Homeworks for SM-I course

Important:

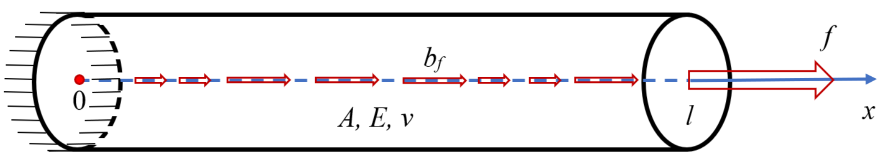
1). Please download the latest grcodes, images, and related chapters before working on the homework.

2). Both pdf files and the source codes must be submitted, or the work will not be marked.

Homework 5: Stressed bars by mechanical and thermal loads

Question 1:

Consider a uniform bar with cross-section area of A, and length l made of linear elastic material. It is subjected to a distributed body-force b_f, as shown in Fig.???. It is fixed at the left-end, meaning that at $x = 0, u = 0$. At $x = l$, it may also be subjected to a concentrated force f. The problem can be treated as one-dimensional (1D), meaning that our concern is mainly the unknown displacement, stress, and strain varying with the coordinate x.



1. Consider only the body force $b_f = q$ (N/m) where q is a constant, derive by hand the formulas for computing the displacement in the x-direction, and normal stress σ_{xx} in the bar.

2. Consider both the body force $b_f = q$ (N/m) and the concentrated force f (N), derive by hand the formulas for computing the displacement in the x-direction, and normal stress σ_{xx}. Compare the solutions with the corresponding ones obtained using the code given in the textbook.

3. Given data, $A = 250mm^2$, $l = 0.5m$, Young's modulus of the material $E = 2.1e^{10}Pa$, $b_f = 5000N/m$, $f = 1500N$, compute and plot the distributions of the displacement and stress along the coordinate x.

In [ ]:

```
# Place curse in this cell, and press Ctrl+Enter to import dependences.
import sys                          # for accessing the computer system
sys.path.append('../grbin/')  # Change to the directory in your system

from commonImports import *       # Import dependences from '../grbin/'
import grcodes as gr                 # Import the module of the author
#importlib.reload(gr)            # When grcodes is modified, reload it

from continuum_mechanics import vector
init_printing(use_unicode=True)      # For latex-like quality printing
np.set_printoptions(precision=4,suppress=True,
       formatter={'float': '{:0.4e}'.format})   # Digits in print-outs
```

1.

$u_{xx}(x) = \frac{-b_f}{EA}$

$u(0) = u_L$

$u(l) = f_R$

$u_x(l) = \frac{f_R}{EA}$

$b_f = q$

$f_R = f$

$u_x(x) = \frac{-qx}{EA} + c_0$

$u(x) = \frac{-qx^2}{2EA} + c_0 x + c_1$

$u(0) = u_L = \frac{-q0^2}{2EA} + c_0 0 + c_1$

$c_1 = u_L$

$u(x) = \frac{-qx^2}{2EA} + c_0 x + u_L$

$u_x(l) = \frac{f_R}{EA} = \frac{-ql}{EA} + c_0$

$\frac{f_R}{EA} + \frac{ql}{EA} = c_0$

$c_0 = \frac{f_R + ql}{EA}$

$u(x) = \frac{-qx^2}{2EA} + \frac{f_R + ql}{EA} x + u_L$

$\sigma_{xx}(x) = \frac{\partial u}{\partial x}$

$\sigma_{xx}(x) = \frac{-qx}{EA} + \frac{f_R + ql}{EA}$

Because there are no point forces, $f_r = 0$. Additionally, there is no displacement at u_L. Therefore;

$$u(x) = \frac{qlx}{AE} - \frac{qx^2}{2AE}$$

$$\sigma_{xx}(x) = \frac{ql}{A} - \frac{qx}{A}$$

Where $u(x)$ is the displacement of the bar, $\sigma_{xx}(x)$ is the stress, q is the body force, l is the length of the bar, A is the cross-sectional area, E is Young's Mudulus, and x is the independent variable representing the location being measured. The maximum displacement of the bar is measured at $x = l$, and the maximum stress is measured at $x = 0$.

2.

Including a point force f at the right end of the bar, the formulas derived above can be used, substituting f for f_R:

$$u(x) = \frac{fx}{AE} + \frac{qlx}{AE} - \frac{qx^2}{2AE}$$

$$\sigma_{xx}(x) = \frac{f}{A} + \frac{ql}{A} - \frac{qx}{A}$$

In [ ]:
```
E, A, l = symbols('E, A, l', nonnegative=True)    #L: Length of the bar
x, q, f = symbols('x, q, f')

u = ((f*x)/(A*E))+((q*l*x)/(A*E))-((q*(x**2))/(2*A*E)), (f/(A))+((q*l)/(A))-((q*x)/
```

In [ ]:
```
usubs=[u[i].subs({E:2.1*(10**10),A:250*(10**-6),l:0.5,q:5000,f:1500}) for i in rang
```

3.

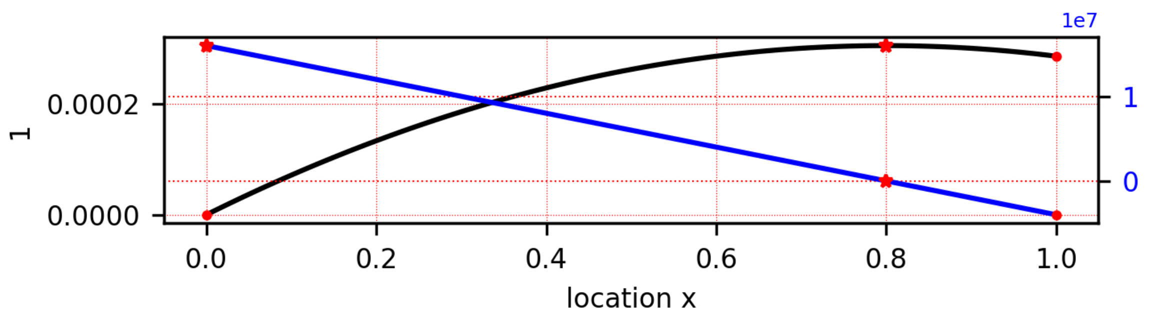
In [ ]:

```python
def plot2curveS(u, xL=0., xR=1., title="f_title"):

    '''Print out maximum values and loctions, as well as stationary
    points, and the values at the stationary points, and boundaries.
    '''
    x = symbols('x')
    dx = 0.01; dxr = dx*10       # x-step
    xi = np.arange(xL, xR+dx, dx)
    uf = sp.lambdify((x), u[0], 'numpy')   #convert Sympy f to numpy f
    yi = uf(xi)
    if type(yi) != np.ndarray:              #in case, uf is a constant
        #type(yi) == int or type(yi) == float: # or len(yi)==1:
        xi = np.arange(xL, xR+dxr, dxr)
        yi = float(yi)*np.ones_like(xi)

    fig, ax1 = plt.subplots(figsize=(5.,1.), dpi=300)
    fs = 8     # fontsize
    color = 'black'
    ax1.set_xlabel('location x', fontsize=fs)
    ax1.set_ylabel(title[0], color=color, fontsize=fs)
    ax1.plot(xi, yi, color=color)
    ax1.grid(color='r',ls=':',lw=.3, which='both') # Use both tick
    ax1.tick_params(axis='x', labelcolor=color, labelsize=fs)
    ax1.tick_params(axis='y', labelcolor=color, labelsize=fs)

    vmax = yi[yi.argmax()]
    max_l = np.argwhere(yi == vmax)
    ax1.plot(xi[max_l],  yi[max_l], 'r*', markersize=4)
    print(f'Maximum {title[0]} value={vmax:.3e}, at x={xi[max_l][0][0]}')

    uf = sp.lambdify((x), u[1], 'numpy') #convert Sympy f to numpy f
    xi = np.arange(xL, xR+dx, dx)
    yi2 = uf(xi)
    if type(yi2) != np.ndarray: # or len(yi2) == 1:
        xi = np.arange(xL, xR+dxr, dxr)
        yi2 = float(yi2)*np.ones_like(xi)

    m1, m2, m3 = np.partition(abs(yi2), 2)[0:3]
    msl=[np.where(abs(yi2)==m1)[0][0],np.where(abs(yi2)==m2)[0][0],
         np.where(abs(yi2)==m3)[0][0]]

    vmax = yi2[yi2.argmax()]
    max_l = np.argwhere(yi2 == vmax)
    print(f'Maximum {title[1]} value={vmax:.3e}, at x={xi[max_l][0][0]}')

    if abs(xi[msl[2]]-xi[msl[1]])<2*dx:
        if abs(yi2[msl[2]]-0.)<abs(yi2[msl[1]]-0.): msl.pop(1)
        else: msl.pop(2)
    if len(msl) > 2:
        if abs(xi[msl[2]]-xi[msl[0]])<2*dx:
            if abs(yi2[msl[2]]-0.)<abs(yi2[msl[0]]-0.): msl.pop(0)
            else: msl.pop(2)
    if len(msl) > 1:
        if abs(xi[msl[1]]-xi[msl[0]])<2*dx:
            if abs(yi2[msl[1]]-0.)<abs(yi2[msl[0]]-0.): msl.pop(0)
            else: msl.pop(1)
```

```
    ax2 = ax1.twinx()  # instantiate second axes sharing the same x-axis

    color = 'blue'
    ax2.set_ylabel(title[1], color=color, fontsize=fs)
    ax2.plot(xi, yi2, color=color)
    ax2.plot(xi[max_l], yi2[max_l], 'r*', markersize=4)
    ax2.plot(xi[msl], yi2[msl], 'r*', markersize=4)
    ax1.plot(xi[msl],  yi[msl], 'r*', markersize=4)
    ax2.plot(xi[0], yi2[0], 'ro', markersize=2)
    ax1.plot(xi[0],  yi[0], 'ro', markersize=2)
    ax2.plot(xi[-1], yi2[-1], 'ro', markersize=2)
    ax1.plot(xi[-1],  yi[-1], 'ro', markersize=2)
    ax2.grid(color='r',ls=':',lw=.5, which='both') # Use both tick
    ax2.tick_params(axis='x', labelcolor=color, labelsize=fs)
    ax2.tick_params(axis='y', labelcolor=color, labelsize=fs)
    np.set_printoptions(formatter={'float': '{: 0.3e}'.format})
    print(f'Extreme {title[0]} values={yi[msl]},\n    at x={xi[msl]}')
    print(f'Critical {title[1]} values={yi2[msl]},\n    at x={xi[msl]}')
    print(f'{title[0]} values at boundary ={yi[0], yi[-1]}')
    print(f'{title[1]} values at boundary ={yi2[0], yi2[-1]}\n')
```

In [ ]:
```
plot2curveS(usubs, title="1.3")           # ux: solution from method 1
#plt.savefig('images/beam_cq.png', dpi=500)  # save the plot to a file
plt.show()
```

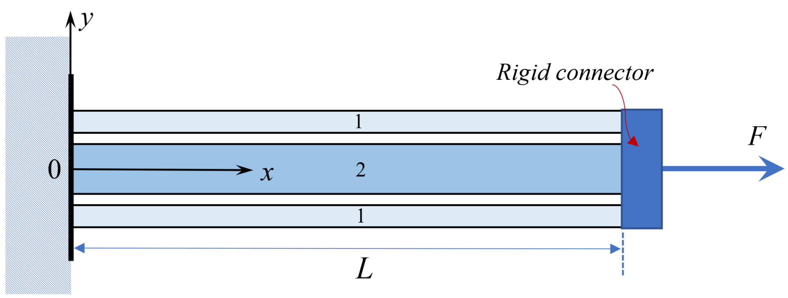
```
Maximum 1 value=3.048e-04, at x=0.8
Maximum . value=1.600e+07, at x=0.0
Extreme 1 values=[ 3.048e-04],
    at x=[ 8.000e-01]
Critical . values=[ 0.000e+00],
    at x=[ 8.000e-01]
1 values at boundary =(0.0, 0.00028571428571428606)
. values at boundary =(16000000.0, -4000000.0)
```



Question 2:

Consider a bar of length L consisting of three thin bars of homogeneous isotropic materials, as shown in Fig.???. The bars are all fixed on a wall at the left-end, and are rigidly connected at the right-ends. The bars on the top and bottom (bar-1) are identical with Young's modulus E_1, thermal expansion coefficient of α_1, and each with area A_1. The bar in the middle (bar-2) is with Young's modulus E_2, area A_2, and thermal expansion coefficient of α_2. Considering only the displacement in the x-direction, and complete the following tasks.

The data are given as $E_1 = 200GPa$, $E_2 = 70GPa$, $A_1 = 5mm^2$, $A_2 = 8mm^2$ and $L = 100mm$, $\alpha_1 = 11.0E^{-6}/^{\circ}C$, $\alpha_2 = 22.0E^{-6}/^{\circ}C$, $\Delta T = 300^{\circ}C$, and $F = 50N$.



1. Consider only external force F applied at the right-end of the bar, derive the formulas for computing the normal stresses and the internal normal forces in each of the three bars, and the elongation of the bar.

2. Consider only temperature change ΔT over the entire bars, derive the formulas for computing the normal stresses and the internal normal forces in each of the three bars, and the elongation of the bar.

3. Using the data given, compute numerical values for normal stresses, internal normal forces in each of the three bars, and the elongation of the bar, when the bar is subjected to both external force and temperature changes.

4. Compute the principal stresses and the maximum stress and their directions in bar-1.

1.

```
In [ ]: E, A, l = symbols('E, A, l', nonnegative=True)   #L: length of the bar
        x, q, bf, c0, c1 = symbols('x, q, b_f, c0, c1')

        fL, fR, uL,uR = symbols('f_L, f_R, u_L, u_R')
        # fL: force at the Left-end; fR: force at the Right-end
        # uL: displacement at the Light-end; uR: displacement at the Right-end

        bf = sp.Function('b_f')(x)                          # 1D body-force (N/m)
        title = ["Displacement", "Stress"]

        # Consider the distrubted force is a constant q
        bf = q #0 #(2*sp.sin(x)+8)/E         # one may try other force funtions
        u = solver1D2(E, A, bf, l, uL, fR, key='force')

        # chech whether the DE is satisfied.
        print(f'Is solution correct? {u[2] == -bf/E/A}')
        u                        # solutions: u; E*u,x; u,xx
```

```
output from solver1D2(): u, σ, u_xx
Is solution correct? True
```

Out[ ]:
$$\left(u_L + \frac{f_R x}{AE} + \frac{lqx}{AE} - \frac{qx^2}{2AE}, \frac{f_R}{A} + \frac{lq}{A} - \frac{qx}{A}, -\frac{q}{AE}\right)$$

In order to use this formula to compute the individual members stresses, first a formula to calculate the total stress is needed:
$\sigma_{xx-t}(x) = \frac{f_R}{A} + \frac{lq}{A} - \frac{qx}{A}$

since there isnt an internal body force:
$\sigma_{xx-t}(x) = f_R(2\frac{1}{A_1} + \frac{1}{A_2})$

Using this, the stress in each bar is equal to the total point force over each bars cross sectional area:
$\sigma_{xx-1}(x) = \frac{f_R}{A_1}$
$\sigma_{xx-2}(x) = \frac{f_R}{A_2}$

Since the bars are rigidly linked at each end, the deformation of each bar is equal, meaning only one formula is needed:
$u(x) = u_L + \frac{f_R x}{AE}$

$u_L = 0$, so:
$u(x) = (f_R x)(2\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2})$

2.

$$\frac{\partial u_T}{\partial x} = \epsilon_{T_{xx}} = \alpha \Delta T$$
$$u_T = \alpha \Delta T x$$
$$u_T = (2\alpha_1 \alpha_2 \Delta T)x$$
$$\sigma_{T_{xx}} = E[\frac{du}{dx} - \alpha \Delta T]$$
$$\sigma_{T_{xx-1}} = E_1[(2\alpha_1 \alpha_2 \Delta T) - \alpha_1 \Delta T]$$
$$\sigma_{T_{xx-2}} = E_2[(2\alpha_1 \alpha_2 \Delta T) - \alpha_2 \Delta T]$$

3.

Total displacement is the sum between displacement from the point force and displacement due to thermal forces. Therefore, total displacement:
$$u(x) = (f_R x)(2\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}) + (2\alpha_1 \alpha_2 \Delta T)x$$

Total thermal stresses of each bar is a function of total displacement, so the thermal stress formulas are updated to the following:
$$\sigma_{T_{xx}} = E[f_R(2\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}) + (2\alpha_1 \alpha_2 \Delta T) - \alpha \Delta T]$$
$$\sigma_{T_{xx-1}} = E_1[f_R(2\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}) + (2\alpha_1 \alpha_2 \Delta T) - \alpha_1 \Delta T]$$
$$\sigma_{T_{xx-2}} = E_2[f_R(2\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}) + (2\alpha_1 \alpha_2 \Delta T) - \alpha_2 \Delta T]$$

The total stress of each bar is the sum of the stress due to the point force and due to thermal displacement, resulting in the following stress formulas:
$$\sigma_{xx-1} = \frac{f_R}{A_1} + E_1[f_R(2\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}) + (2\alpha_1 \alpha_2 \Delta T) - \alpha_1 \Delta T]$$
$$\sigma_{xx-2} = \frac{f_R}{A_2} + E_2[f_R(2\frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}) + (2\alpha_1 \alpha_2 \Delta T) - \alpha_2 \Delta T]$$

In [ ]:
```
E1, E2, A1, A2, a1, a2 = symbols('E1, E2, A1, A2, a1, a2', nonnegative=True)
DT = symbols('DT')

u = (fR*x)*(2*(1/(A1*E1))+(1/(A2*E2)))+(a1*a2*DT)*x

s1 = (fR/A1)+E1*(diff(u,x)-(a1*DT))
s2 = (fR/A2)+E2*(diff(u,x)-(a2*DT))
s1
```

Out[ ]:
$$E_1 \left(DT a_1 a_2 - DT a_1 + f_R \left(\frac{1}{A_2 E_2} + \frac{2}{A_1 E_1} \right) \right) + \frac{f_R}{A_1}$$

In [ ]:
```
usol=u.subs({E1:200*(10**9),E2:70*(10**9),A1:5*(10**-6),A2:8*(1**-6),l:100*(10**-3)
usol
```

Out[ ]:
$$0.000100072689285714x$$

In [ ]:
```
displacement = usol.subs({x:0.5})
print("Total displacement = "+str(displacement))
```

```
Total displacement = 5.00363446428571e-5
```

In [ ]:
```python
s1sol = s1.subs({E1:200*(10**9),E2:70*(10**9),A1:5*(10**-6),A2:8*(1**-6),l:100*(10*
print("Stresses in bar 1 = "+str(s1sol))
```

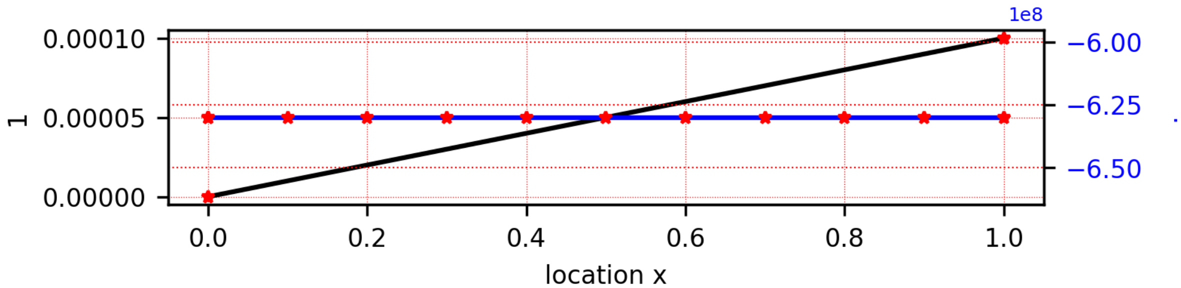
```
Stresses in bar 1 = -629985462.142857
```

In [ ]:
```python
s2sol = s2.subs({E1:200*(10**9),E2:70*(10**9),A1:5*(10**-6),A2:8*(1**-6),l:100*(10*
print("Stresses in each bar 2 = "+str(0.5*s2sol))
```

```
Stresses in each bar 2 = -227497452.750000
```

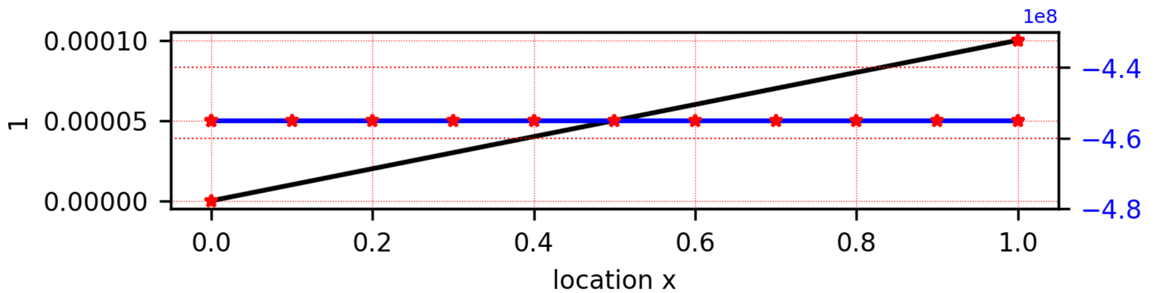
In [ ]:
```python
plot2curveS([usol,s1sol], title="1.3")          # ux: solution from method 1
#plt.savefig('images/beam_cq.png', dpi=500)  # save the plot to a file
plt.show()
```

```
Maximum 1 value=1.001e-04, at x=1.0
Maximum . value=-6.300e+08, at x=0.0
Extreme 1 values=[ 0.000e+00],
    at x=[ 0.000e+00]
Critical . values=[-6.300e+08],
    at x=[ 0.000e+00]
1 values at boundary =(0.0, 0.000100072689285714)
. values at boundary =(-629985462.142857, -629985462.142857)
```



In [ ]:
```python
plot2curveS([usol,s2sol], title="1.3")          # ux: solution from method 1
#plt.savefig('images/beam_cq.png', dpi=500)  # save the plot to a file
plt.show()
```

```
Maximum 1 value=1.001e-04, at x=1.0
Maximum . value=-4.550e+08, at x=0.0
Extreme 1 values=[ 0.000e+00],
    at x=[ 0.000e+00]
Critical . values=[-4.550e+08],
    at x=[ 0.000e+00]
1 values at boundary =(0.0, 0.000100072689285714)
. values at boundary =(-454994905.5, -454994905.5)
```



4.

```
In [ ]: def principalS(S):
            '''Compute the principal stresses and their direction cosines.
            inputs:
               S: given stress tensor, numpy array
            return:
               principal stresses (eigenValues), their direction cosines
               (eigenVectors) ranked by its values. Right-hand-rule is enforced
            '''
            eigenValues, eigenVectors = lg.eig(S)

            #Sort in order
            idx = eigenValues.argsort()[::-1]
            eigenValues = eigenValues[idx]
            eigenVectors = eigenVectors[:,idx]
            print('Pricipal stress (Eigenvalues):\n',eigenValues,'\n')

            # make the first element in the first vector positive (optional):
            #eigenVectors[0,:] = eigenVectors[0,:]/np.sign(eigenVectors[0,0])

            # Determine the sign for given eigenVector-1 and eigenVector-3
            eigenVectors[:,2] = np.cross(eigenVectors[:,0], eigenVectors[:,1])

            angle = np.arccos(eigenVectors[0,0])*180/np.pi       # in degree
            print(f'Principal stress directions:\n{eigenVectors}\n')
            print(f"Possible angles (n1,x)={angle}∘ or {180-angle}∘")

            return eigenValues, eigenVectors
```

```
In [ ]: S1 = np.array([[float(s1sol),0,0],[0,0,0],[0,0,0]])
        S2 = np.array([[float(0.5*s2sol),0,0],[0,0,0],[0,0,0]])

        eigenValues1, eigenVectors1 = principalS(S1)
        eigenValues2, eigenVectors2 = principalS(S2)
```

```
Pricipal stress (Eigenvalues):
 [ 0.000e+00  0.000e+00 -6.300e+08]

Principal stress directions:
[[ 0.000e+00  0.000e+00 -1.000e+00]
 [ 0.000e+00  1.000e+00  0.000e+00]
 [ 1.000e+00  0.000e+00  0.000e+00]]

Possible angles (n1,x)=90.0∘ or 90.0∘
Pricipal stress (Eigenvalues):
 [ 0.000e+00  0.000e+00 -2.275e+08]

Principal stress directions:
[[ 0.000e+00  0.000e+00 -1.000e+00]
 [ 0.000e+00  1.000e+00  0.000e+00]
 [ 1.000e+00  0.000e+00  0.000e+00]]

Possible angles (n1,x)=90.0∘ or 90.0∘
```

In [ ]:

```python
σ11, σ12, σ22, θ = symbols('σ_11, σ_12, σ_22, θ', real = True)
S2D = Matrix([[σ11, σ12], [σ12, σ22]])       # form a 2D stress tensor

T2D = gr.transf_YPRs(θ, about = 'z')[0][:2,:2]    # get T for 2D cases
S_θ = sp.simplify(T2D@S2D@T2D.T)        # perform stress transformation

# Find the maximum of the shear stress in S_θ:
τ_diff_θ = S_θ[0,1].diff(θ)                      # get the derivative it
θ_τmax = Matrix(sp.solve(τ_diff_θ, θ))        # set is to 0, and find θ

anglesmax1 = θ_τmax.subs({σ11:s1sol,σ12:0,σ22:0})

print("Angle of maximum shear stress in beam 1 = %3.2f°" % (anglesmax1[0]*57.29578)

anglesmax2 = θ_τmax.subs({σ11:0.5*s2sol,σ12:0,σ22:0})

print("Angle of maximum shear stress in one of beam 2 = %3.2f°" % (anglesmax2[0]*57
```

```
Angle of maximum shear stress in beam 1 = 45.00°
Angle of maximum shear stress in one of beam 2 = 45.00°
```

In [ ]:

```python
sr  = symbols("s_r", positive = True)
sr_ = θ_τmax[0].args[0].args[1].args[0].args[1]

τ_max = S_θ[0,1].subs(θ, θ_τmax[0]).simplify()
τ_max = τ_max.subs(sr**2, sr_**2).simplify().subs(sr_, sr)\
             .subs(sr_**2, sr**2).simplify().subs(sr, sr_)

τ_max = τ_max.subs(σ11**2-2*σ11*σ22+σ22**2,  (σ11-σ22)**2).simplify()

maxstress1 = τ_max.subs({σ11:s1sol,σ12:0,σ22:0})

print("Maximum shear stress in beam 1 = %3.2f Pa" % (maxstress1))

maxstress2 = τ_max.subs({σ11:0.5*s2sol,σ12:0,σ22:0})

print("Maximum shear stress in beam 1 = %3.2f Pa" % (maxstress2))
```

```
Maximum shear stress in beam 1 = 314992731.07 Pa
Maximum shear stress in beam 1 = 113748726.38 Pa
```