Ex. Solve the differential equation

$$y' = t + y, \quad y(0) = 1, \quad t \in [0, 1]$$

by Euler's method with $h = 0.1$.

Solⁿ $\quad y_{n+1} = y_n + h f(t_n, y_n)$

$h = 0.1$

for $n = 0$

$y_1 \equiv y(0.1) = y_0 + h f(0, y_0) = 1 + 0.1 \times 1$
$= 1 + 0.1 = 1.1$

$\boxed{y_1 = 1.0}$

$y_2 = y_1 + h f(t_1, y_1)$
$= 1.1 + 0.1 * (t_1 + y_1)$
$= 1.1 + 0.1 * (0.1 + 1.1)$
$= 1.1 + 0.1 * 1.2 \quad = 1.1 + 0.12$
$= 1.22$

t_n	y_n
0.1	1.1
0.2	1.22
0.3	1.362
0.4	1.5282
0.5	1.72102
0.6	1.943122
0.7	2.1943122
0.8	2.487178
0.9	2.815895
1.0	3.187485

Ex. Do the same problem with $h = 0.2, 0.3, 0.05$.

Mid point-method

Eg Solve the IVP $u' = -2xu^2$, $u(0) = 1$ using the mid point method, with $h = 0.2$ over the interval $[0,1]$. Use the Taylor series method of second order to compute $u(0.2)$. ①

Mid point-method

$f = -2xu^2$

$$u_{j+1} = u_{j-1} + 2hf_j$$

$$u_{j+1} = u_{j-1} - 4h\,x_j u_j^2 \qquad h = 0.2$$

$$u_{j+1} = u_{j-1} - 0.8\,x_j u_j^2 \quad \text{———} \; (*)$$

$$u_1 = u(x_1) = u(0+h) = u(0) + hu'(0) + \frac{h^2}{2}u''(0) \quad \text{———} \; (1)$$

$$u(0) = 1, \qquad u'(0) = f(0, u_0) = 0$$

$$u' = f(x,u)$$

$$u' = -2xu^2$$

$x_0 = 0$

$x_1 = .2$

$x_2 = .4$

$x_3 = .6$

$x_4 = .8$

$x_5 = 1.0$

$$u'' = -2u^2 - 4xu$$

$$u''(0) = -2u_0^2 - 4x\,u_0$$

$$= -2 - 4.0.\,u_0$$

$$u''(0) = -2$$

$$u_1 = .96 \quad \text{from } (1)$$

$j = 1$, $u_0 = 1$, $u_1 = .96$ $h = 0.2$

$$u_2 = u_0 - .8 \times x_1 u_1^2$$

$$= 1 - .8 \times .2 \times (.96)^2$$

$$= .85244$$

Mid point method

$j=2$

$$u(.6) = u_3 = u_1 - .8 \cdot x_2 u_2^2$$

$$= 0.96 - 0.8 \times 0.4 \times (.85244)^2$$

$$= .7274139$$

Similarly $u_4 = u(.8) = .5905611$

$u_5 = u(1.0) = 0.4901176$

Taylor Series Method

Problems on Taylor's method & Multistep methods.

(1) Consider IVP

$$y' = 2x + 3y, \quad y(0) = 1 \qquad \text{--- (1)}$$

(a) Use Taylor's series second order method to get $y(0.4)$ with step length $h = 0.1$

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' \qquad \text{--- (2)}$$

$$y_n' = 2x_n + 3y_n \qquad \text{--- (A)}$$

$$y_n'' = 2 + 6x_n + 9y_n \qquad \text{--- (B)}$$

$y' = 2x + 3y,$

$y'' = 2 + 3y'$

$= 2 + 3(2x + 3y)$

$y'' = 2 + 6x + 9y$

$h = 0.1 \qquad x_0 = 0$

from (1) $$y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0'' \qquad \text{--- (3)}$$

$y_0 = 1,$

from (A) $y_0' = 2x_0 + 3y_0 = 2 \cdot 0 + 3 \cdot 1 = 3$

from (B) $y_0'' = 2 + 6x_0 + 9y_0 = 2 + 9 = 11$

Then from (3)

$$y_1 = 1 + 0.1 \times 3 + \frac{(.1)^2}{2} \cdot 11$$

$$= 1 + .3 + \frac{.01}{2 \times 1} \times 11$$

$$= 1.3 + .5 \times 11 \times 10^{-2}$$

$$= 1.3 + 5.5 \times 10^{-2}$$

$$y_1 = 1.3 + .055 = 1.355$$

Calculate $y_2 (= 1.855475)$, $y_3 (= 2.5516138)$

$$y_4 = y(x_4) = y(.4) = 3.5109205$$

② Apply Taylor's series method of order p to the problem ②
$y' = y$, $y(0) = 1$ to show that

$$|y_n - y(x_n)| \le \frac{h^p}{(p+1)!} x_n e^{x_n}$$

<u>Sol</u>ⁿ Taylor's series method of order p

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + - \quad + \frac{h^p}{p!} y_n^{(p)}$$

$$y' = y, \qquad y'' = y' = y, \ldots \quad y^p = y$$

so

$$y_{n+1} = \left[1 + h + \frac{h^2}{2!} + \cdots + \frac{h^p}{p!}\right] y_n$$

$$= \left[1 + h + \frac{h^2}{2!} + - + \frac{h^p}{p!}\right]^2 y_{n-1}$$

$$\vdots$$

$$= \left[1 + h + \frac{h^2}{2!} + - \quad + \frac{h^p}{p!}\right]^{(n+1)} y_{n-n}$$

or $$= \left[1 + h + \frac{h^2}{2!} + - \quad + \frac{h^p}{p!}\right]^{n+1} y_0$$

or $$\boxed{y_n = \left[1 + h + \frac{h^2}{2!} + - \quad + \frac{h^p}{p!}\right]^n y_0}$$

<u>Now</u> $y' = y$, $y(0) = 1$

$$\frac{dy}{y} = dx$$

$$\ln y = x + \ln c$$

$$y = c e^x \quad \text{for } y(0) = 1 \Rightarrow c = 1$$

so $y' = y$ with $y(0) = 1$ has exact solution

$$y = e^x$$

So

So $y(x_n) = e^{x_n} = e^{nh} = (e^h)^n$

$$e^h = 1 + h + \frac{h^2}{2!} + \cdots + \frac{h^p}{p!} + \frac{h^{p+1}}{(p+1)!} e^{\theta h} \quad \text{for } 0 < \theta < 1$$

Then

$$y(x_n) - y_n = \left[1 + h + \frac{h^2}{2!} + \cdots + \frac{h^p}{p!} + \frac{h^{p+1}}{(p+1)!} e^{\theta h} \right]^n - \left[1 + h + \frac{h^2}{2!} + \cdots + \frac{h^p}{p!} \right]^n$$

Take $\quad I = 1 + h + \frac{h^2}{2!} + \cdots + \frac{h^p}{p!}$

So

$$y(x_n) - y_n = \left[I + \frac{h^{p+1}}{(p+1)!} e^{\theta h} \right]^n - I^n \quad \text{———} (*)$$

Now we use Binomial ~~ex~~pansion

$$(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \cdots$$

Then from (*)

$$y(x_n) - y_n = \cancel{I^n} + n I^{n-1} \cdot \frac{h^{p+1}}{(p+1)!} e^{\theta h} + O(h^{2(p+1)}) - \cancel{I^n}$$

or

$$y(x_n) - y_n = n I^{n-1} \frac{h^{p+1}}{(p+1)!} e^{\theta h} \quad \text{ignoring } O(h^{2p+2}) \text{ terms}$$

$$= (h.n) \frac{h^p}{(p+1)!} I^{n-1} e^{\theta h}$$

$$= x_n \frac{h^p}{(p+1)!} \left[1 + h + \frac{h^2}{2!} + \cdots + \frac{h^p}{p!} \right]^{n-1} \cdot e^{\theta h}$$

$$\leq x_n \frac{h^p}{(p+1)!} (e^h)^{n-1} \cdot e^{\theta h}$$

$$= x_n \frac{h^p}{(p+1)!} e^{nh} \cdot e^{-h} \cdot e^{\theta h}$$

$$= x_n \frac{h^p}{(p+1)!} e^{x_n} e^{(\theta-1)h}$$

$$|y(x_n) - y_n| \leq x_n \frac{h^p}{(p+1)!} e^{x_n} \cdot e^{(\theta-1)h}$$

But $0 < \theta < 1$ so $e^{(\theta-1)h} = \dfrac{1}{e^{(1-\theta)h}} \leq 1$

$$\Rightarrow \boxed{|y(x_n) - y_n| \leq x_n \frac{h^p}{(p+1)!} e^{x_n}}$$