16 q_0}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}, \quad \left[m,n = 1,3,5,\dots\right]$$

Maximum deflection at $(a/2, b/2)$

$$w_{max} = \frac{16 q_0}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$$

For a square plate, $(a=b)$, taking $(m=1, n=1,3 \text{ and } m=3, n=1,3)$

we get $w_{max} = 0.00406 \frac{q_0 a^4}{D}$

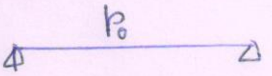
Similarly maximum bending moment with 1st 4 terms,

$$M_{x,max} = M_{y,max} = 0.0469\, q_0 a^2$$

Approximate method (Strip method)

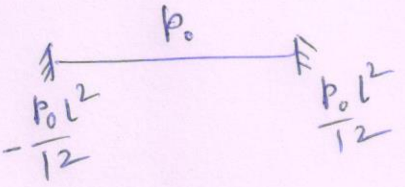
As the name suggest, in this method, plate is assumed to be composed of strips treated as beams.

Useful beam results are:



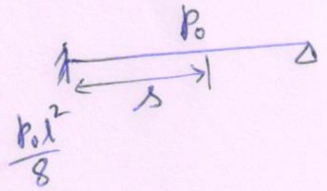
$$BM_{max} = \frac{p_0 l^2}{8} \text{ at mid span}$$

$$\delta_{max} = \frac{5}{384}\frac{p_0 l^4}{EI} \text{ at mid span}$$



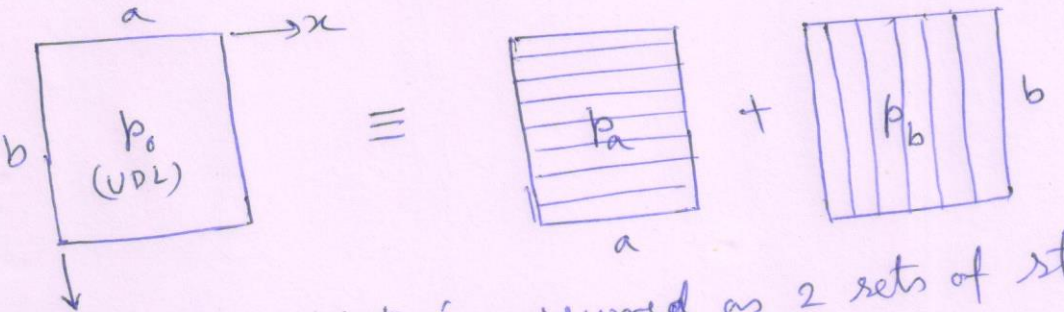
$$BM \text{ at mid span} = \frac{p l^2}{24}$$

$$\delta_{max} \text{ at mid span} = \frac{1}{384}\frac{p_0 l^4}{EI}$$



$$BM_{max}\left(\text{at } s = \frac{5}{8}l\right) = \frac{9}{128}p_0 l^2$$

$$\delta\left(\text{at } \cancel{s = \tfrac{5}{8}l} \text{ mid span}\right) = \frac{1}{192}\frac{p_0 l^4}{EI}$$



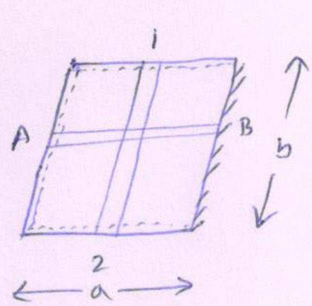
The plate is assumed as 2 sets of strips (beams) spanning in x and y directions.

Total force is carried by the strips (both x and y spans) together, i.e., $p_0 = p_a + p_b$ [p_a = Uniformly distributed pressure on strips spanning in x, p_b = UDP on strips spanning in y]. This is the 1st assumption.

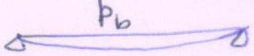
Second assumption is that maximum deflection ~~of~~ from both span are equal, i.e., w_{max} for a span $= w_{max}$ for b span

Let us consider the following example.

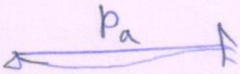
The rectangular beam (shown below) has one side fixed and others are simply supported. Find the solution. Plate carries p_0 (UDP)



strips (1 2) spanning in b can be approximated as



strips spanning in a can be approximated as (AB)



$\therefore$ total pressure or Uniformly distributed pressure

$$p_0 = p_a + p_b \quad \text{—— (i)}$$

mid span deflections are also same, i.e.,

$$\frac{5}{384}\frac{p_b b^4}{EI} = \frac{1}{192}\frac{p_a a^4}{EI} \quad \text{—— (ii)}$$

From (i) and (ii), we get $p_a = \dfrac{5b^4}{2a^4+5b^4}p_0$, $p_b = \dfrac{2a^4}{2a^4+5b^4}p_0$

Now after getting p_a and p_b we can obtain approximate solutions as $\Rightarrow$ Plate deflection at the center $= \dfrac{5}{384}\dfrac{p_b b^4}{D} = \dfrac{5}{192}\dfrac{p_0 a^4 b^4}{(2a^4+5b^4)D}$

$$M_x(\max) = M_x \text{ at } x = \frac{3}{8}l = \frac{45}{128}\frac{p_0 a^2 b^4}{2a^4+5b^4}$$

$$M_y(\max) = M_y \text{ at mid span} = \frac{1}{4}\frac{p_0 a^4 b^2}{2a^4+5b^4}$$

$$\text{BM at fixed support} = \frac{5}{8}\frac{p_0 a^2 b^4}{2a^4+5b^4}$$

For a square plate, $a=b$, the results become

$$w_{max} = 0.00372\frac{p_0 a^4}{D},\ M_x = 0.0502\,p_0 a^2,\ M_y = 0.0357\,p_0 a^2$$

Here, w_{max} is 33% greater the exact value, while the moments are 11% higher.

This approximate method in general produce conservative results. It can be used at the very early stage of a work when calculations are done by hand only.