**Fractals**

Pictures => makes more sense to talk about fractals if we define them and their properties.

Before I get to show you some of the cool properties of fractals, I figured it would be interesting to actually **build one yourself and discover some of their properties** in doing so.

We only need a piece of paper to do this experiment, and I brought a few for you. Now, looking at this sheet of paper, can someone tell me how many dimensions it has?

Great, 2! It has a surface, with length and width – we ignore the thickness because it is very small. Now what would be an example of something that has 3 dimensions? Great, any solid object would do. It has a volume, length, width and height.

Why are we interested in this concept of **dimension**? Well, what happens if we increase the side/radius of an object that is two or three-dimensional? For instance, if we double the length of a square, its surface area increases 4 times = 2 (the ratio of the new to the old length) raised to power 2 (dimension of the space the square resides in). Mass is a multiple of surface area, density times area, so mass increases 4 times too. Similarly, double the radius of a sphere – the volume increases 8 = 2^3 times - and mass is density times volume, so increases 8 times. We’re going to return to this concept of dimension very soon.

**Get to making the fractal – and measurements**

Now we’re going to make a fractal out of this sheet of paper – let’s each take our paper and scrunch it up into a ball. ***(this can be done in pairs)*** Now, we have some sort of object – I am going to refer to it as a crumpled ball. Let’s try to figure out if it is a 2-dimensional or a 3-dimensional object.

It isn’t a flat, 2-dimensional object, but it also doesn’t seem to be solid like a 3D object – for instance, it doesn’t have a volume. It turns out we can estimate the dimension of this object.

We can start by approximating the diameter of our crumpled ball – now, there are probably many ways to do this, but we only want a rough approximation, so let’s take 2 measurements of the diameter of our ball, average them and call that our diameter. Write that number down on another sheet of paper (and make sure you use the same unit of measurement!).

Now un-crumple the ball, cut it in half – I already folded it so that might help. Now the original piece of paper had a mass m, so let’s call the mass of this piece m/2. Crumple it again into a ball, and measure the diameter in the exact same way as before. Make a table on your sheet of paper with the mass and diameter of each trial. You can see that what we’re doing here is similar to our discussion of **dimension** at the beginning – we’re halving the mass to see diameter changes.

And proceed one more time to get a crumpled ball out of a quarter of the paper we started with.

**Power law relation**

It turns out that there is an interesting relationship between the mass m of the paper and the diameter of the paper ball. This is called a power law relationship, where m and D are related by:

where k is an unknown constant. This is a **power law relation**, which means that one quantity (here, m) is a multiple of some power of another quantity (here, diameter D).

**Calculations**

We are interested in the **power law relation between mass and diameter**. It turns out, we do not even need to find out the unknown k. All we are interested in is d, which would give us the dimension of our crumpled ball.

To solve this equation for d, it turns out that it is useful to use these mathematical rules called logarithms. It is not important if you’ve seen how logarithms work or not, all that matters is that by applying the log to the power law equation, we get:

Log(m) = log(k) +d\*log(D).

Another way to get an estimate of d though is the following – we have

m = k\*D1^d for the full paper ball

m/2 = k\*D2^d for the half paper ball

We can divide these to get: 2 = (D1/D2)^d, and thus d = log(2)/(log(D1/D2)). This is a first estimate. Then repeat the calculation for the equation for the half paper ball, and the one for the quarter paper ball – we get another estimate. These won’t be exact because we are estimating widths/diameters, but we get a rough estimate for d. ***- > put them on an axis on the board!***

***Order on calculators:*** First compute D1/D2. Then press log to compute the logarithm of the value. Press MS (memory store) or M+ (add it to memory). Then press C to clear the calculations. Now press 2, then log to compute log(2). Then press divide : and then MR (memory recall) to divide log(2) by the stored value of log(D1/D2). Then press equal, and get the answer needed. Then can press MC (memory clear) to clear the memory for the next calculations.

So let’s try this...pair up with the person next to you, do the calculations and then we’ll come together to see what we get.

**Conclusions - dimension**

We get d approx. 2.5, and definitely 2<d<3, so d is not a whole number. The dimension is thus not quite 2 and not quite 3. And this number will depend on the type of paper we use – if we have a stiffer type of paper, that sheet is likely to have a dimension that is closer to 2 than 3. And you can in fact test this with very different types of materials, say with a taco, slice of pizza or anything else if you so desire.

The dimension of our crumpled ball is thus not a whole number. It turns out that all objects whose dimension is **not a whole number** is a **fractal**. They are said to have a **fractal dimension**, i.e. a dimension that may not be a whole number. So, we in fact built a fractal!

This is in contrast with regular 2-dimensional and 3-dimensional objects, as we saw in the beginning, where the surface area or volume was increased 4, respectively 8 times – in other words, 2^2 or 2^3 times, where 2 and 3 are both whole numbers.

Some of you may still be puzzled by what it means to have a dimension of 2.5, or in general, a **dimension that is not a whole number**. One way to think of the fractal dimension is as measuring the [complexity](http://en.wikipedia.org/wiki/Complexity) of the object (as a ratio of the change in detail to the change in scale). We can also think of this fractal dimension as a measure of the capacity of the pattern (in our case, the crumpled ball) to fill space (that tells how a fractal scales differently from the space it is embedded in). So, a fractal fills space qualitatively and quantitatively differently from how an ordinary geometrical object does. For example, if we obtained a dimension for our ball (made from a different material) equaling 2.1, this means this ball would be filling space in a very similar way to an ordinary surface; but if the dimension of the ball were 2.9, the ball would have folds and flows that fill space very similar to a volume.

**Other conclusions**

But we can notice more. Looking at your un-crumpled piece of paper, is there anything that stands out? If you look closer, you can see some creaks and patterns of creaks that the paper formed in folding. There are some larger such patterns, and then they get smaller, with less clear in-between patterns that are a replica of the larger ones, but at a smaller scale. In other words, your fractal is **self-similar**: that is, the shapes and patterns it creates repeat themselves at smaller and smaller scales.

***Need to bring:*** calculators, rulers, papers already folded, extra papers

Could use 2 types of papers if there are many students at the talk