

Implicit Runge-Kutta Methods

①

Method

$$u_{j+1} = u_j + W_1 K_1 \quad \text{--- ①} \quad y' = f(t, y)$$

$$K_1 = h f(t_j + c_1 h, u_j + a_{11} K_1) \quad \text{--- ②}$$

find W_1 , c_1 and a_{11}

$$\begin{aligned} u(t_{j+1}) &= u(t_j) + h u'(t_j) + \frac{h^2}{2} u''(t_j) + \dots \\ &= u(t_j) + h f(t_j, u(t_j)) + \frac{h^2}{2} (f_t + f f_u)_{t_j} + O(h^3) \end{aligned} \quad \text{--- ③}$$

and

$$K_1 = h f(t_j + c_1 h, u_j + a_{11} K_1)$$

$$K_1 = h [f(t_j, u_j) + (c_1 h f_t + a_{11} K_1 f_u + \dots)_{t_j}]$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h a_{11} K_1 f_u + O(h^3)$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h a_{11} f_u \cdot h f(t_j + c_1 h, u_j + a_{11} K_1) + O(h^3)$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h^2 a_{11} f_u [f(t_j, u_j) + c_1 h f_t + a_{11} K_1 f_u] + O(h^3)$$

$$= h f(t_j, u_j) + c_1 h^2 f_t + h^2 a_{11} f_u f + h^3 [a_{11} c_1 f_t f_u + a_{11} f_u f] + O(h^3)$$

$$K_1 = h f + h^2 [c_1 f_t + a_{11} f_u f] + O(h^3) \quad \text{--- ④}$$

Now putting expressions for $u(t_{j+1})$ and k_1 from (3) & (4) in (1), we get.

(2)

$$u(t_{j+1}) = u(t_j) + W_1 k_1$$

$$u(t_j) + hf + \frac{h^2}{2} (f_t + f_{tt})_{t_j} + O(h^3)$$

$$= u(t_j) + hW_1 f + h^2 [W_1 C f_t + W_1 a_{11} f_{tt}]_{t_j} + O(h^3)$$

This implies

$$W_1 = 1, \quad W_1 C = \frac{1}{2}, \quad W_1 a_{11} = \frac{1}{2}$$

$$\Rightarrow W_1 = 1, \quad C = \frac{1}{2}, \quad a_{11} = \frac{1}{2}$$

Thus second order implicit-Runge-Kutta method is given by

$$u_{j+1} = u_j + k_1$$

$$k_1 = hf(t_j + \frac{1}{2}h, u_j + \frac{1}{2}k_1)$$

Fourth Order implicit-Runge-Kutta Method

$$u_{j+1} = u_j + W_1 k_1 + W_2 k_2 \quad \text{--- (5)}$$

$$k_1 = hf(t_j + c_1 h, u_j + a_{11} k_1 + a_{12} k_2)$$

$$k_2 = hf(t_j + c_2 h, u_j + a_{21} k_1 + a_{22} k_2)$$

Find $W_1, W_2, C_1, C_2, a_{11}, a_{12}, a_{21}, a_{22}$.

Now use Taylor expansions for u_{j+1} , k_1 and k_2 (3)

$$\begin{aligned} u_{j+1} &= u(t_{j+1}) \\ &= u(t_j) + h u'(t_j) + \frac{h^2}{2} u''(t_j) + \frac{h^3}{6} u'''(t_j) \\ &\quad + \frac{h^4}{24} u^{(4)}(t_j) + O(h^5) \end{aligned}$$

and k_1 & k_2 may be written as

$$k_1 = hA_1 + h^2B_1 + h^3C_1 + O(h^4)$$

$$k_2 = hA_2 + h^2B_2 + h^3C_2 + O(h^4)$$

Put all these expressions in (5) and compare the coefficients of h , h^2 , h^3 , h^4 we get all the values as

$$w_1 = \frac{1}{2}, \quad w_2 = \frac{1}{2}, \quad c_1 = \frac{3-\sqrt{3}}{6}, \quad c_2 = \frac{3+\sqrt{3}}{6}$$

$$a_{11} = \frac{1}{4}, \quad a_{12} = \frac{3-2\sqrt{3}}{12}, \quad a_{21} = \frac{3+2\sqrt{3}}{12}, \quad a_{22} = \frac{1}{4}$$

and the method will be as follows:

$$u_{j+1} = u_j + \frac{1}{2} (k_1 + k_2)$$

$$k_1 = h f\left(t_j + \frac{3-\sqrt{3}}{6} h, u_j + \frac{k_1}{4} + \frac{3-2\sqrt{3}}{12} k_2\right)$$

$$k_2 = h f\left(t_j + \frac{3+\sqrt{3}}{6} h, u_j + \frac{3+2\sqrt{3}}{12} k_1 + \frac{1}{4} k_2\right)$$

Prob: Solve the initial value problem

(4)

$$u' = -2tu^2, \quad u(0) = 1$$

with $h = 0.2$ on the interval $[0, 0.4]$. Use the second order implicit Runge-Kutta method.

Soln

$$u_{j+1} = u_j + k_1, \quad j=0,1$$

u_0	u_1	u_2
0	0.2	0.4
t_0	t_1	t_2

$$k_1 = h f(t_j + \frac{h}{2}, u_j + \frac{1}{2}k_1)$$

$$\text{So } k_1 = -h(2t_j + h) \left(u_j + \frac{1}{2}k_1\right)^2 \quad \text{--- (1)}$$

This is an implicit equation in k_1 so we use Newton-Raphson method

$$\text{Take } F(k_1) = k_1 + h(2t_j + h) \left(u_j + \frac{1}{2}k_1\right)^2$$

$$\text{as } h = 0.2$$

$$F(k_1) = k_1 + 0.2(2t_j + 0.2) \left(u_j + \frac{1}{2}k_1\right)^2$$

$$\left(\frac{dF}{dk_1}\right) = F'(k_1) = 1 + 0.2(2t_j + 0.2) \left(u_j + \frac{1}{2}k_1\right)$$

Then Newton's method

$$k_1^{(\Delta+1)} = k_1^{(\Delta)} - \frac{F(k_1^{(\Delta)})}{F'(k_1^{(\Delta)})}, \quad \Delta = 0, 1, 2, \dots$$

$$\text{Take } k_1^{(0)} = h f(t_j; u_j), \quad j=0,1$$

(5)

for $j=0$, $t_0=0$, $u_0=1$, $k_1^{(0)} = hf(t_0, u_0^2)$
 $= -h 2t_0 u_0^2$
 $= 0$ as $t_0=0$

$$f(k_1^{(0)}) = 0.04, \quad f'(k_1^{(0)}) = 1.04$$

$$k_1^{(1)} = -0.03846150$$

$$f(k_1^{(1)}) = 0.00001483, \quad f'(k_1^{(1)}) = 1.03923077$$

$$k_1^{(2)} = -0.03847567$$

$$\text{take } k_1 = k_1^{(2)}$$

$$\text{and } u(0.2) (= u_1) = u_0 + k_1 = 1 - 0.03847567$$

$$u(0.2) = 0.96152433$$

Similarly for $j=1$ find k_1 and u_2

$$k_1 = k_1^{(3)} = -0.09973420$$

$$u(0.4) (= u_2) = u_1 + k_1$$

$$= 0.96152433 + (-0.0997342)$$

$$= \underline{0.86179013}$$